

DECENT ALGEBRAIC SPACES

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1. Introduction

In this chapter we study “local” properties of general algebraic spaces, i.e., those algebraic spaces which aren’t quasi-separated. Quasi-separated algebraic spaces are studied in [Knu71]. It turns out that essentially new phenomena happen, especially regarding points and specializations of points, on more general algebraic spaces. On the other hand, for most basic results on algebraic spaces, one needn’t worry about these phenomena, which is why we have decided to have this material in a separate chapter following the standard development of the theory.

2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf} . And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S . In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Universally bounded fibres

We briefly discuss what it means for a morphism from a scheme to an algebraic space to have universally bounded fibres. Please refer to Morphisms, Section 57 for similar definitions and results on morphisms of schemes.

Definition 3.1. Let S be a scheme. Let X be an algebraic space over S , and let U be a scheme over S . Let $f : U \rightarrow X$ be a morphism over S . We say the *fibres of f are universally bounded*¹ if there exists an integer n such that for all fields k and all morphisms $\mathrm{Spec}(k) \rightarrow X$ the fibre product $\mathrm{Spec}(k) \times_X U$ is a finite scheme over k whose degree over k is $\leq n$.

This definition makes sense because the fibre product $\mathrm{Spec}(k) \times_Y X$ is a scheme. Moreover, if Y is a scheme we recover the notion of Morphisms, Definition 57.1 by virtue of Morphisms, Lemma 57.2.

Lemma 3.2. *Let S be a scheme. Let X be an algebraic space over S . Let $V \rightarrow U$ be a morphism of schemes over S , and let $U \rightarrow X$ be a morphism from U to X . If the fibres of $V \rightarrow U$ and $U \rightarrow X$ are universally bounded, then so are the fibres of $V \rightarrow X$.*

Proof. Let n be an integer which works for $V \rightarrow U$, and let m be an integer which works for $U \rightarrow X$ in Definition 3.1. Let $\mathrm{Spec}(k) \rightarrow X$ be a morphism, where k is a field. Consider the morphisms

$$\mathrm{Spec}(k) \times_X V \longrightarrow \mathrm{Spec}(k) \times_X U \longrightarrow \mathrm{Spec}(k).$$

By assumption the scheme $\mathrm{Spec}(k) \times_X U$ is finite of degree at most m over k , and n is an integer which bounds the degree of the fibres of the first morphism. Hence by Morphisms, Lemma 57.4 we conclude that $\mathrm{Spec}(k) \times_X V$ is finite over k of degree at most nm . \square

Lemma 3.3. *Let S be a scheme. Let $Y \rightarrow X$ be a representable morphism of algebraic spaces over S . Let $U \rightarrow X$ be a morphism from a scheme to X . If the fibres of $U \rightarrow X$ are universally bounded, then the fibres of $U \times_X Y \rightarrow Y$ are universally bounded.*

Proof. This is clear from the definition, and properties of fibre products. (Note that $U \times_X Y$ is a scheme as we assumed $Y \rightarrow X$ representable, so the definition applies.) \square

¹This is probably nonstandard notation.

Lemma 3.4. *Let S be a scheme. Let $g : Y \rightarrow X$ be a representable morphism of algebraic spaces over S . Let $f : U \rightarrow X$ be a morphism from a scheme towards X . Let $f' : U \times_X Y \rightarrow Y$ be the base change of f . If*

$$\mathrm{Im}(|f| : |U| \rightarrow |X|) \subset \mathrm{Im}(|g| : |Y| \rightarrow |X|)$$

and f' has universally bounded fibres, then f has universally bounded fibres.

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibre products $\mathrm{Spec}(k) \times_Y (U \times_X Y)$ as in Definition 3.1 for the morphism f' . We claim that n works for f also. Namely, suppose that $x : \mathrm{Spec}(k) \rightarrow X$ is a morphism from the spectrum of a field. Then either $\mathrm{Spec}(k) \times_X U$ is empty (and there is nothing to prove), or x is in the image of $|f|$. By Properties of Spaces, Lemma 4.3 and the assumption of the lemma we see that this means there exists a field extension k'/k and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k') & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & X \end{array}$$

Hence we see that

$$\mathrm{Spec}(k') \times_Y (U \times_X Y) = \mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} (\mathrm{Spec}(k) \times_X U)$$

Since the scheme $\mathrm{Spec}(k') \times_Y (U \times_X Y)$ is assumed finite of degree $\leq n$ over k' it follows that also $\mathrm{Spec}(k) \times_X U$ is finite of degree $\leq n$ over k as desired. (Some details omitted.) \square

Lemma 3.5. *Let S be a scheme. Let X be an algebraic space over S . Consider a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & \searrow g & \swarrow h \\ & X & \end{array}$$

where U and V are schemes. If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres.

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say $n \geq 0$ is an integer which bounds the degrees of the schemes $\mathrm{Spec}(k) \times_X U$ as in Definition 3.1. We claim n also works for h . Let $\mathrm{Spec}(k) \rightarrow X$ be a morphism from the spectrum of a field to X . Consider the morphism of schemes

$$\mathrm{Spec}(k) \times_X V \longrightarrow \mathrm{Spec}(k) \times_X U$$

It is flat and surjective. By assumption the scheme on the left is finite of degree $\leq n$ over $\mathrm{Spec}(k)$. It follows from Morphisms, Lemma 57.10 that the degree of the scheme on the right is also bounded by n as desired. \square

Lemma 3.6. *Let S be a scheme. Let X be an algebraic space over S , and let U be a scheme over S . Let $\varphi : U \rightarrow X$ be a morphism over S . If the fibres of φ are universally bounded, then there exists an integer n such that each fibre of $|U| \rightarrow |X|$ has at most n elements.*

Proof. The integer n of Definition 3.1 works. Namely, pick $x \in |X|$. Represent x by a morphism $x : \operatorname{Spec}(k) \rightarrow X$. Then we get a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(k) \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \operatorname{Spec}(k) & \xrightarrow{x} & X \end{array}$$

which shows (via Properties of Spaces, Lemma 4.3) that the inverse image of x in $|U|$ is the image of the top horizontal arrow. Since $\operatorname{Spec}(k) \times_X U$ is finite of degree $\leq n$ over k it has at most n points. \square

4. Finiteness conditions and points

In this section we elaborate on the question of when points can be represented by monomorphisms from spectra of fields into the space.

Remark 4.1. Before we give the proof of the next lemma let us recall some facts about étale morphisms of schemes:

- (1) An étale morphism is flat and hence generalizations lift along an étale morphism (Morphisms, Lemmas 36.12 and 25.9).
- (2) An étale morphism is unramified, an unramified morphism is locally quasi-finite, hence fibres are discrete (Morphisms, Lemmas 36.16, 35.10, and 20.6).
- (3) A quasi-compact étale morphism is quasi-finite and in particular has finite fibres (Morphisms, Lemmas 20.9 and 20.10).
- (4) An étale scheme over a field k is a disjoint union of spectra of finite separable field extension of k (Morphisms, Lemma 36.7).

For a general discussion of étale morphisms, please see Étale Morphisms, Section 11.

Lemma 4.2. *Let S be a scheme. Let X be an algebraic space over S . Let $x \in |X|$. The following are equivalent:*

- (1) *there exists a family of schemes U_i and étale morphisms $\varphi_i : U_i \rightarrow X$ such that $\coprod \varphi_i : \coprod U_i \rightarrow X$ is surjective, and such that for each i the fibre of $|U_i| \rightarrow |X|$ over x is finite, and*
- (2) *for every affine scheme U and étale morphism $\varphi : U \rightarrow X$ the fibre of $|U| \rightarrow |X|$ over x is finite.*

Proof. The implication (2) \Rightarrow (1) is trivial. Let $\varphi_i : U_i \rightarrow X$ be a family of étale morphisms as in (1). Let $\varphi : U \rightarrow X$ be an étale morphism from an affine scheme towards X . Consider the fibre product diagrams

$$\begin{array}{ccc} U \times_X U_i & \xrightarrow{p_i} & U_i \\ q_i \downarrow & & \downarrow \varphi_i \\ U & \xrightarrow{\varphi} & X \end{array} \quad \begin{array}{ccc} \coprod U \times_X U_i & \xrightarrow{\coprod p_i} & \coprod U_i \\ \coprod q_i \downarrow & & \downarrow \coprod \varphi_i \\ U & \xrightarrow{\varphi} & X \end{array}$$

Since q_i is étale it is open (see Remark 4.1). Moreover, the morphism $\coprod q_i$ is surjective. Hence there exist finitely many indices i_1, \dots, i_n and a quasi-compact opens $W_{i_j} \subset U \times_X U_{i_j}$ which surject onto U . The morphism p_i is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply

Morphisms, Lemma 57.9 to see the fibres of $p_{i_j}|_{W_{i_j}} : W_{i_j} \rightarrow U_i$ are finite. Hence by Properties of Spaces, Lemma 4.3 and the assumption on φ_i we conclude that the fibre of φ over x is finite. In other words (2) holds. \square

Lemma 4.3. *Let S be a scheme. Let X be an algebraic space over S . Let $x \in |X|$. The following are equivalent:*

- (1) *there exists a scheme U , an étale morphism $\varphi : U \rightarrow X$, and points $u, u' \in U$ mapping to x such that setting $R = U \times_X U$ the fibre of*

$$|R| \rightarrow |U| \times_{|X|} |U|$$

over (u, u') is finite,

- (2) *for every scheme U , étale morphism $\varphi : U \rightarrow X$ and any points $u, u' \in U$ mapping to x setting $R = U \times_X U$ the fibre of*

$$|R| \rightarrow |U| \times_{|X|} |U|$$

over (u, u') is finite,

- (3) *there exists a morphism $\text{Spec}(k) \rightarrow X$ with k a field in the equivalence class of x such that the projections $\text{Spec}(k) \times_X \text{Spec}(k) \rightarrow \text{Spec}(k)$ are étale and quasi-compact, and*
- (4) *there exists a monomorphism $\text{Spec}(k) \rightarrow X$ with k a field in the equivalence class of x .*

Proof. Assume (1), i.e., let $\varphi : U \rightarrow X$ be an étale morphism from a scheme towards X , and let u, u' be points of U lying over x such that the fibre of $|R| \rightarrow |U| \times_{|X|} |U|$ over (u, u') is a finite set. In this proof we think of a point $u = \text{Spec}(\kappa(u))$ as a scheme. Note that $u \rightarrow U$, $u' \rightarrow U$ are monomorphisms (see Schemes, Lemma 23.7), hence $u \times_X u' \rightarrow R = U \times_X U$ is a monomorphism. In this language the assumption really means that $u \times_X u'$ is a scheme whose underlying topological space has finitely many points. Let $\psi : W \rightarrow X$ be an étale morphism from a scheme towards X . Let $w, w' \in W$ be points of W mapping to x . We have to show that $w \times_X w'$ is a scheme whose underlying topological space has finitely many points. Consider the fibre product diagram

$$\begin{array}{ccc} W \times_X U & \xrightarrow{p} & U \\ q \downarrow & & \downarrow \varphi \\ W & \xrightarrow{\psi} & X \end{array}$$

As x is the image of u and u' we may pick points \tilde{w}, \tilde{w}' in $W \times_X U$ with $q(\tilde{w}) = w$, $q(\tilde{w}') = w'$, $u = p(\tilde{w})$ and $u' = p(\tilde{w}')$, see Properties of Spaces, Lemma 4.3. As p, q are étale the field extensions $\kappa(w) \subset \kappa(\tilde{w}) \supset \kappa(u)$ and $\kappa(w') \subset \kappa(\tilde{w}') \supset \kappa(u')$ are finite separable, see Remark 4.1. Then we get a commutative diagram

$$\begin{array}{ccccc} w \times_X w' & \longleftarrow & \tilde{w} \times_X \tilde{w}' & \longrightarrow & u \times_X u' \\ \downarrow & & \downarrow & & \downarrow \\ w \times_X w' & \longleftarrow & \tilde{w} \times_S \tilde{w}' & \longrightarrow & u \times_S u' \end{array}$$

where the squares are fibre product squares. The lower horizontal morphisms are étale and quasi-compact, as any scheme of the form $\text{Spec}(k) \times_S \text{Spec}(k')$ is affine, and by our observations about the field extensions above. Thus we see that the top horizontal arrows are étale and quasi-compact and hence have finite fibres. We

have seen above that $|u \times_X u'|$ is finite, so we conclude that $|w \times_X w'|$ is finite. In other words, (2) holds.

Assume (2). Let $U \rightarrow X$ be an étale morphism from a scheme U such that x is in the image of $|U| \rightarrow |X|$. Let $u \in U$ be a point mapping to x . Then we have seen in the previous paragraph that $u = \text{Spec}(\kappa(u)) \rightarrow X$ has the property that $u \times_X u$ has a finite underlying topological space. On the other hand, the projection maps $u \times_X u \rightarrow u$ are the composition

$$u \times_X u \longrightarrow u \times_X U \longrightarrow u \times_X X = u,$$

i.e., the composition of a monomorphism (the base change of the monomorphism $u \rightarrow U$) by an étale morphism (the base change of the étale morphism $U \rightarrow X$). Hence $u \times_X U$ is a disjoint union of spectra of fields finite separable over $\kappa(u)$ (see Remark 4.1). Since $u \times_X u$ is finite the image of it in $u \times_X U$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. By Schemes, Lemma 23.11 we conclude that $u \times_X u$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. In other words, we see that $u \times_X u \rightarrow u$ is quasi-compact and étale. This means that (3) holds.

Let us prove that (3) implies (4). Let $\text{Spec}(k) \rightarrow X$ be a morphism from the spectrum of a field into X , in the equivalence class of x such that the two projections $t, s : R = \text{Spec}(k) \times_X \text{Spec}(k) \rightarrow \text{Spec}(k)$ are quasi-compact and étale. This means in particular that R is an étale equivalence relation on $\text{Spec}(k)$. By Spaces, Theorem 10.5 we know that the quotient sheaf $X' = \text{Spec}(k)/R$ is an algebraic space. By Groupoids, Lemma 20.6 the map $X' \rightarrow X$ is a monomorphism. Since s, t are quasi-compact, we see that R is quasi-compact and hence Properties of Spaces, Lemma 15.3 applies to X' , and we see that $X' = \text{Spec}(k')$ for some field k' . Hence we get a factorization

$$\text{Spec}(k) \longrightarrow \text{Spec}(k') \longrightarrow X$$

which shows that $\text{Spec}(k') \rightarrow X$ is a monomorphism mapping to $x \in |X|$. In other words (4) holds.

Finally, we prove that (4) implies (1). Let $\text{Spec}(k) \rightarrow X$ be a monomorphism with k a field in the equivalence class of x . Let $U \rightarrow X$ be a surjective étale morphism from a scheme U to X . Let $u \in U$ be a point over x . Since $\text{Spec}(k) \times_X u$ is nonempty, and since $\text{Spec}(k) \times_X u \rightarrow u$ is a monomorphism we conclude that $\text{Spec}(k) \times_X u = u$ (see Schemes, Lemma 23.11). Hence $u \rightarrow U \rightarrow X$ factors through $\text{Spec}(k) \rightarrow X$, here is a picture

$$\begin{array}{ccc} u & \longrightarrow & U \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & X \end{array}$$

Since the right vertical arrow is étale this implies that $\kappa(u)/k$ is a finite separable extension. Hence we conclude that

$$u \times_X u = u \times_{\text{Spec}(k)} u$$

is a finite scheme, and we win by the discussion of the meaning of property (1) in the first paragraph of this proof. \square

Lemma 4.4. *Let S be a scheme. Let X be an algebraic space over S . Let $x \in |X|$. Let U be a scheme and let $\varphi : U \rightarrow X$ be an étale morphism. The following are equivalent:*

- (1) *x is in the image of $|U| \rightarrow |X|$, and setting $R = U \times_X U$ the fibres of both*

$$|U| \longrightarrow |X| \quad \text{and} \quad |R| \longrightarrow |X|$$

over x are finite,
- (2) *there exists a monomorphism $\text{Spec}(k) \rightarrow X$ with k a field in the equivalence class of x , and the fibre product $\text{Spec}(k) \times_X U$ is a finite nonempty scheme over k .*

Proof. Assume (1). This clearly implies the first condition of Lemma 4.3 and hence we obtain a monomorphism $\text{Spec}(k) \rightarrow X$ in the class of x . Taking the fibre product we see that $\text{Spec}(k) \times_X U \rightarrow \text{Spec}(k)$ is a scheme étale over $\text{Spec}(k)$ with finitely many points, hence a finite nonempty scheme over k , i.e., (2) holds.

Assume (2). By assumption x is in the image of $|U| \rightarrow |X|$. The finiteness of the fibre of $|U| \rightarrow |X|$ over x is clear since this fibre is equal to $|\text{Spec}(k) \times_X U|$ by Properties of Spaces, Lemma 4.3. The finiteness of the fibre of $|R| \rightarrow |X|$ above x is also clear since it is equal to the set underlying the scheme

$$(\text{Spec}(k) \times_X U) \times_{\text{Spec}(k)} (\text{Spec}(k) \times_X U)$$

which is finite over k . Thus (1) holds. \square

Lemma 4.5. *Let S be a scheme. Let X be an algebraic space over S . Let $x \in |X|$. The following are equivalent:*

- (1) *for every affine scheme U , any étale morphism $\varphi : U \rightarrow X$ setting $R = U \times_X U$ the fibres of both*

$$|U| \longrightarrow |X| \quad \text{and} \quad |R| \longrightarrow |X|$$

over x are finite,
- (2) *there exist schemes U_i and étale morphisms $U_i \rightarrow X$ such that $\coprod U_i \rightarrow X$ is surjective and for each i , setting $R_i = U_i \times_X U_i$ the fibres of both*

$$|U_i| \longrightarrow |X| \quad \text{and} \quad |R_i| \longrightarrow |X|$$

over x are finite,
- (3) *there exists a monomorphism $\text{Spec}(k) \rightarrow X$ with k a field in the equivalence class of x , and for any affine scheme U and étale morphism $U \rightarrow X$ the fibre product $\text{Spec}(k) \times_X U$ is a finite scheme over k ,*
- (4) *there exists a quasi-compact monomorphism $\text{Spec}(k) \rightarrow X$ with k a field in the equivalence class of x ,*
- (5) *there exists a quasi-compact morphism $\text{Spec}(k) \rightarrow X$ with k a field in the equivalence class of x , and*
- (6) *every morphism $\text{Spec}(k) \rightarrow X$ with k a field in the equivalence class of x is quasi-compact.*

Proof. The equivalence of (1) and (3) follows on applying Lemma 4.4 to every étale morphism $U \rightarrow X$ with U affine. It is clear that (3) implies (2). Assume $U_i \rightarrow X$ and R_i are as in (2). We conclude from Lemma 4.2 that for any affine scheme U and étale morphism $U \rightarrow X$ the fibre of $|U| \rightarrow |X|$ over x is finite. Say this fibre is $\{u_1, \dots, u_n\}$. Then, as Lemma 4.3 (1) applies to $U_i \rightarrow X$ for some i such that x is

in the image of $|U_i| \rightarrow |X|$, we see that the fibre of $|R = U \times_X U| \rightarrow |U| \times_{|X|} |U|$ is finite over (u_a, u_b) , $a, b \in \{1, \dots, n\}$. Hence the fibre of $|R| \rightarrow |X|$ over x is finite. In this way we see that (1) holds. At this point we know that (1), (2), and (3) are equivalent.

If (4) holds, then for any affine scheme U and étale morphism $U \rightarrow X$ the scheme $\text{Spec}(k) \times_X U$ is on the one hand étale over k (hence a disjoint union of spectra of finite separable extensions of k by Remark 4.1) and on the other hand quasi-compact over U (hence quasi-compact). Thus we see that (3) holds. Conversely, if $U_i \rightarrow X$ is as in (2) and $\text{Spec}(k) \rightarrow X$ is a monomorphism as in (3), then

$$\coprod \text{Spec}(k) \times_X U_i \longrightarrow \coprod U_i$$

is quasi-compact (because over each U_i we see that $\text{Spec}(k) \times_X U_i$ is a finite disjoint union spectra of fields). Thus $\text{Spec}(k) \rightarrow X$ is quasi-compact by Morphisms of Spaces, Lemma 8.8.

It is immediate that (4) implies (5). Conversely, let $\text{Spec}(k) \rightarrow X$ be a quasi-compact morphism in the equivalence class of x . Let $U \rightarrow X$ be an étale morphism with U affine. Consider the fibre product

$$\begin{array}{ccc} F & \longrightarrow & U \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & X \end{array}$$

Then $F \rightarrow U$ is quasi-compact, hence F is quasi-compact. On the other hand, $F \rightarrow \text{Spec}(k)$ is étale, hence F is a finite disjoint union of spectra of finite separable extensions of k (Remark 4.1). Since the image of $|F| \rightarrow |U|$ is the fibre of $|U| \rightarrow |X|$ over x (Properties of Spaces, Lemma 4.3), we conclude that the fibre of $|U| \rightarrow |X|$ over x is finite. The scheme $F \times_{\text{Spec}(k)} F$ is also a finite union of spectra of fields because it is also quasi-compact and étale over $\text{Spec}(k)$. There is a monomorphism $F \times_X F \rightarrow F \times_{\text{Spec}(k)} F$, hence $F \times_X F$ is a finite disjoint union of spectra of fields (Schemes, Lemma 23.11). Thus the image of $F \times_X F \rightarrow U \times_X U = R$ is finite. Since this image is the fibre of $|R| \rightarrow |X|$ over x by Properties of Spaces, Lemma 4.3 we conclude that (1) holds. At this point we know that (1) – (5) are equivalent.

It is clear that (6) implies (5). Conversely, assume $\text{Spec}(k) \rightarrow X$ is as in (4) and let $\text{Spec}(k') \rightarrow X$ be another morphism with k' a field in the equivalence class of x . By Properties of Spaces, Lemma 4.11 we have a factorization $\text{Spec}(k') \rightarrow \text{Spec}(k) \rightarrow X$ of the given morphism. This is a composition of quasi-compact morphisms and hence quasi-compact (Morphisms of Spaces, Lemma 8.5) as desired. \square

Lemma 4.6. *Let S be a scheme. Let X be an algebraic space over S . The following are equivalent:*

- (1) *there exist schemes U_i and étale morphisms $U_i \rightarrow X$ such that $\coprod U_i \rightarrow X$ is surjective and each $U_i \rightarrow X$ has universally bounded fibres, and*
- (2) *for every affine scheme U and étale morphism $\varphi : U \rightarrow X$ the fibres of $U \rightarrow X$ are universally bounded.*

Proof. The implication (2) \Rightarrow (1) is trivial. Assume (1). Let $(\varphi_i : U_i \rightarrow X)_{i \in I}$ be a collection of étale morphisms from schemes towards X , covering X , such that

each φ_i has universally bounded fibres. Let $\psi : U \rightarrow X$ be an étale morphism from an affine scheme towards X . For each i consider the fibre product diagram

$$\begin{array}{ccc} U \times_X U_i & \xrightarrow{p_i} & U_i \\ q_i \downarrow & & \downarrow \varphi_i \\ U & \xrightarrow{\psi} & X \end{array}$$

Since q_i is étale it is open (see Remark 4.1). Moreover, we have $U = \bigcup \text{Im}(q_i)$, since the family $(\varphi_i)_{i \in I}$ is surjective. Since U is affine, hence quasi-compact we can find finitely many $i_1, \dots, i_n \in I$ and quasi-compact opens $W_j \subset U \times_X U_{i_j}$ such that $U = \bigcup p_{i_j}(W_j)$. The morphism p_{i_j} is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 57.9 to see the fibres of $p_{i_j}|_{W_j} : W_j \rightarrow U_{i_j}$ are universally bounded. Hence by Lemma 3.2 we see that the fibres of $W_j \rightarrow X$ are universally bounded. Thus also $\coprod_{j=1, \dots, n} W_j \rightarrow X$ has universally bounded fibres. Since $\coprod_{j=1, \dots, n} W_j \rightarrow X$ factors through the surjective étale map $\coprod q_{i_j}|_{W_j} : \coprod_{j=1, \dots, n} W_j \rightarrow U$ we see that the fibres of $U \rightarrow X$ are universally bounded by Lemma 3.5. In other words (2) holds. \square

Lemma 4.7. *Let S be a scheme. Let X be an algebraic space over S . The following are equivalent:*

- (1) *there exists a Zariski covering $X = \bigcup X_i$ and for each i a scheme U_i and a quasi-compact surjective étale morphism $U_i \rightarrow X_i$, and*
- (2) *there exist schemes U_i and étale morphisms $U_i \rightarrow X$ such that the projections $U_i \times_X U_i \rightarrow U_i$ are quasi-compact and $\coprod U_i \rightarrow X$ is surjective.*

Proof. If (1) holds then the morphisms $U_i \rightarrow X_i \rightarrow X$ are étale (combine Morphisms, Lemma 36.3 and Spaces, Lemmas 5.4 and 5.3). Moreover, as $U_i \times_X U_i = U_i \times_{X_i} U_i$, both projections $U_i \times_X U_i \rightarrow U_i$ are quasi-compact.

If (2) holds then let $X_i \subset X$ be the open subspace corresponding to the image of the open map $|U_i| \rightarrow |X|$, see Properties of Spaces, Lemma 4.10. The morphisms $U_i \rightarrow X_i$ are surjective. Hence $U_i \rightarrow X_i$ is surjective étale, and the projections $U_i \times_{X_i} U_i \rightarrow U_i$ are quasi-compact, because $U_i \times_{X_i} U_i = U_i \times_X U_i$. Thus by Spaces, Lemma 11.4 the morphisms $U_i \rightarrow X_i$ are quasi-compact. \square

5. Conditions on algebraic spaces

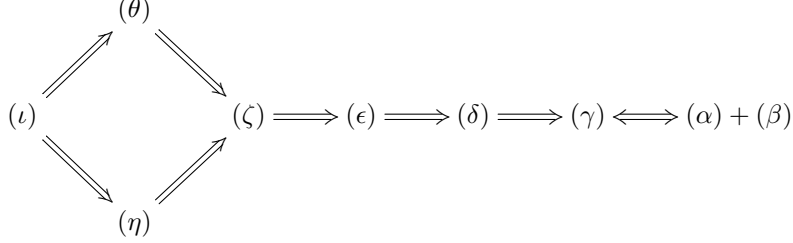
In this section we discuss the relationship between various natural conditions on algebraic spaces we have seen above. Please read Section 6 to get a feeling for the meaning of these conditions.

Lemma 5.1. *Let S be a scheme. Let X be an algebraic space over S . Consider the following conditions on X :*

- (α) *For every $x \in |X|$, the equivalent conditions of Lemma 4.2 hold.*
- (β) *For every $x \in |X|$, the equivalent conditions of Lemma 4.3 hold.*
- (γ) *For every $x \in |X|$, the equivalent conditions of Lemma 4.5 hold.*
- (δ) *The equivalent conditions of Lemma 4.6 hold.*
- (ϵ) *The equivalent conditions of Lemma 4.7 hold.*
- (ζ) *The space X is Zariski locally quasi-separated.*
- (η) *The space X is quasi-separated*
- (θ) *The space X is representable, i.e., X is a scheme.*

(ι) *The space X is a quasi-separated scheme.*

We have



Proof. The implication $(\gamma) \iff (\alpha) + (\beta)$ is immediate. The implications in the diamond on the left are clear from the definitions.

Assume (ζ) , i.e., that X is Zariski locally quasi-separated. Then (ϵ) holds by Properties of Spaces, Lemma 6.6.

Assume (ϵ) . By Lemma 4.7 there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists a scheme U_i and a quasi-compact surjective étale morphism $U_i \rightarrow X_i$. Choose an i and an affine open subscheme $W \subset U_i$. It suffices to show that $W \rightarrow X$ has universally bounded fibres, since then the family of all these morphisms $W \rightarrow X$ covers X . To do this we consider the diagram

$$\begin{array}{ccc}
 W \times_X U_i & \xrightarrow{p} & U_i \\
 q \downarrow & & \downarrow \\
 W & \longrightarrow & X
 \end{array}$$

Since $W \rightarrow X$ factors through X_i we see that $W \times_X U_i = W \times_{X_i} U_i$, and hence q is quasi-compact. Since W is affine this implies that the scheme $W \times_X U_i$ is quasi-compact. Thus we may apply Morphisms, Lemma 57.9 and we conclude that p has universally bounded fibres. From Lemma 3.4 we conclude that $W \rightarrow X$ has universally bounded fibres as well.

Assume (δ) . Let U be an affine scheme, and let $U \rightarrow X$ be an étale morphism. By assumption the fibres of the morphism $U \rightarrow X$ are universally bounded. Thus also the fibres of both projections $R = U \times_X U \rightarrow U$ are universally bounded, see Lemma 3.3. And by Lemma 3.2 also the fibres of $R \rightarrow X$ are universally bounded. Hence for any $x \in X$ the fibres of $|U| \rightarrow |X|$ and $|R| \rightarrow |X|$ over x are finite, see Lemma 3.6. In other words, the equivalent conditions of Lemma 4.5 hold. This proves that $(\delta) \Rightarrow (\gamma)$. \square

Lemma 5.2. *Let S be a scheme. Let \mathcal{P} be one of the properties (α) , (β) , (γ) , (δ) , (ϵ) , (ζ) , or (θ) of algebraic spaces listed in Lemma 5.1. Then if X is an algebraic space over S , and $X = \bigcup X_i$ is a Zariski open covering such that each X_i has \mathcal{P} , then X has \mathcal{P} .*

Proof. Let X be an algebraic space over S , and let $X = \bigcup X_i$ is a Zariski open covering such that each X_i has \mathcal{P} .

The case $\mathcal{P} = (\alpha)$. The condition (α) for X_i means that for every $x \in |X_i|$ and every affine scheme U , and étale morphism $\varphi : U \rightarrow X_i$ the fibre of $\varphi : |U| \rightarrow |X_i|$ over x is finite. Consider $x \in X$, an affine scheme U and an étale morphism $U \rightarrow X$.

Since $X = \bigcup X_i$ is a Zariski open covering there exists a finite affine open covering $U = U_1 \cup \dots \cup U_n$ such that each $U_j \rightarrow X$ factors through some X_{i_j} . By assumption the fibres of $|U_j| \rightarrow |X_{i_j}|$ over x are finite for $j = 1, \dots, n$. Clearly this means that the fibre of $|U| \rightarrow |X|$ over x is finite. This proves the result for (α) .

The case $\mathcal{P} = (\beta)$. The condition (β) for X_i means that every $x \in |X_i|$ is represented by a monomorphism from the spectrum of a field towards X_i . Hence the same follows for X as $X_i \rightarrow X$ is a monomorphism and $X = \bigcup X_i$.

The case $\mathcal{P} = (\gamma)$. Note that $(\gamma) = (\alpha) + (\beta)$ by Lemma 5.1 hence the lemma for (γ) follows from the cases treated above.

The case $\mathcal{P} = (\delta)$. The condition (δ) for X_i means there exist schemes U_{ij} and étale morphisms $U_{ij} \rightarrow X_i$ with universally bounded fibres which cover X_i . These schemes also give an étale surjective morphism $\coprod U_{ij} \rightarrow X$ and $U_{ij} \rightarrow X$ still has universally bounded fibres.

The case $\mathcal{P} = (\epsilon)$. The condition (ϵ) for X_i means we can find a set J_i and morphisms $\varphi_{ij} : U_{ij} \rightarrow X_i$ such that each φ_{ij} is étale, both projections $U_{ij} \times_{X_i} U_{ij} \rightarrow U_{ij}$ are quasi-compact, and $\coprod_{j \in J_i} U_{ij} \rightarrow X_i$ is surjective. In this case the compositions $U_{ij} \rightarrow X_i \rightarrow X$ are étale (combine Morphisms, Lemmas 36.3 and 36.9 and Spaces, Lemmas 5.4 and 5.3). Since $X_i \subset X$ is a subspace we see that $U_{ij} \times_{X_i} U_{ij} = U_{ij} \times_X U_{ij}$, and hence the condition on fibre products is preserved. And clearly $\coprod_{i,j} U_{ij} \rightarrow X$ is surjective. Hence X satisfies (ϵ) .

The case $\mathcal{P} = (\zeta)$. The condition (ζ) for X_i means that X_i is Zariski locally quasi-separated. It is immediately clear that this means X is Zariski locally quasi-separated.

For (θ) , see Properties of Spaces, Lemma 13.1. □

Lemma 5.3. *Let S be a scheme. Let \mathcal{P} be one of the properties (β) , (γ) , (δ) , (ϵ) , or (θ) of algebraic spaces listed in Lemma 5.1. Let X, Y be algebraic spaces over S . Let $X \rightarrow Y$ be a representable morphism. If Y has property \mathcal{P} , so does X .*

Proof. Assume $f : X \rightarrow Y$ is a representable morphism of algebraic spaces, and assume that Y has \mathcal{P} . Let $x \in |X|$, and set $y = f(x) \in |Y|$.

The case $\mathcal{P} = (\beta)$. Condition (β) for Y means there exists a monomorphism $\text{Spec}(k) \rightarrow Y$ representing y . The fibre product $X_y = \text{Spec}(k) \times_Y X$ is a scheme, and x corresponds to a point of X_y , i.e., to a monomorphism $\text{Spec}(k') \rightarrow X_y$. As $X_y \rightarrow X$ is a monomorphism also we see that x is represented by the monomorphism $\text{Spec}(k') \rightarrow X_y \rightarrow X$. In other words (β) holds for X .

The case $\mathcal{P} = (\gamma)$. Since $(\gamma) \Rightarrow (\beta)$ we have seen in the preceding paragraph that y and x can be represented by monomorphisms as in the following diagram

$$\begin{array}{ccc} \text{Spec}(k') & \xrightarrow{x} & X \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \xrightarrow{y} & Y \end{array}$$

Also, by definition of property (γ) via Lemma 4.5 (2) there exist schemes V_i and étale morphisms $V_i \rightarrow Y$ such that $\coprod V_i \rightarrow Y$ is surjective and for each i , setting

$R_i = V_i \times_Y V_i$ the fibres of both

$$|V_i| \longrightarrow |Y| \quad \text{and} \quad |R_i| \longrightarrow |Y|$$

over y are finite. This means that the schemes $(V_i)_y$ and $(R_i)_y$ are finite schemes over $y = \text{Spec}(k)$. As $X \rightarrow Y$ is representable, the fibre products $U_i = V_i \times_Y X$ are schemes. The morphisms $U_i \rightarrow X$ are étale, and $\coprod U_i \rightarrow X$ is surjective. Finally, for each i we have

$$(U_i)_x = (V_i \times_Y X)_x = (V_i)_y \times_{\text{Spec}(k)} \text{Spec}(k')$$

and

$$(U_i \times_X U_i)_x = ((V_i \times_Y X) \times_X (V_i \times_Y X))_x = (R_i)_y \times_{\text{Spec}(k)} \text{Spec}(k')$$

hence these are finite over k' as base changes of the finite schemes $(V_i)_y$ and $(R_i)_y$. This implies that (γ) holds for X , again via the second condition of Lemma 4.5.

The case $\mathcal{P} = (\delta)$. Let $V \rightarrow Y$ be an étale morphism with V an affine scheme. Since Y has property (δ) this morphism has universally bounded fibres. By Lemma 3.3 the base change $V \times_Y X \rightarrow X$ also has universally bounded fibres. Hence the first part of Lemma 4.6 applies and we see that Y also has property (δ) .

The case $\mathcal{P} = (\epsilon)$. We will repeatedly use Spaces, Lemma 5.5. Let $V_i \rightarrow Y$ be as in Lemma 4.7 (2). Set $U_i = X \times_Y V_i$. The morphisms $U_i \rightarrow X$ are étale, and $\coprod U_i \rightarrow X$ is surjective. Because $U_i \times_X U_i = X \times_Y (V_i \times_Y V_i)$ we see that the projections $U_i \times_Y U_i \rightarrow U_i$ are base changes of the projections $V_i \times_Y V_i \rightarrow V_i$, and so quasi-compact as well. Hence X satisfies Lemma 4.7 (2).

The case $\mathcal{P} = (\theta)$. In this case the result is Categories, Lemma 8.3. \square

6. Reasonable and decent algebraic spaces

In Lemma 5.1 we have seen a number of conditions on algebraic spaces related to the behaviour of étale morphisms from affine schemes into X and related to the existence of special étale coverings of X by schemes. We tabulate the different types of conditions here:

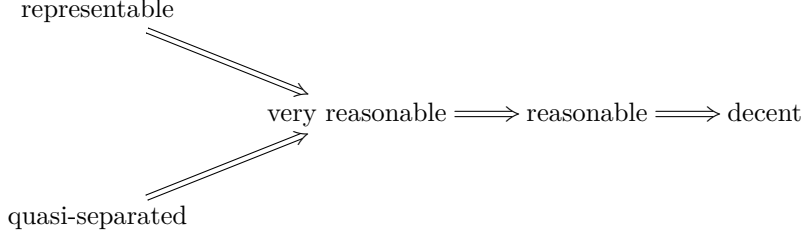
(α)	fibres of étale morphisms from affines are finite
(β)	points come from monomorphisms of spectra of fields
(γ)	points come from quasi-compact monomorphisms of spectra of fields
(δ)	fibres of étale morphisms from affines are universally bounded
(ϵ)	cover by étale morphisms from schemes quasi-compact onto their image

The conditions in the following definition are not exactly conditions on the diagonal of X , but they are in some sense separation conditions on X .

Definition 6.1. Let S be a scheme. Let X be an algebraic space over S .

- (1) We say X is *decent* if for every point $x \in X$ the equivalent conditions of Lemma 4.5 hold, in other words property (γ) of Lemma 5.1 holds.
- (2) We say X is *reasonable* if the equivalent conditions of Lemma 4.6 hold, in other words property (δ) of Lemma 5.1 holds.
- (3) We say X is *very reasonable* if the equivalent conditions of Lemma 4.7 hold, i.e., property (ϵ) of Lemma 5.1 holds.

We have the following implications among these conditions on algebraic spaces:



The notion of a very reasonable algebraic space is obsolete. It was introduced because the assumption was needed to prove some results which are now proven for the class of decent spaces. The class of decent spaces is the largest class of spaces X where one has a good relationship between the topology of $|X|$ and properties of X itself.

Example 6.2. The algebraic space $\mathbf{A}_{\mathbf{Q}}^1/\mathbf{Z}$ constructed in Spaces, Example 14.8 is not decent as its “generic point” cannot be represented by a monomorphism from the spectrum of a field.

Remark 6.3. Reasonable algebraic spaces are technically easier to work with than very reasonable algebraic spaces. For example, if $X \rightarrow Y$ is a quasi-compact étale surjective morphism of algebraic spaces and X is reasonable, then so is Y , see Lemma 17.8 but we don’t know if this is true for the property “very reasonable”. Below we give another technical property enjoyed by reasonable algebraic spaces.

Lemma 6.4. *Let S be a scheme. Let X be a quasi-compact reasonable algebraic space. Then there exists a directed system of quasi-compact and quasi-separated algebraic spaces X_i such that $X = \operatorname{colim}_i X_i$ (colimit in the category of sheaves). Moreover we can arrange it such that*

- (1) *for every quasi-compact scheme T over S we have $\operatorname{colim} X_i(T) = X(T)$,*
- (2) *the transition morphisms $X_i \rightarrow X_{i'}$ of the system and the coprojections $X_i \rightarrow X$ are surjective and étale, and*
- (3) *if X is a scheme, then the algebraic spaces X_i are schemes and the transition morphisms $X_i \rightarrow X_{i'}$ and the coprojections $X_i \rightarrow X$ are local isomorphisms.*

Proof. We sketch the proof. By Properties of Spaces, Lemma 6.3 we have $X = U/R$ with U affine. In this case, reasonable means $U \rightarrow X$ is universally bounded. Hence there exists an integer N such that the “fibres” of $U \rightarrow X$ have degree at most N , see Definition 3.1. Denote $s, t : R \rightarrow U$ and $c : R \times_{s, U, t} R \rightarrow R$ the groupoid structural maps.

Claim: for every quasi-compact open $A \subset R$ there exists an open $R' \subset R$ such that

- (1) $A \subset R'$,
- (2) R' is quasi-compact, and
- (3) $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s, U, t} R'})$ is a groupoid scheme.

Note that $e : U \rightarrow R$ is open as it is a section of the étale morphism $s : R \rightarrow U$, see Étale Morphisms, Proposition 6.1. Moreover U is affine hence quasi-compact. Hence we may replace A by $A \cup e(U) \subset R$, and assume that A contains $e(U)$. Next, we define inductively $A^1 = A$, and

$$A^n = c(A^{n-1} \times_{s, U, t} A) \subset R$$

for $n \geq 2$. Arguing inductively, we see that A^n is quasi-compact for all $n \geq 2$, as the image of the quasi-compact fibre product $A^{n-1} \times_{s,U,t} A$. If k is an algebraically closed field over S , and we consider k -points then

$$A^n(k) = \left\{ (u, u') \in U(k) : \begin{array}{l} \text{there exist } u = u_1, u_2, \dots, u_n \in U(k) \text{ with} \\ (u_i, u_{i+1}) \in A \text{ for all } i = 1, \dots, n-1. \end{array} \right\}$$

But as the fibres of $U(k) \rightarrow X(k)$ have size at most N we see that if $n > N$ then we get a repeat in the sequence above, and we can shorten it proving $A^N = A^n$ for all $n \geq N$. This implies that $R' = A^N$ gives a groupoid scheme $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s,U,t} R'})$, proving the claim above.

Consider the map of sheaves on $(Sch/S)_{fppf}$

$$\text{colim}_{R' \subset R} U/R' \longrightarrow U/R$$

where $R' \subset R$ runs over the quasi-compact open subschemes of R which give étale equivalence relations as above. Each of the quotients U/R' is an algebraic space (see Spaces, Theorem 10.5). Since R' is quasi-compact, and U affine the morphism $R' \rightarrow U \times_{\text{Spec}(\mathbf{Z})} U$ is quasi-compact, and hence U/R' is quasi-separated. Finally, if T is a quasi-compact scheme, then

$$\text{colim}_{R' \subset R} U(T)/R'(T) \longrightarrow U(T)/R(T)$$

is a bijection, since every morphism from T into R ends up in one of the open subrelations R' by the claim above. This clearly implies that the colimit of the sheaves U/R' is U/R . In other words the algebraic space $X = U/R$ is the colimit of the quasi-separated algebraic spaces U/R' .

Properties (1) and (2) follow from the discussion above. If X is a scheme, then if we choose U to be a finite disjoint union of affine opens of X we will obtain (3). Details omitted. \square

Lemma 6.5. *Let S be a scheme. Let X, Y be algebraic spaces over S . Let $X \rightarrow Y$ be a representable morphism. If Y is decent (resp. reasonable), then so is X .*

Proof. Translation of Lemma 5.3. \square

Lemma 6.6. *Let S be a scheme. Let $X \rightarrow Y$ be an étale morphism of algebraic spaces over S . If Y is decent, resp. reasonable, then so is X .*

Proof. Let U be an affine scheme and $U \rightarrow X$ an étale morphism. Set $R = U \times_X U$ and $R' = U \times_Y U$. Note that $R \rightarrow R'$ is a monomorphism.

Let $x \in |X|$. To show that X is decent, we have to show that the fibres of $|U| \rightarrow |X|$ and $|R| \rightarrow |X|$ over x are finite. But if Y is decent, then the fibres of $|U| \rightarrow |Y|$ and $|R'| \rightarrow |Y|$ are finite. Hence the result for “decent”.

To show that X is reasonable, we have to show that the fibres of $U \rightarrow X$ are universally bounded. However, if Y is reasonable, then the fibres of $U \rightarrow Y$ are universally bounded, which immediately implies the same thing for the fibres of $U \rightarrow X$. Hence the result for “reasonable”. \square

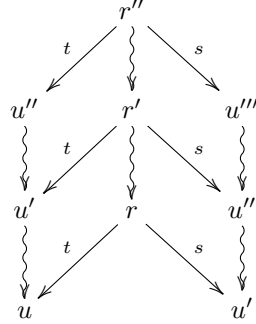
7. Points and specializations

There exists an étale morphism of algebraic spaces $f : X \rightarrow Y$ and a nontrivial specialization between points in a fibre of $|f| : |X| \rightarrow |Y|$, see Examples, Lemma 50.1. If the source of the morphism is a scheme we can avoid this by imposing condition (α) on Y .

Lemma 7.1. *Let S be a scheme. Let X be an algebraic space over S . Let $U \rightarrow X$ be an étale morphism from a scheme to X . Assume $u, u' \in |U|$ map to the same point x of $|X|$, and $u' \rightsquigarrow u$. If the pair (X, x) satisfies the equivalent conditions of Lemma 4.2 then $u = u'$.*

Proof. Assume the pair (X, x) satisfies the equivalent conditions for Lemma 4.2. Let U be a scheme, $U \rightarrow X$ étale, and let $u, u' \in |U|$ map to x of $|X|$, and $u' \rightsquigarrow u$. We may and do replace U by an affine neighbourhood of u . Let $t, s : R = U \times_X U \rightarrow U$ be the étale projection maps.

Pick a point $r \in R$ with $t(r) = u$ and $s(r) = u'$. This is possible by Properties of Spaces, Lemma 4.5. Because generalizations lift along the étale morphism t (Remark 4.1) we can find a specialization $r' \rightsquigarrow r$ with $t(r') = u'$. Set $u'' = s(r')$. Then $u'' \rightsquigarrow u'$. Thus we may repeat and find $r'' \rightsquigarrow r'$ with $t(r'') = u''$. Set $u''' = s(r'')$, and so on. Here is a picture:



In Remark 4.1 we have seen that there are no specializations among points in the fibres of the étale morphism s . Hence if $u^{(n+1)} = u^{(n)}$ for some n , then also $r^{(n)} = r^{(n-1)}$ and hence also (by taking t) $u^{(n)} = u^{(n-1)}$. This then forces the whole tower to collapse, in particular $u = u'$. Thus we see that if $u \neq u'$, then all the specializations are strict and $\{u, u', u'', \dots\}$ is an infinite set of points in U which map to the point x in $|X|$. As we chose U affine this contradicts the second part of Lemma 4.2, as desired. \square

Lemma 7.2. *Let S be a scheme. Let X be an algebraic space over S . Let $U \rightarrow X$ be an étale morphism from a scheme to X . Assume $u, u' \in |U|$ map to the same point x of $|X|$, and $u' \rightsquigarrow u$. If X is locally Noetherian, then $u = u'$.*

Proof. The discussion in Schemes, Section 13 shows that $\mathcal{O}_{U, u'}$ is a localization of the Noetherian local ring $\mathcal{O}_{U, u}$. By Properties of Spaces, Lemma 10.1 we have $\dim(\mathcal{O}_{U, u}) = \dim(\mathcal{O}_{U, u'})$. By dimension theory for Noetherian local rings we conclude $u = u'$. \square

Lemma 7.3. *Let S be a scheme. Let X be an algebraic space over S . Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x' . Assume the pair (X, x') satisfies*

the equivalent conditions of Lemma 4.5. Then for every étale morphism $\varphi : U \rightarrow X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \rightsquigarrow u$ with $\varphi(u') = x'$.

Proof. We may replace U by an affine open neighbourhood of u . Hence we may assume that U is affine. As x is in the image of the open map $|U| \rightarrow |X|$, so is x' . Thus we may replace X by the Zariski open subspace corresponding to the image of $|U| \rightarrow |X|$, see Properties of Spaces, Lemma 4.10. In other words we may assume that $U \rightarrow X$ is surjective and étale. Let $s, t : R = U \times_X U \rightarrow U$ be the projections. By our assumption that (X, x') satisfies the equivalent conditions of Lemma 4.5 we see that the fibres of $|U| \rightarrow |X|$ and $|R| \rightarrow |X|$ over x' are finite. Say $\{u'_1, \dots, u'_n\} \subset U$ and $\{r'_1, \dots, r'_m\} \subset R$ form the complete inverse image of $\{x'\}$. Consider the closed sets

$$T = \overline{\{u'_1\}} \cup \dots \cup \overline{\{u'_n\}} \subset |U|, \quad T' = \overline{\{r'_1\}} \cup \dots \cup \overline{\{r'_m\}} \subset |R|.$$

Trivially we have $s(T') \subset T$. Because R is an equivalence relation we also have $t(T') = s(T')$ as the set $\{r'_j\}$ is invariant under the inverse of R by construction. Let $w \in T$ be any point. Then $u'_i \rightsquigarrow w$ for some i . Choose $r \in R$ with $s(r) = w$. Since generalizations lift along $s : R \rightarrow U$, see Remark 4.1, we can find $r' \rightsquigarrow r$ with $s(r') = u'_i$. Then $r' = r'_j$ for some j and we conclude that $w \in s(T')$. Hence $T = s(T') = t(T')$ is an $|R|$ -invariant closed set in $|U|$. This means T is the inverse image of a closed (!) subset $T'' = \varphi(T)$ of $|X|$, see Properties of Spaces, Lemmas 4.5 and 4.6. Hence $T'' = \overline{\{x'\}}$. Thus T contains some point u_1 mapping to x as $x \in T''$. I.e., we see that for some i there exists a specialization $u'_i \rightsquigarrow u_1$ which maps to the given specialization $x' \rightsquigarrow x$.

To finish the proof, choose a point $r \in R$ such that $s(r) = u$ and $t(r) = u_1$ (using Properties of Spaces, Lemma 4.3). As generalizations lift along t , and $u'_i \rightsquigarrow u_1$ we can find a specialization $r' \rightsquigarrow r$ such that $t(r') = u'_i$. Set $u' = s(r')$. Then $u' \rightsquigarrow u$ and $\varphi(u') = x'$ as desired. \square

Lemma 7.4. *Let S be a scheme. Let $f : Y \rightarrow X$ be a flat morphism of algebraic spaces over S . Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x' . Assume the pair (X, x') satisfies the equivalent conditions of Lemma 4.5 (for example if X is decent, X is quasi-separated, or X is representable). Then for every $y \in |Y|$ with $f(y) = x$, there exists a point $y' \in |Y|$, $y' \rightsquigarrow y$ with $f(y') = x'$.*

Proof. (The parenthetical statement holds by the definition of decent spaces and the implications between the different separation conditions mentioned in Section 6.) Choose a scheme V and a surjective étale morphism $V \rightarrow Y$. Choose $v \in V$ mapping to y . Then we see that it suffices to prove the lemma for $V \rightarrow X$. Thus we may assume Y is a scheme. Choose a scheme U and a surjective étale morphism $U \rightarrow X$. Choose $u \in U$ mapping to x . By Lemma 7.3 we may choose $u' \rightsquigarrow u$ mapping to x' . By Properties of Spaces, Lemma 4.3 we may choose $z \in U \times_X Y$ mapping to y and u . Thus we reduce to the case of the flat morphism of schemes $U \times_X Y \rightarrow U$ which is Morphisms, Lemma 25.9. \square

8. Stratifying algebraic spaces by schemes

In this section we prove that a quasi-compact and quasi-separated algebraic space has a finite stratification by locally closed subspaces each of which is a scheme and

such that the glueing of the parts is by elementary distinguished squares. We first prove a slightly weaker result for reasonable algebraic spaces.

Lemma 8.1. *Let S be a scheme. Let $W \rightarrow X$ be a morphism of a scheme W to an algebraic space X which is flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. There exist reduced closed subspaces*

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \dots \subset Z_n = X$$

such that with $X_r = Z_r \setminus Z_{r-1}$ the stratification $X = \coprod_{r=0, \dots, n} X_r$ is characterized by the following universal property: Given $g : T \rightarrow X$ the projection $W \times_X T \rightarrow T$ is finite locally free of degree r if and only if $g(|T|) \subset |X_r|$.

Proof. Let n be an integer bounding the degrees of the fibres of $W \rightarrow X$. Choose a scheme U and a surjective étale morphism $U \rightarrow X$. Apply More on Morphisms, Lemma 45.3 to $W \times_X U \rightarrow U$. We obtain closed subsets

$$\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset Y_2 \subset \dots \subset Y_n = U$$

characterized by the property stated in the lemma for the morphism $W \times_X U \rightarrow U$. Clearly, the formation of these closed subsets commutes with base change. Setting $R = U \times_X U$ with projection maps $s, t : R \rightarrow U$ we conclude that

$$s^{-1}(Y_r) = t^{-1}(Y_r)$$

as closed subsets of R . In other words the closed subsets $Y_r \subset U$ are R -invariant. This means that $|Y_r|$ is the inverse image of a closed subset $Z_r \subset |X|$. Denote $Z_r \subset X$ also the reduced induced algebraic space structure, see Properties of Spaces, Definition 12.5.

Let $g : T \rightarrow X$ be a morphism of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \rightarrow T$. To prove the final assertion of the lemma it suffices to prove the assertion for the composition $V \rightarrow X$ (by our definition of finite locally free morphisms, see Morphisms of Spaces, Section 46). Similarly, the morphism of schemes $W \times_X V \rightarrow V$ is finite locally free of degree r if and only if the morphism of schemes

$$W \times_X (U \times_X V) \longrightarrow U \times_X V$$

is finite locally free of degree r (see Descent, Lemma 23.30). By construction this happens if and only if $|U \times_X V| \rightarrow |U|$ maps into $|Y_r|$, which is true if and only if $|V| \rightarrow |X|$ maps into $|Z_r|$. \square

Lemma 8.2. *Let S be a scheme. Let $W \rightarrow X$ be a morphism of a scheme W to an algebraic space X which is flat, locally of finite presentation, separated, and locally quasi-finite. Then there exist open subspaces*

$$X = X_0 \supset X_1 \supset X_2 \supset \dots$$

such that a morphism $\text{Spec}(k) \rightarrow X$ where k is a field factors through X_d if and only if $W \times_X \text{Spec}(k)$ has degree $\geq d$ over k .

Proof. Choose a scheme U and a surjective étale morphism $U \rightarrow X$. Apply More on Morphisms, Lemma 45.5 to $W \times_X U \rightarrow U$. We obtain open subschemes

$$U = U_0 \supset U_1 \supset U_2 \supset \dots$$

characterized by the property stated in the lemma for the morphism $W \times_X U \rightarrow U$. Clearly, the formation of these closed subsets commutes with base change. Setting $R = U \times_X U$ with projection maps $s, t : R \rightarrow U$ we conclude that

$$s^{-1}(U_d) = t^{-1}(U_d)$$

as open subschemes of R . In other words the open subschemes $U_d \subset U$ are R -invariant. This means that U_d is the inverse image of an open subspace $X_d \subset X$ (Properties of Spaces, Lemma 12.2). \square

Lemma 8.3. *Let S be a scheme. Let X be a quasi-compact algebraic space over S . There exist open subspaces*

$$\dots \subset U_4 \subset U_3 \subset U_2 \subset U_1 = X$$

with the following properties:

- (1) *setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a separated scheme V_p and a surjective étale morphism $f_p : V_p \rightarrow U_p$ such that $f_p^{-1}(T_p) \rightarrow T_p$ is an isomorphism,*
- (2) *if $x \in |X|$ can be represented by a quasi-compact morphism $\text{Spec}(k) \rightarrow X$ from a field, then $x \in T_p$ for some p .*

Proof. By Properties of Spaces, Lemma 6.3 we can choose an affine scheme U and a surjective étale morphism $U \rightarrow X$. For $p \geq 0$ set

$$W_p = U \times_X \dots \times_X U \setminus \text{all diagonals}$$

where the fibre product has p factors. Since U is separated, the morphism $U \rightarrow X$ is separated and all fibre products $U \times_X \dots \times_X U$ are separated schemes. Since $U \rightarrow X$ is separated the diagonal $U \rightarrow U \times_X U$ is a closed immersion. Since $U \rightarrow X$ is étale the diagonal $U \rightarrow U \times_X U$ is an open immersion, see Morphisms of Spaces, Lemmas 39.10 and 38.9. Similarly, all the diagonal morphisms are open and closed immersions and W_p is an open and closed subscheme of $U \times_X \dots \times_X U$. Moreover, the morphism

$$U \times_X \dots \times_X U \longrightarrow U \times_{\text{Spec}(\mathbf{Z})} \dots \times_{\text{Spec}(\mathbf{Z})} U$$

is locally quasi-finite and separated (Morphisms of Spaces, Lemma 4.5) and its target is an affine scheme. Hence every finite set of points of $U \times_X \dots \times_X U$ is contained in an affine open, see More on Morphisms, Lemma 45.1. Therefore, the same is true for W_p . There is a free action of the symmetric group S_p on W_p over X (because we threw out the fix point locus from $U \times_X \dots \times_X U$). By the above and Properties of Spaces, Proposition 14.1 the quotient $V_p = W_p/S_p$ is a scheme. Since the action of S_p on W_p was over X , there is a morphism $V_p \rightarrow X$. Since $W_p \rightarrow X$ is étale and since $W_p \rightarrow V_p$ is surjective étale, it follows that also $V_p \rightarrow X$ is étale, see Properties of Spaces, Lemma 16.3. Observe that V_p is a separated scheme by Properties of Spaces, Lemma 14.3.

We let $U_p \subset X$ be the open subspace which is the image of $V_p \rightarrow X$. By construction a morphism $\text{Spec}(k) \rightarrow X$ with k algebraically closed, factors through U_p if and only if $U \times_X \text{Spec}(k)$ has $\geq p$ points; as usual observe that $U \times_X \text{Spec}(k)$ is scheme theoretically a disjoint union of (possibly infinitely many) copies of $\text{Spec}(k)$, see Remark 4.1. It follows that the U_p give a filtration of X as stated in the lemma. Moreover, our morphism $\text{Spec}(k) \rightarrow X$ factors through T_p if and only if $U \times_X \text{Spec}(k)$ has exactly p points. In this case we see that $V_p \times_X \text{Spec}(k)$ has exactly one point. Set $Z_p = f_p^{-1}(T_p) \subset V_p$. This is a closed subscheme of V_p . Then $Z_p \rightarrow T_p$

is an étale morphism between algebraic spaces which induces a bijection on k -valued points for any algebraically closed field k . To be sure this implies that $Z_p \rightarrow T_p$ is universally injective, whence an open immersion by Morphisms of Spaces, Lemma 51.2 hence an isomorphism and (1) has been proved.

Let $x : \operatorname{Spec}(k) \rightarrow X$ be a quasi-compact morphism where k is a field. Then the composition $\operatorname{Spec}(\bar{k}) \rightarrow \operatorname{Spec}(k) \rightarrow X$ is quasi-compact as well (Morphisms of Spaces, Lemma 8.5). In this case the scheme $U \times_X \operatorname{Spec}(\bar{k})$ is quasi-compact. In view of the fact (seen above) that it is a disjoint union of copies of $\operatorname{Spec}(\bar{k})$ we find that it has finitely many points. If the number of points is p , then we see that indeed $x \in T_p$ and the proof is finished. \square

Lemma 8.4. *Let S be a scheme. Let X be a quasi-compact, reasonable algebraic space over S . There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \dots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a separated scheme V_p and a surjective étale morphism $f_p : V_p \rightarrow U_p$ such that $f_p^{-1}(T_p) \rightarrow T_p$ is an isomorphism.

Proof. The proof of this lemma is identical to the proof of Lemma 8.3. Let n be an integer bounding the degrees of the fibres of $U \rightarrow X$ which exists as X is reasonable, see Definition 6.1. Then we see that $U_{n+1} = \emptyset$ and the proof is complete. \square

Lemma 8.5. *Let S be a scheme. Let X be a quasi-compact, reasonable algebraic space over S . There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \dots \subset U_1 = X$$

such that each $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) is a scheme.

Proof. Immediate consequence of Lemma 8.4. \square

The following result is almost identical to [GR71, Proposition 5.7.8].

Lemma 8.6. *Let X be a quasi-compact and quasi-separated algebraic space over $\operatorname{Spec}(\mathbf{Z})$. There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \dots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a quasi-compact separated scheme V_p and a surjective étale morphism $f_p : V_p \rightarrow U_p$ such that $f_p^{-1}(T_p) \rightarrow T_p$ is an isomorphism.

Proof. The proof of this lemma is identical to the proof of Lemma 8.3. Observe that a quasi-separated space is reasonable, see Lemma 5.1 and Definition 6.1. Hence we find that $U_{n+1} = \emptyset$ as in Lemma 8.4. At the end of the argument we add that since X is quasi-separated the schemes $U \times_X \dots \times_X U$ are all quasi-compact. Hence the schemes W_p are quasi-compact. Hence the quotients $V_p = W_p/S_p$ by the symmetric group S_p are quasi-compact schemes. \square

The following lemma probably belongs somewhere else.

Lemma 8.7. *Let S be a scheme. Let X be a quasi-separated algebraic space over S . Let $E \subset |X|$ be a subset. Then E is étale locally constructible (Properties of Spaces, Definition 8.2) if and only if E is a locally constructible subset of the topological space $|X|$ (Topology, Definition 15.1).*

Proof. Assume $E \subset |X|$ is a locally constructible subset of the topological space $|X|$. Let $f : U \rightarrow X$ be an étale morphism where U is a scheme. We have to show that $f^{-1}(E)$ is locally constructible in U . The question is local on U and X , hence we may assume that X is quasi-compact, $E \subset |X|$ is constructible, and U is affine. In this case $U \rightarrow X$ is quasi-compact, hence $f : |U| \rightarrow |X|$ is quasi-compact. Observe that retrocompact opens of $|X|$, resp. U are the same thing as quasi-compact opens of $|X|$, resp. U , see Topology, Lemma 27.1. Thus $f^{-1}(E)$ is constructible by Topology, Lemma 15.3.

Conversely, assume E is étale locally constructible. We want to show that E is locally constructible in the topological space $|X|$. The question is local on X , hence we may assume that X is quasi-compact as well as quasi-separated. We will show that in this case E is constructible in $|X|$. Choose open subspaces

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \dots \subset U_1 = X$$

and surjective étale morphisms $f_p : V_p \rightarrow U_p$ inducing isomorphisms $f_p^{-1}(T_p) \rightarrow T_p = U_p \setminus U_{p+1}$ where V_p is a quasi-compact separated scheme as in Lemma 8.6. By definition the inverse image $E_p \subset V_p$ of E is locally constructible in V_p . Then E_p is constructible in V_p by Properties, Lemma 2.5. Thus $E_p \cap |f_p^{-1}(T_p)| = E \cap |T_p|$ is constructible in $|T_p|$ by Topology, Lemma 15.7 (observe that $V_p \setminus f_p^{-1}(T_p)$ is quasi-compact as it is the inverse image of the quasi-compact space U_{p+1} by the quasi-compact morphism f_p). Thus

$$E = (|T_n| \cap E) \cup (|T_{n-1}| \cap E) \cup \dots \cup (|T_1| \cap E)$$

is constructible by Topology, Lemma 15.14. Here we use that $|T_p|$ is constructible in $|X|$ which is clear from what was said above. \square

9. Integral cover by a scheme

Here we prove that given any quasi-compact and quasi-separated algebraic space X , there is a scheme Y and a surjective, integral morphism $Y \rightarrow X$. After we develop some theory about limits of algebraic spaces, we will prove that one can do this with a finite morphism, see Limits of Spaces, Section 16.

Lemma 9.1. *Let S be a scheme. Let $j : V \rightarrow Y$ be a quasi-compact open immersion of algebraic spaces over S . Let $\pi : Z \rightarrow V$ be an integral morphism. Then there exists an integral morphism $\nu : Y' \rightarrow Y$ such that Z is V -isomorphic to the inverse image of V in Y' .*

Proof. Since both j and π are quasi-compact and separated, so is $j \circ \pi$. Let $\nu : Y' \rightarrow Y$ be the normalization of Y in Z , see Morphisms of Spaces, Section 48. Of course ν is integral, see Morphisms of Spaces, Lemma 48.5. The final statement follows formally from Morphisms of Spaces, Lemmas 48.4 and 48.10. \square

Lemma 9.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S .*

- (1) *There exists a surjective integral morphism $Y \rightarrow X$ where Y is a scheme,*
- (2) *given a surjective étale morphism $U \rightarrow X$ we may choose $Y \rightarrow X$ such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \rightarrow X$ factors through U .*

Proof. Part (1) is the special case of part (2) where $U = X$. Choose a surjective étale morphism $U' \rightarrow U$ where U' is a scheme. It is clear that we may replace U by U' and hence we may assume U is a scheme. Since X is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \coprod U_i \rightarrow X$ is surjective. After replacing U by U' again, we see that we may assume U is affine. Since X is quasi-separated, hence reasonable, there exists an integer d bounding the degree of the geometric fibres of $U \rightarrow X$ (see Lemma 5.1). We will prove the lemma by induction on d for all quasi-compact and separated schemes U mapping surjective and étale onto X . If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Morphisms of Spaces, Lemma 52.2 and we obtain a factorization

$$\begin{array}{ccc} U & \xrightarrow{\quad j \quad} & Y \\ & \searrow & \swarrow \pi \\ & X & \end{array}$$

with π integral and j a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in Y . Then $U \times_X Y$ is a quasi-compact, separated scheme (being finite over U) and we have

$$U \times_X Y = U \amalg W$$

Here the first summand is the image of $U \rightarrow U \times_X Y$ (which is closed by Morphisms of Spaces, Lemma 4.6 and open because it is étale as a morphism between algebraic spaces étale over Y) and the second summand is the (open and closed) complement. The image $V \subset Y$ of W is an open subspace containing $Y \setminus U$.

The étale morphism $W \rightarrow Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $|U| \subset |Y|$ is dense, it holds for all geometric points of Y by Lemma 8.1 (the degree of the fibres of a quasi-compact étale morphism does not go up under specialization). Thus we may apply the induction hypothesis to $W \rightarrow V$ and find a surjective integral morphism $Z \rightarrow V$ with Z a scheme, which Zariski locally factors through W . Choose a factorization $Z \rightarrow Z' \rightarrow Y$ with $Z' \rightarrow Y$ integral and $Z \rightarrow Z'$ open immersion (Lemma 9.1). After replacing Z' by the scheme theoretic closure of Z in Z' we may assume that Z is scheme theoretically dense in Z' . After doing this we have $Z' \times_Y V = Z$. Finally, let $T \subset Y$ be the induced closed subspace structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \longrightarrow X$$

This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \rightarrow X$ factors through U . On the other hand, let $z \in Z'$ be a point. If $z \notin Z$, then z maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of z on which the morphism factors through U . If $z \in Z$, then we have an open neighbourhood of z in Z (which is also an open neighbourhood of z in Z') which factors through $W \subset U \times_X Y$ and hence through U . \square

Lemma 9.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S such that $|X|$ has finitely many irreducible components.*

- (1) *There exists a surjective integral morphism $Y \rightarrow X$ where Y is a scheme such that f is finite étale over a quasi-compact dense open $U \subset X$,*

- (2) *given a surjective étale morphism $V \rightarrow X$ we may choose $Y \rightarrow X$ such that for every $y \in Y$ there is an open neighbourhood $W \subset Y$ such that $W \rightarrow X$ factors through V .*

Proof. The proof is the (roughly) same as the proof of Lemma 9.2 with additional technical comments to obtain the dense quasi-compact open U (and unfortunately changes in notation to keep track of U).

Part (1) is the special case of part (2) where $V = X$.

Proof of (2). Choose a surjective étale morphism $V' \rightarrow V$ where V' is a scheme. It is clear that we may replace V by V' and hence we may assume V is a scheme. Since X is quasi-compact, there exist finitely many affine opens $V_i \subset V$ such that $V' = \coprod V_i \rightarrow X$ is surjective. After replacing V by V' again, we see that we may assume V is affine. Since X is quasi-separated, hence reasonable, there exists an integer d bounding the degree of the geometric fibres of $V \rightarrow X$ (see Lemma 5.1).

By induction on $d \geq 1$ we will prove the following induction hypothesis (H_d):

- for any quasi-compact and quasi-separated algebraic space X with finitely many irreducible components, for any $m \geq 0$, for any quasi-compact and separated schemes V_j , $j = 1, \dots, m$, for any étale morphisms $\varphi_j : V_j \rightarrow X$, $j = 1, \dots, m$ such that d bounds the degree of the geometric fibres of $\varphi_j : V_j \rightarrow X$ and $\varphi = \coprod \varphi_j : V = \coprod V_j \rightarrow X$ is surjective, the statement of the lemma holds for $\varphi : V \rightarrow X$.

If $d = 1$, then each φ_j is an open immersion. Hence X is a scheme and the result holds with $Y = V$. Assume $d > 1$, assume (H_{d-1}) and let m , $\varphi : V_j \rightarrow X$, $j = 1, \dots, m$ be as in (H_d).

Let $\eta_1, \dots, \eta_n \in |X|$ be the generic points of the irreducible components of $|X|$. By Properties of Spaces, Proposition 13.3 there is an open subscheme $U \subset X$ with $\eta_1, \dots, \eta_n \in U$. By shrinking U we may assume U affine and by Morphisms, Lemma 51.1 we may assume each $\varphi_j : V_j \rightarrow X$ is finite étale over U . Of course, we see that U is quasi-compact and dense in X and that $\varphi_j^{-1}(U)$ is dense in V_j . In particular each V_j has finitely many irreducible components.

Fix $j \in \{1, \dots, m\}$. As in Morphisms of Spaces, Lemma 52.2 we let Y_j be the normalization of X in V_j . We obtain a factorization

$$\begin{array}{ccc} V_j & \xrightarrow{\quad} & Y_j \\ & \searrow \varphi_j & \swarrow \pi_j \\ & X & \end{array}$$

with π_j integral and $V_j \rightarrow Y_j$ a quasi-compact open immersion. Since Y_j is the normalization of X in V_j , we see from Morphisms of Spaces, Lemmas 48.4 and 48.10 that $\varphi_j^{-1}(U) \rightarrow \pi_j^{-1}(U)$ is an isomorphism. Thus π_j is finite étale over U . Observe that V_j is scheme theoretically dense in Y_j because Y_j is the normalization of X in V_j (follows from the characterization of relative normalization in Morphisms of Spaces, Lemma 48.5). Since V_j is quasi-compact we see that $|V_j| \subset |Y_j|$ is dense, see Morphisms of Spaces, Section 17 (and especially Morphisms of Spaces, Lemma 17.7). It follows that $|Y_j|$ has finitely many irreducible components. Then $V_j \times_X Y_j$ is a quasi-compact, separated scheme (being finite over V_j) and

$$V_j \times_X Y_j = V_j \amalg W_j$$

Here the first summand is the image of $V_j \rightarrow V_j \times_X Y_j$ (which is closed by Morphisms of Spaces, Lemma 4.6 and open because it is étale as a morphism between algebraic spaces étale over Y) and the second summand is the (open and closed) complement.

The étale morphism $W_j \rightarrow Y_j$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $V_j \subset Y_j$ by inspection. Since $|V_j| \subset |Y_j|$ is dense, it holds for all geometric points of Y_j by Lemma 8.1 (the degree of the fibres of a quasi-compact étale morphism does not go up under specialization). By (H_{d-1}) applied to $V_j \amalg W_j \rightarrow Y_j$ we find a surjective integral morphism $Y'_j \rightarrow Y_j$ with Y'_j a scheme, which Zariski locally factors through $V_j \amalg W_j$, and which is finite étale over a quasi-compact dense open $U_j \subset Y_j$. After shrinking U we may and do assume that $\pi_j^{-1}(U) \subset U_j$ (we may and do choose the same U for all j ; some details omitted).

We claim that

$$Y = \coprod_{j=1, \dots, m} Y'_j \longrightarrow X$$

is the solution to our problem. First, this morphism is integral as on each summand we have the composition $Y'_j \rightarrow Y \rightarrow X$ of integral morphisms (Morphisms of Spaces, Lemma 45.4). Second, this morphism Zariski locally factors through $V = \coprod V_j$ because we saw above that each $Y'_j \rightarrow Y_j$ factors Zariski locally through $V_j \amalg W_j = V_j \times_X Y_j$. Finally, since both $Y'_j \rightarrow Y_j$ and $Y_j \rightarrow X$ are finite étale over U , so is the composition. This finishes the proof. \square

10. Schematic locus

In this section we prove that a decent algebraic space has a dense open subspace which is a scheme. We first prove this for reasonable algebraic spaces.

Proposition 10.1. *Let S be a scheme. Let X be an algebraic space over S . If X is reasonable, then there exists a dense open subspace of X which is a scheme.*

Proof. By Properties of Spaces, Lemma 13.1 the question is local on X . Hence we may assume there exists an affine scheme U and a surjective étale morphism $U \rightarrow X$ (Properties of Spaces, Lemma 6.1). Let n be an integer bounding the degrees of the fibres of $U \rightarrow X$ which exists as X is reasonable, see Definition 6.1. We will argue by induction on n that whenever

- (1) $U \rightarrow X$ is a surjective étale morphism whose fibres have degree $\leq n$, and
- (2) U is isomorphic to a locally closed subscheme of an affine scheme

then the schematic locus is dense in X .

Let $X_n \subset X$ be the open subspace which is the complement of the closed subspace $Z_{n-1} \subset X$ constructed in Lemma 8.1 using the morphism $U \rightarrow X$. Let $U_n \subset U$ be the inverse image of X_n . Then $U_n \rightarrow X_n$ is finite locally free of degree n . Hence X_n is a scheme by Properties of Spaces, Proposition 14.1 (and the fact that any finite set of points of U_n is contained in an affine open of U_n , see Properties, Lemma 29.5).

Let $X' \subset X$ be the open subspace such that $|X'|$ is the interior of $|Z_{n-1}|$ in $|X|$ (see Topology, Definition 21.1). Let $U' \subset U$ be the inverse image. Then $U' \rightarrow X'$ is surjective étale and has degrees of fibres bounded by $n-1$. By induction we see that the schematic locus of X' is an open dense $X'' \subset X'$. By elementary topology we see that $X'' \cup X_n \subset X$ is open and dense and we win. \square

Theorem 10.2 (David Rydh). *Let S be a scheme. Let X be an algebraic space over S . If X is decent, then there exists a dense open subspace of X which is a scheme.*

Proof. Assume X is a decent algebraic space for which the theorem is false. By Properties of Spaces, Lemma 13.1 there exists a largest open subspace $X' \subset X$ which is a scheme. Since X' is not dense in X , there exists an open subspace $X'' \subset X$ such that $|X''| \cap |X'| = \emptyset$. Replacing X by X'' we get a nonempty decent algebraic space X which does not contain *any* open subspace which is a scheme.

Choose a nonempty affine scheme U and an étale morphism $U \rightarrow X$. We may and do replace X by the open subscheme corresponding to the image of $|U| \rightarrow |X|$. Consider the sequence of open subspaces

$$X = X_0 \supset X_1 \supset X_2 \dots$$

constructed in Lemma 8.2 for the morphism $U \rightarrow X$. Note that $X_0 = X_1$ as $U \rightarrow X$ is surjective. Let $U = U_0 = U_1 \supset U_2 \dots$ be the induced sequence of open subschemes of U .

Choose a nonempty open affine $V_1 \subset U_1$ (for example $V_1 = U_1$). By induction we will construct a sequence of nonempty affine opens $V_1 \supset V_2 \supset \dots$ with $V_n \subset U_n$. Namely, having constructed V_1, \dots, V_{n-1} we can always choose V_n unless $V_{n-1} \cap U_n = \emptyset$. But if $V_{n-1} \cap U_n = \emptyset$, then the open subspace $X' \subset X$ with $|X'| = \text{Im}(|V_{n-1}| \rightarrow |X|)$ is contained in $|X| \setminus |X_n|$. Hence $V_{n-1} \rightarrow X'$ is an étale morphism whose fibres have degree bounded by $n - 1$. In other words, X' is reasonable (by definition), hence X' contains a nonempty open subscheme by Proposition 10.1. This is a contradiction which shows that we can pick V_n .

By Limits, Lemma 4.3 the limit $V_\infty = \lim V_n$ is a nonempty scheme. Pick a morphism $\text{Spec}(k) \rightarrow V_\infty$. The composition $\text{Spec}(k) \rightarrow V_\infty \rightarrow U \rightarrow X$ has image contained in all X_d by construction. In other words, the fibred $U \times_X \text{Spec}(k)$ has infinite degree which contradicts the definition of a decent space. This contradiction finishes the proof of the theorem. \square

Lemma 10.3. *Let S be a scheme. Let $X \rightarrow Y$ be a surjective finite locally free morphism of algebraic spaces over S . For $y \in |Y|$ the following are equivalent*

- (1) *y is in the schematic locus of Y , and*
- (2) *there exists an affine open $U \subset X$ containing the preimage of y .*

Proof. If $y \in Y$ is in the schematic locus, then it has an affine open neighbourhood $V \subset Y$ and the inverse image U of V in X is an open finite over V , hence affine. Thus (1) implies (2).

Conversely, assume that $U \subset X$ as in (2) is given. Set $R = X \times_Y X$ and denote the projections $s, t : R \rightarrow X$. Consider $Z = R \setminus s^{-1}(U) \cap t^{-1}(U)$. This is a closed subset of R . The image $t(Z)$ is a closed subset of X which can loosely be described as the set of points of X which are R -equivalent to a point of $X \setminus U$. Hence $U' = X \setminus t(Z)$ is an R -invariant, open subspace of X contained in U which contains the fibre of $X \rightarrow Y$ over y . Since $X \rightarrow Y$ is open (Morphisms of Spaces, Lemma 30.6) the image of U' is an open subspace $V' \subset Y$. Since U' is R -invariant and $R = X \times_Y X$, we see that U' is the inverse image of V' (use Properties of Spaces, Lemma 4.3). After replacing Y by V' and X by U' we see that we may assume X is a scheme isomorphic to an open subscheme of an affine scheme.

Assume X is a scheme isomorphic to an open subscheme of an affine scheme. In this case the fppf quotient sheaf X/R is a scheme, see Properties of Spaces, Proposition 14.1. Since Y is a sheaf in the fppf topology, obtain a canonical map $X/R \rightarrow Y$ factoring $X \rightarrow Y$. Since $X \rightarrow Y$ is surjective finite locally free, it is surjective as a map of sheaves (Spaces, Lemma 5.9). We conclude that $X/R \rightarrow Y$ is surjective as a map of sheaves. On the other hand, since $R = X \times_Y X$ as sheaves we conclude that $X/R \rightarrow Y$ is injective as a map of sheaves. Hence $X/R \rightarrow Y$ is an isomorphism and we see that Y is representable. \square

At this point we have several different ways for proving the following lemma.

Lemma 10.4. *Let S be a scheme. Let X be an algebraic space over S . If there exists a finite, étale, surjective morphism $U \rightarrow X$ where U is a scheme, then there exists a dense open subspace of X which is a scheme.*

First proof. The morphism $U \rightarrow X$ is finite locally free. Hence there is a decomposition of X into open and closed subspaces $X_d \subset X$ such that $U \times_X X_d \rightarrow X_d$ is finite locally free of degree d . Thus we may assume $U \rightarrow X$ is finite locally free of degree d . In this case, let $U_i \subset U$, $i \in I$ be the set of affine opens. For each i the morphism $U_i \rightarrow X$ is étale and has universally bounded fibres (namely, bounded by d). In other words, X is reasonable and the result follows from Proposition 10.1. \square

Second proof. The question is local on X (Properties of Spaces, Lemma 13.1), hence may assume X is quasi-compact. Then U is quasi-compact. Then there exists a dense open subscheme $W \subset U$ which is separated (Properties, Lemma 29.3). Set $Z = U \setminus W$. Let $R = U \times_X U$ and $s, t : R \rightarrow U$ the projections. Then $t^{-1}(Z)$ is nowhere dense in R (Topology, Lemma 21.6) and hence $\Delta = s(t^{-1}(Z))$ is an R -invariant closed nowhere dense subset of U (Morphisms, Lemma 48.7). Let $u \in U \setminus \Delta$ be a generic point of an irreducible component. Since these points are dense in $U \setminus \Delta$ and since Δ is nowhere dense, it suffices to show that the image $x \in X$ of u is in the schematic locus of X . Observe that $t(s^{-1}(\{u\})) \subset W$ is a finite set of generic points of irreducible components of W (compare with Properties of Spaces, Lemma 11.1). By Properties, Lemma 29.1 we can find an affine open $V \subset W$ such that $t(s^{-1}(\{u\})) \subset V$. Since $t(s^{-1}(\{u\}))$ is the fibre of $|U| \rightarrow |X|$ over x , we conclude by Lemma 10.3. \square

Third proof. (This proof is essentially the same as the second proof, but uses fewer references.) Assume X is an algebraic space, U a scheme, and $U \rightarrow X$ is a finite étale surjective morphism. Write $R = U \times_X U$ and denote $s, t : R \rightarrow U$ the projections as usual. Note that s, t are surjective, finite and étale. Claim: The union of the R -invariant affine opens of U is topologically dense in U .

Proof of the claim. Let $W \subset U$ be an affine open. Set $W' = t(s^{-1}(W)) \subset U$. Since $s^{-1}(W)$ is affine (hence quasi-compact) we see that $W' \subset U$ is a quasi-compact open. By Properties, Lemma 29.3 there exists a dense open $W'' \subset W'$ which is a separated scheme. Set $\Delta' = W' \setminus W''$. This is a nowhere dense closed subset of W'' . Since $t|_{s^{-1}(W)} : s^{-1}(W) \rightarrow W'$ is open (because it is étale) we see that the inverse image $(t|_{s^{-1}(W)})^{-1}(\Delta') \subset s^{-1}(W)$ is a nowhere dense closed subset (see Topology, Lemma 21.6). Hence, by Morphisms, Lemma 48.7 we see that

$$\Delta = s((t|_{s^{-1}(W)})^{-1}(\Delta'))$$

is a nowhere dense closed subset of W . Pick any point $\eta \in W$, $\eta \notin \Delta$ which is a generic point of an irreducible component of W (and hence of U). By our choices above the finite set $t(s^{-1}(\{\eta\})) = \{\eta_1, \dots, \eta_n\}$ is contained in the separated scheme W'' . Note that the fibres of s are finite discrete spaces, and that generalizations lift along the étale morphism t , see Morphisms, Lemmas 36.12 and 25.9. In this way we see that each η_i is a generic point of an irreducible component of W'' . Thus, by Properties, Lemma 29.1 we can find an affine open $V \subset W''$ such that $\{\eta_1, \dots, \eta_n\} \subset V$. By Groupoids, Lemma 24.1 this implies that η is contained in an R -invariant affine open subscheme of U . The claim follows as W was chosen as an arbitrary affine open of U and because the set of generic points of irreducible components of $W \setminus \Delta$ is dense in W .

Using the claim we can finish the proof. Namely, if $W \subset U$ is an R -invariant affine open, then the restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 19.1 and discussion following it). In particular the maps $R_W \rightarrow W$ are finite étale also. It follows in particular that R_W is affine. Thus we see that W/R_W is a scheme, by Groupoids, Proposition 23.9. On the other hand, W/R_W is an open subspace of X by Spaces, Lemma 10.2. Hence having a dense collection of points contained in R -invariant affine open of U certainly implies that the schematic locus of X (see Properties of Spaces, Lemma 13.1) is open dense in X . \square

11. Residue fields and henselian local rings

For a decent algebraic space we can define the residue field and the henselian local ring at a point. For example, the following lemma tells us the residue field of a point on a decent space is defined.

Lemma 11.1. *Let S be a scheme. Let X be an algebraic space over S . Consider the map*

$$\{\mathrm{Spec}(k) \rightarrow X \text{ monomorphism where } k \text{ is a field}\} \longrightarrow |X|$$

This map is always injective. If X is decent then this map is a bijection.

Proof. We have seen in Properties of Spaces, Lemma 4.12 that the map is an injection in general. By Lemma 5.1 it is surjective when X is decent (actually one can say this is part of the definition of being decent). \square

Let S be a scheme. Let X be an algebraic space over S . If a point $x \in |X|$ can be represented by a monomorphism $\mathrm{Spec}(k) \rightarrow X$, then the field k is unique up to unique isomorphism. For a decent algebraic space such a monomorphism exists for every point by Lemma 11.1 and hence the following definition makes sense.

Definition 11.2. Let S be a scheme. Let X be a decent algebraic space over S . Let $x \in |X|$. The *residue field of X at x* is the unique field $\kappa(x)$ which comes equipped with a monomorphism $\mathrm{Spec}(\kappa(x)) \rightarrow X$ representing x .

Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of decent algebraic spaces over S . Let $x \in |X|$ be a point. Set $y = f(x) \in |Y|$. Then the composition $\mathrm{Spec}(\kappa(x)) \rightarrow Y$ is in the equivalence class defining y and hence factors through $\mathrm{Spec}(\kappa(y)) \rightarrow Y$.

In other words we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\kappa(x)) & \xrightarrow{x} & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\kappa(y)) & \xrightarrow{y} & Y \end{array}$$

The left vertical morphism corresponds to a homomorphism $\kappa(y) \rightarrow \kappa(x)$ of fields. We will often simply call this the homomorphism induced by f .

Lemma 11.3. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of decent algebraic spaces over S . Let $x \in |X|$ be a point with image $y = f(x) \in |Y|$. The following are equivalent*

- (1) *f induces an isomorphism $\kappa(y) \rightarrow \kappa(x)$, and*
- (2) *the induced morphism $\mathrm{Spec}(\kappa(x)) \rightarrow Y$ is a monomorphism.*

Proof. Immediate from the discussion above. \square

The following lemma tells us that the henselian local ring of a point on a decent algebraic space is defined.

Lemma 11.4. *Let S be a scheme. Let X be a decent algebraic space over S . For every point $x \in |X|$ there exists an étale morphism*

$$(U, u) \longrightarrow (X, x)$$

where U is an affine scheme, u is the only point of U lying over x , and the induced homomorphism $\kappa(x) \rightarrow \kappa(u)$ is an isomorphism.

Proof. We may assume that X is quasi-compact by replacing X with a quasi-compact open containing x . Recall that x can be represented by a quasi-compact (mono)morphism from the spectrum of a field (by definition of decent spaces). Thus the lemma follows from Lemma 8.3. \square

Definition 11.5. Let S be a scheme. Let X be an algebraic space over S . Let $x \in X$ be a point. An *elementary étale neighbourhood* is an étale morphism $(U, u) \rightarrow (X, x)$ where U is a scheme, $u \in U$ is a point mapping to x , and the morphism $u = \mathrm{Spec}(\kappa(u)) \rightarrow X$ is a monomorphism. A *morphism of elementary étale neighbourhoods* $(U, u) \rightarrow (U', u')$ is defined as a morphism $U \rightarrow U'$ over X mapping u to u' .

If X is not decent then the category of elementary étale neighbourhoods may be empty.

Lemma 11.6. *Let S be a scheme. Let X be a decent algebraic space over S . Let x be a point of X . The category of elementary étale neighborhoods of (X, x) is cofiltered (see *Categories*, Definition 20.1).*

Proof. The category is nonempty by Lemma 11.4. Suppose that we have two elementary étale neighbourhoods $(U_i, u_i) \rightarrow (X, x)$. Then consider $U = U_1 \times_X U_2$. Since $\mathrm{Spec}(\kappa(u_i)) \rightarrow X$, $i = 1, 2$ are both monomorphisms in the class of x (Lemma 11.3), we see that

$$u = \mathrm{Spec}(\kappa(u_1)) \times_X \mathrm{Spec}(\kappa(u_2))$$

is the spectrum of a field $\kappa(u)$ such that the induced maps $\kappa(u_i) \rightarrow \kappa(u)$ are isomorphisms. Then $u \rightarrow U$ is a point of U and we see that $(U, u) \rightarrow (X, x)$ is an

elementary étale neighbourhood dominating (U_i, u_i) . If $a, b : (U_1, u_1) \rightarrow (U_2, u_2)$ are two morphisms between our elementary étale neighbourhoods, then we consider the scheme

$$U = U_1 \times_{(a,b), (U_2 \times_X U_2), \Delta} U_2$$

Using Properties of Spaces, Lemma 16.6 we see that $U \rightarrow X$ is étale. Moreover, in exactly the same manner as before we see that U has a point u such that $(U, u) \rightarrow (X, x)$ is an elementary étale neighbourhood. Finally, $U \rightarrow U_1$ equalizes a and b and the proof is finished. \square

Definition 11.7. Let S be a scheme. Let X be a decent algebraic space over S . Let $x \in |X|$. The *henselian local ring of X at x* , is

$$\mathcal{O}_{X,x}^h = \operatorname{colim} \Gamma(U, \mathcal{O}_U)$$

where the colimit is over the elementary étale neighbourhoods $(U, u) \rightarrow (X, x)$.

Here is the analogue of Properties of Spaces, Lemma 22.1.

Lemma 11.8. *Let S be a scheme. Let X be a decent algebraic space over S . Let $x \in |X|$. Let $(U, u) \rightarrow (X, x)$ be an elementary étale neighbourhood. Then*

$$\mathcal{O}_{X,x}^h = \mathcal{O}_{U,u}^h$$

In words: the henselian local ring of X at x is equal to the henselization $\mathcal{O}_{U,u}^h$ of the local ring $\mathcal{O}_{U,u}$ of U at u .

Proof. Since the category of elementary étale neighbourhood of (X, x) is cofiltered (Lemma 11.6) we see that the category of elementary étale neighbourhoods of (U, u) is initial in the category of elementary étale neighbourhood of (X, x) . Then the equality follows from More on Morphisms, Lemma 35.5 and Categories, Lemma 17.2 (initial is turned into cofinal because the colimit defining henselian local rings is over the opposite of the category of elementary étale neighbourhoods). \square

Lemma 11.9. *Let S be a scheme. Let X be a decent algebraic space over S . Let \bar{x} be a geometric point of X lying over $x \in |X|$. The étale local ring $\mathcal{O}_{X,\bar{x}}$ of X at \bar{x} (Properties of Spaces, Definition 22.2) is the strict henselization of the henselian local ring $\mathcal{O}_{X,x}^h$ of X at x .*

Proof. Follows from Lemma 11.8, Properties of Spaces, Lemma 22.1 and the fact that $(R^h)^{sh} = R^{sh}$ for a local ring $(R, \mathfrak{m}, \kappa)$ and a given separable algebraic closure κ^{sep} of κ . This equality follows from Algebra, Lemma 154.7. \square

Lemma 11.10. *Let S be a scheme. Let X be a decent algebraic space over S . Let $x \in |X|$. The residue field of the henselian local ring of X at x (Definition 11.7) is the residue field of X at x (Definition 11.2).*

Proof. Choose an elementary étale neighbourhood $(U, u) \rightarrow (X, x)$. Then $\kappa(u) = \kappa(x)$ and $\mathcal{O}_{X,x}^h = \mathcal{O}_{U,u}^h$ (Lemma 11.8). The residue field of $\mathcal{O}_{U,u}^h$ is $\kappa(u)$ by Algebra, Lemma 155.1 (the output of this lemma is the construction/definition of the henselization of a local ring, see Algebra, Definition 155.3). \square

Remark 11.11. Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of decent algebraic spaces over S . Let $x \in |X|$ with image $y \in |Y|$. Choose an elementary étale neighbourhood $(V, v) \rightarrow (Y, y)$ (possible by Lemma 11.4). Then $V \times_Y X$ is an algebraic space étale over X which has a unique point x' mapping to x in

X and to v in V . (Details omitted; use that all points can be represented by monomorphisms from spectra of fields.) Choose an elementary étale neighbourhood $(U, u) \rightarrow (V \times_Y X, x')$. Then we obtain the following commutative diagram

$$\begin{array}{ccccccc}
 \mathrm{Spec}(\mathcal{O}_{X,\bar{x}}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{X,x}^h) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{U,u}) & \longrightarrow & U \longrightarrow X \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow \\
 \mathrm{Spec}(\mathcal{O}_{Y,\bar{y}}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{Y,y}^h) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{V,v}) & \longrightarrow & V \longrightarrow Y
 \end{array}$$

This comes from the identifications $\mathcal{O}_{X,\bar{x}} = \mathcal{O}_{U,u}^{sh}$, $\mathcal{O}_{X,x}^h = \mathcal{O}_{U,u}^h$, $\mathcal{O}_{Y,\bar{y}} = \mathcal{O}_{V,v}^{sh}$, $\mathcal{O}_{Y,y}^h = \mathcal{O}_{V,v}^h$ see in Lemma 11.8 and Properties of Spaces, Lemma 22.1 and the functoriality of the (strict) henselization discussed in Algebra, Sections 154 and 155.

12. Points on decent spaces

In this section we prove some properties of points on decent algebraic spaces. The following lemma shows that specialization of points behaves well on decent algebraic spaces. Spaces, Example 14.9 shows that this is **not** true in general.

Lemma 12.1. *Let S be a scheme. Let X be a decent algebraic space over S . Let $U \rightarrow X$ be an étale morphism from a scheme to X . If $u, u' \in |U|$ map to the same point of $|X|$, and $u' \rightsquigarrow u$, then $u = u'$.*

Proof. Combine Lemmas 5.1 and 7.1. \square

Lemma 12.2. *Let S be a scheme. Let X be a decent algebraic space over S . Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x' . Then for every étale morphism $\varphi : U \rightarrow X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \rightsquigarrow u$ with $\varphi(u') = x'$.*

Proof. Combine Lemmas 5.1 and 7.3. \square

Lemma 12.3. *Let S be a scheme. Let X be a decent algebraic space over S . Then $|X|$ is Kolmogorov (see Topology, Definition 8.6).*

Proof. Let $x_1, x_2 \in |X|$ with $x_1 \rightsquigarrow x_2$ and $x_2 \rightsquigarrow x_1$. We have to show that $x_1 = x_2$. Pick a scheme U and an étale morphism $U \rightarrow X$ such that x_1, x_2 are both in the image of $|U| \rightarrow |X|$. By Lemma 12.2 we can find a specialization $u_1 \rightsquigarrow u_2$ in U mapping to $x_1 \rightsquigarrow x_2$. By Lemma 12.2 we can find $u'_2 \rightsquigarrow u_1$ mapping to $x_2 \rightsquigarrow x_1$. This means that $u'_2 \rightsquigarrow u_2$ is a specialization between points of U mapping to the same point of X , namely x_2 . This is not possible, unless $u'_2 = u_2$, see Lemma 12.1. Hence also $u_1 = u_2$ as desired. \square

Proposition 12.4. *Let S be a scheme. Let X be a decent algebraic space over S . Then the topological space $|X|$ is sober (see Topology, Definition 8.6).*

Proof. We have seen in Lemma 12.3 that $|X|$ is Kolmogorov. Hence it remains to show that every irreducible closed subset $T \subset |X|$ has a generic point. By Properties of Spaces, Lemma 12.3 there exists a closed subspace $Z \subset X$ with $|Z| = |T|$. By definition this means that $Z \rightarrow X$ is a representable morphism of algebraic spaces. Hence Z is a decent algebraic space by Lemma 5.3. By Theorem 10.2 we see that there exists an open dense subspace $Z' \subset Z$ which is a scheme. This means that

$|Z'| \subset T$ is open dense. Hence the topological space $|Z'|$ is irreducible, which means that Z' is an irreducible scheme. By Schemes, Lemma 11.1 we conclude that $|Z'|$ is the closure of a single point $\eta \in T$ and hence also $T = \overline{\{\eta\}}$, and we win. \square

For decent algebraic spaces dimension works as expected.

Lemma 12.5. *Let S be a scheme. Dimension as defined in Properties of Spaces, Section 9 behaves well on decent algebraic spaces X over S .*

- (1) *If $x \in |X|$, then $\dim_x(|X|) = \dim_x(X)$, and*
- (2) *$\dim(|X|) = \dim(X)$.*

Proof. Proof of (1). Choose a scheme U with a point $u \in U$ and an étale morphism $h : U \rightarrow X$ mapping u to x . By definition the dimension of X at x is $\dim_u(|U|)$. Thus we may pick U such that $\dim_x(X) = \dim(|U|)$. Let d be an integer. If $\dim(U) \geq d$, then there exists a sequence of nontrivial specializations $u_d \rightsquigarrow \dots \rightsquigarrow u_0$ in U . Taking the image we find a corresponding sequence $h(u_d) \rightsquigarrow \dots \rightsquigarrow h(u_0)$ each of which is nontrivial by Lemma 12.1. Hence we see that the image of $|U|$ in $|X|$ has dimension at least d . Conversely, suppose that $x_d \rightsquigarrow \dots \rightsquigarrow x_0$ is a sequence of specializations in $|X|$ with x_0 in the image of $|U| \rightarrow |X|$. Then we can lift this to a sequence of specializations in U by Lemma 12.2.

Part (2) is an immediate consequence of part (1), Topology, Lemma 10.2, and Properties of Spaces, Section 9. \square

Lemma 12.6. *Let S be a scheme. Let $X \rightarrow Y$ be a locally quasi-finite morphism of algebraic spaces over S . Let $x \in |X|$ with image $y \in |Y|$. Then the dimension of the local ring of Y at y is \geq to the dimension of the local ring of X at x .*

Proof. The definition of the dimension of the local ring of a point on an algebraic space is given in Properties of Spaces, Definition 10.2. Choose an étale morphism $(V, v) \rightarrow (Y, y)$ where V is a scheme. Choose an étale morphism $U \rightarrow V \times_Y X$ and a point $u \in U$ mapping to $x \in |X|$ and $v \in V$. Then $U \rightarrow V$ is locally quasi-finite and we have to prove that

$$\dim(\mathcal{O}_{V,v}) \geq \dim(\mathcal{O}_{U,u})$$

This is Algebra, Lemma 125.4. \square

Lemma 12.7. *Let S be a scheme. Let $X \rightarrow Y$ be a locally quasi-finite morphism of algebraic spaces over S . Then $\dim(X) \leq \dim(Y)$.*

Proof. This follows from Lemma 12.6 and Properties of Spaces, Lemma 10.3. \square

The following lemma is a tiny bit stronger than Properties of Spaces, Lemma 15.3. We will improve this lemma in Lemma 14.2.

Lemma 12.8. *Let S be a scheme. Let k be a field. Let X be an algebraic space over S and assume that there exists a surjective étale morphism $\mathrm{Spec}(k) \rightarrow X$. If X is decent, then $X \cong \mathrm{Spec}(k')$ where k/k' is a finite separable extension.*

Proof. The assumption implies that $|X| = \{x\}$ is a singleton. Since X is decent we can find a quasi-compact monomorphism $\mathrm{Spec}(k') \rightarrow X$ whose image is x . Then the projection $U = \mathrm{Spec}(k') \times_X \mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k)$ is a monomorphism, whence $U = \mathrm{Spec}(k)$, see Schemes, Lemma 23.11. Hence the projection $\mathrm{Spec}(k) = U \rightarrow \mathrm{Spec}(k')$ is étale and we win. \square

13. Reduced singleton spaces

A *singleton* space is an algebraic space X such that $|X|$ is a singleton. It turns out that these can be more interesting than just being the spectrum of a field, see Spaces, Example 14.7. We develop a tiny bit of machinery to be able to talk about these.

Lemma 13.1. *Let S be a scheme. Let Z be an algebraic space over S . Let k be a field and let $\mathrm{Spec}(k) \rightarrow Z$ be surjective and flat. Then any morphism $\mathrm{Spec}(k') \rightarrow Z$ where k' is a field is surjective and flat.*

Proof. Consider the fibre square

$$\begin{array}{ccc} T & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k') & \longrightarrow & Z \end{array}$$

Note that $T \rightarrow \mathrm{Spec}(k')$ is flat and surjective hence T is not empty. On the other hand $T \rightarrow \mathrm{Spec}(k)$ is flat as k is a field. Hence $T \rightarrow Z$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 31.5 that $\mathrm{Spec}(k') \rightarrow Z$ is flat. It is surjective as by assumption $|Z|$ is a singleton. \square

Lemma 13.2. *Let S be a scheme. Let Z be an algebraic space over S . The following are equivalent*

- (1) *Z is reduced and $|Z|$ is a singleton,*
- (2) *there exists a surjective flat morphism $\mathrm{Spec}(k) \rightarrow Z$ where k is a field, and*
- (3) *there exists a locally of finite type, surjective, flat morphism $\mathrm{Spec}(k) \rightarrow Z$ where k is a field.*

Proof. Assume (1). Let W be a scheme and let $W \rightarrow Z$ be a surjective étale morphism. Then W is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of W . Since W is reduced we have $\mathcal{O}_{W,\eta} = \kappa(\eta)$. It follows that the canonical morphism $\eta = \mathrm{Spec}(\kappa(\eta)) \rightarrow W$ is flat. We see that the composition $\eta \rightarrow Z$ is flat (see Morphisms of Spaces, Lemma 30.3). It is also surjective as $|Z|$ is a singleton. In other words (2) holds.

Assume (2). Let W be a scheme and let $W \rightarrow Z$ be a surjective étale morphism. Choose a field k and a surjective flat morphism $\mathrm{Spec}(k) \rightarrow Z$. Then $W \times_Z \mathrm{Spec}(k)$ is a scheme étale over k . Hence $W \times_Z \mathrm{Spec}(k)$ is a disjoint union of spectra of fields (see Remark 4.1), in particular reduced. Since $W \times_Z \mathrm{Spec}(k) \rightarrow W$ is surjective and flat we conclude that W is reduced (Descent, Lemma 19.1). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme W and an étale morphism $W \rightarrow Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite type by Morphisms of Spaces, Lemmas 23.2 and 39.9. It is also flat and surjective by Lemma 13.1. Hence (3) holds. \square

The following lemma singles out a slightly better class of singleton algebraic spaces than the preceding lemma.

Lemma 13.3. *Let S be a scheme. Let Z be an algebraic space over S . The following are equivalent*

- (1) *Z is reduced, locally Noetherian, and $|Z|$ is a singleton, and*
- (2) *there exists a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \rightarrow Z$ where k is a field.*

Proof. Assume (2) holds. By Lemma 13.2 we see that Z is reduced and $|Z|$ is a singleton. Let W be a scheme and let $W \rightarrow Z$ be a surjective étale morphism. Choose a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \rightarrow Z$. Then $W \times_Z \mathrm{Spec}(k)$ is a scheme étale over k , hence a disjoint union of spectra of fields (see Remark 4.1), hence locally Noetherian. Since $W \times_Z \mathrm{Spec}(k) \rightarrow W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_Z \mathrm{Spec}(k) \rightarrow W\}$ is an fppf covering and we conclude that W is locally Noetherian (Descent, Lemma 16.1). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme W and an étale morphism $W \rightarrow Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because W is locally Noetherian the morphism $w : \mathrm{Spec}(k) \rightarrow W$ is of finite presentation, see Morphisms, Lemma 21.7. Hence the composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 28.2 and 39.8. It is also flat and surjective by Lemma 13.1. Hence (2) holds. \square

Lemma 13.4. *Let S be a scheme. Let $Z' \rightarrow Z$ be a monomorphism of algebraic spaces over S . Assume there exists a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \rightarrow Z$. Then either Z' is empty or $Z' = Z$.*

Proof. We may assume that Z' is nonempty. In this case the fibre product $T = Z' \times_Z \mathrm{Spec}(k)$ is nonempty, see Properties of Spaces, Lemma 4.3. Now T is an algebraic space and the projection $T \rightarrow \mathrm{Spec}(k)$ is a monomorphism. Hence $T = \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemma 10.8. We conclude that $\mathrm{Spec}(k) \rightarrow Z$ factors through Z' . But as $\mathrm{Spec}(k) \rightarrow Z$ is surjective, flat and locally of finite presentation, we see that $\mathrm{Spec}(k) \rightarrow Z$ is surjective as a map of sheaves on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ (see Spaces, Remark 5.2) and we conclude that $Z' = Z$. \square

The following lemma says that to each point of an algebraic space we can associate a canonical reduced, locally Noetherian singleton algebraic space.

Lemma 13.5. *Let S be a scheme. Let X be an algebraic space over S . Let $x \in |X|$. Then there exists a unique monomorphism $Z \rightarrow X$ of algebraic spaces over S such that Z is an algebraic space which satisfies the equivalent conditions of Lemma 13.3 and such that the image of $|Z| \rightarrow |X|$ is $\{x\}$.*

Proof. Choose a scheme U and a surjective étale morphism $U \rightarrow X$. Set $R = U \times_X U$ so that $X = U/R$ is a presentation (see Spaces, Section 9). Set

$$U' = \coprod_{u \in U \text{ lying over } x} \mathrm{Spec}(\kappa(u)).$$

The canonical morphism $U' \rightarrow U$ is a monomorphism. Let

$$R' = U' \times_X U' = R \times_{(U \times_S U)} (U' \times_S U').$$

Because $U' \rightarrow U$ is a monomorphism we see that the projections $s', t' : R' \rightarrow U'$ factor as a monomorphism followed by an étale morphism. Hence, as U' is a disjoint

union of spectra of fields, using Remark 4.1, and using Schemes, Lemma 23.11 we conclude that R' is a disjoint union of spectra of fields and that the morphisms $s', t' : R' \rightarrow U'$ are étale. Hence $Z = U'/R'$ is an algebraic space by Spaces, Theorem 10.5. As R' is the restriction of R by $U' \rightarrow U$ we see $Z \rightarrow X$ is a monomorphism by Groupoids, Lemma 20.6. Since $Z \rightarrow X$ is a monomorphism we see that $|Z| \rightarrow |X|$ is injective, see Morphisms of Spaces, Lemma 10.9. By Properties of Spaces, Lemma 4.3 we see that

$$|U'| = |Z \times_X U'| \rightarrow |Z| \times_{|X|} |U'|$$

is surjective which implies (by our choice of U') that $|Z| \rightarrow |X|$ has image $\{x\}$. We conclude that $|Z|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., we see that Z satisfies the equivalent conditions of Lemma 13.3.

Let us prove uniqueness of $Z \rightarrow X$. Suppose that $Z' \rightarrow X$ is a second such monomorphism of algebraic spaces. Then the projections

$$Z' \longleftarrow Z' \times_X Z \longrightarrow Z$$

are monomorphisms. The algebraic space in the middle is nonempty by Properties of Spaces, Lemma 4.3. Hence the two projections are isomorphisms by Lemma 13.4 and we win. \square

We introduce the following terminology which foreshadows the residual gerbes we will introduce later, see Properties of Stacks, Definition 11.8.

Definition 13.6. Let S be a scheme. Let X be an algebraic space over S . Let $x \in |X|$. The *residual space of X at x* ² is the monomorphism $Z_x \rightarrow X$ constructed in Lemma 13.5.

In particular we know that Z_x is a locally Noetherian, reduced, singleton algebraic space and that there exists a field and a surjective, flat, locally finitely presented morphism

$$\mathrm{Spec}(k) \longrightarrow Z_x.$$

The residual space is often given by a monomorphism from the spectrum of a field.

Lemma 13.7. *Let S be a scheme. Let X be an algebraic space over S . Let $x \in |X|$. The residual space Z_x of X at x is isomorphic to the spectrum of a field if and only if x can be represented by a monomorphism $\mathrm{Spec}(k) \rightarrow X$ where k is a field. If X is decent, this holds for all $x \in |X|$.*

Proof. Since $Z_x \rightarrow X$ is a monomorphism, if $Z_x = \mathrm{Spec}(k)$ for some field k , then x is represented by the monomorphism $\mathrm{Spec}(k) = Z_x \rightarrow X$. Conversely, if $\mathrm{Spec}(k) \rightarrow X$ is a monomorphism which represents x , then $Z_x \times_X \mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k)$ is a monomorphism whose source is nonempty by Properties of Spaces, Lemma 4.3. Hence $Z_x \times_X \mathrm{Spec}(k) = \mathrm{Spec}(k)$ by Morphisms of Spaces, Lemma 10.8. Hence we get a monomorphism $\mathrm{Spec}(k) \rightarrow Z_x$. This is an isomorphism by Lemma 13.4. The final statement follows from Lemma 11.1. \square

The residual space is a regular algebraic space by the following lemma.

Lemma 13.8. *A reduced, locally Noetherian singleton algebraic space Z is regular.*

²This is nonstandard notation.

Proof. Let Z be a reduced, locally Noetherian singleton algebraic space over a scheme S . Let $W \rightarrow Z$ be a surjective étale morphism where W is a scheme. Let k be a field and let $\text{Spec}(k) \rightarrow Z$ be surjective, flat, and locally of finite presentation (see Lemma 13.3). The scheme $T = W \times_Z \text{Spec}(k)$ is étale over k in particular regular, see Remark 4.1. Since $T \rightarrow W$ is locally of finite presentation, flat, and surjective it follows that W is regular, see Descent, Lemma 19.2. By definition this means that Z is regular. \square

Lemma 13.9. *Let S be a scheme. Let $f : Y \rightarrow X$ be a morphism of algebraic spaces over S . Let $x \in |X|$ be a point. Assume*

- (1) $|f|(|Y|)$ is contained in $\{x\} \subset |X|$,
- (2) Y is reduced, and
- (3) X is locally Noetherian.

Then f factors through the residual space Z_x of X at x .

Proof. Preliminary remark: since $Z_x \rightarrow X$ is a monomorphism, it suffices to find a surjective étale morphism $Y' \rightarrow Y$ such that $Y' \rightarrow X$ factors through Z_x . A remark here is that Y' is reduced as well.

Let U be an affine scheme and let $U \rightarrow X$ be an étale morphism such that x is in the image of $|U| \rightarrow |X|$. Since X is locally Noetherian, U is a Noetherian affine scheme. By assumption (1) we see that $Y' = U \times_X Y \rightarrow Y$ is surjective as well as étale. Denote $E \subset |U|$ the set of points mapping to x . There are no nontrivial specializations between the elements of E , see Lemma 7.2. The morphism $Y' \rightarrow U$ maps $|Y'|$ into E . By our construction of Z_x in the proof of Lemma 13.5 we know that $\coprod_{u \in E} u \rightarrow X$ factors through Z_x . Hence it suffices to prove that $Y' \rightarrow U$ factors through $\coprod_{u \in E} u \rightarrow X$. After replacing Y' by an étale covering by a scheme (which we are allowed by our preliminary remark), this follows from Morphisms, Lemma 58.2. \square

Lemma 13.10. *Let S be a scheme. Let $f : Y \rightarrow X$ be a morphism of algebraic spaces over S . Let $x \in |X|$ be a point. Assume*

- (1) $|f|(|Y|)$ is contained in $\{x\} \subset |X|$,
- (2) Y is reduced, and
- (3) x can be represented by a quasi-compact monomorphism $x : \text{Spec}(k) \rightarrow X$ where k is a field (for example if X is decent).

Then f factors through the residual space $Z_x = \text{Spec}(k)$ of X at x .

Proof. By Lemma 13.7 we have $Z_x = \text{Spec}(k)$.

Preliminary remark: since $\text{Spec}(k) \rightarrow X$ is a monomorphism, it suffices to find a surjective étale morphism $Y' \rightarrow Y$ such that $Y' \rightarrow X$ factors through Z_x . A remark here is that Y' is reduced as well.

After replacing X by a quasi-compact open neighbourhood of x , we may assume X quasi-compact. By Lemma 8.3, x is a point of $T \subset U \subset X$ where $T \rightarrow U$ (resp. $U \rightarrow X$) is a closed (resp. open) immersion, and T is a scheme. By Properties of Spaces, Lemma 4.9, f factors through U , so we may assume $U = X$. Then f factors through T because Y is reduced, see Properties of Spaces, Lemma 12.4. So we may assume that $X = T$ is a scheme. By our preliminary remark we may assume Y is a scheme too. This reduces us to Morphisms, Lemma 58.1. \square

Example 13.11. Here is a counter example to Lemmas 13.9 and 13.10 in case X is neither locally Noetherian nor decent. Let k be a field. Let G be an infinite profinite group. Let Y be G viewed as a zero-dimensional affine k -group scheme, i.e., $Y = \operatorname{Spec}(\text{locally constant maps } G \rightarrow k)$. Let Γ be G viewed as a discrete k -group scheme, acting on Y by translations. Put $X = Y/\Gamma$. This is a one-point algebraic space, with projection $q : Y \rightarrow X$. Let $e \in G$ be the origin (any element would do), and view it as a k -point of Y . We get a k -point $x : \operatorname{Spec}(k) \rightarrow X$ which is a monomorphism since it is a section of $X \rightarrow \operatorname{Spec}(k)$. We claim that (although Y is affine and reduced and $|X| = \{x\}$), the morphism q does not factor through any morphism $\operatorname{Spec}(K) \rightarrow X$, where K is a field. Otherwise it would factor through x by Properties of Spaces, Lemma 4.11. Now the pullback of q by x is $\Gamma \rightarrow \operatorname{Spec}(k)$, with the projection $\Gamma \rightarrow Y$ being the orbit map $g \mapsto g \cdot e$. The latter has no section, whence the claim.

Lemma 13.12. *Let S be a scheme. Let X be an algebraic space over S . Let $x \in |X|$ with residual space $Z_x \subset X$. Assume X is locally Noetherian. Then x is a closed point of $|X|$ if and only if the morphism $Z_x \rightarrow X$ is a closed immersion.*

Proof. If $Z_x \rightarrow X$ is a closed immersion, then x is a closed point of $|X|$, see Morphisms of Spaces, Lemma 12.3. Conversely, assume x is a closed point of $|X|$. Let $Z \subset X$ be the reduced closed subspace with $|Z| = \{x\}$ (Properties of Spaces, Lemma 12.3). Then Z is locally Noetherian by Morphisms of Spaces, Lemmas 23.7 and 23.5. Since also Z is reduced and $|Z| = \{x\}$ it $Z = Z_x$ is the residual space by definition. \square

14. Decent spaces

In this section we collect some useful facts on decent spaces.

Lemma 14.1. *Any locally Noetherian decent algebraic space is quasi-separated.*

Proof. Namely, let X be an algebraic space (over some base scheme, for example over \mathbf{Z}) which is decent and locally Noetherian. Let $U \rightarrow X$ and $V \rightarrow X$ be étale morphisms with U and V affine schemes. We have to show that $W = U \times_X V$ is quasi-compact (Properties of Spaces, Lemma 3.3). Since X is locally Noetherian, the schemes U , V are Noetherian and W is locally Noetherian. Since X is decent, the fibres of the morphism $W \rightarrow U$ are finite. Namely, we can represent any $x \in |X|$ by a quasi-compact monomorphism $\operatorname{Spec}(k) \rightarrow X$. Then U_k and V_k are finite disjoint unions of spectra of finite separable extensions of k (Remark 4.1) and we see that $W_k = U_k \times_{\operatorname{Spec}(k)} V_k$ is finite. Let n be the maximum degree of a fibre of $W \rightarrow U$ at a generic point of an irreducible component of U . Consider the stratification

$$U = U_0 \supset U_1 \supset U_2 \supset \dots$$

associated to $W \rightarrow U$ in More on Morphisms, Lemma 45.5. By our choice of n above we conclude that U_{n+1} is empty. Hence we see that the fibres of $W \rightarrow U$ are universally bounded. Then we can apply More on Morphisms, Lemma 45.3 to find a stratification

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \dots \subset Z_n = U$$

by closed subsets such that with $S_r = Z_r \setminus Z_{r-1}$ the morphism $W \times_U S_r \rightarrow S_r$ is finite locally free. Since U is Noetherian, the schemes S_r are Noetherian, whence

the schemes $W \times_U S_r$ are Noetherian, whence $W = \coprod W \times_U S_r$ is quasi-compact as desired. \square

Lemma 14.2. *Let S be a scheme. Let X be a decent algebraic space over S .*

- (1) *If $|X|$ is a singleton then X is a scheme.*
- (2) *If $|X|$ is a singleton and X is reduced, then $X \cong \text{Spec}(k)$ for some field k .*

Proof. Assume $|X|$ is a singleton. It follows immediately from Theorem 10.2 that X is a scheme, but we can also argue directly as follows. Choose an affine scheme U and a surjective étale morphism $U \rightarrow X$. Set $R = U \times_X U$. Then U and R have finitely many points by Lemma 4.5 (and the definition of a decent space). All of these points are closed in U and R by Lemma 12.1. It follows that U and R are affine schemes. We may shrink U to a singleton space. Then U is the spectrum of a henselian local ring, see Algebra, Lemma 153.10. The projections $R \rightarrow U$ are étale, hence finite étale because U is the spectrum of a 0-dimensional henselian local ring, see Algebra, Lemma 153.3. It follows that X is a scheme by Groupoids, Proposition 23.9.

Part (2) follows from (1) and the fact that a reduced singleton scheme is the spectrum of a field. \square

Remark 14.3. We will see in Limits of Spaces, Lemma 15.3 that an algebraic space whose reduction is a scheme is a scheme.

Lemma 14.4. *Let S be a scheme. Let X be a decent algebraic space over S . Consider a commutative diagram*

$$\begin{array}{ccc} \text{Spec}(k) & \xrightarrow{\quad} & X \\ & \searrow & \swarrow \\ & S & \end{array}$$

Assume that the image point $s \in S$ of $\text{Spec}(k) \rightarrow S$ is a closed point and that $\kappa(s) \subset k$ is algebraic. Then the image x of $\text{Spec}(k) \rightarrow X$ is a closed point of $|X|$.

Proof. Suppose that $x \rightsquigarrow x'$ for some $x' \in |X|$. Choose an étale morphism $U \rightarrow X$ where U is a scheme and a point $u' \in U'$ mapping to x' . Choose a specialization $u \rightsquigarrow u'$ in U with u mapping to x in X , see Lemma 12.2. Then u is the image of a point w of the scheme $W = \text{Spec}(k) \times_X U$. Since the projection $W \rightarrow \text{Spec}(k)$ is étale we see that $\kappa(w) \supset k$ is finite. Hence $\kappa(w) \supset \kappa(s)$ is algebraic. Hence $\kappa(u) \supset \kappa(s)$ is algebraic. Thus u is a closed point of U by Morphisms, Lemma 20.2. Thus $u = u'$, whence $x = x'$. \square

Lemma 14.5. *Let S be a scheme. Let X be a decent algebraic space over S . Consider a commutative diagram*

$$\begin{array}{ccc} \text{Spec}(k) & \xrightarrow{\quad} & X \\ & \searrow & \swarrow \\ & S & \end{array}$$

Assume that the image point $s \in S$ of $\text{Spec}(k) \rightarrow S$ is a closed point and that the field extension $k/\kappa(s)$ is finite. Then $\text{Spec}(k) \rightarrow X$ is a finite morphism. If $\kappa(s) = k$ then $\text{Spec}(k) \rightarrow X$ is a closed immersion.

Proof. By Lemma 14.4 the image point $x \in |X|$ is closed. Let $Z \subset X$ be the reduced closed subspace with $|Z| = \{x\}$ (Properties of Spaces, Lemma 12.3). Note that Z is a decent algebraic space by Lemma 6.5. By Lemma 14.2 we see that $Z = \text{Spec}(k')$ for some field k' . Of course $k \supset k' \supset \kappa(s)$. Then $\text{Spec}(k) \rightarrow Z$ is a finite morphism of schemes and $Z \rightarrow X$ is a finite morphism as it is a closed immersion. Hence $\text{Spec}(k) \rightarrow X$ is finite (Morphisms of Spaces, Lemma 45.4). If $k = \kappa(s)$, then $\text{Spec}(k) = Z$ and $\text{Spec}(k) \rightarrow X$ is a closed immersion. \square

Lemma 14.6. *Let S be a scheme. Suppose X is a decent algebraic space over S . Let $x \in |X|$ be a closed point. Then x can be represented by a closed immersion $i : \text{Spec}(k) \rightarrow X$ from the spectrum of a field.*

Proof. We know that x can be represented by a quasi-compact monomorphism $i : \text{Spec}(k) \rightarrow X$ where k is a field (Definition 6.1). Let $U \rightarrow X$ be an étale morphism where U is an affine scheme. As x is closed and X decent, the fibre F of $|U| \rightarrow |X|$ over x consists of closed points (Lemma 12.1). As i is a monomorphism, so is $U_k = U \times_X \text{Spec}(k) \rightarrow U$. In particular, the map $|U_k| \rightarrow F$ is injective. Since U_k is quasi-compact and étale over a field, we see that U_k is a finite disjoint union of spectra of fields (Remark 4.1). Say $U_k = \text{Spec}(k_1) \amalg \dots \amalg \text{Spec}(k_r)$. Since $\text{Spec}(k_i) \rightarrow U$ is a monomorphism, we see that its image u_i has residue field $\kappa(u_i) = k_i$. Since $u_i \in F$ is a closed point we conclude the morphism $\text{Spec}(k_i) \rightarrow U$ is a closed immersion. As the u_i are pairwise distinct, $U_k \rightarrow U$ is a closed immersion. Hence i is a closed immersion (Morphisms of Spaces, Lemma 12.1). This finishes the proof. \square

15. Locally separated spaces

It turns out that a locally separated algebraic space is decent.

Lemma 15.1. *Let A be a ring. Let k be a field. Let \mathfrak{p}_n , $n \geq 1$ be a sequence of pairwise distinct primes of A . Moreover, for each n let $k \rightarrow \kappa(\mathfrak{p}_n)$ be an embedding. Then the closure of the image of*

$$\coprod_{n \neq m} \text{Spec}(\kappa(\mathfrak{p}_n) \otimes_k \kappa(\mathfrak{p}_m)) \longrightarrow \text{Spec}(A \otimes A)$$

meets the diagonal.

Proof. Set $k_n = \kappa(\mathfrak{p}_n)$. We may assume that $A = \prod k_n$. Denote $x_n = \text{Spec}(k_n)$ the open and closed point corresponding to $A \rightarrow k_n$. Then $\text{Spec}(A) = Z \amalg \{x_n\}$ where Z is a nonempty closed subset. Namely, $Z = V(e_n; n \geq 1)$ where e_n is the idempotent of A corresponding to the factor k_n and Z is nonempty as the ideal generated by the e_n is not equal to A . We will show that the closure of the image contains $\Delta(Z)$. The kernel of the map

$$(\prod k_n) \otimes_k (\prod k_m) \longrightarrow \prod_{n \neq m} k_n \otimes_k k_m$$

is the ideal generated by $e_n \otimes e_n$, $n \geq 1$. Hence the closure of the image of the map on spectra is $V(e_n \otimes e_n; n \geq 1)$ whose intersection with $\Delta(\text{Spec}(A))$ is $\Delta(Z)$. Thus it suffices to show that

$$\prod_{n \neq m} \text{Spec}(k_n \otimes_k k_m) \longrightarrow \text{Spec}(\prod_{n \neq m} k_n \otimes_k k_m)$$

has dense image. This follows as the family of ring maps $\prod_{n \neq m} k_n \otimes_k k_m \rightarrow k_n \otimes_k k_m$ is jointly injective. \square

Lemma 15.2 (David Rydh). *A locally separated algebraic space is decent.*

Proof. Let S be a scheme and let X be a locally separated algebraic space over S . We may assume $S = \operatorname{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 3.1. Unadorned fibre products will be over \mathbf{Z} . Let $x \in |X|$. Choose a scheme U , an étale morphism $U \rightarrow X$, and a point $u \in U$ mapping to x in $|X|$. As usual we identify $u = \operatorname{Spec}(\kappa(u))$. As X is locally separated the morphism

$$u \times_X u \rightarrow u \times u$$

is an immersion (Morphisms of Spaces, Lemma 4.5). Hence More on Groupoids, Lemma 11.5 tells us that it is a closed immersion (use Schemes, Lemma 10.4). As $u \times_X u \rightarrow u \times_X U$ is a monomorphism (base change of $u \rightarrow U$) and as $u \times_X U \rightarrow u$ is étale we conclude that $u \times_X u$ is a disjoint union of spectra of fields (see Remark 4.1 and Schemes, Lemma 23.11). Since it is also closed in the affine scheme $u \times u$ we conclude $u \times_X u$ is a finite disjoint union of spectra of fields. Thus x can be represented by a monomorphism $\operatorname{Spec}(k) \rightarrow X$ where k is a field, see Lemma 4.3.

Next, let $U = \operatorname{Spec}(A)$ be an affine scheme and let $U \rightarrow X$ be an étale morphism. To finish the proof it suffices to show that $F = U \times_X \operatorname{Spec}(k)$ is finite. Write $F = \coprod_{i \in I} \operatorname{Spec}(k_i)$ as the disjoint union of finite separable extensions of k . We have to show that I is finite. Set $R = U \times_X U$. As X is locally separated, the morphism $j : R \rightarrow U \times U$ is an immersion. Let $U' \subset U \times U$ be an open such that j factors through a closed immersion $j' : R \rightarrow U'$. Let $e : U \rightarrow R$ be the diagonal map. Using that e is a morphism between schemes étale over U such that $\Delta = j \circ e$ is a closed immersion, we conclude that $R = e(U) \amalg W$ for some open and closed subscheme $W \subset R$. Since j' is a closed immersion we conclude that $j'(W) \subset U'$ is closed and disjoint from $j'(e(U))$. Therefore $\overline{j(W)} \cap \Delta(U) = \emptyset$ in $U \times U$. Note that W contains $\operatorname{Spec}(k_i \otimes_k k_{i'})$ for all $i \neq i', i, i' \in I$. By Lemma 15.1 we conclude that I is finite as desired. \square

16. Valuative criterion

For a quasi-compact morphism from a decent space the valuative criterion is necessary in order for the morphism to be universally closed.

Proposition 16.1. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f is quasi-compact, and X is decent. Then f is universally closed if and only if the existence part of the valuative criterion holds.*

Proof. In Morphisms of Spaces, Lemma 42.1 we have seen one of the implications. To prove the other, assume that f is universally closed. Let

$$\begin{array}{ccc} \operatorname{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(A) & \longrightarrow & Y \end{array}$$

be a diagram as in Morphisms of Spaces, Definition 41.1. Let $X_A = \text{Spec}(A) \times_Y X$, so that we have

$$\begin{array}{ccc} \text{Spec}(K) & \longrightarrow & X_A \\ & \searrow & \downarrow \\ & & \text{Spec}(A) \end{array}$$

By Morphisms of Spaces, Lemma 8.4 we see that $X_A \rightarrow \text{Spec}(A)$ is quasi-compact. Since $X_A \rightarrow X$ is representable, we see that X_A is decent also, see Lemma 5.3. Moreover, as f is universally closed, we see that $X_A \rightarrow \text{Spec}(A)$ is universally closed. Hence we may and do replace X by X_A and Y by $\text{Spec}(A)$.

Let $x' \in |X|$ be the equivalence class of $\text{Spec}(K) \rightarrow X$. Let $y \in |Y| = |\text{Spec}(A)|$ be the closed point. Set $y' = f(x')$; it is the generic point of $\text{Spec}(A)$. Since f is universally closed we see that $f(\overline{\{x'\}})$ contains $\overline{\{y'\}}$, and hence contains y . Let $x \in \overline{\{x'\}}$ be a point such that $f(x) = y$. Let U be a scheme, and $\varphi : U \rightarrow X$ an étale morphism such that there exists a $u \in U$ with $\varphi(u) = x$. By Lemma 7.3 and our assumption that X is decent there exists a specialization $u' \rightsquigarrow u$ on U with $\varphi(u') = x'$. This means that there exists a common field extension $K \subset K' \supset \kappa(u')$ such that

$$\begin{array}{ccc} \text{Spec}(K') & \longrightarrow & U \\ \downarrow & & \downarrow \\ \text{Spec}(K) & \longrightarrow & X \\ & \searrow & \downarrow \\ & & \text{Spec}(A) \end{array}$$

is commutative. This gives the following commutative diagram of rings

$$\begin{array}{ccc} K' & \longleftarrow & \mathcal{O}_{U,u} \\ \uparrow & & \uparrow \\ K & & A \\ & \nwarrow & \uparrow \end{array}$$

By Algebra, Lemma 50.2 we can find a valuation ring $A' \subset K'$ dominating the image of $\mathcal{O}_{U,u}$ in K' . Since by construction $\mathcal{O}_{U,u}$ dominates A we see that A' dominates A also. Hence we obtain a diagram resembling the second diagram of Morphisms of Spaces, Definition 41.1 and the proposition is proved. \square

17. Relative conditions

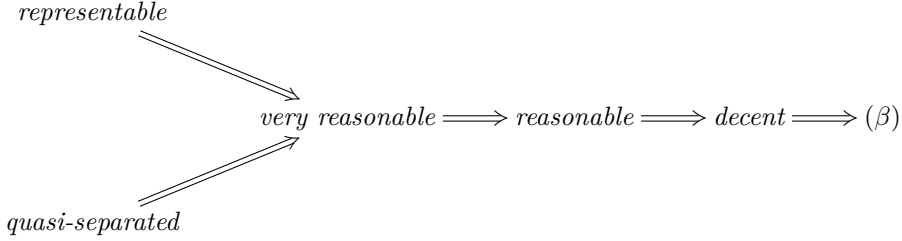
This is a (yet another) technical section dealing with conditions on algebraic spaces having to do with points. It is probably a good idea to skip this section.

Definition 17.1. Let S be a scheme. We say an algebraic space X over S has *property (β)* if X has the corresponding property of Lemma 5.1. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S .

- (1) We say f has *property* (β) if for any scheme T and morphism $T \rightarrow Y$ the fibre product $T \times_Y X$ has property (β) .
- (2) We say f is *decent* if for any scheme T and morphism $T \rightarrow Y$ the fibre product $T \times_Y X$ is a decent algebraic space.
- (3) We say f is *reasonable* if for any scheme T and morphism $T \rightarrow Y$ the fibre product $T \times_Y X$ is a reasonable algebraic space.
- (4) We say f is *very reasonable* if for any scheme T and morphism $T \rightarrow Y$ the fibre product $T \times_Y X$ is a very reasonable algebraic space.

We refer to Remark 17.10 for an informal discussion. It will turn out that the class of very reasonable morphisms is not so useful, but that the classes of decent and reasonable morphisms are useful.

Lemma 17.2. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . We have the following implications among the conditions on f :*



Proof. This is clear from the definitions, Lemma 5.1 and Morphisms of Spaces, Lemma 4.12. \square

Here is another sanity check.

Lemma 17.3. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . If X is decent (resp. is reasonable, resp. has property (β) of Lemma 5.1), then f is decent (resp. reasonable, resp. has property (β)).*

Proof. Let T be a scheme and let $T \rightarrow Y$ be a morphism. Then $T \rightarrow Y$ is representable, hence the base change $T \times_Y X \rightarrow X$ is representable. Hence if X is decent (or reasonable), then so is $T \times_Y X$, see Lemma 6.5. Similarly, for property (β) , see Lemma 5.3. \square

Lemma 17.4. *Having property (β) , being decent, or being reasonable is preserved under arbitrary base change.*

Proof. This is immediate from the definition. \square

Lemma 17.5. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let $\omega \in \{\beta, \text{decent}, \text{reasonable}\}$. Suppose that Y has property (ω) and $f : X \rightarrow Y$ has (ω) . Then X has (ω) .*

Proof. Let us prove the lemma in case $\omega = \beta$. In this case we have to show that any $x \in |X|$ is represented by a monomorphism from the spectrum of a field into X . Let $y = f(x) \in |Y|$. By assumption there exists a field k and a monomorphism $\text{Spec}(k) \rightarrow Y$ representing y . Then x corresponds to a point x' of $\text{Spec}(k) \times_Y X$. By assumption x' is represented by a monomorphism $\text{Spec}(k') \rightarrow \text{Spec}(k) \times_Y X$. Clearly the composition $\text{Spec}(k') \rightarrow X$ is a monomorphism representing x .

Let us prove the lemma in case $\omega = \text{decent}$. Let $x \in |X|$ and $y = f(x) \in |Y|$. By the result of the preceding paragraph we can choose a diagram

$$\begin{array}{ccc} \mathrm{Spec}(k') & \xrightarrow{x} & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(k) & \xrightarrow{y} & Y \end{array}$$

whose horizontal arrows monomorphisms. As Y is decent the morphism y is quasi-compact. As f is decent the algebraic space $\mathrm{Spec}(k) \times_Y X$ is decent. Hence the monomorphism $\mathrm{Spec}(k') \rightarrow \mathrm{Spec}(k) \times_Y X$ is quasi-compact. Then the monomorphism $x : \mathrm{Spec}(k') \rightarrow X$ is quasi-compact as a composition of quasi-compact morphisms (use Morphisms of Spaces, Lemmas 8.4 and 8.5). As the point x was arbitrary this implies X is decent.

Let us prove the lemma in case $\omega = \text{reasonable}$. Choose $V \rightarrow Y$ étale with V an affine scheme. Choose $U \rightarrow V \times_Y X$ étale with U an affine scheme. By assumption $V \rightarrow Y$ has universally bounded fibres. By Lemma 3.3 the morphism $V \times_Y X \rightarrow X$ has universally bounded fibres. By assumption on f we see that $U \rightarrow V \times_Y X$ has universally bounded fibres. By Lemma 3.2 the composition $U \rightarrow X$ has universally bounded fibres. Hence there exists sufficiently many étale morphisms $U \rightarrow X$ from schemes with universally bounded fibres, and we conclude that X is reasonable. \square

Lemma 17.6. *Having property (β) , being decent, or being reasonable is preserved under compositions.*

Proof. Let $\omega \in \{\beta, \text{decent}, \text{reasonable}\}$. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of algebraic spaces over the scheme S . Assume f and g both have property (ω) . Then we have to show that for any scheme T and morphism $T \rightarrow Z$ the space $T \times_Z X$ has (ω) . By Lemma 17.4 this reduces us to the following claim: Suppose that Y is an algebraic space having property (ω) , and that $f : X \rightarrow Y$ is a morphism with (ω) . Then X has (ω) . This is the content of Lemma 17.5. \square

Lemma 17.7. *Let S be a scheme. Let $f : X \rightarrow Y$, $g : Z \rightarrow Y$ be morphisms of algebraic spaces over S . If X and Z are decent (resp. reasonable, resp. have property (β) of Lemma 5.1), then so does $X \times_Y Z$.*

Proof. Namely, by Lemma 17.3 the morphism $X \rightarrow Y$ has the property. Then the base change $X \times_Y Z \rightarrow Z$ has the property by Lemma 17.4. And finally this implies $X \times_Y Z$ has the property by Lemma 17.5. \square

Lemma 17.8. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let $\mathcal{P} \in \{(\beta), \text{decent}, \text{reasonable}\}$. Assume*

- (1) *f is quasi-compact,*
- (2) *f is étale,*
- (3) *$|f| : |X| \rightarrow |Y|$ is surjective, and*
- (4) *the algebraic space X has property \mathcal{P} .*

Then Y has property \mathcal{P} .

Proof. Let us prove this in case $\mathcal{P} = (\beta)$. Let $y \in |Y|$ be a point. We have to show that y can be represented by a monomorphism from a field. Choose a point $x \in |X|$ with $f(x) = y$. By assumption we may represent x by a monomorphism $\mathrm{Spec}(k) \rightarrow X$, with k a field. By Lemma 4.3 it suffices to show that the projections

$\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k)$ are étale and quasi-compact. We can factor the first projection as

$$\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k) \times_Y X \longrightarrow \mathrm{Spec}(k)$$

The first morphism is a monomorphism, and the second is étale and quasi-compact. By Properties of Spaces, Lemma 16.8 we see that $\mathrm{Spec}(k) \times_Y X$ is a scheme. Hence it is a finite disjoint union of spectra of finite separable field extensions of k . By Schemes, Lemma 23.11 we see that the first arrow identifies $\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k)$ with a finite disjoint union of spectra of finite separable field extensions of k . Hence the projection morphism is étale and quasi-compact.

Let us prove this in case $\mathcal{P} = \text{decent}$. We have already seen in the first paragraph of the proof that this implies that every $y \in |Y|$ can be represented by a monomorphism $y : \mathrm{Spec}(k) \rightarrow Y$. Pick such a y . Pick an affine scheme U and an étale morphism $U \rightarrow X$ such that the image of $|U| \rightarrow |Y|$ contains y . By Lemma 4.5 it suffices to show that U_y is a finite scheme over k . The fibre product $X_y = \mathrm{Spec}(k) \times_Y X$ is a quasi-compact étale algebraic space over k . Hence by Properties of Spaces, Lemma 16.8 it is a scheme. So it is a finite disjoint union of spectra of finite separable extensions of k . Say $X_y = \{x_1, \dots, x_n\}$ so x_i is given by $x_i : \mathrm{Spec}(k_i) \rightarrow X$ with $[k_i : k] < \infty$. By assumption X is decent, so the schemes $U_{x_i} = \mathrm{Spec}(k_i) \times_X U$ are finite over k_i . Finally, we note that $U_y = \coprod U_{x_i}$ as a scheme and we conclude that U_y is finite over k as desired.

Let us prove this in case $\mathcal{P} = \text{reasonable}$. Pick an affine scheme V and an étale morphism $V \rightarrow Y$. We have to show the fibres of $V \rightarrow Y$ are universally bounded. The algebraic space $V \times_Y X$ is quasi-compact. Thus we can find an affine scheme W and a surjective étale morphism $W \rightarrow V \times_Y X$, see Properties of Spaces, Lemma 6.3. Here is a picture (solid diagram)

$$\begin{array}{ccccc} W & \longrightarrow & V \times_Y X & \longrightarrow & X & \xleftarrow{\quad x \quad} & \mathrm{Spec}(k) \\ & \searrow & \downarrow & & \downarrow f & \swarrow y & \\ & & V & \longrightarrow & Y & & \end{array}$$

The morphism $W \rightarrow X$ is universally bounded by our assumption that the space X is reasonable. Let n be an integer bounding the degrees of the fibres of $W \rightarrow X$. We claim that the same integer works for bounding the fibres of $V \rightarrow Y$. Namely, suppose $y \in |Y|$ is a point. Then there exists a $x \in |X|$ with $f(x) = y$ (see above). This means we can find a field k and morphisms x, y given as dotted arrows in the diagram above. In particular we get a surjective étale morphism

$$\mathrm{Spec}(k) \times_{x,X} W \rightarrow \mathrm{Spec}(k) \times_{x,X} (V \times_Y X) = \mathrm{Spec}(k) \times_{y,Y} V$$

which shows that the degree of $\mathrm{Spec}(k) \times_{y,Y} V$ over k is less than or equal to the degree of $\mathrm{Spec}(k) \times_{x,X} W$ over k , i.e., $\leq n$, and we win. (This last part of the argument is the same as the argument in the proof of Lemma 3.4. Unfortunately that lemma is not general enough because it only applies to representable morphisms.) \square

Lemma 17.9. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let $\mathcal{P} \in \{(\beta), \text{decent}, \text{reasonable}, \text{very reasonable}\}$. The following are equivalent*

- (1) f is \mathcal{P} ,

- (2) for every affine scheme Z and every morphism $Z \rightarrow Y$ the base change $Z \times_Y X \rightarrow Z$ of f is \mathcal{P} ,
- (3) for every affine scheme Z and every morphism $Z \rightarrow Y$ the algebraic space $Z \times_Y X$ is \mathcal{P} , and
- (4) there exists a Zariski covering $Y = \bigcup Y_i$ such that each morphism $f^{-1}(Y_i) \rightarrow Y_i$ has \mathcal{P} .

If $\mathcal{P} \in \{(\beta), \text{decent}, \text{reasonable}\}$, then this is also equivalent to

- (5) there exists a scheme V and a surjective étale morphism $V \rightarrow Y$ such that the base change $V \times_Y X \rightarrow V$ has \mathcal{P} .

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) are trivial. The implication (3) \Rightarrow (1) can be seen as follows. Let $Z \rightarrow Y$ be a morphism whose source is a scheme over S . Consider the algebraic space $Z \times_Y X$. If we assume (3), then for any affine open $W \subset Z$, the open subspace $W \times_Y X$ of $Z \times_Y X$ has property \mathcal{P} . Hence by Lemma 5.2 the space $Z \times_Y X$ has property \mathcal{P} , i.e., (1) holds. A similar argument (omitted) shows that (4) implies (1).

The implication (1) \Rightarrow (5) is trivial. Let $V \rightarrow Y$ be an étale morphism from a scheme as in (5). Let Z be an affine scheme, and let $Z \rightarrow Y$ be a morphism. Consider the diagram

$$\begin{array}{ccc} Z \times_Y V & \xrightarrow{q} & V \\ p \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

Since p is étale, and hence open, we can choose finitely many affine open subschemes $W_i \subset Z \times_Y V$ such that $Z = \bigcup p(W_i)$. Consider the commutative diagram

$$\begin{array}{ccccc} V \times_Y X & \longleftarrow & (\coprod W_i) \times_Y X & \longrightarrow & Z \times_Y X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & \coprod W_i & \longrightarrow & Z \end{array}$$

We know $V \times_Y X$ has property \mathcal{P} . By Lemma 5.3 we see that $(\coprod W_i) \times_Y X$ has property \mathcal{P} . Note that the morphism $(\coprod W_i) \times_Y X \rightarrow Z \times_Y X$ is étale and quasi-compact as the base change of $\coprod W_i \rightarrow Z$. Hence by Lemma 17.8 we conclude that $Z \times_Y X$ has property \mathcal{P} . \square

Remark 17.10. An informal description of the properties (β) , decent, reasonable, very reasonable was given in Section 6. A morphism has one of these properties if (very) loosely speaking the fibres of the morphism have the corresponding properties. Being decent is useful to prove things about specializations of points on $|X|$. Being reasonable is a bit stronger and technically quite easy to work with.

Here is a lemma we promised earlier which uses decent morphisms.

Lemma 17.11. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f is quasi-compact and decent. (For example if f is representable, or quasi-separated, see Lemma 17.2.) Then f is universally closed if and only if the existence part of the valuative criterion holds.*

Proof. In Morphisms of Spaces, Lemma 42.1 we proved that any quasi-compact morphism which satisfies the existence part of the valuative criterion is universally closed. To prove the other, assume that f is universally closed. In the proof of Proposition 16.1 we have seen that it suffices to show, for any valuation ring A , and any morphism $\text{Spec}(A) \rightarrow Y$, that the base change $f_A : X_A \rightarrow \text{Spec}(A)$ satisfies the existence part of the valuative criterion. By definition the algebraic space X_A has property (γ) and hence Proposition 16.1 applies to the morphism f_A and we win. \square

18. Points of fibres

Let S be a scheme. Consider a cartesian diagram

$$(18.0.1) \quad \begin{array}{ccc} W & \xrightarrow{q} & Z \\ p \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

of algebraic spaces over S . Let $x \in |X|$ and $z \in |Z|$ be points mapping to the same point $y \in |Y|$. We may ask: When is the set

$$(18.0.2) \quad F_{x,z} = \{w \in |W| \text{ such that } p(w) = x \text{ and } q(w) = z\}$$

finite?

Example 18.1. If X, Y, Z are schemes, then the set $F_{x,z}$ is equal to the spectrum of $\kappa(x) \otimes_{\kappa(y)} \kappa(z)$ (Schemes, Lemma 17.5). Thus we obtain a finite set if either $\kappa(y) \subset \kappa(x)$ is finite or if $\kappa(y) \subset \kappa(z)$ is finite. In particular, this is always the case if g is quasi-finite at z (Morphisms, Lemma 20.5).

Example 18.2. Let K be a characteristic 0 field endowed with an automorphism σ of infinite order. Set $Y = \text{Spec}(K)/\mathbf{Z}$ and $X = \mathbf{A}_K^1/\mathbf{Z}$ where \mathbf{Z} acts on K via σ and on $\mathbf{A}_K^1 = \text{Spec}(K[t])$ via $t \mapsto t+1$. Let $Z = \text{Spec}(K)$. Then $W = \mathbf{A}_K^1$. Picture

$$\begin{array}{ccc} \mathbf{A}_K^1 & \xrightarrow{q} & \text{Spec}(K) \\ p \downarrow & & \downarrow g \\ \mathbf{A}_K^1/\mathbf{Z} & \xrightarrow{f} & \text{Spec}(K)/\mathbf{Z} \end{array}$$

Take x corresponding to $t = 0$ and z the unique point of $\text{Spec}(K)$. Then we see that $F_{x,z} = \mathbf{Z}$ as a set.

Lemma 18.3. *In the situation of (18.0.1) if $Z' \rightarrow Z$ is a morphism and $z' \in |Z'|$ maps to z , then the induced map $F_{x,z'} \rightarrow F_{x,z}$ is surjective.*

Proof. Set $W' = X \times_Y Z' = W \times_Z Z'$. Then $|W'| \rightarrow |W| \times_{|Z|} |Z'|$ is surjective by Properties of Spaces, Lemma 4.3. Hence the surjectivity of $F_{x,z'} \rightarrow F_{x,z}$. \square

Lemma 18.4. *In diagram (18.0.1) the set (18.0.2) is finite if f is of finite type and f is quasi-finite at x .*

Proof. The morphism q is quasi-finite at every $w \in F_{x,z}$, see Morphisms of Spaces, Lemma 27.2. Hence the lemma follows from Morphisms of Spaces, Lemma 27.9. \square

Lemma 18.5. *In diagram (18.0.1) the set (18.0.2) is finite if y can be represented by a monomorphism $\mathrm{Spec}(k) \rightarrow Y$ where k is a field and g is quasi-finite at z . (Special case: Y is decent and g is étale.)*

Proof. By Lemma 18.3 applied twice we may replace Z by $Z_k = \mathrm{Spec}(k) \times_Y Z$ and X by $X_k = \mathrm{Spec}(k) \times_Y X$. We may and do replace Y by $\mathrm{Spec}(k)$ as well. Note that $Z_k \rightarrow \mathrm{Spec}(k)$ is quasi-finite at z by Morphisms of Spaces, Lemma 27.2. Choose a scheme V , a point $v \in V$, and an étale morphism $V \rightarrow Z_k$ mapping v to z . Choose a scheme U , a point $u \in U$, and an étale morphism $U \rightarrow X_k$ mapping u to x . Again by Lemma 18.3 it suffices to show $F_{u,v}$ is finite for the diagram

$$\begin{array}{ccc} U \times_{\mathrm{Spec}(k)} V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

The morphism $V \rightarrow \mathrm{Spec}(k)$ is quasi-finite at v (follows from the general discussion in Morphisms of Spaces, Section 22 and the definition of being quasi-finite at a point). At this point the finiteness follows from Example 18.1. The parenthetical remark of the statement of the lemma follows from the fact that on decent spaces points are represented by monomorphisms from fields and from the fact that an étale morphism of algebraic spaces is locally quasi-finite. \square

Lemma 18.6. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let $y \in |Y|$ and assume that y is represented by a quasi-compact monomorphism $\mathrm{Spec}(k) \rightarrow Y$. Then $|X_k| \rightarrow |X|$ is a homeomorphism onto $f^{-1}(\{y\}) \subset |X|$ with induced topology.*

Proof. We will use Properties of Spaces, Lemma 16.7 and Morphisms of Spaces, Lemma 10.9 without further mention. Let $V \rightarrow Y$ be an étale morphism with V affine such that there exists a $v \in V$ mapping to y . Since $\mathrm{Spec}(k) \rightarrow Y$ is quasi-compact there are a finite number of points of V mapping to y (Lemma 4.5). After shrinking V we may assume v is the only one. Choose a scheme U and a surjective étale morphism $U \rightarrow X$. Consider the commutative diagram

$$\begin{array}{ccccc} U & \longleftarrow & U_V & \longleftarrow & U_v \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & X_V & \longleftarrow & X_v \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & V & \longleftarrow & v \end{array}$$

Since $U_v \rightarrow U_V$ identifies U_v with a subset of U_V with the induced topology (Schemes, Lemma 18.5), and since $|U_V| \rightarrow |X_V|$ and $|U_v| \rightarrow |X_v|$ are surjective and open, we see that $|X_v| \rightarrow |X_V|$ is a homeomorphism onto its image (with induced topology). On the other hand, the inverse image of $f^{-1}(\{y\})$ under the open map $|X_V| \rightarrow |X|$ is equal to $|X_v|$. We conclude that $|X_v| \rightarrow f^{-1}(\{y\})$ is open. The morphism $X_v \rightarrow X$ factors through X_k and $|X_k| \rightarrow |X|$ is injective with image $f^{-1}(\{y\})$ by Properties of Spaces, Lemma 4.3. Using $|X_v| \rightarrow |X_k| \rightarrow f^{-1}(\{y\})$ the lemma follows because $X_v \rightarrow X_k$ is surjective. \square

Lemma 18.7. *Let X be an algebraic space locally of finite type over a field k . Let $x \in |X|$. Consider the conditions*

- (1) $\dim_x(|X|) = 0$,
- (2) x is closed in $|X|$ and if $x' \rightsquigarrow x$ in $|X|$ then $x' = x$,
- (3) x is an isolated point of $|X|$,
- (4) $\dim_x(X) = 0$,
- (5) $X \rightarrow \operatorname{Spec}(k)$ is quasi-finite at x .

Then (2), (3), (4), and (5) are equivalent. If X is decent, then (1) is equivalent to the others.

Proof. Parts (4) and (5) are equivalent for example by Morphisms of Spaces, Lemmas 34.7 and 34.8.

Let $U \rightarrow X$ be an étale morphism where U is an affine scheme and let $u \in U$ be a point mapping to x . Moreover, if x is a closed point, e.g., in case (2) or (3), then we may and do assume that u is a closed point. Observe that $\dim_u(U) = \dim_x(X)$ by definition and that this is equal to $\dim(\mathcal{O}_{U,u})$ if u is a closed point, see Algebra, Lemma 114.6.

If $\dim_x(X) > 0$ and u is closed, by the arguments above we can choose a nontrivial specialization $u' \rightsquigarrow u$ in U . Then the transcendence degree of $\kappa(u')$ over k exceeds the transcendence degree of $\kappa(u)$ over k . It follows that the images x and x' in X are distinct, because the transcendence degree of x/k and x'/k are well defined, see Morphisms of Spaces, Definition 33.1. This applies in particular in cases (2) and (3) and we conclude that (2) and (3) imply (4).

Conversely, if $X \rightarrow \operatorname{Spec}(k)$ is locally quasi-finite at x , then $U \rightarrow \operatorname{Spec}(k)$ is locally quasi-finite at u , hence u is an isolated point of U (Morphisms, Lemma 20.6). It follows that (5) implies (2) and (3) as $|U| \rightarrow |X|$ is continuous and open.

Assume X is decent and (1) holds. Then $\dim_x(X) = \dim_x(|X|)$ by Lemma 12.5 and the proof is complete. \square

Lemma 18.8. *Let X be an algebraic space locally of finite type over a field k . Consider the conditions*

- (1) $|X|$ is a finite set,
- (2) $|X|$ is a discrete space,
- (3) $\dim(|X|) = 0$,
- (4) $\dim(X) = 0$,
- (5) $X \rightarrow \operatorname{Spec}(k)$ is locally quasi-finite,

Then (2), (3), (4), and (5) are equivalent. If X is decent, then (1) implies the others.

Proof. Parts (4) and (5) are equivalent for example by Morphisms of Spaces, Lemma 34.7.

Let $U \rightarrow X$ be a surjective étale morphism where U is a scheme.

If $\dim(U) > 0$, then choose a nontrivial specialization $u \rightsquigarrow u'$ in U and the transcendence degree of $\kappa(u)$ over k exceeds the transcendence degree of $\kappa(u')$ over k . It follows that the images x and x' in X are distinct, because the transcendence degree of x/k and x'/k is well defined, see Morphisms of Spaces, Definition 33.1. We conclude that (2) and (3) imply (4).

Conversely, if $X \rightarrow \operatorname{Spec}(k)$ is locally quasi-finite, then U is locally Noetherian (Morphisms, Lemma 15.6) of dimension 0 (Morphisms, Lemma 29.5) and hence is a disjoint union of spectra of Artinian local rings (Properties, Lemma 10.5). Hence U is a discrete topological space, and since $|U| \rightarrow |X|$ is continuous and open, the same is true for $|X|$. In other words, (4) implies (2) and (3).

Assume X is decent and (1) holds. Then we may choose U above to be affine. The fibres of $|U| \rightarrow |X|$ are finite (this is a part of the defining property of decent spaces). Hence U is a finite type scheme over k with finitely many points. Hence U is quasi-finite over k (Morphisms, Lemma 20.7) which by definition means that $X \rightarrow \operatorname{Spec}(k)$ is locally quasi-finite. \square

Lemma 18.9. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $x \in |X|$ with image $y \in |Y|$. Let $F = f^{-1}(\{y\})$ with induced topology from $|X|$. Let k be a field and let $\operatorname{Spec}(k) \rightarrow Y$ be in the equivalence class defining y . Set $X_k = \operatorname{Spec}(k) \times_Y X$. Let $\tilde{x} \in |X_k|$ map to $x \in |X|$. Consider the following conditions*

- (1) $\dim_x(F) = 0$,
- (2) x is isolated in F ,
- (3) x is closed in F and if $x' \rightsquigarrow x$ in F , then $x = x'$,
- (4) $\dim_{\tilde{x}}(|X_k|) = 0$,
- (5) \tilde{x} is isolated in $|X_k|$,
- (6) \tilde{x} is closed in $|X_k|$ and if $\tilde{x}' \rightsquigarrow \tilde{x}$ in $|X_k|$, then $\tilde{x} = \tilde{x}'$,
- (7) $\dim_{\tilde{x}}(X_k) = 0$,
- (8) f is quasi-finite at x .

Then we have

$$(4) \xrightarrow[f \text{ decent}]{} (5) \iff (6) \iff (7) \iff (8)$$

If Y is decent, then conditions (2) and (3) are equivalent to each other and to conditions (5), (6), (7), and (8). If Y and X are decent, then all conditions are equivalent.

Proof. By Lemma 18.7 conditions (5), (6), and (7) are equivalent to each other and to the condition that $X_k \rightarrow \operatorname{Spec}(k)$ is quasi-finite at \tilde{x} . Thus by Morphisms of Spaces, Lemma 27.2 they are also equivalent to (8). If f is decent, then X_k is a decent algebraic space and Lemma 18.7 shows that (4) implies (5).

If Y is decent, then we can pick a quasi-compact monomorphism $\operatorname{Spec}(k') \rightarrow Y$ in the equivalence class of y . In this case Lemma 18.6 tells us that $|X_{k'}| \rightarrow F$ is a homeomorphism. Combined with the arguments given above this implies the remaining statements of the lemma; details omitted. \square

Lemma 18.10. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $y \in |Y|$. Let k be a field and let $\operatorname{Spec}(k) \rightarrow Y$ be in the equivalence class defining y . Set $X_k = \operatorname{Spec}(k) \times_Y X$ and let $F = f^{-1}(\{y\})$ with the induced topology from $|X|$. Consider the following conditions*

- (1) F is finite,
- (2) F is a discrete topological space,
- (3) $\dim(F) = 0$,
- (4) $|X_k|$ is a finite set,

- (5) $|X_k|$ is a discrete space,
- (6) $\dim(|X_k|) = 0$,
- (7) $\dim(X_k) = 0$,
- (8) f is quasi-finite at all points of $|X|$ lying over y .

Then we have

$$(1) \Longleftarrow (4) \xrightarrow[f \text{ decent}]{} (5) \Longleftrightarrow (6) \Longleftrightarrow (7) \Longleftrightarrow (8)$$

If Y is decent, then conditions (2) and (3) are equivalent to each other and to conditions (5), (6), (7), and (8). If Y and X are decent, then (1) implies all the other conditions.

Proof. By Lemma 18.8 conditions (5), (6), and (7) are equivalent to each other and to the condition that $X_k \rightarrow \text{Spec}(k)$ is locally quasi-finite. Thus by Morphisms of Spaces, Lemma 27.2 they are also equivalent to (8). If f is decent, then X_k is a decent algebraic space and Lemma 18.8 shows that (4) implies (5).

The map $|X_k| \rightarrow F$ is surjective by Properties of Spaces, Lemma 4.3 and we see $(4) \Rightarrow (1)$.

If Y is decent, then we can pick a quasi-compact monomorphism $\text{Spec}(k') \rightarrow Y$ in the equivalence class of y . In this case Lemma 18.6 tells us that $|X_{k'}| \rightarrow F$ is a homeomorphism. Combined with the arguments given above this implies the remaining statements of the lemma; details omitted. \square

19. Monomorphisms

Here is another case where monomorphisms are representable. Please see More on Morphisms of Spaces, Section 4 for more information.

Lemma 19.1. *Let S be a scheme. Let Y be a disjoint union of spectra of zero dimensional local rings over S . Let $f : X \rightarrow Y$ be a monomorphism of algebraic spaces over S . Then f is representable, i.e., X is a scheme.*

Proof. This immediately reduces to the case $Y = \text{Spec}(A)$ where A is a zero dimensional local ring, i.e., $\text{Spec}(A) = \{\mathfrak{m}_A\}$ is a singleton. If $X = \emptyset$, then there is nothing to prove. If not, choose a nonempty affine scheme $U = \text{Spec}(B)$ and an étale morphism $U \rightarrow X$. As $|X|$ is a singleton (as a subset of $|Y|$, see Morphisms of Spaces, Lemma 10.9) we see that $U \rightarrow X$ is surjective. Note that $U \times_X U = U \times_Y U = \text{Spec}(B \otimes_A B)$. Thus we see that the ring maps $B \rightarrow B \otimes_A B$ are étale. Since

$$(B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B) = (B/\mathfrak{m}_A B) \otimes_{A/\mathfrak{m}_A} (B/\mathfrak{m}_A B)$$

we see that $B/\mathfrak{m}_A B \rightarrow (B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B)$ is flat and in fact free of rank equal to the dimension of $B/\mathfrak{m}_A B$ as a A/\mathfrak{m}_A -vector space. Since $B \rightarrow B \otimes_A B$ is étale, this can only happen if this dimension is finite (see for example Morphisms, Lemmas 57.8 and 57.9). Every prime of B lies over \mathfrak{m}_A (the unique prime of A). Hence $\text{Spec}(B) = \text{Spec}(B/\mathfrak{m}_A B)$ as a topological space, and this space is a finite discrete set as $B/\mathfrak{m}_A B$ is an Artinian ring, see Algebra, Lemmas 53.2 and 53.6. Hence all prime ideals of B are maximal and $B = B_1 \times \dots \times B_n$ is a product of finitely many local rings of dimension zero, see Algebra, Lemma 53.5. Thus $B \rightarrow B \otimes_A B$ is finite étale as all the local rings B_i are henselian by Algebra, Lemma 15.10. Thus X is an affine scheme by Groupoids, Proposition 23.9. \square

20. Generic points

This section is a continuation of Properties of Spaces, Section 11.

Lemma 20.1. *Let S be a scheme. Let X be a decent algebraic space over S . Let $x \in |X|$. The following are equivalent*

- (1) *x is a generic point of an irreducible component of $|X|$,*
- (2) *for any étale morphism $(Y, y) \rightarrow (X, x)$ of pointed algebraic spaces, y is a generic point of an irreducible component of $|Y|$,*
- (3) *for some étale morphism $(Y, y) \rightarrow (X, x)$ of pointed algebraic spaces, y is a generic point of an irreducible component of $|Y|$,*
- (4) *the dimension of the local ring of X at x is zero, and*
- (5) *x is a point of codimension 0 on X*

Proof. Conditions (4) and (5) are equivalent for any algebraic space by definition, see Properties of Spaces, Definition 10.2. Observe that any Y as in (2) and (3) is decent by Lemma 6.6. Thus it suffices to prove the equivalence of (1) and (4) as then the equivalence with (2) and (3) follows since the dimension of the local ring of Y at y is equal to the dimension of the local ring of X at x . Let $f : U \rightarrow X$ be an étale morphism from an affine scheme and let $u \in U$ be a point mapping to x .

Assume (1). Let $u' \rightsquigarrow u$ be a specialization in U . Then $f(u') = f(u) = x$. By Lemma 12.1 we see that $u' = u$. Hence u is a generic point of an irreducible component of U . Thus $\dim(\mathcal{O}_{U,u}) = 0$ and we see that (4) holds.

Assume (4). The point x is contained in an irreducible component $T \subset |X|$. Since $|X|$ is sober (Proposition 12.4) we T has a generic point x' . Of course $x' \rightsquigarrow x$. Then we can lift this specialization to $u' \rightsquigarrow u$ in U (Lemma 12.2). This contradicts the assumption that $\dim(\mathcal{O}_{U,u}) = 0$ unless $u' = u$, i.e., $x' = x$. \square

Lemma 20.2. *Let S be a scheme. Let X be a decent algebraic space over S . Let $T \subset |X|$ be an irreducible closed subset. Let $\xi \in T$ be the generic point (Proposition 12.4). Then $\text{codim}(T, |X|)$ (Topology, Definition 11.1) is the dimension of the local ring of X at ξ (Properties of Spaces, Definition 10.2).*

Proof. Choose a scheme U , a point $u \in U$, and an étale morphism $U \rightarrow X$ sending u to ξ . Then any sequence of nontrivial specializations $\xi_e \rightsquigarrow \dots \rightsquigarrow \xi_0 = \xi$ can be lifted to a sequence $u_e \rightsquigarrow \dots \rightsquigarrow u_0 = u$ in U by Lemma 12.2. Conversely, any sequence of nontrivial specializations $u_e \rightsquigarrow \dots \rightsquigarrow u_0 = u$ in U maps to a sequence of nontrivial specializations $\xi_e \rightsquigarrow \dots \rightsquigarrow \xi_0 = \xi$ by Lemma 12.1. Because $|X|$ and U are sober topological spaces we conclude that the codimension of T in $|X|$ and of $\overline{\{u\}}$ in U are the same. In this way the lemma reduces to the schemes case which is Properties, Lemma 10.3. \square

Lemma 20.3. *Let S be a scheme. Let X be an algebraic space over S . Assume*

- (1) *every quasi-compact scheme étale over X has finitely many irreducible components, and*
- (2) *every $x \in |X|$ of codimension 0 on X can be represented by a monomorphism $\text{Spec}(k) \rightarrow X$.*

Then X is a reasonable algebraic space.

Proof. Let U be an affine scheme and let $a : U \rightarrow X$ be an étale morphism. We have to show that the fibres of a are universally bounded. By assumption (1) the

scheme U has finitely many irreducible components. Let $u_1, \dots, u_n \in U$ be the generic points of these irreducible components. Let $\{x_1, \dots, x_m\} \subset |X|$ be the image of $\{u_1, \dots, u_n\}$. Each x_j is a point of codimension 0. By assumption (2) we may choose a monomorphism $\text{Spec}(k_j) \rightarrow X$ representing x_j . By Properties of Spaces, Lemma 11.1 we have

$$U \times_X \text{Spec}(k_j) = \coprod_{a(u_i)=x_j} \text{Spec}(\kappa(u_i))$$

This is a scheme finite over $\text{Spec}(k_j)$ of degree $d_j = \sum_{a(u_i)=x_j} [\kappa(u_i) : k_j]$. Set $n = \max d_j$.

Observe that a is separated (Properties of Spaces, Lemma 6.4). Consider the stratification

$$X = X_0 \supset X_1 \supset X_2 \supset \dots$$

associated to $U \rightarrow X$ in Lemma 8.2. By our choice of n above we conclude that X_{n+1} is empty. Namely, if not, then $a^{-1}(X_{n+1})$ is a nonempty open of U and hence would contain one of the x_i . This would mean that X_{n+1} contains $x_j = a(u_i)$ which is impossible. Hence we see that the fibres of $U \rightarrow X$ are universally bounded (in fact by the integer n). \square

Lemma 20.4. *Let S be a scheme. Let X be an algebraic space over S . The following are equivalent*

- (1) *X is decent and $|X|$ has finitely many irreducible components,*
- (2) *every quasi-compact scheme étale over X has finitely many irreducible components, there are finitely many $x \in |X|$ of codimension 0 on X , and each of these can be represented by a monomorphism $\text{Spec}(k) \rightarrow X$,*
- (3) *there exists a dense open $X' \subset X$ which is a scheme, X' has finitely many irreducible components with generic points $\{x'_1, \dots, x'_m\}$, and the morphism $x'_j \rightarrow X$ is quasi-compact for $j = 1, \dots, m$.*

Moreover, if these conditions hold, then X is reasonable and the points $x'_j \in |X|$ are the generic points of the irreducible components of $|X|$.

Proof. In the proof we use Properties of Spaces, Lemma 11.1 without further mention. Assume (1). Then X has a dense open subscheme X' by Theorem 10.2. Since the closure of an irreducible component of $|X'|$ is an irreducible component of $|X|$, we see that $|X'|$ has finitely many irreducible components. Thus (3) holds.

Assume $X' \subset X$ is as in (3). Let $\{x'_1, \dots, x'_m\}$ be the generic points of the irreducible components of X' . Let $a : U \rightarrow X$ be an étale morphism with U a quasi-compact scheme. To prove (2) it suffices to show that U has finitely many irreducible components whose generic points lie over $\{x'_1, \dots, x'_m\}$. It suffices to prove this for the members of a finite affine open cover of U , hence we may and do assume U is affine. Note that $U' = a^{-1}(X') \subset U$ is a dense open. Since $U' \rightarrow X'$ is an étale morphism of schemes, we see the generic points of irreducible components of U' are the points lying over $\{x'_1, \dots, x'_m\}$. Since $x'_j \rightarrow X$ is quasi-compact there are finitely many points of U lying over x'_j (Lemma 4.5). Hence U' has finitely many irreducible components, which implies that the closures of these irreducible components are the irreducible components of U . Thus (2) holds.

Assume (2). This implies (1) and the final statement by Lemma 20.3. (We also use that a reasonable algebraic space is decent, see discussion following Definition 6.1.) \square

21. Generically finite morphisms

This section discusses for morphisms of algebraic spaces the material discussed in Morphisms, Section 51 and Varieties, Section 17 for morphisms of schemes.

Lemma 21.1. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume that f is quasi-separated of finite type. Let $y \in |Y|$ be a point of codimension 0 on Y . The following are equivalent:*

- (1) *the space $|X_k|$ is finite where $\mathrm{Spec}(k) \rightarrow Y$ represents y ,*
- (2) *$X \rightarrow Y$ is quasi-finite at all points of $|X|$ over y ,*
- (3) *there exists an open subspace $Y' \subset Y$ with $y \in |Y'|$ such that $Y' \times_Y X \rightarrow Y'$ is finite.*

If Y is decent these are also equivalent to

- (4) *the set $f^{-1}(\{y\})$ is finite.*

Proof. The equivalence of (1) and (2) follows from Lemma 18.10 (and the fact that a quasi-separated morphism is decent by Lemma 17.2).

Assume the equivalent conditions of (1) and (2). Choose an affine scheme V and an étale morphism $V \rightarrow Y$ mapping a point $v \in V$ to y . Then v is a generic point of an irreducible component of V by Properties of Spaces, Lemma 11.1. Choose an affine scheme U and a surjective étale morphism $U \rightarrow V \times_Y X$. Then $U \rightarrow V$ is of finite type. The morphism $U \rightarrow V$ is quasi-finite at every point lying over v by (2). It follows that the fibre of $U \rightarrow V$ over v is finite (Morphisms, Lemma 20.14). By Morphisms, Lemma 51.1 after shrinking V we may assume that $U \rightarrow V$ is finite. Let

$$R = U \times_{V \times_Y X} U$$

Since f is quasi-separated, we see that $V \times_Y X$ is quasi-separated and hence R is a quasi-compact scheme. Moreover the morphisms $R \rightarrow V$ is quasi-finite as the composition of an étale morphism $R \rightarrow U$ and a finite morphism $U \rightarrow V$. Hence we may apply Morphisms, Lemma 51.1 once more and after shrinking V we may assume that $R \rightarrow V$ is finite as well. This of course implies that the two projections $R \rightarrow V$ are finite étale. It follows that $V/R = V \times_Y X$ is an affine scheme, see Groupoids, Proposition 23.9. By Morphisms, Lemma 41.9 we conclude that $V \times_Y X \rightarrow V$ is proper and by Morphisms, Lemma 44.11 we conclude that $V \times_Y X \rightarrow V$ is finite. Finally, we let $Y' \subset Y$ be the open subspace of Y corresponding to the image of $|V| \rightarrow |Y|$. By Morphisms of Spaces, Lemma 45.3 we conclude that $Y' \times_Y X \rightarrow Y'$ is finite as the base change to V is finite and as $V \rightarrow Y'$ is a surjective étale morphism.

If Y is decent and f is quasi-separated, then we see that X is decent too; use Lemmas 17.2 and 17.5. Hence Lemma 18.10 applies to show that (4) implies (1) and (2). On the other hand, we see that (2) implies (4) by Morphisms of Spaces, Lemma 27.9. \square

Lemma 21.2. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume that f is quasi-separated and locally of finite type and Y quasi-separated. Let $y \in |Y|$ be a point of codimension 0 on Y . The following are equivalent:*

- (1) *the set $f^{-1}(\{y\})$ is finite,*
- (2) *the space $|X_k|$ is finite where $\mathrm{Spec}(k) \rightarrow Y$ represents y ,*

- (3) *there exist open subspaces $X' \subset X$ and $Y' \subset Y$ with $f(X') \subset Y'$, $y \in |Y'|$, and $f^{-1}(\{y\}) \subset |X'|$ such that $f|_{X'} : X' \rightarrow Y'$ is finite.*

Proof. Since quasi-separated algebraic spaces are decent, the equivalence of (1) and (2) follows from Lemma 18.10. To prove that (1) and (2) imply (3) we may and do replace Y by a quasi-compact open containing y . Since $f^{-1}(\{y\})$ is finite, we can find a quasi-compact open subspace of $X' \subset X$ containing the fibre. The restriction $f|_{X'} : X' \rightarrow Y$ is quasi-compact and quasi-separated by Morphisms of Spaces, Lemma 8.10 (this is where we use that Y is quasi-separated). Applying Lemma 21.1 to $f|_{X'} : X' \rightarrow Y$ we see that (3) holds. We omit the proof that (3) implies (2). \square

Lemma 21.3. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f is locally of finite type. Let $X^0 \subset |X|$, resp. $Y^0 \subset |Y|$ denote the set of codimension 0 points of X , resp. Y . Let $y \in Y^0$. The following are equivalent*

- (1) $f^{-1}(\{y\}) \subset X^0$,
- (2) f is quasi-finite at all points lying over y ,
- (3) f is quasi-finite at all $x \in X^0$ lying over y .

Proof. Let V be a scheme and let $V \rightarrow Y$ be a surjective étale morphism. Let U be a scheme and let $U \rightarrow V \times_Y X$ be a surjective étale morphism. Then f is quasi-finite at the image x of a point $u \in U$ if and only if $U \rightarrow V$ is quasi-finite at u . Moreover, $x \in X^0$ if and only if u is the generic point of an irreducible component of U (Properties of Spaces, Lemma 11.1). Thus the lemma reduces to the case of the morphism $U \rightarrow V$, i.e., to Morphisms, Lemma 51.4. \square

Lemma 21.4. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f is locally of finite type. Let $X^0 \subset |X|$, resp. $Y^0 \subset |Y|$ denote the set of codimension 0 points of X , resp. Y . Assume*

- (1) Y is decent,
- (2) X^0 and Y^0 are finite and $f^{-1}(Y^0) = X^0$,
- (3) either f is quasi-compact or f is separated.

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \rightarrow V$ is finite.

Proof. By Lemmas 20.4 and 20.1 we may assume Y is a scheme with finitely many irreducible components. Shrinking further we may assume Y is an irreducible affine scheme with generic point y . Then the fibre of f over y is finite.

Assume f is quasi-compact and Y affine irreducible. Then X is quasi-compact and we may choose an affine scheme U and a surjective étale morphism $U \rightarrow X$. Then $U \rightarrow Y$ is of finite type and the fibre of $U \rightarrow Y$ over y is the set U^0 of generic points of irreducible components of U (Properties of Spaces, Lemma 11.1). Hence U^0 is finite (Morphisms, Lemma 20.14) and after shrinking Y we may assume that $U \rightarrow Y$ is finite (Morphisms, Lemma 51.1). Next, consider $R = U \times_X U$. Since the projection $s : R \rightarrow U$ is étale we see that $R^0 = s^{-1}(U^0)$ lies over y . Since $R \rightarrow U \times_Y U$ is a monomorphism, we conclude that R^0 is finite as $U \times_Y U \rightarrow Y$ is finite. And R is separated (Properties of Spaces, Lemma 6.4). Thus we may shrink Y once more to reach the situation where R is finite over Y (Morphisms, Lemma 51.5). In this case it follows that $X = U/R$ is finite over Y by exactly the

same arguments as given in the proof of Lemma 21.1 (or we can simply apply that lemma because it follows immediately that X is quasi-separated as well).

Assume f is separated and Y affine irreducible. Choose $V \subset Y$ and $U \subset X$ as in Lemma 21.2. Since $f|_U : U \rightarrow V$ is finite, we see that $U \subset f^{-1}(V)$ is closed as well as open (Morphisms of Spaces, Lemmas 40.6 and 45.9). Thus $f^{-1}(V) = U \amalg W$ for some open subspace W of X . However, since U contains all the codimension 0 points of X we conclude that $W = \emptyset$ (Properties of Spaces, Lemma 11.2) as desired. \square

22. Birational morphisms

The following definition of a birational morphism of algebraic spaces seems to be the closest to our definition (Morphisms, Definition 50.1) of a birational morphism of schemes.

Definition 22.1. Let S be a scheme. Let X and Y algebraic spaces over S . Assume X and Y are decent and that $|X|$ and $|Y|$ have finitely many irreducible components. We say a morphism $f : X \rightarrow Y$ is *birational* if

- (1) $|f|$ induces a bijection between the set of generic points of irreducible components of $|X|$ and the set of generic points of the irreducible components of $|Y|$, and
- (2) for every generic point $x \in |X|$ of an irreducible component the local ring map $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is an isomorphism (see clarification below).

Clarification: Since X and Y are decent the topological spaces $|X|$ and $|Y|$ are sober (Proposition 12.4). Hence condition (1) makes sense. Moreover, because we have assumed that $|X|$ and $|Y|$ have finitely many irreducible components, we see that the generic points $x_1, \dots, x_n \in |X|$, resp. $y_1, \dots, y_n \in |Y|$ are contained in any dense open of $|X|$, resp. $|Y|$. In particular, they are contained in the schematic locus of X , resp. Y by Theorem 10.2. Thus we can define \mathcal{O}_{X,x_i} , resp. \mathcal{O}_{Y,y_i} to be the local ring of this scheme at x_i , resp. y_i .

We conclude that if the morphism $f : X \rightarrow Y$ is birational, then there exist dense open subspaces $X' \subset X$ and $Y' \subset Y$ such that

- (1) $f(X') \subset Y'$,
- (2) X' and Y' are representable, and
- (3) $f|_{X'} : X' \rightarrow Y'$ is birational in the sense of Morphisms, Definition 50.1.

However, we do insist that X and Y are decent with finitely many irreducible components. Other ways to characterize decent algebraic spaces with finitely many irreducible components are given in Lemma 20.4. In most cases birational morphisms are isomorphisms over dense opens.

Lemma 22.2. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S which are decent and have finitely many irreducible components. If f is birational then f is dominant.*

Proof. Follows immediately from the definitions. See Morphisms of Spaces, Definition 18.1. \square

Lemma 22.3. *Let S be a scheme. Let $f : X \rightarrow Y$ be a birational morphism of algebraic spaces over S which are decent and have finitely many irreducible components. If $y \in Y$ is the generic point of an irreducible component, then the base change $X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}) \rightarrow \operatorname{Spec}(\mathcal{O}_{Y,y})$ is an isomorphism.*

Proof. Let $X' \subset X$ and $Y' \subset Y$ be the maximal open subspaces which are representable, see Lemma 20.4. By Lemma 21.3 the fibre of f over y is consists of points of codimension 0 of X and is therefore contained in X' . Hence $X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}) = X' \times_{Y'} \operatorname{Spec}(\mathcal{O}_{Y',y})$ and the result follows from Morphisms, Lemma 50.3. \square

Lemma 22.4. *Let S be a scheme. Let $f : X \rightarrow Y$ be a birational morphism of algebraic spaces over S which are decent and have finitely many irreducible components. Assume one of the following conditions is satisfied*

- (1) *f is locally of finite type and Y reduced (i.e., integral),*
- (2) *f is locally of finite presentation.*

Then there exist dense opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \rightarrow V$ is an isomorphism.

Proof. By Lemma 20.4 we may assume that X and Y are schemes. In this case the result is Morphisms, Lemma 50.5. \square

Lemma 22.5. *Let S be a scheme. Let $f : X \rightarrow Y$ be a birational morphism of algebraic spaces over S which are decent and have finitely many irreducible components. Assume*

- (1) *either f is quasi-compact or f is separated, and*
- (2) *either f is locally of finite type and Y is reduced or f is locally of finite presentation.*

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \rightarrow V$ is an isomorphism.

Proof. By Lemma 20.4 we may assume Y is a scheme. By Lemma 21.4 we may assume that f is finite. Then X is a scheme too and the result follows from Morphisms, Lemma 51.6. \square

Lemma 22.6. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S which are decent and have finitely many irreducible components. If f is birational and $V \rightarrow Y$ is an étale morphism with V affine, then $X \times_Y V$ is decent with finitely many irreducible components and $X \times_Y V \rightarrow V$ is birational.*

Proof. The algebraic space $U = X \times_Y V$ is decent (Lemma 6.6). The generic points of V and U are the elements of $|V|$ and $|U|$ which lie over generic points of $|Y|$ and $|X|$ (Lemma 20.1). Since Y is decent we conclude there are finitely many generic points on V . Let $\xi \in |X|$ be a generic point of an irreducible component. By the discussion following Definition 22.1 we have a cartesian square

$$\begin{array}{ccc} \operatorname{Spec}(\mathcal{O}_{X,\xi}) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathcal{O}_{Y,f(\xi)}) & \longrightarrow & Y \end{array}$$

whose horizontal morphisms are monomorphisms identifying local rings and where the left vertical arrow is an isomorphism. It follows that in the diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{X,\xi}) \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{Y,f(\xi)}) \times_Y V & \longrightarrow & V \end{array}$$

the vertical arrow on the left is an isomorphism. The horizontal arrows have image contained in the schematic locus of U and V and identify local rings (some details omitted). Since the image of the horizontal arrows are the points of $|U|$, resp. $|V|$ lying over ξ , resp. $f(\xi)$ we conclude. \square

Lemma 22.7. *Let S be a scheme. Let $f : X \rightarrow Y$ be a birational morphism between algebraic spaces over S which are decent and have finitely many irreducible components. Then the normalizations $X^\nu \rightarrow X$ and $Y^\nu \rightarrow Y$ exist and there is a commutative diagram*

$$\begin{array}{ccc} X^\nu & \longrightarrow & Y^\nu \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

of algebraic spaces over S . The morphism $X^\nu \rightarrow Y^\nu$ is birational.

Proof. By Lemma 20.4 we see that X and Y satisfy the equivalent conditions of Morphisms of Spaces, Lemma 49.1 and the normalizations are defined. By Morphisms of Spaces, Lemma 49.8 the algebraic space X^ν is normal and maps codimension 0 points to codimension 0 points. Since f maps codimension 0 points to codimension 0 points (this is the same as generic points on decent spaces by Lemma 20.1) we obtain from Morphisms of Spaces, Lemma 49.8 a factorization of the composition $X^\nu \rightarrow X \rightarrow Y$ through Y^ν .

Observe that X^ν and Y^ν are decent for example by Lemma 6.5. Moreover the maps $X^\nu \rightarrow X$ and $Y^\nu \rightarrow Y$ induce bijections on irreducible components (see references above) hence X^ν and Y^ν both have a finite number of irreducible components and the map $X^\nu \rightarrow Y^\nu$ induces a bijection between their generic points. To prove that $X^\nu \rightarrow Y^\nu$ is birational, it therefore suffices to show it induces an isomorphism on local rings at these points. To do this we may replace X and Y by open neighbourhoods of their generic points, hence we may assume X and Y are affine irreducible schemes with generic points x and y . Since f is birational the map $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$ is an isomorphism. Let $x^\nu \in X^\nu$ and $y^\nu \in Y^\nu$ be the points lying over x and y . By construction of the normalization we see that $\mathcal{O}_{X^\nu,x^\nu} = \mathcal{O}_{X,x}/\mathfrak{m}_x$ and similarly on Y . Thus the map $\mathcal{O}_{X^\nu,x^\nu} \rightarrow \mathcal{O}_{Y^\nu,y^\nu}$ is an isomorphism as well. \square

Lemma 22.8. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume*

- (1) *X and Y are decent and have finitely many irreducible components,*
- (2) *f is integral and birational,*
- (3) *Y is normal, and*
- (4) *X is reduced.*

Then f is an isomorphism.

Proof. Let $V \rightarrow Y$ be an étale morphism with V affine. It suffices to show that $U = X \times_Y V \rightarrow V$ is an isomorphism. By Lemma 22.6 and its proof we see that U and V are decent and have finitely many irreducible components and that $U \rightarrow V$ is birational. By Properties, Lemma 7.5 V is a finite disjoint union of integral schemes. Thus we may assume V is integral. As f is birational, we see that U is irreducible and reduced, i.e., integral (note that U is a scheme as f is integral, hence representable). Thus we may assume that X and Y are integral schemes and the result follows from the case of schemes, see Morphisms, Lemma 54.8. \square

Lemma 22.9. *Let S be a scheme. Let $f : X \rightarrow Y$ be an integral birational morphism of decent algebraic spaces over S which have finitely many irreducible components. Then there exists a factorization $Y^\nu \rightarrow X \rightarrow Y$ and $Y^\nu \rightarrow X$ is the normalization of X .*

Proof. Consider the map $X^\nu \rightarrow Y^\nu$ of Lemma 22.7. This map is integral by Morphisms of Spaces, Lemma 45.12. Hence it is an isomorphism by Lemma 22.8. \square

23. Jacobson spaces

We have defined the Jacobson property for algebraic spaces in Properties of Spaces, Remark 7.3. For representable algebraic spaces it agrees with the property discussed in Properties, Section 6. The relationship between the Jacobson property and the behaviour of the topological space $|X|$ is not evident for general algebraic spaces $|X|$. However, a decent (for example quasi-separated or locally separated) algebraic space X is Jacobson if and only if $|X|$ is Jacobson (see Lemma 23.4).

Lemma 23.1. *Let S be a scheme. Let X be a Jacobson algebraic space over S . Any algebraic space locally of finite type over X is Jacobson.*

Proof. Let $U \rightarrow X$ be a surjective étale morphism where U is a scheme. Then U is Jacobson (by definition) and for a morphism of schemes $V \rightarrow U$ which is locally of finite type we see that V is Jacobson by the corresponding result for schemes (Morphisms, Lemma 16.9). Thus if $Y \rightarrow X$ is a morphism of algebraic spaces which is locally of finite type, then setting $V = U \times_X Y$ we see that Y is Jacobson by definition. \square

Lemma 23.2. *Let S be a scheme. Let X be a Jacobson algebraic space over S . For $x \in X_{ft-pts}$ and $g : W \rightarrow X$ locally of finite type with W a scheme, if $x \in \text{Im}(|g|)$, then there exists a closed point of W mapping to x .*

Proof. Let $U \rightarrow X$ be an étale morphism with U a scheme and with $u \in U$ closed mapping to x , see Morphisms of Spaces, Lemma 25.3. Observe that W , $W \times_X U$, and U are Jacobson schemes by Lemma 23.1. Hence finite type points on these schemes are the same thing as closed points by Morphisms, Lemma 16.8. The inverse image $T \subset W \times_X U$ of u is a nonempty (as x in the image of $W \rightarrow X$) closed subset. By Morphisms, Lemma 16.7 there is a closed point t of $W \times_X U$ which maps to u . As $W \times_X U \rightarrow W$ is locally of finite type the image of t in W is closed by Morphisms, Lemma 16.8. \square

Lemma 23.3. *Let S be a scheme. Let X be a decent Jacobson algebraic space over S . Then $X_{ft-pts} \subset |X|$ is the set of closed points.*

Proof. If $x \in |X|$ is closed, then we can represent x by a closed immersion $\mathrm{Spec}(k) \rightarrow X$, see Lemma 14.6. Hence x is certainly a finite type point.

Conversely, let $x \in |X|$ be a finite type point. We know that x can be represented by a quasi-compact monomorphism $\mathrm{Spec}(k) \rightarrow X$ where k is a field (Definition 6.1). On the other hand, by definition, there exists a morphism $\mathrm{Spec}(k') \rightarrow X$ which is locally of finite type and represents x (Morphisms, Definition 16.3). We obtain a factorization $\mathrm{Spec}(k') \rightarrow \mathrm{Spec}(k) \rightarrow X$. Let $U \rightarrow X$ be any étale morphism with U affine and consider the morphisms

$$\mathrm{Spec}(k') \times_X U \rightarrow \mathrm{Spec}(k) \times_X U \rightarrow U$$

The quasi-compact scheme $\mathrm{Spec}(k) \times_X U$ is étale over $\mathrm{Spec}(k)$ hence is a finite disjoint union of spectra of fields (Remark 4.1). Moreover, the first morphism is surjective and locally of finite type (Morphisms, Lemma 15.8) hence surjective on finite type points (Morphisms, Lemma 16.6) and the composition (which is locally of finite type) sends finite type points to closed points as U is Jacobson (Morphisms, Lemma 16.8). Thus the image of $\mathrm{Spec}(k) \times_X U \rightarrow U$ is a finite set of closed points hence closed. Since this is true for every affine U and étale morphism $U \rightarrow X$, we conclude that $x \in |X|$ is closed. \square

Lemma 23.4. *Let S be a scheme. Let X be a decent algebraic space over S . Then X is Jacobson if and only if $|X|$ is Jacobson.*

Proof. Assume X is Jacobson and that $T \subset |X|$ is a closed subset. By Morphisms of Spaces, Lemma 25.6 we see that $T \cap X_{\mathrm{ft-pts}}$ is dense in T . By Lemma 23.3 we see that $X_{\mathrm{ft-pts}}$ are the closed points of $|X|$. Thus $|X|$ is indeed Jacobson.

Assume $|X|$ is Jacobson. Let $f : U \rightarrow X$ be an étale morphism with U an affine scheme. We have to show that U is Jacobson. If $x \in |X|$ is closed, then the fibre $F = f^{-1}(\{x\})$ is a finite (by definition of decent) closed (by construction of the topology on $|X|$) subset of U . Since there are no specializations between points of F (Lemma 12.1) we conclude that every point of F is closed in U . If U is not Jacobson, then there exists a non-closed point $u \in U$ such that $\{u\}$ is locally closed (Topology, Lemma 18.3). We will show that $f(u) \in |X|$ is closed; by the above u is closed in U which is a contradiction and finishes the proof. To prove this we may replace U by an affine open neighbourhood of u . Thus we may assume that $\{u\}$ is closed in U . Let $R = U \times_X U$ with projections $s, t : R \rightarrow U$. Then $s^{-1}(\{u\}) = \{r_1, \dots, r_m\}$ is finite (by definition of decent spaces). After replacing U by a smaller affine open neighbourhood of u we may assume that $t(r_j) = u$ for $j = 1, \dots, m$. It follows that $\{u\}$ is an R -invariant closed subset of U . Hence $\{f(u)\}$ is a locally closed subset of X as it is closed in the open $|f|(|U|)$ of $|X|$. Since $|X|$ is Jacobson we conclude that $f(u)$ is closed in $|X|$ as desired. \square

Lemma 23.5. *Let S be a scheme. Let X be a decent locally Noetherian algebraic space over S . Let $x \in |X|$. Then*

$$W = \{x' \in |X| : x' \rightsquigarrow x, x' \neq x\}$$

is a Noetherian, spectral, sober, Jacobson topological space.

Proof. We may replace by any open subspace containing x . Thus we may assume that X is quasi-compact. Then $|X|$ is a Noetherian topological space (Properties

of Spaces, Lemma 24.2). Thus W is a Noetherian topological space (Topology, Lemma 9.2).

Combining Lemma 14.1 with Properties of Spaces, Lemma 15.2 we see that $|X|$ is a spectral topological space. By Topology, Lemma 24.7 we see that $W \cup \{x\}$ is a spectral topological space. Now W is a quasi-compact open of $W \cup \{x\}$ and hence W is spectral by Topology, Lemma 23.5.

Let $E \subset W$ be an irreducible closed subset. Then if $Z \subset |X|$ is the closure of E we see that $x \in Z$. There is a unique generic point $\eta \in Z$ by Proposition 12.4. Of course $\eta \in W$ and hence $\eta \in E$. We conclude that E has a unique generic point, i.e., W is sober.

Let $x' \in W$ be a point such that $\{x'\}$ is locally closed in W . To finish the proof we have to show that x' is a closed point of W . If not, then there exists a nontrivial specialization $x' \rightsquigarrow x'_1$ in W . Let U be an affine scheme, $u \in U$ a point, and let $U \rightarrow X$ be an étale morphism mapping u to x . By Lemma 12.2 we can choose specializations $u' \rightsquigarrow u'_1 \rightsquigarrow u$ mapping to $x' \rightsquigarrow x'_1 \rightsquigarrow x$. Let $\mathfrak{p}' \subset \mathcal{O}_{U,u}$ be the prime ideal corresponding to u' . The existence of the specializations implies that $\dim(\mathcal{O}_{U,u}/\mathfrak{p}') \geq 2$. Hence every nonempty open of $\mathrm{Spec}(\mathcal{O}_{U,u}/\mathfrak{p}')$ is infinite by Algebra, Lemma 61.1. By Lemma 12.1 we obtain a continuous map

$$\mathrm{Spec}(\mathcal{O}_{U,u}/\mathfrak{p}') \setminus \{\mathfrak{m}_u/\mathfrak{p}'\} \longrightarrow W$$

Since the generic point of the LHS maps to x' the image is contained in $\overline{\{x'\}}$. We conclude the inverse image of $\{x'\}$ under the displayed arrow is nonempty open hence infinite. However, the fibres of $U \rightarrow X$ are finite as X is decent and we conclude that $\{x'\}$ is infinite. This contradiction finishes the proof. \square

24. Local irreducibility

We have already defined the geometric number of branches of an algebraic space at a point in Properties of Spaces, Section 23. The number of branches of an algebraic space at a point can only be defined for decent algebraic spaces.

Lemma 24.1. *Let S be a scheme. Let X be a decent algebraic space over S . Let $x \in |X|$ be a point. The following are equivalent*

- (1) *for any elementary étale neighbourhood $(U, u) \rightarrow (X, x)$ the local ring $\mathcal{O}_{U,u}$ has a unique minimal prime,*
- (2) *for any elementary étale neighbourhood $(U, u) \rightarrow (X, x)$ there is a unique irreducible component of U through u ,*
- (3) *for any elementary étale neighbourhood $(U, u) \rightarrow (X, x)$ the local ring $\mathcal{O}_{U,u}$ is unibranch,*
- (4) *the henselian local ring $\mathcal{O}_{X,x}^h$ has a unique minimal prime.*

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in 1-1 correspondence with minimal primes of the local ring of U at u . The ring $\mathcal{O}_{X,x}^h$ is the henselization of $\mathcal{O}_{U,u}$, see discussion following Definition 11.7. In particular (3) and (4) are equivalent by More on Algebra, Lemma 106.3. The equivalence of (2) and (3) follows from More on Morphisms, Lemma 36.2. \square

Definition 24.2. Let S be a scheme. Let X be a decent algebraic space over S . Let $x \in |X|$. We say that X is *unibranched at x* if the equivalent conditions of Lemma 24.1 hold. We say that X is *unibranched* if X is unibranched at every $x \in |X|$.

This is consistent with the definition for schemes (Properties, Definition 15.1).

Lemma 24.3. Let S be a scheme. Let X be a decent algebraic space over S . Let $x \in |X|$ be a point. Let $n \in \{1, 2, \dots\}$ be an integer. The following are equivalent

- (1) for any elementary étale neighbourhood $(U, u) \rightarrow (X, x)$ the number of minimal primes of the local ring $\mathcal{O}_{U,u}$ is $\leq n$ and for at least one choice of (U, u) it is n ,
- (2) for any elementary étale neighbourhood $(U, u) \rightarrow (X, x)$ the number irreducible components of U passing through u is $\leq n$ and for at least one choice of (U, u) it is n ,
- (3) for any elementary étale neighbourhood $(U, u) \rightarrow (X, x)$ the number of branches of U at u is $\leq n$ and for at least one choice of (U, u) it is n ,
- (4) the number of minimal prime ideals of $\mathcal{O}_{X,x}^h$ is n .

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in 1-1 correspondence with minimal primes of the local ring of U at u . The ring $\mathcal{O}_{X,x}$ is the henselization of $\mathcal{O}_{U,u}$, see discussion following Definition 11.7. In particular (3) and (4) are equivalent by More on Algebra, Lemma 106.3. The equivalence of (2) and (3) follows from More on Morphisms, Lemma 36.2. \square

Definition 24.4. Let S be a scheme. Let X be a decent algebraic space over S . Let $x \in |X|$. The *number of branches of X at x* is either $n \in \mathbf{N}$ if the equivalent conditions of Lemma 24.3 hold, or else ∞ .

25. Catenary algebraic spaces

This section extends the material in Properties, Section 11 and Morphisms, Section 17 to algebraic spaces.

Definition 25.1. Let S be a scheme. Let X be a decent algebraic space over S . We say X is *catenary* if $|X|$ is catenary (Topology, Definition 11.4).

If X is representable, then this is equivalent to the corresponding notion for the scheme representing X .

Lemma 25.2. Let S be a locally Noetherian and universally catenary scheme. Let $\delta : S \rightarrow \mathbf{Z}$ be a dimension function. Let X be a decent algebraic space over S such that the structure morphism $X \rightarrow S$ is locally of finite type. Let $\delta_X : |X| \rightarrow \mathbf{Z}$ be the map sending x to $\delta(f(x))$ plus the transcendence degree of $x/f(x)$. Then δ_X is a dimension function on $|X|$.

Proof. Let $\varphi : U \rightarrow X$ be a surjective étale morphism where U is a scheme. Then the similarly defined function δ_U is a dimension function on U by Morphisms, Lemma 52.3. On the other hand, by the definition of relative transcendence degree in (Morphisms of Spaces, Definition 33.1) we see that $\delta_U(u) = \delta_X(\varphi(u))$.

Let $x \rightsquigarrow x'$ be a specialization of points in $|X|$. by Lemma 12.2 we can find a specialization $u \rightsquigarrow u'$ of points of U with $\varphi(u) = x$ and $\varphi(u') = x'$. Moreover, we see that $x = x'$ if and only if $u = u'$, see Lemma 12.1. Thus the fact that

δ_U is a dimension function implies that δ_X is a dimension function, see Topology, Definition 20.1. \square

Lemma 25.3. *Let S be a locally Noetherian and universally catenary scheme. Let X be an algebraic space over S such that X is decent and such that the structure morphism $X \rightarrow S$ is locally of finite type. Then X is catenary.*

Proof. The question is local on S (use Topology, Lemma 11.5). Thus we may assume that S has a dimension function, see Topology, Lemma 20.4. Then we conclude that $|X|$ has a dimension function by Lemma 25.2. Since $|X|$ is sober (Proposition 12.4) we conclude that $|X|$ is catenary by Topology, Lemma 20.2. \square

By Lemma 25.3 the following definition is compatible with the already existing notion for representable algebraic spaces.

Definition 25.4. Let S be a scheme. Let X be a decent and locally Noetherian algebraic space over S . We say X is *universally catenary* if for every morphism $Y \rightarrow X$ of algebraic spaces which is locally of finite type and with Y decent, the algebraic space Y is catenary.

If X is an algebraic space, then the condition “ X is decent and locally Noetherian” is equivalent to “ X is quasi-separated and locally Noetherian”. This is Lemma 14.1. Thus another way to understand the definition above is that X is universally catenary if and only if Y is catenary for all morphisms $Y \rightarrow X$ which are quasi-separated and locally of finite type.

Lemma 25.5. *Let S be a scheme. Let X be a decent, locally Noetherian, and universally catenary algebraic space over S . Then any decent algebraic space locally of finite type over X is universally catenary.*

Proof. This is formal from the definitions and the fact that compositions of morphisms locally of finite type are locally of finite type (Morphisms of Spaces, Lemma 23.2). \square

Lemma 25.6. *Let S be a scheme. Let $f : Y \rightarrow X$ be a surjective finite morphism of decent and locally Noetherian algebraic spaces. Let $\delta : |X| \rightarrow \mathbf{Z}$ be a function. If $\delta \circ |f|$ is a dimension function, then δ is a dimension function.*

Proof. Let $x \mapsto x'$, $x \neq x'$ be a specialization in $|X|$. Choose $y \in |Y|$ with $|f|(y) = x$. Since $|f|$ is closed (Morphisms of Spaces, Lemma 45.9) we find a specialization $y \rightsquigarrow y'$ with $|f|(y') = x'$. Thus we conclude that $\delta(x) = \delta(|f|(y)) > \delta(|f|(y')) = \delta(x')$ (see Topology, Definition 20.1). If $x \rightsquigarrow x'$ is an immediate specialization, then $y \rightsquigarrow y'$ is an immediate specialization too: namely if $y \rightsquigarrow y'' \rightsquigarrow y'$, then $|f|(y'')$ must be either x or x' and there are no nontrivial specializations between points of fibres of $|f|$ by Lemma 18.10. \square

The discussion will be continued in More on Morphisms of Spaces, Section 32.

26. Other chapters

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- (7) Sites and Sheaves
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References

- [GR71] Laurent Gruson and Michel Raynaud, *Critères de platitude et de projectivité*, Invent. math. **13** (1971), 1–89.
- [Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, vol. 203, Springer-Verlag, 1971.