CHOW HOMOLOGY AND CHERN CLASSES

Contents

1. Introduction

In this chapter we discuss Chow homology groups and the construction of Chern classes of vector bundles as elements of operational Chow cohomology groups (everything with **Z**-coefficients).

We start this chapter by giving the shortest possible algebraic proof of the Key Lemma 6.3. We first define the Herbrand quotient (Section 2) and we compute it in some cases (Section 3). Next, we prove some simple algebra lemmas on existence of suitable factorizations after modifications (Section 4). Using these we construct/define the tame symbol in Section 5. Only the most basic properties of the tame symbol are needed to prove the Key Lemma, which we do in Section 6.

Next, we introduce the basic setup we work with in the rest of this chapter in Section 7. To make the material a little bit more challenging we decided to treat a somewhat more general case than is usually done. Namely we assume our schemes X are locally of finite type over a fixed locally Noetherian base scheme which is universally catenary and is endowed with a dimension function. These assumptions suffice to be able to define the Chow homology groups $\mathrm{CH}_*(X)$ and the action of capping with Chern classes on them. This is an indication that we should be able to define these also for algebraic stacks locally of finite type over such a base.

Next, we follow the first few chapters of [Ful98] in order to define cycles, flat pullback, proper pushforward, and rational equivalence, except that we have been less precise about the supports of the cycles involved.

We diverge from the presentation given in [Ful98] by using the Key lemma mentioned above to prove a basic commutativity relation in Section 27. Using this we prove that the operation of intersecting with an invertible sheaf passes through rational equivalence and is commutative, see Section 28. One more application of the Key lemma proves that the Gysin map of an effective Cartier divisor passes through rational equivalence, see Section 30. Having proved this, it is straightforward to define Chern classes of vector bundles, prove additivity, prove the splitting principle, introduce Chern characters, Todd classes, and state the Grothendieck-Riemann-Roch theorem.

There are two appendices. In Appendix A (Section 68) we discuss an alternative (longer) construction of the tame symbol and corresponding proof of the Key Lemma. Finally, in Appendix B (Section 69) we briefly discuss the relationship with K-theory of coherent sheaves and we discuss some blowup lemmas. We suggest the reader look at their introductions for more information.

We will return to the Chow groups $\mathrm{CH}_*(X)$ for smooth projective varieties over algebraically closed fields in the next chapter. Using a moving lemma as in [Sam56], [Che58a], and [Che58b] and Serre's Tor-formula (see [Ser00] or [Ser65]) we will define a ring structure on $\mathrm{CH}_*(X)$. See Intersection Theory, Section 1 ff.

2. Periodic complexes and Herbrand quotients

Of course there is a very general notion of periodic complexes. We can require periodicity of the maps, or periodicity of the objects. We will add these here as needed. For the moment we only need the following cases.

Definition 2.1. Let R be a ring.

(1) A 2-*periodic complex* over R is given by a quadruple (M, N, φ, ψ) consisting of R-modules M, N and R-module maps $\varphi : M \to N$, $\psi : N \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow \ldots$$

is a complex. In this setting we define the *cohomology modules* of the complex to be the R-modules

$$H^0(M, N, \varphi, \psi) = \operatorname{Ker}(\varphi)/\operatorname{Im}(\psi) \quad \text{and} \quad H^1(M, N, \varphi, \psi) = \operatorname{Ker}(\psi)/\operatorname{Im}(\varphi).$$

We say the 2-periodic complex is *exact* if the cohomology groups are zero.

(2) A $(2,1)$-*periodic complex* over R is given by a triple (M, φ, ψ) consisting of an R-module M and R-module maps $\varphi : M \to M$, $\psi : M \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \ldots$$

is a complex. Since this is a special case of a 2-periodic complex we have its *cohomology modules* $H^0(M, \varphi, \psi)$, $H^1(M, \varphi, \psi)$ and a notion of exactness.

In the following we will use any result proved for 2-periodic complexes without further mention for $(2,1)$-periodic complexes. It is clear that the collection of 2-periodic complexes forms a category with morphisms $(f, g) : (M, N, \varphi, \psi) \to (M', N', \varphi', \psi')$ pairs of morphisms $f : M \to M'$ and $g : N \to N'$ such that $\varphi' \circ f = g \circ \varphi$ and $\psi' \circ g = f \circ \psi$. We obtain an abelian category, with kernels and cokernels as in Homology, Lemma 13.3.

Definition 2.2. Let (M, N, φ, ψ) be a 2-periodic complex over a ring R whose cohomology modules have finite length. In this case we define the *multiplicity* of (M, N, φ, ψ) to be the integer

$$e_R(M, N, \varphi, \psi) = \text{length}_R(H^0(M, N, \varphi, \psi)) - \text{length}_R(H^1(M, N, \varphi, \psi))$$

In the case of a $(2,1)$-periodic complex (M, φ, ψ), we denote this by $e_R(M, \varphi, \psi)$ and we will sometimes call this the *(additive) Herbrand quotient*.

If the cohomology groups of (M, φ, ψ) are finite abelian groups, then it is customary to call the *(multiplicative) Herbrand quotient*

$$q(M, \varphi, \psi) = \frac{\#H^0(M, \varphi, \psi)}{\#H^1(M, \varphi, \psi)}$$

In words: the multiplicative Herbrand quotient is the number of elements of H^0 divided by the number of elements of H^1. If R is local and if the residue field of R is finite with q elements, then we see that

$$q(M, \varphi, \psi) = q^{e_R(M, \varphi, \psi)}$$

An example of a $(2,1)$-periodic complex over a ring R is any triple of the form $(M, 0, \psi)$ where M is an R-module and ψ is an R-linear map. If the kernel and cokernel of ψ have finite length, then we obtain

$$e_R(M, 0, \psi) = \text{length}_R(\operatorname{Coker}(\psi)) - \text{length}_R(\operatorname{Ker}(\psi)) \tag{2.2.1}$$

We state and prove the obligatory lemmas on these notations.

Lemma 2.3. *Let R be a ring. Suppose that we have a short exact sequence of 2-periodic complexes*

$$0 \to (M_1, N_1, \varphi_1, \psi_1) \to (M_2, N_2, \varphi_2, \psi_2) \to (M_3, N_3, \varphi_3, \psi_3) \to 0$$

If two out of three have cohomology modules of finite length so does the third and we have

$$e_R(M_2, N_2, \varphi_2, \psi_2) = e_R(M_1, N_1, \varphi_1, \psi_1) + e_R(M_3, N_3, \varphi_3, \psi_3).$$

Proof. We abbreviate $A = (M_1, N_1, \varphi_1, \psi_1)$, $B = (M_2, N_2, \varphi_2, \psi_2)$ and $C = (M_3, N_3, \varphi_3, \psi_3)$. We have a long exact cohomology sequence

$$\ldots \to H^1(C) \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to H^1(C) \to \ldots$$

This gives a finite exact sequence

$$0 \to I \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to K \to 0$$

with $0 \to K \to H^1(C) \to I \to 0$ a filtration. By additivity of the length function (Algebra, Lemma 52.3) we see the result. □

Lemma 2.4. *Let R be a ring. If (M, N, φ, ψ) is a 2-periodic complex such that M, N have finite length, then $e_R(M, N, \varphi, \psi) = length_R(M) - length_R(N)$. In particular, if (M, φ, ψ) is a $(2,1)$-periodic complex such that M has finite length, then $e_R(M, \varphi, \psi) = 0$.*

Proof. This follows from the additity of Lemma 2.3 and the short exact sequence $0 \to (M, 0, 0, 0) \to (M, N, \varphi, \psi) \to (0, N, 0, 0) \to 0$. □

Lemma 2.5. *Let R be a ring. Let $f : (M, \varphi, \psi) \to (M', \varphi', \psi')$ be a map of $(2,1)$-periodic complexes whose cohomology modules have finite length. If $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ have finite length, then $e_R(M, \varphi, \psi) = e_R(M', \varphi', \psi')$.*

Proof. Apply the additivity of Lemma 2.3 and observe that $(\mathrm{Ker}(f), \varphi, \psi)$ and $(\mathrm{Coker}(f), \varphi', \psi')$ have vanishing multiplicity by Lemma 2.4. □

3. Calculation of some multiplicities

To prove equality of certain cycles later on we need to compute some multiplicities. Our main tool, besides the elementary lemmas on multiplicities given in the previous section, will be Algebra, Lemma 121.7.

Lemma 3.1. *Let R be a Noetherian local ring. Let M be a finite R-module. Let $x \in R$. Assume that*

(1) $\dim(\mathit{Supp}(M)) \leq 1$, *and*
(2) $\dim(\mathit{Supp}(M/xM)) \leq 0$.

Write $\mathit{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$. Then

$$e_R(M, 0, x) = \sum\nolimits_{i=1,\ldots,t} ord_{R/\mathfrak{q}_i}(x) length_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

Proof. We first make some preparatory remarks. The result of the lemma holds if M has finite length, i.e., if $t = 0$, because both the left hand side and the right hand side are zero in this case, see Lemma 2.4. Also, if we have a short exact sequence $0 \to M \to M' \to M'' \to 0$ of modules satisfying (1) and (2), then lemma for 2 out of 3 of these implies the lemma for the third by the additivity of length (Algebra, Lemma 52.3) and additivty of multiplicities (Lemma 2.3).

Denote M_i the image of M in $M_{\mathfrak{q}_i}$, so $\mathrm{Supp}(M_i) = \{\mathfrak{m}, \mathfrak{q}_i\}$. The kernel and cokernel of the map $M \to \bigoplus M_i$ have support $\{\mathfrak{m}\}$ and hence have finite length. By our preparatory remarks, it follows that it suffices to prove the lemma for each M_i. Thus we may assume that $\mathrm{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$. In this case we have a finite filtration $M \supset \mathfrak{q}M \supset \mathfrak{q}^2M \supset \ldots \supset \mathfrak{q}^nM = 0$ by Algebra, Lemma 62.4. Again additivity shows that it suffices to prove the lemma in the case M is annihilated by $\mathfrak{q}$. In this case we can view M as a $R/\mathfrak{q}$-module, i.e., we may assume that R is a Noetherian local domain of dimension 1 with fraction field K. Dividing by the torsion submodule, i.e., by the kernel of $M \to M \otimes_R K = V$ (the torsion has finite length hence is handled by our preliminary remarks) we may assume that $M \subset V$ is a lattice (Algebra, Definition 121.3). Then $x : M \to M$ is injective and $\text{length}_R(M/xM) = d(M, xM)$ (Algebra, Definition 121.5). Since $\text{length}_K(V) = \dim_K(V)$ we see that $\det(x : V \to V) = x^{\dim_K(V)}$ and $\text{ord}_R(\det(x : V \to V)) = \dim_K(V)\text{ord}_R(x)$. Thus the desired equality follows from Algebra, Lemma 121.7 in this case. □

Lemma 3.2. *Let R be a Noetherian local ring. Let $x \in R$. If M is a finite Cohen-Macaulay module over R with $\dim(Supp(M)) = 1$ and $\dim(Supp(M/xM)) = 0$, then*

$$\textit{length}_R(M/xM) = \sum\nolimits_i \textit{length}_R(R/(x, \mathfrak{q}_i))\textit{length}_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the support of M. If $I \subset R$ is an ideal such that x is a nonzerodivisor on R/I and $\dim(R/I) = 1$, then

$$\textit{length}_R(R/(x, I)) = \sum\nolimits_i \textit{length}_R(R/(x, \mathfrak{q}_i))\textit{length}_{R_{\mathfrak{q}_i}}((R/I)_{\mathfrak{q}_i})$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are the minimal primes over I.

Proof. These are special cases of Lemma 3.1. □

Here is another case where we can determine the value of a multiplicity.

Lemma 3.3. *Let R be a ring. Let M be an R-module. Let $\varphi : M \to M$ be an endomorphism and $n > 0$ such that $\varphi^n = 0$ and such that $\text{Ker}(\varphi)/\text{Im}(\varphi^{n-1})$ has finite length as an R-module. Then*

$$e_R(M, \varphi^i, \varphi^{n-i}) = 0$$

for $i = 0, \ldots, n$.

Proof. The cases $i = 0, n$ are trivial as $\varphi^0 = \text{id}_M$ by convention. Let us think of M as an $R[t]$-module where multiplication by t is given by φ. Let us write $K_i = \text{Ker}(t^i : M \to M)$ and

$$a_i = \text{length}_R(K_i/t^{n-i}M), \quad b_i = \text{length}_R(K_i/tK_{i+1}), \quad c_i = \text{length}_R(K_1/t^iK_{i+1})$$

Boundary values are $a_0 = a_n = b_0 = c_0 = 0$. The c_i are integers for $i < n$ as K_1/t^iK_{i+1} is a quotient of $K_1/t^{n-1}M$ which is assumed to have finite length. We will use frequently that $K_i \cap t^jM = t^jK_{i+j}$. For $0 < i < n - 1$ we have an exact sequence

$$0 \to K_1/t^{n-i-1}K_{n-i} \to K_{i+1}/t^{n-i-1}M \xrightarrow{t} K_i/t^{n-i}M \to K_i/tK_{i+1} \to 0$$

By induction on i we conclude that a_i and b_i are integers for $i < n$ and that

$$c_{n-i-1} - a_{i+1} + a_i - b_i = 0$$

For $0 < i < n - 1$ there is a short exact sequence

$$0 \to K_i/tK_{i+1} \to K_{i+1}/tK_{i+2} \xrightarrow{t^i} K_1/t^{i+1}K_{i+2} \to K_1/t^iK_{i+1} \to 0$$

which gives

$$b_i - b_{i+1} + c_{i+1} - c_i = 0$$

Since $b_0 = c_0$ we conclude that $b_i = c_i$ for $i < n$. Then we see that

$$a_2 = a_1 + b_{n-2} - b_1, \quad a_3 = a_2 + b_{n-3} - b_2, \quad \ldots$$

It is straighforward to see that this implies $a_i = a_{n-i}$ as desired. □

Lemma 3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R with M finite and with cohomology groups of finite length over R. Let $x \in R$ be such that $\dim(Supp(M/xM)) \leq 0$. Then*

$$e_R(M, x\varphi, \psi) = e_R(M, \varphi, \psi) - e_R(\mathrm{Im}(\varphi), 0, x)$$

and

$$e_R(M, \varphi, x\psi) = e_R(M, \varphi, \psi) + e_R(\mathrm{Im}(\psi), 0, x)$$

Proof. We will only prove the first formula as the second is proved in exactly the same manner. Let $M' = M[x^\infty]$ be the x-power torsion submodule of M. Consider the short exact sequence $0 \to M' \to M \to M'' \to 0$. Then M'' is x-power torsion free (More on Algebra, Lemma 88.4). Since φ, ψ map M' into M' we obtain a short exact sequence

$$0 \to (M', \varphi', \psi') \to (M, \varphi, \psi) \to (M'', \varphi'', \psi'') \to 0$$

of $(2,1)$-periodic complexes. Also, we get a short exact sequence $0 \to M' \cap \mathrm{Im}(\varphi) \to \mathrm{Im}(\varphi) \to \mathrm{Im}(\varphi'') \to 0$. We have $e_R(M', \varphi, \psi) = e_R(M', x\varphi, \psi) = e_R(M' \cap \mathrm{Im}(\varphi), 0, x) = 0$ by Lemma 2.5. By additivity (Lemma 2.3) we see that it suffices to prove the lemma for (M'', φ'', ψ''). This reduces us to the case discussed in the next paragraph.

Assume $x : M \to M$ is injective. In this case $\mathrm{Ker}(x\varphi) = \mathrm{Ker}(\varphi)$. On the other hand we have a short exact sequence

$$0 \to \mathrm{Im}(\varphi)/x\,\mathrm{Im}(\varphi) \to \mathrm{Ker}(\psi)/\mathrm{Im}(x\varphi) \to \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi) \to 0$$

This together with (2.2.1) proves the formula. $\square$

4. Preparation for tame symbols

In this section we put some lemma that will help us define the tame symbol in the next section.

Lemma 4.1. *Let A be a Noetherian ring. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be pairwise distinct maximal ideals of A. For $i = 1, \ldots, r$ let $\varphi_i : A_{\mathfrak{m}_i} \to B_i$ be a ring map whose kernel and cokernel are annihilated by a power of $\mathfrak{m}_i$. Then there exists a ring map $\varphi : A \to B$ such that*

1. *the localization of φ at $\mathfrak{m}_i$ is isomorphic to φ_i, and*
2. *$\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are annihilated by a power of $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$.*

Moreover, if each φ_i is finite, injective, or surjective then so is φ.

Proof. Set $I = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$. Set $A_i = A_{\mathfrak{m}_i}$ and $A' = \prod A_i$. Then $IA' = \prod \mathfrak{m}_i A_i$ and $A \to A'$ is a flat ring map such that $A/I \cong A'/IA'$. Thus we may use More on Algebra, Lemma 89.16 to see that there exists an A-module map $\varphi : A \to B$ with φ_i isomorphic to the localization of φ at $\mathfrak{m}_i$. Then we can use the discussion in More on Algebra, Remark 89.19 to endow B with an A-algebra structure matching the given A-algebra structure on B_i. The final statement of the lemma follows easily from the fact that $\mathrm{Ker}(\varphi)_{\mathfrak{m}_i} \cong \mathrm{Ker}(\varphi_i)$ and $\mathrm{Coker}(\varphi)_{\mathfrak{m}_i} \cong \mathrm{Coker}(\varphi_i)$. $\square$

The following lemma is very similar to Algebra, Lemma 119.3.

Lemma 4.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $a, b \in R$ be nonzerodivisors. There exists a finite ring extension $R \subset R'$ with R'/R annihilated by a power of $\mathfrak{m}$ and nonzerodivisors $t, a', b' \in R'$ such that $a = ta'$ and $b = tb'$ and $R' = a'R' + b'R'$.*

Proof. If a or b is a unit, then the lemma is true with $R = R'$. Thus we may assume $a, b \in \mathfrak{m}$. Set $I = (a, b)$. The idea is to blow up R in I. Instead of doing the algebraic argument we work geometrically. Let $X = \text{Proj}(\bigoplus_{d \geq 0} I^d)$. By Divisors, Lemma 32.4 the morphism $X \to \text{Spec}(R)$ is an isomorphism over the punctured spectrum $U = \text{Spec}(R) \setminus \{\mathfrak{m}\}$. Thus we may and do view U as an open subscheme of X. The morphism $X \to \text{Spec}(R)$ is projective by Divisors, Lemma 32.13. Also, every generic point of X lies in U, for example by Divisors, Lemma 32.10. It follows from Varieties, Lemma 17.2 that $X \to \text{Spec}(R)$ is finite. Thus $X = \text{Spec}(R')$ is affine and $R \to R'$ is finite. We have $R_a \cong R'_a$ as $U = D(a)$. Hence a power of a annihilates the finite R-module R'/R. As $\mathfrak{m} = \sqrt{(a)}$ we see that R'/R is annihilated by a power of $\mathfrak{m}$. By Divisors, Lemma 32.4 we see that IR' is a locally principal ideal. Since R' is semi-local we see that IR' is principal, see Algebra, Lemma 78.7, say $IR' = (t)$. Then we have $a = a't$ and $b = b't$ and everything is clear. $\square$

Lemma 4.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $a, b \in R$ be nonzerodivisors with $a \in \mathfrak{m}$. There exists an integer $n = n(R, a, b)$ such that for a finite ring extension $R \subset R'$ if $b = a^m c$ for some $c \in R'$, then $m \leq n$.*

Proof. Choose a minimal prime $\mathfrak{q} \subset R$. Observe that $\dim(R/\mathfrak{q}) = 1$, in particular $R/\mathfrak{q}$ is not a field. We can choose a discrete valuation ring A dominating $R/\mathfrak{q}$ with the same fraction field, see Algebra, Lemma 119.1. Observe that a and b map to nonzero elements of A as nonzerodivisors in R are not contained in $\mathfrak{q}$. Let v be the discrete valuation on A. Then $v(a) > 0$ as $a \in \mathfrak{m}$. We claim $n = v(b)/v(a)$ works.

Let $R \subset R'$ be given. Set $A' = A \otimes_R R'$. Since $\text{Spec}(R') \to \text{Spec}(R)$ is surjective (Algebra, Lemma 36.17) also $\text{Spec}(A') \to \text{Spec}(A)$ is surjective (Algebra, Lemma 30.3). Pick a prime $\mathfrak{q}' \subset A'$ lying over $(0) \subset A$. Then $A \subset A'' = A'/\mathfrak{q}'$ is a finite extension of rings (again inducing a surjection on spectra). Pick a maximal ideal $\mathfrak{m}'' \subset A''$ lying over the maximal ideal of A and a discrete valuation ring A''' dominating $A''_{\mathfrak{m}''}$ (see lemma cited above). Then $A \to A'''$ is an extension of discrete valuation rings and we have $b = a^m c$ in A'''. Thus $v'''(b) \geq m v'''(a)$. Since $v''' = ev$ where e is the ramification index of A'''/A, we find that $m \leq n$ as desired. $\square$

Lemma 4.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $r \geq 2$ and let $a_1, \ldots, a_r \in A$ be nonzerodivisors not all units. Then there exist*

1. *a finite ring extension $A \subset B$ with B/A annihilated by a power of $\mathfrak{m}$,*
2. *for each maximal ideal $\mathfrak{m}_j \subset B$ a nonzerodivisor $\pi_j \in B_j = B_{\mathfrak{m}_j}$, and*
3. *factorizations $a_i = u_{i,j} \pi_j^{e_{i,j}}$ in B_j with $u_{i,j} \in B_j$ units and $e_{i,j} \geq 0$.*

Proof. Since at least one a_i is not a unit we find that $\mathfrak{m}$ is not an associated prime of A. Moreover, for any $A \subset B$ as in the statement $\mathfrak{m}$ is not an associated prime of B and $\mathfrak{m}_j$ is not an associate prime of B_j. Keeping this in mind will help check the arguments below.

First, we claim that it suffices to prove the lemma for $r = 2$. We will argue this by induction on r; we suggest the reader skip the proof. Suppose we are given $A \subset B$ and π_j in $B_j = B_{\mathfrak{m}_j}$ and factorizations $a_i = u_{i,j} \pi_j^{e_{i,j}}$ for $i = 1, \ldots, r - 1$ in B_j with

$u_{i,j} \in B_j$ units and $e_{i,j} \geq 0$. Then by the case $r = 2$ for π_j and a_r in B_j we can find extensions $B_j \subset C_j$ and for every maximal ideal $\mathfrak{m}_{j,k}$ of C_j a nonzerodivisor $\pi_{j,k} \in C_{j,k} = (C_j)_{\mathfrak{m}_{j,k}}$ and factorizations

$$\pi_j = v_{j,k}\pi_{j,k}^{f_{j,k}} \quad \text{and} \quad a_r = w_{j,k}\pi_{j,k}^{g_{j,k}}$$

as in the lemma. There exists a unique finite extension $B \subset C$ with C/B annihilated by a power of $\mathfrak{m}$ such that $C_j \cong C_{\mathfrak{m}_j}$ for all j, see Lemma 4.1. The maximal ideals of C correspond 1-to-1 to the maximal ideals $\mathfrak{m}_{j,k}$ in the localizations and in these localizations we have

$$a_i = u_{i,j}\pi_j^{e_{i,j}} = u_{i,j}v_{j,k}^{e_{i,j}}\pi_{j,k}^{e_{i,j}f_{j,k}}$$

for $i \leq r - 1$. Since a_r factors correctly too the proof of the induction step is complete.

Proof of the case $r = 2$. We will use induction on

$$\ell = \min(\text{length}_A(A/a_1A),\ \text{length}_A(A/a_2A)).$$

If $\ell = 0$, then either a_1 or a_2 is a unit and the lemma holds with $A = B$. Thus we may and do assume $\ell > 0$.

Suppose we have a finite extension of rings $A \subset A'$ such that A'/A is annihilated by a power of $\mathfrak{m}$ and such that $\mathfrak{m}$ is not an associated prime of A'. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r \subset A'$ be the maximal ideals and set $A'_i = A'_{\mathfrak{m}_i}$. If we can solve the problem for a_1, a_2 in each A'_i, then we can apply Lemma 4.1 to produce a solution for a_1, a_2 in A. Choose $x \in \{a_1, a_2\}$ such that $\ell = \text{length}_A(A/xA)$. By Lemma 2.5 and (2.2.1) we have $\text{length}_A(A/xA) = \text{length}_A(A'/xA')$. On the other hand, we have

$$\text{length}_A(A'/xA') = \sum [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]\text{length}_{A'_i}(A'_i/xA'_i)$$

by Algebra, Lemma 52.12. Since $x \in \mathfrak{m}$ we see that each term on the right hand side is positive. We conclude that the induction hypothesis applies to a_1, a_2 in each A'_i if $r > 1$ or if $r = 1$ and $[\kappa(\mathfrak{m}_1) : \kappa(\mathfrak{m})] > 1$. We conclude that we may assume each A' as above is local with the same residue field as A.

Applying the discussion of the previous paragraph, we may replace A by the ring constructed in Lemma 4.2 for $a_1, a_2 \in A$. Then since A is local we find, after possibly switching a_1 and a_2, that $a_2 \in (a_1)$. Write $a_2 = a_1^m c$ with $m > 0$ maximal. In fact, by Lemma 4.3 we may assume m is maximal even after replacing A by any finite extension $A \subset A'$ as in the previous paragraph. If c is a unit, then we are done. If not, then we replace A by the ring constructed in Lemma 4.2 for $a_1, c \in A$. Then either (1) $c = a_1c'$ or (2) $a_1 = ca'_1$. The first case cannot happen since it would give $a_2 = a_1^{m+1}c'$ contradicting the maximality of m. In the second case we get $a_1 = ca'_1$ and $a_2 = c^{m+1}(a'_1)^m$. Then it suffices to prove the lemma for A and c, a'_1. If a'_1 is a unit we're done and if not, then $\text{length}_A(A/cA) < \ell$ because cA is a strictly bigger ideal than a_1A. Thus we win by induction hypothesis. □

5. Tame symbols

Consider a Noetherian local ring $(A, \mathfrak{m})$ of dimension 1. We denote $Q(A)$ the total ring of fractions of A, see Algebra, Example 9.8. The *tame symbol* will be a map

$$\partial_A(-,-) : Q(A)^* \times Q(A)^* \longrightarrow \kappa(\mathfrak{m})^*$$

satisfying the following properties:

(1) $\partial_A(f, gh) = \partial_A(f, g)\partial_A(f, h)$ for $f, g, h \in Q(A)^*$,
(2) $\partial_A(f, g)\partial_A(g, f) = 1$ for $f, g \in Q(A)^*$,
(3) $\partial_A(f, 1-f) = 1$ for $f \in Q(A)^*$ such that $1 - f \in Q(A)^*$,
(4) $\partial_A(aa', b) = \partial_A(a, b)\partial_A(a', b)$ and $\partial_A(a, bb') = \partial_A(a, b)\partial_A(a, b')$ for $a, a', b, b' \in A$ nonzerodivisors,
(5) $\partial_A(b, b) = (-1)^m$ with $m = \text{length}_A(A/bA)$ for $b \in A$ a nonzerodivisor,
(6) $\partial_A(u, b) = u^m \bmod \mathfrak{m}$ with $m = \text{length}_A(A/bA)$ for $u \in A$ a unit and $b \in A$ a nonzerodivisor, and
(7) $\partial_A(a, b-a)\partial_A(b, b) = \partial_A(b, b-a)\partial_A(a, b)$ for $a, b \in A$ such that $a, b, b-a$ are nonzerodivisors.

Since it is easier to work with elements of A we will often think of ∂_A as a map defined on pairs of nonzerodivisors of A satisfying (4), (5), (6), (7). It is an exercise to see that setting

$$\partial_A(\frac{a}{b}, \frac{c}{d}) = \partial_A(a, c)\partial_A(a, d)^{-1}\partial_A(b, c)^{-1}\partial_A(b, d)$$

we get a well defined map $Q(A)^* \times Q(A)^* \to \kappa(\mathfrak{m})^*$ satisfying (1), (2), (3) as well as the other properties.

We do not claim there is a unique map with these properties. Instead, we will give a recipe for constructing such a map. Namely, given $a_1, a_2 \in A$ nonzerodivisors, we choose a ring extension $A \subset B$ and local factorizations as in Lemma 4.4. Then we define

$$\partial_A(a_1, a_2) = \prod\nolimits_j \text{Norm}_{\kappa(\mathfrak{m}_j)/\kappa(\mathfrak{m})}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak{m}_j)^{m_j} \tag{5.0.1}$$

where $m_j = \text{length}_{B_j}(B_j/\pi_j B_j)$ and the product is taken over the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of B.

Lemma 5.1. *The formula (5.0.1) determines a well defined element of $\kappa(\mathfrak{m})^*$. In other words, the right hand side does not depend on the choice of the local factorizations or the choice of B.*

Proof. Independence of choice of factorizations. Suppose we have a Noetherian 1-dimensional local ring B, elements $a_1, a_2 \in B$, and nonzerodivisors π, θ such that we can write

$$a_1 = u_1\pi^{e_1} = v_1\theta^{f_1}, \quad a_2 = u_2\pi^{e_2} = v_2\theta^{f_2}$$

with $e_i, f_i \geq 0$ integers and u_i, v_i units in B. Observe that this implies

$$a_1^{e_2} = u_1^{e_2}u_2^{-e_1}a_2^{e_1}, \quad a_1^{f_2} = v_1^{f_2}v_2^{-f_1}a_2^{f_1}$$

On the other hand, setting $m = \text{length}_B(B/\pi B)$ and $k = \text{length}_B(B/\theta B)$ we find $e_2 m = \text{length}_B(B/a_2 B) = f_2 k$. Expanding $a_1^{e_2 m} = a_1^{f_2 k}$ using the above we find

$$(u_1^{e_2}u_2^{-e_1})^m = (v_1^{f_2}v_2^{-f_1})^k$$

This proves the desired equality up to signs. To see the signs work out we have to show me_1e_2 is even if and only if kf_1f_2 is even. This follows as both $me_2 = kf_2$ and $me_1 = kf_1$ (same argument as above).

Independence of choice of B. Suppose given two extensions $A \subset B$ and $A \subset B'$ as in Lemma 4.4. Then

$$C = (B \otimes_A B')/(\mathfrak{m}\text{-power torsion})$$

will be a third one. Thus we may assume we have $A \subset B \subset C$ and factorizations over the local rings of B and we have to show that using the same factorizations over the local rings of C gives the same element of $\kappa(\mathfrak{m})$. By transitivity of norms (Fields, Lemma 20.5) this comes down to the following problem: if B is a Noetherian local ring of dimension 1 and $\pi \in B$ is a nonzerodivisor, then

$$\lambda^m = \prod \text{Norm}_{\kappa_k/\kappa}(\lambda)^{m_k}$$

Here we have used the following notation: (1) κ is the residue field of B, (2) λ is an element of κ, (3) $\mathfrak{m}_k \subset C$ are the maximal ideals of C, (4) $\kappa_k = \kappa(\mathfrak{m}_k)$ is the residue field of $C_k = C_{\mathfrak{m}_k}$, (5) $m = \text{length}_B(B/\pi B)$, and (6) $m_k = \text{length}_{C_k}(C_k/\pi C_k)$. The displayed equality holds because $\text{Norm}_{\kappa_k/\kappa}(\lambda) = \lambda^{[\kappa_k:\kappa]}$ as $\lambda \in \kappa$ and because $m = \sum m_k[\kappa_k : \kappa]$. First, we have $m = \text{length}_B(B/xB) = \text{length}_B(C/\pi C)$ by Lemma 2.5 and (2.2.1). Finally, we have $\text{length}_B(C/\pi C) = \sum m_k[\kappa_k : \kappa]$ by Algebra, Lemma 52.12. □

Lemma 5.2. *The tame symbol (5.0.1) satisfies (4), (5), (6), (7) and hence gives a map $\partial_A : Q(A)^* \times Q(A)^* \to \kappa(\mathfrak{m})^*$ satisfying (1), (2), (3).*

Proof. Let us prove (4). Let $a_1, a_2, a_3 \in A$ be nonzerodivisors. Choose $A \subset B$ as in Lemma 4.4 for a_1, a_2, a_3. Then the equality

$$\partial_A(a_1a_2, a_3) = \partial_A(a_1, a_3)\partial_A(a_2, a_3)$$

follows from the equality

$$(-1)^{(e_{1,j}+e_{2,j})e_{3,j}}(u_{1,j}u_{2,j})^{e_{3,j}}u_{3,j}^{-e_{1,j}-e_{2,j}} = (-1)^{e_{1,j}e_{3,j}}u_{1,j}^{e_{3,j}}u_{3,j}^{-e_{1,j}}(-1)^{e_{2,j}e_{3,j}}u_{2,j}^{e_{3,j}}u_{3,j}^{-e_{2,j}}$$

in B_j. Properties (5) and (6) are equally immediate.

Let us prove (7). Let $a_1, a_2, a_1 - a_2 \in A$ be nonzerodivisors and set $a_3 = a_1 - a_2$. Choose $A \subset B$ as in Lemma 4.4 for a_1, a_2, a_3. Then it suffices to show

$$(-1)^{e_{1,j}e_{2,j}+e_{1,j}e_{3,j}+e_{2,j}e_{3,j}+e_{2,j}}u_{1,j}^{e_{2,j}-e_{3,j}}u_{2,j}^{e_{3,j}-e_{1,j}}u_{3,j}^{e_{1,j}-e_{2,j}} \bmod \mathfrak{m}_j = 1$$

This is clear if $e_{1,j} = e_{2,j} = e_{3,j}$. Say $e_{1,j} > e_{2,j}$. Then we see that $e_{3,j} = e_{2,j}$ because $a_3 = a_1 - a_2$ and we see that $u_{3,j}$ has the same residue class as $-u_{2,j}$. Hence the formula is true – the signs work out as well and this verification is the reason for the choice of signs in (5.0.1). The other cases are handled in exactly the same manner. □

Lemma 5.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $A \subset B$ be a finite ring extension with B/A annihilated by a power of $\mathfrak{m}$ and $\mathfrak{m}$ not an associated prime of B. For $a, b \in A$ nonzerodivisors we have*

$$\partial_A(a, b) = \prod Norm_{\kappa(\mathfrak{m}_j)/\kappa(\mathfrak{m})}(\partial_{B_j}(a, b))$$

where the product is over the maximal ideals $\mathfrak{m}_j$ of B and $B_j = B_{\mathfrak{m}_j}$.

Proof. Choose $B_j \subset C_j$ as in Lemma 4.4 for a, b. By Lemma 4.1 we can choose a finite ring extension $B \subset C$ with $C_j \cong C_{\mathfrak{m}_j}$ for all j. Let $\mathfrak{m}_{j,k} \subset C$ be the maximal ideals of C lying over $\mathfrak{m}_j$. Let

$$a = u_{j,k}\pi_{j,k}^{f_{j,k}}, \quad b = v_{j,k}\pi_{j,k}^{g_{j,k}}$$

be the local factorizations which exist by our choice of $C_j \cong C_{\mathfrak m_j}$. By definition we have

$$\partial_A(a, b) = \prod\nolimits_{j,k} \text{Norm}_{\kappa(\mathfrak m_{j,k})/\kappa(\mathfrak m)}((-1)^{f_{j,k}g_{j,k}} u_{j,k}^{g_{j,k}} v_{j,k}^{-f_{j,k}} \bmod \mathfrak m_{j,k})^{m_{j,k}}$$

and

$$\partial_{B_j}(a, b) = \prod\nolimits_{k} \text{Norm}_{\kappa(\mathfrak m_{j,k})/\kappa(\mathfrak m_j)}((-1)^{f_{j,k}g_{j,k}} u_{j,k}^{g_{j,k}} v_{j,k}^{-f_{j,k}} \bmod \mathfrak m_{j,k})^{m_{j,k}}$$

The result follows by transitivity of norms for $\kappa(\mathfrak m_{j,k})/\kappa(\mathfrak m_j)/\kappa(\mathfrak m)$, see Fields, Lemma 20.5. □

Lemma 5.4. *Let $(A, \mathfrak m, \kappa) \to (A', \mathfrak m', \kappa')$ be a local homomorphism of Noetherian local rings. Assume $A \to A'$ is flat and $\dim(A) = \dim(A') = 1$. Set $m = \text{length}_{A'}(A'/\mathfrak m A')$. For $a_1, a_2 \in A$ nonzerodivisors $\partial_A(a_1, a_2)^m$ maps to $\partial_{A'}(a_1, a_2)$ via $\kappa \to \kappa'$.*

Proof. If a_1, a_2 are both units, then $\partial_A(a_1, a_2) = 1$ and $\partial_{A'}(a_1, a_2) = 1$ and the result is true. If not, then we can choose a ring extension $A \subset B$ and local factorizations as in Lemma 4.4. Denote $\mathfrak m_1, \ldots, \mathfrak m_m$ be the maximal ideals of B. Let $\mathfrak m_1, \ldots, \mathfrak m_m$ be the maximal ideals of B with residue fields $\kappa_1, \ldots, \kappa_m$. For each $j \in \{1, \ldots, m\}$ denote $\pi_j \in B_j = B_{\mathfrak m_j}$ a nonzerodivisor such that we have factorizations $a_i = u_{i,j}\pi_j^{e_{i,j}}$ as in the lemma. By definition we have

$$\partial_A(a_1, a_2) = \prod\nolimits_j \text{Norm}_{\kappa_j/\kappa}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak m_j)^{m_j}$$

where $m_j = \text{length}_{B_j}(B_j/\pi_j B_j)$.

Set $B' = A' \otimes_A B$. Since A' is flat over A we see that $A' \subset B'$ is a ring extension with B'/A' annihilated by a power of $\mathfrak m'$. Let

$$\mathfrak m'_{j,l}, \quad l = 1, \ldots, n_j$$

be the maximal ideals of B' lying over $\mathfrak m_j$. Denote $\kappa'_{j,l}$ the residue field of $\mathfrak m'_{j,l}$. Denote $B'_{j,l}$ the localization of B' at $\mathfrak m'_{j,l}$. As factorizations of a_1 and a_2 in $B'_{j,l}$ we use the image of the factorizations $a_i = u_{i,j}\pi_j^{e_{i,j}}$ given to us in B_j. By definition we have

$$\partial_{A'}(a_1, a_2) = \prod\nolimits_{j,l} \text{Norm}_{\kappa'_{j,l}/\kappa'}((-1)^{e_{1,j}e_{2,j}} u_{1,j}^{e_{2,j}} u_{2,j}^{-e_{1,j}} \bmod \mathfrak m'_{j,l})^{m'_{j,l}}$$

where $m'_{j,l} = \text{length}_{B'_{j,l}}(B'_{j,l}/\pi_j B'_{j,l})$.

Comparing the formulae above we see that it suffices to show that for each j and for any unit $u \in B_j$ we have

$$\left(\text{Norm}_{\kappa_j/\kappa}(u \bmod \mathfrak m_j)^{m_j}\right)^m = \prod\nolimits_l \text{Norm}_{\kappa'_{j,l}/\kappa'}(u \bmod \mathfrak m'_{j,l})^{m'_{j,l}} \tag{5.4.1}$$

in κ'. We are going to use the construction of determinants of endomorphisms of finite length modules in More on Algebra, Section 120 to prove this. Set $M = B_j/\pi_j B_j$. By More on Algebra, Lemma 120.2 we have

$$\text{Norm}_{\kappa_j/\kappa}(u \bmod \mathfrak m_j)^{m_j} = \det\nolimits_\kappa(u : M \to M)$$

Thus, by More on Algebra, Lemma 120.3, the left hand side of (5.4.1) is equal to $\det_{\kappa'}(u : M \otimes_A A' \to M \otimes_A A')$. We have an isomorphism

$$M \otimes_A A' = (B_j/\pi_j B_j) \otimes_A A' = \bigoplus\nolimits_l B'_{j,l}/\pi_j B'_{j,l}$$

of A'-modules. Setting $M'_l = B'_{j,l}/\pi_j B'_{j,l}$ we see that $\text{Norm}_{\kappa'_{j,l}/\kappa'}(u \bmod \mathfrak{m}'_{j,l})^{m'_{j,l}} = \det_{\kappa'}(u_j : M'_l \to M'_l)$ by More on Algebra, Lemma 120.2 again. Hence (5.4.1) holds by multiplicativity of the determinant construction, see More on Algebra, Lemma 120.1. □

6. A key lemma

In this section we apply the results above to prove Lemma 6.3. This lemma is a low degree case of the statement that there is a complex for Milnor K-theory similar to the Gersten-Quillen complex in Quillen's K-theory. See Remark 6.4.

Lemma 6.1. *Let $(A, \mathfrak{m})$ be a 2-dimensional Noetherian local ring. Let $t \in \mathfrak{m}$ be a nonzerodivisor. Say $V(t) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$. Let $A_{\mathfrak{q}_i} \subset B_i$ be a finite ring extension with $B_i/A_{\mathfrak{q}_i}$ annihilated by a power of t. Then there exists a finite extension $A \subset B$ of local rings identifying residue fields with $B_i \cong B_{\mathfrak{q}_i}$ and B/A annihilated by a power of t.*

Proof. Choose $n > 0$ such that $B_i \subset t^{-n}A_{\mathfrak{q}_i}$. Let $M \subset t^{-n}A$, resp. $M' \subset t^{-2n}A$ be the A-submodule consisting of elements mapping to B_i in $t^{-n}A_{\mathfrak{q}_i}$, resp. $t^{-2n}A_{\mathfrak{q}_i}$. Then $M \subset M'$ are finite A-modules as A is Noetherian and $M_{\mathfrak{q}_i} = M'_{\mathfrak{q}_i} = B_i$ as localization is exact. Thus M'/M is annihilated by $\mathfrak{m}^c$ for some $c > 0$. Observe that $M \cdot M \subset M'$ under the multiplication $t^{-n}A \times t^{-n}A \to t^{-2n}A$. Hence $B = A + \mathfrak{m}^{c+1}M$ is a finite A-algebra with the correct localizations. We omit the verification that B is local with maximal ideal $\mathfrak{m} + \mathfrak{m}^{c+1}M$. □

Lemma 6.2. *Let $(A, \mathfrak{m})$ be a 2-dimensional Noetherian local ring. Let $a, b \in A$ be nonzerodivisors. Then we have*

$$\sum ord_{A/\mathfrak{q}}(\partial_{A_\mathfrak{q}}(a, b)) = 0$$

where the sum is over the height 1 primes $\mathfrak{q}$ of A.

Proof. If $\mathfrak{q}$ is a height 1 prime of A such that a, b map to a unit of $A_\mathfrak{q}$, then $\partial_{A_\mathfrak{q}}(a, b) = 1$. Thus the sum is finite. In fact, if $V(ab) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$ then the sum is over $i = 1, \ldots, r$. For each i we pick an extension $A_{\mathfrak{q}_i} \subset B_i$ as in Lemma 4.4 for a, b. By Lemma 6.1 with $t = ab$ and the given list of primes we may assume we have a finite local extension $A \subset B$ with B/A annihilated by a power of ab and such that for each i the $B_{\mathfrak{q}_i} \cong B_i$. Observe that if $\mathfrak{q}_{i,j}$ are the primes of B lying over $\mathfrak{q}_i$ then we have

$$\text{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(a, b)) = \sum\nolimits_j \text{ord}_{B/\mathfrak{q}_{i,j}}(\partial_{B_{\mathfrak{q}_{i,j}}}(a, b))$$

by Lemma 5.3 and Algebra, Lemma 121.8. Thus we may replace A by B and reduce to the case discussed in the next paragraph.

Assume for each i there is a nonzerodivisor $\pi_i \in A_{\mathfrak{q}_i}$ and units $u_i, v_i \in A_{\mathfrak{q}_i}$ such that for some integers $e_i, f_i \geq 0$ we have

$$a = u_i \pi_i^{e_i}, \quad b = v_i \pi_i^{f_i}$$

in $A_{\mathfrak{q}_i}$. Setting $m_i = \text{length}_{A_{\mathfrak{q}_i}}(A_{\mathfrak{q}_i}/\pi_i)$ we have $\partial_{A_{\mathfrak{q}_i}}(a, b) = ((-1)^{e_i f_i} u_i^{f_i} v_i^{-e_i})^{m_i}$ by definition. Since a, b are nonzerodivisors the $(2, 1)$-periodic complex $(A/(ab), a, b)$ has vanishing cohomology. Denote M_i the image of $A/(ab)$ in $A_{\mathfrak{q}_i}/(ab)$. Then we have a map

$$A/(ab) \longrightarrow \bigoplus M_i$$

whose kernel and cokernel are supported in $\{\mathfrak{m}\}$ and hence have finite length. Thus we see that

$$\sum e_A(M_i, a, b) = 0$$

by Lemma 2.5. Hence it suffices to show $e_A(M_i, a, b) = -\text{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(a, b))$.

Let us prove this first, in case π_i, u_i, v_i are the images of elements $\pi_i, u_i, v_i \in A$ (using the same symbols should not cause any confusion). In this case we get

$$\begin{aligned} e_A(M_i, a, b) &= e_A(M_i, u_i\pi_i^{e_i}, v_i\pi_i^{f_i}) \\ &= e_A(M_i, \pi_i^{e_i}, \pi_i^{f_i}) - e_A(\pi_i^{e_i}M_i, 0, u_i) + e_A(\pi_i^{f_i}M_i, 0, v_i) \\ &= 0 - f_i m_i \text{ord}_{A/\mathfrak{q}_i}(u_i) + e_i m_i \text{ord}_{A/\mathfrak{q}_i}(v_i) \\ &= -m_i \text{ord}_{A/\mathfrak{q}_i}(u_i^{f_i} v_i^{-e_i}) = -\text{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(a, b)) \end{aligned}$$

The second equality holds by Lemma 3.4. Observe that $M_i \subset (M_i)_{\mathfrak{q}_i} = A_{\mathfrak{q}_i}/(\pi_i^{e_i+f_i})$ and $(\pi_i^{e_i}M_i)_{\mathfrak{q}_i} \cong A_{\mathfrak{q}_i}/\pi_i^{f_i}$ and $(\pi_i^{f_i}M_i)_{\mathfrak{q}_i} \cong A_{\mathfrak{q}_i}/\pi_i^{e_i}$. The 0 in the third equality comes from Lemma 3.3 and the other two terms come from Lemma 3.1. The last two equalities follow from multiplicativity of the order function and from the definition of our tame symbol.

In general, we may first choose $c \in A$, $c \notin \mathfrak{q}_i$ such that $c\pi_i \in A$. After replacing π_i by $c\pi_i$ and u_i by $c^{-e_i}u_i$ and v_i by $c^{-f_i}v_i$ we may and do assume π_i is in A. Next, choose an $c \in A$, $c \notin \mathfrak{q}_i$ with $cu_i, cv_i \in A$. Then we observe that

$$e_A(M_i, ca, cb) = e_A(M_i, a, b) - e_A(aM_i, 0, c) + e_A(bM_i, 0, c)$$

by Lemma 3.1. On the other hand, we have

$$\partial_{A_{\mathfrak{q}_i}}(ca, cb) = c^{m_i(f_i - e_i)}\partial_{A_{\mathfrak{q}_i}}(a, b)$$

in $\kappa(\mathfrak{q}_i)^*$ because c is a unit in $A_{\mathfrak{q}_i}$. The arguments in the previous paragraph show that $e_A(M_i, ca, cb) = -\text{ord}_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(ca, cb))$. Thus it suffices to prove

$$e_A(aM_i, 0, c) = \text{ord}_{A/\mathfrak{q}_i}(c^{m_i f_i}) \quad \text{and} \quad e_A(bM_i, 0, c) = \text{ord}_{A/\mathfrak{q}_i}(c^{m_i e_i})$$

and this follows from Lemma 3.1 by the description (see above) of what happens when we localize at $\mathfrak{q}_i$. ☐

Lemma 6.3 (Key Lemma)**.** *Let A be a 2-dimensional Noetherian local domain with fraction field K. Let $f, g \in K^*$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the height 1 primes $\mathfrak{q}$ of A such that either f or g is not an element of $A_{\mathfrak{q}}^*$. Then we have*

$$\sum\nolimits_{i=1,\ldots,t} ord_{A/\mathfrak{q}_i}(\partial_{A_{\mathfrak{q}_i}}(f, g)) = 0$$

We can also write this as

$$\sum\nolimits_{height(\mathfrak{q})=1} ord_{A/\mathfrak{q}}(\partial_{A_{\mathfrak{q}}}(f, g)) = 0$$

since at any height 1 prime $\mathfrak{q}$ of A where $f, g \in A_{\mathfrak{q}}^*$ we have $\partial_{A_{\mathfrak{q}}}(f, g) = 1$.

Proof. Since the tame symbols $\partial_{A_{\mathfrak{q}}}(f, g)$ are bilinear and the order functions $\text{ord}_{A/\mathfrak{q}}$ are additive it suffices to prove the formula when f and g are elements of A. This case is proven in Lemma 6.2. ☐

Remark 6.4 (Milnor K-theory)**.** For a field k let us denote $K_*^M(k)$ the quotient of the tensor algebra on k^* divided by the two-sided ideal generated by the elements $x \otimes 1 - x$ for $x \in k \setminus \{0, 1\}$. Thus $K_0^M(k) = \mathbf{Z}$, $K_1^M(k) = k^*$, and

$$K_2^M(k) = k^* \otimes_{\mathbf{Z}} k^*/\langle x \otimes 1 - x\rangle$$

If A is a discrete valuation ring with fraction field $F = \text{Frac}(A)$ and residue field κ, there is a tame symbol

$$\partial_A : K_{i+1}^M(F) \to K_i^M(\kappa)$$

defined as in Section 5; see [Kat86]. More generally, this map can be extended to the case where A is an excellent local domain of dimension 1 using normalization and norm maps on K_i^M, see [Kat86]; presumably the method in Section 5 can be used to extend the construction of the tame symbol ∂_A to arbitrary Noetherian local domains A of dimension 1. Next, let X be a Noetherian scheme with a dimension function δ. Then we can use these tame symbols to get the arrows in the following:

$$\bigoplus\nolimits_{\delta(x)=j+1} K_{i+1}^M(\kappa(x)) \longrightarrow \bigoplus\nolimits_{\delta(x)=j} K_i^M(\kappa(x)) \longrightarrow \bigoplus\nolimits_{\delta(x)=j-1} K_{i-1}^M(\kappa(x))$$

However, it is not clear, that the composition is zero, i.e., that we obtain a complex of abelian groups. For excellent X this is shown in [Kat86]. When $i = 1$ and j arbitrary, this follows from Lemma 6.3.

7. Setup

We will throughout work over a locally Noetherian universally catenary base S endowed with a dimension function δ. Although it is likely possible to generalize (parts of) the discussion in the chapter, it seems that this is a good first approximation. It is exactly the generality discussed in [Tho90]. We usually do not assume our schemes are separated or quasi-compact. Many interesting algebraic stacks are non-separated and/or non-quasi-compact and this is a good case study to see how to develop a reasonable theory for those as well. In order to reference these hypotheses we give it a number.

Situation 7.1. Here S is a locally Noetherian, and universally catenary scheme. Moreover, we assume S is endowed with a dimension function $\delta : S \longrightarrow \mathbf{Z}$.

See Morphisms, Definition 17.1 for the notion of a universally catenary scheme, and see Topology, Definition 20.1 for the notion of a dimension function. Recall that any locally Noetherian catenary scheme locally has a dimension function, see Properties, Lemma 11.3. Moreover, there are lots of schemes which are universally catenary, see Morphisms, Lemma 17.5.

Let (S, δ) be as in Situation 7.1. Any scheme X locally of finite type over S is locally Noetherian and catenary. In fact, X has a canonical dimension function

$$\delta = \delta_{X/S} : X \longrightarrow \mathbf{Z}$$

associated to $(f : X \to S, \delta)$ given by the rule $\delta_{X/S}(x) = \delta(f(x)) + \text{trdeg}_{\kappa(f(x))}\kappa(x)$. See Morphisms, Lemma 52.3. Moreover, if $h : X \to Y$ is a morphism of schemes locally of finite type over S, and $x \in X$, $y = h(x)$, then obviously $\delta_{X/S}(x) = \delta_{Y/S}(y) + \text{trdeg}_{\kappa(y)}\kappa(x)$. We will freely use this function and its properties in the following.

Here are the basic examples of setups as above. In fact, the main interest lies in the case where the base is the spectrum of a field, or the case where the base is the spectrum of a Dedekind ring (e.g. $\mathbf{Z}$, or a discrete valuation ring).

Example 7.2. Here $S = \mathrm{Spec}(k)$ and k is a field. We set $\delta(pt) = 0$ where pt indicates the unique point of S. The pair (S, δ) is an example of a situation as in Situation 7.1 by Morphisms, Lemma 17.5.

Example 7.3. Here $S = \mathrm{Spec}(A)$, where A is a Noetherian domain of dimension 1. For example we could consider $A = \mathbf{Z}$. We set $\delta(\mathfrak p) = 0$ if $\mathfrak p$ is a maximal ideal and $\delta(\mathfrak p) = 1$ if $\mathfrak p = (0)$ corresponds to the generic point. This is an example of Situation 7.1 by Morphisms, Lemma 17.5.

Example 7.4. Here S is a Cohen-Macaulay scheme. Then S is universally catenary by Morphisms, Lemma 17.5. We set $\delta(s) = -\dim(\mathcal{O}_{S,s})$. If $s' \leadsto s$ is a nontrivial specialization of points of S, then $\mathcal{O}_{S,s'}$ is the localization of $\mathcal{O}_{S,s}$ at a nonmaximal prime ideal $\mathfrak p \subset \mathcal{O}_{S,s}$, see Schemes, Lemma 13.2. Thus $\dim(\mathcal{O}_{S,s}) = \dim(\mathcal{O}_{S,s'}) + \dim(\mathcal{O}_{S,s}/\mathfrak p) > \dim(\mathcal{O}_{S,s'})$ by Algebra, Lemma 104.4. Hence $\delta(s') > \delta(s)$. If $s' \leadsto s$ is an immediate specialization, then there is no prime ideal strictly between $\mathfrak p$ and $\mathfrak m_s$ and we find $\delta(s') = \delta(s) + 1$. Thus δ is a dimension function. In other words, the pair (S, δ) is an example of Situation 7.1.

If S is Jacobson and δ sends closed points to zero, then δ is the function sending a point to the dimension of its closure.

Lemma 7.5. *Let (S, δ) be as in Situation 7.1. Assume in addition S is a Jacobson scheme, and $\delta(s) = 0$ for every closed point s of S. Let X be locally of finite type over S. Let $Z \subset X$ be an integral closed subscheme and let $\xi \in Z$ be its generic point. The following integers are the same:*

1. $\delta_{X/S}(\xi)$,
2. $\dim(Z)$, *and*
3. $\dim(\mathcal{O}_{Z,z})$ *where z is a closed point of Z.*

Proof. Let $X \to S$, $\xi \in Z \subset X$ be as in the lemma. Since X is locally of finite type over S we see that X is Jacobson, see Morphisms, Lemma 16.9. Hence closed points of X are dense in every closed subset of Z and map to closed points of S. Hence given any chain of irreducible closed subsets of Z we can end it with a closed point of Z. It follows that $\dim(Z) = \sup_z(\dim(\mathcal{O}_{Z,z})$ (see Properties, Lemma 10.3) where $z \in Z$ runs over the closed points of Z. Note that $\dim(\mathcal{O}_{Z,z}) = \delta(\xi) - \delta(z)$ by the properties of a dimension function. For each closed $z \in Z$ the field extension $\kappa(z)/\kappa(f(z))$ is finite, see Morphisms, Lemma 16.8. Hence $\delta_{X/S}(z) = \delta(f(z)) = 0$ for $z \in Z$ closed. It follows that all three integers are equal. □

In the situation of the lemma above the value of δ at the generic point of a closed irreducible subset is the dimension of the irreducible closed subset. However, in general we cannot expect the equality to hold. For example if $S = \mathrm{Spec}(\mathbf{C}[[t]])$ and $X = \mathrm{Spec}(\mathbf{C}((t)))$ then we would get $\delta(x) = 1$ for the unique point of X, but $\dim(X) = 0$. Still we want to think of $\delta_{X/S}$ as giving the dimension of the irreducible closed subschemes. Thus we introduce the following terminology.

Definition 7.6. Let (S, δ) as in Situation 7.1. For any scheme X locally of finite type over S and any irreducible closed subset $Z \subset X$ we define

$$\dim_\delta(Z) = \delta(\xi)$$

where $\xi \in Z$ is the generic point of Z. We will call this the *δ-dimension of Z*. If Z is a closed subscheme of X, then we define $\dim_\delta(Z)$ as the supremum of the δ-dimensions of its irreducible components.

8. Cycles

Since we are not assuming our schemes are quasi-compact we have to be a little careful when defining cycles. We have to allow infinite sums because a rational function may have infinitely many poles for example. In any case, if X is quasi-compact then a cycle is a finite sum as usual.

Definition 8.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) A *cycle on X* is a formal sum
$$\alpha = \sum n_Z[Z]$$
where the sum is over integral closed subschemes $Z \subset X$, each $n_Z \in \mathbf{Z}$, and the collection $\{Z; n_Z \neq 0\}$ is locally finite (Topology, Definition 28.4).

(2) A *k-cycle* on X is a cycle
$$\alpha = \sum n_Z[Z]$$
where $n_Z \neq 0 \Rightarrow \dim_\delta(Z) = k$.

(3) The abelian group of all k-cycles on X is denoted $Z_k(X)$.

In other words, a k-cycle on X is a locally finite formal $\mathbf{Z}$-linear combination of integral closed subschemes of δ-dimension k. Addition of k-cycles $\alpha = \sum n_Z[Z]$ and $\beta = \sum m_Z[Z]$ is given by
$$\alpha + \beta = \sum (n_Z + m_Z)[Z],$$
i.e., by adding the coefficients.

Remark 8.2. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. Then we can write
$$Z_k(X) = \bigoplus\nolimits'_{\delta(x)=k} K_0^M(\kappa(x)) \quad \subset \quad \bigoplus\nolimits_{\delta(x)=k} K_0^M(\kappa(x))$$
with the following notation and conventions:

(1) $K_0^M(\kappa(x)) = \mathbf{Z}$ is the degree 0 part of the Milnor K-theory of the residue field $\kappa(x)$ of the point $x \in X$ (see Remark 6.4), and
(2) the direct sum on the right is over all points $x \in X$ with $\delta(x) = k$,
(3) the notation $\bigoplus'_x$ signifies that we consider the subgroup consisting of locally finite elements; namely, elements $\sum_x n_x$ such that for every quasi-compact open $U \subset X$ the set of $x \in U$ with $n_x \neq 0$ is finite.

Definition 8.3. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. The *support* of a cycle $\alpha = \sum n_Z[Z]$ on X is
$$\text{Supp}(\alpha) = \bigcup\nolimits_{n_Z \neq 0} Z \subset X$$

Since the collection $\{Z; n_Z \neq 0\}$ is locally finite we see that $\text{Supp}(\alpha)$ is a closed subset of X. If α is a k-cycle, then every irreducible component Z of $\text{Supp}(\alpha)$ has δ-dimension k.

Definition 8.4. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. A cycle α on X is *effective* if it can be written as $\alpha = \sum n_Z[Z]$ with $n_Z \geq 0$ for all Z.

The set of all effective cycles is a monoid because the sum of two effective cycles is effective, but it is not a group (unless $X = \emptyset$).

9. Cycle associated to a closed subscheme

Lemma 9.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.*

1. *Let $Z' \subset Z$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then*
$$\text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi} < \infty$$
2. *If $\dim_\delta(Z) \leq k$ and $\xi \in Z$ with $\delta(\xi) = k$, then ξ is a generic point of an irreducible component of Z.*

Proof. Let $Z' \subset Z$, $\xi \in Z'$ be as in (1). Then $\dim(\mathcal{O}_{Z,\xi}) = 0$ (for example by Properties, Lemma 10.3). Hence $\mathcal{O}_{Z,\xi}$ is Noetherian local ring of dimension zero, and hence has finite length over itself (see Algebra, Proposition 60.7). Hence, it also has finite length over $\mathcal{O}_{X,\xi}$, see Algebra, Lemma 52.5.

Assume $\xi \in Z$ and $\delta(\xi) = k$. Consider the closure $Z' = \overline{\{\xi\}}$. It is an irreducible closed subscheme with $\dim_\delta(Z') = k$ by definition. Since $\dim_\delta(Z) = k$ it must be an irreducible component of Z. Hence we see (2) holds. □

Definition 9.2. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.

1. For any irreducible component $Z' \subset Z$ with generic point ξ the integer $m_{Z',Z} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi}$ (Lemma 9.1) is called the *multiplicity of Z' in Z*.
2. Assume $\dim_\delta(Z) \leq k$. The *k-cycle associated to Z* is
$$[Z]_k = \sum m_{Z',Z}[Z']$$
where the sum is over the irreducible components of Z of δ-dimension k. (This is a k-cycle by Divisors, Lemma 26.1.)

It is important to note that we only define $[Z]_k$ if the δ-dimension of Z does not exceed k. In other words, by convention, if we write $[Z]_k$ then this implies that $\dim_\delta(Z) \leq k$.

10. Cycle associated to a coherent sheaf

Lemma 10.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

1. *The collection of irreducible components of the support of $\mathcal{F}$ is locally finite.*
2. *Let $Z' \subset Supp(\mathcal{F})$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then*
$$\text{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi < \infty$$
3. *If $\dim_\delta(Supp(\mathcal{F})) \leq k$ and $\xi \in Z$ with $\delta(\xi) = k$, then ξ is a generic point of an irreducible component of $Supp(\mathcal{F})$.*

Proof. By Cohomology of Schemes, Lemma 9.7 the support Z of $\mathcal{F}$ is a closed subset of X. We may think of Z as a reduced closed subscheme of X (Schemes, Lemma 12.4). Hence (1) follows from Divisors, Lemma 26.1 applied to Z and (3) follows from Lemma 9.1 applied to Z.

Let $\xi \in Z'$ be as in (2). In this case for any specialization $\xi' \rightsquigarrow \xi$ in X we have $\mathcal{F}_{\xi'} = 0$. Recall that the non-maximal primes of $\mathcal{O}_{X,\xi}$ correspond to the points of X specializing to ξ (Schemes, Lemma 13.2). Hence $\mathcal{F}_\xi$ is a finite $\mathcal{O}_{X,\xi}$-module whose support is $\{\mathfrak{m}_\xi\}$. Hence it has finite length by Algebra, Lemma 62.3. $\square$

Definition 10.2. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

(1) For any irreducible component $Z' \subset \text{Supp}(\mathcal{F})$ with generic point ξ the integer $m_{Z',\mathcal{F}} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi$ (Lemma 10.1) is called the *multiplicity of Z' in $\mathcal{F}$*.
(2) Assume $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. The *$k$-cycle associated to $\mathcal{F}$* is
$$[\mathcal{F}]_k = \sum m_{Z',\mathcal{F}}[Z']$$
where the sum is over the irreducible components of $\text{Supp}(\mathcal{F})$ of δ-dimension k. (This is a k-cycle by Lemma 10.1.)

It is important to note that we only define $[\mathcal{F}]_k$ if $\mathcal{F}$ is coherent and the δ-dimension of $\text{Supp}(\mathcal{F})$ does not exceed k. In other words, by convention, if we write $[\mathcal{F}]_k$ then this implies that $\mathcal{F}$ is coherent on X and $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$.

Lemma 10.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme. If $\dim_\delta(Z) \leq k$, then $[Z]_k = [\mathcal{O}_Z]_k$.*

Proof. This is because in this case the multiplicities $m_{Z',Z}$ and $m_{Z',\mathcal{O}_Z}$ agree by definition. $\square$

Lemma 10.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of coherent sheaves on X. Assume that the δ-dimension of the supports of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ is $\leq k$. Then $[\mathcal{G}]_k = [\mathcal{F}]_k + [\mathcal{H}]_k$.*

Proof. Follows immediately from additivity of lengths, see Algebra, Lemma 52.3. $\square$

11. Preparation for proper pushforward

Lemma 11.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume X, Y integral and $\dim_\delta(X) = \dim_\delta(Y)$. Then either $f(X)$ is contained in a proper closed subscheme of Y, or f is dominant and the extension of function fields $R(X)/R(Y)$ is finite.*

Proof. The closure $\overline{f(X)} \subset Y$ is irreducible as X is irreducible (Topology, Lemmas 8.2 and 8.3). If $\overline{f(X)} \neq Y$, then we are done. If $\overline{f(X)} = Y$, then f is dominant and by Morphisms, Lemma 8.6 we see that the generic point η_Y of Y is in the image of f. Of course this implies that $f(\eta_X) = \eta_Y$, where $\eta_X \in X$ is the generic point of X. Since $\delta(\eta_X) = \delta(\eta_Y)$ we see that $R(Y) = \kappa(\eta_Y) \subset \kappa(\eta_X) = R(X)$ is an extension of transcendence degree 0. Hence $R(Y) \subset R(X)$ is a finite extension by Morphisms, Lemma 51.7 (which applies by Morphisms, Lemma 15.8). $\square$

Lemma 11.2. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is quasi-compact, and $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of X. Then $\{\overline{f(Z_i)}\}_{i \in I}$ is a locally finite collection of closed subsets of Y.*

Proof. Let $V \subset Y$ be a quasi-compact open subset. Since f is quasi-compact the open $f^{-1}(V)$ is quasi-compact. Hence the set $\{i \in I \mid Z_i \cap f^{-1}(V) \neq \emptyset\}$ is finite by a simple topological argument which we omit. Since this is the same as the set

$$\{i \in I \mid f(Z_i) \cap V \neq \emptyset\} = \{i \in I \mid \overline{f(Z_i)} \cap V \neq \emptyset\}$$

the lemma is proved. $\square$

12. Proper pushforward

Definition 12.1. Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is proper.

(1) Let $Z \subset X$ be an integral closed subscheme with $\dim_\delta(Z) = k$. We define

$$f_*[Z] = \begin{cases} 0 & \text{if} \quad \dim_\delta(f(Z)) < k, \\ \deg(Z/f(Z))[f(Z)] & \text{if} \quad \dim_\delta(f(Z)) = k. \end{cases}$$

Here we think of $f(Z) \subset Y$ as an integral closed subscheme. The degree of Z over $f(Z)$ is finite if $\dim_\delta(f(Z)) = \dim_\delta(Z)$ by Lemma 11.1.

(2) Let $\alpha = \sum n_Z[Z]$ be a k-cycle on X. The *pushforward* of α as the sum

$$f_*\alpha = \sum n_Z f_*[Z]$$

where each $f_*[Z]$ is defined as above. The sum is locally finite by Lemma 11.2 above.

By definition the proper pushforward of cycles

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

is a homomorphism of abelian groups. It turns $X \mapsto Z_k(X)$ into a covariant functor on the category of schemes locally of finite type over S with morphisms equal to proper morphisms.

Lemma 12.2. *Let (S, δ) be as in Situation 7.1. Let X, Y, and Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms. Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \to Z_k(Z)$.*

Proof. Let $W \subset X$ be an integral closed subscheme of dimension k. Consider $W' = f(W) \subset Y$ and $W'' = g(f(W)) \subset Z$. Since f, g are proper we see that W' (resp. W'') is an integral closed subscheme of Y (resp. Z). We have to show that $g_*(f_*[W]) = (g \circ f)_*[W]$. If $\dim_\delta(W'') < k$, then both sides are zero. If $\dim_\delta(W'') = k$, then we see the induced morphisms

$$W \longrightarrow W' \longrightarrow W''$$

both satisfy the hypotheses of Lemma 11.1. Hence

$$g_*(f_*[W]) = \deg(W/W') \deg(W'/W'')[W''], \quad (g \circ f)_*[W] = \deg(W/W'')[W''].$$

Then we can apply Morphisms, Lemma 51.9 to conclude. $\square$

A closed immersion is proper. If $i : Z \to X$ is a closed immersion then the maps

$$i_* : Z_k(Z) \longrightarrow Z_k(X)$$

are all *injective*.

Lemma 12.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $X_1, X_2 \subset X$ be closed subschemes such that $X = X_1 \cup X_2$ set theoretically. For every $k \in \mathbf{Z}$ the sequence of abelian groups*

$$Z_k(X_1 \cap X_2) \longrightarrow Z_k(X_1) \oplus Z_k(X_2) \longrightarrow Z_k(X) \longrightarrow 0$$

is exact. Here $X_1 \cap X_2$ is the scheme theoretic intersection and the maps are the pushforward maps with one multiplied by -1.

Proof. First assume X is quasi-compact. Then $Z_k(X)$ is a free $\mathbf{Z}$-module with basis given by the elements $[Z]$ where $Z \subset X$ is integral closed of δ-dimension k. The groups $Z_k(X_1)$, $Z_k(X_2)$, $Z_k(X_1 \cap X_2)$ are free on the subset of these Z such that $Z \subset X_1$, $Z \subset X_2$, $Z \subset X_1 \cap X_2$. This immediately proves the lemma in this case. The general case is similar and the proof is omitted. □

Lemma 12.4. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a proper morphism of schemes which are locally of finite type over S.*

(1) *Let $Z \subset X$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then*
$$f_*[Z]_k = [f_*\mathcal{O}_Z]_k.$$
(2) *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then*
$$f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k.$$

Note that the statement makes sense since $f_*\mathcal{F}$ and $f_*\mathcal{O}_Z$ are coherent $\mathcal{O}_Y$-modules by Cohomology of Schemes, Proposition 19.1.

Proof. Part (1) follows from (2) and Lemma 10.3. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$. By Cohomology of Schemes, Lemma 9.7 there exists a closed subscheme $i : Z \to X$ and a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} \cong \mathcal{F}$ and such that the support of $\mathcal{F}$ is Z. Let $Z' \subset Y$ be the scheme theoretic image of $f|_Z : Z \to Y$. Consider the commutative diagram of schemes

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f|_Z"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "i'"] & Y \end{tikzcd}$$

We have $f_*\mathcal{F} = f_*i_*\mathcal{G} = i'_*(f|_Z)_*\mathcal{G}$ by going around the diagram in two ways. Suppose we know the result holds for closed immersions and for $f|_Z$. Then we see that

$$f_*[\mathcal{F}]_k = f_*i_*[\mathcal{G}]_k = (i')_*(f|_Z)_*[\mathcal{G}]_k = (i')_*[(f|_Z)_*\mathcal{G}]_k = [(i')_*(f|_Z)_*\mathcal{G}]_k = [f_*\mathcal{F}]_k$$

as desired. The case of a closed immersion is straightforward (omitted). Note that $f|_Z : Z \to Z'$ is a dominant morphism (see Morphisms, Lemma 6.3). Thus we have reduced to the case where $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant.

Assume $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant. Since f is dominant, for every irreducible component $Z \subset Y$ with generic point η there exists a point

$\xi \in X$ such that $f(\xi) = \eta$. Hence $\delta(\eta) \leq \delta(\xi) \leq k$. Thus we see that in the expressions

$$f_*[\mathcal{F}]_k = \sum n_Z[Z], \quad \text{and} \quad [f_*\mathcal{F}]_k = \sum m_Z[Z].$$

whenever $n_Z \neq 0$, or $m_Z \neq 0$ the integral closed subscheme Z is actually an irreducible component of Y of δ-dimension k. Pick such an integral closed subscheme $Z \subset Y$ and denote η its generic point. Note that for any $\xi \in X$ with $f(\xi) = \eta$ we have $\delta(\xi) \geq k$ and hence ξ is a generic point of an irreducible component of X of δ-dimension k as well (see Lemma 9.1). Since f is quasi-compact and X is locally Noetherian, there can be only finitely many of these and hence $f^{-1}(\{\eta\})$ is finite. By Morphisms, Lemma 51.1 there exists an open neighbourhood $\eta \in V \subset Y$ such that $f^{-1}(V) \to V$ is finite. Replacing Y by V and X by $f^{-1}(V)$ we reduce to the case where Y is affine, and f is finite.

Write $Y = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ (possible as a finite morphism is affine). Then R and A are Noetherian rings and A is finite over R. Moreover $\mathcal{F} = \widetilde{M}$ for some finite A-module M. Note that $f_*\mathcal{F}$ corresponds to M viewed as an R-module. Let $\mathfrak{p} \subset R$ be the minimal prime corresponding to $\eta \in Y$. The coefficient of Z in $[f_*\mathcal{F}]_k$ is clearly $\text{length}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Let $\mathfrak{q}_i$, $i = 1, \ldots, t$ be the primes of A lying over $\mathfrak{p}$. Then $A_\mathfrak{p} = \prod A_{\mathfrak{q}_i}$ since $A_\mathfrak{p}$ is an Artinian ring being finite over the dimension zero local Noetherian ring $R_\mathfrak{p}$. Clearly the coefficient of Z in $f_*[\mathcal{F}]_k$ is

$$\sum\nolimits_{i=1,\ldots,t} [\kappa(\mathfrak{q}_i) : \kappa(\mathfrak{p})] \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$$

Hence the desired equality follows from Algebra, Lemma 52.12. □

13. Preparation for flat pullback

Recall that a morphism $f : X \to Y$ which is locally of finite type is said to have relative dimension r if every nonempty fibre is equidimensional of dimension r. See Morphisms, Definition 29.1.

Lemma 13.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r. For any closed subset $Z \subset Y$ we have*

$$\dim_\delta(f^{-1}(Z)) = \dim_\delta(Z) + r.$$

provided $f^{-1}(Z)$ is nonempty. If Z is irreducible and $Z' \subset f^{-1}(Z)$ is an irreducible component, then Z' dominates Z and $\dim_\delta(Z') = \dim_\delta(Z) + r$.

Proof. It suffices to prove the final statement. We may replace Y by the integral closed subscheme Z and X by the scheme theoretic inverse image $f^{-1}(Z) = Z \times_Y X$. Hence we may assume $Z = Y$ is integral and f is a flat morphism of relative dimension r. Since Y is locally Noetherian the morphism f which is locally of finite type, is actually locally of finite presentation. Hence Morphisms, Lemma 25.10 applies and we see that f is open. Let $\xi \in X$ be a generic point of an irreducible component of X. By the openness of f we see that $f(\xi)$ is the generic point η of $Z = Y$. Note that $\dim_\xi(X_\eta) = r$ by assumption that f has relative dimension r. On the other hand, since ξ is a generic point of X we see that $\mathcal{O}_{X,\xi} = \mathcal{O}_{X_\eta,\xi}$ has only one prime ideal and hence has dimension 0. Thus by Morphisms, Lemma 28.1 we conclude that the transcendence degree of $\kappa(\xi)$ over $\kappa(\eta)$ is r. In other words, $\delta(\xi) = \delta(\eta) + r$ as desired. □

Here is the lemma that we will use to prove that the flat pullback of a locally finite collection of closed subschemes is locally finite.

Lemma 13.2. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of Y. Then $\{f^{-1}(Z_i)\}_{i \in I}$ is a locally finite collection of closed subsets of X.*

Proof. Let $U \subset X$ be a quasi-compact open subset. Since the image $f(U) \subset Y$ is a quasi-compact subset there exists a quasi-compact open $V \subset Y$ such that $f(U) \subset V$. Note that

$$\{i \in I \mid f^{-1}(Z_i) \cap U \neq \emptyset\} \subset \{i \in I \mid Z_i \cap V \neq \emptyset\}.$$

Since the right hand side is finite by assumption we win. □

14. Flat pullback

In the following we use $f^{-1}(Z)$ to denote the *scheme theoretic inverse image* of a closed subscheme $Z \subset Y$ for a morphism of schemes $f : X \to Y$. We recall that the scheme theoretic inverse image is the fibre product

$$\begin{tikzcd} f^{-1}(Z) \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

and it is also the closed subscheme of X cut out by the quasi-coherent sheaf of ideals $f^{-1}(\mathcal{I})\mathcal{O}_X$, if $\mathcal{I} \subset \mathcal{O}_Y$ is the quasi-coherent sheaf of ideals corresponding to Z in Y. (This is discussed in Schemes, Section 4 and Lemma 17.6 and Definition 17.7.)

Definition 14.1. Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r.

(1) Let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. We define $f^*[Z]$ to be the $(k+r)$-cycle on X to the scheme theoretic inverse image
$$f^*[Z] = [f^{-1}(Z)]_{k+r}.$$
This makes sense since $\dim_\delta(f^{-1}(Z)) = k + r$ by Lemma 13.1.

(2) Let $\alpha = \sum n_i[Z_i]$ be a k-cycle on Y. The *flat pullback of α by f* is the sum
$$f^*\alpha = \sum n_i f^*[Z_i]$$
where each $f^*[Z_i]$ is defined as above. The sum is locally finite by Lemma 13.2.

(3) We denote $f^* : Z_k(Y) \to Z_{k+r}(X)$ the map of abelian groups so obtained.

An open immersion is flat. This is an important though trivial special case of a flat morphism. If $U \subset X$ is open then sometimes the pullback by $j : U \to X$ of a cycle is called the *restriction* of the cycle to U. Note that in this case the maps

$$j^* : Z_k(X) \longrightarrow Z_k(U)$$

are all *surjective*. The reason is that given any integral closed subscheme $Z' \subset U$, we can take the closure of Z of Z' in X and think of it as a reduced closed subscheme of

X (see Schemes, Lemma 12.4). And clearly $Z \cap U = Z'$, in other words $j^*[Z] = [Z']$ whence the surjectivity. In fact a little bit more is true.

Lemma 14.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. For every $k \in \mathbf{Z}$ the sequence*

$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. First assume X is quasi-compact. Then $Z_k(X)$ is a free $\mathbf{Z}$-module with basis given by the elements $[Z]$ where $Z \subset X$ is integral closed of δ-dimension k. Such a basis element maps either to the basis element $[Z \cap U]$ or to zero if $Z \subset Y$. Hence the lemma is clear in this case. The general case is similar and the proof is omitted. $\square$

Lemma 14.3. *Let (S, δ) be as in Situation 7.1. Let X, Y, Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of relative dimensions r and s. Then $g \circ f$ is flat of relative dimension $r + s$ and*

$$f^* \circ g^* = (g \circ f)^*$$

as maps $Z_k(Z) \to Z_{k+r+s}(X)$.

Proof. The composition is flat of relative dimension $r + s$ by Morphisms, Lemma 29.3. Suppose that

(1) $W \subset Z$ is a closed integral subscheme of δ-dimension k,
(2) $W' \subset Y$ is a closed integral subscheme of δ-dimension $k + s$ with $W' \subset g^{-1}(W)$, and
(3) $W'' \subset Y$ is a closed integral subscheme of δ-dimension $k + s + r$ with $W'' \subset f^{-1}(W')$.

We have to show that the coefficient n of $[W'']$ in $(g \circ f)^*[W]$ agrees with the coefficient m of $[W'']$ in $f^*(g^*[W])$. That it suffices to check the lemma in these cases follows from Lemma 13.1. Let $\xi'' \in W''$, $\xi' \in W'$ and $\xi \in W$ be the generic points. Consider the local rings $A = \mathcal{O}_{Z,\xi}$, $B = \mathcal{O}_{Y,\xi'}$ and $C = \mathcal{O}_{X,\xi''}$. Then we have local flat ring maps $A \to B$, $B \to C$ and moreover

$$n = \text{length}_C(C/\mathfrak{m}_A C), \quad \text{and} \quad m = \text{length}_C(C/\mathfrak{m}_B C)\text{length}_B(B/\mathfrak{m}_A B)$$

Hence the equality follows from Algebra, Lemma 52.14. $\square$

Lemma 14.4. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r.*

(1) *Let $Z \subset Y$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then we have $\dim_\delta(f^{-1}(Z)) \leq k + r$ and $[f^{-1}(Z)]_{k+r} = f^*[Z]_k$ in $Z_{k+r}(X)$.*
(2) *Let $\mathcal{F}$ be a coherent sheaf on Y with $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then we have $\dim_\delta(Supp(f^*\mathcal{F})) \leq k + r$ and*

$$f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$$

in $Z_{k+r}(X)$.

Proof. The statements on dimensions follow immediately from Lemma 13.1. Part (1) follows from part (2) by Lemma 10.3 and the fact that $f^*\mathcal{O}_Z = \mathcal{O}_{f^{-1}(Z)}$.

Proof of (2). As X, Y are locally Noetherian we may apply Cohomology of Schemes, Lemma 9.1 to see that $\mathcal{F}$ is of finite type, hence $f^*\mathcal{F}$ is of finite type (Modules, Lemma 9.2), hence $f^*\mathcal{F}$ is coherent (Cohomology of Schemes, Lemma 9.1 again). Thus the lemma makes sense. Let $W \subset Y$ be an integral closed subscheme of δ-dimension k, and let $W' \subset X$ be an integral closed subscheme of dimension $k + r$ mapping into W under f. We have to show that the coefficient n of $[W']$ in $f^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W']$ in $[f^*\mathcal{F}]_{k+r}$. Let $\xi \in W$ and $\xi' \in W'$ be the generic points. Let $A = \mathcal{O}_{Y,\xi}$, $B = \mathcal{O}_{X,\xi'}$ and set $M = \mathcal{F}_\xi$ as an A-module. (Note that M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 10.1.) We have $f^*\mathcal{F}_{\xi'} = B \otimes_A M$. Thus we see that

$$n = \text{length}_B(B \otimes_A M) \quad \text{and} \quad m = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}_A B)$$

Thus the equality follows from Algebra, Lemma 52.13. □

15. Push and pull

In this section we verify that proper pushforward and flat pullback are compatible when this makes sense. By the work we did above this is a consequence of cohomology and base change.

Lemma 15.1. *Let (S, δ) be as in Situation 7.1. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a fibre product diagram of schemes locally of finite type over S. Assume $f : X \to Y$ proper and $g : Y' \to Y$ flat of relative dimension r. Then also f' is proper and g' is flat of relative dimension r. For any k-cycle α on X we have

$$g^* f_* \alpha = f'_*(g')^* \alpha$$

in $Z_{k+r}(Y')$.

Proof. The assertion that f' is proper follows from Morphisms, Lemma 41.5. The assertion that g' is flat of relative dimension r follows from Morphisms, Lemmas 29.2 and 25.8. It suffices to prove the equality of cycles when $\alpha = [W]$ for some integral closed subscheme $W \subset X$ of δ-dimension k. Note that in this case we have $\alpha = [\mathcal{O}_W]_k$, see Lemma 10.3. By Lemmas 12.4 and 14.4 it therefore suffices to show that $f'_*(g')^*\mathcal{O}_W$ is isomorphic to $g^*f_*\mathcal{O}_W$. This follows from cohomology and base change, see Cohomology of Schemes, Lemma 5.2. □

Lemma 15.2. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a finite locally free morphism of degree d (see Morphisms, Definition 48.1). Then f is both proper and flat of relative dimension 0, and*

$$f_* f^* \alpha = d\alpha$$

for every $\alpha \in Z_k(Y)$.

Proof. A finite locally free morphism is flat and finite by Morphisms, Lemma 48.2, and a finite morphism is proper by Morphisms, Lemma 44.11. We omit showing that a finite morphism has relative dimension 0. Thus the formula makes sense. To prove it, let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. It suffices to prove the formula for $\alpha = [Z]$. Since the base change of a finite locally free morphism is finite locally free (Morphisms, Lemma 48.4) we see that $f_*f^*\mathcal{O}_Z$ is a finite locally free sheaf of rank d on Z. Hence

$$f_*f^*[Z] = f_*f^*[\mathcal{O}_Z]_k = [f_*f^*\mathcal{O}_Z]_k = d[Z]$$

where we have used Lemmas 14.4 and 12.4. □

16. Preparation for principal divisors

Some of the material in this section partially overlaps with the discussion in Divisors, Section 26.

Lemma 16.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral.*

1. *If $Z \subset X$ is an integral closed subscheme, then the following are equivalent:*
   (a) *Z is a prime divisor,*
   (b) *Z has codimension 1 in X, and*
   (c) $\dim_\delta(Z) = \dim_\delta(X) - 1$.
2. *If Z is an irreducible component of an effective Cartier divisor on X, then* $\dim_\delta(Z) = \dim_\delta(X) - 1$.

Proof. Part (1) follows from the definition of a prime divisor (Divisors, Definition 26.2) and the definition of a dimension function (Topology, Definition 20.1). Let $\xi \in Z$ be the generic point of an irreducible component Z of an effective Cartier divisor $D \subset X$. Then $\dim(\mathcal{O}_{D,\xi}) = 0$ and $\mathcal{O}_{D,\xi} = \mathcal{O}_{X,\xi}/(f)$ for some nonzerodivisor $f \in \mathcal{O}_{X,\xi}$ (Divisors, Lemma 15.2). Then $\dim(\mathcal{O}_{X,\xi}) = 1$ by Algebra, Lemma 60.13. Hence Z is as in (1) by Properties, Lemma 10.3 and the proof is complete. □

Lemma 16.2. *Let $f : X \to Y$ be a morphism of schemes. Let $\xi \in Y$ be a point. Assume that*

1. *X, Y are integral,*
2. *Y is locally Noetherian*
3. *f is proper, dominant and $R(Y) \subset R(X)$ is finite, and*
4. $\dim(\mathcal{O}_{Y,\xi}) = 1$.

Then there exists an open neighbourhood $V \subset Y$ of ξ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. This lemma is a special case of Varieties, Lemma 17.2. Here is a direct argument in this case. By Cohomology of Schemes, Lemma 21.2 it suffices to prove that $f^{-1}(\{\xi\})$ is finite. We replace Y by an affine open, say $Y = \mathrm{Spec}(R)$. Note that R is Noetherian, as Y is assumed locally Noetherian. Since f is proper it is quasi-compact. Hence we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_n$ with each $U_i = \mathrm{Spec}(A_i)$. Note that $R \to A_i$ is a finite type injective homomorphism of domains such that the induced extension of fraction fields is finite. Thus the lemma follows from Algebra, Lemma 113.2. □

17. Principal divisors

The following definition is the analogue of Divisors, Definition 26.5 in our current setup.

Definition 17.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral with $\dim_\delta(X) = n$. Let $f \in R(X)^*$. The *principal divisor associated to* f is the $(n-1)$-cycle

$$\operatorname{div}(f) = \operatorname{div}_X(f) = \sum \operatorname{ord}_Z(f)[Z]$$

defined in Divisors, Definition 26.5. This makes sense because prime divisors have δ-dimension $n-1$ by Lemma 16.1.

In the situation of the definition for $f, g \in R(X)^*$ we have

$$\operatorname{div}_X(fg) = \operatorname{div}_X(f) + \operatorname{div}_X(g)$$

in $Z_{n-1}(X)$. See Divisors, Lemma 26.6. The following lemma will be superseded by the more general Lemma 20.2.

Lemma 17.2. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $g \in R(Y)^*$. Then*

$$f^*(div_Y(g)) = div_X(g)$$

in $Z_{n+r-1}(X)$.

Proof. Note that since f is flat it is dominant so that f induces an embedding $R(Y) \subset R(X)$, and hence we may think of g as an element of $R(X)^*$. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n+r-1$. Let $\xi \in Z$ be its generic point. If $\dim_\delta(f(Z)) > n-1$, then we see that the coefficient of $[Z]$ in the left and right hand side of the equation is zero. Hence we may assume that $Z' = \overline{f(Z)}$ is an integral closed subscheme of Y of δ-dimension $n-1$. Let $\xi' = f(\xi)$. It is the generic point of Z'. Set $A = \mathcal{O}_{Y,\xi'}$, $B = \mathcal{O}_{X,\xi}$. The ring map $A \to B$ is a flat local homomorphism of Noetherian local domains of dimension 1. We have g in the fraction field of A. What we have to show is that

$$\operatorname{ord}_A(g)\operatorname{length}_B(B/\mathfrak{m}_A B) = \operatorname{ord}_B(g).$$

This follows from Algebra, Lemma 52.13 (details omitted). $\square$

18. Principal divisors and pushforward

The first lemma implies that the pushforward of a principal divisor along a generically finite morphism is a principal divisor.

Lemma 18.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(X) = \dim_\delta(Y)$. Let $p : X \to Y$ be a dominant proper morphism. Let $f \in R(X)^*$. Set*

$$g = Nm_{R(X)/R(Y)}(f).$$

Then we have $p_* div(f) = div(g)$.

Proof. Let $Z \subset Y$ be an integral closed subscheme of δ-dimension $n-1$. We want to show that the coefficient of $[Z]$ in $p_*\text{div}(f)$ and $\text{div}(g)$ are equal. We may apply Lemma 16.2 to the morphism $p : X \to Y$ and the generic point $\xi \in Z$. Hence we may replace Y by an affine open neighbourhood of ξ and assume that $p : X \to Y$ is finite. Write $Y = \text{Spec}(R)$ and $X = \text{Spec}(A)$ with p induced by a finite homomorphism $R \to A$ of Noetherian domains which induces an finite field extension L/K of fraction fields. Now we have $f \in L$, $g = \text{Nm}(f) \in K$, and a prime $\mathfrak p \subset R$ with $\dim(R_{\mathfrak p}) = 1$. The coefficient of $[Z]$ in $\text{div}_Y(g)$ is $\text{ord}_{R_{\mathfrak p}}(g)$. The coefficient of $[Z]$ in $p_*\text{div}_X(f)$ is

$$\sum\nolimits_{\mathfrak q \text{ lying over } \mathfrak p} [\kappa(\mathfrak q) : \kappa(\mathfrak p)] \text{ord}_{A_{\mathfrak q}}(f)$$

The desired equality therefore follows from Algebra, Lemma 121.8. □

An important role in the discussion of principal divisors is played by the "universal" principal divisor $[0] - [\infty]$ on $\mathbf{P}^1_S$. To make this more precise, let us denote

$$D_0, D_\infty \subset \mathbf{P}^1_S = \underline{\text{Proj}}_S(\mathcal{O}_S[T_0, T_1]) \tag{18.1.1}$$

the closed subscheme cut out by the section T_1, resp. T_0 of $\mathcal{O}(1)$. These are effective Cartier divisors, see Divisors, Definition 13.1 and Lemma 14.10. The following lemma says that loosely speaking we have "$\text{div}(T_1/T_0) = [D_0] - [D_1]$" and that this is the universal principal divisor.

Lemma 18.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $f \in R(X)^*$. Let $U \subset X$ be a nonempty open such that f corresponds to a section $f \in \Gamma(U, \mathcal{O}_X^*)$. Let $Y \subset X \times_S \mathbf{P}^1_S$ be the closure of the graph of $f : U \to \mathbf{P}^1_S$. Then*

1. *the projection morphism $p : Y \to X$ is proper,*
2. *$p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is an isomorphism,*
3. *the pullbacks $Y_0 = q^{-1}D_0$ and $Y_\infty = q^{-1}D_\infty$ via the morphism $q : Y \to \mathbf{P}^1_S$ are defined (Divisors, Definition 13.12),*
4. *we have*
$$div_Y(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$
5. *we have*
$$div_X(f) = p_* div_Y(f)$$
6. *if we view Y_0 and Y_∞ as closed subschemes of X via the morphism p then we have*
$$div_X(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$

Proof. Since X is integral, we see that U is integral. Hence Y is integral, and $(1, f)(U) \subset Y$ is an open dense subscheme. Also, note that the closed subscheme $Y \subset X \times_S \mathbf{P}^1_S$ does not depend on the choice of the open U, since after all it is the closure of the one point set $\{\eta'\} = \{(1, f)(\eta)\}$ where $\eta \in X$ is the generic point. Having said this let us prove the assertions of the lemma.

For (1) note that p is the composition of the closed immersion $Y \to X \times_S \mathbf{P}^1_S = \mathbf{P}^1_X$ with the proper morphism $\mathbf{P}^1_X \to X$. As a composition of proper morphisms is proper (Morphisms, Lemma 41.4) we conclude.

It is clear that $Y \cap U \times_S \mathbf{P}^1_S = (1, f)(U)$. Thus (2) follows. It also follows that $\dim_\delta(Y) = n$.

Note that $q(\eta') = f(\eta)$ is not contained in D_0 or D_∞ since $f \in R(X)^*$. Hence (3) by Divisors, Lemma 13.13. We obtain $\dim_\delta(Y_0) = n-1$ and $\dim_\delta(Y_\infty) = n-1$ from Lemma 16.1.

Consider the effective Cartier divisor Y_0. At every point $\xi \in Y_0$ we have $f \in \mathcal{O}_{Y,\xi}$ and the local equation for Y_0 is given by f. In particular, if $\delta(\xi) = n-1$ so ξ is the generic point of a integral closed subscheme Z of δ-dimension $n-1$, then we see that the coefficient of $[Z]$ in $\text{div}_Y(f)$ is

$$\text{ord}_Z(f) = \text{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{Y,\xi}/f\mathcal{O}_{Y,\xi}) = \text{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{Y_0,\xi})$$

which is the coefficient of $[Z]$ in $[Y_0]_{n-1}$. A similar argument using the rational function $1/f$ shows that $-[Y_\infty]$ agrees with the terms with negative coefficients in the expression for $\text{div}_Y(f)$. Hence (4) follows.

Note that $D_0 \to S$ is an isomorphism. Hence we see that $X \times_S D_0 \to X$ is an isomorphism as well. Clearly we have $Y_0 = Y \cap X \times_S D_0$ (scheme theoretic intersection) inside $X \times_S \mathbf{P}^1_S$. Hence it is really the case that $Y_0 \to X$ is a closed immersion. It follows that

$$p_*\mathcal{O}_{Y_0} = \mathcal{O}_{Y'_0}$$

where $Y'_0 \subset X$ is the image of $Y_0 \to X$. By Lemma 12.4 we have $p_*[Y_0]_{n-1} = [Y'_0]_{n-1}$. The same is true for D_∞ and Y_∞. Hence (6) is a consequence of (5). Finally, (5) follows immediately from Lemma 18.1. $\square$

The following lemma says that the degree of a principal divisor on a proper curve is zero.

Lemma 18.3. *Let K be any field. Let X be a 1-dimensional integral scheme endowed with a proper morphism $c : X \to \text{Spec}(K)$. Let $f \in K(X)^*$ be an invertible rational function. Then*

$$\sum\nolimits_{x \in X \ closed} [\kappa(x) : K] \, ord_{\mathcal{O}_{X,x}}(f) = 0$$

where ord is as in Algebra, Definition 121.2. In other words, $c_*div(f) = 0$.

Proof. Consider the diagram

$$\begin{tikzcd} Y \arrow[r, "p"'] \arrow[d, "q"'] & X \arrow[d, "c"] \\ \mathbf{P}^1_K \arrow[r, "c'"] & \text{Spec}(K) \end{tikzcd}$$

that we constructed in Lemma 18.2 starting with X and the rational function f over $S = \text{Spec}(K)$. We will use all the results of this lemma without further mention. We have to show that $c_*\text{div}_X(f) = c_*p_*\text{div}_Y(f) = 0$. This is the same as proving that $c'_*q_*\text{div}_Y(f) = 0$. If $q(Y)$ is a closed point of $\mathbf{P}^1_K$ then we see that $\text{div}_X(f) = 0$ and the lemma holds. Thus we may assume that q is dominant. Suppose we can show that $q : Y \to \mathbf{P}^1_K$ is finite locally free of degree d (see Morphisms, Definition 48.1). Since $\text{div}_Y(f) = [q^{-1}D_0]_0 - [q^{-1}D_\infty]_0$ we see (by definition of flat pullback) that $\text{div}_Y(f) = q^*([D_0]_0 - [D_\infty]_0)$. Then by Lemma 15.2 we get $q_*\text{div}_Y(f) = d([D_0]_0 - [D_\infty]_0)$. Since clearly $c'_*[D_0]_0 = c'_*[D_\infty]_0$ we win.

It remains to show that q is finite locally free. (It will automatically have some given degree as $\mathbf{P}^1_K$ is connected.) Since $\dim(\mathbf{P}^1_K) = 1$ we see that q is finite for example by Lemma 16.2. All local rings of $\mathbf{P}^1_K$ at closed points are regular local rings of dimension 1 (in other words discrete valuation rings), since they are localizations of $K[T]$ (see Algebra, Lemma 114.1). Hence for $y \in Y$ closed the local ring $\mathcal{O}_{Y,y}$ will be flat over $\mathcal{O}_{\mathbf{P}^1_K, q(y)}$ as soon as it is torsion free (More on Algebra, Lemma 22.11). This is obviously the case as $\mathcal{O}_{Y,y}$ is a domain and q is dominant. Thus q is flat. Hence q is finite locally free by Morphisms, Lemma 48.2. □

19. Rational equivalence

In this section we define *rational equivalence* on k-cycles. We will allow locally finite sums of images of principal divisors (under closed immersions). This leads to some pretty strange phenomena, see Example 19.5. However, if we do not allow these then we do not know how to prove that capping with Chern classes of line bundles factors through rational equivalence.

Definition 19.1. Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) Given any locally finite collection $\{W_j \subset X\}$ of integral closed subschemes with $\dim_\delta(W_j) = k + 1$, and any $f_j \in R(W_j)^*$ we may consider
$$\sum (i_j)_*\text{div}(f_j) \in Z_k(X)$$
where $i_j : W_j \to X$ is the inclusion morphism. This makes sense as the morphism $\coprod i_j : \coprod W_j \to X$ is proper.
(2) We say that $\alpha \in Z_k(X)$ is *rationally equivalent to zero* if α is a cycle of the form displayed above.
(3) We say $\alpha, \beta \in Z_k(X)$ are *rationally equivalent* and we write $\alpha \sim_{rat} \beta$ if $\alpha - \beta$ is rationally equivalent to zero.
(4) We define
$$\text{CH}_k(X) = Z_k(X)/\sim_{rat}$$
to be the *Chow group of k-cycles on X*. This is sometimes called the *Chow group of k-cycles modulo rational equivalence on X*.

There are many other interesting (adequate) equivalence relations. Rational equivalence is the coarsest one of them all.

Remark 19.2. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. Let us show that we have a presentation
$$\bigoplus\nolimits'_{\delta(x)=k+1} K_1^M(\kappa(x)) \xrightarrow{\partial} \bigoplus\nolimits'_{\delta(x)=k} K_0^M(\kappa(x)) \to \text{CH}_k(X) \to 0$$
Here we use the notation and conventions introduced in Remark 8.2 and in addition

(1) $K_1^M(\kappa(x)) = \kappa(x)^*$ is the degree 1 part of the Milnor K-theory of the residue field $\kappa(x)$ of the point $x \in X$ (see Remark 6.4), and
(2) the differential ∂ is defined as follows: given an element $\xi = \sum_x f_x$ we denote $W_x = \overline{x}$ the integral closed subscheme of X with generic point x and we set
$$\partial(\xi) = \sum (W_x \to X)_*\text{div}(f_x)$$
in $Z_k(X)$ which makes sense as we have seen that the second term of the complex is equal to $Z_k(X)$ by Remark 8.2.

The fact that we obtain a presentation of $\text{CH}_k(X)$ follows immediately by comparing with Definition 19.1.

A very simple but important lemma is the following.

Lemma 19.3. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. Let $k \in \mathbf{Z}$. Suppose $\alpha, \beta \in Z_k(X)$. If $\alpha|_U \sim_{rat} \beta|_U$ then there exist a cycle $\gamma \in Z_k(Y)$ such that*

$$\alpha \sim_{rat} \beta + i_*\gamma.$$

In other words, the sequence

$$\mathrm{CH}_k(Y) \xrightarrow{i_*} \mathrm{CH}_k(X) \xrightarrow{j^*} \mathrm{CH}_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. Let $\{W_j\}_{j \in J}$ be a locally finite collection of integral closed subschemes of U of δ-dimension $k+1$, and let $f_j \in R(W_j)^*$ be elements such that $(\alpha - \beta)|_U = \sum (i_j)_*\mathrm{div}(f_j)$ as in the definition. Set $W_j' \subset X$ equal to the closure of W_j. Suppose that $V \subset X$ is a quasi-compact open. Then also $V \cap U$ is quasi-compact open in U as V is Noetherian. Hence the set $\{j \in J \mid W_j \cap V \neq \emptyset\} = \{j \in J \mid W_j' \cap V \neq \emptyset\}$ is finite since $\{W_j\}$ is locally finite. In other words we see that $\{W_j'\}$ is also locally finite. Since $R(W_j) = R(W_j')$ we see that

$$\alpha - \beta - \sum (i_j')_*\mathrm{div}(f_j)$$

is a cycle supported on Y and the lemma follows (see Lemma 14.2). ☐

Lemma 19.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $X_1, X_2 \subset X$ be closed subschemes such that $X = X_1 \cup X_2$ set theoretically. For every $k \in \mathbf{Z}$ the sequence of abelian groups*

$$\mathrm{CH}_k(X_1 \cap X_2) \longrightarrow \mathrm{CH}_k(X_1) \oplus \mathrm{CH}_k(X_2) \longrightarrow \mathrm{CH}_k(X) \longrightarrow 0$$

is exact. Here $X_1 \cap X_2$ is the scheme theoretic intersection and the maps are the pushforward maps with one multiplied by -1.

Proof. By Lemma 12.3 the arrow $\mathrm{CH}_k(X_1) \oplus \mathrm{CH}_k(X_2) \to \mathrm{CH}_k(X)$ is surjective. Suppose that (α_1, α_2) maps to zero under this map. Write $\alpha_1 = \sum n_{1,i}[W_{1,i}]$ and $\alpha_2 = \sum n_{2,i}[W_{2,i}]$. Then we obtain a locally finite collection $\{W_j\}_{j \in J}$ of integral closed subschemes of X of δ-dimension $k+1$ and $f_j \in R(W_j)^*$ such that

$$\sum n_{1,i}[W_{1,i}] + \sum n_{2,i}[W_{2,i}] = \sum (i_j)_*\mathrm{div}(f_j)$$

as cycles on X where $i_j : W_j \to X$ is the inclusion morphism. Choose a disjoint union decomposition $J = J_1 \amalg J_2$ such that $W_j \subset X_1$ if $j \in J_1$ and $W_j \subset X_2$ if $j \in J_2$. (This is possible because the W_j are integral.) Then we can write the equation above as

$$\sum n_{1,i}[W_{1,i}] - \sum\nolimits_{j \in J_1} (i_j)_*\mathrm{div}(f_j) = -\sum n_{2,i}[W_{2,i}] + \sum\nolimits_{j \in J_2} (i_j)_*\mathrm{div}(f_j)$$

Hence this expression is a cycle (!) on $X_1 \cap X_2$. In other words the element (α_1, α_2) is in the image of the first arrow and the proof is complete. ☐

Example 19.5. Here is a "strange" example. Suppose that S is the spectrum of a field k with δ as in Example 7.2. Suppose that $X = C_1 \cup C_2 \cup \ldots$ is an infinite union of curves $C_j \cong \mathbf{P}^1_k$ glued together in the following way: The point $\infty \in C_j$ is glued transversally to the point $0 \in C_{j+1}$ for $j = 1, 2, 3, \ldots$. Take the point $0 \in C_1$.

This gives a zero cycle $[0] \in Z_0(X)$. The "strangeness" in this situation is that actually $[0] \sim_{rat} 0$! Namely we can choose the rational function $f_j \in R(C_j)$ to be the function which has a simple zero at 0 and a simple pole at ∞ and no other zeros or poles. Then we see that the sum $\sum (i_j)_*\text{div}(f_j)$ is exactly the 0-cycle $[0]$. In fact it turns out that $\text{CH}_0(X) = 0$ in this example. If you find this too bizarre, then you can just make sure your spaces are always quasi-compact (so X does not even exist for you).

Remark 19.6. Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Suppose we have infinite collections $\alpha_i, \beta_i \in Z_k(X)$, $i \in I$ of k-cycles on X. Suppose that the supports of α_i and β_i form locally finite collections of closed subsets of X so that $\sum \alpha_i$ and $\sum \beta_i$ are defined as cycles. Moreover, assume that $\alpha_i \sim_{rat} \beta_i$ for each i. Then it is not clear that $\sum \alpha_i \sim_{rat} \sum \beta_i$. Namely, the problem is that the rational equivalences may be given by locally finite families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ but the union $\{W_{i,j}\}_{i \in I, j \in J_i}$ may not be locally finite.

In many cases in practice, one has a locally finite family of closed subsets $\{T_i\}_{i \in I}$ such that α_i, β_i are supported on T_i and such that $\alpha_i = \beta_i$ in $\text{CH}_k(T_i)$, in other words, the families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ consist of subschemes $W_{i,j} \subset T_i$. In this case it is true that $\sum \alpha_i \sim_{rat} \sum \beta_i$ on X, simply because the family $\{W_{i,j}\}_{i \in I, j \in J_i}$ is automatically locally finite in this case.

20. Rational equivalence and push and pull

In this section we show that flat pullback and proper pushforward commute with rational equivalence.

Lemma 20.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be schemes locally of finite type over S. Assume Y integral with $\dim_\delta(Y) = k$. Let $f : X \to Y$ be a flat morphism of relative dimension r. Then for $g \in R(Y)^*$ we have*

$$f^* div_Y(g) = \sum n_j i_{j,*} div_{X_j}(g \circ f|_{X_j})$$

as $(k + r - 1)$-cycles on X where the sum is over the irreducible components X_j of X and n_j is the multiplicity of X_j in X.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $k + r - 1$. We have to show that the coefficient n of $[Z]$ in $f^*\text{div}(g)$ is equal to the coefficient m of $[Z]$ in $\sum i_{j,*}\text{div}(g \circ f|_{X_j})$. Let Z' be the closure of $f(Z)$ which is an integral closed subscheme of Y. By Lemma 13.1 we have $\dim_\delta(Z') \geq k - 1$. Thus either $Z' = Y$ or Z' is a prime divisor on Y. If $Z' = Y$, then the coefficients n and m are both zero: this is clear for n by definition of f^* and follows for m because $g \circ f|_{X_j}$ is a unit in any point of X_j mapping to the generic point of Y. Hence we may assume that $Z' \subset Y$ is a prime divisor.

We are going to translate the equality of n and m into algebra. Namely, let $\xi' \in Z'$ and $\xi \in Z$ be the generic points. Set $A = \mathcal{O}_{Y,\xi'}$ and $B = \mathcal{O}_{X,\xi}$. Note that A, B are Noetherian, $A \to B$ is flat, local, A is a domain, and $\mathfrak{m}_A B$ is an ideal of definition of the local ring B. The rational function g is an element of the fraction field $Q(A)$ of A. By construction, the closed subschemes X_j which meet ξ correspond 1-to-1 with minimal primes

$$\mathfrak{q}_1, \ldots, \mathfrak{q}_s \subset B$$

The integers n_j are the corresponding lengths

$$n_i = \text{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

The rational functions $g \circ f|_{X_j}$ correspond to the image $g_i \in \kappa(\mathfrak{q}_i)^*$ of $g \in Q(A)$. Putting everything together we see that

$$n = \text{ord}_A(g)\text{length}_B(B/\mathfrak{m}_A B)$$

and that

$$m = \sum \text{ord}_{B/\mathfrak{q}_i}(g_i)\text{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

Writing $g = x/y$ for some nonzero $x, y \in A$ we see that it suffices to prove

$$\text{length}_A(A/(x))\text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(B/xB)$$

(equality uses Algebra, Lemma 52.13) equals

$$\sum\nolimits_{i=1,\ldots,s} \text{length}_{B/\mathfrak{q}_i}(B/(x, \mathfrak{q}_i))\text{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$$

and similarly for y. As $A \to B$ is flat it follows that x is a nonzerodivisor in B. Hence the desired equality follows from Lemma 3.2. □

Lemma 20.2. *Let (S, δ) be as in Situation 7.1. Let X, Y be schemes locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\alpha \sim_{rat} \beta$ be rationally equivalent k-cycles on Y. Then $f^*\alpha \sim_{rat} f^*\beta$ as $(k + r)$-cycles on X.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow Y$$

of closed immersions, with each W_j integral of δ-dimension $k + 1$ and rational functions $g_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j \in J}$ is locally finite on Y. Then we have to show that

$$f^*(\sum i_{j,*}\text{div}(g_j)) = \sum f^* i_{j,*}\text{div}(g_j)$$

is rationally equivalent to zero on X. The sum on the right makes sense as $\{W_j\}$ is locally finite in X by Lemma 13.2.

Consider the fibre products

$$i'_j : W'_j = W_j \times_Y X \longrightarrow X.$$

and denote $f_j : W'_j \to W_j$ the first projection. By Lemma 15.1 we can write the sum above as

$$\sum i'_{j,*}(f_j^*\text{div}(g_j))$$

By Lemma 20.1 we see that each $f_j^*\text{div}(g_j)$ is rationally equivalent to zero on W'_j. Hence each $i'_{j,*}(f_j^*\text{div}(g_j))$ is rationally equivalent to zero. Then the same is true for the displayed sum by the discussion in Remark 19.6. □

Lemma 20.3. *Let (S, δ) be as in Situation 7.1. Let X, Y be schemes locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Suppose $\alpha, \beta \in Z_k(X)$ are rationally equivalent. Then $p_*\alpha$ is rationally equivalent to $p_*\beta$.*

Proof. What do we have to show? Well, suppose we are given a collection
$$i_j : W_j \longrightarrow X$$
of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j \in J}$ is locally finite on X. Then we have to show that
$$p_*\left(\sum i_{j,*}\text{div}(f_j)\right)$$
is rationally equivalent to zero on X.

Note that the sum is equal to
$$\sum p_* i_{j,*}\text{div}(f_j).$$
Let $W'_j \subset Y$ be the integral closed subscheme which is the image of $p \circ i_j$. The collection $\{W'_j\}$ is locally finite in Y by Lemma 11.2. Hence it suffices to show, for a given j, that either $p_* i_{j,*}\text{div}(f_j) = 0$ or that it is equal to $i'_{j,*}\text{div}(g_j)$ for some $g_j \in R(W'_j)^*$.

The arguments above therefore reduce us to the case of a single integral closed subscheme $W \subset X$ of δ-dimension $k+1$. Let $f \in R(W)^*$. Let $W' = p(W)$ as above. We get a commutative diagram of morphisms
$$\begin{tikzcd} W \arrow[r, "i"'] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ W' \arrow[r, "i'"] & Y \end{tikzcd}$$
Note that $p_* i_* \text{div}(f) = i'_*(p')_*\text{div}(f)$ by Lemma 12.2. As explained above we have to show that $(p')_*\text{div}(f)$ is the divisor of a rational function on W' or zero. There are three cases to distinguish.

The case $\dim_\delta(W') < k$. In this case automatically $(p')_*\text{div}(f) = 0$ and there is nothing to prove.

The case $\dim_\delta(W') = k$. Let us show that $(p')_*\text{div}(f) = 0$ in this case. Let $\eta \in W'$ be the generic point. Note that $c : W_\eta \to \text{Spec}(K)$ is a proper integral curve over $K = \kappa(\eta)$ whose function field $K(W_\eta)$ is identified with $R(W)$. Here is a diagram
$$\begin{tikzcd} W_\eta \arrow[r] \arrow[d, "c"'] & W \arrow[d, "p'"] \\ \text{Spec}(K) \arrow[r] & W' \end{tikzcd}$$
Let us denote $f_\eta \in K(W_\eta)^*$ the rational function corresponding to $f \in R(W)^*$. Moreover, the closed points ξ of W_η correspond $1-1$ to the closed integral subschemes $Z = Z_\xi \subset W$ of δ-dimension k with $p'(Z) = W'$. Note that the multiplicity of Z_ξ in $\text{div}(f)$ is equal to $\text{ord}_{\mathcal{O}_{W_\eta,\xi}}(f_\eta)$ simply because the local rings $\mathcal{O}_{W_\eta,\xi}$ and $\mathcal{O}_{W,\xi}$ are identified (as subrings of their fraction fields). Hence we see that the multiplicity of $[W']$ in $(p')_*\text{div}(f)$ is equal to the multiplicity of $[\text{Spec}(K)]$ in $c_*\text{div}(f_\eta)$. By Lemma 18.3 this is zero.

The case $\dim_\delta(W') = k+1$. In this case Lemma 18.1 applies, and we see that indeed $p'_*\text{div}(f) = \text{div}(g)$ for some $g \in R(W')^*$ as desired. $\square$

21. Rational equivalence and the projective line

Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Given any closed subscheme $Z \subset X \times_S \mathbf{P}^1_S = X \times \mathbf{P}^1$ we let Z_0, resp. Z_∞ be the scheme theoretic closed subscheme $Z_0 = \text{pr}_2^{-1}(D_0)$, resp. $Z_\infty = \text{pr}_2^{-1}(D_\infty)$. Here D_0, D_∞ are as in (18.1.1).

Lemma 21.1. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $W \subset X \times_S \mathbf{P}^1_S$ be an integral closed subscheme of δ-dimension $k+1$. Assume $W \neq W_0$, and $W \neq W_\infty$. Then*

1. *W_0, W_∞ are effective Cartier divisors of W,*
2. *W_0, W_∞ can be viewed as closed subschemes of X and*
$$[W_0]_k \sim_{rat} [W_\infty]_k,$$
3. *for any locally finite family of integral closed subschemes $W_i \subset X \times_S \mathbf{P}^1_S$ of δ-dimension $k+1$ with $W_i \neq (W_i)_0$ and $W_i \neq (W_i)_\infty$ we have $\sum([(W_i)_0]_k - [(W_i)_\infty]_k) \sim_{rat} 0$ on X, and*
4. *for any $\alpha \in Z_k(X)$ with $\alpha \sim_{rat} 0$ there exists a locally finite family of integral closed subschemes $W_i \subset X \times_S \mathbf{P}^1_S$ as above such that $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$.*

Proof. Part (1) follows from Divisors, Lemma 13.13 since the generic point of W is not mapped into D_0 or D_∞ under the projection $X \times_S \mathbf{P}^1_S \to \mathbf{P}^1_S$ by assumption.

Since $X \times_S D_0 \to X$ is a closed immersion, we see that W_0 is isomorphic to a closed subscheme of X. Similarly for W_∞. The morphism $p : W \to X$ is proper as a composition of the closed immersion $W \to X \times_S \mathbf{P}^1_S$ and the proper morphism $X \times_S \mathbf{P}^1_S \to X$. By Lemma 18.2 we have $[W_0]_k \sim_{rat} [W_\infty]_k$ as cycles on W. Hence part (2) follows from Lemma 20.3 as clearly $p_*[W_0]_k = [W_0]_k$ and similarly for W_∞.

The only content of statement (3) is, given parts (1) and (2), that the collection $\{(W_i)_0, (W_i)_\infty\}$ is a locally finite collection of closed subschemes of X. This is clear.

Suppose that $\alpha \sim_{rat} 0$. By definition this means there exist integral closed subschemes $V_i \subset X$ of δ-dimension $k+1$ and rational functions $f_i \in R(V_i)^*$ such that the family $\{V_i\}_{i \in I}$ is locally finite in X and such that $\alpha = \sum (V_i \to X)_* \text{div}(f_i)$. Let
$$W_i \subset V_i \times_S \mathbf{P}^1_S \subset X \times_S \mathbf{P}^1_S$$
be the closure of the graph of the rational map f_i as in Lemma 18.2. Then we have that $(V_i \to X)_* \text{div}(f_i)$ is equal to $[(W_i)_0]_k - [(W_i)_\infty]_k$ by that same lemma. Hence the result is clear. $\square$

Lemma 21.2. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let Z be a closed subscheme of $X \times \mathbf{P}^1$. Assume*

1. *$\dim_\delta(Z) \leq k+1$,*
2. *$\dim_\delta(Z_0) \leq k$, $\dim_\delta(Z_\infty) \leq k$, and*
3. *for any embedded point ξ (Divisors, Definition 4.1) of Z either $\xi \notin Z_0 \cup Z_\infty$ or $\delta(\xi) < k$.*

Then $[Z_0]_k \sim_{rat} [Z_\infty]_k$ as k-cycles on X.

Proof. Let $\{W_i\}_{i \in I}$ be the collection of irreducible components of Z which have δ-dimension $k+1$. Write
$$[Z]_{k+1} = \sum n_i [W_i]$$

with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Divisors, Lemma 26.1. We claim that

$$[Z_0]_k = \sum n_i[(W_i)_0]_k$$

and similarly for $[Z_\infty]_k$. If we prove this then the lemma follows from Lemma 21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[Z_0]_k$ is the same as the coefficient m of $[Z']$ in $\sum n_i[(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X \times_S \mathbf{P}^1_S, \xi}$. Let $I \subset A$ be the ideal cutting out Z, in other words so that $A/I = \mathcal{O}_{Z,\xi}$. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence $\dim(A/I) = 1$. Since ξ is not an embedded point by assumption (3) we see that A/I is Cohen-Macaulay. Since $\dim_\delta(Z_0) = k$ we see that $\dim(A/(t, I)) = 0$ which implies that t is a nonzerodivisor on A/I. Finally, the irreducible closed subschemes W_i passing through ξ correspond to the minimal primes $I \subset \mathfrak{q}_i$ over I. The multiplicities n_i correspond to the lengths $\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$. Hence we see that

$$n = \text{length}_A(A/(t, I))$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i))\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 3.2. □

Lemma 21.3. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on $X \times \mathbf{P}^1$. Let $i_0, i_\infty : X \to X \times \mathbf{P}^1$ be the closed immersion such that $i_t(x) = (x, t)$. Denote $\mathcal{F}_0 = i_0^*\mathcal{F}$ and $\mathcal{F}_\infty = i_\infty^*\mathcal{F}$. Assume*

1. $\dim_\delta(\textit{Supp}(\mathcal{F})) \le k + 1$,
2. $\dim_\delta(\textit{Supp}(\mathcal{F}_0)) \le k$, $\dim_\delta(\textit{Supp}(\mathcal{F}_\infty)) \le k$, *and*
3. *for any embedded associated point ξ of $\mathcal{F}$ either $\xi \notin (X \times \mathbf{P}^1)_0 \cup (X \times \mathbf{P}^1)_\infty$ or $\delta(\xi) < k$.*

Then $[\mathcal{F}_0]_k \sim_{rat} [\mathcal{F}_\infty]_k$ as k-cycles on X.

Proof. Let $\{W_i\}_{i \in I}$ be the collection of irreducible components of $\text{Supp}(\mathcal{F})$ which have δ-dimension $k + 1$. Write

$$[\mathcal{F}]_{k+1} = \sum n_i[W_i]$$

with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Lemma 10.1. We claim that

$$[\mathcal{F}_0]_k = \sum n_i[(W_i)_0]_k$$

and similarly for $[\mathcal{F}_\infty]_k$. If we prove this then the lemma follows from Lemma 21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[\mathcal{F}_0]_k$ is the same as the coefficient m of $[Z']$ in $\sum n_i[(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X \times_S \mathbf{P}^1_S, \xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence

$\dim(\text{Supp}(M)) = 1$. Since ξ is not an associated point of $\mathcal{F}$ by assumption (3) we see that M is a Cohen-Macaulay module. Since $\dim_\delta(\text{Supp}(\mathcal{F}_0)) = k$ we see that $\dim(\text{Supp}(M/tM)) = 0$ which implies that t is a nonzerodivisor on M. Finally, the irreducible closed subschemes W_i passing through ξ correspond to the minimal primes $\mathfrak{q}_i$ of $\text{Ass}(M)$. The multiplicities n_i correspond to the lengths $\text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$. Hence we see that

$$n = \text{length}_A(M/tM)$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i)A)\text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 3.2. □

22. Chow groups and envelopes

Here is the definition.

Definition 22.1. Let X be a scheme. An *envelope* is a proper morphism $f : Y \to X$ which is completely decomposed (More on Morphisms, Definition 78.1).

The exact sequence of Lemma 22.4 is the main motivation for the definition.

Lemma 22.2. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. If $f : Y \to X$ and $g : Z \to Y$ are envelopes, then $f \circ g$ is an envelope.*

Proof. Follows from Morphisms, Lemma 41.4 and More on Morphisms, Lemma 78.2. □

Lemma 22.3. *Let (S, δ) be as in Situation 7.1. Let $X' \to X$ be a morphism of schemes locally of finite type over S. If $f : Y \to X$ is an envelope, then the base change $f' : Y' \to X'$ of f is an envelope too.*

Proof. Follows from Morphisms, Lemma 41.5 and More on Morphisms, Lemma 78.3. □

Lemma 22.4. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $f : Y \to X$ be an envelope. Then we have an exact sequence*

$$\text{CH}_k(Y \times_X Y) \xrightarrow{p_* - q_*} \text{CH}_k(Y) \xrightarrow{f_*} \text{CH}_k(X) \to 0$$

for all $k \in \mathbf{Z}$. Here $p, q : Y \times_X Y \to Y$ are the projections.

Proof. Since f is an envelope, f is proper and hence pushforward on cycles and cycle classes is defined, see Sections 12 and 15. Similarly, the morphisms p and q are proper as base changes of f. The composition of the arrows is zero as $f_* \circ p_* = (p \circ f)_* = (q \circ f)_* = f_* \circ q_*$, see Lemma 12.2.

Let us show that $f_* : Z_k(Y) \to Z_k(X)$ is surjective. Namely, suppose that we have $\alpha = \sum n_i[Z_i] \in Z_k(X)$ where $Z_i \subset X$ is a locally finite family of integral closed subschemes. Let $x_i \in Z_i$ be the generic point. Since f is an envelope and hence completely decomposed, there exists a point $y_i \in Y$ with $f(y_i) = x_i$ and with $\kappa(y_i)/\kappa(x_i)$ trivial. Let $W_i \subset Y$ be the integral closed subscheme with generic point y_i. Since f is closed, we see that $f(W_i) = Z_i$. It follows that the family of closed subschemes W_i is locally finite on Y. Since $\kappa(y_i)/\kappa(x_i)$ is trivial we see that $\dim_\delta(W_i) = \dim_\delta(Z_i) = k$. Hence $\beta = \sum n_i[W_i]$ is in $Z_k(Y)$. Finally, since $\kappa(y_i)/\kappa(x_i)$ is trivial, the degree of the dominant morphism $f|_{W_i} : W_i \to Z_i$ is 1 and we conclude that $f_*\beta = \alpha$.

Since $f_* : Z_k(Y) \to Z_k(X)$ is surjective, a fortiori the map $f_* : \mathrm{CH}_k(Y) \to \mathrm{CH}_k(X)$ is surjective.

Let $\beta \in Z_k(Y)$ be an element such that $f_*\beta$ is zero in $\mathrm{CH}_k(X)$. This means we can find a locally finite family of integral closed subschemes $Z_j \subset X$ with $\dim_\delta(Z_j) = k+1$ and $f_j \in R(Z_j)^*$ such that

$$f_*\beta = \sum (Z_j \to X)_* \mathrm{div}(f_j)$$

as cycles where $i_j : Z_j \to X$ is the given closed immersion. Arguing exactly as above, we can find a locally finite family of integral closed subschemes $W_j \subset Y$ with $f(W_j) = Z_j$ and such that $W_j \to Z_j$ is birational, i.e., induces an isomorphism $R(Z_j) = R(W_j)$. Denote $g_j \in R(W_j)^*$ the element corresponding to f_j. Observe that $W_j \to Z_j$ is proper and that $(W_j \to Z_j)_* \mathrm{div}(g_j) = \mathrm{div}(f_j)$ as cycles on Z_j. It follows from this that if we replace β by the rationally equivalent cycle

$$\beta' = \beta - \sum (W_j \to Y)_* \mathrm{div}(g_j)$$

then we find that $f_*\beta' = 0$. (This uses Lemma 12.2.) Thus to finish the proof of the lemma it suffices to show the claim in the following paragraph.

Claim: if $\beta \in Z_k(Y)$ and $f_*\beta = 0$, then $\beta = \delta + p_*\gamma - q_*\gamma$ in $Z_k(Y)$ for some $\gamma \in Z_k(Y \times_X Y)$. Namely, write $\beta = \sum_{j \in J} n_j[W_j]$ with $\{W_j\}_{j \in J}$ a locally finite family of integral closed subschemes of Y with $\dim_\delta(W_j) = k$. Fix an integral closed subscheme $Z \subset X$. Consider the subset $J_Z = \{j \in J : f(W_j) = Z\}$. This is a finite set. There are three cases:

(1) $J_Z = \emptyset$. In this case we set $\gamma_Z = 0$.
(2) $J_Z \neq \emptyset$ and $\dim_\delta(Z) = k$. The condition $f_*\beta = 0$ implies by looking at the coefficient of Z that $\sum_{j \in J_Z} n_j \deg(W_j/Z) = 0$. In this case we choose an integral closed subscheme $W \subset Y$ which maps birationally onto Z (see above). Looking at generic points, we see that $W_j \times_Z W$ has a unique irreducible component $W'_j \subset W_j \times_Z W \subset Y \times_X Y$ mapping birationally to W_j. Then $W'_j \to W$ is dominant and $\deg(W'_j/W) = \deg(W_j/W)$. Thus if we set $\gamma_Z = \sum_{j \in J_Z} n_j[W'_j]$ then we see that $p_*\gamma_Z = \sum_{j \in J_Z} n_j[W_j]$ and $q_*\gamma_Z = \sum_{j \in J_Z} n_j \deg(W'_j/W)[W] = 0$.
(3) $J_Z \neq \emptyset$ and $\dim_\delta(Z) < k$. In this case we choose an integral closed subscheme $W \subset Y$ which maps birationally onto Z (see above). Looking at generic points, we see that $W_j \times_Z W$ has a unique irreducible component $W'_j \subset W_j \times_Z W \subset Y \times_X Y$ mapping birationally to W_j. Then $W'_j \to W$ is dominant and $k = \dim_\delta(W'_j) > \dim_\delta(W) = \dim_\delta(Z)$. Thus if we set $\gamma_Z = \sum_{j \in J_Z} n_j[W'_j]$ then we see that $p_*\gamma_Z = \sum_{j \in J_Z} n_j[W_j]$ and $q_*\gamma_Z = 0$.

Since the family of integral closed subschemes $\{f(W_j)\}$ is locally finite on X (Lemma 11.2) we see that the k-cycle

$$\gamma = \sum\nolimits_{Z \subset X \text{ integral closed}} \gamma_Z$$

on $Y \times_X Y$ is well defined. By our computations above it follows that $p_*\gamma_Z = \beta$ and $q_*\gamma_Z = 0$ which implies what we wanted to prove. $\square$

23. Chow groups and K-groups

In this section we are going to compare K_0 of the category of coherent sheaves to the chow groups.

Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. We denote $\mathit{Coh}(X) = \mathit{Coh}(\mathcal{O}_X)$ the category of coherent sheaves on X. It is an abelian category, see Cohomology of Schemes, Lemma 9.2. For any $k \in \mathbf{Z}$ we let $\mathit{Coh}_{\leq k}(X)$ be the full subcategory of $\mathit{Coh}(X)$ consisting of those coherent sheaves $\mathcal{F}$ having $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$.

Lemma 23.1. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. The categories $\mathit{Coh}_{\leq k}(X)$ are Serre subcategories of the abelian category $\mathit{Coh}(X)$.*

Proof. The definition of a Serre subcategory is Homology, Definition 10.1. The proof of the lemma is straightforward and omitted. □

Lemma 23.2. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. The maps*

$$Z_k(X) \longrightarrow K_0(\mathit{Coh}_{\leq k}(X)/\mathit{Coh}_{\leq k-1}(X)), \quad \sum n_Z[Z] \mapsto \left[\bigoplus\nolimits_{n_Z > 0} \mathcal{O}_Z^{\oplus n_Z}\right] - \left[\bigoplus\nolimits_{n_Z < 0} \mathcal{O}_Z^{\oplus -n_Z}\right]$$

and

$$K_0(\mathit{Coh}_{\leq k}(X)/\mathit{Coh}_{\leq k-1}(X)) \longrightarrow Z_k(X), \quad \mathcal{F} \longmapsto [\mathcal{F}]_k$$

are mutually inverse isomorphisms.

Proof. Note that if $\sum n_Z[Z]$ is in $Z_k(X)$, then the direct sums $\bigoplus_{n_Z > 0} \mathcal{O}_Z^{\oplus n_Z}$ and $\bigoplus_{n_Z < 0} \mathcal{O}_Z^{\oplus -n_Z}$ are coherent sheaves on X since the family $\{Z \mid n_Z > 0\}$ is locally finite on X. The map $\mathcal{F} \to [\mathcal{F}]_k$ is additive on $\mathit{Coh}_{\leq k}(X)$, see Lemma 10.4. And $[\mathcal{F}]_k = 0$ if $\mathcal{F} \in \mathit{Coh}_{\leq k-1}(X)$. By part (1) of Homology, Lemma 11.3 this implies that the second map is well defined too. It is clear that the composition of the first map with the second map is the identity.

Conversely, say we start with a coherent sheaf $\mathcal{F}$ on X. Write $[\mathcal{F}]_k = \sum_{i \in I} n_i[Z_i]$ with $n_i > 0$ and $Z_i \subset X$, $i \in I$ pairwise distinct integral closed subschemes of δ-dimension k. We have to show that

$$[\mathcal{F}] = [\bigoplus\nolimits_{i \in I} \mathcal{O}_{Z_i}^{\oplus n_i}]$$

in $K_0(\mathit{Coh}_{\leq k}(X)/\mathit{Coh}_{\leq k-1}(X))$. Denote $\xi_i \in Z_i$ the generic point. If we set

$$\mathcal{F}' = \text{Ker}(\mathcal{F} \to \bigoplus \xi_{i,*}\mathcal{F}_{\xi_i})$$

then $\mathcal{F}'$ is the maximal coherent submodule of $\mathcal{F}$ whose support has dimension $\leq k-1$. In particular $\mathcal{F}$ and $\mathcal{F}/\mathcal{F}'$ have the same class in $K_0(\mathit{Coh}_{\leq k}(X)/\mathit{Coh}_{\leq k-1}(X))$. Thus after replacing $\mathcal{F}$ by $\mathcal{F}/\mathcal{F}'$ we may and do assume that the kernel $\mathcal{F}'$ displayed above is zero.

For each $i \in I$ we choose a filtration

$$\mathcal{F}_{\xi_i} = \mathcal{F}_i^0 \supset \mathcal{F}_i^1 \supset \ldots \supset \mathcal{F}_i^{n_i} = 0$$

such that the successive quotients are of dimension 1 over the residue field at ξ_i. This is possible as the length of $\mathcal{F}_{\xi_i}$ over $\mathcal{O}_{X,\xi_i}$ is n_i. For $p > n_i$ set $\mathcal{F}_i^p = 0$. For $p \geq 0$ we denote

$$\mathcal{F}^p = \text{Ker}\left(\mathcal{F} \longrightarrow \bigoplus \xi_{i,*}(\mathcal{F}_{\xi_i}/\mathcal{F}_i^p)\right)$$

Then $\mathcal{F}^p$ is coherent, $\mathcal{F}^0 = \mathcal{F}$, and $\mathcal{F}^p/\mathcal{F}^{p+1}$ is isomorphic to a free $\mathcal{O}_{Z_i}$-module of rank 1 (if $n_i > p$) or 0 (if $n_i \leq p$) in an open neighbourhood of ξ_i. Moreover, $\mathcal{F}' = \bigcap \mathcal{F}^p = 0$. Since every quasi-compact open $U \subset X$ contains only a finite number of ξ_i we conclude that $\mathcal{F}^p|_U$ is zero for $p \gg 0$. Hence $\bigoplus_{p\geq 0} \mathcal{F}^p$ is a coherent $\mathcal{O}_X$-module. Consider the short exact sequences

$$0 \to \bigoplus\nolimits_{p>0} \mathcal{F}^p \to \bigoplus\nolimits_{p\geq 0} \mathcal{F}^p \to \bigoplus\nolimits_{p>0} \mathcal{F}^p/\mathcal{F}^{p+1} \to 0$$

and

$$0 \to \bigoplus\nolimits_{p>0} \mathcal{F}^p \to \bigoplus\nolimits_{p\geq 0} \mathcal{F}^p \to \mathcal{F} \to 0$$

of coherent $\mathcal{O}_X$-modules. This already shows that

$$[\mathcal{F}] = [\bigoplus \mathcal{F}^p/\mathcal{F}^{p+1}]$$

in $K_0(\mathit{Coh}_{\leq k}(X)/\mathit{Coh}_{\leq k-1}(X))$. Next, for every $p \geq 0$ and $i \in I$ such that $n_i > p$ we choose a nonzero ideal sheaf $\mathcal{I}_{i,p} \subset \mathcal{O}_{Z_i}$ and a map $\mathcal{I}_{i,p} \to \mathcal{F}^p/\mathcal{F}^{p+1}$ on X which is an isomorphism over the open neighbourhood of ξ_i mentioned above. This is possible by Cohomology of Schemes, Lemma 10.6. Then we consider the short exact sequence

$$0 \to \bigoplus\nolimits_{p\geq 0, i\in I, n_i>p} \mathcal{I}_{i,p} \to \bigoplus \mathcal{F}^p/\mathcal{F}^{p+1} \to \mathcal{Q} \to 0$$

and the short exact sequence

$$0 \to \bigoplus\nolimits_{p\geq 0, i\in I, n_i>p} \mathcal{I}_{i,p} \to \bigoplus\nolimits_{p\geq 0, i\in I, n_i>p} \mathcal{O}_{Z_i} \to \mathcal{Q}' \to 0$$

Observe that both $\mathcal{Q}$ and $\mathcal{Q}'$ are zero in a neighbourhood of the points ξ_i and that they are supported on $\bigcup Z_i$. Hence $\mathcal{Q}$ and $\mathcal{Q}'$ are in $\mathit{Coh}_{\leq k-1}(X)$. Since

$$\bigoplus\nolimits_{i\in I} \mathcal{O}_{Z_i}^{\oplus n_i} \cong \bigoplus\nolimits_{p\geq 0, i\in I, n_i>p} \mathcal{O}_{Z_i}$$

this concludes the proof. □

Lemma 23.3. *Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over (S, δ) as in Situation 7.1. Then $\pi_* : \mathit{Coh}(X) \to \mathit{Coh}(Y)$ is an exact functor which sends $\mathit{Coh}_{\leq k}(X)$ into $\mathit{Coh}_{\leq k}(Y)$ and induces homomorphisms on K_0 of these categories and their quotients. The maps of Lemma 23.2 fit into a commutative diagram*

$$\begin{tikzcd}
Z_k(X) \arrow[r] \arrow[d, "\pi_*"] & K_0(\mathit{Coh}_{\leq k}(X)/\mathit{Coh}_{\leq k-1}(X)) \arrow[r] \arrow[d, "\pi_*"] & Z_k(X) \arrow[d, "\pi_*"] \\
Z_k(Y) \arrow[r] & K_0(\mathit{Coh}_{\leq k}(Y)/\mathit{Coh}_{\leq k-1}(Y)) \arrow[r] & Z_k(Y)
\end{tikzcd}$$

Proof. A finite morphism is affine, hence pushforward of quasi-coherent modules along π is an exact functor by Cohomology of Schemes, Lemma 2.3. A finite morphism is proper, hence π_* sends coherent sheaves to coherent sheaves, see Cohomology of Schemes, Proposition 19.1. The statement on dimensions of supports is clear. Commutativity on the right follows immediately from Lemma 12.4. Since the horizontal arrows are bijections, we find that we have commutativity on the left as well. □

Lemma 23.4. *Let X be a scheme locally of finite type over (S, δ) as in Situation 7.1. There is a canonical map*

$$\mathrm{CH}_k(X) \longrightarrow K_0(\mathit{Coh}_{\leq k+1}(X)/\mathit{Coh}_{\leq k-1}(X))$$

induced by the map $Z_k(X) \to K_0(\mathit{Coh}_{\leq k}(X)/\mathit{Coh}_{\leq k-1}(X))$ from Lemma 23.2.

Proof. We have to show that an element α of $Z_k(X)$ which is rationally equivalent to zero, is mapped to zero in $K_0(\mathit{Coh}_{\leq k+1}(X)/\mathit{Coh}_{\leq k-1}(X))$. Write $\alpha = \sum (i_j)_* \mathrm{div}(f_j)$ as in Definition 19.1. Observe that

$$\pi = \coprod i_j : W = \coprod W_j \longrightarrow X$$

is a finite morphism as each $i_j : W_j \to X$ is a closed immersion and the family of W_j is locally finite in X. Hence we may use Lemma 23.3 to reduce to the case of W. Since W is a disjoint union of integral scheme, we reduce to the case discussed in the next paragraph.

Assume X is integral of δ-dimension $k+1$. Let f be a nonzero rational function on X. Let $\alpha = \mathrm{div}(f)$. We have to show that α is mapped to zero in $K_0(\mathit{Coh}_{\leq k+1}(X)/\mathit{Coh}_{\leq k-1}(X))$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of f, see Divisors, Definition 23.10. Then we have short exact sequences

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$$

and

$$0 \to \mathcal{I} \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_X/f\mathcal{I} \to 0$$

See Divisors, Lemma 23.9. We claim that

$$[\mathcal{O}_X/\mathcal{I}]_k - [\mathcal{O}_X/f\mathcal{I}]_k = \mathrm{div}(f)$$

The claim implies the element $\alpha = \mathrm{div}(f)$ is represented by $[\mathcal{O}_X/\mathcal{I}] - [\mathcal{O}_X/f\mathcal{I}]$ in $K_0(\mathit{Coh}_{\leq k}(X)/\mathit{Coh}_{\leq k-1}(X))$. Then the short exact sequences show that this element maps to zero in $K_0(\mathit{Coh}_{\leq k+1}(X)/\mathit{Coh}_{\leq k-1}(X))$.

To prove the claim, let $Z \subset X$ be an integral closed subscheme of δ-dimension k and let $\xi \in Z$ be its generic point. Then $I = \mathcal{I}_\xi \subset A = \mathcal{O}_{X,\xi}$ is an ideal such that $fI \subset A$. Now the coefficient of $[Z]$ in $\mathrm{div}(f)$ is $\mathrm{ord}_A(f)$. (Of course as usual we identify the function field of X with the fraction field of A.) On the other hand, the coefficient of $[Z]$ in $[\mathcal{O}_X/\mathcal{I}] - [\mathcal{O}_X/f\mathcal{I}]$ is

$$\mathrm{length}_A(A/I) - \mathrm{length}_A(A/fI)$$

Using the distance fuction of Algebra, Definition 121.5 we can rewrite this as

$$d(A, I) - d(A, fI) = d(I, fI) = \mathrm{ord}_A(f)$$

The equalities hold by Algebra, Lemmas 121.6 and 121.7. (Using these lemmas isn't necessary, but convenient.) $\square$

Remark 23.5. Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. We will see later (in Lemma 69.3) that the map

$$\mathrm{CH}_k(X) \longrightarrow K_0(\mathit{Coh}_{k+1}(X)/\mathit{Coh}_{\leq k-1}(X))$$

of Lemma 23.4 is injective. Composing with the canonical map

$$K_0(\mathit{Coh}_{k+1}(X)/\mathit{Coh}_{\leq k-1}(X)) \longrightarrow K_0(\mathit{Coh}(X)/\mathit{Coh}_{\leq k-1}(X))$$

we obtain a canonical map

$$\mathrm{CH}_k(X) \longrightarrow K_0(\textit{Coh}(X)/\textit{Coh}_{\leq k-1}(X)).$$

We have not been able to find a statement or conjecture in the literature as to whether this map should be injective or not. It seems reasonable to expect the kernel of this map to be torsion. We will return to this question (insert future reference).

Lemma 23.6. *Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Denote $\textit{Coh}_Z(X) \subset \textit{Coh}(X)$ the Serre subcategory of coherent $\mathcal{O}_X$-modules whose set theoretic support is contained in Z. Then the exact inclusion functor $\textit{Coh}(Z) \to \textit{Coh}_Z(X)$ induces an isomorphism*

$$K'_0(Z) = K_0(\textit{Coh}(Z)) \longrightarrow K_0(\textit{Coh}_Z(X))$$

Proof. Let $\mathcal{F}$ be an object of $\textit{Coh}_Z(X)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf of Z. Consider the descending filtration

$$\ldots \subset \mathcal{F}^p = \mathcal{I}^p\mathcal{F} \subset \mathcal{F}^{p-1} \subset \ldots \subset \mathcal{F}^0 = \mathcal{F}$$

Exactly as in the proof of Lemma 23.4 this filtration is locally finite and hence $\bigoplus_{p\geq 0} \mathcal{F}^p$, $\bigoplus_{p\geq 1} \mathcal{F}^p$, and $\bigoplus_{p\geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}$ are coherent $\mathcal{O}_X$-modules supported on Z. Hence we get

$$[\mathcal{F}] = [\bigoplus\nolimits_{p\geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}]$$

in $K_0(\textit{Coh}_Z(X))$ exactly as in the proof of Lemma 23.4. Since the coherent module $\bigoplus_{p\geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}$ is annihilated by $\mathcal{I}$ we conclude that $[\mathcal{F}]$ is in the image. Actually, we claim that the map

$$\mathcal{F} \longmapsto c(\mathcal{F}) = [\bigoplus\nolimits_{p\geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}]$$

factors through $K_0(\textit{Coh}_Z(X))$ and is an inverse to the map in the statement of the lemma. To see this all we have to show is that if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence in $\textit{Coh}_Z(X)$, then we get $c(\mathcal{G}) = c(\mathcal{F}) + c(\mathcal{H})$. Observe that for all $q \geq 0$ we have a short exact sequence

$$0 \to (\mathcal{F} \cap \mathcal{I}^q\mathcal{G})/(\mathcal{F} \cap \mathcal{I}^{q+1}\mathcal{G}) \to \mathcal{G}^q/\mathcal{G}^{q+1} \to \mathcal{H}^q/\mathcal{H}^{q+1} \to 0$$

For $p, q \geq 0$ consider the coherent submodule

$$\mathcal{F}^{p,q} = \mathcal{I}^p\mathcal{F} \cap \mathcal{I}^q\mathcal{G}$$

Arguing exactly as above and using that the filtrations $\mathcal{F}^p = \mathcal{I}^p\mathcal{F}$ and $\mathcal{F} \cap \mathcal{I}^q\mathcal{G}$ are locally finite, we find that

$$[\bigoplus\nolimits_{p\geq 0} \mathcal{F}^p/\mathcal{F}^{p+1}] = [\bigoplus\nolimits_{p,q\geq 0} \mathcal{F}^{p,q}/(\mathcal{F}^{p+1,q}+\mathcal{F}^{p,q+1})] = [\bigoplus\nolimits_{q\geq 0} (\mathcal{F}\cap\mathcal{I}^q\mathcal{G})/(\mathcal{F}\cap\mathcal{I}^{q+1}\mathcal{G})]$$

in $K_0(\textit{Coh}(Z))$. Combined with the exact sequences above we obtain the desired result. Some details omitted. $\square$

24. The divisor associated to an invertible sheaf

The following definition is the analogue of Divisors, Definition 27.4 in our current setup.

Definition 24.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the *Weil divisor associated to* s is the $(n-1)$-cycle
$$\text{div}_\mathcal{L}(s) = \sum \text{ord}_{Z,\mathcal{L}}(s)[Z]$$
defined in Divisors, Definition 27.4. This makes sense because Weil divisors have δ-dimension $n-1$ by Lemma 16.1.

(2) We define *Weil divisor associated to* $\mathcal{L}$ as
$$c_1(\mathcal{L}) \cap [X] = \text{class of } \text{div}_\mathcal{L}(s) \in \text{CH}_{n-1}(X)$$
where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Divisors, Lemma 27.3.

Let X and S be as in Definition 24.1 above. Set $n = \dim_\delta(X)$. It is clear from the definitions that $Cl(X) = \text{CH}_{n-1}(X)$ where $Cl(X)$ is the Weil divisor class group of X as defined in Divisors, Definition 26.7. The map
$$\text{Pic}(X) \longrightarrow \text{CH}_{n-1}(X), \quad \mathcal{L} \longmapsto c_1(\mathcal{L}) \cap [X]$$
is the same as the map $\text{Pic}(X) \to Cl(X)$ constructed in Divisors, Equation (27.5.1) for arbitrary locally Noetherian integral schemes. In particular, this map is a homomorphism of abelian groups, it is injective if X is a normal scheme, and an isomorphism if all local rings of X are UFDs. See Divisors, Lemmas 27.6 and 27.7.

There are some cases where it is easy to compute the Weil divisor associated to an invertible sheaf.

Lemma 24.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a nonzero global section. Then*
$$div_\mathcal{L}(s) = [Z(s)]_{n-1}$$
in $Z_{n-1}(X)$ and
$$c_1(\mathcal{L}) \cap [X] = [Z(s)]_{n-1}$$
in $\text{CH}_{n-1}(X)$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. Let $\xi \in Z$ be its generic point. Choose a generator $s_\xi \in \mathcal{L}_\xi$. Write $s = f s_\xi$ for some $f \in \mathcal{O}_{X,\xi}$. By definition of $Z(s)$, see Divisors, Definition 14.8 we see that $Z(s)$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{I}_\xi = (f)$. Hence $\text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{Z(s),\xi}) = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,\xi}/(f)) = \text{ord}_{\mathcal{O}_{X,x}}(f)$ as desired. $\square$

The following lemma will be superseded by the more general Lemma 26.2.

Lemma 24.3. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Let $f : X \to Y$ be a flat morphism of relative dimension r. Then*
$$f^*(c_1(\mathcal{L}) \cap [Y]) = c_1(f^*\mathcal{L}) \cap [X]$$

in $\mathrm{CH}_{n+r-1}(X)$.

Proof. Let s be a nonzero meromorphic section of $\mathcal{L}$. We will show that actually $f^*\mathrm{div}_\mathcal{L}(s) = \mathrm{div}_{f^*\mathcal{L}}(f^*s)$ and hence the lemma holds. To see this let $\xi \in Y$ be a point and let $s_\xi \in \mathcal{L}_\xi$ be a generator. Write $s = gs_\xi$ with $g \in R(Y)^*$. Then there is an open neighbourhood $V \subset Y$ of ξ such that $s_\xi \in \mathcal{L}(V)$ and such that s_ξ generates $\mathcal{L}|_V$. Hence we see that

$$\mathrm{div}_\mathcal{L}(s)|_V = \mathrm{div}_Y(g)|_V.$$

In exactly the same way, since f^*s_ξ generates $f^*\mathcal{L}$ over $f^{-1}(V)$ and since $f^*s = gf^*s_\xi$ we also have

$$\mathrm{div}_\mathcal{L}(f^*s)|_{f^{-1}(V)} = \mathrm{div}_X(g)|_{f^{-1}(V)}.$$

Thus the desired equality of cycles over $f^{-1}(V)$ follows from the corresponding result for pullbacks of principal divisors, see Lemma 17.2. □

25. Intersecting with an invertible sheaf

In this section we study the following construction.

Definition 25.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We define, for every integer k, an operation

$$c_1(\mathcal{L}) \cap - : Z_{k+1}(X) \to \mathrm{CH}_k(X)$$

called *intersection with the first Chern class of $\mathcal{L}$*.

(1) Given an integral closed subscheme $i : W \to X$ with $\dim_\delta(W) = k+1$ we define
$$c_1(\mathcal{L}) \cap [W] = i_*(c_1(i^*\mathcal{L}) \cap [W])$$
where the right hand side is defined in Definition 24.1.

(2) For a general $(k+1)$-cycle $\alpha = \sum n_i[W_i]$ we set
$$c_1(\mathcal{L}) \cap \alpha = \sum n_i c_1(\mathcal{L}) \cap [W_i]$$

Write each $c_1(\mathcal{L}) \cap W_i = \sum_j n_{i,j}[Z_{i,j}]$ with $\{Z_{i,j}\}_j$ a locally finite sum of integral closed subschemes of W_i. Since $\{W_i\}$ is a locally finite collection of integral closed subschemes on X, it follows easily that $\{Z_{i,j}\}_{i,j}$ is a locally finite collection of closed subschemes of X. Hence $c_1(\mathcal{L}) \cap \alpha = \sum n_i n_{i,j}[Z_{i,j}]$ is a cycle. Another, more convenient, way to think about this is to observe that the morphism $\coprod W_i \to X$ is proper. Hence $c_1(\mathcal{L}) \cap \alpha$ can be viewed as the pushforward of a class in $\mathrm{CH}_k(\coprod W_i) = \prod \mathrm{CH}_k(W_i)$. This also explains why the result is well defined up to rational equivalence on X.

The main goal for the next few sections is to show that intersecting with $c_1(\mathcal{L})$ factors through rational equivalence. This is not a triviality.

Lemma 25.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be an invertible sheaves on X. Then*

$$c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap \alpha$$

in $\mathrm{CH}_k(X)$ for every $\alpha \in Z_{k+1}(X)$. Moreover, $c_1(\mathcal{O}_X) \cap \alpha = 0$ for all α.

Proof. The additivity follows directly from Divisors, Lemma 27.5 and the definitions. To see that $c_1(\mathcal{O}_X) \cap \alpha = 0$ consider the section $1 \in \Gamma(X, \mathcal{O}_X)$. This restricts to an everywhere nonzero section on any integral closed subscheme $W \subset X$. Hence $c_1(\mathcal{O}_X) \cap [W] = 0$ as desired. □

Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X, see Divisors, Definition 14.8.

Lemma 25.3. *Let (S, δ) be as in Situation 7.1. Let Y be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Let $s \in \Gamma(Y, \mathcal{L})$. Assume*

1. $\dim_\delta(Y) \leq k + 1$,
2. $\dim_\delta(Z(s)) \leq k$, *and*
3. *for every generic point ξ of an irreducible component of $Z(s)$ of δ-dimension k the multiplication by s induces an injection $\mathcal{O}_{Y,\xi} \to \mathcal{L}_\xi$.*

Write $[Y]_{k+1} = \sum n_i[Y_i]$ where $Y_i \subset Y$ are the irreducible components of Y of δ-dimension $k+1$. Set $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$. Then

$$[Z(s)]_k = \sum n_i[Z(s_i)]_k \tag{25.3.1}$$

as k-cycles on Y.

Proof. Let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be its generic point. We want to compare the coefficient n of $[Z]$ in the expression $\sum n_i[Z(s_i)]_k$ with the coefficient m of $[Z]$ in the expression $[Z(s)]_k$. Choose a generator $s_\xi \in \mathcal{L}_\xi$. Write $A = \mathcal{O}_{Y,\xi}$, $L = \mathcal{L}_\xi$. Then $L = As_\xi$. Write $s = fs_\xi$ for some (unique) $f \in A$. Hypothesis (3) means that $f : A \to A$ is injective. Since $\dim_\delta(Y) \leq k+1$ and $\dim_\delta(Z) = k$ we have $\dim(A) = 0$ or 1. We have

$$m = \text{length}_A(A/(f))$$

which is finite in either case.

If $\dim(A) = 0$, then $f : A \to A$ being injective implies that $f \in A^*$. Hence in this case m is zero. Moreover, the condition $\dim(A) = 0$ means that ξ does not lie on any irreducible component of δ-dimension $k+1$, i.e., $n = 0$ as well.

Now, let $\dim(A) = 1$. Since A is a Noetherian local ring it has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$. These correspond 1-1 with the Y_i passing through ξ'. Moreover $n_i = \text{length}_{A_{\mathfrak{q}_i}}(A_{\mathfrak{q}_i})$. Also, the multiplicity of $[Z]$ in $[Z(s_i)]_k$ is $\text{length}_A(A/(f, \mathfrak{q}_i))$. Hence the equation to prove in this case is

$$\text{length}_A(A/(f)) = \sum \text{length}_{A_{\mathfrak{q}_i}}(A_{\mathfrak{q}_i})\text{length}_A(A/(f, \mathfrak{q}_i))$$

which follows from Lemma 3.2. $\square$

The following lemma is a useful result in order to compute the intersection product of the c_1 of an invertible sheaf and the cycle associated to a closed subscheme. Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X, see Divisors, Definition 14.8.

Lemma 25.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Y \subset X$ be a closed subscheme. Let $s \in \Gamma(Y, \mathcal{L}|_Y)$. Assume*

1. $\dim_\delta(Y) \leq k + 1$,
2. $\dim_\delta(Z(s)) \leq k$, *and*
3. *for every generic point ξ of an irreducible component of $Z(s)$ of δ-dimension k the multiplication by s induces an injection $\mathcal{O}_{Y,\xi} \to (\mathcal{L}|_Y)_\xi$*[1].

[1]For example, this holds if s is a regular section of $\mathcal{L}|_Y$.

Then

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = [Z(s)]_k$$

in $\mathrm{CH}_k(X)$.

Proof. Write

$$[Y]_{k+1} = \sum n_i[Y_i]$$

where $Y_i \subset Y$ are the irreducible components of Y of δ-dimension $k+1$ and $n_i > 0$. By assumption the restriction $s_i = s|_{Y_i} \in \Gamma(Y_i, \mathcal{L}|_{Y_i})$ is not zero, and hence is a regular section. By Lemma 24.2 we see that $[Z(s_i)]_k$ represents $c_1(\mathcal{L}|_{Y_i})$. Hence by definition

$$c_1(\mathcal{L}) \cap [Y]_{k+1} = \sum n_i[Z(s_i)]_k$$

Thus the result follows from Lemma 25.3. □

26. Intersecting with an invertible sheaf and push and pull

In this section we prove that the operation $c_1(\mathcal{L}) \cap -$ commutes with flat pullback and proper pushforward.

Lemma 26.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Assume Y is integral and $n = \dim_\delta(Y)$. Let s be a nonzero meromorphic section of $\mathcal{L}$. Then we have*

$$f^* div_{\mathcal{L}}(s) = \sum n_i div_{f^*\mathcal{L}|_{X_i}}(s_i)$$

in $Z_{n+r-1}(X)$. Here the sum is over the irreducible components $X_i \subset X$ of δ-dimension $n + r$, the section $s_i = f|_{X_i}^*(s)$ is the pullback of s, and $n_i = m_{X_i,X}$ is the multiplicity of X_i in X.

Proof. To prove this equality of cycles, we may work locally on Y. Hence we may assume Y is affine and $s = p/q$ for some nonzero sections $p \in \Gamma(Y, \mathcal{L})$ and $q \in \Gamma(Y, \mathcal{O})$. If we can show both

$$f^*\mathrm{div}_{\mathcal{L}}(p) = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{X_i}}(p_i) \quad \text{and} \quad f^*\mathrm{div}_{\mathcal{O}}(q) = \sum n_i \mathrm{div}_{\mathcal{O}_{X_i}}(q_i)$$

(with obvious notations) then we win by the additivity, see Divisors, Lemma 27.5. Thus we may assume that $s \in \Gamma(Y, \mathcal{L})$. In this case we may apply the equality (25.3.1) to see that

$$[Z(f^*(s))]_{k+r-1} = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{X_i}}(s_i)$$

where $f^*(s) \in f^*\mathcal{L}$ denotes the pullback of s to X. On the other hand we have

$$f^*\mathrm{div}_{\mathcal{L}}(s) = f^*[Z(s)]_{k-1} = [f^{-1}(Z(s))]_{k+r-1},$$

by Lemmas 24.2 and 14.4. Since $Z(f^*(s)) = f^{-1}(Z(s))$ we win. □

Lemma 26.2. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Let α be a k-cycle on Y. Then*

$$f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^*\mathcal{L}) \cap f^*\alpha$$

in $\mathrm{CH}_{k+r-1}(X)$.

Proof. Write $\alpha = \sum n_i[W_i]$. We will show that

$$f^*(c_1(\mathcal{L}) \cap [W_i]) = c_1(f^*\mathcal{L}) \cap f^*[W_i]$$

in $\mathrm{CH}_{k+r-1}(X)$ by producing a rational equivalence on the closed subscheme $f^{-1}(W_i)$ of X. By the discussion in Remark 19.6 this will prove the equality of the lemma is true.

Let $W \subset Y$ be an integral closed subscheme of δ-dimension k. Consider the closed subscheme $W' = f^{-1}(W) = W \times_Y X$ so that we have the fibre product diagram

$$\begin{tikzcd} W' \arrow[r] \arrow[d, "h"'] & X \arrow[d, "f"] \\ W \arrow[r] & Y \end{tikzcd}$$

We have to show that $f^*(c_1(\mathcal{L}) \cap [W]) = c_1(f^*\mathcal{L}) \cap f^*[W]$. Choose a nonzero meromorphic section s of $\mathcal{L}|_W$. Let $W'_i \subset W'$ be the irreducible components of δ-dimension $k + r$. Write $[W']_{k+r} = \sum n_i[W'_i]$ with n_i the multiplicity of W'_i in W' as per definition. So $f^*[W] = \sum n_i[W'_i]$ in $Z_{k+r}(X)$. Since each $W'_i \to W$ is dominant we see that $s_i = s|_{W'_i}$ is a nonzero meromorphic section for each i. By Lemma 26.1 we have the following equality of cycles

$$h^*\mathrm{div}_{\mathcal{L}|_W}(s) = \sum n_i \mathrm{div}_{f^*\mathcal{L}|_{W'_i}}(s_i)$$

in $Z_{k+r-1}(W')$. This finishes the proof since the left hand side is a cycle on W' which pushes to $f^*(c_1(\mathcal{L}) \cap [W])$ in $\mathrm{CH}_{k+r-1}(X)$ and the right hand side is a cycle on W' which pushes to $c_1(f^*\mathcal{L}) \cap f^*[W]$ in $\mathrm{CH}_{k+r-1}(X)$. $\square$

Lemma 26.3. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a proper morphism. Let $\mathcal{L}$ be an invertible sheaf on Y. Let s be a nonzero meromorphic section s of $\mathcal{L}$ on Y. Assume X, Y integral, f dominant, and $\dim_\delta(X) = \dim_\delta(Y)$. Then*

$$f_*\left(div_{f^*\mathcal{L}}(f^*s)\right) = [R(X) : R(Y)] div_{\mathcal{L}}(s).$$

as cycles on Y. In particular

$$f_*(c_1(f^*\mathcal{L}) \cap [X]) = [R(X) : R(Y)]c_1(\mathcal{L}) \cap [Y] = c_1(\mathcal{L}) \cap f_*[X]$$

Proof. The last equation follows from the first since $f_*[X] = [R(X) : R(Y)][Y]$ by definition. It turns out that we can re-use Lemma 18.1 to prove this. Namely, since we are trying to prove an equality of cycles, we may work locally on Y. Hence we may assume that $\mathcal{L} = \mathcal{O}_Y$. In this case s corresponds to a rational function $g \in R(Y)$, and we are simply trying to prove

$$f_*\left(\mathrm{div}_X(g)\right) = [R(X) : R(Y)]\mathrm{div}_Y(g).$$

Comparing with the result of the aforementioned Lemma 18.1 we see this true since $\mathrm{Nm}_{R(X)/R(Y)}(g) = g^{[R(X):R(Y)]}$ as $g \in R(Y)^*$. $\square$

Lemma 26.4. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Let $\alpha \in Z_{k+1}(X)$. Let $\mathcal{L}$ be an invertible sheaf on Y. Then*

$$p_*(c_1(p^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap p_*\alpha$$

in $\mathrm{CH}_k(Y)$.

Proof. Suppose that p has the property that for every integral closed subscheme $W \subset X$ the map $p|_W : W \to Y$ is a closed immersion. Then, by definition of capping with $c_1(\mathcal{L})$ the lemma holds.

We will use this remark to reduce to a special case. Namely, write $\alpha = \sum n_i[W_i]$ with $n_i \neq 0$ and W_i pairwise distinct. Let $W_i' \subset Y$ be the image of W_i (as an integral closed subscheme). Consider the diagram

$$\begin{tikzcd} X' = \coprod W_i \arrow[r, "q"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' = \coprod W_i' \arrow[r, "q'"] & Y. \end{tikzcd}$$

Since $\{W_i\}$ is locally finite on X, and p is proper we see that $\{W_i'\}$ is locally finite on Y and that q, q', p' are also proper morphisms. We may think of $\sum n_i[W_i]$ also as a k-cycle $\alpha' \in Z_k(X')$. Clearly $q_*\alpha' = \alpha$. We have $q_*(c_1(q^*p^*\mathcal{L}) \cap \alpha') = c_1(p^*\mathcal{L}) \cap q_*\alpha'$ and $(q')_*(c_1((q')^*\mathcal{L}) \cap p'_*\alpha') = c_1(\mathcal{L}) \cap q'_*p'_*\alpha'$ by the initial remark of the proof. Hence it suffices to prove the lemma for the morphism p' and the cycle $\sum n_i[W_i]$. Clearly, this means we may assume X, Y integral, $f : X \to Y$ dominant and $\alpha = [X]$. In this case the result follows from Lemma 26.3. $\square$

27. The key formula

Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$ and $\mathcal{N}$ be invertible sheaves on X. Let s be a nonzero meromorphic section of $\mathcal{L}$ and let t be a nonzero meromorphic section of $\mathcal{N}$. Let $Z_i \subset X$, $i \in I$ be a locally finite set of irreducible closed subsets of codimension 1 with the following property: If $Z \not\in \{Z_i\}$ with generic point ξ, then s is a generator for $\mathcal{L}_\xi$ and t is a generator for $\mathcal{N}_\xi$. Such a set exists by Divisors, Lemma 27.2. Then

$$\mathrm{div}_\mathcal{L}(s) = \sum \mathrm{ord}_{Z_i,\mathcal{L}}(s)[Z_i]$$

and similarly

$$\mathrm{div}_\mathcal{N}(t) = \sum \mathrm{ord}_{Z_i,\mathcal{N}}(t)[Z_i]$$

Unwinding the definitions more, we pick for each i generators $s_i \in \mathcal{L}_{\xi_i}$ and $t_i \in \mathcal{N}_{\xi_i}$ where ξ_i is the generic point of Z_i. Then we can write

$$s = f_i s_i \quad \text{and} \quad t = g_i t_i$$

Set $B_i = \mathcal{O}_{X,\xi_i}$. Then by definition

$$\mathrm{ord}_{Z_i,\mathcal{L}}(s) = \mathrm{ord}_{B_i}(f_i) \quad \text{and} \quad \mathrm{ord}_{Z_i,\mathcal{N}}(t) = \mathrm{ord}_{B_i}(g_i)$$

Since t_i is a generator of $\mathcal{N}_{\xi_i}$ we see that its image in the fibre $\mathcal{N}_{\xi_i} \otimes \kappa(\xi_i)$ is a nonzero meromorphic section of $\mathcal{N}|_{Z_i}$. We will denote this image $t_i|_{Z_i}$. From our definitions it follows that

$$c_1(\mathcal{N}) \cap \mathrm{div}_\mathcal{L}(s) = \sum \mathrm{ord}_{B_i}(f_i)(Z_i \to X)_*\mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

and similarly

$$c_1(\mathcal{L}) \cap \mathrm{div}_\mathcal{N}(t) = \sum \mathrm{ord}_{B_i}(g_i)(Z_i \to X)_*\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

in $\mathrm{CH}_{n-2}(X)$. We are going to find a rational equivalence between these two cycles. To do this we consider the tame symbol

$$\partial_{B_i}(f_i, g_i) \in \kappa(\xi_i)^*$$

see Section 5.

Lemma 27.1 (Key formula)**.** *In the situation above the cycle*

$$\sum (Z_i \to X)_* \left(ord_{B_i}(f_i) div_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - ord_{B_i}(g_i) div_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) \right)$$

is equal to the cycle

$$\sum (Z_i \to X)_* div(\partial_{B_i}(f_i, g_i))$$

Proof. First, let us examine what happens if we replace s_i by us_i for some unit u in B_i. Then f_i gets replaced by $u^{-1}f_i$. Thus the first part of the first expression of the lemma is unchanged and in the second part we add

$$-\mathrm{ord}_{B_i}(g_i)\mathrm{div}(u|_{Z_i})$$

(where $u|_{Z_i}$ is the image of u in the residue field) by Divisors, Lemma 27.3 and in the second expression we add

$$\mathrm{div}(\partial_{B_i}(u^{-1}, g_i))$$

by bi-linearity of the tame symbol. These terms agree by property (6) of the tame symbol.

Let $Z \subset X$ be an irreducible closed with $\dim_\delta(Z) = n - 2$. To show that the coefficients of Z of the two cycles of the lemma is the same, we may do a replacement $s_i \mapsto us_i$ as in the previous paragraph. In exactly the same way one shows that we may do a replacement $t_i \mapsto vt_i$ for some unit v of B_i.

Since we are proving the equality of cycles we may argue one coefficient at a time. Thus we choose an irreducible closed $Z \subset X$ with $\dim_\delta(Z) = n - 2$ and compare coefficients. Let $\xi \in Z$ be the generic point and set $A = \mathcal{O}_{X,\xi}$. This is a Noetherian local domain of dimension 2. Choose generators σ and τ for $\mathcal{L}_\xi$ and $\mathcal{N}_\xi$. After shrinking X, we may and do assume σ and τ define trivializations of the invertible sheaves $\mathcal{L}$ and $\mathcal{N}$ over all of X. Because Z_i is locally finite after shrinking X we may assume $Z \subset Z_i$ for all $i \in I$ and that I is finite. Then ξ_i corresponds to a prime $\mathfrak{q}_i \subset A$ of height 1. We may write $s_i = a_i\sigma$ and $t_i = b_i\tau$ for some a_i and b_i units in $A_{\mathfrak{q}_i}$. By the remarks above, it suffices to prove the lemma when $a_i = b_i = 1$ for all i.

Assume $a_i = b_i = 1$ for all i. Then the first expression of the lemma is zero, because we choose σ and τ to be trivializing sections. Write $s = f\sigma$ and $t = g\tau$ with f and g in the fraction field of A. By the previous paragraph we have reduced to the case $f_i = f$ and $g_i = g$ for all i. Moreover, for a height 1 prime $\mathfrak{q}$ of A which is not in $\{\mathfrak{q}_i\}$ we have that both f and g are units in $A_\mathfrak{q}$ (by our choice of the family $\{Z_i\}$ in the discussion preceding the lemma). Thus the coefficient of Z in the second expression of the lemma is

$$\sum\nolimits_i \mathrm{ord}_{A/\mathfrak{q}_i}(\partial_{B_i}(f, g))$$

which is zero by the key Lemma 6.3. $\square$

Remark 27.2. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. We claim that there is a complex

$$\bigoplus\nolimits'_{\delta(x)=k+2} K_2^M(\kappa(x)) \xrightarrow{\partial} \bigoplus\nolimits'_{\delta(x)=k+1} K_1^M(\kappa(x)) \xrightarrow{\partial} \bigoplus\nolimits'_{\delta(x)=k} K_0^M(\kappa(x))$$

Here we use notation and conventions introduced in Remark 19.2 and in addition

1. $K_2^M(\kappa(x))$ is the degree 2 part of the Milnor K-theory of the residue field $\kappa(x)$ of the point $x \in X$ (see Remark 6.4) which is the quotient of $\kappa(x)^* \otimes_{\mathbf{Z}} \kappa(x)^*$ by the subgroup generated by elements of the form $\lambda \otimes (1-\lambda)$ for $\lambda \in \kappa(x) \setminus \{0, 1\}$, and
2. the first differential ∂ is defined as follows: given an element $\xi = \sum_x \alpha_x$ in the first term we set
$$\partial(\xi) = \sum\nolimits_{x \rightsquigarrow x',\ \delta(x')=k+1} \partial_{\mathcal{O}_{W_x, x'}}(\alpha_x)$$
where $\partial_{\mathcal{O}_{W_x,x'}} : K_2^M(\kappa(x)) \to K_1^M(\kappa(x))$ is the tame symbol constructed in Section 5.

We claim that we get a complex, i.e., that $\partial \circ \partial = 0$. To see this it suffices to take an element ξ as above and a point $x'' \in X$ with $\delta(x'') = k$ and check that the coefficient of x'' in the element $\partial(\partial(\xi))$ is zero. Because $\xi = \sum \alpha_x$ is a locally finite sum, we may in fact assume by additivity that $\xi = \alpha_x$ for some $x \in X$ with $\delta(x) = k+2$ and $\alpha_x \in K_2^M(\kappa(x))$. By linearity again we may assume that $\alpha_x = f \otimes g$ for some $f, g \in \kappa(x)^*$. Denote $W \subset X$ the integral closed subscheme with generic point x. If $x'' \notin W$, then it is immediately clear that the coefficient of x in $\partial(\partial(\xi))$ is zero. If $x'' \in W$, then we see that the coefficient of x'' in $\partial(\partial(x))$ is equal to

$$\sum\nolimits_{x \rightsquigarrow x' \rightsquigarrow x'',\ \delta(x')=k+1} \text{ord}_{\mathcal{O}_{\overline{\{x'\}}, x''}}(\partial_{\mathcal{O}_{W,x'}}(f, g))$$

The key algebraic Lemma 6.3 says exactly that this is zero.

Remark 27.3. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$. The complex in Remark 27.2 and the presentation of $\text{CH}_k(X)$ in Remark 19.2 suggests that we can define a first higher Chow group

$$\text{CH}_k^M(X, 1) = H_1(\text{the complex of Remark 27.2})$$

We use the supscript ^{M} to distinguish our notation from the higher chow groups defined in the literature, e.g., in the papers by Spencer Bloch ([Blo86] and [Blo94]). Let $U \subset X$ be open with complement $Y \subset X$ (viewed as reduced closed subscheme). Then we find a split short exact sequence

$$0 \to \bigoplus\nolimits'_{y \in Y, \delta(y)=k+i} K_i^M(\kappa(y)) \to \bigoplus\nolimits'_{x \in X, \delta(x)=k+i} K_i^M(\kappa(x)) \to \bigoplus\nolimits'_{u \in U, \delta(u)=k+i} K_i^M(\kappa(u)) \to 0$$

for $i = 2, 1, 0$ compatible with the boundary maps in the complexes of Remark 27.2. Applying the snake lemma (see Homology, Lemma 13.6) we obtain a six term exact sequence

$$\text{CH}_k^M(Y, 1) \to \text{CH}_k^M(X, 1) \to \text{CH}_k^M(U, 1) \to \text{CH}_k(Y) \to \text{CH}_k(X) \to \text{CH}_k(U) \to 0$$

extending the canonical exact sequence of Lemma 19.3. With some work, one may also define flat pullback and proper pushforward for the first higher chow group $\text{CH}_k^M(X, 1)$. We will return to this later (insert future reference here).

28. Intersecting with an invertible sheaf and rational equivalence

Applying the key lemma we obtain the fundamental properties of intersecting with invertible sheaves. In particular, we will see that $c_1(\mathcal{L}) \cap -$ factors through rational equivalence and that these operations for different invertible sheaves commute.

Lemma 28.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. Choose a nonzero meromorphic section s of $\mathcal{L}$ and a nonzero meromorphic section t of $\mathcal{N}$. Set $\alpha = div_{\mathcal{L}}(s)$ and $\beta = div_{\mathcal{N}}(t)$. Then*

$$c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L}) \cap \beta$$

in $\mathrm{CH}_{n-2}(X)$.

Proof. Immediate from the key Lemma 27.1 and the discussion preceding it. $\square$

Lemma 28.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be invertible on X. The operation $\alpha \mapsto c_1(\mathcal{L}) \cap \alpha$ factors through rational equivalence to give an operation*

$$c_1(\mathcal{L}) \cap - : \mathrm{CH}_{k+1}(X) \to \mathrm{CH}_k(X)$$

Proof. Let $\alpha \in Z_{k+1}(X)$, and $\alpha \sim_{rat} 0$. We have to show that $c_1(\mathcal{L}) \cap \alpha$ as defined in Definition 25.1 is zero. By Definition 19.1 there exists a locally finite family $\{W_j\}$ of integral closed subschemes with $\dim_\delta(W_j) = k + 2$ and rational functions $f_j \in R(W_j)^*$ such that

$$\alpha = \sum (i_j)_* \mathrm{div}_{W_j}(f_j)$$

Note that $p : \coprod W_j \to X$ is a proper morphism, and hence $\alpha = p_*\alpha'$ where $\alpha' \in Z_{k+1}(\coprod W_j)$ is the sum of the principal divisors $\mathrm{div}_{W_j}(f_j)$. By Lemma 26.4 we have $c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to show that each $c_1(\mathcal{L}|_{W_j}) \cap \mathrm{div}_{W_j}(f_j)$ is zero. In other words we may assume that X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$. We can think of f as a regular meromorphic section of the invertible sheaf $\mathcal{N} = \mathcal{O}_X$. Choose a meromorphic section s of $\mathcal{L}$ and denote $\beta = \mathrm{div}_{\mathcal{L}}(s)$. By Lemma 28.1 we conclude that

$$c_1(\mathcal{L}) \cap \alpha = c_1(\mathcal{O}_X) \cap \beta.$$

However, by Lemma 25.2 we see that the right hand side is zero in $\mathrm{CH}_k(X)$ as desired. $\square$

Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be invertible on X. We will denote

$$c_1(\mathcal{L}) \cap - : \mathrm{CH}_{k+1}(X) \to \mathrm{CH}_k(X)$$

the operation $c_1(\mathcal{L}) \cap -$. This makes sense by Lemma 28.2. We will denote $c_1(\mathcal{L})^s \cap -$ the s-fold iterate of this operation for all $s \geq 0$.

Lemma 28.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. For any $\alpha \in \mathrm{CH}_{k+2}(X)$ we have*

$$c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha$$

as elements of $\mathrm{CH}_k(X)$.

Proof. Write $\alpha = \sum m_j[Z_j]$ for some locally finite collection of integral closed subschemes $Z_j \subset X$ with $\dim_\delta(Z_j) = k+2$. Consider the proper morphism $p : \coprod Z_j \to X$. Set $\alpha' = \sum m_j[Z_j]$ as a $(k+2)$-cycle on $\coprod Z_j$. By several applications of Lemma 26.4 we see that $c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap c_1(p^*\mathcal{N}) \cap \alpha')$ and $c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{N}) \cap c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to prove the formula in case X is integral and $\alpha = [X]$. In this case the result follows from Lemma 28.1 and the definitions. $\square$

29. Gysin homomorphisms

In this section we define the gysin map for the zero locus D of a section of an invertible sheaf. An interesting case occurs when D is an effective Cartier divisor, but the generalization to arbitrary D allows us a flexibility to formulate various compatibilities, see Remark 29.7 and Lemmas 29.8, 29.9, and 30.5. These results can be generalized to locally principal closed subschemes endowed with a virtual normal bundle (Remark 29.2) or to pseudo-divisors (Remark 29.3).

Recall that effective Cartier divisors correspond 1-to-1 to isomorphism classes of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf and s is a regular global section, see Divisors, Lemma 14.10. If D corresponds to $(\mathcal{L}, s)$, then $\mathcal{L} = \mathcal{O}_X(D)$. Please keep this in mind while reading this section.

Definition 29.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s)$ be a pair consisting of an invertible sheaf and a global section $s \in \Gamma(X, \mathcal{L})$. Let $D = Z(s)$ be the zero scheme of s, and denote $i : D \to X$ the closed immersion. We define, for every integer k, a *Gysin homomorphism*

$$i^* : Z_{k+1}(X) \to \mathrm{CH}_k(D).$$

by the following rules:

(1) Given a integral closed subscheme $W \subset X$ with $\dim_\delta(W) = k+1$ we define
   (a) if $W \not\subset D$, then $i^*[W] = [D \cap W]_k$ as a k-cycle on D, and
   (b) if $W \subset D$, then $i^*[W] = i'_*(c_1(\mathcal{L}|_W) \cap [W])$, where $i' : W \to D$ is the induced closed immersion.
(2) For a general $(k+1)$-cycle $\alpha = \sum n_j[W_j]$ we set
$$i^*\alpha = \sum n_j i^*[W_j]$$
(3) If D is an effective Cartier divisor, then we denote $D \cdot \alpha = i_*i^*\alpha$ the pushforward of the class $i^*\alpha$ to a class on X.

In fact, as we will see later, this Gysin homomorphism i^* can be viewed as an example of a non-flat pullback. Thus we will sometimes informally call the class $i^*\alpha$ the *pullback* of the class α.

Remark 29.2. Let X be a scheme locally of finite type over S as in Situation 7.1. Let $(D, \mathcal{N}, \sigma)$ be a triple consisting of a locally principal (Divisors, Definition 13.1) closed subscheme $i : D \to X$, an invertible $\mathcal{O}_D$-module $\mathcal{N}$, and a surjection $\sigma : \mathcal{N}^{\otimes -1} \to i^*\mathcal{I}_D$ of $\mathcal{O}_D$-modules[2]. Here $\mathcal{N}$ should be thought of as a *virtual normal bundle of* D *in* X. The construction of $i^* : Z_{k+1}(X) \to \mathrm{CH}_k(D)$ in Definition 29.1 generalizes to such triples, see Section 54.

[2]This condition assures us that if D is an effective Cartier divisor, then $\mathcal{N} = \mathcal{O}_X(D)|_D$.

Remark 29.3. Let X be a scheme locally of finite type over S as in Situation 7.1. In [Ful98] a *pseudo-divisor* on X is defined as a triple $D = (\mathcal{L}, Z, s)$ where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, $Z \subset X$ is a closed subset, and $s \in \Gamma(X \setminus Z, \mathcal{L})$ is a nowhere vanishing section. Similarly to the above, one can define for every α in $\mathrm{CH}_{k+1}(X)$ a product $D \cdot \alpha$ in $\mathrm{CH}_k(Z \cap |\alpha|)$ where $|\alpha|$ is the support of α.

Lemma 29.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 29.1. Let α be a $(k+1)$-cycle on X. Then $i_* i^* \alpha = c_1(\mathcal{L}) \cap \alpha$ in $\mathrm{CH}_k(X)$. In particular, if D is an effective Cartier divisor, then $D \cdot \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$.*

Proof. Write $\alpha = \sum n_j [W_j]$ where $i_j : W_j \to X$ are integral closed subschemes with $\dim_\delta(W_j) = k$. Since D is the zero scheme of s we see that $D \cap W_j$ is the zero scheme of the restriction $s|_{W_j}$. Hence for each j such that $W_j \not\subset D$ we have $c_1(\mathcal{L}) \cap [W_j] = [D \cap W_j]_k$ by Lemma 25.4. So we have

$$c_1(\mathcal{L}) \cap \alpha = \sum\nolimits_{W_j \not\subset D} n_j [D \cap W_j]_k + \sum\nolimits_{W_j \subset D} n_j i_{j,*}(c_1(\mathcal{L})|_{W_j}) \cap [W_j])$$

in $\mathrm{CH}_k(X)$ by Definition 25.1. The right hand side matches (termwise) the push-forward of the class $i^*\alpha$ on D from Definition 29.1. Hence we win. $\square$

Lemma 29.5. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 29.1.*

1. *Let $Z \subset X$ be a closed subscheme such that $\dim_\delta(Z) \leq k+1$ and such that $D \cap Z$ is an effective Cartier divisor on Z. Then $i^*[Z]_{k+1} = [D \cap Z]_k$.*
2. *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(Supp(\mathcal{F})) \leq k+1$ and $s : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ is injective. Then*
$$i^*[\mathcal{F}]_{k+1} = [i^*\mathcal{F}]_k$$
in $\mathrm{CH}_k(D)$.

Proof. Assume $Z \subset X$ as in (1). Then set $\mathcal{F} = \mathcal{O}_Z$. The assumption that $D \cap Z$ is an effective Cartier divisor is equivalent to the assumption that $s : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ is injective. Moreover $[Z]_{k+1} = [\mathcal{F}]_{k+1}$ and $[D \cap Z]_k = [\mathcal{O}_{D \cap Z}]_k = [i^*\mathcal{F}]_k$. See Lemma 10.3. Hence part (1) follows from part (2).

Write $[\mathcal{F}]_{k+1} = \sum m_j [W_j]$ with $m_j > 0$ and pairwise distinct integral closed subschemes $W_j \subset X$ of δ-dimension $k+1$. The assumption that $s : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ is injective implies that $W_j \not\subset D$ for all j. By definition we see that

$$i^*[\mathcal{F}]_{k+1} = \sum m_j [D \cap W_j]_k.$$

We claim that

$$\sum [D \cap W_j]_k = [i^*\mathcal{F}]_k$$

as cycles. Let $Z \subset D$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be its generic point. Let $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$. Let $f \in A$ be an element generating the ideal of D, i.e., such that $\mathcal{O}_{D,\xi} = A/fA$. By assumption $\dim(\mathrm{Supp}(M)) = 1$, the map $f : M \to M$ is injective, and $\mathrm{length}_A(M/fM) < \infty$. Moreover, $\mathrm{length}_A(M/fM)$ is the coefficient of $[Z]$ in $[i^*\mathcal{F}]_k$. On the other hand, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then

$$\sum \mathrm{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \mathrm{ord}_{A/\mathfrak{q}_i}(f)$$

is the coefficient of $[Z]$ in $\sum [D \cap W_j]_k$. Hence we see the equality by Lemma 3.2. $\square$

Remark 29.6. Let $X \to S$, $\mathcal{L}$, s, $i : D \to X$ be as in Definition 29.1 and assume that $\mathcal{L}|_D \cong \mathcal{O}_D$. In this case we can define a canonical map $i^* : Z_{k+1}(X) \to Z_k(D)$ on cycles, by requiring that $i^*[W] = 0$ whenever $W \subset D$ is an integral closed subscheme. The possibility to do this will be useful later on.

Remark 29.7. Let $f : X' \to X$ be a morphism of schemes locally of finite type over S as in Situation 7.1. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 29.1. Then we can set $\mathcal{L}' = f^*\mathcal{L}$, $s' = f^*s$, and $D' = X' \times_X D = Z(s')$. This gives a commutative diagram

$$\begin{tikzcd} D' \arrow[r, "i'"'] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

and we can ask for various compatibilities between i^* and $(i')^*$.

Lemma 29.8. *Let (S, δ) be as in Situation 7.1. Let $f : X' \to X$ be a proper morphism of schemes locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 29.1. Form the diagram*

$$\begin{tikzcd} D' \arrow[r, "i'"'] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

as in Remark 29.7. For any $(k+1)$-cycle α' on X' we have $i^*f_*\alpha' = g_*(i')^*\alpha'$ in $\mathrm{CH}_k(D)$ *(this makes sense as f_* is defined on the level of cycles).*

Proof. Suppose $\alpha = [W']$ for some integral closed subscheme $W' \subset X'$. Let $W = f(W') \subset X$. In case $W' \not\subset D'$, then $W \not\subset D$ and we see that

$$[W' \cap D']_k = \mathrm{div}_{\mathcal{L}'|_{W'}}(s'|_{W'}) \quad \text{and} \quad [W \cap D]_k = \mathrm{div}_{\mathcal{L}|_W}(s|_W)$$

and hence f_* of the first cycle equals the second cycle by Lemma 26.3. Hence the equality holds as cycles. In case $W' \subset D'$, then $W \subset D$ and $f_*(c_1(\mathcal{L}|_{W'}) \cap [W'])$ is equal to $c_1(\mathcal{L}|_W) \cap [W]$ in $\mathrm{CH}_k(W)$ by the second assertion of Lemma 26.3. By Remark 19.6 the result follows for general α'. $\square$

Lemma 29.9. *Let (S, δ) be as in Situation 7.1. Let $f : X' \to X$ be a flat morphism of relative dimension r of schemes locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 29.1. Form the diagram*

$$\begin{tikzcd} D' \arrow[r, "i'"'] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

as in Remark 29.7. For any $(k+1)$-cycle α on X we have $(i')^*f^*\alpha = g^*i^*\alpha$ in $\mathrm{CH}_{k+r}(D')$ *(this makes sense as f^* is defined on the level of cycles).*

Proof. Suppose $\alpha = [W]$ for some integral closed subscheme $W \subset X$. Let $W' = f^{-1}(W) \subset X'$. In case $W \not\subset D$, then $W' \not\subset D'$ and we see that

$$W' \cap D' = g^{-1}(W \cap D)$$

as closed subschemes of D'. Hence the equality holds as cycles, see Lemma 14.4. In case $W \subset D$, then $W' \subset D'$ and $W' = g^{-1}(W)$ with $[W']_{k+1+r} = g^*[W]$ and equality holds in $\mathrm{CH}_{k+r}(D')$ by Lemma 26.2. By Remark 19.6 the result follows for general α'. $\square$

30. Gysin homomorphisms and rational equivalence

In this section we use the key formula to show the Gysin homomorphism factor through rational equivalence. We also prove an important commutativity property.

Lemma 30.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let X be integral and $n = \dim_\delta(X)$. Let $i : D \to X$ be an effective Cartier divisor. Let $\mathcal{N}$ be an invertible $\mathcal{O}_X$-module and let t be a nonzero meromorphic section of $\mathcal{N}$. Then $i^* div_{\mathcal{N}}(t) = c_1(\mathcal{N}|_D) \cap [D]_{n-1}$ in $\mathrm{CH}_{n-2}(D)$.*

Proof. Write $\mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{Z_i, \mathcal{N}}(t)[Z_i]$ for some integral closed subschemes $Z_i \subset X$ of δ-dimension $n-1$. We may assume that the family $\{Z_i\}$ is locally finite, that $t \in \Gamma(U, \mathcal{N}|_U)$ is a generator where $U = X \setminus \bigcup Z_i$, and that every irreducible component of D is one of the Z_i, see Divisors, Lemmas 26.1, 26.4, and 27.2.

Set $\mathcal{L} = \mathcal{O}_X(D)$. Denote $s \in \Gamma(X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{L})$ the canonical section. We will apply the discussion of Section 27 to our current situation. For each i let $\xi_i \in Z_i$ be its generic point. Let $B_i = \mathcal{O}_{X,\xi_i}$. For each i we pick generators $s_i \in \mathcal{L}_{\xi_i}$ and $t_i \in \mathcal{N}_{\xi_i}$ over B_i but we insist that we pick $s_i = s$ if $Z_i \not\subset D$. Write $s = f_i s_i$ and $t = g_i t_i$ with $f_i, g_i \in B_i$. Then $\mathrm{ord}_{Z_i,\mathcal{N}}(t) = \mathrm{ord}_{B_i}(g_i)$. On the other hand, we have $f_i \in B_i$ and

$$[D]_{n-1} = \sum \mathrm{ord}_{B_i}(f_i)[Z_i]$$

because of our choices of s_i. We claim that

$$i^* \mathrm{div}_{\mathcal{N}}(t) = \sum \mathrm{ord}_{B_i}(g_i) \mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i})$$

as cycles. More precisely, the right hand side is a cycle representing the left hand side. Namely, this is clear by our formula for $\mathrm{div}_{\mathcal{N}}(t)$ and the fact that $\mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) = [Z(s_i|_{Z_i})]_{n-2} = [Z_i \cap D]_{n-2}$ when $Z_i \not\subset D$ because in that case $s_i|_{Z_i} = s|_{Z_i}$ is a regular section, see Lemma 24.2. Similarly,

$$c_1(\mathcal{N}) \cap [D]_{n-1} = \sum \mathrm{ord}_{B_i}(f_i) \mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i})$$

The key formula (Lemma 27.1) gives the equality

$$\sum \Big(\mathrm{ord}_{B_i}(f_i) \mathrm{div}_{\mathcal{N}|_{Z_i}}(t_i|_{Z_i}) - \mathrm{ord}_{B_i}(g_i) \mathrm{div}_{\mathcal{L}|_{Z_i}}(s_i|_{Z_i}) \Big) = \sum \mathrm{div}_{Z_i}(\partial_{B_i}(f_i, g_i))$$

of cycles. If $Z_i \not\subset D$, then $f_i = 1$ and hence $\mathrm{div}_{Z_i}(\partial_{B_i}(f_i, g_i)) = 0$. Thus we get a rational equivalence between our specific cycles representing $i^* \mathrm{div}_{\mathcal{N}}(t)$ and $c_1(\mathcal{N}) \cap [D]_{n-1}$ on D. This finishes the proof. $\square$

Lemma 30.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 29.1. The Gysin homomorphism factors through rational equivalence to give a map $i^* : \mathrm{CH}_{k+1}(X) \to \mathrm{CH}_k(D)$.*

Proof. Let $\alpha \in Z_{k+1}(X)$ and assume that $\alpha \sim_{rat} 0$. This means there exists a locally finite collection of integral closed subschemes $W_j \subset X$ of δ-dimension $k+2$

and $f_j \in R(W_j)^*$ such that $\alpha = \sum i_{j,*}\text{div}_{W_j}(f_j)$. Set $X' = \coprod W_i$ and consider the diagram

$$\begin{tikzcd} D' \arrow[r, "i'"] \arrow[d, "q"] & X' \arrow[d, "p"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

of Remark 29.7. Since $X' \to X$ is proper we see that $i^*p_* = q_*(i')^*$ by Lemma 29.8. As we know that q_* factors through rational equivalence (Lemma 20.3), it suffices to prove the result for $\alpha' = \sum \text{div}_{W_j}(f_j)$ on X'. Clearly this reduces us to the case where X is integral and $\alpha = \text{div}(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \text{div}(f)$ for some $f \in R(X)^*$. If $X = D$, then we see that $i^*\alpha$ is equal to $c_1(\mathcal{L}) \cap \alpha$. This is rationally equivalent to zero by Lemma 28.2. If $D \neq X$, then we see that $i^*\text{div}_X(f)$ is equal to $c_1(\mathcal{O}_D) \cap [D]_{n-1}$ in $\text{CH}_{n-2}(D)$ by Lemma 30.1. Of course capping with $c_1(\mathcal{O}_D)$ is the zero map (Lemma 25.2). $\square$

Lemma 30.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 29.1. Then $i^*i_* : \text{CH}_k(D) \to \text{CH}_{k-1}(D)$ sends α to $c_1(\mathcal{L}|_D) \cap \alpha$.*

Proof. This is immediate from the definition of i_* on cycles and the definition of i^* given in Definition 29.1. $\square$

Lemma 30.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 29.1. Let $\mathcal{N}$ be an invertible $\mathcal{O}_X$-module. Then $i^*(c_1(\mathcal{N}) \cap \alpha) = c_1(i^*\mathcal{N}) \cap i^*\alpha$ in $\text{CH}_{k-2}(D)$ for all $\alpha \in \text{CH}_k(X)$.*

Proof. With exactly the same proof as in Lemma 30.2 this follows from Lemmas 26.4, 28.3, and 30.1. $\square$

Lemma 30.5. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ and $(\mathcal{L}', s', i' : D' \to X)$ be two triples as in Definition 29.1. Then the diagram*

$$\begin{tikzcd} \text{CH}_k(X) \arrow[r, "i^*"] \arrow[d, "(i')^*"] & \text{CH}_{k-1}(D) \arrow[d, "j^*"] \\ \text{CH}_{k-1}(D') \arrow[r, "(j')^*"] & \text{CH}_{k-2}(D \cap D') \end{tikzcd}$$

commutes where each of the maps is a gysin map.

Proof. Denote $j : D \cap D' \to D$ and $j' : D \cap D' \to D'$ the closed immersions corresponding to $(\mathcal{L}|_{D'}, s|_{D'}$ and $(\mathcal{L}'_D, s|_D)$. We have to show that $(j')^*i^*\alpha = j^*(i')^*\alpha$ for all $\alpha \in \text{CH}_k(X)$. Let $W \subset X$ be an integral closed subscheme of dimension k. Let us prove the equality in case $\alpha = [W]$. We will deduce it from the key formula.

We let σ be a nonzero meromorphic section of $\mathcal{L}|_W$ which we require to be equal to $s|_W$ if $W \not\subset D$. We let σ' be a nonzero meromorphic section of $\mathcal{L}'|_W$ which we require to be equal to $s'|_W$ if $W \not\subset D'$. Write

$$\text{div}_{\mathcal{L}|_W}(\sigma) = \sum \text{ord}_{Z_i, \mathcal{L}|_W}(\sigma)[Z_i] = \sum n_i[Z_i]$$

and similarly

$$\text{div}_{\mathcal{L}'|_W}(\sigma') = \sum \text{ord}_{Z_i, \mathcal{L}'|_W}(\sigma')[Z_i] = \sum n'_i[Z_i]$$

as in the discussion in Section 27. Then we see that $Z_i \subset D$ if $n_i \neq 0$ and $Z'_i \subset D'$ if $n'_i \neq 0$. For each i, let $\xi_i \in Z_i$ be the generic point. As in Section 27 we choose for each i an element $\sigma_i \in \mathcal{L}_{\xi_i}$, resp. $\sigma'_i \in \mathcal{L}'_{\xi_i}$ which generates over $B_i = \mathcal{O}_{W,\xi_i}$ and which is equal to the image of s, resp. s' if $Z_i \not\subset D$, resp. $Z_i \not\subset D'$. Write $\sigma = f_i\sigma_i$ and $\sigma' = f'_i\sigma'_i$ so that $n_i = \text{ord}_{B_i}(f_i)$ and $n'_i = \text{ord}_{B_i}(f'_i)$. From our definitions it follows that

$$(j')^*i^*[W] = \sum \text{ord}_{B_i}(f_i)\text{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i})$$

as cycles and

$$j^*(i')^*[W] = \sum \text{ord}_{B_i}(f'_i)\text{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})$$

The key formula (Lemma 27.1) now gives the equality

$$\sum \Big(\text{ord}_{B_i}(f_i)\text{div}_{\mathcal{L}'|_{Z_i}}(\sigma'_i|_{Z_i}) - \text{ord}_{B_i}(f'_i)\text{div}_{\mathcal{L}|_{Z_i}}(\sigma_i|_{Z_i})\Big) = \sum \text{div}_{Z_i}(\partial_{B_i}(f_i, f'_i))$$

of cycles. Note that $\text{div}_{Z_i}(\partial_{B_i}(f_i, f'_i)) = 0$ if $Z_i \not\subset D \cap D'$ because in this case either $f_i = 1$ or $f'_i = 1$. Thus we get a rational equivalence between our specific cycles representing $(j')^*i^*[W]$ and $j^*(i')^*[W]$ on $D \cap D' \cap W$. By Remark 19.6 the result follows for general α. □

31. Relative effective Cartier divisors

Relative effective Cartier divisors are defined and studied in Divisors, Section 18. To develop the basic results on Chern classes of vector bundles we only need the case where both the ambient scheme and the effective Cartier divisor are flat over the base.

Lemma 31.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a flat morphism of relative dimension r. Let $i : D \to X$ be a relative effective Cartier divisor (Divisors, Definition 18.2). Let $\mathcal{L} = \mathcal{O}_X(D)$. For any $\alpha \in \text{CH}_{k+1}(Y)$ we have*

$$i^*p^*\alpha = (p|_D)^*\alpha$$

in $\text{CH}_{k+r}(D)$ and

$$c_1(\mathcal{L}) \cap p^*\alpha = i_*((p|_D)^*\alpha)$$

in $\text{CH}_{k+r}(X)$.

Proof. Let $W \subset Y$ be an integral closed subscheme of δ-dimension $k+1$. By Divisors, Lemma 18.1 we see that $D \cap p^{-1}W$ is an effective Cartier divisor on $p^{-1}W$. By Lemma 29.5 we get the first equality in

$$i^*[p^{-1}W]_{k+r+1} = [D \cap p^{-1}W]_{k+r} = [(p|_D)^{-1}(W)]_{k+r}.$$

and the second because $D \cap p^{-1}(W) = (p|_D)^{-1}(W)$ as schemes. Since by definition $p^*[W] = [p^{-1}W]_{k+r+1}$ we see that $i^*p^*[W] = (p|_D)^*[W]$ as cycles. If $\alpha = \sum m_j[W_j]$ is a general $k+1$ cycle, then we get $i^*\alpha = \sum m_j i^*p^*[W_j] = \sum m_j(p|_D)^*[W_j]$ as cycles. This proves then first equality. To deduce the second from the first apply Lemma 29.4. □

32. Affine bundles

For an affine bundle the pullback map is surjective on Chow groups.

Lemma 32.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Assume that for every $y \in Y$, there exists an open neighbourhood $U \subset Y$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is identified with the morphism $U \times \mathbf{A}^r \to U$. Then $f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+r}(X)$ is surjective for all $k \in \mathbf{Z}$.*

Proof. Let $\alpha \in \mathrm{CH}_{k+r}(X)$. Write $\alpha = \sum m_j[W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subschemes of δ-dimension $k + r$. Then the family $\{W_j\}$ is locally finite in X. For any quasi-compact open $V \subset Y$ we see that $f^{-1}(V) \cap W_j$ is nonempty only for finitely many j. Hence the collection $Z_j = \overline{f(W_j)}$ of closures of images is a locally finite collection of integral closed subschemes of Y.

Consider the fibre product diagrams

$$\begin{tikzcd} f^{-1}(Z_j) \arrow[r] \arrow[d, "f_j"'] & X \arrow[d, "f"] \\ Z_j \arrow[r] & Y \end{tikzcd}$$

Suppose that $[W_j] \in Z_{k+r}(f^{-1}(Z_j))$ is rationally equivalent to $f_j^*\beta_j$ for some k-cycle $\beta_j \in \mathrm{CH}_k(Z_j)$. Then $\beta = \sum m_j\beta_j$ will be a k-cycle on Y and $f^*\beta = \sum m_j f_j^*\beta_j$ will be rationally equivalent to α (see Remark 19.6). This reduces us to the case Y integral, and $\alpha = [W]$ for some integral closed subscheme of X dominating Y. In particular we may assume that $d = \dim_\delta(Y) < \infty$.

Hence we can use induction on $d = \dim_\delta(Y)$. If $d < k$, then $\mathrm{CH}_{k+r}(X) = 0$ and the lemma holds. By assumption there exists a dense open $V \subset Y$ such that $f^{-1}(V) \cong V \times \mathbf{A}^r$ as schemes over V. Suppose that we can show that $\alpha|_{f^{-1}(V)} = f^*\beta$ for some $\beta \in Z_k(V)$. By Lemma 14.2 we see that $\beta = \beta'|_V$ for some $\beta' \in Z_k(Y)$. By the exact sequence $\mathrm{CH}_k(f^{-1}(Y \setminus V)) \to \mathrm{CH}_k(X) \to \mathrm{CH}_k(f^{-1}(V))$ of Lemma 19.3 we see that $\alpha - f^*\beta'$ comes from a cycle $\alpha' \in \mathrm{CH}_{k+r}(f^{-1}(Y \setminus V))$. Since $\dim_\delta(Y \setminus V) < d$ we win by induction on d.

Thus we may assume that $X = Y \times \mathbf{A}^r$. In this case we can factor f as

$$X = Y \times \mathbf{A}^r \to Y \times \mathbf{A}^{r-1} \to \ldots \to Y \times \mathbf{A}^1 \to Y.$$

Hence it suffices to do the case $r = 1$. By the argument in the second paragraph of the proof we are reduced to the case $\alpha = [W]$, Y integral, and $W \to Y$ dominant. Again we can do induction on $d = \dim_\delta(Y)$. If $W = Y \times \mathbf{A}^1$, then $[W] = f^*[Y]$. Lastly, $W \subset Y \times \mathbf{A}^1$ is a proper inclusion, then $W \to Y$ induces a finite field extension $R(W)/R(Y)$. Let $P(T) \in R(Y)[T]$ be the monic irreducible polynomial such that the generic fibre of $W \to Y$ is cut out by P in $\mathbf{A}^1_{R(Y)}$. Let $V \subset Y$ be a nonempty open such that $P \in \Gamma(V, \mathcal{O}_Y)[T]$, and such that $W \cap f^{-1}(V)$ is still cut out by P. Then we see that $\alpha|_{f^{-1}(V)} \sim_{rat} 0$ and hence $\alpha \sim_{rat} \alpha'$ for some cycle α' on $(Y \setminus V) \times \mathbf{A}^1$. By induction on the dimension we win. $\square$

Lemma 32.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let*

$$p : L = \underline{\mathrm{Spec}}(\mathit{Sym}^*(\mathcal{L})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+1}(L)$ is an isomorphism for all k.

Proof. For surjectivity see Lemma 32.1. Let $o : X \to L$ be the zero section of $L \to X$, i.e., the morphism corresponding to the surjection $\mathrm{Sym}^*(\mathcal{L}) \to \mathcal{O}_X$ which maps $\mathcal{L}^{\otimes n}$ to zero for all $n > 0$. Then $p \circ o = \mathrm{id}_X$ and $o(X)$ is an effective Cartier divisor on L. Hence by Lemma 31.1 we see that $o^* \circ p^* = \mathrm{id}$ and we conclude that p^* is injective too. ☐

Remark 32.3. We will see later (Lemma 36.3) that if X is a vector bundle of rank r over Y then the pullback map $\mathrm{CH}_k(Y) \to \mathrm{CH}_{k+r}(X)$ is an isomorphism. This is true whenever $X \to Y$ satisfies the assumptions of Lemma 32.1, see [Tot14, Lemma 2.2]. We will sketch a proof in Remark 32.8 using higher chow groups.

Lemma 32.4. *In the situation of Lemma 32.2 denote $o : X \to L$ the zero section (see proof of the lemma). Then we have*

(1) *$o(X)$ is the zero scheme of a regular global section of $p^*\mathcal{L}^{\otimes -1}$,*
(2) *$o_* : \mathrm{CH}_k(X) \to \mathrm{CH}_k(L)$ as o is a closed immersion,*
(3) *$o^* : \mathrm{CH}_{k+1}(L) \to \mathrm{CH}_k(X)$ as $o(X)$ is an effective Cartier divisor,*
(4) *$o^*p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_k(X)$ is the identity map,*
(5) *$o_*\alpha = -p^*(c_1(\mathcal{L}) \cap \alpha)$ for any $\alpha \in \mathrm{CH}_k(X)$, and*
(6) *$o^*o_* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-1}(X)$ is equal to the map $\alpha \mapsto -c_1(\mathcal{L}) \cap \alpha$.*

Proof. Since $p_*\mathcal{O}_L = \mathrm{Sym}^*(\mathcal{L})$ we have $p_*(p^*\mathcal{L}^{\otimes -1}) = \mathrm{Sym}^*(\mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$ by the projection formula (Cohomology, Lemma 54.2) and the section mentioned in (1) is the canonical trivialization $\mathcal{O}_X \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$. We omit the proof that the vanishing locus of this section is precisely $o(X)$. This proves (1).

Parts (2), (3), and (4) we've seen in the course of the proof of Lemma 32.2. Of course (4) is the first formula in Lemma 31.1.

Part (5) follows from the second formula in Lemma 31.1, additivity of capping with c_1 (Lemma 25.2), and the fact that capping with c_1 commutes with flat pullback (Lemma 26.2).

Part (6) follows from Lemma 30.3 and the fact that $o^*p^*\mathcal{L} = \mathcal{L}$. ☐

Lemma 32.5. *Let Y be a scheme. Let $\mathcal{L}_i$, $i = 1, 2$ be invertible $\mathcal{O}_Y$-modules. Let s be a global section of $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$. Denote $i : D \to X$ the zero scheme of s. Then there exists a commutative diagram*

$$\begin{tikzcd} D_1 \arrow[r, "i_1"'] \arrow[d, "p_1"'] & L \arrow[d, "p"] & D_2 \arrow[l, "i_2"] \arrow[d, "p_2"] \\ D \arrow[r, "i"] & Y & D \arrow[l, "i"'] \end{tikzcd}$$

and sections s_i of $p^*\mathcal{L}_i$ such that the following hold:

(1) *$p^*s = s_1 \otimes s_2$,*
(2) *p is of finite type and flat of relative dimension 1,*
(3) *D_i is the zero scheme of s_i,*
(4) *$D_i \cong \underline{\mathrm{Spec}}(Sym^*(\mathcal{L}_{1-i}^{\otimes -1})|_D))$ over D for $i = 1, 2$,*
(5) *$p^{-1}D = D_1 \cup D_2$ (scheme theoretic union),*
(6) *$D_1 \cap D_2$ (scheme theoretic intersection) maps isomorphically to D, and*
(7) *$D_1 \cap D_2 \to D_i$ is the zero section of the line bundle $D_i \to D$ for $i = 1, 2$.*

Moreover, the formation of this diagram and the sections s_i commutes with arbitrary base change.

Proof. Let $p : X \to Y$ be the relative spectrum of the quasi-coherent sheaf of $\mathcal{O}_Y$-algebras

$$\mathcal{A} = \left(\bigoplus\nolimits_{a_1, a_2 \geq 0} \mathcal{L}_1^{\otimes - a_1} \otimes_{\mathcal{O}_Y} \mathcal{L}_2^{\otimes - a_2}\right)/\mathcal{J}$$

where $\mathcal{J}$ is the ideal generated by local sections of the form $st - t$ for t a local section of any summand $\mathcal{L}_1^{\otimes - a_1} \otimes \mathcal{L}_2^{\otimes - a_2}$ with $a_1, a_2 > 0$. The sections s_i viewed as maps $p^*\mathcal{L}_i^{\otimes -1} \to \mathcal{O}_X$ are defined as the adjoints of the maps $\mathcal{L}_i^{\otimes -1} \to \mathcal{A} = p_*\mathcal{O}_X$. For any $y \in Y$ we can choose an affine open $V \subset Y$, say $V = \mathrm{Spec}(B)$, containing y and trivializations $z_i : \mathcal{O}_V \to \mathcal{L}_i^{\otimes -1}|_V$. Observe that $f = s(z_1 z_2) \in A$ cuts out the closed subscheme D. Then clearly

$$p^{-1}(V) = \mathrm{Spec}(B[z_1, z_2]/(z_1 z_2 - f))$$

Since D_i is cut out by z_i everything is clear. $\square$

Lemma 32.6. *In the situation of Lemma 32.5 assume Y is locally of finite type over (S, δ) as in Situation 7.1. Then we have $i_1^* p^* \alpha = p_1^* i^* \alpha$ in $\mathrm{CH}_k(D_1)$ for all $\alpha \in \mathrm{CH}_k(Y)$.*

Proof. Let $W \subset Y$ be an integral closed subscheme of δ-dimension k. We distinguish two cases.

Assume $W \subset D$. Then $i^*[W] = c_1(\mathcal{L}_1) \cap [W] + c_1(\mathcal{L}_2) \cap [W]$ in $\mathrm{CH}_{k-1}(D)$ by our definition of gysin homomorphisms and the additivity of Lemma 25.2. Hence $p_1^* i^*[W] = p_1^*(c_1(\mathcal{L}_1) \cap [W]) + p_1^*(c_1(\mathcal{L}_2) \cap [W])$. On the other hand, we have $p^*[W] = [p^{-1}(W)]_{k+1}$ by construction of flat pullback. And $p^{-1}(W) = W_1 \cup W_2$ (scheme theoretically) where $W_i = p_i^{-1}(W)$ is a line bundle over W by the lemma (since formation of the diagram commutes with base change). Then $[p^{-1}(W)]_{k+1} = [W_1] + [W_2]$ as W_i are integral closed subschemes of L of δ-dimension $k+1$. Hence

$$\begin{aligned} i_1^* p^*[W] &= i_1^*[p^{-1}(W)]_{k+1} \\ &= i_1^*([W_1] + [W_2]) \\ &= c_1(p_1^*\mathcal{L}_1) \cap [W_1] + [W_1 \cap W_2]_k \\ &= c_1(p_1^*\mathcal{L}_1) \cap p_1^*[W] + [W_1 \cap W_2]_k \\ &= p_1^*(c_1(\mathcal{L}_1) \cap [W]) + [W_1 \cap W_2]_k \end{aligned}$$

by construction of gysin homomorphisms, the definition of flat pullback (for the second equality), and compatibility of $c_1 \cap -$ with flat pullback (Lemma 26.2). Since $W_1 \cap W_2$ is the zero section of the line bundle $W_1 \to W$ we see from Lemma 32.4 that $[W_1 \cap W_2]_k = p_1^*(c_1(\mathcal{L}_2) \cap [W])$. Note that here we use the fact that D_1 is the line bundle which is the relative spectrum of the inverse of $\mathcal{L}_2$. Thus we get the same thing as before.

Assume $W \not\subset D$. In this case, both $i_1^* p^*[W]$ and $p_1^* i^*[W]$ are represented by the $k-1$ cycle associated to the scheme theoretic inverse image of W in D_1. $\square$

Lemma 32.7. *In Situation 7.1 let X be a scheme locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 29.1. There exists a commutative* *diagram*

$$\begin{tikzcd} D' \arrow[r, "i'"'] \arrow[d, "p"'] & X' \arrow[d, "g"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

such that

(1) *p and g are of finite type and flat of relative dimension* 1,
(2) $p^* : \mathrm{CH}_k(D) \to \mathrm{CH}_{k+1}(D')$ *is injective for all* k,
(3) $D' \subset X'$ *is the zero scheme of a global section* $s' \in \Gamma(X', \mathcal{O}_{X'})$,
(4) $p^*i^* = (i')^*g^*$ *as maps* $\mathrm{CH}_k(X) \to \mathrm{CH}_k(D')$.

Moreover, these properties remain true after arbitrary base change by morphisms $Y \to X$ which are locally of finite type.

Proof. Observe that $(i')^*$ is defined because we have the triple $(\mathcal{O}_{X'}, s', i' : D' \to X')$ as in Definition 29.1. Thus the statement makes sense.

Set $\mathcal{L}_1 = \mathcal{O}_X$, $\mathcal{L}_2 = \mathcal{L}$ and apply Lemma 32.5 with the section s of $\mathcal{L} = \mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$. Take $D' = D_1$. The results now follow from the lemma, from Lemma 32.6 and injectivity by Lemma 32.2. ☐

Remark 32.8. Let (S, δ) be as in Situation 7.1. Let Y be locally of finite type over S. Let $r \geq 0$. Let $f : X \to Y$ be a morphism of schemes. Assume every $y \in Y$ is contained in an open $V \subset Y$ such that $f^{-1}(V) \cong V \times \mathbf{A}^r$ as schemes over V. In this remark we sketch a proof of the fact that $f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+r}(X)$ is an isomorphism. First, by Lemma 32.1 the map is surjective. Let $\alpha \in \mathrm{CH}_k(Y)$ with $f^*\alpha = 0$. We will prove that $\alpha = 0$.

Step 1. We may assume that $\dim_\delta(Y) < \infty$. (This is immediate in all cases in practice so we suggest the reader skip this step.) Namely, any rational equivalence witnessing that $f^*\alpha = 0$ on X, will use a locally finite collection of integral closed subschemes of dimension $k + r + 1$. Taking the union of the closures of the images of these in Y we get a closed subscheme $Y' \subset Y$ of $\dim_\delta(Y') \leq k + r + 1$ such that α is the image of some $\alpha' \in \mathrm{CH}_k(Y')$ and such that $(f')^*\alpha = 0$ where f' is the base change of f to Y'.

Step 2. Assume $d = \dim_\delta(Y) < \infty$. Then we can use induction on d. If $d < k$, then $\alpha = 0$ and we are done; this is the base case of the induction. In general, our assumption on f shows we can choose a dense open $V \subset Y$ such that $U = f^{-1}(V) = \mathbf{A}^r_V$. Denote $Y' \subset Y$ the complement of V as a reduced closed subscheme and set $X' = f^{-1}(Y')$. Consider

$$\begin{tikzcd} \mathrm{CH}^M_{k+r}(U,1) \arrow[r] & \mathrm{CH}_{k+r}(X') \arrow[r] & \mathrm{CH}_{k+r}(X) \arrow[r] & \mathrm{CH}_{k+r}(U) \arrow[r] & 0 \\ \mathrm{CH}^M_k(V,1) \arrow[r] \arrow[u] & \mathrm{CH}_k(Y') \arrow[r] \arrow[u] & \mathrm{CH}_k(Y) \arrow[r] \arrow[u] & \mathrm{CH}_k(V) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Here we use the first higher Chow groups of V and U and the six term exact sequences constructed in Remark 27.3, as well as flat pullback for these higher chow groups and compatibility of flat pullback with these six term exact sequences. Since $U = \mathbf{A}^r_V$ the vertical map on the right is an isomorphism. The map $\mathrm{CH}_k(Y') \to$

$\mathrm{CH}_{k+r}(X')$ is bijective by induction on d. Hence to finish the argument is suffices to show that

$$\mathrm{CH}^M_k(V, 1) \longrightarrow \mathrm{CH}^M_{k+r}(U, 1)$$

is surjective. Arguing as in the proof of Lemma 32.1 this reduces to Step 3 below.

Step 3. Let F be a field. Then $\mathrm{CH}^M_0(\mathbf{A}^1_F, 1) = 0$. (In the proof of the lemma cited above we proved analogously that $\mathrm{CH}_0(\mathbf{A}^1_F) = 0$.) We have

$$\mathrm{CH}^M_0(\mathbf{A}^1_F, 1) = \mathrm{Coker}\left(\partial : K^M_2(F(T)) \longrightarrow \bigoplus\nolimits_{\mathfrak p \subset F[T] \text{ maximal}} \kappa(\mathfrak p)^*\right)$$

The classical argument for the vanishing of the cokernel is to show by induction on the degree of $\kappa(\mathfrak p)/F$ that the summand corresponding to $\mathfrak p$ is in the image. If $\mathfrak p$ is generated by the irreducible monic polynomial $P(T) \in F[T]$ and if $u \in \kappa(x)^*$ is the residue class of some $Q(T) \in F[T]$ with $\deg(Q) < \deg(P)$ then one shows that $\partial(Q, P)$ produces the element u at $\mathfrak p$ and perhaps some other units at primes dividing Q which have lower degree. This finishes the sketch of the proof.

33. Bivariant intersection theory

In order to intelligently talk about higher Chern classes of vector bundles we introduce bivariant chow classes as in [Ful98]. Our definition differs from [Ful98] in two respects: (1) we work in a different setting, and (2) we only require our bivariant classes commute with the gysin homomorphisms for zero schemes of sections of invertible modules (Section 29). We will see later, in Lemma 54.8, that our bivariant classes commute with all higher codimension gysin homomorphisms and hence satisfy all properties required of them in [Ful98]; see also [Ful98, Theorem 17.1].

Definition 33.1. Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. A *bivariant class c of degree p for f* is given by a rule which assigns to every locally of finite type morphism $Y' \to Y$ and every k a map

$$c \cap - : \mathrm{CH}_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

where $X' = Y' \times_Y X$, satisfying the following conditions

(1) if $Y'' \to Y'$ is a proper, then $c \cap (Y'' \to Y')_*\alpha'' = (X'' \to X')_*(c \cap \alpha'')$ for all α'' on Y'' where $X'' = Y'' \times_Y X$,
(2) if $Y'' \to Y'$ is flat locally of finite type of fixed relative dimension, then $c \cap (Y'' \to Y')^*\alpha' = (X'' \to X')^*(c \cap \alpha')$ for all α' on Y', and
(3) if $(\mathcal{L}', s', i' : D' \to Y')$ is as in Definition 29.1 with pullback $(\mathcal{N}', t', j' : E' \to X')$ to X', then we have $c \cap (i')^*\alpha' = (j')^*(c \cap \alpha')$ for all α' on Y'.

The collection of all bivariant classes of degree p for f is denoted $A^p(X \to Y)$.

Let (S, δ) be as in Situation 7.1. Let $X \to Y$ and $Y \to Z$ be morphisms of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. It is clear that $A^p(X \to Y)$ is an abelian group. Moreover, it is clear that we have a bilinear composition

$$A^p(X \to Y) \times A^q(Y \to Z) \to A^{p+q}(X \to Z)$$

which is associative.

Lemma 33.2. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a flat morphism of relative dimension r between schemes locally of finite type over S. Then the rule that to $Y' \to Y$ assigns $(f')^* : \mathrm{CH}_k(Y') \to \mathrm{CH}_{k+r}(X')$ where $X' = X \times_Y Y'$ is a bivariant class of degree $-r$.*

Proof. This follows from Lemmas 20.2, 14.3, 15.1, and 29.9. □

Lemma 33.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 29.1. Then the rule that to $f : X' \to X$ assigns $(i')^* : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-1}(D')$ where $D' = D \times_X X'$ is a bivariant class of degree 1.*

Proof. This follows from Lemmas 30.2, 29.8, 29.9, and 30.5. □

Lemma 33.4. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes locally of finite type over S. Let $c \in A^p(X \to Z)$ and assume f is proper. Then the rule that to $Z' \to Z$ assigns $\alpha \longmapsto f'_*(c \cap \alpha)$ is a bivariant class denoted $f_* \circ c \in A^p(Y \to Z)$.*

Proof. This follows from Lemmas 12.2, 15.1, and 29.8. □

Remark 33.5. Let (S, δ) be as in Situation 7.1. Let $X \to Y$ and $Y' \to Y$ be morphisms of schemes locally of finite type over S. Let $X' = Y' \times_Y X$. Then there is an obvious restriction map

$$A^p(X \to Y) \longrightarrow A^p(X' \to Y'), \quad c \longmapsto res(c)$$

obtained by viewing a scheme Y'' locally of finite type over Y' as a scheme locally of finite type over Y and settting $res(c) \cap \alpha'' = c \cap \alpha''$ for any $\alpha'' \in \mathrm{CH}_k(Y'')$. This restriction operation is compatible with compositions in an obvious manner.

Remark 33.6. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. For $i = 1, 2$ let $Z_i \to X$ be a morphism of schemes locally of finite type. Let $c_i \in A^{p_i}(Z_i \to X)$, $i = 1, 2$ be bivariant classes. For any $\alpha \in \mathrm{CH}_k(X)$ we can ask whether

$$c_1 \cap c_2 \cap \alpha = c_2 \cap c_1 \cap \alpha$$

in $\mathrm{CH}_{k-p_1-p_2}(Z_1 \times_X Z_2)$. If this is true and if it holds after any base change by $X' \to X$ locally of finite type, then we say c_1 and c_2 *commute*. Of course this is the same thing as saying that

$$res(c_1) \circ c_2 = res(c_2) \circ c_1$$

in $A^{p_1+p_2}(Z_1 \times_X Z_2 \to X)$. Here $res(c_1) \in A^{p_1}(Z_1 \times_X Z_2 \to Z_2)$ is the restriction of c_1 as in Remark 33.5; similarly for $res(c_2)$.

Example 33.7. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ a triple as in Definition 29.1. Let $Z \to X$ be a morphism of schemes locally of finite type and let $c \in A^p(Z \to X)$ be a bivariant class. Then the bivariant gysin class $c' \in A^1(D \to X)$ of Lemma 33.3 commutes with c in the sense of Remark 33.6. Namely, this is a restatement of condition (3) of Definition 33.1.

Remark 33.8. There is a more general type of bivariant class that doesn't seem to be considered in the literature. Namely, suppose we are given a diagram

$$X \longrightarrow Z \longleftarrow Y$$

of schemes locally of finite type over (S, δ) as in Situation 7.1. Let $p \in \mathbf{Z}$. Then we can consider a rule c which assigns to every $Z' \to Z$ locally of finite type maps

$$c \cap - : \mathrm{CH}_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

for all $k \in \mathbf{Z}$ where $X' = X \times_Z Z'$ and $Y' = Z' \times_Z Y$ compatible with

(1) proper pushforward if given $Z'' \to Z'$ proper,
(2) flat pullback if given $Z'' \to Z'$ flat of fixed relative dimension, and
(3) gysin maps if given $D' \subset Z'$ as in Definition 29.1.

We omit the detailed formulations. Suppose we denote the collection of all such operations $A^p(X \to Z \leftarrow Y)$. A simple example of the utility of this concept is when we have a proper morphism $f : X_2 \to X_1$. Then f_* isn't a bivariant operation in the sense of Definition 33.1 but it is in the above generalized sense, namely, $f_* \in A^0(X_1 \to X_1 \leftarrow X_2)$.

34. Chow cohomology and the first Chern class

We will be most interested in $A^p(X) = A^p(X \to X)$, which will always mean the bivariant cohomology classes for id_X. Namely, that is where Chern classes will live.

Definition 34.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. The *Chow cohomology* of X is the graded $\mathbf{Z}$-algebra $A^*(X)$ whose degree p component is $A^p(X \to X)$.

Warning: It is not clear that the $\mathbf{Z}$-algebra structure on $A^*(X)$ is commutative, but we will see that Chern classes live in its center.

Remark 34.2. Let (S, δ) be as in Situation 7.1. Let $f : Y' \to Y$ be a morphism of schemes locally of finite type over S. As a special case of Remark 33.5 there is a canonical $\mathbf{Z}$-algebra map $res : A^*(Y) \to A^*(Y')$. This map is often denoted f^* in the literature.

Lemma 34.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then the rule that to $f : X' \to X$ assigns $c_1(f^*\mathcal{L}) \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-1}(X')$ is a bivariant class of degree 1.*

Proof. This follows from Lemmas 28.2, 26.4, 26.2, and 30.4. □

The lemma above finally allows us to make the following definition.

Definition 34.4. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The *first Chern class* $c_1(\mathcal{L}) \in A^1(X)$ of $\mathcal{L}$ is the bivariant class of Lemma 34.3.

For finite locally free modules we construct the Chern classes in Section 38. Let us prove that $c_1(\mathcal{L})$ is in the center of $A^*(X)$.

Lemma 34.5. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then*

(1) *$c_1(\mathcal{L}) \in A^1(X)$ is in the center of $A^*(X)$ and*
(2) *if $f : X' \to X$ is locally of finite type and $c \in A^*(X' \to X)$, then $c \circ c_1(\mathcal{L}) = c_1(f^*\mathcal{L}) \circ c$.*

Proof. Of course (2) implies (1). Let $p : L \to X$ be as in Lemma 32.2 and let $o : X \to L$ be the zero section. Denote $p' : L' \to X'$ and $o' : X' \to L'$ their base changes. By Lemma 32.4 we have

$$p^*(c_1(\mathcal{L}) \cap \alpha) = -o_*\alpha \quad \text{and} \quad (p')^*(c_1(f^*\mathcal{L}) \cap \alpha') = -o'_*\alpha'$$

Since c is a bivariant class we have

$$\begin{aligned} (p')^*(c \cap c_1(\mathcal{L}) \cap \alpha) &= c \cap p^*(c_1(\mathcal{L}) \cap \alpha) \\ &= -c \cap o_*\alpha \\ &= -o'_*(c \cap \alpha) \\ &= (p')^*(c_1(f^*\mathcal{L}) \cap c \cap \alpha) \end{aligned}$$

Since $(p')^*$ is injective by one of the lemmas cited above we obtain $c \cap c_1(\mathcal{L}) \cap \alpha = c_1(f^*\mathcal{L}) \cap c \cap \alpha$. The same is true after any base change by $Y \to X$ locally of finite type and hence we have the equality of bivariant classes stated in (2). $\square$

Lemma 34.6. *Let (S, δ) be as in Situation 7.1. Let X be a finite type scheme over S which has an ample invertible sheaf. Assume $d = \dim(X) < \infty$ (here we really mean dimension and not δ-dimension). Then for any invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}$ on X we have $c_1(\mathcal{L}_1) \circ \ldots \circ c_1(\mathcal{L}_{d+1}) = 0$ in $A^{d+1}(X)$.*

Proof. We prove this by induction on d. The base case $d = 0$ is true because in this case X is a finite set of closed points and hence every invertible module is trivial. Assume $d > 0$. By Divisors, Lemma 15.12 we can write $\mathcal{L}_{d+1} \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{\otimes -1}$ for some effective Cartier divisors $D, D' \subset X$. Then $c_1(\mathcal{L}_{d+1})$ is the difference of $c_1(\mathcal{O}_X(D))$ and $c_1(\mathcal{O}_X(D'))$ and hence we may assume $\mathcal{L}_{d+1} = \mathcal{O}_X(D)$ for some effective Cartier divisor.

Denote $i : D \to X$ the inclusion morphism and denote $i^* \in A^1(D \to X)$ the bivariant class given by the gysin hommomorphism as in Lemma 33.3. We have $i_* \circ i^* = c_1(\mathcal{L}_{d+1})$ in $A^1(X)$ by Lemma 29.4 (and Lemma 33.4 to make sense of the left hand side). Since $c_1(\mathcal{L}_i)$ commutes with both i_* and i^* (by definition of bivariant classes) we conclude that

$$c_1(\mathcal{L}_1) \circ \ldots \circ c_1(\mathcal{L}_{d+1}) = i_* \circ c_1(\mathcal{L}_1) \circ \ldots \circ c_1(\mathcal{L}_d) \circ i^* = i_* \circ c_1(\mathcal{L}_1|_D) \circ \ldots \circ c_1(\mathcal{L}_d|_D) \circ i^*$$

Thus we conclude by induction on d. Namely, we have $\dim(D) < d$ as none of the generic points of X are in D. $\square$

Remark 34.7. Let (S, δ) be as in Situation 7.1. Let $Z \to X$ be a closed immersion of schemes locally of finite type over S and let $p \geq 0$. In this setting we define

$$A^{(p)}(Z \to X) = \prod\nolimits_{i \leq p-1} A^i(X) \times \prod\nolimits_{i \geq p} A^i(Z \to X).$$

Then $A^{(p)}(Z \to X)$ canonically comes equipped with the structure of a graded algebra. In fact, more generally there is a multiplication

$$A^{(p)}(Z \to X) \times A^{(q)}(Z \to X) \longrightarrow A^{(\max(p,q))}(Z \to X)$$

In order to define these we define maps

$$\begin{aligned} A^i(Z \to X) \times A^j(X) &\to A^{i+j}(Z \to X) \\ A^i(X) \times A^j(Z \to X) &\to A^{i+j}(Z \to X) \\ A^i(Z \to X) \times A^j(Z \to X) &\to A^{i+j}(Z \to X) \end{aligned}$$

For the first we use composition of bivariant classes. For the second we use restriction $A^i(X) \to A^i(Z)$ (Remark 33.5) and composition $A^i(Z) \times A^j(Z \to X) \to A^{i+j}(Z \to X)$. For the third, we send (c, c') to $res(c) \circ c'$ where $res : A^i(Z \to X) \to A^i(Z)$ is the restriction map (see Remark 33.5). We omit the verification that these multiplications are associative in a suitable sense.

Remark 34.8. Let (S, δ) be as in Situation 7.1. Let $Z \to X$ be a closed immersion of schemes locally of finite type over S. Denote $res : A^p(Z \to X) \to A^p(Z)$ the restriction map of Remark 33.5. For $c \in A^p(Z \to X)$ we have $res(c) \cap \alpha = c \cap i_*\alpha$ for $\alpha \in \mathrm{CH}_*(Z)$. Namely $res(c) \cap \alpha = c \cap \alpha$ and compatibility of c with proper pushforward gives $(Z \to Z)_*(c \cap \alpha) = c \cap (Z \to X)_*\alpha$.

35. Lemmas on bivariant classes

In this section we prove some elementary results on bivariant classes. Here is a criterion to see that an operation passes through rational equivalence.

Lemma 35.1. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. Suppose given a rule which assigns to every locally of finite type morphism $Y' \to Y$ and every k a map*

$$c \cap - : Z_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

where $Y' = X' \times_X Y$, satisfying condition (3) of Definition 33.1 whenever $\mathcal{L}'|_{D'} \cong \mathcal{O}_{D'}$. Then $c \cap -$ factors through rational equivalence.

Proof. The statement makes sense because given a triple $(\mathcal{L}, s, i : D \to X)$ as in Definition 29.1 such that $\mathcal{L}|_D \cong \mathcal{O}_D$, then the operation i^* is defined on the level of cycles, see Remark 29.6. Let $\alpha \in Z_k(X')$ be a cycle which is rationally equivalent to zero. We have to show that $c \cap \alpha = 0$. By Lemma 21.1 there exists a cycle $\beta \in Z_{k+1}(X' \times \mathbf{P}^1)$ such that $\alpha = i_0^*\beta - i_\infty^*\beta$ where $i_0, i_\infty : X' \to X' \times \mathbf{P}^1$ are the closed immersions of X' over $0, \infty$. Since these are examples of effective Cartier divisors with trivial normal bundles, we see that $c \cap i_0^*\beta = j_0^*(c \cap \beta)$ and $c \cap i_\infty^*\beta = j_\infty^*(c \cap \beta)$ where $j_0, j_\infty : Y' \to Y' \times \mathbf{P}^1$ are closed immersions as before. Since $j_0^*(c \cap \beta) \sim_{rat} j_\infty^*(c \cap \beta)$ (follows from Lemma 21.1) we conclude. $\square$

Lemma 35.2. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $p \in \mathbf{Z}$. Suppose given a rule which assigns to every locally of finite type morphism $Y' \to Y$ and every k a map*

$$c \cap - : \mathrm{CH}_k(Y') \longrightarrow \mathrm{CH}_{k-p}(X')$$

where $Y' = X' \times_X Y$, satisfying conditions (1), (2) of Definition 33.1 and condition (3) whenever $\mathcal{L}'|_{D'} \cong \mathcal{O}_{D'}$. Then $c \cap -$ is a bivariant class.

Proof. Let $Y' \to Y$ be a morphism of schemes which is locally of finite type. Let $(\mathcal{L}', s', i' : D' \to Y')$ be as in Definition 29.1 with pullback $(\mathcal{N}', t', j' : E' \to X')$ to X'. We have to show that $c \cap (i')^*\alpha' = (j')^*(c \cap \alpha')$ for all $\alpha' \in \mathrm{CH}_k(Y')$.

Denote $g : Y'' \to Y'$ the smooth morphism of relative dimension 1 with $i'' : D'' \to Y''$ and $p : D'' \to D'$ constructed in Lemma 32.7. (Warning: D'' isn't the full inverse image of D'.) Denote $f : X'' \to X'$ and $E'' \subset X''$ their base changes by

$X' \to Y'$. Picture

$$
\begin{tikzcd}
 & X'' \arrow[rr] \arrow[dd, "h"] & & Y'' \arrow[dd, "g"] \\
E'' \arrow[ur, "j''"] \arrow[rr] \arrow[dd, "q"] & & D'' \arrow[ur, "i''"] \arrow[dd] & \\
 & X' \arrow[rr, "p"] & & Y' \\
E' \arrow[ur, "j'"] \arrow[rr] & & D' \arrow[ur, "i'"] &
\end{tikzcd}
$$

By the properties given in the lemma we know that $\beta' = (i')^*\alpha'$ is the unique element of $\mathrm{CH}_{k-1}(D')$ such that $p^*\beta' = (i'')^*g^*\alpha'$. Similarly, we know that $\gamma' = (j')^*(c\cap\alpha')$ is the unique element of $\mathrm{CH}_{k-1-p}(E')$ such that $q^*\gamma' = (j'')^*h^*(c\cap\alpha')$. Since we know that

$$(j'')^*h^*(c \cap \alpha') = (j'')^*(c \cap g^*\alpha') = c \cap (i'')^*g^*\alpha'$$

by our assuptions on c; note that the modified version of (3) assumed in the statement of the lemma applies to i'' and its base change j''. We similarly know that

$$q^*(c \cap \beta') = c \cap p^*\beta'$$

We conclude that $\gamma' = c \cap \beta'$ by the uniqueness pointed out above. $\square$

Here a criterion for when a bivariant class is zero.

Lemma 35.3. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $c \in A^p(X \to Y)$. For $Y'' \to Y' \to Y$ set $X'' = Y'' \times_Y X$ and $X' = Y' \times_Y X$. The following are equivalent*

1. *c is zero,*
2. *$c \cap [Y'] = 0$ in $\mathrm{CH}_*(X')$ for every integral scheme Y' locally of finite type over Y, and*
3. *for every integral scheme Y' locally of finite type over Y, there exists a proper birational morphism $Y'' \to Y'$ such that $c \cap [Y''] = 0$ in $\mathrm{CH}_*(X'')$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear. Assumption (3) implies (2) because $(Y'' \to Y')_*[Y''] = [Y']$ and hence $c \cap [Y'] = (X'' \to X')_*(c \cap [Y''])$ as c is a bivariant class. Assume (2). Let $Y' \to Y$ be locally of finite type. Let $\alpha \in \mathrm{CH}_k(Y')$. Write $\alpha = \sum n_i[Y'_i]$ with $Y'_i \subset Y'$ a locally finite collection of integral closed subschemes of δ-dimension k. Then we see that α is pushforward of the cycle $\alpha' = \sum n_i[Y'_i]$ on $Y'' = \coprod Y'_i$ under the proper morphism $Y'' \to Y'$. By the properties of bivariant classes it suffices to prove that $c\cap\alpha' = 0$ in $\mathrm{CH}_{k-p}(X'')$. We have $\mathrm{CH}_{k-p}(X'') = \prod \mathrm{CH}_{k-p}(X'_i)$ where $X'_i = Y'_i \times_Y X$. This follows immediately from the definitions. The projection maps $\mathrm{CH}_{k-p}(X'') \to \mathrm{CH}_{k-p}(X'_i)$ are given by flat pullback. Since capping with c commutes with flat pullback, we see that it suffices to show that $c\cap[Y'_i]$ is zero in $\mathrm{CH}_{k-p}(X'_i)$ which is true by assumption. $\square$

Lemma 35.4. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Assume we have disjoint union decompositions*

$X = \coprod_{i \in I} X_i$ and $Y = \coprod_{j \in J} Y_j$ by open and closed subschemes and a map $a : I \to J$ of sets such that $f(X_i) \subset Y_{a(i)}$. Then

$$A^p(X \to Y) = \prod\nolimits_{i \in I} A^p(X_i \to Y_{a(i)})$$

Proof. Suppose given an element $(c_i) \in \prod_i A^p(X_i \to Y_{a(i)})$. Then given $\beta \in \mathrm{CH}_k(Y)$ we can map this to the element of $\mathrm{CH}_{k-p}(X)$ whose restriction to X_i is $c_i \cap \beta|_{Y_{a(i)}}$. This works because $\mathrm{CH}_{k-p}(X) = \prod_i \mathrm{CH}_{k-p}(X_i)$. The same construction works after base change by any $Y' \to Y$ locally of finite type and we get $c \in A^p(X \to Y)$. Thus we obtain a map Ψ from the right hand side of the formula to the left hand side of the formula. Conversely, given $c \in A^p(X \to Y)$ and an element $\beta_i \in \mathrm{CH}_k(Y_{a(i)})$ we can consider the element $(c \cap (Y_{a(i)} \to Y)_*\beta_i)|_{X_i}$ in $\mathrm{CH}_{k-p}(X_i)$. The same thing works after base change by any $Y' \to Y$ locally of finite type and we get $c_i \in A^p(X_i \to Y_{a(i)})$. Thus we obtain a map Φ from the left hand side of the formula to the right hand side of the formula. It is immediate that $\Phi \circ \Psi = \mathrm{id}$. For the converse, suppose that $c \in A^p(X \to Y)$ and $\beta \in \mathrm{CH}_k(Y)$. Say $\Phi(c) = (c_i)$. Let $j \in J$. Because c commutes with flat pullback we get

$$(c \cap \beta)|_{\coprod_{a(i)=j} X_i} = c \cap \beta|_{Y_j}$$

Because c commutes with proper pushforward we get

$$(\coprod\nolimits_{a(i)=j} X_i \to X)_*((c \cap \beta)|_{\coprod_{a(i)=j} X_i}) = c \cap (Y_j \to Y)_*\beta|_{Y_j}$$

The left hand side is the cycle on X restricting to $(c \cap \beta)|_{X_i}$ on X_i for $i \in I$ with $a(i) = j$ and 0 else. The right hand side is a cycle on X whose restriction to X_i is $c_i \cap \beta|_{Y_j}$ for $i \in I$ with $a(i) = j$. Thus $c \cap \beta = \Psi((c_i))$ as desired. □

Remark 35.5. Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $X = \coprod_{i \in I} X_i$ and $Y = \coprod_{j \in J} Y_j$ be the decomposition of X and Y into their connected components (the connected components are open as X and Y are locally Noetherian, see Topology, Lemma 9.6 and Properties, Lemma 5.5). Let $a(i) \in J$ be the index such that $f(X_i) \subset Y_{a(i)}$. Then $A^p(X \to Y) = \prod A^p(X_i \to Y_{a(i)})$ by Lemma 35.4. In this setting it is convenient to set

$$A^*(X \to Y)^\wedge = \prod\nolimits_i A^*(X_i \to Y_{a(i)})$$

This "completed" bivariant group is the subset

$$A^*(X \to Y)^\wedge \quad \subset \quad \prod\nolimits_{p \geq 0} A^p(X)$$

consisting of elements $c = (c_0, c_1, c_2, \ldots)$ such that for each connected component X_i the image of c_p in $A^p(X_i \to Y_{a(i)})$ is zero for almost all p. If $Y \to Z$ is a second morphism, then the composition $A^*(X \to Y) \times A^*(Y \to Z) \to A^*(X \to Z)$ extends to a composition $A^*(X \to Y)^\wedge \times A^*(Y \to Z)^\wedge \to A^*(X \to Z)^\wedge$ of completions. We sometimes call $A^*(X)^\wedge = A^*(X \to X)^\wedge$ the *completed bivariant cohomology ring* of X.

Lemma 35.6. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. Let $g : Y' \to Y$ be an envelope (Definition 22.1) and denote $X' = Y' \times_Y X$. Let $p \in \mathbf{Z}$ and let $c' \in A^p(X' \to Y')$. If the two restrictions*

$$res_1(c') = res_2(c') \in A^p(X' \times_X X' \to Y' \times_Y Y')$$

are equal (see proof), then there exists a unique $c \in A^p(X \to Y)$ *whose restriction* $res(c) = c'$ *in* $A^p(X' \to Y')$.

Proof. We have a commutative diagram

$$\begin{tikzcd} X' \times_X X' \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[d, "f''"] & X' \arrow[r, "h"'] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \times_Y Y' \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

The element $res_1(c')$ is the restriction (see Remark 33.5) of c' for the cartesian square with morphisms a, f', p, f'' and the element $res_2(c')$ is the restriction of c' for the cartesian square with morphisms b, f', q, f''. Assume $res_1(c') = res_2(c')$ and let $\beta \in \mathrm{CH}_k(Y)$. By Lemma 22.4 we can find a $\beta' \in \mathrm{CH}_k(Y')$ with $g_*\beta' = \beta$. Then we set

$$c \cap \beta = h_*(c' \cap \beta')$$

To see that this is independent of the choice of β' it suffices to show that $h_*(c' \cap (p_*\gamma - q_*\gamma))$ is zero for $\gamma \in \mathrm{CH}_k(Y' \times_Y Y')$. Since c' is a bivariant class we have

$$h_*(c' \cap (p_*\gamma - q_*\gamma)) = h_*(a_*(c' \cap \gamma) - b_*(c' \cap \gamma)) = 0$$

the last equality since $h_* \circ a_* = h_* \circ b_*$ as $h \circ a = h \circ b$.

Observe that our choice for $c \cap \beta$ is forced by the requirement that $res(c) = c'$ and the compatibility of bivariant classes with proper pushforward.

Of course, in order to define the bivariant class c we need to construct maps $c \cap - : \mathrm{CH}_k(Y_1) \to \mathrm{CH}_{k+p}(Y_1 \times_Y X)$ for any morphism $Y_1 \to Y$ locally of finite type satisfying the conditions listed in Definition 33.1. Denote $Y_1' = Y' \times_Y Y_1$, $X_1 = X \times_Y Y_1$. The morphism $Y_1' \to Y_1$ is an envelope by Lemma 22.3. Hence we can use the base changed diagram

$$\begin{tikzcd} X_1' \times_{X_1} X_1' \arrow[r, shift left, "a_1"] \arrow[r, shift right, "b_1"'] \arrow[d, "f_1''"] & X_1' \arrow[r, "h_1"'] \arrow[d, "f_1'"] & X_1 \arrow[d, "f_1"] \\ Y_1' \times_{Y_1} Y_1' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "q_1"'] & Y_1' \arrow[r, "g_1"] & Y_1 \end{tikzcd}$$

and the same arguments to get a well defined map $c \cap - : \mathrm{CH}_k(Y_1) \to \mathrm{CH}_{k+p}(X_1)$ as before.

Next, we have to check conditions (1), (2), and (3) of Definition 33.1 for c. For example, suppose that $t : Y_2 \to Y_1$ is a proper morphism of schemes locally of finite type over Y. Denote as above the base changes of the first diagram to Y_1, resp. Y_2, by subscripts $_1$, resp. $_2$. Denote $t' : Y_2' \to Y_1'$, $s : X_2 \to X_1$, and $s' : X_2' \to X_1'$ the base changes of t to Y', X, and X'. We have to show that

$$s_*(c \cap \beta_2) = c \cap t_*\beta_2$$

for $\beta_2 \in \mathrm{CH}_k(Y_2)$. Choose $\beta_2' \in \mathrm{CH}_k(Y_2')$ with $g_{2,*}\beta_2' = \beta_2$. Since c' is a bivariant class and the diagrams

$$\begin{tikzcd} X_2' \arrow[r, "h_2"'] \arrow[d, "s'"'] & X_2 \arrow[d, "s"] \\ X_1' \arrow[r, "h_1"] & X_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_2' \arrow[r, "f_2'"'] \arrow[d, "s'"'] & Y_2' \arrow[d, "t'"] \\ X_2' \arrow[r, "f_1'"] & Y_1' \end{tikzcd}$$

are cartesian we have

$$s_*(c \cap \beta_2) = s_*(h_{2,*}(c' \cap \beta_2')) = h_{1,*}s'_*(c' \cap \beta_2') = h_{1,*}(c' \cap (t'_*\beta_2'))$$

and the final expression computes $c \cap t_*\beta_2$ by construction: $t'_*\beta_2' \in \mathrm{CH}_k(Y_1')$ is a class whose image by $g_{1,*}$ is $t_*\beta_2$. This proves condition (1). The other conditions are proved in the same manner and we omit the detailed arguments. ☐

36. Projective space bundle formula

Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Consider a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Our convention is that the *projective bundle associated to* $\mathcal{E}$ is the morphism

$$\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{E})) \xrightarrow{\pi} X$$

over X with $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ normalized so that $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$. In particular there is a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. We will say informally "let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$" to denote the situation where $P = \mathbf{P}(\mathcal{E})$ and $\mathcal{O}_P(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Lemma 36.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For any $\alpha \in \mathrm{CH}_k(X)$ the element*

$$\pi_*\left(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha\right) \in \mathrm{CH}_{k+r-1-s}(X)$$

is 0 if $s < r-1$ and is equal to α when $s = r-1$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Note that $\pi^*[Z] = [\pi^{-1}(Z)]$ as $\pi^{-1}(Z)$ is integral of δ-dimension $r-1$. If $s < r-1$, then by construction $c_1(\mathcal{O}_P(1))^s \cap \pi^*[Z]$ is represented by a $(k+r-1-s)$-cycle supported on $\pi^{-1}(Z)$. Hence the pushforward of this cycle is zero for dimension reasons.

Let $s = r-1$. By the argument given above we see that $\pi_*(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha) = n[Z]$ for some $n \in \mathbf{Z}$. We want to show that $n = 1$. For the same dimension reasons as above it suffices to prove this result after replacing X by $X \setminus T$ where $T \subset Z$ is a proper closed subset. Let ξ be the generic point of Z. We can choose elements $e_1, \ldots, e_{r-1} \in \mathcal{E}_\xi$ which form part of a basis of $\mathcal{E}_\xi$. These give rational sections $s_1, \ldots, s_{r-1}$ of $\mathcal{O}_P(1)|_{\pi^{-1}(Z)}$ whose common zero set is the closure of the image a rational section of $\mathbf{P}(\mathcal{E}|_Z) \to Z$ union a closed subset whose support maps to a proper closed subset T of Z. After removing T from X (and correspondingly $\pi^{-1}(T)$ from P), we see that $s_1, \ldots, s_n$ form a sequence of global sections $s_i \in \Gamma(\pi^{-1}(Z), \mathcal{O}_{\pi^{-1}(Z)}(1))$ whose common zero set is the image of a section

$Z \to \pi^{-1}(Z)$. Hence we see successively that

$$\begin{array}{rcl} \pi^*[Z] & = & [\pi^{-1}(Z)] \\ c_1(\mathcal{O}_P(1)) \cap \pi^*[Z] & = & [Z(s_1)] \\ c_1(\mathcal{O}_P(1))^2 \cap \pi^*[Z] & = & [Z(s_1) \cap Z(s_2)] \\ \ldots & = & \ldots \\ c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*[Z] & = & [Z(s_1) \cap \ldots \cap Z(s_{r-1})] \end{array}$$

by repeated applications of Lemma 25.4. Since the pushforward by π of the image of a section of π over Z is clearly $[Z]$ we see the result when $\alpha = [Z]$. We omit the verification that these arguments imply the result for a general cycle $\alpha = \sum n_j[Z_j]$. □

Lemma 36.2 (Projective space bundle formula)**.** *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. The map*

$$\bigoplus\nolimits_{i=0}^{r-1} \mathrm{CH}_{k+i}(X) \longrightarrow \mathrm{CH}_{k+r-1}(P),$$

$$(\alpha_0, \ldots, \alpha_{r-1}) \longmapsto \pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

is an isomorphism.

Proof. Fix $k \in \mathbf{Z}$. We first show the map is injective. Suppose that $(\alpha_0, \ldots, \alpha_{r-1})$ is an element of the left hand side that maps to zero. By Lemma 36.1 we see that

$$0 = \pi_*(\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}) = \alpha_{r-1}$$

Next, we see that

$$0 = \pi_*(c_1(\mathcal{O}_P(1)) \cap (\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-2} \cap \pi^*\alpha_{r-2})) = \alpha_{r-2}$$

and so on. Hence the map is injective.

It remains to show the map is surjective. Let X_i, $i \in I$ be the irreducible components of X. Then $P_i = \mathbf{P}(\mathcal{E}|_{X_i})$, $i \in I$ are the irreducible components of P. Consider the commutative diagram

$$\begin{tikzcd} \coprod P_i \arrow[r, "p"'] \arrow[d, "\coprod \pi_i"'] & P \arrow[d, "\pi"] \\ \coprod X_i \arrow[r, "q"] & X \end{tikzcd}$$

Observe that p_* is surjective. If $\beta \in \mathrm{CH}_k(\coprod X_i)$ then $\pi^* q_* \beta = p_*(\coprod \pi_i)^*\beta$, see Lemma 15.1. Similarly for capping with $c_1(\mathcal{O}(1))$ by Lemma 26.4. Hence, if the map of the lemma is surjective for each of the morphisms $\pi_i : P_i \to X_i$, then the map is surjective for $\pi : P \to X$. Hence we may assume X is irreducible. Thus $\dim_\delta(X) < \infty$ and in particular we may use induction on $\dim_\delta(X)$.

The result is clear if $\dim_\delta(X) < k$. Let $\alpha \in \mathrm{CH}_{k+r-1}(P)$. For any locally closed subscheme $T \subset X$ denote $\gamma_T : \bigoplus \mathrm{CH}_{k+i}(T) \to \mathrm{CH}_{k+r-1}(\pi^{-1}(T))$ the map

$$\gamma_T(\alpha_0, \ldots, \alpha_{r-1}) = \pi^*\alpha_0 + \ldots + c_1(\mathcal{O}_{\pi^{-1}(T)}(1))^{r-1} \cap \pi^*\alpha_{r-1}.$$

Suppose for some nonempty open $U \subset X$ we have $\alpha|_{\pi^{-1}(U)} = \gamma_U(\alpha_0, \ldots, \alpha_{r-1})$. Then we may choose lifts $\alpha'_i \in \mathrm{CH}_{k+i}(X)$ and we see that $\alpha - \gamma_X(\alpha'_0, \ldots, \alpha'_{r-1})$ is by Lemma 19.3 rationally equivalent to a k-cycle on $P_Y = \mathbf{P}(\mathcal{E}|_Y)$ where $Y = X \setminus U$

as a reduced closed subscheme. Note that $\dim_\delta(Y) < \dim_\delta(X)$. By induction the result holds for $P_Y \to Y$ and hence the result holds for α. Hence we may replace X by any nonempty open of X.

In particular we may assume that $\mathcal{E} \cong \mathcal{O}_X^{\oplus r}$. In this case $\mathbf{P}(\mathcal{E}) = X \times \mathbf{P}^{r-1}$. Let us use the stratification

$$\mathbf{P}^{r-1} = \mathbf{A}^{r-1} \amalg \mathbf{A}^{r-2} \amalg \ldots \amalg \mathbf{A}^0$$

The closure of each stratum is a $\mathbf{P}^{r-1-i}$ which is a representative of $c_1(\mathcal{O}(1))^i \cap [\mathbf{P}^{r-1}]$. Hence P has a similar stratification

$$P = U^{r-1} \amalg U^{r-2} \amalg \ldots \amalg U^0$$

Let P^i be the closure of U^i. Let $\pi^i : P^i \to X$ be the restriction of π to P^i. Let $\alpha \in \mathrm{CH}_{k+r-1}(P)$. By Lemma 32.1 we can write $\alpha|_{U^{r-1}} = \pi^*\alpha_0|_{U^{r-1}}$ for some $\alpha_0 \in \mathrm{CH}_k(X)$. Hence the difference $\alpha - \pi^*\alpha_0$ is the image of some $\alpha' \in \mathrm{CH}_{k+r-1}(P^{r-2})$. By Lemma 32.1 again we can write $\alpha'|_{U^{r-2}} = (\pi^{r-2})^*\alpha_1|_{U^{r-2}}$ for some $\alpha_1 \in \mathrm{CH}_{k+1}(X)$. By Lemma 31.1 we see that the image of $(\pi^{r-2})^*\alpha_1$ represents $c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$. We also see that $\alpha - \pi^*\alpha_0 - c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$ is the image of some $\alpha'' \in \mathrm{CH}_{k+r-1}(P^{r-3})$. And so on. □

Lemma 36.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let*

$$p : E = \underline{\mathrm{Spec}}(Sym^*(\mathcal{E})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+r}(E)$ is an isomorphism for all k.

Proof. (For the case of linebundles, see Lemma 32.2.) For surjectivity see Lemma 32.1. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to the finite locally free sheaf $\mathcal{E} \oplus \mathcal{O}_X$. Let $s \in \Gamma(P, \mathcal{O}_P(1))$ correspond to the global section $(0, 1) \in \Gamma(X, \mathcal{E} \oplus \mathcal{O}_X)$. Let $D = Z(s) \subset P$. Note that $(\pi|_D : D \to X, \mathcal{O}_P(1)|_D)$ is the projective space bundle associated to $\mathcal{E}$. We denote $\pi_D = \pi|_D$ and $\mathcal{O}_D(1) = \mathcal{O}_P(1)|_D$. Moreover, D is an effective Cartier divisor on P. Hence $\mathcal{O}_P(D) = \mathcal{O}_P(1)$ (see Divisors, Lemma 14.10). Also there is an isomorphism $E \cong P \setminus D$. Denote $j : E \to P$ the corresponding open immersion. For injectivity we use that the kernel of

$$j^* : \mathrm{CH}_{k+r}(P) \longrightarrow \mathrm{CH}_{k+r}(E)$$

are the cycles supported in the effective Cartier divisor D, see Lemma 19.3. So if $p^*\alpha = 0$, then $\pi^*\alpha = i_*\beta$ for some $\beta \in \mathrm{CH}_{k+r}(D)$. By Lemma 36.2 we may write

$$\beta = \pi_D^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^{r-1} \cap \pi_D^*\beta_{r-1}.$$

for some $\beta_i \in \mathrm{CH}_{k+i}(X)$. By Lemmas 31.1 and 26.4 this implies

$$\pi^*\alpha = i_*\beta = c_1(\mathcal{O}_P(1)) \cap \pi^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^r \cap \pi^*\beta_{r-1}.$$

Since the rank of $\mathcal{E} \oplus \mathcal{O}_X$ is $r + 1$ this contradicts Lemma 26.4 unless all α and all β_i are zero. □

37. The Chern classes of a vector bundle

We can use the projective space bundle formula to define the Chern classes of a rank r vector bundle in terms of the expansion of $c_1(\mathcal{O}(1))^r$ in terms of the lower powers, see formula (37.1.1). The reason for the signs will be explained later.

Definition 37.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to $\mathcal{E}$.

(1) By Lemma 36.2 there are elements $c_i \in \mathrm{CH}_{n-i}(X)$, $i = 0, \ldots, r$ such that $c_0 = [X]$, and

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^* c_{r-i} = 0. \tag{37.1.1}$$

(2) With notation as above we set $c_i(\mathcal{E}) \cap [X] = c_i$ as an element of $\mathrm{CH}_{n-i}(X)$. We call these the *Chern classes of $\mathcal{E}$ on X*.

(3) The *total Chern class of $\mathcal{E}$ on X* is the combination

$$c(\mathcal{E}) \cap [X] = c_0(\mathcal{E}) \cap [X] + c_1(\mathcal{E}) \cap [X] + \ldots + c_r(\mathcal{E}) \cap [X]$$

which is an element of $\mathrm{CH}_*(X) = \bigoplus_{k \in \mathbf{Z}} \mathrm{CH}_k(X)$.

Let us check that this does not give a new notion in case the vector bundle has rank 1.

Lemma 37.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The first Chern class of $\mathcal{L}$ on X of Definition 37.1 is equal to the Weil divisor associated to $\mathcal{L}$ by Definition 24.1.*

Proof. In this proof we use $c_1(\mathcal{L}) \cap [X]$ to denote the construction of Definition 24.1. Since $\mathcal{L}$ has rank 1 we have $\mathbf{P}(\mathcal{L}) = X$ and $\mathcal{O}_{\mathbf{P}(\mathcal{L})}(1) = \mathcal{L}$ by our normalizations. Hence (37.1.1) reads

$$(-1)^1 c_1(\mathcal{L}) \cap c_0 + (-1)^0 c_1 = 0$$

Since $c_0 = [X]$, we conclude $c_1 = c_1(\mathcal{L}) \cap [X]$ as desired. □

Remark 37.3. We could also rewrite equation 37.1.1 as

$$\sum\nolimits_{i=0}^{r} c_1(\mathcal{O}_P(-1))^i \cap \pi^* c_{r-i} = 0. \tag{37.3.1}$$

but we find it easier to work with the tautological quotient sheaf $\mathcal{O}_P(1)$ instead of its dual.

38. Intersecting with Chern classes

In this section we define Chern classes of vector bundles on X as bivariant classes on X, see Lemma 38.7 and the discussion following this lemma. Our construction follows the familiar pattern of first defining the operation on prime cycles and then summing. In Lemma 38.2 we show that the result is determined by the usual formula on the associated projective bundle. Next, we show that capping with Chern classes passes through rational equivalence, commutes with proper pushforward, commutes with flat pullback, and commutes with the gysin maps for inclusions of effective Cartier divisors. These lemmas could have been avoided by directly using the characterization in Lemma 38.2 and using Lemma 33.4; the reader who wishes to see this worked out should consult Chow Groups of Spaces, Lemma 28.1.

Definition 38.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. We define, for every integer k and any $0 \leq j \leq r$, an operation

$$c_j(\mathcal{E}) \cap - : Z_k(X) \to \mathrm{CH}_{k-j}(X)$$

called *intersection with the jth Chern class of $\mathcal{E}$*.

(1) Given an integral closed subscheme $i : W \to X$ of δ-dimension k we define

$$c_j(\mathcal{E}) \cap [W] = i_*(c_j(i^*\mathcal{E}) \cap [W]) \in \mathrm{CH}_{k-j}(X)$$

where $c_j(i^*\mathcal{E}) \cap [W]$ is as defined in Definition 37.1.

(2) For a general k-cycle $\alpha = \sum n_i[W_i]$ we set

$$c_j(\mathcal{E}) \cap \alpha = \sum n_i c_j(\mathcal{E}) \cap [W_i]$$

If $\mathcal{E}$ has rank 1 then this agrees with our previous definition (Definition 25.1) by Lemma 37.2.

Lemma 38.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For $\alpha \in Z_k(X)$ the elements $c_j(\mathcal{E}) \cap \alpha$ are the unique elements α_j of $\mathrm{CH}_{k-j}(X)$ such that $\alpha_0 = \alpha$ and*

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

holds in the Chow group of P.

Proof. The uniqueness of $\alpha_0, \ldots, \alpha_r$ such that $\alpha_0 = \alpha$ and such that the displayed equation holds follows from the projective space bundle formula Lemma 36.2. The identity holds by definition for $\alpha = [W]$ where W is an integral closed subscheme of X. For a general k-cycle α on X write $\alpha = \sum n_a[W_a]$ with $n_a \neq 0$, and $i_a : W_a \to X$ pairwise distinct integral closed subschemes. Then the family $\{W_a\}$ is locally finite on X. Set $P_a = \pi^{-1}(W_a) = \mathbf{P}(\mathcal{E}|_{W_a})$. Denote $i'_a : P_a \to P$ the corresponding closed immersions. Consider the fibre product diagram

$$\begin{tikzcd} P' \arrow[r, equal] \arrow[d, "\pi'"'] & \coprod P_a \arrow[r, "\coprod i'_a"'] \arrow[d, "\coprod \pi_a"'] & P \arrow[d, "\pi"] \\ X' \arrow[r, equal] & \coprod W_a \arrow[r, "\coprod i_a"] & X \end{tikzcd}$$

The morphism $p : X' \to X$ is proper. Moreover $\pi' : P' \to X'$ together with the invertible sheaf $\mathcal{O}_{P'}(1) = \coprod \mathcal{O}_{P_a}(1)$ which is also the pullback of $\mathcal{O}_P(1)$ is the projective bundle associated to $\mathcal{E}' = p^*\mathcal{E}$. By definition

$$c_j(\mathcal{E}) \cap [\alpha] = \sum i_{a,*}(c_j(\mathcal{E}|_{W_a}) \cap [W_a]).$$

Write $\beta_{a,j} = c_j(\mathcal{E}|_{W_a}) \cap [W_a]$ which is an element of $\mathrm{CH}_{k-j}(W_a)$. We have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_a}(1))^i \cap \pi_a^*(\beta_{a,r-i}) = 0$$

for each a by definition. Thus clearly we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(\beta_{r-i}) = 0$$

with $\beta_j = \sum n_a \beta_{a,j} \in \mathrm{CH}_{k-j}(X')$. Denote $p' : P' \to P$ the morphism $\coprod i'_a$. We have $\pi^* p_* \beta_j = p'_*(\pi')^* \beta_j$ by Lemma 15.1. By the projection formula of Lemma 26.4 we conclude that

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_* \beta_j) = 0$$

Since $p_* \beta_j$ is a representative of $c_j(\mathcal{E}) \cap \alpha$ we win. $\square$

We will consistently use this characterization of Chern classes to prove many more properties.

Lemma 38.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. If $\alpha \sim_{rat} \beta$ are rationally equivalent k-cycles on X then $c_j(\mathcal{E}) \cap \alpha = c_j(\mathcal{E}) \cap \beta$ in $\mathrm{CH}_{k-j}(X)$.*

Proof. By Lemma 38.2 the elements $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, $j \geq 1$ and $\beta_j = c_j(\mathcal{E}) \cap \beta$, $j \geq 1$ are uniquely determined by the *same* equation in the chow group of the projective bundle associated to $\mathcal{E}$. (This of course relies on the fact that flat pullback is compatible with rational equivalence, see Lemma 20.2.) Hence they are equal. $\square$

In other words capping with Chern classes of finite locally free sheaves factors through rational equivalence to give maps

$$c_j(\mathcal{E}) \cap - : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-j}(X).$$

Our next task is to show that Chern classes are bivariant classes, see Definition 33.1.

Lemma 38.4. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $p : X \to Y$ be a proper morphism. Let α be a k-cycle on X. Let $\mathcal{E}$ be a finite locally free sheaf on Y. Then*

$$p_*(c_j(p^*\mathcal{E}) \cap \alpha) = c_j(\mathcal{E}) \cap p_* \alpha$$

Proof. Let $(\pi : P \to Y, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. Then $P_X = X \times_Y P$ is the projective bundle associated to $p^*\mathcal{E}$ and $\mathcal{O}_{P_X}(1)$ is the pullback of $\mathcal{O}_P(1)$. Write $\alpha_j = c_j(p^*\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 38.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi_X^*(\alpha_{r-i}) = 0$$

in the chow group of P_X. Consider the fibre product diagram

$$\begin{tikzcd} P_X \arrow[r, "p'"'] \arrow[d, "\pi_X"'] & P \arrow[d, "\pi"] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

Apply proper pushforward p'_* (Lemma 20.3) to the displayed equality above. Using Lemmas 26.4 and 15.1 we obtain

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_* \alpha_{r-i}) = 0$$

in the chow group of P. By the characterization of Lemma 38.2 we conclude. $\square$

Lemma 38.5. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on Y. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let α be a k-cycle on Y. Then*

$$f^*(c_j(\mathcal{E}) \cap \alpha) = c_j(f^*\mathcal{E}) \cap f^* \alpha$$

Proof. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 38.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to Y, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{tikzcd} P_X = \mathbf{P}(f^*\mathcal{E}) \arrow[r, "f'"'] \arrow[d, "\pi_X"'] & P \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Note that $\mathcal{O}_{P_X}(1)$ is the pullback of $\mathcal{O}_P(1)$. Apply flat pullback $(f')^*$ (Lemma 20.2) to the displayed equation above. By Lemmas 26.2 and 14.3 we see that

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_X}(1))^i \cap \pi_X^*(f^*\alpha_{r-i}) = 0$$

holds in the chow group of P_X. By the characterization of Lemma 38.2 we conclude. $\square$

Lemma 38.6. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\mathcal{L}, s, i : D \to X)$ be as in Definition 29.1. Then $c_j(\mathcal{E}|_D) \cap i^*\alpha = i^*(c_j(\mathcal{E}) \cap \alpha)$ for all $\alpha \in \mathrm{CH}_k(X)$.*

Proof. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 38.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to X, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{tikzcd} P_D = \mathbf{P}(\mathcal{E}|_D) \arrow[r, "i'"'] \arrow[d, "\pi_D"'] & P \arrow[d, "\pi"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

Note that $\mathcal{O}_{P_D}(1)$ is the pullback of $\mathcal{O}_P(1)$. Apply the gysin map $(i')^*$ (Lemma 30.2) to the displayed equation above. Applying Lemmas 30.4 and 29.9 we obtain

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_D}(1))^i \cap \pi_D^*(i^*\alpha_{r-i}) = 0$$

in the chow group of P_D. By the characterization of Lemma 38.2 we conclude. $\square$

At this point we have enough material to be able to prove that capping with Chern classes defines a bivariant class.

Lemma 38.7. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Let $0 \leq p \leq r$. Then the rule that to $f : X' \to X$ assigns $c_p(f^*\mathcal{E}) \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_{k-p}(X')$ is a bivariant class of degree p.*

Proof. Immediate from Lemmas 38.3, 38.4, 38.5, and 38.6 and Definition 33.1. $\square$

This lemma allows us to define the Chern classes of a finite locally free module as follows.

Definition 38.8. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. For $i = 0, \ldots, r$ the *ith Chern class* of $\mathcal{E}$ is the bivariant class $c_i(\mathcal{E}) \in A^i(X)$ of degree i constructed in Lemma 38.7. The *total Chern class* of $\mathcal{E}$ is the formal sum

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \ldots + c_r(\mathcal{E})$$

which is viewed as a nonhomogeneous bivariant class on X.

By the remark following Definition 38.1 if $\mathcal{E}$ is invertible, then this definition agrees with Definition 34.4. Next we see that Chern classes are in the center of the bivariant Chow cohomology ring $A^*(X)$.

Lemma 38.9. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Then*

1. *$c_j(\mathcal{E}) \in A^j(X)$ is in the center of $A^*(X)$ and*
2. *if $f : X' \to X$ is locally of finite type and $c \in A^*(X' \to X)$, then $c \circ c_j(\mathcal{E}) = c_j(f^*\mathcal{E}) \circ c$.*

In particular, if $\mathcal{F}$ is a second locally free $\mathcal{O}_X$-module on X of rank s, then

$$c_i(\mathcal{E}) \cap c_j(\mathcal{F}) \cap \alpha = c_j(\mathcal{F}) \cap c_i(\mathcal{E}) \cap \alpha$$

as elements of $\mathrm{CH}_{k-i-j}(X)$ for all $\alpha \in \mathrm{CH}_k(X)$.

Proof. It is immediate that (2) implies (1). Let $\alpha \in \mathrm{CH}_k(X)$. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 38.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to Y, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Denote $\pi' : P' \to X'$ the base change of π by f. Using Lemma 34.5 and the properties of bivariant classes we obtain

$$\begin{aligned} 0 &= c \cap \left(\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) \right) \\ &= \sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(c \cap \alpha_{r-i}) \end{aligned}$$

in the Chow group of P' (calculation omitted). Hence we see that $c \cap \alpha_j$ is equal to $c_j(f^*\mathcal{E}) \cap (c \cap \alpha)$ by the characterization of Lemma 38.2. This proves the lemma. $\square$

Remark 38.10. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. If the rank of $\mathcal{E}$ is not constant then we can still define the Chern classes of $\mathcal{E}$. Namely, in this case we can write

$$X = X_0 \amalg X_1 \amalg X_2 \amalg \ldots$$

where $X_r \subset X$ is the open and closed subspace where the rank of $\mathcal{E}$ is r. By Lemma 35.4 we have $A^p(X) = \prod A^p(X_r)$. Hence we can define $c_p(\mathcal{E})$ to be the product of the classes $c_p(\mathcal{E}|_{X_r})$ in $A^p(X_r)$. Explicitly, if $X' \to X$ is a morphism locally of finite type, then we obtain by pullback a corresponding decomposition of X' and we find that

$$\mathrm{CH}_*(X') = \prod\nolimits_{r \geq 0} \mathrm{CH}_*(X'_r)$$

by our definitions. Then $c_p(\mathcal{E}) \in A^p(X)$ is the bivariant class which preserves these direct product decompositions and acts by the already defined operations

$c_i(\mathcal{E}|_{X_r}) \cap -$ on the factors. Observe that in this setting it may happen that $c_p(\mathcal{E})$ is nonzero for infinitely many p. It follows that the total chern class is an element

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \ldots \in A^*(X)^\wedge$$

of the completed bivariant cohomology ring, see Remark 35.5. In this setting we define the "rank" of $\mathcal{E}$ to be the element $r(\mathcal{E}) \in A^0(X)$ as the bivariant operation which sends $(\alpha_r) \in \prod \mathrm{CH}_*(X'_r)$ to $(r\alpha_r) \in \prod \mathrm{CH}_*(X'_r)$. Note that it is still true that $c_p(\mathcal{E})$ and $r(\mathcal{E})$ are in the center of $A^*(X)$.

Remark 38.11. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. In general we write $X = \coprod X_r$ as in Remark 38.10. If only a finite number of the X_r are nonempty, then we can set

$$c_{top}(\mathcal{E}) = \sum\nolimits_r c_r(\mathcal{E}|_{X_r}) \in A^*(X) = \bigoplus A^*(X_r)$$

where the equality is Lemma 35.4. If infinitely many X_r are nonempty, we will use the same notation to denote

$$c_{top}(\mathcal{E}) = \prod c_r(\mathcal{E}|_{X_r}) \in \prod A^r(X_r) \subset A^*(X)^\wedge$$

see Remark 35.5 for notation.

39. Polynomial relations among Chern classes

Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}_i$ be a finite collection of finite locally free sheaves on X. By Lemma 38.9 we see that the Chern classes

$$c_j(\mathcal{E}_i) \in A^*(X)$$

generate a commutative (and even central) $\mathbf{Z}$-subalgebra of the Chow cohomology algebra $A^*(X)$. Thus we can say what it means for a polynomial in these Chern classes to be zero, or for two polynomials to be the same. As an example, saying that $c_1(\mathcal{E}_1)^5 + c_2(\mathcal{E}_2)c_3(\mathcal{E}_3) = 0$ means that the operations

$$\mathrm{CH}_k(Y) \longrightarrow \mathrm{CH}_{k-5}(Y), \quad \alpha \longmapsto c_1(\mathcal{E}_1)^5 \cap \alpha + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap \alpha$$

are zero for all morphisms $f : Y \to X$ which are locally of finite type. By Lemma 35.3 this is equivalent to the requirement that given any morphism $f : Y \to X$ where Y is an integral scheme locally of finite type over S the cycle

$$c_1(\mathcal{E}_1)^5 \cap [Y] + c_2(\mathcal{E}_2) \cap c_3(\mathcal{E}_3) \cap [Y]$$

is zero in $\mathrm{CH}_{\dim(Y)-5}(Y)$.

A specific example is the relation

$$c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) = c_1(\mathcal{L}) + c_1(\mathcal{N})$$

proved in Lemma 25.2. More generally, here is what happens when we tensor an arbitrary locally free sheaf by an invertible sheaf.

Lemma 39.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $\mathcal{L}$ be an invertible sheaf on X. Then we have*

$$c_i(\mathcal{E} \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j \quad (39.1.1)$$

in $A^*(X)$.

Proof. This should hold for any triple $(X, \mathcal{E}, \mathcal{L})$. In particular it should hold when X is integral and by Lemma 35.3 it is enough to prove it holds when capping with $[X]$ for such X. Thus assume that X is integral. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ be the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{E} \otimes \mathcal{L}$. Consider the canonical morphism

$$\begin{tikzcd} P \arrow[rr, "g"'] \arrow[rd, "\pi"'] & & P' \arrow[ld, "\pi'"] \\ & X & \end{tikzcd}$$

see Constructions, Lemma 20.1. It has the property that $g^*\mathcal{O}_{P'}(1) = \mathcal{O}_P(1) \otimes \pi^*\mathcal{L}$. This means that we have

$$\sum\nolimits_{i=0}^{r} (-1)^i (\xi + x)^i \cap \pi^*(c_{r-i}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) = 0$$

in $\mathrm{CH}_*(P)$, where ξ represents $c_1(\mathcal{O}_P(1))$ and x represents $c_1(\pi^*\mathcal{L})$. By simple algebra this is equivalent to

$$\sum\nolimits_{i=0}^{r} (-1)^i \xi^i \left(\sum\nolimits_{j=i}^{r} (-1)^{j-i} \binom{j}{i} x^{j-i} \cap \pi^*(c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) \right) = 0$$

Comparing with Equation (37.1.1) it follows from this that

$$c_{r-i}(\mathcal{E}) \cap [X] = \sum\nolimits_{j=i}^{r} \binom{j}{i} (-c_1(\mathcal{L}))^{j-i} \cap c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]$$

Reworking this (getting rid of minus signs, and renumbering) we get the desired relation. □

Some example cases of (39.1.1) are

$$c_1(\mathcal{E} \otimes \mathcal{L}) = c_1(\mathcal{E}) + r c_1(\mathcal{L})$$

$$c_2(\mathcal{E} \otimes \mathcal{L}) = c_2(\mathcal{E}) + (r-1) c_1(\mathcal{E}) c_1(\mathcal{L}) + \binom{r}{2} c_1(\mathcal{L})^2$$

$$c_3(\mathcal{E} \otimes \mathcal{L}) = c_3(\mathcal{E}) + (r-2) c_2(\mathcal{E}) c_1(\mathcal{L}) + \binom{r-1}{2} c_1(\mathcal{E}) c_1(\mathcal{L})^2 + \binom{r}{3} c_1(\mathcal{L})^3$$

40. Additivity of Chern classes

All of the preliminary lemmas follow trivially from the final result.

Lemma 40.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$$

Then we have

$$c_r(\mathcal{E}) = 0, \quad c_j(\mathcal{E}) = c_j(\mathcal{F}), \quad j = 0, \ldots, r-1$$

in $A^*(X)$.

Proof. By Lemma 35.3 it suffices to show that if X is integral then $c_j(\mathcal{E}) \cap [X] = c_j(\mathcal{F}) \cap [X]$. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ denote the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{F}$. The surjection $\mathcal{E} \to \mathcal{F}$ gives rise to a closed immersion

$$i : P' \longrightarrow P$$

over X. Moreover, the element $1 \in \Gamma(X, \mathcal{O}_X) \subset \Gamma(X, \mathcal{E})$ gives rise to a global section $s \in \Gamma(P, \mathcal{O}_P(1))$ whose zero set is exactly P'. Hence P' is an effective Cartier divisor on P such that $\mathcal{O}_P(P') \cong \mathcal{O}_P(1)$. Hence we see that

$$c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha = i_*((\pi')^*\alpha)$$

for any cycle class α on X by Lemma 31.1. By Lemma 38.2 we see that $\alpha_j = c_j(\mathcal{F}) \cap [X]$, $j = 0, \ldots, r-1$ satisfy

$$\sum\nolimits_{j=0}^{r-1} (-1)^j c_1(\mathcal{O}_{P'}(1))^j \cap (\pi')^*\alpha_j = 0$$

Pushing this to P and using the remark above as well as Lemma 26.4 we get

$$\sum\nolimits_{j=0}^{r-1} (-1)^j c_1(\mathcal{O}_P(1))^{j+1} \cap \pi^*\alpha_j = 0$$

By the uniqueness of Lemma 38.2 we conclude that $c_r(\mathcal{E}) \cap [X] = 0$ and $c_j(\mathcal{E}) \cap [X] = \alpha_j = c_j(\mathcal{F}) \cap [X]$ for $j = 0, \ldots, r-1$. Hence the lemma holds. $\square$

Lemma 40.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{L}$ is an invertible sheaf. Then

$$c(\mathcal{E}) = c(\mathcal{L})c(\mathcal{F})$$

in $A^*(X)$.

Proof. This relation really just says that $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$. By Lemma 40.1 we have $c_j(\mathcal{E} \otimes \mathcal{L}^{\otimes -1}) = c_j(\mathcal{F} \otimes \mathcal{L}^{\otimes -1})$ for $j = 0, \ldots, r$ were we set $c_r(\mathcal{F} \otimes \mathcal{L}^{-1}) = 0$ by convention. Applying Lemma 39.1 we deduce

$$\sum_{j=0}^{i} \binom{r-i+j}{j} (-1)^j c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j = \sum_{j=0}^{i} \binom{r-1-i+j}{j} (-1)^j c_{i-j}(\mathcal{F}) c_1(\mathcal{L})^j$$

Setting $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$ gives a "solution" of this equation. The lemma follows if we show that this is the only possible solution. We omit the verification. $\square$

Lemma 40.3. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Suppose that $\mathcal{E}$ sits in an exact sequence*

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

of finite locally free sheaves $\mathcal{E}_i$ of rank r_i. The total Chern classes satisfy

$$c(\mathcal{E}) = c(\mathcal{E}_1)c(\mathcal{E}_2)$$

in $A^*(X)$.

Proof. By Lemma 35.3 we may assume that X is integral and we have to show the identity when capping against $[X]$. By induction on r_1. The case $r_1 = 1$ is Lemma 40.2. Assume $r_1 > 1$. Let $(\pi : P \to X, \mathcal{O}_P(1))$ denote the projective space bundle associated to $\mathcal{E}_1$. Note that

(1) $\pi^* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(P)$ is injective, and
(2) $\pi^*\mathcal{E}_1$ sits in a short exact sequence $0 \to \mathcal{F} \to \pi^*\mathcal{E}_1 \to \mathcal{L} \to 0$ where $\mathcal{L}$ is invertible.

The first assertion follows from the projective space bundle formula and the second follows from the definition of a projective space bundle. (In fact $\mathcal{L} = \mathcal{O}_P(1)$.) Let $Q = \pi^*\mathcal{E}/\mathcal{F}$, which sits in an exact sequence $0 \to \mathcal{L} \to Q \to \pi^*\mathcal{E}_2 \to 0$. By induction we have

$$\begin{aligned} c(\pi^*\mathcal{E}) \cap [P] &= c(\mathcal{F}) \cap c(\pi^*\mathcal{E}/\mathcal{F}) \cap [P] \\ &= c(\mathcal{F}) \cap c(\mathcal{L}) \cap c(\pi^*\mathcal{E}_2) \cap [P] \\ &= c(\pi^*\mathcal{E}_1) \cap c(\pi^*\mathcal{E}_2) \cap [P] \end{aligned}$$

Since $[P] = \pi^*[X]$ we win by Lemma 38.5. □

Lemma 40.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}_i$, $i = 1, \ldots, r$ be invertible $\mathcal{O}_X$-modules on X. Let $\mathcal{E}$ be a locally free rank $\mathcal{O}_X$-module endowed with a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Set $c_1(\mathcal{L}_i) = x_i$. Then

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^{r} (1 + x_i)$$

in $A^*(X)$.

Proof. Apply Lemma 40.2 and induction. □

41. Degrees of zero cycles

We start defining the degree of a zero cycle on a proper scheme over a field. One approach is to define it directly as in Lemma 41.2 and then show it is well defined by Lemma 18.3. Instead we define it as follows.

Definition 41.1. Let k be a field (Example 7.2). Let $p : X \to \text{Spec}(k)$ be proper. The *degree of a zero cycle* on X is given by proper pushforward

$$p_* : \text{CH}_0(X) \to \text{CH}_0(\text{Spec}(k))$$

(Lemma 20.3) combined with the natural isomorphism $\text{CH}_0(\text{Spec}(k)) = \mathbf{Z}$ which maps $[\text{Spec}(k)]$ to 1. Notation: $\deg(\alpha)$.

Let us spell this out further.

Lemma 41.2. *Let k be a field. Let X be proper over k. Let $\alpha = \sum n_i[Z_i]$ be in $Z_0(X)$. Then*

$$\deg(\alpha) = \sum n_i \deg(Z_i)$$

where $\deg(Z_i)$ is the degree of $Z_i \to \text{Spec}(k)$, i.e., $\deg(Z_i) = \dim_k \Gamma(Z_i, \mathcal{O}_{Z_i})$.

Proof. This is the definition of proper pushforward (Definition 12.1). □

Next, we make the connection with degrees of vector bundles over 1-dimensional proper schemes over fields as defined in Varieties, Section 44.

Lemma 41.3. *Let k be a field. Let X be a proper scheme over k of dimension ≤ 1. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module of constant rank. Then*

$$\deg(\mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1)$$

where the left hand side is defined in Varieties, Definition 44.1.

Proof. Let $C_i \subset X$, $i = 1, \ldots, t$ be the irreducible components of dimension 1 with reduced induced scheme structure and let m_i be the multiplicity of C_i in X. Then $[X]_1 = \sum m_i[C_i]$ and $c_1(\mathcal{E}) \cap [X]_1$ is the sum of the pushforwards of the cycles $m_i c_1(\mathcal{E}|_{C_i}) \cap [C_i]$. Since we have a similar decomposition of the degree of $\mathcal{E}$ by Varieties, Lemma 44.6 it suffices to prove the lemma in case X is a proper curve over k.

Assume X is a proper curve over k. By Divisors, Lemma 36.1 there exists a modification $f : X' \to X$ such that $f^*\mathcal{E}$ has a filtration whose successive quotients are invertible $\mathcal{O}_{X'}$-modules. Since $f_*[X']_1 = [X]_1$ we conclude from Lemma 38.4 that

$$\deg(c_1(\mathcal{E}) \cap [X]_1) = \deg(c_1(f^*\mathcal{E}) \cap [X']_1)$$

Since we have a similar relationship for the degree by Varieties, Lemma 44.4 we reduce to the case where $\mathcal{E}$ has a filtration whose successive quotients are invertible $\mathcal{O}_X$-modules. In this case, we may use additivity of the degree (Varieties, Lemma 44.3) and of first Chern classes (Lemma 40.3) to reduce to the case discussed in the next paragraph.

Assume X is a proper curve over k and $\mathcal{E}$ is an invertible $\mathcal{O}_X$-module. By Divisors, Lemma 15.12 we see that $\mathcal{E}$ is isomorphic to $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{\otimes -1}$ for some effective Cartier divisors D, D' on X (this also uses that X is projective, see Varieties, Lemma 43.4 for example). By additivity of degree under tensor product of invertible sheaves (Varieties, Lemma 44.7) and additivity of c_1 under tensor product of invertible sheaves (Lemma 25.2 or 39.1) we reduce to the case $\mathcal{E} = \mathcal{O}_X(D)$. In this case the left hand side gives $\deg(D)$ (Varieties, Lemma 44.9) and the right hand side gives $\deg([D]_0)$ by Lemma 25.4. Since

$$[D]_0 = \sum\nolimits_{x \in D} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D,x})[x] = \sum\nolimits_{x \in D} \text{length}_{\mathcal{O}_{D,x}}(\mathcal{O}_{D,x})[x]$$

by definition, we see

$$\deg([D]_0) = \sum\nolimits_{x \in D} \text{length}_{\mathcal{O}_{D,x}}(\mathcal{O}_{D,x})[\kappa(x) : k] = \dim_k \Gamma(D, \mathcal{O}_D) = \deg(D)$$

The penultimate equality by Algebra, Lemma 52.12 using that D is affine. $\square$

Finally, we can tie everything up with the numerical intersections defined in Varieties, Section 45.

Lemma 41.4. *Let k be a field. Let X be a proper scheme over k. Let $Z \subset X$ be a closed subscheme of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \deg(c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z]_d)$$

where the left hand side is defined in Varieties, Definition 45.3. In particular,

$$\deg_{\mathcal{L}}(Z) = \deg(c_1(\mathcal{L})^d \cap [Z]_d)$$

if $\mathcal{L}$ is an ample invertible $\mathcal{O}_X$-module.

Proof. We will prove this by induction on d. If $d = 0$, then the result is true by Varieties, Lemma 33.3. Assume $d > 0$.

Let $Z_i \subset Z$, $i = 1, \ldots, t$ be the irreducible components of dimension d with reduced induced scheme structure and let m_i be the multiplicity of Z_i in Z. Then $[Z]_d = \sum m_i[Z_i]$ and $c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z]_d$ is the sum of the cycles $m_i c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [Z_i]$. Since we have a similar decomposition for $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ by Varieties, Lemma 45.2 it suffices to prove the lemma in case $Z = X$ is a proper variety of dimension d over k.

By Chow's lemma there exists a birational proper morphism $f : Y \to X$ with Y H-projective over k. See Cohomology of Schemes, Lemma 18.1 and Remark 18.2. Then

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot Y) = (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X)$$

by Varieties, Lemma 45.7 and we have

$$f_*(c_1(f^*\mathcal{L}_1) \cap \ldots \cap c_1(f^*\mathcal{L}_d) \cap [Y]) = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [X]$$

by Lemma 26.4. Thus we may replace X by Y and assume that X is projective over k.

If X is a proper d-dimensional projective variety, then we can write $\mathcal{L}_1 = \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{\otimes -1}$ for some effective Cartier divisors $D, D' \subset X$ by Divisors, Lemma 15.12. By additivity for both sides of the equation (Varieties, Lemma 45.5 and Lemma 25.2) we reduce to the case $\mathcal{L}_1 = \mathcal{O}_X(D)$ for some effective Cartier divisor D. By Varieties, Lemma 45.8 we have

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot D)$$

and by Lemma 25.4 we have

$$c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [X] = c_1(\mathcal{L}_2) \cap \ldots \cap c_1(\mathcal{L}_d) \cap [D]_{d-1}$$

Thus we obtain the result from our induction hypothesis. □

42. Cycles of given codimension

In some cases there is a second grading on the abelian group of all cycles given by codimension.

Lemma 42.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Write $\delta = \delta_{X/S}$ as in Section 7. The following are equivalent*

1. *There exists a decomposition $X = \coprod_{n \in \mathbf{Z}} X_n$ into open and closed subschemes such that $\delta(\xi) = n$ whenever $\xi \in X_n$ is a generic point of an irreducible component of X_n.*
2. *For all $x \in X$ there exists an open neighbourhood $U \subset X$ of x and an integer n such that $\delta(\xi) = n$ whenever $\xi \in U$ is a generic point of an irreducible component of U.*
3. *For all $x \in X$ there exists an integer n_x such that $\delta(\xi) = n_x$ for any generic point ξ of an irreducible component of X containing x.*

The conditions are satisfied if X is either normal or Cohen-Macaulay[3].

Proof. It is clear that (1) ⇒ (2) ⇒ (3). Conversely, if (3) holds, then we set $X_n = \{x \in X \mid n_x = n\}$ and we get a decomposition as in (1). Namely, X_n is open because given x the union of the irreducible components of X passing through x minus the union of the irreducible components of X not passing through x is an open neighbourhood of x. If X is normal, then X is a disjoint union of integral schemes (Properties, Lemma 7.7) and hence the properties hold. If X is Cohen-Macaulay, then $\delta' : X \to \mathbf{Z}$, $x \mapsto -\dim(\mathcal{O}_{X,x})$ is a dimension function on X (see Example 7.4). Since $\delta - \delta'$ is locally constant (Topology, Lemma 20.3) and since $\delta'(\xi) = 0$ for every generic point ξ of X we see that (2) holds. □

[3]In fact, it suffices if X is (S_2). Compare with Local Cohomology, Lemma 3.2.

Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S satisfying the equivalent conditions of Lemma 42.1. For an integral closed subscheme $Z \subset X$ we have the codimension $\text{codim}(Z, X)$ of Z in X, see Topology, Definition 11.1. We define a *codimension p-cycle* to be a cycle $\alpha = \sum n_Z[Z]$ on X such that $n_Z \neq 0 \Rightarrow \text{codim}(Z, X) = p$. The abelian group of all codimension p-cycles is denoted $Z^p(X)$. Let $X = \coprod X_n$ be the decomposition given in Lemma 42.1 part (1). Recalling that our cycles are defined as locally finite sums, it is clear that

$$Z^p(X) = \prod\nolimits_n Z_{n-p}(X_n)$$

Moreover, we see that $\prod_p Z^p(X) = \prod_k Z_k(X)$. We could now define rational equivalence of codimension p cycles on X in exactly the same manner as before and in fact we could redevelop the whole theory from scratch for cycles of a given codimension for X as in Lemma 42.1. However, instead we simply define the *Chow group of codimension p-cycles* as

$$\text{CH}^p(X) = \prod\nolimits_n \text{CH}_{n-p}(X_n)$$

As before we have $\prod_p \text{CH}^p(X) = \prod_k \text{CH}_k(X)$. If X is quasi-compact, then the product in the formula is finite (and hence is a direct sum) and we have $\bigoplus_p \text{CH}^p(X) = \bigoplus_k \text{CH}_k(X)$. If X is quasi-compact and finite dimensional, then only a finite number of these groups is nonzero.

Many of the constructions and results for Chow groups proved above have natural counterparts for the Chow groups $\text{CH}^*(X)$. Each of these is shown by decomposing the relevant schemes into "equidimensional" pieces as in Lemma 42.1 and applying the results already proved for the factors in the product decomposition given above. Let us list some of them.

(1) If $f : X \to Y$ is a flat morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.1 then flat pullback determines a map
$$f^* : \text{CH}^p(Y) \to \text{CH}^p(X)$$
(2) If $f : X \to Y$ is a morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.1 let us say f has *codimension* $r \in \mathbf{Z}$ if for all pairs of irreducible components $Z \subset X$, $W \subset Y$ with $f(Z) \subset W$ we have $\dim_\delta(W) - \dim_\delta(Z) = r$.
(3) If $f : X \to Y$ is a proper morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.1 and f has codimension r, then proper pushforward is a map
$$f_* : \text{CH}^p(X) \to \text{CH}^{p+r}(Y)$$
(4) If $f : X \to Y$ is a morphism of schemes locally of finite type over S and X and Y satisfy the equivalent conditions of Lemma 42.1 and f has codimension r and $c \in A^q(X \to Y)$, then c induces maps
$$c \cap - : \text{CH}^p(Y) \to \text{CH}^{p+q-r}(X)$$
(5) If X is a scheme locally of finite type over S satisfying the equivalent conditions of Lemma 42.1 and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then
$$c_1(\mathcal{L}) \cap - : \text{CH}^p(X) \to \text{CH}^{p+1}(X)$$

(6) If X is a scheme locally of finite type over S satisfying the equivalent conditions of Lemma 42.1 and $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module, then

$$c_i(\mathcal{E}) \cap - : \mathrm{CH}^p(X) \to \mathrm{CH}^{p+i}(X)$$

Warning: the property for a morphism to have codimension r is not preserved by base change.

Remark 42.2. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S satisfying the equivalent conditions of Lemma 42.1. Let $X = \coprod X_n$ be the decomposition into open and closed subschemes such that every irreducible component of X_n has δ-dimension n. In this situation we sometimes set

$$[X] = \sum\nolimits_n [X_n]_n \in \mathrm{CH}^0(X)$$

This class is a kind of "fundamental class" of X in Chow theory.

43. The splitting principle

In our setting it is not so easy to say what the splitting principle exactly says/is. Here is a possible formulation.

Lemma 43.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}_i$ be a finite collection of locally free $\mathcal{O}_X$-modules of rank r_i. There exists a projective flat morphism $\pi : P \to X$ of relative dimension d such that*

1. *for any morphism $f : Y \to X$ the map $\pi_Y^* : \mathrm{CH}_*(Y) \to \mathrm{CH}_{*+d}(Y \times_X P)$ is injective, and*
2. *each $\pi^*\mathcal{E}_i$ has a filtration whose successive quotients $\mathcal{L}_{i,1}, \ldots, \mathcal{L}_{i,r_i}$ are invertible $\mathcal{O}_P$-modules.*

Moreover, when (1) holds the restriction map $A^*(X) \to A^*(P)$ (Remark 34.2) is injective.

Proof. We may assume $r_i \geq 1$ for all i. We will prove the lemma by induction on $\sum (r_i - 1)$. If this integer is 0, then $\mathcal{E}_i$ is invertible for all i and we conclude by taking $\pi = \mathrm{id}_X$. If not, then we can pick an i such that $r_i > 1$ and consider the morphism $\pi_i : P_i = \mathbf{P}(\mathcal{E}_i) \to X$. We have a short exact sequence

$$0 \to \mathcal{F} \to \pi_i^*\mathcal{E}_i \to \mathcal{O}_{P_i}(1) \to 0$$

of finite locally free $\mathcal{O}_{P_i}$-modules of ranks $r_i - 1$, r_i, and 1. Observe that π_i^* is injective on chow groups after any base change by the projective bundle formula (Lemma 36.2). By the induction hypothesis applied to the finite locally free $\mathcal{O}_{P_i}$-modules $\mathcal{F}$ and $\pi_{i'}^*\mathcal{E}_{i'}$ for $i' \neq i$, we find a morphism $\pi : P \to P_i$ with properties stated as in the lemma. Then the composition $\pi_i \circ \pi : P \to X$ does the job. Some details omitted. □

Remark 43.2. The proof of Lemma 43.1 shows that the morphism $\pi : P \to X$ has the following additional properties:

1. π is a finite composition of projective space bundles associated to locally free modules of finite constant rank, and
2. for every $\alpha \in \mathrm{CH}_k(X)$ we have $\alpha = \pi_*(\xi_1 \cap \ldots \cap \xi_d \cap \pi^*\alpha)$ where ξ_i is the first Chern class of some invertible $\mathcal{O}_P$-module.

The second observation follows from the first and Lemma 36.1. We will add more observations here as needed.

Let (S, δ), X, and $\mathcal{E}_i$ be as in Lemma 43.1. The *splitting principle* refers to the practice of symbolically writing

$$c(\mathcal{E}_i) = \prod (1 + x_{i,j})$$

The symbols $x_{i,1}, \ldots, x_{i,r_i}$ are called the *Chern roots* of $\mathcal{E}_i$. In other words, the pth Chern class of $\mathcal{E}_i$ is the pth elementary symmetric function in the Chern roots. The usefulness of the splitting principle comes from the assertion that in order to prove a polynomial relation among Chern classes of the $\mathcal{E}_i$ it is enough to prove the corresponding relation among the Chern roots.

Namely, let $\pi : P \to X$ be as in Lemma 43.1. Recall that there is a canonical $\mathbf{Z}$-algebra map $\pi^* : A^*(X) \to A^*(P)$, see Remark 34.2. The injectivity of π_Y^* on Chow groups for every Y over X, implies that the map $\pi^* : A^*(X) \to A^*(P)$ is injective (details omitted). We have

$$\pi^* c(\mathcal{E}_i) = \prod (1 + c_1(\mathcal{L}_{i,j}))$$

by Lemma 40.4. Thus we may think of the Chern roots $x_{i,j}$ as the elements $c_1(\mathcal{L}_{i,j}) \in A^*(P)$ and the displayed equation as taking place in $A^*(P)$ after applying the injective map $\pi^* : A^*(X) \to A^*(P)$ to the left hand side of the equation.

To see how this works, it is best to give some examples.

Lemma 43.3. *In Situation 7.1 let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$. Then*

$$c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$$

in $A^i(X)$.

Proof. Choose a morphism $\pi : P \to X$ as in Lemma 43.1. By the injectivity of π^* (after any base change) it suffices to prove the relation between the Chern classes of $\mathcal{E}$ and $\mathcal{E}^\vee$ after pulling back to P. Thus we may assume there exist invertible $\mathcal{O}_X$-modules $\mathcal{L}_i$, $i = 1, \ldots, r$ and a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Then we obtain the dual filtration

$$0 = \mathcal{E}_r^\perp \subset \mathcal{E}_1^\perp \subset \mathcal{E}_2^\perp \subset \ldots \subset \mathcal{E}_0^\perp = \mathcal{E}^\vee$$

such that $\mathcal{E}_{i-1}^\perp/\mathcal{E}_i^\perp \cong \mathcal{L}_i^{\otimes -1}$. Set $x_i = c_1(\mathcal{L}_i)$. Then $c_1(\mathcal{L}_i^{\otimes -1}) = -x_i$ by Lemma 25.2. By Lemma 40.4 we have

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^r (1 + x_i) \quad \text{and} \quad c(\mathcal{E}^\vee) = \prod\nolimits_{i=1}^r (1 - x_i)$$

in $A^*(X)$. The result follows from a formal computation which we omit. □

Lemma 43.4. *In Situation 7.1 let X be locally of finite type over S. Let $\mathcal{E}$ and $\mathcal{F}$ be a finite locally free $\mathcal{O}_X$-modules of ranks r and s. Then we have*

$$c_1(\mathcal{E} \otimes \mathcal{F}) = rc_1(\mathcal{F}) + sc_1(\mathcal{E})$$

$$c_2(\mathcal{E} \otimes \mathcal{F}) = rc_2(\mathcal{F}) + sc_2(\mathcal{E}) + \binom{r}{2} c_1(\mathcal{F})^2 + (rs - 1)c_1(\mathcal{F})c_1(\mathcal{E}) + \binom{s}{2} c_1(\mathcal{E})^2$$

and so on in $A^*(X)$.

Proof. Arguing exactly as in the proof of Lemma 43.3 we may assume we have invertible $\mathcal{O}_X$-modules $\mathcal{L}_i$, $i = 1, \ldots, r$ $\mathcal{N}_i$, $i = 1, \ldots, s$ filtrations

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E} \quad \text{and} \quad 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_s = \mathcal{F}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$ and such that $\mathcal{F}_j/\mathcal{F}_{j-1} \cong \mathcal{N}_j$. Ordering pairs (i, j) lexicographically we obtain a filtration

$$0 \subset \ldots \subset \mathcal{E}_i \otimes \mathcal{F}_j + \mathcal{E}_{i-1} \otimes \mathcal{F} \subset \ldots \subset \mathcal{E} \otimes \mathcal{F}$$

with successive quotients

$$\mathcal{L}_1 \otimes \mathcal{N}_1, \mathcal{L}_1 \otimes \mathcal{N}_2, \ldots, \mathcal{L}_1 \otimes \mathcal{N}_s, \mathcal{L}_2 \otimes \mathcal{N}_1, \ldots, \mathcal{L}_r \otimes \mathcal{N}_s$$

By Lemma 40.4 we have

$$c(\mathcal{E}) = \prod (1 + x_i), \quad c(\mathcal{F}) = \prod (1 + y_j), \quad \text{and} \quad c(\mathcal{E} \otimes \mathcal{F}) = \prod (1 + x_i + y_j),$$

in $A^*(X)$. The result follows from a formal computation which we omit. ☐

Remark 43.5. The equalities proven above remain true even when we work with finite locally free $\mathcal{O}_X$-modules whose rank is allowed to be nonconstant. In fact, we can work with polynomials in the rank and the Chern classes as follows. Consider the graded polynomial ring $\mathbf{Z}[r, c_1, c_2, c_3, \ldots]$ where r has degree 0 and c_i has degree i. Let

$$P \in \mathbf{Z}[r, c_1, c_2, c_3, \ldots]$$

be a homogeneous polynomial of degree p. Then for any finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ on X we can consider

$$P(\mathcal{E}) = P(r(\mathcal{E}), c_1(\mathcal{E}), c_2(\mathcal{E}), c_3(\mathcal{E}), \ldots) \in A^p(X)$$

see Remark 38.10 for notation and conventions. To prove relations among these polynomials (for multiple finite locally free modules) we can work locally on X and use the splitting principle as above. For example, we claim that

$$c_2(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) = P(\mathcal{E})$$

where $P = 2rc_2 - (r-1)c_1^2$. Namely, since $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) = \mathcal{E} \otimes \mathcal{E}^\vee$ this follows easily from Lemmas 43.3 and 43.4 above by decomposing X into parts where the rank of $\mathcal{E}$ is constant as in Remark 38.10.

Example 43.6. For every $p \geq 1$ there is a unique homogeneous polynomial $P_p \in \mathbf{Z}[c_1, c_2, c_3, \ldots]$ of degree p such that, for any $n \geq p$ we have

$$P_p(s_1, s_2, \ldots, s_p) = \sum x_i^p$$

in $\mathbf{Z}[x_1, \ldots, x_n]$ where $s_1, \ldots, s_p$ are the elementary symmetric polynomials in $x_1, \ldots, x_n$, so

$$s_i = \sum\nolimits_{1 \leq j_1 < \ldots < j_i \leq n} x_{j_1} x_{j_2} \ldots x_{j_i}$$

The existence of P_p comes from the well known fact that the elementary symmetric functions generate the ring of all symmetric functions over the integers. Another way to characterize $P_p \in \mathbf{Z}[c_1, c_2, c_3, \ldots]$ is that we have

$$\log(1 + c_1 + c_2 + c_3 + \ldots) = \sum\nolimits_{p \geq 1} (-1)^{p-1} \frac{P_p}{p}$$

as formal power series. This is clear by writing $1 + c_1 + c_2 + \ldots = \prod(1 + x_i)$ and applying the power series for the logarithm function. Expanding the left hand side we get

$$\begin{aligned}&(c_1 + c_2 + \ldots) - (1/2)(c_1 + c_2 + \ldots)^2 + (1/3)(c_1 + c_2 + \ldots)^3 - \ldots \\ &= c_1 + (c_2 - (1/2)c_1^2) + (c_3 - c_1c_2 + (1/3)c_1^3) + \ldots\end{aligned}$$

In this way we find that

$$\begin{aligned}P_1 &= c_1, \\ P_2 &= c_1^2 - 2c_2, \\ P_3 &= c_1^3 - 3c_1c_2 + 3c_3, \\ P_4 &= c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4,\end{aligned}$$

and so on. Since the Chern classes of a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ are the elementary symmetric polynomials in the Chern roots x_i, we see that

$$P_p(\mathcal{E}) = \sum x_i^p$$

For convenience we set $P_0 = r$ in $\mathbf{Z}[r, c_1, c_2, c_3, \ldots]$ so that $P_0(\mathcal{E}) = r(\mathcal{E})$ as a bivariant class (as in Remarks 38.10 and 43.5).

44. Chern classes and sections

A brief section whose main result is that we may compute the top Chern class of a finite locally free module using the vanishing locus of a “regular section.

Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Let $f : X' \to X$ be locally of finite type. Let

$$s \in \Gamma(X', f^*\mathcal{E})$$

be a global section of the pullback of $\mathcal{E}$ to X'. Let $Z(s) \subset X'$ be the zero scheme of s. More precisely, we define $Z(s)$ to be the closed subscheme whose quasi-coherent sheaf of ideals is the image of the map $s : f^*\mathcal{E}^\vee \to \mathcal{O}_{X'}$.

Lemma 44.1. *In the situation described just above assume* $\dim_\delta(X') = n$*, that* $f^*\mathcal{E}$ *has constant rank* r*, that* $\dim_\delta(Z(s)) \leq n - r$*, and that for every generic point* $\xi \in Z(s)$ *with* $\delta(\xi) = n - r$ *the ideal of* $Z(s)$ *in* $\mathcal{O}_{X',\xi}$ *is generated by a regular sequence of length* r*. Then*

$$c_r(\mathcal{E}) \cap [X']_n = [Z(s)]_{n-r}$$

in $\mathrm{CH}_*(X')$.

Proof. Since $c_r(\mathcal{E})$ is a bivariant class (Lemma 38.7) we may assume $X = X'$ and we have to show that $c_r(\mathcal{E}) \cap [X]_n = [Z(s)]_{n-r}$ in $\mathrm{CH}_{n-r}(X)$. We will prove the lemma by induction on $r \geq 0$. (The case $r = 0$ is trivial.) The case $r = 1$ is handled by Lemma 25.4. Assume $r > 1$.

Let $\pi : P \to X$ be the projective space bundle associated to $\mathcal{E}$ and consider the short exact sequence

$$0 \to \mathcal{E}' \to \pi^*\mathcal{E} \to \mathcal{O}_P(1) \to 0$$

By the projective space bundle formula (Lemma 36.2) it suffices to prove the equality after pulling back by π. Observe that $\pi^{-1}Z(s) = Z(\pi^*s)$ has δ-dimension $\leq n-1$

and that the assumption on regular sequences at generic points of δ-dimension $n-1$ holds by flat pullback, see Algebra, Lemma 68.5. Let $t \in \Gamma(P, \mathcal{O}_P(1))$ be the image of π^*s. We claim

$$[Z(t)]_{n+r-2} = c_1(\mathcal{O}_P(1)) \cap [P]_{n+r-1}$$

Assuming the claim we finish the proof as follows. The restriction $\pi^*s|_{Z(t)}$ maps to zero in $\mathcal{O}_P(1)|_{Z(t)}$ hence comes from a unique element $s' \in \Gamma(Z(t), \mathcal{E}'|_{Z(t)})$. Note that $Z(s') = Z(\pi^*s)$ as closed subschemes of P. If $\xi \in Z(s')$ is a generic point with $\delta(\xi) = n-1$, then the ideal of $Z(s')$ in $\mathcal{O}_{Z(t),\xi}$ can be generated by a regular sequence of length $r-1$: it is generated by $r-1$ elements which are the images of $r-1$ elements in $\mathcal{O}_{P,\xi}$ which together with a generator of the ideal of $Z(t)$ in $\mathcal{O}_{P,\xi}$ form a regular sequence of length r in $\mathcal{O}_{P,\xi}$. Hence we can apply the induction hypothesis to s' on $Z(t)$ to get $c_{r-1}(\mathcal{E}') \cap [Z(t)]_{n+r-2} = [Z(s')]_{n-1}$. Combining all of the above we obtain

$$\begin{aligned} c_r(\pi^*\mathcal{E}) \cap [P]_{n+r-1} &= c_{r-1}(\mathcal{E}') \cap c_1(\mathcal{O}_P(1)) \cap [P]_{n+r-1} \\ &= c_{r-1}(\mathcal{E}') \cap [Z(t)]_{n+r-2} \\ &= [Z(s')]_{n-1} \\ &= [Z(\pi^*s)]_{n-1} \end{aligned}$$

which is what we had to show.

Proof of the claim. This will follow from an application of the already used Lemma 25.4. We have $\pi^{-1}(Z(s)) = Z(\pi^*s) \subset Z(t)$. On the other hand, for $x \in X$ if $P_x \subset Z(t)$, then $t|_{P_x} = 0$ which implies that s is zero in the fibre $\mathcal{E} \otimes \kappa(x)$, which implies $x \in Z(s)$. It follows that $\dim_\delta(Z(t)) \leq n + (r-1) - 1$. Finally, let $\xi \in Z(t)$ be a generic point with $\delta(\xi) = n + r - 2$. If ξ is not the generic point of the fibre of $P \to X$ it is immediate that a local equation of $Z(t)$ is a nonzerodivisor in $\mathcal{O}_{P,\xi}$ (because we can check this on the fibre by Algebra, Lemma 99.2). If ξ is the generic point of a fibre, then $x = \pi(\xi) \in Z(s)$ and $\delta(x) = n + r - 2 - (r-1) = n-1$. This is a contradiction with $\dim_\delta(Z(s)) \leq n - r$ because $r > 1$ so this case doesn't happen. □

Lemma 44.2. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let*

$$0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{E} \to 0$$

be a short exact sequence of finite locally free $\mathcal{O}_X$-modules. Consider the closed embedding

$$i : N' = \underline{\mathrm{Spec}}_X(Sym((\mathcal{N}')^\vee)) \longrightarrow N = \underline{\mathrm{Spec}}_X(Sym(\mathcal{N}^\vee))$$

For $\alpha \in \mathrm{CH}_k(X)$ we have

$$i_*(p')^*\alpha = p^*(c_{top}(\mathcal{E}) \cap \alpha)$$

where $p' : N' \to X$ and $p : N \to X$ are the structure morphisms.

Proof. Here $c_{top}(\mathcal{E})$ is the bivariant class defined in Remark 38.11. By its very definition, in order to verify the formula, we may assume that $\mathcal{E}$ has constant rank. We may similarly assume $\mathcal{N}'$ and $\mathcal{N}$ have constant ranks, say r' and r, so $\mathcal{E}$ has rank $r - r'$ and $c_{top}(\mathcal{E}) = c_{r-r'}(\mathcal{E})$. Observe that $p^*\mathcal{E}$ has a canonical section

$$s \in \Gamma(N, p^*\mathcal{E}) = \Gamma(X, p_*p^*\mathcal{E}) = \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathrm{Sym}(\mathcal{N}^\vee) \supset \Gamma(X, \mathcal{H}om(\mathcal{N}, \mathcal{E}))$$

corresponding to the surjection $\mathcal{N} \to \mathcal{E}$ given in the statement of the lemma. The vanishing scheme of this section is exactly $N' \subset N$. Let $Y \subset X$ be an integral closed subscheme of δ-dimension n. Then we have

(1) $p^*[Y] = [p^{-1}(Y)]$ since $p^{-1}(Y)$ is integral of δ-dimension $n + r$,
(2) $(p')^*[Y] = [(p')^{-1}(Y)]$ since $(p')^{-1}(Y)$ is integral of δ-dimension $n + r'$,
(3) the restriction of s to $p^{-1}Y$ has vanishing scheme $(p')^{-1}Y$ and the closed immersion $(p')^{-1}Y \to p^{-1}Y$ is a regular immersion (locally cut out by a regular sequence).

We conclude that

$$(p')^*[Y] = c_{r-r'}(p^*\mathcal{E}) \cap p^*[Y] \quad \text{in} \quad \mathrm{CH}_*(N)$$

by Lemma 44.1. This proves the lemma. □

45. The Chern character and tensor products

Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. We define the *Chern character* of a finite locally free $\mathcal{O}_X$-module to be the formal expression

$$ch(\mathcal{E}) = \sum\nolimits_{i=1}^{r} e^{x_i}$$

if the x_i are the Chern roots of $\mathcal{E}$. Writing this as a polynomial in the Chern classes we obtain

$$\begin{aligned} ch(\mathcal{E}) &= r(\mathcal{E}) + c_1(\mathcal{E}) + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) + \frac{1}{6}(c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) + 3c_3(\mathcal{E})) \\ &\quad + \frac{1}{24}(c_1(\mathcal{E})^4 - 4c_1(\mathcal{E})^2c_2(\mathcal{E}) + 4c_1(\mathcal{E})c_3(\mathcal{E}) + 2c_2(\mathcal{E})^2 - 4c_4(\mathcal{E})) + \ldots \\ &= \sum\nolimits_{p=0,1,2,\ldots} \frac{P_p(\mathcal{E})}{p!} \end{aligned}$$

with P_p polynomials in the Chern classes as in Example 43.6. The degree p component of the above is

$$ch_p(\mathcal{E}) = \frac{P_p(\mathcal{E})}{p!} \in A^p(X) \otimes \mathbf{Q}$$

What does it mean that the coefficients are rational numbers? Well this simply means that we think of $ch_p(\mathcal{E})$ as an element of $A^p(X) \otimes \mathbf{Q}$.

Remark 45.1. In the discussion above we have defined the components of the Chern character $ch_p(\mathcal{E}) \in A^p(X) \otimes \mathbf{Q}$ of $\mathcal{E}$ even if the rank of $\mathcal{E}$ is not constant. See Remarks 38.10 and 43.5. Thus the full Chern character of $\mathcal{E}$ is an element of $\prod_{p \geq 0}(A^p(X) \otimes \mathbf{Q})$. If X is quasi-compact and $\dim(X) < \infty$ (usual dimension), then one can show using Lemma 34.6 and the splitting principle that $ch(\mathcal{E}) \in A^*(X) \otimes \mathbf{Q}$.

Lemma 45.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ be a short exact sequence of finite locally free $\mathcal{O}_X$-modules. Then we have the equality*

$$ch(\mathcal{E}) = ch(\mathcal{E}_1) + ch(\mathcal{E}_2)$$

More precisely, we have $P_p(\mathcal{E}) = P_p(\mathcal{E}_1) + P_p(\mathcal{E}_2)$ in $A^p(X)$ where P_p is as in Example 43.6.

Proof. It suffices to prove the more precise statement. By Section 43 this follows because if $x_{1,i}$, $i = 1, \ldots, r_1$ and $x_{2,i}$, $i = 1, \ldots, r_2$ are the Chern roots of $\mathcal{E}_1$ and $\mathcal{E}_2$, then $x_{1,1}, \ldots, x_{1,r_1}, x_{2,1}, \ldots, x_{2,r_2}$ are the Chern roots of $\mathcal{E}$. Hence we get the result from our choice of P_p in Example 43.6. □

Lemma 45.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be finite locally free $\mathcal{O}_X$-modules. Then we have the equality*

$$ch(\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2) = ch(\mathcal{E}_1)ch(\mathcal{E}_2)$$

More precisely, we have

$$P_p(\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2) = \sum\nolimits_{p_1+p_2=p} \binom{p}{p_1} P_{p_1}(\mathcal{E}_1)P_{p_2}(\mathcal{E}_2)$$

in $A^p(X)$ where P_p is as in Example 43.6.

Proof. It suffices to prove the more precise statement. By Section 43 this follows because if $x_{1,i}$, $i = 1, \ldots, r_1$ and $x_{2,i}$, $i = 1, \ldots, r_2$ are the Chern roots of $\mathcal{E}_1$ and $\mathcal{E}_2$, then $x_{1,i} + x_{2,j}$, $1 \leq i \leq r_1$, $1 \leq j \leq r_2$ are the Chern roots of $\mathcal{E}_1 \otimes \mathcal{E}_2$. Hence we get the result from the binomial formula for $(x_{1,i} + x_{2,j})^p$ and the shape of our polynomials P_p in Example 43.6. □

Lemma 45.4. *In Situation 7.1 let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$. Then $ch_i(\mathcal{E}^\vee) = (-1)^i ch_i(\mathcal{E})$ in $A^i(X) \otimes \mathbf{Q}$.*

Proof. Follows from the corresponding result for Chern classes (Lemma 43.3). □

46. Chern classes and the derived category

In this section we define the total Chern class of a perfect object E of the derived category of a scheme X, under the assumption that E may be represented by a finite complex of finite locally free modules on an envelope of X.

Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let

$$\mathcal{E}^a \to \mathcal{E}^{a+1} \to \ldots \to \mathcal{E}^b$$

be a bounded complex of finite locally free $\mathcal{O}_X$-modules of constant rank. Then we define the *total Chern class of the complex* by the formula

$$c(\mathcal{E}^\bullet) = \prod\nolimits_{n=a,\ldots,b} c(\mathcal{E}^n)^{(-1)^n} \in \prod\nolimits_{p \geq 0} A^p(X)$$

Here the inverse is the formal inverse, so

$$(1 + c_1 + c_2 + c_3 + \ldots)^{-1} = 1 - c_1 + c_1^2 - c_2 - c_1^3 + 2c_1c_2 - c_3 + \ldots$$

We will denote $c_p(\mathcal{E}^\bullet) \in A^p(X)$ the degree p part of $c(\mathcal{E}^\bullet)$. We similarly define the *Chern character of the complex* by the formula

$$ch(\mathcal{E}^\bullet) = \sum\nolimits_{n=a,\ldots,b} (-1)^n ch(\mathcal{E}^n) \in \prod\nolimits_{p \geq 0} (A^p(X) \otimes \mathbf{Q})$$

We will denote $ch_p(\mathcal{E}^\bullet) \in A^p(X) \otimes \mathbf{Q}$ the degree p part of $ch(\mathcal{E}^\bullet)$. Finally, for $P_p \in \mathbf{Z}[r, c_1, c_2, c_3, \ldots]$ as in Example 43.6 we define

$$P_p(\mathcal{E}^\bullet) = \sum\nolimits_{n=a,\ldots,b} (-1)^n P_p(\mathcal{E}^n)$$

in $A^p(X)$. Then we have $ch_p(\mathcal{E}^\bullet) = (1/p!)P_p(\mathcal{E}^\bullet)$ as usual. The next lemma shows that these constructions only depends on the image of the complex in the derived category.

Lemma 46.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be an object such that there exists a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing E. Then a slight generalization of the above constructions*

$$c(\mathcal{E}^\bullet) \in \prod\nolimits_{p \geq 0} A^p(X), \quad ch(\mathcal{E}^\bullet) \in \prod\nolimits_{p \geq 0} A^p(X) \otimes \mathbf{Q}, \quad P_p(\mathcal{E}^\bullet) \in A^p(X)$$

are independent of the choice of the complex $\mathcal{E}^\bullet$.

Proof. We prove this for the total Chern class; the other two cases follow by the same arguments using Lemma 45.2 instead of Lemma 40.3.

As in Remark 38.10 in order to define the total chern class $c(\mathcal{E}^\bullet)$ we decompose X into open and closed subschemes

$$X = \coprod\nolimits_{i \in I} X_i$$

such that the rank $\mathcal{E}^n$ is constant on X_i for all n and i. (Since these ranks are locally constant functions on X we can do this.) Since $\mathcal{E}^\bullet$ is locally bounded, we see that only a finite number of the sheaves $\mathcal{E}^n|_{X_i}$ are nonzero for a fixed i. Hence we can define

$$c(\mathcal{E}^\bullet|_{X_i}) = \prod\nolimits_n c(\mathcal{E}^n|_{X_i})^{(-1)^n} \in \prod\nolimits_{p \geq 0} A^p(X_i)$$

as above. By Lemma 35.4 we have $A^p(X) = \prod_i A^p(X_i)$. Hence for each $p \in \mathbf{Z}$ we have a unique element $c_p(\mathcal{E}^\bullet) \in A^p(X)$ restricting to $c_p(\mathcal{E}^\bullet|_{X_i})$ on X_i for all i.

Suppose we have a second locally bounded complex $\mathcal{F}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing E. Let $g : Y \to X$ be a morphism locally of finite type with Y integral. By Lemma 35.3 it suffices to show that with $c(g^*\mathcal{E}^\bullet) \cap [Y]$ is the same as $c(g^*\mathcal{F}^\bullet) \cap [Y]$ and it even suffices to prove this after replacing Y by an integral scheme proper and birational over Y. Then first we conclude that $g^*\mathcal{E}^\bullet$ and $g^*\mathcal{F}^\bullet$ are bounded complexes of finite locally free $\mathcal{O}_Y$-modules of constant rank. Next, by More on Flatness, Lemma 40.3 we may assume that $H^i(Lg^*E)$ is perfect of tor dimension ≤ 1 for all $i \in \mathbf{Z}$. This reduces us to the case discussed in the next paragraph.

Assume X is integral, $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are bounded complexes of finite locally free modules of constant rank, and $H^i(E)$ is a perfect $\mathcal{O}_X$-module of tor dimension ≤ 1 for all $i \in \mathbf{Z}$. We have to show that $c(\mathcal{E}^\bullet) \cap [X]$ is the same as $c(\mathcal{F}^\bullet) \cap [X]$. Denote $d^i_\mathcal{E} : \mathcal{E}^i \to \mathcal{E}^{i+1}$ and $d^i_\mathcal{F} : \mathcal{F}^i \to \mathcal{F}^{i+1}$ the differentials of our complexes. By More on Flatness, Remark 40.4 we know that $\mathrm{Im}(d^i_\mathcal{E})$, $\mathrm{Ker}(d^i_\mathcal{E})$, $\mathrm{Im}(d^i_\mathcal{F})$, and $\mathrm{Ker}(d^i_\mathcal{F})$ are finite locally free $\mathcal{O}_X$-modules for all i. By additivity (Lemma 40.3) we see that

$$c(\mathcal{E}^\bullet) = \prod\nolimits_i c(\mathrm{Ker}(d^i_\mathcal{E}))^{(-1)^i} c(\mathrm{Im}(d^i_\mathcal{E}))^{(-1)^i}$$

and similarly for $\mathcal{F}^\bullet$. Since we have the short exact sequences

$$0 \to \mathrm{Im}(d^i_\mathcal{E}) \to \mathrm{Ker}(d^i_\mathcal{E}) \to H^i(E) \to 0 \quad \text{and} \quad 0 \to \mathrm{Im}(d^i_\mathcal{F}) \to \mathrm{Ker}(d^i_\mathcal{F}) \to H^i(E) \to 0$$

we reduce to the problem stated and solved in the next paragraph.

Assume X is integral and we have two short exact sequences

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{Q} \to 0 \quad \text{and} \quad 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{Q} \to 0$$

with $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{F}$, $\mathcal{F}'$ finite locally free. Problem: show that $c(\mathcal{E})c(\mathcal{E}')^{-1} \cap [X] = c(\mathcal{F})c(\mathcal{F}')^{-1} \cap [X]$. To do this, consider the short exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \oplus \mathcal{F} \to \mathcal{Q} \to 0$$

defining $\mathcal{G}$. Since $\mathcal{Q}$ has tor dimension ≤ 1 we see that $\mathcal{G}$ is finite locally free. A diagram chase shows that the kernel of the surjection $\mathcal{G} \to \mathcal{F}$ maps isomorphically to $\mathcal{E}'$ in $\mathcal{E}$ and the kernel of the surjection $\mathcal{G} \to \mathcal{E}$ maps isomorphically to $\mathcal{F}'$ in $\mathcal{F}$. (Working affine locally this follows from or is equivalent to Schanuel's lemma, see Algebra, Lemma 109.1.) We conclude that

$$c(\mathcal{E})c(\mathcal{F}') = c(\mathcal{G}) = c(\mathcal{F})c(\mathcal{E}')$$

as desired. $\square$

Lemma 46.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Assume there exists an envelope $f : Y \to X$ (Definition 22.1) such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_Y$-modules. Then there exists unique bivariant classes $c(E) \in \prod_{p \geq 0} A^p(X)$, $ch(E) \in \prod_{p \geq 0} A^p(X) \otimes \mathbf{Q}$, and $P_p(E) \in A^p(X)$, independent of the choice of $f : Y \to X$ and $\mathcal{E}^\bullet$, such that the restriction of these classes to Y are equal to $c(\mathcal{E}^\bullet)$, $ch(\mathcal{E}^\bullet)$, and $P_p(\mathcal{E}^\bullet)$.*

Proof. Fix $p \in \mathbf{Z}$. We will prove the lemma for the chern class $c_p(E) \in A^p(X)$ and omit the arguments for the other cases.

Let $g : T \to X$ be a morphism locally of finite type such that there exists a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_T$-modules representing Lg^*E in $D(\mathcal{O}_T)$. The bivariant class $c_p(\mathcal{E}^\bullet) \in A^p(T)$ is independent of the choice of $\mathcal{E}^\bullet$ by Lemma 46.1. Let us write $c_p(Lg^*E) \in A^p(T)$ for this class. For any further morphism $h : T' \to T$ which is locally of finite type, setting $g' = g \circ h$ we see that $L(g')^*E = L(g \circ h)^*E = Lh^*Lg^*E$ is represented by $h^*\mathcal{E}^\bullet$ in $D(\mathcal{O}_{T'})$. We conclude that $c_p(L(g')^*E)$ makes sense and is equal to the restriction (Remark 33.5) of $c_p(Lg^*E)$ to T' (strictly speaking this requires an application of Lemma 38.7).

Let $f : Y \to X$ and $\mathcal{E}^\bullet$ be as in the statement of the lemma. We obtain a bivariant class $c_p(E) \in A^p(X)$ from an application of Lemma 35.6 to $f : Y \to X$ and the class $c' = c_p(Lf^*E)$ we constructed in the previous paragraph. The assumption in the lemma is satisfied because by the discussion in the previous paragraph we have $res_1(c') = c_p(Lg^*E) = res_2(c')$ where $g = f \circ p = f \circ q : Y \times_X Y \to X$.

Finally, suppose that $f' : Y' \to X$ is a second envelope such that $L(f')^*E$ is represented by a bounded complex of finite locally free $\mathcal{O}_{Y'}$-modules. Then it follows that the restrictions of $c_p(Lf^*E)$ and $c_p(L(f')^*E)$ to $Y \times_X Y'$ are equal. Since $Y \times_X Y' \to X$ is an envelope (Lemmas 22.3 and 22.2), we see that our two candidates for $c_p(E)$ agree by the unicity in Lemma 35.6. $\square$

Definition 46.3. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object.

(1) We say the *Chern classes of E are defined*[4] if there exists an envelope $f : Y \to X$ such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex of finite locally free $\mathcal{O}_Y$-modules.

[4] See Lemma 46.4 for some criteria.

(2) If the Chern classes of E are defined, then we define

$$c(E) \in \prod\nolimits_{p \geq 0} A^p(X), \quad ch(E) \in \prod\nolimits_{p \geq 0} A^p(X) \otimes \mathbf{Q}, \quad P_p(E) \in A^p(X)$$

by an application of Lemma 46.2.

This definition applies in many but not all situations envisioned in this chapter, see Lemma 46.4. Perhaps an elementary construction of these bivariant classes for general $E/X/(S, \delta)$ as in the definition exists; we don't know.

Lemma 46.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. If one of the following conditions hold, then the Chern classes of E are defined:*

(1) *there exists an envelope $f : Y \to X$ such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex of finite locally free $\mathcal{O}_Y$-modules,*
(2) *E can be represented by a bounded complex of finite locally free $\mathcal{O}_X$-modules,*
(3) *the irreducible components of X are quasi-compact,*
(4) *X is quasi-compact,*
(5) *there exists a morphism $X \to X'$ of schemes locally of finite type over S such that E is the pullback of a perfect object E' on X' whose chern classes are defined, or*
(6) *add more here.*

Proof. Condition (1) is just Definition 46.3 part (1). Condition (2) implies (1).

As in (3) assume the irreducible components X_i of X are quasi-compact. We view X_i as a reduced integral closed subscheme over X. The morphism $\coprod X_i \to X$ is an envelope. For each i there exists an envelope $X'_i \to X_i$ such that X'_i has an ample family of invertible modules, see More on Morphisms, Proposition 80.3. Observe that $f : Y = \coprod X'_i \to X$ is an envelope; small detail omitted. By Derived Categories of Schemes, Lemma 36.7 each X'_i has the resolution property. Thus the perfect object $L(f|_{X'_i})^*E$ of $D(\mathcal{O}_{X'_i})$ can be represented by a bounded complex of finite locally free $\mathcal{O}_{X'_i}$-modules, see Derived Categories of Schemes, Lemma 37.2. This proves (3) implies (1).

Part (4) implies (3).

Let $g : X \to X'$ and E' be as in part (5). Then there exists an envelope $f' : Y' \to X'$ such that $L(f')^*E'$ is represented by a locally bounded complex $(\mathcal{E}')^\bullet$ of $\mathcal{O}_{Y'}$-modules. Then the base change $f : Y \to X$ is an envelope by Lemma 22.3. Moreover, the pulllback $\mathcal{E}^\bullet = g^*(\mathcal{E}')^\bullet$ represents Lf^*E and we see that the chern classes of E are defined. $\square$

Lemma 46.5. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Assume the Chern classes of E are defined. For $g : W \to X$ locally of finite type with W integral, there exists a commutative diagram*

$$\begin{tikzcd} W' \arrow[rr, "b"] \arrow[dr, "g'"'] & & W \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

with W' integral and $b : W' \to W$ proper birational such that $L(g')^*E$ is represented by a bounded complex $\mathcal{E}^\bullet$ of locally free $\mathcal{O}_{W'}$-modules of constant rank and we have $res(c_p(E)) = c_p(\mathcal{E}^\bullet)$ in $A^p(W')$.

Proof. Choose an envelope $f : Y \to X$ such that Lf^*E is isomorphic in $D(\mathcal{O}_Y)$ to a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_Y$-modules. The base change $Y \times_X W \to W$ of f is an envelope by Lemma 22.3. Choose a point $\xi \in Y \times_X W$ mapping to the generic point of W with the same residue field. Consider the integral closed subscheme $W' \subset Y \times_X W$ with generic point ξ. The restriction of the projection $Y \times_X W \to W$ to W' is a proper birational morphism $b : W' \to W$. Set $g' = g \circ b$. Finally, consider the pullback $(W' \to Y)^*\mathcal{E}^\bullet$. This is a locally bounded complex of finite locally free modules on W'. Since W' is integral it follows that it is bounded and that the terms have constant rank. Finally, by construction $(W' \to Y)^*\mathcal{E}^\bullet$ represents $L(g')^*E$ and by construction its pth chern class gives the restriction of $c_p(E)$ by $W' \to X$. This finishes the proof. □

Lemma 46.6. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be perfect. If the Chern classes of E are defined then*

1. *$c_p(E)$ is in the center of the algebra $A^*(X)$, and*
2. *if $g : X' \to X$ is locally of finite type and $c \in A^*(X' \to X)$, then $c \circ c_p(E) = c_p(Lg^*E) \circ c$.*

Proof. Part (1) follows immediately from part (2). Let $g : X' \to X$ and $c \in A^*(X' \to X)$ be as in (2). To show that $c \circ c_p(E) - c_p(Lg^*E) \circ c = 0$ we use the criterion of Lemma 35.3. Thus we may assume that X is integral and by Lemma 46.5 we may even assume that E is represented by a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules of constant rank. Then we have to show that

$$c \cap c_p(\mathcal{E}^\bullet) \cap [X] = c_p(\mathcal{E}^\bullet) \cap c \cap [X]$$

in $\mathrm{CH}_*(X')$. This is immediate from Lemma 38.9 and the construction of $c_p(\mathcal{E}^\bullet)$ as a polynomial in the chern classes of the locally free modules $\mathcal{E}^n$. □

Lemma 46.7. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let*

$$E_1 \to E_2 \to E_3 \to E_1[1]$$

be a distinguished triangle of perfect objects in $D(\mathcal{O}_X)$. If one of the following conditions holds

1. *there exists an envelope $f : Y \to X$ such that $Lf^*E_1 \to Lf^*E_2$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules,*
2. *$E_1 \to E_2$ can be represented be a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules,*
3. *the irreducible components of X are quasi-compact,*
4. *X is quasi-compact, or*
5. *add more here,*

then the Chern classes of E_1, E_2, E_3 are defined and we have $c(E_2) = c(E_1)c(E_3)$, $ch(E_2) = ch(E_1) + ch(E_3)$, and $P_p(E_2) = P_p(E_1) + P_p(E_3)$.

Proof. Let $f : Y \to X$ be an envelope and let $\alpha^\bullet : \mathcal{E}_1^\bullet \to \mathcal{E}_2^\bullet$ be a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules representing $Lf^*E_1 \to Lf^*E_2$. Then the cone $C(\alpha)^\bullet$ represents Lf^*E_3. Since $C(\alpha)^n = \mathcal{E}_2^n \oplus \mathcal{E}_1^{n+1}$ we see that $C(\alpha)^\bullet$

is a locally bounded complex of finite locally free $\mathcal{O}_Y$-modules. We conclude that the Chern classes of E_1, E_2, E_3 are defined. Moreover, recall that $c_p(E_1)$ is defined as the unique element of $A^p(X)$ which restricts to $c_p(\mathcal{E}_1^\bullet)$ in $A^p(Y)$. Similarly for E_2 and E_3. Hence it suffices to prove $c(\mathcal{E}_2^\bullet) = c(\mathcal{E}_1^\bullet)c(C(\alpha)^\bullet)$ in $\prod_{p \geq 0} A^p(Y)$. In turn, it suffices to prove this after restricting to a connected component of Y. Hence we may assume the complexes $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$ are bounded complexes of finite locally free $\mathcal{O}_Y$-modules of fixed rank. In this case the desired equality follows from the multiplicativity of Lemma 40.3. In the case of ch or P_p we use Lemmas 45.2.

In the previous paragraph we have seen that the lemma holds if condition (1) is satisfied. Since (2) implies (1) this deals with the second case. Assume (3). Arguing exactly as in the proof of Lemma 46.4 we find an envelope $f : Y \to X$ such that Y is a disjoint union $Y = \coprod Y_i$ of quasi-compact (and quasi-separated) schemes each having the resolution property. Then we may represent the restriction of $Lf^*E_1 \to Lf^*E_2$ to Y_i by a map of bounded complexes of finite locally free modules, see Derived Categories of Schemes, Proposition 37.5. In this way we see that condition (3) implies condition (1). Of course condition (4) implies condition (3) and the proof is complete. $\square$

Remark 46.8. The Chern classes of a perfect complex, when defined, satisfy a kind of splitting principle. Namely, suppose that $(S, \delta), X, E$ are as in Definition 46.3 such that the Chern classes of E are defined. Say we want to prove a relation between the bivariant classes $c_p(E)$, $P_p(E)$, and $ch_p(E)$. To do this, we may choose an envelope $f : Y \to X$ and a locally bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules representing E. By the uniqueness in Lemma 46.2 it suffices to prove the desired relation between the bivariant classes $c_p(\mathcal{E}^\bullet)$, $P_p(\mathcal{E}^\bullet)$, and $ch_p(\mathcal{E}^\bullet)$. Thus we may replace X by a connected component of Y and assume that E is represented by a bounded complex $\mathcal{E}^\bullet$ of finite locally free modules of fixed rank. Using the splitting principle (Lemma 43.1) we may assume each $\mathcal{E}^i$ has a filtration whose successive quotients $\mathcal{L}_{i,j}$ are invertible modules. Settting $x_{i,j} = c_1(\mathcal{L}_{i,j})$ we see that

$$c(E) = \prod\nolimits_{i \text{ even}} (1 + x_{i,j}) \prod\nolimits_{i \text{ odd}} (1 + x_{i,j})^{-1}$$

and

$$P_p(E) = \sum\nolimits_{i \text{ even}} (x_{i,j})^p - \sum\nolimits_{i \text{ odd}} (x_{i,j})^p$$

Formally taking the logarithm for the expression for $c(E)$ above we find that

$$\log(c(E)) = \sum (-1)^{p-1} \frac{P_p(E)}{p}$$

Looking at the construction of the polynomials P_p in Example 43.6 it follows that $P_p(E)$ is the exact same expression in the Chern classes of E as in the case of vector bundles, in other words, we have

$$\begin{aligned}
P_1(E) &= c_1(E), \\
P_2(E) &= c_1(E)^2 - 2c_2(E), \\
P_3(E) &= c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E), \\
P_4(E) &= c_1(E)^4 - 4c_1(E)^2c_2(E) + 4c_1(E)c_3(E) + 2c_2(E)^2 - 4c_4(E),
\end{aligned}$$

and so on. On the other hand, the bivariant class $P_0(E) = r(E) = ch_0(E)$ cannot be recovered from the Chern class $c(E)$ of E; the chern class doesn't know about the rank of the complex.

Lemma 46.9. *In Situation 7.1 let X be locally of finite type over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object whose Chern classes are defined. Then $c_i(E^\vee) = (-1)^i c_i(E)$, $P_i(E^\vee) = (-1)^i P_i(E)$, and $ch_i(E^\vee) = (-1)^i ch_i(E)$ in $A^i(X)$.*

Proof. First proof: argue as in the proof of Lemma 46.6 to reduce to the case where E is represented by a bounded complex of finite locally free modules of fixed rank and apply Lemma 43.3. Second proof: use the splitting principle discussed in Remark 46.8 and use that the chern roots of $E^\vee$ are the negatives of the chern roots of E. $\square$

Lemma 46.10. *In Situation 7.1 let X be locally of finite type over S. Let E be a perfect object of $D(\mathcal{O}_X)$ whose Chern classes are defined. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then*

$$c_i(E \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(E) c_1(\mathcal{L})^j$$

provided E has constant rank $r \in \mathbf{Z}$.

Proof. In the case where E is locally free of rank r this is Lemma 39.1. The reader can deduce the lemma from this special case by a formal computation. An alternative is to use the splitting principle of Remark 46.8. In this case one ends up having to prove the following algebra fact: if we write formally

$$\frac{\prod_{a=1,\ldots,n}(1+x_a)}{\prod_{n=1,\ldots,m}(1+y_b)} = 1 + c_1 + c_2 + c_3 + \ldots$$

with c_i homogeneous of degree i in $\mathbf{Z}[x_i, y_j]$ then we have

$$\frac{\prod_{a=1,\ldots,n}(1+x_a+t)}{\prod_{b=1,\ldots,m}(1+y_b+t)} = \sum\nolimits_{i \geq 0} \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j} t^j$$

where $r = n - m$. We omit the details. $\square$

Lemma 46.11. *In Situation 7.1 let X be locally of finite type over S. Let E and F be perfect objects of $D(\mathcal{O}_X)$ whose Chern classes are defined. Then we have*

$$c_1(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F) = r(E)c_1(\mathcal{F}) + r(F)c_1(\mathcal{E})$$

and for $c_2(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F)$ we have the expression

$$r(E)c_2(F) + r(F)c_2(E) + \binom{r(E)}{2} c_1(F)^2 + (r(E)r(F) - 1)c_1(F)c_1(E) + \binom{r(F)}{2} c_1(E)^2$$

and so on for higher Chern classes in $A^*(X)$. Similarly, we have $ch(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F) = ch(E)ch(F)$ in $A^*(X) \otimes \mathbf{Q}$. More precisely, we have

$$P_p(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F) = \sum\nolimits_{p_1+p_2=p} \binom{p}{p_1} P_{p_1}(E) P_{p_2}(F)$$

in $A^p(X)$.

Proof. After choosing an envelope $f : Y \to X$ such that Lf^*E and Lf^*F can be represented by locally bounded complexes of finite locally free $\mathcal{O}_X$-modules this follows by a compuation from the corresponding result for vector bundles in Lemmas 43.4 and 45.3. A better proof is probably to use the splitting principle as in Remark 46.8 and reduce the lemma to computations in polynomial rings which we describe in the next paragraph.

Let A be a commutative ring (for us this will be the subring of the bivariant chow ring of X generated by Chern classes). Let S be a finite set together with maps $\epsilon : S \to \{\pm 1\}$ and $f : S \to A$. Define

$$P_p(S, f, \epsilon) = \sum\nolimits_{s \in S} \epsilon(s) f(s)^p$$

in A. Given a second triple (S', ϵ', f') the equality that has to be shown for P_p is the equality

$$P_p(S \times S', f + f', \epsilon\epsilon') = \sum\nolimits_{p_1 + p_2 = p} \binom{p}{p_1} P_{p_1}(S, f, \epsilon) P_{p_2}(S', f', \epsilon')$$

To see this is true, one reduces to the polynomial ring on variables $S \amalg S'$ and one shows that each term $f(s)^i f'(s')^j$ occurs on the left and right hand side with the same coefficient. To verify the formulas for $c_1(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F)$ and $c_2(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F)$ we use the splitting principle to reduce to checking these formulae in a torsion free ring. Then we use the relationship between $P_j(E)$ and $c_i(E)$ proved in Remark 46.8. For example

$$c_1(E \otimes F) = P_1(E \otimes F) = r(F)P_1(E) + r(E)P_1(F) = r(F)c_1(E) + r(E)c_1(F)$$

the middle equation because $r(E) = P_0(E)$ by definition. Similarly, we have

$$\begin{aligned} & 2c_2(E \otimes F) \\ & = c_1(E \otimes F)^2 - P_2(E \otimes F) \\ & = (r(F)c_1(E) + r(E)c_1(F))^2 - r(F)P_2(E) - P_1(E)P_1(F) - r(E)P_2(F) \\ & = (r(F)c_1(E) + r(E)c_1(F))^2 - r(F)(c_1(E)^2 - 2c_2(E)) - c_1(E)c_1(F) - \\ & \quad r(E)(c_1(F)^2 - 2c_2(F)) \end{aligned}$$

which the reader can verify agrees with the formula in the statement of the lemma up to a factor of 2. ☐

47. A baby case of localized Chern classes

In this section we discuss some properties of the bivariant classes constructed in the following lemma; most of these properties follow immediately from the characterization given in the lemma. We urge the reader to skip the rest of the section.

Lemma 47.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $i_j : X_j \to X$, $j = 1, 2$ be closed immersions such that $X = X_1 \cup X_2$ set theoretically. Let $E_2 \in D(\mathcal{O}_{X_2})$ be a perfect object. Assume*

1. *Chern classes of E_2 are defined,*
2. *the restriction $E_2|_{X_1 \cap X_2}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X_1 \cap X_2}$-module of rank $< p$ sitting in cohomological degree 0.*

Then there is a canonical bivariant class

$$P'_p(E_2),\ \textit{resp. } c'_p(E_2) \in A^p(X_2 \to X)$$

characterized by the property

$$P'_p(E_2) \cap i_{2,*}\alpha_2 = P_p(E_2) \cap \alpha_2 \quad \textit{and} \quad P'_p(E_2) \cap i_{1,*}\alpha_1 = 0,$$

respectively

$$c'_p(E_2) \cap i_{2,*}\alpha_2 = c_p(E_2) \cap \alpha_2 \quad \textit{and} \quad c'_p(E_2) \cap i_{1,*}\alpha_1 = 0$$

for $\alpha_i \in \mathrm{CH}_k(X_i)$ and similarly after any base change $X' \to X$ locally of finite type.

Proof. We are going to use the material of Section 46 without further mention.

Assume $E_2|_{X_1 \cap X_2}$ is zero. Consider a morphism of schemes $X' \to X$ which is locally of finite type and denote $i'_j : X'_j \to X'$ the base change of i_j. By Lemma 19.4 we can write any element $\alpha' \in \mathrm{CH}_k(X')$ as $i'_{1,*}\alpha'_1 + i'_{2,*}\alpha'_2$ where $\alpha'_2 \in \mathrm{CH}_k(X'_2)$ is well defined up to an element in the image of pushforward by $X'_1 \cap X'_2 \to X'_2$. Then we can set $P'_p(E_2) \cap \alpha' = P_p(E_2) \cap \alpha'_2 \in \mathrm{CH}_{k-p}(X'_2)$. This is well defined by our assumption that E_2 restricts to zero on $X_1 \cap X_2$.

If $E_2|_{X_1 \cap X_2}$ is isomorphic to a finite locally free $\mathcal{O}_{X_1 \cap X_2}$-module of rank $< p$ sitting in cohomological degree 0, then $c_p(E_2|_{X_1 \cap X_2}) = 0$ by rank considerations and we can argue in exactly the same manner. □

Lemma 47.2. *In Lemma 47.1 the bivariant class $P'_p(E_2)$, resp. $c'_p(E_2)$ in $A^p(X_2 \to X)$ does not depend on the choice of X_1.*

Proof. Suppose that $X'_1 \subset X$ is another closed subscheme such that $X = X'_1 \cup X_2$ set theoretically and the restriction $E_2|_{X'_1 \cap X_2}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X'_1 \cap X_2}$-module of rank $< p$ sitting in cohomological degree 0. Then $X = (X_1 \cap X'_1) \cup X_2$. Hence we can write any element $\alpha \in \mathrm{CH}_k(X)$ as $i_*\beta + i_{2,*}\alpha_2$ with $\alpha_2 \in \mathrm{CH}_k(X'_2)$ and $\beta \in \mathrm{CH}_k(X_1 \cap X'_1)$. Thus it is clear that $P'_p(E_2) \cap \alpha = P_p(E_2) \cap \alpha_2 \in \mathrm{CH}_{k-p}(X_2)$, resp. $c'_p(E_2) \cap \alpha = c_p(E_2) \cap \alpha_2 \in \mathrm{CH}_{k-p}(X_2)$, is independent of whether we use X_1 or X'_1. Similarly after any base change. □

Lemma 47.3. *In Lemma 47.1 let $X' \to X$ be a morphism which is locally of finite type. Denote $X' = X'_1 \cup X'_2$ and $E'_2 \in D(\mathcal{O}_{X'_2})$ the pullbacks to X'. Then the class $P'_p(E'_2)$, resp. $c'_p(E'_2)$ in $A^p(X'_2 \to X')$ constructed in Lemma 47.1 using $X' = X'_1 \cup X'_2$ and E'_2 is the restriction (Remark 33.5) of the class $P'_p(E_2)$, resp. $c'_p(E_2)$ in $A^p(X_2 \to X)$.*

Proof. Immediate from the characterization of these classes in Lemma 47.1. □

Lemma 47.4. *In Lemma 47.1 say E_2 is the restriction of a perfect $E \in D(\mathcal{O}_X)$ such that $E|_{X_1}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X_1}$-module of rank $< p$ sitting in cohomological degree 0. If Chern classes of E are defined, then $i_{2,*} \circ P'_p(E_2) = P_p(E)$, resp. $i_{2,*} \circ c'_p(E_2) = c_p(E)$ (with $\circ$ as in Lemma 33.4).*

Proof. First, assume $E|_{X_1}$ is zero. With notations as in the proof of Lemma 47.1 the lemma in this case follows from

$$\begin{aligned} P_p(E) \cap \alpha' &= i'_{1,*}(P_p(E) \cap \alpha'_1) + i'_{2,*}(P_p(E) \cap \alpha'_2) \\ &= i'_{1,*}(P_p(E|_{X_1}) \cap \alpha'_1) + i'_{2,*}(P'_p(E_2) \cap \alpha') \\ &= i'_{2,*}(P'_p(E_2) \cap \alpha') \end{aligned}$$

The case where $E|_{X_1}$ is isomorphic to a finite locally free $\mathcal{O}_{X_1}$-module of rank $< p$ sitting in cohomological degree 0 is similar. □

Lemma 47.5. *In Lemma 47.1 suppose we have closed subschemes $X'_2 \subset X_2$ and $X_1 \subset X'_1 \subset X$ such that $X = X'_1 \cup X'_2$ set theoretically. Assume $E_2|_{X'_1 \cap X_2}$ is zero, resp. isomorphic to a finite locally free module of rank $< p$ placed in degree 0. Then we have $(X'_2 \to X_2)_* \circ P'_p(E_2|_{X'_2}) = P'_p(E_2)$, resp. $(X'_2 \to X_2)_* \circ c'_p(E_2|_{X'_2}) = c_p(E_2)$ (with $\circ$ as in Lemma 33.4).*

Proof. This follows immediately from the characterization of these classes in Lemma 47.1. □

Lemma 47.6. *In Lemma 47.1 let $f : Y \to X$ be locally of finite type and say $c \in A^*(Y \to X)$. Then*

$$c \circ P'_p(E_2) = P'_p(Lf_2^*E_2) \circ c \quad \textit{resp.} \quad c \circ c'_p(E_2) = c'_p(Lf_2^*E_2) \circ c$$

in $A^*(Y_2 \to Y)$ where $f_2 : Y_2 \to X_2$ is the base change of f.

Proof. Let $\alpha \in \mathrm{CH}_k(X)$. We may write

$$\alpha = \alpha_1 + \alpha_2$$

with $\alpha_i \in \mathrm{CH}_k(X_i)$; we are omitting the pushforwards by the closed immersions $X_i \to X$. The reader then checks that $c'_p(E_2) \cap \alpha = c_p(E_2) \cap \alpha_2$, $c \cap c'_p(E_2) \cap \alpha = c \cap c_p(E_2) \cap \alpha_2$, $c \cap \alpha = c \cap \alpha_1 + c \cap \alpha_2$, and $c'_p(Lf_2^*E_2) \cap c \cap \alpha = c_p(Lf_2^*E_2) \cap c \cap \alpha_2$. We conclude by Lemma 46.6. □

Lemma 47.7. *In Lemma 47.1 assume $E_2|_{X_1 \cap X_2}$ is zero. Then*

$$\begin{aligned} &P'_1(E_2) = c'_1(E_2), \\ &P'_2(E_2) = c'_1(E_2)^2 - 2c'_2(E_2), \\ &P'_3(E_2) = c'_1(E_2)^3 - 3c'_1(E_2)c'_2(E_2) + 3c'_3(E_2), \\ &P'_4(E_2) = c'_1(E_2)^4 - 4c'_1(E_2)^2c'_2(E_2) + 4c'_1(E_2)c'_3(E_2) + 2c'_2(E_2)^2 - 4c'_4(E_2), \end{aligned}$$

and so on with multiplication as in Remark 34.7.

Proof. The statement makes sense because the zero sheaf has rank < 1 and hence the classes $c'_p(E_2)$ are defined for all $p \geq 1$. The equalities follow immediately from the characterization of the classes produced by Lemma 47.1 and the corresponding result for capping with the Chern classes of E_2 given in Remark 46.8. □

Lemma 47.8. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $i_j : X_j \to X$, $j = 1, 2$ be closed immersions such that $X = X_1 \cup X_2$ set theoretically. Let $E, F \in D(\mathcal{O}_X)$ be perfect objects. Assume*

1. *Chern classes of E and F are defined,*
2. *the restrictions $E|_{X_1 \cap X_2}$ and $F|_{X_1 \cap X_2}$ are isomorphic to a finite locally free $\mathcal{O}_{X_1}$-modules of rank $< p$ and $< q$ sitting in cohomological degree 0.*

With notation as in Remark 34.7 set

$$c^{(p)}(E) = 1 + c_1(E) + \ldots + c_{p-1}(E) + c'_p(E|_{X_2}) + c'_{p+1}(E|_{X_2}) + \ldots \in A^{(p)}(X_2 \to X)$$

with $c'_p(E|_{X_2})$ *as in Lemma 47.1. Similarly for* $c^{(q)}(F)$ *and* $c^{(p+q)}(E \oplus F)$. *Then* $c^{(p+q)}(E \oplus F) = c^{(p)}(E)c^{(q)}(F)$ *in* $A^{(p+q)}(X_2 \to X)$.

Proof. Immediate from the characterization of the classes in Lemma 47.1 and the additivity in Lemma 46.7. $\square$

Lemma 47.9. *Let* (S, δ) *be as in Situation 7.1. Let* X *be locally of finite type over* S. *Let* $i_j : X_j \to X$, $j = 1, 2$ *be closed immersions such that* $X = X_1 \cup X_2$ *set theoretically. Let* $E, F \in D(\mathcal{O}_{X_2})$ *be perfect objects. Assume*

(1) *Chern classes of* E *and* F *are defined,*
(2) *the restrictions* $E|_{X_1 \cap X_2}$ *and* $F|_{X_1 \cap X_2}$ *are zero,*

Denote $P'_p(E), P'_p(F), P'_p(E \oplus F) \in A^p(X_2 \to X)$ *for* $p \geq 0$ *the classes constructed in Lemma 47.1. Then* $P'_p(E \oplus F) = P'_p(E) + P'_p(F)$.

Proof. Immediate from the characterization of the classes in Lemma 47.1 and the additivity in Lemma 46.7. $\square$

Lemma 47.10. *In Lemma 47.1 assume* E_2 *has constant rank* 0. *Let* $\mathcal{L}$ *be an invertible* $\mathcal{O}_X$*-module. Then*

$$c'_i(E_2 \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{-i+j}{j} c'_{i-j}(E_2) c_1(\mathcal{L})^j$$

Proof. The assumption on rank implies that $E_2|_{X_1 \cap X_2}$ is zero. Hence $c'_i(E_2)$ is defined for all $i \geq 1$ and the statement makes sense. The actual equality follows immediately from Lemma 46.10 and the characterization of c'_i in Lemma 47.1. $\square$

Lemma 47.11. *In Situation 7.1 let* X *be locally of finite type over* S. *Let*

$$X = X_1 \cup X_2 = X'_1 \cup X'_2$$

be two ways of writing X *as a set theoretic union of closed subschemes. Let* E, E' *be perfect objects of* $D(\mathcal{O}_X)$ *whose Chern classes are defined. Assume that* $E|_{X_1}$ *and* $E'|_{X'_1}$ *are zero*[5] *for* $i = 1, 2$. *Denote*

(1) $r = P'_0(E) \in A^0(X_2 \to X)$ *and* $r' = P'_0(E') \in A^0(X'_2 \to X)$,
(2) $\gamma_p = c'_p(E|_{X_2}) \in A^p(X_2 \to X)$ *and* $\gamma'_p = c'_p(E'|_{X'_2}) \in A^p(X'_2 \to X)$,
(3) $\chi_p = P'_p(E|_{X_2}) \in A^p(X_2 \to X)$ *and* $\chi'_p = P'_p(E'|_{X'_2}) \in A^p(X'_2 \to X)$

the classes constructed in Lemma 47.1. Then we have

$$c'_1((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E')|_{X_2 \cap X'_2}) = r\gamma'_1 + r'\gamma_1$$

in $A^1(X_2 \cap X'_2 \to X)$ *and*

$$c'_2((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E')|_{X_2 \cap X'_2}) = r\gamma'_2 + r'\gamma_2 + \binom{r}{2}(\gamma'_1)^2 + (rr' - 1)\gamma'_1\gamma_1 + \binom{r'}{2}\gamma_1^2$$

in $A^2(X_2 \cap X'_2 \to X)$ *and so on for higher Chern classes. Similarly, we have*

$$P'_p((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E')|_{X_2 \cap X'_2}) = \sum\nolimits_{p_1 + p_2 = p} \binom{p}{p_1} \chi_{p_1} \chi'_{p_2}$$

[5]Presumably there is a variant of this lemma where we only assume these restrictions are isomorphic to a finite locally free modules of rank $< p$ and $< p'$.

in $A^p(X_2 \cap X_2' \to X)$.

Proof. First we observe that the statement makes sense. Namely, we have $X = (X_2 \cap X_2') \cup Y$ where $Y = (X_1 \cap X_1') \cup (X_1 \cap X_2') \cup (X_2 \cap X_1')$ and the object $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} E'$ restricts to zero on Y. The actual equalities follow from the characterization of our classes in Lemma 47.1 and the equalities of Lemma 46.11. We omit the details. $\square$

48. Gysin at infinity

This section is about the bivariant class constructed in the next lemma. We urge the reader to skip the rest of the section.

Lemma 48.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $b : W \to \mathbf{P}^1_X$ be a proper morphism of schemes which is an isomorphism over $\mathbf{A}^1_X$. Denote $i_\infty : W_\infty \to W$ the inverse image of the divisor $D_\infty \subset \mathbf{P}^1_X$ with complement $\mathbf{A}^1_X$. Then there is a canonical bivariant class*

$$C \in A^0(W_\infty \to X)$$

with the property that $i_{\infty,*}(C \cap \alpha) = i_{0,*}\alpha$ for $\alpha \in \mathrm{CH}_k(X)$ and similarly after any base change by $X' \to X$ locally of finite type.

Proof. Given $\alpha \in \mathrm{CH}_k(X)$ there exists a $\beta \in \mathrm{CH}_{k+1}(W)$ restricting to the flat pullback of α on $b^{-1}(\mathbf{A}^1_X)$, see Lemma 14.2. A second choice of β differs from β by a cycle supported on W_∞, see Lemma 19.3. Since the normal bundle of the effective Cartier divisor $D_\infty \subset \mathbf{P}^1_X$ of (18.1.1) is trivial, the gysin homomorphism i_∞^* kills cycle classes supported on W_∞, see Remark 29.6. Hence setting $C \cap \alpha = i_\infty^*\beta$ is well defined.

Since W_∞ and $W_0 = X \times \{0\}$ are the pullbacks of the rationally equivalent effective Cartier divisors D_0, D_∞ in $\mathbf{P}^1_X$, we see that $i_\infty^*\beta$ and $i_0^*\beta$ map to the same cycle class on W; namely, both represent the class $c_1(\mathcal{O}_{\mathbf{P}^1_X}(1)) \cap \beta$ by Lemma 29.4. By our choice of β we have $i_0^*\beta = \alpha$ as cycles on $W_0 = X \times \{0\}$, see for example Lemma 31.1. Thus we see that $i_{\infty,*}(C \cap \alpha) = i_{0,*}\alpha$ as stated in the lemma.

Observe that the assumptions on b are preserved by any base change by $X' \to X$ locally of finite type. Hence we get an operation $C \cap - : \mathrm{CH}_k(X') \to \mathrm{CH}_k(W'_\infty)$ by the same construction as above. To see that this family of operations defines a bivariant class, we consider the diagram

$$\begin{tikzcd}
 & & \mathrm{CH}_*(X) \arrow[d, "\text{flat pullback}"] & \\
\mathrm{CH}_{*+1}(W_\infty) \arrow[r] \arrow[dr, "0"'] & \mathrm{CH}_{*+1}(W) \arrow[r, "\text{flat pullback}"] \arrow[d, "i_\infty^*"] & \mathrm{CH}_{*+1}(\mathbf{A}^1_X) \arrow[r] \arrow[dl, dotted, "C\cap -"] & 0 \\
 & \mathrm{CH}_*(W_\infty) & &
\end{tikzcd}$$

for X as indicated and the base change of this diagram for any $X' \to X$. We know that flat pullback and i_∞^* are bivariant operations, see Lemmas 33.2 and 33.3. Then a formal argument (involving huge diagrams of schemes and their chow groups) shows that the dotted arrow is a bivariant operation. $\square$

Lemma 48.2. *In Lemma 48.1 let $X' \to X$ be a morphism which is locally of finite type. Denote $b' : W' \to \mathbf{P}^1_{X'}$ and $i'_\infty : W'_\infty \to W'$ the base changes of b and i_∞. Then the class $C' \in A^0(W'_\infty \to X')$ constructed as in Lemma 48.1 using b' is the restriction (Remark 33.5) of C.*

Proof. Immediate from the construction and the fact that a similar statement holds for flat pullback and i_∞^*. □

Lemma 48.3. *In Lemma 48.1 let $g : W' \to W$ be a proper morphism which is an isomorphism over $\mathbf{A}^1_X$. Let $C' \in A^0(W'_\infty \to X)$ and $C \in A^0(W_\infty \to X)$ be the classes constructed in Lemma 48.1. Then $g_{\infty,*} \circ C' = C$ in $A^0(W_\infty \to X)$.*

Proof. Set $b' = b \circ g : W' \to \mathbf{P}^1_X$. Denote $i'_\infty : W'_\infty \to W'$ the inclusion morphism. Denote $g_\infty : W'_\infty \to W_\infty$ the restriction of g. Given $\alpha \in \mathrm{CH}_k(X)$ choose $\beta' \in \mathrm{CH}_{k+1}(W')$ restricting to the flat pullback of α on $(b')^{-1}\mathbf{A}^1_X$. Then $\beta = g_*\beta' \in \mathrm{CH}_{k+1}(W)$ restricts to the flat pullback of α on $b^{-1}\mathbf{A}^1_X$. Then $i_\infty^*\beta = g_{\infty,*}(i'_\infty)^*\beta'$ by Lemma 29.8. This and the corresponding fact after base change by morphisms $X' \to X$ locally of finite type, corresponds to the assertion made in the lemma. □

Lemma 48.4. *In Lemma 48.1 we have $C \circ (W_\infty \to X)_* \circ i_\infty^* = i_\infty^*$.*

Proof. Let $\beta \in \mathrm{CH}_{k+1}(W)$. Denote $i_0 : X = X \times \{0\} \to W$ the closed immersion of the fibre over 0 in $\mathbf{P}^1$. Then $(W_\infty \to X)_* i_\infty^*\beta = i_0^*\beta$ in $\mathrm{CH}_k(X)$ because $i_{\infty,*}i_\infty^*\beta$ and $i_{0,*}i_0^*\beta$ represent the same class on W (for example by Lemma 29.4) and hence pushforward to the same class on X. The restriction of β to $b^{-1}(\mathbf{A}^1_X)$ restricts to the flat pullback of $i_0^*\beta = (W_\infty \to X)_* i_\infty^*\beta$ because we can check this after pullback by i_0, see Lemmas 32.2 and 32.4. Hence we may use β when computing the image of $(W_\infty \to X)_* i_\infty^*\beta$ under C and we get the desired result. □

Lemma 48.5. *In Lemma 48.1 let $f : Y \to X$ be a morphism locally of finite type and $c \in A^*(Y \to X)$. Then $C \circ c = c \circ C$ in $A^*(W_\infty \times_X Y \to X)$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
W_\infty \times_X Y \arrow[r, equal] & W_{Y,\infty} \arrow[r, "i_{Y,\infty}"'] \arrow[d] & W_Y \arrow[r, "b_Y"'] \arrow[d] & \mathbf{P}^1_Y \arrow[r, "p_Y"'] \arrow[d] & Y \arrow[d, "f"] \\
 & W_\infty \arrow[r, "i_\infty"] & W \arrow[r, "b"] & \mathbf{P}^1_X \arrow[r, "p"] & X
\end{tikzcd}$$

with cartesian squares. For an elemnent $\alpha \in \mathrm{CH}_k(X)$ choose $\beta \in \mathrm{CH}_{k+1}(W)$ whose restriction to $b^{-1}(\mathbf{A}^1_X)$ is the flat pullback of α. Then $c \cap \beta$ is a class in $\mathrm{CH}_*(W_Y)$ whose restriction to $b_Y^{-1}(\mathbf{A}^1_Y)$ is the flat pullback of $c \cap \alpha$. Next, we have

$$i_{Y,\infty}^*(c \cap \beta) = c \cap i_\infty^*\beta$$

because c is a bivariant class. This exactly says that $C \cap c \cap \alpha = c \cap C \cap \alpha$. The same argument works after any base change by $X' \to X$ locally of finite type. This proves the lemma. □

49. Preparation for localized Chern classes

In this section we discuss some properties of the bivariant classes constructed in the following lemma. We urge the reader to skip the rest of the section.

Lemma 49.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme. Let*

$$b : W \longrightarrow \mathbf{P}^1_X$$

be a proper morphism of schemes. Let $Q \in D(\mathcal{O}_W)$ be a perfect object. Denote $W_\infty \subset W$ the inverse image of the divisor $D_\infty \subset \mathbf{P}^1_X$ with complement $\mathbf{A}^1_X$. We assume

(A0) *Chern classes of Q are defined (Section 46),*
(A1) *b is an isomorphism over $\mathbf{A}^1_X$,*
(A2) *there exists a closed subscheme $T \subset W_\infty$ containing all points of W_∞ lying over $X \setminus Z$ such that $Q|_T$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p$ sitting in cohomological degree 0.*

Then there exists a canonical bivariant class

$$P'_p(Q), \text{ resp. } c'_p(Q) \in A^p(Z \to X)$$

with $(Z \to X)_* \circ P'_p(Q) = P_p(Q|_{X \times \{0\}})$, resp. $(Z \to X)_* \circ c'_p(Q) = c_p(Q|_{X \times \{0\}})$.

Proof. Denote $E \subset W_\infty$ the inverse image of Z. Then $W_\infty = T \cup E$ and b induces a proper morphism $E \to Z$. Denote $C \in A^0(W_\infty \to X)$ the bivariant class constructed in Lemma 48.1. Denote $P'_p(Q|_E)$, resp. $c'_p(Q|_E)$ in $A^p(E \to W_\infty)$ the bivariant class constructed in Lemma 47.1. This makes sense because $(Q|_E)|_{T \cap E}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{E \cap T}$-module of rank $< p$ sitting in cohomological degree 0 by assumption (A2). Then we define

$$P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C, \text{ resp. } c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

This is a bivariant class, see Lemma 33.4. Since $E \to Z \to X$ is equal to $E \to W_\infty \to W \to X$ we see that

$$\begin{aligned}
(Z \to X)_* \circ c'_p(Q) &= (W \to X)_* \circ i_{\infty,*} \circ (E \to W_\infty)_* \circ c'_p(Q|_E) \circ C \\
&= (W \to X)_* \circ i_{\infty,*} \circ c_p(Q|_{W_\infty}) \circ C \\
&= (W \to X)_* \circ c_p(Q) \circ i_{\infty,*} \circ C \\
&= (W \to X)_* \circ c_p(Q) \circ i_{0,*} \\
&= (W \to X)_* \circ i_{0,*} \circ c_p(Q|_{X \times \{0\}}) \\
&= c_p(Q|_{X \times \{0\}})
\end{aligned}$$

The second equality holds by Lemma 47.4. The third equality because $c_p(Q)$ is a bivariant class. The fourth equality by Lemma 48.1. The fifth equality because $c_p(Q)$ is a bivariant class. The final equality because $(W_0 \to W) \circ (W \to X)$ is the identity on X if we identify W_0 with X as we've done above. The exact same sequence of equations works to prove the property for $P'_p(Q)$. □

Lemma 49.2. *In Lemma 49.1 let $X' \to X$ be a morphism which is locally of finite type. Denote Z', $b' : W' \to \mathbf{P}^1_{X'}$, and $T' \subset W'_\infty$ the base changes of Z, $b : W \to \mathbf{P}^1_X$, and $T \subset W_\infty$. Set $Q' = (W' \to W)^*Q$. Then the class $P'_p(Q')$, resp. $c'_p(Q')$ in $A^p(Z' \to X')$ constructed as in Lemma 49.1 using b', Q', and T' is the restriction (Remark 33.5) of the class $P'_p(Q)$, resp. $c'_p(Q)$ in $A^p(Z \to X)$.*

Proof. Recall that the construction is as follows

$$P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C, \text{ resp. } c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

Thus the lemma follows from the corresponding base change property for C (Lemma 48.2) and the fact that the same base change property holds for the classes constructed in Lemma 47.1 (small detail omitted). $\square$

Lemma 49.3. *In Lemma 49.1 the bivariant class $P'_p(Q)$, resp. $c'_p(Q)$ is independent of the choice of the closed subscheme T. Moreover, given a proper morphism $g : W' \to W$ which is an isomorphism over $\mathbf{A}^1_X$, then setting $Q' = g^*Q$ we have $P'_p(Q) = P'_p(Q')$, resp. $c'_p(Q) = c'_p(Q')$.*

Proof. The independence of T follows immediately from Lemma 47.2.

Let $g : W' \to W$ be a proper morphism which is an isomorphism over $\mathbf{A}^1_X$. Observe that taking $T' = g^{-1}(T) \subset W'_\infty$ is a closed subscheme satisfying (A2) hence the operator $P'_p(Q')$, resp. $c'_p(Q')$ in $A^p(Z \to X)$ corresponding to $b' = b \circ g : W' \to \mathbf{P}^1_X$ and Q' is defined. Denote $E' \subset W'_\infty$ the inverse image of Z in W'_∞. Recall that

$$c'_p(Q') = (E' \to Z)_* \circ c'_p(Q'|_{E'}) \circ C'$$

with $C' \in A^0(W'_\infty \to X)$ and $c'_p(Q'|_{E'}) \in A^p(E' \to W'_\infty)$. By Lemma 48.3 we have $g_{\infty,*} \circ C' = C$. Observe that E' is also the inverse image of E in W'_∞ by g_∞. Since moreover $Q' = g^*Q$ we find that $c'_p(Q'|_{E'})$ is simply the restriction of $c'_p(Q|_E)$ to schemes lying over W'_∞, see Remark 33.5. Thus we obtain

$$\begin{aligned} c'_p(Q') &= (E' \to Z)_* \circ c'_p(Q'|_{E'}) \circ C' \\ &= (E \to Z)_* \circ (E' \to E)_* \circ c'_p(Q|_E) \circ C' \\ &= (E \to Z)_* \circ c'_p(Q|_E) \circ g_{\infty,*} \circ C' \\ &= (E \to Z)_* \circ c'_p(Q|_E) \circ C \\ &= c'_p(Q) \end{aligned}$$

In the third equality we used that $c'_p(Q|_E)$ commutes with proper pushforward as it is a bivariant class. The equality $P'_p(Q) = P'_p(Q')$ is proved in exactly the same way. $\square$

Lemma 49.4. *In Lemma 49.1 assume $Q|_T$ is isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p$. Denote $C \in A^0(W_\infty \to X)$ the class of Lemma 48.1. Then*

$$C \circ c_p(Q|_{X \times \{0\}}) = C \circ (Z \to X)_* \circ c'_p(Q) = c_p(Q|_{W_\infty}) \circ C$$

Proof. The first equality holds because $c_p(Q|_{X \times \{0\}}) = (Z \to X)_* \circ c'_p(Q)$ by Lemma 49.1. We may prove the second equality one cycle class at a time (see Lemma 35.3). Since the construction of the bivariant classes in the lemma is compatible with base change, we may assume we have some $\alpha \in \mathrm{CH}_k(X)$ and we have to show that $C \cap (Z \to X)_*(c'_p(Q) \cap \alpha) = c_p(Q|_{W_\infty}) \cap C \cap \alpha$. Observe that

$$\begin{aligned} C \cap (Z \to X)_*(c'_p(Q) \cap \alpha) &= C \cap (Z \to X)_*(E \to Z)_*(c'_p(Q|_E) \cap C \cap \alpha) \\ &= C \cap (W_\infty \to X)_*(E \to W_\infty)_*(c'_p(Q|_E) \cap C \cap \alpha) \\ &= C \cap (W_\infty \to X)_*(E \to W_\infty)_*(c'_p(Q|_E) \cap i_\infty^*\beta) \\ &= C \cap (W_\infty \to X)_*(c_p(Q|_{W_\infty}) \cap i_\infty^*\beta) \\ &= C \cap (W_\infty \to X)_* i_\infty^*(c_p(Q) \cap \beta) \\ &= i_\infty^*(c_p(Q) \cap \beta) \\ &= c_p(Q|_{W_\infty}) \cap i_\infty^*\beta \\ &= c_p(Q|_{W_\infty}) \cap C \cap \alpha \end{aligned}$$

as desired. For the first equality we used that $c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$ where $E \subset W_\infty$ is the inverse image of Z and $c'_p(Q|_E)$ is the class constructed in Lemma 47.1. The second equality is just the statement that $E \to Z \to X$ is equal to $E \to W_\infty \to X$. For the third equality we choose $\beta \in \mathrm{CH}_{k+1}(W)$ whose restriction to $b^{-1}(\mathbf{A}^1_X)$ is the flat pullback of α so that $C \cap \alpha = i_\infty^*\beta$ by construction. The fourth equality is Lemma 47.4. The fifth equality is the fact that $c_p(Q)$ is a bivariant class and hence commutes with i_∞^*. The sixth equality is Lemma 48.4. The seventh uses again that $c_p(Q)$ is a bivariant class. The final holds as $C \cap \alpha = i_\infty^*\beta$. $\square$

Lemma 49.5. *In Lemma 49.1 let $Y \to X$ be a morphism locally of finite type and let $c \in A^*(Y \to X)$ be a bivariant class. Then*

$$P'_p(Q) \circ c = c \circ P'_p(Q) \quad \textit{resp.} \quad c'_p(Q) \circ c = c \circ c'_p(Q)$$

in $A^*(Y \times_X Z \to X)$.

Proof. Let $E \subset W_\infty$ be the inverse image of Z. Recall that $P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C$, resp. $c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$ where C is as in Lemma 48.1 and $P'_p(Q|_E)$, resp. $c'_p(Q|_E)$ are as in Lemma 47.1. By Lemma 48.5 we see that C commutes with c and by Lemma 47.6 we see that $P'_p(Q|_E)$, resp. $c'_p(Q|_E)$ commutes with c. Since c is a bivariant class it commutes with proper pushforward by $E \to Z$ by definition. This finishes the proof. $\square$

Lemma 49.6. *In Lemma 49.1 assume $Q|_T$ is zero. In $A^*(Z \to X)$ we have*

$$\begin{aligned} P'_1(Q) &= c'_1(Q), \\ P'_2(Q) &= c'_1(Q)^2 - 2c'_2(Q), \\ P'_3(Q) &= c'_1(Q)^3 - 3c'_1(Q)c'_2(Q) + 3c'_3(Q), \\ P'_4(Q) &= c'_1(Q)^4 - 4c'_1(Q)^2c'_2(Q) + 4c'_1(Q)c'_3(Q) + 2c'_2(Q)^2 - 4c'_4(Q), \end{aligned}$$

and so on with multiplication as in Remark 34.7.

Proof. The statement makes sense because the zero sheaf has rank < 1 and hence the classes $c'_p(Q)$ are defined for all $p \geq 1$. In the proof of Lemma 49.1 we have constructed the classes $P'_p(Q)$ and $c'_p(Q)$ using the bivariant class $C \in A^0(W_\infty \to X)$ of Lemma 48.1 and the bivariant classes $P'_p(Q|_E)$ and $c'_p(Q|_E)$ of Lemma 47.1 for the restriction $Q|_E$ of Q to the inverse image E of Z in W_∞. Observe that by

Lemma 47.7 we have the desired relationship between $P'_p(Q|_E)$ and $c'_p(Q|_E)$. Recall that

$$P'_p(Q) = (E \to Z)_* \circ P'_p(Q|_E) \circ C \quad \text{and} \quad c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

To finish the proof it suffices to show the multiplications defined in Remark 34.7 on the classes $a_p = c'_p(Q)$ and on the classes $b_p = c'_p(Q|_E)$ agree:

$$a_{p_1} a_{p_2} \ldots a_{p_r} = (E \to Z)_* \circ b_{p_1} b_{p_2} \ldots b_{p_r} \circ C$$

Some details omitted. If $r = 1$, then this is true. For $r > 1$ note that by Remark 34.8 the multiplication in Remark 34.7 proceeds by inserting $(Z \to X)_*$, resp. $(E \to W_\infty)_*$ in between the factors of the product $a_{p_1} a_{p_2} \ldots a_{p_r}$, resp. $b_{p_1} b_{p_2} \ldots b_{p_r}$ and taking compositions as bivariant classes. Now by Lemma 47.1 we have

$$(E \to W_\infty)_* \circ b_{p_i} = c_{p_i}(Q|_{W_\infty})$$

and by Lemma 49.4 we have

$$C \circ (Z \to X)_* \circ a_{p_i} = c_{p_i}(Q|_{W_\infty}) \circ C$$

for $i = 2, \ldots, r$. A calculation shows that the left and right hand side of the desired equality both simplify to

$$(E \to Z)_* \circ c'_{p_1}(Q|_E) \circ c_{p_2}(Q|_{W_\infty}) \circ \ldots \circ c_{p_r}(Q|_{W_\infty}) \circ C$$

and the proof is complete. ☐

Lemma 49.7. *In Lemma 49.1 assume $Q|_T$ is isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p$. Assume we have another perfect object $Q' \in D(\mathcal{O}_W)$ whose Chern classes are defined with $Q'|_T$ isomorphic to a finite locally free $\mathcal{O}_T$-module of rank $< p'$ placed in cohomological degree 0. With notation as in Remark 34.7 set*

$$c^{(p)}(Q) = 1 + c_1(Q|_{X \times \{0\}}) + \ldots + c_{p-1}(Q|_{X \times \{0\}}) + c'_p(Q) + c'_{p+1}(Q) + \ldots$$

in $A^{(p)}(Z \to X)$ with $c'_i(Q)$ for $i \geq p$ as in Lemma 49.1. Similarly for $c^{(p')}(Q')$ and $c^{(p+p')}(Q \oplus Q')$. Then $c^{(p+p')}(Q \oplus Q') = c^{(p)}(Q)c^{(p')}(Q')$ in $A^{(p+p')}(Z \to X)$.

Proof. Recall that the image of $c'_i(Q)$ in $A^p(X)$ is equal to $c_i(Q|_{X \times \{0\}})$ for $i \geq p$ and similarly for Q' and $Q \oplus Q'$, see Lemma 49.1. Hence the equality in degrees $< p + p'$ follows from the additivity of Lemma 46.7.

Let's take $n \geq p + p'$. As in the proof of Lemma 49.1 let $E \subset W_\infty$ denote the inverse image of Z. Observe that we have the equality

$$c^{(p+p')}(Q|_E \oplus Q'|_E) = c^{(p)}(Q|_E)c^{(p')}(Q'|_E)$$

in $A^{(p+p')}(E \to W_\infty)$ by Lemma 47.8. Since by construction

$$c'_p(Q \oplus Q') = (E \to Z)_* \circ c'_p(Q|_E \oplus Q'|_E) \circ C$$

we conclude that suffices to show for all $i + j = n$ we have

$$(E \to Z)_* \circ c_i^{(p)}(Q|_E)c_j^{(p')}(Q'|_E) \circ C = c_i^{(p)}(Q)c_j^{(p')}(Q')$$

in $A^n(Z \to X)$ where the multiplication is the one from Remark 34.7 on both sides. There are three cases, depending on whether $i \geq p$, $j \geq p'$, or both.

Assume $i \geq p$ and $j \geq p'$. In this case the products are defined by inserting $(E \to W_\infty)_*$, resp. $(Z \to X)_*$ in between the two factors and taking compositions as bivariant classes, see Remark 34.8. In other words, we have to show

$$(E \to Z)_* \circ c'_i(Q|_E) \circ (E \to W_\infty)_* \circ c'_j(Q'|_E) \circ C = c'_i(Q) \circ (Z \to X)_* \circ c'_j(Q')$$

By Lemma 47.1 the left hand side is equal to

$$(E \to Z)_* \circ c'_i(Q|_E) \circ c_j(Q'|_{W_\infty}) \circ C$$

Since $c'_i(Q) = (E \to Z)_* \circ c'_i(Q|_E) \circ C$ the right hand side is equal to

$$(E \to Z)_* \circ c'_i(Q|_E) \circ C \circ (Z \to X)_* \circ c'_j(Q')$$

which is immediately seen to be equal to the above by Lemma 49.4.

Assume $i \geq p$ and $j < p$. Unwinding the products in this case we have to show

$$(E \to Z)_* \circ c'_i(Q|_E) \circ c_j(Q'|_{W_\infty}) \circ C = c'_i(Q) \circ c_j(Q'|_{X \times \{0\}})$$

Again using that $c'_i(Q) = (E \to Z)_* \circ c'_i(Q|_E) \circ C$ we see that it suffices to show $c_j(Q'|_{W_\infty}) \circ C = C \circ c_j(Q'|_{X \times \{0\}})$ which is part of Lemma 49.4.

Assume $i < p$ and $j \geq p'$. Unwinding the products in this case we have to show

$$(E \to Z)_* \circ c_i(Q|_E) \circ c'_j(Q'|_E) \circ C = c_i(Q|_{Z \times \{0\}}) \circ c'_j(Q')$$

However, since $c'_j(Q|_E)$ and $c'_j(Q')$ are bivariant classes, they commute with capping with Chern classes (Lemma 38.9). Hence it suffices to prove

$$(E \to Z)_* \circ c'_j(Q'|_E) \circ c_i(Q|_{W_\infty}) \circ C = c'_j(Q') \circ c_i(Q|_{X \times \{0\}})$$

which we reduces us to the case discussed in the preceding paragraph. □

Lemma 49.8. *In Lemma 49.1 assume $Q|_T$ is zero. Assume we have another perfect object $Q' \in D(\mathcal{O}_W)$ whose Chern classes are defined such that the restriction $Q'|_T$ is zero. In this case the classes $P'_p(Q), P'_p(Q'), P'_p(Q \oplus Q') \in A^p(Z \to X)$ constructed in Lemma 49.1 satisfy $P'_p(Q \oplus Q') = P'_p(Q) + P'_p(Q')$.*

Proof. This follows immediately from the construction of these classes and Lemma 47.9. □

50. Localized Chern classes

Outline of the construction. Let F be a field, let X be a variety over F, let E be a perfect object of $D(\mathcal{O}_X)$, and let $Z \subset X$ be a closed subscheme such that $E|_{X \setminus Z} = 0$. Then we want to construct elements

$$c_p(Z \to X, E) \in A^p(Z \to X)$$

We will do this by constructing a diagram

$$\begin{tikzcd} W \arrow[r, "q"'] \arrow[d, "f"'] & X \\ \mathbf{P}^1_F \end{tikzcd}$$

and a perfect object Q of $D(\mathcal{O}_W)$ such that

(1) f is flat, and f, q are proper; for $t \in \mathbf{P}^1_F$ denote W_t the fibre of f, $q_t : W_t \to X$ the restriction of q, and $Q_t = Q|_{W_t}$,
(2) $q_t : W_t \to X$ is an isomorphism and $Q_t = q_t^* E$ for $t \in \mathbf{A}^1_F$,

(3) $q_\infty : W_\infty \to X$ is an isomorphism over $X \setminus Z$,
(4) if $T \subset W_\infty$ is the closure of $q_\infty^{-1}(X \setminus Z)$ then $Q_\infty|_T$ is zero.

The idea is to think of this as a family $\{(W_t, Q_t)\}$ parametrized by $t \in \mathbf{P}^1$. For $t \neq \infty$ we see that $c_p(Q_t)$ is just $c_p(E)$ on the chow groups of $Q_t = X$. But for $t = \infty$ we see that $c_p(Q_\infty)$ sends classes on Q_∞ to classes supported on $E = q_\infty^{-1}(Z)$ since $Q_\infty|_T = 0$. We think of E as the exceptional locus of $q_\infty : W_\infty \to X$. Since any $\alpha \in \mathrm{CH}_*(X)$ gives rise to a "family" of cycles $\alpha_t \in \mathrm{CH}_*(W_t)$ it makes sense to define $c_p(Z \to X, E) \cap \alpha$ as the pushforward $(E \to Z)_*(c_p(Q_\infty) \cap \alpha_\infty)$.

To make this work there are two main ingredients: (1) the construction of W and Q is a sort of algebraic Macpherson's graph construction; it is done in More on Flatness, Section 44. (2) the construction of the actual class given W and Q is done in Section 49 relying on Sections 48 and 47.

Situation 50.1. Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $i : Z \to X$ be a closed immersion. Let $E \in D(\mathcal{O}_X)$ be an object. Let $p \geq 0$. Assume

(1) E is a perfect object of $D(\mathcal{O}_X)$,
(2) the restriction $E|_{X \setminus Z}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X \setminus Z}$-module of rank $< p$ sitting in cohomological degree 0, and
(3) at least one[6] of the following is true: (a) X is quasi-compact, (b) X has quasi-compact irreducible components, (c) there exists a locally bounded complex of finite locally free $\mathcal{O}_X$-modules representing E, or (d) there exists a morphism $X \to X'$ of schemes locally of finite type over S such that E is the pullback of a perfect object on X' and the irreducible components of X' are quasi-compact.

Lemma 50.2. *In Situation 50.1 there exists a canonical bivariant class*

$$P_p(Z \to X, E) \in A^p(Z \to X), \quad \textit{resp.} \quad c_p(Z \to X, E) \in A^p(Z \to X)$$

with the property that

$$i_* \circ P_p(Z \to X, E) = P_p(E), \quad \textit{resp.} \quad i_* \circ c_p(Z \to X, E) = c_p(E) \tag{50.2.1}$$

as bivariant classes on X (with $\circ$ as in Lemma 33.4).

Proof. The construction of these bivariant classes is as follows. Let

$$b : W \longrightarrow \mathbf{P}^1_X \quad \text{and} \quad T \longrightarrow W_\infty \quad \text{and} \quad Q$$

be the blowing up, the perfect object Q in $D(\mathcal{O}_W)$, and the closed immersion constructed in More on Flatness, Section 44 and Lemma 44.1. Let $T' \subset T$ be the open and closed subscheme such that $Q|_{T'}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{T'}$-module of rank $< p$ sitting in cohomological degree 0. By condition (2) of Situation 50.1 the morphisms

$$T' \to T \to W_\infty \to X$$

are all isomorphisms of schemes over the open subscheme $X \setminus Z$ of X. Below we check the chern classes of Q are defined. Recalling that $Q|_{X \times \{0\}} \cong E$ by construction, we conclude that the bivariant class constructed in Lemma 49.1 using W, b, Q, T' gives us classes

$$P_p(Z \to X, E) = P'_p(Q) \in A^p(Z \to X)$$

[6] Please ignore this technical condition on a first reading; see discussion in Remark 50.5.

and

$$c_p(Z \to X, E) = c'_p(Q) \in A^p(Z \to X)$$

satisfying (50.2.1).

In this paragraph we prove that the chern classes of Q are defined (Definition 46.3); we suggest the reader skip this. If assumption (3)(a) or (3)(b) of Situation 50.1 holds, i.e., if X has quasi-compact irreducible components, then the same is true for W (because $W \to X$ is proper). Hence we conclude that the chern classes of any perfect object of $D(\mathcal{O}_W)$ are defined by Lemma 46.4. If (3)(c) hold, i.e., if E can be represented by a locally bounded complex of finite locally free modules, then the object Q can be represented by a locally bounded complex of finite locally free $\mathcal{O}_W$-modules by part (5) of More on Flatness, Lemma 44.1. Hence the chern classes of Q are defined. Finally, assume (3)(d) holds, i.e., assume we have a morphism $X \to X'$ of schemes locally of finite type over S such that E is the pullback of a perfect object E' on X' and the irreducible components of X' are quasi-compact. Let $b' : W' \to \mathbf{P}^1_{X'}$ and $Q' \in D(\mathcal{O}_{W'})$ be the morphism and perfect object constructed as in More on Flatness, Section 44 starting with the triple $(\mathbf{P}^1_{X'}, (\mathbf{P}^1_{X'})_\infty, L(p')^*E')$. By the discussion above we see that the chern classes of Q' are defined. Since b and b' were constructed via an application of More on Flatness, Lemma 43.6 it follows from More on Flatness, Lemma 43.8 that there exists a morphism $W \to W'$ such that $Q = L(W \to W')^*Q'$. Then it follows from Lemma 46.4 that the chern classes of Q are defined. $\square$

Definition 50.3. With (S, δ), X, $E \in D(\mathcal{O}_X)$, and $i : Z \to X$ as in Situation 50.1.

(1) If the restriction $E|_{X \setminus Z}$ is zero, then for all $p \geq 0$ we define

$$P_p(Z \to X, E) \in A^p(Z \to X)$$

by the construction in Lemma 50.2 and we define the *localized Chern character* by the formula

$$ch(Z \to X, E) = \sum\nolimits_{p=0,1,2,\ldots} \frac{P_p(Z \to X, E)}{p!} \quad \text{in} \quad \prod\nolimits_{p \geq 0} A^p(Z \to X) \otimes \mathbf{Q}$$

(2) If the restriction $E|_{X \setminus Z}$ is isomorphic to a finite locally free $\mathcal{O}_{X \setminus Z}$-module of rank $< p$ sitting in cohomological degree 0, then we define the *localized pth Chern class* $c_p(Z \to X, E)$ by the construction in Lemma 50.2.

In the situation of the definition assume $E|_{X \setminus Z}$ is zero. Then, to be sure, we have the equality

$$i_* \circ ch(Z \to X, E) = ch(E)$$

in $A^*(X) \otimes \mathbf{Q}$ because we have shown the equality (50.2.1) above.

Here is an important sanity check.

Lemma 50.4. *In Situation 50.1 let $f : X' \to X$ be a morphism of schemes which is locally of finite type. Denote $E' = f^*E$ and $Z' = f^{-1}(Z)$. Then the bivariant class of Definition 50.3*

$$P_p(Z' \to X', E') \in A^p(Z' \to X'), \quad \textit{resp.} \quad c_p(Z' \to X', E') \in A^p(Z' \to X')$$

constructed as in Lemma 50.2 using X', Z', E' is the restriction (Remark 33.5) of the bivariant class $P_p(Z \to X, E) \in A^p(Z \to X)$, resp. $c_p(Z \to X, E) \in A^p(Z \to X)$.

Proof. Denote $p : \mathbf{P}^1_X \to X$ and $p' : \mathbf{P}^1_{X'} \to X'$ the structure morphisms. Recall that $b : W \to \mathbf{P}^1_X$ and $b' : W' \to \mathbf{P}^1_{X'}$ are the morphism constructed from the triples $(\mathbf{P}^1_X, (\mathbf{P}^1_X)\infty, p^*E)$ and $(\mathbf{P}^1_{X'}, (\mathbf{P}^1_{X'})\infty, (p')^*E')$ in More on Flatness, Lemma 43.6. Furthermore $Q = L\eta_{\mathcal{I}_\infty} p^*E$ and $Q = L\eta_{\mathcal{I}'_\infty}(p')^*E'$ where $\mathcal{I}_\infty \subset \mathcal{O}_W$ is the ideal sheaf of W_∞ and $\mathcal{I}'_\infty \subset \mathcal{O}_{W'}$ is the ideal sheaf of W'_∞. Next, $h : \mathbf{P}^1_{X'} \to \mathbf{P}^1_X$ is a morphism of schemes such that the pullback of the effective Cartier divisor $(\mathbf{P}^1_X)_\infty$ is the effective Cartier divisor $(\mathbf{P}^1_{X'})_\infty$ and such that $h^*p^*E = (p')^*E'$. By More on Flatness, Lemma 43.8 we obtain a commutative diagram

$$\begin{tikzcd}
W' \arrow[r, "g"'] \arrow[dr, "b'"'] & \mathbf{P}^1_{X'} \times_{\mathbf{P}^1_X} W \arrow[r, "q"'] \arrow[d, "r"] & W \arrow[d, "b"] \\
 & \mathbf{P}^1_{X'} \arrow[r] & \mathbf{P}^1_X
\end{tikzcd}$$

such that W' is the "strict transform" of $\mathbf{P}^1_{X'}$ with respect to b and such that $Q' = (q \circ g)^*Q$. Now recall that $P_p(Z \to X, E) = P'_p(Q)$, resp. $c_p(Z \to X, E) = c'_p(Q)$ where $P'_p(Q)$, resp. $c'_p(Q)$ are constructed in Lemma 49.1 using b, Q, T' where T' is a closed subscheme $T' \subset W_\infty$ with the following two properties: (a) T' contains all points of W_∞ lying over $X \setminus Z$, and (b) $Q|_{T'}$ is zero, resp. isomorphic to a finite locally free module of rank $< p$ placed in degree 0. In the construction of Lemma 49.1 we chose a particular closed subscheme T' with properties (a) and (b) but the precise choice of T' is immaterial, see Lemma 49.3.

Next, by Lemma 49.2 the restriction of the bivariant class $P_p(Z \to X, E) = P'_p(Q)$, resp. $c_p(Z \to X, E) = c_p(Q')$ to X' corresponds to the class $P'_p(q^*Q)$, resp. $c'_p(q^*Q)$ constructed as in Lemma 49.1 using $r : \mathbf{P}^1_{X'} \times_{\mathbf{P}^1_X} W \to \mathbf{P}^1_{X'}$, the complex q^*Q, and the inverse image $q^{-1}(T')$.

Now by the second statement of Lemma 49.3 we have $P'_p(Q') = P'_p(q^*Q)$, resp. $c'_p(q^*Q) = c'_p(Q')$. Since $P_p(Z' \to X', E') = P'_p(Q')$, resp. $c_p(Z' \to X', E') = c'_p(Q')$ we conclude that the lemma is true. □

Remark 50.5. In Situation 50.1 it would have been more natural to replace assumption (3) with the assumption: "the chern classes of E are defined". In fact, combining Lemmas 50.2 and 50.4 with Lemma 35.6 it is easy to extend the definition to this (slightly) more general case. If we ever need this we will do so here.

Lemma 50.6. *In Situation 50.1 we have*

$$P_p(Z \to X, E) \cap i_*\alpha = P_p(E|_Z) \cap \alpha, \quad \textit{resp.} \quad c_p(Z \to X, E) \cap i_*\alpha = c_p(E|_Z) \cap \alpha$$

in $\mathrm{CH}_*(Z)$ *for any* $\alpha \in \mathrm{CH}_*(Z)$.

Proof. We only prove the second equality and we omit the proof of the first. Since $c_p(Z \to X, E)$ is a bivariant class and since the base change of $Z \to X$ by $Z \to X$ is $\mathrm{id} : Z \to Z$ we have $c_p(Z \to X, E) \cap i_*\alpha = c_p(Z \to X, E) \cap \alpha$. By Lemma 50.4 the restriction of $c_p(Z \to X, E)$ to Z (!) is the localized Chern class for $\mathrm{id} : Z \to Z$ and $E|_Z$. Thus the result follows from (50.2.1) with $X = Z$. □

Lemma 50.7. *In Situation 50.1 if* $\alpha \in \mathrm{CH}_k(X)$ *has support disjoint from* Z*, then* $P_p(Z \to X, E) \cap \alpha = 0$*, resp.* $c_p(Z \to X, E) \cap \alpha = 0$.

Proof. This is immediate from the construction of the localized Chern classes. It also follows from the fact that we can compute $c_p(Z \to X, E) \cap \alpha$ by first restricting $c_p(Z \to X, E)$ to the support of α, and then using Lemma 50.4 to see that this restriction is zero. $\square$

Lemma 50.8. *In Situation 50.1 assume $Z \subset Z' \subset X$ where Z' is a closed subscheme of X. Then $P_p(Z' \to X, E) = (Z \to Z')_* \circ P_p(Z \to X, E)$, resp. $c_p(Z' \to X, E) = (Z \to Z')_* \circ c_p(Z \to X, E)$ (with $\circ$ as in Lemma 33.4).*

Proof. The construction of $P_p(Z' \to X, E)$, resp. $c_p(Z' \to X, E)$ in Lemma 50.2 uses the exact same morphism $b : W \to \mathbf{P}^1_X$ and perfect object Q of $D(\mathcal{O}_W)$. Then we can use Lemma 47.5 to conclude. Some details omitted. $\square$

Lemma 50.9. *In Lemma 47.1 say E_2 is the restriction of a perfect $E \in D(\mathcal{O}_X)$ whose restriction to X_1 is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X_1}$-module of rank $< p$ sitting in cohomological degree 0. Then the class $P'_p(E_2)$, resp. $c'_p(E_2)$ of Lemma 47.1 agrees with $P_p(X_2 \to X, E)$, resp. $c_p(X_2 \to X, E)$ of Definition 50.3 provided E satisfies assumption (3) of Situation 50.1.*

Proof. The assumptions on E imply that there is an open $U \subset X$ containing X_1 such that $E|_U$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_U$-module of rank $< p$. See More on Algebra, Lemma 75.6. Let $Z \subset X$ be the complement of U in X endowed with the reduced induced closed subscheme structure. Then $P_p(X_2 \to X, E) = (Z \to X_2)_* \circ P_p(Z \to X, E)$, resp. $c_p(X_2 \to X, E) = (Z \to X_2)_* \circ c_p(Z \to X, E)$ by Lemma 50.8. Now we can prove that $P_p(X_2 \to X, E)$, resp. $c_p(X_2 \to X, E)$ satisfies the characterization of $P'_p(E_2)$, resp. $c'_p(E_2)$ given in Lemma 47.1. Namely, by the relation $P_p(X_2 \to X, E) = (Z \to X_2)_* \circ P_p(Z \to X, E)$, resp. $c_p(X_2 \to X, E) = (Z \to X_2)_* \circ c_p(Z \to X, E)$ just proven and the fact that $X_1 \cap Z = \emptyset$, the composition $P_p(X_2 \to X, E) \circ i_{1,*}$, resp. $c_p(X_2 \to X, E) \circ i_{1,*}$ is zero by Lemma 50.7. On the other hand, $P_p(X_2 \to X, E) \circ i_{2,*} = P_p(E_2)$, resp. $c_p(X_2 \to X, E) \circ i_{2,*} = c_p(E_2)$ by Lemma 50.6. $\square$

51. Two technical lemmas

In this section we develop some additional tools to allow us to work more comfortably with localized Chern classes. The following lemma is a more precise version of something we've already encountered in the proofs of Lemmas 49.6 and 49.7.

Lemma 51.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $b : W \longrightarrow \mathbf{P}^1_X$ be a proper morphism of schemes. Let $n \geq 1$. For $i = 1, \ldots, n$ let $Z_i \subset X$ be a closed subscheme, let $Q_i \in D(\mathcal{O}_W)$ be a perfect object, let $p_i \geq 0$ be an integer, and let $T_i \subset W_\infty$, $i = 1, \ldots, n$ be closed. Denote $W_i = b^{-1}(\mathbf{P}^1_{Z_i})$. Assume*

1. *for $i = 1, \ldots, n$ the assumption of Lemma 49.1 hold for b, Z_i, Q_i, T_i, p_i,*
2. *$Q_i|_{W \setminus W_i}$ is zero, resp. isomorphic to a finite locally free module of rank $< p_i$ placed in cohomological degree 0,*
3. *Q_i on W satisfies assumption (3) of Situation 50.1.*

Then $P'_{p_n}(Q_n) \circ \ldots \circ P'_{p_1}(Q_1)$ is equal to

$$(W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ P'_{p_n}(Q_n|_{W_{n,\infty}}) \circ \ldots \circ P'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C$$

in $A^{p_n+\ldots+p_1}(Z_n \cap \ldots \cap Z_1 \to X)$, *resp.* $c'_{p_n}(Q_n) \circ \ldots \circ c'_{p_1}(Q_1)$ *is equal to*

$$(W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{W_{n,\infty}}) \circ \ldots \circ c'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C$$

in $A^{p_n+\ldots+p_1}(Z_n \cap \ldots \cap Z_1 \to X)$.

Proof. Let us prove the statement on Chern classes by induction on n; the statement on $P_p(-)$ is proved in the exact same manner. The case $n = 1$ is the construction of $c'_{p_1}(Q_1)$ because $W_{1,\infty}$ is the inverse image of Z_1 in W_∞. For $n > 1$ we have by induction that $c'_{p_n}(Q_n) \circ \ldots \circ c'_{p_1}(Q_1)$ is equal to

$$c'_{p_n}(Q_n)\circ(W_{n-1,\infty}\cap\ldots\cap W_{1,\infty} \to Z_{n-1}\cap\ldots\cap Z_1)_*\circ c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1},\infty})\circ\ldots\circ c'_{p_1}(Q_1|_{W_{1,\infty}})\circ C$$

By Lemma 49.2 the restriction of $c'_{p_n}(Q_n)$ to $Z_{n-1} \cap \ldots \cap Z_1$ is computed by the closed subset $Z_n \cap \ldots \cap Z_1$, the morphism $b' : W_{n-1} \cap \ldots \cap W_1 \to \mathbf{P}^1_{Z_{n-1}\cap\ldots\cap Z_1}$ and the restriction of Q_n to $W_{n-1} \cap \ldots \cap W_1$. Observe that $(b')^{-1}(Z_n) = W_n \cap \ldots \cap W_1$ and that $(W_n \cap \ldots \cap W_1)_\infty = W_{n,\infty} \cap \ldots \cap W_{1,\infty}$. Denote $C_{n-1} \in A^0(W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to Z_{n-1} \cap \ldots \cap Z_1)$ the class of Lemma 48.1. We conclude the restriction of $c'_{p_n}(Q_n)$ to $Z_{n-1} \cap \ldots \cap Z_1$ is

$$\begin{aligned}&(W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{(W_n\cap\ldots\cap W_1)_\infty}) \circ C_{n-1}\\ &= (W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{W_{n,\infty}}) \circ C_{n-1}\end{aligned}$$

where the equality follows from Lemma 47.3 (we omit writing the restriction on the right). Hence the above becomes

$$\begin{aligned}(W_{n,\infty} \cap \ldots \cap W_{1,\infty} \to Z_n \cap \ldots \cap Z_1)_* \circ c'_{p_n}(Q_n|_{W_n,\infty})\circ\\ C_{n-1} \circ (W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to Z_{n-1} \cap \ldots \cap Z_1)_*\\ \circ c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1},\infty}) \circ \ldots \circ c'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C\end{aligned}$$

By Lemma 48.4 we know that the composition $C_{n-1} \circ (W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to Z_{n-1} \cap \ldots \cap Z_1)_*$ is the identity on elements in the image of the gysin map

$$(W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to W_{n-1} \cap \ldots \cap W_1)^*$$

Thus it suffices to show that any element in the image of $c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1},\infty})\circ\ldots\circ c'_{p_1}(Q_1|_{W_{1,\infty}}) \circ C$ is in the image of the gysin map. We may write

$$c'_{p_i}(Q_i|_{W_{i,\infty}}) = \text{restriction of } c_{p_i}(W_i \to W, Q_i) \text{ to } W_{i,\infty}$$

by Lemma 50.9 and assumptions (2) and (3) on Q_i in the statement of the lemma. Thus, if $\beta \in \mathrm{CH}_{k+1}(W)$ restricts to the flat pullback of α on $b^{-1}(\mathbf{A}^1_X)$, then

$$\begin{aligned}&c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1},\infty}) \cap \ldots \cap c'_{p_1}(Q_1|_{W_{1,\infty}}) \cap C \cap \alpha\\ &= c'_{p_{n-1}}(Q_{n-1}|_{W_{n-1},\infty}) \cap \ldots \cap c'_{p_1}(Q_1|_{W_{1,\infty}}) \cap i_\infty^*\beta\\ &= c_{p_{n-1}}(W_{n-1} \to W, Q_{n-1}) \cap \ldots \cap c_{p_{n-1}}(W_1 \to W, Q_1) \cap i_\infty^*\beta\\ &= (W_{n-1,\infty} \cap \ldots \cap W_{1,\infty} \to W_{n-1} \cap \ldots \cap W_1)^* \left(c_{p_{n-1}}(W_{n-1} \to W, Q_{n-1}) \cap \ldots \cap c_{p_1}(W_1 \to W, Q_1) \cap \beta\right)\end{aligned}$$

as desired. Namely, for the last equality we use that $c_{p_i}(W_i \to W, Q_i)$ is a bivariant class and hence commutes with i_∞^* by definition. □

The following lemma gives us a tremendous amount of flexibility if we want to compute the localized Chern classes of a complex.

Lemma 51.2. *Assume* $(S, \delta), X, Z, b : W \to \mathbf{P}^1_X, Q, T, p$ *satisfy the assumptions of Lemma 49.1. Let* $F \in D(\mathcal{O}_X)$ *be a perfect object such that*

(1) *the restriction of* Q *to* $b^{-1}(\mathbf{A}^1_X)$ *is isomorphic to the pullback of* F,
(2) $F|_{X \setminus Z}$ *is zero, resp. isomorphic to a finite locally free* $\mathcal{O}_{X\setminus Z}$*-module of rank* $< p$ *sitting in cohomological degree* 0, *and*
(3) Q *on* W *and* F *on* X *satisfy assumption (3) of Situation 50.1.*

Then the class $P'_p(Q)$, *resp.* $c'_p(Q)$ *in* $A^p(Z \to X)$ *constructed in Lemma 49.1 is equal to* $P_p(Z \to X, F)$, *resp.* $c_p(Z \to X, F)$ *from Definition 50.3.*

Proof. The assumptions are preserved by base change with a morphism $X' \to X$ locally of finite type. Hence it suffices to show that $P_p(Z \to X, F) \cap \alpha = P'_p(Q) \cap \alpha$, resp. $c_p(Z \to X, F) \cap \alpha = c'_p(Q) \cap \alpha$ for any $\alpha \in \mathrm{CH}_k(X)$. Choose $\beta \in \mathrm{CH}_{k+1}(W)$ whose restriction to $b^{-1}(\mathbf{A}^1_X)$ is equal to the flat pullback of α as in the construction of C in Lemma 48.1. Denote $W' = b^{-1}(Z)$ and denote $E = W'_\infty \subset W_\infty$ the inverse image of Z by $W_\infty \to X$. The lemma follows from the following sequence of equalities (the case of P_p is similar)

$$\begin{aligned}
c'_p(Q) \cap \alpha &= (E \to Z)_*(c'_p(Q|_E) \cap i_\infty^*\beta) \\
&= (E \to Z)_*(c_p(E \to W_\infty, Q|_{W_\infty}) \cap i_\infty^*\beta) \\
&= (W'_\infty \to Z)_*(c_p(W' \to W, Q) \cap i_\infty^*\beta) \\
&= (W'_\infty \to Z)_*((i'_\infty)^*(c_p(W' \to W, Q) \cap \beta)) \\
&= (W'_\infty \to Z)_*((i'_\infty)^*(c_p(Z' \to X, F) \cap \beta)) \\
&= (W'_0 \to Z)_*((i'_0)^*(c_p(Z' \to X, F) \cap \beta)) \\
&= (W'_0 \to Z)_*(c_p(Z' \to X, F) \cap i_0^*\beta)) \\
&= c_p(Z \to X, F) \cap \alpha
\end{aligned}$$

The first equality is the construction of $c'_p(Q)$ in Lemma 49.1. The second is Lemma 50.9. The base change of $W' \to W$ by $W_\infty \to W$ is the morphism $E = W'_\infty \to W_\infty$. Hence the third equality holds by Lemma 50.4. The fourth equality, in which $i'_\infty : W'_\infty \to W'$ is the inclusion morphism, follows from the fact that $c_p(W' \to W, Q)$ is a bivariant class. For the fith equality, observe that $c_p(W' \to W, Q)$ and $c_p(Z' \to X, F)$ restrict to the same bivariant class in $A^p((b')^{-1} \to b^{-1}(\mathbf{A}^1_X))$ by assumption (1) of the lemma which says that Q and F restrict to the same object of $D(\mathcal{O}_{b^{-1}(\mathbf{A}^1_X)})$; use Lemma 50.4. Since $(i'_\infty)^*$ annihilates cycles supported on W'_∞ (see Remark 29.6) we conclude the fifth equality is true. The sixth equality holds because W'_∞ and W'_0 are the pullbacks of the rationally equivalent effective Cartier divisors D_0, D_∞ in $\mathbf{P}^1_Z$ and hence $i_\infty^*\beta$ and $i_0^*\beta$ map to the same cycle class on W'; namely, both represent the class $c_1(\mathcal{O}_{\mathbf{P}^1_Z}(1)) \cap c_p(Z \to X, F) \cap \beta$ by Lemma 29.4. The seventh equality holds because $c_p(Z \to X, F)$ is a bivariant class. By construction $W'_0 = Z$ and $i_0^*\beta = \alpha$ which explains why the final equality holds. ☐

52. Properties of localized Chern classes

The main results in this section are additivity and multiplicativity for localized Chern classes.

Lemma 52.1. *In Situation 50.1 assume* $E|_{X\setminus Z}$ *is zero. Then*

$P_1(Z \to X, E) = c_1(Z \to X, E)$,

$P_2(Z \to X, E) = c_1(Z \to X, E)^2 - 2c_2(Z \to X, E)$,

$P_3(Z \to X, E) = c_1(Z \to X, E)^3 - 3c_1(Z \to X, E)c_2(Z \to X, E) + 3c_3(Z \to X, E)$, *and so on where the products are taken in the algebra $A^{(1)}(Z \to X)$ of Remark 34.7.*

Proof. The statement makes sense because the zero sheaf has rank < 1 and hence the classes $c_p(Z \to X, E)$ are defined for all $p \geq 1$. The result itself follows immediately from the more general Lemma 49.6 as the localized Chern classes where defined using the procedure of Lemma 49.1 in Section 50. $\square$

Lemma 52.2. *In Situation 50.1 let $Y \to X$ be locally of finite type and $c \in A^*(Y \to X)$. Then*

$$P_p(Z \to X, E) \circ c = c \circ P_p(Z \to X, E),$$

respectively

$$c_p(Z \to X, E) \circ c = c \circ c_p(Z \to X, E)$$

in $A^*(Y \times_X Z \to X)$.

Proof. This follows from Lemma 49.5. More precisely, let

$$b : W \to \mathbf{P}^1_X \quad \text{and} \quad Q \quad \text{and} \quad T' \subset T \subset W_\infty$$

be as in the proof of Lemma 50.2. By definition $c_p(Z \to X, E) = c'_p(Q)$ as bivariant operations where the right hand side is the bivariant class constructed in Lemma 49.1 using W, b, Q, T'. By Lemma 49.5 we have $P'_p(Q) \circ c = c \circ P'_p(Q)$, resp. $c'_p(Q) \circ c = c \circ c'_p(Q)$ in $A^*(Y \times_X Z \to X)$ and we conclude. $\square$

Remark 52.3. In Situation 50.1 it is convenient to define

$$c^{(p)}(Z \to X, E) = 1 + c_1(E) + \ldots + c_{p-1}(E) + c_p(Z \to X, E) + c_{p+1}(Z \to X, E) + \ldots$$

as an element of the algebra $A^{(p)}(Z \to X)$ considered in Remark 34.7.

Lemma 52.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $Z \to X$ be a closed immersion. Let*

$$E_1 \to E_2 \to E_3 \to E_1[1]$$

be a distinguished triangle of perfect objects in $D(\mathcal{O}_X)$. Assume

1. *the restrictions $E_1|_{X \setminus Z}$ and $E_3|_{X \setminus Z}$ are isomorphic to finite locally free $\mathcal{O}_{X \setminus Z}$-modules of rank $< p_1$ and $< p_3$ placed in degree 0, and*
2. *at least one of the following is true: (a) X is quasi-compact, (b) X has quasi-compact irreducible components, (c) $E_3 \to E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules, or (d) there exists an envelope $f : Y \to X$ such that $Lf^*E_3 \to Lf^*E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules.*

With notation as in Remark 52.3 we have

$$c^{(p_1+p_3)}(Z \to X, E_2) = c^{(p_1)}(Z \to X, E_1) c^{(p_3)}(Z \to X, E_3)$$

in $A^{(p_1+p_3)}(Z \to X)$.

Proof. Observe that the assumptions imply that $E_2|_{X \setminus Z}$ is zero, resp. isomorphic to a finite locally free $\mathcal{O}_{X \setminus Z}$-module of rank $< p_1 + p_3$. Thus the statement makes sense.

Let $f : Y \to X$ be an envelope. Expanding the left and right hand sides of the formula in the statement of the lemma we see that we have to prove some equalities of classes in $A^*(X)$ and in $A^*(Z \to X)$. By the uniqueness in Lemma 35.6 it suffices to prove the corresponding relations in $A^*(Y)$ and $A^*(Z \to Y)$. Since moreover the construction of the classes involved is compatible with base change (Lemma 50.4) we may replace X by Y and the distinguished triangle by its pullback.

In the proof of Lemma 46.7 we have seen that conditions (2)(a), (2)(b), and (2)(c) imply condition (2)(d). Combined with the discussion in the previous paragraph we reduce to the case discussed in the next paragraph.

Let $\varphi^\bullet : \mathcal{E}_3^\bullet[-1] \to \mathcal{E}_1^\bullet$ be a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules representing the map $E_3[-1] \to E_1$ in the derived category. Consider the scheme $X' = \mathbf{A}^1 \times X$ with projection $g : X' \to X$. Let $Z' = g^{-1}(Z) = \mathbf{A}^1 \times Z$. Denote t the coordinate on $\mathbf{A}^1$. Consider the cone $\mathcal{C}^\bullet$ of the map of complexes

$$tg^*\varphi^\bullet : g^*\mathcal{E}_3^\bullet[-1] \longrightarrow g^*\mathcal{E}_1^\bullet$$

over X'. We obtain a distinguished triangle

$$g^*\mathcal{E}_1^\bullet \to \mathcal{C}^\bullet \to g^*\mathcal{E}_3^\bullet \to g^*\mathcal{E}_1^\bullet[1]$$

where the first three terms form a termwise split short exact sequence of complexes. Clearly $\mathcal{C}^\bullet$ is a bounded complex of finite locally free $\mathcal{O}_{X'}$-modules whose restriction to $X' \setminus Z'$ is isomorphic to a finite locally free $\mathcal{O}_{X' \setminus Z'}$-module of rank $< p_1 + p_3$ placed in degree 0. Thus we have the localized Chern classes

$$c_p(Z' \to X', \mathcal{C}^\bullet) \in A^p(Z' \to X')$$

for $p \geq p_1 + p_3$. For any $\alpha \in \mathrm{CH}_k(X)$ consider

$$c_p(Z' \to X', \mathcal{C}^\bullet) \cap g^*\alpha \in \mathrm{CH}_{k+1-p}(\mathbf{A}^1 \times X)$$

If we restrict to $t = 0$, then the map $tg^*\varphi^\bullet$ restricts to zero and $\mathcal{C}^\bullet|_{t=0}$ is the direct sum of $\mathcal{E}_1^\bullet$ and $\mathcal{E}_3^\bullet$. By compatibility of localized Chern classes with base change (Lemma 50.4) we conclude that

$$i_0^* \circ c^{(p_1+p_3)}(Z' \to X', \mathcal{C}^\bullet) \circ g^* = c^{(p_1+p_2)}(Z \to X, E_1 \oplus E_3)$$

in $A^{(p_1+p_3)}(Z \to X)$. On the other hand, if we restrict to $t = 1$, then the map $tg^*\varphi^\bullet$ restricts to φ and $\mathcal{C}^\bullet|_{t=1}$ is a bounded complex of finite locally free modules representing E_2. We conclude that

$$i_1^* \circ c^{(p_1+p_3)}(Z' \to X', \mathcal{C}^\bullet) \circ g^* = c^{(p_1+p_2)}(Z \to X, E_2)$$

in $A^{(p_1+p_3)}(Z \to X)$. Since $i_0^* = i_1^*$ by definition of rational equivalence (more precisely this follows from the formulae in Lemma 32.4) we conclude that

$$c^{(p_1+p_2)}(Z \to X, E_2) = c^{(p_1+p_2)}(Z \to X, E_1 \oplus E_3)$$

This reduces us to the case discussed in the next paragraph.

Assume $E_2 = E_1 \oplus E_3$ and the triples (X, Z, E_i) are as in Situation 50.1. For $i = 1, 3$ let

$$b_i : W_i \to \mathbf{P}^1_X \quad \text{and} \quad Q_i \quad \text{and} \quad T'_i \subset T_i \subset W_{i,\infty}$$

be as in the proof of Lemma 50.2. By definition

$$c_p(Z \to X, E_i) = c'_p(Q_i)$$

where the right hand side is the bivariant class constructed in Lemma 49.1 using W_i, b_i, Q_i, T'_i. Set $W = W_1 \times_{b_1, \mathbf{P}^1_X, b_2} W_2$ and consider the cartesian diagram

$$\begin{tikzcd} W \arrow[r, "g_3"] \arrow[d, "g_1"] \arrow[dr, "b"] & W_3 \arrow[d, "b_3"] \\ W_1 \arrow[r, "b_1"] & \mathbf{P}^1_X \end{tikzcd}$$

Of course $b^{-1}(\mathbf{A}^1)$ maps isomorphically to $\mathbf{A}^1_X$. Observe that $T' = g_1^{-1}(T'_1) \cap g_2^{-1}(T'_2)$ still contains all the points of W_∞ lying over $X \setminus Z$. By Lemma 49.3 we may use W, b, $g_i^*\mathcal{Q}_i$, and T' to construct $c_p(Z \to X, E_i)$ for $i = 1, 3$. Also, by the stronger independence given in Lemma 51.2 we may use W, b, $g_1^*Q_1 \oplus g_3^*Q_3$, and T' to compute the classes $c_p(Z \to X, E_2)$. Thus the desired equality follows from Lemma 49.7. □

Lemma 52.5. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $Z \to X$ be a closed immersion. Let*

$$E_1 \to E_2 \to E_3 \to E_1[1]$$

be a distinguished triangle of perfect objects in $D(\mathcal{O}_X)$. Assume

1. *the restrictions $E_1|_{X \setminus Z}$ and $E_3|_{X \setminus Z}$ are zero, and*
2. *at least one of the following is true: (a) X is quasi-compact, (b) X has quasi-compact irreducible components, (c) $E_3 \to E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_X$-modules, or (d) there exists an envelope $f : Y \to X$ such that $Lf^*E_3 \to Lf^*E_1[1]$ can be represented by a map of locally bounded complexes of finite locally free $\mathcal{O}_Y$-modules.*

Then we have

$$P_p(Z \to X, E_2) = P_p(Z \to X, E_1) + P_p(Z \to X, E_3)$$

for all $p \in \mathbf{Z}$ and consequently $ch(Z \to X, E_2) = ch(Z \to X, E_1) + ch(Z \to X, E_3)$.

Proof. The proof is exactly the same as the proof of Lemma 52.4 except it uses Lemma 49.8 at the very end. For $p > 0$ we can deduce this lemma from Lemma 52.4 with $p_1 = p_3 = 1$ and the relationship between $P_p(Z \to X, E)$ and $c_p(Z \to X, E)$ given in Lemma 52.1. The case $p = 0$ can be shown directly (it is only interesting if X has a connected component entirely contained in Z). □

Lemma 52.6. *In Situation 7.1 let X be locally of finite type over S. Let $Z_i \subset X$, $i = 1, 2$ be closed subschemes. Let F_i, $i = 1, 2$ be perfect objects of $D(\mathcal{O}_X)$. Assume for $i = 1, 2$ that $F_i|_{X \setminus Z_i}$ is zero[7] and that F_i on X satisfies assumption (3) of Situation 50.1. Denote $r_i = P_0(Z_i \to X, F_i) \in A^0(Z_i \to X)$. Then we have*

$$c_1(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = r_1 c_1(Z_2 \to X, F_2) + r_2 c_1(Z_1 \to X, F_1)$$

[7]Presumably there is a variant of this lemma where we only assume $F_i|_{X \setminus Z_i}$ is isomorphic to a finite locally free $\mathcal{O}_{X \setminus Z_i}$-module of rank $< p_i$.

in $A^1(Z_1 \cap Z_2 \to X)$ *and*

$$\begin{aligned} c_2(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = {} & r_1 c_2(Z_2 \to X, F_2) + r_2 c_2(Z_1 \to X, F_1) + \\ & \binom{r_1}{2} c_1(Z_2 \to X, F_2)^2 + \\ & (r_1 r_2 - 1) c_1(Z_2 \to X, F_2) c_1(Z_1 \to X, F_1) + \\ & \binom{r_2}{2} c_1(Z_1 \to X, F_1)^2 \end{aligned}$$

in $A^2(Z_1 \cap Z_2 \to X)$ *and so on for higher Chern classes. Similarly, we have*

$$ch(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = ch(Z_1 \to X, F_1) ch(Z_2 \to X, F_2)$$

in $\prod_{p \geq 0} A^p(Z_1 \cap Z_2 \to X) \otimes \mathbf{Q}$*. More precisely, we have*

$$P_p(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = \sum\nolimits_{p_1 + p_2 = p} \binom{p}{p_1} P_{p_1}(Z_1 \to X, F_1) P_{p_2}(Z_2 \to X, F_2)$$

in $A^p(Z_1 \cap Z_2 \to X)$*.*

Proof. Choose proper morphisms $b_i : W_i \to \mathbf{P}^1_X$ and $Q_i \in D(\mathcal{O}_{W_i})$ as well as closed subschemes $T_i \subset W_{i,\infty}$ as in the construction of the localized Chern classes for F_i or more generally as in Lemma 51.2. Choose a commutative diagram

$$\begin{tikzcd} W \arrow[r, "g_2"'] \arrow[d, "g_1"] \arrow[dr, "b"] & W_2 \arrow[d, "b_2"] \\ W_1 \arrow[r, "b_1"] & \mathbf{P}^1_X \end{tikzcd}$$

where all morphisms are proper and isomorphisms over $\mathbf{A}^1_X$. For example, we can take W to be the closure of the graph of the isomorphism between $b_1^{-1}(\mathbf{A}^1_X)$ and $b_2^{-1}(\mathbf{A}^1_X)$. By Lemma 51.2 we may work with W, $b = b_i \circ g_i$, $Lg_i^* Q_i$, and $g_i^{-1}(T_i)$ to construct the localized Chern classes $c_p(Z_i \to X, F_i)$. Thus we reduce to the situation described in the next paragraph.

Assume we have

(1) a proper morphism $b : W \to \mathbf{P}^1_X$ which is an isomorphism over $\mathbf{A}^1_X$,
(2) $E_i \subset W_\infty$ is the inverse image of Z_i,
(3) perfect objects $Q_i \in D(\mathcal{O}_W)$ whose Chern classes are defined, such that
    (a) the restriction of Q_i to $b^{-1}(\mathbf{A}^1_X)$ is the pullback of F_i, and
    (b) there exists a closed subscheme $T_i \subset W_\infty$ containing all points of W_∞ lying over $X \setminus Z_i$ such that $Q_i|_{T_i}$ is zero.

By Lemma 51.2 we have

$$c_p(Z_i \to X, F_i) = c'_p(Q_i) = (E_i \to Z_i)_* \circ c'_p(Q_i|_{E_i}) \circ C$$

and

$$P_p(Z_i \to X, F_i) = P'_p(Q_i) = (E_i \to Z_i)_* \circ P'_p(Q_i|_{E_i}) \circ C$$

for $i = 1, 2$. Next, we observe that $Q = Q_1 \otimes^{\mathbf{L}}_{\mathcal{O}_W} Q_2$ satisfies (3)(a) and (3)(b) for $F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2$ and $T_1 \cup T_2$. Hence we see that

$$c_p(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = (E_1 \cap E_2 \to Z_1 \cap Z_2)_* \circ c'_p(Q|_{E_1 \cap E_2}) \circ C$$

and

$$P_p(Z_1 \cap Z_2 \to X, F_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} F_2) = (E_1 \cap E_2 \to Z_1 \cap Z_2)_* \circ P'_p(Q|_{E_1 \cap E_2}) \circ C$$

by the same lemma. By Lemma 47.11 the classes $c'_p(Q|_{E_1 \cap E_2})$ and $P'_p(Q|_{E_1 \cap E_2})$ can be expanded in the correct manner in terms of the classes $c'_p(Q_i|_{E_i})$ and $P'_p(Q_i|_{E_i})$. Then finally Lemma 51.1 tells us that polynomials in $c'_p(Q_i|_{E_i})$ and $P'_p(Q_i|_{E_i})$ agree with the corresponding polynomials in $c'_p(Q_i)$ and $P'_p(Q_i)$ as desired. $\square$

53. Blowing up at infinity

Let X be a scheme. Let $Z \subset X$ be a closed subscheme cut out by a finite type quasi-coherent sheaf of ideals. Denote $X' \to X$ the blowing up with center Z. Let $b : W \to \mathbf{P}^1_X$ be the blowing up with center $\infty(Z)$. Denote $E \subset W$ the exceptional divisor. There is a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & W \arrow[d, "b"] \\ X \arrow[r, "\infty"] & \mathbf{P}^1_X \end{tikzcd}$$

whose horizontal arrows are closed immersion (Divisors, Lemma 33.2). Denote $E \subset W$ the exceptional divisor and $W_\infty \subset W$ the inverse image of $(\mathbf{P}^1_X)_\infty$. Then the following are true

(1) b is an isomorphism over $\mathbf{A}^1_X \cup \mathbf{P}^1_{X \setminus Z}$,
(2) X' is an effective Cartier divisor on W,
(3) $X' \cap E$ is the exceptional divisor of $X' \to X$,
(4) $W_\infty = X' + E$ as effective Cartier divisors on W,
(5) $E = \underline{\text{Proj}}_Z(\mathcal{C}_{Z/X,*}[S])$ where S is a variable placed in degree 1,
(6) $X' \cap E = \underline{\text{Proj}}_Z(\mathcal{C}_{Z/X,*})$,
(7) $E \setminus X' = E \setminus (X' \cap E) = \underline{\text{Spec}}_Z(\mathcal{C}_{Z/X,*}) = C_Z X$,
(8) there is a closed immersion $\mathbf{P}^1_Z \to W$ whose composition with b is the inclusion morphism $\mathbf{P}^1_Z \to \mathbf{P}^1_X$ and whose base change by ∞ is the composition $Z \to C_Z X \to E \to W_\infty$ where the first arrow is the vertex of the cone.

We recall that $\mathcal{C}_{Z/X,*}$ is the conormal algebra of Z in X, see Divisors, Definition 19.1 and that $C_Z X$ is the normal cone of Z in X, see Divisors, Definition 19.5.

We now give the proof of the numbered assertions above. We strongly urge the reader to work through some examples instead of reading the proofs.

Part (1) follows from the corresponding assertion of Divisors, Lemma 32.4. Observe that $E \subset W$ is an effective Cartier divisor by the same lemma.

Observe that W_∞ is an effective Cartier divisor by Divisors, Lemma 32.11. Since $E \subset W_\infty$ we can write $W_\infty = D + E$ for some effective Cartier divisor D, see Divisors, Lemma 13.8. We will see below that $D = X'$ which will prove (2) and (4).

Since X' is the strict transform of the closed immersion $\infty : X \to \mathbf{P}^1_X$ (see above) it follows that the exceptional divisor of $X' \to X$ is equal to the intersection $X' \cap E$ (for example because both are cut out by the pullback of the ideal sheaf of Z to X'). This proves (3).

The intersection of $\infty(Z)$ with $\mathbf{P}^1_Z$ is the effective Cartier divisor $(\mathbf{P}^1_Z)_\infty$ hence the strict transform of $\mathbf{P}^1_Z$ by the blowing up b maps isomorphically to $\mathbf{P}^1_Z$ (see Divisors, Lemmas 33.2 and 32.7). This gives us the morphism $\mathbf{P}^1_Z \to W$ mentioned in (8). It is a closed immersion as b is separated, see Schemes, Lemma 21.11.

Suppose that $\text{Spec}(A) \subset X$ is an affine open and that $Z \cap \text{Spec}(A)$ corresponds to the finitely generated ideal $I \subset A$. An affine neighbourhood of $\infty(Z \cap \text{Spec}(A))$ is the affine space over A with coordinate $s = T_0/T_1$. Denote $J = (I, s) \subset A[s]$ the ideal generated by I and s. Let $B = A[s] \oplus J \oplus J^2 \oplus \ldots$ be the Rees algebra of $(A[s], J)$. Observe that

$$J^n = I^n \oplus sI^{n-1} \oplus s^2 I^{n-2} \ldots \oplus s^n A \oplus s^{n+1} A \oplus \ldots$$

as an A-submodule of $A[s]$ for all $n \geq 0$. Consider the open subscheme

$$\text{Proj}(B) = \text{Proj}(A[s] \oplus J \oplus J^2 \oplus \ldots) \subset W$$

Finally, denote S the element $s \in J$ viewed as a degree 1 element of B.

Since formation of Proj commutes with base change (Constructions, Lemma 11.6) we see that

$$E = \text{Proj}(B \otimes_{A[s]} A/I) = \text{Proj}((A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots)[S])$$

The verification that $B \otimes_{A[s]} A/I = \bigoplus J^n/J^{n+1}$ is as given follows immediately from our description of the powers J^n above. This proves (5) because the conormal algebra of $Z \cap \text{Spec}(A)$ in $\text{Spec}(A)$ corresponds to the graded A-algebra $A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$ by Divisors, Lemma 19.2.

Recall that $\text{Proj}(B)$ is covered by the affine opens $D_+(S)$ and $D_+(f^{(1)})$ for $f \in I$ which are the spectra of affine blowup algebras $A[s][\frac{J}{s}]$ and $A[s][\frac{J}{f}]$, see Divisors, Lemma 32.2 and Algebra, Definition 70.1. We will describe each of these affine opens and this will finish the proof.

The open $D_+(S)$, i.e., the spectrum of $A[s][\frac{J}{s}]$. It follows from the description of the powers of J above that

$$A[s][\tfrac{J}{s}] = \sum s^{-n} I^n[s] \subset A[s, s^{-1}]$$

The element s is a nonzerodivisor in this ring, defines the exceptional divisor E as well as W_∞. Hence $D \cap D_+(S) = \emptyset$. Finally, the quotient of $A[s][\frac{J}{s}]$ by s is the conormal algebra

$$A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$$

This proves (7).

The open $D_+(f^{(1)})$, i.e., the spectrum of $A[s][\frac{J}{f}]$. It follows from the description of the powers of J above that

$$A[s][\tfrac{J}{f}] = A[\tfrac{I}{f}][\tfrac{s}{f}]$$

where $\frac{s}{f}$ is a variable. The element f is a nonzerodivisor in this ring whose zero scheme defines the exceptional divisor E. Since s defines W_∞ and $s = f \cdot \frac{s}{f}$ we conclude that $\frac{s}{f}$ defines the divisor D constructed above. Then we see that

$$D \cap D_+(f^{(1)}) = \text{Spec}(A[\tfrac{I}{f}])$$

which is the corresponding open of the blowup X' over $\text{Spec}(A)$. Namely, the surjective graded $A[s]$-algebra map $B \to A \oplus I \oplus I^2 \oplus \ldots$ to the Rees algebra of

(A, I) corresponds to the closed immersion $X' \to W$ over $\text{Spec}(A[s])$. This proves $D = X'$ as desired.

Let us prove (6). Observe that the zero scheme of $\frac{s}{f}$ in the previous paragraph is the restriction of the zero scheme of S on the affine open $D_+(f^{(1)})$. Hence we see that $S = 0$ defines $X' \cap E$ on E. Thus (6) follows from (5).

Finally, we have to prove the last part of (8). This is clear because the map $\mathbf{P}^1_Z \to W$ is affine locally given by the surjection

$$B \to B \otimes_{A[s]} A/I = (A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots)[S] \to A/I[S]$$

and the identification $\text{Proj}(A/I[S]) = \text{Spec}(A/I)$. Some details omitted.

54. Higher codimension gysin homomorphisms

Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. In this section we are going to consider triples

$$(Z \to X, \mathcal{N}, \sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X})$$

consisting of a closed immersion $Z \to X$ and a locally free $\mathcal{O}_Z$-module $\mathcal{N}$ and a surjection $\sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ from the dual of $\mathcal{N}$ to the conormal sheaf of Z in X, see Morphisms, Section 31. We will say $\mathcal{N}$ is a *virtual normal sheaf for Z in X*.

Lemma 54.1. *Let (S, δ) be as in Situation 7.1. Let*

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd}$$

be a cartesian diagram of schemes locally of finite type over S whose horizontal arrows are closed immersions. If $\mathcal{N}$ is a virtual normal sheaf for Z in X, then $\mathcal{N}' = g^*\mathcal{N}$ is a virtual normal sheaf for Z' in X'.

Proof. This follows from the surjectivity of the map $g^*\mathcal{C}_{Z/X} \to \mathcal{C}_{Z'/X'}$ proved in Morphisms, Lemma 31.4. $\square$

Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal bundle for a closed immersion $Z \to X$. In this situation we set

$$p : N = \underline{\text{Spec}}_Z(\text{Sym}(\mathcal{N}^\vee)) \longrightarrow Z$$

equal to the vector bundle over Z whose sections correspond to sections of $\mathcal{N}$. In this situation we have canonical closed immersions

$$C_Z X \longrightarrow N_Z X \longrightarrow N$$

The first closed immersion is Divisors, Equation (19.5.1) and the second closed immersion corresponds to the surjection $\text{Sym}(\mathcal{N}^\vee) \to \text{Sym}(\mathcal{C}_{Z/X})$ induced by σ. Let

$$b : W \longrightarrow \mathbf{P}^1_X$$

be the blowing up in $\infty(Z)$ constructed in Section 53. By Lemma 48.1 we have a canonical bivariant class in

$$C \in A^0(W_\infty \to X)$$

Consider the open immersion $j : C_Z X \to W_\infty$ of (7) and the closed immersion $i : C_Z X \to N$ constructed above. By Lemma 36.3 for every $\alpha \in \mathrm{CH}_k(X)$ there exists a unique $\beta \in \mathrm{CH}_*(Z)$ such that

$$i_* j^*(C \cap \alpha) = p^* \beta$$

We set $c(Z \to X, \mathcal{N}) \cap \alpha = \beta$.

Lemma 54.2. *The construction above defines a bivariant class*[8]

$$c(Z \to X, \mathcal{N}) \in A^*(Z \to X)^\wedge$$

and moreover the construction is compatible with base change as in Lemma 54.1. If $\mathcal{N}$ has constant rank r, then $c(Z \to X, \mathcal{N}) \in A^r(Z \to X)$.

Proof. Since both $i_* \circ j^* \circ C$ and p^* are bivariant classes (see Lemmas 33.2 and 33.4) we can use the equation

$$i_* \circ j^* \circ C = p^* \circ c(Z \to X, \mathcal{N})$$

(suitably interpreted) to define $c(Z \to X, \mathcal{N})$ as a bivariant class. This works because p^* is always bijective on chow groups by Lemma 36.3.

Let $X' \to X$, $Z' \to X'$, and $\mathcal{N}'$ be as in Lemma 54.1. Write $c = c(Z \to X, \mathcal{N})$ and $c' = c(Z' \to X', \mathcal{N}')$. The second statement of the lemma means that c' is the restriction of c as in Remark 33.5. Since we claim this is true for all X'/X locally of finite type, a formal argument shows that it suffices to check that $c' \cap \alpha' = c \cap \alpha'$ for $\alpha' \in \mathrm{CH}_k(X')$. To see this, note that we have a commutative diagram

$$\begin{tikzcd}
C_{Z'}X' \arrow[r] \arrow[d] & W'_\infty \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & \mathbf{P}^1_{X'} \arrow[d] \\
C_Z X \arrow[r] & W_\infty \arrow[r] & W \arrow[r] & \mathbf{P}^1_X
\end{tikzcd}$$

which induces closed immersions:

$$W' \to W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'}, \quad W'_\infty \to W_\infty \times_X X', \quad C_{Z'}X' \to C_Z X \times_Z Z'$$

To get $c \cap \alpha'$ we use the class $C \cap \alpha'$ defined using the morphism $W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'} \to \mathbf{P}^1_{X'}$ in Lemma 48.1. To get $c' \cap \alpha'$ on the other hand, we use the class $C' \cap \alpha'$ defined using the morphism $W' \to \mathbf{P}^1_{X'}$. By Lemma 48.3 the pushforward of $C' \cap \alpha'$ by the closed immersion $W'_\infty \to (W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'})_\infty$, is equal to $C \cap \alpha'$. Hence the same is true for the pullbacks to the opens

$$C_{Z'}X' \subset W'_\infty, \quad C_Z X \times_Z Z' \subset (W \times_{\mathbf{P}^1_X} \mathbf{P}^1_{X'})_\infty$$

by Lemma 15.1. Since we have a commutative diagram

$$\begin{tikzcd}
C_{Z'}X' \arrow[r] \arrow[d] & N' \arrow[d, equal] \\
C_Z X \times_Z Z' \arrow[r] & N \times_Z Z'
\end{tikzcd}$$

these classes pushforward to the same class on N' which proves that we obtain the same element $c \cap \alpha' = c' \cap \alpha'$ in $\mathrm{CH}_*(Z')$. $\square$

[8]The notation $A^*(Z \to X)^\wedge$ is discussed in Remark 35.5. If X is quasi-compact, then $A^*(Z \to X)^\wedge = A^*(Z \to X)$.

Lemma 54.3. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for a closed subscheme Z of X. Suppose that we have a short exact sequence $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{E} \to 0$ of finite locally free $\mathcal{O}_Z$-modules such that the given surjection $\sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ factors through a map $\sigma' : (\mathcal{N}')^\vee \to \mathcal{C}_{Z/X}$. Then*

$$c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{E}) \circ c(Z \to X, \mathcal{N}')$$

as bivariant classes.

Proof. Denote $N' \to N$ the closed immersion of vector bundles corresponding to the surjection $\mathcal{N}^\vee \to (\mathcal{N}')^\vee$. Then we have closed immersions

$$C_Z X \to N' \to N$$

Thus the desired relationship between the bivariant classes follows immediately from Lemma 44.2. $\square$

Lemma 54.4. *Let (S, δ) be as in Situation 7.1. Consider a cartesian diagram*

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "g"] & X' \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd}$$

of schemes locally of finite type over S whose horizontal arrows are closed immersions. Let $\mathcal{N}$, resp. $\mathcal{N}'$ be a virtual normal sheaf for $Z \subset X$, resp. $Z' \to X'$. Assume given a short exact sequence $0 \to \mathcal{N}' \to g^*\mathcal{N} \to \mathcal{E} \to 0$ of finite locally free modules on Z' such that the diagram

$$\begin{tikzcd} g^*\mathcal{N}^\vee \arrow[r] \arrow[d] & (\mathcal{N}')^\vee \arrow[d] \\ g^*\mathcal{C}_{Z/X} \arrow[r] & \mathcal{C}_{Z'/X'} \end{tikzcd}$$

commutes. Then we have

$$res(c(Z \to X, \mathcal{N})) = c_{top}(\mathcal{E}) \circ c(Z' \to X', \mathcal{N}')$$

in $A^*(Z' \to X')^\wedge$.

Proof. By Lemma 54.2 we have $res(c(Z \to X, \mathcal{N})) = c(Z' \to X', g^*\mathcal{N})$ and the equality follows from Lemma 54.3. $\square$

Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for a closed subscheme Z of X. Let $Y \to X$ be a morphism which is locally of finite type. Assume $Z \times_X Y \to Y$ is a regular closed immersion, see Divisors, Section 21. In this case the conormal sheaf $\mathcal{C}_{Z \times_X Y/Y}$ is a finite locally free $\mathcal{O}_{Z \times_X Y}$-module and we obtain a short exact sequence

$$0 \to \mathcal{E}^\vee \to \mathcal{N}^\vee|_{Z \times_X Y} \to \mathcal{C}_{Z \times_X Y/Y} \to 0$$

The quotient $\mathcal{N}|_{Y \times_X Z} \to \mathcal{E}$ is called the *excess normal sheaf* of the situation.

Lemma 54.5. *In the situation described just above assume $\dim_\delta(Y) = n$ and that $\mathcal{C}_{Y \times_X Z/Z}$ has constant rank r. Then*

$$c(Z \to X, \mathcal{N}) \cap [Y]_n = c_{top}(\mathcal{E}) \cap [Z \times_X Y]_{n-r}$$

in $\mathrm{CH}_*(Z \times_X Y)$.

Proof. The bivariant class $c_{top}(\mathcal{E}) \in A^*(Z \times_X Y)$ was defined in Remark 38.11. By Lemma 54.2 we may replace X by Y. Thus we may assume $Z \to X$ is a regular closed immersion of codimension r, we have $\dim_\delta(X) = n$, and we have to show that $c(Z \to X, \mathcal{N}) \cap [X]_n = c_{top}(\mathcal{E}) \cap [Z]_{n-r}$ in $\mathrm{CH}_*(Z)$. By Lemma 54.3 we may even assume $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ is an isomorphism. In other words, we have to show $c(Z \to X, \mathcal{C}_{Z/X}^\vee) \cap [X]_n = [Z]_{n-r}$ in $\mathrm{CH}_*(Z)$.

Let us trace through the steps in the definition of $c(Z \to X, \mathcal{C}_{Z/X}^\vee) \cap [X]_n$. Let $b : W \to \mathbf{P}^1_X$ be the blowing up of $\infty(Z)$. We first have to compute $C \cap [X]_n$ where $C \in A^0(W_\infty \to X)$ is the class of Lemma 48.1. To do this, note that $[W]_{n+1}$ is a cycle on W whose restriction to $\mathbf{A}^1_X$ is equal to the flat pullback of $[X]_n$. Hence $C \cap [X]_n$ is equal to $i_\infty^*[W]_{n+1}$. Since W_∞ is an effective Cartier divisor on W we have $i_\infty^*[W]_{n+1} = [W_\infty]_n$, see Lemma 29.5. The restriction of this class to the open $C_Z X \subset W_\infty$ is of course just $[C_Z X]_n$. Because $Z \subset X$ is regularly embedded we have

$$\mathcal{C}_{Z/X,*} = \mathrm{Sym}(\mathcal{C}_{Z/X})$$

as graded $\mathcal{O}_Z$-algebras, see Divisors, Lemma 21.5. Hence $p : N = C_Z X \to Z$ is the structure morphism of the vector bundle associated to the finite locally free module $\mathcal{C}_{Z/X}$ of rank r. Then it is clear that $p^*[Z]_{n-r} = [C_Z X]_n$ and the proof is complete. □

Lemma 54.6. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for a closed subscheme Z of X. Let $Y \to X$ be a morphism which is locally of finite type. Given integers r, n assume*

(1) *$\mathcal{N}$ is locally free of rank r,*
(2) *every irreducible component of Y has δ-dimension n,*
(3) *$\dim_\delta(Z \times_X Y) \leq n - r$, and*
(4) *for $\xi \in Z \times_X Y$ with $\delta(\xi) = n - r$ the local ring $\mathcal{O}_{Y,\xi}$ is Cohen-Macaulay.*

Then $c(Z \to X, \mathcal{N}) \cap [Y]_n = [Z \times_X Y]_{n-r}$ in $\mathrm{CH}_{n-r}(Z \times_X Y)$.

Proof. The statement makes sense as $Z \times_X Y$ is a closed subscheme of Y. Because $\mathcal{N}$ has rank r we know that $c(Z \to X, \mathcal{N}) \cap [Y]_n$ is in $\mathrm{CH}_{n-r}(Z \times_X Y)$. Since $\dim_\delta(Z \cap Y) \leq n - r$ the chow group $\mathrm{CH}_{n-r}(Z \times_X Y)$ is freely generated by the cycle classes of the irreducible components $W \subset Z \times_X Y$ of δ-dimension $n - r$. Let $\xi \in W$ be the generic point. By assumption (2) we see that $\dim(\mathcal{O}_{Y,\xi}) = r$. On the other hand, since $\mathcal{N}$ has rank r and since $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ is surjective, we see that the ideal sheaf of Z is locally cut out by r equations. Hence the quasi-coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$ of $Z \times_X Y$ in Y is locally generated by r elements. Since $\mathcal{O}_{Y,\xi}$ is Cohen-Macaulay of dimension r and since $\mathcal{I}_\xi$ is an ideal of definition (as ξ is a generic point of $Z \times_X Y$) it follows that $\mathcal{I}_\xi$ is generated by a regular sequence (Algebra, Lemma 104.2). By Divisors, Lemma 20.8 we see that $\mathcal{I}$ is generated by a regular sequence over an open neighbourhood $V \subset Y$ of ξ. By our description of $\mathrm{CH}_{n-r}(Z \times_X Y)$ it suffices to show that $c(Z \to X, \mathcal{N}) \cap [V]_n = [Z \times_X V]_{n-r}$ in $\mathrm{CH}_{n-r}(Z \times_X V)$. This follows from Lemma 54.5 because the excess normal sheaf is 0 over V. □

Lemma 54.7. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be a triple as in Definition 29.1. The gysin homomorphism i^* viewed as an element of $A^1(D \to X)$ (see Lemma 33.3) is the* *same as the bivariant class* $c(D \to X, \mathcal{N}) \in A^1(D \to X)$ *constructed using* $\mathcal{N} = i^*\mathcal{L}$ *viewed as a virtual normal sheaf for* D *in* X.

Proof. We will use the criterion of Lemma 35.3. Thus we may assume that X is an integral scheme and we have to show that $i^*[X]$ is equal to $c \cap [X]$. Let $n = \dim_\delta(X)$. As usual, there are two cases.

If $X = D$, then we see that both classes are represented by $c_1(\mathcal{N}) \cap [X]_n$. See Lemma 54.5 and Definition 29.1.

If $D \neq X$, then $D \to X$ is an effective Cartier divisor and in particular a regular closed immersion of codimension 1. Again by Lemma 54.5 we conclude $c(D \to X, \mathcal{N}) \cap [X]_n = [D]_{n-1}$. The same is true by definition for the gysin homomorphism and we conclude once again. $\square$

Lemma 54.8. *Let* (S, δ) *be as in Situation 7.1. Let* X *be a scheme locally of finite type over* S. *Let* $Z \subset X$ *be a closed subscheme with virtual normal sheaf* $\mathcal{N}$. *Let* $Y \to X$ *be locally of finite type and* $c \in A^*(Y \to X)$. *Then* c *and* $c(Z \to X, \mathcal{N})$ *commute (Remark 33.6).*

Proof. To check this we may use Lemma 35.3. Thus we may assume X is an integral scheme and we have to show $c \cap c(Z \to X, \mathcal{N}) \cap [X] = c(Z \to X, \mathcal{N}) \cap c \cap [X]$ in $\mathrm{CH}_*(Z \times_X Y)$.

If $Z = X$, then $c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{N})$ by Lemma 54.5 which commutes with the bivariant class c, see Lemma 38.9.

Assume that Z is not equal to X. By Lemma 35.3 it even suffices to prove the result after blowing up X (in a nonzero ideal). Let us blowup X in the ideal sheaf of Z. This reduces us to the case where Z is an effective Cartier divisor, see Divisors, Lemma 32.4,

If Z is an effective Cartier divisor, then we have

$$c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{E}) \circ i^*$$

where $i^* \in A^1(Z \to X)$ is the gysin homomorphism associated to $i : Z \to X$ (Lemma 33.3) and $\mathcal{E}$ is the dual of the kernel of $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$, see Lemmas 54.3 and 54.7. Then we conclude because Chern classes are in the center of the bivariant ring (in the strong sense formulated in Lemma 38.9) and c commutes with the gysin homomorphism i^* by definition of bivariant classes. $\square$

Let (S, δ) be as in Situation 7.1. Let X be an integral scheme locally of finite type over S of δ-dimension n. Let $Z \subset Y \subset X$ be closed subschemes which are both effective Cartier divisors in X. Denote $o : Y \to C_Y X$ the zero section of the normal line cone of Y in X. As $C_Y X$ is a line bundle over Y we obtain a bivariant class $o^* \in A^1(Y \to C_Y X)$, see Lemma 33.3.

Lemma 54.9. *With notation as above we have*

$$o^*[C_Z X]_n = [C_Z Y]_{n-1}$$

in $\mathrm{CH}_{n-1}(Y \times_{o, C_Y X} C_Z X)$.

Proof. Denote $W \to \mathbf{P}^1_X$ the blowing up of $\infty(Z)$ as in Section 53. Similarly, denote $W' \to \mathbf{P}^1_X$ the blowing up of $\infty(Y)$. Since $\infty(Z) \subset \infty(Y)$ we get an opposite inclusion of ideal sheaves and hence a map of the graded algebras defining these blowups. This produces a rational morphism from W to W' which in fact has a canonical representative

$$W \supset U \longrightarrow W'$$

See Constructions, Lemma 18.1. A local calculation (omitted) shows that U contains at least all points of W not lying over ∞ and the open subscheme $C_Z X$ of the special fibre. After shrinking U we may assume $U_\infty = C_Z X$ and $\mathbf{A}^1_X \subset U$. Another local calculation (omitted) shows that the morphism $U_\infty \to W'_\infty$ induces the canonical morphism $C_Z X \to C_Y X \subset W'_\infty$ of normal cones induced by the inclusion of ideals sheaves coming from $Z \subset Y$. Denote $W'' \subset W$ the strict transform of $\mathbf{P}^1_Y \subset \mathbf{P}^1_X$ in W. Then W'' is the blowing up of $\mathbf{P}^1_Y$ in $\infty(Z)$ by Divisors, Lemma 33.2 and hence $(W'' \cap U)_\infty = C_Z Y$.

Consider the effective Cartier divisor $i : \mathbf{P}^1_Y \to W'$ from (8) and its associated bivariant class $i^* \in A^1(\mathbf{P}^1_Y \to W')$ from Lemma 33.3. We similarly denote $(i'_\infty)^* \in A^1(W'_\infty \to W')$ the gysin map at infinity. Observe that the restriction of i'_∞ (Remark 33.5) to U is the restriction of $i^*_\infty \in A^1(W_\infty \to W)$ to U. On the one hand we have

$$(i'_\infty)^* i^*[U]_{n+1} = i^*_\infty i^*[U]_{n+1} = i^*_\infty[(W'' \cap U)_\infty]_{n+1} = [C_Z Y]_n$$

because i^*_∞ kills all classes supported over ∞, because $i^*[U]$ and $[W'']$ agree as cycles over $\mathbf{A}^1$, and because $C_Z Y$ is the fibre of $W'' \cap U$ over ∞. On the other hand, we have

$$(i'_\infty)^* i^*[U]_{n+1} = i^* i^*_\infty[U]_{n+1} = i^*[U_\infty] = o^*[C_Y X]_n$$

because $(i'_\infty)^*$ and i^* commute (Lemma 30.5) and because the fibre of $i : \mathbf{P}^1_Y \to W'$ over ∞ factors as $o : Y \to C_Y X$ and the open immersion $C_Y X \to W'_\infty$. The lemma follows. $\square$

Lemma 54.10. *Let (S, δ) be as in Situation 7.1. Let $Z \subset Y \subset X$ be closed subschemes of a scheme locally of finite type over S. Let $\mathcal{N}$ be a virtual normal sheaf for $Z \subset X$. Let $\mathcal{N}'$ be a virtual normal sheaf for $Z \subset Y$. Let $\mathcal{N}''$ be a virtual normal sheaf for $Y \subset X$. Assume there is a commutative diagram*

$$\begin{tikzcd}
(\mathcal{N}'')^\vee|_Z \arrow[r] \arrow[d] & \mathcal{N}^\vee \arrow[r] \arrow[d] & (\mathcal{N}')^\vee \arrow[d] \\
\mathcal{C}_{Y/X}|_Z \arrow[r] & \mathcal{C}_{Z/X} \arrow[r] & \mathcal{C}_{Z/Y}
\end{tikzcd}$$

where the sequence at the bottom is from More on Morphisms, Lemma 7.12 and the top sequence is a short exact sequence. Then

$$c(Z \to X, \mathcal{N}) = c(Z \to Y, \mathcal{N}') \circ c(Y \to X, \mathcal{N}'')$$

in $A^*(Z \to X)^\wedge$.

Proof. Observe that the assumptions remain satisfied after any base change by a morphism $X' \to X$ which is locally of finite type (the short exact sequence of virtual normal sheaves is locally split hence remains exact after any base change). Thus to check the equality of bivariant classes we may use Lemma 35.3. Thus we may assume X is an integral scheme and we have to show $c(Z \to X, \mathcal{N}) \cap [X] = c(Z \to Y, \mathcal{N}') \cap c(Y \to X, \mathcal{N}'') \cap [X]$.

If $Y = X$, then we have

$$\begin{aligned} c(Z \to Y, \mathcal{N}') \cap c(Y \to X, \mathcal{N}'') \cap [X] &= c(Z \to Y, \mathcal{N}') \cap c_{top}(\mathcal{N}'') \cap [Y] \\ &= c_{top}(\mathcal{N}''|_Z) \cap c(Z \to Y, \mathcal{N}') \cap [Y] \\ &= c(Z \to X, \mathcal{N}) \cap [X] \end{aligned}$$

The first equality by Lemma 54.3. The second because Chern classes commute with bivariant classes (Lemma 38.9). The third equality by Lemma 54.3.

Assume $Y \neq X$. By Lemma 35.3 it even suffices to prove the result after blowing up X in a nonzero ideal. Let us blowup X in the product of the ideal sheaf of Y and the ideal sheaf of Z. This reduces us to the case where both Y and Z are effective Cartier divisors on X, see Divisors, Lemmas 32.4 and 32.12.

Denote $\mathcal{N}'' \to \mathcal{E}$ the surjection of finite locally free $\mathcal{O}_Z$-modules such that $0 \to \mathcal{E}^\vee \to (\mathcal{N}'')^\vee \to \mathcal{C}_{Y/X} \to 0$ is a short exact sequence. Then $\mathcal{N} \to \mathcal{E}|_Z$ is a surjection as well. Denote $\mathcal{N}_1$ the finite locally free kernel of this map and observe that $\mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ factors through $\mathcal{N}_1$. By Lemma 54.3 we have

$$c(Y \to X, \mathcal{N}'') = c_{top}(\mathcal{E}) \circ c(Y \to X, \mathcal{C}_{Y/X}^\vee)$$

and

$$c(Z \to X, \mathcal{N}) = c_{top}(\mathcal{E}|_Z) \circ c(Z \to X, \mathcal{N}_1)$$

Since Chern classes of bundles commute with bivariant classes (Lemma 38.9) it suffices to prove

$$c(Z \to X, \mathcal{N}_1) = c(Z \to Y, \mathcal{N}') \circ c(Y \to X, \mathcal{C}_{Y/X}^\vee)$$

in $A^*(Z \to X)$. This we may assume that $\mathcal{N}'' = \mathcal{C}_{Y/X}$. This reduces us to the case discussed in the next paragraph.

In this paragraph Z and Y are effective Cartier divisors on X integral of dimension n, we have $\mathcal{N}'' = \mathcal{C}_{Y/X}$. In this case $c(Y \to X, \mathcal{C}_{Y/X}^\vee) \cap [X] = [Y]_{n-1}$ by Lemma 54.5. Thus we have to prove that $c(Z \to X, \mathcal{N}) \cap [X] = c(Z \to Y, \mathcal{N}') \cap [Y]_{n-1}$. Denote N and N' the vector bundles over Z associated to $\mathcal{N}$ and $\mathcal{N}'$. Consider the commutative diagram

$$\begin{tikzcd} N' \arrow[r, "i"'] & N \arrow[r] & (C_Y X) \times_Y Z \\ C_Z Y \arrow[u] \arrow[r] & C_Z X \arrow[u] \end{tikzcd}$$

of cones and vector bundles over Z. Observe that N' is a relative effective Cartier divisor in N over Z and that

$$\begin{tikzcd} N' \arrow[r, "i"'] \arrow[d] & N \arrow[d] \\ Z \arrow[r, "o"] & (C_Y X) \times_Y Z \end{tikzcd}$$

is cartesian where o is the zero section of the line bundle $C_Y X$ over Y. By Lemma 54.9 we have $o^*[C_Z X]_n = [C_Z Y]_{n-1}$ in

$$\mathrm{CH}_{n-1}(Y \times_{o, C_Y X} C_Z X) = \mathrm{CH}_{n-1}(Z \times_{o, (C_Y X) \times_Y Z} C_Z X)$$

By the cartesian property of the square above this implies that

$$i^*[C_Z X]_n = [C_Z Y]_{n-1}$$

in $\mathrm{CH}_{n-1}(N')$. Now observe that $\gamma = c(Z \to X, \mathcal{N}) \cap [X]$ and $\gamma' = c(Z \to Y, \mathcal{N}') \cap [Y]_{n-1}$ are characterized by $p^*\gamma = [C_Z X]_n$ in $\mathrm{CH}_n(N)$ and by $(p')^*\gamma' = [C_Z Y]_{n-1}$ in $\mathrm{CH}_{n-1}(N')$. Hence the proof is finished as $i^* \circ p^* = (p')^*$ by Lemma 31.1. $\square$

Remark 54.11 (Variant for immersions)**.** Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $i : Z \to X$ be an immersion of schemes. In this situation

1. the conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X is defined (Morphisms, Definition 31.1),
2. we say a pair consisting of a finite locally free $\mathcal{O}_Z$-module $\mathcal{N}$ and a surjection $\sigma : \mathcal{N}^\vee \to \mathcal{C}_{Z/X}$ is a virtual normal bundle for the immersion $Z \to X$,
3. choose an open subscheme $U \subset X$ such that $Z \to X$ factors through a closed immersion $Z \to U$ and set $c(Z \to X, \mathcal{N}) = c(Z \to U, \mathcal{N}) \circ (U \to X)^*$.

The bivariant class $c(Z \to X, \mathcal{N})$ does not depend on the choice of the open subscheme U. All of the lemmas have immediate counterparts for this slightly more general construction. We omit the details.

55. Calculating some classes

To get further we need to compute the values of some of the classes we've constructed above.

Lemma 55.1. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Then*

$$\prod\nolimits_{n=0,\ldots,r} c(\wedge^n \mathcal{E})^{(-1)^n} = 1 - (r-1)! c_r(\mathcal{E}) + \ldots$$

Proof. By the splitting principle we can turn this into a calculation in the polynomial ring on the Chern roots $x_1, \ldots, x_r$ of $\mathcal{E}$. See Section 43. Observe that

$$c(\wedge^n \mathcal{E}) = \prod\nolimits_{1 \leq i_1 < \ldots < i_n \leq r} (1 + x_{i_1} + \ldots + x_{i_n})$$

Thus the logarithm of the left hand side of the equation in the lemma is

$$-\sum\nolimits_{p \geq 1} \sum\nolimits_{n=0}^{r} \sum\nolimits_{1 \leq i_1 < \ldots < i_n \leq r} \frac{(-1)^{p+n}}{p} (x_{i_1} + \ldots + x_{i_n})^p$$

Please notice the minus sign in front. However, we have

$$\sum\nolimits_{p \geq 0} \sum\nolimits_{n=0}^{r} \sum\nolimits_{1 \leq i_1 < \ldots < i_n \leq r} \frac{(-1)^{p+n}}{p!} (x_{i_1} + \ldots + x_{i_n})^p = \prod (1 - e^{-x_i})$$

Hence we see that the first nonzero term in our Chern class is in degree r and equal to the predicted value. $\square$

Lemma 55.2. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{C}$ be a locally free $\mathcal{O}_X$-module of rank r. Consider the morphisms*

$$X = \underline{Proj}_X(\mathcal{O}_X[T]) \xrightarrow{i} E = \underline{Proj}_X(Sym^*(\mathcal{C})[T]) \xrightarrow{\pi} X$$

Then $c_t(i_*\mathcal{O}_X) = 0$ for $t = 1, \ldots, r-1$ and in $A^0(C \to E)$ we have

$$p^* \circ \pi_* \circ c_r(i_*\mathcal{O}_X) = (-1)^{r-1}(r-1)! j^*$$

where $j : C \to E$ and $p : C \to X$ are the inclusion and structure morphism of the vector bundle $C = \underline{\mathrm{Spec}}(Sym^*(\mathcal{C}))$.

Proof. The canonical map $\pi^*\mathcal{C} \to \mathcal{O}_E(1)$ vanishes exactly along $i(X)$. Hence the Koszul complex on the map

$$\pi^*\mathcal{C} \otimes \mathcal{O}_E(-1) \to \mathcal{O}_E$$

is a resolution of $i_*\mathcal{O}_X$. In particular we see that $i_*\mathcal{O}_X$ is a perfect object of $D(\mathcal{O}_E)$ whose Chern classes are defined. The vanishing of $c_t(i_*\mathcal{O}_X)$ for $t = 1, \ldots, t-1$ follows from Lemma 55.1. This lemma also gives

$$c_r(i_*\mathcal{O}_X) = -(r-1)!c_r(\pi^*\mathcal{C} \otimes \mathcal{O}_E(-1))$$

On the other hand, by Lemma 43.3 we have

$$c_r(\pi^*\mathcal{C} \otimes \mathcal{O}_E(-1)) = (-1)^r c_r(\pi^*\mathcal{C}^\vee \otimes \mathcal{O}_E(1))$$

and $\pi^*\mathcal{C}^\vee \otimes \mathcal{O}_E(1)$ has a section s vanishing exactly along $i(X)$.

After replacing X by a scheme locally of finite type over X, it suffices to prove that both sides of the equality have the same effect on an element $\alpha \in \mathrm{CH}_*(E)$. Since $C \to X$ is a vector bundle, every cycle class on C is of the form $p^*\beta$ for some $\beta \in \mathrm{CH}_*(X)$ (Lemma 36.3). Hence by Lemma 19.3 we can write $\alpha = \pi^*\beta + \gamma$ where γ is supported on $E \setminus C$. Using the equalities above it suffices to show that

$$p^*(\pi_*(c_r(\pi^*\mathcal{C}^\vee \otimes \mathcal{O}_E(1)) \cap [W])) = j^*[W]$$

when $W \subset E$ is an integral closed subscheme which is either (a) disjoint from C or (b) is of the form $W = \pi^{-1}Y$ for some integral closed subscheme $Y \subset X$. Using the section s and Lemma 44.1 we find in case (a) $c_r(\pi^*\mathcal{C}^\vee \otimes \mathcal{O}_E(1)) \cap [W] = 0$ and in case (b) $c_r(\pi^*\mathcal{C}^\vee \otimes \mathcal{O}_E(1)) \cap [W] = [i(Y)]$. The result follows easily from this; details omitted. □

Lemma 55.3. *Let (S, δ) be as in Situation 7.1. Let $i : Z \to X$ be a regular closed immersion of codimension r between schemes locally of finite type over S. Let $\mathcal{N} = \mathcal{C}_{Z/X}^\vee$ be the normal sheaf. If X is quasi-compact (or has quasi-compact irreducible components), then $c_t(Z \to X, i_*\mathcal{O}_Z) = 0$ for $t = 1, \ldots, r-1$ and*

$$c_r(Z \to X, i_*\mathcal{O}_Z) = (-1)^{r-1}(r-1)!c(Z \to X, \mathcal{N}) \quad in \quad A^r(Z \to X)$$

where $c_t(Z \to X, i_*\mathcal{O}_Z)$ is the localized Chern class of Definition 50.3.

Proof. For any $x \in Z$ we can choose an affine open neighbourhood $\mathrm{Spec}(A) \subset X$ such that $Z \cap \mathrm{Spec}(A) = V(f_1, \ldots, f_r)$ where $f_1, \ldots, f_r \in A$ is a regular sequence. See Divisors, Definition 21.1 and Lemma 20.8. Then we see that the Koszul complex on $f_1, \ldots, f_r$ is a resolution of $A/(f_1, \ldots, f_r)$ for example by More on Algebra, Lemma 30.2. Hence $A/(f_1, \ldots, f_r)$ is perfect as an A-module. It follows that $F = i_*\mathcal{O}_Z$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to $X \setminus Z$ is zero. The assumption that X is quasi-compact (or has quasi-compact irreducible components) means that the localized Chern classes $c_t(Z \to X, i_*\mathcal{O}_Z)$ are defined, see Situation 50.1 and Definition 50.3. All in all we conclude that the statement makes sense.

Denote $b : W \to \mathbf{P}^1_X$ the blowing up in $\infty(Z)$ as in Section 53. By (8) we have a closed immersion

$$i' : \mathbf{P}^1_Z \longrightarrow W$$

We claim that $Q = i'_*\mathcal{O}_{\mathbf{P}^1_Z}$ is a perfect object of $D(\mathcal{O}_W)$ and that F and Q satisfy the assumptions of Lemma 51.2.

Assume the claim. The output of Lemma 51.2 is that we have

$$c_p(Z \to X, F) = c'_p(Q) = (E \to Z)_* \circ c'_p(Q|_E) \circ C$$

for all $p \geq 1$. Observe that $Q|_E$ is equal to the pushforward of the structure sheaf of Z via the morphism $Z \to E$ which is the base change of i' by ∞. Thus the vanishing of $c_t(Z \to X, F)$ for $1 \leq t \leq r-1$ by Lemma 55.2 applied to $E \to Z$. Because $\mathcal{C}_{Z/X} = \mathcal{N}^\vee$ is locally free the bivariant class $c(Z \to X, \mathcal{N})$ is characterized by the relation

$$j^* \circ C = p^* \circ c(Z \to X, \mathcal{N})$$

where $j : C_Z X \to W_\infty$ and $p : C_Z X \to Z$ are the given maps. (Recall $C \in A^0(W_\infty \to X)$ is the class of Lemma 48.1.) Thus the displayed equation in the statement of the lemma follows from the corresponding equation in Lemma 55.2.

Proof of the claim. Let A and $f_1, \ldots, f_r$ be as above. Consider the affine open $\mathrm{Spec}(A[s]) \subset \mathbf{P}^1_X$ as in Section 53. Recall that $s = 0$ defines $(\mathbf{P}^1_X)_\infty$ over this open. Hence over $\mathrm{Spec}(A[s])$ we are blowing up in the ideal generated by the regular sequence $s, f_1, \ldots, f_r$. By More on Algebra, Lemma 31.2 the $r+1$ affine charts are global complete intersections over $A[s]$. The chart corresponding to the affine blowup algebra

$$A[s][f_1/s, \ldots, f_r/s] = A[s, y_1, \ldots, y_r]/(sy_i - f_i)$$

contains $i'(Z \cap \mathrm{Spec}(A))$ as the closed subscheme cut out by $y_1, \ldots, y_r$. Since $y_1, \ldots, y_r, sy_1 - f_1, \ldots, sy_r - f_r$ is a regular sequence in the polynomial ring $A[s, y_1, \ldots, y_r]$ we find that i' is a regular immersion. Some details omitted. As above we conclude that $Q = i'_* \mathcal{O}_{\mathbf{P}^1_Z}$ is a perfect object of $D(\mathcal{O}_W)$. All the other assumptions on F and Q in Lemma 51.2 (and Lemma 49.1) are immediately verified. $\square$

Lemma 55.4. *In the situation of Lemma 55.3 say $\dim_\delta(X) = n$. Then we have*

1. $c_t(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = 0$ *for* $t = 1, \ldots, r-1$,
2. $c_r(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = (-1)^{r-1}(r-1)![Z]_{n-r}$,
3. $ch_t(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = 0$ *for* $t = 0, \ldots, r-1$, *and*
4. $ch_r(Z \to X, i_*\mathcal{O}_Z) \cap [X]_n = [Z]_{n-r}$.

Proof. Parts (1) and (2) follow immediately from Lemma 55.3 combined with Lemma 54.5. Then we deduce parts (3) and (4) using the relationship between $ch_p = (1/p!)P_p$ and c_p given in Lemma 52.1. (Namely, $(-1)^{r-1}(r-1)!ch_r = c_r$ provided $c_1 = c_2 = \ldots = c_{r-1} = 0$.) $\square$

56. An Adams operator

We do the minimal amount of work to define the second adams operator. Let X be a scheme. Recall that *Vect*(X) denotes the category of finite locally free $\mathcal{O}_X$-modules. Moreover, recall that we have constructed a zeroth K-group $K_0(\mathit{Vect}(X))$ associated to this category in Derived Categories of Schemes, Section 38. Finally, $K_0(\mathit{Vect}(X))$ is a ring, see Derived Categories of Schemes, Remark 38.6.

Lemma 56.1. *Let X be a scheme. There is a ring map*

$$\psi^2 : K_0(\mathit{Vect}(X)) \longrightarrow K_0(\mathit{Vect}(X))$$

which sends $[\mathcal{L}]$ to $[\mathcal{L}^{\otimes 2}]$ when $\mathcal{L}$ is invertible and is compatible with pullbacks.

Proof. Let X be a scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. We will consider the element

$$\psi^2(\mathcal{E}) = [\text{Sym}^2(\mathcal{E})] - [\wedge^2(\mathcal{E})]$$

of $K_0(\textit{Vect}(X))$.

Let X be a scheme and consider a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

of finite locally free $\mathcal{O}_X$-modules. Let us think of this as a filtration on $\mathcal{F}$ with 2 steps. The induced filtration on $\text{Sym}^2(\mathcal{F})$ has 3 steps with graded pieces $\text{Sym}^2(\mathcal{E})$, $\mathcal{E} \otimes \mathcal{F}$, and $\text{Sym}^2(\mathcal{G})$. Hence

$$[\text{Sym}^2(\mathcal{F})] = [\text{Sym}^2(\mathcal{E})] + [\mathcal{E} \otimes \mathcal{F}] + [\text{Sym}^2(\mathcal{G})]$$

In exactly the same manner one shows that

$$[\wedge^2(\mathcal{F})] = [\wedge^2(\mathcal{E})] + [\mathcal{E} \otimes \mathcal{F}] + [\wedge^2(\mathcal{G})]$$

Thus we see that $\psi^2(\mathcal{F}) = \psi^2(\mathcal{E}) + \psi^2(\mathcal{G})$. We conclude that we obtain a well defined additive map $\psi^2 : K_0(\textit{Vect}(X)) \to K_0(\textit{Vect}(X))$.

It is clear that this map commutes with pullbacks.

We still have to show that ψ^2 is a ring map. Let X be a scheme and let $\mathcal{E}$ and $\mathcal{F}$ be finite locally free $\mathcal{O}_X$-modules. Observe that there is a short exact sequence

$$0 \to \wedge^2(\mathcal{E}) \otimes \wedge^2(\mathcal{F}) \to \text{Sym}^2(\mathcal{E} \otimes \mathcal{F}) \to \text{Sym}^2(\mathcal{E}) \otimes \text{Sym}^2(\mathcal{F}) \to 0$$

where the first map sends $(e \wedge e') \otimes (f \wedge f')$ to $(e \otimes f)(e' \otimes f') - (e' \otimes f)(e \otimes f')$ and the second map sends $(e \otimes f)(e' \otimes f')$ to $ee' \otimes ff'$. Similarly, there is a short exact sequence

$$0 \to \text{Sym}^2(\mathcal{E}) \otimes \wedge^2(\mathcal{F}) \to \wedge^2(\mathcal{E} \otimes \mathcal{F}) \to \wedge^2(\mathcal{E}) \otimes \text{Sym}^2(\mathcal{F}) \to 0$$

where the first map sends $ee' \otimes f \wedge f'$ to $(e \otimes f) \wedge (e' \otimes f') + (e' \otimes f) \wedge (e \otimes f')$ and the second map sends $(e \otimes f) \wedge (e' \otimes f')$ to $(e \wedge e') \otimes (ff')$. As above this proves the map ψ^2 is multiplicative. Since it is clear that $\psi^2(1) = 1$ this concludes the proof. □

Remark 56.2. Let X be a scheme such that 2 is invertible on X. Then the Adams operator ψ^2 can be defined on the K-group $K_0(X) = K_0(D_{perf}(\mathcal{O}_X))$ (Derived Categories of Schemes, Definition 38.2) in a straightforward manner. Namely, given a perfect complex L on X we get an action of the group $\{\pm 1\}$ on $L \otimes^{\mathbf{L}} L$ by switching the factors. Then we can set

$$\psi^2(L) = [(L \otimes^{\mathbf{L}} L)^+] - [(L \otimes^{\mathbf{L}} L)^-]$$

where $(-)^+$ denotes taking invariants and $(-)^-$ denotes taking anti-invariants (suitably defined). Using exactness of taking invariants and anti-invariants one can argue similarly to the proof of Lemma 56.1 to show that this is well defined. When 2 is not invertible on X the situation is a good deal more complicated and another approach has to be used.

Lemma 56.3. *Let X be a scheme. There is a ring map $\psi^{-1} : K_0(\textit{Vect}(X)) \to K_0(\textit{Vect}(X))$ which sends $[\mathcal{E}]$ to $[\mathcal{E}^\vee]$ when $\mathcal{E}$ is finite locally free and is compatible with pullbacks.*

Proof. The only thing to check is that taking duals is compatible with short exact sequences and with pullbacks. This is clear. $\square$

Remark 56.4. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. The Chern class map defines a canonical map

$$c : K_0(\mathit{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(X)$$

by sending a generator $[\mathcal{E}]$ on the left hand side to $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \ldots$ and extending multiplicatively. Thus $-[\mathcal{E}]$ is sent to the formal inverse $c(\mathcal{E})^{-1}$ which is why we have the infinite product on the right hand side. This is well defined by Lemma 40.3.

Remark 56.5. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. The Chern character map defines a canonical ring map

$$ch : K_0(\mathit{Vect}(X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(X) \otimes \mathbf{Q}$$

by sending a generator $[\mathcal{E}]$ on the left hand side to $ch(\mathcal{E})$ and extending additively. This is well defined by Lemma 45.2 and a ring homomorphism by Lemma 45.3.

Lemma 56.6. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. If ψ^2 is as in Lemma 56.1 and c and ch are as in Remarks 56.4 and 56.5 then we have $c_i(\psi^2(\alpha)) = 2^i c_i(\alpha)$ and $ch_i(\psi^2(\alpha)) = 2^i ch_i(\alpha)$ for all $\alpha \in K_0(\mathit{Vect}(X))$.*

Proof. Observe that the map $\prod_{i \geq 0} A^i(X) \to \prod_{i \geq 0} A^i(X)$ multiplying by 2^i on $A^i(X)$ is a ring map. Hence, since ψ^2 is also a ring map, it suffices to prove the formulas for additive generators of $K_0(\mathit{Vect}(X))$. Thus we may assume $\alpha = [\mathcal{E}]$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. By construction of the Chern classes of $\mathcal{E}$ we immediately reduce to the case where $\mathcal{E}$ has constant rank r, see Remark 38.10. In this case, we can choose a projective smooth morphism $p : P \to X$ such that restriction $A^*(X) \to A^*(P)$ is injective and such that $p^*\mathcal{E}$ has a finite filtration whose graded parts are invertible $\mathcal{O}_P$-modules $\mathcal{L}_j$, see Lemma 43.1. Then $[p^*\mathcal{E}] = \sum [\mathcal{L}_j]$ and hence $\psi^2([p^{\mathcal{E}}]) = \sum [\mathcal{L}_j^{\otimes 2}]$ by definition of ψ^2. Setting $x_j = c_1(\mathcal{L}_j)$ we have

$$c(\alpha) = \prod (1 + x_j) \quad \text{and} \quad c(\psi^2(\alpha)) = \prod (1 + 2x_j)$$

in $\prod A^i(P)$ and we have

$$ch(\alpha) = \sum \exp(x_j) \quad \text{and} \quad ch(\psi^2(\alpha)) = \sum \exp(2x_j)$$

in $\prod A^i(P)$. From these formulas the desired result follows. $\square$

Remark 56.7. Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Consider the strictly full, saturated, triangulated subcategory

$$D_{Z,perf}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

consisting of perfect complexes of $\mathcal{O}_X$-modules whose cohomology sheaves are set-theoretically supported on Z. Denote $\mathit{Coh}_Z(X) \subset \mathit{Coh}(X)$ the Serre subcategory of coherent $\mathcal{O}_X$-modules whose set theoretic support is contained in Z. Observe that given $E \in D_{Z,perf}(\mathcal{O}_X)$ Zariski locally on X only a finite number of the cohomology sheaves $H^i(E)$ are nonzero (and they are all settheoretically supported on Z). Hence we can define

$$K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow K_0(\mathit{Coh}_Z(X)) = K'_0(Z)$$

(equality by Lemma 23.6) by the rule

$$E \longmapsto [\bigoplus\nolimits_{i \in \mathbf{Z}} H^{2i}(E)] - [\bigoplus\nolimits_{i \in \mathbf{Z}} H^{2i+1}(E)]$$

This works because given a distinguished triangle in $D_{Z,perf}(\mathcal{O}_X)$ we have a long exact sequence of cohomology sheaves.

Remark 56.8. Let X, Z, $D_{Z,perf}(\mathcal{O}_X)$ be as in Remark 56.7. Assume X is regular. Then there is a canonical map

$$K_0(\mathit{Coh}(Z)) \longrightarrow K_0(D_{Z,perf}(\mathcal{O}_X))$$

defined as follows. For any coherent $\mathcal{O}_Z$-module $\mathcal{F}$ denote $\mathcal{F}[0]$ the object of $D(\mathcal{O}_X)$ which has $\mathcal{F}$ in degree 0 and is zero in other degrees. Then $\mathcal{F}[0]$ is a perfect complex on X by Derived Categories of Schemes, Lemma 11.8. Hence $\mathcal{F}[0]$ is an object of $D_{Z,perf}(\mathcal{O}_X)$. On the other hand, given a short exact sequence $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of coherent $\mathcal{O}_Z$-modules we obtain a distinguished triangle $\mathcal{F}[0] \to \mathcal{F}'[0] \to \mathcal{F}''[0] \to \mathcal{F}[1]$, see Derived Categories, Section 12. This shows that we obtain a map $K_0(\mathit{Coh}(Z)) \to K_0(D_{Z,perf}(\mathcal{O}_X))$ by sending $[\mathcal{F}]$ to $[\mathcal{F}[0]]$ with apologies for the horrendous notation.

Lemma 56.9. *Let X be a Noetherian regular scheme. Let $Z \subset X$ be a closed subscheme. The maps constructed in Remarks 56.7 and 56.8 are mutually inverse and we get $K'_0(Z) = K_0(D_{Z,perf}(\mathcal{O}_X))$.*

Proof. Clearly the composition

$$K_0(\mathit{Coh}(Z)) \longrightarrow K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow K_0(\mathit{Coh}(Z))$$

is the identity map. Thus it suffices to show the first arrow is surjective. Let E be an object of $D_{Z,perf}(\mathcal{O}_X)$. Recall that $D_{perf}(\mathcal{O}_X) = D^b_{Coh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 11.8. Hence the cohomologies $H^i(E)$ are coherent, can be viewed as objects of $D_{Z,perf}(\mathcal{O}_X)$, and only a finite number are nonzero. Using the distinguished triangles of canonical truncations the reader sees that

$$[E] = \sum (-1)^i [H^i(E)[0]]$$

in $K_0(D_{Z,perf}(\mathcal{O}_X))$. Then it suffices to show that $[\mathcal{F}[0]]$ is in the image of the map for any coherent $\mathcal{O}_X$-module set theoretically supported on Z. Since we can find a finite filtration on $\mathcal{F}$ whose subquotients are $\mathcal{O}_Z$-modules, the proof is complete. $\square$

Remark 56.10. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme and let $D_{Z,perf}(\mathcal{O}_X)$ be as in Remark 56.7. If X is quasi-compact (or more generally the irreducible components of X are quasi-compact), then the localized Chern classes define a canonical map

$$c(Z \to X, -) : K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow A^0(X) \times \prod\nolimits_{i \geq 1} A^i(Z \to X)$$

by sending a generator $[E]$ on the left hand side to

$$c(Z \to X, E) = 1 + c_1(Z \to X, E) + c_2(Z \to X, E) + \ldots$$

and extending multiplicatively (with product on the right hand side as in Remark 34.7). The quasi-compactness condition on X guarantees that the localized chern classes are defined (Situation 50.1 and Definition 50.3) and that these localized chern classes convert distinguished triangles into the corresponding products in the bivariant chow rings (Lemma 52.4).

Remark 56.11. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme and let $D_{Z,perf}(\mathcal{O}_X)$ be as in Remark 56.7. If the irreducible components of X are quasi-compact, then the localized Chern character defines a canonical additive and multiplicative map

$$ch(Z \to X, -) : K_0(D_{Z,perf}(\mathcal{O}_X)) \longrightarrow \prod\nolimits_{i \geq 0} A^i(Z \to X) \otimes \mathbf{Q}$$

by sending a generator $[E]$ on the left hand side to $ch(Z \to X, E)$ and extending additively. Namely, the condition on the irreducible components of X guarantees that the localized chern character is defined (Situation 50.1 and Definition 50.3) and that these localized chern characters convert distinguished triangles into the corresponding sums in the bivariant chow rings (Lemma 52.5). The multiplication on $K_0(D_{Z,perf}(X))$ is defined using derived tensor product (Derived Categories of Schemes, Remark 38.9) hence $ch(Z \to X, \alpha\beta) = ch(Z \to X, \alpha) ch(Z \to X, \beta)$ by Lemma 52.6. If X is quasi-compact, then the map $ch(Z \to X, -)$ has image contained in $A^*(Z \to X) \otimes \mathbf{Q}$; we omit the details.

Remark 56.12. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S and assume X is quasi-compact (or more generally the irreducible components of X are quasi-compact). With $Z = X$ and notation as in Remarks 56.10 and 56.11 we have $D_{Z,perf}(\mathcal{O}_X) = D_{perf}(\mathcal{O}_X)$ and we see that

$$K_0(D_{Z,perf}(\mathcal{O}_X)) = K_0(D_{perf}(\mathcal{O}_X)) = K_0(X)$$

see Derived Categories of Schemes, Definition 38.2. Hence we get

$$c : K_0(X) \to \prod A^i(X) \quad \text{and} \quad ch : K_0(X) \to \prod A^i(X) \otimes \mathbf{Q}$$

as a special case of Remarks 56.10 and 56.11. Of course, instead we could have just directly used Definition 46.3 and Lemmas 46.7 and 46.11 to construct these maps (as this immediately seen to produce the same classes). Recall that there is a canonical map $K_0(\mathit{Vect}(X)) \to K_0(X)$ which sends a finite locally free module to itself viewed as a perfect complex (placed in degree 0), see Derived Categories of Schemes, Section 38. Then the diagram

$$\begin{tikzcd}
K_0((\mathit{Vect}(X)) \arrow[rr] \arrow[dr, "c"'] & & K_0(D_{perf}(\mathcal{O}_X)) = K_0(X) \arrow[dl, "c"] \\
& \prod A^i(X) &
\end{tikzcd}$$

commutes where the south-east arrow is the one constructed in Remark 56.4. Similarly, the diagram

$$\begin{tikzcd}
K_0((\mathit{Vect}(X)) \arrow[rr] \arrow[dr, "ch"'] & & K_0(D_{perf}(\mathcal{O}_X)) = K_0(X) \arrow[dl, "ch"] \\
& \prod A^i(X) \otimes \mathbf{Q} &
\end{tikzcd}$$

commutes where the south-east arrow is the one constructed in Remark 56.5.

57. Chow groups and K-groups revisited

This section is the continuation of Section 23. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. The K-group $K'_0(X) = K_0(Coh(X))$ of coherent sheaves on X has a canonical increasing filtration

$$F_k K'_0(X) = \mathrm{Im}\left(K_0(Coh_{\leq k}(X)) \to K_0(Coh(X)\right)$$

This is called the filtration by dimension of supports. Observe that

$$\mathrm{gr}_k K'_0(X) \subset K'_0(X)/F_{k-1}K'_0(X) = K_0(Coh(X)/Coh_{\leq k-1}(X))$$

where the equality holds by Homology, Lemma 11.3. The discussion in Remark 23.5 shows that there are canonical maps

$$\mathrm{CH}_k(X) \longrightarrow \mathrm{gr}_k K'_0(X)$$

defined by sending the class of an integral closed subscheme $Z \subset X$ of δ-dimension k to the class of $[\mathcal{O}_Z]$ on the right hand side.

Proposition 57.1. *Let (S, δ) be as in Situation 7.1. Assume given a closed immersion $X \to Y$ of schemes locally of finite type over S with Y regular and quasi-compact. Then the composition*

$$K'_0(X) \to K_0(D_{X,perf}(\mathcal{O}_Y)) \to A^*(X \to Y) \otimes \mathbf{Q} \to \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

of the map $\mathcal{F} \mapsto \mathcal{F}[0]$ from Remark 56.8, the map $ch(X \to Y, -)$ from Remark 56.11, and the map $c \mapsto c \cap [Y]$ induces an isomorphism

$$K'_0(X) \otimes \mathbf{Q} \longrightarrow \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

which depends on the choice of Y. Moreover, the canonical map

$$\mathrm{CH}_k(X) \otimes \mathbf{Q} \longrightarrow gr_k K'_0(X) \otimes \mathbf{Q}$$

(see above) is an isomorphism of $\mathbf{Q}$-vector spaces for all $k \in \mathbf{Z}$.

Proof. Since Y is regular, the construction in Remark 56.8 applies. Since Y is quasi-compact, the construction in Remark 56.11 applies. We have that Y is locally equidimensional (Lemma 42.1) and thus the "fundamental cycle" $[Y]$ is defined as an element of $\mathrm{CH}_*(Y)$, see Remark 42.2. Combining this with the map $\mathrm{CH}_k(X) \to \mathrm{gr}_k K'_0(X)$ constructed above we see that it suffices to prove

(1) If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module whose support has δ-dimension $\leq k$, then the composition above sends $[\mathcal{F}]$ into $\bigoplus_{k' \leq k} \mathrm{CH}_{k'}(X) \otimes \mathbf{Q}$.
(2) If $Z \subset X$ is an integral closed subscheme of δ-dimension k, then the composition above sends $[\mathcal{O}_Z]$ to an element whose degree k part is the class of $[Z]$ in $\mathrm{CH}_k(X) \otimes \mathbf{Q}$.

Namely, if this holds, then our maps induce maps $\mathrm{gr}_k K'_0(X) \otimes \mathbf{Q} \to CH_k(X) \otimes \mathbf{Q}$ which are inverse to the canonical maps $\mathrm{CH}_k(X) \otimes \mathbf{Q} \to \mathrm{gr}_k K'_0(X) \otimes \mathbf{Q}$ given above the proposition.

Given a coherent $\mathcal{O}_X$-module $\mathcal{F}$ the composition above sends $[\mathcal{F}]$ to

$$ch(X \to Y, \mathcal{F}[0]) \cap [Y] \in \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

If $\mathcal{F}$ is (set theoretically) supported on a closed subscheme $Z \subset X$, then we have

$$ch(X \to Y, \mathcal{F}[0]) = (Z \to X)_* \circ ch(Z \to Y, \mathcal{F}[0])$$

by Lemma 50.8. We conclude that in this case we end up in the image of $\mathrm{CH}_*(Z) \to \mathrm{CH}_*(X)$. Hence we get condition (1).

Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. The composition above sends $[\mathcal{O}_Z]$ to the element

$$ch(X \to Y, \mathcal{O}_Z[0]) \cap [Y] = (Z \to X)_* ch(Z \to Y, \mathcal{O}_Z[0]) \cap [Y]$$

by the same argument as above. Thus it suffices to prove that the degree k part of $ch(Z \to Y, \mathcal{O}_Z[0]) \cap [Y] \in \mathrm{CH}_*(Z) \otimes \mathbf{Q}$ is $[Z]$. Since $\mathrm{CH}_k(Z) = \mathbf{Z}$, in order to prove this we may replace Y by an open neighbourhood of the generic point ξ of Z. Since the maximal ideal of the regular local ring $\mathcal{O}_{X,\xi}$ is generated by a regular sequence (Algebra, Lemma 106.3) we may assume the ideal of Z is generated by a regular sequence, see Divisors, Lemma 20.8. Thus we deduce the result from Lemma 55.4. $\square$

58. Rational intersection products on regular schemes

We will show that $\mathrm{CH}_*(X) \otimes \mathbf{Q}$ has an intersection product if X is Noetherian, regular, finite dimensional, with affine diagonal. The basis for the construction is the following result (which is a corollary of the proposition in the previous section).

Lemma 58.1. *Let (S, δ) be as in Situation 7.1. Let X be a quasi-compact regular scheme of finite type over S with affine diagonal and $\delta_{X/S} : X \to \mathbf{Z}$ bounded. Then the composition*

$$K_0(\mathit{Vect}(X)) \otimes \mathbf{Q} \longrightarrow A^*(X) \otimes \mathbf{Q} \longrightarrow \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

of the map ch from Remark 56.5 and the map $c \mapsto c \cap [X]$ is an isomorphism.

Proof. We have $K_0'(X) = K_0(X) = K_0(\mathit{Vect}(X))$ by Derived Categories of Schemes, Lemmas 38.4, 36.8, and 38.5. By Remark 56.12 the composition given agrees with the map of Proposition 57.1 for $X = Y$. Thus the result follows from the proposition. $\square$

Let X, S, δ be as in Lemma 58.1. For simplicity let us work with cycles of a given codimension, see Section 42. Let $[X]$ be the fundamental cycle of X, see Remark 42.2. Pick $\alpha \in CH^i(X)$ and $\beta \in CH^j(X)$. By the lemma we can find a unique $\alpha' \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ with $ch(\alpha') \cap [X] = \alpha$. Of course this means that $ch_{i'}(\alpha') \cap [X] = 0$ if $i' \neq i$ and $ch_i(\alpha') \cap [X] = \alpha$. By Lemma 56.6 we see that $\alpha'' = 2^{-i}\psi^2(\alpha')$ is another solution. By uniqueness we get $\alpha'' = \alpha'$ and we conclude that $ch_{i'}(\alpha) = 0$ in $A^{i'}(X) \otimes \mathbf{Q}$ for $i' \neq i$. Then we can define

$$\alpha \cdot \beta = ch(\alpha') \cap \beta = ch_i(\alpha') \cap \beta$$

in $\mathrm{CH}^{i+j}(X) \otimes \mathbf{Q}$ by the property of α' we observed above. This is a symmetric pairing: namely, if we pick $\beta' \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ lifting β, then we get

$$\alpha \cdot \beta = ch(\alpha') \cap \beta = ch(\alpha') \cap ch(\beta') \cap [X]$$

and we know that Chern classes commute. The intersection product is associative for the same reason

$$(\alpha \cdot \beta) \cdot \gamma = ch(\alpha') \cap ch(\beta') \cap ch(\gamma') \cap [X]$$

because we know composition of bivariant classes is associative. Perhaps a better way to formulate this is as follows: there is a unique commutative, associative intersection product on $\mathrm{CH}^*(X)\otimes\mathbf{Q}$ compatible with grading such that the isomorphism $K_0(\mathit{Vect}(X)) \otimes \mathbf{Q} \to \mathrm{CH}^*(X) \otimes \mathbf{Q}$ is an isomorphism of rings.

59. Gysin maps for local complete intersection morphisms

Before reading this section, we suggest the reader read up on regular immersions (Divisors, Section 21) and local complete intersection morphisms (More on Morphisms, Section 62).

Let (S, δ) be as in Situation 7.1. Let $i : X \to Y$ be a regular immersion[9] of schemes locally of finite type over S. In particular, the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free (see Divisors, Lemma 21.5). Hence the normal sheaf

$$\mathcal{N}_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X)$$

is finite locally free as well and we have a surjection $\mathcal{N}_{X/Y}^\vee \to \mathcal{C}_{X/Y}$ (because an isomorphism is also a surjection). The construction in Section 54 gives us a canonical bivariant class

$$i^! = c(X \to Y, \mathcal{N}_{X/Y}) \in A^*(X \to Y)^\wedge$$

We need a couple of lemmas about this notion.

Lemma 59.1. *Let (S, δ) be as in Situation 7.1. Let $i : X \to Y$ and $j : Y \to Z$ be regular immersions of schemes locally of finite type over S. Then $j \circ i$ is a regular immersion and $(j \circ i)^! = i^! \circ j^!$.*

Proof. The first statement is Divisors, Lemma 21.7. By Divisors, Lemma 21.6 there is a short exact sequence

$$0 \to i^*(\mathcal{C}_{Y/Z}) \to \mathcal{C}_{X/Z} \to \mathcal{C}_{X/Y} \to 0$$

Thus the result by the more general Lemma 54.10. $\square$

Lemma 59.2. *Let (S, δ) be as in Situation 7.1. Let $p : P \to X$ be a smooth morphism of schemes locally of finite type over S and let $s : X \to P$ be a section. Then s is a regular immersion and $1 = s^! \circ p^*$ in $A^*(X)^\wedge$ where $p^* \in A^*(P \to X)^\wedge$ is the bivariant class of Lemma 33.2.*

Proof. The first statement is Divisors, Lemma 22.8. It suffices to show that $s^! \cap p^*[Z] = [Z]$ in $\mathrm{CH}_*(X)$ for any integral closed subscheme $Z \subset X$ as the assumptions are preserved by base change by $X' \to X$ locally of finite type. After replacing P by an open neighbourhood of $s(Z)$ we may assume $P \to X$ is smooth of fixed relative dimension r. Say $\dim_\delta(Z) = n$. Then every irreducible component of $p^{-1}(Z)$ has dimension $r + n$ and $p^*[Z]$ is given by $[p^{-1}(Z)]_{n+r}$. Observe that $s(X) \cap p^{-1}(Z) = s(Z)$ scheme theoretically. Hence by the same reference as used above $s(X) \cap p^{-1}(Z)$ is a closed subscheme regularly embedded in $\overline{p}^{-1}(Z)$ of codimension r. We conclude by Lemma 54.5. $\square$

[9]See Divisors, Definition 21.1. Observe that regular immersions are the same thing as Koszul-regular immersions or quasi-regular immersions for locally Noetherian schemes, see Divisors, Lemma 21.3. We will use this without further mention in this section.

Let (S, δ) be as in Situation 7.1. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "i"'] \arrow[dr, "f"'] & & P \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

of schemes locally of finite type over S such that g is smooth and i is a regular immersion. Combining the bivariant class $i^!$ discussed above with the bivariant class $g^* \in A^*(P \to Y)^\wedge$ of Lemma 33.2 we obtain

$$f^! = i^! \circ g^* \in A^*(X \to Y)$$

Observe that the morphism f is a local complete intersection morphism, see More on Morphisms, Definition 62.2. Conversely, if $f : X \to Y$ is a local complete intersection morphism of locally Noetherian schemes and $f = g \circ i$ with g smooth, then i is a regular immersion. We claim that our construction of $f^!$ only depends on the morphism f and not on the choice of factorization $f = g \circ i$.

Lemma 59.3. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. The bivariant class $f^!$ is independent of the choice of the factorization $f = g \circ i$ with g smooth (provided one exists).*

Proof. Given a second such factorization $f = g' \circ i'$ we can consider the smooth morphism $g'' : P \times_Y P' \to Y$, the immersion $i'' : X \to P \times_Y P'$ and the factorization $f = g'' \circ i''$. Thus we may assume that we have a diagram

$$\begin{tikzcd} & P' \arrow[d, "p"] \arrow[dr, "g'"] & \\ X \arrow[ur, "i'"] \arrow[r, "i"] & P \arrow[r, "g"] & Y \end{tikzcd}$$

where p is a smooth morphism. Then $(g')^* = p^* \circ g^*$ (Lemma 14.3) and hence it suffices to show that $i^! = (i')^! \circ p^*$ in $A^*(X \to P)$. Consider the commutative diagram

$$\begin{tikzcd} & X \times_P P' \arrow[r, "j"'] \arrow[d, "\overline{p}"] & P' \arrow[d, "p"] \\ X \arrow[ur, "s"] \arrow[r, "1"] & X \arrow[r, "i"] & P \end{tikzcd}$$

where $s = (1, i')$. Then s and j are regular immersions (by Divisors, Lemma 22.8 and Divisors, Lemma 21.4) and $i' = j \circ s$. By Lemma 59.1 we have $(i')^! = s^! \circ j^!$. Since the square is cartesian, the bivariant class $j^!$ is the restriction (Remark 33.5) of $i^!$ to P', see Lemma 54.2. Since bivariant classes commute with flat pullbacks we find $j^! \circ p^* = \overline{p}^* \circ i^!$. Thus it suffices to show that $s^! \circ \overline{p}^* = \text{id}$ which is done in Lemma 59.2. $\square$

Definition 59.4. Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. We say *the gysin map for f exists* if we can write $f = g \circ i$ with g smooth and i an immersion. In this case we define the *gysin map* $f^! = i^! \circ g^* \in A^*(X \to Y)$ as above.

It follows from the definition that for a regular immersion this agrees with the construction earlier and for a smooth morphism this agrees with flat pullback. In fact, this agreement holds for all syntomic morphisms.

Lemma 59.5. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. If the gysin map exists for f and f is flat, then $f^!$ is equal to the bivariant class of Lemma 33.2.*

Proof. Choose a factorization $f = g \circ i$ with $i : X \to P$ an immersion and $g : P \to Y$ smooth. Observe that for any morphism $Y' \to Y$ which is locally of finite type, the base changes of f', g', i' satisfy the same assumptions (see Morphisms, Lemmas 34.5 and 30.4 and More on Morphisms, Lemma 62.8). Thus we reduce to proving that $f^*[Y] = i^!(g^*[Y])$ in case Y is integral, see Lemma 35.3. Set $n = \dim_\delta(Y)$. After decomposing X and P into connected components we may assume f is flat of relative dimension r and g is smooth of relative dimension t. Then $f^*[Y] = [X]_{n+s}$ and $g^*[Y] = [P]_{n+t}$. On the other hand i is a regular immersion of codimension $t - s$. Thus $i^![P]_{n+t} = [X]_{n+s}$ (Lemma 54.5) and the proof is complete. $\square$

Lemma 59.6. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ and $g : Y \to Z$ be local complete intersection morphisms of schemes locally of finite type over S. Assume the gysin map exists for $g \circ f$ and g. Then the gysin map exists for f and $(g \circ f)^! = f^! \circ g^!$.*

Proof. Observe that $g \circ f$ is a local complete intersection morphism by More on Morphisms, Lemma 62.7 and hence the statement of the lemma makes sense. If $X \to P$ is an immersion of X into a scheme P smooth over Z then $X \to P \times_Z Y$ is an immersion of X into a scheme smooth over Y. This prove the first assertion of the lemma. Let $Y \to P'$ be an immersion of Y into a scheme P' smooth over Z. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & P \times_Z Y \arrow[r, "a"'] \arrow[dl, "p"] & P \times_Z P' \arrow[dl, "q"] \\ Y \arrow[r, "b"'] \arrow[d] & P' \arrow[dl] & \\ Z & & \end{tikzcd}$$

Here the horizontal arrows are regular immersions, the south-west arrows are smooth, and the square is cartesian. Whence $a^! \circ q^* = p^* \circ b^!$ as bivariant classes commute with flat pullback. Combining this fact with Lemmas 59.1 and 14.3 the reader finds the statement of the lemma holds true. Small detail omitted. $\square$

Lemma 59.7. *Let (S, δ) be as in Situation 7.1. Consider a commutative diagram*

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y'' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

of schemes locally of finite type over S with both square cartesian. Assume $f : X \to Y$ is a local complete intersection morphism such that the gysin map exists for f.

Let $c \in A^*(Y'' \to Y')$. Denote $res(f^!) \in A^*(X' \to Y')$ the restriction of $f^!$ to Y' (Remark 33.5). Then c and $res(f^!)$ commute (Remark 33.6).

Proof. Choose a factorization $f = g \circ i$ with g smooth and i an immersion. Since $f^! = i^! \circ g^!$ it suffices to prove the lemma for $g^!$ (which is given by flat pullback) and for $i^!$. The result for flat pullback is part of the definition of a bivariant class. The case of $i^!$ follows immediately from Lemma 54.8. $\square$

Lemma 59.8. *Let (S, δ) be as in Situation 7.1. Consider a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

of schemes locally of finite type over S. Assume

(1) *f is a local complete intersection morphism and the gysin map exists for f,*
(2) *X, X', Y, Y' satisfy the equivalent conditions of Lemma 42.1,*
(3) *for $x' \in X'$ with images x, y', and y in X, Y', and Y we have $n_{x'} - n_{y'} = n_x - n_y$ where $n_{x'}$, n_x, $n_{y'}$, and n_y are as in the lemma, and*
(4) *for every generic point $\xi \in X'$ the local ring $\mathcal{O}_{Y', f'(\xi)}$ is Cohen-Macaulay.*

Then $f^![Y'] = [X']$ where $[Y']$ and $[X']$ are as in Remark 42.2.

Proof. Recall that $n_{x'}$ is the common value of $\delta(\xi)$ where ξ is the generic point of an irreducible component passing through x'. Moreover, the functions $x' \mapsto n_{x'}$, $x \mapsto n_x$, $y' \mapsto n_{y'}$, and $y \mapsto n_y$ are locally constant. Let X'_n, X_n, Y'_n, and Y_n be the open and closed subscheme of X', X, Y', and Y where the function has value n. Recall that $[X'] = \sum [X'_n]_n$ and $[Y'] = \sum [Y'_n]_n$. Having said this, it is clear that to prove the lemma we may replace X' by one of its connected components and X, Y', Y by the connected component that it maps into. Then we know that X', X, Y', and Y are δ-equidimensional in the sense that each irreducible component has the same δ-dimension. Say n', n, m', and m is this common value for X', X, Y', and Y. The last assumption means that $n' - m' = n - m$.

Choose a factorization $f = g \circ i$ where $i : X \to P$ is an immersion and $g : P \to Y$ is smooth. As X is connected, we see that the relative dimension of $P \to Y$ at points of $i(X)$ is constant. Hence after replacing P by an open neighbourhood of $i(X)$, we may assume that $P \to Y$ has constant relative dimension and $i : X \to P$ is a closed immersion. Denote $g' : Y' \times_Y P \to Y'$ the base change of g and denote $i' : X' \to Y' \times_Y P$ the base change of i. It is clear that $g^*[Y] = [P]$ and $(g')^*[Y'] = [Y' \times_Y P]$. Finally, if $\xi' \in X'$ is a generic point, then $\mathcal{O}_{Y' \times_Y P, i'(\xi)}$ is Cohen-Macaulay. Namely, the local ring map $\mathcal{O}_{Y', f'(\xi)} \to \mathcal{O}_{Y' \times_Y P, i'(\xi)}$ is flat with regular fibre (see Algebra, Section 142), a regular local ring is Cohen-Macaulay (Algebra, Lemma 106.3), $\mathcal{O}_{Y', f'(\xi)}$ is Cohen-Macaulay by assumption (4) and we get what we want from Algebra, Lemma 163.3. Thus we reduce to the case discussed in the next paragraph.

Assume f is a regular closed immersion and X', X, Y', and Y are δ-equidimensional of δ-dimensions n', n, m', and m and $m' - n' = m - n$. In this case we obtain the result immediately from Lemma 54.6. $\square$

Remark 59.9. Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a local complete intersection morphism of schemes locally of finite type over S. Assume the gysin map exists for f. Then $f^! \circ c_i(\mathcal{E}) = c_i(f^*\mathcal{E}) \circ f^!$ and similarly for the Chern character, see Lemma 59.7. If X and Y satisfy the equivalent conditions of Lemma 42.1 and Y is Cohen-Macaulay (for example), then $f^![Y] = [X]$ by Lemma 59.8. In this case we also get $f^!(c_i(\mathcal{E}) \cap [Y]) = c_i(f^*\mathcal{E}) \cap [X]$ and similarly for the Chern character.

Lemma 59.10. *Let (S, δ) be as in Situation 7.1. Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes locally of finite type over S. Assume

1. *both f and f' are local complete intersection morphisms, and*
2. *the gysin map exists for f*

Then $\mathcal{C} = \mathrm{Ker}(H^{-1}((g')^* NL_{X/Y}) \to H^{-1}(NL_{X'/Y'}))$ is a finite locally free $\mathcal{O}_{X'}$-module, the gysin map exists for f', and we have

$$res(f^!) = c_{top}(\mathcal{C}^\vee) \circ (f')^!$$

in $A^*(X' \to Y')$.

Proof. The fact that $\mathcal{C}$ is finite locally free follows immediately from More on Algebra, Lemma 85.5. Choose a factorization $f = g \circ i$ with $g : P \to Y$ smooth and i an immersion. Then we can factor $f' = g' \circ i'$ where $g' : P' \to Y'$ and $i' : X' \to P'$ the base changes. Picture

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & P \arrow[r] & Y \end{tikzcd}$$

In particular, we see that the gysin map exists for f'. By More on Morphisms, Lemmas 13.13 we have

$$NL_{X/Y} = \left(\mathcal{C}_{X/P} \to i^*\Omega_{P/Y}\right)$$

where $\mathcal{C}_{X/P}$ is the conormal sheaf of the embedding i. Similarly for the primed version. We have $(g')^* i^*\Omega_{P/Y} = (i')^*\Omega_{P'/Y'}$ because $\Omega_{P/Y}$ pulls back to $\Omega_{P'/Y'}$ by Morphisms, Lemma 32.10. Also, recall that $(g')^*\mathcal{C}_{X/P} \to \mathcal{C}_{X'/P'}$ is surjective, see Morphisms, Lemma 31.4. We deduce that the sheaf $\mathcal{C}$ is canonicallly isomorphic to the kernel of the map $(g')^*\mathcal{C}_{X/P} \to \mathcal{C}_{X'/P'}$ of finite locally free modules. Recall that $i^!$ is defined using $\mathcal{N} = \mathcal{C}_{Z/X}^\vee$ and similarly for $(i')^!$. Thus we have

$$res(i^!) = c_{top}(\mathcal{C}^\vee) \circ (i')^!$$

in $A^*(X' \to P')$ by an application of Lemma 54.4. Since finally we have $f^! = i^! \circ g^*$, $(f')^! = (i')^! \circ (g')^*$, and $(g')^* = res(g^*)$ we conclude. $\square$

Lemma 59.11 (Blow up formula)**.** *Let (S, δ) be as in Situation 7.1. Let $i : Z \to X$ be a regular closed immersion of schemes locally of finite type over S. Let $b : X' \to$*

X be the blowing up with center Z. Picture

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"'] & X' \arrow[d, "b"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

Assume that the gysin map exists for b. Then we have

$$res(b^!) = c_{top}(\mathcal{F}^\vee) \circ \pi^*$$

in $A^*(E \to Z)$ where $\mathcal{F}$ is the kernel of the canonical map $\pi^*\mathcal{C}_{Z/X} \to \mathcal{C}_{E/X'}$.

Proof. Observe that the morphism b is a local complete intersection morphism by More on Algebra, Lemma 31.2 and hence the statement makes sense. Since $Z \to X$ is a regular immersion (and hence a fortiori quasi-regular) we see that $\mathcal{C}_{Z/X}$ is finite locally free and the map $\text{Sym}^*(\mathcal{C}_{Z/X}) \to \mathcal{C}_{Z/X,*}$ is an isomorphism, see Divisors, Lemma 21.5. Since $E = \text{Proj}(\mathcal{C}_{Z/X,*})$ we conclude that $E = \mathbf{P}(\mathcal{C}_{Z/X})$ is a projective space bundle over Z. Thus $E \to Z$ is smooth and certainly a local complete intersection morphism. Thus Lemma 59.10 applies and we see that

$$res(b^!) = c_{top}(\mathcal{C}^\vee) \circ \pi^!$$

with $\mathcal{C}$ as in the statement there. Of course $\pi^* = \pi^!$ by Lemma 59.5. It remains to show that $\mathcal{F}$ is equal to the kernel $\mathcal{C}$ of the map $H^{-1}(j^* NL_{X'/X}) \to H^{-1}(NL_{E/Z})$.

Since $E \to Z$ is smooth we have $H^{-1}(NL_{E/Z}) = 0$, see More on Morphisms, Lemma 13.7. Hence it suffices to show that $\mathcal{F}$ can be identified with $H^{-1}(j^* NL_{X'/X})$. By More on Morphisms, Lemmas 13.11 and 13.9 we have an exact sequence

$$0 \to H^{-1}(j^* NL_{X'/X}) \to H^{-1}(NL_{E/X}) \to \mathcal{C}_{E/X'} \to \ldots$$

By the same lemmas applied to $E \to Z \to X$ we obtain an isomorphism $\pi^*\mathcal{C}_{Z/X} = H^{-1}(\pi^* NL_{Z/X}) \to H^{-1}(NL_{E/X})$. Thus we conclude. $\square$

Lemma 59.12. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S such that both X and Y are quasi-compact, regular, have affine diagonal, and finite dimension. Then f is a local complete intersection morphism. Assume moreover the gysin map exists for f. Then*

$$f^!(\alpha \cdot \beta) = f^!\alpha \cdot f^!\beta$$

in $\text{CH}^*(X) \otimes \mathbf{Q}$ where the intersection product is as in Section 58.

Proof. The first statement follows from More on Morphisms, Lemma 62.11. Observe that $f^![Y] = [X]$, see Lemma 59.8. Write $\alpha = ch(\alpha') \cap [Y]$ and $\beta = ch(\beta') \cap [Y]$ where $\alpha', \beta' \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ as in Section 58. Setting $c = ch(\alpha')$ and $c' = ch(\beta')$

we find $\alpha \cdot \beta = c \cap c' \cap [Y]$ by construction. By Lemma 59.7 we know that $f^!$ commutes with both c and c'. Hence

$$\begin{aligned} f^!(\alpha \cdot \beta) &= f^!(c \cap c' \cap [Y]) \\ &= c \cap c' \cap f^![Y] \\ &= c \cap c' \cap [X] \\ &= (c \cap [X]) \cdot (c' \cap [X]) \\ &= (c \cap f^![Y]) \cdot (c' \cap f^![Y]) \\ &= f^!(\alpha) \cdot f^!(\beta) \end{aligned}$$

as desired. □

Lemma 59.13. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a morphism of schemes locally of finite type over S such that both X and Y are quasi-compact, regular, have affine diagonal, and finite dimension. Then f is a local complete intersection morphism. Assume moreover the gysin map exists for f and that f is proper. Then*

$$f_*(\alpha \cdot f^!\beta) = f_*\alpha \cdot \beta$$

in $\mathrm{CH}^*(Y) \otimes \mathbf{Q}$ where the intersection product is as in Section 58.

Proof. The first statement follows from More on Morphisms, Lemma 62.11. Observe that $f^![Y] = [X]$, see Lemma 59.8. Write $\alpha = ch(\alpha') \cap [X]$ and $\beta = ch(\beta') \cap [Y]$ $\alpha' \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ and $\beta' \in K_0(\mathit{Vect}(Y)) \otimes \mathbf{Q}$ as in Section 58. Set $c = ch(\alpha')$ and $c' = ch(\beta')$. We have

$$\begin{aligned} f_*(\alpha \cdot f^!\beta) &= f_*(c \cap f^!(c' \cap [Y]_e)) \\ &= f_*(c \cap c' \cap f^![Y]_e) \\ &= f_*(c \cap c' \cap [X]_d) \\ &= f_*(c' \cap c \cap [X]_d) \\ &= c' \cap f_*(c \cap [X]_d) \\ &= \beta \cdot f_*(\alpha) \end{aligned}$$

The first equality by the construction of the intersection product. By Lemma 59.7 we know that $f^!$ commutes with c'. The fact that Chern classes are in the center of the bivariant ring justifies switching the order of capping $[X]$ with c and c'. Commuting c' with f_* is allowed as c' is a bivariant class. The final equality is again the construction of the intersection product. □

60. Gysin maps for diagonals

Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ be a smooth morphism of schemes locally of finite type over S. Then the diagonal morphism $\Delta : X \longrightarrow X \times_Y X$ is a regular immersion, see More on Morphisms, Lemma 62.18. Thus we have the gysin map

$$\Delta^! \in A^*(X \to X \times_Y X)^\wedge$$

constructed in Section 59. If $X \to Y$ has constant relative dimension d, then $\Delta^! \in A^d(X \to X \times_Y X)$.

Lemma 60.1. *In the situation above we have $\Delta^! \circ pr_i^! = 1$ in $A^0(X)$.*

Proof. Observe that the projections $\mathrm{pr}_i : X \times_Y X \to X$ are smooth and hence we have gysin maps for these projections as well. Thus the lemma makes sense and is a special case of Lemma 59.6. $\square$

Proposition 60.2. *Let (S, δ) be as in Situation 7.1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes locally of finite type over S. If g is smooth of relative dimension d, then $A^p(X \to Y) = A^{p-d}(X \to Z)$.*

Proof. We will use that smooth morphisms are local complete intersection morphisms whose gysin maps exist (see Section 59). In particular we have $g^! \in A^{-d}(Y \to Z)$. Then we can send $c \in A^p(X \to Y)$ to $c \circ g^! \in A^{p-d}(X \to Z)$.

Conversely, let $c' \in A^{p-d}(X \to Z)$. Denote $res(c')$ the restriction (Remark 33.5) of c' by the morphism $Y \to Z$. Since the diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is cartesian we find $res(c') \in A^{p-d}(X \times_Z Y \to Y)$. Let $\Delta : Y \to Y \times_Z Y$ be the diagonal and denote $res(\Delta^!)$ the restriction of $\Delta^!$ to $X \times_Z Y$ by the morphism $X \times_Z Y \to Y \times_Z Y$. Since the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \times_Z Y \arrow[d] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd}$$

is cartesian we see that $res(\Delta^!) \in A^d(X \to X \times_Z Y)$. Combining these two restrictions we obtain

$$res(\Delta^!) \circ res(c') \in A^p(X \to Y)$$

Thus we have produced maps $A^p(X \to Y) \to A^{p-d}(X \to Z)$ and $A^{p-d}(X \to Z) \to A^p(X \to Y)$. To finish the proof we will show these maps are mutually inverse.

Let us start with $c \in A^p(X \to Y)$. Consider the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] & \\ X \times_Z Y \arrow[r] \arrow[d, "\mathrm{pr}_1"] & Y \times_Z Y \arrow[r, "p_2"'] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

whose squares are carteisan. The lower two square of this diagram show that $res(c \circ g^!) = res(c) \cap p_2^!$ where in this formula $res(c)$ means the restriction of c via p_1. Looking at the upper square of the diagram and using Lemma 59.7 we get $c \circ \Delta^! = res(\Delta^!) \circ res(c)$. We compute

$$\begin{aligned} res(\Delta^!) \circ res(c \circ g^!) &= res(\Delta^!) \circ res(c) \circ p_2^! \\ &= c \circ \Delta^! \circ p_2^! \\ &= c \end{aligned}$$

The final equality by Lemma 60.1.

Conversely, let us start with $c' \in A^{p-d}(X \to Z)$. Looking at the lower rectangle of the diagram above we find $res(c') \circ g^! = \mathrm{pr}_1^! \circ c'$. We compute

$$\begin{aligned} res(\Delta^!) \circ res(c') \circ g^! &= res(\Delta^!) \circ \mathrm{pr}_1^! \circ c' \\ &= c' \end{aligned}$$

The final equality holds because the left two squares of the diagram show that $\mathrm{id} = res(\Delta^! \circ p_1^!) = res(\Delta^!) \circ \mathrm{pr}_1^!$. This finishes the proof. $\square$

61. Exterior product

Let k be a field. In this section we work over $S = \mathrm{Spec}(k)$ with $\delta : S \to \mathbf{Z}$ defined by sending the unique point to 0, see Example 7.2.

Consider a cartesian square

$$\begin{tikzcd} X \times_k Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(k) = S \end{tikzcd}$$

of schemes locally of finite type over k. Then there is a canonical map

$$\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{n+m}(X \times_k Y)$$

which is uniquely determined by the following rule: given integral closed subschemes $X' \subset X$ and $Y' \subset Y$ of dimensions n and m we have

$$[X'] \times [Y'] = [X' \times_k Y']_{n+m}$$

in $\mathrm{CH}_{n+m}(X \times_k Y)$.

Lemma 61.1. *The map* $\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m}(X \times_k Y)$ *is well defined.*

Proof. A first remark is that if $\alpha = \sum n_i[X_i]$ and $\beta = \sum m_j[Y_j]$ with $X_i \subset X$ and $Y_j \subset Y$ locally finite families of integral closed subschemes of dimensions n and m, then $X_i \times_k Y_j$ is a locally finite collection of closed subschemes of $X \times_k Y$ of dimensions $n+m$ and we can indeed consider

$$\alpha \times \beta = \sum n_i m_j [X_i \times_k Y_j]_{n+m}$$

as a $(n+m)$-cycle on $X \times_k Y$. In this way we obtain an additive map $\times : Z_n(X) \otimes_{\mathbf{Z}} Z_m(Y) \to Z_{n+m}(X \times_k Y)$. The problem is to show that this procedure is compatible with rational equivalence.

Let $i : X' \to X$ be the inclusion morphism of an integral closed subscheme of dimension n. Then flat pullback along the morphism $p' : X' \to \mathrm{Spec}(k)$ is an element $(p')^* \in A^{-n}(X' \to \mathrm{Spec}(k))$ by Lemma 33.2 and hence $c' = i_* \circ (p')^* \in A^{-n}(X \to \mathrm{Spec}(k))$ by Lemma 33.4. This produces maps

$$c' \cap - : \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{m+n}(X \times_k Y)$$

which the reader easily sends $[Y']$ to $[X' \times_k Y']_{n+m}$ for any integral closed subscheme $Y' \subset Y$ of dimension m. Hence the construction $([X'], [Y']) \mapsto [X' \times_k Y']_{n+m}$ factors through rational equivalence in the second variable, i.e., gives a well defined map $Z_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m}(X \times_k Y)$. By symmetry the same is true for the other variable and we conclude. $\square$

Lemma 61.2. *Let k be a field. Let X be a scheme locally of finite type over k. Then we have a canonical identification*

$$A^p(X \to \mathrm{Spec}(k)) = \mathrm{CH}_{-p}(X)$$

for all $p \in \mathbf{Z}$.

Proof. Consider the element $[\mathrm{Spec}(k)] \in \mathrm{CH}_0(\mathrm{Spec}(k))$. We get a map $A^p(X \to \mathrm{Spec}(k)) \to \mathrm{CH}_{-p}(X)$ by sending c to $c \cap [\mathrm{Spec}(k)]$.

Conversely, suppose we have $\alpha \in \mathrm{CH}_{-p}(X)$. Then we can define $c_\alpha \in A^p(X \to \mathrm{Spec}(k))$ as follows: given $X' \to \mathrm{Spec}(k)$ and $\alpha' \in \mathrm{CH}_n(X')$ we let

$$c_\alpha \cap \alpha' = \alpha \times \alpha'$$

in $\mathrm{CH}_{n-p}(X \times_k X')$. To show that this is a bivariant class we write $\alpha = \sum n_i[X_i]$ as in Definition 8.1. Consider the composition

$$\coprod X_i \xrightarrow{g} X \to \mathrm{Spec}(k)$$

and denote $f : \coprod X_i \to \mathrm{Spec}(k)$ the composition. Then g is proper and f is flat of relative dimension $-p$. Pullback along f is a bivariant class $f^* \in A^p(\coprod X_i \to \mathrm{Spec}(k))$ by Lemma 33.2. Denote $\nu \in A^0(\coprod X_i)$ the bivariant class which multiplies a cycle by n_i on the ith component. Thus $\nu \circ f^* \in A^p(\coprod X_i \to X)$. Finally, we have a bivariant class

$$g_* \circ \nu \circ f^*$$

by Lemma 33.4. The reader easily verifies that c_α is equal to this class and hence is itself a bivariant class.

To finish the proof we have to show that the two constructions are mutually inverse. Since $c_\alpha \cap [\mathrm{Spec}(k)] = \alpha$ this is clear for one of the two directions. For the other, let $c \in A^p(X \to \mathrm{Spec}(k))$ and set $\alpha = c \cap [\mathrm{Spec}(k)]$. It suffices to prove that

$$c \cap [X'] = c_\alpha \cap [X']$$

when X' is an integral scheme locally of finite type over $\mathrm{Spec}(k)$, see Lemma 35.3. However, then $p' : X' \to \mathrm{Spec}(k)$ is flat of relative dimension $\dim(X')$ and hence $[X'] = (p')^*[\mathrm{Spec}(k)]$. Thus the fact that the bivariant classes c and c_α agree on $[\mathrm{Spec}(k)]$ implies they agree when capped against $[X']$ and the proof is complete. $\square$

Lemma 61.3. *Let k be a field. Let X be a scheme locally of finite type over k. Let $c \in A^p(X \to \mathrm{Spec}(k))$. Let $Y \to Z$ be a morphism of schemes locally of finite type over k. Let $c' \in A^q(Y \to Z)$. Then $c \circ c' = c' \circ c$ in $A^{p+q}(X \times_k Y \to Z)$.*

Proof. In the proof of Lemma 61.2 we have seen that c is given by a combination of proper pushforward, multiplying by integers over connected components, and flat pullback. Since c' commutes with each of these operations by definition of bivariant classes, we conclude. Some details omitted. $\square$

Remark 61.4. The upshot of Lemmas 61.2 and 61.3 is the following. Let k be a field. Let X be a scheme locally of finite type over k. Let $\alpha \in \mathrm{CH}_*(X)$. Let $Y \to Z$ be a morphism of schemes locally of finite type over k. Let $c' \in A^q(Y \to Z)$. Then

$$\alpha \times (c' \cap \beta) = c' \cap (\alpha \times \beta)$$

in $\mathrm{CH}_*(X \times_k Y)$ for any $\beta \in \mathrm{CH}_*(Z)$. Namely, this follows by taking $c = c_\alpha \in A^*(X \to \mathrm{Spec}(k))$ the bivariant class corresponding to α, see proof of Lemma 61.2.

Lemma 61.5. *Exterior product is associative. More precisely, let k be a field, let X, Y, Z be schemes locally of finite type over k, let $\alpha \in \mathrm{CH}_*(X)$, $\beta \in \mathrm{CH}_*(Y)$, $\gamma \in \mathrm{CH}_*(Z)$. Then $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(X \times_k Y \times_k Z)$.*

Proof. Omitted. Hint: associativity of fibre product of schemes. $\square$

62. Intersection products

Let k be a field. In this section we work over $S = \mathrm{Spec}(k)$ with $\delta : S \to \mathbf{Z}$ defined by sending the unique point to 0, see Example 7.2.

Let X be a smooth scheme over k. The bivariant class $\Delta^!$ of Section 60 allows us to define a kind of intersection product on chow groups of schemes locally of finite type over X. Namely, suppose that $Y \to X$ and $Z \to X$ are morphisms of schemes which are locally of finite type. Then observe that

$$Y \times_X Z = (Y \times_k Z) \times_{X \times_k X, \Delta} X$$

Hence we can consider the following sequence of maps

$$\mathrm{CH}_n(Y) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Z) \xrightarrow{\times} \mathrm{CH}_{n+m}(Y \times_k Z) \xrightarrow{\Delta^!} \mathrm{CH}_{n+m-*}(Y \times_X Z)$$

Here the first arrow is the exterior product constructed in Section 61 and the second arrow is the gysin map for the diagonal studied in Section 60. If X is equidimensional of dimension d, then we end up in $\mathrm{CH}_{n+m-d}(Y \times_X Z)$ and in general we can decompose into the parts lying over the open and closed subschemes of X where X has a given dimension. Given $\alpha \in \mathrm{CH}_*(Y)$ and $\beta \in \mathrm{CH}_*(Z)$ we will denote

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta) \in \mathrm{CH}_*(Y \times_X Z)$$

In the special case where $X = Y = Z$ we obtain a multiplication

$$\mathrm{CH}_*(X) \times \mathrm{CH}_*(X) \to \mathrm{CH}_*(X), \quad (\alpha, \beta) \mapsto \alpha \cdot \beta$$

which is called the *intersection product*. We observe that this product is clearly symmetric. Associativity follows from the next lemma.

Lemma 62.1. *The product defined above is associative. More precisely, let k be a field, let X be smooth over k, let Y, Z, W be schemes locally of finite type over X, let $\alpha \in \mathrm{CH}_*(Y)$, $\beta \in \mathrm{CH}_*(Z)$, $\gamma \in \mathrm{CH}_*(W)$. Then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ in $\mathrm{CH}_*(Y \times_X Z \times_X W)$.*

Proof. By Lemma 61.5 we have $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(Y \times_k Z \times_k W)$. Consider the closed immersions

$$\Delta_{12} : X \times_k X \longrightarrow X \times_k X \times_k X, \quad (x, x') \mapsto (x, x, x')$$

and

$$\Delta_{23} : X \times_k X \longrightarrow X \times_k X \times_k X, \quad (x, x') \mapsto (x, x', x')$$

Denote $\Delta_{12}^!$ and $\Delta_{23}^!$ the corresponding bivariant classes; observe that $\Delta_{12}^!$ is the restriction (Remark 33.5) of $\Delta^!$ to $X \times_k X \times_k X$ by the map pr_{12} and that $\Delta_{23}^!$ is the restriction of $\Delta^!$ to $X \times_k X \times_k X$ by the map pr_{23}. Thus clearly the restriction of $\Delta_{12}^!$ by Δ_{23} is $\Delta^!$ and the restriction of $\Delta_{23}^!$ by Δ_{12} is $\Delta^!$ too. Thus by Lemma 54.8 we have

$$\Delta^! \circ \Delta_{12}^! = \Delta^! \circ \Delta_{23}^!$$

Now we can prove the lemma by the following sequence of equalities:

$$\begin{aligned}(\alpha \cdot \beta) \cdot \gamma &= \Delta^!(\Delta^!(\alpha \times \beta) \times \gamma) \\ &= \Delta^!(\Delta^!_{12}((\alpha \times \beta) \times \gamma)) \\ &= \Delta^!(\Delta^!_{23}((\alpha \times \beta) \times \gamma)) \\ &= \Delta^!(\Delta^!_{23}(\alpha \times (\beta \times \gamma)) \\ &= \Delta^!(\alpha \times \Delta^!(\beta \times \gamma)) \\ &= \alpha \cdot (\beta \cdot \gamma)\end{aligned}$$

All equalities are clear from the above except perhaps for the second and penultimate one. The equation $\Delta^!_{23}(\alpha \times (\beta \times \gamma)) = \alpha \times \Delta^!(\beta \times \gamma)$ holds by Remark 61.4. Similarly for the second equation. □

Lemma 62.2. *Let k be a field. Let X be a smooth scheme over k, equidimensional of dimension d. The map*

$$A^p(X) \longrightarrow \mathrm{CH}_{d-p}(X), \quad c \longmapsto c \cap [X]_d$$

is an isomorphism. Via this isomorphism composition of bivariant classes turns into the intersection product defined above.

Proof. Denote $g : X \to \mathrm{Spec}(k)$ the structure morphism. The map is the composition of the isomorphisms

$$A^p(X) \to A^{p-d}(X \to \mathrm{Spec}(k)) \to \mathrm{CH}_{d-p}(X)$$

The first is the isomorphism $c \mapsto c \circ g^*$ of Proposition 60.2 and the second is the isomorphism $c \mapsto c \cap [\mathrm{Spec}(k)]$ of Lemma 61.2. From the proof of Lemma 61.2 we see that the inverse to the second arrow sends $\alpha \in \mathrm{CH}_{d-p}(X)$ to the bivariant class c_α which sends $\beta \in \mathrm{CH}_*(Y)$ for Y locally of finite type over k to $\alpha \times \beta$ in $\mathrm{CH}_*(X \times_k Y)$. From the proof of Proposition 60.2 we see the inverse to the first arrow in turn sends c_α to the bivariant class which sends $\beta \in \mathrm{CH}_*(Y)$ for $Y \to X$ locally of finite type to $\Delta^!(\alpha \times \beta) = \alpha \cdot \beta$. From this the final result of the lemma follows. □

Lemma 62.3. *Let k be a field. Let $f : X \to Y$ be a morphism of schemes smooth over k. Then the gysin map exists for f and $f^!(\alpha \cdot \beta) = f^!\alpha \cdot f^!\beta$.*

Proof. Observe that $X \to X \times_k Y$ is an immersion of X into a scheme smooth over Y. Hence the gysin map exists for f (Definition 59.4). To prove the formula we may decompose X and Y into their connected components, hence we may assume X is smooth over k and equidimensional of dimension d and Y is smooth over k and equidimensional of dimension e. Observe that $f^![Y]_e = [X]_d$ (see for example Lemma 59.8). Write $\alpha = c \cap [Y]_e$ and $\beta = c' \cap [Y]_e$ and hence $\alpha \cdot \beta = c \cap c' \cap [Y]_e$, see Lemma 62.2. By Lemma 59.7 we know that $f^!$ commutes with both c and c'.

Hence

$$\begin{aligned} f^!(\alpha \cdot \beta) &= f^!(c \cap c' \cap [Y]_e) \\ &= c \cap c' \cap f^![Y]_e \\ &= c \cap c' \cap [X]_d \\ &= (c \cap [X]_d) \cdot (c' \cap [X]_d) \\ &= (c \cap f^![Y]_e) \cdot (c' \cap f^![Y]_e) \\ &= f^!(\alpha) \cdot f^!(\beta) \end{aligned}$$

as desired where we have used Lemma 62.2 for X as well.

An alternative proof can be given by proving that $(f \times f)^!(\alpha \times \beta) = f^!\alpha \times f^!\beta$ and using Lemma 59.6. $\square$

Lemma 62.4. *Let k be a field. Let $f : X \to Y$ be a proper morphism of schemes smooth over k. Then the gysin map exists for f and $f_*(\alpha \cdot f^!\beta) = f_*\alpha \cdot \beta$.*

Proof. Observe that $X \to X \times_k Y$ is an immersion of X into a scheme smooth over Y. Hence the gysin map exists for f (Definition 59.4). To prove the formula we may decompose X and Y into their connected components, hence we may assume X is smooth over k and equidimensional of dimension d and Y is smooth over k and equidimensional of dimension e. Observe that $f^![Y]_e = [X]_d$ (see for example Lemma 59.8). Write $\alpha = c \cap [X]_d$ and $\beta = c' \cap [Y]_e$, see Lemma 62.2. We have

$$\begin{aligned} f_*(\alpha \cdot f^!\beta) &= f_*(c \cap f^!(c' \cap [Y]_e)) \\ &= f_*(c \cap c' \cap f^![Y]_e) \\ &= f_*(c \cap c' \cap [X]_d) \\ &= f_*(c' \cap c \cap [X]_d) \\ &= c' \cap f_*(c \cap [X]_d) \\ &= \beta \cdot f_*(\alpha) \end{aligned}$$

The first equality by the result of Lemma 62.2 for X. By Lemma 59.7 we know that $f^!$ commutes with c'. The commutativity of the intersection product justifies switching the order of capping $[X]_d$ with c and c' (via the lemma). Commuting c' with f_* is allowed as c' is a bivariant class. The final equality is again the lemma. $\square$

Lemma 62.5. *Let k be a field. Let X be an integral scheme smooth over k. Let $Y, Z \subset X$ be integral closed subschemes. Set $d = \dim(Y) + \dim(Z) - \dim(X)$. Assume*

1. *$\dim(Y \cap Z) \leq d$, and*
2. *$\mathcal{O}_{Y,\xi}$ and $\mathcal{O}_{Z,\xi}$ are Cohen-Macaulay for every $\xi \in Y \cap Z$ with $\delta(\xi) = d$.*

Then $[Y] \cdot [Z] = [Y \cap Z]_d$ in $\mathrm{CH}_d(X)$.

Proof. Recall that $[Y] \cdot [Z] = \Delta^!([Y \times Z])$ where $\Delta^! = c(\Delta : X \to X \times X, \mathcal{T}_{X/k})$ is a higher codimension gysin map (Section 54) with $\mathcal{T}_{X/k} = \mathcal{H}om(\Omega_{X/k}, \mathcal{O}_X)$ locally free of rank $\dim(X)$. We have the equality of schemes

$$Y \cap Z = X \times_{\Delta, (X \times X)} (Y \times Z)$$

and $\dim(Y \times Z) = \dim(Y) + \dim(Z)$ and hence conditions (1), (2), and (3) of Lemma 54.6 hold. Finally, if $\xi \in Y \cap Z$, then we have a flat local homomorphism

$$\mathcal{O}_{Y,\xi} \longrightarrow \mathcal{O}_{Y \times Z,\xi}$$

whose "fibre" is $\mathcal{O}_{Z,\xi}$. It follows that if both $\mathcal{O}_{Y,\xi}$ and $\mathcal{O}_{Z,\xi}$ are Cohen-Macaulay, then so is $\mathcal{O}_{Y\times Z,\xi}$, see Algebra, Lemma 163.3. In this way we see that all the hypotheses of Lemma 54.6 are satisfied and we conclude. □

Lemma 62.6. *Let k be a field. Let X be a scheme smooth over k. Let $i : Y \to X$ be a regular closed immersion. Let $\alpha \in \mathrm{CH}_*(X)$. If Y is equidimensional of dimension e, then $\alpha \cdot [Y]_e = i_*(i^!(\alpha))$ in $\mathrm{CH}_*(X)$.*

Proof. After decomposing X into connected components we may and do assume X is equidimensional of dimension d. Write $\alpha = c \cap [X]_n$ with $x \in A^*(X)$, see Lemma 62.2. Then

$$i_*(i^!(\alpha)) = i_*(i^!(c \cap [X]_n)) = i_*(c \cap i^![X]_n) = i_*(c \cap [Y]_e) = c \cap i_*[Y]_e = \alpha \cdot [Y]_e$$

The first equality by choice of c. Then second equality by Lemma 59.7. The third because $i^![X]_d = [Y]_e$ in $\mathrm{CH}_*(Y)$ (Lemma 59.8). The fourth because bivariant classes commute with proper pushforward. The last equality by Lemma 62.2. □

Lemma 62.7. *Let k be a field. Let X be a smooth scheme over k which is quasi-compact and has affine diagonal. Then the intersection product on $\mathrm{CH}^*(X)$ constructed in this section agrees after tensoring with $\mathbf{Q}$ with the intersection product constructed in Section 58.*

Proof. Let $\alpha \in \mathrm{CH}^i(X)$ and $\beta \in \mathrm{CH}^j(X)$. Write $\alpha = ch(\alpha') \cap [X]$ and $\beta = ch(\beta') \cap [X]$ $\alpha', \beta' \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ as in Section 58. Set $c = ch(\alpha')$ and $c' = ch(\beta')$. Then the intersection product in Section 58 produces $c \cap c' \cap [X]$. This is the same as $\alpha \cdot \beta$ by Lemma 62.2 (or rather the generalization that $A^i(X) \to \mathrm{CH}^i(X)$, $c \mapsto c \cap [X]$ is an isomorphism for any smooth scheme X over k). □

63. Exterior product over Dedekind domains

Let S be a locally Noetherian scheme which has an open covering by spectra of Dedekind domains. Set $\delta(s) = 0$ for $s \in S$ closed and $\delta(s) = 1$ otherwise. Then (S, δ) is a special case of our general Situation 7.1; see Example 7.3. Observe that S is normal (Algebra, Lemma 120.17) and hence a disjoint union of normal integral schemes (Properties, Lemma 7.7). Thus all of the arguments below reduce to the case where S is irreducible. On the other hand, we allow S to be nonseparated (so S could be the affine line with 0 doubled for example).

Consider a cartesian square

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

of schemes locally of finite type over S. We claim there is a canonical map

$$\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{n+m-1}(X \times_S Y)$$

which is uniquely determined by the following rule: given integral closed subschemes $X' \subset X$ and $Y' \subset Y$ of δ-dimensions n and m we set

(1) $[X'] \times [Y'] = [X' \times_S Y']_{n+m-1}$ if X' or Y' dominates an irreducible component of S,
(2) $[X'] \times [Y'] = 0$ if neither X' nor Y' dominates an irreducible component of S.

Lemma 63.1. *The map* $\times : \mathrm{CH}_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m-1}(X \times_S Y)$ *is well defined.*

Proof. Consider n and m cycles $\alpha = \sum_{i \in I} n_i[X_i]$ and $\beta = \sum_{j \in J} m_j[Y_j]$ with $X_i \subset X$ and $Y_j \subset Y$ locally finite families of integral closed subschemes of δ-dimensions n and m. Let $K \subset I \times J$ be the set of pairs $(i, j) \in I \times J$ such that X_i or Y_j dominates an irreducible component of S. Then $\{X_i \times_S Y_j\}_{(i,j) \in K}$ is a locally finite collection of closed subschemes of $X \times_S Y$ of δ-dimension $n + m - 1$. This means we can indeed consider

$$\alpha \times \beta = \sum\nolimits_{(i,j) \in K} n_i m_j [X_i \times_S Y_j]_{n+m-1}$$

as a $(n + m - 1)$-cycle on $X \times_S Y$. In this way we obtain an additive map $\times : Z_n(X) \otimes_{\mathbf{Z}} Z_m(Y) \to Z_{n+m}(X \times_S Y)$. The problem is to show that this procedure is compatible with rational equivalence.

Let $i : X' \to X$ be the inclusion morphism of an integral closed subscheme of δ-dimension n which dominates an irreducible component of S. Then $p' : X' \to S$ is flat of relative dimension $n - 1$, see More on Algebra, Lemma 22.11. Hence flat pullback along p' is an element $(p')^* \in A^{-n+1}(X' \to S)$ by Lemma 33.2 and hence $c' = i_* \circ (p')^* \in A^{-n+1}(X \to S)$ by Lemma 33.4. This produces maps

$$c' \cap - : \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{m+n-1}(X \times_S Y)$$

which sends $[Y']$ to $[X' \times_S Y']_{n+m-1}$ for any integral closed subscheme $Y' \subset Y$ of δ-dimension m.

Let $i : X' \to X$ be the inclusion morphism of an integral closed subscheme of δ-dimension n such that the composition $X' \to X \to S$ factors through a closed point $s \in S$. Since s is a closed point of the spectrum of a Dedekind domain, we see that s is an effective Cartier divisor on S whose normal bundle is trivial. Denote $c \in A^1(s \to S)$ the gysin homomorphism, see Lemma 33.3. The morphism $p' : X' \to s$ is flat of relative dimension n. Hence flat pullback along p' is an element $(p')^* \in A^{-n}(X' \to S)$ by Lemma 33.2. Thus

$$c' = i_* \circ (p')^* \circ c \in A^{-n}(X \to S)$$

by Lemma 33.4. This produces maps

$$c' \cap - : \mathrm{CH}_m(Y) \longrightarrow \mathrm{CH}_{m+n-1}(X \times_S Y)$$

which for any integral closed subscheme $Y' \subset Y$ of δ-dimension m sends $[Y']$ to either $[X' \times_S Y']_{n+m-1}$ if Y' dominates an irreducible component of S or to 0 if not.

From the previous two paragraphs we conclude the construction $([X'], [Y']) \mapsto [X' \times_S Y']_{n+m-1}$ factors through rational equivalence in the second variable, i.e., gives a well defined map $Z_n(X) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m-1}(X \times_S Y)$. By symmetry the same is true for the other variable and we conclude. $\square$

Lemma 63.2. *Let (S, δ) be as above. Let X be a scheme locally of finite type over S. Then we have a canonical identification*

$$A^p(X \to S) = \mathrm{CH}_{1-p}(X)$$

for all $p \in \mathbf{Z}$.

Proof. Consider the element $[S]_1 \in \mathrm{CH}_1(S)$. We get a map $A^p(X \to S) \to \mathrm{CH}_{1-p}(X)$ by sending c to $c \cap [S]_1$.

Conversely, suppose we have $\alpha \in \mathrm{CH}_{1-p}(X)$. Then we can define $c_\alpha \in A^p(X \to S)$ as follows: given $X' \to S$ and $\alpha' \in \mathrm{CH}_n(X')$ we let

$$c_\alpha \cap \alpha' = \alpha \times \alpha'$$

in $\mathrm{CH}_{n-p}(X \times_S X')$. To show that this is a bivariant class we write $\alpha = \sum_{i \in I} n_i [X_i]$ as in Definition 8.1. In particular the morphism

$$g : \coprod\nolimits_{i \in I} X_i \longrightarrow X$$

is proper. Pick $i \in I$. If X_i dominates an irreducible component of S, then the structure morphism $p_i : X_i \to S$ is flat and we have $\xi_i = p_i^* \in A^p(X_i \to S)$. On the other hand, if p_i factors as $p_i' : X_i \to s_i$ followed by the inclusion $s_i \to S$ of a closed point, then we have $\xi_i = (p_i')^* \circ c_i \in A^p(X_i \to S)$ where $c_i \in A^1(s_i \to S)$ is the gysin homomorphism and $(p_i')^*$ is flat pullback. Observe that

$$A^p(\coprod\nolimits_{i \in I} X_i \to S) = \prod\nolimits_{i \in I} A^p(X_i \to S)$$

Thus we have

$$\xi = \sum n_i \xi_i \in A^p(\coprod\nolimits_{i \in I} X_i \to S)$$

Finally, since g is proper we have a bivariant class

$$g_* \circ \xi \in A^p(X \to S)$$

by Lemma 33.4. The reader easily verifies that c_α is equal to this class (please compare with the proof of Lemma 63.1) and hence is itself a bivariant class.

To finish the proof we have to show that the two constructions are mutually inverse. Since $c_\alpha \cap [S]_1 = \alpha$ this is clear for one of the two directions. For the other, let $c \in A^p(X \to S)$ and set $\alpha = c \cap [S]_1$. It suffices to prove that

$$c \cap [X'] = c_\alpha \cap [X']$$

when X' is an integral scheme locally of finite type over S, see Lemma 35.3. However, either $p' : X' \to S$ is flat of relative dimension $\dim_\delta(X') - 1$ and hence $[X'] = (p')^*[S]_1$ or $X' \to S$ factors as $X' \to s \to S$ and hence $[X'] = (p')^*(s \to S)^*[S]_1$. Thus the fact that the bivariant classes c and c_α agree on $[S]_1$ implies they agree when capped against $[X']$ (since bivariant classes commute with flat pullback and gysin maps) and the proof is complete. $\square$

Lemma 63.3. *Let (S, δ) be as above. Let X be a scheme locally of finite type over S. Let $c \in A^p(X \to S)$. Let $Y \to Z$ be a morphism of schemes locally of finite type over S. Let $c' \in A^q(Y \to Z)$. Then $c \circ c' = c' \circ c$ in $A^{p+q}(X \times_S Y \to X \times_S Z)$.*

Proof. In the proof of Lemma 63.2 we have seen that c is given by a combination of proper pushforward, multiplying by integers over connected components, flat pullback, and gysin maps. Since c' commutes with each of these operations by definition of bivariant classes, we conclude. Some details omitted. $\square$

Remark 63.4. The upshot of Lemmas 63.2 and 63.3 is the following. Let (S, δ) be as above. Let X be a scheme locally of finite type over S. Let $\alpha \in \mathrm{CH}_*(X)$. Let $Y \to Z$ be a morphism of schemes locally of finite type over S. Let $c' \in A^q(Y \to Z)$. Then

$$\alpha \times (c' \cap \beta) = c' \cap (\alpha \times \beta)$$

in $\mathrm{CH}_*(X \times_S Y)$ for any $\beta \in \mathrm{CH}_*(Z)$. Namely, this follows by taking $c = c_\alpha \in A^*(X \to S)$ the bivariant class corresponding to α, see proof of Lemma 63.2.

Lemma 63.5. *Exterior product is associative. More precisely, let (S, δ) be as above, let X, Y, Z be schemes locally of finite type over S, let $\alpha \in \mathrm{CH}_*(X)$, $\beta \in \mathrm{CH}_*(Y)$, $\gamma \in \mathrm{CH}_*(Z)$. Then $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(X \times_S Y \times_S Z)$.*

Proof. Omitted. Hint: associativity of fibre product of schemes. $\square$

64. Intersection products over Dedekind domains

Let S be a locally Noetherian scheme which has an open covering by spectra of Dedekind domains. Set $\delta(s) = 0$ for $s \in S$ closed and $\delta(s) = 1$ otherwise. Then (S, δ) is a special case of our general Situation 7.1; see Example 7.3 and discussion in Section 63.

Let X be a smooth scheme over S. The bivariant class $\Delta^!$ of Section 60 allows us to define a kind of intersection product on chow groups of schemes locally of finite type over X. Namely, suppose that $Y \to X$ and $Z \to X$ are morphisms of schemes which are locally of finite type. Then observe that

$$Y \times_X Z = (Y \times_S Z) \times_{X \times_S X, \Delta} X$$

Hence we can consider the following sequence of maps

$$\mathrm{CH}_n(Y) \otimes_{\mathbf{Z}} \mathrm{CH}_m(Y) \xrightarrow{\times} \mathrm{CH}_{n+m-1}(Y \times_S Z) \xrightarrow{\Delta^!} \mathrm{CH}_{n+m-*}(Y \times_X Z)$$

Here the first arrow is the exterior product constructed in Section 63 and the second arrow is the gysin map for the diagonal studied in Section 60. If X is equidimensional of dimension d, then $X \to S$ is smooth of relative dimension $d - 1$ and hence we end up in $\mathrm{CH}_{n+m-d}(Y \times_X Z)$. In general we can decompose into the parts lying over the open and closed subschemes of X where X has a given dimension. Given $\alpha \in \mathrm{CH}_*(Y)$ and $\beta \in \mathrm{CH}_*(Z)$ we will denote

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta) \in \mathrm{CH}_*(Y \times_X Z)$$

In the special case where $X = Y = Z$ we obtain a multiplication

$$\mathrm{CH}_*(X) \times \mathrm{CH}_*(X) \to \mathrm{CH}_*(X), \quad (\alpha, \beta) \mapsto \alpha \cdot \beta$$

which is called the *intersection product*. We observe that this product is clearly symmetric. Associativity follows from the next lemma.

Lemma 64.1. *The product defined above is associative. More precisely, with (S, δ) as above, let X be smooth over S, let Y, Z, W be schemes locally of finite type over X, let $\alpha \in \mathrm{CH}_*(Y)$, $\beta \in \mathrm{CH}_*(Z)$, $\gamma \in \mathrm{CH}_*(W)$. Then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ in $\mathrm{CH}_*(Y \times_X Z \times_X W)$.*

Proof. By Lemma 63.5 we have $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$ in $\mathrm{CH}_*(Y \times_S Z \times_S W)$. Consider the closed immersions

$$\Delta_{12} : X \times_S X \longrightarrow X \times_S X \times_S X, \quad (x, x') \mapsto (x, x, x')$$

and

$$\Delta_{23} : X \times_S X \longrightarrow X \times_S X \times_S X, \quad (x, x') \mapsto (x, x', x')$$

Denote $\Delta_{12}^!$ and $\Delta_{23}^!$ the corresponding bivariant classes; observe that $\Delta_{12}^!$ is the restriction (Remark 33.5) of $\Delta^!$ to $X \times_S X \times_S X$ by the map pr_{12} and that $\Delta_{23}^!$ is the restriction of $\Delta^!$ to $X \times_S X \times_S X$ by the map pr_{23}. Thus clearly the restriction of $\Delta_{12}^!$ by Δ_{23} is $\Delta^!$ and the restriction of $\Delta_{23}^!$ by Δ_{12} is $\Delta^!$ too. Thus by Lemma 54.8 we have

$$\Delta^! \circ \Delta_{12}^! = \Delta^! \circ \Delta_{23}^!$$

Now we can prove the lemma by the following sequence of equalities:

$$\begin{aligned}
(\alpha \cdot \beta) \cdot \gamma &= \Delta^!(\Delta^!(\alpha \times \beta) \times \gamma) \\
&= \Delta^!(\Delta_{12}^!((\alpha \times \beta) \times \gamma)) \\
&= \Delta^!(\Delta_{23}^!((\alpha \times \beta) \times \gamma)) \\
&= \Delta^!(\Delta_{23}^!(\alpha \times (\beta \times \gamma)) \\
&= \Delta^!(\alpha \times \Delta^!(\beta \times \gamma)) \\
&= \alpha \cdot (\beta \cdot \gamma)
\end{aligned}$$

All equalities are clear from the above except perhaps for the second and penultimate one. The equation $\Delta_{23}^!(\alpha \times (\beta \times \gamma)) = \alpha \times \Delta^!(\beta \times \gamma)$ holds by Remark 61.4. Similarly for the second equation. $\square$

Lemma 64.2. *Let (S, δ) be as above. Let X be a smooth scheme over S, equidimensional of dimension d. The map*

$$A^p(X) \longrightarrow \mathrm{CH}_{d-p}(X), \quad c \longmapsto c \cap [X]_d$$

is an isomorphism. Via this isomorphism composition of bivariant classes turns into the intersection product defined above.

Proof. Denote $g : X \to S$ the structure morphism. The map is the composition of the isomorphisms

$$A^p(X) \to A^{p-d+1}(X \to S) \to \mathrm{CH}_{d-p}(X)$$

The first is the isomorphism $c \mapsto c \circ g^*$ of Proposition 60.2 and the second is the isomorphism $c \mapsto c \cap [S]_1$ of Lemma 63.2. From the proof of Lemma 63.2 we see that the inverse to the second arrow sends $\alpha \in \mathrm{CH}_{d-p}(X)$ to the bivariant class c_α which sends $\beta \in \mathrm{CH}_*(Y)$ for Y locally of finite type over k to $\alpha \times \beta$ in $\mathrm{CH}_*(X \times_k Y)$. From the proof of Proposition 60.2 we see the inverse to the first arrow in turn sends c_α to the bivariant class which sends $\beta \in \mathrm{CH}_*(Y)$ for $Y \to X$ locally of finite type to $\Delta^!(\alpha \times \beta) = \alpha \cdot \beta$. From this the final result of the lemma follows. $\square$

65. Todd classes

A final class associated to a vector bundle $\mathcal{E}$ of rank r is its *Todd class* $Todd(\mathcal{E})$. In terms of the Chern roots $x_1, \ldots, x_r$ it is defined as

$$Todd(\mathcal{E}) = \prod\nolimits_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}}$$

In terms of the Chern classes $c_i = c_i(\mathcal{E})$ we have

$$Todd(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \ldots$$

We have made the appropriate remarks about denominators in the previous section. It is the case that given an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

we have

$$Todd(\mathcal{E}) = Todd(\mathcal{E}_1)Todd(\mathcal{E}_2).$$

66. Grothendieck-Riemann-Roch

Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf $\mathcal{E}$ on X of rank r. Let $f : X \to Y$ be a proper smooth morphism. Assume that $R^if_*\mathcal{E}$ are locally free sheaves on Y of finite rank. The Grothendieck-Riemann-Roch theorem say in this case that

$$f_*(Todd(T_{X/Y})ch(\mathcal{E})) = \sum (-1)^i ch(R^if_*\mathcal{E})$$

Here

$$T_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$$

is the relative tangent bundle of X over Y. If $Y = \text{Spec}(k)$ where k is a field, then we can restate this as

$$\chi(X, \mathcal{E}) = \deg(Todd(T_{X/k})ch(\mathcal{E}))$$

The theorem is more general and becomes easier to prove when formulated in correct generality. We will return to this elsewhere (insert future reference here).

67. Change of base scheme

In this section we explain how to compare theories for different base schemes.

Situation 67.1. Here (S, δ) and (S', δ') are as in Situation 7.1. Furthermore $g : S' \to S$ is a flat morphism of schemes and $c \in \mathbf{Z}$ is an integer such that: for all $s \in S$ and $s' \in S'$ a generic point of an irreducible component of $g^{-1}(\{s\})$ we have $\delta(s') = \delta(s) + c$.

We will see that for a scheme X locally of finite type over S there is a well defined map $\text{CH}_k(X) \to \text{CH}_{k+c}(X \times_S S')$ of Chow groups which (by and large) commutes with the operations we have defined in this chapter.

Lemma 67.2. *In Situation 67.1 let $X \to S$ be locally of finite type. Denote $X' \to S'$ the base change by $S' \to S$. If X is integral with $\dim_\delta(X) = k$, then every irreducible component Z' of X' has $\dim_{\delta'}(Z') = k + c$,*

Proof. The projection $X' \to X$ is flat as a base change of the flat morphism $S' \to S$ (Morphisms, Lemma 25.8). Hence every generic point x' of an irreducible component of X' maps to the generic point $x \in X$ (because generalizations lift along $X' \to X$ by Morphisms, Lemma 25.9). Let $s \in S$ be the image of x. Recall that the scheme $S'_s = S' \times_S s$ has the same underlying topological space as $g^{-1}(\{s\})$ (Schemes, Lemma 18.5). We may view x' as a point of the scheme $S'_s \times_s x$ which comes equipped with a monomorphism $S'_s \times_s x \to S' \times_S X$. Of course, x' is a generic point of an irreducible component of $S'_s \times_s x$ as well. Using the flatness of $\text{Spec}(\kappa(x)) \to \text{Spec}(\kappa(s)) = s$ and arguing as above, we see that x' maps to a generic point s' of an irreducible component of $g^{-1}(\{s\})$. Hence $\delta'(s') = \delta(s) + c$ by assumption. We have $\dim_x(X_s) = \dim_{x'}(X_{s'})$ by Morphisms, Lemma 28.3. Since x is a generic point of an irreducible component X_s (this is an irreducible scheme but we don't need this) and x' is a generic point of an irreducible component of $X'_{s'}$ we conclude that $\text{trdeg}_{\kappa(s)}(\kappa(x)) = \text{trdeg}_{\kappa(s')}(\kappa(x'))$ by Morphisms, Lemma 28.1. Then

$$\delta_{X'/S'}(x') = \delta(s') + \text{trdeg}_{\kappa(s')}(\kappa(x')) = \delta(s) + c + \text{trdeg}_{\kappa(s)}(\kappa(x)) = \delta_{X/S}(x) + c$$

This proves what we want by Definition 7.6. $\square$

In Situation 67.1 let $X \to S$ be locally of finite type. Denote $X' \to S'$ the base change by $g : S' \to S$. There is a unique homomorphism

$$g^* : Z_k(X) \longrightarrow Z_{k+c}(X')$$

which given an integral closed subscheme $Z \subset X$ of δ-dimension k sends $[Z]$ to $[Z \times_S S']_{k+c}$. This makes sense by Lemma 67.2.

Lemma 67.3. *In Situation 67.1 let $X \to S$ locally of finite type and let $X' \to S$ be the base change by $S' \to S$.*

1. *Let $Z \subset X$ be a closed subscheme with $\dim_\delta(Z) \leq k$ and base change $Z' \subset X'$. Then we have $\dim_{\delta'}(Z')) \leq k + c$ and $[Z']_{k+c} = g^*[Z]_k$ in $Z_{k+c}(X')$.*
2. *Let $\mathcal{F}$ be a coherent sheaf on X with $\dim_\delta(Supp(\mathcal{F})) \leq k$ and base change $\mathcal{F}'$ on X'. Then we have $\dim_\delta(Supp(\mathcal{F}')) \leq k + c$ and $g^*[\mathcal{F}]_k = [\mathcal{F}']_{k+c}$ in $Z_{k+c}(X')$.*

Proof. The proof is exactly the same is the proof of Lemma 14.4 and we suggest the reader skip it.

The statements on dimensions follow from Lemma 67.2. Part (1) follows from part (2) by Lemma 10.3 and the fact that the base change of the coherent module $\mathcal{O}_Z$ is $\mathcal{O}_{Z'}$.

Proof of (2). As X, X' are locally Noetherian we may apply Cohomology of Schemes, Lemma 9.1 to see that $\mathcal{F}$ is of finite type, hence $\mathcal{F}'$ is of finite type (Modules, Lemma 9.2), hence $\mathcal{F}'$ is coherent (Cohomology of Schemes, Lemma 9.1 again). Thus the lemma makes sense. Let $W \subset X$ be an integral closed subscheme of δ-dimension k, and let $W' \subset X'$ be an integral closed subscheme of δ'-dimension $k + c$ mapping into W under $X' \to X$. We have to show that the coefficient n of $[W']$ in $g^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W']$ in $[\mathcal{F}']_{k+c}$. Let $\xi \in W$ and $\xi' \in W'$ be the generic points. Let $A = \mathcal{O}_{X,\xi}$, $B = \mathcal{O}_{X',\xi'}$ and set $M = \mathcal{F}_\xi$ as an A-module. (Note that M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 10.1.) We have $\mathcal{F}'_{\xi'} = B \otimes_A M$. Thus we see that

$$n = \text{length}_B(B \otimes_A M) \quad \text{and} \quad m = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}_A B)$$

Thus the equality follows from Algebra, Lemma 52.13. $\square$

Lemma 67.4. *In Situation 67.1 let $X \to S$ be locally of finite type and let $X' \to S'$ be the base change by $S' \to S$. The map $g^* : Z_k(X) \to Z_{k+c}(X')$ above factors through rational equivalence to give a map*

$$g^* : \text{CH}_k(X) \longrightarrow \text{CH}_{k+c}(X')$$

of chow groups.

Proof. Suppose that $\alpha \in Z_k(X)$ is a k-cycle which is rationally equivalent to zero. By Lemma 21.1 there exists a locally finite family of integral closed subschemes $W_i \subset X \times \mathbf{P}^1$ of δ-dimension k not contained in the divisors $(X \times \mathbf{P}^1)_0$ or $(X \times \mathbf{P}^1)_\infty$ of $X \times \mathbf{P}^1$ such that $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$. Thus it suffices to prove for $W \subset X \times \mathbf{P}^1$ integral closed of δ-dimension k not contained in the divisors $(X \times \mathbf{P}^1)_0$ or $(X \times \mathbf{P}^1)_\infty$ of $X \times \mathbf{P}^1$ we have

(1) the base change $W' \subset X' \times \mathbf{P}^1$ satisfies the assumptions of Lemma 21.2 with k replaced by $k+c$, and
(2) $g^*[W_0]_k = [(W')_0]_{k+c}$ and $g^*[W_\infty]_k = [(W')_\infty]_{k+c}$.

Part (2) follows immediately from Lemma 67.3 and the fact that $(W')_0$ is the base change of W_0 (by associativity of fibre products). For part (1), first the statement on dimensions follows from Lemma 67.2. Then let $w' \in (W')_0$ with image $w \in W_0$ and $z \in \mathbf{P}^1_S$. Denote $t \in \mathcal{O}_{\mathbf{P}^1_S,z}$ the usual equation for $0 : S \to \mathbf{P}^1_S$. Since $\mathcal{O}_{W,w} \to \mathcal{O}_{W',w'}$ is flat and since t is a nonzerodivisor on $\mathcal{O}_{W,w}$ (as W is integral and $W \neq W_0$) we see that also t is a nonzerodivisor in $\mathcal{O}_{W',w'}$. Hence W' has no associated points lying on W'_0. ☐

Lemma 67.5. *In Situation 67.1 let $Y \to X \to S$ be locally of finite type and let $Y' \to X' \to S'$ be the base change by $S' \to S$. Assume $f : Y \to X$ is flat of relative dimension r. Then $f' : Y' \to X'$ is flat of relative dimension r and the diagrams*

$$\begin{tikzcd} Z_{k+r}(Y) \arrow[r, "g^*"'] & Z_{k+c+r}(Y') \\ Z_k(X) \arrow[u, "(f')^*"] \arrow[r, "g^*"] & Z_{k+c}(X') \arrow[u, "f^*"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{CH}_{k+r}(Y) \arrow[r, "g^*"'] & \mathrm{CH}_{k+c+r}(Y') \\ \mathrm{CH}_k(X) \arrow[u, "(f')^*"] \arrow[r, "g^*"] & \mathrm{CH}_{k+c}(X') \arrow[u, "f^*"'] \end{tikzcd}$$

of cycle and chow groups commutes.

Proof. It suffices to show the first diagram commutes. To see this, let $Z \subset X$ be an integral closed subscheme of δ-dimension k and denote $Z' \subset X'$ its base change. By construction we have $g^*[Z] = [Z']_{k+c}$. By Lemma 14.4 we have $(f')^*g^*[Z] = [Z' \times_{X'} Y']_{k+c+r}$. Conversely, we have $f^*[Z] = [Z \times_X Y]_{k+r}$ by Definition 14.1. By Lemma 67.3 we have $g^*f^*[Z] = [(Z \times_X Y)']_{k+r+c}$. Since $(Z \times_X Y)' = Z' \times_{X'} Y'$ by associativity of fibre product we conclude. ☐

Lemma 67.6. *In Situation 67.1 let $Y \to X \to S$ be locally of finite type and let $Y' \to X' \to S'$ be the base change by $S' \to S$. Assume $f : Y \to X$ is proper. Then $f' : Y' \to X'$ is proper and the diagram*

$$\begin{tikzcd} Z_k(Y) \arrow[r, "g^*"'] \arrow[d, "f_*"'] & Z_{k+c}(Y') \arrow[d, "f'_*"] \\ Z_k(X) \arrow[r, "g^*"] & Z_{k+c}(X') \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{CH}_k(Y) \arrow[r, "g^*"'] \arrow[d, "f_*"'] & \mathrm{CH}_{k+c}(Y') \arrow[d, "f'_*"] \\ \mathrm{CH}_k(X) \arrow[r, "g^*"] & \mathrm{CH}_{k+c}(X') \end{tikzcd}$$

of cycle and chow groups commutes.

Proof. It suffices to show the first diagram commutes. To see this, let $Z \subset Y$ be an integral closed subscheme of δ-dimension k and denote $Z' \subset X'$ its base change. By construction we have $g^*[Z] = [Z']_{k+c}$. By Lemma 12.4 we have $(f')_*g^*[Z] = [f'_*\mathcal{O}_{Z'}]_{k+c}$. By the same lemma we have $f_*[Z] = [f_*\mathcal{O}_Z]_k$. By Lemma 67.3 we have $g^*f_*[Z] = [(X' \to X)^*f_*\mathcal{O}_Z]_{k+r}$. Thus it suffices to show that

$$(X' \to X)^*f_*\mathcal{O}_Z \cong f'_*\mathcal{O}_{Z'}$$

as coherent modules on X'. As $X' \to X$ is flat and as $\mathcal{O}_{Z'} = (Y' \to Y)^*\mathcal{O}_Z$, this follows from flat base change, see Cohomology of Schemes, Lemma 5.2. $\square$

Lemma 67.7. *In Situation 67.1 let $X \to S$ be locally of finite type and let $X' \to S'$ be the base change by $S' \to S$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module with base change $\mathcal{L}'$ on X'. Then the diagram*

$$\begin{tikzcd} \mathrm{CH}_k(X) \arrow[r, "g^*"'] \arrow[d, "c_1(\mathcal{L})\cap -"'] & \mathrm{CH}_{k+c}(X') \arrow[d, "c_1(\mathcal{L}')\cap -"] \\ \mathrm{CH}_{k-1}(X) \arrow[r, "g^*"] & \mathrm{CH}_{k+c-1}(X') \end{tikzcd}$$

of chow groups commutes.

Proof. Let $p : L \to X$ be the line bundle associated to $\mathcal{L}$ with zero section $o : X \to L$. For $\alpha \in CH_k(X)$ we know that $\beta = c_1(\mathcal{L}) \cap \alpha$ is the unique element of $\mathrm{CH}_{k-1}(X)$ such that $o_*\alpha = -p^*\beta$, see Lemmas 32.2 and 32.4. The same characterization holds after pullback. Hence the lemma follows from Lemmas 67.5 and 67.6. $\square$

Lemma 67.8. *In Situation 67.1 let $X \to S$ be locally of finite type and let $X' \to S'$ be the base change by $S' \to S$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module of rank r with base change $\mathcal{E}'$ on X'. Then the diagram*

$$\begin{tikzcd} \mathrm{CH}_k(X) \arrow[r, "g^*"'] \arrow[d, "c_i(\mathcal{E})\cap -"'] & \mathrm{CH}_{k+c}(X') \arrow[d, "c_i(\mathcal{E}')\cap -"] \\ \mathrm{CH}_{k-i}(X) \arrow[r, "g^*"] & \mathrm{CH}_{k+c-i}(X') \end{tikzcd}$$

of chow groups commutes for all i.

Proof. Set $P = \mathbf{P}(\mathcal{E})$. The base change P' of P is equal to $\mathbf{P}(\mathcal{E}')$. Since we already know that flat pullback and cupping with c_1 of an invertible module commute with base change (Lemmas 67.5 and 67.7) the lemma follows from the characterization of capping with $c_i(\mathcal{E})$ given in Lemma 38.2. $\square$

Lemma 67.9. *Let (S, δ), (S', δ'), (S'', δ'') be as in Situation 7.1. Let $g : S' \to S$ and $g' : S'' \to S'$ be flat morphisms of schemes and let $c, c' \in \mathbf{Z}$ be integers such that $S, \delta, S', \delta', g, c$ and S', δ', S'', g', c' are as in Situation 67.1. Let $X \to S$ be locally of finite type and denote $X' \to S'$ and $X'' \to S''$ the base changes by $S' \to S$ and $S'' \to S$. Then*

1. *$S, \delta, S'', \delta'', g \circ g', c + c'$ is as in Situation 67.1,*
2. *the maps $g^* : Z_k(X) \to Z_{k+c}(X')$ and $(g')^* : Z_{k+c}(X') \to Z_{k+c+c'}(X'')$ of compose to give the map $(g \circ g')^* : Z_k(X) \to Z_{k+c+c'}(X'')$, and*
3. *the maps $g^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+c}(X')$ and $(g')^* : \mathrm{CH}_{k+c}(X') \to \mathrm{CH}_{k+c+c'}(X'')$ of Lemma 67.4 compose to give the map $(g \circ g')^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+c+c'}(X'')$ of Lemma 67.4.*

Proof. Let $s \in S$ and let $s'' \in S''$ be a generic point of an irreducible component of $(g \circ g')^{-1}(\{s\})$. Set $s' = g'(s'')$. Clearly, s'' is a generic point of an irreducible component of $(g')^{-1}(\{s'\})$. Moreover, since g' is flat and hence generalizations lift along g' (Morphisms, Lemma 25.8) we see that also s' is a generic point of an irreducible component of $g^{-1}(\{s\})$. Thus by assumption $\delta'(s') = \delta(s) + c$ and $\delta''(s'') = \delta'(s') + c'$. We conclude $\delta''(s'') = \delta(s) + c + c'$ and the first part of the statement is true.

For the second part, let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Denote $Z' \subset X'$ and $Z'' \subset X''$ the base changes. By definition we have $g^*[Z] = [Z']_{k+c}$. By Lemma 67.3 we have $(g')^*[Z']_{k+c} = [Z'']_{k+c+c'}$. This proves the final statement. $\square$

Lemma 67.10. *In Situation 67.1 assume $c = 0$ and assume that $S' = \lim_{i \in I} S_i$ is a filtered limit of schemes S_i affine over S such that*

(1) *with δ_i equal to $S_i \to S \xrightarrow{\delta} \mathbf{Z}$ the pair (S_i, δ_i) is as in Situation 7.1,*
(2) *$S_i, \delta_i, S, \delta, S \to S_i, c = 0$ is as in Situation 67.1,*
(3) *$S_i, \delta_i, S_{i'}, \delta_{i'}, S_i \to S_{i'}, c = 0$ for $i \geq i'$ is as in Situation 67.1.*

Then for a quasi-compact scheme X of finite type over S with base change X' and X_i by $S' \to S$ and $S_i \to S$ we have $Z_k(X') = \operatorname{colim} Z_k(X_i)$ and $\mathrm{CH}_k(X') = \operatorname{colim} \mathrm{CH}_k(X_i)$.

Proof. By the result of Lemma 67.9 we obtain a system of cycle groups $Z_k(X_i)$ and a system of chow groups $\mathrm{CH}_k(X_i)$ as well as maps $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ and $\operatorname{colim} \mathrm{CH}_i(X_i) \to \mathrm{CH}_k(X')$. We may replace S by a quasi-compact open through which $X \to S$ factors, hence we may and do assume all the schemes occuring in this proof are Noetherian (and hence quasi-compact and quasi-separated).

Let us show that the map $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ is surjective. Namely, let $Z' \subset X'$ be an integral closed subscheme of δ'-dimension k. By Limits, Lemma 10.1 we can find an i and a morphism $Z_i \to X_i$ of finite presentation whose base change is Z'. Afer increasing i we may assume Z_i is a closed subscheme of X_i, see Limits, Lemma 8.5. Then $Z' \to X_i$ factors through Z_i and we may replace Z_i by the scheme theoretic image of $Z' \to X_i$. In this way we see that we may assume Z_i is an integral closed subscheme of X_i. By Lemma 67.2 we conclude that $\dim_{\delta_i}(Z_i) = \dim_{\delta'}(Z') = k$. Thus $Z_k(X_i) \to Z_k(X')$ maps $[Z_i]$ to $[Z']$ and we conclude surjectivity holds.

Let us show that the map $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ is injective. Let $\alpha_i = \sum n_j [Z_j] \in Z_k(X_i)$ be a cycle whose image in $Z_k(X')$ is zero. We may and do assume $Z_j \neq Z_{j'}$ if $j \neq j'$ and $n_j \neq 0$ for all j. Denote $Z'_j \subset X'$ the base change of Z_j. By Lemma 67.2 each irreducible component of Z'_j has δ'-dimension k. Moreover, as Z_j is irreducible and $Z'_j \to Z_j$ is flat (as the base change of $S' \to S$) we see that $Z'_j \to Z_j$ is dominant. Hence if Z'_j is nonempty, then some irreducible component, say Z', of Z'_j dominates Z_j. It follows that Z' cannot be an irreducible component of $Z'_{j'}$ for $j' \neq j$. Hence if Z'_j is nonempty, then we see that $(S' \to S_i)^* \alpha_i = \sum [Z'_j]_r$ is nonzero (as the coefficient of Z' would be nonzero). Thus we see that $Z'_j = \emptyset$ for all j. However, this means that the base change of Z_j by some transition map $S_{i'} \to S_i$ is empty by Limits, Lemma 4.3. Thus α_i dies in the colimit as desired.

The surjectivity of $\operatorname{colim} Z_k(X_i) \to Z_k(X')$ implies that $\operatorname{colim} \mathrm{CH}_k(X_i) \to \mathrm{CH}_k(X')$ is surjective. To finish the proof we show that this map is injective. Let $\alpha_i \in \mathrm{CH}_k(X_i)$ be a cycle whose image $\alpha' \in \mathrm{CH}_k(X')$ is zero. Then there exist integral closed subschemes $W'_l \subset X'$, $l = 1, \ldots, r$ of δ"-dimension $k+1$ and nonzero rational functions f'_l on W'_l such that $\alpha' = \sum_{l=1,\ldots,r} \mathrm{div}_{W'_l}(f'_l)$. Arguing as above we can find an i and integral closed subschemes $W_{i,l} \subset X_i$ of δ_i-dimension $k+1$ whose base change is W'_l. After increasin i we may assume we have rational functions $f_{i,l}$ on $W_{i,l}$. Namely, we may think of f'_l as a section of the structure sheaf over a nonempty open $U'_l \subset W'_l$, we can descend these opens by Limits, Lemma 4.11 and after increasing i we may descend f'_l by Limits, Lemma 4.7. We claim that

$$\alpha_i = \sum\nolimits_{l=1,\ldots,r} \mathrm{div}_{W_{i,l}}(f_{i,l})$$

after possibly increasing i.

To prove the claim, let $Z'_{l,j} \subset W'_l$ be a finite collection of integral closed subschemes of δ'-dimension k such that f'_l is an invertible regular function outside $\bigcup_j Y'_{l,j}$. After increasing i (by the arguments above) we may assume there exist integral closed subschemes $Z_{i,l,j} \subset W_i$ of δ_i-dimension k such that $f_{i,l}$ is an invertible regular function outside $\bigcup_j Z_{i,l,j}$. Then we may write

$$\mathrm{div}_{W'_l}(f'_l) = \sum n_{l,j}[Z'_{l,j}]$$

and

$$\mathrm{div}_{W_{i,l}}(f_{i,l}) = \sum n_{i,l,j}[Z_{i,l,j}]$$

To prove the claim it suffices to show that $n_{l,i} = n_{i,l,j}$. Namely, this will imply that $\beta_i = \alpha_i - \sum_{l=1,\ldots,r} \mathrm{div}_{W_{i,l}}(f_{i,l})$ is a cycle on X_i whose pullback to X' is zero as a cycle! It follows that β_i pulls back to zero as a cycle on $X_{i'}$ for some $i' \geq i$ by an easy argument we omit.

To prove the equality $n_{l,i} = n_{i,l,j}$ we choose a generic point $\xi' \in Z'_{l,j}$ and we denote $\xi \in Z_{i,l,j}$ the image which is a generic point also. Then the local ring map

$$\mathcal{O}_{W_{i,l},\xi} \longrightarrow \mathcal{O}_{W'_l,\xi'}$$

is flat as $W'_l \to W_{i,l}$ is the base change of the flat morphism $S' \to S_i$. We also have $\mathfrak{m}_\xi \mathcal{O}_{W'_l,\xi'} = \mathfrak{m}_{\xi'}$ because $Z_{i,l,j}$ pulls back to $Z'_{l,j}$! Thus the equality of

$$n_{l,j} = \mathrm{ord}_{Z'_{l,j}}(f'_l) = \mathrm{ord}_{\mathcal{O}_{W'_l,\xi'}}(f'_l) \quad \text{and} \quad n_{i,l,j} = \mathrm{ord}_{Z_{i,l,j}}(f_{i,l}) = \mathrm{ord}_{\mathcal{O}_{W_{i,l},\xi}}(f_{i,l})$$

follows from Algebra, Lemma 52.13 and the construction of ord in Algebra, Section 121. ☐

68. Appendix A: Alternative approach to key lemma

In this appendix we first define determinants $\det_\kappa(M)$ of finite length modules M over local rings $(R, \mathfrak{m}, \kappa)$, see Subsection 68.1. The determinant $\det_\kappa(M)$ is a 1-dimensional κ-vector space. We use this in Subsection 68.12 to define the determinant $\det_\kappa(M, \varphi, \psi) \in \kappa^*$ of an exact $(2,1)$-periodic complex (M, φ, ψ) with M of finite length. In Subsection 68.26 we use these determinants to construct a tame symbol $d_R(a,b) = \det_\kappa(R/ab, a, b)$ for a pair of nonzerodivisors $a, b \in R$ when R is Noetherian of dimension 1. Although there is no doubt that

$$d_R(a,b) = \partial_R(a,b)$$

where ∂_R is as in Section 5, we have not (yet) added the verification. The advantage of the tame symbol as constructed in this appendix is that it extends (for example) to pairs of injective endomorphisms φ, ψ of a finite R-module M of dimension 1 such that $\varphi(\psi(M)) = \psi(\varphi(M))$. In Subsection 68.40 we relate Herbrand quotients and determinants. An easy to state version of the main result (Proposition 68.43) is the formula

$$-e_R(M, \varphi, \psi) = \text{ord}_R(\det_K(M_K, \varphi, \psi))$$

when (M, φ, ψ) is a $(2, 1)$-periodic complex whose Herbrand quotient e_R (Definition 2.2) is defined over a 1-dimensonal Noetherian local domain R with fraction field K. We use this proposition to give an alternative proof of the key lemma (Lemma 6.3) for the tame symbol constructed in this appendix, see Lemma 68.46.

68.1. Determinants of finite length modules. The material in this section is related to the material in the paper [KM76] and to the material in the thesis [Ros09].

Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $\varphi : M \to M$ be an R-linear endomorphism of a finite length R-module M. In More on Algebra, Section 120 we have already defined the determinant $\det_\kappa(\varphi)$ (and the trace and the characteristic polynomial) of φ relative to κ. In this section, we will construct a canonical 1-dimensional κ-vector space $\det_\kappa(M)$ such that $\det_\kappa(\varphi : M \to M) : \det_\kappa(M) \to \det_\kappa(M)$ is equal to multiplication by $\det_\kappa(\varphi)$. If M is annihilated by $\mathfrak{m}$, then M can be viewed as a finite dimension κ-vector space and then we have $\det_\kappa(M) = \wedge^n_\kappa(M)$ where $n = \dim_\kappa(M)$. Our construction will generalize this to all finite length modules over R and if R contains its residue field, then the determinant $\det_\kappa(M)$ will be given by the usual determinant in a suitable sense, see Remark 68.9.

Definition 68.2. Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module. Say $l = \text{length}_R(M)$.

(1) Given elements $x_1, \ldots, x_r \in M$ we denote $\langle x_1, \ldots, x_r\rangle = Rx_1 + \ldots + Rx_r$ the R-submodule of M generated by $x_1, \ldots, x_r$.
(2) We will say an l-tuple of elements $(e_1, \ldots, e_l)$ of M is *admissible* if $\mathfrak{m}e_i \subset \langle e_1, \ldots, e_{i-1}\rangle$ for $i = 1, \ldots, l$.
(3) A *symbol* $[e_1, \ldots, e_l]$ will mean $(e_1, \ldots, e_l)$ is an admissible l-tuple.
(4) An *admissible relation* between symbols is one of the following:
   (a) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$, then $[e_1, \ldots, e_l] = 0$,
   (b) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1}\rangle$, then
   $$[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l]$$
   where $\overline{\lambda} \in \kappa^*$ is the image of λ in the residue field, and
   (c) if $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m}e_a \subset \langle e_1, \ldots, e_{a-2}\rangle$ then
   $$[e_1, \ldots, e_l] = -[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l].$$
(5) We define the *determinant of the finite length R-module M* to be
$$\det_\kappa(M) = \left\{\frac{\kappa\text{-vector space generated by symbols}}{\kappa\text{-linear combinations of admissible relations}}\right\}$$

We stress that always $l = \text{length}_R(M)$. We also stress that it does not follow that the symbol $[e_1, \ldots, e_l]$ is additive in the entries (this will typically not be the case). Before we can show that the determinant $\det_\kappa(M)$ actually has dimension 1 we have to show that it has dimension at most 1.

Lemma 68.3. *With notations as above we have* $\dim_\kappa(\det_\kappa(M)) \leq 1$.

Proof. Fix an admissible sequence $(f_1, \ldots, f_l)$ of M such that

$$\text{length}_R(\langle f_1, \ldots, f_i \rangle) = i$$

for $i = 1, \ldots, l$. Such an admissible sequence exists exactly because M has length l. We will show that any element of $\det_\kappa(M)$ is a κ-multiple of the symbol $[f_1, \ldots, f_l]$. This will prove the lemma.

Let $(e_1, \ldots, e_l)$ be an admissible sequence of M. It suffices to show that $[e_1, \ldots, e_l]$ is a multiple of $[f_1, \ldots, f_l]$. First assume that $\langle e_1, \ldots, e_l \rangle \neq M$. Then there exists an $i \in [1, \ldots, l]$ such that $e_i \in \langle e_1, \ldots, e_{i-1} \rangle$. It immediately follows from the first admissible relation that $[e_1, \ldots, e_n] = 0$ in $\det_\kappa(M)$. Hence we may assume that $\langle e_1, \ldots, e_l \rangle = M$. In particular there exists a smallest index $i \in \{1, \ldots, l\}$ such that $f_1 \in \langle e_1, \ldots, e_i \rangle$. This means that $e_i = \lambda f_1 + x$ with $x \in \langle e_1, \ldots, e_{i-1} \rangle$ and $\lambda \in R^*$. By the second admissible relation this means that $[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{i-1}, f_1, e_{i+1}, \ldots, e_l]$. Note that $\mathfrak{m} f_1 = 0$. Hence by applying the third admissible relation $i - 1$ times we see that

$$[e_1, \ldots, e_l] = (-1)^{i-1}\overline{\lambda}[f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l].$$

Note that it is also the case that $\langle f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l \rangle = M$. By induction suppose we have proven that our original symbol is equal to a scalar times

$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l]$$

for some admissible sequence $(f_1, \ldots, f_j, e_{j+1}, \ldots, e_l)$ whose elements generate M, i.e., with $\langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_l \rangle = M$. Then we find the smallest i such that $f_{j+1} \in \langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_i \rangle$ and we go through the same process as above to see that

$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l] = (\text{scalar})[f_1, \ldots, f_j, f_{j+1}, e_{j+1}, \ldots, \hat{e_i}, \ldots, e_l]$$

Continuing in this vein we obtain the desired result. $\square$

Before we show that $\det_\kappa(M)$ always has dimension 1, let us show that it agrees with the usual top exterior power in the case the module is a vector space over κ.

Lemma 68.4. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module which is annihilated by $\mathfrak{m}$. Let $l = \dim_\kappa(M)$. Then the map*

$$\det_\kappa(M) \longrightarrow \wedge^l_\kappa(M), \quad [e_1, \ldots, e_l] \longmapsto e_1 \wedge \ldots \wedge e_l$$

is an isomorphism.

Proof. It is clear that the rule described in the lemma gives a κ-linear map since all of the admissible relations are satisfied by the usual symbols $e_1 \wedge \ldots \wedge e_l$. It is also clearly a surjective map. Since by Lemma 68.3 the left hand side has dimension at most one we see that the map is an isomorphism. $\square$

Lemma 68.5. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module. The determinant $\det_\kappa(M)$ defined above is a κ-vector space of dimension 1. It is generated by the symbol $[f_1, \ldots, f_l]$ for any admissible sequence such that $\langle f_1, \ldots f_l \rangle = M$.*

Proof. We know $\det_\kappa(M)$ has dimension at most 1, and in fact that it is generated by $[f_1, \ldots, f_l]$, by Lemma 68.3 and its proof. We will show by induction on $l = \text{length}(M)$ that it is nonzero. For $l = 1$ it follows from Lemma 68.4. Choose a nonzero element $f \in M$ with $\mathfrak{m}f = 0$. Set $\overline{M} = M/\langle f\rangle$, and denote the quotient map $x \mapsto \overline{x}$. We will define a surjective map

$$\psi : \det_k(M) \to \det_\kappa(\overline{M})$$

which will prove the lemma since by induction the determinant of $\overline{M}$ is nonzero.

We define ψ on symbols as follows. Let $(e_1, \ldots, e_l)$ be an admissible sequence. If $f \notin \langle e_1, \ldots, e_l\rangle$ then we simply set $\psi([e_1, \ldots, e_l]) = 0$. If $f \in \langle e_1, \ldots, e_l\rangle$ then we choose an i minimal such that $f \in \langle e_1, \ldots, e_i\rangle$. We may write $e_i = \lambda f + x$ for some unit $\lambda \in R$ and $x \in \langle e_1, \ldots, e_{i-1}\rangle$. In this case we set

$$\psi([e_1, \ldots, e_l]) = (-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l].$$

Note that it is indeed the case that $(\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l)$ is an admissible sequence in $\overline{M}$, so this makes sense. Let us show that extending this rule κ-linearly to linear combinations of symbols does indeed lead to a map on determinants. To do this we have to show that the admissible relations are mapped to zero.

Type (a) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$. Suppose that $f \in \langle e_1, \ldots, e_i\rangle$ with i minimal. Then $i \neq a$ and $\overline{e}_a \in \langle \overline{e}_1, \ldots, \hat{\overline{e}}_i, \ldots, \overline{e}_{a-1}\rangle$ if $i < a$ or $\overline{e}_a \in \langle \overline{e}_1, \ldots, \overline{e}_{a-1}\rangle$ if $i > a$. Thus the same admissible relation for $\det_\kappa(\overline{M})$ forces the symbol $[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$ to be zero as desired.

Type (b) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1}\rangle$. Suppose that $f \in \langle e_1, \ldots, e_i\rangle$ with i minimal. Say $e_i = \mu f + y$ with $y \in \langle e_1, \ldots, e_{i-1}\rangle$. If $i < a$ then the desired equality is

$$(-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = (-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda\overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. If $i > a$ then the desired equality is

$$(-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = (-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda\overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. The interesting case is when $i = a$. In this case we have $e_a = \lambda e'_a + x = \mu f + y$. Hence also $e'_a = \lambda^{-1}(\mu f + y - x)$. Thus we see that

$$\psi([e_1, \ldots, e_l]) = (-1)^i\overline{\mu}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \psi(\overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l])$$

as desired.

Type (c) relations. Suppose that $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m}e_a \subset \langle e_1, \ldots, e_{a-2}\rangle$. Suppose that $f \in \langle e_1, \ldots, e_i\rangle$ with i minimal. Say $e_i = \lambda f + x$ with $x \in \langle e_1, \ldots, e_{i-1}\rangle$. We distinguish 4 cases:

Case 1: $i < a - 1$. The desired equality is

$$\begin{aligned}&(-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]\\ &= (-1)^{i+1}\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]\end{aligned}$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$.

Case 2: $i > a$. The desired equality is

$$\begin{aligned}&(-1)^i\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]\\ &= (-1)^{i+1}\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]\end{aligned}$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$.

Case 3: $i = a$. We write $e_a = \lambda f + \mu e_{a-1} + y$ with $y \in \langle e_1, \ldots, e_{a-2}\rangle$. Then

$$\psi([e_1, \ldots, e_l]) = (-1)^a\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

by definition. If $\overline{\mu}$ is nonzero, then we have $e_{a-1} = -\mu^{-1}\lambda f + \mu^{-1}e_a - \mu^{-1}y$ and we obtain

$$\psi(-[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]) = (-1)^a\overline{\mu^{-1}\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

by definition. Since in $\overline{M}$ we have $\overline{e}_a = \mu\overline{e}_{a-1} + \overline{y}$ we see the two outcomes are equal by relation (a) for $\det_\kappa(\overline{M})$. If on the other hand $\overline{\mu}$ is zero, then we can write $e_a = \lambda f + y$ with $y \in \langle e_1, \ldots, e_{a-2}\rangle$ and we have

$$\psi(-[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]) = (-1)^a\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which is equal to $\psi([e_1, \ldots, e_l])$.

Case 4: $i = a - 1$. Here we have

$$\psi([e_1, \ldots, e_l]) = (-1)^{a-1}\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \ldots, \overline{e}_l]$$

by definition. If $f \notin \langle e_1, \ldots, e_{a-2}, e_a\rangle$ then

$$\psi(-[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]) = (-1)^{a+1}\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \ldots, \overline{e}_l]$$

Since $(-1)^{a-1} = (-1)^{a+1}$ the two expressions are the same. Finally, assume $f \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$. In this case we see that $e_{a-1} = \lambda f + x$ with $x \in \langle e_1, \ldots, e_{a-2}\rangle$ and $e_a = \mu f + y$ with $y \in \langle e_1, \ldots, e_{a-2}\rangle$ for units $\lambda, \mu \in R$. We conclude that both $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$ and $e_{a-1} \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$. In this case a relation of type (a) applies to both $[e_1, \ldots, e_l]$ and $[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l]$ and the compatibility of ψ with these shown above to see that both

$$\psi([e_1, \ldots, e_l]) \quad \text{and} \quad \psi([e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l])$$

are zero, as desired.

At this point we have shown that ψ is well defined, and all that remains is to show that it is surjective. To see this let $(\overline{f}_2, \ldots, \overline{f}_l)$ be an admissible sequence in $\overline{M}$. We can choose lifts $f_2, \ldots, f_l \in M$, and then $(f, f_2, \ldots, f_l)$ is an admissible sequence in M. Since $\psi([f, f_2, \ldots, f_l]) = [f_2, \ldots, f_l]$ we win. □

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Note that if $\varphi : M \to N$ is an isomorphism of finite length R-modules, then we get an isomorphism

$$\det_\kappa(\varphi) : \det_\kappa(M) \to \det_\kappa(N)$$

simply by the rule

$$\det_\kappa(\varphi)([e_1, \ldots, e_l]) = [\varphi(e_1), \ldots, \varphi(e_l)]$$

for any symbol $[e_1, \ldots, e_l]$ for M. Hence we see that $\det_\kappa$ is a functor

$$(68.5.1) \qquad \left\{\begin{matrix}\text{finite length } R\text{-modules}\\ \text{with isomorphisms}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{1-dimensional } \kappa\text{-vector spaces}\\ \text{with isomorphisms}\end{matrix}\right\}$$

This is typical for a "determinant functor" (see [Knu02]), as is the following additivity property.

Lemma 68.6. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. For every short exact sequence*

$$0 \to K \to L \to M \to 0$$

of finite length R-modules there exists a canonical isomorphism

$$\gamma_{K \to L \to M} : \det_\kappa(K) \otimes_\kappa \det_\kappa(M) \longrightarrow \det_\kappa(L)$$

defined by the rule on nonzero symbols

$$[e_1, \ldots, e_k] \otimes [\overline{f}_1, \ldots, \overline{f}_m] \longrightarrow [e_1, \ldots, e_k, f_1, \ldots, f_m]$$

with the following properties:

(1) *For every isomorphism of short exact sequences, i.e., for every commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "u"] & L \arrow[r] \arrow[d, "v"] & M \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & K' \arrow[r] & L' \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

with short exact rows and isomorphisms u, v, w we have

$$\gamma_{K' \to L' \to M'} \circ (\det_\kappa(u) \otimes \det_\kappa(w)) = \det_\kappa(v) \circ \gamma_{K \to L \to M},$$

(2) *for every commutative square of finite length R-modules with exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

the following diagram is commutative

$$\begin{tikzcd}
\det_\kappa(A) \otimes \det_\kappa(C) \otimes \det_\kappa(G) \otimes \det_\kappa(I) \arrow[r, "\gamma_{A \to B \to C} \otimes \gamma_{G \to H \to I}"'] \arrow[dd, "\epsilon"'] & \det_\kappa(B) \otimes \det_\kappa(H) \arrow[d, "\gamma_{B \to E \to H}"] \\
 & \det_\kappa(E) \\
\det_\kappa(A) \otimes \det_\kappa(G) \otimes \det_\kappa(C) \otimes \det_\kappa(I) \arrow[r, "\gamma_{A \to D \to G} \otimes \gamma_{C \to F \to I}"] & \det_\kappa(D) \otimes \det_\kappa(F) \arrow[u, "\gamma_{D \to E \to F}"']
\end{tikzcd}$$

where ϵ is the switch of the factors in the tensor product times $(-1)^{cg}$ with $c = length_R(C)$ and $g = length_R(G)$, and

(3) *the map* $\gamma_{K\to L\to M}$ *agrees with the usual isomorphism if* $0 \to K \to L \to M \to 0$ *is actually a short exact sequence of* κ*-vector spaces.*

Proof. The significance of taking nonzero symbols in the explicit description of the map $\gamma_{K\to L\to M}$ is simply that if $(e_1, \ldots, e_l)$ is an admissible sequence in K, and $(\overline{f}_1, \ldots, \overline{f}_m)$ is an admissible sequence in M, then it is not guaranteed that $(e_1, \ldots, e_l, f_1, \ldots, f_m)$ is an admissible sequence in L (where of course $f_i \in L$ signifies a lift of $\overline{f}_i$). However, if the symbol $[e_1, \ldots, e_l]$ is nonzero in $\det_\kappa(K)$, then necessarily $K = \langle e_1, \ldots, e_k\rangle$ (see proof of Lemma 68.3), and in this case it is true that $(e_1, \ldots, e_k, f_1, \ldots, f_m)$ is an admissible sequence. Moreover, by the admissible relations of type (b) for $\det_\kappa(L)$ we see that the value of $[e_1, \ldots, e_k, f_1, \ldots, f_m]$ in $\det_\kappa(L)$ is independent of the choice of the lifts f_i in this case also. Given this remark, it is clear that an admissible relation for $e_1, \ldots, e_k$ in K translates into an admissible relation among $e_1, \ldots, e_k, f_1, \ldots, f_m$ in L, and similarly for an admissible relation among the $\overline{f}_1, \ldots, \overline{f}_m$. Thus γ defines a linear map of vector spaces as claimed in the lemma.

By Lemma 68.5 we know $\det_\kappa(L)$ is generated by any single symbol $[x_1, \ldots, x_{k+m}]$ such that $(x_1, \ldots, x_{k+m})$ is an admissible sequence with $L = \langle x_1, \ldots, x_{k+m}\rangle$. Hence it is clear that the map $\gamma_{K\to L\to M}$ is surjective and hence an isomorphism.

Property (1) holds because

$$\begin{aligned} & \det_\kappa(v)([e_1, \ldots, e_k, f_1, \ldots, f_m]) \\ = \; & [v(e_1), \ldots, v(e_k), v(f_1), \ldots, v(f_m)] \\ = \; & \gamma_{K'\to L'\to M'}([u(e_1), \ldots, u(e_k)] \otimes [w(f_1), \ldots, w(f_m)]). \end{aligned}$$

Property (2) means that given a symbol $[\alpha_1, \ldots, \alpha_a]$ generating $\det_\kappa(A)$, a symbol $[\gamma_1, \ldots, \gamma_c]$ generating $\det_\kappa(C)$, a symbol $[\zeta_1, \ldots, \zeta_g]$ generating $\det_\kappa(G)$, and a symbol $[\iota_1, \ldots, \iota_i]$ generating $\det_\kappa(I)$ we have

$$\begin{aligned} & [\alpha_1, \ldots, \alpha_a, \tilde\gamma_1, \ldots, \tilde\gamma_c, \tilde\zeta_1, \ldots, \tilde\zeta_g, \tilde\iota_1, \ldots, \tilde\iota_i] \\ = \; & (-1)^{cg}[\alpha_1, \ldots, \alpha_a, \tilde\zeta_1, \ldots, \tilde\zeta_g, \tilde\gamma_1, \ldots, \tilde\gamma_c, \tilde\iota_1, \ldots, \tilde\iota_i] \end{aligned}$$

(for suitable lifts $\tilde x$ in E) in $\det_\kappa(E)$. This holds because we may use the admissible relations of type (c) cg times in the following order: move the $\tilde\zeta_1$ past the elements $\tilde\gamma_c, \ldots, \tilde\gamma_1$ (allowed since $\mathfrak m\tilde\zeta_1 \subset A$), then move $\tilde\zeta_2$ past the elements $\tilde\gamma_c, \ldots, \tilde\gamma_1$ (allowed since $\mathfrak m\tilde\zeta_2 \subset A + R\tilde\zeta_1$), and so on.

Part (3) of the lemma is obvious. This finishes the proof. □

We can use the maps γ of the lemma to define more general maps γ as follows. Suppose that $(R, \mathfrak m, \kappa)$ is a local ring. Let M be a finite length R-module and suppose we are given a finite filtration (see Homology, Definition 19.1)

$$0 = F^m \subset F^{m-1} \subset \ldots \subset F^{n+1} \subset F^n = M$$

then there is a well defined and canonical isomorphism

$$\gamma_{(M,F)} : \det_\kappa(F^{m-1}/F^m) \otimes_\kappa \ldots \otimes_k \det_\kappa(F^n/F^{n+1}) \longrightarrow \det_\kappa(M)$$

To construct it we use isomorphisms of Lemma 68.6 coming from the short exact sequences $0 \to F^{i-1}/F^i \to M/F^i \to M/F^{i-1} \to 0$. Part (2) of Lemma 68.6 with $G = 0$ shows we obtain the same isomorphism if we use the short exact sequences $0 \to F^i \to F^{i-1} \to F^{i-1}/F^i \to 0$.

Here is another typical result for determinant functors. It is not hard to show. The tricky part is usually to show the existence of a determinant functor.

Lemma 68.7. *Let $(R, \mathfrak{m}, \kappa)$ be any local ring. The functor*

$$\det_\kappa : \left\{ \begin{matrix} \textit{finite length } R\textit{-modules} \\ \textit{with isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} 1\textit{-dimensional } \kappa\textit{-vector spaces} \\ \textit{with isomorphisms} \end{matrix} \right\}$$

endowed with the maps $\gamma_{K \to L \to M}$ is characterized by the following properties

(1) *its restriction to the subcategory of modules annihilated by $\mathfrak{m}$ is isomorphic to the usual determinant functor (see Lemma 68.4), and*
(2) *(1), (2) and (3) of Lemma 68.6 hold.*

Proof. Omitted. □

Lemma 68.8. *Let $(R', \mathfrak{m}') \to (R, \mathfrak{m})$ be a local ring homomorphism which induces an isomorphism on residue fields κ. Then for every finite length R-module the restriction $M_{R'}$ is a finite length R'-module and there is a canonical isomorphism*

$$\det_{R,\kappa}(M) \longrightarrow \det_{R',\kappa}(M_{R'})$$

This isomorphism is functorial in M and compatible with the isomorphisms $\gamma_{K \to L \to M}$ of Lemma 68.6 defined for $\det_{R,\kappa}$ and $\det_{R',\kappa}$.

Proof. If the length of M as an R-module is l, then the length of M as an R'-module (i.e., $M_{R'}$) is l as well, see Algebra, Lemma 52.12. Note that an admissible sequence $x_1, \ldots, x_l$ of M over R is an admissible sequence of M over R' as $\mathfrak{m}'$ maps into $\mathfrak{m}$. The isomorphism is obtained by mapping the symbol $[x_1, \ldots, x_l] \in \det_{R,\kappa}(M)$ to the corresponding symbol $[x_1, \ldots, x_l] \in \det_{R',\kappa}(M)$. It is immediate to verify that this is functorial for isomorphisms and compatible with the isomorphisms γ of Lemma 68.6. □

Remark 68.9. Let $(R, \mathfrak{m}, \kappa)$ be a local ring and assume either the characteristic of κ is zero or it is p and $pR = 0$. Let $M_1, \ldots, M_n$ be finite length R-modules. We will show below that there exists an ideal $I \subset \mathfrak{m}$ annihilating M_i for $i = 1, \ldots, n$ and a section $\sigma : \kappa \to R/I$ of the canonical surjection $R/I \to \kappa$. The restriction $M_{i,\kappa}$ of M_i via σ is a κ-vector space of dimension $l_i = \text{length}_R(M_i)$ and using Lemma 68.8 we see that

$$\det_\kappa(M_i) = \wedge_\kappa^{l_i}(M_{i,\kappa})$$

These isomorphisms are compatible with the isomorphisms $\gamma_{K \to M \to L}$ of Lemma 68.6 for short exact sequences of finite length R-modules annihilated by I. The conclusion is that verifying a property of $\det_\kappa$ often reduces to verifying corresponding properties of the usual determinant on the category finite dimensional vector spaces.

For I we can take the annihilator (Algebra, Definition 40.3) of the module $M = \bigoplus M_i$. In this case we see that $R/I \subset \text{End}_R(M)$ hence has finite length. Thus R/I is an Artinian local ring with residue field κ. Since an Artinian local ring is complete we see that R/I has a coefficient ring by the Cohen structure theorem (Algebra, Theorem 160.8) which is a field by our assumption on R.

Here is a case where we can compute the determinant of a linear map. In fact there is nothing mysterious about this in any case, see Example 68.11 for a random example.

Lemma 68.10. *Let R be a local ring with residue field κ. Let $u \in R^*$ be a unit. Let M be a module of finite length over R. Denote $u_M : M \to M$ the map multiplication by u. Then*

$$\det_\kappa(u_M) : \det_\kappa(M) \longrightarrow \det_\kappa(M)$$

is multiplication by $\overline{u}^l$ where $l = length_R(M)$ and $\overline{u} \in \kappa^*$ is the image of u.

Proof. Denote $f_M \in \kappa^*$ the element such that $\det_\kappa(u_M) = f_M \mathrm{id}_{\det_\kappa(M)}$. Suppose that $0 \to K \to L \to M \to 0$ is a short exact sequence of finite R-modules. Then we see that u_k, u_L, u_M give an isomorphism of short exact sequences. Hence by Lemma 68.6 (1) we conclude that $f_K f_M = f_L$. This means that by induction on length it suffices to prove the lemma in the case of length 1 where it is trivial. $\square$

Example 68.11. Consider the local ring $R = \mathbf{Z}_p$. Set $M = \mathbf{Z}_p/(p^2) \oplus \mathbf{Z}_p/(p^3)$. Let $u : M \to M$ be the map given by the matrix

$$u = \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$$

where $a, b, c, d \in \mathbf{Z}_p$, and $a, d \in \mathbf{Z}_p^*$. In this case $\det_\kappa(u)$ equals multiplication by $a^2 d^3 \bmod p \in \mathbf{F}_p^*$. This can easily be seen by consider the effect of u on the symbol $[p^2 e, pe, pf, e, f]$ where $e = (0, 1) \in M$ and $f = (1, 0) \in M$.

68.12. Periodic complexes and determinants. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2, 1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. We are going to use the determinant construction to define an invariant of this situation. See Subsection 68.1. Let us abbreviate $K_\varphi = \mathrm{Ker}(\varphi)$, $I_\varphi = \mathrm{Im}(\varphi)$, $K_\psi = \mathrm{Ker}(\psi)$, and $I_\psi = \mathrm{Im}(\psi)$. The short exact sequences

$$0 \to K_\varphi \to M \to I_\varphi \to 0, \quad 0 \to K_\psi \to M \to I_\psi \to 0$$

give isomorphisms

$$\gamma_\varphi : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M), \quad \gamma_\psi : \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi) \longrightarrow \det_\kappa(M),$$

see Lemma 68.6. On the other hand the exactness of the complex gives equalities $K_\varphi = I_\psi$, and $K_\psi = I_\varphi$ and hence an isomorphism

$$\sigma : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi)$$

by switching the factors. Using this notation we can define our invariant.

Definition 68.13. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2, 1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. The *determinant of* (M, φ, ψ) is the element

$$\det_\kappa(M, \varphi, \psi) \in \kappa^*$$

such that the composition

$$\det_\kappa(M) \xrightarrow{\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}} \det_\kappa(M)$$

is multiplication by $(-1)^{\mathrm{length}_R(I_\varphi)\mathrm{length}_R(I_\psi)} \det_\kappa(M, \varphi, \psi)$.

Remark 68.14. Here is a more down to earth description of the determinant introduced above. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2, 1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Let us abbreviate $I_\varphi = \mathrm{Im}(\varphi)$, $I_\psi = \mathrm{Im}(\psi)$ as above. Assume that

$\text{length}_R(I_\varphi) = a$ and $\text{length}_R(I_\psi) = b$, so that $a + b = \text{length}_R(M)$ by exactness. Choose admissible sequences $x_1, \ldots, x_a \in I_\varphi$ and $y_1, \ldots, y_b \in I_\psi$ such that the symbol $[x_1, \ldots, x_a]$ generates $\det_\kappa(I_\varphi)$ and the symbol $[x_1, \ldots, x_b]$ generates $\det_\kappa(I_\psi)$. Choose $\tilde{x}_i \in M$ such that $\varphi(\tilde{x}_i) = x_i$. Choose $\tilde{y}_j \in M$ such that $\psi(\tilde{y}_j) = y_j$. Then $\det_\kappa(M, \varphi, \psi)$ is characterized by the equality

$$[x_1, \ldots, x_a, \tilde{y}_1, \ldots, \tilde{y}_b] = (-1)^{ab} \det_\kappa(M, \varphi, \psi)[y_1, \ldots, y_b, \tilde{x}_1, \ldots, \tilde{x}_a]$$

in $\det_\kappa(M)$. This also explains the sign.

Lemma 68.15. *Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Then*

$$\det_\kappa(M, \varphi, \psi) \det_\kappa(M, \psi, \varphi) = 1.$$

Proof. Omitted. ☐

Lemma 68.16. *Let R be a local ring with residue field κ. Let (M, φ, φ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, φ) is exact. Then $length_R(M) = 2 length_R(\text{Im}(\varphi))$ and*

$$\det_\kappa(M, \varphi, \varphi) = (-1)^{length_R(\text{Im}(\varphi))} = (-1)^{\frac{1}{2} length_R(M)}$$

Proof. Follows directly from the sign rule in the definitions. ☐

Lemma 68.17. *Let R be a local ring with residue field κ. Let M be a finite length R-module.*

(1) *if $\varphi : M \to M$ is an isomorphism then $\det_\kappa(M, \varphi, 0) = \det_\kappa(\varphi)$.*
(2) *if $\psi : M \to M$ is an isomorphism then $\det_\kappa(M, 0, \psi) = \det_\kappa(\psi)^{-1}$.*

Proof. Let us prove (1). Set $\psi = 0$. Then we may, with notation as above Definition 68.13, identify $K_\varphi = I_\psi = 0$, $I_\varphi = K_\psi = M$. With these identifications, the map

$$\gamma_\varphi : \kappa \otimes \det_\kappa(M) = \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M)$$

is identified with $\det_\kappa(\varphi^{-1})$. On the other hand the map γ_ψ is identified with the identity map. Hence $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ is equal to $\det_\kappa(\varphi)$ in this case. Whence the result. We omit the proof of (2). ☐

Lemma 68.18. *Let R be a local ring with residue field κ. Suppose that we have a short exact sequence of $(2,1)$-periodic complexes*

$$0 \to (M_1, \varphi_1, \psi_1) \to (M_2, \varphi_2, \psi_2) \to (M_3, \varphi_3, \psi_3) \to 0$$

with all M_i of finite length, and each (M_1, φ_1, ψ_1) exact. Then

$$\det_\kappa(M_2, \varphi_2, \psi_2) = \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3).$$

in κ^*.

Proof. Let us abbreviate $I_{\varphi,i} = \mathrm{Im}(\varphi_i)$, $K_{\varphi,i} = \mathrm{Ker}(\varphi_i)$, $I_{\psi,i} = \mathrm{Im}(\psi_i)$, and $K_{\psi,i} = \mathrm{Ker}(\psi_i)$. Observe that we have a commutative square

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_{\varphi,1} \arrow[r] \arrow[d] & K_{\varphi,2} \arrow[r] \arrow[d] & K_{\varphi,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_{\varphi,1} \arrow[r] \arrow[d] & I_{\varphi,2} \arrow[r] \arrow[d] & I_{\varphi,3} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

of finite length R-modules with exact rows and columns. The top row is exact since it can be identified with the sequence $I_{\psi,1} \to I_{\psi,2} \to I_{\psi,3} \to 0$ of images, and similarly for the bottom row. There is a similar diagram involving the modules $I_{\psi,i}$ and $K_{\psi,i}$. By definition $\det_\kappa(M_2, \varphi_2, \psi_2)$ corresponds, up to a sign, to the composition of the left vertical maps in the following diagram

$$\begin{tikzcd}
\det_\kappa(M_1) \otimes \det_\kappa(M_3) \arrow[r, "\gamma"] \arrow[d, "\gamma^{-1} \otimes \gamma^{-1}"] & \det_\kappa(M_2) \arrow[d, "\gamma^{-1}"] \\
\det_\kappa(K_{\varphi,1}) \otimes \det_\kappa(I_{\varphi,1}) \otimes \det_\kappa(K_{\varphi,3}) \otimes \det_\kappa(I_{\varphi,3}) \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "\sigma\otimes\sigma"] & \det_\kappa(K_{\varphi,2}) \otimes \det_\kappa(I_{\varphi,2}) \arrow[d, "\sigma"] \\
\det_\kappa(K_{\psi,1}) \otimes \det_\kappa(I_{\psi,1}) \otimes \det_\kappa(K_{\psi,3}) \otimes \det_\kappa(I_{\psi,3}) \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "\gamma\otimes\gamma"] & \det_\kappa(K_{\psi,2}) \otimes \det_\kappa(I_{\psi,2}) \arrow[d, "\gamma"] \\
\det_\kappa(M_1) \otimes \det_\kappa(M_3) \arrow[r, "\gamma"] & \det_\kappa(M_2)
\end{tikzcd}$$

The top and bottom squares are commutative up to sign by applying Lemma 68.6 (2). The middle square is trivially commutative (we are just switching factors). Hence we see that $\det_\kappa(M_2, \varphi_2, \psi_2) = \epsilon \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3)$ for some sign ϵ. And the sign can be worked out, namely the outer rectangle in the diagram above commutes up to

$$\begin{aligned}
\epsilon &= (-1)^{\mathrm{length}(I_{\varphi,1})\mathrm{length}(K_{\varphi,3}) + \mathrm{length}(I_{\psi,1})\mathrm{length}(K_{\psi,3})} \\
&= (-1)^{\mathrm{length}(I_{\varphi,1})\mathrm{length}(I_{\psi,3}) + \mathrm{length}(I_{\psi,1})\mathrm{length}(I_{\varphi,3})}
\end{aligned}$$

(proof omitted). It follows easily from this that the signs work out as well. $\square$

Example 68.19. Let k be a field. Consider the ring $R = k[T]/(T^2)$ of dual numbers over k. Denote t the class of T in R. Let $M = R$ and $\varphi = ut$, $\psi = vt$ with $u, v \in k^*$. In this case $\det_k(M)$ has generator $e = [t, 1]$. We identify $I_\varphi = K_\varphi = I_\psi = K_\psi = (t)$. Then $\gamma_\varphi(t \otimes t) = u^{-1}[t, 1]$ (since $u^{-1} \in M$ is a lift of $t \in I_\varphi$) and $\gamma_\psi(t \otimes t) = v^{-1}[t, 1]$ (same reason). Hence we see that $\det_k(M, \varphi, \psi) = -u/v \in k^*$.

Example 68.20. Let $R = \mathbf{Z}_p$ and let $M = \mathbf{Z}_p/(p^l)$. Let $\varphi = p^b u$ and $\varphi = p^a v$ with $a, b \geq 0$, $a + b = l$ and $u, v \in \mathbf{Z}_p^*$. Then a computation as in Example 68.19 shows that

$$\begin{aligned}\det_{\mathbf{F}_p}(\mathbf{Z}_p/(p^l), p^b u, p^a v) &= (-1)^{ab} u^a / v^b \bmod p \\ &= (-1)^{\mathrm{ord}_p(\alpha)\mathrm{ord}_p(\beta)} \frac{\alpha^{\mathrm{ord}_p(\beta)}}{\beta^{\mathrm{ord}_p(\alpha)}} \bmod p\end{aligned}$$

with $\alpha = p^b u, \beta = p^a v \in \mathbf{Z}_p$. See Lemma 68.37 for a more general case (and a proof).

Example 68.21. Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus b}$ be $l = a + b$ dimensional. Let φ and ψ be the following diagonal matrices

$$\varphi = \mathrm{diag}(u_1, \ldots, u_a, 0, \ldots, 0), \quad \psi = \mathrm{diag}(0, \ldots, 0, v_1, \ldots, v_b)$$

with $u_i, v_j \in k^*$. In this case we have

$$\det_k(M, \varphi, \psi) = \frac{u_1 \ldots u_a}{v_1 \ldots v_b}.$$

This can be seen by a direct computation or by computing in case $l = 1$ and using the additivity of Lemma 68.18.

Example 68.22. Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus a}$ be $l = 2a$ dimensional. Let φ and ψ be the following block matrices

$$\varphi = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

with $U, V \in \mathrm{Mat}(a \times a, k)$ invertible. In this case we have

$$\det_k(M, \varphi, \psi) = (-1)^a \frac{\det(U)}{\det(V)}.$$

This can be seen by a direct computation. The case $a = 1$ is similar to the computation in Example 68.19.

Example 68.23. Let $R = k$ be a field. Let $M = k^{\oplus 4}$. Let

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_2 & 0 \end{pmatrix} \quad \varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 \end{pmatrix}$$

with $u_1, u_2, v_1, v_2 \in k^*$. Then we have

$$\det_k(M, \varphi, \psi) = -\frac{u_1 u_2}{v_1 v_2}.$$

Next we come to the analogue of the fact that the determinant of a composition of linear endomorphisms is the product of the determinants. To avoid very long formulae we write $I_\varphi = \mathrm{Im}(\varphi)$, and $K_\varphi = \mathrm{Ker}(\varphi)$ for any R-module map $\varphi : M \to M$. We also denote $\varphi\psi = \varphi \circ \psi$ for a pair of morphisms $\varphi, \psi : M \to M$.

Lemma 68.24. *Let R be a local ring with residue field κ. Let M be a finite length R-module. Let α, β, γ be endomorphisms of M. Assume that*

1. *$I_\alpha = K_{\beta\gamma}$, and similarly for any permutation of α, β, γ,*
2. *$K_\alpha = I_{\beta\gamma}$, and similarly for any permutation of α, β, γ.*

Then

(1) *The triple* $(M, \alpha, \beta\gamma)$ *is an exact* $(2,1)$*-periodic complex.*
(2) *The triple* $(I_\gamma, \alpha, \beta)$ *is an exact* $(2,1)$*-periodic complex.*
(3) *The triple* $(M/K_\beta, \alpha, \gamma)$ *is an exact* $(2,1)$*-periodic complex.*
(4) *We have*

$$\det_\kappa(M, \alpha, \beta\gamma) = \det_\kappa(I_\gamma, \alpha, \beta) \det_\kappa(M/K_\beta, \alpha, \gamma).$$

Proof. It is clear that the assumptions imply part (1) of the lemma.

To see part (1) note that the assumptions imply that $I_{\gamma\alpha} = I_{\alpha\gamma}$, and similarly for kernels and any other pair of morphisms. Moreover, we see that $I_{\gamma\beta} = I_{\beta\gamma} = K_\alpha \subset I_\gamma$ and similarly for any other pair. In particular we get a short exact sequence

$$0 \to I_{\beta\gamma} \to I_\gamma \xrightarrow{\alpha} I_{\alpha\gamma} \to 0$$

and similarly we get a short exact sequence

$$0 \to I_{\alpha\gamma} \to I_\gamma \xrightarrow{\beta} I_{\beta\gamma} \to 0.$$

This proves $(I_\gamma, \alpha, \beta)$ is an exact $(2,1)$-periodic complex. Hence part (2) of the lemma holds.

To see that α, γ give well defined endomorphisms of M/K_β we have to check that $\alpha(K_\beta) \subset K_\beta$ and $\gamma(K_\beta) \subset K_\beta$. This is true because $\alpha(K_\beta) = \alpha(I_{\gamma\alpha}) = I_{\alpha\gamma\alpha} \subset I_{\alpha\gamma} = K_\beta$, and similarly in the other case. The kernel of the map $\alpha : M/K_\beta \to M/K_\beta$ is $K_{\beta\alpha}/K_\beta = I_\gamma/K_\beta$. Similarly, the kernel of $\gamma : M/K_\beta \to M/K_\beta$ is equal to I_α/K_β. Hence we conclude that (3) holds.

We introduce $r = \text{length}_R(K_\alpha)$, $s = \text{length}_R(K_\beta)$ and $t = \text{length}_R(K_\gamma)$. By the exact sequences above and our hypotheses we have $\text{length}_R(I_\alpha) = s + t$, $\text{length}_R(I_\beta) = r + t$, $\text{length}_R(I_\gamma) = r + s$, and $\text{length}(M) = r + s + t$. Choose

(1) an admissible sequence $x_1, \ldots, x_r \in K_\alpha$ generating K_α
(2) an admissible sequence $y_1, \ldots, y_s \in K_\beta$ generating K_β,
(3) an admissible sequence $z_1, \ldots, z_t \in K_\gamma$ generating K_γ,
(4) elements $\tilde{x}_i \in M$ such that $\beta\gamma\tilde{x}_i = x_i$,
(5) elements $\tilde{y}_i \in M$ such that $\alpha\gamma\tilde{y}_i = y_i$,
(6) elements $\tilde{z}_i \in M$ such that $\beta\alpha\tilde{z}_i = z_i$.

With these choices the sequence $y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t$ is an admissible sequence in I_α generating it. Hence, by Remark 68.14 the determinant $D = \det_\kappa(M, \alpha, \beta\gamma)$ is the unique element of κ^* such that

$$\begin{aligned} & [y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_s, \tilde{x}_1, \ldots, \tilde{x}_r] \\ &= (-1)^{r(s+t)} D [x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t] \end{aligned}$$

By the same remark, we see that $D_1 = \det_\kappa(M/K_\beta, \alpha, \gamma)$ is characterized by

$$[y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t, \tilde{x}_1, \ldots, \tilde{x}_r] = (-1)^{rt} D_1 [y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t]$$

By the same remark, we see that $D_2 = \det_\kappa(I_\gamma, \alpha, \beta)$ is characterized by

$$[y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t] = (-1)^{rs} D_2 [x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t]$$

Combining the formulas above we see that $D = D_1 D_2$ as desired. ☐

Lemma 68.25. *Let R be a local ring with residue field κ. Let $\alpha : (M, \varphi, \psi) \to (M', \varphi', \psi')$ be a morphism of $(2,1)$-periodic complexes over R. Assume*

(1) *M, M' have finite length,*

(2) *(M, φ, ψ), (M', φ', ψ') are exact,*
(3) *the maps φ, ψ induce the zero map on $K = \mathrm{Ker}(\alpha)$, and*
(4) *the maps φ, ψ induce the zero map on $Q = \mathrm{Coker}(\alpha)$.*

Denote $N = \alpha(M) \subset M'$. We obtain two short exact sequences of $(2,1)$-periodic complexes

$$0 \to (N, \varphi', \psi') \to (M', \varphi', \psi') \to (Q, 0, 0) \to 0$$
$$0 \to (K, 0, 0) \to (M, \varphi, \psi) \to (N, \varphi', \psi') \to 0$$

which induce two isomorphisms $\alpha_i : Q \to K$, $i = 0, 1$. Then

$$\det_\kappa(M, \varphi, \psi) = \det_\kappa(\alpha_0^{-1} \circ \alpha_1) \det_\kappa(M', \varphi', \psi')$$

In particular, if $\alpha_0 = \alpha_1$, then $\det_\kappa(M, \varphi, \psi) = \det_\kappa(M', \varphi', \psi')$.

Proof. There are (at least) two ways to prove this lemma. One is to produce an enormous commutative diagram using the properties of the determinants. The other is to use the characterization of the determinants in terms of admissible sequences of elements. It is the second approach that we will use.

First let us explain precisely what the maps α_i are. Namely, α_0 is the composition

$$\alpha_0 : Q = H^0(Q, 0, 0) \to H^1(N, \varphi', \psi') \to H^2(K, 0, 0) = K$$

and α_1 is the composition

$$\alpha_1 : Q = H^1(Q, 0, 0) \to H^2(N, \varphi', \psi') \to H^3(K, 0, 0) = K$$

coming from the boundary maps of the short exact sequences of complexes displayed in the lemma. The fact that the complexes (M, φ, ψ), (M', φ', ψ') are exact implies these maps are isomorphisms.

We will use the notation $I_\varphi = \mathrm{Im}(\varphi)$, $K_\varphi = \mathrm{Ker}(\varphi)$ and similarly for the other maps. Exactness for M and M' means that $K_\varphi = I_\psi$ and three similar equalities. We introduce $k = \mathrm{length}_R(K)$, $a = \mathrm{length}_R(I_\varphi)$, $b = \mathrm{length}_R(I_\psi)$. Then we see that $\mathrm{length}_R(M) = a + b$, and $\mathrm{length}_R(N) = a + b - k$, $\mathrm{length}_R(Q) = k$ and $\mathrm{length}_R(M') = a+b$. The exact sequences below will show that also $\mathrm{length}_R(I_{\varphi'}) = a$ and $\mathrm{length}_R(I_{\psi'}) = b$.

The assumption that $K \subset K_\varphi = I_\psi$ means that φ factors through N to give an exact sequence

$$0 \to \alpha(I_\psi) \to N \xrightarrow{\varphi\alpha^{-1}} I_\psi \to 0.$$

Here $\varphi\alpha^{-1}(x') = y$ means $x' = \alpha(x)$ and $y = \varphi(x)$. Similarly, we have

$$0 \to \alpha(I_\varphi) \to N \xrightarrow{\psi\alpha^{-1}} I_\varphi \to 0.$$

The assumption that ψ' induces the zero map on Q means that $I_{\psi'} = K_{\varphi'} \subset N$. This means the quotient $\varphi'(N) \subset I_{\varphi'}$ is identified with Q. Note that $\varphi'(N) = \alpha(I_\varphi)$. Hence we conclude there is an isomorphism

$$\varphi' : Q \to I_{\varphi'}/\alpha(I_\varphi)$$

simply described by $\varphi'(x' \bmod N) = \varphi'(x') \bmod \alpha(I_\varphi)$. In exactly the same way we get

$$\psi' : Q \to I_{\psi'}/\alpha(I_\psi)$$

Finally, note that α_0 is the composition

$$Q \xrightarrow{\varphi'} I_{\varphi'}/\alpha(I_\varphi) \xrightarrow{\psi\alpha^{-1}|_{I_{\varphi'}/\alpha(I_\varphi)}} K$$

and similarly $\alpha_1 = \varphi\alpha^{-1}|_{I_{\psi'}/\alpha(I_\psi)} \circ \psi'$.

To shorten the formulas below we are going to write αx instead of $\alpha(x)$ in the following. No confusion should result since all maps are indicated by Greek letters and elements by Roman letters. We are going to choose

(1) an admissible sequence $z_1, \ldots, z_k \in K$ generating K,
(2) elements $z_i' \in M$ such that $\varphi z_i' = z_i$,
(3) elements $z_i'' \in M$ such that $\psi z_i'' = z_i$,
(4) elements $x_{k+1}, \ldots, x_a \in I_\varphi$ such that $z_1, \ldots, z_k, x_{k+1}, \ldots, x_a$ is an admissible sequence generating I_φ,
(5) elements $\tilde{x}_i \in M$ such that $\varphi\tilde{x}_i = x_i$,
(6) elements $y_{k+1}, \ldots, y_b \in I_\psi$ such that $z_1, \ldots, z_k, y_{k+1}, \ldots, y_b$ is an admissible sequence generating I_ψ,
(7) elements $\tilde{y}_i \in M$ such that $\psi\tilde{y}_i = y_i$, and
(8) elements $w_1, \ldots, w_k \in M'$ such that $w_1 \bmod N, \ldots, w_k \bmod N$ are an admissible sequence in Q generating Q.

By Remark 68.14 the element $D = \det_\kappa(M, \varphi, \psi) \in \kappa^*$ is characterized by

$$\begin{aligned} & [z_1, \ldots, z_k, x_{k+1}, \ldots, x_a, z_1'', \ldots, z_k'', \tilde{y}_{k+1}, \ldots, \tilde{y}_b] \\ = \quad & (-1)^{ab} D[z_1, \ldots, z_k, y_{k+1}, \ldots, y_b, z_1', \ldots, z_k', \tilde{x}_{k+1}, \ldots, \tilde{x}_a] \end{aligned}$$

Note that by the discussion above $\alpha x_{k+1}, \ldots, \alpha x_a, \varphi w_1, \ldots, \varphi w_k$ is an admissible sequence generating $I_{\varphi'}$ and $\alpha y_{k+1}, \ldots, \alpha y_b, \psi w_1, \ldots, \psi w_k$ is an admissible sequence generating $I_{\psi'}$. Hence by Remark 68.14 the element $D' = \det_\kappa(M', \varphi', \psi') \in \kappa^*$ is characterized by

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b, w_1, \ldots, w_k] \\ = \quad & (-1)^{ab} D'[\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a, w_1, \ldots, w_k] \end{aligned}$$

Note how in the first, resp. second displayed formula the first, resp. last k entries of the symbols on both sides are the same. Hence these formulas are really equivalent to the equalities

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z_1'', \ldots, \alpha z_k'', \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b] \\ = \quad & (-1)^{ab} D[\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z_1', \ldots, \alpha z_k', \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a] \end{aligned}$$

and

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b] \\ = \quad & (-1)^{ab} D'[\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a] \end{aligned}$$

in $\det_\kappa(N)$. Note that $\varphi' w_1, \ldots, \varphi' w_k$ and $\alpha z_1'', \ldots, z_k''$ are admissible sequences generating the module $I_{\varphi'}/\alpha(I_\varphi)$. Write

$$[\varphi' w_1, \ldots, \varphi' w_k] = \lambda_0[\alpha z_1'', \ldots, \alpha z_k'']$$

in $\det_\kappa(I_{\varphi'}/\alpha(I_\varphi))$ for some $\lambda_0 \in \kappa^*$. Similarly, write

$$[\psi' w_1, \ldots, \psi' w_k] = \lambda_1[\alpha z_1', \ldots, \alpha z_k']$$

in $\det_\kappa(I_{\psi'}/\alpha(I_\psi))$ for some $\lambda_1 \in \kappa^*$. On the one hand it is clear that

$$\alpha_i([w_1, \ldots, w_k]) = \lambda_i[z_1, \ldots, z_k]$$

for $i = 0, 1$ by our description of α_i above, which means that

$$\det_\kappa(\alpha_0^{-1} \circ \alpha_1) = \lambda_1/\lambda_0$$

and on the other hand it is clear that

$$\begin{aligned} & \lambda_0[\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z_1'', \ldots, \alpha z_k'', \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \\ = \quad & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \end{aligned}$$

and

$$\begin{aligned} & \lambda_1[\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z_1', \ldots, \alpha z_k', \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \\ = \quad & [\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \end{aligned}$$

which imply $\lambda_0 D = \lambda_1 D'$. The lemma follows. ☐

68.26. Symbols. The correct generality for this construction is perhaps the situation of the following lemma.

Lemma 68.27. *Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Assume $\varphi, \psi : M \to M$ are two injective A-module maps, and assume $\varphi(\psi(M)) = \psi(\varphi(M))$, for example if φ and ψ commute. Then $\text{length}_R(M/\varphi\psi M) < \infty$ and $(M/\varphi\psi M, \varphi, \psi)$ is an exact $(2,1)$-periodic complex.*

Proof. Let $\mathfrak{q}$ be a minimal prime of the support of M. Then $M_{\mathfrak{q}}$ is a finite length $A_{\mathfrak{q}}$-module, see Algebra, Lemma 62.3. Hence both φ and ψ induce isomorphisms $M_{\mathfrak{q}} \to M_{\mathfrak{q}}$. Thus the support of $M/\varphi\psi M$ is $\{\mathfrak{m}_A\}$ and hence it has finite length (see lemma cited above). Finally, the kernel of φ on $M/\varphi\psi M$ is clearly $\psi M/\varphi\psi M$, and hence the kernel of φ is the image of ψ on $M/\varphi\psi M$. Similarly the other way since $M/\varphi\psi M = M/\psi\varphi M$ by assumption. ☐

Lemma 68.28. *Let A be a Noetherian local ring. Let $a, b \in A$.*

1. *If M is a finite A-module of dimension 1 such that a, b are nonzerodivisors on M, then $\text{length}_A(M/abM) < \infty$ and $(M/abM, a, b)$ is a $(2,1)$-periodic exact complex.*
2. *If a, b are nonzerodivisors and $\dim(A) = 1$ then $\text{length}_A(A/(ab)) < \infty$ and $(A/(ab), a, b)$ is a $(2,1)$-periodic exact complex.*

In particular, in these cases $\det_\kappa(M/abM, a, b) \in \kappa^*$, resp. $\det_\kappa(A/(ab), a, b) \in \kappa^*$ are defined.

Proof. Follows from Lemma 68.27. ☐

Definition 68.29. Let A be a Noetherian local ring with residue field κ. Let $a, b \in A$. Let M be a finite A-module of dimension 1 such that a, b are nonzerodivisors on M. We define the *symbol associated to M, a, b* to be the element

$$d_M(a, b) = \det_\kappa(M/abM, a, b) \in \kappa^*$$

Lemma 68.30. *Let A be a Noetherian local ring. Let $a, b, c \in A$. Let M be a finite A-module with $\dim(Supp(M)) = 1$. Assume a, b, c are nonzerodivisors on M. Then*

$$d_M(a, bc) = d_M(a, b) d_M(a, c)$$

and $d_M(a, b) d_M(b, a) = 1$.

Proof. The first statement follows from Lemma 68.24 applied to $M/abcM$ and endomorphisms α, β, γ given by multiplication by a, b, c. The second comes from Lemma 68.15. ☐

Definition 68.31. Let A be a Noetherian local domain of dimension 1 with residue field κ. Let K be the fraction field of A. We define the *tame symbol* of A to be the map

$$K^* \times K^* \longrightarrow \kappa^*, \quad (x, y) \longmapsto d_A(x, y)$$

where $d_A(x, y)$ is extended to $K^* \times K^*$ by the multiplicativity of Lemma 68.30.

It is clear that we may extend more generally $d_M(-, -)$ to certain rings of fractions of A (even if A is not a domain).

Lemma 68.32. *Let A be a Noetherian local ring and M a finite A-module of dimension 1. Let $a \in A$ be a nonzerodivisor on M. Then $d_M(a, a) = (-1)^{length_A(M/aM)}$.*

Proof. Immediate from Lemma 68.16. $\square$

Lemma 68.33. *Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Let $b \in A$ be a nonzerodivisor on M, and let $u \in A^*$. Then*

$$d_M(u, b) = u^{length_A(M/bM)} \bmod \mathfrak{m}_A.$$

In particular, if $M = A$, then $d_A(u, b) = u^{ord_A(b)} \bmod \mathfrak{m}_A$.

Proof. Note that in this case $M/ubM = M/bM$ on which multiplication by b is zero. Hence $d_M(u, b) = \det_\kappa(u|_{M/bM})$ by Lemma 68.17. The lemma then follows from Lemma 68.10. $\square$

Lemma 68.34. *Let A be a Noetherian local ring. Let $a, b \in A$. Let*

$$0 \to M \to M' \to M'' \to 0$$

be a short exact sequence of A-modules of dimension 1 such that a, b are nonzerodivisors on all three A-modules. Then

$$d_{M'}(a, b) = d_M(a, b) d_{M''}(a, b)$$

in κ^*.

Proof. It is easy to see that this leads to a short exact sequence of exact $(2, 1)$-periodic complexes

$$0 \to (M/abM, a, b) \to (M'/abM', a, b) \to (M''/abM'', a, b) \to 0$$

Hence the lemma follows from Lemma 68.18. $\square$

Lemma 68.35. *Let A be a Noetherian local ring. Let $\alpha : M \to M'$ be a homomorphism of finite A-modules of dimension 1. Let $a, b \in A$. Assume*

1. *a, b are nonzerodivisors on both M and M', and*
2. $\dim(\mathrm{Ker}(\alpha)), \dim(\mathrm{Coker}(\alpha)) \leq 0$.

Then $d_M(a, b) = d_{M'}(a, b)$.

Proof. If $a \in A^*$, then the equality follows from the equality $\text{length}(M/bM) = \text{length}(M'/bM')$ and Lemma 68.33. Similarly if b is a unit the lemma holds as well (by the symmetry of Lemma 68.30). Hence we may assume that $a, b \in \mathfrak{m}_A$. This in particular implies that $\mathfrak{m}$ is not an associated prime of M, and hence $\alpha : M \to M'$ is injective. This permits us to think of M as a submodule of M'. By assumption M'/M is a finite A-module with support $\{\mathfrak{m}_A\}$ and hence has finite length. Note that for any third module M'' with $M \subset M'' \subset M'$ the maps $M \to M''$ and $M'' \to M'$ satisfy the assumptions of the lemma as well. This reduces us, by induction on the length of M'/M, to the case where $\text{length}_A(M'/M) = 1$. Finally, in this case consider the map

$$\overline{\alpha} : M/abM \longrightarrow M'/abM'.$$

By construction the cokernel Q of $\overline{\alpha}$ has length 1. Since $a, b \in \mathfrak{m}_A$, they act trivially on Q. It also follows that the kernel K of $\overline{\alpha}$ has length 1 and hence also a, b act trivially on K. Hence we may apply Lemma 68.25. Thus it suffices to see that the two maps $\alpha_i : Q \to K$ are the same. In fact, both maps are equal to the map $q = x' \bmod \text{Im}(\overline{\alpha}) \mapsto abx' \in K$. We omit the verification. $\square$

Lemma 68.36. *Let A be a Noetherian local ring. Let M be a finite A-module with $\dim(Supp(M)) = 1$. Let $a, b \in A$ nonzerodivisors on M. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then*

$$d_M(a, b) = \prod\nolimits_{i=1,\ldots,t} d_{A/\mathfrak{q}_i}(a, b)^{length_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})}$$

as elements of κ^*.

Proof. Choose a filtration by A-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_j/M_{j-1} is isomorphic to $A/\mathfrak{p}_j$ for some prime ideal $\mathfrak{p}_j$ of A. See Algebra, Lemma 62.1. For each j we have either $\mathfrak{p}_j = \mathfrak{q}_i$ for some i, or $\mathfrak{p}_j = \mathfrak{m}_A$. Moreover, for a fixed i, the number of j such that $\mathfrak{p}_j = \mathfrak{q}_i$ is equal to $\text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$ by Algebra, Lemma 62.5. Hence $d_{M_j}(a, b)$ is defined for each j and

$$d_{M_j}(a, b) = \begin{cases} d_{M_{j-1}}(a, b) d_{A/\mathfrak{q}_i}(a, b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{q}_i \\ d_{M_{j-1}}(a, b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{m}_A \end{cases}$$

by Lemma 68.34 in the first instance and Lemma 68.35 in the second. Hence the lemma. $\square$

Lemma 68.37. *Let A be a discrete valuation ring with fraction field K. For nonzero $x, y \in K$ we have*

$$d_A(x, y) = (-1)^{ord_A(x) ord_A(y)} \frac{x^{ord_A(y)}}{y^{ord_A(x)}} \bmod \mathfrak{m}_A,$$

in other words the symbol is equal to the usual tame symbol.

Proof. By multiplicativity it suffices to prove this when $x, y \in A$. Let $t \in A$ be a uniformizer. Write $x = t^b u$ and $y = t^b v$ for some $a, b \geq 0$ and $u, v \in A^*$. Set $l = a + b$. Then $t^{l-1}, \ldots, t^b$ is an admissible sequence in $(x)/(xy)$ and $t^{l-1}, \ldots, t^a$ is an admissible sequence in $(y)/(xy)$. Hence by Remark 68.14 we see that $d_A(x, y)$ is characterized by the equation

$$[t^{l-1}, \ldots, t^b, v^{-1}t^{b-1}, \ldots, v^{-1}] = (-1)^{ab} d_A(x, y) [t^{l-1}, \ldots, t^a, u^{-1}t^{a-1}, \ldots, u^{-1}].$$

Hence by the admissible relations for the symbols $[x_1, \ldots, x_l]$ we see that

$$d_A(x, y) = (-1)^{ab} u^a / v^b \bmod \mathfrak{m}_A$$

as desired. $\square$

Lemma 68.38. *Let A be a Noetherian local ring. Let $a, b \in A$. Let M be a finite A-module of dimension 1 on which each of a, b, $b-a$ are nonzerodivisors. Then*

$$d_M(a, b-a)d_M(b,b) = d_M(b, b-a)d_M(a,b)$$

in κ^*.

Proof. By Lemma 68.36 it suffices to show the relation when $M = A/\mathfrak q$ for some prime $\mathfrak q \subset A$ with $\dim(A/\mathfrak q) = 1$.

In case $M = A/\mathfrak q$ we may replace A by $A/\mathfrak q$ and a, b by their images in $A/\mathfrak q$. Hence we may assume $A = M$ and A a local Noetherian domain of dimension 1. The reason is that the residue field κ of A and $A/\mathfrak q$ are the same and that for any $A/\mathfrak q$-module M the determinant taken over A or over $A/\mathfrak q$ are canonically identified. See Lemma 68.8.

It suffices to show the relation when both a, b are in the maximal ideal. Namely, the case where one or both are units follows from Lemmas 68.33 and 68.32.

Choose an extension $A \subset A'$ and factorizations $a = ta'$, $b = tb'$ as in Lemma 4.2. Note that also $b - a = t(b' - a')$ and that $A' = (a', b') = (a', b' - a') = (b' - a', b')$. Here and in the following we think of A' as an A-module and a, b, a', b', t as A-module endomorphisms of A'. We will use the notation $d^A_{A'}(a', b')$ and so on to indicate

$$d^A_{A'}(a', b') = \det\nolimits_\kappa(A'/a'b'A', a', b')$$

which is defined by Lemma 68.27. The upper index ^{A} is used to distinguish this from the already defined symbol $d_{A'}(a', b')$ which is different (for example because it has values in the residue field of A' which may be different from κ). By Lemma 68.35 we see that $d_A(a, b) = d^A_{A'}(a, b)$, and similarly for the other combinations. Using this and multiplicativity we see that it suffices to prove

$$d^A_{A'}(a', b'-a')d^A_{A'}(b', b') = d^A_{A'}(b', b'-a')d^A_{A'}(a', b')$$

Now, since $(a', b') = A'$ and so on we have

$$\begin{array}{rcl} A'/(a'(b'-a')) & \cong & A'/(a') \oplus A'/(b'-a') \\ A'/(b'(b'-a')) & \cong & A'/(b') \oplus A'/(b'-a') \\ A'/(a'b') & \cong & A'/(a') \oplus A'/(b') \end{array}$$

Moreover, note that multiplication by $b' - a'$ on $A/(a')$ is equal to multiplication by b', and that multiplication by $b' - a'$ on $A/(b')$ is equal to multiplication by $-a'$. Using Lemmas 68.17 and 68.18 we conclude

$$\begin{array}{rcl} d^A_{A'}(a', b'-a') & = & \det_\kappa(b'|_{A'/(a')})^{-1} \det_\kappa(a'|_{A'/(b'-a')}) \\ d^A_{A'}(b', b'-a') & = & \det_\kappa(-a'|_{A'/(b')})^{-1} \det_\kappa(b'|_{A'/(b'-a')}) \\ d^A_{A'}(a', b') & = & \det_\kappa(b'|_{A'/(a')})^{-1} \det_\kappa(a'|_{A'/(b')}) \end{array}$$

Hence we conclude that

$$(-1)^{\text{length}_A(A'/(b'))} d^A_{A'}(a', b'-a') = d^A_{A'}(b', b'-a')d^A_{A'}(a', b')$$

the sign coming from the $-a'$ in the second equality above. On the other hand, by Lemma 68.16 we have $d^A_{A'}(b', b') = (-1)^{\text{length}_A(A'/(b'))}$ and the lemma is proved. □

The tame symbol is a Steinberg symbol.

Lemma 68.39. *Let A be a Noetherian local domain of dimension 1 with fraction field K. For $x \in K \setminus \{0, 1\}$ we have*

$$d_A(x, 1-x) = 1$$

Proof. Write $x = a/b$ with $a, b \in A$. The hypothesis implies, since $1-x = (b-a)/b$, that also $b - a \neq 0$. Hence we compute

$$d_A(x, 1-x) = d_A(a, b-a)d_A(a, b)^{-1}d_A(b, b-a)^{-1}d_A(b, b)$$

Thus we have to show that $d_A(a, b-a)d_A(b, b) = d_A(b, b-a)d_A(a, b)$. This is Lemma 68.38. $\square$

68.40. Lengths and determinants. In this section we use the determinant to compare lattices. The key lemma is the following.

Lemma 68.41. *Let R be a Noetherian local ring. Let $\mathfrak{q} \subset R$ be a prime with $\dim(R/\mathfrak{q}) = 1$. Let $\varphi : M \to N$ be a homomorphism of finite R-modules. Assume there exist $x_1, \ldots, x_l \in M$ and $y_1, \ldots, y_l \in M$ with the following properties*

1. *$M = \langle x_1, \ldots, x_l \rangle$,*
2. *$\langle x_1, \ldots, x_i \rangle / \langle x_1, \ldots, x_{i-1} \rangle \cong R/\mathfrak{q}$ for $i = 1, \ldots, l$,*
3. *$N = \langle y_1, \ldots, y_l \rangle$, and*
4. *$\langle y_1, \ldots, y_i \rangle / \langle y_1, \ldots, y_{i-1} \rangle \cong R/\mathfrak{q}$ for $i = 1, \ldots, l$.*

Then φ is injective if and only if $\varphi_{\mathfrak{q}}$ is an isomorphism, and in this case we have

$$\mathit{length}_R(\mathrm{Coker}(\varphi)) = \mathit{ord}_{R/\mathfrak{q}}(f)$$

where $f \in \kappa(\mathfrak{q})$ is the element such that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l]$$

in $\det_{\kappa(\mathfrak{q})}(N_{\mathfrak{q}})$.

Proof. First, note that the lemma holds in case $l = 1$. Namely, in this case x_1 is a basis of M over $R/\mathfrak{q}$ and y_1 is a basis of N over $R/\mathfrak{q}$ and we have $\varphi(x_1) = fy_1$ for some $f \in R$. Thus φ is injective if and only if $f \notin \mathfrak{q}$. Moreover, $\mathrm{Coker}(\varphi) = R/(f, \mathfrak{q})$ and hence the lemma holds by definition of $\mathrm{ord}_{R/q}(f)$ (see Algebra, Definition 121.2).

In fact, suppose more generally that $\varphi(x_i) = f_i y_i$ for some $f_i \in R$, $f_i \notin \mathfrak{q}$. Then the induced maps

$$\langle x_1, \ldots, x_i \rangle / \langle x_1, \ldots, x_{i-1} \rangle \longrightarrow \langle y_1, \ldots, y_i \rangle / \langle y_1, \ldots, y_{i-1} \rangle$$

are all injective and have cokernels isomorphic to $R/(f_i, \mathfrak{q})$. Hence we see that

$$\mathrm{length}_R(\mathrm{Coker}(\varphi)) = \sum \mathrm{ord}_{R/\mathfrak{q}}(f_i).$$

On the other hand it is clear that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f_1 \ldots f_l[y_1, \ldots, y_l]$$

in this case from the admissible relation (b) for symbols. Hence we see the result holds in this case also.

We prove the general case by induction on l. Assume $l > 1$. Let $i \in \{1, \ldots, l\}$ be minimal such that $\varphi(x_1) \in \langle y_1, \ldots, y_i\rangle$. We will argue by induction on i. If $i = 1$, then we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \langle x_1\rangle \arrow[r] \arrow[d] & \langle x_1, \ldots, x_l\rangle \arrow[r] \arrow[d] & \langle x_1, \ldots, x_l\rangle/\langle x_1\rangle \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \langle y_1\rangle \arrow[r] & \langle y_1, \ldots, y_l\rangle \arrow[r] & \langle y_1, \ldots, y_l\rangle/\langle y_1\rangle \arrow[r] & 0 \end{tikzcd}$$

and the lemma follows from the snake lemma and induction on l. Assume now that $i > 1$. Write $\varphi(x_1) = a_1y_1 + \ldots + a_{i-1}y_{i-1} + ay_i$ with $a_j, a \in R$ and $a \notin \mathfrak q$ (since otherwise i was not minimal). Set

$$x'_j = \begin{cases} x_j & \text{if} \quad j = 1 \\ ax_j & \text{if} \quad j \geq 2 \end{cases} \quad \text{and} \quad y'_j = \begin{cases} y_j & \text{if} \quad j < i \\ ay_j & \text{if} \quad j \geq i \end{cases}$$

Let $M' = \langle x'_1, \ldots, x'_l\rangle$ and $N' = \langle y'_1, \ldots, y'_l\rangle$. Since $\varphi(x'_1) = a_1y'_1 + \ldots + a_{i-1}y'_{i-1} + y'_i$ by construction and since for $j > 1$ we have $\varphi(x'_j) = a\varphi(x_i) \in \langle y'_1, \ldots, y'_l\rangle$ we get a commutative diagram of R-modules and maps

$$\begin{tikzcd} M' \arrow[r, "\varphi'"'] \arrow[d] & N' \arrow[d] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

By the result of the second paragraph of the proof we know that $\text{length}_R(M/M') = (l-1)\text{ord}_{R/\mathfrak q}(a)$ and similarly $\text{length}_R(M/M') = (l-i+1)\text{ord}_{R/\mathfrak q}(a)$. By a diagram chase this implies that

$$\text{length}_R(\text{Coker}(\varphi')) = \text{length}_R(\text{Coker}(\varphi)) + i\ \text{ord}_{R/\mathfrak q}(a).$$

On the other hand, it is clear that writing

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l], \quad [\varphi'(x'_1), \ldots, \varphi(x'_l)] = f'[y'_1, \ldots, y'_l]$$

we have $f' = a^i f$. Hence it suffices to prove the lemma for the case that $\varphi(x_1) = a_1y_1 + \ldots a_{i-1}y_{i-1} + y_i$, i.e., in the case that $a = 1$. Next, recall that

$$[y_1, \ldots, y_l] = [y_1, \ldots, y_{i-1}, a_1y_1 + \ldots a_{i-1}y_{i-1} + y_i, y_{i+1}, \ldots, y_l]$$

by the admissible relations for symbols. The sequence $y_1, \ldots, y_{i-1}, a_1y_1 + \ldots + a_{i-1}y_{i-1} + y_i, y_{i+1}, \ldots, y_l$ satisfies the conditions (3), (4) of the lemma also. Hence, we may actually assume that $\varphi(x_1) = y_i$. In this case, note that we have $\mathfrak q x_1 = 0$ which implies also $\mathfrak q y_i = 0$. We have

$$[y_1, \ldots, y_l] = -[y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l]$$

by the third of the admissible relations defining $\det_{\kappa(\mathfrak q)}(N_{\mathfrak q})$. Hence we may replace $y_1, \ldots, y_l$ by the sequence $y'_1, \ldots, y'_l = y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l$ (which also satisfies conditions (3) and (4) of the lemma). Clearly this decreases the invariant i by 1 and we win by induction on i. □

To use the previous lemma we show that often sequences of elements with the required properties exist.

Lemma 68.42. *Let R be a local Noetherian ring. Let $\mathfrak q \subset R$ be a prime ideal. Let M be a finite R-module such that $\mathfrak q$ is one of the minimal primes of the support of M. Then there exist $x_1, \ldots, x_l \in M$ such that*

(1) *the support of* $M/\langle x_1, \ldots, x_l\rangle$ *does not contain* $\mathfrak{q}$, *and*
(2) $\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle \cong R/\mathfrak{q}$ *for* $i = 1, \ldots, l$.

Moreover, in this case $l = \mathit{length}_{R_\mathfrak{q}}(M_\mathfrak{q})$.

Proof. The condition that $\mathfrak{q}$ is a minimal prime in the support of M implies that $l = \text{length}_{R_\mathfrak{q}}(M_\mathfrak{q})$ is finite (see Algebra, Lemma 62.3). Hence we can find $y_1, \ldots, y_l \in M_\mathfrak{q}$ such that $\langle y_1, \ldots, y_i\rangle/\langle y_1, \ldots, y_{i-1}\rangle \cong \kappa(\mathfrak{q})$ for $i = 1, \ldots, l$. We can find $f_i \in R$, $f_i \not\in \mathfrak{q}$ such that $f_i y_i$ is the image of some element $z_i \in M$. Moreover, as R is Noetherian we can write $\mathfrak{q} = (g_1, \ldots, g_t)$ for some $g_j \in R$. By assumption $g_j y_i \in \langle y_1, \ldots, y_{i-1}\rangle$ inside the module $M_\mathfrak{q}$. By our choice of z_i we can find some further elements $f_{ji} \in R$, $f_{ij} \not\in \mathfrak{q}$ such that $f_{ij} g_j z_i \in \langle z_1, \ldots, z_{i-1}\rangle$ (equality in the module M). The lemma follows by taking

$$x_1 = f_{11} f_{12} \ldots f_{1t} z_1, \quad x_2 = f_{11} f_{12} \ldots f_{1t} f_{21} f_{22} \ldots f_{2t} z_2,$$

and so on. Namely, since all the elements f_i, f_{ij} are invertible in $R_\mathfrak{q}$ we still have that $R_\mathfrak{q} x_1 + \ldots + R_\mathfrak{q} x_i / R_\mathfrak{q} x_1 + \ldots + R_\mathfrak{q} x_{i-1} \cong \kappa(\mathfrak{q})$ for $i = 1, \ldots, l$. By construction, $\mathfrak{q} x_i \in \langle x_1, \ldots, x_{i-1}\rangle$. Thus $\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle$ is an R-module generated by one element, annihilated $\mathfrak{q}$ such that localizing at $\mathfrak{q}$ gives a q-dimensional vector space over $\kappa(\mathfrak{q})$. Hence it is isomorphic to $R/\mathfrak{q}$. $\square$

Here is the main result of this section. We will see below the various different consequences of this proposition. The reader is encouraged to first prove the easier Lemma 68.44 his/herself.

Proposition 68.43. *Let* R *be a local Noetherian ring with residue field* κ. *Suppose that* (M, φ, ψ) *is a* $(2, 1)$*-periodic complex over* R. *Assume*

(1) M *is a finite* R*-module,*
(2) *the cohomology modules of* (M, φ, ψ) *are of finite length, and*
(3) $\dim(\mathit{Supp}(M)) = 1$.

Let $\mathfrak{q}_i$, $i = 1, \ldots, t$ *be the minimal primes of the support of* M. *Then we have*[10]

$$-e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} \mathit{ord}_{R/\mathfrak{q}_i}\left(\det\nolimits_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi_{\mathfrak{q}_i}, \psi_{\mathfrak{q}_i})\right)$$

Proof. We first reduce to the case $t = 1$ in the following way. Note that $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$, where $\mathfrak{m} \subset R$ is the maximal ideal. Let M_i denote the image of $M \to M_{\mathfrak{q}_i}$, so $\text{Supp}(M_i) = \{\mathfrak{m}, \mathfrak{q}_i\}$. The map φ (resp. ψ) induces an R-module map $\varphi_i : M_i \to M_i$ (resp. $\psi_i : M_i \to M_i$). Thus we get a morphism of $(2, 1)$-periodic complexes

$$(M, \varphi, \psi) \longrightarrow \bigoplus\nolimits_{i=1,\ldots,t} (M_i, \varphi_i, \psi_i).$$

The kernel and cokernel of this map have support contained in $\{\mathfrak{m}\}$. Hence by Lemma 2.5 we have

$$e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} e_R(M_i, \varphi_i, \psi_i)$$

On the other hand we clearly have $M_{\mathfrak{q}_i} = M_{i,\mathfrak{q}_i}$, and hence the terms of the right hand side of the formula of the lemma are equal to the expressions

$$\text{ord}_{R/\mathfrak{q}_i}\left(\det\nolimits_{\kappa(\mathfrak{q}_i)}(M_{i,\mathfrak{q}_i}, \varphi_{i,\mathfrak{q}_i}, \psi_{i,\mathfrak{q}_i})\right)$$

[10]Obviously we could get rid of the minus sign by redefining $\det_\kappa(M, \varphi, \psi)$ as the inverse of its current value, see Definition 68.13.

In other words, if we can prove the lemma for each of the modules M_i, then the lemma holds. This reduces us to the case $t = 1$.

Assume we have a $(2,1)$-periodic complex (M, φ, ψ) over a Noetherian local ring with M a finite R-module, $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$, and finite length cohomology modules. The proof in this case follows from Lemma 68.41 and careful bookkeeping. Denote $K_\varphi = \text{Ker}(\varphi)$, $I_\varphi = \text{Im}(\varphi)$, $K_\psi = \text{Ker}(\psi)$, and $I_\psi = \text{Im}(\psi)$. Since R is Noetherian these are all finite R-modules. Set

$$a = \text{length}_{R_\mathfrak{q}}(I_{\varphi,\mathfrak{q}}) = \text{length}_{R_\mathfrak{q}}(K_{\psi,\mathfrak{q}}), \quad b = \text{length}_{R_\mathfrak{q}}(I_{\psi,\mathfrak{q}}) = \text{length}_{R_\mathfrak{q}}(K_{\varphi,\mathfrak{q}}).$$

Equalities because the complex becomes exact after localizing at $\mathfrak{q}$. Note that $l = \text{length}_{R_\mathfrak{q}}(M_\mathfrak{q})$ is equal to $l = a + b$.

We are going to use Lemma 68.42 to choose sequences of elements in finite R-modules N with support contained in $\{\mathfrak{m}, \mathfrak{q}\}$. In this case $N_\mathfrak{q}$ has finite length, say $n \in \mathbf{N}$. Let us call a sequence $w_1, \ldots, w_n \in N$ with properties (1) and (2) of Lemma 68.42 a "good sequence". Note that the quotient $N/\langle w_1, \ldots, w_n\rangle$ of N by the submodule generated by a good sequence has support (contained in) $\{\mathfrak{m}\}$ and hence has finite length (Algebra, Lemma 62.3). Moreover, the symbol $[w_1, \ldots, w_n] \in \det_{\kappa(\mathfrak{q})}(N_\mathfrak{q})$ is a generator, see Lemma 68.5.

Having said this we choose good sequences

$$\begin{array}{llll} x_1, \ldots, x_b & \text{in} \quad K_\varphi, & t_1, \ldots, t_a & \text{in} \quad K_\psi, \\ y_1, \ldots, y_a & \text{in} \quad I_\varphi \cap \langle t_1, \ldots t_a\rangle, & s_1, \ldots, s_b & \text{in} \quad I_\psi \cap \langle x_1, \ldots, x_b\rangle. \end{array}$$

We will adjust our choices a little bit as follows. Choose lifts $\tilde{y}_i \in M$ of $y_i \in I_\varphi$ and $\tilde{s}_i \in M$ of $s_i \in I_\psi$. It may not be the case that $\mathfrak{q}\tilde{y}_1 \subset \langle x_1, \ldots, x_b\rangle$ and it may not be the case that $\mathfrak{q}\tilde{s}_1 \subset \langle t_1, \ldots, t_a\rangle$. However, using that $\mathfrak{q}$ is finitely generated (as in the proof of Lemma 68.42) we can find a $d \in R$, $d \notin \mathfrak{q}$ such that $\mathfrak{q}d\tilde{y}_1 \subset \langle x_1, \ldots, x_b\rangle$ and $\mathfrak{q}d\tilde{s}_1 \subset \langle t_1, \ldots, t_a\rangle$. Thus after replacing y_i by dy_i, $\tilde{y}_i$ by $d\tilde{y}_i$, s_i by ds_i and $\tilde{s}_i$ by $d\tilde{s}_i$ we see that we may assume also that $x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_b$ and $t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b$ are good sequences in M.

Finally, we choose a good sequence $z_1, \ldots, z_l$ in the finite R-module

$$\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle \cap \langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle.$$

Note that this is also a good sequence in M.

Since $I_{\varphi,\mathfrak{q}} = K_{\psi,\mathfrak{q}}$ there is a unique element $h \in \kappa(\mathfrak{q})$ such that $[y_1, \ldots, y_a] = h[t_1, \ldots, t_a]$ inside $\det_{\kappa(\mathfrak{q})}(K_{\psi,\mathfrak{q}})$. Similarly, as $I_{\psi,\mathfrak{q}} = K_{\varphi,\mathfrak{q}}$ there is a unique element $h \in \kappa(\mathfrak{q})$ such that $[s_1, \ldots, s_b] = g[x_1, \ldots, x_b]$ inside $\det_{\kappa(\mathfrak{q})}(K_{\varphi,\mathfrak{q}})$. We can also do this with the three good sequences we have in M. All in all we get the following identities

$$\begin{aligned} [y_1, \ldots, y_a] &= h[t_1, \ldots, t_a] \\ [s_1, \ldots, s_b] &= g[x_1, \ldots, x_b] \\ [z_1, \ldots, z_l] &= f_\varphi[x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a] \\ [z_1, \ldots, z_l] &= f_\psi[t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b] \end{aligned}$$

for some $g, h, f_\varphi, f_\psi \in \kappa(\mathfrak{q})$.

Having set up all this notation let us compute $\det_{\kappa(\mathfrak{q})}(M, \varphi, \psi)$. Namely, consider the element $[z_1, \ldots, z_l]$. Under the map $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ of Definition 68.13 we have

$$\begin{array}{rcl} [z_1, \ldots, z_l] & = & f_\varphi[x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a] \\ & \mapsto & f_\varphi[x_1, \ldots, x_b] \otimes [y_1, \ldots, y_a] \\ & \mapsto & f_\varphi h/g[t_1, \ldots, t_a] \otimes [s_1, \ldots, s_b] \\ & \mapsto & f_\varphi h/g[t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b] \\ & = & f_\varphi h/f_\psi g[z_1, \ldots, z_l] \end{array}$$

This means that $\det_{\kappa(\mathfrak{q})}(M_\mathfrak{q}, \varphi_\mathfrak{q}, \psi_\mathfrak{q})$ is equal to $f_\varphi h/f_\psi g$ up to a sign.

We abbreviate the following quantities

$$\begin{array}{rcl} k_\varphi & = & \text{length}_R(K_\varphi/\langle x_1, \ldots, x_b\rangle) \\ k_\psi & = & \text{length}_R(K_\psi/\langle t_1, \ldots, t_a\rangle) \\ i_\varphi & = & \text{length}_R(I_\varphi/\langle y_1, \ldots, y_a\rangle) \\ i_\psi & = & \text{length}_R(I_\psi/\langle s_1, \ldots, s_a\rangle) \\ m_\varphi & = & \text{length}_R(M/\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle) \\ m_\psi & = & \text{length}_R(M/\langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle) \\ \delta_\varphi & = & \text{length}_R(\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle \langle z_1, \ldots, z_l\rangle) \\ \delta_\psi & = & \text{length}_R(\langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle \langle z_1, \ldots, z_l\rangle) \end{array}$$

Using the exact sequences $0 \to K_\varphi \to M \to I_\varphi \to 0$ we get $m_\varphi = k_\varphi + i_\varphi$. Similarly we have $m_\psi = k_\psi + i_\psi$. We have $\delta_\varphi + m_\varphi = \delta_\psi + m_\psi$ since this is equal to the colength of $\langle z_1, \ldots, z_l\rangle$ in M. Finally, we have

$$\delta_\varphi = \text{ord}_{R/\mathfrak{q}}(f_\varphi), \quad \delta_\psi = \text{ord}_{R/\mathfrak{q}}(f_\psi)$$

by our first application of the key Lemma 68.41.

Next, let us compute the multiplicity of the periodic complex

$$\begin{array}{rcl} e_R(M, \varphi, \psi) & = & \text{length}_R(K_\varphi/I_\psi) - \text{length}_R(K_\psi/I_\varphi) \\ & = & \text{length}_R(\langle x_1, \ldots, x_b\rangle/\langle s_1, \ldots, s_b\rangle) + k_\varphi - i_\psi \\ & & -\text{length}_R(\langle t_1, \ldots, t_a\rangle/\langle y_1, \ldots, y_a\rangle) - k_\psi + i_\varphi \\ & = & \text{ord}_{R/\mathfrak{q}}(g/h) + k_\varphi - i_\psi - k_\psi + i_\varphi \\ & = & \text{ord}_{R/\mathfrak{q}}(g/h) + m_\varphi - m_\psi \\ & = & \text{ord}_{R/\mathfrak{q}}(g/h) + \delta_\psi - \delta_\varphi \\ & = & \text{ord}_{R/\mathfrak{q}}(f_\psi g/f_\varphi h) \end{array}$$

where we used the key Lemma 68.41 twice in the third equality. By our computation of $\det_{\kappa(\mathfrak{q})}(M_\mathfrak{q}, \varphi_\mathfrak{q}, \psi_\mathfrak{q})$ this proves the proposition. ☐

In most applications the following lemma suffices.

Lemma 68.44. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module, and let $\psi : M \to M$ be an R-module map. Assume that*

(1) $\text{Ker}(\psi)$ *and* $\text{Coker}(\psi)$ *have finite length, and*
(2) $\dim(Supp(M)) \leq 1$.

Write Supp$(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ *and denote* $f_i \in \kappa(\mathfrak{q}_i)^*$ *the element such that* $\det_{\kappa(\mathfrak{q}_i)}(\psi_{\mathfrak{q}_i}) : \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}) \to \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i})$ *is multiplication by* f_i. *Then we have*

$$\mathit{length}_R(\mathrm{Coker}(\psi)) - \mathit{length}_R(\mathrm{Ker}(\psi)) = \sum\nolimits_{i=1,\ldots,t} \mathit{ord}_{R/\mathfrak{q}_i}(f_i).$$

Proof. Recall that $H^0(M, 0, \psi) = \mathrm{Coker}(\psi)$ and $H^1(M, 0, \psi) = \mathrm{Ker}(\psi)$, see remarks above Definition 2.2. The lemma follows by combining Proposition 68.43 with Lemma 68.17.

Alternative proof. Reduce to the case $\mathrm{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$ as in the proof of Proposition 68.43. Then directly combine Lemmas 68.41 and 68.42 to prove this specific case of Proposition 68.43. There is much less bookkeeping in this case, and the reader is encouraged to work this out. Details omitted. □

68.45. Application to the key lemma. In this section we apply the results above to show the analogue of the key lemma (Lemma 6.3) with the tame symbol d_A constructed above. Please see Remark 6.4 for the relationship with Milnor K-theory.

Lemma 68.46 (Key Lemma)**.** *Let* A *be a* 2*-dimensional Noetherian local domain with fraction field* K. *Let* $f, g \in K^*$. *Let* $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ *be the height* 1 *primes* $\mathfrak{q}$ *of* A *such that either* f *or* g *is not an element of* $A^*_{\mathfrak{q}}$. *Then we have*

$$\sum\nolimits_{i=1,\ldots,t} \mathit{ord}_{A/\mathfrak{q}_i}(d_{A_{\mathfrak{q}_i}}(f, g)) = 0$$

We can also write this as

$$\sum\nolimits_{\mathit{height}(\mathfrak{q})=1} \mathit{ord}_{A/\mathfrak{q}}(d_{A_{\mathfrak{q}}}(f, g)) = 0$$

since at any height one prime $\mathfrak{q}$ *of* A *where* $f, g \in A^*_{\mathfrak{q}}$ *we have* $d_{A_{\mathfrak{q}}}(f, g) = 1$ *by Lemma 68.33.*

Proof. Since the tame symbols $d_{A_{\mathfrak{q}}}(f, g)$ are additive (Lemma 68.30) and the order functions $\mathrm{ord}_{A/\mathfrak{q}}$ are additive (Algebra, Lemma 121.1) it suffices to prove the formula when $f = a \in A$ and $g = b \in A$. In this case we see that we have to show

$$\sum\nolimits_{\mathrm{height}(\mathfrak{q})=1} \mathrm{ord}_{A/\mathfrak{q}}(\det\nolimits_\kappa(A_{\mathfrak{q}}/(ab), a, b)) = 0$$

By Proposition 68.43 this is equivalent to showing that

$$e_A(A/(ab), a, b) = 0.$$

Since the complex $A/(ab) \xrightarrow{a} A/(ab) \xrightarrow{b} A/(ab) \xrightarrow{a} A/(ab)$ is exact we win. □

69. Appendix B: Alternative approaches

In this appendix we first briefly try to connect the material in the main text with K-theory of coherent sheaves. In particular we describe how cupping with c_1 of an invertible module is related to tensoring by this invertible module, see Lemma 69.7. This material is obviously very interesting and deserves a much more detailed and expansive exposition.

69.1. Rational equivalence and K-groups. This section is a continuation of Section 23. The motivation for the following lemma is Homology, Lemma 11.3.

Lemma 69.2. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let*

$$\ldots \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \longrightarrow \ldots$$

be a complex as in Homology, Equation (11.2.1). Assume that

(1) $\dim_\delta(\mathit{Supp}(\mathcal{F})) \leq k+1$.
(2) $\dim_\delta(\mathit{Supp}(H^i(\mathcal{F}, \varphi, \psi))) \leq k$ *for* $i = 0, 1$.

Then we have

$$[H^0(\mathcal{F}, \varphi, \psi)]_k \sim_{rat} [H^1(\mathcal{F}, \varphi, \psi)]_k$$

as k-cycles on X.

Proof. Let $\{W_j\}_{j \in J}$ be the collection of irreducible components of $\text{Supp}(\mathcal{F})$ which have δ-dimension $k+1$. Note that $\{W_j\}$ is a locally finite collection of closed subsets of X by Lemma 10.1. For every j, let $\xi_j \in W_j$ be the generic point. Set

$$f_j = \det_{\kappa(\xi_j)}(\mathcal{F}_{\xi_j}, \varphi_{\xi_j}, \psi_{\xi_j}) \in R(W_j)^*.$$

See Definition 68.13 for notation. We claim that

$$-[H^0(\mathcal{F}, \varphi, \psi)]_k + [H^1(\mathcal{F}, \varphi, \psi)]_k = \sum (W_j \to X)_* \text{div}(f_j)$$

If we prove this then the lemma follows.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z]$ in $[H^0(\mathcal{F}, \varphi, \psi)]_k - [H^1(\mathcal{F}, \varphi, \psi)]_k$ is the same as the coefficient m of $[Z]$ in $\sum (W_j \to X)_* \text{div}(f_j)$. Let $\xi \in Z$ be the generic point. Consider the local ring $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Denote $\varphi, \psi : M \to M$ the action of φ, ψ on the stalk. By our choice of $\xi \in Z$ we have $\delta(\xi) = k$ and hence $\dim(\text{Supp}(M)) = 1$. Finally, the integral closed subschemes W_j passing through ξ correspond to the minimal primes $\mathfrak{q}_i$ of $\text{Supp}(M)$. In each case the element $f_j \in R(W_j)^*$ corresponds to the element $\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi)$ in $\kappa(\mathfrak{q}_i)^*$. Hence we see that

$$n = -e_A(M, \varphi, \psi)$$

and

$$m = \sum \text{ord}_{A/\mathfrak{q}_i}(\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi))$$

Thus the result follows from Proposition 68.43. ☐

Lemma 69.3. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. The map*

$$\text{CH}_k(X) \longrightarrow K_0(\mathit{Coh}_{\leq k+1}(X)/\mathit{Coh}_{\leq k-1}(X))$$

from Lemma 23.4 induces a bijection from $\text{CH}_k(X)$ onto the image $B_k(X)$ of the map

$$K_0(\mathit{Coh}_{\leq k}(X)/\mathit{Coh}_{\leq k-1}(X)) \longrightarrow K_0(\mathit{Coh}_{\leq k+1}(X)/\mathit{Coh}_{\leq k-1}(X)).$$

Proof. By Lemma 23.2 we have $Z_k(X) = K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$ compatible with the map of Lemma 23.4. Thus, suppose we have an element $[A] - [B]$ of $K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$ which maps to zero in $B_k(X)$, i.e., maps to zero in $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$. We have to show that $[A] - [B]$ corresponds to a cycle rationally equivalent to zero on X. Suppose $[A] = [\mathcal{A}]$ and $[B] = [\mathcal{B}]$ for some coherent sheaves $\mathcal{A}, \mathcal{B}$ on X supported in δ-dimension $\leq k$. The assumption that $[A] - [B]$ maps to zero in the group $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$ means that there exists coherent sheaves $\mathcal{A}', \mathcal{B}'$ on X supported in δ-dimension $\leq k-1$ such that $[\mathcal{A}\oplus\mathcal{A}'] - [\mathcal{B}\oplus\mathcal{B}']$ is zero in $K_0(Coh_{k+1}(X))$ (use part (1) of Homology, Lemma 11.3). By part (2) of Homology, Lemma 11.3 this means there exists a $(2,1)$-periodic complex $(\mathcal{F}, \varphi, \psi)$ in the category $Coh_{\leq k+1}(X)$ such that $\mathcal{A}\oplus\mathcal{A}' = H^0(\mathcal{F}, \varphi, \psi)$ and $\mathcal{B} \oplus \mathcal{B}' = H^1(\mathcal{F}, \varphi, \psi)$. By Lemma 69.2 this implies that

$$[\mathcal{A} \oplus \mathcal{A}']_k \sim_{rat} [\mathcal{B} \oplus \mathcal{B}']_k$$

This proves that $[A] - [B]$ maps to a cycle rationally equivalent to zero by the map

$$K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) \longrightarrow Z_k(X)$$

of Lemma 23.2. This is what we had to prove and the proof is complete. $\square$

69.4. Cartier divisors and K-groups. In this section we describe how the intersection with the first Chern class of an invertible sheaf $\mathcal{L}$ corresponds to tensoring with $\mathcal{L} - \mathcal{O}$ in K-groups.

Lemma 69.5. *Let A be a Noetherian local ring. Let M be a finite A-module. Let $a, b \in A$. Assume*

(1) $\dim(A) = 1$,
(2) *both a and b are nonzerodivisors in A,*
(3) *A has no embedded primes,*
(4) *M has no embedded associated primes,*
(5) *$Supp(M) = \text{Spec}(A)$.*

Let $I = \{x \in A \mid x(a/b) \in A\}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes of A. Then $(a/b)IM \subset M$ and

$$length_A(M/(a/b)IM) - length_A(M/IM) = \sum\nolimits_i length_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) ord_{A/\mathfrak{q}_i}(a/b)$$

Proof. Since M has no embedded associated primes, and since the support of M is $\text{Spec}(A)$ we see that $\text{Ass}(M) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$. Hence a, b are nonzerodivisors on M. Note that

$$\begin{aligned} & \text{length}_A(M/(a/b)IM) \\ & = \text{length}_A(bM/aIM) \\ & = \text{length}_A(M/aIM) - \text{length}_A(M/bM) \\ & = \text{length}_A(M/aM) + \text{length}_A(aM/aIM) - \text{length}_A(M/bM) \\ & = \text{length}_A(M/aM) + \text{length}_A(M/IM) - \text{length}_A(M/bM) \end{aligned}$$

as the injective map $b : M \to bM$ maps $(a/b)IM$ to aIM and the injective map $a : M \to aM$ maps IM to aIM. Hence the left hand side of the equation of the lemma is equal to

$$\text{length}_A(M/aM) - \text{length}_A(M/bM).$$

Applying the second formula of Lemma 3.2 with $x = a, b$ respectively and using Algebra, Definition 121.2 of the ord-functions we get the result. $\square$

Lemma 69.6. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a meromorphic section of $\mathcal{L}$. Assume*

(1) $\dim_\delta(X) \leq k+1$,
(2) *X has no embedded points,*
(3) *$\mathcal{F}$ has no embedded associated points,*
(4) *the support of $\mathcal{F}$ is X, and*
(5) *the section s is regular meromorphic.*

In this situation let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s, see Divisors, Definition 23.10. Then we have the following:

(1) *there are short exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{1} & \mathcal{F} & \to & \mathcal{Q}_1 & \to & 0 \\ 0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{s} & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L} & \to & \mathcal{Q}_2 & \to & 0 \end{array}$$

(2) *the coherent sheaves $\mathcal{Q}_1$, $\mathcal{Q}_2$ are supported in δ-dimension $\leq k$,*
(3) *the section s restricts to a regular meromorphic section s_i on every irreducible component X_i of X of δ-dimension $k+1$, and*
(4) *writing $[\mathcal{F}]_{k+1} = \sum m_i[X_i]$ we have*

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i (X_i \to X)_* div_{\mathcal{L}|_{X_i}}(s_i)$$

in $Z_k(X)$, in particular

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$$

in $\mathrm{CH}_k(X)$.

Proof. Recall from Divisors, Lemma 24.5 the existence of injective maps $1 : \mathcal{I}\mathcal{F} \to \mathcal{F}$ and $s : \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ whose cokernels are supported on a closed nowhere dense subsets T. Denote $\mathcal{Q}_i$ there cokernels as in the lemma. We conclude that $\dim_\delta(\mathrm{Supp}(\mathcal{Q}_i)) \leq k$. By Divisors, Lemmas 23.5 and 23.8 the pullbacks s_i are defined and are regular meromorphic sections for $\mathcal{L}|_{X_i}$. The equality of cycles in (4) implies the equality of cycle classes in (4). Hence the only remaining thing to show is that

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i (X_i \to X)_* \mathrm{div}_{\mathcal{L}|_{X_i}}(s_i)$$

holds in $Z_k(X)$. To see this, let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be the generic point. Let $A = \mathcal{O}_{X,\xi}$ and $M = \mathcal{F}_\xi$. Moreover, choose a generator $s_\xi \in \mathcal{L}_\xi$. Then we can write $s = (a/b)s_\xi$ where $a, b \in A$ are nonzerodivisors. In this case $I = \mathcal{I}_\xi = \{x \in A \mid x(a/b) \in A\}$. In this case the coefficient of $[Z]$ in the left hand side is

$$\mathrm{length}_A(M/(a/b)IM) - \mathrm{length}_A(M/IM)$$

and the coefficient of $[Z]$ in the right hand side is

$$\sum \mathrm{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \mathrm{ord}_{A/\mathfrak{q}_i}(a/b)$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the 1-dimensional local ring A. Hence the result follows from Lemma 69.5. □

Lemma 69.7. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume $\dim_\delta(Supp(\mathcal{F})) \leq k+1$. Then the element*

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] \in K_0(\textit{Coh}_{\leq k+1}(X)/\textit{Coh}_{\leq k-1}(X))$$

lies in the subgroup $B_k(X)$ of Lemma 69.3 and maps to the element $c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$ via the map $B_k(X) \to \mathrm{CH}_k(X)$.

Proof. Let

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}' \to 0$$

be the short exact sequence constructed in Divisors, Lemma 4.6. This in particular means that $\mathcal{F}'$ has no embedded associated points. Since the support of $\mathcal{K}$ is nowhere dense in the support of $\mathcal{F}$ we see that $\dim_\delta(\mathrm{Supp}(\mathcal{K})) \leq k$. We may re-apply Divisors, Lemma 4.6 starting with $\mathcal{K}$ to get a short exact sequence

$$0 \to \mathcal{K}'' \to \mathcal{K} \to \mathcal{K}' \to 0$$

where now $\dim_\delta(\mathrm{Supp}(\mathcal{K}'')) < k$ and $\mathcal{K}'$ has no embedded associated points. Suppose we can prove the lemma for the coherent sheaves $\mathcal{F}'$ and $\mathcal{K}'$. Then we see from the equations

$$[\mathcal{F}]_{k+1} = [\mathcal{F}']_{k+1} + [\mathcal{K}']_{k+1} + [\mathcal{K}'']_{k+1}$$

(use Lemma 10.4),

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [\mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}'] + [\mathcal{K}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'] + [\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$$

(use the $\otimes\mathcal{L}$ is exact) and the trivial vanishing of $[\mathcal{K}'']_{k+1}$ and $[\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$ in $K_0(\textit{Coh}_{\leq k+1}(X)/\textit{Coh}_{\leq k-1}(X))$ that the result holds for $\mathcal{F}$. What this means is that we may assume that the sheaf $\mathcal{F}$ has no embedded associated points.

Assume X, $\mathcal{F}$ as in the lemma, and assume in addition that $\mathcal{F}$ has no embedded associated points. Consider the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, the corresponding closed subscheme $i : Z \to X$ and the coherent $\mathcal{O}_Z$-module $\mathcal{G}$ constructed in Divisors, Lemma 4.7. Recall that Z is a locally Noetherian scheme without embedded points, $\mathcal{G}$ is a coherent sheaf without embedded associated points, with $\mathrm{Supp}(\mathcal{G}) = Z$ and such that $i_*\mathcal{G} = \mathcal{F}$. Moreover, set $\mathcal{N} = \mathcal{L}|_Z$.

By Divisors, Lemma 25.4 the invertible sheaf $\mathcal{N}$ has a regular meromorphic section s over Z. Let us denote $\mathcal{J} \subset \mathcal{O}_Z$ the sheaf of denominators of s. By Lemma 69.6 there exist short exact sequences

$$\begin{matrix} 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{1} & \mathcal{G} & \to & \mathcal{Q}_1 & \to & 0 \\ 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{s} & \mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{N} & \to & \mathcal{Q}_2 & \to & 0 \end{matrix}$$

such that $\dim_\delta(\mathrm{Supp}(\mathcal{Q}_i)) \leq k$ and such that the cycle $[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k$ is a representative of $c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}$. We see (using the fact that $i_*(\mathcal{G} \otimes \mathcal{N}) = \mathcal{F} \otimes \mathcal{L}$ by the projection formula, see Cohomology, Lemma 54.2) that

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [i_*\mathcal{Q}_2] - [i_*\mathcal{Q}_1]$$

in $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$. This already shows that $[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}]$ is an element of $B_k(X)$. Moreover we have

$$\begin{aligned}[i_*\mathcal{Q}_2]_k - [i_*\mathcal{Q}_1]_k &= i_*\left([\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k\right) \\ &= i_*\left(c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}\right) \\ &= c_1(\mathcal{L}) \cap i_*[\mathcal{G}]_{k+1} \\ &= c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}\end{aligned}$$

by the above and Lemmas 26.4 and 12.4. And this agree with the image of the element under $B_k(X) \to \mathrm{CH}_k(X)$ by definition. Hence the lemma is proved. □

70. Other chapters

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