LOCAL COHOMOLOGY

Contents

1. Introduction

This chapter continues the study of local cohomology. A reference is [Gro68]. The definition of local cohomology can be found in Dualizing Complexes, Section 9. For Noetherian rings taking local cohomology is the same as deriving a suitable torsion functor as is shown in Dualizing Complexes, Section 10. The relationship with depth can be found in Dualizing Complexes, Section 11.

We discuss finiteness properties of local cohomology leading to a proof of a fairly general version of Grothendieck's finiteness theorem, see Theorem 11.6 and Lemma 12.1 (higher direct images of coherent modules under open immersions). Our methods incorporate a few very slick arguments the reader can find in papers of Faltings, see [Fal78] and [Fal81].

As applications we offer a discussion of Hartshorne-Lichtenbaum vanishing. We also discuss the action of Frobenius and of differential operators on local cohomology.

2. Generalities

The following lemma tells us that the functor $R\Gamma_Z$ is related to cohomology with supports.

Lemma 2.1. *Let A be a ring and let I be a finitely generated ideal. Set $Z = V(I) \subset X = \mathrm{Spec}(A)$. For $K \in D(A)$ corresponding to $\widetilde{K} \in D_{QCoh}(\mathcal{O}_X)$ via Derived Categories of Schemes, Lemma 3.5 there is a functorial isomorphism*

$$R\Gamma_Z(K) = R\Gamma_Z(X, \widetilde{K})$$

where on the left we have Dualizing Complexes, Equation (9.0.1) and on the right we have the functor of Cohomology, Section 34.

Proof. By Cohomology, Lemma 34.5 there exists a distinguished triangle

$$R\Gamma_Z(X, \widetilde{K}) \to R\Gamma(X, \widetilde{K}) \to R\Gamma(U, \widetilde{K}) \to R\Gamma_Z(X, \widetilde{K})[1]$$

where $U = X \setminus Z$. We know that $R\Gamma(X, \widetilde{K}) = K$ by Derived Categories of Schemes, Lemma 3.5. Say $I = (f_1, \ldots, f_r)$. Then we obtain a finite affine open covering $\mathcal{U} : U = D(f_1) \cup \ldots \cup D(f_r)$. By Derived Categories of Schemes, Lemma 9.4 the alternating Čech complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \widetilde{K^\bullet}))$ computes $R\Gamma(U, \widetilde{K})$ where $K^\bullet$ is any complex of A-modules representing K. Working through the definitions we find

$$R\Gamma(U, \widetilde{K}) = \mathrm{Tot}\left(K^\bullet \otimes_A (\prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r})\right)$$

It is clear that $K^\bullet = R\Gamma(X, \widetilde{K^\bullet}) \to R\Gamma(U, \widetilde{K^\bullet})$ is induced by the diagonal map from A into $\prod A_{f_i}$. Hence we conclude that

$$R\Gamma_Z(X, \mathcal{F}^\bullet) = \mathrm{Tot}\left(K^\bullet \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r})\right)$$

By Dualizing Complexes, Lemma 9.1 this complex computes $R\Gamma_Z(K)$ and we see the lemma holds. $\square$

Lemma 2.2. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $X = \mathrm{Spec}(A)$, $Z = V(I)$, $U = X \setminus Z$, and $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. Then*

1. *there exists an A-module M such that $\mathcal{F}$ is the restriction of $\widetilde{M}$ to U,*
2. *given M there is an exact sequence*

$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$

   and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$,

3. *we may take $M = H^0(U, \mathcal{F})$ in which case we have $H^0_Z(M) = H^1_Z(M) = 0$.*

Proof. The existence of M follows from Properties, Lemma 22.1 and the fact that quasi-coherent sheaves on X correspond to A-modules (Schemes, Lemma 7.5). Then we look at the distinguished triangle

$$R\Gamma_Z(X, \widetilde{M}) \to R\Gamma(X, \widetilde{M}) \to R\Gamma(U, \widetilde{M}|_U) \to R\Gamma_Z(X, \widetilde{M})[1]$$

of Cohomology, Lemma 34.5. Since X is affine we have $R\Gamma(X, \widetilde{M}) = M$ by Cohomology of Schemes, Lemma 2.2. By our choice of M we have $\mathcal{F} = \widetilde{M}|_U$ and hence this produces an exact sequence

$$0 \to H^0_Z(X, \widetilde{M}) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(X, \widetilde{M}) \to 0$$

and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(X, \widetilde{M})$ for $p \geq 1$. By Lemma 2.1 we have $H^i_Z(M) = H^i_Z(X, \widetilde{M})$ for all i. Thus (1) and (2) do hold. Finally, setting $M' = H^0(U, \mathcal{F})$ we see that the kernel and cokernel of $M \to M'$ are I-power torsion. Therefore $\widetilde{M}|_U \to \widetilde{M'}|_U$ is an isomorphism and we can indeed use M' as predicted in (3). It goes without saying that we obtain zero for both $H^0_Z(M')$ and $H^0_Z(M')$. $\square$

Lemma 2.3. *Let $I, J \subset A$ be finitely generated ideals of a ring A. If M is an I-power torsion module, then the canonical map*

$$H^i_{V(I) \cap V(J)}(M) \to H^i_{V(J)}(M)$$

is an isomorphism for all i.

Proof. Use the spectral sequence of Dualizing Complexes, Lemma 9.6 to reduce to the statement $R\Gamma_I(M) = M$ which is immediate from the construction of local cohomology in Dualizing Complexes, Section 9. $\square$

Lemma 2.4. *Let $S \subset A$ be a multiplicative set of a ring A. Let M be an A-module with $S^{-1}M = 0$. Then $\text{colim}_{f \in S} H^0_{V(f)}(M) = M$ and $\text{colim}_{f \in S} H^1_{V(f)}(M) = 0$.*

Proof. The statement on H^0 follows directly from the definitions. To see the statement on H^1 observe that $R\Gamma_{V(f)}$ and $H^1_{V(f)}$ commute with colimits. Hence we may assume M is annihilated by some $f \in S$. Then $H^1_{V(ff')}(M) = 0$ for all $f' \in S$ (for example by Lemma 2.3). $\square$

Lemma 2.5. *Let $I \subset A$ be a finitely generated ideal of a ring A. Let $\mathfrak{p}$ be a prime ideal. Let M be an A-module. Let $i \geq 0$ be an integer and consider the map*

$$\Psi : \text{colim}_{f \in A, f \not\in \mathfrak{p}} H^i_{V((I,f))}(M) \longrightarrow H^i_{V(I)}(M)$$

Then

1. $\text{Im}(\Psi)$ *is the set of elements which map to zero in $H^i_{V(I)}(M)_\mathfrak{p}$,*
2. *if $H^{i-1}_{V(I)}(M)_\mathfrak{p} = 0$, then Ψ is injective,*
3. *if $H^{i-1}_{V(I)}(M)_\mathfrak{p} = H^i_{V(I)}(M)_\mathfrak{p} = 0$, then Ψ is an isomorphism.*

Proof. For $f \in A$, $f \not\in \mathfrak{p}$ the spectral sequence of Dualizing Complexes, Lemma 9.6 degenerates to give short exact sequences

$$0 \to H^1_{V(f)}(H^{i-1}_{V(I)}(M)) \to H^i_{V((I,f))}(M) \to H^0_{V(f)}(H^i_{V(I)}(M)) \to 0$$

This proves (1) and part (2) follows from this and Lemma 2.4. Part (3) is a formal consequence. $\square$

Lemma 2.6. *Let $I \subset I' \subset A$ be finitely generated ideals of a Noetherian ring A. Let M be an A-module. Let $i \geq 0$ be an integer. Consider the map*

$$\Psi : H^i_{V(I')}(M) \to H^i_{V(I)}(M)$$

The following are true:

1. *if $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in V(I) \setminus V(I')$, then Ψ is surjective,*

(2) *if* $H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in V(I) \setminus V(I')$*, then* Ψ *is injective,*
(3) *if* $H^{i}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in V(I) \setminus V(I')$*, then* Ψ *is an isomorphism.*

Proof. Proof of (1). Let $\xi \in H^i_{V(I)}(M)$. Since A is Noetherian, there exists a largest ideal $I \subset I'' \subset I'$ such that ξ is the image of some $\xi'' \in H^i_{V(I'')}(M)$. If $V(I'') = V(I')$, then we are done. If not, choose a generic point $\mathfrak{p} \in V(I'')$ not in $V(I')$. Then we have $H^i_{V(I'')}(M)_\mathfrak{p} = H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by assumption. By Lemma 2.5 we can increase I'' which contradicts maximality.

Proof of (2). Let $\xi' \in H^i_{V(I')}(M)$ be in the kernel of Ψ. Since A is Noetherian, there exists a largest ideal $I \subset I'' \subset I'$ such that ξ' maps to zero in $H^i_{V(I'')}(M)$. If $V(I'') = V(I')$, then we are done. If not, then choose a generic point $\mathfrak{p} \in V(I'')$ not in $V(I')$. Then we have $H^{i-1}_{V(I'')}(M)_\mathfrak{p} = H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by assumption. By Lemma 2.5 we can increase I'' which contradicts maximality.

Part (3) is formal from parts (1) and (2). □

3. Hartshorne's connectedness lemma

The title of this section refers to the following result.

Lemma 3.1. *Let A be a Noetherian local ring of depth ≥ 2. Then the punctured spectra of A, A^h, and A^{sh} are connected.*

Proof. Let U be the punctured spectrum of A. If U is disconnected then we see that $\Gamma(U, \mathcal{O}_U)$ has a nontrivial idempotent. But A, being local, does not have a nontrivial idempotent. Hence $A \to \Gamma(U, \mathcal{O}_U)$ is not an isomorphism. By Lemma 2.2 we conclude that either $H^0_\mathfrak{m}(A)$ or $H^1_\mathfrak{m}(A)$ is nonzero. Thus $\text{depth}(A) \leq 1$ by Dualizing Complexes, Lemma 11.1. To see the result for A^h and A^{sh} use More on Algebra, Lemma 45.8. □

Lemma 3.2. *Let A be a Noetherian local ring which is catenary and (S_2). Then* $\text{Spec}(A)$ *is equidimensional.*

Proof. Set $X = \text{Spec}(A)$. Say $d = \dim(A) = \dim(X)$. Inside X consider the union X_1 of the irreducible components of dimension d and the union X_2 of the irreducible components of dimension $< d$. Of course $X = X_1 \cup X_2$. If $X_2 = \emptyset$, then the lemma holds. If not, then $Z = X_1 \cap X_2$ is a nonempty closed subset of X because it contains at least the closed point of X. Hence we can choose a generic point $z \in Z$ of an irreducible component of Z. Recall that the spectrum of $\mathcal{O}_{Z,z}$ is the set of points of X specializing to z. Since z is both contained in an irreducible component of dimension d and in an irreducible component of dimension $< d$ we obtain nontrivial specializations $x_1 \rightsquigarrow z$ and $x_2 \rightsquigarrow z$ such that the closures of x_1 and x_2 have different dimensions. Since X is catenary, this can only happen if at least one of the specializations $x_1 \rightsquigarrow z$ and $x_2 \rightsquigarrow z$ is not immediate! Thus $\dim(\mathcal{O}_{Z,z}) \geq 2$. Therefore $\text{depth}(\mathcal{O}_{Z,z}) \geq 2$ because A is (S_2). However, the punctured spectrum U of $\mathcal{O}_{Z,z}$ is disconnected because the closed subsets $U \cap X_1$ and $U \cap X_2$ are disjoint (by our choice of z) and cover U. This is a contradiction with Lemma 3.1 and the proof is complete. □

4. Cohomological dimension

A quick section about cohomological dimension.

Lemma 4.1. *Let $I \subset A$ be a finitely generated ideal of a ring A. Set $Y = V(I) \subset X = \mathrm{Spec}(A)$. Let $d \geq -1$ be an integer. The following are equivalent*

(1) *$H^i_Y(A) = 0$ for $i > d$,*
(2) *$H^i_Y(M) = 0$ for $i > d$ for every A-module M, and*
(3) *if $d = -1$, then $Y = \emptyset$, if $d = 0$, then Y is open and closed in X, and if $d > 0$ then $H^i(X \setminus Y, \mathcal{F}) = 0$ for $i \geq d$ for every quasi-coherent $\mathcal{O}_{X \setminus Y}$-module $\mathcal{F}$.*

Proof. Observe that $R\Gamma_Y(-)$ has finite cohomological dimension by Dualizing Complexes, Lemma 9.1 for example. Hence there exists an integer i_0 such that $H^i_Y(M) = 0$ for all A-modules M and $i \geq i_0$.

Let us prove that (1) and (2) are equivalent. It is immediate that (2) implies (1). Assume (1). By descending induction on $i > d$ we will show that $H^i_Y(M) = 0$ for all A-modules M. For $i \geq i_0$ we have seen this above. To do the induction step, let $i_0 > i > d$. Choose any A-module M and fit it into a short exact sequence $0 \to N \to F \to M \to 0$ where F is a free A-module. Since $R\Gamma_Y$ is a right adjoint, we see that $H^i_Y(-)$ commutes with direct sums. Hence $H^i_Y(F) = 0$ as $i > d$ by assumption (1). Then we see that $H^i_Y(M) = H^{i+1}_Y(N) = 0$ as desired.

Assume $d = -1$ and (2) holds. Then $0 = H^0_Y(A/I) = A/I \Rightarrow A = I \Rightarrow Y = \emptyset$. Thus (3) holds. We omit the proof of the converse.

Assume $d = 0$ and (2) holds. Set $J = H^0_I(A) = \{x \in A \mid I^n x = 0 \text{ for some } n > 0\}$. Then

$$H^1_Y(A) = \mathrm{Coker}(A \to \Gamma(X \setminus Y, \mathcal{O}_{X \setminus Y})) \quad \text{and} \quad H^1_Y(I) = \mathrm{Coker}(I \to \Gamma(X \setminus Y, \mathcal{O}_{X \setminus Y}))$$

and the kernel of the first map is equal to J. See Lemma 2.2. We conclude from (2) that $I(A/J) = A/J$. Thus we may pick $f \in I$ mapping to 1 in A/J. Then $1 - f \in J$ so $I^n(1 - f) = 0$ for some $n > 0$. Hence $f^n = f^{n+1}$. Then $e = f^n \in I$ is an idempotent. Consider the complementary idempotent $e' = 1 - f^n \in J$. For any element $g \in I$ we have $g^m e' = 0$ for some $m > 0$. Thus I is contained in the radical of ideal $(e) \subset I$. This means $Y = V(I) = V(e)$ is open and closed in X as predicted in (3). Conversely, if $Y = V(I)$ is open and closed, then the functor $H^0_Y(-)$ is exact and has vanshing higher derived functors.

If $d > 0$, then we see immediately from Lemma 2.2 that (2) is equivalent to (3). $\square$

Definition 4.2. Let $I \subset A$ be a finitely generated ideal of a ring A. The smallest integer $d \geq -1$ satisfying the equivalent conditions of Lemma 4.1 is called the *cohomological dimension of I in A* and is denoted $\mathrm{cd}(A, I)$.

Thus we have $\mathrm{cd}(A, I) = -1$ if $I = A$ and $\mathrm{cd}(A, I) = 0$ if I is locally nilpotent or generated by an idempotent. Observe that $\mathrm{cd}(A, I)$ exists by the following lemma.

Lemma 4.3. *Let $I \subset A$ be a finitely generated ideal of a ring A. Then*

(1) *$cd(A, I)$ is at most equal to the number of generators of I,*
(2) *$cd(A, I) \leq r$ if there exist $f_1, \ldots, f_r \in A$ such that $V(f_1, \ldots, f_r) = V(I)$,*
(3) *$cd(A, I) \leq c$ if $\mathrm{Spec}(A) \setminus V(I)$ can be covered by c affine opens.*

Proof. The explicit description for $R\Gamma_Y(-)$ given in Dualizing Complexes, Lemma 9.1 shows that (1) is true. We can deduce (2) from (1) using the fact that $R\Gamma_Z$ depends only on the closed subset Z and not on the choice of the finitely generated ideal $I \subset A$ with $V(I) = Z$. This follows either from the construction of local cohomology in Dualizing Complexes, Section 9 combined with More on Algebra, Lemma 88.6 or it follows from Lemma 2.1. To see (3) we use Lemma 4.1 and the vanishing result of Cohomology of Schemes, Lemma 4.2. $\square$

Lemma 4.4. *Let $I, J \subset A$ be finitely generated ideals of a ring A. Then $cd(A, I + J) \leq cd(A, I) + cd(A, J)$.*

Proof. Use the definition and Dualizing Complexes, Lemma 9.6. $\square$

Lemma 4.5. *Let $A \to B$ be a ring map. Let $I \subset A$ be a finitely generated ideal. Then $cd(B, IB) \leq cd(A, I)$. If $A \to B$ is faithfully flat, then equality holds.*

Proof. Use the definition and Dualizing Complexes, Lemma 9.3. $\square$

Lemma 4.6. *Let $I \subset A$ be a finitely generated ideal of a ring A. Then $cd(A, I) = \max cd(A_\mathfrak{p}, I_\mathfrak{p})$.*

Proof. Let $Y = V(I)$ and $Y' = V(I_\mathfrak{p}) \subset \text{Spec}(A_\mathfrak{p})$. Recall that $R\Gamma_Y(A) \otimes_A A_\mathfrak{p} = R\Gamma_{Y'}(A_\mathfrak{p})$ by Dualizing Complexes, Lemma 9.3. Thus we conclude by Algebra, Lemma 23.1. $\square$

Lemma 4.7. *Let $I \subset A$ be a finitely generated ideal of a ring A. If M is a finite A-module, then $H^i_{V(I)}(M) = 0$ for $i > \dim(\text{Supp}(M))$. In particular, we have $cd(A, I) \leq \dim(A)$.*

Proof. We first prove the second statement. Recall that $\dim(A)$ denotes the Krull dimension. By Lemma 4.6 we may assume A is local. If $V(I) = \emptyset$, then the result is true. If $V(I) \neq \emptyset$, then $\dim(\text{Spec}(A) \setminus V(I)) < \dim(A)$ because the closed point is missing. Observe that $U = \text{Spec}(A) \setminus V(I)$ is a quasi-compact open of the spectral space $\text{Spec}(A)$, hence a spectral space itself. See Algebra, Lemma 26.2 and Topology, Lemma 23.5. Thus Cohomology, Proposition 22.4 implies $H^i(U, \mathcal{F}) = 0$ for $i \geq \dim(A)$ which implies what we want by Lemma 4.1. In the Noetherian case the reader may use Grothendieck's Cohomology, Proposition 20.7.

We will deduce the first statement from the second. Let $\mathfrak{a}$ be the annihilator of the finite A-module M. Set $B = A/\mathfrak{a}$. Recall that $\text{Spec}(B) = \text{Supp}(M)$, see Algebra, Lemma 40.5. Set $J = IB$. Then M is a B-module and $H^i_{V(I)}(M) = H^i_{V(J)}(M)$, see Dualizing Complexes, Lemma 9.2. Since $\text{cd}(B, J) \leq \dim(B) = \dim(\text{Supp}(M))$ by the first part we conclude. $\square$

Lemma 4.8. *Let $I \subset A$ be a finitely generated ideal of a ring A. If $cd(A, I) = 1$ then $\text{Spec}(A) \setminus V(I)$ is nonempty affine.*

Proof. This follows from Lemma 4.1 and Cohomology of Schemes, Lemma 3.1. $\square$

Lemma 4.9. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Then $H^d_\mathfrak{m}(A)$ is nonzero and $cd(A, \mathfrak{m}) = d$.*

Proof. By one of the characterizations of dimension, there exists an ideal of definition for A generated by d elements, see Algebra, Proposition 60.9. Hence $\text{cd}(A, \mathfrak{m}) \leq$

d by Lemma 4.3. Thus $H^d_{\mathfrak m}(A)$ is nonzero if and only if $\text{cd}(A, \mathfrak m) = d$ if and only if $\text{cd}(A, \mathfrak m) \geq d$.

Let $A \to A^\wedge$ be the map from A to its completion. Observe that $A^\wedge$ is a Noetherian local ring of the same dimension as A with maximal ideal $\mathfrak m A^\wedge$. See Algebra, Lemmas 97.6, 97.4, and 97.3 and More on Algebra, Lemma 43.1. By Lemma 4.5 it suffices to prove the lemma for $A^\wedge$.

By the previous paragraph we may assume that A is a complete local ring. Then A has a normalized dualizing complex $\omega_A^\bullet$ (Dualizing Complexes, Lemma 22.4). The local duality theorem (in the form of Dualizing Complexes, Lemma 18.4) tells us $H^d_{\mathfrak m}(A)$ is Matlis dual to $\text{Ext}^{-d}(A, \omega_A^\bullet) = H^{-d}(\omega_A^\bullet)$ which is nonzero for example by Dualizing Complexes, Lemma 16.11. $\square$

Lemma 4.10. *Let $(A, \mathfrak m)$ be a Noetherian local ring. Let $I \subset A$ be a proper ideal. Let $\mathfrak p \subset A$ be a prime ideal such that $V(\mathfrak p) \cap V(I) = \{\mathfrak m\}$. Then $\dim(A/\mathfrak p) \leq$ cd(A, I).*

Proof. By Lemma 4.5 we have $\text{cd}(A, I) \geq \text{cd}(A/\mathfrak p, I(A/\mathfrak p))$. Since $V(I) \cap V(\mathfrak p) = \{\mathfrak m\}$ we have $\text{cd}(A/\mathfrak p, I(A/\mathfrak p)) = \text{cd}(A/\mathfrak p, \mathfrak m/\mathfrak p)$. By Lemma 4.9 this is equal to $\dim(A/\mathfrak p)$. $\square$

Lemma 4.11. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $b : X' \to X = \text{Spec}(A)$ be the blowing up of I. If the fibres of b have dimension $\leq d - 1$, then cd$(A, I) \leq d$.*

Proof. Set $U = X \setminus V(I)$. Denote $j : U \to X'$ the canonical open immersion, see Divisors, Section 32. Since the exceptional divisor is an effective Cartier divisor (Divisors, Lemma 32.4) we see that j is affine, see Divisors, Lemma 13.3. Let $\mathcal F$ be a quasi-coherent $\mathcal O_U$-module. Then $R^p j_*\mathcal F = 0$ for $p > 0$, see Cohomology of Schemes, Lemma 2.3. On the other hand, we have $R^q b_*(j_*\mathcal F) = 0$ for $q \geq d$ by Limits, Lemma 19.2. Thus by the Leray spectral sequence (Cohomology, Lemma 13.8) we conclude that $R^n(b \circ j)_*\mathcal F = 0$ for $n \geq d$. Thus $H^n(U, \mathcal F) = 0$ for $n \geq d$ (by Cohomology, Lemma 13.6). This means that $\text{cd}(A, I) \leq d$ by definition. $\square$

5. More general supports

Let A be a Noetherian ring. Let M be an A-module. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization (Topology, Definition 19.1). Let us define

$$H^0_T(M) = \text{colim}_{Z \subset T} H^0_Z(M)$$

where the colimit is over the directed partially ordered set of closed subsets Z of $\text{Spec}(A)$ contained in T[1]. In other words, an element m of M is in $H^0_T(M) \subset M$ if and only if the support $V(\text{Ann}_R(m))$ of m is contained in T.

Lemma 5.1. *Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. For an A-module M the following are equivalent*

1. *$H^0_T(M) = M$, and*
2. *Supp$(M) \subset T$.*

The category of such A-modules is a Serre subcategory of the category A-modules closed under direct sums.

[1]Since T is stable under specialization we have $T = \bigcup_{Z \subset T} Z$, see Topology, Lemma 19.3.

Proof. The equivalence holds because the support of an element of M is contained in the support of M and conversely the support of M is the union of the supports of its elements. The category of these modules is a Serre subcategory (Homology, Definition 10.1) of Mod_A by Algebra, Lemma 40.9. We omit the proof of the statement on direct sums. $\square$

Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. Let us denote $\text{Mod}_{A,T} \subset \text{Mod}_A$ the Serre subcategory described in Lemma 5.1. Let us denote $D_T(A) \subset D(A)$ the strictly full saturated triangulated subcategory of $D(A)$ (Derived Categories, Lemma 17.1) consisting of complexes of A-modules whose cohomology modules are in $\text{Mod}_{A,T}$. We obtain functors

$$D(\text{Mod}_{A,T}) \to D_T(A) \to D(A)$$

See discussion in Derived Categories, Section 17. Denote $RH^0_T : D(A) \to D(\text{Mod}_{A,T})$ the right derived extension of H^0_T. We will denote

$$R\Gamma_T : D^+(A) \to D^+_T(A),$$

the composition of $RH^0_T : D^+(A) \to D^+(\text{Mod}_{A,T})$ with $D^+(\text{Mod}_{A,T}) \to D^+_T(A)$. If the dimension of A is finite[2], then we will denote

$$R\Gamma_T : D(A) \to D_T(A)$$

the composition of RH^0_T with $D(\text{Mod}_{A,T}) \to D_T(A)$.

Lemma 5.2. *Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. The functor RH^0_T is the right adjoint to the functor $D(Mod_{A,T}) \to D(A)$.*

Proof. This follows from the fact that the functor $H^0_T(-)$ is the right adjoint to the inclusion functor $\text{Mod}_{A,T} \to \text{Mod}_A$, see Derived Categories, Lemma 30.3. $\square$

Lemma 5.3. *Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. For any object K of $D(A)$ we have*

$$H^i(RH^0_T(K)) = \text{colim}_{Z \subset T \ closed} H^i_Z(K)$$

Proof. Let $J^\bullet$ be a K-injective complex representing K. By definition RH^0_T is represented by the complex

$$H^0_T(J^\bullet) = \text{colim}\, H^0_Z(J^\bullet)$$

where the equality follows from our definition of H^0_T. Since filtered colimits are exact the cohomology of this complex in degree i is $\text{colim}\, H^i(H^0_Z(J^\bullet)) = \text{colim}\, H^i_Z(K)$ as desired. $\square$

Lemma 5.4. *Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. The functor $D^+(Mod_{A,T}) \to D^+_T(A)$ is an equivalence.*

Proof. Let M be an object of $\text{Mod}_{A,T}$. Choose an embedding $M \to J$ into an injective A-module. By Dualizing Complexes, Proposition 5.9 the module J is a direct sum of injective hulls of residue fields. Let E be an injective hull of the residue field of $\mathfrak{p}$. Since E is $\mathfrak{p}$-power torsion we see that $H^0_T(E) = 0$ if $\mathfrak{p} \not\in T$ and $H^0_T(E) = E$ if $\mathfrak{p} \in T$. Thus $H^0_T(J)$ is injective as a direct sum of injective hulls (by the proposition) and we have an embedding $M \to H^0_T(J)$. Thus every object M

[2]If $\dim(A) = \infty$ the construction may have unexpected properties on unbounded complexes.

of $\text{Mod}_{A,T}$ has an injective resolution $M \to J^\bullet$ with J^n also in $\text{Mod}_{A,T}$. It follows that $RH^0_T(M) = M$.

Next, suppose that $K \in D^+_T(A)$. Then the spectral sequence

$$R^q H^0_T(H^p(K)) \Rightarrow R^{p+q} H^0_T(K)$$

(Derived Categories, Lemma 21.3) converges and above we have seen that only the terms with $q = 0$ are nonzero. Thus we see that $RH^0_T(K) \to K$ is an isomorphism. Thus the functor $D^+(\text{Mod}_{A,T}) \to D^+_T(A)$ is an equivalence with quasi-inverse given by RH^0_T. $\square$

Lemma 5.5. *Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. If $\dim(A) < \infty$, then functor $D(Mod_{A,T}) \to D_T(A)$ is an equivalence.*

Proof. Say $\dim(A) = d$. Then we see that $H^i_Z(M) = 0$ for $i > d$ for every closed subset Z of $\text{Spec}(A)$, see Lemma 4.7. By Lemma 5.3 we find that H^0_T has bounded cohomological dimension.

Let $K \in D_T(A)$. We claim that $RH^0_T(K) \to K$ is an isomorphism. We know this is true when K is bounded below, see Lemma 5.4. However, since H^0_T has bounded cohomological dimension, we see that the ith cohomology of $RH^0_T(K)$ only depends on $\tau_{\geq -d+i}K$ and we conclude. Thus $D(\text{Mod}_{A,T}) \to D_T(A)$ is an equivalence with quasi-inverse RH^0_T. $\square$

Remark 5.6. Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. The upshot of the discussion above is that $R\Gamma_T : D^+(A) \to D^+_T(A)$ is the right adjoint to the inclusion functor $D^+_T(A) \to D^+(A)$. If $\dim(A) < \infty$, then $R\Gamma_T : D(A) \to D_T(A)$ is the right adjoint to the inclusion functor $D_T(A) \to D(A)$. In both cases we have

$$H^i_T(K) = H^i(R\Gamma_T(K)) = R^i H^0_T(K) = \text{colim}_{Z \subset T \text{ closed}} H^i_Z(K)$$

This follows by combining Lemmas 5.2, 5.3, 5.4, and 5.5.

Lemma 5.7. *Let $A \to B$ be a flat homomorphism of Noetherian rings. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. Let $T' \subset \text{Spec}(B)$ be the inverse image of T. Then the canonical map*

$$R\Gamma_T(K) \otimes^{\mathbf{L}}_A B \longrightarrow R\Gamma_{T'}(K \otimes^{\mathbf{L}}_A B)$$

is an isomorphism for $K \in D^+(A)$. If A and B have finite dimension, then this is true for $K \in D(A)$.

Proof. From the map $R\Gamma_T(K) \to K$ we get a map $R\Gamma_T(K) \otimes^{\mathbf{L}}_A B \to K \otimes^{\mathbf{L}}_A B$. The cohomology modules of $R\Gamma_T(K) \otimes^{\mathbf{L}}_A B$ are supported on T' and hence we get the arrow of the lemma. This arrow is an isomorphism if T is a closed subset of $\text{Spec}(A)$ by Dualizing Complexes, Lemma 9.3. Recall that $H^i_T(K)$ is the colimit of $H^i_Z(K)$ where Z runs over the (directed set of) closed subsets of T, see Lemma 5.3. Correspondingly $H^i_{T'}(K \otimes^{\mathbf{L}}_A B) = \text{colim}\, H^i_{Z'}(K \otimes^{\mathbf{L}}_A B)$ where Z' is the inverse image of Z. Thus the result because $\otimes_A B$ commutes with filtered colimits and there are no higher Tors. $\square$

Lemma 5.8. *Let A be a ring and let $T, T' \subset \mathrm{Spec}(A)$ subsets stable under specialization. For $K \in D^+(A)$ there is a spectral sequence*

$$E_2^{p,q} = H^p_T(H^p_{T'}(K)) \Rightarrow H^{p+q}_{T \cap T'}(K)$$

as in Derived Categories, Lemma 22.2.

Proof. Let E be an object of $D_{T\cap T'}(A)$. Then we have

$$\mathrm{Hom}(E, R\Gamma_T(R\Gamma_{T'}(K))) = \mathrm{Hom}(E, R\Gamma_{T'}(K)) = \mathrm{Hom}(E, K)$$

The first equality by the adjointness property of $R\Gamma_T$ and the second by the adjointness property of $R\Gamma_{T'}$. On the other hand, if $J^\bullet$ is a bounded below complex of injectives representing K, then $H^0_{T'}(J^\bullet)$ is a complex of injective A-modules representing $R\Gamma_{T'}(K)$ and hence $H^0_T(H^0_{T'}(J^\bullet))$ is a complex representing $R\Gamma_T(R\Gamma_{T'}(K))$. Thus $R\Gamma_T(R\Gamma_{T'}(K))$ is an object of $D^+_{T\cap T'}(A)$. Combining these two facts we find that $R\Gamma_{T\cap T'} = R\Gamma_T \circ R\Gamma_{T'}$. This produces the spectral sequence by the lemma referenced in the statement. □

Lemma 5.9. *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Assume A has finite dimension. Then*

$$R\Gamma_T(K) = R\Gamma_T(A) \otimes^{\mathbf{L}}_A K$$

for $K \in D(A)$. For $K, L \in D(A)$ we have

$$R\Gamma_T(K \otimes^{\mathbf{L}}_A L) = K \otimes^{\mathbf{L}}_A R\Gamma_T(L) = R\Gamma_T(K) \otimes^{\mathbf{L}}_A L = R\Gamma_T(K) \otimes^{\mathbf{L}}_A R\Gamma_T(L)$$

If K or L is in $D_T(A)$ then so is $K \otimes^{\mathbf{L}}_A L$.

Proof. By construction we may represent $R\Gamma_T(A)$ by a complex $J^\bullet$ in $\mathrm{Mod}_{A,T}$. Thus if we represent K by a K-flat complex $K^\bullet$ then we see that $R\Gamma_T(A) \otimes^{\mathbf{L}}_A K$ is represented by the complex $\mathrm{Tot}(J^\bullet \otimes_A K^\bullet)$ in $\mathrm{Mod}_{A,T}$. Using the map $R\Gamma_T(A) \to A$ we obtain a map $R\Gamma_T(A) \otimes^{\mathbf{L}}_A K \to K$. Thus by the adjointness property of $R\Gamma_T$ we obtain a canonical map

$$R\Gamma_T(A) \otimes^{\mathbf{L}}_A K \longrightarrow R\Gamma_T(K)$$

factoring the just constructed map. Observe that $R\Gamma_T$ commutes with direct sums in $D(A)$ for example by Lemma 5.3, the fact that directed colimits commute with direct sums, and the fact that usual local cohomology commutes with direct sums (for example by Dualizing Complexes, Lemma 9.1). Thus by More on Algebra, Remark 59.11 it suffices to check the map is an isomorphism for $K = A[k]$ where $k \in \mathbf{Z}$. This is clear.

The final statements follow from the result we've just shown by transitivity of derived tensor products. □

6. Filtrations on local cohomology

Some tricks related to the spectral sequence of Lemma 5.8.

Lemma 6.1. *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let $T' \subset T$ be the set of nonminimal primes in T. Then T' is a subset of* $\mathrm{Spec}(A)$ *stable under specialization and for every A-module M there is an exact sequence*

$$0 \to \mathrm{colim}_{Z,f} H^1_f(H^{i-1}_Z(M)) \to H^i_{T'}(M) \to H^i_T(M) \to \bigoplus\nolimits_{\mathfrak p \in T \setminus T'} H^i_{\mathfrak p A_{\mathfrak p}}(M_{\mathfrak p})$$

where the colimit is over closed subsets $Z \subset T$ and $f \in A$ with $V(f) \cap Z \subset T'$.

Proof. For every Z and f the spectral sequence of Dualizing Complexes, Lemma 9.6 degenerates to give short exact sequences

$$0 \to H^1_f(H^{i-1}_Z(M)) \to H^i_{Z \cap V(f)}(M) \to H^0_f(H^i_Z(M)) \to 0$$

We will use this without further mention below.

Let $\xi \in H^i_T(M)$ map to zero in the direct sum. Then we first write ξ as the image of some $\xi' \in H^i_Z(M)$ for some closed subset $Z \subset T$, see Lemma 5.3. Then ξ' maps to zero in $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p})$ for every $\mathfrak{p} \in Z$, $\mathfrak{p} \notin T'$. Since there are finitely many of these primes, we may choose $f \in A$ not contained in any of these such that f annihilates ξ'. Then ξ' is the image of some $\xi'' \in H^i_{Z'}(M)$ where $Z' = Z \cap V(f)$. By our choice of f we have $Z' \subset T'$ and we get exactness at the penultimate spot.

Let $\xi \in H^i_{T'}(M)$ map to zero in $H^i_T(M)$. Choose closed subsets $Z' \subset Z$ with $Z' \subset T'$ and $Z \subset T$ such that ξ comes from $\xi' \in H^i_{Z'}(M)$ and maps to zero in $H^i_Z(M)$. Then we can find $f \in A$ with $V(f) \cap Z = Z'$ and we conclude. □

Lemma 6.2. *Let A be a Noetherian ring of finite dimension. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let $\{M_n\}_{n \geq 0}$ be an inverse system of A-modules. Let $i \geq 0$ be an integer. Assume that for every m there exists an integer $m'(m) \geq m$ such that for all $\mathfrak{p} \in T$ the induced map*

$$H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{k,\mathfrak{p}}) \longrightarrow H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{m,\mathfrak{p}})$$

is zero for $k \geq m'(m)$. Let $m'' : \mathbf{N} \to \mathbf{N}$ be the $2^{\dim(T)}$-fold self-composition of m'. Then the map $H^i_T(M_k) \to H^i_T(M_m)$ is zero for all $k \geq m''(m)$.

Proof. We first make a general remark: suppose we have an exact sequence

$$(A_n) \to (B_n) \to (C_n)$$

of inverse systems of abelian groups. Suppose that for every m there exists an integer $m'(m) \geq m$ such that

$$A_k \to A_m \quad \text{and} \quad C_k \to C_m$$

are zero for $k \geq m'(m)$. Then for $k \geq m'(m'(m))$ the map $B_k \to B_m$ is zero.

We will prove the lemma by induction on $\dim(T)$ which is finite because $\dim(A)$ is finite. Let $T' \subset T$ be the set of nonminimal primes in T. Then T' is a subset of $\mathrm{Spec}(A)$ stable under specialization and the hypotheses of the lemma apply to T'. Since $\dim(T') < \dim(T)$ we know the lemma holds for T'. For every A-module M there is an exact sequence

$$H^i_{T'}(M) \to H^i_T(M) \to \bigoplus\nolimits_{\mathfrak{p} \in T \setminus T'} H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p})$$

by Lemma 6.1. Thus we conclude by the initial remark of the proof. □

Lemma 6.3. *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let $\{M_n\}_{n \geq 0}$ be an inverse system of A-modules. Let $i \geq 0$ be an integer. Assume the dimension of A is finite and that for every m there exists an integer $m'(m) \geq m$ such that for all $\mathfrak{p} \in T$ we have*

1. *$H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_{k,\mathfrak{p}}) \to H^{i-1}_{\mathfrak{p}A_\mathfrak{p}}(M_{m,\mathfrak{p}})$ is zero for $k \geq m'(m)$, and*
2. *$H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{k,\mathfrak{p}}) \to H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{m,\mathfrak{p}})$ has image $G(\mathfrak{p}, m)$ independent of $k \geq m'(m)$ and moreover $G(\mathfrak{p}, m)$ maps injectively into $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_{0,\mathfrak{p}})$.*

Then there exists an integer m_0 such that for every $m \geq m_0$ there exists an integer $m''(m) \geq m$ such that for $k \geq m''(m)$ the image of $H^i_T(M_k) \to H^i_T(M_m)$ maps injectively into $H^i_T(M_{m_0})$.

Proof. We first make a general remark: suppose we have an exact sequence

$$(A_n) \to (B_n) \to (C_n) \to (D_n)$$

of inverse systems of abelian groups. Suppose that there exists an integer m_0 such that for every $m \geq m_0$ there exists an integer $m'(m) \geq m$ such that the maps

$$\mathrm{Im}(B_k \to B_m) \longrightarrow B_{m_0} \quad \text{and} \quad \mathrm{Im}(D_k \to D_m) \longrightarrow D_{m_0}$$

are injective for $k \geq m'(m)$ and $A_k \to A_m$ is zero for $k \geq m'(m)$. Then for $m \geq m'(m_0)$ and $k \geq m'(m'(m))$ the map

$$\mathrm{Im}(C_k \to C_m) \to C_{m'(m_0)}$$

is injective. Namely, let $c_0 \in C_m$ be the image of $c_3 \in C_k$ and say c_0 maps to zero in $C_{m'(m_0)}$. Picture

$$C_k \to C_{m'(m'(m))} \to C_{m'(m)} \to C_m \to C_{m'(m_0)}, \quad c_3 \mapsto c_2 \mapsto c_1 \mapsto c_0 \mapsto 0$$

We have to show $c_0 = 0$. The image d_3 of c_3 maps to zero in C_{m_0} and hence we see that the image $d_1 \in D_{m'(m)}$ is zero. Thus we can choose $b_1 \in B_{m'(m)}$ mapping to the image c_1. Since c_3 maps to zero in $C_{m'(m_0)}$ we find an element $a_{-1} \in A_{m'(m_0)}$ which maps to the image $b_{-1} \in B_{m'(m_0)}$ of b_1. Since a_{-1} maps to zero in A_{m_0} we conclude that b_1 maps to zero in B_{m_0}. Thus the image $b_0 \in B_m$ is zero which of course implies $c_0 = 0$ as desired.

We will prove the lemma by induction on $\dim(T)$ which is finite because $\dim(A)$ is finite. Let $T' \subset T$ be the set of nonminimal primes in T. Then T' is a subset of $\mathrm{Spec}(A)$ stable under specialization and the hypotheses of the lemma apply to T'. Since $\dim(T') < \dim(T)$ we know the lemma holds for T'. For every A-module M there is an exact sequence

$$0 \to \mathrm{colim}_{Z,f}\, H^1_f(H^{i-1}_Z(M)) \to H^i_{T'}(M) \to H^i_T(M) \to \bigoplus\nolimits_{\mathfrak p \in T \setminus T'} H^i_{\mathfrak p A_{\mathfrak p}}(M_{\mathfrak p})$$

by Lemma 6.1. Thus we conclude by the initial remark of the proof and the fact that we've seen the system of groups

$$\left\{\mathrm{colim}_{Z,f}\, H^1_f(H^{i-1}_Z(M_n))\right\}_{n \geq 0}$$

is pro-zero in Lemma 6.2; this uses that the function $m''(m)$ in that lemma for $H^{i-1}_Z(M)$ is independent of Z. $\square$

7. Finiteness of local cohomology, I

We will follow Faltings approach to finiteness of local cohomology modules, see [Fal78] and [Fal81]. Here is a lemma which shows that it suffices to prove local cohomology modules have an annihilator in order to prove that they are finite modules.

Lemma 7.1. *Let A be a Noetherian ring. Let $T \subset \mathrm{Spec}(A)$ be a subset stable under specialization. Let M be a finite A-module. Let $n \geq 0$. The following are equivalent*

(1) *$H^i_T(M)$ is finite for $i \leq n$,*

(2) *there exists an ideal $J \subset A$ with $V(J) \subset T$ such that J annihilates $H^i_T(M)$ for $i \leq n$.*

If $T = V(I) = Z$ for an ideal $I \subset A$, then these are also equivalent to

(3) *there exists an $e \geq 0$ such that I^e annihilates $H^i_Z(M)$ for $i \leq n$.*

Proof. We prove the equivalence of (1) and (2) by induction on n. For $n = 0$ we have $H^0_T(M) \subset M$ is finite. Hence (1) is true. Since $H^0_T(M) = \operatorname{colim} H^0_{V(J)}(M)$ with J as in (2) we see that (2) is true. Assume that $n > 0$.

Assume (1) is true. Recall that $H^i_J(M) = H^i_{V(J)}(M)$, see Dualizing Complexes, Lemma 10.1. Thus $H^i_T(M) = \operatorname{colim} H^i_J(M)$ where the colimit is over ideals $J \subset A$ with $V(J) \subset T$, see Lemma 5.3. Since $H^i_T(M)$ is finitely generated for $i \leq n$ we can find a $J \subset A$ as in (2) such that $H^i_J(M) \to H^i_T(M)$ is surjective for $i \leq n$. Thus the finite list of generators are J-power torsion elements and we see that (2) holds with J replaced by some power.

Assume we have J as in (2). Let $N = H^0_T(M)$ and $M' = M/N$. By construction of $R\Gamma_T$ we find that $H^i_T(N) = 0$ for $i > 0$ and $H^0_T(N) = N$, see Remark 5.6. Thus we find that $H^0_T(M') = 0$ and $H^i_T(M') = H^i_T(M)$ for $i > 0$. We conclude that we may replace M by M'. Thus we may assume that $H^0_T(M) = 0$. This means that the finite set of associated primes of M are not in T. By prime avoidance (Algebra, Lemma 15.2) we can find $f \in J$ not contained in any of the associated primes of M. Then the long exact local cohomology sequence associated to the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

turns into short exact sequences

$$0 \to H^i_T(M) \to H^i_T(M/fM) \to H^{i+1}_T(M) \to 0$$

for $i < n$. We conclude that J^2 annihilates $H^i_T(M/fM)$ for $i < n$. By induction hypothesis we see that $H^i_T(M/fM)$ is finite for $i < n$. Using the short exact sequence once more we see that $H^{i+1}_T(M)$ is finite for $i < n$ as desired.

We omit the proof of the equivalence of (2) and (3) in case $T = V(I)$. $\square$

The following result of Faltings allows us to prove finiteness of local cohomology at the level of local rings.

Lemma 7.2. *Let A be a Noetherian ring, $I \subset A$ an ideal, M a finite A-module, and $n \geq 0$ an integer. Let $Z = V(I)$. The following are equivalent*

(1) *the modules $H^i_Z(M)$ are finite for $i \leq n$, and*
(2) *for all $\mathfrak{p} \in \operatorname{Spec}(A)$ the modules $H^i_Z(M)_\mathfrak{p}$, $i \leq n$ are finite $A_\mathfrak{p}$-modules.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. We prove the converse by induction on n. The case $n = 0$ is clear because both (1) and (2) are always true in that case.

Assume $n > 0$ and that (2) is true. Let $N = H^0_Z(M)$ and $M' = M/N$. By Dualizing Complexes, Lemma 11.6 we may replace M by M'. Thus we may assume that $H^0_Z(M) = 0$. This means that $\operatorname{depth}_I(M) > 0$ (Dualizing Complexes, Lemma 11.1). Pick $f \in I$ a nonzerodivisor on M and consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

which produces a long exact sequence

$$0 \to H^0_Z(M/fM) \to H^1_Z(M) \to H^1_Z(M) \to H^1_Z(M/fM) \to H^2_Z(M) \to \ldots$$

and similarly after localization. Thus assumption (2) implies that the modules $H^i_Z(M/fM)_\mathfrak{p}$ are finite for $i < n$. Hence by induction assumption $H^i_Z(M/fM)$ are finite for $i < n$.

Let $\mathfrak{p}$ be a prime of A which is associated to $H^i_Z(M)$ for some $i \leq n$. Say $\mathfrak{p}$ is the annihilator of the element $x \in H^i_Z(M)$. Then $\mathfrak{p} \in Z$, hence $f \in \mathfrak{p}$. Thus $fx = 0$ and hence x comes from an element of $H^{i-1}_Z(M/fM)$ by the boundary map δ in the long exact sequence above. It follows that $\mathfrak{p}$ is an associated prime of the finite module $\mathrm{Im}(\delta)$. We conclude that $\mathrm{Ass}(H^i_Z(M))$ is finite for $i \leq n$, see Algebra, Lemma 63.5.

Recall that

$$H^i_Z(M) \subset \prod\nolimits_{\mathfrak{p} \in \mathrm{Ass}(H^i_Z(M))} H^i_Z(M)_\mathfrak{p}$$

by Algebra, Lemma 63.19. Since by assumption the modules on the right hand side are finite and I-power torsion, we can find integers $e_{\mathfrak{p},i} \geq 0$, $i \leq n$, $\mathfrak{p} \in \mathrm{Ass}(H^i_Z(M))$ such that $I^{e_{\mathfrak{p},i}}$ annihilates $H^i_Z(M)_\mathfrak{p}$. We conclude that I^e with $e = \max\{e_{\mathfrak{p},i}\}$ annihilates $H^i_Z(M)$ for $i \leq n$. By Lemma 7.1 we see that $H^i_Z(M)$ is finite for $i \leq n$. $\square$

Lemma 7.3. *Let A be a ring and let $J \subset I \subset A$ be finitely generated ideals. Let $i \geq 0$ be an integer. Set $Z = V(I)$. If $H^i_Z(A)$ is annihilated by J^n for some n, then $H^i_Z(M)$ annihilated by J^m for some $m = m(M)$ for every finitely presented A-module M such that M_f is a finite locally free A_f-module for all $f \in I$.*

Proof. Consider the annihilator $\mathfrak{a}$ of $H^i_Z(M)$. Let $\mathfrak{p} \subset A$ with $\mathfrak{p} \not\in Z$. By assumption there exists an $f \in I$, $f \not\in \mathfrak{p}$ and an isomorphism $\varphi : A_f^{\oplus r} \to M_f$ of A_f-modules. Clearing denominators (and using that M is of finite presentation) we find maps

$$a : A^{\oplus r} \longrightarrow M \quad \text{and} \quad b : M \longrightarrow A^{\oplus r}$$

with $a_f = f^N\varphi$ and $b_f = f^N\varphi^{-1}$ for some N. Moreover we may assume that $a \circ b$ and $b \circ a$ are equal to multiplication by f^{2N}. Thus we see that $H^i_Z(M)$ is annihilated by $f^{2N}J^n$, i.e., $f^{2N}J^n \subset \mathfrak{a}$.

As $U = \mathrm{Spec}(A) \setminus Z$ is quasi-compact we can find finitely many $f_1, \ldots, f_t$ and $N_1, \ldots, N_t$ such that $U = \bigcup D(f_j)$ and $f_j^{2N_j}J^n \subset \mathfrak{a}$. Then $V(I) = V(f_1, \ldots, f_t)$ and since I is finitely generated we conclude $I^M \subset (f_1, \ldots, f_t)$ for some M. All in all we see that $J^m \subset \mathfrak{a}$ for $m \gg 0$, for example $m = M(2N_1 + \ldots + 2N_t)n$ will do. $\square$

Lemma 7.4. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Set $Z = V(I)$. Let $n \geq 0$ be an integer. If $H^i_Z(A)$ is finite for $0 \leq i \leq n$, then the same is true for $H^i_Z(M)$, $0 \leq i \leq n$ for any finite A-module M such that M_f is a finite locally free A_f-module for all $f \in I$.*

Proof. The assumption that $H^i_Z(A)$ is finite for $0 \leq i \leq n$ implies there exists an $e \geq 0$ such that I^e annihilates $H^i_Z(A)$ for $0 \leq i \leq n$, see Lemma 7.1. Then Lemma 7.3 implies that $H^i_Z(M)$, $0 \leq i \leq n$ is annihilated by I^m for some $m = m(M, i)$. We may take the same m for all $0 \leq i \leq n$. Then Lemma 7.1 implies that $H^i_Z(M)$ is finite for $0 \leq i \leq n$ as desired. $\square$

8. Finiteness of pushforwards, I

In this section we discuss the easiest nontrivial case of the finiteness theorem, namely, the finiteness of the first local cohomology or what is equivalent, finiteness of $j_*\mathcal{F}$ where $j : U \to X$ is an open immersion, X is locally Noetherian, and $\mathcal{F}$ is a coherent sheaf on U. Following a method of Kollár ([Kol16] and [Kol15]) we find a necessary and sufficient condition, see Proposition 8.7. The reader who is interested in higher direct images or higher local cohomology groups should skip ahead to Section 12 or Section 11 (which are developed independently of the rest of this section).

Lemma 8.1. *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. For $x \in U$ let $i_x : W_x \to U$ be the integral closed subscheme with generic point x. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. The following are equivalent*

1. *for all $x \in Ass(\mathcal{F})$ the $\mathcal{O}_X$-module $j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent,*
2. *$j_*\mathcal{F}$ is coherent.*

Proof. We first prove that (1) implies (2). Assume (1) holds. The statement is local on X, hence we may assume X is affine. Then U is quasi-compact, hence $\text{Ass}(\mathcal{F})$ is finite (Divisors, Lemma 2.5). Thus we may argue by induction on the number of associated points. Let $x \in U$ be a generic point of an irreducible component of the support of $\mathcal{F}$. By Divisors, Lemma 2.5 we have $x \in \text{Ass}(\mathcal{F})$. By our choice of x we have $\dim(\mathcal{F}_x) = 0$ as $\mathcal{O}_{X,x}$-module. Hence $\mathcal{F}_x$ has finite length as an $\mathcal{O}_{X,x}$-module (Algebra, Lemma 62.3). Thus we may use induction on this length.

Set $\mathcal{G} = j_*i_{x,*}\mathcal{O}_{W_x}$. This is a coherent $\mathcal{O}_X$-module by assumption. We have $\mathcal{G}_x = \kappa(x)$. Choose a nonzero map $\varphi_x : \mathcal{F}_x \to \kappa(x) = \mathcal{G}_x$. By Cohomology of Schemes, Lemma 9.6 there is an open $x \in V \subset U$ and a map $\varphi_V : \mathcal{F}|_V \to \mathcal{G}|_V$ whose stalk at x is φ_x. Choose $f \in \Gamma(X, \mathcal{O}_X)$ which does not vanish at x such that $D(f) \subset V$. By Cohomology of Schemes, Lemma 10.5 (for example) we see that φ_V extends to $f^n\mathcal{F} \to \mathcal{G}|_U$ for some n. Precomposing with multiplication by f^n we obtain a map $\mathcal{F} \to \mathcal{G}|_U$ whose stalk at x is nonzero. Let $\mathcal{F}' \subset \mathcal{F}$ be the kernel. Note that $\text{Ass}(\mathcal{F}') \subset \text{Ass}(\mathcal{F})$, see Divisors, Lemma 2.4. Since $\text{length}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) - 1$ we may apply the induction hypothesis to conclude $j_*\mathcal{F}'$ is coherent. Since $\mathcal{G} = j_*(\mathcal{G}|_U) = j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent, we can consider the exact sequence

$$0 \to j_*\mathcal{F}' \to j_*\mathcal{F} \to \mathcal{G}$$

By Schemes, Lemma 24.1 the sheaf $j_*\mathcal{F}$ is quasi-coherent. Hence the image of $j_*\mathcal{F}$ in $j_*(\mathcal{G}|_U)$ is coherent by Cohomology of Schemes, Lemma 9.3. Finally, $j_*\mathcal{F}$ is coherent by Cohomology of Schemes, Lemma 9.2.

Assume (2) holds. Exactly in the same manner as above we reduce to the case X affine. We pick $x \in \text{Ass}(\mathcal{F})$ and we set $\mathcal{G} = j_*i_{x,*}\mathcal{O}_{W_x}$. Then we choose a nonzero map $\varphi_x : \mathcal{G}_x = \kappa(x) \to \mathcal{F}_x$ which exists exactly because x is an associated point of $\mathcal{F}$. Arguing exactly as above we may assume φ_x extends to an $\mathcal{O}_U$-module map $\varphi : \mathcal{G}|_U \to \mathcal{F}$. Then φ is injective (for example by Divisors, Lemma 2.10) and we find an injective map $\mathcal{G} = j_*(\mathcal{G}|_V) \to j_*\mathcal{F}$. Thus (1) holds. $\square$

Lemma 8.2. *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $X = \mathrm{Spec}(A)$, $Z = V(I)$, $U = X \setminus Z$, and $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Then*

1. *there exists a finite A-module M such that $\mathcal{F}$ is the restriction of $\widetilde{M}$ to U,*
2. *given M there is an exact sequence*
$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$
and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$,
3. *given M and $p \geq 0$ the following are equivalent*
   (a) *$R^p j_* \mathcal{F}$ is coherent,*
   (b) *$H^p(U, \mathcal{F})$ is a finite A-module,*
   (c) *$H^{p+1}_Z(M)$ is a finite A-module,*
4. *if the equivalent conditions in (3) hold for $p = 0$, we may take $M = \Gamma(U, \mathcal{F})$ in which case we have $H^0_Z(M) = H^1_Z(M) = 0$.*

Proof. By Properties, Lemma 22.5 there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}'$ whose restriction to U is isomorphic to $\mathcal{F}$. Say $\mathcal{F}'$ corresponds to the finite A-module M as in (1). Note that $R^p j_* \mathcal{F}$ is quasi-coherent (Cohomology of Schemes, Lemma 4.5) and corresponds to the A-module $H^p(U, \mathcal{F})$. By Lemma 2.1 and the discussion in Cohomology, Sections 21 and 34 we obtain an exact sequence
$$0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$$
and isomorphisms $H^p(U, \mathcal{F}) = H^{p+1}_Z(M)$ for $p \geq 1$. Here we use that $H^j(X, \mathcal{F}') = 0$ for $j > 0$ as X is affine and $\mathcal{F}'$ is quasi-coherent (Cohomology of Schemes, Lemma 2.2). This proves (2). Parts (3) and (4) are straightforward from (2); see also Lemma 2.2. □

Lemma 8.3. *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Assume*

1. *X is Nagata,*
2. *X is universally catenary, and*
3. *for $x \in Ass(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ we have $\dim(\mathcal{O}_{\overline{\{x\}},z}) \geq 2$.*

Then $j_* \mathcal{F}$ is coherent.

Proof. By Lemma 8.1 it suffices to prove $j_* i_{x,*} \mathcal{O}_{W_x}$ is coherent for $x \in \mathrm{Ass}(\mathcal{F})$. Let $\pi : Y \to X$ be the normalization of X in $\mathrm{Spec}(\kappa(x))$, see Morphisms, Section 54. By Morphisms, Lemma 53.14 the morphism π is finite. Since π is finite $\mathcal{G} = \pi_* \mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module by Cohomology of Schemes, Lemma 9.9. Observe that $W_x = U \cap \pi(Y)$. Thus $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ factors through $i_x : W_x \to U$ and we obtain a canonical map
$$i_{x,*} \mathcal{O}_{W_x} \longrightarrow (\pi|_{\pi^{-1}(U)})_* (\mathcal{O}_{\pi^{-1}(U)}) = (\pi_* \mathcal{O}_Y)|_U = \mathcal{G}|_U$$
This map is injective (for example by Divisors, Lemma 2.10). Hence $j_* i_{x,*} \mathcal{O}_{W_x} \subset j_* \mathcal{G}|_U$ and it suffices to show that $j_* \mathcal{G}|_U$ is coherent.

It remains to prove that $j_*(\mathcal{G}|_U)$ is coherent. We claim Divisors, Lemma 5.11 applies to
$$\mathcal{G} \longrightarrow j_*(\mathcal{G}|_U)$$
which finishes the proof. It suffices to show that $\mathrm{depth}(\mathcal{G}_z) \geq 2$ for $z \in Z$. Let $y_1, \ldots, y_n \in Y$ be the points mapping to z. By Algebra, Lemma 72.11 it suffices

to show that $\text{depth}(\mathcal{O}_{Y,y_i}) \geq 2$ for $i = 1, \ldots, n$. If not, then by Properties, Lemma 12.5 we see that $\dim(\mathcal{O}_{Y,y_i}) = 1$ for some i. This is impossible by the dimension formula (Morphisms, Lemma 52.1) for $\pi : Y \to \overline{\{x\}}$ and assumption (3). $\square$

Lemma 8.4. *Let X be an integral locally Noetherian scheme. Let $j : U \to X$ be the inclusion of a nonempty open subscheme with complement Z. Assume that for all $z \in Z$ and any associated prime $\mathfrak{p}$ of the completion $\mathcal{O}_{X,z}^\wedge$ we have $\dim(\mathcal{O}_{X,z}^\wedge/\mathfrak{p}) \geq 2$. Then $j_*\mathcal{O}_U$ is coherent.*

Proof. We may assume X is affine. Using Lemmas 7.2 and 8.2 we reduce to $X = \text{Spec}(A)$ where $(A, \mathfrak{m})$ is a Noetherian local domain and $\mathfrak{m} \in Z$. Then we can use induction on $d = \dim(A)$. (The base case is $d = 0, 1$ which do not happen by our assumption on the local rings.) Set $V = \text{Spec}(A) \setminus \{\mathfrak{m}\}$. Observe that the local rings of V have dimension strictly smaller than d. Repeating the arguments for $j' : U \to V$ we and using induction we conclude that $j'_*\mathcal{O}_U$ is a coherent $\mathcal{O}_V$-module. Pick a nonzero $f \in A$ which vanishes on Z. Since $D(f) \cap V \subset U$ we find an n such that multiplication by f^n on U extends to a map $f^n : j'_*\mathcal{O}_U \to \mathcal{O}_V$ over V (for example by Cohomology of Schemes, Lemma 10.5). This map is injective hence there is an injective map

$$j_*\mathcal{O}_U = j''_*j'_*\mathcal{O}_U \to j''_*\mathcal{O}_V$$

on X where $j'' : V \to X$ is the inclusion morphism. Hence it suffices to show that $j''_*\mathcal{O}_V$ is coherent. In other words, we may assume that X is the spectrum of a local Noetherian domain and that Z consists of the closed point.

Assume $X = \text{Spec}(A)$ with $(A, \mathfrak{m})$ local and $Z = \{\mathfrak{m}\}$. Let $A^\wedge$ be the completion of A. Set $X^\wedge = \text{Spec}(A^\wedge)$, $Z^\wedge = \{\mathfrak{m}^\wedge\}$, $U^\wedge = X^\wedge \setminus Z^\wedge$, and $\mathcal{F}^\wedge = \mathcal{O}_{U^\wedge}$. The ring $A^\wedge$ is universally catenary and Nagata (Algebra, Remark 160.9 and Lemma 162.8). Moreover, condition (3) of Lemma 8.3 for $X^\wedge, Z^\wedge, U^\wedge, \mathcal{F}^\wedge$ holds by assumption! Thus we see that $(U^\wedge \to X^\wedge)_*\mathcal{O}_{U^\wedge}$ is coherent. Since the morphism $c : X^\wedge \to X$ is flat we conclude that the pullback of $j_*\mathcal{O}_U$ is $(U^\wedge \to X^\wedge)_*\mathcal{O}_{U^\wedge}$ (Cohomology of Schemes, Lemma 5.2). Finally, since c is faithfully flat we conclude that $j_*\mathcal{O}_U$ is coherent by Descent, Lemma 7.1. $\square$

Remark 8.5. Let $j : U \to X$ be an open immersion of locally Noetherian schemes. Let $x \in U$. Let $i_x : W_x \to U$ be the integral closed subscheme with generic point x and let $\overline{\{x\}}$ be the closure in X. Then we have a commutative diagram

$$\begin{tikzcd} W_x \arrow[r, "j'"'] \arrow[d, "i_x"'] & \overline{\{x\}} \arrow[d, "i"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

We have $j_*i_{x,*}\mathcal{O}_{W_x} = i_*j'_*\mathcal{O}_{W_x}$. As the left vertical arrow is a closed immersion we see that $j_*i_{x,*}\mathcal{O}_{W_x}$ is coherent if and only if $j'_*\mathcal{O}_{W_x}$ is coherent.

Remark 8.6. Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. If there exists an $x \in \text{Ass}(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ such that $\dim(\mathcal{O}_{\overline{\{x\}},z}) \leq 1$, then $j_*\mathcal{F}$ is not coherent. To prove this we can do a flat base change to the spectrum of $\mathcal{O}_{X,z}$. Let $X' = \overline{\{x\}}$. The assumption implies $\mathcal{O}_{X' \cap U} \subset \mathcal{F}$. Thus it suffices to see that $j_*\mathcal{O}_{X' \cap U}$ is not coherent. This is clear because $X' = \{x, z\}$, hence $j_*\mathcal{O}_{X' \cap U}$

corresponds to $\kappa(x)$ as an $\mathcal{O}_{X,z}$-module which cannot be finite as x is not a closed point.

In fact, the converse of Lemma 8.4 holds true: given an open immersion $j : U \to X$ of integral Noetherian schemes and there exists a $z \in X \setminus U$ and an associated prime $\mathfrak{p}$ of the completion $\mathcal{O}_{X,z}^\wedge$ with $\dim(\mathcal{O}_{X,z}^\wedge/\mathfrak{p}) = 1$, then $j_*\mathcal{O}_U$ is not coherent. Namely, you can pass to the local ring, you can enlarge U to the punctured spectrum, you can pass to the completion, and then the argument above gives the nonfiniteness.

Proposition 8.7 (Kollár)**.** *Let $j : U \to X$ be an open immersion of locally Noetherian schemes with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. The following are equivalent*

1. *$j_*\mathcal{F}$ is coherent,*
2. *for $x \in Ass(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ and any associated prime $\mathfrak{p}$ of the completion $\mathcal{O}_{\overline{\{x\}},z}^\wedge$ we have $\dim(\mathcal{O}_{\overline{\{x\}},z}^\wedge/\mathfrak{p}) \geq 2$.*

Proof. If (2) holds we get (1) by a combination of Lemmas 8.1, Remark 8.5, and Lemma 8.4. If (2) does not hold, then $j_*i_{x,*}\mathcal{O}_{W_x}$ is not finite for some $x \in \text{Ass}(\mathcal{F})$ by the discussion in Remark 8.6 (and Remark 8.5). Thus $j_*\mathcal{F}$ is not coherent by Lemma 8.1. $\square$

Lemma 8.8. *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $Z = V(I)$. Let M be a finite A-module. The following are equivalent*

1. *$H^1_Z(M)$ is a finite A-module, and*
2. *for all $\mathfrak{p} \in Ass(M)$, $\mathfrak{p} \notin Z$ and all $\mathfrak{q} \in V(\mathfrak{p} + I)$ the completion of $(A/\mathfrak{p})_\mathfrak{q}$ does not have associated primes of dimension 1.*

Proof. Follows immediately from Proposition 8.7 via Lemma 8.2. $\square$

The formulation in the following lemma has the advantage that conditions (1) and (2) are inherited by schemes of finite type over X. Moreover, this is the form of finiteness which we will generalize to higher direct images in Section 12.

Lemma 8.9. *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Assume*

1. *X is universally catenary,*
2. *for every $z \in Z$ the formal fibres of $\mathcal{O}_{X,z}$ are (S_1).*

In this situation the following are equivalent

(a) *for $x \in Ass(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ we have $\dim(\mathcal{O}_{\overline{\{x\}},z}) \geq 2$, and*
(b) *$j_*\mathcal{F}$ is coherent.*

Proof. Let $x \in \text{Ass}(\mathcal{F})$. By Proposition 8.7 it suffices to check that $A = \mathcal{O}_{\overline{\{x\}},z}$ satisfies the condition of the proposition on associated primes of its completion if and only if $\dim(A) \geq 2$. Observe that A is universally catenary (this is clear) and that its formal fibres are (S_1) as follows from More on Algebra, Lemma 51.10 and Proposition 51.5. Let $\mathfrak{p}' \subset A^\wedge$ be an associated prime. As $A \to A^\wedge$ is flat, by Algebra, Lemma 65.3, we find that $\mathfrak{p}'$ lies over $(0) \subset A$. The formal fibre $A^\wedge \otimes_A F$ is (S_1) where F is the fraction field of A. We conclude that $\mathfrak{p}'$ is a minimal prime, see Algebra, Lemma 157.2. Since A is universally catenary it is formally catenary by More on Algebra, Proposition 109.5. Hence $\dim(A^\wedge/\mathfrak{p}') = \dim(A)$ which proves the equivalence. $\square$

9. Depth and dimension

Some helper lemmas.

Lemma 9.1. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let $\mathfrak p \in V(I)$ be a prime ideal. Assume $e = depth_{IA_\mathfrak p}(M_\mathfrak p) < \infty$. Then there exists a nonempty open $U \subset V(\mathfrak p)$ such that $depth_{IA_\mathfrak q}(M_\mathfrak q) \geq e$ for all $\mathfrak q \in U$.*

Proof. By definition of depth we have $IM_\mathfrak p \neq M_\mathfrak p$ and there exists an $M_\mathfrak p$-regular sequence $f_1, \ldots, f_e \in IA_\mathfrak p$. After replacing A by a principal localization we may assume $f_1, \ldots, f_e \in I$ form an M-regular sequence, see Algebra, Lemma 68.6. Consider the module $M' = M/IM$. Since $\mathfrak p \in \text{Supp}(M')$ and since the support of a finite module is closed, we find $V(\mathfrak p) \subset \text{Supp}(M')$. Thus for $\mathfrak q \in V(\mathfrak p)$ we get $IM_\mathfrak q \neq M_\mathfrak q$. Hence, using that localization is exact, we see that $\text{depth}_{IA_\mathfrak q}(M_\mathfrak q) \geq e$ for any $\mathfrak q \in V(I)$ by definition of depth. $\square$

Lemma 9.2. *Let A be a Noetherian ring. Let M be a finite A-module. Let $\mathfrak p$ be a prime ideal. Assume $e = depth_{A_\mathfrak p}(M_\mathfrak p) < \infty$. Then there exists a nonempty open $U \subset V(\mathfrak p)$ such that $depth_{A_\mathfrak q}(M_\mathfrak q) \geq e$ for all $\mathfrak q \in U$ and for all but finitely many $\mathfrak q \in U$ we have $depth_{A_\mathfrak q}(M_\mathfrak q) > e$.*

Proof. By definition of depth we have $\mathfrak p M_\mathfrak p \neq M_\mathfrak p$ and there exists an $M_\mathfrak p$-regular sequence $f_1, \ldots, f_e \in \mathfrak p A_\mathfrak p$. After replacing A by a principal localization we may assume $f_1, \ldots, f_e \in \mathfrak p$ form an M-regular sequence, see Algebra, Lemma 68.6. Consider the module $M' = M/(f_1, \ldots, f_e)M$. Since $\mathfrak p \in \text{Supp}(M')$ and since the support of a finite module is closed, we find $V(\mathfrak p) \subset \text{Supp}(M')$. Thus for $\mathfrak q \in V(\mathfrak p)$ we get $\mathfrak q M_\mathfrak q \neq M_\mathfrak q$. Hence, using that localization is exact, we see that $\text{depth}_{A_\mathfrak q}(M_\mathfrak q) \geq e$ for any $\mathfrak q \in V(I)$ by definition of depth. Moreover, as soon as $\mathfrak q$ is not an associated prime of the module M', then the depth goes up. Thus we see that the final statement holds by Algebra, Lemma 63.5. $\square$

Lemma 9.3. *Let X be a Noetherian scheme with dualizing complex $\omega_X^\bullet$. Let $\mathcal F$ be a coherent $\mathcal O_X$-module. Let $k \geq 0$ be an integer. Assume $\mathcal F$ is (S_k). Then there is a finite number of points $x \in X$ such that*

$$depth(\mathcal F_x) = k \quad and \quad \dim(Supp(\mathcal F_x)) > k$$

Proof. We will prove this lemma by induction on k. The base case $k = 0$ says that $\mathcal F$ has a finite number of embedded associated points, which follows from Divisors, Lemma 2.5.

Assume $k > 0$ and the result holds for all smaller k. We can cover X by finitely many affine opens, hence we may assume $X = \text{Spec}(A)$ is affine. Then $\mathcal F$ is the coherent $\mathcal O_X$-module associated to a finite A-module M which satisfies (S_k). We will use Algebra, Lemmas 63.10 and 72.7 without further mention.

Let $f \in A$ be a nonzerodivisor on M. Then M/fM has (S_{k-1}). By induction we see that there are finitely many primes $\mathfrak p \in V(f)$ with $\text{depth}((M/fM)_\mathfrak p) = k - 1$ and $\dim(\text{Supp}((M/fM)_\mathfrak p)) > k - 1$. These are exactly the primes $\mathfrak p \in V(f)$ with $\text{depth}(M_\mathfrak p) = k$ and $\dim(\text{Supp}(M_\mathfrak p)) > k$. Thus we may replace A by A_f and M by M_f in trying to prove the finiteness statement.

Since M satisfies (S_k) and $k > 0$ we see that M has no embedded associated primes (Algebra, Lemma 157.2). Thus $\text{Ass}(M)$ is the set of generic points of the support of M. Thus Dualizing Complexes, Lemma 20.4 shows the set $U = \{\mathfrak q \mid M_\mathfrak q \text{ is Cohen-Macaulay}\}$ is an open containing $\text{Ass}(M)$. By prime avoidance (Algebra, Lemma 15.2) we can pick $f \in A$ with $f \not\in \mathfrak p$ for $\mathfrak p \in \text{Ass}(M)$ such that $D(f) \subset U$. Then f is a nonzerodivisor on M (Algebra, Lemma 63.9). After replacing A by A_f and M by M_f (see above) we find that M is Cohen-Macaulay. Thus for all $\mathfrak q \subset A$ we have $\dim(M_\mathfrak q) = \text{depth}(M_\mathfrak q)$ and hence the set described in the lemma is empty and a fortiori finite. □

Lemma 9.4. *Let $(A, \mathfrak m)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module. Set $E^i = \text{Ext}_A^{-i}(M, \omega_A^\bullet)$. Then*

1. *E^i is a finite A-module nonzero only for $0 \leq i \leq \dim(\text{Supp}(M))$,*
2. *$\dim(\text{Supp}(E^i)) \leq i$,*
3. *$\text{depth}(M)$ is the smallest integer $\delta \geq 0$ such that $E^\delta \neq 0$,*
4. *$\mathfrak p \in \text{Supp}(E^0 \oplus \ldots \oplus E^i) \Leftrightarrow \text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim(A/\mathfrak p) \leq i$,*
5. *the annihilator of E^i is equal to the annihilator of $H^i_\mathfrak m(M)$.*

Proof. Parts (1), (2), and (3) are copies of the statements in Dualizing Complexes, Lemma 16.5. For a prime $\mathfrak p$ of A we have that $(\omega_A^\bullet)_\mathfrak p[-\dim(A/\mathfrak p)]$ is a normalized dualzing complex for $A_\mathfrak p$. See Dualizing Complexes, Lemma 17.3. Thus

$$E^i_\mathfrak p = \text{Ext}^{-i}_A(M, \omega_A^\bullet)_\mathfrak p = \text{Ext}^{-i+\dim(A/\mathfrak p)}_{A_\mathfrak p}(M_\mathfrak p, (\omega_A^\bullet)_\mathfrak p[-\dim(A/\mathfrak p)])$$

is zero for $i - \dim(A/\mathfrak p) < \text{depth}_{A_\mathfrak p}(M_\mathfrak p)$ and nonzero for $i = \dim(A/\mathfrak p) + \text{depth}_{A_\mathfrak p}(M_\mathfrak p)$ by part (3) over $A_\mathfrak p$. This proves part (4). If E is an injective hull of the residue field of A, then we have

$$\text{Hom}_A(H^i_\mathfrak m(M), E) = \text{Ext}^{-i}_A(M, \omega_A^\bullet)^\wedge = (E^i)^\wedge = E^i \otimes_A A^\wedge$$

by the local duality theorem (in the form of Dualizing Complexes, Lemma 18.4). Since $A \to A^\wedge$ is faithfully flat, we find (5) is true by Matlis duality (Dualizing Complexes, Proposition 7.8). □

10. Annihilators of local cohomology, I

This section discusses a result due to Faltings, see [Fal78].

Proposition 10.1. *Let A be a Noetherian ring which has a dualizing complex. Let $T \subset T' \subset \text{Spec}(A)$ be subsets stable under specialization. Let $s \geq 0$ an integer. Let M be a finite A-module. The following are equivalent*

1. *there exists an ideal $J \subset A$ with $V(J) \subset T'$ such that J annihilates $H^i_T(M)$ for $i \leq s$, and*
2. *for all $\mathfrak p \not\in T'$, $\mathfrak q \in T$ with $\mathfrak p \subset \mathfrak q$ we have*

$$\text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$$

Proof. Let $\omega_A^\bullet$ be a dualizing complex. Let δ be its dimension function, see Dualizing Complexes, Section 17. An important role will be played by the finite A-modules

$$E^i = \text{Ext}^i_A(M, \omega_A^\bullet)$$

For $\mathfrak p \subset A$ we will write $H^i_\mathfrak p$ to denote the local cohomology of an $A_\mathfrak p$-module with respect to $\mathfrak p A_\mathfrak p$. Then we see that the $\mathfrak p A_\mathfrak p$-adic completion of

$$(E^i)_\mathfrak p = \text{Ext}^{\delta(\mathfrak p)+i}_{A_\mathfrak p}(M_\mathfrak p, (\omega_A^\bullet)_\mathfrak p[-\delta(\mathfrak p)])$$

is Matlis dual to

$$H_{\mathfrak p}^{-\delta(\mathfrak p)-i}(M_{\mathfrak p})$$

by Dualizing Complexes, Lemma 18.4. In particular we deduce from this the following fact: an ideal $J \subset A$ annihilates $(E^i)_{\mathfrak p}$ if and only if J annihilates $H_{\mathfrak p}^{-\delta(\mathfrak p)-i}(M_{\mathfrak p})$.

Set $T_n = \{\mathfrak p \in T \mid \delta(\mathfrak p) \leq n\}$. As δ is a bounded function, we see that $T_a = \emptyset$ for $a \ll 0$ and $T_b = T$ for $b \gg 0$.

Assume (2). Let us prove the existence of J as in (1). We will use a double induction to do this. For $i \leq s$ consider the induction hypothesis IH_i: $H^a_T(M)$ is annihilated by some $J \subset A$ with $V(J) \subset T'$ for $0 \leq a \leq i$. The case IH_0 is trivial because $H^0_T(M)$ is a submodule of M and hence finite and hence is annihilated by some ideal J with $V(J) \subset T$.

Induction step. Assume IH_{i-1} holds for some $0 < i \leq s$. Pick J' with $V(J') \subset T'$ annihilating $H^a_T(M)$ for $0 \leq a \leq i-1$ (the induction hypothesis guarantees we can do this). We will show by descending induction on n that there exists an ideal J with $V(J) \subset T'$ such that the associated primes of $JH^i_T(M)$ are in T_n. For $n \ll 0$ this implies $JH^i_T(M) = 0$ (Algebra, Lemma 63.7) and hence IH_i will hold. The base case $n \gg 0$ is trivial because $T = T_n$ in this case and all associated primes of $H^i_T(M)$ are in T.

Thus we assume given J with the property for n. Let $\mathfrak q \in T_n$. Let $T_{\mathfrak q} \subset \mathrm{Spec}(A_{\mathfrak q})$ be the inverse image of T. We have $H^j_T(M)_{\mathfrak q} = H^j_{T_{\mathfrak q}}(M_{\mathfrak q})$ by Lemma 5.7. Consider the spectral sequence

$$H^p_{\mathfrak q}(H^q_{T_{\mathfrak q}}(M_{\mathfrak q})) \Rightarrow H^{p+q}_{\mathfrak q}(M_{\mathfrak q})$$

of Lemma 5.8. Below we will find an ideal $J'' \subset A$ with $V(J'') \subset T'$ such that $H^i_{\mathfrak q}(M_{\mathfrak q})$ is annihilated by J'' for all $\mathfrak q \in T_n \setminus T_{n-1}$. Claim: $J(J')^iJ''$ will work for $n-1$. Namely, let $\mathfrak q \in T_n \setminus T_{n-1}$. The spectral sequence above defines a filtration

$$E^{0,i}_\infty = E^{0,i}_{i+2} \subset \ldots \subset E^{0,i}_3 \subset E^{0,i}_2 = H^0_{\mathfrak q}(H^i_{T_{\mathfrak q}}(M_{\mathfrak q}))$$

The module $E^{0,i}_\infty$ is annihilated by J''. The subquotients $E^{0,i}_j/E^{0,i}_{j+1}$ for $i+1 \geq j \geq 2$ are annihilated by J' because the target of $d^{0,i}_j$ is a subquotient of

$$H^j_{\mathfrak q}(H^{i-j+1}_{T_{\mathfrak q}}(M_{\mathfrak q})) = H^j_{\mathfrak q}(H^{i-j+1}_T(M)_{\mathfrak q})$$

and $H^{i-j+1}_T(M)_{\mathfrak q}$ is annihilated by J' by choice of J'. Finally, by our choice of J we have $JH^i_T(M)_{\mathfrak q} \subset H^0_{\mathfrak q}(H^i_T(M)_{\mathfrak q})$ since the non-closed points of $\mathrm{Spec}(A_{\mathfrak q})$ have higher δ values. Thus $\mathfrak q$ cannot be an associated prime of $J(J')^iJ''H^i_T(M)$ as desired.

By our initial remarks we see that J'' should annihilate

$$(E^{-\delta(\mathfrak q)-i})_{\mathfrak q} = (E^{-n-i})_{\mathfrak q}$$

for all $\mathfrak q \in T_n \setminus T_{n-1}$. But if J'' works for one $\mathfrak q$, then it works for all $\mathfrak q$ in an open neighbourhood of $\mathfrak q$ as the modules E^{-n-i} are finite. Since every subset of $\mathrm{Spec}(A)$ is Noetherian with the induced topology (Topology, Lemma 9.2), we conclude that it suffices to prove the existence of J'' for one $\mathfrak q$.

Since the ext modules are finite the existence of J'' is equivalent to

$$\mathrm{Supp}(E^{-n-i}) \cap \mathrm{Spec}(A_{\mathfrak q}) \subset T'.$$

This is equivalent to showing the localization of E^{-n-i} at every $\mathfrak p \subset \mathfrak q$, $\mathfrak p \not\in T'$ is zero. Using local duality over $A_\mathfrak p$ we find that we need to prove that

$$H_\mathfrak p^{i+n-\delta(\mathfrak p)}(M_\mathfrak p) = H_\mathfrak p^{i-\dim((A/\mathfrak p)_\mathfrak q)}(M_\mathfrak p)$$

is zero (this uses that δ is a dimension function). This vanishes by the assumption in the lemma and $i \leq s$ and Dualizing Complexes, Lemma 11.1.

To prove the converse implication we assume (2) does not hold and we work backwards through the arguments above. First, we pick a $\mathfrak q \in T$, $\mathfrak p \subset \mathfrak q$ with $\mathfrak p \not\in T'$ such that

$$i = \text{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) \leq s$$

is minimal. Then $H_\mathfrak p^{i-\dim((A/\mathfrak p)_\mathfrak q)}(M_\mathfrak p)$ is nonzero by the nonvanishing in Dualizing Complexes, Lemma 11.1. Set $n = \delta(\mathfrak q)$. Then there does not exist an ideal $J \subset A$ with $V(J) \subset T'$ such that $J(E^{-n-i})_\mathfrak q = 0$. Thus $H^i_\mathfrak q(M_\mathfrak q)$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. By minimality of i it follows from the spectral sequence displayed above that the module $H^i_T(M)_\mathfrak q$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. Thus $H^i_T(M)$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. This finishes the proof of the proposition. $\square$

Lemma 10.2. *Let I be an ideal of a Noetherian ring A. Let M be a finite A-module, let $\mathfrak p \subset A$ be a prime ideal, and let $s \geq 0$ be an integer. Assume*

1. *A has a dualizing complex,*
2. *$\mathfrak p \not\in V(I)$, and*
3. *for all primes $\mathfrak p' \subset \mathfrak p$ and $\mathfrak q \in V(I)$ with $\mathfrak p' \subset \mathfrak q$ we have*

$$\mathit{depth}_{A_{\mathfrak p'}}(M_{\mathfrak p'}) + \dim((A/\mathfrak p')_\mathfrak q) > s$$

Then there exists an $f \in A$, $f \not\in \mathfrak p$ which annihilates $H^i_{V(I)}(M)$ for $i \leq s$.

Proof. Consider the sets

$$T = V(I) \quad \text{and} \quad T' = \bigcup\nolimits_{f \in A, f \not\in \mathfrak p} V(f)$$

These are subsets of $\text{Spec}(A)$ stable under specialization. Observe that $T \subset T'$ and $\mathfrak p \not\in T'$. Assumption (3) says that hypothesis (2) of Proposition 10.1 holds. Hence we can find $J \subset A$ with $V(J) \subset T'$ such that $JH^i_{V(I)}(M) = 0$ for $i \leq s$. Choose $f \in A$, $f \not\in \mathfrak p$ with $V(J) \subset V(f)$. A power of f annihilates $H^i_{V(I)}(M)$ for $i \leq s$. $\square$

11. Finiteness of local cohomology, II

We continue the discussion of finiteness of local cohomology started in Section 7. Using Faltings Annihilator Theorem we easily prove the following fundamental result.

Proposition 11.1. *Let A be a Noetherian ring which has a dualizing complex. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. Let $s \geq 0$ an integer. Let M be a finite A-module. The following are equivalent*

1. *$H^i_T(M)$ is a finite A-module for $i \leq s$, and*
2. *for all $\mathfrak p \not\in T$, $\mathfrak q \in T$ with $\mathfrak p \subset \mathfrak q$ we have*

$$\mathit{depth}_{A_\mathfrak p}(M_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$$

Proof. Formal consequence of Proposition 10.1 and Lemma 7.1. $\square$

Besides some lemmas for later use, the rest of this section is concerned with the question to what extend the condition in Proposition 11.1 that A has a dualizing complex can be weakened. The answer is roughly that one has to assume the formal fibres of A are (S_n) for sufficiently large n.

Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $X = \mathrm{Spec}(A)$ and $Z = V(I) \subset X$. Let M be a finite A-module. We define

$$(11.1.1) \quad s_{A,I}(M) = \min\{\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) \mid \mathfrak{p} \in X \setminus Z, \mathfrak{q} \in Z, \mathfrak{p} \subset \mathfrak{q}\}$$

Our conventions on depth are that the depth of 0 is ∞ thus we only need to consider primes $\mathfrak{p}$ in the support of M. It will turn out that $s_{A,I}(M)$ is an important invariant of the situation.

Lemma 11.2. *Let $A \to B$ be a finite homomorphism of Noetherian rings. Let $I \subset A$ be an ideal and set $J = IB$. Let M be a finite B-module. If A is universally catenary, then $s_{B,J}(M) = s_{A,I}(M)$.*

Proof. Let $\mathfrak{p} \subset \mathfrak{q} \subset A$ be primes with $I \subset \mathfrak{q}$ and $I \not\subset \mathfrak{p}$. Since $A \to B$ is finite there are finitely many primes $\mathfrak{p}_i$ lying over $\mathfrak{p}$. By Algebra, Lemma 72.11 we have

$$\mathrm{depth}(M_\mathfrak{p}) = \min \mathrm{depth}(M_{\mathfrak{p}_i})$$

Let $\mathfrak{p}_i \subset \mathfrak{q}_{ij}$ be primes lying over $\mathfrak{q}$. By going up for $A \to B$ (Algebra, Lemma 36.22) there is at least one $\mathfrak{q}_{ij}$ for each i. Then we see that

$$\dim((B/\mathfrak{p}_i)_{\mathfrak{q}_{ij}}) = \dim((A/\mathfrak{p})_\mathfrak{q})$$

by the dimension formula, see Algebra, Lemma 113.1. This implies that the minimum of the quantities used to define $s_{B,J}(M)$ for the pairs $(\mathfrak{p}_i, \mathfrak{q}_{ij})$ is equal to the quantity for the pair $(\mathfrak{p}, \mathfrak{q})$. This proves the lemma. $\square$

Lemma 11.3. *Let A be a Noetherian ring which has a dualizing complex. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let A', M' be the I-adic completions of A, M. Let $\mathfrak{p}' \subset \mathfrak{q}'$ be prime ideals of A' with $\mathfrak{q}' \in V(IA')$ lying over $\mathfrak{p} \subset \mathfrak{q}$ in A. Then*

$$\mathit{depth}_{A_{\mathfrak{p}'}}(M'_{\mathfrak{p}'}) \geq \mathit{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$$

and

$$\mathit{depth}_{A_{\mathfrak{p}'}}(M'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) = \mathit{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})$$

Proof. We have

$$\mathrm{depth}(M'_{\mathfrak{p}'}) = \mathrm{depth}(M_\mathfrak{p}) + \mathrm{depth}(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) \geq \mathrm{depth}(M_\mathfrak{p})$$

by flatness of $A \to A'$, see Algebra, Lemma 163.1. Since the fibres of $A \to A'$ are Cohen-Macaulay (Dualizing Complexes, Lemma 23.2 and More on Algebra, Section 51) we see that $\mathrm{depth}(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) = \dim(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'})$. Thus we obtain

$$\begin{aligned}
\mathrm{depth}(M'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) &= \mathrm{depth}(M_\mathfrak{p}) + \dim(A'_{\mathfrak{p}'}/\mathfrak{p}A'_{\mathfrak{p}'}) + \dim((A'/\mathfrak{p}')_{\mathfrak{q}'}) \\
&= \mathrm{depth}(M_\mathfrak{p}) + \dim((A'/\mathfrak{p}A')_{\mathfrak{q}'}) \\
&= \mathrm{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})
\end{aligned}$$

Second equality because A' is catenary and third equality by More on Algebra, Lemma 43.1 as $(A/\mathfrak{p})_\mathfrak{q}$ and $(A'/\mathfrak{p}A')_{\mathfrak{q}'}$ have the same I-adic completions. $\square$

Lemma 11.4. *Let A be a universally catenary Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Then*

$$s_{A,I}(M) \geq s_{A^\wedge, I^\wedge}(M^\wedge)$$

If the formal fibres of A are (S_n), then $\min(n+1, s_{A,I}(M)) \leq s_{A^\wedge, I^\wedge}(M^\wedge)$.

Proof. Write $X = \text{Spec}(A)$, $X^\wedge = \text{Spec}(A^\wedge)$, $Z = V(I) \subset X$, and $Z^\wedge = V(I^\wedge)$. Let $\mathfrak{p}' \subset \mathfrak{q}' \subset A^\wedge$ be primes with $\mathfrak{p}' \notin Z^\wedge$ and $\mathfrak{q}' \in Z^\wedge$. Let $\mathfrak{p} \subset \mathfrak{q}$ be the corresponding primes of A. Then $\mathfrak{p} \notin Z$ and $\mathfrak{q} \in Z$. Picture

$$\begin{tikzcd}
\mathfrak{p}' \arrow[r] \arrow[d, dash] & \mathfrak{q}' \arrow[r] \arrow[d, dash] & A^\wedge \\
\mathfrak{p} \arrow[r] & \mathfrak{q} \arrow[r] & A \arrow[u]
\end{tikzcd}$$

Let us write

$$\begin{aligned}
a &= \dim(A/\mathfrak{p}) = \dim(A^\wedge/\mathfrak{p}A^\wedge), \\
b &= \dim(A/\mathfrak{q}) = \dim(A^\wedge/\mathfrak{q}A^\wedge), \\
a' &= \dim(A^\wedge/\mathfrak{p}'), \\
b' &= \dim(A^\wedge/\mathfrak{q}')
\end{aligned}$$

Equalities by More on Algebra, Lemma 43.1. We also write

$$\begin{aligned}
p &= \dim(A^\wedge_{\mathfrak{p}'}/\mathfrak{p}A^\wedge_{\mathfrak{p}'}) = \dim((A^\wedge/\mathfrak{p}A^\wedge)_{\mathfrak{p}'}) \\
q &= \dim(A^\wedge_{\mathfrak{q}'}/\mathfrak{p}A^\wedge_{\mathfrak{q}'}) = \dim((A^\wedge/\mathfrak{q}A^\wedge)_{\mathfrak{q}'})
\end{aligned}$$

Since A is universally catenary we see that $A^\wedge/\mathfrak{p}A^\wedge = (A/\mathfrak{p})^\wedge$ is equidimensional of dimension a (More on Algebra, Proposition 109.5). Hence $a = a' + p$. Similarly $b = b' + q$. By Algebra, Lemma 163.1 applied to the flat local ring map $A_\mathfrak{p} \to A^\wedge_{\mathfrak{p}'}$ we have

$$\text{depth}(M^\wedge_{\mathfrak{p}'}) = \text{depth}(M_\mathfrak{p}) + \text{depth}(A^\wedge_{\mathfrak{p}'}/\mathfrak{p}A^\wedge_{\mathfrak{p}'})$$

The quantity we are minimizing for $s_{A,I}(M)$ is

$$s(\mathfrak{p}, \mathfrak{q}) = \text{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) = \text{depth}(M_\mathfrak{p}) + a - b$$

(last equality as A is catenary). The quantity we are minimizing for $s_{A^\wedge, I^\wedge}(M^\wedge)$ is

$$s(\mathfrak{p}', \mathfrak{q}') = \text{depth}(M^\wedge_{\mathfrak{p}'}) + \dim((A^\wedge/\mathfrak{p}')_{\mathfrak{q}'}) = \text{depth}(M^\wedge_{\mathfrak{p}'}) + a' - b'$$

(last equality as $A^\wedge$ is catenary). Now we have enough notation in place to start the proof.

Let $\mathfrak{p} \subset \mathfrak{q} \subset A$ be primes with $\mathfrak{p} \notin Z$ and $\mathfrak{q} \in Z$ such that $s_{A,I}(M) = s(\mathfrak{p}, \mathfrak{q})$. Then we can pick $\mathfrak{q}'$ minimal over $\mathfrak{q}A^\wedge$ and $\mathfrak{p}' \subset \mathfrak{q}'$ minimal over $\mathfrak{p}A^\wedge$ (using going down for $A \to A^\wedge$). Then we have four primes as above with $p = 0$ and $q = 0$. Moreover, we have $\text{depth}(A^\wedge_{\mathfrak{p}'}/\mathfrak{p}A^\wedge_{\mathfrak{p}'}) = 0$ also because $p = 0$. This means that $s(\mathfrak{p}', \mathfrak{q}') = s(\mathfrak{p}, \mathfrak{q})$. Thus we get the first inequality.

Assume that the formal fibres of A are (S_n). Then $\text{depth}(A^\wedge_{\mathfrak{p}'}/\mathfrak{p}A^\wedge_{\mathfrak{p}'}) \geq \min(n, p)$. Hence

$$s(\mathfrak{p}', \mathfrak{q}') \geq s(\mathfrak{p}, \mathfrak{q}) + q + \min(n, p) - p \geq s_{A,I}(M) + q + \min(n, p) - p$$

Thus the only way we can get in trouble is if $p > n$. If this happens then

$$\begin{aligned} s(\mathfrak p', \mathfrak q') &= \text{depth}(M^\wedge_{\mathfrak p'}) + \dim((A^\wedge/\mathfrak p')_{\mathfrak q'}) \\ &= \text{depth}(M_{\mathfrak p}) + \text{depth}(A^\wedge_{\mathfrak p'}/\mathfrak p A^\wedge_{\mathfrak p'}) + \dim((A^\wedge/\mathfrak p')_{\mathfrak q'}) \\ &\geq 0 + n + 1 \end{aligned}$$

because $(A^\wedge/\mathfrak p')_{\mathfrak q'}$ has at least two primes. This proves the second inequality. ☐

The method of proof of the following lemma works more generally, but the stronger results one gets will be subsumed in Theorem 11.6 below.

Lemma 11.5. *Let A be a Gorenstein Noetherian local ring. Let $I \subset A$ be an ideal and set $Z = V(I) \subset \text{Spec}(A)$. Let M be a finite A-module. Let $s = s_{A,I}(M)$ as in (11.1.1). Then $H^i_Z(M)$ is finite for $i < s$, but $H^s_Z(M)$ is not finite.*

Proof. Since a Gorenstein local ring has a dualizing complex, this is a special case of Proposition 11.1. It would be helpful to have a short proof of this special case, which will be used in the proof of a general finiteness theorem below. ☐

Observe that the hypotheses of the following theorem are satisfied by excellent Noetherian rings (by definition), by Noetherian rings which have a dualizing complex (Dualizing Complexes, Lemma 17.4 and Dualizing Complexes, Lemma 23.2), and by quotients of regular Noetherian rings.

Theorem 11.6. *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Set $Z = V(I) \subset \text{Spec}(A)$. Let M be a finite A-module. Set $s = s_{A,I}(M)$ as in (11.1.1). Assume that*

1. *A is universally catenary,*
2. *the formal fibres of the local rings of A are Cohen-Macaulay.*

Then $H^i_Z(M)$ is finite for $0 \leq i < s$ and $H^s_Z(M)$ is not finite.

Proof. By Lemma 7.2 we may assume that A is a local ring.

If A is a Noetherian complete local ring, then we can write A as the quotient of a regular complete local ring B by Cohen's structure theorem (Algebra, Theorem 160.8). Using Lemma 11.2 and Dualizing Complexes, Lemma 9.2 we reduce to the case of a regular local ring which is a consequence of Lemma 11.5 because a regular local ring is Gorenstein (Dualizing Complexes, Lemma 21.3).

Let A be a Noetherian local ring. Let $\mathfrak m$ be the maximal ideal. We may assume $I \subset \mathfrak m$, otherwise the lemma is trivial. Let $A^\wedge$ be the completion of A, let $Z^\wedge = V(IA^\wedge)$, and let $M^\wedge = M \otimes_A A^\wedge$ be the completion of M (Algebra, Lemma 97.1). Then $H^i_Z(M) \otimes_A A^\wedge = H^i_{Z^\wedge}(M^\wedge)$ by Dualizing Complexes, Lemma 9.3 and flatness of $A \to A^\wedge$ (Algebra, Lemma 97.2). Hence it suffices to show that $H^i_{Z^\wedge}(M^\wedge)$ is finite for $i < s$ and not finite for $i = s$, see Algebra, Lemma 83.2. Since we know the result is true for $A^\wedge$ it suffices to show that $s_{A,I}(M) = s_{A^\wedge, I^\wedge}(M^\wedge)$. This follows from Lemma 11.4. ☐

Remark 11.7. The astute reader will have realized that we can get away with a slightly weaker condition on the formal fibres of the local rings of A. Namely, in the situation of Theorem 11.6 assume A is universally catenary but make no assumptions on the formal fibres. Suppose we have an n and we want to prove that $H^i_Z(M)$ are finite for $i \leq n$. Then the exact same proof shows that it suffices that $s_{A,I}(M) > n$ and that the formal fibres of local rings of A are (S_n). On the other hand, if we want to show that $H^s_Z(M)$ is not finite where $s = s_{A,I}(M)$, then our arguments prove this if the formal fibres are (S_{s-1}).

12. Finiteness of pushforwards, II

This section is the continuation of Section 8. In this section we reap the fruits of the labor done in Section 11.

Lemma 12.1. *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Let $n \geq 0$ be an integer. Assume*

1. *X is universally catenary,*
2. *for every $z \in Z$ the formal fibres of $\mathcal{O}_{X,z}$ are (S_n).*

In this situation the following are equivalent

- (a) *for $x \in Supp(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$ we have $depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}},z}) > n$,*
- (b) *$R^pj_*\mathcal{F}$ is coherent for $0 \leq p < n$.*

Proof. The statement is local on X, hence we may assume X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = V(I)$. Let M be a finite A-module whose associated coherent $\mathcal{O}_X$-module restricts to $\mathcal{F}$ over U, see Lemma 8.2. This lemma also tells us that $R^pj_*\mathcal{F}$ is coherent if and only if $H^{p+1}_Z(M)$ is a finite A-module. Observe that the minimum of the expressions $\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\mathcal{O}_{\overline{\{x\}},z})$ is the number $s_{A,I}(M)$ of (11.1.1). Having said this the lemma follows from Theorem 11.6 as elucidated by Remark 11.7. $\square$

Lemma 12.2. *Let X be a locally Noetherian scheme. Let $j : U \to X$ be the inclusion of an open subscheme with complement Z. Let $n \geq 0$ be an integer. If $R^pj_*\mathcal{O}_U$ is coherent for $0 \leq p < n$, then the same is true for $R^pj_*\mathcal{F}$, $0 \leq p < n$ for any finite locally free $\mathcal{O}_U$-module $\mathcal{F}$.*

Proof. The question is local on X, hence we may assume X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = V(I)$. Via Lemma 8.2 our lemma follows from Lemma 7.4. $\square$

Lemma 12.3. *Let A be a ring and let $J \subset I \subset A$ be finitely generated ideals. Let $p \geq 0$ be an integer. Set $U = \mathrm{Spec}(A) \setminus V(I)$. If $H^p(U, \mathcal{O}_U)$ is annihilated by J^n for some n, then $H^p(U, \mathcal{F})$ annihilated by J^m for some $m = m(\mathcal{F})$ for every finite locally free $\mathcal{O}_U$-module $\mathcal{F}$.*

Proof. Consider the annihilator $\mathfrak{a}$ of $H^p(U, \mathcal{F})$. Let $u \in U$. There exists an open neighbourhood $u \in U' \subset U$ and an isomorphism $\varphi : \mathcal{O}_{U'}^{\oplus r} \to \mathcal{F}|_{U'}$. Pick $f \in A$ such that $u \in D(f) \subset U'$. There exist maps

$$a : \mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F} \quad \text{and} \quad b : \mathcal{F} \longrightarrow \mathcal{O}_U^{\oplus r}$$

whose restriction to $D(f)$ are equal to $f^N\varphi$ and $f^N\varphi^{-1}$ for some N. Moreover we may assume that $a \circ b$ and $b \circ a$ are equal to multiplication by f^{2N}. This follows from Properties, Lemma 17.3 since U is quasi-compact (I is finitely generated), separated, and $\mathcal{F}$ and $\mathcal{O}_U^{\oplus r}$ are finitely presented. Thus we see that $H^p(U, \mathcal{F})$ is annihilated by $f^{2N}J^n$, i.e., $f^{2N}J^n \subset \mathfrak{a}$.

As U is quasi-compact we can find finitely many $f_1, \ldots, f_t$ and $N_1, \ldots, N_t$ such that $U = \bigcup D(f_i)$ and $f_i^{2N_i}J^n \subset \mathfrak{a}$. Then $V(I) = V(f_1, \ldots, f_t)$ and since I is finitely generated we conclude $I^M \subset (f_1, \ldots, f_t)$ for some M. All in all we see that $J^m \subset \mathfrak{a}$ for $m \gg 0$, for example $m = M(2N_1 + \ldots + 2N_t)n$ will do. $\square$

13. Annihilators of local cohomology, II

We extend the discussion of annihilators of local cohomology in Section 10 to bounded below complexes with finite cohomology modules.

Definition 13.1. Let I be an ideal of a Noetherian ring A. Let $K \in D^+_{Coh}(A)$. We define the *I-depth* of K, denoted $\text{depth}_I(K)$, to be the maximal $m \in \mathbf{Z} \cup \{\infty\}$ such that $H^i_I(K) = 0$ for all $i < m$. If A is local with maximal ideal $\mathfrak m$ then we call $\text{depth}_{\mathfrak m}(K)$ simply the *depth* of K.

This definition does not conflict with Algebra, Definition 72.1 by Dualizing Complexes, Lemma 11.1.

Proposition 13.2. *Let A be a Noetherian ring which has a dualizing complex. Let $T \subset T' \subset \text{Spec}(A)$ be subsets stable under specialization. Let $s \in \mathbf{Z}$. Let K be an object of $D^+_{Coh}(A)$. The following are equivalent*

1. *there exists an ideal $J \subset A$ with $V(J) \subset T'$ such that J annihilates $H^i_T(K)$ for $i \leq s$, and*
2. *for all $\mathfrak p \not\in T'$, $\mathfrak q \in T$ with $\mathfrak p \subset \mathfrak q$ we have*

$$\mathit{depth}_{A_\mathfrak p}(K_\mathfrak p) + \dim((A/\mathfrak p)_\mathfrak q) > s$$

Proof. This lemma is the natural generalization of Proposition 10.1 whose proof the reader should read first. Let $\omega_A^\bullet$ be a dualizing complex. Let δ be its dimension function, see Dualizing Complexes, Section 17. An important role will be played by the finite A-modules

$$E^i = \text{Ext}^i_A(K, \omega_A^\bullet)$$

For $\mathfrak p \subset A$ we will write $H^i_\mathfrak p$ to denote the local cohomology of an object of $D(A_\mathfrak p)$ with respect to $\mathfrak pA_\mathfrak p$. Then we see that the $\mathfrak pA_\mathfrak p$-adic completion of

$$(E^i)_\mathfrak p = \text{Ext}^{\delta(\mathfrak p)+i}_{A_\mathfrak p}(K_\mathfrak p, (\omega_A^\bullet)_\mathfrak p[-\delta(\mathfrak p)])$$

is Matlis dual to

$$H^{-\delta(\mathfrak p)-i}_\mathfrak p(K_\mathfrak p)$$

by Dualizing Complexes, Lemma 18.4. In particular we deduce from this the following fact: an ideal $J \subset A$ annihilates $(E^i)_\mathfrak p$ if and only if J annihilates $H^{-\delta(\mathfrak p)-i}_\mathfrak p(K_\mathfrak p)$.

Set $T_n = \{\mathfrak p \in T \mid \delta(\mathfrak p) \leq n\}$. As δ is a bounded function, we see that $T_a = \emptyset$ for $a \ll 0$ and $T_b = T$ for $b \gg 0$.

Assume (2). Let us prove the existence of J as in (1). We will use a double induction to do this. For $i \leq s$ consider the induction hypothesis IH_i: $H^a_T(K)$ is annihilated by some $J \subset A$ with $V(J) \subset T'$ for $a \leq i$. The case IH_i is trivial for i small enough because K is bounded below.

Induction step. Assume IH_{i-1} holds for some $i \leq s$. Pick J' with $V(J') \subset T'$ annihilating $H^a_T(K)$ for $a \leq i-1$ (the induction hypothesis guarantees we can do this). We will show by descending induction on n that there exists an ideal J with $V(J) \subset T'$ such that the associated primes of $JH^i_T(K)$ are in T_n. For $n \ll 0$ this implies $JH^i_T(K) = 0$ (Algebra, Lemma 63.7) and hence IH_i will hold. The base case $n \gg 0$ is trivial because $T = T_n$ in this case and all associated primes of $H^i_T(K)$ are in T.

Thus we assume given J with the property for n. Let $\mathfrak q \in T_n$. Let $T_{\mathfrak q} \subset \text{Spec}(A_{\mathfrak q})$ be the inverse image of T. We have $H^j_T(K)_{\mathfrak q} = H^j_{T_{\mathfrak q}}(K_{\mathfrak q})$ by Lemma 5.7. Consider the spectral sequence

$$H^p_{\mathfrak q}(H^q_{T_{\mathfrak q}}(K_{\mathfrak q})) \Rightarrow H^{p+q}_{\mathfrak q}(K_{\mathfrak q})$$

of Lemma 5.8. Below we will find an ideal $J'' \subset A$ with $V(J'') \subset T'$ such that $H^i_{\mathfrak q}(K_{\mathfrak q})$ is annihilated by J'' for all $\mathfrak q \in T_n \setminus T_{n-1}$. Claim: $J(J')^iJ''$ will work for $n-1$. Namely, let $\mathfrak q \in T_n \setminus T_{n-1}$. The spectral sequence above defines a filtration

$$E^{0,i}_\infty = E^{0,i}_{i+2} \subset \ldots \subset E^{0,i}_3 \subset E^{0,i}_2 = H^0_{\mathfrak q}(H^i_{T_{\mathfrak q}}(K_{\mathfrak q}))$$

The module $E^{0,i}_\infty$ is annihilated by J''. The subquotients $E^{0,i}_j/E^{0,i}_{j+1}$ for $i+1 \geq j \geq 2$ are annihilated by J' because the target of $d^{0,i}_j$ is a subquotient of

$$H^j_{\mathfrak q}(H^{i-j+1}_{T_{\mathfrak q}}(K_{\mathfrak q})) = H^j_{\mathfrak q}(H^{i-j+1}_T(K)_{\mathfrak q})$$

and $H^{i-j+1}_T(K)_{\mathfrak q}$ is annihilated by J' by choice of J'. Finally, by our choice of J we have $JH^i_T(K)_{\mathfrak q} \subset H^0_{\mathfrak q}(H^i_T(K)_{\mathfrak q})$ since the non-closed points of $\text{Spec}(A_{\mathfrak q})$ have higher δ values. Thus $\mathfrak q$ cannot be an associated prime of $J(J')^iJ''H^i_T(K)$ as desired.

By our initial remarks we see that J'' should annihilate

$$(E^{-\delta(\mathfrak q)-i})_{\mathfrak q} = (E^{-n-i})_{\mathfrak q}$$

for all $\mathfrak q \in T_n \setminus T_{n-1}$. But if J'' works for one $\mathfrak q$, then it works for all $\mathfrak q$ in an open neighbourhood of $\mathfrak q$ as the modules E^{-n-i} are finite. Since every subset of $\text{Spec}(A)$ is Noetherian with the induced topology (Topology, Lemma 9.2), we conclude that it suffices to prove the existence of J'' for one $\mathfrak q$.

Since the ext modules are finite the existence of J'' is equivalent to

$$\text{Supp}(E^{-n-i}) \cap \text{Spec}(A_{\mathfrak q}) \subset T'.$$

This is equivalent to showing the localization of E^{-n-i} at every $\mathfrak p \subset \mathfrak q$, $\mathfrak p \notin T'$ is zero. Using local duality over $A_{\mathfrak p}$ we find that we need to prove that

$$H^{i+n-\delta(\mathfrak p)}_{\mathfrak p}(K_{\mathfrak p}) = H^{i-\dim((A/\mathfrak p)_{\mathfrak q})}_{\mathfrak p}(K_{\mathfrak p})$$

is zero (this uses that δ is a dimension function). This vanishes by the assumption in the lemma and $i \leq s$ and our definition of depth in Definition 13.1.

To prove the converse implication we assume (2) does not hold and we work backwards through the arguments above. First, we pick a $\mathfrak q \in T$, $\mathfrak p \subset \mathfrak q$ with $\mathfrak p \notin T'$ such that

$$i = \text{depth}_{A_{\mathfrak p}}(K_{\mathfrak p}) + \dim((A/\mathfrak p)_{\mathfrak q}) \leq s$$

is minimal. Then $H^{i-\dim((A/\mathfrak p)_{\mathfrak q})}_{\mathfrak p}(K_{\mathfrak p})$ is nonzero by the our definition of depth in Definition 13.1. Set $n = \delta(\mathfrak q)$. Then there does not exist an ideal $J \subset A$ with $V(J) \subset T'$ such that $J(E^{-n-i})_{\mathfrak q} = 0$. Thus $H^i_{\mathfrak q}(K_{\mathfrak q})$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. By minimality of i it follows from the spectral sequence displayed above that the module $H^i_T(K)_{\mathfrak q}$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. Thus $H^i_T(K)$ is not annihilated by an ideal $J \subset A$ with $V(J) \subset T'$. This finishes the proof of the proposition. $\square$

14. Finiteness of local cohomology, III

We extend the discussion of finiteness of local cohomology in Sections 7 and 11 to bounded below complexes with finite cohomology modules.

Lemma 14.1. *Let A be a Noetherian ring. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. Let K be an object of $D^+_{Coh}(A)$. Let $n \in \mathbf{Z}$. The following are equivalent*

1. *$H^i_T(K)$ is finite for $i \leq n$,*
2. *there exists an ideal $J \subset A$ with $V(J) \subset T$ such that J annihilates $H^i_T(K)$ for $i \leq n$.*

If $T = V(I) = Z$ for an ideal $I \subset A$, then these are also equivalent to

3. *there exists an $e \geq 0$ such that I^e annihilates $H^i_Z(K)$ for $i \leq n$.*

Proof. This lemma is the natural generalization of Lemma 7.1 whose proof the reader should read first. Assume (1) is true. Recall that $H^i_J(K) = H^i_{V(J)}(K)$, see Dualizing Complexes, Lemma 10.1. Thus $H^i_T(K) = \text{colim}\, H^i_J(K)$ where the colimit is over ideals $J \subset A$ with $V(J) \subset T$, see Lemma 5.3. Since $H^i_T(K)$ is finitely generated for $i \leq n$ we can find a $J \subset A$ as in (2) such that $H^i_J(K) \to H^i_T(K)$ is surjective for $i \leq n$. Thus the finite list of generators are J-power torsion elements and we see that (2) holds with J replaced by some power.

Let $a \in \mathbf{Z}$ be an integer such that $H^i(K) = 0$ for $i < a$. We prove (2) $\Rightarrow$ (1) by descending induction on a. If $a > n$, then we have $H^i_T(K) = 0$ for $i \leq n$ hence both (1) and (2) are true and there is nothing to prove.

Assume we have J as in (2). Observe that $N = H^a_T(K) = H^0_T(H^a(K))$ is finite as a submodule of the finite A-module $H^a(K)$. If $n = a$ we are done; so assume $a < n$ from now on. By construction of $R\Gamma_T$ we find that $H^i_T(N) = 0$ for $i > 0$ and $H^0_T(N) = N$, see Remark 5.6. Choose a distinguished triangle

$$N[-a] \to K \to K' \to N[-a+1]$$

Then we see that $H^a_T(K') = 0$ and $H^i_T(K) = H^i_T(K')$ for $i > a$. We conclude that we may replace K by K'. Thus we may assume that $H^a_T(K) = 0$. This means that the finite set of associated primes of $H^a(K)$ are not in T. By prime avoidance (Algebra, Lemma 15.2) we can find $f \in J$ not contained in any of the associated primes of $H^a(K)$. Choose a distinguished triangle

$$L \to K \xrightarrow{f} K \to L[1]$$

By construction we see that $H^i(L) = 0$ for $i \leq a$. On the other hand we have a long exact cohomology sequence

$$0 \to H^{a+1}_T(L) \to H^{a+1}_T(K) \xrightarrow{f} H^{a+1}_T(K) \to H^{a+2}_T(L) \to H^{a+2}_T(K) \xrightarrow{f} \ldots$$

which breaks into the identification $H^{a+1}_T(L) = H^{a+1}_T(K)$ and short exact sequences

$$0 \to H^{i-1}_T(K) \to H^i_T(L) \to H^i_T(K) \to 0$$

for $i \leq n$ since $f \in J$. We conclude that J^2 annihilates $H^i_T(L)$ for $i \leq n$. By induction hypothesis applied to L we see that $H^i_T(L)$ is finite for $i \leq n$. Using the short exact sequence once more we see that $H^i_T(K)$ is finite for $i \leq n$ as desired.

We omit the proof of the equivalence of (2) and (3) in case $T = V(I)$. $\square$

Proposition 14.2. *Let A be a Noetherian ring which has a dualizing complex. Let $T \subset \text{Spec}(A)$ be a subset stable under specialization. Let $s \in \mathbf{Z}$. Let $K \in D^+_{Coh}(A)$. The following are equivalent*

1. *$H^i_T(K)$ is a finite A-module for $i \leq s$, and*
2. *for all $\mathfrak{p} \not\in T$, $\mathfrak{q} \in T$ with $\mathfrak{p} \subset \mathfrak{q}$ we have*
$$depth_{A_\mathfrak{p}}(K_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q}) > s$$

Proof. Formal consequence of Proposition 13.2 and Lemma 14.1. $\square$

15. Improving coherent modules

Similar constructions can be found in [DG67] and more recently in [Kol15] and [Kol16].

Lemma 15.1. *Let X be a Noetherian scheme. Let $T \subset X$ be a subset stable under specialization. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there is a unique map $\mathcal{F} \to \mathcal{F}'$ of coherent $\mathcal{O}_X$-modules such that*

1. *$\mathcal{F} \to \mathcal{F}'$ is surjective,*
2. *$\mathcal{F}_x \to \mathcal{F}'_x$ is an isomorphism for $x \not\in T$,*
3. *$depth_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 1$ for $x \in T$.*

If $f : Y \to X$ is a flat morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}'$ is the corresponding quotient for $f^{-1}(T) \subset Y$ and $f^*\mathcal{F}$.

Proof. Condition (3) just means that $\text{Ass}(\mathcal{F}') \cap T = \emptyset$. Thus $\mathcal{F} \to \mathcal{F}'$ is the quotient of $\mathcal{F}$ by the subsheaf of sections whose support is contained in T. This proves uniqueness. The statement on pullbacks follows from Divisors, Lemma 3.1 and the uniqueness.

Existence of $\mathcal{F} \to \mathcal{F}'$. By the uniqueness it suffices to prove the existence and uniqueness locally on X; small detail omitted. Thus we may assume $X = \text{Spec}(A)$ is affine and $\mathcal{F}$ is the coherent module associated to the finite A-module M. Set $M' = M/H^0_T(M)$ with $H^0_T(M)$ as in Section 5. Then $M_\mathfrak{p} = M'_\mathfrak{p}$ for $\mathfrak{p} \not\in T$ which proves (1). On the other hand, we have $H^0_T(M) = \text{colim}\, H^0_Z(M)$ where Z runs over the closed subsets of X contained in T. Thus by Dualizing Complexes, Lemmas 11.6 we have $H^0_T(M') = 0$, i.e., no associated prime of M' is in T. Therefore $\text{depth}(M'_\mathfrak{p}) \geq 1$ for $\mathfrak{p} \in T$. $\square$

Lemma 15.2. *Let $j : U \to X$ be an open immersion of Noetherian schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}' = j_*(\mathcal{F}|_U)$ is coherent. Then $\mathcal{F} \to \mathcal{F}'$ is the unique map of coherent $\mathcal{O}_X$-modules such that*

1. *$\mathcal{F}|_U \to \mathcal{F}'|_U$ is an isomorphism,*
2. *$depth_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 2$ for $x \in X$, $x \not\in U$.*

If $f : Y \to X$ is a flat morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}'$ is the corresponding map for $f^{-1}(U) \subset Y$.

Proof. We have $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x) \geq 2$ by Divisors, Lemma 6.6 part (3). The uniqueness of $\mathcal{F} \to \mathcal{F}'$ follows from Divisors, Lemma 5.11. The compatibility with flat pullbacks follows from flat base change, see Cohomology of Schemes, Lemma 5.2. $\square$

Lemma 15.3. *Let X be a Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume X is universally catenary and the formal* *fibres of local rings have (S_1). Then there exists a unique map $\mathcal{F} \to \mathcal{F}''$ of coherent $\mathcal{O}_X$-modules such that*

(1) *$\mathcal{F}_x \to \mathcal{F}''_x$ is an isomorphism for $x \in X \setminus Z$,*
(2) *$\mathcal{F}_x \to \mathcal{F}''_x$ is surjective and $depth_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) = 1$ for $x \in Z$ such that there exists an immediate specialization $x' \rightsquigarrow x$ with $x' \notin Z$ and $x' \in Ass(\mathcal{F})$,*
(3) *$depth_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$ for the remaining $x \in Z$.*

If $f : Y \to X$ is a Cohen-Macaulay morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}''$ satisfies the same properties with respect to $f^{-1}(Z) \subset Y$.

Proof. Let $\mathcal{F} \to \mathcal{F}'$ be the map constructed in Lemma 15.1 for the subset Z of X. Recall that $\mathcal{F}'$ is the quotient of $\mathcal{F}$ by the subsheaf of sections supported on Z.

We first prove uniqueness. Let $\mathcal{F} \to \mathcal{F}''$ be as in the lemma. We get a factorization $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ since $\text{Ass}(\mathcal{F}'') \cap Z = \emptyset$ by conditions (2) and (3). Let $U \subset X$ be a maximal open subscheme such that $\mathcal{F}'|_U \to \mathcal{F}''|_U$ is an isomorphism. We see that U contains all the points as in (2). Then by Divisors, Lemma 5.11 we conclude that $\mathcal{F}'' = j_*(\mathcal{F}'|_U)$. In this way we get uniqueness (small detail: if we have two of these $\mathcal{F}''$ then we take the intersection of the opens U we get from either).

Proof of existence. Recall that $\text{Ass}(\mathcal{F}') = \{x_1, \ldots, x_n\}$ is finite and $x_i \notin Z$. Let Y_i be the closure of $\{x_i\}$. Let $Z_{i,j}$ be the irreducible components of $Z \cap Y_i$. Observe that $\text{Supp}(\mathcal{F}') \cap Z = \bigcup Z_{i,j}$. Let $z_{i,j} \in Z_{i,j}$ be the generic point. Let

$$d_{i,j} = \dim(\mathcal{O}_{\overline{\{x_i\}}, z_{i,j}})$$

If $d_{i,j} = 1$, then $z_{i,j}$ is one of the points as in (2). Thus we do not need to modify $\mathcal{F}'$ at these points. Furthermore, still assuming $d_{i,j} = 1$, using Lemma 9.2 we can find an open neighbourhood $z_{i,j} \in V_{i,j} \subset X$ such that $\text{depth}_{\mathcal{O}_{X,z}}(\mathcal{F}'_z) \geq 2$ for $z \in Z_{i,j} \cap V_{i,j}$, $z \neq z_{i,j}$. Set

$$Z' = X \setminus \left(X \setminus Z \cup \bigcup\nolimits_{d_{i,j}=1} V_{i,j}) \right)$$

Denote $j' : X \setminus Z' \to X$. By our choice of Z' the assumptions of Lemma 8.9 are satisfied. We conclude by setting $\mathcal{F}'' = j'_*(\mathcal{F}'|_{X \setminus Z'})$ and applying Lemma 15.2.

The final statement follows from the formula for the change in depth along a flat local homomorphism, see Algebra, Lemma 163.1 and the assumption on the fibres of f inherent in f being Cohen-Macaulay. Details omitted. $\square$

Lemma 15.4. *Let X be a Noetherian scheme which locally has a dualizing complex. Let $T' \subset X$ be a subset stable under specialization. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that if $x \rightsquigarrow x'$ is an immediate specialization of points in X with $x' \in T'$ and $x \notin T'$, then $depth(\mathcal{F}_x) \geq 1$. Then there exists a unique map $\mathcal{F} \to \mathcal{F}''$ of coherent $\mathcal{O}_X$-modules such that*

(1) *$\mathcal{F}_x \to \mathcal{F}''_x$ is an isomorphism for $x \notin T'$,*
(2) *$depth_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$ for $x \in T'$.*

If $f : Y \to X$ is a Cohen-Macaulay morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}''$ satisfies the same properties with respect to $f^{-1}(T') \subset Y$.

Proof. Let $\mathcal{F} \to \mathcal{F}'$ be the quotient of $\mathcal{F}$ constructed in Lemma 15.1 using T'. Recall that $\mathcal{F}'$ is the quotient of $\mathcal{F}$ by the subsheaf of sections supported on T'.

Proof of uniqueness. Let $\mathcal{F} \to \mathcal{F}''$ be as in the lemma. We get a factorization $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ since $\text{Ass}(\mathcal{F}'') \cap T' = \emptyset$ by condition (2). Let $U \subset X$ be a maximal open subscheme such that $\mathcal{F}'|_U \to \mathcal{F}''|_U$ is an isomorphism. We see that U contains all the points of T'. Then by Divisors, Lemma 5.11 we conclude that $\mathcal{F}'' = j_*(\mathcal{F}'|_U)$. In this way we get uniqueness (small detail: if we have two of these $\mathcal{F}''$ then we take the intersection of the opens U we get from either).

Proof of existence. We will define

$$\mathcal{F}'' = \text{colim}\, j_*(\mathcal{F}'|_V)$$

where $j : V \to X$ runs over the open subschemes such that $X \setminus V \subset T'$. Observe that the colimit is filtered as T' is stable under specialization. Each of the maps $\mathcal{F}' \to j_*(\mathcal{F}'|_V)$ is injective as $\text{Ass}(\mathcal{F}')$ is disjoint from T'. Thus $\mathcal{F}' \to \mathcal{F}''$ is injective.

Suppose $X = \text{Spec}(A)$ is affine and $\mathcal{F}$ corresponds to the finite A-module M. Then $\mathcal{F}'$ corresponds to $M' = M/H^0_{T'}(M)$, see proof of Lemma 15.1. Applying Lemmas 2.2 and 5.3 we see that $\mathcal{F}''$ corresponds to an A-module M'' which fits into the short exact sequence

$$0 \to M' \to M'' \to H^1_{T'}(M') \to 0$$

By Proposition 11.1 and our condition on immediate specializations in the statement of the lemma we see that M'' is a finite A-module. In this way we see that $\mathcal{F}''$ is coherent.

The final statement follows from the formula for the change in depth along a flat local homomorphism, see Algebra, Lemma 163.1 and the assumption on the fibres of f inherent in f being Cohen-Macaulay. Details omitted. □

Lemma 15.5. *Let X be a Noetherian scheme which locally has a dualizing complex. Let $T' \subset T \subset X$ be subsets stable under specialization such that if $x \rightsquigarrow x'$ is an immediate specialization of points in X and $x' \in T'$, then $x \in T$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there exists a unique map $\mathcal{F} \to \mathcal{F}''$ of coherent $\mathcal{O}_X$-modules such that*

1. *$\mathcal{F}_x \to \mathcal{F}''_x$ is an isomorphism for $x \notin T$,*
2. *$\mathcal{F}_x \to \mathcal{F}''_x$ is surjective and $depth_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 1$ for $x \in T$, $x \notin T'$, and*
3. *$depth_{\mathcal{O}_{X,x}}(\mathcal{F}''_x) \geq 2$ for $x \in T'$.*

If $f : Y \to X$ is a Cohen-Macaulay morphism with Y Noetherian, then $f^*\mathcal{F} \to f^*\mathcal{F}''$ satisfies the same properties with respect to $f^{-1}(T') \subset f^{-1}(T) \subset Y$.

Proof. First, let $\mathcal{F} \to \mathcal{F}'$ be the quotient of $\mathcal{F}$ constructed in Lemma 15.1 using T. Second, let $\mathcal{F}' \to \mathcal{F}''$ be the unique map of coherent modules construction in Lemma 15.4 using T'. Then $\mathcal{F} \to \mathcal{F}''$ is as desired. □

16. Hartshorne-Lichtenbaum vanishing

This vanishing result is the local analogue of Lichtenbaum's theorem that the reader can find in Duality for Schemes, Section 34. This and much else besides can be found in [Har68].

Lemma 16.1. *Let A be a Noetherian ring of dimension d. Let $I \subset I' \subset A$ be ideals. If I' is contained in the Jacobson radical of A and $cd(A, I') < d$, then $cd(A, I) < d$.*

Proof. By Lemma 4.7 we know $\text{cd}(A, I) \leq d$. We will use Lemma 2.6 to show

$$H^d_{V(I')}(A) \to H^d_{V(I)}(A)$$

is surjective which will finish the proof. Pick $\mathfrak{p} \in V(I) \setminus V(I')$. By our assumption on I' we see that $\mathfrak{p}$ is not a maximal ideal of A. Hence $\dim(A_\mathfrak{p}) < d$. Then $H^d_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p}) = 0$ by Lemma 4.7. □

Lemma 16.2. *Let A be a Noetherian ring of dimension d. Let $I \subset A$ be an ideal. If $H^d_{V(I)}(M) = 0$ for some finite A-module whose support contains all the irreducible components of dimension d, then $cd(A, I) < d$.*

Proof. By Lemma 4.7 we know $\text{cd}(A, I) \leq d$. Thus for any finite A-module N we have $H^i_{V(I)}(N) = 0$ for $i > d$. Let us say property $\mathcal{P}$ holds for the finite A-module N if $H^d_{V(I)}(N) = 0$. One of our assumptions is that $\mathcal{P}(M)$ holds. Observe that $\mathcal{P}(N_1 \oplus N_2) \Leftrightarrow (\mathcal{P}(N_1) \wedge \mathcal{P}(N_2))$. Observe that if $N \to N'$ is surjective, then $\mathcal{P}(N) \Rightarrow \mathcal{P}(N')$ as we have the vanishing of $H^{d+1}_{V(I)}$ (see above). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of A with $\dim(A/\mathfrak{p}_i) = d$. Observe that $\mathcal{P}(N)$ holds if the support of N is disjoint from $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ for dimension reasons, see Lemma 4.7. For each i set $M_i = M/\mathfrak{p}_i M$. This is a finite A-module annihilated by $\mathfrak{p}_i$ whose support is equal to $V(\mathfrak{p}_i)$ (here we use the assumption on the support of M). Finally, if $J \subset A$ is an ideal, then we have $\mathcal{P}(JM_i)$ as JM_i is a quotient of a direct sum of copies of M. Thus it follows from Cohomology of Schemes, Lemma 12.8 that $\mathcal{P}$ holds for every finite A-module. □

Lemma 16.3. *Let A be a Noetherian local ring of dimension d. Let $f \in A$ be an element which is not contained in any minimal prime of dimension d. Then $f : H^d_{V(I)}(M) \to H^d_{V(I)}(M)$ is surjective for any finite A-module M and any ideal $I \subset A$.*

Proof. The support of M/fM has dimension $< d$ by our assumption on f. Thus $H^d_{V(I)}(M/fM) = 0$ by Lemma 4.7. Thus $H^d_{V(I)}(fM) \to H^d_{V(I)}(M)$ is surjective. Since by Lemma 4.7 we know $\text{cd}(A, I) \leq d$ we also see that the surjection $M \to fM$, $x \mapsto fx$ induces a surjection $H^d_{V(I)}(M) \to H^d_{V(I)}(fM)$. □

Lemma 16.4. *Let A be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $I \subset A$ be an ideal. If $H^0_{V(I)}(\omega_A^\bullet) = 0$, then $cd(A, I) < \dim(A)$.*

Proof. Set $d = \dim(A)$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset A$ be the minimal primes of dimension d. Recall that the finite A-module $H^{-i}(\omega_A^\bullet)$ is nonzero only for $i \in \{0, \ldots, d\}$ and that the support of $H^{-i}(\omega_A^\bullet)$ has dimension $\leq i$, see Lemma 9.4. Set $\omega_A = H^{-d}(\omega_A^\bullet)$. By prime avoidence (Algebra, Lemma 15.2) we can find $f \in A$, $f \notin \mathfrak{p}_i$ which annihilates $H^{-i}(\omega_A^\bullet)$ for $i < d$. Consider the distinguished triangle

$$\omega_A[d] \to \omega_A^\bullet \to \tau_{\geq -d+1}\omega_A^\bullet \to \omega_A[d+1]$$

See Derived Categories, Remark 12.4. By Derived Categories, Lemma 12.5 we see that f^d induces the zero endomorphism of $\tau_{\geq -d+1}\omega_A^\bullet$. Using the axioms of a triangulated category, we find a map

$$\omega_A^\bullet \to \omega_A[d]$$

whose composition with $\omega_A[d] \to \omega_A^\bullet$ is multiplication by f^d on $\omega_A[d]$. Thus we conclude that f^d annihilates $H^d_{V(I)}(\omega_A)$. By Lemma 16.3 we conlude $H^d_{V(I)}(\omega_A) =$

0. Then we conclude by Lemma 16.2 and the fact that $(\omega_A)_{\mathfrak{p}_i}$ is nonzero (see for example Dualizing Complexes, Lemma 16.11). □

Lemma 16.5. *Let $(A, \mathfrak{m})$ be a complete Noetherian local domain. Let $\mathfrak{p} \subset A$ be a prime ideal of dimension 1. For every $n \geq 1$ there is an $m \geq n$ such that $\mathfrak{p}^{(m)} \subset \mathfrak{p}^n$.*

Proof. Recall that the symbolic power $\mathfrak{p}^{(m)}$ is defined as the kernel of $A \to A_\mathfrak{p}/\mathfrak{p}^m A_\mathfrak{p}$. Since localization is exact we conclude that in the short exact sequence

$$0 \to \mathfrak{a}_n \to A/\mathfrak{p}^n \to A/\mathfrak{p}^{(n)} \to 0$$

the support of $\mathfrak{a}_n$ is contained in $\{\mathfrak{m}\}$. In particular, the inverse system $(\mathfrak{a}_n)$ is Mittag-Leffler as each $\mathfrak{a}_n$ is an Artinian A-module. We conclude that the lemma is equivalent to the requirement that $\lim \mathfrak{a}_n = 0$. Let $f \in \lim \mathfrak{a}_n$. Then f is an element of $A = \lim A/\mathfrak{p}^n$ (here we use that A is complete) which maps to zero in the completion $A_\mathfrak{p}^\wedge$ of $A_\mathfrak{p}$. Since $A_\mathfrak{p} \to A_\mathfrak{p}^\wedge$ is faithfully flat, we see that f maps to zero in $A_\mathfrak{p}$. Since A is a domain we see that f is zero as desired. □

Proposition 16.6. *Let A be a Noetherian local ring with completion $A^\wedge$. Let $I \subset A$ be an ideal such that*

$$\dim V(IA^\wedge + \mathfrak{p}) \geq 1$$

for every minimal prime $\mathfrak{p} \subset A^\wedge$ of dimension $\dim(A)$. Then $cd(A, I) < \dim(A)$.

Proof. Since $A \to A^\wedge$ is faithfully flat we have $H^d_{V(I)}(A) \otimes_A A^\wedge = H^d_{V(IA^\wedge)}(A^\wedge)$ by Dualizing Complexes, Lemma 9.3. Thus we may assume A is complete.

Assume A is complete. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset A$ be the minimal primes of dimension d. Consider the complete local ring $A_i = A/\mathfrak{p}_i$. We have $H^d_{V(I)}(A_i) = H^d_{V(IA_i)}(A_i)$ by Dualizing Complexes, Lemma 9.2. By Lemma 16.2 it suffices to prove the lemma for (A_i, IA_i). Thus we may assume A is a complete local domain.

Assume A is a complete local domain. We can choose a prime ideal $\mathfrak{p} \supset I$ with $\dim(A/\mathfrak{p}) = 1$. By Lemma 16.1 it suffices to prove the lemma for $\mathfrak{p}$.

By Lemma 16.4 it suffices to show that $H^0_{V(\mathfrak{p})}(\omega_A^\bullet) = 0$. Recall that

$$H^0_{V(\mathfrak{p})}(\omega_A^\bullet) = \operatorname{colim} \operatorname{Ext}^0_A(A/\mathfrak{p}^n, \omega_A^\bullet)$$

By Lemma 16.5 we see that the colimit is the same as

$$\operatorname{colim} \operatorname{Ext}^0_A(A/\mathfrak{p}^{(n)}, \omega_A^\bullet)$$

Since $\operatorname{depth}(A/\mathfrak{p}^{(n)}) = 1$ we see that these ext groups are zero by Lemma 9.4 as desired. □

Lemma 16.7. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset A$ be an ideal. Assume A is excellent, normal, and $\dim V(I) \geq 1$. Then $cd(A, I) < \dim(A)$. In particular, if $\dim(A) = 2$, then $\operatorname{Spec}(A) \setminus V(I)$ is affine.*

Proof. By More on Algebra, Lemma 52.6 the completion $A^\wedge$ is normal and hence a domain. Thus the assumption of Proposition 16.6 holds and we conclude. The statement on affineness follows from Lemma 4.8. □

17. Frobenius action

Let p be a prime number. Let A be a ring with $p = 0$ in A. The *Frobenius endomorphism* of A is the map

$$F : A \longrightarrow A, \quad a \longmapsto a^p$$

In this section we prove lemmas on modules which have Frobenius actions.

Lemma 17.1. *Let p be a prime number. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with $p = 0$ in A. Let M be a finite A-module such that $M \otimes_{A,F} A \cong M$. Then M is finite free.*

Proof. Choose a presentation $A^{\oplus m} \to A^{\oplus n} \to M$ which induces an isomorphism $\kappa^{\oplus n} \to M/\mathfrak{m}M$. Let $T = (a_{ij})$ be the matrix of the map $A^{\oplus m} \to A^{\oplus n}$. Observe that $a_{ij} \in \mathfrak{m}$. Applying base change by F, using right exactness of base change, we get a presentation $A^{\oplus m} \to A^{\oplus n} \to M$ where the matrix is $T = (a_{ij}^p)$. Thus we have a presentation with $a_{ij} \in \mathfrak{m}^p$. Repeating this construction we find that for each $e \geq 1$ there exists a presentation with $a_{ij} \in \mathfrak{m}^e$. This implies the fitting ideals (More on Algebra, Definition 8.3) $\text{Fit}_k(M)$ for $k < n$ are contained in $\bigcap_{e \geq 1} \mathfrak{m}^e$. Since this is zero by Krull's intersection theorem (Algebra, Lemma 51.4) we conclude that M is free of rank n by More on Algebra, Lemma 8.7. $\square$

In this section, we say elements $f_1, \ldots, f_r$ of a ring A are *independent* if $\sum a_i f_i = 0$ implies $a_i \in (f_1, \ldots, f_r)$. In other words, with $I = (f_1, \ldots, f_r)$ we have I/I^2 is free over A/I with basis $f_1, \ldots, f_r$.

Lemma 17.2. *Let A be a ring. If $f_1, \ldots, f_{r-1}, f_r g_r$ are independent, then $f_1, \ldots, f_r$ are independent.*

Proof. Say $\sum a_i f_i = 0$. Then $\sum a_i g_r f_i = 0$. Hence $a_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$. Write $a_r = \sum_{i<r} b_i f_i + b f_r g_r$. Then $0 = \sum_{i<r} (a_i + b_i f_r) f_i + b f_r^2 g_r$. Thus $a_i + b_i f_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$ which implies $a_i \in (f_1, \ldots, f_r)$ as desired. $\square$

Lemma 17.3. *Let A be a ring. If $f_1, \ldots, f_{r-1}, f_r g_r$ are independent and if the A-module $A/(f_1, \ldots, f_{r-1}, f_r g_r)$ has finite length, then*

$$\begin{aligned} &\mathit{length}_A(A/(f_1, \ldots, f_{r-1}, f_r g_r)) \\ &= \mathit{length}_A(A/(f_1, \ldots, f_{r-1}, f_r)) + \mathit{length}_A(A/(f_1, \ldots, f_{r-1}, g_r)) \end{aligned}$$

Proof. We claim there is an exact sequence

$$0 \to A/(f_1, \ldots, f_{r-1}, g_r) \xrightarrow{f_r} A/(f_1, \ldots, f_{r-1}, f_r g_r) \to A/(f_1, \ldots, f_{r-1}, f_r) \to 0$$

Namely, if $a f_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$, then $\sum_{i<r} a_i f_i + (a + b g_r) f_r = 0$ for some $b, a_i \in A$. Hence $\sum_{i<r} a_i g_r f_i + (a + b g_r) g_r f_r = 0$ which implies $a + b g_r \in (f_1, \ldots, f_{r-1}, f_r g_r)$ which means that a maps to zero in $A/(f_1, \ldots, f_{r-1}, g_r)$. This proves the claim. To finish use additivity of lengths (Algebra, Lemma 52.3). $\square$

Lemma 17.4. *Let $(A, \mathfrak{m})$ be a local ring. If $\mathfrak{m} = (x_1, \ldots, x_r)$ and $x_1^{e_1}, \ldots, x_r^{e_r}$ are independent for some $e_i > 0$, then $\mathit{length}_A(A/(x_1^{e_1}, \ldots, x_r^{e_r})) = e_1 \ldots e_r$.*

Proof. Use Lemmas 17.2 and 17.3 and induction. $\square$

Lemma 17.5. *Let $\varphi : A \to B$ be a flat ring map. If $f_1, \ldots, f_r \in A$ are independent, then $\varphi(f_1), \ldots, \varphi(f_r) \in B$ are independent.*

Proof. Let $I = (f_1, \ldots, f_r)$ and $J = \varphi(I)B$. By flatness we have $I/I^2 \otimes_A B = J/J^2$. Hence freeness of I/I^2 over A/I implies freeness of J/J^2 over B/J. $\square$

Lemma 17.6 (Kunz)**.** *Let p be a prime number. Let A be a Noetherian ring with $p = 0$. The following are equivalent*

1. *A is regular, and*
2. *$F : A \to A$, $a \mapsto a^p$ is flat.*

Proof. Observe that $\mathrm{Spec}(F) : \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is the identity map. Being regular is defined in terms of the local rings and being flat is something about local rings, see Algebra, Lemma 39.18. Thus we may and do assume A is a Noetherian local ring with maximal ideal $\mathfrak{m}$.

Assume A is regular. Let $x_1, \ldots, x_d$ be a system of parameters for A. Applying F we find $F(x_1), \ldots, F(x_d) = x_1^p, \ldots, x_d^p$, which is a system of parameters for A. Hence F is flat, see Algebra, Lemmas 128.1 and 106.3.

Conversely, assume F is flat. Write $\mathfrak{m} = (x_1, \ldots, x_r)$ with r minimal. Then $x_1, \ldots, x_r$ are independent in the sense defined above. Since F is flat, we see that $x_1^p, \ldots, x_r^p$ are independent, see Lemma 17.5. Hence $\mathrm{length}_A(A/(x_1^p, \ldots, x_r^p)) = p^r$ by Lemma 17.4. Let $\chi(n) = \mathrm{length}_A(A/\mathfrak{m}^n)$ and recall that this is a numerical polynomial of degree $\dim(A)$, see Algebra, Proposition 60.9. Choose $n \gg 0$. Observe that

$$\mathfrak{m}^{pn+pr} \subset F(\mathfrak{m}^n)A \subset \mathfrak{m}^{pn}$$

as can be seen by looking at monomials in $x_1, \ldots, x_r$. We have

$$A/F(\mathfrak{m}^n)A = A/\mathfrak{m}^n \otimes_{A,F} A$$

By flatness of F this has length $\chi(n)\mathrm{length}_A(A/F(\mathfrak{m})A)$ (Algebra, Lemma 52.13) which is equal to $p^r\chi(n)$ by the above. We conclude

$$\chi(pn + pr) \geq p^r\chi(n) \geq \chi(pn)$$

Looking at the leading terms this implies $r = \dim(A)$, i.e., A is regular. $\square$

18. Structure of certain modules

Some results on the structure of certain types of modules over regular local rings. These types of results and much more can be found in [HS93], [Lyu93], [Lyu97].

Lemma 18.1. *Let k be a field of characteristic 0. Let $d \geq 1$. Let $A = k[[x_1, \ldots, x_d]]$ with maximal ideal $\mathfrak{m}$. Let M be an $\mathfrak{m}$-power torsion A-module endowed with additive operators $D_1, \ldots, D_d$ satisfying the leibniz rule*

$$D_i(fz) = \partial_i(f)z + fD_i(z)$$

for $f \in A$ and $z \in M$. Here ∂_i is differentiation with respect to x_i. Then M is isomorphic to a direct sum of copies of the injective hull E of k.

Proof. Choose a set J and an isomorphism $M[\mathfrak{m}] \to \bigoplus_{j \in J} k$. Since $\bigoplus_{j \in J} E$ is injective (Dualizing Complexes, Lemma 3.7) we can extend this isomorphism to an A-module homomorphism $\varphi : M \to \bigoplus_{j \in J} E$. We claim that φ is an isomorphism, i.e., bijective.

Injective. Let $z \in M$ be nonzero. Since M is $\mathfrak{m}$-power torsion we can choose an element $f \in A$ such that $fz \in M[\mathfrak{m}]$ and $fz \neq 0$. Then $\varphi(fz) = f\varphi(z)$ is nonzero, hence $\varphi(z)$ is nonzero.

Surjective. Let $z \in M$. Then $x_1^n z = 0$ for some $n \geq 0$. We will prove that $z \in x_1 M$ by induction on n. If $n = 0$, then $z = 0$ and the result is true. If $n > 0$, then applying D_1 we find $0 = n x_1^{n-1} z + x_1^n D_1(z)$. Hence $x_1^{n-1}(nz + x_1 D_1(z)) = 0$. By induction we get $nz + x_1 D_1(z) \in x_1 M$. Since n is invertible, we conclude $z \in x_1 M$. Thus we see that M is x_1-divisible. If φ is not surjective, then we can choose $e \in \bigoplus_{j \in J} E$ not in M. Arguing as above we may assume $\mathfrak{m}e \subset M$, in particular $x_1 e \in M$. There exists an element $z_1 \in M$ with $x_1 z_1 = x_1 e$. Hence $x_1(z_1 - e) = 0$. Replacing e by $e - z_1$ we may assume e is annihilated by x_1. Thus it suffices to prove that

$$\varphi[x_1] : M[x_1] \longrightarrow \left(\bigoplus\nolimits_{j \in J} E\right)[x_1] = \bigoplus\nolimits_{j \in J} E[x_1]$$

is surjective. If $d = 1$, this is true by construction of φ. If $d > 1$, then we observe that $E[x_1]$ is the injective hull of the residue field of $k[[x_2, \ldots, x_d]]$, see Dualizing Complexes, Lemma 7.1. Observe that $M[x_1]$ as a module over $k[[x_2, \ldots, x_d]]$ is $\mathfrak{m}/(x_1)$-power torsion and comes equipped with operators $D_2, \ldots, D_d$ satisfying the displayed Leibniz rule. Thus by induction on d we conclude that $\varphi[x_1]$ is surjective as desired. $\square$

Lemma 18.2. *Let p be a prime number. Let $(A, \mathfrak{m}, k)$ be a regular local ring with $p = 0$. Denote $F : A \to A$, $a \mapsto a^p$ be the Frobenius endomorphism. Let M be a $\mathfrak{m}$-power torsion module such that $M \otimes_{A,F} A \cong M$. Then M is isomorphic to a direct sum of copies of the injective hull E of k.*

Proof. Choose a set J and an A-module homorphism $\varphi : M \to \bigoplus_{j \in J} E$ which maps $M[\mathfrak{m}]$ isomorphically onto $(\bigoplus_{j \in J} E)[\mathfrak{m}] = \bigoplus_{j \in J} k$. We claim that φ is an isomorphism, i.e., bijective.

Injective. Let $z \in M$ be nonzero. Since M is $\mathfrak{m}$-power torsion we can choose an element $f \in A$ such that $fz \in M[\mathfrak{m}]$ and $fz \neq 0$. Then $\varphi(fz) = f\varphi(z)$ is nonzero, hence $\varphi(z)$ is nonzero.

Surjective. Recall that F is flat, see Lemma 17.6. Let $x_1, \ldots, x_d$ be a minimal system of generators of $\mathfrak{m}$. Denote

$$M_n = M[x_1^{p^n}, \ldots, x_d^{p^n}]$$

the submodule of M consisting of elements killed by $x_1^{p^n}, \ldots, x_d^{p^n}$. So $M_0 = M[\mathfrak{m}]$ is a vector space over k. Also $M = \bigcup M_n$ by our assumption that M is $\mathfrak{m}$-power torsion. Since F^n is flat and $F^n(x_i) = x_i^{p^n}$ we have

$$M_n \cong (M \otimes_{A,F^n} A)[x_1^{p^n}, \ldots, x_d^{p^n}] = M[x_1, \ldots, x_d] \otimes_{A,F^n} A = M_0 \otimes_k A/(x_1^{p^n}, \ldots, x_d^{p^n})$$

Thus M_n is free over $A/(x_1^{p^n}, \ldots, x_d^{p^n})$. A computation shows that every element of $A/(x_1^{p^n}, \ldots, x_d^{p^n})$ annihilated by $x_1^{p^n - 1}$ is divisible by x_1; for example you can use that $A/(x_1^{p^n}, \ldots, x_d^{p^n}) \cong k[x_1, \ldots, x_d]/(x_1^{p^n}, \ldots, x_d^{p^n})$ by Algebra, Lemma 160.10. Thus the same is true for every element of M_n. Since every element of M is in M_n for all $n \gg 0$ and since every element of M is killed by some power of x_1, we conclude that M is x_1-divisible.

Let $x = x_1$. Above we have seen that M is x-divisible. If φ is not surjective, then we can choose $e \in \bigoplus_{j \in J} E$ not in M. Arguing as above we may assume $\mathfrak{m}e \subset M$, in particular $xe \in M$. There exists an element $z_1 \in M$ with $xz_1 = xe$. Hence

$x(z_1 - e) = 0$. Replacing e by $e - z_1$ we may assume e is annihilated by x. Thus it suffices to prove that

$$\varphi[x] : M[x] \longrightarrow \left(\bigoplus\nolimits_{j \in J} E\right)[x] = \bigoplus\nolimits_{j \in J} E[x]$$

is surjective. If $d = 1$, this is true by construction of φ. If $d > 1$, then we observe that $E[x]$ is the injective hull of the residue field of the regular ring A/xA, see Dualizing Complexes, Lemma 7.1. Observe that $M[x]$ as a module over A/xA is $\mathfrak{m}/(x)$-power torsion and we have

$$\begin{aligned} M[x] \otimes_{A/xA,F} A/xA &= M[x] \otimes_{A,F} A \otimes_A A/xA \\ &= (M \otimes_{A,F} A)[x^p] \otimes_A A/xA \\ &\cong M[x^p] \otimes_A A/xA \end{aligned}$$

Argue using flatness of F as before. We claim that $M[x^p] \otimes_A A/xA \to M[x]$, $z \otimes 1 \mapsto x^{p-1}z$ is an isomorphism. This can be seen by proving it for each of the modules M_n, $n > 0$ defined above where it follows by the same result for $A/(x_1^{p^n}, \ldots, x_d^{p^n})$ and $x = x_1$. Thus by induction on $\dim(A)$ we conclude that $\varphi[x]$ is surjective as desired. $\square$

19. Additional structure on local cohomology

Here is a sample result.

Lemma 19.1. *Let A be a ring. Let $I \subset A$ be a finitely generated ideal. Set $Z = V(I)$. For each derivation $\theta : A \to A$ there exists a canonical additive operator D on the local cohomology modules $H^i_Z(A)$ satisfying the Leibniz rule with respect to θ.*

Proof. Let $f_1, \ldots, f_r$ be elements generating I. Recall that $R\Gamma_Z(A)$ is computed by the complex

$$A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}$$

See Dualizing Complexes, Lemma 9.1. Since θ extends uniquely to an additive operator on any localization of A satisfying the Leibniz rule with respect to θ, the lemma is clear. $\square$

Lemma 19.2. *Let p be a prime number. Let A be a ring with $p = 0$. Denote $F : A \to A$, $a \mapsto a^p$ the Frobenius endomorphism. Let $I \subset A$ be a finitely generated ideal. Set $Z = V(I)$. There exists an isomorphism $R\Gamma_Z(A) \otimes^{\mathbf{L}}_{A,F} A \cong R\Gamma_Z(A)$.*

Proof. Follows from Dualizing Complexes, Lemma 9.3 and the fact that $Z = V(f_1^p, \ldots, f_r^p)$ if $I = (f_1, \ldots, f_r)$. $\square$

Lemma 19.3. *Let A be a ring. Let $V \to \mathrm{Spec}(A)$ be quasi-compact, quasi-separated, and étale. For each derivation $\theta : A \to A$ there exists a canonical additive operator D on $H^i(V, \mathcal{O}_V)$ satisfying the Leibniz rule with respect to θ.*

Proof. If V is separated, then we can argue using an affine open covering $V = \bigcup_{j=1,\ldots m} V_j$. Namely, because V is separated we may write $V_{j_0 \ldots j_p} = \mathrm{Spec}(B_{j_0 \ldots j_p})$. See Schemes, Lemma 21.7. Then we find that the A-module $H^i(V, \mathcal{O}_V)$ is the ith cohomology group of the Čech complex

$$\prod B_{j_0} \to \prod B_{j_0 j_1} \to \prod B_{j_0 j_1 j_2} \to \ldots$$

See Cohomology of Schemes, Lemma 2.6. Each $B = B_{j_0 \ldots j_p}$ is an étale A-algebra. Hence $\Omega_B = \Omega_A \otimes_A B$ and we conclude θ extends uniquely to a derivation $\theta_B : B \to B$. These maps define an endomorphism of the Čech complex and define the desired operators on the cohomology groups.

In the general case we use a hypercovering of V by affine opens, exactly as in the first part of the proof of Cohomology of Schemes, Lemma 7.3. We omit the details. $\square$

Remark 19.4. We can upgrade Lemmas 19.1 and 19.3 to include higher order differential operators. If we ever need this we will state and prove a precise lemma here.

Lemma 19.5. *Let p be a prime number. Let A be a ring with $p = 0$. Denote $F : A \to A$, $a \mapsto a^p$ the Frobenius endomorphism. If $V \to \mathrm{Spec}(A)$ is quasi-compact, quasi-separated, and étale, then there exists an isomorphism $R\Gamma(V, \mathcal{O}_V) \otimes^{\mathbf{L}}_{A,F} A \cong R\Gamma(V, \mathcal{O}_V)$.*

Proof. Observe that the relative Frobenius morphism

$$V \longrightarrow V \times_{\mathrm{Spec}(A), \mathrm{Spec}(F)} \mathrm{Spec}(A)$$

of V over A is an isomorphism, see Étale Morphisms, Lemma 14.3. Thus the lemma follows from cohomology and base change, see Derived Categories of Schemes, Lemma 22.5. Observe that since V is étale over A, it is flat over A. $\square$

20. A bit of uniformity, I

The main task of this section is to formulate and prove Lemma 20.2.

Lemma 20.1. *Let R be a ring. Let $M \to M'$ be a map of R-modules with M of finite presentation such that $Tor_1^R(M, N) \to Tor_1^R(M', N)$ is zero for all R-modules N. Then $M \to M'$ factors through a free R-module.*

Proof. We may choose a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & R^{\oplus r} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K' \arrow[r] & \bigoplus_{i \in I} R \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

whose right vertical arrow is the given map. We can factor this map through the short exact sequence

$$0 \to K' \to E \to M \to 0 \tag{20.1.1}$$

which is the pushout of the first short exact sequence by $K \to K'$. By a diagram chase we see that the assumption in the lemma implies that the boundary map $\mathrm{Tor}_1^R(M, N) \to K' \otimes_R N$ induced by (20.1.1) is zero, i.e., the sequence (20.1.1) is universally exact. This implies by Algebra, Lemma 82.4 that (20.1.1) is split (this is where we use that M is of finite presentation). Hence the map $M \to M'$ factors through $\bigoplus_{i \in I} R$ and we win. $\square$

Lemma 20.2. *Let R be a ring. Let $\alpha : M \to M'$ be a map of R-modules. Let $P_\bullet \to M$ and $P'_\bullet \to M'$ be resolutions by projective R-modules. Let $e \geq 0$ be an integer. Consider the following conditions*

(1) *We can find a map of complexes* $a_\bullet : P_\bullet \to P'_\bullet$ *inducing* α *on cohomology with* $a_i = 0$ *for* $i > e$.
(2) *We can find a map of complexes* $a_\bullet : P_\bullet \to P'_\bullet$ *inducing* α *on cohomology with* $a_{e+1} = 0$.
(3) *The map* $\mathrm{Ext}^i_R(M', N) \to \mathrm{Ext}^i_R(M, N)$ *is zero for all* R*-modules* N *and* $i > e$.
(4) *The map* $\mathrm{Ext}^{e+1}_R(M', N) \to \mathrm{Ext}^{e+1}_R(M, N)$ *is zero for all* R*-modules* N.
(5) *Let* $N = \mathrm{Im}(P'_{e+1} \to P'_e)$ *and denote* $\xi \in \mathrm{Ext}^{e+1}_R(M', N)$ *the canonical element (see proof). Then* ξ *maps to zero in* $\mathrm{Ext}^{e+1}_R(M, N)$.
(6) *The map* $Tor^R_i(M, N) \to Tor^R_i(M', N)$ *is zero for all* R*-modules* N *and* $i > e$.
(7) *The map* $Tor^R_{e+1}(M, N) \to Tor^R_{e+1}(M', N)$ *is zero for all* R*-modules* N.

Then we always have the implications

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Rightarrow (6) \Leftrightarrow (7)$$

If M *is* $(-e-1)$*-pseudo-coherent (for example if* R *is Noetherian and* M *is a finite* R*-module), then all conditions are equivalent.*

Proof. It is clear that (2) implies (1). If $a_\bullet$ is as in (1), then we can consider the map of complexes $a'_\bullet : P_\bullet \to P'_\bullet$ with $a'_i = a_i$ for $i \leq e+1$ and $a'_i = 0$ for $i \geq e+1$ to get a map of complexes as in (2). Thus (1) is equivalent to (2).

By the construction of the Ext and Tor functors using resolutions (Algebra, Sections 71 and 75) we see that (1) and (2) imply all of the other conditions.

It is clear that (3) implies (4) implies (5). Let N be as in (5). The canonical map $\tilde{\xi} : P'_{e+1} \to N$ precomposed with $P'_{e+2} \to P'_{e+1}$ is zero. Hence we may consider the class ξ of $\tilde{\xi}$ in

$$\mathrm{Ext}^{e+1}_R(M', N) = \frac{\mathrm{Ker}(\mathrm{Hom}(P'_{e+1}, N \to \mathrm{Hom}(P'_{e+2}, N)}{\mathrm{Im}(\mathrm{Hom}(P'_e, N \to \mathrm{Hom}(P'_{e+1}, N)}$$

Choose a map of complexes $a_\bullet : P_\bullet \to P'_\bullet$ lifting α, see Derived Categories, Lemma 19.6. If ξ maps to zero in $\mathrm{Ext}^{e+1}_R(M', N)$, then we find a map $\varphi : P_e \to N$ such that $\tilde{\xi} \circ a_{e+1} = \varphi \circ d$. Thus we obtain a map of complexes

$$\begin{tikzcd}
\ldots \arrow[r] & P_{e+1} \arrow[r] \arrow[d, "0"] & P_e \arrow[r] \arrow[d, "a_e - \varphi"] & P_{e-1} \arrow[r] \arrow[d, "a_{e-1}"] & \ldots \\
\ldots \arrow[r] & P'_{e+1} \arrow[r] & P'_e \arrow[r] & P'_{e-1} \arrow[r] & \ldots
\end{tikzcd}$$

as in (2). Hence (1) – (5) are equivalent.

The equivalence of (6) and (7) follows from dimension shifting; we omit the details.

Assume M is $(-e-1)$-pseudo-coherent. (The parenthetical statement in the lemma follows from More on Algebra, Lemma 64.17.) We will show that (7) implies (4) which finishes the proof. We will use induction on e. The base case is $e = 0$. Then M is of finite presentation by More on Algebra, Lemma 64.4 and we conclude from Lemma 20.1 that $M \to M'$ factors through a free module. Of course if $M \to M'$

factors through a free module, then $\text{Ext}^i_R(M', N) \to \text{Ext}^i_R(M, N)$ is zero for all $i > 0$ as desired. Assume $e > 0$. We may choose a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & R^{\oplus r} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K' \arrow[r] & \bigoplus_{i \in I} R \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

whose right vertical arrow is the given map. We obtain $\text{Tor}^R_{i+1}(M, N) = \text{Tor}^R_i(K, N)$ and $\text{Ext}^{i+1}_R(M, N) = \text{Ext}^i_R(K, N)$ for $i \geq 1$ and all R-modules N and similarly for M', K'. Hence we see that $\text{Tor}^R_e(K, N) \to \text{Tor}^R_e(K', N)$ is zero for all R-modules N. By More on Algebra, Lemma 64.2 we see that K is $(-e)$-pseudo-coherent. By induction we conclude that $\text{Ext}^e(K', N) \to \text{Ext}^e(K, N)$ is zero for all R-modules N, which gives what we want. □

Lemma 20.3. *Let I be an ideal of a Noetherian ring A. For all $n \geq 1$ there exists an $m > n$ such that the map $A/I^m \to A/I^n$ satisfies the equivalent conditions of Lemma 20.2 with $e = cd(A, I)$.*

Proof. Let $\xi \in \text{Ext}^{e+1}_A(A/I^n, N)$ be the element constructed in Lemma 20.2 part (5). Since $e = \text{cd}(A, I)$ we have $0 = H^{e+1}_Z(N) = H^{e+1}_I(N) = \text{colim}\, \text{Ext}^{e+1}(A/I^m, N)$ by Dualizing Complexes, Lemmas 10.1 and 8.2. Thus we may pick $m \geq n$ such that ξ maps to zero in $\text{Ext}^{e+1}_A(A/I^m, N)$ as desired. □

21. A bit of uniformity, II

Let I be an ideal of a Noetherian ring A. Let M be a finite A-module. Let $i > 0$. By More on Algebra, Lemma 27.3 there exists a $c = c(A, I, M, i)$ such that $\text{Tor}^A_i(M, A/I^n) \to \text{Tor}^A_i(M, A/I^{n-c})$ is zero for all $n \geq c$. In this section, we discuss some results which show that one sometimes can choose a constant c which works for all A-modules M simultaneously (and for a range of indices i). This material is related to uniform Artin-Rees as discussed in [Hun92] and [AHS15].

In Remark 21.9 we will apply this to show that various pro-systems related to derived completion are (or are not) strictly pro-isomorphic.

The following lemma can be significantly strengthened.

Lemma 21.1. *Let I be an ideal of a Noetherian ring A. For every $m \geq 0$ and $i > 0$ there exist a $c = c(A, I, m, i) \geq 0$ such that for every A-module M annihilated by I^m the map*

$$Tor^A_i(M, A/I^n) \to Tor^A_i(M, A/I^{n-c})$$

is zero for all $n \geq c$.

Proof. By induction on i. Base case $i = 1$. The short exact sequence $0 \to I^n \to A \to A/I^n \to 0$ determines an injection $\text{Tor}^A_1(M, A/I^n) \subset I^n \otimes_A M$, see Algebra, Remark 75.9. As M is annihilated by I^m we see that the map $I^n \otimes_A M \to I^{n-m} \otimes_A M$ is zero for $n \geq m$. Hence the result holds with $c = m$.

Induction step. Let $i > 1$ and assume c works for $i-1$. By More on Algebra, Lemma 27.3 applied to $M = A/I^m$ we can choose $c' \geq 0$ such that $\text{Tor}_i(A/I^m, A/I^n) \to$

$\mathrm{Tor}_i(A/I^m, A/I^{n-c'})$ is zero for $n \geq c'$. Let M be annihilated by I^m. Choose a short exact sequence

$$0 \to S \to \bigoplus\nolimits_{i \in I} A/I^m \to M \to 0$$

The corresponding long exact sequence of tors gives an exact sequence

$$\mathrm{Tor}_i^A(\bigoplus\nolimits_{i \in I} A/I^m, A/I^n) \to \mathrm{Tor}_i^A(M, A/I^n) \to \mathrm{Tor}_{i-1}^A(S, A/I^n)$$

for all integers $n \geq 0$. If $n \geq c+c'$, then the map $\mathrm{Tor}_{i-1}^A(S, A/I^n) \to \mathrm{Tor}_{i-1}^A(S, A/I^{n-c})$ is zero and the map $\mathrm{Tor}_i^A(A/I^m, A/I^{n-c}) \to \mathrm{Tor}_i^A(A/I^m, A/I^{n-c-c'})$ is zero. Combined with the short exact sequences this implies the result holds for i with constant $c + c'$. $\square$

Lemma 21.2. *Let $I = (a_1, \ldots, a_t)$ be an ideal of a Noetherian ring A. Set $a = a_1$ and denote $B = A[\frac{I}{a}]$ the affine blowup algebra. There exists a $c > 0$ such that $\mathrm{Tor}_i^A(B, M)$ is annihilated by I^c for all A-modules M and $i \geq t$.*

Proof. Recall that B is the quotient of $A[x_2, \ldots, x_t]/(a_1x_2 - a_2, \ldots, a_1x_t - a_t)$ by its a_1-torsion, see Algebra, Lemma 70.6. Let

$$B_\bullet = \text{Koszul complex on } a_1x_2 - a_2, \ldots, a_1x_t - a_t \text{ over } A[x_2, \ldots, x_t]$$

viewed as a chain complex sitting in degrees $(t-1), \ldots, 0$. The complex $B_\bullet[1/a_1]$ is isomorphic to the Koszul complex on $x_2 - a_2/a_1, \ldots, x_t - a_t/a_1$ which is a regular sequence in $A[1/a_1][x_2, \ldots, x_t]$. Since regular sequences are Koszul regular, we conclude that the augmentation

$$\epsilon : B_\bullet \longrightarrow B$$

is a quasi-isomorphism after inverting a_1. Since the homology modules of the cone $C_\bullet$ on ϵ are finite $A[x_2, \ldots, x_n]$-modules and since $C_\bullet$ is bounded, we conclude that there exists a $c \geq 0$ such that a_1^c annihilates all of these. By Derived Categories, Lemma 12.5 this implies that, after possibly replacing c by a larger integer, that a_1^c is zero on $C_\bullet$ in $D(A)$. The proof is finished once the reader contemplates the distinguished triangle

$$B_\bullet \otimes_A^{\mathbf{L}} M \to B \otimes_A^{\mathbf{L}} M \to C_\bullet \otimes_A^{\mathbf{L}} M$$

Namely, the first term is represented by $B_\bullet \otimes_A M$ which is sitting in homological degrees $(t-1), \ldots, 0$ in view of the fact that the terms in the Koszul complex $B_\bullet$ are free (and hence flat) A-modules. Whence $\mathrm{Tor}_i^A(B, M) = H_i(C_\bullet \otimes_A^{\mathbf{L}} M)$ for $i > t-1$ and this is annihilated by a_1^c. Since $a_1^c B = I^c B$ and since the tor module is a module over B we conclude. $\square$

For the rest of the discussion in this section we fix a Noetherian ring A and an ideal $I \subset A$. We denote

$$p : X \to \mathrm{Spec}(A)$$

the blowing up of $\mathrm{Spec}(A)$ in the ideal I. In other words, X is the Proj of the Rees algebra $\bigoplus_{n \geq 0} I^n$. By Cohomology of Schemes, Lemmas 14.2 and 14.3 we can choose an integer $q(A, I) \geq 0$ such that for all $q \geq q(A, I)$ we have $H^i(X, \mathcal{O}_X(q)) = 0$ for $i > 0$ and $H^0(X, \mathcal{O}_X(q)) = I^q$.

Lemma 21.3. *In the situation above, for $q \geq q(A, I)$ and any A-module M we have*

$$R\Gamma(X, Lp^*\widetilde{M}(q)) \cong M \otimes_A^{\mathbf{L}} I^q$$

in $D(A)$.

Proof. Choose a free resolution $F_\bullet \to M$. Then $\widetilde{F}_\bullet$ is a flat resolution of $\widetilde{M}$. Hence $Lp^*\widetilde{M}$ is given by the complex $p^*\widetilde{F}_\bullet$. Thus $Lp^*\widetilde{M}(q)$ is given by the complex $p^*\widetilde{F}_\bullet(q)$. Since $p^*\widetilde{F}_i(q)$ are right acyclic for $\Gamma(X, -)$ by our choice of $q \geq q(A, I)$ and since we have $\Gamma(X, p^*\widetilde{F}_i(q)) = I^q F_i$ by our choice of $q \geq q(A, I)$, we get that $R\Gamma(X, Lp^*\widetilde{M}(q))$ is given by the complex with terms $I^q F_i$ by Derived Categories of Schemes, Lemma 4.3. The result follows as the complex $I^q F_\bullet$ computes $M \otimes_A^{\mathbf{L}} I^q$ by definition. □

Lemma 21.4. *In the situation above, let t be an upper bound on the number of generators for I. There exists an integer $c = c(A, I) \geq 0$ such that for any A-module M the cohomology sheaves $H^j(Lp^*\widetilde{M})$ are annihilated by I^c for $j \leq -t$.*

Proof. Say $I = (a_1, \ldots, a_t)$. The question is affine local on X. For $1 \leq i \leq t$ let $B_i = A[\frac{I}{a_i}]$ be the affine blowup algebra. Then X has an affine open covering by the spectra of the rings B_i, see Divisors, Lemma 32.2. By the description of derived pullback given in Derived Categories of Schemes, Lemma 3.8 we conclude it suffices to prove that for each i there exists a $c \geq 0$ such that

$$\text{Tor}_j^A(B_i, M)$$

is annihilated by I^c for $j \geq t$. This is Lemma 21.2. □

Lemma 21.5. *In the situation above, let t be an upper bound on the number of generators for I. There exists an integer $c = c(A, I) \geq 0$ such that for any A-module M the tor modules $Tor_i^A(M, A/I^q)$ are annihilated by I^c for $i > t$ and all $q \geq 0$.*

Proof. Let $q(A, I)$ be as above. For $q \geq q(A, I)$ we have

$$R\Gamma(X, Lp^*\widetilde{M}(q)) = M \otimes_A^{\mathbf{L}} I^q$$

by Lemma 21.3. We have a bounded and convergent spectral sequence

$$H^a(X, H^b(Lp^*\widetilde{M}(q))) \Rightarrow \text{Tor}_{-a-b}^A(M, I^q)$$

by Derived Categories of Schemes, Lemma 4.4. Let d be an integer as in Cohomology of Schemes, Lemma 4.4 (actually we can take $d = t$, see Cohomology of Schemes, Lemma 4.2). Then we see that $H^{-i}(X, Lp^*\widetilde{M}(q)) = \text{Tor}_i^A(M, I^q)$ has a finite filtration with at most d steps whose graded are subquotients of the modules

$$H^a(X, H^{-i-a}(Lp^*\widetilde{M})(q)), \quad a = 0, 1, \ldots, d-1$$

If $i \geq t$ then all of these modules are annihilated by I^c where $c = c(A, I)$ is as in Lemma 21.4 because the cohomology sheaves $H^{-i-a}(Lp^*\widetilde{M})$ are all annihilated by I^c by the lemma. Hence we see that $\text{Tor}_i^A(M, I^q)$ is annihilated by I^{dc} for $q \geq q(A, I)$ and $i \geq t$. Using the short exact sequence $0 \to I^q \to A \to A/I^q \to 0$ we find that $\text{Tor}_i(M, A/I^q)$ is annihilated by I^{dc} for $q \geq q(A, I)$ and $i > t$. We conclude that I^m with $m = \max(dc, q(A, I) - 1)$ annihilates $\text{Tor}_i^A(M, A/I^q)$ for all $q \geq 0$ and $i > t$ as desired. □

Lemma 21.6. *Let I be an ideal of a Noetherian ring A. Let $t \geq 0$ be an upper bound on the number of generators of I. There exist $N, c \geq 0$ such that the maps*

$$\mathit{Tor}^A_{t+1}(M, A/I^n) \to \mathit{Tor}^A_{t+1}(M, A/I^{n-c})$$

are zero for any A-module M and all $n \geq N$.

Proof. Let c_1 be the constant found in Lemma 21.5. Please keep in mind that this constant c_1 works for Tor_i for all $i > t$ simultaneously.

Say $I = (a_1, \ldots, a_t)$. For an A-module M we set

$$\ell(M) = \#\{i \mid 1 \leq i \leq t,\ a_i^{c_1} \text{ is zero on } M\}$$

This is an element of $\{0, 1, \ldots, t\}$. We will prove by descending induction on $0 \leq s \leq t$ the following statement H_s: there exist $N, c \geq 0$ such that for every module M with $\ell(M) \geq s$ the maps

$$\mathrm{Tor}^A_{t+1+i}(M, A/I^n) \to \mathrm{Tor}^A_{t+1+i}(M, A/I^{n-c})$$

are zero for $i = 0, \ldots, s$ for all $n \geq N$.

Base case: $s = t$. If $\ell(M) = t$, then M is annihilated by $(a_1^{c_1}, \ldots, a_t^{c_1}\}$ and hence by $I^{t(c_1-1)+1}$. We conclude from Lemma 21.1 that H_t holds by taking $c = N$ to be the maximum of the integers $c(A, I, t(c_1-1)+1, t+1), \ldots, c(A, I, t(c_1-1)+1, 2t+1)$ found in the lemma.

Induction step. Say $0 \leq s < t$ we have N, c as in H_{s+1}. Consider a module M with $\ell(M) = s$. Then we can choose an i such that $a_i^{c_1}$ is nonzero on M. It follows that $\ell(M[a_i^c]) \geq s+1$ and $\ell(M/a_i^{c_1}M) \geq s+1$ and the induction hypothesis applies to them. Consider the exact sequence

$$0 \to M[a_i^{c_1}] \to M \xrightarrow{a_i^{c_1}} M \to M/a_i^{c_1}M \to 0$$

Denote $E \subset M$ the image of the middle arrow. Consider the corresponding diagram of Tor modules

$$\begin{tikzcd}
 & & \mathrm{Tor}_{i+1}(M/a_i^{c_1}M, A/I^q) \arrow[d] \\
\mathrm{Tor}_i(M[a_i^{c_1}], A/I^q) \arrow[r] & \mathrm{Tor}_i(M, A/I^q) \arrow[r] \arrow[dr, "0"] & \mathrm{Tor}_i(E, A/I^q) \arrow[d] \\
 & & \mathrm{Tor}_i(M, A/I^q)
\end{tikzcd}$$

with exact rows and columns (for every q). The south-east arrow is zero by our choice of c_1. We conclude that the module $\mathrm{Tor}_i(M, A/I^q)$ is sandwiched between a quotient module of $\mathrm{Tor}_i(M[a_i^{c_1}], A/I^q)$ and a submodule of $\mathrm{Tor}_{i+1}(M/a_i^{c_1}M, A/I^q)$. Hence we conclude H_s holds with N replaced by $N + c$ and c replaced by $2c$. Some details omitted. □

Proposition 21.7. *Let I be an ideal of a Noetherian ring A. Let $t \geq 0$ be an upper bound on the number of generators of I. There exist $N, c \geq 0$ such that for $n \geq N$ the maps*

$$A/I^n \to A/I^{n-c}$$

satisfy the equivalent conditions of Lemma 20.2 with $e = t$.

Proof. Immediate consequence of Lemmas 21.6 and 20.2. □

Remark 21.8. The paper [AHS15] shows, besides many other things, that if A is local, then Proposition 21.7 also holds with $e = t$ replaced by $e = \dim(A)$. Looking at Lemma 20.3 it is natural to ask whether Proposition 21.7 holds with $e = t$ replaced with $e = \text{cd}(A, I)$. We don't know.

Remark 21.9. Let I be an ideal of a Noetherian ring A. Say $I = (f_1, \ldots, f_r)$. Denote $K_n^\bullet$ the Koszul complex on $f_1^n, \ldots, f_r^n$ as in More on Algebra, Situation 91.15 and denote $K_n \in D(A)$ the corresponding object. Let $M^\bullet$ be a bounded complex of finite A-modules and denote $M \in D(A)$ the corresponding object. Consider the following inverse systems in $D(A)$:

(1) $M^\bullet/I^nM^\bullet$, i.e., the complex whose terms are M^i/I^nM^i,
(2) $M \otimes_A^{\mathbf{L}} A/I^n$,
(3) $M \otimes_A^{\mathbf{L}} K_n$, and
(4) $M \otimes_P^{\mathbf{L}} P/J^n$ (see below).

All of these inverse systems are isomorphic as pro-objects: the isomorphism between (2) and (3) follows from More on Algebra, Lemma 94.1. The isomorphism between (1) and (2) is given in More on Algebra, Lemma 100.3. For the last one, see below.

However, we can ask if these isomorphisms of pro-systems are "strict"; this terminology and question is related to the discussion in [Qui, pages 61, 62]. Namely, given a category $\mathcal{C}$ we can define a "strict pro-category" whose objects are inverse systems (X_n) and whose morphisms $(X_n) \to (Y_n)$ are given by tuples (c, φ_n) consisting of a $c \geq 0$ and morphisms $\varphi_n : X_n \to Y_{n-c}$ for all $n \geq c$ satisfying an obvious compatibility condition and up to a certain equivalence (given essentially by increasing c). Then we ask whether the above inverse systems are isomorphic in this strict pro-category.

This clearly cannot be the case for (1) and (3) even when $M = A[0]$. Namely, the system $H^0(K_n) = A/(f_1^n, \ldots, f_r^n)$ is not strictly pro-isomorphic in the category of modules to the system A/I^n in general. For example, if we take $A = \mathbf{Z}[x_1, \ldots, x_r]$ and $f_i = x_i$, then $H^0(K_n)$ is not annihilated by $I^{r(n-1)}$.[3]

It turns out that the results above show that the natural map from (2) to (1) discussed in More on Algebra, Lemma 100.3 is a strict pro-isomorphism. We will sketch the proof. Using standard arguments involving stupid truncations, we first reduce to the case where $M^\bullet$ is given by a single finite A-module M placed in degree 0. Pick $N, c \geq 0$ as in Proposition 21.7. The proposition implies that for $n \geq N$ we get factorizations

$$M \otimes_A^{\mathbf{L}} A/I^n \to \tau_{\geq -t}(M \otimes_A^{\mathbf{L}} A/I^n) \to M \otimes_A^{\mathbf{L}} A/I^{n-c}$$

of the transition maps in the system (2). On the other hand, by More on Algebra, Lemma 27.3, we can find another constant $c' = c'(M) \geq 0$ such that the maps $\text{Tor}_i^A(M, A/I^{n'}) \to \text{Tor}_i(M, A/I^{n'-c'})$ are zero for $i = 1, 2, \ldots, t$ and $n' \geq c'$. Then it follows from Derived Categories, Lemma 12.5 that the map

$$\tau_{\geq -t}(M \otimes_A^{\mathbf{L}} A/I^{n+tc'}) \to \tau_{\geq -t}(M \otimes_A^{\mathbf{L}} A/I^n)$$

[3]Of course, we can ask whether these pro-systems are isomorphic in a category whose objects are inverse systems and where maps are given by tuples (r, c, φ_n) consisting of $r \geq 1$, $c \geq 0$ and maps $\varphi_n : X_{rn} \to Y_{n-c}$ for $n \geq c$.

factors through $M \otimes_A^{\mathbf{L}} A/I^{n+tc'} \to M/I^{n+tc'}M$. Combined with the previous result we obtain a factorization

$$M \otimes_A^{\mathbf{L}} A/I^{n+tc'} \to M/I^{n+tc'}M \to M \otimes_A^{\mathbf{L}} A/I^{n-c}$$

which gives us what we want. If we ever need this result, we will carefully state it and provide a detailed proof.

For number (4) suppose we have a Noetherian ring P, a ring homomorphism $P \to A$, and an ideal $J \subset P$ such that $I = JA$. By More on Algebra, Section 60 we get a functor $M \otimes_P^{\mathbf{L}} - : D(P) \to D(A)$ and we get an inverse system $M \otimes_P^{\mathbf{L}} P/J^n$ in $D(A)$ as in (4). If P is Noetherian, then the system in (4) is pro-isomorphic to the system in (1) because we can compare with Koszul complexes. If $P \to A$ is finite, then the system (4) is strictly pro-isomorphic to the system (2) because the inverse system $A \otimes_P^{\mathbf{L}} P/J^n$ is strictly pro-isomorphic to the inverse system A/I^n (by the discussion above) and because we have

$$M \otimes_P^{\mathbf{L}} P/J^n = M \otimes_A^{\mathbf{L}} (A \otimes_P^{\mathbf{L}} P/J^n)$$

by More on Algebra, Lemma 60.1.

A standard example in (4) is to take $P = \mathbf{Z}[x_1, \ldots, x_r]$, the map $P \to A$ sending x_i to f_i, and $J = (x_1, \ldots, x_r)$. In this case one shows that

$$M \otimes_P^{\mathbf{L}} P/J^n = M \otimes_{A[x_1,\ldots,x_r]}^{\mathbf{L}} A[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^n$$

and we reduce to one of the cases discussed above (although this case is strictly easier as $A[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^n$ has tor dimension at most r for all n and hence the step using Proposition 21.7 can be avoided). This case is discussed in the proof of [BS13, Proposition 3.5.1].

22. A bit of uniformity, III

In this section we fix a Noetherian ring A and an ideal $I \subset A$. Our goal is to prove Lemma 22.7 which we will use in a later chapter to solve a lifting problem, see Algebraization of Formal Spaces, Lemma 5.3.

Throughout this section we denote

$$p : X \to \mathrm{Spec}(A)$$

the blowing up of $\mathrm{Spec}(A)$ in the ideal I. In other words, X is the Proj of the Rees algebra $\bigoplus_{n \geq 0} I^n$. We also consider the fibre product

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \mathrm{Spec}(A/I) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

Then Y is the exceptional divisor of the blowup and hence an effective Cartier divisor on X such that $\mathcal{O}_X(-1) = \mathcal{O}_X(Y)$. Since taking Proj commutes with base change we have

$$Y = \mathrm{Proj}(\bigoplus\nolimits_{n \geq 0} I^n/I^{n+1}) = \mathrm{Proj}(S)$$

where $S = \mathrm{Gr}_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}$.

We denote $d = d(S) = d(\text{Gr}_I(A)) = d(\bigoplus_{n \geq 0} I^n/I^{n+1})$ the maximum of the dimensions of the fibres of p (and we set it equal to 0 if $X = \emptyset$). This is well defined. In fact, we have

(1) $d \leq t - 1$ if $I = (a_1, \ldots, a_t)$ since then $X \subset \mathbf{P}^{t-1}_A$, and
(2) d is also the maximal dimension of the fibres of $\text{Proj}(S) \to \text{Spec}(S_0)$ provided that Y is nonempty and $d = 0$ if $Y = \emptyset$ (equivalently $S = 0$, equivalently $I = A$).

Hence d only depends on the isomorphism class of $S = \text{Gr}_I(A)$. Observe that $H^i(X, \mathcal{F}) = 0$ for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ and $i > d$ by Cohomology of Schemes, Lemmas 20.9 and 4.6. Of course the same is true for coherent modules on Y.

We denote $d = d(S) = d(\text{Gr}_I(A)) = d(\bigoplus_{n \geq 0} I^n/I^{n+1})$ the integer defined as follows. Note that the algebra $S = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a Noetherian graded ring generated in degree 1 over degree 0. Hence by Cohomology of Schemes, Lemmas 14.2 and 14.3 we can define $q(S)$ as the smallest integer $q(S) \geq 0$ such that for all $q \geq q(S)$ we have $H^i(Y, \mathcal{O}_Y(q)) = 0$ for $1 \leq i \leq d$ and $H^0(Y, \mathcal{O}_Y(q)) = I^q/I^{q+1}$. (If $S = 0$, then $q(S) = 0$.)

For $n \geq 1$ we may consider the effective Cartier divisor nY which we will denote Y_n.

Lemma 22.1. *With $q_0 = q(S)$ and $d = d(S)$ as above, we have*

(1) *for $n \geq 1$, $q \geq q_0$, and $i > 0$ we have $H^i(X, \mathcal{O}_{Y_n}(q)) = 0$,*
(2) *for $n \geq 1$ and $q \geq q_0$ we have $H^0(X, \mathcal{O}_{Y_n}(q)) = I^q/I^{q+n}$,*
(3) *for $q \geq q_0$ and $i > 0$ we have $H^i(X, \mathcal{O}_X(q)) = 0$,*
(4) *for $q \geq q_0$ we have $H^0(X, \mathcal{O}_X(q)) = I^q$.*

Proof. If $I = A$, then X is affine and the statements are trivial. Hence we may and do assume $I \neq A$. Thus Y and X are nonempty schemes.

Let us prove (1) and (2) by induction on n. The base case $n = 1$ is our definition of q_0 as $Y_1 = Y$. Recall that $\mathcal{O}_X(1) = \mathcal{O}_X(-Y)$. Hence we have a short exact sequence

$$0 \to \mathcal{O}_{Y_n}(1) \to \mathcal{O}_{Y_{n+1}} \to \mathcal{O}_Y \to 0$$

Hence for $i > 0$ we find

$$H^i(X, \mathcal{O}_{Y_n}(q+1)) \to H^i(X, \mathcal{O}_{Y_{n+1}}(q)) \to H^i(X, \mathcal{O}_Y(q))$$

and we obtain the desired vanishing of the middle term from the given vanishing of the outer terms. For $i = 0$ we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I^{q+1}/I^{q+1+n} \arrow[r] \arrow[d] & I^q/I^{q+1+n} \arrow[r] \arrow[d] & I^q/I^{q+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(X, \mathcal{O}_{Y_n}(q+1)) \arrow[r] & H^0(X, \mathcal{O}_{Y_{n+1}}(q)) \arrow[r] & H^0(Y, \mathcal{O}_Y(q)) \arrow[r] & 0
\end{tikzcd}$$

with exact rows for $q \geq q_0$ (for the bottom row observe that the next term in the long exact cohomology sequence vanishes for $q \geq q_0$). Since $q \geq q_0$ the left and right vertical arrows are isomorphisms and we conclude the middle one is too.

We omit the proofs of (3) and (4) which are similar. In fact, one can deduce (3) and (4) from (1) and (2) using the theorem on formal functors (but this would be overkill). □

Let us introduce a notation: given $n \geq c \geq 0$ *an* (A, n, c)*-module* is a finite A-module M which is annihilated by I^n and which as an A/I^n-module is I^c/I^n-projective, see More on Algebra, Section 70.

We will use the following abuse of notation: given an A-module M we denote p^*M the quasi-coherent module gotten by pulling back by p the quasi-coherent module $\widetilde{M}$ on $\text{Spec}(A)$ associated to M. For example we have $\mathcal{O}_{Y_n} = p^*(A/I^n)$. For a short exact sequence $0 \to K \to L \to M \to 0$ of A-modules we obtain an exact sequence

$$p^*K \to p^*L \to p^*M \to 0$$

as $\widetilde{\ }$ is an exact functor and p^* is a right exact functor.

Lemma 22.2. *Let $0 \to K \to L \to M \to 0$ be a short exact sequence of A-modules such that K and L are annihilated by I^n and M is an (A, n, c)-module. Then the kernel of $p^*K \to p^*L$ is scheme theoretically supported on Y_c.*

Proof. Let $\text{Spec}(B) \subset X$ be an affine open. The restriction of the exact sequence over $\text{Spec}(B)$ corresponds to the sequence of B-modules

$$K \otimes_A B \to L \otimes_A B \to M \otimes_A B \to 0$$

which is isomorphismic to the sequence

$$K \otimes_{A/I^n} B/I^nB \to L \otimes_{A/I^n} B/I^nB \to M \otimes_{A/I^n} B/I^nB \to 0$$

Hence the kernel of the first map is the image of the module $\text{Tor}_1^{A/I^n}(M, B/I^nB)$. Recall that the exceptional divisor Y is cut out by $I\mathcal{O}_X$. Hence it suffices to show that $\text{Tor}_1^{A/I^n}(M, B/I^nB)$ is annihilated by I^c. Since multiplication by $a \in I^c$ on M factors through a finite free A/I^n-module, this is clear. □

We have the canonical map $\mathcal{O}_X \to \mathcal{O}_X(1)$ which vanishes exactly along Y. Hence for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ we always have canonical maps $\mathcal{F}(q) \to \mathcal{F}(q + n)$ for any $q \in \mathbf{Z}$ and $n \geq 0$.

Lemma 22.3. *Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is scheme theoretically supported on Y_c if and only if the canonical map $\mathcal{F} \to \mathcal{F}(c)$ is zero.*

Proof. This is true because $\mathcal{O}_X \to \mathcal{O}_X(1)$ vanishes exactly along Y. □

Lemma 22.4. *With $q_0 = q(S)$ and $d = d(S)$ as above, suppose we have integers $n \geq c \geq 0$, an (A, n, c)-module M, an index $i \in \{0, 1, \ldots, d\}$, and an integer q. Then we distinguish the following cases*

(1) *In the case $i = d \geq 1$ and $q \geq q_0$ we have $H^d(X, p^*M(q)) = 0$.*
(2) *In the case $i = d - 1 \geq 1$ and $q \geq q_0$ we have $H^{d-1}(X, p^*M(q)) = 0$.*
(3) *In the case $d - 1 > i > 0$ and $q \geq q_0 + (d - 1 - i)c$ the map $H^i(X, p^*M(q)) \to H^i(X, p^*M(q - (d - 1 - i)c))$ is zero.*
(4) *In the case $i = 0$, $d \in \{0, 1\}$, and $q \geq q_0$, there is a surjection*

$$I^qM \longrightarrow H^0(X, p^*M(q))$$

(5) *In the case $i = 0$, $d > 1$, and $q \geq q_0 + (d-1)c$ the map*
$$H^0(X, p^*M(q)) \to H^0(X, p^*M(q-(d-1)c))$$
has image contained in the image of the canonical map $I^{q-(d-1)c}M \to H^0(X, p^*M(q-(d-1)c))$.

Proof. Let M be an (A, n, c)-module. Choose a short exact sequence
$$0 \to K \to (A/I^n)^{\oplus r} \to M \to 0$$
We will use below that K is an (A, n, c)-module, see More on Algebra, Lemma 70.6. Consider the corresponding exact sequence
$$p^*K \to (\mathcal{O}_{Y_n})^{\oplus r} \to p^*M \to 0$$
We split this into short exact sequences
$$0 \to \mathcal{F} \to p^*K \to \mathcal{G} \to 0 \quad \text{and} \quad 0 \to \mathcal{G} \to (\mathcal{O}_{Y_n})^{\oplus r} \to p^*M \to 0$$
By Lemma 22.2 the coherent module $\mathcal{F}$ is scheme theoretically supported on Y_c.

Proof of (1). Assume $d > 0$. We have to prove $H^d(X, p^*M(q)) = 0$ for $q \geq q_0$. By the vanishing of the cohomology of twists of $\mathcal{G}$ in degrees $> d$ and the long exact cohomology sequence associated to the second short exact sequence above, it suffices to prove that $H^d(X, \mathcal{O}_{Y_n}(q)) = 0$. This is true by Lemma 22.1.

Proof of (2). Assume $d > 1$. We have to prove $H^{d-1}(X, p^*M(q)) = 0$ for $q \geq q_0$. Arguing as in the previous paragraph, we see that it suffices to show that $H^d(X, \mathcal{G}(q)) = 0$. Using the first short exact sequence and the vanishing of the cohomology of twists of $\mathcal{F}$ in degrees $> d$ we see that it suffices to show $H^d(X, p^*K(q))$ is zero which is true by (1) and the fact that K is an (A, n, c)-module (see above).

Proof of (3). Let $0 < i < d-1$ and assume the statement holds for $i+1$ except in the case $i = d-2$ we have statement (2). Using the long exact sequence of cohomology associated to the second short exact sequence above we find an injection
$$H^i(X, p^*M(q-(d-1-i)c)) \subset H^{i+1}(X, \mathcal{G}(q-(d-1-i)c))$$
as $q-(d-1-i)c \geq q_0$ gives the vanishing of $H^i(X, \mathcal{O}_{Y_n}(q-(d-1-i)c))$ (see above). Thus it suffices to show that the map $H^{i+1}(X, \mathcal{G}(q)) \to H^{i+1}(X, \mathcal{G}(q-(d-1-i)c))$ is zero. To study this, we consider the maps of exact sequences

$$\begin{tikzcd}
H^{i+1}(X, p^*K(q)) \arrow[r] \arrow[d] & H^{i+1}(X, \mathcal{G}(q)) \arrow[r] \arrow[d] \arrow[dl, dotted] & H^{i+2}(X, \mathcal{F}(q)) \arrow[d] \\
H^{i+1}(X, p^*K(q-c)) \arrow[r] \arrow[d] & H^{i+1}(X, \mathcal{G}(q-c)) \arrow[r] \arrow[d] & H^{i+2}(X, \mathcal{F}(q-c)) \\
H^{i+1}(X, p^*K(q-(d-1-i)c)) \arrow[r] & H^{i+1}(X, \mathcal{G}(q-(d-1-i)c)) &
\end{tikzcd}$$

Since $\mathcal{F}$ is scheme theoretically supported on Y_c we see that the canonical map $\mathcal{G}(q) \to \mathcal{G}(q-c)$ factors through $p^*K(q-c)$ by Lemma 22.3. This gives the dotted arrow in the diagram. (In fact, for the proof it suffices to observe that the vertical arrow on the extreme right is zero in order to get the dotted arrow as a map of sets.) Thus it suffices to show that $H^{i+1}(X, p^*K(q-c)) \to H^{i+1}(X, p^*K(q-(d-1-i)c))$ is zero. If $i = d-2$, then the source of this arrow is zero by (2) as $q-c \geq q_0$ and K

is an (A, n, c)-module. If $i < d - 2$, then as K is an (A, n, c)-module, we get from the induction hypothesis that the map is indeed zero since $q - c - (q - (d-1-i)c) = (d-2-i)c = (d-1-(i+1))c$ and since $q - c \geq q_0 + (d-1-(i+1))c$. In this way we conclude the proof of (3).

Proof of (4). Assume $d \in \{0, 1\}$ and $q \geq q_0$. Then the first short exact sequence gives a surjection $H^1(X, p^*K(q)) \to H^1(X, \mathcal{G}(q))$ and the source of this arrow is zero by case (1). Hence for all $q \in \mathbf{Z}$ we see that the map

$$H^0(X, (\mathcal{O}_{Y_n})^{\oplus r}(q)) \longrightarrow H^0(X, p^*M(q))$$

is surjective. For $q \geq q_0$ the source is equal to $(I^q/I^{q+n})^{\oplus r}$ by Lemma 22.1 and this easily proves the statement.

Proof of (5). Assume $d > 1$. Arguing as in the proof of (4) we see that it suffices to show that the image of

$$H^0(X, p^*M(q)) \longrightarrow H^0(X, p^*M(q - (d-1)c))$$

is contained in the image of

$$H^0(X, (\mathcal{O}_{Y_n})^{\oplus r}(q - (d-1)c)) \longrightarrow H^0(X, p^*M(q - (d-1)c))$$

To show the inclusion above, it suffices to show that for $\sigma \in H^0(X, p^*M(q))$ with boundary $\xi \in H^1(X, \mathcal{G}(q))$ the image of ξ in $H^1(X, \mathcal{G}(q - (d-1)c))$ is zero. This follows by the exact same arguments as in the proof of (3). $\square$

Remark 22.5. Given a pair (M, n) consisting of an integer $n \geq 0$ and a finite A/I^n-module M we set $M^\vee = \mathrm{Hom}_{A/I^n}(M, A/I^n)$. Given a pair $(\mathcal{F}, n)$ consisting of an integer n and a coherent $\mathcal{O}_{Y_n}$-module $\mathcal{F}$ we set

$$\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_{Y_n}}(\mathcal{F}, \mathcal{O}_{Y_n})$$

Given (M, n) as above, there is a canonical map

$$can : p^*(M^\vee) \longrightarrow (p^*M)^\vee$$

Namely, if we choose a presentation $(A/I^n)^{\oplus s} \to (A/I^n)^{\oplus r} \to M \to 0$ then we obtain a presentation $\mathcal{O}_{Y_n}^{\oplus s} \to \mathcal{O}_{Y_n}^{\oplus r} \to p^*M \to 0$. Taking duals we obtain exact sequences

$$0 \to M^\vee \to (A/I^n)^{\oplus r} \to (A/I^n)^{\oplus s}$$

and

$$0 \to (p^*M)^\vee \to \mathcal{O}_{Y_n}^{\oplus r} \to \mathcal{O}_{Y_n}^{\oplus s}$$

Pulling back the first sequence by p we find the desired map *can*. The construction of this map is functorial in the finite A/I^n-module M. The kernel and cokernel of *can* are scheme theoretically supported on Y_c if M is an (A, n, c)-module. Namely, in that case for $a \in I^c$ the map $a : M \to M$ factors through a finite free A/I^n-module for which *can* is an isomorphism. Hence a annihilates the kernel and cokernel of *can*.

Lemma 22.6. *With $q_0 = q(S)$ and $d = d(S)$ as above, let M be an (A, n, c)-module and let $\varphi : M \to I^n/I^{2n}$ be an A-linear map. Assume $n \geq \max(q_0 + (1+d)c, (2+d)c)$ and if $d = 0$ assume $n \geq q_0 + 2c$. Then the composition*

$$M \xrightarrow{\varphi} I^n/I^{2n} \to I^{n-(1+d)c}/I^{2n-(1+d)c}$$

is of the form $\sum a_i \psi_i$ with $a_i \in I^c$ and $\psi_i : M \to I^{n-(2+d)c}/I^{2n-(2+d)c}$.

Proof. The case $d > 1$. Since we have a compatible system of maps $p^*(I^q) \to \mathcal{O}_X(q)$ for $q \geq 0$ there are canonical maps $p^*(I^q/I^{q+\nu}) \to \mathcal{O}_{Y_\nu}(q)$ for $\nu \geq 0$. Using this and pulling back φ we obtain a map

$$\chi : p^*M \longrightarrow \mathcal{O}_{Y_n}(n)$$

such that the composition $M \to H^0(X, p^*M) \to H^0(X, \mathcal{O}_{Y_n}(n))$ is the given homomorphism φ combined with the map $I^n/I^{2n} \to H^0(X, \mathcal{O}_{Y_n}(n))$. Since $\mathcal{O}_{Y_n}(n)$ is invertible on Y_n the linear map χ determines a section

$$\sigma \in \Gamma(X, (p^*M)^\vee(n))$$

with notation as in Remark 22.5. The discussion in Remark 22.5 shows the cokernel and kernel of $can : p^*(M^\vee) \to (p^*M)^\vee$ are scheme theoretically supported on Y_c. By Lemma 22.3 the map $(p^*M)^\vee(n) \to (p^*M)^\vee(n-2c)$ factors through $p^*(M^\vee)(n-2c)$; small detail omitted. Hence the image of σ in $\Gamma(X, (p^*M)^\vee(n-2c))$ comes from an element

$$\sigma' \in \Gamma(X, p^*(M^\vee)(n-2c))$$

By Lemma 22.4 part (5), the fact that $M^\vee$ is an (A, n, c)-module by More on Algebra, Lemma 70.7, and the fact that $n \geq q_0 + (1+d)c$ so $n - 2c \geq q_0 + (d-1)c$ we see that the image of σ' in $H^0(X, p^*M^\vee(n-(1+d)c))$ is the image of an element τ in $I^{n-(1+d)c}M^\vee$. Write $\tau = \sum a_i \tau_i$ with $\tau_i \in I^{n-(2+d)c}M^\vee$; this makes sense as $n - (2+d)c \geq 0$. Then τ_i determines a homomorphism of modules $\psi_i : M \to I^{n-(2+d)c}/I^{2n-(2+d)c}$ using the evaluation map $M \otimes M^\vee \to A/I^n$.

Let us prove that this works[4]. Pick $z \in M$ and let us show that $\varphi(z)$ and $\sum a_i\psi_i(z)$ have the same image in $I^{n-(1+d)c}/I^{2n-(1+d)c}$. First, the element z determines a map $p^*z : \mathcal{O}_{Y_n} \to p^*M$ whose composition with χ is equal to the map $\mathcal{O}_{Y_n} \to \mathcal{O}_{Y_n}(n)$ corresponding to $\varphi(z)$ via the map $I^n/I^{2n} \to \Gamma(\mathcal{O}_{Y_n}(n))$. Next z and p^*z determine evaluation maps $e_z : M^\vee \to A/I^n$ and $e_{p^*z} : (p^*M)^\vee \to \mathcal{O}_{Y_n}$. Since $\chi(p^*z)$ is the section corresponding to $\varphi(z)$ we see that $e_{p^*z}(\sigma)$ is the section corresponding to $\varphi(z)$. Here and below we abuse notation: for a map $a : \mathcal{F} \to \mathcal{G}$ of modules on X we also denote $a : \mathcal{F}(t) \to \mathcal{F}(t)$ the corresponding map of twisted modules. The diagram

$$\begin{tikzcd} p^*(M^\vee) \arrow[r, "p^*e_z"'] \arrow[d, "can"'] & \mathcal{O}_{Y_n} \arrow[d, equal] \\ (p^*M)^\vee \arrow[r, "e_{p^*z}"] & \mathcal{O}_{Y_n} \end{tikzcd}$$

commutes by functoriality of the construction can. Hence $(p^*e_z)(\sigma')$ in $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-2c))$ is the section corresponding to the image of $\varphi(z)$ in I^{n-2c}/I^{2n-2c}. The next step is that σ' maps to the image of $\sum a_i\tau_i$ in $H^0(X, p^*M^\vee(n-(1+d)c))$. This implies that $(p^*e_z)(\sum a_i\tau_i) = \sum a_i p^*e_z(\tau_i)$ in $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-(1+d)c))$ is the section corresponding to the image of $\varphi(z)$ in $I^{n-(1+d)c}/I^{2n-(1+d)c}$. Recall that ψ_i is defined from τ_i using an evaluation map. Hence if we denote

$$\chi_i : p^*M \longrightarrow \mathcal{O}_{Y_n}(n-(2+d)c)$$

the map we get from ψ_i, then we see by the same reasoning as above that the section corresponding to $\psi_i(z)$ is $\chi_i(p^*z) = e_{p^*z}(\chi_i) = p^*e_z(\tau_i)$. Hence we conclude that the image of $\varphi(z)$ in $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-(1+d)c))$ is equal to the image of $\sum a_i\psi_i(z)$.

[4]We hope some reader will suggest a less dirty proof of this fact.

Since $n-(1+d)c \geq q_0$ we have $\Gamma(Y_n, \mathcal{O}_{Y_n}(n-(1+d)c)) = I^{n-(1+d)c}/I^{2n-(1+d)c}$ by Lemma 22.1 and we conclude the desired compatibility is true.

The case $d=1$. Here we argue as above that we get

$$\chi : p^*M \longrightarrow \mathcal{O}_{Y_n}(n), \quad \sigma \in \Gamma(X, (p^*M)^\vee(n)), \quad \sigma' \in \Gamma(X, p^*(M^\vee)(n-2c)),$$

and then we use Lemma 22.4 part (4) to see that σ' is the image of some element $\tau \in I^{n-2c}M^\vee$. The rest of the argument is the same.

The case $d=0$. Argument is exactly the same as in the case $d=1$. $\square$

Lemma 22.7. *With $d = d(S)$ and $q_0 = q(S)$ as above. Then*

1. *for integers $n \geq c \geq 0$ with $n \geq \max(q_0 + (1+d)c, (2+d)c)$,*
2. *for K of $D(A/I^n)$ with $H^i(K) = 0$ for $i \neq -1, 0$ and $H^i(K)$ finite for $i = -1, 0$ such that $\mathrm{Ext}^1_{A/I^c}(K, N)$ is annihilated by I^c for all finite A/I^n-modules N*

the map

$$\mathrm{Ext}^1_{A/I^n}(K, I^n/I^{2n}) \longrightarrow \mathrm{Ext}^1_{A/I^n}(K, I^{n-(1+d)c}/I^{2n-2(1+d)c})$$

is zero.

Proof. The case $d > 0$. Let $K^{-1} \to K^0$ be a complex representing K as in More on Algebra, Lemma 84.5 part (5) with respect to the ideal I^c/I^n in the ring A/I^n. In particular K^{-1} is I^c/I^n-projective as multiplication by elements of I^c/I^n even factor through K^0. By More on Algebra, Lemma 84.4 part (1) we have

$$\mathrm{Ext}^1_{A/I^n}(K, I^n/I^{2n}) = \mathrm{Coker}(\mathrm{Hom}_{A/I^n}(K^0, I^n/I^{2n}) \to \mathrm{Hom}_{A/I^n}(K^{-1}, I^n/I^{2n}))$$

and similarly for other Ext groups. Hence any class ξ in $\mathrm{Ext}^1_{A/I^n}(K, I^n/I^{2n})$ comes from an element $\varphi \in \mathrm{Hom}_{A/I^n}(K^{-1}, I^n/I^{2n})$. Denote φ' the image of φ in $\mathrm{Hom}_{A/I^n}(K^{-1}, I^{n-(1+d)c}/I^{2n-(1+d)c})$. By Lemma 22.6 we can write $\varphi' = \sum a_i\psi_i$ with $a_i \in I^c$ and $\psi_i \in \mathrm{Hom}_{A/I^n}(M, I^{n-(2+d)c}/I^{2n-(2+d)c})$. Choose $h_i : K^0 \to K^{-1}$ such that $a_i \mathrm{id}_{K^{-1}} = h_i \circ d_K^{-1}$. Set $\psi = \sum \psi_i \circ h_i : K^0 \to I^{n-(2+d)c}/I^{2n-(2+d)c}$. Then $\varphi' = \psi \circ \mathrm{d}_K^{-1}$ and we conclude that ξ already maps to zero in $\mathrm{Ext}^1_{A/I^n}(K, I^{n-(1+d)c}/I^{2n-(1+d)c})$ and a fortiori in $\mathrm{Ext}^1_{A/I^n}(K, I^{n-(1+d)c}/I^{2n-2(1+d)c})$.

The case $d = 0$[5]. Let ξ and φ be as above. We consider the diagram

$$\begin{tikzcd} K^0 & & \\ K^{-1} \arrow[u] \arrow[r, "\varphi"] & I^n/I^{2n} \arrow[r] & I^{n-c}/I^{2n-c} \end{tikzcd}$$

Pulling back to X and using the map $p^*(I^n/I^{2n}) \to \mathcal{O}_{Y_n}(n)$ we find a solid diagram

$$\begin{tikzcd} p^*K^0 \arrow[rrd, dotted] & & \\ p^*K^{-1} \arrow[u] \arrow[r] & \mathcal{O}_{Y_n}(n) \arrow[r] & \mathcal{O}_{Y_n}(n-c) \end{tikzcd}$$

[5]The argument given for $d > 0$ works but gives a slightly weaker result.

We can cover X by affine opens $U = \mathrm{Spec}(B)$ such that there exists an $a \in I$ with the following property: $IB = aB$ and a is a nonzerodivisor on B. Namely, we can cover X by spectra of affine blowup algebras, see Divisors, Lemma 32.2. The restriction of $\mathcal{O}_{Y_n}(n) \to \mathcal{O}_{Y_n}(n-c)$ to U is isomorphic to the map of quasi-coherent $\mathcal{O}_U$-modules corresponding to the B-module map $a^c : B/a^nB \to B/a^nB$. Since $a^c : K^{-1} \to K^{-1}$ factors through K^0 we see that the dotted arrow exists over U. In other words, locally on X we can find the dotted arrow! Now the sheaf of dotted arrows fitting into the diagram is principal homogeneous under

$$\mathcal{F} = \mathcal{H}om_{\mathcal{O}_X}(\mathrm{Coker}(p^*K^{-1} \to p^*K^0), \mathcal{O}_{Y_n}(n-c))$$

which is a coherent $\mathcal{O}_X$-module. Hence the obstruction for finding the dotted arrow is an element of $H^1(X, \mathcal{F})$. This cohomology group is zero as $1 > d = 0$, see discussion following the definition of $d = d(S)$. This proves that we can find a dotted arrow $\psi : p^*K^0 \to \mathcal{O}_{Y_n}(n-c)$ fitting into the diagram. Since $n - c \geq q_0$ we find that ψ induces a map $K^0 \to I^{n-c}/I^{2n-c}$. Chasing the diagram we conclude that $\varphi' = \psi \circ \mathrm{d}_K^{-1}$ and the proof is finished as before. □

23. Other chapters

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