SEMISTABLE REDUCTION

Contents

1. Introduction

In this chapter we prove the semistable reduction theorem for curves. We will use the method of Artin and Winters from their paper [AW71].

It turns out that one can prove the semistable reduction theorem for curves without any results on desingularization. Namely, there is a way to establish the existence and projectivity of moduli of semistable curves using Geometric Invariant Theory (GIT) as developed by Mumford, see [MFK94]. This method was championed by Gieseker who proved the full result in his lecture notes [Gie82][1]. This is quite an amazing feat: it seems somewhat counter intuitive that one can prove such a result without ever truly studying families of curves over a positive dimensional base.

Historically the first proof of the semistable reduction theorem for curves can be found in the paper [DM69] by Deligne and Mumford. It proves the theorem by reducing the problem to the case of Abelian varieties which was already known at the time thanks to Grothendieck and others, see [GRR72] and [DK73]). The

[1] Gieseker's lecture notes are written over an algebraically closed field, but the same method works over $\mathbf{Z}$.

semistable reduction theorem for abelian varieties uses the theory of Néron models which in turn rests on a treatment of birational group laws over a base.

The method in the paper by Artin and Winters relies on desingularization of singularities of surfaces to obtain regular models. Given the existence of regular models, the proof consists in analyzing the possibilities for the special fibre and concluding using an inequality for torsion in the Picard group of a 1-dimensional scheme over a field. A similar argument can be found in a paper [Sai87] of Saito who uses étale cohomology directly and who obtains a stronger result in that he can characterize semistable reduction in terms of the action of the inertia on ℓ-adic étale cohomology.

A different approach one can use to prove the theorem is to use rigid analytic geometry techniques. Here we refer the reader to [vdP84] and [AW12].

The paper [Tem10] by Temkin uses valuation theoretic techniques (and proves a lot more besides); also Appendix A of this paper gives a nice overview of the different proofs and the relationship with desingularizations of 2 dimensional schemes.

Another overview paper that the reader may wish to consult is [Abb00] written by Ahmed Abbes.

2. Linear algebra

A couple of lemmas we will use later on.

Lemma 2.1. *Let $A = (a_{ij})$ be a complex $n \times n$ matrix.*

1. *If $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for each i, then $\det(A)$ is nonzero.*
2. *If there exists a real vector $m = (m_1, \ldots, m_n)$ with $m_i > 0$ such that $|a_{ii}m_i| > \sum_{j \neq i} |a_{ij}m_j|$ for each i, then $\det(A)$ is nonzero.*

Proof. If A is as in (1) and $\det(A) = 0$, then there is a nonzero vector z with $Az = 0$. Choose r with $|z_r|$ maximal. Then

$$|a_{rr}z_r| = |\sum\nolimits_{k \neq r} a_{rk}z_k| \leq \sum\nolimits_{k \neq r} |a_{rk}||z_k| \leq |z_r| \sum\nolimits_{k \neq r} |a_{rk}| < |a_{rr}||z_r|$$

which is a contradiction. To prove (2) apply (1) to the matrix $(a_{ij}m_j)$ whose determinant is $m_1 \ldots m_n \det(A)$. ☐

Lemma 2.2. *Let $A = (a_{ij})$ be a real $n \times n$ matrix with $a_{ij} \geq 0$ for $i \neq j$. Let $m = (m_1, \ldots, m_n)$ be a real vector with $m_i > 0$. For $I \subset \{1, \ldots, n\}$ let $x_I \in \mathbf{R}^n$ be the vector whose ith coordinate is m_i if $i \in I$ and 0 otherwise. If*

$$-a_{ii}m_i \geq \sum\nolimits_{j \neq i} a_{ij}m_j \tag{2.2.1}$$

for each i, then $\mathrm{Ker}(A)$ is the vector space spanned by the vectors x_I such that

1. *$a_{ij} = 0$ for $i \in I$, $j \notin I$, and*
2. *equality holds in (2.2.1) for $i \in I$.*

Proof. After replacing a_{ij} by $a_{ij}m_j$ we may assume $m_i = 1$ for all i. If $I \subset \{1, \ldots, n\}$ such that (1) and (2) are true, then a simple computation shows that x_I is in the kernel of A. Conversely, let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ be a nonzero vector in the kernel of A. We will show by induction on the number of nonzero coordinates of x that x is in the span of the vectors x_I satisfying (1) and (2). Let $I \subset \{1, \ldots, n\}$ be the set of indices r with $|x_r|$ maximal. For $r \in I$ we have

$$|a_{rr}x_r| = |\sum\nolimits_{k \neq r} a_{rk}x_k| \leq \sum\nolimits_{k \neq r} a_{rk}|x_k| \leq |x_r| \sum\nolimits_{k \neq r} a_{rk} \leq |a_{rr}||x_r|$$

Thus equality holds everywhere. In particular, we see that $a_{rk} = 0$ if $r \in I$, $k \notin I$ and equality holds in (2.2.1) for $r \in I$. Then we see that we can substract a suitable multiple of x_I from x to decrease the number of nonzero coordinates. $\square$

Lemma 2.3. *Let $A = (a_{ij})$ be a symmetric real $n \times n$ matrix with $a_{ij} \geq 0$ for $i \neq j$. Let $m = (m_1, \ldots, m_n)$ be a real vector with $m_i > 0$. Assume*

1. *$Am = 0$,*
2. *there is no proper nonempty subset $I \subset \{1, \ldots, n\}$ such that $a_{ij} = 0$ for $i \in I$ and $j \notin I$.*

Then $x^t A x \leq 0$ with equality if and only if $x = qm$ for some $q \in \mathbf{R}$.

First proof. After replacing a_{ij} by $a_{ij} m_i m_j$ we may assume $m_i = 1$ for all i. Condition (1) means $-a_{ii} = \sum_{j \neq i} a_{ij}$ for all i. Recall that $x^t A x = \sum_{i,j} x_i a_{ij} x_j$. Then

$$\begin{aligned} \sum\nolimits_{i \neq j} -a_{ij}(x_j - x_i)^2 &= \sum\nolimits_{i \neq j} -a_{ij} x_j^2 + 2a_{ij} x_i x_i - a_{ij} x_i^2 \\ &= \sum\nolimits_{j} a_{jj} x_j^2 + \sum\nolimits_{i \neq j} 2a_{ij} x_i x_i + \sum\nolimits_{j} a_{jj} x_i^2 \\ &= 2x^t A x \end{aligned}$$

This is clearly ≤ 0. If equality holds, then let I be the set of indices i with $x_i \neq x_1$. Then $a_{ij} = 0$ for $i \in I$ and $j \notin I$. Thus $I = \{1, \ldots, n\}$ by condition (2) and x is a multiple of $m = (1, \ldots, 1)$. $\square$

Second proof. The matrix A has real eigenvalues by the spectral theorem. We claim all the eigenvalues are ≤ 0. Namely, since property (1) means $-a_{ii} m_i = \sum_{j \neq i} a_{ij} m_j$ for all i, we find that the matrix $A' = A - \lambda I$ for $\lambda > 0$ satisfies $|a'_{ii} m_i| > \sum a'_{ij} m_j = \sum |a'_{ij} m_j|$ for all i. Hence A' is invertible by Lemma 2.1. This implies that the symmetric bilinear form $x^t A y$ is semi-negative definite, i.e., $x^t A x \leq 0$ for all x. It follows that the kernel of A is equal to the set of vectors x with $x^t A x = 0$. The description of the kernel in Lemma 2.2 gives the final statement of the lemma. $\square$

Lemma 2.4. *Let L be a finite free $\mathbf{Z}$-module endowed with an integral symmetric bilinear positive definite form $\langle\ ,\ \rangle : L \times L \to \mathbf{Z}$. Let $A \subset L$ be a submodule with L/A torsion free. Set $B = \{b \in L \mid \langle a, b \rangle = 0,\ \forall a \in A\}$. Then we have injective maps*

$$A^\#/A \leftarrow L/(A \oplus B) \to B^\#/B$$

whose cokernels are quotients of $L^\#/L$. Here $A^\# = \{a' \in A \otimes \mathbf{Q} \mid \langle a, a' \rangle \in \mathbf{Z},\ \forall a \in A\}$ and similarly for B and L.

Proof. Observe that $L \otimes \mathbf{Q} = A \otimes \mathbf{Q} \oplus B \otimes \mathbf{Q}$ because the form is nondegenerate on A (by positivity). We denote $\pi_B : L \otimes \mathbf{Q} \to B \otimes \mathbf{Q}$ the projection. Observe that $\pi_B(x) \in B^\#$ for $x \in L$ because the form is integral. This gives an exact sequence

$$0 \to A \to L \xrightarrow{\pi_B} B^\# \to Q \to 0$$

where Q is the cokernel of $L \to B^\#$. Observe that Q is a quotient of $L^\#/L$ as the map $L^\# \to B^\#$ is surjective since it is the $\mathbf{Z}$-linear dual to $B \to L$ which is split as a map of $\mathbf{Z}$-modules. Dividing by $A \oplus B$ we get a short exact sequence

$$0 \to L/(A \oplus B) \to B^\#/B \to Q \to 0$$

This proves the lemma. $\square$

Lemma 2.5. *Let L_0, L_1 be a finite free $\mathbf{Z}$-modules endowed with integral symmetric bilinear positive definite forms $\langle\ ,\ \rangle : L_i \times L_i \to \mathbf{Z}$. Let $d : L_0 \to L_1$ and $d^* : L_1 \to L_0$ be adjoint. If $\langle\ ,\ \rangle$ on L_0 is unimodular, then there is an isomorphism*

$$\Phi : \mathrm{Coker}(d^* d)_{torsion} \longrightarrow \mathrm{Im}(d)^{\#} / \mathrm{Im}(d)$$

with notation as in Lemma 2.4.

Proof. Let $x \in L_0$ be an element representing a torsion class in $\mathrm{Coker}(\mathrm{d}^*\mathrm{d})$. Then for some $a > 0$ we can write $ax = \mathrm{d}^*\mathrm{d}(y)$. For any $z \in \mathrm{Im}(\mathrm{d})$, say $z = \mathrm{d}(y')$, we have

$$\langle (1/a)\mathrm{d}(y), z\rangle = \langle (1/a)\mathrm{d}(y), \mathrm{d}(y')\rangle = \langle x, y'\rangle \in \mathbf{Z}$$

Hence $(1/a)\mathrm{d}(y) \in \mathrm{Im}(\mathrm{d})^{\#}$. We define $\Phi(x) = (1/a)\mathrm{d}(y) \bmod \mathrm{Im}(\mathrm{d})$. We omit the proof that Φ is well defined, additive, and injective.

To prove Φ is surjective, let $z \in \mathrm{Im}(\mathrm{d})^{\#}$. Then z defines a linear map $L_0 \to \mathbf{Z}$ by the rule $x \mapsto \langle z, \mathrm{d}(x)\rangle$. Since the pairing on L_0 is unimodular by assumption we can find an $x' \in L_0$ with $\langle x', x\rangle = \langle z, \mathrm{d}(x)\rangle$ for all $x \in L_0$. In particular, we see that x' pairs to zero with $\mathrm{Ker}(\mathrm{d})$. Since $\mathrm{Im}(\mathrm{d}^*\mathrm{d}) \otimes \mathbf{Q}$ is the orthogonal complement of $\mathrm{Ker}(\mathrm{d}) \otimes \mathbf{Q}$ this means that x' defines a torsion class in $\mathrm{Coker}(\mathrm{d}^*\mathrm{d})$. We claim that $\Phi(x') = z$. Namely, write $ax' = \mathrm{d}^*\mathrm{d}(y)$ for some $y \in L_0$ and $a > 0$. For any $x \in L_0$ we get

$$\langle z, \mathrm{d}(x)\rangle = \langle x', x\rangle = \langle (1/a)\mathrm{d}^*\mathrm{d}(y), x\rangle = \langle (1/a)\mathrm{d}(y), \mathrm{d}(x)\rangle$$

Hence $z = \Phi(x')$ and the proof is complete. □

Lemma 2.6. *Let $A = (a_{ij})$ be a symmetric $n \times n$ integer matrix with $a_{ij} \geq 0$ for $i \neq j$. Let $m = (m_1, \ldots, m_n)$ be an integer vector with $m_i > 0$. Assume*

1. *$Am = 0$,*
2. *there is no proper nonempty subset $I \subset \{1, \ldots, n\}$ such that $a_{ij} = 0$ for $i \in I$ and $j \notin I$.*

Let e be the number of pairs (i, j) with $i < j$ and $a_{ij} > 0$. Then for ℓ a prime number coprime with all a_{ij} and m_i we have

$$\dim_{\mathbf{F}_\ell}(\mathrm{Coker}(A)[\ell]) \leq 1 - n + e$$

Proof. By Lemma 2.3 the rank of A is $n - 1$. The composition

$$\mathbf{Z}^{\oplus n} \xrightarrow{\mathrm{diag}(m_1,\ldots,m_n)} \mathbf{Z}^{\oplus n} \xrightarrow{(a_{ij})} \mathbf{Z}^{\oplus n} \xrightarrow{\mathrm{diag}(m_1,\ldots,m_n)} \mathbf{Z}^{\oplus n}$$

has matrix $a_{ij}m_im_j$. Since the cokernel of the first and last maps are torsion of order prime to ℓ by our restriction on ℓ we see that it suffices to prove the lemma for the matrix with entries $a_{ij}m_im_j$. Thus we may assume $m = (1, \ldots, 1)$.

Assume $m = (1, \ldots, 1)$. Set $V = \{1, \ldots, n\}$ and $E = \{(i, j) \mid i < j \text{ and } a_{ij} > 0\}$. For $e = (i, j) \in E$ set $a_e = a_{ij}$. Define maps $s, t : E \to V$ by setting $s(i, j) = i$ and $t(i, j) = j$. Set $\mathbf{Z}(V) = \bigoplus_{i \in V} \mathbf{Z}i$ and $\mathbf{Z}(E) = \bigoplus_{e \in E} \mathbf{Z}e$. We define symmetric positive definite integer valued pairings on $\mathbf{Z}(V)$ and $\mathbf{Z}(E)$ by setting

$$\langle i, i\rangle = 1 \text{ for } i \in V, \quad \langle e, e\rangle = a_e \text{ for } e \in E$$

and all other pairings zero. Consider the maps

$$\mathrm{d} : \mathbf{Z}(V) \to \mathbf{Z}(E), \quad i \longmapsto \sum\nolimits_{e \in E,\ s(e) = i} e - \sum\nolimits_{e \in E,\ t(e) = i} e$$

and

$$\mathrm{d}^*(e) = a_e(s(e) - t(e))$$

A computation shows that

$$\langle d(x), y\rangle = \langle x, \mathrm{d}^*(y)\rangle$$

in other words, d and d^* are adjoint. Next we compute

$$\begin{aligned}\mathrm{d}^*\mathrm{d}(i) &= \mathrm{d}^*(\textstyle\sum_{e\in E,\ s(e)=i} e - \sum_{e\in E,\ t(e)=i} e) \\ &= \textstyle\sum_{e\in E,\ s(e)=i} a_e(s(e)-t(e)) - \sum_{e\in E,\ t(e)=i} a_e(s(e)-t(e))\end{aligned}$$

The coefficient of i in $\mathrm{d}^*\mathrm{d}(i)$ is

$$\textstyle\sum_{e\in E,\ s(e)=i} a_e + \sum_{e\in E,\ t(e)=i} a_e = -a_{ii}$$

because $\sum_j a_{ij} = 0$ and the coefficient of $j \neq i$ in $\mathrm{d}^*\mathrm{d}(i)$ is $-a_{ij}$. Hence $\mathrm{Coker}(A) = \mathrm{Coker}(\mathrm{d}^*\mathrm{d})$.

Consider the inclusion

$$\mathrm{Im}(\mathrm{d}) \oplus \mathrm{Ker}(\mathrm{d}^*) \subset \mathbf{Z}(E)$$

The left hand side is an orthogonal direct sum. Clearly $\mathbf{Z}(E)/\,\mathrm{Ker}(\mathrm{d}^*)$ is torsion free. We claim $\mathbf{Z}(E)/\,\mathrm{Im}(\mathrm{d})$ is torsion free as well. Namely, say $x = \sum x_e e \in \mathbf{Z}(E)$ and $a > 1$ are such that $ax = \mathrm{d}y$ for some $y = \sum y_i i \in \mathbf{Z}(V)$. Then $ax_e = y_{s(e)} - y_{t(e)}$. By property (2) we conclude that all y_i have the same congruence class modulo a. Hence we can write $y = ay' + (y_1, y_1, \ldots, y_1)$. Since $\mathrm{d}(y_1, y_1, \ldots, y_1) = 0$ we conclude that $x = \mathrm{d}(y')$ which is what we had to show.

Hence we may apply Lemma 2.4 to get injective maps

$$\mathrm{Im}(\mathrm{d})^{\#}/\,\mathrm{Im}(\mathrm{d}) \leftarrow \mathbf{Z}(E)/(\mathrm{Im}(\mathrm{d}) \oplus \mathrm{Ker}(\mathrm{d}^*)) \rightarrow \mathrm{Ker}(\mathrm{d}^*)^{\#}/\,\mathrm{Ker}(\mathrm{d}^*)$$

whose cokernels are annihilated by the product of the a_e (which is prime to ℓ). Since $\mathrm{Ker}(\mathrm{d}^*)$ is a lattice of rank $1 - n + e$ we see that the proof is complete if we prove that there exists an isomorphism

$$\Phi : M_{torsion} \longrightarrow \mathrm{Im}(\mathrm{d})^{\#}/\,\mathrm{Im}(\mathrm{d})$$

This is proved in Lemma 2.5. □

3. Numerical types

Part of the arguments will involve the combinatorics of the following data structures.

Definition 3.1. A *numerical type* T is given by

$$n, m_i, a_{ij}, w_i, g_i$$

where $n \geq 1$ is an integer and m_i, a_{ij}, w_i, g_i are integers for $1 \leq i, j \leq n$ subject to the following conditions

1. $m_i > 0$, $w_i > 0$, $g_i \geq 0$,
2. the matrix $A = (a_{ij})$ is symmetric and $a_{ij} \geq 0$ for $i \neq j$,
3. there is no proper nonempty subset $I \subset \{1, \ldots, n\}$ such that $a_{ij} = 0$ for $i \in I$, $j \notin I$,
4. for each i we have $\sum_j a_{ij} m_j = 0$, and
5. $w_i | a_{ij}$.

This is obviously a somewhat annoying type of structure to work with, but it is exactly what shows up in special fibres of proper regular models of smooth geometrically connected curves. Of course we only care about these types up to reordering the indices.

Definition 3.2. We say two numerical types n, m_i, a_{ij}, w_i, g_i and $n', m'_i, a'_{ij}, w'_i, g'_i$ are *equivalent types* if there exists a permutation σ of $\{1, \ldots, n\}$ such that $m_i = m'_{\sigma(i)}$, $a_{ij} = a'_{\sigma(i)\sigma(j)}$, $w_i = w'_{\sigma(i)}$, and $g_i = g'_{\sigma(i)}$.

A numerical type has a genus.

Lemma 3.3. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type. Then the expression*

$$g = 1 + \sum m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$$

is an integer.

Proof. To prove g is an integer we have to show that $\sum a_{ii}m_i$ is even. This we can see by computing modulo 2 as follows

$$\begin{aligned}
\sum\nolimits_i a_{ii}m_i &\equiv \sum\nolimits_{i,\ m_i \text{ odd}} a_{ii}m_i \\
&\equiv \sum\nolimits_{i,\ m_i \text{ odd}} \sum\nolimits_{j \neq i} a_{ij}m_j \\
&\equiv \sum\nolimits_{i,\ m_i \text{ odd}} \sum\nolimits_{j \neq i,\ m_j \text{ odd}} a_{ij}m_j \\
&\equiv \sum\nolimits_{i<j,\ m_i \text{ and } m_j \text{ odd}} a_{ij}(m_i + m_j) \\
&\equiv 0
\end{aligned}$$

where we have used that $a_{ij} = a_{ji}$ and that $\sum_j a_{ij}m_j = 0$ for all i. □

Definition 3.4. We say n, m_i, a_{ij}, w_i, g_i is a *numerical type of genus g* if $g = 1 + \sum m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ is the integer from Lemma 3.3.

We will prove below (Lemma 3.14) that the genus is almost always ≥ 0. But you can have numerical types with negative genus.

Lemma 3.5. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. If $n = 1$, then $a_{11} = 0$ and $g = 1 + m_1w_1(g_1 - 1)$. Moreover, we can classify all such numerical types as follows*

1. *If $g < 0$, then $g_1 = 0$ and there are finitely many possible numerical types of genus g with $n = 1$ corresponding to factorizations $m_1w_1 = 1 - g$.*
2. *If $g = 0$, then $m_1 = 1$, $w_1 = 1$, $g_1 = 0$ as in Lemma 6.1.*
3. *If $g = 1$, then we conclude $g_1 = 1$ but m_1, w_1 can be arbitrary positive integers; this is case (1) of Lemma 6.2.*
4. *If $g > 1$, then $g_1 > 1$ and there are finitely many possible numerical types of genus g with $n = 1$ corresponding to factorizations $m_1w_1(g_1 - 1) = g - 1$.*

Proof. The lemma proves itself. □

Lemma 3.6. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. If $n > 1$, then $a_{ii} < 0$ for all i.*

Proof. Lemma 2.3 applies to the matrix A. □

Lemma 3.7. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Assume $n > 1$. If i is such that the contribution $m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ to the genus g is < 0, then $g_i = 0$ and $a_{ii} = -w_i$.*

Proof. Follows immediately from Lemma 3.6 and $w_i > 0$, $g_i \geq 0$, and $w_i | a_{ii}$. $\square$

Definition 3.8. Let n, m_i, a_{ij}, w_i, g_i be a numerical type. We say i is a (-1)-*index* if $g_i = 0$ and $a_{ii} = -w_i$.

We can "contract" (-1)-indices.

Lemma 3.9. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. Assume n is a (-1)-index. Then there is a numerical type T' given by $n', m'_i, a'_{ij}, w'_i, g'_i$ with*

(1) $n' = n - 1$,
(2) $m'_i = m_i$,
(3) $a'_{ij} = a_{ij} - a_{in}a_{jn}/a_{nn}$,
(4) $w'_i = w_i/2$ *if* a_{in}/w_n *even and* a_{in}/w_i *odd and* $w'_i = w_i$ *else,*
(5) $g'_i = \frac{w_i}{w'_i}(g_i - 1) + 1 + \frac{a_{in}^2 - w_n a_{in}}{2w'_i w_n}$.

Moreover, we have $g = g'$.

Proof. Observe that $n > 1$ for example by Lemma 3.5 and hence $n' \geq 1$. We check conditions (1) – (5) of Definition 3.1 for $n', m'_i, a'_{ij}, w'_i, g'_i$.

Condition (1) is immediate.

Condition (2). Symmetry of $A' = (a'_{ij})$ is immediate and since $a_{nn} < 0$ by Lemma 3.6 we see that $a'_{ij} \geq a_{ij} \geq 0$ if $i \neq j$.

Condition (3). Suppose that $I \subset \{1, \ldots, n-1\}$ such that $a'_{ii'} = 0$ for $i \in I$ and $i' \in \{1, \ldots, n-1\} \setminus I$. Then we see that for each $i \in I$ and $i' \in I'$ we have $a_{in}a_{i'n} = 0$. Thus either $a_{in} = 0$ for all $i \in I$ and $I \subset \{1, \ldots, n\}$ is a contradiction for property (3) for T, or $a_{i'n} = 0$ for all $i' \in \{1, \ldots, n-1\} \setminus I$ and $I \cup \{n\} \subset \{1, \ldots, n\}$ is a contradiction for property (3) of T. Hence (3) holds for T'.

Condition (4). We compute

$$\sum\nolimits_{j=1}^{n-1} a'_{ij}m_j = \sum\nolimits_{j=1}^{n-1}(a_{ij}m_j - \frac{a_{in}a_{jn}m_j}{a_{nn}}) = -a_{in}m_n - \frac{a_{in}}{a_{nn}}(-a_{nn}m_n) = 0$$

as desired.

Condition (5). We have to show that w'_i divides $a_{in}a_{jn}/a_{nn}$. This is clear because $a_{nn} = -w_n$ and $w_n | a_{jn}$ and $w_i | a_{in}$.

To show that $g = g'$ we first write

$$\begin{aligned} g &= 1 + \sum\nolimits_{i=1}^{n} m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii}) \\ &= 1 + \sum\nolimits_{i=1}^{n-1} m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii}) - \frac{1}{2}m_n w_n \\ &= 1 + \sum\nolimits_{i=1}^{n-1} m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii} - \frac{1}{2}a_{in}) \end{aligned}$$

Comparing with the expression for g' we see that it suffices if

$$w'_i(g'_i - 1) - \frac{1}{2}a'_{ii} = w_i(g_i - 1) - \frac{1}{2}a_{in} - \frac{1}{2}a_{ii}$$

for $i \leq n-1$. In other words, we have

$$g_i' = \frac{2w_i(g_i - 1) - a_{in} - a_{ii} + a_{ii}' + 2w_i'}{2w_i'} = \frac{w_i}{w_i'}(g_i - 1) + 1 + \frac{a_{in}^2 - w_n a_{in}}{2w_i' w_n}$$

It is elementary to check that this is an integer ≥ 0 if we choose w_i' as in (4). □

Lemma 3.10. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type. Let e be the number of pairs (i,j) with $i < j$ and $a_{ij} > 0$. Then the expression $g_{top} = 1 - n + e$ is ≥ 0.*

Proof. If not, then $e < n - 1$ which means there exists an i such that $a_{ij} = 0$ for all $j \neq i$. This contradicts assumption (3) of Definition 3.1. □

Definition 3.11. Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. The *topological genus of T* is the nonnegative integer $g_{top} = 1 - n + e$ from Lemma 3.10.

We want to bound the genus by the topological genus. However, this will not always be the case, for example for numerical types with $n = 1$ as in Lemma 3.5. But it will be true for minimal numerical types which are defined as follows.

Definition 3.12. We say the numerical type n, m_i, a_{ij}, w_i, g_i of genus g is *minimal* if there does not exist an i with $g_i = 0$ and $a_{ii} = -w_i$, in other words, if there does not exist a (-1)-index.

We will prove that the genus g of a minimal type with $n > 1$ is greater than or equal to $\max(1, g_{top})$.

Lemma 3.13. *If n, m_i, a_{ij}, w_i, g_i is a minimal numerical type with $n > 1$, then $g \geq 1$.*

Proof. This is true because $g = 1 + \sum \Phi_i$ with $\Phi_i = m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ nonnegative by Lemma 3.7 and the definition of minimal types. □

Lemma 3.14. *If n, m_i, a_{ij}, w_i, g_i is a minimal numerical type with $n > 1$, then $g \geq g_{top}$.*

Proof. The reader who is only interested in the case of numerical types associated to proper regular models can skip this proof as we will reprove this in the geometric situation later. We can write

$$g_{top} = 1 - n + \frac{1}{2}\sum\nolimits_{a_{ij}>0} 1 = 1 + \sum\nolimits_i (-1 + \frac{1}{2}\sum\nolimits_{j \neq i,\ a_{ij}>0} 1)$$

On the other hand, we have

$$\begin{aligned} g &= 1 + \sum m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii}) \\ &= 1 + \sum m_i w_i g_i - \sum m_i w_i + \frac{1}{2}\sum\nolimits_{i \neq j} a_{ij} m_j \\ &= 1 + \sum\nolimits_i m_i w_i(-1 + g_i + \frac{1}{2}\sum\nolimits_{j \neq i} \frac{a_{ij}}{w_i}) \end{aligned}$$

The first equality is the definition, the second equality uses that $\sum a_{ij} m_j = 0$, and the last equality uses that uses $a_{ij} = a_{ji}$ and switching order of summation. Comparing with the formula for g_{top} we conclude that the lemma holds if

$$\Psi_i = m_i w_i(-1 + g_i + \frac{1}{2}\sum\nolimits_{j \neq i} \frac{a_{ij}}{w_i}) - (-1 + \frac{1}{2}\sum\nolimits_{j \neq i,\ a_{ij}>0} 1)$$

is ≥ 0 for each i. However, this may not be the case. Let us analyze for which indices we can have $\Psi_i < 0$. First, observe that

$$(-1 + g_i + \frac{1}{2}\sum\nolimits_{j \neq i} \frac{a_{ij}}{w_i}) \geq (-1 + \frac{1}{2}\sum\nolimits_{j \neq i,\ a_{ij} > 0} 1)$$

because a_{ij}/w_i is a nonnegative integer. Since $m_i w_i$ is a positive integer we conclude that $\Psi_i \geq 0$ as soon as either $m_i w_i = 1$ or the left hand side of the inequality is ≥ 0 which happens if $g_i > 0$, or $a_{ij} > 0$ for at least two indices j, or if there is a j with $a_{ij} > w_i$. Thus

$$P = \{i : \Psi_i < 0\}$$

is the set of indices i such that $m_i w_i > 1$, $g_i = 0$, $a_{ij} > 0$ for a unique j, and $a_{ij} = w_i$ for this j. Moreover

$$i \in P \Rightarrow \Psi_i = \frac{1}{2}(-m_i w_i + 1)$$

The strategy of proof is to show that given $i \in P$ we can borrow a bit from Ψ_j where j is the neighbour of i, i.e., $a_{ij} > 0$. However, this won't quite work because j may be an index with $\Psi_j = 0$.

Consider the set

$$Z = \{j : g_j = 0 \text{ and } j \text{ has exactly two neighbours } i, k \text{ with } a_{ij} = w_j = a_{jk}\}$$

For $j \in Z$ we have $\Psi_j = 0$. We will consider sequences $M = (i, j_1, \ldots, j_s)$ where $s \geq 0$, $i \in P$, $j_1, \ldots, j_s \in Z$, and $a_{ij_1} > 0, a_{j_1 j_2} > 0, \ldots, a_{j_{s-1} j_s} > 0$. If our numerical type consists of two indices which are in P or more generally if our numerical type consists of two indices which are in P and all other indices in Z, then $g_{top} = 0$ and we win by Lemma 3.13. We may and do discard these cases.

Let $M = (i, j_1, \ldots, j_s)$ be a maximal sequence and let k be the second neighbour of j_s. (If $s = 0$, then k is the unique neighbour of i.) By maximality $k \notin Z$ and by what we just said $k \notin P$. Observe that $w_i = a_{ij_1} = w_{j_1} = a_{j_1 j_2} = \ldots = w_{j_s} = a_{j_s k}$. Looking at the definition of a numerical type we see that

$$\begin{aligned} m_i a_{ii} + m_{j_1} w_i &= 0, \\ m_i w_i + m_{j_1} a_{j_1 j_1} + m_{j_2} w_i &= 0, \\ &\ldots\ldots \\ m_{j_{s-1}} w_i + m_{j_s} a_{j_s j_s} + m_k w_i &= 0 \end{aligned}$$

The first equality implies $m_{j_1} \geq 2m_i$ because the numerical type is minimal. Then the second equality implies $m_{j_2} \geq 3m_i$, and so on. In any case, we conclude that $m_k \geq 2m_i$ (including when $s = 0$).

Let k be an index such that we have a $t > 0$ and pairwise distinct maximal sequences $M_1, \ldots, M_t$ as above, with $M_b = (i_b, j_{b,1}, \ldots, j_{b,s_b})$ such that k is a neighbour of j_{b,s_b} for $b = 1, \ldots, t$. We will show that $\Phi_j + \sum_{b=1,\ldots,t} \Phi_{i_b} \geq 0$. This will finish the proof of the lemma by what we said above. Let M be the union of the indices occurring in M_b, $b = 1, \ldots, t$. We write

$$\Psi_k = -\sum\nolimits_{b=1,\ldots,t} \Psi_{i_b} + \Psi'_k$$

where

$$\Psi'_k = m_k w_k \left(-1 + g_k + \frac{1}{2}\sum\nolimits_{b=1,\ldots t}(\frac{a_{kj_{b,s_b}}}{w_k} - \frac{m_{i_b}w_{i_b}}{m_k w_k}) + \frac{1}{2}\sum\nolimits_{l\neq k,\ l\notin M}\frac{a_{kl}}{w_k}\right) - \left(-1 + \frac{1}{2}\sum\nolimits_{l\neq k,\ l\notin M,\ a_{kl}>0} 1\right)$$

Assume $\Psi'_k < 0$ to get a contradiction. If the set $\{l : l \neq k,\ l \notin M,\ a_{kl} > 0\}$ is empty, then $\{1, \ldots, n\} = M \cup \{k\}$ and $g_{top} = 0$ because $e = n - 1$ in this case and the result holds by Lemma 3.13. Thus we may assume there is at least one such l which contributes $(1/2)a_{kl}/w_k \geq 1/2$ to the sum inside the first brackets. For each $b = 1, \ldots, t$ we have

$$\frac{a_{kj_{b,s_b}}}{w_k} - \frac{m_{i_b}w_{i_b}}{m_k w_k} = \frac{w_{i_b}}{w_k}(1 - \frac{m_{i_b}}{m_k})$$

This expression is $\geq \frac{1}{2}$ because $m_k \geq 2m_{i_b}$ by the previous paragraph and is ≥ 1 if $w_k < w_{i_b}$. It follows that $\Psi'_k < 0$ implies $g_k = 0$. If $t \geq 2$ or $t = 1$ and $w_k < w_{i_1}$, then $\Psi'_k \geq 0$ (here we use the existence of an l as shown above) which is a contradiction too. Thus $t = 1$ and $w_k = w_{i_1}$. If there at least two nonzero terms in the sum over l or if there is one such k and $a_{kl} > w_k$, then $\Psi'_k \geq 0$ as well. The final possibility is that $t = 1$ and there is one l with $a_{kl} = w_k$. This is disallowed as this would mean $k \in Z$ contradicting the maximality of M_1. □

Lemma 3.15. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Assume $n > 1$. If i is such that the contribution $m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$ to the genus g is 0, then $g_i = 0$ and $a_{ii} = -2w_i$.*

Proof. Follows immediately from Lemma 3.6 and $w_i > 0$, $g_i \geq 0$, and $w_i | a_{ii}$. □

It turns out that the indices satisfying this relation play an important role in the structure of minimal numerical types. Hence we give them a name.

Definition 3.16. Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. We say i is a (-2)-*index* if $g_i = 0$ and $a_{ii} = -2w_i$.

Given a minimal numerical type of genus g the (-2)-indices are exactly the indices which do not contribute a positive number to the genus in the formula

$$g = 1 + \sum m_i(w_i(g_i - 1) - \frac{1}{2}a_{ii})$$

Thus it will be somewhat tricky to bound the quantities associated with (-2)-indices as we will see later.

Remark 3.17. Let n, m_i, a_{ij}, w_i, g_i be a minimal numerical type with $n > 1$. Equality $g = g_{top}$ can hold in Lemma 3.14. For example, if $m_i = w_i = 1$ and $g_i = 0$ for all i and $a_{ij} \in \{0, 1\}$ for $i < j$.

4. The Picard group of a numerical type

Here is the definition.

Definition 4.1. Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. The *Picard group of* T is the cokernel of the matrix (a_{ij}/w_i), more precisely

$$\mathrm{Pic}(T) = \mathrm{Coker}\left(\mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n}, \quad e_i \mapsto \sum \frac{a_{ij}}{w_j}e_j\right)$$

where e_i denotes the ith standard basis vector for $\mathbf{Z}^{\oplus n}$.

Lemma 4.2. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. The Picard group of T is a finitely generated abelian group of rank* 1.

Proof. If $n = 1$, then $A = (a_{ij})$ is the zero matrix and the result is clear. For $n > 1$ the matrix A has rank $n - 1$ by either Lemma 2.2 or Lemma 2.3. Of course the rank is not affected by scaling the rows by $1/w_i$. This proves the lemma. □

Lemma 4.3. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. Then* $\mathrm{Pic}(T) \subset \mathrm{Coker}(A)$ *where $A = (a_{ij})$.*

Proof. Since $\mathrm{Pic}(T)$ is the cokernel of (a_{ij}/w_i) we see that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}^{\oplus n} \arrow[r, "A"'] & \mathbf{Z}^{\oplus n} \arrow[r] & \mathrm{Coker}(A) \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{Z}^{\oplus n} \arrow[r, "(a_{ij}/w_i)"] \arrow[u, "\mathrm{id}"] & \mathbf{Z}^{\oplus n} \arrow[r] \arrow[u, "\mathrm{diag}(w_1,\ldots,w_n)"'] & \mathrm{Pic}(T) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with exact rows. By the snake lemma we conclude that $\mathrm{Pic}(T) \subset \mathrm{Coker}(A)$. □

Lemma 4.4. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. Assume n is a (-1)-index. Let T' be the numerical type constructed in Lemma 3.9. There exists an injective map*

$$\mathrm{Pic}(T) \to \mathrm{Pic}(T')$$

whose cokernel is an elementary abelian 2*-group.*

Proof. Recall that $n' = n - 1$. Let e_i, resp., e'_i be the ith basis vector of $\mathbf{Z}^{\oplus n}$, resp. $\mathbf{Z}^{\oplus n-1}$. First we denote

$$q : \mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n-1}, \quad e_n \mapsto 0 \text{ and } e_i \mapsto e'_i \text{ for } i \leq n-1$$

and we set

$$p : \mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n-1}, \quad e_n \mapsto \sum\nolimits_{j=1}^{n-1} \frac{a_{nj}}{w'_j} e'_j \text{ and } e_i \mapsto \frac{w_i}{w'_i} e'_i \text{ for } i \leq n-1$$

A computation (which we omit) shows there is a commutative diagram

$$\begin{tikzcd}
\mathbf{Z}^{\oplus n} \arrow[r, "(a_{ij}/w_i)"'] \arrow[d, "q"'] & \mathbf{Z}^{\oplus n} \arrow[d, "p"] \\
\mathbf{Z}^{\oplus n'} \arrow[r, "(a'_{ij}/w'_i)"] & \mathbf{Z}^{\oplus n'}
\end{tikzcd}$$

Since the cokernel of the top arrow is $\mathrm{Pic}(T)$ and the cokernel of the bottom arrow is $\mathrm{Pic}(T')$, we obtain the desired homomorphism of Picard groups. Since $\frac{w_i}{w'_i} \in \{1, 2\}$ we see that the cokernel of $\mathrm{Pic}(T) \to \mathrm{Pic}(T')$ is annihilated by 2 (because $2e'_i$ is in the image of p for all $i \leq n - 1$). Finally, we show $\mathrm{Pic}(T) \to \mathrm{Pic}(T')$ is injective. Let $L = (l_1, \ldots, l_n)$ be a representative of an element of $\mathrm{Pic}(T)$ mapping to zero in $\mathrm{Pic}(T')$. Since q is surjective, a diagram chase shows that we can assume L is in the kernel of p. This means that $l_n a_{ni}/w'_i + l_i w_i/w'_i = 0$, i.e., $l_i = -a_{ni}/w_i l_n$. Thus L is the image of $-l_n e_n$ under the map (a_{ij}/w_j) and the lemma is proved. □

Lemma 4.5. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type T. If the genus g of T is ≤ 0, then* $\mathrm{Pic}(T) = \mathbf{Z}$.

Proof. By induction on n. If $n = 1$, then the assertion is clear. If $n > 1$, then T is not minimal by Lemma 3.13. After replacing T by an equivalent type we may assume n is a (-1)-index. By Lemma 4.4 we find $\text{Pic}(T) \subset \text{Pic}(T')$. By Lemma 3.9 we see that the genus of T' is equal to the genus of T and we conclude by induction. $\square$

5. Classification of proper subgraphs

In this section we assume given a numerical type n, m_i, a_{ij}, w_i, g_i of genus g. We will find a complete list of possible "subgraphs" consisting entirely of (-2)-indices (Definition 3.16) and at the same time we classify all possible minimal numerical types of genus 1. In other words, in this section we prove Proposition 5.17 and Lemma 6.2

Our strategy will be as follows. Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Let $I \subset \{1, \ldots, n\}$ be a subset consisting of (-2)-indices such that there does not exist a nonempty proper subset $J \subset I$ with $a_{jj'} = 0$ for $j \in J$, $j' \in I \setminus J$. We work by induction on the cardinality $|I|$ of I. If $I = \{i\}$ consists of 1 index, then the only constraints on m_i, a_{ii}, and w_i are $w_i | a_{ii}$ from Definition 3.1 and $a_{ii} < 0$ from Lemma 3.6 and this will serve as our base case. In the induction step we first apply the induction hypothesis to subsets $I' \subset I$ of size $|I'| < |I|$. This will put some constraints on the possible m_i, a_{ij}, w_i, $i, j \in I$. In particular, since $|I'| < |I| \leq n$ it will follow from $\sum a_{ij} m_j = 0$ and Lemma 2.3 that the sub matrices $(a_{ij})_{i,j \in I'}$ are negative definite and their determinant will have sign $(-1)^m$. For each possibility left over we compute the determinant of $(a_{ij})_{i,j \in I}$. If the determinant has sign $-(-1)^{|I|}$ then this case can be discarded because Sylvester's theorem tells us the matrix $(a_{ij})_{i,j \in I}$ is not negative semi-definite. If the determinant has sign $(-1)^{|I|}$, then $|I| < n$ and we (tentatively) conclude this case can occur as a possible proper subgraph and we list it in one of the lemmas in this section. If the determinant is 0, then we must have $|I| = n$ (by Lemma 2.3 again) and $g = 0$. In these cases we actually find all possible m_i, a_{ij}, w_i, $i, j \in I$ and list them in Lemma 6.2. After completing the argument we obtain all possible minimal numerical types of genus 1 with $n > 1$ because each of these necessarily consists entirely of (-2)-indices (and hence will show up in the induction process) by the formula for the genus and the remarks in the previous section. At the very end of the day the reader can go through the list of possibilities given in Lemma 6.2 to see that all configurations of proper subgraphs listed in this section as possible do in fact occur already for numerical types of genus 1.

Suppose that i and j are (-2)-indices with $a_{ij} > 0$. Since the matrix $A = (a_{ij})$ is semi-negative definite by Lemma 2.3 we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} \\ a_{ij} & -2w_j \end{pmatrix}$$

is negative definite unless $n = 2$. The case $n = 2$ can happen: then the determinant $4w_1w_2 - a_{12}^2$ is zero. Using that $\text{lcm}(w_1, w_2)$ divides a_{12} the reader easily finds that the only possibilities are

$$(w_1, w_2, a_{12}) = (w, w, 2w), (w, 4w, 4w), \text{ or } (4w, w, 4w)$$

Observe that the case $(4w, w, 4w)$ is obtained from the case $(w, 4w, 4w)$ by switching the indices i, j. In these cases $g = 1$. This leads to cases (2) and (3) of Lemma

6.2. Assuming $n > 2$ we see that the determinant $4w_iw_j - a_{ij}^2$ of the displayed matrix is > 0 and we conclude that $a_{ij}^2/w_iw_j < 4$. On the other hand, we know that $\mathrm{lcm}(w_i, w_j)|a_{ij}$ and hence a_{ij}^2/w_iw_j is an integer. Thus $a_{ij}^2/w_iw_j \in \{1, 2, 3\}$ and $w_i|w_j$ or vice versa. This leads to the following possibilities

$$(w_1, w_2, a_{12}) = (w, w, w), (w, 2w, 2w), (w, 3w, 3w), (2w, w, 2w), \text{ or } (3w, w, 3w)$$

Observe that the case $(2w, w, 2w)$ is obtained from the case $(w, 2w, 2w)$ by switching the indices i, j and similarly for the cases $(3w, w, 3w)$ and $(w, 3w, 3w)$. The first three solutions lead to cases (1), (2), and (3) of Lemma 5.1. In this lemma we wrote out the consequences for the integers m_i and m_j using that $\sum_l a_{kl}m_l = 0$ for each k in particular implies $a_{ii}m_i + a_{ij}m_j \leq 0$ for $k = i$ and $a_{ij}m_i + a_{jj}m_j \leq 0$ for $k = j$.

Lemma 5.1. *Classification of proper subgraphs of the form*

$$\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$$

If $n > 2$, then given a pair i, j of (-2)-indices with $a_{ij} > 0$, then up to ordering we have the m's, a's, w's

(1) *are given by*
$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \begin{pmatrix} -2w & w \\ w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \end{pmatrix}$$
with w arbitrary and $2m_1 \geq m_2$ and $2m_2 \geq m_1$, or

(2) *are given by*
$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w \\ 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \end{pmatrix}$$
with w arbitrary and $m_1 \geq m_2$ and $2m_2 \geq m_1$, or

(3) *are given by*
$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \begin{pmatrix} -2w & 3w \\ 3w & -6w \end{pmatrix}, \quad \begin{pmatrix} w \\ 3w \end{pmatrix}$$
with w arbitrary and $2m_1 \geq 3m_2$ and $2m_2 \geq m_1$.

Proof. See discussion above. □

Suppose that i, j, and k are three (-2)-indices with $a_{ij} > 0$ and $a_{jk} > 0$. In other words, the index i "meets" j and j "meets" k. We will use without further mention that each pair (i, j), (i, k), and (j, k) is as listed in Lemma 5.1. Since the matrix $A = (a_{ij})$ is semi-negative definite by Lemma 2.3 we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} & a_{ik} \\ a_{ij} & -2w_j & a_{jk} \\ a_{ik} & a_{jk} & -2w_k \end{pmatrix}$$

is negative definite unless $n = 3$. The case $n = 3$ can happen: then the determinant[2] of the matrix is zero and we obtain the equation

$$4 = \frac{a_{ij}^2}{w_iw_j} + \frac{a_{jk}^2}{w_jw_k} + \frac{a_{ik}^2}{w_iw_k} + \frac{a_{ij}a_{ik}a_{jk}}{w_iw_jw_k}$$

[2] It is $-8w_iw_jw_k + 2a_{ij}^2w_k + 2a_{jk}^2w_i + 2a_{ik}^2w_j + 2a_{ij}a_{jk}a_{ik}$.

of integers. The last term on the right in this equation is determined by the others because

$$\left(\frac{a_{ij}a_{ik}a_{jk}}{w_iw_jw_k}\right)^2 = \frac{a_{ij}^2}{w_iw_j}\frac{a_{jk}^2}{w_jw_k}\frac{a_{ik}^2}{w_iw_k}$$

Since we have seen above that $\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}$ are in $\{1,2,3\}$ and $\frac{a_{ik}^2}{w_iw_k}$ in $\{0,1,2,3\}$, we conclude that the only possibilities are

$$\left(\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{ik}^2}{w_iw_k}\right) = (1,1,1), (1,3,0), (2,2,0), \text{ or } (3,1,0)$$

Observe that the case $(3,1,0)$ is obtained from the case $(1,3,0)$ by reversing the order the indices i,j,k. In each of these cases $g=1$; the reader can find these as cases (4), (5), (6), (7), (8), (9) of Lemma 6.2 with one case corresponding to $(1,1,1)$, two cases corresponding to $(1,3,0)$, and three cases corresponding to $(2,2,0)$. Assuming $n>3$ we obtain the inequality

$$4 > \frac{a_{ij}^2}{w_iw_j} + \frac{a_{ik}^2}{w_iw_k} + \frac{a_{jk}^2}{w_jw_k} + \frac{a_{ij}a_{ik}a_{jk}}{w_iw_jw_k}$$

of integers. Using the restrictions on the numbers given above we see that the only possibilities are

$$\left(\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{ik}^2}{w_iw_k}\right) = (1,1,0), (1,2,0), \text{ or } (2,1,0)$$

in particular $a_{ik}=0$ (recall we are assuming $a_{ij}>0$ and $a_{jk}>0$). Observe that the case $(2,1,0)$ is obtained from the case $(1,2,0)$ by reversing the ordering of the indices i,j,k. The first two solutions lead to cases (1), (2), and (3) of Lemma 5.2 where we also wrote out the consequences for the integers m_i, m_j, and m_k.

Lemma 5.2. *Classification of proper subgraphs of the form*

$$\bullet \text{———} \bullet \text{———} \bullet$$

If $n>3$, then given a triple i,j,k of (-2)-indices with at least two a_{ij}, a_{ik}, a_{jk} nonzero, then up to ordering we have the m's, a's, w's

(1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 \\ w & -2w & w \\ 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2$, or

(2) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 \\ w & -2w & 2w \\ 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ 2w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + 2m_3$, $2m_3 \geq m_2$, or

(3) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 \\ 2w & -4w & 2w \\ 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $m_3 \geq m_2$.

Proof. See discussion above. $\square$

Suppose that i, j, k, and l are four (-2)-indices with $a_{ij} > 0$, $a_{jk} > 0$, and $a_{kl} > 0$. In other words, the index i "meets" j, j "meets" k, and k "meets" l. Then we see from Lemma 5.2 that $a_{ik} = a_{jl} = 0$. Since the matrix $A = (a_{ij})$ is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} & 0 & a_{il} \\ a_{ij} & -2w_j & a_{jk} & 0 \\ 0 & a_{jk} & -2w_k & a_{kl} \\ a_{il} & 0 & a_{kl} & -2w_l \end{pmatrix}$$

is negative definite unless $n = 4$. The case $n = 4$ can happen: then the determinant[3] of the matrix is zero and we obtain the equation

$$16 + \frac{a_{ij}^2}{w_iw_j}\frac{a_{kl}^2}{w_kw_l} + \frac{a_{jk}^2}{w_jw_k}\frac{a_{il}^2}{w_iw_l} = 4\frac{a_{ij}^2}{w_iw_j} + 4\frac{a_{jk}^2}{w_jw_k} + 4\frac{a_{kl}^2}{w_kw_l} + 4\frac{a_{il}^2}{w_iw_l} + 2\frac{a_{ij}a_{il}a_{jk}a_{kl}}{w_iw_jw_kw_l}$$

of nonnegative integers. The last term on the right in this equation is determined by the others because

$$\left(\frac{a_{ij}a_{il}a_{jk}a_{kl}}{w_iw_jw_kw_l}\right)^2 = \frac{a_{ij}^2}{w_iw_j}\frac{a_{jk}^2}{w_jw_k}\frac{a_{kl}^2}{w_kw_l}\frac{a_{il}^2}{w_iw_l}$$

Since we have seen above that $\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{kl}^2}{w_kw_l}$ are in $\{1, 2\}$ and $\frac{a_{il}^2}{w_iw_l}$ in $\{0, 1, 2\}$, we conclude that the only possible solutions are

$$(\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{kl}^2}{w_kw_l}, \frac{a_{il}^2}{w_iw_l}) = (1, 1, 1, 1) \text{ or } (2, 1, 2, 0)$$

and case $g = 1$; the reader can find these as cases (10), (11), (12), and (13) of Lemma 6.2. Assuming $n > 4$ we obtain the inequality

$$16 + \frac{a_{ij}^2}{w_iw_j}\frac{a_{kl}^2}{w_kw_l} + \frac{a_{jk}^2}{w_jw_k}\frac{a_{il}^2}{w_iw_l} > 4\frac{a_{ij}^2}{w_iw_j} + 4\frac{a_{jk}^2}{w_jw_k} + 4\frac{a_{kl}^2}{w_kw_l} + 4\frac{a_{il}^2}{w_iw_l} + 2\frac{a_{ij}a_{il}a_{jk}a_{kl}}{w_iw_jw_kw_l}$$

of nonnegative integers. Using the restrictions on the numbers given above we see that the only possibilities are

$$(\frac{a_{ij}^2}{w_iw_j}, \frac{a_{jk}^2}{w_jw_k}, \frac{a_{kl}^2}{w_kw_l}, \frac{a_{il}^2}{w_iw_l}) = (1, 1, 1, 0), (1, 1, 2, 0), (1, 2, 1, 0), \text{ or } (2, 1, 1, 0)$$

in particular $a_{il} = 0$ (recall that we assumed the other three to be nonzero). Observe that the case $(2, 1, 1, 0)$ is obtained from the case $(1, 1, 2, 0)$ by reversing the ordering of the indices i, j, k, l. The first three solutions lead to cases (1), (2), (3), and (4) of Lemma 5.3 where we also wrote out the consequences for the integers m_i, m_j, m_k, and m_l.

Lemma 5.3. *Classification of proper subgraphs of the form*

$$\bullet - \bullet - \bullet - \bullet$$

If $n > 4$, then given four (-2)-indices i, j, k, l with a_{ij}, a_{jk}, a_{kl} nonzero, then up to ordering we have the m's, a's, w's

[3] It is $16w_iw_jw_kw_l - 4a_{ij}^2w_kw_l - 4a_{jk}^2w_iw_l - 4a_{kl}^2w_iw_j - 4a_{il}^2w_jw_k + a_{ij}^2a_{kl}^2 + a_{jk}^2a_{il}^2 - 2a_{ij}a_{il}a_{jk}a_{kl}$.

(1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 \\ w & -2w & w & 0 \\ 0 & w & -2w & w \\ 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, *and* $2m_4 \geq m_3$, *or*

(2) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 \\ w & -2w & w & 0 \\ 0 & w & -2w & 2w \\ 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ 2w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + 2m_4$, *and* $2m_4 \geq m_3$, *or*

(3) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 \\ 2w & -4w & 2w & 0 \\ 0 & 2w & -4w & 2w \\ 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, *and* $m_4 \geq m_3$, *or*

(4) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 \\ w & -2w & 2w & 0 \\ 0 & 2w & -4w & 2w \\ 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ 2w \\ 2w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + 2m_3$, $2m_3 \geq m_2 + m_4$, *and* $2m_4 \geq m_3$.

Proof. See discussion above. □

Suppose that i, j, k, and l are four (-2)-indices with $a_{ij} > 0$, $a_{ij} > 0$, and $a_{il} > 0$. In other words, the index i "meets" the indices j, k, l. Then we see from Lemma 5.2 that $a_{jk} = a_{jl} = a_{kl} = 0$. Since the matrix $A = (a_{ij})$ is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_i & a_{ij} & a_{ik} & a_{il} \\ a_{ij} & -2w_j & 0 & 0 \\ a_{ik} & 0 & -2w_k & 0 \\ a_{il} & 0 & 0 & -2w_l \end{pmatrix}$$

is negative definite unless $n = 4$. The case $n = 4$ can happen: then the determinant[4] of the matrix is zero and we obtain the equation

$$4 = \frac{a_{ij}^2}{w_i w_j} + \frac{a_{ik}^2}{w_i w_k} + \frac{a_{il}^2}{w_j w_l}$$

of nonnegative integers. Since we have seen above that $\frac{a_{ij}^2}{w_i w_j}, \frac{a_{ik}^2}{w_i w_k}, \frac{a_{il}^2}{w_i w_l}$ are in $\{1, 2\}$, we conclude that the only possibilities are up to reordering: $4 = 1 + 1 + 2$.

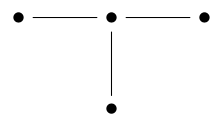
[4]It is $16 w_i w_j w_k w_l - 4a_{ij}^2 w_k w_l - 4a_{ik}^2 w_j w_l - 4a_{il}^2 w_j w_k$.

In each of these cases $g = 1$; the reader can find these as cases (14) and (15) of Lemma 6.2. Assuming $n > 4$ we obtain the inequality

$$4 > \frac{a_{ij}^2}{w_i w_j} + \frac{a_{ik}^2}{w_i w_k} + \frac{a_{il}^2}{w_j w_l}$$

of nonnegative integers. This implies that $\frac{a_{ij}^2}{w_i w_j} = \frac{a_{ik}^2}{w_i w_k} = \frac{a_{il}^2}{w_j w_l} = 1$ and that $w_i = w_j = w_k = w_l$. This leads to case (1) of Lemma 5.4 where we also wrote out the consequences for the integers m_i, m_j, m_k, and m_l.

Lemma 5.4. *Classification of proper subgraphs of the form*



If $n > 4$, then given four (-2)-indices i, j, k, l with a_{ij}, a_{ik}, a_{il} nonzero, then up to ordering we have the m's, a's, w's

(1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w & w \\ w & -2w & 0 & 0 \\ w & 0 & -2w & 0 \\ w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2 + m_3 + m_4$, $2m_2 \geq m_1$, $2m_3 \geq m_1$, $2m_4 \geq m_1$. Observe that this implies $m_1 \geq \max(m_2, m_3, m_4)$.

Proof. See discussion above. □

Suppose that h, i, j, k, and l are five (-2)-indices with $a_{hi} > 0$, $a_{ij} > 0$, $a_{jk} > 0$, and $a_{kl} > 0$. In other words, the index h "meets" i, i "meets" j, j "meets" k, and k "meets" l. Then we can apply Lemmas 5.2 and 5.3 to see that $a_{hj} = a_{hk} = a_{ik} = a_{il} = a_{jl} = 0$ and that the fractions $\frac{a_{hi}^2}{w_h w_i}, \frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}, \frac{a_{kl}^2}{w_k w_l}$ are in $\{1, 2\}$ and the fraction $\frac{a_{hl}^2}{w_h w_l} \in \{0, 1, 2\}$. Since the matrix $A = (a_{ij})$ is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_h & a_{hi} & 0 & 0 & a_{hl} \\ a_{hi} & -2w_i & a_{ij} & 0 & 0 \\ 0 & a_{ij} & -2w_j & a_{jk} & 0 \\ 0 & 0 & a_{jk} & -2w_k & a_{kl} \\ a_{hl} & 0 & 0 & a_{kl} & -2w_l \end{pmatrix}$$

is negative definite unless $n = 5$. The case $n = 5$ can happen: then the determinant[5] of the matrix is zero and we obtain the equation

$$\begin{aligned} 16 + \frac{a_{hi}^2}{w_h w_i}\frac{a_{jk}^2}{w_j w_k} + \frac{a_{hi}^2}{w_h w_i}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{ij}^2}{w_i w_j}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{ij}^2}{w_i w_j} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{jk}^2}{w_j w_k} \\ = 4\frac{a_{hi}^2}{w_h w_i} + 4\frac{a_{ij}^2}{w_i w_j} + 4\frac{a_{jk}^2}{w_j w_k} + 4\frac{a_{kl}^2}{w_k w_l} + 4\frac{a_{hl}^2}{w_h w_l} + \frac{a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}}{w_h w_i w_j w_k w_l} \end{aligned}$$

[5]It is $-32 w_h w_i w_j w_k w_l + 8a_{hi}^2 w_j w_k w_l + 8a_{ij}^2 w_h w_k w_l + 8a_{jk}^2 w_h w_i w_l + 8a_{kl}^2 w_h w_i w_j + 8a_{hl}^2 w_i w_j w_k - 2a_{hi}^2 a_{jk}^2 w_l - 2a_{hi}^2 a_{kl}^2 w_j - 2a_{ij}^2 a_{kl}^2 w_h - 2a_{hl}^2 a_{ij}^2 w_k - 2a_{hl}^2 a_{jk}^2 w_i + 2a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}$.

of nonnegative integers. The last term on the right in this equation is determined by the others because

$$\left(\frac{a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}}{w_h w_i w_j w_k w_l}\right)^2 = \frac{a_{hi}^2}{w_h w_i}\frac{a_{ij}^2}{w_i w_j}\frac{a_{jk}^2}{w_j w_k}\frac{a_{kl}^2}{w_k w_l}\frac{a_{hl}^2}{w_h w_l}$$

We conclude the only possible solutions are

$$(\frac{a_{hi}^2}{w_h w_i}, \frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}, \frac{a_{kl}^2}{w_k w_l}, \frac{a_{hl}^2}{w_h w_l}) = (1,1,1,1,1), (1,1,2,1,0), (1,2,1,1,0), \text{ or } (2,1,1,2,0)$$

Observe that the case $(1,2,1,1,0)$ is obtained from the case $(1,1,2,1,0)$ by reversing the order of the indices h,i,j,k,l. In these cases $g = 1$; the reader can find these as cases (16), (17), (18), (19), (20), and (21) of Lemma 6.2 with one case corresponding to $(1,1,1,1,1)$, two cases corresponding to $(1,1,2,1,0)$, and three cases corresponding to $(2,1,1,2,0)$. Assuming $n > 5$ we obtain the inequality

$$\begin{aligned} 16 &+ \frac{a_{hi}^2}{w_h w_i}\frac{a_{jk}^2}{w_j w_k} + \frac{a_{hi}^2}{w_h w_i}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{ij}^2}{w_i w_j}\frac{a_{kl}^2}{w_k w_l} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{ij}^2}{w_i w_j} + \frac{a_{hl}^2}{w_h w_l}\frac{a_{jk}^2}{w_j w_k} \\ &> 4\frac{a_{hi}^2}{w_h w_i} + 4\frac{a_{ij}^2}{w_i w_j} + 4\frac{a_{jk}^2}{w_j w_k} + 4\frac{a_{kl}^2}{w_k w_l} + 4\frac{a_{hl}^2}{w_h w_l} + \frac{a_{hi}a_{ij}a_{jk}a_{kl}a_{hl}}{w_h w_i w_j w_k w_l} \end{aligned}$$

of nonnegative integers. Using the restrictions on the numbers given above we see that the only possibilities are

$$(\frac{a_{hi}^2}{w_h w_i}, \frac{a_{ij}^2}{w_i w_j}, \frac{a_{jk}^2}{w_j w_k}, \frac{a_{kl}^2}{w_k w_l}, \frac{a_{hl}^2}{w_h w_l}) = (1,1,1,1,0), (1,1,1,2,0), \text{ or } (2,1,1,1,0)$$

in particular $a_{hl} = 0$ (recall that we assumed the other four to be nonzero). Observe that the case $(1,1,1,2,0)$ is obtained from the case $(2,1,1,1,0)$ by reversing the order of the indices h,i,j,k,l. The first two solutions lead to cases (1), (2), and (3) of Lemma 5.5 where we also wrote out the consequences for the integers m_h, m_i, m_j, m_k, and m_l.

Lemma 5.5. *Classification of proper subgraphs of the form*

$$\bullet - \bullet - \bullet - \bullet - \bullet$$

If $n > 5$, then given five (-2)-indices h,i,j,k,l with $a_{hi}, a_{ij}, a_{jk}, a_{kl}$ nonzero, then up to ordering we have the m's, a's, w's

(1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & w \\ 0 & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5$, and $2m_5 \geq m_4$, or

(2) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & 2w \\ 0 & 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ 2w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + 2m_4$, $2m_4 \geq m_3 + m_5$, *and* $2m_5 \geq m_4$, *or*

(3) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -4w & 2w \\ 0 & 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ 2w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5$, *and* $m_4 \geq m_3$.

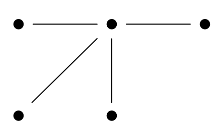
Proof. See discussion above. □

Suppose that h, i, j, k, and l are five (-2)-indices with $a_{hi} > 0$, $a_{hj} > 0$, $a_{hk} > 0$, and $a_{hl} > 0$. In other words, the index h "meets" the indices i, j, k, l. Then we see from Lemma 5.2 that $a_{ij} = a_{ik} = a_{il} = a_{jk} = a_{jl} = a_{kl} = 0$ and by Lemma 5.4 that $w_h = w_i = w_j = w_k = w_l = w$ for some integer $w > 0$ and $a_{hi} = a_{hj} = a_{hk} = a_{hl} = -2w$. The corresponding matrix

$$\begin{pmatrix} -2w & w & w & w & w \\ w & -2w & 0 & 0 & 0 \\ w & 0 & -2w & 0 & 0 \\ w & 0 & 0 & -2w & 0 \\ w & 0 & 0 & 0 & -2w \end{pmatrix}$$

is singular. Hence this can only happen if $n = 5$ and $g = 1$. The reader can find this as case (22) Lemma 6.2.

Lemma 5.6. *Nonexistence of proper subgraphs of the form*



If $n > 5$, *there do* **not** *exist five* (-2)*-indices* h, i, j, k *with* $a_{hi} > 0$, $a_{hj} > 0$, $a_{hk} > 0$, *and* $a_{hl} > 0$.

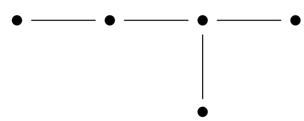
Proof. See discussion above. □

Suppose that h, i, j, k, and l are five (-2)-indices with $a_{hi} > 0$, $a_{ij} > 0$, $a_{jk} > 0$, and $a_{jl} > 0$. In other words, the index h "meets" i and the index j "meets" the indices i, k, l. Then we see from Lemma 5.4 that $a_{ik} = a_{il} = a_{kl} = 0$, $w_i = w_j = w_k = w_l = w$, and $a_{ij} = a_{jk} = a_{jl} = w$ for some integer w. Applying Lemma 5.3 to the four tuples h, i, j, k and h, i, j, l we see that $a_{hj} = a_{hk} = a_{hl} = 0$, that $w_h = \frac{1}{2}w$, w, or $2w$, and that correspondingly $a_{hi} = w$, w, or $2w$. Since A is semi-negative definite we see that the matrix

$$\begin{pmatrix} -2w_h & a_{hi} & 0 & 0 & 0 \\ a_{hi} & -2w & w & 0 & 0 \\ 0 & w & -2w & w & w \\ 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & -2w \end{pmatrix}$$

is negative definite unless $n = 5$. The reader computes that the determinant of the matrix is 0 when $w_h = \frac{1}{2}w$ or $2w$. This leads to cases (23) and (24) of Lemma 6.2. For $w_h = w$ we obtain case (1) of Lemma 5.7.

Lemma 5.7. *Classification of proper subgraphs of the form*



If $n > 5$, then given five (-2)-indices h, i, j, k, l with $a_{hi}, a_{ij}, a_{jk}, a_{jl}$ nonzero, then up to ordering we have the m's, a's, w's

(1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & w \\ 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4 + m_5$, $2m_4 \geq m_3$, and $2m_5 \geq m_3$.

Proof. See discussion above. $\square$

Suppose that $t > 5$ and $i_1, \ldots, i_t$ are t distinct (-2)-indices such that $a_{i_j i_{j+1}}$ is nonzero for $j = 1, \ldots, t-1$. We will prove by induction on t that if $n = t$ this leads to possibilities (25), (26), (27), (28) of Lemma 6.2 and if $n > t$ to cases (1), (2), and (3) of Lemma 5.8. First, if $a_{i_1 i_t}$ is nonzero, then it is clear from the result of Lemma 5.5 that $w_{i_1} = \ldots = w_{i_t} = w$ and that $a_{i_j i_{j+1}} = w$ for $j = 1, \ldots, t-1$ and $a_{i_1 i_t} = w$. Then the vector $(1, \ldots, 1)$ is in the kernel of the corresponding $t \times t$ matrix. Thus we must have $n = t$ and we see that the genus is 1 and that we are in case (25) of Lemma 6.2. Thus we may assume $a_{i_1 i_t} = 0$. By induction hypothesis (or Lemma 5.5 if $t = 6$) we see that $a_{i_j i_k} = 0$ if $k > j + 1$. Moreover, we have $w_{i_1} = \ldots = w_{i_{t-1}} = w$ for some integer w and $w_{i_1}, w_{i_t} \in \{\frac{1}{2}w, w, 2w\}$. Moreover, the value of w_{i_1}, resp. w_{i_t} being $\frac{1}{2}w$, w, or $2w$ implies that the value of $a_{i_1 i_2}$, resp. $a_{i_{t-1} i_t}$ is w, w, or $2w$. This gives 9 possibilities. In each case it is easy to decide what happens:

(1) if $(w_{i_1}, w_{i_t}) = (\frac{1}{2}w, \frac{1}{2}w)$, then we are in case (27) of Lemma 6.2,
(2) if $(w_{i_1}, w_{i_t}) = (\frac{1}{2}w, w)$ or $(w, \frac{1}{2}w)$ then we are in case (3) of Lemma 5.8,
(3) if $(w_{i_1}, w_{i_t}) = (\frac{1}{2}w, 2w)$ or $(2w, \frac{1}{2}w)$ then we are in case (26) of Lemma 6.2,
(4) if $(w_{i_1}, w_{i_t}) = (w, w)$ then we are in case (1) of Lemma 5.8,
(5) if $(w_{i_1}, w_{i_t}) = (w, 2w)$ or $(2w, w)$ then we are in case (2) of Lemma 5.8, and
(6) if $(w_{i_1}, w_{i_t}) = (2w, 2w)$ then we are in case (28) of Lemma 6.2.

Lemma 5.8. *Classification of proper subgraphs of the form*

Let $t > 5$ and $n > t$. Then given t distinct (-2)-indices $i_1, \ldots, i_t$ such that $a_{i_j i_{j+1}}$ is nonzero for $j = 1, \ldots, t-1$, then up to reversing the order of these indices we have the a's and w's

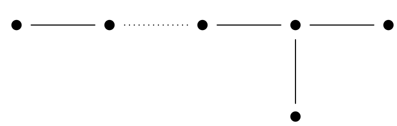
(1) *are given by $w_{i_1} = w_{i_2} = \ldots = w_{i_t} = w$, $a_{i_j i_{j+1}} = w$, and $a_{i_j i_k} = 0$ if $k > j+1$, or*
(2) *are given by $w_{i_1} = w_{i_2} = \ldots = w_{i_{t-1}} = w$, $w_{j_t} = 2w$, $a_{i_j i_{j+1}} = w$ for $j < t-1$, $a_{i_{t-1} i_t} = 2w$, and $a_{i_j i_k} = 0$ if $k > j+1$, or*
(3) *are given by $w_{i_1} = w_{i_2} = \ldots = w_{i_{t-1}} = 2w$, $w_{j_t} = w$, $a_{i_j i_{j+1}} = 2w$, and $a_{i_{t-1} i_t} = 2w$, and $a_{i_j i_k} = 0$ if $k > j+1$.*

Proof. See discussion above. □

Suppose that $t > 4$ and $i_1, \ldots, i_{t+1}$ are $t+1$ distinct (-2)-indices such that $a_{i_j i_{j+1}} > 0$ for $j = 1, \ldots, t-1$ and such that $a_{j_{t-1} j_{t+1}} > 0$. See picture in Lemma 5.9. We will prove by induction on t that if $n = t+1$ this leads to possibilities (29) and (30) of Lemma 6.2 and if $n > t+1$ to case (1) of Lemma 5.9. By induction hypothesis (or Lemma 5.7 in case $t = 5$) we see that $a_{i_j i_k}$ is zero outside of the required nonvanishing ones for $j, k \geq 2$. Moreover, we see that $w_2 = \ldots = w_{t+1} = w$ for some integer w and that the nonvanishing $a_{i_j i_k}$ for $j, k \geq 2$ are equal to w. Applying Lemma 5.8 (or Lemma 5.5 if $t = 5$) to the sequence $i_1, \ldots, i_t$ and to the sequence $i_1, \ldots, i_{t-1}, i_{t+1}$ we conclude that $a_{i_1 i_j} = 0$ for $j \geq 3$ and that w_1 is equal to $\frac{1}{2}w$, w, or $2w$ and that correspondingly $a_{i_1 i_2}$ is $w, w, 2w$. This gives 3 possibilities. In each case it is easy to decide what happens:

(1) If $w_1 = \frac{1}{2}w$, then we are in case (30) of Lemma 6.2.
(2) If $w_1 = w$, then we are in case (1) of Lemma 5.9.
(3) If $w_1 = 2w$, then we are in case (29) of Lemma 6.2.

Lemma 5.9. *Classification of proper subgraphs of the form*



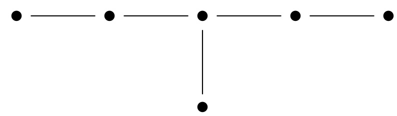
Let $t > 4$ and $n > t+1$. Then given $t+1$ distinct (-2)-indices $i_1, \ldots, i_{t+1}$ such that $a_{i_j i_{j+1}}$ is nonzero for $j = 1, \ldots, t-1$ and $a_{i_{t-1} i_{t+1}}$ is nonzero, then we have the a's and w's

(1) *are given by $w_{i_1} = w_{i_2} = \ldots = w_{i_{t+1}} = w$, $a_{i_j i_{j+1}} = w$ for $j = 1, \ldots, t-1$, $a_{i_{t-1} i_{t+1}} = w$ and $a_{i_j i_k} = 0$ for other pairs (j, k) with $j > k$.*

Proof. See discussion above. □

Suppose we are given 6 distinct (-2)-indices g, h, i, j, k, l such that $a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{il}$ are nonzero. See picture in Lemma 5.10. Then we can apply Lemma 5.7 to see that we must be in the situation of Lemma 5.10. Since the determinant is $3w^6 > 0$ we conclude that in this case it never happens that $n = 6$!

Lemma 5.10. *Classification of proper subgraphs of the form*



Let $n > 6$. *Then given* 6 *distinct* (-2)*-indices* $i_1, \ldots, i_6$ *such that* $a_{12}, a_{23}, a_{34}, a_{45}, a_{36}$ *are nonzero, then we have the m's, a's, and w's*

(1) *are given by*

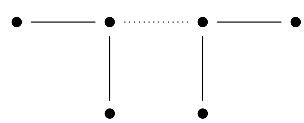
$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & w \\ 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4 + m_6$, $2m_4 \geq m_3 + m_5$, $2m_5 \geq m_3$, *and* $2m_6 \geq m_3$.

Proof. See discussion above. □

Suppose that $t \geq 4$ and $i_0, \ldots, i_{t+1}$ are $t+2$ distinct (-2)-indices such that $a_{i_j i_{j+1}} > 0$ for $j = 1, \ldots, t-1$ and $a_{i_0 i_2} > 0$ and $a_{i_{t-1} i_{t+1}} > 0$. See picture in Lemma 5.11. Then we can apply Lemmas 5.7 and 5.9 to see that all other $a_{i_j i_k}$ for $j < k$ are zero and that $w_{i_0} = \ldots = w_{i_{t+1}} = w$ for some integer w and that the required nonzero off diagonal entries of A are equal to w. A computation shows that the determinant of the corresponding matrix is zero. Hence $n = t+2$ and we are in case (31) of Lemma 6.2.

Lemma 5.11. *Nonexistence of proper subgraphs of the form*



Assume $t \geq 4$ *and* $n > t+2$. *There do* **not** *exist* $t+2$ *distinct* (-2)*-indices* $i_0, \ldots, i_{t+1}$ *such that* $a_{i_j i_{j+1}} > 0$ *for* $j = 1, \ldots, t-1$ *and* $a_{i_0 i_2} > 0$ *and* $a_{i_{t-1} i_{t+1}} > 0$.

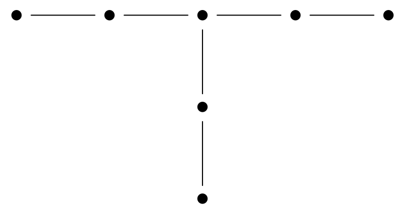
Proof. See discussion above. □

Suppose we are given 7 distinct (-2)-indices f, g, h, i, j, k, l such that the numbers $a_{fg}, a_{gh}, a_{ij}, a_{jh}, a_{kl}, a_{lh}$ are nonzero. See picture in Lemma 5.12. Then we can apply Lemma 5.7 to see that the corresponding matrix is

$$\begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & w & -2w & 0 & w & 0 & w \\ 0 & 0 & 0 & -2w & w & 0 & 0 \\ 0 & 0 & w & w & -2w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2w & w \\ 0 & 0 & w & 0 & 0 & w & -2w \end{pmatrix}$$

Since the determinant is 0 we conclude that we must have $n = 7$ and $g = 1$ and we get case (32) of Lemma 6.2.

Lemma 5.12. *Nonexistence of proper subgraphs of the form*

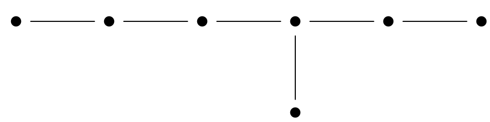


Assume $n > 7$. There do **not** *exist 7 distinct (-2)-indices f, g, h, i, j, k, l such that $a_{fg}, a_{gh}, a_{ij}, a_{jh}, a_{kl}, a_{lh}$ are nonzero.*

Proof. See discussion above. □

Suppose we are given 7 distinct (-2)-indices f, g, h, i, j, k, l such that the numbers $a_{fg}, a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{il}$ are nonzero. See picture in Lemma 5.13. Then we can apply Lemmas 5.7 and 5.9 to see that we must be in the situation of Lemma 5.13. Since the determinant is $-8w^7 > 0$ we conclude that in this case it never happens that $n = 7$!

Lemma 5.13. *Classification of proper subgraphs of the form*



Let $n > 7$. Then given 7 distinct (-2)-indices $i_1, \ldots, i_7$ such that $a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{47}$ are nonzero, then we have the m's, a's, and w's

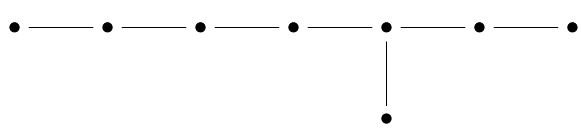
(1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & w \\ 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}$$

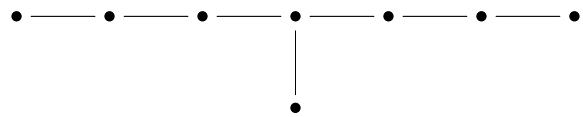
with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5 + m_7$, $2m_5 \geq m_4 + m_6$, $2m_6 \geq m_5$, and $2m_7 \geq m_4$.

Proof. See discussion above. □

Suppose we are given 8 distinct (-2)-indices whose pattern of nonzero entries a_{ij} of the matrix A looks like

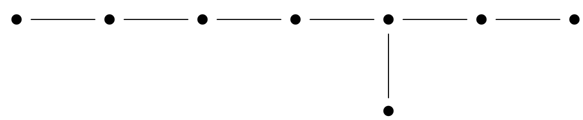


or like



Arguing exactly as in the proof of Lemma 5.13 we see that the first pattern leads to case (1) in Lemma 5.14 and does not lead to a new case in Lemma 6.2. Arguing exactly as in the proof of Lemma 5.12 we see that the second pattern does not occur if $n > 8$, but leads to case (33) in Lemma 6.2 when $n = 8$.

Lemma 5.14. *Classification of proper subgraphs of the form*



Let $n > 8$. *Then given* 8 *distinct* (-2)*-indices* $i_1, \ldots, i_8$ *such that* $a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{65}, a_{57}$ *are nonzero, then we have the m's, a's, and w's*

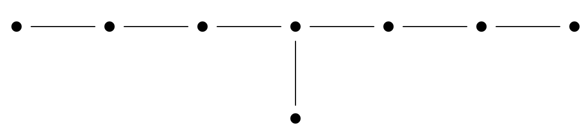
(1) *are given by*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & 0 & 0 \\ 0 & 0 & 0 & w & -2w & w & 0 & w \\ 0 & 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}$$

with $2m_1 \geq m_2$, $2m_2 \geq m_1 + m_3$, $2m_3 \geq m_2 + m_4$, $2m_4 \geq m_3 + m_5$, $2m_5 \geq m_4 + m_6 + m_8$, $2m_6 \geq m_5 + m_7$, $2m_7 \geq m_6$, *and* $2m_8 \geq m_5$.

Proof. See discussion above. □

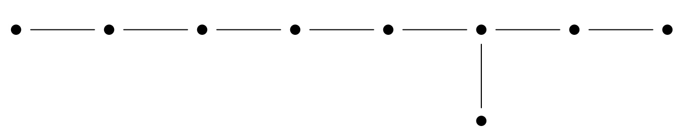
Lemma 5.15. *Nonexistence of proper subgraphs of the form*



Assume $n > 8$. *There do* **not** *exist* 8 *distinct* (-2)*-indices* e, f, g, h, i, j, k, l *such that* $a_{ef}, a_{fg}, a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{lh}$ *are nonzero.*

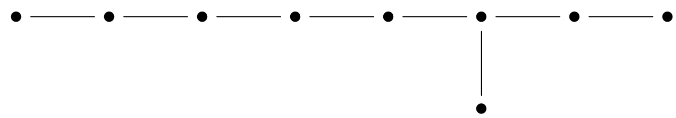
Proof. See discussion above. □

Suppose we are given 9 distinct (-2)-indices whose pattern of nonzero entries a_{ij} of the matrix A looks like



Arguing exactly as in the proof of Lemma 5.12 we see that this pattern does not occur if $n > 9$, but leads to case (34) in Lemma 6.2 when $n = 9$.

Lemma 5.16. *Nonexistence of proper subgraphs of the form*



Assume $n > 9$. *There do* **not** *exist* 9 *distinct* (-2)*-indices* $d, e, f, g, h, i, j, k, l$ *such that* $a_{de}, a_{ef}, a_{fg}, a_{gh}, a_{hi}, a_{ij}, a_{jk}, a_{lh}$ *are nonzero.*

Proof. See discussion above. ☐

Collecting all the information together we find the following.

Proposition 5.17. *Let* n, m_i, a_{ij}, w_i, g_i *be a numerical type of genus* g. *Let* $I \subset \{1, \ldots, n\}$ *be a proper subset of cardinality* ≥ 2 *consisting of* (-2)*-indices such that there does not exist a nonempty proper subset* $I' \subset I$ *with* $a_{i'i} = 0$ *for* $i' \in I$, $i \in I \setminus I'$. *Then up to reordering the* m_i*'s,* a_{ij}*'s,* w_i*'s for* $i, j \in I$ *are as listed in Lemmas 5.1, 5.2, 5.3, 5.4, 5.5, 5.7, 5.8, 5.9, 5.10, 5.13, or 5.14.*

Proof. This follows from the discussion above; see discussion at the start of Section 5. ☐

6. Classification of minimal type for genus zero and one

The title of the section explains it all.

Lemma 6.1 (Genus zero)**.** *The only minimal numerical type of genus zero is* $n = 1$, $m_1 = 1$, $a_{11} = 0$, $w_1 = 1$, $g_1 = 0$.

Proof. Follows from Lemmas 3.13 and 3.5. ☐

Lemma 6.2 (Genus one)**.** *The minimal numerical types of genus one are up to equivalence*

(1) $n = 1$, $a_{11} = 0$, $g_1 = 1$, $m_1, w_1 \geq 1$ *arbitrary,*

(2) $n = 2$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w \\ 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

(3) $n = 2$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 4w \\ 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

(4) $n = 3$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w \\ w & -2w & w \\ w & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

(5) *$n = 3$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 \\ w & -2w & 3w \\ 0 & 3w & -6w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ 3w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(6) *$n = 3$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \end{pmatrix}, \quad \begin{pmatrix} -6w & 3w & 0 \\ 3w & -6w & 3w \\ 0 & 3w & -2w \end{pmatrix}, \quad \begin{pmatrix} 3w \\ 3w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(7) *$n = 3$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 \\ 2w & -4w & 4w \\ 0 & 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(8) *$n = 3$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 \\ 2w & -4w & 2w \\ 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(9) *$n = 3$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 \\ 2w & -2w & 2w \\ 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(10) *$n = 4$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & w \\ w & -2w & w & 0 \\ 0 & w & -2w & w \\ w & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(11) *$n = 4$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} 2m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 \\ 2w & -4w & 2w & 0 \\ 0 & 2w & -4w & 4w \\ 0 & 0 & 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(12) *$n = 4$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 \\ 2w & -4w & 2w & 0 \\ 0 & 2w & -4w & 2w \\ 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(13) *$n = 4$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 \\ 2w & -2w & w & 0 \\ 0 & w & -2w & 2w \\ 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(14) *$n = 4$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} 2m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w & 2w \\ w & -2w & 0 & 0 \\ w & 0 & -2w & 0 \\ 2w & 0 & 0 & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(15) *$n = 4$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} 2m \\ m \\ m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 2w & 2w \\ 2w & -4w & 0 & 0 \\ 2w & 0 & -4w & 0 \\ 2w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(16) *$n = 5$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & w \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & w \\ w & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(17) *$n = 5$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 \\ 0 & w & -2w & 2w & 0 \\ 0 & 0 & 2w & -4w & 2w \\ 0 & 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ 2w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(18) *$n = 5$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 4m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -2w & w \\ 0 & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ 2w \\ 2w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(19) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ 2m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -4w & 4w \\ 0 & 0 & 0 & 4w & -8w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 2w \\ 4w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

(20) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 0 \\ 0 & 0 & 2w & -4w & 2w \\ 0 & 0 & 0 & 2w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 2w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

(21) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ 2m \\ 2m \\ 2m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & 0 \\ 0 & 0 & w & -2w & 2w \\ 0 & 0 & 0 & 2w & -4w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ w \\ w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

(22) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ m \\ m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & w & w & w \\ w & -2w & 0 & 0 & 0 \\ w & 0 & -2w & 0 & 0 \\ w & 0 & 0 & -2w & 0 \\ w & 0 & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

(23) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ 2m \\ 2m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -4w & 2w & 0 & 0 & 0 \\ 2w & -2w & w & 0 & 0 \\ 0 & w & -2w & w & w \\ 0 & 0 & w & -2w & 0 \\ 0 & 0 & w & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} 2w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

(24) $n = 5$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} 2m \\ 2m \\ 2m \\ m \\ m \end{pmatrix}, \quad \begin{pmatrix} -2w & 2w & 0 & 0 & 0 \\ 2w & -4w & 2w & 0 & 0 \\ 0 & 2w & -4w & 2w & 2w \\ 0 & 0 & 2w & -4w & 0 \\ 0 & 0 & 2w & 0 & -4w \end{pmatrix}, \quad \begin{pmatrix} w \\ 2w \\ 2w \\ 2w \\ 2w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

(25) $n \geq 6$ *and we have an* n*-cycle generalizing (16):*
   (a) $m_1 = \ldots = m_n = m$,

(b) $a_{12} = \ldots = a_{(n-1)n} = w$, $a_{1n} = w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
(c) $w_1 = \ldots = w_n = w$

with w *and* m *arbitrary,*

(26) $n \geq 6$ *and we have a chain generalizing (19):*
(a) $m_1 = \ldots = m_{n-1} = 2m$, $m_n = m$,
(b) $a_{12} = \ldots = a_{(n-2)(n-1)} = 2w$, $a_{(n-1)n} = 4w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
(c) $w_1 = w$, $w_2 = \ldots = w_{n-1} = 2w$, $w_n = 4w$

with w *and* m *arbitrary,*

(27) $n \geq 6$ *and we have a chain generalizing (20):*
(a) $m_1 = \ldots = m_n = m$,
(b) $a_{12} = \ldots = a_{(n-1)n} = w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
(c) $w_1 = w$, $w_2 = \ldots = w_{n-1} = 2w$, $w_n = w$

with w *and* m *arbitrary,*

(28) $n \geq 6$ *and we have a chain generalizing (21):*
(a) $m_1 = w$, $w_2 = \ldots = m_{n-1} = 2m$, $m_n = m$,
(b) $a_{12} = 2w$, $a_{23} = \ldots = a_{(n-2)(n-1)} = w$, $a_{(n-1)n} = 2w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
(c) $w_1 = 2w$, $w_2 = \ldots = w_{n-1} = w$, $w_n = 2w$

with w *and* m *arbitrary,*

(29) $n \geq 6$ *and we have a type generalizing (23):*
(a) $m_1 = m$, $m_2 = \ldots = m_{n-3} = 2m$, $m_{n-1} = m_n = m$,
(b) $a_{12} = 2w$, $a_{23} = \ldots = a_{(n-2)(n-1)} = w$, $a_{(n-2)n} = w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
(c) $w_1 = 2w$, $w_2 = \ldots = w_n = w$

with w *and* m *arbitrary,*

(30) $n \geq 6$ *and we have a type generalizing (24):*
(a) $m_1 = \ldots = m_{n-3} = 2m$, $m_{n-1} = m_n = m$,
(b) $a_{12} = \ldots = a_{(n-2)(n-1)} = 2w$, $a_{(n-2)n} = 2w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
(c) $w_1 = w$, $w_2 = \ldots = w_n = 2w$

with w *and* m *arbitrary,*

(31) $n \geq 6$ *and we have a type generalizing (22):*
(a) $m_1 = m_2 = m$, $m_3 = \ldots = m_{n-2} = 2m$, $m_{n-1} = m_n = m$,
(b) $a_{13} = w$, $a_{23} = \ldots = a_{(n-2)(n-1)} = w$, $a_{(n-2)n} = w$, *and for other* $i < j$ *we have* $a_{ij} = 0$,
(c) $w_1 = \ldots = w_n = w$,

with w *and* m *arbitrary,*

(32) $n = 7$, *and* m_i, a_{ij}, w_i, g_i *given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \\ m \\ 2m \\ m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & w & -2w & 0 & w & 0 & w \\ 0 & 0 & 0 & -2w & w & 0 & 0 \\ 0 & 0 & w & w & -2w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2w & w \\ 0 & 0 & w & 0 & 0 & w & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w *and* m *arbitrary,*

(33) *$n = 8$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 4m \\ 3m \\ 2m \\ m \\ 2m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & 0 & w \\ 0 & 0 & 0 & w & -2w & w & 0 & 0 \\ 0 & 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & w & 0 & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary,

(34) *$n = 9$, and m_i, a_{ij}, w_i, g_i given by*

$$\begin{pmatrix} m \\ 2m \\ 3m \\ 4m \\ 5m \\ 6m \\ 4m \\ 2m \\ 3m \end{pmatrix}, \quad \begin{pmatrix} -2w & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & -2w & w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w & -2w & w & 0 & w \\ 0 & 0 & 0 & 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w & -2w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & -2w \end{pmatrix}, \quad \begin{pmatrix} w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \\ w \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with w and m arbitrary.

Proof. This is proved in Section 5. See discussion at the start of Section 5. □

7. Bounding invariants of numerical types

In our proof of semistable reduction for curves we'll use a bound on Picard groups of numerical types of genus g which we will prove in this section.

Lemma 7.1. *Let n, m_i, a_{ij}, w_i, g_i be a numerical type of genus g. Given i, j with $a_{ij} > 0$ we have $m_i a_{ij} \leq m_j |a_{jj}|$ and $m_i w_i \leq m_j |a_{jj}|$.*

Proof. For every index j we have $m_j a_{jj} + \sum_{i \neq j} m_i a_{ij} = 0$. Thus if we have an upper bound on $|a_{jj}|$ and m_j, then we also get an upper bound on the nonzero (and hence positive) a_{ij} as well as m_i. Recalling that w_i divides a_{ij}, the reader easily sees the lemma is correct. □

Lemma 7.2. *Fix $g \geq 2$. For every minimal numerical type n, m_i, a_{ij}, w_i, g_i of genus g with $n > 1$ we have*

1. *the set $J \subset \{1, \ldots, n\}$ of non-(-2)-indices has at most $2g - 2$ elements,*
2. *for $j \in J$ we have $g_j < g$,*
3. *for $j \in J$ we have $m_j |a_{jj}| \leq 6g - 6$, and*
4. *for $j \in J$ and $i \in \{1, \ldots, n\}$ we have $m_i a_{ij} \leq 6g - 6$.*

Proof. Recall that $g = 1 + \sum m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$. For $j \in J$ the contribution $m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$ to the genus g is > 0 and hence $\geq 1/2$. This uses Lemma 3.7, Definition 3.8, Definition 3.12, Lemma 3.15, and Definition 3.16; we will use these results without further mention in the following. Thus J has at most $2(g-1)$ elements. This proves (1).

Recall that $-a_{ii} > 0$ for all i by Lemma 3.6. Hence for $j \in J$ the contribution $m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$ to the genus g is $> m_j w_j(g_j - 1)$. Thus

$$g - 1 > m_j w_j(g_j - 1) \Rightarrow g_j < (g - 1)/m_j w_j + 1$$

This indeed implies $g_j < g$ which proves (2).

For $j \in J$ if $g_j > 0$, then the contribution $m_j(w_j(g_j - 1) - \frac{1}{2}a_{jj})$ to the genus g is $\geq -\frac{1}{2}m_j a_{jj}$ and we immediately conclude that $m_j|a_{jj}| \leq 2(g - 1)$. Otherwise $a_{jj} = -kw_j$ for some integer $k \geq 3$ (because $j \in J$) and we get

$$m_j w_j(-1 + \frac{k}{2}) \leq g - 1 \Rightarrow m_j w_j \leq \frac{2(g-1)}{k-2}$$

Plugging this back into $a_{jj} = -km_j w_j$ we obtain

$$m_j|a_{jj}| \leq 2(g - 1)\frac{k}{k-2} \leq 6(g - 1)$$

This proves (3).

Part (4) follows from Lemma 7.1 and (3). □

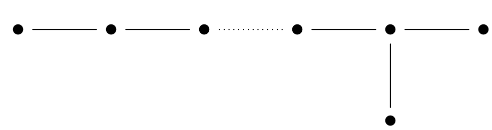
Lemma 7.3. *Fix $g \geq 2$. For every minimal numerical type n, m_i, a_{ij}, w_i, g_i of genus g we have $m_i|a_{ij}| \leq 768g$.*

Proof. By Lemma 7.1 it suffices to show $m_i|a_{ii}| \leq 768g$ for all i. Let $J \subset \{1, \ldots, n\}$ be the set of non-(-2)-indices as in Lemma 7.2. Observe that J is nonempty as $g \geq 2$. Also $m_j|a_{jj}| \leq 6g$ for $j \in J$ by the lemma.

Suppose we have $j \in J$ and a sequence $i_1, \ldots, i_7$ of (-2)-indices such that a_{ji_1} and $a_{i_1i_2}$, $a_{i_2i_3}$, $a_{i_3i_4}$, $a_{i_4i_5}$, $a_{i_5i_6}$, and $a_{i_6i_7}$ are nonzero. Then we see from Lemma 7.1 that $m_{i_1}w_{i_1} \leq 6g$ and $m_{i_1}a_{ji_1} \leq 6g$. Because i_1 is a (-2)-index, we have $a_{i_1i_1} = -2w_{i_1}$ and we conclude that $m_{i_1}|a_{i_1i_1}| \leq 12g$. Repeating the argument we conclude that $m_{i_2}w_{i_2} \leq 12g$ and $m_{i_2}a_{i_1i_2} \leq 12g$. Then $m_{i_2}|a_{i_2i_2}| \leq 24g$ and so on. Eventually we conclude that $m_{i_k}|a_{i_ki_k}| \leq 2^k(6g) \leq 768g$ for $k = 1, \ldots, 7$.

Let $I \subset \{1, \ldots, n\} \setminus J$ be a maximal connected subset. In other words, there does not exist a nonempty proper subset $I' \subset I$ such that $a_{i'i} = 0$ for $i' \in I'$ and $i \in I \setminus I'$ and I is maximal with this property. In particular, since a numerical type is connected by definition, we see that there exists a $j \in J$ and $i \in I$ with $a_{ij} > 0$. Looking at the classification of such I in Proposition 5.17 and using the result of the previous paragraph, we see that $w_i|a_{ii}| \leq 768g$ for all $i \in I$ unless I is as described in Lemma 5.8 or Lemma 5.9. Thus we may assume the nonvanishing of $a_{ii'}$, $i, i' \in I$ has either the shape

(which has 3 subcases as detailed in Lemma 5.8) or the shape



We will prove the bound holds for the first subcase of Lemma 5.8 and leave the other cases to reader (the argument is almost exactly the same in those cases).

After renumbering we may assume $I = \{1, \ldots, t\} \subset \{1, \ldots, n\}$ and there is an integer w such that

$$w = w_1 = \ldots = w_t = a_{12} = \ldots = a_{(t-1)t} = -\frac{1}{2}a_{i_1 i_2} = \ldots = -\frac{1}{2}a_{(t-1)t}$$

The equalities $a_{ii}m_i + \sum_{j \neq i} a_{ij}m_j = 0$ imply that we have

$$2m_2 \geq m_1 + m_3, \ldots, 2m_{t-1} \geq m_{t-2} + m_t$$

Equality holds in $2m_i \geq m_{i-1} + m_{i+1}$ if and only if i does not "meet" any indices besides $i-1$ and $i+1$. And if i does meet another index, then this index is in J (by maximality of I). In particular, the map $\{1, \ldots, t\} \to \mathbf{Z}$, $i \mapsto m_i$ is concave.

Let $m = \max(m_i, i \in \{1, \ldots, t\})$. Then $m_i|a_{ii}| \leq 2mw$ for $i \leq t$ and our goal is to show that $2mw \leq 768g$. Let s, resp. s' in $\{1, \ldots, t\}$ be the smallest, resp. biggest index with $m_s = m = m_{s'}$. By concavity we see that $m_i = m$ for $s \leq i \leq s'$. If $s > 1$, then we do not have equality in $2m_s \geq m_{s-1} + m_{s+1}$ and we see that s meets an index from J. In this case $2mw \leq 12g$ by the result of the second paragraph of the proof. Similarly, if $s' < t$, then s' meets an index from J and we get $2mw \leq 12g$ as well. But if $s = 1$ and $s' = t$, then we conclude that $a_{ij} = 0$ for all $j \in J$ and $i \in \{2, \ldots, t-1\}$. But as we've seen that there must be a pair $(i, j) \in I \times J$ with $a_{ij} > 0$, we conclude that this happens either with $i = 1$ or with $i = t$ and we conclude $2mw \leq 12g$ in the same manner as before (as $m_1 = m = m_t$ in this case). □

Proposition 7.4. *Let $g \geq 2$. For every numerical type T of genus g and prime number $\ell > 768g$ we have*

$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(T)[\ell] \leq g$$

where $\mathrm{Pic}(T)$ *is as in Definition 4.1. If T is minimal, then we even have*

$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(T)[\ell] \leq g_{top} \leq g$$

where g_{top} as in Definition 3.11.

Proof. Say T is given by n, m_i, a_{ij}, w_i, g_i. If T is not minimal, then there exists a (-1)-index. After replacing T by an equivalent type we may assume n is a (-1)-index. Applying Lemma 4.4 we find $\mathrm{Pic}(T) \subset \mathrm{Pic}(T')$ where T' is a numerical type of genus g (Lemma 3.9) with $n-1$ indices. Thus we conclude by induction on n provided we prove the lemma for minimal numerical types.

Assume that T is a minimal numerical type of genus ≥ 2. Observe that $g_{top} \leq g$ by Lemma 3.14. If $A = (a_{ij})$ then since $\mathrm{Pic}(T) \subset \mathrm{Coker}(A)$ by Lemma 4.3. Thus it suffices to prove the lemma for $\mathrm{Coker}(A)$. By Lemma 7.3 we see that $m_i|a_{ij}| \leq 768g$ for all i, j. Hence the result by Lemma 2.6. □

8. Models

In this chapter R will be a discrete valuation ring and K will be its fraction field. If needed we will denote $\pi \in R$ a uniformizer and $k = R/(\pi)$ its residue field.

Let V be an algebraic K-scheme (Varieties, Definition 20.1). A *model* for V will mean a flat finite type[6] morphism $X \to \mathrm{Spec}(R)$ endowed with an isomorphism $V \to X_K = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$. We often will identify V and the generic fibre

[6]Occasionally it is useful to allow models to be locally of finite type over R, but we'll cross that bridge when we come to it.

X_K of X and just write $V = X_K$. The special fibre is $X_k = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$. A *morphism of models* $X \to X'$ *for* V is a morphism $X \to X'$ of schemes over R which induces the identity on V.

We will say X *is a proper model of* V if X is a model of V and the structure morphism $X \to \mathrm{Spec}(R)$ is proper. Similarly for separated models, smooth models, and add more here. We will say X *is a regular model of* V if X is a model of V and X is a regular scheme. Similarly for normal models, reduced models, and add more here.

Let $R \subset R'$ be an extension of discrete valuation rings (More on Algebra, Definition 111.1). This induces an extension K'/K of fraction fields. Given an algebraic scheme V over K, denote V' the base change $V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K')$. Then there is a functor

$$\text{models for } V \text{ over } R \longrightarrow \text{models for } V' \text{ over } R'$$

sending X to $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$.

Lemma 8.1. *Let $V_1 \to V_2$ be a closed immersion of algebraic schemes over K. If X_2 is a model for V_2, then the scheme theoretic image of $V_1 \to X_2$ is a model for V_1.*

Proof. Using Morphisms, Lemma 6.3 and Example 6.4 this boils down to the following algebra statement. Let A_1 be a finite type R-algebra flat over R. Let $A_1 \otimes_R K \to B_2$ be a surjection. Then $A_2 = A_1/\mathrm{Ker}(A_1 \to B_2)$ is a finite type R-algebra flat over R such that $B_2 = A_2 \otimes_R K$. We omit the detailed proof; use More on Algebra, Lemma 22.11 to prove that A_2 is flat. $\square$

Lemma 8.2. *Let X be a model of a geometrically normal variety V over K. Then the normalization $\nu : X^\nu \to X$ is finite and the base change of X^ν to the completion $R^\wedge$ is the normalization of the base change of X. Moreover, for each $x \in X^\nu$ the completion of $\mathcal{O}_{X^\nu,x}$ is normal.*

Proof. Observe that $R^\wedge$ is a discrete valuation ring (More on Algebra, Lemma 43.5). Set $Y = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R^\wedge)$. Since $R^\wedge$ is a discrete valuation ring, we see that

$$Y \setminus Y_k = Y \times_{\mathrm{Spec}(R^\wedge)} \mathrm{Spec}(K^\wedge) = V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^\wedge)$$

where $K^\wedge$ is the fraction field of $R^\wedge$. Since V is geometrically normal, we find that this is a normal scheme. Hence the first part of the lemma follows from Resolution of Surfaces, Lemma 11.6.

To prove the second part we may assume X and Y are normal (by the first part). If x is in the generic fibre, then $\mathcal{O}_{X,x} = \mathcal{O}_{V,x}$ is a normal local ring essentially of finite type over a field. Such a ring is excellent (More on Algebra, Proposition 52.3). If x is a point of the special fibre with image $y \in Y$, then $\mathcal{O}_{X,x}^\wedge = \mathcal{O}_{Y,y}^\wedge$ by Resolution of Surfaces, Lemma 11.1. In this case $\mathcal{O}_{Y,y}$ is a excellent normal local domain by the same reference as before as $R^\wedge$ is excellent. If B is a excellent local normal domain, then the completion $B^\wedge$ is normal (as $B \to B^\wedge$ is regular and More on Algebra, Lemma 42.2 applies). This finishes the proof. $\square$

Lemma 8.3. *Let X be a model of a smooth curve C over K. Then there exists a resolution of singularities of X and any resolution is a model of C.*

Proof. We check condition (4) of Lipman's theorem (Resolution of Surfaces, Theorem 14.5) hold. This is clear from Lemma 8.2 except for the statement that X^ν has finitely many singular points. To see this we can use that R is J-2 by More on Algebra, Proposition 48.7 and hence the nonsingular locus is open in X^ν. Since X^ν is normal of dimension ≤ 2, the singular points are closed, hence closedness of the singular locus means there are finitely many of them (as X is quasi-compact). Observe that any resolution of X is a modification of X (Resolution of Surfaces, Definition 14.1). This will be an isomorphism over the normal locus of X by Varieties, Lemma 17.3. Since the set of normal points includes $C = X_K$ we conclude any resolution is a model of C. $\square$

Definition 8.4. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. A *minimal model* will be a regular, proper model X for C such that X does not contain an exceptional curve of the first kind (Resolution of Surfaces, Section 16).

Really such a thing should be called a minimal regular proper model or even a relatively minimal regular projective model. But as long as we stick to models over discrete valuation rings (as we will in this chapter), no confusion should arise.

Minimal models always exist (Proposition 8.6) and are unique when the genus is > 0 (Lemma 10.1).

Lemma 8.5. *Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. If X is a regular proper model for C, then there exists a sequence of morphisms*

$$X = X_m \to X_{m-1} \to \ldots \to X_1 \to X_0$$

of proper regular models of C, such that each morphism is a contraction of an exceptional curve of the first kind, and such that X_0 is a minimal model.

Proof. By Resolution of Surfaces, Lemma 16.11 we see that X is projective over R. Hence X has an ample invertible sheaf by More on Morphisms, Lemma 50.1 (we will use this below). Let $E \subset X$ be an exceptional curve of the first kind. See Resolution of Surfaces, Section 16. By Resolution of Surfaces, Lemma 16.8 we can contract E by a morphism $X \to X'$ such that X' is regular and is projective over R. Clearly, the number of irreducible components of X'_k is exactly one less than the number of irreducible components of X_k. Thus we can only perform a finite number of these contractions until we obtain a minimal model. $\square$

Proposition 8.6. *Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. A minimal model exists.*

Proof. Choose a closed immersion $C \to \mathbf{P}^n_K$. Let X be the scheme theoretic image of $C \to \mathbf{P}^n_R$. Then $X \to \text{Spec}(R)$ is a projective model of C by Lemma 8.1. By Lemma 8.3 there exists a resolution of singularities $X' \to X$ and X' is a model for C. Then $X' \to \text{Spec}(R)$ is proper as a composition of proper morphisms. Then we may apply Lemma 8.5 to obtain a minimal model. $\square$

9. The geometry of a regular model

In this section we describe the geometry of a proper regular model X of a smooth projective curve C over K with $H^0(C, \mathcal{O}_C) = K$.

Lemma 9.1. *Let X be a regular model of a smooth curve C over K.*

(1) *the special fibre X_k is an effective Cartier divisor on X,*

(2) *each irreducible component C_i of X_k is an effective Cartier divisor on X,*
(3) $X_k = \sum m_i C_i$ *(sum of effective Cartier divisors) where m_i is the multiplicity of C_i in X_k,*
(4) $\mathcal{O}_X(X_k) \cong \mathcal{O}_X$.

Proof. Recall that R is a discrete valuation ring with uniformizer π and residue field $k = R/(\pi)$. Because $X \to \text{Spec}(R)$ is flat, the element π is a nonzerodivisor affine locally on X (see More on Algebra, Lemma 22.11). Thus if $U = \text{Spec}(A) \subset X$ is an affine open, then

$$X_K \cap U = U_k = \text{Spec}(A \otimes_R k) = \text{Spec}(A/\pi A)$$

and π is a nonzerodivisor in A. Hence $X_k = V(\pi)$ is an effective Cartier divisor by Divisors, Lemma 13.2. Hence (1) is true.

The discussion above shows that the pair $(\mathcal{O}_X(X_k), 1)$ is isomorphic to the pair $(\mathcal{O}_X, \pi)$ which proves (4).

By Divisors, Lemma 15.11 there exist pairwise distinct integral effective Cartier divisors $D_i \subset X$ and integers $a_i \geq 0$ such that $X_k = \sum a_i D_i$. We can throw out those divisors D_i such that $a_i = 0$. Then it is clear (from the definition of addition of effective Cartier divisors) that $X_k = \bigcup D_i$ set theoretically. Thus $C_i = D_i$ are the irreducible components of X_k which proves (2). Let ξ_i be the generic point of C_i. Then $\mathcal{O}_{X,\xi_i}$ is a discrete valuation ring (Divisors, Lemma 15.4). The uniformizer $\pi_i \in \mathcal{O}_{X,\xi_i}$ is a local equation for C_i and the image of π is a local equation for X_k. Since $X_k = \sum a_i C_i$ we see that π and $\pi_i^{a_i}$ generate the same ideal in $\mathcal{O}_{X,\xi_i}$. On the other hand, the multiplicity of C_i in X_k is

$$m_i = \text{length}_{\mathcal{O}_{C_i,\xi_i}} \mathcal{O}_{X_k,\xi_i} = \text{length}_{\mathcal{O}_{C_i,\xi_i}} \mathcal{O}_{X,\xi_i}/(\pi) = \text{length}_{\mathcal{O}_{C_i,\xi_i}} \mathcal{O}_{X,\xi_i}/(\pi_i^{a_i}) = a_i$$

See Chow Homology, Definition 9.2. Thus $a_i = m_i$ and (3) is proved. $\square$

Lemma 9.2. *Let X be a regular model of a smooth curve C over K. Then*

(1) $X \to \text{Spec}(R)$ *is a Gorenstein morphism of relative dimension* 1,
(2) *each of the irreducible components C_i of X_k is Gorenstein.*

Proof. Since $X \to \text{Spec}(R)$ is flat, to prove (1) it suffices to show that the fibres are Gorenstein (Duality for Schemes, Lemma 25.3). The generic fibre is a smooth curve, which is regular and hence Gorenstein (Duality for Schemes, Lemma 24.3). For the special fibre X_k we use that it is an effective Cartier divisor on a regular (hence Gorenstein) scheme and hence Gorenstein for example by Dualizing Complexes, Lemma 21.6. The curves C_i are Gorenstein by the same argument. $\square$

Situation 9.3. Let R be a discrete valuation ring with fraction field K, residue field k, and uniformizer π. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Let X be a regular proper model of C. Let $C_1, \ldots, C_n$ be the irreducible components of the special fibre X_k. Write $X_k = \sum m_i C_i$ as in Lemma 9.1.

Lemma 9.4. *In Situation 9.3 the special fibre X_k is connected.*

Proof. Consequence of More on Morphisms, Lemma 53.6. $\square$

Lemma 9.5. *In Situation 9.3 there is an exact sequence*

$$0 \to \mathbf{Z} \to \mathbf{Z}^{\oplus n} \to \text{Pic}(X) \to \text{Pic}(C) \to 0$$

where the first map sends 1 *to* $(m_1, \ldots, m_n)$ *and the second maps sends the* i*th basis vector to* $\mathcal{O}_X(C_i)$.

Proof. Observe that $C \subset X$ is an open subscheme. The restriction map $\text{Pic}(X) \to \text{Pic}(C)$ is surjective by Divisors, Lemma 28.3. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module such that there is an isomorphism $s : \mathcal{O}_C \to \mathcal{L}|_C$. Then s is a regular meromorphic section of $\mathcal{L}$ and we see that $\text{div}_\mathcal{L}(s) = \sum a_i C_i$ for some $a_i \in \mathbf{Z}$ (Divisors, Definition 27.4). By Divisors, Lemma 27.6 (and the fact that X is normal) we conclude that $\mathcal{L} = \mathcal{O}_X(\sum a_i C_i)$. Finally, suppose that $\mathcal{O}_X(\sum a_i C_i) \cong \mathcal{O}_X$. Then there exists an element g of the function field of X with $\text{div}_X(g) = \sum a_i C_i$. In particular the rational function g has no zeros or poles on the generic fibre C of X. Since C is a normal scheme this implies $g \in H^0(C, \mathcal{O}_C) = K$. Thus $g = \pi^a u$ for some $a \in \mathbf{Z}$ and $u \in R^*$. We conclude that $\text{div}_X(g) = a \sum m_i C_i$ and the proof is complete. $\square$

In Situation 9.3 for every invertible $\mathcal{O}_X$-module $\mathcal{L}$ and every i we get an integer

$$\deg(\mathcal{L}|_{C_i}) = \chi(C_i, \mathcal{L}|_{C_i}) - \chi(C_i, \mathcal{O}_{C_i})$$

by taking the degree of the restriction of $\mathcal{L}$ to C_i relative to the ground field k[7] as in Varieties, Section 44.

Lemma 9.6. *In Situation 9.3 given* $\mathcal{L}$ *an invertible* $\mathcal{O}_X$*-module and* $a = (a_1, \ldots, a_n) \in \mathbf{Z}^{\oplus n}$ *we define*

$$\langle a, \mathcal{L} \rangle = \sum a_i \deg(\mathcal{L}|_{C_i})$$

Then $\langle , \rangle$ *is bilinear and for* $b = (b_1, \ldots, b_n) \in \mathbf{Z}^{\oplus n}$ *we have*

$$\left\langle a, \mathcal{O}_X(\sum b_i C_i) \right\rangle = \left\langle b, \mathcal{O}_X(\sum a_i C_i) \right\rangle$$

Proof. Bilinearity is immediate from the definition and Varieties, Lemma 44.7. To prove symmetry it suffices to assume a and b are standard basis vectors in $\mathbf{Z}^{\oplus n}$. Hence it suffices to prove that

$$\deg(\mathcal{O}_X(C_j)|_{C_i}) = \deg(\mathcal{O}_X(C_i)|_{C_j})$$

for all $1 \leq i, j \leq n$. If $i = j$ there is nothing to prove. If $i \neq j$, then the canonical section 1 of $\mathcal{O}_X(C_j)$ restricts to a nonzero (hence regular) section of $\mathcal{O}_X(C_j)|_{C_i}$ whose zero scheme is exactly $C_i \cap C_j$ (scheme theoretic intersection). In other words, $C_i \cap C_j$ is an effective Cartier divisor on C_i and

$$\deg(\mathcal{O}_X(C_j)|_{C_i}) = \deg(C_i \cap C_j)$$

by Varieties, Lemma 44.9. By symmetry we obtain the same (!) formula for the other side and the proof is complete. $\square$

In Situation 9.3 it is often convenient to think of $\mathbf{Z}^{\oplus n}$ as the free abelian group on the set $\{C_1, \ldots, C_n\}$. We will indicate an element of this group as $\sum a_i C_i$; here we think of this as a formal sum although equivalently we may (and we sometimes do) think of such a sum as a Weil divisor on X supported on the special fibre X_k. Now Lemma 9.6 allows us to define a symmetric bilinear form $(\cdot)$ on this free abelian group by the rule

$$\left(\sum a_i C_i \cdot \sum b_j C_j\right) = \left\langle a, \mathcal{O}_X(\sum b_j C_j) \right\rangle = \left\langle b, \mathcal{O}_X(\sum a_i C_i) \right\rangle \tag{9.6.1}$$

[7] Observe that it may happen that the field $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ is strictly bigger than k. In this case every invertible module on C_i has degree (as defined above) divisible by $[\kappa_i : k]$.

We will prove some properties of this bilinear form.

Lemma 9.7. *In Situation 9.3 the symmetric bilinear form (9.6.1) has the following properties*

(1) $(C_i \cdot C_j) \geq 0$ *if* $i \neq j$ *with equality if and only if* $C_i \cap C_j = \emptyset$*,*
(2) $(\sum m_i C_i \cdot C_j) = 0$*,*
(3) *there is no nonempty proper subset* $I \subset \{1, \ldots, n\}$ *such that* $(C_i \cdot C_j) = 0$ *for* $i \in I$*,* $j \notin I$*.*
(4) $(\sum a_i C_i \cdot \sum a_i C_i) \leq 0$ *with equality if and only if there exists a* $q \in \mathbf{Q}$ *such that* $a_i = qm_i$ *for* $i = 1, \ldots, n$*,*

Proof. In the proof of Lemma 9.6 we saw that $(C_i \cdot C_j) = \deg(C_i \cap C_j)$ if $i \neq j$. This is ≥ 0 and > 0 if and only if $C_i \cap C_j \neq \emptyset$. This proves (1).

Proof of (2). This is true because by Lemma 9.1 the invertible sheaf associated to $\sum m_i C_i$ is trivial and the trivial sheaf has degree zero.

Proof of (3). This is expressing the fact that X_k is connected (Lemma 9.4) via the description of the intersection products given in the proof of (1).

Part (4) follows from (1), (2), and (3) by Lemma 2.3. □

Lemma 9.8. *In Situation 9.3 set* $d = \gcd(m_1, \ldots, m_n)$ *and let* $D = \sum (m_i/d) C_i$ *as an effective Cartier divisor. Then* $\mathcal{O}_X(D)$ *has order dividing* d *in* $\text{Pic}(X)$ *and* $\mathcal{C}_{D/X}$ *an invertible* $\mathcal{O}_D$*-module of order dividing* d *in* $\text{Pic}(D)$*.*

Proof. We have

$$\mathcal{O}_X(D)^{\otimes d} = \mathcal{O}_X(dD) = \mathcal{O}_X(X_k) = \mathcal{O}_X$$

by Lemma 9.1. We conclude as $\mathcal{C}_{D/X}$ is the pullback of $\mathcal{O}_X(-D)$. □

Lemma 9.9. *In Situation 9.3 let* $d = \gcd(m_1, \ldots, m_n)$*. Let* $D = \sum (m_i/d) C_i$ *as an effective Cartier divisor. Then there exists a sequence of effective Cartier divisors*

$$(X_k)_{red} = Z_0 \subset Z_1 \subset \ldots \subset Z_m = D$$

such that $Z_j = Z_{j-1} + C_{i_j}$ *for some* $i_j \in \{1, \ldots, n\}$ *for* $j = 1, \ldots, m$ *and such that* $H^0(Z_j, \mathcal{O}_{Z_j})$ *is a field finite over* k *for* $j = 0, \ldots m$*.*

Proof. The reduction $D_{red} = (X_k)_{red} = \sum C_i$ is connected (Lemma 9.4) and proper over k. Hence $H^0(D_{red}, \mathcal{O})$ is a field and a finite extension of k by Varieties, Lemma 9.3. Thus the result for $Z_0 = D_{red} = (X_k)_{red}$ is true. Suppose that we have already constructed

$$(X_k)_{red} = Z_0 \subset Z_1 \subset \ldots \subset Z_t \subset D$$

with $Z_j = Z_{j-1} + C_{i_j}$ for some $i_j \in \{1, \ldots, n\}$ for $j = 1, \ldots, t$ and such that $H^0(Z_j, \mathcal{O}_{Z_j})$ is a field finite over k for $j = 0, \ldots, t$. Write $Z_t = \sum a_i C_i$ with $1 \leq a_i \leq m_i/d$. If $a_i = m_i/d$ for all i, then $Z_t = D$ and the lemma is proved. If not, then $a_i < m_i/d$ for some i and it follows that $(Z_t \cdot Z_t) < 0$ by Lemma 9.7. This means that $(D - Z_t \cdot Z_t) > 0$ because $(D \cdot Z_t) = 0$ by the lemma. Thus we can find an i with $a_i < m_i/d$ such that $(C_i \cdot Z_t) > 0$. Set $Z_{t+1} = Z_t + C_i$ and $i_{t+1} = i$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(-Z_t)|_{C_i} \to \mathcal{O}_{Z_{t+1}} \to \mathcal{O}_{Z_t} \to 0$$

of Divisors, Lemma 14.3. By our choice of i we see that $\mathcal{O}_X(-Z_t)|_{C_i}$ is an invertible sheaf of negative degree on the proper curve C_i, hence it has no nonzero global sections (Varieties, Lemma 44.12). We conclude that $H^0(\mathcal{O}_{Z_{t+1}}) \subset H^0(\mathcal{O}_{Z_t})$ is a field (this is clear but also follows from Algebra, Lemma 36.18) and a finite extension of k. Thus we have extended the sequence. Since the process must stop, for example because $t \leq \sum(m_i/d - 1)$, this finishes the proof. $\square$

Lemma 9.10. *In Situation 9.3 let* $d = \gcd(m_1, \ldots, m_n)$. *Let* $D = \sum(m_i/d)C_i$ *as an effective Cartier divisor on* X. *Then*

$$1 - g_C = d[\kappa : k](1 - g_D)$$

where g_C *is the genus of* C, g_D *is the genus of* D, *and* $\kappa = H^0(D, \mathcal{O}_D)$.

Proof. By Lemma 9.9 we see that κ is a field and a finite extension of k. Since also $H^0(C, \mathcal{O}_C) = K$ we see that the genus of C and D are defined (see Algebraic Curves, Definition 8.1) and we have $g_C = \dim_K H^1(C, \mathcal{O}_C)$ and $g_D = \dim_\kappa H^1(D, \mathcal{O}_D)$. By Derived Categories of Schemes, Lemma 32.2 we have

$$1 - g_C = \chi(C, \mathcal{O}_C) = \chi(X_k, \mathcal{O}_{X_k}) = \dim_k H^0(X_k, \mathcal{O}_{X_k}) - \dim_k H^1(X_k, \mathcal{O}_{X_k})$$

We claim that

$$\chi(X_k, \mathcal{O}_{X_k}) = d\chi(D, \mathcal{O}_D)$$

This will prove the lemma because

$$\chi(D, \mathcal{O}_D) = \dim_k H^0(D, \mathcal{O}_D) - \dim_k H^1(D, \mathcal{O}_D) = [\kappa : k](1 - g_D)$$

Observe that $X_k = dD$ as an effective Cartier divisor. To prove the claim we prove by induction on $1 \leq r \leq d$ that $\chi(rD, \mathcal{O}_{rD}) = r\chi(D, \mathcal{O}_D)$. The base case $r = 1$ is trivial. If $1 \leq r < d$, then we consider the short exact sequence

$$0 \to \mathcal{O}_X(rD)|_D \to \mathcal{O}_{(r+1)D} \to \mathcal{O}_{rD} \to 0$$

of Divisors, Lemma 14.3. By additivity of Euler characteristics (Varieties, Lemma 33.2) it suffices to prove that $\chi(D, \mathcal{O}_X(rD)|_D) = \chi(D, \mathcal{O}_D)$. This is true because $\mathcal{O}_X(rD)|_D$ is a torsion element of $\mathrm{Pic}(D)$ (Lemma 9.8) and because the degree of a line bundle is additive (Varieties, Lemma 44.7) hence zero for torsion invertible sheaves. $\square$

Lemma 9.11. *In Situation 9.3 given a pair of indices* i, j *such that* C_i *and* C_j *are exceptional curves of the first kind and* $C_i \cap C_j \neq \emptyset$, *then* $n = 2$, $m_1 = m_2 = 1$, $C_1 \cong \mathbf{P}^1_k$, $C_2 \cong \mathbf{P}^1_k$, C_1 *and* C_2 *meet in a* k*-rational point, and* C *has genus* 0.

Proof. Choose isomorphisms $C_i = \mathbf{P}^1_{\kappa_i}$ and $C_j = \mathbf{P}^1_{\kappa_j}$. The scheme $C_i \cap C_j$ is a nonempty effective Cartier divisor in both C_i and C_j. Hence

$$(C_i \cdot C_j) = \deg(C_i \cap C_j) \geq \max([\kappa_i : k], [\kappa_j : k])$$

The first equality was shown in the proof of Lemma 9.6. On the other hand, the self intersection $(C_i \cdot C_i)$ is equal to the degree of $\mathcal{O}_X(C_i)$ on C_i which is $-[\kappa_i : k]$ as C_i is an exceptional curve of the first kind. Similarly for C_j. By Lemma 9.7

$$0 \geq (C_i + C_j)^2 = -[\kappa_i : k] + 2(C_i \cdot C_j) - [\kappa_j : k]$$

This implies that $[\kappa_i : k] = \deg(C_i \cap C_j) = [\kappa_j : k]$ and that we have $(C_i + C_j)^2 = 0$. Looking at the lemma again we conclude that $n = 2$, $\{1, 2\} = \{i, j\}$, and $m_1 = m_2$. Moreover, the scheme theoretic intersection $C_i \cap C_j$ consists of a single point p with residue field κ and $\kappa_i \to \kappa \leftarrow \kappa_j$ are isomorphisms. Let $D = C_1 + C_2$ as effective

Cartier divisor on X. Observe that D is the scheme theoretic union of C_1 and C_2 (Divisors, Lemma 13.10) hence we have a short exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_p \to 0$$

by Morphisms, Lemma 4.6. Since we know the cohomology of $C_i \cong \mathbf{P}^1_\kappa$ (Cohomology of Schemes, Lemma 8.1) we conclude from the long exact cohomology sequence that $H^0(D, \mathcal{O}_D) = \kappa$ and $H^1(D, \mathcal{O}_D) = 0$. By Lemma 9.10 we conclude

$$1 - g_C = d[\kappa : k](1 - 0)$$

where $d = m_1 = m_2$. It follows that $g_C = 0$ and $d = m_1 = m_2 = 1$ and $\kappa = k$. □

10. Uniqueness of the minimal model

If the genus of the generic fibre is positive, then minimal models are unique (Lemma 10.1) and consequently have a suitable mapping property (Lemma 10.2).

Lemma 10.1. *Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus > 0. There is a unique minimal model for C.*

Proof. We have already proven the hard part of the lemma which is the existence of a minimal model (whose proof relies on resolution of surface singularities), see Proposition 8.6. To prove uniqueness, suppose that X and Y are two minimal models. By Resolution of Surfaces, Lemma 17.2 there exists a diagram of S-morphisms

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n = Y_m \to \ldots \to Y_1 \to Y_0 = Y$$

where each morphism is a blowup in a closed point. The exceptional fibre of the morphism $X_n \to X_{n-1}$ is an exceptional curve of the first kind E. We claim that E is contracted to a point under the morphism $X_n = Y_m \to Y$. If this is true, then $X_n \to Y$ factors through X_{n-1} by Resolution of Surfaces, Lemma 16.1. In this case the morphism $X_{n-1} \to Y$ is still a sequence of contractions of exceptional curves by Resolution of Surfaces, Lemma 17.1. Hence by induction on n we conclude. (The base case $n = 0$ means that there is a sequence of contractions $X = Y_m \to \ldots \to Y_1 \to Y_0 = Y$ ending with Y. However as X is a minimal model it contains no exceptional curves of the first kind, hence $m = 0$ and $X = Y$.)

Proof of the claim. We will show by induction on m that any exceptional curve of the first kind $E \subset Y_m$ is mapped to a point by the morphism $Y_m \to Y$. If $m = 0$ this is clear because Y is a minimal model. If $m > 0$, then either $Y_m \to Y_{m-1}$ contracts E (and we're done) or the exceptional fibre $E' \subset Y_m$ of $Y_m \to Y_{m-1}$ is a second exceptional curve of the first kind. Since both E and E' are irreducible components of the special fibre and since $g_C > 0$ by assumption, we conclude that $E \cap E' = \emptyset$ by Lemma 9.11. Then the image of E in Y_{m-1} is an exceptional curve of the first kind (this is clear because the morphism $Y_m \to Y_{m-1}$ is an isomorphism in a neighbourhood of E). By induction we see that $Y_{m-1} \to Y$ contracts this curve and the proof is complete. □

Lemma 10.2. *Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus > 0. Let X be the minimal model for C (Lemma 10.1). Let Y be a regular proper model for C. Then there is a unique morphism of models $Y \to X$ which is a sequence of contractions of exceptional curves of the first kind.*

Proof. The existence and properties of the morphism $X \to Y$ follows immediately from Lemma 8.5 and the uniqueness of the minimal model. The morphism $Y \to X$ is unique because $C \subset Y$ is scheme theoretically dense and X is separated (see Morphisms, Lemma 7.10). $\square$

Example 10.3. If the genus of C is 0, then minimal models are indeed nonunique. Namely, consider the closed subscheme

$$X \subset \mathbf{P}^2_R$$

defined by $T_1T_2 - \pi T_0^2 = 0$. More precisely X is defined as $\text{Proj}(R[T_0, T_1, T_2]/(T_1T_2 - \pi T_0^2))$. Then the special fibre X_k is a union of two exceptional curves C_1, C_2 both isomorphic to $\mathbf{P}^1_k$ (exactly as in Lemma 9.11). Projection from $(0 : 1 : 0)$ defines a morphism $X \to \mathbf{P}^1_R$ contracting C_2 and inducing an isomorphism of C_1 with the special fiber of $\mathbf{P}^1_R$. Projection from $(0 : 0 : 1)$ defines a morphism $X \to \mathbf{P}^1_R$ contracting C_1 and inducing an isomorphism of C_2 with the special fiber of $\mathbf{P}^1_R$. More precisely, these morphisms correspond to the graded R-algebra maps

$$R[T_0, T_1] \longrightarrow R[T_0, T_1, T_2]/(T_1T_2 - \pi T_0^2) \longleftarrow R[T_0, T_2]$$

In Lemma 12.4 we will study this phenomenon.

11. A formula for the genus

There is one more restriction on the combinatorial structure coming from a proper regular model.

Lemma 11.1. *In Situation 9.3 suppose we have an effective Cartier divisors $D, D' \subset X$ such that $D' = D + C_i$ for some $i \in \{1, \ldots, n\}$ and $D' \subset X_k$. Then*

$$\chi(X_k, \mathcal{O}_{D'}) - \chi(X_k, \mathcal{O}_D) = \chi(X_k, \mathcal{O}_X(-D)|_{C_i}) = -(D \cdot C_i) + \chi(C_i, \mathcal{O}_{C_i})$$

Proof. The second equality follows from the definition of the bilinear form $(\cdot)$ in (9.6.1) and Lemma 9.6. To see the first equality we distinguish two cases. Namely, if $C_i \not\subset D$, then D' is the scheme theoretic union of D and C_i (by Divisors, Lemma 13.10) and we get a short exact sequence

$$0 \to \mathcal{O}_{D'} \to \mathcal{O}_D \times \mathcal{O}_{C_i} \to \mathcal{O}_{D \cap C_i} \to 0$$

by Morphisms, Lemma 4.6. Since we also have an exact sequence

$$0 \to \mathcal{O}_X(-D)|_{C_i} \to \mathcal{O}_{C_i} \to \mathcal{O}_{D \cap C_i} \to 0$$

(Divisors, Remark 14.11) we conclude that the claim holds by additivity of euler characteristics (Varieties, Lemma 33.2). On the other hand, if $C_i \subset D$ then we get an exact sequence

$$0 \to \mathcal{O}_X(-D)|_{C_i} \to \mathcal{O}_{D'} \to \mathcal{O}_D \to 0$$

by Divisors, Lemma 14.3 and we immediately see the lemma holds. $\square$

Lemma 11.2. *In Situation 9.3 we have*

$$g_C = 1 + \sum\nolimits_{i=1,\ldots,n} m_i \left([\kappa_i : k](g_i - 1) - \frac{1}{2}(C_i \cdot C_i) \right)$$

where $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$, g_i is the genus of C_i, and g_C is the genus of C.

Proof. Our basic tool will be Derived Categories of Schemes, Lemma 32.2 which shows that

$$1 - g_C = \chi(C, \mathcal{O}_C) = \chi(X_k, \mathcal{O}_{X_k})$$

Choose a sequence of effective Cartier divisors

$$X_k = D_m \supset D_{m-1} \supset \ldots \supset D_1 \supset D_0 = \emptyset$$

such that $D_{j+1} = D_j + C_{i_j}$ for each j. (It is clear that we can choose such a sequence by decreasing one nonzero multiplicity of D_{j+1} one step at a time.) Applying Lemma 11.1 starting with $\chi(\mathcal{O}_{D_0}) = 0$ we get

$$\begin{aligned}
1 - g_C &= \chi(X_k, \mathcal{O}_{X_k}) \\
&= \sum\nolimits_j \left(-(D_j \cdot C_{i_j}) + \chi(C_{i_j}, \mathcal{O}_{C_{i_j}}) \right) \\
&= -\sum\nolimits_j (C_{i_1} + C_{i_2} + \ldots + C_{i_{j-1}} \cdot C_{i_j}) + \sum\nolimits_j \chi(C_{i_j}, \mathcal{O}_{C_{i_j}}) \\
&= -\frac{1}{2} \sum\nolimits_{j \neq j'} (C_{i_{j'}} \cdot C_{i_j}) + \sum m_i \chi(C_i, \mathcal{O}_{C_i}) \\
&= \frac{1}{2} \sum m_i (C_i \cdot C_i) + \sum m_i \chi(C_i, \mathcal{O}_{C_i})
\end{aligned}$$

Perhaps the last equality deserves some explanation. Namely, since $\sum_j C_{i_j} = \sum m_i C_i$ we have $(\sum_j C_{i_j} \cdot \sum_j C_{i_j}) = 0$ by Lemma 9.7. Thus we see that

$$0 = \sum\nolimits_{j \neq j'} (C_{i_{j'}} \cdot C_{i_j}) + \sum m_i (C_i \cdot C_i)$$

by splitting this product into "nondiagonal" and "diagonal" terms. Note that κ_i is a field finite over k by Varieties, Lemma 26.2. Hence the genus of C_i is defined and we have $\chi(C_i, \mathcal{O}_{C_i}) = [\kappa_i : k](1 - g_i)$. Putting everything together and rearranging terms we get

$$g_C = -\frac{1}{2} \sum m_i (C_i \cdot C_i) + \sum m_i [\kappa_i : k](g_i - 1) + 1$$

which is what the lemma says too. $\square$

Lemma 11.3. *In Situation 9.3 with $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ and g_i the genus of C_i the data*

$$n, m_i, (C_i \cdot C_j), [\kappa_i : k], g_i$$

is a numerical type of genus equal to the genus of C.

Proof. (In the proof of Lemma 11.2 we have seen that the quantities used in the statement of the lemma are well defined.) We have to verify the conditions (1) – (5) of Definition 3.1.

Condition (1) is immediate.

Condition (2). Symmetry of the matrix $(C_i \cdot C_j)$ follows from Equation (9.6.1) and Lemma 9.6. Nonnegativity of $(C_i \cdot C_j)$ for $i \neq j$ is part (1) of Lemma 9.7.

Condition (3) is part (3) of Lemma 9.7.

Condition (4) is part (2) of Lemma 9.7.

Condition (5) follows from the fact that $(C_i \cdot C_j)$ is the degree of an invertible module on C_i which is divisible by $[\kappa_i : k]$, see Varieties, Lemma 44.10.

The genus formula proved in Lemma 11.2 tells us that the numerical type has the genus as stated, see Definition 3.4. □

Definition 11.4. In Situation 9.3 the *numerical type associated to X* is the numerical type described in Lemma 11.3.

Now we match minimality of the model with minimality of the type.

Lemma 11.5. *In Situation 9.3. The following are equivalent*

1. *X is a minimal model, and*
2. *the numerical type associated to X is minimal.*

Proof. If the numerical type is minimal, then there is no i with $g_i = 0$ and $(C_i \cdot C_i) = -[\kappa_i : k]$, see Definition 3.12. Certainly, this implies that none of the curves C_i are exceptional curves of the first kind.

Conversely, suppose that the numerical type is not minimal. Then there exists an i such that $g_i = 0$ and $(C_i \cdot C_i) = -[\kappa_i : k]$. We claim this implies that C_i is an exceptional curve of the first kind. Namely, the invertible sheaf $\mathcal{O}_X(-C_i)|_{C_i}$ has degree $-(C_i \cdot C_i) = [\kappa_i : k]$ when C_i is viewed as a proper curve over k, hence has degree 1 when C_i is viewed as a proper curve over κ_i. Applying Algebraic Curves, Proposition 10.4 we conclude that $C_i \cong \mathbf{P}^1_{\kappa_i}$ as schemes over κ_i. Since the Picard group of $\mathbf{P}^1$ over a field is $\mathbf{Z}$, we see that the normal sheaf of C_i in X is isomorphic to $\mathcal{O}_{\mathbf{P}_{\kappa_i}}(-1)$ and the proof is complete. □

Remark 11.6. Not every numerical type comes from a model for the silly reason that there exist numerical types whose genus is negative. There exist a minimal numerical types of positive genus which are not the numerical type associated to a model (over some dvr) of a smooth projective geometrically irreducible curve (over the fraction field of the dvr). A simple example is $n = 1$, $m_1 = 1$, $a_{11} = 0$, $w_1 = 6$, $g_1 = 1$. Namely, in this case the special fibre X_k would not be geometrically connected because it would live over an extension κ of k of degree 6. This is a contradiction with the fact that the generic fibre is geometrically connected (see More on Morphisms, Lemma 53.6). Similarly, $n = 2$, $m_1 = m_2 = 1$, $-a_{11} = -a_{22} = a_{12} = a_{21} = 6$, $w_1 = w_2 = 6$, $g_1 = g_2 = 1$ would be an example for the same reason (details omitted). But if the gcd of the w_i is 1 we do not have an example.

Lemma 11.7. *In Situation 9.3 assume C has a K-rational point. Then*

1. *X_k has a k-rational point x which is a smooth point of X_k over k,*
2. *if $x \in C_i$, then $H^0(C_i, \mathcal{O}_{C_i}) = k$ and $m_i = 1$, and*
3. *$H^0(X_k, \mathcal{O}_{X_k}) = k$ and X_k has genus equal to the genus of C.*

Proof. Since $X \to \text{Spec}(R)$ is proper, the K-rational point extends to a morphism $a : \text{Spec}(R) \to X$ by the valuative criterion of properness (Morphisms, Lemma 42.1). Let $x \in X$ be the image under a of the closed point of $\text{Spec}(R)$. Then a corresponds to an R-algebra homomorphism $\psi : \mathcal{O}_{X,x} \to R$ (see Schemes, Section 13). It follows that $\pi \notin \mathfrak{m}_x^2$ (since the image of π in R is not in $\mathfrak{m}_R^2$). Hence $\mathcal{O}_{X_k,x} = \mathcal{O}_{X,x}/\pi\mathcal{O}_{X,x}$ is regular (Algebra, Lemma 106.3). Then $X_k \to \text{Spec}(k)$ is smooth at x by Algebra, Lemma 140.5. It follows that x is contained in a unique irreducible component C_i of X_k, that $\mathcal{O}_{C_i,x} = \mathcal{O}_{X_k,x}$, and that $m_i = 1$. The fact that C_i has a k-rational point implies that the field $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ (Varieties, Lemma 26.2) is equal to k. This proves (1). We have $H^0(X_k, \mathcal{O}_{X_k}) = k$ because

$H^0(X_k, \mathcal{O}_{X_k})$ is a field extension of k (Lemma 9.9) which maps to $H^0(C_i, \mathcal{O}_{C_i}) = k$. The genus equality follows from Lemma 9.10. $\square$

Lemma 11.8. *In Situation 9.3 assume X is a minimal model,* $\gcd(m_1, \ldots, m_n) = 1$*, and $H^0((X_k)_{red}, \mathcal{O}) = k$. Then the map*

$$H^1(X_k, \mathcal{O}_{X_k}) \to H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}})$$

is surjective and has a nontrivial kernel as soon as $(X_k)_{red} \neq X_k$.

Proof. By vanishing of cohomology in degrees ≥ 2 over X_k (Cohomology, Proposition 20.7) any surjection of abelian sheaves on X_k induces a surjection on H^1. Consider the sequence

$$(X_k)_{red} = Z_0 \subset Z_1 \subset \ldots \subset Z_m = X_k$$

of Lemma 9.9. Since the field maps $H^0(Z_j, \mathcal{O}_{Z_j}) \to H^0((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) = k$ are injective we conclude that $H^0(Z_j, \mathcal{O}_{Z_j}) = k$ for $j = 0, \ldots, m$. It follows that $H^0(X_k, \mathcal{O}_{X_k}) \to H^0(Z_{m-1}, \mathcal{O}_{Z_{m-1}})$ is surjective. Let $C = C_{i_m}$. Then $X_k = Z_{m-1} + C$. Let $\mathcal{L} = \mathcal{O}_X(-Z_{m-1})|_C$. Then $\mathcal{L}$ is an invertible $\mathcal{O}_C$-module. As in the proof of Lemma 9.9 there is an exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_{X_k} \to \mathcal{O}_{Z_{m-1}} \to 0$$

of coherent sheaves on X_k. We conclude that we get a short exact sequence

$$0 \to H^1(C, \mathcal{L}) \to H^1(X_k, \mathcal{O}_{X_k}) \to H^1(Z_{m-1}, \mathcal{O}_{Z_{m-1}}) \to 0$$

The degree of $\mathcal{L}$ on C over k is

$$(C \cdot -Z_{m-1}) = (C \cdot C - X_k) = (C \cdot C)$$

Set $\kappa = H^0(C, \mathcal{O}_C)$ and $w = [\kappa : k]$. By definition of the degree of an invertible sheaf we see that

$$\chi(C, \mathcal{L}) = \chi(C, \mathcal{O}_C) + (C \cdot C) = w(1 - g_C) + (C \cdot C)$$

where g_C is the genus of C. This expression is < 0 as X is minimal and hence C is not an exceptional curve of the first kind (see proof of Lemma 11.5). Thus $\dim_k H^1(C, \mathcal{L}) > 0$ which finishes the proof. $\square$

Lemma 11.9. *In Situation 9.3 assume X_k has a k-rational point x which is a smooth point of $X_k \to \mathrm{Spec}(k)$. Then*

$$\dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) \geq g_{top} + g_{geom}(X_k/k)$$

where g_{geom} is as in Algebraic Curves, Section 18 and g_{top} is the topological genus (Definition 3.11) of the numerical type associated to X_k (Definition 11.4).

Proof. We are going to prove the inequality

$$\dim_k H^1(D, \mathcal{O}_D) \geq g_{top}(D) + g_{geom}(D/k)$$

for all connected reduced effective Cartier divisors $D \subset (X_k)_{red}$ containing x by induction on the number of irreducible components of D. Here $g_{top}(D) = 1 - m + e$ where m is the number of irreducible components of D and e is the number of unordered pairs of components of D which meet.

Base case: D has one irreducible component. Then $D = C_i$ is the unique irreducible component containing x. In this case $\dim_k H^1(D, \mathcal{O}_D) = g_i$ and $g_{top}(D) = 0$. Since C_i has a k-rational smooth point it is geometrically integral (Varieties, Lemma

25.10). It follows that g_i is the genus of $C_{i,\overline{k}}$ (Algebraic Curves, Lemma 8.2). It also follows that $g_{geom}(D/k)$ is the genus of the normalization $C^\nu_{i,\overline{k}}$ of $C_{i,\overline{k}}$. Applying Algebraic Curves, Lemma 18.4 to the normalization morphism $C^\nu_{i,\overline{k}} \to C_{i,\overline{k}}$ we get

$$\text{genus of } C_{i,\overline{k}} \geq \text{genus of } C^\nu_{i,\overline{k}} \tag{11.9.1}$$

Combining the above we conclude that $\dim_k H^1(D, \mathcal{O}_D) \geq g_{top}(D) + g_{geom}(D/k)$ in this case.

Induction step. Suppose we have D with more than 1 irreducible component. Then we can write $D = C_i + D'$ where $x \in D'$ and D' is still connected. This is an exercise in graph theory we leave to the reader (hint: let C_i be the component of D which is farthest from x). We compute how the invariants change. As $x \in D'$ we have $H^0(D, \mathcal{O}_D) = H^0(D', \mathcal{O}_{D'}) = k$. Looking at the short exact sequence of sheaves

$$0 \to \mathcal{O}_D \to \mathcal{O}_{C_i} \oplus \mathcal{O}_{D'} \to \mathcal{O}_{C_i \cap D'} \to 0$$

(Morphisms, Lemma 4.6) and using additivity of euler characteristics we find

$$\begin{aligned} \dim_k H^1(D, \mathcal{O}_D) - \dim_k H^1(D', \mathcal{O}_{D'}) &= -\chi(\mathcal{O}_{C_i}) + \chi(\mathcal{O}_{C_i \cap D'}) \\ &= w_i(g_i - 1) + \sum\nolimits_{C_j \subset D'} a_{ij} \end{aligned}$$

Here as in Lemma 11.3 we set $w_i = [\kappa_i : k]$, $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$, g_i is the genus of C_i, and $a_{ij} = (C_i \cdot C_j)$. We have

$$g_{top}(D) - g_{top}(D') = -1 + \sum\nolimits_{C_j \subset D' \text{ meeting } C_i} 1$$

We have

$$g_{geom}(D/k) - g_{geom}(D'/k) = g_{geom}(C_i/k)$$

by Algebraic Curves, Lemma 18.1. Combining these with our induction hypothesis, we conclude that it suffices to show that

$$w_i g_i - g_{geom}(C_i/k) + \sum\nolimits_{C_j \subset D' \text{ meets } C_i} (a_{ij} - 1) - (w_i - 1)$$

is nonnegative. In fact, we have

$$w_i g_i \geq [\kappa_i : k]_s g_i \geq g_{geom}(C_i/k) \tag{11.9.2}$$

The second inequality by Algebraic Curves, Lemma 18.5. On the other hand, since w_i divides a_{ij} (Varieties, Lemma 44.10) it is clear that

$$\sum\nolimits_{C_j \subset D' \text{ meets } C_i} (a_{ij} - 1) - (w_i - 1) \geq 0 \tag{11.9.3}$$

because there is at least one $C_j \subset D'$ which meets C_i. □

Lemma 11.10. *If equality holds in Lemma 11.9 then*

(1) *the unique irreducible component of X_k containing x is a smooth projective geometrically irreducible curve over k,*
(2) *if $C \subset X_k$ is another irreducible component, then $\kappa = H^0(C, \mathcal{O}_C)$ is a finite separable extension of k, C has a κ-rational point, and C is smooth over κ*

Proof. Looking over the proof of Lemma 11.9 we see that in order to get equality, the inequalities (11.9.1), (11.9.2), and (11.9.3) have to be equalities.

Let C_i be the irreducible component containing x. Equality in (11.9.1) shows via Algebraic Curves, Lemma 18.4 that $C^\nu_{i,\overline{k}} \to C_{i,\overline{k}}$ is an isomorphism. Hence $C_{i,\overline{k}}$ is smooth and part (1) holds.

Next, let $C_i \subset X_k$ be another irreducible component. Then we may assume we have $D = D' + C_i$ as in the induction step in the proof of Lemma 11.9. Equality in (11.9.2) immediately implies that κ_i/k is finite separable. Equality in (11.9.3) implies either $a_{ij} = 1$ for some j or that there is a unique $C_j \subset D'$ meeting C_i and $a_{ij} = w_i$. In both cases we find that C_i has a κ_i-rational point c and $c = C_i \cap C_j$ scheme theoretically. Since $\mathcal{O}_{X,c}$ is a regular local ring, this implies that the local equations of C_i and C_j form a regular system of parameters in the local ring $\mathcal{O}_{X,c}$. Then $\mathcal{O}_{C_i,c}$ is regular by (Algebra, Lemma 106.3). We conclude that $C_i \to \text{Spec}(\kappa_i)$ is smooth at c (Algebra, Lemma 140.5). It follows that C_i is geometrically integral over κ_i (Varieties, Lemma 25.10). To finish we have to show that C_i is smooth over κ_i. Observe that

$$C_{i,\overline{k}} = C_i \times_{\text{Spec}(k)} \text{Spec}(\overline{k}) = \coprod\nolimits_{\kappa_i \to \overline{k}} C_i \times_{\text{Spec}(\kappa_i)} \text{Spec}(\overline{k})$$

where there are $[\kappa_i : k]$-summands. Thus if C_i is not smooth over κ_i, then each of these curves is not smooth, then these curves are not normal and the normalization morphism drops the genus (Algebraic Curves, Lemma 18.4) which is disallowed because it would drop the geometric genus of C_i/k contradicting $[\kappa_i : k]g_i = g_{geom}(C_i/k)$. $\square$

12. Blowing down exceptional curves

The following lemma tells us what happens with the intersection numbers when we contract an exceptional curve of the first kind in a regular proper model. We put this here mostly to compare with the numerical contractions introduced in Lemma 3.9. We will compare the geometric and numerical contractions in Remark 12.3.

Lemma 12.1. *In Situation 9.3 assume that C_n is an exceptional curve of the first kind. Let $f : X \to X'$ be the contraction of C_n. Let $C'_i = f(C_i)$. Write $X'_k = \sum m'_i C'_i$. Then X', C'_i, $i = 1, \ldots, n' = n-1$, and $m'_i = m_i$ is as in Situation 9.3 and we have*

1. *for $i, j < n$ we have $(C'_i \cdot C'_j) = (C_i \cdot C_j) - (C_i \cdot C_n)(C_j \cdot C_n)/(C_n \cdot C_n)$,*
2. *for $i < n$ if $C_i \cap C_n \neq \emptyset$, then there are maps $\kappa_i \leftarrow \kappa'_i \to \kappa_n$.*

Here $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ and $\kappa'_i = H^0(C'_i, \mathcal{O}_{C'_i})$.

Proof. By Resolution of Surfaces, Lemma 16.8 we can contract C_n by a morphism $f : X \to X'$ such that X' is regular and is projective over R. Thus we see that X' is as in Situation 9.3. Let $x \in X'$ be the image of C_n. Since f defines an isomorphism $X \setminus C_n \to X' \setminus \{x\}$ it is clear that $m'_i = m_i$ for $i < n$.

Part (2) of the lemma is immediately clear from the existence of the morphisms $C_i \to C'_i$ and $C_n \to x \to C'_i$.

By Divisors, Lemma 32.11 the pullback $f^{-1}C'_i$ is defined. By Divisors, Lemma 15.11 we see that $f^{-1}C'_i = C_i + e_i C_n$ for some $e_i \geq 0$. Since $\mathcal{O}_X(C_i + e_i C_n) = \mathcal{O}_X(f^{-1}C'_i) = f^*\mathcal{O}_{X'}(C'_i)$ (Divisors, Lemma 14.5) and since the pullback of an invertible sheaf restricts to the trivial invertible sheaf on C_n we see that

$$0 = \deg_{C_n}(\mathcal{O}_X(C_i + e_i C_n)) = (C_i + e_i C_n \cdot C_n) = (C_i \cdot C_n) + e_i(C_n \cdot C_n)$$

As $f_j = f|_{C_j} : C_j \to C_j$ is a proper birational morphism of proper curves over k, we see that $\deg_{C'_j}(\mathcal{O}_{X'}(C'_i)|_{C'_j})$ is the same as $\deg_{C_j}(f_j^*\mathcal{O}_{X'}(C'_i)|_{C'_j})$ (Varieties, Lemma 44.4). Looking at the commutative diagram

$$\begin{tikzcd} C_j \arrow[r] \arrow[d, "f_j"'] & X \arrow[d, "f"] \\ C'_j \arrow[r] & X' \end{tikzcd}$$

and using Divisors, Lemma 14.5 we see that

$$(C'_i \cdot C'_j) = \deg_{C'_j}(\mathcal{O}_{X'}(C'_i)|_{C'_j}) = \deg_{C_j}(\mathcal{O}_X(C_i + e_iC_n)) = (C_i + e_iC_n \cdot C_j)$$

Plugging in the formula for e_i found above we see that (1) holds. $\square$

Remark 12.2. In the situation of Lemma 12.1 we can also say exactly how the genus g_i of C_i and the genus g'_i of C'_i are related. The formula is

$$g'_i = \frac{w_i}{w'_i}(g_i - 1) + 1 + \frac{(C_i \cdot C_n)^2 - w_n(C_i \cdot C_n)}{2w'_iw_n}$$

where $w_i = [\kappa_i : k]$, $w_n = [\kappa_n : k]$, and $w'_i = [\kappa'_i : k]$. To prove this we consider the short exact sequence

$$0 \to \mathcal{O}_{X'}(-C'_i) \to \mathcal{O}_{X'} \to \mathcal{O}_{C'_i} \to 0$$

and its pullback to X which reads

$$0 \to \mathcal{O}_X(-C'_i - e_iC_n) \to \mathcal{O}_X \to \mathcal{O}_{C_i+e_iC_n} \to 0$$

with e_i as in the proof of Lemma 12.1. Since $Rf_*f^*\mathcal{L} = \mathcal{L}$ for any invertible module $\mathcal{L}$ on X' (details omitted), we conclude that

$$Rf_*\mathcal{O}_{C_i+e_iC_n} = \mathcal{O}_{C'_i}$$

as complexes of coherent sheaves on X'_k. Hence both sides have the same Euler characteristic and this agrees with the Euler characteristic of $\mathcal{O}_{C_i+e_iC_n}$ on X_k. Using the exact sequence

$$0 \to \mathcal{O}_{C_i+e_iC_n} \to \mathcal{O}_{C_i} \oplus \mathcal{O}_{e_iC_n} \to \mathcal{O}_{C_i \cap e_iC_n} \to 0$$

and further filtering $\mathcal{O}_{e_iC_n}$ (details omitted) we find

$$\chi(\mathcal{O}_{C'_i}) = \chi(\mathcal{O}_{C_i}) - \binom{e_i+1}{2}(C_n \cdot C_n) - e_i(C_i \cdot C_n)$$

Since $e_i = -(C_i \cdot C_n)/(C_n \cdot C_n)$ and $(C_n \cdot C_n) = -w_n$ this leads to the formula stated at the start of this remark. If we ever need this we will formulate this as a lemma and provide a detailed proof.

Remark 12.3. Let $f : X \to X'$ be as in Lemma 12.1. Let n, m_i, a_{ij}, w_i, g_i be the numerical type associated to X and let $n', m'_i, a'_{ij}, w'_i, g'_i$ be the numerical type associated to X'. It is clear from Lemma 12.1 and Remark 12.2 that this agrees with the contraction of numerical types in Lemma 3.9 except for the value of w'_i. In the geometric situation w'_i is some positive integer dividing both w_i and w_n. In the numerical case we chose w'_i to be the largest possible integer dividing w_i such that g'_i (as given by the formula) is an integer. This works well in the numerical setting in that it helps compare the Picard groups of the numerical types, see Lemma 4.4

(although only injectivity is every used in the following and this injectivity works as well for smaller w'_i).

Lemma 12.4. *Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus 0. If there is more than one minimal model for C, then the special fibre of every minimal model is isomorphic to $\mathbf{P}^1_k$.*

This lemma can be improved to say that the birational transformation between two nonisomorphic minimal models can be factored as a sequence of elementary transformations as in Example 10.3. If we ever need this, we will precisely formulate and prove this here.

Proof. Let X be some minimal model of C. The numerical type associated to X has genus 0 and is minimal (Definition 11.4 and Lemma 11.5). Hence by Lemma 6.1 we see that X_k is reduced, irreducible, has $H^0(X_k, \mathcal{O}_{X_k}) = k$, and has genus 0. Let Y be a second minimal model for C which is not isomorphic to X. By Resolution of Surfaces, Lemma 17.2 there exists a diagram of S-morphisms

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n = Y_m \to \ldots \to Y_1 \to Y_0 = Y$$

where each morphism is a blowup in a closed point. We will prove the lemma by induction on m. The base case is $m = 0$; it is true in this case because we assumed that Y is minimal hence this would mean $n = 0$, but X is not isomorphic to Y, so this does not happen, i.e., there is nothing to check.

Before we continue, note that $n + 1 = m + 1$ is equal to the number of irreducible components of the special fibre of $X_n = Y_m$ because both X_k and Y_k are irreducible. Another observation we will use below is that if $X' \to X''$ is a morphism of regular proper models for C, then $X' \to X''$ is an isomorphism over an open set of X'' whose complement is a finite set of closed points of the special fibre of X'', see Varieties, Lemma 17.3. In fact, any such $X' \to X''$ is a sequence of blowing ups in closed points (Resolution of Surfaces, Lemma 17.1) and the number of blowups is the difference in the number of irreducible components of the special fibres of X' and X''.

Let $E_i \subset Y_i$, $m \geq i \geq 1$ be the curve which is contracted by the morphism $Y_i \to Y_{i-1}$. Let i be the biggest index such that E_i has multiplicity > 1 in the special fibre of Y_i. Then the further blowups $Y_m \to \ldots \to Y_{i+1} \to Y_i$ are isomorphisms over E_i since otherwise E_j for some $j > i$ would have multiplicity > 1. Let $E \subset Y_m$ be the inverse image of E_i. By what we just said $E \subset Y_m$ is an exceptional curve of the first kind. Let $Y_m \to Y'$ be the contraction of E (which exists by Resolution of Surfaces, Lemma 16.9). The morphism $Y_m \to X$ has to contract E, because X_k is reduced. Hence there are morphisms $Y' \to Y$ and $Y' \to X$ (by Resolution of Surfaces, Lemma 16.1) which are compositions of at most $n-1 = m-1$ contractions of exceptional curves (see discussion above). We win by induction on m. Upshot: we may assume that the special fibres of all of the curves X_i and Y_i are reduced.

Since the fibres of X_i and Y_i are reduced, it has to be the case that the blowups $X_i \to X_{i-1}$ and $Y_i \to Y_{i-1}$ happen in closed points which are regular points of the special fibres. Namely, if X'' is a regular model for C and if $x \in X''$ is a closed point of the special fibre, and $\pi \in \mathfrak{m}_x^2$, then the exceptional fibre E of the blowup $X' \to X''$ at x has multiplicity at least 2 in the special fibre of X' (local computation omitted). Hence $\mathcal{O}_{X''_k, x} = \mathcal{O}_{X'', x}/\pi$ is regular (Algebra, Lemma 106.3) as claimed. In particular x is a Cartier divisor on the unique irreducible component Z' of X''_k it lies on (Varieties, Lemma 43.8). It follows that the strict transform $Z \subset X'$ of Z' maps isomorphically to Z' (use Divisors, Lemmas 33.2 and 32.7). In other words, if an irreducible component Z of X_i is not contracted under the map $X_i \to X_j$ $(i > j)$ then it maps isomorphically to its image.

Now we are ready to prove the lemma. Let $E \subset Y_m$ be the exceptional curve of the first kind which is contracted by the morphism $Y_m \to Y_{m-1}$. If E is contracted by the morphism $Y_m = X_n \to X$, then there is a factorization $Y_{m-1} \to X$ (Resolution of Surfaces, Lemma 16.1) and moreover $Y_{m-1} \to X$ is a sequence of blowups in closed points (Resolution of Surfaces, Lemma 17.1). In this case we lower m and we win by induction. Finally, assume that E is not contracted by the morphism $Y_m \to X$. Then $E \to X_k$ is surjective as X_k is irreducible and by the above this means it is an isomorphism. Hence X_k is isomorphic to a projective line as desired. □

13. Picard groups of models

Assume $R, K, k, \pi, C, X, n, C_1, \ldots, C_n, m_1, \ldots, m_n$ are as in Situation 9.3. In Lemma 9.5 we found an exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z}^{\oplus n} \to \mathrm{Pic}(X) \to \mathrm{Pic}(C) \to 0$$

We want to use this sequence to study the ℓ-torsion in the Picard groups for suitable primes ℓ.

Lemma 13.1. *In Situation 9.3 let $d = \gcd(m_1, \ldots, m_n)$. If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module which*

1. *restricts to the trivial invertible module on C, and*
2. *has degree 0 on each C_i,*

then $\mathcal{L}^{\otimes d} \cong \mathcal{O}_X$.

Proof. By Lemma 9.5 we have $\mathcal{L} \cong \mathcal{O}_X(\sum a_i C_i)$ for some $a_i \in \mathbf{Z}$. The degree of $\mathcal{L}|_{C_j}$ is $\sum_j a_j (C_i \cdot C_j)$. In particular $(\sum a_i C_i \cdot \sum a_i C_i) = 0$. Hence we see from Lemma 9.7 that $(a_1, \ldots, a_n) = q(m_1, \ldots, m_n)$ for some $q \in \mathbf{Q}$. Thus $\mathcal{L} = \mathcal{O}_X(lD)$ for some $l \in \mathbf{Z}$ where $D = \sum (m_i/d) C_i$ is as in Lemma 9.8 and we conclude. □

Lemma 13.2. *In Situation 9.3 let T be the numerical type associated to X. There exists a canonical map*

$$\mathrm{Pic}(C) \to \mathrm{Pic}(T)$$

whose kernel is exactly those invertible modules on C which are the restriction of invertible modules $\mathcal{L}$ on X with $\deg_{C_i}(\mathcal{L}|_{C_i}) = 0$ for $i = 1, \ldots, n$.

Proof. Recall that $w_i = [\kappa_i : k]$ where $\kappa_i = H^0(C_i, \mathcal{O}_{C_i})$ and recall that the degree of any invertible module on C_i is divisible by w_i (Varieties, Lemma 44.10). Thus we can consider the map

$$\frac{\deg}{w} : \mathrm{Pic}(X) \to \mathbf{Z}^{\oplus n}, \quad \mathcal{L} \mapsto (\frac{\deg(\mathcal{L}|_{C_1})}{w_1}, \ldots, \frac{\deg(\mathcal{L}|_{C_n})}{w_n})$$

The image of $\mathcal{O}_X(C_j)$ under this map is

$$((C_j \cdot C_1)/w_1, \ldots, (C_j \cdot C_n)/w_n) = (a_{1j}/w_1, \ldots, a_{nj}/w_n)$$

which is exactly the image of the jth basis vector under the map $(a_{ij}/w_i) : \mathbf{Z}^{\oplus n} \to \mathbf{Z}^{\oplus n}$ defining the Picard group of T, see Definition 4.1. Thus the canonical map of the lemma comes from the commutative diagram

$$\begin{tikzcd} \mathbf{Z}^{\oplus n} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "\frac{\deg}{w}"] & \mathrm{Pic}(C) \arrow[r] \arrow[d] & 0 \\ \mathbf{Z}^{\oplus n} \arrow[r, "(a_{ij}/w_i)"] & \mathbf{Z}^{\oplus n} \arrow[r] & \mathrm{Pic}(T) \arrow[r] & 0 \end{tikzcd}$$

with exact rows (top row by Lemma 9.5). The description of the kernel is clear. ☐

Lemma 13.3. *In Situation 9.3 let $d = \gcd(m_1, \ldots, m_n)$ and let T be the numerical type associated to X. Let $h \geq 1$ be an integer prime to d. There exists an exact sequence*

$$0 \to \mathrm{Pic}(X)[h] \to \mathrm{Pic}(C)[h] \to \mathrm{Pic}(T)[h]$$

Proof. Taking h-torsion in the exact sequence of Lemma 9.5 we obtain the exactness of $0 \to \mathrm{Pic}(X)[h] \to \mathrm{Pic}(C)[h]$ because h is prime to d. Using the map of Lemma 13.2 we get a map $\mathrm{Pic}(C)[h] \to \mathrm{Pic}(T)[h]$ which annihilates elements of $\mathrm{Pic}(X)[h]$. Conversely, if $\xi \in \mathrm{Pic}(C)[h]$ maps to zero in $\mathrm{Pic}(T)[h]$, then we can find an invertible $\mathcal{O}_X$-module $\mathcal{L}$ with $\deg(\mathcal{L}|_{C_i}) = 0$ for all i whose restriction to C is ξ. Then $\mathcal{L}^{\otimes h}$ is d-torsion by Lemma 13.1. Let d' be an integer such that $dd' \equiv 1 \bmod h$. Such an integer exists because h and d are coprime. Then $\mathcal{L}^{\otimes dd'}$ is an h-torsion invertible sheaf on X whose restriction to C is ξ. ☐

Lemma 13.4. *In Situation 9.3 let h be an integer prime to the characteristic of k. Then the map*

$$\mathrm{Pic}(X)[h] \longrightarrow \mathrm{Pic}((X_k)_{red})[h]$$

is injective.

Proof. Observe that $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)$ is a finite order thickening of $(X_k)_{red}$ (this follows for example from Cohomology of Schemes, Lemma 10.2). Thus the canonical map $\mathrm{Pic}(X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)) \to \mathrm{Pic}((X_k)_{red})$ identifies h torsion by More on Morphisms, Lemma 4.2 and our assumption on h. Thus if $\mathcal{L}$ is an h-torsion invertible sheaf on X which restricts to the trivial sheaf on $(X_k)_{red}$ then $\mathcal{L}$ restricts to the trivial sheaf on $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)$ for all n. We find

$$\begin{aligned} H^0(X, \mathcal{L})^\wedge &= \lim H^0(X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n), \mathcal{L}|_{X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)}) \\ &\cong \lim H^0(X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n), \mathcal{O}_{X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)}) \\ &= R^\wedge \end{aligned}$$

using the theorem on formal functions (Cohomology of Schemes, Theorem 20.5) for the first and last equality and for example More on Algebra, Lemma 100.5 for the middle isomorphism. Since $H^0(X, \mathcal{L})$ is a finite R-module and R is a discrete valuation ring, this means that $H^0(X, \mathcal{L})$ is free of rank 1 as an R-module. Let $s \in H^0(X, \mathcal{L})$ be a basis element. Then tracing back through the isomorphisms above we see that $s|_{X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\pi^n)}$ is a trivialization for all n. Since the vanishing locus of s is closed in X and $X \to \mathrm{Spec}(R)$ is proper we conclude that the vanishing locus of s is empty as desired. ☐

14. Semistable reduction

In this section we carefully define what we mean by semistable reduction.

Example 14.1. Let R be a discrete valuation ring with uniformizer π. Given $n \geq 0$, consider the ring map

$$R \longrightarrow A = R[x, y]/(xy - \pi^n)$$

Set $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$. If $n = 0$, then $X \to S$ is smooth. For all n the morphism $X \to S$ is at-worst-nodal of relative dimension 1 as defined in Algebraic Curves, Section 20. If $n = 1$, then X is regular, but if $n > 1$, then X is not regular as (x, y) no longer generate the maximal ideal $\mathfrak{m} = (\pi, x, y)$. To ameliorate the situation in case $n > 1$ we consider the blowup $b : X' \to X$ of X in $\mathfrak{m}$. See Divisors, Section 32. By construction X' is covered by three affine pieces corresponding to the blowup algebras $A[\frac{\mathfrak{m}}{\pi}]$, $A[\frac{\mathfrak{m}}{x}]$, and $A[\frac{\mathfrak{m}}{y}]$.

The algebra $A[\frac{\mathfrak{m}}{\pi}]$ has generators $x' = x/\pi$ and $y' = y/\pi$ and $x'y' = \pi^{n-2}$. Thus this part of X' is the spectrum of $R[x', y'](x'y' - \pi^{n-2})$.

The algebra $A[\frac{\mathfrak{m}}{x}]$ has generators x, $u = \pi/x$ subject to the relation $xu - \pi$. Note that this ring contains $y/x = \pi^n/x^2 = u^2\pi^{n-2}$. Thus this part of X' is regular.

By symmetry the case of the algebra $A[\frac{\mathfrak{m}}{y}]$ is the same as the case of $A[\frac{\mathfrak{m}}{x}]$.

Thus we see that $X' \to S$ is at-worst-nodal of relative dimension 1 and that X' is regular, except for one point which has an affine open neighbourhood exactly as above but with n replaced by $n - 2$. Using induction on n we conclude that there is a sequence of blowing ups in closed points

$$X_{\lfloor n/2 \rfloor} \to \ldots \to X_1 \to X_0 = X$$

such that $X_{\lfloor n/2 \rfloor} \to S$ is at-worst-nodal of relative dimension 1 and $X_{\lfloor n/2 \rfloor}$ is regular.

Lemma 14.2. *Let R be a discrete valuation ring. Let X be a scheme which is at-worst-nodal of relative dimension 1 over R. Let $x \in X$ be a point of the special fibre of X over R. Then there exists a commutative diagram*

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[dl] \\
\mathrm{Spec}(R) & \mathrm{Spec}(R') \arrow[l] &
\end{tikzcd}$$

where $R \subset R'$ is an étale extension of discrete valuation rings, the morphism $U \to X$ is étale, the morphism $U \to \mathrm{Spec}(A)$ is étale, there is a point $x' \in U$ mapping to x, and

$$A = R'[u, v]/(uv) \quad \text{or} \quad A = R'[u, v]/(uv - \pi^n)$$

where $n \geq 0$ and $\pi \in R'$ is a uniformizer.

Proof. We have already proved this lemma in much greater generality, see Algebraic Curves, Lemma 20.12. All we have to do here is to translate the statement given there into the statement given above.

First, if the morphism $X \to \mathrm{Spec}(R)$ is smooth at x, then we can find an étale morphism $U \to \mathbf{A}^1_R = \mathrm{Spec}(R[u])$ for some affine open neighbourhood $U \subset X$ of

x. This is Morphisms, Lemma 36.20. After replacing the coordinate u by $u+1$ if necessary, we may assume that x maps to a point in the standard open $D(u) \subset \mathbf{A}^1_R$. Then $D(u) = \mathrm{Spec}(A)$ with $A = R[u,v]/(uv-1)$ and we see that the result is true in this case.

Next, assume that x is a singular point of the fibre. Then we may apply Algebraic Curves, Lemma 20.12 to get a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[rr] \arrow[dr] & & W \arrow[r] \arrow[dl] & \mathrm{Spec}(\mathbf{Z}[u,v,a]/(uv-a)) \arrow[d] \\ \mathrm{Spec}(R) & & V \arrow[ll] \arrow[rr] & & \mathrm{Spec}(\mathbf{Z}[a]) \end{tikzcd}$$

with all the properties mentioned in the statement of the cited lemma. Let $x' \in U$ be the point mapping to x promised by the lemma. First we shrink V to an affine neighbourhood of the image of x'. Say $V = \mathrm{Spec}(R')$. Then $R \to R'$ is étale. Since R is a discrete valuation ring, we see that R' is a finite product of quasi-local Dedekind domains (use More on Algebra, Lemma 44.4). Hence (for example using prime avoidance) we find a standard open $D(f) \subset V = \mathrm{Spec}(R')$ containing the image of x' such that R'_f is a discrete valuation ring. Replacing R' by R'_f we reach the situation where $V = \mathrm{Spec}(R')$ with $R \subset R'$ an étale extension of discrete valuation rings (extensions of discrete valuation rings are defined in More on Algebra, Definition 111.1).

The morphism $V \to \mathrm{Spec}(\mathbf{Z}[a])$ is determined by the image h of a in R'. Then $W = \mathrm{Spec}(R'[u,v]/(uv-h))$. Thus the lemma holds with $A = R'[u,v]/(uv-h)$. If $h = 0$ then we clearly obtain the first case mentioned in the lemma. If $h \neq 0$ then we may write $h = \epsilon\pi^n$ for some $n \geq 0$ where ϵ is a unit of R'. Changing coordinates $u_{new} = \epsilon u$ and $v_{new} = v$ we obtain the second isomorphism type of A listed in the lemma. $\square$

Lemma 14.3. *Let R be a discrete valuation ring. Let X be a quasi-compact scheme which is at-worst-nodal of relative dimension 1 with smooth generic fibre over R. Then there exists $m \geq 0$ and a sequence*

$$X_m \to \ldots \to X_1 \to X_0 = X$$

such that

1. *$X_{i+1} \to X_i$ is the blowing up of a closed point x_i where X_i is singular,*
2. *$X_i \to \mathrm{Spec}(R)$ is at-worst-nodal of relative dimension 1,*
3. *X_m is regular.*

A slightly stronger statement (also true) would be that no matter how you blow up in singular points you eventually end up with a resolution and all the intermediate blowups are at-worst-nodal of relative dimension 1 over R.

Proof. Since X is quasi-compact we see that the special fibre X_k is quasi-compact. Since the singularities of X_k are at-worst-nodal, we see that X_k has a finite number of nodes and is otherwise smooth over k. As $X \to \mathrm{Spec}(R)$ is flat with smooth generic fibre it follows that X is smooth over R except at the finite number of nodes of X_k (use Morphisms, Lemma 34.14). It follows that X is regular at every point except for possibly the nodes of its special fibre (see Algebra, Lemma 163.10). Let $x \in X$ be such a node. Choose a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[dl] \\ \mathrm{Spec}(R) & \mathrm{Spec}(R') \arrow[l] & \end{tikzcd}$$

as in Lemma 14.2. Observe that the case $A = R'[u,v]/(uv)$ cannot occur, as this would mean that the generic fibre of X/R is singular (tiny detail omitted). Thus $A = R'[u,v]/(uv - \pi^n)$ for some $n \geq 0$. Since x is a singular point, we have $n \geq 2$, see discussion in Example 14.1.

After shrinking U we may assume there is a unique point $u \in U$ mapping to x. Let $w \in \mathrm{Spec}(A)$ be the image of u. We may also assume that u is the unique point of U mapping to w. Since the two horizontal arrows are étale we see that u, viewed as a closed subscheme of U, is the scheme theoretic inverse image of $x \in X$ and the scheme theoretic inverse image of $w \in \mathrm{Spec}(A)$. Since blowing up commutes with flat base change (Divisors, Lemma 32.3) we find a commutative diagram

$$\begin{tikzcd} X' \arrow[d] & U' \arrow[l] \arrow[r] \arrow[d] & W' \arrow[d] \\ X & U \arrow[l] \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

with cartesian squares where the vertical arrows are the blowing up of x, u, w in $X, U, \mathrm{Spec}(A)$. The scheme W' was described in Example 14.1. We saw there that W' at-worst-nodal of relative dimension 1 over R'. Thus W' is at-worst-nodal of relative dimension 1 over R (Algebraic Curves, Lemma 20.7). Hence U' is at-worst-nodal of relative dimension 1 over R (see Algebraic Curves, Lemma 20.8). Since $X' \to X$ is an isomorphism over the complement of x, we conclude the same thing is true of X'/R (by Algebraic Curves, Lemma 20.8 again).

Finally, we need to argue that after doing a finite number of these blowups we arrive at a regular model X_m. This is rather clear because the "invariant" n decreases by 2 under the blowup described above, see computation in Example 14.1. However, as we want to avoid precisely defining this invariant and establishing its properties, we in stead argue as follows. If $n = 2$, then W' is regular and hence X' is regular at all points lying over x and we have decreased the number of singular points of X by 1. If $n > 2$, then the unique singular point w' of W' lying over w has $\kappa(w) = \kappa(w')$. Hence U' has a unique singular point u' lying over u with $\kappa(u) = \kappa(u')$. Clearly, this implies that X' has a unique singular point x' lying over x, namely the image of u'. Thus we can argue exactly as above that we get a commutative diagram

$$\begin{tikzcd} X'' \arrow[d] & U'' \arrow[l] \arrow[r] \arrow[d] & W'' \arrow[d] \\ X' & U' \arrow[l] \arrow[r] & W' \end{tikzcd}$$

with cartesian squares where the vertical arrows are the blowing up of x', u', w' in X', U', W'. Continuing like this we get a compatible sequence of blowups which stops after $\lfloor n/2 \rfloor$ steps. At the completion of this process the scheme $X^{(\lfloor n/2 \rfloor)}$ will have one fewer singular point than X. Induction on the number of singular points completes the proof. $\square$

Lemma 14.4. *Let R be a discrete valuation ring with fraction field K and residue field k. Assume $X \to \text{Spec}(R)$ is at-worst-nodal of relative dimension 1 over R. Let $X \to X'$ be the contraction of an exceptional curve $E \subset X$ of the first kind. Then X' is at-worst-nodal of relative dimension 1 over R.*

Proof. Namely, let $x' \in X'$ be the image of E. Then the only issue is to see that $X' \to \text{Spec}(R)$ is at-worst-nodal of relative dimension 1 in a neighbourhood of x'. The closed fibre of $X \to \text{Spec}(R)$ is reduced, hence $\pi \in R$ vanishes to order 1 on E. This immediately implies that π viewed as an element of $\mathfrak{m}_{x'} \subset \mathcal{O}_{X',x'}$ but is not in $\mathfrak{m}_{x'}^2$. Since $\mathcal{O}_{X',x'}$ is regular of dimension 2 (by definition of contractions in Resolution of Surfaces, Section 16), this implies that $\mathcal{O}_{X'_k,x'}$ is regular of dimension 1 (Algebra, Lemma 106.3). On the other hand, the curve E has to meet at least one other component, say C of the closed fibre X_k. Say $x \in E \cap C$. Then x is a node of the special fibre X_k and hence $\kappa(x)/k$ is finite separable, see Algebraic Curves, Lemma 19.7. Since $x \mapsto x'$ we conclude that $\kappa(x')/k$ is finite separable. By Algebra, Lemma 140.5 we conclude that $X'_k \to \text{Spec}(k)$ is smooth in an open neighbourhood of x'. Combined with flatness, this proves that $X' \to \text{Spec}(R)$ is smooth in a neighbourhood of x' (Morphisms, Lemma 34.14). This finishes the proof as a smooth morphism of relative dimension 1 is at-worst-nodal of relative dimension 1 (Algebraic Curves, Lemma 20.3). $\square$

Lemma 14.5. *Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. The following are equivalent*

1. *there exists a proper model of C which is at-worst-nodal of relative dimension 1 over R,*
2. *there exists a minimal model of C which is at-worst-nodal of relative dimension 1 over R, and*
3. *any minimal model of C is at-worst-nodal of relative dimension 1 over R.*

Proof. To make sense out of this statement, recall that a minimal model is defined as a regular proper model without exceptional curves of the first kind (Definition 8.4), that minimal models exist (Proposition 8.6), and that minimal models are unique if the genus of C is > 0 (Lemma 10.1). Keeping this in mind the implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are clear.

Assume (1). Let X be a proper model of C which is at-worst-nodal of relative dimension 1 over R. Applying Lemma 14.3 we see that we may assume X is regular as well. Let

$$X = X_m \to X_{m-1} \to \ldots \to X_1 \to X_0$$

be as in Lemma 8.5. By Lemma 14.4 and induction this implies X_0 is at-worst-nodal of relative dimension 1 over R.

To finish the proof we have to show that (2) implies (3). This is clear if the genus of C is > 0, since then the minimal model is unique (see discussion above). On the other hand, if the minimal model is not unique, then the morphism $X \to \text{Spec}(R)$ is smooth for any minimal model as its special fibre will be isomorphic to $\mathbf{P}^1_k$ by Lemma 12.4. $\square$

Definition 14.6. Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. We say that C has *semistable reduction* if the equivalent conditions of Lemma 14.5 are satisfied.

Lemma 14.7. *Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. The following are equivalent*

1. *there exists a proper smooth model for C,*
2. *there exists a minimal model for C which is smooth over R,*
3. *any minimal model is smooth over R.*

Proof. If X is a smooth proper model, then the special fibre is connected (Lemma 9.4) and smooth, hence irreducible. This immediately implies that it is minimal. Thus (1) implies (2). To finish the proof we have to show that (2) implies (3). This is clear if the genus of C is > 0, since then the minimal model is unique (Lemma 10.1). On the other hand, if the minimal model is not unique, then the morphism $X \to \text{Spec}(R)$ is smooth for any minimal model as its special fibre will be isomorphic to $\mathbf{P}^1_k$ by Lemma 12.4. □

Definition 14.8. Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. We say that C has *good reduction* if the equivalent conditions of Lemma 14.7 are satisfied.

15. Semistable reduction in genus zero

In this section we prove the semistable reduction theorem (Theorem 18.1) for genus zero curves.

Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. If the genus of C is 0, then C is isomorphic to a conic, see Algebraic Curves, Lemma 10.3. Thus there exists a finite separable extension K'/K of degree at most 2 such that $C(K') \neq \emptyset$, see Algebraic Curves, Lemma 9.4. Let $R' \subset K'$ be the integral closure of R, see discussion in More on Algebra, Remark 111.6. We will show that $C_{K'}$ has semistable reduction over $R'_\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$ of R' (of course in the current case there are at most two such ideals). After replacing R by $R'_\mathfrak{m}$ and C by $C_{K'}$ we reduce to the case discussed in the next paragraph.

In this paragraph R is a discrete valuation ring with fraction field K, C is a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$, of genus 0, and C has a K-rational point. In this case $C \cong \mathbf{P}^1_K$ by Algebraic Curves, Proposition 10.4. Thus we can use $\mathbf{P}^1_R$ as a model and we see that C has both good and semistable reduction.

Example 15.1. Let $R = \mathbf{R}[[\pi]]$ and consider the scheme

$$X = V(T_1^2 + T_2^2 - \pi T_0^2) \subset \mathbf{P}^2_R$$

The base change of X to $\mathbf{C}[[\pi]]$ is isomorphic to the scheme defined in Example 10.3 because we have the factorization $T_1^2 + T_2^2 = (T_1 + iT_2)(T_1 - iT_2)$ over $\mathbf{C}$. Thus X is regular and its special fibre is irreducible yet singular, hence X is the unique minimal model of its generic fibre (use Lemma 12.4). It follows that an extension is needed even in genus 0.

16. Semistable reduction in genus one

In this section we prove the semistable reduction theorem (Theorem 18.1) for curves of genus one. We suggest the reader first read the proof in the case of genus ≥ 2 (Section 17). We are going to use as much as possible the classification of minimal numerical types of genus 1 given in Lemma 6.2.

Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Assume the genus of C is 1. Choose a prime $\ell \geq 7$ different from the characteristic of k. Choose a finite separable extension K'/K of such that $C(K') \neq \emptyset$ and such that $\text{Pic}(C_{K'})[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2}$. See Algebraic Curves, Lemma 17.2. Let $R' \subset K'$ be the integral closure of R, see discussion in More on Algebra, Remark 111.6. We may replace R by $R'_\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in R' and C by $C_{K'}$. This reduces us to the case discussed in the next paragraph.

In the rest of this section R is a discrete valuation ring with fraction field K, C is a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$, with genus 1, having a K-rational point, and with $\text{Pic}(C)[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2}$ for some prime $\ell \geq 7$ different from the characteristic of k. We will prove that C has semistable reduction.

Let X be a minimal model for C, see Proposition 8.6. Let $T = (n, m_i, (a_{ij}), w_i, g_i)$ be the numerical type associated to X (Definition 11.4). Then T is a minimal numerical type (Lemma 11.5). As C has a rational point, there exists an i such that $m_i = w_i = 1$ by Lemma 11.7. Looking at the classification of minimal numerical types of genus 1 in Lemma 6.2 we see that $m = w = 1$ and that cases (3), (6), (7), (9), (11), (13), (15), (18), (19), (21), (24), (26), (28), (30) are disallowed (because there is no index where both w_i and m_i is equal to 1). Let e be the number of pairs (i, j) with $i < j$ and $a_{ij} > 0$. For the remaining cases we have

(A) $e = n - 1$ for cases (1), (2), (5), (8), (12), (14), (17), (20), (22), (23), (27), (29), (31), (32), (33), and (34), and
(B) $e = n$ for cases (4), (10), (16), and (25).

We will argue these cases separately.

Case (A). In this case $\text{Pic}(T)[\ell]$ is trivial (the Picard group of a numerical type is defined in Section 4). The vanishing follows as $\text{Pic}(T) \subset \text{Coker}(A)$ (Lemma 4.3) and $\text{Coker}(A)[\ell] = 0$ by Lemma 2.6 and the fact that ℓ was chosen relatively prime to a_{ij} and m_i. By Lemmas 13.3 and 13.4 we conclude that there is an embedding

$$(\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2} \subset \text{Pic}((X_k)_{red})[\ell].$$

By Algebraic Curves, Lemma 18.6 we obtain

$$2 \leq \dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) + g_{geom}((X_k)_{red}/k)$$

By Algebraic Curves, Lemmas 18.1 and 18.5 we see that $g_{geom}((X_k)_{red}/k) \leq \sum w_i g_i$. The assumptions of Lemma 11.8 hold by Lemma 11.7 and we conclude that we have $\dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) \leq g = 1$. Combining these we see

$$2 \leq 1 + \sum w_i g_i$$

Looking at the list we conclude that the numerical type is given by $n = 1$, $w_1 = m_1 = g_1 = 1$. Because we have equality everywhere we see that $g_{geom}(C_1/k) = 1$. On the other hand, we know that C_1 has a k-rational point x such that $C_1 \to \text{Spec}(k)$ is smooth at x. It follows that C_1 is geometrically integral (Varieties, Lemma 25.10). Thus $g_{geom}(C_1/k) = 1$ is both equal to the genus of the normalization of $C_{1,\overline{k}}$ and the genus of $C_{1,\overline{k}}$. It follows that the normalization morphism $C^\nu_{1,\overline{k}} \to C_{1,\overline{k}}$ is an isomorphism (Algebraic Curves, Lemma 18.4). We conclude that C_1 is smooth over k as desired.

Case (B). Here we only conclude that there is an embedding

$$\mathbf{Z}/\ell\mathbf{Z} \subset \mathrm{Pic}(X_k)[\ell]$$

From the classification of types we see that $m_i = w_i = 1$ and $g_i = 0$ for each i. Thus each C_i is a genus zero curve over k. Moreover, for each i there is a j such that $C_i \cap C_j$ is a k-rational point. Then it follows that $C_i \cong \mathbf{P}^1_k$ by Algebraic Curves, Proposition 10.4. In particular, since X_k is the scheme theoretic union of the C_i we see that $X_{\overline{k}}$ is the scheme theoretic union of the $C_{i,\overline{k}}$. Hence $X_{\overline{k}}$ is a reduced connected proper scheme of dimension 1 over $\overline{k}$ with $\dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) = 1$. Also, by Varieties, Lemma 30.3 and the above we still have

$$\dim_{\mathbf{F}_\ell}(\mathrm{Pic}(X_{\overline{k}}) \geq 1$$

By Algebraic Curves, Proposition 17.3 we see that $X_{\overline{k}}$ has at only multicross singularities. But since X_k is Gorenstein (Lemma 9.2), so is $X_{\overline{k}}$ (Duality for Schemes, Lemma 25.1). We conclude $X_{\overline{k}}$ is at-worst-nodal by Algebraic Curves, Lemma 16.4. This finishes the proof in case (B).

Example 16.1. Let k be an algebraically closed field. Let Z be a smooth projective curve over k of positive genus g. Let $n \geq 1$ be an integer prime to the characteristic of k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Z$-module of order n, see Algebraic Curves, Lemma 17.1. Pick an isomorphism $\varphi : \mathcal{L}^{\otimes n} \to \mathcal{O}_Z$. Set $R = k[[\pi]]$ with fraction field $K = k((\pi))$. Denote Z_R the base change of Z to R. Let $\mathcal{L}_R$ be the pullback of $\mathcal{L}$ to Z_R. Consider the finite flat morphism

$$p : X \longrightarrow Z_R$$

such that

$$p_*\mathcal{O}_X = \mathrm{Sym}^*_{\mathcal{O}_{Z_R}}(\mathcal{L}_R)/(\varphi - \pi) = \mathcal{O}_{Z_R} \oplus \mathcal{L}_R \oplus \mathcal{L}_R^{\otimes 2} \oplus \ldots \oplus \mathcal{L}_R^{\otimes n-1}$$

More precisely, if $U = \mathrm{Spec}(A) \subset Z$ is an affine open such that $\mathcal{L}|_U$ is trivialized by a section s with $\varphi(s^{\otimes n}) = f$ (with f a unit), then

$$p^{-1}(U_R) = \mathrm{Spec}\left((A \otimes_R R[[\pi]])[x]/(x^n - \pi f)\right)$$

The reader verifies that the morphism $X_K \to Z_K$ of generic fibres is finite étale. Looking at the description of the structure sheaf we see that $H^0(X, \mathcal{O}_X) = R$ and $H^0(X_K, \mathcal{O}_{X_K}) = K$. By Riemann-Hurwitz (Algebraic Curves, Lemma 12.4) the genus of X_K is $n(g-1)+1$. In particular X_K has genus 1, if Z has genus 1. On the other hand, the scheme X is regular by the local equation above and the special fibre X_k is n times the reduced special fibre as an effective Cartier divisor. It follows that any finite extension K'/K over which X_K attains semistable reduction has to ramify with ramification index at least n (some details omitted). Thus there does not exist a universal bound for the degree of an extension over which a genus 1 curve attains semistable reduction.

17. Semistable reduction in genus at least two

In this section we prove the semistable reduction theorem (Theorem 18.1) for curves of genus ≥ 2. Fix $g \geq 2$.

Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Assume the genus of C is g. Choose a prime $\ell > 768g$ different from the characteristic of k. Choose a finite separable extension K'/K of such that $C(K') \neq \emptyset$ and such that $\text{Pic}(C_{K'})[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2g}$. See Algebraic Curves, Lemma 17.2. Let $R' \subset K'$ be the integral closure of R, see discussion in More on Algebra, Remark 111.6. We may replace R by $R'_\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in R' and C by $C_{K'}$. This reduces us to the case discussed in the next paragraph.

In the rest of this section R is a discrete valuation ring with fraction field K, C is a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$, with genus g, having a K-rational point, and with $\text{Pic}(C)[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2g}$ for some prime $\ell \geq 768g$ different from the characteristic of k. We will prove that C has semistable reduction.

In the rest of this section we will use without further mention that the conclusions of Lemma 11.7 are true.

Let X be a minimal model for C, see Proposition 8.6. Let $T = (n, m_i, (a_{ij}), w_i, g_i)$ be the numerical type associated to X (Definition 11.4). Then T is a minimal numerical type of genus g (Lemma 11.5). By Proposition 7.4 we have

$$\dim_{\mathbf{F}_\ell} \text{Pic}(T)[\ell] \leq g_{top}$$

By Lemmas 13.3 and 13.4 we conclude that there is an embedding

$$(\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2g - g_{top}} \subset \text{Pic}((X_k)_{red})[\ell].$$

By Algebraic Curves, Lemma 18.6 we obtain

$$2g - g_{top} \leq \dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) + g_{geom}(X_k/k)$$

By Lemmas 11.8 and 11.9 we have

$$g \geq \dim_k H^1((X_k)_{red}, \mathcal{O}_{(X_k)_{red}}) \geq g_{top} + g_{geom}(X_k/k)$$

Elementary number theory tells us that the only way these 3 inequalities can hold is if they are all equalities. Looking at Lemma 11.8 we conclude that $m_i = 1$ for all i. Looking at Lemma 11.10 we conclude that every irreducible component of X_k is smooth over k.

In particular, since X_k is the scheme theoretic union of its irreducible components C_i we see that $X_{\overline{k}}$ is the scheme theoretic union of the $C_{i,\overline{k}}$. Hence $X_{\overline{k}}$ is a reduced connected proper scheme of dimension 1 over $\overline{k}$ with $\dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) = g$. Also, by Varieties, Lemma 30.3 and the above we still have

$$\dim_{\mathbf{F}_\ell}(\text{Pic}(X_{\overline{k}})[\ell]) \geq 2g - g_{top} = \dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) + g_{geom}(X_{\overline{k}})$$

By Algebraic Curves, Proposition 17.3 we see that $X_{\overline{k}}$ has at only multicross singularities. But since X_k is Gorenstein (Lemma 9.2), so is $X_{\overline{k}}$ (Duality for Schemes, Lemma 25.1). We conclude $X_{\overline{k}}$ is at-worst-nodal by Algebraic Curves, Lemma 16.4. This finishes the proof.

18. Semistable reduction for curves

In this section we finish the proof of the theorem. For $g \geq 2$ let $768g < \ell' < \ell$ be the first two primes $> 768g$ and set

$$B_g = (2g-2)(\ell^{2g})! \tag{18.0.1}$$

The precise form of B_g is unimportant; the point we are trying to make is that it depends only on g.

Theorem 18.1. *Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. Then there exists an extension of discrete valuation rings $R \subset R'$ which induces a finite separable extension of fraction fields K'/K such that $C_{K'}$ has semistable reduction. More precisely, we have the following*

1. *If the genus of C is zero, then there exists a degree 2 separable extension K'/K such that $C_{K'} \cong \mathbf{P}^1_{K'}$ and hence $C_{K'}$ is isomorphic to the generic fibre of the smooth projective scheme $\mathbf{P}^1_{R'}$ over the integral closure R' of R in K'.*
2. *If the genus of C is one, then there exists a finite separable extension K'/K such that $C_{K'}$ has semistable reduction over $R'_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of the integral closure R' of R in K'. Moreover, the special fibre of the (unique) minimal model of $C_{K'}$ over $R'_\mathfrak{m}$ is either a smooth genus one curve or a cycle of rational curves.*
3. *If the genus g of C is greater than one, then there exists a finite separable extension K'/K of degree at most B_g (18.0.1) such that $C_{K'}$ has semistable reduction over $R'_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of the integral closure R' of R in K'.*

Proof. For the case of genus zero, see Section 15. For the case of genus one, see Section 16. For the case of genus greater than one, see Section 17. To see that we have a bound on the degree $[K' : K]$ you can use the bound on the degree of the extension needed to make all ℓ or ℓ' torsion visible proved in Algebraic Curves, Lemma 17.2. (The reason for using ℓ and ℓ' is that we need to avoid the characteristic of the residue field k.) $\square$

Remark 18.2 (Improving the bound)**.** Results in the literature suggest that one can improve the bound given in the statement of Theorem 18.1. For example, in [DM69] it is shown that semistable reduction of C and its Jacobian are the same thing if the residue field is perfect and presumably this is true for general residue fields as well. For an abelian variety we have semistable reduction if the action of Galois on the ℓ-torsion is trivial for any $\ell \geq 3$ not equal to the residue characteristic. Thus we can presumably choose $\ell = 5$ in the formula (18.0.1) for B_g (but the proof would take a lot more work; if we ever need this we will make a precise statement and provide a proof here).

19. Other chapters

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