MORE ÉTALE COHOMOLOGY

Contents

1. Introduction

This chapter is the second in a series of chapter on the étale cohomology of schemes. To read the first chapter, please visit Étale Cohomology, Section 1.

The split with the previous chapter is roughly speaking that anything concerning "shriek functors" (cohomology with compact support and its right adjoint) and anything using this material goes into this chapter.

2. Growing sections

In this section we discuss results of the following type.

Lemma 2.1. *Let X be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $\varphi : U' \to U$ be a morphism of $X_{étale}$. Let $Z' \subset U'$ be a closed subscheme such that $Z' \to U' \to U$ is a closed immersion with image $Z \subset U$. Then there is a canonical bijection*

$$\{s \in \mathcal{F}(U) \mid Supp(s) \subset Z\} = \{s' \in \mathcal{F}(U') \mid Supp(s') \subset Z'\}$$

which is given by restriction if $\varphi^{-1}(Z) = Z'$.

Proof. Consider the closed subscheme $Z'' = \varphi^{-1}(Z)$ of U'. Then $Z' \subset Z''$ is closed because Z' is closed in U'. On the other hand, $Z' \to Z''$ is an étale morphism (as a morphism between schemes étale over Z) and hence open. Thus $Z'' = Z' \amalg T$ for some closed subset T. The open covering $U' = (U' \setminus T) \cup (U' \setminus Z')$ shows that

$$\{s' \in \mathcal{F}(U') \mid \mathrm{Supp}(s') \subset Z'\} = \{s' \in \mathcal{F}(U' \setminus T) \mid \mathrm{Supp}(s') \subset Z'\}$$

and the étale covering $\{U' \setminus T \to U, U \setminus Z \to U\}$ shows that

$$\{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset Z\} = \{s' \in \mathcal{F}(U' \setminus T) \mid \mathrm{Supp}(s') \subset Z'\}$$

This finishes the proof. □

Lemma 2.2. *Let X be a scheme. Let $Z \subset X$ be a locally closed subscheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Given $U, U' \subset X$ open containing Z as a closed subscheme, there is a canonical bijection*

$$\{s \in \mathcal{F}(U) \mid Supp(s) \subset Z\} = \{s \in \mathcal{F}(U') \mid Supp(s) \subset Z\}$$

which is given by restriction if $U' \subset U$.

Proof. Since Z is a closed subscheme of $U \cap U'$, it suffices to prove the lemma when $U' \subset U$. Then it is a special case of Lemma 2.1. □

Let us introduce a bit of nonstandard notation which will stand us in good stead later. Namely, in the situation of Lemma 2.2 above, let us denote

$$H_Z(\mathcal{F}) = \{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset Z\}$$

where $U \subset X$ is any choice of open subscheme containing Z as a closed subscheme. The reader who is troubled by the lack of precision this entails may choose $U = X \setminus \partial Z$ where $\partial Z = \overline{Z} \setminus Z$ is the "boundary" of Z in X. However, in many of the arguments below the flexibility of choosing different opens will play a role. Here are some properties of this construction:

(1) If $Z \subset Z'$ are locally closed subschemes of X and Z is closed in Z', then there is a natural injective map
$$H_Z(\mathcal{F}) \to H_{Z'}(\mathcal{F}).$$
(2) If $f : Y \to X$ is a morphism of schemes and $Z \subset X$ is a locally closed subscheme, then there is a natural pullback map $f^* : H_Z(\mathcal{F}) \to H_{f^{-1}Z}(f^{-1}\mathcal{F})$.

It will be convenient to extend our notation to the following situation: suppose that we have $W \in X_{\acute{e}tale}$ and a locally closed subscheme $Z \subset W$. Then we will denote

$$H_Z(\mathcal{F}) = \{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset Z\} = H_Z(\mathcal{F}|_{W_{\acute{e}tale}})$$

where $U \subset W$ is any choice of open subscheme containing Z as a closed subscheme, exactly as above[1].

[1] In fact, Lemma 2.1 shows, given Z over X which is isomorphic to a locally closed subscheme of some object W of $X_{\acute{e}tale}$, that the choice of W is irrelevant.

3. Sections with compact support

A reference for this section is [AGV71, Exposee XVII, Section 6]. Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. In this section we define a functor $f_! : Ab(X_{\acute{e}tale}) \to Ab(Y_{\acute{e}tale})$ by taking $f_!\mathcal{F} \subset f_*\mathcal{F}$ to be the subsheaf of sections which have proper support relative to Y (suitably defined).

Warning: The functor $f_!$ is the zeroth cohomology sheaf of a functor $Rf_!$ on the derived category (insert future reference), but $Rf_!$ is not the derived functor of $f_!$.

Lemma 3.1. *Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. The rule*

$$Y_{\acute{e}tale} \longrightarrow Ab, \quad V \longmapsto \{s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V) \mid Supp(s) \subset X_V \text{ is proper over } V\}$$

is an abelian subsheaf of $f_*\mathcal{F}$.

Warning: This sheaf isn't the "correct one" if f is not separated.

Proof. Recall that the support of a section is closed (Étale Cohomology, Lemma 31.4) hence the material in Cohomology of Schemes, Section 26 applies. By the lemma above and Cohomology of Schemes, Lemma 26.6 we find that our subset of $f_*\mathcal{F}(V)$ is a subgroup. By Cohomology of Schemes, Lemma 26.4 we see that our rule defines a sub presheaf. Finally, suppose that we have $s \in f_*\mathcal{F}(V)$ and an étale covering $\{V_i \to V\}$ such that $s|_{V_i}$ has support proper over V_i. Observe that the support of $s|_{V_i}$ is the inverse image of the support of $s|_V$ (use the characterization of the support in terms of stalks and Étale Cohomology, Lemma 36.2). Whence the support of s is proper over V by Descent, Lemma 25.5. This proves that our rule satisfies the sheaf condition. $\square$

Lemma 3.2. *Let $j : U \to X$ be a separated étale morphism. Let $\mathcal{F}$ be an abelian sheaf on $U_{\acute{e}tale}$. The image of the injective map $j_!\mathcal{F} \to j_*\mathcal{F}$ of Étale Cohomology, Lemma 70.6 is the subsheaf of Lemma 3.1.*

An alternative would be to move this lemma later and prove this using the descrition of the stalks of both sheaves.

Proof. The construction of $j_!\mathcal{F} \to j_*\mathcal{F}$ in the proof of Étale Cohomology, Lemma 70.6 is via the construction of a map $j_{p!}\mathcal{F} \to j_*\mathcal{F}$ of presheaves whose image is clearly contained in the subsheaf of Lemma 3.1. Hence since $j_!\mathcal{F}$ is the sheafification of $j_{p!}\mathcal{F}$ we conclude the image of $j_!\mathcal{F} \to j_*\mathcal{F}$ is contained in this subsheaf. Conversely, let $s \in j_*\mathcal{F}(V)$ have support Z proper over V. Then $Z \to V$ is finite with closed image $Z' \subset V$, see More on Morphisms, Lemma 44.1. The restriction of s to $V \setminus Z'$ is zero and the zero section is contained in the image of $j_!\mathcal{F} \to j_*\mathcal{F}$. On the other hand, if $v \in Z'$, then we can find an étale neighbourhood $(V', v') \to (V, v)$ such that we have a decomposition $U_{V'} = W \amalg U'_1 \amalg \ldots \amalg U'_n$ into open and closed subschemes with $U'_i \to V'$ an isomorphism and with $T_{V'} \subset U'_1 \amalg \ldots \amalg U'_n$, see Étale Morphisms, Lemma 18.2. Inverting the isomorphisms $U'_i \to V'$ we obtain n morphisms $\varphi'_i : V' \to U$ and sections s'_i over V' by pulling back s. Then the section $\sum(\varphi'_i, s'_i)$ of $j_{p!}\mathcal{F}$ over V', see formula for $j_{p!}\mathcal{F}(V')$ in proof of Étale Cohomology, Lemma 70.6, maps to the restriction of s to V' by construction. We conclude that s is étale locally in the image of $j_!\mathcal{F} \to j_*\mathcal{F}$ and the proof is complete. $\square$

Definition 3.3. Let $f : X \to Y$ be a morphism of schemes which is separated (!) and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. The subsheaf $f_!\mathcal{F} \subset f_*\mathcal{F}$ constructed in Lemma 3.1 is called the *direct image with compact support*.

By Lemma 3.2 this does not conflict with Étale Cohomology, Definition 70.1 as we have agreement when both definitions apply. Here is a sanity check.

Lemma 3.4. *Let $f : X \to Y$ be a proper morphism of schemes. Then $f_! = f_*$.*

Proof. Immediate from the construction of $f_!$. □

A very useful observation is the following.

Remark 3.5 (Covariance with respect to open embeddings)**.** Let $f : X \to Y$ be morphism of schemes which is separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $X' \subset X$ be an open subscheme. Denote $f' : X' \to Y$ the restriction of f. There is a canonical injective map

$$f'_!(\mathcal{F}|_{X'}) \longrightarrow f_!\mathcal{F}$$

Namely, let $V \in Y_{étale}$ and consider a section $s' \in f'_*(\mathcal{F}|_{X'})(V) = \mathcal{F}(X' \times_Y V)$ with support Z' proper over V. Then Z' is closed in $X \times_Y V$ as well, see Cohomology of Schemes, Lemma 26.5. Thus there is a unique section $s \in \mathcal{F}(X \times_Y V) = f_*\mathcal{F}(V)$ whose restriction to $X' \times_Y V$ is s' and whose restriction to $X \times_Y V \setminus Z'$ is zero, see Lemma 2.2. This construction is compatible with restriction maps and hence induces the desired map of sheaves $f'_!(\mathcal{F}|_{X'}) \to f_!\mathcal{F}$ which is clearly injective. By construction we obtain a commutative diagram

$$\begin{tikzcd}
f'_!(\mathcal{F}|_{X'}) \arrow[r] \arrow[d] & f_!\mathcal{F} \arrow[d] \\
f'_*(\mathcal{F}|_{X'}) & f_*\mathcal{F} \arrow[l]
\end{tikzcd}$$

functorial in $\mathcal{F}$. It is clear that for $X'' \subset X'$ open with $f'' = f|_{X''} : X'' \to Y$ the composition of the canonical maps $f''_!\mathcal{F}|_{X''} \to f'_!\mathcal{F}|_{X'} \to f_!\mathcal{F}$ just constructed is the canonical map $f''_!\mathcal{F}|_{X''} \to f_!\mathcal{F}$.

Lemma 3.6. *Let Y be a scheme. Let $j : X \to \overline{X}$ be an open immersion of schemes over Y with $\overline{X}$ proper over Y. Denote $f : X \to Y$ and $\overline{f} : \overline{X} \to Y$ the structure morphisms. For $\mathcal{F} \in Ab(X_{étale})$ there is a canonical isomorphism (see proof)*

$$f_!\mathcal{F} \longrightarrow \overline{f}_!j_!\mathcal{F}$$

As we have $\overline{f}_! = \overline{f}_*$ by Lemma 3.4 we obtain $\overline{f}_* \circ j_! = f_!$ as functors $Ab(X_{étale}) \to Ab(Y_{étale})$.

Proof. We have $(j_!\mathcal{F})|_X = \mathcal{F}$, see Étale Cohomology, Lemma 70.4. Thus the displayed arrow is the injective map $f_!(\mathcal{G}|_X) \to \overline{f}_!\mathcal{G}$ of Remark 3.5 for $\mathcal{G} = j_!\mathcal{F}$. The explicit nature of this map implies that it now suffices to show: if $V \in Y_{étale}$ and $s \in \overline{f}_!\mathcal{G}(V) = \overline{f}_*\mathcal{G}(V) = \mathcal{G}(\overline{X}_V)$ is a section, then the support of s is contained in the open $X_V \subset \overline{X}_V$. This is immediate from the fact that the stalks of $\mathcal{G}$ are zero at geometric points of $\overline{X} \setminus X$. □

We want to relate the stalks of $f_!\mathcal{F}$ to sections with compact support on fibres. In order to state this, we need a definition.

Definition 3.7. Let X be a separated scheme locally of finite type over a field k. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. We let $H^0_c(X, \mathcal{F}) \subset H^0(X, \mathcal{F})$ be the set of sections whose support is proper over k. Elements of $H^0_c(X, \mathcal{F})$ are called *sections with compact support.*

Warning: This definition isn't the "correct one" if X isn't separated over k.

Lemma 3.8. *Let X be a proper scheme over a field k. Then $H^0_c(X, \mathcal{F}) = H^0(X, \mathcal{F})$.*

Proof. Immediate from the construction of H^0_c. $\square$

Remark 3.9 (Open embeddings and compactly supported sections)**.** Let X be a separated scheme locally of finite type over a field k. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Exactly as in Remark 3.5 for $X' \subset X$ open there is an injective map

$$H^0_c(X', \mathcal{F}|_{X'}) \longrightarrow H^0_c(X, \mathcal{F})$$

and these maps turn H^0_c into a "cosheaf" on the Zariski site of X.

Lemma 3.10. *Let k be a field. Let $j : X \to \overline{X}$ be an open immersion of schemes over k with $\overline{X}$ proper over k. For $\mathcal{F} \in Ab(X_{\acute{e}tale})$ there is a canonical isomorphism (see proof)*

$$H^0_c(X, \mathcal{F}) \longrightarrow H^0_c(\overline{X}, j_!\mathcal{F}) = H^0(\overline{X}, j_!\mathcal{F})$$

where we have the equality on the right by Lemma 3.8.

Proof. We have $(j_!\mathcal{F})|_X = \mathcal{F}$, see Étale Cohomology, Lemma 70.4. Thus the displayed arrow is the injective map $H^0_c(X, \mathcal{G}|_X) \to H^0_c(\overline{X}, \mathcal{G})$ of Remark 3.9 for $\mathcal{G} = j_!\mathcal{F}$. The explicit nature of this map implies that it now suffices to show: if $s \in H^0(\overline{X}, \mathcal{G})$ is a section, then the support of s is contained in the open X. This is immediate from the fact that the stalks of $\mathcal{G}$ are zero at geometric points of $\overline{X} \setminus X$. $\square$

Lemma 3.11. *Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Then there is a canonical isomorphism*

$$(f_!\mathcal{F})_{\overline{y}} \longrightarrow H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$$

for any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$.

Proof. Recall that $(f_*\mathcal{F})_{\overline{y}} = \mathrm{colim}\, f_*\mathcal{F}(V)$ where the colimit is over the étale neighbourhoods $(V, \overline{v})$ of $\overline{y}$. If $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$, then we can pullback s to a section of $\mathcal{F}$ over $(X_V)_{\overline{v}} = X_{\overline{y}}$. Thus we obtain a canonical map

$$c_{\overline{y}} : (f_*\mathcal{F})_{\overline{y}} \longrightarrow H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$$

We claim that this map induces a bijection between the subgroups $(f_!\mathcal{F})_{\overline{y}}$ and $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. The claim implies the lemma, but is a little bit more precise in that it describes the identification of the lemma as given by pullbacks of sections of $\mathcal{F}$ to the geometric fibre of f.

Observe that any element $s \in (f_!\mathcal{F})_{\overline{y}} \subset (f_*\mathcal{F})_{\overline{y}}$ is mapped by $c_{\overline{y}}$ to an element of $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \subset H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. This is true because taking the support of a section commutes with pullback and because properness is preserved by base change. This at least produces the map in the statement of the lemma. To prove that it is an isomorphism we may work Zariski locally on Y and hence we may and do assume Y is affine.

An observation that we will use below is that given an open subscheme $X' \subset X$ and if $f' = f|_{X'}$, then we obtain a commutative diagram

$$\begin{tikzcd} (f'_!(\mathcal{F}|_{X'}))_{\overline{y}} \arrow[r] \arrow[d] & H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}}) \arrow[d] \\ (f_!\mathcal{F})_{\overline{y}} \arrow[r] & H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \end{tikzcd}$$

where the horizontal arrows are the maps constructed above and the vertical arrows are given in Remarks 3.5 and 3.9. The reason is that given an étale neighbourhood $(V, \overline{v})$ of $\overline{y}$ and a section $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$ whose support Z happens to be contained in X'_V and is proper over V, so that s gives rise to an element of both $(f'_!(\mathcal{F}|_{X'}))_{\overline{y}}$ and $(f_!\mathcal{F})_{\overline{y}}$ which correspond via the vertical arrow of the diagram, then these elements are mapped via the horizontal arrows to the pullback $s|_{X_{\overline{y}}}$ of s to $X_{\overline{y}}$ whose support $Z_{\overline{y}}$ is contained in $X'_{\overline{y}}$ and hence this restriction gives rise to a compatible pair of elements of $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}})$ and $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$.

Suppose $s \in (f_!\mathcal{F})_{\overline{y}}$ maps to zero in $H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Say s corresponds to $s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V)$ with support Z proper over V. We may assume that V is affine and hence Z is quasi-compact. Then we may choose a quasi-compact open $X' \subset X$ containing the image of Z. Then Z is contained in X'_V and hence s is the image of an element $s' \in f'_!(\mathcal{F}|_{X'})(V)$ where $f' = f|_{X'}$ as in the previous paragraph. Then s' maps to zero in $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}})$. Hence in order to prove injectivity, we may replace X by X', i.e., we may assume X is quasi-compact. We will prove this case below.

Suppose that $t \in H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Then the support of t is contained in a quasi-compact open subscheme $W \subset X_{\overline{y}}$. Hence we can find a quasi-compact open subscheme $X' \subset X$ such that $X'_{\overline{y}}$ contains W. Then it is clear that t is contained in the image of the injective map $H^0_c(X'_{\overline{y}}, \mathcal{F}|_{X'_{\overline{y}}}) \to H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}})$. Hence in order to show surjectivity, we may replace X by X', i.e., we may assume X is quasi-compact. We will prove this case below.

In this last paragraph of the proof we prove the lemma in case X is quasi-compact and Y is affine. By More on Flatness, Theorem 33.8 there exists a compactification $j : X \to \overline{X}$ over Y. Set $\mathcal{G} = j_!\mathcal{F}$ so that $\mathcal{F} = \mathcal{G}|_X$ by Étale Cohomology, Lemma 70.4. By the disussion above we get a commutative diagram

$$\begin{tikzcd} (f_!\mathcal{F})_{\overline{y}} \arrow[r] \arrow[d] & H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[d] \\ (\overline{f}_!\mathcal{G})_{\overline{y}} \arrow[r] & H^0_c(\overline{X}_{\overline{y}}, \mathcal{G}|_{\overline{X}_{\overline{y}}}) \end{tikzcd}$$

By Lemmas 3.6 and 3.10 the vertical maps are isomorphisms. This reduces us to the case of the proper morphism $\overline{X} \to Y$. For a proper morphism our map is an isomorphism by Lemmas 3.4 and 3.8 and proper base change for pushforwards, see Étale Cohomology, Lemma 91.4. $\square$

Lemma 3.12. *Consider a cartesian square*

$$
\begin{tikzcd}
X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}
$$

of schemes with f separated and locally of finite type. For any abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $f'_!(g')^{-1}\mathcal{F} = g^{-1}f_!\mathcal{F}$.

Proof. In great generality there is a pullback map $g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$, see Sites, Section 45. We claim that this map sends $g^{-1}f_!\mathcal{F}$ into the subsheaf $f'_!(g')^{-1}\mathcal{F}$ and induces the isomorphism in the lemma.

Choose a geometric point $\overline{y}' : \mathrm{Spec}(k) \to Y'$ and denote $\overline{y} = g \circ \overline{y}'$ the image in Y. There is a commutative diagram

$$
\begin{tikzcd}
(f_*\mathcal{F})_{\overline{y}} \arrow[r] \arrow[d] & H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[d] \\
(f'_*(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[r] & H^0(X'_{\overline{y}'}, (g')^{-1}\mathcal{F}|_{X'_{\overline{y}'}})
\end{tikzcd}
$$

where the horizontal maps were used in the proof of Lemma 3.11 and the vertical maps are the pullback maps above. The diagram commutes because each of the four maps in question is given by pulling back local sections along a morphism of schemes and the underlying diagram of morphisms of schemes commutes. Since the diagram in the statement of the lemma is cartesian we have $X'_{\overline{y}'} = X_{\overline{y}}$. Hence by Lemma 3.11 and its proof we obtain a commutative diagram

$$
\begin{tikzcd}
(f_*\mathcal{F})_{\overline{y}} \arrow[rrr] \arrow[ddd] & & & H^0(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[ddd] \\
& (f_!\mathcal{F})_{\overline{y}} \arrow[ul] \arrow[r] \arrow[d, dotted] & H^0_c(X_{\overline{y}}, \mathcal{F}|_{X_{\overline{y}}}) \arrow[ur] \arrow[d] & \\
& (f'_!(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[dl] \arrow[r] & H^0_c(X'_{\overline{y}'}, (g')^{-1}\mathcal{F}|_{X'_{\overline{y}'}}) \arrow[dr] & \\
(f'_*(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[rrr] & & & H^0(X'_{\overline{y}'}, (g')^{-1}\mathcal{F}|_{X'_{\overline{y}'}})
\end{tikzcd}
$$

where the horizontal arrows of the inner square are isomorphisms and the two right vertical arrows are equalities. Also, the se, sw, ne, nw arrows are injective. It follows that there is a unique bijective dotted arrow fitting into the diagram. We conclude that $g^{-1}f_!\mathcal{F} \subset g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$ is mapped into the subsheaf $f'_!(g')^{-1}\mathcal{F} \subset f'_*(g')^{-1}\mathcal{F}$ because this is true on stalks, see Étale Cohomology, Theorem 29.10. The same theorem then implies that the induced map is an isomorphism and the proof is complete. □

Lemma 3.13. *Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms of schemes which are separated and locally of finite type. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Then $g_! f_! \mathcal{F} = (g \circ f)_! \mathcal{F}$ as subsheaves of $(g \circ f)_* \mathcal{F}$.*

Proof. We strongly urge the reader to prove this for themselves. Let $W \in Z_{\acute{e}tale}$ and $s \in (g \circ f)_* \mathcal{F}(W) = \mathcal{F}(X_W)$. Denote $T \subset X_W$ the support of s; this is a closed subset. Observe that s is a section of $(g \circ f)_! \mathcal{F}$ if and only if T is proper over W. We have $f_! \mathcal{F} \subset f_* \mathcal{F}$ and hence $g_! f_! \mathcal{F} \subset g_! f_* \mathcal{F} \subset g_* f_* \mathcal{F}$. On the other hand, s is a section of $g_! f_! \mathcal{F}$ if and only if (a) T is proper over Y_W and (b) the support T' of s viewed as section of $f_! \mathcal{F}$ is proper over W. If (a) holds, then the image of T in Y_W is closed and since $f_! \mathcal{F} \subset f_* \mathcal{F}$ we see that $T' \subset Y_W$ is the image of T (details omitted; look at stalks).

The conclusion is that we have to show a closed subset $T \subset X_W$ is proper over W if and only if T is proper over Y_W and the image of T in Y_W is proper over W. Let us endow T with the reduced induced closed subscheme structure. If T is proper over W, then $T \to Y_W$ is proper by Morphisms, Lemma 41.7 and the image of T in Y_W is proper over W by Cohomology of Schemes, Lemma 26.5. Conversely, if T is proper over Y_W and the image of T in Y_W is proper over W, then the morphism $T \to W$ is proper as a composition of proper morphisms (here we endow the closed image of T in Y_W with its reduced induced scheme structure to turn the question into one about morphisms of schemes), see Morphisms, Lemma 41.4. $\square$

Remark 3.14. The isomorphisms between functors constructed above satisfy the following two properties:

(1) Let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to T$ be composable morphisms of schemes which are separated and locally of finite type. Then the diagram

$$\begin{tikzcd} (h \circ g \circ f)_! \arrow[r] \arrow[d] & (h \circ g)_! \circ f_! \arrow[d] \\ h_! \circ (g \circ f)_! \arrow[r] & h_! \circ g_! \circ f_! \end{tikzcd}$$

commutes where the arrows are those of Lemma 3.13.

(2) Suppose that we have a diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "c"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "b"'] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "a"] & Z \end{tikzcd}$$

with both squares cartesian and f and g separated and locally of finite type. Then the diagram

$$\begin{tikzcd} a^{-1} \circ (g \circ f)_! \arrow[rr] \arrow[d] & & (g' \circ f')_! \circ c^{-1} \arrow[d] \\ a^{-1} \circ g_! \circ f_! \arrow[r] & g'_! \circ b^{-1} \circ f_! \arrow[r] & g'_! \circ f'_! \circ c^{-1} \end{tikzcd}$$

commutes where the horizontal arrows are those of Lemma 3.12 the arrows are those of Lemma 3.13.

Part (1) holds true because we have a similar commutative diagram for pushforwards. Part (2) holds by the very general compatibility of base change maps for pushforwards (Sites, Remark 45.3) and the fact that the isomorphisms in Lemmas 3.12 and 3.13 are constructed using the corresponding maps fo pushforwards.

Lemma 3.15. *Let $f : X \to Y$ be morphism of schemes which is separated and locally of finite type. Let $X = \bigcup_{i \in I} X_i$ be an open covering such that for all $i, j \in I$ there exists a k with $X_i \cup X_j \subset X_k$. Denote $f_i : X_i \to Y$ the restriction of f. Then*

$$f_!\mathcal{F} = \mathrm{colim}_{i \in I}\, f_{i,!}(\mathcal{F}|_{X_i})$$

functorially in $\mathcal{F} \in Ab(X_{étale})$ where the transition maps are the ones constructed in Remark 3.5.

Proof. It suffices to show that the canonical map from right to left is a bijection when evaluated on a quasi-compact object V of $Y_{étale}$. Observe that the colimit on the right hand side is directed and has injective transition maps. Thus we can use Sites, Lemma 17.7 to evaluate the colimit. Hence, the statement comes down to the observation that a closed subset $Z \subset X_V$ proper over V is quasi-compact and hence is contained in $X_{i,V}$ for some i. $\square$

Lemma 3.16. *Let $f : X \to Y$ be a morphism of schemes which is separated and locally of finite type. Then functor $f_!$ commutes with direct sums.*

Proof. Let $\mathcal{F} = \bigoplus \mathcal{F}_i$. To show that the map $\bigoplus f_!\mathcal{F}_i \to f_!\mathcal{F}$ is an isomorphism, it suffices to show that these sheaves have the same sections over a quasi-compact object V of $Y_{étale}$. Replacing Y by V it suffices to show $H^0(Y, f_!\mathcal{F}) \subset H^0(X, \mathcal{F})$ is equal to $\bigoplus H^0(Y, f_!\mathcal{F}_i) \subset \bigoplus H^0(X, \mathcal{F}_i) \subset H^0(X, \bigoplus \mathcal{F}_i)$. In this case, by writing X as the union of its quasi-compact opens and using Lemma 3.15 we reduce to the case where X is quasi-compact as well. Then $H^0(X, \mathcal{F}) = \bigoplus H^0(X, \mathcal{F}_i)$ by Étale Cohomology, Theorem 51.3. Looking at supports of sections the reader easily concludes. $\square$

Lemma 3.17. *Let $f : X \to Y$ be a morphism of schemes which is separated and locally quasi-finite. Then*

(1) *for $\mathcal{F}$ in $Ab(X_{étale})$ and a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have*

$$(f_!\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}}$$

functorially in $\mathcal{F}$, and

(2) *the functor $f_!$ is exact.*

Proof. The functor $f_!$ is left exact by construction. Right exactness may be checked on stalks (Étale Cohomology, Theorem 29.10). Thus it suffices to prove part (1).

Let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. The scheme $X_{\overline{y}}$ has a discrete underlying topological space (Morphisms, Lemma 20.8) and all the residue fields at the points are equal to k (as finite extensions of k). Hence $\{\overline{x} : \mathrm{Spec}(k) \to X : f(\overline{x}) = \overline{y}\}$ is equal to the set of points of $X_{\overline{y}}$. Thus the computation of the stalk follows from the more general Lemma 3.11. $\square$

4. Sections with finite support

In this section we extend the construction of Section 3 to not necessarily separated locally quasi-finite morphisms.

Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Given V in $Y_{\acute{e}tale}$ denote $X_V = X \times_Y V$ the base change. We are going to consider the group of finite formal sums

$$s = \sum\nolimits_{i=1,\ldots,n} (Z_i, s_i) \tag{4.0.1}$$

where $Z_i \subset X_V$ is a locally closed subscheme such that the morphism $Z_i \to V$ is finite[2] and where $s_i \in H_{Z_i}(\mathcal{F})$. Here, as in Section 2, we set

$$H_{Z_i}(\mathcal{F}) = \{s_i \in \mathcal{F}(U_i) \mid \mathrm{Supp}(s_i) \subset Z_i\}$$

where $U_i \subset X_V$ is an open subscheme containing Z_i as a closed subscheme. We are going to consider these formal sums modulo the following relations

(1) $(Z, s) + (Z, s') = (Z, s + s')$,
(2) $(Z, s) = (Z', s)$ if $Z \subset Z'$.

Observe that the second relation makes sense: since $Z \to V$ is finite and $Z' \to V$ is separated, the inclusion $Z \to Z'$ is closed and we can use the map discussed in (1).

Let us denote $f_{p!}\mathcal{F}(V)$ the quotient of the abelian group of formal sums (4.0.1) by these relations. The first relation tells us that $f_{p!}\mathcal{F}(V)$ is a quotient of the direct sum of the abelian groups $H_Z(\mathcal{F})$ over all locally closed subschemes $Z \subset X_V$ finite over V. The second relation tells us that we are really taking the colimit

$$f_{p!}\mathcal{F}(V) = \mathrm{colim}_Z\, H_Z(\mathcal{F}) \tag{4.0.2}$$

This formula will be a convenient abstract way to think about our construction.

Next, we observe that there is a natural way to turn this construction into a presheaf $f_{p!}\mathcal{F}$ of abelian groups on $Y_{\acute{e}tale}$. Namely, given $V' \to V$ in $Y_{\acute{e}tale}$ we obtain the base change morphism $X_{V'} \to X_V$. If $Z \subset X_V$ is a locally closed subscheme finite over V, then the scheme theoretic inverse image $Z' \subset X_{V'}$ is finite over V'. Moreover, if $U \subset X_V$ is an open such that Z is closed in U, then the inverse image $U' \subset X_{V'}$ is an open such that Z' is closed in U'. Hence the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ of $\mathcal{F}$ sends $H_Z(\mathcal{F})$ into $H_{Z'}(\mathcal{F})$; this is a special case of the functoriality discussed in (2) above. Clearly, these maps are compatible with inclusions $Z_1 \subset Z_2$ of such locally closed subschemes of X_V and we obtain a map

$$f_{p!}\mathcal{F}(V) = \mathrm{colim}_Z\, H_Z(\mathcal{F}) \longrightarrow \mathrm{colim}_{Z'}\, H_{Z'}(\mathcal{F}) = f_{p!}\mathcal{F}(V')$$

These maps indeed turn $f_{p!}\mathcal{F}$ into a presheaf of abelian groups on $Y_{\acute{e}tale}$. We omit the details.

A final observation is that the construction of $f_{p!}\mathcal{F}$ is functorial in $\mathcal{F}$ in $Ab(X_{\acute{e}tale})$. We conclude that given a locally quasi-finite morphism $f : X \to Y$ we have constructed a functor

$$f_{p!} : Ab(X_{\acute{e}tale}) \longrightarrow PAb(Y_{\acute{e}tale})$$

from the category of abelian sheaves on $X_{\acute{e}tale}$ to the category of abelian presheaves on $Y_{\acute{e}tale}$. Before we define $f_!$ as the sheafification of this functor, let us check that

[2]Since f is locally quasi-finite, the morphism $Z_i \to V$ is finite if and only if it is proper.

it agrees with the construction in Section 3 and with the construction in Étale Cohomology, Section 70 when both apply.

Lemma 4.1. *Let $f : X \to Y$ be a separated and locally quasi-finite morphism of schemes. Functorially in $\mathcal{F} \in Ab(X_{étale})$ there is a canonical isomorphism(!)*

$$f_{p!}\mathcal{F} \longrightarrow f_!\mathcal{F}$$

of abelian presheaves which identifies the sheaf $f_!\mathcal{F}$ of Definition 3.3 with the presheaf $f_{p!}\mathcal{F}$ constructed above.

Proof. Let V be an object of $Y_{étale}$. If $Z \subset X_V$ is locally closed and finite over V, then, since f is separated, we see that the morphism $Z \to X_V$ is a closed immersion. Moreover, if Z_i, $i = 1, \ldots, n$ are closed subschemes of X_V finite over V, then $Z_1 \cup \ldots \cup Z_n$ (scheme theoretic union) is a closed subscheme finite over V. Hence in this case the colimit (4.0.2) defining $f_{p!}\mathcal{F}(V)$ is directed and we find that $f_{!p}\mathcal{F}(V)$ is simply equal to the set of sections of $\mathcal{F}(X_V)$ whose support is finite over V. Since any closed subset of X_V which is proper over V is actually finite over V (as f is locally quasi-finite) we conclude that this is equal to $f_!\mathcal{F}(V)$ by its very definition. □

Lemma 4.2. *Let $f : X \to Y$ be a morphism of schemes which is locally quasi-finite. Let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. Functorially in $\mathcal{F}$ in $Ab(X_{étale})$ we have*

$$(f_{p!}\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

Proof. Recall that the stalk at $\overline{y}$ of a presheaf is defined by the usual colimit over étale neighbourhoods $(V, \overline{v})$ of $\overline{y}$, see Étale Cohomology, Definition 29.6. Accordingly suppose $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (4.0.1) is an element of $f_{p!}\mathcal{F}(V)$ where $(V, \overline{v})$ is an étale neighbourhood of $\overline{y}$. Then since

$$X_{\overline{y}} = (X_V)_{\overline{v}} \supset Z_{i,\overline{v}}$$

and since s_i is a section of $\mathcal{F}$ on an open neighbourhood of Z_i in X_V we can send s to

$$\sum\nolimits_{i=1,\ldots,n} \sum\nolimits_{\overline{x} \in Z_{i,\overline{v}}} (\text{class of } s_i \text{ in } \mathcal{F}_{\overline{x}}) \quad \in \quad \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

We omit the verification that this is compatible with restriction maps and that the relations (1) $(Z, s) + (Z, s') - (Z, s + s')$ and (2) $(Z, s) - (Z', s)$ if $Z \subset Z'$ are sent to zero. Thus we obtain a map

$$(f_{p!}\mathcal{F})_{\overline{y}} \longrightarrow \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

Let us prove this arrow is surjective. For this it suffices to pick an $\overline{x}$ with $f(\overline{x}) = \overline{y}$ and prove that an element s in the summand $\mathcal{F}_{\overline{x}}$ is in the image. Let s correspond to the element $s \in \mathcal{F}(U)$ where $(U, \overline{u})$ is an étale neighbourhood of $\overline{x}$. Since f is locally quasi-finite, the morphism $U \to Y$ is locally quasi-finite too. By More on Morphisms, Lemma 41.3 we can find an étale neighbourhood $(V, \overline{v})$ of $\overline{y}$, an open subscheme

$$W \subset U \times_Y V,$$

and a geometric point $\overline{w}$ mapping to $\overline{u}$ and $\overline{v}$ such that $W \to V$ is finite and $\overline{w}$ is the only geometric point of W mapping to $\overline{v}$. (We omit the translation between the language of geometric points we are currently using and the language of points and residue field extensions used in the statement of the lemma.) Observe that $W \to X_V = X \times_Y V$ is étale. Choose an affine open neighbourhood $W' \subset X_V$ of the image $\overline{w}'$ of $\overline{w}$. Since $\overline{w}$ is the only point of W over $\overline{v}$ and since $W \to V$ is closed, after replacing V by an open neighbourhood of $\overline{v}$, we may assume $W \to X_V$ maps into W'. Then $W \to W'$ is finite and étale and there is a unique geometric point $\overline{w}$ of W lying over $\overline{w}'$. It follows that $W \to W'$ is an open immersion over an open neighbourhood of $\overline{w}'$ in W', see Étale Morphisms, Lemma 14.2. Shrinking V and W' we may assume $W \to W'$ is an isomorphism. Thus s may be viewed as a section s' of $\mathcal{F}$ over the open subscheme $W' \subset X_V$ which is finite over V. Hence by definition (W', s') defines an element of $j_{p!}\mathcal{F}(V)$ which maps to s as desired.

Let us prove the arrow is injective. To do this, let $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (4.0.1) be an element of $f_{p!}\mathcal{F}(V)$ where $(V, \overline{v})$ is an étale neighbourhood of $\overline{y}$. Assume s maps to zero under the map constructed above. First, after replacing $(V, \overline{v})$ by an étale neighbourhood of itself, we may assume there exist decompositions $Z_i = Z_{i,1} \amalg \ldots \amalg Z_{i,m_i}$ into open and closed subschemes such that each $Z_{i,j}$ has exactly one geometric point over $\overline{v}$. Say under the obvious direct sum decomposition

$$H_{Z_i}(\mathcal{F}) = \bigoplus H_{Z_{i,j}}(\mathcal{F})$$

the element s_i corresponds to $\sum s_{i,j}$. We may use relations (1) and (2) to replace s by $\sum_{i=1,\ldots,n}\sum_{j=1,\ldots,m_i}(Z_{i,j}, s_{i,j})$. In other words, we may assume Z_i has a unique geometric point lying over $\overline{v}$. Let $\overline{x}_1, \ldots, \overline{x}_m$ be the geometric points of X over $\overline{y}$ corresponding to the geometric points of our Z_i over $\overline{v}$; note that for one $j \in \{1, \ldots, m\}$ there may be multiple indices i for which $\overline{x}_j$ corresponds to a point of Z_i. By More on Morphisms, Lemma 41.3 applied to both $X_V \to V$ after replacing $(V, \overline{v})$ by an étale neighbourhood of itself we may assume there exist open subschemes

$$W_j \subset X \times_Y V, \quad j = 1, \ldots, m$$

and a geometric point $\overline{w}_j$ of W_j mapping to $\overline{x}_j$ and $\overline{v}$ such that $W_j \to V$ is finite and $\overline{w}_j$ is the only geometric point of W_j mapping to $\overline{v}$. After shrinking V we may assume $Z_i \subset W_j$ for some j and we have the map $H_{Z_i}(\mathcal{F}) \to H_{W_j}(\mathcal{F})$. Thus by the relation (2) we see that our element is equivalent to an element of the form

$$\sum\nolimits_{j=1,\ldots,m}(W_j, t_j)$$

for some $t_j \in H_{W_j}(\mathcal{F})$. Clearly, this element is mapped simply to the class of t_j in the summand $\mathcal{F}_{\overline{x}_j}$. Since s maps to zero, we find that t_j maps to zero in $\mathcal{F}_{\overline{x}_j}$. This implies that t_j restricts to zero on an open neighbourhood of $\overline{w}_j$ in W_j, see Étale Cohomology, Lemma 31.2. Shrinking V once more we obtain $t_j = 0$ for all j as desired. □

Lemma 4.3. *Let $f = j : U \to X$ be an étale of schemes. Denote $j_{p!}$ the construction of Étale Cohomology, Equation (70.1.1) and denote $f_{p!}$ the construction above. Functorially in $\mathcal{F} \in Ab(X_{étale})$ there is a canonical map*

$$j_{p!}\mathcal{F} \longrightarrow f_{p!}\mathcal{F}$$

of abelian presheaves which identifies the sheaf $j_!\mathcal{F} = (j_{p!}\mathcal{F})^\#$ of Étale Cohomology, Definition 70.1 with $(f_{p!}\mathcal{F})^\#$.

Proof. Please read the proof of Étale Cohomology, Lemma 70.6 before reading the proof of this lemma. Let V be an object of $X_{étale}$. Recall that

$$j_{p!}\mathcal{F}(V) = \bigoplus\nolimits_{\varphi : V \to U} \mathcal{F}(V \xrightarrow{\varphi} U)$$

Given φ we obtain an open subscheme $Z_\varphi \subset U_V = U \times_X V$, namely, the image of the graph of φ. Via φ we obtain an isomorphism $V \to Z_\varphi$ over U and we can think of an element

$$s_\varphi \in \mathcal{F}(V \xrightarrow{\varphi} U) = \mathcal{F}(Z_\varphi) = H_{Z_\varphi}(\mathcal{F})$$

as a section of $\mathcal{F}$ over Z_φ. Since $Z_\varphi \subset U_V$ is open, we actually have $H_{Z_\varphi}(\mathcal{F}) = \mathcal{F}(Z_\varphi)$ and we can think of s_φ as an element of $H_{Z_\varphi}(\mathcal{F})$. Having said this, our map $j_{p!}\mathcal{F} \to f_{p!}\mathcal{F}$ is defined by the rule

$$\sum\nolimits_{i=1,\ldots,n} s_{\varphi_i} \longmapsto \sum\nolimits_{i=1,\ldots,n} (Z_{\varphi_i}, s_{\varphi_i})$$

with right hand side a sum as in (4.0.1). We omit the verification that this is compatible with restriction mappings and functorial in $\mathcal{F}$.

To finish the proof, we claim that given a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ there is a commutative diagram

$$\begin{tikzcd}
(j_{p!}\mathcal{F})_{\overline{y}} \arrow[r] \arrow[d] & \bigoplus_{j(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}} \arrow[d, equal] \\
(f_{p!}\mathcal{F})_{\overline{y}} \arrow[r] & \bigoplus_{f(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}}
\end{tikzcd}$$

where the top horizontal arrow is constructed in the proof of Étale Cohomology, Proposition 70.3, the bottom horizontal arrow is constructed in the proof of Lemma 4.2, the right vertical arrow is the obvious equality, and the left veritical arrow is the map defined in the previous paragraph on stalks. The claim follows in a straightforward manner from the explicit description of all of the arrows involved here and in the references given. Since the horizontal arrows are isomorphisms we conclude so is the left vertical arrow. Hence we find that our map induces an isomorphism on sheafifications by Étale Cohomology, Theorem 29.10. □

Definition 4.4. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. We define the *direct image with compact support* to be the functor

$$f_! : Ab(X_{étale}) \longrightarrow Ab(Y_{étale})$$

defined by the formula $f_!\mathcal{F} = (f_{p!}\mathcal{F})^\#$, i.e., $f_!\mathcal{F}$ is the sheafification of the presheaf $f_{p!}\mathcal{F}$ constructed above.

By Lemma 4.1 this does not conflict with Definition 3.3 (when both definitions apply) and by Lemma 4.3 this does not conflict with Étale Cohomology, Definition 70.1 (when both definitions apply).

Lemma 4.5. *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Then*

(1) *for $\mathcal{F}$ in $Ab(X_{étale})$ and a geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have*

$$(f_!\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x}) = \overline{y}} \mathcal{F}_{\overline{x}}$$

functorially in $\mathcal{F}$, and

(2) *the functor $f_! : Ab(X_{étale}) \to Ab(Y_{étale})$ is exact and commutes with direct sums.*

Proof. The formula for the stalks is immediate (and in fact equivalent) to Lemma 4.2. The exactness of the functor follows immediately from this and the fact that exactness may be checked on stalks, see Étale Cohomology, Theorem 29.10. $\square$

Remark 4.6 (Covariance with respect to open embeddings)**.** Let $f : X \to Y$ be locally quasi-finite morphism of schemes. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $X' \subset X$ be an open subscheme and denote $f' : X' \to Y$ the restriction of f. We claim there is a canonical map

$$f'_!(\mathcal{F}|_{X'}) \longrightarrow f_!\mathcal{F}$$

Namely, this map will be the sheafification of a canonical map

$$f'_{p!}(\mathcal{F}|_{X'}) \to f_{p!}\mathcal{F}$$

constructed as follows. Let $V \in Y_{étale}$ and consider a section $s' = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$ as in (4.0.1) defining an element of $f'_{p!}(\mathcal{F}|_{X'})(V)$. Then $Z'_i \subset X'_V$ may also be viewed as a locally closed subscheme of X_V and we have $H_{Z'_i}(\mathcal{F}|_{X'}) = H_{Z'_i}(\mathcal{F})$. We will map s' to the exact same sum $s = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$ but now viewed as an element of $f_{p!}\mathcal{F}(V)$. We omit the verification that this construction is compatible with restriction mappings and functorial in $\mathcal{F}$. This construction has the following properties:

(1) The maps $f'_{p!}\mathcal{F}' \to f_{p!}\mathcal{F}$ and $f'_!\mathcal{F}' \to f_!\mathcal{F}$ are compatible with the description of stalks given in Lemmas 4.2 and 4.5.
(2) If f is separated, then the map $f'_{p!}\mathcal{F}' \to f_{p!}\mathcal{F}$ is the same as the map constructed in Remark 3.5 via the isomorphism in Lemma 4.1.
(3) If $X'' \subset X'$ is another open, then the composition of $f''_{p!}(\mathcal{F}|_{X''}) \to f'_{p!}(\mathcal{F}|_{X'}) \to f_{p!}\mathcal{F}$ is the map $f''_{p!}(\mathcal{F}|_{X''}) \to f_{p!}\mathcal{F}$ for the inclusion $X'' \subset X$. Sheafifying we conclude the same holds true for $f''_!(\mathcal{F}|_{X''}) \to f'_!(\mathcal{F}|_{X'}) \to f_!\mathcal{F}$.
(4) The map $f'_!\mathcal{F}' \to f_!\mathcal{F}$ is injective because we can check this on stalks.

All of these statements are easily proven by representing elements as finite sums as above and considering what happens to these elements.

Lemma 4.7. *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $X = \bigcup_{i \in I} X_i$ be an open covering. Then there exists an exact complex*

$$\ldots \to \bigoplus\nolimits_{i_0,i_1,i_2} f_{i_0i_1i_2,!}\mathcal{F}|_{X_{i_0i_1i_2}} \to \bigoplus\nolimits_{i_0,i_1} f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \to \bigoplus\nolimits_{i_0} f_{i_0,!}\mathcal{F}|_{X_{i_0}} \to f_!\mathcal{F} \to 0$$

functorial in $\mathcal{F} \in Ab(X_{étale})$, see proof for details.

Proof. Here as usual we set $X_{i_0\ldots i_p} = X_{i_0} \cap \ldots \cap X_{i_p}$ and we denote $f_{i_0\ldots i_p}$ the restriction of f to $X_{i_0\ldots i_p}$. The maps in the complex are the maps constructed in Remark 4.6 with sign rules as in the Čech complex. Exactness follows easily from the description of stalks in Lemma 4.5. Details omitted. $\square$

Remark 4.8 (Alternative construction)**.** Lemma 4.7 gives an alternative construction of the functor $f_!$ for locally quasi-finite morphisms f. Namely, given a locally quasi-finite morphism $f : X \to Y$ of schemes we can choose an open covering $X = \bigcup_{i \in I} X_i$ such that each $f_i : X_i \to Y$ is separated. For example choose an affine open covering of X. Then we can define $f_!\mathcal{F}$ as the cokernel of the penultimate map of the complex of the lemma, i.e.,

$$f_!\mathcal{F} = \text{Coker}\left(\bigoplus\nolimits_{i_0,i_1} f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \to \bigoplus\nolimits_{i_0} f_{i_0,!}\mathcal{F}|_{X_{i_0}}\right)$$

where we can use the construction of $f_{i_0,!}$ and $f_{i_0i_1,!}$ in Section 3 because the morphisms f_{i_0} and $f_{i_0i_1}$ are separated. One can then compute the stalks of $f_!$ (using the separated case, namely Lemma 3.17) and obtain the result of Lemma 4.5. Having done so all the other results of this section can be deduced from this as well.

Remark 4.9. Let $g : Y' \to Y$ be a morphism of schemes. For an abelian presheaf $\mathcal{G}'$ on $Y'_{\acute{e}tale}$ let us denote $g_*\mathcal{G}'$ the presheaf $V \mapsto \mathcal{G}'(Y' \times_Y V)$. If $\alpha : \mathcal{G} \to g_*\mathcal{G}'$ is a map of abelian presheaves on $Y_{\acute{e}tale}$, then there is a unique map $\alpha^\# : \mathcal{G}^\# \to g_*((\mathcal{G}')^\#)$ of abelian sheaves on $Y_{\acute{e}tale}$ such that the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\alpha"'] \arrow[d] & g_*\mathcal{G}' \arrow[d] \\ \mathcal{G}^\# \arrow[r, "\alpha^\#"] & g_*((\mathcal{G}')^\#) \end{tikzcd}$$

is commutative where the vertical maps come from the canonical maps $\mathcal{G} \to \mathcal{G}^\#$ and $\mathcal{G}' \to (\mathcal{G}')^\#$. If $\alpha' : g^{-1}\mathcal{G}^\# \to (\mathcal{G}')^\#$ is the map adjoint to $\alpha^\#$, then for a geometric point $\overline{y}' : \text{Spec}(k) \to Y'$ with image $\overline{y} = g \circ \overline{y}'$ in Y, the map

$$\alpha'_{\overline{y}'} : \mathcal{G}_{\overline{y}} = (\mathcal{G}^\#)_{\overline{y}} = (g^{-1}\mathcal{G}^\#)_{\overline{y}'} \longrightarrow (\mathcal{G}')^\#_{\overline{y}'} = \mathcal{G}'_{\overline{y}'}$$

is given by mapping the class in the stalk of a section s of $\mathcal{G}$ over an étale neighbourhood $(V, \overline{v})$ to the class of the section $\alpha(s)$ in $g_*\mathcal{G}'(V) = \mathcal{G}'(Y' \times_Y V)$ over the étale neighbourhood $(Y' \times_Y V, (\overline{y}', \overline{v}))$ in the stalk of $\mathcal{G}'$ at $\overline{y}'$.

Lemma 4.10. *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes with f locally quasi-finite. There is an isomorphism $g^{-1}f_!\mathcal{F} \to f'_!(g')^{-1}\mathcal{F}$ functorial for $\mathcal{F}$ in $Ab(X_{\acute{e}tale})$ which is compatible with the descriptions of stalks given in Lemma 4.5 (see proof for the precise statement).

Proof. With conventions as in Remark 4.9 we will explicitly construct a map

$$c : f_{p!}\mathcal{F} \longrightarrow g_*f'_{p!}(g')^{-1}\mathcal{F}$$

of abelian presheaves on $Y_{\acute{e}tale}$. By the discussion in Remark 4.9 this will determine a canonical map $g^{-1}f_!\mathcal{F} \to f'_!(g')^{-1}\mathcal{F}$. Finally, we will show this map induces isomorphisms on stalks and conclude by Étale Cohomology, Theorem 29.10.

Construction of the map c. Let $V \in Y_{\acute{e}tale}$ and consider a section $s = \sum_{i=1,\ldots,n}(Z_i, s_i)$ as in (4.0.1) defining an element of $f_{p!}\mathcal{F}(V)$. The value of $g_*f'_{p!}(g')^{-1}\mathcal{F}$ at V is $f'_{p!}(g')^{-1}\mathcal{F}(V')$ where $V' = V \times_Y Y'$. Denote $Z'_i \subset X'_{V'}$ the base change of Z_i to V'. By (2) there is a pullback map $H_{Z_i}(\mathcal{F}) \to H_{Z'_i}((g')^{-1}\mathcal{F})$. Denoting $s'_i \in H_{Z'_i}((g')^{-1}\mathcal{F})$ the image of s_i under pullback, we set $c(s) = \sum_{i=1,\ldots,n}(Z'_i, s'_i)$

as in (4.0.1) defining an element of $f'_{p!}(g')^{-1}\mathcal{F}(V')$. We omit the verification that this construction is compatible the relations (1) and (2) and compatible with restriction mappings. The construction is clearly functorial in $\mathcal{F}$.

Let $\overline{y}' : \mathrm{Spec}(k) \to Y'$ be a geometric point with image $\overline{y} = g \circ \overline{y}'$ in Y. Observe that $X'_{\overline{y}'} = X_{\overline{y}}$ by transitivity of fibre products. Hence g' produces a bijection $\{f'(\overline{x}') = \overline{y}'\} \to \{f(\overline{x}) = \overline{y}\}$ and if $\overline{x}'$ maps to $\overline{x}$, then $((g')^{-1}\mathcal{F})_{\overline{x}'} = \mathcal{F}_{\overline{x}}$ by Étale Cohomology, Lemma 36.2. Now we claim that the diagram

$$\begin{tikzcd} (g^{-1}f_!\mathcal{F})_{\overline{y}'} \arrow[r, equal] \arrow[d] & (f_!\mathcal{F})_{\overline{y}} \arrow[r] \arrow[dl] & \bigoplus_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}} \arrow[d] \\ (f'_!(g')^{-1}\mathcal{F})_{\overline{y}'} \arrow[rr] & & \bigoplus_{f'(\overline{x}')=\overline{y}'} (g')^{-1}\mathcal{F}_{\overline{x}'} \end{tikzcd}$$

commutes where the horizontal arrows are given in the proof of Lemma 4.2 and where the right vertical arrow is an equality by what we just said above. The southwest arrow is described in Remark 4.9 as the pullback map, i.e., simply given by our construction c above. Then the simple description of the image of a sum $\sum(Z_i, z_i)$ in the stalk at $\overline{x}$ given in the proof of Lemma 4.2 immediately shows the diagram commutes. This finishes the proof of the lemma. □

Lemma 4.11. *Let $f' : X \to Y'$ and $g : Y' \to Y$ be composable morphisms of schemes with f' and $f = g \circ f'$ locally quasi-finite and g separated and locally of finite type. Then there is a canonical isomorphism of functors $g_! \circ f'_! = f_!$. This isomorphism is compatible with*

- (a) *covariance with respect to open embeddings as in Remarks 3.5 and 4.6,*
- (b) *the base change isomorphisms of Lemmas 4.10 and 3.12, and*
- (c) *equal to the isomorphism of Lemma 3.13 via the identifications of Lemma 4.1 in case f' is separated.*

Proof. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. With conventions as in Remark 4.9 we will explicitly construct a map

$$c : f_{p!}\mathcal{F} \longrightarrow g_* f'_{p!}\mathcal{F}$$

of abelian presheaves on $Y_{\acute{e}tale}$. By the discussion in Remark 4.9 this will determine a canonical map $c^\# : f_!\mathcal{F} \to g_* f'_!\mathcal{F}$. We will show that $c^\#$ has image contained in the subsheaf $g_! f'_!\mathcal{F}$, thereby obtaining a map $c' : f_!\mathcal{F} \to g_! f'_!\mathcal{F}$. Next, we will prove (a), (b), and (c) that. Finally, part (b) will allow us to show that c' is an isomorphism.

Construction of the map c. Let $V \in Y_{\acute{e}tale}$ and let $s = \sum(Z_i, s_i)$ be a sum as in (4.0.1) defining an element of $f_{p!}\mathcal{F}(V)$. Recall that $Z_i \subset X_V = X \times_Y V$ is a locally closed subscheme finite over V. Setting $V' = Y' \times_Y V$ we get $X_{V'} = X \times_{Y'} V' = X_V$. Hence $Z_i \subset X_{V'}$ is locally closed and Z_i is finite over V' because g is separated (Morphisms, Lemma 44.14). Hence we may set $c(s) = \sum(Z_i, s_i)$ but now viewed as an element of $f'_{p!}\mathcal{F}(V') = (g_* f'_{p!}\mathcal{F})(V)$. The construction is clearly compatible with relations (1) and (2) and compatible with restriction mappings and hence we obtain the map c.

Observe that in the discussion above our section $c(s) = \sum(Z_i, s_i)$ of $f'_!\mathcal{F}$ over V' restricts to zero on $V' \setminus \mathrm{Im}(\coprod Z_i \to V')$. Since $\mathrm{Im}(\coprod Z_i \to V')$ is proper over V

(for example by Morphisms, Lemma 41.10) we conclude that $c(s)$ defines a section of $g_! f'_! \mathcal{F} \subset g_* f'_! \mathcal{F}$ over V. Since every local section of $f_! \mathcal{F}$ locally comes from a local section of $f_{p!}\mathcal{F}$ we conclude that the image of $c^\#$ is contained in $g_! f'_! \mathcal{F}$. Thus we obtain an induced map $c' : f_! \mathcal{F} \to g_! f'_! \mathcal{F}$ factoring $c^\#$ as predicted in the first paragraph of the proof.

Proof of (a). Let $Y'_1 \subset Y'$ be an open subscheme and set $X_1 = (f')^{-1}(W')$. We obtain a diagram

$$\begin{tikzcd} X_1 \arrow[r, "a"'] \arrow[d, "f'_1"'] \arrow[dd, bend right, "f_1"'] & X \arrow[d, "f'"] \arrow[dd, bend left, "f"] \\ Y'_1 \arrow[r, "b'"'] \arrow[d, "g_1"'] & Y' \arrow[d, "g"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

where the horizontal arrows are open immersions. Then our claim is that the diagram

$$\begin{tikzcd} f_{1,!}\mathcal{F}|_{X_1} \arrow[r, "c'_1"'] \arrow[dd] & g_{1,!}f'_{1,!}\mathcal{F}|_{X_1} \arrow[d, equal] & \\ & g_{1,!}(f'_!\mathcal{F})|_{Y'_1} \arrow[d] & \\ f_!\mathcal{F} \arrow[r, "c'"] & g_! f'_! \mathcal{F} \arrow[r] & g_* f'_! \mathcal{F} \end{tikzcd}$$

commutes where the left vertical arrow is Remark 4.6 and the right vertical arrow is Remark 3.5. The equality sign in the diagram comes about because f'_1 is the restriction of f' to Y'_1 and our construction of $f'_!$ is local on the base. Finally, to prove the commutativity we choose an object V of $Y_{étale}$ and a formal sum $s_1 = \sum (Z_{1,i}, s_{1,i})$ as in (4.0.1) defining an element of $f_{1,p!}\mathcal{F}|_{X_1}(V)$. Recall this means $Z_{1,i} \subset X_1 \times_Y V$ is locally closed finite over V and $s_{1,i} \in H_{Z_{1,i}}(\mathcal{F})$. Then we chase this section across the maps involved, but we only need to show we end up with the same element of $g_* f'_! \mathcal{F}(V) = f'_! \mathcal{F}(Y' \times_Y V)$. Going around both sides of the diagram the reader immediately sees we end up with the element $\sum (Z_{1,i}, s_{1,i})$ where now $Z_{1,i}$ is viewed as a locally closed subscheme of $X \times_{Y'} (Y' \times_Y V) = X \times_Y V$ finite over $Y' \times_Y V$.

Proof of (b). Let $b : Y_1 \to Y$ be a morphism of schemes. Let us form the commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "a"'] \arrow[d, "f'_1"'] \arrow[dd, bend right, "f_1"'] & X \arrow[d, "f'"] \arrow[dd, bend left, "f"] \\ Y'_1 \arrow[r, "b'"'] \arrow[d, "g_1"'] & Y' \arrow[d, "g"] \\ Y_1 \arrow[r, "b"] & Y \end{tikzcd}$$

with cartesian squares. We claim that our construction is compatible with the base change maps of Lemmas 4.10 and 3.12, i.e., that the top rectangle of the diagram

$$\begin{tikzcd}
b^{-1}f_!\mathcal{F} \arrow[rr] \arrow[d, "b^{-1}c'"'] & & f_{1,!}a^{-1}\mathcal{F} \arrow[d, "c'_1"] \\
b^{-1}g_!f'_!\mathcal{F} \arrow[r] \arrow[d] & g_{1,!}(b')^{-1}f'_!\mathcal{F} \arrow[r] \arrow[d] & g_{1,!}f'_{1,!}a^{-1}\mathcal{F} \arrow[d] \\
b^{-1}g_*f'_!\mathcal{F} \arrow[r] & g_{1,*}(b')^{-1}f'_!\mathcal{F} \arrow[r] & g_{1,*}f'_{1,!}a^{-1}\mathcal{F}
\end{tikzcd}$$

commutes. The verification of this is completely routine and we urge the reader to skip it. Since the arrows going from the middle row down to the bottom row are injective, it suffices to show that the outer diagram commutes. To show this it suffices to take a local section of $b^{-1}f_!\mathcal{F}$ and show we end up with the same local section of $g_{1,*}f'_{1,!}a^{-1}\mathcal{F}$ going around either way. However, in fact it suffices to check this for local sections which are of the the pullback by b of a section $s = \sum(Z_i, s_i)$ of $f_{p!}\mathcal{F}(V)$ as above (since such pullbacks generate the abelian sheaf $b^{-1}f_!\mathcal{F}$). Denote V_1, V'_1, and $Z_{1,i}$ the base change of V, $V' = Y' \times_Y V$, Z_i by $Y_1 \to Y$. Recall that Z_i is a locally closed subscheme of $X_V = X_{V'}$ and hence $Z_{1,i}$ is a locally closed subscheme of $(X_1)_{V_1} = (X_1)_{V'_1}$. Then $b^{-1}c'$ sends the pullback of s to the pullback of the local section $c(s)\sum(Z_i, s_i)$ viewed as an element of $f'_{p!}\mathcal{F}(V') = (g_*f'_{p!}\mathcal{F})(V)$. The composition of the bottom two base change maps simply maps this to $\sum(Z_{i,1}, s_{1,i})$ viewed as an element of $f'_{1,p!}a^{-1}\mathcal{F}(V'_1) = g_{1,*}f'_{1,p!}a^{-1}\mathcal{F}(V_1)$. On the other hand, the base change map at the top of the diagram sends the pullback of s to $\sum(Z_{1,i}, s_{1,i})$ viewed as an element of $f_{1,!}a^{-1}\mathcal{F}(V_1)$. Then finally c'_1 by its very construction does indeed map this to $\sum(Z_{i,1}, s_{1,i})$ viewed as an element of $f'_{1,p!}a^{-1}\mathcal{F}(V'_1) = g_{1,*}f'_{1,p!}a^{-1}\mathcal{F}(V_1)$ and the commutativity has been verified.

Proof of (c). This follows from comparing the definitions for both maps; we omit the details.

To finish the proof it suffices to show that the pullback of c' via any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ is an isomorphism. Namely, pulling back by $\overline{y}$ is the same thing as taking stalks and $\overline{y}$ (Étale Cohomology, Remark 56.6) and hence we can invoke Étale Cohomology, Theorem 29.10. By the compatibility (b) just shown, we conclude that we may assume Y is the spectrum of k and we have to show that c' is an isomorphism. To do this it suffices to show that the induced map

$$\bigoplus\nolimits_{x \in X} \mathcal{F}_x = H^0(Y, f_!\mathcal{F}) \longrightarrow H^0(Y, g_!f'_!\mathcal{F}) = H^0_c(Y', f'_!\mathcal{F})$$

is an isomorphism. The equalities hold by Lemmas 4.5 and 3.11. Recall that X is a disjoint union of spectra of Artinian local rings with residue field k, see Varieties, Lemma 20.2. Since the left and right hand side commute with direct sums (details omitted) we may assume that $\mathcal{F}$ is a skyscraper sheaf x_*A supported at some $x \in X$. Then $f'_!\mathcal{F}$ is the skyscraper sheaf at the image y' of x in Y by Lemma 4.5. In this case it is obvious that our construction produces the identity map $A \to H^0_c(Y', y'_*A) = A$ as desired. $\square$

Lemma 4.12. *Let $f : X \to Y$ and $g : Y \to Z$ be composable locally quasi-finite morphisms of schemes. Then there is a canonical isomorphism of functors*

$$(g \circ f)_! \longrightarrow g_! \circ f_!$$

These isomorphisms satisfy the following properties:

1. *If f and g are separated, then the isomorphism agrees with Lemma 3.13.*
2. *If g is separated, then the isomorphism agrees with Lemma 4.11.*
3. *For a geometric point $\overline{z} : \mathrm{Spec}(k) \to Z$ the diagram*

$$\begin{tikzcd} ((g \circ f)_!\mathcal{F})_{\overline{z}} \arrow[rr] \arrow[d] & & \bigoplus_{g(f(\overline{x}))=\overline{z}} \mathcal{F}_{\overline{x}} \arrow[d, equal] \\ (g_! f_! \mathcal{F})_{\overline{z}} \arrow[r] & \bigoplus_{g(\overline{y})=\overline{z}} (f_!\mathcal{F})_{\overline{y}} \arrow[r] & \bigoplus_{g(f(\overline{x}))=\overline{z}} \mathcal{F}_{\overline{x}} \end{tikzcd}$$

   is commutative where the horizontal arrows are given by Lemma 4.5.
4. *Let $h : Z \to T$ be a third locally quasi-finite morphism of schemes. Then the diagram*

$$\begin{tikzcd} (h \circ g \circ f)_! \arrow[r] \arrow[d] & (h \circ g)_! \circ f_! \arrow[d] \\ h_! \circ (g \circ f)_! \arrow[r] & h_! \circ g_! \circ f_! \end{tikzcd}$$

   commutes.
5. *Suppose that we have a diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "c"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "b"'] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "a"] & Z \end{tikzcd}$$

   with both squares cartesian and f and g locally quasi-finite. Then the diagram

$$\begin{tikzcd} a^{-1} \circ (g \circ f)_! \arrow[rr] \arrow[d] & & (g' \circ f')_! \circ c^{-1} \arrow[d] \\ a^{-1} \circ g_! \circ f_! \arrow[r] & g'_! \circ b^{-1} \circ f_! \arrow[r] & g'_! \circ f'_! \circ c^{-1} \end{tikzcd}$$

   commutes where the horizontal arrows are those of Lemma 4.10.

Proof. If f and g are separated, then this is a special case of Lemma 3.13. If g is separated, then this is a special case of Lemma 4.11 which moreover agrees with the case where f and g are separated.

Construction in the general case. Choose an open covering $Y = \bigcup Y_i$ such that the restriction $g_i : Y_i \to Z$ of g is separated. Set $X_i = f^{-1}(Y_i)$ and denote $f_i : X_i \to Y_i$

the restriction of f. Also denote $h = g \circ f$ and $h_i : X_i \to Z$ the restriction of h. Consider the following diagram

$$\begin{tikzcd}
\bigoplus_{i_0,i_1} h_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \arrow[r] \arrow[d] & \bigoplus_{i_0} h_{i_0,!}\mathcal{F}|_{X_{i_0}} \arrow[r] \arrow[d] & h_!\mathcal{F} \arrow[r] \arrow[dd, dotted] & 0 \\
\bigoplus_{i_0,i_1} g_{i_0i_1,!}f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}} \arrow[r] \arrow[d] & \bigoplus_{i_0} g_{i_0,!}f_{i_0,!}\mathcal{F}|_{X_{i_0}} \arrow[d] & & \\
\bigoplus_{i_0,i_1} g_{i_0i_1,!}(f_!\mathcal{F})|_{Y_{i_0i_1}} \arrow[r] & \bigoplus_{i_0} g_{i_0,!}(f_!\mathcal{F})|_{Y_{i_0}} \arrow[r] & g_!f_!\mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

By Lemma 4.7 the top and bottom row in the diagram are exact. By Lemma 4.11 the top left square commutes. The vertical arrows in the lower left square come about because $(f_!\mathcal{F})|_{Y_{i_0i_1}} = f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}}$ and $(f_!\mathcal{F})|_{Y_{i_0}} = f_{i_0,!}\mathcal{F}|_{X_{i_0}}$ as the construction of $f_!$ is local on the base. Moreover, these equalities are (of course) compatible with the identifications $((f_!\mathcal{F})|_{Y_{i_0}})|_{Y_{i_0i_1}} = (f_!\mathcal{F})|_{Y_{i_0i_1}}$ and $(f_{i_0,!}\mathcal{F}|_{X_{i_0}})|_{Y_{i_0i_1}} = f_{i_0i_1,!}\mathcal{F}|_{X_{i_0i_1}}$ which are used (together with the covariance for open embeddings for $Y_{i_0i_1} \subset Y_{i_0}$) to define the horizontal maps of the lower left square. Thus this square commutes as well. In this way we conclude there is a unique dotted arrow as indicated in the diagram and moreover this arrow is an isomorphism.

Proof of properties (1) – (5). Fix the open covering $Y = \bigcup Y_i$. Observe that if $Y \to Z$ happens to be separated, then we get a dotted arrow fitting into the huge diagram above by using the map of Lemma 4.11 (by the very properties of that lemma). This proves (2) and hence also (1) by the compatibility of the maps of Lemma 4.11 and Lemma 3.13. Next, for any scheme Z' over Z, we obtain the compatibility in (5) for the map $(g' \circ f')_! \to g'_! \circ f'_!$ constructed using the open covering $Y' = \bigcup b^{-1}(Y_i)$. This is clear from the corresponding compatibility of the maps constructed in Lemma 4.11. In particular, we can consider a geometric point $\overline{z} : \mathrm{Spec}(k) \to Z$. Since $X_{\overline{z}} \to Y_{\overline{z}} \to \mathrm{Spec}(k)$ are separated maps, we find that the base change of $(g \circ f)_!\mathcal{F} \to g_!f_!\mathcal{F}$ by $\overline{z}$ is equal to the map of Lemma 3.13. The reader then immediately sees that we obtain property (3). Of course, property (3) guarantees that our transformation of functors $(g \circ f)_! \to g_! \circ f_!$ constructed using the open covering $Y = \bigcup Y_i$ doesn't depend on the choice of this open covering. Finally, property (4) follows by looking at what happens on stalks using the already proven property (3). $\square$

5. Weightings and trace maps for locally quasi-finite morphisms

A reference for this section is [AGV71, Exposee XVII, Proposition 6.2.5].

Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}$ be a weighting of f, see More on Morphisms, Definition 75.2. Let $\mathcal{F}$ be an abelian sheaf on $Y_{étale}$. In this section we will show that there exists map

$$\mathrm{Tr}_{f,w,\mathcal{F}} : f_!f^{-1}\mathcal{F} \longrightarrow \mathcal{F}$$

of abelian sheaves on $Y_{étale}$ characterized by the following property: on stalks at a geometric point $\overline{y}$ of Y we obtain the map

$$\bigoplus\nolimits_{f(\overline{x})=\overline{y}} w(\overline{x}) : (f_!f^{-1}\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{y}} \longrightarrow \mathcal{F}_{\overline{y}}$$

Here as indicated the arrow is given by multiplication by the integer $w(\overline{x})$ on the summand corresponding to $\overline{x}$. The equality on the left of the arrow follows from Lemma 4.5 combined with Étale Cohomology, Lemma 36.2.

If the morphism $f : X \to Y$ is flat, locally quasi-finite, and locally of finite presentation, then there exists a canonical weighting and we obtain a canonical trace map whose formation is compatible with base change, see Example 5.5. If Y is a locally Noetherian unibranch scheme and $f : X \to Y$ is locally quasi-finite, then we can also define a (natural) weighting for f and we have trace maps in this case as well, see Example 5.7.

Lemma 5.1. *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let Λ be a ring. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{étale}$ and let $\mathcal{G}$ be a sheaf of Λ-modues on $Y_{étale}$. There is a canonical isomorphism*

$$can : f_!\mathcal{F} \otimes_\Lambda \mathcal{G} \longrightarrow f_!(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})$$

of sheaves of Λ-modules on $Y_{étale}$.

Proof. Recall that $f_!\mathcal{F} = (f_{p!}\mathcal{F})^\#$ by Definition 4.4 where $f_{p!}\mathcal{F}$ is the presheaf constructed in Section 4. Thus in order to construct the arrow it suffices to construct a map

$$f_{p!}\mathcal{F} \otimes_{p,\Lambda} \mathcal{G} \longrightarrow f_{p!}(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})$$

of presheaves on $Y_{étale}$. Here the symbol $\otimes_{p,\Lambda}$ denotes the presheaf tensor product, see Modules on Sites, Section 26. Let V be an object of $Y_{étale}$. Recall that

$$f_{p!}\mathcal{F}(V) = \text{colim}_Z\, H_Z(\mathcal{F}) \quad \text{and} \quad f_{p!}(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})(V) = \text{colim}_Z\, H_Z(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})$$

See Section 4. Our map will be defined on pure tensors by the rule

$$(Z, s) \otimes t \longmapsto (Z, s \otimes f^{-1}t)$$

(for notation see below) and extended by linearity to all of $(f_{p!}\mathcal{F} \otimes_{p,\Lambda} \mathcal{G})(V) = f_{p!}\mathcal{F}(V) \otimes_\Lambda \mathcal{G}(V)$. Here the notation used is as follows

1. $Z \subset X_V$ is a locally closed subscheme finite over V,
2. $s \in H_Z(\mathcal{F})$ which means that $s \in \mathcal{F}(U)$ with $\text{Supp}(s) \subset Z$ for some $U \subset X_V$ open such that $Z \subset U$ is closed, and
3. $t \in \mathcal{G}(V)$ with image $f^{-1}t \in f^{-1}\mathcal{G}(U)$.

Since the support of $s \in \mathcal{F}(U)$ is contained in Z it is clear that the support of $s \otimes f^{-1}t$ is contained in Z as well. Thus considering the pair $(Z, s \otimes f^{-1}t)$ makes sense. It is immediate that the construction commutes with the transition maps in the colimit $\text{colim}_Z\, H_Z(\mathcal{F})$ and that it is compatible with restriction mappings. Finally, it is equally clear that the construction is compatible with the identifications of stalks of $f_!$ in Lemma 4.5. In other words, the map *can* we've produced on stalks at a geometric point $\overline{y}$ fits into a commutative diagram

$$\begin{tikzcd}
(f_!\mathcal{F} \otimes_\Lambda \mathcal{G})_{\overline{y}} \arrow[r, "can_{\overline{y}}"'] \arrow[d] & f_!(\mathcal{F} \otimes_\Lambda f^{-1}\mathcal{G})_{\overline{y}} \arrow[d] \\
(\bigoplus \mathcal{F}_{\overline{x}}) \otimes_\Lambda \mathcal{G}_{\overline{y}} \arrow[r] & \bigoplus (\mathcal{F}_{\overline{x}} \otimes_\Lambda \mathcal{G}_{\overline{y}})
\end{tikzcd}$$

where the direct sums are over the geometric points $\overline{x}$ lying over $\overline{y}$, where the vertical arrows are the identifications of Lemma 4.5, and where the lower horizontal arrow is the obvious isomorphism. We conclude that *can* is an isomorphism as desired. $\square$

Lemma 5.2. *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}$ be a weighting of f. For any abelian sheaf $\mathcal{F}$ on Y there exists a unique trace map $Tr_{f,w,\mathcal{F}} : f_! f^{-1}\mathcal{F} \to \mathcal{F}$ having the prescribed behaviour on stalks.*

Proof. By Lemma 5.1 we have an identification $f_! f^{-1}\mathcal{F} = f_! \underline{\mathbf{Z}} \otimes \mathcal{F}$ compatible with the description of stalks of these sheaves at geometric points. Hence it suffices to produce the map

$$\mathrm{Tr}_{f,w,\underline{\mathbf{Z}}} : f_!\underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}}$$

having the prescribed behaviour on stalks. By Definition 4.4 we have $f_!\underline{\mathbf{Z}} = (f_{p!}\underline{\mathbf{Z}})^\#$ where $f_{p!}\underline{\mathbf{Z}}$ is the presheaf constructed in Section 4. Thus it suffices to construct a map

$$f_{p!}\underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}}$$

of presheaves on $Y_{étale}$. Let V be an object of $Y_{étale}$. Recall from Section 4 that

$$f_{p!}\underline{\mathbf{Z}}(V) = \mathrm{colim}_Z\, H_Z(\underline{\mathbf{Z}})$$

Here the colimit is over the (partially ordered) collection of locally closed subschemes $Z \subset X_V$ which are finite over V. For each such Z we will define a map

$$H_Z(\underline{\mathbf{Z}}) \longrightarrow \underline{\mathbf{Z}}(V)$$

compatible with the maps defining the colimit.

Let $Z \subset X_V$ be locally closed and finite over V. Choose an open $U \subset X_V$ containing Z as a closed subset. An element s of $H_Z(\underline{\mathbf{Z}})$ is a section $s \in \underline{\mathbf{Z}}(U)$ whose support is contained in Z. Let $U_n \subset U$ be the open and closed subset where the value of s is $n \in \mathbf{Z}$. By the support condition we see that $Z \cap U_n = U_n$ for $n \neq 0$. Hence for $n \neq 0$, the open U_n is also closed in Z (as the complement of all the others) and we conclude that $U_n \to V$ is finite as Z is finite over V. By the very definition of a weighting this means the function $\int_{U_n \to V} w|_{U_n}$ is locally constant on V and we may view it as an element of $\underline{\mathbf{Z}}(V)$. Our construction sends (Z, s) to the element

$$\sum\nolimits_{n \in \mathbf{Z},\ n \neq 0} n \left(\int_{U_n \to V} w|_{U_n}\right) \quad \in \quad \underline{\mathbf{Z}}(V)$$

The sum is locally finite on V and hence makes sense; details omitted (in the whole discussion the reader may first choose affine opens and make sure all the schemes occuring in the argument are quasi-compact so the sum is finite). We omit the verification that this construction is compatible with the maps in the colimit and with the restriction mappings defining $f_{p!}\underline{\mathbf{Z}}$.

Let $\overline{y}$ be a geometric point of Y lying over the point $y \in Y$. Taking stalks at $\overline{y}$ the construction above determines a map

$$(f_!\underline{\mathbf{Z}})_{\overline{y}} = \bigoplus\nolimits_{f(\overline{x}) = \overline{y}} \mathbf{Z} \longrightarrow \mathbf{Z} = \underline{\mathbf{Z}}_{\overline{y}}$$

To finish the proof we will show this map is given by multiplication by $w(\overline{x})$ on the summand corresponding to $\overline{x}$. Namely, pick $\overline{x}$ lying over $\overline{y}$. We can find an étale neighbourhood $(V, \overline{v}) \to (Y, \overline{y})$ such that X_V contains an open U finite over V such that only the geometric point $\overline{x}$ is in U and not the other geometric points of X lifting $\overline{y}$. This follows from More on Morphisms, Lemma 41.3; some details omitted. Then $(U, 1)$ defines a section of $f_!\underline{\mathbf{Z}}$ over V which maps to 1 in the summand corresponding to $\overline{x}$ and zero in the other summands (see proof of Lemma

4.2) and our construction above sends $(U, 1)$ to $\int_{U \to V} w|_U$ which is constant with value $w(\overline{x})$ in a neighbourhood of $\overline{v}$ as desired. $\square$

Lemma 5.3. *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}$ be a weighting of f. The trace maps constructed above have the following properties:*

1. *$Tr_{f,w,\mathcal{F}}$ is functorial in $\mathcal{F}$,*
2. *$Tr_{f,w,\mathcal{F}}$ is compatible with arbitrary base change,*
3. *given a ring Λ and K in $D(Y_{étale}, \Lambda)$ we obtain $Tr_{f,w,K} : f_! f^{-1} K \to K$ functorial in K and compatible with arbitrary base change.*

Proof. Part (1) either follows from the construction of the trace map in the proof of Lemma 5.2 or more simply because the characterization of the map forces it to be true on all stalks. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of schemes. Then the function $w' = w \circ g' : X' \to \mathbf{Z}$ is a weighting of f' by More on Morphisms, Lemma 75.3. Statement (2) means that the diagram

$$\begin{tikzcd} g^{-1} f_! f^{-1} \mathcal{F} \arrow[r, "g^{-1}\mathrm{Tr}_{f,w,\mathcal{F}}"'] \arrow[d, equal] & g^{-1}\mathcal{F} \arrow[d, equal] \\ f'_!(f')^{-1} g^{-1}\mathcal{F} \arrow[r, "\mathrm{Tr}_{f',w',g^{-1}\mathcal{F}}"] & g^{-1}\mathcal{F} \end{tikzcd}$$

is commutative where the left vertical equality is given by

$$g^{-1} f_! f^{-1} \mathcal{F} = f'_!(g')^{-1} f^{-1} \mathcal{F} = f'_!(f')^{-1} g^{-1} \mathcal{F}$$

with first equality sign given by Lemma 4.10 (base change for lower shriek). The commutativity of this diagram follows from the characterization of the action of our trace maps on stalks and the fact that the base change map of Lemma 4.10 respects the descriptions of stalks.

Given parts (1) and (2), part (3) follows as the functors $f^{-1} : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$ and $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ are obtained by applying f^{-1} and $f_!$ to any complexes of modules representing the objects in question. $\square$

Lemma 5.4. *Let $f : X \to Y$ and $g : Y \to Z$ be locally quasi-finite morphisms. Let $w_f : X \to \mathbf{Z}$ be a weighting of f and let $w_g : Y \to \mathbf{Z}$ be a weighting of g. For $K \in D(Z_{étale}, \Lambda)$ the composition*

$$(g \circ f)_!(g \circ f)^{-1} K = g_! f_! f^{-1} g^{-1} K \xrightarrow{g_! Tr_{f,w_f,g^{-1}K}} g_! g^{-1} K \xrightarrow{Tr_{g,w_g,K}} K$$

is equal to $Tr_{g \circ f, w_{g\circ f}, K}$ where $w_{g \circ f}(x) = w_f(x) w_g(f(x))$.

Proof. We have $(g \circ f)_! = g_! \circ f_!$ by Lemma 4.12. In More on Morphisms, Lemma 75.5 we have seen that $w_{g \circ f}$ is a weighting for $g \circ f$ so the statement makes sense. To check equality compute on stalks. Details omitted. $\square$

Example 5.5 (Trace for flat quasi-finite)**.** Let $f : X \to Y$ be a morphism of schemes which is flat, locally quasi-finite, and locally of finite presentation. Then we obtain a canonical positive weighting $w : X \to \mathbf{Z}$ by setting

$$w(x) = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x})[\kappa(x) : \kappa(f(x))]_i$$

See More on Morphisms, Lemma 75.7. Thus by Lemmas 5.2 and 5.3 for f we obtain trace maps

$$\text{Tr}_{f,K} : f_! f^{-1}K \longrightarrow K$$

functorial for K in $D(Y_{étale}, \Lambda)$ and compatible with arbitrary base change. Note that any base change $f' : X' \to Y'$ of f satisfies the same properties and that w restricts to the canonical weighting for f'.

Remark 5.6. Let $j : U \to X$ be an étale morphism of schemes. Then the trace map $\text{Tr} : j_! j^{-1}K \to K$ of Example 5.5 is equal to the counit for the adjunction between $j_!$ and j^{-1}. We already used the terminology "trace" for this counit in Étale Cohomology, Section 66.

Example 5.7 (Trace for quasi-finite over normal)**.** Let Y be a geometrically unibranch and locally Noetherian scheme, for example Y could be a normal variety. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Then there exists a positive weighting $w : X \to \mathbf{Z}$ for f which is roughly defined by sending x to the "generic separable degree" of $\mathcal{O}^{sh}_{X,x}$ over $\mathcal{O}^{sh}_{Y,f(x)}$. See More on Morphisms, Lemma 75.8. Thus by Lemmas 5.2 and 5.3 for f and w we obtain trace maps

$$\text{Tr}_{f,w,K} : f_! f^{-1}K \longrightarrow K$$

functorial for K in $D(Y_{étale}, \Lambda)$ and compatible with arbitrary base change. However, in this case, given a base change $f' : X' \to Y'$ of f the restriction of w to X' in general does not have a "natural" interpretation in terms of the morphism f'.

6. Upper shriek for locally quasi-finite morphisms

For a locally quasi-finite morphism $f : X \to Y$ of schemes, the functor $f_! : Ab(X_{étale}) \to Ab(Y_{étale})$ commutes with direct sums and is exact, see Lemma 4.5. This suggests that it has a right adjoint which we will denote $f^!$.

Warning: This functor is the non-derived version!

Lemma 6.1. *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes.*

1. *The functor $f_! : Ab(X_{étale}) \to Ab(Y_{étale})$ has a right adjoint $f^! : Ab(Y_{étale}) \to Ab(X_{étale})$.*
2. *We have $f^!(\overline{y}_* A) = \prod_{f(\overline{x}) = \overline{y}} \overline{x}_* A$.*
3. *If Λ is a ring, then the functor $f_! : Mod(X_{étale}, \Lambda) \to Mod(Y_{étale}, \Lambda)$ has a right adjoint $f^! : Mod(Y_{étale}, \Lambda) \to Mod(X_{étale}, \Lambda)$ which agrees with $f^!$ on underlying abelian sheaves.*

Proof. Proof of (1). Let $E \subset \text{Ob}(Ab(Y_{étale}))$ be the class consisting of products of skyscraper sheaves. We claim that

(a) every $\mathcal{G}$ in $Ab(Y_{étale})$ is a subsheaf of an element of E, and
(b) for every $\mathcal{G} \in E$ there exists an object $\mathcal{H}$ of $Ab(X_{étale})$ such that $\text{Hom}(f_!\mathcal{F}, \mathcal{G}) = \text{Hom}(\mathcal{F}, \mathcal{H})$ functorially in $\mathcal{F}$.

Once the claim has been verified, the dual of Homology, Lemma 29.6 produces the adjoint functor $f^!$.

Part (a) is true because we can map $\mathcal{G}$ to the sheaf $\prod \overline{y}_*\mathcal{G}_{\overline{y}}$ where the product is over all geometric points of Y. This is an injection by Étale Cohomology, Theorem 29.10. (This is the first step in the Godement resolution when done in the setting of abelian sheaves on topological spaces.)

Part (b) and part (2) of the lemma can be seen as follows. Suppose that $\mathcal{G} = \prod \overline{y}_*A_{\overline{y}}$ for some abelian groups $A_{\overline{y}}$. Then

$$\mathrm{Hom}(f_!\mathcal{F}, \mathcal{G}) = \prod \mathrm{Hom}(f_!\mathcal{F}, \overline{y}_*A_{\overline{y}})$$

Thus it suffices to find abelian sheaves $\mathcal{H}_{\overline{y}}$ on $X_{étale}$ representing the functors $\mathcal{F} \mapsto \mathrm{Hom}(f_!\mathcal{F}, \overline{y}_*A_{\overline{y}})$ and to take $\mathcal{H} = \prod \mathcal{H}_{\overline{y}}$. This reduces us to the case $\mathcal{H} = \overline{y}_*A$ for some fixed geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ and some fixed abelian group A. We claim that in this case $\mathcal{H} = \prod_{f(\overline{x})=\overline{y}} \overline{x}_*A$ works. This will finish the proof of parts (1) and (2) of the lemma. Namely, we have

$$\mathrm{Hom}(f_!\mathcal{F}, \overline{y}_*A) = \mathrm{Hom}_{Ab}((f_!\mathcal{F})_{\overline{y}}, A) = \mathrm{Hom}_{Ab}(\bigoplus\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}, A)$$

by the description of stalks in Lemma 4.5 on the one hand and on the other hand we have

$$\mathrm{Hom}(\mathcal{F}, \mathcal{H}) = \prod\nolimits_{f(\overline{x})=\overline{y}} \mathrm{Hom}(\mathcal{F}, \overline{x}_*A) = \prod\nolimits_{f(\overline{x})=\overline{y}} \mathrm{Hom}_{Ab}(\mathcal{F}_{\overline{x}}, A)$$

We leave it to the reader to identify these as functors of $\mathcal{F}$.

Proof of part (3). Observe that an object $Mod(X_{étale}, \Lambda)$ is the same thing as an object $\mathcal{F}$ of $Ab(X_{étale})$ together with a map $\Lambda \to \mathrm{End}(\mathcal{F})$. Hence the functors $f_!$ and $f^!$ in (1) define functors $f_!$ and $f^!$ as in (3). A straightforward computation shows that they are adjoints. $\square$

Lemma 6.2. *Let $j : U \to X$ be an étale morphism. Then $j^! = j^{-1}$.*

Proof. This is true because $j_!$ as defined in Section 4 agrees with $j_!$ as defined in Étale Cohomology, Section 70, see Lemma 4.3. Finally, in Étale Cohomology, Section 70 the functor $j_!$ is defined as the left adjoint of j^{-1} and hence we conclude by uniqueness of adjoint functors. $\square$

Lemma 6.3. *Let $f : X \to Y$ and $g : Y \to Z$ be separated and locally quasi-finite morphisms. There is a canonical isomorphism $(g \circ f)^! \to f^! \circ g^!$. Given a third locally quasi-finite morphism $h : Z \to T$ the diagram*

$$\begin{tikzcd}
(h \circ g \circ f)^! \arrow[r] \arrow[d] & f^! \circ (h \circ g)^! \arrow[d] \\
(g \circ f)^! \circ h^! \arrow[r] & f^! \circ g^! \circ h^!
\end{tikzcd}$$

commutes.

Proof. By uniqueness of adjoint functors, this immediately translates into the corresponding (dual) statement for the functors $f_!$. See Lemma 4.12. $\square$

Lemma 6.4. *Let $j : U \to X$ and $j' : V \to U$ be étale morphisms. The isomorphism $(j \circ j')^{-1} = (j')^{-1} \circ j^{-1}$ and the isomorphism $(j \circ j')^! = (j')^! \circ j^!$ of Lemma 6.3 agree via the isomorphism of Lemma 6.2.*

Proof. Omitted. □

Lemma 6.5. *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes with f locally quasi-finite. For any abelian sheaf $\mathcal{F}$ on $Y'_{étale}$ we have $(g')_*(f')^!\mathcal{F} = f^! g_*\mathcal{F}$.

Proof. By uniqueness of adjoint functors, this follows from the corresponding (dual) statement for the functors $f_!$. See Lemma 4.10. □

Remark 6.6. The material in this section can be generalized to sheaves of pointed sets. Namely, for a site $\mathcal{C}$ denote $Sh^*(\mathcal{C})$ the category of sheaves of pointed sets. The constructions in this and the preceding section apply, mutatis mutandis, to sheaves of pointed sets. Thus given a locally quasi-finite morphism $f : X \to Y$ of schemes we obtain an adjoint pair of functors

$$f_! : Sh^*(X_{étale}) \longrightarrow Sh^*(Y_{étale}) \quad \text{and} \quad f^! : Sh^*(Y_{étale}) \longrightarrow Sh^*(X_{étale})$$

such that for every geometric point $\overline{y}$ of Y there are isomorphisms

$$(f_!\mathcal{F})_{\overline{y}} = \coprod\nolimits_{f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}$$

(coproduct taken in the category of pointed sets) functorial in $\mathcal{F} \in Sh^*(X_{étale})$ and isomorphisms

$$f^!(\overline{y}_*S) = \prod\nolimits_{f(\overline{x})=\overline{y}} \overline{x}_*S$$

functorial in the pointed set S. If $F : Ab(X_{étale}) \to Sh^*(X_{étale})$ and $F : Ab(Y_{étale}) \to Sh^*(Y_{étale})$ denote the forgetful functors, compatibility between the constructions will guarantee the existence of canonical maps

$$f_!F(\mathcal{F}) \longrightarrow F(f_!\mathcal{F})$$

functorial in $\mathcal{F} \in Ab(X_{étale})$ and

$$F(f^!\mathcal{G}) \longrightarrow f^!F(\mathcal{G})$$

functorial in $\mathcal{G} \in Ab(Y_{étale})$ which produce the obvious maps on stalks, resp. skyscraper sheaves. In fact, the transformation $F \circ f^! \to f^! \circ F$ is an isomorphism (because $f^!$ commutes with products).

7. Derived upper shriek for locally quasi-finite morphisms

We can take the derived versions of the functors in Section 6 and obtain the following.

Lemma 7.1. *Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let Λ be a ring. The functors $f_!$ and $f^!$ of Definition 4.4 and Lemma 6.1 induce adjoint functors $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ and $Rf^! : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$ on derived categories.*

In the separated case the functor $f_!$ is defined in Section 3.

Proof. This follows immediately from Derived Categories, Lemma 30.3, the fact that $f_!$ is exact (Lemma 4.5) and hence $Lf_! = f_!$ and the fact that we have enough K-injective complexes of Λ-modules on $Y_{étale}$ so that $Rf^!$ is defined. □

Remark 7.2. Let $f : X \to Y$ be a locally quasi-finite morphism of schemes. Let Λ be a ring. The functor $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ of Lemma 7.1 sends complexes with torsion cohomology sheaves to complexes with torsion cohomology sheaves. This is immediate from the description of the stalks of $f_!$, see Lemma 4.5.

Lemma 7.3. *Let X be a scheme. Let $X = U \cup V$ with U and V open. Let Λ be a ring. Let $K \in D(X_{étale}, \Lambda)$. There is a distinguished triangle*

$$j_{U\cap V!}K|_{U\cap V} \to j_{U!}K|_U \oplus j_{V!}K|_V \to K \to j_{U\cap V!}K|_{U\cap V}[1]$$

in $D(X_{étale}, \Lambda)$ with obvious notation.

Proof. Since the restriction functors and the lower shriek functors we use are exact, it suffices to show for any abelian sheaf $\mathcal{F}$ on $X_{étale}$ the sequence

$$0 \to j_{U\cap V!}\mathcal{F}|_{U\cap V} \to j_{U!}\mathcal{F}|_U \oplus j_{V!}\mathcal{F}|_V \to \mathcal{F} \to 0$$

is exact. This can be seen by looking at stalks. □

Lemma 7.4. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. Let Λ be a ring. Let $K \in D(X_{étale}, \Lambda)$. There is a distinguished triangle*

$$j_!j^{-1}K \to K \to i_*i^{-1}K \to j_!j^{-1}K[1]$$

in $D(X_{étale}, \Lambda)$.

Proof. Immediate consequence of Étale Cohomology, Lemma 70.8 and the fact that the functors $j_!$, j^{-1}, i_*, i^{-1} are exact and hence their derived versions are computed by applying these functors to any complex of sheaves representing K. □

8. Preliminaries to derived lower shriek via compactifications

In this section we prove some lemmas on the existence of certain natural isomorphisms of functors which follow immediately from proper base change.

Lemma 8.1. *Consider a commutative diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with f and f' proper and g and g' separated and locally quasi-finite. Let Λ be a ring. Functorially in $K \in D(X'_{étale}, \Lambda)$ there is a canonical map

$$g_!Rf'_*K \longrightarrow Rf_*(g'_!K)$$

in $D(Y_{étale}, \Lambda)$. This map is an isomorphism if (a) K is bounded below and has torsion cohomology sheaves, or (b) Λ is a torsion ring.

Proof. Represent K by a K-injective complex $\mathcal{J}^\bullet$ of sheaves of Λ-modules on $X'_{\acute{e}tale}$. Choose a quasi-isomorphism $g'_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $X_{\acute{e}tale}$. Then we can consider the map

$$g_!f'_*\mathcal{J}^\bullet = g_!f'_!\mathcal{J}^\bullet = f_!g'_!\mathcal{J}^\bullet = f_*g'_!\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$$

where the first and third equality come from Lemma 3.4 and the second equality comes from Lemma 3.13 which tells us that both $g_! \circ f'_!$ and $f_! \circ g'_!$ are equal to $(g \circ f')_! = (f \circ g')_!$ as subsheaves of $(g \circ f')_* = (f \circ g')_*$.

Assume Λ is torsion, i.e., we are in case (b). With notation as above, it suffices to show that $f_*g'_!\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$ is an isomorphism. The question is local on Y. Hence we may assume that the dimension of fibres of f is bounded, see Morphisms, Lemma 28.5. Then we see that Rf_* has finite cohomological dimension, see Étale Cohomology, Lemma 92.2. Hence by Derived Categories, Lemma 32.2, if we show that $R^qf_*(g'_!\mathcal{J}) = 0$ for $q > 0$ and any injective sheaf of Λ-modules $\mathcal{J}$ on $X'_{\acute{e}tale}$, then the result follows.

The stalk of $R^qf_*(g'_!\mathcal{J})$ at a geometric point $\overline{y}$ is equal to $H^q(X_{\overline{y}}, (g'_!\mathcal{J})|_{X_{\overline{y}}})$ by Étale Cohomology, Lemma 91.13. Since formation of $g'_!$ commutes with base change (Lemma 3.12) this is equal to

$$H^q(X_{\overline{y}}, g'_{\overline{y},!}(\mathcal{J}|_{X'_{\overline{y}}}))$$

where $g'_{\overline{y}} : X'_{\overline{y}} \to X_{\overline{y}}$ is the induced morphism between geometric fibres. Since $Y' \to Y$ is locally quasi-finite, we see that $X'_{\overline{y}}$ is a disjoint union of the fibres $X'_{\overline{y}'}$ at geometric points $\overline{y}'$ of Y' lying over $\overline{y}$. Denote $g'_{\overline{y}'} : X'_{\overline{y}'} \to X_{\overline{y}}$ the restriction of $g'_{\overline{y}}$ to $X'_{\overline{y}'}$. Thus the previous cohomology group is equal to

$$H^q(X_{\overline{y}}, \bigoplus\nolimits_{\overline{y}'/\overline{y}} g'_{\overline{y}',!}(\mathcal{J}|_{X'_{\overline{y}'}}))$$

for example by Lemma 3.15 (but it is also obvious from the definition of $g'_{\overline{y},!}$ in Section 3). Since taking étale cohomology over $X_{\overline{y}}$ commutes with direct sums (Étale Cohomology, Theorem 51.3) we conclude it suffices to show that

$$H^q(X_{\overline{y}}, g'_{\overline{y}',!}(\mathcal{J}|_{X'_{\overline{y}'}}))$$

is zero. Observe that $g_{\overline{y}'} : X'_{\overline{y}'} \to X_{\overline{y}}$ is a morphism between proper scheme over $\overline{y}$ and hence is proper itself. As it is locally quasi-finite as well we conclude that $g_{\overline{y}'}$ is finite. Thus we see that $g'_{\overline{y}',!} = g'_{\overline{y}',*} = Rg'_{\overline{y}',*}$. By Leray we conlude that we have to show

$$H^q(X'_{\overline{y}'}, \mathcal{J}|_{X'_{\overline{y}'}})$$

is zero. As Λ is torsion, this follows from proper base change (Étale Cohomology, Lemma 91.13) as the higher direct images of $\mathcal{J}$ under f' are zero.

Proof in case (a). We will deduce this from case (b) by standard arguments. We will show that the induced map $g_!R^pf'_*K \to R^pf_*(g'_!K)$ is an isomorphism for all $p \in \mathbf{Z}$. Fix an integer $p_0 \in \mathbf{Z}$. Let a be an integer such that $H^j(K) = 0$ for $j < a$. We will prove $g_!R^pf'_*K \to R^pf_*(g'_!K)$ is an isomorphism for $p \leq p_0$ by descending induction on a. If $a > p_0$, then we see that the left and right hand side of the map are zero for $p \leq p_0$ by trivial vanishing, see Derived Categories, Lemma 16.1 (and use that $g_!$ and $g'_!$ are exact functors). Assume $a \leq p_0$. Consider the distinguished triangle

$$H^a(K)[-a] \to K \to \tau_{\geq a+1}K$$

By induction we have the result for $\tau_{\geq a+1}K$. In the next paragraph, we will prove the result for $H^a(K)[-a]$. Then five lemma applied to the map between long exact sequence of cohomology sheaves associated to the map of distinguished triangles

$$\begin{tikzcd} g_!Rf'_*(H^a(K)[-a]) \arrow[r] \arrow[d] & g_!Rf'_*K \arrow[r] \arrow[d] & g_!Rf'_*\tau_{\geq a+1}K \arrow[d] \\ Rf_*(g'_!(H^a(K)[-a])) \arrow[r] & Rf_*(g'_!K) \arrow[r] & Rf_*(g'_!\tau_{|geqa+1}K) \end{tikzcd}$$

gives the result for K. Some details omitted.

Let $\mathcal{F}$ be a torsion abelian sheaf on $X'_{étale}$. To finish the proof we show that $g_!Rf'_*\mathcal{F} \to R^pf_*(g'_!\mathcal{F})$ is an isomorphism for all p. We can write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[n] = \text{Ker}(n : \mathcal{F} \to \mathcal{F})$. We have the isomorphism for $\mathcal{F}[n]$ by case (b). Since the functors $g_!$, $g'_!$, R^pf_*, $R^pf'_*$ commute with filtered colimits (follows from Lemma 3.17 and Étale Cohomology, Lemma 51.8) the proof is complete. □

Lemma 8.2. *Consider a commutative diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "k"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "l"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "m"] & Z \end{tikzcd}$$

with f, f', g and g' proper and k, l, and m separated and locally quasi-finite. Then the isomorphisms of Lemma 8.1 for the two squares compose to give the isomorphism for the outer rectangle (see proof for a precise statement).

Proof. The statement means that if we write $R(g \circ f)_* = Rg_* \circ Rf_*$ and $R(g' \circ f')_* = Rg'_* \circ Rf'_*$, then the isomorphism $m_! \circ Rg'_* \circ Rf'_* \to Rg_* \circ Rf_* \circ k_!$ of the outer rectangle is equal to the composition

$$m_! \circ Rg'_* \circ Rf'_* \to Rg_* \circ l_! \circ Rf'_* \to Rg_* \circ Rf_* \circ k_!$$

of the two maps of the squares in the diagram. To prove this choose a K-injective complex $\mathcal{J}^\bullet$ of Λ-modules on $X'_{étale}$ and a quasi-isomorphism $k_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of Λ-modules on $X_{étale}$. The proof of Lemma 8.1 shows that the canonical map

$$a : l_!f'_*\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$$

is a quasi-isomorphism and this quasi-isomorphism produces the second arrow on applying Rg_*. By Cohomology on Sites, Lemma 20.10 the complex $f_*\mathcal{I}^\bullet$, resp. $f'_*\mathcal{J}^\bullet$ is a K-injective complex of Λ-modules on $Y_{étale}$, resp. $Y'_{étale}$. (Using this is cheating and could be avoided.) In particular, the same reasoning gives that the canonical map

$$b : m_!g'_*f'_*\mathcal{J}^\bullet \to g_*f_*\mathcal{I}^\bullet$$

is a quasi-isomorphism and this quasi-isomorphism represents the first arrow. Finally, the proof of Lemma 8.1 show that $g_*l_!f'_!\mathcal{J}^\bullet$ represents $Rg_*(l_!f'_*\mathcal{J}^\bullet)$ because $f'_*\mathcal{J}^\bullet$ is K-injective. Hence $Rg_*(a) = g_*(a)$ and the composition $g_*(a) \circ b$ is the arrow of Lemma 8.1 for the rectangle. □

Lemma 8.3. *Consider a commutative diagram of schemes*

$$\begin{tikzcd} X'' \arrow[r, "g'"] \arrow[d, "f''"] & X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "h'"] & Y' \arrow[r, "h"] & Y \end{tikzcd}$$

with f, f', and f'' proper and g, g', h, and h' separated and locally quasi-finite. Then the isomorphisms of Lemma 8.1 for the two squares compose to give the isomorphism for the outer rectangle (see proof for a precise statement).

Proof. The statement means that if we write $(h \circ h')_! = h_! \circ h'_!$ and $(g \circ g')_! = g_! \circ g'_!$ using the equalities of Lemma 3.13, then the isomorphism $h_! \circ h'_! \circ Rf''_* \to Rf_* \circ g_! \circ g'_!$ of the outer rectangle is equal to the composition

$$h_! \circ h'_! \circ Rf''_* \to h_! \circ Rf'_* \circ g'_! \to Rf_* \circ g_! \circ g'_!$$

of the two maps of the squares in the diagram. To prove this choose a K-injective complex $\mathcal{I}^\bullet$ of Λ-modules on $X''_{\acute{e}tale}$ and a quasi-isomorphism $g'_!\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ to a K-injective complex $\mathcal{J}^\bullet$ of Λ-modules on $X'_{\acute{e}tale}$. Next, choose a quasi-isomorphism $g_!\mathcal{J}^\bullet \to \mathcal{K}^\bullet$ to a K-injective complex $\mathcal{K}^\bullet$ of Λ-modules on $X_{\acute{e}tale}$. The proof of Lemma 8.1 shows that the canonical maps

$$h'_!f''_*\mathcal{I}^\bullet \to f'_*\mathcal{J}^\bullet \quad \text{and} \quad h_!f'_*\mathcal{J}^\bullet \to f_*\mathcal{K}^\bullet$$

are quasi-isomorphisms and these quasi-isomorphisms define the first and second arrow above. Since $g_!$ is an exact functor (Lemma 3.17) we find that $g_!g'_!\mathcal{I}^\bullet \to \mathcal{K}^\bullet$ is a quasi-ismorphism and hence the canonical map

$$h_!h'_!f''_*\mathcal{I}^\bullet \to f_*\mathcal{K}^\bullet$$

is a quasi-isomorphism and represents the map for the outer rectangle in the derived category. Clearly this map is the composition of the other two and the proof is complete. □

Remark 8.4. Consider a commutative diagram

$$\begin{tikzcd} X'' \arrow[r, "k'"] \arrow[d, "f''"] & X' \arrow[r, "k"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "l'"] \arrow[d, "g''"] & Y' \arrow[r, "l"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ Z'' \arrow[r, "m'"] & Z' \arrow[r, "m"] & Z \end{tikzcd}$$

of schemes whose vertical arrows are proper and whose horizontal arrows are separated and locally quasi-finite. Let us label the squares of the diagram A, B, C, D as follows

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

Then the maps of Lemma 8.1 for the squares are (where we use $Rf_* = f_*$, etc)

$$\gamma_A : l'_! \circ f''_* \to f'_* \circ k'_! \quad \gamma_B : l_! \circ f'_* \to f_* \circ k_!$$
$$\gamma_C : m'_! \circ g''_* \to g'_* \circ l'_! \quad \gamma_D : m_! \circ g'_* \to g_* \circ l_!$$

For the 2×1 and 1×2 rectangles we have four further maps

$$\gamma_{A+B} : (l \circ l')_! \circ f''_* \to f_* \circ (k \circ k')_*$$
$$\gamma_{C+D} : (m \circ m')_! \circ g''_* \to g_* \circ (l \circ l')_!$$
$$\gamma_{A+C} : m'_! \circ (g'' \circ f'')_* \to (g' \circ f')_* \circ k'_!$$
$$\gamma_{B+D} : m_! \circ (g' \circ f')_* \to (g \circ f)_* \circ k_!$$

By Lemma 8.3 we have

$$\gamma_{A+B} = \gamma_B \circ \gamma_A, \quad \gamma_{C+D} = \gamma_D \circ \gamma_C$$

and by Lemma 8.2 we have

$$\gamma_{A+C} = \gamma_A \circ \gamma_C, \quad \gamma_{B+D} = \gamma_B \circ \gamma_D$$

Here it would be more correct to write $\gamma_{A+B} = (\gamma_B \star \mathrm{id}_{k'_!}) \circ (\mathrm{id}_{l_!} \star \gamma_A)$ with notation as in Categories, Section 28 and similarly for the others. Having said all of this we find (a priori) two transformations

$$m_! \circ m'_! \circ g''_* \circ f''_* \longrightarrow g_* \circ f_* \circ k_! \circ k'_!$$

namely

$$\gamma_B \circ \gamma_D \circ \gamma_A \circ \gamma_C = \gamma_{B+D} \circ \gamma_{A+C}$$

and

$$\gamma_B \circ \gamma_A \circ \gamma_D \circ \gamma_C = \gamma_{A+B} \circ \gamma_{C+D}$$

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

$$\begin{tikzcd}
m_! \circ g'_* \circ l'_! \circ f''_* \arrow[r, "\gamma_D"'] \arrow[d, "\gamma_A"'] & g_* \circ l_! \circ l'_! \circ f''_* \arrow[d, "\gamma_A"] \\
m_! \circ g'_* \circ f'_* \circ k'_! \arrow[r, "\gamma_D"] & g_* \circ l_! \circ f'_* \circ k'_!
\end{tikzcd}$$

commutes. This is true because the squares A and D meet in only one point, more precisely by Categories, Lemma 28.2 or more simply the discussion preceding Categories, Definition 28.1.

Lemma 8.5. *Let $b : Y_1 \to Y$ be a morphism of schemes. Consider a commutative diagram of schemes*

$$\begin{tikzcd}
X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd} \quad \textit{and let} \quad \begin{tikzcd}
X'_1 \arrow[r, "g'_1"'] \arrow[d, "f'_1"'] & X_1 \arrow[d, "f_1"] \\
Y'_1 \arrow[r, "g_1"] & Y_1
\end{tikzcd}$$

be the base change by b. Assume f and f' proper and g and g' separated and locally quasi-finite. For a ring Λ and K in $D(X'_{\acute{e}tale}, \Lambda)$ there is commutative diagram

$$\begin{tikzcd} b^{-1}g_!Rf'_*K \arrow[r] \arrow[d] & g_{1,!}(b')^{-1}Rf'_*K \arrow[r] & g_{1,!}Rf'_{1,*}(a')^{-1}K \arrow[d] \\ b^{-1}Rf_*g'_!K \arrow[r] & Rf_{1,*}a^{-1}g'_!K \arrow[r] & Rf_{1,*}g'_{1,!}(a')^{-1}K \end{tikzcd}$$

in $D(Y_{1,\acute{e}tale}, \Lambda)$ where $a : X_1 \to X$, $a' : X'_1 \to X'$, $b' : Y'_1 \to Y'$ are the projections, the vertical maps are the arrows of Lemma 8.1 and the horizontal arrows are the base change map (from Étale Cohomology, Section 86) and the base change map of Lemma 3.12.

Proof. Represent K by a K-injective complex $\mathcal{J}^\bullet$ of sheaves of Λ-modules on $X'_{\acute{e}tale}$. Choose a quasi-isomorphism $g'_!\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $X_{\acute{e}tale}$. The proof of Lemma 8.1 constructs $g_!Rf'_*K \to Rf_*g'_!K$ as

$$g_!f'_*\mathcal{J}^\bullet = g_!f'_!\mathcal{J}^\bullet = f_!g'_!\mathcal{J}^\bullet = f_*g'_!\mathcal{J}^\bullet \to f_*\mathcal{I}^\bullet$$

Choose a quasi-isomorphism $(a')^{-1}\mathcal{J}^\bullet \to \mathcal{J}_1^\bullet$ to a K-injective complex $\mathcal{J}_1^\bullet$ of sheaves of Λ-modules on $X'_{1,\acute{e}tale}$. Then we can pick a diagram of complexes

$$\begin{tikzcd} g'_{1,!}\mathcal{J}_1^\bullet \arrow[rr] & & \mathcal{I}_1^\bullet \\ g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \arrow[u] \arrow[r, equal] & a^{-1}g'_!\mathcal{J}^\bullet \arrow[r] & a^{-1}\mathcal{I}^\bullet \arrow[u] \end{tikzcd}$$

commuting up to homotopy where all arrows are quasi-isomorphisms, the equality comes from Lemma 3.4, and $\mathcal{I}_1^\bullet$ is a K-injective complex of sheaves of Λ-modules on $X_{1,\acute{e}tale}$. The map $g_{1,!}Rf'_{1,*}(a')^{-1}K \to Rf_{1,*}g'_{1,!}(a')^{-1}K$ is given by

$$g_{1,!}f'_{1,*}\mathcal{J}_1^\bullet = g_{1,!}f'_{1,!}\mathcal{J}_1^\bullet = f_{1,!}g'_{1,!}\mathcal{J}_1^\bullet = f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \to f_{1,*}\mathcal{I}_1^\bullet$$

The identifications across the 3 equal signs in both arrows are compatible with pullback maps, i.e., the diagram

$$\begin{tikzcd} b^{-1}g_!f'_*\mathcal{J}^\bullet \arrow[r] \arrow[d, equal] & g_{1,!}(b')^{-1}f'_*\mathcal{J}^\bullet \arrow[r] & g_{1,!}f'_{1,*}(a')^{-1}\mathcal{J}^\bullet \arrow[d, equal] \\ b^{-1}f_*g'_!\mathcal{J}^\bullet \arrow[r] & f_{1,*}a^{-1}g'_!\mathcal{J}^\bullet \arrow[r] & f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \end{tikzcd}$$

of complexes of abelian sheaves commutes. To show this it is enough to show the diagram commutes with $g_!, g_{1,!}, g'_!, g'_{1,!}$ replaced by $g_*, g_{1,*}, g'_*, g'_{1,*}$ (because the shriek functors are defined as subfunctors of the $*$ functors and the base change maps are defined in a manner compatible with this, see proof of Lemma 3.12). For this new diagram the commutativity follows from the compatibility of pullback maps with horizontal and vertical stacking of diagrams, see Sites, Remarks 45.3 and 45.4 so that going around the diagram in either direction is the pullback map for the base change of $f \circ g' = g \circ f'$ by b. Since of course

$$\begin{tikzcd} g_{1,!}f'_{1,*}(a')^{-1}\mathcal{J}^\bullet \arrow[r] \arrow[d, equal] & g_{1,!}f'_{1,*}\mathcal{J}_1^\bullet \arrow[d, equal] \\ f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \arrow[r] & f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \end{tikzcd}$$

commutes, to finish the proof it suffices to show that

$$\begin{tikzcd} b^{-1}f_*g'_!\mathcal{J}^\bullet \arrow[r] \arrow[d] & f_{1,*}a^{-1}g'_!\mathcal{J}^\bullet \arrow[r] \arrow[d] & f_{1,*}g'_{1,!}(a')^{-1}\mathcal{J}^\bullet \arrow[r] & f_{1,*}g'_{1,!}\mathcal{J}_1^\bullet \arrow[d] \\ b^{-1}f_*\mathcal{I}^\bullet \arrow[r] & f_{1,*}a^{-1}\mathcal{I}^\bullet \arrow[rr] & & f_{1,*}\mathcal{I}_1^\bullet \end{tikzcd}$$

commutes in the derived category, which holds by our choice of maps earlier. □

Lemma 8.6. *Consider a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rd, "g"'] & Y \arrow[d, "h"] \\ & Z \end{tikzcd}$$

with f and g locally quasi-finite and h proper. Let Λ be a ring. Funtorially in $K \in D(X_{étale}, \Lambda)$ there is a canonical map

$$g_!K \longrightarrow Rh_*(f_!K)$$

in $D(Z_{étale}, \Lambda)$. This map is an isomorphism if (a) K is bounded below and has torsion cohomology sheaves, or (b) Λ is a torsion ring.

Proof. This is a special case of Lemma 8.1 if f and g are separated. We urge the reader to skip the proof in the general case as we'll mainly use the case where f and g are separated.

Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules on $X_{étale}$. Choose a quasi-isomorphism $f_!\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex $\mathcal{I}^\bullet$ of sheaves of Λ-modules on $Y_{étale}$. Consider the map

$$g_!\mathcal{K}^\bullet = h_!f_!\mathcal{K}^\bullet = h_*f_!\mathcal{K}^\bullet \longrightarrow h_*\mathcal{I}^\bullet$$

where the equalities are Lemmas 4.11 and 3.4. This map of complexes determines the map $g_!K \to Rh_*(f_!K)$ of the statement of the lemma.

Assume Λ is torsion, i.e., we are in case (b). To check the map is an isomorphism we may work locally on Z. Hence we may assume that the dimension of fibres of h is bounded, see Morphisms, Lemma 28.5. Then we see that Rh_* has finite cohomological dimension, see Étale Cohomology, Lemma 92.2. Hence by Derived Categories, Lemma 32.2, if we show that $R^qh_*(f_!\mathcal{F}) = 0$ for $q > 0$ and any sheaf $\mathcal{F}$ of Λ-modules on $X_{étale}$, then $h_*f_!\mathcal{K}^\bullet \to h_*\mathcal{I}^\bullet$ is a quasi-isomorphism.

Observe that $\mathcal{G} = f_!\mathcal{F}$ is a sheaf of Λ-modules on Y whose stalks are nonzero only at points $y \in Y$ such that $\kappa(y)/\kappa(h(y))$ is a finite extension. This follows from the description of stalks of $f_!\mathcal{F}$ in Lemma 4.5 and the fact that both f and g are locally quasi-finite. Hence by the proper base change theorem (Étale Cohomology, Lemma 91.13) it suffices to show that $H^q(Y_{\overline{z}}, \mathcal{H}) = 0$ where $\mathcal{H}$ is a sheaf on the proper scheme $Y_{\overline{z}}$ over $\kappa(\overline{z})$ whose support is contained in the set of closed points. Thus the required vanishing by Étale Cohomology, Lemma 97.3.

Case (a) follows from case (b) by the exact same argument as used in the proof of Lemma 8.1 (using Lemma 4.5 instead of Lemma 3.17). $\square$

9. Derived lower shriek via compactifications

Let $f : X \to Y$ be a finite type separated morphism of schemes with Y quasi-compact and quasi-separated. Choose a compactification $j : X \to \overline{X}$ over Y, see More on Flatness, Theorem 33.8. Let Λ be a ring. Denote $D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ the strictly full saturated triangulated subcategory of $D(X_{\acute{e}tale}, \Lambda)$ consisting of objects K which are bounded below and whose cohomology sheaves are torsion. We will consider the functor

$$Rf_! = R\overline{f}_* \circ j_! : D^+_{tors}(X_{\acute{e}tale}, \Lambda) \longrightarrow D^+_{tors}(Y_{\acute{e}tale}, \Lambda)$$

where $\overline{f} : \overline{X} \to Y$ is the structure morphism. This makes sense: the functor $j_!$ sends $D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ into $D^+_{tors}(\overline{X}_{\acute{e}tale}, \Lambda)$ by Remark 7.2 and $R\overline{f}_*$ sends $D^+_{tors}(\overline{X}_{\acute{e}tale}, \Lambda)$ into $D^+_{tors}(Y_{\acute{e}tale}, \Lambda)$ by Étale Cohomology, Lemma 78.2. If Λ is a torsion ring, then we define

$$Rf_! = R\overline{f}_* \circ j_! : D(X_{\acute{e}tale}, \Lambda) \longrightarrow D(Y_{\acute{e}tale}, \Lambda)$$

Here is the obligatory lemma.

Lemma 9.1. *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. The functors $Rf_!$ constructed above are, up to canonical isomorphism, independent of the choice of the compactification.*

Proof. We will prove this for the functor $Rf_! : D(X_{\acute{e}tale}, \Lambda) \to D(Y_{\acute{e}tale}, \Lambda)$ when Λ is a torsion ring; the case of the functor $Rf_! : D^+_{tors}(X_{\acute{e}tale}, \Lambda) \to D^+_{tors}(Y_{\acute{e}tale}, \Lambda)$ is proved in exactly the same way.

Consider the category of compactifications of X over Y, which is cofiltered according to More on Flatness, Theorem 33.8 and Lemmas 32.1 and 32.2. To every choice of a compactification

$$j : X \to \overline{X}, \quad \overline{f} : \overline{X} \to Y$$

the construction above associates the functor $R\overline{f}_* \circ j_! : D(X_{\acute{e}tale}, \Lambda) \to D(Y_{\acute{e}tale}, \Lambda)$. Let's be a little more explicit. Given a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules on $X_{\acute{e}tale}$, we choose a quasi-isomorphism $j_!\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex of sheaves of Λ-modules on $\overline{X}_{\acute{e}tale}$. Then our functor sends $\mathcal{K}^\bullet$ to $\overline{f}_*\mathcal{I}^\bullet$.

Suppose given a morphism $g : \overline{X}_1 \to \overline{X}_2$ between compactifications $j_i : X \to \overline{X}_i$ over Y. Then we get an isomorphism

$$R\overline{f}_{2,*} \circ j_{2,!} = R\overline{f}_{2,*} \circ Rg_* \circ j_{1,!} = R\overline{f}_{1,*} \circ j_{1,!}$$

using Lemma 8.6 in the first equality.

To finish the proof, since the category of compactifications of X over Y is cofiltered, it suffices to show compositions of morphisms of compactifications of X over Y are turned into compositions of isomorphisms of functors[3]. To do this, suppose that

[3]Namely, if $\alpha, \beta : F \to G$ are morphisms of functors and $\gamma : G \to H$ is an isomorphism of functors such that $\gamma \circ \alpha = \gamma \circ \beta$, then we conclude $\alpha = \beta$.

$j_3 : X \to \overline{X}_3$ is a third compactification and that $h : \overline{X}_2 \to \overline{X}_3$ is a morphism of compactifications. Then we have to show that the composition

$$R\overline{f}_{3,*} \circ j_{3,!} = R\overline{f}_{3,*} \circ Rh_* \circ j_{2,!} = R\overline{f}_{2,*} \circ j_{2,!} = R\overline{f}_{2,*} \circ Rg_* \circ j_{1,!} = R\overline{f}_{1,*} \circ j_{1,!}$$

is equal to the isomorphism of functors constructed using simply j_3, $g \circ h$, and j_1. A calculation shows that it suffices to prove that the composition of the maps

$$j_{3,!} \to Rh_* \circ j_{2,!} \to Rh_* \circ Rg_* \circ j_{1,!}$$

of Lemma 8.6 agrees with the corresponding map $j_{3,!} \to R(h \circ g)_* \circ j_{1,!}$ via the identification $R(h \circ g)_* = Rh_* \circ Rg_*$. Since the map of Lemma 8.6 is a special case of the map of Lemma 8.1 (as j_1 and j_2 are separated) this follows immediately from Lemma 8.2. □

Lemma 9.2. *Let $f : X \to Y$ and $g : Y \to Z$ be separated morphisms of finite type of quasi-compact and quasi-separated schemes. Then there is a canonical isomorphism $Rg_! \circ Rf_! \to R(g \circ f)_!$.*

Proof. Choose a compactification $i : Y \to \overline{Y}$ of Y over Z. Choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 33.8 and Lemma 32.2 twice. Let U be the inverse image of Y in $\overline{X}$ so that we get the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "j"'] \arrow[d, "f"'] & U \arrow[r, "j'"'] \arrow[dl, "f'"] & \overline{X} \arrow[dl, "\overline{f}"] \\
Y \arrow[r, "i"'] \arrow[d, "g"'] & \overline{Y} \arrow[dl, "\overline{g}"] & \\
Z & &
\end{tikzcd}$$

Then we have

$$\begin{aligned}
R(g \circ f)_! &= R(\overline{g} \circ \overline{f})_* \circ (j' \circ j)_! \\
&= R\overline{g}_* \circ R\overline{f}_* \circ j'_! \circ j_! \\
&= R\overline{g}_* \circ i_! \circ Rf'_* \circ j_! \\
&= Rg_! \circ Rf_!
\end{aligned}$$

The first equality is the definition of $R(g \circ f)_!$. The second equality uses the identifications $R(\overline{g} \circ \overline{f})_* = R\overline{g}_* \circ R\overline{f}_*$ and $(j' \circ j)_! = j'_! \circ j_!$ of Lemma 3.13. The identification $i_! \circ Rf'_* \to R\overline{f}_* \circ j_!$ used in the third equality is Lemma 8.1. The final fourth equality is the definition of $Rg_!$ and $Rf_!$. To finish the proof we show that this isomorphism is independent of choices made.

Suppose we have two diagrams

$$\begin{tikzcd}
X \arrow[r, "j_1"'] \arrow[d] & U_1 \arrow[r, "j'_1"'] \arrow[dl, "f_1"] & \overline{X}_1 \arrow[dl, "\overline{f}_1"] \\
Y \arrow[r, "i_1"'] \arrow[d] & \overline{Y}_1 \arrow[dl, "\overline{g}_1"] & \\
Z & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r, "j_2"'] \arrow[d] & U_2 \arrow[r, "j'_2"'] \arrow[dl, "f_2"] & \overline{X}_2 \arrow[dl, "\overline{f}_2"] \\
Y \arrow[r, "i_2"'] \arrow[d] & \overline{Y}_2 \arrow[dl, "\overline{g}_2"] & \\
Z & &
\end{tikzcd}$$

We can first choose a compactification $i : Y \to \overline{Y}$ of Y over Z which dominates both $\overline{Y}_1$ and $\overline{Y}_2$, see More on Flatness, Lemma 32.1. By More on Flatness, Lemma 32.3 and Categories, Lemmas 27.13 and 27.14 we can choose a compactification $X \to \overline{X}$ of X over $\overline{Y}$ with morphisms $\overline{X} \to \overline{X}_1$ and $\overline{X} \to \overline{X}_2$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_1$ is equal to the composition $\overline{X} \to \overline{X}_1 \to \overline{Y}_1$ and such that the composition $\overline{X} \to \overline{Y} \to \overline{Y}_2$ is equal to the composition $\overline{X} \to \overline{X}_2 \to \overline{Y}_2$. Thus we see that it suffices to compare the maps determined by our diagrams when we have a commutative diagram as follows

$$\begin{tikzcd}
X \arrow[r, "j_1"'] \arrow[d, equal] & U_1 \arrow[r, "j_1'"'] \arrow[d, "h'"] \arrow[ddl] & \overline{X}_1 \arrow[d, "h"] \arrow[ddl] \\
X \arrow[r, "j_2"] \arrow[d] & U_2 \arrow[r, "j_2'"] \arrow[ddl] & \overline{X}_2 \arrow[ddl] \\
Y \arrow[r, "i_1"] \arrow[d, equal] & \overline{Y}_1 \arrow[d, "k"] & \\
Y \arrow[r, "i_2"] \arrow[d] & \overline{Y}_2 \arrow[dl] & \\
Z & &
\end{tikzcd}$$

Each of the squares

$$\begin{tikzcd}
X \arrow[r, "j_1"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "A"] & U_1 \arrow[d, "h'"] \\
X \arrow[r, "j_2"] & U_2
\end{tikzcd}
\quad
\begin{tikzcd}
U_2 \arrow[r, "j_2'"'] \arrow[d, "f_2"'] \arrow[dr, phantom, "B"] & \overline{X}_2 \arrow[d, "\overline{f}_2"] \\
Y \arrow[r, "i_2"] & \overline{Y}_2
\end{tikzcd}
\quad
\begin{tikzcd}
U_1 \arrow[r, "j_1'"'] \arrow[d, "f_1"'] \arrow[dr, phantom, "C"] & \overline{X}_1 \arrow[d, "\overline{f}_1"] \\
Y \arrow[r, "i_1"] & \overline{Y}_1
\end{tikzcd}
\quad
\begin{tikzcd}
Y \arrow[r, "i_1"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "D"] & \overline{Y}_1 \arrow[d, "k"] \\
Y \arrow[r, "i_2"] & \overline{Y}_2
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r, "j_1' \circ j_1"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "E"] & \overline{X}_1 \arrow[d, "h"] \\
X \arrow[r, "j_2"] & \overline{X}_2
\end{tikzcd}$$

gives rise to an isomorphism as follows

$$\begin{aligned}
\gamma_A &: j_{2,!} \to Rh'_* \circ j_{1,!} \\
\gamma_B &: i_{2,!} \circ Rf_{2,*} \to R\overline{f}_{2,*} \circ j'_{2,!} \\
\gamma_C &: i_{1,!} \circ Rf_{1,*} \to R\overline{f}_{1,*} \circ j'_{1,!} \\
\gamma_D &: i_{2,!} \to Rk_* \circ i_{1,!} \\
\gamma_E &: j_{2,!} \to Rh_* \circ (j'_1 \circ j_1)_!
\end{aligned}$$

by applying the map from Lemma 8.1 (which is the same as the map in Lemma 8.6 in case the left vertical arrow is the identity). Let us write

$$\begin{aligned}
F_1 &= Rf_{1,*} \circ j_{1,!} \\
F_2 &= Rf_{2,*} \circ j_{2,!} \\
G_1 &= R\overline{g}_{1,*} \circ i_{1,!} \\
G_2 &= R\overline{g}_{2,*} \circ i_{2,!} \\
C_1 &= R(\overline{g}_1 \circ \overline{f}_1)_* \circ (j'_1 \circ j_1)_! \\
C_2 &= R(\overline{g}_2 \circ \overline{f}_2)_* \circ (j'_2 \circ j_2)_!
\end{aligned}$$

The construction given in the first paragraph of the proof and in Lemma 9.1 uses

(1) γ_C for the map $G_1 \circ F_1 \to C_1$,
(2) γ_B for the map $G_2 \circ F_2 \to C_2$,
(3) γ_A for the map $F_2 \to F_1$,
(4) γ_D for the map $G_2 \to G_1$, and
(5) γ_E for the map $C_2 \to C_1$.

This implies that we have to show that the diagram

$$\begin{tikzcd} C_2 \arrow[rr, "\gamma_E"'] & & C_1 \\ G_2 \circ F_2 \arrow[u, "\gamma_B"] \arrow[rr, "\gamma_D \circ \gamma_A"] & & G_1 \circ F_1 \arrow[u, "\gamma_C"'] \end{tikzcd}$$

is commutative. We will use Lemmas 8.2 and 8.3 and with (abuse of) notation as in Remark 8.4 (in particular dropping $\star$ products with identity transformations from the notation). We can write $\gamma_E = \gamma_F \circ \gamma_A$ where

$$\begin{tikzcd} U_1 \arrow[r, "j'_1"'] \arrow[d, "h'"'] \arrow[dr, phantom, "F"] & \overline{X}_1 \arrow[d, "h"] \\ U_2 \arrow[r, "j'_2"] & \overline{X}_2 \end{tikzcd}$$

Thus we see that

$$\gamma_E \circ \gamma_B = \gamma_F \circ \gamma_A \circ \gamma_B = \gamma_F \circ \gamma_B \circ \gamma_A$$

the last equality because the two squares A and B only intersect in one point (similar to the last argument in Remark 8.4). Thus it suffices to prove that $\gamma_C \circ \gamma_D = \gamma_F \circ \gamma_B$. Since both of these are equal to the map for the square

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & \overline{X}_1 \arrow[d] \\ Y \arrow[r] & \overline{Y}_2 \end{tikzcd}$$

we conclude. ☐

Lemma 9.3. *Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$ be separated morphisms of finite type of quasi-compact and quasi-separated schemes. Then the diagram*

$$\begin{tikzcd} Rh_! \circ Rg_! \circ Rf_! \arrow[r, "\gamma_C"'] \arrow[d, "\gamma_A"] & R(h \circ g)_! \circ Rf_! \arrow[d, "\gamma_{A+B}"'] \\ Rh_! \circ R(g \circ f)_! \arrow[r, "\gamma_{B+C}"] & R(h \circ g \circ f)_! \end{tikzcd}$$

of isomorphisms of Lemma 9.2 commutes (for the meaning of the γ's see proof).

Proof. To do this we choose a compactification $\overline{Z}$ of Z over T, then a compactification $\overline{Y}$ of Y over $\overline{Z}$, and then a compactification $\overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 33.8 and Lemma 32.2. Let $W \subset \overline{Y}$ be the inverse image of Z

under $\overline{Y} \to \overline{Z}$ and let $U \subset V \subset \overline{X}$ be the inverse images of $Y \subset W$ under $\overline{X} \to \overline{Y}$. This produces the following diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & U \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & V \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \overline{X} \arrow[d] \\
Y \arrow[r] \arrow[d, "g"'] & Y \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] \arrow[dr, phantom, "C"] & \overline{Y} \arrow[d] \\
Z \arrow[r] \arrow[d, "h"'] & Z \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \overline{Z} \arrow[d] \\
T \arrow[r] & T \arrow[r] & T \arrow[r] & T
\end{tikzcd}$$

Without introducing tons of notation but arguing exactly as in the proof of Lemma 9.2 we see that the maps in the first displayed diagram use the maps of Lemma 8.1 for the rectangles $A + B$, $B + C$, A, and C as indicated in the diagram in the statement of the lemma. Since by Lemmas 8.2 and 8.3 we have $\gamma_{A+B} = \gamma_B \circ \gamma_A$ and $\gamma_{B+C} = \gamma_B \circ \gamma_C$ we conclude that the desired equality holds provided $\gamma_A \circ \gamma_C = \gamma_C \circ \gamma_A$. This is true because the two squares A and C only intersect in one point (similar to the last argument in Remark 8.4). ☐

Lemma 9.4. *Consider a cartesian square*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

of quasi-compact and quasi-separated schemes with f separated and of finite type. Then there is a canonical isomorphism

$$g^{-1} \circ Rf_! \to Rf'_! \circ (g')^{-1}$$

Moreover, these isomorphisms are compatible with the isomorphisms of Lemma 9.2.

Proof. Choose a compactification $j : X \to \overline{X}$ over Y and denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Let $j' : X' \to \overline{X}'$ and $\overline{f}' : \overline{X}' \to Y'$ denote the base changes of j and $\overline{f}$. Since $Rf_! = R\overline{f}_* \circ j_!$ and $Rf'_! = R\overline{f}'_* \circ j'_!$ the isomorphism can be constructed via

$$g^{-1} \circ R\overline{f}_* \circ j_! \to R\overline{f}'_* \circ (\overline{g}')^{-1} \circ j_! \to R\overline{f}'_* \circ j'_! \circ (g')^{-1}$$

where the first arrow is the isomorphism given to us by the proper base change theorem (Étale Cohomology, Lemma 91.12 in the bounded below torsion case and Étale Cohomology, Lemma 92.3 in the case that Λ is torsion) and the second arrow is the isomorphism of Lemma 3.12.

To finish the proof we have to show two things: first we have to show that the isomorphism of functors so obtained does not depend on the choice of the compactification and second we have to show that if we vertically stack two base change diagrams as in the lemma, then these base change isomorphisms are compatible with the isomorphisms of Lemma 9.2. A straightforward argument which we omit shows that both follow if we can show that the isomorphisms

(1) $Rg_* \circ Rf_* = R(g \circ f)_*$ for $f : X \to Y$ and $g : Y \to Z$ proper,
(2) $g_! \circ f_! = (g \circ f)_!$ for $f : X \to Y$ and $g : Y \to Z$ separated and quasi-finite, and
(3) $g_! \circ Rf'_* = Rf_* \circ g'_!$ for $f : X \to Y$ and $f' : X' \to Y'$ proper and $g : Y' \to Y$ and $g' : X' \to X$ separated and quasi-finite with $f \circ g' = g \circ f'$

are compatible with base change. This holds for (1) by Cohomology on Sites, Remark 19.4, for (2) by Remark 3.14, and (3) by Lemma 8.5. □

Remark 9.5. Let $f : X \to Y$ be a finite type separated morphism of schemes with Y quasi-compact and quasi-separated. Below we will construct a map

$$Rf_!K \longrightarrow Rf_*K$$

functorial for K in $D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ or $D(X_{\acute{e}tale}, \Lambda)$ if Λ is torsion. This transformation of functors in both cases is compatible with

(1) the isomorphism $Rg_! \circ Rf_! \to R(g \circ f)_!$ of Lemma 9.2 and the isomorphism $Rg_* \circ Rf_* \to R(g \circ f)_*$ of Cohomology on Sites, Lemma 19.2 and
(2) the isomorphism $g^{-1} \circ Rf_! \to Rf'_! \circ (g')^{-1}$ of Lemma 9.4 and the base change map of Cohomology on Sites, Remark 19.3.

Namely, choose a compactification $j : X \to \overline{X}$ over Y and denote $\overline{f} : \overline{X} \to Y$ the structure morphism. Since $Rf_! = R\overline{f}_* \circ j_!$ and $Rf_* = R\overline{f}_* \circ Rj_*$ it suffices to construct a transformation of functors $j_! \to Rj_*$. For this we use the canonical transformation $j_! \to j_*$ of Étale Cohomology, Lemma 70.6. We omit the proof that the resulting transformation is independent of the choice of compactification and we omit the proof of the compatibilities (1) and (2).

10. Properties of derived lower shriek

Here are some properties of derived lower shriek.

Lemma 10.1. *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a ring.*

(1) *Let $K_i \in D^+_{tors}(X_{\acute{e}tale}, \Lambda)$, $i \in I$ be a family of objects. Assume given $a \in \mathbf{Z}$ such that $H^n(K_i) = 0$ for $n < a$ and $i \in I$. Then $Rf_!(\bigoplus_i K_i) = \bigoplus_i Rf_!K_i$.*
(2) *If Λ is torsion, then the functor $Rf_! : D(X_{\acute{e}tale}, \Lambda) \to D(Y_{\acute{e}tale}, \Lambda)$ commutes with direct sums.*

Proof. By construction it suffices to prove this when f is an open immersion and when f is a proper morphism. For any open immersion $j : U \to X$ of schemes, the functor $j_! : D(U_{\acute{e}tale}) \to D(X_{\acute{e}tale})$ is a left adjoint to pullback $j^{-1} : D(X_{\acute{e}tale}) \to D(U_{\acute{e}tale})$ and hence commutes with direct sums, see Cohomology on Sites, Lemma 20.8. In the proper case we have $Rf_! = Rf_*$ and we get the result from Étale Cohomology, Lemma 52.6 in the bounded belo case and from Étale Cohomology, Lemma 96.4 in the case that our coefficient ring Λ is a torsion ring. □

Lemma 10.2. *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a ring. The functors $Rf_!$ constructed in Section 9 are bounded in the following sense: There exists an integer N such that for $E \in D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ or $E \in D(X_{\acute{e}tale}, \Lambda)$ if Λ is torsion, we have*

(1) *$H^i(Rf_!(\tau_{\leq a}E) \to H^i(Rf_!(E))$ is an isomorphism for $i \leq a$,*
(2) *$H^i(Rf_!(E)) \to H^i(Rf_!(\tau_{\geq b-N}E))$ is an isomorphism for $i \geq b$,*

(3) *if* $H^i(E) = 0$ *for* $i \notin [a, b]$ *for some* $-\infty \leq a \leq b \leq \infty$, *then* $H^i(Rf_!(E)) = 0$ *for* $i \notin [a, b + N]$.

Proof. Assume Λ is torsion and consider the functor $Rf_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$. By construction it suffices to prove this when f is an open immersion and when f is a proper morphism. For any open immersion $j : U \to X$ of schemes, the functor $j_! : D(U_{étale}) \to D(X_{étale})$ is exact and hence the statement holds with $N = 0$ in this case. If f is proper then $Rf_! = Rf_*$, i.e., it is a right derived functor. Hence the bound on the left by Derived Categories, Lemma 16.1. Moreover in this case $f_* : Mod(X_{étale}, \Lambda) \to Mod(Y_{étale}, \Lambda)$ has bounded cohomological dimension by Morphisms, Lemma 28.5 and Étale Cohomology, Lemma 92.2. Thus we conclude by Derived Categories, Lemma 32.2.

Next, assume Λ is arbitrary and let us consider the functor $Rf_! : D^+_{tors}(X_{étale}, \Lambda) \to D^+_{tors}(Y_{étale}, \Lambda)$. Again we immediately reduce to the case where f is proper and $Rf_! = Rf_*$. Again part (1) is immediate. To show part (3) we can use induction on $b - a$, the distinguished triangles of trunctions, and Étale Cohomology, Lemma 92.2. Part (2) follows from (3). Details omitted. □

Lemma 10.3. *Let* $f : X \to Y$ *be a quasi-finite separated morphism of quasi-compact and quasi-separated schemes. Then the functors* $Rf_!$ *constructed in Section 9 agree with the restriction of the functor* $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ *constructed in Section 7 to their common domains of definition.*

Proof. By Zariski's main theorem (More on Morphisms, Lemma 43.3) we can find an open immersion $j : X \to \overline{X}$ and a finite morphism $\overline{f} : \overline{X} \to Y$ with $f = \overline{f} \circ j$. By construction we have $Rf_! = R\overline{f}_* \circ j_!$. Since $\overline{f}$ is finite, we have $R\overline{f}_* = \overline{f}_*$ by Étale Cohomology, Proposition 55.2. The lemma follows because $\overline{f}_* \circ j_! = f_!$ for example by Lemma 3.6. □

Lemma 10.4. *Let* $f : X \to Y$ *be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let* U *and* V *be quasi-compact opens of* X *such that* $X = U \cup V$. *Denote* $a : U \to Y$, $b : V \to Y$ *and* $c : U \cap V \to Y$ *the restrictions of* f. *Let* Λ *be a ring. For* K *in* $D^+_{tors}(X_{étale}, \Lambda)$ *or* $K \in D(X_{étale}, \Lambda)$ *if* Λ *is torsion, we have a distinguished triangle*

$$Rc_!(K|_{U \cap V}) \to Ra_!(K|_U) \oplus Rb_!(K|_V) \to Rf_!K \to Rc_!(K|_{U \cap V})[1]$$

in $D(Y_{étale}, \Lambda)$.

Proof. This follows from Lemma 7.3, the fact that $Rf_! \circ Rj_{U!} = Ra_!$ by Lemma 9.2, and the fact that $Rj_{U!} = j_{U!}$ by Lemma 10.3. □

Lemma 10.5. *Let* $f : X \to Y$ *be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let* U *be a quasi-compact open of* X *with complement* $Z \subset X$. *Denote* $g : U \to Y$ *and* $h : Z \to Y$ *the restrictions of* f. *Let* Λ *be a ring. For* K *in* $D^+_{tors}(X_{étale}, \Lambda)$ *or* $K \in D(X_{étale}, \Lambda)$ *if* Λ *is torsion, we have a distinguished triangle*

$$Rg_!(K|_U) \to Rf_!K \to Rh_!(K|_Z) \to Rg_!(K|_U)[1]$$

in $D(Y_{étale}, \Lambda)$.

Proof. This follows from Lemma 7.4, the fact that $Rf_! \circ Rj_! = Rg_!$ and $Rf_! \circ Ri_!$ by Lemma 9.2, and the fact that $Rj_! = j_!$ and $Ri_! = i_! = i_*$ by Lemma 10.3. □

Lemma 10.6. *Let $f' : X' \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let $i : X \to X'$ be a thickening and denote $f = f' \circ i$. Let Λ be a ring. For K' in $D^+_{tors}(X'_{\acute{e}tale}, \Lambda)$ or $K' \in D(X'_{\acute{e}tale}, \Lambda)$ if Λ is torsion, we have $Rf_! i^{-1}K' = Rf'_! K'$.*

Proof. This is true because i^{-1} and $i_* = i_!$ inverse equivalences of categories by the topological invariance of the small étale topos (Étale Cohomology, Theorem 45.2) and we can apply Lemma 9.2. $\square$

Lemma 10.7. *Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. Let $E \in D(X_{\acute{e}tale}, \Lambda)$ and $K \in D(Y_{\acute{e}tale}, \Lambda)$. Then*

$$Rf_! E \otimes^{\mathbf{L}}_\Lambda K = Rf_!(E \otimes^{\mathbf{L}}_\Lambda f^{-1}K)$$

in $D(Y_{\acute{e}tale}, \Lambda)$.

Proof. Choose $j : X \to \overline{X}$ and $\overline{f} : \overline{X} \to Y$ as in the construction of $Rf_!$. We have $j_! E \otimes^{\mathbf{L}}_\Lambda \overline{f}^{-1} K = j_!(E \otimes^{\mathbf{L}}_\Lambda f^{-1}K)$ by Cohomology on Sites, Lemma 20.9. Then we get the result by applying Étale Cohomology, Lemma 96.6 and using that $f^{-1} = j^{-1} \circ \overline{f}^{-1}$ and $Rf_! = R\overline{f}_* j_!$. $\square$

Remark 10.8. Let $\Lambda_1 \to \Lambda_2$ be a homomorphism of torsion rings. Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. The diagram

$$\begin{tikzcd} D(X_{\acute{e}tale}, \Lambda_2) \arrow[r, "res"'] \arrow[d, "Rf_!"'] & D(X_{\acute{e}tale}, \Lambda_1) \arrow[d, "Rf_!"] \\ D(Y_{\acute{e}tale}, \Lambda_2) \arrow[r, "res"] & D(Y_{\acute{e}tale}, \Lambda_1) \end{tikzcd}$$

commutes where *res* is the "restriction" functor which turns a Λ_2-module into a Λ_1-module using the given ring map. Writing $Rf_! = R\overline{f}_* \circ j_!$ for a factorization $f = \overline{f} \circ j$ as in Section 9, we see that the result holds for $j_!$ by inspection and for $R\overline{f}_*$ by Cohomology on Sites, Lemma 20.7. On the other hand, also the diagram

$$\begin{tikzcd} D(X_{\acute{e}tale}, \Lambda_1) \arrow[r, "-\otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_2"'] \arrow[d, "Rf_!"'] & D(X_{\acute{e}tale}, \Lambda_2) \arrow[d, "Rf_!"] \\ D(Y_{\acute{e}tale}, \Lambda_1) \arrow[r, "-\otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_2"] & D(Y_{\acute{e}tale}, \Lambda_2) \end{tikzcd}$$

is commutative as follows from Lemma 10.7.

Remark 10.9. Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring and let K and L be objects of $D(X_{\acute{e}tale}, \Lambda)$. We claim there is a canonical map

$$\alpha : Rf_* R\mathcal{H}om_\Lambda(K, L) \longrightarrow R\mathcal{H}om_\Lambda(Rf_!K, Rf_!L)$$

functorial in K and L. Namely, choose $j : X \to \overline{X}$ and $\overline{f} : \overline{X} \to Y$ as in the construction of $Rf_!$. We first define a map

$$\beta : Rj_* R\mathcal{H}om_\Lambda(K, L) \longrightarrow R\mathcal{H}om_\Lambda(j_!K, j_!L)$$

By the construction of internal hom in the derived category, this is the same thing as defining a map

$$\beta' : Rj_* R\mathcal{H}om_\Lambda(K, L) \otimes^{\mathbf{L}}_\Lambda j_! K \longrightarrow j_! L$$

See Cohomology on Sites, Section 35. The source of β' is equal to

$$j_! \left(R\mathcal{H}om_\Lambda(K, L) \otimes^{\mathbf{L}}_\Lambda K \right)$$

by Cohomology on Sites, Lemma 20.9. Hence we can set $\beta' = j_!\beta''$ where $\beta'' : R\mathcal{H}om_\Lambda(K, L) \otimes^{\mathbf{L}}_\Lambda K \to L$ corresponds to the identity on $R\mathcal{H}om_\Lambda(K, L)$ via the universal property of internal hom mentioned above. By Cohomology on Sites, Remark 35.10 we have a canonical map

$$\gamma : R\overline{f}_* R\mathcal{H}om_\Lambda(j_!K, j_!L) \longrightarrow R\mathcal{H}om_\Lambda(R\overline{f}_* j_! K, R\overline{f}_* j_! L)$$

Since $Rf_! = R\overline{f}_* j_!$ and $Rf_* = R\overline{f}_* Rj_*$ (by Leray) we obtain the desired map $\alpha = \gamma \circ R\overline{f}_*\beta$.

11. Derived upper shriek

We obtain $Rf^!$ by a Brown representability theorem.

Lemma 11.1. *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring. The functor $Rf_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ has a right adjoint $Rf^! : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$.*

Proof. This follows from Injectives, Proposition 15.2 and Lemma 10.1 above. $\square$

Lemma 11.2. *Let $f : X \to Y$ be a separated quasi-finite morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring. The functor $Rf^! : D(Y_{étale}, \Lambda) \to D(X_{étale}, \Lambda)$ of Lemma 11.1 is the same as the functor $Rf^!$ of Lemma 7.1.*

Proof. Follows from uniqueness of adjoints as $Rf_! = f_!$ by Lemma 10.3. $\square$

Lemma 11.3. *Let $j : U \to X$ be a separated étale morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion coefficient ring. The functor $Rj^! : D(X_{étale}, \Lambda) \to D(U_{étale}, \Lambda)$ is equal to j^{-1}.*

Proof. This is true because both $Rj^!$ and j^{-1} are right adjoints to $Rj_! = j_!$. See for example Lemmas 11.2 and 6.2. $\square$

Lemma 11.4. *Let $f : X \to Y$ be a finite type separated morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. The functor $Rf^!$ sends $D^+(Y_{étale}, \Lambda)$ into $D^+(X_{étale}, \Lambda)$. More precisely, there exists an integer $N \geq 0$ such that if $K \in D(Y_{étale}, \Lambda)$ has $H^i(K) = 0$ for $i < a$ then $H^i(Rf^!K) = 0$ for $i < a - N$.*

Proof. Let N be the integer found in Lemma 10.2. By construction, for $K \in D(Y_{étale}, \Lambda)$ and $L \in\in D(X_{étale}, \Lambda)$ we have $\mathrm{Hom}_X(L, Rf^!K) = \mathrm{Hom}_Y(Rf_!L, K)$. Suppose $H^i(K) = 0$ for $i < a$. Then we take $L = \tau_{\leq a-N-1} Rf^!K$. By Lemma 10.2 the complex $Rf_!L$ has vanishing cohomology sheaves in degrees $\leq a - 1$. Hence $\mathrm{Hom}_Y(Rf_!L, K) = 0$ by Derived Categories, Lemma 27.3. Hence the canonical map $\tau_{\leq a-N-1} Rf^!K \to Rf^!K$ is zero which implies $H^i(Rf^!K) = 0$ for $i \leq a - N - 1$. $\square$

Let $f : X \to Y$ be a separated finite type morphism of quasi-separated and quasi-compact schemes. Let Λ be a torsion coefficient ring. For every $K \in D(Y_{étale}, \Lambda)$ and $L \in D(X_{étale}, \Lambda)$ we obtain a canonical map

$$Rf_* R\mathcal{H}om_\Lambda(L, Rf^! K) \longrightarrow R\mathcal{H}om_\Lambda(Rf_! L, K) \tag{11.4.1}$$

Namely, this map is constructed as the composition

$$Rf_* R\mathcal{H}om_\Lambda(L, Rf^! K) \to R\mathcal{H}om_\Lambda(Rf_! L, Rf_! Rf^! K) \to R\mathcal{H}om_\Lambda(Rf_! L, K)$$

where the first arrow is Remark 10.9 and the second arrow is the counit $Rf_! Rf^! K \to K$ of the adjunction.

Lemma 11.5. *Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. For every $K \in D(Y_{étale}, \Lambda)$ and $L \in D(X_{étale}, \Lambda)$ the map (11.4.1)*

$$Rf_* R\mathcal{H}om_\Lambda(L, Rf^! K) \longrightarrow R\mathcal{H}om_\Lambda(Rf_! L, K)$$

is an isomorphism.

Proof. To prove the lemma we have to show that for any $M \in D(Y_{étale}, \Lambda)$ the map (11.4.1) induces an bijection

$$\mathrm{Hom}_Y(M, Rf_* R\mathcal{H}om_\Lambda(L, Rf^! K)) \longrightarrow \mathrm{Hom}_Y(M, R\mathcal{H}om_\Lambda(Rf_! L, K))$$

To see this we use the following string of equalities

$$\begin{aligned} \mathrm{Hom}_Y(M, Rf_* R\mathcal{H}om_\Lambda(L, Rf^! K)) &= \mathrm{Hom}_X(f^{-1}M, R\mathcal{H}om_\Lambda(L, Rf^! K)) \\ &= \mathrm{Hom}_X(f^{-1}M \otimes_\Lambda^{\mathbf{L}} L, Rf^! K) \\ &= \mathrm{Hom}_Y(Rf_!(f^{-1}M \otimes_\Lambda^{\mathbf{L}} L), K) \\ &= \mathrm{Hom}_Y(M \otimes_\Lambda^{\mathbf{L}} Rf_! L, K) \\ &= \mathrm{Hom}_Y(M, R\mathcal{H}om_\Lambda(Rf_! L, K)) \end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 19.1. The second equality by Cohomology on Sites, Lemma 35.2. The third equality by construction of $Rf^!$. The fourth equality by Lemma 10.7 (this is the important step). The fifth by Cohomology on Sites, Lemma 35.2. □

Lemma 11.6. *Let $f : X \to Y$ be a separated finite type morphism of quasi-separated and quasi-compact schemes. Let Λ be a torsion ring. For every $K \in D(Y_{étale}, \Lambda)$ and $L \in D(X_{étale}, \Lambda)$ the map (11.4.1) induces an isomorphism*

$$R\,\mathrm{Hom}_X(L, Rf^! K) \longrightarrow R\,\mathrm{Hom}_Y(Rf_! L, K)$$

of global derived homs.

Proof. By the construction in Cohomology on Sites, Section 36 we have

$$R\,\mathrm{Hom}_X(L, Rf^! K) = R\Gamma(X, R\mathcal{H}om_\Lambda(L, Rf^! K)) = R\Gamma(Y, Rf_* R\mathcal{H}om_\Lambda(L, Rf^! K))$$

(the second equality by Leray) and

$$R\,\mathrm{Hom}_Y(Rf_! L, K) = R\Gamma(Y, R\mathcal{H}om_\Lambda(Rf_! L, K))$$

Thus the lemma is a consequence of Lemma 11.5. □

Lemma 11.7. *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of quasi-compact and quasi-separated schemes with f separated and of finite type. Then we have $Rf^! \circ Rg_* = Rg'_* \circ R(f')^!$.

Proof. By uniqueness of adjoint functors this follows from base change for derived lower shriek: we have $g^{-1} \circ Rf_! = Rf'_! \circ (g')^{-1}$ by Lemma 9.4. $\square$

Remark 11.8. Let $\Lambda_1 \to \Lambda_2$ be a homomorphism of torsion rings. Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. The diagram

$$\begin{tikzcd} D(X_{\acute{e}tale}, \Lambda_2) \arrow[r, "res"] & D(X_{\acute{e}tale}, \Lambda_1) \\ D(Y_{\acute{e}tale}, \Lambda_2) \arrow[r, "res"] \arrow[u, "Rf^!"] & D(Y_{\acute{e}tale}, \Lambda_1) \arrow[u, "Rf^!"'] \end{tikzcd}$$

commutes where res is the "restriction" functor which turns a Λ_2-module into a Λ_1-module using the given ring map. This holds by uniquenss of adjoints, the second commutative diagram of Remark 10.8 and because we have

$$\mathrm{Hom}_{\Lambda_2}(K_1 \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_2, K_2) = \mathrm{Hom}_{\Lambda_1}(K_1, res(K_2))$$

This equality either for objects living over $X_{\acute{e}tale}$ or on $Y_{\acute{e}tale}$ is a very special case of Cohomology on Sites, Lemma 19.1.

12. Compactly supported cohomology

Let k be a field. Let Λ be a ring. Let X be a separated scheme of finite type over k with structure morphism $f : X \to \mathrm{Spec}(k)$. In Section 9 we have defined the functor $Rf_! : D^+_{tors}(X_{\acute{e}tale}, \Lambda) \to D^+_{tors}(\mathrm{Spec}(k), \Lambda)$ and the functor $Rf_! : D(X_{\acute{e}tale}, \Lambda) \to D(\mathrm{Spec}(k), \Lambda)$ if Λ is a torsion ring. Composing with the global sections functor on $\mathrm{Spec}(k)$ we obtain what we will call the compactly supported cohomology.

Definition 12.1. Let X be a separated scheme of finite type over a field k. Let Λ be a ring. Let K be an object of $D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ or of $D(X_{\acute{e}tale}, \Lambda)$ in case Λ is torsion. The *cohomology of K with compact support* or the *compactly supported cohomology of K* is

$$R\Gamma_c(X, K) = R\Gamma(\mathrm{Spec}(k), Rf_!K)$$

where $f : X \to \mathrm{Spec}(k)$ is the structure morphism. We will write $H^i_c(X, K) = H^i(R\Gamma_c(X, K))$.

We will check that this definition doesn't conflict with Definition 3.7 by Lemma 12.3. The utility of this definition lies in the following result.

Lemma 12.2. *Let $f : X \to Y$ be a finite type separated morphism of schemes with Y quasi-compact and quasi-separated. Let K be an object of $D^+_{tors}(X_{\acute{e}tale}, \Lambda)$ or of $D(X_{\acute{e}tale}, \Lambda)$ in case Λ is torsion. Then there is a canonical isomorphism*

$$(Rf_!K)_{\overline{y}} \longrightarrow R\Gamma_c(X_{\overline{y}}, K|_{X_{\overline{y}}})$$

in $D(\Lambda)$ for any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$.

Proof. Immediate consequence of Lemma 9.4 and the definitions. ☐

Lemma 12.3. *Let X be a separated scheme of finite type over a field k. If $\mathcal{F}$ is a torsion abelian sheaf, then the abelian group $H^0_c(X, \mathcal{F})$ defined in Definition 3.7 agrees with the abelian group $H^0_c(X, \mathcal{F})$ defined in Definition 12.1.*

Proof. Choose a compactification $j : X \to \overline{X}$ over k. In both cases the group is defined as $H^0(\overline{X}, j_!\mathcal{F})$. This is true for the first version by Lemma 3.10 and for the second version by construction. ☐

Lemma 12.4. *Let k be an algebraically closed field. Let X be a separated scheme of finite type type over k of dimension ≤ 1. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on X which is torsion. Then $H^q_c(X, \mathcal{F})$ is a finite Λ-module.*

Proof. This is a consequence of Étale Cohomology, Theorem 84.7. Namely, choose a compactification $j : X \to \overline{X}$. After replacing $\overline{X}$ by the scheme theoretic closure of X, we see that we may assume $\dim(\overline{X}) \leq 1$. Then $H^q_c(X, \mathcal{F}) = H^q(\overline{X}, j_!\mathcal{F})$ and the theorem applies. ☐

Remark 12.5 (Covariance of compactly supported cohomology)**.** Let k be a field. Let $f : X \to Y$ be a morphism of separated schemes of finite type over k. If X, Y, and f satisfies one of the following conditions

1. f is étale, or
2. f is flat and quasi-finite, or
3. f is quasi-finite and Y is geometrically unibranch, or
4. f is quasi-finite and there exists a weighting $w : X \to \mathbf{Z}$ of f

then compactly supported cohomology is covariant with respect to f. More precisely, let Λ be a ring. Let K be an object of $D^+_{tors}(Y_{\acute{e}tale}, \Lambda)$ or of $D(Y_{\acute{e}tale}, \Lambda)$ in case Λ is torsion. Under one of the assumptions (1) – (4) there is a canonical map

$$\mathrm{Tr}_{f,w,K} : f_! f^{-1} K \longrightarrow K$$

See Section 5 for the existence of the trace map and Examples 5.5 and 5.7 for cases (2) and (3). If $p : X \to \mathrm{Spec}(k)$ and $q : Y \to \mathrm{Spec}(k)$ denote the structure morphisms, then we have $Rq_! \circ f_! = Rp_!$ by Lemma 9.2 and the fact that $Rf_! = f_!$ for the quasi-finite separated morphism f by Lemma 10.3. Hence we can look at the map

$$\begin{aligned} R\Gamma_c(X, f^{-1}K) &= R\Gamma(\mathrm{Spec}(k), Rp_! f^{-1}K) \\ &= R\Gamma(\mathrm{Spec}(k), Rq_! f_! f^{-1}K) \\ &\xrightarrow{Rq_! \mathrm{Tr}_{f,w,K}} R\Gamma(\mathrm{Spec}(k), Rq_! K) \\ &= R\Gamma_c(Y, K) \end{aligned}$$

In particular, if Λ is a torsion ring, then we obtain an arrow

$$\mathrm{Tr}_f : R\Gamma_c(X, \Lambda) \longrightarrow R\Gamma_c(Y, \Lambda)$$

This map has lots of additional properties, for example it is compatible with taking ground field extensions.

13. A constructibility result

We "compute" the cohomology of a smooth projective family of curves with constant coefficients.

Lemma 13.1. *Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module viewed as an $\mathcal{O}_S$-module on $S_{étale}$. Let $F : \mathcal{E} \to \mathcal{E}$ be a homomorphism of abelian sheaves on $S_{étale}$ such that $F(ae) = a^pF(e)$ for local sections a, e of $\mathcal{O}_S$, $\mathcal{E}$ on $S_{étale}$. Then*

$$\mathrm{Coker}(F - 1 : \mathcal{E} \to \mathcal{E})$$

is zero and

$$\mathrm{Ker}(F - 1 : \mathcal{E} \to \mathcal{E})$$

is a constructible abelian sheaf on $S_{étale}$.

This lemma is a generalization of Étale Cohomology, Lemma 63.2.

Proof. We may assume $S = \mathrm{Spec}(A)$ where A is an $\mathbf{F}_p$-algebra and that $\mathcal{E}$ is the quasi-coherent module associated to the free A-module $Ae_1 \oplus \ldots \oplus Ae_n$. We write $F(e_i) = \sum a_{ij}e_j$.

Surjectivity of $F - 1$. It suffices to show that any element $\sum a_ie_i$, $a_i \in A$ is in the image of $F - 1$ after replacing A by a faithfully flat étale extension. Observe that

$$F(\sum x_ie_i) - \sum x_ie_i = \sum x_i^p a_{ij}e_j - \sum x_ie_i$$

Consider the A-algebra

$$A' = A[x_1, \ldots, x_n]/(a_i + x_i - \sum\nolimits_j a_{ji}x_j^p)$$

A computation shows that $\mathrm{d}x_i$ is zero in $\Omega_{A'/A}$ and hence $\Omega_{A'/A} = 0$. Since A' is of finite type over A, this implies that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is unramified and hence is quasi-finite. Since A' is generated by n elements and cut out by n equations, we conclude that A' is a global relative complete intersection over A. Thus A' is flat over A and we conclude that $A \to A'$ is étale (as a flat and unramified ring map). Finally, the reader can show that $A \to A'$ is faithfully flat by verifying directly that all geometric fibres of $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ are nonempty, however this also follows from Étale Cohomology, Lemma 63.2. Finally, the element $\sum x_ie_i \in A'e_1 \oplus \ldots \oplus A'e_n$ maps to $\sum a_ie_i$ by $F - 1$.

Constructibility of the kernel. The calculations above show that $\mathrm{Ker}(F - 1)$ is represented by the scheme

$$\mathrm{Spec}(A[x_1, \ldots, x_n]/(x_i - \sum\nolimits_j a_{ji}x_j^p))$$

over $S = \mathrm{Spec}(A)$. Since this is a scheme affine and étale over S we obtain the result from Étale Cohomology, Lemma 73.1. □

Lemma 13.2. *Let $f : X \to S$ be a proper smooth morphism of schemes with geometrically connected fibres of dimension 1. Let ℓ be a prime number. Then $R^qf_*\underline{\mathbf{Z}/\ell\mathbf{Z}}$ is a constructible.*

Proof. We may assume S is affine. Say $S = \text{Spec}(A)$. Then, if we write $A = \bigcup A_i$ as the union of its finite type $\mathbf{Z}$-subalgebras, we can find an i and a morphism $f_i : X_i \to S_i = \text{Spec}(A_i)$ of finite type whose base change to S is $f : X \to S$, see Limits, Lemma 10.1. After increasing i we may assume $f_i : X_i \to S_i$ is smooth, proper, and of relative dimension 1, see Limits, Lemmas 13.1 8.9, and 18.4. By More on Morphisms, Lemma 53.8 we obtain an open subscheme $U_i \subset S_i$ such that the fibres of $f_i : X_i \to S_i$ over U_i are geometrically connected. Then $S \to S_i$ maps into U_i. We may replace $X \to S$ by $f_i : f_i^{-1}(U_i) \to U_i$ to reduce to the case discussed in the next paragraph.

Assume S is Noetherian. We may write $S = U \cup Z$ where U is the open subscheme defined by the nonvanishing of ℓ and $Z = V(\ell) \subset S$. Since the formation of $R^q f_* \underline{\mathbf{Z}/\ell\mathbf{Z}}$ commutes with arbtrary base change (Étale Cohomology, Theorem 91.11), it suffices to prove the result over U and over Z. Thus we reduce to the following two cases: (a) ℓ is invertible on S and (b) ℓ is zero on S.

Case (a). We claim that in this case the sheaves $R^q f_* \underline{\mathbf{Z}/\ell\mathbf{Z}}$ are finite locally constant on S. First, by proper base change (in the form of Étale Cohomology, Lemma 91.13) and by finiteness (Étale Cohomology, Theorem 83.10) we see that the stalks of $R^q f_* \underline{\mathbf{Z}/\ell\mathbf{Z}}$ are finite. By Étale Cohomology, Lemma 94.4 all specialization maps are isomorphisms. We conclude the claim holds by Étale Cohomology, Lemma 75.6.

Case (b). Here $\ell = p$ is a prime and S is a scheme over $\text{Spec}(\mathbf{F}_p)$. By the same references as above we already know that the stalks of $R^q f_* \underline{\mathbf{Z}/p\mathbf{Z}}$ are finite and zero for $q \geq 2$. It follows from Étale Cohomology, Lemma 39.3 that $f_* \underline{\mathbf{Z}/p\mathbf{Z}} = \underline{\mathbf{Z}/p\mathbf{Z}}$. It remains to prove that $R^1 f_* \underline{\mathbf{Z}/p\mathbf{Z}}$ is constructible. Consider the Artin-Schreyer sequence

$$0 \to \underline{\mathbf{Z}/p\mathbf{Z}} \to \mathcal{O}_X \xrightarrow{F-1} \mathcal{O}_X \to 0$$

See Étale Cohomology, Section 63. Recall that $f_* \mathcal{O}_X = \mathcal{O}_S$ and $R^1 f_* \mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module of rank equal to the genera of the fibres of $X \to S$, see Algebraic Curves, Lemma 20.13. We conclude that we have a short exact sequence

$$0 \to \text{Coker}(F-1 : \mathcal{O}_S \to \mathcal{O}_S) \to R^1 f_* \underline{\mathbf{Z}/p\mathbf{Z}} \to \text{Ker}(F-1 : R^1 f_* \mathcal{O}_X \to R^1 f_* \mathcal{O}_X) \to 0$$

Applying Lemma 13.1 we win. $\square$

Lemma 13.3. *Let $f : X \to S$ be a proper smooth morphism of schemes with geometrically connected fibres of dimension 1. Let Λ be a Noetherian ring. Let M be a finite Λ-module annihilated by an integer $n > 0$. Then $R^q f_* \underline{M}$ is a constructible sheaf of Λ-modules on S.*

Proof. If $n = \ell n'$ for some prime number ℓ, then we get a short exact sequence $0 \to M[\ell] \to M \to M' \to 0$ of finite Λ-modules and M' is annihilated by n'. This produces a corresponding short exact sequence of constant sheaves, which in turn gives rise to an exact sequence

$$R^{q-1} f_* \underline{M'} \to R^q f_* \underline{M[n]} \to R^q f_* \underline{M} \to R^q f_* \underline{M'} \to R^{q+1} f_* \underline{M[n]}$$

Thus, if we can show the result in case M is annihilated by a prime number, then by induction on n we win by Étale Cohomology, Lemma 71.6.

Let ℓ be a prime number such that ℓ annihilates M. Then we can replace Λ by the $\mathbf{F}_\ell$-algebra $\Lambda/\ell\Lambda$. Namely, the sheaf $R^q f_* \underline{M}$ where $\underline{M}$ is viewed as a sheaf of

Λ-modules is the same as the sheaf $R^qf_*\underline{M}$ computed by viewing $\underline{M}$ as a sheaf of $\Lambda/\ell\Lambda$-modules, see Cohomology on Sites, Lemma 20.7.

Assume ℓ be a prime number such that ℓ annihilates M and Λ. Let us reduce to the case where M is a finite free Λ-module. Namely, choose a resolution

$$\ldots \to \Lambda^{\oplus m_2} \to \Lambda^{\oplus m_1} \to \Lambda^{\oplus m_0} \to M \to 0$$

Recall that f_* has finite cohomological dimension on sheaves of Λ-modules, see Étale Cohomology, Lemma 92.2 and Derived Categories, Lemma 32.2. Thus we see that $R^qf_*\underline{M}$ is the qth cohomology sheaf of the object

$$Rf_*(\underline{\Lambda^{\oplus m_a}} \to \ldots \to \underline{\Lambda^{\oplus m_0}})$$

in $D(S_{étale}, \Lambda)$ for some integer a large enough. Using the first spectral sequence of Derived Categories, Lemma 21.3 (or alternatively using an argument with truncations) we conclude that it suffices to prove that $R^qf_*\underline{\Lambda})$ is constructible.

At this point we can finally use that

$$(Rf_*\underline{\mathbf{Z}/\ell\mathbf{Z}}) \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell\mathbf{Z}} \underline{\Lambda} = Rf_*\underline{\Lambda}$$

by Étale Cohomology, Lemma 96.6. Since any module over the field $\mathbf{Z}/\ell\mathbf{Z}$ is flat we obtain

$$(R^qf_*\underline{\mathbf{Z}/\ell\mathbf{Z}}) \otimes_{\mathbf{Z}/\ell\mathbf{Z}} \underline{\Lambda} = R^qf_*\underline{\Lambda}$$

Hence it suffices to prove the result for $R^qf_*\underline{\mathbf{Z}/\ell\mathbf{Z}}$ by Étale Cohomology, Lemma 71.10. This case is Lemma 13.2. □

14. Complexes with constructible cohomology

We continue the discussion started in Étale Cohomology, Section 76. In particular, for a scheme X and a Noetherian ring Λ we denote $D_c(X_{étale}, \Lambda)$ the strictly full saturated triangulated subcategory of $D(X_{étale}, \Lambda)$ consisting of objects whose cohomology sheaves are constructible sheaves of Λ-modules.

Lemma 14.1. *Let $f : X \to Y$ be a morphism of schemes which is locally quasi-finite and of finite presentation. The functor $f_! : D(X_{étale}, \Lambda) \to D(Y_{étale}, \Lambda)$ of Lemma 7.1 sends $D_c(X_{étale}, \Lambda)$ into $D_c(Y_{étale}, \Lambda)$.*

Proof. Since the functor $f_!$ is exact, it suffices to show that $f_!\mathcal{F}$ is constructible for any constructible sheaf $\mathcal{F}$ of Λ-modules on $X_{étale}$. The question is local on Y and hence we may and do assume Y is affine. Then X is quasi-compact and quasi-separated, see Morphisms, Definition 21.1. Say $X = \bigcup_{i=1,\ldots,n} X_i$ is a finite affine open covering. By Lemma 4.7 we see that it suffices to show that $f_{i,!}\mathcal{F}|_{X_i}$ and $f_{ii',!}\mathcal{F}|_{X_i \cap X_{i'}}$ are constructible where $f_i : X_i \to Y$ and $f_{ii'} : X_i \cap X_{i'} \to Y$ are the restrictions of f. Since X_i and $X_i \cap X_{i'}$ are quasi-compact and separated this means we may assume f is separated. By Zariski's main theorem (in the form of More on Morphisms, Lemma 43.4) we can choose a factorization $f = g \circ j$ where $j : X \to X'$ is an open immersion and $g : X' \to Y$ is finite and of finite presentation. Then $f_! = g_! \circ j_!$ by Lemma 3.13. By Étale Cohomology, Lemma 73.1 we see that $j_!\mathcal{F}$ is constructible on X'. The morphism g is finite hence $g_! = g_*$ by Lemma 3.4. Thus $f_!\mathcal{F} = g_!j_!\mathcal{F} = g_*j_!\mathcal{F}$ is constructible by Étale Cohomology, Lemma 73.9. □

Lemma 14.2. *Let S be a Noetherian affine scheme of finite dimension. Let $f : X \to S$ be a separated, affine, smooth morphism of relative dimension 1. Let Λ be a Noetherian ring which is torsion. Let M be a finite Λ-module. Then $Rf_!\underline{M}$ has constructible cohomology sheaves.*

Proof. We will prove the result by induction on $d = \dim(S)$.

Base case. If $d = 0$, then the only thing to show is that the stalks of $R^qf_!\underline{M}$ are finite Λ-modules. If $\overline{s}$ is a geometric point of S, then we have $(R^qf_!\underline{M})_{\overline{s}} = H^q_c(X_{\overline{s}}, \underline{M})$ by Lemma 12.2. This is a finite Λ-module by Lemma 12.4.

Induction step. It suffices to find a dense open $U \subset S$ such that $Rf_!\underline{M}|_U$ has constructible cohomology sheaves. Namely, the restriction of $Rf_!\underline{M}$ to the complement $S \setminus U$ will have constructible cohomology sheaves by induction and the fact that formation of $Rf_!\underline{M}$ commutes with all base change (Lemma 9.4). In fact, let $\eta \in S$ be a generic point of an irreducible component of S. Then it suffices to find an open neighbourhood U of η such that the restriction of $Rf_!\underline{M}$ to U is constructible. This is what we will do in the next paragraph.

Given a generic point $\eta \in S$ we choose a diagram

$$\begin{tikzcd}
\overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n \arrow[rd] & Y_1 \amalg \ldots \amalg Y_n \arrow[l, "j"'] \arrow[r, "\nu"'] \arrow[d] & X_V \arrow[r] \arrow[d] & X_U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
 & T_1 \amalg \ldots \amalg T_n \arrow[r] & V \arrow[r] & U \arrow[r] & S
\end{tikzcd}$$

as in More on Morphisms, Lemma 56.1. We will show that $Rf_!\underline{M}|_U$ is constructible. First, since $V \to U$ is finite and surjective, it suffices to show that the pullback to V is constructible, see Étale Cohomology, Lemma 73.3. Since formation of $Rf_!$ commutes with base change, we see that it suffices to show that $R(X_V \to V)_!\underline{M}$ is constructible. Let $W \subset X_V$ be the open subscheme given to us by More on Morphisms, Lemma 56.1 part (4). Let $Z \subset X_V$ be the reduced induced scheme structure on the complement of W in X_V. Then the fibres of $Z \to V$ have dimension 0 (as W is dense in the fibres) and hence $Z \to V$ is quasi-finite. From the distinguished triangle

$$R(W \to V)_!\underline{M} \to R(X_V \to V)_!\underline{M} \to R(Z \to V)_!\underline{M} \to \ldots$$

of Lemma 10.5 and from Lemma 14.1 we conclude that it suffices to show that $R(W \to V)_!\underline{M}$ has constructible cohomology sheaves. Next, we have

$$R(W \to V)_!\underline{M} = R(\nu^{-1}(W) \to V)_!\underline{M}$$

because the morphism $\nu : \nu^{-1}(W) \to W$ is a thickening and we may apply Lemma 10.6. Next, we let $Z' \subset \coprod \overline{Y}_i$ denote the complement of the open $j(\nu^{-1}(W))$. Again $Z' \to V$ is quasi-finite. Again use the distinguished triangle

$$R(\nu^{-1}(W) \to V)_!\underline{M} \to R(\coprod \overline{Y}_i \to V)_!\underline{M} \to R(Z' \to V)_!\underline{M} \to \ldots$$

to conclude that it suffices to prove

$$R(\coprod \overline{Y}_i \to V)_!\underline{M} = \bigoplus\nolimits_i R(\overline{Y}_i \to V)_!\underline{M} = \bigoplus\nolimits_i R(T_i \to V)_! R(\overline{Y}_i \to T_i)_!\underline{M}$$

has constructible cohomology sheaves (second equality by Lemma 9.2). The result for $R(\overline{Y}_i \to T_i)_!\underline{M}$ is Lemma 13.3 and we win because $T_i \to V$ is finite étale and we can apply Lemma 14.1. $\square$

Lemma 14.3. *Let Y be a Noetherian affine scheme of finite dimension. Let Λ be a Noetherian ring which is torsion. Let $\mathcal{F}$ be a finite type, locally constant sheaf of Λ-modules on an open subscheme $U \subset \mathbf{A}^1_Y$. Then $Rf_!\mathcal{F}$ has constructible cohomology sheaves where $f : U \to Y$ is the structure morphism.*

Proof. We may decompose Λ as a product $\Lambda = \Lambda_1 \times \ldots \times \Lambda_r$ where Λ_i is ℓ_i-primary for some prime ℓ_i. Thus we may assume there exists a prime ℓ and an integer $n > 0$ such that ℓ^n annihilates Λ (and hence $\mathcal{F}$).

Since U is Noetherian, we see that U has finitely many connected components. Thus we may assume U is connected. Let $g : U' \to U$ be the finite étale covering constructed in Étale Cohomology, Lemma 66.4. The discussion in Étale Cohomology, Section 66 gives maps

$$\mathcal{F} \to g_*g^{-1}\mathcal{F} \to \mathcal{F}$$

whose composition is an isomorphism. Hence it suffices to prove the result for $g_*g^{-1}\mathcal{F}$. On the other hand, we have $Rf_!g_*g^{-1}\mathcal{F} = R(f \circ g)_!g^{-1}\mathcal{F}$ by Lemma 9.2. Since $g^{-1}\mathcal{F}$ has a finite filtration by constant sheaves of Λ-modules of the form $\underline{M}$ for some finite Λ-module M (by our choice of g) this reduces us to the case proved in Lemma 14.2. □

Lemma 14.4. *Let Y be an affine scheme. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $\mathbf{A}^1_Y$ which is torsion. Then $Rf_!\mathcal{F}$ has constructible cohomology sheaves where $f : \mathbf{A}^1_Y \to Y$ is the structure morphism.*

Proof. Say $\mathcal{F}$ is annihilated by $n > 0$. Then we can replace Λ by $\Lambda/n\Lambda$ without changing $Rf_!\mathcal{F}$. Thus we may and do assume Λ is a torsion ring.

Say $Y = \text{Spec}(R)$. Then, if we write $R = \bigcup R_i$ as the union of its finite type $\mathbf{Z}$-subalgebras, we can find an i such that $\mathcal{F}$ is the pullback of a constructible sheaf of Λ-modules on $\mathbf{A}^1_{R_i}$, see Étale Cohomology, Lemma 73.10. Hence we may assume Y is a Noetherian scheme of finite dimension.

Assume Y is a Noetherian scheme of finite dimension $d = \dim(Y)$ and Λ is torsion. We will prove the result by induction on d.

Base case. If $d = 0$, then the only thing to show is that the stalks of $R^qf_!\mathcal{F}$ are finite Λ-modules. If $\overline{y}$ is a geometric point of Y, then we have $(R^qf_!\mathcal{F})_{\overline{y}} = H^q_c(X_{\overline{y}}, \mathcal{F})$ by Lemma 12.2. This is a finite Λ-module by Lemma 12.4.

Induction step. It suffices to find a dense open $V \subset Y$ such that $Rf_!\mathcal{F}|_V$ has constructible cohomology sheaves. Namely, the restriction of $Rf_!\mathcal{F}$ to the complement $Y \setminus V$ will have constructible cohomology sheaves by induction and the fact that formation of $Rf_!\mathcal{F}$ commutes with all base change (Lemma 9.4). By definition of constructible sheaves of Λ-modules, there is a dense open subscheme $U \subset \mathbf{A}^1_Y$ such that $\mathcal{F}|_U$ is a finite type, locally constant sheaf of Λ-modules. Denote $Z \subset \mathbf{A}^1_Y$ the complement (viewed as a reduced closed subscheme). Note that U contains all the generic points of the fibres of $\mathbf{A}^1_Y \to Y$ over the generic points $\xi_1, \ldots, \xi_n$ of the irreducible components of Y. Hence $Z \to Y$ has finite fibres over $\xi_1, \ldots, \xi_n$. After replacing Y by a dense open (which is allowed), we may assume $Z \to Y$ is finite, see Morphisms, Lemma 51.1. By the distinguished triangle of Lemma 10.5 and the result for $Z \to Y$ (Lemma 14.1) we reduce to showing that $R(U \to Y)_!\mathcal{F}$ has constructible cohomology sheaves. This is Lemma 14.3. □

Theorem 14.5. *Let $f : X \to Y$ be a separated morphism of finite presentation of quasi-compact and quasi-separated schemes. Let Λ be a Noetherian ring. Let K be an object of $D^+_{tors,c}(X_{étale}, \Lambda)$ or of $D_c(X_{étale}, \Lambda)$ in case Λ is torsion. Then $Rf_!K$ has constructible cohomology sheaves, i.e., $Rf_!K$ is in $D^+_{tors,c}(Y_{étale}, \Lambda)$ or in $D_c(Y_{étale}, \Lambda)$ in case Λ is torsion.*

Proof. The question is local on Y hence we may and do assume Y is affine. By the induction principle and Lemma 10.4 we reduce to the case where X is also affine.

Assume X and Y are affine. Since X is of finite presentation, we can choose a closed immersion $i : X \to \mathbf{A}^n_Y$ which is of finite presentation. If $p : \mathbf{A}^n_Y \to Y$ denotes the structure morphism, then we see that $Rf_! = Rp_! \circ Ri_!$ by Lemma 9.2. By Lemma 14.1 we have the result for $Ri_! = i_!$. Hence we may assume f is the projection morphism $\mathbf{A}^n_Y \to Y$. Since we can view f as the composition

$$X = \mathbf{A}^n_Y \to \mathbf{A}^{n-1}_Y \to \mathbf{A}^{n-2}_S \to \ldots \to \mathbf{A}^1_Y \to Y$$

we may assume $n = 1$.

Assume Y is affine and $X = \mathbf{A}^1_Y$. Since $Rf_!$ has finite cohomological dimension (Lemma 10.2) we may assume K is bounded below. Using the first spectral sequence of Derived Categories, Lemma 21.3 (or alternatively using an argument with truncations), we reduce to showing the result of Lemma 14.4. ☐

15. Applications

In this section we give some applications of Theorem 14.5.

Lemma 15.1. *Let k be an algebraically closed field. Let X be a finite type separated scheme over k. Let Λ be a Noetherian ring. Let K be an object of $D^+_{tors,c}(X_{étale}, \Lambda)$ or of $D_c(X_{étale}, \Lambda)$ in case Λ is torsion. Then $H^i_c(X, K)$ is a finite Λ-module for all $i \in \mathbf{Z}$.*

Proof. Immediate consequence of Theorem 14.5 and the definition of compactly supported cohomology in Section 12. ☐

Proposition 15.2. *Let $f : X \to S$ be a smooth proper morphism of schemes. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a finite type, locally constant sheaf of Λ-modules on $X_{étale}$ such that for every geometric point $\overline{x}$ of X the stalk $\mathcal{F}_{\overline{x}}$ is annihilated by an integer $n > 0$ prime to the residue characteristic of $\overline{x}$. Then $R^if_*\mathcal{F}$ is a finite type, locally constant sheaf of Λ-modules on $S_{étale}$ for all $i \in \mathbf{Z}$.*

Proof. The question is local on S and hence we may assume S is affine. For a point x of X denote $n_x \geq 1$ the smallest integer annihilating $\mathcal{F}_{\overline{x}}$ for some (equivalently any) geometric point $\overline{x}$ of X lying over x. Since X is quasi-compact (being proper over affine) there exists a finite étale covering $\{U_j \to X\}_{j=1,\ldots,m}$ such that $\mathcal{F}|_{U_j}$ is constant. Since $U_j \to X$ is open, we conclude that the function $x \mapsto n_x$ is locally constant and takes finitely many values. Accordingly we obtain a finite decomposition $X = X_1 \amalg \ldots \amalg X_N$ into open and closed subschemes such that $n_x = n$ if and only if $x \in X_n$. Then it suffices to prove the lemma for the induced morphisms $X_n \to S$ and the restriction of $\mathcal{F}$ to X_n. Thus we may and do assume there exists an integer $n > 0$ such that $\mathcal{F}$ is annihilated by n and such that n is prime to the residue characteristics of all residue fields of X.

Since f is smooth and proper the image $f(X) \subset S$ is open and closed. Hence we may replace S by $f(X)$ and assume $f(X) = S$. In particular, we see that we may assume n is invertible in the ring defining the affine scheme S.

In this paragraph we reduce to the case where S is Noetherian. Write $S = \mathrm{Spec}(A)$ for some $\mathbf{Z}[1/n]$-algebra A. Write $A = \bigcup A_i$ as the union of its finite type $\mathbf{Z}[1/n]$-subalgebras. We can find an i and a morphism $f_i : X_i \to S_i = \mathrm{Spec}(A_i)$ of finite type whose base change to S is $f : X \to S$, see Limits, Lemma 10.1. After increasing i we may assume $f_i : X_i \to S_i$ is smooth and proper, see Limits, Lemmas 13.1 8.9, and 18.4. By Étale Cohomology, Lemma 73.11 we see that there exists an i and a finite type, locally constant sheaf of Λ-modules $\mathcal{F}_i$ whose pullback to X is isomorphic to $\mathcal{F}$. As $\mathcal{F}$ is annihilated by n, we may replace $\mathcal{F}_i$ by $\mathrm{Ker}(n : \mathcal{F}_i \to \mathcal{F}_i)$ and assume the same thing is true for $\mathcal{F}_i$. This reduces us to the case discussed in the next paragraph.

Assume we have an integer $n \geq 1$, the base scheme S is Noetherian and lives over $\mathbf{Z}[1/n]$, and $\mathcal{F}$ is n-torsion. By Theorem 14.5 the sheaves $R^i f_* \mathcal{F}$ are constructible sheaves of Λ-modules. By Étale Cohomology, Lemma 94.3 the specialization maps of $R^i f_* \mathcal{F}$ are always isomorphisms. We conclude by Étale Cohomology, Lemma 75.6. $\square$

16. More on derived upper shriek

Let Λ be a torsion ring. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[rr, "j"'] \arrow[dr, "g"'] & & U' \arrow[dl, "g'"] \\ & Y & \end{tikzcd}$$

of quasi-compact and quasi-separated schemes with g and g' separated and of finite type and with j étale. This induces a canonical map

$$Rg_! \Lambda \longrightarrow Rg'_! \Lambda$$

in $D(Y_{étale}, \Lambda)$. Namely, by Lemmas 9.2 and 10.3 we have $Rg_! = Rg'_! \circ j_!$. On the other hand, since $j_!$ is left adjoint to j^{-1} we have the counit $\mathrm{Tr}_j : j_! \Lambda = j_! j^{-1} \Lambda \to \Lambda$; we also call this the trace map for j, see Remark 5.6. The map above is constructed as the composition

$$Rg_! \Lambda = Rg'_! j_! \Lambda \xrightarrow{Rg'_! \mathrm{Tr}_j} Rg'_! \Lambda$$

Given a second étale morphism $j' : U' \to U''$ for some $g'' : U'' \to Y$ separated and of finite type the composition

$$Rg_! \Lambda \longrightarrow Rg'_! \Lambda \longrightarrow Rg''_! \Lambda$$

of the maps for j and j' is equal to the map $Rg_! \Lambda \longrightarrow Rg''_! \Lambda$ constructed for $j' \circ j$. This follows from the corresponding statement on trace maps, see Lemma 5.4 for a more general case.

Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Then we obtain a functor

$$X_{affine, étale} \longrightarrow \left\{ \begin{matrix} \text{schemes separated of finite type over } Y \\ \text{with étale morphisms between them} \end{matrix} \right\}$$

Thus the construction above determines a functor $X_{affine,étale}^{opp} \to D(Y_{étale}, \Lambda)$ sending U to $R(U \to Y)_!\Lambda$.

Lemma 16.1. *Let $f : X \to Y$ be a separated finite type morphism of quasi-compact and quasi-separated schemes. Let Λ be a torsion ring. Let $K \in D(Y_{étale}, \Lambda)$. For $n \in \mathbf{Z}$ the cohomology sheaf $H^n(Rf^!K)$ restricted to $X_{affine,étale}$ is the sheaf associated to the presheaf*

$$U \longmapsto \mathrm{Hom}_Y(R(U \to Y)_!\Lambda, K[n])$$

See discussion above for the functorial nature of $R(U \to Y)_!\Lambda$.

Proof. Let $j : U \to X$ be an object of $X_{affine,étale}$ and set $g = f \circ j$. Recall that $\mathrm{Hom}_X(j_!\Lambda, M[n]) = H^n(U, M)$ for any M in $D(X_{étale}, \Lambda)$. Then $H^n(Rf^!K)$ is the sheaf associated to the presheaf

$$U \mapsto H^n(U, Rf^!K) = \mathrm{Hom}_X(j_!\Lambda, Rf^!K[n]) = \mathrm{Hom}_Y(Rf_!j_!\Lambda, K[n] = \mathrm{Hom}_Y(Rg_!\Lambda, K[n])$$

We omit the verification that the transition maps are given by the transition maps between the objects $Rg_!\Lambda = R(U \to Y)_!\Lambda$ we constructed above. $\square$

17. Other chapters

References

[AGV71] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier, *Theorie de topos et cohomologie etale des schemas I, II, III*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer, 1971.