ALGEBRAIC SPACES OVER FIELDS

Contents

1. Introduction

This chapter is the analogue of the chapter on varieties in the setting of algebraic spaces. A reference for algebraic spaces is [Knu71].

2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Generically finite morphisms

This section continues the discussion in Decent Spaces, Section 21 and the analogue for morphisms of algebraic spaces of Varieties, Section 17.

Lemma 3.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y is locally Noetherian. Let $y \in |Y|$ be a point of codimension ≤ 1 on Y. Let $X^0 \subset |X|$ be the set of points of codimension 0 on X. Assume in addition one of the following conditions is satisfied*

(1) *for every $x \in X^0$ the transcendence degree of $x/f(x)$ is 0,*
(2) *for every $x \in X^0$ with $f(x) \rightsquigarrow y$ the transcendence degree of $x/f(x)$ is 0,*
(3) *f is quasi-finite at every $x \in X^0$,*
(4) *f is quasi-finite at a dense set of points of $|X|$,*
(5) *add more here.*

Then f is quasi-finite at every point of X lying over y.

Proof. We want to reduce the proof to the case of schemes. To do this we choose a commutative diagram

$$
\begin{array}{ccc}
U & \longrightarrow & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
V & \longrightarrow & Y
\end{array}
$$

where U, V are schemes and where the horizontal arrows are étale and surjective. Pick $v \in V$ mapping to y. Observe that V is locally Noetherian and that $\dim(\mathcal{O}_{V,v}) \leq 1$ (see Properties of Spaces, Definitions 10.2 and Remark 7.3). The fibre U_v of $U \to V$ over v surjects onto $f^{-1}(\{y\}) \subset |X|$. The inverse image of X^0 in U is exactly the set of generic points of irreducible components of U (Properties of Spaces, Lemma 11.1). If $\eta \in U$ is such a point with image $x \in X^0$, then the transcendence degree of $x/f(x)$ is the transcendence degree of $\kappa(\eta)$ over $\kappa(g(\eta))$ (Morphisms of Spaces, Definition 33.1). Observe that $U \to V$ is quasi-finite at $u \in U$ if and only if f is quasi-finite at the image of u in X.

Case (1). Here case (1) of Varieties, Lemma 17.1 applies and we conclude that $U \to V$ is quasi-finite at all points of U_v. Hence f is quasi-finite at every point lying over y.

Case (2). Let $u \in U$ be a generic point of an irreducible component whose image in V specializes to v. Then the image $x \in X^0$ of u has the property that $f(x) \rightsquigarrow y$. Hence we see that case (2) of Varieties, Lemma 17.1 applies and we conclude as before.

Case (3) follows from case (3) of Varieties, Lemma 17.1.

In case (4), since $|U| \to |X|$ is open, we see that the set of points where $U \to V$ is quasi-finite is dense as well. Hence case (4) of Varieties, Lemma 17.1 applies. $\square$

Lemma 3.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is proper and Y is locally Noetherian. Let $y \in Y$ be a point of codimension ≤ 1 in Y. Let $X^0 \subset |X|$ be the set of points of codimension 0 on X. Assume in addition one of the following conditions is satisfied*

(1) *for every $x \in X^0$ the transcendence degree of $x/f(x)$ is 0,*
(2) *for every $x \in X^0$ with $f(x) \rightsquigarrow y$ the transcendence degree of $x/f(x)$ is 0,*
(3) *f is quasi-finite at every $x \in X^0$,*
(4) *f is quasi-finite at a dense set of points of $|X|$,*
(5) *add more here.*

Then there exists an open subspace $Y' \subset Y$ containing y such that $Y' \times_Y X \to Y'$ is finite.

Proof. By Lemma 3.1 the morphism f is quasi-finite at every point lying over y. Let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point lying over y. Then $|X_{\overline{y}}|$ is a discrete space (Decent Spaces, Lemma 18.10). Since $X_{\overline{y}}$ is quasi-compact as f is proper we conclude that $|X_{\overline{y}}|$ is finite. Thus we can apply Cohomology of Spaces, Lemma 23.2 to conclude. $\square$

Lemma 3.3. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $f : Y \to X$ be a birational proper morphism of algebraic spaces with Y reduced. Let $U \subset X$ be the maximal open over which f is an isomorphism. Then U contains*

    (1) *every point of codimension 0 in X,*
    (2) *every $x \in |X|$ of codimension 1 on X such that the local ring of X at x is normal (Properties of Spaces, Remark 7.6), and*
    (3) *every $x \in |X|$ such that the fibre of $|Y| \to |X|$ over x is finite and such that the local ring of X at x is normal.*

Proof. Part (1) follows from Decent Spaces, Lemma 22.5 (and the fact that the Noetherian algebraic spaces X and Y are quasi-separated and hence decent). Part (2) follows from part (3) and Lemma 3.2 (and the fact that finite morphisms have finite fibres). Let $x \in |X|$ be as in (3). By Cohomology of Spaces, Lemma 23.2 (which applies by Decent Spaces, Lemma 18.10) we may assume f is finite. Choose an affine scheme X' and an étale morphism $X' \to X$ and a point $x' \in X$ mapping to x. It suffices to show there exists an open neighbourhood U' of $x' \in X'$ such that $Y \times_X X' \to X'$ is an isomorphism over U' (namely, then U contains the image of U' in X, see Spaces, Lemma 5.6). Then $Y \times_X X' \to X$ is a finite birational (Decent Spaces, Lemma 22.6) morphism. Since a finite morphism is affine we reduce to the case of a finite birational morphism of Noetherian affine schemes $Y \to X$ and $x \in X$ such that $\mathcal{O}_{X,x}$ is a normal domain. This is treated in Varieties, Lemma 17.3. $\square$

4. Integral algebraic spaces

We have not yet defined the notion of an integral algebraic space. The problem is that being integral is not an étale local property of schemes. We could use the property, that X is reduced and $|X|$ is irreducible, given in Properties, Lemma 3.4 to define integral algebraic spaces. In this case the algebraic space described in Spaces, Example 14.9 would be integral which does not seem right. To avoid this type of pathology we will in addition assume that X is a decent algebraic space, although perhaps a weaker alternative exists.

Definition 4.1. Let S be a scheme. We say an algebraic space X over S is *integral* if it is reduced, decent, and $|X|$ is irreducible.

In this case the irreducible topological space $|X|$ is sober (Decent Spaces, Proposition 12.4). Hence it has a unique generic point x. In fact, in Decent Spaces, Lemma 20.4 we characterized decent algebraic spaces with finitely many irreducible components. Applying that lemma we see that an algebraic space X is integral if it is reduced, has an irreducible dense open subscheme X' with generic point x' and the morphism $x' \to X$ is quasi-compact.

Lemma 4.2. *Let S be a scheme. Let X be an integral algebraic space over S. Let $\eta \in |X|$ be the generic point of X. There are canonical identifications*

$$R(X) = \mathcal{O}^h_{X,\eta} = \kappa(\eta)$$

where $R(X)$ is the ring of rational functions defined in Morphisms of Spaces, Definition 47.3, $\kappa(\eta)$ is the residue field defined in Decent Spaces, Definition 11.2, and $\mathcal{O}^h_{X,\eta}$ is the henselian local ring defined in Decent Spaces, Definition 11.5. In particular, these rings are fields.

Proof. Since X is a scheme in an open neighbourhood of η (see discussion above), this follows immediately from the corresponding result for schemes, see Morphisms, Lemma 49.5. We also use: the henselianization of a field is itself and that our definitions of these objects for algebraic spaces are compatible with those for schemes. Details omitted. $\square$

This leads to the following definition.

Definition 4.3. Let S be a scheme. Let X be an integral algebraic space over S. The *function field*, or the *field of rational functions* of X is the field $R(X)$ of Lemma 4.2.

We may occasionally indicate this field $k(X)$ instead of $R(X)$.

Lemma 4.4. *Let S be a scheme. Let X be an integral algebraic space over S. Then $\Gamma(X, \mathcal{O}_X)$ is a domain.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. If $f, g \in R$ are nonzero and $fg = 0$ then $X = V(f) \cup V(g)$ where $V(f)$ denotes the closed subspace of X cut out by f. Since X is irreducible, we see that either $V(f) = X$ or $V(g) = X$. Then either $f = 0$ or $g = 0$ by Properties of Spaces, Lemma 21.4. $\square$

Here is a lemma about normal integral algebraic spaces.

Lemma 4.5. *Let S be a scheme. Let X be a normal integral algebraic space over S. For every $x \in |X|$ there exists a normal integral affine scheme U and an étale morphism $U \to X$ such that x is in the image.*

Proof. Choose an affine scheme U and an étale morphism $U \to X$ such that x is in the image. Let u_i, $i \in I$ be the generic points of irreducible components of U. Then each u_i maps to the generic point of X (Decent Spaces, Lemma 20.1). By our definition of a decent space (Decent Spaces, Definition 6.1), we see that I is finite. Hence $U = \mathrm{Spec}(A)$ where A is a normal ring with finitely many minimal primes. Thus $A = \prod_{i \in I} A_i$ is a product of normal domains by Algebra, Lemma 37.16. Then $U = \coprod U_i$ with $U_i = \mathrm{Spec}(A_i)$ and x is in the image of $U_i \to X$ for some i. This proves the lemma. $\square$

Lemma 4.6. *Let S be a scheme. Let X be a normal integral algebraic space over S. Then $\Gamma(X, \mathcal{O}_X)$ is a normal domain.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. Then R is a domain by Lemma 4.4. Let $f = a/b$ be an element of the fraction field of R which is integral over R. For any $U \to X$ étale with U a scheme there is at most one $f_U \in \Gamma(U, \mathcal{O}_U)$ with $b|_U f_U = a|_U$. Namely, U is reduced and the generic points of U map to the generic point of X which implies that $b|_U$ is a nonzerodivisor. For every $x \in |X|$ we choose $U \to X$ as in Lemma

4.5. Then there is a unique $f_U \in \Gamma(U, \mathcal{O}_U)$ with $b|_U f_U = a|_U$ because $\Gamma(U, \mathcal{O}_U)$ is a normal domain by Properties, Lemma 7.9. By the uniqueness mentioned above these f_U glue and define a global section f of the structure sheaf, i.e., of R. $\square$

Lemma 4.7. *Let S be a scheme. Let X be a decent algebraic space over S. There are canonical bijections between the following sets:*
   (1) *the set of points of X, i.e., $|X|$,*
   (2) *the set of irreducible closed subsets of $|X|$,*
   (3) *the set of integral closed subspaces of X.*
The bijection from (1) to (2) sends x to $\overline{\{x\}}$. The bijection from (3) to (2) sends Z to $|Z|$.

Proof. Our map defines a bijection between (1) and (2) as $|X|$ is sober by Decent Spaces, Proposition 12.4. Given $T \subset |X|$ closed and irreducible, there is a unique reduced closed subspace $Z \subset X$ such that $|Z| = T$, namely, Z is the reduced induced subspace structure on T, see Properties of Spaces, Definition 12.5. This is an integral algebraic space because it is decent, reduced, and irreducible. $\square$

5. Morphisms between integral algebraic spaces

The following lemma characterizes dominant morphisms of finite degree between integral algebraic spaces.

Lemma 5.1. *Let S be a scheme. Let X, Y be integral algebraic spaces over S Let $x \in |X|$ and $y \in |Y|$ be the generic points. Let $f : X \to Y$ be locally of finite type. Assume f is dominant (Morphisms of Spaces, Definition 18.1). The following are equivalent:*
   (1) *the transcendence degree of x/y is 0,*
   (2) *the extension $\kappa(x)/\kappa(y)$ (see proof) is finite,*
   (3) *there exist nonempty affine opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is finite,*
   (4) *f is quasi-finite at x, and*
   (5) *x is the only point of $|X|$ mapping to y.*
If f is separated or if f is quasi-compact, then these are also equivalent to
   (6) *there exists a nonempty affine open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.*

Proof. By elementary topology, we see that $f(x) = y$ as f is dominant. Let $Y' \subset Y$ be the schematic locus of Y and let $X' \subset f^{-1}(Y')$ be the schematic locus of $f^{-1}(Y')$. By the discussion above, using Decent Spaces, Proposition 12.4 and Theorem 10.2, we see that $x \in |X'|$ and $y \in |Y'|$. Then $f|_{X'} : X' \to Y'$ is a morphism of integral schemes which is locally of finite type. Thus we see that (1), (2), (3) are equivalent by Morphisms, Lemma 51.7.

Condition (4) implies condition (1) by Morphisms of Spaces, Lemma 33.3 applied to $X \to Y \to Y$. On the other hand, condition (3) implies condition (4) as a finite morphism is quasi-finite and as $x \in U$ because x is the generic point. Thus (1) – (4) are equivalent.

Assume the equivalent conditions (1) – (4). Suppose that $x' \mapsto y$. Then $x \rightsquigarrow x'$ is a specialization in the fibre of $|X| \to |Y|$ over y. If $x' \neq x$, then f is not quasi-finite at x by Decent Spaces, Lemma 18.9. Hence $x = x'$ and (5) holds. Conversely, if

(5) holds, then (5) holds for the morphism of schemes $X' \to Y'$ (see above) and we can use Morphisms, Lemma 51.7 to see that (1) holds.

Observe that (6) implies the equivalent conditions (1) – (5) without any further assumptions on f. To finish the proof we have to show the equivalent conditions (1) – (5) imply (6). This follows from Decent Spaces, Lemma 21.4. $\qquad\square$

Definition 5.2. Let S be a scheme. Let X and Y be integral algebraic spaces over S. Let $f : X \to Y$ be locally of finite type and dominant. Assume any of the equivalent conditions (1) – (5) of Lemma 5.1. Let $x \in |X|$ and $y \in |Y|$ be the generic points. Then the positive integer

$$\deg(X/Y) = [\kappa(x) : \kappa(y)]$$

is called the *degree of X over Y*.

Lemma 5.3. *Let S be a scheme. Let X, Y, Z be integral algebraic spaces over S. Let $f : X \to Y$ and $g : Y \to Z$ be dominant morphisms locally of finite type. Assume any of the equivalent conditions (1) – (5) of Lemma 5.1 hold for f and g. Then*

$$\deg(X/Z) = \deg(X/Y) \deg(Y/Z).$$

Proof. This comes from the multiplicativity of degrees in towers of finite extensions of fields, see Fields, Lemma 7.7. $\qquad\square$

6. Weil divisors

This section is the analogue of Divisors, Section 26.

We will introduce Weil divisors and rational equivalence of Weil divisors for locally Noetherian integral algebraic spaces. Since we are not assuming our algebraic spaces are quasi-compact we have to be a little careful when defining Weil divisors. We have to allow infinite sums of prime divisors because a rational function may have infinitely many poles for example. In the quasi-compact case our Weil divisors are finite sums as usual. Here is a basic lemma we will often use to prove collections of closed subspaces are locally finite.

Lemma 6.1. *Let S be a scheme and let X be a locally Noetherian algebraic space over S. If $T \subset |X|$ is a closed subset, then the collection of irreducible components of T is locally finite.*

Proof. The topological space $|X|$ is locally Noetherian (Properties of Spaces, Lemma 24.2). A Noetherian topological space has a finite number of irreducible components and a subspace of a Noetherian space is Noetherian (Topology, Lemma 9.2). Thus the lemma follows from the definition of locally finite (Topology, Definition 28.4). $\qquad\square$

Let S be a scheme. Let X be a decent algebraic space over S. Let Z be an integral closed subspace of X and let $\xi \in |Z|$ be the generic point. Then the codimension of $|Z|$ in $|X|$ is equal to the dimension of the local ring of X at ξ by Decent Spaces, Lemma 20.2. Recall that we also indicate this by saying that ξ *is a point of codimension* 1 *on X*, see Properties of Spaces, Definition 10.2.

Definition 6.2. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S.

(1) A *prime divisor* is an integral closed subspace $Z \subset X$ of codimension 1, i.e., the generic point of $|Z|$ is a point of codimension 1 on X.
(2) A *Weil divisor* is a formal sum $D = \sum n_Z Z$ where the sum is over prime divisors of X and the collection $\{|Z| : n_Z \neq 0\}$ is locally finite in $|X|$ (Topology, Definition 28.4).

The group of all Weil divisors on X is denoted $\mathrm{Div}(X)$.

Our next task is to define the Weil divisor associated to a rational function. In order to do this we need to define the order of vanishing of a rational function on a locally Noetherian integral algebraic space X along a prime divisor Z. Let $\xi \in |Z|$ be the generic point. Here we run into the problem that the local ring $\mathcal{O}_{X,\xi}$ doesn't exist and the henselian local ring $\mathcal{O}_{X,\xi}^h$ may not be a domain, see Example 6.11. To get around this we use the following lemma.

Lemma 6.3. *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $Z \subset X$ be a prime divisor and let $\xi \in |Z|$ be the generic point. Then the henselian local ring $\mathcal{O}_{X,\xi}^h$ is a reduced 1-dimensional Noetherian local ring and there is a canonical injective map*

$$R(X) \longrightarrow Q(\mathcal{O}_{X,\xi}^h)$$

from the function field $R(X)$ of X into the total ring of fractions.

Proof. We will use the results of Decent Spaces, Section 11. Let $(U, u) \to (X, \xi)$ be an elementary étale neighbourhood. Observe that U is locally Noetherian and reduced. Thus $\mathcal{O}_{U,u}$ is a 1-dimensional (by our definition of prime divisors) reduced Noetherian ring. After replacing U by an affine open neighbourhood of u we may assume U is Noetherian and affine. After replacing U by a smaller open, we may assume every irreducible component of U passes through u. Since $U \to X$ is open and X irreducible, $U \to X$ is dominant. Hence we obtain a ring map $R(X) \to R(U)$ by composing rational maps, see Morphisms of Spaces, Section 47. Since $R(X)$ is a field, this map is injective. By our choice of U we see that $R(U)$ is the total quotient ring $Q(\mathcal{O}_{U,u})$, see Morphisms, Lemma 49.5 and Algebra, Lemma 25.4.

At this point we have proved all the statements in the lemma with $\mathcal{O}_{U,u}$ in stead of $\mathcal{O}_{X,\xi}^h$. However, $\mathcal{O}_{X,\xi}^h$ is the henselization of $\mathcal{O}_{U,u}$. Thus $\mathcal{O}_{X,\xi}^h$ is a 1-dimensional reduced Noetherian ring, see More on Algebra, Lemmas 45.4, 45.7, and 45.3. Since $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\xi}^h$ is faithfully flat by More on Algebra, Lemma 45.1 it sends nonzerodivisors to nonzerodivisors. Therefore we obtain a canonical map $Q(\mathcal{O}_{U,u}) \to Q(\mathcal{O}_{X,\xi}^h)$ and we obtain our map. We omit the verification that the map is independent of the choice of $(U, u) \to (X, x)$; a slightly better approach would be to first observe that $\mathrm{colim}\, Q(\mathcal{O}_{U,u}) = Q(\mathcal{O}_{X,\xi}^h)$. $\square$

Definition 6.4. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. For every prime divisor $Z \subset X$ we define the *order of vanishing of f along Z* as the integer

$$\mathrm{ord}_Z(f) = \mathrm{length}_{\mathcal{O}_{X,\xi}^h}(\mathcal{O}_{X,\xi}^h/a\mathcal{O}_{X,\xi}^h) - \mathrm{length}_{\mathcal{O}_{X,\xi}^h}(\mathcal{O}_{X,\xi}^h/b\mathcal{O}_{X,\xi}^h)$$

where $a, b \in \mathcal{O}_{X,\xi}^h$ are nonzerodivisors such that the image of f in $Q(\mathcal{O}_{X,\xi}^h)$ (Lemma 6.3) is equal to a/b. This is well defined by Algebra, Lemma 121.1.

If $\mathcal{O}_{X,\xi}^h$ happens to be a domain, then we obtain

$$\mathrm{ord}_Z(f) = \mathrm{ord}_{\mathcal{O}_{X,\xi}^h}(f)$$

where the right hand side is the notion of Algebra, Definition 121.2. Note that for $f, g \in R(X)^*$ we have

$$\mathrm{ord}_Z(fg) = \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g).$$

Of course it can happen that $\mathrm{ord}_Z(f) < 0$. In this case we say that f has a *pole* along Z and that $-\mathrm{ord}_Z(f) > 0$ is the *order of pole of f along Z*. It is important to note that the condition $\mathrm{ord}_Z(f) \geq 0$ is **not** equivalent to the condition $f \in \mathcal{O}_{X,\xi}^h$ unless the local ring $\mathcal{O}_{X,\xi}$ is a discrete valuation ring.

Lemma 6.5. *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. If the prime divisor $Z \subset X$ meets the schematic locus of X, then the order of vanishing $\mathrm{ord}_Z(f)$ of Definition 6.4 agrees with the order of vanishing of Divisors, Definition 26.3.*

Proof. After shrinking X we may assume X is an integral Noetherian scheme. If $\xi \in Z$ denotes the generic point, then we find that $\mathcal{O}_{X,\xi}^h$ is the henselization of $\mathcal{O}_{X,\xi}$ (Decent Spaces, Lemma 11.8). To prove the lemma it suffices and is necessary to show that

$$\mathrm{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{X,\xi}/a\mathcal{O}_{X,\xi}) = \mathrm{length}_{\mathcal{O}_{X,\xi}^h}(\mathcal{O}_{X,\xi}^h/a\mathcal{O}_{X,\xi}^h)$$

This follows immediately from Algebra, Lemma 52.13 (and the fact that $\mathcal{O}_{X,\xi} \to \mathcal{O}_{X,\xi}^h$ is a flat local ring homomorphism of local Noetherian rings). $\square$

Lemma 6.6. *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. Then the collections*

$$\{Z \subset X \mid Z \text{ a prime divisor with generic point } \xi \text{ and } f \text{ not in } \mathcal{O}_{X,\xi}\}$$

and

$$\{Z \subset X \mid Z \text{ a prime divisor and } \mathrm{ord}_Z(f) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subspace $U \subset X$ such that f corresponds to a section of $\Gamma(U, \mathcal{O}_X^*)$. Hence the prime divisors which can occur in the sets of the lemma all correspond to irreducible components of $|X| \setminus |U|$. Hence Lemma 6.1 gives the desired result. $\square$

This lemma allows us to make the following definition.

Definition 6.7. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f \in R(X)^*$. The *principal Weil divisor associated to f* is the Weil divisor

$$\mathrm{div}(f) = \mathrm{div}_X(f) = \sum \mathrm{ord}_Z(f)[Z]$$

where the sum is over prime divisors and $\mathrm{ord}_Z(f)$ is as in Definition 6.4. This makes sense by Lemma 6.6.

Lemma 6.8. *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $f, g \in R(X)^*$. Then*

$$div_X(fg) = div_X(f) + div_X(g)$$

as Weil divisors on X.

Proof. This is clear from the additivity of the ord functions. $\square$

We see from the lemma above that the collection of principal Weil divisors form a subgroup of the group of all Weil divisors. This leads to the following definition.

Definition 6.9. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. The *Weil divisor class group* of X is the quotient of the group of Weil divisors by the subgroup of principal Weil divisors. Notation: $\mathrm{Cl}(X)$.

By construction we obtain an exact complex

(6.9.1) $$R(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \to \mathrm{Cl}(X) \to 0$$

which we can think of as a presentation of $\mathrm{Cl}(X)$. Our next task is to relate the Weil divisor class group to the Picard group.

Example 6.10. This is a continuation of Morphisms of Spaces, Example 53.3. Consider the algebraic space $X = \mathbf{A}^1_k/\{t \sim -t \mid t \neq 0\}$. This is a smooth algebraic space over the field k. There is a universal homeomorphism

$$X \longrightarrow \mathbf{A}^1_k = \mathrm{Spec}(k[t])$$

which is an isomorphism over $\mathbf{A}^1_k \setminus \{0\}$. We conclude that X is Noetherian and integral. Since $\dim(X) = 1$, we see that the prime divisors of X are the closed points of X. Consider the unique closed point $x \in |X|$ lying over $0 \in \mathbf{A}^1_k$. Since $X \setminus \{x\}$ maps isomorphically to $\mathbf{A}^1 \setminus \{0\}$ we see that the classes in $\mathrm{Cl}(X)$ of closed points different from x are zero. However, the divisor of t on X is $2[x]$. We conclude that $\mathrm{Cl}(X) = \mathbf{Z}/2\mathbf{Z}$.

Example 6.11. Let k be a field. Let

$$U = \mathrm{Spec}(k[x,y]/(xy))$$

be the union of the coordinate axes in $\mathbf{A}^2_k$. Denote $\Delta : U \to U \times_k U$ the diagonal and $\Delta' : U \to U \times_k U$ the map $u \mapsto (u, \sigma(u))$ where $\sigma : U \to U$, $(x,y) \mapsto (y,x)$ is the automorphism flipping the coordinate axes. Set

$$R = \Delta(U) \amalg \Delta'(U \setminus \{0_U\})$$

where $0_U \in U$ is the origin. It is easy to see that R is an étale equivalence relation on U. The quotient $X = U/R$ is an algebraic space. The morphism $U \to \mathbf{A}^1_k$, $(x,y) \mapsto x + y$ is R-invariant and hence defines a morphism

$$X \longrightarrow \mathbf{A}^1_k$$

This morphism is a universal homeomorphism and an isomorphism over $\mathbf{A}^1_k \setminus \{0\}$. It follows that X is integral and Noetherian. Exactly as in Example 6.10 the reader shows that $\mathrm{Cl}(X) = \mathbf{Z}/2\mathbf{Z}$ with generator corresponding to the unique closed point $x \in |X|$ mapping to $0 \in \mathbf{A}^1_k$. However, in this case the henselian local ring of X at x isn't a domain, as it is the henselization of $\mathcal{O}_{U,0_U}$.

7. The Weil divisor class associated to an invertible module

In this section we go through exactly the same progression as in Section 6 to define a canonical map $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ on a locally Noetherian integral algebraic space.

Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. By Divisors on Spaces, Lemma 10.11 there exists a regular meromorphic section $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$. In fact, by Divisors on Spaces,

Lemma 10.8 this is the same thing as a nonzero element in $\mathcal{L}_\eta$ where $\eta \in |X|$ is the generic point. The same lemma tells us that if $\mathcal{L} = \mathcal{O}_X$, then s is the same thing as a nonzero rational function on X (so what we will do below matches the construction in Section 6).

Let $Z \subset X$ be a prime divisor and let $\xi \in |Z|$ be the generic point. We are going to define the order of vanishing of s along Z. Consider the canonical morphism

$$c_\xi : \mathrm{Spec}(\mathcal{O}_{X,\xi}^h) \longrightarrow X$$

whose source is the spectrum of the henselian local ring of X as ξ (Decent Spaces, Definition 11.7). The pullback $\mathcal{L}_\xi = c_\xi^* \mathcal{L}$ is an invertible module and hence trivial; choose a generator s_ξ of $\mathcal{L}_\xi$. Since c_ξ is flat, pullbacks of meromorphic functions and (regular) sections are defined for c_ξ, see Divisors on Spaces, Definition 10.6 and Lemmas 10.7 and 10.10. Thus we get

$$c_\xi^*(s) = f s_\xi$$

for some nonzerodivisor $f \in Q(\mathcal{O}_{X,\xi}^h)$. Here we are using Divisors, Lemma 24.2 to identify the space of meromorphic sections of $\mathcal{L}_\xi \cong \mathcal{O}_{\mathrm{Spec}(\mathcal{O}_{X,\xi}^h)}$ in terms of the total ring of fractions of $\mathcal{O}_{X,\xi}^h$. Let us agree to denote this element

$$s/s_\xi = f \in Q(\mathcal{O}_{X,\xi}^h)$$

Observe that $f = s/s_\xi$ is replaced by uf where $u \in \mathcal{O}_{X,\xi}^h$ is a unit if we change our choice of s_ξ.

Definition 7.1. Let S be a scheme. Let X be a locally Noetherian integral algebraic algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a regular meromorphic section of $\mathcal{L}$. For every prime divisor $Z \subset X$ with generic point $\xi \in |Z|$ we define the *order of vanishing of s along Z* as the integer

$$\mathrm{ord}_{Z,\mathcal{L}}(s) = \mathrm{length}_{\mathcal{O}_{X,\xi}^h}(\mathcal{O}_{X,\xi}^h/a\mathcal{O}_{X,\xi}^h) - \mathrm{length}_{\mathcal{O}_{X,\xi}^h}(\mathcal{O}_{X,\xi}^h/b\mathcal{O}_{X,\xi}^h)$$

where $a, b \in \mathcal{O}_{X,\xi}^h$ are nonzerodivisors such that the element s/s_ξ of $Q(\mathcal{O}_{X,\xi}^h)$ constructed above is equal to a/b. This is well defined by the above and Algebra, Lemma 121.1.

As explained above, a regular meromorphic section s of $\mathcal{O}_X$ can be written $s = f \cdot 1$ where f is a nonzero rational function on X and we have $\mathrm{ord}_Z(f) = \mathrm{ord}_{Z,\mathcal{O}_X}(s)$. As in the case of principal divisors we have the following lemma.

Lemma 7.2. *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \mathcal{K}_X(\mathcal{L})$ be a regular (i.e., nonzero) meromorphic section of $\mathcal{L}$. Then the sets*

$$\{Z \subset X \mid Z \text{ a prime divisor with generic point } \xi \text{ and } s \text{ not in } \mathcal{L}_\xi\}$$

and

$$\{Z \subset X \mid Z \text{ is a prime divisor and } \mathrm{ord}_{Z,\mathcal{L}}(s) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subspace $U \subset X$ such that s corresponds to a section of $\Gamma(U, \mathcal{L})$ which generates $\mathcal{L}$ over U. Hence the prime divisors which can occur in the sets of the lemma all correspond to irreducible components of $|X| \setminus |U|$. Hence Lemma 6.1. gives the desired result. $\square$

Lemma 7.3. *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s, s' \in \mathcal{K}_X(\mathcal{L})$ be nonzero meromorphic sections of $\mathcal{L}$. Then $f = s/s'$ is an element of $R(X)^*$ and we have*

$$\sum ord_{Z,\mathcal{L}}(s)[Z] = \sum ord_{Z,\mathcal{L}}(s')[Z] + div(f)$$

as Weil divisors.

Proof. This is clear from the definitions. Note that Lemma 7.2 guarantees that the sums are indeed Weil divisors. □

Definition 7.4. Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the *Weil divisor associated to s* as

$$\mathrm{div}_{\mathcal{L}}(s) = \sum \mathrm{ord}_{Z,\mathcal{L}}(s)[Z] \in \mathrm{Div}(X)$$

where the sum is over prime divisors. This is well defined by Lemma 7.2.
(2) We define *Weil divisor class associated to $\mathcal{L}$* as the image of $\mathrm{div}_{\mathcal{L}}(s)$ in $\mathrm{Cl}(X)$ where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Lemma 7.3.

As expected this construction is additive in the invertible module.

Lemma 7.5. *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. Let $\mathcal{L}$, $\mathcal{N}$ be invertible $\mathcal{O}_X$-modules. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$ and*

$$div_{\mathcal{L}\otimes\mathcal{N}}(st) = div_{\mathcal{L}}(s) + div_{\mathcal{N}}(t)$$

in $Div(X)$. In particular, the Weil divisor class of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sum of the Weil divisor classes of $\mathcal{L}$ and $\mathcal{N}$.

Proof. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$. Let $Z \subset X$ be a prime divisor. Let $\xi \in |Z|$ be its generic point. Choose generators $s_\xi \in \mathcal{L}_\xi$, and $t_\xi \in \mathcal{N}_\xi$ with notation as described earlier in this section. Then $s_\xi \otimes t_\xi$ is a generator for $(\mathcal{L} \otimes \mathcal{N})_\xi$. So $st/(s_\xi t_\xi) = (s/s_\xi)(t/t_\xi)$ in $Q(\mathcal{O}_{X,\xi}^h)$. Applying the additivity of Algebra, Lemma 121.1 we conclude that

$$\mathrm{div}_{\mathcal{L}\otimes\mathcal{N},Z}(st) = \mathrm{div}_{\mathcal{L},Z}(s) + \mathrm{div}_{\mathcal{N},Z}(t)$$

Some details omitted. □

Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. By the constructions and lemmas above we obtain a homomorphism of abelian groups

(7.5.1) $$\mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X)$$

which assigns to an invertible module its Weil divisor class.

Lemma 7.6. *Let S be a scheme. Let X be a locally Noetherian integral algebraic space over S. If X is normal, then the map (7.5.1) $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ is injective.*

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose associated Weil divisor class is trivial. Let s be a regular meromorphic section of $\mathcal{L}$. The assumption means that $\mathrm{div}_{\mathcal{L}}(s) = \mathrm{div}(f)$ for some $f \in R(X)^*$. Then we see that $t = f^{-1}s$ is a regular meromorphic section of $\mathcal{L}$ with $\mathrm{div}_{\mathcal{L}}(t) = 0$, see Lemma 7.3. We claim that t defines a trivialization of $\mathcal{L}$. The claim finishes the proof of the lemma. Our proof of the claim is a bit awkward as we don't yet have a lot of theory at our dispposal; we suggest the reader skip the proof.

We may check our claim étale locally. Let $U \in X_{\acute{e}tale}$ be affine such that $\mathcal{L}|_U$ is trivial. Say $s_U \in \Gamma(U, \mathcal{L}|_U)$ is a trivialization. By Properties, Lemma 7.5 we may also assume U is integral. Write $U = \mathrm{Spec}(A)$ as the spectrum of a normal Noetherian domain A with fraction field K. We may write $t|_U = fs_U$ for some element f of K, see Divisors on Spaces, Lemma 10.4 for example. Let $\mathfrak{p} \subset A$ be a height one prime corresponding to a codimension 1 point $u \in U$ which maps to a codimension 1 point $\xi \in |X|$. Choose a trivialization s_ξ of $\mathcal{L}_\xi$ as in the beginning of this section. Choose a geometric point $\overline{u}$ of U lying over u. Then

$$(\mathcal{O}_{X,\xi}^h)^{sh} = \mathcal{O}_{X,\overline{u}} = \mathcal{O}_{U,u}^{sh} = (A_{\mathfrak{p}})^{sh}$$

see Decent Spaces, Lemmas 11.9 and Properties of Spaces, Lemma 22.1. The normality of X shows that all of these are discrete valuation rings. The trivializations s_U and s_ξ differ by a unit as sections of $\mathcal{L}$ pulled back to $\mathrm{Spec}(\mathcal{O}_{X,\overline{u}})$. Write $t = f_\xi s_\xi$ with $f_\xi \in Q(\mathcal{O}_{X,\xi}^h)$. We conclude that f_ξ and f differ by a unit in $Q(\mathcal{O}_{X,\overline{u}})$. If $Z \subset X$ denotes the prime divisor corresponding to ξ (Lemma 4.7), then $0 = \mathrm{ord}_{Z,\mathcal{L}}(t) = \mathrm{ord}_{\mathcal{O}_{X,\xi}^h}(f_\xi)$ and since $\mathcal{O}_{X,\xi}^h$ is a discrete valuation ring we see that f_ξ is a unit. Thus f is a unit in $\mathcal{O}_{X,\overline{u}}$ and hence in particular $f \in A_{\mathfrak{p}}$. This implies $f \in A$ by Algebra, Lemma 157.6. We conclude that $t \in \Gamma(X, \mathcal{L})$. Repeating the argument with t^{-1} viewed as a meromorphic section of $\mathcal{L}^{\otimes -1}$ finishes the proof. $\square$

8. Modifications and alterations

Using our notion of an integral algebraic space we can define a modification as follows.

Definition 8.1. Let S be a scheme. Let X be an integral algebraic space over S. A *modification of X* is a birational proper morphism $f : X' \to X$ of algebraic spaces over S with X' integral.

For birational morphisms of algebraic spaces, see Decent Spaces, Definition 22.1.

Lemma 8.2. *Let $f : X' \to X$ be a modification as in Definition 8.1. There exists a nonempty open $U \subset X$ such that $f^{-1}(U) \to U$ is an isomorphism.*

Proof. By Lemma 5.1 there exists a nonempty $U \subset X$ such that $f^{-1}(U) \to U$ is finite. By generic flatness (Morphisms of Spaces, Proposition 32.1) we may assume $f^{-1}(U) \to U$ is flat and of finite presentation. So $f^{-1}(U) \to U$ is finite locally free (Morphisms of Spaces, Lemma 46.6). Since f is birational, the degree of X' over X is 1. Hence $f^{-1}(U) \to U$ is finite locally free of degree 1, in other words it is an isomorphism. $\square$

Definition 8.3. Let S be a scheme. Let X be an integral algebraic space over S. An *alteration of X* is a proper dominant morphism $f : Y \to X$ of algebraic spaces

over S with Y integral such that $f^{-1}(U) \to U$ is finite for some nonempty open $U \subset X$.

If $f : Y \to X$ is a dominant and proper morphism between integral algebraic spaces, then it is an alteration as soon as the induced extension of residue fields in generic points is finite. Here is the precise statement.

Lemma 8.4. *Let S be a scheme. Let $f : X \to Y$ be a proper dominant morphism of integral algebraic spaces over S. Then f is an alteration if and only if any of the equivalent conditions (1) – (6) of Lemma 5.1 hold.*

Proof. Immediate consequence of the lemma referenced in the statement. $\square$

Lemma 8.5. *Let S be a scheme. Let $f : X \to Y$ be a proper surjective morphism of algebraic spaces over S. Assume Y is integral. Then there exists an integral closed subspace $X' \subset X$ such that $f' = f|_{X'} : X' \to Y$ is an alteration.*

Proof. Let $V \subset Y$ be a nonempty open affine (Decent Spaces, Theorem 10.2). Let $\eta \in V$ be the generic point. Then X_η is a nonempty proper algebraic space over η. Choose a closed point $x \in |X_\eta|$ (exists because $|X_\eta|$ is a quasi-compact, sober topological space, see Decent Spaces, Proposition 12.4 and Topology, Lemma 12.8.) Let X' be the reduced induced closed subspace structure on $\overline{\{x\}} \subset |X|$ (Properties of Spaces, Definition 12.5. Then $f' : X' \to Y$ is surjective as the image contains η. Also f' is proper as a composition of a closed immersion and a proper morphism. Finally, the fibre X'_η has a single point; to see this use Decent Spaces, Lemma 18.6 for both $X \to Y$ and $X' \to Y$ and the point η. Since Y is decent and $X' \to Y$ is separated we see that X' is decent (Decent Spaces, Lemmas 17.2 and 17.5). Thus f' is an alteration by Lemma 8.4. $\square$

9. Schematic locus

We have already proven a number of results on the schematic locus of an algebraic space. Here is a list of references:

(1) Properties of Spaces, Sections 13 and 14,
(2) Decent Spaces, Section 10,
(3) Properties of Spaces, Lemma 15.3 $\Leftarrow$ Decent Spaces, Lemma 12.8 $\Leftarrow$ Decent Spaces, Lemma 14.2,
(4) Limits of Spaces, Section 15, and
(5) Limits of Spaces, Section 17.

There are some cases where certain types of morphisms of algebraic spaces are automatically representable, for example separated, locally quasi-finite morphisms (Morphisms of Spaces, Lemma 51.1), and flat monomorphisms (More on Morphisms of Spaces, Lemma 4.1). In Section 10 we will study what happens with the schematic locus under extension of base field.

Lemma 9.1. *Let S be a scheme. Let X be an algebraic space over S. Assume X satisfies at least one of the following conditions*

(1) *X is quasi-separated and $\dim(X) = 0$,*
(2) *X is locally of finite type over a field k and $\dim(X) = 0$,*
(3) *X is Noetherian and $\dim(X) = 0$, or*
(4) *add more here.*

Then X is a separated scheme and any quasi-compact open of X is affine.

Proof. If we prove that any quasi-compact open of X is affine, then X is a separated scheme. Thus we may assume X is quasi-compact and we aim to show that X is affine. Cases (2) and (3) follow immediately from case (1) but we will give a separate proofs of (2) and (3) as these proofs use significantly less theory.

Proof of (3). Let U be an affine scheme and let $U \to X$ be an étale morphism. Set $R = U \times_X U$. The two projection morphisms $s, t : R \to U$ are étale morphisms of schemes. By Properties of Spaces, Definition 9.2 we see that $\dim(U) = 0$ and $\dim(R) = 0$. Since R is a locally Noetherian scheme of dimension 0, we see that R is a disjoint union of spectra of Artinian local rings (Properties, Lemma 10.5). Since we assumed that X is Noetherian (so quasi-separated) we conclude that R is quasi-compact. Hence R is an affine scheme (use Schemes, Lemma 6.8). The étale morphisms $s, t : R \to U$ induce finite residue field extensions. Hence s and t are finite by Algebra, Lemma 54.4 (small detail omitted). Thus Groupoids, Proposition 23.9 shows that $X = U/R$ is an affine scheme.

Proof of (2) – almost identical to the proof of (3). Let U be an affine scheme and let $U \to X$ be a surjective étale morphism. Set $R = U \times_X U$. The two projection morphisms $s, t : R \to U$ are étale morphisms of schemes. By Properties of Spaces, Definition 9.2 we see that $\dim(U) = 0$ and similarly $\dim(R) = 0$. On the other hand, the morphism $U \to \mathrm{Spec}(k)$ is locally of finite type as the composition of the étale morphism $U \to X$ and $X \to \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemmas 23.2 and 39.9. Similarly, $R \to \mathrm{Spec}(k)$ is locally of finite type. Hence by Varieties, Lemma 20.2 we see that U and R are disjoint unions of spectra of local Artinian k-algebras finite over k. The same thing is therefore true of $U \times_{\mathrm{Spec}(k)} U$. As

$$R = U \times_X U \longrightarrow U \times_{\mathrm{Spec}(k)} U$$

is a monomorphism, we see that R is a finite(!) union of spectra of finite k-algebras. It follows that R is affine, see Schemes, Lemma 6.8. Applying Varieties, Lemma 20.2 once more we see that R is finite over k. Hence s, t are finite, see Morphisms, Lemma 44.14. Thus Groupoids, Proposition 23.9 shows that $X = U/R$ is an affine scheme.

Cohomological proof of (1). By Cohomology of Spaces, Lemma 10.1 we have vanishing of higher cohomology groups for all quasi-coherent sheaves $\mathcal{F}$ on X. Hence X is affine (in particular a scheme) by Cohomology of Spaces, Proposition 16.7.

Geometric proof of (1). Choose a stratification

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and étale morphisms $f_p : V_p \to U_p$ as in Decent Spaces, Lemma 8.6 (we will use all their properties below). Then $\dim(V_p) = 0$ by our definition of dimension of algebraic spaces. Thus Properties, Lemma 10.6 applies to each V_p. Then $f_p^{-1}(U_{p+1}) \subset V_p$ is quasi-compact open and hence is affine as well as closed. It follows that $|T_p| \subset |U_p|$ (see locus citatus) is open as well as closed. Hence X is a disjoint union of open and closed subspaces whose reduced structures are schemes. It follows that X is a scheme (Limits of Spaces, Lemma 15.3). Then the proof is finished by the case of schemes that we already referenced above. $\square$

The following lemma tells us that a quasi-separated algebraic space is a scheme away from codimension 1.

Lemma 9.2. *Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let $x \in |X|$. The following are equivalent*

(1) *x is a point of codimension 0 on X,*
(2) *the local ring of X at x has dimension 0, and*
(3) *x is a generic point of an irreducible component of $|X|$.*

If true, then there exists an open subspace of X containing x which is a scheme.

Proof. The equivalence of (1), (2), and (3) follows from Decent Spaces, Lemma 20.1 and the fact that a quasi-separated algebraic space is decent (Decent Spaces, Section 6). However in the next paragraph we will give a more elementary proof of the equivalence.

Note that (1) and (2) are equivalent by definition (Properties of Spaces, Definition 10.2). To prove the equivalence of (1) and (3) we may assume X is quasi-compact. Choose

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and $f_i : V_i \to U_i$ as in Decent Spaces, Lemma 8.6. Say $x \in U_i$, $x \notin U_{i+1}$. Then $x = f_i(y)$ for a unique $y \in V_i$. If (1) holds, then y is a generic point of an irreducible component of V_i (Properties of Spaces, Lemma 11.1). Since $f_i^{-1}(U_{i+1})$ is a quasi-compact open of V_i not containing y, there is an open neighbourhood $W \subset V_i$ of y disjoint from $f_i^{-1}(V_i)$ (see Properties, Lemma 2.2 or more simply Algebra, Lemma 26.4). Then $f_i|_W : W \to X$ is an isomorphism onto its image and hence $x = f_i(y)$ is a generic point of $|X|$. Conversely, assume (3) holds. Then f_i maps $\overline{\{y\}}$ onto the irreducible component $\overline{\{x\}}$ of $|U_i|$. Since $|f_i|$ is bijective over $\overline{\{x\}}$, it follows that $\overline{\{y\}}$ is an irreducible component of U_i. Thus x is a point of codimension 0.

The final statement of the lemma is Properties of Spaces, Proposition 13.3. $\square$

The following lemma says that a separated locally Noetherian algebraic space is a scheme in codimension 1, i.e., away from codimension 2.

Lemma 9.3. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. If X is separated, locally Noetherian, and the dimension of the local ring of X at x is ≤ 1 (Properties of Spaces, Definition 10.2), then there exists an open subspace of X containing x which is a scheme.*

Proof. (Please see the remark below for a different approach avoiding the material on finite groupoids.) We can replace X by an quasi-compact neighbourhood of x, hence we may assume X is quasi-compact, separated, and Noetherian. There exists a scheme U and a finite surjective morphism $U \to X$, see Limits of Spaces, Proposition 16.1. Let $R = U \times_X U$. Then $j : R \to U \times_S U$ is an equivalence relation and we obtain a groupoid scheme (U, R, s, t, c) over S with s, t finite and U Noetherian and separated. Let $\{u_1, \ldots, u_n\} \subset U$ be the set of points mapping to x. Then $\dim(\mathcal{O}_{U,u_i}) \leq 1$ by Decent Spaces, Lemma 12.6.

By More on Groupoids, Lemma 14.10 there exists an R-invariant affine open $W \subset U$ containing the orbit $\{u_1, \ldots, u_n\}$. Since $U \to X$ is finite surjective the continuous map $|U| \to |X|$ is closed surjective, hence submersive by Topology, Lemma 6.5. Thus $f(W)$ is open and there is an open subspace $X' \subset X$ with $f : W \to X'$ a surjective finite morphism. Then X' is an affine scheme by Cohomology of Spaces, Lemma 17.3 and the proof is finished. $\square$

Remark 9.4. Here is a sketch of a proof of Lemma 9.3 which avoids using More on Groupoids, Lemma 14.10.

Step 1. We may assume X is a reduced Noetherian separated algebraic space (for example by Cohomology of Spaces, Lemma 17.3 or by Limits of Spaces, Lemma 15.3) and we may choose a finite surjective morphism $Y \to X$ where Y is a Noetherian scheme (by Limits of Spaces, Proposition 16.1).

Step 2. After replacing X by an open neighbourhood of x, there exists a birational finite morphism $X' \to X$ and a closed subscheme $Y' \subset X' \times_X Y$ such that $Y' \to X'$ is surjective finite locally free. Namely, because X is reduced there is a dense open subspace $U \subset X$ over which Y is flat (Morphisms of Spaces, Proposition 32.1). Then we can choose a U-admissible blowup $b : \tilde{X} \to X$ such that the strict transform $\tilde{Y}$ of Y is flat over $\tilde{X}$, see More on Morphisms of Spaces, Lemma 39.1. (An alternative is to use Hilbert schemes if one wants to avoid using the result on blowups). Then we let $X' \subset \tilde{X}$ be the scheme theoretic closure of $b^{-1}(U)$ and $Y' = X' \times_{\tilde{X}} \tilde{Y}$. Since x is a codimension 1 point, we see that $X' \to X$ is finite over a neighbourhood of x (Lemma 3.2).

Step 3. After shrinking X to a smaller neighbourhood of x we get that X' is a scheme. This holds because Y' is a scheme and $Y' \to X'$ being finite locally free and because every finite set of codimension 1 points of Y' is contained in an affine open. Use Properties of Spaces, Proposition 14.1 and Varieties, Proposition 42.7.

Step 4. There exists an affine open $W' \subset X'$ containing all points lying over x which is the inverse image of an open subspace of X. To prove this let $Z \subset X$ be the closure of the set of points where $X' \to X$ is not an isomorphism. We may assume $x \in Z$ otherwise we are already done. Then x is a generic point of an irreducible component of Z and after shrinking X we may assume Z is an affine scheme (Lemma 9.2). Then the inverse image $Z' \subset X'$ is an affine scheme as well. Say $x_1, \ldots, x_n \in Z'$ are the points mapping to x. Then we can find an affine open W' in X' whose intersection with Z' is the inverse image of a principal open of Z containing x. Namely, we first pick an affine open $W' \subset X'$ containing $x_1, \ldots, x_n$ using Varieties, Proposition 42.7. Then we pick a principal open $D(f) \subset Z$ containing x whose inverse image $D(f|_{Z'})$ is contained in $W' \cap Z'$. Then we pick $f' \in \Gamma(W', \mathcal{O}_{W'})$ restricting to $f|_{Z'}$ and we replace W' by $D(f') \subset W'$. Since $X' \to X$ is an isomorphism away from $Z' \to Z$ the choice of W' guarantees that the image $W \subset X$ of W' is open with inverse image W' in X'.

Step 5. Then $W' \to W$ is a finite surjective morphism and W is a scheme by Cohomology of Spaces, Lemma 17.3 and the proof is complete.

10. Schematic locus and field extension

It can happen that a nonrepresentable algebraic space over a field k becomes representable (i.e., a scheme) after base change to an extension of k. See Spaces, Example 14.2. In this section we address this issue.

Lemma 10.1. *Let k be a field. Let X be an algebraic space over k. If there exists a purely inseparable field extension k'/k such that $X_{k'}$ is a scheme, then X is a scheme.*

Proof. The morphism $X_{k'} \to X$ is integral, surjective, and universally injective. Hence this lemma follows from Limits of Spaces, Lemma 15.4. $\qquad\qquad\square$

Lemma 10.2. *Let k be a field with algebraic closure $\overline{k}$. Let X be a quasi-separated algebraic space over k.*

(1) *If there exists a field extension K/k such that X_K is a scheme, then $X_{\overline{k}}$ is a scheme.*

(2) *If X is quasi-compact and there exists a field extension K/k such that X_K is a scheme, then $X_{k'}$ is a scheme for some finite separable extension k' of k.*

Proof. Since every algebraic space is the union of its quasi-compact open subspaces, we see that the first part of the lemma follows from the second part (some details omitted). Thus we assume X is quasi-compact and we assume given an extension K/k with X_K representable. Write $K = \bigcup A$ as the colimit of finitely generated k-subalgebras A. By Limits of Spaces, Lemma 5.11 we see that X_A is a scheme for some A. Choose a maximal ideal $\mathfrak{m} \subset A$. By the Hilbert Nullstellensatz (Algebra, Theorem 34.1) the residue field $k' = A/\mathfrak{m}$ is a finite extension of k. Thus we see that $X_{k'}$ is a scheme. If $k' \supset k$ is not separable, let $k'/k''/k$ be the subextension found in Fields, Lemma 14.6. Since k'/k'' is purely inseparable, by Lemma 10.1 the algebraic space $X_{k''}$ is a scheme. Since $k''|k$ is separable the proof is complete. $\square$

Lemma 10.3. *Let k'/k be a finite Galois extension with Galois group G. Let X be an algebraic space over k. Then G acts freely on the algebraic space $X_{k'}$ and $X = X_{k'}/G$ in the sense of Properties of Spaces, Lemma 34.1.*

Proof. Omitted. Hints: First show that $\operatorname{Spec}(k) = \operatorname{Spec}(k')/G$. Then use compatibility of taking quotients with base change. $\square$

Lemma 10.4. *Let S be a scheme. Let X be an algebraic space over S and let G be a finite group acting freely on X. Set $Y = X/G$ as in Properties of Spaces, Lemma 34.1. For $y \in |Y|$ the following are equivalent*

(1) *y is in the schematic locus of Y, and*

(2) *there exists an affine open $U \subset X$ containing the preimage of y.*

Proof. It follows from the construction of $Y = X/G$ in Properties of Spaces, Lemma 34.1 that the morphism $X \to Y$ is surjective and étale. Of course we have $X \times_Y X = X \times G$ hence the morphism $X \to Y$ is even finite étale. It is also surjective. Thus the lemma follows from Decent Spaces, Lemma 10.3. $\square$

Lemma 10.5. *Let k be a field. Let X be a quasi-separated algebraic space over k. If there exists a purely transcendental field extension K/k such that X_K is a scheme, then X is a scheme.*

Proof. Since every algebraic space is the union of its quasi-compact open subspaces, we may assume X is quasi-compact (some details omitted). Recall (Fields, Definition 26.1) that the assumption on the extension K/k signifies that K is the fraction field of a polynomial ring (in possibly infinitely many variables) over k. Thus $K = \bigcup A$ is the union of subalgebras each of which is a localization of a finite polynomial algebra over k. By Limits of Spaces, Lemma 5.11 we see that X_A is a scheme for some A. Write

$$A = k[x_1, \ldots, x_n][1/f]$$

for some nonzero $f \in k[x_1, \ldots, x_n]$.

If k is infinite then we can finish the proof as follows: choose $a_1, \ldots, a_n \in k$ with $f(a_1, \ldots, a_n) \neq 0$. Then $(a_1, \ldots, a_n)$ define an k-algebra map $A \to k$ mapping x_i to a_i and $1/f$ to $1/f(a_1, \ldots, a_n)$. Thus the base change $X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \cong X$ is a scheme as desired.

In this paragraph we finish the proof in case k is finite. In this case we write $X = \lim X_i$ with X_i of finite presentation over k and with affine transition morphisms (Limits of Spaces, Lemma 10.2). Using Limits of Spaces, Lemma 5.11 we see that $X_{i,A}$ is a scheme for some i. Thus we may assume $X \to \mathrm{Spec}(k)$ is of finite presentation. Let $x \in |X|$ be a closed point. We may represent x by a closed immersion $\mathrm{Spec}(\kappa) \to X$ (Decent Spaces, Lemma 14.6). Then $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(k)$ is of finite type, hence κ is a finite extension of k (by the Hilbert Nullstellensatz, see Algebra, Theorem 34.1; some details omitted). Say $[\kappa : k] = d$. Choose an integer $n \gg 0$ prime to d and let k'/k be the extension of degree n. Then k'/k is Galois with $G = \mathrm{Aut}(k'/k)$ cyclic of order n. If n is large enough there will be k-algebra homomorphism $A \to k'$ by the same reason as above. Then $X_{k'}$ is a scheme and $X = X_{k'}/G$ (Lemma 10.3). On the other hand, since n and d are relatively prime we see that

$$\mathrm{Spec}(\kappa) \times_X X_{k'} = \mathrm{Spec}(\kappa) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k') = \mathrm{Spec}(\kappa \otimes_k k')$$

is the spectrum of a field. In other words, the fibre of $X_{k'} \to X$ over x consists of a single point. Thus by Lemma 10.4 we see that x is in the schematic locus of X as desired. $\qquad\square$

Remark 10.6. Let k be a finite field. Let K/k be a geometrically irreducible field extension. Then K is the limit of geometrically irreducible finite type k-algebras A. Given A the estimates of Lang and Weil [LW54], show that for $n \gg 0$ there exists an k-algebra homomorphism $A \to k'$ with k'/k of degree n. Analyzing the argument given in the proof of Lemma 10.5 we see that if X is a quasi-separated algebraic space over k and X_K is a scheme, then X is a scheme. If we ever need this result we will precisely formulate it and prove it here.

Lemma 10.7. *Let k be a field with algebraic closure $\overline{k}$. Let X be an algebraic space over k such that*

(1) *X is decent and locally of finite type over k,*
(2) *$X_{\overline{k}}$ is a scheme, and*
(3) *any finite set of $\overline{k}$-rational points of $X_{\overline{k}}$ is contained in an affine.*

Then X is a scheme.

Proof. If K/k is an extension, then the base change X_K is decent (Decent Spaces, Lemma 6.5) and locally of finite type over K (Morphisms of Spaces, Lemma 23.3). By Lemma 10.1 it suffices to prove that X becomes a scheme after base change to the perfection of k, hence we may assume k is a perfect field (this step isn't strictly necessary, but makes the other arguments easier to think about). By covering X by quasi-compact opens we see that it suffices to prove the lemma in case X is quasi-compact (small detail omitted). In this case $|X|$ is a sober topological space (Decent Spaces, Proposition 12.4). Hence it suffices to show that every closed point in $|X|$ is contained in the schematic locus of X (use Properties of Spaces, Lemma 13.1 and Topology, Lemma 12.8).

Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 14.6 we can find a closed immersion $\mathrm{Spec}(l) \to X$ representing x. Then $\mathrm{Spec}(l) \to \mathrm{Spec}(k)$ is of finite type (Morphisms of Spaces, Lemma 23.2) and we conclude that l is a finite extension of k by the Hilbert Nullstellensatz (Algebra, Theorem 34.1). It is separable because k is perfect. Thus the scheme

$$\mathrm{Spec}(l) \times_X X_{\overline{k}} = \mathrm{Spec}(l) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}) = \mathrm{Spec}(l \otimes_k \overline{k})$$

is the disjoint union of a finite number of $\overline{k}$-rational points. By assumption (3) we can find an affine open $W \subset X_{\overline{k}}$ containing these points.

By Lemma 10.2 we see that $X_{k'}$ is a scheme for some finite extension k'/k. After enlarging k' we may assume that there exists an affine open $U' \subset X_{k'}$ whose base change to $\overline{k}$ recovers W (use that $X_{\overline{k}}$ is the limit of the schemes $X_{k''}$ for $k' \subset k'' \subset \overline{k}$ finite and use Limits, Lemmas 4.11 and 4.13). We may assume that k'/k is a Galois extension (take the normal closure Fields, Lemma 16.3 and use that k is perfect). Set $G = \mathrm{Gal}(k'/k)$. By construction the G-invariant closed subscheme $\mathrm{Spec}(l) \times_X X_{k'}$ is contained in U'. Thus x is in the schematic locus by Lemmas 10.3 and 10.4. $\square$

The following two lemmas should go somewhere else. Please compare the next lemma to Decent Spaces, Lemma 18.8.

Lemma 10.8. *Let k be a field. Let X be an algebraic space over k. The following are equivalent*

1. *X is locally quasi-finite over k,*
2. *X is locally of finite type over k and has dimension 0,*
3. *X is a scheme and is locally quasi-finite over k,*
4. *X is a scheme and is locally of finite type over k and has dimension 0, and*
5. *X is a disjoint union of spectra of Artinian local k-algebras A over k with $\dim_k(A) < \infty$.*

Proof. Because we are over a field relative dimension of X/k is the same as the dimension of X. Hence by Morphisms of Spaces, Lemma 34.6 we see that (1) and (2) are equivalent. Hence it follows from Lemma 9.1 (and trivial implications) that (1) – (4) are equivalent. Finally, Varieties, Lemma 20.2 shows that (1) – (4) are equivalent with (5). $\square$

Lemma 10.9. *Let k be a field. Let $f : X \to Y$ be a monomorphism of algebraic spaces over k. If Y is locally quasi-finite over k so is X.*

Proof. Assume Y is locally quasi-finite over k. By Lemma 10.8 we see that $Y = \coprod \mathrm{Spec}(A_i)$ where each A_i is an Artinian local ring finite over k. By Decent Spaces, Lemma 19.1 we see that X is a scheme. Consider $X_i = f^{-1}(\mathrm{Spec}(A_i))$. Then X_i has either one or zero points. If X_i has zero points there is nothing to prove. If X_i has one point, then $X_i = \mathrm{Spec}(B_i)$ with B_i a zero dimensional local ring and $A_i \to B_i$ is an epimorphism of rings. In particular $A_i/\mathfrak{m}_{A_i} = B_i/\mathfrak{m}_{A_i} B_i$ and we see that $A_i \to B_i$ is surjective by Nakayama's lemma, Algebra, Lemma 20.1 (because $\mathfrak{m}_{A_i}$ is a nilpotent ideal!). Thus B_i is a finite local k-algebra, and we conclude by Lemma 10.8 that $X \to \mathrm{Spec}(k)$ is locally quasi-finite. $\square$

11. Geometrically reduced algebraic spaces

If X is a reduced algebraic space over a field, then it can happen that X becomes nonreduced after extending the ground field. This does not happen for geometrically reduced algebraic spaces.

Definition 11.1. Let k be a field. Let X be an algebraic space over k.

(1) Let $x \in |X|$ be a point. We say X is *geometrically reduced at* x if $\mathcal{O}_{X,\overline{x}}$ is geometrically reduced over k.
(2) We say X is *geometrically reduced* over k if X is geometrically reduced at every point of X.

Observe that if X is geometrically reduced at x, then the local ring of X at x is reduced (Properties of Spaces, Lemma 22.6). Similarly, if X is geometrically reduced over k, then X is reduced (by Properties of Spaces, Lemma 21.4). The following lemma in particular implies this definition does not clash with the corresponding property for schemes over a field.

Lemma 11.2. *Let k be a field. Let X be an algebraic space over k. Let $x \in |X|$. The following are equivalent*

(1) *X is geometrically reduced at x,*
(2) *for some étale neighbourhood $(U, u) \to (X, x)$ where U is a scheme, U is geometrically reduced at u,*
(3) *for any étale neighbourhood $(U, u) \to (X, x)$ where U is a scheme, U is geometrically reduced at u.*

Proof. Recall that the local ring $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{U,u}$, see Properties of Spaces, Lemma 22.1. By Varieties, Lemma 6.2 we find that U is geometrically reduced at u if and only if $\mathcal{O}_{U,u}$ is geometrically reduced over k. Thus we have to show: if A is a local k-algebra, then A is geometrically reduced over k if and only if A^{sh} is geometrically reduced over k. We check this using the definition of geometrically reduced algebras (Algebra, Definition 43.1). Let K/k be a field extension. Since $A \to A^{sh}$ is faithfully flat (More on Algebra, Lemma 45.1) we see that $A \otimes_k K \to A^{sh} \otimes_k K$ is faithfully flat (Algebra, Lemma 39.7). Hence if $A^{sh} \otimes_k K$ is reduced, so is $A \otimes_k K$ by Algebra, Lemma 164.2. Conversely, recall that A^{sh} is a colimit of étale A-algebra, see Algebra, Lemma 155.2. Thus $A^{sh} \otimes_k K$ is a filtered colimit of étale $A \otimes_k K$-algebras. We conclude by Algebra, Lemma 163.7. $\square$

Lemma 11.3. *Let k be a field. Let X be an algebraic space over k. The following are equivalent*

(1) *X is geometrically reduced,*
(2) *for some surjective étale morphism $U \to X$ where U is a scheme, U is geometrically reduced,*
(3) *for any étale morphism $U \to X$ where U is a scheme, U is geometrically reduced.*

Proof. Immediate from the definitions and Lemma 11.2. $\square$

The notion isn't interesting in characteristic zero.

Lemma 11.4. *Let X be an algebraic space over a perfect field k (for example k has characteristic zero).*

(1) *For $x \in |X|$, if $\mathcal{O}_{X,\overline{x}}$ is reduced, then X is geometrically reduced at x.*
(2) *If X is reduced, then X is geometrically reduced over k.*

Proof. The first statement follows from Algebra, Lemma 43.6 and the definition of a perfect field (Algebra, Definition 45.1). The second statement follows from the first. $\square$

Lemma 11.5. *Let k be a field of characteristic $p > 0$. Let X be an algebraic space over k. The following are equivalent*

(1) *X is geometrically reduced over k,*
(2) *$X_{k'}$ is reduced for every field extension k'/k,*
(3) *$X_{k'}$ is reduced for every finite purely inseparable field extension k'/k,*
(4) *$X_{k^{1/p}}$ is reduced,*
(5) *$X_{k^{perf}}$ is reduced, and*
(6) *$X_{\overline{k}}$ is reduced.*

Proof. Choose a surjective étale morphism $U \to X$ where U is a scheme. Via Lemma 11.3 the lemma follows from the result for U over k. See Varieties, Lemma 6.4. $\square$

Lemma 11.6. *Let k be a field. Let X be an algebraic space over k. Let k'/k be a field extension. Let $x \in |X|$ be a point and let $x' \in |X_{k'}|$ be a point lying over x. The following are equivalent*

(1) *X is geometrically reduced at x,*
(2) *$X_{k'}$ is geometrically reduced at x'.*

In particular, X is geometrically reduced over k if and only if $X_{k'}$ is geometrically reduced over k'.

Proof. Choose an étale morphism $U \to X$ where U is a scheme and a point $u \in U$ mapping to $x \in |X|$. By Properties of Spaces, Lemma 4.3 we may choose a point $u' \in U_{k'} = U \times_X X_{k'}$ mapping to both u and x'. By Lemma 11.2 the lemma follows from the lemma for U, u, u' which is Varieties, Lemma 6.6. $\square$

Lemma 11.7. *Let k be a field. Let $f : X \to Y$ be a morphism of algebraic spaces over k. Let $x \in |X|$ be a point with image $y \in |Y|$.*

(1) *if f is étale at x, then X is geometrically reduced at $x \Leftrightarrow Y$ is geometrically reduced at y,*
(2) *if f is surjective étale, then X is geometrically reduced $\Leftrightarrow Y$ is geometrically reduced.*

Proof. Part (1) is clear because $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{Y,\overline{y}}$ if f is étale at x. Part (2) follows immediately from part (1). $\square$

12. Geometrically connected algebraic spaces

If X is a connected algebraic space over a field, then it can happen that X becomes disconnected after extending the ground field. This does not happen for geometrically connected algebraic spaces.

Definition 12.1. Let X be an algebraic space over the field k. We say X is *geometrically connected* over k if the base change $X_{k'}$ is connected for every field extension k' of k.

By convention a connected topological space is nonempty; hence a fortiori geometrically connected algebraic spaces are nonempty.

Lemma 12.2. *Let X be an algebraic space over the field k. Let k'/k be a field extension. Then X is geometrically connected over k if and only if $X_{k'}$ is geometrically connected over k'.*

Proof. If X is geometrically connected over k, then it is clear that $X_{k'}$ is geometrically connected over k'. For the converse, note that for any field extension k''/k there exists a common field extension k'''/k' and k'''/k'. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the connectedness of $X_{k'''}$ implies the connectedness of $X_{k''}$. Thus if $X_{k'}$ is geometrically connected over k' then X is geometrically connected over k. $\square$

Lemma 12.3. *Let k be a field. Let X, Y be algebraic spaces over k. Assume X is geometrically connected over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between connected components.

Proof. Let $y \in |Y|$ be represented by a morphism $\mathrm{Spec}(K) \to Y$ where K is a field. The fibre of $|X \times_k Y| \to |Y|$ over y is the image of $|X_K| \to |X \times_k Y|$ by Properties of Spaces, Lemma 4.3. Thus these fibres are connected by our assumption that X is geometrically connected. By Morphisms of Spaces, Lemma 6.6 the map $|p|$ is open. Thus we may apply Topology, Lemma 7.6 to conclude. $\square$

Lemma 12.4. *Let k'/k be an extension of fields. Let X be an algebraic space over k. Assume k separably algebraically closed. Then the morphism $X_{k'} \to X$ induces a bijection of connected components. In particular, X is geometrically connected over k if and only if X is connected.*

Proof. Since k is separably algebraically closed we see that k' is geometrically connected over k, see Algebra, Lemma 48.4. Hence $Z = \mathrm{Spec}(k')$ is geometrically connected over k by Varieties, Lemma 7.5. Since $X_{k'} = Z \times_k X$ the result is a special case of Lemma 12.3. $\square$

Lemma 12.5. *Let k be a field. Let X be an algebraic space over k. Let $\overline{k}$ be a separable algebraic closure of k. Then X is geometrically connected if and only if the base change $X_{\overline{k}}$ is connected.*

Proof. Assume $X_{\overline{k}}$ is connected. Let k'/k be a field extension. There exists a field extension $\overline{k}'/\overline{k}$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 12.4 we see that $X_{\overline{k}'}$ is connected. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is connected as desired. $\square$

Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. For example $\overline{k}$ could be the separable algebraic closure of k. For any $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we get a corresponding automorphism $\mathrm{Spec}(\sigma) : \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$. Note that $\mathrm{Spec}(\sigma) \circ \mathrm{Spec}(\tau) = \mathrm{Spec}(\tau \circ \sigma)$. Hence we get an action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$$

of the opposite group on the scheme $\mathrm{Spec}(\overline{k})$. Let X be an algebraic space over k. Since $X_{\overline{k}} = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} X$ by definition we see that the action above induces a canonical action

$$(12.5.1) \qquad \mathrm{Gal}(\overline{k}/k)^{opp} \times X_{\overline{k}} \longrightarrow X_{\overline{k}}.$$

Lemma 12.6. *Let k be a field. Let X be an algebraic space over k. Let $\overline{k}$ be a (possibly infinite) Galois extension of k. Let $V \subset X_{\overline{k}}$ be a quasi-compact open. Then*

(1) *there exists a finite subextension $\overline{k}/k'/k$ and a quasi-compact open $V' \subset X_{k'}$ such that $V = (V')_{\overline{k}}$,*

(2) *there exists an open subgroup $H \subset \mathrm{Gal}(\overline{k}/k)$ such that $\sigma(V) = V$ for all $\sigma \in H$.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a quasi-compact open $W \subset U_{\overline{k}}$ whose image in $X_{\overline{k}}$ is V. This is possible because $|U_{\overline{k}}| \to |X_{\overline{k}}|$ is continuous and because $|U_{\overline{k}}|$ has a basis of quasi-compact opens. We can apply Varieties, Lemma 7.9 to $W \subset U_{\overline{k}}$ to obtain the lemma. $\square$

Lemma 12.7. *Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. Let X be an algebraic space over k. Let $\overline{T} \subset |X_{\overline{k}}|$ have the following properties*

(1) *$\overline{T}$ is a closed subset of $|X_{\overline{k}}|$,*

(2) *for every $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we have $\sigma(\overline{T}) = \overline{T}$.*

Then there exists a closed subset $T \subset |X|$ whose inverse image in $|X_{k'}|$ is $\overline{T}$.

Proof. Let $T \subset |X|$ be the image of $\overline{T}$. Since $|X_{\overline{k}}| \to |X|$ is surjective, the statement means that T is closed and that its inverse image is $\overline{T}$. Choose a scheme U and a surjective étale morphism $U \to X$. By the case of schemes (see Varieties, Lemma 7.10) there exists a closed subset $T' \subset |U|$ whose inverse image in $|U_{\overline{k}}|$ is the inverse image of $\overline{T}$. Since $|U_{\overline{k}}| \to |X_{\overline{k}}|$ is surjective, we see that T' is the inverse image of T via $|U| \to |X|$. By our construction of the topology on $|X|$ this means that T is closed. In the same manner one sees that $\overline{T}$ is the inverse image of T. $\square$

Lemma 12.8. *Let k be a field. Let X be an algebraic space over k. The following are equivalent*

(1) *X is geometrically connected,*

(2) *for every finite separable field extension k'/k the algebraic space $X_{k'}$ is connected.*

Proof. This proof is identical to the proof of Varieties, Lemma 7.11 except that we replace Varieties, Lemma 7.7 by Lemma 12.5, we replace Varieties, Lemma 7.9 by Lemma 12.6, and we replace Varieties, Lemma 7.10 by Lemma 12.7. We urge the reader to read that proof in stead of this one.

It follows immediately from the definition that (1) implies (2). Assume that X is not geometrically connected. Let $k \subset \overline{k}$ be a separable algebraic closure of k. By Lemma 12.5 it follows that $X_{\overline{k}}$ is disconnected. Say $X_{\overline{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open, closed, and nonempty algebraic subspaces of $X_{\overline{k}}$.

Suppose that $W \subset X$ is any quasi-compact open subspace. Then $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are open and closed subspaces of $W_{\overline{k}}$. In particular $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are quasi-compact, and by Lemma 12.6 both $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are defined over

a finite subextension and invariant under an open subgroup of $\mathrm{Gal}(\overline{k}/k)$. We will use this without further mention in the following.

Pick $W_0 \subset X$ quasi-compact open subspace such that both $W_{0,\overline{k}} \cap \overline{U}$ and $W_{0,\overline{k}} \cap \overline{V}$ are nonempty. Choose a finite subextension $\overline{k}/k'/k$ and a decomposition $W_{0,k'} = U_0' \amalg V_0'$ into open and closed subsets such that $W_{0,\overline{k}} \cap \overline{U} = (U_0')_{\overline{k}}$ and $W_{0,\overline{k}} \cap \overline{V} = (V_0')_{\overline{k}}$. Let $H = \mathrm{Gal}(\overline{k}/k') \subset \mathrm{Gal}(\overline{k}/k)$. In particular $\sigma(W_{0,\overline{k}} \cap \overline{U}) = W_{0,\overline{k}} \cap \overline{U}$ and similarly for $\overline{V}$.

Having chosen W_0, k' as above, for every quasi-compact open subspace $W \subset X$ we set
$$U_W = \bigcap\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{U}), \quad V_W = \bigcup\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{V}).$$
Now, since $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are fixed by an open subgroup of $\mathrm{Gal}(\overline{k}/k)$ we see that the union and intersection above are finite. Hence U_W and V_W are both open and closed subspaces. Also, by construction $W_{\overline{k}} = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open subspaces, then $W_{\overline{k}} \cap U_{W'} = U_W$ and $W_{\overline{k}} \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X_{\overline{k}} = U \amalg V$ is a disjoint union of open and closed subsets. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $W_0 \cap \overline{U}$ by construction. Finally, $U, V \subset X_{\overline{k}}$ are closed and H-invariant by construction. Hence by Lemma 12.7 we have $U = (U')_{\overline{k}}$, and $V = (V')_{\overline{k}}$ for some closed $U', V' \subset X_{k'}$. Clearly $X_{k'} = U' \amalg V'$ and we see that $X_{k'}$ is disconnected as desired. $\qquad\square$

13. Geometrically irreducible algebraic spaces

Spaces, Example 14.9 shows that it is best not to think about irreducible algebraic spaces in complete generality[1]. For decent (for example quasi-separated) algebraic spaces this kind of disaster doesn't happen. Thus we make the following definition only under the assumption that our algebraic space is decent.

Definition 13.1. Let k be a field. Let X be a decent algebraic space over k. We say X is *geometrically irreducible* if the topological space $|X_{k'}|$ is irreducible[2] for any field extension k' of k.

Observe that $X_{k'}$ is a decent algebraic space (Decent Spaces, Lemma 6.5). Hence the topological space $|X_{k'}|$ is sober. Decent Spaces, Proposition 12.4.

14. Geometrically integral algebraic spaces

Recall that integral algebraic spaces are by definition decent, see Section 4.

Definition 14.1. Let X be an algebraic space over the field k. We say X is *geometrically integral* over k if the algebraic space $X_{k'}$ is integral (Definition 4.1) for every field extension k' of k.

In particular X is a decent algebraic space. We can relate this to being geometrically reduced and geometrically irreducible as follows.

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[1]To be sure, if we say "the algebraic space X is irreducible", we probably mean to say "the topological space $|X|$ is irreducible".

[2]An irreducible space is nonempty.

Lemma 14.2. *Let k be a field. Let X be a decent algebraic space over k. Then X is geometrically integral over k if and only if X is both geometrically reduced and geometrically irreducible over k.*

Proof. This is an immediate consequence of the definitions because our notion of integral (in the presence of decency) is equivalent to reduced and irreducible. $\square$

Lemma 14.3. *Let k be a field. Let X be a proper algebraic space over k.*
  (1) *$A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra,*
  (2) *$A = \prod_{i=1,\ldots,n} A_i$ is a product of Artinian local k-algebras, one factor for each connected component of $|X|$,*
  (3) *if X is reduced, then $A = \prod_{i=1,\ldots,n} k_i$ is a product of fields, each a finite extension of k,*
  (4) *if X is geometrically reduced, then k_i is finite separable over k,*
  (5) *if X is geometrically connected, then A is geometrically irreducible over k,*
  (6) *if X is geometrically irreducible, then A is geometrically irreducible over k,*
  (7) *if X is geometrically reduced and connected, then $A = k$, and*
  (8) *if X is geometrically integral, then $A = k$.*

Proof. By Cohomology of Spaces, Lemma 20.3 we see that $A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra. This proves (1).

Then A is a product of local rings by Algebra, Lemma 53.2 and Algebra, Proposition 60.7. If $X = Y \amalg Z$ with Y and Z open subspaces of X, then we obtain an idempotent $e \in A$ by taking the section of $\mathcal{O}_X$ which is 1 on Y and 0 on Z. Conversely, if $e \in A$ is an idempotent, then we get a corresponding decomposition of $|X|$. Finally, as $|X|$ is a Noetherian topological space (by Morphisms of Spaces, Lemma 28.6 and Properties of Spaces, Lemma 24.2) its connected components are open. Hence the connected components of $|X|$ correspond 1-to-1 with primitive idempotents of A. This proves (2).

If X is reduced, then A is reduced (Properties of Spaces, Lemma 21.4). Hence the local rings $A_i = k_i$ are reduced and therefore fields (for example by Algebra, Lemma 25.1). This proves (3).

If X is geometrically reduced, then same thing is true for $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ (see Cohomology of Spaces, Lemma 11.2 for equality). This implies that $k_i \otimes_k \overline{k}$ is a product of fields and hence k_i/k is separable for example by Algebra, Lemmas 44.1 and 44.3. This proves (4).

If X is geometrically connected, then $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ is a zero dimensional local ring by part (2) and hence its spectrum has one point, in particular it is irreducible. Thus A is geometrically irreducible. This proves (5). Of course (5) implies (6).

If X is geometrically reduced and connected, then $A = k_1$ is a field and the extension k_1/k is finite separable and geometrically irreducible. However, then $k_1 \otimes_k \overline{k}$ is a product of $[k_1 : k]$ copies of $\overline{k}$ and we conclude that $k_1 = k$. This proves (7). Of course (7) implies (8). $\square$

Lemma 14.4. *Let k be a field. Let X be a proper integral algebraic space over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{\otimes -1})$ are both nonzero, then $\mathcal{L} \cong \mathcal{O}_X$.*

Proof. Let $s \in H^0(X, \mathcal{L})$ and $t \in H^0(X, \mathcal{L}^{\otimes -1})$ be nonzero sections. Let $x \in |X|$ be a point in the support of s. Choose an affine étale neighbourhood $(U, u) \to (X, x)$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Then $s|_U$ corresponds to a nonzero regular function on the reduced (because X is reduced) scheme U and hence is nonvanishing in a generic point of an irreducible component of U. By Decent Spaces, Lemma 20.1 we conclude that the generic point η of $|X|$ is in the support of s. The same is true for t. Then of course st must be nonzero because the local ring of X at η is a field (by aforementioned lemma the local ring has dimension zero, as X is reduced the local ring is reduced, and Algebra, Lemma 25.1). However, we have seen that $K = H^0(X, \mathcal{O}_X)$ is a field in Lemma 14.3. Thus st is everywhere nonzero and we see that $s : \mathcal{O}_X \to \mathcal{L}$ is an isomorphism. $\qquad\square$

15. Dimension

In this section we continue the discussion about dimension. Here is a list of previous material:

(1) dimension is defined in Properties of Spaces, Section 9,
(2) dimension of local ring is defined in Properties of Spaces, Section 10,
(3) a couple of results in Properties of Spaces, Lemmas 22.4 and 22.5,
(4) relative dimension is defined in Morphisms of Spaces, Section 33,
(5) results on dimension of fibres in Morphisms of Spaces, Section 34,
(6) a weak form of the dimension formula Morphisms of Spaces, Section 35,
(7) a result on smoothness and dimension Morphisms of Spaces, Lemma 37.10,
(8) dimension is $\dim(|X|)$ for decent spaces Decent Spaces, Lemma 12.5,
(9) quasi-finite maps and dimension Decent Spaces, Lemmas 12.6 and 12.7.

In More on Morphisms of Spaces, Section 31 we will discuss jumping of dimension in fibres of a finite type morphism.

Lemma 15.1. *Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces. Then $\dim(X) \leq \dim(Y)$. If f is surjective then $\dim(X) = \dim(Y)$.*

Proof. Choose $V \to Y$ surjective étale with V a scheme. Then $U = X \times_Y V$ is a scheme and $U \to V$ is integral (and surjective if f is surjective). By Properties of Spaces, Lemma 22.5 we have $\dim(X) = \dim(U)$ and $\dim(Y) = \dim(V)$. Thus the result follows from the case of schemes which is Morphisms, Lemma 44.9. $\qquad\square$

Lemma 15.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

(1) *Y is locally Noetherian,*
(2) *X and Y are integral algebraic spaces,*
(3) *f is dominant, and*
(4) *f is locally of finite type.*

If $x \in |X|$ and $y \in |Y|$ are the generic points, then

$$\dim(X) \leq \dim(Y) + \text{transcendence degree of } x/y.$$

If f is proper, then equality holds.

Proof. Recall that $|X|$ and $|Y|$ are irreducible sober topological spaces, see discussion following Definition 4.1. Thus the fact that f is dominant means that $|f|$ maps x to y. Moreover, $x \in |X|$ is the unique point at which the local ring of X has

dimension 0, see Decent Spaces, Lemma 20.1. By Morphisms of Spaces, Lemma 35.1 we see that the dimension of the local ring of X at any point $x' \in |X|$ is at most the dimension of the local ring of Y at $y' = f(x')$ plus the transcendence degree of x/y. Since the dimension of X, resp. dimension of Y is the supremum of the dimensions of the local rings at x', resp. y' (Properties of Spaces, Lemma 10.3) we conclude the inequality holds.

Assume f is proper. Let $V \subset Y$ be a nonempty quasi-compact open subspace. If we can prove the equality for the morphism $f^{-1}(V) \to V$, then we get the equality for $X \to Y$. Thus we may assume that X and Y are quasi-compact. Observe that X is quasi-separated as a locally Noetherian decent algebraic space, see Decent Spaces, Lemma 14.1. Thus we may choose $Y' \to Y$ finite surjective where Y' is a scheme, see Limits of Spaces, Proposition 16.1. After replacing Y' by a suitable closed subscheme, we may assume Y' is integral, see for example the more general Lemma 8.5. By the same lemma, we may choose a closed subspace $X' \subset X \times_Y Y'$ such that X' is integral and $X' \to X$ is finite surjective. Now X' is also locally Noetherian (Morphisms of Spaces, Lemma 23.5) and we can use Limits of Spaces, Proposition 16.1 once more to choose a finite surjective morphism $X'' \to X'$ with X'' a scheme. As before we may assume that X'' is integral. Picture

$$
\begin{array}{ccc}
X'' & \longrightarrow & X \\
\downarrow & & \downarrow f \\
Y' & \longrightarrow & Y
\end{array}
$$

By Lemma 15.1 we have $\dim(X'') = \dim(X)$ and $\dim(Y') = \dim(Y)$. Since X and Y have open neighbourhoods of x, resp. y which are schemes, we readily see that the generic points $x'' \in X''$, resp. $y' \in Y'$ are the unique points mapping to x, resp. y and that the residue field extensions $\kappa(x'')/\kappa(x)$ and $\kappa(y')/\kappa(y)$ are finite. This implies that the transcendence degree of x''/y' is the same as the transcendence degree of x/y. Thus the equality follows from the case of schemes whicn is Morphisms, Lemma 52.4. $\qquad \square$

16. Spaces smooth over fields

This section is the analogue of Varieties, Section 25.

Lemma 16.1. *Let k be a field. Let X be an algebraic space smooth over k. Then X is a regular algebraic space.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The morphism $U \to \operatorname{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 39.6 and 37.2). Hence U is regular by Varieties, Lemma 25.3. By Properties of Spaces, Definition 7.2 this means that X is regular. $\qquad \square$

Lemma 16.2. *Let k be a field. Let X be an algebraic space smooth over $\operatorname{Spec}(k)$. The set of $x \in |X|$ which are image of morphisms $\operatorname{Spec}(k') \to X$ with $k' \supset k$ finite separable is dense in $|X|$.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The morphism $U \to \operatorname{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 39.6 and 37.2).

Hence we can apply Varieties, Lemma 25.6 to see that the closed points of U whose residue fields are finite separable over k are dense. This implies the lemma by our definition of the topology on $|X|$. $\qquad\square$

17. Euler characteristics

In this section we prove some elementary properties of Euler characteristics of coherent sheaves on algebraic spaces proper over fields.

Definition 17.1. Let k be a field. Let X be a proper algebraic over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. In this situation the *Euler characteristic of $\mathcal{F}$* is the integer

$$\chi(X, \mathcal{F}) = \sum\nolimits_i (-1)^i \dim_k H^i(X, \mathcal{F}).$$

For justification of the formula see below.

In the situation of the definition only a finite number of the vector spaces $H^i(X, \mathcal{F})$ are nonzero (Cohomology of Spaces, Lemma 7.3) and each of these spaces is finite dimensional (Cohomology of Spaces, Lemma 20.3). Thus $\chi(X, \mathcal{F}) \in \mathbf{Z}$ is well defined. Observe that this definition depends on the field k and not just on the pair $(X, \mathcal{F})$.

Lemma 17.2. *Let k be a field. Let X be a proper algebraic space over k. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent modules on X. Then*

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$$

Proof. Consider the long exact sequence of cohomology

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1) \to \ldots$$

associated to the short exact sequence of the lemma. The rank-nullity theorem in linear algebra shows that

$$0 = \dim H^0(X, \mathcal{F}_1) - \dim H^0(X, \mathcal{F}_2) + \dim H^0(X, \mathcal{F}_3) - \dim H^1(X, \mathcal{F}_1) + \ldots$$

This immediately implies the lemma. $\qquad\square$

Lemma 17.3. *Let k be a field. Let $f : Y \to X$ be a morphism of algebraic spaces proper over k. Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module. Then*

$$\chi(Y, \mathcal{G}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{G})$$

Proof. The formula makes sense: the sheaves $R^i f_* \mathcal{G}$ are coherent and only a finite number of them are nonzero, see Cohomology of Spaces, Lemmas 20.2 and 8.1. By Cohomology on Sites, Lemma 14.5 there is a spectral sequence with

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{G})$$

converging to $H^{p+q}(Y, \mathcal{G})$. By finiteness of cohomology on X we see that only a finite number of $E_2^{p,q}$ are nonzero and each $E_2^{p,q}$ is a finite dimensional vector space. It follows that the same is true for $E_r^{p,q}$ for $r \geq 2$ and that

$$\sum (-1)^{p+q} \dim_k E_r^{p,q}$$

is independent of r. Since for r large enough we have $E_r^{p,q} = E_\infty^{p,q}$ and since convergence means there is a filtration on $H^n(Y, \mathcal{G})$ whose graded pieces are $E_\infty^{p,q}$ with $p + 1 = n$ (this is the meaning of convergence of the spectral sequence), we conclude. $\qquad\square$

18. Numerical intersections

In this section we play around with the Euler characteristic of coherent sheaves on proper algebraic spaces to obtain numerical intersection numbers for invertible modules. Our main tool will be the following lemma.

Lemma 18.1. *Let k be a field. Let X be a proper algebraic space over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. The map*

$$(n_1, \ldots, n_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most the dimension of the scheme theoretic support of $\mathcal{F}$.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Then $\mathcal{F} = i_*\mathcal{G}$ for some coherent $\mathcal{O}_Z$-module $\mathcal{G}$ (Cohomology of Spaces, Lemma 12.7) and we have

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) = \chi(Z, \mathcal{G} \otimes i^*\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes i^*\mathcal{L}_r^{\otimes n_r})$$

by the projection formula (Cohomology on Sites, Lemma 50.1) and Cohomology of Spaces, Lemma 8.3. Since $|Z| = \mathrm{Supp}(\mathcal{F})$ we see that it suffices to show

$$P_{\mathcal{F}}(n_1, \ldots, n_r) : (n_1, \ldots, n_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most $\dim(X)$. Let us say property $\mathcal{P}$ holds for the coherent $\mathcal{O}_X$-module $\mathcal{F}$ if the above is true.

We will prove this statement by devissage, more precisely we will check conditions (1), (2), and (3) of Cohomology of Spaces, Lemma 14.6 are satisfied.

Verification of condition (1). Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. By Lemma 17.2 we have

$$P_{\mathcal{F}_2}(n_1, \ldots, n_r) = P_{\mathcal{F}_1}(n_1, \ldots, n_r) + P_{\mathcal{F}_3}(n_1, \ldots, n_r)$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then so does the third.

Condition (2) follows because $P_{\mathcal{F}^{\oplus m}}(n_1, \ldots, n_r) = m P_{\mathcal{F}}(n_1, \ldots, n_r)$.

Proof of (3). Let $i : Z \to X$ be a reduced closed subspace with $|Z|$ irreducible. We have to find a coherent module $\mathcal{G}$ on X whose support is Z such that $\mathcal{P}$ holds for $\mathcal{G}$. We will give two constructions: one using Chow's lemma and one using a finite cover by a scheme.

Proof existence $\mathcal{G}$ using a finite cover by a scheme. Choose $\pi : Z' \to Z$ finite surjective where Z' is a scheme, see Limits of Spaces, Proposition 16.1. Set $\mathcal{G} = i_*\pi_*\mathcal{O}_{Z'} = (i \circ \pi)_*\mathcal{O}_{Z'}$. Note that Z' is proper over k and that the support of $\mathcal{G}$ is Y (details omitted). We have

$$R(\pi \circ i)_*(\mathcal{O}_{Z'}) = \mathcal{G} \quad \text{and} \quad R(\pi \circ i)_*(\pi^* i^*(\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})) = \mathcal{G} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}$$

The first equality holds because $i \circ \pi$ is affine (Cohomology of Spaces, Lemma 8.2) and the second equality follows from the first and the projection formula (Cohomology on Sites, Lemma 50.1). Using Leray (Cohomology on Sites, Lemma 14.6) we obtain

$$P_{\mathcal{G}}(n_1, \ldots, n_r) = \chi(Z', \pi^* i^*(\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}))$$

By the case of schemes (Varieties, Lemma 45.1) this is a numerical polynomial in $n_1, \ldots, n_r$ of degree at most $\dim(Z')$. We conclude because $\dim(Z') \leq \dim(Z) \leq \dim(X)$. The first inequality follows from Decent Spaces, Lemma 12.7.

Proof existence $\mathcal{G}$ using Chow's lemma. We apply Cohomology of Spaces, Lemma 18.1 to the morphism $Z \to \mathrm{Spec}(k)$. Thus we get a surjective proper morphism $f : Y \to Z$ over $\mathrm{Spec}(k)$ where Y is a closed subscheme of $\mathbf{P}_k^m$ for some m. After replacing Y by a closed subscheme we may assume that Y is integral and $f : Y \to Z$ is an alteration, see Lemma 8.5. Denote $\mathcal{O}_Y(n)$ the pullback of $\mathcal{O}_{\mathbf{P}_k^m}(n)$. Pick $n > 0$ such that $R^p f_* \mathcal{O}_Y(n) = 0$ for $p > 0$, see Cohomology of Spaces, Lemma 20.1. We claim that $\mathcal{G} = i_* f_* \mathcal{O}_Y(n)$ satisfies $\mathcal{P}$. Namely, by the case of schemes (Varieties, Lemma 45.1) we know that

$$(n_1, \ldots, n_r) \longmapsto \chi(Y, \mathcal{O}_Y(n) \otimes f^* i^* (\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}))$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most $\dim(Y)$. On the other hand, by the projection formula (Cohomology on Sites, Lemma 50.1)

$$i_* Rf_* \left(\mathcal{O}_Y(n) \otimes f^* i^* (\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) \right) = i_* Rf_* \mathcal{O}_Y(n) \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}$$
$$= \mathcal{G} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}$$

the last equality by our choice of n. By Leray (Cohomology on Sites, Lemma 14.6) we get

$$\chi(Y, \mathcal{O}_Y(n) \otimes f^* i^* (\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})) = P_{\mathcal{G}}(n_1, \ldots, n_r)$$

and we conclude because $\dim(Y) \leq \dim(Z) \leq \dim(X)$. The first inequality holds by Morphisms of Spaces, Lemma 35.2 and the fact that $Y \to Z$ is an alteration (and hence the induced extension of residue fields in generic points is finite). $\qquad\square$

The following lemma roughly shows that the leading coefficient only depends on the length of the coherent module in the generic points of its support.

Lemma 18.2. *Let k be a field. Let X be a proper algebraic space over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. Let $d = \dim(Supp(\mathcal{F}))$. Let $Z_i \subset X$ be the irreducible components of $Supp(\mathcal{F})$ of dimension d. Let $\overline{x}_i$ be a geometric generic point of Z_i and set $m_i = length_{\mathcal{O}_{X,\overline{x}_i}}(\mathcal{F}_{\overline{x}_i})$. Then*

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum_i m_i\ \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree $< d$.

Proof. We first prove a slightly weaker statement. Namely, say $\dim(X) = N$ and let $X_i \subset X$ be the irreducible components of dimension N. Let $\overline{x}_i$ be a geometric generic point of X_i. The étale local ring $\mathcal{O}_{X,\overline{x}_i}$ is Noetherian of dimension 0, hence for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ the length

$$m_i(\mathcal{F}) = length_{\mathcal{O}_{X,\overline{x}_i}}(\mathcal{F}_{\overline{x}_i})$$

is an integer ≥ 0. We claim that

$$E(\mathcal{F}) = \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum_i m_i(\mathcal{F})\ \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree $< N$. We will prove this using Cohomology of Spaces, Lemma 14.6. For any short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ we have $E(\mathcal{F}) = E(\mathcal{F}') + E(\mathcal{F}'')$. This follows from additivity of Euler characteristics (Lemma 17.2) and additivity of lengths (Algebra, Lemma 52.3).

This immediately implies properties (1) and (2) of Cohomology of Spaces, Lemma 14.6. Finally, property (3) holds because for $\mathcal{G} = \mathcal{O}_Z$ for any $Z \subset X$ irreducible reduced closed subspace. Namely, if $Z = Z_{i_0}$ for some i_0, then $m_i(\mathcal{G}) = \delta_{i_0 i}$ and we conclude $E(\mathcal{G}) = 0$. If $Z \neq Z_i$ for any i, then $m_i(\mathcal{G}) = 0$ for all i, $\dim(Z) < N$ and we get the result from Lemma 18.1.

Proof of the statement as in the lemma. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Then $\mathcal{F} = i_* \mathcal{G}$ for some coherent $\mathcal{O}_Z$-module $\mathcal{G}$ (Cohomology of Spaces, Lemma 12.7) and we have

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) = \chi(Z, \mathcal{G} \otimes i^* \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes i^* \mathcal{L}_r^{\otimes n_r})$$

by the projection formula (Cohomology on Sites, Lemma 50.1) and Cohomology of Spaces, Lemma 8.3. Since $|Z| = \operatorname{Supp}(\mathcal{F})$ we see that $Z_i \subset Z$ for all i and we see that these are the irreducible components of Z of dimension d. We may and do think of $\overline{x}_i$ as a geometric point of Z. The map $i^\sharp : \mathcal{O}_X \to i_* \mathcal{O}_Z$ determines a surjection

$$\mathcal{O}_{X, \overline{x}_i} \to \mathcal{O}_{Z, \overline{x}_i}$$

Via this map we have an isomorphism of modules $\mathcal{G}_{\overline{x}_i} = \mathcal{F}_{\overline{x}_i}$ as $\mathcal{F} = i_* \mathcal{G}$. This implies that

$$m_i = \operatorname{length}_{\mathcal{O}_{X, \overline{x}_i}}(\mathcal{F}_{\overline{x}_i}) = \operatorname{length}_{\mathcal{O}_{Z, \overline{x}_i}}(\mathcal{G}_{\overline{x}_i})$$

Thus we see that the expression in the lemma is equal to

$$\chi(Z, \mathcal{G} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum_i m_i \ \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

and the result follows from the discussion in the first paragraph (applied with Z instead of X). $\square$

Definition 18.3. Let k be a field. Let X be a proper algebraic space over k. Let $i : Z \to X$ be a closed subspace of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. We define the *intersection number* $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ as the coefficient of $n_1 \ldots n_d$ in the numerical polynomial

$$\chi(X, i_* \mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

In the special case that $\mathcal{L}_1 = \ldots = \mathcal{L}_d = \mathcal{L}$ we write $(\mathcal{L}^d \cdot Z)$.

The displayed equality in the definition follows from the projection formula (Cohomology, Section 54) and Cohomology of Schemes, Lemma 2.4. We prove a few lemmas for these intersection numbers.

Lemma 18.4. *In the situation of Definition 18.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is an integer.*

Proof. Any numerical polynomial of degree e in $n_1, \ldots, n_d$ can be written uniquely as a $\mathbf{Z}$-linear combination of the functions $\binom{n_1}{k_1}\binom{n_2}{k_2} \ldots \binom{n_d}{k_d}$ with $k_1 + \ldots + k_d \leq e$. Apply this with $e = d$. Left as an exercise. $\square$

Lemma 18.5. *In the situation of Definition 18.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is additive: if $\mathcal{L}_i = \mathcal{L}_i' \otimes \mathcal{L}_i''$, then we have*

$$(\mathcal{L}_1 \cdots \mathcal{L}_i \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_1 \cdots \mathcal{L}_i' \cdots \mathcal{L}_d \cdot Z) + (\mathcal{L}_1 \cdots \mathcal{L}_i'' \cdots \mathcal{L}_d \cdot Z)$$

Proof. This is true because by Lemma 18.1 the function

$$(n_1, \ldots, n_{i-1}, n_i', n_i'', n_{i+1}, \ldots, n_d) \mapsto \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes (\mathcal{L}_i')^{\otimes n_i'} \otimes (\mathcal{L}_i'')^{\otimes n_i''} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

is a numerical polynomial of total degree at most d in $d+1$ variables. $\square$

Lemma 18.6. *In the situation of Definition 18.3 let $Z_i \subset Z$ be the irreducible components of dimension d. Let $m_i = \text{length}_{\mathcal{O}_{X,\overline{x}_i}}(\mathcal{O}_{Z,\overline{x}_i})$ where $\overline{x}_i$ is a geometric generic point of Z_i. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \sum m_i (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z_i)$$

Proof. Immediate from Lemma 18.2 and the definitions. $\square$

Lemma 18.7. *Let k be a field. Let $f : Y \to X$ be a morphism of algebraic spaces proper over k. Let $Z \subset Y$ be an integral closed subspace of dimension d and let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then*

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot Z) = \deg(f|_Z : Z \to f(Z))(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot f(Z))$$

where $\deg(Z \to f(Z))$ is as in Definition 5.2 or 0 if $\dim(f(Z)) < d$.

Proof. In the statement $f(Z) \subset X$ is the scheme theoretic image of f and it is also the reduced induced algebraic space structure on the closed subset $f(|Z|) \subset X$, see Morphisms of Spaces, Lemma 16.4. Then Z and $f(Z)$ are reduced, proper (hence decent) algebraic spaces over k, whence integral (Definition 4.1). The left hand side is computed using the coefficient of $n_1 \ldots n_d$ in the function

$$\chi(Y, \mathcal{O}_Z \otimes f^*\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes f^*\mathcal{L}_d^{\otimes n_d}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$$

The equality follows from Lemma 17.3 and the projection formula (Cohomology, Lemma 54.2). If $f(Z)$ has dimension $< d$, then the right hand side is a polynomial of total degree $< d$ by Lemma 18.1 and the result is true. Assume $\dim(f(Z)) = d$. Then by dimension theory (Lemma 15.2) we find that the equivalent conditions (1) – (5) of Lemma 5.1 hold. Thus $\deg(Z \to f(Z))$ is well defined. By the already used Lemma 5.1 we find $f : Z \to f(Z)$ is finite over a nonempty open V of $f(Z)$; after possibly shrinking V we may assume V is a scheme. Let $\xi \in V$ be the generic point. Thus $\deg(f : Z \to f(Z))$ the length of the stalk of $f_*\mathcal{O}_Z$ at ξ over $\mathcal{O}_{X,\xi}$ and the stalk of $R^i f_* \mathcal{O}_X$ at ξ is zero for $i > 0$ (for example by Cohomology of Spaces, Lemma 4.1). Thus the terms $\chi(X, R^i f_* \mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ with $i > 0$ have total degree $< d$ and

$$\chi(X, f_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \deg(f : Z \to f(Z))\chi(f(Z), \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_{f(Z)})$$

modulo a polynomial of total degree $< d$ by Lemma 18.2. The desired result follows. $\square$

Lemma 18.8. *Let k be a field. Let X be a proper algebraic space over k. Let $Z \subset X$ be a closed subspace of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Assume there exists an effective Cartier divisor $D \subset Z$ such that $\mathcal{L}_1|_Z \cong \mathcal{O}_Z(D)$. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot D)$$

Proof. We may replace X by Z and $\mathcal{L}_i$ by $\mathcal{L}_i|_Z$. Thus we may assume $X = Z$ and $\mathcal{L}_1 = \mathcal{O}_X(D)$. Then $\mathcal{L}_1^{-1}$ is the ideal sheaf of D and we can consider the short exact sequence

$$0 \to \mathcal{L}_1^{\otimes -1} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

Set $P(n_1, \ldots, n_d) = \chi(X, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ and $Q(n_1, \ldots, n_d) = \chi(D, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_D)$. We conclude from additivity (Lemma 17.2) that

$$P(n_1, \ldots, n_d) - P(n_1 - 1, n_2, \ldots, n_d) = Q(n_1, \ldots, n_d)$$

Because the total degree of P is at most d, we see that the coefficient of $n_1 \ldots n_d$ in P is equal to the coefficient of $n_2 \ldots n_d$ in Q. □

19. Other chapters

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