RESOLUTION OF SURFACES

Contents

1. Introduction

This chapter discusses resolution of singularities of surfaces following Lipman [Lip78] and mostly following the exposition of Artin in [Art86]. The main result (Theorem 14.5) tells us that a Noetherian 2-dimensional scheme Y has a resolution of singularities when it has a finite normalization $Y^\nu \to Y$ with finitely many singular points $y_i \in Y^\nu$ and for each i the completion $\mathcal{O}^\wedge_{Y^\nu, y_i}$ is normal.

To be sure, if Y is a 2-dimensional scheme of finite type over a quasi-excellent base ring R (for example a field or a Dedekind domain with fraction field of characteristic 0 such as $\mathbf{Z}$) then the normalization of Y is finite, has finitely many singular points, and the completions of the local rings are normal. See the discussion in More on Algebra, Sections 47, 50, and 52 and More on Algebra, Lemma 42.2. Thus such a Y has a resolution of singularities.

A rough outline of the proof is as follows. Let A be a Noetherian local domain of dimension 2. The steps of the proof are as follows

- N replace A by its normalization,
- V prove Grauert-Riemenschneider,

B show there is a maximum g of the lengths of $H^1(X, \mathcal{O}_X)$ over all normal modifications $X \to \text{Spec}(A)$ and reduce to the case $g = 0$,
R we say A defines a rational singularity if $g = 0$ and in this case after a finite number of blowups we may assume A is Gorenstein and $g = 0$,
D we say A defines a rational double point if $g = 0$ and A is Gorenstein and in this case we explicitly resolve singularities.

Each of these steps needs assumptions on the ring A. We will discuss each of these in turn.

Ad N: Here we need to assume that A has a finite normalization (this is not automatic). Throughout most of the chapter we will assume that our scheme is Nagata if we need to know some normalization is finite. However, being Nagata is a slightly stronger condition than is given to us in the statement of the theorem. A solution to this (slight) problem would have been to use that our ring A is formally unramified (i.e., its completion is reduced) and to use Lemma 11.5. However, the way our proof works, it turns out it is easier to use Lemma 11.6 to lift finiteness of the normalization over the completion to finiteness of the normalization over A.

Ad V: This is Proposition 7.8 and it roughly states that for a normal modification $f : X \to \text{Spec}(A)$ one has $R^1 f_* \omega_X = 0$ where ω_X is the dualizing module of X/A (Remark 7.7). In fact, by duality the result is equivalent to a statement (Lemma 7.6) about the object $Rf_*\mathcal{O}_X$ in the *derived category* $D(A)$. Having said this, the proof uses the standard fact that components of the special fibre have positive conormal sheaves (Lemma 7.4).

Ad B: This is in some sense the most subtle part of the proof. In the end we only need to use the output of this step when A is a complete Noetherian local ring, although the writeup is a bit more general. The terminology is set in Definition 8.3. If g (as defined above) is bounded, then a straightforward argument shows that we can find a normal modification $X \to \text{Spec}(A)$ such that all singular points of X are rational singularities, see Lemma 8.5. We show that given a finite extension $A \subset B$, then g is bounded for B if it is bounded for A in the following two cases: (1) if the fraction field extension is separable, see Lemma 8.5 and (2) if the fraction field extension has degree p, the characteristic is p, and A is regular and complete, see Lemma 8.10.

Ad R: Here we reduce the case $g = 0$ to the Gorenstein case. A marvellous fact, which makes everything work, is that the blowing up of a rational surface singularity is normal, see Lemma 9.4.

Ad D: The resolution of rational double points proceeds more or less by hand, see Section 12. A rational double point is a hypersurface singularity (this is true but we don't prove it as we don't need it). The local equation looks like

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2 = \sum a_{ijk}x_ix_jx_k$$

Using that the quadratic part cannot be zero because the multiplicity is 2 and remains 2 after any blowup and the fact that every blowup is normal one quickly achieves a resolution. One twist is that we do not have an invariant which decreases every blowup, but we rely on the material on formal arcs from Section 10 to demonstrate that the process stops.

To put everything together some additional work has to be done. The main kink is that we want to lift a resolution of the completion $A^\wedge$ to a resolution of $\text{Spec}(A)$. In order to do this we first show that if a resolution exists, then there is a resolution by normalized blowups (Lemma 14.3). A sequence of normalized blowups can be lifted from the completion by Lemma 11.7. We then use this even in the proof of resolution of complete local rings A because our strategy works by induction on the degree of a finite inclusion $A_0 \subset A$ with A_0 regular, see Lemma 14.4. With a stronger result in B (such as is proved in Lipman's paper) this step could be avoided.

2. A trace map in positive characteristic

Some of the results in this section can be deduced from the much more general discussion on traces on differential forms in de Rham Cohomology, Section 19. See Remark 2.3 for a discussion.

We fix a prime number p. Let R be an $\mathbf{F}_p$-algebra. Given an $a \in R$ set $S = R[x]/(x^p - a)$. Define an R-linear map

$$\text{Tr}_x : \Omega_{S/R} \longrightarrow \Omega_R$$

by the rule

$$x^i \text{d}x \longmapsto \begin{cases} 0 & \text{if} \quad 0 \leq i \leq p-2, \\ \text{d}a & \text{if} \quad i = p-1 \end{cases}$$

This makes sense as $\Omega_{S/R}$ is a free R-module with basis $x^i\text{d}x$, $0 \leq i \leq p-1$. The following lemma implies that the trace map is well defined, i.e., independent of the choice of the coordinate x.

Lemma 2.1. *Let $\varphi : R[x]/(x^p - a) \to R[y]/(y^p - b)$ be an R-algebra homomorphism. Then $Tr_x = Tr_y \circ \varphi$.*

Proof. Say $\varphi(x) = \lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1}$ with $\lambda_i \in R$. The condition that mapping x to $\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1}$ induces an R-algebra homomorphism $R[x]/(x^p - a) \to R[y]/(y^p - b)$ is equivalent to the condition that

$$a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$$

in the ring R. Consider the polynomial ring

$$R_{univ} = \mathbf{F}_p[b, \lambda_0, \ldots, \lambda_{p-1}]$$

with the element $a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$ Consider the universal algebra map $\varphi_{univ} : R_{univ}[x]/(x^p - a) \to R_{univ}[y]/(y^p - b)$ given by mapping x to $\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1}$. We obtain a canonical map

$$R_{univ} \longrightarrow R$$

sending b, λ_i to b, λ_i. By construction we get a commutative diagram

$$\begin{tikzcd} R_{univ}[x]/(x^p - a) \arrow[r] \arrow[d, "\varphi_{univ}"'] & R[x]/(x^p - a) \arrow[d, "\varphi"] \\ R_{univ}[y]/(y^p - b) \arrow[r] & R[y]/(y^p - b) \end{tikzcd}$$

and the horizontal arrows are compatible with the trace maps. Hence it suffices to prove the lemma for the map φ_{univ}. Thus we may assume $R = \mathbf{F}_p[b, \lambda_0, \ldots, \lambda_{p-1}]$

is a polynomial ring. We will check the lemma holds in this case by evaluating $\mathrm{Tr}_y(\varphi(x)^i \mathrm{d}\varphi(x))$ for $i = 0, \ldots, p-1$.

The case $0 \leq i \leq p-2$. Expand

$$(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^i (\lambda_1 + 2\lambda_2 y + \ldots + (p-1)\lambda_{p-1} y^{p-2})$$

in the ring $R[y]/(y^p - b)$. We have to show that the coefficient of y^{p-1} is zero. For this it suffices to show that the expression above as a polynomial in y has vanishing coefficients in front of the powers y^{pk-1}. Then we write our polynomial as

$$\frac{\mathrm{d}}{(i+1)\mathrm{d}y}(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^{i+1}$$

and indeed the coefficients of y^{kp-1} are all zero.

The case $i = p-1$. Expand

$$(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^{p-1} (\lambda_1 + 2\lambda_2 y + \ldots + (p-1)\lambda_{p-1} y^{p-2})$$

in the ring $R[y]/(y^p - b)$. To finish the proof we have to show that the coefficient of y^{p-1} times $\mathrm{d}b$ is $\mathrm{d}a$. Here we use that R is S/pS where $S = \mathbf{Z}[b, \lambda_0, \ldots, \lambda_{p-1}]$. Then the above, as a polynomial in y, is equal to

$$\frac{\mathrm{d}}{p\mathrm{d}y}(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^p$$

Since $\frac{\mathrm{d}}{\mathrm{d}y}(y^{pk}) = pky^{pk-1}$ it suffices to understand the coefficients of y^{pk} in the polynomial $(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^p$ modulo p. The sum of these terms gives

$$\lambda_0^p + \lambda_1^p y^p + \ldots + \lambda_{p-1}^p y^{p(p-1)} \bmod p$$

Whence we see that we obtain after applying the operator $\frac{\mathrm{d}}{p\mathrm{d}y}$ and after reducing modulo $y^p - b$ the value

$$\lambda_1^p + 2\lambda_2^p b + \ldots + (p-1)\lambda_{p-1} b^{p-2}$$

for the coefficient of y^{p-1} we wanted to compute. Now because $a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$ in R we obtain that

$$\mathrm{d}a = (\lambda_1^p + 2\lambda_2^p b + \ldots + (p-1)\lambda_{p-1}^p b^{p-2})\mathrm{d}b$$

in R. This proves that the coefficient of y^{p-1} is as desired. $\square$

Lemma 2.2. *Let $\mathbf{F}_p \subset \Lambda \subset R \subset S$ be ring extensions and assume that S is isomorphic to $R[x]/(x^p - a)$ for some $a \in R$. Then there are canonical R-linear maps*

$$Tr : \Omega^{t+1}_{S/\Lambda} \longrightarrow \Omega^{t+1}_{R/\Lambda}$$

for $t \geq 0$ such that

$$\eta_1 \wedge \ldots \wedge \eta_t \wedge x^i \, dx \longmapsto \begin{cases} 0 & \text{if} \quad 0 \leq i \leq p-2, \\ \eta_1 \wedge \ldots \wedge \eta_t \wedge da & \text{if} \quad i = p-1 \end{cases}$$

for $\eta_i \in \Omega_{R/\Lambda}$ and such that Tr annihilates the image of $S \otimes_R \Omega^{t+1}_{R/\Lambda} \to \Omega^{t+1}_{S/\Lambda}$.

Proof. For $t = 0$ we use the composition

$$\Omega_{S/\Lambda} \to \Omega_{S/R} \to \Omega_R \to \Omega_{R/\Lambda}$$

where the second map is Lemma 2.1. There is an exact sequence

$$H_1(L_{S/R}) \xrightarrow{\delta} \Omega_{R/\Lambda} \otimes_R S \to \Omega_{S/\Lambda} \to \Omega_{S/R} \to 0$$

(Algebra, Lemma 134.4). The module $\Omega_{S/R}$ is free over S with basis $\mathrm{d}x$ and the module $H_1(L_{S/R})$ is free over S with basis $x^p - a$ which δ maps to $-\mathrm{d}a \otimes 1$ in $\Omega_{R/\Lambda} \otimes_R S$. In particular, if we set

$$M = \mathrm{Coker}(R \to \Omega_{R/\Lambda}, 1 \mapsto -\mathrm{d}a)$$

then we see that $\mathrm{Coker}(\delta) = M \otimes_R S$. We obtain a canonical map

$$\Omega^{t+1}_{S/\Lambda} \to \wedge^t_S(\mathrm{Coker}(\delta)) \otimes_S \Omega_{S/R} = \wedge^t_R(M) \otimes_R \Omega_{S/R}$$

Now, since the image of the map $\mathrm{Tr} : \Omega_{S/R} \to \Omega_{R/\Lambda}$ of Lemma 2.1 is contained in $R\mathrm{d}a$ we see that wedging with an element in the image annihilates $\mathrm{d}a$. Hence there is a canonical map

$$\wedge^t_R(M) \otimes_R \Omega_{S/R} \to \Omega^{t+1}_{R/\Lambda}$$

mapping $\overline{\eta}_1 \wedge \ldots \wedge \overline{\eta}_t \wedge \omega$ to $\eta_1 \wedge \ldots \wedge \eta_t \wedge \mathrm{Tr}(\omega)$. □

Remark 2.3. Let $\mathbf{F}_p \subset \Lambda \subset R \subset S$ and Tr be as in Lemma 2.2. By de Rham Cohomology, Proposition 19.3 there is a canonical map of complexes

$$\Theta_{S/R} : \Omega^\bullet_{S/\Lambda} \longrightarrow \Omega^\bullet_{R/\Lambda}$$

The computation in de Rham Cohomology, Example 19.4 shows that $\Theta_{S/R}(x^i\mathrm{d}x) = \mathrm{Tr}_x(x^i\mathrm{d}x)$ for all i. Since $\mathrm{Trace}_{S/R} = \Theta^0_{S/R}$ is identically zero and since

$$\Theta_{S/R}(a \wedge b) = a \wedge \Theta_{S/R}(b)$$

for $a \in \Omega^i_{R/\Lambda}$ and $b \in \Omega^j_{S/\Lambda}$ it follows that $\mathrm{Tr} = \Theta_{S/R}$. The advantage of using Tr is that it is a good deal more elementary to construct.

Lemma 2.4. *Let S be a scheme over $\mathbf{F}_p$. Let $f : Y \to X$ be a finite morphism of Noetherian normal integral schemes over S. Assume*

(1) *the extension of function fields is purely inseparable of degree p, and*
(2) *$\Omega_{X/S}$ is a coherent $\mathcal{O}_X$-module (for example if X is of finite type over S).*

For $i \geq 1$ there is a canonical map

$$Tr : f_*\Omega^i_{Y/S} \longrightarrow (\Omega^i_{X/S})^{**}$$

whose stalk in the generic point of X recovers the trace map of Lemma 2.2.

Proof. The exact sequence $f^*\Omega_{X/S} \to \Omega_{Y/S} \to \Omega_{Y/X} \to 0$ shows that $\Omega_{Y/S}$ and hence $f_*\Omega_{Y/S}$ are coherent modules as well. Thus it suffices to prove the trace map in the generic point extends to stalks at $x \in X$ with $\dim(\mathcal{O}_{X,x}) = 1$, see Divisors, Lemma 12.14. Thus we reduce to the case discussed in the next paragraph.

Assume $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ with A a discrete valuation ring and B finite over A. Since the induced extension L/K of fraction fields is purely inseparable, we see that B is local too. Hence B is a discrete valuation ring too. Then either

(1) B/A has ramification index p and hence $B = A[x]/(x^p - a)$ where $a \in A$ is a uniformizer, or
(2) $\mathfrak{m}_B = \mathfrak{m}_A B$ and the residue field $B/\mathfrak{m}_A B$ is purely inseparable of degree p over $\kappa_A = A/\mathfrak{m}_A$. Choose any $x \in B$ whose residue class is not in κ_A and then we'll have $B = A[x]/(x^p - a)$ where $a \in A$ is a unit.

Let $\mathrm{Spec}(\Lambda) \subset S$ be an affine open such that X maps into $\mathrm{Spec}(\Lambda)$. Then we can apply Lemma 2.2 to see that the trace map extends to $\Omega^i_{B/\Lambda} \to \Omega^i_{A/\Lambda}$ for all $i \geq 1$. □

3. Quadratic transformations

In this section we study what happens when we blow up a nonsingular point on a surface. We hesitate the formally define such a morphism as a *quadratic transformation* as on the one hand often other names are used and on the other hand the phrase "quadratic transformation" is sometimes used with a different meaning.

Lemma 3.1. *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \mathrm{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$ wotj exceptional divisor E. There is a closed immersion*

$$r : X \longrightarrow \mathbf{P}^1_S$$

over S such that

1. *$r|_E : E \to \mathbf{P}^1_\kappa$ is an isomorphism,*
2. *$\mathcal{O}_X(E) = \mathcal{O}_X(-1) = r^*\mathcal{O}_{\mathbf{P}^1}(-1)$, and*
3. *$\mathcal{C}_{E/X} = (r|_E)^*\mathcal{O}_{\mathbf{P}^1}(1)$ and $\mathcal{N}_{E/X} = (r|_E)^*\mathcal{O}_{\mathbf{P}^1}(-1)$.*

Proof. As A is regular of dimension 2 we can write $\mathfrak{m} = (x, y)$. Then x and y placed in degree 1 generate the Rees algebra $\bigoplus_{n \geq 0} \mathfrak{m}^n$ over A. Recall that $X = \mathrm{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n)$, see Divisors, Lemma 32.2. Thus the surjection

$$A[T_0, T_1] \longrightarrow \bigoplus\nolimits_{n \geq 0} \mathfrak{m}^n, \quad T_0 \mapsto x,\ T_1 \mapsto y$$

of graded A-algebras induces a closed immersion $r : X \to \mathbf{P}^1_S = \mathrm{Proj}(A[T_0, T_1])$ such that $\mathcal{O}_X(1) = r^*\mathcal{O}_{\mathbf{P}^1_S}(1)$, see Constructions, Lemma 11.5. This proves (2) because $\mathcal{O}_X(E) = \mathcal{O}_X(-1)$ by Divisors, Lemma 32.4.

To prove (1) note that

$$\left(\bigoplus\nolimits_{n \geq 0} \mathfrak{m}^n\right) \otimes_A \kappa = \bigoplus\nolimits_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \kappa[\overline{x}, \overline{y}]$$

a polynomial algebra, see Algebra, Lemma 106.1. This proves that the fibre of $X \to S$ over $\mathrm{Spec}(\kappa)$ is equal to $\mathrm{Proj}(\kappa[\overline{x}, \overline{y}]) = \mathbf{P}^1_\kappa$, see Constructions, Lemma 11.6. Recall that E is the closed subscheme of X defined by $\mathfrak{m}\mathcal{O}_X$, i.e., $E = X_\kappa$. By our choice of the morphism r we see that $r|_E$ in fact produces the identification of $E = X_\kappa$ with the special fibre of $\mathbf{P}^1_S \to S$.

Part (3) follows from (1) and (2) and Divisors, Lemma 14.2. □

Lemma 3.2. *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \mathrm{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Then X is an irreducible regular scheme.*

Proof. Observe that X is integral by Divisors, Lemma 32.9 and Algebra, Lemma 106.2. To see X is regular it suffices to check that $\mathcal{O}_{X,x}$ is regular for closed points $x \in X$, see Properties, Lemma 9.2. Let $x \in X$ be a closed point. Since f is proper x maps to $\mathfrak{m}$, i.e., x is a point of the exceptional divisor E. Then E is an effective Cartier divisor and $E \cong \mathbf{P}^1_\kappa$. Thus if $g \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is a local equation for E, then $\mathcal{O}_{X,x}/(g) \cong \mathcal{O}_{\mathbf{P}^1_\kappa,x}$. Since $\mathbf{P}^1_\kappa$ is covered by two affine opens which are the spectrum of a polynomial ring over κ, we see that $\mathcal{O}_{\mathbf{P}^1_\kappa,x}$ is regular by Algebra, Lemma 114.1. We conclude by Algebra, Lemma 106.7. □

Lemma 3.3. *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \mathrm{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Then $\mathrm{Pic}(X) = \mathbf{Z}$ generated by $\mathcal{O}_X(E)$.*

Proof. Recall that $E = \mathbf{P}^1_\kappa$ has Picard group $\mathbf{Z}$ with generator $\mathcal{O}(1)$, see Divisors, Lemma 28.5. By Lemma 3.1 the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(E)$ restricts to $\mathcal{O}(-1)$. Hence $\mathcal{O}_X(E)$ generates an infinite cyclic group in $\text{Pic}(X)$. Since A is regular it is a UFD, see More on Algebra, Lemma 121.2. Then the punctured spectrum $U = S \setminus \{\mathfrak{m}\} = X \setminus E$ has trivial Picard group, see Divisors, Lemma 28.4. Hence for every invertible $\mathcal{O}_X$-module $\mathcal{L}$ there is an isomorphism $s : \mathcal{O}_U \to \mathcal{L}|_U$. Then s is a regular meromorphic section of $\mathcal{L}$ and we see that $\text{div}_\mathcal{L}(s) = nE$ for some $n \in \mathbf{Z}$ (Divisors, Definition 27.4). By Divisors, Lemma 27.6 (and the fact that X is normal by Lemma 3.2) we conclude that $\mathcal{L} = \mathcal{O}_X(nE)$. □

Lemma 3.4. *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

1. *$H^p(X, \mathcal{F}) = 0$ for $p \notin \{0, 1\}$,*
2. *$H^1(X, \mathcal{O}_X(n)) = 0$ for $n \geq -1$,*
3. *$H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ or $\mathcal{F}(1)$ is globally generated,*
4. *$H^0(X, \mathcal{O}_X(n)) = \mathfrak{m}^{\max(0,n)}$,*
5. *$\text{length}_A H^1(X, \mathcal{O}_X(n)) = -n(-n-1)/2$ if $n < 0$.*

Proof. If $\mathfrak{m} = (x, y)$, then X is covered by the spectra of the affine blowup algebras $A[\frac{\mathfrak{m}}{x}]$ and $A[\frac{\mathfrak{m}}{y}]$ because x and y placed in degree 1 generate the Rees algebra $\bigoplus \mathfrak{m}^n$ over A. See Divisors, Lemma 32.2 and Constructions, Lemma 8.9. Since X is separated by Constructions, Lemma 8.8 we see that cohomology of quasi-coherent sheaves vanishes in degrees ≥ 2 by Cohomology of Schemes, Lemma 4.2.

Let $i : E \to X$ be the exceptional divisor, see Divisors, Definition 32.1. Recall that $\mathcal{O}_X(-E) = \mathcal{O}_X(1)$ is f-relatively ample, see Divisors, Lemma 32.4. Hence we know that $H^1(X, \mathcal{O}_X(-nE)) = 0$ for some $n > 0$, see Cohomology of Schemes, Lemma 16.2. Consider the filtration

$$\mathcal{O}_X(-nE) \subset \mathcal{O}_X(-(n-1)E) \subset \ldots \subset \mathcal{O}_X(-E) \subset \mathcal{O}_X \subset \mathcal{O}_X(E)$$

The successive quotients are the sheaves

$$\mathcal{O}_X(-tE)/\mathcal{O}_X(-(t+1)E) = \mathcal{O}_X(t)/\mathcal{I}(t) = i_*\mathcal{O}_E(t)$$

where $\mathcal{I} = \mathcal{O}_X(-E)$ is the ideal sheaf of E. By Lemma 3.1 we have $E = \mathbf{P}^1_\kappa$ and $\mathcal{O}_E(1)$ indeed corresponds to the usual Serre twist of the structure sheaf on $\mathbf{P}^1$. Hence the cohomology of $\mathcal{O}_E(t)$ vanishes in degree 1 for $t \geq -1$, see Cohomology of Schemes, Lemma 8.1. Since this is equal to $H^1(X, i_*\mathcal{O}_E(t))$ (by Cohomology of Schemes, Lemma 2.4) we find that $H^1(X, \mathcal{O}_X(-(t+1)E)) \to H^1(X, \mathcal{O}_X(-tE))$ is surjective for $t \geq -1$. Hence

$$0 = H^1(X, \mathcal{O}_X(-nE)) \longrightarrow H^1(X, \mathcal{O}_X(-tE)) = H^1(X, \mathcal{O}_X(t))$$

is surjective for $t \geq -1$ which proves (2).

Let $\mathcal{F}$ be globally generated. This means there exists a short exact sequence

$$0 \to \mathcal{G} \to \bigoplus\nolimits_{i \in I} \mathcal{O}_X \to \mathcal{F} \to 0$$

Note that $H^1(X, \bigoplus_{i \in I} \mathcal{O}_X) = \bigoplus_{i \in I} H^1(X, \mathcal{O}_X)$ by Cohomology, Lemma 19.1. By part (2) we have $H^1(X, \mathcal{O}_X) = 0$. If $\mathcal{F}(1)$ is globally generated, then we can find a surjection $\bigoplus_{i \in I} \mathcal{O}_X(-1) \to \mathcal{F}$ and argue in a similar fashion. In other words, part (3) follows from part (2).

For part (4) we note that for all n large enough we have $\Gamma(X, \mathcal{O}_X(n)) = \mathfrak{m}^n$, see Cohomology of Schemes, Lemma 14.3. If $n \geq 0$, then we can use the short exact sequence

$$0 \to \mathcal{O}_X(n) \to \mathcal{O}_X(n-1) \to i_*\mathcal{O}_E(n-1) \to 0$$

and the vanishing of H^1 for the sheaf on the left to get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}^{\max(0,n)} \arrow[r] \arrow[d] & \mathfrak{m}^{\max(0,n-1)} \arrow[r] \arrow[d] & \mathfrak{m}^{\max(0,n)}/\mathfrak{m}^{\max(0,n-1)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X, \mathcal{O}_X(n)) \arrow[r] & \Gamma(X, \mathcal{O}_X(n-1)) \arrow[r] & \Gamma(E, \mathcal{O}_E(n-1)) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. In fact, the rows are exact also for $n < 0$ because in this case the groups on the right are zero. In the proof of Lemma 3.1 we have seen that the right vertical arrow is an isomorphism (details omitted). Hence if the left vertical arrow is an isomorphism, so is the middle one. In this way we see that (4) holds by descending induction on n.

Finally, we prove (5) by descending induction on n and the sequences

$$0 \to \mathcal{O}_X(n) \to \mathcal{O}_X(n-1) \to i_*\mathcal{O}_E(n-1) \to 0$$

Namely, for $n \geq -1$ we already know $H^1(X, \mathcal{O}_X(n)) = 0$. Since

$$H^1(X, i_*\mathcal{O}_E(-2)) = H^1(E, \mathcal{O}_E(-2)) = H^1(\mathbf{P}^1_\kappa, \mathcal{O}(-2)) \cong \kappa$$

by Cohomology of Schemes, Lemma 8.1 which has length 1 as an A-module, we conclude from the long exact cohomology sequence that (5) holds for $n = -2$. And so on and so forth. $\square$

Lemma 3.5. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension 2. Let $f : X \to S = \mathrm{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Let $\mathfrak{m}^n \subset I \subset \mathfrak{m}$ be an ideal. Let $d \geq 0$ be the largest integer such that*

$$I\mathcal{O}_X \subset \mathcal{O}_X(-dE)$$

where E is the exceptional divisor. Set $\mathcal{I}' = I\mathcal{O}_X(dE) \subset \mathcal{O}_X$. Then $d > 0$, the sheaf $\mathcal{O}_X/\mathcal{I}'$ is supported in finitely many closed points $x_1, \ldots, x_r$ of X, and

$$\begin{aligned} \mathit{length}_A(A/I) &> \mathit{length}_A \Gamma(X, \mathcal{O}_X/\mathcal{I}') \\ &\geq \sum\nolimits_{i=1,\ldots,r} \mathit{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i}) \end{aligned}$$

Proof. Since $I \subset \mathfrak{m}$ we see that every element of I vanishes on E. Thus we see that $d \geq 1$. On the other hand, since $\mathfrak{m}^n \subset I$ we see that $d \leq n$. Consider the short exact sequence

$$0 \to I\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_X/I\mathcal{O}_X \to 0$$

Since $I\mathcal{O}_X$ is globally generated, we see that $H^1(X, I\mathcal{O}_X) = 0$ by Lemma 3.4. Hence we obtain a surjection $A/I \to \Gamma(X, \mathcal{O}_X/I\mathcal{O}_X)$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(-dE)/I\mathcal{O}_X \to \mathcal{O}_X/I\mathcal{O}_X \to \mathcal{O}_X/\mathcal{O}_X(-dE) \to 0$$

By Divisors, Lemma 15.8 we see that $\mathcal{O}_X(-dE)/I\mathcal{O}_X$ is supported in finitely many closed points of X. In particular, this coherent sheaf has vanishing higher cohomology groups (detail omitted). Thus in the following diagram

$$\begin{tikzcd} & & A/I \arrow[d] & & \\ 0 \arrow[r] & \Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) \arrow[r] & \Gamma(X, \mathcal{O}_X/I\mathcal{O}_X) \arrow[r] & \Gamma(X, \mathcal{O}_X/\mathcal{O}_X(-dE)) \arrow[r] & 0 \end{tikzcd}$$

the bottom row is exact and the vertical arrow surjective. We have

$$\text{length}_A \Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) < \text{length}_A(A/I)$$

since $\Gamma(X, \mathcal{O}_X/\mathcal{O}_X(-dE))$ is nonzero. Namely, the image of $1 \in \Gamma(X, \mathcal{O}_X)$ is nonzero as $d > 0$.

To finish the proof we translate the results above into the statements of the lemma. Since $\mathcal{O}_X(dE)$ is invertible we have

$$\mathcal{O}_X/\mathcal{I}' = \mathcal{O}_X(-dE)/I\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(dE).$$

Thus $\mathcal{O}_X/\mathcal{I}'$ and $\mathcal{O}_X(-dE)/I\mathcal{O}_X$ are supported in the same set of finitely many closed points, say $x_1, \ldots, x_r \in E \subset X$. Moreover we obtain

$$\Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) = \bigoplus \mathcal{O}_X(-dE)_{x_i}/I\mathcal{O}_{X,x_i} \cong \bigoplus \mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i} = \Gamma(X, \mathcal{O}_X/\mathcal{I}')$$

because an invertible module over a local ring is trivial. Thus we obtain the strict inequality. We also get the second because

$$\text{length}_A(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i}) \geq \text{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i})$$

as is immediate from the definition of length. □

Lemma 3.6. *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Then $\Omega_{X/S} = i_*\Omega_{E/\kappa}$, where $i : E \to X$ is the immersion of the exceptional divisor.*

Proof. Writing $\mathbf{P}^1 = \mathbf{P}^1_S$, let $r : X \to \mathbf{P}^1$ be as in Lemma 3.1. Then we have an exact sequence

$$\mathcal{C}_{X/\mathbf{P}^1} \to r^*\Omega_{\mathbf{P}^1/S} \to \Omega_{X/S} \to 0$$

see Morphisms, Lemma 32.15. Since $\Omega_{\mathbf{P}^1/S}|_E = \Omega_{E/\kappa}$ by Morphisms, Lemma 32.10 it suffices to see that the first arrow defines a surjection onto the kernel of the canonical map $r^*\Omega_{\mathbf{P}^1/S} \to i_*\Omega_{E/\kappa}$. This we can do locally. With notation as in the proof of Lemma 3.1 on an affine open of X the morphism f corresponds to the ring map

$$A \to A[t]/(xt - y)$$

where $x, y \in \mathfrak{m}$ are generators. Thus $\text{d}(xt - y) = x\text{d}t$ and $y\text{d}t = t \cdot x\text{d}t$ which proves what we want. □

4. Dominating by quadratic transformations

Using the result above we can prove that blowups in points dominate any modification of a regular 2 dimensional scheme.

Let X be a scheme. Let $x \in X$ be a closed point. As usual, we view $i : x = \mathrm{Spec}(\kappa(x)) \to X$ as a closed subscheme. The *blowing up $X' \to X$ of X at x* is the blowing up of X in the closed subscheme $x \subset X$. Observe that if X is locally Noetherian, then $X' \to X$ is projective (in particular proper) by Divisors, Lemma 32.13.

Lemma 4.1. *Let X be a Noetherian scheme. Let $T \subset X$ be a finite set of closed points x such that $\mathcal{O}_{X,x}$ is regular of dimension 2 for $x \in T$. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals such that $\mathcal{O}_X/\mathcal{I}$ is supported on T. Then there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf.

Proof. Say $T = \{x_1, \ldots, x_r\}$. Denote I_i the stalk of $\mathcal{I}$ at x_i. Set

$$n_i = \mathrm{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/I_i)$$

This is finite as $\mathcal{O}_X/\mathcal{I}$ is supported on T and hence $\mathcal{O}_{X,x_i}/I_i$ has support equal to $\{\mathfrak{m}_{x_i}\}$ (see Algebra, Lemma 62.3). We are going to use induction on $\sum n_i$. If $n_i = 0$ for all i, then $\mathcal{I} = \mathcal{O}_X$ and we are done.

Suppose $n_i > 0$. Let $X' \to X$ be the blowing up of X in x_i (see discussion above the lemma). Since $\mathrm{Spec}(\mathcal{O}_{X,x_i}) \to X$ is flat we see that $X' \times_X \mathrm{Spec}(\mathcal{O}_{X,x_i})$ is the blowup of the ring $\mathcal{O}_{X,x_i}$ in the maximal ideal, see Divisors, Lemma 32.3. Hence the square in the commutative diagram

$$\begin{tikzcd}
\mathrm{Proj}(\bigoplus_{d \geq 0} \mathfrak{m}_{x_i}^d) \arrow[r] \arrow[d] & X' \arrow[d] \\
\mathrm{Spec}(\mathcal{O}_{X,x_i}) \arrow[r] & X
\end{tikzcd}$$

is cartesian. Let $E \subset X'$ and $E' \subset \mathrm{Proj}(\bigoplus_{d \geq 0} \mathfrak{m}_{x_i}^d)$ be the exceptional divisors. Let $d \geq 1$ be the integer found in Lemma 3.5 for the ideal $\mathcal{I}_i \subset \mathcal{O}_{X,x_i}$. Since the horizontal arrows in the diagram are flat, since $E' \to E$ is surjective, and since E' is the pullback of E, we see that

$$\mathcal{I}\mathcal{O}_{X'} \subset \mathcal{O}_{X'}(-dE)$$

(some details omitted). Set $\mathcal{I}' = \mathcal{I}\mathcal{O}_{X'}(dE) \subset \mathcal{O}_{X'}$. Then we see that $\mathcal{O}_{X'}/\mathcal{I}'$ is supported in finitely many closed points $T' \subset |X'|$ because this holds over $X \setminus \{x_i\}$ and for the pullback to $\mathrm{Proj}(\bigoplus_{d \geq 0} \mathfrak{m}_{x_i}^d)$. The final assertion of Lemma 3.5 tells us that the sum of the lengths of the stalks $\mathcal{O}_{X',x'}/\mathcal{I}'\mathcal{O}_{X',x'}$ for x' lying over x_i is $< n_i$. Hence the sum of the lengths has decreased.

By induction hypothesis, there exists a sequence

$$X'_n \to \ldots \to X'_1 \to X'$$

of blowups at closed points lying over T' such that $\mathcal{I}'\mathcal{O}_{X'_n}$ is invertible. Since $\mathcal{I}'\mathcal{O}_{X'}(-dE) = \mathcal{I}\mathcal{O}_{X'}$, we see that $\mathcal{I}\mathcal{O}_{X'_n} = \mathcal{I}'\mathcal{O}_{X'_n}(-d(f')^{-1}E)$ where $f' : X'_n \to$

X' is the composition. Note that $(f')^{-1}E$ is an effective Cartier divisor by Divisors, Lemma 32.11. Thus we are done by Divisors, Lemma 13.7. $\square$

Lemma 4.2. *Let X be a Noetherian scheme. Let $T \subset X$ be a finite set of closed points x such that $\mathcal{O}_{X,x}$ is a regular local ring of dimension 2. Let $f : Y \to X$ be a proper morphism of schemes which is an isomorphism over $U = X \setminus T$. Then there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T and a factorization $X_n \to Y \to X$ of the composition.

Proof. By More on Flatness, Lemma 31.4 there exists a U-admissible blowup $X' \to X$ which dominates $Y \to X$. Hence we may assume there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{O}_X/\mathcal{I}$ is supported on T and such that Y is the blowing up of X in $\mathcal{I}$. By Lemma 4.1 there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf. By the universal property of blowing up (Divisors, Lemma 32.5) we find the desired factorization. $\square$

Lemma 4.3. *Let S be a scheme. Let X be a scheme over S which is regular and has dimension 2. Let Y be a proper scheme over S. Given an S-rational map $f : U \to Y$ from X to Y there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

and an S-morphism $f_n : X_n \to Y$ such that $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point not lying over U and f_n and f agree.

Proof. We may assume U contains every point of codimension 1, see Morphisms, Lemma 42.5. Hence the complement $T \subset X$ of U is a finite set of closed points whose local rings are regular of dimension 2. Applying Divisors, Lemma 36.2 we find a proper morphism $p : X' \to X$ which is an isomorphism over U and a morphism $f' : X' \to Y$ agreeing with f over U. Apply Lemma 4.2 to the morphism $p : X' \to X$. The composition $X_n \to X' \to Y$ is the desired morphism. $\square$

5. Dominating by normalized blowups

In this section we prove that a modification of a surface can be dominated by a sequence of normalized blowups in points.

Definition 5.1. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $x \in X$ be a closed point. The *normalized blowup of X at x* is the composition $X'' \to X' \to X$ where $X' \to X$ is the blowup of X in x and $X'' \to X'$ is the normalization of X'.

Here the normalization $X'' \to X'$ is defined as the scheme X' has an open covering by opens which have finitely many irreducible components by Divisors, Lemma 32.10. See Morphisms, Definition 54.1 for the definition of the normalization.

In general the normalized blowing up need not be proper even when X is Noetherian. Recall that a scheme is Nagata if it has an open covering by affines which are spectra of Nagata rings (Properties, Definition 13.1).

Lemma 5.2. *In Definition 5.1 if X is Nagata, then the normalized blowing up of X at x is normal, Nagata, and proper over X.*

Proof. The blowup morphism $X' \to X$ is proper (as X is locally Noetherian we may apply Divisors, Lemma 32.13). Thus X' is Nagata (Morphisms, Lemma 18.1). Therefore the normalization $X'' \to X'$ is finite (Morphisms, Lemma 54.10) and we conclude that $X'' \to X$ is proper as well (Morphisms, Lemmas 44.11 and 41.4). It follows that the normalized blowing up is a normal (Morphisms, Lemma 54.5) Nagata algebraic space. $\square$

In the following lemma we need to assume X is Noetherian in order to make sure that it has finitely many irreducible components. Then the properness of $f : Y \to X$ assures that Y has finitely many irreducible components too and it makes sense to require f to be birational (Morphisms, Definition 50.1).

Lemma 5.3. *Let X be a scheme which is Noetherian, Nagata, and has dimension 2. Let $f : Y \to X$ be a proper birational morphism. Then there exists a commutative diagram*

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] & \ldots \arrow[r] & X_1 \arrow[r] & X_0 \arrow[d] \\ Y \arrow[rrrr] & & & & X \end{tikzcd}$$

where $X_0 \to X$ is the normalization and where $X_{i+1} \to X_i$ is the normalized blowing up of X_i at a closed point.

Proof. We will use the results of Morphisms, Sections 18, 52, and 54 without further mention. We may replace Y by its normalization. Let $X_0 \to X$ be the normalization. The morphism $Y \to X$ factors through X_0. Thus we may assume that both X and Y are normal.

Assume X and Y are normal. The morphism $f : Y \to X$ is an isomorphism over an open which contains every point of codimension 0 and 1 in Y and every point of Y over which the fibre is finite, see Varieties, Lemma 17.3. Hence there is a finite set of closed points $T \subset X$ such that f is an isomorphism over $X \setminus T$. For each $x \in T$ the fibre Y_x is a proper geometrically connected scheme of dimension 1 over $\kappa(x)$, see More on Morphisms, Lemma 53.6. Thus

$$BadCurves(f) = \{C \subset Y \text{ closed} \mid \dim(C) = 1, f(C) = \text{a point}\}$$

is a finite set. We will prove the lemma by induction on the number of elements of $BadCurves(f)$. The base case is the case where $BadCurves(f)$ is empty, and in that case f is an isomorphism.

Fix $x \in T$. Let $X' \to X$ be the normalized blowup of X at x and let Y' be the normalization of $Y \times_X X'$. Picture

$$\begin{tikzcd} Y' \arrow[r, "f'"'] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Let $x' \in X'$ be a closed point lying over x such that the fibre $Y'_{x'}$ has dimension ≥ 1. Let $C' \subset Y'$ be an irreducible component of $Y'_{x'}$, i.e., $C' \in BadCurves(f')$. Since $Y' \to Y \times_X X'$ is finite we see that C' must map to an irreducible component

$C \subset Y_x$. If is clear that $C \in BadCurves(f)$. Since $Y' \to Y$ is birational and hence an isomorphism over points of codimension 1 in Y, we see that we obtain an injective map

$$BadCurves(f') \longrightarrow BadCurves(f)$$

Thus it suffices to show that after a finite number of these normalized blowups we get rid at of at least one of the bad curves, i.e., the displayed map is not surjective.

We will get rid of a bad curve using an argument due to Zariski. Pick $C \in BadCurves(f)$ lying over our x. Denote $\mathcal{O}_{Y,C}$ the local ring of Y at the generic point of C. Choose an element $u \in \mathcal{O}_{X,C}$ whose image in the residue field $R(C)$ is transcendental over $\kappa(x)$ (we can do this because $R(C)$ has transcendence degree 1 over $\kappa(x)$ by Varieties, Lemma 20.3). We can write $u = a/b$ with $a, b \in \mathcal{O}_{X,x}$ as $\mathcal{O}_{Y,C}$ and $\mathcal{O}_{X,x}$ have the same fraction fields. By our choice of u it must be the case that $a, b \in \mathfrak{m}_x$. Hence

$$N_{u,a,b} = \min\{\mathrm{ord}_{\mathcal{O}_{Y,C}}(a), \mathrm{ord}_{\mathcal{O}_{Y,C}}(b)\} > 0$$

Thus we can do descending induction on this integer. Let $X' \to X$ be the normalized blowing up of x and let Y' be the normalization of $X' \times_X Y$ as above. We will show that if C is the image of some bad curve $C' \subset Y'$ lying over $x' \in X'$, then there exists a choice of $a', b' \mathcal{O}_{X',x'}$ such that $N_{u,a',b'} < N_{u,a,b}$. This will finish the proof. Namely, since $X' \to X$ factors through the blowing up, we see that there exists a nonzero element $d \in \mathfrak{m}_{x'}$ such that $a = a'd$ and $b = b'd$ (namely, take d to be the local equation for the exceptional divisor of the blowup). Since $Y' \to Y$ is an isomorphism over an open containing the generic point of C (seen above) we see that $\mathcal{O}_{Y',C'} = \mathcal{O}_{Y,C}$. Hence

$$\mathrm{ord}_{\mathcal{O}_{Y,C}}(a) = \mathrm{ord}_{\mathcal{O}_{Y',C'}}(a'd) = \mathrm{ord}_{\mathcal{O}_{Y',C'}}(a') + \mathrm{ord}_{\mathcal{O}_{Y',C'}}(d) > \mathrm{ord}_{\mathcal{O}_{Y',C'}}(a')$$

Similarly for b and the proof is complete. □

Lemma 5.4. *Let S be a scheme. Let X be a scheme over S which is Noetherian, Nagata, and has dimension 2. Let Y be a proper scheme over S. Given an S-rational map $f : U \to Y$ from X to Y there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 \to X$$

and an S-morphism $f_n : X_n \to Y$ such that $X_0 \to X$ is the normalization, $X_{i+1} \to X_i$ is the normalized blowing up of X_i at a closed point, and f_n and f agree.

Proof. Applying Divisors, Lemma 36.2 we find a proper morphism $p : X' \to X$ which is an isomorphism over U and a morphism $f' : X' \to Y$ agreeing with f over U. Apply Lemma 5.3 to the morphism $p : X' \to X$. The composition $X_n \to X' \to Y$ is the desired morphism. □

6. Modifying over local rings

Let S be a scheme. Let $s_1, \ldots, s_n \in S$ be pairwise distinct closed points. Assume that the open embedding

$$U = S \setminus \{s_1, \ldots, s_n\} \longrightarrow S$$

is quasi-compact. Denote $FP_{S,\{s_1,\ldots,s_n\}}$ the category of morphisms $f : X \to S$ of finite presentation which induce an isomorphism $f^{-1}(U) \to U$. Morphisms are morphisms of schemes over S. For each i set $S_i = \mathrm{Spec}(\mathcal{O}_{S,s_i})$ and let $V_i = S_i \setminus \{s_i\}$. Denote FP_{S_i,s_i} the category of morphisms $g_i : Y_i \to S_i$ of finite presentation which induce an isomorphism $g_i^{-1}(V_i) \to V_i$. Morphisms are morphisms over S_i. Base change defines an functor

$$F : FP_{S,\{s_1,\ldots,s_n\}} \longrightarrow FP_{S_1,s_1} \times \ldots \times FP_{S_n,s_n} \tag{6.0.1}$$

To reduce at least some of the problems in this chapter to the case of local rings we have the following lemma.

Lemma 6.1. *The functor F (6.0.1) is an equivalence.*

Proof. For $n = 1$ this is Limits, Lemma 21.1. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $g_i : Y_i \to S_i$ are objects of FP_{S_i,s_i}. Then by the case $n = 1$ we can find $f_i' : X_i' \to S$ of finite presentation which are isomorphisms over $S \setminus \{s_i\}$ and whose base change to S_i is g_i. Then we can set

$$f : X = X_1' \times_S \ldots \times_S X_n' \to S$$

This is an object of $FP_{S,\{s_1,\ldots,s_n\}}$ whose base change by $S_i \to S$ recovers g_i. Thus the functor is essentially surjective. We omit the proof of fully faithfulness. $\square$

Lemma 6.2. *Let S, s_i, S_i be as in (6.0.1). If $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F, then f is separated, proper, finite, if and only if g_i is so for $i = 1, \ldots, n$.*

Proof. Follows from Limits, Lemma 21.2. $\square$

Lemma 6.3. *Let S, s_i, S_i be as in (6.0.1). If $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F, then $X_{s_i} \cong (Y_i)_{s_i}$ as schemes over $\kappa(s_i)$.*

Proof. This is clear. $\square$

Lemma 6.4. *Let S, s_i, S_i be as in (6.0.1) and assume $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F. Then there exists a factorization*

$$X = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = S$$

of f where $Z_{j+1} \to Z_j$ is the blowing up of Z_j at a closed point z_j lying over $\{s_1, \ldots, s_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = S_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over s_i.

Proof. Let's start with a sequence of blowups $Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = S$. The first morphism $Z_1 \to S$ is given by blowing up one of the s_i, say s_1. Applying F to $Z_1 \to S$ we find a blowup $Z_{1,1} \to S_1$ at s_1 is the blowing up at s_1 and otherwise $Z_{i,0} = S_i$ for $i > 1$. In the next step, we either blow up one of the s_i, $i \geq 2$ on Z_1 or we pick a closed point z_1 of the fibre of $Z_1 \to S$ over s_1. In the first case it is clear what to do and in the second case we use that $(Z_1)_{s_1} \cong (Z_{1,1})_{s_1}$ (Lemma 6.3) to get a closed point $z_{1,1} \in Z_{1,1}$ corresponding to z_1. Then we set $Z_{1,2} \to Z_{1,1}$ equal to the blowing up in $z_{1,1}$. Continuing in this manner we construct the factorizations of each g_i.

Conversely, given sequences of blowups $Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = S_i$ we construct the sequence of blowing ups of S in exactly the same manner. $\square$

Here is the analogue of Lemma 6.4 for normalized blowups.

Lemma 6.5. *Let S, s_i, S_i be as in (6.0.1) and assume $f : X \to S$ corresponds to $g_i : Y_i \to S_i$ under F. Assume every quasi-compact open of S has finitely many irreducible components. Then there exists a factorization*

$$X = Z_m \to Z_{m-1} \to \ldots \to Z_1 \to Z_0 = S$$

of f where $Z_{j+1} \to Z_j$ is the normalized blowing up of Z_j at a closed point z_j lying over $\{x_1, \ldots, x_n\}$ if and only if for each i there exists a factorization

$$Y_i = Z_{i,m_i} \to Z_{i,m_i-1} \to \ldots \to Z_{i,1} \to Z_{i,0} = S_i$$

of g_i where $Z_{i,j+1} \to Z_{i,j}$ is the normalized blowing up of $Z_{i,j}$ at a closed point $z_{i,j}$ lying over s_i.

Proof. The assumption on S is used to assure us (successively) that the schemes we are normalizing have locally finitely many irreducible components so that the statement makes sense. Having said this the lemma follows by the exact same argument as used to prove Lemma 6.4. □

7. Vanishing

In this section we will often work in the following setting. Recall that a modification is a proper birational morphism between integral schemes (Morphisms, Definition 51.11).

Situation 7.1. Here $(A, \mathfrak{m}, \kappa)$ be a local Noetherian normal domain of dimension 2. Let s be the closed point of $S = \mathrm{Spec}(A)$ and $U = S \setminus \{s\}$. Let $f : X \to S$ be a modification. We denote $C_1, \ldots, C_r$ the irreducible components of the special fibre X_s of f.

By Varieties, Lemma 17.3 the morphism f defines an isomorphism $f^{-1}(U) \to U$. The special fibre X_s is proper over $\mathrm{Spec}(\kappa)$ and has dimension at most 1 by Varieties, Lemma 19.3. By Stein factorization (More on Morphisms, Lemma 53.6) we have $f_*\mathcal{O}_X = \mathcal{O}_S$ and the special fibre X_s is geometrically connected over κ. If X_s has dimension 0, then f is finite (More on Morphisms, Lemma 44.2) and hence an isomorphism (Morphisms, Lemma 54.8). We will discard this uninteresting case and we conclude that $\dim(C_i) = 1$ for $i = 1, \ldots, r$.

Lemma 7.2. *In Situation 7.1 there exists a U-admissible blowup $X' \to S$ which dominates X.*

Proof. This is a special case of More on Flatness, Lemma 31.4. □

Lemma 7.3. *In Situation 7.1 there exists a nonzero $f \in \mathfrak{m}$ such that for every $i = 1, \ldots, r$ there exist*

1. *a closed point $x_i \in C_i$ with $x_i \notin C_j$ for $j \neq i$,*
2. *a factorization $f = g_i f_i$ of f in $\mathcal{O}_{X,x_i}$ such that $g_i \in \mathfrak{m}_{x_i}$ maps to a nonzero element of $\mathcal{O}_{C_i,x_i}$.*

Proof. We will use the observations made following Situation 7.1 without further mention. Pick a closed point $x_i \in C_i$ which is not in C_j for $j \neq i$. Pick $g_i \in \mathfrak{m}_{x_i}$ which maps to a nonzero element of $\mathcal{O}_{C_i,x_i}$. Since the fraction field of A is the fraction field of $\mathcal{O}_{X_i,x_i}$ we can write $g_i = a_i/b_i$ for some $a_i, b_i \in A$. Take $f = \prod a_i$. □

Lemma 7.4. *In Situation 7.1 assume X is normal. Let $Z \subset X$ be a nonempty effective Cartier divisor such that $Z \subset X_s$ set theoretically. Then the conormal sheaf of Z is not trivial. More precisely, there exists an i such that $C_i \subset Z$ and $\deg(\mathcal{C}_{Z/X}|_{C_i}) > 0$.*

Proof. We will use the observations made following Situation 7.1 without further mention. Let f be a function as in Lemma 7.3. Let $\xi_i \in C_i$ be the generic point. Let $\mathcal{O}_i$ be the local ring of X at ξ_i. Then $\mathcal{O}_i$ is a discrete valuation ring. Let e_i be the valuation of f in $\mathcal{O}_i$, so $e_i > 0$. Let $h_i \in \mathcal{O}_i$ be a local equation for Z and let d_i be its valuation. Then $d_i \geq 0$. Choose and fix i with d_i/e_i maximal (then $d_i > 0$ as Z is not empty). Replace f by f^{d_i} and Z by $e_i Z$. This is permissible, by the relation $\mathcal{O}_X(e_i Z) = \mathcal{O}_X(Z)^{\otimes e_i}$, the relation between the conormal sheaf and $\mathcal{O}_X(Z)$ (see Divisors, Lemmas 14.4 and 14.2, and since the degree gets multiplied by e_i, see Varieties, Lemma 44.7. Let $\mathcal{I}$ be the ideal sheaf of Z so that $\mathcal{C}_{Z/X} = \mathcal{I}|_Z$. Consider the image $\overline{f}$ of f in $\Gamma(Z, \mathcal{O}_Z)$. By our choices above we see that $\overline{f}$ vanishes in the generic points of irreducible components of Z (these are all generic points of C_j as Z is contained in the special fibre). On the other hand, Z is (S_1) by Divisors, Lemma 15.6. Thus the scheme Z has no embedded associated points and we conclude that $\overline{f} = 0$ (Divisors, Lemmas 4.3 and 5.6). Hence f is a global section of $\mathcal{I}$ which generates $\mathcal{I}_{\xi_i}$ by construction. Thus the image s_i of f in $\Gamma(C_i, \mathcal{I}|_{C_i})$ is nonzero. However, our choice of f guarantees that s_i has a zero at x_i. Hence the degree of $\mathcal{I}|_{C_i}$ is > 0 by Varieties, Lemma 44.12. $\square$

Lemma 7.5. *In Situation 7.1 assume X is normal and A Nagata. The map*

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(f^{-1}(U), \mathcal{O}_X)$$

is injective.

Proof. Let $0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X \to 0$ be the extension corresponding to a nontrivial element ξ of $H^1(X, \mathcal{O}_X)$ (Cohomology, Lemma 5.1). Let $\pi : P = \mathbf{P}(\mathcal{E}) \to X$ be the projective bundle associated to $\mathcal{E}$. The surjection $\mathcal{E} \to \mathcal{O}_X$ defines a section $\sigma : X \to P$ whose conormal sheaf is isomorphic to $\mathcal{O}_X$ (Divisors, Lemma 31.6). If the restriction of ξ to $f^{-1}(U)$ is trivial, then we get a map $\mathcal{E}|_{f^{-1}(U)} \to \mathcal{O}_{f^{-1}(U)}$ splitting the injection $\mathcal{O}_X \to \mathcal{E}$. This defines a second section $\sigma' : f^{-1}(U) \to P$ disjoint from σ. Since ξ is nontrivial we conclude that σ' cannot extend to all of X and be disjoint from σ. Let $X' \subset P$ be the scheme theoretic image of σ' (Morphisms, Definition 6.2). Picture

$$\begin{tikzcd}
 & X' \arrow[r] \arrow[dr, "g"'] & P \arrow[d, shift right, "\pi"'] \\
f^{-1}(U) \arrow[ur, "\sigma'"'] \arrow[rr] & & X \arrow[u, shift right, bend right, "\sigma"']
\end{tikzcd}$$

The morphism $P \setminus \sigma(X) \to X$ is affine. If $X' \cap \sigma(X) = \emptyset$, then $X' \to X$ is both affine and proper, hence finite (Morphisms, Lemma 44.11), hence an isomorphism (as X is normal, see Morphisms, Lemma 54.8). This is impossible as mentioned above.

Let X^ν be the normalization of X'. Since A is Nagata, we see that $X^\nu \to X'$ is finite (Morphisms, Lemmas 54.10 and 18.2). Let $Z \subset X^\nu$ be the pullback of the effective Cartier divisor $\sigma(X) \subset P$. By the above we see that Z is not empty and is contained in the closed fibre of $X^\nu \to S$. Since $P \to X$ is smooth, we see that $\sigma(X)$ is an effective Cartier divisor (Divisors, Lemma 22.8). Hence $Z \subset X^\nu$ is an effective Cartier divisor too. Since the conormal sheaf of $\sigma(X)$ in P is $\mathcal{O}_X$, the conormal sheaf of Z in X^ν (which is a priori invertible) is $\mathcal{O}_Z$ by Morphisms, Lemma 31.4. This is impossible by Lemma 7.4 and the proof is complete. □

Lemma 7.6. *In Situation 7.1 assume X is normal and A Nagata. Then*

$$\mathrm{Hom}_{D(A)}(\kappa[-1], Rf_*\mathcal{O}_X)$$

is zero. This uses $D(A) = D_{QCoh}(\mathcal{O}_S)$ to think of $Rf_*\mathcal{O}_X$ as an object of $D(A)$.

Proof. By adjointness of Rf_* and Lf^* such a map is the same thing as a map $\alpha : Lf^*\kappa[-1] \to \mathcal{O}_X$. Note that

$$H^i(Lf^*\kappa[-1]) = \begin{cases} 0 & \text{if} \quad i > 1 \\ \mathcal{O}_{X_s} & \text{if} \quad i = 1 \\ \text{some } \mathcal{O}_{X_s}\text{-module} & \text{if} \quad i \leq 0 \end{cases}$$

Since $\mathrm{Hom}(H^0(Lf^*\kappa[-1]), \mathcal{O}_X) = 0$ as $\mathcal{O}_X$ is torsion free, the spectral sequence for Ext (Cohomology on Sites, Example 32.1) implies that $\mathrm{Hom}_{D(\mathcal{O}_X)}(Lf^*\kappa[-1], \mathcal{O}_X)$ is equal to $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{X_s}, \mathcal{O}_X)$. We conclude that $\alpha : Lf^*\kappa[-1] \to \mathcal{O}_X$ is given by an extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_{X_s} \to 0$$

By Lemma 7.5 the pullback of this extension via the surjection $\mathcal{O}_X \to \mathcal{O}_{X_s}$ is zero (since this pullback is clearly split over $f^{-1}(U)$). Thus $1 \in \mathcal{O}_{X_s}$ lifts to a global section s of $\mathcal{E}$. Multiplying s by the ideal sheaf $\mathcal{I}$ of X_s we obtain an $\mathcal{O}_X$-module map $c_s : \mathcal{I} \to \mathcal{O}_X$. Applying f_* we obtain an A-linear map $f_*c_s : \mathfrak{m} \to A$. Since A is a Noetherian normal local domain this map is given by multiplication by an element $a \in A$. Changing s into $s - a$ we find that s is annihilated by $\mathcal{I}$ and the extension is trivial as desired. □

Remark 7.7. Let X be an integral Noetherian normal scheme of dimension 2. In this case the following are equivalent

1. X has a dualizing complex $\omega_X^\bullet$,
2. there is a coherent $\mathcal{O}_X$-module ω_X such that $\omega_X[n]$ is a dualizing complex, where n can be any integer.

This follows from the fact that X is Cohen-Macaulay (Properties, Lemma 12.7) and Duality for Schemes, Lemma 23.1. In this situation we will say that ω_X is a *dualizing module* in accordance with Duality for Schemes, Section 22. In particular, when A is a Noetherian normal local domain of dimension 2, then we say *A has a dualizing module* ω_A if the above is true. In this case, if $X \to \mathrm{Spec}(A)$ is a normal modification, then X has a dualizing module too, see Duality for Schemes, Example 22.1. In this situation we always denote ω_X the dualizing module normalized with respect to ω_A, i.e., such that $\omega_X[2]$ is the dualizing complex normalized relative to $\omega_A[2]$. See Duality for Schemes, Section 20.

The Grauert-Riemenschneider vanishing of the next proposition is a formal consequence of Lemma 7.6 and the general theory of duality.

Proposition 7.8 (Grauert-Riemenschneider)**.** *In Situation 7.1 assume*

1. *X is a normal scheme,*
2. *A is Nagata and has a dualizing complex $\omega_A^\bullet$.*

Let ω_X be the dualizing module of X (Remark 7.7). Then $R^1 f_*\omega_X = 0$.

Proof. In this proof we will use the identification $D(A) = D_{QCoh}(\mathcal{O}_S)$ to identify quasi-coherent $\mathcal{O}_S$-modules with A-modules. Moreover, we may assume that $\omega_A^\bullet$ is normalized, see Dualizing Complexes, Section 16. Since X is a Noetherian normal 2-dimensional scheme it is Cohen-Macaulay (Properties, Lemma 12.7). Thus $\omega_X^\bullet = \omega_X[2]$ (Duality for Schemes, Lemma 23.1 and the normalization in Duality for Schemes, Example 22.1). If the proposition is false, then we can find a nonzero map $R^1 f_*\omega_X \to \kappa$. In other words we obtain a nonzero map $\alpha : Rf_*\omega_X^\bullet \to \kappa[1]$. Applying $R\operatorname{Hom}_A(-, \omega_A^\bullet)$ we get a nonzero map

$$\beta : \kappa[-1] \longrightarrow Rf_*\mathcal{O}_X$$

which is impossible by Lemma 7.6. To see that $R\operatorname{Hom}_A(-, \omega_A^\bullet)$ does what we said, first note that

$$R\operatorname{Hom}_A(\kappa[1], \omega_A^\bullet) = R\operatorname{Hom}_A(\kappa, \omega_A^\bullet)[-1] = \kappa[-1]$$

as $\omega_A^\bullet$ is normalized and we have

$$R\operatorname{Hom}_A(Rf_*\omega_X^\bullet, \omega_A^\bullet) = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(\omega_X^\bullet, \omega_X^\bullet) = Rf_*\mathcal{O}_X$$

The first equality by Duality for Schemes, Example 3.9 and the fact that $\omega_X^\bullet = f^!\omega_A^\bullet$ by construction, and the second equality because $\omega_X^\bullet$ is a dualizing complex for X (which goes back to Duality for Schemes, Lemma 17.7). □

8. Boundedness

In this section we begin the discussion which will lead to a reduction to the case of rational singularities for 2-dimensional schemes.

Lemma 8.1. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian normal local domain of dimension 2. Consider a commutative diagram*

$$\begin{tikzcd} X' \arrow[rr, "g"] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & \operatorname{Spec}(A) & \end{tikzcd}$$

where f and f' are modifications as in Situation 7.1 and X normal. Then we have a short exact sequence

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(X', \mathcal{O}_{X'}) \to H^0(X, R^1 g_*\mathcal{O}_{X'}) \to 0$$

Also $\dim(Supp(R^1 g_*\mathcal{O}_{X'})) = 0$ and $R^1 g_*\mathcal{O}_{X'}$ is generated by global sections.

Proof. We will use the observations made following Situation 7.1 without further mention. As X is normal and g is dominant and birational, we have $g_*\mathcal{O}_{X'} = \mathcal{O}_X$, see for example More on Morphisms, Lemma 53.6. Since the fibres of g have dimension ≤ 1, we have $R^p g_*\mathcal{O}_{X'} = 0$ for $p > 1$, see for example Cohomology of Schemes, Lemma 20.9. The support of $R^1 g_*\mathcal{O}_{X'}$ is contained in the set of points of X where the fibres of g' have dimension ≥ 1. Thus it is contained in the set of images of those irreducible components $C' \subset X'_s$ which map to points of X_s which is a finite set of closed points (recall that $X'_s \to X_s$ is a morphism of proper 1-dimensional schemes over κ). Then $R^1 g_*\mathcal{O}_{X'}$ is globally generated by Cohomology of Schemes, Lemma 9.10. Using the morphism $f : X \to S$ and the references above we find that $H^p(X, \mathcal{F}) = 0$ for $p > 1$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$. Hence the short exact sequence of the lemma is a consequence of the Leray spectral sequence for g and $\mathcal{O}_{X'}$, see Cohomology, Lemma 13.4. □

Lemma 8.2. *Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2. Let $a \in A$ be nonzero. There exists an integer N such that for every modification $f : X \to \mathrm{Spec}(A)$ with X normal the A-module*

$$M_{X,a} = \mathrm{Coker}(A \longrightarrow H^0(Z, \mathcal{O}_Z))$$

where $Z \subset X$ is cut out by a has length bounded by N.

Proof. By the short exact sequence $0 \to \mathcal{O}_X \xrightarrow{a} \mathcal{O}_X \to \mathcal{O}_Z \to 0$ we see that

$$M_{X,a} = H^1(X, \mathcal{O}_X)[a] \tag{8.2.1}$$

Here $N[a] = \{n \in N \mid an = 0\}$ for an A-module N. Thus if a divides b, then $M_{X,a} \subset M_{X,b}$. Suppose that for some $c \in A$ the modules $M_{X,c}$ have bounded length. Then for every X we have an exact sequence

$$0 \to M_{X,c} \to M_{X,c^2} \to M_{X,c}$$

where the second arrow is given by multiplication by c. Hence we see that M_{X,c^2} has bounded length as well. Thus it suffices to find a $c \in A$ for which the lemma is true such that a divides c^n for some $n > 0$. By More on Algebra, Lemma 125.6 we may assume $A/(a)$ is a reduced ring.

Assume that $A/(a)$ is reduced. Let $A/(a) \subset B$ be the normalization of $A/(a)$ in its quotient ring. Because A is Nagata, we see that $\mathrm{Coker}(A \to B)$ is finite. We claim the length of this finite module is a bound. To see this, consider $f : X \to \mathrm{Spec}(A)$ as in the lemma and let $Z' \subset Z$ be the scheme theoretic closure of $Z \cap f^{-1}(U)$. Then $Z' \to \mathrm{Spec}(A/(a))$ is finite for example by Varieties, Lemma 17.2. Hence $Z' = \mathrm{Spec}(B')$ with $A/(a) \subset B' \subset B$. On the other hand, we claim the map

$$H^0(Z, \mathcal{O}_Z) \to H^0(Z', \mathcal{O}_{Z'})$$

is injective. Namely, if $s \in H^0(Z, \mathcal{O}_Z)$ is in the kernel, then the restriction of s to $f^{-1}(U) \cap Z$ is zero. Hence the image of s in $H^1(X, \mathcal{O}_X)$ vanishes in $H^1(f^{-1}(U), \mathcal{O}_X)$. By Lemma 7.5 we see that s comes from an element $\tilde{s}$ of A. But by assumption $\tilde{s}$ maps to zero in B' which implies that $s = 0$. Putting everything together we see that $M_{X,a}$ is a subquotient of B'/A, namely not every element of B' extends to a global section of $\mathcal{O}_Z$, but in any case the length of $M_{X,a}$ is bounded by the length of B/A. □

In some cases, resolution of singularities reduces to the case of rational singularities.

Definition 8.3. Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2.

(1) We say A *defines a rational singularity* if for every normal modification $X \to \mathrm{Spec}(A)$ we have $H^1(X, \mathcal{O}_X) = 0$.
(2) We say that *reduction to rational singularities is possible for A* if the length of the A-modules
$$H^1(X, \mathcal{O}_X)$$
is bounded for all modifications $X \to \mathrm{Spec}(A)$ with X normal.

The meaning of the language in (2) is explained by Lemma 8.5. The following lemma says roughly speaking that local rings of modifications of $\mathrm{Spec}(A)$ with A defining a rational singularity also define rational singularities.

Lemma 8.4. *Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2 which defines a rational singularity. Let $A \subset B$ be a local extension of domains with the same fraction field which is essentially of finite type such that $\dim(B) = 2$ and B normal. Then B defines a rational singularity.*

Proof. Choose a finite type A-algebra C such that $B = C_\mathfrak{q}$ for some prime $\mathfrak{q} \subset C$. After replacing C by the image of C in B we may assume that C is a domain with fraction field equal to the fraction field of A. Then we can choose a closed immersion $\text{Spec}(C) \to \mathbf{A}^n_A$ and take the closure in $\mathbf{P}^n_A$ to conclude that B is isomorphic to $\mathcal{O}_{X,x}$ for some closed point $x \in X$ of a projective modification $X \to \text{Spec}(A)$. (Morphisms, Lemma 52.1, shows that $\kappa(x)$ is finite over κ and then Morphisms, Lemma 20.2 shows that x is a closed point.) Let $\nu : X^\nu \to X$ be the normalization. Since A is Nagata the morphism ν is finite (Morphisms, Lemma 54.10). Thus X^ν is projective over A by More on Morphisms, Lemma 50.2. Since $B = \mathcal{O}_{X,x}$ is normal, we see that $\mathcal{O}_{X,x} = (\nu_*\mathcal{O}_{X^\nu})_x$. Hence there is a unique point $x^\nu \in X^\nu$ lying over x and $\mathcal{O}_{X^\nu,x^\nu} = \mathcal{O}_{X,x}$. Thus we may assume X is normal and projective over A. Let $Y \to \text{Spec}(\mathcal{O}_{X,x}) = \text{Spec}(B)$ be a modification with Y normal. We have to show that $H^1(Y, \mathcal{O}_Y) = 0$. By Limits, Lemma 21.1 we can find a morphism of schemes $g : X' \to X$ which is an isomorphism over $X \setminus \{x\}$ such that $X' \times_X \text{Spec}(\mathcal{O}_{X,x})$ is isomorphic to Y. Then g is a modification as it is proper by Limits, Lemma 21.2. The local ring of X' at a point of x' is either isomorphic to the local ring of X at $g(x')$ if $g(x') \neq x$ and if $g(x') = x$, then the local ring of X' at x' is isomorphic to the local ring of Y at the corresponding point. Hence we see that X' is normal as both X and Y are normal. Thus $H^1(X', \mathcal{O}_{X'}) = 0$ by our assumption on A. By Lemma 8.1 we have $R^1g_*\mathcal{O}_{X'} = 0$. Clearly this means that $H^1(Y, \mathcal{O}_Y) = 0$ as desired. $\square$

Lemma 8.5. *Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2. If reduction to rational singularities is possible for A, then there exists a finite sequence of normalized blowups*

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \text{Spec}(A)$$

in closed points such that for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ defines a rational singularity. In particular $X \to \text{Spec}(A)$ is a modification and X is a normal scheme projective over A.

Proof. We choose a modification $X \to \text{Spec}(A)$ with X normal which maximizes the length of $H^1(X, \mathcal{O}_X)$. By Lemma 8.1 for any further modification $g : X' \to X$ with X' normal we have $R^1g_*\mathcal{O}_{X'} = 0$ and $H^1(X, \mathcal{O}_X) = H^1(X', \mathcal{O}_{X'})$.

Let $x \in X$ be a closed point. We will show that $\mathcal{O}_{X,x}$ defines a rational singularity. Let $Y \to \text{Spec}(\mathcal{O}_{X,x})$ be a modification with Y normal. We have to show that $H^1(Y, \mathcal{O}_Y) = 0$. By Limits, Lemma 21.1 we can find a morphism of schemes $g : X' \to X$ which is an isomorphism over $X \setminus \{x\}$ such that $X' \times_X \text{Spec}(\mathcal{O}_{X,x})$ is isomorphic to Y. Then g is a modification as it is proper by Limits, Lemma 21.2. The local ring of X' at a point of x' is either isomorphic to the local ring of X at $g(x')$ if $g(x') \neq x$ and if $g(x') = x$, then the local ring of X' at x' is isomorphic to the local ring of Y at the corresponding point. Hence we see that X' is normal as both X and Y are normal. By maximality we have $R^1g_*\mathcal{O}_{X'} = 0$ (see first paragraph). Clearly this means that $H^1(Y, \mathcal{O}_Y) = 0$ as desired.

The conclusion is that we've found one normal modification X of $\mathrm{Spec}(A)$ such that the local rings of X at closed points all define rational singularities. Then we choose a sequence of normalized blowups $X_n \to \ldots \to X_1 \to \mathrm{Spec}(A)$ such that X_n dominates X, see Lemma 5.3. For a closed point $x' \in X_n$ mapping to $x \in X$ we can apply Lemma 8.4 to the ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_n,x'}$ to see that $\mathcal{O}_{X_n,x'}$ defines a rational singularity. □

Lemma 8.6. *Let $A \to B$ be a finite injective local ring map of local normal Nagata domains of dimension 2. Assume that the induced extension of fraction fields is separable. If reduction to rational singularities is possible for A then it is possible for B.*

Proof. Let n be the degree of the fraction field extension L/K. Let $\mathrm{Trace}_{L/K} : L \to K$ be the trace. Since the extension is finite separable the trace pairing $(h, g) \mapsto \mathrm{Trace}_{L/K}(fg)$ is a nondegenerate bilinear form on L over K. See Fields, Lemma 20.7. Pick $b_1, \ldots, b_n \in B$ which form a basis of L over K. By the above $d = \det(\mathrm{Trace}_{L/K}(b_i b_j)) \in A$ is nonzero.

Let $Y \to \mathrm{Spec}(B)$ be a modification with Y normal. We can find a U-admissible blowup X' of $\mathrm{Spec}(A)$ such that the strict transform Y' of Y is finite over X', see More on Flatness, Lemma 31.2. Picture

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y \arrow[r] & \mathrm{Spec}(B) \arrow[d] \\ X' \arrow[rr] & & \mathrm{Spec}(A) \end{tikzcd}$$

After replacing X' and Y' by their normalizations we may assume that X' and Y' are normal modifications of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$. In this way we reduce to the case where there exists a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"'] \arrow[d, "\pi"'] & \mathrm{Spec}(B) \arrow[d] \\ X \arrow[r, "f"] & \mathrm{Spec}(A) \end{tikzcd}$$

with X and Y normal modifications of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ and π finite.

The trace map on L over K extends to a map of $\mathcal{O}_X$-modules $\mathrm{Trace} : \pi_*\mathcal{O}_Y \to \mathcal{O}_X$. Consider the map

$$\Phi : \pi_*\mathcal{O}_Y \longrightarrow \mathcal{O}_X^{\oplus n}, \quad s \longmapsto (\mathrm{Trace}(b_1 s), \ldots, \mathrm{Trace}(b_n s))$$

This map is injective (because it is injective in the generic point) and there is a map

$$\mathcal{O}_X^{\oplus n} \longrightarrow \pi_*\mathcal{O}_Y, \quad (s_1, \ldots, s_n) \longmapsto \sum b_i s_i$$

whose composition with Φ has matrix $\mathrm{Trace}(b_i b_j)$. Hence the cokernel of Φ is annihilated by d. Thus we see that we have an exact sequence

$$H^0(X, \mathrm{Coker}(\Phi)) \to H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)^{\oplus n}$$

Since the right hand side is bounded by assumption, it suffices to show that the d-torsion in $H^1(Y, \mathcal{O}_Y)$ is bounded. This is the content of Lemma 8.2 and (8.2.1). □

Lemma 8.7. *Let A be a Nagata regular local ring of dimension 2. Then A defines a rational singularity.*

Proof. (The assumption that A be Nagata is not necessary for this proof, but we've only defined the notion of rational singularity in the case of Nagata 2-dimensional normal local domains.) Let $X \to \mathrm{Spec}(A)$ be a modification with X normal. By Lemma 4.2 we can dominate X by a scheme X_n which is the last in a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$$

of blowing ups in closed points. By Lemma 3.2 the schemes X_i are regular, in particular normal (Algebra, Lemma 157.5). By Lemma 8.1 we have $H^1(X, \mathcal{O}_X) \subset H^1(X_n, \mathcal{O}_{X_n})$. Thus it suffices to prove $H^1(X_n, \mathcal{O}_{X_n}) = 0$. Using Lemma 8.1 again, we see that it suffices to prove $R^1(X_i \to X_{i-1})_*\mathcal{O}_{X_i} = 0$ for $i = 1, \ldots, n$. This follows from Lemma 3.4. □

Lemma 8.8. *Let A be a local normal Nagata domain of dimension 2 which has a dualizing complex $\omega_A^\bullet$. If there exists a nonzero $d \in A$ such that for all normal modifications $X \to \mathrm{Spec}(A)$ the cokernel of the trace map*

$$\Gamma(X, \omega_X) \to \omega_A$$

is annihilated by d, then reduction to rational singularities is possible for A.

Proof. For $X \to \mathrm{Spec}(A)$ as in the statement we have to bound $H^1(X, \mathcal{O}_X)$. Let ω_X be the dualizing module of X as in the statement of Grauert-Riemenschneider (Proposition 7.8). The trace map is the map $Rf_*\omega_X \to \omega_A$ described in Duality for Schemes, Section 7. By Grauert-Riemenschneider we have $Rf_*\omega_X = f_*\omega_X$ thus the trace map indeed produces a map $\Gamma(X, \omega_X) \to \omega_A$. By duality we have $Rf_*\omega_X = R\,\mathrm{Hom}_A(Rf_*\mathcal{O}_X, \omega_A)$ (this uses that $\omega_X[2]$ is the dualizing complex on X normalized relative to $\omega_A[2]$, see Duality for Schemes, Lemma 20.9 or more directly Section 19 or even more directly Example 3.9). The distinguished triangle

$$A \to Rf_*\mathcal{O}_X \to R^1f_*\mathcal{O}_X[-1] \to A[1]$$

is transformed by $R\,\mathrm{Hom}_A(-, \omega_A)$ into the short exact sequence

$$0 \to f_*\omega_X \to \omega_A \to \mathrm{Ext}^2_A(R^1f_*\mathcal{O}_X, \omega_A) \to 0$$

(and $\mathrm{Ext}^i_A(R^1f_*\mathcal{O}_X, \omega_A) = 0$ for $i \neq 2$; this will follow from the discussion below as well). Since $R^1f_*\mathcal{O}_X$ is supported in $\{\mathfrak{m}\}$, the local duality theorem tells us that

$$\mathrm{Ext}^2_A(R^1f_*\mathcal{O}_X, \omega_A) = \mathrm{Ext}^0_A(R^1f_*\mathcal{O}_X, \omega_A[2]) = \mathrm{Hom}_A(R^1f_*\mathcal{O}_X, E)$$

is the Matlis dual of $R^1f_*\mathcal{O}_X$ (and the other ext groups are zero), see Dualizing Complexes, Lemma 18.4. By the equivalence of categories inherent in Matlis duality (Dualizing Complexes, Proposition 7.8), if $R^1f_*\mathcal{O}_X$ is not annihilated by d, then neither is the Ext^2 above. Hence we see that $H^1(X, \mathcal{O}_X)$ is annihilated by d. Thus the required boundedness follows from Lemma 8.2 and (8.2.1). □

Lemma 8.9. *Let p be a prime number. Let A be a regular local ring of dimension 2 and characteristic p. Let $A_0 \subset A$ be a subring such that Ω_{A/A_0} is free of rank $r < \infty$. Set $\omega_A = \Omega^r_{A/A_0}$. If $X \to \mathrm{Spec}(A)$ is the result of a sequence of blowups in closed points, then there exists a map*

$$\varphi_X : (\Omega^r_{X/\mathrm{Spec}(A_0)})^{**} \longrightarrow \omega_X$$

extending the given identification in the generic point.

Proof. Observe that A is Gorenstein (Dualizing Complexes, Lemma 21.3) and hence the invertible module ω_A does indeed serve as a dualizing module. Moreover, any X as in the lemma has an invertible dualizing module ω_X as X is regular (hence Gorenstein) and proper over A, see Remark 7.7 and Lemma 3.2. Suppose we have constructed the map $\varphi_X : (\Omega^r_{X/A_0})^{**} \to \omega_X$ and suppose that $b : X' \to X$ is a blowup in a closed point. Set $\Omega^r_X = (\Omega^r_{X/A_0})^{**}$ and $\Omega^r_{X'} = (\Omega^r_{X'/A_0})^{**}$. Since $\omega_{X'} = b^!(\omega_X)$ a map $\Omega^r_{X'} \to \omega_{X'}$ is the same thing as a map $Rb_*(\Omega^r_{X'}) \to \omega_X$. See discussion in Remark 7.7 and Duality for Schemes, Section 19. Thus in turn it suffices to produce a map

$$Rb_*(\Omega^r_{X'}) \longrightarrow \Omega^r_X$$

The sheaves $\Omega^r_{X'}$ and Ω^r_X are invertible, see Divisors, Lemma 12.15. Consider the exact sequence

$$b^*\Omega_{X/A_0} \to \Omega_{X'/A_0} \to \Omega_{X'/X} \to 0$$

A local calculation shows that $\Omega_{X'/X}$ is isomorphic to an invertible module on the exceptional divisor E, see Lemma 3.6. It follows that either

$$\Omega^r_{X'} \cong (b^*\Omega^r_X)(E) \quad \text{or} \quad \Omega^r_{X'} \cong b^*\Omega^r_X$$

see Divisors, Lemma 15.13. (The second possibility never happens in characteristic zero, but can happen in characteristic p.) In both cases we see that $R^1b_*(\Omega^r_{X'}) = 0$ and $b_*(\Omega^r_{X'}) = \Omega^r_X$ by Lemma 3.4. $\square$

Lemma 8.10. *Let p be a prime number. Let A be a complete regular local ring of dimension 2 and characteristic p. Let L/K be a degree p inseparable extension of the fraction field K of A. Let $B \subset L$ be the integral closure of A. Then reduction to rational singularities is possible for B.*

Proof. We have $A = k[[x, y]]$. Write $L = K[x]/(x^p - f)$ for some $f \in A$ and denote $g \in B$ the congruence class of x, i.e., the element such that $g^p = f$. By Algebra, Lemma 158.2 we see that $\mathrm{d}f$ is nonzero in $\Omega_{K/\mathbf{F}_p}$. By More on Algebra, Lemma 46.5 there exists a subfield $k^p \subset k' \subset k$ with $p^e = [k : k'] < \infty$ such that $\mathrm{d}f$ is nonzero in Ω_{K/K_0} where K_0 is the fraction field of $A_0 = k'[[x^p, y^p]] \subset A$. Then

$$\Omega_{A/A_0} = A \otimes_k \Omega_{k/k'} \oplus A\mathrm{d}x \oplus A\mathrm{d}y$$

is finite free of rank $e + 2$. Set $\omega_A = \Omega^{e+2}_{A/A_0}$. Consider the canonical map

$$\mathrm{Tr} : \Omega^{e+2}_{B/A_0} \longrightarrow \Omega^{e+2}_{A/A_0} = \omega_A$$

of Lemma 2.4. By duality this determines a map

$$c : \Omega^{e+2}_{B/A_0} \to \omega_B = \mathrm{Hom}_A(B, \omega_A)$$

Claim: the cokernel of c is annihilated by a nonzero element of B.

Since $\mathrm{d}f$ is nonzero in Ω_{A/A_0} we can find $\eta_1, \ldots, \eta_{e+1} \in \Omega_{A/A_0}$ such that $\theta = \eta_1 \wedge \ldots \wedge \eta_{e+1} \wedge \mathrm{d}f$ is nonzero in $\omega_A = \Omega^{e+2}_{A/A_0}$. To prove the claim we will construct elements ω_i of Ω^{e+2}_{B/A_0}, $i = 0, \ldots, p-1$ which are mapped to $\varphi_i \in \omega_B = \mathrm{Hom}_A(B, \omega_A)$ with $\varphi_i(g^j) = \delta_{ij}\theta$ for $j = 0, \ldots, p-1$. Since $\{1, g, \ldots, g^{p-1}\}$ is a basis for L/K this proves the claim. We set $\eta = \eta_1 \wedge \ldots \wedge \eta_{e+1}$ so that $\theta = \eta \wedge \mathrm{d}f$. Set $\omega_i = \eta \wedge g^{p-1-i}\mathrm{d}g$. Then by construction we have

$$\varphi_i(g^j) = \mathrm{Tr}(g^j \eta \wedge g^{p-1-i}\mathrm{d}g) = \mathrm{Tr}(\eta \wedge g^{p-1-i+j}\mathrm{d}g) = \delta_{ij}\theta$$

by the explicit description of the trace map in Lemma 2.2.

Let $Y \to \mathrm{Spec}(B)$ be a normal modification. Exactly as in the proof of Lemma 8.6 we can reduce to the case where Y is finite over a modification X of $\mathrm{Spec}(A)$. By Lemma 4.2 we may even assume $X \to \mathrm{Spec}(A)$ is the result of a sequence of blowing ups in closed points. Picture:

$$\begin{tikzcd} Y \arrow[r, "g"'] \arrow[d, "\pi"'] & \mathrm{Spec}(B) \arrow[d] \\ X \arrow[r, "f"] & \mathrm{Spec}(A) \end{tikzcd}$$

We may apply Lemma 2.4 to π and we obtain the first arrow in

$$\pi_*(\Omega^{e+2}_{Y/A_0}) \xrightarrow{\mathrm{Tr}} (\Omega^{e+2}_{X/A_0})^{**} \xrightarrow{\varphi_X} \omega_X$$

and the second arrow is from Lemma 8.9 (because f is a sequence of blowups in closed points). By duality for the finite morphism π this corresponds to a map

$$c_Y : \Omega^{e+2}_{Y/A_0} \longrightarrow \omega_Y$$

extending the map c above. Hence we see that the image of $\Gamma(Y, \omega_Y) \to \omega_B$ contains the image of c. By our claim we see that the cokernel is annihilated by a fixed nonzero element of B. We conclude by Lemma 8.8. $\square$

9. Rational singularities

In this section we reduce from rational singular points to Gorenstein rational singular points. See [Lip69] and [Mat70].

Situation 9.1. Here $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2 which defines a rational singularity. Let s be the closed point of $S = \mathrm{Spec}(A)$ and $U = S \setminus \{s\}$. Let $f : X \to S$ be a modification with X normal. We denote $C_1, \ldots, C_r$ the irreducible components of the special fibre X_s of f.

Lemma 9.2. *In Situation 9.1. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

1. *$H^p(X, \mathcal{F}) = 0$ for $p \notin \{0, 1\}$, and*
2. *$H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ is globally generated.*

Proof. Part (1) follows from Cohomology of Schemes, Lemma 20.9. If $\mathcal{F}$ is globally generated, then there is a surjection $\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$. By part (1) and the long exact sequence of cohomology this induces a surjection on H^1. Since $H^1(X, \mathcal{O}_X) = 0$ as S has a rational singularity, and since $H^1(X, -)$ commutes with direct sums (Cohomology, Lemma 19.1) we conclude. $\square$

Lemma 9.3. *In Situation 9.1 assume $E = X_s$ is an effective Cartier divisor. Let $\mathcal{I}$ be the ideal sheaf of E. Then $H^0(X, \mathcal{I}^n) = \mathfrak{m}^n$ and $H^1(X, \mathcal{I}^n) = 0$.*

Proof. We have $H^0(X, \mathcal{O}_X) = A$, see discussion following Situation 7.1. Then $\mathfrak{m} \subset H^0(X, \mathcal{I}) \subset H^0(X, \mathcal{O}_X)$. The second inclusion is not an equality as $X_s \neq \emptyset$. Thus $H^0(X, \mathcal{I}) = \mathfrak{m}$. As $\mathcal{I}^n = \mathfrak{m}^n \mathcal{O}_X$ our Lemma 9.2 shows that $H^1(X, \mathcal{I}^n) = 0$.

Choose generators $x_1, \ldots, x_{\mu+1}$ of $\mathfrak{m}$. These define global sections of $\mathcal{I}$ which generate it. Hence a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus \mu+1} \to \mathcal{I} \to 0$$

Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module of rank μ and $\mathcal{F} \otimes \mathcal{I}$ is globally generated by Constructions, Lemma 13.9. Hence $\mathcal{F} \otimes \mathcal{I}^n$ is globally generated for all $n \geq 1$. Thus for $n \geq 2$ we can consider the exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{I}^{n-1} \to (\mathcal{I}^{n-1})^{\oplus \mu+1} \to \mathcal{I}^n \to 0$$

Applying the long exact sequence of cohomology using that $H^1(X, \mathcal{F} \otimes \mathcal{I}^{n-1}) = 0$ by Lemma 9.2 we obtain that every element of $H^0(X, \mathcal{I}^n)$ is of the form $\sum x_i a_i$ for some $a_i \in H^0(X, \mathcal{I}^{n-1})$. This shows that $H^0(X, \mathcal{I}^n) = \mathfrak{m}^n$ by induction. □

Lemma 9.4. *In Situation 9.1 the blowup of* $\text{Spec}(A)$ *in* $\mathfrak{m}$ *is normal.*

Proof. Let $X' \to \text{Spec}(A)$ be the blowup, in other words

$$X' = \text{Proj}(A \oplus \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \ldots).$$

is the Proj of the Rees algebra. This in particular shows that X' is integral and that $X' \to \text{Spec}(A)$ is a projective modification. Let X be the normalization of X'. Since A is Nagata, we see that $\nu : X \to X'$ is finite (Morphisms, Lemma 54.10). Let $E' \subset X'$ be the exceptional divisor and let $E \subset X$ be the inverse image. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be their ideal sheaves. Recall that $\mathcal{I}' = \mathcal{O}_{X'}(1)$ (Divisors, Lemma 32.13). Observe that $\mathcal{I} = \nu^*\mathcal{I}'$ and that E is an effective Cartier divisor (Divisors, Lemma 13.13). We are trying to show that ν is an isomorphism. As ν is finite, it suffices to show that $\mathcal{O}_{X'} \to \nu_*\mathcal{O}_X$ is an isomorphism. If not, then we can find an $n \geq 0$ such that

$$H^0(X', (\mathcal{I}')^n) \neq H^0(X', (\nu_*\mathcal{O}_X) \otimes (\mathcal{I}')^n)$$

for example because we can recover quasi-coherent $\mathcal{O}_{X'}$-modules from their associated graded modules, see Properties, Lemma 28.3. By the projection formula we have

$$H^0(X', (\nu_*\mathcal{O}_X) \otimes (\mathcal{I}')^n) = H^0(X, \nu^*(\mathcal{I}')^n) = H^0(X, \mathcal{I}^n) = \mathfrak{m}^n$$

the last equality by Lemma 9.3. On the other hand, there is clearly an injection $\mathfrak{m}^n \to H^0(X', (\mathcal{I}')^n)$. Since $H^0(X', (\mathcal{I}')^n)$ is torsion free we conclude equality holds for all n, hence $X = X'$. □

Lemma 9.5. *In Situation 9.1. Let X be the blowup of* $\text{Spec}(A)$ *in* $\mathfrak{m}$. *Let $E \subset X$ be the exceptional divisor. With $\mathcal{O}_X(1) = \mathcal{I}$ as usual and $\mathcal{O}_E(1) = \mathcal{O}_X(1)|_E$ we have*

1. *E is a proper Cohen-Macaulay curve over κ.*
2. *$\mathcal{O}_E(1)$ is very ample*
3. $\deg(\mathcal{O}_E(1)) \geq 1$ *and equality holds only if A is a regular local ring,*
4. *$H^1(E, \mathcal{O}_E(n)) = 0$ for $n \geq 0$, and*
5. *$H^0(E, \mathcal{O}_E(n)) = \mathfrak{m}^n/\mathfrak{m}^{n+1}$ for $n \geq 0$.*

Proof. Since $\mathcal{O}_X(1)$ is very ample by construction, we see that its restriction to the special fibre E is very ample as well. By Lemma 9.4 the scheme X is normal. Then E is Cohen-Macaulay by Divisors, Lemma 15.6. Lemma 9.3 applies and we obtain (4) and (5) from the exact sequences

$$0 \to \mathcal{I}^{n+1} \to \mathcal{I}^n \to i_*\mathcal{O}_E(n) \to 0$$

and the long exact cohomology sequence. In particular, we see that

$$\deg(\mathcal{O}_E(1)) = \chi(E, \mathcal{O}_E(1)) - \chi(E, \mathcal{O}_E) = \dim(\mathfrak{m}/\mathfrak{m}^2) - 1$$

by Varieties, Definition 44.1. Thus (3) follows as well. □

Lemma 9.6. *In Situation 9.1 assume A has a dualizing complex $\omega_A^\bullet$. With ω_X the dualizing module of X, the trace map $H^0(X, \omega_X) \to \omega_A$ is an isomorphism and consequently there is a canonical map $f^*\omega_A \to \omega_X$.*

Proof. By Grauert-Riemenschneider (Proposition 7.8) we see that $Rf_*\omega_X = f_*\omega_X$. By duality we have a short exact sequence

$$0 \to f_*\omega_X \to \omega_A \to \mathrm{Ext}^2_A(R^1 f_*\mathcal{O}_X, \omega_A) \to 0$$

(for example see proof of Lemma 8.8) and since A defines a rational singularity we obtain $f_*\omega_X = \omega_A$. □

Lemma 9.7. *In Situation 9.1 assume A has a dualizing complex $\omega_A^\bullet$ and is not regular. Let X be the blowup of* $\mathrm{Spec}(A)$ *in* $\mathfrak{m}$ *with exceptional divisor $E \subset X$. Let ω_X be the dualizing module of X. Then*

1. $\omega_E = \omega_X|_E \otimes \mathcal{O}_E(-1)$,
2. $H^1(X, \omega_X(n)) = 0$ *for* $n \geq 0$,
3. *the map $f^*\omega_A \to \omega_X$ of Lemma 9.6 is surjective.*

Proof. We will use the results of Lemma 9.5 without further mention. Observe that $\omega_E = \omega_X|_E \otimes \mathcal{O}_E(-1)$ by Duality for Schemes, Lemmas 14.2 and 9.7. Thus $\omega_X|_E = \omega_E(1)$. Consider the short exact sequences

$$0 \to \omega_X(n+1) \to \omega_X(n) \to i_*\omega_E(n+1) \to 0$$

By Algebraic Curves, Lemma 6.4 we see that $H^1(E, \omega_E(n+1)) = 0$ for $n \geq 0$. Thus we see that the maps

$$\ldots \to H^1(X, \omega_X(2)) \to H^1(X, \omega_X(1)) \to H^1(X, \omega_X)$$

are surjective. Since $H^1(X, \omega_X(n))$ is zero for $n \gg 0$ (Cohomology of Schemes, Lemma 16.2) we conclude that (2) holds.

By Algebraic Curves, Lemma 6.7 we see that $\omega_X|_E = \omega_E \otimes \mathcal{O}_E(1)$ is globally generated. Since we seen above that $H^1(X, \omega_X(1)) = 0$ the map $H^0(X, \omega_X) \to H^0(E, \omega_X|_E)$ is surjective. We conclude that ω_X is globally generated hence (3) holds because $\Gamma(X, \omega_X) = \omega_A$ is used in Lemma 9.6 to define the map. □

Lemma 9.8. *Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension* 2 *which defines a rational singularity. Assume A has a dualizing complex. Then there exists a finite sequence of blowups in singular closed points*

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$$

such that X_i is normal for each i and such that the dualizing sheaf ω_X of X is an invertible $\mathcal{O}_X$-module.

Proof. The dualizing module ω_A is a finite A-module whose stalk at the generic point is invertible. Namely, $\omega_A \otimes_A K$ is a dualizing module for the fraction field K of A, hence has rank 1. Thus there exists a blowup $b : Y \to \mathrm{Spec}(A)$ such that the strict transform of ω_A with respect to b is an invertible $\mathcal{O}_Y$-module, see Divisors, Lemma 35.3. By Lemma 5.3 we can choose a sequence of normalized blowups

$$X_n \to X_{n-1} \to \ldots \to X_1 \to \mathrm{Spec}(A)$$

such that X_n dominates Y. By Lemma 9.4 and arguing by induction each $X_i \to X_{i-1}$ is simply a blowing up.

We claim that ω_{X_n} is invertible. Since ω_{X_n} is a coherent $\mathcal{O}_{X_n}$-module, it suffices to see its stalks are invertible modules. If $x \in X_n$ is a regular point, then this is clear from the fact that regular schemes are Gorenstein (Dualizing Complexes, Lemma 21.3). If x is a singular point of X_n, then each of the images $x_i \in X_i$ of x is a singular point (because the blowup of a regular point is regular by Lemma 3.2). Consider the canonical map $f_n^*\omega_A \to \omega_{X_n}$ of Lemma 9.6. For each i the morphism $X_{i+1} \to X_i$ is either a blowup of x_i or an isomorphism at x_i. Since x_i is always a singular point, it follows from Lemma 9.7 and induction that the maps $f_i^*\omega_A \to \omega_{X_i}$ is always surjective on stalks at x_i. Hence

$$(f_n^*\omega_A)_x \longrightarrow \omega_{X_n,x}$$

is surjective. On the other hand, by our choice of b the quotient of $f_n^*\omega_A$ by its torsion submodule is an invertible module $\mathcal{L}$. Moreover, the dualizing module is torsion free (Duality for Schemes, Lemma 22.3). It follows that $\mathcal{L}_x \cong \omega_{X_n,x}$ and the proof is complete. $\square$

10. Formal arcs

Let X be a locally Noetherian scheme. In this section we say that a *formal arc* in X is a morphism $a : T \to X$ where T is the spectrum of a complete discrete valuation ring R whose residue field κ is identified with the residue field of the image p of the closed point of $\text{Spec}(R)$. Let us say that the formal arc a is *centered at p* in this case. We say the formal arc $T \to X$ is *nonsingular* if the induced map $\mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathfrak{m}_R/\mathfrak{m}_R^2$ is surjective.

Let $a : T \to X$, $T = \text{Spec}(R)$ be a nonsingular formal arc centered at a closed point p of X. Assume X is locally Noetherian. Let $b : X_1 \to X$ be the blowing up of X at x. Since a is nonsingular, we see that there is an element $f \in \mathfrak{m}_p$ which maps to a uniformizer in R. In particular, we find that the generic point of T maps to a point of X not equal to p. In other words, with K the fraction field of R, the restriction of a defines a morphism $\text{Spec}(K) \to X \setminus \{p\}$. Since the morphism b is proper and an isomorphism over $X \setminus \{x\}$ we can apply the valuative criterion of properness to obtain a unique morphism a_1 making the following diagram commute

$$\begin{tikzcd} T \arrow[r, "a_1"'] \arrow[dr, "a"'] & X_1 \arrow[d, "b"] \\ & X \end{tikzcd}$$

Let $p_1 \in X_1$ be the image of the closed point of T. Observe that p_1 is a closed point as it is a $\kappa = \kappa(p)$-rational point on the fibre of $X_1 \to X$ over x. Since we have a factorization

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X_1,p_1} \to R$$

we see that a_1 is a nonsingular formal arc as well.

We can repeat the process and obtain a sequence of blowing ups

$$\begin{tikzcd} T \arrow[d, "a"'] \arrow[dr, "a_1"'] \arrow[drr, "a_2"'] \arrow[drrr, "a_3"] & & & & \\ (X,p) & (X_1,p_1) \arrow[l] & (X_2,p_2) \arrow[l] & (X_3,p_3) \arrow[l] & \ldots \arrow[l] \end{tikzcd}$$

This kind of sequence of blowups can be characterized as follows.

Lemma 10.1. *Let X be a locally Noetherian scheme. Let*

$$(X, p) = (X_0, p_0) \leftarrow (X_1, p_1) \leftarrow (X_2, p_2) \leftarrow (X_3, p_3) \leftarrow \ldots$$

be a sequence of blowups such that

(1) *p_i is closed, maps to p_{i-1}, and $\kappa(p_i) = \kappa(p_{i-1})$,*
(2) *there exists an $x_1 \in \mathfrak{m}_p$ whose image in $\mathfrak{m}_{p_i}$, $i > 0$ defines the exceptional divisor $E_i \subset X_i$.*

Then the sequence is obtained from a nonsingular arc $a : T \to X$ as above.

Proof. Let us write $\mathcal{O}_n = \mathcal{O}_{X_n, p_n}$ and $\mathcal{O} = \mathcal{O}_{X,p}$. Denote $\mathfrak{m} \subset \mathcal{O}$ and $\mathfrak{m}_n \subset \mathcal{O}_n$ the maximal ideals.

We claim that $x_1^t \not\in \mathfrak{m}_n^{t+1}$. Namely, if this were the case, then in the local ring $\mathcal{O}_{n+1}$ the element x_1^t would be in the ideal of $(t+1)E_{n+1}$. This contradicts the assumption that x_1 defines E_{n+1}.

For every n choose generators $y_{n,1}, \ldots, y_{n,t_n}$ for $\mathfrak{m}_n$. As $\mathfrak{m}_n \mathcal{O}_{n+1} = x_1 \mathcal{O}_{n+1}$ by assumption (2), we can write $y_{n,i} = a_{n,i} x_1$ for some $a_{n,i} \in \mathcal{O}_{n+1}$. Since the map $\mathcal{O}_n \to \mathcal{O}_{n+1}$ defines an isomorphism on residue fields by (1) we can choose $c_{n,i} \in \mathcal{O}_n$ having the same residue class as $a_{n,i}$. Then we see that

$$\mathfrak{m}_n = (x_1, z_{n,1}, \ldots, z_{n,t_n}), \quad z_{n,i} = y_{n,i} - c_{n,i} x_1$$

and the elements $z_{n,i}$ map to elements of $\mathfrak{m}_{n+1}^2$ in $\mathcal{O}_{n+1}$.

Let us consider

$$J_n = \text{Ker}(\mathcal{O} \to \mathcal{O}_n / \mathfrak{m}_n^{n+1})$$

We claim that $\mathcal{O}/J_n$ has length $n+1$ and that $\mathcal{O}/(x_1) + J_n$ equals the residue field. For $n = 0$ this is immediate. Assume the statement holds for n. Let $f \in J_n$. Then in $\mathcal{O}_n$ we have

$$f = a x_1^{n+1} + x_1^n A_1(z_{n,i}) + x_1^{n-1} A_2(z_{n,i}) + \ldots + A_{n+1}(z_{n,i})$$

for some $a \in \mathcal{O}_n$ and some A_i homogeneous of degree i with coefficients in $\mathcal{O}_n$. Since $\mathcal{O} \to \mathcal{O}_n$ identifies residue fields, we may choose $a \in \mathcal{O}$ (argue as in the construction of $z_{n,i}$ above). Taking the image in $\mathcal{O}_{n+1}$ we see that f and $a x_1^{n+1}$ have the same image modulo $\mathfrak{m}_{n+1}^{n+2}$. Since $x_n^{n+1} \not\in \mathfrak{m}_{n+1}^{n+2}$ it follows that J_n / J_{n+1} has length 1 and the claim is true.

Consider $R = \lim \mathcal{O}/J_n$. This is a quotient of the $\mathfrak{m}$-adic completion of $\mathcal{O}$ hence it is a complete Noetherian local ring. On the other hand, it is not finite length and x_1 generates the maximal ideal. Thus R is a complete discrete valuation ring. The map $\mathcal{O} \to R$ lifts to a local homomorphism $\mathcal{O}_n \to R$ for every n. There are two ways to show this: (1) for every n one can use a similar procedure to construct $\mathcal{O}_n \to R_n$ and then one can show that $\mathcal{O} \to \mathcal{O}_n \to R_n$ factors through an isomorphism $R \to R_n$, or (2) one can use Divisors, Lemma 32.6 to show that $\mathcal{O}_n$ is a localization of a repeated affine blowup algebra to explicitly construct a map $\mathcal{O}_n \to R$. Having said this it is clear that our sequence of blowups comes from the nonsingular arc $a : T = \text{Spec}(R) \to X$. $\square$

The following lemma is a kind of Néron desingularization lemma.

Lemma 10.2. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local domain of dimension 2. Let $A \to R$ be a surjection onto a complete discrete valuation ring. This defines a nonsingular arc $a : T = \mathrm{Spec}(R) \to \mathrm{Spec}(A)$. Let*

$$\mathrm{Spec}(A) = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \ldots$$

be the sequence of blowing ups constructed from a. If $A_\mathfrak{p}$ is a regular local ring where $\mathfrak{p} = \mathrm{Ker}(A \to R)$, then for some i the scheme X_i is regular at x_i.

Proof. Let $x_1 \in \mathfrak{m}$ map to a uniformizer of R. Observe that $\kappa(\mathfrak{p}) = K$ is the fraction field of R. Write $\mathfrak{p} = (x_2, \ldots, x_r)$ with r minimal. If $r = 2$, then $\mathfrak{m} = (x_1, x_2)$ and A is regular and the lemma is true. Assume $r > 2$. After renumbering if necessary, we may assume that x_2 maps to a uniformizer of $A_\mathfrak{p}$. Then $\mathfrak{p}/\mathfrak{p}^2 + (x_2)$ is annihilated by a power of x_1. For $i > 2$ we can find $n_i \geq 0$ and $a_i \in A$ such that

$$x_1^{n_i} x_i - a_i x_2 = \sum\nolimits_{2 \leq j \leq k} a_{jk} x_j x_k$$

for some $a_{jk} \in A$. If $n_i = 0$ for some i, then we can remove x_i from the list of generators of $\mathfrak{p}$ and we win by induction on r. If for some i the element a_i is a unit, then we can remove x_2 from the list of generators of $\mathfrak{p}$ and we win in the same manner. Thus either $a_i \in \mathfrak{p}$ or $a_i = u_i x_1^{m_1} \bmod \mathfrak{p}$ for some $m_1 > 0$ and unit $u_i \in A$. Thus we have either

$$x_1^{n_i} x_i = \sum\nolimits_{2 \leq j \leq k} a_{jk} x_j x_k \quad \text{or} \quad x_1^{n_i} x_i - u_i x_1^{m_i} x_2 = \sum\nolimits_{2 \leq j \leq k} a_{jk} x_j x_k$$

We will prove that after blowing up the integers n_i, m_i decrease which will finish the proof.

Let us see what happens with these equations on the affine blowup algebra $A' = A[\mathfrak{m}/x_1]$. As $\mathfrak{m} = (x_1, \ldots, x_r)$ we see that A' is generated over R by $y_i = x_i/x_1$ for $i \geq 2$. Clearly $A \to R$ extends to $A' \to R$ with kernel $(y_2, \ldots, y_r)$. Then we see that either

$$x_1^{n_i - 1} y_i = \sum\nolimits_{2 \leq j \leq k} a_{jk} y_j y_k \quad \text{or} \quad x_1^{n_i - 1} y_i - u_i x_1^{m_1 - 1} y_2 = \sum\nolimits_{2 \leq j \leq k} a_{jk} y_j y_k$$

and the proof is complete. $\square$

11. Base change to the completion

The following simple lemma will turn out to be a useful tool in what follows.

Lemma 11.1. *Let $(A, \mathfrak{m}, \kappa)$ be a local ring with finitely generated maximal ideal $\mathfrak{m}$. Let X be a scheme over A. Let $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ where $A^\wedge$ is the $\mathfrak{m}$-adic completion of A. For a point $q \in Y$ with image $p \in X$ lying over the closed point of $\mathrm{Spec}(A)$ the local ring map $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,q}$ induces an isomorphism on completions.*

Proof. We may assume X is affine. Then we may write $X = \mathrm{Spec}(B)$. Let $\mathfrak{q} \subset B' = B \otimes_A A^\wedge$ be the prime corresponding to q and let $\mathfrak{p} \subset B$ be the prime ideal corresponding to p. By Algebra, Lemma 96.3 we have

$$B'/(\mathfrak{m}^\wedge)^n B' = A^\wedge/(\mathfrak{m}^\wedge)^n \otimes_A B = A/\mathfrak{m}^n \otimes_A B = B/\mathfrak{m}^n B$$

for all n. Since $\mathfrak{m}B \subset \mathfrak{p}$ and $\mathfrak{m}^\wedge B' \subset \mathfrak{q}$ we see that $B/\mathfrak{p}^n$ and $B'/\mathfrak{q}^n$ are both quotients of the ring displayed above by the nth power of the same prime ideal. The lemma follows. $\square$

Lemma 11.2. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $X \to \mathrm{Spec}(A)$ be a morphism which is locally of finite type. Set $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$. Let $y \in Y$ with image $x \in X$. Then*

1. *if $\mathcal{O}_{Y,y}$ is regular, then $\mathcal{O}_{X,x}$ is regular,*
2. *if y is in the closed fibre, then $\mathcal{O}_{Y,y}$ is regular $\Leftrightarrow$ $\mathcal{O}_{X,x}$ is regular, and*
3. *If X is proper over A, then X is regular if and only if Y is regular.*

Proof. Since $A \to A^\wedge$ is faithfully flat (Algebra, Lemma 97.3), we see that $Y \to X$ is flat. Hence (1) by Algebra, Lemma 164.4. Lemma 11.1 shows the morphism $Y \to X$ induces an isomorphism on complete local rings at points of the special fibres. Thus (2) by More on Algebra, Lemma 43.4. If X is proper over A, then Y is proper over $A^\wedge$ (Morphisms, Lemma 41.5) and we see every closed point of X and Y lies in the closed fibre. Thus we see that Y is a regular scheme if and only if X is so by Properties, Lemma 9.2. $\square$

Lemma 11.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with completion $A^\wedge$. Let $U \subset \mathrm{Spec}(A)$ and $U^\wedge \subset \mathrm{Spec}(A^\wedge)$ be the punctured spectra. If $Y \to \mathrm{Spec}(A^\wedge)$ is a $U^\wedge$-admissible blowup, then there exists a U-admissible blowup $X \to \mathrm{Spec}(A)$ such that $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$.*

Proof. By definition there exists an ideal $J \subset A^\wedge$ such that $V(J) = \{\mathfrak{m}A^\wedge\}$ and such that Y is the blowup of $S^\wedge$ in the closed subscheme defined by J, see Divisors, Definition 34.1. Since $A^\wedge$ is Noetherian this implies $\mathfrak{m}^n A^\wedge \subset J$ for some n. Since $A^\wedge/\mathfrak{m}^n A^\wedge = A/\mathfrak{m}^n$ we find an ideal $\mathfrak{m}^n \subset I \subset A$ such that $J = IA^\wedge$. Let $X \to S$ be the blowup in I. Since $A \to A^\wedge$ is flat we conclude that the base change of X is Y by Divisors, Lemma 32.3. $\square$

Lemma 11.4. *Let $(A, \mathfrak{m}, \kappa)$ be a Nagata local normal domain of dimension 2. Assume A defines a rational singularity and that the completion $A^\wedge$ of A is normal. Then*

1. *$A^\wedge$ defines a rational singularity, and*
2. *if $X \to \mathrm{Spec}(A)$ is the blowing up in $\mathfrak{m}$, then for a closed point $x \in X$ the completion $\mathcal{O}_{X,x}$ is normal.*

Proof. Let $Y \to \mathrm{Spec}(A^\wedge)$ be a modification with Y normal. We have to show that $H^1(Y, \mathcal{O}_Y) = 0$. By Varieties, Lemma 17.3 $Y \to \mathrm{Spec}(A^\wedge)$ is an isomorphism over the punctured spectrum $U^\wedge = \mathrm{Spec}(A^\wedge) \setminus \{\mathfrak{m}^\wedge\}$. By Lemma 7.2 there exists a $U^\wedge$-admissible blowup $Y' \to \mathrm{Spec}(A^\wedge)$ dominating Y. By Lemma 11.3 we find there exists a U-admissible blowup $X \to \mathrm{Spec}(A)$ whose base change to $A^\wedge$ dominates Y. Since A is Nagata, we can replace X by its normalization after which $X \to \mathrm{Spec}(A)$ is a normal modification (but possibly no longer a U-admissible blowup). Then $H^1(X, \mathcal{O}_X) = 0$ as A defines a rational singularity. It follows that $H^1(X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge), \mathcal{O}_{X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)}) = 0$ by flat base change (Cohomology of Schemes, Lemma 5.2 and flatness of $A \to A^\wedge$ by Algebra, Lemma 97.2). We find that $H^1(Y, \mathcal{O}_Y) = 0$ by Lemma 8.1.

Finally, let $X \to \mathrm{Spec}(A)$ be the blowing up of $\mathrm{Spec}(A)$ in $\mathfrak{m}$. Then $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$ is the blowing up of $\mathrm{Spec}(A^\wedge)$ in $\mathfrak{m}^\wedge$. By Lemma 9.4 we see that both Y and X are normal. On the other hand, $A^\wedge$ is excellent (More on Algebra, Proposition 52.3) hence every affine open in Y is the spectrum of an excellent normal domain (More on Algebra, Lemma 52.2). Thus for $y \in Y$ the ring map

$\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}^\wedge$ is regular and by More on Algebra, Lemma 42.2 we find that $\mathcal{O}_{Y,y}^\wedge$ is normal. If $x \in X$ is a closed point of the special fibre, then there is a unique closed point $y \in Y$ lying over x. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ induces an isomorphism on completions (Lemma 11.1) we conclude. □

Lemma 11.5. *Let $(A, \mathfrak{m})$ be a local Noetherian ring. Let X be a scheme over A. Assume*

1. *A is analytically unramified (Algebra, Definition 162.9),*
2. *X is locally of finite type over A, and*
3. *$X \to \mathrm{Spec}(A)$ is étale at the generic points of irreducible components of X.*

Then the normalization of X is finite over X.

Proof. Since A is analytically unramified it is reduced by Algebra, Lemma 162.10. Since the normalization of X depends only on the reduction of X, we may replace X by its reduction X_{red}; note that $X_{red} \to X$ is an isomorphism over the open U where $X \to \mathrm{Spec}(A)$ is étale because U is reduced (Descent, Lemma 18.1) hence condition (3) remains true after this replacement. In addition we may and do assume that $X = \mathrm{Spec}(B)$ is affine.

The map

$$K = \prod\nolimits_{\mathfrak{p} \subset A \text{ minimal}} \kappa(\mathfrak{p}) \longrightarrow K^\wedge = \prod\nolimits_{\mathfrak{p}^\wedge \subset A^\wedge \text{ minimal}} \kappa(\mathfrak{p}^\wedge)$$

is injective because $A \to A^\wedge$ is faithfully flat (Algebra, Lemma 97.3) hence induces a surjective map between sets of minimal primes (by going down for flat ring maps, see Algebra, Section 41). Both sides are finite products of fields as our rings are Noetherian. Let $L = \prod_{\mathfrak{q} \subset B \text{ minimal}} \kappa(\mathfrak{q})$. Our assumption (3) implies that $L = B \otimes_A K$ and that $K \to L$ is a finite étale ring map (this is true because $A \to B$ is generically finite, for example use Algebra, Lemma 122.10 or the more detailed results in Morphisms, Section 51). Since B is reduced we see that $B \subset L$. This implies that

$$C = B \otimes_A A^\wedge \subset L \otimes_A A^\wedge = L \otimes_K K^\wedge = M$$

Then M is the total ring of fractions of C and is a finite product of fields as a finite separable algebra over $K^\wedge$. It follows that C is reduced and that its normalization C' is the integral closure of C in M. The normalization B' of B is the integral closure of B in L. By flatness of $A \to A^\wedge$ we obtain an injective map $B' \otimes_A A^\wedge \to M$ whose image is contained in C'. Picture

$$B' \otimes_A A^\wedge \longrightarrow C'$$

As $A^\wedge$ is Nagata (by Algebra, Lemma 162.8), we see that C' is finite over $C = B \otimes_A A^\wedge$ (see Algebra, Lemmas 162.8 and 162.2). As C is Noetherian, we conclude that $B' \otimes_A A^\wedge$ is finite over $C = B \otimes_A A^\wedge$. Therefore by faithfully flat descent (Algebra, Lemma 83.2) we see that B' is finite over B which is what we had to show. □

Lemma 11.6. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $X \to \mathrm{Spec}(A)$ be a morphism which is locally of finite type. Set $Y = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$. If the complement of the special fibre in Y is normal, then the normalization $X^\nu \to X$ is finite and the base change of X^ν to $\mathrm{Spec}(A^\wedge)$ recovers the normalization of Y.*

Proof. There is an immediate reduction to the case where $X = \text{Spec}(B)$ is affine with B a finite type A-algebra. Set $C = B \otimes_A A^\wedge$ so that $Y = \text{Spec}(C)$. Since $A \to A^\wedge$ is faithfully flat, for any prime $\mathfrak{q} \subset B$ there exists a prime $\mathfrak{r} \subset C$ lying over $\mathfrak{q}$. Then $B_\mathfrak{q} \to C_\mathfrak{r}$ is faithfully flat. Hence if $\mathfrak{q}$ does not lie over $\mathfrak{m}$, then $C_\mathfrak{r}$ is normal by assumption on Y and we conclude that $B_\mathfrak{q}$ is normal by Algebra, Lemma 164.3. In this way we see that X is normal away from the special fibre.

Recall that the complete Noetherian local ring $A^\wedge$ is Nagata (Algebra, Lemma 162.8). Hence the normalization $Y^\nu \to Y$ is finite (Morphisms, Lemma 54.10) and an isomorphism away from the special fibre. Say $Y^\nu = \text{Spec}(C')$. Then $C \to C'$ is finite and an isomorphism away from $V(\mathfrak{m}C)$. Since $B \to C$ is flat and induces an isomorphism $B/\mathfrak{m}B \to C/\mathfrak{m}C$ there exists a finite ring map $B \to B'$ whose base change to C recovers $C \to C'$. See More on Algebra, Lemma 89.16 and Remark 89.19. Thus we find a finite morphism $X' \to X$ which is an isomorphism away from the special fibre and whose base change recovers $Y^\nu \to Y$. By the discussion in the first paragraph we see that X' is normal at points not on the special fibre. For a point $x \in X'$ on the special fibre we have a corresponding point $y \in Y^\nu$ and a flat map $\mathcal{O}_{X',x} \to \mathcal{O}_{Y^\nu,y}$. Since $\mathcal{O}_{Y^\nu,y}$ is normal, so is $\mathcal{O}_{X',x}$, see Algebra, Lemma 164.3. Thus X' is normal and it follows that it is the normalization of X. □

Lemma 11.7. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local domain whose completion $A^\wedge$ is normal. Then given any sequence*

$$Y_n \to Y_{n-1} \to \ldots \to Y_1 \to \text{Spec}(A^\wedge)$$

of normalized blowups, there exists a sequence of (proper) normalized blowups

$$X_n \to X_{n-1} \to \ldots \to X_1 \to \text{Spec}(A)$$

whose base change to $A^\wedge$ recovers the given sequence.

Proof. Given the sequence $Y_n \to \ldots \to Y_1 \to Y_0 = \text{Spec}(A^\wedge)$ we inductively construct $X_n \to \ldots \to X_1 \to X_0 = \text{Spec}(A)$. The base case is $i = 0$. Given X_i whose base change is Y_i, let $Y_i' \to Y_i$ be the blowing up in the closed point $y_i \in Y_i$ such that Y_{i+1} is the normalization of Y_i. Since the closed fibres of Y_i and X_i are isomorphic, the point y_i corresponds to a closed point x_i on the special fibre of X_i. Let $X_i' \to X_i$ be the blowup of X_i in x_i. Then the base change of X_i' to $\text{Spec}(A^\wedge)$ is isomorphic to Y_i'. By Lemma 11.6 the normalization $X_{i+1} \to X_i'$ is finite and its base change to $\text{Spec}(A^\wedge)$ is isomorphic to Y_{i+1}. □

12. Rational double points

In Section 9 we argued that resolution of 2-dimensional rational singularities reduces to the Gorenstein case. A Gorenstein rational surface singularity is a rational double point. We will resolve them by explicit computations.

According to the discussion in Examples, Section 19 there exists a normal Noetherian local domain A whose completion is isomorphic to $\mathbf{C}[[x, y, z]]/(z^2)$. In this case one could say that A has a rational double point singularity, but on the other hand, $\text{Spec}(A)$ does not have a resolution of singularities. This kind of behaviour cannot occur if A is a Nagata ring, see Algebra, Lemma 162.13.

However, it gets worse as there exists a local normal Nagata domain A whose completion is $\mathbf{C}[[x, y, z]]/(yz)$ and another whose completion is $\mathbf{C}[[x, y, z]]/(y^2 - z^3)$.

This is Example 2.5 of [Nis12]. This is why we need to assume the completion of our ring is normal in this section.

Situation 12.1. Here $(A, \mathfrak{m}, \kappa)$ be a Nagata local normal domain of dimension 2 which defines a rational singularity, whose completion is normal, and which is Gorenstein. We assume A is not regular.

The arguments in this section will show that repeatedly blowing up singular points resolves $\mathrm{Spec}(A)$ in this situation. We will need the following lemma in the course of the proof.

Lemma 12.2. *Let κ be a field. Let $I \subset \kappa[x, y]$ be an ideal. Let*

$$a + bx + cy + dx^2 + exy + fy^2 \in I^2$$

for some $a, b, c, d, e, f \in k$ not all zero. If the colength of I in $\kappa[x, y]$ is > 1, then $a + bx + cy + dx^2 + exy + fy^2 = j(g + hx + iy)^2$ for some $j, g, h, i \in \kappa$.

Proof. Consider the partial derivatives $b + 2dx + ey$ and $c + ex + 2fy$. By the Leibniz rules these are contained in I. If one of these is nonzero, then after a linear change of coordinates, i.e., of the form $x \mapsto \alpha + \beta x + \gamma y$ and $y \mapsto \delta + \epsilon x + \zeta y$, we may assume that $x \in I$. Then we see that $I = (x)$ or $I = (x, F)$ with F a monic polynomial of degree ≥ 2 in y. In the first case the statement is clear. In the second case observe that we can write any element in I^2 in the form

$$A(x, y)x^2 + B(y)xF + C(y)F^2$$

for some $A(x, y) \in \kappa[x, y]$ and $B, C \in \kappa[y]$. Thus

$$a + bx + cy + dx^2 + exy + fy^2 = A(x, y)x^2 + B(y)xF + C(y)F^2$$

and by degree reasons we see that $B = C = 0$ and A is a constant.

To finish the proof we need to deal with the case that both partial derivatives are zero. This can only happen in characteristic 2 and then we get

$$a + dx^2 + fy^2 \in I^2$$

We may assume f is nonzero (if not, then switch the roles of x and y). After dividing by f we obtain the case where the characteristic of κ is 2 and

$$a + dx^2 + y^2 \in I^2$$

If a and d are squares in κ, then we are done. If not, then there exists a derivation $\theta : \kappa \to \kappa$ with $\theta(a) \neq 0$ or $\theta(d) \neq 0$, see Algebra, Lemma 158.2. We can extend this to a derivation of $\kappa[x, y]$ by setting $\theta(x) = \theta(y) = 0$. Then we find that

$$\theta(a) + \theta(d)x^2 \in I$$

The case $\theta(d) = 0$ is absurd. Thus we may assume that $\alpha + x^2 \in I$ for some $\alpha \in \kappa$. Combining with the above we find that $a + \alpha d + y^2 \in I$. Hence

$$J = (\alpha + x^2, a + \alpha d + y^2) \subset I$$

with codimension at most 2. Observe that J/J^2 is free over $\kappa[x, y]/J$ with basis $\alpha + x^2$ and $a + \alpha d + y^2$. Thus $a + dx^2 + y^2 = 1 \cdot (a + \alpha d + y^2) + d \cdot (\alpha + x^2) \in I^2$ implies that the inclusion $J \subset I$ is strict. Thus we find a nonzero element of the form $g + hx + iy + jxy$ in I. If $j = 0$, then I contains a linear form and we can conclude as in the first paragraph. Thus $j \neq 0$ and $\dim_\kappa(I/J) = 1$ (otherwise we could find an element as above in I with $j = 0$). We conclude that I has the form

$(\alpha + x^2, \beta + y^2, g + hx + iy + jxy)$ with $j \neq 0$ and has colength 3. In this case $a + dx^2 + y^2 \in I^2$ is impossible. This can be shown by a direct computation, but we prefer to argue as follows. Namely, to prove this statement we may assume that κ is algebraically closed. Then we can do a coordinate change $x \mapsto \sqrt{\alpha} + x$ and $y \mapsto \sqrt{\beta} + y$ and assume that $I = (x^2, y^2, g' + h'x + i'y + jxy)$ with the same j. Then $g' = h' = i' = 0$ otherwise the colength of I is not 3. Thus we get $I = (x^2, y^2, xy)$ and the result is clear. $\square$

Let $(A, \mathfrak{m}, \kappa)$ be as in Situation 12.1. Let $X \to \text{Spec}(A)$ be the blowing up of $\mathfrak{m}$ in $\text{Spec}(A)$. By Lemma 9.4 we see that X is normal. All singularities of X are rational singularities by Lemma 8.4. Since $\omega_A = A$ we see from Lemma 9.7 that $\omega_X \cong \mathcal{O}_X$ (see discussion in Remark 7.7 for conventions). Thus all singularities of X are Gorenstein. Moreover, the local rings of X at closed point have normal completions by Lemma 11.4. In other words, by blowing up $\text{Spec}(A)$ we obtain a normal surface X whose singular points are as in Situation 12.1. We will use this below without further mention. (Note: we will see in the course of the discussion below that there are finitely many of these singular points.)

Let $E \subset X$ be the exceptional divisor. We have $\omega_E = \mathcal{O}_E(-1)$ by Lemma 9.7. By Lemma 9.5 we have $\kappa = H^0(E, \mathcal{O}_E)$. Thus E is a Gorenstein curve and by Riemann-Roch as discussed in Algebraic Curves, Section 5 we have

$$\chi(E, \mathcal{O}_E) = 1 - g = -(1/2)\deg(\omega_E) = (1/2)\deg(\mathcal{O}_E(1))$$

where $g = \dim_\kappa H^1(E, \mathcal{O}_E) \geq 0$. Since $\deg(\mathcal{O}_E(1))$ is positive by Varieties, Lemma 44.15 we find that $g = 0$ and $\deg(\mathcal{O}_E(1)) = 2$. It follows that we have

$$\dim_\kappa(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 2n + 1$$

by Lemma 9.5 and Riemann-Roch on E.

Choose $x_1, x_2, x_3 \in \mathfrak{m}$ which map to a basis of $\mathfrak{m}/\mathfrak{m}^2$. Because $\dim_\kappa(\mathfrak{m}^2/\mathfrak{m}^3) = 5$ the images of $x_i x_j$, $i \geq j$ in this κ-vector space satisfy a relation. In other words, we can find $a_{ij} \in A$, $i \geq j$, not all contained in $\mathfrak{m}$, such that

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2 = \sum a_{ijk}x_ix_jx_k$$

for some $a_{ijk} \in A$ where $i \leq j \leq k$. Denote $a \mapsto \overline{a}$ the map $A \to \kappa$. The quadratic form $q = \sum \overline{a}_{ij} t_i t_j \in \kappa[t_1, t_2, t_3]$ is well defined up to multiplication by an element of κ^* by our choices. If during the course of our arguments we find that $\overline{a}_{ij} = 0$ in κ, then we can subsume the term $a_{ij}x_ix_j$ in the right hand side and assume $a_{ij} = 0$; this operation changes the a_{ijk} but not the other $a_{i'j'}$.

The blowing up is covered by 3 affine charts corresponding to the "variables" x_1, x_2, x_3. By symmetry it suffices to study one of the charts. To do this let

$$A' = A[\mathfrak{m}/x_1]$$

be the affine blowup algebra (as in Algebra, Section 70). Since x_1, x_2, x_3 generate $\mathfrak{m}$ we see that A' is generated by $y_2 = x_2/x_1$ and $y_3 = x_3/x_1$ over A. We will occasionally use $y_1 = 1$ to simplify formulas. Moreover, looking at our relation above we find that

$$a_{11} + a_{12}y_2 + a_{13}y_3 + a_{22}y_2^2 + a_{23}y_2y_3 + a_{33}y_3^2 = x_1(\sum a_{ijk}y_iy_jy_k)$$

in A'. Recall that $x_1 \in A'$ defines the exceptional divisor E on our affine open of X which is therefore scheme theoretically given by

$$\kappa[y_2, y_3]/(\overline{a}_{11} + \overline{a}_{12}y_2 + \overline{a}_{13}y_3 + \overline{a}_{22}y_2^2 + \overline{a}_{23}y_2y_3 + \overline{a}_{33}y_3^2)$$

In other words, $E \subset \mathbf{P}^2_\kappa = \text{Proj}(\kappa[t_1, t_2, t_3])$ is the zero scheme of the quadratic form q introduced above.

The quadratic form q is an important invariant of the singularity defined by A. Let us say we are in **case II** if q is a square of a linear form times an element of κ^* and in **case I** otherwise. Observe that we are in case II exactly if, after changing our choice of x_1, x_2, x_3, we have

$$x_3^2 = \sum a_{ijk}x_ix_jx_k$$

in the local ring A.

Let $\mathfrak{m}' \subset A'$ be a maximal ideal lying over $\mathfrak{m}$ with residue field κ'. In other words, $\mathfrak{m}'$ corresponds to a closed point $p \in E$ of the exceptional divisor. Recall that the surjection

$$\kappa[y_2, y_3] \to \kappa'$$

has kernel generated by two elements $f_2, f_3 \in \kappa[y_2, y_3]$ (see for example Algebra, Example 27.3 or the proof of Algebra, Lemma 114.1). Let $z_2, z_3 \in A'$ map to f_2, f_3 in $\kappa[y_2, y_3]$. Then we see that $\mathfrak{m}' = (x_1, z_2, z_3)$ because x_2 and x_3 become divisible by x_1 in A'.

Claim. If X is singular at p, then $\kappa' = \kappa$ or we are in case II. Namely, if $A'_{\mathfrak{m}'}$ is singular, then $\dim_{\kappa'} \mathfrak{m}'/(\mathfrak{m}')^2 = 3$ which implies that $\dim_{\kappa'} \overline{\mathfrak{m}}'/(\overline{\mathfrak{m}}')^2 = 2$ where $\overline{m}'$ is the maximal ideal of $\mathcal{O}_{E,p} = \mathcal{O}_{X,p}/x_1\mathcal{O}_{X,p}$. This implies that

$$q(1, y_2, y_3) = \overline{a}_{11} + \overline{a}_{12}y_2 + \overline{a}_{13}y_3 + \overline{a}_{22}y_2^2 + \overline{a}_{23}y_2y_3 + \overline{a}_{33}y_3^2 \in (f_2, f_3)^2$$

otherwise there would be a relation between the classes of z_2 and z_3 in $\overline{\mathfrak{m}}'/(\overline{\mathfrak{m}}')^2$. The claim now follows from Lemma 12.2.

Resolution in case I. By the claim any singular point of X is κ-rational. Pick such a singular point p. We may choose our $x_1, x_2, x_3 \in \mathfrak{m}$ such that p lies on the chart described above and has coordinates $y_2 = y_3 = 0$. Since it is a singular point arguing as in the proof of the claim we find that $q(1, y_2, y_3) \in (y_2, y_3)^2$. Thus we can choose $a_{11} = a_{12} = a_{13} = 0$ and $q(t_1, t_2, t_3) = q(t_2, t_3)$. It follows that

$$E = V(q) \subset \mathbf{P}^1_\kappa$$

either is the union of two distinct lines meeting at p or is a degree 2 curve with a unique κ-rational point (small detail omitted; use that q is not a square of a linear form up to a scalar). In both cases we conclude that X has a unique singular point p which is κ-rational. We need a bit more information in this case. First, looking at higher terms in the expression above, we find that $\overline{a}_{111} = 0$ because p is singular. Then we can write $a_{111} = b_{111}x_1 \bmod (x_2, x_3)$ for some $b_{111} \in A$. Then the quadratic form at p for the generators x_1, y_2, y_3 of $\mathfrak{m}'$ is

$$q' = \overline{b}_{111}t_1^2 + \overline{a}_{112}t_1t_2 + \overline{a}_{113}t_1t_3 + \overline{a}_{22}t_2^2 + \overline{a}_{23}t_2t_3 + \overline{a}_{33}t_3^2$$

We see that $E' = V(q')$ intersects the line $t_1 = 0$ in either two points or one point of degree 2. We conclude that p lies in case I.

Suppose that the blowing up $X' \to X$ of X at p again has a singular point p'. Then we see that p' is a κ-rational point and we can blow up to get $X'' \to X'$. If this process does not stop we get a sequence of blowings up

$$\mathrm{Spec}(A) \leftarrow X \leftarrow X' \leftarrow X'' \leftarrow \ldots$$

We want to show that Lemma 10.1 applies to this situation. To do this we have to say something about the choice of the element x_1 of $\mathfrak{m}$. Suppose that A is in case I and that X has a singular point. Then we will say that $x_1 \in \mathfrak{m}$ is a *good coordinate* if for any (equivalently some) choice of x_2, x_3 the quadratic form $q(t_1, t_2, t_3)$ has the property that $q(0, t_2, t_3)$ is not a scalar times a square. We have seen above that a good coordinate exists. If x_1 is a good coordinate, then the singular point $p \in E$ of X does not lie on the hypersurface $t_1 = 0$ because either this does not have a rational point or if it does, then it is not singular on X. Observe that this is equivalent to the statement that the image of x_1 in $\mathcal{O}_{X,p}$ cuts out the exceptional divisor E. Now the computations above show that if x_1 is a good coordinate for A, then $x_1 \in \mathfrak{m}'\mathcal{O}_{X,p}$ is a good coordinate for p. This of course uses that the notion of good coordinate does not depend on the choice of x_2, x_3 used to do the computation. Hence x_1 maps to a good coordinate at p', p'', etc. Thus Lemma 10.1 applies and our sequence of blowing ups comes from a nonsingular arc $A \to R$. Then the map $A^\wedge \to R$ is a surjection. Since the completion of A is normal, we conclude by Lemma 10.2 that after a finite number of blowups

$$\mathrm{Spec}(A^\wedge) \leftarrow X^\wedge \leftarrow (X')^\wedge \leftarrow \ldots$$

the resulting scheme $(X^{(n)})^\wedge$ is regular. Since $(X^{(n)})^\wedge \to X^{(n)}$ induces isomorphisms on complete local rings (Lemma 11.1) we conclude that the same is true for $X^{(n)}$.

Resolution in case II. Here we have

$$x_3^2 = \sum a_{ijk} x_i x_j x_k$$

in A for some choice of generators x_1, x_2, x_3 of $\mathfrak{m}$. Then $q = t_3^2$ and $E = 2C$ where C is a line. Recall that in A' we get

$$y_3^2 = x_1\left(\sum a_{ijk} y_i y_j y_k\right)$$

Since we know that X is normal, we get a discrete valuation ring $\mathcal{O}_{X,\xi}$ at the generic point ξ of C. The element $y_3 \in A'$ maps to a uniformizer of $\mathcal{O}_{X,\xi}$. Since x_1 scheme theoretically cuts out E which is C with multiplicity 2, we see that x_1 is a unit times y_3^2 in $\mathcal{O}_{X,\xi}$. Looking at our equality above we conclude that

$$h(y_2) = \overline{a}_{111} + \overline{a}_{112} y_2 + \overline{a}_{122} y_2^2 + \overline{a}_{222} y_2^3$$

must be nonzero in the residue field of ξ. Now, suppose that $p \in C$ defines a singular point. Then y_3 is zero at p and p must correspond to a zero of h by the reasoning used in proving the claim above. If h does not have a double zero at p, then the quadratic form q' at p is not a square and we conclude that p falls in case I which we have treated above[1]. Since the degree of h is 3 we get at most

[1]The maximal ideal at p in A' is generated by y_3, x_1 and a third element g whose image in $\kappa[y_2]$ is the prime divisor of h corresponding to p. If this prime divisor doesn't divide h twice, then we see that the quadratic form at p looks like

$$y_3^2 - x_1((\mathit{something})x_1 + (\mathit{something})y_3 + (\mathit{unit})g)$$

and this can never be a square in $\kappa[y_3, x_1, g]$.

one singular point $p \in C$ falling into case II which is moreover κ-rational. After changing our choice of x_1, x_2, x_3 we may assume this is the point $y_2 = y_3 = 0$. Then $h = \overline{a}_{122}y_2^2 + \overline{a}_{222}y_2^3$. Moreover, it still has to be the case that $\overline{a}_{113} = 0$ for the quadratic form q' to have the right shape. Thus the local ring $\mathcal{O}_{X,p}$ defines a singularity as in the next paragraph.

The final case we treat is the case where we can choose our generators x_1, x_2, x_3 of $\mathfrak{m}$ such that

$$x_3^2 + x_1(ax_2^2 + bx_2x_3 + cx_3^2) \in \mathfrak{m}^4$$

for some $a, b, c \in A$. This is a subclass of case II. If $\overline{a} = 0$, then we can write $a = a_1x_1 + a_2x_2 + a_3x_3$ and we get after blowing up

$$y_3^2 + x_1(a_1x_1y_2^2 + a_2x_1y_2^3 + a_3x_1y_2^2y_3 + by_2y_3 + cy_3^2) = x_1^2(\sum a_{ijkl}y_iy_jy_ky_l)$$

This means that X is not normal[2] a contradiction. By the result of the previous paragraph, if the blowup X has a singular point p which falls in case II, then there is only one and it is κ-rational. Computing the affine blowup algebras $A[\frac{\mathfrak{m}}{x_2}]$ and $A[\frac{\mathfrak{m}}{x_3}]$ the reader easily sees that p cannot be contained the corresponding opens of X. Thus p is in the spectrum of $A[\frac{\mathfrak{m}}{x_1}]$. Doing the blowing up as before we see that p must be the point with coordinates $y_2 = y_3 = 0$ and the new equation looks like

$$y_3^2 + x_1(ay_2^2 + by_2y_3 + cy_3^2) \in (\mathfrak{m}')^4$$

which has the same shape as before and has the property that x_1 defines the exceptional divisor. Thus if the process does not stop we get an infinite sequence of blowups and on each of these x_1 defines the exceptional divisor in the local ring of the singular point. Thus we can finish the proof using Lemmas 10.1 and 10.2 and the same reasoning as before.

Lemma 12.3. *Let $(A, \mathfrak{m}, \kappa)$ be a local normal Nagata domain of dimension 2 which defines a rational singularity, whose completion is normal, and which is Gorenstein. Then there exists a finite sequence of blowups in singular closed points*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(A)$$

such that X_n is regular and such that each intervening schemes X_i is normal with finitely many singular points of the same type.

Proof. This is exactly what was proved in the discussion above. □

13. Implied properties

In this section we prove that for a Noetherian integral scheme the existence of a regular alteration has quite a few consequences. This section should be skipped by those not interested in "bad" Noetherian rings.

Lemma 13.1. *Let Y be a Noetherian integral scheme. Assume there exists an alteration $f : X \to Y$ with X regular. Then the normalization $Y^\nu \to Y$ is finite and Y has a dense open which is regular.*

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[2]Namely, the equation shows that you get something singular along the 1-dimensional locus $x_1 = y_3 = 0$ which cannot happen for a normal surface.

Proof. It suffices to prove this when $Y = \mathrm{Spec}(A)$ where A is a Noetherian domain. Let B be the integral closure of A in its fraction field. Set $C = \Gamma(X, \mathcal{O}_X)$. By Cohomology of Schemes, Lemma 19.2 we see that C is a finite A-module. As X is normal (Properties, Lemma 9.4) we see that C is normal domain (Properties, Lemma 7.9). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian.

There exists a nonempty open $V \subset Y$ such that $f^{-1}V \to V$ is finite, see Morphisms, Definition 51.12. After shrinking V we may assume that $f^{-1}V \to V$ is flat (Morphisms, Proposition 27.1). Thus $f^{-1}V \to V$ is faithfully flat. Then V is regular by Algebra, Lemma 164.4. ☐

Lemma 13.2. *Let $(A, \mathfrak{m})$ be a local Noetherian ring. Let $B \subset C$ be finite A-algebras. Assume that (a) B is a normal ring, and (b) the $\mathfrak{m}$-adic completion $C^\wedge$ is a normal ring. Then $B^\wedge$ is a normal ring.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[d] \\ B^\wedge \arrow[r] & C^\wedge \end{tikzcd}$$

Recall that $\mathfrak{m}$-adic completion on the category of finite A-modules is exact because it is given by tensoring with the flat A-algebra $A^\wedge$ (Algebra, Lemma 97.2). We will use Serre's criterion (Algebra, Lemma 157.4) to prove that the Noetherian ring $B^\wedge$ is normal. Let $\mathfrak{q} \subset B^\wedge$ be a prime lying over $\mathfrak{p} \subset B$. If $\dim(B_\mathfrak{p}) \geq 2$, then $\mathrm{depth}(B_\mathfrak{p}) \geq 2$ and since $B_\mathfrak{p} \to B^\wedge_\mathfrak{q}$ is flat we find that $\mathrm{depth}(B^\wedge_\mathfrak{q}) \geq 2$ (Algebra, Lemma 163.2). If $\dim(B_\mathfrak{p}) \leq 1$, then $B_\mathfrak{p}$ is either a discrete valuation ring or a field. In that case $C_\mathfrak{p}$ is faithfully flat over $B_\mathfrak{p}$ (because it is finite and torsion free). Hence $B^\wedge_\mathfrak{p} \to C^\wedge_\mathfrak{p}$ is faithfully flat and the same holds after localizing at $\mathfrak{q}$. As $C^\wedge$ and hence any localization is (S_2) we conclude that $B^\wedge_\mathfrak{p}$ is (S_2) by Algebra, Lemma 164.5. All in all we find that (S_2) holds for $B^\wedge$. To prove that $B^\wedge$ is (R_1) we only have to consider primes $\mathfrak{q} \subset B^\wedge$ with $\dim(B^\wedge_\mathfrak{q}) \leq 1$. Since $\dim(B^\wedge_\mathfrak{q}) = \dim(B_\mathfrak{p}) + \dim(B^\wedge_\mathfrak{q}/\mathfrak{p}B^\wedge_\mathfrak{q})$ by Algebra, Lemma 112.6 we find that $\dim(B_\mathfrak{p}) \leq 1$ and we see that $B^\wedge_\mathfrak{q} \to C^\wedge_\mathfrak{q}$ is faithfully flat as before. We conclude using Algebra, Lemma 164.6. ☐

Lemma 13.3. *Let $(A, \mathfrak{m}, \kappa)$ be a local Noetherian domain. Assume there exists an alteration $f : X \to \mathrm{Spec}(A)$ with X regular. Then*

1. *there exists a nonzero $f \in A$ such that A_f is regular,*
2. *the integral closure B of A in its fraction field is finite over A,*
3. *the $\mathfrak{m}$-adic completion of B is a normal ring, i.e., the completions of B at its maximal ideals are normal domains, and*
4. *the generic formal fibre of A is regular.*

Proof. Parts (1) and (2) follow from Lemma 13.1. We have to redo part of the proof of that lemma in order to set up notation for the proof of (3). Set $C = \Gamma(X, \mathcal{O}_X)$. By Cohomology of Schemes, Lemma 19.2 we see that C is a finite A-module. As X is normal (Properties, Lemma 9.4) we see that C is normal domain (Properties, Lemma 7.9). Thus $B \subset C$ and we conclude that B is finite over A as A is Noetherian. By Lemma 13.2 in order to prove (3) it suffices to show that the $\mathfrak{m}$-adic completion $C^\wedge$ is normal.

By Algebra, Lemma 97.8 the completion $C^\wedge$ is the product of the completions of C at the prime ideals of C lying over $\mathfrak m$. There are finitely many of these and these are the maximal ideals $\mathfrak m_1, \ldots, \mathfrak m_r$ of C. (The corresponding result for B explains the final statement of the lemma.) Thus replacing A by $C_{\mathfrak m_i}$ and X by $X_i = X \times_{\text{Spec}(C)} \text{Spec}(C_{\mathfrak m_i})$ we reduce to the case discussed in the next paragraph. (Note that $\Gamma(X_i, \mathcal{O}) = C_{\mathfrak m_i}$ by Cohomology of Schemes, Lemma 5.2.)

Here A is a Noetherian local normal domain and $f : X \to \text{Spec}(A)$ is a regular alteration with $\Gamma(X, \mathcal{O}_X) = A$. We have to show that the completion $A^\wedge$ of A is a normal domain. By Lemma 11.2 $Y = X \times_{\text{Spec}(A)} \text{Spec}(A^\wedge)$ is regular. Since $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ by Cohomology of Schemes, Lemma 5.2, we conclude that $A^\wedge$ is normal as before. Namely, Y is normal by Properties, Lemma 9.4. It is connected because $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is local. Hence Y is normal and integral (as connected and normal implies integral for Noetherian schemes). Thus $\Gamma(Y, \mathcal{O}_Y) = A^\wedge$ is a normal domain by Properties, Lemma 7.9. This proves (3).

Proof of (4). Let $\eta \in \text{Spec}(A)$ denote the generic point and denote by a subscript η the base change to η. Since f is an alteration, the scheme X_η is finite and faithfully flat over η. Since $Y = X \times_{\text{Spec}(A)} \text{Spec}(A^\wedge)$ is regular by Lemma 11.2 we see that Y_η is regular (as a limit of opens in Y). Then $Y_\eta \to \text{Spec}(A^\wedge \otimes_A \kappa(\eta))$ is finite faithfully flat onto the generic formal fibre. We conclude by Algebra, Lemma 164.4. □

14. Resolution

Here is a definition.

Definition 14.1. Let Y be a Noetherian integral scheme. A *resolution of singularities* of Y is a modification $f : X \to Y$ such that X is regular.

In the case of surfaces we sometimes want a bit more information.

Definition 14.2. Let Y be a 2-dimensional Noetherian integral scheme. We say Y has a *resolution of singularities by normalized blowups* if there exists a sequence

$$Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y_0 \to Y$$

where

(1) Y_i is proper over Y for $i = 0, \ldots, n$,
(2) $Y_0 \to Y$ is the normalization,
(3) $Y_i \to Y_{i-1}$ is a normalized blowup for $i = 1, \ldots, n$, and
(4) Y_n is regular.

Observe that condition (1) implies that the normalization Y_0 of Y is finite over Y and that the normalizations used in the normalized blowing ups are finite as well.

Lemma 14.3. *Let $(A, \mathfrak m, \kappa)$ be a Noetherian local ring. Assume A is normal and has dimension 2. If $\text{Spec}(A)$ has a resolution of singularities, then $\text{Spec}(A)$ has a resolution by normalized blowups.*

Proof. By Lemma 13.3 the completion $A^\wedge$ of A is normal. By Lemma 11.2 we see that $\text{Spec}(A^\wedge)$ has a resolution. By Lemma 11.7 any sequence $Y_n \to Y_{n-1} \to \ldots \to \text{Spec}(A^\wedge)$ of normalized blowups of comes from a sequence of normalized blowups $X_n \to \ldots \to \text{Spec}(A)$. Moreover if Y_n is regular, then X_n is regular by Lemma 11.2. Thus it suffices to prove the lemma in case A is complete.

Assume in addition A is a complete. We will use that A is Nagata (Algebra, Proposition 162.16), excellent (More on Algebra, Proposition 52.3), and has a dualizing complex (Dualizing Complexes, Lemma 22.4). Moreover, the same is true for any ring essentially of finite type over A. If B is a excellent local normal domain, then the completion $B^\wedge$ is normal (as $B \to B^\wedge$ is regular and More on Algebra, Lemma 42.2 applies). We will use this without further mention in the rest of the proof.

Let $X \to \text{Spec}(A)$ be a resolution of singularities. Choose a sequence of normalized blowing ups

$$Y_n \to Y_{n-1} \to \ldots \to Y_1 \to \text{Spec}(A)$$

dominating X (Lemma 5.3). The morphism $Y_n \to X$ is an isomorphism away from finitely many points of X. Hence we can apply Lemma 4.2 to find a sequence of blowing ups

$$X_m \to X_{m-1} \to \ldots \to X$$

in closed points such that X_m dominates Y_n. Diagram

$$\begin{tikzcd} & Y_n \arrow[rr] \arrow[dr] & & \text{Spec}(A) \\ X_m \arrow[ur] \arrow[rr] & & X \arrow[ur] & \end{tikzcd}$$

To prove the lemma it suffices to show that a finite number of normalized blowups of Y_n produce a regular scheme. By our diagram above we see that Y_n has a resolution (namely X_m). As Y_n is a normal surface this implies that Y_n has at most finitely many singularities $y_1, \ldots, y_t$ (because $X_m \to Y_n$ is an isomorphism away from the fibres of dimension 1, see Varieties, Lemma 17.3).

Let $x_a \in X$ be the image of y_a. Then $\mathcal{O}_{X,x_a}$ is regular and hence defines a rational singularity (Lemma 8.7). Apply Lemma 8.4 to $\mathcal{O}_{X,x_a} \to \mathcal{O}_{Y_n,y_a}$ to see that $\mathcal{O}_{Y_n,y_a}$ defines a rational singularity. By Lemma 9.8 there exists a finite sequence of blowups in singular closed points

$$Y_{a,n_a} \to Y_{a,n_a-1} \to \ldots \to \text{Spec}(\mathcal{O}_{Y_n,y_a})$$

such that Y_{a,n_a} is Gorenstein, i.e., has an invertible dualizing module. By (the essentially trivial) Lemma 6.4 with $n' = \sum n_a$ these sequences correspond to a sequence of blowups

$$Y_{n+n'} \to Y_{n+n'-1} \to \ldots \to Y_n$$

such that $Y_{n+n'}$ is normal and the local rings of $Y_{n+n'}$ are Gorenstein. Using the references given above we can dominate $Y_{n+n'}$ by a sequence of blowups $X_{m+m'} \to \ldots \to X_m$ dominating $Y_{n+n'}$ as in the following

$$\begin{tikzcd} & Y_{n+n'} \arrow[rr] & & Y_n \arrow[rr] \arrow[dr] & & \text{Spec}(A) \\ X_{m+m'} \arrow[ur] \arrow[rr] & & X_m \arrow[ur] \arrow[rr] & & X \arrow[ur] & \end{tikzcd}$$

Thus again $Y_{n+n'}$ has a finite number of singular points $y'_1, \ldots, y'_s$, but this time the singularities are rational double points, more precisely, the local rings $\mathcal{O}_{Y_{n+n'},y'_b}$ are as in Lemma 12.3. Arguing exactly as above we conclude that the lemma is true. $\square$

Lemma 14.4. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian complete local ring. Assume A is a normal domain of dimension 2. Then $\mathrm{Spec}(A)$ has a resolution of singularities.*

Proof. A Noetherian complete local ring is J-2 (More on Algebra, Proposition 48.7), Nagata (Algebra, Proposition 162.16), excellent (More on Algebra, Proposition 52.3), and has a dualizing complex (Dualizing Complexes, Lemma 22.4). Moreover, the same is true for any ring essentially of finite type over A. If B is a excellent local normal domain, then the completion $B^\wedge$ is normal (as $B \to B^\wedge$ is regular and More on Algebra, Lemma 42.2 applies). In other words, the local rings which we encounter in the rest of the proof will have the required "excellency" properties required of them.

Choose $A_0 \subset A$ with A_0 a regular complete local ring and $A_0 \to A$ finite, see Algebra, Lemma 160.11. This induces a finite extension of fraction fields K/K_0. We will argue by induction on $[K : K_0]$. The base case is when the degree is 1 in which case $A_0 = A$ and the result is true.

Suppose there is an intermediate field $K_0 \subset L \subset K$, $K_0 \neq L \neq K$. Let $B \subset A$ be the integral closure of A_0 in L. By induction we choose a resolution of singularities $Y \to \mathrm{Spec}(B)$. Let X be the normalization of $Y \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$. Picture:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d] \\ Y \arrow[r] & \mathrm{Spec}(B) \end{tikzcd}$$

Since A is J-2 the regular locus of X is open. Since X is a normal surface we conclude that X has at worst finitely many singular points $x_1, \ldots, x_n$ which are closed points with $\dim(\mathcal{O}_{X,x_i}) = 2$. For each i let $y_i \in Y$ be the image. Since $\mathcal{O}_{Y,y_i}^\wedge \to \mathcal{O}_{X,x_i}^\wedge$ is finite of smaller degree than before we conclude by induction hypothesis that $\mathcal{O}_{X,x_i}^\wedge$ has resolution of singularities. By Lemma 14.3 there is a sequence

$$Z_{i,n_i}^\wedge \to \ldots \to Z_{i,1}^\wedge \to \mathrm{Spec}(\mathcal{O}_{X,x_i}^\wedge)$$

of normalized blowups with $Z_{i,n_i}^\wedge$ regular. By Lemma 11.7 there is a corresponding sequence of normalized blowing ups

$$Z_{i,n_i} \to \ldots \to Z_{i,1} \to \mathrm{Spec}(\mathcal{O}_{X,x_i})$$

Then Z_{i,n_i} is a regular scheme by Lemma 11.2. By Lemma 6.5 we can fit these normalized blowing ups into a corresponding sequence

$$Z_n \to Z_{n-1} \to \ldots \to Z_1 \to X$$

and of course Z_n is regular too (look at the local rings). This proves the induction step.

Assume there is no intermediate field $K_0 \subset L \subset K$ with $K_0 \neq L \neq K$. Then either K/K_0 is separable or the characteristic to K is p and $[K : K_0] = p$. Then either Lemma 8.6 or 8.10 implies that reduction to rational singularities is possible. By Lemma 8.5 we conclude that there exists a normal modification $X \to \mathrm{Spec}(A)$ such that for every singular point x of X the local ring $\mathcal{O}_{X,x}$ defines a rational singularity. Since A is J-2 we find that X has finitely many singular points $x_1, \ldots, x_n$. By Lemma 9.8 there exists a finite sequence of blowups in singular closed points

$$X_{i,n_i} \to X_{i,n_i-1} \to \ldots \to \mathrm{Spec}(\mathcal{O}_{X,x_i})$$

such that X_{i,n_i} is Gorenstein, i.e., has an invertible dualizing module. By (the essentially trivial) Lemma 6.4 with $n = \sum n_a$ these sequences correspond to a sequence of blowups

$$X_n \to X_{n-1} \to \ldots \to X$$

such that X_n is normal and the local rings of X_n are Gorenstein. Again X_n has a finite number of singular points $x'_1, \ldots, x'_s$, but this time the singularities are rational double points, more precisely, the local rings $\mathcal{O}_{X_n,x'_i}$ are as in Lemma 12.3. Arguing exactly as above we conclude that the lemma is true. □

We finally come to the main theorem of this chapter.

Theorem 14.5 (Lipman)**.** *Let Y be a two dimensional integral Noetherian scheme. The following are equivalent*

1. *there exists an alteration $X \to Y$ with X regular,*
2. *there exists a resolution of singularities of Y,*
3. *Y has a resolution of singularities by normalized blowups,*
4. *the normalization $Y^\nu \to Y$ is finite, Y^ν has finitely many singular points $y_1, \ldots, y_m$, and for each y_i the completion of $\mathcal{O}_{Y^\nu,y_i}$ is normal.*

Proof. The implications (3) ⇒ (2) ⇒ (1) are immediate.

Let $X \to Y$ be an alteration with X regular. Then $Y^\nu \to Y$ is finite by Lemma 13.1. Consider the factorization $f : X \to Y^\nu$ from Morphisms, Lemma 54.5. The morphism f is finite over an open $V \subset Y^\nu$ containing every point of codimension ≤ 1 in Y^ν by Varieties, Lemma 17.2. Then f is flat over V by Algebra, Lemma 128.1 and the fact that a normal local ring of dimension ≤ 2 is Cohen-Macaulay by Serre's criterion (Algebra, Lemma 157.4). Then V is regular by Algebra, Lemma 164.4. As Y^ν is Noetherian we conclude that $Y^\nu \setminus V = \{y_1, \ldots, y_m\}$ is finite. By Lemma 13.3 the completion of $\mathcal{O}_{Y^\nu,y_i}$ is normal. In this way we see that (1) ⇒ (4).

Assume (4). We have to prove (3). We may immediately replace Y by its normalization. Let $y_1, \ldots, y_m \in Y$ be the singular points. Applying Lemmas 14.4 and 14.3 we find there exists a finite sequence of normalized blowups

$$Y_{i,n_i} \to Y_{i,n_i-1} \to \ldots \to \mathrm{Spec}(\mathcal{O}^\wedge_{Y,y_i})$$

such that Y_{i,n_i} is regular. By Lemma 11.7 there is a corresponding sequence of normalized blowing ups

$$X_{i,n_i} \to \ldots \to X_{i,1} \to \mathrm{Spec}(\mathcal{O}_{Y,y_i})$$

Then X_{i,n_i} is a regular scheme by Lemma 11.2. By Lemma 6.5 we can fit these normalized blowing ups into a corresponding sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to Y$$

and of course X_n is regular too (look at the local rings). This completes the proof. □

15. Embedded resolution

Given a curve on a surface there is a blowing up which turns the curve into a strict normal crossings divisor. In this section we will use that a one dimensional locally Noetherian scheme is normal if and only if it is regular (Algebra, Lemma 119.7). We will also use that any point on a locally Noetherian scheme specializes to a closed point (Properties, Lemma 5.9).

Lemma 15.1. *Let Y be a one dimensional integral Noetherian scheme. The following are equivalent*

1. *there exists an alteration $X \to Y$ with X regular,*
2. *there exists a resolution of singularities of Y,*
3. *there exists a finite sequence $Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y$ of blowups in closed points with Y_n regular, and*
4. *the normalization $Y^\nu \to Y$ is finite.*

Proof. The implications (3) ⇒ (2) ⇒ (1) are immediate. The implication (1) ⇒ (4) follows from Lemma 13.1. Observe that a normal one dimensional scheme is regular hence the implication (4) ⇒ (2) is clear as well. Thus it remains to show that the equivalent conditions (1), (2), and (4) imply (3).

Let $f : X \to Y$ be a resolution of singularities. Since the dimension of Y is one we see that f is finite by Varieties, Lemma 17.2. We will construct factorizations

$$X \to \ldots \to Y_2 \to Y_1 \to Y$$

where $Y_i \to Y_{i-1}$ is a blowing up of a closed point and not an isomorphism as long as Y_{i-1} is not regular. Each of these morphisms will be finite (by the same reason as above) and we will get a corresponding system

$$f_*\mathcal{O}_X \supset \ldots \supset f_{2,*}\mathcal{O}_{Y_2} \supset f_{1,*}\mathcal{O}_{Y_1} \supset \mathcal{O}_Y$$

where $f_i : Y_i \to Y$ is the structure morphism. Since Y is Noetherian, this increasing sequence of coherent submodules must stabilize (Cohomology of Schemes, Lemma 10.1) which proves that for some n the scheme Y_n is regular as desired. To construct Y_i given Y_{i-1} we pick a singular closed point $y_{i-1} \in Y_{i-1}$ and we let $Y_i \to Y_{i-1}$ be the corresponding blowup. Since X is regular of dimension 1 (and hence the local rings at closed points are discrete valuation rings and in particular PIDs), the ideal sheaf $\mathfrak{m}_{y_{i-1}} \cdot \mathcal{O}_X$ is invertible. By the universal property of blowing up (Divisors, Lemma 32.5) this gives us a factorization $X \to Y_i$. Finally, $Y_i \to Y_{i-1}$ is not an isomorphism as $\mathfrak{m}_{y_{i-1}}$ is not an invertible ideal. ☐

Lemma 15.2. *Let X be a Noetherian scheme. Let $Y \subset X$ be an integral closed subscheme of dimension 1 satisfying the equivalent conditions of Lemma 15.1. Then there exists a finite sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X$$

of blowups in closed points such that the strict transform of Y in X_n is a regular curve.

Proof. Let $Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y$ be the sequence of blowups given to us by Lemma 15.1. Let $X_n \to X_{n-1} \to \ldots \to X_1 \to X$ be the corresponding sequence of blowups of X. This works because the strict transform is the blowup by Divisors, Lemma 33.2. ☐

Let X be a locally Noetherian scheme. Let $Y, Z \subset X$ be closed subschemes. Let $p \in Y \cap Z$ be a closed point. Assume that Y is integral of dimension 1 and that the generic point of Y is not contained in Z. In this situation we can consider the invariant

$$m_p(Y \cap Z) = \text{length}_{\mathcal{O}_{X,p}}(\mathcal{O}_{Y \cap Z,p}) \tag{15.2.1}$$

This is an integer ≥ 1. Namely, if $I, J \subset \mathcal{O}_{X,p}$ are the ideals corresponding to Y, Z, then we see that $\mathcal{O}_{Y\cap Z,p} = \mathcal{O}_{X,p}/I + J$ has support equal to $\{\mathfrak{m}_p\}$ because we assumed that $Y \cap Z$ does not contain the unique point of Y specializing to p. Hence the length is finite by Algebra, Lemma 62.3.

Lemma 15.3. *In the situation above let $X' \to X$ be the blowing up of X in p. Let $Y', Z' \subset X'$ be the strict transforms of Y, Z. If $\mathcal{O}_{Y,p}$ is regular, then*

1. *$Y' \to Y$ is an isomorphism,*
2. *Y' meets the exceptional fibre $E \subset X'$ in one point q and $m_q(Y \cap E) = 1$,*
3. *if $q \in Z'$ too, then $m_q(Y \cap Z') < m_p(Y \cap Z)$.*

Proof. Since $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,p}$ is surjective and $\mathcal{O}_{Y,p}$ is a discrete valuation ring, we can pick an element $x_1 \in \mathfrak{m}_p$ mapping to a uniformizer in $\mathcal{O}_{Y,p}$. Choose an affine open $U = \text{Spec}(A)$ containing p such that $x_1 \in A$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to p. Let $I, J \subset A$ be the ideals defining Y, Z in $\text{Spec}(A)$. After shrinking U we may assume that $\mathfrak{m} = I + (x_1)$, in other words, that $V(x_1) \cap U \cap Y = \{p\}$ scheme theoretically. We conclude that p is an effective Cartier divisor on Y and since Y' is the blowing up of Y in p (Divisors, Lemma 33.2) we see that $Y' \to Y$ is an isomorphism by Divisors, Lemma 32.7. The relationship $\mathfrak{m} = I + (x_1)$ implies that $\mathfrak{m}^n \subset I + (x_1^n)$ hence we can define a map

$$\psi : A[\tfrac{\mathfrak{m}}{x_1}] \longrightarrow A/I$$

by sending $y/x_1^n \in A[\frac{\mathfrak{m}}{x_1}]$ to the class of a in A/I where a is chosen such that $y \equiv ax_1^n \bmod I$. Then ψ corresponds to the morphism of $Y \cap U$ into X' over U given by $Y' \cong Y$. Since the image of x_1 in $A[\frac{\mathfrak{m}}{x_1}]$ cuts out the exceptional divisor we conclude that $m_q(Y', E) = 1$. Finally, since $J \subset \mathfrak{m}$ implies that the ideal $J' \subset A[\frac{\mathfrak{m}}{x_1}]$ certainly contains the elements f/x_1 for $f \in J$. Thus if we choose $f \in J$ whose image $\overline{f}$ in A/I has minimal valuation equal to $m_p(Y \cap Z)$, then we see that $\psi(f/x_1) = \overline{f}/x_1$ in A/I has valuation one less proving the last part of the lemma. ☐

Lemma 15.4. *Let X be a Noetherian scheme. Let $Y_i \subset X$, $i = 1, \ldots, n$ be an integral closed subschemes of dimension 1 each satisfying the equivalent conditions of Lemma 15.1. Then there exists a finite sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X$$

of blowups in closed points such that the strict transform $Y_i' \subset X_n$ of Y_i in X_n are pairwise disjoint regular curves.

Proof. It follows from Lemma 15.2 that we may assume Y_i is a regular curve for $i = 1, \ldots, n$. For every $i \neq j$ and $p \in Y_i \cap Y_j$ we have the invariant $m_p(Y_i \cap Y_j)$ (15.2.1). If the maximum of these numbers is > 1, then we can decrease it (Lemma 15.3) by blowing up in all the points p where the maximum is attained. If the maximum is 1 then we can separate the curves using the same lemma by blowing up in all these points p. ☐

When our curve is contained on a regular surface we often want to turn it into a divisor with normal crossings.

Lemma 15.5. *Let X be a regular scheme of dimension 2. Let $Z \subset X$ be a proper closed subscheme. There exists a sequence*

$$X_n \to \ldots \to X_1 \to X$$

of blowing ups in closed points such that the inverse image Z_n of Z in X_n is an effective Cartier divisor.

Proof. Let $D \subset Z$ be the largest effective Cartier divisor contained in Z. Then $\mathcal{I}_Z \subset \mathcal{I}_D$ and the quotient is supported in closed points by Divisors, Lemma 15.8. Thus we can write $\mathcal{I}_Z = \mathcal{I}_{Z'}\mathcal{I}_D$ where $Z' \subset X$ is a closed subscheme which set theoretically consists of finitely many closed points. Applying Lemma 4.1 we find a sequence of blowups as in the statement of our lemma such that $\mathcal{I}_{Z'}\mathcal{O}_{X_n}$ is invertible. This proves the lemma. $\square$

Lemma 15.6. *Let X be a regular scheme of dimension 2. Let $Z \subset X$ be a proper closed subscheme such that every irreducible component $Y \subset Z$ of dimension 1 satisfies the equivalent conditions of Lemma 15.1. Then there exists a sequence*

$$X_n \to \ldots \to X_1 \to X$$

of blowups in closed points such that the inverse image Z_n of Z in X_n is an effective Cartier divisor supported on a strict normal crossings divisor.

Proof. Let $X' \to X$ be a blowup in a closed point p. Then the inverse image $Z' \subset X'$ of Z is supported on the strict transform of Z and the exceptional divisor. The exceptional divisor is a regular curve (Lemma 3.1) and the strict transform Y' of each irreducible component Y is either equal to Y or the blowup of Y at p. Thus in this process we do not produce additional singular components of dimension 1. Thus it follows from Lemmas 15.5 and 15.4 that we may assume Z is an effective Cartier divisor and that all irreducible components Y of Z are regular. (Of course we cannot assume the irreducible components are pairwise disjoint because in each blowup of a point of Z we add a new irreducible component to Z, namely the exceptional divisor.)

Assume Z is an effective Cartier divisor whose irreducible components Y_i are regular. For every $i \neq j$ and $p \in Y_i \cap Y_j$ we have the invariant $m_p(Y_i \cap Y_j)$ (15.2.1). If the maximum of these numbers is > 1, then we can decrease it (Lemma 15.3) by blowing up in all the points p where the maximum is attained (note that the "new" invariants $m_{q_i}(Y_i' \cap E)$ are always 1). If the maximum is 1 then, if $p \in Y_1 \cap \ldots \cap Y_r$ for some $r > 2$ and not any of the others (for example), then after blowing up p we see that $Y_1', \ldots, Y_r'$ do not meet in points above p and $m_{q_i}(Y_i', E) = 1$ where $Y_i' \cap E = \{q_i\}$. Thus continuing to blowup points where more than 3 of the components of Z meet, we reach the situation where for every closed point $p \in X$ there is either (a) no curves Y_i passing through p, (b) exactly one curve Y_i passing through p and $\mathcal{O}_{Y_i,p}$ is regular, or (c) exactly two curves Y_i, Y_j passing through p, the local rings $\mathcal{O}_{Y_i,p}$, $\mathcal{O}_{Y_j,p}$ are regular and $m_p(Y_i \cap Y_j) = 1$. This means that $\sum Y_i$ is a strict normal crossings divisor on the regular surface X, see Étale Morphisms, Lemma 21.2. $\square$

16. Contracting exceptional curves

Let X be a Noetherian scheme. Let $E \subset X$ be a closed subscheme with the following properties

(1) E is an effective Cartier divisor on X,
(2) there exists a field k and an isomorphism $\mathbf{P}^1_k \to E$ of schemes,
(3) the normal sheaf $\mathcal{N}_{E/X}$ pulls back to $\mathcal{O}_{\mathbf{P}^1}(-1)$.

Such a closed subscheme is called an *exceptional curve of the first kind.*

Let X' be a Noetherian scheme and let $x \in X'$ be a closed point such that $\mathcal{O}_{X',x}$ is regular of dimension 2. Let $b : X \to X'$ be the blowing up of X' at x. In this case the exceptional fibre $E \subset X$ is an exceptional curve of the first kind. This follows from Lemma 3.1.

Question: Is every exceptional curve of the first kind obtained as the fibre of a blowing up as above? In other words, does there always exist a proper morphism of schemes $X \to X'$ such that E maps to a closed point $x \in X'$, such that $\mathcal{O}_{X',x}$ is regular of dimension 2, and such that X is the blowing up of X' at x. If true we say *there exists a contraction of E.*

Lemma 16.1. *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. If a contraction $X \to X'$ of E exists, then it has the following universal property: for every morphism $\varphi : X \to Y$ such that $\varphi(E)$ is a point, there is a unique factorization $X \to X' \to Y$ of φ.*

Proof. Let $b : X \to X'$ be a contraction of E. As a topological space X' is the quotient of X by the relation identifying all points of E to one point. Namely, b is proper (Divisors, Lemma 32.13 and Morphisms, Lemma 43.5) and surjective, hence defines a submersive map of topological spaces (Topology, Lemma 6.5). On the other hand, the canonical map $\mathcal{O}_{X'} \to b_*\mathcal{O}_X$ is an isomorphism. Namely, this is clear over the complement of the image point $x \in X'$ of E and on stalks at x the map is an isomorphism by part (4) of Lemma 3.4. Thus the pair $(X', \mathcal{O}_{X'})$ is constructed from X by taking the quotient as a topological space and endowing this with $b_*\mathcal{O}_X$ as structure sheaf.

Given φ we can let $\varphi' : X' \to Y$ be the unique map of topological spaces such that $\varphi = \varphi' \circ b$. Then the map

$$\varphi^\sharp : \varphi^{-1}\mathcal{O}_Y = b^{-1}((\varphi')^{-1}\mathcal{O}_Y) \to \mathcal{O}_X$$

is adjoint to a map

$$(\varphi')^\sharp : (\varphi')^{-1}\mathcal{O}_Y \to b_*\mathcal{O}_X = \mathcal{O}_{X'}$$

Then $(\varphi', (\varphi')^\sharp)$ is a morphism of ringed spaces from X' to Y such that we get the desired factorization. Since φ is a morphism of locally ringed spaces, it follows that φ' is too. Namely, the only thing to check is that the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X',x}$ is local, where $y \in Y$ is the image of E under φ. This is true because an element $f \in \mathfrak{m}_y$ pulls back to a function on X which is zero in every point of E hence the pull back of f to X' is a function defined on a neighbourhood of x in X' with the same property. Then it is clear that this function must vanish at x as desired. $\square$

Lemma 16.2. *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. If there exists a contraction of E, then it is unique up to unique isomorphism.*

Proof. This is immediate from the universal property of Lemma 16.1. $\square$

Lemma 16.3. *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. Let $E_n = nE$ and denote $\mathcal{O}_n$ its structure sheaf. Then*

$$A = \lim H^0(E_n, \mathcal{O}_n)$$

is a complete local Noetherian regular local ring of dimension 2 *and* $\mathrm{Ker}(A \to H^0(E_n, \mathcal{O}_n))$ *is the nth power of its maximal ideal.*

Proof. Recall that there exists an isomorphism $\mathbf{P}^1_k \to E$ such that the normal sheaf of E in X pulls back to $\mathcal{O}(-1)$. Then $H^0(E, \mathcal{O}_E) = k$. We will denote $\mathcal{O}_n(iE)$ the restriction of the invertible sheaf $\mathcal{O}_X(iE)$ to E_n for all $n \geq 1$ and $i \in \mathbf{Z}$. Recall that $\mathcal{O}_X(-nE)$ is the ideal sheaf of E_n. Hence for $d \geq 0$ we obtain a short exact sequence

$$0 \to \mathcal{O}_E(-(d+n)E) \to \mathcal{O}_{n+1}(-dE) \to \mathcal{O}_n(-dE) \to 0$$

Since $\mathcal{O}_E(-(d+n)E) = \mathcal{O}_{\mathbf{P}^1_k}(d+n)$ the first cohomology group vanishes for all $d \geq 0$ and $n \geq 1$. We conclude that the transition maps of the system $H^0(E_n, \mathcal{O}_n(-dE))$ are surjective. For $d = 0$ we get an inverse system of surjections of rings such that the kernel of each transition map is a nilpotent ideal. Hence $A = \lim H^0(E_n, \mathcal{O}_n)$ is a local ring with residue field k and maximal ideal

$$\lim \mathrm{Ker}(H^0(E_n, \mathcal{O}_n) \to H^0(E, \mathcal{O}_E)) = \lim H^0(E_n, \mathcal{O}_n(-E))$$

Pick x, y in this kernel mapping to a k-basis of $H^0(E, \mathcal{O}_E(-E)) = H^0(\mathbf{P}^1_k, \mathcal{O}(1))$. Then $x^d, x^{d-1}y, \ldots, y^d$ are elements of $\lim H^0(E_n, \mathcal{O}_n(-dE))$ which map to a basis of $H^0(E, \mathcal{O}_E(-dE)) = H^0(\mathbf{P}^1_k, \mathcal{O}(d))$. In this way we see that A is separated and complete with respect to the linear topology defined by the kernels

$$I_n = \mathrm{Ker}(A \longrightarrow H^0(E_n, \mathcal{O}_n))$$

We have $x, y \in I_1$, $I_d I_{d'} \subset I_{d+d'}$ and I_d/I_{d+1} is a free k-module on $x^d, x^{d-1}y, \ldots, y^d$. We will show that $I_d = (x, y)^d$. Namely, if $z_e \in I_e$ with $e \geq d$, then we can write

$$z_e = a_{e,0}x^d + a_{e,1}x^{d-1}y + \ldots + a_{e,d}y^d + z_{e+1}$$

where $a_{e,j} \in (x, y)^{e-d}$ and $z_{e+1} \in I_{e+1}$ by our description of I_d/I_{d+1}. Thus starting with some $z = z_d \in I_d$ we can do this inductively

$$z = \sum\nolimits_{e \geq d} \sum\nolimits_j a_{e,j} x^{d-j} y^j$$

with some $a_{e,j} \in (x, y)^{e-d}$. Then $a_j = \sum_{e \geq d} a_{e,j}$ exists (by completeness and the fact that $a_{e,j} \in I_{e-d}$) and we have $z = \sum a_{e,j} x^{d-j} y^j$. Hence $I_d = (x, y)^d$. Thus A is (x, y)-adically complete. Then A is Noetherian by Algebra, Lemma 97.5. It is clear that the dimension is 2 by the description of $(x, y)^d/(x, y)^{d+1}$ and Algebra, Proposition 60.9. Since the maximal ideal is generated by two elements it is regular. □

Lemma 16.4. *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. If there exists a morphism $f : X \to Y$ such that*

1. *Y is Noetherian,*
2. *f is proper,*
3. *f maps E to a point y of Y,*
4. *f is quasi-finite at every point not in E,*

Then there exists a contraction of E and it is the Stein factorization of f.

Proof. We apply More on Morphisms, Theorem 53.4 to get a Stein factorization $X \to X' \to Y$. Then $X \to X'$ satisfies all the hypotheses of the lemma (some details omitted). Thus after replacing Y by X' we may in addition assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and that the fibres of f are geometrically connected.

Assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and that the fibres of f are geometrically connected. Note that $y \in Y$ is a closed point as f is closed and E is closed. The restriction

$f^{-1}(Y \setminus \{y\}) \to Y \setminus \{y\}$ of f is a finite morphism (More on Morphisms, Lemma 44.1). Hence this restriction is an isomorphism since $f_*\mathcal{O}_X = \mathcal{O}_Y$ since finite morphisms are affine. To prove that $\mathcal{O}_{Y,y}$ is regular of dimension 2 we consider the isomorphism

$$\mathcal{O}_{Y,y}^\wedge \longrightarrow \lim H^0(X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n), \mathcal{O})$$

of Cohomology of Schemes, Lemma 20.7. Let $E_n = nE$ as in Lemma 16.3. Observe that

$$E_n \subset X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n)$$

because $E \subset X_y = X \times_Y \mathrm{Spec}(\kappa(y))$. On the other hand, since $E = f^{-1}(\{y\})$ set theoretically (because the fibres of f are geometrically connected), we see that the scheme theoretic fibre X_y is scheme theoretically contained in E_n for some $n > 0$. Namely, apply Cohomology of Schemes, Lemma 10.2 to the coherent $\mathcal{O}_X$-module $\mathcal{F} = \mathcal{O}_{X_y}$ and the ideal sheaf $\mathcal{I}$ of E and use that $\mathcal{I}^n$ is the ideal sheaf of E_n. This shows that

$$X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^m) \subset E_{nm}$$

Thus the inverse limit displayed above is equal to $\lim H^0(E_n, \mathcal{O}_n)$ which is a regular two dimensional local ring by Lemma 16.3. Hence $\mathcal{O}_{Y,y}$ is a two dimensional regular local ring because its completion is so (More on Algebra, Lemma 43.4 and 43.1).

We still have to prove that $f : X \to Y$ is the blowup $b : Y' \to Y$ of Y at y. We encourage the reader to find her own proof. First, we note that Lemma 16.3 also implies that $X_y = E$ scheme theoretically. Since the ideal sheaf of E is invertible, this shows that $f^{-1}\mathfrak{m}_y \cdot \mathcal{O}_X$ is invertible. Hence we obtain a factorization

$$X \to Y' \to Y$$

of the morphism f by the universal property of blowing up, see Divisors, Lemma 32.5. Recall that the exceptional fibre of $E' \subset Y'$ is an exceptional curve of the first kind by Lemma 3.1. Let $g : E \to E'$ be the induced morphism. Because for both E' and E the conormal sheaf is generated by (pullbacks of) a and b, we see that the canonical map $g^*\mathcal{C}_{E'/Y'} \to \mathcal{C}_{E/X}$ (Morphisms, Lemma 31.3) is surjective. Since both are invertible, this map is an isomorphism. Since $\mathcal{C}_{E/X}$ has positive degree, it follows that g cannot be a constant morphism. Hence g has finite fibres. Hence g is a finite morphism (same reference as above). However, since Y' is regular (and hence normal) at all points of E' and since $X \to Y'$ is birational and an isomorphism away from E', we conclude that $X \to Y'$ is an isomorphism by Varieties, Lemma 17.3. □

Lemma 16.5. *Let $b : X \to X'$ be the contraction of an exceptional curve of the first kind $E \subset X$. Then there is a short exact sequence*

$$0 \to \mathrm{Pic}(X') \to \mathrm{Pic}(X) \to \mathbf{Z} \to 0$$

where the first map is pullback by b and the second map sends $\mathcal{L}$ to the degree of $\mathcal{L}$ on the exceptional curve E. The sequence is split by the map $n \mapsto \mathcal{O}_X(-nE)$.

Proof. Since $E = \mathbf{P}^1_k$ we see that the Picard group of E is $\mathbf{Z}$, see Divisors, Lemma 28.5. Hence we can think of the last map as $\mathcal{L} \mapsto \mathcal{L}|_E$. The degree of the restriction of $\mathcal{O}_X(E)$ to E is -1 by definition of exceptional curves of the first kind. Combining these remarks we see that it suffices to show that $\mathrm{Pic}(X') \to \mathrm{Pic}(X)$ is injective with image the invertible sheaves restricting to $\mathcal{O}_E$ on E.

Given an invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ we claim the map $\mathcal{L}' \to b_*b^*\mathcal{L}'$ is an isomorphism. This is clear everywhere except possibly at the image point $x \in X'$ of E. To check it is an isomorphism on stalks at x we may replace X' by an open neighbourhood at x and assume $\mathcal{L}'$ is $\mathcal{O}_{X'}$. Then we have to show that the map $\mathcal{O}_{X'} \to b_*\mathcal{O}_X$ is an isomorphism. This follows from Lemma 3.4 part (4).

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module with $\mathcal{L}|_E = \mathcal{O}_E$. Then we claim (1) $b_*\mathcal{L}$ is invertible and (2) $b^*b_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. Statements (1) and (2) are clear over $X' \setminus \{x\}$. Thus it suffices to prove (1) and (2) after base change to $\mathrm{Spec}(\mathcal{O}_{X',x})$. Computing b_* commutes with flat base change (Cohomology of Schemes, Lemma 5.2) and similarly for b^* and formation of the adjunction map. But if X' is the spectrum of a regular local ring then $\mathcal{L}$ is trivial by the description of the Picard group in Lemma 3.3. Thus the claim is proved.

Combining the claims proved in the previous two paragraphs we see that the map $\mathcal{L} \mapsto b_*\mathcal{L}$ is an inverse to the map

$$\mathrm{Pic}(X') \longrightarrow \mathrm{Ker}(\mathrm{Pic}(X) \to \mathrm{Pic}(E))$$

and the lemma is proved. $\square$

Remark 16.6. Let $b : X \to X'$ be the contraction of an exceptional curve of the first kind $E \subset X$. From Lemma 16.5 we obtain an identification

$$\mathrm{Pic}(X) = \mathrm{Pic}(X') \oplus \mathbf{Z}$$

where $\mathcal{L}$ corresponds to the pair $(\mathcal{L}', n)$ if and only if $\mathcal{L} = (b^*\mathcal{L}')(-nE)$, i.e., $\mathcal{L}(nE) = b^*\mathcal{L}'$. In fact the proof of Lemma 16.5 shows that $\mathcal{L}' = b_*\mathcal{L}(nE)$. Of course the assignment $\mathcal{L} \mapsto \mathcal{L}'$ is a group homomorphism.

Lemma 16.7. *Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let n be the integer such that $\mathcal{L}|_E$ has degree n viewed as an invertible module on $\mathbf{P}^1$. Then*

1. *If $H^1(X, \mathcal{L}) = 0$ and $n \geq 0$, then $H^1(X, \mathcal{L}(iE)) = 0$ for $0 \leq i \leq n+1$.*
2. *If $n \leq 0$, then $H^1(X, \mathcal{L}) \subset H^1(X, \mathcal{L}(E))$.*

Proof. Observe that $\mathcal{L}|_E = \mathcal{O}(n)$ by Divisors, Lemma 28.5. Use induction, the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{L} \to \mathcal{L}(E) \to \mathcal{L}(E)|_E \to 0,$$

and use the fact that $H^1(\mathbf{P}^1, \mathcal{O}(d)) = 0$ for $d \geq -1$ and $H^0(\mathbf{P}^1, \mathcal{O}(d)) = 0$ for $d \leq -1$. Some details omitted. $\square$

Lemma 16.8. *Let $S = \mathrm{Spec}(R)$ be an affine Noetherian scheme. Let $X \to S$ be a proper morphism. Let $\mathcal{L}$ be an ample invertible sheaf on X. Let $E \subset X$ be an exceptional curve of the first kind. Then*

1. *there exists a contraction $b : X \to X'$ of E,*
2. *X' is proper over S, and*
3. *the invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ is ample with $\mathcal{L}'$ as in Remark 16.6.*

Proof. Let n be the degree of $\mathcal{L}|_E$ as in Lemma 16.7. Observe that $n > 0$ as $\mathcal{L}$ is ample on E (Varieties, Lemma 44.14 and Properties, Lemma 26.3). After replacing $\mathcal{L}$ by a power we may assume $H^i(X, \mathcal{L}^{\otimes e}) = 0$ for all $i > 0$ and $e > 0$, see Cohomology of Schemes, Lemma 17.1. Finally, after replacing $\mathcal{L}$ by another power we may assume there exist global sections $t_0, \ldots, t_n$ of $\mathcal{L}$ which define a closed immersion $\psi : X \to \mathbf{P}^n_S$, see Morphisms, Lemma 39.4.

Set $\mathcal{M} = \mathcal{L}(nE)$. Then $\mathcal{M}|_E \cong \mathcal{O}_E$. Since we have the short exact sequence

$$0 \to \mathcal{M}(-E) \to \mathcal{M} \to \mathcal{O}_E \to 0$$

and since $H^1(X, \mathcal{M}(-E))$ is zero (by Lemma 16.7 and the fact that $n > 0$) we can pick a section s_{n+1} of $\mathcal{M}$ which generates $\mathcal{M}|_E$. Finally, denote $s_0, \ldots, s_n$ the sections of $\mathcal{M}$ we get from the sections $t_0, \ldots, t_n$ of $\mathcal{L}$ chosen above via $\mathcal{L} \subset \mathcal{L}(nE) = \mathcal{M}$. Combined the sections $s_0, \ldots, s_n, s_{n+1}$ generate $\mathcal{M}$ in every point of X and therefore define a morphism

$$\varphi : X \longrightarrow \mathbf{P}^{n+1}_S$$

over S, see Constructions, Lemma 13.1.

Below we will check the conditions of Lemma 16.4. Once this is done we see that the Stein factorization $X \to X' \to \mathbf{P}^{n+1}_S$ of φ is the desired contraction which proves (1). Moreover, the morphism $X' \to \mathbf{P}^{n+1}_S$ is finite hence X' is proper over S (Morphisms, Lemmas 44.11 and 41.4). This proves (2). Observe that X' has an ample invertible sheaf. Namely the pullback $\mathcal{M}'$ of $\mathcal{O}_{\mathbf{P}^{n+1}_S}(1)$ is ample by Morphisms, Lemma 37.7. Observe that $\mathcal{M}'$ pulls back to $\mathcal{M}$ on X (by Constructions, Lemma 13.1). Finally, $\mathcal{M} = \mathcal{L}(nE)$. Since in the arguments above we have replaced the original $\mathcal{L}$ by a positive power we conclude that the invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ mentioned in (3) of the lemma is ample on X' by Properties, Lemma 26.2.

Easy observations: $\mathbf{P}^{n+1}_S$ is Noetherian and φ is proper. Details omitted.

Next, we observe that any point of $U = X \setminus E$ is mapped to the open subscheme W of $\mathbf{P}^{n+1}_S$ where one of the first $n+1$ homogeneous coordinates is nonzero. On the other hand, any point of E is mapped to a point where the first $n+1$ homogeneous coordinates are all zero, in particular into the complement of W. Moreover, it is clear that there is a factorization

$$U = \varphi^{-1}(W) \xrightarrow{\varphi|_U} W \xrightarrow{\text{pr}} \mathbf{P}^n_S$$

of $\psi|_U$ where pr is the projection using the first $n+1$ coordinates and $\psi : X \to \mathbf{P}^n_S$ is the embedding chosen above. It follows that $\varphi|_U : U \to W$ is quasi-finite.

Finally, we consider the map $\varphi|_E : E \to \mathbf{P}^{n+1}_S$. Observe that for any point $x \in E$ the image $\varphi(x)$ has its first $n+1$ coordinates equal to zero, i.e., the morphism $\varphi|_E$ factors through the closed subscheme $\mathbf{P}^0_S \cong S$. The morphism $E \to S = \text{Spec}(R)$ factors as $E \to \text{Spec}(H^0(E, \mathcal{O}_E)) \to \text{Spec}(R)$ by Schemes, Lemma 6.4. Since by assumption $H^0(E, \mathcal{O}_E)$ is a field we conclude that E maps to a point in $S \subset \mathbf{P}^{n+1}_S$ which finishes the proof. □

Lemma 16.9. *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of finite type. Let $E \subset X$ be an exceptional curve of the first kind which is in a fibre of f.*

1. *If X is projective over S, then there exists a contraction $X \to X'$ of E and X' is projective over S.*
2. *If X is quasi-projective over S, then there exists a contraction $X \to X'$ of E and X' is quasi-projective over S.*

Proof. Both cases follow from Lemma 16.8 using standard results on ample invertible modules and (quasi-)projective morphisms.

Proof of (1). Projectivity of f means that f is proper and there exists an f-ample invertible module $\mathcal{L}$, see Morphisms, Lemma 43.13 and Definition 40.1. Let $U \subset S$ be an affine open containing the image of E. By Lemma 16.8 there exists a contraction $c : f^{-1}(U) \to V'$ of E and an ample invertible module $\mathcal{N}'$ on V' whose pullback to $f^{-1}(U)$ is equal to $\mathcal{L}(nE)|_{f^{-1}(U)}$. Let $v \in V'$ be the closed point such that c is the blowing up of v. Then we can glue V' and $X \setminus E$ along $f^{-1}(U) \setminus E = V' \setminus \{v\}$ to get a scheme X' over S. The morphisms c and $\mathrm{id}_{X \setminus E}$ glue to a morphism $b : X \to X'$ which is the contraction of E. The inverse image of U in X' is proper over U. On the other hand, the restriction of $X' \to S$ to the complement of the image of v in S is isomorphic to the restriction of $X \to S$ to that open. Hence $X' \to S$ is proper (as being proper is local on the base by Morphisms, Lemma 41.3). Finally, $\mathcal{N}'$ and $\mathcal{L}|_{X \setminus E}$ restrict to isomorphic invertible modules over $f^{-1}(U) \setminus E = V' \setminus \{v\}$ and hence glue to an invertible module $\mathcal{L}'$ over X'. The restriction of $\mathcal{L}'$ to the inverse image of U in X' is ample because this is true for $\mathcal{N}'$. For affine opens of S avoiding the image of v, we see that the same is true because it holds for $\mathcal{L}$. Thus $\mathcal{L}'$ is $(X' \to S)$-relatively ample by Morphisms, Lemma 37.4 and (1) is proved.

Proof of (2). We can write X as an open subscheme of a scheme $\overline{X}$ projective over S by Morphisms, Lemma 43.12. By (1) there is a contraction $b : \overline{X} \to \overline{X}'$ and $\overline{X}'$ is projective over S. Then we let $X' \subset \overline{X}$ be the image of $X \to \overline{X}'$; this is an open as b is an isomorphism away from E. Then $X \to X'$ is the desired contraction. Note that X' is quasi-projective over S as it has an S-relatively ample invertible module by the construction in the proof of part (1). □

Lemma 16.10. *Let S be a Noetherian scheme. Let $f : X \to S$ be a separated morphism of finite type with X regular of dimension 2. Then X is quasi-projective over S.*

Proof. By Chow's lemma (Cohomology of Schemes, Lemma 18.1) there exists a proper morphism $\pi : X' \to X$ which is an isomorphism over a dense open $U \subset X$ such that $X' \to S$ is H-quasi-projective. By Lemma 4.3 there exists a sequence of blowups in closed points

$$X_n \to \ldots \to X_1 \to X_0 = X$$

and an S-morphism $X_n \to X'$ extending the rational map $U \to X'$. Observe that $X_n \to X$ is projective by Divisors, Lemma 32.13 and Morphisms, Lemma 43.14. This implies that $X_n \to X'$ is projective by Morphisms, Lemma 43.15. Hence $X_n \to S$ is quasi-projective by Morphisms, Lemma 40.3 (and the fact that a projective morphism is quasi-projective, see Morphisms, Lemma 43.10). By Lemma 16.9 (and uniqueness of contractions Lemma 16.2) we conclude that $X_{n-1}, \ldots, X_0 = X$ are quasi-projective over S as desired. □

Lemma 16.11. *Let S be a Noetherian scheme. Let $f : X \to S$ be a proper morphism with X regular of dimension 2. Then X is projective over S.*

Proof. This follows from Lemma 16.10 and Morphisms, Lemma 43.13. □

17. Factorization birational maps

Proper birational morphisms between nonsingular surfaces are given by sequences of quadratic transforms.

Lemma 17.1. *Let $f : X \to Y$ be a proper birational morphism between integral Noetherian schemes regular of dimension 2. Then f is a sequence of blowups in closed points.*

Proof. Let $V \subset Y$ be the maximal open over which f is an isomorphism. Then V contains all codimension 1 points of V (Varieties, Lemma 17.3). Let $y \in Y$ be a closed point not contained in V. Then we want to show that f factors through the blowup $b : Y' \to Y$ of Y at y. Namely, if this is true, then at least one (and in fact exactly one) component of the fibre $f^{-1}(y)$ will map isomorphically onto the exceptional curve in Y' and the number of curves in fibres of $X \to Y'$ will be strictly less that the number of curves in fibres of $X \to Y$, so we conclude by induction. Some details omitted.

By Lemma 4.3 we know that there exists a sequence of blowing ups

$$X' = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

in closed points lying over the fibre $f^{-1}(y)$ and a morphism $X' \to Y'$ such that

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is commutative. We want to show that the morphism $X' \to Y'$ factors through X and hence we can use induction on n to reduce to the case where $X' \to X$ is the blowup of X in a closed point $x \in X$ mapping to y.

Let $E \subset X'$ be the exceptional fibre of the blowing up $X' \to X$. If E maps to a point in Y', then we obtain the desired factorization by Lemma 16.1. We will prove that if this is not the case we obtain a contradiction. Namely, if $f'(E)$ is not a point, then $E' = f'(E)$ must be the exceptional curve in Y'. Picture

$$\begin{tikzcd} E \arrow[r] \arrow[d, "g"'] & X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ E' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

Arguing as before f' is an isomorphism in an open neighbourhood of the generic point of E'. Hence $g : E \to E'$ is a finite birational morphism. Then the inverse of g (a rational map) is everywhere defined by Morphisms, Lemma 42.5 and g is an isomorphism. Consider the map

$$g^*\mathcal{C}_{E'/Y'} \longrightarrow \mathcal{C}_{E/X'}$$

of Morphisms, Lemma 31.3. Since the source and target are invertible modules of degree 1 on $E = E' = \mathbf{P}^1_\kappa$ and since the map is nonzero (as f' is an isomorphism in the generic point of E) we conclude it is an isomorphism. By Morphisms, Lemma 32.18 we conclude that $\Omega_{X'/Y'}|_E = 0$. This means that f' is unramified at every point of E (Morphisms, Lemma 35.14). Hence f' is quasi-finite at every point of E (Morphisms, Lemma 35.10). Hence the maximal open $V' \subset Y'$ over which f' is an isomorphism contains E' by Varieties, Lemma 17.3. This in turn implies that the inverse image of y in X' is E'. Hence the inverse image of y in X is x. Hence $x \in X$ is in the maximal open over which f is an isomorphism by Varieties, Lemma 17.3. This is a contradiction as we assumed that y is not in this open. □

Lemma 17.2. *Let S be a Noetherian scheme. Let X and Y be proper integral schemes over S which are regular of dimension 2. Then X and Y are S-birational if and only if there exists a diagram of S-morphisms*

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n = Y_m \to \ldots \to Y_1 \to Y_0 = Y$$

where each morphism is a blowup in a closed point.

Proof. Let $U \subset X$ be open and let $f : U \to Y$ be the given S-rational map (which is invertible as an S-rational map). By Lemma 4.3 we can factor f as $X_n \to \ldots \to X_1 \to X_0 = X$ and $f_n : X_n \to Y$. Since X_n is proper over S and Y separated over S the morphism f_n is proper. Clearly f_n is birational. Hence f_n is a composition of contractions by Lemma 17.1. We omit the proof of the converse. □

18. Other chapters

Preliminaries

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Schemes

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(51) Local Cohomology
(52) Algebraic and Formal Geometry
(53) Algebraic Curves
(54) Resolution of Surfaces
(55) Semistable Reduction
(56) Functors and Morphisms

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