HYPERCOVERINGS

Contents

1. Introduction

Let $\mathcal{C}$ be a site, see Sites, Definition 6.2. Let X be an object of $\mathcal{C}$. Given an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we would like to compute its cohomology groups

$$H^i(X, \mathcal{F}).$$

According to our general definitions (Cohomology on Sites, Section 2) this cohomology group is computed by choosing an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \dots$ and setting

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1) \to \Gamma(X, \mathcal{I}^2) \to \dots)$$

The goal of this chapter is to show that we may also compute these cohomology groups without choosing an injective resolution (in the case that $\mathcal{C}$ has fibre products). To do this we will use hypercoverings.

A hypercovering in a site is a generalization of a covering, see [AGV71, Exposé V, Sec. 7]. Given a hypercovering K of an object X, there is a Čech to cohomology spectral sequence expressing the cohomology of an abelian sheaf $\mathcal{F}$ over X in terms of the cohomology of the sheaf over the components K_n of K. It turns out that there are always enough hypercoverings, so that taking the colimit over all hypercoverings, the spectral sequence degenerates and the cohomology of $\mathcal{F}$ over X is computed by the colimit of the Čech cohomology groups.

A more general gadget one can consider is a simplicial augmentation where one has cohomological descent, see [AGV71, Exposé Vbis]. A nice manuscript on cohomological descent is the text by Brian Conrad, see https://math.stanford.edu/

`~conrad/papers/hypercover.pdf`. We will come back to these issue in the chapter on simplicial spaces where we will show, for example, that proper hypercoverings of "locally compact" topological spaces are of cohomological descent (Simplicial Spaces, Section 25). Our method of attack will be to reduce this statement to the Čech to cohomology spectral sequence constructed in this chapter.

2. Semi-representable objects

In order to start we make the following definition. The letters "SR" stand for Semi-Representable.

Definition 2.1. Let $\mathcal{C}$ be a category. We denote $\mathrm{SR}(\mathcal{C})$ the category of *semi-representable objects* defined as follows

(1) objects are families of objects $\{U_i\}_{i \in I}$, and
(2) morphisms $\{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ are given by a map $\alpha : I \to J$ and for each $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ of $\mathcal{C}$.

Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. The category of *semi-representable objects over X* is the category $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$.

This definition is essentially equivalent to [AGV71, Exposé V, Subsection 7.3.0]. Note that this is a "big" category. We will later "bound" the size of the index sets I that we need for hypercoverings of X. We can then redefine $\mathrm{SR}(\mathcal{C}, X)$ to become a category. Let's spell out the objects and morphisms $\mathrm{SR}(\mathcal{C}, X)$:

(1) objects are families of morphisms $\{U_i \to X\}_{i \in I}$, and
(2) morphisms $\{U_i \to X\}_{i \in I} \to \{V_j \to X\}_{j \in J}$ are given by a map $\alpha : I \to J$ and for each $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ over X.

There is a forgetful functor $\mathrm{SR}(\mathcal{C}, X) \to \mathrm{SR}(\mathcal{C})$.

Definition 2.2. Let $\mathcal{C}$ be a category. We denote F the functor *which associates a presheaf to a semi-representable object.* In a formula

$$F : \mathrm{SR}(\mathcal{C}) \quad \longrightarrow \quad PSh(\mathcal{C})$$

$$\{U_i\}_{i \in I} \quad \longmapsto \quad \amalg_{i \in I} h_{U_i}$$

where h_U denotes the representable presheaf associated to the object U.

Given a morphism $U \to X$ we obtain a morphism $h_U \to h_X$ of representable presheaves. Thus we often think of F on $\mathrm{SR}(\mathcal{C}, X)$ as a functor into the category of presheaves of sets over h_X, namely $PSh(\mathcal{C})/h_X$. Here is a picture:

$$\begin{array}{ccc}
\mathrm{SR}(\mathcal{C}, X) & \xrightarrow{\ \ F\ \ } & PSh(\mathcal{C})/h_X \\
\downarrow & & \downarrow \\
\mathrm{SR}(\mathcal{C}) & \xrightarrow{\ \ F\ \ } & PSh(\mathcal{C})
\end{array}$$

Next we discuss the existence of limits in the category of semi-representable objects.

Lemma 2.3. *Let $\mathcal{C}$ be a category.*

(1) *the category $SR(\mathcal{C})$ has coproducts and F commutes with them,*
(2) *the functor $F : SR(\mathcal{C}) \to PSh(\mathcal{C})$ commutes with limits,*
(3) *if $\mathcal{C}$ has fibre products, then $SR(\mathcal{C})$ has fibre products,*
(4) *if $\mathcal{C}$ has products of pairs, then $SR(\mathcal{C})$ has products of pairs,*
(5) *if $\mathcal{C}$ has equalizers, so does $SR(\mathcal{C})$, and*

(6) *if $\mathcal{C}$ has a final object, so does $SR(\mathcal{C})$.*

Let $X \in \mathrm{Ob}(\mathcal{C})$.

(1) *the category $SR(\mathcal{C}, X)$ has coproducts and F commutes with them,*
(2) *if $\mathcal{C}$ has fibre products, then $SR(\mathcal{C}, X)$ has finite limits and $F : SR(\mathcal{C}, X) \to PSh(\mathcal{C})/h_X$ commutes with them.*

Proof. Proof of the results on $SR(\mathcal{C})$. Proof of (1). The coproduct of $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ is $\{U_i\}_{i \in I} \amalg \{V_j\}_{j \in J}$, in other words, the family of objects whose index set is $I \amalg J$ and for an element $k \in I \amalg J$ gives U_i if $k = i \in I$ and gives V_j if $k = j \in J$. Similarly for coproducts of families of objects. It is clear that F commutes with these.

Proof of (2). For U in $\mathrm{Ob}(\mathcal{C})$ consider the object $\{U\}$ of $SR(\mathcal{C})$. It is clear that $\mathrm{Mor}_{SR(\mathcal{C})}(\{U\}, K)) = F(K)(U)$ for $K \in \mathrm{Ob}(SR(\mathcal{C}))$. Since limits of presheaves are computed at the level of sections (Sites, Section 4) we conclude that F commutes with limits.

Proof of (3). Suppose given a morphism $(\alpha, f_i) : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ and a morphism $(\beta, g_k) : \{W_k\}_{k \in K} \to \{V_j\}_{j \in J}$. The fibred product of these morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k\}_{(i,j,k) \in I \times J \times K \text{ such that } j = \alpha(i) = \beta(k)}$$

The fibre products exist if $\mathcal{C}$ has fibre products.

Proof of (4). The product of $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ is $\{U_i \times V_j\}_{i \in I, j \in J}$. The products exist if $\mathcal{C}$ has products.

Proof of (5). The equalizer of two maps $(\alpha, f_i), (\alpha', f_i') : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ is

$$\{\mathrm{Eq}(f_i, f_i' : U_i \to V_{\alpha(i)})\}_{i \in I, \ \alpha(i) = \alpha'(i)}$$

The equalizers exist if $\mathcal{C}$ has equalizers.

Proof of (6). If X is a final object of $\mathcal{C}$, then $\{X\}$ is a final object of $SR(\mathcal{C})$.

Proof of the statements about $SR(\mathcal{C}, X)$. These follow from the results above applied to the category $\mathcal{C}/X$ using that $SR(\mathcal{C}/X) = SR(\mathcal{C}, X)$ and that $PSh(\mathcal{C}/X) = PSh(\mathcal{C})/h_X$ (Sites, Lemma 25.4 applied to $\mathcal{C}$ endowed with the chaotic topology). However we also argue directly as follows. It is clear that the coproduct of $\{U_i \to X\}_{i \in I}$ and $\{V_j \to X\}_{j \in J}$ is $\{U_i \to X\}_{i \in I} \amalg \{V_j \to X\}_{j \in J}$ and similarly for coproducts of families of families of morphisms with target X. The object $\{X \to X\}$ is a final object of $SR(\mathcal{C}, X)$. Suppose given a morphism $(\alpha, f_i) : \{U_i \to X\}_{i \in I} \to \{V_j \to X\}_{j \in J}$ and a morphism $(\beta, g_k) : \{W_k \to X\}_{k \in K} \to \{V_j \to X\}_{j \in J}$. The fibred product of these morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k \to X\}_{(i,j,k) \in I \times J \times K \text{ such that } j = \alpha(i) = \beta(k)}$$

The fibre products exist by the assumption that $\mathcal{C}$ has fibre products. Thus $SR(\mathcal{C}, X)$ has finite limits, see Categories, Lemma 18.4. We omit verifying the statements on the functor F in this case. $\square$

3. Hypercoverings

If we assume our category is a site, then we can make the following definition.

Definition 3.1. Let $\mathcal{C}$ be a site. Let $f = (\alpha, f_i) : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ be a morphism in the category $\mathrm{SR}(\mathcal{C})$. We say that f is a *covering* if for every $j \in J$ the family of morphisms $\{U_i \to V_j\}_{i \in I, \alpha(i)=j}$ is a covering for the site $\mathcal{C}$. Let X be an object of $\mathcal{C}$. A morphism $K \to L$ in $\mathrm{SR}(\mathcal{C}, X)$ is a *covering* if its image in $\mathrm{SR}(\mathcal{C})$ is a covering.

Lemma 3.2. *Let $\mathcal{C}$ be a site.*
  (1) *A composition of coverings in $\mathrm{SR}(\mathcal{C})$ is a covering.*
  (2) *If $K \to L$ is a covering in $\mathrm{SR}(\mathcal{C})$ and $L' \to L$ is a morphism, then $L' \times_L K$ exists and $L' \times_L K \to L'$ is a covering.*
  (3) *If $\mathcal{C}$ has products of pairs, and $A \to B$ and $K \to L$ are coverings in $\mathrm{SR}(\mathcal{C})$, then $A \times K \to B \times L$ is a covering.*

Let $X \in \mathrm{Ob}(\mathcal{C})$. Then (1) and (2) holds for $\mathrm{SR}(\mathcal{C}, X)$ and (3) holds if $\mathcal{C}$ has fibre products.

Proof. Part (1) is immediate from the axioms of a site. Part (2) follows by the construction of fibre products in $\mathrm{SR}(\mathcal{C})$ in the proof of Lemma 2.3 and the requirement that the morphisms in a covering of $\mathcal{C}$ are representable. Part (3) follows by thinking of $A \times K \to B \times L$ as the composition $A \times K \to B \times K \to B \times L$ and hence a composition of basechanges of coverings. The final statement follows because $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$. $\square$

By Lemma 2.3 and Simplicial, Lemma 19.2 the coskeleton of a truncated simplicial object of $\mathrm{SR}(\mathcal{C}, X)$ exists if $\mathcal{C}$ has fibre products. Hence the following definition makes sense.

Definition 3.3. Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. A *hypercovering of X* is a simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ such that

  (1) The object K_0 is a covering of X for the site $\mathcal{C}$.
  (2) For every $n \geq 0$ the canonical morphism

$$K_{n+1} \longrightarrow (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$$

   is a covering in the sense defined above.

Condition (1) makes sense since each object of $\mathrm{SR}(\mathcal{C}, X)$ is after all a family of morphisms with target X. It could also be formulated as saying that the morphism of K_0 to the final object of $\mathrm{SR}(\mathcal{C}, X)$ is a covering.

Example 3.4 (Čech hypercoverings). Let $\mathcal{C}$ be a site with fibre products. Let $\{U_i \to X\}_{i \in I}$ be a covering of $\mathcal{C}$. Set $K_0 = \{U_i \to X\}_{i \in I}$. Then K_0 is a 0-truncated simplicial object of $\mathrm{SR}(\mathcal{C}, X)$. Hence we may form

$$K = \mathrm{cosk}_0 K_0.$$

Clearly K passes condition (1) of Definition 3.3. Since all the morphisms $K_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$ are isomorphisms by Simplicial, Lemma 19.10 it also passes condition (2). Note that the terms K_n are the usual

$$K_n = \{U_{i_0} \times_X U_{i_1} \times_X \ldots \times_X U_{i_n} \to X\}_{(i_0, i_1, \ldots, i_n) \in I^{n+1}}$$

A hypercovering of X of this form is called a *Čech hypercovering* of X.

Example 3.5 (Hypercovering by a simplicial object of the site)**.** Let $\mathcal{C}$ be a site with fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$. Let U be a simplicial object of $\mathcal{C}$. As usual we denote $U_n = U([n])$. Finally, assume given an augmentation

$$a : U \to X$$

In this situation we can consider the simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ with terms $K_n = \{U_n \to X\}$. Then K is a hypercovering of X in the sense of Definition 3.3 if and only if the following three conditions[1] hold:

(1) $\{U_0 \to X\}$ is a covering of $\mathcal{C}$,
(2) $\{U_1 \to U_0 \times_X U_0\}$ is a covering of $\mathcal{C}$,
(3) $\{U_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n U)_{n+1}\}$ is a covering of $\mathcal{C}$ for $n \geq 1$.

We omit the straightforward verification.

Example 3.6 (Čech hypercovering associated to a cover)**.** Let $\mathcal{C}$ be a site with fibre products. Let $U \to X$ be a morphism of $\mathcal{C}$ such that $\{U \to X\}$ is a covering of $\mathcal{C}^2$. Consider the simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ with terms

$$K_n = \{U \times_X U \times_X \ldots \times_X U \to X\} \quad (n+1 \text{ factors})$$

Then K is a hypercovering of X. This example is a special case of both Example 3.4 and of Example 3.5.

Lemma 3.7. *Let $\mathcal{C}$ be a site with fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. The collection of all hypercoverings of X forms a set.*

Proof. Since $\mathcal{C}$ is a site, the set of all coverings of X forms a set. Thus we see that the collection of possible K_0 forms a set. Suppose we have shown that the collection of all possible $K_0, \ldots, K_n$ form a set. Then it is enough to show that given $K_0, \ldots, K_n$ the collection of all possible K_{n+1} forms a set. And this is clearly true since we have to choose K_{n+1} among all possible coverings of $(\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$. $\square$

Remark 3.8. The lemma does not just say that there is a cofinal system of choices of hypercoverings that is a set, but that really the hypercoverings form a set.

The category of presheaves on $\mathcal{C}$ has finite (co)limits. Hence the functors cosk_n exists for presheaves of sets.

Lemma 3.9. *Let $\mathcal{C}$ be a site with fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. Let K be a hypercovering of X. Consider the simplicial object $F(K)$ of $PSh(\mathcal{C})$, endowed with its augmentation to the constant simplicial presheaf h_X.*

(1) *The morphism of presheaves $F(K)_0 \to h_X$ becomes a surjection after sheafification.*
(2) *The morphism*

$$(d_0^1, d_1^1) : F(K)_1 \longrightarrow F(K)_0 \times_{h_X} F(K)_0$$

becomes a surjection after sheafification.
(3) *For every $n \geq 1$ the morphism*

$$F(K)_{n+1} \longrightarrow (cosk_n \, sk_n F(K))_{n+1}$$

turns into a surjection after sheafification.

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[1]As $\mathcal{C}$ has fibre products, the category $\mathcal{C}/X$ has all finite limits. Hence the required coskeleta exist by Simplicial, Lemma 19.2.

[2]A morphism of $\mathcal{C}$ with this property is sometimes called a "cover".

Proof. We will use the fact that if $\{U_i \to U\}_{i \in I}$ is a covering of the site $\mathcal{C}$, then the morphism

$$\amalg_{i \in I} h_{U_i} \to h_U$$

becomes surjective after sheafification, see Sites, Lemma 12.4. Thus the first assertion follows immediately.

For the second assertion, note that according to Simplicial, Example 19.1 the simplicial object $\mathrm{cosk}_0 \mathrm{sk}_0 K$ has terms $K_0 \times \ldots \times K_0$. Thus according to the definition of a hypercovering we see that $(d_0^1, d_1^1) : K_1 \to K_0 \times K_0$ is a covering. Hence (2) follows from the claim above and the fact that F transforms products into fibred products over h_X.

For the third, we claim that $\mathrm{cosk}_n \mathrm{sk}_n F(K) = F(\mathrm{cosk}_n \mathrm{sk}_n K)$ for $n \geq 1$. To prove this, denote temporarily F' the functor $\mathrm{SR}(\mathcal{C}, X) \to PSh(\mathcal{C})/h_X$. By Lemma 2.3 the functor F' commutes with finite limits. By our description of the cosk_n functor in Simplicial, Section 12 we see that $\mathrm{cosk}_n \mathrm{sk}_n F'(K) = F'(\mathrm{cosk}_n \mathrm{sk}_n K)$. Recall that the category used in the description of $(\mathrm{cosk}_n U)_m$ in Simplicial, Lemma 19.2 is the category $(\Delta/[m])_{\leq n}^{opp}$. It is an amusing exercise to show that $(\Delta/[m])_{\leq n}$ is a connected category (see Categories, Definition 16.1) as soon as $n \geq 1$. Hence, Categories, Lemma 16.2 shows that $\mathrm{cosk}_n \mathrm{sk}_n F'(K) = \mathrm{cosk}_n \mathrm{sk}_n F(K)$. Whence the claim. Property (2) follows from this, because now we see that the morphism in (2) is the result of applying the functor F to a covering as in Definition 3.1, and the result follows from the first fact mentioned in this proof. $\square$

4. Acyclicity

Let $\mathcal{C}$ be a site. For a presheaf of sets $\mathcal{F}$ we denote $\mathbf{Z}_{\mathcal{F}}$ the presheaf of abelian groups defined by the rule

$$\mathbf{Z}_{\mathcal{F}}(U) = \text{free abelian group on } \mathcal{F}(U).$$

We will sometimes call this the *free abelian presheaf on* $\mathcal{F}$. Of course the construction $\mathcal{F} \mapsto \mathbf{Z}_{\mathcal{F}}$ is a functor and it is left adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Of course the sheafification $\mathbf{Z}_{\mathcal{F}}^{\#}$ is a sheaf of abelian groups, and the functor $\mathcal{F} \mapsto \mathbf{Z}_{\mathcal{F}}^{\#}$ is a left adjoint as well. We sometimes call $\mathbf{Z}_{\mathcal{F}}^{\#}$ the *free abelian sheaf on* $\mathcal{F}$.

For an object X of the site $\mathcal{C}$ we denote $\mathbf{Z}_X$ the free abelian presheaf on h_X, and we denote $\mathbf{Z}_X^{\#}$ its sheafification.

Definition 4.1. Let $\mathcal{C}$ be a site. Let K be a simplicial object of $PSh(\mathcal{C})$. By the above we get a simplicial object $\mathbf{Z}_K^{\#}$ of $Ab(\mathcal{C})$. We can take its associated complex of abelian presheaves $s(\mathbf{Z}_K^{\#})$, see Simplicial, Section 23. The *homology of K* is the homology of the complex of abelian sheaves $s(\mathbf{Z}_K^{\#})$.

In other words, the *ith homology $H_i(K)$ of K* is the sheaf of abelian groups $H_i(K) = H_i(s(\mathbf{Z}_K^{\#}))$. In this section we worry about the homology in case K is a hypercovering of an object X of $\mathcal{C}$.

Lemma 4.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{F} \to \mathcal{G}$ be a morphism of presheaves of sets. Denote K the simplicial object of $PSh(\mathcal{C})$ whose nth term is the $(n+1)$st fibre*

product of $\mathcal{F}$ over $\mathcal{G}$, see Simplicial, Example 3.5. Then, if $\mathcal{F} \to \mathcal{G}$ is surjective after sheafification, we have

$$H_i(K) = \begin{cases} 0 & \text{if} \quad i > 0 \\ \mathbf{Z}_{\mathcal{G}}^{\#} & \text{if} \quad i = 0 \end{cases}$$

The isomorphism in degree 0 is given by the morphism $H_0(K) \to \mathbf{Z}_{\mathcal{G}}^{\#}$ coming from the map $(\mathbf{Z}_K^{\#})_0 = \mathbf{Z}_{\mathcal{F}}^{\#} \to \mathbf{Z}_{\mathcal{G}}^{\#}$.

Proof. Let $\mathcal{G}' \subset \mathcal{G}$ be the image of the morphism $\mathcal{F} \to \mathcal{G}$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Set $A = \mathcal{F}(U)$ and $B = \mathcal{G}'(U)$. Then the simplicial set $K(U)$ is equal to the simplicial set with n-simplices given by

$$A \times_B A \times_B \ldots \times_B A \ (n+1 \text{ factors}).$$

By Simplicial, Lemma 32.3 the morphism $K(U) \to B$ is a trivial Kan fibration. Thus it is a homotopy equivalence (Simplicial, Lemma 30.8). Hence applying the functor "free abelian group on" to this we deduce that

$$\mathbf{Z}_K(U) \longrightarrow \mathbf{Z}_B$$

is a homotopy equivalence. Note that $s(\mathbf{Z}_B)$ is the complex

$$\ldots \to \bigoplus_{b \in B} \mathbf{Z} \xrightarrow{0} \bigoplus_{b \in B} \mathbf{Z} \xrightarrow{1} \bigoplus_{b \in B} \mathbf{Z} \xrightarrow{0} \bigoplus_{b \in B} \mathbf{Z} \to 0$$

see Simplicial, Lemma 23.3. Thus we see that $H_i(s(\mathbf{Z}_K(U))) = 0$ for $i > 0$, and $H_0(s(\mathbf{Z}_K(U))) = \bigoplus_{b \in B} \mathbf{Z} = \bigoplus_{s \in \mathcal{G}'(U)} \mathbf{Z}$. These identifications are compatible with restriction maps.

We conclude that $H_i(s(\mathbf{Z}_K)) = 0$ for $i > 0$ and $H_0(s(\mathbf{Z}_K)) = \mathbf{Z}_{\mathcal{G}'}$, where here we compute homology groups in $PAb(\mathcal{C})$. Since sheafification is an exact functor we deduce the result of the lemma. Namely, the exactness implies that $H_0(s(\mathbf{Z}_K))^{\#} = H_0(s(\mathbf{Z}_K^{\#}))$, and similarly for other indices. $\square$

Lemma 4.3. *Let $\mathcal{C}$ be a site. Let $f : L \to K$ be a morphism of simplicial objects of $PSh(\mathcal{C})$. Let $n \geq 0$ be an integer. Assume that*

(1) *For $i < n$ the morphism $L_i \to K_i$ is an isomorphism.*
(2) *The morphism $L_n \to K_n$ is surjective after sheafification.*
(3) *The canonical map $L \to \mathrm{cosk}_n \mathrm{sk}_n L$ is an isomorphism.*
(4) *The canonical map $K \to \mathrm{cosk}_n \mathrm{sk}_n K$ is an isomorphism.*

Then $H_i(f) : H_i(L) \to H_i(K)$ is an isomorphism.

Proof. This proof is exactly the same as the proof of Lemma 4.2 above. Namely, we first let $K_n' \subset K_n$ be the sub presheaf which is the image of the map $L_n \to K_n$. Assumption (2) means that the sheafification of K_n' is equal to the sheafification of K_n. Moreover, since $L_i = K_i$ for all $i < n$ we see that get an n-truncated simplicial presheaf U by taking $U_0 = L_0 = K_0, \ldots, U_{n-1} = L_{n-1} = K_{n-1}, U_n = K_n'$. Denote $K' = \mathrm{cosk}_n U$, a simplicial presheaf. Because we can construct K_m' as a finite limit, and since sheafification is exact, we see that $(K_m')^{\#} = K_m$. In other words, $(K')^{\#} = K^{\#}$. We conclude, by exactness of sheafification once more, that $H_i(K) = H_i(K')$. Thus it suffices to prove the lemma for the morphism $L \to K'$, in other words, we may assume that $L_n \to K_n$ is a surjective morphism of *presheaves*!

In this case, for any object U of $\mathcal{C}$ we see that the morphism of simplicial sets

$$L(U) \longrightarrow K(U)$$

satisfies all the assumptions of Simplicial, Lemma 32.1. Hence it is a trivial Kan fibration. In particular it is a homotopy equivalence (Simplicial, Lemma 30.8). Thus

$$\mathbf{Z}_L(U) \longrightarrow \mathbf{Z}_K(U)$$

is a homotopy equivalence too. This for all U. The result follows. $\square$

Lemma 4.4. *Let $\mathcal{C}$ be a site. Let K be a simplicial presheaf. Let $\mathcal{G}$ be a presheaf. Let $K \to \mathcal{G}$ be an augmentation of K towards $\mathcal{G}$. Assume that*

(1) *The morphism of presheaves $K_0 \to \mathcal{G}$ becomes a surjection after sheafification.*
(2) *The morphism*
$$(d_0^1, d_1^1) : K_1 \longrightarrow K_0 \times_{\mathcal{G}} K_0$$
becomes a surjection after sheafification.
(3) *For every $n \geq 1$ the morphism*
$$K_{n+1} \longrightarrow (cosk_n \, sk_n K)_{n+1}$$
turns into a surjection after sheafification.

Then $H_i(K) = 0$ for $i > 0$ and $H_0(K) = \mathbf{Z}_{\mathcal{G}}^{\#}$.

Proof. Denote $K^n = \mathrm{cosk}_n \mathrm{sk}_n K$ for $n \geq 1$. Define K^0 as the simplicial object with terms $(K^0)_n$ equal to the $(n+1)$-fold fibred product $K_0 \times_{\mathcal{G}} \ldots \times_{\mathcal{G}} K_0$, see Simplicial, Example 3.5. We have morphisms

$$K \longrightarrow \ldots \to K^n \to K^{n-1} \to \ldots \to K^1 \to K^0.$$

The morphisms $K \to K^i$, $K^j \to K^i$ for $j \geq i \geq 1$ come from the universal properties of the cosk_n functors. The morphism $K^1 \to K^0$ is the canonical morphism from Simplicial, Remark 20.4. We also recall that $K^0 \to \mathrm{cosk}_1 \mathrm{sk}_1 K^0$ is an isomorphism, see Simplicial, Lemma 20.3.

By Lemma 4.2 we see that $H_i(K^0) = 0$ for $i > 0$ and $H_0(K^0) = \mathbf{Z}_{\mathcal{G}}^{\#}$.

Pick $n \geq 1$. Consider the morphism $K^n \to K^{n-1}$. It is an isomorphism on terms of degree $< n$. Note that $K^n \to \mathrm{cosk}_n \mathrm{sk}_n K^n$ and $K^{n-1} \to \mathrm{cosk}_n \mathrm{sk}_n K^{n-1}$ are isomorphisms. Note that $(K^n)_n = K_n$ and that $(K^{n-1})_n = (\mathrm{cosk}_{n-1} \mathrm{sk}_{n-1} K)_n$. Hence by assumption, we have that $(K^n)_n \to (K^{n-1})_n$ is a morphism of presheaves which becomes surjective after sheafification. By Lemma 4.3 we conclude that $H_i(K^n) = H_i(K^{n-1})$. Combined with the above this proves the lemma. $\square$

Lemma 4.5. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. The homology of the simplicial presheaf $F(K)$ is 0 in degrees > 0 and equal to $\mathbf{Z}_X^{\#}$ in degree 0.*

Proof. Combine Lemmas 4.4 and 3.9. $\square$

5. Čech cohomology and hypercoverings

Let $\mathcal{C}$ be a site. Consider a presheaf of abelian groups $\mathcal{F}$ on the site $\mathcal{C}$. It defines a functor

$$\begin{aligned} \mathcal{F} : \mathrm{SR}(\mathcal{C})^{opp} &\longrightarrow Ab \\ \{U_i\}_{i \in I} &\longmapsto \prod_{i \in I} \mathcal{F}(U_i) \end{aligned}$$

Thus a simplicial object K of $\mathrm{SR}(\mathcal{C})$ is turned into a cosimplicial object $\mathcal{F}(K)$ of Ab. The cochain complex $s(\mathcal{F}(K))$ associated to $\mathcal{F}(K)$ (Simplicial, Section 25) is called the Čech complex of $\mathcal{F}$ with respect to the simplicial object K. We set

$$\check{H}^i(K, \mathcal{F}) = H^i(s(\mathcal{F}(K))).$$

and we call it the ith Čech cohomology group of $\mathcal{F}$ with respect to K. In this section we prove analogues of some of the results for Čech cohomology of open coverings proved in Cohomology, Sections 9, 10 and 11.

Lemma 5.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then $\check{H}^0(K, \mathcal{F}) = \mathcal{F}(X)$.*

Proof. We have

$$\check{H}^0(K, \mathcal{F}) = \mathrm{Ker}(\mathcal{F}(K_0) \longrightarrow \mathcal{F}(K_1))$$

Write $K_0 = \{U_i \to X\}$. It is a covering in the site $\mathcal{C}$. As well, we have that $K_1 \to K_0 \times K_0$ is a covering in $\mathrm{SR}(\mathcal{C}, X)$. Hence we may write $K_1 = \amalg_{i_0, i_1 \in I} \{V_{i_0 i_1 j} \to X\}$ so that the morphism $K_1 \to K_0 \times K_0$ is given by coverings $\{V_{i_0 i_1 j} \to U_{i_0} \times_X U_{i_1}\}$ of the site $\mathcal{C}$. Thus we can further identify

$$\check{H}^0(K, \mathcal{F}) = \mathrm{Ker}(\prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i_0 i_1 j} \mathcal{F}(V_{i_0 i_1 j}))$$

with obvious map. The sheaf property of $\mathcal{F}$ implies that $\check{H}^0(K, \mathcal{F}) = H^0(X, \mathcal{F})$. $\square$

In fact this property characterizes the abelian sheaves among all abelian presheaves on $\mathcal{C}$ of course. The analogue of Cohomology, Lemma 5.2 in this case is the following.

Lemma 5.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{I}$ be an injective sheaf of abelian groups on $\mathcal{C}$. Then*

$$\check{H}^p(K, \mathcal{I}) = \begin{cases} \mathcal{I}(X) & if \quad p = 0 \\ 0 & if \quad p > 0 \end{cases}$$

Proof. Observe that for any object $Z = \{U_i \to X\}$ of $\mathrm{SR}(\mathcal{C}, X)$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$
\begin{aligned}
\mathcal{F}(Z) &= \prod \mathcal{F}(U_i) \\
&= \prod \mathrm{Mor}_{PSh(\mathcal{C})}(h_{U_i}, \mathcal{F}) \\
&= \mathrm{Mor}_{PSh(\mathcal{C})}(F(Z), \mathcal{F}) \\
&= \mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_{F(Z)}, \mathcal{F}) \\
&= \mathrm{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}_{F(Z)}^{\#}, \mathcal{F})
\end{aligned}
$$

Thus we see, for any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ that we have

$$(5.2.1) \qquad s(\mathcal{F}(K)) = \mathrm{Hom}_{Ab(\mathcal{C})}(s(\mathbf{Z}_{F(K)}^{\#}), \mathcal{F})$$

see Definition 4.1 for notation. The complex of sheaves $s(\mathbf{Z}_{F(K)}^{\#})$ is quasi-isomorphic to $\mathbf{Z}_X^{\#}$ if K is a hypercovering, see Lemma 4.5. We conclude that if $\mathcal{I}$ is an injective abelian sheaf, and K a hypercovering, then the complex $s(\mathcal{I}(K))$ is acyclic except possibly in degree 0. In other words, we have

$$\check{H}^i(K, \mathcal{I}) = 0$$

for $i > 0$. Combined with Lemma 5.1 the lemma is proved. $\square$

Next we come to the analogue of Cohomology on Sites, Lemma 10.6. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Recall that $\underline{H}^i(\mathcal{F})$ indicates the presheaf of abelian groups on $\mathcal{C}$ which is defined by the rule $\underline{H}^i(\mathcal{F}) : U \longmapsto H^i(U, \mathcal{F})$. We extend this to $\mathrm{SR}(\mathcal{C})$ as in the introduction to this section.

Lemma 5.3. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. There is a map*

$$s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D^+(Ab)$ functorial in $\mathcal{F}$, which induces natural transformations

$$\check{H}^i(K, -) \longrightarrow H^i(X, -)$$

as functors $Ab(\mathcal{C}) \to Ab$. Moreover, there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$ and in the hypercovering K.

Proof. We could prove this by the same method as employed in the corresponding lemma in the chapter on cohomology. Instead let us prove this by a double complex argument.

Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category of abelian sheaves on $\mathcal{C}$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \mathcal{I}^q(K_p)$$

where the differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential on the complex $s(\mathcal{I}^q(K))$ associated to the cosimplicial abelian group $\mathcal{I}^p(K)$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential $\mathcal{I}^q \to \mathcal{I}^{q+1}$. Denote $\mathrm{Tot}(A^{\bullet,\bullet})$ the total complex associated to the double complex $A^{\bullet,\bullet}$, see Homology, Section 18. We will use the two spectral sequences $('E_r, 'd_r)$ and $(''E_r, ''d_r)$ associated to this double complex, see Homology, Section 25.

By Lemma 5.2 the complexes $s(\mathcal{I}^q(K))$ are acyclic in positive degrees and have H^0 equal to $\mathcal{I}^q(X)$. Hence by Homology, Lemma 25.4 the natural map

$$\mathcal{I}^\bullet(X) \longrightarrow \mathrm{Tot}(A^{\bullet,\bullet})$$

is a quasi-isomorphism of complexes of abelian groups. In particular we conclude that $H^n(\mathrm{Tot}(A^{\bullet,\bullet})) = H^n(X, \mathcal{F})$.

The map $s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$ of the lemma is the composition of the map $s(\mathcal{F}(K)) \to \mathrm{Tot}(A^{\bullet,\bullet})$ followed by the inverse of the displayed quasi-isomorphism above. This works because $\mathcal{I}^\bullet(X)$ is a representative of $R\Gamma(X, \mathcal{F})$.

Consider the spectral sequence $('E_r, 'd_r)_{r \geq 0}$. By Homology, Lemma 25.1 we see that

$$'E_2^{p,q} = H_I^p(H_{II}^q(A^{\bullet,\bullet}))$$

In other words, we first take cohomology with respect to d_2 which gives the groups $'E_1^{p,q} = \underline{H}^q(\mathcal{F})(K_p)$. Hence it is indeed the case (by the description of the differential $'d_1$) that $'E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$. By the above and Homology, Lemma 25.3 we see that this converges to $H^n(X, \mathcal{F})$ as desired.

We omit the proof of the statements regarding the functoriality of the above constructions in the abelian sheaf $\mathcal{F}$ and the hypercovering K. $\square$

6. Hypercoverings a la Verdier

The astute reader will have noticed that all we need in order to get the Čech to cohomology spectral sequence for a hypercovering of an object X, is the conclusion of Lemma 3.9. Therefore the following definition makes sense.

Definition 6.1. Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has equalizers and fibre products. Let $\mathcal{G}$ be a presheaf of sets. A *hypercovering of $\mathcal{G}$* is a simplicial object K of $\text{SR}(\mathcal{C})$ endowed with an augmentation $F(K) \to \mathcal{G}$ such that

(1) $F(K_0) \to \mathcal{G}$ becomes surjective after sheafification,
(2) $F(K_1) \to F(K_0) \times_{\mathcal{G}} F(K_0)$ becomes surjective after sheafification, and
(3) $F(K_{n+1}) \longrightarrow F((\text{cosk}_n\text{sk}_n K)_{n+1})$ for $n \geq 1$ becomes surjective after sheafification.

We say that a simplicial object K of $\text{SR}(\mathcal{C})$ is a *hypercovering* if K is a hypercovering of the final object $*$ of $PSh(\mathcal{C})$.

The assumption that $\mathcal{C}$ has fibre products and equalizers guarantees that $\text{SR}(\mathcal{C})$ has fibre products and equalizers and F commutes with these (Lemma 2.3) which suffices to define the coskeleton functors used (see Simplicial, Remark 19.11 and Categories, Lemma 18.2). If $\mathcal{C}$ is general, we can replace the condition (3) by the condition that $F(K_{n+1}) \longrightarrow ((\text{cosk}_n\text{sk}_n F(K))_{n+1})$ for $n \geq 1$ becomes surjective after sheafification and the results of this section remain valid.

Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. In the previous section, we defined the Čech complex of $\mathcal{F}$ with respect to a simplicial object K of $\text{SR}(\mathcal{C})$. Next, given a presheaf $\mathcal{G}$ we set

$$H^0(\mathcal{G}, \mathcal{F}) = \text{Mor}_{PSh(\mathcal{C})}(\mathcal{G}, \mathcal{F}) = \text{Mor}_{Sh(\mathcal{C})}(\mathcal{G}^{\#}, \mathcal{F}) = H^0(\mathcal{G}^{\#}, \mathcal{F})$$

with notation as in Cohomology on Sites, Section 13. This is a left exact functor and its higher derived functors (briefly studied in Cohomology on Sites, Section 13) are denoted $H^i(\mathcal{G}, \mathcal{F})$. We will show that given a hypercovering K of $\mathcal{G}$, there is a Čech to cohomology spectral sequence converging to the cohomology $H^i(\mathcal{G}, \mathcal{F})$. Note that if $\mathcal{G} = *$, then $H^i(*, \mathcal{F}) = H^i(\mathcal{C}, \mathcal{F})$ recovers the cohomology of $\mathcal{F}$ on the site $\mathcal{C}$.

Lemma 6.2. *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then $\check{H}^0(K, \mathcal{F}) = H^0(\mathcal{G}, \mathcal{F})$.*

Proof. This follows from the definition of $H^0(\mathcal{G}, \mathcal{F})$ and the fact that

$$F(K_1) \underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}} F(K_0) \longrightarrow \mathcal{G}$$

becomes an coequalizer diagram after sheafification. $\square$

Lemma 6.3. *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{I}$ be an injective sheaf of abelian groups on $\mathcal{C}$. Then*

$$\check{H}^p(K, \mathcal{I}) = \begin{cases} H^0(\mathcal{G}, \mathcal{I}) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

Proof. By (5.2.1) we have

$$s(\mathcal{F}(K)) = \text{Hom}_{Ab(\mathcal{C})}(s(\mathbf{Z}^{\#}_{F(K)}), \mathcal{F})$$

The complex $s(\mathbf{Z}^{\#}_{F(K)})$ is quasi-isomorphic to $\mathbf{Z}^{\#}_{\mathcal{G}}$, see Lemma 4.4. We conclude that if $\mathcal{I}$ is an injective abelian sheaf, then the complex $s(\mathcal{I}(K))$ is acyclic except possibly in degree 0. In other words, we have $\check{H}^i(K, \mathcal{I}) = 0$ for $i > 0$. Combined with Lemma 6.2 the lemma is proved. $\square$

Lemma 6.4. *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. There is a map*

$$s(\mathcal{F}(K)) \longrightarrow R\Gamma(\mathcal{G}, \mathcal{F})$$

in $D^+(Ab)$ functorial in $\mathcal{F}$, which induces a natural transformation

$$\check{H}^i(K, -) \longrightarrow H^i(\mathcal{G}, -)$$

of functors $Ab(\mathcal{C}) \to Ab$. Moreover, there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(\mathcal{G}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$ and in the hypercovering K.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category of abelian sheaves on $\mathcal{C}$. Consider the double complex $A^{\bullet, \bullet}$ with terms

$$A^{p,q} = \mathcal{I}^q(K_p)$$

where the differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential $\mathcal{I}^p \to \mathcal{I}^{p+1}$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential on the complex $s(\mathcal{I}^p(K))$ associated to the cosimplicial abelian group $\mathcal{I}^p(K)$ as explained above. We will use the two spectral sequences $('E_r, 'd_r)$ and $(''E_r, ''d_r)$ associated to this double complex, see Homology, Section 25.

By Lemma 6.3 the complexes $s(\mathcal{I}^p(K))$ are acyclic in positive degrees and have H^0 equal to $H^0(\mathcal{G}, \mathcal{I}^p)$. Hence by Homology, Lemma 25.4 and its proof the spectral sequence $('E_r, 'd_r)$ degenerates, and the natural map

$$H^0(\mathcal{G}, \mathcal{I}^\bullet) \longrightarrow \text{Tot}(A^{\bullet, \bullet})$$

is a quasi-isomorphism of complexes of abelian groups. The map $s(\mathcal{F}(K)) \longrightarrow R\Gamma(\mathcal{G}, \mathcal{F})$ of the lemma is the composition of the natural map $s(\mathcal{F}(K)) \to \text{Tot}(A^{\bullet, \bullet})$ followed by the inverse of the displayed quasi-isomorphism above. This works because $H^0(\mathcal{G}, \mathcal{I}^\bullet)$ is a representative of $R\Gamma(\mathcal{G}, \mathcal{F})$.

Consider the spectral sequence $(''E_r, ''d_r)_{r \geq 0}$. By Homology, Lemma 25.1 we see that

$$''E_2^{p,q} = H_{II}^p(H_I^q(A^{\bullet, \bullet}))$$

In other words, we first take cohomology with respect to d_1 which gives the groups $''E_1^{p,q} = \underline{H}^p(\mathcal{F})(K_q)$. Hence it is indeed the case (by the description of the differential $''d_1$) that $''E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$. Since this spectral sequence converges to the cohomology of $\text{Tot}(A^{\bullet, \bullet})$ the proof is finished. $\square$

Lemma 6.5. *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Let $\mathcal{F}$ be an abelian sheaf. There is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

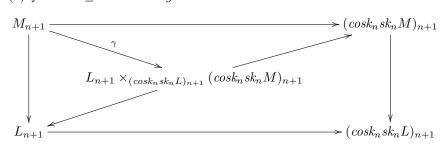*converging to the global cohomology groups $H^{p+q}(\mathcal{F})$.*

Proof. This is a special case of Lemma 6.4. □

7. Covering hypercoverings

Here are some ways to construct hypercoverings. We note that since the category $SR(\mathcal{C}, X)$ has fibre products the category of simplicial objects of $SR(\mathcal{C}, X)$ has fibre products as well, see Simplicial, Lemma 7.2.

Lemma 7.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K, L, M be simplicial objects of $SR(\mathcal{C}, X)$. Let $a : K \to L$, $b : M \to L$ be morphisms. Assume*

(1) *K is a hypercovering of X,*
(2) *the morphism $M_0 \to L_0$ is a covering, and*
(3) *for all $n \geq 0$ in the diagram*

$$M_{n+1} \longrightarrow (cosk_n sk_n M)_{n+1}$$
$$\gamma$$
$$L_{n+1} \times_{(cosk_n sk_n L)_{n+1}} (cosk_n sk_n M)_{n+1}$$
$$L_{n+1} \longrightarrow (cosk_n sk_n L)_{n+1}$$
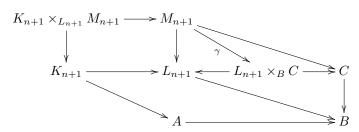
the arrow γ is a covering.

Then the fibre product $K \times_L M$ is a hypercovering of X.

Proof. The morphism $(K \times_L M)_0 = K_0 \times_{L_0} M_0 \to K_0$ is a base change of a covering by (2), hence a covering, see Lemma 3.2. And $K_0 \to \{X \to X\}$ is a covering by (1). Thus $(K \times_L M)_0 \to \{X \to X\}$ is a covering by Lemma 3.2. Hence $K \times_L M$ satisfies the first condition of Definition 3.3.

We still have to check that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K \times_L M)_{n+1} \longrightarrow (cosk_n sk_n (K \times_L M))_{n+1}$$

is a covering for all $n \geq 0$. We abbreviate as follows: $A = (cosk_n sk_n K)_{n+1}$, $B = (cosk_n sk_n L)_{n+1}$, and $C = (cosk_n sk_n M)_{n+1}$. The functor $cosk_n sk_n$ commutes with fibre products, see Simplicial, Lemma 19.13. Thus the right hand side above is equal to $A \times_B C$. Consider the following commutative diagram

$$K_{n+1} \times_{L_{n+1}} M_{n+1} \longrightarrow M_{n+1}$$
$$\gamma$$
$$K_{n+1} \longrightarrow L_{n+1} \longleftarrow L_{n+1} \times_B C \longrightarrow C$$
$$A \longrightarrow B$$

This diagram shows that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1}$$

Now, $K_{n+1} \times_B C \to A \times_B C$ is a base change of the covering $K_{n+1} \to A$ via the morphism $A \times_B C \to A$, hence is a covering. By assumption (3) the morphism γ is a covering. Hence the morphism

$$(K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1} \longrightarrow K_{n+1} \times_B C$$

is a covering as a base change of a covering. The lemma follows as a composition of coverings is a covering. $\qquad\square$

Lemma 7.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. If K, L are hypercoverings of X, then $K \times L$ is a hypercovering of X.*

Proof. You can either verify this directly, or use Lemma 7.1 above and check that $L \to \{X \to X\}$ has property (3). $\qquad\square$

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Since the category $\mathrm{SR}(\mathcal{C}, X)$ has coproducts and finite limits, it is permissible to speak about the objects $U \times K$ and $\mathrm{Hom}(U, K)$ for certain simplicial sets U (for example those with finitely many nondegenerate simplices) and any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$. See Simplicial, Sections 13 and 17.

Lemma 7.3. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $k \geq 0$ be an integer. Let $u : Z \to K_k$ be a covering in $SR(\mathcal{C}, X)$. Then there exists a morphism of hypercoverings $f : L \to K$ such that $L_k \to K_k$ factors through u.*

Proof. Denote $Y = K_k$. Let $C[k]$ be the cosimplicial set defined in Simplicial, Example 5.6. We will use the description of $\mathrm{Hom}(C[k], Y)$ and $\mathrm{Hom}(C[k], Z)$ given in Simplicial, Lemma 15.2. There is a canonical morphism $K \to \mathrm{Hom}(C[k], Y)$ corresponding to id : $K_k = Y \to Y$. Consider the morphism $\mathrm{Hom}(C[k], Z) \to \mathrm{Hom}(C[k], Y)$ which on degree n terms is the morphism

$$\prod\nolimits_{\alpha:[k]\to[n]} Z \longrightarrow \prod\nolimits_{\alpha:[k]\to[n]} Y$$

using the given morphism $Z \to Y$ on each factor. Set

$$L = K \times_{\mathrm{Hom}(C[k],Y)} \mathrm{Hom}(C[k], Z).$$

The morphism $L_k \to K_k$ sits in to a commutative diagram

$$
\begin{array}{ccccc}
L_k & \longrightarrow & \prod_{\alpha:[k]\to[k]} Z & \xrightarrow{\mathrm{pr}_{\mathrm{id}_{[k]}}} & Z \\
\downarrow & & \downarrow & & \downarrow \\
K_k & \longrightarrow & \prod_{\alpha:[k]\to[k]} Y & \xrightarrow{\mathrm{pr}_{\mathrm{id}_{[k]}}} & Y
\end{array}
$$

Since the composition of the two bottom arrows is the identity we conclude that we have the desired factorization.

We still have to show that L is a hypercovering of X. To see this we will use Lemma 7.1. Condition (1) is satisfied by assumption. For (2), the morphism

$$\mathrm{Hom}(C[k], Z)_0 \to \mathrm{Hom}(C[k], Y)_0$$

is a covering because it is isomorphic to $Z \to Y$ as there is only one morphism $[k] \to [0]$.

Let us consider condition (3) for $n = 0$. Then, since $(\mathrm{cosk}_0 T)_1 = T \times T$ (Simplicial, Example 19.1) and since $\mathrm{Hom}(C[k], Z)_1 = \prod_{\alpha:[k]\to[1]} Z$ we obtain the diagram

$$\begin{array}{ccc} \prod_{\alpha:[k]\to[1]} Z & \longrightarrow & Z \times Z \\ \downarrow & & \downarrow \\ \prod_{\alpha:[k]\to[1]} Y & \longrightarrow & Y \times Y \end{array}$$

with horizontal arrows corresponding to the projection onto the factors corresponding to the two nonsurjective α. Thus the arrow γ is the morphism

$$\prod_{\alpha:[k]\to[1]} Z \longrightarrow \prod_{\alpha:[k]\to[1] \text{ not onto}} Z \times \prod_{\alpha:[k]\to[1] \text{ onto}} Y$$

which is a product of coverings and hence a covering by Lemma 3.2.

Let us consider condition (3) for $n > 0$. We claim there is an injective map $\tau : S' \to S$ of finite sets, such that for any object T of $\mathrm{SR}(\mathcal{C}, X)$ the morphism

(7.3.1) $$\mathrm{Hom}(C[k], T)_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n \mathrm{Hom}(C[k], T))_{n+1}$$

is isomorphic to the projection $\prod_{s\in S} T \to \prod_{s'\in S'} T$ functorially in T. If this is true, then we see, arguing as in the previous paragraph, that the arrow γ is the morphism

$$\prod_{s\in S} Z \longrightarrow \prod_{s\in S'} Z \times \prod_{s\notin \tau(S')} Y$$

which is a product of coverings and hence a covering by Lemma 3.2. By construction, we have $\mathrm{Hom}(C[k], T)_{n+1} = \prod_{\alpha:[k]\to[n+1]} T$ (see Simplicial, Lemma 15.2). Correspondingly we take $S = \mathrm{Map}([k], [n+1])$. On the other hand, Simplicial, Lemma 19.5, provides a description of points of $(\mathrm{cosk}_n \mathrm{sk}_n \mathrm{Hom}(C[k], T))_{n+1}$ as sequences $(f_0, \ldots, f_{n+1})$ of points of $\mathrm{Hom}(C[k], T)_n$ satisfying $d_{j-1}^n f_i = d_i^n f_j$ for $0 \le i < j \le n + 1$. We can write $f_i = (f_{i,\alpha})$ with $f_{i,\alpha}$ a point of T and $\alpha \in \mathrm{Map}([k], [n])$. The conditions translate into

$$f_{i, \delta_{j-1}^n \circ \beta} = f_{j, \delta_i^n \circ \beta}$$

for any $0 \le i < j \le n + 1$ and $\beta : [k] \to [n-1]$. Thus we see that

$$S' = \{0, \ldots, n+1\} \times \mathrm{Map}([k], [n])/\sim$$

where the equivalence relation is generated by the equivalences

$$(i, \delta_{j-1}^n \circ \beta) \sim (j, \delta_i^n \circ \beta)$$

for $0 \le i < j \le n + 1$ and $\beta : [k] \to [n-1]$. A computation (omitted) shows that the morphism (7.3.1) corresponds to the map $S' \to S$ which sends (i, α) to $\delta_i^{n+1} \circ \alpha \in S$. (It may be a comfort to the reader to see that this map is well defined by part (1) of Simplicial, Lemma 2.3.) To finish the proof it suffices to show that if $\alpha, \alpha' : [k] \to [n]$ and $0 \le i < j \le n + 1$ are such that

$$\delta_i^{n+1} \circ \alpha = \delta_j^{n+1} \circ \alpha'$$

then we have $\alpha = \delta_{j-1}^n \circ \beta$ and $\alpha' = \delta_i^n \circ \beta$ for some $\beta : [k] \to [n-1]$. This is easy to see and omitted. $\qquad\square$

Lemma 7.4. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $n \geq 0$ be an integer. Let $u : \mathcal{F} \to F(K_n)$ be a morphism of presheaves which becomes surjective on sheafification. Then there exists a morphism of hypercoverings $f : L \to K$ such that $F(f_n) : F(L_n) \to F(K_n)$ factors through u.*

Proof. Write $K_n = \{U_i \to X\}_{i \in I}$. Thus the map u is a morphism of presheaves of sets $u : \mathcal{F} \to \amalg h_{u_i}$. The assumption on u means that for every $i \in I$ there exists a covering $\{U_{ij} \to U_i\}_{j \in I_i}$ of the site $\mathcal{C}$ and a morphism of presheaves $t_{ij} : h_{U_{ij}} \to \mathcal{F}$ such that $u \circ t_{ij}$ is the map $h_{U_{ij}} \to h_{U_i}$ coming from the morphism $U_{ij} \to U_i$. Set $J = \amalg_{i \in I} I_i$, and let $\alpha : J \to I$ be the obvious map. For $j \in J$ denote $V_j = U_{\alpha(j)j}$. Set $Z = \{V_j \to X\}_{j \in J}$. Finally, consider the morphism $u' : Z \to K_n$ given by $\alpha : J \to I$ and the morphisms $V_j = U_{\alpha(j)j} \to U_{\alpha(j)}$ above. Clearly, this is a covering in the category $\mathrm{SR}(\mathcal{C}, X)$, and by construction $F(u') : F(Z) \to F(K_n)$ factors through u. Thus the result follows from Lemma 7.3 above. $\qquad\square$

8. Adding simplices

In this section we prove some technical lemmas which we will need later. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. As we pointed out in Section 7 above, the objects $U \times K$ and $\mathrm{Hom}(U, K)$ for certain simplicial sets U and any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ are defined. See Simplicial, Sections 13 and 17.

Lemma 8.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U has finitely many nondegenerate simplices. Suppose $n \geq 0$ and $x \in V_n$, $x \notin U_n$ are such that*

(1) *$V_i = U_i$ for $i < n$,*
(2) *$V_n = U_n \cup \{x\}$,*
(3) *any $z \in V_j$, $z \notin U_j$ for $j > n$ is degenerate.*

Then the morphism

$$\mathrm{Hom}(V, K)_0 \longrightarrow \mathrm{Hom}(U, K)_0$$

of $SR(\mathcal{C}, X)$ is a covering.

Proof. If $n = 0$, then it follows easily that $V = U \amalg \Delta[0]$ (see below). In this case $\mathrm{Hom}(V, K)_0 = \mathrm{Hom}(U, K)_0 \times K_0$. The result, in this case, then follows from Lemma 3.2.

Let $a : \Delta[n] \to V$ be the morphism associated to x as in Simplicial, Lemma 11.3. Let us write $\partial \Delta[n] = i_{(n-1)!}\mathrm{sk}_{n-1}\Delta[n]$ for the $(n-1)$-skeleton of $\Delta[n]$. Let $b : \partial \Delta[n] \to U$ be the restriction of a to the $(n-1)$ skeleton of $\Delta[n]$. By Simplicial, Lemma 21.7 we have $V = U \amalg_{\partial \Delta[n]} \Delta[n]$. By Simplicial, Lemma 17.5 we get that

$$
\begin{CD}
\mathrm{Hom}(V, K)_0 @>>> \mathrm{Hom}(U, K)_0 \\
@VVV @VVV \\
\mathrm{Hom}(\Delta[n], K)_0 @>>> \mathrm{Hom}(\partial\Delta[n], K)_0
\end{CD}
$$

is a fibre product square. Thus it suffices to show that the bottom horizontal arrow is a covering. By Simplicial, Lemma 21.11 this arrow is identified with

$$K_n \to (\mathrm{cosk}_{n-1}\mathrm{sk}_{n-1}K)_n$$

and hence is a covering by definition of a hypercovering. $\square$

Lemma 8.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U and V have finitely many nondegenerate simplices. Then the morphism*

$$\mathrm{Hom}(V, K)_0 \longrightarrow \mathrm{Hom}(U, K)_0$$

of $SR(\mathcal{C}, X)$ is a covering.

Proof. By Lemma 8.1 above, it suffices to prove a simple lemma about inclusions of simplicial sets $U \subset V$ as in the lemma. And this is exactly the result of Simplicial, Lemma 21.8. $\square$

Lemma 8.3. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Then*

(1) K_n *is a covering of X for each $n \geq 0$,*
(2) $d_i^n : K_n \to K_{n-1}$ *is a covering for all $n \geq 1$ and $0 \leq i \leq n$.*

Proof. Recall that K_0 is a covering of X by Definition 3.3 and that this is equivalent to saying that $K_0 \to \{X \to X\}$ is a covering in the sense of Definition 3.1. Hence (1) follows from (2) because it will prove that the composition $K_n \to K_{n-1} \to \ldots \to K_0 \to \{X \to X\}$ is a covering by Lemma 3.2.

Proof of (2). Observe that $\mathrm{Mor}(\Delta[n], K)_0 = K_n$ by Simplicial, Lemma 17.4. Therefore (2) follows from Lemma 8.2 applied to the $n + 1$ different inclusions $\Delta[n-1] \to \Delta[n]$. $\square$

Remark 8.4. A useful special case of Lemmas 8.2 and 8.3 is the following. Suppose we have a category $\mathcal{C}$ having fibre products. Let $P \subset \mathrm{Arrows}(\mathcal{C})$ be a subset stable under base change, stable under composition, and containing all isomorphisms. Then one says a *P-hypercovering* is an augmentation $a : U \to X$ from a simplicial object of $\mathcal{C}$ such that

(1) $U_0 \to X$ is in P,
(2) $U_1 \to U_0 \times_X U_0$ is in P,
(3) $U_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n U)_{n+1}$ is in P for $n \geq 1$.

The category $\mathcal{C}/X$ has all finite limits, hence the coskeleta used in the formulation above exist (see Categories, Lemma 18.4). Then we claim that the morphisms $U_n \to X$ and $d_i^n : U_n \to U_{n-1}$ are in P. This follows from the aforementioned lemmas by turning $\mathcal{C}$ into a site whose coverings are $\{f : V \to U\}$ with $f \in P$ and taking K given by $K_n = \{U_n \to X\}$.

9. Homotopies

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let L be a simplicial object of $SR(\mathcal{C}, X)$. According to Simplicial, Lemma 17.4 there exists an object $\mathrm{Hom}(\Delta[1], L)$ in the category $\mathrm{Simp}(SR(\mathcal{C}, X))$ which represents the functor

$$T \longmapsto \mathrm{Mor}_{\mathrm{Simp}(SR(\mathcal{C}, X))}(\Delta[1] \times T, L)$$

There is a canonical morphism

$$\mathrm{Hom}(\Delta[1], L) \to L \times L$$

coming from $e_i : \Delta[0] \to \Delta[1]$ and the identification $\mathrm{Hom}(\Delta[0], L) = L$.

Lemma 9.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let L be a simplicial object of $SR(\mathcal{C}, X)$. Let $n \geq 0$. Consider the commutative diagram*

(9.1.1)
$$
\begin{array}{ccc}
\mathrm{Hom}(\Delta[1], L)_{n+1} & \longrightarrow & (cosk_n sk_n \, \mathrm{Hom}(\Delta[1], L))_{n+1} \\
\downarrow & & \downarrow \\
(L \times L)_{n+1} & \longrightarrow & (cosk_n sk_n (L \times L))_{n+1}
\end{array}
$$

coming from the morphism defined above. We can identify the terms in this diagram as follows, where $\partial\Delta[n+1] = i_{n!}sk_n\Delta[n+1]$ is the n-skeleton of the $(n+1)$-simplex:

$$
\begin{array}{rcl}
\mathrm{Hom}(\Delta[1], L)_{n+1} & = & \mathrm{Hom}(\Delta[1] \times \Delta[n+1], L)_0 \\
(cosk_n sk_n \, \mathrm{Hom}(\Delta[1], L))_{n+1} & = & \mathrm{Hom}(\Delta[1] \times \partial\Delta[n+1], L)_0 \\
(L \times L)_{n+1} & = & \mathrm{Hom}((\Delta[n+1] \amalg \Delta[n+1], L)_0 \\
(cosk_n sk_n (L \times L))_{n+1} & = & \mathrm{Hom}(\partial\Delta[n+1] \amalg \partial\Delta[n+1], L)_0
\end{array}
$$

and the morphism between these objects of $SR(\mathcal{C}, X)$ come from the commutative diagram of simplicial sets

(9.1.2)
$$
\begin{array}{ccc}
\Delta[1] \times \Delta[n+1] & \longleftarrow & \Delta[1] \times \partial\Delta[n+1] \\
\uparrow & & \uparrow \\
\Delta[n+1] \amalg \Delta[n+1] & \longleftarrow & \partial\Delta[n+1] \amalg \partial\Delta[n+1]
\end{array}
$$

Moreover the fibre product of the bottom arrow and the right arrow in (9.1.1) is equal to

$$
\mathrm{Hom}(U, L)_0
$$

where $U \subset \Delta[1] \times \Delta[n+1]$ is the smallest simplicial subset such that both $\Delta[n+1] \amalg \Delta[n+1]$ and $\Delta[1] \times \partial\Delta[n+1]$ map into it.

Proof. The first and third equalities are Simplicial, Lemma 17.4. The second and fourth follow from the cited lemma combined with Simplicial, Lemma 21.11. The last assertion follows from the fact that U is the push-out of the bottom and right arrow of the diagram (9.1.2), via Simplicial, Lemma 17.5. To see that U is equal to this push-out it suffices to see that the intersection of $\Delta[n+1] \amalg \Delta[n+1]$ and $\Delta[1] \times \partial\Delta[n+1]$ in $\Delta[1] \times \Delta[n+1]$ is equal to $\partial\Delta[n+1] \amalg \partial\Delta[n+1]$. This we leave to the reader. $\square$

Lemma 9.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K, L be hypercoverings of X. Let $a, b : K \to L$ be morphisms of hypercoverings. There exists a morphism of hypercoverings $c : K' \to K$ such that $a \circ c$ is homotopic to $b \circ c$.*

Proof. Consider the following commutative diagram

$$
\begin{array}{ccc}
K' \overset{def}{=\!=\!=} K \times_{(L \times L)} \mathrm{Hom}(\Delta[1], L) & \longrightarrow & \mathrm{Hom}(\Delta[1], L) \\
{\scriptstyle c} \searrow \quad \downarrow & & \downarrow \\
K & \xrightarrow{\quad (a,b) \quad} & L \times L
\end{array}
$$

By the functorial property of $\mathrm{Hom}(\Delta[1], L)$ the composition of the horizontal morphisms corresponds to a morphism $K' \times \Delta[1] \to L$ which defines a homotopy

between $c \circ a$ and $c \circ b$. Thus if we can show that K' is a hypercovering of X, then we obtain the lemma. To see this we will apply Lemma 7.1 to the pair of morphisms $K \to L \times L$ and $\mathrm{Hom}(\Delta[1], L) \to L \times L$. Condition (1) of Lemma 7.1 is satisfied. Condition (2) of Lemma 7.1 is true because $\mathrm{Hom}(\Delta[1], L)_0 = L_1$, and the morphism $(d_0^1, d_1^1) : L_1 \to L_0 \times L_0$ is a covering of $\mathrm{SR}(\mathcal{C}, X)$ by our assumption that L is a hypercovering. To prove condition (3) of Lemma 7.1 we use Lemma 9.1 above. According to this lemma the morphism γ of condition (3) of Lemma 7.1 is the morphism

$$\mathrm{Hom}(\Delta[1] \times \Delta[n+1], L)_0 \longrightarrow \mathrm{Hom}(U, L)_0$$

where $U \subset \Delta[1] \times \Delta[n+1]$. According to Lemma 8.2 this is a covering and hence the claim has been proven. $\square$

Remark 9.3. Note that the crux of the proof is to use Lemma 8.2. This lemma is completely general and does not care about the exact shape of the simplicial sets (as long as they have only finitely many nondegenerate simplices). It seems altogether reasonable to expect a result of the following kind: Given any morphism $a : K \times \partial\Delta[k] \to L$, with K and L hypercoverings, there exists a morphism of hypercoverings $c : K' \to K$ and a morphism $g : K' \times \Delta[k] \to L$ such that $g|_{K' \times \partial\Delta[k]} = a \circ (c \times \mathrm{id}_{\partial\Delta[k]})$. In other words, the category of hypercoverings is in a suitable sense contractible.

10. Cohomology and hypercoverings

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Let K, L be hypercoverings of X. If $a, b : K \to L$ are homotopic maps, then $\mathcal{F}(a), \mathcal{F}(b) : \mathcal{F}(K) \to \mathcal{F}(L)$ are homotopic maps, see Simplicial, Lemma 28.4. Hence have the same effect on cohomology groups of the associated cochain complexes, see Simplicial, Lemma 28.6. We are going to use this to define the colimit over all hypercoverings.

Let us temporarily denote $\mathrm{HC}(\mathcal{C}, X)$ the category whose objects are hypercoverings of X and whose morphisms are maps between hypercoverings of X up to homotopy. We have seen that this is a category and not a "big" category, see Lemma 3.7. The opposite to $\mathrm{HC}(\mathcal{C}, X)$ will be the index category for our diagram, see Categories, Section 14 for terminology. Consider the diagram

$$\check{H}^i(-, \mathcal{F}) : \mathrm{HC}(\mathcal{C}, X)^{opp} \longrightarrow Ab.$$

By Lemmas 7.2 and 9.2 and the remark on homotopies above, this diagram is directed, see Categories, Definition 19.1. Thus the colimit

$$\check{H}^i_{\mathrm{HC}}(X, \mathcal{F}) = \mathrm{colim}_{K \in \mathrm{HC}(\mathcal{C}, X)} \check{H}^i(K, \mathcal{F})$$

has a particularly simple description (see location cited).

Theorem 10.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let $i \geq 0$. The functors*

$$\begin{aligned} Ab(\mathcal{C}) &\longrightarrow Ab \\ \mathcal{F} &\longmapsto H^i(X, \mathcal{F}) \\ \mathcal{F} &\longmapsto \check{H}^i_{HC}(X, \mathcal{F}) \end{aligned}$$

are canonically isomorphic.

Proof using spectral sequences. Suppose that $\xi \in H^p(X, \mathcal{F})$ for some $p \geq 0$. Let us show that ξ is in the image of the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ of Lemma 5.3 for some hypercovering K of X.

This is true if $p = 0$ by Lemma 5.1. If $p = 1$, choose a Čech hypercovering K of X as in Example 3.4 starting with a covering $K_0 = \{U_i \to X\}$ in the site $\mathcal{C}$ such that $\xi|_{U_i} = 0$, see Cohomology on Sites, Lemma 7.3. It follows immediately from the spectral sequence in Lemma 5.3 that ξ comes from an element of $\check{H}^1(K, \mathcal{F})$ in this case. In general, choose any hypercovering K of X such that ξ maps to zero in $\underline{H}^p(\mathcal{F})(K_0)$ (using Example 3.4 and Cohomology on Sites, Lemma 7.3 again). By the spectral sequence of Lemma 5.3 the obstruction for ξ to come from an element of $\check{H}^p(K, \mathcal{F})$ is a sequence of elements $\xi_1, \ldots, \xi_{p-1}$ with $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathcal{F}))$ (more precisely the images of the ξ_q in certain subquotients of these groups).

We can inductively replace the hypercovering K by refinements such that the obstructions $\xi_1, \ldots, \xi_{p-1}$ restrict to zero (and not just the images in the subquotients – so no subtlety here). Indeed, suppose we have already managed to reach the situation where $\xi_{q+1}, \ldots, \xi_{p-1}$ are zero. Note that $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathcal{F}))$ is the class of some element

$$\tilde{\xi}_q \in \underline{H}^q(\mathcal{F})(K_{p-q}) = \prod H^q(U_i, \mathcal{F})$$

if $K_{p-q} = \{U_i \to X\}_{i \in I}$. Let $\xi_{q,i}$ be the component of $\tilde{\xi}_q$ in $H^q(U_i, \mathcal{F})$. As $q \geq 1$ we can use Cohomology on Sites, Lemma 7.3 yet again to choose coverings $\{U_{i,j} \to U_i\}$ of the site such that each restriction $\xi_{q,i}|_{U_{i,j}} = 0$. Consider the object $Z = \{U_{i,j} \to X\}$ of the category $\mathrm{SR}(\mathcal{C}, X)$ and its obvious morphism $u : Z \to K_{p-q}$. It is clear that u is a covering, see Definition 3.1. By Lemma 7.3 there exists a morphism $L \to K$ of hypercoverings of X such that $L_{p-q} \to K_{p-q}$ factors through u. Then clearly the image of ξ_q in $\underline{H}^q(\mathcal{F})(L_{p-q})$. is zero. Since the spectral sequence of Lemma 5.3 is functorial this means that after replacing K by L we reach the situation where $\xi_q, \ldots, \xi_{p-1}$ are all zero. Continuing like this we end up with a hypercovering where they are all zero and hence ξ is in the image of the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$.

Suppose that K is a hypercovering of X, that $\xi \in \check{H}^p(K, \mathcal{F})$ and that the image of ξ under the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ of Lemma 5.3 is zero. To finish the proof of the theorem we have to show that there exists a morphism of hypercoverings $L \to K$ such that ξ restricts to zero in $\check{H}^p(L, \mathcal{F})$. By the spectral sequence of Lemma 5.3 the vanishing of the image of ξ in $H^p(X, \mathcal{F})$ means that there exist elements $\xi_1, \ldots, \xi_{p-2}$ with $\xi_q \in \check{H}^{p-1-q}(K, \underline{H}^q(\mathcal{F}))$ (more precisely the images of these in certain subquotients) such that the images $d_{q+1}^{p-1-q,q}\xi_q$ (in the spectral sequence) add up to ξ. Hence by exactly the same mechanism as above we can find a morphism of hypercoverings $L \to K$ such that the restrictions of the elements ξ_q, $q = 1, \ldots, p - 2$ in $\check{H}^{p-1-q}(L, \underline{H}^q(\mathcal{F}))$ are zero. Then it follows that ξ is zero since the morphism $L \to K$ induces a morphism of spectral sequences according to Lemma 5.3. $\square$

Proof without using spectral sequences. We have seen the result for $i = 0$, see Lemma 5.1. We know that the functors $H^i(X, -)$ form a universal δ-functor, see Derived Categories, Lemma 20.4. In order to prove the theorem it suffices to show that the sequence of functors $\check{H}_{HC}^i(X, -)$ forms a δ-functor. Namely we know

that Čech cohomology is zero on injective sheaves (Lemma 5.2) and then we can apply Homology, Lemma 12.4.

Let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$. Choose a hypercovering K of X and an element $\sigma \in \mathcal{H}(K_p)$ representing ξ in cohomology. There is a corresponding exact sequence of complexes

$$0 \to s(\mathcal{F}(K)) \to s(\mathcal{G}(K)) \to s(\mathcal{H}(K))$$

but we are not assured that there is a zero on the right also and this is the only thing that prevents us from defining $\delta(\xi)$ by a simple application of the snake lemma. Recall that

$$\mathcal{H}(K_p) = \prod \mathcal{H}(U_i)$$

if $K_p = \{U_i \to X\}$. Let $\sigma = \prod \sigma_i$ with $\sigma_i \in \mathcal{H}(U_i)$. Since $\mathcal{G} \to \mathcal{H}$ is a surjection of sheaves we see that there exist coverings $\{U_{i,j} \to U_i\}$ such that $\sigma_i|_{U_{i,j}}$ is the image of some element $\tau_{i,j} \in \mathcal{G}(U_{i,j})$. Consider the object $Z = \{U_{i,j} \to X\}$ of the category $\mathrm{SR}(\mathcal{C}, X)$ and its obvious morphism $u : Z \to K_p$. It is clear that u is a covering, see Definition 3.1. By Lemma 7.3 there exists a morphism $L \to K$ of hypercoverings of X such that $L_p \to K_p$ factors through u. After replacing K by L we may therefore assume that σ is the image of an element $\tau \in \mathcal{G}(K_p)$. Note that $d(\sigma) = 0$, but not necessarily $d(\tau) = 0$. Thus $d(\tau) \in \mathcal{F}(K_{p+1})$ is a cocycle. In this situation we define $\delta(\xi)$ as the class of the cocycle $d(\tau)$ in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$.

At this point there are several things to verify: (a) $\delta(\xi)$ does not depend on the choice of τ, (b) $\delta(\xi)$ does not depend on the choice of the hypercovering $L \to K$ such that σ lifts, and (c) $\delta(\xi)$ does not depend on the initial hypercovering and σ chosen to represent ξ. We omit the verification of (a), (b), and (c); the independence of the choices of the hypercoverings really comes down to Lemmas 7.2 and 9.2. We also omit the verification that δ is functorial with respect to morphisms of short exact sequences of abelian sheaves on $\mathcal{C}$.

Finally, we have to verify that with this definition of δ our short exact sequence of abelian sheaves above leads to a long exact sequence of Čech cohomology groups. First we show that if $\delta(\xi) = 0$ (with ξ as above) then ξ is the image of some element $\xi' \in \check{H}^p_{HC}(X, \mathcal{G})$. Namely, if $\delta(\xi) = 0$, then, with notation as above, we see that the class of $d(\tau)$ is zero in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$. Hence there exists a morphism of hypercoverings $L \to K$ such that the restriction of $d(\tau)$ to an element of $\mathcal{F}(L_{p+1})$ is equal to $d(\upsilon)$ for some $\upsilon \in \mathcal{F}(L_p)$. This implies that $\tau|_{L_p} + \upsilon$ form a cocycle, and determine a class $\xi' \in \check{H}^p(L, \mathcal{G})$ which maps to ξ as desired.

We omit the proof that if $\xi' \in \check{H}^{p+1}_{HC}(X, \mathcal{F})$ maps to zero in $\check{H}^{p+1}_{HC}(X, \mathcal{G})$, then it is equal to $\delta(\xi)$ for some $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$. $\qquad\square$

Next, we deduce Verdier's case of Theorem 10.1 by a sleight of hand.

Proposition 10.2. *Let $\mathcal{C}$ be a site with fibre products and products of pairs. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Let $i \geq 0$. Then*

(1) *for every $\xi \in H^i(\mathcal{F})$ there exists a hypercovering K such that ξ is in the image of the canonical map $\check{H}^i(K, \mathcal{F}) \to H^i(\mathcal{F})$, and*

(2) *if K, L are hypercoverings and $\xi_K \in \check{H}^i(K, \mathcal{F})$, $\xi_L \in \check{H}^i(L, \mathcal{F})$ are elements mapping to the same element of $H^i(\mathcal{F})$, then there exists a hypercovering M and morphisms $M \to K$ and $M \to L$ such that ξ_K and ξ_L map to the same element of $\check{H}^i(M, \mathcal{F})$.*

In other words, modulo set theoretical issues, the cohomology groups of $\mathcal{F}$ on $\mathcal{C}$ are the colimit of the Čech cohomology groups of $\mathcal{F}$ over all hypercoverings.

Proof. This result is a trivial consequence of Theorem 10.1. Namely, we can artificially replace $\mathcal{C}$ with a slightly bigger site $\mathcal{C}'$ such that (I) $\mathcal{C}'$ has a final object X and (II) hypercoverings in $\mathcal{C}$ are more or less the same thing as hypercoverings of X in $\mathcal{C}'$. But due to the nature of things, there is quite a bit of bookkeeping to do.

Let us call a family of morphisms $\{U_i \to U\}$ in $\mathcal{C}$ with fixed target a *weak covering* if the sheafification of the map $\coprod_{i \in I} h_{U_i} \to h_U$ becomes surjective. We construct a new site $\mathcal{C}'$ as follows

(1) as a category set $\mathrm{Ob}(\mathcal{C}') = \mathrm{Ob}(\mathcal{C}) \amalg \{X\}$ and add a unique morphism to X from every object of $\mathcal{C}'$,
(2) $\mathcal{C}'$ has fibre products as fibre products and products of pairs exist in $\mathcal{C}$,
(3) coverings of $\mathcal{C}'$ are weak coverings of $\mathcal{C}$ together with those $\{U_i \to X\}_{i \in I}$ such that either $U_i = X$ for some i, or $U_i \neq X$ for all i and the map $\coprod h_{U_i} \to *$ of presheaves on $\mathcal{C}$ becomes surjective after sheafification on $\mathcal{C}$,
(4) we apply Sets, Lemma 11.1 to restrict the coverings to obtain our site $\mathcal{C}'$.

Then $Sh(\mathcal{C}') = Sh(\mathcal{C})$ because the inclusion functor $\mathcal{C} \to \mathcal{C}'$ is a special cocontinuous functor (see Sites, Definition 29.2). We omit the straightforward verifications.

Choose a covering $\{U_i \to X\}$ of $\mathcal{C}'$ such that U_i is an object of $\mathcal{C}$ for all i (possible because $\mathcal{C} \to \mathcal{C}'$ is special cocontinuous). Then $K_0 = \{U_i \to X\}$ is a covering in the site $\mathcal{C}'$ constructed above. We view K_0 as an object of $\mathrm{SR}(\mathcal{C}', X)$ and we set $K_{init} = \mathrm{cosk}_0(K_0)$. Then K_{init} is a hypercovering of X, see Example 3.4. Note that every $K_{init,n}$ has the shape $\{W_j \to X\}$ with $W_j \in \mathrm{Ob}(\mathcal{C})$.

Proof of (1). Choose $\xi \in H^i(\mathcal{F}) = H^i(X, \mathcal{F}')$ where $\mathcal{F}'$ is the abelian sheaf on $\mathcal{C}'$ corresponding to $\mathcal{F}$ on $\mathcal{C}$. By Theorem 10.1 there exists a morphism of hypercoverings $K' \to K_{init}$ of X in $\mathcal{C}'$ such that ξ comes from an element of $\check{H}^i(K', \mathcal{F})$. Write $K'_n = \{U_{n,j} \to X\}$. Now since K'_n maps to $K_{init,n}$ we see that $U_{n,j}$ is an object of $\mathcal{C}$. Hence we can define a simplicial object K of $\mathrm{SR}(\mathcal{C})$ by setting $K_n = \{U_{n,j}\}$. Since coverings in $\mathcal{C}'$ consisting of families of morphisms of $\mathcal{C}$ are weak coverings, we see that K is a hypercovering in the sense of Definition 6.1. Finally, since $\mathcal{F}'$ is the unique sheaf on $\mathcal{C}'$ whose restriction to $\mathcal{C}$ is equal to $\mathcal{F}$ we see that the Čech complexes $s(\mathcal{F}(K))$ and $s(\mathcal{F}'(K'))$ are identical and (1) follows. (Compatibility with map into cohomology groups omitted.)

Proof of (2). Let K and L be hypercoverings in $\mathcal{C}$. Let K' and L' be the simplicial objects of $\mathrm{SR}(\mathcal{C}', X)$ gotten from K and L by the functor $\mathrm{SR}(\mathcal{C}) \to \mathrm{SR}(\mathcal{C}', X)$, $\{U_i\} \mapsto \{U_i \to X\}$. As before we have equality of Čech complexes and hence we obtain $\xi_{K'}$ and $\xi_{L'}$ mapping to the same cohomology class of $\mathcal{F}'$ over $\mathcal{C}'$. After possibly enlarging our choice of coverings in $\mathcal{C}'$ (due to a set theoretical issue) we may assume that K' and L' are hypercoverings of X in $\mathcal{C}'$; this is true by our definition of hypercoverings in Definition 6.1 and the fact that weak coverings in $\mathcal{C}$ give coverings in $\mathcal{C}'$. By Theorem 10.1 there exists a hypercovering M' of X in

$\mathcal{C}'$ and morphisms $M' \to K'$, $M' \to L'$, and $M' \to K_{init}$ such that $\xi_{K'}$ and $\xi_{L'}$ restrict to the same element of $\check{H}^i(M', \mathcal{F})$. Unwinding this statement as above we find that (2) is true. $\square$

11. Hypercoverings of spaces

The theory above is mildly interesting even in the case of topological spaces. In this case we can work out what a hypercovering is and see what the result actually says.

Let X be a topological space. Consider the site X_{Zar} of Sites, Example 6.4. Recall that an object of X_{Zar} is simply an open of X and that morphisms of X_{Zar} correspond simply to inclusions. So what is a hypercovering of X for the site X_{Zar}?

Let us first unwind Definition 2.1. An object of $\mathrm{SR}(X_{Zar}, X)$ is simply given by a set I and for each $i \in I$ an open $U_i \subset X$. Let us denote this by $\{U_i\}_{i \in I}$ since there can be no confusion about the morphism $U_i \to X$. A morphism $\{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ between two such objects is given by a map of sets $\alpha : I \to J$ such that $U_i \subset V_{\alpha(i)}$ for all $i \in I$. When is such a morphism a covering? This is the case if and only if for every $j \in J$ we have $V_j = \bigcup_{i \in I, \ \alpha(i)=j} U_i$ (and is a covering in the site X_{Zar}).

Using the above we get the following description of a hypercovering in the site X_{Zar}. A hypercovering of X in X_{Zar} is given by the following data

(1) a simplicial set I (see Simplicial, Section 11), and
(2) for each $n \geq 0$ and every $i \in I_n$ an open set $U_i \subset X$.

We will denote such a collection of data by the notation $(I, \{U_i\})$. In order for this to be a hypercovering of X we require the following properties

- for $i \in I_n$ and $0 \leq a \leq n$ we have $U_i \subset U_{d_a^n(i)}$,
- for $i \in I_n$ and $0 \leq a \leq n$ we have $U_i = U_{s_a^n(i)}$,
- we have

$$(11.0.1) \qquad X = \bigcup_{i \in I_0} U_i,$$

- for every $i_0, i_1 \in I_0$, we have

$$(11.0.2) \qquad U_{i_0} \cap U_{i_1} = \bigcup_{i \in I_1, \ d_0^1(i)=i_0, \ d_1^1(i)=i_1} U_i,$$

- for every $n \geq 1$ and every $(i_0, \dots, i_{n+1}) \in (I_n)^{n+2}$ such that $d_{b-1}^n(i_a) = d_a^n(i_b)$ for all $0 \leq a < b \leq n+1$ we have

$$(11.0.3) \qquad U_{i_0} \cap \dots \cap U_{i_{n+1}} = \bigcup_{i \in I_{n+1}, \ d_a^{n+1}(i)=i_a, \ a=0,\dots,n+1} U_i,$$

- each of the open coverings (11.0.1), (11.0.2), and (11.0.3) is an element of $\mathrm{Cov}(X_{Zar})$ (this is a set theoretic condition, bounding the size of the index sets of the coverings).

Conditions (11.0.1) and (11.0.2) should be familiar from the chapter on sheaves on spaces for example, and condition (11.0.3) is the natural generalization.

Remark 11.1. One feature of this description is that if one of the multiple intersections $U_{i_0} \cap \dots \cap U_{i_{n+1}}$ is empty then the covering on the right hand side may be the empty covering. Thus it is not automatically the case that the maps $I_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n I)_{n+1}$ are surjective. This means that the geometric realization of I may be an interesting (non-contractible) space.

In fact, let $I'_n \subset I_n$ be the subset consisting of those simplices $i \in I_n$ such that $U_i \neq \emptyset$. It is easy to see that $I' \subset I$ is a subsimplicial set, and that $(I', \{U_i\})$ is a hypercovering. Hence we can always refine a hypercovering to a hypercovering where none of the opens U_i is empty.

Remark 11.2. Let us repackage this information in yet another way. Namely, suppose that $(I, \{U_i\})$ is a hypercovering of the topological space X. Given this data we can construct a simplicial topological space $U_\bullet$ by setting

$$U_n = \coprod_{i \in I_n} U_i,$$

and where for given $\varphi : [n] \to [m]$ we let morphisms $U(\varphi) : U_n \to U_m$ be the morphism coming from the inclusions $U_i \subset U_{\varphi(i)}$ for $i \in I_n$. This simplicial topological space comes with an augmentation $\epsilon : U_\bullet \to X$. With this morphism the simplicial space $U_\bullet$ becomes a hypercovering of X along which one has cohomological descent in the sense of [AGV71, Exposé Vbis]. In other words, $H^n(U_\bullet, \epsilon^* \mathcal{F}) = H^n(X, \mathcal{F})$. (Insert future reference here to cohomology over simplicial spaces and cohomological descent formulated in those terms.) Suppose that $\mathcal{F}$ is an abelian sheaf on X. In this case the spectral sequence of Lemma 5.3 becomes the spectral sequence with E_1-term

$$E_1^{p,q} = H^q(U_p, \epsilon_q^* \mathcal{F}) \Rightarrow H^{p+q}(U_\bullet, \epsilon^* \mathcal{F}) = H^{p+q}(X, \mathcal{F})$$

comparing the total cohomology of $\epsilon^* \mathcal{F}$ to the cohomology groups of $\mathcal{F}$ over the pieces of $U_\bullet$. (Insert future reference to this spectral sequence here.)

In topology we often want to find hypercoverings of X which have the property that all the U_i come from a given basis for the topology of X and that all the coverings (11.0.2) and (11.0.3) are from a given cofinal collection of coverings. Here are two example lemmas.

Lemma 11.3. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X. There exists a hypercovering $(I, \{U_i\})$ of X such that each U_i is an element of $\mathcal{B}$.*

Proof. Let $n \geq 0$. Let us say that an *n-truncated hypercovering* of X is given by an n-truncated simplicial set I and for each $i \in I_a$, $0 \leq a \leq n$ an open U_i of X such that the conditions defining a hypercovering hold whenever they make sense. In other words we require the inclusion relations and covering conditions only when all simplices that occur in them are a-simplices with $a \leq n$. The lemma follows if we can prove that given a n-truncated hypercovering $(I, \{U_i\})$ with all $U_i \in \mathcal{B}$ we can extend it to an $(n+1)$-truncated hypercovering without adding any a-simplices for $a \leq n$. This we do as follows. First we consider the $(n+1)$-truncated simplicial set I' defined by $I' = \mathrm{sk}_{n+1}(\mathrm{cosk}_n I)$. Recall that

$$I'_{n+1} = \left\{ \begin{array}{c} (i_0, \ldots, i_{n+1}) \in (I_n)^{n+2} \text{ such that} \\ d_{b-1}^n(i_a) = d_a^n(i_b) \text{ for all } 0 \leq a < b \leq n+1 \end{array} \right\}$$

If $i' \in I'_{n+1}$ is degenerate, say $i' = s_a^n(i)$ then we set $U_{i'} = U_i$ (this is forced on us anyway by the second condition). We also set $J_{i'} = \{i'\}$ in this case. If $i' \in I'_{n+1}$ is nondegenerate, say $i' = (i_0, \ldots, i_{n+1})$, then we choose a set $J_{i'}$ and an open covering

$$(11.3.1) \qquad\qquad U_{i_0} \cap \ldots \cap U_{i_{n+1}} = \bigcup_{i \in J_{i'}} U_i,$$

with $U_i \in \mathcal{B}$ for $i \in J_{i'}$. Set

$$I_{n+1} = \coprod_{i' \in I'_{n+1}} J_{i'}$$

There is a canonical map $\pi : I_{n+1} \to I'_{n+1}$ which is a bijection over the set of degenerate simplices in I'_{n+1} by construction. For $i \in I_{n+1}$ we define $d^{n+1}_a(i) = d^{n+1}_a(\pi(i))$. For $i \in I_n$ we define $s^n_a(i) \in I_{n+1}$ as the unique simplex lying over the degenerate simplex $s^n_a(i) \in I'_{n+1}$. We omit the verification that this defines an $(n+1)$-truncated hypercovering of X. $\square$

Lemma 11.4. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X. Assume that*

  (1) *X is quasi-compact,*
  (2) *each $U \in \mathcal{B}$ is quasi-compact open, and*
  (3) *the intersection of any two quasi-compact opens in X is quasi-compact.*

Then there exists a hypercovering $(I, \{U_i\})$ of X with the following properties

  (1) *each U_i is an element of the basis $\mathcal{B}$,*
  (2) *each of the I_n is a finite set, and in particular*
  (3) *each of the coverings (11.0.1), (11.0.2), and (11.0.3) is finite.*

Proof. This follows directly from the construction in the proof of Lemma 11.3 if we choose finite coverings by elements of $\mathcal{B}$ in (11.3.1). Details omitted. $\square$

12. Constructing hypercoverings

Let $\mathcal{C}$ be a site. In this section we will think of a simplicial object of $\mathrm{SR}(\mathcal{C})$ as follows. As usual, we set $K_n = K([n])$ and we denote $K(\varphi) : K_n \to K_m$ the morphism associated to $\varphi : [m] \to [n]$. We may write $K_n = \{U_{n,i}\}_{i \in I_n}$. For $\varphi : [m] \to [n]$ the morphism $K(\varphi) : K_n \to K_m$ is given by a map $\alpha(\varphi) : I_n \to I_m$ and morphisms $f_{\varphi,i} : U_{n,i} \to U_{m,\alpha(\varphi)(i)}$ for $i \in I_n$. The fact that K is a simplicial object of $\mathrm{SR}(\mathcal{C})$ implies that $(I_n, \alpha(\varphi))$ is a simplicial set and that $f_{\psi,\alpha(\varphi)(i)} \circ f_{\varphi,i} = f_{\varphi \circ \psi, i}$ when $\psi : [l] \to [m]$.

Lemma 12.1. *Let $\mathcal{C}$ be a site. Let K be an r-truncated simplicial object of $\mathrm{SR}(\mathcal{C})$. The following are equivalent*

  (1) *K is split (Simplicial, Definition 18.1),*
  (2) *$f_{\varphi,i} : U_{n,i} \to U_{m,\alpha(\varphi)(i)}$ is an isomorphism for $r \geq n \geq 0$, $\varphi : [m] \to [n]$ surjective, $i \in I_n$, and*
  (3) *$f_{\sigma^n_j,i} : U_{n,i} \to U_{n+1,\alpha(\sigma^n_j)(i)}$ is an isomorphism for $0 \leq j \leq n < r$, $i \in I_n$.*

The same holds for simplicial objects if in (2) and (3) we set $r = \infty$.

Proof. The splitting of a simplicial set is unique and is given by the nondegenerate indices $N(I_n)$ in each degree n, see Simplicial, Lemma 18.2. The coproduct of two objects $\{U_i\}_{i \in I}$ and $\{U_j\}_{j \in J}$ of $\mathrm{SR}(\mathcal{C})$ is given by $\{U_l\}_{l \in I \amalg J}$ with obvious notation. Hence a splitting of K must be given by $N(K_n) = \{U_i\}_{i \in N(I_n)}$. The equivalence of (1) and (2) now follows by unwinding the definitions. The equivalence of (2) and (3) follows from the fact that any surjection $\varphi : [m] \to [n]$ is a composition of morphisms σ^k_j with $k = n, n+1, \ldots, m-1$. $\square$

Lemma 12.2. *Let $\mathcal{C}$ be a site with fibre products. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

(1) *any object U of $\mathcal{C}$ has a covering $\{U_j \to U\}_{j \in J}$ with $U_j \in \mathcal{B}$, and*
(2) *if $\{U_j \to U\}_{j \in J}$ is a covering with $U_j \in \mathcal{B}$ and $\{U' \to U\}$ is a morphism with $U' \in \mathcal{B}$, then $\{U_j \to U\}_{j \in J} \amalg \{U' \to U\}$ is a covering.*

Then for any X in $\mathcal{C}$ there is a hypercovering K of X such that $K_n = \{U_{n,i}\}_{i \in I_n}$ with $U_{n,i} \in \mathcal{B}$ for all $i \in I_n$.

Proof. A warmup for this proof is the proof of Lemma 11.3 and we encourage the reader to read that proof first.

First we replace $\mathcal{C}$ by the site $\mathcal{C}/X$. After doing so we may assume that X is the final object of $\mathcal{C}$ and that $\mathcal{C}$ has all finite limits (Categories, Lemma 18.4).

Let $n \geq 0$. Let us say that an *n-truncated $\mathcal{B}$-hypercovering of X* is given by an n-truncated simplicial object K of $\mathrm{SR}(\mathcal{C})$ such that for $i \in I_a$, $0 \leq a \leq n$ we have $U_{a,i} \in \mathcal{B}$ and such that K_0 is a covering of X and $K_{a+1} \to (\mathrm{cosk}_a \mathrm{sk}_a K)_{a+1}$ for $a = 0, \ldots, n-1$ is a covering as in Definition 3.1.

Since X has a covering $\{U_{0,i} \to X\}_{i \in I_0}$ with $U_i \in \mathcal{B}$ by assumption, we get a 0-truncated $\mathcal{B}$-hypercovering of X. Observe that any 0-truncated $\mathcal{B}$-hypercovering of X is split, see Lemma 12.1.

The lemma follows if we can prove for $n \geq 0$ that given a split n-truncated $\mathcal{B}$-hypercovering K of X we can extend it to a split $(n+1)$-truncated $\mathcal{B}$-hypercovering of X.

Construction of the extension. Consider the $(n+1)$-truncated simplicial object $K' = \mathrm{sk}_{n+1}(\mathrm{cosk}_n K)$ of $\mathrm{SR}(\mathcal{C})$. Write

$$K'_{n+1} = \{U'_{n+1,i}\}_{i \in I'_{n+1}}$$

Since $K = \mathrm{sk}_n K'$ we have $K_a = K'_a$ for $0 \leq a \leq n$. For every $i' \in I'_{n+1}$ we choose a covering

(12.2.1) $\{g_{n+1,j} : U_{n+1,j} \to U'_{n+1,i'}\}_{j \in J_{i'}}$

with $U_{n+1,j} \in \mathcal{B}$ for $j \in J_{i'}$. This is possible by our assumption on $\mathcal{B}$ in the lemma. For $0 \leq m \leq n$ denote $N_m \subset I_m$ the subset of nondegenerate indices. We set

$$I_{n+1} = \coprod_{\varphi:[n+1]\to[m] \text{ surjective, } 0 \leq m \leq n} N_m \amalg \coprod_{i' \in I'_{n+1}} J_{i'}$$

For $j \in I_{n+1}$ we set

$$U_{n+1,j} = \begin{cases} U_{m,i} & \text{if} \quad j = (\varphi, i) \quad \text{where} \quad \varphi : [n+1] \to [m], i \in N_m \\ U_{n+1,j} & \text{if} \quad j \in J_{i'} \quad \text{where} \quad i' \in I'_{n+1} \end{cases}$$

with obvious notation. We set $K_{n+1} = \{U_{n+1,j}\}_{j \in I_{n+1}}$. By construction $U_{n+1,j}$ is an element of $\mathcal{B}$ for all $j \in I_{n+1}$. Let us define compatible maps

$$I_{n+1} \to I'_{n+1} \quad \text{and} \quad K_{n+1} \to K'_{n+1}$$

Namely, the first map is given by $(\varphi, i) \mapsto \alpha'(\varphi)(i)$ and $(j \in J_{i'}) \mapsto i'$. For the second map we use the morphisms

$$f'_{\varphi,i} : U_{m,i} \to U'_{n+1,\alpha'(\varphi)(i)} \quad \text{and} \quad g_{n+1,j} : U_{n+1,j} \to U'_{n+1,i'}$$

We claim the morphism

$$K_{n+1} \to K'_{n+1} = (\mathrm{cosk}_n \mathrm{sk}_n K')_{n+1} = (\mathrm{cosk}_n K)_{n+1}$$

is a covering as in Definition 3.1. Namely, if $i' \in I'_{n+1}$, then either i' is nondegenerate and the inverse image of i' in I_{n+1} is equal to $J_{i'}$ and we get a covering of $U'_{n+1,i'}$ by our choice (12.2.1), or i' is degenerate and the inverse image of i' in I_{n+1} is $J_{i'} \amalg \{(\varphi, i)\}$ for a unique pair (φ, i) and we get a covering by our choice (12.2.1) and assumption (2) of the lemma.

To finish the proof we have to define the morphisms $K(\varphi) : K_{n+1} \to K_m$ corresponding to morphisms $\varphi : [m] \to [n+1]$, $0 \le m \le n$ and the morphisms $K(\varphi) : K_m \to K_{n+1}$ corresponding to morphisms $\varphi : [n+1] \to [m]$, $0 \le m \le n$ satisfying suitable composition relations. For the first kind we use the composition

$$K_{n+1} \to K'_{n+1} \xrightarrow{K'(\varphi)} K'_m = K_m$$

to define $K(\varphi) : K_{n+1} \to K_m$. For the second kind, suppose given $\varphi : [n+1] \to [m]$, $0 \le m \le n$. We define the corresponding morphism $K(\varphi) : K_m \to K_{n+1}$ as follows:

(1) for $i \in I_m$ there is a unique surjective map $\psi : [m] \to [m_0]$ and a unique $i_0 \in I_{m_0}$ nondegenerate such that $\alpha(\psi)(i_0) = i^3$,
(2) we set $\varphi_0 = \psi_0 \circ \varphi : [n+1] \to [m_0]$ and we map $i \in I_m$ to $(\varphi_0, i_0) \in I_{n+1}$, in other words, $\alpha(\varphi)(i) = (\varphi_0, i_0)$, and
(3) the morphism $f_{\varphi,i} : U_{m,i} \to U_{n+1,\alpha(\varphi)(i)} = U_{m_0,i_0}$ is the inverse of the isomorphism $f_{\psi,i_0} : U_{m_0,i_0} \to U_{m,i}$ (see Lemma 12.1).

We omit the straightforward but cumbersome verification that this defines a split $(n+1)$-truncated $\mathcal{B}$-hypercovering of X extending the given n-truncated one. In fact, everything is clear from the above, except for the verification that the morphisms $K(\varphi)$ compose correctly for all $\varphi : [a] \to [b]$ with $0 \le a, b \le n+1$. $\square$

Lemma 12.3. *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume that any object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$. Then there is a hypercovering K such that $K_n = \{U_i\}_{i \in I_n}$ with $U_i \in \mathcal{B}$ for all $i \in I_n$.*

Proof. This proof is almost the same as the proof of Lemma 12.2. We will only explain the differences.

Let $n \ge 1$. Let us say that an *n-truncated $\mathcal{B}$-hypercovering* is given by an n-truncated simplicial object K of $\mathrm{SR}(\mathcal{C})$ such that for $i \in I_a$, $0 \le a \le n$ we have $U_{a,i} \in \mathcal{B}$ and such that

(1) $F(K_0)^\# \to *$ is surjective,
(2) $F(K_1)^\# \to F(K_0)^\# \times F(K_0)^\#$ is surjective,
(3) $F(K_{a+1})^\# \to F((\mathrm{cosk}_a \mathrm{sk}_a K)_{a+1})^\#$ for $a = 1, \ldots, n-1$ is surjective.

We first explicitly construct a split 1-truncated $\mathcal{B}$-hypercovering.

Take $I_0 = \mathcal{B}$ and $K_0 = \{U\}_{U \in \mathcal{B}}$. Then (1) holds by our assumption on $\mathcal{B}$. Set

$$\Omega = \{(U, V, W, a, b) \mid U, V, W \in \mathcal{B}, a : U \to V, b : U \to W\}$$

Then we set $I_1 = I_0 \amalg \Omega$. For $i \in I_1$ we set $U_{1,i} = U_{0,i}$ if $i \in I_0$ and $U_{1,i} = U$ if $i = (U, V, W, a, b) \in \Omega$. The map $K(\sigma_0^0) : K_0 \to K_1$ corresponds to the inclusion $\alpha(\sigma_0^0) : I_0 \to I_1$ and the identity $f_{\sigma_0^0, i} : U_{0,i} \to U_{1,i}$ on objects. The maps $K(\delta_0^1), K(\delta_1^1) : K_1 \to K_0$ correspond to the two maps $I_1 \to I_0$ which are the identity on $I_0 \subset I_1$ and map $(U, V, W, a, b) \in \Omega \subset I_1$ to V, resp. W. The

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[3]For example, if i is nondegenerate, then $m = m_0$ and $\psi = \mathrm{id}_{[m]}$.

corresponding morphisms $f_{\delta_0^1,i}, f_{\delta_1^1,i} : U_{1,i} \to U_{0,i}$ are the identity if $i \in I_0$ and a, b in case $i = (U, V, W, a, b) \in \Omega$. The reason that (2) holds is that any section of $F(K_0)^\# \times F(K_0)^\#$ over an object U of $\mathcal{C}$ comes, after replacing U by the members of a covering, from a map $U \to F(K_0) \times F(K_0)$. This in turn means we have $V, W \in \mathcal{B}$ and two morphisms $U \to V$ and $U \to W$. Further replacing U by the members of a covering we may assume $U \in \mathcal{B}$ as desired.

The lemma follows if we can prove that given a split n-truncated $\mathcal{B}$-hypercovering K for $n \geq 1$ we can extend it to a split $(n+1)$-truncated $\mathcal{B}$-hypercovering. Here the argument proceeds exactly as in the proof of Lemma 12.2. We omit the precise details, except for the following comments. First, we do not need assumption (2) in the proof of the current lemma as we do not need the morphism $K_{n+1} \to (\mathrm{cosk}_n K)_{n+1}$ to be covering; we only need it to induce a surjection on associated sheaves of sets which follows from Sites, Lemma 12.4. Second, the assumption that $\mathcal{C}$ has fibre products and equalizers guarantees that $\mathrm{SR}(\mathcal{C})$ has fibre products and equalizers and F commutes with these (Lemma 2.3). This suffices assure us the coskeleton functors used exist (see Simplicial, Remark 19.11 and Categories, Lemma 18.2). $\qquad\square$

Lemma 12.4. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the functor $u : \mathcal{D} \to \mathcal{C}$. Assume $\mathcal{D}$ and $\mathcal{C}$ have equalizers and fibre products and u commutes with them. If a simplicial object K of $SR(\mathcal{D})$ is a hypercovering, then $u(K)$ is a hypercovering.*

Proof. If we write $K_n = \{U_{n,i}\}_{i \in I_n}$ as in the introduction to this section, then $u(K)$ is the object of $\mathrm{SR}(\mathcal{C})$ given by $u(K_n) = \{u(U_i)\}_{i \in I_n}$. By Sites, Lemma 13.5 we have $f^{-1}h_U^\# = h_{u(U)}^\#$ for $U \in \mathrm{Ob}(\mathcal{D})$. This means that $f^{-1}F(K_n)^\# = F(u(K_n))^\#$ for all n. Let us check the conditions (1), (2), (3) for $u(K)$ to be a hypercovering from Definition 6.1. Since f^{-1} is an exact functor, we find that

$$F(u(K_0))^\# = f^{-1}F(K_0)^\# \to f^{-1}* = *$$

is surjective as a pullback of a surjective map and we get (1). Similarly,

$$F(u(K_1))^\# = f^{-1}F(K_1)^\# \to f^{-1}(F(K_0) \times F(K_0))^\# = F(u(K_0))^\# \times F(u(K_0))^\#$$

is surjective as a pullback and we get (2). For condition (3), in order to conclude by the same method it suffices if

$$F((\mathrm{cosk}_n \mathrm{sk}_n u(K))_{n+1})^\# = f^{-1}F((\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1})^\#$$

The above shows that $f^{-1}F(-) = F(u(-))$. Thus it suffices to show that u commutes with the limits used in defining $(\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$ for $n \geq 1$. By Simplicial, Remark 19.11 these limits are finite connected limits and u commutes with these by assumption. $\qquad\square$

Lemma 12.5. *Let $\mathcal{C}, \mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a continuous functor. Assume $\mathcal{D}$ and $\mathcal{C}$ have fibre products and u commutes with them. Let $Y \in \mathcal{D}$ and $K \in SR(\mathcal{D}, Y)$ a hypercovering of Y. Then $u(K)$ is a hypercovering of $u(Y)$.*

Proof. This is easier than the proof of Lemma 12.4 because the notion of being a hypercovering of an object is stronger, see Definitions 3.3 and 3.1. Namely, u sends coverings to coverings by the definition of a morphism of sites. It suffices to check u commutes with the limits used in defining $(\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$ for $n \geq 1$. This is clear because the induced functor $\mathcal{D}/Y \to \mathcal{C}/X$ commutes with all finite limits (and source and target have all finite limits by Categories, Lemma 18.4). $\qquad\square$

Lemma 12.6. *Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

(1) *$\mathcal{C}$ has fibre products,*
(2) *for all $X \in \mathrm{Ob}(\mathcal{C})$ there exists a finite covering $\{U_i \to X\}_{i \in I}$ with $U_i \in \mathcal{B}$,*
(3) *if $\{U_i \to X\}_{i \in I}$ is a finite covering with $U_i \in \mathcal{B}$ and $U \to X$ is a morphism with $U \in \mathcal{B}$, then $\{U_i \to X\}_{i \in I} \amalg \{U \to X\}$ is a covering.*

Then for every X there exists a hypercovering K of X such that each $K_n = \{U_{n,i} \to X\}_{i \in I_n}$ with I_n finite and $U_{n,i} \in \mathcal{B}$.

Proof. This lemma is the analogue of Lemma 11.4 for sites. To prove the lemma we follow exactly the proof of Lemma 12.2 paying attention to the following two points

(a) We choose our initial covering $\{U_{0,i} \to X\}_{i \in I_0}$ with $U_{0,i} \in \mathcal{B}$ such that the index set I_0 is finite, and
(b) in choosing the coverings (12.2.1) we choose $J_{i'}$ finite.

The reader sees easily that with these modifications we end up with finite index sets I_n for all n. $\qquad\square$

Remark 12.7. Let $\mathcal{C}$ be a site. Let K and L be objects of $\mathrm{SR}(\mathcal{C})$. Write $K = \{U_i\}_{i \in I}$ and $L = \{V_j\}_{j \in J}$. Assume $U = \coprod_{i \in I} U_i$ and $V = \coprod_{j \in J} V_j$ exist. Then we get

$$\mathrm{Mor}_{\mathrm{SR}(\mathcal{C})}(K, L) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(U, V)$$

as follows. Given $f : K \to L$ given by $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$ we obtain a transformation of functors

$$\mathrm{Mor}_{\mathcal{C}}(V, -) = \prod_{j \in J} \mathrm{Mor}_{\mathcal{C}}(V_j, -) \to \prod_{i \in I} \mathrm{Mor}_{\mathcal{C}}(U_i, -) = \mathrm{Mor}_{\mathcal{C}}(U, -)$$

sending $(g_j)_{j \in J}$ to $(g_{\alpha(i)} \circ f_i)_{i \in I}$. Hence the Yoneda lemma produces the corresponding map $U \to V$. Of course, $U \to V$ maps the summand U_i into the summand $V_{\alpha(i)}$ via the morphism f_i.

Remark 12.8. Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has fibre products and equalizers and let K be a hypercovering. Write $K_n = \{U_{n,i}\}_{i \in I_n}$. Suppose that

(a) $U_n = \coprod_{i \in I_n} U_{n,i}$ exists, and
(b) $\coprod_{i \in I_n} h_{U_{n,i}} \to h_{U_n}$ induces an isomorphism on sheafifications.

Then we get another simplicial object L of $\mathrm{SR}(\mathcal{C})$ with $L_n = \{U_n\}$, see Remark 12.7. Now we claim that L is a hypercovering. To see this we check conditions (1), (2), (3) of Definition 6.1. Condition (1) follows from (b) and (1) for K. Condition (2) follows in exactly the same way. Condition (3) follows because

$$F((\mathrm{cosk}_n \mathrm{sk}_n L)_{n+1})^{\#} = ((\mathrm{cosk}_n \mathrm{sk}_n F(L)^{\#})_{n+1})$$
$$= ((\mathrm{cosk}_n \mathrm{sk}_n F(K)^{\#})_{n+1})$$
$$= F((\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1})^{\#}$$

for $n \geq 1$ and hence the condition for K implies the condition for L exactly as in (1) and (2). Note that F commutes with connected limits and sheafification is exact proving the first and last equality; the middle equality follows as $F(K)^{\#} = F(L)^{\#}$ by (b).

Remark 12.9. Let $\mathcal{C}$ be a site. Let $X \in \mathrm{Ob}(\mathcal{C})$. Assume $\mathcal{C}$ has fibre products and let K be a hypercovering of X. Write $K_n = \{U_{n,i}\}_{i \in I_n}$. Suppose that

(a) $U_n = \coprod_{i \in I_n} U_{n,i}$ exists,

(b) given morphisms $(\alpha, f_i) : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ and $(\beta, g_k) : \{W_k\}_{k \in K} \to \{V_j\}_{j \in J}$ in $\mathrm{SR}(\mathcal{C})$ such that $U = \coprod U_i$, $V = \coprod V_j$, and $W = \coprod W_j$ exist, then $U \times_V W = \coprod_{(i,j,k), \alpha(i) = j = \beta(k)} U_i \times_{V_j} W_k$,

(c) if $(\alpha, f_i) : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ is a covering in the sense of Definition 3.1 and $U = \coprod U_i$ and $V = \coprod V_j$ exist, then the corresponding morphism $U \to V$ of Remark 12.7 is a covering of $\mathcal{C}$.

Then we get another simplicial object L of $\mathrm{SR}(\mathcal{C})$ with $L_n = \{U_n\}$, see Remark 12.7. Now we claim that L is a hypercovering of X. To see this we check conditions (1), (2) of Definition 3.3. Condition (1) follows from (c) and (1) for K because (1) for K says $K_0 = \{U_{0,i}\}_{i \in I_0}$ is a covering of $\{X\}$ in the sense of Definition 3.1. Condition (2) follows because $\mathcal{C}/X$ has all finite limits hence $\mathrm{SR}(\mathcal{C}/X)$ has all finite limits, and condition (b) says the construction of "taking disjoint unions" commutes with these finite limits. Thus the morphism

$$L_{n+1} \longrightarrow (\mathrm{cosk}_n \mathrm{sk}_n L)_{n+1}$$

is a covering as it is the consequence of applying our "taking disjoint unions" functor to the morphism

$$K_{n+1} \longrightarrow (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$$

which is assumed to be a covering in the sense of Definition 3.1 by (2) for K. This makes sense because property (b) in particular assures us that if we start with a finite diagram of semi-representable objects over X for which we can take disjoint unions, then the limit of the diagram in $\mathrm{SR}(\mathcal{C}/X)$ still is a semi-representable object over X for which we can take disjoint unions.

13. Other chapters

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