

COHOMOLOGY OF SCHEMES

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1. Introduction

In this chapter we first prove a number of results on the cohomology of quasi-coherent sheaves. A fundamental reference is [DG67]. Having done this we will elaborate on cohomology of coherent sheaves in the Noetherian setting. See [Ser55].

2. Čech cohomology of quasi-coherent sheaves

Let X be a scheme. Let $U \subset X$ be an affine open. Recall that a *standard open covering* of U is a covering of the form $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ where $f_1, \dots, f_n \in \Gamma(U, \mathcal{O}_X)$ generate the unit ideal, see Schemes, Definition 5.2.

Lemma 2.1. *Let X be a scheme. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ be a standard open covering of an affine open of X . Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. Write $U = \text{Spec}(A)$ for some ring A . In other words, f_1, \dots, f_n are elements of A which generate the unit ideal of A . Write $\mathcal{F}|_U = \widetilde{M}$ for some A -module M . Clearly the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is identified with the complex

$$\prod_{i_0} M_{f_{i_0}} \rightarrow \prod_{i_0 i_1} M_{f_{i_0} f_{i_1}} \rightarrow \prod_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \rightarrow \dots$$

We are asked to show that the extended complex

$$(2.1.1) \quad 0 \rightarrow M \rightarrow \prod_{i_0} M_{f_{i_0}} \rightarrow \prod_{i_0 i_1} M_{f_{i_0} f_{i_1}} \rightarrow \prod_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \rightarrow \dots$$

(whose truncation we have studied in Algebra, Lemma 24.1) is exact. It suffices to show that (2.1.1) is exact after localizing at a prime \mathfrak{p} , see Algebra, Lemma 23.1. In fact we will show that the extended complex localized at \mathfrak{p} is homotopic to zero.

There exists an index i such that $f_i \notin \mathfrak{p}$. Choose and fix such an element i_{fix} . Note that $M_{f_{i_{\text{fix}}}, \mathfrak{p}} = M_{\mathfrak{p}}$. Similarly for a localization at a product $f_{i_0} \dots f_{i_p}$ and \mathfrak{p} we can drop any f_{i_j} for which $i_j = i_{\text{fix}}$. Let us define a homotopy

$$h : \prod_{i_0 \dots i_{p+1}} M_{f_{i_0} \dots f_{i_{p+1}}, \mathfrak{p}} \longrightarrow \prod_{i_0 \dots i_p} M_{f_{i_0} \dots f_{i_p}, \mathfrak{p}}$$

by the rule

$$h(s)_{i_0 \dots i_p} = s_{i_{\text{fix}} i_0 \dots i_p}$$

(This is “dual” to the homotopy in the proof of Cohomology, Lemma 10.4.) In other words, $h : \prod_{i_0} M_{f_{i_0}, \mathfrak{p}} \rightarrow M_{\mathfrak{p}}$ is projection onto the factor $M_{f_{i_{\text{fix}}}, \mathfrak{p}} = M_{\mathfrak{p}}$ and in general the map h equal projection onto the factors $M_{f_{i_{\text{fix}}} f_{i_1} \dots f_{i_{p+1}}, \mathfrak{p}} = M_{f_{i_1} \dots f_{i_{p+1}}, \mathfrak{p}}$. We compute

$$\begin{aligned} (dh + hd)(s)_{i_0 \dots i_p} &= \sum_{j=0}^p (-1)^j h(s)_{i_0 \dots \hat{i}_j \dots i_p} + d(s)_{i_{\text{fix}} i_0 \dots i_p} \\ &= \sum_{j=0}^p (-1)^j s_{i_{\text{fix}} i_0 \dots \hat{i}_j \dots i_p} + s_{i_0 \dots i_p} + \sum_{j=0}^p (-1)^{j+1} s_{i_{\text{fix}} i_0 \dots \hat{i}_j \dots i_p} \\ &= s_{i_0 \dots i_p} \end{aligned}$$

This proves the identity map is homotopic to zero as desired. \square

The following lemma says in particular that for any affine scheme X and any quasi-coherent sheaf \mathcal{F} on X we have

$$H^p(X, \mathcal{F}) = 0$$

for all $p > 0$.

Lemma 2.2. *Let X be a scheme. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. For any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. We are going to apply Cohomology, Lemma 11.9. As our basis \mathcal{B} for the topology of X we are going to use the affine opens of X . As our set Cov of open coverings we are going to use the standard open coverings of affine opens of X . Next we check that conditions (1), (2) and (3) of Cohomology, Lemma 11.9 hold. Note that the intersection of standard opens in an affine is another standard open. Hence property (1) holds. The coverings form a cofinal system of open coverings of any element of \mathcal{B} , see Schemes, Lemma 5.1. Hence (2) holds. Finally, condition (3) of the lemma follows from Lemma 2.1. \square

Here is a relative version of the vanishing of cohomology of quasi-coherent sheaves on affines.

Lemma 2.3. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. If f is affine then $R^i f_* \mathcal{F} = 0$ for all $i > 0$.*

Proof. According to Cohomology, Lemma 7.3 the sheaf $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$. By assumption, whenever V is affine we have that $f^{-1}(V)$ is affine, see Morphisms, Definition 11.1. By Lemma 2.2 we conclude that $H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) = 0$ whenever V is affine. Since S has a basis consisting of affine opens we win. \square

Lemma 2.4. *Let $f : X \rightarrow S$ be an affine morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then $H^i(X, \mathcal{F}) = H^i(S, f_* \mathcal{F})$ for all $i \geq 0$.*

Proof. Follows from Lemma 2.3 and the Leray spectral sequence. See Cohomology, Lemma 13.6. \square

The following two lemmas explain when Čech cohomology can be used to compute cohomology of quasi-coherent modules.

Lemma 2.5. *Let X be a scheme. The following are equivalent*

- (1) *X has affine diagonal $\Delta : X \rightarrow X \times X$,*
- (2) *for $U, V \subset X$ affine open, the intersection $U \cap V$ is affine, and*
- (3) *there exists an open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ such that $U_{i_0 \dots i_p}$ is affine open for all $p \geq 0$ and all $i_0, \dots, i_p \in I$.*

In particular this holds if X is separated.

Proof. Assume X has affine diagonal. Let $U, V \subset X$ be affine opens. Then $U \cap V = \Delta^{-1}(U \times V)$ is affine. Thus (2) holds. It is immediate that (2) implies (3). Conversely, if there is a covering of X as in (3), then $X \times X = \bigcup U_i \times U_{i'}$ is an affine open covering, and we see that $\Delta^{-1}(U_i \times U_{i'}) = U_i \cap U_{i'}$ is affine. Then Δ is an affine morphism by Morphisms, Lemma 11.3. The final assertion follows from Schemes, Lemma 21.7. \square

Lemma 2.6. *Let X be a scheme. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering such that $U_{i_0 \dots i_p}$ is affine open for all $p \geq 0$ and all $i_0, \dots, i_p \in I$. In this case for any quasi-coherent sheaf \mathcal{F} we have*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$$

as $\Gamma(X, \mathcal{O}_X)$ -modules for all p .

Proof. In view of Lemma 2.2 this is a special case of Cohomology, Lemma 11.6. \square

3. Vanishing of cohomology

We have seen that on an affine scheme the higher cohomology groups of any quasi-coherent sheaf vanish (Lemma 2.2). It turns out that this also characterizes affine schemes. We give two versions.

Lemma 3.1. *Let X be a scheme. Assume that*

- (1) *X is quasi-compact,*
- (2) *for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we have $H^1(X, \mathcal{I}) = 0$.*

Then X is affine.

Proof. Let $x \in X$ be a closed point. Let $U \subset X$ be an affine open neighbourhood of x . Write $U = \text{Spec}(A)$ and let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x . Set $Z = X \setminus U$ and $Z' = Z \cup \{x\}$. By Schemes, Lemma 12.4 there are quasi-coherent sheaves of ideals \mathcal{I} , resp. \mathcal{I}' cutting out the reduced closed subschemes Z , resp. Z' . Consider the short exact sequence

$$0 \rightarrow \mathcal{I}' \rightarrow \mathcal{I} \rightarrow \mathcal{I}/\mathcal{I}' \rightarrow 0.$$

Since x is a closed point of X and $x \notin Z$ we see that \mathcal{I}/\mathcal{I}' is supported at x . In fact, the restriction of \mathcal{I}/\mathcal{I}' to U corresponds to the A -module A/\mathfrak{m} . Hence we see that $\Gamma(X, \mathcal{I}/\mathcal{I}') = A/\mathfrak{m}$. Since by assumption $H^1(X, \mathcal{I}') = 0$ we see there exists a global section $f \in \Gamma(X, \mathcal{I})$ which maps to the element $1 \in A/\mathfrak{m}$ as a section of \mathcal{I}/\mathcal{I}' . Clearly we have $x \in X_f \subset U$. This implies that $X_f = D(f_A)$ where f_A is the image of f in $A = \Gamma(U, \mathcal{O}_X)$. In particular X_f is affine.

Consider the union $W = \bigcup X_f$ over all $f \in \Gamma(X, \mathcal{O}_X)$ such that X_f is affine. Obviously W is open in X . By the arguments above every closed point of X is contained in W . The closed subset $X \setminus W$ of X is also quasi-compact (see Topology, Lemma 12.3). Hence it has a closed point if it is nonempty (see Topology, Lemma 12.8). This would contradict the fact that all closed points are in W . Hence we conclude $X = W$.

Choose finitely many $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that $X = X_{f_1} \cup \dots \cup X_{f_n}$ and such that each X_{f_i} is affine. This is possible as we've seen above. By Properties, Lemma 27.3 to finish the proof it suffices to show that f_1, \dots, f_n generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus n} \xrightarrow{f_1, \dots, f_n} \mathcal{O}_X \longrightarrow 0$$

The arrow defined by f_1, \dots, f_n is surjective since the opens X_{f_i} cover X . We let \mathcal{F} be the kernel of this surjective map. Observe that \mathcal{F} has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_n = \mathcal{F}$$

so that each subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to a quasi-coherent sheaf of ideals. Namely we can take \mathcal{F}_i to be the intersection of \mathcal{F} with the first i direct summands of $\mathcal{O}_X^{\oplus n}$. The assumption of the lemma implies that $H^1(X, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ for all i . This implies that $H^1(X, \mathcal{F}_2) = 0$ because it is sandwiched between $H^1(X, \mathcal{F}_1)$ and $H^1(X, \mathcal{F}_2/\mathcal{F}_1)$. Continuing like this we deduce that $H^1(X, \mathcal{F}) = 0$. Therefore we conclude that the map

$$\bigoplus_{i=1, \dots, n} \Gamma(X, \mathcal{O}_X) \xrightarrow{f_1, \dots, f_n} \Gamma(X, \mathcal{O}_X)$$

is surjective as desired. \square

Note that if X is a Noetherian scheme then every quasi-coherent sheaf of ideals is automatically a coherent sheaf of ideals and a finite type quasi-coherent sheaf of ideals. Hence the preceding lemma and the next lemma both apply in this case.

Lemma 3.2. *Let X be a scheme. Assume that*

- (1) X is quasi-compact,
- (2) X is quasi-separated, and
- (3) $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals \mathcal{I} of finite type.

Then X is affine.

Proof. By Properties, Lemma 22.3 every quasi-coherent sheaf of ideals is a directed colimit of quasi-coherent sheaves of ideals of finite type. By Cohomology, Lemma 19.1 taking cohomology on X commutes with directed colimits. Hence we see that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals on X . In other words we see that Lemma 3.1 applies. \square

We can use the arguments given above to find a sufficient condition to see when an invertible sheaf is ample. However, we warn the reader that this condition is not necessary.

Lemma 3.3. *Let X be a scheme. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Assume that*

- (1) X is quasi-compact,
- (2) for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ there exists an $n \geq 1$ such that $H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$.

Then \mathcal{L} is ample.

Proof. This is proved in exactly the same way as Lemma 3.1. Let $x \in X$ be a closed point. Let $U \subset X$ be an affine open neighbourhood of x such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Write $U = \text{Spec}(A)$ and let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x . Set $Z = X \setminus U$ and $Z' = Z \cup \{x\}$. By Schemes, Lemma 12.4 there are quasi-coherent sheaves of ideals \mathcal{I} , resp. \mathcal{I}' cutting out the reduced closed subschemes Z , resp. Z' . Consider the short exact sequence

$$0 \rightarrow \mathcal{I}' \rightarrow \mathcal{I} \rightarrow \mathcal{I}/\mathcal{I}' \rightarrow 0.$$

For every $n \geq 1$ we obtain a short exact sequence

$$0 \rightarrow \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \rightarrow \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \rightarrow \mathcal{I}/\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \rightarrow 0.$$

By our assumption we may pick n such that $H^1(X, \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$. Since x is a closed point of X and $x \notin Z$ we see that \mathcal{I}/\mathcal{I}' is supported at x . In fact, the restriction of \mathcal{I}/\mathcal{I}' to U corresponds to the A -module A/\mathfrak{m} . Since \mathcal{L} is trivial on U we see that the restriction of $\mathcal{I}/\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ to U also corresponds to the A -module A/\mathfrak{m} . Hence we see that $\Gamma(X, \mathcal{I}/\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = A/\mathfrak{m}$. By our choice of n we see there exists a global section $s \in \Gamma(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ which maps to the element $1 \in A/\mathfrak{m}$. Clearly we have $x \in X_s \subset U$ because s vanishes at points of Z . This implies that $X_s = D(f)$ where $f \in A$ is the image of s in $A \cong \Gamma(U, \mathcal{L}^{\otimes n})$. In particular X_s is affine.

Consider the union $W = \bigcup X_s$ over all $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ for $n \geq 1$ such that X_s is affine. Obviously W is open in X . By the arguments above every closed point of X is contained in W . The closed subset $X \setminus W$ of X is also quasi-compact (see Topology, Lemma 12.3). Hence it has a closed point if it is nonempty (see Topology, Lemma 12.8). This would contradict the fact that all closed points are

in W . Hence we conclude $X = W$. This means that \mathcal{L} is ample by Properties, Definition 26.1. \square

There is a variant of Lemma 3.3 with finite type ideal sheaves which we will formulate and prove here if we ever need it.

Lemma 3.4. *Let $f : X \rightarrow Y$ be a quasi-compact morphism with X and Y quasi-separated. If $R^1 f_* \mathcal{I} = 0$ for every quasi-coherent sheaf of ideals \mathcal{I} on X , then f is affine.*

Proof. Let $V \subset Y$ be an affine open subscheme. We have to show that $U = f^{-1}(V)$ is affine. The inclusion morphism $V \rightarrow Y$ is quasi-compact by Schemes, Lemma 21.14. Hence the base change $U \rightarrow X$ is quasi-compact, see Schemes, Lemma 19.3. Thus any quasi-coherent sheaf of ideals \mathcal{I} on U extends to a quasi-coherent sheaf of ideals on X , see Properties, Lemma 22.1. Since the formation of $R^1 f_*$ is local on Y (Cohomology, Section 7) we conclude that $R^1(U \rightarrow V)_* \mathcal{I} = 0$ by the assumption in the lemma. Hence by the Leray Spectral sequence (Cohomology, Lemma 13.4) we conclude that $H^1(U, \mathcal{I}) = H^1(V, (U \rightarrow V)_* \mathcal{I})$. Since $(U \rightarrow V)_* \mathcal{I}$ is quasi-coherent by Schemes, Lemma 24.1, we have $H^1(V, (U \rightarrow V)_* \mathcal{I}) = 0$ by Lemma 2.2. Thus we find that U is affine by Lemma 3.1. \square

4. Quasi-coherence of higher direct images

We have seen that the higher cohomology groups of a quasi-coherent module on an affine are zero. For (quasi-)separated quasi-compact schemes X this implies vanishing of cohomology groups of quasi-coherent sheaves beyond a certain degree. However, it may not be the case that X has finite cohomological dimension, because that is defined in terms of vanishing of cohomology of *all* \mathcal{O}_X -modules.

Lemma 4.1 (Induction Principle). *Let X be a quasi-compact and quasi-separated scheme. Let P be a property of the quasi-compact opens of X . Assume that*

- (1) *P holds for every affine open of X ,*
- (2) *if U is quasi-compact open, V affine open, P holds for U , V , and $U \cap V$, then P holds for $U \cup V$.*

Then P holds for every quasi-compact open of X and in particular for X .

Proof. First we argue by induction that P holds for *separated* quasi-compact opens $W \subset X$. Namely, such an open can be written as $W = U_1 \cup \dots \cup U_n$ and we can do induction on n using property (2) with $U = U_1 \cup \dots \cup U_{n-1}$ and $V = U_n$. This is allowed because $U \cap V = (U_1 \cap U_n) \cup \dots \cup (U_{n-1} \cap U_n)$ is also a union of $n-1$ affine open subschemes by Schemes, Lemma 21.7 applied to the affine opens U_i and U_n of W . Having said this, for any quasi-compact open $W \subset X$ we can do induction on the number of affine opens needed to cover W using the same trick as before and using that the quasi-compact open $U_i \cap U_n$ is separated as an open subscheme of the affine scheme U_n . \square

Lemma 4.2. *Let X be a quasi-compact scheme with affine diagonal (for example if X is separated). Let $t = t(X)$ be the minimal number of affine opens needed to cover X . Then $H^n(X, \mathcal{F}) = 0$ for all $n \geq t$ and all quasi-coherent sheaves \mathcal{F} .*

Proof. First proof. By induction on t . If $t = 1$ the result follows from Lemma 2.2. If $t > 1$ write $X = U \cup V$ with V affine open and $U = U_1 \cup \dots \cup U_{t-1}$ a union of $t-1$ open affines. Note that in this case $U \cap V = (U_1 \cap V) \cup \dots \cup (U_{t-1} \cap V)$ is also a union

of $t-1$ affine open subschemes. Namely, since the diagonal is affine, the intersection of two affine opens is affine, see Lemma 2.5. We apply the Mayer-Vietoris long exact sequence

$$0 \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \rightarrow H^0(U \cap V, \mathcal{F}) \rightarrow H^1(X, \mathcal{F}) \rightarrow \dots$$

see Cohomology, Lemma 8.2. By induction we see that the groups $H^i(U, \mathcal{F})$, $H^i(V, \mathcal{F})$, $H^i(U \cap V, \mathcal{F})$ are zero for $i \geq t-1$. It follows immediately that $H^i(X, \mathcal{F})$ is zero for $i \geq t$.

Second proof. Let $\mathcal{U} : X = \bigcup_{i=1}^t U_i$ be a finite affine open covering. Since X has affine diagonal the multiple intersections $U_{i_0 \dots i_p}$ are all affine, see Lemma 2.5. By Lemma 2.6 the Čech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ agree with the cohomology groups. By Cohomology, Lemma 23.6 the Čech cohomology groups may be computed using the alternating Čech complex $\check{C}_{alt}^\bullet(\mathcal{U}, \mathcal{F})$. As the covering consists of t elements we see immediately that $\check{C}_{alt}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p \geq t$. Hence the result follows. \square

Lemma 4.3. *Let X be a quasi-compact scheme with affine diagonal (for example if X is separated). Then*

- (1) *given a quasi-coherent \mathcal{O}_X -module \mathcal{F} there exists an embedding $\mathcal{F} \rightarrow \mathcal{F}'$ of quasi-coherent \mathcal{O}_X -modules such that $H^p(X, \mathcal{F}') = 0$ for all $p \geq 1$, and*
- (2) *$\{H^n(X, -)\}_{n \geq 0}$ is a universal δ -functor from $QCoh(\mathcal{O}_X)$ to Ab .*

Proof. Let $X = \bigcup U_i$ be a finite affine open covering. Set $U = \coprod U_i$ and denote $j : U \rightarrow X$ the morphism inducing the given open immersions $U_i \rightarrow X$. Since U is an affine scheme and X has affine diagonal, the morphism j is affine, see Morphisms, Lemma 11.11. For every \mathcal{O}_X -module \mathcal{F} there is a canonical map $\mathcal{F} \rightarrow j_* j^* \mathcal{F}$. This map is injective as can be seen by checking on stalks: if $x \in U_i$, then we have a factorization

$$\mathcal{F}_x \rightarrow (j_* j^* \mathcal{F})_x \rightarrow (j^* \mathcal{F})_{x'} = \mathcal{F}_x$$

where $x' \in U$ is the point x viewed as a point of $U_i \subset U$. Now if \mathcal{F} is quasi-coherent, then $j^* \mathcal{F}$ is quasi-coherent on the affine scheme U hence has vanishing higher cohomology by Lemma 2.2. Then $H^p(X, j_* j^* \mathcal{F}) = 0$ for $p > 0$ by Lemma 2.4 as j is affine. This proves (1). Finally, we see that the map $H^p(X, \mathcal{F}) \rightarrow H^p(X, j_* j^* \mathcal{F})$ is zero and part (2) follows from Homology, Lemma 12.4. \square

Lemma 4.4. *Let X be a quasi-compact quasi-separated scheme. Let $X = U_1 \cup \dots \cup U_t$ be an open covering with each U_i quasi-compact and separated (for example affine). Set*

$$d = \max_{I \subset \{1, \dots, t\}} \left(|I| + t \left(\bigcap_{i \in I} U_i \right) - 1 \right)$$

where $t(U)$ is the minimal number of affines needed to cover the scheme U . Then $H^n(X, \mathcal{F}) = 0$ for all $n \geq d$ and all quasi-coherent sheaves \mathcal{F} .

Proof. Note that since X is quasi-separated and U_i quasi-compact the numbers $t(\bigcap_{i \in I} U_i)$ are finite. Proof using induction on t . If $t = 1$ then the result follows from Lemma 4.2. If $t > 1$, write $X = U \cup V$ with $U = U_1 \cup \dots \cup U_{t-1}$ and $V = U_t$. We apply the Mayer-Vietoris long exact sequence

$$0 \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \rightarrow H^0(U \cap V, \mathcal{F}) \rightarrow H^1(X, \mathcal{F}) \rightarrow \dots$$

see Cohomology, Lemma 8.2. Since V is affine, we have $H^i(V, \mathcal{F}) = 0$ for $i \geq 0$. By induction hypothesis we have $H^i(U, \mathcal{F}) = 0$ for

$$i \geq \max_{I \subset \{1, \dots, t-1\}} \left(|I| + t \left(\bigcap_{i \in I} U_i \right) - 1 \right)$$

and the bound on the right is less than the bound in the statement of the lemma. Finally we may use our induction hypothesis for the open $U \cap V = (U_1 \cap U_t) \cup \dots \cup (U_{t-1} \cap U_t)$ to get the vanishing of $H^i(U \cap V, \mathcal{F}) = 0$ for

$$i \geq \max_{I \subset \{1, \dots, t-1\}} \left(|I| + t \left(U_t \cap \bigcap_{i \in I} U_i \right) - 1 \right)$$

Since the bound on the right is at least 1 less than the bound in the statement of the lemma, the lemma follows. \square

Lemma 4.5. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact.*

- (1) *For any quasi-coherent \mathcal{O}_X -module \mathcal{F} the higher direct images $R^p f_* \mathcal{F}$ are quasi-coherent on S .*
- (2) *If S is quasi-compact, there exists an integer $n = n(X, S, f)$ such that $R^p f_* \mathcal{F} = 0$ for all $p \geq n$ and any quasi-coherent sheaf \mathcal{F} on X .*
- (3) *In fact, if S is quasi-compact we can find $n = n(X, S, f)$ such that for every morphism of schemes $S' \rightarrow S$ we have $R^p(f')_* \mathcal{F}' = 0$ for $p \geq n$ and any quasi-coherent sheaf \mathcal{F}' on X' . Here $f' : X' = S' \times_S X \rightarrow S'$ is the base change of f .*

Proof. We first prove (1). Note that under the hypotheses of the lemma the sheaf $R^0 f_* \mathcal{F} = f_* \mathcal{F}$ is quasi-coherent by Schemes, Lemma 24.1. Using Cohomology, Lemma 7.4 we see that forming higher direct images commutes with restriction to open subschemes. Since being quasi-coherent is local on S we reduce to the case discussed in the next paragraph.

Proof of (1) in case S is affine. We will use the induction principle. Since f quasi-compact and quasi-separated we see that X is quasi-compact and quasi-separated. For $U \subset X$ quasi-compact open and $a = f|_U$ we let $P(U)$ be the property that $R^p a_* \mathcal{F}$ is quasi-coherent on S for all quasi-coherent modules \mathcal{F} on U and all $p \geq 0$. Since $P(X)$ is (1), it suffices to prove conditions (1) and (2) of Lemma 4.1 hold. If U is affine, then $P(U)$ holds because $R^p a_* \mathcal{F} = 0$ for $p \geq 1$ (by Lemma 2.3 and Morphisms, Lemma 11.12) and we've already observed the result holds for $p = 0$ in the first paragraph. Next, let $U \subset X$ be a quasi-compact open, $V \subset X$ an affine open, and assume $P(U)$, $P(V)$, $P(U \cap V)$ hold. Let $a = f|_U$, $b = f|_V$, $c = f|_{U \cap V}$, and $g = f|_{U \cup V}$. Then for any quasi-coherent $\mathcal{O}_{U \cup V}$ -module \mathcal{F} we have the relative Mayer-Vietoris sequence

$$0 \rightarrow g_* \mathcal{F} \rightarrow a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \rightarrow c_*(\mathcal{F}|_{U \cap V}) \rightarrow R^1 g_* \mathcal{F} \rightarrow \dots$$

see Cohomology, Lemma 8.3. By $P(U)$, $P(V)$, $P(U \cap V)$ we see that $R^p a_*(\mathcal{F}|_U)$, $R^p b_*(\mathcal{F}|_V)$ and $R^p c_*(\mathcal{F}|_{U \cap V})$ are all quasi-coherent. Using the results on quasi-coherent sheaves in Schemes, Section 24 this implies that each of the sheaves $R^p g_* \mathcal{F}$ is quasi-coherent since it sits in the middle of a short exact sequence with a cokernel of a map between quasi-coherent sheaves on the left and a kernel of a map between quasi-coherent sheaves on the right. Whence $P(U \cup V)$ and the proof of (1) is complete.

Next, we prove (3) and a fortiori (2). Choose a finite affine open covering $S = \bigcup_{j=1, \dots, m} S_j$. For each j choose a finite affine open covering $f^{-1}(S_j) = \bigcup_{i=1, \dots, t_j} U_{ji}$. Let

$$d_j = \max_{I \subset \{1, \dots, t_j\}} \left(|I| + t \left(\bigcap_{i \in I} U_{ji} \right) \right)$$

be the integer found in Lemma 4.4. We claim that $n(X, S, f) = \max d_j$ works.

Namely, let $S' \rightarrow S$ be a morphism of schemes and let \mathcal{F}' be a quasi-coherent sheaf on $X' = S' \times_S X$. We want to show that $R^p f'_* \mathcal{F}' = 0$ for $p \geq n(X, S, f)$. Since this question is local on S' we may assume that S' is affine and maps into S_j for some j . Then $X' = S' \times_{S_j} f^{-1}(S_j)$ is covered by the open affines $S' \times_{S_j} U_{ji}$, $i = 1, \dots, t_j$ and the intersections

$$\bigcap_{i \in I} S' \times_{S_j} U_{ji} = S' \times_{S_j} \bigcap_{i \in I} U_{ji}$$

are covered by the same number of affines as before the base change. Applying Lemma 4.4 we get $H^p(X', \mathcal{F}') = 0$. By the first part of the proof we already know that each $R^q f'_* \mathcal{F}'$ is quasi-coherent hence has vanishing higher cohomology groups on our affine scheme S' , thus we see that $H^0(S', R^p f'_* \mathcal{F}') = H^p(X', \mathcal{F}') = 0$ by Cohomology, Lemma 13.6. Since $R^p f'_* \mathcal{F}'$ is quasi-coherent we conclude that $R^p f'_* \mathcal{F}' = 0$. \square

Lemma 4.6. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact. Assume S is affine. For any quasi-coherent \mathcal{O}_X -module \mathcal{F} we have*

$$H^q(X, \mathcal{F}) = H^0(S, R^q f_* \mathcal{F})$$

for all $q \in \mathbf{Z}$.

Proof. Consider the Leray spectral sequence $E_2^{p,q} = H^p(S, R^q f_* \mathcal{F})$ converging to $H^{p+q}(X, \mathcal{F})$, see Cohomology, Lemma 13.4. By Lemma 4.5 we see that the sheaves $R^q f_* \mathcal{F}$ are quasi-coherent. By Lemma 2.2 we see that $E_2^{p,q} = 0$ when $p > 0$. Hence the spectral sequence degenerates at E_2 and we win. See also Cohomology, Lemma 13.6 (2) for the general principle. \square

5. Cohomology and base change, I

Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Suppose further that $g : S' \rightarrow S$ is any morphism of schemes. Denote $X' = X_{S'} = S' \times_S X$ the base change of X and denote $f' : X' \rightarrow S'$ the base change of f . Also write $g' : X' \rightarrow X$ the projection, and set $\mathcal{F}' = (g')^* \mathcal{F}$. Here is a diagram representing the situation:

$$(5.0.1) \quad \begin{array}{ccccc} \mathcal{F}' = (g')^* \mathcal{F} & & X' & \xrightarrow{g'} & X & & \mathcal{F} \\ & & \downarrow f' & & \downarrow f & & \\ & & S' & \xrightarrow{g} & S & & Rf_* \mathcal{F} \\ & Rf'_* \mathcal{F}' & & & & & \end{array}$$

Here is the simplest case of the base change property we have in mind.

Lemma 5.1. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Assume f is affine. In this case $f_* \mathcal{F} \cong Rf_* \mathcal{F}$ is a quasi-coherent sheaf, and for every base change diagram (5.0.1) we have*

$$g^* f_* \mathcal{F} = f'_* (g')^* \mathcal{F}.$$

Proof. The vanishing of higher direct images is Lemma 2.3. The statement is local on S and S' . Hence we may assume $X = \operatorname{Spec}(A)$, $S = \operatorname{Spec}(R)$, $S' = \operatorname{Spec}(R')$ and $\mathcal{F} = \widetilde{M}$ for some A -module M . We use Schemes, Lemma 7.3 to describe pullbacks and pushforwards of \mathcal{F} . Namely, $X' = \operatorname{Spec}(R' \otimes_R A)$ and \mathcal{F}' is the quasi-coherent sheaf associated to $(R' \otimes_R A) \otimes_A M$. Thus we see that the lemma boils down to the equality

$$(R' \otimes_R A) \otimes_A M = R' \otimes_R M$$

as R' -modules. \square

In many situations it is sufficient to know about the following special case of cohomology and base change. It follows immediately from the stronger results in Section 7, but since it is so important it deserves its own proof.

Lemma 5.2 (Flat base change). *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module with pullback $\mathcal{F}' = (g')^* \mathcal{F}$. Assume that g is flat and that f is quasi-compact and quasi-separated. For any $i \geq 0$

- (1) *the base change map of Cohomology, Lemma 17.1 is an isomorphism*

$$g^* R^i f_* \mathcal{F} \longrightarrow R^i f'_* \mathcal{F}',$$

- (2) *if $S = \operatorname{Spec}(A)$ and $S' = \operatorname{Spec}(B)$, then $H^i(X, \mathcal{F}) \otimes_A B = H^i(X', \mathcal{F}')$.*

Proof. Using Cohomology, Lemma 17.1 in (1) is allowed since g' is flat by Morphisms, Lemma 25.8. Having said this, part (1) follows from part (2). Namely, part (1) is local on S' and hence we may assume S and S' are affine. In other words, we have $S = \operatorname{Spec}(A)$ and $S' = \operatorname{Spec}(B)$ as in (2). Then since $R^i f_* \mathcal{F}$ is quasi-coherent (Lemma 4.5), it is the quasi-coherent \mathcal{O}_S -module associated to the A -module $H^0(S, R^i f_* \mathcal{F}) = H^i(X, \mathcal{F})$ (equality by Lemma 4.6). Similarly, $R^i f'_* \mathcal{F}'$ is the quasi-coherent $\mathcal{O}_{S'}$ -module associated to the B -module $H^i(X', \mathcal{F}')$. Since pullback by g corresponds to $- \otimes_A B$ on modules (Schemes, Lemma 7.3) we see that it suffices to prove (2).

Let $A \rightarrow B$ be a flat ring homomorphism. Let X be a quasi-compact and quasi-separated scheme over A . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Set $X_B = X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B)$ and denote \mathcal{F}_B the pullback of \mathcal{F} . We are trying to show that the map

$$H^i(X, \mathcal{F}) \otimes_A B \longrightarrow H^i(X_B, \mathcal{F}_B)$$

(given by the reference in the statement of the lemma) is an isomorphism.

In case X is separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \dots \cup U_t$ and recall that

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F}),$$

see Lemma 2.6. If $\mathcal{U}_B : X_B = (U_1)_B \cup \dots \cup (U_t)_B$ we obtain by base change, then it is still the case that each $(U_i)_B$ is affine and that X_B is separated. Thus we obtain

$$\check{H}^p(\mathcal{U}_B, \mathcal{F}_B) = H^p(X_B, \mathcal{F}_B).$$

We have the following relation between the Čech complexes

$$\check{C}^\bullet(\mathcal{U}_B, \mathcal{F}_B) = \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B$$

as follows from Lemma 5.1. Since $A \rightarrow B$ is flat, the same thing remains true on taking cohomology.

In case X is quasi-separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \dots \cup U_t$. We will use the Čech-to-cohomology spectral sequence Cohomology, Lemma 11.5. The reader who wishes to avoid this spectral sequence can use Mayer-Vietoris and induction on t as in the proof of Lemma 4.5. The spectral sequence has E_2 -page $E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ and converges to $H^{p+q}(X, \mathcal{F})$. Similarly, we have a spectral sequence with E_2 -page $E_2^{p,q} = \check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B))$ which converges to $H^{p+q}(X_B, \mathcal{F}_B)$. Since the intersections $U_{i_0 \dots i_p}$ are quasi-compact and separated, the result of the second paragraph of the proof gives $\check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B)) = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \otimes_A B$. Using that $A \rightarrow B$ is flat we conclude that $H^i(X, \mathcal{F}) \otimes_A B \rightarrow H^i(X_B, \mathcal{F}_B)$ is an isomorphism for all i and we win. \square

Lemma 5.3 (Finite locally free base change). *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} Y & \xrightarrow{h} & X \\ g \downarrow & & \downarrow f \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module with pullback $\mathcal{G} = h^*\mathcal{F}$. If B is a finite locally free A -module, then $H^i(X, \mathcal{F}) \otimes_A B = H^i(Y, \mathcal{G})$.

Warning: Do not use this lemma unless you understand the difference between this and Lemma 5.2.

Proof. In case X is separated, choose an affine open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and recall that

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F}),$$

see Lemma 2.6. Let $\mathcal{V} : Y = \bigcup_{i \in I} g^{-1}(U_i)$ be the corresponding affine open covering of Y . The opens $V_i = g^{-1}(U_i) = U_i \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ are affine and Y is separated. Thus we obtain

$$\check{H}^p(\mathcal{V}, \mathcal{G}) = H^p(Y, \mathcal{G}).$$

We claim the map of Čech complexes

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B \longrightarrow \check{C}^\bullet(\mathcal{V}, \mathcal{G})$$

is an isomorphism. Namely, as B is finitely presented as an A -module we see that tensoring with B over A commutes with products, see Algebra, Proposition 89.3. Thus it suffices to show that the maps $\Gamma(U_{i_0 \dots i_p}, \mathcal{F}) \otimes_A B \rightarrow \Gamma(V_{i_0 \dots i_p}, \mathcal{G})$ are isomorphisms which follows from Lemma 5.1. Since $A \rightarrow B$ is flat, the same thing remains true on taking cohomology.

In the general case we argue in exactly the same way using affine open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and the corresponding covering $\mathcal{V} : Y = \bigcup_{i \in I} V_i$ with $V_i = g^{-1}(U_i)$ as above. We will use the Čech-to-cohomology spectral sequence Cohomology, Lemma 11.5. The spectral sequence has E_2 -page $E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$

and converges to $H^{p+q}(X, \mathcal{F})$. Similarly, we have a spectral sequence with E_2 -page $E_2^{p,q} = \check{H}^p(\mathcal{V}, \underline{H}^q(\mathcal{G}))$ which converges to $H^{p+q}(Y, \mathcal{G})$. Since the intersections $U_{i_0 \dots i_p}$ are separated, the result of the previous paragraph gives isomorphisms $\Gamma(U_{i_0 \dots i_p}, \underline{H}^q(\mathcal{F})) \otimes_A B \rightarrow \Gamma(U_{i_0 \dots i_p}, \underline{H}^q(\mathcal{G}))$. Using that $- \otimes_A B$ commutes with products and is exact, we conclude that $\check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \otimes_A B \rightarrow \check{H}^p(\mathcal{V}, \underline{H}^q(\mathcal{G}))$ is an isomorphism. Using that $A \rightarrow B$ is flat we conclude that $H^i(X, \mathcal{F}) \otimes_A B \rightarrow H^i(Y, \mathcal{G})$ is an isomorphism for all i and we win. \square

6. Colimits and higher direct images

General results of this nature can be found in Cohomology, Section 19, Sheaves, Lemma 29.1, and Modules, Lemma 22.8.

Lemma 6.1. *Let $f : X \rightarrow S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F} = \text{colim } \mathcal{F}_i$ be a filtered colimit of quasi-coherent sheaves on X . Then for any $p \geq 0$ we have*

$$R^p f_* \mathcal{F} = \text{colim } R^p f_* \mathcal{F}_i.$$

Proof. Recall that $R^p f_* \mathcal{F}$ is the sheaf associated to $U \mapsto H^p(f^{-1}U, \mathcal{F})$, see Cohomology, Lemma 7.3. Recall that the colimit is the sheaf associated to the presheaf colimit (taking colimits over opens). Hence we can apply Cohomology, Lemma 19.1 to $H^p(f^{-1}U, -)$ where U is affine to conclude. (Because the basis of affine opens in $f^{-1}U$ satisfies the assumptions of that lemma.) \square

7. Cohomology and base change, II

Let $f : X \rightarrow S$ be a morphism of schemes and let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. If f is quasi-compact and quasi-separated we would like to represent $Rf_* \mathcal{F}$ by a complex of quasi-coherent sheaves on S . This follows from the fact that the sheaves $R^i f_* \mathcal{F}$ are quasi-coherent if S is quasi-compact and has affine diagonal, using that $D_{QCoh}(S)$ is equivalent to $D(QCoh(\mathcal{O}_S))$, see Derived Categories of Schemes, Proposition 7.5.

In this section we will use a different approach which produces an explicit complex having a good base change property. The construction is particularly easy if f and S are separated, or more generally have affine diagonal. Since this is the case which by far the most often used we treat it separately.

Lemma 7.1. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Assume X is quasi-compact and X and S have affine diagonal (e.g., if X and S are separated). In this case we can compute $Rf_* \mathcal{F}$ as follows:*

- (1) Choose a finite affine open covering $\mathcal{U} : X = \bigcup_{i=1, \dots, n} U_i$.
- (2) For $i_0, \dots, i_p \in \{1, \dots, n\}$ denote $f_{i_0 \dots i_p} : U_{i_0 \dots i_p} \rightarrow S$ the restriction of f to the intersection $U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$.
- (3) Set $\mathcal{F}_{i_0 \dots i_p}$ equal to the restriction of \mathcal{F} to $U_{i_0 \dots i_p}$.
- (4) Set

$$\check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) = \bigoplus_{i_0 \dots i_p} f_{i_0 \dots i_p}^* \mathcal{F}_{i_0 \dots i_p}$$

and define differentials $d : \check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) \rightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, f, \mathcal{F})$ as in Cohomology, Equation (9.0.1).

Then the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ is a complex of quasi-coherent sheaves on S which comes equipped with an isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf_*\mathcal{F}$$

in $D^+(S)$. This isomorphism is functorial in the quasi-coherent sheaf \mathcal{F} .

Proof. Consider the resolution $\mathcal{F} \rightarrow \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of Cohomology, Lemma 24.1. We have an equality of complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = f_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of quasi-coherent \mathcal{O}_S -modules. The morphisms $j_{i_0\dots i_p} : U_{i_0\dots i_p} \rightarrow X$ and the morphisms $f_{i_0\dots i_p} : U_{i_0\dots i_p} \rightarrow S$ are affine by Morphisms, Lemma 11.11 and Lemma 2.5. Hence $R^q j_{i_0\dots i_p*} \mathcal{F}_{i_0\dots i_p}$ as well as $R^q f_{i_0\dots i_p*} \mathcal{F}_{i_0\dots i_p}$ are zero for $q > 0$ (Lemma 2.3). Using $f \circ j_{i_0\dots i_p} = f_{i_0\dots i_p}$ and the spectral sequence of Cohomology, Lemma 13.8 we conclude that $R^q f_*(j_{i_0\dots i_p*} \mathcal{F}_{i_0\dots i_p}) = 0$ for $q > 0$. Since the terms of the complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ are finite direct sums of the sheaves $j_{i_0\dots i_p*} \mathcal{F}_{i_0\dots i_p}$ we conclude using Leray's acyclicity lemma (Derived Categories, Lemma 16.7) that

$$Rf_*\mathcal{F} = f_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$$

as desired. \square

Next, we are going to consider what happens if we do a base change.

Lemma 7.2. *With notation as in diagram (5.0.1). Assume $f : X \rightarrow S$ and \mathcal{F} satisfy the hypotheses of Lemma 7.1. Choose a finite affine open covering $\mathcal{U} : X = \bigcup U_i$ of X . There is a canonical isomorphism*

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf'_*\mathcal{F}'$$

in $D^+(S')$. Moreover, if $S' \rightarrow S$ is affine, then in fact

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$

with $\mathcal{U}' : X' = \bigcup U'_i$ where $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ is also affine.

Proof. In fact we may define $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ no matter whether S' is affine over S or not. Let $\mathcal{U}' : X' = \bigcup U'_i$ be the induced covering of X' . In this case we claim that

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$

with $\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$ defined in exactly the same manner as in Lemma 7.1. This is clear from the case of affine morphisms (Lemma 5.1) by working locally on S' . Moreover, exactly as in the proof of Lemma 7.1 one sees that there is an isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}') \longrightarrow Rf'_*\mathcal{F}'$$

in $D^+(S')$ since the morphisms $U'_i \rightarrow X'$ and $U'_i \rightarrow S'$ are still affine (being base changes of affine morphisms). Details omitted. \square

The lemma above says that the complex

$$\mathcal{K}^\bullet = \check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$$

is a bounded below complex of quasi-coherent sheaves on S which *universally* computes the higher direct images of $f : X \rightarrow S$. This is something about this particular complex and it is not preserved by replacing $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ by a quasi-isomorphic complex in general! In other words, this is not a statement that makes sense in the derived category. The reason is that the pullback $g^*\mathcal{K}^\bullet$ is *not* equal to the derived pullback $Lg^*\mathcal{K}^\bullet$ of \mathcal{K}^\bullet in general!

Here is a more general case where we can prove this statement. We remark that the condition of S being separated is harmless in most applications, since this is usually used to prove some local property of the total derived image. The proof is significantly more involved and uses hypercoverings; it is a nice example of how you can use them sometimes.

Lemma 7.3. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Assume that f is quasi-compact and quasi-separated and that S is quasi-compact and separated. There exists a bounded below complex \mathcal{K}^\bullet of quasi-coherent \mathcal{O}_S -modules with the following property: For every morphism $g : S' \rightarrow S$ the complex $g^*\mathcal{K}^\bullet$ is a representative for $Rf'_*\mathcal{F}'$ with notation as in diagram (5.0.1).*

Proof. (If f is separated as well, please see Lemma 7.2.) The assumptions imply in particular that X is quasi-compact and quasi-separated as a scheme. Let \mathcal{B} be the set of affine opens of X . By Hypercoverings, Lemma 11.4 we can find a hypercovering $K = (I, \{U_i\})$ such that each I_n is finite and each U_i is an affine open of X . By Hypercoverings, Lemma 5.3 there is a spectral sequence with E_2 -page

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. Note that $\check{H}^p(K, \underline{H}^q(\mathcal{F}))$ is the p th cohomology group of the complex

$$\prod_{i \in I_0} H^q(U_i, \mathcal{F}) \rightarrow \prod_{i \in I_1} H^q(U_i, \mathcal{F}) \rightarrow \prod_{i \in I_2} H^q(U_i, \mathcal{F}) \rightarrow \dots$$

Since each U_i is affine we see that this is zero unless $q = 0$ in which case we obtain

$$\prod_{i \in I_0} \mathcal{F}(U_i) \rightarrow \prod_{i \in I_1} \mathcal{F}(U_i) \rightarrow \prod_{i \in I_2} \mathcal{F}(U_i) \rightarrow \dots$$

Thus we conclude that $R\Gamma(X, \mathcal{F})$ is computed by this complex.

For any n and $i \in I_n$ denote $f_i : U_i \rightarrow S$ the restriction of f to U_i . As S is separated and U_i is affine this morphism is affine. Consider the complex of quasi-coherent sheaves

$$\mathcal{K}^\bullet = (\prod_{i \in I_0} f_{i,*}\mathcal{F}|_{U_i} \rightarrow \prod_{i \in I_1} f_{i,*}\mathcal{F}|_{U_i} \rightarrow \prod_{i \in I_2} f_{i,*}\mathcal{F}|_{U_i} \rightarrow \dots)$$

on S . As in Hypercoverings, Lemma 5.3 we obtain a map $\mathcal{K}^\bullet \rightarrow Rf_*\mathcal{F}$ in $D(\mathcal{O}_S)$ by choosing an injective resolution of \mathcal{F} (details omitted). Consider any affine scheme V and a morphism $g : V \rightarrow S$. Then the base change X_V has a hypercovering $K_V = (I, \{U_{i,V}\})$ obtained by base change. Moreover, $g^*f_{i,*}\mathcal{F} = f_{i,V,*}(g')^*\mathcal{F}|_{U_{i,V}}$. Thus the arguments above prove that $\Gamma(V, g^*\mathcal{K}^\bullet)$ computes $R\Gamma(X_V, (g')^*\mathcal{F})$. This finishes the proof of the lemma as it suffices to prove the equality of complexes Zariski locally on S' . \square

The following lemma is a variant to flat base change.

Lemma 7.4. *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let \mathcal{G} be a quasi-coherent $\mathcal{O}_{S'}$ -module flat over S . Assume f is quasi-compact and quasi-separated. For any $i \geq 0$ there is an identification

$$\mathcal{G} \otimes_{\mathcal{O}_{S'}} g^* R^i f_* \mathcal{F} = R^i f'_* ((f')^* \mathcal{G} \otimes_{\mathcal{O}_{X'}} (g')^* \mathcal{F})$$

Proof. Let us construct a map from left to right. First, we have the base change map $Lg^* Rf_* \mathcal{F} \rightarrow Rf'_* L(g')^* \mathcal{F}$. There is also the adjunction map $\mathcal{G} \rightarrow Rf'_* L(f')^* \mathcal{G}$. Using the relative cup product we obtain

$$\begin{aligned} \mathcal{G} \otimes_{\mathcal{O}_{S'}}^{\mathbf{L}} Lg^* Rf_* \mathcal{F} &\rightarrow Rf'_* L(f')^* \mathcal{G} \otimes_{\mathcal{O}_{S'}}^{\mathbf{L}} Rf'_* L(g')^* \mathcal{F} \\ &\rightarrow Rf'_* \left(L(f')^* \mathcal{G} \otimes_{\mathcal{O}_{X'}}^{\mathbf{L}} L(g')^* \mathcal{F} \right) \\ &\rightarrow Rf'_* ((f')^* \mathcal{G} \otimes_{\mathcal{O}_{X'}} (g')^* \mathcal{F}) \end{aligned}$$

where for the middle arrow we used the relative cup product, see Cohomology, Remark 28.7. The source of the composition is

$$\mathcal{G} \otimes_{\mathcal{O}_{S'}}^{\mathbf{L}} Lg^* Rf_* \mathcal{F} = \mathcal{G} \otimes_{g^{-1}\mathcal{O}_S}^{\mathbf{L}} g^{-1} Rf_* \mathcal{F}$$

by Cohomology, Lemma 27.4. Since \mathcal{G} is flat as a sheaf of $g^{-1}\mathcal{O}_S$ -modules and since g^{-1} is an exact functor, this is a complex whose i th cohomology sheaf is $\mathcal{G} \otimes_{g^{-1}\mathcal{O}_S} g^{-1} R^i f_* \mathcal{F} = \mathcal{G} \otimes_{\mathcal{O}_{S'}} g^* R^i f_* \mathcal{F}$. In this way we obtain global maps from left to right in the equality of the lemma. To show this map is an isomorphism we may work locally on S' . Thus we may and do assume that S and S' are affine schemes.

Proof in case S and S' are affine. Say $S = \text{Spec}(A)$ and $S' = \text{Spec}(B)$ and say \mathcal{G} corresponds to the B -module N which is assumed to be A -flat. Since S is affine, X is quasi-compact and quasi-separated. We will use a hypercovering argument to finish the proof; if X is separated or has affine diagonal, then you can use a Čech covering. Let \mathcal{B} be the set of affine opens of X . By Hypercoverings, Lemma 11.4 we can find a hypercovering $K = (I, \{U_i\})$ of X such that each I_n is finite and each U_i is an affine open of X . By Hypercoverings, Lemma 5.3 there is a spectral sequence with E_2 -page

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. Since each U_i is affine and \mathcal{F} is quasi-coherent the value of $\underline{H}^q(\mathcal{F})$ is zero on U_i for $q > 0$. Thus the spectral sequence degenerates and we conclude that the cohomology modules $H^q(X, \mathcal{F})$ are computed by

$$\prod_{i \in I_0} \mathcal{F}(U_i) \rightarrow \prod_{i \in I_1} \mathcal{F}(U_i) \rightarrow \prod_{i \in I_2} \mathcal{F}(U_i) \rightarrow \dots$$

Next, note that the base change of our hypercovering to S' is a hypercovering of $X' = S' \times_S X$. The schemes $S' \times_S U_i$ are affine too and we have

$$((f')^* \mathcal{G} \otimes_{\mathcal{O}_{S'}} (g')^* \mathcal{F})(S' \times_S U_i) = N \otimes_A \mathcal{F}(U_i)$$

In this way we conclude that the cohomology modules $H^q(X', (f')^* \mathcal{G} \otimes_{\mathcal{O}_{S'}} (g')^* \mathcal{F})$ are computed by

$$N \otimes_A \left(\prod_{i \in I_0} \mathcal{F}(U_i) \rightarrow \prod_{i \in I_1} \mathcal{F}(U_i) \rightarrow \prod_{i \in I_2} \mathcal{F}(U_i) \rightarrow \dots \right)$$

Since N is flat over A , we conclude that

$$H^q(X', (f')^* \mathcal{G} \otimes_{\mathcal{O}_{S'}} (g')^* \mathcal{F}) = N \otimes_A H^q(X, \mathcal{F})$$

Since this is the translation into algebra of the statement we had to show the proof is complete. \square

8. Cohomology of projective space

In this section we compute the cohomology of the twists of the structure sheaf on \mathbf{P}_S^n over a scheme S . Recall that \mathbf{P}_S^n was defined as the fibre product $\mathbf{P}_S^n = S \times_{\text{Spec}(\mathbf{Z})} \mathbf{P}_{\mathbf{Z}}^n$ in Constructions, Definition 13.2. It was shown to be equal to

$$\mathbf{P}_S^n = \text{Proj}_S(\mathcal{O}_S[T_0, \dots, T_n])$$

in Constructions, Lemma 21.5. In particular, projective space is a particular case of a projective bundle. If $S = \text{Spec}(R)$ is affine then we have

$$\mathbf{P}_S^n = \mathbf{P}_R^n = \text{Proj}(R[T_0, \dots, T_n]).$$

All these identifications are compatible and compatible with the constructions of the twisted structure sheaves $\mathcal{O}_{\mathbf{P}_S^n}(d)$.

Before we state the result we need some notation. Let R be a ring. Recall that $R[T_0, \dots, T_n]$ is a graded R -algebra where each T_i is homogeneous of degree 1. Denote $(R[T_0, \dots, T_n])_d$ the degree d summand. It is a finite free R -module of rank $\binom{n+d}{d}$ when $d \geq 0$ and zero else. It has a basis consisting of monomials $T_0^{e_0} \dots T_n^{e_n}$ with $\sum e_i = d$. We will also use the following notation: $R[\frac{1}{T_0}, \dots, \frac{1}{T_n}]$ denotes the \mathbf{Z} -graded ring with $\frac{1}{T_i}$ in degree -1 . In particular the \mathbf{Z} -graded $R[\frac{1}{T_0}, \dots, \frac{1}{T_n}]$ module

$$\frac{1}{T_0 \dots T_n} R[\frac{1}{T_0}, \dots, \frac{1}{T_n}]$$

which shows up in the statement below is zero in degrees $\geq -n$, is free on the generator $\frac{1}{T_0 \dots T_n}$ in degree $-n-1$ and is free of rank $(-1)^n \binom{n+d}{d}$ for $d \leq -n-1$.

Lemma 8.1. *Let R be a ring. Let $n \geq 0$ be an integer. We have*

$$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}_R^n}(d)) = \begin{cases} (R[T_0, \dots, T_n])_d & \text{if } q = 0 \\ 0 & \text{if } q \neq 0, n \\ \left(\frac{1}{T_0 \dots T_n} R[\frac{1}{T_0}, \dots, \frac{1}{T_n}] \right)_d & \text{if } q = n \end{cases}$$

as R -modules.

Proof. We will use the standard affine open covering

$$\mathcal{U} : \mathbf{P}_R^n = \bigcup_{i=0}^n D_+(T_i)$$

to compute the cohomology using the Čech complex. This is permissible by Lemma 2.6 since any intersection of finitely many affine $D_+(T_i)$ is also a standard affine open (see Constructions, Section 8). In fact, we can use the alternating or ordered Čech complex according to Cohomology, Lemmas 23.3 and 23.6.

The ordering we will use on $\{0, \dots, n\}$ is the usual one. Hence the complex we are looking at has terms

$$\check{C}_{ord}^p(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R^n}(d)) = \bigoplus_{i_0 < \dots < i_p} (R[T_0, \dots, T_n, \frac{1}{T_{i_0} \dots T_{i_p}}])_d$$

Moreover, the maps are given by the usual formula

$$d(s)_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0 \dots \hat{i}_j \dots i_{p+1}}$$

see Cohomology, Section 23. Note that each term of this complex has a natural \mathbf{Z}^{n+1} -grading. Namely, we get this by declaring a monomial $T_0^{e_0} \dots T_n^{e_n}$ to be homogeneous with weight $(e_0, \dots, e_n) \in \mathbf{Z}^{n+1}$. It is clear that the differential given above respects the grading. In a formula we have

$$\check{\mathcal{C}}_{ord}^\bullet(\mathcal{U}, \mathcal{O}_{\mathbf{P}^R}(d)) = \bigoplus_{\vec{e} \in \mathbf{Z}^{n+1}} \check{\mathcal{C}}^\bullet(\vec{e})$$

where not all summands on the right hand side occur (see below). Hence in order to compute the cohomology modules of the complex it suffices to compute the cohomology of the graded pieces and take the direct sum at the end.

Fix $\vec{e} = (e_0, \dots, e_n) \in \mathbf{Z}^{n+1}$. In order for this weight to occur in the complex above we need to assume $e_0 + \dots + e_n = d$ (if not then it occurs for a different twist of the structure sheaf of course). Assuming this, set

$$NEG(\vec{e}) = \{i \in \{0, \dots, n\} \mid e_i < 0\}.$$

With this notation the weight \vec{e} summand $\check{\mathcal{C}}^\bullet(\vec{e})$ of the Čech complex above has the following terms

$$\check{\mathcal{C}}^p(\vec{e}) = \bigoplus_{i_0 < \dots < i_p, \ NEG(\vec{e}) \subset \{i_0, \dots, i_p\}} R \cdot T_0^{e_0} \dots T_n^{e_n}$$

In other words, the terms corresponding to $i_0 < \dots < i_p$ such that $NEG(\vec{e})$ is not contained in $\{i_0 \dots i_p\}$ are zero. The differential of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is still given by the exact same formula as above.

Suppose that $NEG(\vec{e}) = \{0, \dots, n\}$, i.e., that all exponents e_i are negative. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has only one term, namely $\check{\mathcal{C}}^n(\vec{e}) = R \cdot \frac{1}{T_0^{-e_0} \dots T_n^{-e_n}}$. Hence in this case

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot \frac{1}{T_0^{-e_0} \dots T_n^{-e_n}} & \text{if } q = n \\ 0 & \text{if } \text{else} \end{cases}$$

The direct sum of all of these terms clearly gives the value

$$\left(\frac{1}{T_0 \dots T_n} R \left[\frac{1}{T_0}, \dots, \frac{1}{T_n} \right] \right)_d$$

in degree n as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

Assume $NEG(\vec{e}) = \emptyset$. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has a summand R corresponding to all $i_0 < \dots < i_p$. Let us compare the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ to another complex. Namely, consider the affine open covering

$$\mathcal{V} : \text{Spec}(R) = \bigcup_{i \in \{0, \dots, n\}} V_i$$

where $V_i = \text{Spec}(R)$ for all i . Consider the alternating Čech complex

$$\check{\mathcal{C}}_{ord}^\bullet(\mathcal{V}, \mathcal{O}_{\text{Spec}(R)})$$

By the same reasoning as above this computes the cohomology of the structure sheaf on $\text{Spec}(R)$. Hence we see that $H^p(\check{\mathcal{C}}_{ord}^\bullet(\mathcal{V}, \mathcal{O}_{\text{Spec}(R)})) = R$ if $p = 0$ and is 0 whenever $p > 0$. For these facts, see Lemma 2.1 and its proof. Note that also $\check{\mathcal{C}}_{ord}^\bullet(\mathcal{V}, \mathcal{O}_{\text{Spec}(R)})$ has a summand R for every $i_0 < \dots < i_p$ and has exactly the same

differential as $\check{\mathcal{C}}^\bullet(\vec{e})$. In other words these complexes are isomorphic complexes and hence have the same cohomology. We conclude that

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot T_0^{e_0} \dots T_n^{e_n} & \text{if } q = 0 \\ 0 & \text{if } \text{else} \end{cases}$$

in the case that $NEG(\vec{e}) = \emptyset$. The direct sum of all of these terms clearly gives the value

$$(R[T_0, \dots, T_n])_d$$

in degree 0 as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

To finish the proof of the lemma we have to show that the complexes $\check{\mathcal{C}}^\bullet(\vec{e})$ are acyclic when $NEG(\vec{e})$ is neither empty nor equal to $\{0, \dots, n\}$. Pick an index $i_{\text{fix}} \notin NEG(\vec{e})$ (such an index exists). Consider the map

$$h : \check{\mathcal{C}}^{p+1}(\vec{e}) \rightarrow \check{\mathcal{C}}^p(\vec{e})$$

given by the rule that for $i_0 < \dots < i_p$ we have

$$h(s)_{i_0 \dots i_p} = \begin{cases} 0 & \text{if } p \notin \{0, \dots, n-1\} \\ 0 & \text{if } i_{\text{fix}} \in \{i_0, \dots, i_p\} \\ s_{i_{\text{fix}} i_0 \dots i_p} & \text{if } i_{\text{fix}} < i_0 \\ (-1)^a s_{i_0 \dots i_{a-1} i_{\text{fix}} i_a \dots i_p} & \text{if } i_{a-1} < i_{\text{fix}} < i_a \\ (-1)^p s_{i_0 \dots i_p} & \text{if } i_p < i_{\text{fix}} \end{cases}$$

Please compare with the proof of Lemma 2.1. This makes sense because we have

$$NEG(\vec{e}) \subset \{i_0, \dots, i_p\} \Leftrightarrow NEG(\vec{e}) \subset \{i_{\text{fix}}, i_0, \dots, i_p\}$$

The exact same (combinatorial) computation¹ as in the proof of Lemma 2.1 shows that

$$(hd + dh)(s)_{i_0 \dots i_p} = s_{i_0 \dots i_p}$$

Hence we see that the identity map of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is homotopic to zero which implies that it is acyclic. \square

In the following lemma we are going to use the pairing of free R -modules

$$R[T_0, \dots, T_n] \times \frac{1}{T_0 \dots T_n} R\left[\frac{1}{T_0}, \dots, \frac{1}{T_n}\right] \longrightarrow R$$

which is defined by the rule

$$(f, g) \longmapsto \text{coefficient of } \frac{1}{T_0 \dots T_n} \text{ in } fg.$$

¹For example, suppose that $i_0 < \dots < i_p$ is such that $i_{\text{fix}} \notin \{i_0, \dots, i_p\}$ and that $i_{a-1} < i_{\text{fix}} < i_a$ for some $1 \leq a \leq p$. Then we have

$$\begin{aligned} & (dh + hd)(s)_{i_0 \dots i_p} \\ &= \sum_{j=0}^p (-1)^j h(s)_{i_0 \dots \hat{i}_j \dots i_p} + (-1)^a d(s)_{i_0 \dots i_{a-1} i_{\text{fix}} i_a \dots i_p} \\ &= \sum_{j=0}^{a-1} (-1)^{j+a-1} s_{i_0 \dots \hat{i}_j \dots i_{a-1} i_{\text{fix}} i_a \dots i_p} + \sum_{j=a}^p (-1)^{j+a} s_{i_0 \dots i_{a-1} i_{\text{fix}} i_a \dots \hat{i}_j \dots i_p} + \\ & \quad \sum_{j=0}^{a-1} (-1)^{a+j} s_{i_0 \dots \hat{i}_j \dots i_{a-1} i_{\text{fix}} i_a \dots i_p} + (-1)^{2a} s_{i_0 \dots i_p} + \sum_{j=a}^p (-1)^{a+j+1} s_{i_0 \dots i_{a-1} i_{\text{fix}} i_a \dots \hat{i}_j \dots i_p} \\ &= s_{i_0 \dots i_p} \end{aligned}$$

as desired. The other cases are similar.

In other words, the basis element $T_0^{e_0} \dots T_n^{e_n}$ pairs with the basis element $T_0^{d_0} \dots T_n^{d_n}$ to give 1 if and only if $e_i + d_i = -1$ for all i , and pairs to zero in all other cases. Using this pairing we get an identification

$$\left(\frac{1}{T_0 \dots T_n} R \left[\frac{1}{T_0}, \dots, \frac{1}{T_n} \right] \right)_d = \text{Hom}_R((R[T_0, \dots, T_n])_{-n-1-d}, R)$$

Thus we can reformulate the result of Lemma 8.1 as saying that

$$(8.1.1) \quad H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}_R^n}(d)) = \begin{cases} (R[T_0, \dots, T_n])_d & \text{if } q = 0 \\ 0 & \text{if } q \neq 0, n \\ \text{Hom}_R((R[T_0, \dots, T_n])_{-n-1-d}, R) & \text{if } q = n \end{cases}$$

Lemma 8.2. *The identifications of Equation (8.1.1) are compatible with base change w.r.t. ring maps $R \rightarrow R'$. Moreover, for any $f \in R[T_0, \dots, T_n]$ homogeneous of degree m the map multiplication by f*

$$\mathcal{O}_{\mathbf{P}_R^n}(d) \longrightarrow \mathcal{O}_{\mathbf{P}_R^n}(d+m)$$

induces the map on the cohomology group via the identifications of Equation (8.1.1) which is multiplication by f for H^0 and the contragredient of multiplication by f

$$(R[T_0, \dots, T_n])_{-n-1-(d+m)} \longrightarrow (R[T_0, \dots, T_n])_{-n-1-d}$$

on H^n .

Proof. Suppose that $R \rightarrow R'$ is a ring map. Let \mathcal{U} be the standard affine open covering of \mathbf{P}_R^n , and let \mathcal{U}' be the standard affine open covering of $\mathbf{P}_{R'}^n$. Note that \mathcal{U}' is the pullback of the covering \mathcal{U} under the canonical morphism $\mathbf{P}_{R'}^n \rightarrow \mathbf{P}_R^n$. Hence there is a map of Čech complexes

$$\gamma : \check{C}_{ord}^\bullet(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R^n}(d)) \longrightarrow \check{C}_{ord}^\bullet(\mathcal{U}', \mathcal{O}_{\mathbf{P}_{R'}^n}(d))$$

which is compatible with the map on cohomology by Cohomology, Lemma 15.1. It is clear from the computations in the proof of Lemma 8.1 that this map of Čech complexes is compatible with the identifications of the cohomology groups in question. (Namely the basis elements for the Čech complex over R simply map to the corresponding basis elements for the Čech complex over R' .) Whence the first statement of the lemma.

Now fix the ring R and consider two homogeneous polynomials $f, g \in R[T_0, \dots, T_n]$ both of the same degree m . Since cohomology is an additive functor, it is clear that the map induced by multiplication by $f + g$ is the same as the sum of the maps induced by multiplication by f and the map induced by multiplication by g . Moreover, since cohomology is a functor, a similar result holds for multiplication by a product fg where f, g are both homogeneous (but not necessarily of the same degree). Hence to verify the second statement of the lemma it suffices to prove this when $f = x \in R$ or when $f = T_i$. In the case of multiplication by an element $x \in R$ the result follows since every cohomology group or complex in sight has the structure of an R -module or complex of R -modules. Finally, we consider the case of multiplication by T_i as a $\mathcal{O}_{\mathbf{P}_R^n}$ -linear map

$$\mathcal{O}_{\mathbf{P}_R^n}(d) \longrightarrow \mathcal{O}_{\mathbf{P}_R^n}(d+1)$$

The statement on H^0 is clear. For the statement on H^n consider multiplication by T_i as a map on Čech complexes

$$\check{C}_{ord}^\bullet(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R^n}(d)) \longrightarrow \check{C}_{ord}^\bullet(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R^n}(d+1))$$

We are going to use the notation introduced in the proof of Lemma 8.1. We consider the effect of multiplication by T_i in terms of the decompositions

$$\check{\mathcal{C}}_{ord}^\bullet(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus_{\vec{e} \in \mathbf{Z}^{n+1}, \sum e_i = d} \check{\mathcal{C}}^\bullet(\vec{e})$$

and

$$\check{\mathcal{C}}_{ord}^\bullet(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d+1)) = \bigoplus_{\vec{e} \in \mathbf{Z}^{n+1}, \sum e_i = d+1} \check{\mathcal{C}}^\bullet(\vec{e})$$

It is clear that it maps the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e})$ to the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e} + \vec{b}_i)$ where $\vec{b}_i = (0, \dots, 0, 1, 0, \dots, 0)$ the i th basis vector. In other words, it maps the summand of H^n corresponding to \vec{e} with $e_i < 0$ and $\sum e_i = d$ to the summand of H^n corresponding to $\vec{e} + \vec{b}_i$ (which is zero if $e_i + b_i \geq 0$). It is easy to see that this corresponds exactly to the action of the contragredient of multiplication by T_i as a map

$$(R[T_0, \dots, T_n])_{-n-1-(d+1)} \longrightarrow (R[T_0, \dots, T_n])_{-n-1-d}$$

This proves the lemma. \square

Before we state the relative version we need some notation. Namely, recall that $\mathcal{O}_S[T_0, \dots, T_n]$ is a graded \mathcal{O}_S -module where each T_i is homogeneous of degree 1. Denote $(\mathcal{O}_S[T_0, \dots, T_n])_d$ the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S .

Lemma 8.3. *Let S be a scheme. Let $n \geq 0$ be an integer. Consider the structure morphism*

$$f : \mathbf{P}_S^n \longrightarrow S.$$

We have

$$R^q f_*(\mathcal{O}_{\mathbf{P}_S^n}(d)) = \begin{cases} (\mathcal{O}_S[T_0, \dots, T_n])_d & \text{if } q = 0 \\ 0 & \text{if } q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}((\mathcal{O}_S[T_0, \dots, T_n])_{-n-1-d}, \mathcal{O}_S) & \text{if } q = n \end{cases}$$

Proof. Omitted. Hint: This follows since the identifications in (8.1.1) are compatible with affine base change by Lemma 8.2. \square

Next we state the version for projective bundles associated to finite locally free sheaves. Let S be a scheme. Let \mathcal{E} be a finite locally free \mathcal{O}_S -module of constant rank $n+1$, see Modules, Section 14. In this case we think of $\text{Sym}(\mathcal{E})$ as a graded \mathcal{O}_S -module where \mathcal{E} is the graded part of degree 1. And $\text{Sym}^d(\mathcal{E})$ is the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S . Recall that our normalization is that

$$\pi : \mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\text{Sym}(\mathcal{E})) \longrightarrow S$$

and that there are natural maps $\text{Sym}^d(\mathcal{E}) \rightarrow \pi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$.

Lemma 8.4. *Let S be a scheme. Let $n \geq 1$. Let \mathcal{E} be a finite locally free \mathcal{O}_S -module of constant rank $n+1$. Consider the structure morphism*

$$\pi : \mathbf{P}(\mathcal{E}) \longrightarrow S.$$

We have

$$R^q \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) = \begin{cases} \text{Sym}^d(\mathcal{E}) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}(\text{Sym}^{-n-1-d}(\mathcal{E}) \otimes_{\mathcal{O}_S} \wedge^{n+1} \mathcal{E}, \mathcal{O}_S) & \text{if } q = n \end{cases}$$

These identifications are compatible with base change and isomorphism between locally free sheaves.

Proof. Consider the canonical map

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

and twist down by 1 to get

$$\pi^*(\mathcal{E})(-1) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}$$

This is a surjective map from a locally free rank $n + 1$ sheaf onto the structure sheaf. Hence the corresponding Koszul complex is exact (More on Algebra, Lemma 28.5). In other words there is an exact complex

$$0 \rightarrow \pi^*(\wedge^{n+1} \mathcal{E})(-n-1) \rightarrow \dots \rightarrow \pi^*(\wedge^i \mathcal{E})(-i) \rightarrow \dots \rightarrow \pi^* \mathcal{E}(-1) \rightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})} \rightarrow 0$$

We will think of the term $\pi^*(\wedge^i \mathcal{E})(-i)$ as being in degree $-i$. We are going to compute the higher direct images of this acyclic complex using the first spectral sequence of Derived Categories, Lemma 21.3. Namely, we see that there is a spectral sequence with terms

$$E_1^{p,q} = R^q \pi_* (\pi^*(\wedge^{-p} \mathcal{E})(p))$$

converging to zero! By the projection formula (Cohomology, Lemma 54.2) we have

$$E_1^{p,q} = \wedge^{-p} \mathcal{E} \otimes_{\mathcal{O}_S} R^q \pi_* (\mathcal{O}_{\mathbf{P}(\mathcal{E})}(p)).$$

Note that locally on S the sheaf \mathcal{E} is trivial, i.e., isomorphic to $\mathcal{O}_S^{\oplus n+1}$, hence locally on S the morphism $\mathbf{P}(\mathcal{E}) \rightarrow S$ can be identified with $\mathbf{P}_S^n \rightarrow S$. Hence locally on S we can use the result of Lemmas 8.1, 8.2, or 8.3. It follows that $E_1^{p,q} = 0$ unless (p, q) is $(0, 0)$ or $(-n-1, n)$. The nonzero terms are

$$E_1^{0,0} = \pi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})} = \mathcal{O}_S$$

$$E_1^{-n-1,n} = R^n \pi_* (\pi^*(\wedge^{n+1} \mathcal{E})(-n-1)) = \wedge^{n+1} \mathcal{E} \otimes_{\mathcal{O}_S} R^n \pi_* (\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1))$$

Hence there can only be one nonzero differential in the spectral sequence namely the map $d_{n+1}^{-n-1,n} : E_{n+1}^{-n-1,n} \rightarrow E_{n+1}^{0,0}$ which has to be an isomorphism (because the spectral sequence converges to the 0 sheaf). Thus $E_1^{p,q} = E_{n+1}^{p,q}$ and we obtain a canonical isomorphism

$$\wedge^{n+1} \mathcal{E} \otimes_{\mathcal{O}_S} R^n \pi_* (\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)) = R^n \pi_* (\pi^*(\wedge^{n+1} \mathcal{E})(-n-1)) \xrightarrow{d_{n+1}^{-n-1,n}} \mathcal{O}_S$$

Since $\wedge^{n+1} \mathcal{E}$ is an invertible sheaf, this implies that $R^n \pi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$ is invertible as well and canonically isomorphic to the inverse of $\wedge^{n+1} \mathcal{E}$. In other words we have proved the case $d = -n-1$ of the lemma.

Working locally on S we see immediately from the computation of cohomology in Lemmas 8.1, 8.2, or 8.3 the statements on vanishing of the lemma. Moreover the result on $R^0 \pi_*$ is clear as well, since there are canonical maps $\text{Sym}^d(\mathcal{E}) \rightarrow \pi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ for all d . It remains to show that the description of $R^n \pi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ is correct for $d < -n-1$. In order to do this we consider the map

$$\pi^*(\text{Sym}^{-d-n-1}(\mathcal{E})) \otimes_{\mathcal{O}_{\mathbf{P}(\mathcal{E})}} \mathcal{O}_{\mathbf{P}(\mathcal{E})}(d) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$$

Applying $R^n \pi_*$ and the projection formula (see above) we get a map

$$\text{Sym}^{-d-n-1}(\mathcal{E}) \otimes_{\mathcal{O}_S} R^n \pi_* (\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) \longrightarrow R^n \pi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1) = (\wedge^{n+1} \mathcal{E})^{\otimes -1}$$

(the last equality we have shown above). Again by the local calculations of Lemmas 8.1, 8.2, or 8.3 it follows that this map induces a perfect pairing between $R^n \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d))$ and $\mathrm{Sym}^{-d-n-1}(\mathcal{E}) \otimes \wedge^{n+1}(\mathcal{E})$ as desired. \square

9. Coherent sheaves on locally Noetherian schemes

We have defined the notion of a coherent module on any ringed space in Modules, Section 12. Although it is possible to consider coherent sheaves on non-Noetherian schemes we will always assume the base scheme is locally Noetherian when we consider coherent sheaves. Here is a characterization of coherent sheaves on locally Noetherian schemes.

Lemma 9.1. *Let X be a locally Noetherian scheme. Let \mathcal{F} be an \mathcal{O}_X -module. The following are equivalent*

- (1) \mathcal{F} is coherent,
- (2) \mathcal{F} is a quasi-coherent, finite type \mathcal{O}_X -module,
- (3) \mathcal{F} is a finitely presented \mathcal{O}_X -module,
- (4) for any affine open $\mathrm{Spec}(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A -module, and
- (5) there exists an affine open covering $X = \bigcup U_i$, $U_i = \mathrm{Spec}(A_i)$ such that each $\mathcal{F}|_{U_i} = \widetilde{M_i}$ with M_i a finite A_i -module.

In particular \mathcal{O}_X is coherent, any invertible \mathcal{O}_X -module is coherent, and more generally any finite locally free \mathcal{O}_X -module is coherent.

Proof. The implications (1) \Rightarrow (2) and (1) \Rightarrow (3) hold in general, see Modules, Lemma 12.2. If \mathcal{F} is finitely presented then \mathcal{F} is quasi-coherent, see Modules, Lemma 11.2. Hence also (3) \Rightarrow (2).

Assume \mathcal{F} is a quasi-coherent, finite type \mathcal{O}_X -module. By Properties, Lemma 16.1 we see that on any affine open $\mathrm{Spec}(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A -module. Since A is Noetherian we see that M has a finite resolution

$$A^{\oplus m} \rightarrow A^{\oplus n} \rightarrow M \rightarrow 0.$$

Hence \mathcal{F} is of finite presentation by Properties, Lemma 16.2. In other words (2) \Rightarrow (3).

By Modules, Lemma 12.5 it suffices to show that \mathcal{O}_X is coherent in order to show that (3) implies (1). Thus we have to show: given any open $U \subset X$ and any finite collection of sections $f_i \in \mathcal{O}_X(U)$, $i = 1, \dots, n$ the kernel of the map $\bigoplus_{i=1, \dots, n} \mathcal{O}_U \rightarrow \mathcal{O}_U$ is of finite type. Since being of finite type is a local property it suffices to check this in a neighbourhood of any $x \in U$. Thus we may assume $U = \mathrm{Spec}(A)$ is affine. In this case $f_1, \dots, f_n \in A$ are elements of A . Since A is Noetherian, see Properties, Lemma 5.2 the kernel K of the map $\bigoplus_{i=1, \dots, n} A \rightarrow A$ is a finite A -module. See for example Algebra, Lemma 51.1. As the functor $\widetilde{}$ is exact, see Schemes, Lemma 5.4 we get an exact sequence

$$\widetilde{K} \rightarrow \bigoplus_{i=1, \dots, n} \mathcal{O}_U \rightarrow \mathcal{O}_U$$

and by Properties, Lemma 16.1 again we see that \widetilde{K} is of finite type. We conclude that (1), (2) and (3) are all equivalent.

It follows from Properties, Lemma 16.1 that (2) implies (4). It is trivial that (4) implies (5). The discussion in Schemes, Section 24 show that (5) implies that \mathcal{F} is

quasi-coherent and it is clear that (5) implies that \mathcal{F} is of finite type. Hence (5) implies (2) and we win. \square

Lemma 9.2. *Let X be a locally Noetherian scheme. The category of coherent \mathcal{O}_X -modules is abelian. More precisely, the kernel and cokernel of a map of coherent \mathcal{O}_X -modules are coherent. Any extension of coherent sheaves is coherent.*

Proof. This is a restatement of Modules, Lemma 12.4 in a particular case. \square

The following lemma does not always hold for the category of coherent \mathcal{O}_X -modules on a general ringed space X .

Lemma 9.3. *Let X be a locally Noetherian scheme. Let \mathcal{F} be a coherent \mathcal{O}_X -module. Any quasi-coherent submodule of \mathcal{F} is coherent. Any quasi-coherent quotient module of \mathcal{F} is coherent.*

Proof. We may assume that X is affine, say $X = \text{Spec}(A)$. Properties, Lemma 5.2 implies that A is Noetherian. Lemma 9.1 turns this into algebra. The algebraic counter part of the lemma is that a quotient, or a submodule of a finite A -module is a finite A -module, see for example Algebra, Lemma 51.1. \square

Lemma 9.4. *Let X be a locally Noetherian scheme. Let \mathcal{F}, \mathcal{G} be coherent \mathcal{O}_X -modules. The \mathcal{O}_X -modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent.*

Proof. It is shown in Modules, Lemma 22.6 that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent. The result for tensor products is Modules, Lemma 16.6 \square

Lemma 9.5. *Let X be a locally Noetherian scheme. Let \mathcal{F}, \mathcal{G} be coherent \mathcal{O}_X -modules. Let $\varphi : \mathcal{G} \rightarrow \mathcal{F}$ be a homomorphism of \mathcal{O}_X -modules. Let $x \in X$.*

- (1) *If $\mathcal{F}_x = 0$ then there exists an open neighbourhood $U \subset X$ of x such that $\mathcal{F}|_U = 0$.*
- (2) *If $\varphi_x : \mathcal{G}_x \rightarrow \mathcal{F}_x$ is injective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is injective.*
- (3) *If $\varphi_x : \mathcal{G}_x \rightarrow \mathcal{F}_x$ is surjective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is surjective.*
- (4) *If $\varphi_x : \mathcal{G}_x \rightarrow \mathcal{F}_x$ is bijective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is an isomorphism.*

Proof. See Modules, Lemmas 9.4, 9.5, and 12.6. \square

Lemma 9.6. *Let X be a locally Noetherian scheme. Let \mathcal{F}, \mathcal{G} be coherent \mathcal{O}_X -modules. Let $x \in X$. Suppose $\psi : \mathcal{G}_x \rightarrow \mathcal{F}_x$ is a map of $\mathcal{O}_{X,x}$ -modules. Then there exists an open neighbourhood $U \subset X$ of x and a map $\varphi : \mathcal{G}|_U \rightarrow \mathcal{F}|_U$ such that $\varphi_x = \psi$.*

Proof. In view of Lemma 9.1 this is a reformulation of Modules, Lemma 22.4. \square

Lemma 9.7. *Let X be a locally Noetherian scheme. Let \mathcal{F} be a coherent \mathcal{O}_X -module. Then $\text{Supp}(\mathcal{F})$ is closed, and \mathcal{F} comes from a coherent sheaf on the scheme theoretic support of \mathcal{F} , see Morphisms, Definition 5.5.*

Proof. Let $i : Z \rightarrow X$ be the scheme theoretic support of \mathcal{F} and let \mathcal{G} be the finite type quasi-coherent sheaf on Z such that $i_*\mathcal{G} \cong \mathcal{F}$. Since $Z = \text{Supp}(\mathcal{F})$ we see that the support is closed. The scheme Z is locally Noetherian by Morphisms, Lemmas 15.5 and 15.6. Finally, \mathcal{G} is a coherent \mathcal{O}_Z -module by Lemma 9.1 \square

Lemma 9.8. *Let $i : Z \rightarrow X$ be a closed immersion of locally Noetherian schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z . The functor i_* induces an equivalence between the category of coherent \mathcal{O}_X -modules annihilated by \mathcal{I} and the category of coherent \mathcal{O}_Z -modules.*

Proof. The functor is fully faithful by Morphisms, Lemma 4.1. Let \mathcal{F} be a coherent \mathcal{O}_X -module annihilated by \mathcal{I} . By Morphisms, Lemma 4.1 we can write $\mathcal{F} = i_*\mathcal{G}$ for some quasi-coherent sheaf \mathcal{G} on Z . By Modules, Lemma 13.3 we see that \mathcal{G} is of finite type. Hence \mathcal{G} is coherent by Lemma 9.1. Thus the functor is also essentially surjective as desired. \square

Lemma 9.9. *Let $f : X \rightarrow Y$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Assume f is finite and Y locally Noetherian. Then $R^pf_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F}$ is coherent if \mathcal{F} is coherent.*

Proof. The higher direct images vanish by Lemma 2.3 and because a finite morphism is affine (by definition). Note that the assumptions imply that also X is locally Noetherian (see Morphisms, Lemma 15.6) and hence the statement makes sense. Let $\text{Spec}(A) = V \subset Y$ be an affine open subset. By Morphisms, Definition 44.1 we see that $f^{-1}(V) = \text{Spec}(B)$ with $A \rightarrow B$ finite. Lemma 9.1 turns the statement of the lemma into the following algebra fact: If M is a finite B -module, then M is also finite viewed as a A -module, see Algebra, Lemma 7.2. \square

In the situation of the lemma also the higher direct images are coherent since they vanish. We will show that this is always the case for a proper morphism between locally Noetherian schemes (Proposition 19.1).

Lemma 9.10. *Let X be a locally Noetherian scheme. Let \mathcal{F} be a coherent sheaf with $\dim(\text{Supp}(\mathcal{F})) \leq 0$. Then \mathcal{F} is generated by global sections and $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Proof. By Lemma 9.7 we see that $\mathcal{F} = i_*\mathcal{G}$ where $i : Z \rightarrow X$ is the inclusion of the scheme theoretic support of \mathcal{F} and where \mathcal{G} is a coherent \mathcal{O}_Z -module. Since the dimension of Z is 0, we see Z is a disjoint union of affines (Properties, Lemma 10.5). Hence \mathcal{G} is globally generated and the higher cohomology groups of \mathcal{G} are zero (Lemma 2.2). Hence $\mathcal{F} = i_*\mathcal{G}$ is globally generated. Since the cohomologies of \mathcal{F} and \mathcal{G} agree (Lemma 2.4 applies as a closed immersion is affine) we conclude that the higher cohomology groups of \mathcal{F} are zero. \square

Lemma 9.11. *Let X be a scheme. Let $j : U \rightarrow X$ be the inclusion of an open. Let $T \subset X$ be a closed subset contained in U . If \mathcal{F} is a coherent \mathcal{O}_U -module with $\text{Supp}(\mathcal{F}) \subset T$, then $j_*\mathcal{F}$ is a coherent \mathcal{O}_X -module.*

Proof. Consider the open covering $X = U \cup (X \setminus T)$. Then $j_*\mathcal{F}|_U = \mathcal{F}$ is coherent and $j_*\mathcal{F}|_{X \setminus T} = 0$ is also coherent. Hence $j_*\mathcal{F}$ is coherent. \square

10. Coherent sheaves on Noetherian schemes

In this section we mention some properties of coherent sheaves on Noetherian schemes.

Lemma 10.1. *Let X be a Noetherian scheme. Let \mathcal{F} be a coherent \mathcal{O}_X -module. The ascending chain condition holds for quasi-coherent submodules of \mathcal{F} . In other words, given any sequence*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}$$

of quasi-coherent submodules, then $\mathcal{F}_n = \mathcal{F}_{n+1} = \dots$ for some $n \geq 0$.

Proof. Choose a finite affine open covering. On each member of the covering we get stabilization by Algebra, Lemma 51.1. Hence the lemma follows. \square

Lemma 10.2. *Let X be a Noetherian scheme. Let \mathcal{F} be a coherent sheaf on X . Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals corresponding to a closed subscheme $Z \subset X$. Then there is some $n \geq 0$ such that $\mathcal{I}^n \mathcal{F} = 0$ if and only if $\text{Supp}(\mathcal{F}) \subset Z$ (set theoretically).*

Proof. This follows immediately from Algebra, Lemma 62.4 because X has a finite covering by spectra of Noetherian rings. \square

Lemma 10.3 (Artin-Rees). *Let X be a Noetherian scheme. Let \mathcal{F} be a coherent sheaf on X . Let $\mathcal{G} \subset \mathcal{F}$ be a quasi-coherent subsheaf. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then there exists a $c \geq 0$ such that for all $n \geq c$ we have*

$$\mathcal{I}^{n-c}(\mathcal{I}^c \mathcal{F} \cap \mathcal{G}) = \mathcal{I}^n \mathcal{F} \cap \mathcal{G}$$

Proof. This follows immediately from Algebra, Lemma 51.2 because X has a finite covering by spectra of Noetherian rings. \square

Lemma 10.4. *Let X be a Noetherian scheme. Every quasi-coherent \mathcal{O}_X -module is the filtered colimit of its coherent submodules.*

Proof. This is a reformulation of Properties, Lemma 22.3 in view of the fact that a finite type quasi-coherent \mathcal{O}_X -module is coherent by Lemma 9.1. \square

Lemma 10.5. *Let X be a Noetherian scheme. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let \mathcal{G} be a coherent \mathcal{O}_X -module. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Denote $Z \subset X$ the corresponding closed subscheme and set $U = X \setminus Z$. There is a canonical isomorphism*

$$\text{colim}_n \text{Hom}_{\mathcal{O}_X}(\mathcal{I}^n \mathcal{G}, \mathcal{F}) \longrightarrow \text{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular we have an isomorphism

$$\text{colim}_n \text{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}).$$

Proof. We first prove the second map is an isomorphism. It is injective by Properties, Lemma 25.3. Since \mathcal{F} is the union of its coherent submodules, see Properties, Lemma 22.3 (and Lemma 9.1) we may and do assume that \mathcal{F} is coherent to prove surjectivity. Let \mathcal{F}_n denote the quasi-coherent subsheaf of \mathcal{F} consisting of sections annihilated by \mathcal{I}^n , see Properties, Lemma 25.3. Since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ we see that $\mathcal{F}_n = \mathcal{F}_{n+1} = \dots$ for some $n \geq 0$ by Lemma 10.1. Set $\mathcal{H} = \mathcal{F}_n$ for this n . By Artin-Rees (Lemma 10.3) there exists a $c \geq 0$ such that $\mathcal{I}^m \mathcal{F} \cap \mathcal{H} \subset \mathcal{I}^{m-c} \mathcal{H}$. Picking $m = n + c$ we get $\mathcal{I}^m \mathcal{F} \cap \mathcal{H} \subset \mathcal{I}^n \mathcal{H} = 0$. Thus if we set $\mathcal{F}' = \mathcal{I}^m \mathcal{F}$ then we see that $\mathcal{F}' \cap \mathcal{F}_n = 0$ and $\mathcal{F}'|_U = \mathcal{F}|_U$. Note in particular that the subsheaf $(\mathcal{F}')_N$ of sections annihilated by \mathcal{I}^N is zero for all $N \geq 0$. Hence by Properties, Lemma 25.3 we deduce that the top horizontal arrow in the following commutative diagram is a bijection:

$$\begin{array}{ccc} \text{colim}_n \text{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}') & \longrightarrow & \Gamma(U, \mathcal{F}') \\ \downarrow & & \downarrow \\ \text{colim}_n \text{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{F}) \end{array}$$

Since also the right vertical arrow is a bijection we conclude that the bottom horizontal arrow is surjective as desired.

Next, we prove the first arrow of the lemma is a bijection. By Lemma 9.1 the sheaf \mathcal{G} is of finite presentation and hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is quasi-coherent, see Schemes, Section 24. By definition we have

$$\mathcal{H}(U) = \text{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U)$$

Pick a ψ in the right hand side of the first arrow of the lemma, i.e., $\psi \in \mathcal{H}(U)$. The result just proved applies to \mathcal{H} and hence there exists an $n \geq 0$ and an $\varphi : \mathcal{I}^n \rightarrow \mathcal{H}$ which recovers ψ on restriction to U . By Modules, Lemma 22.1 φ corresponds to a map

$$\varphi : \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}.$$

This is almost what we want except that the source of the arrow is the tensor product of \mathcal{I}^n and \mathcal{G} and not the product. We will show that, at the cost of increasing n , the difference is irrelevant. Consider the short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G} \rightarrow \mathcal{I}^n \mathcal{G} \rightarrow 0$$

where \mathcal{K} is defined as the kernel. Note that $\mathcal{I}^n \mathcal{K} = 0$ (proof omitted). By Artin-Rees again we see that

$$\mathcal{K} \cap \mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) = 0$$

for some m large enough. In other words we see that

$$\mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) \longrightarrow \mathcal{I}^{n+m} \mathcal{G}$$

is an isomorphism. Let φ' be the restriction of φ to this submodule thought of as a map $\mathcal{I}^{n+m} \mathcal{G} \rightarrow \mathcal{F}$. Then φ' gives an element of the left hand side of the first arrow of the lemma which maps to ψ via the arrow. In other words we have proved surjectivity of the arrow. We omit the proof of injectivity. \square

Lemma 10.6. *Let X be a locally Noetherian scheme. Let \mathcal{F}, \mathcal{G} be coherent \mathcal{O}_X -modules. Let $U \subset X$ be open and let $\varphi : \mathcal{F}|_U \rightarrow \mathcal{G}|_U$ be an \mathcal{O}_U -module map. Then there exists a coherent submodule $\mathcal{F}' \subset \mathcal{F}$ agreeing with \mathcal{F} over U such that φ extends to $\varphi' : \mathcal{F}' \rightarrow \mathcal{G}$.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals cutting out the reduced induced scheme structure on $X \setminus U$. If X is Noetherian, then Lemma 10.5 tells us that we can take $\mathcal{F}' = \mathcal{I}^n \mathcal{F}$ for some n . The general case will follow from this using Zorn's lemma.

Consider the set of triples $(U', \mathcal{F}', \varphi')$ where $U \subset U' \subset X$ is open, $\mathcal{F}' \subset \mathcal{F}|_{U'}$ is a coherent subsheaf agreeing with \mathcal{F} over U , and $\varphi' : \mathcal{F}' \rightarrow \mathcal{G}|_{U'}$ restricts to φ over U . We say $(U'', \mathcal{F}'', \varphi'') \geq (U', \mathcal{F}', \varphi')$ if and only if $U'' \supset U'$, $\mathcal{F}''|_{U'} = \mathcal{F}'$, and $\varphi''|_{U'} = \varphi'$. It is clear that if we have a totally ordered collection of triples $(U_i, \mathcal{F}_i, \varphi_i)$, then we can glue the \mathcal{F}_i to a subsheaf \mathcal{F}' of \mathcal{F} over $U' = \bigcup U_i$ and extend φ to a map $\varphi' : \mathcal{F}' \rightarrow \mathcal{G}|_{U'}$. Hence any totally ordered subset of triples has an upper bound. Finally, suppose that $(U', \mathcal{F}', \varphi')$ is any triple but $U' \neq X$. Then we can choose an affine open $W \subset X$ which is not contained in U' . By the result of the first paragraph we can extend the subsheaf $\mathcal{F}'|_{W \cap U'}$ and the restriction $\varphi'|_{W \cap U'}$ to some subsheaf $\mathcal{F}'' \subset \mathcal{F}|_W$ and map $\varphi'' : \mathcal{F}'' \rightarrow \mathcal{G}|_W$. Of course the agreement between (\mathcal{F}', φ') and $(\mathcal{F}'', \varphi'')$ over $W \cap U'$ exactly means that we can extend this

to a triple $(U' \cup W, \mathcal{F}''', \varphi''')$. Hence any maximal triple $(U', \mathcal{F}', \varphi')$ (which exist by Zorn's lemma) must have $U' = X$ and the proof is complete. \square

11. Depth

In this section we talk a little bit about depth and property (S_k) for coherent modules on locally Noetherian schemes. Note that we have already discussed this notion for locally Noetherian schemes in Properties, Section 12.

Definition 11.1. Let X be a locally Noetherian scheme. Let \mathcal{F} be a coherent \mathcal{O}_X -module. Let $k \geq 0$ be an integer.

- (1) We say \mathcal{F} has *depth k at a point x of X* if $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = k$.
- (2) We say X has *depth k at a point x of X* if $\text{depth}(\mathcal{O}_{X,x}) = k$.
- (3) We say \mathcal{F} has property (S_k) if

$$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq \min(k, \dim(\text{Supp}(\mathcal{F}_x)))$$

for all $x \in X$.

- (4) We say X has property (S_k) if \mathcal{O}_X has property (S_k) .

Any coherent sheaf satisfies condition (S_0) . Condition (S_1) is equivalent to having no embedded associated points, see Divisors, Lemma 4.3.

Lemma 11.2. Let X be a locally Noetherian scheme. Let \mathcal{F}, \mathcal{G} be coherent \mathcal{O}_X -modules and $x \in X$.

- (1) If \mathcal{G}_x has depth ≥ 1 , then $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$ has depth ≥ 1 .
- (2) If \mathcal{G}_x has depth ≥ 2 , then $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$ has depth ≥ 2 .

Proof. Observe that $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a coherent \mathcal{O}_X -module by Lemma 9.4. Coherent modules are of finite presentation (Lemma 9.1) hence taking stalks commutes with taking Hom and Hom , see Modules, Lemma 22.4. Thus we reduce to the case of finite modules over local rings which is More on Algebra, Lemma 23.10. \square

Lemma 11.3. Let X be a locally Noetherian scheme. Let \mathcal{F}, \mathcal{G} be coherent \mathcal{O}_X -modules.

- (1) If \mathcal{G} has property (S_1) , then $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has property (S_1) .
- (2) If \mathcal{G} has property (S_2) , then $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has property (S_2) .

Proof. Follows immediately from Lemma 11.2 and the definitions. \square

We have seen in Properties, Lemma 12.3 that a locally Noetherian scheme is Cohen-Macaulay if and only if (S_k) holds for all k . Thus it makes sense to introduce the following definition, which is equivalent to the condition that all stalks are Cohen-Macaulay modules.

Definition 11.4. Let X be a locally Noetherian scheme. Let \mathcal{F} be a coherent \mathcal{O}_X -module. We say \mathcal{F} is *Cohen-Macaulay* if and only if (S_k) holds for all $k \geq 0$.

Lemma 11.5. Let X be a regular scheme. Let \mathcal{F} be a coherent \mathcal{O}_X -module. The following are equivalent

- (1) \mathcal{F} is Cohen-Macaulay and $\text{Supp}(\mathcal{F}) = X$,
- (2) \mathcal{F} is finite locally free of rank > 0 .

Proof. Let $x \in X$. If (2) holds, then \mathcal{F}_x is a free $\mathcal{O}_{X,x}$ -module of rank > 0 . Hence $\text{depth}(\mathcal{F}_x) = \dim(\mathcal{O}_{X,x})$ because a regular local ring is Cohen-Macaulay (Algebra, Lemma 106.3). Conversely, if (1) holds, then \mathcal{F}_x is a maximal Cohen-Macaulay module over $\mathcal{O}_{X,x}$ (Algebra, Definition 103.8). Hence \mathcal{F}_x is free by Algebra, Lemma 106.6. \square

12. Devissage of coherent sheaves

Let X be a Noetherian scheme. Consider an integral closed subscheme $i : Z \rightarrow X$. It is often convenient to consider coherent sheaves of the form $i_*\mathcal{G}$ where \mathcal{G} is a coherent sheaf on Z . In particular we are interested in these sheaves when \mathcal{G} is a torsion free rank 1 sheaf. For example \mathcal{G} could be a nonzero sheaf of ideals on Z , or even more specifically $\mathcal{G} = \mathcal{O}_Z$.

Throughout this section we will use that a coherent sheaf is the same thing as a finite type quasi-coherent sheaf and that a quasi-coherent subquotient of a coherent sheaf is coherent, see Section 9. The support of a coherent sheaf is closed, see Modules, Lemma 9.6.

Lemma 12.1. *Let X be a Noetherian scheme. Let \mathcal{F} be a coherent sheaf on X . Suppose that $\text{Supp}(\mathcal{F}) = Z \cup Z'$ with Z, Z' closed. Then there exists a short exact sequence of coherent sheaves*

$$0 \rightarrow \mathcal{G}' \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$$

with $\text{Supp}(\mathcal{G}') \subset Z'$ and $\text{Supp}(\mathcal{G}) \subset Z$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals defining the reduced induced closed subscheme structure on Z , see Schemes, Lemma 12.4. Consider the subsheaves $\mathcal{G}'_n = \mathcal{I}^n \mathcal{F}$ and the quotients $\mathcal{G}_n = \mathcal{F}/\mathcal{I}^n \mathcal{F}$. For each n we have a short exact sequence

$$0 \rightarrow \mathcal{G}'_n \rightarrow \mathcal{F} \rightarrow \mathcal{G}_n \rightarrow 0$$

For every point x of $Z' \setminus Z$ we have $\mathcal{I}_x = \mathcal{O}_{X,x}$ and hence $\mathcal{G}_{n,x} = 0$. Thus we see that $\text{Supp}(\mathcal{G}_n) \subset Z$. Note that $X \setminus Z'$ is a Noetherian scheme. Hence by Lemma 10.2 there exists an n such that $\mathcal{G}'_n|_{X \setminus Z'} = \mathcal{I}^n \mathcal{F}|_{X \setminus Z'} = 0$. For such an n we see that $\text{Supp}(\mathcal{G}'_n) \subset Z'$. Thus setting $\mathcal{G}' = \mathcal{G}'_n$ and $\mathcal{G} = \mathcal{G}_n$ works. \square

Lemma 12.2. *Let X be a Noetherian scheme. Let $i : Z \rightarrow X$ be an integral closed subscheme. Let $\xi \in Z$ be the generic point. Let \mathcal{F} be a coherent sheaf on X . Assume that \mathcal{F}_ξ is annihilated by \mathfrak{m}_ξ . Then there exist an integer $r \geq 0$ and a coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ and an injective map of coherent sheaves*

$$i_*(\mathcal{I}^{\oplus r}) \rightarrow \mathcal{F}$$

which is an isomorphism in a neighbourhood of ξ .

Proof. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Z . Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of local sections of \mathcal{F} which are annihilated by \mathcal{J} . It is a quasi-coherent sheaf by Properties, Lemma 24.2. Moreover, $\mathcal{F}'_\xi = \mathcal{F}_\xi$ because $\mathcal{J}_\xi = \mathfrak{m}_\xi$ and part (3) of Properties, Lemma 24.2. By Lemma 9.5 we see that $\mathcal{F}' \rightarrow \mathcal{F}$ induces an isomorphism in a neighbourhood of ξ . Hence we may replace \mathcal{F} by \mathcal{F}' and assume that \mathcal{F} is annihilated by \mathcal{J} .

Assume $\mathcal{J}\mathcal{F} = 0$. By Lemma 9.8 we can write $\mathcal{F} = i_*\mathcal{G}$ for some coherent sheaf \mathcal{G} on Z . Suppose we can find a morphism $\mathcal{I}^{\oplus r} \rightarrow \mathcal{G}$ which is an isomorphism in a

neighbourhood of the generic point ξ of Z . Then applying i_* (which is left exact) we get the result of the lemma. Hence we have reduced to the case $X = Z$.

Suppose $Z = X$ is an integral Noetherian scheme with generic point ξ . Note that $\mathcal{O}_{X,\xi} = \kappa(\xi)$ is the function field of X in this case. Since \mathcal{F}_ξ is a finite \mathcal{O}_ξ -module we see that $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is finite. Hence the sheaves $\mathcal{O}_X^{\oplus r}$ and \mathcal{F} have isomorphic stalks at ξ . By Lemma 9.6 there exists a nonempty open $U \subset X$ and a morphism $\psi : \mathcal{O}_X^{\oplus r}|_U \rightarrow \mathcal{F}|_U$ which is an isomorphism at ξ , and hence an isomorphism in a neighbourhood of ξ by Lemma 9.5. By Schemes, Lemma 12.4 there exists a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose associated closed subscheme $Z \subset X$ is the complement of U . By Lemma 10.5 there exists an $n \geq 0$ and a morphism $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) \rightarrow \mathcal{F}$ which recovers our ψ over U . Since $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) = (\mathcal{I}^n)^{\oplus r}$ we get a map as in the lemma. It is injective because X is integral and it is injective at the generic point of X (easy proof omitted). \square

Lemma 12.3. *Let X be a Noetherian scheme. Let \mathcal{F} be a coherent sheaf on X . There exists a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \dots, m$ there exist an integral closed subscheme $Z_j \subset X$ and a nonzero coherent sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (Z_j \rightarrow X)_* \mathcal{I}_j$$

Proof. Consider the collection

$$\mathcal{T} = \left\{ Z \subset X \text{ closed such that there exists a coherent sheaf } \mathcal{F} \right. \\ \left. \text{with } \text{Supp}(\mathcal{F}) = Z \text{ for which the lemma is wrong} \right\}$$

We are trying to show that \mathcal{T} is empty. If not, then because X is Noetherian we can choose a minimal element $Z \in \mathcal{T}$. This means that there exists a coherent sheaf \mathcal{F} on X whose support is Z and for which the lemma does not hold. Clearly $Z \neq \emptyset$ since the only sheaf whose support is empty is the zero sheaf for which the lemma does hold (with $m = 0$).

If Z is not irreducible, then we can write $Z = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than Z . Then we can apply Lemma 12.1 to get a short exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{G}_2 \rightarrow 0$$

with $\text{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of Z each of \mathcal{G}_i has a filtration as in the statement of the lemma. By considering the induced filtration on \mathcal{F} we arrive at a contradiction. Hence we conclude that Z is irreducible.

Suppose Z is irreducible. Let \mathcal{J} be the sheaf of ideals cutting out the reduced induced closed subscheme structure of Z , see Schemes, Lemma 12.4. By Lemma 10.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n \mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{J}^n \mathcal{F} \subset \mathcal{J}^{n-1} \mathcal{F} \subset \dots \subset \mathcal{J} \mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by \mathcal{J} . Hence if each of these subquotients has a filtration as in the statement of the lemma then also \mathcal{F} does. In other words we may assume that \mathcal{J} does annihilate \mathcal{F} .

In the case where Z is irreducible and $\mathcal{J}\mathcal{F} = 0$ we can apply Lemma 12.2. This gives a short exact sequence

$$0 \rightarrow i_*(\mathcal{I}^{\oplus r}) \rightarrow \mathcal{F} \rightarrow \mathcal{Q} \rightarrow 0$$

where \mathcal{Q} is defined as the quotient. Since \mathcal{Q} is zero in a neighbourhood of ξ by the lemma just cited we see that the support of \mathcal{Q} is strictly smaller than Z . Hence we see that \mathcal{Q} has a filtration of the desired type by minimality of Z . But then clearly \mathcal{F} does too, which is our final contradiction. \square

Lemma 12.4. *Let X be a Noetherian scheme. Let \mathcal{P} be a property of coherent sheaves on X . Assume*

- (1) *For any short exact sequence of coherent sheaves*

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2 \rightarrow 0$$

if \mathcal{F}_i , $i = 1, 2$ have property \mathcal{P} then so does \mathcal{F} .

- (2) *For every integral closed subscheme $Z \subset X$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have \mathcal{P} for $i_*\mathcal{I}$.*

Then property \mathcal{P} holds for every coherent sheaf on X .

Proof. First note that if \mathcal{F} is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property \mathcal{P} , then so does \mathcal{F} . This follows from the property (1) for \mathcal{P} . On the other hand, by Lemma 12.3 we can filter any \mathcal{F} with successive subquotients as in (2). Hence the lemma follows. \square

Lemma 12.5. *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ . Let \mathcal{P} be a property of coherent sheaves on X with support contained in Z_0 such that*

- (1) *For any short exact sequence of coherent sheaves if two out of three of them have property \mathcal{P} then so does the third.*
- (2) *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have \mathcal{P} for $(Z \rightarrow X)_*\mathcal{I}$.*
- (3) *There exists some coherent sheaf \mathcal{G} on X such that*
 - (a) *$\text{Supp}(\mathcal{G}) = Z_0$,*
 - (b) *\mathcal{G}_ξ is annihilated by \mathfrak{m}_ξ ,*
 - (c) *$\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, and*
 - (d) *property \mathcal{P} holds for \mathcal{G} .*

Then property \mathcal{P} holds for every coherent sheaf \mathcal{F} on X whose support is contained in Z_0 .

Proof. First note that if \mathcal{F} is a coherent sheaf with support contained in Z_0 with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property \mathcal{P} , then so does \mathcal{F} . Or, if \mathcal{F} has property \mathcal{P} and all but one of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property \mathcal{P} then so does the last one. This follows from assumption (1).

As a first application we conclude that any coherent sheaf whose support is strictly contained in Z_0 has property \mathcal{P} . Namely, such a sheaf has a filtration (see Lemma 12.3) whose subquotients have property \mathcal{P} according to (2).

Let \mathcal{G} be as in (3). By Lemma 12.2 there exist a sheaf of ideals \mathcal{I} on Z_0 , an integer $r \geq 1$, and a short exact sequence

$$0 \rightarrow ((Z_0 \rightarrow X)_*\mathcal{I})^{\oplus r} \rightarrow \mathcal{G} \rightarrow \mathcal{Q} \rightarrow 0$$

where the support of \mathcal{Q} is strictly contained in Z_0 . By (3)(c) we see that $r = 1$. Since \mathcal{Q} has property \mathcal{P} too we conclude that $(Z_0 \rightarrow X)_*\mathcal{I}$ has property \mathcal{P} .

Next, suppose that $\mathcal{I}' \neq 0$ is another quasi-coherent sheaf of ideals on Z_0 . Then we can consider the intersection $\mathcal{I}'' = \mathcal{I}' \cap \mathcal{I}$ and we get two short exact sequences

$$0 \rightarrow (Z_0 \rightarrow X)_*\mathcal{I}'' \rightarrow (Z_0 \rightarrow X)_*\mathcal{I} \rightarrow \mathcal{Q} \rightarrow 0$$

and

$$0 \rightarrow (Z_0 \rightarrow X)_*\mathcal{I}'' \rightarrow (Z_0 \rightarrow X)_*\mathcal{I}' \rightarrow \mathcal{Q}' \rightarrow 0.$$

Note that the support of the coherent sheaves \mathcal{Q} and \mathcal{Q}' are strictly contained in Z_0 . Hence \mathcal{Q} and \mathcal{Q}' have property \mathcal{P} (see above). Hence we conclude using (1) that $(Z_0 \rightarrow X)_*\mathcal{I}''$ and $(Z_0 \rightarrow X)_*\mathcal{I}'$ both have \mathcal{P} as well.

The final step of the proof is to note that any coherent sheaf \mathcal{F} on X whose support is contained in Z_0 has a filtration (see Lemma 12.3 again) whose subquotients all have property \mathcal{P} by what we just said. \square

Lemma 12.6. *Let X be a Noetherian scheme. Let \mathcal{P} be a property of coherent sheaves on X such that*

- (1) *For any short exact sequence of coherent sheaves if two out of three of them have property \mathcal{P} then so does the third.*
- (2) *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf \mathcal{G} such that*
 - (a) *$\text{Supp}(\mathcal{G}) = Z$,*
 - (b) *\mathcal{G}_ξ is annihilated by \mathfrak{m}_ξ ,*
 - (c) *$\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, and*
 - (d) *property \mathcal{P} holds for \mathcal{G} .*

Then property \mathcal{P} holds for every coherent sheaf on X .

Proof. According to Lemma 12.4 it suffices to show that for all integral closed subschemes $Z \subset X$ and all quasi-coherent ideal sheaves $\mathcal{I} \subset \mathcal{O}_Z$ we have \mathcal{P} for $(Z \rightarrow X)_*\mathcal{I}$. If this fails, then since X is Noetherian there is a minimal integral closed subscheme $Z_0 \subset X$ such that \mathcal{P} fails for $(Z_0 \rightarrow X)_*\mathcal{I}_0$ for some quasi-coherent sheaf of ideals $\mathcal{I}_0 \subset \mathcal{O}_{Z_0}$, but \mathcal{P} does hold for $(Z \rightarrow X)_*\mathcal{I}$ for all integral closed subschemes $Z \subset Z_0$, $Z \neq Z_0$ and quasi-coherent ideal sheaves $\mathcal{I} \subset \mathcal{O}_Z$. Since we have the existence of \mathcal{G} for Z_0 by part (2), according to Lemma 12.5 this cannot happen. \square

Lemma 12.7. *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ . Let \mathcal{P} be a property of coherent sheaves on X such that*

- (1) *For any short exact sequence of coherent sheaves*

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2 \rightarrow 0$$

if \mathcal{F}_i , $i = 1, 2$ have property \mathcal{P} then so does \mathcal{F} .

- (2) *If \mathcal{P} holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for \mathcal{F} .*
- (3) *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have \mathcal{P} for $(Z \rightarrow X)_*\mathcal{I}$.*

- (4) *There exists some coherent sheaf \mathcal{G} such that*
- (a) *$\text{Supp}(\mathcal{G}) = Z_0$,*
 - (b) *\mathcal{G}_ξ is annihilated by \mathfrak{m}_ξ , and*
 - (c) *for every quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and such that \mathcal{P} holds for \mathcal{G}' .*

Then property \mathcal{P} holds for every coherent sheaf \mathcal{F} on X whose support is contained in Z_0 .

Proof. Note that if \mathcal{F} is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property \mathcal{P} , then so does \mathcal{F} . This follows from assumption (1).

As a first application we conclude that any coherent sheaf whose support is strictly contained in Z_0 has property \mathcal{P} . Namely, such a sheaf has a filtration (see Lemma 12.3) whose subquotients have property \mathcal{P} according to (3).

Let us denote $i : Z_0 \rightarrow X$ the closed immersion. Consider a coherent sheaf \mathcal{G} as in (4). By Lemma 12.2 there exists a sheaf of ideals \mathcal{I} on Z_0 and a short exact sequence

$$0 \rightarrow i_*\mathcal{I}^{\oplus r} \rightarrow \mathcal{G} \rightarrow \mathcal{Q} \rightarrow 0$$

where the support of \mathcal{Q} is strictly contained in Z_0 . In particular $r > 0$ and \mathcal{I} is nonzero because the support of \mathcal{G} is equal to Z_0 . Let $\mathcal{I}' \subset \mathcal{I}$ be any nonzero quasi-coherent sheaf of ideals on Z_0 contained in \mathcal{I} . Then we also get a short exact sequence

$$0 \rightarrow i_*(\mathcal{I}')^{\oplus r} \rightarrow \mathcal{G} \rightarrow \mathcal{Q}' \rightarrow 0$$

where \mathcal{Q}' has support properly contained in Z_0 . Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals cutting out the support of \mathcal{Q}' (for example the ideal corresponding to the reduced induced closed subscheme structure on the support of \mathcal{Q}'). Then $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. By Lemma 10.2 we see that $\mathcal{J}^n \mathcal{Q}' = 0$ for some n . Hence $\mathcal{J}^n \mathcal{G} \subset i_*(\mathcal{I}')^{\oplus r}$. By assumption (4)(c) of the lemma we see there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}^n \mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ for which property \mathcal{P} holds. Hence we get a short exact sequence

$$0 \rightarrow \mathcal{G}' \rightarrow i_*(\mathcal{I}')^{\oplus r} \rightarrow \mathcal{Q}'' \rightarrow 0$$

where \mathcal{Q}'' has support properly contained in Z_0 . Thus by our initial remarks and property (1) of the lemma we conclude that $i_*(\mathcal{I}')^{\oplus r}$ satisfies \mathcal{P} . Hence we see that $i_*\mathcal{I}'$ satisfies \mathcal{P} by (2). Finally, for an arbitrary quasi-coherent sheaf of ideals $\mathcal{I}'' \subset \mathcal{O}_{Z_0}$ we can set $\mathcal{I}' = \mathcal{I}'' \cap \mathcal{I}$ and we get a short exact sequence

$$0 \rightarrow i_*(\mathcal{I}') \rightarrow i_*(\mathcal{I}'') \rightarrow \mathcal{Q}''' \rightarrow 0$$

where \mathcal{Q}''' has support properly contained in Z_0 . Hence we conclude that property \mathcal{P} holds for $i_*\mathcal{I}''$ as well.

The final step of the proof is to note that any coherent sheaf \mathcal{F} on X whose support is contained in Z_0 has a filtration (see Lemma 12.3 again) whose subquotients all have property \mathcal{P} by what we just said. \square

Lemma 12.8. *Let X be a Noetherian scheme. Let \mathcal{P} be a property of coherent sheaves on X such that*

- (1) For any short exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2 \rightarrow 0$$

if \mathcal{F}_i , $i = 1, 2$ have property \mathcal{P} then so does \mathcal{F} .

- (2) If \mathcal{P} holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for \mathcal{F} .
 (3) For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf \mathcal{G} such that
 (a) $\text{Supp}(\mathcal{G}) = Z$,
 (b) \mathcal{G}_ξ is annihilated by \mathfrak{m}_ξ , and
 (c) for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{I}_\xi = \mathcal{O}_{X,\xi}$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{I}\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and such that \mathcal{P} holds for \mathcal{G}' .

Then property \mathcal{P} holds for every coherent sheaf on X .

Proof. Follows from Lemma 12.7 in exactly the same way that Lemma 12.6 follows from Lemma 12.5. \square

13. Finite morphisms and affines

In this section we use the results of the preceding sections to show that the image of a Noetherian affine scheme under a finite morphism is affine. We will see later that this result holds more generally (see Limits, Lemma 11.1 and Proposition 11.2).

Lemma 13.1. *Let $f : Y \rightarrow X$ be a morphism of schemes. Assume f is finite, surjective and X locally Noetherian. Let $Z \subset X$ be an integral closed subscheme with generic point ξ . Then there exists a coherent sheaf \mathcal{F} on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by \mathfrak{m}_ξ .*

Proof. Note that Y is locally Noetherian by Morphisms, Lemma 15.6. Because f is surjective the fibre Y_ξ is not empty. Pick $\xi' \in Y$ mapping to ξ . Let $Z' = \overline{\{\xi'\}}$. We may think of $Z' \subset Y$ as a reduced closed subscheme, see Schemes, Lemma 12.4. Hence the sheaf $\mathcal{F} = (Z' \rightarrow Y)_*\mathcal{O}_{Z'}$ is a coherent sheaf on Y (see Lemma 9.9). Look at the commutative diagram

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & Y \\ f' \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & X \end{array}$$

We see that $f_*\mathcal{F} = i_*f'_*\mathcal{O}_{Z'}$. Hence the stalk of $f_*\mathcal{F}$ at ξ is the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ . Note that since Z' is integral with generic point ξ' we have that ξ' is the only point of Z' lying over ξ , see Algebra, Lemmas 36.3 and 36.20. Hence the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ equal $\mathcal{O}_{Z',\xi'} = \kappa(\xi')$. In particular the stalk of $f_*\mathcal{F}$ at ξ is not zero. This combined with the fact that $f_*\mathcal{F}$ is of the form $i_*f'_*(\text{something})$ implies the lemma. \square

Lemma 13.2. *Let $f : Y \rightarrow X$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on Y . Let \mathcal{I} be a quasi-coherent sheaf of ideals on X . If the morphism f is affine then $\mathcal{I}f_*\mathcal{F} = f_*(f^{-1}\mathcal{I}\mathcal{F})$.*

Proof. The notation means the following. Since f^{-1} is an exact functor we see that $f^{-1}\mathcal{I}$ is a sheaf of ideals of $f^{-1}\mathcal{O}_X$. Via the map $f^\# : f^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_Y$ this acts on \mathcal{F} . Then $f^{-1}\mathcal{I}\mathcal{F}$ is the subsheaf generated by sums of local sections of the

form as where a is a local section of $f^{-1}\mathcal{I}$ and s is a local section of \mathcal{F} . It is a quasi-coherent \mathcal{O}_Y -submodule of \mathcal{F} because it is also the image of a natural map $f^*\mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{F} \rightarrow \mathcal{F}$.

Having said this the proof is straightforward. Namely, the question is local and hence we may assume X is affine. Since f is affine we see that Y is affine too. Thus we may write $Y = \operatorname{Spec}(B)$, $X = \operatorname{Spec}(A)$, $\mathcal{F} = \widetilde{M}$, and $\mathcal{I} = \widetilde{I}$. The assertion of the lemma in this case boils down to the statement that

$$I(M_A) = ((IB)M)_A$$

where M_A indicates the A -module associated to the B -module M . \square

Lemma 13.3. *Let $f : Y \rightarrow X$ be a morphism of schemes. Assume*

- (1) f finite,
- (2) f surjective,
- (3) Y affine, and
- (4) X Noetherian.

Then X is affine.

Proof. We will prove that under the assumptions of the lemma for any coherent \mathcal{O}_X -module \mathcal{F} we have $H^1(X, \mathcal{F}) = 0$. This will in particular imply that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals of \mathcal{O}_X . Then it follows that X is affine from either Lemma 3.1 or Lemma 3.2.

Let \mathcal{P} be the property of coherent sheaves \mathcal{F} on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow H^1(X, \mathcal{F}) = 0.$$

We are going to apply Lemma 12.8. Thus we have to verify (1), (2) and (3) of that lemma for \mathcal{P} . Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves. Property (2) follows since $H^1(X, -)$ is an additive functor. To see (3) let $Z \subset X$ be an integral closed subscheme with generic point ξ . Let \mathcal{F} be a coherent sheaf on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by \mathfrak{m}_ξ , see Lemma 13.1. We claim that taking $\mathcal{G} = f_*\mathcal{F}$ works. We only have to verify part (3)(c) of Lemma 12.8. Hence assume that $\mathcal{J} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. A finite morphism is affine hence by Lemma 13.2 we see that $\mathcal{J}\mathcal{G} = f_*(f^{-1}\mathcal{J}\mathcal{F})$. Also, as pointed out in the proof of Lemma 13.2 the sheaf $f^{-1}\mathcal{J}\mathcal{F}$ is a quasi-coherent \mathcal{O}_Y -module. Since Y is affine we see that $H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0$, see Lemma 2.2. Since f is finite, hence affine, we see that

$$H^1(X, \mathcal{J}\mathcal{G}) = H^1(X, f_*(f^{-1}\mathcal{J}\mathcal{F})) = H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0$$

by Lemma 2.4. Hence the quasi-coherent subsheaf $\mathcal{G}' = \mathcal{J}\mathcal{G}$ satisfies \mathcal{P} . This verifies property (3)(c) of Lemma 12.8 as desired. \square

14. Coherent sheaves on Proj, I

In this section we discuss coherent sheaves on $\operatorname{Proj}(A)$ where A is a Noetherian graded ring generated by A_1 over A_0 . In the next section we discuss what happens if A is not generated by degree 1 elements. First, we formulate an all-in-one result for projective space over a Noetherian ring.

Lemma 14.1. *Let R be a Noetherian ring. Let $n \geq 0$ be an integer. For every coherent sheaf \mathcal{F} on \mathbf{P}_R^n we have the following:*

- (1) *There exists an $r \geq 0$ and $d_1, \dots, d_r \in \mathbf{Z}$ and a surjection*

$$\bigoplus_{j=1, \dots, r} \mathcal{O}_{\mathbf{P}_R^n}(d_j) \longrightarrow \mathcal{F}.$$

- (2) *We have $H^i(\mathbf{P}_R^n, \mathcal{F}) = 0$ unless $0 \leq i \leq n$.*
 (3) *For any i the cohomology group $H^i(\mathbf{P}_R^n, \mathcal{F})$ is a finite R -module.*
 (4) *If $i > 0$, then $H^i(\mathbf{P}_R^n, \mathcal{F}(d)) = 0$ for all d large enough.*
 (5) *For any $k \in \mathbf{Z}$ the graded $R[T_0, \dots, T_n]$ -module*

$$\bigoplus_{d \geq k} H^0(\mathbf{P}_R^n, \mathcal{F}(d))$$

is a finite $R[T_0, \dots, T_n]$ -module.

Proof. We will use that $\mathcal{O}_{\mathbf{P}_R^n}(1)$ is an ample invertible sheaf on the scheme \mathbf{P}_R^n . This follows directly from the definition since \mathbf{P}_R^n covered by the standard affine opens $D_+(T_i)$. Hence by Properties, Proposition 26.13 every finite type quasi-coherent $\mathcal{O}_{\mathbf{P}_R^n}$ -module is a quotient of a finite direct sum of tensor powers of $\mathcal{O}_{\mathbf{P}_R^n}(1)$. On the other hand coherent sheaves and finite type quasi-coherent sheaves are the same thing on projective space over R by Lemma 9.1. Thus we see (1).

Projective n -space \mathbf{P}_R^n is covered by $n + 1$ affines, namely the standard opens $D_+(T_i)$, $i = 0, \dots, n$, see Constructions, Lemma 13.3. Hence we see that for any quasi-coherent sheaf \mathcal{F} on \mathbf{P}_R^n we have $H^i(\mathbf{P}_R^n, \mathcal{F}) = 0$ for $i \geq n + 1$, see Lemma 4.2. Hence (2) holds.

Let us prove (3) and (4) simultaneously for all coherent sheaves on \mathbf{P}_R^n by descending induction on i . Clearly the result holds for $i \geq n + 1$ by (2). Suppose we know the result for $i + 1$ and we want to show the result for i . (If $i = 0$, then part (4) is vacuous.) Let \mathcal{F} be a coherent sheaf on \mathbf{P}_R^n . Choose a surjection as in (1) and denote \mathcal{G} the kernel so that we have a short exact sequence

$$0 \rightarrow \mathcal{G} \rightarrow \bigoplus_{j=1, \dots, r} \mathcal{O}_{\mathbf{P}_R^n}(d_j) \rightarrow \mathcal{F} \rightarrow 0$$

By Lemma 9.2 we see that \mathcal{G} is coherent. The long exact cohomology sequence gives an exact sequence

$$H^i(\mathbf{P}_R^n, \bigoplus_{j=1, \dots, r} \mathcal{O}_{\mathbf{P}_R^n}(d_j)) \rightarrow H^i(\mathbf{P}_R^n, \mathcal{F}) \rightarrow H^{i+1}(\mathbf{P}_R^n, \mathcal{G}).$$

By induction assumption the right R -module is finite and by Lemma 8.1 the left R -module is finite. Since R is Noetherian it follows immediately that $H^i(\mathbf{P}_R^n, \mathcal{F})$ is a finite R -module. This proves the induction step for assertion (3). Since $\mathcal{O}_{\mathbf{P}_R^n}(d)$ is invertible we see that twisting on \mathbf{P}_R^n is an exact functor (since you get it by tensoring with an invertible sheaf, see Constructions, Definition 10.1). This means that for all $d \in \mathbf{Z}$ the sequence

$$0 \rightarrow \mathcal{G}(d) \rightarrow \bigoplus_{j=1, \dots, r} \mathcal{O}_{\mathbf{P}_R^n}(d_j + d) \rightarrow \mathcal{F}(d) \rightarrow 0$$

is short exact. The resulting cohomology sequence is

$$H^i(\mathbf{P}_R^n, \bigoplus_{j=1, \dots, r} \mathcal{O}_{\mathbf{P}_R^n}(d_j + d)) \rightarrow H^i(\mathbf{P}_R^n, \mathcal{F}(d)) \rightarrow H^{i+1}(\mathbf{P}_R^n, \mathcal{G}(d)).$$

By induction assumption we see the module on the right is zero for $d \gg 0$ and by the computation in Lemma 8.1 the module on the left is zero as soon as $d \geq -\min\{d_j\}$ and $i \geq 1$. Hence the induction step for assertion (4). This concludes the proof of (3) and (4).

In order to prove (5) note that for all sufficiently large d the map

$$H^0(\mathbf{P}_R^n, \bigoplus_{j=1, \dots, r} \mathcal{O}_{\mathbf{P}_R^n}(d_j + d)) \rightarrow H^0(\mathbf{P}_R^n, \mathcal{F}(d))$$

is surjective by the vanishing of $H^1(\mathbf{P}_R^n, \mathcal{G}(d))$ we just proved. In other words, the module

$$M_k = \bigoplus_{d \geq k} H^0(\mathbf{P}_R^n, \mathcal{F}(d))$$

is for k large enough a quotient of the corresponding module

$$N_k = \bigoplus_{d \geq k} H^0(\mathbf{P}_R^n, \bigoplus_{j=1, \dots, r} \mathcal{O}_{\mathbf{P}_R^n}(d_j + d))$$

When k is sufficiently small (e.g. $k < -d_j$ for all j) then

$$N_k = \bigoplus_{j=1, \dots, r} R[T_0, \dots, T_n](d_j)$$

by our computations in Section 8. In particular it is finitely generated. Suppose $k \in \mathbf{Z}$ is arbitrary. Choose $k_- \ll k \ll k_+$. Consider the diagram

$$\begin{array}{ccc} N_{k_-} & \longleftarrow & N_{k_+} \\ & & \downarrow \\ M_k & \longleftarrow & M_{k_+} \end{array}$$

where the vertical arrow is the surjective map above and the horizontal arrows are the obvious inclusion maps. By what was said above we see that N_{k_-} is a finitely generated $R[T_0, \dots, T_n]$ -module. Hence N_{k_+} is a finitely generated $R[T_0, \dots, T_n]$ -module because it is a submodule of a finitely generated module and the ring $R[T_0, \dots, T_n]$ is Noetherian. Since the vertical arrow is surjective we conclude that M_{k_+} is a finitely generated $R[T_0, \dots, T_n]$ -module. The quotient M_k/M_{k_+} is finite as an R -module since it is a finite direct sum of the finite R -modules $H^0(\mathbf{P}_R^n, \mathcal{F}(d))$ for $k \leq d < k_+$. Note that we use part (3) for $i = 0$ here. Hence M_k/M_{k_+} is a fortiori a finite $R[T_0, \dots, T_n]$ -module. In other words, we have sandwiched M_k between two finite $R[T_0, \dots, T_n]$ -modules and we win. \square

Lemma 14.2. *Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0 . Set $X = \text{Proj}(A)$. Then X is a Noetherian scheme. Let \mathcal{F} be a coherent \mathcal{O}_X -module.*

(1) *There exists an $r \geq 0$ and $d_1, \dots, d_r \in \mathbf{Z}$ and a surjection*

$$\bigoplus_{j=1, \dots, r} \mathcal{O}_X(d_j) \longrightarrow \mathcal{F}.$$

(2) *For any p the cohomology group $H^p(X, \mathcal{F})$ is a finite A_0 -module.*

(3) *If $p > 0$, then $H^p(X, \mathcal{F}(d)) = 0$ for all d large enough.*

(4) *For any $k \in \mathbf{Z}$ the graded A -module*

$$\bigoplus_{d \geq k} H^0(X, \mathcal{F}(d))$$

is a finite A -module.

Proof. By assumption there exists a surjection of graded A_0 -algebras

$$A_0[T_0, \dots, T_n] \longrightarrow A$$

where $\deg(T_j) = 1$ for $j = 0, \dots, n$. By Constructions, Lemma 11.5 this defines a closed immersion $i : X \rightarrow \mathbf{P}_{A_0}^n$ such that $i^* \mathcal{O}_{\mathbf{P}_{A_0}^n}(1) = \mathcal{O}_X(1)$. In particular, X is

Noetherian as a closed subscheme of the Noetherian scheme $\mathbf{P}_{A_0}^n$. We claim that the results of the lemma for \mathcal{F} follow from the corresponding results of Lemma 14.1 for the coherent sheaf $i_*\mathcal{F}$ (Lemma 9.8) on $\mathbf{P}_{A_0}^n$. For example, by this lemma there exists a surjection

$$\bigoplus_{j=1,\dots,r} \mathcal{O}_{\mathbf{P}_{A_0}^n}(d_j) \longrightarrow i_*\mathcal{F}.$$

By adjunction this corresponds to a map $\bigoplus_{j=1,\dots,r} \mathcal{O}_X(d_j) \longrightarrow \mathcal{F}$ which is surjective as well. The statements on cohomology follow from the fact that $H^p(X, \mathcal{F}(d)) = H^p(\mathbf{P}_{A_0}^n, i_*\mathcal{F}(d))$ by Lemma 2.4. \square

Lemma 14.3. *Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0 . Let M be a finite graded A -module. Set $X = \text{Proj}(A)$ and let \widetilde{M} be the quasi-coherent \mathcal{O}_X -module on X associated to M . The maps*

$$M_n \longrightarrow \Gamma(X, \widetilde{M}(n))$$

from Constructions, Lemma 10.3 are isomorphisms for all sufficiently large n .

Proof. Because M is a finite A -module we see that \widetilde{M} is a finite type \mathcal{O}_X -module, i.e., a coherent \mathcal{O}_X -module. Set $N = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M}(n))$. We have to show that the map $M \rightarrow N$ of graded A -modules is an isomorphism in all sufficiently large degrees. By Properties, Lemma 28.5 we have a canonical isomorphism $\widetilde{N} \rightarrow \widetilde{M}$ such that the induced maps $N_n \rightarrow N_n = \Gamma(X, \widetilde{N}(n))$ are the identity maps. Thus we have maps $\widetilde{M} \rightarrow \widetilde{N} \rightarrow \widetilde{M}$ such that for all n the diagram

$$\begin{array}{ccccc} M_n & \longrightarrow & N_n & & \\ \downarrow & & \downarrow & \searrow & \\ \Gamma(X, \widetilde{M}(n)) & \longrightarrow & \Gamma(X, \widetilde{N}(n)) & \xrightarrow{\cong} & \Gamma(X, \widetilde{M}(n)) \end{array}$$

is commutative. This means that the composition

$$M_n \rightarrow \Gamma(X, \widetilde{M}(n)) \rightarrow \Gamma(X, \widetilde{N}(n)) \rightarrow \Gamma(X, \widetilde{M}(n))$$

is equal to the canonical map $M_n \rightarrow \Gamma(X, \widetilde{M}(n))$. Clearly this implies that the composition $\widetilde{M} \rightarrow \widetilde{N} \rightarrow \widetilde{M}$ is the identity. Hence $\widetilde{M} \rightarrow \widetilde{N}$ is an isomorphism. Let $K = \text{Ker}(M \rightarrow N)$ and $Q = \text{Coker}(M \rightarrow N)$. Recall that the functor $M \mapsto \widetilde{M}$ is exact, see Constructions, Lemma 8.4. Hence we see that $\widetilde{K} = 0$ and $\widetilde{Q} = 0$. Recall that A is a Noetherian ring, M is a finitely generated A -module, and N is a graded A -module such that $N' = \bigoplus_{n \geq 0} N_n$ is finitely generated by the last part of Lemma 14.2. Hence $K' = \bigoplus_{n \geq 0} K_n$ and $Q' = \bigoplus_{n \geq 0} Q_n$ are finite A -modules. Observe that $\widetilde{Q} = \widetilde{Q'}$ and $\widetilde{K} = \widetilde{K'}$. Thus to finish the proof it suffices to show that a finite A -module K with $\widetilde{K} = 0$ has only finitely many nonzero homogeneous parts K_d with $d \geq 0$. To do this, let $x_1, \dots, x_r \in K$ be homogeneous generators say sitting in degrees d_1, \dots, d_r . Let $f_1, \dots, f_n \in A_1$ be elements generating A over A_0 . For each i and j there exists an $n_{ij} \geq 0$ such that $f_i^{n_{ij}} x_j = 0$ in $K_{d_j+n_{ij}}$: if not then $x_i/f_i^{d_i} \in K_{(f_i)}$ would not be zero, i.e., \widetilde{K} would not be zero. Then we see that K_d is zero for $d > \max_j(d_j + \sum_i n_{ij})$ as every element of K_d is a sum of terms where each term is a monomials in the f_i times one of the x_j of total degree d . \square

Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0 . Recall that $A_+ = \bigoplus_{n>0} A_n$ is the irrelevant ideal. Let M be a graded A -module. Recall that M is an A_+ -power torsion module if for all $x \in M$ there is an $n \geq 1$ such that $(A_+)^n x = 0$, see More on Algebra, Definition 88.1. If M is finitely generated, then we see that this is equivalent to $M_n = 0$ for $n \gg 0$. Sometimes A_+ -power torsion modules are called torsion modules. Sometimes a graded A -module M is called torsion free if $x \in M$ with $(A_+)^n x = 0$, $n > 0$ implies $x = 0$. Denote Mod_A the category of graded A -modules, Mod_A^{fg} the full subcategory of finitely generated ones, and $\text{Mod}_{A,torsion}^{fg}$ the full subcategory of modules M such that $M_n = 0$ for $n \gg 0$.

Proposition 14.4. *Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0 . Set $X = \text{Proj}(A)$. The functor $M \mapsto \widetilde{M}$ induces an equivalence*

$$\text{Mod}_A^{fg} / \text{Mod}_{A,torsion}^{fg} \longrightarrow \text{Coh}(\mathcal{O}_X)$$

whose quasi-inverse is given by $\mathcal{F} \mapsto \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$.

Proof. The subcategory $\text{Mod}_{A,torsion}^{fg}$ is a Serre subcategory of Mod_A^{fg} , see Homology, Definition 10.1. This is clear from the description of objects given above but it also follows from More on Algebra, Lemma 88.5. Hence the quotient category on the left of the arrow is defined in Homology, Lemma 10.6. To define the functor of the proposition, it suffices to show that the functor $M \mapsto \widetilde{M}$ sends torsion modules to 0. This is clear because for any $f \in A_+$ homogeneous the module M_f is zero and hence the value $M_{(f)}$ of \widetilde{M} on $D_+(f)$ is zero too.

By Lemma 14.2 the proposed quasi-inverse makes sense. Namely, the lemma shows that $\mathcal{F} \mapsto \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$ is a functor $\text{Coh}(\mathcal{O}_X) \rightarrow \text{Mod}_A^{fg}$ which we can compose with the quotient functor $\text{Mod}_A^{fg} \rightarrow \text{Mod}_A^{fg} / \text{Mod}_{A,torsion}^{fg}$.

By Lemma 14.3 the composite left to right to left is isomorphic to the identity functor.

Finally, let \mathcal{F} be a coherent \mathcal{O}_X -module. Set $M = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ viewed as a graded A -module, so that our functor sends \mathcal{F} to $M_{\geq 0} = \bigoplus_{n \geq 0} M_n$. By Properties, Lemma 28.5 the canonical map $\widetilde{M} \rightarrow \mathcal{F}$ is an isomorphism. Since the inclusion map $M_{\geq 0} \rightarrow M$ defines an isomorphism $\widetilde{M}_{\geq 0} \rightarrow \widetilde{M}$ we conclude that the composite right to left to right is isomorphic to the identity functor as well. \square

15. Coherent sheaves on Proj, II

In this section we discuss coherent sheaves on $\text{Proj}(A)$ where A is a Noetherian graded ring. Most of the results will be deduced by sleight of hand from the corresponding result in the previous section where we discussed what happens if A is generated by degree 1 elements.

Lemma 15.1. *Let A be a Noetherian graded ring. Set $X = \text{Proj}(A)$. Then X is a Noetherian scheme. Let \mathcal{F} be a coherent \mathcal{O}_X -module.*

- (1) *There exists an $r \geq 0$ and $d_1, \dots, d_r \in \mathbf{Z}$ and a surjection*

$$\bigoplus_{j=1, \dots, r} \mathcal{O}_X(d_j) \longrightarrow \mathcal{F}.$$

- (2) For any p the cohomology group $H^p(X, \mathcal{F})$ is a finite A_0 -module.
- (3) If $p > 0$, then $H^p(X, \mathcal{F}(d)) = 0$ for all d large enough.
- (4) For any $k \in \mathbf{Z}$ the graded A -module

$$\bigoplus_{d \geq k} H^0(X, \mathcal{F}(d))$$

is a finite A -module.

Proof. We will prove this by reducing the statement to Lemma 14.2. By Algebra, Lemmas 58.2 and 58.1 the ring A_0 is Noetherian and A is generated over A_0 by finitely many elements f_1, \dots, f_r homogeneous of positive degree. Let d be a sufficiently divisible integer. Set $A' = A^{(d)}$ with notation as in Algebra, Section 56. Then A' is generated over $A'_0 = A_0$ by elements of degree 1, see Algebra, Lemma 56.2. Thus Lemma 14.2 applies to $X' = \text{Proj}(A')$.

By Constructions, Lemma 11.8 there exist an isomorphism of schemes $i : X \rightarrow X'$ and isomorphisms $\mathcal{O}_X(nd) \rightarrow i^* \mathcal{O}_{X'}(n)$ compatible with the map $A' \rightarrow A$ and the maps $A_n \rightarrow H^0(X, \mathcal{O}_X(n))$ and $A'_n \rightarrow H^0(X', \mathcal{O}_{X'}(n))$. Thus Lemma 14.2 implies X is Noetherian and that (1) and (2) hold. To see (3) and (4) we can use that for any fixed k, p , and q we have

$$\bigoplus_{dn+q \geq k} H^p(X, \mathcal{F}(dn+q)) = \bigoplus_{dn+q \geq k} H^p(X', (i_* \mathcal{F}(q))(n))$$

by the compatibilities above. If $p > 0$, we have the vanishing of the right hand side for k depending on q large enough by Lemma 14.2. Since there are only a finite number of congruence classes of integers modulo d , we see that (3) holds for \mathcal{F} on X . If $p = 0$, then we have that the right hand side is a finite A' -module by Lemma 14.2. Using the finiteness of congruence classes once more, we find that $\bigoplus_{n \geq k} H^0(X, \mathcal{F}(n))$ is a finite A' -module too. Since the A' -module structure comes from the A -module structure (by the compatibilities mentioned above), we conclude it is finite as an A -module as well. \square

Lemma 15.2. *Let A be a Noetherian graded ring and let d be the lcm of generators of A over A_0 . Let M be a finite graded A -module. Set $X = \text{Proj}(A)$ and let \widetilde{M} be the quasi-coherent \mathcal{O}_X -module on X associated to M . Let $k \in \mathbf{Z}$.*

- (1) $N' = \bigoplus_{n \geq k} H^0(X, \widetilde{M}(n))$ is a finite A -module,
- (2) $N = \bigoplus_{n \geq k} H^0(X, \widetilde{M}(n))$ is a finite A -module,
- (3) there is a canonical map $N \rightarrow N'$,
- (4) if k is small enough there is a canonical map $M \rightarrow N'$,
- (5) the map $M_n \rightarrow N'_n$ is an isomorphism for $n \gg 0$,
- (6) $N_n \rightarrow N'_n$ is an isomorphism for $d|n$.

Proof. The map $N \rightarrow N'$ in (3) comes from Constructions, Equation (10.1.5) by taking global sections.

By Constructions, Equation (10.1.6) there is a map of graded A -modules $M \rightarrow \bigoplus_{n \in \mathbf{Z}} H^0(X, \widetilde{M}(n))$. If the generators of M sit in degrees $\geq k$, then the image is contained in the submodule $N' \subset \bigoplus_{n \in \mathbf{Z}} H^0(X, \widetilde{M}(n))$ and we get the map in (4).

By Algebra, Lemmas 58.2 and 58.1 the ring A_0 is Noetherian and A is generated over A_0 by finitely many elements f_1, \dots, f_r homogeneous of positive degree. Let

$d = \text{lcm}(\deg(f_i))$. Then we see that (6) holds for example by Constructions, Lemma 10.4.

Because M is a finite A -module we see that \widetilde{M} is a finite type \mathcal{O}_X -module, i.e., a coherent \mathcal{O}_X -module. Thus part (2) follows from Lemma 15.1.

We will deduce (1) from (2) using a trick. For $q \in \{0, \dots, d-1\}$ write

$${}^qN = \bigoplus_{n+q \geq k} H^0(X, \widetilde{M(q)}(n))$$

By part (2) these are finite A -modules. The Noetherian ring A is finite over $A^{(d)} = \bigoplus_{n \geq 0} A_{dn}$, because it is generated by f_i over $A^{(d)}$ and $f_i^d \in A^{(d)}$. Hence qN is a finite $A^{(d)}$ -module. Moreover, $A^{(d)}$ is Noetherian (follows from Algebra, Lemma 57.9). It follows that the $A^{(d)}$ -submodule ${}^qN^{(d)} = \bigoplus_{n \in \mathbf{Z}} {}^qN_{dn}$ is a finite module over $A^{(d)}$. Using the isomorphisms $\widetilde{M(dn+q)} = \widetilde{M(q)}(dn)$ we can write

$$N' = \bigoplus_{q \in \{0, \dots, d-1\}} \bigoplus_{dn+q \geq k} H^0(X, \widetilde{M(q)}(dn)) = \bigoplus_{q \in \{0, \dots, d-1\}} {}^qN^{(d)}$$

Thus N' is finite over $A^{(d)}$ and a fortiori finite over A . Thus (1) is true.

Let K be a finite A -module such that $\widetilde{K} = 0$. We claim that $K_n = 0$ for $d|n$ and $n \gg 0$. Arguing as above we see that $K^{(d)}$ is a finite $A^{(d)}$ -module. Let $x_1, \dots, x_m \in K$ be homogeneous generators of $K^{(d)}$ over $A^{(d)}$, say sitting in degrees d_1, \dots, d_m with $d|d_j$. For each i and j there exists an $n_{ij} \geq 0$ such that $f_i^{n_{ij}} x_j = 0$ in $K_{d_j+n_{ij}}$: if not then $x_j/f_i^{d_i/\deg(f_i)} \in K_{(f_i)}$ would not be zero, i.e., \widetilde{K} would not be zero. Here we use that $\deg(f_i)|d|d_j$ for all i, j . We conclude that K_n is zero for n with $d|n$ and $n > \max_j(d_j + \sum_i n_{ij} \deg(f_i))$ as every element of K_n is a sum of terms where each term is a monomial in the f_i times one of the x_j of total degree n .

To finish the proof, we have to show that $M \rightarrow N'$ is an isomorphism in all sufficiently large degrees. The map $N \rightarrow N'$ induces an isomorphism $\widetilde{N} \rightarrow \widetilde{N}'$ because on the affine opens $D_+(f_i) = D_+(f_i^d)$ the corresponding modules are isomorphic: $N_{(f_i)} \cong N_{(f_i^d)} \cong N'_{(f_i^d)} \cong N'_{(f_i)}$ by property (6). By Properties, Lemma 28.5 we have a canonical isomorphism $\widetilde{N} \rightarrow \widetilde{M}$. The composition $\widetilde{N} \rightarrow \widetilde{M} \rightarrow \widetilde{N}'$ is the isomorphism above (proof omitted; hint: look on standard affine opens to check this). Thus the map $M \rightarrow N'$ induces an isomorphism $\widetilde{M} \rightarrow \widetilde{N}'$. Let $K = \text{Ker}(M \rightarrow N')$ and $Q = \text{Coker}(M \rightarrow N')$. Recall that the functor $M \mapsto \widetilde{M}$ is exact, see Constructions, Lemma 8.4. Hence we see that $\widetilde{K} = 0$ and $\widetilde{Q} = 0$. By the result of the previous paragraph we see that $K_n = 0$ and $Q_n = 0$ for $d|n$ and $n \gg 0$. At this point we finally see the advantage of using N' over N : the functor $M \rightsquigarrow N'$ is compatible with shifts (immediate from the construction). Thus, repeating the whole argument with M replaced by $M(q)$ we find that $K_n = 0$ and $Q_n = 0$ for $n \equiv q \pmod{d}$ and $n \gg 0$. Since there are only finitely many congruence classes modulo n the proof is finished. \square

Let A be a Noetherian graded ring. Recall that $A_+ = \bigoplus_{n > 0} A_n$ is the irrelevant ideal. By Algebra, Lemmas 58.2 and 58.1 the ring A_0 is Noetherian and A is generated over A_0 by finitely many elements f_1, \dots, f_r homogeneous of positive

degree. Let $d = \text{lcm}(\deg(f_i))$. Let M be a graded A -module. In this situation we say a homogeneous element $x \in M$ is *irrelevant*² if

$$(A_+x)_{nd} = 0 \text{ for all } n \gg 0$$

If $x \in M$ is homogeneous and irrelevant and $f \in A$ is homogeneous, then fx is irrelevant too. Hence the set of irrelevant elements generate a graded submodule $M_{\text{irrelevant}} \subset M$. We will say M is *irrelevant* if every homogeneous element of M is irrelevant, i.e., if $M_{\text{irrelevant}} = M$. If M is finitely generated, then we see that this is equivalent to $M_{nd} = 0$ for $n \gg 0$. Denote Mod_A the category of graded A -modules, Mod_A^{fg} the full subcategory of finitely generated ones, and $\text{Mod}_{A,\text{irrelevant}}^{fg}$ the full subcategory of irrelevant modules.

Proposition 15.3. *Let A be a Noetherian graded ring. Set $X = \text{Proj}(A)$. The functor $M \mapsto \widetilde{M}$ induces an equivalence*

$$\text{Mod}_A^{fg} / \text{Mod}_{A,\text{irrelevant}}^{fg} \longrightarrow \text{Coh}(\mathcal{O}_X)$$

whose quasi-inverse is given by $\mathcal{F} \mapsto \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$.

Proof. We urge the reader to read the proof in the case where A is generated in degree 1 first, see Proposition 14.4. Let $f_1, \dots, f_r \in A$ be homogeneous elements of positive degree which generate A over A_0 . Let d be the lcm of the degrees d_i of f_i . Let M be a finite A -module. Let us show that \widetilde{M} is zero if and only if M is an irrelevant graded A -module (as defined above the statement of the proposition). Namely, let $x \in M$ be a homogeneous element. Choose $k \in \mathbf{Z}$ sufficiently small and let $N \rightarrow N'$ and $M \rightarrow N'$ be as in Lemma 15.2. We may also pick l sufficiently large such that $M_n \rightarrow N_n$ is an isomorphism for $n \geq l$. If \widetilde{M} is zero, then $N = 0$. Thus for any $f \in A_+$ homogeneous with $\deg(f) + \deg(x) = nd$ and $nd > l$ we see that fx is zero because $N_{nd} \rightarrow N'_{nd}$ and $M_{nd} \rightarrow N'_{nd}$ are isomorphisms. Hence x is irrelevant. Conversely, assume M is irrelevant. Then M_{nd} is zero for $n \gg 0$ (see discussion above proposition). Clearly this implies that $M_{(f_i)} = M_{(f_i^{d/\deg(f_i)})} = 0$, whence $\widetilde{M} = 0$ by construction.

It follows that the subcategory $\text{Mod}_{A,\text{irrelevant}}^{fg}$ is a Serre subcategory of Mod_A^{fg} as the kernel of the exact functor $M \mapsto \widetilde{M}$, see Homology, Lemma 10.4 and Constructions, Lemma 8.4. Hence the quotient category on the left of the arrow is defined in Homology, Lemma 10.6. To define the functor of the proposition, it suffices to show that the functor $M \mapsto \widetilde{M}$ sends irrelevant modules to 0 which we have shown above.

By Lemma 15.1 the proposed quasi-inverse makes sense. Namely, the lemma shows that $\mathcal{F} \mapsto \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$ is a functor $\text{Coh}(\mathcal{O}_X) \rightarrow \text{Mod}_A^{fg}$ which we can compose with the quotient functor $\text{Mod}_A^{fg} \rightarrow \text{Mod}_A^{fg} / \text{Mod}_{A,\text{irrelevant}}^{fg}$.

By Lemma 15.2 the composite left to right to left is isomorphic to the identity functor. Namely, let M be a finite graded A -module and let $k \in \mathbf{Z}$ sufficiently small and let $N \rightarrow N'$ and $M \rightarrow N'$ be as in Lemma 15.2. Then the kernel and cokernel of $M \rightarrow N'$ are nonzero in only finitely many degrees, hence are irrelevant. Moreover, the kernel and cokernel of the map $N \rightarrow N'$ are zero in all sufficiently

²This is nonstandard notation.

large degrees divisible by d , hence these are irrelevant modules too. Thus $M \rightarrow N'$ and $N \rightarrow N'$ are both isomorphisms in the quotient category, as desired.

Finally, let \mathcal{F} be a coherent \mathcal{O}_X -module. Set $M = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ viewed as a graded A -module, so that our functor sends \mathcal{F} to $M_{\geq 0} = \bigoplus_{n \geq 0} M_n$. By Properties, Lemma 28.5 the canonical map $\widetilde{M} \rightarrow \mathcal{F}$ is an isomorphism. Since the inclusion map $M_{\geq 0} \rightarrow M$ defines an isomorphism $\widetilde{M}_{\geq 0} \rightarrow \widetilde{M}$ we conclude that the composite right to left to right is isomorphic to the identity functor as well. \square

16. Higher direct images along projective morphisms

We first state and prove a result for when the base is affine and then we deduce some results for projective morphisms.

Lemma 16.1. *Let R be a Noetherian ring. Let $X \rightarrow \operatorname{Spec}(R)$ be a proper morphism. Let \mathcal{L} be an ample invertible sheaf on X . Let \mathcal{F} be a coherent \mathcal{O}_X -module.*

- (1) *The graded ring $A = \bigoplus_{d \geq 0} H^0(X, \mathcal{L}^{\otimes d})$ is a finitely generated R -algebra.*
- (2) *There exists an $r \geq 0$ and $d_1, \dots, d_r \in \mathbf{Z}$ and a surjection*

$$\bigoplus_{j=1, \dots, r} \mathcal{L}^{\otimes d_j} \longrightarrow \mathcal{F}.$$

- (3) *For any p the cohomology group $H^p(X, \mathcal{F})$ is a finite R -module.*
- (4) *If $p > 0$, then $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all d large enough.*
- (5) *For any $k \in \mathbf{Z}$ the graded A -module*

$$\bigoplus_{d \geq k} H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$$

is a finite A -module.

Proof. By Morphisms, Lemma 39.4 there exists a $d > 0$ and an immersion $i : X \rightarrow \mathbf{P}_R^n$ such that $\mathcal{L}^{\otimes d} \cong i^* \mathcal{O}_{\mathbf{P}_R^n}(1)$. Since X is proper over R the morphism i is a closed immersion (Morphisms, Lemma 41.7). Thus we have $H^i(X, \mathcal{G}) = H^i(\mathbf{P}_R^n, i_* \mathcal{G})$ for any quasi-coherent sheaf \mathcal{G} on X (by Lemma 2.4 and the fact that closed immersions are affine, see Morphisms, Lemma 11.9). Moreover, if \mathcal{G} is coherent, then $i_* \mathcal{G}$ is coherent as well (Lemma 9.8). We will use these facts without further mention.

Proof of (1). Set $S = R[T_0, \dots, T_n]$ so that $\mathbf{P}_R^n = \operatorname{Proj}(S)$. Observe that A is an S -algebra (but the ring map $S \rightarrow A$ is not a homomorphism of graded rings because S_n maps into A_{dn}). By the projection formula (Cohomology, Lemma 54.2) we have

$$i_*(\mathcal{L}^{\otimes nd+q}) = i_*(\mathcal{L}^{\otimes q}) \otimes_{\mathcal{O}_{\mathbf{P}_R^n}} \mathcal{O}_{\mathbf{P}_R^n}(n)$$

for all $n \in \mathbf{Z}$. We conclude that $\bigoplus_{n \geq 0} A_{nd+q}$ is a finite graded S -module by Lemma 14.1. Since $A = \bigoplus_{q \in \{0, \dots, d-1\}} \bigoplus_{n \geq 0} A_{nd+q}$ we see that A is finite as an S -algebra, hence (1) is true.

Proof of (2). This follows from Properties, Proposition 26.13.

Proof of (3). Apply Lemma 14.1 and use $H^p(X, \mathcal{F}) = H^p(\mathbf{P}_R^n, i_* \mathcal{F})$.

Proof of (4). Fix $p > 0$. By the projection formula we have

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes nd+q}) = i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes q}) \otimes_{\mathcal{O}_{\mathbf{P}_R^n}} \mathcal{O}_{\mathbf{P}_R^n}(n)$$

for all $n \in \mathbf{Z}$. By Lemma 14.1 we conclude that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{nd+q}) = 0$ for $n \gg 0$. Since there are only finitely many congruence classes of integers modulo d this proves (4).

Proof of (5). Fix an integer k . Set $M = \bigoplus_{n \geq k} H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. Arguing as above we conclude that $\bigoplus_{nd+q \geq k} M_{nd+q}$ is a finite graded S -module. Since $M = \bigoplus_{q \in \{0, \dots, d-1\}} \bigoplus_{nd+q \geq k} M_{nd+q}$ we see that M is finite as an S -module. Since the S -module structure factors through the ring map $S \rightarrow A$, we conclude that M is finite as an A -module. \square

Lemma 16.2. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let \mathcal{L} be an invertible sheaf on X . Assume that*

- (1) S is Noetherian,
- (2) f is proper,
- (3) \mathcal{F} is coherent, and
- (4) \mathcal{L} is relatively ample on X/S .

Then there exists an n_0 such that for all $n \geq n_0$ we have

$$R^p f_* (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$$

for all $p > 0$.

Proof. Choose a finite affine open covering $S = \bigcup V_j$ and set $X_j = f^{-1}(V_j)$. Clearly, if we solve the question for each of the finitely many systems $(X_j \rightarrow V_j, \mathcal{L}|_{X_j}, \mathcal{F}|_{V_j})$ then the result follows. Thus we may assume S is affine. In this case the vanishing of $R^p f_* (\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is equivalent to the vanishing of $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$, see Lemma 4.6. Thus the required vanishing follows from Lemma 16.1 (which applies because \mathcal{L} is ample on X by Morphisms, Lemma 39.4). \square

Lemma 16.3. *Let S be a locally Noetherian scheme. Let $f : X \rightarrow S$ be a locally projective morphism. Let \mathcal{F} be a coherent \mathcal{O}_X -module. Then $R^i f_* \mathcal{F}$ is a coherent \mathcal{O}_S -module for all $i \geq 0$.*

Proof. We first remark that a locally projective morphism is proper (Morphisms, Lemma 43.5) and hence of finite type. In particular X is locally Noetherian (Morphisms, Lemma 15.6) and hence the statement makes sense. Moreover, by Lemma 4.5 the sheaves $R^p f_* \mathcal{F}$ are quasi-coherent.

Having said this the statement is local on S (for example by Cohomology, Lemma 7.4). Hence we may assume $S = \text{Spec}(R)$ is the spectrum of a Noetherian ring, and X is a closed subscheme of \mathbf{P}_R^n for some n , see Morphisms, Lemma 43.4. In this case, the sheaves $R^p f_* \mathcal{F}$ are the quasi-coherent sheaves associated to the R -modules $H^p(X, \mathcal{F})$, see Lemma 4.6. Hence it suffices to show that R -modules $H^p(X, \mathcal{F})$ are finite R -modules (Lemma 9.1). This follows from Lemma 16.1 (because the restriction of $\mathcal{O}_{\mathbf{P}_R^n}(1)$ to X is ample on X). \square

17. Ample invertible sheaves and cohomology

Here is a criterion for ampleness on proper schemes over affine bases in terms of vanishing of cohomology after twisting.

Lemma 17.1. *Let R be a Noetherian ring. Let $f : X \rightarrow \text{Spec}(R)$ be a proper morphism. Let \mathcal{L} be an invertible \mathcal{O}_X -module. The following are equivalent*

- (1) \mathcal{L} is ample on X (this is equivalent to many other things, see *Properties, Proposition 26.13 and Morphisms, Lemma 39.4*),
- (2) for every coherent \mathcal{O}_X -module \mathcal{F} there exists an $n_0 \geq 0$ such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$, and
- (3) for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, there exists an $n \geq 1$ such that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$.

Proof. The implication (1) \Rightarrow (2) follows from Lemma 16.1. The implication (2) \Rightarrow (3) is trivial. The implication (3) \Rightarrow (1) is Lemma 3.3. \square

Lemma 17.2. *Let R be a Noetherian ring. Let $f : Y \rightarrow X$ be a morphism of schemes proper over R . Let \mathcal{L} be an invertible \mathcal{O}_X -module. Assume f is finite and surjective. Then \mathcal{L} is ample if and only if $f^*\mathcal{L}$ is ample.*

Proof. The pullback of an ample invertible sheaf by a quasi-affine morphism is ample, see *Morphisms, Lemma 37.7*. This proves one of the implications as a finite morphism is affine by definition.

Assume that $f^*\mathcal{L}$ is ample. Let P be the following property on coherent \mathcal{O}_X -modules \mathcal{F} : there exists an n_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$. We will prove that P holds for any coherent \mathcal{O}_X -module \mathcal{F} , which implies \mathcal{L} is ample by Lemma 17.1. We are going to apply Lemma 12.8. Thus we have to verify (1), (2) and (3) of that lemma for P . Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves and the fact that tensoring with an invertible sheaf is an exact functor. Property (2) follows since $H^p(X, -)$ is an additive functor. To see (3) let $Z \subset X$ be an integral closed subscheme with generic point ξ . Let \mathcal{F} be a coherent sheaf on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by \mathfrak{m}_ξ , see Lemma 13.1. We claim that taking $\mathcal{G} = f_*\mathcal{F}$ works. We only have to verify part (3)(c) of Lemma 12.8. Hence assume that $\mathcal{J} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. A finite morphism is affine hence by Lemma 13.2 we see that $\mathcal{J}\mathcal{G} = f_*(f^{-1}\mathcal{J}\mathcal{F})$. Also, as pointed out in the proof of Lemma 13.2 the sheaf $f^{-1}\mathcal{J}\mathcal{F}$ is a coherent \mathcal{O}_Y -module. As \mathcal{L} is ample we see from Lemma 17.1 that there exists an n_0 such that

$$H^p(Y, f^{-1}\mathcal{J}\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0,$$

for $n \geq n_0$ and $p > 0$. Since f is finite, hence affine, we see that

$$\begin{aligned} H^p(X, \mathcal{J}\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) &= H^p(X, f_*(f^{-1}\mathcal{J}\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \\ &= H^p(X, f_*(f^{-1}\mathcal{J}\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n})) \\ &= H^p(Y, f^{-1}\mathcal{J}\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0 \end{aligned}$$

Here we have used the projection formula (*Cohomology, Lemma 54.2*) and Lemma 2.4. Hence the quasi-coherent subsheaf $\mathcal{G}' = \mathcal{J}\mathcal{G}$ satisfies P . This verifies property (3)(c) of Lemma 12.8 as desired. \square

Cohomology is functorial. In particular, given a ringed space X , an invertible \mathcal{O}_X -module \mathcal{L} , a section $s \in \Gamma(X, \mathcal{L})$ we get maps

$$H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}), \quad \xi \longmapsto s\xi$$

induced by the map $\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ which is multiplication by s . We set $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ as a graded ring, see *Modules, Definition 25.7*. Given a sheaf of

\mathcal{O}_X -modules \mathcal{F} and an integer $p \geq 0$ we set

$$H_*^p(X, \mathcal{L}, \mathcal{F}) = \bigoplus_{n \in \mathbf{Z}} H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

This is a graded $\Gamma_*(X, \mathcal{L})$ -module by the multiplication defined above. Warning: the notation $H_*^p(X, \mathcal{L}, \mathcal{F})$ is nonstandard.

Lemma 17.3. *Let X be a scheme. Let \mathcal{L} be an invertible sheaf on X . Let $s \in \Gamma(X, \mathcal{L})$. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. If X is quasi-compact and quasi-separated, the canonical map*

$$H_*^p(X, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow H^p(X_s, \mathcal{F})$$

which maps ξ/s^n to $s^{-n}\xi$ is an isomorphism.

Proof. Note that for $p = 0$ this is Properties, Lemma 17.2. We will prove the statement using the induction principle (Lemma 4.1) where for $U \subset X$ quasi-compact open we let $P(U)$ be the property: for all $p \geq 0$ the map

$$H_*^p(U, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow H^p(U_s, \mathcal{F})$$

is an isomorphism.

If U is affine, then both sides of the arrow displayed above are zero for $p > 0$ by Lemma 2.2 and Properties, Lemma 26.4 and the statement is true. If P is true for U , V , and $U \cap V$, then we can use the Mayer-Vietoris sequences (Cohomology, Lemma 8.2) to obtain a map of long exact sequences

$$\begin{array}{ccccccc} H_*^{p-1}(U \cap V, \mathcal{L}, \mathcal{F})_{(s)} & \longrightarrow & H_*^p(U \cup V, \mathcal{L}, \mathcal{F})_{(s)} & \longrightarrow & H_*^p(U, \mathcal{L}, \mathcal{F})_{(s)} \oplus H_*^p(V, \mathcal{L}, \mathcal{F})_{(s)} \\ \downarrow & & \downarrow & & \downarrow \\ H^{p-1}(U_s \cap V_s, \mathcal{F}) & \longrightarrow & H^p(U_s \cup V_s, \mathcal{F}) & \longrightarrow & H^p(U_s, \mathcal{F}) \oplus H^p(V_s, \mathcal{F}) \end{array}$$

(only a snippet shown). Observe that $U_s \cap V_s = (U \cap V)_s$ and that $U_s \cup V_s = (U \cup V)_s$. Thus the left and right vertical maps are isomorphisms (as well as one more to the right and one more to the left which are not shown in the diagram). We conclude that $P(U \cup V)$ holds by the 5-lemma (Homology, Lemma 5.20). This finishes the proof. \square

Lemma 17.4. *Let X be a scheme. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Assume that*

- (1) X is quasi-compact and quasi-separated, and
- (2) X_s is affine.

Then for every quasi-coherent \mathcal{O}_X -module \mathcal{F} and every $p > 0$ and all $\xi \in H^p(X, \mathcal{F})$ there exists an $n \geq 0$ such that $s^n \xi = 0$ in $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$.

Proof. Recall that $H^p(X_s, \mathcal{G})$ is zero for every quasi-coherent module \mathcal{G} by Lemma 2.2. Hence the lemma follows from Lemma 17.3. \square

For a more general version of the following lemma see Limits, Lemma 11.4.

Lemma 17.5. *Let $i : Z \rightarrow X$ be a closed immersion of Noetherian schemes inducing a homeomorphism of underlying topological spaces. Let \mathcal{L} be an invertible sheaf on X . Then $i^* \mathcal{L}$ is ample on Z , if and only if \mathcal{L} is ample on X .*

Proof. If \mathcal{L} is ample, then $i^*\mathcal{L}$ is ample for example by Morphisms, Lemma 37.7. Assume $i^*\mathcal{L}$ is ample. We have to show that \mathcal{L} is ample on X . Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals cutting out the closed subscheme Z . Since $i(Z) = X$ set theoretically we see that $\mathcal{I}^n = 0$ for some n by Lemma 10.2. Consider the sequence

$$X = Z_n \supset Z_{n-1} \supset Z_{n-2} \supset \dots \supset Z_1 = Z$$

of closed subschemes cut out by $0 = \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \dots \subset \mathcal{I}$. Then each of the closed immersions $Z_i \rightarrow Z_{i-1}$ is defined by a coherent sheaf of ideals of square zero. In this way we reduce to the case that $\mathcal{I}^2 = 0$.

Consider the short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow i_*\mathcal{O}_Z \rightarrow 0$$

of quasi-coherent \mathcal{O}_X -modules. Tensoring with $\mathcal{L}^{\otimes n}$ we obtain short exact sequences

$$(17.5.1) \quad 0 \rightarrow \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \rightarrow \mathcal{L}^{\otimes n} \rightarrow i_*i^*\mathcal{L}^{\otimes n} \rightarrow 0$$

As $\mathcal{I}^2 = 0$, we can use Morphisms, Lemma 4.1 to think of \mathcal{I} as a quasi-coherent \mathcal{O}_Z -module and then $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} = \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n}$ with obvious abuse of notation. Moreover, the cohomology of this sheaf over Z is canonically the same as the cohomology of this sheaf over X (as i is a homeomorphism).

Let $x \in X$ be a point and denote $z \in Z$ the corresponding point. Because $i^*\mathcal{L}$ is ample there exists an n and a section $s \in \Gamma(Z, i^*\mathcal{L}^{\otimes n})$ with $z \in Z_s$ and with Z_s affine. The obstruction to lifting s to a section of $\mathcal{L}^{\otimes n}$ over X is the boundary

$$\xi = \partial s \in H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = H^1(Z, \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n})$$

coming from the short exact sequence of sheaves (17.5.1). If we replace s by s^{e+1} then ξ is replaced by $\partial(s^{e+1}) = (e+1)s^e\xi$ in $H^1(Z, \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes(e+1)n})$ because the boundary map for

$$0 \rightarrow \bigoplus_{m \geq 0} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m} \rightarrow \bigoplus_{m \geq 0} \mathcal{L}^{\otimes m} \rightarrow \bigoplus_{m \geq 0} i_*i^*\mathcal{L}^{\otimes m} \rightarrow 0$$

is a derivation by Cohomology, Lemma 25.5. By Lemma 17.4 we see that $s^e\xi$ is zero for e large enough. Hence, after replacing s by a power, we can assume s is the image of a section $s' \in \Gamma(X, \mathcal{L}^{\otimes n})$. Then $X_{s'}$ is an open subscheme and $Z_s \rightarrow X_{s'}$ is a surjective closed immersion of Noetherian schemes with Z_s affine. Hence X_s is affine by Lemma 13.3 and we conclude that \mathcal{L} is ample. \square

For a more general version of the following lemma see Limits, Lemma 11.5.

Lemma 17.6. *Let $i : Z \rightarrow X$ be a closed immersion of Noetherian schemes inducing a homeomorphism of underlying topological spaces. Then X is quasi-affine if and only if Z is quasi-affine.*

Proof. Recall that a scheme is quasi-affine if and only if the structure sheaf is ample, see Properties, Lemma 27.1. Hence if Z is quasi-affine, then \mathcal{O}_Z is ample, hence \mathcal{O}_X is ample by Lemma 17.5, hence X is quasi-affine. A proof of the converse, which can also be seen in an elementary way, is gotten by reading the argument just given backwards. \square

Lemma 17.7. *Let X be a scheme. Let \mathcal{L} be an ample invertible \mathcal{O}_X -module. Let n_0 be an integer. If $H^p(X, \mathcal{L}^{\otimes -n}) = 0$ for $n \geq n_0$ and $p > 0$, then X is affine.*

Proof. We claim $H^p(X, \mathcal{F}) = 0$ for every quasi-coherent \mathcal{O}_X -module and $p > 0$. Since X is quasi-compact by Properties, Definition 26.1 the claim finishes the proof by Lemma 3.1. The scheme X is separated by Properties, Lemma 26.8. Say X is covered by $e + 1$ affine opens. Then $H^p(X, \mathcal{F}) = 0$ for $p > e$, see Lemma 4.2. Thus we may use descending induction on p to prove the claim. Writing \mathcal{F} as a filtered colimit of finite type quasi-coherent modules (Properties, Lemma 22.3) and using Cohomology, Lemma 19.1 we may assume \mathcal{F} is of finite type. Then we can choose $n > n_0$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated, see Properties, Proposition 26.13. This means there is a short exact sequence

$$0 \rightarrow \mathcal{F}' \rightarrow \bigoplus_{i \in I} \mathcal{L}^{\otimes -n} \rightarrow \mathcal{F} \rightarrow 0$$

for some set I (in fact we can choose I finite). By induction hypothesis we have $H^{p+1}(X, \mathcal{F}') = 0$ and by assumption (combined with the already used commutation of cohomology with colimits) we have $H^p(X, \bigoplus_{i \in I} \mathcal{L}^{\otimes -n}) = 0$. From the long exact cohomology sequence we conclude that $H^p(X, \mathcal{F}) = 0$ as desired. \square

Lemma 17.8. *Let X be a quasi-affine scheme. If $H^p(X, \mathcal{O}_X) = 0$ for $p > 0$, then X is affine.*

Proof. Since \mathcal{O}_X is ample by Properties, Lemma 27.1 this follows from Lemma 17.7. \square

18. Chow's Lemma

In this section we prove Chow's lemma in the Noetherian case (Lemma 18.1). In Limits, Section 12 we prove some variants for the non-Noetherian case.

Lemma 18.1. *Let S be a Noetherian scheme. Let $f : X \rightarrow S$ be a separated morphism of finite type. Then there exist an $n \geq 0$ and a diagram*

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \longrightarrow & \mathbf{P}_S^n \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

where $X' \rightarrow \mathbf{P}_S^n$ is an immersion, and $\pi : X' \rightarrow X$ is proper and surjective. Moreover, we may arrange it such that there exists a dense open subscheme $U \subset X$ such that $\pi^{-1}(U) \rightarrow U$ is an isomorphism.

Proof. All of the schemes we will encounter during the rest of the proof are going to be of finite type over the Noetherian scheme S and hence Noetherian (see Morphisms, Lemma 15.6). All morphisms between them will automatically be quasi-compact, locally of finite type and quasi-separated, see Morphisms, Lemma 15.8 and Properties, Lemmas 5.4 and 5.8.

The scheme X has only finitely many irreducible components (Properties, Lemma 5.7). Say $X = X_1 \cup \dots \cup X_r$ is the decomposition of X into irreducible components. Let $\eta_i \in X_i$ be the generic point. For every point $x \in X$ there exists an affine open $U_x \subset X$ which contains x and each of the generic points η_i . See Properties, Lemma 29.4. Since X is quasi-compact, we can find a finite affine open covering $X = U_1 \cup \dots \cup U_m$ such that each U_i contains η_1, \dots, η_r . In particular we conclude that the open $U = U_1 \cap \dots \cap U_m \subset X$ is a dense open. This and the fact that the U_i are affine opens covering X are all that we will use below.

Let $X^* \subset X$ be the scheme theoretic closure of $U \rightarrow X$, see Morphisms, Definition 6.2. Let $U_i^* = X^* \cap U_i$. Note that U_i^* is a closed subscheme of U_i . Hence U_i^* is affine. Since U is dense in X the morphism $X^* \rightarrow X$ is a surjective closed immersion. It is an isomorphism over U . Hence we may replace X by X^* and U_i by U_i^* and assume that U is scheme theoretically dense in X , see Morphisms, Definition 7.1.

By Morphisms, Lemma 39.3 we can find an immersion $j_i : U_i \rightarrow \mathbf{P}_S^{n_i}$ for each i . By Morphisms, Lemma 7.7 we can find closed subschemes $Z_i \subset \mathbf{P}_S^{n_i}$ such that $j_i : U_i \rightarrow Z_i$ is a scheme theoretically dense open immersion. Note that $Z_i \rightarrow S$ is proper, see Morphisms, Lemma 43.5. Consider the morphism

$$j = (j_1|_U, \dots, j_m|_U) : U \longrightarrow \mathbf{P}_S^{n_1} \times_S \dots \times_S \mathbf{P}_S^{n_m}.$$

By the lemma cited above we can find a closed subscheme Z of $\mathbf{P}_S^{n_1} \times_S \dots \times_S \mathbf{P}_S^{n_m}$ such that $j : U \rightarrow Z$ is an open immersion and such that U is scheme theoretically dense in Z . The morphism $Z \rightarrow S$ is proper. Consider the i th projection

$$\text{pr}_i|_Z : Z \longrightarrow \mathbf{P}_S^{n_i}.$$

This morphism factors through Z_i (see Morphisms, Lemma 6.6). Denote $p_i : Z \rightarrow Z_i$ the induced morphism. This is a proper morphism, see Morphisms, Lemma 41.7 for example. At this point we have that $U \subset U_i \subset Z_i$ are scheme theoretically dense open immersions. Moreover, we can think of Z as the scheme theoretic image of the “diagonal” morphism $U \rightarrow Z_1 \times_S \dots \times_S Z_m$.

Set $V_i = p_i^{-1}(U_i)$. Note that $p_i|_{V_i} : V_i \rightarrow U_i$ is proper. Set $X' = V_1 \cup \dots \cup V_m$. By construction X' has an immersion into the scheme $\mathbf{P}_S^{n_1} \times_S \dots \times_S \mathbf{P}_S^{n_m}$. Thus by the Segre embedding (see Constructions, Lemma 13.6) we see that X' has an immersion into a projective space over S .

We claim that the morphisms $p_i|_{V_i} : V_i \rightarrow U_i$ glue to a morphism $X' \rightarrow X$. Namely, it is clear that $p_i|_U$ is the identity map from U to U . Since $U \subset X'$ is scheme theoretically dense by construction, it is also scheme theoretically dense in the open subscheme $V_i \cap V_j$. Thus we see that $p_i|_{V_i \cap V_j} = p_j|_{V_i \cap V_j}$ as morphisms into the separated S -scheme X , see Morphisms, Lemma 7.10. We denote the resulting morphism $\pi : X' \rightarrow X$.

We claim that $\pi^{-1}(U_i) = V_i$. Since $\pi|_{V_i} = p_i|_{V_i}$ it follows that $V_i \subset \pi^{-1}(U_i)$. Consider the diagram

$$\begin{array}{ccc} V_i & \xrightarrow{\quad} & \pi^{-1}(U_i) \\ & \searrow p_i|_{V_i} & \downarrow \\ & & U_i \end{array}$$

Since $V_i \rightarrow U_i$ is proper we see that the image of the horizontal arrow is closed, see Morphisms, Lemma 41.7. Since $V_i \subset \pi^{-1}(U_i)$ is scheme theoretically dense (as it contains U) we conclude that $V_i = \pi^{-1}(U_i)$ as claimed.

This shows that $\pi^{-1}(U_i) \rightarrow U_i$ is identified with the proper morphism $p_i|_{V_i} : V_i \rightarrow U_i$. Hence we see that X has a finite affine covering $X = \bigcup U_i$ such that the restriction of π is proper on each member of the covering. Thus by Morphisms, Lemma 41.3 we see that π is proper.

Finally we have to show that $\pi^{-1}(U) = U$. To see this we argue in the same way as above using the diagram

$$\begin{array}{ccc} U & \longrightarrow & \pi^{-1}(U) \\ & \searrow & \downarrow \\ & & U \end{array}$$

and using that $\text{id}_U : U \rightarrow U$ is proper and that U is scheme theoretically dense in $\pi^{-1}(U)$. \square

Remark 18.2. In the situation of Chow's Lemma 18.1:

- (1) The morphism π is actually H-projective (hence projective, see Morphisms, Lemma 43.3) since the morphism $X' \rightarrow \mathbf{P}_S^n \times_S X = \mathbf{P}_X^n$ is a closed immersion (use the fact that π is proper, see Morphisms, Lemma 41.7).
- (2) We may assume that $\pi^{-1}(U)$ is scheme theoretically dense in X' . Namely, we can simply replace X' by the scheme theoretic closure of $\pi^{-1}(U)$. In this case we can think of U as a scheme theoretically dense open subscheme of X' . See Morphisms, Section 6.
- (3) If X is reduced then we may choose X' reduced. This is clear from (2).

19. Higher direct images of coherent sheaves

In this section we prove the fundamental fact that the higher direct images of a coherent sheaf under a proper morphism are coherent.

Proposition 19.1. *Let S be a locally Noetherian scheme. Let $f : X \rightarrow S$ be a proper morphism. Let \mathcal{F} be a coherent \mathcal{O}_X -module. Then $R^i f_* \mathcal{F}$ is a coherent \mathcal{O}_S -module for all $i \geq 0$.*

Proof. Since the problem is local on S we may assume that S is a Noetherian scheme. Since a proper morphism is of finite type we see that in this case X is a Noetherian scheme also. Consider the property \mathcal{P} of coherent sheaves on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow R^p f_* \mathcal{F} \text{ is coherent for all } p \geq 0$$

We are going to use the result of Lemma 12.6 to prove that \mathcal{P} holds for every coherent sheaf on X .

Let

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

be a short exact sequence of coherent sheaves on X . Consider the long exact sequence of higher direct images

$$R^{p-1} f_* \mathcal{F}_3 \rightarrow R^p f_* \mathcal{F}_1 \rightarrow R^p f_* \mathcal{F}_2 \rightarrow R^p f_* \mathcal{F}_3 \rightarrow R^{p+1} f_* \mathcal{F}_1$$

Then it is clear that if 2-out-of-3 of the sheaves \mathcal{F}_i have property \mathcal{P} , then the higher direct images of the third are sandwiched in this exact complex between two coherent sheaves. Hence these higher direct images are also coherent by Lemma 9.2 and 9.3. Hence property \mathcal{P} holds for the third as well.

Let $Z \subset X$ be an integral closed subscheme. We have to find a coherent sheaf \mathcal{F} on X whose support is contained in Z , whose stalk at the generic point ξ of Z is a 1-dimensional vector space over $\kappa(\xi)$ such that \mathcal{P} holds for \mathcal{F} . Denote $g = f|_Z : Z \rightarrow S$ the restriction of f . Suppose we can find a coherent sheaf \mathcal{G}

on Z such that (a) \mathcal{G}_ξ is a 1-dimensional vector space over $\kappa(\xi)$, (b) $R^p g_* \mathcal{G} = 0$ for $p > 0$, and (c) $g_* \mathcal{G}$ is coherent. Then we can consider $\mathcal{F} = (Z \rightarrow X)_* \mathcal{G}$. As $Z \rightarrow X$ is a closed immersion we see that $(Z \rightarrow X)_* \mathcal{G}$ is coherent on X and $R^p(Z \rightarrow X)_* \mathcal{G} = 0$ for $p > 0$ (Lemma 9.9). Hence by the relative Leray spectral sequence (Cohomology, Lemma 13.8) we will have $R^p f_* \mathcal{F} = R^p g_* \mathcal{G} = 0$ for $p > 0$ and $f_* \mathcal{F} = g_* \mathcal{G}$ is coherent. Finally $\mathcal{F}_\xi = ((Z \rightarrow X)_* \mathcal{G})_\xi = \mathcal{G}_\xi$ which verifies the condition on the stalk at ξ . Hence everything depends on finding a coherent sheaf \mathcal{G} on Z which has properties (a), (b), and (c).

We can apply Chow's Lemma 18.1 to the morphism $Z \rightarrow S$. Thus we get a diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\pi} & Z' & \xrightarrow{i} & \mathbf{P}_S^m \\ & \searrow g & \downarrow g' & \swarrow & \\ & & S & & \end{array}$$

as in the statement of Chow's lemma. Also, let $U \subset Z$ be the dense open subscheme such that $\pi^{-1}(U) \rightarrow U$ is an isomorphism. By the discussion in Remark 18.2 we see that $i' = (i, \pi) : Z' \rightarrow \mathbf{P}_Z^m$ is a closed immersion. Hence

$$\mathcal{L} = i^* \mathcal{O}_{\mathbf{P}_S^m}(1) \cong (i')^* \mathcal{O}_{\mathbf{P}_Z^m}(1)$$

is g' -relatively ample and π -relatively ample (for example by Morphisms, Lemma 39.7). Hence by Lemma 16.2 there exists an $n \geq 0$ such that both $R^p \pi_* \mathcal{L}^{\otimes n} = 0$ for all $p > 0$ and $R^p (g')_* \mathcal{L}^{\otimes n} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_* \mathcal{L}^{\otimes n}$. Property (a) holds because $\pi_* \mathcal{L}^{\otimes n}|_U$ is an invertible sheaf (as $\pi^{-1}(U) \rightarrow U$ is an isomorphism). Properties (b) and (c) hold because by the relative Leray spectral sequence (Cohomology, Lemma 13.8) we have

$$E_2^{p,q} = R^p g_* R^q \pi_* \mathcal{L}^{\otimes n} \Rightarrow R^{p+q} (g')_* \mathcal{L}^{\otimes n}$$

and by choice of n the only nonzero terms in $E_2^{p,q}$ are those with $q = 0$ and the only nonzero terms of $R^{p+q} (g')_* \mathcal{L}^{\otimes n}$ are those with $p = q = 0$. This implies that $R^p g_* \mathcal{G} = 0$ for $p > 0$ and that $g_* \mathcal{G} = (g')_* \mathcal{L}^{\otimes n}$. Finally, applying the previous Lemma 16.3 we see that $g_* \mathcal{G} = (g')_* \mathcal{L}^{\otimes n}$ is coherent as desired. \square

Lemma 19.2. *Let $S = \operatorname{Spec}(A)$ with A a Noetherian ring. Let $f : X \rightarrow S$ be a proper morphism. Let \mathcal{F} be a coherent \mathcal{O}_X -module. Then $H^i(X, \mathcal{F})$ is a finite A -module for all $i \geq 0$.*

Proof. This is just the affine case of Proposition 19.1. Namely, by Lemmas 4.5 and 4.6 we know that $R^i f_* \mathcal{F}$ is the quasi-coherent sheaf associated to the A -module $H^i(X, \mathcal{F})$ and by Lemma 9.1 this is a coherent sheaf if and only if $H^i(X, \mathcal{F})$ is an A -module of finite type. \square

Lemma 19.3. *Let A be a Noetherian ring. Let B be a finitely generated graded A -algebra. Let $f : X \rightarrow \operatorname{Spec}(A)$ be a proper morphism. Set $\mathcal{B} = f^* \tilde{B}$. Let \mathcal{F} be a quasi-coherent graded \mathcal{B} -module of finite type.*

- (1) *For every $p \geq 0$ the graded B -module $H^p(X, \mathcal{F})$ is a finite B -module.*
- (2) *If \mathcal{L} is an ample invertible \mathcal{O}_X -module, then there exists an integer d_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$ and $d \geq d_0$.*

Proof. To prove this we consider the fibre product diagram

$$\begin{array}{ccc} X' = \operatorname{Spec}(B) \times_{\operatorname{Spec}(A)} X & \xrightarrow{\pi} & X \\ f' \downarrow & & \downarrow f \\ \operatorname{Spec}(B) & \longrightarrow & \operatorname{Spec}(A) \end{array}$$

Note that f' is a proper morphism, see Morphisms, Lemma 41.5. Also, B is a finitely generated A -algebra, and hence Noetherian (Algebra, Lemma 31.1). This implies that X' is a Noetherian scheme (Morphisms, Lemma 15.6). Note that X' is the relative spectrum of the quasi-coherent \mathcal{O}_X -algebra \mathcal{B} by Constructions, Lemma 4.6. Since \mathcal{F} is a quasi-coherent \mathcal{B} -module we see that there is a unique quasi-coherent $\mathcal{O}_{X'}$ -module \mathcal{F}' such that $\pi_* \mathcal{F}' = \mathcal{F}$, see Morphisms, Lemma 11.6. Since \mathcal{F} is finite type as a \mathcal{B} -module we conclude that \mathcal{F}' is a finite type $\mathcal{O}_{X'}$ -module (details omitted). In other words, \mathcal{F}' is a coherent $\mathcal{O}_{X'}$ -module (Lemma 9.1). Since the morphism $\pi : X' \rightarrow X$ is affine we have

$$H^p(X, \mathcal{F}) = H^p(X', \mathcal{F}')$$

by Lemma 2.4. Thus (1) follows from Lemma 19.2. Given \mathcal{L} as in (2) we set $\mathcal{L}' = \pi^* \mathcal{L}$. Note that \mathcal{L}' is ample on X' by Morphisms, Lemma 37.7. By the projection formula (Cohomology, Lemma 54.2) we have $\pi_*(\mathcal{F}' \otimes \mathcal{L}') = \mathcal{F} \otimes \mathcal{L}$. Thus part (2) follows by the same reasoning as above from Lemma 16.2. \square

20. The theorem on formal functions

In this section we study the behaviour of cohomology of sequences of sheaves either of the form $\{I^n \mathcal{F}\}_{n \geq 0}$ or of the form $\{\mathcal{F}/I^n \mathcal{F}\}_{n \geq 0}$ as n varies.

Here and below we use the following notation. Given a morphism of schemes $f : X \rightarrow Y$, a quasi-coherent sheaf \mathcal{F} on X , and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$ we denote $\mathcal{I}^n \mathcal{F}$ the quasi-coherent subsheaf generated by products of local sections of $f^{-1}(\mathcal{I}^n)$ and \mathcal{F} . In a formula

$$\mathcal{I}^n \mathcal{F} = \operatorname{Im}(f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow \mathcal{F}).$$

Note that there are natural maps

$$f^{-1}(\mathcal{I}^n) \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{I}^m \mathcal{F} \rightarrow f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{I}^m \mathcal{F} \rightarrow \mathcal{I}^{n+m} \mathcal{F}$$

Hence a section of \mathcal{I}^n will give rise to a map $R^p f_*(\mathcal{I}^m \mathcal{F}) \rightarrow R^p f_*(\mathcal{I}^{n+m} \mathcal{F})$ by functoriality of higher direct images. Localizing and then sheafifying we see that there are \mathcal{O}_Y -module maps

$$\mathcal{I}^n \otimes_{\mathcal{O}_Y} R^p f_*(\mathcal{I}^m \mathcal{F}) \rightarrow R^p f_*(\mathcal{I}^{n+m} \mathcal{F}).$$

In other words we see that $\bigoplus_{n \geq 0} R^p f_*(\mathcal{I}^n \mathcal{F})$ is a graded $\bigoplus_{n \geq 0} \mathcal{I}^n$ -module.

If $Y = \operatorname{Spec}(A)$ and $\mathcal{I} = \tilde{I}$ we denote $\mathcal{I}^n \mathcal{F}$ simply $I^n \mathcal{F}$. The maps introduced above give $M = \bigoplus H^p(X, I^n \mathcal{F})$ the structure of a graded $S = \bigoplus I^n$ -module. If f is proper, A is Noetherian and \mathcal{F} is coherent, then this turns out to be a module of finite type.

Lemma 20.1. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Set $B = \bigoplus_{n \geq 0} I^n$. Let $f : X \rightarrow \operatorname{Spec}(A)$ be a proper morphism. Let \mathcal{F} be a coherent sheaf on X . Then for every $p \geq 0$ the graded B -module $\bigoplus_{n \geq 0} H^p(X, I^n \mathcal{F})$ is a finite B -module.*

Proof. Let $\mathcal{B} = \bigoplus I^n \mathcal{O}_X = f^* \tilde{B}$. Then $\bigoplus I^n \mathcal{F}$ is a finite type graded \mathcal{B} -module. Hence the result follows from Lemma 19.3 part (1). \square

Lemma 20.2. *Given a morphism of schemes $f : X \rightarrow Y$, a quasi-coherent sheaf \mathcal{F} on X , and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$. Assume Y locally Noetherian, f proper, and \mathcal{F} coherent. Then*

$$\mathcal{M} = \bigoplus_{n \geq 0} R^p f_*(\mathcal{I}^n \mathcal{F})$$

is a graded $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ -module which is quasi-coherent and of finite type.

Proof. The statement is local on Y , hence this reduces to the case where Y is affine. In the affine case the result follows from Lemma 20.1. Details omitted. \square

Lemma 20.3. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \rightarrow \text{Spec}(A)$ be a proper morphism. Let \mathcal{F} be a coherent sheaf on X . Then for every $p \geq 0$ there exists an integer $c \geq 0$ such that*

- (1) *the multiplication map $I^{n-c} \otimes H^p(X, I^c \mathcal{F}) \rightarrow H^p(X, I^n \mathcal{F})$ is surjective for all $n \geq c$,*
- (2) *the image of $H^p(X, I^{n+m} \mathcal{F}) \rightarrow H^p(X, I^n \mathcal{F})$ is contained in the submodule $I^{m-e} H^p(X, I^n \mathcal{F})$ where $e = \max(0, c - n)$ for $n + m \geq c$, $n, m \geq 0$,*
- (3) *we have*

$$\text{Ker}(H^p(X, I^n \mathcal{F}) \rightarrow H^p(X, \mathcal{F})) = \text{Ker}(H^p(X, I^n \mathcal{F}) \rightarrow H^p(X, I^{n-c} \mathcal{F}))$$

for $n \geq c$,

- (4) *there are maps $I^n H^p(X, \mathcal{F}) \rightarrow H^p(X, I^{n-c} \mathcal{F})$ for $n \geq c$ such that the compositions*

$$H^p(X, I^n \mathcal{F}) \rightarrow I^{n-c} H^p(X, \mathcal{F}) \rightarrow H^p(X, I^{n-2c} \mathcal{F})$$

and

$$I^n H^p(X, \mathcal{F}) \rightarrow H^p(X, I^{n-c} \mathcal{F}) \rightarrow I^{n-2c} H^p(X, \mathcal{F})$$

for $n \geq 2c$ are the canonical ones, and

- (5) *the inverse systems $(H^p(X, I^n \mathcal{F}))$ and $(I^n H^p(X, \mathcal{F}))$ are pro-isomorphic.*

Proof. Write $M_n = H^p(X, I^n \mathcal{F})$ for $n \geq 1$ and $M_0 = H^p(X, \mathcal{F})$ so that we have maps $\dots \rightarrow M_3 \rightarrow M_2 \rightarrow M_1 \rightarrow M_0$. Setting $B = \bigoplus_{n \geq 0} I^n$, then $M = \bigoplus_{n \geq 0} M_n$ is a finite graded B -module, see Lemma 20.1. Observe that the products $B_n \otimes M_m \rightarrow M_{n+m}$, $a \otimes m \mapsto a \cdot m$ are compatible with the maps in our inverse system in the sense that the diagrams

$$\begin{array}{ccc} B_n \otimes_A M_m & \longrightarrow & M_{n+m} \\ \downarrow & & \downarrow \\ B_n \otimes_A M_{m'} & \longrightarrow & M_{n+m'} \end{array}$$

commute for $n, m' \geq 0$ and $m \geq m'$.

Proof of (1). Choose $d_1, \dots, d_t \geq 0$ and $x_i \in M_{d_i}$ such that M is generated by x_1, \dots, x_t over B . For any $c \geq \max\{d_i\}$ we conclude that $B_{n-c} \cdot M_c = M_n$ for $n \geq c$ and we conclude (1) is true.

Proof of (2). Let c be as in the proof of (1). Let $n + m \geq c$. We have $M_{n+m} = B_{n+m-c} \cdot M_c$. If $c > n$ then we use $M_c \rightarrow M_n$ and the compatibility of products with transition maps pointed out above to conclude that the image of $M_{n+m} \rightarrow M_n$ is

contained in $I^{n+m-c}M_n$. If $c \leq n$, then we write $M_{n+m} = B_m \cdot B_{n-c} \cdot M_c = B_m \cdot M_n$ to see that the image is contained in $I^m M_n$. This proves (2).

Let $K_n \subset M_n$ be the kernel of the map $M_n \rightarrow M_0$. The compatibility of products with transition maps pointed out above shows that $K = \bigoplus K_n \subset M$ is a graded B -submodule. As B is Noetherian and M is a finitely generated graded B -module, this shows that K is a finitely generated graded B -module. Choose $d'_1, \dots, d'_{t'} \geq 0$ and $y_i \in K_{d'_i}$ such that K is generated by $y_1, \dots, y_{t'}$ over B . Set $c = \max(d'_i, d'_j)$. Since $y_i \in \text{Ker}(M_{d'_i} \rightarrow M_0)$ we see that $B_n \cdot y_i \subset \text{Ker}(M_{n+d'_i} \rightarrow M_n)$. In this way we see that $K_n = \text{Ker}(M_n \rightarrow M_{n-c})$ for $n \geq c$. This proves (3).

Consider the following commutative solid diagram

$$\begin{array}{ccccc} I^n \otimes_A M_0 & \longrightarrow & I^n M_0 & \longrightarrow & M_0 \\ \downarrow & & \downarrow & & \downarrow \\ M_n & \longrightarrow & M_{n-c} & \longrightarrow & M_0 \end{array}$$

Since the kernel of the surjective arrow $I^n \otimes_A M_0 \rightarrow I^n M_0$ maps into K_n by the above we obtain the dotted arrow and the composition $I^n M_0 \rightarrow M_{n-c} \rightarrow M_0$ is the canonical map. Then clearly the composition $I^n M_0 \rightarrow M_{n-c} \rightarrow I^{n-2c} M_0$ is the canonical map for $n \geq 2c$. Consider the composition $M_n \rightarrow I^{n-c} M_0 \rightarrow M_{n-2c}$. The first map sends an element of the form $a \cdot m$ with $a \in I^{n-c}$ and $m \in M_c$ to am' where m' is the image of m in M_0 . Then the second map sends this to $a \cdot m'$ in M_{n-2c} and we see (4) is true.

Part (5) is an immediate consequence of (4) and the definition of morphisms of pro-objects. \square

In the situation of Lemmas 20.1 and 20.3 consider the inverse system

$$\mathcal{F}/I\mathcal{F} \leftarrow \mathcal{F}/I^2\mathcal{F} \leftarrow \mathcal{F}/I^3\mathcal{F} \leftarrow \dots$$

We would like to know what happens to the cohomology groups. Here is a first result.

Lemma 20.4. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \rightarrow \text{Spec}(A)$ be a proper morphism. Let \mathcal{F} be a coherent sheaf on X . Fix $p \geq 0$. There exists a $c \geq 0$ such that*

- (1) *for all $n \geq c$ we have*

$$\text{Ker}(H^p(X, \mathcal{F}) \rightarrow H^p(X, \mathcal{F}/I^n \mathcal{F})) \subset I^{n-c} H^p(X, \mathcal{F}).$$

- (2) *the inverse system*

$$(H^p(X, \mathcal{F}/I^n \mathcal{F}))_{n \in \mathbf{N}}$$

satisfies the Mittag-Leffler condition (see Homology, Definition 31.2), and

- (3) *we have*

$$\text{Im}(H^p(X, \mathcal{F}/I^k \mathcal{F}) \rightarrow H^p(X, \mathcal{F}/I^n \mathcal{F})) = \text{Im}(H^p(X, \mathcal{F}) \rightarrow H^p(X, \mathcal{F}/I^n \mathcal{F}))$$

for all $k \geq n + c$.

Proof. Let $c = \max\{c_p, c_{p+1}\}$, where c_p, c_{p+1} are the integers found in Lemma 20.3 for H^p and H^{p+1} .

Let us prove part (1). Consider the short exact sequence

$$0 \rightarrow I^n \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/I^n \mathcal{F} \rightarrow 0$$

From the long exact cohomology sequence we see that

$$\text{Ker}(H^p(X, \mathcal{F}) \rightarrow H^p(X, \mathcal{F}/I^n \mathcal{F})) = \text{Im}(H^p(X, I^n \mathcal{F}) \rightarrow H^p(X, \mathcal{F}))$$

Hence by Lemma 20.3 part (2) we see that this is contained in $I^{n-c} H^p(X, \mathcal{F})$ for $n \geq c$.

Note that part (3) implies part (2) by definition of the Mittag-Leffler systems.

Let us prove part (3). Fix an n . Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I^n \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/I^n \mathcal{F} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & I^{n+m} \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/I^{n+m} \mathcal{F} \longrightarrow 0 \end{array}$$

This gives rise to the following commutative diagram

$$\begin{array}{ccccccc} H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}/I^n \mathcal{F}) & \xrightarrow{\delta} & H^{p+1}(X, I^n \mathcal{F}) & \longrightarrow & H^{p+1}(X, \mathcal{F}) \\ \uparrow 1 & & \uparrow \gamma & & \uparrow \alpha & & \uparrow 1 \\ H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}/I^{n+m} \mathcal{F}) & \longrightarrow & H^{p+1}(X, I^{n+m} \mathcal{F}) & \xrightarrow{\beta} & H^{p+1}(X, \mathcal{F}) \end{array}$$

with exact rows. By Lemma 20.3 part (4) the kernel of β is equal to the kernel of α for $m \geq c$. By a diagram chase this shows that the image of γ is contained in the kernel of δ which shows that part (3) is true (set $k = n + m$ to get it). \square

Theorem 20.5 (Theorem on formal functions). *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \rightarrow \text{Spec}(A)$ be a proper morphism. Let \mathcal{F} be a coherent sheaf on X . Fix $p \geq 0$. The system of maps*

$$H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}/I^n \mathcal{F})$$

define an isomorphism of limits

$$H^p(X, \mathcal{F})^\wedge \longrightarrow \lim_n H^p(X, \mathcal{F}/I^n \mathcal{F})$$

where the left hand side is the completion of the A -module $H^p(X, \mathcal{F})$ with respect to the ideal I , see Algebra, Section 96. Moreover, this is in fact a homeomorphism for the limit topologies.

Proof. This follows from Lemma 20.4 as follows. Set $M = H^p(X, \mathcal{F})$, $M_n = H^p(X, \mathcal{F}/I^n \mathcal{F})$, and denote $N_n = \text{Im}(M \rightarrow M_n)$. By Lemma 20.4 parts (2) and (3) we see that (M_n) is a Mittag-Leffler system with $N_n \subset M_n$ equal to the image of M_k for all $k \gg n$. It follows that $\lim M_n = \lim N_n$ as topological modules (with limit topologies). On the other hand, the N_n form an inverse system of quotients of the module M and hence $\lim N_n$ is the completion of M with respect to the topology given by the kernels $K_n = \text{Ker}(M \rightarrow N_n)$. By Lemma 20.4 part (1) we have $K_n \subset I^{n-c} M$ and since $N_n \subset M_n$ is annihilated by I^n we have $I^n M \subset K_n$. Thus the topology defined using the submodules K_n as a fundamental system of open neighbourhoods of 0 is the same as the I -adic topology and we find that

the induced map $M^\wedge = \varprojlim M/I^n M \rightarrow \varprojlim N_n = \varprojlim M_n$ is an isomorphism of topological modules³. \square

Lemma 20.6. *Let A be a ring. Let $I \subset A$ be an ideal. Assume A is Noetherian and complete with respect to I . Let $f : X \rightarrow \operatorname{Spec}(A)$ be a proper morphism. Let \mathcal{F} be a coherent sheaf on X . Then*

$$H^p(X, \mathcal{F}) = \varprojlim_n H^p(X, \mathcal{F}/I^n \mathcal{F})$$

for all $p \geq 0$.

Proof. This is a reformulation of the theorem on formal functions (Theorem 20.5) in the case of a complete Noetherian base ring. Namely, in this case the A -module $H^p(X, \mathcal{F})$ is finite (Lemma 19.2) hence I -adically complete (Algebra, Lemma 97.1) and we see that completion on the left hand side is not necessary. \square

Lemma 20.7. *Given a morphism of schemes $f : X \rightarrow Y$ and a quasi-coherent sheaf \mathcal{F} on X . Assume*

- (1) Y locally Noetherian,
- (2) f proper, and
- (3) \mathcal{F} coherent.

Let $y \in Y$ be a point. Consider the infinitesimal neighbourhoods

$$\begin{array}{ccc} X_n = \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \times_Y X & \xrightarrow{i_n} & X \\ f_n \downarrow & & \downarrow f \\ \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) & \xrightarrow{c_n} & Y \end{array}$$

of the fibre $X_1 = X_y$ and set $\mathcal{F}_n = i_n^ \mathcal{F}$. Then we have*

$$(R^p f_* \mathcal{F})_y^\wedge \cong \varprojlim_n H^p(X_n, \mathcal{F}_n)$$

as $\mathcal{O}_{Y,y}^\wedge$ -modules.

Proof. This is just a reformulation of a special case of the theorem on formal functions, Theorem 20.5. Let us spell it out. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : \operatorname{Spec}(\mathcal{O}_{Y,y}) \rightarrow Y$, see Schemes, Equation (13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \rightarrow \operatorname{Spec}(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^* R^p f_* \mathcal{F} = R^p f'_* \mathcal{F}'$ by Lemma 5.2. Moreover, the infinitesimal neighbourhoods of the fibre X_y and X'_y are identified (verification omitted; hint: the morphisms c_n factor through c).

Hence we may assume that $Y = \operatorname{Spec}(A)$ is the spectrum of a Noetherian local ring A with maximal ideal \mathfrak{m} and that $y \in Y$ corresponds to the closed point (i.e., to \mathfrak{m}). In particular it follows that

$$(R^p f_* \mathcal{F})_y = \Gamma(Y, R^p f_* \mathcal{F}) = H^p(X, \mathcal{F}).$$

³To be sure, the limit topology on M^\wedge is the same as its I -adic topology as follows from Algebra, Lemma 96.3. See More on Algebra, Section 36.

In this case also, the morphisms c_n are each closed immersions. Hence their base changes i_n are closed immersions as well. Note that $i_{n,*}\mathcal{F}_n = i_{n,*}i_n^*\mathcal{F} = \mathcal{F}/\mathfrak{m}^n\mathcal{F}$. By the Leray spectral sequence for i_n , and Lemma 9.9 we see that

$$H^p(X_n, \mathcal{F}_n) = H^p(X, i_{n,*}\mathcal{F}_n) = H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F})$$

Hence we may indeed apply the theorem on formal functions to compute the limit in the statement of the lemma and we win. \square

Here is a lemma which we will generalize later to fibres of dimension > 0 , namely the next lemma.

Lemma 20.8. *Let $f : X \rightarrow Y$ be a morphism of schemes. Let $y \in Y$. Assume*

- (1) *Y locally Noetherian,*
- (2) *f is proper, and*
- (3) *$f^{-1}(\{y\})$ is finite.*

Then for any coherent sheaf \mathcal{F} on X we have $(R^p f_ \mathcal{F})_y = 0$ for all $p > 0$.*

Proof. The fibre X_y is finite, and by Morphisms, Lemma 20.7 it is a finite discrete space. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same. Hence each of the schemes X_n is affine according to Schemes, Lemma 11.8. Hence it follows that $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > 0$. Hence we see that $(R^p f_* \mathcal{F})_y^\wedge = 0$ by Lemma 20.7. Note that $R^p f_* \mathcal{F}$ is coherent by Proposition 19.1 and hence $R^p f_* \mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$ -module. By Nakayama's lemma (Algebra, Lemma 20.1) if the completion of a finite module over a local ring is zero, then the module is zero. Whence $(R^p f_* \mathcal{F})_y = 0$. \square

Lemma 20.9. *Let $f : X \rightarrow Y$ be a morphism of schemes. Let $y \in Y$. Assume*

- (1) *Y locally Noetherian,*
- (2) *f is proper, and*
- (3) *$\dim(X_y) = d$.*

Then for any coherent sheaf \mathcal{F} on X we have $(R^p f_ \mathcal{F})_y = 0$ for all $p > d$.*

Proof. The fibre X_y is of finite type over $\text{Spec}(\kappa(y))$. Hence X_y is a Noetherian scheme by Morphisms, Lemma 15.6. Hence the underlying topological space of X_y is Noetherian, see Properties, Lemma 5.5. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same as that of X_y . Hence $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > d$ by Cohomology, Proposition 20.7. Hence we see that $(R^p f_* \mathcal{F})_y^\wedge = 0$ by Lemma 20.7 for $p > d$. Note that $R^p f_* \mathcal{F}$ is coherent by Proposition 19.1 and hence $R^p f_* \mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$ -module. By Nakayama's lemma (Algebra, Lemma 20.1) if the completion of a finite module over a local ring is zero, then the module is zero. Whence $(R^p f_* \mathcal{F})_y = 0$. \square

21. Applications of the theorem on formal functions

We will add more here as needed. For the moment we need the following characterization of finite morphisms in the Noetherian case.

Lemma 21.1. *(For a more general version see More on Morphisms, Lemma 44.1.) Let $f : X \rightarrow S$ be a morphism of schemes. Assume S is locally Noetherian. The following are equivalent*

- (1) *f is finite, and*
- (2) *f is proper with finite fibres.*

Proof. A finite morphism is proper according to Morphisms, Lemma 44.11. A finite morphism is quasi-finite according to Morphisms, Lemma 44.10. A quasi-finite morphism has finite fibres, see Morphisms, Lemma 20.10. Hence a finite morphism is proper and has finite fibres.

Assume f is proper with finite fibres. We want to show f is finite. In fact it suffices to prove f is affine. Namely, if f is affine, then it follows that f is integral by Morphisms, Lemma 44.7 whereupon it follows from Morphisms, Lemma 44.4 that f is finite.

To show that f is affine we may assume that S is affine, and our goal is to show that X is affine too. Since f is proper we see that X is separated and quasi-compact. Hence we may use the criterion of Lemma 3.2 to prove that X is affine. To see this let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type ideal sheaf. In particular \mathcal{I} is a coherent sheaf on X . By Lemma 20.8 we conclude that $R^1 f_* \mathcal{I}_s = 0$ for all $s \in S$. In other words, $R^1 f_* \mathcal{I} = 0$. Hence we see from the Leray Spectral Sequence for f that $H^1(X, \mathcal{I}) = H^1(S, f_* \mathcal{I})$. Since S is affine, and $f_* \mathcal{I}$ is quasi-coherent (Schemes, Lemma 24.1) we conclude $H^1(S, f_* \mathcal{I}) = 0$ from Lemma 2.2 as desired. Hence $H^1(X, \mathcal{I}) = 0$ as desired. \square

As a consequence we have the following useful result.

Lemma 21.2. *(For a more general version see More on Morphisms, Lemma 44.2.)*

Let $f : X \rightarrow S$ be a morphism of schemes. Let $s \in S$. Assume

- (1) *S is locally Noetherian,*
- (2) *f is proper, and*
- (3) *$f^{-1}(\{s\})$ is a finite set.*

Then there exists an open neighbourhood $V \subset S$ of s such that $f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V$ is finite.

Proof. The morphism f is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 20.7. By Morphisms, Lemma 56.2 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since f is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of s with $Z \cap V = \emptyset$. Then $f^{-1}(V) \rightarrow V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 20.9), hence has finite fibres (Morphisms, Lemma 20.10), hence is finite by Lemma 21.1. \square

Lemma 21.3. *Let $f : X \rightarrow Y$ be a proper morphism of schemes with Y Noetherian. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let \mathcal{F} be a coherent \mathcal{O}_X -module. Let $y \in Y$ be a point such that \mathcal{L}_y is ample on X_y . Then there exists a d_0 such that for all $d \geq d_0$ we have*

$$R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = 0 \text{ for } p > 0$$

and the map

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y \longrightarrow H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$$

is surjective.

Proof. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : \text{Spec}(\mathcal{O}_{Y,y}) \rightarrow Y$, see Schemes, Equation (13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \rightarrow \text{Spec}(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^* R^p f_* \mathcal{F} = R^p f'_* \mathcal{F}'$ by Lemma 5.2. Moreover, the fibres X_y and X'_y are identified. Hence we may assume that $Y = \text{Spec}(A)$ is the spectrum of a Noetherian local ring $(A, \mathfrak{m}, \kappa)$ and $y \in Y$ corresponds to

\mathfrak{m} . In this case $R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ for all $p \geq 0$. Denote $f_y : X_y \rightarrow \text{Spec}(\kappa)$ the projection.

Let $B = \text{Gr}_{\mathfrak{m}}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$. Consider the sheaf $\mathcal{B} = f_y^* \tilde{B}$ of quasi-coherent graded \mathcal{O}_{X_y} -algebras. We will use notation as in Section 20 with I replaced by \mathfrak{m} . Since X_y is the closed subscheme of X cut out by $\mathfrak{m}\mathcal{O}_X$ we may think of $\mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F}$ as a coherent \mathcal{O}_{X_y} -module, see Lemma 9.8. Then $\bigoplus_{n \geq 0} \mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F}$ is a quasi-coherent graded \mathcal{B} -module of finite type because it is generated in degree zero over \mathcal{B} and because the degree zero part is $\mathcal{F}_y = \mathcal{F} / \mathfrak{m}\mathcal{F}$ which is a coherent \mathcal{O}_{X_y} -module. Hence by Lemma 19.3 part (2) we see that

$$H^p(X_y, \mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d}) = 0$$

for all $p > 0$, $d \geq d_0$, and $n \geq 0$. By Lemma 2.4 this is the same as the statement that $H^p(X, \mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$.

Consider the short exact sequences

$$0 \rightarrow \mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \rightarrow \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \rightarrow \mathcal{F} / \mathfrak{m}^n \mathcal{F} \rightarrow 0$$

of coherent \mathcal{O}_X -modules. Tensoring with $\mathcal{L}^{\otimes d}$ is an exact functor and we obtain short exact sequences

$$0 \rightarrow \mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \rightarrow \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \rightarrow \mathcal{F} / \mathfrak{m}^n \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \rightarrow 0$$

Using the long exact cohomology sequence and the vanishing above we conclude (using induction) that

- (1) $H^p(X, \mathcal{F} / \mathfrak{m}^n \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$, and
- (2) $H^0(X, \mathcal{F} / \mathfrak{m}^n \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \rightarrow H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective for all $d \geq d_0$ and $n \geq 1$.

By the theorem on formal functions (Theorem 20.5) we find that the \mathfrak{m} -adic completion of $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. Since $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is a finite A -module by Lemma 19.2 it follows from Nakayama's lemma (Algebra, Lemma 20.1) that $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. For $p = 0$ we deduce from Lemma 20.4 part (3) that $H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \rightarrow H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective, which gives the final statement of the lemma. \square

Lemma 21.4. (For a more general version see *More on Morphisms*, Lemma 50.3.) Let $f : X \rightarrow Y$ be a proper morphism of schemes with Y Noetherian. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $y \in Y$ be a point such that \mathcal{L}_y is ample on X_y . Then there is an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)/V$.

Proof. Pick d_0 as in Lemma 21.3 for $\mathcal{F} = \mathcal{O}_X$. Pick $d \geq d_0$ so that we can find $r \geq 0$ and sections $s_{y,0}, \dots, s_{y,r} \in H^0(X_y, \mathcal{L}_y^{\otimes d})$ which define a closed immersion

$$\varphi_y = \varphi_{\mathcal{L}_y^{\otimes d}, (s_{y,0}, \dots, s_{y,r})} : X_y \rightarrow \mathbf{P}_{\kappa(y)}^r.$$

This is possible by Morphisms, Lemma 39.4 but we also use Morphisms, Lemma 41.7 to see that φ_y is a closed immersion and Constructions, Section 13 for the description of morphisms into projective space in terms of invertible sheaves and sections. By our choice of d_0 , after replacing Y by an open neighbourhood of y , we can choose $s_0, \dots, s_r \in H^0(X, \mathcal{L}^{\otimes d})$ mapping to $s_{y,0}, \dots, s_{y,r}$. Let $X_{s_i} \subset X$ be the open subset where s_i is a generator of $\mathcal{L}^{\otimes d}$. Since the $s_{y,i}$ generate $\mathcal{L}_y^{\otimes d}$ we see that $X_y \subset U = \bigcup X_{s_i}$. Since $X \rightarrow Y$ is closed, we see that there is an open

neighbourhood $y \in V \subset Y$ such that $f^{-1}(V) \subset U$. After replacing Y by V we may assume that the s_i generate $\mathcal{L}^{\otimes d}$. Thus we obtain a morphism

$$\varphi = \varphi_{\mathcal{L}^{\otimes d}, (s_0, \dots, s_r)} : X \longrightarrow \mathbf{P}_Y^r$$

with $\mathcal{L}^{\otimes d} \cong \varphi^* \mathcal{O}_{\mathbf{P}_Y^r}(1)$ whose base change to y gives φ_y .

We will finish the proof by a sleight of hand; the “correct” proof proceeds by directly showing that φ is a closed immersion after base changing to an open neighbourhood of y . Namely, by Lemma 21.2 we see that φ is a finite over an open neighbourhood of the fibre $\mathbf{P}_{\kappa(y)}^r$ of $\mathbf{P}_Y^r \rightarrow Y$ above y . Using that $\mathbf{P}_Y^r \rightarrow Y$ is closed, after shrinking Y we may assume that φ is finite. Then $\mathcal{L}^{\otimes d} \cong \varphi^* \mathcal{O}_{\mathbf{P}_Y^r}(1)$ is ample by the very general Morphisms, Lemma 37.7. \square

22. Cohomology and base change, III

In this section we prove the simplest case of a very general phenomenon that will be discussed in Derived Categories of Schemes, Section 22. Please see Remark 22.2 for a translation of the following lemma into algebra.

Lemma 22.1. *Let A be a Noetherian ring and set $S = \operatorname{Spec}(A)$. Let $f : X \rightarrow S$ be a proper morphism of schemes. Let \mathcal{F} be a coherent \mathcal{O}_X -module flat over S . Then*

- (1) *$R\Gamma(X, \mathcal{F})$ is a perfect object of $D(A)$, and*
- (2) *for any ring map $A \rightarrow A'$ the base change map*

$$R\Gamma(X, \mathcal{F}) \otimes_A^{\mathbf{L}} A' \longrightarrow R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

is an isomorphism.

Proof. Choose a finite affine open covering $X = \bigcup_{i=1, \dots, n} U_i$. By Lemmas 7.1 and 7.2 the Čech complex $K^\bullet = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$ satisfies

$$K^\bullet \otimes_A A' = R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

for all ring maps $A \rightarrow A'$. Let $K_{alt}^\bullet = \check{C}_{alt}^\bullet(\mathcal{U}, \mathcal{F})$ be the alternating Čech complex. By Cohomology, Lemma 23.6 there is a homotopy equivalence $K_{alt}^\bullet \rightarrow K^\bullet$ of A -modules. In particular, we have

$$K_{alt}^\bullet \otimes_A A' = R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

as well. Since \mathcal{F} is flat over A we see that each K_{alt}^n is flat over A (see Morphisms, Lemma 25.2). Since moreover K_{alt}^\bullet is bounded above (this is why we switched to the alternating Čech complex) $K_{alt}^\bullet \otimes_A A' = K_{alt}^\bullet \otimes_A^{\mathbf{L}} A'$ by the definition of derived tensor products (see More on Algebra, Section 59). By Lemma 19.2 the cohomology groups $H^i(K_{alt}^\bullet)$ are finite A -modules. As K_{alt}^\bullet is bounded, we conclude that K_{alt}^\bullet is pseudo-coherent, see More on Algebra, Lemma 64.17. Given any A -module M set $A' = A \oplus M$ where M is a square zero ideal, i.e., $(a, m) \cdot (a', m') = (aa', am' + a'm)$. By the above we see that $K_{alt}^\bullet \otimes_A^{\mathbf{L}} A'$ has cohomology in degrees $0, \dots, n$. Hence $K_{alt}^\bullet \otimes_A^{\mathbf{L}} M$ has cohomology in degrees $0, \dots, n$. Hence K_{alt}^\bullet has finite Tor dimension, see More on Algebra, Definition 66.1. We win by More on Algebra, Lemma 74.2. \square

Remark 22.2. A consequence of Lemma 22.1 is that there exists a finite complex of finite projective A -modules M^\bullet such that we have

$$H^i(X_{A'}, \mathcal{F}_{A'}) = H^i(M^\bullet \otimes_A A')$$

functorially in A' . The condition that \mathcal{F} is flat over A is essential, see [Har98].

23. Coherent formal modules

As we do not yet have the theory of formal schemes to our disposal, we develop a bit of language that replaces the notion of a “coherent module on a Noetherian adic formal scheme”.

Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. We will consider inverse systems (\mathcal{F}_n) of coherent \mathcal{O}_X -modules such that

- (1) \mathcal{F}_n is annihilated by \mathcal{I}^n , and
- (2) the transition maps induce isomorphisms $\mathcal{F}_{n+1}/\mathcal{I}^n \mathcal{F}_{n+1} \rightarrow \mathcal{F}_n$.

A morphism of such inverse systems is defined as usual. Let us denote the category of these inverse systems with $\text{Coh}(X, \mathcal{I})$. We are going to proceed by proving a bunch of lemmas about objects in this category. In fact, most of the lemmas that follow are straightforward consequences of the following description of the category in the affine case.

Lemma 23.1. *If $X = \text{Spec}(A)$ is the spectrum of a Noetherian ring and \mathcal{I} is the quasi-coherent sheaf of ideals associated to the ideal $I \subset A$, then $\text{Coh}(X, \mathcal{I})$ is equivalent to the category of finite A^\wedge -modules where A^\wedge is the completion of A with respect to I .*

Proof. Let $\text{Mod}_{A, I}^{fg}$ be the category of inverse systems (M_n) of finite A -modules satisfying: (1) M_n is annihilated by I^n and (2) $M_{n+1}/I^n M_{n+1} = M_n$. By the correspondence between coherent sheaves on X and finite A -modules (Lemma 9.1) it suffices to show $\text{Mod}_{A, I}^{fg}$ is equivalent to the category of finite A^\wedge -modules. To see this it suffices to prove that given an object (M_n) of $\text{Mod}_{A, I}^{fg}$ the module

$$M = \lim M_n$$

is a finite A^\wedge -module and that $M/I^n M = M_n$. As the transition maps are surjective, we see that $M \rightarrow M_1$ is surjective. Pick $x_1, \dots, x_t \in M$ which map to generators of M_1 . This induces a map of systems $(A/I^n)^{\oplus t} \rightarrow M_n$. By Nakayama’s lemma (Algebra, Lemma 20.1) these maps are surjective. Let $K_n \subset (A/I^n)^{\oplus t}$ be the kernel. Property (2) implies that $K_{n+1} \rightarrow K_n$ is surjective, in particular the system (K_n) satisfies the Mittag-Leffler condition. By Homology, Lemma 31.3 we obtain an exact sequence $0 \rightarrow K \rightarrow (A^\wedge)^{\oplus t} \rightarrow M \rightarrow 0$ with $K = \lim K_n$. Hence M is a finite A^\wedge -module. As $K \rightarrow K_n$ is surjective it follows that

$$M/I^n M = \text{Coker}(K \rightarrow (A/I^n)^{\oplus t}) = (A/I^n)^{\oplus t} / K_n = M_n$$

as desired. \square

Lemma 23.2. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals.*

- (1) *The category $\text{Coh}(X, \mathcal{I})$ is abelian.*
- (2) *For $U \subset X$ open the restriction functor $\text{Coh}(X, \mathcal{I}) \rightarrow \text{Coh}(U, \mathcal{I}|_U)$ is exact.*
- (3) *Exactness in $\text{Coh}(X, \mathcal{I})$ may be checked by restricting to the members of an open covering of X .*

Proof. Let $\alpha = (\alpha_n) : (\mathcal{F}_n) \rightarrow (\mathcal{G}_n)$ be a morphism of $\text{Coh}(X, \mathcal{I})$. The cokernel of α is the inverse system $(\text{Coker}(\alpha_n))$ (details omitted). To describe the kernel let

$$\mathcal{K}'_{l, m} = \text{Im}(\text{Ker}(\alpha_l) \rightarrow \mathcal{F}_m)$$

for $l \geq m$. We claim:

- (a) the inverse system $(\mathcal{K}'_{l,m})_{l \geq m}$ is eventually constant, say with value \mathcal{K}'_m ,
- (b) the system $(\mathcal{K}'_m/\mathcal{I}^n \mathcal{K}'_m)_{m \geq n}$ is eventually constant, say with value \mathcal{K}_n ,
- (c) the system (\mathcal{K}_n) forms an object of $\text{Coh}(X, \mathcal{I})$, and
- (d) this object is the kernel of α .

To see (a), (b), and (c) we may work affine locally, say $X = \text{Spec}(A)$ and \mathcal{I} corresponds to the ideal $I \subset A$. By Lemma 23.1 α corresponds to a map $f : M \rightarrow N$ of finite A^\wedge -modules. Denote $K = \text{Ker}(f)$. Note that A^\wedge is a Noetherian ring (Algebra, Lemma 97.6). Choose an integer $c \geq 0$ such that $K \cap I^n M \subset I^{n-c} K$ for $n \geq c$ (Algebra, Lemma 51.2) and which satisfies Algebra, Lemma 51.3 for the map f and the ideal $I^\wedge = IA^\wedge$. Then $\mathcal{K}'_{l,m}$ corresponds to the A -module

$$K'_{l,m} = \frac{a^{-1}(I^l N) + I^m M}{I^m M} = \frac{K + I^{l-c} f^{-1}(I^c N) + I^m M}{I^m M} = \frac{K + I^m M}{I^m M}$$

where the last equality holds if $l \geq m + c$. So \mathcal{K}'_m corresponds to the A -module $K/K \cap I^m M$ and $\mathcal{K}'_m/\mathcal{I}^n \mathcal{K}'_m$ corresponds to

$$\frac{K}{K \cap I^m M + I^n K} = \frac{K}{I^n K}$$

for $m \geq n + c$ by our choice of c above. Hence \mathcal{K}_n corresponds to $K/I^n K$.

We prove (d). It is clear from the description on affines above that the composition $(\mathcal{K}_n) \rightarrow (\mathcal{F}_n) \rightarrow (\mathcal{G}_n)$ is zero. Let $\beta : (\mathcal{H}_n) \rightarrow (\mathcal{F}_n)$ be a morphism such that $\alpha \circ \beta = 0$. Then $\mathcal{H}_l \rightarrow \mathcal{F}_l$ maps into $\text{Ker}(\alpha_l)$. Since $\mathcal{H}_m = \mathcal{H}_l/\mathcal{I}^m \mathcal{H}_l$ for $l \geq m$ we obtain a system of maps $\mathcal{H}_m \rightarrow \mathcal{K}'_{l,m}$. Thus a map $\mathcal{H}_m \rightarrow \mathcal{K}'_m$. Since $\mathcal{H}_n = \mathcal{H}_m/\mathcal{I}^n \mathcal{H}_m$ we obtain a system of maps $\mathcal{H}_n \rightarrow \mathcal{K}'_m/\mathcal{I}^n \mathcal{K}'_m$ and hence a map $\mathcal{H}_n \rightarrow \mathcal{K}_n$ as desired.

To finish the proof of (1) we still have to show that $\text{Coim} = \text{Im}$ in $\text{Coh}(X, \mathcal{I})$. We have seen above that taking kernels and cokernels commutes, over affines, with the description of $\text{Coh}(X, \mathcal{I})$ as a category of modules. Since $\text{Im} = \text{Coim}$ holds in the category of modules this gives $\text{Coim} = \text{Im}$ in $\text{Coh}(X, \mathcal{I})$. Parts (2) and (3) of the lemma are immediate from our construction of kernels and cokernels. \square

Lemma 23.3. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. A map $(\mathcal{F}_n) \rightarrow (\mathcal{G}_n)$ is surjective in $\text{Coh}(X, \mathcal{I})$ if and only if $\mathcal{F}_1 \rightarrow \mathcal{G}_1$ is surjective.*

Proof. Omitted. Hint: Look on affine opens, use Lemma 23.1, and use Algebra, Lemma 20.1. \square

Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. There is a functor

$$(23.3.1) \quad \text{Coh}(\mathcal{O}_X) \longrightarrow \text{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

which associates to the coherent \mathcal{O}_X -module \mathcal{F} the object $\mathcal{F}^\wedge = (\mathcal{F}/\mathcal{I}^n \mathcal{F})$ of $\text{Coh}(X, \mathcal{I})$.

Lemma 23.4. *The functor (23.3.1) is exact.*

Proof. It suffices to check this locally on X . Hence we may assume X is affine, i.e., we have a situation as in Lemma 23.1. The functor is the functor $\text{Mod}_A^{fg} \rightarrow \text{Mod}_{A^\wedge}^{fg}$ which associates to a finite A -module M the completion M^\wedge . Thus the result follows from Algebra, Lemma 97.2. \square

Lemma 23.5. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let \mathcal{F}, \mathcal{G} be coherent \mathcal{O}_X -modules. Set $\mathcal{H} = \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$. Then*

$$\lim H^0(X, \mathcal{H}/\mathcal{I}^n \mathcal{H}) = \text{Mor}_{\text{Coh}(X, \mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge).$$

Proof. To prove this we may work affine locally on X . Hence we may assume $X = \text{Spec}(A)$ and \mathcal{F}, \mathcal{G} given by finite A -module M and N . Then \mathcal{H} corresponds to the finite A -module $H = \text{Hom}_A(M, N)$. The statement of the lemma becomes the statement

$$H^\wedge = \text{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$$

via the equivalence of Lemma 23.1. By Algebra, Lemma 97.2 (used 3 times) we have

$$H^\wedge = \text{Hom}_A(M, N) \otimes_A A^\wedge = \text{Hom}_{A^\wedge}(M \otimes_A A^\wedge, N \otimes_A A^\wedge) = \text{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$$

where the second equality uses that A^\wedge is flat over A (see More on Algebra, Lemma 65.4). The lemma follows. \square

Let X be a Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. We say an object (\mathcal{F}_n) of $\text{Coh}(X, \mathcal{I})$ is \mathcal{I} -power torsion or is annihilated by a power of \mathcal{I} if there exists a $c \geq 1$ such that $\mathcal{F}_n = \mathcal{F}_c$ for all $n \geq c$. If this is the case we will say that (\mathcal{F}_n) is annihilated by \mathcal{I}^c . If $X = \text{Spec}(A)$ is affine, then, via the equivalence of Lemma 23.1, these objects corresponds exactly to the finite A -modules annihilated by a power of I or by I^c .

Lemma 23.6. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let \mathcal{G} be a coherent \mathcal{O}_X -module. Let (\mathcal{F}_n) an object of $\text{Coh}(X, \mathcal{I})$.*

- (1) *If $\alpha : (\mathcal{F}_n) \rightarrow \mathcal{G}^\wedge$ is a map whose kernel and cokernel are annihilated by a power of \mathcal{I} , then there exists a unique (up to unique isomorphism) triple (\mathcal{F}, a, β) where*
 - (a) \mathcal{F} is a coherent \mathcal{O}_X -module,
 - (b) $a : \mathcal{F} \rightarrow \mathcal{G}$ is an \mathcal{O}_X -module map whose kernel and cokernel are annihilated by a power of \mathcal{I} ,
 - (c) $\beta : (\mathcal{F}_n) \rightarrow \mathcal{F}^\wedge$ is an isomorphism, and
 - (d) $\alpha = a^\wedge \circ \beta$.
- (2) *If $\alpha : \mathcal{G}^\wedge \rightarrow (\mathcal{F}_n)$ is a map whose kernel and cokernel are annihilated by a power of \mathcal{I} , then there exists a unique (up to unique isomorphism) triple (\mathcal{F}, a, β) where*
 - (a) \mathcal{F} is a coherent \mathcal{O}_X -module,
 - (b) $a : \mathcal{G} \rightarrow \mathcal{F}$ is an \mathcal{O}_X -module map whose kernel and cokernel are annihilated by a power of \mathcal{I} ,
 - (c) $\beta : \mathcal{F}^\wedge \rightarrow (\mathcal{F}_n)$ is an isomorphism, and
 - (d) $\alpha = \beta \circ a^\wedge$.

Proof. Proof of (1). The uniqueness implies it suffices to construct (\mathcal{F}, a, β) Zariski locally on X . Thus we may assume $X = \text{Spec}(A)$ and \mathcal{I} corresponds to the ideal $I \subset A$. In this situation Lemma 23.1 applies. Let M' be the finite A^\wedge -module corresponding to (\mathcal{F}_n) . Let N be the finite A -module corresponding to \mathcal{G} . Then α corresponds to a map

$$\varphi : M' \longrightarrow N^\wedge$$

whose kernel and cokernel are annihilated by I^t for some t . Recall that $N^\wedge = N \otimes_A A^\wedge$ (Algebra, Lemma 97.1). By More on Algebra, Lemma 89.16 there is an

A -module map $\psi : M \rightarrow N$ whose kernel and cokernel are I -power torsion and an isomorphism $M \otimes_A A^\wedge = M'$ compatible with φ . As N and M' are finite modules, we conclude that M is a finite A -module, see More on Algebra, Remark 89.19. Hence $M \otimes_A A^\wedge = M^\wedge$. We omit the verification that the triple $(M, N \rightarrow M, M^\wedge \rightarrow M')$ so obtained is unique up to unique isomorphism.

The proof of (2) is exactly the same and we omit it. \square

Lemma 23.7. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Any object of $\text{Coh}(X, \mathcal{I})$ which is annihilated by a power of \mathcal{I} is in the essential image of (23.3.1). Moreover, if \mathcal{F}, \mathcal{G} are in $\text{Coh}(\mathcal{O}_X)$ and either \mathcal{F} or \mathcal{G} is annihilated by a power of \mathcal{I} , then the maps*

$$\begin{array}{ccc} \text{Hom}_X(\mathcal{F}, \mathcal{G}) & & \text{Ext}_X(\mathcal{F}, \mathcal{G}) \\ \downarrow & & \downarrow \\ \text{Hom}_{\text{Coh}(X, \mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge) & & \text{Ext}_{\text{Coh}(X, \mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge) \end{array}$$

are isomorphisms.

Proof. Suppose (\mathcal{F}_n) is an object of $\text{Coh}(X, \mathcal{I})$ which is annihilated by \mathcal{I}^c for some $c \geq 1$. Then $\mathcal{F}_n \rightarrow \mathcal{F}_c$ is an isomorphism for $n \geq c$. Hence if we set $\mathcal{F} = \mathcal{F}_c$, then we see that $\mathcal{F}^\wedge \cong (\mathcal{F}_n)$. This proves the first assertion.

Let \mathcal{F}, \mathcal{G} be objects of $\text{Coh}(\mathcal{O}_X)$ such that either \mathcal{F} or \mathcal{G} is annihilated by \mathcal{I}^c for some $c \geq 1$. Then $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is a coherent \mathcal{O}_X -module annihilated by \mathcal{I}^c . Hence we see that

$$\text{Hom}_X(\mathcal{G}, \mathcal{F}) = H^0(X, \mathcal{H}) = \lim H^0(X, \mathcal{H}/\mathcal{I}^n \mathcal{H}) = \text{Mor}_{\text{Coh}(X, \mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge).$$

see Lemma 23.5. This proves the statement on homomorphisms.

The notation Ext refers to extensions as defined in Homology, Section 6. The injectivity of the map on Ext 's follows immediately from the bijectivity of the map on Hom 's. For surjectivity, assume \mathcal{F} is annihilated by a power of \mathcal{I} . Then part (1) of Lemma 23.6 shows that given an extension

$$0 \rightarrow \mathcal{G}^\wedge \rightarrow (\mathcal{E}_n) \rightarrow \mathcal{F}^\wedge \rightarrow 0$$

in $\text{Coh}(U, I\mathcal{O}_U)$ the morphism $\mathcal{G}^\wedge \rightarrow (\mathcal{E}_n)$ is isomorphic to $\mathcal{G} \rightarrow \mathcal{E}^\wedge$ for some $\mathcal{G} \rightarrow \mathcal{E}$ in $\text{Coh}(\mathcal{O}_U)$. Similarly in the other case. \square

Lemma 23.8. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If (\mathcal{F}_n) is an object of $\text{Coh}(X, \mathcal{I})$ then $\bigoplus \text{Ker}(\mathcal{F}_{n+1} \rightarrow \mathcal{F}_n)$ is a finite type, graded, quasi-coherent $\bigoplus \mathcal{I}^n/\mathcal{I}^{n+1}$ -module.*

Proof. The question is local on X hence we may assume X is affine, i.e., we have a situation as in Lemma 23.1. In this case, if (\mathcal{F}_n) corresponds to the finite A^\wedge module M , then $\bigoplus \text{Ker}(\mathcal{F}_{n+1} \rightarrow \mathcal{F}_n)$ corresponds to $\bigoplus I^n M/I^{n+1} M$ which is clearly a finite module over $\bigoplus I^n/I^{n+1}$. \square

Lemma 23.9. *Let $f : X \rightarrow Y$ be a morphism of Noetherian schemes. Let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals and set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Then there is a right exact functor*

$$f^* : \text{Coh}(Y, \mathcal{J}) \longrightarrow \text{Coh}(X, \mathcal{I})$$

which sends (\mathcal{G}_n) to $(f^*\mathcal{G}_n)$. If f is flat, then f^* is an exact functor.

Proof. Since $f^* : \text{Coh}(\mathcal{O}_Y) \rightarrow \text{Coh}(\mathcal{O}_X)$ is right exact we have

$$f^* \mathcal{G}_n = f^*(\mathcal{G}_{n+1}/\mathcal{I}^n \mathcal{G}_{n+1}) = f^* \mathcal{G}_{n+1}/f^{-1} \mathcal{I}^n f^* \mathcal{G}_{n+1} = f^* \mathcal{G}_{n+1}/\mathcal{J}^n f^* \mathcal{G}_{n+1}$$

hence the pullback of a system is a system. The construction of cokernels in the proof of Lemma 23.2 shows that $f^* : \text{Coh}(Y, \mathcal{J}) \rightarrow \text{Coh}(X, \mathcal{I})$ is always right exact. If f is flat, then $f^* : \text{Coh}(\mathcal{O}_Y) \rightarrow \text{Coh}(\mathcal{O}_X)$ is an exact functor. It follows from the construction of kernels in the proof of Lemma 23.2 that in this case $f^* : \text{Coh}(Y, \mathcal{J}) \rightarrow \text{Coh}(X, \mathcal{I})$ also transforms kernels into kernels. \square

Lemma 23.10. *Let $f : X' \rightarrow X$ be a morphism of Noetherian schemes. Let $Z \subset X$ be a closed subscheme and denote $Z' = f^{-1}Z$ the scheme theoretic inverse image. Let $\mathcal{I} \subset \mathcal{O}_X$, $\mathcal{I}' \subset \mathcal{O}_{X'}$ be the corresponding quasi-coherent sheaves of ideals. If f is flat and the induced morphism $Z' \rightarrow Z$ is an isomorphism, then the pullback functor $f^* : \text{Coh}(X, \mathcal{I}) \rightarrow \text{Coh}(X', \mathcal{I}')$ (Lemma 23.9) is an equivalence.*

Proof. If X and X' are affine, then this follows immediately from More on Algebra, Lemma 89.3. To prove it in general we let $Z_n \subset X$, $Z'_n \subset X'$ be the n th infinitesimal neighbourhoods of Z , Z' . The induced morphism $Z_n \rightarrow Z'_n$ is a homeomorphism on underlying topological spaces. On the other hand, if $z' \in Z'$ maps to $z \in Z$, then the ring map $\mathcal{O}_{X,z} \rightarrow \mathcal{O}_{X',z'}$ is flat and induces an isomorphism $\mathcal{O}_{X,z}/\mathcal{I}_z \rightarrow \mathcal{O}_{X',z'}/\mathcal{I}'_{z'}$. Hence it induces an isomorphism $\mathcal{O}_{X,z}/\mathcal{I}_z^n \rightarrow \mathcal{O}_{X',z'}/(\mathcal{I}'_{z'})^n$ for all $n \geq 1$ for example by More on Algebra, Lemma 89.2. Thus $Z'_n \rightarrow Z_n$ is an isomorphism of schemes. Thus f^* induces an equivalence between the category of coherent \mathcal{O}_X -modules annihilated by \mathcal{I}^n and the category of coherent $\mathcal{O}_{X'}$ -modules annihilated by $(\mathcal{I}')^n$, see Lemma 9.8. This clearly implies the lemma. \square

Lemma 23.11. *Let X be a Noetherian scheme. Let $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. If $V(\mathcal{I}) = V(\mathcal{J})$ is the same closed subset of X , then $\text{Coh}(X, \mathcal{I})$ and $\text{Coh}(X, \mathcal{J})$ are equivalent.*

Proof. First, assume $X = \text{Spec}(A)$ is affine. Let $I, J \subset A$ be the ideals corresponding to \mathcal{I}, \mathcal{J} . Then $V(I) = V(J)$ implies we have $I^c \subset J$ and $J^d \subset I$ for some $c, d \geq 1$ by elementary properties of the Zariski topology (see Algebra, Section 17 and Lemma 32.5). Hence the I -adic and J -adic completions of A agree, see Algebra, Lemma 96.9. Thus the equivalence follows from Lemma 23.1 in this case.

In general, using what we said above and the fact that X is quasi-compact, to choose $c, d \geq 1$ such that $\mathcal{I}^c \subset \mathcal{J}$ and $\mathcal{J}^d \subset \mathcal{I}$. Then given an object (\mathcal{F}_n) in $\text{Coh}(X, \mathcal{I})$ we claim that the inverse system

$$(\mathcal{F}_{cn}/\mathcal{J}^n \mathcal{F}_{cn})$$

is in $\text{Coh}(X, \mathcal{J})$. This may be checked on the members of an affine covering; we omit the details. In the same manner we can construct an object of $\text{Coh}(X, \mathcal{I})$ starting with an object of $\text{Coh}(X, \mathcal{J})$. We omit the verification that these constructions define mutually quasi-inverse functors. \square

24. Grothendieck's existence theorem, I

In this section we discuss Grothendieck's existence theorem for the projective case. We will use the notion of coherent formal modules developed in Section 23. The reader who is familiar with formal schemes is encouraged to read the statement and proof of the theorem in [DG67].

Lemma 24.1. *Let A be Noetherian ring complete with respect to an ideal I . Let $f : X \rightarrow \operatorname{Spec}(A)$ be a proper morphism. Let $\mathcal{I} = I\mathcal{O}_X$. Then the functor (23.3.1) is fully faithful.*

Proof. Let \mathcal{F}, \mathcal{G} be coherent \mathcal{O}_X -modules. Then $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is a coherent \mathcal{O}_X -module, see Modules, Lemma 22.6. By Lemma 23.5 the map

$$\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n \mathcal{H}) \rightarrow \operatorname{Mor}_{\operatorname{Coh}(X, \mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective. Hence fully faithfulness of (23.3.1) follows from the theorem on formal functions (Lemma 20.6) for the coherent sheaf \mathcal{H} . \square

Lemma 24.2. *Let A be Noetherian ring and $I \subset A$ an ideal. Let $f : X \rightarrow \operatorname{Spec}(A)$ be a proper morphism and let \mathcal{L} be an f -ample invertible sheaf. Let $\mathcal{I} = I\mathcal{O}_X$. Let (\mathcal{F}_n) be an object of $\operatorname{Coh}(X, \mathcal{I})$. Then there exists an integer d_0 such that*

$$H^1(X, \operatorname{Ker}(\mathcal{F}_{n+1} \rightarrow \mathcal{F}_n) \otimes \mathcal{L}^{\otimes d}) = 0$$

for all $n \geq 0$ and all $d \geq d_0$.

Proof. Set $B = \bigoplus I^n/I^{n+1}$ and $\mathcal{B} = \bigoplus \mathcal{I}^n/\mathcal{I}^{n+1} = f^* \tilde{B}$. By Lemma 23.8 the graded quasi-coherent \mathcal{B} -module $\mathcal{G} = \bigoplus \operatorname{Ker}(\mathcal{F}_{n+1} \rightarrow \mathcal{F}_n)$ is of finite type. Hence the lemma follows from Lemma 19.3 part (2). \square

Lemma 24.3. *Let A be Noetherian ring complete with respect to an ideal I . Let $f : X \rightarrow \operatorname{Spec}(A)$ be a projective morphism. Let $\mathcal{I} = I\mathcal{O}_X$. Then the functor (23.3.1) is an equivalence.*

Proof. We have already seen that (23.3.1) is fully faithful in Lemma 24.1. Thus it suffices to show that the functor is essentially surjective.

We first show that every object (\mathcal{F}_n) of $\operatorname{Coh}(X, \mathcal{I})$ is the quotient of an object in the image of (23.3.1). Let \mathcal{L} be an f -ample invertible sheaf on X . Choose d_0 as in Lemma 24.2. Choose a $d \geq d_0$ such that $\mathcal{F}_1 \otimes \mathcal{L}^{\otimes d}$ is globally generated by some sections $s_{1,1}, \dots, s_{t,1}$. Since the transition maps of the system

$$H^0(X, \mathcal{F}_{n+1} \otimes \mathcal{L}^{\otimes d}) \longrightarrow H^0(X, \mathcal{F}_n \otimes \mathcal{L}^{\otimes d})$$

are surjective by the vanishing of H^1 we can lift $s_{1,1}, \dots, s_{t,1}$ to a compatible system of global sections $s_{1,n}, \dots, s_{t,n}$ of $\mathcal{F}_n \otimes \mathcal{L}^{\otimes d}$. These determine a compatible system of maps

$$(s_{1,n}, \dots, s_{t,n}) : (\mathcal{L}^{\otimes -d})^{\oplus t} \longrightarrow \mathcal{F}_n$$

Using Lemma 23.3 we deduce that we have a surjective map

$$((\mathcal{L}^{\otimes -d})^{\oplus t})^\wedge \longrightarrow (\mathcal{F}_n)$$

as desired.

The result of the previous paragraph and the fact that $\operatorname{Coh}(X, \mathcal{I})$ is abelian (Lemma 23.2) implies that every object of $\operatorname{Coh}(X, \mathcal{I})$ is a cokernel of a map between objects coming from $\operatorname{Coh}(\mathcal{O}_X)$. As (23.3.1) is fully faithful and exact by Lemmas 24.1 and 23.4 we conclude. \square

25. Grothendieck's existence theorem, II

In this section we discuss Grothendieck's existence theorem in the proper case. Before we give the statement and proof, we need to develop a bit more theory regarding the categories $Coh(X, \mathcal{I})$ of coherent formal modules introduced in Section 23.

Remark 25.1. Let X be a Noetherian scheme and let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $\alpha : (\mathcal{F}_n) \rightarrow (\mathcal{G}_n)$ be a morphism of $Coh(X, \mathcal{I})$. Given an affine open $\text{Spec}(A) = U \subset X$ with $\mathcal{I}|_U, \mathcal{K}|_U$ corresponding to ideals $I, K \subset A$ denote $\alpha_U : M \rightarrow N$ of finite A^\wedge -modules which corresponds to $\alpha|_U$ via Lemma 23.1. We claim the following are equivalent

- (1) there exists an integer $t \geq 1$ such that $\text{Ker}(\alpha_n)$ and $\text{Coker}(\alpha_n)$ are annihilated by \mathcal{K}^t for all $n \geq 1$,
- (2) for any affine open $\text{Spec}(A) = U \subset X$ as above the modules $\text{Ker}(\alpha_U)$ and $\text{Coker}(\alpha_U)$ are annihilated by K^t for some integer $t \geq 1$, and
- (3) there exists a finite affine open covering $X = \bigcup U_i$ such that the conclusion of (2) holds for α_{U_i} .

If these equivalent conditions hold we will say that α is a *map whose kernel and cokernel are annihilated by a power of \mathcal{K}* . To see the equivalence we use the following commutative algebra fact: suppose given an exact sequence

$$0 \rightarrow T \rightarrow M \rightarrow N \rightarrow Q \rightarrow 0$$

of A -modules with T and Q annihilated by K^t for some ideal $K \subset A$. Then for every $f, g \in K^t$ there exists a canonical map " fg " : $N \rightarrow M$ such that $M \rightarrow N \rightarrow M$ is equal to multiplication by fg . Namely, for $y \in N$ we can pick $x \in M$ mapping to fy in N and then we can set " fg "(y) = gx . Thus it is clear that $\text{Ker}(M/JM \rightarrow N/JN)$ and $\text{Coker}(M/JM \rightarrow N/JN)$ are annihilated by K^{2t} for any ideal $J \subset A$.

Applying the commutative algebra fact to α_{U_i} and $J = I^n$ we see that (3) implies (1). Conversely, suppose (1) holds and $M \rightarrow N$ is equal to α_U . Then there is a $t \geq 1$ such that $\text{Ker}(M/I^n M \rightarrow N/I^n N)$ and $\text{Coker}(M/I^n M \rightarrow N/I^n N)$ are annihilated by K^t for all n . We obtain maps " fg " : $N/I^n N \rightarrow M/I^n M$ which in the limit induce a map $N \rightarrow M$ as N and M are I -adically complete. Since the composition with $N \rightarrow M \rightarrow N$ is multiplication by fg we conclude that fg annihilates T and Q . In other words T and Q are annihilated by K^{2t} as desired.

Lemma 25.2. *Let X be a Noetherian scheme. Let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $X_e \subset X$ be the closed subscheme cut out by \mathcal{K}^e . Let $\mathcal{I}_e = \mathcal{I}\mathcal{O}_{X_e}$. Let (\mathcal{F}_n) be an object of $Coh(X, \mathcal{I})$. Assume*

- (1) *the functor $Coh(\mathcal{O}_{X_e}) \rightarrow Coh(X_e, \mathcal{I}_e)$ is an equivalence for all $e \geq 1$, and*
- (2) *there exists a coherent sheaf \mathcal{H} on X and a map $\alpha : (\mathcal{F}_n) \rightarrow \mathcal{H}^\wedge$ whose kernel and cokernel are annihilated by a power of \mathcal{K} .*

Then (\mathcal{F}_n) is in the essential image of (23.3.1).

Proof. During this proof we will use without further mention that for a closed immersion $i : Z \rightarrow X$ the functor i_* gives an equivalence between the category of coherent modules on Z and coherent modules on X annihilated by the ideal sheaf of Z , see Lemma 9.8. In particular we may identify $Coh(\mathcal{O}_{X_e})$ with the category of coherent \mathcal{O}_X -modules annihilated by \mathcal{K}^e and $Coh(X_e, \mathcal{I}_e)$ as the full subcategory of

$Coh(X, \mathcal{I})$ of objects annihilated by \mathcal{K}^e . Moreover (1) tells us these two categories are equivalent under the completion functor (23.3.1).

Applying this equivalence we get a coherent \mathcal{O}_X -module \mathcal{G}_e annihilated by \mathcal{K}^e corresponding to the system $(\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$. The maps $\mathcal{F}_n/\mathcal{K}^{e+1}\mathcal{F}_n \rightarrow \mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n$ correspond to canonical maps $\mathcal{G}_{e+1} \rightarrow \mathcal{G}_e$ which induce isomorphisms $\mathcal{G}_{e+1}/\mathcal{K}^e\mathcal{G}_{e+1} \rightarrow \mathcal{G}_e$. Hence (\mathcal{G}_e) is an object of $Coh(X, \mathcal{K})$. The map α induces a system of maps

$$\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \longrightarrow \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$$

whence maps $\mathcal{G}_e \rightarrow \mathcal{H}/\mathcal{K}^e\mathcal{H}$ (by the equivalence of categories again). Let $t \geq 1$ be an integer, which exists by assumption (2), such that \mathcal{K}^t annihilates the kernel and cokernel of all the maps $\mathcal{F}_n \rightarrow \mathcal{H}/\mathcal{I}^n\mathcal{H}$. Then \mathcal{K}^{2t} annihilates the kernel and cokernel of the maps $\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \rightarrow \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$, see Remark 25.1. Whereupon we conclude that \mathcal{K}^{4t} annihilates the kernel and the cokernel of the maps

$$\mathcal{G}_e \longrightarrow \mathcal{H}/\mathcal{K}^e\mathcal{H},$$

see Remark 25.1. We apply Lemma 23.6 to obtain a coherent \mathcal{O}_X -module \mathcal{F} , a map $a : \mathcal{F} \rightarrow \mathcal{H}$ and an isomorphism $\beta : (\mathcal{G}_e) \rightarrow (\mathcal{F}/\mathcal{K}^e\mathcal{F})$ in $Coh(X, \mathcal{K})$. Working backwards, for a given n the triple $(\mathcal{F}/\mathcal{I}^n\mathcal{F}, a \bmod \mathcal{I}^n, \beta \bmod \mathcal{I}^n)$ is a triple as in the lemma for the morphism $\alpha_n \bmod \mathcal{K}^e : (\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n) \rightarrow (\mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H})$ of $Coh(X, \mathcal{K})$. Thus the uniqueness in Lemma 23.6 gives a canonical isomorphism $\mathcal{F}/\mathcal{I}^n\mathcal{F} \rightarrow \mathcal{F}_n$ compatible with all the morphisms in sight. This finishes the proof of the lemma. \square

Lemma 25.3. *Let Y be a Noetherian scheme. Let $\mathcal{J}, \mathcal{K} \subset \mathcal{O}_Y$ be quasi-coherent sheaves of ideals. Let $f : X \rightarrow Y$ be a proper morphism which is an isomorphism over $V = Y \setminus V(\mathcal{K})$. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Let (\mathcal{G}_n) be an object of $Coh(Y, \mathcal{J})$, let \mathcal{F} be a coherent \mathcal{O}_X -module, and let $\beta : (f^*\mathcal{G}_n) \rightarrow \mathcal{F}^\wedge$ be an isomorphism in $Coh(X, \mathcal{I})$. Then there exists a map*

$$\alpha : (\mathcal{G}_n) \longrightarrow (f_*\mathcal{F})^\wedge$$

in $Coh(Y, \mathcal{J})$ whose kernel and cokernel are annihilated by a power of \mathcal{K} .

Proof. Since f is a proper morphism we see that $f_*\mathcal{F}$ is a coherent \mathcal{O}_Y -module (Proposition 19.1). Thus the statement of the lemma makes sense. Consider the compositions

$$\gamma_n : \mathcal{G}_n \rightarrow f_*f^*\mathcal{G}_n \rightarrow f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F}).$$

Here the first map is the adjunction map and the second is $f_*\beta_n$. We claim that there exists a unique α as in the lemma such that the compositions

$$\mathcal{G}_n \xrightarrow{\alpha_n} f_*\mathcal{F}/\mathcal{J}^n f_*\mathcal{F} \rightarrow f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F})$$

equal γ_n for all n . Because of the uniqueness we may assume that $Y = \text{Spec}(B)$ is affine. Let $J \subset B$ corresponds to the ideal \mathcal{J} . Set

$$M_n = H^0(X, \mathcal{F}/\mathcal{I}^n\mathcal{F}) \quad \text{and} \quad M = H^0(X, \mathcal{F})$$

By Lemma 20.4 and Theorem 20.5 the inverse limit of the modules M_n equals the completion $M^\wedge = \lim M/J^n M$. Set $N_n = H^0(Y, \mathcal{G}_n)$ and $N = \lim N_n$. Via the equivalence of categories of Lemma 23.1 the finite B^\wedge modules N and M^\wedge correspond to (\mathcal{G}_n) and $f_*\mathcal{F}^\wedge$. It follows from this that α has to be the morphism of $Coh(Y, \mathcal{J})$ corresponding to the homomorphism

$$\lim \gamma_n : N = \lim_n N_n \longrightarrow \lim M_n = M^\wedge$$

of finite B^\wedge -modules.

We still have to show that the kernel and cokernel of α are annihilated by a power of \mathcal{K} . Set $Y' = \operatorname{Spec}(B^\wedge)$ and $X' = Y' \times_Y X$. Let \mathcal{K}' , \mathcal{J}' , \mathcal{G}'_n and \mathcal{I}' , \mathcal{F}' be the pullback of \mathcal{K} , \mathcal{J} , \mathcal{G}_n and \mathcal{I} , \mathcal{F} , to Y' and X' . The projection morphism $f' : X' \rightarrow Y'$ is the base change of f by $Y' \rightarrow Y$. Note that $Y' \rightarrow Y$ is a flat morphism of schemes as $B \rightarrow B^\wedge$ is flat by Algebra, Lemma 97.2. Hence $f'_*\mathcal{F}'$, resp. $f'_*(f')^*\mathcal{G}'_n$ is the pullback of $f_*\mathcal{F}$, resp. $f_*f^*\mathcal{G}_n$ to Y' by Lemma 5.2. The uniqueness of our construction shows the pullback of α to Y' is the corresponding map α' constructed for the situation on Y' . Moreover, to check that the kernel and cokernel of α are annihilated by \mathcal{K}^t it suffices to check that the kernel and cokernel of α' are annihilated by $(\mathcal{K}')^t$. Namely, to see this we need to check this for kernels and cokernels of the maps α_n and α'_n (see Remark 25.1) and the ring map $B \rightarrow B^\wedge$ induces an equivalence of categories between modules annihilated by J^n and $(J')^n$, see More on Algebra, Lemma 89.3. Thus we may assume B is complete with respect to J .

Assume $Y = \operatorname{Spec}(B)$ is affine, \mathcal{J} corresponds to the ideal $J \subset B$, and B is complete with respect to J . In this case (\mathcal{G}_n) is in the essential image of the functor $\operatorname{Coh}(\mathcal{O}_Y) \rightarrow \operatorname{Coh}(Y, \mathcal{J})$. Say \mathcal{G} is a coherent \mathcal{O}_Y -module such that $(\mathcal{G}_n) = \mathcal{G}^\wedge$. Note that $f^*(\mathcal{G}^\wedge) = (f^*\mathcal{G})^\wedge$. Hence Lemma 24.1 tells us that β comes from an isomorphism $b : f^*\mathcal{G} \rightarrow \mathcal{F}$ and α is the completion functor applied to

$$\mathcal{G} \rightarrow f_*f^*\mathcal{G} \cong f_*\mathcal{F}$$

Hence we are trying to verify that the kernel and cokernel of the adjunction map $c : \mathcal{G} \rightarrow f_*f^*\mathcal{G}$ are annihilated by a power of \mathcal{K} . However, since the restriction $f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V$ is an isomorphism we see that $c|_V$ is an isomorphism. Thus the coherent sheaves $\operatorname{Ker}(c)$ and $\operatorname{Coker}(c)$ are supported on $V(\mathcal{K})$ hence are annihilated by a power of \mathcal{K} (Lemma 10.2) as desired. \square

The following proposition is the form of Grothendieck's existence theorem which is most often used in practice.

Proposition 25.4. *Let A be a Noetherian ring complete with respect to an ideal I . Let $f : X \rightarrow \operatorname{Spec}(A)$ be a proper morphism of schemes. Set $\mathcal{I} = I\mathcal{O}_X$. Then the functor (23.3.1) is an equivalence.*

Proof. We have already seen that (23.3.1) is fully faithful in Lemma 24.1. Thus it suffices to show that the functor is essentially surjective.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that every object (\mathcal{F}_n) annihilated by \mathcal{K} is in the essential image. We want to show (0) is in Ξ . If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals \mathcal{K} not in Ξ , see Lemma 10.1. After replacing X by the closed subscheme of X corresponding to \mathcal{K} we may assume that every nonzero \mathcal{K} is in Ξ . (This uses the correspondence by coherent modules annihilated by \mathcal{K} and coherent modules on the closed subscheme corresponding to \mathcal{K} , see Lemma 9.8.) Let (\mathcal{F}_n) be an object of $\operatorname{Coh}(X, \mathcal{I})$. We will show that this object is in the essential image of the functor (23.3.1), thereby completing the proof of the proposition.

Apply Chow's lemma (Lemma 18.1) to find a proper surjective morphism $f : X' \rightarrow X$ which is an isomorphism over a dense open $U \subset X$ such that X' is projective over A . Let \mathcal{K} be the quasi-coherent sheaf of ideals cutting out the reduced complement

$X \setminus U$. By the projective case of Grothendieck's existence theorem (Lemma 24.3) there exists a coherent module \mathcal{F}' on X' such that $(\mathcal{F}')^\wedge \cong (f^*\mathcal{F}_n)$. By Proposition 19.1 the \mathcal{O}_X -module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 25.3 there exists a morphism $(\mathcal{F}_n) \rightarrow \mathcal{H}^\wedge$ of $\text{Coh}(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of \mathcal{K} . The powers \mathcal{K}^e are all in Ξ so that (23.3.1) is an equivalence for the closed subschemes $X_e = V(\mathcal{K}^e)$. We conclude by Lemma 25.2. \square

26. Being proper over a base

This is just a short section to point out some useful features of closed subsets proper over a base and finite type, quasi-coherent modules with support proper over a base.

Lemma 26.1. *Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let $Z \subset X$ be a closed subset. The following are equivalent*

- (1) *the morphism $Z \rightarrow S$ is proper if Z is endowed with the reduced induced closed subscheme structure (Schemes, Definition 12.5),*
- (2) *for some closed subscheme structure on Z the morphism $Z \rightarrow S$ is proper,*
- (3) *for any closed subscheme structure on Z the morphism $Z \rightarrow S$ is proper.*

Proof. The implications (3) \Rightarrow (1) and (1) \Rightarrow (2) are immediate. Thus it suffices to prove that (2) implies (3). We urge the reader to find their own proof of this fact. Let Z' and Z'' be closed subscheme structures on Z such that $Z' \rightarrow S$ is proper. We have to show that $Z'' \rightarrow S$ is proper. Let $Z''' = Z' \cup Z''$ be the scheme theoretic union, see Morphisms, Definition 4.4. Then Z''' is another closed subscheme structure on Z . This follows for example from the description of scheme theoretic unions in Morphisms, Lemma 4.6. Since $Z'' \rightarrow Z'''$ is a closed immersion it suffices to prove that $Z''' \rightarrow S$ is proper (see Morphisms, Lemmas 41.6 and 41.4). The morphism $Z' \rightarrow Z'''$ is a bijective closed immersion and in particular surjective and universally closed. Then the fact that $Z' \rightarrow S$ is separated implies that $Z''' \rightarrow S$ is separated, see Morphisms, Lemma 41.11. Moreover $Z''' \rightarrow S$ is locally of finite type as $X \rightarrow S$ is locally of finite type (Morphisms, Lemmas 15.5 and 15.3). Since $Z' \rightarrow S$ is quasi-compact and $Z' \rightarrow Z'''$ is a homeomorphism we see that $Z''' \rightarrow S$ is quasi-compact. Finally, since $Z' \rightarrow S$ is universally closed, we see that the same thing is true for $Z''' \rightarrow S$ by Morphisms, Lemma 41.9. This finishes the proof. \square

Definition 26.2. Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let $Z \subset X$ be a closed subset. We say *Z is proper over S* if the equivalent conditions of Lemma 26.1 are satisfied.

The lemma used in the definition above is false if the morphism $f : X \rightarrow S$ is not locally of finite type. Therefore we urge the reader not to use this terminology if f is not locally of finite type.

Lemma 26.3. *Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let $Y \subset Z \subset X$ be closed subsets. If Z is proper over S , then the same is true for Y .*

Proof. Omitted. \square

Lemma 26.4. *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{\quad g' \quad} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{\quad g \quad} & S \end{array}$$

with f locally of finite type. If Z is a closed subset of X proper over S , then $(g')^{-1}(Z)$ is a closed subset of X' proper over S' .

Proof. Observe that the statement makes sense as f' is locally of finite type by Morphisms, Lemma 15.4. Endow Z with the reduced induced closed subscheme structure. Denote $Z' = (g')^{-1}(Z)$ the scheme theoretic inverse image (Schemes, Definition 17.7). Then $Z' = X' \times_X Z = (S' \times_S X) \times_X Z = S' \times_S Z$ is proper over S' as a base change of Z over S (Morphisms, Lemma 41.5). \square

Lemma 26.5. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of schemes which are locally of finite type over S .*

- (1) *If Y is separated over S and $Z \subset X$ is a closed subset proper over S , then $f(Z)$ is a closed subset of Y proper over S .*
- (2) *If f is universally closed and $Z \subset X$ is a closed subset proper over S , then $f(Z)$ is a closed subset of Y proper over S .*
- (3) *If f is proper and $Z \subset Y$ is a closed subset proper over S , then $f^{-1}(Z)$ is a closed subset of X proper over S .*

Proof. Proof of (1). Assume Y is separated over S and $Z \subset X$ is a closed subset proper over S . Endow Z with the reduced induced closed subscheme structure and apply Morphisms, Lemma 41.10 to $Z \rightarrow Y$ over S to conclude.

Proof of (2). Assume f is universally closed and $Z \subset X$ is a closed subset proper over S . Endow Z and $Z' = f(Z)$ with their reduced induced closed subscheme structures. We obtain an induced morphism $Z \rightarrow Z'$. Denote $Z'' = f^{-1}(Z')$ the scheme theoretic inverse image (Schemes, Definition 17.7). Then $Z'' \rightarrow Z'$ is universally closed as a base change of f (Morphisms, Lemma 41.5). Hence $Z \rightarrow Z'$ is universally closed as a composition of the closed immersion $Z \rightarrow Z''$ and $Z'' \rightarrow Z'$ (Morphisms, Lemmas 41.6 and 41.4). We conclude that $Z' \rightarrow S$ is separated by Morphisms, Lemma 41.11. Since $Z \rightarrow S$ is quasi-compact and $Z \rightarrow Z'$ is surjective we see that $Z' \rightarrow S$ is quasi-compact. Since $Z' \rightarrow S$ is the composition of $Z' \rightarrow Y$ and $Y \rightarrow S$ we see that $Z' \rightarrow S$ is locally of finite type (Morphisms, Lemmas 15.5 and 15.3). Finally, since $Z \rightarrow S$ is universally closed, we see that the same thing is true for $Z' \rightarrow S$ by Morphisms, Lemma 41.9. This finishes the proof.

Proof of (3). Assume f is proper and $Z \subset Y$ is a closed subset proper over S . Endow Z with the reduced induced closed subscheme structure. Denote $Z' = f^{-1}(Z)$ the scheme theoretic inverse image (Schemes, Definition 17.7). Then $Z' \rightarrow Z$ is proper as a base change of f (Morphisms, Lemma 41.5). Whence $Z' \rightarrow S$ is proper as the composition of $Z' \rightarrow Z$ and $Z \rightarrow S$ (Morphisms, Lemma 41.4). This finishes the proof. \square

Lemma 26.6. *Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let $Z_i \subset X$, $i = 1, \dots, n$ be closed subsets. If Z_i , $i = 1, \dots, n$ are proper over S , then the same is true for $Z_1 \cup \dots \cup Z_n$.*

Proof. Endow Z_i with their reduced induced closed subscheme structures. The morphism

$$Z_1 \amalg \dots \amalg Z_n \longrightarrow X$$

is finite by Morphisms, Lemmas 44.12 and 44.13. As finite morphisms are universally closed (Morphisms, Lemma 44.11) and since $Z_1 \amalg \dots \amalg Z_n$ is proper over S we conclude by Lemma 26.5 part (2) that the image $Z_1 \cup \dots \cup Z_n$ is proper over S . \square

Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let \mathcal{F} be a finite type, quasi-coherent \mathcal{O}_X -module. Then the support $\text{Supp}(\mathcal{F})$ of \mathcal{F} is a closed subset of X , see Morphisms, Lemma 5.3. Hence it makes sense to say “the support of \mathcal{F} is proper over S ”.

Lemma 26.7. *Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let \mathcal{F} be a finite type, quasi-coherent \mathcal{O}_X -module. The following are equivalent*

- (1) *the support of \mathcal{F} is proper over S ,*
- (2) *the scheme theoretic support of \mathcal{F} (Morphisms, Definition 5.5) is proper over S , and*
- (3) *there exists a closed subscheme $Z \subset X$ and a finite type, quasi-coherent \mathcal{O}_Z -module \mathcal{G} such that (a) $Z \rightarrow S$ is proper, and (b) $(Z \rightarrow X)_* \mathcal{G} = \mathcal{F}$.*

Proof. The support $\text{Supp}(\mathcal{F})$ of \mathcal{F} is a closed subset of X , see Morphisms, Lemma 5.3. Hence we can apply Definition 26.2. Since the scheme theoretic support of \mathcal{F} is a closed subscheme whose underlying closed subset is $\text{Supp}(\mathcal{F})$ we see that (1) and (2) are equivalent by Definition 26.2. It is clear that (2) implies (3). Conversely, if (3) is true, then $\text{Supp}(\mathcal{F}) \subset Z$ (an inclusion of closed subsets of X) and hence $\text{Supp}(\mathcal{F})$ is proper over S for example by Lemma 26.3. \square

Lemma 26.8. *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{\quad g' \quad} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{\quad g \quad} & S \end{array}$$

with f locally of finite type. Let \mathcal{F} be a finite type, quasi-coherent \mathcal{O}_X -module. If the support of \mathcal{F} is proper over S , then the support of $(g')^ \mathcal{F}$ is proper over S' .*

Proof. Observe that the statement makes sense because $(g')^* \mathcal{F}$ is of finite type by Modules, Lemma 9.2. We have $\text{Supp}((g')^* \mathcal{F}) = (g')^{-1}(\text{Supp}(\mathcal{F}))$ by Morphisms, Lemma 5.3. Thus the lemma follows from Lemma 26.4. \square

Lemma 26.9. *Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let \mathcal{F}, \mathcal{G} be finite type, quasi-coherent \mathcal{O}_X -module.*

- (1) *If the supports of \mathcal{F}, \mathcal{G} are proper over S , then the same is true for $\mathcal{F} \oplus \mathcal{G}$, for any extension of \mathcal{G} by \mathcal{F} , for $\text{Im}(u)$ and $\text{Coker}(u)$ given any \mathcal{O}_X -module map $u : \mathcal{F} \rightarrow \mathcal{G}$, and for any quasi-coherent quotient of \mathcal{F} or \mathcal{G} .*
- (2) *If S is locally Noetherian, then the category of coherent \mathcal{O}_X -modules with support proper over S is a Serre subcategory (Homology, Definition 10.1) of the abelian category of coherent \mathcal{O}_X -modules.*

Proof. Proof of (1). Let Z, Z' be the support of \mathcal{F} and \mathcal{G} . Then all the sheaves mentioned in (1) have support contained in $Z \cup Z'$. Thus the assertion itself is clear from Lemmas 26.3 and 26.6 provided we check that these sheaves are finite type and quasi-coherent. For quasi-coherence we refer the reader to Schemes, Section 24. For “finite type” we suggest the reader take a look at Modules, Section 9.

Proof of (2). The proof is the same as the proof of (1). Note that the assertions make sense as X is locally Noetherian by Morphisms, Lemma 15.6 and by the description of the category of coherent modules in Section 9. \square

Lemma 26.10. *Let S be a locally Noetherian scheme. Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let \mathcal{F} be a coherent \mathcal{O}_X -module with support proper over S . Then $R^p f_* \mathcal{F}$ is a coherent \mathcal{O}_S -module for all $p \geq 0$.*

Proof. By Lemma 26.7 there exists a closed immersion $i : Z \rightarrow X$ and a finite type, quasi-coherent \mathcal{O}_Z -module \mathcal{G} such that (a) $g = f \circ i : Z \rightarrow S$ is proper, and (b) $i_* \mathcal{G} = \mathcal{F}$. We see that $R^p g_* \mathcal{G}$ is coherent on S by Proposition 19.1. On the other hand, $R^q i_* \mathcal{G} = 0$ for $q > 0$ (Lemma 9.9). By Cohomology, Lemma 13.8 we get $R^p f_* \mathcal{F} = R^p g_* \mathcal{G}$ which concludes the proof. \square

Lemma 26.11. *Let S be a Noetherian scheme. Let $f : X \rightarrow S$ be a finite type morphism. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The following are Serre subcategories of $\text{Coh}(X, \mathcal{I})$*

- (1) *the full subcategory of $\text{Coh}(X, \mathcal{I})$ consisting of those objects (\mathcal{F}_n) such that the support of \mathcal{F}_1 is proper over S ,*
- (2) *the full subcategory of $\text{Coh}(X, \mathcal{I})$ consisting of those objects (\mathcal{F}_n) such that there exists a closed subscheme $Z \subset X$ proper over S with $\mathcal{I}_Z \mathcal{F}_n = 0$ for all $n \geq 1$.*

Proof. We will use the criterion of Homology, Lemma 10.2. Moreover, we will use that if $0 \rightarrow (\mathcal{G}_n) \rightarrow (\mathcal{F}_n) \rightarrow (\mathcal{H}_n) \rightarrow 0$ is a short exact sequence of $\text{Coh}(X, \mathcal{I})$, then (a) $\mathcal{G}_n \rightarrow \mathcal{F}_n \rightarrow \mathcal{H}_n \rightarrow 0$ is exact for all $n \geq 1$ and (b) \mathcal{G}_n is a quotient of $\text{Ker}(\mathcal{F}_m \rightarrow \mathcal{H}_m)$ for some $m \geq n$. See proof of Lemma 23.2.

Proof of (1). Let (\mathcal{F}_n) be an object of $\text{Coh}(X, \mathcal{I})$. Then $\text{Supp}(\mathcal{F}_n) = \text{Supp}(\mathcal{F}_1)$ for all $n \geq 1$. Hence by remarks (a) and (b) above we see that for any short exact sequence $0 \rightarrow (\mathcal{G}_n) \rightarrow (\mathcal{F}_n) \rightarrow (\mathcal{H}_n) \rightarrow 0$ of $\text{Coh}(X, \mathcal{I})$ we have $\text{Supp}(\mathcal{G}_1) \cup \text{Supp}(\mathcal{H}_1) = \text{Supp}(\mathcal{F}_1)$. This proves that the category defined in (1) is a Serre subcategory of $\text{Coh}(X, \mathcal{I})$.

Proof of (2). Here we argue the same way. Let $0 \rightarrow (\mathcal{G}_n) \rightarrow (\mathcal{F}_n) \rightarrow (\mathcal{H}_n) \rightarrow 0$ be a short exact sequence of $\text{Coh}(X, \mathcal{I})$. If $Z \subset X$ is a closed subscheme and \mathcal{I}_Z annihilates \mathcal{F}_n for all n , then \mathcal{I}_Z annihilates \mathcal{G}_n and \mathcal{H}_n for all n by (a) and (b) above. Hence if $Z \rightarrow S$ is proper, then we conclude that the category defined in (2) is closed under taking sub and quotient objects inside of $\text{Coh}(X, \mathcal{I})$. Finally, suppose that $Z \subset X$ and $Y \subset X$ are closed subschemes proper over S such that $\mathcal{I}_Z \mathcal{G}_n = 0$ and $\mathcal{I}_Y \mathcal{H}_n = 0$ for all $n \geq 1$. Then it follows from (a) above that $\mathcal{I}_{Z \cup Y} = \mathcal{I}_Z \cdot \mathcal{I}_Y$ annihilates \mathcal{F}_n for all n . By Lemma 26.6 (and via Definition 26.2 which tells us we may choose an arbitrary scheme structure used on the union) we see that $Z \cup Y \rightarrow S$ is proper and the proof is complete. \square

27. Grothendieck's existence theorem, III

To state the general version of Grothendieck's existence theorem we introduce a bit more notation. Let A be a Noetherian ring complete with respect to an ideal I . Let $f : X \rightarrow \operatorname{Spec}(A)$ be a separated finite type morphism of schemes. Set $\mathcal{I} = I\mathcal{O}_X$. In this situation we let

$$\operatorname{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$$

be the full subcategory of $\operatorname{Coh}(\mathcal{O}_X)$ consisting of those coherent \mathcal{O}_X -modules whose support is proper over $\operatorname{Spec}(A)$. This is a Serre subcategory of $\operatorname{Coh}(\mathcal{O}_X)$, see Lemma 26.9. Similarly, we let

$$\operatorname{Coh}_{\text{support proper over } A}(X, \mathcal{I})$$

be the full subcategory of $\operatorname{Coh}(X, \mathcal{I})$ consisting of those objects (\mathcal{F}_n) such that the support of \mathcal{F}_1 is proper over $\operatorname{Spec}(A)$. This is a Serre subcategory of $\operatorname{Coh}(X, \mathcal{I})$ by Lemma 26.11 part (1). Since the support of a quotient module is contained in the support of the module, it follows that (23.3.1) induces a functor

$$(27.0.1) \quad \operatorname{Coh}_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow \operatorname{Coh}_{\text{support proper over } A}(X, \mathcal{I})$$

We are now ready to state the main theorem of this section.

Theorem 27.1 (Grothendieck's existence theorem). *Let A be a Noetherian ring complete with respect to an ideal I . Let X be a separated, finite type scheme over A . Then the functor (27.0.1)*

$$\operatorname{Coh}_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow \operatorname{Coh}_{\text{support proper over } A}(X, \mathcal{I})$$

is an equivalence.

Proof. We will use the equivalence of categories of Lemma 9.8 without further mention. For a closed subscheme $Z \subset X$ proper over A in this proof we will say a coherent module on X is “supported on Z ” if it is annihilated by the ideal sheaf of Z or equivalently if it is the pushforward of a coherent module on Z . By Proposition 25.4 we know that the result is true for the functor between coherent modules and systems of coherent modules supported on Z . Hence it suffices to show that every object of $\operatorname{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$ and every object of $\operatorname{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ is supported on a closed subscheme $Z \subset X$ proper over A . This holds by definition for objects of $\operatorname{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$. We will prove this statement for objects of $\operatorname{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ using the method of proof of Proposition 25.4. We urge the reader to read that proof first.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that the statement holds for every object (\mathcal{F}_n) of $\operatorname{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by \mathcal{K} . We want to show (0) is in Ξ . If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals \mathcal{K} not in Ξ , see Lemma 10.1. After replacing X by the closed subscheme of X corresponding to \mathcal{K} we may assume that every nonzero \mathcal{K} is in Ξ . Let (\mathcal{F}_n) be an object of $\operatorname{Coh}_{\text{support proper over } A}(X, \mathcal{I})$. We will show that this object is supported on a closed subscheme $Z \subset X$ proper over A , thereby completing the proof of the theorem.

Apply Chow's lemma (Lemma 18.1) to find a proper surjective morphism $f : Y \rightarrow X$ which is an isomorphism over a dense open $U \subset X$ such that Y is H-quasi-projective over A . Choose an open immersion $j : Y \rightarrow Y'$ with Y' projective over

A , see Morphisms, Lemma 43.11. Observe that

$$\mathrm{Supp}(f^*\mathcal{F}_n) = f^{-1}\mathrm{Supp}(\mathcal{F}_n) = f^{-1}\mathrm{Supp}(\mathcal{F}_1)$$

The first equality by Morphisms, Lemma 5.3. By assumption and Lemma 26.5 part (3) we see that $f^{-1}\mathrm{Supp}(\mathcal{F}_1)$ is proper over A . Hence the image of $f^{-1}\mathrm{Supp}(\mathcal{F}_1)$ under j is closed in Y' by Lemma 26.5 part (1). Thus $\mathcal{F}'_n = j_*f^*\mathcal{F}_n$ is coherent on Y' by Lemma 9.11. It follows that (\mathcal{F}'_n) is an object of $\mathrm{Coh}(Y', \mathcal{O}_{Y'})$. By the projective case of Grothendieck's existence theorem (Lemma 24.3) there exists a coherent $\mathcal{O}_{Y'}$ -module \mathcal{F}' and an isomorphism $(\mathcal{F}')^\wedge \cong (\mathcal{F}'_n)$ in $\mathrm{Coh}(Y', \mathcal{O}_{Y'})$. Since $\mathcal{F}'/I\mathcal{F}' = \mathcal{F}'_1$ we see that

$$\mathrm{Supp}(\mathcal{F}') \cap V(I\mathcal{O}_{Y'}) = \mathrm{Supp}(\mathcal{F}'_1) = j(f^{-1}\mathrm{Supp}(\mathcal{F}_1))$$

The structure morphism $p' : Y' \rightarrow \mathrm{Spec}(A)$ is proper, hence $p'(\mathrm{Supp}(\mathcal{F}') \setminus j(Y))$ is closed in $\mathrm{Spec}(A)$. A nonempty closed subset of $\mathrm{Spec}(A)$ contains a point of $V(I)$ as I is contained in the Jacobson radical of A by Algebra, Lemma 96.6. The displayed equation shows that $\mathrm{Supp}(\mathcal{F}') \cap (p')^{-1}V(I) \subset j(Y)$ hence we conclude that $\mathrm{Supp}(\mathcal{F}') \subset j(Y)$. Thus $\mathcal{F}'|_Y = j^*\mathcal{F}'$ is supported on a closed subscheme Z' of Y proper over A and $(\mathcal{F}'|_Y)^\wedge = (f^*\mathcal{F}_n)$.

Let \mathcal{K} be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By Proposition 19.1 the \mathcal{O}_X -module $\mathcal{H} = f_*(\mathcal{F}'|_Y)$ is coherent and by Lemma 25.3 there exists a morphism $\alpha : (\mathcal{F}_n) \rightarrow \mathcal{H}^\wedge$ of $\mathrm{Coh}(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power \mathcal{K}^t of \mathcal{K} . We obtain an exact sequence

$$0 \rightarrow \mathrm{Ker}(\alpha) \rightarrow (\mathcal{F}_n) \rightarrow \mathcal{H}^\wedge \rightarrow \mathrm{Coker}(\alpha) \rightarrow 0$$

in $\mathrm{Coh}(X, \mathcal{I})$. If $Z_0 \subset X$ is the scheme theoretic support of \mathcal{H} , then it is clear that $Z_0 \subset f(Z')$ set-theoretically. Hence Z_0 is proper over A by Lemma 26.3 and Lemma 26.5 part (2). Hence \mathcal{H}^\wedge is in the subcategory defined in Lemma 26.11 part (2) and a fortiori in $\mathrm{Coh}_{\mathrm{support\ proper\ over\ } A}(X, \mathcal{I})$. We conclude that $\mathrm{Ker}(\alpha)$ and $\mathrm{Coker}(\alpha)$ are in $\mathrm{Coh}_{\mathrm{support\ proper\ over\ } A}(X, \mathcal{I})$ by Lemma 26.11 part (1). By induction hypothesis, more precisely because \mathcal{K}^t is in Ξ , we see that $\mathrm{Ker}(\alpha)$ and $\mathrm{Coker}(\alpha)$ are in the subcategory defined in Lemma 26.11 part (2). Since this is a Serre subcategory by the lemma, we conclude that the same is true for (\mathcal{F}_n) which is what we wanted to show. \square

Remark 27.2 (Unwinding Grothendieck's existence theorem). Let A be a Noetherian ring complete with respect to an ideal I . Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \rightarrow S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Picture:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \dots & X \\ \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \dots & S \end{array}$$

In this situation we consider systems $(\mathcal{F}_n, \varphi_n)$ where

- (1) \mathcal{F}_n is a coherent \mathcal{O}_{X_n} -module,
- (2) $\varphi_n : i_n^*\mathcal{F}_{n+1} \rightarrow \mathcal{F}_n$ is an isomorphism, and
- (3) $\mathrm{Supp}(\mathcal{F}_1)$ is proper over S_1 .

Theorem 27.1 says that the completion functor

$$\begin{array}{ccc} \text{coherent } \mathcal{O}_X\text{-modules } \mathcal{F} & \longrightarrow & \text{systems } (\mathcal{F}_n) \\ \text{with support proper over } A & & \text{as above} \end{array}$$

is an equivalence of categories. In the special case that X is proper over A we can omit the conditions on the supports.

28. Grothendieck's algebraization theorem

Our first result is a translation of Grothendieck's existence theorem in terms of closed subschemes and finite morphisms.

Lemma 28.1. *Let A be a Noetherian ring complete with respect to an ideal I . Write $S = \operatorname{Spec}(A)$ and $S_n = \operatorname{Spec}(A/I^n)$. Let $X \rightarrow S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{array}{ccccccc} Z_1 & \longrightarrow & Z_2 & \longrightarrow & Z_3 & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \dots \end{array}$$

of schemes with cartesian squares. Assume that

- (1) $Z_1 \rightarrow X_1$ is a closed immersion, and
- (2) $Z_1 \rightarrow S_1$ is proper.

Then there exists a closed immersion of schemes $Z \rightarrow X$ such that $Z_n = Z \times_S S_n$. Moreover, Z is proper over S .

Proof. Let's write $j_n : Z_n \rightarrow X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of j_n to X_1 is j_1 . Thus Morphisms, Lemma 45.7 shows that j_n is a closed immersion. Set $\mathcal{F}_n = j_{n,*} \mathcal{O}_{Z_n}$, so that j_n^\sharp is a surjection $\mathcal{O}_{X_n} \rightarrow \mathcal{F}_n$. Again using that the squares are cartesian we see that the pullback of \mathcal{F}_{n+1} to X_n is \mathcal{F}_n . Hence Grothendieck's existence theorem, as reformulated in Remark 27.2, tells us there exists a map $\mathcal{O}_X \rightarrow \mathcal{F}$ of coherent \mathcal{O}_X -modules whose restriction to X_n recovers $\mathcal{O}_{X_n} \rightarrow \mathcal{F}_n$. Moreover, the support of \mathcal{F} is proper over S . As the completion functor is exact (Lemma 23.4) we see that the cokernel \mathcal{Q} of $\mathcal{O}_X \rightarrow \mathcal{F}$ has vanishing completion. Since \mathcal{F} has support proper over S and so does \mathcal{Q} this implies that $\mathcal{Q} = 0$ for example because the functor (27.0.1) is an equivalence by Grothendieck's existence theorem. Thus $\mathcal{F} = \mathcal{O}_X / \mathcal{J}$ for some quasi-coherent sheaf of ideals \mathcal{J} . Setting $Z = V(\mathcal{J})$ finishes the proof. \square

In the following lemma it is actually enough to assume that $Y_1 \rightarrow X_1$ is finite as it will imply that $Y_n \rightarrow X_n$ is finite too (see More on Morphisms, Lemma 3.3).

Lemma 28.2. *Let A be a Noetherian ring complete with respect to an ideal I . Write $S = \operatorname{Spec}(A)$ and $S_n = \operatorname{Spec}(A/I^n)$. Let $X \rightarrow S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{array}{ccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \dots \end{array}$$

of schemes with cartesian squares. Assume that

- (1) $Y_n \rightarrow X_n$ is a finite morphism, and
- (2) $Y_1 \rightarrow S_1$ is proper.

Then there exists a finite morphism of schemes $Y \rightarrow X$ such that $Y_n = Y \times_S S_n$. Moreover, Y is proper over S .

Proof. Let's write $f_n : Y_n \rightarrow X_n$ for the vertical morphisms. Set $\mathcal{F}_n = f_{n,*} \mathcal{O}_{Y_n}$. This is a coherent \mathcal{O}_{X_n} -module as f_n is finite (Lemma 9.9). Using that the squares are cartesian we see that the pullback of \mathcal{F}_{n+1} to X_n is \mathcal{F}_n . Hence Grothendieck's existence theorem, as reformulated in Remark 27.2, tells us there exists a coherent \mathcal{O}_X -module \mathcal{F} whose restriction to X_n recovers \mathcal{F}_n . Moreover, the support of \mathcal{F} is proper over S . As the completion functor is fully faithful (Theorem 27.1) we see that the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{F}_n \rightarrow \mathcal{F}_n$ fit together to give an algebra structure on \mathcal{F} . Setting $Y = \underline{\text{Spec}}_X(\mathcal{F})$ finishes the proof. \square

Lemma 28.3. *Let A be a Noetherian ring complete with respect to an ideal I . Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let X, Y be schemes over S . For $n \geq 1$ we set $X_n = X \times_S S_n$ and $Y_n = Y \times_S S_n$. Suppose given a compatible system of commutative diagrams*

$$\begin{array}{ccccc}
 & & X_{n+1} & \xrightarrow{g_{n+1}} & Y_{n+1} \\
 & \nearrow & \searrow & & \nearrow \\
 X_n & \xrightarrow{g_n} & Y_n & & S_{n+1} \\
 & \searrow & \nearrow & \nearrow & \\
 & & S_n & &
 \end{array}$$

Assume that

- (1) $X \rightarrow S$ is proper, and
- (2) $Y \rightarrow S$ is separated of finite type.

Then there exists a unique morphism of schemes $g : X \rightarrow Y$ over S such that g_n is the base change of g to S_n .

Proof. The morphisms $(1, g_n) : X_n \rightarrow X_n \times_S Y_n$ are closed immersions because $Y_n \rightarrow S_n$ is separated (Schemes, Lemma 21.11). Thus by Lemma 28.1 there exists a closed subscheme $Z \subset X \times_S Y$ proper over S whose base change to S_n recovers $X_n \subset X_n \times_S Y_n$. The first projection $p : Z \rightarrow X$ is a proper morphism (as Z is proper over S , see Morphisms, Lemma 41.7) whose base change to S_n is an isomorphism for all n . In particular, $p : Z \rightarrow X$ is finite over an open neighbourhood of X_0 by Lemma 21.2. As X is proper over S this open neighbourhood is all of X and we conclude $p : Z \rightarrow X$ is finite. Applying the equivalence of Proposition 25.4 we see that $p_* \mathcal{O}_Z = \mathcal{O}_X$ as this is true modulo I^n for all n . Hence p is an isomorphism and we obtain the morphism g as the composition $X \cong Z \rightarrow Y$. We omit the proof of uniqueness. \square

In order to prove an “abstract” algebraization theorem we need to assume we have an ample invertible sheaf, as the result is false without such an assumption.

Theorem 28.4 (Grothendieck's algebraization theorem). *Let A be a Noetherian ring complete with respect to an ideal I . Set $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$.*

Consider a commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \dots \end{array}$$

of schemes with cartesian squares. Suppose given $(\mathcal{L}_n, \varphi_n)$ where each \mathcal{L}_n is an invertible sheaf on X_n and $\varphi_n : i_n^* \mathcal{L}_{n+1} \rightarrow \mathcal{L}_n$ is an isomorphism. If

- (1) $X_1 \rightarrow S_1$ is proper, and
- (2) \mathcal{L}_1 is ample on X_1

then there exists a proper morphism of schemes $X \rightarrow S$ and an ample invertible \mathcal{O}_X -module \mathcal{L} and isomorphisms $X_n \cong X \times_S S_n$ and $\mathcal{L}_n \cong \mathcal{L}|_{X_n}$ compatible with the morphisms i_n and φ_n .

Proof. Since the squares in the diagram are cartesian and since the morphisms $S_n \rightarrow S_{n+1}$ are closed immersions, we see that the morphisms i_n are closed immersions too. In particular we may think of X_m as a closed subscheme of X_n for $m < n$. In fact X_m is the closed subscheme cut out by the quasi-coherent sheaf of ideals $I^m \mathcal{O}_{X_n}$. Moreover, the underlying topological spaces of the schemes X_1, X_2, X_3, \dots are all identified, hence we may (and do) think of sheaves \mathcal{O}_{X_n} as living on the same underlying topological space; similarly for coherent \mathcal{O}_{X_n} -modules. Set

$$\mathcal{F}_n = \text{Ker}(\mathcal{O}_{X_{n+1}} \rightarrow \mathcal{O}_{X_n})$$

so that we obtain short exact sequences

$$0 \rightarrow \mathcal{F}_n \rightarrow \mathcal{O}_{X_{n+1}} \rightarrow \mathcal{O}_{X_n} \rightarrow 0$$

By the above we have $\mathcal{F}_n = I^n \mathcal{O}_{X_{n+1}}$. It follows \mathcal{F}_n is a coherent sheaf on X_{n+1} annihilated by I , hence we may (and do) think of it as a coherent module \mathcal{O}_{X_1} -module. Observe that for $m > n$ the sheaf

$$I^n \mathcal{O}_{X_m} / I^{n+1} \mathcal{O}_{X_m}$$

maps isomorphically to \mathcal{F}_n under the map $\mathcal{O}_{X_m} \rightarrow \mathcal{O}_{X_{n+1}}$. Hence given $n_1, n_2 \geq 0$ we can pick an $m > n_1 + n_2$ and consider the multiplication map

$$I^{n_1} \mathcal{O}_{X_m} \times I^{n_2} \mathcal{O}_{X_m} \longrightarrow I^{n_1+n_2} \mathcal{O}_{X_m} \rightarrow \mathcal{F}_{n_1+n_2}$$

This induces an \mathcal{O}_{X_1} -bilinear map

$$\mathcal{F}_{n_1} \times \mathcal{F}_{n_2} \longrightarrow \mathcal{F}_{n_1+n_2}$$

which in turn defines the structure of a graded \mathcal{O}_{X_1} -algebra on $\mathcal{F} = \bigoplus_{n \geq 0} \mathcal{F}_n$.

Set $B = \bigoplus I^n / I^{n+1}$; this is a finitely generated graded A/I -algebra. Set $\mathcal{B} = (X_1 \rightarrow S_1)^* \tilde{B}$. The discussion above provides us with a canonical surjection

$$\mathcal{B} \longrightarrow \mathcal{F}$$

of graded \mathcal{O}_{X_1} -algebras. In particular we see that \mathcal{F} is a finite type quasi-coherent graded \mathcal{B} -module. By Lemma 19.3 we can find an integer d_0 such that $H^1(X_1, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$ for all $d \geq d_0$. Pick a $d \geq d_0$ such that there exist sections $s_{0,1}, \dots, s_{N,1} \in \Gamma(X_1, \mathcal{L}_1^{\otimes d})$ which induce an immersion

$$\psi_1 : X_1 \rightarrow \mathbf{P}_{S_1}^N$$

over S_1 , see Morphisms, Lemma 39.4. As X_1 is proper over S_1 we see that ψ_1 is a closed immersion, see Morphisms, Lemma 41.7 and Schemes, Lemma 10.4. We are going to “lift” ψ_1 to a compatible system of closed immersions of X_n into \mathbf{P}^N .

Upon tensoring the short exact sequences of the first paragraph of the proof by $\mathcal{L}_{n+1}^{\otimes d}$ we obtain short exact sequences

$$0 \rightarrow \mathcal{F}_n \otimes \mathcal{L}_{n+1}^{\otimes d} \rightarrow \mathcal{L}_{n+1}^{\otimes d} \rightarrow \mathcal{L}_{n+1}^{\otimes d} \rightarrow 0$$

Using the isomorphisms φ_n we obtain isomorphisms $\mathcal{L}_{n+1} \otimes \mathcal{O}_{X_l} = \mathcal{L}_l$ for $l \leq n$. Whence the sequence above becomes

$$0 \rightarrow \mathcal{F}_n \otimes \mathcal{L}_1^{\otimes d} \rightarrow \mathcal{L}_{n+1}^{\otimes d} \rightarrow \mathcal{L}_n^{\otimes d} \rightarrow 0$$

The vanishing of $H^1(X, \mathcal{F}_n \otimes \mathcal{L}_1^{\otimes d})$ implies we can inductively lift $s_{0,1}, \dots, s_{N,1} \in \Gamma(X_1, \mathcal{L}_1^{\otimes d})$ to sections $s_{0,n}, \dots, s_{N,n} \in \Gamma(X_n, \mathcal{L}_n^{\otimes d})$. Thus we obtain a commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \dots \\ \psi_1 \downarrow & & \psi_2 \downarrow & & \psi_3 \downarrow & & \\ \mathbf{P}_{S_1}^N & \longrightarrow & \mathbf{P}_{S_2}^N & \longrightarrow & \mathbf{P}_{S_3}^N & \longrightarrow & \dots \end{array}$$

where $\psi_n = \varphi_{(\mathcal{L}_n, (s_{0,n}, \dots, s_{N,n}))}$ in the notation of Constructions, Section 13. As the squares in the statement of the theorem are cartesian we see that the squares in the above diagram are cartesian. We win by applying Lemma 28.1. \square

29. Other chapters

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References

- [DG67] Jean Dieudonné and Alexander Grothendieck, *Éléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32** (1961–1967).
- [Har98] Robin Hartshorne, *Coherent functors*, Adv. Math. **140** (1998), no. 1, 44–94.
- [Ser55] Jean-Pierre Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278.