DESCENT AND ALGEBRAIC SPACES

Contents

1. Introduction

In the chapter on topologies on algebraic spaces (see Topologies on Spaces, Section 1) we introduced étale, fppf, smooth, syntomic and fpqc coverings of algebraic spaces. In this chapter we discuss what kind of structures over algebraic spaces can be descended through such coverings. See for example [Gro95a], [Gro95b], [Gro95e], [Gro95f], [Gro95c], and [Gro95d].

2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Descent data for quasi-coherent sheaves

This section is the analogue of Descent, Section 2 for algebraic spaces. It makes sense to read that section first.

Definition 3.1. Let S be a scheme. Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target X.

(1) A *descent datum* $(\mathcal{F}_i, \varphi_{ij})$ *for quasi-coherent sheaves* with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on X_i for each $i \in I$, an isomorphism of quasi-coherent $\mathcal{O}_{X_i \times_X X_j}$-modules $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \to \mathrm{pr}_1^*\mathcal{F}_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[rr, "\mathrm{pr}_{02}^*\varphi_{ik}"'] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & \mathrm{pr}_2^*\mathcal{F}_k \\ & \mathrm{pr}_1^*\mathcal{F}_j \arrow[ur, "\mathrm{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$

of $\mathcal{O}_{X_i \times_X X_j \times_X X_k}$-modules commutes. This is called the *cocycle condition*.

(2) A *morphism* $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ *of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of $\mathcal{O}_{X_i}$-modules $\psi_i : \mathcal{F}_i \to \mathcal{F}'_i$ such that all the diagrams

$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[r, "\varphi_{ij}"'] \arrow[d, "\mathrm{pr}_0^*\psi_i"'] & \mathrm{pr}_1^*\mathcal{F}_j \arrow[d, "\mathrm{pr}_1^*\psi_j"] \\ \mathrm{pr}_0^*\mathcal{F}'_i \arrow[r, "\varphi'_{ij}"] & \mathrm{pr}_1^*\mathcal{F}'_j \end{tikzcd}$$

commute.

Lemma 3.2. *Let S be a scheme. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be families of morphisms of algebraic spaces over S with fixed targets. Let $(g, \alpha : I \to J, (g_i)) : \mathcal{U} \to \mathcal{V}$ be a morphism of families of maps with fixed target, see Sites, Definition 8.1. Let $(\mathcal{F}_j, \varphi_{jj'})$ be a descent datum for quasi-coherent sheaves with respect to the family $\{V_j \to V\}_{j \in J}$. Then*

(1) *The system*
$$\left(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum with respect to the family $\{U_i \to U\}_{i \in I}$.

(2) *This construction is functorial in the descent datum $(\mathcal{F}_j, \varphi_{jj'})$.*

(3) *Given a second morphism $(g', \alpha' : I \to J, (g'_i))$ of families of maps with fixed target with $g = g'$ there exists a functorial isomorphism of descent data*
$$(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}) \cong ((g'_i)^*\mathcal{F}_{\alpha'(i)}, (g'_i \times g'_{i'})^*\varphi_{\alpha'(i)\alpha'(i')}).$$

Proof. Omitted. Hint: The maps $g_i^*\mathcal{F}_{\alpha(i)} \to (g'_i)^*\mathcal{F}_{\alpha'(i)}$ which give the isomorphism of descent data in part (3) are the pullbacks of the maps $\varphi_{\alpha(i)\alpha'(i)}$ by the morphisms $(g_i, g'_i) : U_i \to V_{\alpha(i)} \times_V V_{\alpha'(i)}$. $\square$

Let $g : U \to V$ be a morphism of algebraic spaces. The lemma above tells us that there is a well defined pullback functor between the categories of descent data relative to families of maps with target V and U provided there is a morphism between those families of maps which "lives over g".

Definition 3.3. Let S be a scheme. Let $\{U_i \to U\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target.

(1) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. We call the unique descent on $\mathcal{F}$ datum with respect to the covering $\{U \to U\}$ the *trivial descent datum*.
(2) The pullback of the trivial descent datum to $\{U_i \to U\}$ is called the *canonical descent datum*. Notation: $(\mathcal{F}|_{U_i}, can)$.
(3) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given family is said to be *effective* if there exists a quasi-coherent sheaf $\mathcal{F}$ on U such that $(\mathcal{F}_i, \varphi_{ij})$ is isomorphic to $(\mathcal{F}|_{U_i}, can)$.

Lemma 3.4. *Let S be a scheme. Let U be an algebraic space over S. Let $\{U_i \to U\}$ be a Zariski covering of U, see Topologies on Spaces, Definition 3.1. Any descent datum on quasi-coherent sheaves for the family $\mathcal{U} = \{U_i \to U\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_U$-modules to the category of descent data with respect to $\{U_i \to U\}$ is fully faithful.*

Proof. Omitted. □

4. Fpqc descent of quasi-coherent sheaves

The main application of flat descent for modules is the corresponding descent statement for quasi-coherent sheaves with respect to fpqc-coverings.

Proposition 4.1. *Let S be a scheme. Let $\{X_i \to X\}$ be an fpqc covering of algebraic spaces over S, see Topologies on Spaces, Definition 9.1. Any descent datum on quasi-coherent sheaves for $\{X_i \to X\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of descent data with respect to $\{X_i \to X\}$ is fully faithful.*

Proof. This is more or less a formal consequence of the corresponding result for schemes, see Descent, Proposition 5.2. Here is a strategy for a proof:

(1) The fact that $\{X_i \to X\}$ is a refinement of the trivial covering $\{X \to X\}$ gives, via Lemma 3.2, a functor $QCoh(\mathcal{O}_X) \to DD(\{X_i \to X\})$ from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of descent data for the given family.
(2) In order to prove the proposition we will construct a quasi-inverse functor $back : DD(\{X_i \to X\}) \to QCoh(\mathcal{O}_X)$.
(3) Applying again Lemma 3.2 we see that there is a functor $DD(\{X_i \to X\}) \to DD(\{T_j \to X\})$ if $\{T_j \to X\}$ is a refinement of the given family. Hence in order to construct the functor *back* we may assume that each X_i is a scheme, see Topologies on Spaces, Lemma 9.5. This reduces us to the case where all the X_i are schemes.
(4) A quasi-coherent sheaf on X is by definition a quasi-coherent $\mathcal{O}_X$-module on $X_{étale}$. Now for any $U \in \mathrm{Ob}(X_{étale})$ we get an fppf covering $\{U_i \times_X X_i \to U\}$ by schemes and a morphism $g : \{U_i \times_X X_i \to U\} \to \{X_i \to X\}$ of coverings lying over $U \to X$. Given a descent datum $\xi = (\mathcal{F}_i, \varphi_{ij})$ we obtain a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_{\xi,U}$ corresponding to the pullback $g^*\xi$

of Lemma 3.2 to the covering of U and using effectivity for fppf covering of schemes, see Descent, Proposition 5.2.

(5) Check that $\xi \mapsto \mathcal{F}_{\xi,U}$ is functorial in ξ. Omitted.
(6) Check that $\xi \mapsto \mathcal{F}_{\xi,U}$ is compatible with morphisms $U \to U'$ of the site $X_{étale}$, so that the system of sheaves $\mathcal{F}_{\xi,U}$ corresponds to a quasi-coherent $\mathcal{F}_\xi$ on $X_{étale}$, see Properties of Spaces, Lemma 29.3. Details omitted.
(7) Check that *back* $: \xi \mapsto \mathcal{F}_\xi$ is quasi-inverse to the functor constructed in (1). Omitted.

This finishes the proof. □

5. Quasi-coherent modules and affines

Let S be a scheme. Let X be an algebraic space over S. Recall that $X_{affine,étale}$ is the full subcategory of $X_{étale}$ whose objects are affine turned into a site by declaring the coverings to be the standard étale coverings. See Properties of Spaces, Definition 18.5. By Properties of Spaces, Lemma 18.6 we have an equivalence of topoi $g : Sh(X_{affine,étale}) \to Sh(X_{étale})$ whose pullback functor is given by restriction. Recall that $\mathcal{O}_X$ denotes the structure sheaf on $X_{étale}$. Then we obtain an equivalence

$$(Sh(X_{affine,étale}), \mathcal{O}_X|_{X_{affine,étale}}) \longrightarrow (Sh(X_{étale}), \mathcal{O}_X) \tag{5.0.1}$$

of ringed topoi. We will often write $\mathcal{O}_X$ in stead of $\mathcal{O}_X|_{X_{affine,étale}}$. Having said this we can compare quasi-coherent modules as well.

Lemma 5.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules on $X_{affine,étale}$. The following are equivalent*

(1) *for every morphism $U \to U'$ of $X_{affine,étale}$ the map $\mathcal{F}(U') \otimes_{\mathcal{O}_X(U')} \mathcal{O}_X(U) \to \mathcal{F}(U)$ is an isomorphism,*
(2) *$\mathcal{F}$ is a quasi-coherent module on the ringed site $(X_{affine,étale}, \mathcal{O}_X)$ in the sense of Modules on Sites, Definition 23.1,*
(3) *$\mathcal{F}$ corresponds to a quasi-coherent module on X via the equivalence (5.0.1),*

Proof. Assume (1) holds. To show that $\mathcal{F}$ is a sheaf, let $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ be a covering of $X_{affine,étale}$. The sheaf condition for $\mathcal{F}$ and $\mathcal{U}$, by our assumption on $\mathcal{F}$, reduces to showing that

$$0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U_i) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U_i \times_U U_j)$$

is exact. This is true because $\mathcal{O}_X(U) \to \prod \mathcal{O}_X(U_i)$ is faithfully flat (by Descent, Lemma 9.1 and the fact that coverings in $X_{affine,étale}$ are standard étale coverings) and we may apply Descent, Lemma 3.6. Next, we show that $\mathcal{F}$ is quasi-coherent on $X_{affine,étale}$. Namely, for U in $X_{affine,étale}$, set $R = \mathcal{O}_X(U)$ and choose a presentation

$$\bigoplus\nolimits_{k \in K} R \longrightarrow \bigoplus\nolimits_{l \in L} R \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

by free R-modules. By property (1) and the right exactness of tensor product we see that for every morphism $U' \to U$ in $X_{affine,étale}$ we obtain a presentation

$$\bigoplus\nolimits_{k \in K} \mathcal{O}_X(U') \longrightarrow \bigoplus\nolimits_{l \in L} \mathcal{O}_X(U') \longrightarrow \mathcal{F}(U') \longrightarrow 0$$

In other words, we see that the restriction of $\mathcal{F}$ to the localized category $X_{affine,etale}/U$ has a presentation

$$\bigoplus\nolimits_{k\in K} \mathcal{O}_X|_{X_{affine,\acute{e}tale}/U} \longrightarrow \bigoplus\nolimits_{l\in L} \mathcal{O}_X|_{X_{affine,\acute{e}tale}/U} \longrightarrow \mathcal{F}|_{X_{affine,\acute{e}tale}/U} \longrightarrow 0$$

as required to show that $\mathcal{F}$ is quasi-coherent. With apologies for the horrible notation, this finishes the proof that (1) implies (2).

Since the notion of a quasi-coherent module is intrinsic (Modules on Sites, Lemma 23.2) we see that the equivalence (5.0.1) induces an equivalence between categories of quasi-coherent modules. Thus we have the equivalence of (2) and (3).

Let us assume (3) and prove (1). Namely, let $\mathcal{G}$ be a quasi-coherent module on X corresponding to $\mathcal{F}$. Let $h : U \to U' \to X$ be a morphism of $X_{affine,\acute{e}tale}$. Denote $f : U \to X$ and $f' : U' \to X$ the structure morphisms, so that $f = f' \circ h$. We have $\mathcal{F}(U') = \Gamma(U', (f')^*\mathcal{G})$ and $\mathcal{F}(U) = \Gamma(U, f^*\mathcal{G}) = \Gamma(U, h^*(f')^*\mathcal{G})$. Hence (1) holds by Schemes, Lemma 7.3. $\square$

6. Descent of finiteness properties of modules

This section is the analogue for the case of algebraic spaces of Descent, Section 7. The goal is to show that one can check a quasi-coherent module has a certain finiteness conditions by checking on the members of a covering. We will repeatedly use the following proof scheme. Suppose that X is an algebraic space, and that $\{X_i \to X\}$ is a fppf (resp. fpqc) covering. Let $U \to X$ be a surjective étale morphism such that U is a scheme. Then there exists an fppf (resp. fpqc) covering $\{Y_j \to X\}$ such that

(1) $\{Y_j \to X\}$ is a refinement of $\{X_i \to X\}$,
(2) each Y_j is a scheme, and
(3) each morphism $Y_j \to X$ factors though U, and
(4) $\{Y_j \to U\}$ is an fppf (resp. fpqc) covering of U.

Namely, first refine $\{X_i \to X\}$ by an fppf (resp. fpqc) covering such that each X_i is a scheme, see Topologies on Spaces, Lemma 7.4, resp. Lemma 9.5. Then set $Y_i = U \times_X X_i$. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is of finite type, of finite presentation, etc if and only if the quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}|_U$ is of finite type, of finite presentation, etc. Hence we can use the existence of the refinement $\{Y_j \to X\}$ to reduce the proof of the following lemmas to the case of schemes. We will indicate this by saying that "*the result follows from the case of schemes by étale localization*".

Lemma 6.1. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite type $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 7.1, by étale localization. $\square$

Lemma 6.2. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is an $\mathcal{O}_{X_i}$-module of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. This follows from the case of schemes, see Descent, Lemma 7.3, by étale localization. $\square$

Lemma 6.3. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a flat $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 7.5, by étale localization. □

Lemma 6.4. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite locally free $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 7.6, by étale localization. □

The definition of a locally projective quasi-coherent sheaf can be found in Properties of Spaces, Section 31. It is also proved there that this notion is preserved under pullback.

Lemma 6.5. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a locally projective $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a locally projective $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 7.7, by étale localization. □

We also add here two results which are related to the results above, but are of a slightly different nature.

Lemma 6.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is a finite morphism. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite type.*

Proof. As f is finite it is representable. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $U = V \times_Y X$ is a scheme with a surjective étale morphism towards X and a finite morphism $\psi : U \to V$ (the base change of f). Since $\psi_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ the result of the lemma follows immediately from the schemes version which is Descent, Lemma 7.9. □

Lemma 6.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite presentation.*

Proof. As f is finite it is representable. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $U = V \times_Y X$ is a scheme with a surjective étale morphism towards X and a finite morphism $\psi : U \to V$ (the base change of f). Since $\psi_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ the result of the lemma follows immediately from the schemes version which is Descent, Lemma 7.10. □

7. Fpqc coverings

This section is the analogue of Descent, Section 13. At the moment we do not know if all of the material for fpqc coverings of schemes holds also for algebraic spaces.

Lemma 7.1. *Let S be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Suppose that for each i we have an open subspace $W_i \subset T_i$ such that for all $i, j \in I$ we have $pr_0^{-1}(W_i) = pr_1^{-1}(W_j)$ as open subspaces of $T_i \times_T T_j$. Then there exists a unique open subspace $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i.*

Proof. By Topologies on Spaces, Lemma 9.5 we may assume each T_i is a scheme. Choose a scheme U and a surjective étale morphism $U \to T$. Then $\{T_i \times_T U \to U\}$ is an fpqc covering of U and $T_i \times_T U$ is a scheme for each i. Hence we see that the collection of opens $W_i \times_T U$ comes from a unique open subscheme $W' \subset U$ by Descent, Lemma 13.6. As $U \to X$ is open we can define $W \subset X$ the Zariski open which is the image of W', see Properties of Spaces, Section 4. We omit the verification that this works, i.e., that W_i is the inverse image of W for each i. $\square$

Lemma 7.2. *Let S be a scheme. Let $\{T_i \to T\}$ be an fpqc covering of algebraic spaces over S, see Topologies on Spaces, Definition 9.1. Then given an algebraic space B over S the sequence*

$$\mathrm{Mor}_S(T, B) \longrightarrow \prod_i \mathrm{Mor}_S(T_i, B) \rightrightarrows \prod_{i,j} \mathrm{Mor}_S(T_i \times_T T_j, B)$$

is an equalizer diagram. In other words, every representable functor on the category of algebraic spaces over S satisfies the sheaf condition for fpqc coverings.

Proof. We know this is true if $\{T_i \to T\}$ is an fpqc covering of schemes, see Properties of Spaces, Proposition 17.1. This is the key fact and we encourage the reader to skip the rest of the proof which is formal. Choose a scheme U and a surjective étale morphism $U \to T$. Let U_i be a scheme and let $U_i \to T_i \times_T U$ be a surjective étale morphism. Then $\{U_i \to U\}$ is an fpqc covering. This follows from Topologies on Spaces, Lemmas 9.3 and 9.4. By the above we have the result for $\{U_i \to U\}$.

What this means is the following: Suppose that $b_i : T_i \to B$ is a family of morphisms with $b_i \circ \mathrm{pr}_0 = b_j \circ \mathrm{pr}_1$ as morphisms $T_i \times_T T_j \to B$. Then we let $a_i : U_i \to B$ be the composition of $U_i \to T_i$ with b_i. By what was said above we find a unique morphism $a : U \to B$ such that a_i is the composition of a with $U_i \to U$. The uniqueness guarantees that $a \circ \mathrm{pr}_0 = a \circ \mathrm{pr}_1$ as morphisms $U \times_T U \to B$. Then since $T = U/(U \times_T U)$ as a sheaf, we find that a comes from a unique morphism $b : T \to B$. Chasing diagrams we find that b is the morphism we are looking for. $\square$

8. Descent of finiteness and smoothness properties of morphisms

The following type of lemma is occasionally useful.

Lemma 8.1. *Let S be a scheme. Let $X \to Y \to Z$ be morphism of algebraic spaces. Let P be one of the following properties of morphisms of algebraic spaces over S: flat, locally finite type, locally finite presentation. Assume that $X \to Z$ has P and that $X \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$. Then $Y \to Z$ is P.*

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By assumption we can find an fppf covering $\{V_i \to V\}$ and lifts $V_i \to X$ of the morphism $V_i \to Y$. Since $U \to X$ is surjective étale we see that over the members of the fppf covering $\{V_i \times_X U \to V\}$

we have lifts into U. Hence $U \to V$ induces a surjection of sheaves on $(Sch/S)_{fppf}$. By our definition of what it means to have property P for a morphism of algebraic spaces (see Morphisms of Spaces, Definition 30.1, Definition 23.1, and Definition 28.1) we see that $U \to W$ has P and we have to show $V \to W$ has P. Thus we reduce the question to the case of morphisms of schemes which is treated in Descent, Lemma 14.8. □

A more standard case of the above lemma is the following. (The version with "flat" follows from Morphisms of Spaces, Lemma 31.5.)

Lemma 8.2. *Let S be a scheme. Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume that f is surjective, flat, and locally of finite presentation and assume that p is locally of finite presentation (resp. locally of finite type). Then q is locally of finite presentation (resp. locally of finite type).

Proof. Since $\{X \to Y\}$ is an fppf covering, it induces a surjection of fppf sheaves (Topologies on Spaces, Lemma 7.5) and the lemma is a special case of Lemma 8.1. On the other hand, an easier argument is to deduce it from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemmas 23.4 and 28.4). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 14.3. □

Lemma 8.3. *Let S be a scheme. Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume that

1. *f is surjective, and syntomic (resp. smooth, resp. étale),*
2. *p is syntomic (resp. smooth, resp. étale).*

Then q is syntomic (resp. smooth, resp. étale).

Proof. We deduce this from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemmas 36.4, 37.4, and 39.2). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 14.4. □

Actually we can strengthen this result as follows.

Lemma 8.4. *Let S be a scheme. Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume that

1. *f is surjective, flat, and locally of finite presentation,*
2. *p is smooth (resp. étale).*

Then q is smooth (resp. étale).

Proof. We deduce this from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemmas 37.4 and 39.2). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 14.5. $\square$

Lemma 8.5. *Let S be a scheme. Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume that

1. *f is surjective, flat, and locally of finite presentation,*
2. *p is syntomic.*

Then both q and f are syntomic.

Proof. We deduce this from the analogue for schemes. Namely, the problem is étale local on B and Y (Morphisms of Spaces, Lemma 36.4). Hence we may assume that B and Y are affine schemes. Since $|X| \to |Y|$ is open (Morphisms of Spaces, Lemma 30.6), we can choose an affine scheme U and an étale morphism $U \to X$ such that the composition $U \to Y$ is surjective. In this case the result follows from Descent, Lemma 14.7. $\square$

9. Descending properties of spaces

In this section we put some results of the following kind.

Lemma 9.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. If f is flat at x and X is geometrically unibranch at x, then Y is geometrically unibranch at $f(x)$.*

Proof. Consider the map of étale local rings $\mathcal{O}_{Y,f(\overline{x})} \to \mathcal{O}_{X,\overline{x}}$. By Morphisms of Spaces, Lemma 30.8 this is flat. Hence if $\mathcal{O}_{X,\overline{x}}$ has a unique minimal prime, so does $\mathcal{O}_{Y,f(\overline{x})}$ (by going down, see Algebra, Lemma 39.19). $\square$

Lemma 9.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is flat and surjective and X is reduced, then Y is reduced.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective and flat, the morphism of schemes $U \to V$ is surjective and flat. In this way we reduce the problem to the case of schemes (as reducedness of X and Y is defined in terms of reducedness of U and V, see Properties of Spaces, Section 7). The case of schemes is Descent, Lemma 19.1. $\square$

Lemma 9.3. *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is locally Noetherian, then Y is locally Noetherian.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective, flat, and locally of finite presentation the morphism of schemes $U \to V$ is surjective, flat, and locally of finite presentation. In this way we reduce the problem to the case of schemes (as being locally Noetherian for X and Y is defined in terms of being locally Noetherian of U and V, see Properties of Spaces, Section 7). In the case of schemes the result follows from Descent, Lemma 16.1. $\square$

Lemma 9.4. *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is regular, then Y is regular.*

Proof. By Lemma 9.3 we know that Y is locally Noetherian. Choose a scheme V and a surjective étale morphism $V \to Y$. It suffices to prove that the local rings of V are all regular local rings, see Properties, Lemma 9.2. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective and flat the morphism of schemes $U \to V$ is surjective and flat. By assumption U is a regular scheme in particular all of its local rings are regular (by the lemma above). Hence the lemma follows from Algebra, Lemma 110.9. $\square$

Lemma 9.5. *Let $f : X \to Y$ be a smooth morphism of algebraic spaces. If Y is reduced, then X is reduced. If f is surjective and X is reduced, then Y is reduced.*

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes, the vertical arrows are surjective and étale, and $U \to X \times_Y V$ is surjective étale. Observe that X is a reduced algebraic space if and only if U is a reduced scheme by our definition of reduced algebraic spaces in Properties of Spaces, Section 7. Similarly for Y and V. The morphism $U \to V$ is a smooth morphism of schemes, see Morphisms of Spaces, Lemma 37.4. Since being reduced is local for the smooth topology for schemes (Descent, Lemma 18.1) we see that U is reduced if V is reduced. On the other hand, if $X \to Y$ is surjective, then $U \to V$ is surjective and in this case if U is reduced, then V is reduced. $\square$

10. Descending properties of morphisms

In this section we introduce the notion of when a property of morphisms of algebraic spaces is local on the target in a topology. Please compare with Descent, Section 22.

Definition 10.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale\}$. We say $\mathcal{P}$ is *τ local on the base*, or *τ local on the target*, or *local on the base for the τ-topology* if for any τ-covering $\{Y_i \to Y\}_{i \in I}$ of algebraic spaces and any morphism of algebraic spaces $f : X \to Y$ we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } Y_i \times_Y X \to Y_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the target then it is preserved by base changes by morphisms which occur in τ-coverings. Here is a formal statement.

Lemma 10.2. *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the target. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $Y' \to Y$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, the base change $f' : Y' \times_Y X \to Y'$ of f has property $\mathcal{P}$.*

Proof. This is true because we can fit $Y' \to Y$ into a family of morphisms which forms a τ-covering. $\square$

A simple often used consequence of the above is that if $f : X \to Y$ has property $\mathcal{P}$ which is τ-local on the target and $f(X) \subset V$ for some open subspace $V \subset Y$, then also the induced morphism $X \to V$ has $\mathcal{P}$. Proof: The base change f by $V \to Y$ gives $X \to V$.

Lemma 10.3. *Let S be a scheme. Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the target. For any morphism of algebraic spaces $f : X \to Y$ over S there exists a largest open subspace $W(f) \subset Y$ such that the restriction $X_{W(f)} \to W(f)$ has $\mathcal{P}$. Moreover,*

1. *if $g : Y' \to Y$ is a morphism of algebraic spaces which is flat and locally of finite presentation, syntomic, smooth, or étale and the base change $f' : X_{Y'} \to Y'$ has $\mathcal{P}$, then g factors through $W(f)$,*
2. *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale, then $W(f') = g^{-1}(W(f))$, and*
3. *if $\{g_i : Y_i \to Y\}$ is a τ-covering, then $g_i^{-1}(W(f)) = W(f_i)$, where f_i is the base change of f by $Y_i \to Y$.*

Proof. Consider the union $W_{set} \subset |Y|$ of the images $g(|Y'|) \subset |Y|$ of morphisms $g : Y' \to Y$ with the properties:

1. g is flat and locally of finite presentation, syntomic, smooth, or étale, and
2. the base change $Y' \times_{g,Y} X \to Y'$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms of Spaces, Lemma 30.6) we see that W_{set} is an open subset of $|Y|$. Denote $W \subset Y$ the open subspace whose underlying set of points is W_{set}, see Properties of Spaces, Lemma 4.8. Since $\mathcal{P}$ is local in the τ topology the restriction $X_W \to W$ has property $\mathcal{P}$ because we are given a covering $\{Y' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence and proves that $W(f)$ has property (1). To see property (2) note that $W(f') \supset g^{-1}(W(f))$ because $\mathcal{P}$ is stable under base change by flat and locally of finite presentation, syntomic, smooth, or étale morphisms, see Lemma 10.2. On the other hand, if $Y'' \subset Y'$ is an open such that $X_{Y''} \to Y''$ has property $\mathcal{P}$, then $Y'' \to Y$ factors through W by construction, i.e., $Y'' \subset g^{-1}(W(f))$. This proves (2). Assertion (3) follows from (2) because each morphism $Y_i \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale by our definition of a τ-covering. $\square$

Lemma 10.4. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume*

1. *if $X_i \to Y_i$, $i = 1, 2$ have property $\mathcal{P}$ so does $X_1 \amalg X_2 \to Y_1 \amalg Y_2$,*
2. *a morphism of algebraic spaces $f : X \to Y$ has property $\mathcal{P}$ if and only if for every affine scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f has property $\mathcal{P}$, and*
3. *for any surjective flat morphism of affine schemes $Z' \to Z$ over S and a morphism $f : X \to Z$ from an algebraic space to Z we have*
$$f' : Z' \times_Z X \to Z' \text{ has } \mathcal{P} \Rightarrow f \text{ has } \mathcal{P}.$$

Then $\mathcal{P}$ is fpqc local on the base.

Proof. If $\mathcal{P}$ has property (2), then it is automatically stable under any base change. Hence the direct implication in Definition 10.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $f_i : Y_i \times_Y X \to Y_i$ has property $\mathcal{P}$. Our goal is to show that f has $\mathcal{P}$. Let Z be an affine scheme, and let $Z \to Y$ be a morphism. By (2) it suffices to show that the morphism of algebraic spaces $Z \times_Y X \to Z$ has $\mathcal{P}$. Since $\{Y_i \to Y\}_{i \in I}$ is an fpqc covering we know there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to Y_{i_j}$ over Y for suitable indices $i_j \in I$. Since f_{i_j} has $\mathcal{P}$ we see that

$$Z_j \times_Y X = Z_j \times_{Y_{i_j}} (Y_{i_j} \times_Y X) \longrightarrow Z_j$$

has $\mathcal{P}$ as a base change of f_{i_j} (see first remark of the proof). Set $Z' = \coprod_{j=1,\ldots,n} Z_j$, so that $Z' \to Z$ is a flat and surjective morphism of affine schemes over S. By (1) we conclude that $Z' \times_Y X \to Z'$ has property $\mathcal{P}$. Since this is the base change of the morphism $Z \times_Y X \to Z$ by the morphism $Z' \to Z$ we conclude that $Z \times_Y X \to Z$ has property $\mathcal{P}$ as desired. $\square$

11. Descending properties of morphisms in the fpqc topology

In this section we find a large number of properties of morphisms of algebraic spaces which are local on the base in the fpqc topology. Please compare with Descent, Section 23 for the case of morphisms of schemes.

Lemma 11.1. *Let S be a scheme. The property $\mathcal{P}(f) =$"f is quasi-compact" is fpqc local on the base on algebraic spaces over S.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 8.8. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is quasi-compact. We have to show that f is quasi-compact. To see this, using Morphisms of Spaces, Lemma 8.8 again, it is enough to show that for every affine scheme Y and morphism $Y \to Z$ the fibre product $Y \times_Z X$ is quasi-compact. Here is a picture:

(11.1.1)
$$\begin{tikzcd}
Y \times_Z Z' \times_Z X \arrow[rr] \arrow[dr] \arrow[dd] & & Z' \times_Z X \arrow[dr] \arrow[dd, "f'" near start] & \\
& Y \times_Z X \arrow[rr, crossing over] & & X \arrow[dd, "f"] \\
Y \times_Z Z' \arrow[rr] \arrow[dr] & & Z' \arrow[dr] & \\
& Y \arrow[rr] \arrow[from=uu, crossing over] & & Z
\end{tikzcd}$$

Note that all squares are cartesian and the bottom square consists of affine schemes. The assumption that f' is quasi-compact combined with the fact that $Y \times_Z Z'$ is affine implies that $Y \times_Z Z' \times_Z X$ is quasi-compact. Since

$$Y \times_Z Z' \times_Z X \longrightarrow Y \times_Z X$$

is surjective as a base change of $Z' \to Z$ we conclude that $Y \times_Z X$ is quasi-compact, see Morphisms of Spaces, Lemma 8.6. This finishes the proof. $\square$

Lemma 11.2. *Let S be a scheme. The property $\mathcal{P}(f) =$"f is quasi-separated" is fpqc local on the base on algebraic spaces over S.*

Proof. A base change of a quasi-separated morphism is quasi-separated, see Morphisms of Spaces, Lemma 4.4. Hence the direct implication in Definition 10.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is quasi-separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is quasi-compact. The base change of a fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 9.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 11.1 that Δ is quasi-compact, i.e., f is quasi-separated. $\square$

Lemma 11.3. *Let S be a scheme. The property $\mathcal{P}(f) =$"f is universally closed" is fpqc local on the base on algebraic spaces over S.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 9.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is universally closed. We have to show that f is universally closed. To see this, using Morphisms of Spaces, Lemma 9.5 again, it is enough to show that for every affine scheme Y and morphism $Y \to Z$ the map $|Y \times_Z X| \to |Y|$ is closed. Consider the cube (11.1.1). The assumption that f' is universally closed implies that $|Y \times_Z Z' \times_Z X| \to |Y \times_Z Z'|$ is closed. As $Y \times_Z Z' \to Y$ is quasi-compact, surjective, and flat as a base change of $Z' \to Z$ we see the map $|Y \times_Z Z'| \to |Y|$ is submersive, see Morphisms, Lemma 25.12. Moreover the map

$$|Y \times_Z Z' \times_Z X| \longrightarrow |Y \times_Z Z'| \times_{|Y|} |Y \times_Z X|$$

is surjective, see Properties of Spaces, Lemma 4.3. It follows by elementary topology that $|Y \times_Z X| \to |Y|$ is closed. $\square$

Lemma 11.4. *Let S be a scheme. The property $\mathcal{P}(f) =$"f is universally open" is fpqc local on the base on algebraic spaces over S.*

Proof. The proof is the same as the proof of Lemma 11.3. $\square$

Lemma 11.5. *The property $\mathcal{P}(f) =$"f is universally submersive" is fpqc local on the base.*

Proof. The proof is the same as the proof of Lemma 11.3. $\square$

Lemma 11.6. *The property $\mathcal{P}(f) =$"f is surjective" is fpqc local on the base.*

Proof. Omitted. (Hint: Use Properties of Spaces, Lemma 4.3.) $\square$

Lemma 11.7. *The property $\mathcal{P}(f) =$"f is universally injective" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 9.5. Let $Z' \to Z$ be a flat surjective morphism of affine schemes over S and let $f : X \to Z$ be a morphism from an algebraic space to Z. Assume that the base change $f' : X' \to Z'$ is universally injective. Let K be a field, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. As $Z' \to Z$ is surjective there exists a field extension K'/K and a morphism $\mathrm{Spec}(K') \to Z'$ such that the following solid diagram commutes

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[dd] \arrow[rd, dashed, "{a',b'}"'] \arrow[rrd] & & \\
& X' \arrow[r] \arrow[d] & Z' \arrow[d] \\
\mathrm{Spec}(K) \arrow[r, "{a,b}"] & X \arrow[r] & Z
\end{tikzcd}$$

As the square is cartesian we get the two dotted arrows a', b' making the diagram commute. Since $X' \to Z'$ is universally injective we get $a' = b'$. This forces $a = b$ as $\{\mathrm{Spec}(K') \to \mathrm{Spec}(K)\}$ is an fpqc covering, see Properties of Spaces, Proposition 17.1. Hence f is universally injective as desired. $\square$

Lemma 11.8. *The property $\mathcal{P}(f) =$"f is a universal homeomorphism" is fpqc local on the base.*

Proof. This can be proved in exactly the same manner as Lemma 11.3. Alternatively, one can use that a map of topological spaces is a homeomorphism if and only if it is injective, surjective, and open. Thus a universal homeomorphism is the same thing as a surjective, universally injective, and universally open morphism. See Morphisms of Spaces, Lemma 5.5 and Morphisms of Spaces, Definitions 19.3, 5.2, 6.2, 53.2. Thus the lemma follows from Lemmas 11.6, 11.7, and 11.4. $\square$

Lemma 11.9. *The property $\mathcal{P}(f) =$"f is locally of finite type" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 23.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally of finite type. We have to show that f is locally of finite type. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 23.4 again, it is enough to show that $U \to Z$ is locally of finite type. Since f' is locally of finite type, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally of finite type. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally of finite type by Descent, Lemma 23.10 as desired. □

Lemma 11.10. *The property $\mathcal{P}(f) =$"f is locally of finite presentation" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 28.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally of finite presentation. We have to show that f is locally of finite presentation. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 28.4 again, it is enough to show that $U \to Z$ is locally of finite presentation. Since f' is locally of finite presentation, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally of finite presentation. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally of finite presentation by Descent, Lemma 23.11 as desired. □

Lemma 11.11. *The property $\mathcal{P}(f) =$"f is of finite type" is fpqc local on the base.*

Proof. Combine Lemmas 11.1 and 11.9. □

Lemma 11.12. *The property $\mathcal{P}(f) =$"f is of finite presentation" is fpqc local on the base.*

Proof. Combine Lemmas 11.1, 11.2 and 11.10. □

Lemma 11.13. *The property $\mathcal{P}(f) =$"f is flat" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 30.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is flat. We have to show that f is flat. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 30.5 again, it is enough to show that $U \to Z$ is flat. Since f' is flat, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is flat. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is flat by Descent, Lemma 23.15 as desired. □

Lemma 11.14. *The property $\mathcal{P}(f) =$"f is an open immersion" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 12.1. Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is an open immersion. We have to show that $X \to Z$ is an open immersion. Note that $|X'| \subset |Z'|$ corresponds to an open subscheme $U' \subset Z'$ (isomorphic to X') with the property that $\text{pr}_0^{-1}(U') = \text{pr}_1^{-1}(U')$ as open subschemes of $Z' \times_Z Z'$. Hence there exists an open subscheme $U \subset Z$ such that $X' = (Z' \to Z)^{-1}(U)$, see Descent, Lemma 13.6. By Properties of Spaces, Proposition 17.1 we see that X satisfies the sheaf condition for the fpqc topology. Now we have the fpqc covering $\mathcal{U} = \{U' \to U\}$ and the element $U' \to X' \to X \in \check{H}^0(\mathcal{U}, X)$. By the sheaf condition we obtain a morphism $U \to X$ such that

$$\begin{tikzcd} U' \arrow[r] \arrow[d, "\cong"] \arrow[dd, bend right] & U \arrow[d] \arrow[dd, bend left] \\ X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

is commutative. On the other hand, we know that for any scheme T over S and T-valued point $T \to X$ the composition $T \to X \to Z$ is a morphism such that $Z' \times_Z T \to Z'$ factors through U'. Clearly this means that $T \to Z$ factors through U. In other words the map of sheaves $U \to X$ is bijective and we win. $\square$

Lemma 11.15. *The property $\mathcal{P}(f) =$"f is an isomorphism" is fpqc local on the base.*

Proof. Combine Lemmas 11.6 and 11.14. $\square$

Lemma 11.16. *The property $\mathcal{P}(f) =$"f is affine" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 20.3. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is affine. Let X' be a scheme representing $Z' \times_Z X$. We obtain a canonical isomorphism

$$\varphi : X' \times_Z Z' \longrightarrow Z' \times_Z X'$$

since both schemes represent the algebraic space $Z' \times_Z Z' \times_Z X$. This is a descent datum for $X'/Z'/Z$, see Descent, Definition 34.1 (verification omitted, compare with Descent, Lemma 39.1). Since $X' \to Z'$ is affine this descent datum is effective, see Descent, Lemma 37.1. Thus there exists a scheme $Y \to Z$ over Z and an isomorphism $\psi : Z' \times_Z Y \to X'$ compatible with descent data. Of course $Y \to Z$ is affine (by construction or by Descent, Lemma 23.18). Note that $\mathcal{Y} = \{Z' \times_Z Y \to Y\}$ is a fpqc covering, and interpreting ψ as an element of $X(Z' \times_Z Y)$ we see that $\psi \in \check{H}^0(\mathcal{Y}, X)$. By the sheaf condition for X with respect to this covering (see Properties of Spaces, Proposition 17.1) we obtain a morphism $Y \to X$. By construction the base change of this to Z' is an isomorphism, hence an isomorphism by Lemma 11.15. This proves that X is representable by an affine scheme and we win. □

Lemma 11.17. *The property $\mathcal{P}(f) =$"f is a closed immersion" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 12.1. Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is a closed immersion. We have to show that $X \to Z$ is a closed immersion. The morphism $X' \to Z'$ is affine. Hence by Lemma 11.16 we see that X is a scheme and $X \to Z$ is affine. It follows from Descent, Lemma 23.19 that $X \to Z$ is a closed immersion as desired. □

Lemma 11.18. *The property $\mathcal{P}(f) =$"f is separated" is fpqc local on the base.*

Proof. A base change of a separated morphism is separated, see Morphisms of Spaces, Lemma 4.4. Hence the direct implication in Definition 10.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is a closed immersion. The base change of a fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 9.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 11.17 that Δ is a closed immersion, i.e., f is separated. □

Lemma 11.19. *The property $\mathcal{P}(f) =$"f is proper" is fpqc local on the base.*

Proof. The lemma follows by combining Lemmas 11.3, 11.18 and 11.11. □

Lemma 11.20. *The property $\mathcal{P}(f) =$"f is quasi-affine" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 21.3. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is quasi-affine. Let X' be a scheme representing $Z' \times_Z X$. We obtain a canonical isomorphism

$$\varphi : X' \times_Z Z' \longrightarrow Z' \times_Z X'$$

since both schemes represent the algebraic space $Z' \times_Z Z' \times_Z X$. This is a descent datum for $X'/Z'/Z$, see Descent, Definition 34.1 (verification omitted, compare with Descent, Lemma 39.1). Since $X' \to Z'$ is quasi-affine this descent datum is effective, see Descent, Lemma 38.1. Thus there exists a scheme $Y \to Z$ over Z and an isomorphism $\psi : Z' \times_Z Y \to X'$ compatible with descent data. Of course $Y \to Z$ is quasi-affine (by construction or by Descent, Lemma 23.20). Note that $\mathcal{Y} = \{Z' \times_Z Y \to Y\}$ is a fpqc covering, and interpreting ψ as an element of $X(Z' \times_Z Y)$ we see that $\psi \in \check{H}^0(\mathcal{Y}, X)$. By the sheaf condition for X (see Properties of Spaces, Proposition 17.1) we obtain a morphism $Y \to X$. By construction the base change of this to Z' is an isomorphism, hence an isomorphism by Lemma 11.15. This proves that X is representable by a quasi-affine scheme and we win. □

Lemma 11.21. *The property $\mathcal{P}(f) =$"f is a quasi-compact immersion" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemmas 12.1 and 8.8. Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is a quasi-compact immersion. We have to show that $X \to Z$ is a closed immersion. The morphism $X' \to Z'$ is quasi-affine. Hence by Lemma 11.20 we see that X is a scheme and $X \to Z$ is quasi-affine. It follows from Descent, Lemma 23.21 that $X \to Z$ is a quasi-compact immersion as desired. □

Lemma 11.22. *The property $\mathcal{P}(f) =$"f is integral" is fpqc local on the base.*

Proof. An integral morphism is the same thing as an affine, universally closed morphism. See Morphisms of Spaces, Lemma 45.7. Hence the lemma follows on combining Lemmas 11.3 and 11.16. □

Lemma 11.23. *The property $\mathcal{P}(f) =$"f is finite" is fpqc local on the base.*

Proof. An finite morphism is the same thing as an integral, morphism which is locally of finite type. See Morphisms of Spaces, Lemma 45.6. Hence the lemma follows on combining Lemmas 11.9 and 11.22. □

Lemma 11.24. *The properties $\mathcal{P}(f) =$"f is locally quasi-finite" and $\mathcal{P}(f) =$"f is quasi-finite" are fpqc local on the base.*

Proof. We have already seen that "quasi-compact" is fpqc local on the base, see Lemma 11.1. Hence it is enough to prove the lemma for "locally quasi-finite". We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 27.6. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally quasi-finite. We have to show that f is locally quasi-finite. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 27.6 again, it is enough to show that $U \to Z$ is locally quasi-finite. Since f' is locally quasi-finite, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally quasi-finite. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally quasi-finite by Descent, Lemma 23.24 as desired. □

Lemma 11.25. *The property $\mathcal{P}(f) =$"f is syntomic" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 36.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is syntomic. We have to show that f is syntomic. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 36.4 again, it is enough to show that $U \to Z$ is syntomic. Since f' is syntomic, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is syntomic. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is syntomic by Descent, Lemma 23.26 as desired. $\square$

Lemma 11.26. *The property $\mathcal{P}(f) =$"f is smooth" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 37.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is smooth. We have to show that f is smooth. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 37.4 again, it is enough to show that $U \to Z$ is smooth. Since f' is smooth, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is smooth. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is smooth by Descent, Lemma 23.27 as desired. $\square$

Lemma 11.27. *The property $\mathcal{P}(f) =$"f is unramified" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 38.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is unramified. We have to show that f is unramified. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 38.5 again, it is enough to show that $U \to Z$ is unramified. Since f' is unramified, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is unramified. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is unramified by Descent, Lemma 23.28 as desired. $\square$

Lemma 11.28. *The property $\mathcal{P}(f) =$"f is étale" is fpqc local on the base.*

Proof. We will use Lemma 10.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 39.2. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is étale. We have to show that f is étale. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 39.2 again, it is enough to show that $U \to Z$ is étale. Since f' is étale, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is étale. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is étale by Descent, Lemma 23.29 as desired. $\square$

Lemma 11.29. *The property $\mathcal{P}(f) =$"f is finite locally free" is fpqc local on the base.*

Proof. Being finite locally free is equivalent to being finite, flat and locally of finite presentation (Morphisms of Spaces, Lemma 46.6). Hence this follows from Lemmas 11.23, 11.13, and 11.10. $\square$

Lemma 11.30. *The property $\mathcal{P}(f) =$"f is a monomorphism" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. Let $\{Y_i \to Y\}$ be an fpqc covering, and assume each of the base changes $f_i : X_i \to Y_i$ of f is a monomorphism. We have to show that f is a monomorphism.

First proof. Note that f is a monomorphism if and only if $\Delta : X \to X \times_Y X$ is an isomorphism. By applying this to f_i we see that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is an isomorphism. The base change of an fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 9.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 11.15 that Δ is an isomorphism, i.e., f is a monomorphism.

Second proof. Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. If we can show that $V \times_Y X \to V$ is a monomorphism, then it follows that $X \to Y$ is a monomorphism. Namely, given any cartesian diagram of sheaves

$$\begin{tikzcd} \mathcal{F} \arrow[r, "a"'] \arrow[d, "b"'] & \mathcal{G} \arrow[d, "c"] \\ \mathcal{H} \arrow[r, "d"] & \mathcal{I} \end{tikzcd} \qquad \mathcal{F} = \mathcal{H} \times_{\mathcal{I}} \mathcal{G}$$

if c is a surjection of sheaves, and a is injective, then also d is injective. This reduces the problem to the case where Y is a scheme. Moreover, in this case we may assume that the algebraic spaces Y_i are schemes also, since we can always refine the covering to place ourselves in this situation, see Topologies on Spaces, Lemma 9.5.

Assume $\{Y_i \to Y\}$ is an fpqc covering of schemes. Let $a, b : T \to X$ be two morphisms such that $f \circ a = f \circ b$. We have to show that $a = b$. Since f_i is a monomorphism we see that $a_i = b_i$, where $a_i, b_i : Y_i \times_Y T \to X_i$ are the base changes. In particular the compositions $Y_i \times_Y T \to T \to X$ are equal. Since $\{Y_i \times_Y T \to T\}$ is an fpqc covering we deduce that $a = b$ from Properties of Spaces, Proposition 17.1. $\square$

12. Descending properties of morphisms in the fppf topology

In this section we find some properties of morphisms of algebraic spaces for which we could not (yet) show they are local on the base in the fpqc topology which, however, are local on the base in the fppf topology.

Lemma 12.1. *The property $\mathcal{P}(f) =$"f is an immersion" is fppf local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. Let $\{Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Let $f_i : X_i \to Y_i$ be the base change of f.

If f is an immersion, then each f_i is an immersion by Spaces, Lemma 12.3. This proves the direct implication in Definition 10.1.

Conversely, assume each f_i is an immersion. By Morphisms of Spaces, Lemma 10.7 this implies each f_i is separated. By Morphisms of Spaces, Lemma 27.7 this implies each f_i is locally quasi-finite. Hence we see that f is locally quasi-finite and separated, by applying Lemmas 11.18 and 11.24. By Morphisms of Spaces, Lemma 51.1 this implies that f is representable!

By Morphisms of Spaces, Lemma 12.1 it suffices to show that for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ is an immersion. By Topologies on Spaces, Lemma 7.4 we can find an fppf covering $\{Z_i \to Z\}$ by schemes which refines the pullback of the covering $\{Y_i \to Y\}$ to Z. Hence we see that $Z \times_Y X \to Z$ (which is a morphism of schemes according to the result of the preceding paragraph) becomes an immersion after pulling back to the members of an fppf (by schemes) of Z. Hence $Z \times_Y X \to Z$ is an immersion by the result for schemes, see Descent, Lemma 24.1. $\square$

Lemma 12.2. *The property $\mathcal{P}(f)$ =“f is locally separated” is fppf local on the base.*

Proof. A base change of a locally separated morphism is locally separated, see Morphisms of Spaces, Lemma 4.4. Hence the direct implication in Definition 10.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fppf covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is locally separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is an immersion. The base change of a fppf covering is an fppf covering, see Topologies on Spaces, Lemma 7.3 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fppf covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 12.1 that Δ is a immersion, i.e., f is locally separated. $\square$

13. Application of descent of properties of morphisms

This section is the analogue of Descent, Section 25.

Lemma 13.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\{g_i : Y_i \to Y\}_{i \in I}$ be an fpqc covering. Let $f_i : X_i \to Y_i$ be the base change of f and let $\mathcal{L}_i$ be the pullback of $\mathcal{L}$ to X_i. The following are equivalent*

1. *$\mathcal{L}$ is ample on X/Y, and*
2. *$\mathcal{L}_i$ is ample on X_i/Y_i for every $i \in I$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Divisors on Spaces, Lemma 14.3. Assume (2). To check $\mathcal{L}$ is ample on X/Y we may work étale localy on Y, see Divisors on Spaces, Lemma 14.6. Thus we may assume that Y is a scheme and then we may in turn assume each Y_i is a scheme too, see Topologies on Spaces, Lemma 9.5. In other words, we may assume that $\{Y_i \to Y\}$ is an fpqc covering of schemes.

By Divisors on Spaces, Lemma 14.4 we see that $X_i \to Y_i$ is representable (i.e., X_i is a scheme), quasi-compact, and separated. Hence f is quasi-compact and separated by Lemmas 11.1 and 11.18. This means that $\mathcal{A} = \bigoplus_{d \geq 0} f_*\mathcal{L}^{\otimes d}$ is a quasi-coherent graded $\mathcal{O}_Y$-algebra (Morphisms of Spaces, Lemma 11.2). Moreover, the formation of $\mathcal{A}$ commutes with flat base change by Cohomology of Spaces, Lemma 11.2. In particular, if we set $\mathcal{A}_i = \bigoplus_{d \geq 0} f_{i,*}\mathcal{L}_i^{\otimes d}$ then we have $\mathcal{A}_i = g_i^*\mathcal{A}$. It follows that the natural maps $\psi_d : f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d}$ of $\mathcal{O}_X$ pullback to give the natural maps $\psi_{i,d} : f_i^*(\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ of $\mathcal{O}_{X_i}$-modules. Since $\mathcal{L}_i$ is ample on X_i/Y_i we see that for any point $x_i \in X_i$, there exists a $d \geq 1$ such that $f_i^*(\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ is surjective on stalks at x_i. This follows either directly from the definition of a relatively ample module or from Morphisms, Lemma 37.4. If $x \in |X|$, then we can choose an i and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X_i,\overline{x}_i}$ is flat hence faithfully flat, we conclude that for every $x \in |X|$ there exists a $d \geq 1$ such that $f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d}$ is surjective on stalks at x. This implies that the open subset $U(\psi) \subset X$ of Divisors on Spaces, Lemma 13.1 corresponding to the map $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ of graded $\mathcal{O}_X$-algebras is equal to X. Consider the corresponding morphism

$$r_{\mathcal{L},\psi} : X \longrightarrow \underline{\mathrm{Proj}}_Y(\mathcal{A})$$

It is clear from the above that the base change of $r_{\mathcal{L},\psi}$ to Y_i is the morphism $r_{\mathcal{L}_i,\psi_i}$ which is an open immersion by Morphisms, Lemma 37.4. Hence $r_{\mathcal{L},\psi}$ is an open immersion by Lemma 11.14. Hence X is a scheme and we conclude $\mathcal{L}$ is ample on X/Y by Morphisms, Lemma 37.4. □

Lemma 13.2. *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. There exists an open subspace $V \subset Y$ characterized by the following property: A morphism $Y' \to Y$ of algebraic spaces factors through V if and only if the pullback $\mathcal{L}'$ of $\mathcal{L}$ to $X' = Y' \times_Y X$ is ample on X'/Y' (as in Divisors on Spaces, Definition 14.1).*

Proof. Suppose that the lemma holds whenever Y is a scheme. Let U be a scheme and let $U \to Y$ be a surjective étale morphism. Let $R = U \times_Y U$ with projections $t, s : R \to U$. Denote $X_U = U \times_Y X$ and $\mathcal{L}_U$ the pullback. Then we get an open subscheme $V' \subset U$ as in the lemma for $(X_U \to U, \mathcal{L}_U)$. By the functorial characterization we see that $s^{-1}(V') = t^{-1}(V')$. Thus there is an open subspace $V \subset Y$ such that V' is the inverse image of V in U. In particular $V' \to V$ is surjective étale and we conclude that $\mathcal{L}_V$ is ample on X_V/V (Divisors on Spaces, Lemma 14.6). Now, if $Y' \to Y$ is a morphism such that $\mathcal{L}'$ is ample on X'/Y', then $U \times_Y Y' \to Y'$ must factor through V' and we conclude that $Y' \to Y$ factors through V. Hence $V \subset Y$ is as in the statement of the lemma. In this way we reduce to the case dealt with in the next paragraph.

Assume Y is a scheme. Since the question is local on Y we may assume Y is an affine scheme. We will show the following:

(A) If $\mathrm{Spec}(k) \to Y$ is a morphism such that $\mathcal{L}_k$ is ample on X_k/k, then there is an open neighbourhood $V \subset Y$ of the image of $\mathrm{Spec}(k) \to Y$ such that $\mathcal{L}_V$ is ample on X_V/V.

It is clear that (A) implies the truth of the lemma.

Let $X \to Y$, $\mathcal{L}$, $\mathrm{Spec}(k) \to Y$ be as in (A). By Lemma 13.1 we may assume that $k = \kappa(y)$ is the residue field of a point y of Y.

As Y is affine we can find a directed set I and an inverse system of morphisms $X_i \to Y_i$ of algebraic spaces with Y_i of finite presentation over $\mathbf{Z}$, with affine transition morphisms $X_i \to X_{i'}$ and $Y_i \to Y_{i'}$, with $X_i \to Y_i$ proper and of finite presentation, and such that $X \to Y = \lim(X_i \to Y_i)$. See Limits of Spaces, Lemma 12.2. After shrinking I we may assume Y_i is an (affine) scheme for all i, see Limits of Spaces, Lemma 5.10. After shrinking I we can assume we have a compatible system of invertible $\mathcal{O}_{X_i}$-modules $\mathcal{L}_i$ pulling back to $\mathcal{L}$, see Limits of Spaces, Lemma 7.3. Let $y_i \in Y_i$ be the image of y. Then $\kappa(y) = \text{colim}\, \kappa(y_i)$. Hence $X_y = \lim X_{i,y_i}$ and after shrinking I we may assume X_{i,y_i} is a scheme for all i, see Limits of Spaces, Lemma 5.11. Hence for some i we have $\mathcal{L}_{i,y_i}$ is ample on X_{i,y_i} by Limits, Lemma 4.15. By Divisors on Spaces, Lemma 15.3 we find an open neigbourhood $V_i \subset Y_i$ of y_i such that $\mathcal{L}_i$ restricted to $f_i^{-1}(V_i)$ is ample relative to V_i. Letting $V \subset Y$ be the inverse image of V_i finishes the proof (hints: use Morphisms, Lemma 37.9 and the fact that $X \to Y \times_{Y_i} X_i$ is affine and the fact that the pullback of an ample invertible sheaf by an affine morphism is ample by Morphisms, Lemma 37.7). □

14. Properties of morphisms local on the source

In this section we define what it means for a property of morphisms of algebraic spaces to be local on the source. Please compare with Descent, Section 26.

Definition 14.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. We say $\mathcal{P}$ is τ *local on the source*, or *local on the source for the τ-topology* if for any morphism $f : X \to Y$ of algebraic spaces over S, and any τ-covering $\{X_i \to X\}_{i \in I}$ of algebraic spaces we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the source then it is preserved by precomposing with morphisms which occur in τ-coverings. Here is a formal statement.

Lemma 14.2. *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the source. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $a : X' \to X$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. smooth, resp. étale, the composition $f \circ a : X' \to Y$ has property $\mathcal{P}$.*

Proof. This is true because we can fit $X' \to X$ into a family of morphisms which forms a τ-covering. □

Lemma 14.3. *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. Suppose that $\mathcal{P}$ is a property of morphisms of schemes over S which is étale local on the source-and-target. Denote $\mathcal{P}_{spaces}$ the corresponding property of morphisms of algebraic spaces over S, see Morphisms of Spaces, Definition 22.2. If $\mathcal{P}$ is local on the source for the τ-topology, then $\mathcal{P}_{spaces}$ is local on the source for the τ-topology.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{X_i \to X\}_{i \in I}$ be a τ-covering of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. For each i choose a scheme U_i and a surjective étale morphism $U_i \to X_i \times_X U$.

Note that $\{X_i \times_X U \to U\}_{i \in I}$ is a τ-covering. Note that each $\{U_i \to X_i \times_X U\}$ is an étale covering, hence a τ-covering. Hence $\{U_i \to U\}_{i \in I}$ is a τ-covering of algebraic spaces over S. But since U and each U_i is a scheme we see that $\{U_i \to U\}_{i \in I}$ is a τ-covering of schemes over S.

Now we have

$$\begin{aligned} f \text{ has } \mathcal{P}_{spaces} &\Leftrightarrow U \to V \text{ has } \mathcal{P} \\ &\Leftrightarrow \text{each } U_i \to V \text{ has } \mathcal{P} \\ &\Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}_{spaces}. \end{aligned}$$

the first and last equivalence by the definition of $\mathcal{P}_{spaces}$ the middle equivalence because we assumed $\mathcal{P}$ is local on the source in the τ-topology. $\square$

15. Properties of morphisms local in the fpqc topology on the source

Here are some properties of morphisms that are fpqc local on the source.

Lemma 15.1. *The property $\mathcal{P}(f)=$"f is flat" is fpqc local on the source.*

Proof. Follows from Lemma 14.3 using Morphisms of Spaces, Definition 30.1 and Descent, Lemma 27.1. $\square$

16. Properties of morphisms local in the fppf topology on the source

Here are some properties of morphisms that are fppf local on the source.

Lemma 16.1. *The property $\mathcal{P}(f)=$"f is locally of finite presentation" is fppf local on the source.*

Proof. Follows from Lemma 14.3 using Morphisms of Spaces, Definition 28.1 and Descent, Lemma 28.1. $\square$

Lemma 16.2. *The property $\mathcal{P}(f)=$"f is locally of finite type" is fppf local on the source.*

Proof. Follows from Lemma 14.3 using Morphisms of Spaces, Definition 23.1 and Descent, Lemma 28.2. $\square$

Lemma 16.3. *The property $\mathcal{P}(f)=$"f is open" is fppf local on the source.*

Proof. Follows from Lemma 14.3 using Morphisms of Spaces, Definition 6.2 and Descent, Lemma 28.3. $\square$

Lemma 16.4. *The property $\mathcal{P}(f)=$"f is universally open" is fppf local on the source.*

Proof. Follows from Lemma 14.3 using Morphisms of Spaces, Definition 6.2 and Descent, Lemma 28.4. $\square$

17. Properties of morphisms local in the syntomic topology on the source

Here are some properties of morphisms that are syntomic local on the source.

Lemma 17.1. *The property $\mathcal{P}(f)=$"f is syntomic" is syntomic local on the source.*

Proof. Follows from Lemma 14.3 using Morphisms of Spaces, Definition 36.1 and Descent, Lemma 29.1. $\square$

18. Properties of morphisms local in the smooth topology on the source

Here are some properties of morphisms that are smooth local on the source.

Lemma 18.1. *The property $\mathcal{P}(f) =$"f is smooth" is smooth local on the source.*

Proof. Follows from Lemma 14.3 using Morphisms of Spaces, Definition 37.1 and Descent, Lemma 30.1. ☐

19. Properties of morphisms local in the étale topology on the source

Here are some properties of morphisms that are étale local on the source.

Lemma 19.1. *The property $\mathcal{P}(f) =$"f is étale" is étale local on the source.*

Proof. Follows from Lemma 14.3 using Morphisms of Spaces, Definition 39.1 and Descent, Lemma 31.1. ☐

Lemma 19.2. *The property $\mathcal{P}(f) =$"f is locally quasi-finite" is étale local on the source.*

Proof. Follows from Lemma 14.3 using Morphisms of Spaces, Definition 27.1 and Descent, Lemma 31.2. ☐

Lemma 19.3. *The property $\mathcal{P}(f) =$"f is unramified" is étale local on the source.*

Proof. Follows from Lemma 14.3 using Morphisms of Spaces, Definition 38.1 and Descent, Lemma 31.3. ☐

20. Properties of morphisms smooth local on source-and-target

Let $\mathcal{P}$ be a property of morphisms of algebraic spaces. There is an intuitive meaning to the phrase "$\mathcal{P}$ is smooth local on the source and target". However, it turns out that this notion is not the same as asking $\mathcal{P}$ to be both smooth local on the source and smooth local on the target. We have discussed a similar phenomenon (for the étale topology and the category of schemes) in great detail in Descent, Section 32 (for a quick overview take a look at Descent, Remark 32.8). However, there is an important difference between the case of the smooth and the étale topology. To see this difference we encourage the reader to ponder the difference between Descent, Lemma 32.4 and Lemma 20.2 as well as the difference between Descent, Lemma 32.5 and Lemma 20.3. Namely, in the étale setting the choice of the étale "covering" of the target is immaterial, whereas in the smooth setting it is not.

Definition 20.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. We say $\mathcal{P}$ is *smooth local on source-and-target* if

1. (stable under precomposing with smooth maps) if $f : X \to Y$ is smooth and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
2. (stable under smooth base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is smooth, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
3. (locality) given a morphism $f : X \to Y$ the following are equivalent
    (a) f has $\mathcal{P}$,

(b) for every $x \in |X|$ there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with smooth vertical arrows and $u \in |U|$ with $a(u) = x$ such that h has $\mathcal{P}$.

The above serves as our definition. In the lemmas below we will show that this is equivalent to $\mathcal{P}$ being smooth local on the target, smooth local on the source, and stable under post-composing by smooth morphisms.

Lemma 20.2. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is smooth local on source-and-target. Then*

(1) *$\mathcal{P}$ is smooth local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target,*
(3) *$\mathcal{P}$ is stable under postcomposing with smooth morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is smooth, then $g \circ f$ has $\mathcal{P}$.*

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{X_i \to X\}_{i \in I}$ be a smooth covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $|x| \in X$ we can find an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is a smooth morphism of pairs, and $\mathrm{id}_Y : Y \to Y$ is a smooth morphism, and h_i is as in part (3) of Definition 20.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 20.1 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}_{i \in I}$ be a smooth covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in |X|$ we pick an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is a smooth morphism of pairs, $Y_i \to Y$ is smooth, and h_i is as in part (3) of Definition 20.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 20.1 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is smooth. For every $x \in |X|$ we can think of $(X, x) \to (X, x)$ as a smooth morphism of pairs, $Y \to Z$ is a smooth morphism, and $h = f$ is as in part (3) of Definition 20.1. Thus we see that $g \circ f$ has $\mathcal{P}$. □

The following lemma is the analogue of Morphisms, Lemma 14.4.

Lemma 20.3. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is smooth local on source-and-target. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(a) *f has property $\mathcal{P}$,*
(b) *for every $x \in |X|$ there exists a smooth morphism of pairs $a : (U, u) \to (X, x)$, a smooth morphism $b : V \to Y$, and a morphism $h : U \to V$ such that $f \circ a = b \circ h$ and h has $\mathcal{P}$,*

(c) *for some commutative diagram*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with a, b smooth and a surjective the morphism h has $\mathcal{P}$,

(d) *for any commutative diagram*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with b smooth and $U \to X \times_Y V$ smooth the morphism h has $\mathcal{P}$,

(e) *there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,*

(f) *there exists a smooth covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,*

(g) *there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ a smooth covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.*

Proof. The equivalence of (a) and (b) is part of Definition 20.1. The equivalence of (a) and (e) is Lemma 20.2 part (2). The equivalence of (a) and (f) is Lemma 20.2 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (b). If (b) holds, then for any $x \in |X|$ we can choose a smooth morphism of pairs $a_x : (U_x, u_x) \to (X, x)$, a smooth morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. Then $h = \coprod h_x : \coprod U_x \to \coprod V_x$ with $a = \coprod a_x$ and $b = \coprod b_x$ is a diagram as in (c). (Note that h has property $\mathcal{P}$ as $\{V_x \to \coprod V_x\}$ is a smooth covering and $\mathcal{P}$ is smooth local on the target.) Thus (b) is equivalent to (c).

Now we know that (a), (b), (c), (e), (f), and (g) are equivalent. Suppose (a) holds. Let U, V, a, b, h be as in (d). Then $X \times_Y V \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under smooth base change, whence $U \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under precomposing with smooth morphisms. Conversely, if (d) holds, then setting $U = X$ and $V = Y$ we see that f has $\mathcal{P}$. □

Lemma 20.4. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume*

1. *$\mathcal{P}$ is smooth local on the source,*
2. *$\mathcal{P}$ is smooth local on the target, and*
3. *$\mathcal{P}$ is stable under postcomposing with smooth morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $Y \to Z$ is a smooth morphism then $X \to Z$ has $\mathcal{P}$.*

Then $\mathcal{P}$ is smooth local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 14.2 we see that $\mathcal{P}$ is stable under precomposing with smooth morphisms. By Lemma 10.2 we see that $\mathcal{P}$ is stable under smooth base change. Hence it suffices to prove part (3) of Definition 20.1 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of algebraic spaces over S which satisfies Definition 20.1 part (3)(b). In other words, for every $x \in X$ there exists a smooth morphism $a_x : U_x \to X$, a point $u_x \in |U_x|$ mapping to x, a smooth morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with a, b smooth, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is smooth local on the target. Because a is surjective and $\mathcal{P}$ is smooth local on the source, it suffices to prove that $b \circ h$ has $\mathcal{P}$. This follows as we assumed that $\mathcal{P}$ is stable under postcomposing with a smooth morphism and as b is smooth. $\square$

Remark 20.5. Using Lemma 20.4 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are smooth local on the source-and-target. In each case we list the lemma which implies the property is smooth local on the source and the lemma which implies the property is smooth local on the target. In each case the third assumption of Lemma 20.4 is trivial to check, and we omit it. Here is the list:

(1) flat, see Lemmas 15.1 and 11.13,
(2) locally of finite presentation, see Lemmas 16.1 and 11.10,
(3) locally finite type, see Lemmas 16.2 and 11.9,
(4) universally open, see Lemmas 16.4 and 11.4,
(5) syntomic, see Lemmas 17.1 and 11.25,
(6) smooth, see Lemmas 18.1 and 11.26,
(7) add more here as needed.

21. Properties of morphisms étale-smooth local on source-and-target

This section is the analogue of Section 20 for properties of morphisms which are étale local on the source and smooth local on the target. We give this property a ridiculously long name in order to avoid using it too much.

Definition 21.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. We say $\mathcal{P}$ is *étale-smooth local on source-and-target* if

(1) (stable under precomposing with étale maps) if $f : X \to Y$ is étale and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
(2) (stable under smooth base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is smooth, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
(3) (locality) given a morphism $f : X \to Y$ the following are equivalent
    (a) f has $\mathcal{P}$,

(b) for every $x \in |X|$ there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with b smooth and $U \to X \times_Y V$ étale and $u \in |U|$ with $a(u) = x$ such that h has $\mathcal{P}$.

The above serves as our definition. In the lemmas below we will show that this is equivalent to $\mathcal{P}$ being étale local on the target, smooth local on the source, and stable under post-composing by étale morphisms.

Lemma 21.2. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is étale-smooth local on source-and-target. Then*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target,*
(3) *$\mathcal{P}$ is stable under postcomposing with étale morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$, and*
(4) *$\mathcal{P}$ has a permanence property: given $f : X \to Y$ and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$, then f has $\mathcal{P}$.*

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{X_i \to X\}_{i \in I}$ be an étale covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $|x| \in X$ we can find an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of pairs, and $\mathrm{id}_Y : Y \to Y$ is a smooth morphism, and h_i is as in part (3) of Definition 21.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 21.1 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}_{i \in I}$ be a smooth covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in |X|$ we pick an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $X_i \to X \times_Y Y_i$ is an étale morphism (because it is an isomorphism), $Y_i \to Y$ is smooth, and h_i is as in part (3) of Definition 20.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 20.1 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale. The morphism $X \to Y \times_Z X$ is étale as a morphism between algebraic spaces étale over X (Properties of Spaces, Lemma 16.6). Also $Y \to Z$ is étale hence a smooth morphism. Thus the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "g \circ f"] & Z \end{tikzcd}$$

works for every $x \in |X|$ in part (3) of Definition 20.1 and we conclude that $g \circ f$ has $\mathcal{P}$.

Proof of (4). Let $f : X \to Y$ be a morphism and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$. Then by Definition 21.1 part (2) we see that $\mathrm{pr}_Y : Y \times_Z X \to Y$ has $\mathcal{P}$.

But the morphism $(f, 1) : X \to Y \times_Z X$ is étale as a section to the étale projection $\text{pr}_X : Y \times_Z X \to X$, see Morphisms of Spaces, Lemma 39.11. Hence $f = \text{pr}_Y \circ (f, 1)$ has $\mathcal{P}$ by Definition 21.1 part (1). □

Lemma 21.3. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is etale-smooth local on source-and-target. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(a) *f has property $\mathcal{P}$,*
(b) *for every $x \in |X|$ there exists a smooth morphism $b : V \to Y$, an étale morphism $a : U \to V \times_Y X$, and a point $u \in |U|$ mapping to x such that $U \to V$ has $\mathcal{P}$,*
(c) *for some commutative diagram*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with b smooth, $U \to V \times_Y X$ étale, and a surjective the morphism h has $\mathcal{P}$,
(d) *for any commutative diagram*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with b smooth and $U \to X \times_Y V$ étale, the morphism h has $\mathcal{P}$,
(e) *there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,*
(f) *there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,*
(g) *there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ an étale covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.*

Proof. The equivalence of (a) and (b) is part of Definition 21.1. The equivalence of (a) and (e) is Lemma 21.2 part (2). The equivalence of (a) and (f) is Lemma 21.2 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (b). If (b) holds, then for any $x \in |X|$ we can choose a smooth morphism a smooth morphism $b_x : V_x \to Y$, an étale morphism $U_x \to V_x \times_Y X$, and $u_x \in |U_x|$ mapping to x such that $U_x \to V_x$ has $\mathcal{P}$. Then $h = \coprod h_x : \coprod U_x \to \coprod V_x$ with $a = \coprod a_x$ and $b = \coprod b_x$ is a diagram as in (c). (Note that h has property $\mathcal{P}$ as $\{V_x \to \coprod V_x\}$ is a smooth covering and $\mathcal{P}$ is smooth local on the target.) Thus (b) is equivalent to (c).

Now we know that (a), (b), (c), (e), (f), and (g) are equivalent. Suppose (a) holds. Let U, V, a, b, h be as in (d). Then $X \times_Y V \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under smooth base change, whence $U \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under precomposing with étale morphisms. Conversely, if (d) holds, then setting $U = X$ and $V = Y$ we see that f has $\mathcal{P}$. □

Lemma 21.4. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target, and*
(3) *$\mathcal{P}$ is stable under postcomposing with open immersions: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is an open embedding then $X \to Z$ has $\mathcal{P}$.*

Then $\mathcal{P}$ is étale-smooth local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 14.2 we see that $\mathcal{P}$ is stable under precomposing with étale morphisms. By Lemma 10.2 we see that $\mathcal{P}$ is stable under smooth base change. Hence it suffices to prove part (3) of Definition 20.1 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of algebraic spaces over S which satisfies Definition 20.1 part (3)(b). In other words, for every $x \in X$ there exists a smooth morphism $b_x : V_x \to Y$, an étale morphism $U_x \to V_x \times_Y X$, and a point $u_x \in |U_x|$ mapping to x such that $h_x : U_x \to V_x$ has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$.

Let $a_x : U_x \to X$ be the composition $U_x \to V_x \times_Y X \to X$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with b smooth, $U \to V \times_Y X$ étale, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is smooth local on the target. In the next paragraph we prove that we may assume U, V, X, Y are schemes; we encourage the reader to skip it.

Let X, Y, U, V, a, b, f, h be as in the previous paragraph. We have to show f has $\mathcal{P}$. Let $X' \to X$ be a surjective étale morphism with X_i a scheme. Set $U' = X' \times_X U$. Then $U' \to X'$ is surjective and $U' \to X' \times_Y V$ is étale. Since $\mathcal{P}$ is étale local on the source, we see that $U' \to V$ has $\mathcal{P}$ and that it suffices to show that $X' \to Y$ has $\mathcal{P}$. In other words, we may assume that X is a scheme. Next, choose a surjective étale morphism $Y' \to Y$ with Y' a scheme. Set $V' = V \times_Y Y'$, $X' = X \times_Y Y'$, and $U' = U \times_Y Y'$. Then $U' \to X'$ is surjective and $U' \to X' \times_{Y'} V'$ is étale. Since $\mathcal{P}$ is smooth local on the target, we see that $U' \to V'$ has $\mathcal{P}$ and that it suffices to prove $X' \to Y'$ has $\mathcal{P}$. Thus we may assume both X and Y are schemes. Choose a surjective étale morphism $V' \to V$ with V' a scheme. Set $U' = U \times_V V'$. Then $U' \to X$ is surjective and $U' \to X \times_Y V'$ is étale. Since $\mathcal{P}$ is smooth local on the source, we see that $U' \to V'$ has $\mathcal{P}$. Thus we may replace U, V by U', V' and assume X, Y, V are schemes. Finally, we replace U by a scheme surjective étale over U and we see that we may assume U, V, X, Y are all schemes.

If U, V, X, Y are schemes, then f has $\mathcal{P}$ by Descent, Lemma 32.11. $\square$

Remark 21.5. Using Lemma 21.4 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are smooth local on the source-and-target. In each case we list the lemma which implies the property is etale local on the source and the lemma which implies the property is smooth local on the target. In each case the third assumption of Lemma 21.4 is trivial to check, and we omit it. Here is the list:

(1) étale, see Lemmas 19.1 and 11.28,

(2) locally quasi-finite, see Lemmas 19.2 and 11.24,
(3) unramified, see Lemmas 19.3 and 11.27, and
(4) add more here as needed.

Of course any property listed in Remark 20.5 is a fortiori an example that could be listed here.

22. Descent data for spaces over spaces

This section is the analogue of Descent, Section 34 for algebraic spaces. Most of the arguments in this section are formal relying only on the definition of a descent datum.

Definition 22.1. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S.

(1) Let $V \to Y$ be a morphism of algebraic spaces. A *descent datum for* $V/Y/X$ is an isomorphism $\varphi : V \times_X Y \to Y \times_X V$ of algebraic spaces over $Y \times_X Y$ satisfying the *cocycle condition* that the diagram

$$\begin{tikzcd} V \times_X Y \times_X Y \arrow[rr, "\varphi_{02}"] \arrow[dr, "\varphi_{01}"'] & & Y \times_X Y \times_X V \\ & Y \times_X V \times_X Y \arrow[ur, "\varphi_{12}"'] & \end{tikzcd}$$

commutes (with obvious notation).

(2) We also say that the pair $(V/Y, \varphi)$ is a *descent datum relative to* $Y \to X$.
(3) A *morphism* $f : (V/Y, \varphi) \to (V'/Y, \varphi')$ *of descent data relative to* $Y \to X$ is a morphism $f : V \to V'$ of algebraic spaces over Y such that the diagram

$$\begin{tikzcd} V \times_X Y \arrow[r, "\varphi"'] \arrow[d, "f \times \mathrm{id}_Y"'] & Y \times_X V \arrow[d, "\mathrm{id}_Y \times f"] \\ V' \times_X Y \arrow[r, "\varphi'"] & Y \times_X V' \end{tikzcd}$$

commutes.

Remark 22.2. Let S be a scheme. Let $Y \to X$ be a morphism of algebraic spaces over S. Let $(V/Y, \varphi)$ be a descent datum relative to $Y \to X$. We may think of the isomorphism φ as an isomorphism

$$(Y \times_X Y) \times_{\mathrm{pr}_0, Y} V \longrightarrow (Y \times_X Y) \times_{\mathrm{pr}_1, Y} V$$

of algebraic spaces over $Y \times_X Y$. So loosely speaking one may think of φ as a map $\varphi : \mathrm{pr}_0^*V \to \mathrm{pr}_1^*V$[1]. The cocycle condition then says that $\mathrm{pr}_{02}^*\varphi = \mathrm{pr}_{12}^*\varphi \circ \mathrm{pr}_{01}^*\varphi$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

Here is the definition in case you have a family of morphisms with fixed target.

Definition 22.3. Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target X.

[1] Unfortunately, we have chosen the "wrong" direction for our arrow here. In Definitions 22.1 and 22.3 we should have the opposite direction to what was done in Definition 3.1 by the general principle that "functions" and "spaces" are dual.

(1) A *descent datum* (V_i, φ_{ij}) *relative to the family* $\{X_i \to X\}$ is given by an algebraic space V_i over X_i for each $i \in I$, an isomorphism $\varphi_{ij} : V_i \times_X X_j \to X_i \times_X V_j$ of algebraic spaces over $X_i \times_X X_j$ for each pair $(i,j) \in I^2$ such that for every triple of indices $(i,j,k) \in I^3$ the diagram

$$\begin{tikzcd} V_i \times_X X_j \times_X X_k \arrow[rr, "\mathrm{pr}_{02}^*\varphi_{ik}"] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & X_i \times_X X_j \times_X V_k \\ & X_i \times_X V_j \times_X X_k \arrow[ur, "\mathrm{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$

of algebraic spaces over $X_i \times_X X_j \times_X X_k$ commutes (with obvious notation).

(2) A *morphism* $\psi : (V_i, \varphi_{ij}) \to (V_i', \varphi_{ij}')$ *of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms $\psi_i : V_i \to V_i'$ of algebraic spaces over X_i such that all the diagrams

$$\begin{tikzcd} V_i \times_X X_j \arrow[r, "\varphi_{ij}"'] \arrow[d, "\psi_i \times \mathrm{id}"'] & X_i \times_X V_j \arrow[d, "\mathrm{id} \times \psi_j"] \\ V_i' \times_X X_j \arrow[r, "\varphi_{ij}'"] & X_i \times_X V_j' \end{tikzcd}$$

commute.

Remark 22.4. Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target X. Let (V_i, φ_{ij}) be a descent datum relative to $\{X_i \to X\}$. We may think of the isomorphisms φ_{ij} as isomorphisms

$$(X_i \times_X X_j) \times_{\mathrm{pr}_0, X_i} V_i \longrightarrow (X_i \times_X X_j) \times_{\mathrm{pr}_1, X_j} V_j$$

of algebraic spaces over $X_i \times_X X_j$. So loosely speaking one may think of φ_{ij} as an isomorphism $\mathrm{pr}_0^* V_i \to \mathrm{pr}_1^* V_j$ over $X_i \times_X X_j$. The cocycle condition then says that $\mathrm{pr}_{02}^*\varphi_{ik} = \mathrm{pr}_{12}^*\varphi_{jk} \circ \mathrm{pr}_{01}^*\varphi_{ij}$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

The reason we will usually work with the version of a family consisting of a single morphism is the following lemma.

Lemma 22.5. *Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target X. Set $Y = \coprod_{i \in I} X_i$. There is a canonical equivalence of categories*

$$\begin{matrix} \textit{category of descent data} \\ \textit{relative to the family } \{X_i \to X\}_{i \in I} \end{matrix} \longrightarrow \begin{matrix} \textit{category of descent data} \\ \textit{relative to } Y/X \end{matrix}$$

which maps (V_i, φ_{ij}) to (V, φ) with $V = \coprod_{i \in I} V_i$ and $\varphi = \coprod \varphi_{ij}$.

Proof. Observe that $Y \times_X Y = \coprod_{ij} X_i \times_X X_j$ and similarly for higher fibre products. Giving a morphism $V \to Y$ is exactly the same as giving a family $V_i \to X_i$. And giving a descent datum φ is exactly the same as giving a family φ_{ij}. $\square$

Lemma 22.6. *Pullback of descent data. Let S be a scheme.*

(1) *Let*

$$\begin{tikzcd} Y' \arrow[r, "f"'] \arrow[d, "a'"'] & Y \arrow[d, "a"] \\ X' \arrow[r, "h"] & X \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. The construction

$$(V \to Y, \varphi) \longmapsto f^*(V \to Y, \varphi) = (V' \to Y', \varphi')$$

where $V' = Y' \times_Y V$ and where φ' is defined as the composition

$$\begin{tikzcd}
V' \times_{X'} Y' \arrow[r, equal] & (Y' \times_Y V) \times_{X'} Y' \arrow[r, equal] & (Y' \times_{X'} Y') \times_{Y \times_X Y} (V \times_X Y) \arrow[d, "id \times \varphi"] \\
Y' \times_{X'} V' \arrow[r, equal] & Y' \times_{X'} (Y' \times_Y V) \arrow[r, equal] & (Y' \times_X Y') \times_{Y \times_X Y} (Y \times_X V)
\end{tikzcd}$$

defines a functor from the category of descent data relative to $Y \to X$ to the category of descent data relative to $Y' \to X'$.

(2) *Given two morphisms $f_i : Y' \to Y$, $i = 0, 1$ making the diagram commute the functors f_0^* and f_1^* are canonically isomorphic.*

Proof. We omit the proof of (1), but we remark that the morphism φ' is the morphism $(f \times f)^*\varphi$ in the notation introduced in Remark 22.2. For (2) we indicate which morphism $f_0^*V \to f_1^*V$ gives the functorial isomorphism. Namely, since f_0 and f_1 both fit into the commutative diagram we see there is a unique morphism $r : Y' \to Y \times_X Y$ with $f_i = \text{pr}_i \circ r$. Then we take

$$\begin{aligned}
f_0^*V &= Y' \times_{f_0, Y} V \\
&= Y' \times_{\text{pr}_0 \circ r, Y} V \\
&= Y' \times_{r, Y \times_X Y} (Y \times_X Y) \times_{\text{pr}_0, Y} V \\
&\xrightarrow{\varphi} Y' \times_{r, Y \times_X Y} (Y \times_X Y) \times_{\text{pr}_1, Y} V \\
&= Y' \times_{\text{pr}_1 \circ r, Y} V \\
&= Y' \times_{f_1, Y} V \\
&= f_1^*V
\end{aligned}$$

We omit the verification that this works. $\square$

Definition 22.7. With $S, X, X', Y, Y', f, a, a', h$ as in Lemma 22.6 the functor

$$(V, \varphi) \longmapsto f^*(V, \varphi)$$

constructed in that lemma is called the *pullback functor* on descent data.

Lemma 22.8. *Let S be a scheme. Let $\mathcal{U}' = \{X'_i \to X'\}_{i \in I'}$ and $\mathcal{U} = \{X_j \to X\}_{i \in I}$ be families of morphisms with fixed target. Let $\alpha : I' \to I$, $g : X' \to X$ and $g_i : X'_i \to X_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 8.1.*

(1) *Let (V_i, φ_{ij}) be a descent datum relative to the family $\mathcal{U}$. The system*
$$\left(g_i^* V_{\alpha(i)}, (g_i \times g_j)^* \varphi_{\alpha(i)\alpha(j)}\right)$$
(with notation as in Remark 22.4) is a descent datum relative to $\mathcal{U}'$.

(2) *This construction defines a functor between the category of descent data relative to $\mathcal{U}$ and the category of descent data relative to $\mathcal{U}'$.*

(3) *Given a second $\beta : I' \to I$, $h : X' \to X$ and $h'_i : X'_i \to X_{\beta(i)}$ morphism of families of maps with fixed target, then if $g = h$ the two resulting functors between descent data are canonically isomorphic.*

(4) *These functors agree, via Lemma 22.5, with the pullback functors constructed in Lemma 22.6.*

Proof. This follows from Lemma 22.6 via the correspondence of Lemma 22.5. $\square$

Definition 22.9. With $\mathcal{U}' = \{X'_i \to X'\}_{i \in I'}$, $\mathcal{U} = \{X_i \to X\}_{i \in I}$, $\alpha : I' \to I$, $g : X' \to X$, and $g_i : X'_i \to X_{\alpha(i)}$ as in Lemma 22.8 the functor

$$(V_i, \varphi_{ij}) \longmapsto (g_i^* V_{\alpha(i)}, (g_i \times g_j)^* \varphi_{\alpha(i)\alpha(j)})$$

constructed in that lemma is called the *pullback functor* on descent data.

If $\mathcal{U}$ and $\mathcal{U}'$ have the same target X, and if $\mathcal{U}'$ refines $\mathcal{U}$ (see Sites, Definition 8.1) but no explicit pair (α, g_i) is given, then we can still talk about the pullback functor since we have seen in Lemma 22.8 that the choice of the pair does not matter (up to a canonical isomorphism).

Definition 22.10. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S.

(1) Given an algebraic space U over X we have the *trivial descent datum* of U relative to $\mathrm{id} : X \to X$, namely the identity morphism on U.
(2) By Lemma 22.6 we get a *canonical descent datum* on $Y \times_X U$ relative to $Y \to X$ by pulling back the trivial descent datum via f. We often denote $(Y \times_X U, can)$ this descent datum.
(3) A descent datum (V, φ) relative to Y/X is called *effective* if (V, φ) is isomorphic to the canonical descent datum $(Y \times_X U, can)$ for some algebraic space U over X.

Thus being effective means there exists an algebraic space U over X and an isomorphism $\psi : V \to Y \times_X U$ over Y such that φ is equal to the composition

$$V \times_X Y \xrightarrow{\psi \times \mathrm{id}_Y} Y \times_X U \times_S Y = Y \times_X Y \times_X U \xrightarrow{\mathrm{id}_Y \times \psi^{-1}} Y \times_X V$$

There is a slight problem here which is that this definition (in spirit) conflicts with the definition given in Descent, Definition 34.10 in case Y and X are schemes. However, it will always be clear from context which version we mean.

Definition 22.11. Let S be a scheme. Let $\{X_i \to X\}$ be a family of morphisms of algebraic spaces over S with fixed target X.

(1) Given an algebraic space U over X we have a *canonical descent datum* on the family of algebraic spaces $X_i \times_X U$ by pulling back the trivial descent datum for U relative to $\{\mathrm{id} : S \to S\}$. We denote this descent datum $(X_i \times_X U, can)$.
(2) A descent datum (V_i, φ_{ij}) relative to $\{X_i \to S\}$ is called *effective* if there exists an algebraic space U over X such that (V_i, φ_{ij}) is isomorphic to $(X_i \times_X U, can)$.

23. Descent data in terms of sheaves

This section is the analogue of Descent, Section 39. It is slightly different as algebraic spaces are already sheaves.

Lemma 23.1. *Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering of algebraic spaces over S (Topologies on Spaces, Definition 7.1). There is an equivalence of categories*

$$\left\{ \begin{matrix} \textit{descent data } (V_i, \varphi_{ij}) \\ \textit{relative to } \{X_i \to X\} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{sheaves } F \textit{ on } (Sch/S)_{fppf} \textit{ endowed} \\ \textit{with a map } F \to X \textit{ such that each} \\ X_i \times_X F \textit{ is an algebraic space} \end{matrix} \right\}.$$

Moreover,

(1) *the algebraic space $X_i \times_X F$ on the right hand side corresponds to V_i on the left hand side, and*
(2) *the sheaf F is an algebraic space*[2] *if and only if the corresponding descent datum (X_i, φ_{ij}) is effective.*

Proof. Let us construct the functor from right to left. Let $F \to X$ be a map of sheaves on $(Sch/S)_{fppf}$ such that each $V_i = X_i \times_X F$ is an algebraic space. We have the projection $V_i \to X_i$. Then both $V_i \times_X X_j$ and $X_i \times_X V_j$ represent the sheaf $X_i \times_X F \times_X X_j$ and hence we obtain an isomorphism

$$\varphi_{ii'} : V_i \times_X X_j \to X_i \times_X V_j$$

It is straightforward to see that the maps φ_{ij} are morphisms over $X_i \times_X X_j$ and satisfy the cocycle condition. The functor from right to left is given by this construction $F \mapsto (V_i, \varphi_{ij})$.

Let us construct a functor from left to right. The isomorphisms φ_{ij} give isomorphisms

$$\varphi_{ij} : V_i \times_X X_j \longrightarrow X_i \times_X V_j$$

over $X_i \times X_j$. Set F equal to the coequalizer in the following diagram

$$\coprod_{i,i'} V_i \times_X X_j \underset{\mathrm{pr}_1 \circ \varphi_{ij}}{\overset{\mathrm{pr}_0}{\rightrightarrows}} \coprod_i V_i \longrightarrow F$$

The cocycle condition guarantees that F comes with a map $F \to X$ and that $X_i \times_X F$ is isomorphic to V_i. The functor from left to right is given by this construction $(V_i, \varphi_{ij}) \mapsto F$.

We omit the verification that these constructions are mutually quasi-inverse functors. The final statements (1) and (2) follow from the constructions. $\square$

24. Other chapters

[2]We will see later that this is always the case if I is not too large, see Bootstrap, Lemma 11.3.

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