

INTERSECTION THEORY

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1. Introduction

In this chapter we construct the intersection product on the Chow groups modulo rational equivalence on a nonsingular projective variety over an algebraically closed field. Our tools are Serre's Tor formula (see [Ser65, Chapter V]), reduction to the diagonal, and the moving lemma.

We first recall cycles and how to construct proper pushforward and flat pullback of cycles. Next, we introduce rational equivalence of cycles which gives us the Chow

groups $\mathrm{CH}_*(X)$. Proper pushforward and flat pullback factor through rational equivalence to give operations on Chow groups. This takes up Sections 3, 4, 5, 6, 7, 8, 9, 10, and 11. For proofs we mostly refer to the chapter on Chow homology where these results have been proven in the setting of schemes locally of finite type over a universally catenary Noetherian base, see Chow Homology, Section 7 ff.

Since we work on a nonsingular projective X any irreducible component of the intersection $V \cap W$ of two irreducible closed subvarieties has dimension at least $\dim(V) + \dim(W) - \dim(X)$. We say V and W intersect properly if equality holds for every irreducible component Z . In this case we define the intersection multiplicity $e_Z = e(X, V \cdot W, Z)$ by the formula

$$e_Z = \sum_i (-1)^i \mathrm{length}_{\mathcal{O}_{X,Z}} \mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{W,Z}, \mathcal{O}_{V,Z})$$

We need to do a little bit of commutative algebra to show that these intersection multiplicities agree with intuition in simple cases, namely, that sometimes

$$e_Z = \mathrm{length}_{\mathcal{O}_{X,Z}} \mathcal{O}_{V \cap W, Z},$$

in other words, only Tor_0 contributes. This happens when V and W are Cohen-Macaulay in the generic point of Z or when W is cut out by a regular sequence in $\mathcal{O}_{X,Z}$ which also defines a regular sequence on $\mathcal{O}_{V,Z}$. However, Example 14.4 shows that higher tors are necessary in general. Moreover, there is a relationship with the Samuel multiplicity. These matters are discussed in Sections 13, 14, 15, 16, and 17.

Reduction to the diagonal is the statement that we can intersect V and W by intersecting $V \times W$ with the diagonal in $X \times X$. This innocuous statement, which is clear on the level of scheme theoretic intersections, reduces an intersection of a general pair of closed subschemes, to the case where one of the two is locally cut out by a regular sequence. We use this, following Serre, to obtain positivity of intersection multiplicities. Moreover, reduction to the diagonal leads to additivity of intersection multiplicities, associativity, and a projection formula. This can be found in Sections 18, 19, 20, 21, and 22.

Finally, we come to the moving lemmas and applications. There are two parts to the moving lemma. The first is that given closed subvarieties

$$Z \subset X \subset \mathbf{P}^N$$

with X nonsingular, we can find a subvariety $C \subset \mathbf{P}^N$ intersecting X properly such that

$$C \cdot X = [Z] + \sum m_j [Z_j]$$

and such that the other components Z_j are “more general” than Z . The second part is that one can move $C \subset \mathbf{P}^N$ over a rational curve to a subvariety in general position with respect to any given list of subvarieties. Combined these results imply that it suffices to define the intersection product of cycles on X which intersect properly which was done above. Of course this only leads to an intersection product on $\mathrm{CH}_*(X)$ if one can show, as we do in the text, that these products pass through rational equivalence. This and some applications are discussed in Sections 23, 24, 25, 26, 27, and 28.

2. Conventions

We fix an algebraically closed ground field \mathbf{C} of any characteristic. All schemes and varieties are over \mathbf{C} and all morphisms are over \mathbf{C} . A variety X is *nonsingular* if X is a regular scheme (see Properties, Definition 9.1). In our case this means that the morphism $X \rightarrow \operatorname{Spec}(\mathbf{C})$ is smooth (see Varieties, Lemma 12.6).

3. Cycles

Let X be a variety. A *closed subvariety* of X is an integral closed subscheme $Z \subset X$. A *k-cycle* on X is a finite formal sum $\sum n_i [Z_i]$ where each Z_i is a closed subvariety of dimension k . Whenever we use the notation $\alpha = \sum n_i [Z_i]$ for a k -cycle we always assume the subvarieties Z_i are pairwise distinct and $n_i \neq 0$ for all i . In this case the *support* of α is the closed subset

$$\operatorname{Supp}(\alpha) = \bigcup Z_i \subset X$$

of dimension k . The group of k -cycles is denoted $Z_k(X)$. See Chow Homology, Section 8.

4. Cycle associated to closed subscheme

Suppose that X is a variety and that $Z \subset X$ be a closed subscheme with $\dim(Z) \leq k$. Let Z_i be the irreducible components of Z of dimension k and let n_i be the *multiplicity of Z_i in Z* defined as

$$n_i = \operatorname{length}_{\mathcal{O}_{X, Z_i}} \mathcal{O}_{Z, Z_i}$$

where \mathcal{O}_{X, Z_i} , resp. \mathcal{O}_{Z, Z_i} is the local ring of X , resp. Z at the generic point of Z_i . We define the k -cycle associated to Z to be the k -cycle

$$[Z]_k = \sum n_i [Z_i].$$

See Chow Homology, Section 9.

5. Cycle associated to a coherent sheaf

Suppose that X is a variety and that \mathcal{F} is a coherent \mathcal{O}_X -module with $\dim(\operatorname{Supp}(\mathcal{F})) \leq k$. Let Z_i be the irreducible components of $\operatorname{Supp}(\mathcal{F})$ of dimension k and let n_i be the *multiplicity of Z_i in \mathcal{F}* defined as

$$n_i = \operatorname{length}_{\mathcal{O}_{X, Z_i}} \mathcal{F}_{\xi_i}$$

where \mathcal{O}_{X, Z_i} is the local ring of X at the generic point ξ_i of Z_i and \mathcal{F}_{ξ_i} is the stalk of \mathcal{F} at this point. We define the k -cycle associated to \mathcal{F} to be the k -cycle

$$[\mathcal{F}]_k = \sum n_i [Z_i].$$

See Chow Homology, Section 10. Note that, if $Z \subset X$ is a closed subscheme with $\dim(Z) \leq k$, then $[Z]_k = [\mathcal{O}_Z]_k$ by definition.

6. Proper pushforward

Suppose that $f : X \rightarrow Y$ is a proper morphism of varieties. Let $Z \subset X$ be a k -dimensional closed subvariety. We define $f_*[Z]$ to be 0 if $\dim(f(Z)) < k$ and $d \cdot [f(Z)]$ if $\dim(f(Z)) = k$ where

$$d = [\mathbf{C}(Z) : \mathbf{C}(f(Z))] = \deg(Z/f(Z))$$

is the degree of the dominant morphism $Z \rightarrow f(Z)$, see Morphisms, Definition 51.8. Let $\alpha = \sum n_i [Z_i]$ be a k -cycle on X . The *pushforward* of α is the sum $f_*\alpha = \sum n_i f_*[Z_i]$ where each $f_*[Z_i]$ is defined as above. This defines a homomorphism

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

See Chow Homology, Section 12.

Lemma 6.1. *Suppose that $f : X \rightarrow Y$ is a proper morphism of varieties. Let \mathcal{F} be a coherent sheaf with $\dim(\text{Supp}(\mathcal{F})) \leq k$, then $f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k$. In particular, if $Z \subset X$ is a closed subscheme of dimension $\leq k$, then $f_*[Z]_k = [f_*\mathcal{O}_Z]_k$.*

Proof. See Chow Homology, Lemma 12.4. □

Lemma 6.2. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be proper morphisms of varieties. Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \rightarrow Z_k(Z)$.*

Proof. Special case of Chow Homology, Lemma 12.2. □

7. Flat pullback

Suppose that $f : X \rightarrow Y$ is a flat morphism of varieties. By Morphisms, Lemma 28.2 every fibre of f has dimension $r = \dim(X) - \dim(Y)$ ¹. Let $Z \subset X$ be a k -dimensional closed subvariety. We define $f^*[Z]$ to be the $(k+r)$ -cycle associated to the scheme theoretic inverse image: $f^*[Z] = [f^{-1}(Z)]_{k+r}$. Let $\alpha = \sum n_i [Z_i]$ be a k -cycle on Y . The *pullback* of α is the sum $f^*\alpha = \sum n_i f^*[Z_i]$ where each $f^*[Z_i]$ is defined as above. This defines a homomorphism

$$f^* : Z_k(Y) \longrightarrow Z_{k+r}(X)$$

See Chow Homology, Section 14.

Lemma 7.1. *Let $f : X \rightarrow Y$ be a flat morphism of varieties. Set $r = \dim(X) - \dim(Y)$. Then $f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$ if \mathcal{F} is a coherent sheaf on Y and the dimension of the support of \mathcal{F} is at most k .*

Proof. See Chow Homology, Lemma 14.4. □

Lemma 7.2. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be flat morphisms of varieties. Then $g \circ f$ is flat and $f^* \circ g^* = (g \circ f)^*$ as maps $Z_k(Z) \rightarrow Z_{k+\dim(X)-\dim(Z)}(X)$.*

Proof. Special case of Chow Homology, Lemma 14.3. □

¹Conversely, if $f : X \rightarrow Y$ is a dominant morphism of varieties, X is Cohen-Macaulay, Y is nonsingular, and all fibres have the same dimension r , then f is flat. This follows from Algebra, Lemma 128.1 and Varieties, Lemma 20.4 showing $\dim(X) = \dim(Y) + r$.

8. Rational Equivalence

We are going to define rational equivalence in a way which at first glance may seem different from what you are used to, or from what is in [Ful98, Chapter I] or Chow Homology, Section 19. However, in Section 9 we will show that the two notions agree.

Let X be a variety. Let $W \subset X \times \mathbf{P}^1$ be a closed subvariety of dimension $k + 1$. Let a, b be distinct closed points of \mathbf{P}^1 . Assume that $X \times a$, $X \times b$ and W intersect properly:

$$\dim(W \cap X \times a) \leq k, \quad \dim(W \cap X \times b) \leq k.$$

This is true as soon as $W \rightarrow \mathbf{P}^1$ is dominant or if W is contained in a fibre of the projection over a closed point different from a or b (this is an uninteresting case which we will discard). In this situation the scheme theoretic fibre W_a of the morphism $W \rightarrow \mathbf{P}^1$ is equal to the scheme theoretic intersection $W \cap X \times a$ in $X \times \mathbf{P}^1$. Identifying $X \times a$ and $X \times b$ with X we may think of the fibres W_a and W_b as closed subschemes of X of dimension $\leq k^2$. A basic example of a rational equivalence is

$$[W_a]_k \sim_{rat} [W_b]_k$$

The cycles $[W_a]_k$ and $[W_b]_k$ are easy to compute in practice (given W) because they are obtained by proper intersection with a Cartier divisor (we will see this in Section 17). Since the automorphism group of \mathbf{P}^1 is 2-transitive we may move the pair of closed points a, b to any pair we like. A traditional choice is to choose $a = 0$ and $b = \infty$.

More generally, let $\alpha = \sum n_i [W_i]$ be a $(k + 1)$ -cycle on $X \times \mathbf{P}^1$. Let a_i, b_i be pairs of distinct closed points of \mathbf{P}^1 . Assume that $X \times a_i$, $X \times b_i$ and W_i intersect properly, in other words, each W_i, a_i, b_i satisfies the condition discussed above. A cycle *rationally equivalent to zero* is any cycle of the form

$$\sum n_i ([W_{i,a_i}]_k - [W_{i,b_i}]_k).$$

This is indeed a k -cycle. The collection of k -cycles rationally equivalent to zero is an additive subgroup of the group of k -cycles. We say two k -cycles are *rationally equivalent*, notation $\alpha \sim_{rat} \alpha'$, if $\alpha - \alpha'$ is a cycle rationally equivalent to zero.

We define

$$\mathrm{CH}_k(X) = Z_k(X) / \sim_{rat}$$

to be the *Chow group of k -cycles on X* . We will see in Lemma 9.1 that this agrees with the Chow group as defined in Chow Homology, Definition 19.1.

9. Rational equivalence and rational functions

Let X be a variety. Let $W \subset X$ be a subvariety of dimension $k + 1$. Let $f \in \mathbf{C}(W)^*$ be a nonzero rational function on W . For every subvariety $Z \subset W$ of dimension k one can define the order of vanishing $\mathrm{ord}_{W,Z}(f)$ of f at Z . If f is an element of the local ring $\mathcal{O}_{W,Z}$, then one has

$$\mathrm{ord}_{W,Z}(f) = \mathrm{length}_{\mathcal{O}_{X,Z}} \mathcal{O}_{W,Z} / f \mathcal{O}_{W,Z}$$

²We will sometimes think of W_a as a closed subscheme of $X \times \mathbf{P}^1$ and sometimes as a closed subscheme of X . It should always be clear from context which point of view is taken.

where $\mathcal{O}_{X,Z}$, resp. $\mathcal{O}_{W,Z}$ is the local ring of X , resp. W at the generic point of Z . In general one extends the definition by multiplicativity. The *principal divisor associated to f* is

$$\operatorname{div}_W(f) = \sum \operatorname{ord}_{W,Z}(f)[Z]$$

in $Z_k(W)$. Since $W \subset X$ is a closed subvariety we may think of $\operatorname{div}_W(f)$ as a cycle on X . See Chow Homology, Section 17.

Lemma 9.1. *Let X be a variety. Let $W \subset X$ be a subvariety of dimension $k+1$. Let $f \in \mathbf{C}(W)^*$ be a nonzero rational function on W . Then $\operatorname{div}_W(f)$ is rationally equivalent to zero on X . Conversely, these principal divisors generate the abelian group of cycles rationally equivalent to zero on X .*

Proof. The first assertion follows from Chow Homology, Lemma 18.2. More precisely, let $W' \subset X \times \mathbf{P}^1$ be the closure of the graph of f . Then $\operatorname{div}_W(f) = [W'_0]_k - [W'_\infty]_k$ in $Z_k(W) \subset Z_k(X)$, see part (6) of Chow Homology, Lemma 18.2.

For the second, let $W' \subset X \times \mathbf{P}^1$ be a closed subvariety of dimension $k+1$ which dominates \mathbf{P}^1 . We will show that $[W'_0]_k - [W'_\infty]_k$ is a principal divisor which will finish the proof. Let $W \subset X$ be the image of W' under the projection to X . Then $W \subset X$ is a closed subvariety and $W' \rightarrow W$ is proper and dominant with fibres of dimension 0 or 1. If $\dim(W) = k$, then $W' = W \times \mathbf{P}^1$ and we see that $[W'_0]_k - [W'_\infty]_k = [W] - [W] = 0$. If $\dim(W) = k+1$, then $W' \rightarrow W$ is generically finite³. Let f denote the projection $W' \rightarrow \mathbf{P}^1$ viewed as an element of $\mathbf{C}(W')^*$. Let $g = \operatorname{Nm}(f) \in \mathbf{C}(W)^*$ be the norm. By Chow Homology, Lemma 18.1 we have

$$\operatorname{div}_W(g) = \operatorname{pr}_{X,*} \operatorname{div}_{W'}(f)$$

Since $\operatorname{div}_{W'}(f) = [W'_0]_k - [W'_\infty]_k$ by Chow Homology, Lemma 18.2 the proof is complete. \square

10. Proper pushforward and rational equivalence

Suppose that $f : X \rightarrow Y$ is a proper morphism of varieties. Let $\alpha \sim_{rat} 0$ be a k -cycle on X rationally equivalent to 0. Then the pushforward of α is rationally equivalent to zero: $f_*\alpha \sim_{rat} 0$. See Chapter I of [Ful98] or Chow Homology, Lemma 20.3.

Therefore we obtain a commutative diagram

$$\begin{array}{ccc} Z_k(X) & \longrightarrow & \operatorname{CH}_k(X) \\ f_* \downarrow & & \downarrow f_* \\ Z_k(Y) & \longrightarrow & \operatorname{CH}_k(Y) \end{array}$$

of groups of k -cycles.

³If $W' \rightarrow W$ is birational, then the result follows from Chow Homology, Lemma 18.2. Our task is to show that even if $W' \rightarrow W$ has degree > 1 the basic rational equivalence $[W'_0]_k \sim_{rat} [W'_\infty]_k$ comes from a principal divisor on a subvariety of X .

11. Flat pullback and rational equivalence

Suppose that $f : X \rightarrow Y$ is a flat morphism of varieties. Set $r = \dim(X) - \dim(Y)$. Let $\alpha \sim_{rat} 0$ be a k -cycle on Y rationally equivalent to 0. Then the pullback of α is rationally equivalent to zero: $f^*\alpha \sim_{rat} 0$. See Chapter I of [Ful98] or Chow Homology, Lemma 20.2.

Therefore we obtain a commutative diagram

$$\begin{array}{ccc} Z_{k+r}(X) & \longrightarrow & CH_{k+r}(X) \\ f^* \uparrow & & \uparrow f^* \\ Z_k(Y) & \longrightarrow & CH_k(Y) \end{array}$$

of groups of k -cycles.

12. The short exact sequence for an open

Let X be a variety and let $U \subset X$ be an open subvariety. Let $X \setminus U = \bigcup Z_i$ be the decomposition into irreducible components⁴. Then for each $k \geq 0$ there exists a commutative diagram

$$\begin{array}{ccccccc} \bigoplus Z_k(Z_i) & \longrightarrow & Z_k(X) & \longrightarrow & Z_k(U) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus CH_k(Z_i) & \longrightarrow & CH_k(X) & \longrightarrow & CH_k(U) & \longrightarrow & 0 \end{array}$$

with exact rows. Here the vertical arrows are the canonical quotient maps. The left horizontal arrows are given by proper pushforward along the closed immersions $Z_i \rightarrow X$. The right horizontal arrows are given by flat pullback along the open immersion $j : U \rightarrow X$. Since we have seen that these maps factor through rational equivalence we obtain the commutativity of the squares. The top row is exact simply because every subvariety of X is either contained in some Z_i or has irreducible intersection with U . The bottom row is exact because every principal divisor $\text{div}_W(f)$ on U is the restriction of a principal divisor on X . More precisely, if $W \subset U$ is a $(k+1)$ -dimensional closed subvariety and $f \in \mathbf{C}(W)^*$, then denote \overline{W} the closure of W in X . Then $W \subset \overline{W}$ is an open immersion, so $\mathbf{C}(W) = \mathbf{C}(\overline{W})$ and we may think of f as a nonconstant rational function on \overline{W} . Then clearly

$$j^* \text{div}_{\overline{W}}(f) = \text{div}_W(f)$$

in $Z_k(X)$. The exactness of the lower row follows easily from this. For details see Chow Homology, Lemma 19.3.

13. Proper intersections

First a few lemmas to get dimension estimates.

Lemma 13.1. *Let X and Y be varieties. Then $X \times Y$ is a variety and $\dim(X \times Y) = \dim(X) + \dim(Y)$.*

Proof. The scheme $X \times Y = X \times_{\text{Spec}(\mathbf{C})} Y$ is a variety by Varieties, Lemma 3.3. The statement on dimension is Varieties, Lemma 20.5. \square

⁴Since in this chapter we only consider Chow groups of varieties, we are prohibited from taking $Z_k(X \setminus U)$ and $CH_k(X \setminus U)$, hence the approach using the varieties Z_i .

Recall that a regular immersion $i : X \rightarrow Y$ of schemes is a closed immersion whose corresponding sheaf of ideals is locally generated by a regular sequence, see Divisors, Section 21. Moreover, the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free of rank equal to the length of the regular sequence. Let us say i is a *regular immersion of codimension c* if $\mathcal{C}_{X/Y}$ is locally free of rank c .

More generally, recall (More on Morphisms, Section 62) that $f : X \rightarrow Y$ is a local complete intersection morphism if we can cover X by opens U such that we can factor $f|_U$ as

$$\begin{array}{ccc} U & \xrightarrow{\quad i \quad} & \mathbf{A}_Y^n \\ & \searrow & \swarrow \\ & Y & \end{array}$$

where i is a Koszul regular immersion (if Y is locally Noetherian this is the same as asking i to be a regular immersion, see Divisors, Lemma 21.3). Let us say that f is a *local complete intersection morphism of relative dimension r* if for any factorization as above, the closed immersion i has conormal sheaf of rank $n - r$ (in other words if i is a Koszul-regular immersion of codimension $n - r$ which in the Noetherian case just means it is regular immersion of codimension $n - r$).

Lemma 13.2. *Let $f : X \rightarrow Y$ be a morphism of varieties.*

- (1) *If $Z \subset Y$ is a subvariety dimension d and f is a regular immersion of codimension c , then every irreducible component of $f^{-1}(Z)$ has dimension $\geq d - c$.*
- (2) *If $Z \subset Y$ is a subvariety of dimension d and f is a local complete intersection morphism of relative dimension r , then every irreducible component of $f^{-1}(Z)$ has dimension $\geq d + r$.*

Proof. Proof of (1). We may work locally, hence we may assume that $Y = \text{Spec}(A)$ and $X = V(f_1, \dots, f_c)$ where f_1, \dots, f_c is a regular sequence in A . If $Z = \text{Spec}(A/\mathfrak{p})$, then we see that $f^{-1}(Z) = \text{Spec}(A/\mathfrak{p} + (f_1, \dots, f_c))$. If V is an irreducible component of $f^{-1}(Z)$, then we can choose a closed point $v \in V$ not contained in any other irreducible component of $f^{-1}(Z)$. Then

$$\dim(Z) = \dim \mathcal{O}_{Z,v} \quad \text{and} \quad \dim(V) = \dim \mathcal{O}_{V,v} = \dim \mathcal{O}_{Z,v}/(f_1, \dots, f_c)$$

The first equality for example by Algebra, Lemma 116.1 and the second equality by our choice of closed point. The result now follows from the fact that dividing by one element in the maximal ideal decreases the dimension by at most 1, see Algebra, Lemma 60.13.

Proof of (2). Choose a factorization as in the definition of a local complete intersection and apply (1). Some details omitted. \square

Lemma 13.3. *Let X be a nonsingular variety. Then the diagonal $\Delta : X \rightarrow X \times X$ is a regular immersion of codimension $\dim(X)$.*

Proof. In fact, any closed immersion between nonsingular projective varieties is a regular immersion, see Divisors, Lemma 22.11. \square

The following lemma demonstrates how reduction to the diagonal works.

Lemma 13.4. *Let X be a nonsingular variety and let $W, V \subset X$ be closed subvarieties with $\dim(W) = s$ and $\dim(V) = r$. Then every irreducible component Z of $V \cap W$ has dimension $\geq r + s - \dim(X)$.*

Proof. Since $V \cap W = \Delta^{-1}(V \times W)$ (scheme theoretically) we conclude by Lemmas 13.3 and 13.2. \square

This lemma suggests the following definition.

Definition 13.5. Let X be a nonsingular variety.

- (1) Let $W, V \subset X$ be closed subvarieties with $\dim(W) = s$ and $\dim(V) = r$. We say that W and V *intersect properly* if $\dim(V \cap W) \leq r + s - \dim(X)$.
- (2) Let $\alpha = \sum n_i [W_i]$ be an s -cycle, and $\beta = \sum_j m_j [V_j]$ be an r -cycle on X . We say that α and β *intersect properly* if W_i and V_j intersect properly for all i and j .

14. Intersection multiplicities using Tor formula

A basic fact we will use frequently is that given sheaves of modules \mathcal{F}, \mathcal{G} on a ringed space (X, \mathcal{O}_X) and a point $x \in X$ we have

$$\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x = \mathrm{Tor}_p^{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

as $\mathcal{O}_{X,x}$ -modules. This can be seen in several ways from our construction of derived tensor products in Cohomology, Section 26, for example it follows from Cohomology, Lemma 26.4. Moreover, if X is a scheme and \mathcal{F} and \mathcal{G} are quasi-coherent, then the modules $\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are quasi-coherent too, see Derived Categories of Schemes, Lemma 3.9. More important for our purposes is the following result.

Lemma 14.1. *Let X be a locally Noetherian scheme.*

- (1) *If \mathcal{F} and \mathcal{G} are coherent \mathcal{O}_X -modules, then $\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is too.*
- (2) *If L and K are in $D_{\mathrm{Coh}}^-(\mathcal{O}_X)$, then so is $L \otimes_{\mathcal{O}_X}^{\mathbf{L}} K$.*

Proof. Let us explain how to prove (1) in a more elementary way and part (2) using previously developed general theory.

Proof of (1). Since formation of Tor commutes with localization we may assume X is affine. Hence $X = \mathrm{Spec}(A)$ for some Noetherian ring A and \mathcal{F}, \mathcal{G} correspond to finite A -modules M and N (Cohomology of Schemes, Lemma 9.1). By Derived Categories of Schemes, Lemma 3.9 we may compute the Tor's by first computing the Tor's of M and N over A , and then taking the associated \mathcal{O}_X -module. Since the modules $\mathrm{Tor}_p^A(M, N)$ are finite by Algebra, Lemma 75.7 we conclude.

By Derived Categories of Schemes, Lemma 10.3 the assumption is equivalent to asking L and K to be (locally) pseudo-coherent. Then $L \otimes_{\mathcal{O}_X}^{\mathbf{L}} K$ is pseudo-coherent by Cohomology, Lemma 47.5. \square

Lemma 14.2. *Let X be a nonsingular variety. Let \mathcal{F}, \mathcal{G} be coherent \mathcal{O}_X -modules. The \mathcal{O}_X -module $\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent, has stalk at x equal to $\mathrm{Tor}_p^{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$, is supported on $\mathrm{Supp}(\mathcal{F}) \cap \mathrm{Supp}(\mathcal{G})$, and is nonzero only for $p \in \{0, \dots, \dim(X)\}$.*

Proof. The result on stalks was discussed above and it implies the support condition. The Tor's are coherent by Lemma 14.1. The vanishing of negative Tor's is immediate from the construction. The vanishing of Tor_p for $p > \dim(X)$ can be

seen as follows: the local rings $\mathcal{O}_{X,x}$ are regular (as X is nonsingular) of dimension $\leq \dim(X)$ (Algebra, Lemma 116.1), hence $\mathcal{O}_{X,x}$ has finite global dimension $\leq \dim(X)$ (Algebra, Lemma 110.8) which implies that Tor-groups of modules vanish beyond the dimension (More on Algebra, Lemma 66.19). \square

Let X be a nonsingular variety and $W, V \subset X$ be closed subvarieties with $\dim(W) = s$ and $\dim(V) = r$. Assume V and W intersect properly. In this case Lemma 13.4 tells us all irreducible components of $V \cap W$ have dimension equal to $r + s - \dim(X)$. The sheaves $\mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V)$ are coherent, supported on $V \cap W$, and zero if $j < 0$ or $j > \dim(X)$ (Lemma 14.2). We define the *intersection product* as

$$W \cdot V = \sum_i (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V)]_{r+s-\dim(X)}.$$

We stress that this makes sense only because of our assumption that V and W intersect properly. This fact will necessitate a moving lemma in order to define the intersection product in general.

With this notation, the cycle $V \cdot W$ is a formal linear combination $\sum e_Z Z$ of the irreducible components Z of the intersection $V \cap W$. The integers e_Z are called the *intersection multiplicities*

$$e_Z = e(X, V \cdot W, Z) = \sum_i (-1)^i \mathrm{length}_{\mathcal{O}_{X,Z}} \mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{W,Z}, \mathcal{O}_{V,Z})$$

where $\mathcal{O}_{X,Z}$, resp. $\mathcal{O}_{W,Z}$, resp. $\mathcal{O}_{V,Z}$ denotes the local ring of X , resp. W , resp. V at the generic point of Z . These alternating sums of lengths of Tor's satisfy many good properties, as we will see later on.

In the case of transversal intersections, the intersection number is 1.

Lemma 14.3. *Let X be a nonsingular variety. Let $V, W \subset X$ be closed subvarieties which intersect properly. Let Z be an irreducible component of $V \cap W$ and assume that the multiplicity (in the sense of Section 4) of Z in the closed subscheme $V \cap W$ is 1. Then $e(X, V \cdot W, Z) = 1$ and V and W are smooth in a general point of Z .*

Proof. Let $(A, \mathfrak{m}, \kappa) = (\mathcal{O}_{X,\xi}, \mathfrak{m}_\xi, \kappa(\xi))$ where $\xi \in Z$ is the generic point. Then $\dim(A) = \dim(X) - \dim(Z)$, see Varieties, Lemma 20.3. Let $I, J \subset A$ cut out the trace of V and W in $\mathrm{Spec}(A)$. Set $\bar{I} = I + \mathfrak{m}^2/\mathfrak{m}^2$. Then $\dim_\kappa \bar{I} \leq \dim(X) - \dim(V)$ with equality if and only if A/I is regular (this follows from the lemma cited above and the definition of regular rings, see Algebra, Definition 60.10 and the discussion preceding it). Similarly for \bar{J} . If the multiplicity is 1, then $\mathrm{length}_A(A/I + J) = 1$, hence $I + J = \mathfrak{m}$, hence $\bar{I} + \bar{J} = \mathfrak{m}/\mathfrak{m}^2$. Then we get equality everywhere (because the intersection is proper). Hence we find $f_1, \dots, f_a \in I$ and $g_1, \dots, g_b \in J$ such that $\bar{f}_1, \dots, \bar{g}_b$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$. Then f_1, \dots, g_b is a regular system of parameters and a regular sequence (Algebra, Lemma 106.3). The same lemma shows $A/(f_1, \dots, f_a)$ is a regular local ring of dimension $\dim(X) - \dim(V)$, hence $A/(f_1, \dots, f_a) \rightarrow A/I$ is an isomorphism (if the kernel is nonzero, then the dimension of A/I is strictly less, see Algebra, Lemmas 106.2 and 60.13). We conclude $I = (f_1, \dots, f_a)$ and $J = (g_1, \dots, g_b)$ by symmetry. Thus the Koszul complex $K_\bullet(A, f_1, \dots, f_a)$ on f_1, \dots, f_a is a resolution of A/I , see More on Algebra, Lemma 30.2. Hence

$$\begin{aligned} \mathrm{Tor}_p^A(A/I, A/J) &= H_p(K_\bullet(A, f_1, \dots, f_a) \otimes_A A/J) \\ &= H_p(K_\bullet(A/J, f_1 \bmod J, \dots, f_a \bmod J)) \end{aligned}$$

Since we've seen above that $f_1 \bmod J, \dots, f_a \bmod J$ is a regular system of parameters in the regular local ring A/J we conclude that there is only one cohomology group, namely $H_0 = A/(I + J) = \kappa$. This finishes the proof. \square

Example 14.4. In this example we show that it is necessary to use the higher tors in the formula for the intersection multiplicities above. Let X be a nonsingular variety of dimension 4. Let $p \in X$ be a closed point. Let $V, W \subset X$ be closed subvarieties in X . Assume that there is an isomorphism

$$\mathcal{O}_{X,p}^\wedge \cong \mathbf{C}[[x, y, z, w]]$$

such that the ideal of V is (xz, xw, yz, yw) and the ideal of W is $(x - z, y - w)$. Then a computation shows that

$$\text{length } \mathbf{C}[[x, y, z, w]]/(xz, xw, yz, yw, x - z, y - w) = 3$$

On the other hand, the multiplicity $e(X, V \cdot W, p) = 2$ as can be seen from the fact that formal locally V is the union of two smooth planes $x = y = 0$ and $z = w = 0$ at p , each of which has intersection multiplicity 1 with the plane $x - z = y - w = 0$ (Lemma 14.3). To make an actual example, take a general morphism $f : \mathbf{P}^2 \rightarrow \mathbf{P}^4$ given by 5 homogeneous polynomials of degree > 1 . The image $V \subset \mathbf{P}^4 = X$ will have singularities of the type described above, because there will be $p_1, p_2 \in \mathbf{P}^2$ with $f(p_1) = f(p_2)$. To find W take a general plane passing through such a point.

15. Algebraic multiplicities

Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let M be a finite A -module and let $I \subset A$ be an ideal of definition (Algebra, Definition 59.1). Recall that the function

$$\chi_{I,M}(n) = \text{length}_A(M/I^n M) = \sum_{p=0, \dots, n-1} \text{length}_A(I^p M/I^{p+1} M)$$

is a numerical polynomial (Algebra, Proposition 59.5). The degree of this polynomial is equal to $\dim(\text{Supp}(M))$ by Algebra, Lemma 62.6.

Definition 15.1. In the situation above, if $d \geq \dim(\text{Supp}(M))$, then we set $e_I(M, d)$ equal to 0 if $d > \dim(\text{Supp}(M))$ and equal to $d!$ times the leading coefficient of the numerical polynomial $\chi_{I,M}$ so that

$$\chi_{I,M}(n) \sim e_I(M, d) \frac{n^d}{d!} + \text{lower order terms}$$

The *multiplicity of M for the ideal of definition I* is $e_I(M) = e_I(M, \dim(\text{Supp}(M)))$.

We have the following properties of these multiplicities.

Lemma 15.2. *Let A be a Noetherian local ring. Let $I \subset A$ be an ideal of definition. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of finite A -modules. Let $d \geq \dim(\text{Supp}(M))$. Then*

$$e_I(M, d) = e_I(M', d) + e_I(M'', d)$$

Proof. Immediate from the definitions and Algebra, Lemma 59.10. \square

Lemma 15.3. *Let A be a Noetherian local ring. Let $I \subset A$ be an ideal of definition. Let M be a finite A -module. Let $d \geq \dim(\text{Supp}(M))$. Then*

$$e_I(M, d) = \sum \text{length}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) e_I(A/\mathfrak{p}, d)$$

where the sum is over primes $\mathfrak{p} \subset A$ with $\dim(A/\mathfrak{p}) = d$.

Proof. Both the left and side and the right hand side are additive in short exact sequences of modules of dimension $\leq d$, see Lemma 15.2 and Algebra, Lemma 52.3. Hence by Algebra, Lemma 62.1 it suffices to prove this when $M = A/\mathfrak{q}$ for some prime \mathfrak{q} of A with $\dim(A/\mathfrak{q}) \leq d$. This case is obvious. \square

Lemma 15.4. *Let P be a polynomial of degree r with leading coefficient a . Then*

$$r!a = \sum_{i=0, \dots, r} (-1)^i \binom{r}{i} P(t-i)$$

for any t .

Proof. Let us write Δ the operator which to a polynomial P associates the polynomial $\Delta(P) = P(t) - P(t-1)$. We claim that

$$\Delta^r(P) = \sum_{i=0, \dots, r} (-1)^i \binom{r}{i} P(t-i)$$

This is true for $r = 0, 1$ by inspection. Assume it is true for r . Then we compute

$$\begin{aligned} \Delta^{r+1}(P) &= \sum_{i=0, \dots, r} (-1)^i \binom{r}{i} \Delta(P)(t-i) \\ &= \sum_{n=-r, \dots, 0} (-1)^i \binom{r}{i} (P(t-i) - P(t-i-1)) \end{aligned}$$

Thus the claim follows from the equality

$$\binom{r+1}{i} = \binom{r}{i} + \binom{r}{i-1}$$

The lemma follows from the fact that $\Delta(P)$ is of degree $r-1$ with leading coefficient ra if the degree of P is r . \square

An important fact is that one can compute the multiplicity in terms of the Koszul complex. Recall that if R is a ring and $f_1, \dots, f_r \in R$, then $K_\bullet(f_1, \dots, f_r)$ denotes the Koszul complex, see More on Algebra, Section 28.

Theorem 15.5. *Let A be a Noetherian local ring. Let $I = (f_1, \dots, f_r) \subset A$ be an ideal of definition. Let M be a finite A -module. Then*

$$e_I(M, r) = \sum (-1)^i \text{length}_A H_i(K_\bullet(f_1, \dots, f_r) \otimes_A M)$$

Proof. Let us change the Koszul complex $K_\bullet(f_1, \dots, f_r)$ into a cochain complex K^\bullet by setting $K^n = K_{-n}(f_1, \dots, f_r)$. Then K^\bullet is sitting in degrees $-r, \dots, 0$ and $H^i(K^\bullet \otimes_A M) = H_{-i}(K_\bullet(f_1, \dots, f_r) \otimes_A M)$. The statement of the theorem makes sense as the modules $H^i(K^\bullet \otimes_A M)$ are annihilated by f_1, \dots, f_r (More on Algebra, Lemma 28.6) hence have finite length. Define a filtration on the complex K^\bullet by setting

$$F^p(K^n \otimes_A M) = I^{\max(0, p+n)}(K^n \otimes_A M), \quad p \in \mathbf{Z}$$

Since $f_i I^p \subset I^{p+1}$ this is a filtration by subcomplexes. Thus we have a filtered complex and we obtain a spectral sequence, see Homology, Section 24. We have

$$E_0 = \bigoplus_{p,q} E_0^{p,q} = \bigoplus_{p,q} \text{gr}^p(K^{p+q} \otimes_A M) = \text{Gr}_I(K^\bullet \otimes_A M)$$

Since K^n is finite free we have

$$\text{Gr}_I(K^\bullet \otimes_A M) = \text{Gr}_I(K^\bullet) \otimes_{\text{Gr}_I(A)} \text{Gr}_I(M)$$

Note that $\mathrm{Gr}_I(K^\bullet)$ is the Koszul complex over $\mathrm{Gr}_I(A)$ on the elements $\bar{f}_1, \dots, \bar{f}_r \in I/I^2$. A simple calculation (omitted) shows that the differential d_0 on E_0 agrees with the differential coming from the Koszul complex. Since $\mathrm{Gr}_I(M)$ is a finite $\mathrm{Gr}_I(A)$ -module and since $\mathrm{Gr}_I(A)$ is Noetherian (as a quotient of $A/I[x_1, \dots, x_r]$ with $x_i \mapsto \bar{f}_i$), the cohomology module $E_1 = \bigoplus E_1^{p,q}$ is a finite $\mathrm{Gr}_I(A)$ -module. However, as above E_1 is annihilated by $\bar{f}_1, \dots, \bar{f}_r$. We conclude E_1 has finite length. In particular we find that $\mathrm{Gr}_F^p(K^\bullet \otimes M)$ is acyclic for $p \gg 0$.

Next, we check that the spectral sequence above converges using Homology, Lemma 24.10. The required equalities follow easily from the Artin-Rees lemma in the form stated in Algebra, Lemma 51.3. Thus we see that

$$\begin{aligned} \sum (-1)^i \mathrm{length}_A(H^i(K^\bullet \otimes_A M)) &= \sum (-1)^{p+q} \mathrm{length}_A(E_\infty^{p,q}) \\ &= \sum (-1)^{p+q} \mathrm{length}_A(E_1^{p,q}) \end{aligned}$$

because as we've seen above the length of E_1 is finite (of course this uses additivity of lengths). Pick t so large that $\mathrm{Gr}_F^p(K^\bullet \otimes M)$ is acyclic for $p \geq t$ (see above). Using additivity again we see that

$$\sum (-1)^{p+q} \mathrm{length}_A(E_1^{p,q}) = \sum_n \sum_{p \leq t} (-1)^n \mathrm{length}_A(\mathrm{gr}^p(K^n \otimes_A M))$$

This is equal to

$$\sum_{n=-r, \dots, 0} (-1)^n \binom{r}{|n|} \chi_{I,M}(t+n)$$

by our choice of filtration above and the definition of $\chi_{I,M}$ in Algebra, Section 59. The lemma follows from Lemma 15.4 and the definition of $e_I(M, r)$. \square

Remark 15.6 (Trivial generalization). Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let M be a finite A -module. Let $I \subset A$ be an ideal. The following are equivalent

- (1) $I' = I + \mathrm{Ann}(M)$ is an ideal of definition (Algebra, Definition 59.1),
- (2) the image \bar{I} of I in $\bar{A} = A/\mathrm{Ann}(M)$ is an ideal of definition,
- (3) $\mathrm{Supp}(M/IM) \subset \{\mathfrak{m}\}$,
- (4) $\dim(\mathrm{Supp}(M/IM)) \leq 0$, and
- (5) $\mathrm{length}_A(M/IM) < \infty$.

This follows from Algebra, Lemma 62.3 (details omitted). If this is the case we have $M/I^n M = M/(I')^n M$ for all n and $M/I^n M = M/\bar{I}^n M$ for all n if M is viewed as an \bar{A} -module. Thus we can define

$$\chi_{I,M}(n) = \mathrm{length}_A(M/I^n M) = \sum_{p=0, \dots, n-1} \mathrm{length}_A(I^p M/I^{p+1} M)$$

and we get

$$\chi_{I,M}(n) = \chi_{I',M}(n) = \chi_{\bar{I},M}(n)$$

for all n by the equalities above. All the results of Algebra, Section 59 and all the results in this section, have analogues in this setting. In particular we can define multiplicities $e_I(M, d)$ for $d \geq \dim(\mathrm{Supp}(M))$ and we have

$$\chi_{I,M}(n) \sim e_I(M, d) \frac{n^d}{d!} + \text{lower order terms}$$

as in the case where I is an ideal of definition.

16. Computing intersection multiplicities

In this section we discuss some cases where the intersection multiplicities can be computed by different means. Here is a first example.

Lemma 16.1. *Let X be a nonsingular variety and $W, V \subset X$ closed subvarieties which intersect properly. Let Z be an irreducible component of $V \cap W$ with generic point ξ . Assume that $\mathcal{O}_{W,\xi}$ and $\mathcal{O}_{V,\xi}$ are Cohen-Macaulay. Then*

$$e(X, V \cdot W, Z) = \text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{V \cap W, \xi})$$

where $V \cap W$ is the scheme theoretic intersection. In particular, if both V and W are Cohen-Macaulay, then $V \cdot W = [V \cap W]_{\dim(V) + \dim(W) - \dim(X)}$.

Proof. Set $A = \mathcal{O}_{X,\xi}$, $B = \mathcal{O}_{V,\xi}$, and $C = \mathcal{O}_{W,\xi}$. By Auslander-Buchsbaum (Algebra, Proposition 111.1) we can find a finite free resolution $F_\bullet \rightarrow B$ of length

$$\text{depth}(A) - \text{depth}(B) = \dim(A) - \dim(B) = \dim(C)$$

First equality as A and B are Cohen-Macaulay and the second as V and W intersect properly. Then $F_\bullet \otimes_A C$ is a complex of finite free modules representing $B \otimes_A^{\mathbf{L}} C$ hence has cohomology modules with support in $\{\mathfrak{m}_A\}$. By the Acyclicity lemma (Algebra, Lemma 102.8) which applies as C is Cohen-Macaulay we conclude that $F_\bullet \otimes_A C$ has nonzero cohomology only in degree 0. This finishes the proof. \square

Lemma 16.2. *Let A be a Noetherian local ring. Let $I = (f_1, \dots, f_r)$ be an ideal generated by a regular sequence. Let M be a finite A -module. Assume that $\dim(\text{Supp}(M/IM)) = 0$. Then*

$$e_I(M, r) = \sum (-1)^i \text{length}_A(\text{Tor}_i^A(A/I, M))$$

Here $e_I(M, r)$ is as in Remark 15.6.

Proof. Since f_1, \dots, f_r is a regular sequence the Koszul complex $K_\bullet(f_1, \dots, f_r)$ is a resolution of A/I over A , see More on Algebra, Lemma 30.7. Thus the right hand side is equal to

$$\sum (-1)^i \text{length}_A H_i(K_\bullet(f_1, \dots, f_r) \otimes_A M)$$

Now the result follows immediately from Theorem 15.5 if I is an ideal of definition. In general, we replace A by $\bar{A} = A/\text{Ann}(M)$ and f_1, \dots, f_r by $\bar{f}_1, \dots, \bar{f}_r$ which is allowed because

$$K_\bullet(f_1, \dots, f_r) \otimes_A M = K_\bullet(\bar{f}_1, \dots, \bar{f}_r) \otimes_{\bar{A}} M$$

Since $e_I(M, r) = e_{\bar{I}}(M, r)$ where $\bar{I} = (\bar{f}_1, \dots, \bar{f}_r) \subset \bar{A}$ is an ideal of definition the result follows from Theorem 15.5 in this case as well. \square

Lemma 16.3. *Let X be a nonsingular variety. Let $W, V \subset X$ be closed subvarieties which intersect properly. Let Z be an irreducible component of $V \cap W$ with generic point ξ . Suppose the ideal of V in $\mathcal{O}_{X,\xi}$ is cut out by a regular sequence $f_1, \dots, f_c \in \mathcal{O}_{X,\xi}$. Then $e(X, V \cdot W, Z)$ is equal to $c!$ times the leading coefficient in the Hilbert polynomial*

$$t \mapsto \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{W,\xi} / (f_1, \dots, f_c)^t, \quad t \gg 0.$$

In particular, this coefficient is > 0 .

Proof. The equality

$$e(X, V \cdot W, Z) = e_{(f_1, \dots, f_c)}(\mathcal{O}_{W, \xi}, c)$$

follows from the more general Lemma 16.2. To see that $e_{(f_1, \dots, f_c)}(\mathcal{O}_{W, \xi}, c)$ is > 0 or equivalently that $e_{(f_1, \dots, f_c)}(\mathcal{O}_{W, \xi}, c)$ is the leading coefficient of the Hilbert polynomial it suffices to show that the dimension of $\mathcal{O}_{W, \xi}$ is c , because the degree of the Hilbert polynomial is equal to the dimension by Algebra, Proposition 60.9. Say $\dim(V) = r$, $\dim(W) = s$, and $\dim(X) = n$. Then $\dim(Z) = r + s - n$ as the intersection is proper. Thus the transcendence degree of $\kappa(\xi)$ over \mathbf{C} is $r + s - n$, see Algebra, Lemma 116.1. We have $r + c = n$ because V is cut out by a regular sequence in a neighbourhood of ξ , see Divisors, Lemma 20.8 and then Lemma 13.2 applies (for example). Thus

$$\dim(\mathcal{O}_{W, \xi}) = s - (r + s - n) = s - ((n - c) + s - n) = c$$

the first equality by Algebra, Lemma 116.3. \square

Lemma 16.4. *In Lemma 16.3 assume that $c = 1$, i.e., V is an effective Cartier divisor. Then*

$$e(X, V \cdot W, Z) = \text{length}_{\mathcal{O}_{X, \xi}}(\mathcal{O}_{W, \xi} / f_1 \mathcal{O}_{W, \xi}).$$

Proof. In this case the image of f_1 in $\mathcal{O}_{W, \xi}$ is nonzero by properness of intersection, hence a nonzerodivisor. Moreover, $\mathcal{O}_{W, \xi}$ is a Noetherian local domain of dimension 1. Thus

$$\text{length}_{\mathcal{O}_{X, \xi}}(\mathcal{O}_{W, \xi} / f_1^t \mathcal{O}_{W, \xi}) = t \text{length}_{\mathcal{O}_{X, \xi}}(\mathcal{O}_{W, \xi} / f_1 \mathcal{O}_{W, \xi})$$

for all $t \geq 1$, see Algebra, Lemma 121.1. This proves the lemma. \square

Lemma 16.5. *In Lemma 16.3 assume that the local ring $\mathcal{O}_{W, \xi}$ is Cohen-Macaulay. Then we have*

$$e(X, V \cdot W, Z) = \text{length}_{\mathcal{O}_{X, \xi}}(\mathcal{O}_{W, \xi} / f_1 \mathcal{O}_{W, \xi} + \dots + f_c \mathcal{O}_{W, \xi}).$$

Proof. This follows immediately from Lemma 16.1. Alternatively, we can deduce it from Lemma 16.3. Namely, by Algebra, Lemma 104.2 we see that f_1, \dots, f_c is a regular sequence in $\mathcal{O}_{W, \xi}$. Then Algebra, Lemma 69.2 shows that f_1, \dots, f_c is a quasi-regular sequence. This easily implies the length of $\mathcal{O}_{W, \xi} / (f_1, \dots, f_c)^t$ is

$$\binom{c+t}{c} \text{length}_{\mathcal{O}_{X, \xi}}(\mathcal{O}_{W, \xi} / f_1 \mathcal{O}_{W, \xi} + \dots + f_c \mathcal{O}_{W, \xi}).$$

Looking at the leading coefficient we conclude. \square

17. Intersection product using Tor formula

Let X be a nonsingular variety. Let $\alpha = \sum n_i [W_i]$ be an r -cycle and $\beta = \sum_j m_j [V_j]$ be an s -cycle on X . Assume that α and β intersect properly, see Definition 13.5. In this case we define

$$\alpha \cdot \beta = \sum_{i,j} n_i m_j W_i \cdot V_j.$$

where $W_i \cdot V_j$ is as defined in Section 14. If $\beta = [V]$ where V is a closed subvariety of dimension s , then we sometimes write $\alpha \cdot \beta = \alpha \cdot V$.

Lemma 17.1. *Let X be a nonsingular variety. Let $a, b \in \mathbf{P}^1$ be distinct closed points. Let $k \geq 0$.*

- (1) If $W \subset X \times \mathbf{P}^1$ is a closed subvariety of dimension $k + 1$ which intersects $X \times a$ properly, then
 - (a) $[W_a]_k = W \cdot X \times a$ as cycles on $X \times \mathbf{P}^1$, and
 - (b) $[W_a]_k = pr_{X,*}(W \cdot X \times a)$ as cycles on X .
- (2) Let α be a $(k + 1)$ -cycle on $X \times \mathbf{P}^1$ which intersects $X \times a$ and $X \times b$ properly. Then $pr_{X,*}(\alpha \cdot X \times a - \alpha \cdot X \times b)$ is rationally equivalent to zero.
- (3) Conversely, any k -cycle which is rationally equivalent to 0 is of this form.

Proof. First we observe that $X \times a$ is an effective Cartier divisor in $X \times \mathbf{P}^1$ and that W_a is the scheme theoretic intersection of W with $X \times a$. Hence the equality in (1)(a) is immediate from the definitions and the calculation of intersection multiplicity in case of a Cartier divisor given in Lemma 16.4. Part (1)(b) holds because $W_a \rightarrow X \times \mathbf{P}^1 \rightarrow X$ maps isomorphically onto its image which is how we viewed W_a as a closed subscheme of X in Section 8. Parts (2) and (3) are formal consequences of part (1) and the definitions. \square

For transversal intersections of closed subschemes the intersection multiplicity is 1.

Lemma 17.2. *Let X be a nonsingular variety. Let $r, s \geq 0$ and let $Y, Z \subset X$ be closed subschemes with $\dim(Y) \leq r$ and $\dim(Z) \leq s$. Assume $[Y]_r = \sum n_i [Y_i]$ and $[Z]_s = \sum m_j [Z_j]$ intersect properly. Let T be an irreducible component of $Y_{i_0} \cap Z_{j_0}$ for some i_0 and j_0 and assume that the multiplicity (in the sense of Section 4) of T in the closed subscheme $Y \cap Z$ is 1. Then*

- (1) *the coefficient of T in $[Y]_r \cdot [Z]_s$ is 1,*
- (2) *Y and Z are nonsingular at the generic point of Z ,*
- (3) *$n_{i_0} = 1$, $m_{j_0} = 1$, and*
- (4) *T is not contained in Y_i or Z_j for $i \neq i_0$ and $j \neq j_0$.*

Proof. Set $n = \dim(X)$, $a = n - r$, $b = n - s$. Observe that $\dim(T) = r + s - n = n - a - b$ by the assumption that the intersections are transversal. Let $(A, \mathfrak{m}, \kappa) = (\mathcal{O}_{X, \xi}, \mathfrak{m}_\xi, \kappa(\xi))$ where $\xi \in T$ is the generic point. Then $\dim(A) = a + b$, see Varieties, Lemma 20.3. Let $I_0, I, J_0, J \subset A$ cut out the trace of Y_{i_0} , Y , Z_{j_0} , Z in $\text{Spec}(A)$. Then $\dim(A/I) = \dim(A/I_0) = b$ and $\dim(A/J) = \dim(A/J_0) = a$ by the same reference. Set $\bar{I} = I + \mathfrak{m}^2/\mathfrak{m}^2$. Then $I \subset I_0 \subset \mathfrak{m}$ and $J \subset J_0 \subset \mathfrak{m}$ and $I + J = \mathfrak{m}$. By Lemma 14.3 and its proof we see that $I_0 = (f_1, \dots, f_a)$ and $J_0 = (g_1, \dots, g_b)$ where f_1, \dots, f_a is a regular system of parameters for the regular local ring A . Since $I + J = \mathfrak{m}$, the map

$$I \oplus J \rightarrow \mathfrak{m}/\mathfrak{m}^2 = \kappa f_1 \oplus \dots \oplus \kappa f_a \oplus \kappa g_1 \oplus \dots \oplus \kappa g_b$$

is surjective. We conclude that we can find $f'_1, \dots, f'_a \in I$ and $g'_1, \dots, g'_b \in J$ whose residue classes in $\mathfrak{m}/\mathfrak{m}^2$ are equal to the residue classes of f_1, \dots, f_a and g_1, \dots, g_b . Then f'_1, \dots, g'_b is a regular system of parameters of A . By Algebra, Lemma 106.3 we find that $A/(f'_1, \dots, f'_a)$ is a regular local ring of dimension b . Thus any nontrivial quotient of $A/(f'_1, \dots, f'_a)$ has strictly smaller dimension (Algebra, Lemmas 106.2 and 60.13). Hence $I = (f'_1, \dots, f'_a) = I_0$. By symmetry $J = J_0$. This proves (2), (3), and (4). Finally, the coefficient of T in $[Y]_r \cdot [Z]_s$ is the coefficient of T in $Y_{i_0} \cdot Z_{j_0}$ which is 1 by Lemma 14.3. \square

18. Exterior product

Let X and Y be varieties. Let V , resp. W be a closed subvariety of X , resp. Y . The product $V \times W$ is a closed subvariety of $X \times Y$ (Lemma 13.1). For a k -cycle

$\alpha = \sum n_i [V_i]$ and a l -cycle $\beta = \sum m_j [V_j]$ on Y we define the *exterior product* of α and β to be the cycle $\alpha \times \beta = \sum n_i m_j [V_i \times W_j]$. Exterior product defines a \mathbf{Z} -linear map

$$Z_r(X) \otimes_{\mathbf{Z}} Z_s(Y) \longrightarrow Z_{r+s}(X \times Y)$$

Let us prove that exterior product factors through rational equivalence.

Lemma 18.1. *Let X and Y be varieties. Let $\alpha \in Z_r(X)$ and $\beta \in Z_s(Y)$. If $\alpha \sim_{rat} 0$ or $\beta \sim_{rat} 0$, then $\alpha \times \beta \sim_{rat} 0$.*

Proof. By linearity and symmetry in X and Y , it suffices to prove this when $\alpha = [V]$ for some subvariety $V \subset X$ of dimension s and $\beta = [W_a]_s - [W_b]_s$ for some closed subvariety $W \subset Y \times \mathbf{P}^1$ of dimension $s+1$ which intersects $Y \times a$ and $Y \times b$ properly. In this case the lemma follows if we can prove

$$[(V \times W)_a]_{r+s} = [V] \times [W_a]_s$$

and similarly with a replaced by b . Namely, then we see that $\alpha \times \beta = [(V \times W)_a]_{r+s} - [(V \times W)_b]_{r+s}$ as desired. To see the displayed equality we note the equality

$$V \times W_a = (V \times W)_a$$

of schemes. The projection $V \times W_a \rightarrow W_a$ induces a bijection of irreducible components (see for example Varieties, Lemma 8.4). Let $W' \subset W_a$ be an irreducible component with generic point ζ . Then $V \times W'$ is the corresponding irreducible component of $V \times W_a$ (see Lemma 13.1). Let ξ be the generic point of $V \times W'$. We have to show that

$$\text{length}_{\mathcal{O}_{Y,\zeta}}(\mathcal{O}_{W_a,\zeta}) = \text{length}_{\mathcal{O}_{X \times Y,\xi}}(\mathcal{O}_{V \times W_a,\xi})$$

In this formula we may replace $\mathcal{O}_{Y,\zeta}$ by $\mathcal{O}_{W_a,\zeta}$ and we may replace $\mathcal{O}_{X \times Y,\xi}$ by $\mathcal{O}_{V \times W_a,\xi}$ (see Algebra, Lemma 52.5). As $\mathcal{O}_{W_a,\zeta} \rightarrow \mathcal{O}_{V \times W_a,\xi}$ is flat, by Algebra, Lemma 52.13 it suffices to show that

$$\text{length}_{\mathcal{O}_{V \times W_a,\xi}}(\mathcal{O}_{V \times W_a,\xi} / \mathfrak{m}_{\zeta} \mathcal{O}_{V \times W_a,\xi}) = 1$$

This is true because the quotient on the right is the local ring $\mathcal{O}_{V \times W',\xi}$ of a variety at a generic point hence equal to $\kappa(\xi)$. \square

We conclude that exterior product defines a commutative diagram

$$\begin{array}{ccc} Z_r(X) \otimes_{\mathbf{Z}} Z_s(Y) & \longrightarrow & Z_{r+s}(X \times Y) \\ \downarrow & & \downarrow \\ \text{CH}_r(X) \otimes_{\mathbf{Z}} \text{CH}_s(Y) & \longrightarrow & \text{CH}_{r+s}(X \times Y) \end{array}$$

for any pair of varieties X and Y . For nonsingular varieties we can think of the exterior product as an intersection product of pullbacks.

Lemma 18.2. *Let X and Y be nonsingular varieties. Let $\alpha \in Z_r(X)$ and $\beta \in Z_s(Y)$. Then*

- (1) $pr_Y^*(\beta) = [X] \times \beta$ and $pr_X^*(\alpha) = \alpha \times [Y]$,
- (2) $\alpha \times [Y]$ and $[X] \times \beta$ intersect properly on $X \times Y$, and
- (3) we have $\alpha \times \beta = (\alpha \times [Y]) \cdot ([X] \times \beta) = pr_Y^*(\alpha) \cdot pr_X^*(\beta)$ in $Z_{r+s}(X \times Y)$.

Proof. By linearity we may assume $\alpha = [V]$ and $\beta = [W]$. Then (1) says that $\text{pr}_Y^{-1}(W) = X \times W$ and $\text{pr}_X^{-1}(V) = V \times Y$. This is clear. Part (2) holds because $X \times W \cap V \times Y = V \times W$ and $\dim(V \times W) = r + s$ by Lemma 13.1.

Proof of (3). Let ξ be the generic point of $V \times W$. Since the projections $X \times W \rightarrow W$ is smooth as a base change of $X \rightarrow \text{Spec}(\mathbf{C})$, we see that $X \times W$ is nonsingular at every point lying over the generic point of W , in particular at ξ . Similarly for $V \times Y$. Hence $\mathcal{O}_{X \times W, \xi}$ and $\mathcal{O}_{V \times Y, \xi}$ are Cohen-Macaulay local rings and Lemma 16.1 applies. Since $V \times Y \cap X \times W = V \times W$ scheme theoretically the proof is complete. \square

19. Reduction to the diagonal

Let X be a nonsingular variety. We will use Δ to denote either the diagonal morphism $\Delta : X \rightarrow X \times X$ or the image $\Delta \subset X \times X$. Reduction to the diagonal is the statement that intersection products on X can be reduced to intersection products of exterior products with the diagonal on $X \times X$.

Lemma 19.1. *Let X be a nonsingular variety.*

- (1) *If \mathcal{F} and \mathcal{G} are coherent \mathcal{O}_X -modules, then there are canonical isomorphisms*

$$\text{Tor}_i^{\mathcal{O}_{X \times X}}(\mathcal{O}_\Delta, \text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times X}} \text{pr}_2^* \mathcal{G}) = \Delta_* \text{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

- (2) *If K and M are in $D_{Q\text{Coh}}(\mathcal{O}_X)$, then there is a canonical isomorphism*

$$L\Delta^* \left(L\text{pr}_1^* K \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} L\text{pr}_2^* M \right) = K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M$$

in $D_{Q\text{Coh}}(\mathcal{O}_X)$ and a canonical isomorphism

$$\mathcal{O}_\Delta \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} L\text{pr}_1^* K \otimes_{\mathcal{O}_{X \times X}}^{\mathbf{L}} L\text{pr}_2^* M = \Delta_*(K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$$

in $D_{Q\text{Coh}}(X \times X)$.

Proof. Let us explain how to prove (1) in a more elementary way and part (2) using more general theory. As (2) implies (1) the reader can skip the proof of (1).

Proof of (1). Choose an affine open $\text{Spec}(A) \subset X$. Then A is a Noetherian \mathbf{C} -algebra and \mathcal{F}, \mathcal{G} correspond to finite A -modules M and N (Cohomology of Schemes, Lemma 9.1). By Derived Categories of Schemes, Lemma 3.9 we may compute Tor_i over \mathcal{O}_X by first computing the Tor 's of M and N over A , and then taking the associated \mathcal{O}_X -module. For the Tor_i over $\mathcal{O}_{X \times X}$ we compute the tor of A and $M \otimes_{\mathbf{C}} N$ over $A \otimes_{\mathbf{C}} A$ and then take the associated $\mathcal{O}_{X \times X}$ -module. Hence on this affine patch we have to prove that

$$\text{Tor}_i^{A \otimes_{\mathbf{C}} A}(A, M \otimes_{\mathbf{C}} N) = \text{Tor}_i^A(M, N)$$

To see this choose resolutions $F_\bullet \rightarrow M$ and $G_\bullet \rightarrow N$ by finite free A -modules (Algebra, Lemma 71.1). Note that $\text{Tot}(F_\bullet \otimes_{\mathbf{C}} G_\bullet)$ is a resolution of $M \otimes_{\mathbf{C}} N$ as it computes Tor groups over \mathbf{C} ! Of course the terms of $F_\bullet \otimes_{\mathbf{C}} G_\bullet$ are finite free $A \otimes_{\mathbf{C}} A$ -modules. Hence the left hand side of the displayed equation is the module

$$H_i(A \otimes_{A \otimes_{\mathbf{C}} A} \text{Tot}(F_\bullet \otimes_{\mathbf{C}} G_\bullet))$$

and the right hand side is the module

$$H_i(\text{Tot}(F_\bullet \otimes_A G_\bullet))$$

Since $A \otimes_{A \otimes_{\mathbb{C}} A} (F_p \otimes_{\mathbb{C}} G_q) = F_p \otimes_A G_q$ we see that these modules are equal. This defines an isomorphism over the affine open $\text{Spec}(A) \times \text{Spec}(A)$ (which is good enough for the application to equality of intersection numbers). We omit the proof that these isomorphisms glue.

Proof of (2). The second statement follows from the first by the projection formula as stated in Derived Categories of Schemes, Lemma 22.1. To see the first, represent K and M by K-flat complexes \mathcal{K}^\bullet and \mathcal{M}^\bullet . Since pullback and tensor product preserve K-flat complexes (Cohomology, Lemmas 26.5 and 26.8) we see that it suffices to show

$$\Delta^* \text{Tot}(\text{pr}_1^* \mathcal{K}^\bullet \otimes_{\mathcal{O}_{X \times X}} \text{pr}_2^* \mathcal{M}^\bullet) = \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)$$

Thus it suffices to see that there are canonical isomorphisms

$$\Delta^*(\text{pr}_1^* \mathcal{K} \otimes_{\mathcal{O}_{X \times X}} \text{pr}_2^* \mathcal{M}) \longrightarrow \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{M}$$

whenever \mathcal{K} and \mathcal{M} are \mathcal{O}_X -modules (not necessarily quasi-coherent or flat). We omit the details. \square

Lemma 19.2. *Let X be a nonsingular variety. Let α , resp. β be an r -cycle, resp. s -cycle on X . Assume α and β intersect properly. Then*

- (1) $\alpha \times \beta$ and $[\Delta]$ intersect properly
- (2) we have $\Delta_*(\alpha \cdot \beta) = [\Delta] \cdot \alpha \times \beta$ as cycles on $X \times X$,
- (3) if X is proper, then $\text{pr}_{1,*}([\Delta] \cdot \alpha \times \beta) = \alpha \cdot \beta$, where $\text{pr}_1 : X \times X \rightarrow X$ is the projection.

Proof. By linearity it suffices to prove this when $\alpha = [V]$ and $\beta = [W]$ for some closed subvarieties $V \subset X$ and $W \subset Y$ which intersect properly. Recall that $V \times W$ is a closed subvariety of dimension $r+s$. Observe that scheme theoretically we have $V \cap W = \Delta^{-1}(V \times W)$ as well as $\Delta(V \cap W) = \Delta \cap V \times W$. This proves (1).

Proof of (2). Let $Z \subset V \cap W$ be an irreducible component with generic point ξ . We have to show that the coefficient of Z in $\alpha \cdot \beta$ is the same as the coefficient of $\Delta(Z)$ in $[\Delta] \cdot \alpha \times \beta$. The first is given by the integer

$$\sum (-1)^i \text{length}_{\mathcal{O}_{X,\xi}} \text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)_\xi$$

and the second by the integer

$$\sum (-1)^i \text{length}_{\mathcal{O}_{X \times Y, \Delta(\xi)}} \text{Tor}_i^{\mathcal{O}_{X \times Y}}(\mathcal{O}_\Delta, \mathcal{O}_{V \times W})_{\Delta(\xi)}$$

However, by Lemma 19.1 we have

$$\text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)_\xi \cong \text{Tor}_i^{\mathcal{O}_{X \times Y}}(\mathcal{O}_\Delta, \mathcal{O}_{V \times W})_{\Delta(\xi)}$$

as $\mathcal{O}_{X \times X, \Delta(\xi)}$ -modules. Thus equality of lengths (by Algebra, Lemma 52.5 to be precise).

Part (2) implies (3) because $\text{pr}_{1,*} \circ \Delta_* = \text{id}$ by Lemma 6.2. \square

Proposition 19.3. *Let X be a nonsingular variety. Let $V \subset X$ and $W \subset Y$ be closed subvarieties which intersect properly. Let $Z \subset V \cap W$ be an irreducible component. Then $e(X, V \cdot W, Z) > 0$.*

Proof. By Lemma 19.2 we have

$$e(X, V \cdot W, Z) = e(X \times X, \Delta \cdot V \times W, \Delta(Z))$$

Since $\Delta : X \rightarrow X \times X$ is a regular immersion (see Lemma 13.3), we see that $e(X \times X, \Delta \cdot V \times W, \Delta(Z))$ is a positive integer by Lemma 16.3. \square

The following is a key lemma in the development of the theory as is done in this chapter. Essentially, this lemma tells us that the intersection numbers have a suitable additivity property.

Lemma 19.4. *Let X be a nonsingular variety. Let \mathcal{F} and \mathcal{G} be coherent sheaves on X with $\dim(\text{Supp}(\mathcal{F})) \leq r$, $\dim(\text{Supp}(\mathcal{G})) \leq s$, and $\dim(\text{Supp}(\mathcal{F}) \cap \text{Supp}(\mathcal{G})) \leq r + s - \dim X$. In this case $[\mathcal{F}]_r$ and $[\mathcal{G}]_s$ intersect properly and*

$$[\mathcal{F}]_r \cdot [\mathcal{G}]_s = \sum (-1)^p [\text{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]_{r+s-\dim(X)}.$$

Proof. The statement that $[\mathcal{F}]_r$ and $[\mathcal{G}]_s$ intersect properly is immediate. Since we are proving an equality of cycles we may work locally on X . (Observe that the formation of the intersection product of cycles, the formation of Tor-sheaves, and forming the cycle associated to a coherent sheaf, each commute with restriction to open subschemes.) Thus we may and do assume that X is affine.

Denote

$$RHS(\mathcal{F}, \mathcal{G}) = [\mathcal{F}]_r \cdot [\mathcal{G}]_s \quad \text{and} \quad LHS(\mathcal{F}, \mathcal{G}) = \sum (-1)^p [\text{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]_{r+s-\dim(X)}$$

Consider a short exact sequence

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

of coherent sheaves on X with $\text{Supp}(\mathcal{F}_i) \subset \text{Supp}(\mathcal{F})$, then both $LHS(\mathcal{F}_i, \mathcal{G})$ and $RHS(\mathcal{F}_i, \mathcal{G})$ are defined for $i = 1, 2, 3$ and we have

$$RHS(\mathcal{F}_2, \mathcal{G}) = RHS(\mathcal{F}_1, \mathcal{G}) + RHS(\mathcal{F}_3, \mathcal{G})$$

and similarly for LHS. Namely, the support condition guarantees that everything is defined, the short exact sequence and additivity of lengths gives

$$[\mathcal{F}_2]_r = [\mathcal{F}_1]_r + [\mathcal{F}_3]_r$$

(Chow Homology, Lemma 10.4) which implies additivity for RHS. The long exact sequence of Tors

$$\dots \rightarrow \text{Tor}_1(\mathcal{F}_3, \mathcal{G}) \rightarrow \text{Tor}_0(\mathcal{F}_1, \mathcal{G}) \rightarrow \text{Tor}_0(\mathcal{F}_2, \mathcal{G}) \rightarrow \text{Tor}_0(\mathcal{F}_3, \mathcal{G}) \rightarrow 0$$

and additivity of lengths as before implies additivity for LHS.

By Algebra, Lemma 62.1 and the fact that X is affine, we can find a filtration of \mathcal{F} whose graded pieces are structure sheaves of closed subvarieties of $\text{Supp}(\mathcal{F})$. The additivity shown in the previous paragraph, implies that it suffices to prove $LHS = RHS$ with \mathcal{F} replaced by \mathcal{O}_V where $V \subset \text{Supp}(\mathcal{F})$. By symmetry we can do the same for \mathcal{G} . This reduces us to proving that

$$LHS(\mathcal{O}_V, \mathcal{O}_W) = RHS(\mathcal{O}_V, \mathcal{O}_W)$$

where $W \subset \text{Supp}(\mathcal{G})$ is a closed subvariety. If $\dim(V) = r$ and $\dim(W) = s$, then this equality is the **definition** of $V \cdot W$. On the other hand, if $\dim(V) <$

r or $\dim(W) < s$, i.e., $[V]_r = 0$ or $[W]_s = 0$, then we have to prove that $RHS(\mathcal{O}_V, \mathcal{O}_W) = 0$ ⁵.

Let $Z \subset V \cap W$ be an irreducible component of dimension $r+s-\dim(X)$. This is the maximal dimension of a component and it suffices to show that the coefficient of Z in RHS is zero. Let $\xi \in Z$ be the generic point. Write $A = \mathcal{O}_{X,\xi}$, $B = \mathcal{O}_{X \times X, \Delta(\xi)}$, and $C = \mathcal{O}_{V \times W, \Delta(\xi)}$. By Lemma 19.1 we have

$$\text{coeff of } Z \text{ in } RHS(\mathcal{O}_V, \mathcal{O}_W) = \sum (-1)^i \text{length}_B \text{Tor}_i^B(A, C)$$

Since $\dim(V) < r$ or $\dim(W) < s$ we have $\dim(V \times W) < r + s$ which implies $\dim(C) < \dim(X)$ (small detail omitted). Moreover, the kernel I of $B \rightarrow A$ is generated by a regular sequence of length $\dim(X)$ (Lemma 13.3). Hence vanishing by Lemma 16.2 because the Hilbert function of C with respect to I has degree $\dim(C) < n$ by Algebra, Proposition 60.9. \square

Remark 19.5. Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring. Let M and N be nonzero finite A -modules such that $M \otimes_A N$ is supported in $\{\mathfrak{m}\}$. Then

$$\chi(M, N) = \sum (-1)^i \text{length}_A \text{Tor}_i^A(M, N)$$

is finite. Let $r = \dim(\text{Supp}(M))$ and $s = \dim(\text{Supp}(N))$. In [Ser65] it is shown that $r + s \leq \dim(A)$ and the following conjectures are made:

- (1) if $r + s < \dim(A)$, then $\chi(M, N) = 0$, and
- (2) if $r + s = \dim(A)$, then $\chi(M, N) > 0$.

The arguments that prove Lemma 19.4 and Proposition 19.3 can be leveraged (as is done in Serre's text) to show that (1) and (2) are true if A contains a field. Currently, conjecture (1) is known in general and it is known that $\chi(M, N) \geq 0$ in general (Gabber). Positivity is, as far as we know, still an open problem.

20. Associativity of intersections

It is clear that proper intersections as defined above are commutative. Using the key Lemma 19.4 we can prove that (proper) intersection products are associative.

Lemma 20.1. *Let X be a nonsingular variety. Let U, V, W be closed subvarieties. Assume that U, V, W intersect properly pairwise and that $\dim(U \cap V \cap W) \leq \dim(U) + \dim(V) + \dim(W) - 2\dim(X)$. Then*

$$U \cdot (V \cdot W) = (U \cdot V) \cdot W$$

as cycles on X .

Proof. We are going to use Lemma 19.4 without further mention. This implies that

$$\begin{aligned} V \cdot W &= \sum (-1)^i [\text{Tor}_i(\mathcal{O}_V, \mathcal{O}_W)]_{b+c-n} \\ U \cdot (V \cdot W) &= \sum (-1)^{i+j} [\text{Tor}_j(\mathcal{O}_U, \text{Tor}_i(\mathcal{O}_V, \mathcal{O}_W))]_{a+b+c-2n} \\ U \cdot V &= \sum (-1)^i [\text{Tor}_i(\mathcal{O}_U, \mathcal{O}_V)]_{a+b-n} \\ (U \cdot V) \cdot W &= \sum (-1)^{i+j} [\text{Tor}_j(\text{Tor}_i(\mathcal{O}_U, \mathcal{O}_V), \mathcal{O}_W)]_{a+b+c-2n} \end{aligned}$$

⁵The reader can see that this is not a triviality by taking $r = s = 1$ and X a nonsingular surface and $V = W$ a closed point x of X . In this case there are 3 nonzero Tors of lengths 1, 2, 1 at x .

where $\dim(U) = a$, $\dim(V) = b$, $\dim(W) = c$, $\dim(X) = n$. The assumptions in the lemma guarantee that the coherent sheaves in the formulae above satisfy the required bounds on dimensions of supports in order to make sense of these. Now consider the object

$$K = \mathcal{O}_U \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_V \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_W$$

of the derived category $D_{\text{Coh}}(\mathcal{O}_X)$. We claim that the expressions obtained above for $U \cdot (V \cdot W)$ and $(U \cdot V) \cdot W$ are equal to

$$\sum (-1)^k [H^k(K)]_{a+b+c-2n}$$

This will prove the lemma. By symmetry it suffices to prove one of these equalities. To do this we represent \mathcal{O}_U and $\mathcal{O}_V \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_W$ by K-flat complexes M^\bullet and L^\bullet and use the spectral sequence associated to the double complex $M^\bullet \otimes_{\mathcal{O}_X} L^\bullet$ in Homology, Section 25. This is a spectral sequence with E_2 page

$$E_2^{p,q} = \text{Tor}_{-p}(\mathcal{O}_U, \text{Tor}_{-q}(\mathcal{O}_V, \mathcal{O}_W))$$

converging to $H^{p+q}(K)$ (details omitted; compare with More on Algebra, Example 62.4). Since lengths are additive in short exact sequences we see that the result is true. \square

21. Flat pullback and intersection products

Short discussion of the interaction between intersections and flat pullback.

Lemma 21.1. *Let $f : X \rightarrow Y$ be a flat morphism of nonsingular varieties. Set $e = \dim(X) - \dim(Y)$. Let \mathcal{F} and \mathcal{G} be coherent sheaves on Y with $\dim(\text{Supp}(\mathcal{F})) \leq r$, $\dim(\text{Supp}(\mathcal{G})) \leq s$, and $\dim(\text{Supp}(\mathcal{F}) \cap \text{Supp}(\mathcal{G})) \leq r + s - \dim(Y)$. In this case the cycles $[f^*\mathcal{F}]_{r+e}$ and $[f^*\mathcal{G}]_{s+e}$ intersect properly and*

$$f^*([\mathcal{F}]_r \cdot [\mathcal{G}]_s) = [f^*\mathcal{F}]_{r+e} \cdot [f^*\mathcal{G}]_{s+e}$$

Proof. The statement that $[f^*\mathcal{F}]_{r+e}$ and $[f^*\mathcal{G}]_{s+e}$ intersect properly is immediate from the assumption that f has relative dimension e . By Lemmas 19.4 and 7.1 it suffices to show that

$$f^* \text{Tor}_i^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) = \text{Tor}_i^{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G})$$

as \mathcal{O}_X -modules. This follows from Cohomology, Lemma 27.3 and the fact that f^* is exact, so $Lf^*\mathcal{F} = f^*\mathcal{F}$ and similarly for \mathcal{G} . \square

Lemma 21.2. *Let $f : X \rightarrow Y$ be a flat morphism of nonsingular varieties. Let α be a r -cycle on Y and β an s -cycle on Y . Assume that α and β intersect properly. Then $f^*\alpha$ and $f^*\beta$ intersect properly and $f^*(\alpha \cdot \beta) = f^*\alpha \cdot f^*\beta$.*

Proof. By linearity we may assume that $\alpha = [V]$ and $\beta = [W]$ for some closed subvarieties $V, W \subset Y$ of dimension r, s . Say f has relative dimension e . Then the lemma is a special case of Lemma 21.1 because $[V] = [\mathcal{O}_V]_r$, $[W] = [\mathcal{O}_W]_s$, $f^*[V] = [f^{-1}(V)]_{r+e} = [f^*\mathcal{O}_V]_{r+e}$, and $f^*[W] = [f^{-1}(W)]_{s+e} = [f^*\mathcal{O}_W]_{s+e}$. \square

22. Projection formula for flat proper morphisms

Short discussion of the projection formula for flat proper morphisms.

Lemma 22.1. *Let $f : X \rightarrow Y$ be a flat proper morphism of nonsingular varieties. Set $e = \dim(X) - \dim(Y)$. Let α be an r -cycle on X and let β be a s -cycle on Y . Assume that α and $f^*(\beta)$ intersect properly. Then $f_*(\alpha)$ and β intersect properly and*

$$f_*(\alpha) \cdot \beta = f_*(\alpha \cdot f^*\beta)$$

Proof. By linearity we reduce to the case where $\alpha = [V]$ and $\beta = [W]$ for some closed subvariety $V \subset X$ and $W \subset Y$ of dimension r and s . Then $f^{-1}(W)$ has pure dimension $s + e$. We assume the cycles $[V]$ and $f^*[W]$ intersect properly. We will use without further mention the fact that $V \cap f^{-1}(W) \rightarrow f(V) \cap W$ is surjective.

Let a be the dimension of the generic fibre of $V \rightarrow f(V)$. If $a > 0$, then $f_*[V] = 0$. In particular $f_*\alpha$ and β intersect properly. To finish this case we have to show that $f_*([V] \cdot f^*[W]) = 0$. However, since every fibre of $V \rightarrow f(V)$ has dimension $\geq a$ (see Morphisms, Lemma 28.4) we conclude that every irreducible component Z of $V \cap f^{-1}(W)$ has fibres of dimension $\geq a$ over $f(Z)$. This certainly implies what we want.

Assume that $V \rightarrow f(V)$ is generically finite. Let $Z \subset f(V) \cap W$ be an irreducible component. Let $Z_i \subset V \cap f^{-1}(W)$, $i = 1, \dots, t$ be the irreducible components of $V \cap f^{-1}(W)$ dominating Z . By assumption each Z_i has dimension $r + s + e - \dim(X) = r + s - \dim(Y)$. Hence $\dim(Z) \leq r + s - \dim(Y)$. Thus we see that $f(V)$ and W intersect properly, $\dim(Z) = r + s - \dim(Y)$, and each $Z_i \rightarrow Z$ is generically finite. In particular, it follows that $V \rightarrow f(V)$ has finite fibre over the generic point ξ of Z . Thus $V \rightarrow Y$ is finite in an open neighbourhood of ξ , see Cohomology of Schemes, Lemma 21.2. Using a very general projection formula for derived tensor products, we get

$$Rf_*(\mathcal{O}_V \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*\mathcal{O}_W) = Rf_*\mathcal{O}_V \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{O}_W$$

see Derived Categories of Schemes, Lemma 22.1. Since f is flat, we see that $Lf^*\mathcal{O}_W = f^*\mathcal{O}_W$. Since $f|_V$ is finite in an open neighbourhood of ξ we have

$$(Rf_*\mathcal{F})_\xi = (f_*\mathcal{F})_\xi$$

for any coherent sheaf on X whose support is contained in V (see Cohomology of Schemes, Lemma 20.8). Thus we conclude that

$$(22.1.1) \quad \left(f_* \operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, f^*\mathcal{O}_W) \right)_\xi = \left(\operatorname{Tor}_i^{\mathcal{O}_Y}(f_*\mathcal{O}_V, \mathcal{O}_W) \right)_\xi$$

for all i . Since $f^*[W] = [f^*\mathcal{O}_W]_{s+e}$ by Lemma 7.1 we have

$$[V] \cdot f^*[W] = \sum (-1)^i [\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, f^*\mathcal{O}_W)]_{r+s-\dim(Y)}$$

by Lemma 19.4. Applying Lemma 6.1 we find

$$f_*([V] \cdot f^*[W]) = \sum (-1)^i [f_* \operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, f^*\mathcal{O}_W)]_{r+s-\dim(Y)}$$

Since $f_*[V] = [f_*\mathcal{O}_V]_r$ by Lemma 6.1 we have

$$[f_*V] \cdot [W] = \sum (-1)^i [\operatorname{Tor}_i^{\mathcal{O}_X}(f_*\mathcal{O}_V, \mathcal{O}_W)]_{r+s-\dim(Y)}$$

again by Lemma 19.4. Comparing the formula for $f_*([V] \cdot f^*[W])$ with the formula for $f_*[V] \cdot [W]$ and looking at the coefficient of Z by taking lengths of stalks at ξ , we see that (22.1.1) finishes the proof. \square

Lemma 22.2. *Let $X \rightarrow P$ be a closed immersion of nonsingular varieties. Let $C' \subset P \times \mathbf{P}^1$ be a closed subvariety of dimension $r + 1$. Assume*

- (1) *the fibre $C = C'_0$ has dimension r , i.e., $C' \rightarrow \mathbf{P}^1$ is dominant,*
- (2) *C' intersects $X \times \mathbf{P}^1$ properly,*
- (3) *$[C]_r$ intersects X properly.*

Then setting $\alpha = [C]_r \cdot X$ viewed as cycle on X and $\beta = C' \cdot X \times \mathbf{P}^1$ viewed as cycle on $X \times \mathbf{P}^1$, we have

$$\alpha = pr_{X,*}(\beta \cdot X \times 0)$$

as cycles on X where $pr_X : X \times \mathbf{P}^1 \rightarrow X$ is the projection.

Proof. Let $pr : P \times \mathbf{P}^1 \rightarrow P$ be the projection. Since we are proving an equality of cycles it suffices to think of α , resp. β as a cycle on P , resp. $P \times \mathbf{P}^1$ and prove the result for pushing forward by pr . Because $pr^*X = X \times \mathbf{P}^1$ and pr defines an isomorphism of C'_0 onto C the projection formula (Lemma 22.1) gives

$$pr_*([C'_0]_r \cdot X \times \mathbf{P}^1) = [C]_r \cdot X = \alpha$$

On the other hand, we have $[C'_0]_r = C' \cdot P \times 0$ as cycles on $P \times \mathbf{P}^1$ by Lemma 17.1. Hence

$$[C'_0]_r \cdot X \times \mathbf{P}^1 = (C' \cdot P \times 0) \cdot X \times \mathbf{P}^1 = (C' \cdot X \times \mathbf{P}^1) \cdot P \times 0$$

by associativity (Lemma 20.1) and commutativity of the intersection product. It remains to show that the intersection product of $C' \cdot X \times \mathbf{P}^1$ with $P \times 0$ on $P \times \mathbf{P}^1$ is equal (as a cycle) to the intersection product of β with $X \times 0$ on $X \times \mathbf{P}^1$. Write $C' \cdot X \times \mathbf{P}^1 = \sum n_k [E_k]$ and hence $\beta = \sum n_k [E_k]$ for some subvarieties $E_k \subset X \times \mathbf{P}^1 \subset P \times \mathbf{P}^1$. Then both intersections are equal to $\sum m_k [E_{k,0}]$ by Lemma 17.1 applied twice. This finishes the proof. \square

23. Projections

Recall that we are working over a fixed algebraically closed ground field \mathbf{C} . If V is a finite dimensional vector space over \mathbf{C} then we set

$$\mathbf{P}(V) = \text{Proj}(\text{Sym}(V))$$

where $\text{Sym}(V)$ is the symmetric algebra on V over \mathbf{C} . See Constructions, Example 21.2. The normalization is chosen such that $V = \Gamma(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1))$. Of course we have $\mathbf{P}(V) \cong \mathbf{P}_{\mathbf{C}}^n$ if $\dim(V) = n + 1$. We note that $\mathbf{P}(V)$ is a nonsingular projective variety.

Let $p \in \mathbf{P}(V)$ be a closed point. The point p corresponds to a surjection $V \rightarrow L_p$ of vector spaces where $\dim(L_p) = 1$, see Constructions, Lemma 12.3. Let us denote $W_p = \text{Ker}(V \rightarrow L_p)$. *Projection from p* is the morphism

$$r_p : \mathbf{P}(V) \setminus \{p\} \longrightarrow \mathbf{P}(W_p)$$

of Constructions, Lemma 11.1. Here is a lemma to warm up.

Lemma 23.1. *Let V be a vector space of dimension $n + 1$. Let $X \subset \mathbf{P}(V)$ be a closed subscheme. If $X \neq \mathbf{P}(V)$, then there is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ the restriction of the projection r_p defines a finite morphism $r_p|_X : X \rightarrow \mathbf{P}(W_p)$.*

Proof. We claim the lemma holds with $U = \mathbf{P}(V) \setminus X$. For a closed point p of U we indeed obtain a morphism $r_p|_X : X \rightarrow \mathbf{P}(W_p)$. This morphism is proper because X is a proper scheme (Morphisms, Lemmas 43.5 and 41.7). On the other hand, the fibres of r_p are affine lines as can be seen by a direct calculation. Hence the fibres of $r_p|_X$ are proper and affine, whence finite (Morphisms, Lemma 44.11). Finally, a proper morphism with finite fibres is finite (Cohomology of Schemes, Lemma 21.1). \square

Lemma 23.2. *Let V be a vector space of dimension $n + 1$. Let $X \subset \mathbf{P}(V)$ be a closed subvariety. Let $x \in X$ be a nonsingular point.*

- (1) *If $\dim(X) < n - 1$, then there is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ the morphism $r_p|_X : X \rightarrow \mathbf{P}(W_p)$ is an isomorphism over an open neighbourhood of $r_p(x)$.*
- (2) *If $\dim(X) = n - 1$, then there is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ the morphism $r_p|_X : X \rightarrow \mathbf{P}(W_p)$ is étale at x .*

Proof. Proof of (1). Note that if $x, y \in X$ have the same image under r_p then p is on the line \overline{xy} . Consider the finite type scheme

$$T = \{(y, p) \mid y \in X \setminus \{x\}, p \in \mathbf{P}(V), p \in \overline{xy}\}$$

and the morphisms $T \rightarrow X$ and $T \rightarrow \mathbf{P}(V)$ given by $(y, p) \mapsto y$ and $(y, p) \mapsto p$. Since each fibre of $T \rightarrow X$ is a line, we see that the dimension of T is $\dim(X) + 1 < \dim(\mathbf{P}(V))$. Hence $T \rightarrow \mathbf{P}(V)$ is not surjective. On the other hand, consider the finite type scheme

$$T' = \{p \mid p \in \mathbf{P}(V) \setminus \{x\}, \overline{xp} \text{ tangent to } X \text{ at } x\}$$

Then the dimension of T' is $\dim(X) < \dim(\mathbf{P}(V))$. Thus the morphism $T' \rightarrow \mathbf{P}(V)$ is not surjective either. Let $U \subset \mathbf{P}(V) \setminus X$ be nonempty open and disjoint from these images; such a U exists because the images of T and T' in $\mathbf{P}(V)$ are constructible by Morphisms, Lemma 22.2. Then for $p \in U$ closed the projection $r_p|_X : X \rightarrow \mathbf{P}(W_p)$ is injective on the tangent space at x and $r_p^{-1}(\{r_p(x)\}) = \{x\}$. This means that r_p is unramified at x (Varieties, Lemma 16.8), finite by Lemma 23.1, and $r_p^{-1}(\{r_p(x)\}) = \{x\}$ thus Étale Morphisms, Lemma 7.3 applies and there is an open neighbourhood R of $r_p(x)$ in $\mathbf{P}(W_p)$ such that $(r_p|_X)^{-1}(R) \rightarrow R$ is a closed immersion which proves (1).

Proof of (2). In this case we still conclude that the morphism $T' \rightarrow \mathbf{P}(V)$ is not surjective. Arguing as above we conclude that for $U \subset \mathbf{P}(V)$ avoiding X and the image of T' , the projection $r_p|_X : X \rightarrow \mathbf{P}(W_p)$ is étale at x and finite. \square

Lemma 23.3. *Let V be a vector space of dimension $n + 1$. Let $Y, Z \subset \mathbf{P}(V)$ be closed subvarieties. There is a nonempty Zariski open $U \subset \mathbf{P}(V)$ such that for all closed points $p \in U$ we have*

$$Y \cap r_p^{-1}(r_p(Z)) = (Y \cap Z) \cup E$$

with $E \subset Y$ closed and $\dim(E) \leq \dim(Y) + \dim(Z) + 1 - n$.

Proof. Set $Y' = Y \setminus Y \cap Z$. Let $y \in Y'$, $z \in Z$ be closed points with $r_p(y) = r_p(z)$. Then p is on the line \overline{yz} passing through y and z . Consider the finite type scheme

$$T = \{(y, z, p) \mid y \in Y', z \in Z, p \in \overline{yz}\}$$

and the morphism $T \rightarrow \mathbf{P}(V)$ given by $(y, z, p) \mapsto p$. Observe that T is irreducible and that $\dim(T) = \dim(Y) + \dim(Z) + 1$. Hence the general fibre of $T \rightarrow \mathbf{P}(V)$ has dimension at most $\dim(Y) + \dim(Z) + 1 - n$, more precisely, there exists a nonempty open $U \subset \mathbf{P}(V) \setminus (Y \cup Z)$ over which the fibre has dimension at most $\dim(Y) + \dim(Z) + 1 - n$ (Varieties, Lemma 20.4). Let $p \in U$ be a closed point and let $F \subset T$ be the fibre of $T \rightarrow \mathbf{P}(V)$ over p . Then

$$(Y \cap r_p^{-1}(r_p(Z))) \setminus (Y \cap Z)$$

is the image of $F \rightarrow Y$, $(y, z, p) \mapsto y$. Again by Varieties, Lemma 20.4 the closure of the image of $F \rightarrow Y$ has dimension at most $\dim(Y) + \dim(Z) + 1 - n$. \square

Lemma 23.4. *Let V be a vector space. Let $B \subset \mathbf{P}(V)$ be a closed subvariety of codimension ≥ 2 . Let $p \in \mathbf{P}(V)$ be a closed point, $p \notin B$. Then there exists a line $\ell \subset \mathbf{P}(V)$ with $\ell \cap B = \emptyset$. Moreover, these lines sweep out an open subset of $\mathbf{P}(V)$.*

Proof. Consider the image of B under the projection $r_p : \mathbf{P}(V) \rightarrow \mathbf{P}(W_p)$. Since $\dim(W_p) = \dim(V) - 1$, we see that $r_p(B)$ has codimension ≥ 1 in $\mathbf{P}(W_p)$. For any $q \in \mathbf{P}(V)$ with $r_p(q) \notin r_p(B)$ we see that the line $\ell = \overline{pq}$ connecting p and q works. \square

Lemma 23.5. *Let V be a vector space. Let $G = \mathrm{PGL}(V)$. Then $G \times \mathbf{P}(V) \rightarrow \mathbf{P}(V)$ is doubly transitive.*

Proof. Omitted. Hint: This follows from the fact that $\mathrm{GL}(V)$ acts doubly transitive on pairs of linearly independent vectors. \square

Lemma 23.6. *Let k be a field. Let $n \geq 1$ be an integer and let $x_{ij}, 1 \leq i, j \leq n$ be variables. Then*

$$\det \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & \cdots & x_{nn} \end{pmatrix}$$

is an irreducible element of the polynomial ring $k[x_{ij}]$.

Proof. Let V be an n dimensional vector space. Translating into geometry the lemma signifies that the variety C of non-invertible linear maps $V \rightarrow V$ is irreducible. Let W be a vector space of dimension $n - 1$. By elementary linear algebra, the morphism

$$\mathrm{Hom}(W, V) \times \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(V, V), \quad (\psi, \varphi) \longmapsto \psi \circ \varphi$$

has image C . Since the source is irreducible, so is the image. \square

Let V be a vector space of dimension $n + 1$. Set $E = \mathrm{End}(V)$. Let $E^\vee = \mathrm{Hom}(E, \mathbf{C})$ be the dual vector space. Write $\mathbf{P} = \mathbf{P}(E^\vee)$. There is a canonical linear map

$$V \longrightarrow V \otimes_{\mathbf{C}} E^\vee = \mathrm{Hom}(E, V)$$

sending $v \in V$ to the map $g \mapsto g(v)$ in $\text{Hom}(E, V)$. Recall that we have a canonical map $E^\vee \rightarrow \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1))$ which is an isomorphism. Hence we obtain a canonical map

$$\psi : V \otimes \mathcal{O}_{\mathbf{P}} \rightarrow V \otimes \mathcal{O}_{\mathbf{P}}(1)$$

of sheaves of modules on \mathbf{P} which on global sections recovers the given map. Recall that a projective bundle $\mathbf{P}(\mathcal{E})$ is defined as the relative Proj of the symmetric algebra on \mathcal{E} , see Constructions, Definition 21.1. We are going to study the rational map between $\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}(1))$ and $\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}})$ associated to ψ . By Constructions, Lemma 16.10 we have a canonical isomorphism

$$\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}) = \mathbf{P} \times \mathbf{P}(V)$$

By Constructions, Lemma 20.1 we see that

$$\mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}(1)) = \mathbf{P}(V \otimes \mathcal{O}_{\mathbf{P}}) = \mathbf{P} \times \mathbf{P}(V)$$

Combining this with Constructions, Lemma 18.1 we obtain

$$(23.6.1) \quad \mathbf{P} \times \mathbf{P}(V) \supset U(\psi) \xrightarrow{r_\psi} \mathbf{P} \times \mathbf{P}(V)$$

To understand this better we work out what happens on fibres over \mathbf{P} . Let $g \in E$ be nonzero. This defines a nonzero map $E^\vee \rightarrow \mathbf{C}$, hence a point $[g] \in \mathbf{P}$. On the other hand, g defines a \mathbf{C} -linear map $g : V \rightarrow V$. Hence we obtain, by Constructions, Lemma 11.1 a map

$$\mathbf{P}(V) \supset U(g) \xrightarrow{r_g} \mathbf{P}(V)$$

What we will use below is that $U(g)$ is the fibre $U(\psi)_{[g]}$ and that r_g is the fibre of r_ψ over the point $[g]$. Another observation we will use is that the complement of $U(g)$ in $\mathbf{P}(V)$ is the image of the closed immersion

$$\mathbf{P}(\text{Coker}(g)) \longrightarrow \mathbf{P}(V)$$

and the image of r_g is the image of the closed immersion

$$\mathbf{P}(\text{Im}(g)) \longrightarrow \mathbf{P}(V)$$

Lemma 23.7. *With notation as above. Let X, Y be closed subvarieties of $\mathbf{P}(V)$ which intersect properly such that $X \neq \mathbf{P}(V)$ and $X \cap Y \neq \emptyset$. For a general line $\ell \subset \mathbf{P}$ with $[\text{id}_V] \in \ell$ we have*

- (1) $X \subset U_g$ for all $[g] \in \ell$,
- (2) $g(X)$ intersects Y properly for all $[g] \in \ell$.

Proof. Let $B \subset \mathbf{P}$ be the set of “bad” points, i.e., those points $[g]$ that violate either (1) or (2). Note that $[\text{id}_V] \notin B$ by assumption. Moreover, B is closed. Hence it suffices to prove that $\dim(B) \leq \dim(\mathbf{P}) - 2$ (Lemma 23.4).

First, consider the open $G = \text{PGL}(V) \subset \mathbf{P}$ consisting of points $[g]$ such that $g : V \rightarrow V$ is invertible. Since G acts doubly transitively on $\mathbf{P}(V)$ (Lemma 23.5) we see that

$$T = \{(x, y, [g]) \mid x \in X, y \in Y, [g] \in G, r_g(x) = y\}$$

is a locally trivial fibration over $X \times Y$ with fibre equal to the stabilizer of a point in G . Hence T is a variety. Observe that the fibre of $T \rightarrow G$ over $[g]$ is $r_g(X) \cap Y$. The morphism $T \rightarrow G$ is surjective, because any translate of X intersects Y (note that by the assumption that X and Y intersect properly and that $X \cap Y \neq \emptyset$ we see that $\dim(X) + \dim(Y) \geq \dim(\mathbf{P}(V))$ and then Varieties, Lemma 34.3 implies all translates of X intersect Y). Since the dimensions of fibres of a dominant morphism

of varieties do not jump in codimension 1 (Varieties, Lemma 20.4) we conclude that $B \cap G$ has codimension ≥ 2 .

Next we look at the complement $Z = \mathbf{P} \setminus G$. This is an irreducible variety because the determinant is an irreducible polynomial (Lemma 23.6). Thus it suffices to prove that B does not contain the generic point of Z . For a general point $[g] \in Z$ the cokernel $V \rightarrow \text{Coker}(g)$ has dimension 1, hence $U(g)$ is the complement of a point. Since $X \neq \mathbf{P}(V)$ we see that for a general $[g] \in Z$ we have $X \subset U(g)$. Moreover, the morphism $r_g|_X : X \rightarrow r_g(X)$ is finite, hence $\dim(r_g(X)) = \dim(X)$. On the other hand, for such a g the image of r_g is the closed subspace $H = \mathbf{P}(\text{Im}(g)) \subset \mathbf{P}(V)$ which has codimension 1. For general point of Z we see that $H \cap Y$ has dimension 1 less than Y (compare with Varieties, Lemma 35.3). Thus we see that we have to show that $r_g(X)$ and $H \cap Y$ intersect properly in H . For a fixed choice of H , we can by postcomposing g by an automorphism, move $r_g(X)$ by an arbitrary automorphism of $H = \mathbf{P}(\text{Im}(g))$. Thus we can argue as above to conclude that the intersection of $H \cap Y$ with $r_g(X)$ is proper for general g with given $H = \mathbf{P}(\text{Im}(g))$. Some details omitted. \square

24. Moving Lemma

The moving lemma states that given an r -cycle α and an s -cycle β there exists $\alpha', \alpha' \sim_{\text{rat}} \alpha$ such that α' and β intersect properly (Lemma 24.3). See [Sam56], [Che58a], [Che58b]. The key to this is Lemma 24.1; the reader may find this lemma in the form stated in [Ful98, Example 11.4.1] and find a proof in [Rob72].

Lemma 24.1. *Let $X \subset \mathbf{P}^N$ be a nonsingular closed subvariety. Let $n = \dim(X)$ and $0 \leq d, d' < n$. Let $Z \subset X$ be a closed subvariety of dimension d and $T_i \subset X$, $i \in I$ be a finite collection of closed subvarieties of dimension d' . Then there exists a subvariety $C \subset \mathbf{P}^N$ such that C intersects X properly and such that*

$$C \cdot X = Z + \sum_{j \in J} m_j Z_j$$

where $Z_j \subset X$ are irreducible of dimension d , distinct from Z , and

$$\dim(Z_j \cap T_i) \leq \dim(Z \cap T_i)$$

with strict inequality if Z does not intersect T_i properly in X .

Proof. Write $\mathbf{P}^N = \mathbf{P}(V_N)$ so $\dim(V_N) = N + 1$ and set $X_N = X$. We are going to choose a sequence of projections from points

$$\begin{aligned} r_N : \mathbf{P}(V_N) \setminus \{p_N\} &\rightarrow \mathbf{P}(V_{N-1}), \\ r_{N-1} : \mathbf{P}(V_{N-1}) \setminus \{p_{N-1}\} &\rightarrow \mathbf{P}(V_{N-2}), \\ &\dots, \\ r_{n+1} : \mathbf{P}(V_{n+1}) \setminus \{p_{n+1}\} &\rightarrow \mathbf{P}(V_n) \end{aligned}$$

as in Section 23. At each step we will choose $p_N, p_{N-1}, \dots, p_{n+1}$ in a suitable Zariski open set. Pick a closed point $x \in Z \subset X$. For every i pick closed points $x_{it} \in T_i \cap Z$, at least one in each irreducible component of $T_i \cap Z$. Taking the composition we obtain a morphism

$$\pi = (r_{n+1} \circ \dots \circ r_N)|_X : X \longrightarrow \mathbf{P}(V_n)$$

which has the following properties

- (1) π is finite,

- (2) π is étale at x and all x_{it} ,
- (3) $\pi|_Z : Z \rightarrow \pi(Z)$ is an isomorphism over an open neighbourhood of $\pi(x_{it})$,
- (4) $T_i \cap \pi^{-1}(\pi(Z)) = (T_i \cap Z) \cup E_i$ with $E_i \subset T_i$ closed and $\dim(E_i) \leq d + d' + 1 - (n + 1) = d + d' - n$.

It follows in a straightforward manner from Lemmas 23.1, 23.2, and 23.3 and induction that we can do this; observe that the last projection is from $\mathbf{P}(V_{n+1})$ and that $\dim(V_{n+1}) = n + 2$ which explains the inequality in (4).

Let $C \subset \mathbf{P}(V_N)$ be the scheme theoretic closure of $(r_{n+1} \circ \dots \circ r_N)^{-1}(\pi(Z))$. Because π is étale at the point x of Z , we see that the closed subscheme $C \cap X$ contains Z with multiplicity 1 (local calculation omitted). Hence by Lemma 17.2 we conclude that

$$C \cdot X = [Z] + \sum m_j [Z_j]$$

for some subvarieties $Z_j \subset X$ of dimension d . Note that

$$C \cap X = \pi^{-1}(\pi(Z))$$

set theoretically. Hence $T_i \cap Z_j \subset T_i \cap \pi^{-1}(\pi(Z)) \subset T_i \cap Z \cup E_i$. For any irreducible component of $T_i \cap Z$ contained in E_i we have the desired dimension bound. Finally, let V be an irreducible component of $T_i \cap Z_j$ which is contained in $T_i \cap Z$. To finish the proof it suffices to show that V does not contain any of the points x_{it} , because then $\dim(V) < \dim(Z \cap T_i)$. To show this it suffices to show that $x_{it} \notin Z_j$ for all i, t, j .

Set $Z' = \pi(Z)$ and $Z'' = \pi^{-1}(Z')$, scheme theoretically. By condition (3) we can find an open $U \subset \mathbf{P}(V_n)$ containing $\pi(x_{it})$ such that $\pi^{-1}(U) \cap Z \rightarrow U \cap Z'$ is an isomorphism. In particular, $Z \rightarrow Z'$ is a local isomorphism at x_{it} . On the other hand, $Z'' \rightarrow Z'$ is étale at x_{it} by condition (2). Hence the closed immersion $Z \rightarrow Z''$ is étale at x_{it} (Morphisms, Lemma 36.18). Thus $Z = Z''$ in a Zariski neighbourhood of x_{it} which proves the assertion. \square

The actual moving is done using the following lemma.

Lemma 24.2. *Let $C \subset \mathbf{P}^N$ be a closed subvariety. Let $X \subset \mathbf{P}^N$ be subvariety and let $T_i \subset X$ be a finite collection of closed subvarieties. Assume that C and X intersect properly. Then there exists a closed subvariety $C' \subset \mathbf{P}^N \times \mathbf{P}^1$ such that*

- (1) $C' \rightarrow \mathbf{P}^1$ is dominant,
- (2) $C'_0 = C$ scheme theoretically,
- (3) C' and $X \times \mathbf{P}^1$ intersect properly,
- (4) C'_∞ properly intersects each of the given T_i .

Proof. If $C \cap X = \emptyset$, then we take the constant family $C' = C \times \mathbf{P}^1$. Thus we may and do assume $C \cap X \neq \emptyset$.

Write $\mathbf{P}^N = \mathbf{P}(V)$ so $\dim(V) = N + 1$. Let $E = \text{End}(V)$. Let $E^\vee = \text{Hom}(E, \mathbf{C})$. Set $\mathbf{P} = \mathbf{P}(E^\vee)$ as in Lemma 23.7. Choose a general line $\ell \subset \mathbf{P}$ passing through id_V . Set $C' \subset \ell \times \mathbf{P}(V)$ equal to the closed subscheme having fibre $r_g(C)$ over $[g] \in \ell$. More precisely, C' is the image of

$$\ell \times C \subset \mathbf{P} \times \mathbf{P}(V)$$

under the morphism (23.6.1). By Lemma 23.7 this makes sense, i.e., $\ell \times C \subset U(\psi)$. The morphism $\ell \times C \rightarrow C'$ is finite and $C'_{[g]} = r_g(C)$ set theoretically for all $[g] \in \ell$. Parts (1) and (2) are clear with $0 = [\text{id}_V] \in \ell$. Part (3) follows from the fact that

$r_g(C)$ and X intersect properly for all $[g] \in \ell$. Part (4) follows from the fact that a general point $\infty = [g] \in \ell$ is a general point of \mathbf{P} and for such as point $r_g(C) \cap T$ is proper for any closed subvariety T of $\mathbf{P}(V)$. Details omitted. \square

Lemma 24.3. *Let X be a nonsingular projective variety. Let α be an r -cycle and β be an s -cycle on X . Then there exists an r -cycle α' such that $\alpha' \sim_{\text{rat}} \alpha$ and such that α' and β intersect properly.*

Proof. Write $\beta = \sum n_i [T_i]$ for some subvarieties $T_i \subset X$ of dimension s . By linearity we may assume that $\alpha = [Z]$ for some irreducible closed subvariety $Z \subset X$ of dimension r . We will prove the lemma by induction on the maximum e of the integers

$$\dim(Z \cap T_i)$$

The base case is $e = r + s - \dim(X)$. In this case Z intersects β properly and the lemma is trivial.

Induction step. Assume that $e > r + s - \dim(X)$. Choose an embedding $X \subset \mathbf{P}^N$ and apply Lemma 24.1 to find a closed subvariety $C \subset \mathbf{P}^N$ such that $C \cdot X = [Z] + \sum m_j [Z_j]$ and such that the induction hypothesis applies to each Z_j . Next, apply Lemma 24.2 to C, X, T_i to find $C' \subset \mathbf{P}^N \times \mathbf{P}^1$. Let $\gamma = C' \cdot X \times \mathbf{P}^1$ viewed as a cycle on $X \times \mathbf{P}^1$. By Lemma 22.2 we have

$$[Z] + \sum m_j [Z_j] = \text{pr}_{X,*}(\gamma \cdot X \times 0)$$

On the other hand the cycle $\gamma_\infty = \text{pr}_{X,*}(\gamma \cdot X \times \infty)$ is supported on $C'_\infty \cap X$ hence intersects β transversally. Thus we see that $[Z] \sim_{\text{rat}} -\sum m_j [Z_j] + \gamma_\infty$ by Lemma 17.1. Since by induction each $[Z_j]$ is rationally equivalent to a cycle which properly intersects β this finishes the proof. \square

25. Intersection products and rational equivalence

With definitions as above we show that the intersection product is well defined modulo rational equivalence. We first deal with a special case.

Lemma 25.1. *Let X be a nonsingular variety. Let $W \subset X \times \mathbf{P}^1$ be an $(s+1)$ -dimensional subvariety dominating \mathbf{P}^1 . Let W_a , resp. W_b be the fibre of $W \rightarrow \mathbf{P}^1$ over a , resp. b . Let V be a r -dimensional subvariety of X such that V intersects both W_a and W_b properly. Then $[V] \cdot [W_a]_r \sim_{\text{rat}} [V] \cdot [W_b]_r$.*

Proof. We have $[W_a]_r = \text{pr}_{X,*}(W \cdot X \times a)$ and similarly for $[W_b]_r$, see Lemma 17.1. Thus we reduce to showing

$$V \cdot \text{pr}_{X,*}(W \cdot X \times a) \sim_{\text{rat}} V \cdot \text{pr}_{X,*}(W \cdot X \times b).$$

Applying the projection formula Lemma 22.1 we get

$$V \cdot \text{pr}_{X,*}(W \cdot X \times a) = \text{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times a))$$

and similarly for b . Thus we reduce to showing

$$\text{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times a)) \sim_{\text{rat}} \text{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times b))$$

If $V \times \mathbf{P}^1$ intersects W properly, then associativity for the intersection multiplicities (Lemma 20.1) gives $V \times \mathbf{P}^1 \cdot (W \cdot X \times a) = (V \times \mathbf{P}^1 \cdot W) \cdot X \times a$ and similarly for b . Thus we reduce to showing

$$\text{pr}_{X,*}((V \times \mathbf{P}^1 \cdot W) \cdot X \times a) \sim_{\text{rat}} \text{pr}_{X,*}((V \times \mathbf{P}^1 \cdot W) \cdot X \times b)$$

which is true by Lemma 17.1.

The argument above does not quite work. The obstruction is that we do not know that $V \times \mathbf{P}^1$ and W intersect properly. We only know that V and W_a and V and W_b intersect properly. Let $Z_i, i \in I$ be the irreducible components of $V \times \mathbf{P}^1 \cap W$. Then we know that $\dim(Z_i) \geq r+1+s+1-n-1 = r+s+1-n$ where $n = \dim(X)$, see Lemma 13.4. Since we have assumed that V and W_a intersect properly, we see that $\dim(Z_{i,a}) = r+s-n$ or $Z_{i,a} = \emptyset$. On the other hand, if $Z_{i,a} \neq \emptyset$, then $\dim(Z_{i,a}) \geq \dim(Z_i) - 1 = r+s-n$. It follows that $\dim(Z_i) = r+s+1-n$ if Z_i meets $X \times a$ and in this case $Z_i \rightarrow \mathbf{P}^1$ is surjective. Thus we may write $I = I' \amalg I''$ where I' is the set of $i \in I$ such that $Z_i \rightarrow \mathbf{P}^1$ is surjective and I'' is the set of $i \in I$ such that Z_i lies over a closed point $t_i \in \mathbf{P}^1$ with $t_i \neq a$ and $t_i \neq b$. Consider the cycle

$$\gamma = \sum_{i \in I'} e_i [Z_i]$$

where we take

$$e_i = \sum_p (-1)^{p \cdot \text{length}} \text{Tor}_p^{\mathcal{O}_{X \times \mathbf{P}^1, Z_i}} (\mathcal{O}_{V \times \mathbf{P}^1, Z_i}, \mathcal{O}_{W, Z_i})$$

We will show that γ can be used as a replacement for the intersection product of $V \times \mathbf{P}^1$ and W .

We will show this using associativity of intersection products in exactly the same way as above. Let $U = \mathbf{P}^1 \setminus \{t_i, i \in I''\}$. Note that $X \times a$ and $X \times b$ are contained in $X \times U$. The subvarieties

$$V \times U, \quad W_U, \quad X \times a \quad \text{of} \quad X \times U$$

intersect transversally pairwise by our choice of U and moreover $\dim(V \times U \cap W_U \cap X \times a) = \dim(V \cap W_a)$ has the expected dimension. Thus we see that

$$V \times U \cdot (W_U \cdot X \times a) = (V \times U \cdot W_U) \cdot X \times a$$

as cycles on $X \times U$ by Lemma 20.1. By construction γ restricts to the cycle $V \times U \cdot W_U$ on $X \times U$. Trivially, $V \times \mathbf{P}^1 \cdot (W \times X \times a)$ restricts to $V \times U \cdot (W_U \cdot X \times a)$ on $X \times U$. Hence

$$V \times \mathbf{P}^1 \cdot (W \cdot X \times a) = \gamma \cdot X \times a$$

as cycles on $X \times \mathbf{P}^1$ (because both sides are contained in $X \times U$ and are equal after restricting to $X \times U$ by what was said before). Since we have the same for b we conclude

$$\begin{aligned} V \cdot [W_a] &= \text{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times a)) \\ &= \text{pr}_{X,*}(\gamma \cdot X \times a) \\ &\sim_{\text{rat}} \text{pr}_{X,*}(\gamma \cdot X \times b) \\ &= \text{pr}_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times b)) \\ &= V \cdot [W_b] \end{aligned}$$

The first and the last equality by the first paragraph of the proof, the second and penultimate equalities were shown in this paragraph, and the middle equivalence is Lemma 17.1. \square

Theorem 25.2. *Let X be a nonsingular projective variety. Let α , resp. β be an r , resp. s cycle on X . Assume that α and β intersect properly so that $\alpha \cdot \beta$ is defined. Finally, assume that $\alpha \sim_{\text{rat}} 0$. Then $\alpha \cdot \beta \sim_{\text{rat}} 0$.*

Proof. Pick a closed immersion $X \subset \mathbf{P}^N$. By linearity it suffices to prove the result when $\beta = [Z]$ for some s -dimensional closed subvariety $Z \subset X$ which intersects α properly. The condition $\alpha \sim_{rat} 0$ means there are finitely many $(r+1)$ -dimensional closed subvarieties $W_i \subset X \times \mathbf{P}^1$ such that

$$\alpha = \sum [W_{i,a_i}]_r - [W_{i,b_i}]_r$$

for some pairs of points a_i, b_i of \mathbf{P}^1 . Let W_{i,a_i}^t and W_{i,b_i}^t be the irreducible components of W_{i,a_i} and W_{i,b_i} . We will use induction on the maximum d of the integers

$$\dim(Z \cap W_{i,a_i}^t), \quad \dim(Z \cap W_{i,b_i}^t)$$

The main problem in the rest of the proof is that although we know that Z intersects α properly, it may not be the case that Z intersects the “intermediate” varieties W_{i,a_i}^t and W_{i,b_i}^t properly, i.e., it may happen that $d > r + s - \dim(X)$.

Base case: $d = r + s - \dim(X)$. In this case all the intersections of Z with the W_{i,a_i}^t and W_{i,b_i}^t are proper and the desired result follows from Lemma 25.1, because it applies to show that $[Z] \cdot [W_{i,a_i}]_r \sim_{rat} [Z] \cdot [W_{i,b_i}]_r$ for each i .

Induction step: $d > r + s - \dim(X)$. Apply Lemma 24.1 to $Z \subset X$ and the family of subvarieties $\{W_{i,a_i}^t, W_{i,b_i}^t\}$. Then we find a closed subvariety $C \subset \mathbf{P}^N$ intersecting X properly such that

$$C \cdot X = [Z] + \sum m_j [Z_j]$$

and such that

$$\dim(Z_j \cap W_{i,a_i}^t) \leq \dim(Z \cap W_{i,a_i}^t), \quad \dim(Z_j \cap W_{i,b_i}^t) \leq \dim(Z \cap W_{i,b_i}^t)$$

with strict inequality if the right hand side is $> r + s - \dim(X)$. This implies two things: (a) the induction hypothesis applies to each Z_j , and (b) $C \cdot X$ and α intersect properly (because α is a linear combination of those $[W_{i,a_i}^t]$ and $[W_{i,b_i}^t]$ which intersect Z properly). Next, pick $C' \subset \mathbf{P}^N \times \mathbf{P}^1$ as in Lemma 24.2 with respect to C , X , and W_{i,a_i}^t, W_{i,b_i}^t . Write $C' \cdot X \times \mathbf{P}^1 = \sum n_k [E_k]$ for some subvarieties $E_k \subset X \times \mathbf{P}^1$ of dimension $s+1$. Note that $n_k > 0$ for all k by Proposition 19.3. By Lemma 22.2 we have

$$[Z] + \sum m_j [Z_j] = \sum n_k [E_{k,0}]_s$$

Since $E_{k,0} \subset C \cap X$ we see that $[E_{k,0}]_s$ and α intersect properly. On the other hand, the cycle

$$\gamma = \sum n_k [E_{k,\infty}]_s$$

is supported on $C'_\infty \cap X$ and hence properly intersects each W_{i,a_i}^t, W_{i,b_i}^t . Thus by the base case and linearity, we see that

$$\gamma \cdot \alpha \sim_{rat} 0$$

As we have seen that $E_{k,0}$ and $E_{k,\infty}$ intersect α properly Lemma 25.1 applied to $E_k \subset X \times \mathbf{P}^1$ and α gives

$$[E_{k,0}] \cdot \alpha \sim_{rat} [E_{k,\infty}] \cdot \alpha$$

Putting everything together we have

$$\begin{aligned}
[Z] \cdot \alpha &= \left(\sum n_k [E_{k,0}]_r - \sum m_j [Z_j] \right) \cdot \alpha \\
&\sim_{rat} \sum n_k [E_{k,0}] \cdot \alpha \quad (\text{by induction hypothesis}) \\
&\sim_{rat} \sum n_k [E_{k,\infty}] \cdot \alpha \quad (\text{by the lemma}) \\
&= \gamma \cdot \alpha \\
&\sim_{rat} 0 \quad (\text{by base case})
\end{aligned}$$

This finishes the proof. \square

Remark 25.3. Lemma 24.3 and Theorem 25.2 also hold for nonsingular quasi-projective varieties with the same proof. The only change is that one needs to prove the following version of the moving Lemma 24.1: Let $X \subset \mathbf{P}^N$ be a closed subvariety. Let $n = \dim(X)$ and $0 \leq d, d' < n$. Let $X^{reg} \subset X$ be the open subset of nonsingular points. Let $Z \subset X^{reg}$ be a closed subvariety of dimension d and $T_i \subset X^{reg}$, $i \in I$ be a finite collection of closed subvarieties of dimension d' . Then there exists a subvariety $C \subset \mathbf{P}^N$ such that C intersects X properly and such that

$$(C \cdot X)|_{X^{reg}} = Z + \sum_{j \in J} m_j Z_j$$

where $Z_j \subset X^{reg}$ are irreducible of dimension d , distinct from Z , and

$$\dim(Z_j \cap T_i) \leq \dim(Z \cap T_i)$$

with strict inequality if Z does not intersect T_i properly in X^{reg} .

26. Chow rings

Let X be a nonsingular projective variety. We define the intersection product

$$\mathrm{CH}_r(X) \times \mathrm{CH}_s(X) \longrightarrow \mathrm{CH}_{r+s-\dim(X)}(X), \quad (\alpha, \beta) \longmapsto \alpha \cdot \beta$$

as follows. Let $\alpha \in Z_r(X)$ and $\beta \in Z_s(X)$. If α and β intersect properly, we use the definition given in Section 17. If not, then we choose $\alpha \sim_{rat} \alpha'$ as in Lemma 24.3 and we set

$$\alpha \cdot \beta = \text{class of } \alpha' \cdot \beta \in \mathrm{CH}_{r+s-\dim(X)}(X)$$

This is well defined and passes through rational equivalence by Theorem 25.2. The intersection product on $\mathrm{CH}_*(X)$ is commutative (this is clear), associative (Lemma 20.1) and has a unit $[X] \in \mathrm{CH}_{\dim(X)}(X)$.

We often use $\mathrm{CH}^c(X) = \mathrm{CH}_{\dim X - c}(X)$ to denote the Chow group of cycles of codimension c , see Chow Homology, Section 42. The intersection product defines a product

$$\mathrm{CH}^k(X) \times \mathrm{CH}^l(X) \longrightarrow \mathrm{CH}^{k+l}(X)$$

which is commutative, associative, and has a unit $1 = [X] \in \mathrm{CH}^0(X)$.

27. Pullback for a general morphism

Let $f : X \rightarrow Y$ be a morphism of nonsingular projective varieties. We define

$$f^* : \mathrm{CH}_k(Y) \rightarrow \mathrm{CH}_{k+\dim X - \dim Y}(X)$$

by the rule

$$f^*(\alpha) = pr_{X,*}(\Gamma_f \cdot pr_Y^*(\alpha))$$

where $\Gamma_f \subset X \times Y$ is the graph of f . Note that in this generality, it is defined only on cycle classes and not on cycles. With the notation CH^* introduced in Section 26 we may think of pullback as a map

$$f^* : \mathrm{CH}^*(Y) \rightarrow \mathrm{CH}^*(X)$$

in other words, it is a map of graded abelian groups.

Lemma 27.1. *Let $f : X \rightarrow Y$ be a morphism of nonsingular projective varieties. The pullback map on chow groups satisfies:*

- (1) $f^* : \mathrm{CH}^*(Y) \rightarrow \mathrm{CH}^*(X)$ is a ring map,
- (2) $(g \circ f)^* = f^* \circ g^*$ for a composable pair f, g ,
- (3) the projection formula holds: $f_*(\alpha) \cdot \beta = f_*(\alpha \cdot f^*\beta)$, and
- (4) if f is flat then it agrees with the previous definition.

Proof. All of these follow readily from the results above.

For (1) it suffices to show that $pr_{X,*}(\Gamma_f \cdot \alpha \cdot \beta) = pr_{X,*}(\Gamma_f \cdot \alpha) \cdot pr_{X,*}(\Gamma_f \cdot \beta)$ for cycles α, β on $X \times Y$. If α is a cycle on $X \times Y$ which intersects Γ_f properly, then it is easy to see that

$$\Gamma_f \cdot \alpha = \Gamma_f \cdot pr_X^*(pr_{X,*}(\Gamma_f \cdot \alpha))$$

as cycles because Γ_f is a graph. Thus we get the first equality in

$$\begin{aligned} pr_{X,*}(\Gamma_f \cdot \alpha \cdot \beta) &= pr_{X,*}(\Gamma_f \cdot pr_X^*(pr_{X,*}(\Gamma_f \cdot \alpha)) \cdot \beta) \\ &= pr_{X,*}(pr_X^*(pr_{X,*}(\Gamma_f \cdot \alpha)) \cdot (\Gamma_f \cdot \beta)) \\ &= pr_{X,*}(\Gamma_f \cdot \alpha) \cdot pr_{X,*}(\Gamma_f \cdot \beta) \end{aligned}$$

the last step by the projection formula in the flat case (Lemma 22.1).

If $g : Y \rightarrow Z$ then property (2) follows formally from the observation that

$$\Gamma = pr_{X \times Y}^* \Gamma_f \cdot pr_{Y \times Z}^* \Gamma_g$$

in $Z_*(X \times Y \times Z)$ where $\Gamma = \{(x, f(x), g(f(x)))\}$ and maps isomorphically to $\Gamma_{g \circ f}$ in $X \times Z$. The equality follows from the scheme theoretic equality and Lemma 14.3.

For (3) we use the projection formula for flat maps twice

$$\begin{aligned} f_*(\alpha \cdot pr_{X,*}(\Gamma_f \cdot pr_Y^*(\beta))) &= f_*(pr_{X,*}(pr_X^* \alpha \cdot \Gamma_f \cdot pr_Y^*(\beta))) \\ &= pr_{Y,*}(pr_X^* \alpha \cdot \Gamma_f \cdot pr_Y^*(\beta)) \\ &= pt_{Y,*}(pr_X^* \alpha \cdot \Gamma_f) \cdot \beta \\ &= f_*(\alpha) \cdot \beta \end{aligned}$$

where in the last equality we use the remark on graphs made above. This proves (3).

Property (4) rests on identifying the intersection product $\Gamma_f \cdot pr_Y^* \alpha$ in the case f is flat. Namely, in this case if $V \subset Y$ is a closed subvariety, then every generic point ξ of the scheme $f^{-1}(V) \cong \Gamma_f \cap pr_Y^{-1}(V)$ lies over the generic point of V . Hence

the local ring of $pr_Y^{-1}(V) = X \times V$ at ξ is Cohen-Macaulay. Since $\Gamma_f \subset X \times Y$ is a regular immersion (as a morphism of smooth projective varieties) we find that

$$\Gamma_f \cdot pr_Y^*[V] = [\Gamma_f \cap pr_Y^{-1}(V)]_d$$

with d the dimension of $\Gamma_f \cap pr_Y^{-1}(V)$, see Lemma 16.5. Since $\Gamma_f \cap pr_Y^{-1}(V)$ maps isomorphically to $f^{-1}(V)$ we conclude. \square

28. Pullback of cycles

Suppose that X and Y be nonsingular projective varieties, and let $f : X \rightarrow Y$ be a morphism. Suppose that $Z \subset Y$ is a closed subvariety. Let $f^{-1}(Z)$ be the scheme theoretic inverse image:

$$\begin{array}{ccc} f^{-1}(Z) & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is a fibre product diagram of schemes. In particular $f^{-1}(Z) \subset X$ is a closed subscheme of X . In this case we always have

$$\dim f^{-1}(Z) \geq \dim Z + \dim X - \dim Y.$$

If equality holds in the formula above, then $f^*[Z] = [f^{-1}(Z)]_{\dim Z + \dim X - \dim Y}$ provided that the scheme Z is Cohen-Macaulay at the images of the generic points of $f^{-1}(Z)$. This follows by identifying $f^{-1}(Z)$ with the scheme theoretic intersection of Γ_f and $X \times Z$ and using Lemma 16.5. Details are similar to the proof of part (4) of Lemma 27.1 above.

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