

THE GEOMETRY OF ALGEBRAIC STACKS

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1. Introduction

This chapter discusses a few geometric properties of algebraic stacks. The initial versions of Sections 3 and 5 were written by Matthew Emerton and Toby Gee and can be found in their original form in [EG17].

2. Versal rings

In this section we elucidate the relationship between deformation rings and local rings on algebraic stacks of finite type over a locally Noetherian base.

Situation 2.1. Here \mathcal{X} is an algebraic stack locally of finite type over a locally Noetherian scheme S .

Here is the definition.

Definition 2.2. In Situation 2.1 let $x_0 : \mathrm{Spec}(k) \rightarrow \mathcal{X}$ be a morphism, where k is a finite type field over S . A *versal ring to \mathcal{X} at x_0* is a complete Noetherian local S -algebra A with residue field k such that there exists a versal formal object (A, ξ_n, f_n) as in Artin's Axioms, Definition 12.1 with $\xi_1 \cong x_0$ (a 2-isomorphism).

We want to prove that versal rings exist and are unique up to smooth factors. To do this, we will use the predeformation categories of Artin's Axioms, Section 3. These are always deformation categories in our situation.

Lemma 2.3. *In Situation 2.1 let $x_0 : \mathrm{Spec}(k) \rightarrow \mathcal{X}$ be a morphism, where k is a finite type field over S . Then $\mathcal{F}_{\mathcal{X},k,x_0}$ is a deformation category and $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional k -vector spaces.*

Proof. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that $\mathrm{Spec}(k) \rightarrow S$ factors through it. By Artin's Axioms, Section 3 we obtain a predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$ over the category \mathcal{C}_Λ . (As pointed out in locus citatus this category only depends on the morphism $\mathrm{Spec}(k) \rightarrow S$ and not on the choice of Λ .) By Artin's Axioms, Lemmas 6.1 and 5.2 $\mathcal{F}_{\mathcal{X},k,x_0}$ is actually a deformation category. By Artin's Axioms,

Lemma 8.1 we find that $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\text{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional k -vector spaces. \square

Lemma 2.4. *In Situation 2.1 let $x_0 : \text{Spec}(k) \rightarrow \mathcal{X}$ be a morphism, where k is a finite type field over S . Then a versal ring to \mathcal{X} at x_0 exists. Given a pair A, A' of these, then $A \cong A'[[t_1, \dots, t_r]]$ or $A' \cong A[[t_1, \dots, t_r]]$ as S -algebras for some r .*

Proof. By Lemma 2.3 and Formal Deformation Theory, Lemma 13.4 (note that the assumptions of this lemma hold by Formal Deformation Theory, Lemmas 16.6 and Definition 16.8). By the uniqueness result of Formal Deformation Theory, Lemma 14.5 there exists a “minimal” versal ring A of \mathcal{X} at x_0 such that any other versal ring of \mathcal{X} at x_0 is isomorphic to $A[[t_1, \dots, t_r]]$ for some r . This clearly implies the second statement. \square

Lemma 2.5. *In Situation 2.1 let $x_0 : \text{Spec}(k) \rightarrow \mathcal{X}$ be a morphism, where k is a finite type field over S . Let l/k be a finite extension of fields and denote $x_{l,0} : \text{Spec}(l) \rightarrow \mathcal{X}$ the induced morphism. Given a versal ring A to \mathcal{X} at x_0 there exists a versal ring A' to \mathcal{X} at $x_{l,0}$ such that there is a S -algebra map $A \rightarrow A'$ which induces the given field extension l/k and is formally smooth in the $\mathfrak{m}_{A'}$ -adic topology.*

Proof. Follows immediately from Artin’s Axioms, Lemma 7.1 and Formal Deformation Theory, Lemma 29.6. (We also use that \mathcal{X} satisfies (RS) by Artin’s Axioms, Lemma 5.2.) \square

Lemma 2.6. *In Situation 2.1 let $x : U \rightarrow \mathcal{X}$ be a morphism where U is a scheme locally of finite type over S . Let $u_0 \in U$ be a finite type point. Set $k = \kappa(u_0)$ and denote $x_0 : \text{Spec}(k) \rightarrow \mathcal{X}$ the induced map. The following are equivalent*

- (1) x is versal at u_0 (Artin’s Axioms, Definition 12.2),
- (2) $\hat{x} : \mathcal{F}_{U,k,u_0} \rightarrow \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth,
- (3) the formal object associated to $x|_{\text{Spec}(\mathcal{O}_{U,u_0}^\wedge)}$ is versal, and
- (4) there is an open neighbourhood $U' \subset U$ of x such that $x|_{U'} : U' \rightarrow \mathcal{X}$ is smooth.

Moreover, in this case the completion $\mathcal{O}_{U,u_0}^\wedge$ is a versal ring to \mathcal{X} at x_0 .

Proof. Since $U \rightarrow S$ is locally of finite type (as a composition of such morphisms), we see that $\text{Spec}(k) \rightarrow S$ is of finite type (again as a composition). Thus the statement makes sense. The equivalence of (1) and (2) is the definition of x being versal at u_0 . The equivalence of (1) and (3) is Artin’s Axioms, Lemma 12.3. Thus (1), (2), and (3) are equivalent.

If $x|_{U'}$ is smooth, then the functor $\hat{x} : \mathcal{F}_{U,k,u_0} \rightarrow \mathcal{F}_{\mathcal{X},k,x_0}$ is smooth by Artin’s Axioms, Lemma 3.2. Thus (4) implies (1), (2), and (3). For the converse, assume x is versal at u_0 . Choose a surjective smooth morphism $y : V \rightarrow \mathcal{X}$ where V is a scheme. Set $Z = V \times_{\mathcal{X}} U$ and pick a finite type point $z_0 \in |Z|$ lying over u_0 (this is possible by Morphisms of Spaces, Lemma 25.5). By Artin’s Axioms, Lemma 12.6 the morphism $Z \rightarrow V$ is smooth at z_0 . By definition we can find an open neighbourhood $W \subset Z$ of z_0 such that $W \rightarrow V$ is smooth. Since $Z \rightarrow U$ is open, let $U' \subset U$ be the image of W . Then we see that $U' \rightarrow \mathcal{X}$ is smooth by our definition of smooth morphisms of stacks.

The final statement follows from the definitions as $\mathcal{O}_{U,u_0}^\wedge$ prorepresents \mathcal{F}_{U,k,u_0} . \square

Lemma 2.7. *In Situation 2.1. Let $x_0 : \mathrm{Spec}(k) \rightarrow \mathcal{X}$ be a morphism such that $\mathrm{Spec}(k) \rightarrow S$ is of finite type with image s . Let A be a versal ring to \mathcal{X} at x_0 . The following are equivalent*

- (1) x_0 is in the smooth locus of $\mathcal{X} \rightarrow S$ (Morphisms of Stacks, Lemma 33.6),
- (2) $\mathcal{O}_{S,s} \rightarrow A$ is formally smooth in the \mathfrak{m}_A -adic topology, and
- (3) $\mathcal{F}_{\mathcal{X},k,x_0}$ is unobstructed.

Proof. The equivalence of (2) and (3) follows immediately from Formal Deformation Theory, Lemma 9.4.

Note that $\mathcal{O}_{S,s} \rightarrow A$ is formally smooth in the \mathfrak{m}_A -adic topology if and only if $\mathcal{O}_{S,s} \rightarrow A' = A[[t_1, \dots, t_r]]$ is formally smooth in the $\mathfrak{m}_{A'}$ -adic topology. Hence (2) does not depend on the choice of our versal ring by Lemma 2.4. Next, let l/k be a finite extension and choose $A \rightarrow A'$ as in Lemma 2.5. If $\mathcal{O}_{S,s} \rightarrow A$ is formally smooth in the \mathfrak{m}_A -adic topology, then $\mathcal{O}_{S,s} \rightarrow A'$ is formally smooth in the $\mathfrak{m}_{A'}$ -adic topology, see More on Algebra, Lemma 37.7. Conversely, if $\mathcal{O}_{S,s} \rightarrow A'$ is formally smooth in the $\mathfrak{m}_{A'}$ -adic topology, then $\mathcal{O}_{S,s}^\wedge \rightarrow A'$ and $A \rightarrow A'$ are regular (More on Algebra, Proposition 49.2) and hence $\mathcal{O}_{S,s}^\wedge \rightarrow A$ is regular (More on Algebra, Lemma 41.7), hence $\mathcal{O}_{S,s} \rightarrow A$ is formally smooth in the \mathfrak{m}_A -adic topology (same lemma as before). Thus the equivalence of (2) and (1) holds for k and x_0 if and only if it holds for l and $x_{0,l}$.

Choose a scheme U and a smooth morphism $U \rightarrow \mathcal{X}$ such that $\mathrm{Spec}(k) \times_{\mathcal{X}} U$ is nonempty. Choose a finite extension l/k and a point $w_0 : \mathrm{Spec}(l) \rightarrow \mathrm{Spec}(k) \times_{\mathcal{X}} U$. Let $u_0 \in U$ be the image of w_0 . We may apply the above to l/k and to $l/\kappa(u_0)$ to see that we can reduce to u_0 . Thus we may assume $A = \mathcal{O}_{U,u_0}^\wedge$, see Lemma 2.6. Observe that x_0 is in the smooth locus of $\mathcal{X} \rightarrow S$ if and only if u_0 is in the smooth locus of $U \rightarrow S$, see for example Morphisms of Stacks, Lemma 33.6. Thus the equivalence of (1) and (2) follows from More on Algebra, Lemma 38.6. \square

We recall a consequence of Artin approximation.

Lemma 2.8. *In Situation 2.1. Let $x_0 : \mathrm{Spec}(k) \rightarrow \mathcal{X}$ be a morphism such that $\mathrm{Spec}(k) \rightarrow S$ is of finite type with image s . Let A be a versal ring to \mathcal{X} at x_0 . If $\mathcal{O}_{S,s}$ is a G -ring, then we may find a smooth morphism $U \rightarrow \mathcal{X}$ whose source is a scheme and a point $u_0 \in U$ with residue field k , such that*

- (1) $\mathrm{Spec}(k) \rightarrow U \rightarrow \mathcal{X}$ coincides with the given morphism x_0 ,
- (2) there is an isomorphism $\mathcal{O}_{U,u_0}^\wedge \cong A$.

Proof. Let (ξ_n, f_n) be the versal formal object over A . By Artin's Axioms, Lemma 9.5 we know that $\xi = (A, \xi_n, f_n)$ is effective. By assumption \mathcal{X} is locally of finite presentation over S (use Morphisms of Stacks, Lemma 27.5), and hence limit preserving by Limits of Stacks, Proposition 3.8. Thus Artin approximation as in Artin's Axioms, Lemma 12.7 shows that we may find a morphism $U \rightarrow \mathcal{X}$ with source a finite type S -scheme, containing a point $u_0 \in U$ of residue field k satisfying (1) and (2) such that $U \rightarrow \mathcal{X}$ is versal at u_0 . By Lemma 2.6 after shrinking U we may assume $U \rightarrow \mathcal{X}$ is smooth. \square

Remark 2.9 (Upgrading versal rings). In Situation 2.1 let $x_0 : \mathrm{Spec}(k) \rightarrow \mathcal{X}$ be a morphism, where k is a finite type field over S . Let A be a versal ring to \mathcal{X} at

x_0 . By Artin's Axioms, Lemma 9.5 our versal formal object in fact comes from a morphism

$$\mathrm{Spec}(A) \longrightarrow \mathcal{X}$$

over S . Moreover, the results above each can be upgraded to be compatible with this morphism. Here is a list:

- (1) in Lemma 2.4 the isomorphism $A \cong A'[[t_1, \dots, t_r]]$ or $A' \cong A[[t_1, \dots, t_r]]$ may be chosen compatible with these morphisms,
- (2) in Lemma 2.5 the homomorphism $A \rightarrow A'$ may be chosen compatible with these morphisms,
- (3) in Lemma 2.6 the morphism $\mathrm{Spec}(\mathcal{O}_{U, u_0}^\wedge) \rightarrow \mathcal{X}$ is the composition of the canonical map $\mathrm{Spec}(\mathcal{O}_{U, u_0}^\wedge) \rightarrow U$ and the given map $U \rightarrow \mathcal{X}$,
- (4) in Lemma 2.8 the isomorphism $\mathcal{O}_{U, u_0}^\wedge \cong A$ may be chosen so $\mathrm{Spec}(A) \rightarrow \mathcal{X}$ corresponds to the canonical map in the item above.

In each case the statement follows from the fact that our maps are compatible with versal formal elements; we note however that the implied diagrams are 2-commutative only up to a (noncanonical) choice of a 2-arrow. Still, this means that the implied map $A' \rightarrow A$ or $A \rightarrow A'$ in (1) is well defined up to formal homotopy, see Formal Deformation Theory, Lemma 28.3.

Lemma 2.10. *In Situation 2.1 let $x_0 : \mathrm{Spec}(k) \rightarrow \mathcal{X}$ be a morphism, where k is a finite type field over S . Let A be a versal ring to \mathcal{X} at x_0 . Then the morphism $\mathrm{Spec}(A) \rightarrow \mathcal{X}$ of Remark 2.9 is flat.*

Proof. If the local ring of S at the image point is a G-ring, then this follows immediately from Lemma 2.8 and the fact that the map from a Noetherian local ring to its completion is flat. In general we prove it as follows.

Step I. If A and A' are two versal rings to \mathcal{X} at x_0 , then the result is true for A if and only if it is true for A' . Namely, after possible swapping A and A' , we may assume there is a formally smooth map $\varphi : A \rightarrow A'$ such that the composition

$$\mathrm{Spec}(A') \rightarrow \mathrm{Spec}(A) \rightarrow \mathcal{X}$$

is the morphism $\mathrm{Spec}(A') \rightarrow \mathcal{X}$, see Lemma 2.4 and Remark 2.9. Since $A \rightarrow A'$ is faithfully flat we obtain the equivalence from Morphisms of Stacks, Lemmas 25.2 and 25.5.

Step II. Let l/k be a finite extension of fields. Let $x_{l,0} : \mathrm{Spec}(l) \rightarrow \mathcal{X}$ be the induced morphism. Let A be a versal ring to \mathcal{X} at x_0 and let $A \rightarrow A'$ be as in Lemma 2.5. Then again the composition

$$\mathrm{Spec}(A') \rightarrow \mathrm{Spec}(A) \rightarrow \mathcal{X}$$

is the morphism $\mathrm{Spec}(A') \rightarrow \mathcal{X}$, see Remark 2.9. Arguing as before and using step I to see choice of versal rings is irrelevant, we see that the lemma holds for x_0 if and only if it holds for $x_{l,0}$.

Step III. Choose a scheme U and a surjective smooth morphism $U \rightarrow \mathcal{X}$. Then we can choose a finite type point z_0 on $Z = U \times_{\mathcal{X}} x_0$ (this is a nonempty algebraic space). Let $u_0 \in U$ be the image of z_0 in U . Choose a scheme and a surjective étale map $W \rightarrow Z$ such that z_0 is the image of a closed point $w_0 \in W$ (see Morphisms of Spaces, Section 25). Since $W \rightarrow \mathrm{Spec}(k)$ and $W \rightarrow U$ are of finite type, we see that $\kappa(w_0)/k$ and $\kappa(w_0)/\kappa(u_0)$ are finite extensions of fields (see Morphisms,

Section 16). Applying Step II twice we may replace x_0 by $u_0 \rightarrow U \rightarrow \mathcal{X}$. Then we see our morphism is the composition

$$\mathrm{Spec}(\mathcal{O}_{U,u_0}^\wedge) \rightarrow U \rightarrow \mathcal{X}$$

The first arrow is flat because completion of Noetherian local rings are flat (Algebra, Lemma 97.2) and the second arrow is flat as a smooth morphism is flat. The composition is flat as composition preserves flatness. \square

Remark 2.11. In Situation 2.1 let $x_0 : \mathrm{Spec}(k) \rightarrow \mathcal{X}$ be a morphism, where k is a finite type field over S . By Lemma 2.3 and Formal Deformation Theory, Theorem 26.4 we know that $\mathcal{F}_{\mathcal{X},k,x_0}$ has a presentation by a smooth prorepresentable groupoid in functors on \mathcal{C}_Λ . Unwinding the definitions, this means we can choose

- (1) a Noetherian complete local Λ -algebra A with residue field k and a versal formal object ξ of $\mathcal{F}_{\mathcal{X},k,x_0}$ over A ,
- (2) a Noetherian complete local Λ -algebra B with residue field k and an isomorphism

$$\underline{B}|_{\mathcal{C}_\Lambda} \longrightarrow \underline{A}|_{\mathcal{C}_\Lambda} \times_{\xi, \mathcal{F}_{\mathcal{X},k,x_0}, \xi} \underline{A}|_{\mathcal{C}_\Lambda}$$

The projections correspond to formally smooth maps $t : A \rightarrow B$ and $s : A \rightarrow B$ (because ξ is versal). There is a map $c : B \rightarrow B \hat{\otimes}_{s,A,t} B$ which turns (A, B, s, t, c) into a cogroupoid in the category of Noetherian complete local Λ -algebras with residue field k (on prorepresentable functors this map is constructed in Formal Deformation Theory, Lemma 25.2). Finally, the cited theorem tells us that ξ induces an equivalence

$$[\underline{A}|_{\mathcal{C}_\Lambda} / \underline{B}|_{\mathcal{C}_\Lambda}] \longrightarrow \mathcal{F}_{\mathcal{X},k,x_0}$$

of groupoids cofibred over \mathcal{C}_Λ . In fact, we also get an equivalence

$$[\underline{A}/\underline{B}] \longrightarrow \widehat{\mathcal{F}}_{\mathcal{X},k,x_0}$$

of groupoids cofibred over the completed category $\widehat{\mathcal{C}}_\Lambda$ (see discussion in Formal Deformation Theory, Section 22 as to why this works). Of course A is a versal ring to \mathcal{X} at x_0 .

3. Multiplicities of components of algebraic stacks

If X is a locally Noetherian scheme, then we may write X (thought of simply as a topological space) as a union of irreducible components, say $X = \bigcup T_i$. Each irreducible component is the closure of a unique generic point ξ_i , and the local ring \mathcal{O}_{X,ξ_i} is a local Artin ring. We may define the *multiplicity of X along T_i* or the *multiplicity of T_i in X* by

$$m_{T_i,X} = \mathrm{length}_{\mathcal{O}_{X,\xi_i}} \mathcal{O}_{X,\xi_i}$$

In other words, it is the length of the local Artinian ring. Please compare with Chow Homology, Section 9.

Our goal here is to generalise this definition to locally Noetherian algebraic stacks. If \mathcal{X} is a stack, then its topological space $|\mathcal{X}|$ (see Properties of Stacks, Definition 4.8) is locally Noetherian (Morphisms of Stacks, Lemma 8.3). The irreducible components of $|\mathcal{X}|$ are sometimes referred to as the irreducible components of \mathcal{X} . If \mathcal{X} is quasi-separated, then $|\mathcal{X}|$ is sober (Morphisms of Stacks, Lemma 30.3), but it

need not be in the non-quasi-separated case. Consider for example the non-quasi-separated algebraic space $X = \mathbf{A}_{\mathbf{C}}^1/\mathbf{Z}$. Furthermore, there is no structure sheaf on $|\mathcal{X}|$ whose stalks can be used to define multiplicities.

Lemma 3.1. *Let $f : U \rightarrow \mathcal{X}$ be a smooth morphism from a scheme to a locally Noetherian algebraic stack. The closure of the image of any irreducible component of $|U|$ is an irreducible component of $|\mathcal{X}|$. If $U \rightarrow \mathcal{X}$ is surjective, then all irreducible components of $|\mathcal{X}|$ are obtained in this way.*

Proof. The map $|U| \rightarrow |\mathcal{X}|$ is continuous and open by Properties of Stacks, Lemma 4.7. Let $T \subset |U|$ be an irreducible component. Since U is locally Noetherian, we can find a nonempty affine open $W \subset U$ contained in T . Then $f(T) \subset |\mathcal{X}|$ is irreducible and contains the nonempty open subset $f(W)$. Thus the closure of $f(T)$ is irreducible and contains a nonempty open. It follows that this closure is an irreducible component.

Assume $U \rightarrow \mathcal{X}$ is surjective and let $Z \subset |\mathcal{X}|$ be an irreducible component. Choose a Noetherian open subset V of $|\mathcal{X}|$ meeting Z . After removing the other irreducible components from V we may assume that $V \subset Z$. Take an irreducible component of the nonempty open $f^{-1}(V) \subset |U|$ and let $T \subset |U|$ be its closure. This is an irreducible component of $|U|$ and the closure of $f(T)$ must agree with Z by our choice of T . \square

The preceding lemma applies in particular in the case of smooth morphisms between locally Noetherian schemes. This particular case is implicitly invoked in the statement of the following lemma.

Lemma 3.2. *Let $U \rightarrow X$ be a smooth morphism of locally Noetherian schemes. Let T' be an irreducible component of U . Let T be the irreducible component of X obtained as the closure of the image of T' . Then $m_{T',U} = m_{T,X}$.*

Proof. Write ξ' for the generic point of T' , and ξ for the generic point of T . Let $A = \mathcal{O}_{X,\xi}$ and $B = \mathcal{O}_{U,\xi'}$. We need to show that $\text{length}_A A = \text{length}_B B$. Since $A \rightarrow B$ is a flat local homomorphism of rings (since smooth morphisms are flat), we have

$$\text{length}_A(A) \text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(B)$$

by Algebra, Lemma 52.13. Thus it suffices to show $\mathfrak{m}_A B = \mathfrak{m}_B$, or equivalently, that $B/\mathfrak{m}_A B$ is reduced. Since $U \rightarrow X$ is smooth, so is its base change $U_\xi \rightarrow \text{Spec } \kappa(\xi)$. As U_ξ is a smooth scheme over a field, it is reduced, and thus so its local ring at any point (Varieties, Lemma 25.4). In particular,

$$B/\mathfrak{m}_A B = \mathcal{O}_{U,\xi'}/\mathfrak{m}_{X,\xi} \mathcal{O}_{U,\xi'} = \mathcal{O}_{U_\xi,\xi'}$$

is reduced, as required. \square

Using this result, we may show that there exists a good notion of multiplicity by looking smooth locally.

Lemma 3.3. *Let $U_1 \rightarrow \mathcal{X}$ and $U_2 \rightarrow \mathcal{X}$ be two smooth morphisms from schemes to a locally Noetherian algebraic stack \mathcal{X} . Let T'_1 and T'_2 be irreducible components of $|U_1|$ and $|U_2|$ respectively. Assume the closures of the images of T'_1 and T'_2 are the same irreducible component T of $|\mathcal{X}|$. Then $m_{T'_1,U_1} = m_{T'_2,U_2}$.*

Proof. Let V_1 and V_2 be dense subsets of T'_1 and T'_2 , respectively, that are open in U_1 and U_2 respectively (see proof of Lemma 3.1). The images of $|V_1|$ and $|V_2|$ in $|\mathcal{X}|$ are non-empty open subsets of the irreducible subset T , and therefore have non-empty intersection. By Properties of Stacks, Lemma 4.3, the map $|V_1 \times_{\mathcal{X}} V_2| \rightarrow |V_1| \times_{|\mathcal{X}|} |V_2|$ is surjective. Consequently $V_1 \times_{\mathcal{X}} V_2$ is a non-empty algebraic space; we may therefore choose an étale surjection $V \rightarrow V_1 \times_{\mathcal{X}} V_2$ whose source is a (non-empty) scheme. If we let T' be any irreducible component of V , then Lemma 3.1 shows that the closure of the image of T' in U_1 (respectively U_2) is equal to T'_1 (respectively T'_2).

Applying Lemma 3.2 twice we find that

$$m_{T'_1, U_1} = m_{T', V} = m_{T'_2, U_2},$$

as required. \square

At this point we have done enough work to show the following definition makes sense.

Definition 3.4. Let \mathcal{X} be a locally Noetherian algebraic stack. Let $T \subset |\mathcal{X}|$ be an irreducible component. The *multiplicity* of T in \mathcal{X} is defined as $m_{T, \mathcal{X}} = m_{T', U}$ where $f : U \rightarrow \mathcal{X}$ is a smooth morphism from a scheme and $T' \subset |U|$ is an irreducible component with $f(T') \subset T$.

This is independent of the choice of $f : U \rightarrow \mathcal{X}$ and the choice of the irreducible component T' mapping to T by Lemmas 3.1 and 3.3.

As a closing remark, we note that it is sometimes convenient to think of an irreducible component of \mathcal{X} as a closed substack. To this end, if $T \subset |\mathcal{X}|$ is an irreducible component, then we may consider the unique reduced closed substack $\mathcal{T} \subset \mathcal{X}$ with $|\mathcal{T}| = T$, see Properties of Stacks, Definition 10.4. If \mathcal{X} is quasi-separated, then an irreducible component is an integral stack; see Morphisms of Stacks, Section 50 for further discussion.

4. Formal branches and multiplicities

It will be convenient to have a comparison between the notion of multiplicity of an irreducible component given by Definition 3.4 and the related notion of multiplicities of irreducible components of (the spectra of) versal rings of \mathcal{X} at finite type points.

In Situation 2.1 let $x_0 : \mathrm{Spec}(k) \rightarrow \mathcal{X}$ be a morphism, where k is a finite type field over S . Let A, A' be versal rings to \mathcal{X} at x_0 . After possibly swapping A and A' , we know there is a formally smooth¹ map $\varphi : A \rightarrow A'$ compatible with versal formal objects, see Lemma 2.4 and Remark 2.9. Moreover, φ is well defined up to formal homotopy, see Formal Deformation Theory, Lemma 28.3. In particular, we find that $\varphi(\mathfrak{p})A'$ is a well defined ideal of A' by Formal Deformation Theory, Lemma 28.4. Since $A \rightarrow A'$ is formally smooth, in fact $\varphi(\mathfrak{p})A'$ is a minimal prime of A' and every minimal prime of A' is of this form for a unique minimal prime $\mathfrak{p} \subset A$ (all of this is easy to prove by writing A' as a power series ring over A). Therefore, recalling that minimal primes correspond to irreducible components, the following definition makes sense.

¹In the sense that A' becomes isomorphic to a power series ring over A .

Definition 4.1. Let \mathcal{X} be an algebraic stack locally of finite type over a locally Noetherian scheme S . Let $x_0 : \operatorname{Spec}(k) \rightarrow \mathcal{X}$ is a morphism where k is a field of finite type over S . The *formal branches of \mathcal{X} through x_0* is the set of irreducible components of $\operatorname{Spec}(A)$ for any choice of versal ring to \mathcal{X} at x_0 identified for different choices of A by the procedure described above.

Suppose in the situation of Definition 4.1 we are given a finite extension l/k . Set $x_{l,0} : \operatorname{Spec}(l) \rightarrow \mathcal{X}$ equal to the composition of $\operatorname{Spec}(l) \rightarrow \operatorname{Spec}(k)$ with x_0 . Let $A \rightarrow A'$ be as in Lemma 2.5. Since $A \rightarrow A'$ is faithfully flat, the morphism

$$\operatorname{Spec}(A') \rightarrow \operatorname{Spec}(A)$$

sends (generic points of) irreducible components to (generic points of) irreducible components. This will be a surjective map, but in general this map will not be a bijection. In other words, we obtain a surjective map

$$\text{formal branches of } \mathcal{X} \text{ through } x_{l,0} \longrightarrow \text{formal branches of } \mathcal{X} \text{ through } x_0$$

It turns out that if l/k is purely inseparable, then the map is injective as well (we'll add a precise statement and proof here if we ever need this).

Lemma 4.2. *In the situation of Definition 4.1 there is a canonical surjection from the set of formal branches of \mathcal{X} through x_0 to the set of irreducible components of $|\mathcal{X}|$ containing x_0 in $|\mathcal{X}|$.*

Proof. Let A be as in Definition 4.1 and let $\operatorname{Spec}(A) \rightarrow \mathcal{X}$ be as in Remark 2.9. We claim that the generic point of an irreducible component of $\operatorname{Spec}(A)$ maps to a generic point of an irreducible component of $|\mathcal{X}|$. Choose a scheme U and a surjective smooth morphism $U \rightarrow \mathcal{X}$. Consider the diagram

$$\begin{array}{ccc} \operatorname{Spec}(A) \times_{\mathcal{X}} U & \xrightarrow{q} & U \\ p \downarrow & & \downarrow f \\ \operatorname{Spec}(A) & \xrightarrow{j} & \mathcal{X} \end{array}$$

By Lemma 2.10 we see that j is flat. Hence q is flat. On the other hand, f is surjective smooth hence p is surjective smooth. This implies that any generic point $\eta \in \operatorname{Spec}(A)$ of an irreducible component is the image of a codimension 0 point η' of the algebraic space $\operatorname{Spec}(A) \times_{\mathcal{X}} U$ (see Properties of Spaces, Section 11 for notation and use going down on étale local rings). Since q is flat, $q(\eta')$ is a codimension 0 point of U (same argument). Since U is a scheme, $q(\eta')$ is the generic point of an irreducible component of U . Thus the closure of the image of $q(\eta')$ in $|\mathcal{X}|$ is an irreducible component by Lemma 3.1 as claimed.

Clearly the claim provides a mechanism for defining the desired map. To see that it is surjective, we choose $u_0 \in U$ mapping to x_0 in $|\mathcal{X}|$. Choose an affine open $U' \subset U$ neighbourhood of u_0 . After shrinking U' we may assume every irreducible component of U' passes through u_0 . Then we may replace \mathcal{X} by the open substack corresponding to the image of $|U'| \rightarrow |\mathcal{X}|$. Thus we may assume U is affine has a point u_0 mapping to $x_0 \in |\mathcal{X}|$ and every irreducible component of U passes through u_0 . By Properties of Stacks, Lemma 4.3 there is a point $t \in |\operatorname{Spec}(A) \times_{\mathcal{X}} U|$ mapping to the closed point of $\operatorname{Spec}(A)$ and to u_0 . Using going down for the flat local ring homomorphisms

$$A \longrightarrow \mathcal{O}_{\operatorname{Spec}(A) \times_{\mathcal{X}} U, \bar{t}} \longleftarrow \mathcal{O}_{U, u_0}$$

we see that every minimal prime of \mathcal{O}_{U,u_0} is the image of a minimal prime of the local ring in the middle and such a minimal prime maps to a minimal prime of A . This proves the surjectivity. Some details omitted. \square

Let A be a Noetherian complete local ring. Then the irreducible components of $\text{Spec}(A)$ have multiplicities, see introduction to Section 3. If $A' = A[[t_1, \dots, t_r]]$, then the morphism $\text{Spec}(A') \rightarrow \text{Spec}(A)$ induces a bijection on irreducible components preserving multiplicities (we omit the easy proof). This and the discussion preceding Definition 4.1 mean that the following definition makes sense.

Definition 4.3. Let \mathcal{X} be an algebraic stack locally of finite type over a locally Noetherian scheme S . Let $x_0 : \text{Spec}(k) \rightarrow \mathcal{X}$ is a morphism where k is a field of finite type over S . The *multiplicity of a formal branch of \mathcal{X} through x_0* is the multiplicity of the corresponding irreducible component of $\text{Spec}(A)$ for any choice of versal ring to \mathcal{X} at x_0 (see discussion above).

Lemma 4.4. *Let \mathcal{X} be an algebraic stack locally of finite type over a locally Noetherian scheme S . Let $x_0 : \text{Spec}(k) \rightarrow \mathcal{X}$ is a morphism where k is a field of finite type over S with image $s \in S$. If $\mathcal{O}_{S,s}$ is a G-ring, then the map of Lemma 4.2 preserves multiplicities.*

Proof. By Lemma 2.8 we may assume there is a smooth morphism $U \rightarrow \mathcal{X}$ where U is a scheme and a k -valued point u_0 of U such that $\mathcal{O}_{U,u_0}^\wedge$ is a versal ring to \mathcal{X} at x_0 . By construction of our map in the proof of Lemma 4.2 (which simplifies greatly because $A = \mathcal{O}_{U,u_0}^\wedge$) we find that it suffices to show: the multiplicity of an irreducible component of U passing through u_0 is the same as the multiplicity of any irreducible component of $\text{Spec}(\mathcal{O}_{U,u_0}^\wedge)$ mapping into it.

Translated into commutative algebra we find the following: Let $C = \mathcal{O}_{U,u_0}$. This is essentially of finite type over $\mathcal{O}_{S,s}$ and hence is a G-ring (More on Algebra, Proposition 50.10). Then $A = C^\wedge$. Therefore $C \rightarrow A$ is a regular ring map. Let $\mathfrak{q} \subset C$ be a minimal prime and let $\mathfrak{p} \subset A$ be a minimal prime lying over \mathfrak{q} . Then

$$R = C_{\mathfrak{p}} \longrightarrow A_{\mathfrak{p}} = R'$$

is a regular ring map of Artinian local rings. For such a ring map it is always the case that

$$\text{length}_R R = \text{length}_{R'} R'$$

This is what we have to show because the left hand side is the multiplicity of our component on U and the right hand side is the multiplicity of our component on $\text{Spec}(A)$. To see the equality, first we use that

$$\text{length}_R(R) \text{length}_{R'}(R'/\mathfrak{m}_R R') = \text{length}_{R'}(R')$$

by Algebra, Lemma 52.13. Thus it suffices to show $\mathfrak{m}_R R' = \mathfrak{m}_{R'}$, which is a consequence of being a regular homomorphism of zero dimensional local rings. \square

5. Dimension theory of algebraic stacks

The main results on the dimension theory of algebraic stacks in the literature that we are aware of are those of [Oss15], which makes a study of the notions of codimension and relative dimension. We make a more detailed examination of the notion of the dimension of an algebraic stack at a point, and prove various results relating the dimension of the fibres of a morphism at a point in the source to the dimension

of its source and target. We also prove a result (Lemma 6.4 below) which allow us (under suitable hypotheses) to compute the dimension of an algebraic stack at a point in terms of a versal ring.

While we haven't always tried to optimise our results, we have largely tried to avoid making unnecessary hypotheses. However, in some of our results, in which we compare certain properties of an algebraic stack to the properties of a versal ring to this stack at a point, we have restricted our attention to the case of algebraic stacks that are locally finitely presented over a locally Noetherian scheme base, all of whose local rings are G -rings. This gives us the convenience of having Artin approximation available to compare the geometry of the versal ring to the geometry of the stack itself. However, this restrictive hypothesis may not be necessary for the truth of all of the various statements that we prove. Since it is satisfied in the applications that we have in mind, though, we have been content to make it when it helps.

If X is a scheme, then we define the dimension $\dim(X)$ of X to be the Krull dimension of the topological space underlying X , while if x is a point of X , then we define the dimension $\dim_x(X)$ of X at x to be the minimum of the dimensions of the open subsets U of X containing x , see Properties, Definition 10.1. One has the relation $\dim(X) = \sup_{x \in X} \dim_x(X)$, see Properties, Lemma 10.2. If X is locally Noetherian, then $\dim_x(X)$ coincides with the supremum of the dimensions at x of the irreducible components of X passing through x .

If X is an algebraic space and $x \in |X|$, then we define $\dim_x X = \dim_u U$, where U is any scheme admitting an étale surjection $U \rightarrow X$, and $u \in U$ is any point lying over x , see Properties of Spaces, Definition 9.1. We set $\dim(X) = \sup_{x \in |X|} \dim_x(X)$, see Properties of Spaces, Definition 9.2.

Remark 5.1. In general, the dimension of the algebraic space X at a point x may not coincide with the dimension of the underlying topological space $|X|$ at x . E.g. if k is a field of characteristic zero and $X = \mathbf{A}_k^1/\mathbf{Z}$, then X has dimension 1 (the dimension of \mathbf{A}_k^1) at each of its points, while $|X|$ has the indiscrete topology, and hence is of Krull dimension zero. On the other hand, in Algebraic Spaces, Example 14.9 there is given an example of an algebraic space which is of dimension 0 at each of its points, while $|X|$ is irreducible of Krull dimension 1, and admits a generic point (so that the dimension of $|X|$ at any of its points is 1); see also the discussion of this example in Properties of Spaces, Section 9.

On the other hand, if X is a *decent* algebraic space, in the sense of Decent Spaces, Definition 6.1 (in particular, if X is quasi-separated; see Decent Spaces, Section 6) then in fact the dimension of X at x does coincide with the dimension of $|X|$ at x ; see Decent Spaces, Lemma 12.5.

In order to define the dimension of an algebraic stack, it will be useful to first have the notion of the relative dimension, at a point in the source, of a morphism whose source is an algebraic space, and whose target is an algebraic stack. The definition is slightly involved, just because (unlike in the case of schemes) the points of an algebraic stack, or an algebraic space, are not describable as morphisms from the spectrum of a field, but only as equivalence classes of such.

Definition 5.2. If $f : T \rightarrow \mathcal{X}$ is a locally of finite type morphism from an algebraic space to an algebraic stack, and if $t \in |T|$ is a point with image $x \in |\mathcal{X}|$, then we

define the *relative dimension* of f at t , denoted $\dim_t(T_x)$, as follows: choose a morphism $\mathrm{Spec} k \rightarrow \mathcal{X}$, with source the spectrum of a field, which represents x , and choose a point $t' \in |T \times_{\mathcal{X}} \mathrm{Spec} k|$ mapping to t under the projection to $|T|$ (such a point t' exists, by Properties of Stacks, Lemma 4.3); then

$$\dim_t(T_x) = \dim_{t'}(T \times_{\mathcal{X}} \mathrm{Spec} k).$$

Note that since T is an algebraic space and \mathcal{X} is an algebraic stack, the fibre product $T \times_{\mathcal{X}} \mathrm{Spec} k$ is an algebraic space, and so the quantity on the right hand side of this proposed definition is in fact defined (see discussion above).

Remark 5.3. (1) One easily verifies (for example, by using the invariance of the relative dimension of locally of finite type morphisms of schemes under base-change; see for example Morphisms, Lemma 28.3) that $\dim_t(T_x)$ is well-defined, independently of the choices used to compute it.

(2) In the case that \mathcal{X} is also an algebraic space, it is straightforward to confirm that this definition agrees with the definition of relative dimension given in Morphisms of Spaces, Definition 33.1.

We next recall the following lemma, on which our study of the dimension of a locally Noetherian algebraic stack is founded.

Lemma 5.4. *If $f : U \rightarrow X$ is a smooth morphism of locally Noetherian algebraic spaces, and if $u \in |U|$ with image $x \in |X|$, then*

$$\dim_u(U) = \dim_x(X) + \dim_u(U_x)$$

where $\dim_u(U_x)$ is defined via Definition 5.2.

Proof. See Morphisms of Spaces, Lemma 37.10 noting that the definition of $\dim_u(U_x)$ used here coincides with the definition used there, by Remark 5.3 (2). \square

Lemma 5.5. *If \mathcal{X} is a locally Noetherian algebraic stack and $x \in |\mathcal{X}|$. Let $U \rightarrow \mathcal{X}$ be a smooth morphism from an algebraic space to \mathcal{X} , let u be any point of $|U|$ mapping to x . Then we have*

$$\dim_x(\mathcal{X}) = \dim_u(U) - \dim_u(U_x)$$

where the relative dimension $\dim_u(U_x)$ is defined by Definition 5.2 and the dimension of \mathcal{X} at x is as in Properties of Stacks, Definition 12.2.

Proof. Lemma 5.4 can be used to verify that the right hand side $\dim_u(U) + \dim_u(U_x)$ is independent of the choice of the smooth morphism $U \rightarrow \mathcal{X}$ and $u \in |U|$. We omit the details. In particular, we may assume U is a scheme. In this case we can compute $\dim_u(U_x)$ by choosing the representative of x to be the composite $\mathrm{Spec} \kappa(u) \rightarrow U \rightarrow \mathcal{X}$, where the first morphism is the canonical one with image $u \in U$. Then, if we write $R = U \times_{\mathcal{X}} U$, and let $e : U \rightarrow R$ denote the diagonal morphism, the invariance of relative dimension under base-change shows that $\dim_u(U_x) = \dim_{e(u)}(R_u)$. Thus we see that the right hand side is equal to $\dim_u(U) - \dim_{e(u)}(R_u) = \dim_x(\mathcal{X})$ as desired. \square

Remark 5.6. For Deligne–Mumford stacks which are suitably decent (e.g. quasi-separated), it will again be the case that $\dim_x(\mathcal{X})$ coincides with the topologically defined quantity $\dim_x |\mathcal{X}|$. However, for more general Artin stacks, this will typically not be the case. For example, if $\mathcal{X} = [\mathbf{A}^1/\mathbf{G}_m]$ (over some field, with the

quotient being taken with respect to the usual multiplication action of \mathbf{G}_m on \mathbf{A}^1), then $|\mathcal{X}|$ has two points, one the specialisation of the other (corresponding to the two orbits of \mathbf{G}_m on \mathbf{A}^1), and hence is of dimension 1 as a topological space; but $\dim_x(\mathcal{X}) = 0$ for both points $x \in |\mathcal{X}|$. (An even more extreme example is given by the classifying space $[\mathrm{Spec} k/\mathbf{G}_m]$, whose dimension at its unique point is equal to -1 .)

We can now extend Definition 5.2 to the context of (locally finite type) morphisms between (locally Noetherian) algebraic stacks.

Definition 5.7. If $f : \mathcal{T} \rightarrow \mathcal{X}$ is a locally of finite type morphism between locally Noetherian algebraic stacks, and if $t \in |\mathcal{T}|$ is a point with image $x \in |\mathcal{X}|$, then we define the *relative dimension* of f at t , denoted $\dim_t(\mathcal{T}_x)$, as follows: choose a morphism $\mathrm{Spec} k \rightarrow \mathcal{X}$, with source the spectrum of a field, which represents x , and choose a point $t' \in |\mathcal{T} \times_{\mathcal{X}} \mathrm{Spec} k|$ mapping to t under the projection to $|\mathcal{T}|$ (such a point t' exists, by Properties of Stacks, Lemma 4.3; then

$$\dim_t(\mathcal{T}_x) = \dim_{t'}(\mathcal{T} \times_{\mathcal{X}} \mathrm{Spec} k).$$

Note that since \mathcal{T} is an algebraic stack and \mathcal{X} is an algebraic stack, the fibre product $\mathcal{T} \times_{\mathcal{X}} \mathrm{Spec} k$ is an algebraic stack, which is locally Noetherian by Morphisms of Stacks, Lemma 17.5. Thus the quantity on the right side of this proposed definition is defined by Properties of Stacks, Definition 12.2.

Remark 5.8. Standard manipulations show that $\dim_t(\mathcal{T}_x)$ is well-defined, independently of the choices made to compute it.

We now establish some basic properties of relative dimension, which are obvious generalisations of the corresponding statements in the case of morphisms of schemes.

Lemma 5.9. *Suppose given a Cartesian square of morphisms of locally Noetherian stacks*

$$\begin{array}{ccc} \mathcal{T}' & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow \\ \mathcal{X}' & \longrightarrow & \mathcal{X} \end{array}$$

in which the vertical morphisms are locally of finite type. If $t' \in |\mathcal{T}'|$, with images $t, x',$ and x in $|\mathcal{T}|, |\mathcal{X}'|,$ and $|\mathcal{X}|$ respectively, then $\dim_{t'}(\mathcal{T}'_{x'}) = \dim_t(\mathcal{T}_x)$.

Proof. Both sides can (by definition) be computed as the dimension of the same fibre product. \square

Lemma 5.10. *If $f : \mathcal{U} \rightarrow \mathcal{X}$ is a smooth morphism of locally Noetherian algebraic stacks, and if $u \in |\mathcal{U}|$ with image $x \in |\mathcal{X}|$, then*

$$\dim_u(\mathcal{U}) = \dim_x(\mathcal{X}) + \dim_u(\mathcal{U}_x).$$

Proof. Choose a smooth surjective morphism $V \rightarrow \mathcal{U}$ whose source is a scheme, and let $v \in |V|$ be a point mapping to u . Then the composite $V \rightarrow \mathcal{U} \rightarrow \mathcal{X}$ is also smooth, and by Lemma 5.4 we have $\dim_x(\mathcal{X}) = \dim_v(V) - \dim_v(V_x)$, while $\dim_u(\mathcal{U}) = \dim_v(V) - \dim_v(V_u)$. Thus

$$\dim_u(\mathcal{U}) - \dim_x(\mathcal{X}) = \dim_v(V_x) - \dim_v(V_u).$$

Choose a representative $\text{Spec } k \rightarrow \mathcal{X}$ of x and choose a point $v' \in |V \times_{\mathcal{X}} \text{Spec } k|$ lying over v , with image u' in $|\mathcal{U} \times_{\mathcal{X}} \text{Spec } k|$; then by definition $\dim_u(\mathcal{U}_x) = \dim_{u'}(\mathcal{U} \times_{\mathcal{X}} \text{Spec } k)$, and $\dim_v(V_x) = \dim_{v'}(V \times_{\mathcal{X}} \text{Spec } k)$.

Now $V \times_{\mathcal{X}} \text{Spec } k \rightarrow \mathcal{U} \times_{\mathcal{X}} \text{Spec } k$ is a smooth surjective morphism (being the base-change of such a morphism) whose source is an algebraic space (since V and $\text{Spec } k$ are schemes, and \mathcal{X} is an algebraic stack). Thus, again by definition, we have

$$\begin{aligned} \dim_{u'}(\mathcal{U} \times_{\mathcal{X}} \text{Spec } k) &= \dim_{v'}(V \times_{\mathcal{X}} \text{Spec } k) - \dim_{v'}(V \times_{\mathcal{X}} \text{Spec } k)_{u'} \\ &= \dim_v(V_x) - \dim_{v'}((V \times_{\mathcal{X}} \text{Spec } k)_{u'}). \end{aligned}$$

Now $V \times_{\mathcal{X}} \text{Spec } k \cong V \times_{\mathcal{U}}(\mathcal{U} \times_{\mathcal{X}} \text{Spec } k)$, and so Lemma 5.9 shows that $\dim_{v'}((V \times_{\mathcal{X}} \text{Spec } k)_{u'}) = \dim_v(V_x)$. Putting everything together, we find that

$$\dim_u(\mathcal{U}) - \dim_x(\mathcal{X}) = \dim_u(\mathcal{U}_x),$$

as required. \square

Lemma 5.11. *Let $f : \mathcal{T} \rightarrow \mathcal{X}$ be a locally of finite type morphism of algebraic stacks.*

- (1) *The function $t \mapsto \dim_t(\mathcal{T}_{f(t)})$ is upper semi-continuous on $|\mathcal{T}|$.*
- (2) *If f is smooth, then the function $t \mapsto \dim_t(\mathcal{T}_{f(t)})$ is locally constant on $|\mathcal{T}|$.*

Proof. Suppose to begin with that \mathcal{T} is a scheme T , let $U \rightarrow \mathcal{X}$ be a smooth surjective morphism whose source is a scheme, and let $T' = T \times_{\mathcal{X}} U$. Let $f' : T' \rightarrow U$ be the pull-back of f over U , and let $g : T' \rightarrow T$ be the projection.

Lemma 5.9 shows that $\dim_{t'}(T'_{f'(t')}) = \dim_{g(t')}(T_{f(g(t'))})$, for $t' \in T'$, while, since g is smooth and surjective (being the base-change of a smooth surjective morphism) the map induced by g on underlying topological spaces is continuous and open (by Properties of Spaces, Lemma 4.6), and surjective. Thus it suffices to note that part (1) for the morphism f' follows from Morphisms of Spaces, Lemma 34.4, and part (2) from either of Morphisms, Lemma 29.4 or Morphisms, Lemma 34.12 (each of which gives the result for schemes, from which the analogous results for algebraic spaces can be deduced exactly as in Morphisms of Spaces, Lemma 34.4).

Now return to the general case, and choose a smooth surjective morphism $h : V \rightarrow \mathcal{T}$ whose source is a scheme. If $v \in V$, then, essentially by definition, we have

$$\dim_{h(v)}(\mathcal{T}_{f(h(v))}) = \dim_v(V_{f(h(v))}) - \dim_v(V_{h(v)}).$$

Since V is a scheme, we have proved that the first of the terms on the right hand side of this equality is upper semi-continuous (and even locally constant if f is smooth), while the second term is in fact locally constant. Thus their difference is upper semi-continuous (and locally constant if f is smooth), and hence the function $\dim_{h(v)}(\mathcal{T}_{f(h(v))})$ is upper semi-continuous on $|V|$ (and locally constant if f is smooth). Since the morphism $|V| \rightarrow |\mathcal{T}|$ is open and surjective, the lemma follows. \square

Before continuing with our development, we prove two lemmas related to the dimension theory of schemes.

To put the first lemma in context, we note that if X is a finite dimensional scheme, then since $\dim X$ is defined to equal the supremum of the dimensions $\dim_x X$, there exists a point $x \in X$ such that $\dim_x X = \dim X$. The following lemma shows that we may furthermore take the point x to be of finite type.

Lemma 5.12. *If X is a finite dimensional scheme, then there exists a closed (and hence finite type) point $x \in X$ such that $\dim_x X = \dim X$.*

Proof. Let $d = \dim X$, and choose a maximal strictly decreasing chain of irreducible closed subsets of X , say

$$(5.12.1) \quad Z_0 \supset Z_1 \supset \dots \supset Z_d.$$

The subset Z_d is a minimal irreducible closed subset of X , and thus any point of Z_d is a generic point of Z_d . Since the underlying topological space of the scheme X is sober, we conclude that Z_d is a singleton, consisting of a single closed point $x \in X$. If U is any neighbourhood of x , then the chain

$$U \cap Z_0 \supset U \cap Z_1 \supset \dots \supset U \cap Z_d = Z_d = \{x\}$$

is then a strictly descending chain of irreducible closed subsets of U , showing that $\dim U \geq d$. Thus we find that $\dim_x X \geq d$. The other inequality being obvious, the lemma is proved. \square

The next lemma shows that $\dim_x X$ is a *constant* function on an irreducible scheme satisfying some mild additional hypotheses.

Lemma 5.13. *If X is an irreducible, Jacobson, catenary, and locally Noetherian scheme of finite dimension, then $\dim U = \dim X$ for every non-empty open subset U of X . Equivalently, $\dim_x X$ is a constant function on X .*

Proof. The equivalence of the two claims follows directly from the definitions. Suppose, then, that $U \subset X$ is a non-empty open subset. Certainly $\dim U \leq \dim X$, and we have to show that $\dim U \geq \dim X$. Write $d = \dim X$, and choose a maximal strictly decreasing chain of irreducible closed subsets of X , say

$$X = Z_0 \supset Z_1 \supset \dots \supset Z_d.$$

Since X is Jacobson, the minimal irreducible closed subset Z_d is equal to $\{x\}$ for some closed point x .

If $x \in U$, then

$$U = U \cap Z_0 \supset U \cap Z_1 \supset \dots \supset U \cap Z_d = \{x\}$$

is a strictly decreasing chain of irreducible closed subsets of U , and so we conclude that $\dim U \geq d$, as required. Thus we may suppose that $x \notin U$.

Consider the flat morphism $\mathrm{Spec} \mathcal{O}_{X,x} \rightarrow X$. The non-empty (and hence dense) open subset U of X pulls back to an open subset $V \subset \mathrm{Spec} \mathcal{O}_{X,x}$. Replacing U by a non-empty quasi-compact, and hence Noetherian, open subset, we may assume that the inclusion $U \rightarrow X$ is a quasi-compact morphism. Since the formation of scheme-theoretic images of quasi-compact morphisms commutes with flat base-change Morphisms, Lemma 25.16 we see that V is dense in $\mathrm{Spec} \mathcal{O}_{X,x}$, and so in particular non-empty, and of course $x \notin V$. (Here we use x also to denote the closed point of $\mathrm{Spec} \mathcal{O}_{X,x}$, since its image is equal to the given point $x \in X$.) Now $\mathrm{Spec} \mathcal{O}_{X,x} \setminus \{x\}$ is Jacobson Properties, Lemma 6.4 and hence V contains a closed point z of $\mathrm{Spec} \mathcal{O}_{X,x} \setminus \{x\}$. The closure in X of the image of z is then an irreducible closed subset Z of X containing x , whose intersection with U is non-empty, and for which there is no irreducible closed subset properly contained in Z and properly containing $\{x\}$ (because pull-back to $\mathrm{Spec} \mathcal{O}_{X,x}$ induces a bijection between irreducible closed subsets of X containing x and irreducible closed subsets of $\mathrm{Spec} \mathcal{O}_{X,x}$). Since $U \cap Z$ is a non-empty closed subset of U , it contains a point

u that is closed in X (since X is Jacobson), and since $U \cap Z$ is a non-empty (and hence dense) open subset of the irreducible set Z (which contains a point not lying in U , namely x), the inclusion $\{u\} \subset U \cap Z$ is proper.

As X is catenary, the chain

$$X = Z_0 \supset Z \supset \{x\} = Z_d$$

can be refined to a chain of length $d + 1$, which must then be of the form

$$X = Z_0 \supset W_1 \supset \dots \supset W_{d-1} = Z \supset \{x\} = Z_d.$$

Since $U \cap Z$ is non-empty, we then find that

$$U = U \cap Z_0 \supset U \cap W_1 \supset \dots \supset U \cap W_{d-1} = U \cap Z \supset \{u\}$$

is a strictly decreasing chain of irreducible closed subsets of U of length $d + 1$, showing that $\dim U \geq d$, as required. \square

We will prove a stack-theoretic analogue of Lemma 5.13 in Lemma 5.17 below, but before doing so, we have to introduce an additional definition, necessitated by the fact that the notion of a scheme being catenary is not an étale local one (see the example of Algebra, Remark 164.8 which makes it difficult to define what it means for an algebraic space or algebraic stack to be catenary (see the discussion of [Oss15, page 3]). For certain aspects of dimension theory, the following definition seems to provide a good substitute for the missing notion of a catenary algebraic stack.

Definition 5.14. We say that a locally Noetherian algebraic stack \mathcal{X} is *pseudo-catenary* if there exists a smooth and surjective morphism $U \rightarrow \mathcal{X}$ whose source is a universally catenary scheme.

Example 5.15. If \mathcal{X} is locally of finite type over a universally catenary locally Noetherian scheme S , and $U \rightarrow \mathcal{X}$ is a smooth surjective morphism whose source is a scheme, then the composite $U \rightarrow \mathcal{X} \rightarrow S$ is locally of finite type, and so U is universally catenary Morphisms, Lemma 17.2. Thus \mathcal{X} is pseudo-catenary.

The following lemma shows that the property of being pseudo-catenary passes through finite-type morphisms.

Lemma 5.16. *If \mathcal{X} is a pseudo-catenary locally Noetherian algebraic stack, and if $\mathcal{Y} \rightarrow \mathcal{X}$ is a locally of finite type morphism, then there exists a smooth surjective morphism $V \rightarrow \mathcal{Y}$ whose source is a universally catenary scheme; thus \mathcal{Y} is again pseudo-catenary.*

Proof. By assumption we may find a smooth surjective morphism $U \rightarrow \mathcal{X}$ whose source is a universally catenary scheme. The base-change $U \times_{\mathcal{X}} \mathcal{Y}$ is then an algebraic stack; let $V \rightarrow U \times_{\mathcal{X}} \mathcal{Y}$ be a smooth surjective morphism whose source is a scheme. The composite $V \rightarrow U \times_{\mathcal{X}} \mathcal{Y} \rightarrow \mathcal{Y}$ is then smooth and surjective (being a composite of smooth and surjective morphisms), while the morphism $V \rightarrow U \times_{\mathcal{X}} \mathcal{Y} \rightarrow U$ is locally of finite type (being a composite of morphisms that are locally finite type). Since U is universally catenary, we see that V is universally catenary (by Morphisms, Lemma 17.2), as claimed. \square

We now study the behaviour of the function $\dim_x(\mathcal{X})$ on $|\mathcal{X}|$ (for some locally Noetherian stack \mathcal{X}) with respect to the irreducible components of $|\mathcal{X}|$, as well as various related topics.

Lemma 5.17. *If \mathcal{X} is a Jacobson, pseudo-catenary, and locally Noetherian algebraic stack for which $|\mathcal{X}|$ is irreducible, then $\dim_x(\mathcal{X})$ is a constant function on $|\mathcal{X}|$.*

Proof. It suffices to show that $\dim_x(\mathcal{X})$ is locally constant on $|\mathcal{X}|$, since it will then necessarily be constant (as $|\mathcal{X}|$ is connected, being irreducible). Since \mathcal{X} is pseudo-catenary, we may find a smooth surjective morphism $U \rightarrow \mathcal{X}$ with U being a universally catenary scheme. If $\{U_i\}$ is a cover of U by quasi-compact open subschemes, we may replace U by $\coprod U_i$, and it suffices to show that the function $u \mapsto \dim_{f(u)}(\mathcal{X})$ is locally constant on U_i . Since we check this for one U_i at a time, we now drop the subscript, and write simply U rather than U_i . Since U is quasi-compact, it is the union of a finite number of irreducible components, say $T_1 \cup \dots \cup T_n$. Note that each T_i is Jacobson, catenary, and locally Noetherian, being a closed subscheme of the Jacobson, catenary, and locally Noetherian scheme U .

By Lemma 5.4, we have $\dim_{f(u)}(\mathcal{X}) = \dim_u(U) - \dim_u(U_{f(u)})$. Lemma 5.11 (2) shows that the second term in the right hand expression is locally constant on U , as f is smooth, and hence we must show that $\dim_u(U)$ is locally constant on U . Since $\dim_u(U)$ is the maximum of the dimensions $\dim_u T_i$, as T_i ranges over the components of U containing u , it suffices to show that if a point u lies on two distinct components, say T_i and T_j (with $i \neq j$), then $\dim_u T_i = \dim_u T_j$, and then to note that $t \mapsto \dim_t T$ is a constant function on an irreducible Jacobson, catenary, and locally Noetherian scheme T (as follows from Lemma 5.13).

Let $V = T_i \setminus (\bigcup_{i' \neq i} T_{i'})$ and $W = T_j \setminus (\bigcup_{i' \neq j} T_{i'})$. Then each of V and W is a non-empty open subset of U , and so each has non-empty open image in $|\mathcal{X}|$. As $|\mathcal{X}|$ is irreducible, these two non-empty open subsets of $|\mathcal{X}|$ have a non-empty intersection. Let x be a point lying in this intersection, and let $v \in V$ and $w \in W$ be points mapping to x . We then find that

$$\dim T_i = \dim V = \dim_v(U) = \dim_x(\mathcal{X}) + \dim_v(U_x)$$

and similarly that

$$\dim T_j = \dim W = \dim_w(U) = \dim_x(\mathcal{X}) + \dim_w(U_x).$$

Since $u \mapsto \dim_u(U_{f(u)})$ is locally constant on U , and since $T_i \cup T_j$ is connected (being the union of two irreducible, hence connected, sets that have non-empty intersection), we see that $\dim_v(U_x) = \dim_w(U_x)$, and hence, comparing the preceding two equations, that $\dim T_i = \dim T_j$, as required. \square

Lemma 5.18. *If $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed immersion of locally Noetherian algebraic stacks, and if $z \in |\mathcal{Z}|$ has image $x \in |\mathcal{X}|$, then $\dim_z(\mathcal{Z}) \leq \dim_x(\mathcal{X})$.*

Proof. Choose a smooth surjective morphism $U \rightarrow \mathcal{X}$ whose source is a scheme; the base-changed morphism $V = U \times_{\mathcal{X}} \mathcal{Z} \rightarrow \mathcal{Z}$ is then also smooth and surjective, and the projection $V \rightarrow U$ is a closed immersion. If $v \in |V|$ maps to $z \in |\mathcal{Z}|$, and if we let u denote the image of v in $|U|$, then clearly $\dim_v(V) \leq \dim_u(U)$, while $\dim_v(V_z) = \dim_u(U_x)$, by Lemma 5.9. Thus

$$\dim_z(\mathcal{Z}) = \dim_v(V) - \dim_v(V_z) \leq \dim_u(U) - \dim_u(U_x) = \dim_x(\mathcal{X}),$$

as claimed. \square

Lemma 5.19. *If \mathcal{X} is a locally Noetherian algebraic stack, and if $x \in |\mathcal{X}|$, then $\dim_x(\mathcal{X}) = \sup_{\mathcal{T}} \{\dim_x(\mathcal{T})\}$, where \mathcal{T} runs over all the irreducible components of $|\mathcal{X}|$ passing through x (endowed with their induced reduced structure).*

Proof. Lemma 5.18 shows that $\dim_x(\mathcal{T}) \leq \dim_x(\mathcal{X})$ for each irreducible component \mathcal{T} passing through the point x . Thus to prove the lemma, it suffices to show that

$$(5.19.1) \quad \dim_x(\mathcal{X}) \leq \sup_{\mathcal{T}} \{\dim_x(\mathcal{T})\}.$$

Let $U \rightarrow \mathcal{X}$ be a smooth cover by a scheme. If T is an irreducible component of U then we let \mathcal{T} denote the closure of its image in \mathcal{X} , which is an irreducible component of \mathcal{X} . Let $u \in U$ be a point mapping to x . Then we have $\dim_x(\mathcal{X}) = \dim_u U - \dim_u U_x = \sup_T \dim_u T - \dim_u U_x$, where the supremum is over the irreducible components of U passing through u . Choose a component T for which the supremum is achieved, and note that $\dim_x(\mathcal{T}) = \dim_u T - \dim_u T_x$. The desired inequality (5.19.1) now follows from the evident inequality $\dim_u T_x \leq \dim_u U_x$. (Note that if $\text{Spec } k \rightarrow \mathcal{X}$ is a representative of x , then $T \times_{\mathcal{X}} \text{Spec } k$ is a closed subspace of $U \times_{\mathcal{X}} \text{Spec } k$.) \square

Lemma 5.20. *If \mathcal{X} is a locally Noetherian algebraic stack, and if $x \in |\mathcal{X}|$, then for any open substack \mathcal{V} of \mathcal{X} containing x , there is a finite type point $x_0 \in |\mathcal{V}|$ such that $\dim_{x_0}(\mathcal{X}) = \dim_x(\mathcal{V})$.*

Proof. Choose a smooth surjective morphism $f : U \rightarrow \mathcal{X}$ whose source is a scheme, and consider the function $u \mapsto \dim_{f(u)}(\mathcal{X})$; since the morphism $|U| \rightarrow |\mathcal{X}|$ induced by f is open (as f is smooth) as well as surjective (by assumption), and takes finite type points to finite type points (by the very definition of the finite type points of $|\mathcal{X}|$), it suffices to show that for any $u \in U$, and any open neighbourhood of u , there is a finite type point u_0 in this neighbourhood such that $\dim_{f(u_0)}(\mathcal{X}) = \dim_{f(u)}(\mathcal{X})$. Since, with this reformulation of the problem, the surjectivity of f is no longer required, we may replace U by the open neighbourhood of the point u in question, and thus reduce to the problem of showing that for each $u \in U$, there is a finite type point $u_0 \in U$ such that $\dim_{f(u_0)}(\mathcal{X}) = \dim_{f(u)}(\mathcal{X})$. By Lemma 5.4 $\dim_{f(u)}(\mathcal{X}) = \dim_u(U) - \dim_u(U_{f(u)})$, while $\dim_{f(u_0)}(\mathcal{X}) = \dim_{u_0}(U) - \dim_{u_0}(U_{f(u_0)})$. Since f is smooth, the expression $\dim_{u_0}(U_{f(u_0)})$ is locally constant as u_0 varies over U (by Lemma 5.11 (2)), and so shrinking U further around u if necessary, we may assume it is constant. Thus the problem becomes to show that we may find a finite type point $u_0 \in U$ for which $\dim_{u_0}(U) = \dim_u(U)$. Since by definition $\dim_u U$ is the minimum of the dimensions $\dim V$, as V ranges over the open neighbourhoods V of u in U , we may shrink U down further around u so that $\dim_u U = \dim U$. The existence of desired point u_0 then follows from Lemma 5.12. \square

Lemma 5.21. *Let $\mathcal{T} \hookrightarrow \mathcal{X}$ be a locally of finite type monomorphism of algebraic stacks, with \mathcal{X} (and thus also \mathcal{T}) being Jacobson, pseudo-catenary, and locally Noetherian. Suppose further that \mathcal{T} is irreducible of some (finite) dimension d , and that \mathcal{X} is reduced and of dimension less than or equal to d . Then there is a non-empty open substack \mathcal{V} of \mathcal{T} such that the induced monomorphism $\mathcal{V} \hookrightarrow \mathcal{X}$ is an open immersion which identifies \mathcal{V} with an open subset of an irreducible component of \mathcal{X} .*

Proof. Choose a smooth surjective morphism $f : U \rightarrow \mathcal{X}$ with source a scheme, necessarily reduced since \mathcal{X} is, and write $U' = \mathcal{T} \times_{\mathcal{X}} U$. The base-changed morphism $U' \rightarrow U$ is a monomorphism of algebraic spaces, locally of finite type, and thus representable Morphisms of Spaces, Lemma 51.1 and 27.10; since U is a scheme, so is U' . The projection $f' : U' \rightarrow \mathcal{T}$ is again a smooth surjection. Let $u' \in U'$, with image $u \in U$. Lemma 5.9 shows that $\dim_{u'}(U'_{f'(u')}) = \dim_u(U_{f(u)})$, while $\dim_{f'(u')}(\mathcal{T}) = d \geq \dim_{f(u)}(\mathcal{X})$ by Lemma 5.17 and our assumptions on \mathcal{T} and \mathcal{X} . Thus we see that

$$(5.21.1) \quad \dim_{u'}(U') = \dim_{u'}(U'_{f'(u')}) + \dim_{f'(u')}(\mathcal{T}) \geq \dim_u(U_{f(u)}) + \dim_{f(u)}(\mathcal{X}) = \dim_u(U).$$

Since $U' \rightarrow U$ is a monomorphism, locally of finite type, it is in particular unramified, and so by the étale local structure of unramified morphisms Étale Morphisms, Lemma 17.3, we may find a commutative diagram

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U \end{array}$$

in which the scheme V' is non-empty, the vertical arrows are étale, and the upper horizontal arrow is a closed immersion. Replacing V by a quasi-compact open subset whose image has non-empty intersection with the image of U' , and replacing V' by the preimage of V , we may further assume that V (and thus V') is quasi-compact. Since V is also locally Noetherian, it is thus Noetherian, and so is the union of finitely many irreducible components.

Since étale morphisms preserve pointwise dimension Descent, Lemma 21.2 we deduce from (5.21.1) that for any point $v' \in V'$, with image $v \in V$, we have $\dim_{v'}(V') \geq \dim_v(V)$. In particular, the image of V' can't be contained in the intersection of two distinct irreducible components of V , and so we may find at least one irreducible open subset of V which has non-empty intersection with V' ; replacing V by this subset, we may assume that V is integral (being both reduced and irreducible). From the preceding inequality on dimensions, we conclude that the closed immersion $V' \hookrightarrow V$ is in fact an isomorphism. If we let W denote the image of V' in U' , then W is a non-empty open subset of U' (as étale morphisms are open), and the induced monomorphism $W \rightarrow U$ is étale (since it is so étale locally on the source, i.e. after pulling back to V'), and hence is an open immersion (being an étale monomorphism). Thus, if we let \mathcal{V} denote the image of W in \mathcal{T} , then \mathcal{V} is a dense (equivalently, non-empty) open substack of \mathcal{T} , whose image is dense in an irreducible component of \mathcal{X} . Finally, we note that the morphism $\mathcal{V} \rightarrow \mathcal{X}$ is smooth (since its composite with the smooth morphism $W \rightarrow \mathcal{V}$ is smooth), and also a monomorphism, and thus is an open immersion. \square

Lemma 5.22. *Let $f : \mathcal{T} \rightarrow \mathcal{X}$ be a locally of finite type morphism of Jacobson, pseudo-catenary, and locally Noetherian algebraic stacks, whose source is irreducible and whose target is quasi-separated, and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ denote the scheme-theoretic image of \mathcal{T} . Then for all $t \in |T|$, we have that $\dim_t(\mathcal{T}_{f(t)}) \geq \dim \mathcal{T} - \dim \mathcal{Z}$, and there is a non-empty (equivalently, dense) open subset of $|T|$ over which equality holds.*

Proof. Replacing \mathcal{X} by \mathcal{Z} , we may and do assume that f is scheme theoretically dominant, and also that \mathcal{X} is irreducible. By the upper semi-continuity of fibre dimensions (Lemma 5.11 (1)), it suffices to prove that the equality $\dim_t(\mathcal{T}_{f(t)}) = \dim \mathcal{T} - \dim \mathcal{Z}$ holds for t lying in some non-empty open substack of \mathcal{T} . For this reason, in the argument we are always free to replace \mathcal{T} by a non-empty open substack.

Let $T' \rightarrow \mathcal{T}$ be a smooth surjective morphism whose source is a scheme, and let T be a non-empty quasi-compact open subset of T' . Since \mathcal{Y} is quasi-separated, we find that $T \rightarrow \mathcal{Y}$ is quasi-compact (by Morphisms of Stacks, Lemma 7.7, applied to the morphisms $T \rightarrow \mathcal{Y} \rightarrow \text{Spec } \mathbf{Z}$). Thus, if we replace \mathcal{T} by the image of T in \mathcal{T} , then we may assume (appealing to Morphisms of Stacks, Lemma 7.6 that the morphism $f : \mathcal{T} \rightarrow \mathcal{X}$ is quasi-compact.

If we choose a smooth surjection $U \rightarrow \mathcal{X}$ with U a scheme, then Lemma 3.1 ensures that we may find an irreducible open subset V of U such that $V \rightarrow \mathcal{X}$ is smooth and scheme-theoretically dominant. Since scheme-theoretic dominance for quasi-compact morphisms is preserved by flat base-change, the base-change $\mathcal{T} \times_{\mathcal{X}} V \rightarrow V$ of the scheme-theoretically dominant morphism f is again scheme-theoretically dominant. We let Z denote a scheme admitting a smooth surjection onto this fibre product; then $Z \rightarrow \mathcal{T} \times_{\mathcal{X}} V \rightarrow V$ is again scheme-theoretically dominant. Thus we may find an irreducible component C of Z which scheme-theoretically dominates V . Since the composite $Z \rightarrow \mathcal{T} \times_{\mathcal{X}} V \rightarrow \mathcal{T}$ is smooth, and since \mathcal{T} is irreducible, Lemma 3.1 shows that any irreducible component of the source has dense image in $|\mathcal{T}|$. We now replace C by a non-empty open subset W which is disjoint from every other irreducible component of Z , and then replace \mathcal{T} and \mathcal{X} by the images of W and V (and apply Lemma 5.17 to see that this doesn't change the dimension of either \mathcal{T} or \mathcal{X}). If we let \mathcal{W} denote the image of the morphism $W \rightarrow \mathcal{T} \times_{\mathcal{X}} V$, then \mathcal{W} is open in $\mathcal{T} \times_{\mathcal{X}} V$ (since the morphism $W \rightarrow \mathcal{T} \times_{\mathcal{X}} V$ is smooth), and is irreducible (being the image of an irreducible scheme). Thus we end up with a commutative diagram

$$\begin{array}{ccccc} W & \longrightarrow & \mathcal{W} & \longrightarrow & V \\ & \searrow & \downarrow & & \downarrow \\ & & \mathcal{T} & \longrightarrow & \mathcal{X} \end{array}$$

in which W and V are schemes, the vertical arrows are smooth and surjective, the diagonal arrows and the left-hand upper horizontal arrow are smooth, and the induced morphism $\mathcal{W} \rightarrow \mathcal{T} \times_{\mathcal{X}} V$ is an open immersion. Using this diagram, together with the definitions of the various dimensions involved in the statement of the lemma, we will reduce our verification of the lemma to the case of schemes, where it is known.

Fix $w \in |W|$ with image $w' \in |\mathcal{W}|$, image $t \in |\mathcal{T}|$, image v in $|V|$, and image x in $|\mathcal{X}|$. Essentially by definition (using the fact that \mathcal{W} is open in $\mathcal{T} \times_{\mathcal{X}} V$, and that the fibre of a base-change is the base-change of the fibre), we obtain the equalities

$$\dim_v V_x = \dim_{w'} \mathcal{W}_t$$

and

$$\dim_t \mathcal{T}_x = \dim_{w'} \mathcal{W}_v.$$

By Lemma 5.4 (the diagonal arrow and right-hand vertical arrow in our diagram realise W and V as smooth covers by schemes of the stacks \mathcal{T} and \mathcal{X}), we find that

$$\dim_t \mathcal{T} = \dim_w W - \dim_w W_t$$

and

$$\dim_x \mathcal{X} = \dim_v V - \dim_v V_x.$$

Combining the equalities, we find that

$$\dim_t \mathcal{T}_x - \dim_t \mathcal{T} + \dim_x \mathcal{X} = \dim_{w'} \mathcal{W}_v - \dim_w W + \dim_w W_t + \dim_v V - \dim_{w'} \mathcal{W}_t$$

Since $W \rightarrow \mathcal{W}$ is a smooth surjection, the same is true if we base-change over the morphism $\mathrm{Spec} \kappa(v) \rightarrow V$ (thinking of $W \rightarrow \mathcal{W}$ as a morphism over V), and from this smooth morphism we obtain the first of the following two equalities

$$\dim_w W_v - \dim_{w'} \mathcal{W}_v = \dim_w (W_v)_{w'} = \dim_w W_{w'};$$

the second equality follows via a direct comparison of the two fibres involved. Similarly, if we think of $W \rightarrow \mathcal{W}$ as a morphism of schemes over \mathcal{T} , and base-change over some representative of the point $t \in |\mathcal{T}|$, we obtain the equalities

$$\dim_w W_t - \dim_{w'} \mathcal{W}_t = \dim_w (W_t)_{w'} = \dim_w W_{w'}.$$

Putting everything together, we find that

$$\dim_t \mathcal{T}_x - \dim_t \mathcal{T} + \dim_x \mathcal{X} = \dim_w W_v - \dim_w W + \dim_v V.$$

Our goal is to show that the left-hand side of this equality vanishes for a non-empty open subset of t . As w varies over a non-empty open subset of W , its image $t \in |\mathcal{T}|$ varies over a non-empty open subset of $|\mathcal{T}|$ (as $W \rightarrow \mathcal{T}$ is smooth).

We are therefore reduced to showing that if $W \rightarrow V$ is a scheme-theoretically dominant morphism of irreducible locally Noetherian schemes that is locally of finite type, then there is a non-empty open subset of points $w \in W$ such that $\dim_w W_v = \dim_w W - \dim_v V$ (where v denotes the image of w in V). This is a standard fact, whose proof we recall for the convenience of the reader.

We may replace W and V by their underlying reduced subschemes without altering the validity (or not) of this equation, and thus we may assume that they are in fact integral schemes. Since $\dim_w W_v$ is locally constant on W , replacing W by a non-empty open subset if necessary, we may assume that $\dim_w W_v$ is constant, say equal to d . Choosing this open subset to be affine, we may also assume that the morphism $W \rightarrow V$ is in fact of finite type. Replacing V by a non-empty open subset if necessary (and then pulling back W over this open subset; the resulting pull-back is non-empty, since the flat base-change of a quasi-compact and scheme-theoretically dominant morphism remains scheme-theoretically dominant), we may furthermore assume that W is flat over V . The morphism $W \rightarrow V$ is thus of relative dimension d in the sense of Morphisms, Definition 29.1 and it follows from Morphisms, Lemma 29.6 that $\dim_w (W) = \dim_v (V) + d$, as required. \square

Remark 5.23. We note that in the context of the preceding lemma, it need not be that $\dim \mathcal{T} \geq \dim \mathcal{Z}$; this does not contradict the inequality in the statement of the lemma, because the fibres of the morphism f are again algebraic stacks, and so may have negative dimension. This is illustrated by taking k to be a field, and applying the lemma to the morphism $[\mathrm{Spec} k/\mathbf{G}_m] \rightarrow \mathrm{Spec} k$.

If the morphism f in the statement of the lemma is assumed to be quasi-DM (in the sense of Morphisms of Stacks, Definition 4.1; e.g. morphisms that are representable by algebraic spaces are quasi-DM), then the fibres of the morphism over points of the target are quasi-DM algebraic stacks, and hence are of non-negative dimension. In this case, the lemma implies that indeed $\dim \mathcal{T} \geq \dim \mathcal{Z}$. In fact, we obtain the following more general result.

Lemma 5.24. *Let $f : \mathcal{T} \rightarrow \mathcal{X}$ be a locally of finite type morphism of Jacobson, pseudo-catenary, and locally Noetherian algebraic stacks which is quasi-DM, whose source is irreducible and whose target is quasi-separated, and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ denote the scheme-theoretic image of \mathcal{T} . Then $\dim \mathcal{Z} \leq \dim \mathcal{T}$, and furthermore, exactly one of the following two conditions holds:*

- (1) *for every finite type point $t \in |\mathcal{T}|$, we have $\dim_t(\mathcal{T}_{f(t)}) > 0$, in which case $\dim \mathcal{Z} < \dim \mathcal{T}$; or*
- (2) *\mathcal{T} and \mathcal{Z} are of the same dimension.*

Proof. As was observed in the preceding remark, the dimension of a quasi-DM stack is always non-negative, from which we conclude that $\dim_t \mathcal{T}_{f(t)} \geq 0$ for all $t \in |\mathcal{T}|$, with the equality

$$\dim_t \mathcal{T}_{f(t)} = \dim_t \mathcal{T} - \dim_{f(t)} \mathcal{Z}$$

holding for a dense open subset of points $t \in |\mathcal{T}|$. □

6. The dimension of the local ring

An algebraic stack doesn't really have local rings in the usual sense, but we can define the dimension of the local ring as follows.

Lemma 6.1. *Let \mathcal{X} be a locally Noetherian algebraic stack. Let $U \rightarrow \mathcal{X}$ be a smooth morphism and let $u \in U$. Then*

$$\dim(\mathcal{O}_{U, \bar{u}}) - \dim(\mathcal{O}_{R_u, e(\bar{u})}) = 2 \dim(\mathcal{O}_{U, \bar{u}}) - \dim(\mathcal{O}_{R, e(\bar{u})})$$

Here $R = U \times_{\mathcal{X}} U$ with projections $s, t : R \rightarrow U$ and diagonal $e : U \rightarrow R$ and R_u is the fibre of $s : R \rightarrow U$ over u .

Proof. This is true because $s : \mathcal{O}_{U, \bar{u}} \rightarrow \mathcal{O}_{R, e(\bar{u})}$ is a flat local homomorphism of Noetherian local rings and hence

$$\dim(\mathcal{O}_{R, e(\bar{u})}) = \dim(\mathcal{O}_{U, \bar{u}}) + \dim(\mathcal{O}_{R_u, e(\bar{u})})$$

by Algebra, Lemma 112.7. □

Lemma 6.2. *Let \mathcal{X} be a locally Noetherian algebraic stack. Let $x \in |\mathcal{X}|$ be a finite type point (Morphisms of Stacks, Definition 18.2). Let $d \in \mathbf{Z}$. The following are equivalent*

- (1) *there exists a scheme U , a smooth morphism $U \rightarrow \mathcal{X}$, and a finite type point $u \in U$ mapping to x such that $2 \dim(\mathcal{O}_{U, \bar{u}}) - \dim(\mathcal{O}_{R_u, e(\bar{u})}) = d$, and*
- (2) *for any scheme U , a smooth morphism $U \rightarrow \mathcal{X}$, and finite type point $u \in U$ mapping to x we have $2 \dim(\mathcal{O}_{U, \bar{u}}) - \dim(\mathcal{O}_{R_u, e(\bar{u})}) = d$.*

Here $R = U \times_{\mathcal{X}} U$ with projections $s, t : R \rightarrow U$ and diagonal $e : U \rightarrow R$ and R_u is the fibre of $s : R \rightarrow U$ over u .

Proof. Suppose we have two smooth neighbourhoods (U, u) and (U', u') of x with u and u' finite type points. After shrinking U and U' we may assume that u and u' are closed points (by definition of finite type points). Then we choose a surjective étale morphism $W \rightarrow U \times_{\mathcal{X}} U'$. Let W_u be the fibre of $W \rightarrow U$ over u and let $W_{u'}$ be the fibre of $W \rightarrow U'$ over u' . Since u and u' map to the same point of $|\mathcal{X}|$ we see that $W_u \cap W_{u'}$ is nonempty. Hence we may choose a closed point $w \in W$ mapping to both u and u' . This reduces us to the discussion in the next paragraph.

Assume $(U', u') \rightarrow (U, u)$ is a smooth morphism of smooth neighbourhoods of x with u and u' closed points. Goal: prove the invariant defined for (U, u) is the same as the invariant defined for (U', u') . To see this observe that $\mathcal{O}_{U,u} \rightarrow \mathcal{O}_{U',u'}$ is a flat local homomorphism of Noetherian local rings and hence

$$\dim(\mathcal{O}_{U',\bar{u}'}) = \dim(\mathcal{O}_{U,\bar{u}}) + \dim(\mathcal{O}_{U',\bar{u}'})$$

by Algebra, Lemma 112.7. (We omit working through all the steps to relate properties of local rings and their strict henselizations, see More on Algebra, Section 45). On the other hand we have

$$R' = U' \times_{U,t} R \times_{s,U} U'$$

Thus we see that

$$\dim(\mathcal{O}_{R',e(\bar{u}')}) = \dim(\mathcal{O}_{R,e(\bar{u})}) + \dim(\mathcal{O}_{U'_u \times_u U'_u,(\bar{u}',\bar{u}')})$$

To prove the lemma it suffices to show that

$$\dim(\mathcal{O}_{U'_u \times_u U'_u,(\bar{u}',\bar{u}')}) = 2 \dim(\mathcal{O}_{U'_u,\bar{u}'})$$

Observe that this isn't always true (example: if U'_u is a curve and u' is the generic point of this curve). However, we know that u' is a closed point of the algebraic space U'_u locally of finite type over u . In this case the equality holds because, first $\dim_{(u',u')}(U'_u \times_u U'_u) = 2 \dim_{u'}(U'_u)$ by Varieties, Lemma 20.5 and second the agreement of dimension with dimension of local rings in closed points of locally algebraic schemes, see Varieties, Lemma 20.3. We omit the translation of these results for schemes into the language of algebraic spaces. \square

Definition 6.3. Let \mathcal{X} be a locally Noetherian algebraic stack. Let $x \in |\mathcal{X}|$ be a finite type point. The *dimension of the local ring of \mathcal{X} at x* is $d \in \mathbf{Z}$ if the equivalent conditions of Lemma 6.2 are satisfied.

To be sure, this is motivated by Lemma 6.1 and Properties of Stacks, Definition 12.2. We close this section by establishing a formula allowing us to compute $\dim_x(\mathcal{X})$ in terms of properties of the versal ring to \mathcal{X} at x .

Lemma 6.4. *Suppose that \mathcal{X} is an algebraic stack, locally of finite type over a locally Noetherian scheme S . Let $x_0 : \mathrm{Spec}(k) \rightarrow \mathcal{X}$ be a morphism where k is a field of finite type over S . Represent $\mathcal{F}_{\mathcal{X},k,x_0}$ as in Remark 2.11 by a cogroupoid (A, B, s, t, c) of Noetherian complete local S -algebras with residue field k . Then*

$$\text{the dimension of the local ring of } \mathcal{X} \text{ at } x_0 = 2 \dim A - \dim B$$

Proof. Let $s \in S$ be the image of x_0 . If $\mathcal{O}_{S,s}$ is a G-ring (a condition that is almost always satisfied in practice), then we can prove the lemma as follows. By Lemma 2.8, we may find a smooth morphism $U \rightarrow \mathcal{X}$, whose source is a scheme, containing a point $u_0 \in U$ of residue field k , such that induced morphism $\mathrm{Spec}(k) \rightarrow U \rightarrow \mathcal{X}$

coincides with x_0 and such that $A = \mathcal{O}_{U, u_0}^\wedge$. Write $R = U \times_{\mathcal{X}} U$. Then we may identify $\mathcal{O}_{R, e(u_0)}^\wedge$ with B . Hence the equality follows from the definitions.

In the rest of this proof we explain how to prove the lemma in general, but we urge the reader to skip this.

First let us show that the right hand side is independent of the choice of (A, B, s, t, c) . Namely, suppose that (A', B', s', t', c') is a second choice. Since A and A' are versal rings to \mathcal{X} at x_0 , we can choose, after possibly switching A and A' , a formally smooth map $A \rightarrow A'$ compatible with the given versal formal objects ξ and ξ' over A and A' . Recall that $\widehat{\mathcal{C}}_\Lambda$ has coproducts and that these are given by completed tensor product over Λ , see Formal Deformation Theory, Lemma 4.4. Then B prorepresents the functor of isomorphisms between the two pushforwards of ξ to $A \widehat{\otimes}_\Lambda A$. Similarly for B' . We conclude that

$$B' = B \otimes_{(A \widehat{\otimes}_\Lambda A)} (A' \widehat{\otimes}_\Lambda A')$$

It is straightforward to see that

$$A \widehat{\otimes}_\Lambda A \longrightarrow A \widehat{\otimes}_\Lambda A' \longrightarrow A' \widehat{\otimes}_\Lambda A'$$

is formally smooth of relative dimension equal to 2 times the relative dimension of the formally smooth map $A \rightarrow A'$. (This follows from general principles, but also because in this particular case A' is a power series ring over A in r variables.) Hence $B \rightarrow B'$ is formally smooth of relative dimension $2(\dim(A') - \dim(A))$ as desired.

Next, let l/k be a finite extension. let $x_{l,0} : \mathrm{Spec}(l) \rightarrow \mathcal{X}$ be the induced point. We claim that the right hand side of the formula is the same for x_0 as it is for $x_{l,0}$. This can be shown by choosing $A \rightarrow A'$ as in Lemma 2.5 and arguing exactly as in the preceding paragraph. We omit the details.

Finally, arguing as in the proof of Lemma 2.10 we can use the compatibilities in the previous two paragraphs to reduce to the case (discussed in the first paragraph) where A is the complete local ring of U at u_0 for some scheme smooth over \mathcal{X} and finite type point u_0 . Details omitted. \square

7. Other chapters

Preliminaries

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- (2) Conventions
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- (4) Categories
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- (6) Sheaves on Spaces
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Schemes

- (26) Schemes

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