

CHOW GROUPS OF SPACES

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1. Introduction

In this chapter we first discuss Chow groups of algebraic spaces. Having defined these, we define Chern classes of vector bundles as operators on these chow groups.

The strategy will be entirely the same as the strategy in the case of schemes. Therefore we urge the reader to take a look at the introduction (Chow Homology, Section 1) of the corresponding chapter for schemes.

Some related papers: [EG98] and [Kre99].

2. Setup

We first fix the category of algebraic spaces we will be working with. Please keep in mind throughout this chapter that “decent + locally Noetherian” is the same as “quasi-separated + locally Noetherian” according to Decent Spaces, Lemma 14.1.

Situation 2.1. Here S is a scheme and B is an algebraic space over S . We assume B is quasi-separated, locally Noetherian, and universally catenary (Decent Spaces, Definition 25.4). Moreover, we assume given a dimension function $\delta : |B| \rightarrow \mathbf{Z}$. We say X/B is *good* if X is an algebraic space over B whose structure morphism $f : X \rightarrow B$ is quasi-separated and locally of finite type. In this case we define

$$\delta = \delta_{X/B} : |X| \rightarrow \mathbf{Z}$$

as the map sending x to $\delta(f(x))$ plus the transcendence degree of $x/f(x)$ (Morphisms of Spaces, Definition 33.1). This is a dimension function by More on Morphisms of Spaces, Lemma 32.2.

A special case is when $S = B$ is a scheme and (S, δ) is as in Chow Homology, Situation 7.1. Thus B might be the spectrum of a field (Chow Homology, Example 7.2) or $B = \text{Spec}(\mathbf{Z})$ (Chow Homology, Example 7.3).

Many lemma, proposition, theorems, definitions on algebraic spaces are easier in the setting of Situation 2.1 because the algebraic spaces we are working with are quasi-separated (and thus a fortiori decent) and locally Noetherian. We will sprinkle this chapter with remarks such as the following to point this out.

Remark 2.2. In Situation 2.1 if X/B is good, then $|X|$ is a sober topological space. See Properties of Spaces, Lemma 15.1 or Decent Spaces, Proposition 12.4. We will use this without further mention to choose generic points of irreducible closed subsets of $|X|$.

Remark 2.3. In Situation 2.1 if X/B is good, then X is integral (Spaces over Fields, Definition 4.1) if and only if X is reduced and $|X|$ is irreducible. Moreover, for any point $\xi \in |X|$ there is a unique integral closed subspace $Z \subset X$ such that ξ is the generic point of the closed subset $|Z| \subset |X|$, see Spaces over Fields, Lemma 4.7.

If B is Jacobson and δ sends closed points to zero, then δ is the function sending a point to the dimension of its closure.

Lemma 2.4. *In Situation 2.1 assume B is Jacobson and that $\delta(b) = 0$ for every closed point b of $|B|$. Let X/B be good. If $Z \subset X$ is an integral closed subspace with generic point $\xi \in |Z|$, then the following integers are the same:*

- (1) $\delta(\xi) = \delta_{X/B}(\xi)$,
- (2) $\dim(|Z|)$,
- (3) $\text{codim}(\{z\}, |Z|)$ for $z \in |Z|$ closed,
- (4) the dimension of the local ring of Z at z for $z \in |Z|$ closed, and
- (5) $\dim(\mathcal{O}_{Z, \bar{z}})$ for $z \in |Z|$ closed.

Proof. Let X, Z, ξ be as in the lemma. Since X is locally of finite type over B we see that X is Jacobson, see Decent Spaces, Lemma 23.1. Hence $X_{\text{ft-pts}} \subset |X|$ is the set of closed points by Decent Spaces, Lemma 23.3. Given a chain $T_0 \supset \dots \supset T_e$ of irreducible closed subsets of $|Z|$ we have $T_e \cap X_{\text{ft-pts}}$ nonempty by Morphisms of Spaces, Lemma 25.6. Thus we can always assume such a chain ends with $T_e = \{z\}$ for some $z \in |Z|$ closed. It follows that $\dim(Z) = \sup_z \text{codim}(\{z\}, |Z|)$ where z runs over the closed points of $|Z|$. We have $\text{codim}(\{z\}, Z) = \delta(\xi) - \delta(z)$ by Topology, Lemma 20.2. By Morphisms of Spaces, Lemma 25.4 the image of z is a finite type point of B , i.e., a closed point of $|B|$. By Morphisms of Spaces, Lemma 33.4 the transcendence degree of z/b is 0. We conclude that $\delta(z) = \delta(b) = 0$ by assumption. Thus we obtain equality

$$\dim(|Z|) = \text{codim}(\{z\}, Z) = \delta(\xi)$$

for all $z \in |Z|$ closed. Finally, we have that $\text{codim}(\{z\}, Z)$ is equal to the dimension of the local ring of Z at z by Decent Spaces, Lemma 20.2 which in turn is equal to $\dim(\mathcal{O}_{Z, \bar{z}})$ by Properties of Spaces, Lemma 22.4. \square

In the situation of the lemma above the value of δ at the generic point of a closed irreducible subset is the dimension of the irreducible closed subset. This motivates the following definition.

Definition 2.5. In Situation 2.1 for any good X/B and any irreducible closed subset $T \subset |X|$ we define

$$\dim_\delta(T) = \delta(\xi)$$

where $\xi \in T$ is the generic point of T . We will call this the δ -dimension of T . If $T \subset |X|$ is any closed subset, then we define $\dim_\delta(T)$ as the supremum of the δ -dimensions of the irreducible components of T . If Z is a closed subspace of X , then we set $\dim_\delta(Z) = \dim_\delta(|Z|)$.

Of course this just means that $\dim_\delta(T) = \sup\{\delta(t) \mid t \in T\}$.

3. Cycles

This is the analogue of Chow Homology, Section 8

Since we are not assuming our spaces are quasi-compact we have to be a little careful when defining cycles. We have to allow infinite sums because a rational function may have infinitely many poles for example. In any case, if X is quasi-compact then a cycle is a finite sum as usual.

Definition 3.1. In Situation 2.1 let X/B be good. Let $k \in \mathbf{Z}$.

- (1) A *cycle on X* is a formal sum

$$\alpha = \sum n_Z [Z]$$

where the sum is over integral closed subspaces $Z \subset X$, each $n_Z \in \mathbf{Z}$, and $\{|Z|; n_Z \neq 0\}$ is a locally finite collection of subsets of $|X|$ (Topology, Definition 28.4).

- (2) A *k -cycle on X* is a cycle

$$\alpha = \sum n_Z [Z]$$

where $n_Z \neq 0 \Rightarrow \dim_\delta(Z) = k$.

- (3) The abelian group of all k -cycles on X is denoted $Z_k(X)$.

In other words, a k -cycle on X is a locally finite formal \mathbf{Z} -linear combination of integral closed subspaces (Remark 2.3) of δ -dimension k . Addition of k -cycles $\alpha = \sum n_Z[Z]$ and $\beta = \sum m_Z[Z]$ is given by

$$\alpha + \beta = \sum (n_Z + m_Z)[Z],$$

i.e., by adding the coefficients.

4. Multiplicities

A section with a few simple results on lengths and multiplicities.

Lemma 4.1. *Let S be a scheme and let X be an algebraic space over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $x \in |X|$. Let $d \in \{0, 1, 2, \dots, \infty\}$. The following are equivalent*

- (1) $\text{length}_{\mathcal{O}_{X,\bar{x}}} \mathcal{F}_{\bar{x}} = d$
- (2) *for some étale morphism $U \rightarrow X$ with U a scheme and $u \in U$ mapping to x we have $\text{length}_{\mathcal{O}_{U,u}} (\mathcal{F}|_U)_u = d$*
- (3) *for any étale morphism $U \rightarrow X$ with U a scheme and $u \in U$ mapping to x we have $\text{length}_{\mathcal{O}_{U,u}} (\mathcal{F}|_U)_u = d$*

Proof. Let $U \rightarrow X$ and $u \in U$ be as in (2) or (3). Then we know that $\mathcal{O}_{X,\bar{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$ and that

$$\mathcal{F}_{\bar{x}} = (\mathcal{F}|_U)_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\bar{x}}$$

See Properties of Spaces, Lemmas 22.1 and 29.4. Thus the equality of the lengths follows from Algebra, Lemma 52.13 the fact that $\mathcal{O}_{U,u} \rightarrow \mathcal{O}_{X,\bar{x}}$ is flat and the fact that $\mathcal{O}_{X,\bar{x}}/\mathfrak{m}_u \mathcal{O}_{X,\bar{x}}$ is equal to the residue field of $\mathcal{O}_{X,\bar{x}}$. These facts about strict henselizations can be found in More on Algebra, Lemma 45.1. \square

Definition 4.2. Let S be a scheme and let X be an algebraic space over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $x \in |X|$. Let $d \in \{0, 1, 2, \dots, \infty\}$. We say \mathcal{F} has length d at x if the equivalent conditions of Lemma 4.1 are satisfied.

Lemma 4.3. *Let S be a scheme. Let $i : Y \rightarrow X$ be a closed immersion of algebraic spaces over S . Let \mathcal{G} be a quasi-coherent \mathcal{O}_Y -module. Let $y \in |Y|$ with image $x \in |X|$. Let $d \in \{0, 1, 2, \dots, \infty\}$. The following are equivalent*

- (1) \mathcal{G} has length d at y , and
- (2) $i_* \mathcal{G}$ has length d at x .

Proof. Choose an étale morphism $f : U \rightarrow X$ with U a scheme and $u \in U$ mapping to x . Set $V = Y \times_X U$. Denote $g : V \rightarrow Y$ and $j : V \rightarrow U$ the projections. Then $j : V \rightarrow U$ is a closed immersion and there is a unique point $v \in V$ mapping to $y \in |Y|$ and $u \in U$ (use Properties of Spaces, Lemma 4.3 and Spaces, Lemma 12.3). We have $j_*(\mathcal{G}|_V) = (i_* \mathcal{G})|_U$ as modules on the scheme V and j_* the “usual” pushforward of modules for the morphism of schemes j , see discussion surrounding Cohomology of Spaces, Equation (3.0.1). In this way we reduce to the case of schemes: if $i : Y \rightarrow X$ is a closed immersion of schemes, then

$$(i_* \mathcal{G})_x = \mathcal{G}_y$$

as modules over $\mathcal{O}_{X,x}$ where the module structure on the right hand side is given by the surjection $i_y^\# : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$. Thus equality by Algebra, Lemma 52.5. \square

Lemma 4.4. *Let S be a scheme and let X be a locally Noetherian algebraic space over S . Let \mathcal{F} be a coherent \mathcal{O}_X -module. Let $x \in |X|$. The following are equivalent*

- (1) *for some étale morphism $U \rightarrow X$ with U a scheme and $u \in U$ mapping to x we have u is a generic point of an irreducible component of $\text{Supp}(\mathcal{F}|_U)$,*
- (2) *for any étale morphism $U \rightarrow X$ with U a scheme and $u \in U$ mapping to x we have u is a generic point of an irreducible component of $\text{Supp}(\mathcal{F}|_U)$,*
- (3) *the length of \mathcal{F} at x is finite and nonzero.*

If X is decent (equivalently quasi-separated) then these are also equivalent to

- (4) *x is a generic point of an irreducible component of $\text{Supp}(\mathcal{F})$.*

Proof. Assume $f : U \rightarrow X$ is an étale morphism with U a scheme and $u \in U$ maps to x . Then $\mathcal{F}|_U = f^*\mathcal{F}$ is a coherent \mathcal{O}_U -module on the locally Noetherian scheme U and in particular $(\mathcal{F}|_U)_u$ is a finite $\mathcal{O}_{U,u}$ -module, see Cohomology of Spaces, Lemma 12.2 and Cohomology of Schemes, Lemma 9.1. Recall that the support of $\mathcal{F}|_U$ is a closed subset of U (Morphisms, Lemma 5.3) and that the support of $(\mathcal{F}|_U)_u$ is the pullback of the support of $\mathcal{F}|_U$ by the morphism $\text{Spec}(\mathcal{O}_{U,u}) \rightarrow U$. Thus u is a generic point of an irreducible component of $\text{Supp}(\mathcal{F}|_U)$ if and only if the support of $(\mathcal{F}|_U)_u$ is equal to the maximal ideal of $\mathcal{O}_{U,u}$. Now the equivalence of (1), (2), (3) follows from Algebra, Lemma 62.3.

If X is decent we choose an étale morphism $f : U \rightarrow X$ and a point $u \in U$ mapping to x . The support of \mathcal{F} pulls back to the support of $\mathcal{F}|_U$, see Morphisms of Spaces, Lemma 15.2. Also, specializations $x' \rightsquigarrow x$ in $|X|$ lift to specializations $u' \rightsquigarrow u$ in U and any nontrivial specialization $u' \rightsquigarrow u$ in U maps to a nontrivial specialization $f(u') \rightsquigarrow f(u)$ in $|X|$, see Decent Spaces, Lemmas 12.2 and 12.1. Using that $|X|$ and U are sober topological spaces (Decent Spaces, Proposition 12.4 and Schemes, Lemma 11.1) we conclude x is a generic point of the support of \mathcal{F} if and only if u is a generic point of the support of $\mathcal{F}|_U$. We conclude (4) is equivalent to (1).

The parenthetical statement follows from Decent Spaces, Lemma 14.1. □

Lemma 4.5. *In Situation 2.1 let X/B be good. Let $T \subset |X|$ be a closed subset and $t \in T$. If $\dim_\delta(T) \leq k$ and $\delta(t) = k$, then t is a generic point of an irreducible component of T .*

Proof. We know t is contained in an irreducible component $T' \subset T$. Let $t' \in T'$ be the generic point. Then $k \geq \delta(t') \geq \delta(t)$. Since δ is a dimension function we see that $t = t'$. □

5. Cycle associated to a closed subspace

This section is the analogue of Chow Homology, Section 9.

Remark 5.1. In Situation 2.1 let X/B be good. Let $Y \subset X$ be a closed subspace. By Remarks 2.2 and 2.3 there are 1-to-1 correspondences between

- (1) irreducible components T of $|Y|$,
- (2) generic points of irreducible components of $|Y|$, and
- (3) integral closed subspaces $Z \subset Y$ with the property that $|Z|$ is an irreducible component of $|Y|$.

In this chapter we will call Z as in (3) an *irreducible component of Y* and we will call $\xi \in |Z|$ its *generic point*.

Definition 5.2. In Situation 2.1 let X/B be good. Let $Y \subset X$ be a closed subspace.

- (1) For an irreducible component $Z \subset Y$ with generic point ξ the length of \mathcal{O}_Y at ξ (Definition 4.2) is called the *multiplicity of Z in Y* . By Lemma 4.4 applied to \mathcal{O}_Y on Y this is a positive integer.
- (2) Assume $\dim_\delta(Y) \leq k$. The *k -cycle associated to Y* is

$$[Y]_k = \sum m_{Z,Y} [Z]$$

where the sum is over the irreducible components Z of Y of δ -dimension k and $m_{Z,Y}$ is the multiplicity of Z in Y . This is a k -cycle by Spaces over Fields, Lemma 6.1.

It is important to note that we only define $[Y]_k$ if the δ -dimension of Y does not exceed k . In other words, by convention, if we write $[Y]_k$ then this implies that $\dim_\delta(Y) \leq k$.

6. Cycle associated to a coherent sheaf

This is the analogue of Chow Homology, Section 10.

Definition 6.1. In Situation 2.1 let X/B be good. Let \mathcal{F} be a coherent \mathcal{O}_X -module.

- (1) For an integral closed subspace $Z \subset X$ with generic point ξ such that $|Z|$ is an irreducible component of $\text{Supp}(\mathcal{F})$ the length of \mathcal{F} at ξ (Definition 4.2) is called the *multiplicity of Z in \mathcal{F}* . By Lemma 4.4 this is a positive integer.
- (2) Assume $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. The *k -cycle associated to \mathcal{F}* is

$$[\mathcal{F}]_k = \sum m_{Z,\mathcal{F}} [Z]$$

where the sum is over the integral closed subspaces $Z \subset X$ corresponding to irreducible components of $\text{Supp}(\mathcal{F})$ of δ -dimension k and $m_{Z,\mathcal{F}}$ is the multiplicity of Z in \mathcal{F} . This is a k -cycle by Spaces over Fields, Lemma 6.1.

It is important to note that we only define $[\mathcal{F}]_k$ if \mathcal{F} is coherent and the δ -dimension of $\text{Supp}(\mathcal{F})$ does not exceed k . In other words, by convention, if we write $[\mathcal{F}]_k$ then this implies that \mathcal{F} is coherent on X and $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$.

Lemma 6.2. In Situation 2.1 let X/B be good. Let \mathcal{F} be a coherent \mathcal{O}_X -module with $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. Let Z be an integral closed subspace of X with $\dim_\delta(Z) = k$. Let $\xi \in |Z|$ be the generic point. Then the coefficient of Z in $[\mathcal{F}]_k$ is the length of \mathcal{F} at ξ .

Proof. Observe that $|Z|$ is an irreducible component of $\text{Supp}(\mathcal{F})$ if and only if $\xi \in \text{Supp}(\mathcal{F})$, see Lemma 4.5. Moreover, the length of \mathcal{F} at ξ is zero if $\xi \notin \text{Supp}(\mathcal{F})$. Combining this with Definition 6.1 we conclude. \square

Lemma 6.3. In Situation 2.1 let X/B be good. Let $Y \subset X$ be a closed subspace. If $\dim_\delta(Y) \leq k$, then $[Y]_k = [i_* \mathcal{O}_Y]_k$ where $i : Y \rightarrow X$ is the inclusion morphism.

Proof. Let Z be an integral closed subspace of X with $\dim_\delta(Z) = k$. If $Z \not\subset Y$ the Z has coefficient zero in both $[Y]_k$ and $[i_* \mathcal{O}_Y]_k$. If $Z \subset Y$, then the generic point of Z may be viewed as a point $y \in |Y|$ whose image $x \in |X|$. Then the coefficient of Z in $[Y]_k$ is the length of \mathcal{O}_Y at y and the coefficient of Z in $[i_* \mathcal{O}_Y]_k$ is the length of $i_* \mathcal{O}_Y$ at x . Thus the equality of the coefficients follows from Lemma 4.3. \square

Lemma 6.4. *In Situation 2.1 let X/B be good. Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ be a short exact sequence of coherent \mathcal{O}_X -modules. Assume that the δ -dimension of the supports of \mathcal{F} , \mathcal{G} , and \mathcal{H} are $\leq k$. Then $[\mathcal{G}]_k = [\mathcal{F}]_k + [\mathcal{H}]_k$.*

Proof. Let Z be an integral closed subspace of X with $\dim_\delta(Z) = k$. It suffices to show that the coefficients of Z in $[\mathcal{G}]_k$, $[\mathcal{F}]_k$, and $[\mathcal{H}]_k$ satisfy the corresponding additivity. By Lemma 6.2 it suffices to show

$$\text{the length of } \mathcal{G} \text{ at } x = \text{the length of } \mathcal{F} \text{ at } x + \text{the length of } \mathcal{H} \text{ at } x$$

for any $x \in |X|$. Looking at Definition 4.2 this follows immediately from additivity of lengths, see Algebra, Lemma 52.3. \square

7. Preparation for proper pushforward

This section is the analogue of Chow Homology, Section 11.

Lemma 7.1. *In Situation 2.1 let $X, Y/B$ be good and let $f : X \rightarrow Y$ be a morphism over B . If $Z \subset X$ is an integral closed subspace, then there exists a unique integral closed subspace $Z' \subset Y$ such that there is a commutative diagram*

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Z' & \longrightarrow & Y \end{array}$$

with $Z \rightarrow Z'$ dominant. If f is proper, then $Z \rightarrow Z'$ is proper and surjective.

Proof. Let $\xi \in |Z|$ be the generic point. Let $Z' \subset Y$ be the integral closed subspace whose generic point is $\xi' = f(\xi)$, see Remark 2.3. Since $\xi \in |f^{-1}(Z')| = |f|^{-1}(|Z'|)$ by Properties of Spaces, Lemma 4.3 and since Z is the reduced with $|Z| = \{\overline{\xi}\}$ we see that $Z \subset f^{-1}(Z')$ as closed subspaces of X (see Properties of Spaces, Lemma 12.4). Thus we obtain our morphism $Z \rightarrow Z'$. This morphism is dominant as the generic point of Z maps to the generic point of Z' . Uniqueness of Z' is clear. If f is proper, then $Z \rightarrow Y$ is proper as a composition of proper morphisms (Morphisms of Spaces, Lemmas 40.3 and 40.5). Then we conclude that $Z \rightarrow Z'$ is proper by Morphisms of Spaces, Lemma 40.6. Surjectivity then follows as the image of a proper morphism is closed. \square

Remark 7.2. In Situation 2.1 let X/B be good. Every $x \in |X|$ can be represented by a (unique) monomorphism $\text{Spec}(k) \rightarrow X$ where k is a field, see Decent Spaces, Lemma 11.1. Then k is the *residue field* of x and is denoted $\kappa(x)$. Recall that X has a dense open subscheme $U \subset X$ (Properties of Spaces, Proposition 13.3). If $x \in U$, then $\kappa(x)$ agrees with the residue field of x on U as a scheme. See Decent Spaces, Section 11.

Remark 7.3. In Situation 2.1 let X/B be good. Assume X is integral. In this case the *function field* $R(X)$ of X is defined and is equal to the residue field of X at its generic point. See Spaces over Fields, Definition 4.3. Combining this with Remark 2.3 we find that for any $x \in X$ the residue field $\kappa(x)$ is the function field of the unique integral closed subspace $Z \subset X$ whose generic point is x .

Lemma 7.4. *In Situation 2.1 let $X, Y/B$ be good and let $f : X \rightarrow Y$ be a morphism over B . Assume X, Y integral and $\dim_\delta(X) = \dim_\delta(Y)$. Then either f*

factors through a proper closed subspace of Y , or f is dominant and the extension of function fields $R(X)/R(Y)$ is finite.

Proof. By Lemma 7.1 there is a unique integral closed subspace $Z \subset Y$ such that f factors through a dominant morphism $X \rightarrow Z$. Then $Z = Y$ if and only if $\dim_\delta(Z) = \dim_\delta(Y)$. On the other hand, by our construction of dimension functions (see Situation 2.1) we have $\dim_\delta(X) = \dim_\delta(Z) + r$ where r the transcendence degree of the extension $R(X)/R(Z)$. Combining this with Spaces over Fields, Lemma 5.1 the lemma follows. \square

Lemma 7.5. *In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a morphism over B . Assume f is quasi-compact, and $\{T_i\}_{i \in I}$ is a locally finite collection of closed subsets of $|X|$. Then $\{\overline{|f|(T_i)}\}_{i \in I}$ is a locally finite collection of closed subsets of $|Y|$.*

Proof. Let $V \subset |Y|$ be a quasi-compact open subset. Then $|f|^{-1}(V) \subset |X|$ is quasi-compact by Morphisms of Spaces, Lemma 8.3. Hence the set $\{i \in I : T_i \cap |f|^{-1}(V) \neq \emptyset\}$ is finite by a simple topological argument which we omit. Since this is the same as the set

$$\{i \in I : |f|(T_i) \cap V \neq \emptyset\} = \{i \in I : \overline{|f|(T_i)} \cap V \neq \emptyset\}$$

the lemma is proved. \square

8. Proper pushforward

This section is the analogue of Chow Homology, Section 12.

Definition 8.1. In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a morphism over B . Assume f is proper.

- (1) Let $Z \subset X$ be an integral closed subspace with $\dim_\delta(Z) = k$. Let $Z' \subset Y$ be the image of Z as in Lemma 7.1. We define

$$f_*[Z] = \begin{cases} 0 & \text{if } \dim_\delta(Z') < k, \\ \deg(Z/Z')[Z'] & \text{if } \dim_\delta(Z') = k. \end{cases}$$

The degree of Z over Z' is defined and finite if $\dim_\delta(Z') = \dim_\delta(Z)$ by Lemma 7.4 and Spaces over Fields, Definition 5.2.

- (2) Let $\alpha = \sum n_Z [Z]$ be a k -cycle on X . The *pushforward* of α as the sum

$$f_*\alpha = \sum n_Z f_*[Z]$$

where each $f_*[Z]$ is defined as above. The sum is locally finite by Lemma 7.5 above.

By definition the proper pushforward of cycles

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

is a homomorphism of abelian groups. It turns $X \mapsto Z_k(X)$ into a covariant functor on the category whose object are good algebraic spaces over B and whose morphisms are proper morphisms over B .

Lemma 8.2. *In Situation 2.1 let $X, Y, Z/B$ be good. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be proper morphisms over B . Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \rightarrow Z_k(Z)$.*

Proof. Let $W \subset X$ be an integral closed subspace of dimension k . Consider the integral closed subspaces $W' \subset Y$ and $W'' \subset Z$ we get by applying Lemma 7.1 to f and W and then to g and W' . Then $W \rightarrow W'$ and $W' \rightarrow W''$ are surjective and proper. We have to show that $g_*(f_*[W]) = (f \circ g)_*[W]$. If $\dim_\delta(W'') < k$, then both sides are zero. If $\dim_\delta(W'') = k$, then we see $W \rightarrow W'$ and $W' \rightarrow W''$ both satisfy the hypotheses of Lemma 7.4. Hence

$$g_*(f_*[W]) = \deg(W/W') \deg(W'/W'')[W''], \quad (f \circ g)_*[W] = \deg(W/W'')[W''].$$

Then we can apply Spaces over Fields, Lemma 5.3 to conclude. \square

Lemma 8.3. *In Situation 2.1 let $f : X \rightarrow Y$ be a proper morphism of good algebraic spaces over B .*

(1) *Let $Z \subset X$ be a closed subspace with $\dim_\delta(Z) \leq k$. Then*

$$f_*[Z]_k = [f_*\mathcal{O}_Z]_k.$$

(2) *Let \mathcal{F} be a coherent sheaf on X such that $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. Then*

$$f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k.$$

Note that the statement makes sense since $f_\mathcal{F}$ and $f_*\mathcal{O}_Z$ are coherent \mathcal{O}_Y -modules by Cohomology of Spaces, Lemma 20.2.*

Proof. Part (1) follows from (2) and Lemma 6.3. Let \mathcal{F} be a coherent sheaf on X . Assume that $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. By Cohomology of Spaces, Lemma 12.7 there exists a closed immersion $i : Z \rightarrow X$ and a coherent \mathcal{O}_Z -module \mathcal{G} such that $i_*\mathcal{G} \cong \mathcal{F}$ and such that the support of \mathcal{F} is Z . Let $Z' \subset Y$ be the scheme theoretic image of $f|_Z : Z \rightarrow Y$, see Morphisms of Spaces, Definition 16.2. Consider the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ f|_Z \downarrow & & \downarrow f \\ Z' & \xrightarrow{i'} & Y \end{array}$$

of algebraic spaces over B . Observe that $f|_Z$ is surjective (follows from Morphisms of Spaces, Lemma 16.3 and the fact that $|f|$ is closed) and proper (follows from Morphisms of Spaces, Lemmas 40.3, 40.5, and 40.6). We have $f_*\mathcal{F} = f_*i_*\mathcal{G} = i'_*(f|_Z)_*\mathcal{G}$ by going around the diagram in two ways. Suppose we know the result holds for closed immersions and for $f|_Z$. Then we see that

$$f_*[\mathcal{F}]_k = f_*i_*[\mathcal{G}]_k = (i')_*((f|_Z)_*[\mathcal{G}]_k) = (i')_*[(f|_Z)_*\mathcal{G}]_k = [(i')_*((f|_Z)_*\mathcal{G})]_k = [f_*\mathcal{F}]_k$$

as desired. The case of a closed immersion follows from Lemma 4.3 and the definitions. Thus we have reduced to the case where $\dim_\delta(X) \leq k$ and $f : X \rightarrow Y$ is proper and surjective.

Assume $\dim_\delta(X) \leq k$ and $f : X \rightarrow Y$ is proper and surjective. For every irreducible component $Z \subset Y$ with generic point η there exists a point $\xi \in X$ such that $f(\xi) = \eta$. Hence $\delta(\eta) \leq \delta(\xi) \leq k$. Thus we see that in the expressions

$$f_*[\mathcal{F}]_k = \sum n_Z[Z], \quad \text{and} \quad [f_*\mathcal{F}]_k = \sum m_Z[Z].$$

whenever $n_Z \neq 0$, or $m_Z \neq 0$ the integral closed subspace Z is actually an irreducible component of Y of δ -dimension k (see Lemma 4.5). Pick such an integral closed subspace $Z \subset Y$ and denote η its generic point. Note that for any $\xi \in X$ with $f(\xi) = \eta$ we have $\delta(\xi) \geq k$ and hence ξ is a generic point of an irreducible

component of X of δ -dimension k as well (see Lemma 4.5). By Spaces over Fields, Lemma 3.2 there exists an open subspace $\eta \in V \subset Y$ such that $f^{-1}(V) \rightarrow V$ is finite. Since η is a generic point of an irreducible component of $|Y|$ we may assume V is an affine scheme, see Properties of Spaces, Proposition 13.3. Replacing Y by V and X by $f^{-1}(V)$ we reduce to the case where Y is affine, and f is finite. In particular X and Y are schemes and we reduce to the corresponding result for schemes, see Chow Homology, Lemma 12.4 (applied with $S = Y$). \square

9. Preparation for flat pullback

This section is the analogue of Chow Homology, Section 13.

Recall that a morphism of algebraic spaces is said to have relative dimension r if étale locally on the source and the target we get a morphism of schemes which has relative dimension r . The precise definition is equivalent, but in fact slightly different, see Morphisms of Spaces, Definition 33.2.

Lemma 9.1. *In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a morphism over B . Assume f is flat of relative dimension r . For any closed subset $T \subset |Y|$ we have*

$$\dim_\delta(|f|^{-1}(T)) = \dim_\delta(T) + r.$$

provided $|f|^{-1}(T)$ is nonempty. If $Z \subset Y$ is an integral closed subscheme and $Z' \subset f^{-1}(Z)$ is an irreducible component, then Z' dominates Z and $\dim_\delta(Z') = \dim_\delta(Z) + r$.

Proof. Since the δ -dimension of a closed subset is the supremum of the δ -dimensions of the irreducible components, it suffices to prove the final statement. We may replace Y by the integral closed subscheme Z and X by $f^{-1}(Z) = Z \times_Y X$. Hence we may assume $Z = Y$ is integral and f is a flat morphism of relative dimension r . Since Y is locally Noetherian the morphism f which is locally of finite type, is actually locally of finite presentation. Hence Morphisms of Spaces, Lemma 30.6 applies and we see that f is open. Let $\xi \in X$ be a generic point of an irreducible component of X . By the openness of f we see that $f(\xi)$ is the generic point η of $Z = Y$. Thus Z' dominates $Z = Y$. Finally, we see that ξ and η are in the schematic locus of X and Y by Properties of Spaces, Proposition 13.3. Since ξ is a generic point of X we see that $\mathcal{O}_{X,\xi} = \mathcal{O}_{X_\eta,\xi}$ has only one prime ideal and hence has dimension 0 (we may use usual local rings as ξ and η are in the schematic loci of X and Y). Thus by Morphisms of Spaces, Lemma 34.1 (and the definition of morphisms of given relative dimension) we conclude that the transcendence degree of $\kappa(\xi)$ over $\kappa(\eta)$ is r . In other words, $\delta(\xi) = \delta(\eta) + r$ as desired. \square

Here is the lemma that we will use to prove that the flat pullback of a locally finite collection of closed subschemes is locally finite.

Lemma 9.2. *In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a morphism over B . Assume $\{T_i\}_{i \in I}$ is a locally finite collection of closed subsets of $|Y|$. Then $\{|f|^{-1}(T_i)\}_{i \in I}$ is a locally finite collection of closed subsets of X .*

Proof. Let $U \subset |X|$ be a quasi-compact open subset. Since the image $|f|(U) \subset |Y|$ is a quasi-compact subset there exists a quasi-compact open $V \subset |Y|$ such that $|f|(U) \subset V$. Note that

$$\{i \in I : |f|^{-1}(T_i) \cap U \neq \emptyset\} \subset \{i \in I : T_i \cap V \neq \emptyset\}.$$

Since the right hand side is finite by assumption we win. \square

10. Flat pullback

This section is the analogue of Chow Homology, Section 14.

Let S be a scheme and let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let $Z \subset Y$ be a closed subspace. In this chapter we will sometimes use the terminology *scheme theoretic inverse image* for the inverse image $f^{-1}(Z)$ of Z constructed in Morphisms of Spaces, Definition 13.2. The scheme theoretic inverse image is the fibre product

$$\begin{array}{ccc} f^{-1}(Z) & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

If $\mathcal{I} \subset \mathcal{O}_Y$ is the quasi-coherent sheaf of ideals corresponding to Z in Y , then $f^{-1}(\mathcal{I})\mathcal{O}_X$ is the quasi-coherent sheaf of ideals corresponding to $f^{-1}(Z)$ in X .

Definition 10.1. In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a morphism over B . Assume f is flat of relative dimension r .

- (1) Let $Z \subset Y$ be an integral closed subspace of δ -dimension k . We define $f^*[Z]$ to be the $(k+r)$ -cycle on X associated to the scheme theoretic inverse image

$$f^*[Z] = [f^{-1}(Z)]_{k+r}.$$

This makes sense since $\dim_\delta(f^{-1}(Z)) = k + r$ by Lemma 9.1.

- (2) Let $\alpha = \sum n_i [Z_i]$ be a k -cycle on Y . The *flat pullback of α by f* is the sum

$$f^*\alpha = \sum n_i f^*[Z_i]$$

where each $f^*[Z_i]$ is defined as above. The sum is locally finite by Lemma 9.2.

- (3) We denote $f^* : Z_k(Y) \rightarrow Z_{k+r}(X)$ the map of abelian groups so obtained.

An open immersion is flat. This is an important though trivial special case of a flat morphism. If $U \subset X$ is open then sometimes the pullback by $j : U \rightarrow X$ of a cycle is called the *restriction* of the cycle to U . Note that in this case the maps

$$j^* : Z_k(X) \longrightarrow Z_k(U)$$

are all *surjective*. The reason is that given any integral closed subspace $Z' \subset U$, we can take the closure of Z' in X and think of it as a reduced closed subspace of X (see Properties of Spaces, Definition 12.5). And clearly $Z \cap U = Z'$, in other words $j^*[Z] = [Z']$ whence the surjectivity. In fact a little bit more is true.

Lemma 10.2. In Situation 2.1 let X/B be good. Let $U \subset X$ be an open subspace. Let Y be the reduced closed subspace of X with $|Y| = |X| \setminus |U|$ and denote $i : Y \rightarrow X$ the inclusion morphism. For every $k \in \mathbf{Z}$ the sequence

$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. Surjectivity of j^* we saw above. First assume X is quasi-compact. Then $Z_k(X)$ is a free \mathbf{Z} -module with basis given by the elements $[Z]$ where $Z \subset X$ is integral closed of δ -dimension k . Such a basis element maps either to the basis element $[Z \cap U]$ of $Z_k(U)$ or to zero if $Z \subset Y$. Hence the lemma is clear in this case. The general case is similar and the proof is omitted. \square

Lemma 10.3. *In Situation 2.1 let $f : X \rightarrow Y$ be an étale morphism of good algebraic spaces over B . If $Z \subset Y$ is an integral closed subspace, then $f^*[Z] = \sum [Z']$ where the sum is over the irreducible components (Remark 5.1) of $f^{-1}(Z)$.*

Proof. The meaning of the lemma is that the coefficient of $[Z']$ is 1. This follows from the fact that $f^{-1}(Z)$ is a reduced algebraic space because it is étale over the integral algebraic space Z . \square

Lemma 10.4. *In Situation 2.1 let $X, Y, Z/B$ be good. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be flat morphisms of relative dimensions r and s over B . Then $g \circ f$ is flat of relative dimension $r + s$ and*

$$f^* \circ g^* = (g \circ f)^*$$

as maps $Z_k(Z) \rightarrow Z_{k+r+s}(X)$.

Proof. The composition is flat of relative dimension $r + s$ by Morphisms of Spaces, Lemmas 34.2 and 30.3. Suppose that

- (1) $A \subset Z$ is a closed integral subspace of δ -dimension k ,
- (2) $A' \subset Y$ is a closed integral subspace of δ -dimension $k + s$ with $A' \subset g^{-1}(A)$, and
- (3) $A'' \subset X$ is a closed integral subspace of δ -dimension $k + s + r$ with $A'' \subset f^{-1}(A')$.

We have to show that the coefficient n of $[A'']$ in $(g \circ f)^*[A]$ agrees with the coefficient m of $[A'']$ in $f^*(g^*[A])$. We may choose a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

where U, V, W are schemes, the vertical arrows are étale, and there exist points $u \in U$, $v \in V$, $w \in W$ such that $u \mapsto v \mapsto w$ and such that u, v, w map to the generic points of A'', A', A . (Details omitted.) Then we have flat local ring homomorphisms $\mathcal{O}_{W,w} \rightarrow \mathcal{O}_{V,v}$, $\mathcal{O}_{V,v} \rightarrow \mathcal{O}_{U,u}$, and repeatedly using Lemma 4.1 we find

$$n = \text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/\mathfrak{m}_w \mathcal{O}_{U,u})$$

and

$$m = \text{length}_{\mathcal{O}_{V,v}}(\mathcal{O}_{V,v}/\mathfrak{m}_w \mathcal{O}_{V,v}) \text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/\mathfrak{m}_v \mathcal{O}_{U,u})$$

Hence the equality follows from Algebra, Lemma 52.14. \square

Lemma 10.5. *In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a flat morphism of relative dimension r .*

- (1) *Let $Z \subset Y$ be a closed subspace with $\dim_\delta(Z) \leq k$. Then we have $\dim_\delta(f^{-1}(Z)) \leq k + r$ and $[f^{-1}(Z)]_{k+r} = f^*[Z]_k$ in $Z_{k+r}(X)$.*

- (2) Let \mathcal{F} be a coherent sheaf on Y with $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. Then we have $\dim_\delta(\text{Supp}(f^*\mathcal{F})) \leq k + r$ and

$$f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$$

in $Z_{k+r}(X)$.

Proof. Part (1) follows from part (2) by Lemma 6.3 and the fact that $f^*\mathcal{O}_Z = \mathcal{O}_{f^{-1}(Z)}$.

Proof of (2). As X, Y are locally Noetherian we may apply Cohomology of Spaces, Lemma 12.2 to see that \mathcal{F} is of finite type, hence $f^*\mathcal{F}$ is of finite type (Modules on Sites, Lemma 23.4), hence $f^*\mathcal{F}$ is coherent (Cohomology of Spaces, Lemma 12.2 again). Thus the lemma makes sense. Let $W \subset Y$ be an integral closed subspace of δ -dimension k , and let $W' \subset X$ be an integral closed subspace of dimension $k + r$ mapping into W under f . We have to show that the coefficient n of $[W']$ in $f^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W']$ in $[f^*\mathcal{F}]_{k+r}$. We may choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and there exist points $u \in U$, $v \in V$ such that $u \mapsto v$ and such that u, v map to the generic points of W', W . (Details omitted.) Consider the stalk $M = (\mathcal{F}|_V)_v$ as an $\mathcal{O}_{V,v}$ -module. (Note that M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 4.4.) We have $(f^*\mathcal{F}|_U)_u = \mathcal{O}_{U,u} \otimes_{\mathcal{O}_{V,v}} M$. Thus we see that

$$n = \text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u} \otimes_{\mathcal{O}_{V,v}} M) \quad \text{and} \quad m = \text{length}_{\mathcal{O}_{V,v}}(M) \text{length}_{\mathcal{O}_{V,v}}(\mathcal{O}_{U,u}/\mathfrak{m}_v \mathcal{O}_{U,u})$$

Thus the equality follows from Algebra, Lemma 52.13. \square

11. Push and pull

This section is the analogue of Chow Homology, Section 14.

In this section we verify that proper pushforward and flat pullback are compatible when this makes sense. By the work we did above this is a consequence of cohomology and base change.

Lemma 11.1. *In Situation 2.1 let*

$$\begin{array}{ccc} X' & \xrightarrow{\quad g' \quad} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\quad g \quad} & Y \end{array}$$

be a fibre product diagram of good algebraic spaces over B . Assume $f : X \rightarrow Y$ proper and $g : Y' \rightarrow Y$ flat of relative dimension r . Then also f' is proper and g' is flat of relative dimension r . For any k -cycle α on X we have

$$g^* f_* \alpha = f'_*(g')^* \alpha$$

in $Z_{k+r}(Y')$.

Proof. The assertion that f' is proper follows from Morphisms of Spaces, Lemma 40.3. The assertion that g' is flat of relative dimension r follows from Morphisms of Spaces, Lemmas 34.3 and 30.4. It suffices to prove the equality of cycles when $\alpha = [W]$ for some integral closed subspace $W \subset X$ of δ -dimension k . Note that in this case we have $\alpha = [\mathcal{O}_W]_k$, see Lemma 6.3. By Lemmas 8.3 and 10.5 it therefore suffices to show that $f'_*(g')^*\mathcal{O}_W$ is isomorphic to $g^*f_*\mathcal{O}_W$. This follows from cohomology and base change, see Cohomology of Spaces, Lemma 11.2. \square

Lemma 11.2. *In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a finite locally free morphism of degree d (see Morphisms of Spaces, Definition 46.2). Then f is both proper and flat of relative dimension 0, and*

$$f_*f^*\alpha = d\alpha$$

for every $\alpha \in Z_k(Y)$.

Proof. A finite locally free morphism is flat and finite by Morphisms of Spaces, Lemma 46.6, and a finite morphism is proper by Morphisms of Spaces, Lemma 45.9. We omit showing that a finite morphism has relative dimension 0. Thus the formula makes sense. To prove it, let $Z \subset Y$ be an integral closed subscheme of δ -dimension k . It suffices to prove the formula for $\alpha = [Z]$. Since the base change of a finite locally free morphism is finite locally free (Morphisms of Spaces, Lemma 46.5) we see that $f_*f^*\mathcal{O}_Z$ is a finite locally free sheaf of rank d on Z . Thus clearly $f_*f^*\mathcal{O}_Z$ has length d at the generic point of Z . Hence

$$f_*f^*[Z] = f_*f^*[\mathcal{O}_Z]_k = [f_*f^*\mathcal{O}_Z]_k = d[Z]$$

where we have used Lemmas 10.5 and 8.3. \square

12. Preparation for principal divisors

This section is the analogue of Chow Homology, Section 16. Some of the material in this section partially overlaps with the discussion in Spaces over Fields, Section 6.

Lemma 12.1. *In Situation 2.1 let X/B be good. Assume X is integral.*

- (1) *If $Z \subset X$ is an integral closed subspace, then the following are equivalent:*
 - (a) *Z is a prime divisor,*
 - (b) *$|Z|$ has codimension 1 in $|X|$, and*
 - (c) *$\dim_\delta(Z) = \dim_\delta(X) - 1$.*
- (2) *If Z is an irreducible component of an effective Cartier divisor on X , then $\dim_\delta(Z) = \dim_\delta(X) - 1$.*

Proof. Part (1) follows from the definition of a prime divisor (Spaces over Fields, Definition 6.2), Decent Spaces, Lemma 20.2, and the definition of a dimension function (Topology, Definition 20.1).

Let $D \subset X$ be an effective Cartier divisor. Let $Z \subset D$ be an irreducible component and let $\xi \in |Z|$ be the generic point. Choose an étale neighbourhood $(U, u) \rightarrow (X, \xi)$ where $U = \text{Spec}(A)$ and $D \times_X U$ is cut out by a nonzerodivisor $f \in A$, see Divisors on Spaces, Lemma 6.2. Then u is a generic point of $V(f)$ by Decent Spaces, Lemma 20.1. Hence $\mathcal{O}_{U,u}$ has dimension 1 by Krull's Hauptidealsatz (Algebra, Lemma 60.11). Thus ξ is a codimension 1 point on X and Z is a prime divisor as desired. \square

13. Principal divisors

This section is the analogue of Chow Homology, Section 17. The following definition is the analogue of Spaces over Fields, Definition 6.7 in our current setup.

Definition 13.1. In Situation 2.1 let X/B be good. Assume X is integral with $\dim_\delta(X) = n$. Let $f \in R(X)^*$. The *principal divisor associated to f* is the $(n-1)$ -cycle

$$\operatorname{div}(f) = \operatorname{div}_X(f) = \sum \operatorname{ord}_Z(f)[Z]$$

defined in Spaces over Fields, Definition 6.7. This makes sense because prime divisors have δ -dimension $n-1$ by Lemma 12.1.

In the situation of the definition for $f, g \in R(X)^*$ we have

$$\operatorname{div}_X(fg) = \operatorname{div}_X(f) + \operatorname{div}_X(g)$$

in $Z_{n-1}(X)$. See Spaces over Fields, Lemma 6.8. The following lemma will allow us to reduce statements about principal divisors to the case of schemes.

Lemma 13.2. *In Situation 2.1 let $f : X \rightarrow Y$ be an étale morphism of good algebraic spaces over B . Assume Y is integral. Let $g \in R(Y)^*$. As cycles on X we have*

$$f^*(\operatorname{div}_Y(g)) = \sum_{X'} (X' \rightarrow X)_* \operatorname{div}_{X'}(g \circ f|_{X'})$$

where the sum is over the irreducible components of X (Remark 5.1).

Proof. The map $|X| \rightarrow |Y|$ is open. The set of irreducible components of $|X|$ is locally finite in $|X|$. We conclude that $f|_{X'} : X' \rightarrow Y$ is dominant for every irreducible component $X' \subset X$. Thus $g \circ f|_{X'}$ is defined (Morphisms of Spaces, Section 47), hence $\operatorname{div}_{X'}(g \circ f|_{X'})$ is defined. Moreover, the sum is locally finite and we find that the right hand side indeed is a cycle on X . The left hand side is defined by Definition 10.1 and the fact that an étale morphism is flat of relative dimension 0.

Since f is étale we see that $\delta_X(x) = \delta_Y(f(x))$ for all $x \in |X|$. Thus if $\dim_\delta(Y) = n$, then $\dim_\delta(X') = n$ for every irreducible component X' of X (since generic points of X map to the generic point of Y , see above). Thus both left and right hand side are $(n-1)$ -cycles.

Let $Z \subset X$ be an integral closed subspace with $\dim_\delta(Z) = n-1$. To prove the equality, we need to show that the coefficients of Z are the same. Let $Z' \subset Y$ be the integral closed subspace constructed in Lemma 7.1. Then $\dim_\delta(Z') = n-1$ too. Let $\xi \in |Z|$ be the generic point. Then $\xi' = f(\xi) \in |Z'|$ is the generic point. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(\mathcal{O}_{X,\xi}^h) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathcal{O}_{Y,\xi'}^h) & \longrightarrow & Y \end{array}$$

of Decent Spaces, Remark 11.11. We have to be slightly careful as the reduced Noetherian local rings $\mathcal{O}_{X,\xi}^h$ and $\mathcal{O}_{Y,\xi'}^h$ need not be domains. Thus we work with total rings of fractions $Q(-)$ rather than fraction fields. By definition, to get the

coefficient of Z' in $\text{div}_Y(g)$ we write the image of g in $Q(\mathcal{O}_{Y,\xi'}^h)$ as a/b with $a, b \in \mathcal{O}_{Y,\xi'}^h$ nonzerodivisors and we take

$$\text{ord}_{Z'}(g) = \text{length}_{\mathcal{O}_{Y,\xi'}^h}(\mathcal{O}_{Y,\xi'}^h/a\mathcal{O}_{Y,\xi'}^h) - \text{length}_{\mathcal{O}_{Y,\xi'}^h}(\mathcal{O}_{Y,\xi'}^h/b\mathcal{O}_{Y,\xi'}^h)$$

Observe that the coefficient of Z in $f^*\text{div}_Y(G)$ is the same integer, see Lemma 10.3. Suppose that $\xi \in X'$. Then we can consider the maps

$$\mathcal{O}_{Y,\xi'}^h \rightarrow \mathcal{O}_{X,\xi}^h \rightarrow \mathcal{O}_{X',\xi}^h$$

The first arrow is flat and the second arrow is a surjective map of reduced local Noetherian rings of dimension 1. Therefore both these maps send nonzerodivisors to nonzerodivisors and we conclude the coefficient of Z' in $\text{div}_{X'}(g \circ f|_{X'})$ is

$$\text{ord}_Z(g \circ f|_{X'}) = \text{length}_{\mathcal{O}_{X',\xi}^h}(\mathcal{O}_{X',\xi}^h/a\mathcal{O}_{X',\xi}^h) - \text{length}_{\mathcal{O}_{X',\xi}^h}(\mathcal{O}_{X',\xi}^h/b\mathcal{O}_{X',\xi}^h)$$

by the same prescription as above. Thus it suffices to show

$$\text{length}_{\mathcal{O}_{Y,\xi'}^h}(\mathcal{O}_{Y,\xi'}^h/a\mathcal{O}_{Y,\xi'}^h) = \sum_{\xi \in |X'|} \text{length}_{\mathcal{O}_{X',\xi}^h}(\mathcal{O}_{X',\xi}^h/a\mathcal{O}_{X',\xi}^h)$$

First, since the ring map $\mathcal{O}_{Y,\xi'}^h \rightarrow \mathcal{O}_{X,\xi}^h$ is flat and unramified, we have

$$\text{length}_{\mathcal{O}_{Y,\xi'}^h}(\mathcal{O}_{Y,\xi'}^h/a\mathcal{O}_{Y,\xi'}^h) = \text{length}_{\mathcal{O}_{X,\xi}^h}(\mathcal{O}_{X,\xi}^h/a\mathcal{O}_{X,\xi}^h)$$

by Algebra, Lemma 52.13. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ be the nonmaximal primes of $\mathcal{O}_{X,\xi}^h$ and set $R_j = \mathcal{O}_{X,\xi}^h/\mathfrak{q}_j$. For X' as above, denote $J(X') \subset \{1, \dots, t\}$ the set of indices such that \mathfrak{q}_j corresponds to a point of X' , i.e., such that under the surjection $\mathcal{O}_{X,\xi}^h \rightarrow \mathcal{O}_{X',\xi}^h$ the prime \mathfrak{q}_j corresponds to a prime of $\mathcal{O}_{X',\xi}^h$. By Chow Homology, Lemma 3.2 we get

$$\text{length}_{\mathcal{O}_{X,\xi}^h}(\mathcal{O}_{X,\xi}^h/a\mathcal{O}_{X,\xi}^h) = \sum_j \text{length}_{R_j}(R_j/aR_j)$$

and

$$\text{length}_{\mathcal{O}_{X',\xi}^h}(\mathcal{O}_{X',\xi}^h/a\mathcal{O}_{X',\xi}^h) = \sum_{j \in J(X')} \text{length}_{R_j}(R_j/aR_j)$$

Thus the result of the lemma holds because $\{1, \dots, t\}$ is the disjoint union of the sets $J(X')$: each point of codimension 0 on X lies on a unique X' . \square

14. Principal divisors and pushforward

This section is the analogue of Chow Homology, Section 18.

Lemma 14.1. *In Situation 2.1 let $X, Y/B$ be good. Assume X, Y are integral and $n = \dim_\delta(X) = \dim_\delta(Y)$. Let $p : X \rightarrow Y$ be a dominant proper morphism. Let $f \in R(X)^*$. Set*

$$g = Nm_{R(X)/R(Y)}(f).$$

Then we have $p_\text{div}(f) = \text{div}(g)$.*

Proof. We are going to deduce this from the case of schemes by étale localization. Let $Z \subset Y$ be an integral closed subspace of δ -dimension $n-1$. We want to show that the coefficient of $[Z]$ in $p_*\text{div}(f)$ and $\text{div}(g)$ are equal. Apply Spaces over Fields, Lemma 3.2 to the morphism $p : X \rightarrow Y$ and the generic point $\xi \in |Z|$. We find that we may replace Y by an open subspace containing ξ and assume that $p : X \rightarrow Y$ is finite. Pick an étale neighbourhood $(V, v) \rightarrow (Y, \xi)$ where V is an

affine scheme. By Lemma 10.3 it suffices to prove the equality of cycles after pulling back to V . Set $U = V \times_Y X$ and consider the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{a} & X \\ p' \downarrow & & \downarrow p \\ V & \xrightarrow{b} & Y \end{array}$$

Let $V_j \subset V$, $j = 1, \dots, m$ be the irreducible components of V . For each i , let $U_{j,i}$, $i = 1, \dots, n_j$ be the irreducible components of U dominating V_j . Denote $p'_{j,i} : U_{j,i} \rightarrow V_j$ the restriction of $p' : U \rightarrow V$. By the case of schemes (Chow Homology, Lemma 18.1) we see that

$$p'_{j,i,*} \operatorname{div}_{U_{j,i}}(f_{j,i}) = \operatorname{div}_{V_j}(g_{j,i})$$

where $f_{j,i}$ is the restriction of f to $U_{j,i}$ and $g_{j,i}$ is the norm of $f_{j,i}$ along the finite extension $R(U_{j,i})/R(V_j)$. We have

$$\begin{aligned} b^* p_* \operatorname{div}_X(f) &= p'_* a^* \operatorname{div}_X(f) \\ &= p'_* \left(\sum_{j,i} (U_{j,i} \rightarrow U)_* \operatorname{div}_{U_{j,i}}(f_{j,i}) \right) \\ &= \sum_{j,i} (V_j \rightarrow V)_* p'_{j,i,*} \operatorname{div}_{U_{j,i}}(f_{j,i}) \\ &= \sum_j (V_j \rightarrow V)_* \left(\sum_i \operatorname{div}_{V_j}(g_{j,i}) \right) \\ &= \sum_j (V_j \rightarrow V)_* \operatorname{div}_{V_j} \left(\prod_i g_{j,i} \right) \end{aligned}$$

by Lemmas 11.1, 13.2, and 8.2. To finish the proof, using Lemma 13.2 again, it suffices to show that

$$g \circ b|_{V_j} = \prod_i g_{j,i}$$

as elements of the function field of V_j . In terms of fields this is the following statement: let L/K be a finite extension. Let M/K be a finite separable extension. Write $M \otimes_K L = \prod M_i$. Then for $t \in L$ with images $t_i \in M_i$ the image of $\operatorname{Norm}_{L/K}(t)$ in M is $\prod \operatorname{Norm}_{M_i/M}(t_i)$. We omit the proof. \square

15. Rational equivalence

This section is the analogue of Chow Homology, Section 19. In this section we define *rational equivalence* on k -cycles. We will allow locally finite sums of images of principal divisors (under closed immersions). This leads to some pretty strange phenomena (see examples in the chapter on schemes). However, if we do not allow these then we do not know how to prove that capping with Chern classes of line bundles factors through rational equivalence.

Definition 15.1. In Situation 2.1 let X/B be good. Let $k \in \mathbf{Z}$.

- (1) Given any locally finite collection $\{W_j \subset X\}$ of integral closed subspaces with $\dim_\delta(W_j) = k + 1$, and any $f_j \in R(W_j)^*$ we may consider

$$\sum (i_j)_* \operatorname{div}(f_j) \in Z_k(X)$$

where $i_j : W_j \rightarrow X$ is the inclusion morphism. This makes sense as the morphism $\coprod i_j : \coprod W_j \rightarrow X$ is proper.

- (2) We say that $\alpha \in Z_k(X)$ is *rationally equivalent to zero* if α is a cycle of the form displayed above.

- (3) We say $\alpha, \beta \in Z_k(X)$ are *rationally equivalent* and we write $\alpha \sim_{rat} \beta$ if $\alpha - \beta$ is rationally equivalent to zero.
- (4) We define

$$CH_k(X) = Z_k(X) / \sim_{rat}$$

to be the *Chow group of k -cycles on X* . This is sometimes called the *Chow group of k -cycles modulo rational equivalence on X* .

There are many other interesting equivalence relations. Rational equivalence is the coarsest of them all. A very simple but important lemma is the following.

Lemma 15.2. *In Situation 2.1 let X/B be good. Let $U \subset X$ be an open subspace. Let Y be the reduced closed subspace of X with $|Y| = |X| \setminus |U|$ and denote $i : Y \rightarrow X$ the inclusion morphism. Let $k \in \mathbf{Z}$. Suppose $\alpha, \beta \in Z_k(X)$. If $\alpha|_U \sim_{rat} \beta|_U$ then there exist a cycle $\gamma \in Z_k(Y)$ such that*

$$\alpha \sim_{rat} \beta + i_* \gamma.$$

In other words, the sequence

$$CH_k(Y) \xrightarrow{i_*} CH_k(X) \xrightarrow{j^*} CH_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. Let $\{W_j\}_{j \in J}$ be a locally finite collection of integral closed subspaces of U of δ -dimension $k+1$, and let $f_j \in R(W_j)^*$ be elements such that $(\alpha - \beta)|_U = \sum (i_j)_* \text{div}(f_j)$ as in the definition. Let $W'_j \subset X$ be the corresponding integral closed subspace of X , i.e., having the same generic point as W_j . Suppose that $V \subset X$ is a quasi-compact open. Then also $V \cap U$ is quasi-compact open in U as V is Noetherian. Hence the set $\{j \in J \mid W_j \cap V \neq \emptyset\} = \{j \in J \mid W'_j \cap V \neq \emptyset\}$ is finite since $\{W_j\}$ is locally finite. In other words we see that $\{W'_j\}$ is also locally finite. Since $R(W_j) = R(W'_j)$ we see that

$$\alpha - \beta - \sum (i'_j)_* \text{div}(f_j)$$

is a cycle on X whose restriction to U is zero. The lemma follows by applying Lemma 10.2. \square

Remark 15.3. In Situation 2.1 let X/B be good. Suppose we have infinite collections $\alpha_i, \beta_i \in Z_k(X)$, $i \in I$ of k -cycles on X . Suppose that the supports of α_i and β_i form locally finite collections of closed subsets of X so that $\sum \alpha_i$ and $\sum \beta_i$ are defined as cycles. Moreover, assume that $\alpha_i \sim_{rat} \beta_i$ for each i . Then it is not clear that $\sum \alpha_i \sim_{rat} \sum \beta_i$. Namely, the problem is that the rational equivalences may be given by locally finite families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ but the union $\{W_{i,j}\}_{i \in I, j \in J_i}$ may not be locally finite.

In many cases in practice, one has a locally finite family of closed subsets $\{T_i\}_{i \in I}$ of $|X|$ such that α_i, β_i are supported on T_i and such that $\alpha_i \sim_{rat} \beta_i$ “on” T_i . More precisely, the families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ consist of integral closed subspaces $W_{i,j}$ with $|W_{i,j}| \subset T_i$. In this case it is true that $\sum \alpha_i \sim_{rat} \sum \beta_i$ on X , simply because the family $\{W_{i,j}\}_{i \in I, j \in J_i}$ is automatically locally finite in this case.

16. Rational equivalence and push and pull

This section is the analogue of Chow Homology, Section 20. In this section we show that flat pullback and proper pushforward commute with rational equivalence.

Lemma 16.1. *In Situation 2.1 let $X, Y/B$ be good. Assume Y integral with $\dim_\delta(Y) = k$. Let $f : X \rightarrow Y$ be a flat morphism of relative dimension r . Then for $g \in R(Y)^*$ we have*

$$f^* \operatorname{div}_Y(g) = \sum m_{X',X}(X' \rightarrow X)_* \operatorname{div}_{X'}(g \circ f|_{X'})$$

as $(k+r-1)$ -cycles on X where the sum is over the irreducible components X' of X and $m_{X',X}$ is the multiplicity of X' in X .

Proof. Observe that any irreducible component of X dominates Y (Lemma 9.1) and hence the composition $g \circ f|_{X'}$ is defined (Morphisms of Spaces, Section 47). We will reduce this to the case of schemes. Choose a scheme V and a surjective étale morphism $V \rightarrow Y$. Choose a scheme U and a surjective étale morphism $U \rightarrow V \times_Y X$. Picture

$$\begin{array}{ccc} U & \xrightarrow{a} & X \\ h \downarrow & & \downarrow f \\ V & \xrightarrow{b} & Y \end{array}$$

Since a is surjective and étale it follows from Lemma 10.3 that it suffices to prove the equality of cycles after pulling back by a . We can use Lemma 13.2 to write

$$b^* \operatorname{div}_Y(g) = \sum (V' \rightarrow V)_* \operatorname{div}_{V'}(g \circ b|_{V'})$$

where the sum is over the irreducible components V' of V . Using Lemma 11.1 we find

$$h^* b^* \operatorname{div}_Y(g) = \sum (V' \times_V U \rightarrow U)_* (h')^* \operatorname{div}_{V'}(g \circ b|_{V'})$$

where $h' : V' \times_V U \rightarrow V'$ is the projection. We may apply the lemma in the case of schemes (Chow Homology, Lemma 20.1) to the morphism $h' : V' \times_V U \rightarrow V'$ to see that we have

$$(h')^* \operatorname{div}_{V'}(g \circ b|_{V'}) = \sum m_{U', V' \times_V U}(U' \rightarrow V' \times_V U)_* \operatorname{div}_{U'}(g \circ b|_{V'} \circ h'|_{U'})$$

where the sum is over the irreducible components U' of $V' \times_V U$. Each U' occurring in this sum is an irreducible component of U and conversely every irreducible component U' of U is an irreducible component of $V' \times_V U$ for a unique irreducible component $V' \subset V$. Given an irreducible component $U' \subset U$, denote $\overline{a(U')} \subset X$ the “image” in X (Lemma 7.1); this is an irreducible component of X for example by Lemma 9.1. The multiplicity $m_{U', V' \times_V U}$ is equal to the multiplicity $m_{\overline{a(U')}, X}$. This follows from the equality $h^* a^*[Y] = b^* f^*[Y]$ (Lemma 10.4), the definitions, and Lemma 10.3. Combining all of what we just said we obtain

$$a^* f^* \operatorname{div}_Y(g) = h^* b^* \operatorname{div}_Y(g) = \sum m_{\overline{a(U')}, X}(U' \rightarrow U)_* \operatorname{div}_{U'}(g \circ (f \circ a)|_{U'})$$

Next, we analyze what happens with the right hand side of the formula in the statement of the lemma if we pullback by a . First, we use Lemma 11.1 to get

$$a^* \sum m_{X',X}(X' \rightarrow X)_* \operatorname{div}_{X'}(g \circ f|_{X'}) = \sum m_{X',X}(X' \times_X U \rightarrow U)_* (a')^* \operatorname{div}_{X'}(g \circ f|_{X'})$$

where $a' : X' \times_X U \rightarrow X'$ is the projection. By Lemma 13.2 we get

$$(a')^* \operatorname{div}_{X'}(g \circ f|_{X'}) = \sum (U' \rightarrow X' \times_X U)_* \operatorname{div}_{U'}(g \circ (f \circ a)|_{U'})$$

where the sum is over the irreducible components U' of $X' \times_X U$. These U' are irreducible components of U and in fact are exactly the irreducible components of U such that $\overline{a(U')} = X'$. Comparing with what we obtained above we conclude. \square

Lemma 16.2. *In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a flat morphism of relative dimension r . Let $\alpha \sim_{\text{rat}} \beta$ be rationally equivalent k -cycles on Y . Then $f^* \alpha \sim_{\text{rat}} f^* \beta$ as $(k+r)$ -cycles on X .*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow Y$$

of closed immersions, with each W_j integral of δ -dimension $k+1$ and rational functions $g_j \in R(W_j)^*$. Moreover, assume that the collection $\{|i_j|(|W_j|)\}_{j \in J}$ is locally finite in $|Y|$. Then we have to show that

$$f^* \left(\sum i_{j,*} \operatorname{div}(g_j) \right) = \sum f^* i_{j,*} \operatorname{div}(g_j)$$

is rationally equivalent to zero on X . The sum on the right makes sense by Lemma 9.2.

Consider the fibre products

$$i'_j : W'_j = W_j \times_Y X \longrightarrow X.$$

and denote $f_j : W'_j \rightarrow W_j$ the first projection. By Lemma 11.1 we can write the sum above as

$$\sum i'_{j,*} (f_j^* \operatorname{div}(g_j))$$

By Lemma 16.1 we see that each $f_j^* \operatorname{div}(g_j)$ is rationally equivalent to zero on W'_j . Hence each $i'_{j,*} (f_j^* \operatorname{div}(g_j))$ is rationally equivalent to zero. Then the same is true for the displayed sum by the discussion in Remark 15.3. \square

Lemma 16.3. *In Situation 2.1 let $X, Y/B$ be good. Let $p : X \rightarrow Y$ be a proper morphism. Suppose $\alpha, \beta \in Z_k(X)$ are rationally equivalent. Then $p_* \alpha$ is rationally equivalent to $p_* \beta$.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow X$$

of closed immersions, with each W_j integral of δ -dimension $k+1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j \in J}$ is locally finite on X . Then we have to show that

$$p_* \left(\sum i_{j,*} \operatorname{div}(f_j) \right)$$

is rationally equivalent to zero on X .

Note that the sum is equal to

$$\sum p_* i_{j,*} \operatorname{div}(f_j).$$

Let $W'_j \subset Y$ be the integral closed subspace which is the image of $p \circ i_j$, see Lemma 7.1. The collection $\{W'_j\}$ is locally finite in Y by Lemma 7.5. Hence it suffices to

show, for a given j , that either $p_* i_{j,*} \operatorname{div}(f_j) = 0$ or that it is equal to $i'_{j,*} \operatorname{div}(g_j)$ for some $g_j \in R(W'_j)^*$.

The arguments above therefore reduce us to the case of a since integral closed subspace $W \subset X$ of δ -dimension $k + 1$. Let $f \in R(W)^*$. Let $W' = p(W)$ as above. We get a commutative diagram of morphisms

$$\begin{array}{ccc} W & \xrightarrow{i} & X \\ p' \downarrow & & \downarrow p \\ W' & \xrightarrow{i'} & Y \end{array}$$

Note that $p_* i_* \operatorname{div}(f) = i'_*(p')_* \operatorname{div}(f)$ by Lemma 8.2. As explained above we have to show that $(p')_* \operatorname{div}(f)$ is the divisor of a rational function on W' or zero. There are three cases to distinguish.

The case $\dim_\delta(W') < k$. In this case automatically $(p')_* \operatorname{div}(f) = 0$ and there is nothing to prove.

The case $\dim_\delta(W') = k$. Let us show that $(p')_* \operatorname{div}(f) = 0$ in this case. Since $(p')_* \operatorname{div}(f)$ is a k -cycle, we see that $(p')_* \operatorname{div}(f) = n[W']$ for some $n \in \mathbf{Z}$. In order to prove that $n = 0$ we may replace W' by a nonempty open subspace. In particular, we may and do assume that W' is a scheme. Let $\eta \in W'$ be the generic point. Let $K = \kappa(\eta) = R(W')$ be the function field. Consider the base change diagram

$$\begin{array}{ccc} W_\eta & \longrightarrow & W \\ c \downarrow & & \downarrow p' \\ \operatorname{Spec}(K) & \xrightarrow{\eta} & W' \end{array}$$

Observe that c is proper. Also $|W_\eta|$ has dimension 1: use Decent Spaces, Lemma 18.6 to identify $|W_\eta|$ as the subspace of $|W|$ of points mapping to η and note that since $\dim_\delta(W) = k + 1$ and $\delta(\eta) = k$ points of W_η must have δ -value either k or $k + 1$. Thus the local rings have dimension ≤ 1 by Decent Spaces, Lemma 20.2. By Spaces over Fields, Lemma 9.3 we find that W_η is a scheme. Since $\operatorname{Spec}(K)$ is the limit of the nonempty affine open subschemes of W' we conclude that we may assume that W is a scheme by Limits of Spaces, Lemma 5.11. Then finally by the case of schemes (Chow Homology, Lemma 20.3) we find that $n = 0$.

The case $\dim_\delta(W') = k + 1$. In this case Lemma 14.1 applies, and we see that indeed $p'_* \operatorname{div}(f) = \operatorname{div}(g)$ for some $g \in R(W')^*$ as desired. \square

17. The divisor associated to an invertible sheaf

This section is the analogue of Chow Homology, Section 24. The following definition is the analogue of Spaces over Fields, Definition 7.4 in our current setup.

Definition 17.1. In Situation 2.1 let X/B be good. Assume X is integral and $n = \dim_\delta(X)$. Let \mathcal{L} be an invertible \mathcal{O}_X -module.

- (1) For any nonzero meromorphic section s of \mathcal{L} we define the *Weil divisor associated to s* is the $(n - 1)$ -cycle

$$\operatorname{div}_{\mathcal{L}}(s) = \sum \operatorname{ord}_{Z, \mathcal{L}}(s)[Z]$$

defined in Spaces over Fields, Definition 7.4. This makes sense because Weil divisors have δ -dimension $n - 1$ by Lemma 12.1.

(2) We define *Weil divisor associated to \mathcal{L}* as

$$c_1(\mathcal{L}) \cap [X] = \text{class of } \text{div}_{\mathcal{L}}(s) \in \text{CH}_{n-1}(X)$$

where s is any nonzero meromorphic section of \mathcal{L} over X . This is well defined by Spaces over Fields, Lemma 7.3.

The zero scheme of a nonzero section is an effective Cartier divisor whose Weil divisor class computes the Weil divisor associated to the invertible module.

Lemma 17.2. *In Situation 2.1 let X/B be good. Assume X is integral and $n = \dim_{\delta}(X)$. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $s \in \Gamma(X, \mathcal{L})$ be a nonzero global section. Then*

$$\text{div}_{\mathcal{L}}(s) = [Z(s)]_{n-1}$$

in $Z_{n-1}(X)$ and

$$c_1(\mathcal{L}) \cap [X] = [Z(s)]_{n-1}$$

in $\text{CH}_{n-1}(X)$.

Proof. Let $Z \subset X$ be an integral closed subspace of δ -dimension $n - 1$. Let $\xi \in |Z|$ be its generic point. To prove the first equality we compare the coefficients of Z on both sides. Choose an elementary étale neighbourhood $(U, u) \rightarrow (X, \xi)$, see Decent Spaces, Section 11 and recall that $\mathcal{O}_{X, \xi}^h = \mathcal{O}_{U, u}^h$ in this case. After replacing U by an open neighbourhood of u we may assume there is a trivializing section s_U of $\mathcal{L}|_U$. Write $s|_U = f s_U$ for some $f \in \Gamma(U, \mathcal{O}_U)$. Then $Z \times_X U$ is equal to $V(f)$ as a closed subscheme of U , see Divisors on Spaces, Definition 7.6. As in Spaces over Fields, Section 7 denote \mathcal{L}_{ξ} the pullback of \mathcal{L} under the canonical morphism $c_{\xi} : \text{Spec}(\mathcal{O}_{X, \xi}^h) \rightarrow X$. Denote s_{ξ} the pullback of s_U ; it is a trivialization of \mathcal{L}_{ξ} . Then we see that $c_{\xi}^*(s) = f s_{\xi}$. The coefficient of Z in $[Z(s)]_{n-1}$ is by definition

$$\text{length}_{\mathcal{O}_{U, u}}(\mathcal{O}_{U, u}/f\mathcal{O}_{U, u})$$

Since $\mathcal{O}_{U, u} \rightarrow \mathcal{O}_{X, \xi}^h$ is flat and identifies residue fields this is equal to

$$\text{length}_{\mathcal{O}_{X, \xi}^h}(\mathcal{O}_{X, \xi}^h/f\mathcal{O}_{X, \xi}^h)$$

by Algebra, Lemma 52.13. This final quantity is equal to $\text{ord}_{Z, \mathcal{L}}(s)$ by Spaces over Fields, Definition 7.1, i.e., to the coefficient of Z in $\text{div}_{\mathcal{L}}(s)$ as desired. \square

Lemma 17.3. *In Situation 2.1 let X/B be good. Let \mathcal{L} be an invertible \mathcal{O}_X -module. The morphism*

$$q : T = \underline{\text{Spec}} \left(\bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n} \right) \longrightarrow X$$

has the following properties:

- (1) q is surjective, smooth, affine, of relative dimension 1,
- (2) there is an isomorphism $\alpha : q^* \mathcal{L} \cong \mathcal{O}_T$,
- (3) formation of $(q : T \rightarrow X, \alpha)$ commutes with base change,
- (4) $q^* : Z_k(X) \rightarrow Z_{k+1}(T)$ is injective,
- (5) if $Z \subset X$ is an integral closed subspace, then $q^{-1}(Z) \subset T$ is an integral closed subspace,
- (6) if $Z \subset X$ is a closed subspace of X of δ -dimension $\leq k$, then $q^{-1}(Z)$ is a closed subspace of T of δ -dimension $\leq k + 1$ and $q^*[Z]_k = [q^{-1}(Z)]_{k+1}$,

- (7) if $\xi' \in |T|$ is the generic point of the fibre of $|T| \rightarrow |X|$ over ξ , then the ring map $\mathcal{O}_{X,\xi}^h \rightarrow \mathcal{O}_{T,\xi'}^h$ is flat, we have $\mathfrak{m}_{\xi'}^h = \mathfrak{m}_{\xi}^h \mathcal{O}_{T,\xi'}^h$, and the residue field extension is purely transcendental of transcendence degree 1, and
- (8) add more here as needed.

Proof. Let $U \rightarrow X$ be an étale morphism such that $\mathcal{L}|_U$ is trivial. Then $T \times_X U \rightarrow U$ is isomorphic to the projection morphism $\mathbf{G}_m \times U \rightarrow U$, where \mathbf{G}_m is the multiplicative group scheme, see Groupoids, Example 5.1. Thus (1) is clear.

To see (2) observe that $q_* q^* \mathcal{L} = \bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n+1}$. Thus there is an obvious isomorphism $q_* q^* \mathcal{L} \rightarrow q_* \mathcal{O}_T$ of $q_* \mathcal{O}_T$ -modules. By Morphisms of Spaces, Lemma 20.10 this determines an isomorphism $q^* \mathcal{L} \rightarrow \mathcal{O}_T$.

Part (3) holds because forming the relative spectrum commutes with arbitrary base change and the same thing is clearly true for the isomorphism α .

Part (4) follows immediately from (1) and the definitions.

Part (5) follows from the fact that if Z is an integral algebraic space, then $\mathbf{G}_m \times Z$ is an integral algebraic space.

Part (6) follows from the fact that lengths are preserved: if (A, \mathfrak{m}) is a local ring and $B = A[x]_{\mathfrak{m}A[x]}$ and if M is an A -module, then $\text{length}_A(M) = \text{length}_B(M \otimes_A B)$. This implies that if \mathcal{F} is a coherent \mathcal{O}_X -module and $\xi \in |X|$ with $\xi' \in |T|$ the generic point of the fibre over ξ , then the length of \mathcal{F} at ξ is the same as the length of $q^* \mathcal{F}$ at ξ' . Tracing through the definitions this gives (6) and more.

The map in part (7) comes from Decent Spaces, Remark 11.11. However, in our case we have

$$\text{Spec}(\mathcal{O}_{X,\xi}^h) \times_X T = \mathbf{G}_m \times \text{Spec}(\mathcal{O}_{X,\xi}^h) = \text{Spec}(\mathcal{O}_{X,\xi}^h[t, t^{-1}])$$

and ξ' corresponds to the generic point of the special fibre of this over $\text{Spec}(\mathcal{O}_{X,\xi}^h)$. Thus $\mathcal{O}_{T,\xi'}^h$ is the henselization of the localization of $\mathcal{O}_{X,\xi}^h[t, t^{-1}]$ at the corresponding prime. Part (7) follows from this and some commutative algebra; details omitted. \square

Lemma 17.4. *In Situation 2.1 let X/B be good. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Assume X is integral. Let s be a nonzero meromorphic section of \mathcal{L} . Let $q : T \rightarrow X$ be the morphism of Lemma 17.3. Then*

$$q^* \text{div}_{\mathcal{L}}(s) = \text{div}_T(q^*(s))$$

where we view the pullback $q^*(s)$ as a nonzero meromorphic function on T using the isomorphism $q^* \mathcal{L} \rightarrow \mathcal{O}_T$

Proof. Observe that $\text{div}_T(q^*(s)) = \text{div}_{\mathcal{O}_T}(q^*(s))$ by the compatibility between the constructions given in Spaces over Fields, Sections 6 and 7. We will show the agreement with $\text{div}_{\mathcal{O}_T}(q^*(s))$ in this proof. We will use all the properties of $q : T \rightarrow X$ stated in Lemma 17.3 without further mention. Let $Z \subset T$ be a prime divisor. Then either $Z \rightarrow X$ is dominant or $Z = q^{-1}(Z')$ for some prime divisor $Z' \subset X$. If $Z \rightarrow X$ is dominant, then the coefficient of Z in either side of the equality of the lemma is zero. Thus we may assume $Z = q^{-1}(Z')$ where $Z' \subset X$ is

a prime divisor. Let $\xi' \in |Z'|$ and $\xi \in |Z|$ be the generic points. Then we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{T,\xi}^h) & \xrightarrow{c_\xi} & T \\ h \downarrow & & \downarrow q \\ \mathrm{Spec}(\mathcal{O}_{X,\xi'}^h) & \xrightarrow{c_{\xi'}} & X \end{array}$$

see Decent Spaces, Remark 11.11. Choose a trivialization $s_{\xi'}$ of $\mathcal{L}_{\xi'} = c_{\xi'}^* \mathcal{L}$. Then we can use the pullback s_ξ of $s_{\xi'}$ via h as our trivialization of $\mathcal{L}_\xi = c_\xi^* q^* \mathcal{L}$. Write $s/s_{\xi'} = a/b$ for $a, b \in \mathcal{O}_{X,\xi'}$ nonzerodivisors. By definition the coefficient of Z' in $\mathrm{div}_\mathcal{L}(s)$ is

$$\mathrm{length}_{\mathcal{O}_{X,\xi'}^h}(\mathcal{O}_{X,\xi'}^h/a\mathcal{O}_{X,\xi'}^h) - \mathrm{length}_{\mathcal{O}_{X,\xi'}^h}(\mathcal{O}_{X,\xi'}^h/b\mathcal{O}_{X,\xi'}^h)$$

Since $Z = q^{-1}(Z')$, this is also the coefficient of Z in $q^* \mathrm{div}_\mathcal{L}(s)$. Since $\mathcal{O}_{X,\xi'}^h \rightarrow \mathcal{O}_{T,\xi}^h$ is flat the elements a, b map to nonzerodivisors in $\mathcal{O}_{T,\xi}^h$. Thus $q^*(s)/s_\xi = a/b$ in the total quotient ring of $\mathcal{O}_{T,\xi}^h$. By definition the coefficient of Z in $\mathrm{div}_T(q^*(s))$ is

$$\mathrm{length}_{\mathcal{O}_{T,\xi}^h}(\mathcal{O}_{T,\xi}^h/a\mathcal{O}_{T,\xi}^h) - \mathrm{length}_{\mathcal{O}_{T,\xi}^h}(\mathcal{O}_{T,\xi}^h/b\mathcal{O}_{T,\xi}^h)$$

The proof is finished because these lengths are the same as before by Algebra, Lemma 52.13 and the fact that $\mathfrak{m}_\xi^h = \mathfrak{m}_{\xi'}^h \mathcal{O}_{T,\xi}^h$ shown in Lemma 17.3. \square

18. Intersecting with an invertible sheaf

This section is the analogue of Chow Homology, Section 25. In this section we study the following construction.

Definition 18.1. In Situation 2.1 let X/B be good. Let \mathcal{L} be an invertible \mathcal{O}_X -module. We define, for every integer k , an operation

$$c_1(\mathcal{L}) \cap - : Z_{k+1}(X) \rightarrow \mathrm{CH}_k(X)$$

called *intersection with the first Chern class of \mathcal{L}* .

- (1) Given an integral closed subspace $i : W \rightarrow X$ with $\dim_\delta(W) = k + 1$ we define

$$c_1(\mathcal{L}) \cap [W] = i_*(c_1(i^* \mathcal{L}) \cap [W])$$

where the right hand side is defined in Definition 17.1.

- (2) For a general $(k + 1)$ -cycle $\alpha = \sum n_i [W_i]$ we set

$$c_1(\mathcal{L}) \cap \alpha = \sum n_i c_1(\mathcal{L}) \cap [W_i]$$

Write each $c_1(\mathcal{L}) \cap W_i = \sum_j n_{i,j} [Z_{i,j}]$ with $\{Z_{i,j}\}_j$ a locally finite sum of integral closed subspaces of W_i . Since $\{W_i\}$ is a locally finite collection of integral closed subspaces on X , it follows easily that $\{Z_{i,j}\}_{i,j}$ is a locally finite collection of closed subspaces of X . Hence $c_1(\mathcal{L}) \cap \alpha = \sum n_i n_{i,j} [Z_{i,j}]$ is a cycle. Another, often more convenient, way to think about this is to observe that the morphism $\coprod W_i \rightarrow X$ is proper. Hence $c_1(\mathcal{L}) \cap \alpha$ can be viewed as the pushforward of a class in $\mathrm{CH}_k(\coprod W_i) = \prod \mathrm{CH}_k(W_i)$. This also explains why the result is well defined up to rational equivalence on X .

The main goal for the next few sections is to show that intersecting with $c_1(\mathcal{L})$ factors through rational equivalence. This is not a triviality.

Lemma 18.2. *In Situation 2.1 let X/B be good. Let \mathcal{L}, \mathcal{N} be invertible sheaves on X . Then*

$$c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap \alpha$$

in $\mathrm{CH}_k(X)$ for every $\alpha \in Z_{k-1}(X)$. Moreover, $c_1(\mathcal{O}_X) \cap \alpha = 0$ for all α .

Proof. The additivity follows directly from Spaces over Fields, Lemma 7.5 and the definitions. To see that $c_1(\mathcal{O}_X) \cap \alpha = 0$ consider the section $1 \in \Gamma(X, \mathcal{O}_X)$. This restricts to an everywhere nonzero section on any integral closed subspace $W \subset X$. Hence $c_1(\mathcal{O}_X) \cap [W] = 0$ as desired. \square

Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on an algebraic space X , see Divisors on Spaces, Definition 7.6.

Lemma 18.3. *In Situation 2.1 let Y/B be good. Let \mathcal{L} be an invertible \mathcal{O}_Y -module. Let $s \in \Gamma(Y, \mathcal{L})$ be a regular section and assume $\dim_\delta(Y) \leq k+1$. Write $[Y]_{k+1} = \sum n_i [Y_i]$ where $Y_i \subset Y$ are the irreducible components of Y of δ -dimension $k+1$. Set $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$. Then*

$$(18.3.1) \quad [Z(s)]_k = \sum n_i [Z(s_i)]_k$$

as k -cycles on Y .

Proof. Let $\varphi : V \rightarrow Y$ be a surjective étale morphism where V is a scheme. It suffices to prove the equality after pulling back by φ , see Lemma 10.3. That same lemma tells us that $\varphi^*[Y_i] = [\varphi^{-1}(Y_i)] = \sum [V_{i,j}]$ where $V_{i,j}$ are the irreducible components of V lying over Y_i . Hence if we first apply the case of schemes (Chow Homology, Lemma 25.3) to φ^*s_i on $Y_i \times_Y V$ we find that $\varphi^*[Z(s_i)]_k = [Z(\varphi^*s_i)] = \sum [Z(s_{i,j})]_k$ where $s_{i,j}$ is the pullback of s to $V_{i,j}$. Applying the case of schemes to φ^*s we get

$$\varphi^*[Z(s)]_k = [Z(\varphi^*s)]_k = \sum n_i [Z(s_{i,j})]_k$$

by our remark on multiplicities above. Combining all of the above the proof is complete. \square

The following lemma is a useful result in order to compute the intersection product of the c_1 of an invertible sheaf and the cycle associated to a closed subscheme. Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X , see Divisors, Definition 14.8.

Lemma 18.4. *In Situation 2.1 let X/B be good. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $Y \subset X$ be a closed subscheme with $\dim_\delta(Y) \leq k+1$ and let $s \in \Gamma(Y, \mathcal{L}|_Y)$ be a regular section. Then*

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = [Z(s)]_k$$

in $\mathrm{CH}_k(X)$.

Proof. Write

$$[Y]_{k+1} = \sum n_i [Y_i]$$

where $Y_i \subset Y$ are the irreducible components of Y of δ -dimension $k+1$ and $n_i > 0$. By assumption the restriction $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$ is not zero, and hence is a regular section. By Lemma 17.2 we see that $[Z(s_i)]_k$ represents $c_1(\mathcal{L}|_{Y_i})$. Hence by definition

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = \sum n_i [Z(s_i)]_k$$

Thus the result follows from Lemma 18.3. \square

19. Intersecting with an invertible sheaf and push and pull

This section is the analogue of Chow Homology, Section 26. In this section we prove that the operation $c_1(\mathcal{L}) \cap -$ commutes with flat pullback and proper pushforward.

Lemma 19.1. *In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a flat morphism of relative dimension r . Let \mathcal{L} be an invertible sheaf on Y . Assume Y is integral and $n = \dim_\delta(Y)$. Let s be a nonzero meromorphic section of \mathcal{L} . Then we have*

$$f^* \operatorname{div}_{\mathcal{L}}(s) = \sum n_i \operatorname{div}_{f^* \mathcal{L}|_{X_i}}(s_i)$$

in $Z_{n+r-1}(X)$. Here the sum is over the irreducible components $X_i \subset X$ of δ -dimension $n + r$, the section $s_i = f|_{X_i}^*(s)$ is the pullback of s , and $n_i = m_{X_i, X}$ is the multiplicity of X_i in X .

Proof. Using sleight of hand we will deduce this from Lemma 16.1. (An alternative is to redo the proof of that lemma in the setting of meromorphic sections of invertible modules.) Namely, let $q : T \rightarrow Y$ be the morphism of Lemma 17.3 constructed using \mathcal{L} on Y . We will use all the properties of T stated in this lemma. Consider the fibre product diagram

$$\begin{array}{ccc} T' & \xrightarrow{\quad} & X \\ h \downarrow & q' \searrow & \downarrow f \\ T & \xrightarrow{\quad q \quad} & Y \end{array}$$

Then $q' : T' \rightarrow X$ is the morphism constructed using $f^* \mathcal{L}$ on X . Then it suffices to prove

$$(q')^* f^* \operatorname{div}_{\mathcal{L}}(s) = \sum n_i (q')^* \operatorname{div}_{f^* \mathcal{L}|_{X_i}}(s_i)$$

Observe that $T'_i = q'^{-1}(X_i)$ are the irreducible components of T' and that n_i is the multiplicity of T'_i in T' . The left hand side is equal to

$$h^* q^* \operatorname{div}_{\mathcal{L}}(s) = h^* \operatorname{div}_T(q^*(s))$$

by Lemma 17.4 (and Lemma 10.4). On the other hand, denoting $q'_i : T'_i \rightarrow X_i$ the restriction of q' we find that Lemma 17.4 also tells us the right hand side is equal to

$$\sum n_i \operatorname{div}_{T'_i}((q'_i)^*(s_i))$$

In these two formulas the expressions $q^*(s)$ and $(q'_i)^*(s_i)$ represent the rational functions corresponding to the pulled back meromorphic sections of $q^* \mathcal{L}$ and $(q'_i)^* f^* \mathcal{L}|_{X_i}$ via the isomorphism $\alpha : q^* \mathcal{L} \rightarrow \mathcal{O}_T$ and its pullbacks to spaces over T . With this convention it is clear that $(q'_i)^*(s_i)$ is the composition of the rational function $q^*(s)$ on T and the morphism $h|_{T'_i} : T'_i \rightarrow T$. Thus Lemma 16.1 exactly says that

$$h^* \operatorname{div}_T(q^*(s)) = \sum n_i \operatorname{div}_{T'_i}((q'_i)^*(s_i))$$

as desired. \square

Lemma 19.2. *In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a flat morphism of relative dimension r . Let \mathcal{L} be an invertible sheaf on Y . Let α be a k -cycle on Y . Then*

$$f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^* \mathcal{L}) \cap f^* \alpha$$

in $\operatorname{CH}_{k+r-1}(X)$.

Proof. Write $\alpha = \sum n_i [W_i]$. We will show that

$$f^*(c_1(\mathcal{L}) \cap [W_i]) = c_1(f^*\mathcal{L}) \cap f^*[W_i]$$

in $\text{CH}_{k+r-1}(X)$ by producing a rational equivalence on the closed subspace $f^{-1}(W_i)$ of X . By the discussion in Remark 15.3 this will prove the equality of the lemma is true.

Let $W \subset Y$ be an integral closed subspace of δ -dimension k . Consider the closed subspace $W' = f^{-1}(W) = W \times_Y X$ so that we have the fibre product diagram

$$\begin{array}{ccc} W' & \longrightarrow & X \\ h \downarrow & & \downarrow f \\ W & \longrightarrow & Y \end{array}$$

We have to show that $f^*(c_1(\mathcal{L}) \cap [W]) = c_1(f^*\mathcal{L}) \cap f^*[W]$. Choose a nonzero meromorphic section s of $\mathcal{L}|_W$. Let $W'_i \subset W'$ be the irreducible components of δ -dimension $k+r$. Write $[W']_{k+r} = \sum n_i [W'_i]$ with n_i the multiplicity of W'_i in W' as per definition. So $f^*[W] = \sum n_i [W'_i]$ in $Z_{k+r}(X)$. Since each $W'_i \rightarrow W$ is dominant we see that $s_i = s|_{W'_i}$ is a nonzero meromorphic section for each i . By Lemma 19.1 we have the following equality of cycles

$$h^* \text{div}_{\mathcal{L}|_W}(s) = \sum n_i \text{div}_{f^*\mathcal{L}|_{W'_i}}(s_i)$$

in $Z_{k+r-1}(W')$. This finishes the proof since the left hand side is a cycle on W' which pushes to $f^*(c_1(\mathcal{L}) \cap [W])$ in $\text{CH}_{k+r-1}(X)$ and the right hand side is a cycle on W' which pushes to $c_1(f^*\mathcal{L}) \cap f^*[W]$ in $\text{CH}_{k+r-1}(X)$. \square

Lemma 19.3. *In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a proper morphism. Let \mathcal{L} be an invertible sheaf on Y . Assume X, Y integral, f dominant, and $\dim_\delta(X) = \dim_\delta(Y)$. Let s be a nonzero meromorphic section s of \mathcal{L} on Y . Then*

$$f_*(\text{div}_{f^*\mathcal{L}}(f^*s)) = [R(X) : R(Y)] \text{div}_{\mathcal{L}}(s).$$

as cycles on Y . In particular

$$f_*(c_1(f^*\mathcal{L}) \cap [X]) = c_1(\mathcal{L}) \cap f_*[Y].$$

Proof. The last equation follows from the first since $f_*[X] = [R(X) : R(Y)][Y]$ by definition. Proof of the first equation. Let $q : T \rightarrow Y$ be the morphism of Lemma 17.3 constructed using \mathcal{L} on Y . We will use all the properties of T stated in this lemma. Consider the fibre product diagram

$$\begin{array}{ccc} T' & \longrightarrow & X \\ h \downarrow & & \downarrow f \\ T & \xrightarrow{q} & Y \end{array}$$

Then $q' : T' \rightarrow X$ is the morphism constructed using $f^*\mathcal{L}$ on X . It suffices to prove the equality after pulling back to T' . The left hand side pulls back to

$$\begin{aligned} q^* f_*(\text{div}_{f^*\mathcal{L}}(f^*s)) &= h_*(q')^* \text{div}_{f^*\mathcal{L}}(f^*s) \\ &= h_* \text{div}_{(q')^* f^*\mathcal{L}}((q')^* f^*s) \\ &= h_* \text{div}_{h^* q^* \mathcal{L}}(h^* q^* s) \end{aligned}$$

The first equality by Lemma 11.1. The second by Lemma 19.1 using that T' is integral. The third because the displayed diagram commutes. The right hand side pulls back to

$$[R(X) : R(Y)]q^*\text{div}_{\mathcal{L}}(s) = [R(T') : R(T)]\text{div}_{q^*\mathcal{L}}(q^*s)$$

This follows from Lemma 19.1, the fact that T is integral, and the equality $[R(T') : R(T)] = [R(X) : R(Y)]$ whose proof we omit (it follows from Lemma 11.1 but that would be a silly way to prove the equality). Thus it suffices to prove the lemma for $h : T' \rightarrow T$, the invertible module $q^*\mathcal{L}$ and the section q^*s . Since $q^*\mathcal{L} = \mathcal{O}_T$ we reduce to the case where $\mathcal{L} \cong \mathcal{O}$ discussed in the next paragraph.

Assume that $\mathcal{L} = \mathcal{O}_Y$. In this case s corresponds to a rational function $g \in R(Y)$. Using the embedding $R(Y) \subset R(X)$ we may think of g as a rational on X and we are simply trying to prove

$$f_*(\text{div}_X(g)) = [R(X) : R(Y)]\text{div}_Y(g).$$

Comparing with the result of Lemma 14.1 we see this true since $\text{Nm}_{R(X)/R(Y)}(g) = g^{[R(X):R(Y)]}$ as $g \in R(Y)^*$. \square

Lemma 19.4. *In Situation 2.1 let $X, Y/B$ be good. Let $p : X \rightarrow Y$ be a proper morphism. Let $\alpha \in Z_{k+1}(X)$. Let \mathcal{L} be an invertible sheaf on Y . Then*

$$p_*(c_1(p^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap p_*\alpha$$

in $\text{CH}_k(Y)$.

Proof. Suppose that p has the property that for every integral closed subspace $W \subset X$ the map $p|_W : W \rightarrow Y$ is a closed immersion. Then, by definition of capping with $c_1(\mathcal{L})$ the lemma holds.

We will use this remark to reduce to a special case. Namely, write $\alpha = \sum n_i[W_i]$ with $n_i \neq 0$ and W_i pairwise distinct. Let $W'_i \subset Y$ be the “image” of W_i as in Lemma 7.1. Consider the diagram

$$\begin{array}{ccc} X' = \coprod W_i & \xrightarrow{q} & X \\ p' \downarrow & & \downarrow p \\ Y' = \coprod W'_i & \xrightarrow{q'} & Y. \end{array}$$

Since $\{W_i\}$ is locally finite on X , and p is proper we see that $\{W'_i\}$ is locally finite on Y and that q, q', p' are also proper morphisms. We may think of $\sum n_i[W_i]$ also as a k -cycle $\alpha' \in Z_k(X')$. Clearly $q_*\alpha' = \alpha$. We have $q_*(c_1(q^*p^*\mathcal{L}) \cap \alpha') = c_1(p^*\mathcal{L}) \cap q_*\alpha'$ and $(q')_*(c_1((q')^*\mathcal{L}) \cap p'_*\alpha') = c_1(\mathcal{L}) \cap q'_*p'_*\alpha'$ by the initial remark of the proof. Hence it suffices to prove the lemma for the morphism p' and the cycle $\sum n_i[W_i]$. Clearly, this means we may assume X, Y integral, $f : X \rightarrow Y$ dominant and $\alpha = [X]$. In this case the result follows from Lemma 19.3. \square

20. The key formula

This section is the analogue of Chow Homology, Section 27. We strongly urge the reader to read the proof in that case first.

In Situation 2.1 let X/B be good. Assume X is integral and $\dim_{\delta}(X) = n$. Let \mathcal{L} and \mathcal{N} be invertible \mathcal{O}_X -modules. Let s be a nonzero meromorphic section of \mathcal{L}

and let t be a nonzero meromorphic section of \mathcal{N} . Let $Z \subset X$ be a prime divisor with generic point $\xi \in |Z|$. Consider the morphism

$$c_\xi : \text{Spec}(\mathcal{O}_{X,\xi}^h) \longrightarrow X$$

used in Spaces over Fields, Section 7. We denote \mathcal{L}_ξ and \mathcal{N}_ξ the pullbacks of \mathcal{L} and \mathcal{N} by c_ξ ; we often think of \mathcal{L}_ξ and \mathcal{N}_ξ as the rank 1 free $\mathcal{O}_{X,\xi}^h$ -modules they give rise to. Note that the pullback of s , resp. t is a regular meromorphic section of \mathcal{L}_ξ , resp. \mathcal{N}_ξ .

Let $Z_i \subset X$, $i \in I$ be a locally finite set of prime divisors with the following property: If $Z \notin \{Z_i\}$, then s is a generator for \mathcal{L}_ξ and t is a generator for \mathcal{N}_ξ . Such a set exists by Spaces over Fields, Lemma 7.2. Then

$$\text{div}_{\mathcal{L}}(s) = \sum \text{ord}_{Z_i, \mathcal{L}}(s)[Z_i]$$

and similarly

$$\text{div}_{\mathcal{N}}(t) = \sum \text{ord}_{Z_i, \mathcal{N}}(t)[Z_i]$$

Unwinding the definitions more, we pick for each i generators $s_i \in \mathcal{L}_{\xi_i}$ and $t_i \in \mathcal{N}_{\xi_i}$ where ξ_i is the generic point of Z_i . Then we can write

$$s = f_i s_i \quad \text{and} \quad t = g_i t_i$$

with f_i, g_i invertible elements of the total ring of fractions $Q(\mathcal{O}_{X,\xi_i}^h)$. We abbreviate $B_i = \mathcal{O}_{X,\xi_i}^h$. Let us denote

$$\text{ord}_{B_i} : Q(B_i)^* \longrightarrow \mathbf{Z}, \quad a/b \longmapsto \text{length}_{B_i}(B_i/aB_i) - \text{length}_{B_i}(B_i/bB_i)$$

In other words, we temporarily extend Algebra, Definition 121.2 to these reduced Noetherian local rings of dimension 1. Then by definition

$$\text{ord}_{Z_i, \mathcal{L}}(s) = \text{ord}_{B_i}(f_i) \quad \text{and} \quad \text{ord}_{Z_i, \mathcal{N}}(t) = \text{ord}_{B_i}(g_i)$$

Since ξ_i is the generic point of Z_i we see that the residue field $\kappa(\xi_i)$ is the function field of Z_i . Moreover $\kappa(\xi_i)$ is the residue field of B_i , see Decent Spaces, Lemma 11.10. Since t_i is a generator of \mathcal{N}_{ξ_i} we see that its image in the fibre $\mathcal{N}_{\xi_i} \otimes_{B_i} \kappa(\xi_i)$ is a nonzero meromorphic section of $\mathcal{N}|_{Z_i}$. We will denote this image $t_i|_{Z_i}$. From our definitions it follows that

$$c_1(\mathcal{N}) \cap \text{div}_{\mathcal{L}}(s) = \sum \text{ord}_{B_i}(f_i)(Z_i \rightarrow X)_* \text{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

and similarly

$$c_1(\mathcal{L}) \cap \text{div}_{\mathcal{N}}(t) = \sum \text{ord}_{B_i}(g_i)(Z_i \rightarrow X)_* \text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

in $\text{CH}_{n-2}(X)$. We are going to find a rational equivalence between these two cycles. To do this we consider the tame symbol

$$\partial_{B_i}(f_i, g_i) \in \kappa(\xi_i)^* = R(Z_i)^*$$

see Chow Homology, Section 5.

Lemma 20.1 (Key formula). *In the situation above the cycle*

$$\sum (Z_i \rightarrow X)_* \left(\text{ord}_{B_i}(f_i) \text{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - \text{ord}_{B_i}(g_i) \text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) \right)$$

is equal to the cycle

$$\sum (Z_i \rightarrow X)_* \text{div}(\partial_{B_i}(f_i, g_i))$$

Proof. The strategy of the proof will be: first reduce to the case where \mathcal{L} and \mathcal{N} are trivial invertible modules, then change our choices of local trivializations, and then finally use étale localization to reduce to the case of schemes¹.

First step. Let $q : T \rightarrow X$ be the morphism constructed in Lemma 17.3. We will use all properties stated in that lemma without further mention. In particular, it suffices to show that the cycles are equal after pulling back by q . Denote s' and t' the pullbacks of s and t to meromorphic sections of $q^*\mathcal{L}$ and $q^*\mathcal{N}$. Denote $Z'_i = q^{-1}(Z_i)$, denote $\xi'_i \in |Z'_i|$ the generic point, denote $B'_i = \mathcal{O}_{T, \xi'_i}^h$, denote $\mathcal{L}_{\xi'_i}$ and $\mathcal{N}_{\xi'_i}$ the pullbacks of \mathcal{L} and \mathcal{N} to $\text{Spec}(B'_i)$. Recall that we have commutative diagrams

$$\begin{array}{ccc} \text{Spec}(B'_i) & \xrightarrow{c_{\xi'_i}} & T \\ \downarrow & & \downarrow q \\ \text{Spec}(B_i) & \xrightarrow{c_{\xi_i}} & X \end{array}$$

see Decent Spaces, Remark 11.11. Denote s'_i and t'_i the pullbacks of s_i and t_i which are generators of $\mathcal{L}_{\xi'_i}$ and $\mathcal{N}_{\xi'_i}$. Then we have

$$s' = f'_i s'_i \quad \text{and} \quad t' = g'_i t'_i$$

where f'_i and g'_i are the images of f_i, g_i under the map $Q(B_i) \rightarrow Q(B'_i)$ induced by $B_i \rightarrow B'_i$. By Algebra, Lemma 52.13 we have

$$\text{ord}_{B_i}(f_i) = \text{ord}_{B'_i}(f'_i) \quad \text{and} \quad \text{ord}_{B_i}(g_i) = \text{ord}_{B'_i}(g'_i)$$

By Lemma 19.1 applied to $q : Z'_i \rightarrow Z_i$ we have

$$q^* \text{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) = \text{div}_{q^*\mathcal{N}|_{Z'_i}}(t'_i|_{Z'_i}) \quad \text{and} \quad q^* \text{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) = \text{div}_{q^*\mathcal{L}|_{Z'_i}}(s'_i|_{Z'_i})$$

This already shows that the first cycle in the statement of the lemma pulls back to the corresponding cycle for s', t', Z'_i, s'_i, t'_i . To see the same is true for the second, note that by Chow Homology, Lemma 5.4 we have

$$\partial_{B_i}(f_i, g_i) \mapsto \partial_{B'_i}(f'_i, g'_i) \quad \text{via} \quad \kappa(\xi_i) \rightarrow \kappa(\xi'_i)$$

Hence the same lemma as before shows that

$$q^* \text{div}(\partial_{B_i}(f_i, g_i)) = \text{div}(\partial_{B'_i}(f'_i, g'_i))$$

Since $q^*\mathcal{L} \cong \mathcal{O}_T$ we find that it suffices to prove the equality in case \mathcal{L} is trivial. Exchanging the roles of \mathcal{L} and \mathcal{N} we see that we may similarly assume \mathcal{N} is trivial. This finishes the proof of the first step.

Second step. Assume $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{N} = \mathcal{O}_X$. Denote 1 the trivializing section of \mathcal{L} . Then $s_i = u \cdot 1$ for some unit $u \in B_i$. Let us examine what happens if we replace s_i by 1. Then f_i gets replaced by $u f_i$. Thus the first part of the first expression of the lemma is unchanged and in the second part we add

$$\text{ord}_{B_i}(g_i) \text{div}(u|_{Z_i})$$

where $u|_{Z_i}$ is the image of u in the residue field by Spaces over Fields, Lemma 7.3 and in the second expression we add

$$\text{div}(\partial_{B_i}(u, g_i))$$

¹It is possible that a shorter proof can be given by immediately applying étale localization.

by bi-linearity of the tame symbol. These terms agree by the property of the tame symbol given in Chow Homology, Equation (6).

Let $Y \subset X$ be an integral closed subspace with $\dim_\delta(Y) = n - 2$. To show that the coefficients of Y of the two cycles of the lemma is the same, we may do a replacement of s_i by 1 as in the previous paragraph. In exactly the same way one shows that we may do a replacement of t_i by 1. Since there are only a finite number of Z_i such that $Y \subset Z_i$ we may assume $s_i = 1$ and $t_i = 1$ for all these Z_i .

Third step. Here we prove the coefficients of Y in the cycles of the lemma agree for an integral closed subspace Y with $\dim_\delta(Y) = n - 2$ such that moreover $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{N} = \mathcal{O}_X$ and $s_i = 1$ and $t_i = 1$ for all Z_i such that $Y \subset Z_i$. After replacing X by a smaller open subspace we may in fact assume that s_i and t_i are equal to 1 for all i . In this case the first cycle is zero. Our task is to show that the coefficient of Y in the second cycle is zero as well.

First, since $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{N} = \mathcal{O}_X$ we may and do think of s, t as rational functions f, g on X . Since s_i and t_i are equal to 1 we find that f_i , resp. g_i is the image of f , resp. g in $Q(B_i)$ for all i . Let $\zeta \in |Y|$ be the generic point. Choose an étale neighbourhood

$$(U, u) \longrightarrow (X, \zeta)$$

and denote $Y' = \overline{\{u\}} \subset U$. Since an étale morphism is flat, we can pullback f and g to regular meromorphic functions on U which we will also denote f and g . For every prime divisor $Y \subset Z \subset X$ the scheme $Z \times_X U$ is a union of prime divisors of U . Conversely, given a prime divisor $Y' \subset Z' \subset U$, there is a prime divisor $Y \subset Z \subset X$ such that Z' is a component of $Z \times_X U$. Given such a pair (Z, Z') the ring map

$$\mathcal{O}_{X, \xi}^h \rightarrow \mathcal{O}_{U, \xi'}^h$$

is étale (in fact it is finite étale). Hence we find that

$$\partial_{\mathcal{O}_{X, \xi}^h}(f, g) \mapsto \partial_{\mathcal{O}_{U, \xi'}^h}(f, g) \quad \text{via} \quad \kappa(\xi) \rightarrow \kappa(\xi')$$

by Chow Homology, Lemma 5.4. Thus Lemma 13.2 applies to show

$$(Z \times_X U \rightarrow Z)^* \operatorname{div}_Z(\partial_{\mathcal{O}_{X, \xi}^h}(f, g)) = \sum_{Z' \subset Z \times_X U} \operatorname{div}_{Z'}(\partial_{\mathcal{O}_{U, \xi'}^h}(f, g))$$

Since flat pullback commutes with pushforward along closed immersions (Lemma 11.1) we see that it suffices to prove that the coefficient of Y' in

$$\sum_{Z' \subset U} (Z' \rightarrow U)_* \operatorname{div}_{Z'}(\partial_{\mathcal{O}_{U, \xi'}^h}(f, g))$$

is zero.

Let $A = \mathcal{O}_{U, u}$. Then $f, g \in Q(A)^*$. Thus we can write $f = a/b$ and $g = c/d$ with $a, b, c, d \in A$ nonzerodivisors. The coefficient of Y' in the expression above is

$$\sum_{\mathfrak{q} \subset A \text{ height } 1} \operatorname{ord}_{A/\mathfrak{q}}(\partial_{A/\mathfrak{q}}(f, g))$$

By bilinearity of ∂_A it suffices to prove

$$\sum_{\mathfrak{q} \subset A \text{ height } 1} \operatorname{ord}_{A/\mathfrak{q}}(\partial_{A/\mathfrak{q}}(a, c))$$

is zero and similarly for the other pairs (a, d) , (b, c) , and (b, d) . This is true by Chow Homology, Lemma 6.2. \square

21. Intersecting with an invertible sheaf and rational equivalence

This section is the analogue of Chow Homology, Section 28. Applying the key lemma we obtain the fundamental properties of intersecting with invertible sheaves. In particular, we will see that $c_1(\mathcal{L}) \cap -$ factors through rational equivalence and that these operations for different invertible sheaves commute.

Lemma 21.1. *In Situation 2.1 let X/B be good. Assume X integral and $\dim_\delta(X) = n$. Let \mathcal{L}, \mathcal{N} be invertible on X . Choose a nonzero meromorphic section s of \mathcal{L} and a nonzero meromorphic section t of \mathcal{N} . Set $\alpha = \text{div}_{\mathcal{L}}(s)$ and $\beta = \text{div}_{\mathcal{N}}(t)$. Then*

$$c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L}) \cap \beta$$

in $\text{CH}_{n-2}(X)$.

Proof. Immediate from the key Lemma 20.1 and the discussion preceding it. \square

Lemma 21.2. *In Situation 2.1 let X/B be good. Let \mathcal{L} be invertible on X . The operation $\alpha \mapsto c_1(\mathcal{L}) \cap \alpha$ factors through rational equivalence to give an operation*

$$c_1(\mathcal{L}) \cap - : \text{CH}_{k+1}(X) \rightarrow \text{CH}_k(X)$$

Proof. Let $\alpha \in Z_{k+1}(X)$, and $\alpha \sim_{\text{rat}} 0$. We have to show that $c_1(\mathcal{L}) \cap \alpha$ as defined in Definition 18.1 is zero. By Definition 15.1 there exists a locally finite family $\{W_j\}$ of integral closed subspaces with $\dim_\delta(W_j) = k + 2$ and rational functions $f_j \in R(W_j)^*$ such that

$$\alpha = \sum (i_j)_* \text{div}_{W_j}(f_j)$$

Note that $p : \coprod W_j \rightarrow X$ is a proper morphism, and hence $\alpha = p_* \alpha'$ where $\alpha' \in Z_{k+1}(\coprod W_j)$ is the sum of the principal divisors $\text{div}_{W_j}(f_j)$. By Lemma 19.4 we have $c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^* \mathcal{L}) \cap \alpha')$. Hence it suffices to show that each $c_1(\mathcal{L}|_{W_j}) \cap \text{div}_{W_j}(f_j)$ is zero. In other words we may assume that X is integral and $\alpha = \text{div}_X(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \text{div}_X(f)$ for some $f \in R(X)^*$. We can think of f as a regular meromorphic section of the invertible sheaf $\mathcal{N} = \mathcal{O}_X$. Choose a meromorphic section s of \mathcal{L} and denote $\beta = \text{div}_{\mathcal{L}}(s)$. By Lemma 21.1 we conclude that

$$c_1(\mathcal{L}) \cap \alpha = c_1(\mathcal{O}_X) \cap \beta.$$

However, by Lemma 18.2 we see that the right hand side is zero in $\text{CH}_k(X)$ as desired. \square

In Situation 2.1 let X/B be good. Let \mathcal{L} be invertible on X . We will denote

$$c_1(\mathcal{L})^s \cap - : \text{CH}_{k+s}(X) \rightarrow \text{CH}_k(X)$$

the operation $c_1(\mathcal{L}) \cap -$. This makes sense by Lemma 21.2. We will denote $c_1(\mathcal{L}^s \cap -$ the s -fold iterate of this operation for all $s \geq 0$.

Lemma 21.3. *In Situation 2.1 let X/B be good. Let \mathcal{L}, \mathcal{N} be invertible on X . For any $\alpha \in \text{CH}_{k+2}(X)$ we have*

$$c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha$$

as elements of $\text{CH}_k(X)$.

Proof. Write $\alpha = \sum m_j [Z_j]$ for some locally finite collection of integral closed subspaces $Z_j \subset X$ with $\dim_\delta(Z_j) = k + 2$. Consider the proper morphism $p : \coprod Z_j \rightarrow X$. Set $\alpha' = \sum m_j [Z_j]$ as a $(k + 2)$ -cycle on $\coprod Z_j$. By several applications of Lemma 19.4 we see that $c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap c_1(p^*\mathcal{N}) \cap \alpha')$ and $c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{N}) \cap c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to prove the formula in case X is integral and $\alpha = [X]$. In this case the result follows from Lemma 21.1 and the definitions. \square

22. Intersecting with effective Cartier divisors

This section is the analogue of Chow Homology, Section 29. Please read the introduction of that section for motivation.

Recall that effective Cartier divisors correspond 1-to-1 to isomorphism classes of pairs (\mathcal{L}, s) where \mathcal{L} is an invertible sheaf and s is a global section, see Divisors on Spaces, Lemma 7.8. If D corresponds to (\mathcal{L}, s) , then $\mathcal{L} = \mathcal{O}_X(D)$. Please keep this in mind while reading this section.

Definition 22.1. In Situation 2.1 let X/B be good. Let (\mathcal{L}, s) be a pair consisting of an invertible sheaf and a global section $s \in \Gamma(X, \mathcal{L})$. Let $D = Z(s)$ be the vanishing locus of s , and denote $i : D \rightarrow X$ the closed immersion. We define, for every integer k , a (refined) *Gysin homomorphism*

$$i^* : Z_{k+1}(X) \rightarrow \text{CH}_k(D).$$

by the following rules:

- (1) Given an integral closed subspace $W \subset X$ with $\dim_\delta(W) = k + 1$ we define
 - (a) if $W \not\subset D$, then $i^*[W] = [D \cap W]_k$ as a k -cycle on D , and
 - (b) if $W \subset D$, then $i^*[W] = i'_*(c_1(\mathcal{L}|_W) \cap [W])$, where $i' : W \rightarrow D$ is the induced closed immersion.
- (2) For a general $(k + 1)$ -cycle $\alpha = \sum n_j [W_j]$ we set

$$i^*\alpha = \sum n_j i^*[W_j]$$

- (3) If D is an effective Cartier divisor, then we denote $D \cdot \alpha = i_* i^* \alpha$ the pushforward of the class to a class on X .

In fact, as we will see later, this Gysin homomorphism i^* can be viewed as an example of a non-flat pullback. Thus we will sometimes informally call the class $i^*\alpha$ the *pullback* of the class α .

Remark 22.2. Let $S, B, X, \mathcal{L}, s, i : D \rightarrow X$ be as in Definition 22.1 and assume that $\mathcal{L}|_D \cong \mathcal{O}_D$. In this case we can define a canonical map $i^* : Z_{k+1}(X) \rightarrow Z_k(D)$ on cycles, by requiring that $i^*[W] = 0$ whenever $W \subset D$. The possibility to do this will be useful later on.

Remark 22.3. Let $f : X' \rightarrow X$ be a morphism of good algebraic spaces over B as in Situation 2.1. Let $(\mathcal{L}, s, i : D \rightarrow X)$ be a triple as in Definition 22.1. Then we can set $\mathcal{L}' = f^*\mathcal{L}$, $s' = f^*s$, and $D' = X' \times_X D = Z(s')$. This gives a commutative diagram

$$\begin{array}{ccc} D' & \xrightarrow{\quad i' \quad} & X' \\ g \downarrow & & \downarrow f \\ D & \xrightarrow{\quad i \quad} & X \end{array}$$

and we can ask for various compatibilities between i^* and $(i')^*$.

Lemma 22.4. *In Situation 2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \rightarrow X)$ be as in Definition 22.1. Let α be a $(k+1)$ -cycle on X . Then $i_* i^* \alpha = c_1(\mathcal{L}) \cap \alpha$ in $\text{CH}_k(X)$. In particular, if D is an effective Cartier divisor, then $D \cdot \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$.*

Proof. Write $\alpha = \sum n_j [W_j]$ where $i_j : W_j \rightarrow X$ are integral closed subspaces with $\dim_\delta(W_j) = k$. Since D is the vanishing locus of s we see that $D \cap W_j$ is the vanishing locus of the restriction $s|_{W_j}$. Hence for each j such that $W_j \not\subset D$ we have $c_1(\mathcal{L}) \cap [W_j] = [D \cap W_j]_k$ by Lemma 18.4. So we have

$$c_1(\mathcal{L}) \cap \alpha = \sum_{W_j \not\subset D} n_j [D \cap W_j]_k + \sum_{W_j \subset D} n_j i_{j,*} (c_1(\mathcal{L})|_{W_j}) \cap [W_j]$$

in $\text{CH}_k(X)$ by Definition 18.1. The right hand side matches (termwise) the push-forward of the class $i^* \alpha$ on D from Definition 22.1. Hence we win. \square

Lemma 22.5. *In Situation 2.1. Let $f : X' \rightarrow X$ be a proper morphism of good algebraic spaces over B . Let $(\mathcal{L}, s, i : D \rightarrow X)$ be as in Definition 22.1. Form the diagram*

$$\begin{array}{ccc} D' & \xrightarrow{\quad} & X' \\ g \downarrow & i' \searrow & \downarrow f \\ D & \xrightarrow{\quad} & X \end{array}$$

as in Remark 22.3. For any $(k+1)$ -cycle α' on X' we have $i^* f_* \alpha' = g_* (i')^* \alpha'$ in $\text{CH}_k(D)$ (this makes sense as f_* is defined on the level of cycles).

Proof. Suppose $\alpha = [W']$ for some integral closed subspace $W' \subset X'$. Let $W \subset X$ be the “image” of W' as in Lemma 7.1. In case $W' \not\subset D'$, then $W \not\subset D$ and we see that

$$[W' \cap D']_k = \text{div}_{\mathcal{L}'|_{W'}}(s'|_{W'}) \quad \text{and} \quad [W \cap D]_k = \text{div}_{\mathcal{L}|_W}(s|_W)$$

and hence f_* of the first cycle equals the second cycle by Lemma 19.3. Hence the equality holds as cycles. In case $W' \subset D'$, then $W \subset D$ and $f_*(c_1(\mathcal{L}|_{W'}) \cap [W'])$ is equal to $c_1(\mathcal{L}|_W) \cap [W]$ in $\text{CH}_k(W)$ by the second assertion of Lemma 19.3. By Remark 15.3 the result follows for general α' . \square

Lemma 22.6. *In Situation 2.1. Let $f : X' \rightarrow X$ be a flat morphism of relative dimension r of good algebraic spaces over B . Let $(\mathcal{L}, s, i : D \rightarrow X)$ be as in Definition 22.1. Form the diagram*

$$\begin{array}{ccc} D' & \xrightarrow{\quad} & X' \\ g \downarrow & i' \searrow & \downarrow f \\ D & \xrightarrow{\quad} & X \end{array}$$

as in Remark 22.3. For any $(k+1)$ -cycle α on X we have $(i')^* f^* \alpha = g^* i^* \alpha$ in $\text{CH}_{k+r}(D)$ (this makes sense as f^* is defined on the level of cycles).

Proof. Suppose $\alpha = [W]$ for some integral closed subspace $W \subset X$. Let $W' = f^{-1}(W) \subset X'$. In case $W \not\subset D$, then $W' \not\subset D'$ and we see that

$$W' \cap D' = g^{-1}(W \cap D)$$

as closed subspaces of D' . Hence the equality holds as cycles, see Lemma 10.5. In case $W \subset D$, then $W' \subset D'$ and $W' = g^{-1}(W)$ with $[W']_{k+1+r} = g^*[W]$ and

equality holds in $\mathrm{CH}_{k+r}(D')$ by Lemma 19.2. By Remark 15.3 the result follows for general α' . \square

Lemma 22.7. *In Situation 2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \rightarrow X)$ be as in Definition 22.1. Let $Z \subset X$ be a closed subscheme such that $\dim_\delta(Z) \leq k+1$ and such that $D \cap Z$ is an effective Cartier divisor on Z . Then $i^*([Z]_{k+1}) = [D \cap Z]_k$.*

Proof. The assumption means that $s|_Z$ is a regular section of $\mathcal{L}|_Z$. Thus $D \cap Z = Z(s)$ and we get

$$[D \cap Z]_k = \sum n_i [Z(s_i)]_k$$

as cycles where $s_i = s|_{Z_i}$, the Z_i are the irreducible components of δ -dimension $k+1$, and $[Z]_{k+1} = \sum n_i [Z_i]$. See Lemma 18.3. We have $D \cap Z_i = Z(s_i)$. Comparing with the definition of the gysin map we conclude. \square

23. Gysin homomorphisms

This section is the analogue of Chow Homology, Section 30. In this section we use the key formula to show the Gysin homomorphism factor through rational equivalence.

Lemma 23.1. *In Situation 2.1 let X/B be good. Assume X integral and $n = \dim_\delta(X)$. Let $i : D \rightarrow X$ be an effective Cartier divisor. Let \mathcal{N} be an invertible \mathcal{O}_X -module and let t be a nonzero meromorphic section of \mathcal{N} . Then $i^* \mathrm{div}_{\mathcal{N}}(t) = c_1(\mathcal{N}) \cap [D]_{n-1}$ in $\mathrm{CH}_{n-2}(D)$.*

Proof. Write $\mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{Z_i, \mathcal{N}}(t) [Z_i]$ for some integral closed subspaces $Z_i \subset X$ of δ -dimension $n-1$. We may assume that the family $\{Z_i\}$ is locally finite, that $t \in \Gamma(U, \mathcal{N}|_U)$ is a generator where $U = X \setminus \bigcup Z_i$, and that every irreducible component of D is one of the Z_i , see Spaces over Fields, Lemmas 6.1, 6.6, and 7.2.

Set $\mathcal{L} = \mathcal{O}_X(D)$. Denote $s \in \Gamma(X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{L})$ the canonical section. We will apply the discussion of Section 20 to our current situation. For each i let $\xi_i \in |Z_i|$ be its generic point. Let $B_i = \mathcal{O}_{X, \xi_i}^h$. For each i we pick generators s_i of \mathcal{L}_{ξ_i} and t_i of \mathcal{N}_{ξ_i} over B_i but we insist that we pick $s_i = s$ if $Z_i \not\subset D$. Write $s = f_i s_i$ and $t = g_i t_i$ with $f_i, g_i \in B_i$. Then $\mathrm{ord}_{Z_i, \mathcal{N}}(t) = \mathrm{ord}_{B_i}(g_i)$. On the other hand, we have $f_i \in B_i$ and

$$[D]_{n-1} = \sum \mathrm{ord}_{B_i}(f_i) [Z_i]$$

because of our choices of s_i . We claim that

$$i^* \mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{B_i}(g_i) \mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

as cycles. More precisely, the right hand side is a cycle representing the left hand side. Namely, this is clear by our formula for $\mathrm{div}_{\mathcal{N}}(t)$ and the fact that $\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) = [Z(s_i|_{Z_i})]_{n-2} = [Z_i \cap D]_{n-2}$ when $Z_i \not\subset D$ because in that case $s_i|_{Z_i} = s|_{Z_i}$ is a regular section, see Lemma 17.2. Similarly,

$$c_1(\mathcal{N}) \cap [D]_{n-1} = \sum \mathrm{ord}_{B_i}(f_i) \mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

The key formula (Lemma 20.1) gives the equality

$$\sum \left(\mathrm{ord}_{B_i}(f_i) \mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - \mathrm{ord}_{B_i}(g_i) \mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) \right) = \sum \mathrm{div}_{Z_i}(\partial_{B_i}(f_i, g_i))$$

of cycles. If $Z_i \not\subset D$, then $f_i = 1$ and hence $\text{div}_{Z_i}(\partial_{B_i}(f_i, g_i)) = 0$. Thus we get a rational equivalence between our specific cycles representing $i^* \text{div}_{\mathcal{N}}(t)$ and $c_1(\mathcal{N}) \cap [D]_{n-1}$ on D . This finishes the proof. \square

Lemma 23.2. *In Situation 2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \rightarrow X)$ be as in Definition 22.1. The Gysin homomorphism factors through rational equivalence to give a map $i^* : \text{CH}_{k+1}(X) \rightarrow \text{CH}_k(D)$.*

Proof. Let $\alpha \in Z_{k+1}(X)$ and assume that $\alpha \sim_{\text{rat}} 0$. This means there exists a locally finite collection of integral closed subspaces $W_j \subset X$ of δ -dimension $k+2$ and $f_j \in R(W_j)^*$ such that $\alpha = \sum i_{j,*} \text{div}_{W_j}(f_j)$. Set $X' = \coprod W_i$ and consider the diagram

$$\begin{array}{ccc} D' & \xrightarrow{i'} & X' \\ q \downarrow & & \downarrow p \\ D & \xrightarrow{i} & X \end{array}$$

of Remark 22.3. Since $X' \rightarrow X$ is proper we see that $i^* p_* = q_*(i')^*$ by Lemma 22.5. As we know that q_* factors through rational equivalence (Lemma 16.3), it suffices to prove the result for $\alpha' = \sum \text{div}_{W_j}(f_j)$ on X' . Clearly this reduces us to the case where X is integral and $\alpha = \text{div}(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \text{div}(f)$ for some $f \in R(X)^*$. If $X = D$, then we see that $i^* \alpha$ is equal to $c_1(\mathcal{L}) \cap \alpha$. This is rationally equivalent to zero by Lemma 21.2. If $D \neq X$, then we see that $i^* \text{div}_X(f)$ is equal to $c_1(\mathcal{O}_D) \cap [D]_{n-1}$ in $\text{CH}_k(D)$ by Lemma 23.1. Of course capping with $c_1(\mathcal{O}_D)$ is the zero map. \square

Lemma 23.3. *In Situation 2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \rightarrow X)$ be a triple as in Definition 22.1. Let \mathcal{N} be an invertible \mathcal{O}_X -module. Then $i^*(c_1(\mathcal{N}) \cap \alpha) = c_1(i^* \mathcal{N}) \cap i^* \alpha$ in $\text{CH}_{k-2}(D)$ for all $\alpha \in \text{CH}_k(X)$.*

Proof. With exactly the same proof as in Lemma 23.2 this follows from Lemmas 19.4, 21.3, and 23.1. \square

Lemma 23.4. *In Situation 2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \rightarrow X)$ and $(\mathcal{L}', s', i' : D' \rightarrow X)$ be two triples as in Definition 22.1. Then the diagram*

$$\begin{array}{ccc} \text{CH}_k(X) & \xrightarrow{i^*} & \text{CH}_{k-1}(D) \\ (i')^* \downarrow & & \downarrow \\ \text{CH}_{k-1}(D') & \xrightarrow{} & \text{CH}_{k-2}(D \cap D') \end{array}$$

commutes where each of the maps is a gysin map.

Proof. Denote $j : D \cap D' \rightarrow D$ and $j' : D \cap D' \rightarrow D'$ the closed immersions corresponding to $(\mathcal{L}|_{D'}, s|_{D'})$ and $(\mathcal{L}'_D, s|_D)$. We have to show that $(j')^* i^* \alpha = j^* (i')^* \alpha$ for all $\alpha \in \text{CH}_k(X)$. Let $W \subset X$ be an integral closed subscheme of dimension k . We will prove the equality in case $\alpha = [W]$. The general case will then follow from the observation in Remark 15.3 (and the specific shape of our rational equivalence produced below). We will deduce the equality for $\alpha = [W]$ from the key formula.

We let σ be a nonzero meromorphic section of $\mathcal{L}|_W$ which we require to be equal to $s|_W$ if $W \not\subset D$. We let σ' be a nonzero meromorphic section of $\mathcal{L}'|_W$ which we

require to be equal to $s'|_W$ if $W \not\subset D'$. Write

$$\operatorname{div}_{\mathcal{L}|_W}(\sigma) = \sum \operatorname{ord}_{Z_i, \mathcal{L}|_W}(\sigma)[Z_i] = \sum n_i[Z_i]$$

and similarly

$$\operatorname{div}_{\mathcal{L}'|_W}(\sigma') = \sum \operatorname{ord}_{Z_i, \mathcal{L}'|_W}(\sigma')[Z_i] = \sum n'_i[Z_i]$$

as in the discussion in Section 20. Then we see that $Z_i \subset D$ if $n_i \neq 0$ and $Z'_i \subset D'$ if $n'_i \neq 0$. For each i , let $\xi_i \in |Z_i|$ be the generic point. As in Section 20 we choose for each i an element $\sigma_i \in \mathcal{L}_{\xi_i}$, resp. $\sigma'_i \in \mathcal{L}'_{\xi_i}$ which generates over $B_i = \mathcal{O}_{W, \xi_i}^h$ and which is equal to the image of s , resp. s' if $Z_i \not\subset D$, resp. $Z_i \not\subset D'$. Write $\sigma = f_i \sigma_i$ and $\sigma' = f'_i \sigma'_i$ so that $n_i = \operatorname{ord}_{B_i}(f_i)$ and $n'_i = \operatorname{ord}_{B_i}(f'_i)$. From our definitions it follows that

$$(j')^* i^* [W] = \sum \operatorname{ord}_{B_i}(f_i) \operatorname{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i})$$

as cycles and

$$j^*(i')^* [W] = \sum \operatorname{ord}_{B_i}(f'_i) \operatorname{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})$$

The key formula (Lemma 20.1) now gives the equality

$$\sum \left(\operatorname{ord}_{B_i}(f_i) \operatorname{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i}) - \operatorname{ord}_{B_i}(f'_i) \operatorname{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i}) \right) = \sum \operatorname{div}_{Z_i}(\partial_{B_i}(f_i, f'_i))$$

of cycles. Note that $\operatorname{div}_{Z_i}(\partial_{B_i}(f_i, f'_i)) = 0$ if $Z_i \not\subset D \cap D'$ because in this case either $f_i = 1$ or $f'_i = 1$. Thus we get a rational equivalence between our specific cycles representing $(j')^* i^* [W]$ and $j^*(i')^* [W]$ on $D \cap D' \cap W$. \square

24. Relative effective Cartier divisors

This section is the analogue of Chow Homology, Section 31. Relative effective Cartier divisors are defined in Divisors on Spaces, Section 9. To develop the basic results on Chern classes of vector bundles we only need the case where both the ambient scheme and the effective Cartier divisor are flat over the base.

Lemma 24.1. *In Situation 2.1. Let $X, Y/B$ be good. Let $p : X \rightarrow Y$ be a flat morphism of relative dimension r . Let $i : D \rightarrow X$ be a relative effective Cartier divisor (Divisors on Spaces, Definition 9.2). Let $\mathcal{L} = \mathcal{O}_X(D)$. For any $\alpha \in \operatorname{CH}_{k+1}(Y)$ we have*

$$i^* p^* \alpha = (p|_D)^* \alpha$$

in $\operatorname{CH}_{k+r}(D)$ and

$$c_1(\mathcal{L}) \cap p^* \alpha = i_*((p|_D)^* \alpha)$$

in $\operatorname{CH}_{k+r}(X)$.

Proof. Let $W \subset Y$ be an integral closed subspace of δ -dimension $k+1$. By Divisors on Spaces, Lemma 9.1 we see that $D \cap p^{-1}W$ is an effective Cartier divisor on $p^{-1}W$. By Lemma 22.7 we get the first equality in

$$i^* [p^{-1}W]_{k+r+1} = [D \cap p^{-1}W]_{k+r} = [(p|_D)^{-1}(W)]_{k+r}.$$

and the second because $D \cap p^{-1}(W) = (p|_D)^{-1}(W)$ as algebraic spaces. Since by definition $p^*[W] = [p^{-1}W]_{k+r+1}$ we see that $i^* p^*[W] = (p|_D)^*[W]$ as cycles. If $\alpha = \sum m_j [W_j]$ is a general $k+1$ cycle, then we get $i^* \alpha = \sum m_j i^* p^*[W_j] = \sum m_j (p|_D)^*[W_j]$ as cycles. This proves then first equality. To deduce the second from the first apply Lemma 22.4. \square

25. Affine bundles

This section is the analogue of Chow Homology, Section 32. For an affine bundle the pullback map is surjective on Chow groups.

Lemma 25.1. *In Situation 2.1 let $X, Y/B$ be good. Let $f : X \rightarrow Y$ be a quasi-compact flat morphism over B of relative dimension r . Assume that for every $y \in Y$ we have $X_y \cong \mathbf{A}_{\kappa(y)}^r$. Then $f^* : \mathrm{CH}_k(Y) \rightarrow \mathrm{CH}_{k+r}(X)$ is surjective for all $k \in \mathbf{Z}$.*

Proof. Let $\alpha \in \mathrm{CH}_{k+r}(X)$. Write $\alpha = \sum m_j [W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subspaces of δ -dimension $k+r$. Then the family $\{W_j\}$ is locally finite in X . Let $Z_j \subset Y$ be the integral closed subspace such that we obtain a dominant morphism $W_j \rightarrow Z_j$ as in Lemma 7.1. For any quasi-compact open $V \subset Y$ we see that $f^{-1}(V) \cap W_j$ is nonempty only for finitely many j . Hence the collection Z_j of closures of images is a locally finite collection of integral closed subspaces of Y .

Consider the fibre product diagrams

$$\begin{array}{ccc} f^{-1}(Z_j) & \longrightarrow & X \\ f_j \downarrow & & \downarrow f \\ Z_j & \longrightarrow & Y \end{array}$$

Suppose that $[W_j] \in Z_{k+r}(f^{-1}(Z_j))$ is rationally equivalent to $f_j^* \beta_j$ for some k -cycle $\beta_j \in \mathrm{CH}_k(Z_j)$. Then $\beta = \sum m_j \beta_j$ will be a k -cycle on Y and $f^* \beta = \sum m_j f_j^* \beta_j$ will be rationally equivalent to α (see Remark 15.3). This reduces us to the case Y integral, and $\alpha = [W]$ for some integral closed subscheme of X dominating Y . In particular we may assume that $d = \dim_\delta(Y) < \infty$.

Hence we can use induction on $d = \dim_\delta(Y)$. If $d < k$, then $\mathrm{CH}_{k+r}(X) = 0$ and the lemma holds; this is the base case of the induction. Consider a nonempty open $V \subset Y$. Suppose that we can show that $\alpha|_{f^{-1}(V)} = f^* \beta$ for some $\beta \in Z_k(V)$. By Lemma 10.2 we see that $\beta = \beta'|_V$ for some $\beta' \in Z_k(Y)$. By the exact sequence $\mathrm{CH}_k(f^{-1}(Y \setminus V)) \rightarrow \mathrm{CH}_k(X) \rightarrow \mathrm{CH}_k(f^{-1}(V))$ of Lemma 15.2 we see that $\alpha - f^* \beta'$ comes from a cycle $\alpha' \in \mathrm{CH}_{k+r}(f^{-1}(Y \setminus V))$. Since $\dim_\delta(Y \setminus V) < d$ we win by induction on d .

In particular, by replacing Y by a suitable open we may assume Y is a scheme with generic point η . The isomorphism $Y_\eta \cong \mathbf{A}_\eta^r$ extends to an isomorphism over a nonempty open $V \subset Y$, see Limits of Spaces, Lemma 7.1. This reduces us to the case of schemes which is Chow Homology, Lemma 32.1. \square

Lemma 25.2. *In Situation 2.1 let X/B be good. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let*

$$p : L = \underline{\mathrm{Spec}}(\mathrm{Sym}^*(\mathcal{L})) \longrightarrow X$$

be the associated vector bundle over X . Then $p^ : \mathrm{CH}_k(X) \rightarrow \mathrm{CH}_{k+1}(L)$ is an isomorphism for all k .*

Proof. For surjectivity see Lemma 25.1. Let $o : X \rightarrow L$ be the zero section of $L \rightarrow X$, i.e., the morphism corresponding to the surjection $\mathrm{Sym}^*(\mathcal{L}) \rightarrow \mathcal{O}_X$ which maps $\mathcal{L}^{\otimes n}$ to zero for all $n > 0$. Then $p \circ o = \mathrm{id}_X$ and $o(X)$ is an effective Cartier

divisor on L . Hence by Lemma 24.1 we see that $o^* \circ p^* = \text{id}$ and we conclude that p^* is injective too. \square

26. Bivariant intersection theory

This section is the analogue of Chow Homology, Section 33. In order to intelligently talk about higher Chern classes of vector bundles we introduce the following notion, following [FM81]. It follows from [Ful98, Theorem 17.1] that our definition agrees with that of [Ful98] modulo the caveat that we are working in different settings.

Definition 26.1. In Situation 2.1 let $f : X \rightarrow Y$ be a morphism of good algebraic spaces over B . Let $p \in \mathbf{Z}$. A *bivariant class of degree p for f* is given by a rule which assigns to every morphism $Y' \rightarrow Y$ of good algebraic spaces over B and every k a map

$$c \cap - : \text{CH}_k(Y') \longrightarrow \text{CH}_{k-p}(X')$$

where $X' = Y' \times_Y X$, satisfying the following conditions

- (1) if $Y'' \rightarrow Y'$ is a proper morphism, then $c \cap (Y'' \rightarrow Y')_* \alpha'' = (X'' \rightarrow X')_*(c \cap \alpha'')$ for all α'' on Y'' ,
- (2) if $Y'' \rightarrow Y'$ a morphism of good algebraic spaces over B which is flat of relative dimension r , then $c \cap (Y'' \rightarrow Y')^* \alpha' = (X'' \rightarrow X')^*(c \cap \alpha')$ for all α' on Y' ,
- (3) if $(\mathcal{L}', s', i' : D' \rightarrow Y')$ is as in Definition 22.1 with pullback $(\mathcal{N}', t', j' : E' \rightarrow X')$ to X' , then we have $c \cap (i')^* \alpha' = (j')^*(c \cap \alpha')$ for all α' on Y' .

The collection of all bivariant classes of degree p for f is denoted $A^p(X \rightarrow Y)$.

In Situation 2.1 let $X \rightarrow Y$ and $Y \rightarrow Z$ be morphisms of good algebraic spaces over B . Let $p \in \mathbf{Z}$. It is clear that $A^p(X \rightarrow Y)$ is an abelian group. Moreover, it is clear that we have a bilinear composition

$$A^p(X \rightarrow Y) \times A^q(Y \rightarrow Z) \rightarrow A^{p+q}(X \rightarrow Z)$$

which is associative. We will be most interested in $A^p(X) = A^p(X \rightarrow X)$, which will always mean the bivariant cohomology classes for id_X . Namely, that is where Chern classes will live.

Definition 26.2. In Situation 2.1 let X/B be good. The *Chow cohomology* of X is the graded \mathbf{Z} -algebra $A^*(X)$ whose degree p component is $A^p(X \rightarrow X)$.

Warning: It is not clear that the \mathbf{Z} -algebra structure on $A^*(X)$ is commutative, but we will see that Chern classes live in its center.

Remark 26.3. In Situation 2.1 let $f : X \rightarrow Y$ be a morphism of good algebraic spaces over B . Then there is a canonical \mathbf{Z} -algebra map $A^*(Y) \rightarrow A^*(X)$. Namely, given $c \in A^p(Y)$ and $X' \rightarrow X$, then we can let f^*c be defined by the map $c \cap - : \text{CH}_k(X') \rightarrow \text{CH}_{k-p}(X')$ which is given by thinking of X' as an algebraic space over Y .

Lemma 26.4. In Situation 2.1 let X/B be good. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Then the rule that to $f : X' \rightarrow X$ assigns $c_1(f^*\mathcal{L}) \cap - : \text{CH}_k(X') \rightarrow \text{CH}_{k-1}(X')$ is a bivariant class of degree 1.

Proof. This follows from Lemmas 21.2, 19.4, 19.2, and 23.3. \square

Lemma 26.5. *In Situation 2.1 let $f : X \rightarrow Y$ be a morphism of good algebraic spaces over B which is flat of relative dimension r . Then the rule that to $Y' \rightarrow Y$ assigns $(f')^* : \mathrm{CH}_k(Y') \rightarrow \mathrm{CH}_{k+r}(X')$ where $X' = X \times_Y Y'$ is a bivariant class of degree $-r$.*

Proof. This follows from Lemmas 16.2, 10.4, 11.1, and 22.6. \square

Lemma 26.6. *In Situation 2.1 let X/B be good. Let $(\mathcal{L}, s, i : D \rightarrow X)$ be a triple as in Definition 22.1. Then the rule that to $f : X' \rightarrow X$ assigns $(i')^* : \mathrm{CH}_k(X') \rightarrow \mathrm{CH}_{k-1}(D')$ where $D' = D \times_X X'$ is a bivariant class of degree 1.*

Proof. This follows from Lemmas 23.2, 22.5, 22.6, and 23.4. \square

Lemma 26.7. *In Situation 2.1 let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of good algebraic spaces over B . Let $c \in A^p(X \rightarrow Z)$ and assume f is proper. Then the rule that to $X' \rightarrow X$ assigns $\alpha \mapsto f_*(c \cap \alpha)$ is a bivariant class of degree p .*

Proof. This follows from Lemmas 8.2, 11.1, and 22.5. \square

Here we see that $c_1(\mathcal{L})$ is in the center of $A^*(X)$.

Lemma 26.8. *In Situation 2.1 let X/B be good. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Then $c_1(\mathcal{L}) \in A^1(X)$ commutes with every element $c \in A^p(X)$.*

Proof. Let $p : L \rightarrow X$ be as in Lemma 25.2 and let $o : X \rightarrow L$ be the zero section. Observe that $p^*\mathcal{L}^{\otimes -1}$ has a canonical section whose vanishing locus is exactly the effective Cartier divisor $o(X)$. Let $\alpha \in \mathrm{CH}_k(X)$. Then we see that

$$p^*(c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) = c_1(p^*\mathcal{L}^{\otimes -1}) \cap p^*\alpha = o_*o^*p^*\alpha$$

by Lemmas 19.2 and 24.1. Since c is a bivariant class we have

$$\begin{aligned} p^*(c \cap c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) &= c \cap p^*(c_1(\mathcal{L}^{\otimes -1}) \cap \alpha) \\ &= c \cap o_*o^*p^*\alpha \\ &= o_*o^*p^*(c \cap \alpha) \\ &= p^*(c_1(\mathcal{L}^{\otimes -1}) \cap c \cap \alpha) \end{aligned}$$

(last equality by the above applied to $c \cap \alpha$). Since p^* is injective by a lemma cited above we get that $c_1(\mathcal{L}^{\otimes -1})$ is in the center of $A^*(X)$. This proves the lemma. \square

Here a criterion for when a bivariant class is zero.

Lemma 26.9. *In Situation 2.1 let X/B be good. Let $c \in A^p(X)$. Then c is zero if and only if $c \cap [Y] = 0$ in $\mathrm{CH}_*(Y)$ for every integral algebraic space Y locally of finite type over X .*

Proof. The if direction is clear. For the converse, assume that $c \cap [Y] = 0$ in $\mathrm{CH}_*(Y)$ for every integral algebraic space Y locally of finite type over X . Let $X' \rightarrow X$ be locally of finite type. Let $\alpha \in \mathrm{CH}_k(X')$. Write $\alpha = \sum n_i[Y_i]$ with $Y_i \subset X'$ a locally finite collection of integral closed subschemes of δ -dimension k . Then we see that α is pushforward of the cycle $\alpha' = \sum n_i[Y_i]$ on $X'' = \coprod Y_i$ under the proper morphism $X'' \rightarrow X'$. By the properties of bivariant classes it suffices to prove that $c \cap \alpha' = 0$ in $\mathrm{CH}_{k-p}(X'')$. We have $\mathrm{CH}_{k-p}(X'') = \prod \mathrm{CH}_{k-p}(Y_i)$ as follows immediately from the definitions. The projection maps $\mathrm{CH}_{k-p}(X'') \rightarrow \mathrm{CH}_{k-p}(Y_i)$ are given by flat pullback. Since capping with c commutes with flat pullback, we see that it suffices to show that $c \cap [Y_i]$ is zero in $\mathrm{CH}_{k-p}(Y_i)$ which is true by assumption. \square

27. Projective space bundle formula

In Situation 2.1 let X/B be good. Consider a finite locally free \mathcal{O}_X -module \mathcal{E} of rank r . Our convention is that the *projective bundle associated to \mathcal{E}* is the morphism

$$\mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_X(\text{Sym}^*(\mathcal{E})) \xrightarrow{\pi} X$$

over X with $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ normalized so that $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$. In particular there is a surjection $\pi^*\mathcal{E} \rightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. We will say informally “let $(\pi : P \rightarrow X, \mathcal{O}_P(1))$ be the projective bundle associated to \mathcal{E} ” to denote the situation where $P = \mathbf{P}(\mathcal{E})$ and $\mathcal{O}_P(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Lemma 27.1. *In Situation 2.1 let X/B be good. Let \mathcal{E} be a finite locally free \mathcal{O}_X -module \mathcal{E} of rank r . Let $(\pi : P \rightarrow X, \mathcal{O}_P(1))$ be the projective bundle associated to \mathcal{E} . For any $\alpha \in \text{CH}_k(X)$ the element*

$$\pi_*(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha) \in \text{CH}_{k+r-1-s}(X)$$

is 0 if $s < r - 1$ and is equal to α when $s = r - 1$.

Proof. Let $Z \subset X$ be an integral closed subspace of δ -dimension k . We will prove the lemma for $\alpha = [Z]$. We omit the argument deducing the general case from this special case; hint: argue as in Remark 15.3.

Let $P_Z = P \times_X Z$ be the base change; of course $\pi_Z : P_Z \rightarrow Z$ is the projective bundle associated to $\mathcal{E}|_Z$ and $\mathcal{O}_P(1)$ pulls back to the corresponding invertible module on P_Z . Since $c_1(\mathcal{O}_P(1)) \cap -$, and π^* are bivariant classes by Lemmas 26.4 and 26.5 we see that

$$\pi_*(c_1(\mathcal{O}_P(1))^s \cap \pi^*[Z]) = (Z \rightarrow X)_* \pi_{Z,*} (c_1(\mathcal{O}_{P_Z}(1))^s \cap \pi_Z^*[Z])$$

Hence it suffices to prove the lemma in case X is integral and $\alpha = [X]$.

Assume X is integral, $\dim_\delta(X) = k$, and $\alpha = [X]$. Note that $\pi^*[X] = [P]$ as P is integral of δ -dimension $r - 1$. If $s < r - 1$, then by construction $c_1(\mathcal{O}_P(1))^s \cap [P]$ a $(k + r - 1 - s)$ -cycle. Hence the pushforward of this cycle is zero for dimension reasons.

Let $s = r - 1$. By the argument given above we see that $\pi_*(c_1(\mathcal{O}_P(1))^s \cap [P]) = n[X]$ for some $n \in \mathbf{Z}$. We want to show that $n = 1$. For the same dimension reasons as above it suffices to prove this result after replacing X by a dense open. Thus we may assume X is a scheme and the result follows from Chow Homology, Lemma 36.1. \square

Lemma 27.2 (Projective space bundle formula). *Let (S, δ) be as in Situation 2.1. Let X be locally of finite type over S . Let \mathcal{E} be a finite locally free \mathcal{O}_X -module \mathcal{E} of rank r . Let $(\pi : P \rightarrow X, \mathcal{O}_P(1))$ be the projective bundle associated to \mathcal{E} . The map*

$$\bigoplus_{i=0}^{r-1} \text{CH}_{k+i}(X) \longrightarrow \text{CH}_{k+r-1}(P),$$

$$(\alpha_0, \dots, \alpha_{r-1}) \mapsto \pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \dots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

is an isomorphism.

Proof. Fix $k \in \mathbf{Z}$. We first show the map is injective. Suppose that $(\alpha_0, \dots, \alpha_{r-1})$ is an element of the left hand side that maps to zero. By Lemma 27.1 we see that

$$0 = \pi_*(\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \dots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}) = \alpha_{r-1}$$

Next, we see that

$$0 = \pi_*(c_1(\mathcal{O}_P(1)) \cap (\pi^* \alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^* \alpha_1 + \dots + c_1(\mathcal{O}_P(1))^{r-2} \cap \pi^* \alpha_{r-2})) = \alpha_{r-2}$$

and so on. Hence the map is injective.

To prove the map is surjective, we will argue exactly as in the proof of Lemma 25.1 to reduce to the case of schemes. We urge the reader to skip the proof.

Let $\beta \in \text{CH}_{k+r-1}(P)$. Write $\beta = \sum m_j [W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subspaces of δ -dimension $k+r$. Then the family $\{W_j\}$ is locally finite in P . Let $Z_j \subset X$ be the “image” of W_j as in Lemma 7.1. For any quasi-compact open $U \subset X$ we see that $\pi^{-1}(U) \cap W_j$ is nonempty only for finitely many j . Hence the collection Z_j of images is a locally finite collection of integral closed subspaces of X .

Consider the fibre product diagrams

$$\begin{array}{ccc} P_j & \longrightarrow & P \\ \pi_j \downarrow & & \downarrow \pi \\ Z_j & \longrightarrow & X \end{array}$$

Suppose that $[W_j] \in Z_{k+r-1}(P_j)$ is rationally equivalent to

$$\pi_j^* \alpha_{j,0} + c_1(\mathcal{O}(1)) \cap \pi_j^* \alpha_{j,1} + \dots + c_1(\mathcal{O}(1))^{r-1} \cap \pi_j^* \alpha_{j,r-1}$$

for some $(k+i)$ -cycle $\alpha_{j,i} \in \text{CH}_{k+i}(Z_j)$. Then $\alpha_i = \sum m_j \beta_{j,i}$ will be a $(k+i)$ -cycle on X and

$$\pi^* \alpha_0 + c_1(\mathcal{O}(1)) \cap \pi^* \alpha_1 + \dots + c_1(\mathcal{O}(1))^{r-1} \cap \pi^* \alpha_{r-1}$$

will be rationally equivalent to β (see Remark 15.3). This reduces us to the case X integral, and $\alpha = [W]$ for some integral closed subscheme of P dominating X . In particular we may assume that $d = \dim_\delta(X) < \infty$.

Hence we can use induction on $d = \dim_\delta(X)$. If $d < k$, then $\text{CH}_{k+r-1}(X) = 0$ and the lemma holds; this is the base case of the induction. Consider a nonempty open $U \subset X$. Suppose that we can show that

$$\beta|_{\pi^{-1}(U)} = \pi^* \alpha_0 + c_1(\mathcal{O}(1)) \cap \pi^* \alpha_1 + \dots + c_1(\mathcal{O}(1))^{r-1} \cap \pi^* \alpha_{r-1}$$

for some $\alpha_i \in Z_{k+i}(U)$. By Lemma 10.2 we see that $\alpha_i = \alpha'_i|_U$ for some $\alpha'_i \in Z_{k+i}(X)$. By the exact sequences $\text{CH}_{k+i}(\pi^{-1}(X \setminus U)) \rightarrow \text{CH}_{k+i}(P) \rightarrow \text{CH}_{k+i}(\pi^{-1}(U))$ of Lemma 15.2 we see that

$$\beta - (\pi^* \alpha'_0 + c_1(\mathcal{O}(1)) \cap \pi^* \alpha'_1 + \dots + c_1(\mathcal{O}(1))^{r-1} \cap \pi^* \alpha'_{r-1})$$

comes from a cycle $\beta' \in \text{CH}_{k+r}(\pi^{-1}(X \setminus U))$. Since $\dim_\delta(X \setminus U) < d$ we win by induction on d .

In particular, by replacing X by a suitable open we may assume X is a scheme and we have reduced our problem to Chow Homology, Lemma 36.2. \square

Lemma 27.3. *In Situation 2.1 let X/B be good. Let \mathcal{E} be a finite locally free sheaf of rank r on X . Let*

$$p : E = \underline{\text{Spec}}(\text{Sym}^*(\mathcal{E})) \longrightarrow X$$

be the associated vector bundle over X . Then $p^ : \text{CH}_k(X) \rightarrow \text{CH}_{k+r}(E)$ is an isomorphism for all k .*

Proof. (For the case of linebundles, see Lemma 25.2.) For surjectivity see Lemma 25.1. Let $(\pi : P \rightarrow X, \mathcal{O}_P(1))$ be the projective space bundle associated to the finite locally free sheaf $\mathcal{E} \oplus \mathcal{O}_X$. Let $s \in \Gamma(P, \mathcal{O}_P(1))$ correspond to the global section $(0, 1) \in \Gamma(X, \mathcal{E} \oplus \mathcal{O}_X)$. Let $D = Z(s) \subset P$. Note that $(\pi|_D : D \rightarrow X, \mathcal{O}_P(1)|_D)$ is the projective space bundle associated to \mathcal{E} . We denote $\pi_D = \pi|_D$ and $\mathcal{O}_D(1) = \mathcal{O}_P(1)|_D$. Moreover, D is an effective Cartier divisor on P . Hence $\mathcal{O}_P(D) = \mathcal{O}_P(1)$ (see Divisors on Spaces, Lemma 7.8). Also there is an isomorphism $E \cong P \setminus D$. Denote $j : E \rightarrow P$ the corresponding open immersion. For injectivity we use that the kernel of

$$j^* : \mathrm{CH}_{k+r}(P) \longrightarrow \mathrm{CH}_{k+r}(E)$$

are the cycles supported in the effective Cartier divisor D , see Lemma 15.2. So if $p^*\alpha = 0$, then $\pi^*\alpha = i_*\beta$ for some $\beta \in \mathrm{CH}_{k+r}(D)$. By Lemma 27.2 we may write

$$\beta = \pi_D^*\beta_0 + \dots + c_1(\mathcal{O}_D(1))^{r-1} \cap \pi_D^*\beta_{r-1}.$$

for some $\beta_i \in \mathrm{CH}_{k+i}(X)$. By Lemmas 24.1 and 19.4 this implies

$$\pi^*\alpha = i_*\beta = c_1(\mathcal{O}_P(1)) \cap \pi^*\beta_0 + \dots + c_1(\mathcal{O}_D(1))^r \cap \pi^*\beta_{r-1}.$$

Since the rank of $\mathcal{E} \oplus \mathcal{O}_X$ is $r+1$ this contradicts Lemma 19.4 unless all α and all β_i are zero. \square

28. The Chern classes of a vector bundle

This section is the analogue of Chow Homology, Sections 37 and 38. However, contrary to what is done there, we directly define the Chern classes of a vector bundle as bivariant classes. This saves a considerable amount of work.

Lemma 28.1. *In Situation 2.1 let X/B be good. Let \mathcal{E} be a finite locally free sheaf of rank r on X . Let $(\pi : P \rightarrow X, \mathcal{O}_P(1))$ be the projective space bundle associated to \mathcal{E} . For every morphism $X' \rightarrow X$ of good algebraic spaces over B there are unique maps*

$$c_i(\mathcal{E}) \cap - : \mathrm{CH}_k(X') \longrightarrow \mathrm{CH}_{k-i}(X'), \quad i = 0, \dots, r$$

such that for $\alpha \in \mathrm{CH}_k(X')$ we have $c_0(\mathcal{E}) \cap \alpha = \alpha$ and

$$\sum_{i=0, \dots, r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^* (c_{r-i}(\mathcal{E}) \cap \alpha) = 0$$

where $\pi' : P' \rightarrow X'$ is the base change of π . Moreover, these maps define a bivariant class $c_i(\mathcal{E})$ of degree i on X .

Proof. Uniqueness and existence of the maps $c_i(\mathcal{E}) \cap -$ follows immediately from Lemma 27.2 and the given description of $c_0(\mathcal{E})$. For every $i \in \mathbf{Z}$ the rule which to every morphism $X' \rightarrow X$ of good algebraic spaces over B assigns the map

$$t_i(\mathcal{E}) \cap - : \mathrm{CH}_k(X') \longrightarrow \mathrm{CH}_{k-i}(X'), \quad \alpha \longmapsto \pi'_*(c_1(\mathcal{O}_{P'}(1))^{r-1+i} \cap (\pi')^*\alpha)$$

is a bivariant class² by Lemmas 26.4, 26.5, and 26.7. By Lemma 27.1 we have $t_i(\mathcal{E}) = 0$ for $i < 0$ and $t_0(\mathcal{E}) = 1$. Applying pushforward to the equation in the statement of the lemma we find from Lemma 27.1 that

$$(-1)^r t_1(\mathcal{E}) + (-1)^{r-1} c_1(\mathcal{E}) = 0$$

²Up to signs these are the Segre classes of \mathcal{E} .

In particular we find that $c_1(\mathcal{E})$ is a bivariant class. If we multiply the equation in the statement of the lemma by $c_1(\mathcal{O}_{P'}(1))$ and push the result forward to X' we find

$$(-1)^r t_2(\mathcal{E}) + (-1)^{r-1} t_1(\mathcal{E}) \cap c_1(\mathcal{E}) + (-1)^{r-2} c_2(\mathcal{E}) = 0$$

As before we conclude that $c_2(\mathcal{E})$ is a bivariant class. And so on. \square

Definition 28.2. In Situation 2.1 let X/B be good. Let \mathcal{E} be a finite locally free sheaf of rank r on X . For $i = 0, \dots, r$ the i th Chern class of \mathcal{E} is the bivariant class $c_i(\mathcal{E}) \in A^i(X)$ of degree i constructed in Lemma 28.1. The total Chern class of \mathcal{E} is the formal sum

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \dots + c_r(\mathcal{E})$$

which is viewed as a nonhomogeneous bivariant class on X .

For convenience we often set $c_i(\mathcal{E}) = 0$ for $i > r$ and $i < 0$. By definition we have $c_0(\mathcal{E}) = 1 \in A^0(X)$. Here is a sanity check.

Lemma 28.3. In Situation 2.1 let X/B be good. Let \mathcal{L} be an invertible \mathcal{O}_X -module. The first Chern class of \mathcal{L} on X of Definition 28.2 is equal to the bivariant class of Lemma 26.4.

Proof. Namely, in this case $P = \mathbf{P}(\mathcal{L}) = X$ with $\mathcal{O}_P(1) = \mathcal{L}$ by our normalization of the projective bundle, see Section 27. Hence the equation in Lemma 28.1 reads

$$(-1)^0 c_1(\mathcal{L})^0 \cap c_1^{new}(\mathcal{L}) \cap \alpha + (-1)^1 c_1(\mathcal{L})^1 \cap c_0^{new}(\mathcal{L}) \cap \alpha = 0$$

where $c_i^{new}(\mathcal{L})$ is as in Definition 28.2. Since $c_0^{new}(\mathcal{L}) = 1$ and $c_1(\mathcal{L})^0 = 1$ we conclude. \square

Next we see that Chern classes are in the center of the bivariant Chow cohomology ring $A^*(X)$.

Lemma 28.4. In Situation 2.1 let X/B be good. Let \mathcal{E} be a locally free \mathcal{O}_X -module of rank r . Then $c_j(\mathcal{E}) \in A^j(X)$ commutes with every element $c \in A^p(X)$. In particular, if \mathcal{F} is a second locally free \mathcal{O}_X -module on X of rank s , then

$$c_i(\mathcal{E}) \cap c_j(\mathcal{F}) \cap \alpha = c_j(\mathcal{F}) \cap c_i(\mathcal{E}) \cap \alpha$$

as elements of $\mathrm{CH}_{k-i-j}(X)$ for all $\alpha \in \mathrm{CH}_k(X)$.

Proof. Let $X' \rightarrow X$ be a morphism of good algebraic spaces over B . Let $\alpha \in \mathrm{CH}_k(X')$. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 28.1 we have

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi' : P' \rightarrow X', \mathcal{O}_{P'}(1))$ associated to $(X' \rightarrow X)^* \mathcal{E}$. Applying $c \cap -$ and using Lemma 26.8 and the properties of bivariant classes we obtain

$$\sum_{i=0}^r (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap \pi^*(c \cap \alpha_{r-i}) = 0$$

in the Chow group of P' . Hence we see that $c \cap \alpha_j$ is equal to $c_j(\mathcal{E}) \cap (c \cap \alpha)$ by the uniqueness in Lemma 28.1. This proves the lemma. \square

Remark 28.5. In Situation 2.1 let X/B be good. Let \mathcal{E} be a finite locally free \mathcal{O}_X -module. If the rank of \mathcal{E} is not constant then we can still define the Chern classes of \mathcal{E} . Namely, in this case we can write

$$X = X_0 \amalg X_1 \amalg X_2 \amalg \dots$$

where $X_r \subset X$ is the open and closed subspace where the rank of \mathcal{E} is r . If $X' \rightarrow X$ is a morphism of good algebraic spaces over B , then we obtain by pullback a corresponding decomposition of X' and we find that

$$\mathrm{CH}_*(X') = \prod_{r \geq 0} \mathrm{CH}_*(X'_r)$$

by our definitions. Then we simply define $c_i(\mathcal{E})$ to be the bivariant class which preserves these direct product decompositions and acts by the already defined operations $c_i(\mathcal{E}|_{X_r}) \cap -$ on the factors. Observe that in this setting it may happen that $c_i(\mathcal{E})$ is nonzero for infinitely many i .

29. Polynomial relations among Chern classes

In Situation 2.1 let X/B be good. Let \mathcal{E}_i be a finite collection of finite locally free \mathcal{O}_X -modules. By Lemma 28.4 we see that the Chern classes

$$c_j(\mathcal{E}_i) \in A^*(X)$$

generate a commutative (and even central) \mathbf{Z} -subalgebra of the Chow cohomology $A^*(X)$. Thus we can say what it means for a polynomial in these Chern classes to be zero, or for two polynomials to be the same. As an example, saying that $c_1(\mathcal{E}_1)^5 + c_2(\mathcal{E}_2)c_3(\mathcal{E}_3) = 0$ means that the operations

$$\mathrm{CH}_k(Y) \longrightarrow \mathrm{CH}_{k-5}(Y), \quad \alpha \longmapsto c_1(\mathcal{E}_1)^5 \cap \alpha + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap \alpha$$

are zero for all morphisms $f : Y \rightarrow X$ of good algebraic spaces over B . By Lemma 26.9 this is equivalent to the requirement that given any morphism $f : Y \rightarrow X$ where Y is an integral algebraic space locally of finite type over X the cycle

$$c_1(\mathcal{E}_1)^5 \cap [Y] + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap [Y]$$

is zero in $\mathrm{CH}_{\dim(Y)-5}(Y)$.

A specific example is the relation

$$c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) = c_1(\mathcal{L}) + c_1(\mathcal{N})$$

proved in Lemma 18.2. More generally, here is what happens when we tensor an arbitrary locally free sheaf by an invertible sheaf.

Lemma 29.1. *In Situation 2.1 let X/B be good. Let \mathcal{E} be a finite locally free sheaf of rank r on X . Let \mathcal{L} be an invertible sheaf on X . Then we have*

$$(29.1.1) \quad c_i(\mathcal{E} \otimes \mathcal{L}) = \sum_{j=0}^i \binom{r-i+j}{j} c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j$$

in $A^*(X)$.

Proof. The proof is identical to the proof of Chow Homology, Lemma 39.1 replacing the lemmas used there by Lemmas 26.9 and 28.1. \square

30. Additivity of Chern classes

This section is the analogue of Chow Homology, Section 40.

Lemma 30.1. *In Situation 2.1 let X/B be good. Let \mathcal{E}, \mathcal{F} be finite locally free sheaves on X of ranks $r, r-1$ which fit into a short exact sequence*

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

Then we have

$$c_r(\mathcal{E}) = 0, \quad c_j(\mathcal{E}) = c_j(\mathcal{F}), \quad j = 0, \dots, r-1$$

in $A^(X)$.*

Proof. The proof is identical to the proof of Chow Homology, Lemma 40.1 replacing the lemmas used there by Lemmas 26.9, 24.1, 19.4, and 28.1. \square

Lemma 30.2. *In Situation 2.1 let X/B be good. Let \mathcal{E}, \mathcal{F} be finite locally free sheaves on X of ranks $r, r-1$ which fit into a short exact sequence*

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

where \mathcal{L} is an invertible sheaf. Then

$$c(\mathcal{E}) = c(\mathcal{L})c(\mathcal{F})$$

in $A^(X)$.*

Proof. The proof is identical to the proof of Chow Homology, Lemma 40.2 replacing the lemmas used there by Lemmas 30.1 and 29.1. \square

Lemma 30.3. *In Situation 2.1 let X/B be good. Suppose that \mathcal{E} sits in an exact sequence*

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow 0$$

of finite locally free sheaves \mathcal{E}_i of rank r_i . The total Chern classes satisfy

$$c(\mathcal{E}) = c(\mathcal{E}_1)c(\mathcal{E}_2)$$

in $A^(X)$.*

Proof. The proof is identical to the proof of Chow Homology, Lemma 40.3 replacing the lemmas used there by Lemmas 26.9, 30.2, and 28.1. \square

Lemma 30.4. *In Situation 2.1 let X/B be good. Let $\mathcal{L}_i, i = 1, \dots, r$ be invertible \mathcal{O}_X -modules. Let \mathcal{E} be a locally free rank \mathcal{O}_X -module endowed with a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Set $c_1(\mathcal{L}_i) = x_i$. Then

$$c(\mathcal{E}) = \prod_{i=1}^r (1 + x_i)$$

in $A^(X)$.*

Proof. Apply Lemma 30.2 and induction. \square

31. The splitting principle

This section is the analogue of Chow Homology, Section 40.

Lemma 31.1. *In Situation 2.1 let X/B be good. Let \mathcal{E}_i be a finite collection of locally free \mathcal{O}_X -modules of rank r_i . There exists a projective flat morphism $\pi : P \rightarrow X$ of relative dimension d such that*

- (1) *for any morphism $f : Y \rightarrow X$ of good algebraic spaces over B the map $\pi_Y^* : \mathrm{CH}_*(Y) \rightarrow \mathrm{CH}_{*+d}(Y \times_X P)$ is injective, and*
- (2) *each $\pi^* \mathcal{E}_i$ has a filtration whose successive quotients $\mathcal{L}_{i,1}, \dots, \mathcal{L}_{i,r_i}$ are invertible \mathcal{O}_P -modules.*

Proof. We prove this by induction on the integer $r = \sum r_i$. If $r = 0$ we can take $\pi = \mathrm{id}_X$. If $r_i = 1$ for all i , then we can also take $\pi = \mathrm{id}_X$. Assume that $r_{i_0} > 1$ for some i_0 . Let $(\pi : P \rightarrow X, \mathcal{O}_P(1))$ be the projective bundle associated to \mathcal{E}_{i_0} . The canonical map $\pi^* \mathcal{E}_{i_0} \rightarrow \mathcal{O}_P(1)$ is surjective and hence its kernel \mathcal{E}'_{i_0} is finite locally free of rank $r_{i_0} - 1$. Observe that π_Y^* is injective for any morphism $f : Y \rightarrow X$ of good algebraic spaces over B , see Lemma 27.2. Thus it suffices to prove the lemma for P and the locally free sheaves $\pi^* \mathcal{E}_i$. However, because we have the subbundle $\mathcal{E}_{i_0} \subset \pi^* \mathcal{E}_{i_0}$ with invertible quotient, it now suffices to prove the lemma for the collection $\{\mathcal{E}_i\}_{i \neq i_0} \cup \{\mathcal{E}'_{i_0}\}$. This decreases r by 1 and we win by induction hypothesis. \square

Rather than explaining what the splitting principle says, let us use it in the proof of some lemmas.

Lemma 31.2. *In Situation 2.1 let X/B be good. Let \mathcal{E} be a finite locally free \mathcal{O}_X -module with dual \mathcal{E}^\vee . Then*

$$c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$$

in $A^(X)$.*

Proof. Choose a morphism $\pi : P \rightarrow X$ as in Lemma 31.1. By the injectivity of π^* (after any base change) it suffices to prove the relation between the Chern classes of \mathcal{E} and \mathcal{E}^\vee after pulling back to P . Thus we may assume there exist invertible \mathcal{O}_X -modules \mathcal{L}_i , $i = 1, \dots, r$ and a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Then we obtain the dual filtration

$$0 = \mathcal{E}_r^\perp \subset \mathcal{E}_{r-1}^\perp \subset \mathcal{E}_{r-2}^\perp \subset \dots \subset \mathcal{E}_0^\perp = \mathcal{E}^\vee$$

such that $\mathcal{E}_{i-1}^\perp/\mathcal{E}_i^\perp \cong \mathcal{L}_i^{\otimes -1}$. Set $x_i = c_1(\mathcal{L}_i)$. Then $c_1(\mathcal{L}_i^{\otimes -1}) = -x_i$ by Lemma 18.2. By Lemma 30.4 we have

$$c(\mathcal{E}) = \prod_{i=1}^r (1 + x_i) \quad \text{and} \quad c(\mathcal{E}^\vee) = \prod_{i=1}^r (1 - x_i)$$

in $A^*(X)$. The result follows from a formal computation which we omit. \square

Lemma 31.3. *In Situation 2.1 let X/B be good. Let \mathcal{E} and \mathcal{F} be a finite locally free \mathcal{O}_X -modules of ranks r and s . Then we have*

$$c_1(\mathcal{E} \otimes \mathcal{F}) = r c_1(\mathcal{F}) + s c_1(\mathcal{E})$$

$$c_2(\mathcal{E} \otimes \mathcal{F}) = r^2 c_2(\mathcal{F}) + r s c_1(\mathcal{F}) c_1(\mathcal{E}) + s^2 c_2(\mathcal{E})$$

and so on (see proof).

Proof. Arguing exactly as in the proof of Lemma 31.2 we may assume we have invertible \mathcal{O}_X -modules \mathcal{L}_i , $i = 1, \dots, r$ \mathcal{N}_i , $i = 1, \dots, s$ filtrations

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_r = \mathcal{E} \quad \text{and} \quad 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_s = \mathcal{F}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$ and such that $\mathcal{F}_j/\mathcal{F}_{j-1} \cong \mathcal{N}_j$. Ordering pairs (i, j) lexicographically we obtain a filtration

$$0 \subset \dots \subset \mathcal{E}_i \otimes \mathcal{F}_j + \mathcal{E}_{i-1} \otimes \mathcal{F} \subset \dots \subset \mathcal{E} \otimes \mathcal{F}$$

with successive quotients

$$\mathcal{L}_1 \otimes \mathcal{N}_1, \mathcal{L}_1 \otimes \mathcal{N}_2, \dots, \mathcal{L}_1 \otimes \mathcal{N}_s, \mathcal{L}_2 \otimes \mathcal{N}_1, \dots, \mathcal{L}_r \otimes \mathcal{N}_s$$

By Lemma 30.4 we have

$$c(\mathcal{E}) = \prod (1 + x_i), \quad c(\mathcal{F}) = \prod (1 + y_j), \quad \text{and} \quad c(\mathcal{E} \otimes \mathcal{F}) = \prod (1 + x_i + y_j),$$

in $A^*(X)$. The result follows from a formal computation which we omit. \square

32. Degrees of zero cycles

This section is the analogue of Chow Homology, Section 41. We start with defining the degree of a zero cycle on a proper algebraic space over a field.

Definition 32.1. Let k be a field. Let $p : X \rightarrow \text{Spec}(k)$ be a proper morphism of algebraic spaces. The *degree of a zero cycle* on X is given by proper pushforward

$$p_* : \text{CH}_0(X) \longrightarrow \text{CH}_0(\text{Spec}(k)) \longrightarrow \mathbf{Z}$$

(Lemma 16.3) composed with the natural isomorphism $\text{CH}_0(\text{Spec}(k)) \rightarrow \mathbf{Z}$ which maps $[\text{Spec}(k)]$ to 1. Notation: $\deg(\alpha)$.

Let us spell this out further.

Lemma 32.2. Let k be a field. Let X be a proper algebraic space over k . Let $\alpha = \sum n_i [Z_i]$ be in $Z_0(X)$. Then

$$\deg(\alpha) = \sum n_i \deg(Z_i)$$

where $\deg(Z_i)$ is the degree of $Z_i \rightarrow \text{Spec}(k)$, i.e., $\deg(Z_i) = \dim_k \Gamma(Z_i, \mathcal{O}_{Z_i})$.

Proof. This is the definition of proper pushforward (Definition 8.1). \square

Lemma 32.3. Let k be a field. Let X be a proper algebraic space over k . Let $Z \subset X$ be a closed subspace of dimension d . Let $\mathcal{L}_1, \dots, \mathcal{L}_d$ be invertible \mathcal{O}_X -modules. Then

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \deg(c_1(\mathcal{L}_1) \cap \dots \cap c_1(\mathcal{L}_d) \cap [Z]_d)$$

where the left hand side is defined in *Spaces over Fields*, Definition 18.3.

Proof. Let $Z_i \subset Z$, $i = 1, \dots, t$ be the irreducible components of dimension d . Let m_i be the multiplicity of Z_i in Z . Then $[Z]_d = \sum m_i [Z_i]$ and $c_1(\mathcal{L}_1) \cap \dots \cap c_1(\mathcal{L}_d) \cap [Z]_d$ is the sum of the cycles $m_i c_1(\mathcal{L}_1) \cap \dots \cap c_1(\mathcal{L}_d) \cap [Z_i]$. Since we have a similar decomposition for $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ by *Spaces over Fields*, Lemma 18.2 it suffices to prove the lemma in case $Z = X$ is a proper integral algebraic space over k .

By Chow's lemma there exists a proper morphism $f : X' \rightarrow X$ which is an isomorphism over a dense open $U \subset X$ such that X' is a scheme. See *More on Morphisms*

of Spaces, Lemma 40.5. Then X' is a proper scheme over k . After replacing X' by the scheme theoretic closure of $f^{-1}(U)$ we may assume that X' is integral. Then

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot X') = (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X)$$

by Spaces over Fields, Lemma 18.7 and we have

$$f_*(c_1(f^*\mathcal{L}_1) \cap \dots \cap c_1(f^*\mathcal{L}_d) \cap [Y]) = c_1(\mathcal{L}_1) \cap \dots \cap c_1(\mathcal{L}_d) \cap [X]$$

by Lemma 19.4. Thus we may replace X by X' and assume that X is a proper scheme over k . This case was proven in Chow Homology, Lemma 41.4. \square

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