DIFFERENTIAL GRADED ALGEBRA

Contents

1. Introduction

In this chapter we talk about differential graded algebras, modules, categories, etc. A basic reference is [Kel94]. A survey paper is [Kel06].

Since we do not worry about length of exposition in the Stacks project we first develop the material in the setting of categories of differential graded modules. After that we redo the constructions in the setting of differential graded modules over differential graded categories.

2. Conventions

In this chapter we hold on to the convention that *ring* means commutative ring with 1. If R is a ring, then an R-*algebra* A will be an R-module A endowed with an R-bilinear map $A \times A \to A$ (multiplication) such that multiplication is associative and has a unit. In other words, these are unital associative R-algebras such that the structure map $R \to A$ maps into the center of A.

Sign rules. In this chapter we will work with graded algebras and graded modules often equipped with differentials. The sign rules on underlying complexes will always be (compatible with) those introduced in More on Algebra, Section 72. This will occasionally cause the multiplicative structure to be twisted in unexpected ways especially when considering left modules or the relationship between left and right modules.

3. Differential graded algebras

Just the definitions.

Definition 3.1. Let R be a commutative ring. A *differential graded algebra over R* is either

(1) a chain complex $A_\bullet$ of R-modules endowed with R-bilinear maps $A_n \times A_m \to A_{n+m}$, $(a, b) \mapsto ab$ such that
$$\mathrm{d}_{n+m}(ab) = \mathrm{d}_n(a)b + (-1)^n a \mathrm{d}_m(b)$$
and such that $\bigoplus A_n$ becomes an associative and unital R-algebra, or

(2) a cochain complex $A^\bullet$ of R-modules endowed with R-bilinear maps $A^n \times A^m \to A^{n+m}$, $(a, b) \mapsto ab$ such that
$$\mathrm{d}^{n+m}(ab) = \mathrm{d}^n(a)b + (-1)^n a \mathrm{d}^m(b)$$
and such that $\bigoplus A^n$ becomes an associative and unital R-algebra.

We often just write $A = \bigoplus A_n$ or $A = \bigoplus A^n$ and think of this as an associative unital R-algebra endowed with a $\mathbf{Z}$-grading and an R-linear operator d whose square is zero and which satisfies the Leibniz rule as explained above. In this case we often say "Let (A, d) be a differential graded algebra".

The Leibniz rule relating differentials and multiplication on a differential graded R-algebra A exactly means that the multiplication map defines a map of cochain complexes
$$\mathrm{Tot}(A^\bullet \otimes_R A^\bullet) \to A^\bullet$$
Here $A^\bullet$ denote the underlying cochain complex of A.

Definition 3.2. A *homomorphism of differential graded algebras* $f : (A, \mathrm{d}) \to (B, \mathrm{d})$ is an algebra map $f : A \to B$ compatible with the gradings and d.

Definition 3.3. A differential graded algebra (A, d) is *commutative* if $ab = (-1)^{nm}ba$ for a in degree n and b in degree m. We say A is *strictly commutative* if in addition $a^2 = 0$ for $\deg(a)$ odd.

The following definition makes sense in general but is perhaps "correct" only when tensoring commutative differential graded algebras.

Definition 3.4. Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. The *tensor product differential graded algebra* of A and B is the algebra $A \otimes_R B$ with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(a')\deg(b)}aa' \otimes bb'$$

endowed with differential d defined by the rule $\mathrm{d}(a \otimes b) = \mathrm{d}(a) \otimes b + (-1)^m a \otimes \mathrm{d}(b)$ where $m = \deg(a)$.

Lemma 3.5. *Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. Denote $A^\bullet$, $B^\bullet$ the underlying cochain complexes. As cochain complexes of R-modules we have*

$$(A \otimes_R B)^\bullet = Tot(A^\bullet \otimes_R B^\bullet).$$

Proof. Recall that the differential of the total complex is given by $\mathrm{d}_1^{p,q} + (-1)^p \mathrm{d}_2^{p,q}$ on $A^p \otimes_R B^q$. And this is exactly the same as the rule for the differential on $A \otimes_R B$ in Definition 3.4. $\qquad\square$

4. Differential graded modules

Our default in this chapter is right modules; we discuss left modules in Section 11.

Definition 4.1. Let R be a ring. Let (A, d) be a differential graded algebra over R. A (right) *differential graded module M* over A is a right A-module M which has a grading $M = \bigoplus M^n$ and a differential d such that $M^n A^m \subset M^{n+m}$, such that $\mathrm{d}(M^n) \subset M^{n+1}$, and such that

$$\mathrm{d}(ma) = \mathrm{d}(m)a + (-1)^n m\mathrm{d}(a)$$

for $a \in A$ and $m \in M^n$. A *homomorphism of differential graded modules $f : M \to N$* is an A-module map compatible with gradings and differentials. The category of (right) differential graded A-modules is denoted $\mathrm{Mod}_{(A,\mathrm{d})}$.

Note that we can think of M as a cochain complex $M^\bullet$ of (right) R-modules. Namely, for $r \in R$ we have $\mathrm{d}(r) = 0$ and r maps to a degree 0 element of A, hence $\mathrm{d}(mr) = \mathrm{d}(m)r$.

The Leibniz rule relating differentials and multiplication on a differential graded R-module M over a differential graded R-algebra A exactly means that the multiplication map defines a map of cochain complexes

$$\mathrm{Tot}(M^\bullet \otimes_R A^\bullet) \to M^\bullet$$

Here $A^\bullet$ and $M^\bullet$ denote the underlying cochain complexes of A and M.

Lemma 4.2. *Let (A, d) be a differential graded algebra. The category $Mod_{(A,d)}$ is abelian and has arbitrary limits and colimits.*

Proof. Kernels and cokernels commute with taking underlying A-modules. Similarly for direct sums and colimits. In other words, these operations in $\text{Mod}_{(A,\text{d})}$ commute with the forgetful functor to the category of A-modules. This is not the case for products and limits. Namely, if N_i, $i \in I$ is a family of differential graded A-modules, then the product $\prod N_i$ in $\text{Mod}_{(A,\text{d})}$ is given by setting $(\prod N_i)^n = \prod N_i^n$ and $\prod N_i = \bigoplus_n (\prod N_i)^n$. Thus we see that the product does commute with the forgetful functor to the category of graded A-modules. A category with products and equalizers has limits, see Categories, Lemma 14.11. $\qquad\square$

Thus, if (A, d) is a differential graded algebra over R, then there is an exact functor

$$\text{Mod}_{(A,\text{d})} \longrightarrow \text{Comp}(R)$$

of abelian categories. For a differential graded module M the cohomology groups $H^n(M)$ are defined as the cohomology of the corresponding complex of R-modules. Therefore, a short exact sequence $0 \to K \to L \to M \to 0$ of differential graded modules gives rise to a long exact sequence

$$(4.2.1) \qquad\qquad H^n(K) \to H^n(L) \to H^n(M) \to H^{n+1}(K)$$

of cohomology modules, see Homology, Lemma 13.12.

Moreover, from now on we borrow all the terminology used for complexes of modules. For example, we say that a differential graded A-module M is *acyclic* if $H^k(M) = 0$ for all $k \in \mathbf{Z}$. We say that a homomorphism $M \to N$ of differential graded A-modules is a *quasi-isomorphism* if it induces isomorphisms $H^k(M) \to H^k(N)$ for all $k \in \mathbf{Z}$. And so on and so forth.

Definition 4.3. Let (A, d) be a differential graded algebra. Let M be a differential graded module whose underlying complex of R-modules is $M^\bullet$. For any $k \in \mathbf{Z}$ we define the *k-shifted module $M[k]$* as follows

(1) the underlying complex of R-modules of $M[k]$ is $M^\bullet[k]$, i.e., we have $M[k]^n = M^{n+k}$ and $\text{d}_{M[k]} = (-1)^k \text{d}_M$ and

(2) as A-module the multiplication

$$(M[k])^n \times A^m \longrightarrow (M[k])^{n+m}$$

is equal to the given multiplication $M^{n+k} \times A^m \to M^{n+k+m}$.

For a morphism $f : M \to N$ of differential graded A-modules we let $f[k] : M[k] \to N[k]$ be the map equal to f on underlying A-modules. This defines a functor $[k] : \text{Mod}_{(A,\text{d})} \to \text{Mod}_{(A,\text{d})}$.

Let us check that with this choice the Leibniz rule is satisfied. Let $x \in M[k]^n = M^{n+k}$ and $a \in A^m$ and denoting $\cdot_{M[k]}$ the product in $M[k]$ then we see

$$\begin{aligned}
\text{d}_{M[k]}(x \cdot_{M[k]} a) &= (-1)^k \text{d}_M(xa) \\
&= (-1)^k \text{d}_M(x)a + (-1)^{k+n+k} x \text{d}(a) \\
&= \text{d}_{M[k]}(x)a + (-1)^n x \text{d}(a) \\
&= \text{d}_{M[k]}(x) \cdot_{M[k]} a + (-1)^n x \cdot_{M[k]} \text{d}(a)
\end{aligned}$$

This is what we want as x has degree n as a homogeneous element of $M[k]$. We also observe that with these choices we may think of the multiplication map as the map of complexes

$$\text{Tot}(M^\bullet[k] \otimes_R A^\bullet) \to \text{Tot}(M^\bullet \otimes_R A^\bullet)[k] \to M^\bullet[k]$$

where the first arrow is More on Algebra, Section 72 (7) which in this case does not involve a sign. (In fact, we could have deduced that the Liebniz rule holds from this observation.)

The remarks in Homology, Section 14 apply. In particular, we will identify the cohomology groups of all shifts $M[k]$ without the intervention of signs.

At this point we have enough structure to talk about *triangles*, see Derived Categories, Definition 3.1. In fact, our next goal is to develop enough theory to be able to state and prove that the homotopy category of differential graded modules is a triangulated category. First we define the homotopy category.

5. The homotopy category

Our homotopies take into account the A-module structure and the grading, but not the differential (of course).

Definition 5.1. Let (A, d) be a differential graded algebra. Let $f, g : M \to N$ be homomorphisms of differential graded A-modules. A *homotopy between f and g* is an A-module map $h : M \to N$ such that

(1) $h(M^n) \subset N^{n-1}$ for all n, and
(2) $f(x) - g(x) = d_N(h(x)) + h(d_M(x))$ for all $x \in M$.

If a homotopy exists, then we say f and g are *homotopic*.

Thus h is compatible with the A-module structure and the grading but not with the differential. If $f = g$ and h is a homotopy as in the definition, then h defines a morphism $h : M \to N[-1]$ in $\mathrm{Mod}_{(A,d)}$.

Lemma 5.2. *Let (A, d) be a differential graded algebra. Let $f, g : L \to M$ be homomorphisms of differential graded A-modules. Suppose given further homomorphisms $a : K \to L$, and $c : M \to N$. If $h : L \to M$ is an A-module map which defines a homotopy between f and g, then $c \circ h \circ a$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Immediate from Homology, Lemma 13.7. $\square$

This lemma allows us to define the homotopy category as follows.

Definition 5.3. Let (A, d) be a differential graded algebra. The *homotopy category*, denoted $K(\mathrm{Mod}_{(A,d)})$, is the category whose objects are the objects of $\mathrm{Mod}_{(A,d)}$ and whose morphisms are homotopy classes of homomorphisms of differential graded A-modules.

The notation $K(\mathrm{Mod}_{(A,d)})$ is not standard but at least is consistent with the use of $K(-)$ in other places of the Stacks project.

Lemma 5.4. *Let (A, d) be a differential graded algebra. The homotopy category $K(\mathrm{Mod}_{(A,d)})$ has direct sums and products.*

Proof. Omitted. Hint: Just use the direct sums and products as in Lemma 4.2. This works because we saw that these functors commute with the forgetful functor to the category of graded A-modules and because $\prod$ is an exact functor on the category of families of abelian groups. $\square$

6. Cones

We introduce cones for the category of differential graded modules.

Definition 6.1. Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded A-modules. The *cone* of f is the differential graded A-module $C(f)$ given by $C(f) = L \oplus K$ with grading $C(f)^n = L^n \oplus K^{n+1}$ and differential

$$d_{C(f)} = \begin{pmatrix} \mathrm{d}_L & f \\ 0 & -\mathrm{d}_K \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : L \to C(f)$ and $p : C(f) \to K[1]$ induced by the obvious maps $L \to C(f)$ and $C(f) \to K$.

The formation of the cone triangle is functorial in the following sense.

Lemma 6.2. *Let (A, d) be a differential graded algebra. Suppose that*

$$\begin{array}{ccc} K_1 & \xrightarrow{\ f_1\ } & L_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ K_2 & \xrightarrow{\ f_2\ } & L_2 \end{array}$$

is a diagram of homomorphisms of differential graded A-modules which is commutative up to homotopy. Then there exists a morphism $c : C(f_1) \to C(f_2)$ which gives rise to a morphism of triangles

$$(a, b, c) : (K_1, L_1, C(f_1), f_1, i_1, p_1) \to (K_1, L_1, C(f_1), f_2, i_2, p_2)$$

in $K(Mod_{(A,d)})$.

Proof. Let $h : K_1 \to L_2$ be a homotopy between $f_2 \circ a$ and $b \circ f_1$. Define c by the matrix

$$c = \begin{pmatrix} b & h \\ 0 & a \end{pmatrix} : L_1 \oplus K_1 \to L_2 \oplus K_2$$

A matrix computation show that c is a morphism of differential graded modules. It is trivial that $c \circ i_1 = i_2 \circ b$, and it is trivial also to check that $p_2 \circ c = a \circ p_1$. $\square$

7. Admissible short exact sequences

An admissible short exact sequence is the analogue of termwise split exact sequences in the setting of differential graded modules.

Definition 7.1. Let (A, d) be a differential graded algebra.
  (1) A homomorphism $K \to L$ of differential graded A-modules is an *admissible monomorphism* if there exists a graded A-module map $L \to K$ which is left inverse to $K \to L$.
  (2) A homomorphism $L \to M$ of differential graded A-modules is an *admissible epimorphism* if there exists a graded A-module map $M \to L$ which is right inverse to $L \to M$.
  (3) A short exact sequence $0 \to K \to L \to M \to 0$ of differential graded A-modules is an *admissible short exact sequence* if it is split as a sequence of graded A-modules.

Thus the splittings are compatible with all the data except for the differentials. Given an admissible short exact sequence we obtain a triangle; this is the reason that we require our splittings to be compatible with the A-module structure.

Lemma 7.2. *Let (A, d) be a differential graded algebra. Let $0 \to K \to L \to M \to 0$ be an admissible short exact sequence of differential graded A-modules. Let $s : M \to L$ and $\pi : L \to K$ be splittings such that $\mathrm{Ker}(\pi) = \mathrm{Im}(s)$. Then we obtain a morphism*

$$\delta = \pi \circ d_L \circ s : M \to K[1]$$

of $\mathrm{Mod}_{(A,d)}$ which induces the boundary maps in the long exact sequence of cohomology (4.2.1).

Proof. The map $\pi \circ d_L \circ s$ is compatible with the A-module structure and the gradings by construction. It is compatible with differentials by Homology, Lemmas 14.10. Let R be the ring that A is a differential graded algebra over. The equality of maps is a statement about R-modules. Hence this follows from Homology, Lemmas 14.10 and 14.11. $\square$

Lemma 7.3. *Let (A, d) be a differential graded algebra. Let*

$$
\begin{array}{ccc}
K & \xrightarrow{\ f\ } & L \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
M & \xrightarrow{\ g\ } & N
\end{array}
$$

be a diagram of homomorphisms of differential graded A-modules commuting up to homotopy.

(1) *If f is an admissible monomorphism, then b is homotopic to a homomorphism which makes the diagram commute.*

(2) *If g is an admissible epimorphism, then a is homotopic to a morphism which makes the diagram commute.*

Proof. Let $h : K \to N$ be a homotopy between bf and ga, i.e., $bf - ga = dh + hd$. Suppose that $\pi : L \to K$ is a graded A-module map left inverse to f. Take $b' = b - dh\pi - h\pi d$. Suppose $s : N \to M$ is a graded A-module map right inverse to g. Take $a' = a + dsh + shd$. Computations omitted. $\square$

Lemma 7.4. *Let (A, d) be a differential graded algebra. Let $\alpha : K \to L$ be a homomorphism of differential graded A-modules. There exists a factorization*

$$K \xrightarrow{\ \tilde{\alpha}\ } \tilde{L} \xrightarrow{\ \pi\ } L$$
$$\underbrace{\phantom{K \xrightarrow{\ \ } \tilde{L} \xrightarrow{\ \ } L}}_{\alpha}$$

in $\mathrm{Mod}_{(A,d)}$ such that

(1) *$\tilde{\alpha}$ is an admissible monomorphism (see Definition 7.1),*

(2) *there is a morphism $s : L \to \tilde{L}$ such that $\pi \circ s = id_L$ and such that $s \circ \pi$ is homotopic to $id_{\tilde{L}}$.*

Proof. The proof is identical to the proof of Derived Categories, Lemma 9.6. Namely, we set $\tilde{L} = L \oplus C(1_K)$ and we use elementary properties of the cone construction. $\square$

Lemma 7.5. *Let (A, d) be a differential graded algebra. Let $L_1 \to L_2 \to \ldots \to L_n$ be a sequence of composable homomorphisms of differential graded A-modules.*

There exists a commutative diagram

$$
\begin{array}{ccccccc}
L_1 & \longrightarrow & L_2 & \longrightarrow & \ldots & \longrightarrow & L_n \\
\uparrow & & \uparrow & & & & \uparrow \\
M_1 & \longrightarrow & M_2 & \longrightarrow & \ldots & \longrightarrow & M_n
\end{array}
$$

in $Mod_{(A,d)}$ such that each $M_i \to M_{i+1}$ is an admissible monomorphism and each $M_i \to L_i$ is a homotopy equivalence.

Proof. The case $n = 1$ is without content. Lemma 7.4 is the case $n = 2$. Suppose we have constructed the diagram except for M_n. Apply Lemma 7.4 to the composition $M_{n-1} \to L_{n-1} \to L_n$. The result is a factorization $M_{n-1} \to M_n \to L_n$ as desired. $\square$

Lemma 7.6. *Let (A, d) be a differential graded algebra. Let $0 \to K_i \to L_i \to M_i \to 0$, $i = 1, 2, 3$ be admissible short exact sequence of differential graded A-modules. Let $b : L_1 \to L_2$ and $b' : L_2 \to L_3$ be homomorphisms of differential graded modules such that*

$$
\begin{array}{ccc}
K_1 \longrightarrow L_1 \longrightarrow M_1 \\
\scriptstyle 0 \downarrow \qquad \scriptstyle b \downarrow \qquad \scriptstyle 0 \downarrow \\
K_2 \longrightarrow L_2 \longrightarrow M_2
\end{array}
\quad and \quad
\begin{array}{ccc}
K_2 \longrightarrow L_2 \longrightarrow M_2 \\
\scriptstyle 0 \downarrow \qquad \scriptstyle b' \downarrow \qquad \scriptstyle 0 \downarrow \\
K_3 \longrightarrow L_3 \longrightarrow M_3
\end{array}
$$

commute up to homotopy. Then $b' \circ b$ is homotopic to 0.

Proof. By Lemma 7.3 we can replace b and b' by homotopic maps such that the right square of the left diagram commutes and the left square of the right diagram commutes. In other words, we have $\mathrm{Im}(b) \subset \mathrm{Im}(K_2 \to L_2)$ and $\mathrm{Ker}((b')^n) \supset \mathrm{Im}(K_2 \to L_2)$. Then $b \circ b' = 0$ as a map of modules. $\square$

8. Distinguished triangles

The following lemma produces our distinguished triangles.

Lemma 8.1. *Let (A, d) be a differential graded algebra. Let $0 \to K \to L \to M \to 0$ be an admissible short exact sequence of differential graded A-modules. The triangle*

$$(8.1.1) \qquad\qquad\qquad K \to L \to M \xrightarrow{\delta} K[1]$$

with δ as in Lemma 7.2 is, up to canonical isomorphism in $K(Mod_{(A,d)})$, independent of the choices made in Lemma 7.2.

Proof. Namely, let (s', π') be a second choice of splittings as in Lemma 7.2. Then we claim that δ and δ' are homotopic. Namely, write $s' = s + \alpha \circ h$ and $\pi' = \pi + g \circ \beta$ for some unique homomorphisms of A-modules $h : M \to K$ and $g : M \to K$ of degree -1. Then $g = -h$ and g is a homotopy between δ and δ'. The computations are done in the proof of Homology, Lemma 14.12. $\square$

Definition 8.2. Let (A, d) be a differential graded algebra.
  (1) If $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded A-modules, then the *triangle associated to $0 \to K \to L \to M \to 0$* is the triangle (8.1.1) of $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

(2) A triangle of $K(\text{Mod}_{(A,\text{d})})$ is called a *distinguished triangle* if it is isomorphic to a triangle associated to an admissible short exact sequence of differential graded A-modules.

9. Cones and distinguished triangles

Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded A-modules. Then $(K, L, C(f), f, i, p)$ forms a triangle:

$$K \to L \to C(f) \to K[1]$$

in $\text{Mod}_{(A,\text{d})}$ and hence in $K(\text{Mod}_{(A,\text{d})})$. Cones are **not** distinguished triangles in general, but the difference is a sign or a rotation (your choice). Here are two precise statements.

Lemma 9.1. *Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded modules. The triangle $(L, C(f), K[1], i, p, f[1])$ is the triangle associated to the admissible short exact sequence*

$$0 \to L \to C(f) \to K[1] \to 0$$

coming from the definition of the cone of f.

Proof. Immediate from the definitions. $\square$

Lemma 9.2. *Let (A, d) be a differential graded algebra. Let $\alpha : K \to L$ and $\beta : L \to M$ define an admissible short exact sequence*

$$0 \to K \to L \to M \to 0$$

of differential graded A-modules. Let $(K, L, M, \alpha, \beta, \delta)$ be the associated triangle. Then the triangles

$$(M[-1], K, L, \delta[-1], \alpha, \beta) \quad and \quad (M[-1], K, C(\delta[-1]), \delta[-1], i, p)$$

are isomorphic.

Proof. Using a choice of splittings we write $L = K \oplus M$ and we identify α and β with the natural inclusion and projection maps. By construction of δ we have

$$d_B = \begin{pmatrix} d_K & \delta \\ 0 & d_M \end{pmatrix}$$

On the other hand the cone of $\delta[-1] : M[-1] \to K$ is given as $C(\delta[-1]) = K \oplus M$ with differential identical with the matrix above! Whence the lemma. $\square$

Lemma 9.3. *Let (A, d) be a differential graded algebra. Let $f_1 : K_1 \to L_1$ and $f_2 : K_2 \to L_2$ be homomorphisms of differential graded A-modules. Let*

$$(a, b, c) : (K_1, L_1, C(f_1), f_1, i_1, p_1) \longrightarrow (K_1, L_1, C(f_1), f_2, i_2, p_2)$$

be any morphism of triangles of $K(\text{Mod}_{(A,d)})$. If a and b are homotopy equivalences then so is c.

Proof. Let $a^{-1} : K_2 \to K_1$ be a homomorphism of differential graded A-modules which is inverse to a in $K(\text{Mod}_{(A,\text{d})})$. Let $b^{-1} : L_2 \to L_1$ be a homomorphism of differential graded A-modules which is inverse to b in $K(\text{Mod}_{(A,\text{d})})$. Let $c' : C(f_2) \to C(f_1)$ be the morphism from Lemma 6.2 applied to $f_1 \circ a^{-1} = b^{-1} \circ f_2$. If we can show that $c \circ c'$ and $c' \circ c$ are isomorphisms in $K(\text{Mod}_{(A,\text{d})})$ then we win. Hence it suffices to prove the following: Given a morphism of triangles $(1, 1, c) : (K, L, C(f), f, i, p)$

in $K(\mathrm{Mod}_{(A,\mathrm{d})})$ the morphism c is an isomorphism in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. By assumption the two squares in the diagram

$$
\begin{array}{ccccc}
L & \longrightarrow & C(f) & \longrightarrow & K[1] \\
{\scriptstyle 1}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle 1}\downarrow \\
L & \longrightarrow & C(f) & \longrightarrow & K[1]
\end{array}
$$

commute up to homotopy. By construction of $C(f)$ the rows form admissible short exact sequences. Thus we see that $(c-1)^2 = 0$ in $K(\mathrm{Mod}_{(A,\mathrm{d})})$ by Lemma 7.6. Hence c is an isomorphism in $K(\mathrm{Mod}_{(A,\mathrm{d})})$ with inverse $2 - c$. $\qquad\square$

The following lemma shows that the collection of triangles of the homotopy category given by cones and the distinguished triangles are the same up to isomorphisms, at least up to sign!

Lemma 9.4. *Let (A, d) be a differential graded algebra.*

(1) *Given an admissible short exact sequence $0 \to K \xrightarrow{\alpha} L \to M \to 0$ of differential graded A-modules there exists a homotopy equivalence $C(\alpha) \to M$ such that the diagram*

$$
\begin{array}{ccccccc}
K & \longrightarrow & L & \longrightarrow & C(\alpha) & \xrightarrow{\ -p\ } & K[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K & \xrightarrow{\ \alpha\ } & L & \xrightarrow{\ \beta\ } & M & \xrightarrow{\ \delta\ } & K[1]
\end{array}
$$

defines an isomorphism of triangles in $K(\mathrm{Mod}_{(A,d)})$.

(2) *Given a morphism of complexes $f : K \to L$ there exists an isomorphism of triangles*

$$
\begin{array}{ccccccc}
K & \longrightarrow & \tilde{L} & \longrightarrow & M & \xrightarrow{\ \delta\ } & K[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K & \longrightarrow & L & \longrightarrow & C(f) & \xrightarrow{\ -p\ } & K[1]
\end{array}
$$

where the upper triangle is the triangle associated to a admissible short exact sequence $K \to \tilde{L} \to M$.

Proof. Proof of (1). We have $C(\alpha) = L \oplus K$ and we simply define $C(\alpha) \to M$ via the projection onto L followed by β. This defines a morphism of differential graded modules because the compositions $K^{n+1} \to L^{n+1} \to M^{n+1}$ are zero. Choose splittings $s : M \to L$ and $\pi : L \to K$ with $\mathrm{Ker}(\pi) = \mathrm{Im}(s)$ and set $\delta = \pi \circ \mathrm{d}_L \circ s$ as usual. To get a homotopy inverse we take $M \to C(\alpha)$ given by $(s, -\delta)$. This is compatible with differentials because δ^n can be characterized as the unique map $M^n \to K^{n+1}$ such that $\mathrm{d} \circ s^n - s^{n+1} \circ \mathrm{d} = \alpha \circ \delta^n$, see proof of Homology, Lemma 14.10. The composition $M \to C(f) \to M$ is the identity. The composition $C(f) \to M \to C(f)$ is equal to the morphism

$$
\begin{pmatrix}
s \circ \beta & 0 \\
-\delta \circ \beta & 0
\end{pmatrix}
$$

To see that this is homotopic to the identity map use the homotopy $h : C(\alpha) \to C(\alpha)$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} : C(\alpha) = L \oplus K \to L \oplus K = C(\alpha)$$

It is trivial to verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s \\ -\delta \end{pmatrix} \begin{pmatrix} \beta & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{d} & \alpha \\ 0 & -\mathrm{d} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d} & \alpha \\ 0 & -\mathrm{d} \end{pmatrix}$$

To finish the proof of (1) we have to show that the morphisms $-p : C(\alpha) \to K[1]$ (see Definition 6.1) and $C(\alpha) \to M \to K[1]$ agree up to homotopy. This is clear from the above. Namely, we can use the homotopy inverse $(s, -\delta) : M \to C(\alpha)$ and check instead that the two maps $M \to K[1]$ agree. And note that $p \circ (s, -\delta) = -\delta$ as desired.

Proof of (2). We let $\tilde{f} : K \to \tilde{L}$, $s : L \to \tilde{L}$ and $\pi : L \to L$ be as in Lemma 7.4. By Lemmas 6.2 and 9.3 the triangles $(K, L, C(f), i, p)$ and $(K, \tilde{L}, C(\tilde{f}), \tilde{i}, \tilde{p})$ are isomorphic. Note that we can compose isomorphisms of triangles. Thus we may replace L by $\tilde{L}$ and f by $\tilde{f}$. In other words we may assume that f is an admissible monomorphism. In this case the result follows from part (1). $\square$

10. The homotopy category is triangulated

We first prove that it is pre-triangulated.

Lemma 10.1. *Let (A, d) be a differential graded algebra. The homotopy category $K(Mod_{(A,d)})$ with its natural translation functors and distinguished triangles is a pre-triangulated category.*

Proof. Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Also, any triangle $(K, K, 0, 1, 0, 0)$ is distinguished since $0 \to K \to K \to 0 \to 0$ is an admissible short exact sequence. Finally, given any homomorphism $f : K \to L$ of differential graded A-modules the triangle $(K, L, C(f), f, i, -p)$ is distinguished by Lemma 9.4.

Proof of TR2. Let (X, Y, Z, f, g, h) be a triangle. Assume $(Y, Z, X[1], g, h, -f[1])$ is distinguished. Then there exists an admissible short exact sequence $0 \to K \to L \to M \to 0$ such that the associated triangle $(K, L, M, \alpha, \beta, \delta)$ is isomorphic to $(Y, Z, X[1], g, h, -f[1])$. Rotating back we see that (X, Y, Z, f, g, h) is isomorphic to $(M[-1], K, L, -\delta[-1], \alpha, \beta)$. It follows from Lemma 9.2 that the triangle $(M[-1], K, L, \delta[-1], \alpha, \beta)$ is isomorphic to $(M[-1], K, C(\delta[-1]), \delta[-1], i, p)$. Precomposing the previous isomorphism of triangles with -1 on Y it follows that (X, Y, Z, f, g, h) is isomorphic to $(M[-1], K, C(\delta[-1]), \delta[-1], i, -p)$. Hence it is distinguished by Lemma 9.4. On the other hand, suppose that (X, Y, Z, f, g, h) is distinguished. By Lemma 9.4 this means that it is isomorphic to a triangle of the form $(K, L, C(f), f, i, -p)$ for some morphism f of $Mod_{(A,d)}$. Then the rotated triangle $(Y, Z, X[1], g, h, -f[1])$ is isomorphic to $(L, C(f), K[1], i, -p, -f[1])$ which is isomorphic to the triangle $(L, C(f), K[1], i, p, f[1])$. By Lemma 9.1 this triangle is distinguished. Hence $(Y, Z, X[1], g, h, -f[1])$ is distinguished as desired.

Proof of TR3. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be distinguished triangles of $K(\mathcal{A})$ and let $a : X \to X'$ and $b : Y \to Y'$ be morphisms such that $f' \circ a = b \circ f$. By Lemma 9.4 we may assume that $(X, Y, Z, f, g, h) = (X, Y, C(f), f, i, -p)$

and $(X', Y', Z', f', g', h') = (X', Y', C(f'), f', i', -p')$. At this point we simply apply Lemma 6.2 to the commutative diagram given by f, f', a, b. $\qquad\square$

Before we prove TR4 in general we prove it in a special case.

Lemma 10.2. *Let (A, d) be a differential graded algebra. Suppose that $\alpha : K \to L$ and $\beta : L \to M$ are admissible monomorphisms of differential graded A-modules. Then there exist distinguished triangles $(K, L, Q_1, \alpha, p_1, d_1)$, $(K, M, Q_2, \beta \circ \alpha, p_2, d_2)$ and $(L, M, Q_3, \beta, p_3, d_3)$ for which TR4 holds.*

Proof. Say $\pi_1 : L \to K$ and $\pi_3 : M \to L$ are homomorphisms of graded A-modules which are left inverse to α and β. Then also $K \to M$ is an admissible monomorphism with left inverse $\pi_2 = \pi_1 \circ \pi_3$. Let us write Q_1, Q_2 and Q_3 for the cokernels of $K \to L$, $K \to M$, and $L \to M$. Then we obtain identifications (as graded A-modules) $Q_1 = \mathrm{Ker}(\pi_1)$, $Q_3 = \mathrm{Ker}(\pi_3)$ and $Q_2 = \mathrm{Ker}(\pi_2)$. Then $L = K \oplus Q_1$ and $M = L \oplus Q_3$ as graded A-modules. This implies $M = K \oplus Q_1 \oplus Q_3$. Note that $\pi_2 = \pi_1 \circ \pi_3$ is zero on both Q_1 and Q_3. Hence $Q_2 = Q_1 \oplus Q_3$. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & K & \to & L & \to & Q_1 & \to & 0 \\
  &   & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & M & \to & Q_2 & \to & 0 \\
  &   & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & M & \to & Q_3 & \to & 0 \\
\end{array}
$$

The rows of this diagram are admissible short exact sequences, and hence determine distinguished triangles by definition. Moreover downward arrows in the diagram above are compatible with the chosen splittings and hence define morphisms of triangles

$$(K \to L \to Q_1 \to K[1]) \longrightarrow (K \to M \to Q_2 \to K[1])$$

and

$$(K \to M \to Q_2 \to K[1]) \longrightarrow (L \to M \to Q_3 \to L[1]).$$

Note that the splittings $Q_3 \to M$ of the bottom sequence in the diagram provides a splitting for the split sequence $0 \to Q_1 \to Q_2 \to Q_3 \to 0$ upon composing with $M \to Q_2$. It follows easily from this that the morphism $\delta : Q_3 \to Q_1[1]$ in the corresponding distinguished triangle

$$(Q_1 \to Q_2 \to Q_3 \to Q_1[1])$$

is equal to the composition $Q_3 \to L[1] \to Q_1[1]$. Hence we get a structure as in the conclusion of axiom TR4. $\qquad\square$

Here is the final result.

Proposition 10.3. *Let (A, d) be a differential graded algebra. The homotopy category $K(Mod_{(A,d)})$ of differential graded A-modules with its natural translation functors and distinguished triangles is a triangulated category.*

Proof. We know that $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is a pre-triangulated category. Hence it suffices to prove TR4 and to prove it we can use Derived Categories, Lemma 4.15. Let $K \to L$ and $L \to M$ be composable morphisms of $K(\mathrm{Mod}_{(A,\mathrm{d})})$. By Lemma 7.5 we may assume that $K \to L$ and $L \to M$ are admissible monomorphisms. In this case the result follows from Lemma 10.2. $\qquad\square$

11. Left modules

Everything we have said sofar has an analogue in the setting of left differential graded modules, except that one has to take care with some sign rules.

Let (A, d) be a differential graded R-algebra. Exactly analogous to right modules, we define a *left differential graded A-module M* as a left A-module M which has a grading $M = \bigoplus M^n$ and a differential d, such that $A^n M^m \subset M^{n+m}$, such that $\mathrm{d}(M^n) \subset M^{n+1}$, and such that

$$\mathrm{d}(am) = \mathrm{d}(a)m + (-1)^{\deg(a)} a \mathrm{d}(m)$$

for homogeneous elements $a \in A$ and $m \in M$. As before this Leibniz rule exactly signifies that the multiplication defines a map of complexes

$$\mathrm{Tot}(A^\bullet \otimes_R M^\bullet) \to M^\bullet$$

Here $A^\bullet$ and $M^\bullet$ denote the complexes of R-modules underlying A and M.

Definition 11.1. Let R be a ring. Let (A, d) be a differential graded algebra over R. The *opposite differential graded algebra* is the differential graded algebra (A^{opp}, d) over R where $A^{opp} = A$ as a graded R-module, $\mathrm{d} = \mathrm{d}$, and multiplication is given by

$$a \cdot_{opp} b = (-1)^{\deg(a) \deg(b)} ba$$

for homogeneous elements $a, b \in A$.

This makes sense because

$$
\begin{aligned}
\mathrm{d}(a \cdot_{opp} b) &= (-1)^{\deg(a)\deg(b)} \mathrm{d}(ba) \\
&= (-1)^{\deg(a)\deg(b)} \mathrm{d}(b)a + (-1)^{\deg(a)\deg(b)+\deg(b)} b\mathrm{d}(a) \\
&= (-1)^{\deg(a)} a \cdot_{opp} \mathrm{d}(b) + \mathrm{d}(a) \cdot_{opp} b
\end{aligned}
$$

as desired. In terms of underlying complexes of R-modules this means that the diagram

$$
\begin{array}{ccc}
\mathrm{Tot}(A^\bullet \otimes_R A^\bullet) & \xrightarrow{\text{multiplication of } A^{opp}} & A^\bullet \\
{\scriptstyle \text{commutativity constraint}} \downarrow & & \downarrow {\scriptstyle \text{id}} \\
\mathrm{Tot}(A^\bullet \otimes_R A^\bullet) & \xrightarrow{\text{multiplication of } A} & A^\bullet
\end{array}
$$

commutes. Here the commutativity constraint on the symmetric monoidal category of complexes of R-modules is given in More on Algebra, Section 72.

Let (A, d) be a differential graded algebra over R. Let M be a left differential graded A-module. We will denote M^{opp} the module M viewed as a right A^{opp}-module with multiplication $\cdot_{opp}$ defined by the rule

$$m \cdot_{opp} a = (-1)^{\deg(a) \deg(m)} am$$

for a and m homogeneous. This is compatible with differentials because we could have used the diagram

$$
\begin{array}{ccc}
\mathrm{Tot}(M^\bullet \otimes_R A^\bullet) & \xrightarrow{\text{multiplication on } M^{opp}} & M^\bullet \\
{\scriptstyle \text{commutativity constraint}} \downarrow & & \downarrow {\scriptstyle \text{id}} \\
\mathrm{Tot}(A^\bullet \otimes_R M^\bullet) & \xrightarrow{\text{multiplication on } M} & M^\bullet
\end{array}
$$

to define the multiplication $\cdot_{opp}$ on M^{opp}. To see that it is an associative multiplication we compute for homogeneous elements $a, b \in A$ and $m \in M$ that

$$
\begin{aligned}
m \cdot_{opp} (a \cdot_{opp} b) &= (-1)^{\deg(a)\deg(b)} m \cdot_{opp} (ba) \\
&= (-1)^{\deg(a)\deg(b)+\deg(ab)\deg(m)} bam \\
&= (-1)^{\deg(a)\deg(b)+\deg(ab)\deg(m)+\deg(b)\deg(am)} (am) \cdot_{opp} b \\
&= (-1)^{\deg(a)\deg(b)+\deg(ab)\deg(m)+\deg(b)\deg(am)+\deg(a)\deg(m)} (m \cdot_{opp} a) \cdot_{opp} b \\
&= (m \cdot_{opp} a) \cdot_{opp} b
\end{aligned}
$$

Of course, we could have been shown this using the compatibility between the associativity and commutativity constraint on the symmetric monoidal category of complexes of R-modules as well.

Lemma 11.2. *Let (A, d) be a differential graded R-algebra. The functor $M \mapsto M^{opp}$ from the category of left differential graded A-modules to the category of right differential graded A^{opp}-modules is an equivalence.*

Proof. Omitted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Mext, we come to shifts. Let (A, d) be a differential graded algebra. Let M be a left differential graded A-module whose underlying complex of R-modules is denoted $M^\bullet$. For any $k \in \mathbf{Z}$ we define the *k-shifted module $M[k]$* as follows

(1) the underlying complex of R-modules of $M[k]$ is $M^\bullet[k]$
(2) as A-module the multiplication

$$
A^n \times (M[k])^m \longrightarrow (M[k])^{n+m}
$$

   is equal to $(-1)^{nk}$ times the given multiplication $A^n \times M^{m+k} \to M^{n+m+k}$.

Let us check that with this choice the Leibniz rule is satisfied. Let $a \in A^n$ and $x \in M[k]^m = M^{m+k}$ and denoting $\cdot_{M[k]}$ the product in $M[k]$ then we see

$$
\begin{aligned}
\mathrm{d}_{M[k]}(a \cdot_{M[k]} x) &= (-1)^{k+nk} \mathrm{d}_M(ax) \\
&= (-1)^{k+nk} \mathrm{d}(a)x + (-1)^{k+nk+n} a \mathrm{d}_M(x) \\
&= \mathrm{d}(a) \cdot_{M[k]} x + (-1)^{nk+n} a \mathrm{d}_{M[k]}(x) \\
&= \mathrm{d}(a) \cdot_{M[k]} x + (-1)^n a \cdot_{M[k]} \mathrm{d}_{M[k]}(x)
\end{aligned}
$$

This is what we want as a has degree n as a homogeneous element of A. We also observe that with these choices we may think of the multiplication map as the map of complexes

$$
\mathrm{Tot}(A^\bullet \otimes_R M^\bullet[k]) \to \mathrm{Tot}(A^\bullet \otimes_R M^\bullet)[k] \to M^\bullet[k]
$$

where the first arrow is More on Algebra, Section 72 (7) which in this case involves exactly the sign we chose above. (In fact, we could have deduced that the Liebniz rule holds from this observation.)

With the rule above we have canonical identifications

$$
(M[k])^{opp} = M^{opp}[k]
$$

of right differential graded A^{opp}-modules defined without the intervention of signs, in other words, the equivalence of Lemma 11.2 is compatible with shift functors.

Our choice above necessitates the following definition.

Definition 11.3. Let R be a ring. Let A be a **Z**-graded R-algebra.

 (1) Given a right graded A-module M we define the *kth shifted A-module $M[k]$* as the same as a right A-module but with grading $(M[k])^n = M^{n+k}$.
 (2) Given a left graded A-module M we define the *kth shifted A-module $M[k]$* as the module with grading $(M[k])^n = M^{n+k}$ and multiplication $A^n \times (M[k])^m \to (M[k])^{n+m}$ equal to $(-1)^{nk}$ times the given multiplication $A^n \times M^{m+k} \to M^{n+m+k}$.

Let (A, d) be a differential graded algebra. Let $f, g : M \to N$ be homomorphisms of left differential graded A-modules. A *homotopy between f and g* is a graded A-module map $h : M \to N[-1]$ (observe the shift!) such that

$$f(x) - g(x) = \mathrm{d}_N(h(x)) + h(\mathrm{d}_M(x))$$

for all $x \in M$. If a homotopy exists, then we say f and g are *homotopic*. Thus h is compatible with the A-module structure (with the shifted one on N) and the grading (with shifted grading on N) but not with the differential. If $f = g$ and h is a homotopy, then h defines a morphism $h : M \to N[-1]$ of left differential graded A-modules.

With the rule above we find that $f, g : M \to N$ are homotopic if and only if the induced morphisms $f^{opp}, g^{opp} : M^{opp} \to N^{opp}$ are homotopic as right differential graded A^{opp}-module homomorphisms (with the same homotopy).

The homotopy category, cones, admissible short exact sequences, distinguished triangles are all defined in exactly the same manner as for right differential graded modules (and everything agrees on underlying complexes of R-modules with the constructions for complexes of R-modules). In this manner we obtain the analogue of Proposition 10.3 for left modules as well, or we can deduce it by working with right modules over the opposite algebra.

12. Tensor product

Let R be a ring. Let A be an R-algebra (see Section 2). Given a right A-module M and a left A-module N there is a *tensor product*

$$M \otimes_A N$$

This tensor product is a module over R. As an R-module $M \otimes_A N$ is generated by symbols $x \otimes y$ with $x \in M$ and $y \in N$ subject to the relations

$$(x_1 + x_2) \otimes y - x_1 \otimes y - x_2 \otimes y,$$
$$x \otimes (y_1 + y_2) - x \otimes y_1 - x \otimes y_2,$$
$$xa \otimes y - x \otimes ay$$

for $a \in A$, $x, x_1, x_2 \in M$ and $y, y_1, y_2 \in N$. We list some properties of the tensor product

In each variable the tensor product is right exact, in fact commutes with direct sums and arbitrary colimits.

The tensor product $M \otimes_A N$ is the receptacle of the universal A-bilinear map $M \times N \to M \otimes_A N$, $(x, y) \mapsto x \otimes y$. In a formula

$$\mathrm{Bilinear}_A(M \times N, Q) = \mathrm{Hom}_R(M \otimes_A N, Q)$$

for any R-module Q.

If A is a $\mathbf{Z}$-graded algebra and M, N are graded A-modules then $M \otimes_A N$ is a graded R-module. Then nth graded piece $(M \otimes_A N)^n$ of $M \otimes_A N$ is equal to

$$\mathrm{Coker}\left(\bigoplus\nolimits_{r+t+s=n} M^r \otimes_R A^t \otimes_R N^s \to \bigoplus\nolimits_{p+q=n} M^p \otimes_R N^q\right)$$

where the map sends $x \otimes a \otimes y$ to $x \otimes ay - xa \otimes y$ for $x \in M^r$, $y \in N^s$, and $a \in A^t$ with $r + s + t = n$. In this case the map $M \times N \to M \otimes_A N$ is A-bilinear and compatible with gradings and universal in the sense that

$$\mathrm{GradedBilinear}_A(M \times N, Q) = \mathrm{Hom}_{\mathrm{graded}\ R\text{-modules}}(M \otimes_A N, Q)$$

for any graded R-module Q with an obvious notion of graded bilinar map.

If (A, d) is a differential graded algebra and M and N are left and right differential graded A-modules, then $M \otimes_A N$ is a differential graded R-module with differential

$$\mathrm{d}(x \otimes y) = \mathrm{d}(x) \otimes y + (-1)^{\deg(x)} x \otimes \mathrm{d}(y)$$

for $x \in M$ and $y \in N$ homogeneous. In this case the map $M \times N \to M \otimes_A N$ is A-bilinear, compatible with gradings, and compatible with differentials and universal in the sense that

$$\mathrm{DifferentialGradedBilinear}_A(M \times N, Q) = \mathrm{Hom}_{\mathrm{Comp}(R)}(M \otimes_A N, Q)$$

for any differential graded R-module Q with an obvious notion of differential graded bilinar map.

13. Hom complexes and differential graded modules

We urge the reader to skip this section.

Let R be a ring and let $M^\bullet$ be a complex of R-modules. Consider the complex of R-modules

$$E^\bullet = \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)$$

introduced in More on Algebra, Section 71. By More on Algebra, Lemma 71.3 there is a canonical composition law

$$\mathrm{Tot}(E^\bullet \otimes_R E^\bullet) \to E^\bullet$$

which is a map of complexes. Thus we see that $E^\bullet$ with this multiplication is a differential graded R-algebra which we will denote (E, d). Moreover, viewing $M^\bullet$ as $\mathrm{Hom}^\bullet(R, M^\bullet)$ we see that composition defines a multiplication

$$\mathrm{Tot}(E^\bullet \otimes_R M^\bullet) \to M^\bullet$$

which turns $M^\bullet$ into a **left** differential graded E-module which we will denote M.

Lemma 13.1. *In the situation above, let A be a differential graded R-algebra. To give a left A-module structure on M is the same thing as giving a homomorphism $A \to E$ of differential graded R-algebras.*

Proof. Proof omitted. Observe that no signs intervene in this correspondence. $\square$

We continue with the discussion above and we assume given another complex $N^\bullet$ of R-modules. Consider the complex of R-modules $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ introduced in More on Algebra, Section 71. As above we see that composition

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(M^\bullet, N^\bullet) \otimes_R E^\bullet) \to \mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$$

defines a multiplication which turns $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ into a **right** differential graded E-module. Using Lemma 13.1 we conclude that given a left differential graded A-module M and a complex of R-modules $N^\bullet$ there is a canonical right differential graded A-module whose underlying complex of R-modules is $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ and where multiplication

$$\mathrm{Hom}^n(M^\bullet, N^\bullet) \times A^m \longrightarrow \mathrm{Hom}^{n+m}(M^\bullet, N^\bullet)$$

sends $f = (f_{p,q})_{p+q=n}$ with $f_{p,q} \in \mathrm{Hom}(M^{-q}, N^p)$ and $a \in A^m$ to the element $f \cdot a = (f_{p,q} \circ a)$ where $f_{p,q} \circ a$ is the map

$$M^{-q-m} \xrightarrow{a} M^{-q} \xrightarrow{f_{p,q}} N^p, \quad x \longmapsto f_{p,q}(ax)$$

without the intervention of signs. Let us use the notation $\mathrm{Hom}(M, N^\bullet)$ to denote this right differential graded A-module.

Lemma 13.2. *Let R be a ring. Let (A, d) be a differential graded R-algebra. Let M' be a right differential graded A-module and let M be a left differential graded A-module. Let $N^\bullet$ be a complex of R-modules. Then we have*

$$\mathrm{Hom}_{Mod_{(A,d)}}(M', \mathrm{Hom}(M, N^\bullet)) = \mathrm{Hom}_{Comp(R)}(M' \otimes_A M, N^\bullet)$$

where $M \otimes_A M$ is viewed as a complex of R-modules as in Section 12.

Proof. Let us show that both sides correspond to graded A-bilinear maps

$$M' \times M \longrightarrow N^\bullet$$

compatible with differentials. We have seen this is true for the right hand side in Section 12. Given an element g of the left hand side, the equality of More on Algebra, Lemma 71.1 determines a map of complexes of R-modules $g' : \mathrm{Tot}(M' \otimes_R M) \to N^\bullet$. In other words, we obtain a graded R-bilinear map $g'' : M' \times M \to N^\bullet$ compatible with differentials. The A-linearity of g translates immediately into A-bilinarity of g''. $\qquad\square$

Let R, $M^\bullet$, $E^\bullet$, E, and M be as above. However, now suppose given a differential graded R-algebra A and a **right** differential graded A-module structure on M. Then we can consider the map

$$\mathrm{Tot}(A^\bullet \otimes_R M^\bullet) \xrightarrow{\psi} \mathrm{Tot}(A^\bullet \otimes_R M^\bullet) \to M^\bullet$$

where the first arrow is the commutativity constraint on the differential graded category of complexes of R-modules. This corresponds to a map

$$\tau : A^\bullet \longrightarrow E^\bullet$$

of complexes of R-modules. Recall that $E^n = \prod_{p+q=n} \mathrm{Hom}_R(M^{-q}, M^p)$ and write $\tau(a) = (\tau_{p,q}(a))_{p+q=n}$ for $a \in A^n$. Then we see

$$\tau_{p,q}(a) : M^{-q} \longrightarrow M^p, \quad x \longmapsto (-1)^{\deg(a)\deg(x)} xa = (-1)^{-nq} xa$$

This is not compatible with the product on A as the readed should expect from the discussion in Section 11. Namely, we have

$$\tau(aa') = (-1)^{\deg(a)\deg(a')} \tau(a')\tau(a)$$

We conclude the following lemma is true

Lemma 13.3. *In the situation above, let A be a differential graded R-algebra. To give a right A-module structure on M is the same thing as giving a homomorphism $\tau : A \to E^{opp}$ of differential graded R-algebras.*

Proof. See discussion above and note that the construction of τ from the multiplication map $M^n \times A^m \to M^{n+m}$ uses signs. $\square$

Let R, $M^\bullet$, $E^\bullet$, E, A and M be as above and let a right differential graded A-module structure on M be given as in the lemma. In this case there is a canonical left differential graded A-module whose underlying complex of R-modules is $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$. Namely, for multiplication we can use

$$
\mathrm{Tot}(A^\bullet \otimes_R \mathrm{Hom}^\bullet(M^\bullet, N^\bullet)) \xrightarrow{\psi} \mathrm{Tot}(\mathrm{Hom}^\bullet(M^\bullet, N^\bullet) \otimes_R A^\bullet)
$$
$$
\xrightarrow{\tau} \mathrm{Tot}(\mathrm{Hom}^\bullet(M^\bullet, N^\bullet) \otimes_R \mathrm{Hom}^\bullet(M^\bullet, M^\bullet))
$$
$$
\to \mathrm{Tot}(\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)
$$

The first arrow uses the commutativity constraint on the category of complexes of R-modules, the second arrow is described above, and the third arrow is the composition law for the Hom complex. Each map is a map of complexes, hence the result is a map of complexes. In fact, this construction turns $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ into a left differential graded A-module (associativity of the multiplication can be shown using the symmetric monoidal structure or by a direct calculation using the formulae below). Let us explicate the multiplication

$$
A^n \times \mathrm{Hom}^m(M^\bullet, N^\bullet) \longrightarrow \mathrm{Hom}^{n+m}(M^\bullet, N^\bullet)
$$

It sends $a \in A^n$ and $f = (f_{p,q})_{p+q=m}$ with $f_{p,q} \in \mathrm{Hom}(M^{-q}, N^p)$ to the element $a \cdot f$ with constituents

$$
(-1)^{nm} f_{p,q} \circ \tau_{-q,q+n}(a) = (-1)^{nm-n(q+n)} f_{p,q} \circ a = (-1)^{np+n} f_{p,q} \circ a
$$

in $\mathrm{Hom}_R(M^{-q-n}, N^p)$ where $f_{p,q} \circ a$ is the map

$$
M^{-q-n} \xrightarrow{a} M^{-q} \xrightarrow{f_{p,q}} N^p, \quad x \longmapsto f_{p,q}(xa)
$$

Here a sign of $(-1)^{np+n}$ does intervene. Let us use the notation $\mathrm{Hom}(M, N^\bullet)$ to denote this left differential graded A-module.

Lemma 13.4. *Let R be a ring. Let (A, d) be a differential graded R-algebra. Let M be a right differential graded A-module and let M' be a left differential graded A-module. Let $N^\bullet$ be a complex of R-modules. Then we have*

$$
\mathrm{Hom}_{left\ diff\ graded\ A\text{-}modules}(M', \mathrm{Hom}(M, N^\bullet)) = \mathrm{Hom}_{Comp(R)}(M \otimes_A M', N^\bullet)
$$

where $M \otimes_A M'$ is viewed as a complex of R-modules as in Section 12.

Proof. Let us show that both sides correspond to graded A-bilinear maps

$$
M \times M' \longrightarrow N^\bullet
$$

compatible with differentials. We have seen this is true for the right hand side in Section 12. Given an element g of the left hand side, the equality of More on Algebra, Lemma 71.1 determines a map of complexes $g' : \mathrm{Tot}(M' \otimes_R M) \to N^\bullet$. We precompose with the commutativity constraint to get

$$
\mathrm{Tot}(M \otimes_R M') \xrightarrow{\psi} \mathrm{Tot}(M' \otimes_R M) \xrightarrow{g'} N^\bullet
$$

which corresponds to a graded R-bilinear map $g'' : M \times M' \to N^\bullet$ compatible with differentials. The A-linearity of g translates immediately into A-bilinarity of g''. Namely, say $x \in M^e$ and $x' \in (M')^{e'}$ and $a \in A^n$. Then on the one hand we have

$$
\begin{aligned}
g''(x, ax') &= (-1)^{e(n+e')} g'(ax' \otimes x) \\
&= (-1)^{e(n+e')} g(ax')(x) \\
&= (-1)^{e(n+e')} (a \cdot g(x'))(x) \\
&= (-1)^{e(n+e')+n(n+e+e')+n} g(x')(xa)
\end{aligned}
$$

and on the other hand we have

$$
g''(xa, x') = (-1)^{(e+n)e'} g'(x' \otimes xa) = (-1)^{(e+n)e'} g(x')(xa)
$$

which is the same thing by a trivial mod 2 calculation of the exponents. $\square$

Remark 13.5. Let R be a ring. Let A be a differential graded R-algebra. Let M be a left differential graded A-module. Let $N^\bullet$ be a complex of R-modules. The constructions above produce a right differential graded A-module $\operatorname{Hom}(M, N^\bullet)$ and then a leftt differential graded A-module $\operatorname{Hom}(\operatorname{Hom}(M, N^\bullet), N^\bullet)$. We claim there is an evaluation map

$$
ev : M \longrightarrow \operatorname{Hom}(\operatorname{Hom}(M, N^\bullet), N^\bullet)
$$

in the category of left differential graded A-modules. To define it, by Lemma 13.2 it suffices to construct an A-bilinear pairing

$$
\operatorname{Hom}(M, N^\bullet) \times M \longrightarrow N^\bullet
$$

compatible with grading and differentials. For this we take

$$
(f, x) \longmapsto f(x)
$$

We leave it to the reader to verify this is compatible with grading, differentials, and A-bilinear. The map ev on underlying complexes of R-modules is More on Algebra, Item (17).

Remark 13.6. Let R be a ring. Let A be a differential graded R-algebra. Let M be a right differential graded A-module. Let $N^\bullet$ be a complex of R-modules. The constructions above produce a left differential graded A-module $\operatorname{Hom}(M, N^\bullet)$ and then a right differential graded A-module $\operatorname{Hom}(\operatorname{Hom}(M, N^\bullet), N^\bullet)$. We claim there is an evaluation map

$$
ev : M \longrightarrow \operatorname{Hom}(\operatorname{Hom}(M, N^\bullet), N^\bullet)
$$

in the category of right differential graded A-modules. To define it, by Lemma 13.2 it suffices to construct an A-bilinear pairing

$$
M \times \operatorname{Hom}(M, N^\bullet) \longrightarrow N^\bullet
$$

compatible with grading and differentials. For this we take

$$
(x, f) \longmapsto (-1)^{\deg(x)\deg(f)} f(x)
$$

We leave it to the reader to verify this is compatible with grading, differentials, and A-bilinear. The map ev on underlying complexes of R-modules is More on Algebra, Item (17).

Remark 13.7. Let R be a ring. Let A be a differential graded R-algebra. Let $M^\bullet$ and $N^\bullet$ be complexes of R-modules. Let $k \in \mathbf{Z}$ and consider the isomorphism

$$\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)[-k] \longrightarrow \mathrm{Hom}^\bullet(M^\bullet[k], N^\bullet)$$

of complexes of R-modules defined in More on Algebra, Item (18). If $M^\bullet$ has the structure of a left, resp. right differential graded A-module, then this is a map of right, resp. left differential graded A-modules (with the module structures as defined in this section). We omit the verification; we warn the reader that the A-module structure on the shift of a left graded A-module is defined using a sign, see Definition 11.3.

14. Projective modules over algebras

In this section we discuss projective modules over algebras analogous to Algebra, Section 77. This section should probably be moved somewhere else.

Let R be a ring and let A be an R-algebra, see Section 2 for our conventions. It is clear that A is a projective right A-module since $\mathrm{Hom}_A(A, M) = M$ for any right A-module M (and thus $\mathrm{Hom}_A(A, -)$ is exact). Conversely, let P be a projective right A-module. Then we can choose a surjection $\bigoplus_{i \in I} A \to P$ by choosing a set $\{p_i\}_{i \in I}$ of generators of P over A. Since P is projective there is a left inverse to the surjection, and we find that P is isomorphic to a direct summand of a free module, exactly as in the commutative case (Algebra, Lemma 77.2).

We conclude

(1) the category of A-modules has enough projectives,
(2) A is a projective A-module,
(3) every A-module is a quotient of a direct sum of copies of A,
(4) every projective A-module is a direct summand of a direct sum of copies of A.

15. Projective modules over graded algebras

In this section we discuss projective graded modules over graded algebras analogous to Algebra, Section 77.

Let R be a ring. Let A be a $\mathbf{Z}$-graded algebra over R. Section 2 for our conventions. Let Mod_A denote the category of graded right A-modules. For an integer k let $A[k]$ denote the shift of A. For a graded right A-module we have

$$\mathrm{Hom}_{\mathrm{Mod}_A}(A[k], M) = M^{-k}$$

As the functor $M \mapsto M^{-k}$ is exact on Mod_A we conclude that $A[k]$ is a projective object of Mod_A. Conversely, suppose that P is a projective object of Mod_A. By choosing a set of homogeneous generators of P as an A-module, we can find a surjection

$$\bigoplus_{i \in I} A[k_i] \longrightarrow P$$

Thus we conclude that a projective object of Mod_A is a direct summand of a direct sum of the shifts $A[k]$.

We conclude

(1) the category of graded A-modules has enough projectives,
(2) $A[k]$ is a projective A-module for every $k \in \mathbf{Z}$,

(3) every graded A-module is a quotient of a direct sum of copies of the modules $A[k]$ for varying k,
(4) every projective A-module is a direct summand of a direct sum of copies of the modules $A[k]$ for varying k.

16. Projective modules and differential graded algebras

If (A, d) is a differential graded algebra and P is an object of $\mathrm{Mod}_{(A,\mathrm{d})}$ then we say P *is projective as a graded A-module* or sometimes P *is graded projective* to mean that P is a projective object of the abelian category Mod_A of graded A-modules as in Section 15.

Lemma 16.1. *Let (A, d) be a differential graded algebra. Let $M \to P$ be a surjective homomorphism of differential graded A-modules. If P is projective as a graded A-module, then $M \to P$ is an admissible epimorphism.*

Proof. This is immediate from the definitions. $\square$

Lemma 16.2. *Let (A, d) be a differential graded algebra. Then we have*
$$\mathrm{Hom}_{Mod_{(A,d)}}(A[k], M) = \mathrm{Ker}(d : M^{-k} \to M^{-k+1})$$
and
$$\mathrm{Hom}_{K(Mod_{(A,d)})}(A[k], M) = H^{-k}(M)$$
for any differential graded A-module M.

Proof. Immediate from the definitions. $\square$

17. Injective modules over algebras

In this section we discuss injective modules over algebras analogous to More on Algebra, Section 55. This section should probably be moved somewhere else.

Let R be a ring and let A be an R-algebra, see Section 2 for our conventions. For a right A-module M we set
$$M^\vee = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$$
which we think of as a left A-module by the multiplication $(af)(x) = f(xa)$. Namely, $((ab)f)(x) = f(xab) = (bf)(xa) = (a(bf))(x)$. Conversely, if M is a left A-module, then $M^\vee$ is a right A-module. Since $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group (More on Algebra, Lemma 54.1), the functor $M \mapsto M^\vee$ is exact (More on Algebra, Lemma 55.6). Moreover, the evaluation map $M \to (M^\vee)^\vee$ is injective for all modules M (More on Algebra, Lemma 55.7).

We claim that $A^\vee$ is an injective right A-module. Namely, given a right A-module N we have
$$\mathrm{Hom}_A(N, A^\vee) = \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})) = N^\vee$$
and we conclude because the functor $N \mapsto N^\vee$ is exact. The second equality holds because
$$\mathrm{Hom}_{\mathbf{Z}}(N, \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})) = \mathrm{Hom}_{\mathbf{Z}}(N \otimes_{\mathbf{Z}} A, \mathbf{Q}/\mathbf{Z})$$
by Algebra, Lemma 12.8. Inside this module A-linearity exactly picks out the bilinear maps $\varphi : N \times A \to \mathbf{Q}/\mathbf{Z}$ which have the same value on $x \otimes a$ and $xa \otimes 1$, i.e., come from elements of $N^\vee$.

Finally, for every right A-module M we can choose a surjection $\bigoplus_{i \in I} A \to M^\vee$ to get an injection $M \to (M^\vee)^\vee \to \prod_{i \in I} A^\vee$.

We conclude

(1) the category of A-modules has enough injectives,
(2) $A^\vee$ is an injective A-module, and
(3) every A-module injects into a product of copies of $A^\vee$.

18. Injective modules over graded algebras

In this section we discuss injective graded modules over graded algebras analogous to More on Algebra, Section 55.

Let R be a ring. Let A be a $\mathbf{Z}$-graded algebra over R. Section 2 for our conventions. If M is a graded R-module we set

$$M^\vee = \bigoplus_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(M^{-n}, \mathbf{Q}/\mathbf{Z}) = \bigoplus_{n \in \mathbf{Z}} (M^{-n})^\vee$$

as a graded R-module (no signs in the actions of R on the homogeneous parts). If M has the structure of a left graded A-module, then we define a right graded A-module structure on $M^\vee$ by letting $a \in A^m$ act by

$$(M^{-n})^\vee \to (M^{-n-m})^\vee, \quad f \mapsto f \circ a$$

as in Section 13. If M has the structure of a right graded A-module, then we define a left graded A-module structure on $M^\vee$ by letting $a \in A^n$ act by

$$(M^{-m})^\vee \to (M^{-m-n})^\vee, \quad f \mapsto (-1)^{nm} f \circ a$$

as in Section 13 (the sign is forced on us because we want to use the same formula for the case when working with differential graded modules — if you only care about graded modules, then you can omit the sign here). On the category of (left or right) graded A-modules the functor $M \mapsto M^\vee$ is exact (check on graded pieces). Moreover, there is an injective evaluation map

$$ev : M \longrightarrow (M^\vee)^\vee, \quad ev^n = (-1)^n \text{ the evaluation map } M^n \to ((M^n)^\vee)^\vee$$

of graded R-modules, see More on Algebra, Item (17). This evaluation map is a left, resp. right A-module homomorphism if M is a left, resp. right A-module, see Remarks 13.5 and 13.6. Finally, given $k \in \mathbf{Z}$ there is a canonical isomorphism

$$M^\vee[-k] \longrightarrow (M[k])^\vee$$

of graded R-modules which uses a sign and which, if M is a left, resp. right A-module, is an isomorphism of right, resp. left A-modules. See Remark 13.7.

We claim that $A^\vee$ is an injective object of the category Mod_A of graded right A-modules. Namely, given a graded right A-module N we have

$$\mathrm{Hom}_{\mathrm{Mod}_A}(N, A^\vee) = \mathrm{Hom}_{\mathrm{Comp}(\mathbf{Z})}(N \otimes_A A, \mathbf{Q}/\mathbf{Z}) = (N^0)^\vee$$

by Lemma 13.2 (applied to the case where all the differentials are zero). We conclude because the functor $N \mapsto (N^0)^\vee = (N^\vee)^0$ is exact.

Finally, for every graded right A-module M we can choose a surjection of graded left A-modules

$$\bigoplus_{i \in I} A[k_i] \to M^\vee$$

where $A[k_i]$ denotes the shift of A by $k_i \in \mathbf{Z}$. We do this by choosing homogeneous generators for $M^\vee$. In this way we get an injection

$$M \to (M^\vee)^\vee \to \prod A[k_i]^\vee = \prod A^\vee[-k_i]$$

Observe that the products in the formula above are products in the category of graded modules (in other words, take products in each degree and then take the direct sum of the pieces).

We conclude that

(1) the category of graded A-modules has enough injectives,
(2) for every $k \in \mathbf{Z}$ the module $A^\vee[k]$ is injective, and
(3) every A-module injects into a product in the category of graded modules of copies of shifts $A^\vee[k]$.

19. Injective modules and differential graded algebras

If (A, d) is a differential graded algebra and I is an object of $\mathrm{Mod}_{(A,\mathrm{d})}$ then we say I *is injective as a graded A-module* or sometimes I *is graded injective* to mean that I is a injective object of the abelian category Mod_A of graded A-modules.

Lemma 19.1. *Let (A, d) be a differential graded algebra. Let $I \to M$ be an injective homomorphism of differential graded A-modules. If I is graded injective, then $I \to M$ is an admissible monomorphism.*

Proof. This is immediate from the definitions. $\square$

Let (A, d) be a differential graded algebra. If M is a left, resp. right differential graded A-module, then

$$M^\vee = \mathrm{Hom}^\bullet(M^\bullet, \mathbf{Q}/\mathbf{Z})$$

with A-module structure constructed in Section 18 is a right, resp. left differential graded A-module by the discussion in Section 13. By Remarks 13.5 and 13.6 there evaluation map of Section 18

$$M \longrightarrow (M^\vee)^\vee$$

is a homomorphism of left, resp. right differential graded A-modules

Lemma 19.2. *Let (A, d) be a differential graded algebra. If M is a left differential graded A-module and N is a right differential graded A-module, then*

$$\mathrm{Hom}_{Mod_{(A,d)}}(N, M^\vee) = \mathrm{Hom}_{Comp(\mathbf{Z})}(N \otimes_A M, \mathbf{Q}/\mathbf{Z})$$
$$= DifferentialGradedBilinear_A(N \times M, \mathbf{Q}/\mathbf{Z})$$

Proof. The first equality is Lemma 13.2 and the second equality was shown in Section 12. $\square$

Lemma 19.3. *Let (A, d) be a differential graded algebra. Then we have*

$$\mathrm{Hom}_{Mod_{(A,d)}}(M, A^\vee[k]) = \mathrm{Ker}(d : (M^\vee)^k \to (M^\vee)^{k+1})$$

and

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(M, A^\vee[k]) = H^k(M^\vee)$$

as functors in the differential graded A-module M.

Proof. This is clear from the discussion above. $\square$

20. P-resolutions

This section is the analogue of Derived Categories, Section 29.

Let (A, d) be a differential graded algebra. Let P be a differential graded A-module. We say P *has property (P)* if it there exists a filtration

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset P$$

by differential graded submodules such that

(1) $P = \bigcup F_pP$,
(2) the inclusions $F_iP \to F_{i+1}P$ are admissible monomorphisms,
(3) the quotients $F_{i+1}P/F_iP$ are isomorphic as differential graded A-modules to a direct sum of $A[k]$.

In fact, condition (2) is a consequence of condition (3), see Lemma 16.1. Moreover, the reader can verify that as a graded A-module P will be isomorphic to a direct sum of shifts of A.

Lemma 20.1. *Let (A, d) be a differential graded algebra. Let P be a differential graded A-module. If $F_\bullet$ is a filtration as in property (P), then we obtain an admissible short exact sequence*

$$0 \to \bigoplus F_iP \to \bigoplus F_iP \to P \to 0$$

of differential graded A-modules.

Proof. The second map is the direct sum of the inclusion maps. The first map on the summand F_iP of the source is the sum of the identity $F_iP \to F_iP$ and the negative of the inclusion map $F_iP \to F_{i+1}P$. Choose homomorphisms $s_i : F_{i+1}P \to F_iP$ of graded A-modules which are left inverse to the inclusion maps. Composing gives maps $s_{j,i} : F_jP \to F_iP$ for all $j > i$. Then a left inverse of the first arrow maps $x \in F_jP$ to $(s_{j,0}(x), s_{j,1}(x), \ldots, s_{j,j-1}(x), 0, \ldots)$ in $\bigoplus F_iP$. $\qquad\square$

The following lemma shows that differential graded modules with property (P) are the dual notion to K-injective modules (i.e., they are K-projective in some sense). See Derived Categories, Definition 31.1.

Lemma 20.2. *Let (A, d) be a differential graded algebra. Let P be a differential graded A-module with property (P). Then*

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(P, N) = 0$$

for all acyclic differential graded A-modules N.

Proof. We will use that $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is a triangulated category (Proposition 10.3). Let $F_\bullet$ be a filtration on P as in property (P). The short exact sequence of Lemma 20.1 produces a distinguished triangle. Hence by Derived Categories, Lemma 4.2 it suffices to show that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(F_iP, N) = 0$$

for all acyclic differential graded A-modules N and all i. Each of the differential graded modules F_iP has a finite filtration by admissible monomorphisms, whose graded pieces are direct sums of shifts $A[k]$. Thus it suffices to prove that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(A[k], N) = 0$$

for all acyclic differential graded A-modules N and all k. This follows from Lemma 16.2. $\qquad\square$

Lemma 20.3. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $P \to M$ of differential graded A-modules with the following properties*

(1) *$P \to M$ is surjective,*
(2) *$\mathrm{Ker}(d_P) \to \mathrm{Ker}(d_M)$ is surjective, and*
(3) *P sits in an admissible short exact sequence $0 \to P' \to P \to P'' \to 0$ where P', P'' are direct sums of shifts of A.*

Proof. Let P_k be the free A-module with generators x, y in degrees k and $k+1$. Define the structure of a differential graded A-module on P_k by setting $d(x) = y$ and $d(y) = 0$. For every element $m \in M^k$ there is a homomorphism $P_k \to M$ sending x to m and y to $d(m)$. Thus we see that there is a surjection from a direct sum of copies of P_k to M. This clearly produces $P \to M$ having properties (1) and (3). To obtain property (2) note that if $m \in \mathrm{Ker}(d_M)$ has degree k, then there is a map $A[k] \to M$ mapping 1 to m. Hence we can achieve (2) by adding a direct sum of copies of shifts of A. $\square$

Lemma 20.4. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $P \to M$ of differential graded A-modules such that*

(1) *$P \to M$ is a quasi-isomorphism, and*
(2) *P has property (P).*

Proof. Set $M = M_0$. We inductively choose short exact sequences

$$0 \to M_{i+1} \to P_i \to M_i \to 0$$

where the maps $P_i \to M_i$ are chosen as in Lemma 20.3. This gives a "resolution"

$$\ldots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to M \to 0$$

Then we set

$$P = \bigoplus_{i \geq 0} P_i$$

as an A-module with grading given by $P^n = \bigoplus_{a+b=n} P^b_{-a}$ and differential (as in the construction of the total complex associated to a double complex) by

$$d_P(x) = f_{-a}(x) + (-1)^a d_{P_{-a}}(x)$$

for $x \in P^b_{-a}$. With these conventions P is indeed a differential graded A-module. Recalling that each P_i has a two step filtration $0 \to P'_i \to P_i \to P''_i \to 0$ we set

$$F_{2i} P = \bigoplus_{i \geq j \geq 0} P_j \subset \bigoplus_{i \geq 0} P_i = P$$

and we add P'_{i+1} to $F_{2i} P$ to get F_{2i+1}. These are differential graded submodules and the successive quotients are direct sums of shifts of A. By Lemma 16.1 we see that the inclusions $F_i P \to F_{i+1} P$ are admissible monomorphisms. Finally, we have to show that the map $P \to M$ (given by the augmentation $P_0 \to M$) is a quasi-isomorphism. This follows from Homology, Lemma 26.2. $\square$

21. I-resolutions

This section is the dual of the section on P-resolutions.

Let (A, d) be a differential graded algebra. Let I be a differential graded A-module. We say *I has property (I)* if it there exists a filtration

$$I = F_0 I \supset F_1 I \supset F_2 I \supset \ldots \supset 0$$

by differential graded submodules such that

  (1) $I = \lim I/F_p I$,
  (2) the maps $I/F_{i+1}I \to I/F_i I$ are admissible epimorphisms,
  (3) the quotients $F_i I/F_{i+1}I$ are isomorphic as differential graded A-modules to products of the modules $A^\vee[k]$ constructed in Section 19.

In fact, condition (2) is a consequence of condition (3), see Lemma 19.1. The reader can verify that as a graded module I will be isomorphic to a product of $A^\vee[k]$.

Lemma 21.1. *Let (A, d) be a differential graded algebra. Let I be a differential graded A-module. If $F_\bullet$ is a filtration as in property (I), then we obtain an admissible short exact sequence*

$$0 \to I \to \prod I/F_i I \to \prod I/F_i I \to 0$$

of differential graded A-modules.

Proof. Omitted. Hint: This is dual to Lemma 20.1. $\square$

The following lemma shows that differential graded modules with property (I) are the analogue of K-injective modules. See Derived Categories, Definition 31.1.

Lemma 21.2. *Let (A, d) be a differential graded algebra. Let I be a differential graded A-module with property (I). Then*

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(N, I) = 0$$

for all acyclic differential graded A-modules N.

Proof. We will use that $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is a triangulated category (Proposition 10.3). Let $F_\bullet$ be a filtration on I as in property (I). The short exact sequence of Lemma 21.1 produces a distinguished triangle. Hence by Derived Categories, Lemma 4.2 it suffices to show that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(N, I/F_i I) = 0$$

for all acyclic differential graded A-modules N and all i. Each of the differential graded modules $I/F_i I$ has a finite filtration by admissible monomorphisms, whose graded pieces are products of $A^\vee[k]$. Thus it suffices to prove that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(N, A^\vee[k]) = 0$$

for all acyclic differential graded A-modules N and all k. This follows from Lemma 19.3 and the fact that $(-)^\vee$ is an exact functor. $\square$

Lemma 21.3. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $M \to I$ of differential graded A-modules with the following properties*

  (1) $M \to I$ *is injective,*
  (2) $\mathrm{Coker}(d_M) \to \mathrm{Coker}(d_I)$ *is injective, and*

(3) I sits in an admissible short exact sequence $0 \to I' \to I \to I'' \to 0$ where I', I'' are products of shifts of $A^\vee$.

Proof. We will use the functors $N \mapsto N^\vee$ (from left to right differential graded modules and from right to left differential graded modules) constructed in Section 19 and all of their properties. For every $k \in \mathbf{Z}$ let Q_k be the free left A-module with generators x, y in degrees k and $k+1$. Define the structure of a left differential graded A-module on Q_k by setting $\mathrm{d}(x) = y$ and $\mathrm{d}(y) = 0$. Arguing exactly as in the proof of Lemma 20.3 we find a surjection

$$\bigoplus_{i \in I} Q_{k_i} \longrightarrow M^\vee$$

of left differential graded A-modules. Then we can consider the injection

$$M \to (M^\vee)^\vee \to \left(\bigoplus_{i \in I} Q_{k_i}\right)^\vee = \prod_{i \in I} I_{k_i}$$

where $I_k = Q^\vee_{-k}$ is the "dual" right differential graded A-module. Further, the short exact sequence $0 \to A[-k-1] \to Q_k \to A[-k] \to 0$ produces a short exact sequence $0 \to A^\vee[k] \to I_k \to A^\vee[k+1] \to 0$.

The result of the previous paragraph produces $M \to I$ having properties (1) and (3). To obtain property (2), suppose $\overline{m} \in \mathrm{Coker}(\mathrm{d}_M)$ is a nonzero element of degree k. Pick a map $\lambda : M^k \to \mathbf{Q}/\mathbf{Z}$ which vanishes on $\mathrm{Im}(M^{k-1} \to M^k)$ but not on m. By Lemma 19.3 this corresponds to a homomorphism $M \to A^\vee[k]$ of differential graded A-modules which does not vanish on m. Hence we can achieve (2) by adding a product of copies of shifts of $A^\vee$. $\qquad\square$

Lemma 21.4. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $M \to I$ of differential graded A-modules such that*

(1) *$M \to I$ is a quasi-isomorphism, and*
(2) *I has property (I).*

Proof. Set $M = M_0$. We inductively choose short exact sequences

$$0 \to M_i \to I_i \to M_{i+1} \to 0$$

where the maps $M_i \to I_i$ are chosen as in Lemma 21.3. This gives a "resolution"

$$0 \to M \to I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} I_1 \to \dots$$

Denote I the differential graded A-module with graded parts

$$I^n = \prod_{i \geq 0} I_i^{n-i}$$

and differential defined by

$$\mathrm{d}_I(x) = f_i(x) + (-1)^i \mathrm{d}_{I_i}(x)$$

for $x \in I_i^{n-i}$. With these conventions I is indeed a differential graded A-module. Recalling that each I_i has a two step filtration $0 \to I_i' \to I_i \to I_i'' \to 0$ we set

$$F_{2i} I^n = \prod_{j \geq i} I_j^{n-j} \subset \prod_{i \geq 0} I_i^{n-i} = I^n$$

and we add a factor I_{i+1}' to $F_{2i}I$ to get $F_{2i+1}I$. These are differential graded submodules and the successive quotients are products of shifts of $A^\vee$. By Lemma 19.1 we see that the inclusions $F_{i+1}I \to F_iI$ are admissible monomorphisms. Finally,

we have to show that the map $M \to I$ (given by the augmentation $M \to I_0$) is a quasi-isomorphism. This follows from Homology, Lemma 26.3.   □

22. The derived category

Recall that the notions of acyclic differential graded modules and quasi-isomorphism of differential graded modules make sense (see Section 4).

Lemma 22.1. *Let (A, d) be a differential graded algebra. The full subcategory Ac of $K(Mod_{(A,d)})$ consisting of acyclic modules is a strictly full saturated triangulated subcategory of $K(Mod_{(A,d)})$. The corresponding saturated multiplicative system (see Derived Categories, Lemma 6.10) of $K(Mod_{(A,d)})$ is the class Qis of quasi-isomorphisms. In particular, the kernel of the localization functor*

$$Q : K(Mod_{(A,d)}) \to Qis^{-1}K(Mod_{(A,d)})$$

is Ac. Moreover, the functor H^0 factors through Q.

Proof. We know that H^0 is a homological functor by the long exact sequence of homology (4.2.1). The kernel of H^0 is the subcategory of acyclic objects and the arrows with induce isomorphisms on all H^i are the quasi-isomorphisms. Thus this lemma is a special case of Derived Categories, Lemma 6.11.

Set theoretical remark. The construction of the localization in Derived Categories, Proposition 5.6 assumes the given triangulated category is "small", i.e., that the underlying collection of objects forms a set. Let V_α be a partial universe (as in Sets, Section 5) containing (A, d) and where the cofinality of α is bigger than $\aleph_0$ (see Sets, Proposition 7.2). Then we can consider the category $Mod_{(A,d),\alpha}$ of differential graded A-modules contained in V_α. A straightforward check shows that all the constructions used in the proof of Proposition 10.3 work inside of $Mod_{(A,d),\alpha}$ (because at worst we take finite direct sums of differential graded modules). Thus we obtain a triangulated category $Qis_\alpha^{-1}K(Mod_{(A,d),\alpha})$. We will see below that if $\beta > \alpha$, then the transition functors

$$Qis_\alpha^{-1}K(Mod_{(A,d),\alpha}) \longrightarrow Qis_\beta^{-1}K(Mod_{(A,d),\beta})$$

are fully faithful as the morphism sets in the quotient categories are computed by maps in the homotopy categories from P-resolutions (the construction of a P-resolution in the proof of Lemma 20.4 takes countable direct sums as well as direct sums indexed over subsets of the given module). The reader should therefore think of the category of the lemma as the union of these subcategories.   □

Taking into account the set theoretical remark at the end of the proof of the preceding lemma we define the derived category as follows.

Definition 22.2. Let (A, d) be a differential graded algebra. Let Ac and Qis be as in Lemma 22.1. The *derived category of* (A, d) is the triangulated category

$$D(A, d) = K(Mod_{(A,d)})/Ac = Qis^{-1}K(Mod_{(A,d)}).$$

We denote $H^0 : D(A, d) \to Mod_R$ the unique functor whose composition with the quotient functor gives back the functor H^0 defined above.

Here is the promised lemma computing morphism sets in the derived category.

Lemma 22.3. *Let (A, d) be a differential graded algebra. Let M and N be differential graded A-modules.*

(1) *Let $P \to M$ be a P-resolution as in Lemma 20.4. Then*
$$\mathrm{Hom}_{D(A,d)}(M,N) = \mathrm{Hom}_{K(Mod_{(A,d)})}(P,N)$$

(2) *Let $N \to I$ be an I-resolution as in Lemma 21.4. Then*
$$\mathrm{Hom}_{D(A,d)}(M,N) = \mathrm{Hom}_{K(Mod_{(A,d)})}(M,I)$$

Proof. Let $P \to M$ be as in (1). Since $P \to M$ is a quasi-isomorphism we see that
$$\mathrm{Hom}_{D(A,\mathrm{d})}(P,N) = \mathrm{Hom}_{D(A,\mathrm{d})}(M,N)$$
by definition of the derived category. A morphism $f : P \to N$ in $D(A,\mathrm{d})$ is equal to $s^{-1}f'$ where $f' : P \to N'$ is a morphism and $s : N \to N'$ is a quasi-isomorphism. Choose a distinguished triangle
$$N \to N' \to Q \to N[1]$$
As s is a quasi-isomorphism, we see that Q is acyclic. Thus $\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(P,Q[k]) = 0$ for all k by Lemma 20.2. Since $\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(P,-)$ is cohomological, we conclude that we can lift $f' : P \to N'$ uniquely to a morphism $f : P \to N$. This finishes the proof.

The proof of (2) is dual to that of (1) using Lemma 21.2 in stead of Lemma 20.2. $\square$

Lemma 22.4. *Let (A, d) be a differential graded algebra. Then*

(1) *$D(A, d)$ has both direct sums and products,*
(2) *direct sums are obtained by taking direct sums of differential graded modules,*
(3) *products are obtained by taking products of differential graded modules.*

Proof. We will use that $\mathrm{Mod}_{(A,\mathrm{d})}$ is an abelian category with arbitrary direct sums and products, and that these give rise to direct sums and products in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. See Lemmas 4.2 and 5.4.

Let M_j be a family of differential graded A-modules. Consider the graded direct sum $M = \bigoplus M_j$ which is a differential graded A-module with the obvious. For a differential graded A-module N choose a quasi-isomorphism $N \to I$ where I is a differential graded A-module with property (I). See Lemma 21.4. Using Lemma 22.3 we have
$$\mathrm{Hom}_{D(A,\mathrm{d})}(M,N) = \mathrm{Hom}_{K(A,\mathrm{d})}(M,I)$$
$$= \prod \mathrm{Hom}_{K(A,\mathrm{d})}(M_j,I)$$
$$= \prod \mathrm{Hom}_{D(A,\mathrm{d})}(M_j,N)$$
whence the existence of direct sums in $D(A,\mathrm{d})$ as given in part (2) of the lemma.

Let M_j be a family of differential graded A-modules. Consider the product $M = \prod M_j$ of differential graded A-modules. For a differential graded A-module N choose a quasi-isomorphism $P \to N$ where P is a differential graded A-module with property (P). See Lemma 20.4. Using Lemma 22.3 we have
$$\mathrm{Hom}_{D(A,\mathrm{d})}(N,M) = \mathrm{Hom}_{K(A,\mathrm{d})}(P,M)$$
$$= \prod \mathrm{Hom}_{K(A,\mathrm{d})}(P,M_j)$$
$$= \prod \mathrm{Hom}_{D(A,\mathrm{d})}(N,M_j)$$
whence the existence of direct sums in $D(A,\mathrm{d})$ as given in part (3) of the lemma. $\square$

Remark 22.5. Let R be a ring. Let (A, d) be a differential graded R-algebra. Using P-resolutions we can sometimes reduce statements about general objects of $D(A, \mathrm{d})$ to statements about $A[k]$. Namely, let T be a property of objects of $D(A, \mathrm{d})$ and assume that

(1) if K_i, $i \in I$ is a family of objects of $D(A, \mathrm{d})$ and $T(K_i)$ holds for all $i \in I$, then $T(\bigoplus K_i)$,

(2) if $K \to L \to M \to K[1]$ is a distinguished triangle of $D(A, \mathrm{d})$ and T holds for two, then T holds for the third object, and

(3) $T(A[k])$ holds for all $k \in \mathbf{Z}$.

Then T holds for all objects of $D(A, \mathrm{d})$. This is clear from Lemmas 20.1 and 20.4.

23. The canonical delta-functor

Let (A, d) be a differential graded algebra. Consider the functor $\mathrm{Mod}_{(A,\mathrm{d})} \to K(\mathrm{Mod}_{(A,\mathrm{d})})$. This functor is **not** a δ-functor in general. However, it turns out that the functor $\mathrm{Mod}_{(A,\mathrm{d})} \to D(A, \mathrm{d})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to K \xrightarrow{a} L \xrightarrow{b} M \to 0$$

in the abelian category $\mathrm{Mod}_{(A,\mathrm{d})}$. Consider the cone $C(a)$ of the morphism a. We have $C(a) = L \oplus K$ and we define $q : C(a) \to M$ via the projection to L followed by b. Hence a homomorphism of differential graded A-modules

$$q : C(a) \longrightarrow M.$$

It is clear that $q \circ i = b$ where i is as in Definition 6.1. Note that, as a is injective, the kernel of q is identified with the cone of id_K which is acyclic. Hence we see that q is a quasi-isomorphism. According to Lemma 9.4 the triangle

$$(K, L, C(a), a, i, -p)$$

is a distinguished triangle in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. As the localization functor $K(\mathrm{Mod}_{(A,\mathrm{d})}) \to D(A, \mathrm{d})$ is exact we see that $(K, L, C(a), a, i, -p)$ is a distinguished triangle in $D(A, \mathrm{d})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(A, \mathrm{d})$. Hence we deduce that

$$(K, L, M, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(A, \mathrm{d})$. This suggests the following lemma.

Lemma 23.1. *Let (A, d) be a differential graded algebra. The functor $Mod_{(A,d)} \to D(A, d)$ defined has the natural structure of a δ-functor, with*

$$\delta_{K \to L \to M} = -p \circ q^{-1}$$

with p and q as explained above.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show functoriality of this construction, see Derived Categories, Definition 3.6. This follows from Lemma 6.2 with a bit of work. Compare with Derived Categories, Lemma 12.1. $\square$

Lemma 23.2. *Let (A, d) be a differential graded algebra. Let M_n be a system of differential graded modules. Then the derived colimit $\mathrm{hocolim} M_n$ in $D(A, d)$ is represented by the differential graded module $\mathrm{colim}\, M_n$.*

Proof. Set $M = \operatorname{colim} M_n$. We have an exact sequence of differential graded modules

$$0 \to \bigoplus M_n \to \bigoplus M_n \to M \to 0$$

by Derived Categories, Lemma 33.6 (applied the underlying complexes of abelian groups). The direct sums are direct sums in $D(\mathcal{A})$ by Lemma 22.4. Thus the result follows from the definition of derived colimits in Derived Categories, Definition 33.1 and the fact that a short exact sequence of complexes gives a distinguished triangle (Lemma 23.1). $\square$

24. Linear categories

Just the definitions.

Definition 24.1. Let R be a ring. An R-*linear category* $\mathcal{A}$ is a category where every morphism set is given the structure of an R-module and where for $x, y, z \in \operatorname{Ob}(\mathcal{A})$ composition law

$$\operatorname{Hom}_{\mathcal{A}}(y, z) \times \operatorname{Hom}_{\mathcal{A}}(x, y) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(x, z)$$

is R-bilinear.

Thus composition determines an R-linear map

$$\operatorname{Hom}_{\mathcal{A}}(y, z) \otimes_R \operatorname{Hom}_{\mathcal{A}}(x, y) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(x, z)$$

of R-modules. Note that we do not assume R-linear categories to be additive.

Definition 24.2. Let R be a ring. A *functor of R-linear categories*, or an R-*linear functor* is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \operatorname{Hom}_{\mathcal{A}}(x, y) \to \operatorname{Hom}_{\mathcal{B}}(F(x), F(y))$ is a homomorphism of R-modules.

25. Graded categories

Just some definitions.

Definition 25.1. Let R be a ring. A *graded category* $\mathcal{A}$ *over R* is a category where every morphism set is given the structure of a graded R-module and where for $x, y, z \in \operatorname{Ob}(\mathcal{A})$ composition is R-bilinear and induces a homomorphism

$$\operatorname{Hom}_{\mathcal{A}}(y, z) \otimes_R \operatorname{Hom}_{\mathcal{A}}(x, y) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(x, z)$$

of graded R-modules (i.e., preserving degrees).

In this situation we denote $\operatorname{Hom}^i_{\mathcal{A}}(x, y)$ the degree i part of the graded object $\operatorname{Hom}_{\mathcal{A}}(x, y)$, so that

$$\operatorname{Hom}_{\mathcal{A}}(x, y) = \bigoplus_{i \in \mathbf{Z}} \operatorname{Hom}^i_{\mathcal{A}}(x, y)$$

is the direct sum decomposition into graded parts.

Definition 25.2. Let R be a ring. A *functor of graded categories over R*, or a *graded functor* is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \operatorname{Hom}_{\mathcal{A}}(x, y) \to \operatorname{Hom}_{\mathcal{A}}(F(x), F(y))$ is a homomorphism of graded R-modules.

Given a graded category we are often interested in the corresponding "usual" category of maps of degree 0. Here is a formal definition.

Definition 25.3. Let R be a ring. Let $\mathcal{A}$ be a graded category over R. We let $\mathcal{A}^0$ be the category with the same objects as $\mathcal{A}$ and with

$$\operatorname{Hom}_{\mathcal{A}^0}(x,y) = \operatorname{Hom}_{\mathcal{A}}^0(x,y)$$

the degree 0 graded piece of the graded module of morphisms of $\mathcal{A}$.

Definition 25.4. Let R be a ring. Let $\mathcal{A}$ be a graded category over R. A direct sum (x,y,z,i,j,p,q) in $\mathcal{A}$ (notation as in Homology, Remark 3.6) is a *graded direct sum* if i,j,p,q are homogeneous of degree 0.

Example 25.5 (Graded category of graded objects)**.** Let $\mathcal{B}$ be an additive category. Recall that we have defined the category $\operatorname{Gr}(\mathcal{B})$ of graded objects of $\mathcal{B}$ in Homology, Definition 16.1. In this example, we will construct a graded category $\operatorname{Gr}^{gr}(\mathcal{B})$ over $R = \mathbf{Z}$ whose associated category $\operatorname{Gr}^{gr}(\mathcal{B})^0$ recovers $\operatorname{Gr}(\mathcal{B})$. As objects of $\operatorname{Gr}^{gr}(\mathcal{B})$ we take graded objects of $\mathcal{B}$. Then, given graded objects $A = (A^i)$ and $B = (B^i)$ of $\mathcal{B}$ we set

$$\operatorname{Hom}_{\operatorname{Gr}^{gr}(\mathcal{B})}(A,B) = \bigoplus\nolimits_{n \in \mathbf{Z}} \operatorname{Hom}^n(A,B)$$

where the graded piece of degree n is the abelian group of homogeneous maps of degree n from A to B. Explicitly we have

$$\operatorname{Hom}^n(A,B) = \prod\nolimits_{p+q=n} \operatorname{Hom}_{\mathcal{B}}(A^{-q}, B^p)$$

(observe reversal of indices and observe that we have a product here and not a direct sum). In other words, a degree n morphism f from A to B can be seen as a system $f = (f_{p,q})$ where $p,q \in \mathbf{Z}$, $p+q=n$ with $f_{p,q} : A^{-q} \to B^p$ a morphism of $\mathcal{B}$. Given graded objects A, B, C of $\mathcal{B}$ composition of morphisms in $\operatorname{Gr}^{gr}(\mathcal{B})$ is defined via the maps

$$\operatorname{Hom}^m(B,C) \times \operatorname{Hom}^n(A,B) \longrightarrow \operatorname{Hom}^{n+m}(A,C)$$

by simple composition $(g,f) \mapsto g \circ f$ of homogeneous maps of graded objects. In terms of components we have

$$(g \circ f)_{p,r} = g_{p,q} \circ f_{-q,r}$$

where q is such that $p+q=m$ and $-q+r=n$.

Example 25.6 (Graded category of graded modules)**.** Let A be a $\mathbf{Z}$-graded algebra over a ring R. We will construct a graded category $\operatorname{Mod}_A^{gr}$ over R whose associated category $(\operatorname{Mod}_A^{gr})^0$ is the category of graded A-modules. As objects of $\operatorname{Mod}_A^{gr}$ we take right graded A-modules (see Section 14). Given graded A-modules L and M we set

$$\operatorname{Hom}_{\operatorname{Mod}_A^{gr}}(L,M) = \bigoplus\nolimits_{n \in \mathbf{Z}} \operatorname{Hom}^n(L,M)$$

where $\operatorname{Hom}^n(L,M)$ is the set of right A-module maps $L \to M$ which are homogeneous of degree n, i.e., $f(L^i) \subset M^{i+n}$ for all $i \in \mathbf{Z}$. In terms of components, we have that

$$\operatorname{Hom}^n(L,M) \subset \prod\nolimits_{p+q=n} \operatorname{Hom}_R(L^{-q}, M^p)$$

(observe reversal of indices) is the subset consisting of those $f = (f_{p,q})$ such that

$$f_{p,q}(ma) = f_{p-i,q+i}(m)a$$

for $a \in A^i$ and $m \in L^{-q-i}$. For graded A-modules K, L, M we define composition in $\operatorname{Mod}_A^{gr}$ via the maps

$$\operatorname{Hom}^m(L,M) \times \operatorname{Hom}^n(K,L) \longrightarrow \operatorname{Hom}^{n+m}(K,M)$$

by simple composition of right A-module maps: $(g, f) \mapsto g \circ f$.

Remark 25.7. Let R be a ring. Let $\mathcal{D}$ be an R-linear category endowed with a collection of R-linear functors $[n] : \mathcal{D} \to \mathcal{D}$, $x \mapsto x[n]$ indexed by $n \in \mathbf{Z}$ such that $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}_{\mathcal{D}}$ (equality as functors). This allows us to construct a graded category $\mathcal{D}^{gr}$ over R with the same objects of $\mathcal{D}$ setting

$$\mathrm{Hom}_{\mathcal{D}^{gr}}(x, y) = \bigoplus_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{D}}(x, y[n])$$

for x, y in $\mathcal{D}$. Observe that $(\mathcal{D}^{gr})^0 = \mathcal{D}$ (see Definition 25.3). Moreover, the graded category $\mathcal{D}^{gr}$ inherits R-linear graded functors $[n]$ satisfying $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}_{\mathcal{D}^{gr}}$ with the property that

$$\mathrm{Hom}_{\mathcal{D}^{gr}}(x, y[n]) = \mathrm{Hom}_{\mathcal{D}^{gr}}(x, y)[n]$$

as graded R-modules compatible with composition of morphisms.

Conversely, suppose given a graded category $\mathcal{A}$ over R endowed with a collection of R-linear graded functors $[n]$ satisfying $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}_{\mathcal{A}}$ which are moreover equipped with isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(x, y[n]) = \mathrm{Hom}_{\mathcal{A}}(x, y)[n]$$

as graded R-modules compatible with composition of morphisms. Then the reader easily shows that $\mathcal{A} = (\mathcal{A}^0)^{gr}$.

Here are two examples of the relationship $\mathcal{D} \leftrightarrow \mathcal{A}$ we established above:

(1) Let $\mathcal{B}$ be an additive category. If $\mathcal{D} = \mathrm{Gr}(\mathcal{B})$, then $\mathcal{A} = \mathrm{Gr}^{gr}(\mathcal{B})$ as in Example 25.5.
(2) If A is a graded ring and $\mathcal{D} = \mathrm{Mod}_A$ is the category of graded right A-modules, then $\mathcal{A} = \mathrm{Mod}_A^{gr}$, see Example 25.6.

26. Differential graded categories

Note that if R is a ring, then R is a differential graded algebra over itself (with $R = R^0$ of course). In this case a differential graded R-module is the same thing as a complex of R-modules. In particular, given two differential graded R-modules M and N we denote $M \otimes_R N$ the differential graded R-module corresponding to the total complex associated to the double complex obtained by the tensor product of the complexes of R-modules associated to M and N.

Definition 26.1. Let R be a ring. A *differential graded category $\mathcal{A}$ over R* is a category where every morphism set is given the structure of a differential graded R-module and where for $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition is R-bilinear and induces a homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(y, z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x, z)$$

of differential graded R-modules.

The final condition of the definition signifies the following: if $f \in \mathrm{Hom}_{\mathcal{A}}^n(x, y)$ and $g \in \mathrm{Hom}_{\mathcal{A}}^m(y, z)$ are homogeneous of degrees n and m, then

$$\mathrm{d}(g \circ f) = \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f)$$

in $\mathrm{Hom}_{\mathcal{A}}^{n+m+1}(x, z)$. This follows from the sign rule for the differential on the total complex of a double complex, see Homology, Definition 18.3.

Definition 26.2. Let R be a ring. A *functor of differential graded categories over* R is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{A}}(F(x), F(y))$ is a homomorphism of differential graded R-modules.

Given a differential graded category we are often interested in the corresponding categories of complexes and homotopy category. Here is a formal definition.

Definition 26.3. Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. Then we let

(1) the *category of complexes of* $\mathcal{A}$[1] be the category $\mathrm{Comp}(\mathcal{A})$ whose objects are the same as the objects of $\mathcal{A}$ and with

$$\mathrm{Hom}_{\mathrm{Comp}(\mathcal{A})}(x, y) = \mathrm{Ker}(d : \mathrm{Hom}_{\mathcal{A}}^0(x, y) \to \mathrm{Hom}_{\mathcal{A}}^1(x, y))$$

(2) the *homotopy category of* $\mathcal{A}$ be the category $K(\mathcal{A})$ whose objects are the same as the objects of $\mathcal{A}$ and with

$$\mathrm{Hom}_{K(\mathcal{A})}(x, y) = H^0(\mathrm{Hom}_{\mathcal{A}}(x, y))$$

Our use of the symbol $K(\mathcal{A})$ is nonstandard, but at least is compatible with the use of $K(-)$ in other chapters of the Stacks project.

Definition 26.4. Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. A direct sum (x, y, z, i, j, p, q) in $\mathcal{A}$ (notation as in Homology, Remark 3.6) is a *differential graded direct sum* if i, j, p, q are homogeneous of degree 0 and closed, i.e., $\mathrm{d}(i) = 0$, etc.

Lemma 26.5. *Let R be a ring. A functor $F : \mathcal{A} \to \mathcal{B}$ of differential graded categories over R induces functors $\mathrm{Comp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{B})$ and $K(\mathcal{A}) \to K(\mathcal{B})$.*

Proof. Omitted. $\qquad\square$

Example 26.6 (Differential graded category of complexes)**.** Let $\mathcal{B}$ be an additive category. We will construct a differential graded category $\mathrm{Comp}^{dg}(\mathcal{B})$ over $R = \mathbf{Z}$ whose associated category of complexes is $\mathrm{Comp}(\mathcal{B})$ and whose associated homotopy category is $K(\mathcal{B})$. As objects of $\mathrm{Comp}^{dg}(\mathcal{B})$ we take complexes of $\mathcal{B}$. Given complexes $A^\bullet$ and $B^\bullet$ of $\mathcal{B}$, we sometimes also denote $A^\bullet$ and $B^\bullet$ the corresponding graded objects of $\mathcal{B}$ (i.e., forget about the differential). Using this abuse of notation, we set

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Gr}^{gr}(\mathcal{B})}(A^\bullet, B^\bullet) = \bigoplus_{n \in \mathbf{Z}} \mathrm{Hom}^n(A, B)$$

as a graded $\mathbf{Z}$-module with notation and definitions as in Example 25.5. In other words, the nth graded piece is the abelian group of homogeneous morphism of degree n of graded objects

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \prod_{p+q=n} \mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^p)$$

Observe reversal of indices and observe we have a direct product and not a direct sum. For an element $f \in \mathrm{Hom}^n(A^\bullet, B^\bullet)$ of degree n we set

$$\mathrm{d}(f) = \mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A$$

The sign is exactly as in More on Algebra, Section 72. To make sense of this we think of d_B and d_A as maps of graded objects of $\mathcal{B}$ homogeneous of degree 1 and

---

[1]This may be nonstandard terminology.

we use composition in the category $\mathrm{Gr}^{gr}(\mathcal{B})$ on the right hand side. In terms of components, if $f = (f_{p,q})$ with $f_{p,q} : A^{-q} \to B^p$ we have

$$(26.6.1) \qquad \mathrm{d}(f_{p,q}) = \mathrm{d}_B \circ f_{p,q} - (-1)^{p+q} f_{p,q} \circ \mathrm{d}_A$$

Note that the first term of this expression is in $\mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^{p+1})$ and the second term is in $\mathrm{Hom}_{\mathcal{B}}(A^{-q-1}, B^p)$. The reader checks that

(1) d has square zero,
(2) an element f in $\mathrm{Hom}^n(A^\bullet, B^\bullet)$ has $\mathrm{d}(f) = 0$ if and only if the morphism $f : A^\bullet \to B^\bullet[n]$ of graded objects of $\mathcal{B}$ is actually a map of complexes,
(3) in particular, the category of complexes of $\mathrm{Comp}^{dg}(\mathcal{B})$ is equal to $\mathrm{Comp}(\mathcal{B})$,
(4) the morphism of complexes defined by f as in (2) is homotopy equivalent to zero if and only if $f = \mathrm{d}(g)$ for some $g \in \mathrm{Hom}^{n-1}(A^\bullet, B^\bullet)$.
(5) in particular, we obtain a canonical isomorphism

$$\mathrm{Hom}_{K(\mathcal{B})}(A^\bullet, B^\bullet) \longrightarrow H^0(\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet))$$

and the homotopy category of $\mathrm{Comp}^{dg}(\mathcal{B})$ is equal to $K(\mathcal{B})$.

Given complexes $A^\bullet$, $B^\bullet$, $C^\bullet$ we define composition

$$\mathrm{Hom}^m(B^\bullet, C^\bullet) \times \mathrm{Hom}^n(A^\bullet, B^\bullet) \longrightarrow \mathrm{Hom}^{n+m}(A^\bullet, C^\bullet)$$

by composition $(g, f) \mapsto g \circ f$ in the graded category $\mathrm{Gr}^{gr}(\mathcal{B})$, see Example 25.5. This defines a map of differential graded modules

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(B^\bullet, C^\bullet) \otimes_R \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, C^\bullet)$$

as required in Definition 26.1 because

$$\begin{aligned}
\mathrm{d}(g \circ f) &= \mathrm{d}_C \circ g \circ f - (-1)^{n+m} g \circ f \circ \mathrm{d}_A \\
&= (\mathrm{d}_C \circ g - (-1)^m g \circ \mathrm{d}_B) \circ f + (-1)^m g \circ (\mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A) \\
&= \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f)
\end{aligned}$$

as desired.

Lemma 26.7. *Let $F : \mathcal{B} \to \mathcal{B}'$ be an additive functor between additive categories. Then F induces a functor of differential graded categories*

$$F : \mathrm{Comp}^{dg}(\mathcal{B}) \to \mathrm{Comp}^{dg}(\mathcal{B}')$$

of Example 26.6 inducing the usual functors on the category of complexes and the homotopy categories.

Proof. Omitted. $\square$

Example 26.8 (Differential graded category of differential graded modules)**.** Let (A, d) be a differential graded algebra over a ring R. We will construct a differential graded category $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ over R whose category of complexes is $\mathrm{Mod}_{(A,\mathrm{d})}$ and whose homotopy category is $K(\mathrm{Mod}_{(A,\mathrm{d})})$. As objects of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ we take the differential graded A-modules. Given differential graded A-modules L and M we set

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L, M) = \mathrm{Hom}_{\mathrm{Mod}^{gr}_A}(L, M) = \bigoplus \mathrm{Hom}^n(L, M)$$

as a graded R-module where the right hand side is defined as in Example 25.6. In other words, the nth graded piece $\mathrm{Hom}^n(L, M)$ is the R-module of right A-module maps homogeneous of degree n. For an element $f \in \mathrm{Hom}^n(L, M)$ we set

$$\mathrm{d}(f) = \mathrm{d}_M \circ f - (-1)^n f \circ \mathrm{d}_L$$

To make sense of this we think of d_M and d_L as graded R-module maps and we use composition of graded R-module maps. It is clear that $\mathrm{d}(f)$ is homogeneous of degree $n + 1$ as a graded R-module map, and it is A-linear because

$$
\begin{aligned}
\mathrm{d}(f)(xa) &= \mathrm{d}_M(f(x)a) - (-1)^n f(\mathrm{d}_L(xa)) \\
&= \mathrm{d}_M(f(x))a + (-1)^{\deg(x)+n} f(x)\mathrm{d}(a) - (-1)^n f(\mathrm{d}_L(x))a - (-1)^{n+\deg(x)} f(x)\mathrm{d}(a) \\
&= \mathrm{d}(f)(x)a
\end{aligned}
$$

as desired (observe that this calculation would not work without the sign in the definition of our differential on Hom). Similar formulae to those of Example 26.6 hold for the differential of f in terms of components. The reader checks (in the same way as in Example 26.6) that

(1) d has square zero,
(2) an element f in $\mathrm{Hom}^n(L, M)$ has $\mathrm{d}(f) = 0$ if and only if $f : L \to M[n]$ is a homomorphism of differential graded A-modules,
(3) in particular, the category of complexes of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ is $\mathrm{Mod}_{(A,\mathrm{d})}$,
(4) the homomorphism defined by f as in (2) is homotopy equivalent to zero if and only if $f = \mathrm{d}(g)$ for some $g \in \mathrm{Hom}^{n-1}(L, M)$.
(5) in particular, we obtain a canonical isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(L, M) \longrightarrow H^0(\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L, M))$$

and the homotopy category of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ is $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

Given differential graded A-modules K, L, M we define composition

$$\mathrm{Hom}^m(L, M) \times \mathrm{Hom}^n(K, L) \longrightarrow \mathrm{Hom}^{n+m}(K, M)$$

by composition of homogeneous right A-module maps $(g, f) \mapsto g \circ f$. This defines a map of differential graded modules

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L, M) \otimes_R \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(K, L) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(K, M)$$

as required in Definition 26.1 because

$$
\begin{aligned}
\mathrm{d}(g \circ f) &= \mathrm{d}_M \circ g \circ f - (-1)^{n+m} g \circ f \circ \mathrm{d}_K \\
&= (\mathrm{d}_M \circ g - (-1)^m g \circ \mathrm{d}_L) \circ f + (-1)^m g \circ (\mathrm{d}_L \circ f - (-1)^n f \circ \mathrm{d}_K) \\
&= \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f)
\end{aligned}
$$

as desired.

Lemma 26.9. *Let $\varphi : (A, d) \to (E, d)$ be a homomorphism of differential graded algebras. Then φ induces a functor of differential graded categories*

$$F : Mod^{dg}_{(E,d)} \longrightarrow Mod^{dg}_{(A,d)}$$

of Example 26.8 inducing obvious restriction functors on the categories of differential graded modules and homotopy categories.

Proof. Omitted. $\square$

Lemma 26.10. *Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. Let x be an object of $\mathcal{A}$. Let*

$$(E, d) = \mathrm{Hom}_{\mathcal{A}}(x, x)$$

be the differential graded R-algebra of endomorphisms of x. We obtain a functor

$$\mathcal{A} \longrightarrow Mod^{dg}_{(E,d)}, \quad y \longmapsto \mathrm{Hom}_{\mathcal{A}}(x, y)$$

of differential graded categories by letting E act on $\mathrm{Hom}_{\mathcal{A}}(x, y)$ via composition in $\mathcal{A}$. This functor induces functors

$$Comp(\mathcal{A}) \to Mod_{(A,d)} \quad and \quad K(\mathcal{A}) \to K(Mod_{(A,d)})$$

by an application of Lemma 26.5.

Proof. This lemma proves itself. $\qquad\square$

27. Obtaining triangulated categories

In this section we discuss the most general setup to which the arguments proving Derived Categories, Proposition 10.3 and Proposition 10.3 apply.

Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. To make our argument work, we impose some axioms on $\mathcal{A}$:

- (A) $\mathcal{A}$ has a zero object and differential graded direct sums of two objects (as in Definition 26.4).
- (B) there are functors $[n] : \mathcal{A} \longrightarrow \mathcal{A}$ of differential graded categories such that $[0] = \mathrm{id}_{\mathcal{A}}$ and $[n + m] = [n] \circ [m]$ and given isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(x, y[n]) = \mathrm{Hom}_{\mathcal{A}}(x, y)[n]$$

  of differential graded R-modules compatible with composition.

Given our differential graded category $\mathcal{A}$ we say

- (1) a sequence $x \to y \to z$ of morphisms of $Comp(\mathcal{A})$ is an *admissible short exact sequence* if there exists an isomorphism $y \cong x \oplus z$ in the underlying graded category such that $x \to z$ and $y \to z$ are (co)projections.
- (2) a morphism $x \to y$ of $Comp(\mathcal{A})$ is an *admissible monomorphism* if it extends to an admissible short exact sequence $x \to y \to z$.
- (3) a morphism $y \to z$ of $Comp(\mathcal{A})$ is an *admissible epimorphism* if it extends to an admissible short exact sequence $x \to y \to z$.
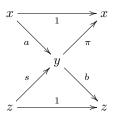
The next lemma tells us an admissible short exact sequence gives a triangle, provided we have axioms (A) and (B).

Lemma 27.1. *Let $\mathcal{A}$ be a differential graded category satisfying axioms (A) and (B). Given an admissible short exact sequence $x \to y \to z$ we obtain (see proof) a triangle*

$$x \to y \to z \to x[1]$$

in $Comp(\mathcal{A})$ with the property that any two compositions in $z[-1] \to x \to y \to z \to x[1]$ are zero in $K(\mathcal{A})$.

Proof. Choose a diagram

giving the isomorphism of graded objects $y \cong x \oplus z$ as in the definition of an admissible short exact sequence. Here are some equations that hold in this situation

(1) $1 = \pi a$ and hence $\mathrm{d}(\pi)a = 0$,

(2) $1 = bs$ and hence $b\mathrm{d}(s) = 0$,

(3) $1 = a\pi + sb$ and hence $a\mathrm{d}(\pi) + \mathrm{d}(s)b = 0$,

(4) $\pi s = 0$ and hence $\mathrm{d}(\pi)s + \pi\mathrm{d}(s) = 0$,

(5) $\mathrm{d}(s) = a\pi\mathrm{d}(s)$ because $\mathrm{d}(s) = (a\pi + sb)\mathrm{d}(s)$ and $b\mathrm{d}(s) = 0$,

(6) $\mathrm{d}(\pi) = \mathrm{d}(\pi)sb$ because $\mathrm{d}(\pi) = \mathrm{d}(\pi)(a\pi + sb)$ and $\mathrm{d}(\pi)a = 0$,

(7) $\mathrm{d}(\pi\mathrm{d}(s)) = 0$ because if we postcompose it with the monomorphism a we get $\mathrm{d}(a\pi\mathrm{d}(s)) = \mathrm{d}(\mathrm{d}(s)) = 0$, and

(8) $\mathrm{d}(\mathrm{d}(\pi)s) = 0$ as by (4) it is the negative of $\mathrm{d}(\pi\mathrm{d}(s))$ which is 0 by (7).

We've used repeatedly that $\mathrm{d}(a) = 0$, $\mathrm{d}(b) = 0$, and that $\mathrm{d}(1) = 0$. By (7) we see that

$$\delta = \pi\mathrm{d}(s) = -\mathrm{d}(\pi)s : z \to x[1]$$

is a morphism in $\mathrm{Comp}(\mathcal{A})$. By (5) we see that the composition $a\delta = a\pi\mathrm{d}(s) = \mathrm{d}(s)$ is homotopic to zero. By (6) we see that the composition $\delta b = -\mathrm{d}(\pi)sb = \mathrm{d}(-\pi)$ is homotopic to zero. $\square$

Besides axioms (A) and (B) we need an axiom concerning the existence of cones. We formalize everything as follows.

Situation 27.2. Here R is a ring and $\mathcal{A}$ is a differential graded category over R having axioms (A), (B), and

(C) given an arrow $f : x \to y$ of degree 0 with $\mathrm{d}(f) = 0$ there exists an admissible short exact sequence $y \to c(f) \to x[1]$ in $\mathrm{Comp}(\mathcal{A})$ such that the map $x[1] \to y[1]$ of Lemma 27.1 is equal to $f[1]$.

We will call $c(f)$ a *cone* of the morphism f. If (A), (B), and (C) hold, then cones are functorial in a weak sense.

Lemma 27.3. *In Situation 27.2 suppose that*

is a diagram of $\mathrm{Comp}(\mathcal{A})$ commutative up to homotopy. Then there exists a morphism $c : c(f_1) \to c(f_2)$ which gives rise to a morphism of triangles

$$(a, b, c) : (x_1, y_1, c(f_1)) \to (x_1, y_1, c(f_1))$$

in $K(\mathcal{A})$.

Proof. The assumption means there exists a morphism $h : x_1 \to y_2$ of degree -1 such that $\mathrm{d}(h) = bf_1 - f_2a$. Choose isomorphisms $c(f_i) = y_i \oplus x_i[1]$ of graded objects compatible with the morphisms $y_i \to c(f_i) \to x_i[1]$. Let's denote $a_i : y_i \to c(f_i)$, $b_i : c(f_i) \to x_i[1]$, $s_i : x_i[1] \to c(f_i)$, and $\pi_i : c(f_i) \to y_i$ the given morphisms. Recall that $x_i[1] \to y_i[1]$ is given by $\pi_i\mathrm{d}(s_i)$. By axiom (C) this means that

$$f_i = \pi_i\mathrm{d}(s_i) = -\mathrm{d}(\pi_i)s_i$$

(we identify $\mathrm{Hom}(x_i, y_i)$ with $\mathrm{Hom}(x_i[1], y_i[1])$ using the shift functor $[1]$). Set $c = a_2 b \pi_1 + s_2 ab_1 + a_2 hb$. Then, using the equalities found in the proof of Lemma 27.1 we obtain

$$
\begin{aligned}
\mathrm{d}(c) &= a_2 b \mathrm{d}(\pi_1) + \mathrm{d}(s_2)ab_1 + a_2\mathrm{d}(h)b_1 \\
&= -a_2 bf_1 b_1 + a_2 f_2 ab_1 + a_2(bf_1 - f_2a)b_1 \\
&= 0
\end{aligned}
$$

(where we have used in particular that $\mathrm{d}(\pi_1) = \mathrm{d}(\pi_1)s_1 b_1 = f_1 b_1$ and $\mathrm{d}(s_2) = a_2\pi_2\mathrm{d}(s_2) = a_2 f_2$). Thus c is a degree 0 morphism $c : c(f_1) \to c(f_2)$ of $\mathcal{A}$ compatible with the given morphisms $y_i \to c(f_i) \to x_i[1]$. $\qquad\square$

In Situation 27.2 we say that a triangle (x, y, z, f, g, h) in $K(\mathcal{A})$ is a *distinguished triangle* if there exists an admissible short exact sequence $x' \to y' \to z'$ such that (x, y, z, f, g, h) is isomorphic as a triangle in $K(\mathcal{A})$ to the triangle $(x', y', z', x' \to y', y' \to z', \delta)$ constructed in Lemma 27.1. We will show below that

$$\boxed{K(\mathcal{A}) \text{ is a triangulated category}}$$

This result, although not as general as one might think, applies to a number of natural generalizations of the cases covered so far in the Stacks project. Here are some examples:

(1) Let $(X, \mathcal{O}_X)$ be a ringed space. Let (A, d) be a sheaf of differential graded $\mathcal{O}_X$-algebras. Let $\mathcal{A}$ be the differential graded category of differential graded A-modules. Then $K(\mathcal{A})$ is a triangulated category.

(2) Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (A, d) be a sheaf of differential graded $\mathcal{O}$-algebras. Let $\mathcal{A}$ be the differential graded category of differential graded A-modules. Then $K(\mathcal{A})$ is a triangulated category. See Differential Graded Sheaves, Proposition 22.4.

(3) Two examples with a different flavor may be found in Examples, Section 69.

The following simple lemma is a key to the construction.

Lemma 27.4. *In Situation 27.2 given any object x of $\mathcal{A}$, and the cone $C(1_x)$ of the identity morphism $1_x : x \to x$, the identity morphism on $C(1_x)$ is homotopic to zero.*

Proof. Consider the admissible short exact sequence given by axiom (C).

$$x \underset{\pi}{\overset{a}{\rightleftarrows}} C(1_x) \underset{s}{\overset{b}{\rightleftarrows}} x[1]$$

Then by Lemma 27.1, identifying hom-sets under shifting, we have $1_x = \pi d(s) = -d(\pi)s$ where s is regarded as a morphism in $\mathrm{Hom}_{\mathcal{A}}^{-1}(x, C(1_x))$. Therefore $a =$

$a\pi d(s) = d(s)$ using formula (5) of Lemma 27.1, and $b = -d(\pi)sb = -d(\pi)$ by formula (6) of Lemma 27.1. Hence

$$1_{C(1_x)} = a\pi + sb = d(s)\pi - sd(\pi) = d(s\pi)$$

since s is of degree -1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A more general version of the above lemma will appear in Lemma 27.13. The following lemma is the analogue of Lemma 7.3.

Lemma 27.5. *In Situation 27.2 given a diagram*

$$
\begin{array}{ccc}
x & \xrightarrow{\ f\ } & y \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
z & \xrightarrow{\ g\ } & w
\end{array}
$$

in Comp$(\mathcal{A})$ commuting up to homotopy. Then

   (1) *If f is an admissible monomorphism, then b is homotopic to a morphism b' which makes the diagram commute.*
   (2) *If g is an admissible epimorphism, then a is homotopic to a morphism a' which makes the diagram commute.*

Proof. To prove (1), observe that the hypothesis implies that there is some $h \in \mathrm{Hom}_{\mathcal{A}}(x, w)$ of degree -1 such that $bf - ga = d(h)$. Since f is an admissible monomorphism, there is a morphism $\pi : y \to x$ in the category $\mathcal{A}$ of degree 0. Let $b' = b - d(h\pi)$. Then

$$
\begin{aligned}
b'f = bf - d(h\pi)f &= bf - d(h\pi f) \quad (\text{since } d(f) = 0) \\
&= bf - d(h) \\
&= ga
\end{aligned}
$$

as desired. The proof for (2) is omitted. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following lemma is the analogue of Lemma 7.4.

Lemma 27.6. *In Situation 27.2 let $\alpha : x \to y$ be a morphism in Comp$(\mathcal{A})$. Then there exists a factorization in Comp$(\mathcal{A})$:*

$$x \xrightarrow{\ \tilde{\alpha}\ } \tilde{y} \underset{s}{\overset{\pi}{\rightleftarrows}} y$$

such that

   (1) *$\tilde{\alpha}$ is an admissible monomorphism, and $\pi\tilde{\alpha} = \alpha$.*
   (2) *There exists a morphism $s : y \to \tilde{y}$ in Comp$(\mathcal{A})$ such that $\pi s = 1_y$ and $s\pi$ is homotopic to $1_{\tilde{y}}$.*

Proof. By axiom (A), we may let $\tilde{y}$ be the differential graded direct sum of y and $C(1_x)$, i.e., there exists a diagram

$$y \underset{\pi}{\overset{s}{\rightleftarrows}} y \oplus C(1_x) \underset{t}{\overset{p}{\rightleftarrows}} C(1_x)$$

where all morphisms are of degree zero, and in Comp$(\mathcal{A})$. Let $\tilde{y} = y \oplus C(1_x)$. Then $1_{\tilde{y}} = s\pi + tp$. Consider now the diagram

$$x \xrightarrow{\ \tilde{\alpha}\ } \tilde{y} \underset{s}{\overset{\pi}{\rightleftarrows}} y$$

where $\tilde{\alpha}$ is induced by the morphism $x \xrightarrow{\alpha} y$ and the natural morphism $x \to C(1_x)$ fitting in the admissible short exact sequence

$$x \rightleftarrows C(1_x) \rightleftarrows x[1]$$

So the morphism $C(1_x) \to x$ of degree 0 in this diagram, together with the zero morphism $y \to x$, induces a degree-0 morphism $\beta : \tilde{y} \to x$. Then $\tilde{\alpha}$ is an admissible monomorphism since it fits into the admissible short exact sequence

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \longrightarrow x[1]$$

Furthermore, $\pi\tilde{\alpha} = \alpha$ by the construction of $\tilde{\alpha}$, and $\pi s = 1_y$ by the first diagram. It remains to show that $s\pi$ is homotopic to $1_{\tilde{y}}$. Write 1_x as $d(h)$ for some degree -1 map. Then, our last statement follows from

$$
\begin{aligned}
1_{\tilde{y}} - s\pi &= tp \\
&= t(dh)p \quad \text{(by Lemma 27.4)} \\
&= d(thp)
\end{aligned}
$$

since $dt = dp = 0$, and t is of degree zero. $\qquad\square$

The following lemma is the analogue of Lemma 7.5.

Lemma 27.7. *In Situation 27.2 let $x_1 \to x_2 \to \ldots \to x_n$ be a sequence of composable morphisms in $\mathrm{Comp}(\mathcal{A})$. Then there exists a commutative diagram in $\mathrm{Comp}(\mathcal{A})$:*

$$
\begin{array}{ccccccc}
x_1 & \longrightarrow & x_2 & \longrightarrow & \ldots & \longrightarrow & x_n \\
\uparrow & & \uparrow & & & & \uparrow \\
y_1 & \longrightarrow & y_2 & \longrightarrow & \ldots & \longrightarrow & y_n
\end{array}
$$

such that each $y_i \to y_{i+1}$ is an admissible monomorphism and each $y_i \to x_i$ is a homotopy equivalence.

Proof. The case for $n = 1$ is trivial: one simply takes $y_1 = x_1$ and the identity morphism on x_1 is in particular a homotopy equivalence. The case $n = 2$ is given by Lemma 27.6. Suppose we have constructed the diagram up to x_{n-1}. We apply Lemma 27.6 to the composition $y_{n-1} \to x_{n-1} \to x_n$ to obtain y_n. Then $y_{n-1} \to y_n$ will be an admissible monomorphism, and $y_n \to x_n$ a homotopy equivalence. $\qquad\square$

The following lemma is the analogue of Lemma 7.6.

Lemma 27.8. *In Situation 27.2 let $x_i \to y_i \to z_i$ be morphisms in $\mathcal{A}$ $(i = 1, 2, 3)$ such that $x_2 \to y_2 \to z_2$ is an admissible short exact sequence. Let $b : y_1 \to y_2$ and $b' : y_2 \to y_3$ be morphisms in $\mathrm{Comp}(\mathcal{A})$ such that*

$$
\begin{array}{ccccc}
x_1 & \longrightarrow & y_1 & \longrightarrow & z_1 \\
{\scriptstyle 0}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle 0}\downarrow \\
x_2 & \longrightarrow & y_2 & \longrightarrow & z_2
\end{array}
\quad and \quad
\begin{array}{ccccc}
x_2 & \longrightarrow & y_2 & \longrightarrow & z_2 \\
{\scriptstyle 0}\downarrow & & {\scriptstyle b'}\downarrow & & {\scriptstyle 0}\downarrow \\
x_3 & \longrightarrow & y_3 & \longrightarrow & z_3
\end{array}
$$

commute up to homotopy. Then $b' \circ b$ is homotopic to 0.

Proof. By Lemma 27.5, we can replace b and b' by homotopic maps $\tilde{b}$ and $\tilde{b}'$, such that the right square of the left diagram commutes and the left square of the right diagram commutes. Say $b = \tilde{b} + d(h)$ and $b' = \tilde{b}' + d(h')$ for degree -1 morphisms h and h' in $\mathcal{A}$. Hence

$$b'b = \tilde{b}'\tilde{b} + d(\tilde{b}'h + h'\tilde{b} + h'd(h))$$

since $d(\tilde{b}) = d(\tilde{b}') = 0$, i.e. $b'b$ is homotopic to $\tilde{b}'\tilde{b}$. We now want to show that $\tilde{b}'\tilde{b} = 0$. Because $x_2 \xrightarrow{f} y_2 \xrightarrow{g} z_2$ is an admissible short exact sequence, there exist degree 0 morphisms $\pi : y_2 \to x_2$ and $s : z_2 \to y_2$ such that $\mathrm{id}_{y_2} = f\pi + sg$. Therefore

$$\tilde{b}'\tilde{b} = \tilde{b}'(f\pi + sg)\tilde{b} = 0$$

since $g\tilde{b} = 0$ and $\tilde{b}'f = 0$ as consequences of the two commuting squares. $\square$

The following lemma is the analogue of Lemma 8.1.

Lemma 27.9. *In Situation 27.2 let $0 \to x \to y \to z \to 0$ be an admissible short exact sequence in $\mathrm{Comp}(\mathcal{A})$. The triangle*

$$x \longrightarrow y \longrightarrow z \xrightarrow{\delta} x[1]$$

with $\delta : z \to x[1]$ as defined in Lemma 27.1 is up to canonical isomorphism in $K(\mathcal{A})$, independent of the choices made in Lemma 27.1.

Proof. Suppose δ is defined by the splitting

$$x \underset{\pi}{\overset{a}{\rightleftarrows}} y \underset{s}{\overset{b}{\rightleftarrows}} z$$

and δ' is defined by the splitting with π', s' in place of π, s. Then

$$s' - s = (a\pi + sb)(s' - s) = a\pi s'$$

since $bs' = bs = 1_z$ and $\pi s = 0$. Similarly,

$$\pi' - \pi = (\pi' - \pi)(a\pi + sb) = \pi' sb$$

Since $\delta = \pi d(s)$ and $\delta' = \pi' d(s')$ as constructed in Lemma 27.1, we may compute

$$\delta' = \pi' d(s') = (\pi + \pi' sb)d(s + a\pi s') = \delta + d(\pi s')$$

using $\pi a = 1_x$, $ba = 0$, and $\pi' sbd(s') = \pi' sba\pi d(s') = 0$ by formula (5) in Lemma 27.1. $\square$

The following lemma is the analogue of Lemma 9.1.

Lemma 27.10. *In Situation 27.2 let $f : x \to y$ be a morphism in $\mathrm{Comp}(\mathcal{A})$. The triangle $(y, c(f), x[1], i, p, f[1])$ is the triangle associated to the admissible short exact sequence*

$$y \longrightarrow c(f) \longrightarrow x[1]$$

where the cone $c(f)$ is defined as in Lemma 27.1.

Proof. This follows from axiom (C). $\square$

The following lemma is the analogue of Lemma 9.2.

Lemma 27.11. *In Situation 27.2 let $\alpha : x \to y$ and $\beta : y \to z$ define an admissible short exact sequence*

$$x \longrightarrow y \longrightarrow z$$

in $\mathrm{Comp}(\mathcal{A})$. Let $(x, y, z, \alpha, \beta, \delta)$ be the associated triangle in $K(\mathcal{A})$. Then, the triangles

$$(z[-1], x, y, \delta[-1], \alpha, \beta) \quad and \quad (z[-1], x, c(\delta[-1]), \delta[-1], i, p)$$

are isomorphic.

Proof. We have a diagram of the form

$$
\begin{array}{ccccccc}
z[-1] & \xrightarrow{\delta[-1]} & x & \underset{\tilde{\alpha}}{\overset{\alpha}{\rightleftarrows}} & y & \underset{\tilde{\beta}}{\overset{\beta}{\rightleftarrows}} & z \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \vdots & & \downarrow{\scriptstyle 1} \\
z[-1] & \xrightarrow{\delta[-1]} & x & \underset{\tilde{i}}{\overset{i}{\rightleftarrows}} & c(\delta[-1]) & \underset{\tilde{p}}{\overset{p}{\rightleftarrows}} & z
\end{array}
$$

with splittings to α, β, i, and p given by $\tilde{\alpha}, \tilde{\beta}, \tilde{i}$, and $\tilde{p}$ respectively. Define a morphism $y \to c(\delta[-1])$ by $i\tilde{\alpha} + \tilde{p}\beta$ and a morphism $c(\delta[-1]) \to y$ by $\alpha\tilde{i} + \tilde{\beta}p$. Let us first check that these define morphisms in $\mathrm{Comp}(\mathcal{A})$. We remark that by identities from Lemma 27.1, we have the relation $\delta[-1] = \tilde{\alpha}d(\tilde{\beta}) = -d(\tilde{\alpha})\tilde{\beta}$ and the relation $\delta[-1] = \tilde{i}d(\tilde{p})$. Then

$$
\begin{aligned}
d(\tilde{\alpha}) &= d(\tilde{\alpha})\tilde{\beta}\beta \\
&= -\delta[-1]\beta
\end{aligned}
$$

where we have used equation (6) of Lemma 27.1 for the first equality and the preceeding remark for the second. Similarly, we obtain $d(\tilde{p}) = i\delta[-1]$. Hence

$$
\begin{aligned}
d(i\tilde{\alpha} + \tilde{p}\beta) &= d(i)\tilde{\alpha} + id(\tilde{\alpha}) + d(\tilde{p})\beta + \tilde{p}d(\beta) \\
&= id(\tilde{\alpha}) + d(\tilde{p})\beta \\
&= -i\delta[-1]\beta + i\delta[-1]\beta \\
&= 0
\end{aligned}
$$

so $i\tilde{\alpha} + \tilde{p}\beta$ is indeed a morphism of $\mathrm{Comp}(\mathcal{A})$. By a similar calculation, $\alpha\tilde{i} + \tilde{\beta}p$ is also a morphism of $\mathrm{Comp}(\mathcal{A})$. It is immediate that these morphisms fit in the commutative diagram. We compute:

$$
\begin{aligned}
(i\tilde{\alpha} + \tilde{p}\beta)(\alpha\tilde{i} + \tilde{\beta}p) &= i\tilde{\alpha}\alpha\tilde{i} + i\tilde{\alpha}\tilde{\beta}p + \tilde{p}\beta\alpha\tilde{i} + \tilde{p}\beta\tilde{\beta}p \\
&= i\tilde{i} + \tilde{p}p \\
&= 1_{c(\delta[-1])}
\end{aligned}
$$

where we have freely used the identities of Lemma 27.1. Similarly, we compute $(\alpha\tilde{i} + \tilde{\beta}p)(i\tilde{\alpha} + \tilde{p}\beta) = 1_y$, so we conclude $y \cong c(\delta[-1])$. Hence, the two triangles in question are isomorphic. $\square$

The following lemma is the analogue of Lemma 9.3.

Lemma 27.12. *In Situation 27.2 let $f_1 : x_1 \to y_1$ and $f_2 : x_2 \to y_2$ be morphisms in $\mathrm{Comp}(\mathcal{A})$. Let*

$$(a, b, c) : (x_1, y_1, c(f_1), f_1, i_1, p_1) \to (x_2, y_2, c(f_2), f_2, i_1, p_1)$$

be any morphism of triangles in $K(\mathcal{A})$. If a and b are homotopy equivalences, then so is c.

Proof. Since a and b are homotopy equivalences, they are invertible in $K(\mathcal{A})$ so let a^{-1} and b^{-1} denote their inverses in $K(\mathcal{A})$, giving us a commutative diagram

$$
\begin{array}{ccccc}
x_2 & \xrightarrow{f_2} & y_2 & \xrightarrow{i_2} & c(f_2) \\
\downarrow{\scriptstyle a^{-1}} & & \downarrow{\scriptstyle b^{-1}} & & \downarrow{\scriptstyle c'} \\
x_1 & \xrightarrow{f_1} & y_1 & \xrightarrow{i_1} & c(f_1)
\end{array}
$$

where the map c' is defined via Lemma 27.3 applied to the left commutative box of the above diagram. Since the diagram commutes in $K(\mathcal{A})$, it suffices by Lemma 27.8 to prove the following: given a morphism of triangle $(1, 1, c) : (x, y, c(f), f, i, p) \to (x, y, c(f), f, i, p)$ in $K(\mathcal{A})$, the map c is an isomorphism in $K(\mathcal{A})$. We have the commutative diagrams in $K(\mathcal{A})$:

$$
\begin{array}{ccccc}
y & \longrightarrow & c(f) & \longrightarrow & x[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle 1} \\
y & \longrightarrow & c(f) & \longrightarrow & x[1]
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccccc}
y & \longrightarrow & c(f) & \longrightarrow & x[1] \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle c-1} & & \downarrow{\scriptstyle 0} \\
y & \longrightarrow & c(f) & \longrightarrow & x[1]
\end{array}
$$

Since the rows are admissible short exact sequences, we obtain the identity $(c-1)^2 = 0$ by Lemma 27.8, from which we conclude that $2 - c$ is inverse to c in $K(\mathcal{A})$ so that c is an isomorphism. $\qquad\square$

The following lemma is the analogue of Lemma 9.4.

Lemma 27.13. *In Situation 27.2.*

(1) *Given an admissible short exact sequence $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$. Then there exists a homotopy equivalence $e : C(\alpha) \to z$ such that the diagram*

(27.13.1)
$$
\begin{array}{ccccccc}
x & \xrightarrow{\alpha} & y & \xrightarrow{b} & C(\alpha) & \xrightarrow{-c} & x[1] \\
\downarrow & & \downarrow & & \vdots{\scriptstyle e} & & \downarrow \\
x & \xrightarrow{\alpha} & y & \xrightarrow{\beta} & z & \xrightarrow{\delta} & x[1]
\end{array}
$$

defines an isomorphism of triangles in $K(\mathcal{A})$. Here $y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]$ is the admissible short exact sequence given as in axiom (C).

(2) *Given a morphism $\alpha : x \to y$ in $Comp(\mathcal{A})$, let $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to y$ be the factorization given as in Lemma 27.6, where the admissible monomorphism $x \xrightarrow{\tilde{\alpha}} y$ extends to the admissible short exact sequence*

$$
x \xrightarrow{\tilde{\alpha}} \tilde{y} \longrightarrow z
$$

Then there exists an isomorphism of triangles

$$
\begin{array}{ccccccc}
x & \xrightarrow{\tilde{\alpha}} & \tilde{y} & \longrightarrow & z & \xrightarrow{\delta} & x[1] \\
\downarrow & & \downarrow & & \vdots{\scriptstyle e} & & \downarrow \\
x & \xrightarrow{\alpha} & y & \longrightarrow & C(\alpha) & \xrightarrow{-c} & x[1]
\end{array}
$$

where the upper triangle is the triangle associated to the sequence $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to z$.

Proof. For (1), we consider the more complete diagram, *without* the sign change on c:

$$x \underset{\pi}{\overset{\alpha}{\rightleftarrows}} y \underset{p}{\overset{b}{\rightleftarrows}} C(\alpha) \underset{\sigma}{\overset{c}{\rightleftarrows}} x[1] \underset{\pi}{\overset{\alpha}{\rightleftarrows}} y[1]$$

$$x \underset{\pi}{\overset{\alpha}{\rightleftarrows}} y \underset{s}{\overset{\beta}{\rightleftarrows}} z \xrightarrow{\delta} x[1]$$

where the admissible short exact sequence $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ is given the splitting π, s, and the admissible short exact sequence $y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]$ is given the splitting p, σ. Note that (identifying hom-sets under shifting)

$$\alpha = pd(\sigma) = -d(p)\sigma, \quad \delta = \pi d(s) = -d(\pi)s$$

by the construction in Lemma 27.1.

We define $e = \beta p$ and $f = bs - \sigma\delta$. We first check that they are morphisms in $\mathrm{Comp}(\mathcal{A})$. To show that $d(e) = \beta d(p)$ vanishes, it suffices to show that $\beta d(p)b$ and $\beta d(p)\sigma$ both vanish, whereas

$$\beta d(p)b = \beta d(pb) = \beta d(1_y) = 0, \quad \beta d(p)\sigma = -\beta\alpha = 0$$

Similarly, to check that $d(f) = bd(s) - d(\sigma)\delta$ vanishes, it suffices to check the post-compositions by p and c both vanish, whereas

$$pbd(s) - pd(\sigma)\delta = d(s) - \alpha\delta = d(s) - \alpha\pi d(s) = 0$$
$$cbd(s) - cd(\sigma)\delta = -cd(\sigma)\delta = -d(c\sigma)\delta = 0$$

The commutativity of left two squares of the diagram 27.13.1 follows directly from definition. Before we prove the commutativity of the right square (up to homotopy), we first check that e is a homotopy equivalence. Clearly,

$$ef = \beta p(bs - \sigma\delta) = \beta s = 1_z$$

To check that fe is homotopic to $1_{C(\alpha)}$, we first observe

$$b\alpha = bpd(\alpha) = d(\sigma), \quad \alpha c = -d(p)\sigma c = -d(p), \quad d(\pi)p = d(\pi)s\beta p = -\delta\beta p$$

Using these identities, we compute

$$\begin{aligned}
1_{C(\alpha)} =&bp + \sigma c \quad \text{(from } y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]) \\
=&b(\alpha\pi + s\beta)p + \sigma(\pi\alpha)c \quad \text{(from } x \xrightarrow{\alpha} y \xrightarrow{\beta} z) \\
=&d(\sigma)\pi p + bs\beta p - \sigma\pi d(p) \quad \text{(by the first two identities above)} \\
=&d(\sigma)\pi p + bs\beta p - \sigma\delta\beta p + \sigma\delta\beta p - \sigma\pi d(p) \\
=&(bs - \sigma\delta)\beta p + d(\sigma)\pi p - \sigma d(\pi)p - \sigma\pi d(p) \quad \text{(by the third identity above)} \\
=&fe + d(\sigma\pi p)
\end{aligned}$$

since $\sigma \in \mathrm{Hom}^{-1}(x, C(\alpha))$ (cf. proof of Lemma 27.4). Hence e and f are homotopy inverses. Finally, to check that the right square of diagram 27.13.1 commutes up to homotopy, it suffices to check that $-cf = \delta$. This follows from

$$-cf = -c(bs - \sigma\delta) = c\sigma\delta = \delta$$

since $cb = 0$.

For (2), consider the factorization $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to y$ given as in Lemma 27.6, so the second morphism is a homotopy equivalence. By Lemmas 27.3 and 27.12, there exists an isomorphism of triangles between

$$x \xrightarrow{\alpha} y \to C(\alpha) \to x[1] \quad \text{and} \quad x \xrightarrow{\tilde{\alpha}} \tilde{y} \to C(\tilde{\alpha}) \to x[1]$$

Since we can compose isomorphisms of triangles, by replacing α by $\tilde{\alpha}$, y by $\tilde{y}$, and $C(\alpha)$ by $C(\tilde{\alpha})$, we may assume α is an admissible monomorphism. In this case, the result follows from (1). $\qquad\square$

The following lemma is the analogue of Lemma 10.1.

Lemma 27.14. *In Situation 27.2 the homotopy category $K(\mathcal{A})$ with its natural translation functors and distinguished triangles is a pre-triangulated category.*

Proof. We will verify each of TR1, TR2, and TR3.

Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Since

$$x \xrightarrow{\ 1_x\ } x \longrightarrow 0$$

is an admissible short exact sequence, $(x, x, 0, 1_x, 0, 0)$ is a distinguished triangle. Moreover, given a morphism $\alpha : x \to y$ in $\text{Comp}(\mathcal{A})$, the triangle given by $(x, y, c(\alpha), \alpha, i, -p)$ is distinguished by Lemma 27.13.

Proof of TR2. Let $(x, y, z, \alpha, \beta, \gamma)$ be a triangle and suppose $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is distinguished. Then there exists an admissible short exact sequence $0 \to x' \to y' \to z' \to 0$ such that the associated triangle $(x', y', z', \alpha', \beta', \gamma')$ is isomorphic to $(y, z, x[1], \beta, \gamma, -\alpha[1])$. After rotating, we conclude that $(x, y, z, \alpha, \beta, \gamma)$ is isomorphic to $(z'[-1], x', y', \gamma'[-1], \alpha', \beta')$. By Lemma 27.11, we deduce that $(z'[-1], x', y', \gamma'[-1], \alpha', \beta')$ is isomorphic to $(z'[-1], x', c(\gamma'[-1]), \gamma'[-1], i, p)$. Composing the two isomorphisms with sign changes as indicated in the following diagram:

$$
\begin{array}{ccccccc}
x & \xrightarrow{\ \alpha\ } & y & \xrightarrow{\ \beta\ } & z & \xrightarrow{\ \gamma\ } & x[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
z'[-1] & \xrightarrow{-\gamma'[-1]} & x & \xrightarrow{\ \alpha'\ } & y' & \xrightarrow{\ \beta'\ } & z' \\
{\scriptstyle -1_{z'[-1]}}\downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle -1_{z'}} \\
z'[-1] & \xrightarrow{\gamma'[-1]} & x & \xrightarrow{\ \alpha'\ } & c(\gamma'[-1]) & \xrightarrow{\ -p\ } & z'
\end{array}
$$

We conclude that $(x, y, z, \alpha, \beta, \gamma)$ is distinguished by Lemma 27.13 (2). Conversely, suppose that $(x, y, z, \alpha, \beta, \gamma)$ is distinguished, so that by Lemma 27.13 (1), it is isomorphic to a triangle of the form $(x', y', c(\alpha'), \alpha', i, -p)$ for some morphism $\alpha' : x' \to y'$ in $\text{Comp}(\mathcal{A})$. The rotated triangle $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is isomorphic to the triangle $(y', c(\alpha'), x'[1], i, -p, -\alpha'[1])$ which is isomorphic to $(y', c(\alpha'), x'[1], i, p, \alpha'[1])$. By Lemma 27.10, this triangle is distinguished, from which it follows that $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is distinguished.

Proof of TR3: Suppose $(x, y, z, \alpha, \beta, \gamma)$ and $(x', y', z', \alpha', \beta', \gamma')$ are distinguished triangles of $\text{Comp}(\mathcal{A})$ and let $f : x \to x'$ and $g : y \to y'$ be morphisms such that $\alpha' \circ f = g \circ \alpha$. By Lemma 27.13, we may assume that $(x, y, z, \alpha, \beta, \gamma) =$

$(x, y, c(\alpha), \alpha, i, -p)$ and $(x', y', z', \alpha', \beta', \gamma') = (x', y', c(\alpha'), \alpha', i', -p')$. Now apply Lemma 27.3 and we are done. $\square$

The following lemma is the analogue of Lemma 10.2.

Lemma 27.15. *In Situation 27.2 given admissible monomorphisms $x \xrightarrow{\alpha} y$, $y \xrightarrow{\beta} z$ in $\mathcal{A}$, there exist distinguished triangles $(x, y, q_1, \alpha, p_1, \delta_1)$, $(x, z, q_2, \beta\alpha, p_2, \delta_2)$ and $(y, z, q_3, \beta, p_3, \delta_3)$ for which TR4 holds.*

Proof. Given admissible monomorphisms $x \xrightarrow{\alpha} y$ and $y \xrightarrow{\beta} z$, we can find distinguished triangles, via their extensions to admissible short exact sequences,

$$x \underset{\pi_1}{\overset{\alpha}{\rightleftarrows}} y \underset{s_1}{\overset{p_1}{\rightleftarrows}} q_1 \xrightarrow{\delta_1} x[1]$$

$$x \underset{\pi_1\pi_3}{\overset{\beta\alpha}{\rightleftarrows}} z \underset{s_2}{\overset{p_2}{\rightleftarrows}} q_2 \xrightarrow{\delta_2} x[1]$$

$$y \underset{\pi_3}{\overset{\beta}{\rightleftarrows}} z \underset{s_3}{\overset{p_3}{\rightleftarrows}} q_3 \xrightarrow{\delta_3} x[1]$$

In these diagrams, the maps δ_i are defined as $\delta_i = \pi_i d(s_i)$ analogous to the maps defined in Lemma 27.1. They fit in the following solid commutative diagram

where we have defined the dashed arrows as indicated. Clearly, their composition $p_3 s_2 p_2 \beta s_1 = 0$ since $s_2 p_2 = 0$. We claim that they both are morphisms of $\mathrm{Comp}(\mathcal{A})$. We can check this using equations in Lemma 27.1:

$$d(p_2 \beta s_1) = p_2 \beta d(s_1) = p_2 \beta \alpha \pi_1 d(s_1) = 0$$

since $p_2 \beta \alpha = 0$, and

$$d(p_3 s_2) = p_3 d(s_2) = p_3 \beta \alpha \pi_1 \pi_3 d(s_2) = 0$$

since $p_3 \beta = 0$. To check that $q_1 \to q_2 \to q_3$ is an admissible short exact sequence, it remains to show that in the underlying graded category, $q_2 = q_1 \oplus q_3$ with the above two morphisms as coprojection and projection. To do this, observe that in the underlying graded category $\mathcal{C}$, there hold

$$y = x \oplus q_1, \quad z = y \oplus q_3 = x \oplus q_1 \oplus q_3$$

where $\pi_1\pi_3$ gives the projection morphism onto the first factor: $x \oplus q_1 \oplus q_3 \to z$. By axiom (A) on $\mathcal{A}$, $\mathcal{C}$ is an additive category, hence we may apply Homology, Lemma 3.10 and conclude that

$$\mathrm{Ker}(\pi_1\pi_3) = q_1 \oplus q_3$$

in $\mathcal{C}$. Another application of Homology, Lemma 3.10 to $z = x \oplus q_2$ gives $\mathrm{Ker}(\pi_1\pi_3) = q_2$. Hence $q_2 \cong q_1 \oplus q_3$ in $\mathcal{C}$. It is clear that the dashed morphisms defined above give coprojection and projection.

Finally, we have to check that the morphism $\delta : q_3 \to q_1[1]$ induced by the admissible short exact sequence $q_1 \to q_2 \to q_3$ agrees with $p_1\delta_3$. By the construction in Lemma 27.1, the morphism δ is given by

$$\begin{aligned}
p_1\pi_3 s_2 d(p_2 s_3) &= p_1\pi_3 s_2 p_2 d(s_3) \\
&= p_1\pi_3(1 - \beta\alpha\pi_1\pi_3)d(s_3) \\
&= p_1\pi_3 d(s_3) \quad (\text{since } \pi_3\beta = 0) \\
&= p_1\delta_3
\end{aligned}$$

as desired. The proof is complete. $\qquad\square$

Putting everything together we finally obtain the analogue of Proposition 10.3.

Proposition 27.16. *In Situation 27.2 the homotopy category $K(\mathcal{A})$ with its natural translation functors and distinguished triangles is a triangulated category.*

Proof. By Lemma 27.14 we know that $K(\mathcal{A})$ is pre-triangulated. Combining Lemmas 27.7 and 27.15 with Derived Categories, Lemma 4.15, we conclude that $K(\mathcal{A})$ is a triangulated category. $\qquad\square$

Lemma 27.17. *Let R be a ring. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between differential graded categories over R satisfying axioms (A), (B), and (C) such that $F(x[1]) = F(x)[1]$. Then F induces an exact functor $K(\mathcal{A}) \to K(\mathcal{B})$ of triangulated categories.*

Proof. Namely, if $x \to y \to z$ is an admissible short exact sequence in $\mathrm{Comp}(\mathcal{A})$, then $F(x) \to F(y) \to F(z)$ is an admissible short exact sequence in $\mathrm{Comp}(\mathcal{B})$. Moreover, the "boundary" morphism $\delta = \pi\mathrm{d}(s) : z \to x[1]$ constructed in Lemma 27.1 produces the morphism $F(\delta) : F(z) \to F(x[1]) = F(x)[1]$ which is equal to the boundary map $F(\pi)\mathrm{d}(F(s))$ for the admissible short exact sequence $F(x) \to F(y) \to F(z)$. $\qquad\square$

28. Bimodules

We continue the discussion started in Section 12.

Definition 28.1. Bimodules. Let R be a ring.

(1) Let A and B be R-algebras. An (A, B)-*bimodule* is an R-module M equipped with R-bilinear maps

$$A \times M \to M, (a, x) \mapsto ax \quad \text{and} \quad M \times B \to M, (x, b) \mapsto xb$$

such that the following hold
  (a) $a'(ax) = (a'a)x$ and $(xb)b' = x(bb')$,
  (b) $a(xb) = (ax)b$, and
  (c) $1x = x = x1$.

(2) Let A and B be **Z**-graded R-algebras. A *graded (A, B)-bimodule* is an (A, B)-bimodule M which has a grading $M = \bigoplus M^n$ such that $A^n M^m \subset M^{n+m}$ and $M^n B^m \subset M^{n+m}$.

(3) Let A and B be differential graded R-algebras. A *differential graded (A, B)-bimodule* is a graded (A, B)-bimodule which comes equipped with a differential $d : M \to M$ homogeneous of degree 1 such that $d(ax) = d(a)x + (-1)^{\deg(a)} a d(x)$ and $d(xb) = d(x)b + (-1)^{\deg(x)} x d(b)$ for homogeneous elements $a \in A$, $x \in M$, $b \in B$.

Observe that a differential graded (A, B)-bimodule M is the same thing as a right differential graded B-module which is also a left differential graded A-module such that the grading and differentials agree and such that the A-module structure commutes with the B-module structure. Here is a precise statement.

Lemma 28.2. *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let M be a right differential graded B-module. There is a 1-to-1 correspondence between (A, B)-bimodule structures on M compatible with the given differential graded B-module structure and homomorphisms*

$$A \longrightarrow \mathrm{Hom}_{Mod^{dg}_{(B,d)}}(M, M)$$

of differential graded R-algebras.

Proof. Let $\mu : A \times M \to M$ define a left differential graded A-module structure on the underlying complex of R-modules $M^\bullet$ of M. By Lemma 13.1 the structure μ corresponds to a map $\gamma : A \to \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)$ of differential graded R-algebras. The assertion of the lemma is simply that μ commutes with the B-action, if and only if γ ends up inside

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,d)}}(M, M) \subset \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)$$

We omit the detailed calculation. $\square$

Let M be a differential graded (A, B)-bimodule. Recall from Section 11 that the left differential graded A-module structure corresponds to a right differential graded A^{opp}-module structure. Since the A and B module structures commute this gives M the structure of a differential graded $A^{opp} \otimes_R B$-module:

$$x \cdot (a \otimes b) = (-1)^{\deg(a)\deg(x)} axb$$

Conversely, if we have a differential graded $A^{opp} \otimes_R B$-module M, then we can use the formula above to get a differential graded (A, B)-bimodule.

Lemma 28.3. *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. The construction above defines an equivalence of categories*

$$\begin{array}{ccc} \textit{differential graded} & & \textit{right differential graded} \\ (A, B)\textit{-bimodules} & \longleftrightarrow & A^{opp} \otimes_R B\textit{-modules} \end{array}$$

Proof. Immediate from discussion the above. $\square$

Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let P be a differential graded (A, B)-bimodule. We say *P has property (P)* if it there exists a filtration

$$0 = F_{-1}P \subset F_0 P \subset F_1 P \subset \ldots \subset P$$

by differential graded (A, B)-bimodules such that

(1) $P = \bigcup F_p P$,
(2) the inclusions $F_i P \to F_{i+1} P$ are split as graded (A, B)-bimodule maps,
(3) the quotients $F_{i+1} P / F_i P$ are isomorphic as differential graded (A, B)-bimodules
    to a direct sum of $(A \otimes_R B)[k]$.

Lemma 28.4. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let M be a differential graded (A, B)-bimodule. There exists a homomorphism $P \to M$ of differential graded (A, B)-bimodules which is a quasi-isomorphism such that P has property (P) as defined above.*

Proof. Immediate from Lemmas 28.3 and 20.4. $\square$

Lemma 28.5. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let P be a differential graded (A, B)-bimodule having property (P) with corresponding filtration $F_\bullet$, then we obtain a short exact sequence*

$$0 \to \bigoplus F_i P \to \bigoplus F_i P \to P \to 0$$

of differential graded (A, B)-bimodules which is split as a sequence of graded (A, B)-bimodules.

Proof. Immediate from Lemmas 28.3 and 20.1. $\square$

29. Bimodules and tensor product

Let R be a ring. Let A and B be R-algebras. Let M be a right A-module. Let N be a (A, B)-bimodule. Then $M \otimes_A N$ is a right B-module.

If in the situation of the previous paragraph A and B are **Z**-graded algebras, M is a graded A-module, and N is a graded (A, B)-bimodule, then $M \otimes_A N$ is a right graded B-module. The construction is functorial in M and defines a functor

$$- \otimes_A N : \mathrm{Mod}_A^{gr} \longrightarrow \mathrm{Mod}_B^{gr}$$

of graded categories as in Example 25.6. Namely, if M and M' are graded A-modules and $f : M \to M'$ is an A-module homomorphism homogeneous of degree n, then $f \otimes \mathrm{id}_N : M \otimes_A N \to M' \otimes_A N$ is a B-module homomorphism homogeneous of degree n.

If in the situation of the previous paragraph (A, d) and (B, d) are differential graded algebras, M is a differential graded A-module, and N is a differential graded (A, B)-bimodule, then $M \otimes_A N$ is a right differential graded B-module.

Lemma 29.1. *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Then $M \mapsto M \otimes_A N$ defines a functor*

$$- \otimes_A N : Mod_{(A,d)}^{dg} \longrightarrow Mod_{(B,d)}^{dg}$$

of differential graded categories. This functor induces functors

$$Mod_{(A,d)} \to Mod_{(B,d)} \quad and \quad K(Mod_{(A,d)}) \to K(Mod_{(B,d)})$$

by an application of Lemma 26.5.

Proof. Above we have seen how the construction defines a functor of underlying graded categories. Thus it suffices to show that the construction is compatible with differentials. Let M and M' be differential graded A-modules and let $f : M \to M'$ be an A-module homomorphism which is homogeneous of degree n. Then we have

$$\mathrm{d}(f) = \mathrm{d}_{M'} \circ f - (-1)^n f \circ \mathrm{d}_M$$

On the other hand, we have

$$\mathrm{d}(f \otimes \mathrm{id}_N) = \mathrm{d}_{M' \otimes_A N} \circ (f \otimes \mathrm{id}_N) - (-1)^n (f \otimes \mathrm{id}_N) \circ \mathrm{d}_{M \otimes_A N}$$

Applying this to an element $x \otimes y$ with $x \in M$ and $y \in N$ homogeneous we get

$$
\begin{aligned}
\mathrm{d}(f \otimes \mathrm{id}_N)(x \otimes y) =& \mathrm{d}_{M'}(f(x)) \otimes y + (-1)^{n+\deg(x)} f(x) \otimes \mathrm{d}_N(y) \\
& - (-1)^n f(\mathrm{d}_M(x)) \otimes y - (-1)^{n+\deg(x)} f(x) \otimes \mathrm{d}_N(y) \\
=& \mathrm{d}(f)(x \otimes y)
\end{aligned}
$$

Thus we see that $\mathrm{d}(f) \otimes \mathrm{id}_N = \mathrm{d}(f \otimes \mathrm{id}_N)$ and the proof is complete. $\qquad\square$

Remark 29.2. Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Let M be a right differential graded A-module. Then for every $k \in \mathbf{Z}$ there is an isomorphism

$$(M \otimes_A N)[k] \longrightarrow M[k] \otimes_A N$$

of right differential graded B-modules defined without the intervention of signs, see More on Algebra, Section 72.

If we have a ring R and R-algebras A, B, and C, a right A-module M, an (A, B)-bimodule N, and a (B, C)-bimodule N', then $N \otimes_B N'$ is a (A, C)-bimodule and we have

$$(M \otimes_A N) \otimes_B N' = M \otimes_A (N \otimes_B N')$$

This equality continuous to hold in the graded and in the differential graded case. See More on Algebra, Section 72 for sign rules.

30. Bimodules and internal hom

Let R be a ring. If A is an R-algebra (see our conventions in Section 2) and M, M' are right A-modules, then we define

$$\mathrm{Hom}_A(M, M') = \{ f : M \to M' \mid f \text{ is } A\text{-linear} \}$$

as usual.

Let R-be a ring. Let A and B be R-algebras. Let N be an (A, B)-bimodule. Let N' be a right B-module. In this situation we will think of

$$\mathrm{Hom}_B(N, N')$$

as a right A-module using precomposition.

Let R-be a ring. Let A and B be $\mathbf{Z}$-graded R-algebras. Let N be a graded (A, B)-bimodule. Let N' be a right graded B-module. In this situation we will think of the graded R-module

$$\mathrm{Hom}_{\mathrm{Mod}_B^{gr}}(N, N')$$

defined in Example 25.6 as a right graded A-module using precomposition. The construction is functorial in N' and defines a functor

$$\mathrm{Hom}_{\mathrm{Mod}_B^{gr}}(N, -) : \mathrm{Mod}_B^{gr} \longrightarrow \mathrm{Mod}_A^{gr}$$

of graded categories as in Example 25.6. Namely, if N_1 and N_2 are graded B-modules and $f : N_1 \to N_2$ is a B-module homomorphism homogeneous of degree n, then the induced map $\mathrm{Hom}_{\mathrm{Mod}_B^{gr}}(N, N_1) \to \mathrm{Hom}_{\mathrm{Mod}_B^{gr}}(N, N_2)$ is an A-module homomorphism homogeneous of degree n.

Let R be a ring. Let A and B be differential $\mathbf{Z}$-graded R-algebras. Let N be a differential graded (A, B)-bimodule. Let N' be a right differential graded B-module. In this situation we will think of the differential graded R-module

$$\mathrm{Hom}_{\mathrm{Mod}_{(B,\mathrm{d})}^{dg}}(N, N')$$

defined in Example 26.8 as a right differential graded A-module using precomposition as in the graded case. This is compatible with differentials because multiplication is the composition

$$\mathrm{Hom}_{\mathrm{Mod}_B^{dg}}(N, N') \otimes_R A \to \mathrm{Hom}_{\mathrm{Mod}_B^{dg}}(N, N') \otimes_R \mathrm{Hom}_{\mathrm{Mod}_B^{dg}}(N, N) \to \mathrm{Hom}_{\mathrm{Mod}_B^{dg}}(N, N')$$

The first arrow uses the map of Lemma 28.2 and the second arrow is the composition in the differential graded category $\mathrm{Mod}_{(B,\mathrm{d})}^{dg}$.

Lemma 30.1. *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. The construction above defines a functor*

$$\mathrm{Hom}_{Mod_{(B,d)}^{dg}}(N, -) : Mod_{(B,d)}^{dg} \longrightarrow Mod_{(A,d)}^{dg}$$

of differential graded categories. This functor induces functors

$$Mod_{(B,d)} \to Mod_{(A,d)} \quad and \quad K(Mod_{(B,d)}) \to K(Mod_{(A,d)})$$

by an application of Lemma 26.5.

Proof. Above we have seen how the construction defines a functor of underlying graded categories. Thus it suffices to show that the construction is compatible with differentials. Let N_1 and N_2 be differential graded B-modules. Write

$$H_{12} = \mathrm{Hom}_{\mathrm{Mod}_{(B,\mathrm{d})}^{dg}}(N_1, N_2), \quad H_1 = \mathrm{Hom}_{\mathrm{Mod}_{(B,\mathrm{d})}^{dg}}(N, N_1), \quad H_2 = \mathrm{Hom}_{\mathrm{Mod}_{(B,\mathrm{d})}^{dg}}(N, N_2)$$

Consider the composition

$$c : H_{12} \otimes_R H_1 \longrightarrow H_2$$

in the differential graded category $\mathrm{Mod}_{(B,\mathrm{d})}^{dg}$. Let $f : N_1 \to N_2$ be a B-module homomorphism which is homogeneous of degree n, in other words, $f \in H_{12}^n$. The functor in the lemma sends f to $c_f : H_1 \to H_2$, $g \mapsto c(f, g)$. Simlarly for $\mathrm{d}(f)$. On the other hand, the differential on

$$\mathrm{Hom}_{\mathrm{Mod}_{(A,\mathrm{d})}^{dg}}(H_1, H_2)$$

sends c_f to $\mathrm{d}_{H_2} \circ c_f - (-1)^n c_f \circ \mathrm{d}_{H_1}$. As c is a morphism of complexes of R-modules we have $\mathrm{d}c(f, g) = c(\mathrm{d}f, g) + (-1)^n c(f, \mathrm{d}g)$. Hence we see that

$$\begin{aligned}
(\mathrm{d}c_f)(g) &= \mathrm{d}c(f, g) - (-1)^n c(f, \mathrm{d}g) \\
&= c(\mathrm{d}f, g) + (-1)^n c(f, \mathrm{d}g) - (-1)^n c(f, \mathrm{d}g) \\
&= c(\mathrm{d}f, g) = c_{\mathrm{d}f}(g)
\end{aligned}$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 30.2. Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Let N' be a right differential graded B-module. Then for every $k \in \mathbf{Z}$ there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}_B^{gr}}(N, N')[k] \longrightarrow \mathrm{Hom}_{\mathrm{Mod}_B^{gr}}(N, N'[k])$$

of right differential graded A-modules defined without the intervention of signs, see More on Algebra, Section 72.

Lemma 30.3. *Let R be a ring. Let A and B be R-algebras. Let M be a right A-module, N an (A, B)-bimodule, and N' a right B-module. Then we have a canonical isomorphism*

$$\mathrm{Hom}_B(M \otimes_A N, N') = \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, N'))$$

of R-modules. If A, B, M, N, N' are compatibly graded, then we have a canonical isomorphism

$$\mathrm{Hom}_{Mod_B^{gr}}(M \otimes_A N, N') = \mathrm{Hom}_{Mod_A^{gr}}(M, \mathrm{Hom}_{Mod_B^{gr}}(N, N'))$$

of graded R-modules If A, B, M, N, N' are compatibly differential graded, then we have a canonical isomorphism

$$\mathrm{Hom}_{Mod_{(B,d)}^{dg}}(M \otimes_A N, N') = \mathrm{Hom}_{Mod_{(A,d)}^{dg}}(M, \mathrm{Hom}_{Mod_{(B,d)}^{dg}}(N, N'))$$

of complexes of R-modules.

Proof. Omitted. Hint: in the ungraded case interpret both sides as A-bilinear maps $\psi : M \times N \to N'$ which are B-linear on the right. In the (differential) graded case, use the isomorphism of More on Algebra, Lemma 71.1 and check it is compatible with the module structures. Alternatively, use the isomorphism of Lemma 13.2 and show that it is compatible with the B-module structures. $\square$

31. Derived Hom

This section is analogous to More on Algebra, Section 73.

Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Consider the functor

$$(31.0.1) \qquad \mathrm{Hom}_{\mathrm{Mod}_{(B,\mathrm{d})}^{dg}}(N, -) : \mathrm{Mod}_{(B,\mathrm{d})} \longrightarrow \mathrm{Mod}_{(A,\mathrm{d})}$$

of Section 30.

Lemma 31.1. *The functor (31.0.1) defines an exact functor $K(Mod_{(B,d)}) \to K(Mod_{(A,d)})$ of triangulated categories.*

Proof. Via Lemma 30.1 and Remark 30.2 this follows from the general principle of Lemma 27.17. $\square$

Recall that we have an exact functor of triangulated categories

$$\mathrm{Hom}_{\mathrm{Mod}_{(B,\mathrm{d})}^{dg}}(N, -) : K(\mathrm{Mod}_{(B,\mathrm{d})}) \to K(\mathrm{Mod}_{(A,\mathrm{d})})$$

see Lemma 31.1. Consider the diagram

We would like to construct a dotted arrow as the *right derived functor* of the composition F. (*Warning*: in most interesting cases the diagram will not commute.) Namely, in the general setting of Derived Categories, Section 14 we want to compute the right derived functor of F with respect to the multiplicative system of quasi-isomorphisms in $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

Lemma 31.2. *In the situation above, the right derived functor of F exists. We denote it $R\operatorname{Hom}(N, -) : D(B, d) \to D(A, d)$.*

Proof. We will use Derived Categories, Lemma 14.15 to prove this. As our collection $\mathcal{I}$ of objects we will use the objects with property (I). Property (1) was shown in Lemma 21.4. Property (2) holds because if $s : I \to I'$ is a quasi-isomorphism of modules with property (I), then s is a homotopy equivalence by Lemma 22.3. $\quad\square$

Lemma 31.3. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let $f : N \to N'$ be a homomorphism of differential graded (A, B)-bimodules. Then f induces a morphism of functors*

$$- \circ f : R\operatorname{Hom}(N', -) \longrightarrow R\operatorname{Hom}(N, -)$$

If f is a quasi-isomorphism, then $f \circ -$ is an isomorphism of functors.

Proof. Write $\mathcal{B} = \mathrm{Mod}^{dg}_{(B,\mathrm{d})}$ the differential graded category of differential graded B-modules, see Example 26.8. Let I be a differential graded B-module with property (I). Then $f \circ - : \operatorname{Hom}_{\mathcal{B}}(N', I) \to \operatorname{Hom}_{\mathcal{B}}(N, I)$ is a map of differential graded A-modules. Moreover, this is functorial with respect to I. Since the functors $R\operatorname{Hom}(N', -)$ and $R\operatorname{Hom}(N, -)$ are computed by applying $\operatorname{Hom}_{\mathcal{B}}$ into objects with property (I) (Lemma 31.2) we obtain a transformation of functors as indicated.

Assume that f is a quasi-isomorphism. Let $F_\bullet$ be the given filtration on I. Since $I = \lim I/F_p I$ we see that $\operatorname{Hom}_{\mathcal{B}}(N', I) = \lim \operatorname{Hom}_{\mathcal{B}}(N', I/F_p I)$ and $\operatorname{Hom}_{\mathcal{B}}(N, I) = \lim \operatorname{Hom}_{\mathcal{B}}(N, I/F_p I)$. Since the transition maps in the system $I/F_p I$ are split as graded modules, we see that the transition maps in the systems $\operatorname{Hom}_{\mathcal{B}}(N', I/F_p I)$ and $\operatorname{Hom}_{\mathcal{B}}(N, I/F_p I)$ are surjective. Hence $\operatorname{Hom}_{\mathcal{B}}(N', I)$, resp. $\operatorname{Hom}_{\mathcal{B}}(N, I)$ viewed as a complex of abelian groups computes $R\lim$ of the system of complexes $\operatorname{Hom}_{\mathcal{B}}(N', I/F_p I)$, resp. $\operatorname{Hom}_{\mathcal{B}}(N, I/F_p I)$. See More on Algebra, Lemma 86.1. Thus it suffices to prove each

$$\operatorname{Hom}_{\mathcal{B}}(N', I/F_p I) \to \operatorname{Hom}_{\mathcal{B}}(N, I/F_p I)$$

is a quasi-isomorphism. Since the surjections $I/F_{p+1} I \to I/F_p I$ are split as maps of graded B-modules we see that

$$0 \to \operatorname{Hom}_{\mathcal{B}}(N', F_p I/F_{p+1} I) \to \operatorname{Hom}_{\mathcal{B}}(N', I/F_{p+1} I) \to \operatorname{Hom}_{\mathcal{B}}(N', I/F_p I) \to 0$$

is a short exact sequence of differential graded A-modules. There is a similar sequence for N and f induces a map of short exact sequences. Hence by induction on p (starting with $p = 0$ when $I/F_0 I = 0$) we conclude that it suffices to show that the map $\operatorname{Hom}_{\mathcal{B}}(N', F_p I/F_{p+1} I) \to \operatorname{Hom}_{\mathcal{B}}(N, F_p I/F_{p+1} I)$ is a quasi-isomorphism. Since $F_p I/F_{p+1} I$ is a product of shifts of $A^\vee$ it suffice to prove $\operatorname{Hom}_{\mathcal{B}}(N', B^\vee[k]) \to \operatorname{Hom}_{\mathcal{B}}(N, B^\vee[k])$ is a quasi-isomorphism. By Lemma 19.3 it suffices to show $(N')^\vee \to N^\vee$ is a quasi-isomorphism. This is true because f is a quasi-isomorphism and $(\ )^\vee$ is an exact functor. $\quad\square$

Lemma 31.4. *Let (A, d) and (B, d) be differential graded algebras over a ring R. Let N be a differential graded (A, B)-bimodule. Then for every $n \in \mathbf{Z}$ there are isomorphisms*

$$H^n(R\operatorname{Hom}(N, M)) = \operatorname{Ext}^n_{D(B,d)}(N, M)$$

of R-modules functorial in M. It is also functorial in N with respect to the operation described in Lemma 31.3.

Proof. In the proof of Lemma 31.2 we have seen

$$R\operatorname{Hom}(N, M) = \operatorname{Hom}_{\operatorname{Mod}^{dg}_{(B,d)}}(N, I)$$

as a differential graded A-module where $M \to I$ is a quasi-isomorphism of M into a differential graded B-module with property (I). Hence this complex has the correct cohomology modules by Lemma 22.3. We omit a discussion of the functorial nature of these identifications. $\square$

Lemma 31.5. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. If $\operatorname{Hom}_{D(B,d)}(N, N') = \operatorname{Hom}_{K(\operatorname{Mod}_{(B,d)})}(N, N')$ for all $N' \in K(B, d)$, for example if N has property (P) as a differential graded B-module, then*

$$R\operatorname{Hom}(N, M) = \operatorname{Hom}_{\operatorname{Mod}^{dg}_{(B,d)}}(N, M)$$

functorially in M in $D(B, d)$.

Proof. By construction (Lemma 31.2) to find $R\operatorname{Hom}(N, M)$ we choose a quasi-isomorphism $M \to I$ where I is a differential graded B-module with property (I) and we set $R\operatorname{Hom}(N, M) = \operatorname{Hom}_{\operatorname{Mod}^{dg}_{(B,d)}}(N, I)$. By assumption the map

$$\operatorname{Hom}_{\operatorname{Mod}^{dg}_{(B,d)}}(N, M) \longrightarrow \operatorname{Hom}_{\operatorname{Mod}^{dg}_{(B,d)}}(N, I)$$

induced by $M \to I$ is a quasi-isomorphism, see discussion in Example 26.8. This proves the lemma. If N has property (P) as a B-module, then we see that the assumption is satisfied by Lemma 22.3. $\square$

32. Variant of derived Hom

Let $\mathcal{A}$ be an abelian category. Consider the differential graded category $\operatorname{Comp}^{dg}(\mathcal{A})$ of complexes of $\mathcal{A}$, see Example 26.6. Let $K^\bullet$ be a complex of $\mathcal{A}$. Set

$$(E, d) = \operatorname{Hom}_{\operatorname{Comp}^{dg}(\mathcal{A})}(K^\bullet, K^\bullet)$$

and consider the functor of differential graded categories

$$\operatorname{Comp}^{dg}(\mathcal{A}) \longrightarrow \operatorname{Mod}^{dg}_{(E,d)}, \quad X^\bullet \longmapsto \operatorname{Hom}_{\operatorname{Comp}^{dg}(\mathcal{A})}(K^\bullet, X^\bullet)$$

of Lemma 26.10.

Lemma 32.1. *In the situation above. If the right derived functor $R\operatorname{Hom}(K^\bullet, -)$ of $\operatorname{Hom}(K^\bullet, -) : K(\mathcal{A}) \to D(Ab)$ is everywhere defined on $D(\mathcal{A})$, then we obtain a canonical exact functor*

$$R\operatorname{Hom}(K^\bullet, -) : D(\mathcal{A}) \longrightarrow D(E, d)$$

of triangulated categories which reduces to the usual one on taking associated complexes of abelian groups.

Proof. Note that we have an associated functor $K(\mathcal{A}) \to K(\mathrm{Mod}_{(E,\mathrm{d})})$ by Lemma 26.10. We claim this functor is an exact functor of triangulated categories. Namely, let $f : A^\bullet \to B^\bullet$ be a map of complexes of $\mathcal{A}$. Then a computation shows that

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, C(f)^\bullet) = C\left(\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, A^\bullet) \to \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, B^\bullet)\right)$$

where the right hand side is the cone in $\mathrm{Mod}_{(E,\mathrm{d})}$ defined earlier in this chapter. This shows that our functor is compatible with cones, hence with distinguished triangles. Let $X^\bullet$ be an object of $K(\mathcal{A})$. Consider the category of quasi-isomorphisms $s : X^\bullet \to Y^\bullet$. We are given that the functor $(s : X^\bullet \to Y^\bullet) \mapsto \mathrm{Hom}_{\mathcal{A}}(K^\bullet, Y^\bullet)$ is essentially constant when viewed in $D(Ab)$. But since the forgetful functor $D(E,\mathrm{d}) \to D(Ab)$ is compatible with taking cohomology, the same thing is true in $D(E,\mathrm{d})$. This proves the lemma. $\square$

Warning: Although the lemma holds as stated and may be useful as stated, the differential algebra E isn't the "correct" one unless $H^n(E) = \mathrm{Ext}^n_{D(\mathcal{A})}(K^\bullet, K^\bullet)$ for all $n \in \mathbf{Z}$.

33. Derived tensor product

This section is analogous to More on Algebra, Section 60.

Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Consider the functor

$$(33.0.1) \qquad \mathrm{Mod}_{(A,\mathrm{d})} \longrightarrow \mathrm{Mod}_{(B,\mathrm{d})}, \quad M \longmapsto M \otimes_A N$$

defined in Section 29.

Lemma 33.1. *The functor (33.0.1) defines an exact functor of triangulated categories $K(Mod_{(A,d)}) \to K(Mod_{(B,d)})$.*

Proof. Via Lemma 29.1 and Remark 29.2 this follows from the general principle of Lemma 27.17. $\square$

At this point we can consider the diagram

$$\begin{array}{ccc}
K(\mathrm{Mod}_{(A,\mathrm{d})}) & \xrightarrow{\ -\otimes_A N\ } & K(\mathrm{Mod}_{(B,\mathrm{d})}) \\
\downarrow & \searrow^{F} & \downarrow \\
D(A, \mathrm{d}) & \dashrightarrow & D(B, \mathrm{d})
\end{array}$$

The dotted arrow that we will construct below will be the *left derived functor* of the composition F. (*Warning*: the diagram will not commute.) Namely, in the general setting of Derived Categories, Section 14 we want to compute the left derived functor of F with respect to the multiplicative system of quasi-isomorphisms in $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

Lemma 33.2. *In the situation above, the left derived functor of F exists. We denote it $- \otimes^{\mathbf{L}}_A N : D(A, d) \to D(B, d)$.*

Proof. We will use Derived Categories, Lemma 14.15 to prove this. As our collection $\mathcal{P}$ of objects we will use the objects with property (P). Property (1) was shown in Lemma 20.4. Property (2) holds because if $s : P \to P'$ is a quasi-isomorphism of modules with property (P), then s is a homotopy equivalence by Lemma 22.3. $\square$

Lemma 33.3. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let $f : N \to N'$ be a homomorphism of differential graded (A, B)-bimodules. Then f induces a morphism of functors*

$$1 \otimes f : - \otimes_A^{\mathbf{L}} N \longrightarrow - \otimes_A^{\mathbf{L}} N'$$

If f is a quasi-isomorphism, then $1 \otimes f$ is an isomorphism of functors.

Proof. Let M be a differential graded A-module with property (P). Then $1 \otimes f : M \otimes_A N \to M \otimes_A N'$ is a map of differential graded B-modules. Moreover, this is functorial with respect to M. Since the functors $- \otimes_A^{\mathbf{L}} N$ and $- \otimes_A^{\mathbf{L}} N'$ are computed by tensoring on objects with property (P) (Lemma 33.2) we obtain a transformation of functors as indicated.

Assume that f is a quasi-isomorphism. Let $F_\bullet$ be the given filtration on M. Observe that $M \otimes_A N = \operatorname{colim} F_i(M) \otimes_A N$ and $M \otimes_A N' = \operatorname{colim} F_i(M) \otimes_A N'$. Hence it suffices to show that $F_n(M) \otimes_A N \to F_n(M) \otimes_A N'$ is a quasi-isomorphism (filtered colimits are exact, see Algebra, Lemma 8.8). Since the inclusions $F_n(M) \to F_{n+1}(M)$ are split as maps of graded A-modules we see that

$$0 \to F_n(M) \otimes_A N \to F_{n+1}(M) \otimes_A N \to F_{n+1}(M)/F_n(M) \otimes_A N \to 0$$

is a short exact sequence of differential graded B-modules. There is a similar sequence for N' and f induces a map of short exact sequences. Hence by induction on n (starting with $n = -1$ when $F_{-1}(M) = 0$) we conclude that it suffices to show that the map $F_{n+1}(M)/F_n(M) \otimes_A N \to F_{n+1}(M)/F_n(M) \otimes_A N'$ is a quasi-isomorphism. This is true because $F_{n+1}(M)/F_n(M)$ is a direct sum of shifts of A and the result is true for $A[k]$ as $f : N \to N'$ is a quasi-isomorphism. $\square$

Lemma 33.4. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule which has property (P) as a left differential graded A-module. Then $M \otimes_A^{\mathbf{L}} N$ is computed by $M \otimes_A N$ for all differential graded A-modules M.*

Proof. Let $f : M \to M'$ be a homomorphism of differential graded A-modules which is a quasi-isomorphism. We claim that $f \otimes \operatorname{id} : M \otimes_A N \to M' \otimes_A N$ is a quasi-isomorphism. If this is true, then by the construction of the derived tensor product in the proof of Lemma 33.2 we obtain the desired result. The construction of the map $f \otimes \operatorname{id}$ only depends on the left differential graded A-module structure on N. Moreover, we have $M \otimes_A N = N \otimes_{A^{opp}} M = N \otimes_{A^{opp}}^{\mathbf{L}} M$ because N has property (P) as a differential graded A^{opp}-module. Hence the claim follows from Lemma 33.3. $\square$

Lemma 33.5. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. Then the functor*

$$- \otimes_A^{\mathbf{L}} N : D(A, d) \longrightarrow D(B, d)$$

of Lemma 33.2 is a left adjoint to the functor

$$R \operatorname{Hom}(N, -) : D(B, d) \longrightarrow D(A, d)$$

of Lemma 31.2.

Proof. This follows from Derived Categories, Lemma 30.1 and the fact that $- \otimes_A N$ and $\operatorname{Hom}_{\operatorname{Mod}_{(B,\mathrm{d})}^{dg}}(N, -)$ are adjoint by Lemma 30.3. $\square$

Example 33.6. Let R be a ring. Let $(A, \mathrm{d}) \to (B, \mathrm{d})$ be a homomorphism of differential graded R-algebras. Then we can view B as a differential graded (A, B)-bimodule and we get a functor

$$- \otimes_A B : D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$$

By Lemma 33.5 the left adjoint of this is the functor $R\operatorname{Hom}(B, -)$. For a differential graded B-module let us denote N_A the differential graded A-module obtained from N by restriction via $A \to B$. Then we clearly have a canonical isomorphism

$$\operatorname{Hom}_{\operatorname{Mod}^{dg}_{(B,\mathrm{d})}}(B, N) \longrightarrow N_A, \quad f \longmapsto f(1)$$

functorial in the B-module N. Thus we see that $R\operatorname{Hom}(B, -)$ is the restriction functor and we obtain

$$\operatorname{Hom}_{D(A,\mathrm{d})}(M, N_A) = \operatorname{Hom}_{D(B,\mathrm{d})}(M \otimes_A^{\mathbf{L}} B, N)$$

bifunctorially in M and N exactly as in the case of commutative rings. Finally, observe that restriction is a tensor functor as well, since $N_A = N \otimes_B {}_B B_A = N \otimes_B^{\mathbf{L}} {}_B B_A$ where ${}_B B_A$ is B viewed as a differential graded (B, A)-bimodule.

Lemma 33.7. *With notation and assumptions as in Lemma 33.5. Assume*

(1) *N defines a compact object of $D(B, d)$, and*
(2) *the map $H^k(A) \to \operatorname{Hom}_{D(B,d)}(N, N[k])$ is an isomorphism for all $k \in \mathbf{Z}$.*

Then the functor $- \otimes_A^{\mathbf{L}} N$ is fully faithful.

Proof. Our functor has a left adjoint given by $R\operatorname{Hom}(N, -)$ by Lemma 33.5. By Categories, Lemma 24.4 it suffices to show that for a differential graded A-module M the map

$$M \longrightarrow R\operatorname{Hom}(N, M \otimes_A^{\mathbf{L}} N)$$

is an isomorphism in $D(A, \mathrm{d})$. For this it suffices to show that

$$H^n(M) \longrightarrow \operatorname{Ext}^n_{D(B,\mathrm{d})}(N, M \otimes_A^{\mathbf{L}} N)$$

is an isomorphism, see Lemma 31.4. Since N is a compact object the right hand side commutes with direct sums. Thus by Remark 22.5 it suffices to prove this map is an isomorphism for $M = A[k]$. Since $(A[k] \otimes_A^{\mathbf{L}} N) = N[k]$ by Remark 29.2, assumption (2) on N is that the result holds for these. $\qquad\square$

Lemma 33.8. *Let $R \to R'$ be a ring map. Let (A, d) be a differential graded R-algebra. Let (A', d) be the base change, i.e., $A' = A \otimes_R R'$. If A is K-flat as a complex of R-modules, then*

(1) *$- \otimes_A^{\mathbf{L}} A' : D(A, d) \to D(A', d)$ is equal to the right derived functor of*

$$K(A, d) \longrightarrow K(A', d), \quad M \longmapsto M \otimes_R R'$$

(2) *the diagram*

$$
\begin{array}{ccc}
D(A, d) & \xrightarrow{\ -\otimes_A^{\mathbf{L}} A'\ } & D(A', d) \\
{\scriptstyle restriction}\downarrow & & \downarrow{\scriptstyle restriction} \\
D(R) & \xrightarrow{\ -\otimes_R^{\mathbf{L}} R'\ } & D(R')
\end{array}
$$

commutes, and
(3) *if M is K-flat as a complex of R-modules, then the differential graded A'-module $M \otimes_R R'$ represents $M \otimes_A^{\mathbf{L}} A'$.*

Proof. For any differential graded A-module M there is a canonical map

$$c_M : M \otimes_R R' \longrightarrow M \otimes_A A'$$

Let P be a differential graded A-module with property (P). We claim that c_P is an isomorphism and that P is K-flat as a complex of R-modules. This will prove all the results stated in the lemma by formal arguments using the definition of derived tensor product in Lemma 33.2 and More on Algebra, Section 59.

Let $F_\bullet$ be the filtration on P showing that P has property (P). Note that c_A is an isomorphism and A is K-flat as a complex of R-modules by assumption. Hence the same is true for direct sums of shifts of A (you can use More on Algebra, Lemma 59.8 to deal with direct sums if you like). Hence this holds for the complexes $F_{p+1}P/F_pP$. Since the short exact sequences

$$0 \to F_pP \to F_{p+1}P \to F_{p+1}P/F_pP \to 0$$

are split exact as sequences of graded modules, we can argue by induction that c_{F_pP} is an isomorphism for all p and that F_pP is K-flat as a complex of R-modules (use More on Algebra, Lemma 59.5). Finally, using that $P = \operatorname{colim} F_pP$ we conclude that c_P is an isomorphism and that P is K-flat as a complex of R-modules (use More on Algebra, Lemma 59.8). $\square$

Lemma 33.9. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let T be a differential graded (A, B)-bimodule. Assume*

(1) *T defines a compact object of $D(B, d)$, and*
(2) *$S = \operatorname{Hom}_{Mod^{dg}_{(B,d)}}(T, B)$ represents $R\operatorname{Hom}(T, B)$ in $D(A, d)$.*

Then S has a structure of a differential graded (B, A)-bimodule and there is an isomorphism

$$N \otimes_B^{\mathbf{L}} S \longrightarrow R\operatorname{Hom}(T, N)$$

functorial in N in $D(B, d)$.

Proof. Write $\mathcal{B} = \operatorname{Mod}^{dg}_{(B,\mathrm{d})}$. The right A-module structure on S comes from the map $A \to \operatorname{Hom}_\mathcal{B}(T, T)$ and the composition $\operatorname{Hom}_\mathcal{B}(T, B) \otimes \operatorname{Hom}_\mathcal{B}(T, T) \to \operatorname{Hom}_\mathcal{B}(T, B)$ defined in Example 26.8. Using this multiplication a second time there is a map

$$c_N : N \otimes_B S = \operatorname{Hom}_\mathcal{B}(B, N) \otimes_B \operatorname{Hom}_\mathcal{B}(T, B) \longrightarrow \operatorname{Hom}_\mathcal{B}(T, N)$$

functorial in N. Given N we can choose quasi-isomorphisms $P \to N \to I$ where P, resp. I is a differential graded B-module with property (P), resp. (I). Then using c_N we obtain a map $P \otimes_B S \to \operatorname{Hom}_\mathcal{B}(T, I)$ between the objects representing $S \otimes_B^{\mathbf{L}} N$ and $R\operatorname{Hom}(T, N)$. Clearly this defines a transformation of functors c as in the lemma.

To prove that c is an isomorphism of functors, we may assume N is a differential graded B-module which has property (P). Since T defines a compact object in $D(B, \mathrm{d})$ and since both sides of the arrow define exact functors of triangulated categories, we reduce using Lemma 20.1 to the case where N has a finite filtration whose graded pieces are direct sums of $B[k]$. Using again that both sides of the arrow are exact functors of triangulated categories and compactness of T we reduce to the case $N = B[k]$. Assumption (2) is exactly the assumption that c is an isomorphism in this case. $\square$

34. Composition of derived tensor products

We encourage the reader to skip this section.

Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-module. We denote N_B the bimodule N viewed as a differential graded B-module (forgetting about the A-structure). There is a canonical map

$$(34.0.1) \qquad\qquad N_B \otimes_B^{\mathbf{L}} N' \longrightarrow (N \otimes_B N')_C$$

in $D(C, \mathrm{d})$. Here $(N \otimes_B N')_C$ denotes the (A, C)-bimodule $N \otimes_B N'$ viewed as a differential graded C-module. Namely, this map comes from the fact that the derived tensor product always maps to the plain tensor product (as it is a left derived functor).

Lemma 34.1. *Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-module. Assume (34.0.1) is an isomorphism. Then the composition*

$$D(A, d) \xrightarrow{\ -\otimes_A^{\mathbf{L}} N\ } D(B, d) \xrightarrow{\ -\otimes_B^{\mathbf{L}} N'\ } D(C, d)$$

is isomorphic to $- \otimes_A^{\mathbf{L}} N''$ with $N'' = N \otimes_B N'$ viewed as (A, C)-bimodule.

Proof. Let us define a transformation of functors

$$(- \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} N' \longrightarrow - \otimes_A^{\mathbf{L}} N''$$

To do this, let M be a differential graded A-module with property (P). According to the construction of the functor $- \otimes_A^{\mathbf{L}} N''$ of the proof of Lemma 33.2 the plain tensor product $M \otimes_A N''$ represents $M \otimes_A^{\mathbf{L}} N''$ in $D(C, \mathrm{d})$. Then we write

$$M \otimes_A N'' = M \otimes_A (N \otimes_B N') = (M \otimes_A N) \otimes_B N'$$

The module $M \otimes_A N$ represents $M \otimes_A^{\mathbf{L}} N$ in $D(B, \mathrm{d})$. Choose a quasi-isomorphism $Q \to M \otimes_A N$ where Q is a differential graded B-module with property (P). Then $Q \otimes_B N'$ represents $(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} N'$ in $D(C, \mathrm{d})$. Thus we can define our map via

$$(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} N' = Q \otimes_B N' \to M \otimes_A N \otimes_B N' = M \otimes_A^{\mathbf{L}} N''$$

The construction of this map is functorial in M and compatible with distinguished triangles and direct sums; we omit the details. Consider the property T of objects M of $D(A, \mathrm{d})$ expressing that this map is an isomorphism. Then

1. if T holds for M_i then T holds for $\bigoplus M_i$,
2. if T holds for 2-out-of-3 in a distinguished triangle, then it holds for the third, and
3. T holds for $A[k]$ because here we obtain a shift of the map (34.0.1) which we have assumed is an isomorphism.

Thus by Remark 22.5 property T always holds and the proof is complete. $\qquad\square$

Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. We temporarily denote $(A \otimes_R B)_B$ the differential graded algebra $A \otimes_R B$ viewed as a (right) differential graded B-module, and $_B(B \otimes_R C)_C$ the differential graded algebra $B \otimes_R C$ viewed as a differential graded (B, C)-bimodule. Then there is a canonical map

$$(34.1.1) \qquad (A \otimes_R B)_B \otimes_B^{\mathbf{L}} {}_B(B \otimes_R C)_C \longrightarrow (A \otimes_R B \otimes_R C)_C$$

in $D(C, \mathrm{d})$ where $(A \otimes_R B \otimes_R C)_C$ denotes the differential graded R-algebra $A \otimes_R B \otimes_R C$ viewed as a (right) differential graded C-module. Namely, this map comes from the identification

$$(A \otimes_R B)_B \otimes_B {}_B(B \otimes_R C)_C = (A \otimes_R B \otimes_R C)_C$$

and the fact that the derived tensor product always maps to the plain tensor product (as it is a left derived functor).

Lemma 34.2. *Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Assume that (34.1.1) is an isomorphism. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-bimodule. Then the composition*

$$D(A, d) \xrightarrow{\ -\otimes_A^{\mathbf{L}} N\ } D(B, d) \xrightarrow{\ -\otimes_B^{\mathbf{L}} N'\ } D(C, d)$$

is isomorphic to $- \otimes_A^{\mathbf{L}} N''$ for a differential graded (A, C)-bimodule N'' described in the proof.

Proof. By Lemma 33.3 we may replace N and N' by quasi-isomorphic bimodules. Thus we may assume N, resp. N' has property (P) as differential graded (A, B)-bimodule, resp. (B, C)-bimodule, see Lemma 28.4. We claim the lemma holds with the (A, C)-bimodule $N'' = N \otimes_B N'$. To prove this, it suffices to show that

$$N_B \otimes_B^{\mathbf{L}} N' \longrightarrow (N \otimes_B N')_C$$

is an isomorphism in $D(C, \mathrm{d})$, see Lemma 34.1.

Let $F_\bullet$ be the filtration on N as in property (P) for bimodules. By Lemma 28.5 there is a short exact sequence

$$0 \to \bigoplus F_i N \to \bigoplus F_i N \to N \to 0$$

of differential graded (A, B)-bimodules which is split as a sequence of graded (A, B)-bimodules. A fortiori this is an admissible short exact sequence of differential graded B-modules and this produces a distinguished triangle

$$\bigoplus F_i N_B \to \bigoplus F_i N_B \to N_B \to \bigoplus F_i N_B[1]$$

in $D(B, \mathrm{d})$. Using that $- \otimes_B^{\mathbf{L}} N'$ is an exact functor of triangulated categories and commutes with direct sums and using that $- \otimes_B N'$ transforms admissible exact sequences into admissible exact sequences and commutes with direct sums we reduce to proving that

$$(F_p N)_B \otimes_B^{\mathbf{L}} N' \longrightarrow (F_p N)_B \otimes_B N'$$

is a quasi-isomorphism for all p. Repeating the argument with the short exact sequences of (A, B)-bimodules

$$0 \to F_p N \to F_{p+1} N \to F_{p+1} N / F_p N \to 0$$

which are split as graded (A, B)-bimodules we reduce to showing the same statement for $F_{p+1} N / F_p N$. Since these modules are direct sums of shifts of $(A \otimes_R B)_B$ we reduce to showing that

$$(A \otimes_R B)_B \otimes_B^{\mathbf{L}} N' \longrightarrow (A \otimes_R B)_B \otimes_B N'$$

is a quasi-isomorphism.

Choose a filtration $F_\bullet$ on N' as in property (P) for bimodules. Choose a quasi-isomorphism $P \to (A \otimes_R B)_B$ of differential graded B-modules where P has property (P). We have to show that $P \otimes_B N' \to (A \otimes_R B)_B \otimes_B N'$ is a quasi-isomorphism because $P \otimes_B N'$ represents $(A \otimes_R B)_B \otimes_B^{\mathbf{L}} N'$ in $D(C, \mathrm{d})$ by the construction in Lemma 33.2. As $N' = \mathrm{colim}\, F_p N'$ we find that it suffices to show that $P \otimes_B F_p N' \to (A \otimes_R B)_B \otimes_B F_p N'$ is a quasi-isomorphism. Using the short exact sequences $0 \to F_p N' \to F_{p+1} N' \to F_{p+1} N'/F_p N' \to 0$ which are split as graded (B, C)-bimodules we reduce to showing $P \otimes_B F_{p+1} N'/F_p N' \to (A \otimes_R B)_B \otimes_B F_{p+1} N'/F_p N'$ is a quasi-isomorphism for all p. Then finally using that $F_{p+1} N'/F_p N'$ is a direct sum of shifts of ${}_B(B \otimes_R C)_C$ we conclude that it suffices to show that

$$P \otimes_B {}_B(B \otimes_R C)_C \to (A \otimes_R B)_B \otimes_B {}_B(B \otimes_R C)_C$$

is a quasi-isomorphism. Since $P \to (A \otimes_R B)_B$ is a resolution by a module satisfying property (P) this map of differential graded C-modules represents the morphism (34.1.1) in $D(C, \mathrm{d})$ and the proof is complete. $\qquad\square$

Lemma 34.3. *Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. If C is K-flat as a complex of R-modules, then (34.1.1) is an isomorphism and the conclusion of Lemma 34.2 is valid.*

Proof. Choose a quasi-isomorphism $P \to (A \otimes_R B)_B$ of differential graded B-modules, where P has property (P). Then we have to show that

$$P \otimes_B (B \otimes_R C) \longrightarrow (A \otimes_R B) \otimes_B (B \otimes_R C)$$

is a quasi-isomorphism. Equivalently we are looking at

$$P \otimes_R C \longrightarrow A \otimes_R B \otimes_R C$$

This is a quasi-isomorphism if C is K-flat as a complex of R-modules by More on Algebra, Lemma 59.2. $\qquad\square$

35. Variant of derived tensor product

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Then we have the functors

$$\mathrm{Comp}(\mathcal{O}) \to K(\mathcal{O}) \to D(\mathcal{O})$$

and as we've seen above we have differential graded enhancement $\mathrm{Comp}^{dg}(\mathcal{O})$. Namely, this is the differential graded category of Example 26.6 associated to the abelian category $\mathit{Mod}(\mathcal{O})$. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules in other words, an object of $\mathrm{Comp}^{dg}(\mathcal{O})$. Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(K^\bullet, K^\bullet)$$

This is a differential graded $\mathbf{Z}$-algebra. We claim there is an analogue of the derived base change in this situation.

Lemma 35.1. *In the situation above there is a functor*

$$- \otimes_E K^\bullet : \mathit{Mod}^{dg}_{(E, d)} \longrightarrow \mathrm{Comp}^{dg}(\mathcal{O})$$

of differential graded categories. This functor sends E to $K^\bullet$ and commutes with direct sums.

Proof. Let M be a differential graded E-module. For every object U of $\mathcal{C}$ the complex $K^\bullet(U)$ is a left differential graded E-module as well as a right $\mathcal{O}(U)$-module. The actions commute, so we have a bimodule. Thus, by the constructions in Sections 12 and 28 we can form the tensor product

$$M \otimes_E K^\bullet(U)$$

which is a differential graded $\mathcal{O}(U)$-module, i.e., a complex of $\mathcal{O}(U)$-modules. This construction is functorial with respect to U, hence we can sheafify to get a complex of $\mathcal{O}$-modules which we denote

$$M \otimes_E K^\bullet$$

Moreover, for each U the construction determines a functor $\mathrm{Mod}^{dg}_{(E,\mathrm{d})} \to \mathrm{Comp}^{dg}(\mathcal{O}(U))$ of differential graded categories by Lemma 29.1. It is therefore clear that we obtain a functor as stated in the lemma. $\square$

Lemma 35.2. *The functor of Lemma 35.1 defines an exact functor of triangulated categories $K(Mod_{(E,d)}) \to K(\mathcal{O})$.*

Proof. The functor induces a functor between homotopy categories by Lemma 26.5. We have to show that $- \otimes_E K^\bullet$ transforms distinguished triangles into distinguished triangles. Suppose that $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded E-modules. Let $s : M \to L$ be a graded E-module homomorphism which is left inverse to $L \to M$. Then s defines a map $M \otimes_E K^\bullet \to L \otimes_E K^\bullet$ of graded $\mathcal{O}$-modules (i.e., respecting $\mathcal{O}$-module structure and grading, but not differentials) which is left inverse to $L \otimes_E K^\bullet \to M \otimes_E K^\bullet$. Thus we see that

$$0 \to K \otimes_E K^\bullet \to L \otimes_E K^\bullet \to M \otimes_E K^\bullet \to 0$$

is a termwise split short exact sequences of complexes, i.e., a defines a distinguished triangle in $K(\mathcal{O})$. $\square$

Lemma 35.3. *The functor $K(Mod_{(E,d)}) \to K(\mathcal{O})$ of Lemma 35.2 has a left derived version defined on all of $D(E,d)$. We denote it $- \otimes^{\mathbf{L}}_E K^\bullet : D(E,d) \to D(\mathcal{O})$.*

Proof. We will use Derived Categories, Lemma 14.15 to prove this. As our collection $\mathcal{P}$ of objects we will use the objects with property (P). Property (1) was shown in Lemma 20.4. Property (2) holds because if $s : P \to P'$ is a quasi-isomorphism of modules with property (P), then s is a homotopy equivalence by Lemma 22.3. $\square$

Lemma 35.4. *Let R be a ring. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of commutative R-algebras. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. The functor of Lemma 35.3 has the following property: For every M, N in $D(E,d)$ there is a canonical map*

$$R \operatorname{Hom}(M, N) \longrightarrow R \operatorname{Hom}_{\mathcal{O}}(M \otimes^{\mathbf{L}}_E K^\bullet, N \otimes^{\mathbf{L}}_E K^\bullet)$$

in $D(R)$ which on cohomology modules gives the maps

$$\operatorname{Ext}^n_{D(E,d)}(M, N) \to \operatorname{Ext}^n_{D(\mathcal{O})}(M \otimes^{\mathbf{L}}_E K^\bullet, N \otimes^{\mathbf{L}}_E K^\bullet)$$

induced by the functor $- \otimes^{\mathbf{L}}_E K^\bullet$.

Proof. The right hand side of the arrow is the global derived hom introduced in Cohomology on Sites, Section 36 which has the correct cohomology modules. For the left hand side we think of M as a (R, A)-bimodule and we have the derived Hom introduced in Section 31 which also has the correct cohomology modules.

To prove the lemma we may assume M and N are differential graded E-modules with property (P); this does not change the left hand side of the arrow by Lemma 31.3. By Lemma 31.5 this means that the left hand side of the arrow becomes $\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(M, N)$. In Lemmas 35.1, 35.2, and 35.3 we have constructed a functor

$$- \otimes_E K^\bullet : \mathrm{Mod}^{dg}_{(E,\mathrm{d})} \longrightarrow \mathrm{Comp}^{dg}(\mathcal{O})$$

of differential graded categories and we have shown that $- \otimes^{\mathbf{L}}_E K^\bullet$ is computed by evaluating this functor on differential graded E-modules with property (P). Hence we obtain a map of complexes of R-modules

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(M, N) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(M \otimes_E K^\bullet, N \otimes_E K^\bullet)$$

For any complexes of $\mathcal{O}$-modules $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ there is a canonical map

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) \longrightarrow R\,\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Combining these maps we obtain the desired map of the lemma. $\qquad\square$

Lemma 35.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. Then the functor*

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \longrightarrow D(\mathcal{O})$$

of Lemma 35.3 is a left adjoint of the functor

$$R\,\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}) \longrightarrow D(E, d)$$

of Lemma 32.1.

Proof. The statement means that we have

$$\mathrm{Hom}_{D(E,\mathrm{d})}(M, R\,\mathrm{Hom}(K^\bullet, L^\bullet)) = \mathrm{Hom}_{D(\mathcal{O})}(M \otimes^{\mathbf{L}}_E K^\bullet, L^\bullet)$$

bifunctorially in M and $L^\bullet$. To see this we may replace M by a differential graded E-module P with property (P). We also may replace $L^\bullet$ by a K-injective complex of $\mathcal{O}$-modules $I^\bullet$. The computation of the derived functors given in the lemmas referenced in the statement combined with Lemma 22.3 translates the above into

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(E,\mathrm{d})})}(P, \mathrm{Hom}_{\mathcal{B}}(K^\bullet, I^\bullet)) = \mathrm{Hom}_{K(\mathcal{O})}(P \otimes_E K^\bullet, I^\bullet)$$

where $\mathcal{B} = \mathrm{Comp}^{dg}(\mathcal{O})$. There is an evaluation map from right to left functorial in P and $I^\bullet$ (details omitted). Choose a filtration $F_\bullet$ on P as in the definition of property (P). By Lemma 20.1 and the fact that both sides of the equation are homological functors in P on $K(\mathrm{Mod}_{(E,\mathrm{d})})$ we reduce to the case where P is replaced by the differential graded E-module $\bigoplus F_i P$. Since both sides turn direct sums in the variable P into direct products we reduce to the case where P is one of the differential graded E-modules $F_i P$. Since each $F_i P$ has a finite filtration (given by admissible monomorphisms) whose graded pieces are graded projective E-modules we reduce to the case where P is a graded projective E-module. In this case we clearly have

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(E,\mathrm{d})}}(P, \mathrm{Hom}_{\mathcal{B}}(K^\bullet, I^\bullet)) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(P \otimes_E K^\bullet, I^\bullet)$$

as graded $\mathbf{Z}$-modules (because this statement reduces to the case $P = E[k]$ where it is obvious). As the isomorphism is compatible with differentials we conclude. $\quad\square$

Lemma 35.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. Assume*

(1) $K^\bullet$ *represents a compact object of $D(\mathcal{O})$, and*
(2) $E = \mathrm{Hom}_{Comp^{dg}(\mathcal{O})}(K^\bullet, K^\bullet)$ *computes the ext groups of $K^\bullet$ in $D(\mathcal{O})$.*

Then the functor

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \longrightarrow D(\mathcal{O})$$

of Lemma 35.3 is fully faithful.

Proof. Because our functor has a left adjoint given by $R\,\mathrm{Hom}(K^\bullet, -)$ by Lemma 35.5 it suffices to show for a differential graded E-module M that the map

$$H^0(M) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(K^\bullet, M \otimes^{\mathbf{L}}_E K^\bullet)$$

is an isomorphism. We may assume that $M = P$ is a differential graded E-module which has property (P). Since $K^\bullet$ defines a compact object, we reduce using Lemma 20.1 to the case where P has a finite filtration whose graded pieces are direct sums of $E[k]$. Again using compactness we reduce to the case $P = E[k]$. The assumption on $K^\bullet$ is that the result holds for these. $\qquad\square$

36. Characterizing compact objects

Compact objects of additive categories are defined in Derived Categories, Definition 37.1. In this section we characterize compact objects of the derived category of a differential graded algebra.

Remark 36.1. Let (A, d) be a differential graded algebra. Is there a characterization of those differential graded A-modules P for which we have

$$\mathrm{Hom}_{K(A,\mathrm{d})}(P, M) = \mathrm{Hom}_{D(A,\mathrm{d})}(P, M)$$

for all differential graded A-modules M? Let $\mathcal{D} \subset K(A, \mathrm{d})$ be the full subcategory whose objects are the objects P satisfying the above. Then $\mathcal{D}$ is a strictly full saturated triangulated subcategory of $K(A, \mathrm{d})$. If P is projective as a graded A-module, then to see where P is an object of $\mathcal{D}$ it is enough to check that $\mathrm{Hom}_{K(A,\mathrm{d})}(P, M) = 0$ whenever M is acyclic. However, in general it is not enough to assume that P is projective as a graded A-module. Example: take $A = R = k[\epsilon]$ where k is a field and $k[\epsilon] = k[x]/(x^2)$ is the ring of dual numbers. Let P be the object with $P^n = R$ for all $n \in \mathbf{Z}$ and differential given by multiplication by ϵ. Then $\mathrm{id}_P \in \mathrm{Hom}_{K(A,\mathrm{d})}(P, P)$ is a nonzero element but P is acyclic.

Remark 36.2. Let (A, d) be a differential graded algebra. Let us say a differential graded A-module M is *finite* if M is generated, as a right A-module, by finitely many elements. If P is a differential graded A-module which is finite graded projective, then we can ask: Does P give a compact object of $D(A, \mathrm{d})$? Presumably, this is not true in general, but we do not know a counter example. However, if P is also an object of the category $\mathcal{D}$ of Remark 36.1, then this is the case (this follows from the fact that direct sums in $D(A, \mathrm{d})$ are given by direct sums of modules; details omitted).

Lemma 36.3. *Let (A, d) be a differential graded algebra. Let E be a compact object of $D(A, d)$. Let P be a differential graded A-module which has a finite filtration*

$$0 = F_{-1}P \subset F_0 P \subset F_1 P \subset \ldots \subset F_n P = P$$

by differential graded submodules such that

$$F_{i+1}P/F_i P \cong \bigoplus\nolimits_{j \in J_i} A[k_{i,j}]$$

as differential graded A-modules for some sets J_i and integers $k_{i,j}$. Let $E \to P$ be a morphism of $D(A, d)$. Then there exists a differential graded submodule $P' \subset P$ such that $F_{i+1}P \cap P'/(F_iP \cap P')$ is equal to $\bigoplus_{j \in J'_i} A[k_{i,j}]$ for some finite subsets $J'_i \subset J_i$ and such that $E \to P$ factors through P'.

Proof. We will prove by induction on $-1 \le m \le n$ that there exists a differential graded submodule $P' \subset P$ such that

(1) $F_m P \subset P'$,
(2) for $i \ge m$ the quotient $F_{i+1}P \cap P'/(F_iP \cap P')$ is isomorphic to $\bigoplus_{j \in J'_i} A[k_{i,j}]$ for some finite subsets $J'_i \subset J_i$, and
(3) $E \to P$ factors through P'.

The base case is $m = n$ where we can take $P' = P$.

Induction step. Assume P' works for m. For $i \ge m$ and $j \in J'_i$ let $x_{i,j} \in F_{i+1}P \cap P'$ be a homogeneous element of degree $k_{i,j}$ whose image in $F_{i+1}P \cap P'/(F_iP \cap P')$ is the generator in the summand corresponding to $j \in J_i$. The $x_{i,j}$ generate $P'/F_m P$ as an A-module. Write

$$\mathrm{d}(x_{i,j}) = \sum x_{i',j'} a_{i,j}^{i',j'} + y_{i,j}$$

with $y_{i,j} \in F_m P$ and $a_{i,j}^{i',j'} \in A$. There exists a finite subset $J'_{m-1} \subset J_{m-1}$ such that each $y_{i,j}$ maps to an element of the submodule $\bigoplus_{j \in J'_{m-1}} A[k_{m-1,j}]$ of $F_m P/F_{m-1}P$. Let $P'' \subset F_m P$ be the inverse image of $\bigoplus_{j \in J'_{m-1}} A[k_{m-1,j}]$ under the map $F_m P \to F_m P/F_{m-1}P$. Then we see that the A-submodule

$$P'' + \sum x_{i,j} A$$

is a differential graded submodule of the type we are looking for. Moreover

$$P'/(P'' + \sum x_{i,j} A) = \bigoplus_{j \in J_{m-1} \setminus J'_{m-1}} A[k_{m-1,j}]$$

Since E is compact, the composition of the given map $E \to P'$ with the quotient map, factors through a finite direct subsum of the module displayed above. Hence after enlarging J'_{m-1} we may assume $E \to P'$ factors through $P'' + \sum x_{i,j} A$ as desired. $\square$

It is not true that every compact object of $D(A, \mathrm{d})$ comes from a finite graded projective differential graded A-module, see Examples, Section 68.

Proposition 36.4. *Let (A, d) be a differential graded algebra. Let E be an object of $D(A, d)$. Then the following are equivalent*

(1) *E is a compact object,*
(2) *E is a direct summand of an object of $D(A, d)$ which is represented by a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_iP/F_{i-1}P$ are finite direct sums of shifts of A.*

Proof. Assume E is compact. By Lemma 20.4 we may assume that E is represented by a differential graded A-module P with property (P). Consider the distinguished triangle

$$\bigoplus F_i P \to \bigoplus F_i P \to P \xrightarrow{\delta} \bigoplus F_i P[1]$$

coming from the admissible short exact sequence of Lemma 20.1. Since E is compact we have $\delta = \sum_{i=1,\ldots,n} \delta_i$ for some $\delta_i : P \to F_i P[1]$. Since the composition of δ with the map $\bigoplus F_i P[1] \to \bigoplus F_i P[1]$ is zero (Derived Categories, Lemma 4.1) it follows that $\delta = 0$ (follows as $\bigoplus F_i P \to \bigoplus F_i P$ maps the summand $F_i P$ via the difference of id and the inclusion map into $F_{i-1} P$). Thus we see that the identity on E factors through $\bigoplus F_i P$ in $D(A, \mathrm{d})$ (by Derived Categories, Lemma 4.11). Next, we use that P is compact again to see that the map $E \to \bigoplus F_i P$ factors through $\bigoplus_{i=1,\ldots,n} F_i P$ for some n. In other words, the identity on E factors through $\bigoplus_{i=1,\ldots,n} F_i P$. By Lemma 36.3 we see that the identity of E factors as $E \to P \to E$ where P is as in part (2) of the statement of the lemma. In other words, we have proven that (1) implies (2).

Assume (2). By Derived Categories, Lemma 37.2 it suffices to show that P gives a compact object. Observe that P has property (P), hence we have

$$\mathrm{Hom}_{D(A,\mathrm{d})}(P, M) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, M)$$

for any differential graded module M by Lemma 22.3. As direct sums in $D(A, \mathrm{d})$ are given by direct sums of graded modules (Lemma 22.4) we reduce to showing that $\mathrm{Hom}_{K(A,\mathrm{d})}(P, M)$ commutes with direct sums. Using that $K(A, \mathrm{d})$ is a triangulated category, that Hom is a cohomological functor in the first variable, and the filtration on P, we reduce to the case that P is a finite direct sum of shifts of A. Thus we reduce to the case $P = A[k]$ which is clear. $\square$

Lemma 36.5. *Let (A, d) be a differential graded algebra. For every compact object E of $D(A, d)$ there exist integers $a \leq b$ such that $\mathrm{Hom}_{D(A,d)}(E, M) = 0$ if $H^i(M) = 0$ for $i \in [a, b]$.*

Proof. Observe that the collection of objects of $D(A, \mathrm{d})$ for which such a pair of integers exists is a saturated, strictly full triangulated subcategory of $D(A, \mathrm{d})$. Thus by Proposition 36.4 it suffices to prove this when E is represented by a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_i P / F_{i-1} P$ are finite direct sums of shifts of A. Using the compatibility with triangles, we see that it suffices to prove it for $P = A$. In this case $\mathrm{Hom}_{D(A,\mathrm{d})}(A, M) = H^0(M)$ and the result holds with $a = b = 0$. $\square$

If (A, d) is just an algebra placed in degree 0 with zero differential or more generally lives in only a finite number of degrees, then we do obtain the more precise description of compact objects.

Lemma 36.6. *Let (A, d) be a differential graded algebra. Assume that $A^n = 0$ for $|n| \gg 0$. Let E be an object of $D(A, d)$. The following are equivalent*

(1) *E is a compact object, and*
(2) *E can be represented by a differential graded A-module P which is finite projective as a graded A-module and satisfies $\mathrm{Hom}_{K(A,d)}(P, M) = \mathrm{Hom}_{D(A,d)}(P, M)$ for every differential graded A-module M.*

Proof. Let $\mathcal{D} \subset K(A, \mathrm{d})$ be the triangulated subcategory discussed in Remark 36.1. Let P be an object of $\mathcal{D}$ which is finite projective as a graded A-module. Then P represents a compact object of $D(A, \mathrm{d})$ by Remark 36.2.

To prove the converse, let E be a compact object of $D(A, \mathrm{d})$. Fix $a \leq b$ as in Lemma 36.5. After decreasing a and increasing b if necessary, we may also assume

that $H^i(E) = 0$ for $i \notin [a, b]$ (this follows from Proposition 36.4 and our assumption on A). Moreover, fix an integer $c > 0$ such that $A^n = 0$ if $|n| \geq c$.

By Proposition 36.4 we see that E is a direct summand, in $D(A, \mathrm{d})$, of a differential graded A-module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_i P / F_{i-1} P$ are finite direct sums of shifts of A. In particular, P has property (P) and we have $\mathrm{Hom}_{D(A,\mathrm{d})}(P, M) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, M)$ for any differential graded module M by Lemma 22.3. In other words, P is an object of the triangulated subcategory $\mathcal{D} \subset K(A, \mathrm{d})$ discussed in Remark 36.1. Note that P is finite free as a graded A-module.

Choose $n > 0$ such that $b + 4c - n < a$. Represent the projector onto E by an endomorphism $\varphi : P \to P$ of differential graded A-modules. Consider the distinguished triangle

$$P \xrightarrow{1-\varphi} P \to C \to P[1]$$

in $K(A, \mathrm{d})$ where C is the cone of the first arrow. Then C is an object of $\mathcal{D}$, we have $C \cong E \oplus E[1]$ in $D(A, \mathrm{d})$, and C is a finite graded free A-module. Next, consider a distinguished triangle

$$C[1] \to C \to C' \to C[2]$$

in $K(A, \mathrm{d})$ where C' is the cone on a morphism $C[1] \to C$ representing the composition

$$C[1] \cong E[1] \oplus E[2] \to E[1] \to E \oplus E[1] \cong C$$

in $D(A, \mathrm{d})$. Then we see that C' represents $E \oplus E[2]$. Continuing in this manner we see that we can find a differential graded A-module P which is an object of $\mathcal{D}$, is a finite free as a graded A-module, and represents $E \oplus E[n]$.

Choose a basis x_i, $i \in I$ of homogeneous elements for P as an A-module. Let $d_i = \deg(x_i)$. Let P_1 be the A-submodule of P generated by x_i and $\mathrm{d}(x_i)$ for $d_i \leq a - c - 1$. Let P_2 be the A-submodule of P generated by x_i and $\mathrm{d}(x_i)$ for $d_i \geq b - n + c$. We observe

    (1) P_1 and P_2 are differential graded submodules of P,
    (2) $P_1^t = 0$ for $t \geq a$,
    (3) $P_1^t = P^t$ for $t \leq a - 2c$,
    (4) $P_2^t = 0$ for $t \leq b - n$,
    (5) $P_2^t = P^t$ for $t \geq b - n + 2c$.

As $b - n + 2c \geq a - 2c$ by our choice of n we obtain a short exact sequence of differential graded A-modules

$$0 \to P_1 \cap P_2 \to P_1 \oplus P_2 \xrightarrow{\pi} P \to 0$$

Since P is projective as a graded A-module this is an admissible short exact sequence (Lemma 16.1). Hence we obtain a boundary map $\delta : P \to (P_1 \cap P_2)[1]$ in $K(A, \mathrm{d})$, see Lemma 7.2. Since $P = E \oplus E[n]$ and since $P_1 \cap P_2$ lives in degrees $(b - n, a)$ we find that $\mathrm{Hom}_{D(A,\mathrm{d})}(E \oplus E[n], (P_1 \cap P_2)[1])$ is zero. Therefore $\delta = 0$ as a morphism in $K(A, \mathrm{d})$ as P is an object of $\mathcal{D}$. By Derived Categories, Lemma 4.11 we can find a map $s : P \to P_1 \oplus P_2$ such that $\pi \circ s = \mathrm{id}_P + \mathrm{d}h + h\mathrm{d}$ for some $h : P \to P$ of degree -1. Since $P_1 \oplus P_2 \to P$ is surjective and since P is projective as a graded A-module we can choose a homogeneous lift $\tilde{h} : P \to P_1 \oplus P_2$ of h. Then we change s into $s + \mathrm{d}\tilde{h} + \tilde{h}\mathrm{d}$ to get $\pi \circ s = \mathrm{id}_P$. This means we obtain a direct sum decomposition $P = s^{-1}(P_1) \oplus s^{-1}(P_2)$. Since $s^{-1}(P_2)$ is equal to P in degrees $\geq b - n + 2c$ we see

that $s^{-1}(P_2) \to P \to E$ is a quasi-isomorphism, i.e., an isomorphism in $D(A, \mathrm{d})$. This finishes the proof. $\square$

37. Equivalences of derived categories

Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. A natural question that arises in nature is what it means that $D(A, \mathrm{d})$ is equivalent to $D(B, \mathrm{d})$ as an R-linear triangulated category. This is a rather subtle question and it will turn out it isn't always the correct question to ask. Nonetheless, in this section we collection some conditions that guarantee this is the case.

We strongly urge the reader to take a look at the groundbreaking paper [Ric89] on this topic.

Lemma 37.1. *Let R be a ring. Let $(A, d) \to (B, d)$ be a homomorphism of differential graded algebras over R, which induces an isomorphism on cohomology algebras. Then*

$$- \otimes_A^{\mathbf{L}} B : D(A, d) \to D(B, d)$$

gives an R-linear equivalence of triangulated categories with quasi-inverse the restriction functor $N \mapsto N_A$.

Proof. By Lemma 33.7 the functor $M \longmapsto M \otimes_A^{\mathbf{L}} B$ is fully faithful. By Lemma 33.5 the functor $N \longmapsto R\operatorname{Hom}(B, N) = N_A$ is a right adjoint, see Example 33.6. It is clear that the kernel of $R\operatorname{Hom}(B, -)$ is zero. Hence the result follows from Derived Categories, Lemma 7.2. $\square$

When we analyze the proof above we see that we obtain the following generalization for free.

Lemma 37.2. *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Assume that*

(1) *N defines a compact object of $D(B, d)$,*
(2) *if $N' \in D(B, d)$ and $\operatorname{Hom}_{D(B,d)}(N, N'[n]) = 0$ for $n \in \mathbf{Z}$, then $N' = 0$, and*
(3) *the map $H^k(A) \to \operatorname{Hom}_{D(B,d)}(N, N[k])$ is an isomorphism for all $k \in \mathbf{Z}$.*

Then

$$- \otimes_A^{\mathbf{L}} N : D(A, d) \to D(B, d)$$

gives an R-linear equivalence of triangulated categories.

Proof. By Lemma 33.7 the functor $M \longmapsto M \otimes_A^{\mathbf{L}} N$ is fully faithful. By Lemma 33.5 the functor $N' \longmapsto R\operatorname{Hom}(N, N')$ is a right adjoint. By assumption (3) the kernel of $R\operatorname{Hom}(N, -)$ is zero. Hence the result follows from Derived Categories, Lemma 7.2. $\square$

Remark 37.3. In Lemma 37.2 we can replace condition (2) by the condition that N is a classical generator for $D_{compact}(B, d)$, see Derived Categories, Proposition 37.6. Moreover, if we knew that $R\operatorname{Hom}(N, B)$ is a compact object of $D(A, \mathrm{d})$, then it suffices to check that N is a weak generator for $D_{compact}(B, \mathrm{d})$. We omit the proof; we will add it here if we ever need it in the Stacks project.

Sometimes the B-module P in the lemma below is called an "(A, B)-tilting complex".

Lemma 37.4. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Assume that $A = H^0(A)$. The following are equivalent*

(1) *$D(A, d)$ and $D(B, d)$ are equivalent as R-linear triangulated categories, and*
(2) *there exists an object P of $D(B, d)$ such that*
    (a) *P is a compact object of $D(B, d)$,*
    (b) *if $N \in D(B, d)$ with $\operatorname{Hom}_{D(B,d)}(P, N[i]) = 0$ for $i \in \mathbf{Z}$, then $N = 0$,*
    (c) *$\operatorname{Hom}_{D(B,d)}(P, P[i]) = 0$ for $i \neq 0$ and equal to A for $i = 0$.*

The equivalence $D(A, d) \to D(B, d)$ constructed in (2) sends A to P.

Proof. Let $F : D(A, d) \to D(B, d)$ be an equivalence. Then F maps compact objects to compact objects. Hence $P = F(A)$ is compact, i.e., (2)(a) holds. Conditions (2)(b) and (2)(c) are immediate from the fact that F is an equivalence.

Let P be an object as in (2). Represent P by a differential graded module with property (P). Set

$$(E, d) = \operatorname{Hom}_{\operatorname{Mod}^{dg}_{(B,d)}}(P, P)$$

Then $H^0(E) = A$ and $H^k(E) = 0$ for $k \neq 0$ by Lemma 22.3 and assumption (2)(c). Viewing P as a (E, B)-bimodule and using Lemma 37.2 and assumption (2)(b) we obtain an equivalence

$$D(E, d) \to D(B, d)$$

sending E to P. Let $E' \subset E$ be the differential graded R-subalgebra with

$$(E')^i = \begin{cases} E^i & \text{if } i < 0 \\ \operatorname{Ker}(E^0 \to E^1) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Then there are quasi-isomorphisms of differential graded algebras $(A, d) \leftarrow (E', d) \to (E, d)$. Thus we obtain equivalences

$$D(A, d) \leftarrow D(E', d) \to D(E, d) \to D(B, d)$$

by Lemma 37.1. $\square$

Remark 37.5. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Suppose given an R-linear equivalence

$$F : D(A, d) \longrightarrow D(B, d)$$

of triangulated categories. Set $N = F(A)$. Then N is a differential graded B-module. Since F is an equivalence and A is a compact object of $D(A, d)$, we conclude that N is a compact object of $D(B, d)$. Since A generates $D(A, d)$ and F is an equivalence, we see that N generates $D(B, d)$. Finally, $H^k(A) = \operatorname{Hom}_{D(A,d)}(A, A[k])$ and as F an equivalence we see that F induces an isomorphism $H^k(A) = \operatorname{Hom}_{D(B,d)}(N, N[k])$ for all k. In order to conclude that there is an equivalence $D(A, d) \longrightarrow D(B, d)$ which arises from the construction in Lemma 37.2 all we need is a left A-module structure on N compatible with derivation and commuting with the given right B-module structure. In fact, it suffices to do this after replacing N by a quasi-isomorphic differential graded B-module. The module structure can be constructed in certain cases. For example, if we assume that F can be lifted to a differential graded functor

$$F^{dg} : \operatorname{Mod}^{dg}_{(A,d)} \longrightarrow \operatorname{Mod}^{dg}_{(B,d)}$$

(for notation see Example 26.8) between the associated differential graded categories, then this holds. Another case is discussed in the proposition below.

Proposition 37.6. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let $F : D(A, d) \to D(B, d)$ be an R-linear equivalence of triangulated categories. Assume that*

(1) $A = H^0(A)$, *and*
(2) B *is K-flat as a complex of R-modules.*

Then there exists an (A, B)-bimodule N as in Lemma 37.2.

Proof. As in Remark 37.5 above, we set $N = F(A)$ in $D(B, \mathrm{d})$. We may assume that N is a differential graded B-module with property (P). Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B, \mathrm{d})}} (N, N)$$

Then $H^0(E) = A$ and $H^k(E) = 0$ for $k \neq 0$ by Lemma 22.3. Moreover, by the discussion in Remark 37.5 and by Lemma 37.2 we see that N as a (E, B)-bimodule induces an equivalence $- \otimes^{\mathbf{L}}_E N : D(E, \mathrm{d}) \to D(B, \mathrm{d})$. Let $E' \subset E$ be the differential graded R-subalgebra with

$$(E')^i = \begin{cases} E^i & \text{if } i < 0 \\ \mathrm{Ker}(E^0 \to E^1) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Then there are quasi-isomorphisms of differential graded algebras $(A, \mathrm{d}) \leftarrow (E', \mathrm{d}) \to (E, \mathrm{d})$. Thus we obtain equivalences

$$D(A, \mathrm{d}) \leftarrow D(E', \mathrm{d}) \to D(E, \mathrm{d}) \to D(B, \mathrm{d})$$

by Lemma 37.1. Note that the quasi-inverse $D(A, \mathrm{d}) \to D(E', \mathrm{d})$ of the left vertical arrow is given by $M \mapsto M \otimes^{\mathbf{L}}_A A$ where A is viewed as a (A, E')-bimodule, see Example 33.6. On the other hand the functor $D(E', \mathrm{d}) \to D(B, \mathrm{d})$ is given by $M \mapsto M \otimes^{\mathbf{L}}_{E'} N$ where N is as above. We conclude by Lemma 34.3. $\square$

Remark 37.7. Let A, B, F, N be as in Proposition 37.6. It is not clear that F and the functor $G(-) = - \otimes^{\mathbf{L}}_A N$ are isomorphic. By construction there is an isomorphism $N = G(A) \to F(A)$ in $D(B, \mathrm{d})$. It is straightforward to extend this to a functorial isomorphism $G(M) \to F(M)$ for M is a differential graded A-module which is graded projective (e.g., a sum of shifts of A). Then one can conclude that $G(M) \cong F(M)$ when M is a cone of a map between such modules. We don't know whether more is true in general.

Lemma 37.8. *Let R be a ring. Let A and B be R-algebras. The following are equivalent*

(1) *there is an R-linear equivalence $D(A) \to D(B)$ of triangulated categories,*
(2) *there exists an object P of $D(B)$ such that*
    (a) *P can be represented by a finite complex of finite projective B-modules,*
    (b) *if $K \in D(B)$ with $\mathrm{Ext}^i_B(P, K) = 0$ for $i \in \mathbf{Z}$, then $K = 0$, and*
    (c) *$\mathrm{Ext}^i_B(P, P) = 0$ for $i \neq 0$ and equal to A for $i = 0$.*

Moreover, if B is flat as an R-module, then this is also equivalent to

(3) *there exists an (A, B)-bimodule N such that $- \otimes^{\mathbf{L}}_A N : D(A) \to D(B)$ is an equivalence.*

Proof. The equivalence of (1) and (2) is a special case of Lemma 37.4 combined with the result of Lemma 36.6 characterizing compact objects of $D(B)$ (small detail omitted). The equivalence with (3) if B is R-flat follows from Proposition 37.6. □

Remark 37.9. Let R be a ring. Let A and B be R-algebras. If $D(A)$ and $D(B)$ are equivalent as R-linear triangulated categories, then the centers of A and B are isomorphic as R-algebras. In particular, if A and B are commutative, then $A \cong B$. The rather tricky proof can be found in [Ric89, Proposition 9.2] or [KZ98, Proposition 6.3.2]. Another approach might be to use Hochschild cohomology (see remark below).

Remark 37.10. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras which are derived equivalent, i.e., such that there exists an R-linear equivalence $D(A, \mathrm{d}) \to D(B, \mathrm{d})$ of triangulated categories. We would like to show that certain invariants of (A, d) and (B, d) coincide. In many situations one has more control of the situation. For example, it may happen that there is an equivalence of the form

$$- \otimes_A \Omega : D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$$

for some differential graded (A, B)-bimodule Ω (this happens in the situation of Proposition 37.6 and is often true if the equivalence comes from a geometric construction). If also the quasi-inverse of our functor is given as

$$- \otimes_B^{\mathbf{L}} \Omega' : D(B, \mathrm{d}) \longrightarrow D(A, \mathrm{d})$$

for a differential graded (B, A)-bimodule Ω' (and as before such a module Ω' often exists in practice). In this case we can consider the functor

$$D(A^{opp} \otimes_R A, \mathrm{d}) \longrightarrow D(B^{opp} \otimes_R B, \mathrm{d}), \quad M \longmapsto \Omega' \otimes_A^{\mathbf{L}} M \otimes_A^{\mathbf{L}} \Omega$$

on derived categories of bimodules (use Lemma 28.3 to turn bimodules into right modules). Observe that this functor sends the (A, A)-bimodule A to the (B, B)-bimodule B. Under suitable conditions (e.g., flatness of A, B, Ω over R, etc) this functor will be an equivalence as well. If this is the case, then it follows that we have isomorphisms of Hochschild cohomology groups

$$HH^i(A, \mathrm{d}) = \mathrm{Hom}_{D(A^{opp} \otimes_R A, \mathrm{d})}(A, A[i]) \longrightarrow \mathrm{Hom}_{D(B^{opp} \otimes_R B, \mathrm{d})}(B, B[i]) = HH^i(B, \mathrm{d}).$$

For example, if $A = H^0(A)$, then $HH^0(A, \mathrm{d})$ is equal to the center of A, and this gives a conceptual proof of the result mentioned in Remark 37.9. If we ever need this remark we will provide a precise statement with a detailed proof here.

38. Resolutions of differential graded algebras

Let R be a ring. Under our assumptions the free R-algebra $R\langle S \rangle$ on a set S is the algebra with R-basis the expressions

$$s_1 s_2 \ldots s_n$$

where $n \geq 0$ and $s_1, \ldots, s_n \in S$ is a sequence of elements of S. Multiplication is given by concatenation

$$(s_1 s_2 \ldots s_n) \cdot (s_1' s_2' \ldots s_m') = s_1 \ldots s_n s_1' \ldots s_m'$$

This algebra is characterized by the property that the map

$$\mathrm{Mor}_{R\text{-alg}}(R\langle S \rangle, A) \to \mathrm{Map}(S, A), \quad \varphi \longmapsto (s \mapsto \varphi(s))$$

is a bijection for every R-algebra A.

In the category of graded R-algebras our set S should come with a grading, which we think of as a map deg $: S \to \mathbf{Z}$. Then $R\langle S \rangle$ has a grading such that the monomials have degree

$$\deg(s_1 s_2 \ldots s_n) = \deg(s_1) + \ldots + \deg(s_n)$$

In this setting we have

$$\mathrm{Mor}_{\text{graded } R\text{-alg}}(R\langle S \rangle, A) \to \mathrm{Map}_{\text{graded sets}}(S, A), \quad \varphi \longmapsto (s \mapsto \varphi(s))$$

is a bijection for every graded R-algebra A.

If A is a graded R-algebra and S is a graded set, then we can similarly form $A\langle S \rangle$. Elements of $A\langle S \rangle$ are sums of elements of the form

$$a_0 s_1 a_1 s_2 \ldots a_{n-1} s_n a_n$$

with $a_i \in A$ modulo the relations that these expressions are R-multilinear in $(a_0, \ldots, a_n)$. Thus for every sequence $s_1, \ldots, s_n$ of elements of S there is an inclusion

$$A \otimes_R \ldots \otimes_R A \subset A\langle S \rangle$$

and the algebra is the direct sum of these. With this definition the reader shows that the map

$$\mathrm{Mor}_{\text{graded } R\text{-alg}}(A\langle S \rangle, B) \to \mathrm{Mor}_{\text{graded } R\text{-alg}}(A, B) \times \mathrm{Map}_{\text{graded sets}}(S, B),$$

sending φ to $(\varphi|_A, (s \mapsto \varphi(s)))$ is a bijection for every graded R-algebra A. We observe that if A was a free graded R-algebra, then so is $A\langle S \rangle$.

Suppose that A is a differential graded R-algebra and that S is a graded set. Suppose moreover for every $s \in S$ we are given a homogeneous element $f_s \in A$ with $\deg(f_s) = \deg(s) + 1$ and $\mathrm{d}f_s = 0$. Then there exists a unique structure of differential graded algebra on $A\langle S \rangle$ with $\mathrm{d}(s) = f_s$. For example, given $a, b, c \in A$ and $s, t \in S$ we would define

$$\mathrm{d}(asbtc) = \mathrm{d}(a)sbtc + (-1)^{\deg(a)} a f_s btc + (-1)^{\deg(a)+\deg(s)} as\mathrm{d}(b)tc$$

$$+ (-1)^{\deg(a)+\deg(s)+\deg(b)} asb f_t c + (-1)^{\deg(a)+\deg(s)+\deg(b)+\deg(t)} asbt\mathrm{d}(c)$$

We omit the details.

Lemma 38.1. *Let R be a ring. Let (B, d) be a differential graded R-algebra. There exists a quasi-isomorphism $(A, d) \to (B, d)$ of differential graded R-algebras with the following properties*

    (1) *A is K-flat as a complex of R-modules,*

    (2) *A is a free graded R-algebra.*

Proof. First we claim we can find $(A_0, \mathrm{d}) \to (B, \mathrm{d})$ having (1) and (2) inducing a surjection on cohomology. Namely, take a graded set S and for each $s \in S$ a homogeneous element $b_s \in \mathrm{Ker}(d : B \to B)$ of degree $\deg(s)$ such that the classes $\bar{b}_s$ in $H^*(B)$ generate $H^*(B)$ as an R-module. Then we can set $A_0 = R\langle S \rangle$ with zero differential and $A_0 \to B$ given by mapping s to b_s.

Given $A_0 \to B$ inducing a surjection on cohomology we construct a sequence

$$A_0 \to A_1 \to A_2 \to \ldots B$$

by induction. Given $A_n \to B$ we set S_n be a graded set and for each $s \in S_n$ we let $a_s \in \mathrm{Ker}(A_n \to A_n)$ be a homogeneous element of degree $\deg(s) + 1$ mapping to a

class $\bar{a}_s$ in $H^*(A_n)$ which maps to zero in $H^*(B)$. We choose S_n large enough so that the elements $\bar{a}_s$ generate $\mathrm{Ker}(H^*(A_n) \to H^*(B))$ as an R-module. Then we set

$$A_{n+1} = A_n\langle S_n\rangle$$

with differential given by $\mathrm{d}(s) = a_s$ see discussion above. Then each (A_n, d) satisfies (1) and (2), we omit the details. The map $H^*(A_n) \to H^*(B)$ is surjective as this was true for $n = 0$.

It is clear that $A = \mathrm{colim}\, A_n$ is a free graded R-algebra. It is K-flat by More on Algebra, Lemma 59.8. The map $H^*(A) \to H^*(B)$ is an isomorphism as it is surjective and injective: every element of $H^*(A)$ comes from an element of $H^*(A_n)$ for some n and if it dies in $H^*(B)$, then it dies in $H^*(A_{n+1})$ hence in $H^*(A)$. $\square$

As an application we prove the "correct" version of Lemma 34.2.

Lemma 38.2. *Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Assume $A \otimes_R C$ represents $A \otimes_R^{\mathbf{L}} C$ in $D(R)$. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-bimodule. Then the composition*

$$D(A, d) \xrightarrow{\ -\otimes_A^{\mathbf{L}} N\ } D(B, d) \xrightarrow{\ -\otimes_B^{\mathbf{L}} N'\ } D(C, d)$$

is isomorphic to $-\otimes_A^{\mathbf{L}} N''$ for some differential graded (A, C)-bimodule N''.

Proof. Using Lemma 38.1 we choose a quasi-isomorphism $(B', \mathrm{d}) \to (B, \mathrm{d})$ with B' K-flat as a complex of R-modules. By Lemma 37.1 the functor $-\otimes_{B'}^{\mathbf{L}} B : D(B', \mathrm{d}) \to D(B, \mathrm{d})$ is an equivalence with quasi-inverse given by restriction. Note that restriction is canonically isomorphic to the functor $-\otimes_B^{\mathbf{L}} B : D(B, \mathrm{d}) \to D(B', \mathrm{d})$ where B is viewed as a (B, B')-bimodule. Thus it suffices to prove the lemma for the compositions

$$D(A) \to D(B) \to D(B'), \quad D(B') \to D(B) \to D(C), \quad D(A) \to D(B') \to D(C).$$

The first one is Lemma 34.3 because B' is K-flat as a complex of R-modules. The second one is true because $B \otimes_B^{\mathbf{L}} N' = N' = B \otimes_B N'$ and hence Lemma 34.1 applies. Thus we reduce to the case where B is K-flat as a complex of R-modules.

Assume B is K-flat as a complex of R-modules. It suffices to show that (34.1.1) is an isomorphism, see Lemma 34.2. Choose a quasi-isomorphism $L \to A$ where L is a differential graded R-module which has property (P). Then it is clear that $P = L \otimes_R B$ has property (P) as a differential graded B-module. Hence we have to show that $P \to A \otimes_R B$ induces a quasi-isomorphism

$$P \otimes_B (B \otimes_R C) \longrightarrow (A \otimes_R B) \otimes_B (B \otimes_R C)$$

We can rewrite this as

$$P \otimes_R B \otimes_R C \longrightarrow A \otimes_R B \otimes_R C$$

Since B is K-flat as a complex of R-modules, it follows from More on Algebra, Lemma 59.2 that it is enough to show that

$$P \otimes_R C \to A \otimes_R C$$

is a quasi-isomorphism, which is exactly our assumption. $\square$

The following lemma does not really belong in this section, but there does not seem to be a good natural spot for it.

Lemma 38.3. *Let (A, d) be a differential graded algebra with $H^i(A)$ countable for each i. Let M be an object of $D(A, d)$. Then the following are equivalent*

(1) $M = hocolim E_n$ *with E_n compact in $D(A, d)$, and*
(2) $H^i(M)$ *is countable for each i.*

Proof. Assume (1) holds. Then we have $H^i(M) = \operatorname{colim} H^i(E_n)$ by Derived Categories, Lemma 33.8. Thus it suffices to prove that $H^i(E_n)$ is countable for each n. By Proposition 36.4 we see that E_n is isomorphic in $D(A, \mathrm{d})$ to a direct summand of a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_j P / F_{j-1} P$ are finite direct sums of shifts of A. By assumption the groups $H^i(F_j P / F_{j-1} P)$ are countable. Arguing by induction on the length of the filtration and using the long exact cohomology sequence we conclude that (2) is true. The interesting implication is the other one.

We claim there is a countable differential graded subalgebra $A' \subset A$ such that the inclusion map $A' \to A$ defines an isomorphism on cohomology. To construct A' we choose countable differential graded subalgebras

$$A_1 \subset A_2 \subset A_3 \subset \dots$$

such that (a) $H^i(A_1) \to H^i(A)$ is surjective, and (b) for $n > 1$ the kernel of the map $H^i(A_{n-1}) \to H^i(A_n)$ is the same as the kernel of the map $H^i(A_{n-1}) \to H^i(A)$. To construct A_1 take any countable collection of cochains $S \subset A$ generating the cohomology of A (as a ring or as a graded abelian group) and let A_1 be the differential graded subalgebra of A generated by S. To construct A_n given A_{n-1} for each cochain $a \in A_{n-1}^i$ which maps to zero in $H^i(A)$ choose $s_a \in A^{i-1}$ with $\mathrm{d}(s_a) = a$ and let A_n be the differential graded subalgebra of A generated by A_{n-1} and the elements s_a. Finally, take $A' = \bigcup A_n$.

By Lemma 37.1 the restriction map $D(A, \mathrm{d}) \to D(A', \mathrm{d})$, $M \mapsto M_{A'}$ is an equivalence. Since the cohomology groups of M and $M_{A'}$ are the same, we see that it suffices to prove the implication $(2) \Rightarrow (1)$ for (A', d).

Assume A is countable. By the exact same type of argument as given above we see that for M in $D(A, \mathrm{d})$ the following are equivalent: $H^i(M)$ is countable for each i and M can be represented by a countable differential graded module. Hence in order to prove the implication $(2) \Rightarrow (1)$ we reduce to the situation described in the next paragraph.

Assume A is countable and that M is a countable differential graded module over A. We claim there exists a homomorphism $P \to M$ of differential graded A-modules such that

(1) $P \to M$ is a quasi-isomorphism,
(2) P has property (P), and
(3) P is countable.

Looking at the proof of the construction of P-resolutions in Lemma 20.4 we see that it suffices to show that we can prove Lemma 20.3 in the setting of countable differential graded modules. This is immediate from the proof.

Assume that A is countable and that M is a countable differential graded module with property (P). Choose a filtration

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset P$$

by differential graded submodules such that we have

(1) $P = \bigcup F_pP$,
(2) $F_iP \to F_{i+1}P$ is an admissible monomorphism,
(3) isomorphisms of differential graded modules $F_iP/F_{i-1}P \to \bigoplus_{j \in J_i} A[k_j]$ for some sets J_i and integers k_j.

Of course J_i is countable for each i. For each i and $j \in J_i$ choose $x_{i,j} \in F_iP$ of degree k_j whose image in $F_iP/F_{i-1}P$ generates the summand corresponding to j.

Claim: Given n and finite subsets $S_i \subset J_i$, $i = 1, \ldots, n$ there exist finite subsets $S_i \subset T_i \subset J_i$, $i = 1, \ldots, n$ such that $P' = \bigoplus_{i \leq n} \bigoplus_{j \in T_i} Ax_{i,j}$ is a differential graded submodule of P. This was shown in the proof of Lemma 36.3 but it is also easily shown directly: the elements $x_{i,j}$ freely generate P as a right A-module. The structure of P shows that

$$\mathrm{d}(x_{i,j}) = \sum\nolimits_{i' < i} x_{i',j'} a_{i',j'}$$

where of course the sum is finite. Thus given $S_0, \ldots, S_n$ we can first choose $S_0 \subset S'_0, \ldots, S_{n-1} \subset S'_{n-1}$ with $\mathrm{d}(x_{n,j}) \in \bigoplus_{i' < n, j' \in S'_{i'}} x_{i',j'} A$ for all $j \in S_n$. Then by induction on n we can choose $S'_0 \subset T_0, \ldots, S'_{n-1} \subset T_{n-1}$ to make sure that $\bigoplus_{i' < n, j' \in T_{i'}} x_{i',j'} A$ is a differential graded A-submodule. Setting $T_n = S_n$ we find that $P' = \bigoplus_{i \leq n, j \in T_i} x_{i,j} A$ is as desired.

From the claim it is clear that $P = \bigcup P'_n$ is a countable rising union of P'_n as above. By construction each P'_n is a differential graded module with property (P) such that the filtration is finite and the succesive quotients are finite direct sums of shifts of A. Hence P'_n defines a compact object of $D(A, \mathrm{d})$, see for example Proposition 36.4. Since $P = \mathrm{hocolim} P'_n$ in $D(A, \mathrm{d})$ by Lemma 23.2 the proof of the implication (2) $\Rightarrow$ (1) is complete. $\square$

39. Other chapters

Preliminaries

Schemes

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