PROPERTIES OF ALGEBRAIC STACKS

Contents

1. Introduction

Please see Algebraic Stacks, Section 1 for a brief introduction to algebraic stacks, and please read some of that chapter for our foundations of algebraic stacks. The intent is that in that chapter we are careful to distinguish between schemes, algebraic spaces, algebraic stacks, and starting with this chapter we employ the customary abuse of language where all of these concepts are used interchangeably.

The goal of this chapter is to introduce some basic notions and properties of algebraic stacks. A fundamental reference for the case of quasi-separated algebraic stacks with representable diagonal is [LMB00].

2. Conventions and abuse of language

We choose a big fppf site Sch_{fppf}. All schemes are contained in Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

We also fix a base scheme S, by the conventions above an element of Sch_{fppf}. The reader who is only interested in the absolute case can take $S = \mathrm{Spec}(\mathbf{Z})$.

Here are our conventions regarding algebraic stacks:

(1) When we say *algebraic stack* we will mean an algebraic stacks over S, i.e., a category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ which satisfies the conditions of Algebraic Stacks, Definition 12.1.

(2) We will say $f : \mathcal{X} \to \mathcal{Y}$ is a *morphism of algebraic stacks* to indicate a 1-morphism of algebraic stacks over S, i.e., a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, see Algebraic Stacks, Definition 12.3.

(3) A 2-*morphism* $\alpha : f \to g$ will indicate a 2-morphism in the 2-category of algebraic stacks over S, see Algebraic Stacks, Definition 12.3.

(4) Given morphisms $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ of algebraic stacks we abusively call the 2-fibre product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ the *fibre product*.

(5) We will write $\mathcal{X} \times_S \mathcal{Y}$ for the product of the algebraic stacks $\mathcal{X}$, $\mathcal{Y}$.

(6) We will often abuse notation and say two algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ are *isomorphic* if they are equivalent in this 2-category.

Here are our conventions regarding algebraic spaces.

(1) If we say X is an *algebraic space* then we mean that X is an algebraic space over S, i.e., X is a presheaf on $(Sch/S)_{fppf}$ which satisfies the conditions of Spaces, Definition 6.1.

(2) A *morphism of algebraic spaces* $f : X \to Y$ is a morphism of algebraic spaces over S as defined in Spaces, Definition 6.3.

(3) We will **not** distinguish between an algebraic space X and the algebraic stack $\mathcal{S}_X \to (Sch/S)_{fppf}$ it gives rise to, see Algebraic Stacks, Lemma 13.1.

(4) In particular, a *morphism* $f : X \to \mathcal{Y}$ from X to an algebraic stack $\mathcal{Y}$ means a morphism $f : \mathcal{S}_X \to \mathcal{Y}$ of algebraic stacks. Similarly for morphisms $\mathcal{Y} \to X$.

(5) Moreover, given an algebraic stack $\mathcal{X}$ we say $\mathcal{X}$ *is an algebraic space* to indicate that $\mathcal{X}$ is representable by an algebraic space, see Algebraic Stacks, Definition 8.1.

(6) We will use the following notational convention: If we indicate an algebraic stack by a roman capital (such as $X, Y, Z, A, B, \ldots$) then it will be the case that its inertia stack is trivial, and hence it is an algebraic space, see Algebraic Stacks, Proposition 13.3.

Here are our conventions regarding schemes.

(1) If we say X is a *scheme* then we mean that X is a scheme over S, i.e., X is an object of $(Sch/S)_{fppf}$.

(2) By a *morphism of schemes* we mean a morphism of schemes over S.

(3) We will **not** distinguish between a scheme X and the algebraic stack $\mathcal{S}_X \to (Sch/S)_{fppf}$ it gives rise to, see Algebraic Stacks, Lemma 13.1.

(4) In particular, a *morphism* $f : X \to \mathcal{Y}$ from a scheme X to an algebraic stack $\mathcal{Y}$ means a morphism $f : \mathcal{S}_X \to \mathcal{Y}$ of algebraic stacks. Similarly for morphisms $\mathcal{Y} \to X$.

(5) Moreover, given an algebraic stack $\mathcal{X}$ we say $\mathcal{X}$ *is a scheme* to indicate that $\mathcal{X}$ is representable, see Algebraic Stacks, Section 4.

Here are our conventions regarding morphisms of algebraic stacks:

(1) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *representable*, or *representable by schemes* if for every scheme T and morphism $T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is a scheme. See Algebraic Stacks, Section 6.

(2) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *representable by algebraic spaces* if for every scheme T and morphism $T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space. See Algebraic Stacks, Definition 9.1. In this case

$Z \times_\mathcal{Y} \mathcal{X}$ is an algebraic space whenever $Z \to \mathcal{Y}$ is a morphism whose source is an algebraic space, see Algebraic Stacks, Lemma 9.8.
(3) We may abuse notation and say that a diagram of algebraic stacks commutes if the diagram is 2-commutative in the 2-category of algebraic stacks.

Note that every morphism $X \to \mathcal{Y}$ from an algebraic space to an algebraic stack is representable by algebraic spaces, see Algebraic Stacks, Lemma 10.11. We will use this basic result without further mention.

3. Properties of morphisms representable by algebraic spaces

We will study properties of (arbitrary) morphisms of algebraic stacks in its own chapter. For morphisms representable by algebraic spaces we know what it means to be surjective, smooth, or étale, etc. This applies in particular to morphisms $X \to \mathcal{Y}$ from algebraic spaces to algebraic stacks. In this section, we recall how this works, we list the properties to which this applies, and we prove a few easy lemmas.

Our first lemma says a morphism is representable by algebraic spaces if it is so after a base change by a flat, locally finitely presented, surjective morphism.

Lemma 3.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let W be an algebraic space and let $W \to \mathcal{Y}$ be surjective, locally of finite presentation, and flat. The following are equivalent*

(1) *f is representable by algebraic spaces, and*
(2) *$W \times_\mathcal{Y} \mathcal{X}$ is an algebraic space.*

Proof. The implication $(1) \Rightarrow (2)$ is Algebraic Stacks, Lemma 9.8. Conversely, let $W \to \mathcal{Y}$ be as in (2). To prove (1) it suffices to show that f is faithful on fibre categories, see Algebraic Stacks, Lemma 15.2. Assumption (2) implies in particular that $W \times_\mathcal{Y} \mathcal{X} \to W$ is faithful. Hence the faithfulness of f follows from Stacks, Lemma 6.9. $\square$

Let P be a property of morphisms of algebraic spaces which is fppf local on the target and preserved by arbitrary base change. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then we say *f has property P* if and only if for every scheme T and morphism $T \to \mathcal{Y}$ the morphism of algebraic spaces $T \times_\mathcal{Y} \mathcal{X} \to T$ has property P, see Algebraic Stacks, Definition 10.1.

It turns out that if $f : \mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces and has property P, then for any morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the base change $\mathcal{Y}' \times_\mathcal{Y} \mathcal{X} \to \mathcal{Y}'$ has property P, see Algebraic Stacks, Lemmas 9.7 and 10.6. If the property P is preserved under compositions, then this holds also in the setting of morphisms of algebraic stacks representable by algebraic spaces, see Algebraic Stacks, Lemmas 9.9 and 10.5. Moreover, in this case products $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$ of morphisms representable by algebraic spaces having property $\mathcal{P}$ have property $\mathcal{P}$, see Algebraic Stacks, Lemma 10.8.

Finally, if we have two properties P, P' of morphisms of algebraic spaces which are fppf local on the target and preserved by arbitrary base change and if $P(f) \Rightarrow P'(f)$ for every morphism f, then the same implication holds for the corresponding property of morphisms of algebraic stacks representable by algebraic spaces, see

Algebraic Stacks, Lemma 10.9. We will use this *without further mention* in the following and in the following chapters.

The discussion above applies to each of the following properties of morphisms of algebraic spaces

(1) quasi-compact, see Morphisms of Spaces, Lemma 8.4 and Descent on Spaces, Lemma 11.1,
(2) quasi-separated, see Morphisms of Spaces, Lemma 4.4 and Descent on Spaces, Lemma 11.2,
(3) universally closed, see Morphisms of Spaces, Lemma 9.3 and Descent on Spaces, Lemma 11.3,
(4) universally open, see Morphisms of Spaces, Lemma 6.3 and Descent on Spaces, Lemma 11.4,
(5) universally submersive, see Morphisms of Spaces, Lemma 7.3 and Descent on Spaces, Lemma 11.5,
(6) universal homeomorphism, see Morphisms of Spaces, Lemma 53.4 and Descent on Spaces, Lemma 11.8,
(7) surjective, see Morphisms of Spaces, Lemma 5.5 and Descent on Spaces, Lemma 11.6,
(8) universally injective, see Morphisms of Spaces, Lemma 19.5 and Descent on Spaces, Lemma 11.7,
(9) locally of finite type, see Morphisms of Spaces, Lemma 23.3 and Descent on Spaces, Lemma 11.9,
(10) locally of finite presentation, see Morphisms of Spaces, Lemma 28.3 and Descent on Spaces, Lemma 11.10,
(11) finite type, see Morphisms of Spaces, Lemma 23.3 and Descent on Spaces, Lemma 11.11,
(12) finite presentation, see Morphisms of Spaces, Lemma 28.3 and Descent on Spaces, Lemma 11.12,
(13) flat, see Morphisms of Spaces, Lemma 30.4 and Descent on Spaces, Lemma 11.13,
(14) open immersion, see Morphisms of Spaces, Section 12 and Descent on Spaces, Lemma 11.14,
(15) isomorphism, see Descent on Spaces, Lemma 11.15,
(16) affine, see Morphisms of Spaces, Lemma 20.5 and Descent on Spaces, Lemma 11.16,
(17) closed immersion, see Morphisms of Spaces, Section 12 and Descent on Spaces, Lemma 11.17,
(18) separated, see Morphisms of Spaces, Lemma 4.4 and Descent on Spaces, Lemma 11.18,
(19) proper, see Morphisms of Spaces, Lemma 40.3 and Descent on Spaces, Lemma 11.19,
(20) quasi-affine, see Morphisms of Spaces, Lemma 21.5 and Descent on Spaces, Lemma 11.20,
(21) integral, see Morphisms of Spaces, Lemma 45.5 and Descent on Spaces, Lemma 11.22,
(22) finite, see Morphisms of Spaces, Lemma 45.5 and Descent on Spaces, Lemma 11.23,

(23) (locally) quasi-finite, see Morphisms of Spaces, Lemma 27.4 and Descent on Spaces, Lemma 11.24,
(24) syntomic, see Morphisms of Spaces, Lemma 36.3 and Descent on Spaces, Lemma 11.25,
(25) smooth, see Morphisms of Spaces, Lemma 37.3 and Descent on Spaces, Lemma 11.26,
(26) unramified, see Morphisms of Spaces, Lemma 38.4 and Descent on Spaces, Lemma 11.27,
(27) étale, see Morphisms of Spaces, Lemma 39.4 and Descent on Spaces, Lemma 11.28,
(28) finite locally free, see Morphisms of Spaces, Lemma 46.5 and Descent on Spaces, Lemma 11.29,
(29) monomorphism, see Morphisms of Spaces, Lemma 10.5 and Descent on Spaces, Lemma 11.30,
(30) immersion, see Morphisms of Spaces, Section 12 and Descent on Spaces, Lemma 12.1,
(31) locally separated, see Morphisms of Spaces, Lemma 4.4 and Descent on Spaces, Lemma 12.2,

Lemma 3.2. *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. The following are equivalent:*

(1) *f has P,*
(2) *for every algebraic space Z and morphism $Z \to \mathcal{Y}$ the morphism $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has P.*

Proof. The implication $(2) \Rightarrow (1)$ is immediate. Assume (1). Let $Z \to \mathcal{Y}$ be as in (2). Choose a scheme U and a surjective étale morphism $U \to Z$. By assumption the morphism $U \times_{\mathcal{Y}} \mathcal{X} \to U$ has P. But the diagram

$$
\begin{array}{ccc}
U \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & Z \times_{\mathcal{Y}} \mathcal{X} \\
\downarrow & & \downarrow \\
U & \longrightarrow & Z
\end{array}
$$

is cartesian, hence the right vertical arrow has P as $\{U \to Z\}$ is an fppf covering. $\square$

The following lemma tells us it suffices to check P after a base change by a surjective, flat, locally finitely presented morphism.

Lemma 3.3. *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let W be an algebraic space and let $W \to \mathcal{Y}$ be surjective, locally of finite presentation, and flat. Set $V = W \times_{\mathcal{Y}} \mathcal{X}$. Then*

$$(f \ has \ P) \Leftrightarrow (the \ projection \ V \to W \ has \ P).$$

Proof. The implication from left to right follows from Lemma 3.2. Assume $V \to W$ has P. Let T be a scheme, and let $T \to \mathcal{Y}$ be a morphism. Consider the

commutative diagram

$$\begin{array}{ccc} T \times_{\mathcal{Y}} \mathcal{X} & \longleftarrow T \times_{\mathcal{Y}} V \longrightarrow & V \\ \downarrow & \downarrow & \downarrow \\ T & \longleftarrow T \times_{\mathcal{Y}} W \longrightarrow & W \end{array}$$

of algebraic spaces. The squares are cartesian. The bottom left morphism is a surjective, flat morphism which is locally of finite presentation, hence $\{T\times_{\mathcal{Y}}V \to T\}$ is an fppf covering. Hence the fact that the right vertical arrow has property P implies that the left vertical arrow has property P. □

Lemma 3.4. *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Set $\mathcal{W} = \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. Then*

$$(f \text{ has } P) \Leftrightarrow (the \text{ projection } \mathcal{W} \to \mathcal{Z} \text{ has } P).$$

Proof. Choose an algebraic space W and a morphism $W \to \mathcal{Z}$ which is surjective, flat, and locally of finite presentation. By the discussion above the composition $W \to \mathcal{Y}$ is also surjective, flat, and locally of finite presentation. Denote $V = W \times_{\mathcal{Z}} \mathcal{W} = V \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 3.3 we see that f has $\mathcal{P}$ if and only if $V \to W$ does and that $\mathcal{W} \to \mathcal{Z}$ has $\mathcal{P}$ if and only if $V \to W$ does. The lemma follows. □

Lemma 3.5. *Let P be a property of morphisms of algebraic spaces as above. Let $\tau \in \{\acute{e}tale, smooth, syntomic, fppf\}$. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks representable by algebraic spaces. Assume*

(1) *$\mathcal{X} \to \mathcal{Y}$ is surjective and étale, smooth, syntomic, or flat and locally of finite presentation,*
(2) *the composition has P, and*
(3) *P is local on the source in the τ topology.*

Then $\mathcal{Y} \to \mathcal{Z}$ has property P.

Proof. Let Z be a scheme and let $Z \to \mathcal{Z}$ be a morphism. Set $X = \mathcal{X} \times_{\mathcal{Z}} Z$, $Y = \mathcal{Y} \times_{\mathcal{Z}} Z$. By (1) $\{X \to Y\}$ is a τ covering of algebraic spaces and by (2) $X \to Z$ has property P. By (3) this implies that $Y \to Z$ has property P and we win. □

Lemma 3.6. *Let $g : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Let $[U/R] \to \mathcal{X}$ be a presentation. Set $U' = U \times_{\mathcal{X}} \mathcal{X}'$, and $R' = R \times_{\mathcal{X}} \mathcal{X}'$. Then there exists a groupoid in algebraic spaces of the form (U', R', s', t', c'), a presentation $[U'/R'] \to \mathcal{X}'$, and the diagram*

$$\begin{array}{ccc} [U'/R'] & \longrightarrow & \mathcal{X}' \\ {\scriptstyle [pr]}\downarrow & & \downarrow {\scriptstyle g} \\ [U/R] & \longrightarrow & \mathcal{X} \end{array}$$

is 2-commutative where the morphism $[pr]$ comes from a morphism of groupoids $pr : (U', R', s', t', c') \to (U, R, s, t, c)$.

Proof. Since $U \to \mathcal{Y}$ is surjective and smooth, see Algebraic Stacks, Lemma 17.2 the base change $U' \to \mathcal{X}'$ is also surjective and smooth. Hence, by Algebraic Stacks, Lemma 16.2 it suffices to show that $R' = U' \times_{\mathcal{X}'} U'$ in order to get a smooth groupoid (U', R', s', t', c') and a presentation $[U'/R'] \to \mathcal{X}'$. Using that $R = V \times_{\mathcal{Y}} V$ (see Groupoids in Spaces, Lemma 22.2) this follows from

$$R' = U \times_{\mathcal{X}} U \times_{\mathcal{X}} \mathcal{X}' = (U \times_{\mathcal{X}} \mathcal{X}') \times_{\mathcal{X}'} (U \times_{\mathcal{X}} \mathcal{X}')$$

see Categories, Lemmas 31.8 and 31.10. Clearly the projection morphisms $U' \to U$ and $R' \to R$ give the desired morphism of groupoids pr $: (U', R', s', t', c') \to (U, R, s, t, c)$. Hence the morphism [pr] of quotient stacks by Groupoids in Spaces, Lemma 21.1.

We still have to show that the diagram 2-commutes. It is clear that the diagram

$$\begin{array}{ccc} U' & \xrightarrow{\ f'\ } & \mathcal{X}' \\ {\scriptstyle \mathrm{pr}_U}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{\ f\ } & \mathcal{X} \end{array}$$

2-commutes where $\mathrm{pr}_U : U' \to U$ is the projection. There is a canonical 2-arrow $\tau : f \circ t \to f \circ s$ in $\mathrm{Mor}(R, \mathcal{X})$ coming from $R = U \times_{\mathcal{X}} U$, $t = \mathrm{pr}_0$, and $s = \mathrm{pr}_1$. Using the isomorphism $R' \to U' \times_{\mathcal{X}'} U'$ we get similarly an isomorphism $\tau' : f' \circ t' \to f' \circ s'$. Note that $g \circ f' \circ t' = f \circ t \circ \mathrm{pr}_R$ and $g \circ f' \circ s' = f \circ s \circ \mathrm{pr}_R$, where $\mathrm{pr}_R : R' \to R$ is the projection. Thus it makes sense to ask if

$$(3.6.1) \qquad\qquad \tau \star \mathrm{id}_{\mathrm{pr}_R} = \mathrm{id}_g \star \tau'.$$

Now we make two claims: (1) if Equation (3.6.1) holds, then the diagram 2-commutes, and (2) Equation (3.6.1) holds. We omit the proof of both claims. Hints: part (1) follows from the construction of $f = f_{can}$ and $f' = f'_{can}$ in Algebraic Stacks, Lemma 16.1. Part (2) follows by carefully working through the definitions. $\qquad\square$
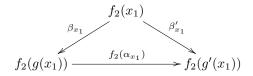
Remark 3.7. Let $\mathcal{Y}$ be an algebraic stack. Consider the following 2-category:

(1) An object is a morphism $f : \mathcal{X} \to \mathcal{Y}$ which is representable by algebraic spaces,
(2) a 1-morphism $(g, \beta) : (f_1 : \mathcal{X}_1 \to \mathcal{Y}) \to (f_2 : \mathcal{X}_2 \to \mathcal{Y})$ consists of a morphism $g : \mathcal{X}_1 \to \mathcal{X}_2$ and a 2-morphism $\beta : f_1 \to f_2 \circ g$, and
(3) a 2-morphism between $(g, \beta), (g', \beta') : (f_1 : \mathcal{X}_1 \to \mathcal{Y}) \to (f_2 : \mathcal{X}_2 \to \mathcal{Y})$ is a 2-morphism $\alpha : g \to g'$ such that $(\mathrm{id}_{f_2} \star \alpha) \circ \beta = \beta'$.

Let us denote this 2-category $Spaces/\mathcal{Y}$ by analogy with the notation of Topologies on Spaces, Section 2. Now we claim that in this 2-category the morphism categories

$$\mathrm{Mor}_{Spaces/\mathcal{Y}}((f_1 : \mathcal{X}_1 \to \mathcal{Y}), (f_2 : \mathcal{X}_2 \to \mathcal{Y}))$$

are all setoids. Namely, a 2-morphism α is a rule which to each object x_1 of $\mathcal{X}_1$ assigns an isomorphism $\alpha_{x_1} : g(x_1) \longrightarrow g'(x_1)$ in the relevant fibre category of $\mathcal{X}_2$ such that the diagram

$$\begin{array}{ccc} & f_2(x_1) & \\ {\scriptstyle \beta_{x_1}}\swarrow & & \searrow{\scriptstyle \beta'_{x_1}} \\ f_2(g(x_1)) & \xrightarrow{\ f_2(\alpha_{x_1})\ } & f_2(g'(x_1)) \end{array}$$

commutes. But since f_2 is faithful (see Algebraic Stacks, Lemma 15.2) this means that if α_{x_1} exists, then it is unique! In other words the 2-category *Spaces*/$\mathcal{Y}$ is very close to being a category. Namely, if we replace 1-morphisms by isomorphism classes of 1-morphisms we obtain a category. We will often perform this replacement without further mention.

4. Points of algebraic stacks

Let $\mathcal{X}$ be an algebraic stack. Let K, L be two fields and let $p : \mathrm{Spec}(K) \to \mathcal{X}$ and $q : \mathrm{Spec}(L) \to \mathcal{X}$ be morphisms. We say that p and q are *equivalent* if there exists a field Ω and a 2-commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(\Omega) & \longrightarrow & \mathrm{Spec}(L) \\
\downarrow & & \downarrow{\scriptstyle q} \\
\mathrm{Spec}(K) & \xrightarrow{\ p\ } & \mathcal{X}.
\end{array}
$$

Lemma 4.1. *The notion above does indeed define an equivalence relation on morphisms from spectra of fields into the algebraic stack $\mathcal{X}$.*

Proof. It is clear that the relation is reflexive and symmetric. Hence we have to prove that it is transitive. This comes down to the following: Given a diagram

$$
\begin{array}{ccccc}
\mathrm{Spec}(\Omega) & \xrightarrow{\ b\ } & \mathrm{Spec}(L) & \xleftarrow{\ b'\ } & \mathrm{Spec}(\Omega') \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle a'} \\
\mathrm{Spec}(K) & \xrightarrow{\ p\ } & \mathcal{X} & \xleftarrow{\ p'\ } & \mathrm{Spec}(K')
\end{array}
$$

with both squares 2-commutative we have to show that p is equivalent to p'. By the 2-Yoneda lemma (see Algebraic Stacks, Section 5) the morphisms p, p', and q are given by objects x, x', and y in the fibre categories of $\mathcal{X}$ over $\mathrm{Spec}(K)$, $\mathrm{Spec}(K')$, and $\mathrm{Spec}(L)$. The 2-commutativity of the squares means that there are isomorphisms $\alpha : a^*x \to b^*y$ and $\alpha' : (a')^*x' \to (b')^*y$ in the fibre categories of $\mathcal{X}$ over $\mathrm{Spec}(\Omega)$ and $\mathrm{Spec}(\Omega')$. Choose any field Ω'' and embeddings $\Omega \to \Omega''$ and $\Omega' \to \Omega''$ agreeing on L. Then we can extend the diagram above to

$$
\begin{array}{ccccc}
& & \mathrm{Spec}(\Omega'') & & \\
& {\scriptstyle c}\swarrow & \downarrow{\scriptstyle q'} & \searrow{\scriptstyle c'} & \\
\mathrm{Spec}(\Omega) & \xrightarrow{\ b\ } & \mathrm{Spec}(L) & \xleftarrow{\ b'\ } & \mathrm{Spec}(\Omega') \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle a'} \\
\mathrm{Spec}(K) & \xrightarrow{\ p\ } & \mathcal{X} & \xleftarrow{\ p'\ } & \mathrm{Spec}(K')
\end{array}
$$

with commutative triangles and

$$(q')^*(\alpha')^{-1} \circ (q')^*\alpha : (a \circ c)^*x \longrightarrow (a' \circ c')^*x'$$

is an isomorphism in the fibre category over $\mathrm{Spec}(\Omega'')$. Hence p is equivalent to p' as desired. $\square$

Definition 4.2. Let $\mathcal{X}$ be an algebraic stack. A *point* of $\mathcal{X}$ is an equivalence class of morphisms from spectra of fields into $\mathcal{X}$. The set of points of $\mathcal{X}$ is denoted $|\mathcal{X}|$.

This agrees with our definition of points of algebraic spaces, see Properties of Spaces, Definition 4.1. Moreover, for a scheme we recover the usual notion of points, see Properties of Spaces, Lemma 4.2. If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks then there is an induced map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ which maps a representative $x : \mathrm{Spec}(K) \to \mathcal{X}$ to the representative $f \circ x : \mathrm{Spec}(K) \to \mathcal{Y}$. This is well defined: namely 2-isomorphic 1-morphisms remain 2-isomorphic after pre- or post-composing by a 1-morphism because you can horizontally pre- or post-compose by the identity of the given 1-morphism. This holds in any (strict) $(2, 1)$-category. If

$$
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{W} & \longrightarrow & \mathcal{Z}
\end{array}
$$

is a 2-commutative diagram of algebraic stacks, then the diagram of sets

$$
\begin{array}{ccc}
|\mathcal{X}| & \longrightarrow & |\mathcal{Y}| \\
\downarrow & & \downarrow \\
|\mathcal{W}| & \longrightarrow & |\mathcal{Z}|
\end{array}
$$

is commutative. In particular, if $\mathcal{X} \to \mathcal{Y}$ is an equivalence then $|\mathcal{X}| \to |\mathcal{Y}|$ is a bijection.

Lemma 4.3. *Let*

$$
\begin{array}{ccc}
\mathcal{Z} \times_\mathcal{Y} \mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{Z} & \longrightarrow & \mathcal{Y}
\end{array}
$$

be a fibre product of algebraic stacks. Then the map of sets of points

$$
|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \longrightarrow |\mathcal{Z}| \times_{|\mathcal{Y}|} |\mathcal{X}|
$$

is surjective.

Proof. Namely, suppose given fields K, L and morphisms $\mathrm{Spec}(K) \to \mathcal{X}$, $\mathrm{Spec}(L) \to \mathcal{Z}$, then the assumption that they agree as elements of $|\mathcal{Y}|$ means that there is a common extension M/K and M/L such that $\mathrm{Spec}(M) \to \mathrm{Spec}(K) \to \mathcal{X} \to \mathcal{Y}$ and $\mathrm{Spec}(M) \to \mathrm{Spec}(L) \to \mathcal{Z} \to \mathcal{Y}$ are 2-isomorphic. And this is exactly the condition which says you get a morphism $\mathrm{Spec}(M) \to \mathcal{Z} \times_\mathcal{Y} \mathcal{X}$. $\square$

Lemma 4.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent:*

(1) *$|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is surjective, and*
(2) *f is surjective (in the sense of Section 3).*

Proof. Assume (1). Let $T \to \mathcal{Y}$ be a morphism whose source is a scheme. To prove (2) we have to show that the morphism of algebraic spaces $T \times_\mathcal{Y} \mathcal{X} \to T$ is surjective. By Morphisms of Spaces, Definition 5.2 this means we have to show that $|T \times_\mathcal{Y} \mathcal{X}| \to |T|$ is surjective. Applying Lemma 4.3 we see that this follows from (1).

Conversely, assume (2). Let $y : \mathrm{Spec}(K) \to \mathcal{Y}$ be a morphism from the spectrum of a field into $\mathcal{Y}$. By assumption the morphism $\mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X} \to \mathrm{Spec}(K)$ of

algebraic spaces is surjective. By Morphisms of Spaces, Definition 5.2 this means there exists a field extension K'/K and a morphism $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X}$ such that the left square of the diagram

$$
\begin{array}{ccccc}
\mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(K) & =\!=\!= & \mathrm{Spec}(K) & \xrightarrow{\ y\ } & \mathcal{Y}
\end{array}
$$

is commutative. This shows that $|X| \to |\mathcal{Y}|$ is surjective. $\square$

Here is a lemma explaining how to compute the set of points in terms of a presentation.

Lemma 4.5. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X} = [U/R]$ be a presentation of $\mathcal{X}$, see Algebraic Stacks, Definition 16.5. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}|$ is the quotient of $|U|$ by this equivalence relation.*

Proof. The assumption means that we have a smooth groupoid (U, R, s, t, c) in algebraic spaces, and an equivalence $f : [U/R] \to \mathcal{X}$. We may assume $\mathcal{X} = [U/R]$. The induced morphism $p : U \to \mathcal{X}$ is smooth and surjective, see Algebraic Stacks, Lemma 17.2. Hence $|U| \to |\mathcal{X}|$ is surjective by Lemma 4.4. Note that $R = U \times_{\mathcal{X}} U$, see Groupoids in Spaces, Lemma 22.2. Hence Lemma 4.3 implies the map

$$|R| \longrightarrow |U| \times_{|\mathcal{X}|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|\mathcal{X}|$. Combining these two statements we get the result of the lemma. $\square$

Remark 4.6. The result of Lemma 4.5 can be generalized as follows. Let $\mathcal{X}$ be an algebraic stack. Let U be an algebraic space and let $f : U \to \mathcal{X}$ be a surjective morphism (which makes sense by Section 3). Let $R = U \times_{\mathcal{X}} U$, let (U, R, s, t, c) be the groupoid in algebraic spaces, and let $f_{can} : [U/R] \to \mathcal{X}$ be the canonical morphism as constructed in Algebraic Stacks, Lemma 16.1. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}| = |U|/|R|$. The proof of Lemma 4.5 works without change. (Of course in general $[U/R]$ is not an algebraic stack, and in general f_{can} is not an isomorphism.)

Lemma 4.7. *There exists a unique topology on the sets of points of algebraic stacks with the following properties:*
  (1) *for every morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ the map $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous, and*
  (2) *for every morphism $U \to \mathcal{X}$ which is flat and locally of finite presentation with U an algebraic space the map of topological spaces $|U| \to |\mathcal{X}|$ is continuous and open.*

Proof. Choose a morphism $p : U \to \mathcal{X}$ which is surjective, flat, and locally of finite presentation with U an algebraic space. Such exist by the definition of an algebraic stack, as a smooth morphism is flat and locally of finite presentation (see Morphisms of Spaces, Lemmas 37.5 and 37.7). We define a topology on $|\mathcal{X}|$ by the rule: $W \subset |\mathcal{X}|$ is open if and only if $|p|^{-1}(W)$ is open in $|U|$. To show that this is independent of the choice of p, let $p' : U' \to \mathcal{X}$ be another morphism which

is surjective, flat, locally of finite presentation from an algebraic space to $\mathcal{X}$. Set $U'' = U \times_{\mathcal{X}} U'$ so that we have a 2-commutative diagram

$$\begin{array}{ccc} U'' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

As $U \to \mathcal{X}$ and $U' \to \mathcal{X}$ are surjective, flat, locally of finite presentation we see that $U'' \to U'$ and $U'' \to U$ are surjective, flat and locally of finite presentation, see Lemma 3.2. Hence the maps $|U''| \to |U'|$ and $|U''| \to |U|$ are continuous, open and surjective, see Morphisms of Spaces, Definition 5.2 and Lemma 30.6. This clearly implies that our definition is independent of the choice of $p : U \to \mathcal{X}$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. By Algebraic Stacks, Lemma 15.1 we can find a 2-commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\;a\;} & V \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ \mathcal{X} & \xrightarrow{\;f\;} & \mathcal{Y} \end{array}$$

with surjective smooth vertical arrows. Consider the associated commutative diagram

$$\begin{array}{ccc} |U| & \xrightarrow{\;|a|\;} & |V| \\ {\scriptstyle |x|}\downarrow & & \downarrow{\scriptstyle |y|} \\ |\mathcal{X}| & \xrightarrow{\;|f|\;} & |\mathcal{Y}| \end{array}$$

of sets. If $W \subset |\mathcal{Y}|$ is open, then by the definition above this means exactly that $|y|^{-1}(W)$ is open in $|V|$. Since $|a|$ is continuous we conclude that $|a|^{-1}|y|^{-1}(W) = |x|^{-1}|f|^{-1}(W)$ is open in $|W|$ which means by definition that $|f|^{-1}(W)$ is open in $|\mathcal{X}|$. Thus $|f|$ is continuous.

Finally, we have to show that if U is an algebraic space, and $U \to \mathcal{X}$ is flat and locally of finite presentation, then $|U| \to |\mathcal{X}|$ is open. Let $V \to \mathcal{X}$ be surjective, flat, and locally of finite presentation with V an algebraic space. Consider the commutative diagram

$$\begin{array}{ccccc} |U \times_{\mathcal{X}} V| & \xrightarrow{\;e\;} & |U| \times_{|\mathcal{X}|} |V| & \xrightarrow{\;d\;} & |V| \\ & {\scriptstyle f}\searrow & {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle b} \\ & & |U| & \xrightarrow{\;a\;} & |\mathcal{X}| \end{array}$$

Now the morphism $U \times_{\mathcal{X}} V \to U$ is surjective, i.e, $f : |U \times_{\mathcal{X}} V| \to |U|$ is surjective. The left top horizontal arrow is surjective, see Lemma 4.3. The morphism $U \times_{\mathcal{X}} V \to V$ is flat and locally of finite presentation, hence $d \circ e : |U \times_{\mathcal{X}} V| \to |V|$ is open, see Morphisms of Spaces, Lemma 30.6. Pick $W \subset |U|$ open. The properties above imply that $b^{-1}(a(W)) = (d \circ e)(f^{-1}(W))$ is open, which by construction means that $a(W)$ is open as desired. $\qquad\square$

Definition 4.8. Let $\mathcal{X}$ be an algebraic stack. The underlying *topological space* of $\mathcal{X}$ is the set of points $|\mathcal{X}|$ endowed with the topology constructed in Lemma 4.7.

This definition does not conflict with the already existing topology on $|\mathcal{X}|$ if $\mathcal{X}$ is an algebraic space.

Lemma 4.9. *Let $\mathcal{X}$ be an algebraic stack. Every point of $|\mathcal{X}|$ has a fundamental system of quasi-compact open neighbourhoods. In particular $|\mathcal{X}|$ is locally quasi-compact in the sense of Topology, Definition 13.1.*

Proof. This follows formally from the fact that there exists a scheme U and a surjective, open, continuous map $U \to |\mathcal{X}|$ of topological spaces. Namely, if $U \to \mathcal{X}$ is surjective and smooth, then Lemma 4.7 guarantees that $|U| \to |\mathcal{X}|$ is continuous, surjective, and open. $\square$

5. Surjective morphisms

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. In Section 3 we have already defined what it means for f to be surjective. In Lemma 4.4 we have seen that this is equivalent to requiring $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ to be surjective. This clears the way for the following definition.

Definition 5.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *surjective* if the map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ of associated topological spaces is surjective.

Here are some lemmas.

Lemma 5.2. *The composition of surjective morphisms is surjective.*

Proof. Omitted. $\square$

Lemma 5.3. *The base change of a surjective morphism is surjective.*

Proof. Omitted. Hint: Use Lemma 4.3. $\square$

Lemma 5.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Y}' \to \mathcal{Y}$ be a surjective morphism of algebraic stacks. If the base change $f' : \mathcal{Y}' \times_\mathcal{Y} \mathcal{X} \to \mathcal{Y}'$ of f is surjective, then f is surjective.*

Proof. Immediate from Lemma 4.3. $\square$

Lemma 5.5. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is surjective so is $\mathcal{Y} \to \mathcal{Z}$.*

Proof. Immediate. $\square$

6. Quasi-compact algebraic stacks

The following definition is equivalent with the definition for algebraic spaces by Properties of Spaces, Lemma 5.2.

Definition 6.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *quasi-compact* if and only if $|\mathcal{X}|$ is quasi-compact.

Lemma 6.2. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:*
  (1) *$\mathcal{X}$ is quasi-compact,*
  (2) *there exists a surjective smooth morphism $U \to \mathcal{X}$ with U an affine scheme,*
  (3) *there exists a surjective smooth morphism $U \to \mathcal{X}$ with U a quasi-compact scheme,*

(4) *there exists a surjective smooth morphism $U \to \mathcal{X}$ with U a quasi-compact algebraic space, and*
(5) *there exists a surjective morphism $\mathcal{U} \to \mathcal{X}$ of algebraic stacks such that $\mathcal{U}$ is quasi-compact.*

Proof. We will use Lemma 4.4. Suppose $\mathcal{U}$ and $\mathcal{U} \to \mathcal{X}$ are as in (5). Then since $|\mathcal{U}| \to |\mathcal{X}|$ is surjective and continuous we conclude that $|\mathcal{X}|$ is quasi-compact. Thus (5) implies (1). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are immediate. Assume (1), i.e., $\mathcal{X}$ is quasi-compact, i.e., that $|\mathcal{X}|$ is quasi-compact. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then since $|U| \to |\mathcal{X}|$ is open we see that there exists a quasi-compact open $U' \subset U$ such that $|U'| \to |X|$ is surjective (and still smooth). Choose a finite affine open covering $U' = U_1 \cup \ldots \cup U_n$. Then $U_1 \amalg \ldots \amalg U_n \to \mathcal{X}$ is a surjective smooth morphism whose source is an affine scheme (Schemes, Lemma 6.8). Hence (2) holds. $\square$

Lemma 6.3. *A finite disjoint union of quasi-compact algebraic stacks is a quasi-compact algebraic stack.*

Proof. This is clear from the corresponding topological fact. $\square$

7. Properties of algebraic stacks defined by properties of schemes

Any smooth local property of schemes gives rise to a corresponding property of algebraic stacks via the following lemma. Note that a property of schemes which is smooth local is also étale local as any étale covering is also a smooth covering. Hence for a smooth local property P of schemes we know what it means to say that an algebraic space has P, see Properties of Spaces, Section 7.

Lemma 7.1. *Let $\mathcal{P}$ be a property of schemes which is local in the smooth topology, see Descent, Definition 15.1. Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *for some scheme U and some surjective smooth morphism $U \to \mathcal{X}$ the scheme U has property $\mathcal{P}$,*
(2) *for every scheme U and every smooth morphism $U \to \mathcal{X}$ the scheme U has property $\mathcal{P}$,*
(3) *for some algebraic space U and some surjective smooth morphism $U \to \mathcal{X}$ the algebraic space U has property $\mathcal{P}$, and*
(4) *for every algebraic space U and every smooth morphism $U \to \mathcal{X}$ the algebraic space U has property $\mathcal{P}$.*

If $\mathcal{X}$ is a scheme this is equivalent to $\mathcal{P}(U)$. If $\mathcal{X}$ is an algebraic space this is equivalent to X having property $\mathcal{P}$.

Proof. Let $U \to \mathcal{X}$ surjective and smooth with U an algebraic space. Let $V \to \mathcal{X}$ be a smooth morphism with V an algebraic space. Choose schemes U' and V' and surjective étale morphisms $U' \to U$ and $V' \to V$. Finally, choose a scheme W and a surjective étale morphism $W \to V' \times_{\mathcal{X}} U'$. Then $W \to V'$ and $W \to U'$ are smooth morphisms of schemes as compositions of étale and smooth morphisms of algebraic spaces, see Morphisms of Spaces, Lemmas 39.6 and 37.2. Moreover, $W \to V'$ is surjective as $U' \to \mathcal{X}$ is surjective. Hence, we have

$$\mathcal{P}(U) \Leftrightarrow \mathcal{P}(U') \Rightarrow \mathcal{P}(W) \Rightarrow \mathcal{P}(V') \Leftrightarrow \mathcal{P}(V)$$

where the equivalences are by definition of property $\mathcal{P}$ for algebraic spaces, and the two implications come from Descent, Definition 15.1. This proves (3) $\Rightarrow$ (4).

The implications (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (3), and (4) $\Rightarrow$ (2) are immediate. $\square$

Definition 7.2. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{P}$ be a property of schemes which is local in the smooth topology. We say $\mathcal{X}$ *has property* $\mathcal{P}$ if any of the equivalent conditions of Lemma 7.1 hold.

Remark 7.3. Here is a list of properties which are local for the smooth topology (keep in mind that the fpqc, fppf, and syntomic topologies are stronger than the smooth topology):

(1) locally Noetherian, see Descent, Lemma 16.1,
(2) Jacobson, see Descent, Lemma 16.2,
(3) locally Noetherian and (S_k), see Descent, Lemma 17.1,
(4) Cohen-Macaulay, see Descent, Lemma 17.2,
(5) reduced, see Descent, Lemma 18.1,
(6) normal, see Descent, Lemma 18.2,
(7) locally Noetherian and (R_k), see Descent, Lemma 18.3,
(8) regular, see Descent, Lemma 18.4,
(9) Nagata, see Descent, Lemma 18.5.

Any smooth local property of germs of schemes gives rise to a corresponding property of algebraic stacks. Note that a property of germs which is smooth local is also étale local. Hence for a smooth local property of germs of schemes P we know what it means to say that an algebraic space X has property P at $x \in |X|$, see Properties of Spaces, Section 7.

Lemma 7.4. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point of $\mathcal{X}$. Let $\mathcal{P}$ be a property of germs of schemes which is smooth local, see Descent, Definition 21.1. The following are equivalent*

(1) *for any smooth morphism $U \to \mathcal{X}$ with U a scheme and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$,*
(2) *for some smooth morphism $U \to \mathcal{X}$ with U a scheme and some $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$,*
(3) *for any smooth morphism $U \to \mathcal{X}$ with U an algebraic space and $u \in |U|$ with $a(u) = x$ the algebraic space U has property $\mathcal{P}$ at u, and*
(4) *for some smooth morphism $U \to \mathcal{X}$ with U a an algebraic space and some $u \in |U|$ with $a(u) = x$ the algebraic space U has property $\mathcal{P}$ at u.*

If $\mathcal{X}$ is representable, then this is equivalent to $\mathcal{P}(\mathcal{X}, x)$. If $\mathcal{X}$ is an algebraic space then this is equivalent to $\mathcal{X}$ having property $\mathcal{P}$ at x.

Proof. Let $a : U \to \mathcal{X}$ and $u \in |U|$ as in (3). Let $b : V \to \mathcal{X}$ be another smooth morphism with V an algebraic space and $v \in |V|$ with $b(v) = x$ also. Choose a scheme U', an étale morphism $U' \to U$ and $u' \in U'$ mapping to u. Choose a scheme V', an étale morphism $V' \to V$ and $v' \in V'$ mapping to v. By Lemma 4.3 there exists a point $\overline{w} \in |V' \times_{\mathcal{X}} U'|$ mapping to u' and v'. Choose a scheme W and a surjective étale morphism $W \to V' \times_{\mathcal{X}} U'$. We may choose a $w \in |W|$ mapping to $\overline{w}$ (see Properties of Spaces, Lemma 4.4). Then $W \to V'$ and $W \to U'$ are smooth morphisms of schemes as compositions of étale and smooth morphisms of algebraic

spaces, see Morphisms of Spaces, Lemmas 39.6 and 37.2. Hence

$$\mathcal{P}(U, u) \Leftrightarrow \mathcal{P}(U', u') \Leftrightarrow \mathcal{P}(W, w) \Leftrightarrow \mathcal{P}(V', v') \Leftrightarrow \mathcal{P}(V, v)$$

The outer two equivalences by Properties of Spaces, Definition 7.5 and the other two by what it means to be a smooth local property of germs of schemes. This proves (4) $\Rightarrow$ (3).

The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (4), and (3) $\Rightarrow$ (1) are immediate. $\square$

Definition 7.5. Let $\mathcal{P}$ be a property of germs of schemes which is smooth local. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. We say $\mathcal{X}$ *has property* $\mathcal{P}$ *at* x if any of the equivalent conditions of Lemma 7.4 holds.

8. Monomorphisms of algebraic stacks

We define a monomorphism of algebraic stacks in the following way. We will see in Lemma 8.4 that this is compatible with the corresponding 2-category theoretic notion.

Definition 8.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is a *monomorphism* if it is representable by algebraic spaces and a monomorphism in the sense of Section 3.

First some basic lemmas.

Lemma 8.2. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a monomorphism. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is a monomorphism.*

Proof. This follows from the general discussion in Section 3. $\square$

Lemma 8.3. *Compositions of monomorphisms of algebraic stacks are monomorphisms.*

Proof. This follows from the general discussion in Section 3 and Morphisms of Spaces, Lemma 10.4. $\square$

Lemma 8.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

(1) *f is a monomorphism,*
(2) *f is fully faithful,*
(3) *the diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an equivalence, and*
(4) *there exists an algebraic space W and a surjective, flat morphism $W \to \mathcal{Y}$ which is locally of finite presentation such that $V = \mathcal{X} \times_{\mathcal{Y}} W$ is an algebraic space, and the morphism $V \to W$ is a monomorphism of algebraic spaces.*

Proof. The equivalence of (1) and (4) follows from the general discussion in Section 3 and in particular Lemmas 3.1 and 3.3.

The equivalence of (2) and (3) is Categories, Lemma 35.10.

Assume the equivalent conditions (2) and (3). Then f is representable by algebraic spaces according to Algebraic Stacks, Lemma 15.2. Moreover, the 2-Yoneda lemma combined with the fully faithfulness implies that for every scheme T the functor

$$\mathrm{Mor}(T, \mathcal{X}) \longrightarrow \mathrm{Mor}(T, \mathcal{Y})$$

is fully faithful. Hence given a morphism $y : T \to \mathcal{Y}$ there exists up to unique 2-isomorphism at most one morphism $x : T \to \mathcal{X}$ such that $y \cong f \circ x$. In particular, given a morphism of schemes $h : T' \to T$ there exists at most one lift $\tilde{h} : T' \to T \times_{\mathcal{Y}} \mathcal{X}$ of h. Thus $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is a monomorphism of algebraic spaces, which proves that (1) holds.

Finally, assume that (1) holds. Then for any scheme T and morphism $y : T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is a monomorphism. Hence there exists up to unique isomorphism exactly one pair (x, α) where $x : T \to \mathcal{X}$ is a morphism and $\alpha : f \circ x \to y$ is a 2-morphism. Applying the 2-Yoneda lemma this says exactly that f is fully faithful, i.e., that (2) holds. $\qquad\square$

Lemma 8.5. *A monomorphism of algebraic stacks induces an injective map of sets of points.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of algebraic stacks. Suppose that $x_i : \mathrm{Spec}(K_i) \to \mathcal{X}$ be morphisms such that $f \circ x_1$ and $f \circ x_2$ define the same element of $|\mathcal{Y}|$. Applying the definition we find a common extension Ω with corresponding morphisms $c_i : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(K_i)$ and a 2-isomorphism $\beta : f \circ x_1 \circ c_1 \to f \circ x_1 \circ c_2$. As f is fully faithful, see Lemma 8.4, we can lift β to an isomorphism $\alpha : x_1 \circ c_1 \to x_1 \circ c_2$. Hence x_1 and x_2 define the same point of $|\mathcal{X}|$ as desired. $\qquad\square$

Lemma 8.6. *Let $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{X}'$ is a monomorphism then the canonical diagram*

$$
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{X}' & \longrightarrow & \mathcal{X}' \times_{\mathcal{Y}} \mathcal{X}'
\end{array}
$$

is a fibre product square.

Proof. We have $\mathcal{X} = \mathcal{X} \times_{\mathcal{X}'} \mathcal{X}$ by Lemma 8.4. Thus the result by applying Categories, Lemma 31.13. $\qquad\square$

9. Immersions of algebraic stacks

Immersions of algebraic stacks are defined as follows.

Definition 9.1. Immersions.
  (1) A morphism of algebraic stacks is called an *open immersion* if it is representable, and an open immersion in the sense of Section 3.
  (2) A morphism of algebraic stacks is called a *closed immersion* if it is representable, and a closed immersion in the sense of Section 3.
  (3) A morphism of algebraic stacks is called an *immersion* if it is representable, and an immersion in the sense of Section 3.

This is not the most convenient way to think about immersions for us. For us it is a little bit more convenient to think of an immersion as a morphism of algebraic stacks which is representable by algebraic spaces and is an immersion in the sense of Section 3. Similarly for closed and open immersions. Since this is clearly equivalent to the notion just defined we shall use this characterization without further mention. We prove a few simple lemmas about this notion.

Lemma 9.2. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a (closed, resp. open) immersion. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is a (closed, resp. open) immersion.*

Proof. This follows from the general discussion in Section 3. $\square$

Lemma 9.3. *Compositions of immersions of algebraic stacks are immersions. Similarly for closed immersions and open immersions.*

Proof. This follows from the general discussion in Section 3 and Spaces, Lemma 12.2. $\square$

Lemma 9.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. let W be an algebraic space and let $W \to \mathcal{Y}$ be a surjective, flat morphism which is locally of finite presentation. The following are equivalent:*

(1) *f is an (open, resp. closed) immersion, and*
(2) *$V = W \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and $V \to W$ is an (open, resp. closed) immersion.*

Proof. This follows from the general discussion in Section 3 and in particular Lemmas 3.1 and 3.3. $\square$

Lemma 9.5. *An immersion is a monomorphism.*

Proof. See Morphisms of Spaces, Lemma 10.7. $\square$

Lemma 9.6. *If $f : \mathcal{X} \to \mathcal{Y}$ is an immersion, then $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is a homeomorphism onto a locally closed subset. If f is a closed, resp. open immersion, then $|f|$ is closed, resp. open.*

Proof. Omitted. $\square$

The following two lemmas explain how to think about immersions in terms of presentations.

Lemma 9.7. *Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces. Let $i : \mathcal{Z} \to [U/R]$ be an immersion. Then there exists an R-invariant locally closed subspace $Z \subset U$ and a presentation $[Z/R_Z] \to \mathcal{Z}$ where R_Z is the restriction of R to Z such that*

$$
\begin{array}{ccc}
[Z/R_Z] & \longrightarrow & \mathcal{Z} \\
 & \searrow & \downarrow i \\
 & & [U/R]
\end{array}
$$

is 2-commutative. If i is a closed (resp. open) immersion then Z is a closed (resp. open) subspace of U.

Proof. By Lemma 3.6 we get a commutative diagram

$$
\begin{array}{ccc}
[U'/R'] & \longrightarrow & \mathcal{Z} \\
 & \searrow & \downarrow \\
 & & [U/R]
\end{array}
$$

where $U' = \mathcal{Z} \times_{[U/R]} U$ and $R' = \mathcal{Z} \times_{[U/R]} R$. Since $\mathcal{Z} \to [U/R]$ is an immersion we see that $U' \to U$ is an immersion of algebraic spaces. Let $Z \subset U$ be the locally
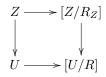
closed subspace such that $U' \to U$ factors through Z and induces an isomorphism $U' \to Z$. It is clear from the construction of R' that $R' = U' \times_{U,t} R = R \times_{s,U} U'$. This implies that $Z \cong U'$ is R-invariant and that the image of $R' \to R$ identifies R' with the restriction $R_Z = s^{-1}(Z) = t^{-1}(Z)$ of R to Z. Hence the lemma holds. $\square$

Lemma 9.8. *Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces. Let $\mathcal{X} = [U/R]$ be the associated algebraic stack, see Algebraic Stacks, Theorem 17.3. Let $Z \subset U$ be an R-invariant locally closed subspace. Then*

$$[Z/R_Z] \longrightarrow [U/R]$$

is an immersion of algebraic stacks, where R_Z is the restriction of R to Z. If $Z \subset U$ is open (resp. closed) then the morphism is an open (resp. closed) immersion of algebraic stacks.

Proof. Recall that by Groupoids in Spaces, Definition 18.1 (see also discussion following the definition) we have $R_Z = s^{-1}(Z) = t^{-1}(Z)$ as locally closed subspaces of R. Hence the two morphisms $R_Z \to Z$ are smooth as base changes of s and t. Hence $(Z, R_Z, s|_{R_Z}, t|_{R_Z}, c|_{R_Z \times_{s,Z,t} R_Z})$ is a smooth groupoid in algebraic spaces, and we see that $[Z/R_Z]$ is an algebraic stack, see Algebraic Stacks, Theorem 17.3. The assumptions of Groupoids in Spaces, Lemma 25.3 are all satisfied and it follows that we have a 2-fibre square

$$
\begin{array}{ccc}
Z & \longrightarrow & [Z/R_Z] \\
\downarrow & & \downarrow \\
U & \longrightarrow & [U/R]
\end{array}
$$

It follows from this and Lemma 3.1 that $[Z/R_Z] \to [U/R]$ is representable by algebraic spaces, whereupon it follows from Lemma 3.3 that the right vertical arrow is an immersion (resp. closed immersion, resp. open immersion) if and only if the left vertical arrow is. $\square$

We can define open, closed, and locally closed substacks as follows.

Definition 9.9. Let $\mathcal{X}$ be an algebraic stack.
  (1) An *open substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is an open immersion.
  (2) A *closed substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is a closed immersion.
  (3) A *locally closed substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is an immersion.

This definition should be used with caution. Namely, if $f : \mathcal{X} \to \mathcal{Y}$ is an equivalence of algebraic stacks and $\mathcal{X}' \subset \mathcal{X}$ is an open substack, then it is not necessarily the case that the subcategory $f(\mathcal{X}')$ is an open substack of $\mathcal{Y}$. The problem is that it may not be a *strictly* full subcategory; but this is also the only problem. Here is a formal statement.

Lemma 9.10. *For any immersion $i : \mathcal{Z} \to \mathcal{X}$ there exists a unique locally closed substack $\mathcal{X}' \subset \mathcal{X}$ such that i factors as the composition of an equivalence $i' : \mathcal{Z} \to \mathcal{X}'$ followed by the inclusion morphism $\mathcal{X}' \to \mathcal{X}$. If i is a closed (resp. open) immersion, then $\mathcal{X}'$ is a closed (resp. open) substack of $\mathcal{X}$.*

Proof. Omitted. $\square$

Lemma 9.11. *Let $[U/R] \to \mathcal{X}$ be a presentation of an algebraic stack. There is a canonical bijection*

locally closed substacks $\mathcal{Z}$ of $\mathcal{X} \longrightarrow R$-invariant locally closed subspaces Z of U

which sends $\mathcal{Z}$ to $U \times_{\mathcal{X}} \mathcal{Z}$. Moreover, a morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Z}$ if and only if $\mathcal{Y} \times_{\mathcal{X}} U \to U$ factors through Z. Similarly for closed substacks and open substacks.

Proof. By Lemmas 9.7 and 9.8 we find that the map is a bijection. If $\mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Z}$ then of course the base change $\mathcal{Y} \times_{\mathcal{X}} U \to U$ factors through Z. Converse, suppose that $\mathcal{Y} \to \mathcal{X}$ is a morphism such that $\mathcal{Y} \times_{\mathcal{X}} U \to U$ factors through Z. We will show that for every scheme T and morphism $T \to \mathcal{Y}$, given by an object y of the fibre category of $\mathcal{Y}$ over T, the object y is in fact in the fibre category of $\mathcal{Z}$ over T. Namely, the fibre product $T \times_{\mathcal{X}} U$ is an algebraic space and $T \times_{\mathcal{X}} U \to T$ is a surjective smooth morphism. Hence there is an fppf covering $\{T_i \to T\}$ such that $T_i \to T$ factors through $T \times_{\mathcal{X}} U \to T$ for all i. Then $T_i \to \mathcal{X}$ factors through $\mathcal{Y} \times_{\mathcal{X}} U$ and hence through $Z \subset U$. Thus $y|_{T_i}$ is an object of $\mathcal{Z}$ (as Z is the fibre product of U with $\mathcal{Z}$ over $\mathcal{X}$). Since $\mathcal{Z}$ is a strictly full substack, we conclude that y is an object of $\mathcal{Z}$ as desired. $\square$

Lemma 9.12. *Let $\mathcal{X}$ be an algebraic stack. The rule $\mathcal{U} \mapsto |\mathcal{U}|$ defines an inclusion preserving bijection between open substacks of $\mathcal{X}$ and open subsets of $|\mathcal{X}|$.*

Proof. Choose a presentation $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 16.2. By Lemma 9.11 we see that open substacks correspond to R-invariant open subschemes of U. On the other hand Lemmas 4.5 and 4.7 guarantee these correspond bijectively to open subsets of $|\mathcal{X}|$. $\square$

Lemma 9.13. *Let $\mathcal{X}$ be an algebraic stack. Let U be an algebraic space and $U \to \mathcal{X}$ a surjective smooth morphism. For an open immersion $V \hookrightarrow U$, there exists an algebraic stack $\mathcal{Y}$, an open immersion $\mathcal{Y} \to \mathcal{X}$, and a surjective smooth morphism $V \to \mathcal{Y}$.*

Proof. We define a category fibred in groupoids $\mathcal{Y}$ by letting the fiber category $\mathcal{Y}_T$ over an object T of $(Sch/S)_{fppf}$ be the full subcategory of $\mathcal{X}_T$ consisting of all $y \in \mathrm{Ob}(\mathcal{X}_T)$ such that the projection morphism $V \times_{\mathcal{X},y} T \to T$ surjective. Now for any morphism $x : T \to \mathcal{X}$, the 2-fibred product $T \times_{x,\mathcal{X}} \mathcal{Y}$ has fiber category over T' consisting of triples $(f : T' \to T, y \in \mathcal{X}_{T'}, f^*x \simeq y)$ such that $V \times_{\mathcal{X},y} T' \to T'$ is surjective. Note that $T \times_{x,\mathcal{X}} \mathcal{Y}$ is fibered in setoids since $\mathcal{Y} \to \mathcal{X}$ is faithful (see Stacks, Lemma 6.7). Now the isomorphism $f^*x \simeq y$ gives the diagram

$$
\begin{array}{ccccc}
V \times_{\mathcal{X},y} T' & \longrightarrow & V \times_{\mathcal{X},x} T & \longrightarrow & V \\
\downarrow & & \downarrow & & \downarrow \\
T' & \xrightarrow{\ f\ } & T & \xrightarrow{\ x\ } & \mathcal{X}
\end{array}
$$

where both squares are cartesian. The morphism $V \times_{\mathcal{X},x} T \to T$ is smooth by base change, and hence open. Let $T_0 \subset T$ be its image. From the cartesian squares we deduce that $V \times_{\mathcal{X},y} T' \to T'$ is surjective if and only if f lands in T_0. Therefore $T \times_{x,\mathcal{X}} \mathcal{Y}$ is representable by T_0, so the inclusion $\mathcal{Y} \to \mathcal{X}$ is an open immersion. By Algebraic Stacks, Lemma 15.5 we conclude that $\mathcal{Y}$ is an algebraic stack. Lastly

if we denote the morphism $V \to \mathcal{X}$ by g, we have $V \times_{\mathcal{X}} V \to V$ is surjective (the diagonal gives a section). Hence g is in the image of $\mathcal{Y}_V \to \mathcal{X}_V$, i.e., we obtain a morphism $g' : V \to \mathcal{Y}$ fitting into the commutative diagram
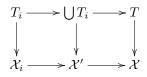
$$
\begin{array}{ccc}
V & \longrightarrow & U \\
\downarrow{\scriptstyle g'} & & \downarrow \\
\mathcal{Y} & \longrightarrow & \mathcal{X}
\end{array}
$$

Since $V \times_{g,\mathcal{X}} \mathcal{Y} \to V$ is a monomorphism, it is in fact an isomorphism since $(1, g')$ defines a section. Therefore $g' : V \to \mathcal{Y}$ is a smooth morphism, as it is the base change of the smooth morphism $g : V \to \mathcal{X}$. It is surjective by our construction of $\mathcal{Y}$ which finishes the proof of the lemma. $\qquad\square$

Lemma 9.14. *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{X}_i \subset \mathcal{X}$ a collection of open substacks indexed by $i \in I$. Then there exists an open substack, which we denote $\bigcup_{i \in I} \mathcal{X}_i \subset \mathcal{X}$, such that the $\mathcal{X}_i$ are open substacks covering it.*

Proof. We define a fibred subcategory $\mathcal{X}' = \bigcup_{i \in I} \mathcal{X}_i$ by letting the fiber category over an object T of $(Sch/S)_{fppf}$ be the full subcategory of $\mathcal{X}_T$ consisting of all $x \in \mathrm{Ob}(\mathcal{X}_T)$ such that the morphism $\coprod_{i \in I}(\mathcal{X}_i \times_{\mathcal{X}} T) \to T$ is surjective. Let $x_i \in \mathrm{Ob}((\mathcal{X}_i)_T)$. Then $(x_i, 1)$ gives a section of $\mathcal{X}_i \times_{\mathcal{X}} T \to T$, so we have an isomorphism. Thus $\mathcal{X}_i \subset \mathcal{X}'$ is a full subcategory. Now let $x \in \mathrm{Ob}(\mathcal{X}_T)$. Then $\mathcal{X}_i \times_{\mathcal{X}} T$ is representable by an open subscheme $T_i \subset T$. The 2-fibred product $\mathcal{X}' \times_{\mathcal{X}} T$ has fiber over T' consisting of $(y \in \mathcal{X}_{T'}, f : T' \to T, f^*x \simeq y)$ such that $\coprod(\mathcal{X}_i \times_{\mathcal{X},y} T') \to T'$ is surjective. The isomorphism $f^*x \simeq y$ induces an isomorphism $\mathcal{X}_i \times_{\mathcal{X},y} T' \simeq T_i \times_T T'$. Then the $T_i \times_T T'$ cover T' if and only if f lands in $\bigcup T_i$. Therefore we have a diagram

$$
\begin{array}{ccccc}
T_i & \longrightarrow & \bigcup T_i & \longrightarrow & T \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_i & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X}
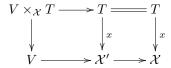\end{array}
$$

with both squares cartesian. By Algebraic Stacks, Lemma 15.5 we conclude that $\mathcal{X}' \subset \mathcal{X}$ is algebraic and an open substack. It is also clear from the cartesian squares above that the morphism $\coprod_{i \in I} \mathcal{X}_i \to \mathcal{X}'$ which finishes the proof of the lemma. $\quad\square$

Lemma 9.15. *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{X}' \subset \mathcal{X}$ a quasi-compact open substack. Suppose that we have a collection of open substacks $\mathcal{X}_i \subset \mathcal{X}$ indexed by $i \in I$ such that $\mathcal{X}' \subset \bigcup_{i \in I} \mathcal{X}_i$, where we define the union as in Lemma 9.14. Then there exists a finite subset $I' \subset I$ such that $\mathcal{X}' \subset \bigcup_{i \in I'} \mathcal{X}_i$.*

Proof. Since $\mathcal{X}$ is algebraic, there exists a scheme U with a surjective smooth morphism $U \to \mathcal{X}$. Let $U_i \subset U$ be the open subscheme representing $\mathcal{X}_i \times_{\mathcal{X}} U$ and $U' \subset U$ the open subscheme representing $\mathcal{X}' \times_{\mathcal{X}} U$. By hypothesis, $U' \subset \bigcup_{i \in I} U_i$. From the proof of Lemma 6.2, there is a quasi-compact open $V \subset U'$ such that $V \to \mathcal{X}'$ is a surjective smooth morphism. Therefore there exists a finite subset $I' \subset I$ such that $V \subset \bigcup_{i \in I'} U_i$. We claim that $\mathcal{X}' \subset \bigcup_{i \in I'} \mathcal{X}_i$. Take $x \in \mathrm{Ob}(\mathcal{X}'_T)$ for $T \in \mathrm{Ob}((Sch/S)_{fppf})$. Since $\mathcal{X}' \to \mathcal{X}$ is a monomorphism, we have cartesian

squares

$$
\begin{array}{ccc}
V \times_{\mathcal{X}} T & \longrightarrow T == T \\
\downarrow & \quad \downarrow x \quad \downarrow x \\
V & \longrightarrow \mathcal{X}' \longrightarrow \mathcal{X}
\end{array}
$$

By base change, $V \times_{\mathcal{X}} T \to T$ is surjective. Therefore $\bigcup_{i \in I'} U_i \times_{\mathcal{X}} T \to T$ is also surjective. Let $T_i \subset T$ be the open subscheme representing $\mathcal{X}_i \times_{\mathcal{X}} T$. By a formal argument, we have a Cartesian square

$$
\begin{array}{ccc}
U_i \times_{\mathcal{X}_i} T_i & \longrightarrow & U \times_{\mathcal{X}} T \\
\downarrow & & \downarrow \\
T_i & \longrightarrow & T
\end{array}
$$

where the vertical arrows are surjective by base change. Since $U_i \times_{\mathcal{X}_i} T_i \simeq U_i \times_{\mathcal{X}} T$, we find that $\bigcup_{i \in I'} T_i = T$. Hence x is an object of $(\bigcup_{i \in I'} \mathcal{X}_i)_T$ by definition of the union. Observe that the inclusion $\mathcal{X}' \subset \bigcup_{i \in I'} \mathcal{X}_i$ is automatically an open substack. $\qquad \square$

Lemma 9.16. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X}_i$, $i \in I$ be a set of open substacks of $\mathcal{X}$. Assume*

(1) *$\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$, and*
(2) *each $\mathcal{X}_i$ is an algebraic space.*

Then $\mathcal{X}$ is an algebraic space.

Proof. Apply Stacks, Lemma 6.10 to the morphism $\coprod_{i \in I} \mathcal{X}_i \to \mathcal{X}$ and the morphism id $: \mathcal{X} \to \mathcal{X}$ to see that $\mathcal{X}$ is a stack in setoids. Hence $\mathcal{X}$ is an algebraic space, see Algebraic Stacks, Proposition 13.3. $\qquad \square$

Lemma 9.17. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X}_i$, $i \in I$ be a set of open substacks of $\mathcal{X}$. Assume*

(1) *$\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$, and*
(2) *each $\mathcal{X}_i$ is a scheme*

Then $\mathcal{X}$ is a scheme.

Proof. By Lemma 9.16 we see that $\mathcal{X}$ is an algebraic space. Since any algebraic space has a largest open subspace which is a scheme, see Properties of Spaces, Lemma 13.1 we see that $\mathcal{X}$ is a scheme. $\qquad \square$

The following lemma is the analogue of More on Groupoids, Lemma 6.1.

Lemma 9.18. *Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be properties of morphisms of algebraic spaces. Assume*

(1) *$\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are fppf local on the target and stable under arbitrary base change,*
(2) *smooth $\Rightarrow \mathcal{R}$,*
(3) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$ there exists a largest open subspace $W(\mathcal{P}, f) \subset X$ such that $f|_{W(\mathcal{P}, f)}$ has $\mathcal{P}$, and*
(4) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$, and any morphism $Y' \to Y$ which has $\mathcal{R}$ we have $Y' \times_Y W(\mathcal{P}, f) = W(\mathcal{P}, f')$, where $f' : X_{Y'} \to Y'$ is the base change of f.*

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Assume f has $\mathcal{Q}$. Then

(A) *there exists a largest open substack $\mathcal{X}' \subset \mathcal{X}$ such that $f|_{\mathcal{X}'}$ has $\mathcal{P}$, and*
(B) *if $\mathcal{Z} \to \mathcal{Y}$ is a morphism of algebraic stacks representable by algebraic spaces which has $\mathcal{R}$ then $\mathcal{Z} \times_\mathcal{Y} \mathcal{X}'$ is the largest open substack of $\mathcal{Z} \times_\mathcal{Y} \mathcal{X}$ over which the base change $id_\mathcal{Z} \times f$ has property $\mathcal{P}$.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = V \times_\mathcal{Y} \mathcal{X}$ and let $f' : U \to V$ be the base change of f. The morphism of algebraic spaces $f' : U \to V$ has property $\mathcal{Q}$. Thus we obtain the open $W(\mathcal{P}, f') \subset U$ by assumption (3). Note that $U \times_\mathcal{X} U = (V \times_\mathcal{Y} V) \times_\mathcal{Y} \mathcal{X}$ hence the morphism $f'' : U \times_\mathcal{X} U \to V \times_\mathcal{Y} V$ is the base change of f via either projection $V \times_\mathcal{Y} V \to V$. By our choice of V these projections are smooth, hence have property $\mathcal{R}$ by (2). Thus by (4) we see that the inverse images of $W(\mathcal{P}, f')$ under the two projections $\mathrm{pr}_i : U \times_\mathcal{X} U \to U$ agree. In other words, $W(\mathcal{P}, f')$ is an R-invariant subspace of U (where $R = U \times_\mathcal{X} U$). Let $\mathcal{X}'$ be the open substack of $\mathcal{X}$ corresponding to $W(\mathcal{P}, f)$ via Lemma 9.7. By construction $W(\mathcal{P}, f') = \mathcal{X}' \times_\mathcal{Y} V$ hence $f|_{\mathcal{X}'}$ has property $\mathcal{P}$ by Lemma 3.3. Also, $\mathcal{X}'$ is the largest open substack such that $f|_{\mathcal{X}'}$ has $\mathcal{P}$ as the same maximality holds for $W(\mathcal{P}, f)$. This proves (A).

Finally, if $\mathcal{Z} \to \mathcal{Y}$ is a morphism of algebraic stacks representable by algebraic spaces which has $\mathcal{R}$ then we set $T = V \times_\mathcal{Y} \mathcal{Z}$ and we see that $T \to V$ is a morphism of algebraic spaces having property $\mathcal{R}$. Set $f'_T : T \times_V U \to T$ the base change of f'. By (4) again we see that $W(\mathcal{P}, f'_T)$ is the inverse image of $W(\mathcal{P}, f)$ in $T \times_V U$. This implies (B); some details omitted. □

Remark 9.19. Warning: Lemma 9.18 should be used with care. For example, it applies to $\mathcal{P} =$"flat", $\mathcal{Q} =$"empty", and $\mathcal{R} =$"flat and locally of finite presentation". But given a morphism of algebraic spaces $f : X \to Y$ the largest open subspace $W \subset X$ such that $f|_W$ is flat is *not* the set of points where f is flat!

Remark 9.20. Notwithstanding the warning in Remark 9.19 there are some cases where Lemma 9.18 can be used without causing ambiguity. We give a list. In each case we omit the verification of assumptions (1) and (2) and we give references which imply (3) and (4). Here is the list:

(1) $\mathcal{Q} =$"locally of finite type", $\mathcal{R} = \emptyset$, and $\mathcal{P} =$"relative dimension $\leq d$". See Morphisms of Spaces, Definition 33.2 and Morphisms of Spaces, Lemmas 34.4 and 34.3.
(2) $\mathcal{Q} =$"locally of finite type", $\mathcal{R} = \emptyset$, and $\mathcal{P} =$"locally quasi-finite". This is the case $d = 0$ of the previous item, see Morphisms of Spaces, Lemma 34.6. On the other hand, properties (3) and (4) are spelled out in Morphisms of Spaces, Lemma 34.7.
(3) $\mathcal{Q} =$"locally of finite type", $\mathcal{R} = \emptyset$, and $\mathcal{P} =$"unramified". This is Morphisms of Spaces, Lemma 38.10.
(4) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"flat". See More on Morphisms of Spaces, Theorem 22.1 and Lemma 22.2. Note that here $W(\mathcal{P}, f)$ is always exactly the set of points where the morphism f is flat because we only consider this open when f has $\mathcal{Q}$ (see loc.cit.).
(5) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"étale". This follows on combining (3) and (4) because an unramified morphism which is flat and locally of finite presentation is étale, see Morphisms of Spaces, Lemma 39.12.

(6) Add more here as needed (compare with the longer list at More on Groupoids, Remark 6.3).

10. Reduced algebraic stacks

We have already defined reduced algebraic stacks in Section 7.

Lemma 10.1. *Let $\mathcal{X}$ be an algebraic stack. Let $T \subset |\mathcal{X}|$ be a closed subset. There exists a unique closed substack $\mathcal{Z} \subset \mathcal{X}$ with the following properties: (a) we have $|\mathcal{Z}| = T$, and (b) $\mathcal{Z}$ is reduced.*

Proof. Let $U \to \mathcal{X}$ be a surjective smooth morphism, where U is an algebraic space. Set $R = U \times_{\mathcal{X}} U$, so that there is a presentation $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 16.2. As usual we denote $s, t : R \to U$ the two smooth projection morphisms. By Lemma 4.5 we see that T corresponds to a closed subset $T' \subset |U|$ such that $|s|^{-1}(T') = |t|^{-1}(T')$. Let $Z \subset U$ be the reduced induced algebraic space structure on T', see Properties of Spaces, Definition 12.5. The fibre products $Z \times_{U,t} R$ and $R \times_{s,U} Z$ are closed subspaces of R (Spaces, Lemma 12.3). The projections $Z \times_{U,t} R \to Z$ and $R \times_{s,U} Z \to Z$ are smooth by Morphisms of Spaces, Lemma 37.3. Thus as Z is reduced, it follows that $Z \times_{U,t} R$ and $R \times_{s,U} Z$ are reduced, see Remark 7.3. Since

$$|Z \times_{U,t} R| = |t|^{-1}(T') = |s|^{-1}(T') = R \times_{s,U} Z$$

we conclude from the uniqueness in Properties of Spaces, Lemma 12.3 that $Z \times_{U,t} R = R \times_{s,U} Z$. Hence Z is an R-invariant closed subspace of U. By the correspondence of Lemma 9.11 we obtain a closed substack $\mathcal{Z} \subset \mathcal{X}$ with $Z = \mathcal{Z} \times_{\mathcal{X}} U$. Then $[Z/R_Z] \to \mathcal{Z}$ is a presentation (Lemma 9.7). Then $|\mathcal{Z}| = |Z|/|R_Z| = |T'|/\sim$ is the given closed subset T. We omit the proof of unicity. $\square$

Lemma 10.2. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}' \subset \mathcal{X}$ is a closed substack, $\mathcal{X}$ is reduced and $|\mathcal{X}'| = |\mathcal{X}|$, then $\mathcal{X}' = \mathcal{X}$.*

Proof. Choose a presentation $[U/R] \to \mathcal{X}$ with U a scheme. As $\mathcal{X}$ is reduced, we see that U is reduced (by definition of reduced algebraic stacks). By Lemma 9.11 $\mathcal{X}'$ corresponds to an R-invariant closed subscheme $Z \subset U$. But now $|Z| \subset |U|$ is the inverse image of $|\mathcal{X}'|$, and hence $|Z| = |U|$. Hence Z is a closed subscheme of U whose underlying sets of points agree. By Schemes, Lemma 12.7 the map $\mathrm{id}_U : U \to U$ factors through $Z \to U$, and hence $Z = U$, i.e., $\mathcal{X}' = \mathcal{X}$. $\square$

Lemma 10.3. *Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks. Let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack Assume $\mathcal{Y}$ is reduced. A morphism $f : \mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Z}$ if and only if $f(|\mathcal{Y}|) \subset |\mathcal{Z}|$.*

Proof. Assume $f(|\mathcal{Y}|) \subset |\mathcal{Z}|$. Consider $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Y}$. There is an equivalence $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Y}'$ where $\mathcal{Y}'$ is a closed substack of $\mathcal{Y}$, see Lemmas 9.2 and 9.10. Using Lemmas 4.3, 8.5, and 9.5 we see that $|\mathcal{Y}'| = |\mathcal{Y}|$. Hence we have reduced the lemma to Lemma 10.2. $\square$

Definition 10.4. Let $\mathcal{X}$ be an algebraic stack. Let $Z \subset |\mathcal{X}|$ be a closed subset. An *algebraic stack structure on Z* is given by a closed substack $\mathcal{Z}$ of $\mathcal{X}$ with $|\mathcal{Z}|$ equal to Z. The *reduced induced algebraic stack structure* on Z is the one constructed in Lemma 10.1. The *reduction $\mathcal{X}_{red}$ of $\mathcal{X}$* is the reduced induced algebraic stack structure on $|\mathcal{X}|$.

In fact we can use this to define the reduced induced algebraic stack structure on a locally closed subset.

Remark 10.5. Let X be an algebraic stack. Let $T \subset |\mathcal{X}|$ be a locally closed subset. Let ∂T be the boundary of T in the topological space $|\mathcal{X}|$. In a formula
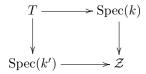
$$\partial T = \overline{T} \setminus T.$$

Let $\mathcal{U} \subset \mathcal{X}$ be the open substack of X with $|\mathcal{U}| = |\mathcal{X}| \setminus \partial T$, see Lemma 9.12. Let $\mathcal{Z}$ be the reduced closed substack of $\mathcal{U}$ with $|\mathcal{Z}| = T$ obtained by taking the reduced induced closed subspace structure, see Definition 10.4. By construction $\mathcal{Z} \to \mathcal{U}$ is a closed immersion of algebraic stacks and $\mathcal{U} \to \mathcal{X}$ is an open immersion, hence $\mathcal{Z} \to \mathcal{X}$ is an immersion of algebraic stacks by Lemma 9.3. Note that $\mathcal{Z}$ is a reduced algebraic stack and that $|\mathcal{Z}| = T$ as subsets of $|X|$. We sometimes say $\mathcal{Z}$ is the *reduced induced substack structure* on T.

11. Residual gerbes

In the Stacks project we would like to define the *residual gerbe* of an algebraic stack $\mathcal{X}$ at a point $x \in |\mathcal{X}|$ to be a monomorphism of algebraic stacks $m_x : \mathcal{Z}_x \to \mathcal{X}$ where $\mathcal{Z}_x$ is a reduced algebraic stack having a unique point which is mapped by m_x to x. It turns out that there are many issues with this notion; existence is not clear in general and neither is uniqueness. We resolve the uniqueness issue by imposing a slightly stronger condition on the algebraic stacks $\mathcal{Z}_x$. We discuss this in more detail by working through a few simple lemmas regarding reduced algebraic stacks having a unique point.

Lemma 11.1. *Let $\mathcal{Z}$ be an algebraic stack. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be surjective and flat. Then any morphism $\mathrm{Spec}(k') \to \mathcal{Z}$ where k' is a field is surjective and flat.*

Proof. Consider the fibre square

$$
\begin{array}{ccc}
T & \longrightarrow & \mathrm{Spec}(k) \\
\downarrow & & \downarrow \\
\mathrm{Spec}(k') & \longrightarrow & \mathcal{Z}
\end{array}
$$

Note that $T \to \mathrm{Spec}(k')$ is flat and surjective hence T is not empty. On the other hand $T \to \mathrm{Spec}(k)$ is flat as k is a field. Hence $T \to \mathcal{Z}$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 31.5 (via the discussion in Section 3) that $\mathrm{Spec}(k') \to \mathcal{Z}$ is flat. It is clear that it is surjective as by assumption $|\mathcal{Z}|$ is a singleton. $\square$

Lemma 11.2. *Let $\mathcal{Z}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{Z}$ is reduced and $|\mathcal{Z}|$ is a singleton,*
(2) *there exists a surjective flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field, and*
(3) *there exists a locally of finite type, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field.*

Proof. Assume (1). Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Then W is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of W. Since W is reduced we have $\mathcal{O}_{W,\eta} = \kappa(\eta)$. It follows that the canonical morphism $\eta = \mathrm{Spec}(\kappa(\eta)) \to W$ is flat. We see that the composition

$\eta \to \mathcal{Z}$ is flat (see Morphisms of Spaces, Lemma 30.3). It is also surjective as $|\mathcal{Z}|$ is a singleton. In other words (2) holds.

Assume (2). Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Choose a field k and a surjective flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then $W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is an algebraic space smooth over k, hence regular (see Spaces over Fields, Lemma 16.1) and in particular reduced. Since $W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W$ is surjective and flat we conclude that W is reduced (Descent on Spaces, Lemma 9.2). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme W and a smooth morphism $W \to \mathcal{Z}$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow \mathcal{Z}$$

is locally of finite type by Morphisms of Spaces, Lemmas 23.2 and 37.6. It is also flat and surjective by Lemma 11.1. Hence (3) holds. $\square$

The following lemma singles out a slightly better class of singleton algebraic stacks than the preceding lemma.

Lemma 11.3. *Let $\mathcal{Z}$ be an algebraic stack. The following are equivalent*
   (1) *$\mathcal{Z}$ is reduced, locally Noetherian, and $|\mathcal{Z}|$ is a singleton, and*
   (2) *there exists a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field.*

Proof. Assume (2) holds. By Lemma 11.2 we see that $\mathcal{Z}$ is reduced and $|\mathcal{Z}|$ is a singleton. Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Choose a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then $W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is an algebraic space smooth over k, hence locally Noetherian (see Morphisms of Spaces, Lemma 23.5). Since $W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W\}$ is an fppf covering and we conclude that W is locally Noetherian (Descent on Spaces, Lemma 9.3). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme W and a smooth morphism $W \to \mathcal{Z}$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because W is locally Noetherian the morphism $w : \mathrm{Spec}(k) \to W$ is of finite presentation, see Morphisms, Lemma 21.7. Hence the composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow \mathcal{Z}$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 28.2 and 37.5. It is also flat and surjective by Lemma 11.1. Hence (2) holds. $\square$

Lemma 11.4. *Let $\mathcal{Z}' \to \mathcal{Z}$ be a monomorphism of algebraic stacks. Assume there exists a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then either $\mathcal{Z}'$ is empty or $\mathcal{Z}' \to \mathcal{Z}$ is an equivalence.*

Proof. We may assume that $\mathcal{Z}'$ is nonempty. In this case the fibre product $T = \mathcal{Z}' \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is nonempty, see Lemma 4.3. Now T is an algebraic space and the projection $T \to \mathrm{Spec}(k)$ is a monomorphism. Hence $T = \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemma 10.8. We conclude that $\mathrm{Spec}(k) \to \mathcal{Z}$ factors through $\mathcal{Z}'$. Suppose the morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}$ is given by the object ξ over $\mathrm{Spec}(k)$. We have just seen that ξ is isomorphic to an object ξ' of $\mathcal{Z}'$ over $\mathrm{Spec}(k)$. Since z is surjective, flat,

and locally of finite presentation we see that every object of $\mathcal{Z}$ over any scheme is fppf locally isomorphic to a pullback of ξ, hence also to a pullback of ξ'. By descent of objects for stacks in groupoids this implies that $\mathcal{Z}' \to \mathcal{Z}$ is essentially surjective (as well as fully faithful, see Lemma 8.4). Hence we win. $\square$

Lemma 11.5. *Let $\mathcal{Z}$ be an algebraic stack. Assume $\mathcal{Z}$ satisfies the equivalent conditions of Lemma 11.2. Then there exists a unique strictly full subcategory $\mathcal{Z}' \subset \mathcal{Z}$ such that $\mathcal{Z}'$ is an algebraic stack which satisfies the equivalent conditions of Lemma 11.3. The inclusion morphism $\mathcal{Z}' \to \mathcal{Z}$ is a monomorphism of algebraic stacks.*

Proof. The last part is immediate from the first part and Lemma 8.4. Pick a field k and a morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ which is surjective, flat, and locally of finite type. Set $U = \mathrm{Spec}(k)$ and $R = U \times_{\mathcal{Z}} U$. The projections $s, t : R \to U$ are locally of finite type. Since U is the spectrum of a field, it follows that s, t are flat and locally of finite presentation (by Morphisms of Spaces, Lemma 28.7). We see that $\mathcal{Z}' = [U/R]$ is an algebraic stack by Criteria for Representability, Theorem 17.2. By Algebraic Stacks, Lemma 16.1 we obtain a canonical morphism

$$f : \mathcal{Z}' \longrightarrow \mathcal{Z}$$

which is fully faithful. Hence this morphism is representable by algebraic spaces, see Algebraic Stacks, Lemma 15.2 and a monomorphism, see Lemma 8.4. By Criteria for Representability, Lemma 17.1 the morphism $U \to \mathcal{Z}'$ is surjective, flat, and locally of finite presentation. Hence $\mathcal{Z}'$ is an algebraic stack which satisfies the equivalent conditions of Lemma 11.3. By Algebraic Stacks, Lemma 12.4 we may replace $\mathcal{Z}'$ by its essential image in $\mathcal{Z}$. Hence we have proved all the assertions of the lemma except for the uniqueness of $\mathcal{Z}' \subset \mathcal{Z}$. Suppose that $\mathcal{Z}'' \subset \mathcal{Z}$ is a second such algebraic stack. Then the projections

$$\mathcal{Z}' \longleftarrow \mathcal{Z}' \times_{\mathcal{Z}} \mathcal{Z}'' \longrightarrow \mathcal{Z}''$$

are monomorphisms. The algebraic stack in the middle is nonempty by Lemma 4.3. Hence the two projections are isomorphisms by Lemma 11.4 and we win. $\square$

Example 11.6. Here is an example where the morphism constructed in Lemma 11.5 isn't an isomorphism. This example shows that imposing that residual gerbes are locally Noetherian is necessary in Definition 11.8. In fact, the example is even an algebraic space! Let $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be the absolute Galois group of $\mathbf{Q}$ with the pro-finite topology. Let

$$U = \mathrm{Spec}(\overline{\mathbf{Q}}) \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(\overline{\mathbf{Q}}) = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \times \mathrm{Spec}(\overline{\mathbf{Q}})$$

(we omit a precise explanation of the meaning of the last equal sign). Let G denote the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with the discrete topology viewed as a constant group scheme over $\mathrm{Spec}(\overline{\mathbf{Q}})$, see Groupoids, Example 5.6. Then G acts freely and transitively on U. Let $X = U/G$, see Spaces, Definition 14.4. Then X is a non-noetherian reduced algebraic space with exactly one point. Furthermore, X has a (locally) finite type point:

$$x : \mathrm{Spec}(\overline{\mathbf{Q}}) \longrightarrow U \longrightarrow X$$

Indeed, every point of U is actually closed! As X is an algebraic space over $\overline{\mathbf{Q}}$ it follows that x is a monomorphism. So x is the morphism constructed in Lemma

11.5 but x is not an isomorphism. In fact $\mathrm{Spec}(\overline{\mathbf{Q}}) \to X$ is the residual gerbe of X at x.

It will turn out later that under mild assumptions on the algebraic stack $\mathcal{X}$ the equivalent conditions of the following lemma are satisfied for every point $x \in |\mathcal{X}|$ (see Morphisms of Stacks, Section 31).

Lemma 11.7. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point. The following are equivalent*

    (1) *there exists an algebraic stack $\mathcal{Z}$ and a monomorphism $\mathcal{Z} \to \mathcal{X}$ such that $|\mathcal{Z}|$ is a singleton and such that the image of $|\mathcal{Z}|$ in $|\mathcal{X}|$ is x,*

    (2) *there exists a reduced algebraic stack $\mathcal{Z}$ and a monomorphism $\mathcal{Z} \to \mathcal{X}$ such that $|\mathcal{Z}|$ is a singleton and such that the image of $|\mathcal{Z}|$ in $|\mathcal{X}|$ is x,*

    (3) *there exists an algebraic stack $\mathcal{Z}$, a monomorphism $f : \mathcal{Z} \to \mathcal{X}$, and a surjective flat morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field such that $x = f(z)$.*

Moreover, if these conditions hold, then there exists a unique strictly full subcategory $\mathcal{Z}_x \subset \mathcal{X}$ such that $\mathcal{Z}_x$ is a reduced, locally Noetherian algebraic stack and $|\mathcal{Z}_x|$ is a singleton which maps to x via the map $|\mathcal{Z}_x| \to |\mathcal{X}|$.

Proof. If $\mathcal{Z} \to \mathcal{X}$ is as in (1), then $\mathcal{Z}_{red} \to \mathcal{X}$ is as in (2). (See Section 10 for the notion of the reduction of an algebraic stack.) Hence (1) implies (2). It is immediate that (2) implies (1). The equivalence of (2) and (3) is immediate from Lemma 11.2.

At this point we've seen the equivalence of (1) – (3). Pick a monomorphism $f : \mathcal{Z} \to \mathcal{X}$ as in (2). Note that this implies that f is fully faithful, see Lemma 8.4. Denote $\mathcal{Z}' \subset \mathcal{X}$ the essential image of the functor f. Then $f : \mathcal{Z} \to \mathcal{Z}'$ is an equivalence and hence $\mathcal{Z}'$ is an algebraic stack, see Algebraic Stacks, Lemma 12.4. Apply Lemma 11.5 to get a strictly full subcategory $\mathcal{Z}_x \subset \mathcal{Z}'$ as in the statement of the lemma. This proves all the statements of the lemma except for uniqueness.

In order to prove the uniqueness suppose that $\mathcal{Z}_x \subset \mathcal{X}$ and $\mathcal{Z}'_x \subset \mathcal{X}$ are two strictly full subcategories as in the statement of the lemma. Then the projections

$$\mathcal{Z}'_x \longleftarrow \mathcal{Z}'_x \times_{\mathcal{X}} \mathcal{Z}_x \longrightarrow \mathcal{Z}_x$$

are monomorphisms. The algebraic stack in the middle is nonempty by Lemma 4.3. Hence the two projections are isomorphisms by Lemma 11.4 and we win. $\square$

Having explained the above we can now make the following definition.

Definition 11.8. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$.

    (1) We say the *residual gerbe of $\mathcal{X}$ at x exists* if the equivalent conditions (1), (2), and (3) of Lemma 11.7 hold.

    (2) If the residual gerbe of $\mathcal{X}$ at x exists, then the *residual gerbe of $\mathcal{X}$ at x*[1] is the strictly full subcategory $\mathcal{Z}_x \subset \mathcal{X}$ constructed in Lemma 11.7.

---

[1]This clashes with [LMB00] in spirit, but not in fact. Namely, in Chapter 11 they associate to any point on any quasi-separated algebraic stack a gerbe (not necessarily algebraic) which they call the residual gerbe. We will see in Morphisms of Stacks, Lemma 31.1 that on a quasi-separated algebraic stack every point has a residual gerbe in our sense which is then equivalent to theirs. For more information on this topic see [Ryd10, Appendix B].

In particular we know that $\mathcal{Z}_x$ (if it exists) is a locally Noetherian, reduced algebraic stack and that there exists a field and a surjective, flat, locally finitely presented morphism

$$\mathrm{Spec}(k) \longrightarrow \mathcal{Z}_x.$$

We will see in Morphisms of Stacks, Lemma 28.12 that $\mathcal{Z}_x$ is a gerbe. Existence of residual gerbes is discussed in Morphisms of Stacks, Section 31.

Example 11.9. Let X be a scheme and let $x \in X$ be a point. Then the monomorphism $x \to X$ is the residual gerbe of X at x where we, as usual, identify x with the scheme $x = \mathrm{Spec}(\kappa(x))$. If X is an algebraic space and $x \in |X|$, then the residual gerbe at x (which is called the residual space) always exists, see Decent Spaces, Section 13.

The residual gerbe, if it exists, is a regular algebraic stack by the following lemma.

Lemma 11.10. *A reduced, locally Noetherian algebraic stack $\mathcal{Z}$ such that $|\mathcal{Z}|$ is a singleton is regular.*

Proof. Let $W \to \mathcal{Z}$ be a surjective smooth morphism where W is a scheme. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be surjective, flat, and locally of finite presentation (see Lemma 11.3). The algebraic space $T = W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is smooth over k in particular regular, see Spaces over Fields, Lemma 16.1. Since $T \to W$ is locally of finite presentation, flat, and surjective it follows that W is regular, see Descent on Spaces, Lemma 9.4. By definition this means that $\mathcal{Z}$ is regular. $\square$

Lemma 11.11. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Assume that the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ exists. Let $f : \mathrm{Spec}(K) \to \mathcal{X}$ be a morphism where K is a field in the equivalence class of x. Then f factors through the inclusion morphism $\mathcal{Z}_x \to \mathcal{X}$.*

Proof. Choose a field k and a surjective flat locally finite presentation morphism $\mathrm{Spec}(k) \to \mathcal{Z}_x$. Set $T = \mathrm{Spec}(K) \times_{\mathcal{X}} \mathcal{Z}_x$. By Lemma 4.3 we see that T is nonempty. As $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism we see that $T \to \mathrm{Spec}(K)$ is a monomorphism. Hence by Morphisms of Spaces, Lemma 10.8 we see that $T = \mathrm{Spec}(K)$ which proves the lemma. $\square$

Lemma 11.12. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Let $\mathcal{Z}$ be an algebraic stack satisfying the equivalent conditions of Lemma 11.3 and let $\mathcal{Z} \to \mathcal{X}$ be a monomorphism such that the image of $|\mathcal{Z}| \to |\mathcal{X}|$ is x. Then the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and $\mathcal{Z} \to \mathcal{X}$ factors as $\mathcal{Z} \to \mathcal{Z}_x \to \mathcal{X}$ where the first arrow is an equivalence.*

Proof. Let $\mathcal{Z}_x \subset \mathcal{X}$ be the full subcategory corresponding to the essential image of the functor $\mathcal{Z} \to \mathcal{X}$. Then $\mathcal{Z} \to \mathcal{Z}_x$ is an equivalence, hence $\mathcal{Z}_x$ is an algebraic stack, see Algebraic Stacks, Lemma 12.4. Since $\mathcal{Z}_x$ inherits all the properties of $\mathcal{Z}$ from this equivalence it is clear from the uniqueness in Lemma 11.7 that $\mathcal{Z}_x$ is the residual gerbe of $\mathcal{X}$ at x. $\square$

Lemma 11.13. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. If the residual gerbes $\mathcal{Z}_x \subset \mathcal{X}$ and $\mathcal{Z}_y \subset \mathcal{Y}$ of x and y exist,*

then f induces a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \longleftarrow & \mathcal{Z}_x \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathcal{Y} & \longleftarrow & \mathcal{Z}_y \end{array}$$

Proof. Choose a field k and a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}_x$. The morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ factors through $\mathcal{Z}_y$ by Lemma 11.11. Thus $\mathcal{Z}_x \times_{\mathcal{Y}} \mathcal{Z}_y$ is a nonempty substack of $\mathcal{Z}_x$ hence equal to $\mathcal{Z}_x$ by Lemma 11.4. $\square$

Lemma 11.14. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. Assume the residual gerbes $\mathcal{Z}_x \subset \mathcal{X}$ and $\mathcal{Z}_y \subset \mathcal{Y}$ of x and y exist and that there exists a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x such that*

$$\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathrm{Spec}(k)$$

is an isomorphism. Then $\mathcal{Z}_x \to \mathcal{Z}_y$ is an isomorphism.

Proof. Let k'/k be an extension of fields. Then

$$\mathrm{Spec}(k') \times_{\mathcal{X}} \mathrm{Spec}(k') \longrightarrow \mathrm{Spec}(k') \times_{\mathcal{Y}} \mathrm{Spec}(k')$$

is the base change of the morphism in the lemma by the faithfully flat morphism $\mathrm{Spec}(k' \otimes k') \to \mathrm{Spec}(k \otimes k)$. Thus the property described in the lemma is independent of the choice of the morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x. Thus we may assume that $\mathrm{Spec}(k) \to \mathcal{Z}_x$ is surjective, flat, and locally of finite presentation. In this situation we have

$$\mathcal{Z}_x = [\mathrm{Spec}(k)/R]$$

with $R = \mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k)$. See proof of Lemma 11.5. Since also $R = \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathrm{Spec}(k)$ we conclude that the morphism $\mathcal{Z}_x \to \mathcal{Z}_y$ of Lemma 11.13 is fully faithful by Algebraic Stacks, Lemma 16.1. We conclude for example by Lemma 11.12. $\square$

12. Dimension of a stack

We can define the dimension of an algebraic stack $\mathcal{X}$ at a point x, using the notion of dimension of an algebraic space at a point (Properties of Spaces, Definition 9.1). In the following lemma the output may be ∞ either because $\mathcal{X}$ is not quasi-compact or because we run into the phenomenon described in Examples, Section 15.

Lemma 12.1. *Let $\mathcal{X}$ be a locally Noetherian algebraic stack over a scheme S. Let $x \in |\mathcal{X}|$ be a point of $\mathcal{X}$. Let $[U/R] \to \mathcal{X}$ be a presentation (Algebraic Stacks, Definition 16.5) where U is a scheme. Let $u \in U$ be a point that maps to x. Let $e : U \to R$ be the "identity" map and let $s : R \to U$ be the "source" map, which is a smooth morphism of algebraic spaces. Let R_u be the fiber of $s : R \to U$ over u. The element*

$$\dim_x(\mathcal{X}) = \dim_u(U) - \dim_{e(u)}(R_u) \in \mathbf{Z} \cup \infty$$

is independent of the choice of presentation and the point u over x.

Proof. Since $R \to U$ is smooth, the scheme R_u is smooth over $\kappa(u)$ and hence has finite dimension. On the other hand, the scheme U is locally Noetherian, but this does not guarantee that $\dim_u(U)$ is finite. Thus the difference is an element of $\mathbf{Z} \cup \{\infty\}$.

Let $[U'/R'] \to \mathcal{X}$ and $u' \in U'$ be a second presentation where U' is a scheme and u' maps to x. Consider the algebraic space $P = U \times_{\mathcal{X}} U'$. By Lemma 4.3 there exists a $p \in |P|$ mapping to u and u'. Since $P \to U$ and $P \to U'$ are smooth we see that $\dim_p(P) = \dim_u(U) + \dim_p(P_u)$ and $\dim_p(P) = \dim_{u'}(U') + \dim_p(P_{u'})$, see Morphisms of Spaces, Lemma 37.10. Note that

$$R'_{u'} = \mathrm{Spec}(\kappa(u')) \times_{\mathcal{X}} U' \quad \text{and} \quad P_u = \mathrm{Spec}(\kappa(u)) \times_{\mathcal{X}} U'$$

Let us represent $p \in |P|$ by a morphism $\mathrm{Spec}(\Omega) \to P$. Since p maps to both u and u' it induces a 2-morphism between the compositions $\mathrm{Spec}(\Omega) \to \mathrm{Spec}(\kappa(u')) \to \mathcal{X}$ and $\mathrm{Spec}(\Omega) \to \mathrm{Spec}(\kappa(u)) \to \mathcal{X}$ which in turn defines an isomorphism

$$\mathrm{Spec}(\Omega) \times_{\mathrm{Spec}(\kappa(u'))} R'_{u'} \cong \mathrm{Spec}(\Omega) \times_{\mathrm{Spec}(\kappa(u))} P_u$$

as algebraic spaces over $\mathrm{Spec}(\Omega)$ mapping the Ω-rational point $(1, e'(u'))$ to $(1, p)$ (some details omitted). We conclude that

$$\dim_{e'(u')}(R'_{u'}) = \dim_p(P_u)$$

by Morphisms of Spaces, Lemma 34.3. By symmetry we have $\dim_{e(u)}(R_u) = \dim_p(P_{u'})$. Putting everything together we obtain the independence of choices. $\square$

We can use the lemma above to make the following definition.

Definition 12.2. Let $\mathcal{X}$ be a locally Noetherian algebraic stack over a scheme S. Let $x \in |\mathcal{X}|$ be a point of $\mathcal{X}$. Let $[U/R] \to \mathcal{X}$ be a presentation (Algebraic Stacks, Definition 16.5) where U is a scheme and let $u \in U$ be a point that maps to x. We define the *dimension of $\mathcal{X}$ at x* to be the element $\dim_x(\mathcal{X}) \in \mathbf{Z} \cup \infty$ such that

$$\dim_x(\mathcal{X}) = \dim_u(U) - \dim_{e(u)}(R_u).$$

with notation as in Lemma 12.1.

The dimension of a stack at a point agrees with the usual notion when $\mathcal{X}$ is a scheme (Topology, Definition 10.1), or more generally when $\mathcal{X}$ is a locally Noetherian algebraic space (Properties of Spaces, Definition 9.1).

Definition 12.3. Let S be a scheme. Let $\mathcal{X}$ be a locally Noetherian algebraic stack over S. The *dimension* $\dim(\mathcal{X})$ of $\mathcal{X}$ is defined to be

$$\dim(\mathcal{X}) = \sup_{x \in |\mathcal{X}|} \dim_x(\mathcal{X})$$

This definition of dimension agrees with the usual notion if $\mathcal{X}$ is a scheme (Properties, Lemma 10.2) or an algebraic space (Properties of Spaces, Definition 9.2).

Remark 12.4. If $\mathcal{X}$ is a nonempty stack of finite type over a field, then $\dim(\mathcal{X})$ is an integer. For an arbitrary locally Noetherian algebraic stack $\mathcal{X}$, $\dim(\mathcal{X})$ is in $Z \cup \{\pm\infty\}$, and $\dim(\mathcal{X}) = -\infty$ if and only if $\mathcal{X}$ is empty.

Example 12.5. Let X be a scheme of finite type over a field k, and let G be a group scheme of finite type over k which acts on X. Then the dimension of the quotient stack $[X/G]$ is equal to $\dim(X) - \dim(G)$. In particular, the dimension of the classifying stack $BG = [\mathrm{Spec}(k)/G]$ is $- \dim(G)$. Thus the dimension of an algebraic stack can be a negative integer, in contrast to what happens for schemes or algebraic spaces.

13. Local irreducibility

We have defined the geometric number of branches of a scheme at a point in Properties, Section 15 and for an algebraic space at a point in Properties of Spaces, Section 23. Let $n \in \mathbf{N}$. For a local ring A set

$$P_n(A) = \text{the number of geometric branches of } A \text{ is } n$$

For a smooth ring map $A \to B$ and a prime ideal $\mathfrak{q}$ of B lying over $\mathfrak{p}$ of A we have

$$P_n(A_\mathfrak{p}) \Leftrightarrow P_n(B_\mathfrak{q})$$

by More on Algebra, Lemma 106.8. As in Properties of Spaces, Remark 7.6 we may use P_n to define an étale local property $\mathcal{P}_n$ of germs (U, u) of schemes by setting $\mathcal{P}_n(U, u) = P_n(\mathcal{O}_{U,u})$. The corresponding property $\mathcal{P}_n$ of an algebraic space X at a point x (see Properties of Spaces, Definition 7.5) is just the property "the number of geometric branches of X at x is n", see Properties of Spaces, Definition 23.4. Moreover, the property $\mathcal{P}_n$ is smooth local, see Descent, Definition 21.1. This follows either from the equivalence displayed above or More on Morphisms, Lemma 36.4. Thus Definition 7.5 applies and we obtain a notion for algebraic stacks at a point.

Definition 13.1. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$.
1. The *number of geometric branches of $\mathcal{X}$ at x* is either $n \in \mathbf{N}$ if the equivalent conditions of Lemma 7.4 hold for $\mathcal{P}_n$ defined above, or else ∞.
2. We say $\mathcal{X}$ is *geometrically unibranch at x* if the number of geometric branches of $\mathcal{X}$ at x is 1.

14. Finiteness conditions and points

This section is the analogue of Decent Spaces, Section 4 for points of algebraic stacks.

Lemma 14.1. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point. The following are equivalent*
1. *some morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x is quasi-compact, and*
2. *any morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x is quasi-compact.*

Proof. Let $\mathrm{Spec}(k) \to \mathcal{X}$ be in the equivalence class of x. Let k'/k be a field extension. Then we have to show that $\mathrm{Spec}(k) \to \mathcal{X}$ is quasi-compact if and only if $\mathrm{Spec}(k') \to \mathcal{X}$ is quasi-compact. This follows from Morphisms of Spaces, Lemma 8.6 and the principle of Algebraic Stacks, Lemma 10.9. $\square$

Sometimes people say that a point $x \in |\mathcal{X}|$ satisfying the equivalent conditions of Lemma 14.1 is a "*quasi-compact point*".

15. Other chapters

Preliminaries

References

[LMB00] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 39, Springer-Verlag, 2000.

[Ryd10] David Rydh, *étale dévissage, descent and pushouts of stacks*, math.AG/1005.2171v1 (2010).