BOOTSTRAP

Contents

1. Introduction

In this chapter we use the material from the preceding sections to give criteria under which a presheaf of sets on the category of schemes is an algebraic space. Some of this material comes from the work of Artin, see [Art69b], [Art70], [Art73], [Art71b], [Art71a], [Art69a], [Art69c], and [Art74]. However, our method will be to use as much as possible arguments similar to those of the paper by Keel and Mori, see [KM97].

2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Morphisms representable by algebraic spaces

Here we define the notion of one presheaf being relatively representable by algebraic spaces over another, and we prove some properties of this notion.

Definition 3.1. Let S be a scheme contained in Sch_{fppf}. Let F, G be presheaves on Sch_{fppf}/S. We say a morphism $a : F \to G$ is *representable by algebraic spaces* if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi : U \to G$ the fiber product $U \times_{\xi,G} F$ is an algebraic space.

Here is a sanity check.

Lemma 3.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is representable by algebraic spaces.*

Proof. This is formal. It relies on the fact that the category of algebraic spaces over S has fibre products, see Spaces, Lemma 7.3. $\square$

Lemma 3.3. *Let S be a scheme. Let*

$$
\begin{array}{ccc}
G' \times_G F & \longrightarrow & F \\
\downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} \\
G' & \longrightarrow & G
\end{array}
$$

be a fibre square of presheaves on $(Sch/S)_{fppf}$. If a is representable by algebraic spaces so is a'.

Proof. Omitted. Hint: This is formal. $\square$

Lemma 3.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. If G is a sheaf, then so is F.*

Proof. (Same as the proof of Spaces, Lemma 3.5.) Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(Sch/S)_{fppf}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma,G,a} F$ is a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\mathrm{id}_T, s) \in F'(T)$. Clearly s is the section of F we are looking for. $\square$

Lemma 3.5. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. Then $\Delta_{F/G} : F \to F \times_G F$ is representable by algebraic spaces.*

Proof. (Same as the proof of Spaces, Lemma 3.6.) Let U be a scheme. Let $\xi = (\xi_1, \xi_2) \in (F \times_G F)(U)$. Set $\xi' = a(\xi_1) = a(\xi_2) \in G(U)$. By assumption there exist an algebraic space V and a morphism $V \to U$ representing the fibre product $U \times_{\xi',G} F$. In particular, the elements ξ_1, ξ_2 give morphisms $f_1, f_2 : U \to V$ over U. Because V represents the fibre product $U \times_{\xi',G} F$ and because $\xi' = a \circ \xi_1 = a \circ \xi_2$ we see that if $g : U' \to U$ is a morphism then

$$
g^*\xi_1 = g^*\xi_2 \Leftrightarrow f_1 \circ g = f_2 \circ g.
$$

In other words, we see that $U \times_{\xi, F \times_G F} F$ is represented by $V \times_{\Delta, V \times V, (f_1, f_2)} U$ which is an algebraic space. $\square$

The proof of Lemma 3.6 below is actually slightly tricky. Namely, we cannot use the argument of the proof of Spaces, Lemma 11.3 because we do not yet know that a composition of transformations representable by algebraic spaces is representable by algebraic spaces. In fact, we will use this lemma to prove that statement.

Lemma 3.6. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to$ Sets. Let $a : F \to G$ be representable by algebraic spaces. If G is an algebraic space, then so is F.*

Proof. We have seen in Lemma 3.4 that F is a sheaf.

Let U be a scheme and let $U \to G$ be a surjective étale morphism. In this case $U \times_G F$ is an algebraic space. Let W be a scheme and let $W \to U \times_G F$ be a surjective étale morphism.

First we claim that $W \to F$ is representable. To see this let X be a scheme and let $X \to F$ be a morphism. Then

$$W \times_F X = W \times_{U \times_G F} U \times_G F \times_F X = W \times_{U \times_G F} (U \times_G X)$$

Since both $U \times_G F$ and G are algebraic spaces we see that this is a scheme.

Next, we claim that $W \to F$ is surjective and étale (this makes sense now that we know it is representable). This follows from the formula above since both $W \to U \times_G F$ and $U \to G$ are étale and surjective, hence $W \times_{U \times_G F} (U \times_G X) \to U \times_G X$ and $U \times_G X \to X$ are surjective and étale, and the composition of surjective étale morphisms is surjective and étale.

Set $R = W \times_F W$. By the above R is a scheme and the projections $t, s : R \to W$ are étale. It is clear that R is an equivalence relation, and $W \to F$ is a surjection of sheaves. Hence R is an étale equivalence relation and $F = W/R$. Hence F is an algebraic space by Spaces, Theorem 10.5. $\square$

Lemma 3.7. *Let S be a scheme. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$. Suppose $a : F \to G$ is representable by algebraic spaces. If X is an algebraic space over S, and $X \to G$ is a map of presheaves then $X \times_G F$ is an algebraic space.*

Proof. By Lemma 3.3 the transformation $X \times_G F \to X$ is representable by algebraic spaces. Hence it is an algebraic space by Lemma 3.6. $\square$

Lemma 3.8. *Let S be a scheme. Let*

$$F \xrightarrow{\ a\ } G \xrightarrow{\ b\ } H$$

be maps of presheaves on $(Sch/S)_{fppf}$. If a and b are representable by algebraic spaces, so is $b \circ a$.

Proof. Let T be a scheme over S, and let $T \to H$ be a morphism. By assumption $T \times_H G$ is an algebraic space. Hence by Lemma 3.7 we see that $T \times_H F = (T \times_H G) \times_G F$ is an algebraic space as well. $\square$

Lemma 3.9. *Let S be a scheme. Let $F_i, G_i : (Sch/S)^{opp}_{fppf} \to$ Sets, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable by algebraic spaces. Then*

$$a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$$

is a representable by algebraic spaces.

Proof. Write $a_1 \times a_2$ as the composition $F_1 \times F_2 \to G_1 \times F_2 \to G_1 \times G_2$. The first arrow is the base change of a_1 by the map $G_1 \times F_2 \to G_1$, and the second arrow is the base change of a_2 by the map $G_1 \times G_2 \to G_2$. Hence this lemma is a formal consequence of Lemmas 3.8 and 3.3. $\square$

Lemma 3.10. *Let S be a scheme. Let $a : F \to G$ and $b : G \to H$ be transformations of functors $(Sch/S)_{fppf}^{opp} \to Sets$. Assume*

(1) $\Delta : G \to G \times_H G$ *is representable by algebraic spaces, and*
(2) $b \circ a : F \to H$ *is representable by algebraic spaces.*

Then a is representable by algebraic spaces.

Proof. Let U be a scheme over S and let $\xi \in G(U)$. Then

$$U \times_{\xi, G, a} F = (U \times_{b(\xi), H, b \circ a} F) \times_{(\xi, a), (G \times_H G), \Delta} G$$

Hence the result using Lemma 3.7. $\qquad\qquad\square$

Lemma 3.11. *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. Assume*

(1) F *is a sheaf for the Zariski topology on $(Sch/S)_{fppf}$,*
(2) *there exists an index set I and subfunctors $F_i \subset F$ such that*
    (a) *each F_i is an fppf sheaf,*
    (b) *each $F_i \to F$ is representable by algebraic spaces,*
    (c) $\coprod F_i \to F$ *becomes surjective after fppf sheafification.*

Then F is an fppf sheaf.

Proof. Let $T \in \mathrm{Ob}((Sch/S)_{fppf})$ and let $s \in F(T)$. By (2)(c) there exists an fppf covering $\{T_j \to T\}$ such that $s|_{T_j}$ is a section of $F_{\alpha(j)}$ for some $\alpha(j) \in I$. Let $W_j \subset T$ be the image of $T_j \to T$ which is an open subscheme Morphisms, Lemma 25.10. By (2)(b) we see $F_{\alpha(j)} \times_{F, s|_{W_j}} W_j \to W_j$ is a monomorphism of algebraic spaces through which T_j factors. Since $\{T_j \to W_j\}$ is an fppf covering, we conclude that $F_{\alpha(j)} \times_{F, s|_{W_j}} W_j = W_j$, in other words $s|_{W_j} \in F_{\alpha(j)}(W_j)$. Hence we conclude that $\coprod F_i \to F$ is surjective for the Zariski topology.

Let $\{T_j \to T\}$ be an fppf covering in $(Sch/S)_{fppf}$. Let $s, s' \in F(T)$ with $s|_{T_j} = s'|_{T_j}$ for all j. We want to show that s, s' are equal. As F is a Zariski sheaf by (1) we may work Zariski locally on T. By the result of the previous paragraph we may assume there exist i such that $s \in F_i(T)$. Then we see that $s'|_{T_j}$ is a section of F_i. By (2)(b) we see $F_i \times_{F, s'} T \to T$ is a monomorphism of algebraic spaces through which all of the T_j factor. Hence we conclude that $s' \in F_i(T)$. Since F_i is a sheaf for the fppf topology we conclude that $s = s'$.

Let $\{T_j \to T\}$ be an fppf covering in $(Sch/S)_{fppf}$ and let $s_j \in F(T_j)$ such that $s_j|_{T_j \times_T T_{j'}} = s_{j'}|_{T_j \times_T T_{j'}}$. By assumption (2)(b) we may refine the covering and assume that $s_j \in F_{\alpha(j)}(T_j)$ for some $\alpha(j) \in I$. Let $W_j \subset T$ be the image of $T_j \to T$ which is an open subscheme Morphisms, Lemma 25.10. Then $\{T_j \to W_j\}$ is an fppf covering. Since $F_{\alpha(j)}$ is a sub presheaf of F we see that the two restrictions of s_j to $T_j \times_{W_j} T_j$ agree as elements of $F_{\alpha(j)}(T_j \times_{W_j} T_j)$. Hence, the sheaf condition for $F_{\alpha(j)}$ implies there exists a $s'_j \in F_{\alpha(j)}(W_j)$ whose restriction to T_j is s_j. For a pair of indices j and j' the sections $s'_j|_{W_j \cap W_{j'}}$ and $s'_{j'}|_{W_j \cap W_{j'}}$ of F agree by the result of the previous paragraph. This finishes the proof by the fact that F is a Zariski sheaf. $\qquad\square$

4. Properties of maps of presheaves representable by algebraic spaces

Here is the definition that makes this work.

Definition 4.1. Let S be a scheme. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$ which is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

(1) is preserved under any base change, and
(2) is fppf local on the base, see Descent on Spaces, Definition 10.1.

In this case we say that a has *property* $\mathcal{P}$ if for every scheme U and $\xi : U \to G$ the resulting morphism of algebraic spaces $U \times_G F \to U$ has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

The definition above applies[1] for example to the properties of being "surjective", "quasi-compact", "étale", "smooth", "flat", "separated", "(locally) of finite type", "(locally) quasi-finite", "(locally) of finite presentation", "affine", "proper", and "a closed immersion". In other words, a is *surjective* (resp. *quasi-compact, étale, smooth, flat, separated, (locally) of finite type, (locally) quasi-finite, (locally) of finite presentation, proper, a closed immersion*) if for every scheme T and map $\xi : T \to G$ the morphism of algebraic spaces $T \times_{\xi,G} F \to T$ is surjective (resp. quasi-compact, étale, flat, separated, (locally) of finite type, (locally) quasi-finite, (locally) of finite presentation, proper, a closed immersion).

Next, we check consistency with the already existing notions. By Lemma 3.2 any morphism between algebraic spaces over S is representable by algebraic spaces. And by Morphisms of Spaces, Lemma 5.3 (resp. 8.8, 39.2, 37.4, 30.5, 4.12, 23.4, 27.6, 28.4, 20.3, 40.2, 12.1) the definition of surjective (resp. quasi-compact, étale, smooth, flat, separated, (locally) of finite type, (locally) quasi-finite, (locally) of finite presentation, affine, proper, closed immersion) above agrees with the already existing definition of morphisms of algebraic spaces.

Some formal lemmas follow.

Lemma 4.2. *Let S be a scheme. Let $\mathcal{P}$ be a property as in Definition 4.1. Let*

$$
\begin{array}{ccc}
G' \times_G F & \longrightarrow & F \\
\downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} \\
G' & \longrightarrow & G
\end{array}
$$

be a fibre square of presheaves on $(Sch/S)_{fppf}$. If a is representable by algebraic spaces and has $\mathcal{P}$ so does a'.

Proof. Omitted. Hint: This is formal. □

Lemma 4.3. *Let S be a scheme. Let $\mathcal{P}$ be a property as in Definition 4.1, and assume $\mathcal{P}$ is stable under composition. Let*

$$
F \xrightarrow{\ a\ } G \xrightarrow{\ b\ } H
$$

---

[1]Being preserved under base change holds by Morphisms of Spaces, Lemmas 5.5, 8.4, 39.4, 37.3, 30.4, 4.4, 23.3, 27.4, 28.3, 20.5, 40.3, and Spaces, Lemma 12.3. Being fppf local on the base holds by Descent on Spaces, Lemmas 11.6, 11.1, 11.28, 11.26, 11.13, 11.18, 11.11, 11.24, 11.10, 11.16, 11.19, and 11.17.

be maps of presheaves on $(Sch/S)_{fppf}$. If a, b are representable by algebraic spaces and has $\mathcal{P}$ so does $b \circ a$.

Proof. Omitted. Hint: See Lemma 3.8 and use stability under composition. $\square$

Lemma 4.4. *Let S be a scheme. Let $F_i, G_i : (Sch/S)_{fppf}^{opp} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable by algebraic spaces. Let $\mathcal{P}$ be a property as in Definition 4.1 which is stable under composition. If a_1 and a_2 have property $\mathcal{P}$ so does $a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$.*

Proof. Note that the lemma makes sense by Lemma 3.9. Proof omitted. $\square$

Lemma 4.5. *Let S be a scheme. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a transformation of functors representable by algebraic spaces. Let $\mathcal{P}$, $\mathcal{P}'$ be properties as in Definition 4.1. Suppose that for any morphism $f : X \to Y$ of algebraic spaces over S we have $\mathcal{P}(f) \Rightarrow \mathcal{P}'(f)$. If a has property $\mathcal{P}$, then a has property $\mathcal{P}'$.*

Proof. Formal. $\square$

Lemma 4.6. *Let S be a scheme. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$ be sheaves. Let $a : F \to G$ be representable by algebraic spaces, flat, locally of finite presentation, and surjective. Then $a : F \to G$ is surjective as a map of sheaves.*

Proof. Let T be a scheme over S and let $g : T \to G$ be a T-valued point of G. By assumption $T' = F \times_G T$ is an algebraic space and the morphism $T' \to T$ is a flat, locally of finite presentation, and surjective morphism of algebraic spaces. Let $U \to T'$ be a surjective étale morphism, where U is a scheme. Then by the definition of flat morphisms of algebraic spaces the morphism of schemes $U \to T$ is flat. Similarly for "locally of finite presentation". The morphism $U \to T$ is surjective also, see Morphisms of Spaces, Lemma 5.3. Hence we see that $\{U \to T\}$ is an fppf covering such that $g|_U \in G(U)$ comes from an element of $F(U)$, namely the map $U \to T' \to F$. This proves the map is surjective as a map of sheaves, see Sites, Definition 11.1. $\square$

5. Bootstrapping the diagonal

In this section we prove that the diagonal of a sheaf F on $(Sch/S)_{fppf}$ is representable as soon as there exists an "fppf cover" of F by a scheme or by an algebraic space, see Lemma 5.3.

Lemma 5.1. *Let S be a scheme. If F is a presheaf on $(Sch/S)_{fppf}$. The following are equivalent:*

(1) *$\Delta_F : F \to F \times F$ is representable by algebraic spaces,*
(2) *for every scheme T any map $T \to F$ is representable by algebraic spaces, and*
(3) *for every algebraic space X any map $X \to F$ is representable by algebraic spaces.*

Proof. Assume (1). Let $X \to F$ be as in (3). Let T be a scheme, and let $T \to F$ be a morphism. Then we have

$$T \times_F X = (T \times_S X) \times_{F \times F, \Delta} F$$

which is an algebraic space by Lemma 3.7 and (1). Hence $X \to F$ is representable, i.e., (3) holds. The implication (3) $\Rightarrow$ (2) is trivial. Assume (2). Let T be a scheme, and let $(a, b) : T \to F \times F$ be a morphism. Then

$$F \times_{\Delta_F, F \times F} T = (T \times_{a, F, b} T) \times_{T \times T, \Delta_T} T$$

which is an algebraic space by assumption. Hence Δ_F is representable by algebraic spaces, i.e., (1) holds. $\qquad \square$

In particular if F is a presheaf satisfying the equivalent conditions of the lemma, then for any morphism $X \to F$ where X is an algebraic space it makes sense to say that $X \to F$ is surjective (resp. étale, flat, locally of finite presentation) by using Definition 4.1.

Before we actually do the bootstrap we prove a fun lemma.

Lemma 5.2. *Let S be a scheme. Let*

$$
\begin{array}{ccc}
E & \xrightarrow{\ a\ } & F \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
H & \xrightarrow{\ b\ } & G
\end{array}
$$

be a cartesian diagram of sheaves on $(Sch/S)_{fppf}$, so $E = H \times_G F$. If

   (1) *g is representable by algebraic spaces, surjective, flat, and locally of finite presentation, and*
   (2) *a is representable by algebraic spaces, separated, and locally quasi-finite*

then b is representable (by schemes) as well as separated and locally quasi-finite.

Proof. Let T be a scheme, and let $T \to G$ be a morphism. We have to show that $T \times_G H$ is a scheme, and that the morphism $T \times_G H \to T$ is separated and locally quasi-finite. Thus we may base change the whole diagram to T and assume that G is a scheme. In this case F is an algebraic space. Let U be a scheme, and let $U \to F$ be a surjective étale morphism. Then $U \to F$ is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 39.7 and 39.8. By Lemma 3.8 $U \to G$ is surjective, flat and locally of finite presentation also. Note that the base change $E \times_F U \to U$ of a is still separated and locally quasi-finite (by Lemma 4.2). Hence we may replace the upper part of the diagram of the lemma by $E \times_F U \to U$. In other words, we may assume that $F \to G$ is a surjective, flat morphism of schemes which is locally of finite presentation. In particular, $\{F \to G\}$ is an fppf covering of schemes. By Morphisms of Spaces, Proposition 50.2 we conclude that E is a scheme also. By Descent, Lemma 39.1 the fact that $E = H \times_G F$ means that we get a descent datum on E relative to the fppf covering $\{F \to G\}$. By More on Morphisms, Lemma 57.1 this descent datum is effective. By Descent, Lemma 39.1 again this implies that H is a scheme. By Descent, Lemmas 23.6 and 23.24 it now follows that b is separated and locally quasi-finite. $\qquad \square$

Here is the result that the section title refers to.

Lemma 5.3. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Assume that*

   (1) *the presheaf F is a sheaf,*

(2) *there exists an algebraic space X and a map $X \to F$ which is representable by algebraic spaces, surjective, flat and locally of finite presentation.*

Then Δ_F is representable (by schemes).

Proof. Let $U \to X$ be a surjective étale morphism from a scheme towards X. Then $U \to X$ is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 39.7 and 39.8. By Lemma 4.3 the composition $U \to F$ is representable by algebraic spaces, surjective, flat and locally of finite presentation also. Thus we see that $R = U \times_F U$ is an algebraic space, see Lemma 3.7. The morphism of algebraic spaces $R \to U \times_S U$ is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism, see Morphisms of Spaces, Lemma 10.2). Since $U \to F$ is locally of finite presentation, both morphisms $R \to U$ are locally of finite presentation, see Lemma 4.2. Hence $R \to U \times_S U$ is locally of finite type (use Morphisms of Spaces, Lemmas 28.5 and 23.6). Altogether this means that $R \to U \times_S U$ is a monomorphism which is locally of finite type, hence a separated and locally quasi-finite morphism, see Morphisms of Spaces, Lemma 27.10.

Now we are ready to prove that Δ_F is representable. Let T be a scheme, and let $(a, b) : T \to F \times F$ be a morphism. Set

$$T' = (U \times_S U) \times_{F \times F} T.$$

Note that $U \times_S U \to F \times F$ is representable by algebraic spaces, surjective, flat and locally of finite presentation by Lemma 4.4. Hence T' is an algebraic space, and the projection morphism $T' \to T$ is surjective, flat, and locally of finite presentation. Consider $Z = T \times_{F \times F} F$ (this is a sheaf) and

$$Z' = T' \times_{U \times_S U} R = T' \times_T Z.$$

We see that Z' is an algebraic space, and $Z' \to T'$ is separated and locally quasi-finite by the discussion in the first paragraph of the proof which showed that R is an algebraic space and that the morphism $R \to U \times_S U$ has those properties. Hence we may apply Lemma 5.2 to the diagram

$$
\begin{array}{ccc}
Z' & \longrightarrow & T' \\
\downarrow & & \downarrow \\
Z & \longrightarrow & T
\end{array}
$$

and we conclude. $\qquad\square$

Here is a variant of the result above.

Lemma 5.4. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Let X be a scheme and let $X \to F$ be representable by algebraic spaces and locally quasi-finite. Then $X \to F$ is representable (by schemes).*

Proof. Let T be a scheme and let $T \to F$ be a morphism. We have to show that the algebraic space $X \times_F T$ is representable by a scheme. Consider the morphism

$$X \times_F T \longrightarrow X \times_{\mathrm{Spec}(\mathbf{Z})} T$$

Since $X \times_F T \to T$ is locally quasi-finite, so is the displayed arrow (Morphisms of Spaces, Lemma 27.8). On the other hand, the displayed arrow is a monomorphism

and hence separated (Morphisms of Spaces, Lemma 10.3). Thus $X \times_F T$ is a scheme by Morphisms of Spaces, Proposition 50.2. □

6. Bootstrap

We warn the reader right away that the result of this section will be superseded by the stronger Theorem 10.1. On the other hand, the theorem in this section is quite a bit easier to prove and still provides quite a bit of insight into how things work, especially for those readers mainly interested in Deligne-Mumford stacks.

In Spaces, Section 6 we defined an algebraic space as a sheaf in the fppf topology whose diagonal is representable, and such that there exist a surjective étale morphism from a scheme towards it. In this section we show that a sheaf in the fppf topology whose diagonal is representable by algebraic spaces and which has an étale surjective covering by an algebraic space is also an algebraic space. In other words, the category of algebraic spaces is an enlargement of the category of schemes by those fppf sheaves F which have a representable diagonal and an étale covering by a scheme. The result of this section says that doing the same process again starting with the category of algebraic spaces, does not lead to yet another category.

Another motivation for the material in this section is that it will guarantee later that a Deligne-Mumford stack whose inertia stack is trivial is equivalent to an algebraic space, see Algebraic Stacks, Lemma 13.2.

Here is the main result of this section (as we mentioned above this will be superseded by the stronger Theorem 10.1).

Theorem 6.1. *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Assume that*

(1) *the presheaf F is a sheaf,*
(2) *the diagonal morphism $F \to F \times F$ is representable by algebraic spaces, and*
(3) *there exists an algebraic space X and a map $X \to F$ which is surjective, and étale.*

or assume that

(a) *the presheaf F is a sheaf, and*
(b) *there exists an algebraic space X and a map $X \to F$ which is representable by algebraic paces, surjective, and étale.*

Then F is an algebraic space.

Proof. We will use the remarks directly below Definition 4.1 without further mention.

Assume (1), (2), and (3) and let $X \to F$ be as in (3). By Lemma 5.1 the morphism $X \to F$ is representable by algebraic spaces. Thus we see that (a) and (b) hold.

Assume (a) and (b) and let $X \to F$ be as in (b). Let $U \to X$ be a surjective étale morphism from a scheme towards X. By Lemma 3.8 the transformation $U \to F$ is representable by algebraic spaces, surjective, and étale. Hence to prove that F is an algebraic space boils down to proving that Δ_F is representable (Spaces, Definition 6.1). This follows immediately from Lemma 5.3. On the other hand we can circumvent this lemma and show directly F is an algebraic space as in the next paragraph.

Namely, let U be a scheme and let $U \to F$ be representable by algebraic spaces, surjective, and étale. Consider the fibre product $R = U \times_F U$. Both projections $R \to U$ are representable by algebraic spaces, surjective, and étale (Lemma 4.2). In particular R is an algebraic space by Lemma 3.6. The morphism of algebraic spaces $R \to U \times_S U$ is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism). Since $R \to U$ is étale, we see that $R \to U$ is locally quasi-finite, see Morphisms of Spaces, Lemma 39.5. We conclude that also $R \to U \times_S U$ is locally quasi-finite by Morphisms of Spaces, Lemma 27.8. Hence Morphisms of Spaces, Proposition 50.2 applies and R is a scheme. By Lemma 4.6 the map $U \to F$ is a surjection of sheaves. Thus $F = U/R$. We conclude that F is an algebraic space by Spaces, Theorem 10.5. □

7. Finding opens

First we prove a lemma which is a slight improvement and generalization of Spaces, Lemma 10.2 to quotient sheaves associated to groupoids.

Lemma 7.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism. Assume*

(1) *the composition*

$$U' \times_{g,U,t} R \overset{h}{\underset{pr_1}{\longrightarrow}} R \overset{}{\underset{s}{\longrightarrow}} U$$

has an open image $W \subset U$, and
(2) *the resulting map $h : U' \times_{g,U,t} R \to W$ defines a surjection of sheaves in the fppf topology.*

Let $R' = R|_{U'}$ be the restriction of R to U'. Then the map of quotient sheaves

$$U'/R' \to U/R$$

in the fppf topology is representable, and is an open immersion.

Proof. Note that W is an R-invariant open subscheme of U. This is true because the set of points of W is the set of points of U which are equivalent in the sense of Groupoids, Lemma 3.4 to a point of $g(U') \subset U$ (the lemma applies as $j : R \to U \times_S U$ is a pre-equivalence relation by Groupoids, Lemma 13.2). Also $g : U' \to U$ factors through W. Let $R|_W$ be the restriction of R to W. Then it follows that R' is also the restriction of $R|_W$ to U'. Hence we can factor the map of sheaves of the lemma as

$$U'/R' \longrightarrow W/R|_W \longrightarrow U/R$$

By Groupoids, Lemma 20.6 we see that the first arrow is an isomorphism of sheaves. Hence it suffices to show the lemma in case g is the immersion of an R-invariant open into U.

Assume $U' \subset U$ is an R-invariant open and g is the inclusion morphism. Set $F = U/R$ and $F' = U'/R'$. By Groupoids, Lemma 20.5 or 20.6 the map $F' \to F$ is injective. Let $\xi \in F(T)$. We have to show that $T \times_{\xi,F} F'$ is representable by an open subscheme of T. There exists an fppf covering $\{f_i : T_i \to T\}$ such that $\xi|_{T_i}$ is the image via $U \to U/R$ of a morphism $a_i : T_i \to U$. Set $V_i = a_i^{-1}(U')$. We claim that $V_i \times_T T_j = T_i \times_T V_j$ as open subschemes of $T_i \times_T T_j$.

As $a_i \circ \mathrm{pr}_0$ and $a_j \circ \mathrm{pr}_1$ are morphisms $T_i \times_T T_j \to U$ which both map to the section $\xi|_{T_i \times_T T_j} \in F(T_i \times_T T_j)$ we can find an fppf covering $\{f_{ijk} : T_{ijk} \to T_i \times_T T_j\}$ and morphisms $r_{ijk} : T_{ijk} \to R$ such that

$$a_i \circ \mathrm{pr}_0 \circ f_{ijk} = s \circ r_{ijk}, \quad a_j \circ \mathrm{pr}_1 \circ f_{ijk} = t \circ r_{ijk},$$

see Groupoids, Lemma 20.4. Since U' is R-invariant we have $s^{-1}(U') = t^{-1}(U')$ and hence $f_{ijk}^{-1}(V_i \times_T T_j) = f_{ijk}^{-1}(T_i \times_T V_j)$. As $\{f_{ijk}\}$ is surjective this implies the claim above. Hence by Descent, Lemma 13.6 there exists an open subscheme $V \subset T$ such that $f_i^{-1}(V) = V_i$. We claim that V represents $T \times_{\xi,F} F'$.

As a first step, we will show that $\xi|_V$ lies in $F'(V) \subset F(V)$. Namely, the family of morphisms $\{V_i \to V\}$ is an fppf covering, and by construction we have $\xi|_{V_i} \in F'(V_i)$. Hence by the sheaf property of F' we get $\xi|_V \in F'(V)$. Finally, let $T' \to T$ be a morphism of schemes and that $\xi|_{T'} \in F'(T')$. To finish the proof we have to show that $T' \to T$ factors through V. We can find a fppf covering $\{T_j' \to T'\}_{j \in J}$ and morphisms $b_j : T_j' \to U'$ such that $\xi|_{T_j'}$ is the image via $U' \to U/R$ of b_j. Clearly, it is enough to show that the compositions $T_j' \to T$ factor through V. Hence we may assume that $\xi|_{T'}$ is the image of a morphism $b : T' \to U'$. Now, it is enough to show that $T' \times_T T_i \to T_i$ factors through V_i. Over the scheme $T' \times_T T_i$ the restriction of ξ is the image of two elements of $(U/R)(T' \times_T T_i)$, namely $a_i \circ \mathrm{pr}_1$, and $b \circ \mathrm{pr}_0$, the second of which factors through the R-invariant open U'. Hence by Groupoids, Lemma 20.4 there exists a covering $\{h_k : Z_k \to T' \times_T T_i\}$ and morphisms $r_k : Z_k \to R$ such that $a_i \circ \mathrm{pr}_1 \circ h_k = s \circ r_k$ and $b \circ \mathrm{pr}_0 \circ h_k = t \circ r_k$. As U' is an R-invariant open the fact that b has image in U' then implies that each $a_i \circ \mathrm{pr}_1 \circ h_k$ has image in U'. It follows from this that $T' \times_T T_i \to T_i$ has image in V_i by definition of V_i which concludes the proof. $\square$

8. Slicing equivalence relations

In this section we explain how to "improve" a given equivalence relation by slicing. This is not a kind of "étale slicing" that you may be used to but a much coarser kind of slicing.

Lemma 8.1. *Let S be a scheme. Let $j : R \to U \times_S U$ be an equivalence relation on schemes over S. Assume $s, t : R \to U$ are flat and locally of finite presentation. Then there exists an equivalence relation $j' : R' \to U' \times_S U'$ on schemes over S, and an isomorphism*

$$U'/R' \longrightarrow U/R$$

induced by a morphism $U' \to U$ which maps R' into R such that $s', t' : R \to U$ are flat, locally of finite presentation and locally quasi-finite.

Proof. We will prove this lemma in several steps. We will use without further mention that an equivalence relation gives rise to a groupoid scheme and that the restriction of an equivalence relation is an equivalence relation, see Groupoids, Lemmas 3.2, 13.3, and 18.3.

Step 1: We may assume that $s, t : R \to U$ are locally of finite presentation and Cohen-Macaulay morphisms. Namely, as in More on Groupoids, Lemma 8.1 let $g : U' \to U$ be the open subscheme such that $t^{-1}(U') \subset R$ is the maximal open

over which $s : R \to U$ is Cohen-Macaulay, and denote R' the restriction of R to U'. By the lemma cited above we see that

$$t^{-1}(U') \mathrel{=\!=\!=} U' \times_{g,U,t} R \xrightarrow[\mathrm{pr}_1]{\overset{h}{\longrightarrow}} R \xrightarrow{s} U$$

is surjective. Since h is flat and locally of finite presentation, we see that $\{h\}$ is a fppf covering. Hence by Groupoids, Lemma 20.6 we see that $U'/R' \to U/R$ is an isomorphism. By the construction of U' we see that s', t' are Cohen-Macaulay and locally of finite presentation.

Step 2. Assume s, t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a point of finite type. By More on Groupoids, Lemma 12.4 there exists an affine scheme U' and a morphism $g : U' \to U$ such that

(1) g is an immersion,
(2) $u \in U'$,
(3) g is locally of finite presentation,
(4) h is flat, locally of finite presentation and locally quasi-finite, and
(5) the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation and locally quasi-finite.

Here we have used the notation introduced in More on Groupoids, Situation 12.1.

Step 3. For each point $u \in U$ which is of finite type choose a $g_u : U'_u \to U$ as in Step 2 and denote R'_u the restriction of R to U'_u. Denote $h_u = s \circ \mathrm{pr}_1 : U'_u \times_{g_u,U,t} R \to U$. Set $U' = \coprod_{u \in U} U'_u$, and $g = \coprod g_u$. Let R' be the restriction of R to U' as above. We claim that the pair (U', g) works[2]. Note that

$$R' = \coprod_{u_1,u_2 \in U} (U'_{u_1} \times_{g_{u_1},U,t} R) \times_R (R \times_{s,U,g_{u_2}} U'_{u_2})$$

$$= \coprod_{u_1,u_2 \in U} (U'_{u_1} \times_{g_{u_1},U,t} R) \times_{h_{u_1},U,g_{u_2}} U'_{u_2}$$

Hence the projection $s' : R' \to U' = \coprod U'_{u_2}$ is flat, locally of finite presentation and locally quasi-finite as a base change of $\coprod h_{u_1}$. Finally, by construction the morphism $h : U' \times_{g,U,t} R \to U$ is equal to $\coprod h_u$ hence its image contains all points of finite type of U. Since each h_u is flat and locally of finite presentation we conclude that h is flat and locally of finite presentation. In particular, the image of h is open (see Morphisms, Lemma 25.10) and since the set of points of finite type is dense (see Morphisms, Lemma 16.7) we conclude that the image of h is U. This implies that $\{h\}$ is an fppf covering. By Groupoids, Lemma 20.6 this means that $U'/R' \to U/R$ is an isomorphism. This finishes the proof of the lemma. $\qquad\square$

9. Quotient by a subgroupoid

We need one more lemma before we can do our final bootstrap. Let us discuss what is going on in terms of "plain" groupoids before embarking on the scheme theoretic version.

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[2]Here we should check that U' is not too large, i.e., that it is isomorphic to an object of the category Sch_{fppf}, see Section 2. This is a purely set theoretical matter; let us use the notion of size of a scheme introduced in Sets, Section 9. Note that each U'_u has size at most the size of U and that the cardinality of the index set is at most the cardinality of $|U|$ which is bounded by the size of U. Hence U' is isomorphic to an object of Sch_{fppf} by Sets, Lemma 9.9 part (6).

Let $\mathcal{C}$ be a groupoid, see Categories, Definition 2.5. As discussed in Groupoids, Section 13 this corresponds to a quintuple $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c)$. Suppose we are given a subset $P \subset \mathrm{Arrows}$ such that $(\mathrm{Ob}, P, s|_P, t|_P, c|_P)$ is also a groupoid and such that there are no nontrivial automorphisms in P. Then we can construct the quotient groupoid $(\overline{\mathrm{Ob}}, \overline{\mathrm{Arrows}}, \overline{s}, \overline{t}, \overline{c})$ as follows:

(1) $\overline{\mathrm{Ob}} = \mathrm{Ob}/P$ is the set of P-isomorphism classes,
(2) $\overline{\mathrm{Arrows}} = P\backslash\mathrm{Arrows}/P$ is the set of arrows in $\mathcal{C}$ up to pre-composing and post-composing by arrows of P,
(3) the source and target maps $\overline{s}, \overline{t} : P\backslash\mathrm{Arrows}/P \to \mathrm{Ob}/P$ are induced by s, t,
(4) composition is defined by the rule $\overline{c}(\overline{a}, \overline{b}) = \overline{c(a, b)}$ which is well defined.

In fact, it turns out that the original groupoid $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c)$ is canonically isomorphic to the restriction (see discussion in Groupoids, Section 18) of the groupoid $(\overline{\mathrm{Ob}}, \overline{\mathrm{Arrows}}, \overline{s}, \overline{t}, \overline{c})$ via the quotient map $g : \mathrm{Ob} \to \overline{\mathrm{Ob}}$. Recall that this means that

$$\mathrm{Arrows} = \mathrm{Ob} \times_{g, \overline{\mathrm{Ob}}, \overline{t}} \overline{\mathrm{Arrows}} \times_{\overline{s}, \overline{\mathrm{Ob}}, g} \mathrm{Ob}$$

which holds as P has no nontrivial automorphisms. We omit the details.

The following lemma holds in much greater generality, but this is the version we use in the proof of the final bootstrap (after which we can more easily prove the more general versions of this lemma).

Lemma 9.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $P \to R$ be monomorphism of schemes. Assume that*

(1) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ *is a groupoid scheme,*
(2) $s|_P, t|_P : P \to U$ *are finite locally free,*
(3) $j|_P : P \to U \times_S U$ *is a monomorphism.*
(4) U *is affine, and*
(5) $j : R \to U \times_S U$ *is separated and locally quasi-finite,*

Then U/P is representable by an affine scheme $\overline{U}$, the quotient morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. Moreover, R is the restriction of a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ on $\overline{U}$ via the quotient morphism $U \to \overline{U}$.

Proof. Conditions (1), (2), (3), and (4) and Groupoids, Proposition 23.9 imply the affine scheme $\overline{U}$ representing U/P exists, the morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. The identification $P = U \times_{\overline{U}} U$ is such that $t|_P = \mathrm{pr}_0$ and $s|_P = \mathrm{pr}_1$, and such that composition is equal to $\mathrm{pr}_{02} : U \times_{\overline{U}} U \times_{\overline{U}} U \to U \times_{\overline{U}} U$. A product of finite locally free morphisms is finite locally free (see Spaces, Lemma 5.7 and Morphisms, Lemmas 48.4 and 48.3). To get $\overline{R}$ we are going to descend the scheme R via the finite locally free morphism $U \times_S U \to \overline{U} \times_S \overline{U}$. Namely, note that

$$(U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) = P \times_S P$$

by the above. Thus giving a descent datum (see Descent, Definition 34.1) for $R/U \times_S U/\overline{U} \times_S \overline{U}$ consists of an isomorphism

$$\varphi : R \times_{(U \times_S U), t \times t} (P \times_S P) \longrightarrow (P \times_S P) \times_{s \times s, (U \times_S U)} R$$

over $P \times_S P$ satisfying a cocycle condition. We define φ on T-valued points by the rule

$$\varphi : (r, (p, p')) \longmapsto ((p, p'), p^{-1} \circ r \circ p')$$

where the composition is taken in the groupoid category $(U(T), R(T), s, t, c)$. This makes sense because for $(r, (p, p'))$ to be a T-valued point of the source of φ it needs to be the case that $t(r) = t(p)$ and $s(r) = t(p')$. Note that this map is an isomorphism with inverse given by $((p, p'), r') \mapsto (p \circ r' \circ (p')^{-1}, (p, p'))$. To check the cocycle condition we have to verify that $\varphi_{02} = \varphi_{12} \circ \varphi_{01}$ as maps over

$$(U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) = (P \times_S P) \times_{s \times s, (U \times_S U), t \times t} (P \times_S P)$$

By explicit calculation we see that

$$
\begin{array}{llll}
\varphi_{02} & (r, (p_1, p_1'), (p_2, p_2')) & \mapsto & ((p_1, p_1'), (p_2, p_2'), (p_1 \circ p_2)^{-1} \circ r \circ (p_1' \circ p_2')) \\
\varphi_{01} & (r, (p_1, p_1'), (p_2, p_2')) & \mapsto & ((p_1, p_1'), p_1^{-1} \circ r \circ p_1', (p_2, p_2')) \\
\varphi_{12} & ((p_1, p_1'), r, (p_2, p_2')) & \mapsto & ((p_1, p_1'), (p_2, p_2'), p_2^{-1} \circ r \circ p_2')
\end{array}
$$

(with obvious notation) which implies what we want. As j is separated and locally quasi-finite by (5) we may apply More on Morphisms, Lemma 57.1 to get a scheme $\overline{R} \to \overline{U} \times_S \overline{U}$ and an isomorphism

$$R \to \overline{R} \times_{(\overline{U} \times_S \overline{U})} (U \times_S U)$$

which identifies the descent datum φ with the canonical descent datum on $\overline{R} \times_{(\overline{U} \times_S \overline{U})} (U \times_S U)$, see Descent, Definition 34.10.

Since $U \times_S U \to \overline{U} \times_S \overline{U}$ is finite locally free we conclude that $R \to \overline{R}$ is finite locally free as a base change. Hence $R \to \overline{R}$ is surjective as a map of sheaves on $(Sch/S)_{fppf}$. Our choice of φ implies that given T-valued points $r, r' \in R(T)$ these have the same image in $\overline{R}$ if and only if $p^{-1} \circ r \circ p'$ for some $p, p' \in P(T)$. Thus $\overline{R}$ represents the sheaf

$$T \longmapsto \overline{R(T)} = P(T) \backslash R(T) / P(T)$$

with notation as in the discussion preceding the lemma. Hence we can define the groupoid structure on $(\overline{U} = U/P, \overline{R} = P \backslash R/P)$ exactly as in the discussion of the "plain" groupoid case. It follows from this that (U, R, s, t, c) is the pullback of this groupoid structure via the morphism $U \to \overline{U}$. This concludes the proof. $\square$

10. Final bootstrap

The following result goes quite a bit beyond the earlier results.

Theorem 10.1. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Any one of the following conditions implies that F is an algebraic space:*

(1) *$F = U/R$ where (U, R, s, t, c) is a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,*

(2) *$F = U/R$ where (U, R, s, t, c) is a groupoid scheme over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,*

(3) *F is a sheaf and there exists an algebraic space U and a morphism $U \to F$ which is representable by algebraic spaces, surjective, flat and locally of finite presentation,*

(4) *F is a sheaf and there exists a scheme U and a morphism $U \to F$ which is representable by algebraic spaces or schemes, surjective, flat and locally of finite presentation,*

(5) F is a sheaf, Δ_F is representable by algebraic spaces, and there exists an algebraic space U and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation, or

(6) F is a sheaf, Δ_F is representable, and there exists a scheme U and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation.

Proof. Trivial observations: (6) is a special case of (5) and (4) is a special case of (3). We first prove that cases (5) and (3) reduce to case (1). Namely, by bootstrapping the diagonal Lemma 5.3 we see that (3) implies (5). In case (5) we set $R = U \times_F U$ which is an algebraic space by assumption. Moreover, by assumption both projections $s, t : R \to U$ are surjective, flat and locally of finite presentation. The map $j : R \to U \times_S U$ is clearly an equivalence relation. By Lemma 4.6 the map $U \to F$ is a surjection of sheaves. Thus $F = U/R$ which reduces us to case (1).

Next, we show that (1) reduces to (2). Namely, let (U, R, s, t, c) be a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. Choose a scheme U' and a surjective étale morphism $U' \to U$. Let $R' = R|_{U'}$ be the restriction of R to U'. By Groupoids in Spaces, Lemma 19.6 we see that $U/R = U'/R'$. Since $s', t' : R' \to U'$ are also flat and locally of finite presentation (see More on Groupoids in Spaces, Lemma 8.1) this reduces us to the case where U is a scheme. As j is an equivalence relation we see that j is a monomorphism. As $s : R \to U$ is locally of finite presentation we see that $j : R \to U \times_S U$ is locally of finite type, see Morphisms of Spaces, Lemma 23.6. By Morphisms of Spaces, Lemma 27.10 we see that j is locally quasi-finite and separated. Hence if U is a scheme, then R is a scheme by Morphisms of Spaces, Proposition 50.2. Thus we reduce to proving the theorem in case (2).

Assume $F = U/R$ where (U, R, s, t, c) is a groupoid scheme over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. By Lemma 8.1 we reduce to that case where s, t are flat, locally of finite presentation, and locally quasi-finite. Let $U = \bigcup_{i \in I} U_i$ be an affine open covering (with index set I of cardinality $\leq$ than the size of U to avoid set theoretic problems later – most readers can safely ignore this remark). Let $(U_i, R_i, s_i, t_i, c_i)$ be the restriction of R to U_i. It is clear that s_i, t_i are still flat, locally of finite presentation, and locally quasi-finite as R_i is the open subscheme $s^{-1}(U_i) \cap t^{-1}(U_i)$ of R and s_i, t_i are the restrictions of s, t to this open. By Lemma 7.1 (or the simpler Spaces, Lemma 10.2) the map $U_i/R_i \to U/R$ is representable by open immersions. Hence if we can show that $F_i = U_i/R_i$ is an algebraic space, then $\coprod_{i \in I} F_i$ is an algebraic space by Spaces, Lemma 8.4. As $U = \bigcup U_i$ is an open covering it is clear that $\coprod F_i \to F$ is surjective. Thus it follows that U/R is an algebraic space, by Spaces, Lemma 8.5. In this way we reduce to the case where U is affine and s, t are flat, locally of finite presentation, and locally quasi-finite and j is an equivalence.
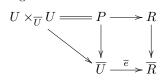
Assume (U, R, s, t, c) is a groupoid scheme over S, with U affine, such that s, t are flat, locally of finite presentation, and locally quasi-finite, and j is an equivalence relation. Choose $u \in U$. We apply More on Groupoids in Spaces, Lemma 15.13 to $u \in U, R, s, t, c$. We obtain an affine scheme U', an étale morphism $g : U' \to U$,

a point $u' \in U'$ with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ is quasi-split over u'. Note that the image $g(U')$ is open as g is étale and contains u. Hence, repeatedly applying the lemma, we can find finitely many points $u_i \in U$, $i = 1, \ldots, n$, affine schemes U'_i, étale morphisms $g_i : U'_i \to U$, points $u'_i \in U'_i$ with $g(u'_i) = u_i$ such that (a) each restriction R'_i is quasi-split over some point in U'_i and (b) $U = \bigcup_{i=1,\ldots,n} g_i(U'_i)$. Now we rerun the last part of the argument in the preceding paragraph: Using Lemma 7.1 (or the simpler Spaces, Lemma 10.2) the map $U'_i/R'_i \to U/R$ is representable by open immersions. If we can show that $F_i = U'_i/R'_i$ is an algebraic space, then $\coprod_{i \in I} F_i$ is an algebraic space by Spaces, Lemma 8.4. As $\{g_i : U'_i \to U\}$ is an étale covering it is clear that $\coprod F_i \to F$ is surjective. Thus it follows that U/R is an algebraic space, by Spaces, Lemma 8.5. In this way we reduce to the case where U is affine and s, t are flat, locally of finite presentation, and locally quasi-finite, j is an equivalence, and R is quasi-split over u for some $u \in U$.

Assume (U, R, s, t, c) is a groupoid scheme over S, with U affine, $u \in U$ such that s, t are flat, locally of finite presentation, and locally quasi-finite and $j = (t, s) :$ $R \to U \times_S U$ is an equivalence relation and R is quasi-split over u. Let $P \subset R$ be a quasi-splitting of R over u. By Lemma 9.1 we see that (U, R, s, t, c) is the restriction of a groupoid $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ by a surjective finite locally free morphism $U \to \overline{U}$ such that $P = U \times_{\overline{U}} U$. Note that s admits a factorization

$$R = U \times_{\overline{U}, \overline{t}} \overline{R} \times_{\overline{s}, \overline{U}} U \xrightarrow{\mathrm{pr}_{23}} \overline{R} \times_{\overline{s}, \overline{U}} U \xrightarrow{\mathrm{pr}_2} U$$

The map pr_2 is the base change of $\overline{s}$, and the map pr_{23} is a base change of the surjective finite locally free map $U \to \overline{U}$. Since s is flat, locally of finite presentation, and locally quasi-finite and since pr_{23} is surjective finite locally free (as a base change of such), we conclude that pr_2 is flat, locally of finite presentation, and locally quasi-finite by Descent, Lemmas 27.1 and 28.1 and Morphisms, Lemma 20.18. Since pr_2 is the base change of the morphism $\overline{s}$ by $U \to \overline{U}$ and $\{U \to \overline{U}\}$ is an fppf covering we conclude $\overline{s}$ is flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas 23.15, 23.11, and 23.24. The same goes for $\overline{t}$. Consider the commutative diagram

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since P is open in R (by definition of a quasi-splitting) we conclude that $\overline{e}$ is an open immersion by Descent, Lemma 23.16. An application of Groupoids, Lemma 20.5 shows that $U/R = \overline{U}/\overline{R}$. Hence we have reduced to the case where (U, R, s, t, c) is a groupoid scheme over S, with U affine, $u \in U$ such that s, t are flat, locally of finite presentation, and locally quasi-finite and $j = (t, s) : R \to U \times_S U$ is an equivalence relation and $e : U \to R$ is an open immersion!

But of course, if e is an open immersion and s, t are flat and locally of finite presentation then the morphisms t, s are étale. For example you can see this by applying More on Groupoids, Lemma 4.1 which shows that $\Omega_{R/U} = 0$ which in turn implies that $s, t : R \to U$ is G-unramified (see Morphisms, Lemma 35.2), which in

turn implies that s, t are étale (see Morphisms, Lemma 36.16). And if s, t are étale then finally U/R is an algebraic space by Spaces, Theorem 10.5. □

11. Applications

As a first application we obtain the following fundamental fact:

A sheaf which is fppf locally an algebraic space is an algebraic space.

This is the content of the following lemma. Note that assumption (2) is equivalent to the condition that $F|_{(Sch/S_i)_{fppf}}$ is an algebraic space, see Spaces, Lemma 16.4. Assumption (3) is a set theoretic condition which may be ignored by those not worried about set theoretic questions.

Lemma 11.1. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Let $\{S_i \to S\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Assume that*

  (1) *F is a sheaf,*
  (2) *each $F_i = h_{S_i} \times F$ is an algebraic space, and*
  (3) *$\coprod_{i \in I} F_i$ is an algebraic space (see Spaces, Lemma 8.4).*

Then F is an algebraic space.

Proof. Consider the morphism $\coprod F_i \to F$. This is the base change of $\coprod S_i \to S$ via $F \to S$. Hence it is representable, locally of finite presentation, flat and surjective by our definition of an fppf covering and Lemma 4.2. Thus Theorem 10.1 applies to show that F is an algebraic space. □

Here is a special case of Lemma 11.1 where we do not need to worry about set theoretical issues.

Lemma 11.2. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Let $\{S_i \to S\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Assume that*

  (1) *F is a sheaf,*
  (2) *each $F_i = h_{S_i} \times F$ is an algebraic space, and*
  (3) *the morphisms $F_i \to S_i$ are of finite type.*

Then F is an algebraic space.

Proof. We will use Lemma 11.1 above. To do this we will show that the assumption that F_i is of finite type over S_i to prove that the set theoretic condition in the lemma is satisfied (after perhaps refining the given covering of S a bit). We suggest the reader skip the rest of the proof.

If $S_i' \to S_i$ is a morphism of schemes then

$$h_{S_i'} \times F = h_{S_i'} \times_{h_{S_i}} h_{S_i} \times F = h_{S_i'} \times_{h_{S_i}} F_i$$

is an algebraic space of finite type over S_i', see Spaces, Lemma 7.3 and Morphisms of Spaces, Lemma 23.3. Thus we may refine the given covering. After doing this we may assume: (a) each S_i is affine, and (b) the cardinality of I is at most the cardinality of the set of points of S. (Since to cover all of S it is enough that each point is in the image of $S_i \to S$ for some i.)

Since each S_i is affine and each F_i of finite type over S_i we conclude that F_i is quasi-compact. Hence by Properties of Spaces, Lemma 6.3 we can find an affine $U_i \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U_i \to F_i$. The fact that

$F_i \to S_i$ is locally of finite type then implies that $U_i \to S_i$ is locally of finite type, and in particular $U_i \to S$ is locally of finite type. By Sets, Lemma 9.7 we conclude that $\mathrm{size}(U_i) \le \mathrm{size}(S)$. Since also $|I| \le \mathrm{size}(S)$ we conclude that $\coprod_{i \in I} U_i$ is isomorphic to an object of $(Sch/S)_{fppf}$ by Sets, Lemma 9.5 and the construction of Sch. This implies that $\coprod F_i$ is an algebraic space by Spaces, Lemma 8.4 and we win. $\square$

As a second application we obtain

> Any fppf descent datum for algebraic spaces is effective.

This holds modulo set theoretical difficulties; as an example result we offer the following lemma.

Lemma 11.3. *Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering of algebraic spaces over S.*

  (1) *If I is countable[3], then any descent datum for algebraic spaces relative to $\{X_i \to X\}$ is effective.*
  (2) *Any descent datum (Y_i, φ_{ij}) relative to $\{X_i \to X\}_{i \in I}$ (Descent on Spaces, Definition 22.3) with $Y_i \to X_i$ of finite type is effective.*

Proof. Proof of (1). By Descent on Spaces, Lemma 23.1 this translates into the statement that an fppf sheaf F endowed with a map $F \to X$ is an algebraic space provided that each $F \times_X X_i$ is an algebraic space. The restriction on the cardinality of I implies that coproducts of algebraic spaces indexed by I are algebraic spaces, see Spaces, Lemma 8.4 and Sets, Lemma 9.9. The morphism

$$\coprod F \times_X X_i \longrightarrow F$$

is representable by algebraic spaces (as the base change of $\coprod X_i \to X$, see Lemma 3.3), and surjective, flat, and locally of finite presentation (as the base change of $\coprod X_i \to X$, see Lemma 4.2). Hence part (1) follows from Theorem 10.1.

Proof of (2). First we apply Descent on Spaces, Lemma 23.1 to obtain an fppf sheaf F endowed with a map $F \to X$ such that $F \times_X X_i = Y_i$ for all $i \in I$. Our goal is to show that F is an algebraic space. Choose a scheme U and a surjective étale morphism $U \to X$. Then $F' = U \times_X F \to F$ is representable, surjective, and étale as the base change of $U \to X$. By Theorem 10.1 it suffices to show that $F' = U \times_X F$ is an algebraic space. We may choose an fppf covering $\{U_j \to U\}_{j \in J}$ where U_j is a scheme refining the fppf covering $\{X_i \times_X U \to U\}_{i \in I}$, see Topologies on Spaces, Lemma 7.4. Thus we get a map $a : J \to I$ and for each j a morphism $U_j \to X_{a(j)}$ over X. Then we see that $U_j \times_U F' = U_j \times_{X_{a(j)}} Y_{a(j)}$ is of finite type over U_j. Hence F' is an algebraic space by Lemma 11.2. $\square$

Here is a different type of application.

Lemma 11.4. *Let S be a scheme. Let $a : F \to G$ and $b : G \to H$ be transformations of functors $(Sch/S)_{fppf}^{opp} \to Sets$. Assume*

  (1) *F, G, H are sheaves,*

---

[3] The restriction on countablility can be ignored by those who do not care about set theoretical issues. We can allow larger index sets here if we can bound the size of the algebraic spaces which we are descending. See for example Lemma 11.2.

(2) $a : F \to G$ is representable by algebraic spaces, flat, locally of finite presentation, and surjective, and

(3) $b \circ a : F \to H$ is representable by algebraic spaces.

Then b is representable by algebraic spaces.

Proof. Let U be a scheme over S and let $\xi \in H(U)$. We have to show that $U \times_{\xi,H} G$ is an algebraic space. On the other hand, we know that $U \times_{\xi,H} F$ is an algebraic space and that $U \times_{\xi,H} F \to U \times_{\xi,H} G$ is representable by algebraic spaces, flat, locally of finite presentation, and surjective as a base change of the morphism a (see Lemma 4.2). Thus the result follows from Theorem 10.1. $\square$

Lemma 11.5. *Assume $B \to S$ and (U, R, s, t, c) are as in Groupoids in Spaces, Definition 20.1 (1). For any scheme T over S and objects x, y of $[U/R]$ over T the sheaf $Isom(x, y)$ on $(Sch/T)_{fppf}$ is an algebraic space.*

Proof. By Groupoids in Spaces, Lemma 22.3 there exists an fppf covering $\{T_i \to T\}_{i \in I}$ such that $Isom(x, y)|_{(Sch/T_i)_{fppf}}$ is an algebraic space for each i. By Spaces, Lemma 16.4 this means that each $F_i = h_{S_i} \times Isom(x, y)$ is an algebraic space. Thus to prove the lemma we only have to verify the set theoretic condition that $\coprod F_i$ is an algebraic space of Lemma 11.1 above to conclude. To do this we use Spaces, Lemma 8.4 which requires showing that I and the F_i are not "too large". We suggest the reader skip the rest of the proof.

Choose $U' \in \mathrm{Ob}(Sch/S)_{fppf}$ and a surjective étale morphism $U' \to U$. Let R' be the restriction of R to U'. Since $[U/R] = [U'/R']$ we may, after replacing U by U', assume that U is a scheme. (This step is here so that the fibre products below are over a scheme.)

Note that if we refine the covering $\{T_i \to T\}$ then it remains true that each F_i is an algebraic space. Hence we may assume that each T_i is affine. Since $T_i \to T$ is locally of finite presentation, this then implies that $\mathrm{size}(T_i) \leq \mathrm{size}(T)$, see Sets, Lemma 9.7. We may also assume that the cardinality of the index set I is at most the cardinality of the set of points of T since to get a covering it suffices to check that each point of T is in the image. Hence $|I| \leq \mathrm{size}(T)$. Choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $W \to R$. Note that in the proof of Groupoids in Spaces, Lemma 22.3 we showed that F_i is representable by $T_i \times_{(y_i, x_i), U \times_B U} R$ for some $x_i, y_i : T_i \to U$. Hence now we see that $V_i = T_i \times_{(y_i, x_i), U \times_B U} W$ is a scheme which comes with an étale surjection $V_i \to F_i$. By Sets, Lemma 9.6 we see that

$$\mathrm{size}(V_i) \leq \max\{\mathrm{size}(T_i), \mathrm{size}(W)\} \leq \max\{\mathrm{size}(T), \mathrm{size}(W)\}$$

Hence, by Sets, Lemma 9.5 we conclude that

$$\mathrm{size}(\coprod\nolimits_{i \in I} V_i) \leq \max\{|I|, \mathrm{size}(T), \mathrm{size}(W)\}.$$

Hence we conclude by our construction of Sch that $\coprod_{i \in I} V_i$ is isomorphic to an object V of $(Sch/S)_{fppf}$. This verifies the hypothesis of Spaces, Lemma 8.4 and we win. $\square$

Lemma 11.6. *Let S be a scheme. Consider an algebraic space F of the form $F = U/R$ where (U, R, s, t, c) is a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. Then $U \to F$ is surjective, flat, and locally of finite presentation.*

Proof. This is almost but not quite a triviality. Namely, by Groupoids in Spaces, Lemma 19.5 and the fact that j is a monomorphism we see that $R = U \times_F U$. Choose a scheme W and a surjective étale morphism $W \to F$. As $U \to F$ is a surjection of sheaves we can find an fppf covering $\{W_i \to W\}$ and maps $W_i \to U$ lifting the morphisms $W_i \to F$. Then we see that

$$W_i \times_F U = W_i \times_U U \times_F U = W_i \times_{U,t} R$$

and the projection $W_i \times_F U \to W_i$ is the base change of $t : R \to U$ hence flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 30.4 and 28.3. Hence by Descent on Spaces, Lemmas 11.13 and 11.10 we see that $U \to F$ is flat and locally of finite presentation. It is surjective by Spaces, Remark 5.2. $\square$

Lemma 11.7. *Let S be a scheme. Let $X \to B$ be a morphism of algebraic spaces over S. Let G be a group algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. If*

    (1) *a is a free action, and*
    (2) *$G \to B$ is flat and locally of finite presentation,*

then X/G (see Groupoids in Spaces, Definition 19.1) is an algebraic space, the morphism $X \to X/G$ is surjective, flat, and locally of finite presentation, and X is an fppf G-torsor over X/G.

Proof. The fact that X/G is an algebraic space is immediate from Theorem 10.1 and the definitions. Namely, $X/G = X/R$ where $R = G \times_B X$. The morphisms $s, t : G \times_B X \to X$ are flat and locally of finite presentation (clear for s as a base change of $G \to B$ and by symmetry using the inverse it follows for t) and the morphism $j : G \times_B X \to X \times_B X$ is a monomorphism by Groupoids in Spaces, Lemma 8.3 as the action is free. The morphism $X \to X/G$ is surjective, flat, and locally of finite presentation by Lemma 11.6. To see that $X \to X/G$ is an fppf G-torsor (Groupoids in Spaces, Definition 9.3) we have to show that $G \times_S X \to X \times_{X/G} X$ is an isomorphism and that $X \to X/G$ fppf locally has sections. The second part is clear from the properties of $X \to X/G$ already shown. The map $G \times_S X \to X \times_{X/G} X$ is injective (as a map of fppf sheaves) as the action is free. Finally, the map is also surjective as a map of sheaves by Groupoids in Spaces, Lemma 19.5. This finishes the proof. $\square$

Lemma 11.8. *Let $\{S_i \to S\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Let G be a group algebraic space over S, and denote $G_i = G_{S_i}$ the base changes. Suppose given*

    (1) *for each $i \in I$ an fppf G_i-torsor X_i over S_i, and*
    (2) *for each $i, j \in I$ a $G_{S_i \times_S S_j}$-equivariant isomorphism $\varphi_{ij} : X_i \times_S S_j \to S_i \times_S X_j$ satisfying the cocycle condition over every $S_i \times_S S_j \times_S S_j$.*

Then there exists an fppf G-torsor X over S whose base change to S_i is isomorphic to X_i such that we recover the descent datum φ_{ij}.

Proof. We may think of X_i as a sheaf on $(Sch/S_i)_{fppf}$, see Spaces, Section 16. By Sites, Section 26 the descent datum (X_i, φ_{ij}) is effective in the sense that there exists a unique sheaf X on $(Sch/S)_{fppf}$ which recovers the algebraic spaces X_i after restricting back to $(Sch/S_i)_{fppf}$. Hence we see that $X_i = h_{S_i} \times X$. By Lemma 11.1 we see that X is an algebraic space, modulo verifying that $\coprod X_i$ is an algebraic space which we do at the end of the proof. By the equivalence of categories in Sites, Lemma 26.5 the action maps $G_i \times_{S_i} X_i \to X_i$ glue to give a map $a : G \times_S X \to X$.

Now we have to show that a is an action and that X is a pseudo-torsor, and fppf locally trivial (see Groupoids in Spaces, Definition 9.3). These may be checked fppf locally, and hence follow from the corresponding properties of the actions $G_i \times_{S_i} X_i \to X_i$. Hence the lemma is true.

We suggest the reader skip the rest of the proof, which is purely set theoretical. Pick coverings $\{S_{ij} \to S_j\}_{j \in J_i}$ of $(Sch/S)_{fppf}$ which trivialize the G_i torsors X_i (possible by assumption, and Topologies, Lemma 7.7 part (1)). Then $\{S_{ij} \to S\}_{i \in I, j \in J_i}$ is a covering of $(Sch/S)_{fppf}$ and hence we may assume that each X_i is the trivial torsor! Of course we may also refine the covering further, hence we may assume that each S_i is affine and that the index set I has cardinality bounded by the cardinality of the set of points of S. Choose $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to G$. Then we see that $U_i = U \times_S S_i$ comes with an étale surjective morphism to $X_i \cong G_i$. By Sets, Lemma 9.6 we see $\mathrm{size}(U_i) \le \max\{\mathrm{size}(U), \mathrm{size}(S_i)\}$. By Sets, Lemma 9.7 we have $\mathrm{size}(S_i) \le \mathrm{size}(S)$. Hence we see that $\mathrm{size}(U_i) \le \max\{\mathrm{size}(U), \mathrm{size}(S)\}$ for all $i \in I$. Together with the bound on $|I|$ we found above we conclude from Sets, Lemma 9.5 that $\mathrm{size}(\coprod U_i) \le \max\{\mathrm{size}(U), \mathrm{size}(S)\}$. Hence Spaces, Lemma 8.4 applies to show that $\coprod X_i$ is an algebraic space which is what we had to prove. $\square$

12. Algebraic spaces in the étale topology

Let S be a scheme. Instead of working with sheaves over the big fppf site $(Sch/S)_{fppf}$ we could work with sheaves over the big étale site $(Sch/S)_{étale}$. All of the material in Algebraic Spaces, Sections 3 and 5 makes sense for sheaves over $(Sch/S)_{étale}$. Thus we get a second notion of algebraic spaces by working in the étale topology. This notion is (a priori) weaker then the notion introduced in Algebraic Spaces, Definition 6.1 since a sheaf in the fppf topology is certainly a sheaf in the étale topology. However, the notions are equivalent as is shown by the following lemma.

Lemma 12.1. *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 11.1). Let S be an object of Sch_α. Let*

$$F : (Sch_\alpha/S)^{opp} \longrightarrow Sets$$

be a presheaf with the following properties:
  (1) *F is a sheaf for the étale topology,*
  (2) *the diagonal $\Delta : F \to F \times F$ is representable, and*
  (3) *there exists $U \in \mathrm{Ob}(Sch_\alpha/S)$ and $U \to F$ which is surjective and étale.*
Then F is an algebraic space in the sense of Algebraic Spaces, Definition 6.1.

Proof. Note that properties (2) and (3) of the lemma and the corresponding properties (2) and (3) of Algebraic Spaces, Definition 6.1 are independent of the topology. This is true because these properties involve only the notion of a fibre product of presheaves, maps of presheaves, the notion of a representable transformation of functors, and what it means for such a transformation to be surjective and étale. Thus all we have to prove is that an étale sheaf F with properties (2) and (3) is also an fppf sheaf.

To do this, let $R = U \times_F U$. By (2) the presheaf R is representable by a scheme and by (3) the projections $R \to U$ are étale. Thus $j : R \to U \times_S U$ is an étale equivalence relation. Moreover $U \to F$ identifies F as the quotient of U by R for the étale topology: (a) if $T \to F$ is a morphism, then $\{T \times_F U \to T\}$ is an étale covering, hence $U \to F$ is a surjection of sheaves for the étale topology, (b)

if $a, b : T \to U$ map to the same section of F, then $(a, b) : T \to R$ hence a and b have the same image in the quotient of U by R for the étale topology. Next, let U/R denote the quotient sheaf in the fppf topology which is an algebraic space by Spaces, Theorem 10.5. Thus we have morphisms (transformations of functors)

$$U \to F \to U/R.$$

By the aforementioned Spaces, Theorem 10.5 the composition is representable, surjective, and étale. Hence for any scheme T and morphism $T \to U/R$ the fibre product $V = T \times_{U/R} U$ is a scheme surjective and étale over T. In other words, $\{V \to U\}$ is an étale covering. This proves that $U \to U/R$ is surjective as a map of sheaves in the étale topology. It follows that $F \to U/R$ is surjective as a map of sheaves in the étale topology. On the other hand, the map $F \to U/R$ is injective (as a map of presheaves) since $R = U \times_{U/R} U$ again by Spaces, Theorem 10.5. It follows that $F \to U/R$ is an isomorphism of étale sheaves, see Sites, Lemma 11.2 which concludes the proof. $\qquad\square$

There is also an analogue of Spaces, Lemma 11.1.

Lemma 12.2. *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 11.1). Let S be an object of Sch_α. Let*

$$F : (Sch_\alpha/S)^{opp} \longrightarrow Sets$$

be a presheaf with the following properties:

(1) *F is a sheaf for the étale topology,*
(2) *there exists an algebraic space U over S and a map $U \to F$ which is representable by algebraic spaces, surjective, and étale.*

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 6.1.

Proof. Set $R = U \times_F U$. This is an algebraic space as $U \to F$ is assumed representable by algebraic spaces. The projections $s, t : R \to U$ are étale morphisms of algebraic spaces as $U \to F$ is assumed étale. The map $j = (t, s) : R \to U \times_S U$ is a monomorphism and an equivalence relation as $R = U \times_F U$. By Theorem 10.1 the fppf quotient sheaf $F' = U/R$ is an algebraic space. The morphism $U \to F'$ is surjective, flat, and locally of finite presentation by Lemma 11.6. The map $R \to U \times_{F'} U$ is surjective as a map of fppf sheaves by Groupoids in Spaces, Lemma 19.5 and since j is a monomorphism it is an isomorphism. Hence the base change of $U \to F'$ by $U \to F'$ is étale, and we conclude that $U \to F'$ is étale by Descent on Spaces, Lemma 11.28. Thus $U \to F'$ is surjective as a map of étale sheaves. This means that F' is equal to the quotient sheaf U/R in the étale topology (small check omitted). Hence we obtain a canonical factorization $U \to F' \to F$ and $F' \to F$ is an injective map of sheaves. On the other hand, $U \to F$ is surjective as a map of étale sheaves and hence so is $F' \to F$. This means that $F' = F$ and the proof is complete. $\qquad\square$

In fact, it suffices to have a smooth cover by a scheme and it suffices to assume the diagonal is representable by algebraic spaces.

Lemma 12.3. *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 11.1). Let S be an object of Sch_α.*

$$F : (Sch_\alpha/S)^{opp} \longrightarrow Sets$$

be a presheaf with the following properties:

(1) *F is a sheaf for the étale topology,*
(2) *the diagonal $\Delta : F \to F \times F$ is representable by algebraic spaces, and*
(3) *there exists $U \in \mathrm{Ob}(Sch_\alpha/S)$ and $U \to F$ which is surjective and smooth.*

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 6.1.

Proof. The proof mirrors the proof of Lemma 12.1. Let $R = U \times_F U$. By (2) the presheaf R is an algebraic space and by (3) the projections $R \to U$ are smooth and surjective. Denote (U, R, s, t, c) the groupoid associated to the equivalence relation $j : R \to U \times_S U$ (see Groupoids in Spaces, Lemma 11.3). By Theorem 10.1 we see that $X = U/R$ (quotient in the fppf-topology) is an algebraic space. Using that the smooth topology and the étale topology have the same sheaves (by More on Morphisms, Lemma 38.7) we see the map $U \to F$ identifies F as the quotient of U by R for the smooth topology (details omitted). Thus we have morphisms (transformations of functors)

$$U \to F \to X.$$

By Lemma 11.6 we see that $U \to X$ is surjective, flat and locally of finite presentation. By Groupoids in Spaces, Lemma 19.5 (and the fact that j is a monomorphism) we have $R = U \times_X U$. By Descent on Spaces, Lemma 11.26 we conclude that $U \to X$ is smooth and surjective (as the projections $R \to U$ are smooth and surjective and $\{U \to X\}$ is an fppf covering). Hence for any scheme T and morphism $T \to X$ the fibre product $T \times_X U$ is an algebraic space surjective and smooth over T. Choose a scheme V and a surjective étale morphism $V \to T \times_X U$. Then $\{V \to T\}$ is a smooth covering such that $V \to T \to X$ lifts to a morphism $V \to U$. This proves that $U \to X$ is surjective as a map of sheaves in the smooth topology. It follows that $F \to X$ is surjective as a map of sheaves in the smooth topology. On the other hand, the map $F \to X$ is injective (as a map of presheaves) since $R = U \times_X U$. It follows that $F \to X$ is an isomorphism of smooth (= étale) sheaves, see Sites, Lemma 11.2 which concludes the proof. $\square$

Finally, here is the analogue of Spaces, Lemma 11.1 with a smooth morphism covering the space.

Lemma 12.4. *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 11.1). Let S be an object of Sch_α. Let*

$$F : (Sch_\alpha/S)^{opp} \longrightarrow Sets$$

be a presheaf with the following properties:

(1) *F is a sheaf for the étale topology,*
(2) *there exists an algebraic space U over S and a map $U \to F$ which is representable by algebraic spaces, surjective, and smooth.*

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 6.1.

Proof. The proof is identical to the proof of Lemma 12.2. Set $R = U \times_F U$. This is an algebraic space as $U \to F$ is assumed representable by algebraic spaces. The projections $s, t : R \to U$ are smooth morphisms of algebraic spaces as $U \to F$ is assumed smooth. The map $j = (t, s) : R \to U \times_S U$ is a monomorphism and an equivalence relation as $R = U \times_F U$. By Theorem 10.1 the fppf quotient sheaf $F' = U/R$ is an algebraic space. The morphism $U \to F'$ is surjective, flat, and locally of finite presentation by Lemma 11.6. The map $R \to U \times_{F'} U$ is surjective as a map of fppf sheaves by Groupoids in Spaces, Lemma 19.5 and since j is a

monomorphism it is an isomorphism. Hence the base change of $U \to F'$ by $U \to F'$ is smooth, and we conclude that $U \to F'$ is smooth by Descent on Spaces, Lemma 11.26. Thus $U \to F'$ is surjective as a map of étale sheaves (as the smooth topology is equal to the étale topology by More on Morphisms, Lemma 38.7). This means that F' is equal to the quotient sheaf U/R in the étale topology (small check omitted). Hence we obtain a canonical factorization $U \to F' \to F$ and $F' \to F$ is an injective map of sheaves. On the other hand, $U \to F$ is surjective as a map of étale sheaves (as the smooth topology is the same as the étale topology) and hence so is $F' \to F$. This means that $F' = F$ and the proof is complete. $\qquad\square$

13. Other chapters

Preliminaries

- (1) Introduction
- (2) Conventions
- (3) Set Theory
- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces
- (7) Sites and Sheaves
- (8) Stacks
- (9) Fields
- (10) Commutative Algebra
- (11) Brauer Groups
- (12) Homological Algebra
- (13) Derived Categories
- (14) Simplicial Methods
- (15) More on Algebra
- (16) Smoothing Ring Maps
- (17) Sheaves of Modules
- (18) Modules on Sites
- (19) Injectives
- (20) Cohomology of Sheaves
- (21) Cohomology on Sites
- (22) Differential Graded Algebra
- (23) Divided Power Algebra
- (24) Differential Graded Sheaves
- (25) Hypercoverings

Schemes

- (26) Schemes
- (27) Constructions of Schemes
- (28) Properties of Schemes
- (29) Morphisms of Schemes
- (30) Cohomology of Schemes
- (31) Divisors
- (32) Limits of Schemes
- (33) Varieties
- (34) Topologies on Schemes
- (35) Descent
- (36) Derived Categories of Schemes
- (37) More on Morphisms
- (38) More on Flatness
- (39) Groupoid Schemes
- (40) More on Groupoid Schemes
- (41) Étale Morphisms of Schemes

Topics in Scheme Theory

- (42) Chow Homology
- (43) Intersection Theory
- (44) Picard Schemes of Curves
- (45) Weil Cohomology Theories
- (46) Adequate Modules
- (47) Dualizing Complexes
- (48) Duality for Schemes
- (49) Discriminants and Differents
- (50) de Rham Cohomology
- (51) Local Cohomology
- (52) Algebraic and Formal Geometry
- (53) Algebraic Curves
- (54) Resolution of Surfaces
- (55) Semistable Reduction
- (56) Functors and Morphisms
- (57) Derived Categories of Varieties
- (58) Fundamental Groups of Schemes
- (59) Étale Cohomology
- (60) Crystalline Cohomology
- (61) Pro-étale Cohomology
- (62) Relative Cycles
- (63) More Étale Cohomology
- (64) The Trace Formula

Algebraic Spaces

- (65) Algebraic Spaces
- (66) Properties of Algebraic Spaces
- (67) Morphisms of Algebraic Spaces
- (68) Decent Algebraic Spaces

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