# THE COTANGENT COMPLEX

## Contents

## 1. Introduction

The goal of this chapter is to construct the cotangent complex of a ring map, of a morphism of schemes, and of a morphism of algebraic spaces. Some references are the notes [Qui], the paper [Qui70], and the books [And67] and [Ill72].

## 2. Advice for the reader

In writing this chapter we have tried to minimize the use of simplicial techniques. We view the choice of a *resolution* $P_\bullet$ of a ring $B$ over a ring $A$ as a tool to calculating the *homology* of abelian sheaves on the category $\mathcal{C}_{B/A}$, see Remark 5.5. This is similar to the role played by a "good cover" to compute cohomology using the Čech complex. To read a bit on homology on categories, please visit Cohomology on Sites, Section 39. The derived lower shriek functor $L\pi_!$ is to homology what $R\Gamma(\mathcal{C}_{B/A}, -)$ is to cohomology. The category $\mathcal{C}_{B/A}$, studied in Section 4, is the opposite of the category of factorizations $A \to P \to B$ where $P$ is a polynomial algebra over $A$. This category comes with maps of sheaves of rings

$$\underline{A} \longrightarrow \mathcal{O} \longrightarrow \underline{B}$$

where over the object $U = (P \to B)$ we have $\mathcal{O}(U) = P$. It turns out that we obtain the cotangent complex of $B$ over $A$ as

$$L_{B/A} = L\pi_!(\Omega_{\mathcal{O}/\underline{A}} \otimes_\mathcal{O} \underline{B})$$

see Lemma 4.3. We have consistently tried to use this point of view to prove the basic properties of cotangent complexes of ring maps. In particular, all of the results can be proven without relying on the existence of standard resolutions, although we have not done so. The theory is quite satisfactory, except that perhaps the proof of the fundamental triangle (Proposition 7.4) uses just a little bit more theory on derived lower shriek functors. To provide the reader with an alternative, we give a rather complete sketch of an approach to this result based on simple properties of standard resolutions in Remarks 7.5 and 7.6.

Our approach to the cotangent complex for morphisms of ringed topoi, morphisms of schemes, morphisms of algebraic spaces, etc is to deduce as much as possible from the case of "plain ring maps" discussed above.

## 3. The cotangent complex of a ring map

Let $A$ be a ring. Let $Alg_A$ be the category of $A$-algebras. Consider the pair of adjoint functors $(U, V)$ where $V : Alg_A \to Sets$ is the forgetful functor and $U : Sets \to Alg_A$ assigns to a set $E$ the polynomial algebra $A[E]$ on $E$ over $A$. Let $X_\bullet$ be the simplicial object of $\text{Fun}(Alg_A, Alg_A)$ constructed in Simplicial, Section 34.

Consider an $A$-algebra $B$. Denote $P_\bullet = X_\bullet(B)$ the resulting simplicial $A$-algebra. Recall that $P_0 = A[B]$, $P_1 = A[A[B]]$, and so on. In particular each term $P_n$ is a polynomial $A$-algebra. Recall also that there is an augmentation

$$\epsilon : P_\bullet \longrightarrow B$$

where we view $B$ as a constant simplicial $A$-algebra.

**Definition 3.1.** Let $A \to B$ be a ring map. The *standard resolution of $B$ over $A$* is the augmentation $\epsilon : P_\bullet \to B$ with terms

$$P_0 = A[B], \quad P_1 = A[A[B]], \quad \dots$$

and maps as constructed above.

It will turn out that we can use the standard resolution to compute left derived functors in certain settings.

**Definition 3.2.** The *cotangent complex* $L_{B/A}$ of a ring map $A \to B$ is the complex of $B$-modules associated to the simplicial $B$-module

$$\Omega_{P_\bullet/A} \otimes_{P_\bullet, \epsilon} B$$

where $\epsilon : P_\bullet \to B$ is the standard resolution of $B$ over $A$.

In Simplicial, Section 23 we associate a chain complex to a simplicial module, but here we work with cochain complexes. Thus the term $L_{B/A}^{-n}$ in degree $-n$ is the $B$-module $\Omega_{P_n/A} \otimes_{P_n, \epsilon_n} B$ and $L_{B/A}^m = 0$ for $m > 0$.

**Remark 3.3.** Let $A \to B$ be a ring map. Let $\mathcal{A}$ be the category of arrows $\psi : C \to B$ of $A$-algebras and let $\mathcal{S}$ be the category of maps $E \to B$ where $E$ is a set. There are adjoint functors $V : \mathcal{A} \to \mathcal{S}$ (the forgetful functor) and $U : \mathcal{S} \to \mathcal{A}$ which sends $E \to B$ to $A[E] \to B$. Let $X_\bullet$ be the simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ constructed in Simplicial, Section 34. The diagram

$$\begin{array}{ccc} \mathcal{A} & \rightleftarrows & \mathcal{S} \\ \downarrow & & \downarrow \\ Alg_A & \rightleftarrows & Sets \end{array}$$
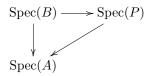
commutes. It follows that $X_\bullet(\mathrm{id}_B : B \to B)$ is equal to the standard resolution of $B$ over $A$.

**Lemma 3.4.** *Let $A_i \to B_i$ be a system of ring maps over a directed index set $I$. Then* $\mathrm{colim}\, L_{B_i/A_i} = L_{\mathrm{colim}\, B_i / \mathrm{colim}\, A_i}$.

**Proof.** This is true because the forgetful functor $V : A\text{-}Alg \to Sets$ and its adjoint $U : Sets \to A\text{-}Alg$ commute with filtered colimits. Moreover, the functor $B/A \mapsto \Omega_{B/A}$ does as well (Algebra, Lemma 131.5). $\square$

## 4. Simplicial resolutions and derived lower shriek

Let $A \to B$ be a ring map. Consider the category whose objects are $A$-algebra maps $\alpha : P \to B$ where $P$ is a polynomial algebra over $A$ (in some set[1] of variables) and whose morphisms $s : (\alpha : P \to B) \to (\alpha' : P' \to B)$ are $A$-algebra homomorphisms $s : P \to P'$ with $\alpha' \circ s = \alpha$. Let $\mathcal{C} = \mathcal{C}_{B/A}$ denote the **opposite** of this category. The reason for taking the opposite is that we want to think of objects $(P, \alpha)$ as corresponding to the diagram of affine schemes

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(P) \\ \downarrow & \swarrow & \\ \mathrm{Spec}(A) & & \end{array}$$

We endow $\mathcal{C}$ with the chaotic topology (Sites, Example 6.6), i.e., we endow $\mathcal{C}$ with the structure of a site where coverings are given by identities so that all presheaves are sheaves. Moreover, we endow $\mathcal{C}$ with two sheaves of rings. The first is the sheaf

---

[1] It suffices to consider sets of cardinality at most the cardinality of $B$.

$\mathcal{O}$ which sends to object $(P, \alpha)$ to $P$. Then second is the constant sheaf $B$, which we will denote $\underline{B}$. We obtain the following diagram of morphisms of ringed topoi

(4.0.1)
$$
\begin{array}{ccc}
(Sh(\mathcal{C}), \underline{B}) & \xrightarrow{\ i\ } & (Sh(\mathcal{C}), \mathcal{O}) \\
{\scriptstyle \pi}\downarrow & & \\
(Sh(*), B) & &
\end{array}
$$

The morphism $i$ is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{B}$ is the obvious map. The map $\pi$ is as in Cohomology on Sites, Example 39.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{B})$ left adjoint to $Ri_* = i_* : D(\underline{B}) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{B}) \longrightarrow D(B)$ left adjoint to $\pi^* = \pi^{-1} : D(B) \to D(\underline{B})$.

**Lemma 4.1.** *With notation as above let $P_\bullet$ be a simplicial $A$-algebra endowed with an augmentation $\epsilon : P_\bullet \to B$. Assume each $P_n$ is a polynomial algebra over $A$ and $\epsilon$ is a trivial Kan fibration on underlying simplicial sets. Then*

$$
L\pi_!(\mathcal{F}) = \mathcal{F}(P_\bullet, \epsilon)
$$

*in $D(Ab)$, resp. $D(B)$ functorially in $\mathcal{F}$ in $Ab(\mathcal{C})$, resp. $Mod(\underline{B})$.*

**Proof.** We will use the criterion of Cohomology on Sites, Lemma 39.7 to prove this. Given an object $U = (Q, \beta)$ of $\mathcal{C}$ we have to show that

$$
S_\bullet = \mathrm{Mor}_{\mathcal{C}}((Q, \beta), (P_\bullet, \epsilon))
$$

is homotopy equivalent to a singleton. Write $Q = A[E]$ for some set $E$ (this is possible by our choice of the category $\mathcal{C}$). We see that

$$
S_\bullet = \mathrm{Mor}_{Sets}((E, \beta|_E), (P_\bullet, \epsilon))
$$

Let $*$ be the constant simplicial set on a singleton. For $b \in B$ let $F_{b,\bullet}$ be the simplicial set defined by the cartesian diagram

$$
\begin{array}{ccc}
F_{b,\bullet} & \longrightarrow & P_\bullet \\
\downarrow & & {\scriptstyle \epsilon}\downarrow \\
* & \xrightarrow{\ b\ } & B
\end{array}
$$

With this notation $S_\bullet = \prod_{e \in E} F_{\beta(e),\bullet}$. Since we assumed $\epsilon$ is a trivial Kan fibration we see that $F_{b,\bullet} \to *$ is a trivial Kan fibration (Simplicial, Lemma 30.3). Thus $S_\bullet \to *$ is a trivial Kan fibration (Simplicial, Lemma 30.6). Therefore $S_\bullet$ is homotopy equivalent to $*$ (Simplicial, Lemma 30.8). $\square$

In particular, we can use the standard resolution of $B$ over $A$ to compute derived lower shriek.

**Lemma 4.2.** *Let $A \to B$ be a ring map. Let $\epsilon : P_\bullet \to B$ be the standard resolution of $B$ over $A$. Let $\pi$ be as in (4.0.1). Then*

$$
L\pi_!(\mathcal{F}) = \mathcal{F}(P_\bullet, \epsilon)
$$

*in $D(Ab)$, resp. $D(B)$ functorially in $\mathcal{F}$ in $Ab(\mathcal{C})$, resp. $Mod(\underline{B})$.*

**First proof.** We will apply Lemma 4.1. Since the terms $P_n$ are polynomial algebras we see the first assumption of that lemma is satisfied. The second assumption is proved as follows. By Simplicial, Lemma 34.3 the map $\epsilon$ is a homotopy equivalence of underlying simplicial sets. By Simplicial, Lemma 31.9 this implies $\epsilon$ induces a quasi-isomorphism of associated complexes of abelian groups. By Simplicial, Lemma 31.8 this implies that $\epsilon$ is a trivial Kan fibration of underlying simplicial sets. $\square$

**Second proof.** We will use the criterion of Cohomology on Sites, Lemma 39.7. Let $U = (Q, \beta)$ be an object of $\mathcal{C}$. We have to show that

$$S_\bullet = \mathrm{Mor}_\mathcal{C}((Q, \beta), (P_\bullet, \epsilon))$$

is homotopy equivalent to a singleton. Write $Q = A[E]$ for some set $E$ (this is possible by our choice of the category $\mathcal{C}$). Using the notation of Remark 3.3 we see that

$$S_\bullet = \mathrm{Mor}_\mathcal{S}((E \to B), i(P_\bullet \to B))$$

By Simplicial, Lemma 34.3 the map $i(P_\bullet \to B) \to i(B \to B)$ is a homotopy equivalence in $\mathcal{S}$. Hence $S_\bullet$ is homotopy equivalent to

$$\mathrm{Mor}_\mathcal{S}((E \to B), (B \to B)) = \{*\}$$

as desired. $\square$

**Lemma 4.3.** *Let $A \to B$ be a ring map. Let $\pi$ and $i$ be as in (4.0.1). There is a canonical isomorphism*

$$L_{B/A} = L\pi_!(Li^*\Omega_{\mathcal{O}/A}) = L\pi_!(i^*\Omega_{\mathcal{O}/A}) = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_\mathcal{O} \underline{B})$$

*in $D(B)$.*

**Proof.** For an object $\alpha : P \to B$ of the category $\mathcal{C}$ the module $\Omega_{P/A}$ is a free $P$-module. Thus $\Omega_{\mathcal{O}/A}$ is a flat $\mathcal{O}$-module. Hence $Li^*\Omega_{\mathcal{O}/A} = i^*\Omega_{\mathcal{O}/A}$ is the sheaf of $\underline{B}$-modules which associates to $\alpha : P \to A$ the $B$-module $\Omega_{P/A} \otimes_{P,\alpha} B$. By Lemma 4.2 we see that the right hand side is computed by the value of this sheaf on the standard resolution which is our definition of the left hand side (Definition 3.2). $\square$

**Lemma 4.4.** *If $A \to B$ is a ring map, then $L\pi_!(\pi^{-1}M) = M$ with $\pi$ as in (4.0.1).*

**Proof.** This follows from Lemma 4.1 which tells us $L\pi_!(\pi^{-1}M)$ is computed by $(\pi^{-1}M)(P_\bullet, \epsilon)$ which is the constant simplicial object on $M$. $\square$

**Lemma 4.5.** *If $A \to B$ is a ring map, then $H^0(L_{B/A}) = \Omega_{B/A}$.*

**Proof.** We will prove this by a direct calculation. We will use the identification of Lemma 4.3. There is clearly a map from $\Omega_{\mathcal{O}/A} \otimes \underline{B}$ to the constant sheaf with value $\Omega_{B/A}$. Thus this map induces a map

$$H^0(L_{B/A}) = H^0(L\pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B})) = \pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B}) \to \Omega_{B/A}$$

By choosing an object $P \to B$ of $\mathcal{C}_{B/A}$ with $P \to B$ surjective we see that this map is surjective (by Algebra, Lemma 131.6). To show that it is injective, suppose that $P \to B$ is an object of $\mathcal{C}_{B/A}$ and that $\xi \in \Omega_{P/A} \otimes_P B$ is an element which maps to zero in $\Omega_{B/A}$. We first choose factorization $P \to P' \to B$ such that $P' \to B$ is surjective and $P'$ is a polynomial algebra over $A$. We may replace $P$ by $P'$. If $B = P/I$, then the kernel $\Omega_{P/A} \otimes_P B \to \Omega_{B/A}$ is the image of $I/I^2$ (Algebra,

Lemma 131.9). Say $\xi$ is the image of $f \in I$. Then we consider the two maps $a, b : P' = P[x] \to P$, the first of which maps $x$ to 0 and the second of which maps $x$ to $f$ (in both cases $P[x] \to B$ maps $x$ to zero). We see that $\xi$ and 0 are the image of $dx \otimes 1$ in $\Omega_{P'/A} \otimes_{P'} B$. Thus $\xi$ and 0 have the same image in the colimit (see Cohomology on Sites, Example 39.1) $\pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B})$ as desired.                              $\square$

**Lemma 4.6.** *If $B$ is a polynomial algebra over the ring $A$, then with $\pi$ as in (4.0.1) we have that $\pi_!$ is exact and $\pi_! \mathcal{F} = \mathcal{F}(B \to B)$.*

**Proof.** This follows from Lemma 4.1 which tells us the constant simplicial algebra on $B$ can be used to compute $L\pi_!$.                              $\square$

**Lemma 4.7.** *If $B$ is a polynomial algebra over the ring $A$, then $L_{B/A}$ is quasi-isomorphic to $\Omega_{B/A}[0]$.*

**Proof.** Immediate from Lemmas 4.3 and 4.6.                              $\square$

## 5. Constructing a resolution

In the Noetherian finite type case we can construct a "small" simplicial resolution for finite type ring maps.

**Lemma 5.1.** *Let $A$ be a Noetherian ring. Let $A \to B$ be a finite type ring map. Let $\mathcal{A}$ be the category of $A$-algebra maps $C \to B$. Let $n \geq 0$ and let $P_\bullet$ be a simplicial object of $\mathcal{A}$ such that*

(1) *$P_\bullet \to B$ is a trivial Kan fibration of simplicial sets,*
(2) *$P_k$ is finite type over $A$ for $k \leq n$,*
(3) *$P_\bullet = cosk_n sk_n P_\bullet$ as simplicial objects of $\mathcal{A}$.*

*Then $P_{n+1}$ is a finite type $A$-algebra.*

**Proof.** Although the proof we give of this lemma is straightforward, it is a bit messy. To clarify the idea we explain what happens for low $n$ before giving the proof in general. For example, if $n = 0$, then (3) means that $P_1 = P_0 \times_B P_0$. Since the ring map $P_0 \to B$ is surjective, this is of finite type over $A$ by More on Algebra, Lemma 5.1.

If $n = 1$, then (3) means that

$$P_2 = \{(f_0, f_1, f_2) \in P_1^3 \mid d_0 f_0 = d_0 f_1, \ d_1 f_0 = d_0 f_2, \ d_1 f_1 = d_1 f_2\}$$

where the equalities take place in $P_0$. Observe that the triple

$$(d_0 f_0, d_1 f_0, d_1 f_1) = (d_0 f_1, d_0 f_2, d_1 f_2)$$

is an element of the fibre product $P_0 \times_B P_0 \times_B P_0$ over $B$ because the maps $d_i : P_1 \to P_0$ are morphisms over $B$. Thus we get a map

$$\psi : P_2 \longrightarrow P_0 \times_B P_0 \times_B P_0$$

The fibre of $\psi$ over an element $(g_0, g_1, g_2) \in P_0 \times_B P_0 \times_B P_0$ is the set of triples $(f_0, f_1, f_2)$ of 1-simplices with $(d_0, d_1)(f_0) = (g_0, g_1)$, $(d_0, d_1)(f_1) = (g_0, g_2)$, and

$(d_0, d_1)(f_2) = (g_1, g_2)$. As $P_\bullet \to B$ is a trivial Kan fibration the map $(d_0, d_1) :$ $P_1 \to P_0 \times_B P_0$ is surjective. Thus we see that $P_2$ fits into the cartesian diagram

$$
\begin{array}{ccc}
P_2 & \longrightarrow & P_1^3 \\
\downarrow & & \downarrow \\
P_0 \times_B P_0 \times_B P_0 & \longrightarrow & (P_0 \times_B P_0)^3
\end{array}
$$

By More on Algebra, Lemma 5.2 we conclude. The general case is similar, but requires a bit more notation.

The case $n > 1$. By Simplicial, Lemma 19.14 the condition $P_\bullet = \mathrm{cosk}_n \mathrm{sk}_n P_\bullet$ implies the same thing is true in the category of simplicial $A$-algebras and hence in the category of sets (as the forgetful functor from $A$-algebras to sets commutes with limits). Thus

$$P_{n+1} = \mathrm{Mor}(\Delta[n+1], P_\bullet) = \mathrm{Mor}(\mathrm{sk}_n \Delta[n+1], \mathrm{sk}_n P_\bullet)$$

by Simplicial, Lemma 11.3 and Equation (19.0.1). We will prove by induction on $1 \le k < m \le n + 1$ that the ring

$$Q_{k,m} = \mathrm{Mor}(\mathrm{sk}_k \Delta[m], \mathrm{sk}_k P_\bullet)$$

is of finite type over $A$. The case $k = 1$, $1 < m \le n + 1$ is entirely similar to the discussion above in the case $n = 1$. Namely, there is a cartesian diagram

$$
\begin{array}{ccc}
Q_{1,m} & \longrightarrow & P_1^N \\
\downarrow & & \downarrow \\
P_0 \times_B \ldots \times_B P_0 & \longrightarrow & (P_0 \times_B P_0)^N
\end{array}
$$

where $N = \binom{m+1}{2}$. We conclude as before.

Let $1 \le k_0 \le n$ and assume $Q_{k,m}$ is of finite type over $A$ for all $1 \le k \le k_0$ and $k < m \le n + 1$. For $k_0 + 1 < m \le n + 1$ we claim there is a cartesian square

$$
\begin{array}{ccc}
Q_{k_0+1,m} & \longrightarrow & P_{k_0+1}^N \\
\downarrow & & \downarrow \\
Q_{k_0,m} & \longrightarrow & Q_{k_0,k_0+1}^N
\end{array}
$$

where $N$ is the number of nondegenerate $(k_0+1)$-simplices of $\Delta[m]$. Namely, to see this is true, think of an element of $Q_{k_0+1,m}$ as a function $f$ from the $(k_0+1)$-skeleton of $\Delta[m]$ to $P_\bullet$. We can restrict $f$ to the $k_0$-skeleton which gives the left vertical map of the diagram. We can also restrict to each nondegenerate $(k_0 + 1)$-simplex which gives the top horizontal arrow. Moreover, to give such an $f$ is the same thing as giving its restriction to $k_0$-skeleton and to each nondegenerate $(k_0 + 1)$-face, provided these agree on the overlap, and this is exactly the content of the diagram. Moreover, the fact that $P_\bullet \to B$ is a trivial Kan fibration implies that the map

$$P_{k_0} \to Q_{k_0,k_0+1} = \mathrm{Mor}(\partial \Delta[k_0 + 1], P_\bullet)$$

is surjective as every map $\partial \Delta[k_0 + 1] \to B$ can be extended to $\Delta[k_0 + 1] \to B$ for $k_0 \ge 1$ (small argument about constant simplicial sets omitted). Since by induction

hypothesis the rings $Q_{k_0,m}$, $Q_{k_0,k_0+1}$ are finite type $A$-algebras, so is $Q_{k_0+1,m}$ by More on Algebra, Lemma 5.2 once more. $\square$

**Proposition 5.2.** *Let $A$ be a Noetherian ring. Let $A \to B$ be a finite type ring map. There exists a simplicial $A$-algebra $P_\bullet$ with an augmentation $\epsilon : P_\bullet \to B$ such that each $P_n$ is a polynomial algebra of finite type over $A$ and such that $\epsilon$ is a trivial Kan fibration of simplicial sets.*

**Proof.** Let $\mathcal{A}$ be the category of $A$-algebra maps $C \to B$. In this proof our simplicial objects and skeleton and coskeleton functors will be taken in this category.

Choose a polynomial algebra $P_0$ of finite type over $A$ and a surjection $P_0 \to B$. As a first approximation we take $P_\bullet = \text{cosk}_0(P_0)$. In other words, $P_\bullet$ is the simplicial $A$-algebra with terms $P_n = P_0 \times_A \ldots \times_A P_0$. (In the final paragraph of the proof this simplicial object will be denoted $P_\bullet^0$.) By Simplicial, Lemma 32.3 the map $P_\bullet \to B$ is a trivial Kan fibration of simplicial sets. Also, observe that $P_\bullet = \text{cosk}_0\text{sk}_0 P_\bullet$.

Suppose for some $n \geq 0$ we have constructed $P_\bullet$ (in the final paragraph of the proof this will be $P_\bullet^n$) such that

   (a) $P_\bullet \to B$ is a trivial Kan fibration of simplicial sets,
   (b) $P_k$ is a finitely generated polynomial algebra for $0 \leq k \leq n$, and
   (c) $P_\bullet = \text{cosk}_n\text{sk}_n P_\bullet$

By Lemma 5.1 we can find a finitely generated polynomial algebra $Q$ over $A$ and a surjection $Q \to P_{n+1}$. Since $P_n$ is a polynomial algebra the $A$-algebra maps $s_i : P_n \to P_{n+1}$ lift to maps $s_i' : P_n \to Q$. Set $d_j' : Q \to P_n$ equal to the composition of $Q \to P_{n+1}$ and $d_j : P_{n+1} \to P_n$. We obtain a truncated simplicial object $P_\bullet'$ of $\mathcal{A}$ by setting $P_k' = P_k$ for $k \leq n$ and $P_{n+1}' = Q$ and morphisms $d_i' = d_i$ and $s_i' = s_i$ in degrees $k \leq n-1$ and using the morphisms $d_j'$ and $s_i'$ in degree $n$. Extend this to a full simplicial object $P_\bullet'$ of $\mathcal{A}$ using $\text{cosk}_{n+1}$. By functoriality of the coskeleton functors there is a morphism $P_\bullet' \to P_\bullet$ of simplicial objects extending the given morphism of $(n+1)$-truncated simplicial objects. (This morphism will be denoted $P_\bullet^{n+1} \to P_\bullet^n$ in the final paragraph of the proof.)

Note that conditions (b) and (c) are satisfied for $P_\bullet'$ with $n$ replaced by $n+1$. We claim the map $P_\bullet' \to P_\bullet$ satisfies assumptions (1), (2), (3), and (4) of Simplicial, Lemmas 32.1 with $n+1$ instead of $n$. Conditions (1) and (2) hold by construction. By Simplicial, Lemma 19.14 we see that we have $P_\bullet = \text{cosk}_{n+1}\text{sk}_{n+1} P_\bullet$ and $P_\bullet' = \text{cosk}_{n+1}\text{sk}_{n+1} P_\bullet'$ not only in $\mathcal{A}$ but also in the category of $A$-algebras, whence in the category of sets (as the forgetful functor from $A$-algebras to sets commutes with all limits). This proves (3) and (4). Thus the lemma applies and $P_\bullet' \to P_\bullet$ is a trivial Kan fibration. By Simplicial, Lemma 30.4 we conclude that $P_\bullet' \to B$ is a trivial Kan fibration and (a) holds as well.

To finish the proof we take the inverse limit $P_\bullet = \lim P_\bullet^n$ of the sequence of simplicial algebras

$$\ldots \to P_\bullet^2 \to P_\bullet^1 \to P_\bullet^0$$

constructed above. The map $P_\bullet \to B$ is a trivial Kan fibration by Simplicial, Lemma 30.5. However, the construction above stabilizes in each degree to a fixed finitely generated polynomial algebra as desired. $\square$

**Lemma 5.3.** *Let $A$ be a Noetherian ring. Let $A \to B$ be a finite type ring map. Let $\pi$, $\underline{B}$ be as in (4.0.1). If $\mathcal{F}$ is an $\underline{B}$-module such that $\mathcal{F}(P, \alpha)$ is a finite $B$-module*

*for all* $\alpha : P = A[x_1, \ldots, x_n] \to B$, *then the cohomology modules of* $L\pi_!(\mathcal{F})$ *are finite B-modules.*

**Proof.** By Lemma 4.1 and Proposition 5.2 we can compute $L\pi_!(\mathcal{F})$ by a complex constructed out of the values of $\mathcal{F}$ on finite type polynomial algebras. $\square$

**Lemma 5.4.** *Let $A$ be a Noetherian ring. Let $A \to B$ be a finite type ring map. Then $H^n(L_{B/A})$ is a finite B-module for all $n \in \mathbf{Z}$.*

**Proof.** Apply Lemmas 4.3 and 5.3. $\square$

**Remark 5.5** (Resolutions)**.** Let $A \to B$ be any ring map. Let us call an augmented simplicial $A$-algebra $\epsilon : P_\bullet \to B$ a *resolution of B over A* if each $P_n$ is a polynomial algebra and $\epsilon$ is a trivial Kan fibration of simplicial sets. If $P_\bullet \to B$ is an augmentation of a simplicial $A$-algebra with each $P_n$ a polynomial algebra surjecting onto $B$, then the following are equivalent

(1) $\epsilon : P_\bullet \to B$ is a resolution of $B$ over $A$,
(2) $\epsilon : P_\bullet \to B$ is a quasi-isomorphism on associated complexes,
(3) $\epsilon : P_\bullet \to B$ induces a homotopy equivalence of simplicial sets.

To see this use Simplicial, Lemmas 30.8, 31.9, and 31.8. A resolution $P_\bullet$ of $B$ over $A$ gives a cosimplicial object $U_\bullet$ of $\mathcal{C}_{B/A}$ as in Cohomology on Sites, Lemma 39.7 and it follows that

$$L\pi_!\mathcal{F} = \mathcal{F}(P_\bullet)$$

functorially in $\mathcal{F}$, see Lemma 4.1. The (formal part of the) proof of Proposition 5.2 shows that resolutions exist. We also have seen in the first proof of Lemma 4.2 that the standard resolution of $B$ over $A$ is a resolution (so that this terminology doesn't lead to a conflict). However, the argument in the proof of Proposition 5.2 shows the existence of resolutions without appealing to the simplicial computations in Simplicial, Section 34. Moreover, for *any* choice of resolution we have a canonical isomorphism

$$L_{B/A} = \Omega_{P_\bullet/A} \otimes_{P_\bullet,\epsilon} B$$

in $D(B)$ by Lemma 4.3. The freedom to choose an arbitrary resolution can be quite useful.

**Lemma 5.6.** *Let $A \to B$ be a ring map. Let $\pi, \mathcal{O}, \underline{B}$ be as in (4.0.1). For any $\mathcal{O}$-module $\mathcal{F}$ we have*

$$L\pi_!(\mathcal{F}) = L\pi_!(Li^*\mathcal{F}) = L\pi_!(\mathcal{F} \otimes_{\mathcal{O}}^{\mathbf{L}} \underline{B})$$

*in $D(Ab)$.*

**Proof.** It suffices to verify the assumptions of Cohomology on Sites, Lemma 39.12 hold for $\mathcal{O} \to \underline{B}$ on $\mathcal{C}_{B/A}$. We will use the results of Remark 5.5 without further mention. Choose a resolution $P_\bullet$ of $B$ over $A$ to get a suitable cosimplicial object $U_\bullet$ of $\mathcal{C}_{B/A}$. Since $P_\bullet \to B$ induces a quasi-isomorphism on associated complexes of abelian groups we see that $L\pi_!\mathcal{O} = B$. On the other hand $L\pi_!\underline{B}$ is computed by $\underline{B}(U_\bullet) = B$. This verifies the second assumption of Cohomology on Sites, Lemma 39.12 and we are done with the proof. $\square$

**Lemma 5.7.** *Let $A \to B$ be a ring map. Let $\pi, \mathcal{O}, \underline{B}$ be as in (4.0.1). We have*

$$L\pi_!(\mathcal{O}) = L\pi_!(\underline{B}) = B \quad and \quad L_{B/A} = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) = L\pi_!(\Omega_{\mathcal{O}/A})$$

*in $D(Ab)$.*

**Proof.** This is just an application of Lemma 5.6 (and the first equality on the right is Lemma 4.3). $\square$

Here is a special case of the fundamental triangle that is easy to prove.

**Lemma 5.8.** *Let $A \to B \to C$ be ring maps. If $B$ is a polynomial algebra over $A$, then there is a distinguished triangle $L_{B/A} \otimes_B^{\mathbf{L}} C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes_B^{\mathbf{L}} C[1]$ in $D(C)$.*

**Proof.** We will use the observations of Remark 5.5 without further mention. Choose a resolution $\epsilon : P_\bullet \to C$ of $C$ over $B$ (for example the standard resolution). Since $B$ is a polynomial algebra over $A$ we see that $P_\bullet$ is also a resolution of $C$ over $A$. Hence $L_{C/A}$ is computed by $\Omega_{P_\bullet/A} \otimes_{P_\bullet,\epsilon} C$ and $L_{C/B}$ is computed by $\Omega_{P_\bullet/B} \otimes_{P_\bullet,\epsilon} C$. Since for each $n$ we have the short exact sequence $0 \to \Omega_{B/A} \otimes_B P_n \to \Omega_{P_n/A} \to \Omega_{P_n/B}$ (Algebra, Lemma 138.9) and since $L_{B/A} = \Omega_{B/A}[0]$ (Lemma 4.7) we obtain the result. $\square$

**Example 5.9.** Let $A \to B$ be a ring map. In this example we will construct an "explicit" resolution $P_\bullet$ of $B$ over $A$ of length 2. To do this we follow the procedure of the proof of Proposition 5.2, see also the discussion in Remark 5.5.

We choose a surjection $P_0 = A[u_i] \to B$ where $u_i$ is a set of variables. Choose generators $f_t \in P_0$, $t \in T$ of the ideal $\mathrm{Ker}(P_0 \to B)$. We choose $P_1 = A[u_i, x_t]$ with face maps $d_0$ and $d_1$ the unique $A$-algebra maps with $d_j(u_i) = u_i$ and $d_0(x_t) = 0$ and $d_1(x_t) = f_t$. The map $s_0 : P_0 \to P_1$ is the unique $A$-algebra map with $s_0(u_i) = u_i$. It is clear that

$$P_1 \xrightarrow{d_0 - d_1} P_0 \to B \to 0$$

is exact, in particular the map $(d_0, d_1) : P_1 \to P_0 \times_B P_0$ is surjective. Thus, if $P_\bullet$ denotes the 1-truncated simplicial $A$-algebra given by $P_0$, $P_1$, $d_0$, $d_1$, and $s_0$, then the augmentation $\mathrm{cosk}_1(P_\bullet) \to B$ is a trivial Kan fibration. The next step of the procedure in the proof of Proposition 5.2 is to choose a polynomial algebra $P_2$ and a surjection

$$P_2 \longrightarrow \mathrm{cosk}_1(P_\bullet)_2$$

Recall that

$$\mathrm{cosk}_1(P_\bullet)_2 = \{(g_0, g_1, g_2) \in P_1^3 \mid d_0(g_0) = d_0(g_1), d_1(g_0) = d_0(g_2), d_1(g_1) = d_1(g_2)\}$$

Thinking of $g_i \in P_1$ as a polynomial in $x_t$ the conditions are

$$g_0(0) = g_1(0), \quad g_0(f_t) = g_2(0), \quad g_1(f_t) = g_2(f_t)$$

Thus $\mathrm{cosk}_1(P_\bullet)_2$ contains the elements $y_t = (x_t, x_t, f_t)$ and $z_t = (0, x_t, x_t)$. Every element $G$ in $\mathrm{cosk}_1(P_\bullet)_2$ is of the form $G = H + (0, 0, g)$ where $H$ is in the image of $A[u_i, y_t, z_t] \to \mathrm{cosk}_1(P_\bullet)_2$. Here $g \in P_1$ is a polynomial with vanishing constant term such that $g(f_t) = 0$ in $P_0$. Observe that

(1) $g = x_t x_{t'} - f_t x_{t'}$ and
(2) $g = \sum r_t x_t$ with $r_t \in P_0$ if $\sum r_t f_t = 0$ in $P_0$

are elements of $P_1$ of the desired form. Let

$$Rel = \mathrm{Ker}(\bigoplus\nolimits_{t \in T} P_0 \longrightarrow P_0), \quad (r_t) \longmapsto \sum r_t f_t$$

We set $P_2 = A[u_i, y_t, z_t, v_r, w_{t,t'}]$ where $r = (r_t) \in Rel$, with map

$$P_2 \longrightarrow \mathrm{cosk}_1(P_\bullet)_2$$

given by $y_t \mapsto (x_t, x_t, f_t)$, $z_t \mapsto (0, x_t, x_t)$, $v_r \mapsto (0, 0, \sum r_t x_t)$, and $w_{t,t'} \mapsto (0, 0, x_t x_{t'} - f_t x_{t'})$. A calculation (omitted) shows that this map is surjective. Our choice of the map displayed above determines the maps $d_0, d_1, d_2 : P_2 \to P_1$. Finally, the procedure in the proof of Proposition 5.2 tells us to choose the maps $s_0, s_1 : P_1 \to P_2$ lifting the two maps $P_1 \to \mathrm{cosk}_1(P_\bullet)_2$. It is clear that we can take $s_i$ to be the unique $A$-algebra maps determined by $s_0(x_t) = y_t$ and $s_1(x_t) = z_t$.

## 6. Functoriality

In this section we consider a commutative square

$$(6.0.1) \qquad \begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ \\ A & \longrightarrow & A' \end{array}$$

of ring maps. We claim there is a canonical $B$-linear map of complexes

$$L_{B/A} \longrightarrow L_{B'/A'}$$

associated to this diagram. Namely, if $P_\bullet \to B$ is the standard resolution of $B$ over $A$ and $P'_\bullet \to B'$ is the standard resolution of $B'$ over $A'$, then there is a canonical map $P_\bullet \to P'_\bullet$ of simplicial $A$-algebras compatible with the augmentations $P_\bullet \to B$ and $P'_\bullet \to B'$. This can be seen in terms of the construction of standard resolutions in Simplicial, Section 34 but in the special case at hand it probably suffices to say simply that the maps

$$P_0 = A[B] \longrightarrow A'[B'] = P'_0, \quad P_1 = A[A[B]] \longrightarrow A'[A'[B']] = P'_1,$$

and so on are given by the given maps $A \to A'$ and $B \to B'$. The desired map $L_{B/A} \to L_{B'/A'}$ then comes from the associated maps $\Omega_{P_n/A} \to \Omega_{P'_n/A'}$.

Another description of the functoriality map can be given as follows. Let $\mathcal{C} = \mathcal{C}_{B/A}$ and $\mathcal{C}' = \mathcal{C}'_{B'/A}$ be the categories considered in Section 4. There is a functor

$$u : \mathcal{C} \longrightarrow \mathcal{C}', \quad (P, \alpha) \longmapsto (P \otimes_A A', c \circ (\alpha \otimes 1))$$

where $c : B \otimes_A A' \to B'$ is the obvious map. As discussed in Cohomology on Sites, Example 39.3 we obtain a morphism of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ and a commutative diagram of maps of ringed topoi

$$(6.0.2) \qquad \begin{array}{ccccc} (Sh(\mathcal{C}'), \underline{B}) & \xleftarrow{\ h\ } & (Sh(\mathcal{C}'), \underline{B'}) & \xleftarrow{\ g\ } & (Sh(\mathcal{C}), \underline{B'}) \\ \pi \downarrow & & \pi \downarrow & & \pi' \downarrow \\ (Sh(*), B) & \xleftarrow{\ f\ } & (Sh(*), B') & \longleftarrow & (Sh(*), B') \end{array}$$

Here $h$ is the identity on underlying topoi and given by the ring map $B \to B'$ on sheaves of rings. By Cohomology on Sites, Remark 38.7 given $\mathcal{F}$ on $\mathcal{C}$ and $\mathcal{F}'$ on $\mathcal{C}'$ and a transformation $t : \mathcal{F} \to g^{-1}\mathcal{F}'$ we obtain a canonical map $L\pi_!(\mathcal{F}) \to L\pi'_!(\mathcal{F}')$. If we apply this to the sheaves

$$\mathcal{F} : (P, \alpha) \mapsto \Omega_{P/A} \otimes_P B, \quad \mathcal{F}' : (P', \alpha') \mapsto \Omega_{P'/A'} \otimes_{P'} B',$$

and the transformation $t$ given by the canonical maps

$$\Omega_{P/A} \otimes_P B \longrightarrow \Omega_{P \otimes_A A'/A'} \otimes_{P \otimes_A A'} B'$$

to get a canonical map

$$L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) \longrightarrow L\pi'_!(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B}')$$

By Lemma 4.3 this gives $L_{B/A} \to L_{B'/A'}$. We omit the verification that this map agrees with the map defined above in terms of simplicial resolutions.

**Lemma 6.1.** *Assume (6.0.1) induces a quasi-isomorphism $B \otimes_A^{\mathbf{L}} A' = B'$. Then, with notation as in (6.0.2) and $\mathcal{F}' \in Ab(\mathcal{C}')$, we have $L\pi_!(g^{-1}\mathcal{F}') = L\pi'_!(\mathcal{F}')$.*

**Proof.** We use the results of Remark 5.5 without further mention. We will apply Cohomology on Sites, Lemma 39.8. Let $P_\bullet \to B$ be a resolution. If we can show that $u(P_\bullet) = P_\bullet \otimes_A A' \to B'$ is a quasi-isomorphism, then we are done. The complex of $A$-modules $s(P_\bullet)$ associated to $P_\bullet$ (viewed as a simplicial $A$-module) is a free $A$-module resolution of $B$. Namely, $P_n$ is a free $A$-module and $s(P_\bullet) \to B$ is a quasi-isomorphism. Thus $B \otimes_A^{\mathbf{L}} A'$ is computed by $s(P_\bullet) \otimes_A A' = s(P_\bullet \otimes_A A')$. Therefore the assumption of the lemma signifies that $\epsilon' : P_\bullet \otimes_A A' \to B'$ is a quasi-isomorphism. $\square$

The following lemma in particular applies when $A \to A'$ is flat and $B' = B \otimes_A A'$ (flat base change).

**Lemma 6.2.** *If (6.0.1) induces a quasi-isomorphism $B \otimes_A^{\mathbf{L}} A' = B'$, then the functoriality map induces an isomorphism*

$$L_{B/A} \otimes_B^{\mathbf{L}} B' \longrightarrow L_{B'/A'}$$

**Proof.** We will use the notation introduced in Equation (6.0.2). We have

$$L_{B/A} \otimes_B^{\mathbf{L}} B' = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) \otimes_B^{\mathbf{L}} B' = L\pi_!(Lh^*(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}))$$

the first equality by Lemma 4.3 and the second by Cohomology on Sites, Lemma 39.6. Since $\Omega_{\mathcal{O}/A}$ is a flat $\mathcal{O}$-module, we see that $\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}$ is a flat $\underline{B}$-module. Thus $Lh^*(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) = \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}'$ which is equal to $g^{-1}(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B}')$ by inspection. we conclude by Lemma 6.1 and the fact that $L_{B'/A'}$ is computed by $L\pi'_!(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B}')$. $\square$

**Remark 6.3.** Suppose that we are given a square (6.0.1) such that there exists an arrow $\kappa : B \to A'$ making the diagram commute:

$$
\begin{array}{ccc}
B & \xrightarrow{\quad\beta\quad} & B' \\
\uparrow & {\scriptstyle\kappa}\searrow & \uparrow \\
A & \xrightarrow{\quad\alpha\quad} & A'
\end{array}
$$

In this case we claim the functoriality map $P_\bullet \to P'_\bullet$ is homotopic to the composition $P_\bullet \to B \to A' \to P'_\bullet$. Namely, using $\kappa$ the functoriality map factors as

$$P_\bullet \to P_{A'/A',\bullet} \to P'_\bullet$$

where $P_{A'/A',\bullet}$ is the standard resolution of $A'$ over $A'$. Since $A'$ is the polynomial algebra on the empty set over $A'$ we see from Simplicial, Lemma 34.3 that the augmentation $\epsilon_{A'/A'} : P_{A'/A',\bullet} \to A'$ is a homotopy equivalence of simplicial rings. Observe that the homotopy inverse map $c : A' \to P_{A'/A',\bullet}$ constructed in the proof

of that lemma is just the structure morphism, hence we conclude what we want because the two compositions

$$P_\bullet \longrightarrow P_{A'/A',\bullet} \overset{\mathrm{id}}{\underset{co\epsilon_{A'/A'}}{\rightrightarrows}} P_{A'/A',\bullet} \longrightarrow P'_\bullet$$

are the two maps discussed above and these are homotopic (Simplicial, Remark 26.5). Since the second map $P_\bullet \to P'_\bullet$ induces the zero map $\Omega_{P_\bullet/A} \to \Omega_{P'_\bullet/A'}$ we conclude that the functoriality map $L_{B/A} \to L_{B'/A'}$ is homotopic to zero in this case.

**Lemma 6.4.** *Let $A \to B$ and $A \to C$ be ring maps. Then the map $L_{B \times C/A} \to L_{B/A} \oplus L_{C/A}$ is an isomorphism in $D(B \times C)$.*

**Proof.** Although this lemma can be deduced from the fundamental triangle we will give a direct and elementary proof of this now. Factor the ring map $A \to B \times C$ as $A \to A[x] \to B \times C$ where $x \mapsto (1,0)$. By Lemma 5.8 we have a distinguished triangle

$$L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (B \times C) \to L_{B \times C/A} \to L_{B \times C/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (B \times C)[1]$$

in $D(B \times C)$. Similarly we have the distinguished triangles

$$L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} B \to L_{B/A} \to L_{B/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} B[1]$$
$$L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} C \to L_{C/A} \to L_{C/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} C[1]$$

Thus it suffices to prove the result for $B \times C$ over $A[x]$. Note that $A[x] \to A[x, x^{-1}]$ is flat, that $(B \times C) \otimes_{A[x]} A[x, x^{-1}] = B \otimes_{A[x]} A[x, x^{-1}]$, and that $C \otimes_{A[x]} A[x, x^{-1}] = 0$. Thus by base change (Lemma 6.2) the map $L_{B \times C/A[x]} \to L_{B/A[x]} \oplus L_{C/A[x]}$ becomes an isomorphism after inverting $x$. In the same way one shows that the map becomes an isomorphism after inverting $x - 1$. This proves the lemma. $\square$

## 7. The fundamental triangle

In this section we consider a sequence of ring maps $A \to B \to C$. It is our goal to show that this triangle gives rise to a distinguished triangle

$$(7.0.1) \qquad L_{B/A} \otimes^{\mathbf{L}}_B C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes^{\mathbf{L}}_B C[1]$$

in $D(C)$. This will be proved in Proposition 7.4. For an alternative approach see Remark 7.5.

Consider the category $\mathcal{C}_{C/B/A}$ wich is the **opposite** of the category whose objects are $(P \to B, Q \to C)$ where

(1) $P$ is a polynomial algebra over $A$,
(2) $P \to B$ is an $A$-algebra homomorphism,
(3) $Q$ is a polynomial algebra over $P$, and
(4) $Q \to C$ is a $P$-algebra-homomorphism.

We take the opposite as we want to think of $(P \to B, Q \to C)$ as corresponding to the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(C) & \longrightarrow & \mathrm{Spec}(Q) \\
\downarrow & & \downarrow \\
\mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(P) \\
\downarrow & \swarrow & \\
\mathrm{Spec}(A) & &
\end{array}
$$

Let $\mathcal{C}_{B/A}$, $\mathcal{C}_{C/A}$, $\mathcal{C}_{C/B}$ be the categories considered in Section 4. There are functors

$$
\begin{aligned}
u_1 &: \mathcal{C}_{C/B/A} \to \mathcal{C}_{B/A}, & (P \to B, Q \to C) &\mapsto (P \to B) \\
u_2 &: \mathcal{C}_{C/B/A} \to \mathcal{C}_{C/A}, & (P \to B, Q \to C) &\mapsto (Q \to C) \\
u_3 &: \mathcal{C}_{C/B/A} \to \mathcal{C}_{C/B}, & (P \to B, Q \to C) &\mapsto (Q \otimes_P B \to C)
\end{aligned}
$$

These functors induce corresponding morphisms of topoi $g_i$. Let us denote $\mathcal{O}_i = g_i^{-1}\mathcal{O}$ so that we get morphisms of ringed topoi

$$
\begin{aligned}
g_1 &: (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_1) \longrightarrow (Sh(\mathcal{C}_{B/A}), \mathcal{O}) \\
g_2 &: (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_2) \longrightarrow (Sh(\mathcal{C}_{C/A}), \mathcal{O}) \\
g_3 &: (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_3) \longrightarrow (Sh(\mathcal{C}_{C/B}), \mathcal{O})
\end{aligned}
$$

(7.0.2)

Let us denote $\pi : Sh(\mathcal{C}_{C/B/A}) \to Sh(*)$, $\pi_1 : Sh(\mathcal{C}_{B/A}) \to Sh(*)$, $\pi_2 : Sh(\mathcal{C}_{C/A}) \to Sh(*)$, and $\pi_3 : Sh(\mathcal{C}_{C/B}) \to Sh(*)$, so that $\pi = \pi_i \circ g_i$. We will obtain our distinguished triangle from the identification of the cotangent complex in Lemma 4.3 and the following lemmas.

**Lemma 7.1.** *With notation as in (7.0.2) set*

$$
\begin{aligned}
\Omega_1 &= \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B} \ \text{on } \mathcal{C}_{B/A} \\
\Omega_2 &= \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{C} \ \text{on } \mathcal{C}_{C/A} \\
\Omega_3 &= \Omega_{\mathcal{O}/B} \otimes_{\mathcal{O}} \underline{C} \ \text{on } \mathcal{C}_{C/B}
\end{aligned}
$$

*Then we have a canonical short exact sequence of sheaves of $\underline{C}$-modules*

$$
0 \to g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C} \to g_2^{-1}\Omega_2 \to g_3^{-1}\Omega_3 \to 0
$$

*on $\mathcal{C}_{C/B/A}$.*

**Proof.** Recall that $g_i^{-1}$ is gotten by simply precomposing with $u_i$. Given an object $U = (P \to B, Q \to C)$ we have a split short exact sequence

$$
0 \to \Omega_{P/A} \otimes Q \to \Omega_{Q/A} \to \Omega_{Q/P} \to 0
$$

for example by Algebra, Lemma 138.9. Tensoring with $C$ over $Q$ we obtain a short exact sequence

$$
0 \to \Omega_{P/A} \otimes C \to \Omega_{Q/A} \otimes C \to \Omega_{Q/P} \otimes C \to 0
$$

We have $\Omega_{P/A} \otimes C = \Omega_{P/A} \otimes B \otimes C$ whence this is the value of $g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C}$ on $U$. The module $\Omega_{Q/A} \otimes C$ is the value of $g_2^{-1}\Omega_2$ on $U$. We have $\Omega_{Q/P} \otimes C = \Omega_{Q \otimes_P B/B} \otimes C$ by Algebra, Lemma 131.12 hence this is the value of $g_3^{-1}\Omega_3$ on $U$. Thus the short exact sequence of the lemma comes from assigning to $U$ the last displayed short exact sequence. $\square$

**Lemma 7.2.** *With notation as in (7.0.2) suppose that $C$ is a polynomial algebra over $B$. Then $L\pi_!(g_3^{-1}\mathcal{F}) = L\pi_{3,!}\mathcal{F} = \pi_{3,!}\mathcal{F}$ for any abelian sheaf $\mathcal{F}$ on $\mathcal{C}_{C/B}$*

**Proof.** Write $C = B[E]$ for some set $E$. Choose a resolution $P_\bullet \to B$ of $B$ over $A$. For every $n$ consider the object $U_n = (P_n \to B, P_n[E] \to C)$ of $\mathcal{C}_{C/B/A}$. Then $U_\bullet$ is a cosimplicial object of $\mathcal{C}_{C/B/A}$. Note that $u_3(U_\bullet)$ is the constant cosimplicial object of $\mathcal{C}_{C/B}$ with value $(C \to C)$. We will prove that the object $U_\bullet$ of $\mathcal{C}_{C/B/A}$ satisfies the hypotheses of Cohomology on Sites, Lemma 39.7. This implies the lemma as it shows that $L\pi_!(g_3^{-1}\mathcal{F})$ is computed by the constant simplicial abelian group $\mathcal{F}(C \to C)$ which is the value of $L\pi_{3,!}\mathcal{F} = \pi_{3,!}\mathcal{F}$ by Lemma 4.6.

Let $U = (\beta : P \to B, \gamma : Q \to C)$ be an object of $\mathcal{C}_{C/B/A}$. We may write $P = A[S]$ and $Q = A[S \amalg T]$ by the definition of our category $\mathcal{C}_{C/B/A}$. We have to show that

$$\text{Mor}_{\mathcal{C}_{C/B/A}}(U_\bullet, U)$$

is homotopy equivalent to a singleton simplicial set $*$. Observe that this simplicial set is the product

$$\prod_{s \in S} F_s \times \prod_{t \in T} F'_t$$

where $F_s$ is the corresponding simplicial set for $U_s = (A[\{s\}] \to B, A[\{s\}] \to C)$ and $F'_t$ is the corresponding simplicial set for $U_t = (A \to B, A[\{t\}] \to C)$. Namely, the object $U$ is the product $\prod U_s \times \prod U_t$ in $\mathcal{C}_{C/B/A}$. It suffices each $F_s$ and $F'_t$ is homotopy equivalent to $*$, see Simplicial, Lemma 26.10. The case of $F_s$ follows as $P_\bullet \to B$ is a trivial Kan fibration (as a resolution) and $F_s$ is the fibre of this map over $\beta(s)$. (Use Simplicial, Lemmas 30.3 and 30.8). The case of $F'_t$ is more interesting. Here we are saying that the fibre of

$$P_\bullet[E] \longrightarrow C = B[E]$$

over $\gamma(t) \in C$ is homotopy equivalent to a point. In fact we will show this map is a trivial Kan fibration. Namely, $P_\bullet \to B$ is a trivial can fibration. For any ring $R$ we have

$$R[E] = \text{colim}_{\Sigma \subset \text{Map}(E, \mathbf{Z}_{\geq 0}) \text{ finite}} \prod_{I \in \Sigma} R$$

(filtered colimit). Thus the displayed map of simplicial sets is a filtered colimit of trivial Kan fibrations, whence a trivial Kan fibration by Simplicial, Lemma 30.7. $\square$

**Lemma 7.3.** *With notation as in (7.0.2) we have $Lg_{i,!} \circ g_i^{-1} = \text{id}$ for $i = 1, 2, 3$ and hence also $L\pi_! \circ g_i^{-1} = L\pi_{i,!}$ for $i = 1, 2, 3$.*

**Proof.** Proof for $i = 1$. We claim the functor $\mathcal{C}_{C/B/A}$ is a fibred category over $\mathcal{C}_{B/A}$ Namely, suppose given $(P \to B, Q \to C)$ and a morphism $(P' \to B) \to (P \to B)$ of $\mathcal{C}_{B/A}$. Recall that this means we have an $A$-algebra homomorphism $P \to P'$ compatible with maps to $B$. Then we set $Q' = Q \otimes_P P'$ with induced map to $C$ and the morphism

$$(P' \to B, Q' \to C) \longrightarrow (P \to B, Q \to C)$$

in $\mathcal{C}_{C/B/A}$ (note reversal arrows again) is strongly cartesian in $\mathcal{C}_{C/B/A}$ over $\mathcal{C}_{B/A}$. Moreover, observe that the fibre category of $u_1$ over $P \to B$ is the category $\mathcal{C}_{C/P}$. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{B/A}$. Since we have a fibred category we may apply Cohomology on Sites, Lemma 40.2. Thus $L_n g_{1,!} g_1^{-1} \mathcal{F}$ is the (pre)sheaf which assigns to $U \in \text{Ob}(\mathcal{C}_{B/A})$ the $n$th homology of $g_1^{-1}\mathcal{F}$ restricted to the fibre category

over $U$. Since these restrictions are constant the desired result follows from Lemma 4.4 via our identifications of fibre categories above.

The case $i = 2$. We claim $\mathcal{C}_{C/B/A}$ is a fibred category over $\mathcal{C}_{C/A}$ is a fibred category. Namely, suppose given $(P \to B, Q \to C)$ and a morphism $(Q' \to C) \to (Q \to C)$ of $\mathcal{C}_{C/A}$. Recall that this means we have a $B$-algebra homomorphism $Q \to Q'$ compatible with maps to $C$. Then

$$(P \to B, Q' \to C) \longrightarrow (P \to B, Q \to C)$$

is strongly cartesian in $\mathcal{C}_{C/B/A}$ over $\mathcal{C}_{C/A}$. Note that the fibre category of $u_2$ over $Q \to C$ has an final (beware reversal arrows) object, namely, $(A \to B, Q \to C)$. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{C/A}$. Since we have a fibred category we may apply Cohomology on Sites, Lemma 40.2. Thus $L_n g_{2,!} g_2^{-1} \mathcal{F}$ is the (pre)sheaf which assigns to $U \in \mathrm{Ob}(\mathcal{C}_{C/A})$ the $n$th homology of $g_1^{-1}\mathcal{F}$ restricted to the fibre category over $U$. Since these restrictions are constant the desired result follows from Cohomology on Sites, Lemma 39.5 because the fibre categories all have final objects.

The case $i = 3$. In this case we will apply Cohomology on Sites, Lemma 40.3 to $u = u_3 : \mathcal{C}_{C/B/A} \to \mathcal{C}_{C/B}$ and $\mathcal{F}' = g_3^{-1}\mathcal{F}$ for some abelian sheaf $\mathcal{F}$ on $\mathcal{C}_{C/B}$. Suppose $U = (\overline{Q} \to C)$ is an object of $\mathcal{C}_{C/B}$. Then $\mathcal{I}_U = \mathcal{C}_{\overline{Q}/B/A}$ (again beware of reversal of arrows). The sheaf $\mathcal{F}'_U$ is given by the rule $(P \to B, Q \to \overline{Q}) \mapsto \mathcal{F}(Q \otimes_P B \to C)$. In other words, this sheaf is the pullback of a sheaf on $\mathcal{C}_{\overline{Q}/C}$ via the morphism $Sh(\mathcal{C}_{\overline{Q}/B/A}) \to Sh(\mathcal{C}_{\overline{Q}/B})$. Thus Lemma 7.2 shows that $H_n(\mathcal{I}_U, \mathcal{F}'_U) = 0$ for $n > 0$ and equal to $\mathcal{F}(\overline{Q} \to C)$ for $n = 0$. The aforementioned Cohomology on Sites, Lemma 40.3 implies that $Lg_{3,!}(g_3^{-1}\mathcal{F}) = \mathcal{F}$ and the proof is done. $\square$

**Proposition 7.4.** *Let $A \to B \to C$ be ring maps. There is a canonical distinguished triangle*

$$L_{B/A} \otimes_B^{\mathbf{L}} C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes_B^{\mathbf{L}} C[1]$$

*in $D(C)$.*

**Proof.** Consider the short exact sequence of sheaves of Lemma 7.1 and apply the derived functor $L\pi_!$ to obtain a distinguished triangle

$$L\pi_!(g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C}) \to L\pi_!(g_2^{-1}\Omega_2) \to L\pi_!(g_3^{-1}\Omega_3) \to L\pi_!(g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C})[1]$$

in $D(C)$. Using Lemmas 7.3 and 4.3 we see that the second and third terms agree with $L_{C/A}$ and $L_{C/B}$ and the first one equals

$$L\pi_{1,!}(\Omega_1 \otimes_{\underline{B}} \underline{C}) = L\pi_{1,!}(\Omega_1) \otimes_B^{\mathbf{L}} C = L_{B/A} \otimes_B^{\mathbf{L}} C$$

The first equality by Cohomology on Sites, Lemma 39.6 (and flatness of $\Omega_1$ as a sheaf of modules over $\underline{B}$) and the second by Lemma 4.3. $\square$

**Remark 7.5.** We sketch an alternative, perhaps simpler, proof of the existence of the fundamental triangle. Let $A \to B \to C$ be ring maps and assume that $B \to C$ is injective. Let $P_\bullet \to B$ be the standard resolution of $B$ over $A$ and let $Q_\bullet \to C$ be the standard resolution of $C$ over $B$. Picture

$$
\begin{array}{ccccccccc}
P_\bullet : & & A[A[A[B]]] & \rightrightarrows & A[A[B]] & \rightleftarrows & A[B] & \longrightarrow & B \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q_\bullet : & & A[A[A[C]]] & \rightrightarrows & A[A[C]] & \rightleftarrows & A[C] & \longrightarrow & C
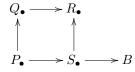\end{array}
$$

Observe that since $B \to C$ is injective, the ring $Q_n$ is a polynomial algebra over $P_n$ for all $n$. Hence we obtain a cosimplicial object in $\mathcal{C}_{C/B/A}$ (beware reversal arrows). Now set $\overline{Q}_\bullet = Q_\bullet \otimes_{P_\bullet} B$. The key to the proof of Proposition 7.4 is to show that $\overline{Q}_\bullet$ is a resolution of $C$ over $B$. This follows from Cohomology on Sites, Lemma 39.12 applied to $\mathcal{C} = \Delta$, $\mathcal{O} = P_\bullet$, $\mathcal{O}' = B$, and $\mathcal{F} = Q_\bullet$ (this uses that $Q_n$ is flat over $P_n$; see Cohomology on Sites, Remark 39.11 to relate simplicial modules to sheaves). The key fact implies that the distinguished triangle of Proposition 7.4 is the distinguished triangle associated to the short exact sequence of simplicial $C$-modules

$$0 \to \Omega_{P_\bullet/A} \otimes_{P_\bullet} C \to \Omega_{Q_\bullet/A} \otimes_{Q_\bullet} C \to \Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C \to 0$$

which is deduced from the short exact sequences $0 \to \Omega_{P_n/A} \otimes_{P_n} Q_n \to \Omega_{Q_n/A} \to \Omega_{Q_n/P_n} \to 0$ of Algebra, Lemma 138.9. Namely, by Remark 5.5 and the key fact the complex on the right hand side represents $L_{C/B}$ in $D(C)$.

If $B \to C$ is not injective, then we can use the above to get a fundamental triangle for $A \to B \to B \times C$. Since $L_{B \times C/B} \to L_{B/B} \oplus L_{C/B}$ and $L_{B \times C/A} \to L_{B/A} \oplus L_{C/A}$ are quasi-isomorphism in $D(B \times C)$ (Lemma 6.4) this induces the desired distinguished triangle in $D(C)$ by tensoring with the flat ring map $B \times C \to C$.

**Remark 7.6.** Let $A \to B \to C$ be ring maps with $B \to C$ injective. Recall the notation $P_\bullet$, $Q_\bullet$, $\overline{Q}_\bullet$ of Remark 7.5. Let $R_\bullet$ be the standard resolution of $C$ over $B$. In this remark we explain how to get the canonical identification of $\Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C$ with $L_{C/B} = \Omega_{R_\bullet/B} \otimes_{R_\bullet} C$. Let $S_\bullet \to B$ be the standard resolution of $B$ over $B$. Note that the functoriality map $S_\bullet \to R_\bullet$ identifies $R_n$ as a polynomial algebra over $S_n$ because $B \to C$ is injective. For example in degree 0 we have the map $B[B] \to B[C]$, in degree 1 the map $B[B[B]] \to B[B[C]]$, and so on. Thus $\overline{R}_\bullet = R_\bullet \otimes_{S_\bullet} B$ is a simplicial polynomial algebra over $B$ as well and it follows (as in Remark 7.5) from Cohomology on Sites, Lemma 39.12 that $\overline{R}_\bullet \to C$ is a resolution. Since we have a commutative diagram

$$
\begin{array}{ccc}
Q_\bullet & \longrightarrow & R_\bullet \\
\uparrow & & \uparrow \\
P_\bullet & \longrightarrow S_\bullet \longrightarrow & B
\end{array}
$$

we obtain a canonical map $\overline{Q}_\bullet = Q_\bullet \otimes_{P_\bullet} B \to \overline{R}_\bullet$. Thus the maps

$$L_{C/B} = \Omega_{R_\bullet/B} \otimes_{R_\bullet} C \longrightarrow \Omega_{\overline{R}_\bullet/B} \otimes_{\overline{R}_\bullet} C \longleftarrow \Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C$$

are quasi-isomorphisms (Remark 5.5) and composing one with the inverse of the other gives the desired identification.

## 8. Localization and étale ring maps

In this section we study what happens if we localize our rings. Let $A \to A' \to B$ be ring maps such that $B = B \otimes_A^{\mathbf{L}} A'$. This happens for example if $A' = S^{-1}A$ is the localization of $A$ at a multiplicative subset $S \subset A$. In this case for an abelian sheaf $\mathcal{F}'$ on $\mathcal{C}_{B/A'}$ the homology of $g^{-1}\mathcal{F}'$ over $\mathcal{C}_{B/A}$ agrees with the homology of $\mathcal{F}'$ over $\mathcal{C}_{B/A'}$, see Lemma 6.1 for a precise statement.

**Lemma 8.1.** *Let $A \to A' \to B$ be ring maps such that $B = B \otimes_A^{\mathbf{L}} A'$. Then $L_{B/A} = L_{B/A'}$ in $D(B)$.*

**Proof.** According to the discussion above (i.e., using Lemma 6.1) and Lemma 4.3 we have to show that the sheaf given by the rule $(P \to B) \mapsto \Omega_{P/A} \otimes_P B$ on $\mathcal{C}_{B/A}$ is the pullback of the sheaf given by the rule $(P \to B) \mapsto \Omega_{P/A'} \otimes_P B$. The pullback functor $g^{-1}$ is given by precomposing with the functor $u : \mathcal{C}_{B/A} \to \mathcal{C}_{B/A'}$, $(P \to B) \mapsto (P \otimes_A A' \to B)$. Thus we have to show that

$$\Omega_{P/A} \otimes_P B = \Omega_{P \otimes_A A'/A'} \otimes_{(P \otimes_A A')} B$$

By Algebra, Lemma 131.12 the right hand side is equal to

$$(\Omega_{P/A} \otimes_A A') \otimes_{(P \otimes_A A')} B$$

Since $P$ is a polynomial algebra over $A$ the module $\Omega_{P/A}$ is free and the equality is obvious. $\square$

**Lemma 8.2.** *Let $A \to B$ be a ring map such that $B = B \otimes_A^{\mathbf{L}} B$. Then $L_{B/A} = 0$ in $D(B)$.*

**Proof.** This is true because $L_{B/A} = L_{B/B} = 0$ by Lemmas 8.1 and 4.7. $\square$

**Lemma 8.3.** *Let $A \to B$ be a ring map such that $Tor_i^A(B, B) = 0$ for $i > 0$ and such that $L_{B/B \otimes_A B} = 0$. Then $L_{B/A} = 0$ in $D(B)$.*

**Proof.** By Lemma 6.2 we see that $L_{B/A} \otimes_B^{\mathbf{L}} (B \otimes_A B) = L_{B \otimes_A B/B}$. Now we use the distinguished triangle (7.0.1)

$$L_{B \otimes_A B/B} \otimes_{(B \otimes_A B)}^{\mathbf{L}} B \to L_{B/B} \to L_{B/B \otimes_A B} \to L_{B \otimes_A B/B} \otimes_{(B \otimes_A B)}^{\mathbf{L}} B[1]$$

associated to the ring maps $B \to B \otimes_A B \to B$ and the vanishing of $L_{B/B}$ (Lemma 4.7) and $L_{B/B \otimes_A B}$ (assumed) to see that

$$0 = L_{B \otimes_A B/B} \otimes_{(B \otimes_A B)}^{\mathbf{L}} B = L_{B/A} \otimes_B^{\mathbf{L}} (B \otimes_A B) \otimes_{(B \otimes_A B)}^{\mathbf{L}} B = L_{B/A}$$

as desired. $\square$

**Lemma 8.4.** *The cotangent complex $L_{B/A}$ is zero in each of the following cases:*

(1) *$A \to B$ and $B \otimes_A B \to B$ are flat, i.e., $A \to B$ is weakly étale (More on Algebra, Definition 104.1),*

(2) *$A \to B$ is a flat epimorphism of rings,*

(3) *$B = S^{-1}A$ for some multiplicative subset $S \subset A$,*

(4) *$A \to B$ is unramified and flat,*

(5) *$A \to B$ is étale,*

(6) *$A \to B$ is a filtered colimit of ring maps for which the cotangent complex vanishes,*

(7) *$B$ is a henselization of a local ring of $A$,*

(8) *$B$ is a strict henselization of a local ring of $A$, and*

(9) *add more here.*

**Proof.** In case (1) we may apply Lemma 8.2 to the surjective flat ring map $B \otimes_A B \to B$ to conclude that $L_{B/B \otimes_A B} = 0$ and then we use Lemma 8.3 to conclude. The cases (2) – (5) are each special cases of (1). Part (6) follows from Lemma 3.4. Parts (7) and (8) follows from the fact that (strict) henselizations are filtered colimits of étale ring extensions of $A$, see Algebra, Lemmas 155.7 and 155.11. $\square$

**Lemma 8.5.** *Let $A \to B \to C$ be ring maps such that $L_{C/B} = 0$. Then $L_{C/A} = L_{B/A} \otimes_B^{\mathbf{L}} C$.*

**Proof.** This is a trivial consequence of the distinguished triangle (7.0.1). □

**Lemma 8.6.** *Let $A \to B$ be ring maps and $S \subset A$, $T \subset B$ multiplicative subsets such that $S$ maps into $T$. Then $L_{T^{-1}B/S^{-1}A} = L_{B/A} \otimes_B T^{-1}B$ in $D(T^{-1}B)$.*

**Proof.** Lemma 8.5 shows that $L_{T^{-1}B/A} = L_{B/A} \otimes_B T^{-1}B$ and Lemma 8.1 shows that $L_{T^{-1}B/A} = L_{T^{-1}B/S^{-1}A}$. □

**Lemma 8.7.** *Let $A \to B$ be a local ring homomorphism of local rings. Let $A^h \to B^h$, resp. $A^{sh} \to B^{sh}$ be the induced maps of henselizations, resp. strict henselizations. Then*

$$L_{B^h/A^h} = L_{B^h/A} = L_{B/A} \otimes_B^{\mathbf{L}} B^h \quad resp. \quad L_{B^{sh}/A^{sh}} = L_{B^{sh}/A} = L_{B/A} \otimes_B^{\mathbf{L}} B^{sh}$$

*in $D(B^h)$, resp. $D(B^{sh})$.*

**Proof.** The complexes $L_{A^h/A}$, $L_{A^{sh}/A}$, $L_{B^h/B}$, and $L_{B^{sh}/B}$ are all zero by Lemma 8.4. Using the fundamental distinguished triangle (7.0.1) for $A \to B \to B^h$ we obtain $L_{B^h/A} = L_{B/A} \otimes_B^{\mathbf{L}} B^h$. Using the fundamental triangle for $A \to A^h \to B^h$ we obtain $L_{B^h/A^h} = L_{B^h/A}$. Similarly for strict henselizations. □

## 9. Smooth ring maps

Let $C \to B$ be a surjection of rings with kernel $I$. Let us call such a ring map "weakly quasi-regular" if $I/I^2$ is a flat $B$-module and $\text{Tor}_*^C(B, B)$ is the exterior algebra on $I/I^2$. The generalization to "smooth ring maps" of what is done in Lemma 8.4 for "étale ring maps" is to look at flat ring maps $A \to B$ such that the multiplication map $B \otimes_A B \to B$ is weakly quasi-regular. For the moment we just stick to smooth ring maps.

**Lemma 9.1.** *If $A \to B$ is a smooth ring map, then $L_{B/A} = \Omega_{B/A}[0]$.*

**Proof.** We have the agreement in cohomological degree 0 by Lemma 4.5. Thus it suffices to prove the other cohomology groups are zero. It suffices to prove this locally on $\text{Spec}(B)$ as $L_{B_g/A} = (L_{B/A})_g$ for $g \in B$ by Lemma 8.5. Thus we may assume that $A \to B$ is standard smooth (Algebra, Lemma 137.10), i.e., that we can factor $A \to B$ as $A \to A[x_1, \ldots, x_n] \to B$ with $A[x_1, \ldots, x_n] \to B$ étale. In this case Lemmas 8.4 and Lemma 8.5 show that $L_{B/A} = L_{A[x_1, \ldots, x_n]/A} \otimes B$ whence the conclusion by Lemma 4.7. □

## 10. Positive characteristic

In this section we fix a prime number $p$. If $A$ is a ring with $p = 0$ in $A$, then $F_A : A \to A$ denotes the Frobenius endomorphism $a \mapsto a^p$.

**Lemma 10.1.** *Let $A \to B$ be a ring map with $p = 0$ in $A$. Let $P_\bullet$ be the standard resolution of $B$ over $A$. The map $P_\bullet \to P_\bullet$ induced by the diagram*

$$
\begin{array}{ccc}
B & \xrightarrow{F_B} & B \\
\uparrow & & \uparrow \\
A & \xrightarrow{F_A} & A
\end{array}
$$

*discussed in Section 6 is homotopic to the Frobenius endomorphism $P_\bullet \to P_\bullet$ given by Frobenius on each $P_n$.*

**Proof.** Let $\mathcal{A}$ be the category of $\mathbf{F}_p$-algebra maps $A \to B$. Let $\mathcal{S}$ be the category of pairs $(A, E)$ where $A$ is an $\mathbf{F}_p$-algebra and $E$ is a set. Consider the adjoint functors

$$V : \mathcal{A} \to \mathcal{S}, \quad (A \to B) \mapsto (A, B)$$

and

$$U : \mathcal{S} \to \mathcal{A}, \quad (A, E) \mapsto (A \to A[E])$$

Let $X$ be the simplicial object in in the category of functors from $\mathcal{A}$ to $\mathcal{A}$ constructed in Simplicial, Section 34. It is clear that $P_\bullet = X(A \to B)$ because if we fix $A$ then.

Set $Y = U \circ V$. Recall that $X$ is constructed from $Y$ and certain maps and has terms $X_n = Y \circ \ldots \circ Y$ with $n + 1$ terms; the construction is given in Simplicial, Example 33.1 and please see proof of Simplicial, Lemma 34.2 for details.

Let $f : \mathrm{id}_\mathcal{A} \to \mathrm{id}_\mathcal{A}$ be the Frobenius endomorphism of the identity functor. In other words, we set $f_{A \to B} = (F_A, F_B) : (A \to B) \to (A \to B)$. Then our two maps on $X(A \to B)$ are given by the natural transformations $f \star 1_X$ and $1_X \star f$. Details omitted. Thus we conclude by Simplicial, Lemma 33.6. $\square$

**Lemma 10.2.** *Let $p$ be a prime number. Let $A \to B$ be a ring homomorphism and assume that $p = 0$ in $A$. The map $L_{B/A} \to L_{B/A}$ of Section 6 induced by the Frobenius maps $F_A$ and $F_B$ is homotopic to zero.*

**Proof.** Let $P_\bullet$ be the standard resolution of $B$ over $A$. By Lemma 10.1 the map $P_\bullet \to P_\bullet$ induced by $F_A$ and $F_B$ is homotopic to the map $F_{P_\bullet} : P_\bullet \to P_\bullet$ given by Frobenius on each term. Hence we obtain what we want as clearly $F_{P_\bullet}$ induces the zero zero map $\Omega_{P_n/A} \to \Omega_{P_n/A}$ (since the derivative of a $p$th power is zero). $\square$

**Lemma 10.3.** *Let $p$ be a prime number. Let $A \to B$ be a ring homomorphism and assume that $p = 0$ in $A$. If $A$ and $B$ are perfect, then $L_{B/A}$ is zero in $D(B)$.*

**Proof.** The map $(F_A, F_B) : (A \to B) \to (A \to B)$ is an isomorphism hence induces an isomorphism on $L_{B/A}$ and on the other hand induces zero on $L_{B/A}$ by Lemma 10.2. $\square$

## 11. Comparison with the naive cotangent complex

The naive cotangent complex was introduced in Algebra, Section 134.

**Remark 11.1.** Let $A \to B$ be a ring map. Working on $\mathcal{C}_{B/A}$ as in Section 4 let $\mathcal{J} \subset \mathcal{O}$ be the kernel of $\mathcal{O} \to \underline{B}$. Note that $L\pi_!(\mathcal{J}) = 0$ by Lemma 5.7. Set $\Omega = \Omega_{\mathcal{O}/A} \otimes_\mathcal{O} \underline{B}$ so that $L_{B/A} = L\pi_!(\Omega)$ by Lemma 4.3. It follows that $L\pi_!(\mathcal{J} \to \Omega) = L\pi_!(\Omega) = L_{B/A}$. Thus, for any object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we obtain a map

(11.1.1)                                $(J \to \Omega_{P/A} \otimes_P B) \longrightarrow L_{B/A}$

where $J = \mathrm{Ker}(P \to B)$ in $D(A)$, see Cohomology on Sites, Remark 39.4. Continuing in this manner, note that $L\pi_!(\mathcal{J} \otimes^{\mathbf{L}}_\mathcal{O} \underline{B}) = L\pi_!(\mathcal{J}) = 0$ by Lemma 5.6. Since $\mathrm{Tor}^\mathcal{O}_0(\mathcal{J}, \underline{B}) = \mathcal{J}/\mathcal{J}^2$ the spectral sequence

$$H_p(\mathcal{C}_{B/A}, \mathrm{Tor}^\mathcal{O}_q(\mathcal{J}, \underline{B})) \Rightarrow H_{p+q}(\mathcal{C}_{B/A}, \mathcal{J} \otimes^{\mathbf{L}}_\mathcal{O} \underline{B}) = 0$$

(dual of Derived Categories, Lemma 21.3) implies that $H_0(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) = 0$ and $H_1(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) = 0$. It follows that the complex of $\underline{B}$-modules $\mathcal{J}/\mathcal{J}^2 \to \Omega$

satisfies $\tau_{\geq -1}L\pi_!(\mathcal{J}/\mathcal{J}^2 \to \Omega) = \tau_{\geq -1}L_{B/A}$. Thus, for any object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we obtain a map

(11.1.2) $$(J/J^2 \to \Omega_{P/A} \otimes_P B) \longrightarrow \tau_{\geq -1}L_{B/A}$$

in $D(B)$, see Cohomology on Sites, Remark 39.4.

The first case is where we have a surjection of rings.

**Lemma 11.2.** *Let $A \to B$ be a surjective ring map with kernel $I$. Then $H^0(L_{B/A}) = 0$ and $H^{-1}(L_{B/A}) = I/I^2$. This isomorphism comes from the map (11.1.2) for the object $(A \to B)$ of $\mathcal{C}_{B/A}$.*

**Proof.** We will show below (using the surjectivity of $A \to B$) that there exists a short exact sequence

$$0 \to \pi^{-1}(I/I^2) \to \mathcal{J}/\mathcal{J}^2 \to \Omega \to 0$$

of sheaves on $\mathcal{C}_{B/A}$. Taking $L\pi_!$ and the associated long exact sequence of homology, and using the vanishing of $H_1(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2)$ and $H_0(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2)$ shown in Remark 11.1 we obtain what we want using Lemma 4.4.

What is left is to verify the local statement mentioned above. For every object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we can choose an isomorphism $P = A[E]$ such that the map $P \to B$ maps each $e \in E$ to zero. Then $J = \mathcal{J}(U) \subset P = \mathcal{O}(U)$ is equal to $J = IP + (e; e \in E)$. The value on $U$ of the short sequence of sheaves above is the sequence

$$0 \to I/I^2 \to J/J^2 \to \Omega_{P/A} \otimes_P B \to 0$$

Verification omitted (hint: the only tricky point is that $IP \cap J^2 = IJ$; which follows for example from More on Algebra, Lemma 30.9). $\qquad\square$

**Lemma 11.3.** *Let $A \to B$ be a ring map. Then $\tau_{\geq -1}L_{B/A}$ is canonically quasi-isomorphic to the naive cotangent complex.*

**Proof.** Consider $P = A[B] \to B$ with kernel $I$. The naive cotangent complex $NL_{B/A}$ of $B$ over $A$ is the complex $I/I^2 \to \Omega_{P/A} \otimes_P B$, see Algebra, Definition 134.1. Observe that in (11.1.2) we have already constructed a canonical map

$$c : NL_{B/A} \longrightarrow \tau_{\geq -1}L_{B/A}$$

Consider the distinguished triangle (7.0.1)

$$L_{P/A} \otimes_P^{\mathbf{L}} B \to L_{B/A} \to L_{B/P} \to (L_{P/A} \otimes_P^{\mathbf{L}} B)[1]$$

associated to the ring maps $A \to A[B] \to B$. We know that $L_{P/A} = \Omega_{P/A}[0] = NL_{P/A}$ in $D(P)$ (Lemma 4.7 and Algebra, Lemma 134.3) and that $\tau_{\geq -1}L_{B/P} = I/I^2[1] = NL_{B/P}$ in $D(B)$ (Lemma 11.2 and Algebra, Lemma 134.6). To show $c$ is a quasi-isomorphism it suffices by Algebra, Lemma 134.4 and the long exact cohomology sequence associated to the distinguished triangle to show that the maps $L_{P/A} \to L_{B/A} \to L_{B/P}$ are compatible on cohomology groups with the corresponding maps $NL_{P/A} \to NL_{B/A} \to NL_{B/P}$ of the naive cotangent complex. We omit the verification. $\qquad\square$

**Remark 11.4.** We can make the comparison map of Lemma 11.3 explicit in the following way. Let $P_\bullet$ be the standard resolution of $B$ over $A$. Let $I = \mathrm{Ker}(A[B] \to$

$B$). Recall that $P_0 = A[B]$. The map of the lemma is given by the commutative diagram

$$
\begin{array}{ccccccc}
L_{B/A} & \dots \longrightarrow & \Omega_{P_2/A} \otimes_{P_2} B & \longrightarrow & \Omega_{P_1/A} \otimes_{P_1} B & \longrightarrow & \Omega_{P_0/A} \otimes_{P_0} B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
NL_{B/A} & \dots \longrightarrow & 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega_{P_0/A} \otimes_{P_0} B
\end{array}
$$

We construct the downward arrow with target $I/I^2$ by sending $\mathrm{d}f \otimes b$ to the class of $(d_0(f) - d_1(f))b$ in $I/I^2$. Here $d_i : P_1 \to P_0$, $i = 0, 1$ are the two face maps of the simplicial structure. This makes sense as $d_0 - d_1$ maps $P_1$ into $I = \mathrm{Ker}(P_0 \to B)$. We omit the verification that this rule is well defined. Our map is compatible with the differential $\Omega_{P_1/A} \otimes_{P_1} B \to \Omega_{P_0/A} \otimes_{P_0} B$ as this differential maps $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f)) \otimes b$. Moreover, the differential $\Omega_{P_2/A} \otimes_{P_2} B \to \Omega_{P_1/A} \otimes_{P_1} B$ maps $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f) + d_2(f)) \otimes b$ which are annihilated by our downward arrow. Hence a map of complexes. We omit the verification that this is the same as the map of Lemma 11.3.

**Remark 11.5.** Adopt notation as in Remark 11.1. The arguments given there show that the differential

$$
H_2(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) \longrightarrow H_0(\mathcal{C}_{B/A}, \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{J}, \underline{B}))
$$

of the spectral sequence is an isomorphism. Let $\mathcal{C}'_{B/A}$ denote the full subcategory of $\mathcal{C}_{B/A}$ consisting of surjective maps $P \to B$. The agreement of the cotangent complex with the naive cotangent complex (Lemma 11.3) shows that we have an exact sequence of sheaves

$$
0 \to \underline{H_1(L_{B/A})} \to \mathcal{J}/\mathcal{J}^2 \xrightarrow{\mathrm{d}} \Omega \to \underline{H_2(L_{B/A})} \to 0
$$

on $\mathcal{C}'_{B/A}$. It follows that $\mathrm{Ker}(d)$ and $\mathrm{Coker}(d)$ on the whole category $\mathcal{C}_{B/A}$ have vanishing higher homology groups, since these are computed by the homology groups of constant simplicial abelian groups by Lemma 4.1. Hence we conclude that

$$
H_n(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) \to H_n(L_{B/A})
$$

is an isomorphism for all $n \geq 2$. Combined with the remark above we obtain the formula $H_2(L_{B/A}) = H_0(\mathcal{C}_{B/A}, \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{J}, \underline{B}))$.

## 12. A spectral sequence of Quillen

In this section we discuss a spectral sequence relating derived tensor product to the cotangent complex.

**Lemma 12.1.** *Notation and assumptions as in Cohomology on Sites, Example 39.1. Assume $\mathcal{C}$ has a cosimplicial object as in Cohomology on Sites, Lemma 39.7. Let $\mathcal{F}$ be a flat $\underline{B}$-module such that $H_0(\mathcal{C}, \mathcal{F}) = 0$. Then $H_l(\mathcal{C}, Sym_{\underline{B}}^k(\mathcal{F})) = 0$ for $l < k$.*

**Proof.** We drop the subscript $\underline{B}$ from tensor products, wedge powers, and symmetric powers. We will prove the lemma by induction on $k$. The cases $k = 0, 1$ follow from the assumptions. If $k > 1$ consider the exact complex

$$
\dots \to \wedge^2 \mathcal{F} \otimes \mathrm{Sym}^{k-2}\mathcal{F} \to \mathcal{F} \otimes \mathrm{Sym}^{k-1}\mathcal{F} \to \mathrm{Sym}^k\mathcal{F} \to 0
$$

with differentials as in the Koszul complex. If we think of this as a resolution of $\mathrm{Sym}^k \mathcal{F}$, then this gives a first quadrant spectral sequence

$$E_1^{p,q} = H_p(\mathcal{C}, \wedge^{q+1} \mathcal{F} \otimes \mathrm{Sym}^{k-q-1} \mathcal{F}) \Rightarrow H_{p+q}(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F}))$$

By Cohomology on Sites, Lemma 39.10 we have

$$L\pi_!(\wedge^{q+1} \mathcal{F} \otimes \mathrm{Sym}^{k-q-1} \mathcal{F}) = L\pi_!(\wedge^{q+1} \mathcal{F}) \otimes_B^{\mathbf{L}} L\pi_!(\mathrm{Sym}^{k-q-1} \mathcal{F}))$$

It follows (from the construction of derived tensor products) that the induction hypothesis combined with the vanishing of $H_0(\mathcal{C}, \wedge^{q+1}(\mathcal{F})) = 0$ will prove what we want. This is true because $\wedge^{q+1}(\mathcal{F})$ is a quotient of $\mathcal{F}^{\otimes q+1}$ and $H_0(\mathcal{C}, \mathcal{F}^{\otimes q+1})$ is a quotient of $H_0(\mathcal{C}, \mathcal{F})^{\otimes q+1}$ which is zero. $\qquad\square$

**Remark 12.2.** In the situation of Lemma 12.1 one can show that $H_k(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F})) = \wedge_B^k(H_1(\mathcal{C}, \mathcal{F}))$. Namely, it can be deduced from the proof that $H_k(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F}))$ is the $S_k$-coinvariants of

$$H^{-k}(L\pi_!(\mathcal{F}) \otimes_B^{\mathbf{L}} L\pi_!(\mathcal{F}) \otimes_B^{\mathbf{L}} \ldots \otimes_B^{\mathbf{L}} L\pi_!(\mathcal{F})) = H_1(\mathcal{C}, \mathcal{F})^{\otimes k}$$

Thus our claim is that this action is given by the usual action of $S_k$ on the tensor product multiplied by the sign character. To prove this one has to work through the sign conventions in the definition of the total complex associated to a multi-complex. We omit the verification.

**Lemma 12.3.** *Let $A$ be a ring. Let $P = A[E]$ be a polynomial ring. Set $I = (e; e \in E) \subset P$. The maps $\mathrm{Tor}_i^P(A, I^{n+1}) \to \mathrm{Tor}_i^P(A, I^n)$ are zero for all $i$ and $n$.*

**Proof.** Denote $x_e \in P$ the variable corresponding to $e \in E$. A free resolution of $A$ over $P$ is given by the Koszul complex $K_\bullet$ on the $x_e$. Here $K_i$ has basis given by wedges $e_1 \wedge \ldots \wedge e_i$, $e_1, \ldots, e_i \in E$ and $d(e) = x_e$. Thus $K_\bullet \otimes_P I^n = I^n K_\bullet$ computes $\mathrm{Tor}_i^P(A, I^n)$. Observe that everything is graded with $\deg(x_e) = 1$, $\deg(e) = 1$, and $\deg(a) = 0$ for $a \in A$. Suppose $\xi \in I^{n+1} K_i$ is a cocycle homogeneous of degree $m$. Note that $m \geq i + 1 + n$. Then $\xi = d\eta$ for some $\eta \in K_{i+1}$ as $K_\bullet$ is exact in degrees $> 0$. (The case $i = 0$ is left to the reader.) Now $\deg(\eta) = m \geq i + 1 + n$. Hence writing $\eta$ in terms of the basis we see the coordinates are in $I^n$. Thus $\xi$ maps to zero in the homology of $I^n K_\bullet$ as desired. $\qquad\square$

**Theorem 12.4** (Quillen spectral sequence). *Let $A \to B$ be a surjective ring map. Consider the sheaf $\Omega = \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}$ of $\underline{B}$-modules on $\mathcal{C}_{B/A}$, see Section 4. Then there is a spectral sequence with $E_1$-page*

$$E_1^{p,q} = H_{-p-q}(\mathcal{C}_{B/A}, \mathrm{Sym}_{\underline{B}}^p(\Omega)) \Rightarrow \mathrm{Tor}_{-p-q}^A(B, B)$$

*with $d_r$ of bidegree $(r, -r + 1)$. Moreover, $H_i(\mathcal{C}_{B/A}, \mathrm{Sym}_{\underline{B}}^k(\Omega)) = 0$ for $i < k$.*

**Proof.** Let $I \subset A$ be the kernel of $A \to B$. Let $\mathcal{J} \subset \mathcal{O}$ be the kernel of $\mathcal{O} \to \underline{B}$. Then $I\mathcal{O} \subset \mathcal{J}$. Set $\mathcal{K} = \mathcal{J}/I\mathcal{O}$ and $\overline{\mathcal{O}} = \mathcal{O}/I\mathcal{O}$.

For every object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we can choose an isomorphism $P = A[E]$ such that the map $P \to B$ maps each $e \in E$ to zero. Then $J = \mathcal{J}(U) \subset P = \mathcal{O}(U)$ is equal to $J = IP + (e; e \in E)$. Moreover $\overline{\mathcal{O}}(U) = B[E]$ and $K = \mathcal{K}(U) = (e; e \in E)$ is the ideal generated by the variables in the polynomial ring $B[E]$. In particular it is clear that

$$K/K^2 \xrightarrow{\mathrm{d}} \Omega_{P/A} \otimes_P B$$

is a bijection. In other words, $\Omega = \mathcal{K}/\mathcal{K}^2$ and $\mathrm{Sym}_B^k(\Omega) = \mathcal{K}^k/\mathcal{K}^{k+1}$. Note that $\pi_!(\Omega) = \Omega_{B/A} = 0$ (Lemma 4.5) as $A \to B$ is surjective (Algebra, Lemma 131.4). By Lemma 12.1 we conclude that

$$H_i(\mathcal{C}_{B/A}, \mathcal{K}^k/\mathcal{K}^{k+1}) = H_i(\mathcal{C}_{B/A}, \mathrm{Sym}_{\underline{B}}^k(\Omega)) = 0$$

for $i < k$. This proves the final statement of the theorem.

The approach to the theorem is to note that

$$B \otimes_A^{\mathbf{L}} B = L\pi_!(\mathcal{O}) \otimes_A^{\mathbf{L}} B = L\pi_!(\mathcal{O} \otimes_{\underline{A}}^{\mathbf{L}} \underline{B}) = L\pi_!(\overline{\mathcal{O}})$$

The first equality by Lemma 5.7, the second equality by Cohomology on Sites, Lemma 39.6, and the third equality as $\mathcal{O}$ is flat over $\underline{A}$. The sheaf $\overline{\mathcal{O}}$ has a filtration

$$\ldots \subset \mathcal{K}^3 \subset \mathcal{K}^2 \subset \mathcal{K} \subset \overline{\mathcal{O}}$$

This induces a filtration $F$ on a complex $C$ representing $L\pi_!(\overline{\mathcal{O}})$ with $F^p C$ representing $L\pi_!(\mathcal{K}^p)$ (construction of $C$ and $F$ omitted). Consider the spectral sequence of Homology, Section 24 associated to $(C, F)$. It has $E_1$-page

$$E_1^{p,q} = H_{-p-q}(\mathcal{C}_{B/A}, \mathcal{K}^p/\mathcal{K}^{p+1}) \quad \Rightarrow \quad H_{-p-q}(\mathcal{C}_{B/A}, \overline{\mathcal{O}}) = \mathrm{Tor}_{-p-q}^A(B, B)$$

and differentials $E_r^{p,q} \to E_r^{p+r,q-r+1}$. To show convergence we will show that for every $k$ there exists a $c$ such that $H_i(\mathcal{C}_{B/A}, \mathcal{K}^n) = 0$ for $i < k$ and $n > c^2$.

Given $k \geq 0$ set $c = k^2$. We claim that

$$H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) \to H_i(\mathcal{C}_{B/A}, \mathcal{K}^n)$$

is zero for $i < k$ and all $n \geq 0$. Note that $\mathcal{K}^n/\mathcal{K}^{n+c}$ has a finite filtration whose successive quotients $\mathcal{K}^m/\mathcal{K}^{m+1}$, $n \leq m < n + c$ have $H_i(\mathcal{C}_{B/A}, \mathcal{K}^m/\mathcal{K}^{m+1}) = 0$ for $i < n$ (see above). Hence the claim implies $H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) = 0$ for $i < k$ and all $n \geq k$ which is what we need to show.

Proof of the claim. Recall that for any $\mathcal{O}$-module $\mathcal{F}$ the map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}}^{\mathbf{L}} B$ induces an isomorphism on applying $L\pi_!$, see Lemma 5.6. Consider the map

$$\mathcal{K}^{n+k} \otimes_{\mathcal{O}}^{\mathbf{L}} B \longrightarrow \mathcal{K}^n \otimes_{\mathcal{O}}^{\mathbf{L}} B$$

We claim that this map induces the zero map on cohomology sheaves in degrees $0, -1, \ldots, -k+1$. If this second claim holds, then the $k$-fold composition

$$\mathcal{K}^{n+c} \otimes_{\mathcal{O}}^{\mathbf{L}} B \longrightarrow \mathcal{K}^n \otimes_{\mathcal{O}}^{\mathbf{L}} B$$

factors through $\tau_{\leq -k}\mathcal{K}^n \otimes_{\mathcal{O}}^{\mathbf{L}} B$ hence induces zero on $H_i(\mathcal{C}_{B/A}, -) = L_i\pi_!(-)$ for $i < k$, see Derived Categories, Lemma 12.5. By the remark above this means the same thing is true for $H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) \to H_i(\mathcal{C}_{B/A}, \mathcal{K}^n)$ which proves the (first) claim.

Proof of the second claim. The statement is local, hence we may work over an object $U = (P \to B)$ as above. We have to show the maps

$$\mathrm{Tor}_i^P(B, K^{n+k}) \to \mathrm{Tor}_i^P(B, K^n)$$

are zero for $i < k$. There is a spectral sequence

$$\mathrm{Tor}_a^P(P/IP, \mathrm{Tor}_b^{P/IP}(B, K^n)) \Rightarrow \mathrm{Tor}_{a+b}^P(B, K^n),$$

---

[2]A posteriori the "correct" vanishing $H_i(\mathcal{C}_{B/A}, \mathcal{K}^n) = 0$ for $i < n$ can be concluded.

see More on Algebra, Example 62.2. Thus it suffices to prove the maps

$$\operatorname{Tor}_i^{P/IP}(B, K^{n+1}) \to \operatorname{Tor}_i^{P/IP}(B, K^n)$$

are zero for all $i$. This is Lemma 12.3. $\qquad\square$

**Remark 12.5.** In the situation of Theorem 12.4 let $I = \operatorname{Ker}(A \to B)$. Then $H^{-1}(L_{B/A}) = H_1(\mathcal{C}_{B/A}, \Omega) = I/I^2$, see Lemma 11.2. Hence $H_k(\mathcal{C}_{B/A}, \operatorname{Sym}^k(\Omega)) = \wedge_B^k(I/I^2)$ by Remark 12.2. Thus the $E_1$-page looks like

$$
\begin{array}{llll}
B & & & \\
0 & & & \\
0 & I/I^2 & & \\
0 & H^{-2}(L_{B/A}) & & \\
0 & H^{-3}(L_{B/A}) & \wedge^2(I/I^2) & \\
0 & H^{-4}(L_{B/A}) & H_3(\mathcal{C}_{B/A}, \operatorname{Sym}^2(\Omega)) & \\
0 & H^{-5}(L_{B/A}) & H_4(\mathcal{C}_{B/A}, \operatorname{Sym}^2(\Omega)) & \wedge^3(I/I^2)
\end{array}
$$

with horizontal differential. Thus we obtain edge maps $\operatorname{Tor}_i^A(B, B) \to H^{-i}(L_{B/A})$, $i > 0$ and $\wedge_B^i(I/I^2) \to \operatorname{Tor}_i^A(B, B)$. Finally, we have $\operatorname{Tor}_1^A(B, B) = I/I^2$ and there is a five term exact sequence

$$\operatorname{Tor}_3^A(B, B) \to H^{-3}(L_{B/A}) \to \wedge_B^2(I/I^2) \to \operatorname{Tor}_2^A(B, B) \to H^{-2}(L_{B/A}) \to 0$$

of low degree terms.

**Remark 12.6.** Let $A \to B$ be a ring map. Let $P_\bullet$ be a resolution of $B$ over $A$ (Remark 5.5). Set $J_n = \operatorname{Ker}(P_n \to B)$. Note that

$$\operatorname{Tor}_2^{P_n}(B, B) = \operatorname{Tor}_1^{P_n}(J_n, B) = \operatorname{Ker}(J_n \otimes_{P_n} J_n \to J_n^2).$$

Hence $H_2(L_{B/A})$ is canonically equal to

$$\operatorname{Coker}(\operatorname{Tor}_2^{P_1}(B, B) \to \operatorname{Tor}_2^{P_0}(B, B))$$

by Remark 11.5. To make this more explicit we choose $P_2$, $P_1$, $P_0$ as in Example 5.9. We claim that

$$\operatorname{Tor}_2^{P_1}(B, B) = \wedge^2(\bigoplus_{t \in T} B) \oplus \bigoplus_{t \in T} J_0 \oplus \operatorname{Tor}_2^{P_0}(B, B)$$

Namely, the basis elements $x_t \wedge x_{t'}$ of the first summand corresponds to the element $x_t \otimes x_{t'} - x_{t'} \otimes x_t$ of $J_1 \otimes_{P_1} J_1$. For $f \in J_0$ the element $x_t \otimes f$ of the second summand corresponds to the element $x_t \otimes s_0(f) - s_0(f) \otimes x_t$ of $J_1 \otimes_{P_1} J_1$. Finally, the map $\operatorname{Tor}_2^{P_0}(B, B) \to \operatorname{Tor}_2^{P_1}(B, B)$ is given by $s_0$. The map $d_0 - d_1 : \operatorname{Tor}_2^{P_1}(B, B) \to \operatorname{Tor}_2^{P_0}(B, B)$ is zero on the last summand, maps $x_t \otimes f$ to $f \otimes f_t - f_t \otimes f$, and maps $x_t \wedge x_{t'}$ to $f_t \otimes f_{t'} - f_{t'} \otimes f_t$. All in all we conclude that there is an exact sequence

$$\wedge_B^2(J_0/J_0^2) \to \operatorname{Tor}_2^{P_0}(B, B) \to H^{-2}(L_{B/A}) \to 0$$

In this way we obtain a direct proof of a consequence of Quillen's spectral sequence discussed in Remark 12.5.

## 13. Comparison with Lichtenbaum-Schlessinger

Let $A \to B$ be a ring map. In [LS67] there is a fairly explicit determination of $\tau_{\geq -2} L_{B/A}$ which is often used in calculations of versal deformation spaces of singularities. The construction follows. Choose a polynomial algebra $P$ over $A$ and a surjection $P \to B$ with kernel $I$. Choose generators $f_t$, $t \in T$ for $I$ which induces a surjection $F = \bigoplus_{t \in T} P \to I$ with $F$ a free $P$-module. Let $Rel \subset F$ be the kernel of $F \to I$, in other words $Rel$ is the set of relations among the $f_t$. Let $TrivRel \subset Rel$ be the submodule of trivial relations, i.e., the submodule of $Rel$ generated by the elements $(\ldots, f_{t'}, 0, \ldots, 0, -f_t, 0, \ldots)$. Consider the complex of $B$-modules

(13.0.1) $$Rel/TrivRel \longrightarrow F \otimes_P B \longrightarrow \Omega_{P/A} \otimes_P B$$

where the last term is placed in degree 0. The first map is the obvious one and the second map sends the basis element corresponding to $t \in T$ to $\mathrm{d}f_t \otimes 1$.

**Definition 13.1.** Let $A \to B$ be a ring map. Let $M$ be a $(B, B)$-bimodule over $A$. An *A-biderivation* is an $A$-linear map $\lambda : B \to M$ such that $\lambda(xy) = x\lambda(y) + \lambda(x)y$.

For a polynomial algebra the biderivations are easy to describe.

**Lemma 13.2.** *Let $P = A[S]$ be a polynomial ring over $A$. Let $M$ be a $(P, P)$-bimodule over $A$. Given $m_s \in M$ for $s \in S$, there exists a unique $A$-biderivation $\lambda : P \to M$ mapping $s$ to $m_s$ for $s \in S$.*

**Proof.** We set

$$\lambda(s_1 \ldots s_t) = \sum s_1 \ldots s_{i-1} m_{s_i} s_{i+1} \ldots s_t$$

in $M$. Extending by $A$-linearity we obtain a biderivation.                                   $\square$

Here is the comparison statement. The reader may also read about this in [And74, page 206, Proposition 12] or in the paper [DRGV92] which extends the complex (13.0.1) by one term and the comparison to $\tau_{\geq -3}$.

**Lemma 13.3.** *In the situation above denote $L$ the complex (13.0.1). There is a canonical map $L_{B/A} \to L$ in $D(B)$ which induces an isomorphism $\tau_{\geq -2} L_{B/A} \to L$ in $D(B)$.*

**Proof.** Let $P_\bullet \to B$ be a resolution of $B$ over $A$ (Remark 5.5). We will identify $L_{B/A}$ with $\Omega_{P_\bullet/A} \otimes B$. To construct the map we make some choices.

Choose an $A$-algebra map $\psi : P_0 \to P$ compatible with the given maps $P_0 \to B$ and $P \to B$.

Write $P_1 = A[S]$ for some set $S$. For $s \in S$ we may write

$$\psi(d_0(s) - d_1(s)) = \sum p_{s,t} f_t$$

for some $p_{s,t} \in P$. Think of $F = \bigoplus_{t \in T} P$ as a $(P_1, P_1)$-bimodule via the maps $(\psi \circ d_0, \psi \circ d_1)$. By Lemma 13.2 we obtain a unique $A$-biderivation $\lambda : P_1 \to F$ mapping $s$ to the vector with coordinates $p_{s,t}$. By construction the composition

$$P_1 \longrightarrow F \longrightarrow P$$

sends $f \in P_1$ to $\psi(d_0(f) - d_1(f))$ because the map $f \mapsto \psi(d_0(f) - d_1(f))$ is an $A$-biderivation agreeing with the composition on generators.

For $g \in P_2$ we claim that $\lambda(d_0(g) - d_1(g) + d_2(g))$ is an element of $Rel$. Namely, by the last remark of the previous paragraph the image of $\lambda(d_0(g) - d_1(g) + d_2(g))$ in $P$ is

$$\psi((d_0 - d_1)(d_0(g) - d_1(g) + d_2(g)))$$

which is zero by Simplicial, Section 23).

The choice of $\psi$ determines a map

$$\mathrm{d}\psi \otimes 1 : \Omega_{P_0/A} \otimes B \longrightarrow \Omega_{P/A} \otimes B$$

Composing $\lambda$ with the map $F \to F \otimes B$ gives a usual $A$-derivation as the two $P_1$-module structures on $F \otimes B$ agree. Thus $\lambda$ determines a map

$$\overline{\lambda} : \Omega_{P_1/A} \otimes B \longrightarrow F \otimes B$$

Finally, We obtain a $B$-linear map

$$q : \Omega_{P_2/A} \otimes B \longrightarrow Rel/TrivRel$$

by mapping $\mathrm{d}g$ to the class of $\lambda(d_0(g) - d_1(g) + d_2(g))$ in the quotient.

The diagram

$$\begin{array}{ccccccc}
\Omega_{P_3/A} \otimes B & \longrightarrow & \Omega_{P_2/A} \otimes B & \longrightarrow & \Omega_{P_1/A} \otimes B & \longrightarrow & \Omega_{P_0/A} \otimes B \\
\downarrow & & \downarrow q & & \downarrow \overline{\lambda} & & \downarrow \mathrm{d}\psi \otimes 1 \\
0 & \longrightarrow & Rel/TrivRel & \longrightarrow & F \otimes B & \longrightarrow & \Omega_{P/A} \otimes B
\end{array}$$

commutes (calculation omitted) and we obtain the map of the lemma. By Remark 11.4 and Lemma 11.3 we see that this map induces isomorphisms $H_1(L_{B/A}) \to H_1(L)$ and $H_0(L_{B/A}) \to H_0(L)$.

It remains to see that our map $L_{B/A} \to L$ induces an isomorphism $H_2(L_{B/A}) \to H_2(L)$. Choose a resolution of $B$ over $A$ with $P_0 = P = A[u_i]$ and then $P_1$ and $P_2$ as in Example 5.9. In Remark 12.6 we have constructed an exact sequence

$$\wedge_B^2(J_0/J_0^2) \to \mathrm{Tor}_2^{P_0}(B, B) \to H^{-2}(L_{B/A}) \to 0$$

where $P_0 = P$ and $J_0 = \mathrm{Ker}(P \to B) = I$. Calculating the Tor group using the short exact sequences $0 \to I \to P \to B \to 0$ and $0 \to Rel \to F \to I \to 0$ we find that $\mathrm{Tor}_2^P(B, B) = \mathrm{Ker}(Rel \otimes B \to F \otimes B)$. The image of the map $\wedge_B^2(I/I^2) \to \mathrm{Tor}_2^P(B, B)$ under this identification is exactly the image of $TrivRel \otimes B$. Thus we see that $H_2(L_{B/A}) \cong H_2(L)$.

Finally, we have to check that our map $L_{B/A} \to L$ actually induces this isomorphism. We will use the notation and results discussed in Example 5.9 and Remarks 12.6 and 11.5 without further mention. Pick an element $\xi$ of $\mathrm{Tor}_2^{P_0}(B, B) = \mathrm{Ker}(I \otimes_P I \to I^2)$. Write $\xi = \sum h_{t',t} f_{t'} \otimes f_t$ for some $h_{t',t} \in P$. Tracing through the exact sequences above we find that $\xi$ corresponds to the image in $Rel \otimes B$ of the element $r \in Rel \subset F = \bigoplus_{t \in T} P$ with $t$th coordinate $r_t = \sum_{t' \in T} h_{t',t} f_{t'}$. On the other hand, $\xi$ corresponds to the element of $H_2(L_{B/A}) = H_2(\Omega)$ which is the image via $\mathrm{d} : H_2(\mathcal{J}/\mathcal{J}^2) \to H_2(\Omega)$ of the boundary of $\xi$ under the 2-extension

$$0 \to \mathrm{Tor}_2^{\mathcal{O}}(\underline{B}, \underline{B}) \to \mathcal{J} \otimes_{\mathcal{O}} \mathcal{J} \to \mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to 0$$

We compute the successive transgressions of our element. First we have

$$\xi = (d_0 - d_1)(-\sum s_0(h_{t',t} f_{t'}) \otimes x_t)$$

and next we have

$$\sum s_0(h_{t',t}f_{t'})x_t = d_0(v_r) - d_1(v_r) + d_2(v_r)$$

by our choice of the variables $v$ in Example 5.9. We may choose our map $\lambda$ above such that $\lambda(u_i) = 0$ and $\lambda(x_t) = -e_t$ where $e_t \in F$ denotes the basis vector corresponding to $t \in T$. Hence the construction of our map $q$ above sends $dv_r$ to

$$\lambda(\sum s_0(h_{t',t}f_{t'})x_t) = \sum_t \left(\sum_{t'} h_{t',t}f_{t'}\right) e_t$$

matching the image of $\xi$ in $Rel \otimes B$ (the two minus signs we found above cancel out). This agreement finishes the proof.  $\square$

**Remark 13.4** (Functoriality of the Lichtenbaum-Schlessinger complex)**.** Consider a commutative square

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

of ring maps. Choose a factorization

$$\begin{array}{ccccc} A' & \longrightarrow & P' & \longrightarrow & B' \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & P & \longrightarrow & B \end{array}$$

with $P$ a polynomial algebra over $A$ and $P'$ a polynomial algebra over $A'$. Choose generators $f_t$, $t \in T$ for $\mathrm{Ker}(P \to B)$. For $t \in T$ denote $f_t'$ the image of $f_t$ in $P'$. Choose $f_s' \in P'$ such that the elements $f_t'$ for $t \in T' = T \amalg S$ generate the kernel of $P' \to B'$. Set $F = \bigoplus_{t \in T} P$ and $F' = \bigoplus_{t' \in T'} P'$. Let $Rel = \mathrm{Ker}(F \to P)$ and $Rel' = \mathrm{Ker}(F' \to P')$ where the maps are given by multiplication by $f_t$, resp. $f_t'$ on the coordinates. Finally, set $TrivRel$, resp. $TrivRel'$ equal to the submodule of $Rel$, resp. $TrivRel$ generated by the elements $(\ldots, f_{t'}, 0, \ldots, 0, -f_t, 0, \ldots)$ for $t, t' \in T$, resp. $T'$. Having made these choices we obtain a canonical commutative diagram

$$\begin{array}{ccccccc} L' : & & Rel'/TrivRel' & \longrightarrow & F' \otimes_{P'} B' & \longrightarrow & \Omega_{P'/A'} \otimes_{P'} B' \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ L : & & Rel/TrivRel & \longrightarrow & F \otimes_P B & \longrightarrow & \Omega_{P/A} \otimes_P B \end{array}$$

Moreover, tracing through the choices made in the proof of Lemma 13.3 the reader sees that one obtains a commutative diagram

$$\begin{array}{ccc} L_{B'/A'} & \longrightarrow & L' \\ \uparrow & & \uparrow \\ L_{B/A} & \longrightarrow & L \end{array}$$

## 14. The cotangent complex of a local complete intersection

If $A \to B$ is a local complete intersection map, then $L_{B/A}$ is a perfect complex. The key to proving this is the following lemma.

**Lemma 14.1.** *Let $A = \mathbf{Z}[x_1, \ldots, x_n] \to B = \mathbf{Z}$ be the ring map which sends $x_i$ to $0$ for $i = 1, \ldots, n$. Let $I = (x_1, \ldots, x_n) \subset A$. Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.*

**Proof.** There are several ways to prove this. For example one can explicitly construct a resolution of $B$ over $A$ and compute. We will use (7.0.1). Namely, consider the distinguished triangle

$$L_{\mathbf{Z}[x_1, \ldots, x_n]/\mathbf{Z}} \otimes_{\mathbf{Z}[x_1, \ldots, x_n]} \mathbf{Z} \to L_{\mathbf{Z}/\mathbf{Z}} \to L_{\mathbf{Z}/\mathbf{Z}[x_1, \ldots, x_n]} \to L_{\mathbf{Z}[x_1, \ldots, x_n]/\mathbf{Z}} \otimes_{\mathbf{Z}[x_1, \ldots, x_n]} \mathbf{Z}[1]$$

The complex $L_{\mathbf{Z}[x_1, \ldots, x_n]/\mathbf{Z}}$ is quasi-isomorphic to $\Omega_{\mathbf{Z}[x_1, \ldots, x_n]/\mathbf{Z}}$ by Lemma 4.7. The complex $L_{\mathbf{Z}/\mathbf{Z}}$ is zero in $D(\mathbf{Z})$ by Lemma 8.4. Thus we see that $L_{B/A}$ has only one nonzero cohomology group which is as described in the lemma by Lemma 11.2. □

**Lemma 14.2.** *Let $A \to B$ be a surjective ring map whose kernel $I$ is generated by a Koszul-regular sequence (for example a regular sequence). Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.*

**Proof.** Let $f_1, \ldots, f_r \in I$ be a Koszul regular sequence generating $I$. Consider the ring map $\mathbf{Z}[x_1, \ldots, x_r] \to A$ sending $x_i$ to $f_i$. Since $x_1, \ldots, x_r$ is a regular sequence in $\mathbf{Z}[x_1, \ldots, x_r]$ we see that the Koszul complex on $x_1, \ldots, x_r$ is a free resolution of $\mathbf{Z} = \mathbf{Z}[x_1, \ldots, x_r]/(x_1, \ldots, x_r)$ over $\mathbf{Z}[x_1, \ldots, x_r]$ (see More on Algebra, Lemma 30.2). Thus the assumption that $f_1, \ldots, f_r$ is Koszul regular exactly means that $B = A \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1, \ldots, x_r]} \mathbf{Z}$. Hence $L_{B/A} = L_{\mathbf{Z}/\mathbf{Z}[x_1, \ldots, x_r]} \otimes^{\mathbf{L}}_{\mathbf{Z}} B$ by Lemmas 6.2 and 14.1. □

**Lemma 14.3.** *Let $A \to B$ be a surjective ring map whose kernel $I$ is Koszul. Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.*

**Proof.** Locally on $\operatorname{Spec}(A)$ the ideal $I$ is generated by a Koszul regular sequence, see More on Algebra, Definition 32.1. Hence this follows from Lemma 6.2. □

**Proposition 14.4.** *Let $A \to B$ be a local complete intersection map. Then $L_{B/A}$ is a perfect complex with tor amplitude in $[-1, 0]$.*
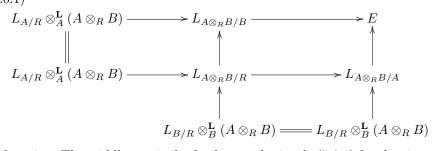
**Proof.** Choose a surjection $P = A[x_1, \ldots, x_n] \to B$ with kernel $J$. By Lemma 11.3 we see that $J/J^2 \to \bigoplus B dx_i$ is quasi-isomorphic to $\tau_{\geq -1} L_{B/A}$. Note that $J/J^2$ is finite projective (More on Algebra, Lemma 32.3), hence $\tau_{\geq -1} L_{B/A}$ is a perfect complex with tor amplitude in $[-1, 0]$. Thus it suffices to show that $H^i(L_{B/A}) = 0$ for $i \notin [-1, 0]$. This follows from (7.0.1)

$$L_{P/A} \otimes^{\mathbf{L}}_P B \to L_{B/A} \to L_{B/P} \to L_{P/A} \otimes^{\mathbf{L}}_P B[1]$$

and Lemma 14.3 to see that $H^i(L_{B/P})$ is zero unless $i \in \{-1, 0\}$. (We also use Lemma 4.7 for the term on the left.) □

## 15. Tensor products and the cotangent complex

Let $R$ be a ring and let $A$, $B$ be $R$-algebras. In this section we discuss $L_{A \otimes_R B / R}$. Most of the information we want is contained in the following diagram
(15.0.1)

$$\begin{array}{ccccc}
L_{A/R} \otimes_A^{\mathbf{L}} (A \otimes_R B) & \longrightarrow & L_{A \otimes_R B / B} & \longrightarrow & E \\
\| & & \uparrow & & \uparrow \\
L_{A/R} \otimes_A^{\mathbf{L}} (A \otimes_R B) & \longrightarrow & L_{A \otimes_R B / R} & \longrightarrow & L_{A \otimes_R B / A} \\
& & \uparrow & & \uparrow \\
& & L_{B/R} \otimes_B^{\mathbf{L}} (A \otimes_R B) & = & L_{B/R} \otimes_B^{\mathbf{L}} (A \otimes_R B)
\end{array}$$

Explanation: The middle row is the fundamental triangle (7.0.1) for the ring maps $R \to A \to A \otimes_R B$. The middle column is the fundamental triangle (7.0.1) for the ring maps $R \to B \to A \otimes_R B$. Next, $E$ is an object of $D(A \otimes_R B)$ which "fits" into the upper right corner, i.e., which turns both the top row and the right column into distinguished triangles. Such an $E$ exists by Derived Categories, Proposition 4.23 applied to the lower left square (with 0 placed in the missing spot). To be more explicit, we could for example define $E$ as the cone (Derived Categories, Definition 9.1) of the map of complexes

$$L_{A/R} \otimes_A^{\mathbf{L}} (A \otimes_R B) \oplus L_{B/R} \otimes_B^{\mathbf{L}} (A \otimes_R B) \longrightarrow L_{A \otimes_R B / R}$$

and get the two maps with target $E$ by an application of TR3. In the Tor independent case the object $E$ is zero.

**Lemma 15.1.** *If $A$ and $B$ are Tor independent $R$-algebras, then the object $E$ in (15.0.1) is zero. In this case we have*

$$L_{A \otimes_R B / R} = L_{A/R} \otimes_A^{\mathbf{L}} (A \otimes_R B) \oplus L_{B/R} \otimes_B^{\mathbf{L}} (A \otimes_R B)$$

*which is represented by the complex $L_{A/R} \otimes_R B \oplus L_{B/R} \otimes_R A$ of $A \otimes_R B$-modules.*

**Proof.** The first two statements are immediate from Lemma 6.2. The last statement follows as $L_{A/R}$ is a complex of free $A$-modules, hence $L_{A/R} \otimes_A^{\mathbf{L}} (A \otimes_R B)$ is represented by $L_{A/R} \otimes_A (A \otimes_R B) = L_{A/R} \otimes_R B$     $\square$

In general we can say this about the object $E$.

**Lemma 15.2.** *Let $R$ be a ring and let $A$, $B$ be $R$-algebras. The object $E$ in (15.0.1) satisfies*

$$H^i(E) = \begin{cases} 0 & \text{if} \quad i \geq -1 \\ Tor_1^R(A, B) & \text{if} \quad i = -2 \end{cases}$$

**Proof.** We use the description of $E$ as the cone on $L_{B/R} \otimes_B^{\mathbf{L}} (A \otimes_R B) \to L_{A \otimes_R B / A}$. By Lemma 13.3 the canonical truncations $\tau_{\geq -2} L_{B/R}$ and $\tau_{\geq -2} L_{A \otimes_R B / A}$ are computed by the Lichtenbaum-Schlessinger complex (13.0.1). These isomorphisms are compatible with functoriality (Remark 13.4). Thus in this proof we work with the Lichtenbaum-Schlessinger complexes.

Choose a polynomial algebra $P$ over $R$ and a surjection $P \to B$. Choose generators $f_t \in P$, $t \in T$ of the kernel of this surjection. Let $Rel \subset F = \bigoplus_{t \in T} P$ be the kernel of the map $F \to P$ which maps the basis vector corresponding to $t$ to $f_t$. Set

$P_A = A \otimes_R P$ and $F_A = A \otimes_R F = P_A \otimes_P F$. Let $Rel_A$ be the kernel of the map $F_A \to P_A$. Using the exact sequence

$$0 \to Rel \to F \to P \to B \to 0$$

and standard short exact sequences for Tor we obtain an exact sequence

$$A \otimes_R Rel \to Rel_A \to \mathrm{Tor}_1^R(A, B) \to 0$$

Note that $P_A \to A \otimes_R B$ is a surjection whose kernel is generated by the elements $1 \otimes f_t$ in $P_A$. Denote $TrivRel_A \subset Rel_A$ the $P_A$-submodule generated by the elements $(\dots, 1 \otimes f_{t'}, 0, \dots, 0, -1 \otimes f_t \otimes 1, 0, \dots)$. Since $TrivRel \otimes_R A \to TrivRel_A$ is surjective, we find a canonical exact sequence

$$A \otimes_R (Rel/TrivRel) \to Rel_A/TrivRel_A \to \mathrm{Tor}_1^R(A, B) \to 0$$

The map of Lichtenbaum-Schlessinger complexes is given by the diagram

$$
\begin{array}{ccccc}
Rel_A/TrivRel_A & \longrightarrow & F_A \otimes_{P_A} (A \otimes_R B) & \longrightarrow & \Omega_{P_A/A \otimes_R B} \otimes_{P_A} (A \otimes_R B) \\
\Big\uparrow{\scriptstyle -2} & & \Big\uparrow{\scriptstyle -1} & & \Big\uparrow{\scriptstyle 0} \\
Rel/TrivRel & \longrightarrow & F \otimes_P B & \longrightarrow & \Omega_{P/A} \otimes_P B
\end{array}
$$

Note that vertical maps $-1$ and $-0$ induce an isomorphism after applying the functor $A \otimes_R - = P_A \otimes_P -$ to the source and the vertical map $-2$ gives exactly the map whose cokernel is the desired Tor module as we saw above. $\qquad \square$

## 16. Deformations of ring maps and the cotangent complex

This section is the continuation of Deformation Theory, Section 2 which we urge the reader to read first. We start with a surjective ring map $A' \to A$ whose kernel is an ideal $I$ of square zero. Moreover we assume given a ring map $A \to B$, a $B$-module $N$, and an $A$-module map $c : I \to N$. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

(16.0.1)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & ? & \longrightarrow & B & \longrightarrow & 0 \\
& & {\scriptstyle c}\Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \\
0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

and moreover how unique the solution is (if it exists). More precisely, we look for a surjection of $A'$-algebras $B' \to B$ whose kernel is an ideal of square zero and is identified with $N$ such that $A' \to B'$ induces the given map $c$. We will say $B'$ is a *solution* to (16.0.1).

**Lemma 16.1.** *In the situation above we have*

(1) *There is a canonical element $\xi \in \mathrm{Ext}_B^2(L_{B/A}, N)$ whose vanishing is a sufficient and necessary condition for the existence of a solution to (16.0.1).*

(2) *If there exists a solution, then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}_B^1(L_{B/A}, N)$.*

(3) *Given a solution $B'$, the set of automorphisms of $B'$ fitting into (16.0.1) is canonically isomorphic to $\mathrm{Ext}_B^0(L_{B/A}, N)$.*

**Proof.** Via the identifications $NL_{B/A} = \tau_{\geq -1} L_{B/A}$ (Lemma 11.3) and $H^0(L_{B/A}) = \Omega_{B/A}$ (Lemma 4.5) we have seen parts (2) and (3) in Deformation Theory, Lemmas 2.1 and 2.2.

Proof of (1). Roughly speaking, this follows from the discussion in Deformation Theory, Remark 2.8 by replacing the naive cotangent complex by the full cotangent complex. Here is a more detailed explanation. By Deformation Theory, Lemma 2.7 and Remark 2.8 there exists an element

$$\xi' \in \mathrm{Ext}^1_A(NL_{A/A'}, N) = \mathrm{Ext}^1_B(NL_{A/A'} \otimes^{\mathbf{L}}_A B, N) = \mathrm{Ext}^1_B(L_{A/A'} \otimes^{\mathbf{L}}_A B, N)$$

(for the equalities see Deformation Theory, Remark 2.8 and use that $NL_{A'/A} = \tau_{\geq -1} L_{A'/A}$) such that a solution exists if and only if this element is in the image of the map

$$\mathrm{Ext}^1_B(NL_{B/A'}, N) = \mathrm{Ext}^1_B(L_{B/A'}, N) \longrightarrow \mathrm{Ext}^1_B(L_{A/A'} \otimes^{\mathbf{L}}_A B, N)$$

The distinguished triangle (7.0.1) for $A' \to A \to B$ gives rise to a long exact sequence

$$\ldots \to \mathrm{Ext}^1_B(L_{B/A'}, N) \to \mathrm{Ext}^1_B(L_{A/A'} \otimes^{\mathbf{L}}_A B, N) \to \mathrm{Ext}^2_B(L_{B/A}, N) \to \ldots$$

Hence taking $\xi$ the image of $\xi'$ works. $\qquad\square$

## 17. The Atiyah class of a module

Let $A \to B$ be a ring map. Let $M$ be a $B$-module. Let $P \to B$ be an object of $\mathcal{C}_{B/A}$ (Section 4). Consider the extension of principal parts

$$0 \to \Omega_{P/A} \otimes_P M \to P^1_{P/A}(M) \to M \to 0$$

see Algebra, Lemma 133.6. This sequence is functorial in $P$ by Algebra, Remark 133.7. Thus we obtain a short exact sequence of sheaves of $\mathcal{O}$-modules

$$0 \to \Omega_{\mathcal{O}/\underline{A}} \otimes_{\mathcal{O}} \underline{M} \to P^1_{\mathcal{O}/\underline{A}}(M) \to \underline{M} \to 0$$

on $\mathcal{C}_{B/A}$. We have $L\pi_!(\Omega_{\mathcal{O}/\underline{A}} \otimes_{\mathcal{O}} \underline{M}) = L_{B/A} \otimes_B M = L_{B/A} \otimes^{\mathbf{L}}_B M$ by Lemma 4.2 and the flatness of the terms of $L_{B/A}$. We have $L\pi_!(\underline{M}) = M$ by Lemma 4.4. Thus a distinguished triangle

$$(17.0.1) \qquad L_{B/A} \otimes^{\mathbf{L}}_B M \to L\pi_! \left( P^1_{\mathcal{O}/\underline{A}}(M) \right) \to M \to L_{B/A} \otimes^{\mathbf{L}}_B M[1]$$

in $D(B)$. Here we use Cohomology on Sites, Remark 39.13 to get a distinguished triangle in $D(B)$ and not just in $D(A)$.

**Definition 17.1.** Let $A \to B$ be a ring map. Let $M$ be a $B$-module. The map $M \to L_{B/A} \otimes^{\mathbf{L}}_B M[1]$ in (17.0.1) is called the *Atiyah class* of $M$.

## 18. The cotangent complex

In this section we discuss the cotangent complex of a map of sheaves of rings on a site. In later sections we specialize this to obtain the cotangent complex of a morphism of ringed topoi, a morphism of ringed spaces, a morphism of schemes, a morphism of algebraic space, etc.

Let $\mathcal{C}$ be a site and let $Sh(\mathcal{C})$ denote the associated topos. Let $\mathcal{A}$ denote a sheaf of rings on $\mathcal{C}$. Let $\mathcal{A}$-$Alg$ be the category of $\mathcal{A}$-algebras. Consider the pair of adjoint functors $(U, V)$ where $V : \mathcal{A}$-$Alg \to Sh(\mathcal{C})$ is the forgetful functor and $U : Sh(\mathcal{C}) \to \mathcal{A}$-$Alg$ assigns to a sheaf of sets $\mathcal{E}$ the polynomial algebra $\mathcal{A}[\mathcal{E}]$ on

$\mathcal{E}$ over $\mathcal{A}$. Let $X_\bullet$ be the simplicial object of $\text{Fun}(\mathcal{A}\text{-}Alg, \mathcal{A}\text{-}Alg)$ constructed in Simplicial, Section 34.

Now assume that $\mathcal{A} \to \mathcal{B}$ is a homomorphism of sheaves of rings. Then $\mathcal{B}$ is an object of the category $\mathcal{A}\text{-}Alg$. Denote $\mathcal{P}_\bullet = X_\bullet(\mathcal{B})$ the resulting simplicial $\mathcal{A}$-algebra. Recall that $\mathcal{P}_0 = \mathcal{A}[\mathcal{B}]$, $\mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]]$, and so on. Recall also that there is an augmentation

$$\epsilon : \mathcal{P}_\bullet \longrightarrow \mathcal{B}$$

where we view $\mathcal{B}$ as a constant simplicial $\mathcal{A}$-algebra.

**Definition 18.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The *standard resolution of $\mathcal{B}$ over $\mathcal{A}$* is the augmentation $\epsilon : \mathcal{P}_\bullet \to \mathcal{B}$ with terms

$$\mathcal{P}_0 = \mathcal{A}[\mathcal{B}], \quad \mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]], \quad \dots$$

and maps as constructed above.

With this definition in hand the cotangent complex of a map of sheaves of rings is defined as follows. We will use the module of differentials as defined in Modules on Sites, Section 33.

**Definition 18.2.** Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The *cotangent complex $L_{\mathcal{B}/\mathcal{A}}$* is the complex of $\mathcal{B}$-modules associated to the simplicial module

$$\Omega_{\mathcal{P}_\bullet/\mathcal{A}} \otimes_{\mathcal{P}_\bullet, \epsilon} \mathcal{B}$$

where $\epsilon : \mathcal{P}_\bullet \to \mathcal{B}$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$. We usually think of $L_{\mathcal{B}/\mathcal{A}}$ as an object of $D(\mathcal{B})$.

These constructions satisfy a functoriality similar to that discussed in Section 6. Namely, given a commutative diagram

$$(18.2.1) \qquad \begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array}$$

of sheaves of rings on $\mathcal{C}$ there is a canonical $\mathcal{B}$-linear map of complexes

$$L_{\mathcal{B}/\mathcal{A}} \longrightarrow L_{\mathcal{B}'/\mathcal{A}'}$$

constructed as follows. If $\mathcal{P}_\bullet \to \mathcal{B}$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$ and $\mathcal{P}'_\bullet \to \mathcal{B}'$ is the standard resolution of $\mathcal{B}'$ over $\mathcal{A}'$, then there is a canonical map $\mathcal{P}_\bullet \to \mathcal{P}'_\bullet$ of simplicial $\mathcal{A}$-algebras compatible with the augmentations $\mathcal{P}_\bullet \to \mathcal{B}$ and $\mathcal{P}'_\bullet \to \mathcal{B}'$. The maps

$$\mathcal{P}_0 = \mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{A}'[\mathcal{B}'] = \mathcal{P}'_0, \quad \mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]] \longrightarrow \mathcal{A}'[\mathcal{A}'[\mathcal{B}']] = \mathcal{P}'_1$$

and so on are given by the given maps $\mathcal{A} \to \mathcal{A}'$ and $\mathcal{B} \to \mathcal{B}'$. The desired map $L_{\mathcal{B}/\mathcal{A}} \to L_{\mathcal{B}'/\mathcal{A}'}$ then comes from the associated maps on sheaves of differentials.

**Lemma 18.3.** *Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $f^{-1}L_{\mathcal{B}/\mathcal{A}} = L_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.*

**Proof.** The diagram

$$
\begin{array}{ccc}
\mathcal{A}\text{-}Alg & \xrightarrow{\phantom{xxxx}} & Sh(\mathcal{C}) \\
{\scriptstyle f^{-1}}\downarrow & & \downarrow{\scriptstyle f^{-1}} \\
f^{-1}\mathcal{A}\text{-}Alg & \xrightarrow{\phantom{xxxx}} & Sh(\mathcal{D})
\end{array}
$$

commutes. $\square$

**Lemma 18.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $H^i(L_{\mathcal{B}/\mathcal{A}})$ is the sheaf associated to the presheaf $U \mapsto H^i(L_{\mathcal{B}(U)/\mathcal{A}(U)})$.*

**Proof.** Let $\mathcal{C}'$ be the site we get by endowing $\mathcal{C}$ with the chaotic topology (presheaves are sheaves). There is a morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ where $f_*$ is the inclusion of sheaves into presheaves and $f^{-1}$ is sheafification. By Lemma 18.3 it suffices to prove the result for $\mathcal{C}'$, i.e., in case $\mathcal{C}$ has the chaotic topology.

If $\mathcal{C}$ carries the chaotic topology, then $L_{\mathcal{B}/\mathcal{A}}(U)$ is equal to $L_{\mathcal{B}(U)/\mathcal{A}(U)}$ because

$$
\begin{array}{ccc}
\mathcal{A}\text{-}Alg & \xrightarrow{\phantom{xxxx}} & Sh(\mathcal{C}) \\
{\scriptstyle \text{sections over } U}\downarrow & & \downarrow{\scriptstyle \text{sections over } U} \\
\mathcal{A}(U)\text{-}Alg & \xrightarrow{\phantom{xxxx}} & Sets
\end{array}
$$

commutes. $\square$

**Remark 18.5.** It is clear from the proof of Lemma 18.4 that for any $U \in \mathrm{Ob}(\mathcal{C})$ there is a canonical map $L_{\mathcal{B}(U)/\mathcal{A}(U)} \to L_{\mathcal{B}/\mathcal{A}}(U)$ of complexes of $\mathcal{B}(U)$-modules. Moreover, these maps are compatible with restriction maps and the complex $L_{\mathcal{B}/\mathcal{A}}$ is the sheafification of the rule $U \mapsto L_{\mathcal{B}(U)/\mathcal{A}(U)}$.

**Lemma 18.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $H^0(L_{\mathcal{B}/\mathcal{A}}) = \Omega_{\mathcal{B}/\mathcal{A}}$.*

**Proof.** Follows from Lemmas 18.4 and 4.5 and Modules on Sites, Lemma 33.4. $\square$

**Lemma 18.7.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \to \mathcal{B}'$ be homomorphisms of sheaves of rings on $\mathcal{C}$. Then*

$$
L_{\mathcal{B}\times\mathcal{B}'/\mathcal{A}} \longrightarrow L_{\mathcal{B}/\mathcal{A}} \oplus L_{\mathcal{B}'/\mathcal{A}}
$$

*is an isomorphism in $D(\mathcal{B} \times \mathcal{B}')$.*

**Proof.** By Lemma 18.4 it suffices to prove this for ring maps. In the case of rings this is Lemma 6.4. $\square$

The fundamental triangle for the cotangent complex of sheaves of rings is an easy consequence of the result for homomorphisms of rings.

**Lemma 18.8.** *Let $\mathcal{D}$ be a site. Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be homomorphisms of sheaves of rings on $\mathcal{D}$. There is a canonical distinguished triangle*

$$
L_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}}^{\mathbf{L}} \mathcal{C} \to L_{\mathcal{C}/\mathcal{A}} \to L_{\mathcal{C}/\mathcal{B}} \to L_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}}^{\mathbf{L}} \mathcal{C}[1]
$$

*in $D(\mathcal{C})$.*

**Proof.** We will use the method described in Remarks 7.5 and 7.6 to construct the triangle; we will freely use the results mentioned there. As in those remarks we first construct the triangle in case $\mathcal{B} \to \mathcal{C}$ is an injective map of sheaves of rings. In this case we set

(1) $\mathcal{P}_\bullet$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$,
(2) $\mathcal{Q}_\bullet$ is the standard resolution of $\mathcal{C}$ over $\mathcal{A}$,
(3) $\mathcal{R}_\bullet$ is the standard resolution of $\mathcal{C}$ over $\mathcal{B}$,
(4) $\mathcal{S}_\bullet$ is the standard resolution of $\mathcal{B}$ over $\mathcal{B}$,
(5) $\overline{\mathcal{Q}}_\bullet = \mathcal{Q}_\bullet \otimes_{\mathcal{P}_\bullet} \mathcal{B}$, and
(6) $\overline{\mathcal{R}}_\bullet = \mathcal{R}_\bullet \otimes_{\mathcal{S}_\bullet} \mathcal{B}$.

The distinguished triangle is the distinguished triangle associated to the short exact sequence of simplicial $\mathcal{C}$-modules

$$0 \to \Omega_{\mathcal{P}_\bullet/\mathcal{A}} \otimes_{\mathcal{P}_\bullet} \mathcal{C} \to \Omega_{\mathcal{Q}_\bullet/\mathcal{A}} \otimes_{\mathcal{Q}_\bullet} \mathcal{C} \to \Omega_{\overline{\mathcal{Q}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{Q}}_\bullet} \mathcal{C} \to 0$$

The first two terms are equal to the first two terms of the triangle of the statement of the lemma. The identification of the last term with $L_{\mathcal{C}/\mathcal{B}}$ uses the quasi-isomorphisms of complexes

$$L_{\mathcal{C}/\mathcal{B}} = \Omega_{\mathcal{R}_\bullet/\mathcal{B}} \otimes_{\mathcal{R}_\bullet} \mathcal{C} \longrightarrow \Omega_{\overline{\mathcal{R}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{R}}_\bullet} \mathcal{C} \longleftarrow \Omega_{\overline{\mathcal{Q}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{Q}}_\bullet} \mathcal{C}$$

All the constructions used above can first be done on the level of presheaves and then sheafified. Hence to prove sequences are exact, or that map are quasi-isomorphisms it suffices to prove the corresponding statement for the ring maps $\mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$ which are known. This finishes the proof in the case that $\mathcal{B} \to \mathcal{C}$ is injective.

In general, we reduce to the case where $\mathcal{B} \to \mathcal{C}$ is injective by replacing $\mathcal{C}$ by $\mathcal{B} \times \mathcal{C}$ if necessary. This is possible by the argument given in Remark 7.5 by Lemma 18.7. $\qquad\square$

**Lemma 18.9.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. If $p$ is a point of $\mathcal{C}$, then $(L_{\mathcal{B}/\mathcal{A}})_p = L_{\mathcal{B}_p/\mathcal{A}_p}$.*

**Proof.** This is a special case of Lemma 18.3. $\qquad\square$

For the construction of the naive cotangent complex and its properties we refer to Modules on Sites, Section 35.

**Lemma 18.10.** *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. There is a canonical map $L_{\mathcal{B}/\mathcal{A}} \to NL_{\mathcal{B}/\mathcal{A}}$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1} L_{\mathcal{B}/\mathcal{A}}$.*

**Proof.** Let $\mathcal{P}_\bullet$ be the standard resolution of $\mathcal{B}$ over $\mathcal{A}$. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{A}[\mathcal{B}] \to \mathcal{B})$. Recall that $\mathcal{P}_0 = \mathcal{A}[\mathcal{B}]$. The map of the lemma is given by the commutative diagram

$$
\begin{array}{ccccccc}
L_{\mathcal{B}/\mathcal{A}} & \dots \longrightarrow & \Omega_{\mathcal{P}_2/\mathcal{A}} \otimes_{\mathcal{P}_2} \mathcal{B} & \longrightarrow & \Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B} & \longrightarrow & \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
NL_{\mathcal{B}/\mathcal{A}} & \dots \longrightarrow & 0 & \longrightarrow & \mathcal{I}/\mathcal{I}^2 & \longrightarrow & \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B}
\end{array}
$$

We construct the downward arrow with target $\mathcal{I}/\mathcal{I}^2$ by sending a local section $\mathrm{d}f \otimes b$ to the class of $(d_0(f) - d_1(f))b$ in $\mathcal{I}/\mathcal{I}^2$. Here $d_i : \mathcal{P}_1 \to \mathcal{P}_0$, $i = 0, 1$ are the two face maps of the simplicial structure. This makes sense as $d_0 - d_1$ maps $\mathcal{P}_1$ into $\mathcal{I} = \mathrm{Ker}(\mathcal{P}_0 \to \mathcal{B})$. We omit the verification that this rule is well defined. Our map

is compatible with the differential $\Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B} \to \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B}$ as this differential maps a local section $df \otimes b$ to $d(d_0(f) - d_1(f)) \otimes b$. Moreover, the differential $\Omega_{\mathcal{P}_2/\mathcal{A}} \otimes_{\mathcal{P}_2} \mathcal{B} \to \Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B}$ maps a local section $df \otimes b$ to $d(d_0(f) - d_1(f) + d_2(f)) \otimes b$ which are annihilated by our downward arrow. Hence a map of complexes.

To see that our map induces an isomorphism on the cohomology sheaves $H^0$ and $H^{-1}$ we argue as follows. Let $\mathcal{C}'$ be the site with the same underlying category as $\mathcal{C}$ but endowed with the chaotic topology. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be the morphism of topoi whose pullback functor is sheafification. Let $\mathcal{A}' \to \mathcal{B}'$ be the given map, but thought of as a map of sheaves of rings on $\mathcal{C}'$. The construction above gives a map $L_{\mathcal{B}'/\mathcal{A}'} \to NL_{\mathcal{B}'/\mathcal{A}'}$ on $\mathcal{C}'$ whose value over any object $U$ of $\mathcal{C}'$ is just the map

$$L_{\mathcal{B}(U)/\mathcal{A}(U)} \to NL_{\mathcal{B}(U)/\mathcal{A}(U)}$$

of Remark 11.4 which induces an isomorphism on $H^0$ and $H^{-1}$. Since $f^{-1}L_{\mathcal{B}'/\mathcal{A}'} = L_{\mathcal{B}/\mathcal{A}}$ (Lemma 18.3) and $f^{-1} NL_{\mathcal{B}'/\mathcal{A}'} = NL_{\mathcal{B}/\mathcal{A}}$ (Modules on Sites, Lemma 35.3) the lemma is proved. $\square$

## 19. The Atiyah class of a sheaf of modules

Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. Let $\mathcal{P}_\bullet \to \mathcal{B}$ be the standard resolution of $\mathcal{B}$ over $\mathcal{A}$ (Section 18). For every $n \geq 0$ consider the extension of principal parts

(19.0.1) $$0 \to \Omega_{\mathcal{P}_n/\mathcal{A}} \otimes_{\mathcal{P}_n} \mathcal{F} \to \mathcal{P}^1_{\mathcal{P}_n/\mathcal{A}}(\mathcal{F}) \to \mathcal{F} \to 0$$

see Modules on Sites, Lemma 34.6. The functoriality of this construction (Modules on Sites, Remark 34.7) tells us (19.0.1) is the degree $n$ part of a short exact sequence of simplicial $\mathcal{P}_\bullet$-modules (Cohomology on Sites, Section 41). Using the functor $L\pi_! : D(\mathcal{P}_\bullet) \to D(\mathcal{B})$ of Cohomology on Sites, Remark 41.3 (here we use that $\mathcal{P}_\bullet \to \mathcal{A}$ is a resolution) we obtain a distinguished triangle

(19.0.2) $$L_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}}^{\mathbf{L}} \mathcal{F} \to L\pi_! \left( \mathcal{P}^1_{\mathcal{P}_\bullet/\mathcal{A}}(\mathcal{F}) \right) \to \mathcal{F} \to L_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}}^{\mathbf{L}} \mathcal{F}[1]$$

in $D(\mathcal{B})$.

**Definition 19.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. The map $\mathcal{F} \to L_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}}^{\mathbf{L}} \mathcal{F}[1]$ in (19.0.2) is called the *Atiyah class* of $\mathcal{F}$.

## 20. The cotangent complex of a morphism of ringed spaces

The cotangent complex of a morphism of ringed spaces is defined in terms of the cotangent complex we defined above.

**Definition 20.1.** Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. The *cotangent complex* $L_f$ of $f$ is $L_f = L_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}$. We will also use the notation $L_f = L_{X/S} = L_{\mathcal{O}_X/\mathcal{O}_S}$.

More precisely, this means that we consider the cotangent complex (Definition 18.2) of the homomorphism $f^\sharp : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$ of sheaves of rings on the site associated to the topological space $X$ (Sites, Example 6.4).

**Lemma 20.2.** *Let* $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ *be a morphism of ringed spaces. Then* $H^0(L_{X/S}) = \Omega_{X/S}$.

**Proof.** Special case of Lemma 18.6. $\square$

**Lemma 20.3.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Then there is a canonical distinguished triangle*

$$Lf^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to Lf^* L_{Y/Z}[1]$$

*in $D(\mathcal{O}_X)$.*

**Proof.** Set $h = g \circ f$ so that $h^{-1}\mathcal{O}_Z = f^{-1}g^{-1}\mathcal{O}_Z$. By Lemma 18.3 we have $f^{-1}L_{Y/Z} = L_{f^{-1}\mathcal{O}_Y/h^{-1}\mathcal{O}_Z}$ and this is a complex of flat $f^{-1}\mathcal{O}_Y$-modules. Hence the distinguished triangle above is an example of the distinguished triangle of Lemma 18.8 with $\mathcal{A} = h^{-1}\mathcal{O}_Z$, $\mathcal{B} = f^{-1}\mathcal{O}_Y$, and $\mathcal{C} = \mathcal{O}_X$. $\qquad\square$

**Lemma 20.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical map $L_{X/Y} \to NL_{X/Y}$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1} L_{X/Y}$.*

**Proof.** Special case of Lemma 18.10. $\qquad\square$

## 21. Deformations of ringed spaces and the cotangent complex

This section is the continuation of Deformation Theory, Section 7 which we urge the reader to read first. We briefly recall the setup. We have a first order thickening $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$ of ringed spaces with $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, and an $f$-map $c : \mathcal{J} \to \mathcal{G}$ of sheaves of modules. We ask whether we can find the question mark fitting into the following diagram

(21.0.1)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0
\end{array}
$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ and a morphism of thickenings $(f, f')$ as in Deformation Theory, Equation (3.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map $c$. We will say $X'$ is a *solution* to (21.0.1).

**Lemma 21.1.** *In the situation above we have*
  (1) *There is a canonical element $\xi \in \mathrm{Ext}^2_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$ whose vanishing is a sufficient and necessary condition for the existence of a solution to (21.0.1).*
  (2) *If there exists a solution, then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$.*
  (3) *Given a solution $X'$, the set of automorphisms of $X'$ fitting into (21.0.1) is canonically isomorphic to $\mathrm{Ext}^0_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$.*

**Proof.** Via the identifications $NL_{X/S} = \tau_{\geq -1}L_{X/S}$ (Lemma 20.4) and $H^0(L_{X/S}) = \Omega_{X/S}$ (Lemma 20.2) we have seen parts (2) and (3) in Deformation Theory, Lemmas 7.1 and 7.3.

Proof of (1). Roughly speaking, this follows from the discussion in Deformation Theory, Remark 7.9 by replacing the naive cotangent complex by the full cotangent complex. Here is a more detailed explanation. By Deformation Theory, Lemma 7.8 there exists an element

$$\xi' \in \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{S/S'}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* L_{S/S'}, \mathcal{G})$$

such that a solution exists if and only if this element is in the image of the map

$$\text{Ext}^1_{\mathcal{O}_X}(NL_{X/S'}, \mathcal{G}) = \text{Ext}^1_{\mathcal{O}_X}(L_{X/S'}, \mathcal{G}) \longrightarrow \text{Ext}^1_{\mathcal{O}_X}(Lf^*L_{S/S'}, \mathcal{G})$$

The distinguished triangle of Lemma 20.3 for $X \to S \to S'$ gives rise to a long exact sequence

$$\ldots \to \text{Ext}^1_{\mathcal{O}_X}(L_{X/S'}, \mathcal{G}) \to \text{Ext}^1_{\mathcal{O}_X}(Lf^*L_{S/S'}, \mathcal{G}) \to \text{Ext}^2_{\mathcal{O}_X}(L_{X/S}, \mathcal{G}) \to \ldots$$

Hence taking $\xi$ the image of $\xi'$ works.                                    $\square$

## 22. The cotangent complex of a morphism of ringed topoi

The cotangent complex of a morphism of ringed topoi is defined in terms of the cotangent complex we defined above.

**Definition 22.1.** Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. The *cotangent complex* $L_f$ of $f$ is $L_f = L_{\mathcal{O}_\mathcal{C}/f^{-1}\mathcal{O}_\mathcal{D}}$. We sometimes write $L_f = L_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{D}}$.

This definition applies to many situations, but it doesn't always produce the thing one expects. For example, if $f : X \to Y$ is a morphism of schemes, then $f$ induces a morphism of big étale sites $f_{big} : (Sch/X)_{étale} \to (Sch/Y)_{étale}$ which is a morphism of ringed topoi (Descent, Remark 8.4). However, $L_{f_{big}} = 0$ since $(f_{big})^\sharp$ is an isomorphism. On the other hand, if we take $L_f$ where we think of $f$ as a morphism between the underlying Zariski ringed topoi, then $L_f$ does agree with the cotangent complex $L_{X/Y}$ (as defined below) whose zeroth cohomology sheaf is $\Omega_{X/Y}$.

**Lemma 22.2.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. Then $H^0(L_f) = \Omega_f$.*

**Proof.** Special case of Lemma 18.6.                                    $\square$

**Lemma 22.3.** *Let $f : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $g : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. Then there is a canonical distinguished triangle*

$$Lf^*L_g \to L_{g \circ f} \to L_f \to Lf^*L_g[1]$$

*in $D(\mathcal{O}_1)$.*

**Proof.** Set $h = g \circ f$ so that $h^{-1}\mathcal{O}_3 = f^{-1}g^{-1}\mathcal{O}_3$. By Lemma 18.3 we have $f^{-1}L_g = L_{f^{-1}\mathcal{O}_2/h^{-1}\mathcal{O}_3}$ and this is a complex of flat $f^{-1}\mathcal{O}_2$-modules. Hence the distinguished triangle above is an example of the distinguished triangle of Lemma 18.8 with $\mathcal{A} = h^{-1}\mathcal{O}_3$, $\mathcal{B} = f^{-1}\mathcal{O}_2$, and $\mathcal{C} = \mathcal{O}_1$.                                    $\square$

**Lemma 22.4.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. There is a canonical map $L_f \to NL_f$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1}L_f$.*

**Proof.** Special case of Lemma 18.10.                                    $\square$

## 23. Deformations of ringed topoi and the cotangent complex

This section is the continuation of Deformation Theory, Section 13 which we urge the reader to read first. We briefly recall the setup. We have a first order thickening $t : (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ of ringed topoi with $\mathcal{J} = \mathrm{Ker}(t^{\sharp})$, a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, an $\mathcal{O}$-module $\mathcal{G}$, and a map $f^{-1}\mathcal{J} \to \mathcal{G}$ of sheaves of $f^{-1}\mathcal{O}_{\mathcal{B}}$-modules. We ask whether we can find the question mark fitting into the following diagram

$$(23.0.1) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & f^{-1}\mathcal{J} & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}'} & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}} & \longrightarrow & 0 \end{array}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and a morphism of thickenings $(f, f')$ as in Deformation Theory, Equation (9.1.1) where $\mathrm{Ker}(i^{\sharp})$ is identified with $\mathcal{G}$ such that $(f')^{\sharp}$ induces the given map $c$. We will say $(Sh(\mathcal{C}'), \mathcal{O}')$ is a *solution* to (23.0.1).

**Lemma 23.1.** *In the situation above we have*
  (1) *There is a canonical element $\xi \in \mathrm{Ext}^2_{\mathcal{O}}(L_f, \mathcal{G})$ whose vanishing is a sufficient and necessary condition for the existence of a solution to (23.0.1).*
  (2) *If there exists a solution, then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}}(L_f, \mathcal{G})$.*
  (3) *Given a solution $X'$, the set of automorphisms of $X'$ fitting into (23.0.1) is canonically isomorphic to $\mathrm{Ext}^0_{\mathcal{O}}(L_f, \mathcal{G})$.*

**Proof.** Via the identifications $NL_f = \tau_{\geq -1} L_f$ (Lemma 22.4) and $H^0(L_f) = \Omega_f$ (Lemma 22.2) we have seen parts (2) and (3) in Deformation Theory, Lemmas 13.1 and 13.3.

Proof of (1). To match notation with Deformation Theory, Section 13 we will write $NL_f = NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}$ and $L_f = L_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}$ and similarly for the morphisms $t$ and $t \circ f$. By Deformation Theory, Lemma 13.8 there exists an element

$$\xi' \in \mathrm{Ext}^1_{\mathcal{O}}(Lf^* NL_{\mathcal{O}_{\mathcal{B}}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}}(Lf^* L_{\mathcal{O}_{\mathcal{B}}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$$

such that a solution exists if and only if this element is in the image of the map

$$\mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}}(L_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}}(Lf^* L_{\mathcal{O}_{\mathcal{B}}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G})$$

The distinguished triangle of Lemma 22.3 for $f$ and $t$ gives rise to a long exact sequence

$$\ldots \to \mathrm{Ext}^1_{\mathcal{O}}(L_{\mathcal{O}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}}(Lf^* L_{\mathcal{O}_{\mathcal{B}}/\mathcal{O}_{\mathcal{B}'}}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}}(L_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$$

Hence taking $\xi$ the image of $\xi'$ works. $\qquad \square$

## 24. The cotangent complex of a morphism of schemes

As promised above we define the cotangent complex of a morphism of schemes as follows.

**Definition 24.1.** Let $f : X \to Y$ be a morphism of schemes. The *cotangent complex $L_{X/Y}$ of $X$ over $Y$* is the cotangent complex of $f$ as a morphism of ringed spaces (Definition 20.1).

In particular, the results of Section 20 apply to cotangent complexes of morphisms of schemes. The next lemma shows this definition is compatible with the definition for ring maps and it also implies that $L_{X/Y}$ is an object of $D_{QCoh}(\mathcal{O}_X)$.

**Lemma 24.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $U = \mathrm{Spec}(A) \subset X$ and $V = \mathrm{Spec}(B) \subset Y$ be affine opens such that $f(U) \subset V$. There is a canonical map*

$$\widetilde{L_{B/A}} \longrightarrow L_{X/Y}|_U$$

*of complexes which is an isomorphism in $D(\mathcal{O}_U)$. This map is compatible with restricting to smaller affine opens of $X$ and $Y$.*

**Proof.** By Remark 18.5 there is a canonical map of complexes $L_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)} \to L_{X/Y}(U)$ of $B = \mathcal{O}_X(U)$-modules, which is compatible with further restrictions. Using the canonical map $A \to f^{-1}\mathcal{O}_Y(U)$ we obtain a canonical map $L_{B/A} \to L_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)}$ of complexes of $B$-modules. Using the universal property of the $\widetilde{\phantom{x}}$ functor (see Schemes, Lemma 7.1) we obtain a map as in the statement of the lemma. We may check this map is an isomorphism on cohomology sheaves by checking it induces isomorphisms on stalks. This follows immediately from Lemmas 18.9 and 8.6 (and the description of the stalks of $\mathcal{O}_X$ and $f^{-1}\mathcal{O}_Y$ at a point $\mathfrak{p} \in \mathrm{Spec}(B)$ as $B_\mathfrak{p}$ and $A_\mathfrak{q}$ where $\mathfrak{q} = A \cap \mathfrak{p}$; references used are Schemes, Lemma 5.4 and Sheaves, Lemma 21.5). $\qquad\square$

**Lemma 24.3.** *Let $\Lambda$ be a ring. Let $X$ be a scheme over $\Lambda$. Then*

$$L_{X/\mathrm{Spec}(\Lambda)} = L_{\mathcal{O}_X/\underline{\Lambda}}$$

*where $\underline{\Lambda}$ is the constant sheaf with value $\Lambda$ on $X$.*

**Proof.** Let $p : X \to \mathrm{Spec}(\Lambda)$ be the structure morphism. Let $q : \mathrm{Spec}(\Lambda) \to (*, \Lambda)$ be the obvious morphism. By the distinguished triangle of Lemma 20.3 it suffices to show that $L_q = 0$. To see this it suffices to show for $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$ that

$$(L_q)_\mathfrak{p} = L_{\mathcal{O}_{\mathrm{Spec}(\Lambda),\mathfrak{p}}/\Lambda} = L_{\Lambda_\mathfrak{p}/\Lambda}$$

(Lemma 18.9) is zero which follows from Lemma 8.4. $\qquad\square$

## 25. The cotangent complex of a scheme over a ring

Let $\Lambda$ be a ring and let $X$ be a scheme over $\Lambda$. Write $L_{X/\mathrm{Spec}(\Lambda)} = L_{X/\Lambda}$ which is justified by Lemma 24.3. In this section we give a description of $L_{X/\Lambda}$ similar to Lemma 4.3. Namely, we construct a category $\mathcal{C}_{X/\Lambda}$ fibred over $X_{Zar}$ and endow it with a sheaf of (polynomial) $\Lambda$-algebras $\mathcal{O}$ such that

$$L_{X/\Lambda} = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_\mathcal{O} \underline{\mathcal{O}}_X).$$

We will later use the category $\mathcal{C}_{X/\Lambda}$ to construct a naive obstruction theory for the stack of coherent sheaves.

Let $\Lambda$ be a ring. Let $X$ be a scheme over $\Lambda$. Let $\mathcal{C}_{X/\Lambda}$ be the category whose objects are commutative diagrams

(25.0.1)

$$\begin{array}{ccc} X & \longleftarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\Lambda) & \longleftarrow & \mathbf{A} \end{array}$$

of schemes where

(1) $U$ is an open subscheme of $X$,
(2) there exists an isomorphism $\mathbf{A} = \mathrm{Spec}(P)$ where $P$ is a polynomial algebra over $\Lambda$ (on some set of variables).

In other words, $\mathbf{A}$ is an (infinite dimensional) affine space over $\mathrm{Spec}(\Lambda)$. Morphisms are given by commutative diagrams. Recall that $X_{Zar}$ denotes the small Zariski site $X$. There is a forgetful functor

$$u : \mathcal{C}_{X/\Lambda} \to X_{Zar}, \ (U \to \mathbf{A}) \mapsto U$$

Observe that the fibre category over $U$ is canonically equivalent to the category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ introduced in Section 4.

**Lemma 25.1.** *In the situation above the category $\mathcal{C}_{X/\Lambda}$ is fibred over $X_{Zar}$.*

**Proof.** Given an object $U \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ and a morphism $U' \to U$ of $X_{Zar}$ consider the object $U' \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ where $U' \to \mathbf{A}$ is the composition of $U \to \mathbf{A}$ and $U' \to U$. The morphism $(U' \to \mathbf{A}) \to (U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ is strongly cartesian over $X_{Zar}$. $\square$

We endow $\mathcal{C}_{X/\Lambda}$ with the topology inherited from $X_{Zar}$ (see Stacks, Section 10). The functor $u$ defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/\Lambda}) \to Sh(X_{Zar})$. The site $\mathcal{C}_{X/\Lambda}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to \mathbf{A}) \mapsto \Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$.
(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to \mathbf{A}) \mapsto \mathcal{O}_X(U)$.
(3) The constant sheaf $\underline{\Lambda}$.

We obtain morphisms of ringed topoi

(25.1.1)
$$
\begin{array}{ccc}
(Sh(\mathcal{C}_{X/\Lambda}), \underline{\mathcal{O}}_X) & \xrightarrow{\ i\ } & (Sh(\mathcal{C}_{X/\Lambda}), \mathcal{O}) \\
\pi \downarrow & & \\
(Sh(X_{Zar}), \mathcal{O}_X) & &
\end{array}
$$

The morphism $i$ is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map $\pi$ is a special case of Cohomology on Sites, Situation 38.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$. We can compute $L\pi_!$ thanks to our earlier work.

**Remark 25.2.** In the situation above, for every $U \subset X$ open let $P_{\bullet,U}$ be the standard resolution of $\mathcal{O}_X(U)$ over $\Lambda$. Set $\mathbf{A}_{n,U} = \mathrm{Spec}(P_{n,U})$. Then $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of the fibre category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ of $\mathcal{C}_{X/\Lambda}$ over $U$. Moreover, as discussed in Remark 5.5 we have that $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ as in Cohomology on Sites, Lemma 39.7. Since the construction $U \mapsto \mathbf{A}_{\bullet,U}$ is functorial in $U$, given any (abelian) sheaf $\mathcal{F}$ on $\mathcal{C}_{X/\Lambda}$ we obtain a complex of presheaves

$$U \longmapsto \mathcal{F}(\mathbf{A}_{\bullet,U})$$

whose cohomology groups compute the homology of $\mathcal{F}$ on the fibre category. We conclude by Cohomology on Sites, Lemma 40.2 that the sheafification computes $L_n\pi_!(\mathcal{F})$. In other words, the complex of sheaves whose term in degree $-n$ is the sheafification of $U \mapsto \mathcal{F}(\mathbf{A}_{n,U})$ computes $L\pi_!(\mathcal{F})$.

With this remark out of the way we can state the main result of this section.

**Lemma 25.3.** *In the situation above there is a canonical isomorphism*

$$L_{X/\Lambda} = L\pi_!(Li^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(i^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$$

*in $D(\mathcal{O}_X)$.*

**Proof.** We first observe that for any object $(U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ the value of the sheaf $\mathcal{O}$ is a polynomial algebra over $\Lambda$. Hence $\Omega_{\mathcal{O}/\underline{\Lambda}}$ is a flat $\mathcal{O}$-module and we conclude the second and third equalities of the statement of the lemma hold.

By Remark 25.2 the object $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ is computed as the sheafification of the complex of presheaves

$$U \mapsto \left(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X\right)(\mathbf{A}_{\bullet,U}) = \Omega_{P_{\bullet,U}/\Lambda} \otimes_{P_{\bullet,U}} \mathcal{O}_X(U) = L_{\mathcal{O}_X(U)/\Lambda}$$

using notation as in Remark 25.2. Now Remark 18.5 shows that $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ computes the cotangent complex of the map of rings $\underline{\Lambda} \to \mathcal{O}_X$ on $X$. This is what we want by Lemma 24.3. $\qquad\square$

## 26. The cotangent complex of a morphism of algebraic spaces

We define the cotangent complex of a morphism of algebraic spaces using the associated morphism between the small étale sites.

**Definition 26.1.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. The *cotangent complex $L_{X/Y}$ of $X$ over $Y$* is the cotangent complex of the morphism of ringed topoi $f_{small}$ between the small étale sites of $X$ and $Y$ (see Properties of Spaces, Lemma 21.3 and Definition 22.1).

In particular, the results of Section 22 apply to cotangent complexes of morphisms of algebraic spaces. The next lemmas show this definition is compatible with the definition for ring maps and for schemes and that $L_{X/Y}$ is an object of $D_{QCoh}(\mathcal{O}_X)$.

**Lemma 26.2.** *Let $S$ be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\ g\ } & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

*of algebraic spaces over $S$ with $p$ and $q$ étale. Then there is a canonical identification $L_{X/Y}|_{U_{étale}} = L_{U/V}$ in $D(\mathcal{O}_U)$.*

**Proof.** Formation of the cotangent complex commutes with pullback (Lemma 18.3) and we have $p_{small}^{-1}\mathcal{O}_X = \mathcal{O}_U$ and $g_{small}^{-1}\mathcal{O}_{V_{étale}} = p_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}}$ because $q_{small}^{-1}\mathcal{O}_{Y_{étale}} = \mathcal{O}_{V_{étale}}$ (Properties of Spaces, Lemma 26.1). Tracing through the definitions we conclude that $L_{X/Y}|_{U_{étale}} = L_{U/V}$. $\qquad\square$

**Lemma 26.3.** *Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume $X$ and $Y$ representable by schemes $X_0$ and $Y_0$. Then there is a canonical identification $L_{X/Y} = \epsilon^* L_{X_0/Y_0}$ in $D(\mathcal{O}_X)$ where $\epsilon$ is as in Derived Categories of Spaces, Section 4 and $L_{X_0/Y_0}$ is as in Definition 24.1.*

**Proof.** Let $f_0 : X_0 \to Y_0$ be the morphism of schemes corresponding to $f$. There is a canonical map $\epsilon^{-1} f_0^{-1} \mathcal{O}_{Y_0} \to f_{small}^{-1} \mathcal{O}_Y$ compatible with $\epsilon^\sharp : \epsilon^{-1} \mathcal{O}_{X_0} \to \mathcal{O}_X$ because there is a commutative diagram

$$
\begin{array}{ccc}
X_{0,Zar} & \xleftarrow{\ \epsilon\ } & X_{\acute{e}tale} \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\
Y_{0,Zar} & \xleftarrow{\ \epsilon\ } & Y_{\acute{e}tale}
\end{array}
$$

see Derived Categories of Spaces, Remark 6.3. Thus we obtain a canonical map

$$\epsilon^{-1} L_{X_0/Y_0} = \epsilon^{-1} L_{\mathcal{O}_{X_0}/f_0^{-1}\mathcal{O}_{Y_0}} = L_{\epsilon^{-1}\mathcal{O}_{X_0}/\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0}} \longrightarrow L_{\mathcal{O}_X/f_{small}^{-1}\mathcal{O}_Y} = L_{X/Y}$$

by the functoriality discussed in Section 18 and Lemma 18.3. To see that the induced map $\epsilon^* L_{X_0/Y_0} \to L_{X/Y}$ is an isomorphism we may check on stalks at geometric points (Properties of Spaces, Theorem 19.12). We will use Lemma 18.9 to compute the stalks. Let $\overline{x} : \mathrm{Spec}(k) \to X_0$ be a geometric point lying over $x \in X_0$, with $\overline{y} = f \circ \overline{x}$ lying over $y \in Y_0$. Then

$$L_{X/Y,\overline{x}} = L_{\mathcal{O}_{X,\overline{x}}/\mathcal{O}_{Y,\overline{y}}}$$

and

$$(\epsilon^* L_{X_0/Y_0})_{\overline{x}} = L_{X_0/Y_0,x} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}} = L_{\mathcal{O}_{X_0,x}/\mathcal{O}_{Y_0,y}} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}}$$

Some details omitted (hint: use that the stalk of a pullback is the stalk at the image point, see Sites, Lemma 34.2, as well as the corresponding result for modules, see Modules on Sites, Lemma 36.4). Observe that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X_0,x}$ and similarly for $\mathcal{O}_{Y,\overline{y}}$ (Properties of Spaces, Lemma 22.1). Thus the result follows from Lemma 8.7. $\qquad\square$

**Lemma 26.4.** *Let $\Lambda$ be a ring. Let $X$ be an algebraic space over $\Lambda$. Then*

$$L_{X/\mathrm{Spec}(\Lambda)} = L_{\mathcal{O}_X/\underline{\Lambda}}$$

*where $\underline{\Lambda}$ is the constant sheaf with value $\Lambda$ on $X_{\acute{e}tale}$.*

**Proof.** Let $p : X \to \mathrm{Spec}(\Lambda)$ be the structure morphism. Let $q : \mathrm{Spec}(\Lambda)_{\acute{e}tale} \to (*, \Lambda)$ be the obvious morphism. By the distinguished triangle of Lemma 22.3 it suffices to show that $L_q = 0$. To see this it suffices to show (Properties of Spaces, Theorem 19.12) for a geometric point $\overline{t} : \mathrm{Spec}(k) \to \mathrm{Spec}(\Lambda)$ that

$$(L_q)_{\overline{t}} = L_{\mathcal{O}_{\mathrm{Spec}(\Lambda)_{\acute{e}tale},\overline{t}}/\Lambda}$$

(Lemma 18.9) is zero. Since $\mathcal{O}_{\mathrm{Spec}(\Lambda)_{\acute{e}tale},\overline{t}}$ is a strict henselization of a local ring of $\Lambda$ (Properties of Spaces, Lemma 22.1) this follows from Lemma 8.4. $\qquad\square$

## 27. The cotangent complex of an algebraic space over a ring

Let $\Lambda$ be a ring and let $X$ be an algebraic space over $\Lambda$. Write $L_{X/\mathrm{Spec}(\Lambda)} = L_{X/\Lambda}$ which is justified by Lemma 26.4. In this section we give a description of $L_{X/\Lambda}$ similar to Lemma 4.3. Namely, we construct a category $\mathcal{C}_{X/\Lambda}$ fibred over $X_{\acute{e}tale}$ and endow it with a sheaf of (polynomial) $\Lambda$-algebras $\mathcal{O}$ such that

$$L_{X/\Lambda} = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X).$$

We will later use the category $\mathcal{C}_{X/\Lambda}$ to construct a naive obstruction theory for the stack of coherent sheaves.

Let $\Lambda$ be a ring. Let $X$ be an algebraic space over $\Lambda$. Let $\mathcal{C}_{X/\Lambda}$ be the category whose objects are commutative diagrams

(27.0.1)

$$
\begin{array}{ccc}
X & \longleftarrow & U \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\Lambda) & \longleftarrow & \mathbf{A}
\end{array}
$$

of schemes where

(1) $U$ is a scheme,
(2) $U \to X$ is étale,
(3) there exists an isomorphism $\mathbf{A} = \mathrm{Spec}(P)$ where $P$ is a polynomial algebra over $\Lambda$ (on some set of variables).

In other words, $\mathbf{A}$ is an (infinite dimensional) affine space over $\mathrm{Spec}(\Lambda)$. Morphisms are given by commutative diagrams. Recall that $X_{\acute{e}tale}$ denotes the small étale site of $X$ whose objects are schemes étale over $X$. There is a forgetful functor

$$
u : \mathcal{C}_{X/\Lambda} \to X_{\acute{e}tale}, \quad (U \to \mathbf{A}) \mapsto U
$$

Observe that the fibre category over $U$ is canonically equivalent to the category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ introduced in Section 4.

**Lemma 27.1.** *In the situation above the category $\mathcal{C}_{X/\Lambda}$ is fibred over $X_{\acute{e}tale}$.*

**Proof.** Given an object $U \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ and a morphism $U' \to U$ of $X_{\acute{e}tale}$ consider the object $U' \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ where $U' \to \mathbf{A}$ is the composition of $U \to \mathbf{A}$ and $U' \to U$. The morphism $(U' \to \mathbf{A}) \to (U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ is strongly cartesian over $X_{\acute{e}tale}$. $\square$

We endow $\mathcal{C}_{X/\Lambda}$ with the topology inherited from $X_{\acute{e}tale}$ (see Stacks, Section 10). The functor $u$ defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/\Lambda}) \to Sh(X_{\acute{e}tale})$. The site $\mathcal{C}_{X/\Lambda}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to \mathbf{A}) \mapsto \Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$.
(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to \mathbf{A}) \mapsto \mathcal{O}_X(U)$.
(3) The constant sheaf $\underline{\Lambda}$.

We obtain morphisms of ringed topoi

(27.1.1)

$$
\begin{array}{ccc}
(Sh(\mathcal{C}_{X/\Lambda}), \underline{\mathcal{O}}_X) & \xrightarrow{\ i\ } & (Sh(\mathcal{C}_{X/\Lambda}), \mathcal{O}) \\
{\scriptstyle \pi} \downarrow & & \\
(Sh(X_{\acute{e}tale}), \mathcal{O}_X) & &
\end{array}
$$

The morphism $i$ is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map $\pi$ is a special case of Cohomology on Sites, Situation 38.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$. We can compute $L\pi_!$ thanks to our earlier work.

**Remark 27.2.** In the situation above, for every object $U \to X$ of $X_{\acute{e}tale}$ let $P_{\bullet,U}$ be the standard resolution of $\mathcal{O}_X(U)$ over $\Lambda$. Set $\mathbf{A}_{n,U} = \mathrm{Spec}(P_{n,U})$. Then $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of the fibre category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ of $\mathcal{C}_{X/\Lambda}$ over $U$. Moreover, as discussed in Remark 5.5 we have that $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ as in

Cohomology on Sites, Lemma 39.7. Since the construction $U \mapsto \mathbf{A}_{\bullet,U}$ is functorial in $U$, given any (abelian) sheaf $\mathcal{F}$ on $\mathcal{C}_{X/\Lambda}$ we obtain a complex of presheaves

$$U \longmapsto \mathcal{F}(\mathbf{A}_{\bullet,U})$$

whose cohomology groups compute the homology of $\mathcal{F}$ on the fibre category. We conclude by Cohomology on Sites, Lemma 40.2 that the sheafification computes $L_n \pi_!(\mathcal{F})$. In other words, the complex of sheaves whose term in degree $-n$ is the sheafification of $U \mapsto \mathcal{F}(\mathbf{A}_{n,U})$ computes $L\pi_!(\mathcal{F})$.

With this remark out of the way we can state the main result of this section.

**Lemma 27.3.** *In the situation above there is a canonical isomorphism*

$$L_{X/\Lambda} = L\pi_!(Li^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(i^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$$

*in* $D(\mathcal{O}_X)$.

**Proof.** We first observe that for any object $(U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ the value of the sheaf $\mathcal{O}$ is a polynomial algebra over $\Lambda$. Hence $\Omega_{\mathcal{O}/\underline{\Lambda}}$ is a flat $\mathcal{O}$-module and we conclude the second and third equalities of the statement of the lemma hold.

By Remark 27.2 the object $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ is computed as the sheafification of the complex of presheaves

$$U \mapsto \left(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X\right)(\mathbf{A}_{\bullet,U}) = \Omega_{P_{\bullet,U}/\Lambda} \otimes_{P_{\bullet,U}} \mathcal{O}_X(U) = L_{\mathcal{O}_X(U)/\Lambda}$$

using notation as in Remark 27.2. Now Remark 18.5 shows that $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ computes the cotangent complex of the map of rings $\underline{\Lambda} \to \mathcal{O}_X$ on $X_{\acute{e}tale}$. This is what we want by Lemma 26.4. $\square$

## 28. Fibre products of algebraic spaces and the cotangent complex

Let $S$ be a scheme. Let $X \to B$ and $Y \to B$ be morphisms of algebraic spaces over $S$. Consider the fibre product $X \times_B Y$ with projection morphisms $p : X \times_B Y \to X$ and $q : X \times_B Y \to Y$. In this section we discuss $L_{X \times_B Y/B}$. Most of the information we want is contained in the following diagram

$$
\begin{array}{ccccc}
Lp^*L_{X/B} & \longrightarrow & L_{X \times_B Y/Y} & \longrightarrow & E \\
\| & & \uparrow & & \uparrow \\
Lp^*L_{X/B} & \longrightarrow & L_{X \times_B Y/B} & \longrightarrow & L_{X \times_B Y/X} \\
& & \uparrow & & \uparrow \\
& & Lq^*L_{Y/B} & = = = = & Lq^*L_{Y/B}
\end{array}
$$

(28.0.1)

Explanation: The middle row is the fundamental triangle of Lemma 22.3 for the morphisms $X \times_B Y \to X \to B$. The middle column is the fundamental triangle for the morphisms $X \times_B Y \to Y \to B$. Next, $E$ is an object of $D(\mathcal{O}_{X \times_B Y})$ which "fits" into the upper right corner, i.e., which turns both the top row and the right column into distinguished triangles. Such an $E$ exists by Derived Categories, Proposition 4.23 applied to the lower left square (with 0 placed in the missing spot). To be more explicit, we could for example define $E$ as the cone (Derived Categories, Definition 9.1) of the map of complexes

$$Lp^*L_{X/B} \oplus Lq^*L_{Y/B} \longrightarrow L_{X \times_B Y/B}$$

and get the two maps with target $E$ by an application of TR3. In the Tor independent case the object $E$ is zero.

**Lemma 28.1.** *In the situation above, if $X$ and $Y$ are Tor independent over $B$, then the object $E$ in (28.0.1) is zero. In this case we have*

$$L_{X \times_B Y/B} = Lp^* L_{X/B} \oplus Lq^* L_{Y/B}$$

**Proof.** Choose a scheme $W$ and a surjective étale morphism $W \to B$. Choose a scheme $U$ and a surjective étale morphism $U \to X \times_B W$. Choose a scheme $V$ and a surjective étale morphism $V \to Y \times_B W$. Then $U \times_W V \to X \times_B Y$ is surjective étale too. Hence it suffices to prove that the restriction of $E$ to $U \times_W V$ is zero. By Lemma 26.3 and Derived Categories of Spaces, Lemma 20.3 this reduces us to the case of schemes. Taking suitable affine opens we reduce to the case of affine schemes. Using Lemma 24.2 we reduce to the case of a tensor product of rings, i.e., to Lemma 15.1.                                                                   $\square$

In general we can say the following about the object $E$.

**Lemma 28.2.** *Let $S$ be a scheme. Let $X \to B$ and $Y \to B$ be morphisms of algebraic spaces over $S$. The object $E$ in (28.0.1) satisfies $H^i(E) = 0$ for $i = 0, -1$ and for a geometric point $(\overline{x}, \overline{y}) : \mathrm{Spec}(k) \to X \times_B Y$ we have*

$$H^{-2}(E)_{(\overline{x}, \overline{y})} = Tor_1^R(A, B) \otimes_{A \otimes_R B} C$$

*where $R = \mathcal{O}_{B, \overline{b}}$, $A = \mathcal{O}_{X, \overline{x}}$, $B = \mathcal{O}_{Y, \overline{y}}$, and $C = \mathcal{O}_{X \times_B Y, (\overline{x}, \overline{y})}$.*

**Proof.** The formation of the cotangent complex commutes with taking stalks and pullbacks, see Lemmas 18.9 and 18.3. Note that $C$ is a henselization of $A \otimes_R B$. $L_{C/R} = L_{A \otimes_R B/R} \otimes_{A \otimes_R B} C$ by the results of Section 8. Thus the stalk of $E$ at our geometric point is the cone of the map $L_{A/R} \otimes C \to L_{A \otimes_R B/R} \otimes C$. Therefore the results of the lemma follow from the case of rings, i.e., Lemma 15.2.                                       $\square$

## 29. Other chapters

Preliminaries

Schemes

## References

[And67]    Michel André, *Méthode simpliciale en algèbre homologique et algèbre commutative*,
           Lecture Notes in Mathematics, Vol. 32, Springer-Verlag, Berlin, 1967.
[And74]    _____, *Homologie des algèbres commutatives*, Springer-Verlag, Berlin, 1974, Die
           Grundlehren der mathematischen Wissenschaften, Band 206.
[DRGV92]   José Luís Doncel, Alfredo Rodríguez-Grandjeán, and Maria Jesús Vale, *On the homol-
           ogy of commutative algebras*, J. Pure Appl. Algebra **79** (1992), no. 2, 131–157.
[Ill72]    Luc Illusie, *Complexe cotangent et déformations I and II*, Lecture Notes in Mathe-
           matics, Vol. 239 and 283, Springer-Verlag, Berlin, 1971/1972.
[LS67]     S. Lichtenbaum and M. Schlessinger, *The cotangent complex of a morphism*, Trans.
           Amer. Math. Soc. **128** (1967), 41–70.
[Qui]      Daniel Quillen, *Homology of commutative rings*, Unpublished, pp. 1–81.
[Qui70]    _____, *On the (co-) homology of commutative rings*, Applications of Categorical
           Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), Amer. Math. Soc.,
           Providence, R.I., 1970, pp. 65–87.