FUNDAMENTAL GROUPS OF SCHEMES

Contents

1. Introduction

In this chapter we discuss Grothendieck's fundamental group of a scheme and applications. A foundational reference is [Gro71]. A nice introduction is [Len]. Other references [Mur67] and [GM71].

2. Schemes étale over a point

In this section we describe schemes étale over the spectrum of a field. Before we state the result we introduce the category of G-sets for a topological group G.

Definition 2.1. Let G be a topological group. A *G-set*, sometimes called a *discrete G-set*, is a set X endowed with a left action $a : G \times X \to X$ such that a is continuous when X is given the discrete topology and $G \times X$ the product topology. A *morphism of G-sets* $f : X \to Y$ is simply any G-equivariant map from X to Y. The category of G-sets is denoted *G-Sets*.

The condition that $a : G \times X \to X$ is continuous signifies simply that the stabilizer of any $x \in X$ is open in G. If G is an abstract group G (i.e., a group but not a topological group) then this agrees with our preceding definition (see for example Sites, Example 6.5) provided we endow G with the discrete topology.

Recall that if L/K is an infinite Galois extension then the Galois group $G = \text{Gal}(L/K)$ comes endowed with a canonical topology, see Fields, Section 22.

Lemma 2.2. *Let K be a field. Let K^{sep} be a separable closure of K. Consider the profinite group $G = Gal(K^{sep}/K)$. The functor*

$$\begin{matrix} \textit{schemes étale over } K & \longrightarrow & G\textit{-Sets} \\ X/K & \longmapsto & \text{Mor}_{\text{Spec}(K)}(\text{Spec}(K^{sep}), X) \end{matrix}$$

is an equivalence of categories.

Proof. A scheme X over K is étale over K if and only if $X \cong \coprod_{i \in I} \text{Spec}(K_i)$ with each K_i a finite separable extension of K (Morphisms, Lemma 36.7). The functor of the lemma associates to X the G-set

$$\coprod\nolimits_i \text{Hom}_K(K_i, K^{sep})$$

with its natural left G-action. Each element has an open stabilizer by definition of the topology on G. Conversely, any G-set S is a disjoint union of its orbits. Say $S = \coprod S_i$. Pick $s_i \in S_i$ and denote $G_i \subset G$ its open stabilizer. By Galois theory (Fields, Theorem 22.4) the fields $(K^{sep})^{G_i}$ are finite separable field extensions of K, and hence the scheme

$$\coprod\nolimits_i \text{Spec}((K^{sep})^{G_i})$$

is étale over K. This gives an inverse to the functor of the lemma. Some details omitted. $\square$

Remark 2.3. Under the correspondence of Lemma 2.2, the coverings in the small étale site $\text{Spec}(K)_{étale}$ of K correspond to surjective families of maps in *G-Sets*.

3. Galois categories

In this section we discuss some of the material the reader can find in [Gro71, Exposé V, Sections 4, 5, and 6].

Let $F : \mathcal{C} \to \textit{Sets}$ be a functor. Recall that by our conventions categories have a set of objects and for any pair of objects a set of morphisms. There is a canonical injective map

$$\text{Aut}(F) \longrightarrow \prod\nolimits_{X \in \text{Ob}(\mathcal{C})} \text{Aut}(F(X)) \tag{3.0.1}$$

For a set E we endow $\text{Aut}(E)$ with the compact open topology, see Topology, Example 30.2. Of course this is the discrete topology when E is finite, which is the case of interest in this section[1]. We endow $\text{Aut}(F)$ with the topology induced from the product topology on the right hand side of (3.0.1). In particular, the action maps

$$\text{Aut}(F) \times F(X) \longrightarrow F(X)$$

are continuous when $F(X)$ is given the discrete topology because this is true for the action maps $\text{Aut}(E) \times E \to E$ for any set E. The universal property of our topology on $\text{Aut}(F)$ is the following: suppose that G is a topological group and $G \to \text{Aut}(F)$ is a group homomorphism such that the induced actions $G \times F(X) \to F(X)$ are continuous for all $X \in \text{Ob}(\mathcal{C})$ where $F(X)$ has the discrete topology. Then $G \to \text{Aut}(F)$ is continuous.

The following lemma tells us that the group of automorphisms of a functor to the category of finite sets is automatically a profinite group.

Lemma 3.1. *Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \textit{Sets}$ be a functor. The map (3.0.1) identifies $\text{Aut}(F)$ with a closed subgroup of $\prod_{X \in \text{Ob}(\mathcal{C})} \text{Aut}(F(X))$. In particular, if $F(X)$ is finite for all X, then $\text{Aut}(F)$ is a profinite group.*

Proof. Let $\xi = (\gamma_X) \in \prod \text{Aut}(F(X))$ be an element not in $\text{Aut}(F)$. Then there exists a morphism $f : X \to X'$ of $\mathcal{C}$ and an element $x \in F(X)$ such that $F(f)(\gamma_X(x)) \neq \gamma_{X'}(F(f)(x))$. Consider the open neighbourhood $U = \{\gamma \in \text{Aut}(F(X)) \mid \gamma(x) = \gamma_X(x)\}$ of γ_X and the open neighbourhood $U' = \{\gamma' \in \text{Aut}(F(X')) \mid \gamma'(F(f)(x)) = \gamma_{X'}(F(f)(x))\}$. Then $U \times U' \times \prod_{X'' \neq X, X'} \text{Aut}(F(X''))$ is an open neighbourhood of ξ not meeting $\text{Aut}(F)$. The final statement follows from the fact that $\prod \text{Aut}(F(X))$ is a profinite space if each $F(X)$ is finite. $\square$

Example 3.2. Let G be a topological group. An important example will be the forgetful functor

$$\textit{Finite-G-Sets} \longrightarrow \textit{Sets} \tag{3.2.1}$$

where *Finite-G-Sets* is the full subcategory of *G-Sets* whose objects are the finite G-sets. The category *G-Sets* of G-sets is defined in Definition 2.1.

Let G be a topological group. The *profinite completion* of G will be the profinite group

$$G^\wedge = \lim\nolimits_{U \subset G \text{ open, normal, finite index}} G/U$$

with its profinite topology. Observe that the limit is cofiltered as a finite intersection of open, normal subgroups of finite index is another. The universal property of the

[1] When we discuss the pro-étale fundamental group the general case will be of interest.

profinite completion is that any continuous map $G \to H$ to a profinite group H factors canonically as $G \to G^\wedge \to H$.

Lemma 3.3. *Let G be a topological group. The automorphism group of the functor (3.2.1) endowed with its profinite topology from Lemma 3.1 is the profinite completion of G.*

Proof. Denote F_G the functor (3.2.1). Any morphism $X \to Y$ in *Finite-G-Sets* commutes with the action of G. Thus any $g \in G$ defines an automorphism of F_G and we obtain a canonical homomorphism $G \to \text{Aut}(F_G)$ of groups. Observe that any finite G-set X is a finite disjoint union of G-sets of the form G/H_i with canonical G-action where $H_i \subset G$ is an open subgroup of finite index. Then $U_i = \bigcap gH_ig^{-1}$ is open, normal, and has finite index. Moreover U_i acts trivially on G/H_i hence $U = \bigcap U_i$ acts trivially on $F_G(X)$. Hence the action $G \times F_G(X) \to F_G(X)$ is continuous. By the universal property of the topology on $\text{Aut}(F_G)$ the map $G \to \text{Aut}(F_G)$ is continuous. By Lemma 3.1 and the universal property of profinite completion there is an induced continuous group homomorphism

$$G^\wedge \longrightarrow \text{Aut}(F_G)$$

Moreover, since G/U acts faithfully on G/U this map is injective. If the image is dense, then the map is surjective and hence a homeomorphism by Topology, Lemma 17.8.

Let $\gamma \in \text{Aut}(F_G)$ and let $X \in \text{Ob}(\mathcal{C})$. We will show there is a $g \in G$ such that γ and g induce the same action on $F_G(X)$. This will finish the proof. As before we see that X is a finite disjoint union of G/H_i. With U_i and U as above, the finite G-set $Y = G/U$ surjects onto G/H_i for all i and hence it suffices to find $g \in G$ such that γ and g induce the same action on $F_G(G/U) = G/U$. Let $e \in G$ be the neutral element and say that $\gamma(eU) = g_0U$ for some $g_0 \in G$. For any $g_1 \in G$ the morphism

$$R_{g_1} : G/U \longrightarrow G/U, \quad gU \longmapsto gg_1U$$

of *Finite-G-Sets* commutes with the action of γ. Hence

$$\gamma(g_1U) = \gamma(R_{g_1}(eU)) = R_{g_1}(\gamma(eU)) = R_{g_1}(g_0U) = g_0g_1U$$

Thus we see that $g = g_0$ works. □

Recall that an exact functor is one which commutes with all finite limits and finite colimits. In particular such a functor commutes with equalizers, coequalizers, fibred products, pushouts, etc.

Lemma 3.4. *Let G be a topological group. Let F : Finite-G-Sets $\to$ Sets be an exact functor with $F(X)$ finite for all X. Then F is isomorphic to the functor (3.2.1).*

Proof. Let X be a nonempty object of *Finite-G-Sets*. The diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \{*\} \arrow[d] \\ \{*\} \arrow[r] & \{*\} \end{tikzcd}$$

is cocartesian. Hence we conclude that $F(X)$ is nonempty. Let $U \subset G$ be an open, normal subgroup with finite index. Observe that

$$G/U \times G/U = \coprod\nolimits_{gU \in G/U} G/U$$

where the summand corresponding to gU corresponds to the orbit of (eU, gU) on the left hand side. Then we see that

$$F(G/U) \times F(G/U) = F(G/U \times G/U) = \coprod\nolimits_{gU \in G/U} F(G/U)$$

Hence $|F(G/U)| = |G/U|$ as $F(G/U)$ is nonempty. Thus we see that

$$\lim\nolimits_{U \subset G \text{ open, normal, finite idex}} F(G/U)$$

is nonempty (Categories, Lemma 21.7). Pick $\gamma = (\gamma_U)$ an element in this limit. Denote F_G the functor (3.2.1). We can identify F_G with the functor

$$X \longmapsto \operatorname{colim}_U \operatorname{Mor}(G/U, X)$$

where $f : G/U \to X$ corresponds to $f(eU) \in X = F_G(X)$ (details omitted). Hence the element γ determines a well defined map

$$t : F_G \longrightarrow F$$

Namely, given $x \in X$ choose U and $f : G/U \to X$ sending eU to x and then set $t_X(x) = F(f)(\gamma_U)$. We will show that t induces a bijective map $t_{G/U} : F_G(G/U) \to F(G/U)$ for any U. This implies in a straightforward manner that t is an isomorphism (details omitted). Since $|F_G(G/U)| = |F(G/U)|$ it suffices to show that $t_{G/U}$ is surjective. The image contains at least one element, namely $t_{G/U}(eU) = F(\mathrm{id}_{G/U})(\gamma_U) = \gamma_U$. For $g \in G$ denote $R_g : G/U \to G/U$ right multiplication. Then set of fixed points of $F(R_g) : F(G/U) \to F(G/U)$ is equal to $F(\emptyset) = \emptyset$ if $g \notin U$ because F commutes with equalizers. It follows that if $g_1, \ldots, g_{|G/U|}$ is a system of representatives for G/U, then the elements $F(R_{g_i})(\gamma_U)$ are pairwise distinct and hence fill out $F(G/U)$. Then

$$t_{G/U}(g_i U) = F(R_{g_i})(\gamma_U)$$

and the proof is complete. ☐

Example 3.5. Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \textit{Sets}$ be a functor such that $F(X)$ is finite for all $X \in \mathrm{Ob}(\mathcal{C})$. By Lemma 3.1 we see that $G = \mathrm{Aut}(F)$ comes endowed with the structure of a profinite topological group in a canonical manner. We obtain a functor

$$\mathcal{C} \longrightarrow \textit{Finite-G-Sets}, \quad X \longmapsto F(X) \tag{3.5.1}$$

where $F(X)$ is endowed with the induced action of G. This action is continuous by our construction of the topology on $\mathrm{Aut}(F)$.

The purpose of defining Galois categories is to single out those pairs $(\mathcal{C}, F)$ for which the functor (3.5.1) is an equivalence. Our definition of a Galois category is as follows.

Definition 3.6. Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \textit{Sets}$ be a functor. The pair $(\mathcal{C}, F)$ is a *Galois category* if

1. (1) $\mathcal{C}$ has finite limits and finite colimits,
2. (2) every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects,
3. (3) $F(X)$ is finite for all $X \in \mathrm{Ob}(\mathcal{C})$, and

(4) F reflects isomorphisms[2] and is exact[3].

Here we say $X \in \text{Ob}(\mathcal{C})$ is connected if it is not initial and for any monomorphism $Y \to X$ either Y is initial or $Y \to X$ is an isomorphism.

Warning: This definition is not the same (although eventually we'll see it is equivalent) as the definition given in most references. Namely, in [Gro71, Exposé V, Definition 5.1] a Galois category is defined to be a category equivalent to *Finite-G-Sets* for some profinite group G. Then Grothendieck characterizes Galois categories by a list of axioms (G1) – (G6) which are weaker than our axioms above. The motivation for our choice is to stress the existence of finite limits and finite colimits and exactness of the functor F. The price we'll pay for this later is that we'll have to work a bit harder to apply the results of this section.

Lemma 3.7. *Let $(\mathcal{C}, F)$ be a Galois category. Let $X \to Y \in \mathit{Arrows}(\mathcal{C})$. Then*

(1) *F is faithful,*
(2) *$X \to Y$ is a monomorphism $\Leftrightarrow F(X) \to F(Y)$ is injective,*
(3) *$X \to Y$ is an epimorphism $\Leftrightarrow F(X) \to F(Y)$ is surjective,*
(4) *an object A of $\mathcal{C}$ is initial if and only if $F(A) = \emptyset$,*
(5) *an object Z of $\mathcal{C}$ is final if and only if $F(Z)$ is a singleton,*
(6) *if X and Y are connected, then $X \to Y$ is an epimorphism,*
(7) *if X is connected and $a, b : X \to Y$ are two morphisms then $a = b$ as soon as $F(a)$ and $F(b)$ agree on one element of $F(X)$,*
(8) *if $X = \coprod_{i=1,\ldots,n} X_i$ and $Y = \coprod_{j=1,\ldots,m} Y_j$ where X_i, Y_j are connected, then there is map $\alpha : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $X \to Y$ comes from a collection of morphisms $X_i \to Y_{\alpha(i)}$.*

Proof. Proof of (1). Suppose $a, b : X \to Y$ with $F(a) = F(b)$. Let E be the equalizer of a and b. Then $F(E) = F(X)$ and we see that $E = X$ because F reflects isomorphisms.

Proof of (2). This is true because F turns the morphism $X \to X \times_Y X$ into the map $F(X) \to F(X) \times_{F(Y)} F(X)$ and F reflects isomorphisms.

Proof of (3). This is true because F turns the morphism $Y \amalg_X Y \to Y$ into the map $F(Y) \amalg_{F(X)} F(Y) \to F(Y)$ and F reflects isomorphisms.

Proof of (4). There exists an initial object A and certainly $F(A) = \emptyset$. On the other hand, if X is an object with $F(X) = \emptyset$, then the unique map $A \to X$ induces a bijection $F(A) \to F(X)$ and hence $A \to X$ is an isomorphism.

Proof of (5). There exists a final object Z and certainly $F(Z)$ is a singleton. On the other hand, if X is an object with $F(X)$ a singleton, then the unique map $X \to Z$ induces a bijection $F(X) \to F(Z)$ and hence $X \to Z$ is an isomorphism.

Proof of (6). The equalizer E of the two maps $Y \to Y \amalg_X Y$ is not an initial object of $\mathcal{C}$ because $X \to Y$ factors through E and $F(X) \neq \emptyset$. Hence $E = Y$ and we conclude.

Proof of (7). The equalizer E of a and b comes with a monomorphism $E \to X$ and $F(E) \subset F(X)$ is the set of elements where $F(a)$ and $F(b)$ agree. To finish use that either E is initial or $E = X$.

[2]Namely, given a morphism f of $\mathcal{C}$ if $F(f)$ is an isomorphism, then f is an isomorphism.

[3]This means that F commutes with finite limits and colimits, see Categories, Section 23.

Proof of (8). For each i, j we see that $E_{ij} = X_i \times_Y Y_j$ is either initial or equal to X_i. Picking $s \in F(X_i)$ we see that $E_{ij} = X_i$ if and only if s maps to an element of $F(Y_j) \subset F(Y)$, hence this happens for a unique $j = \alpha(i)$. □

By the lemma above we see that, given a connected object X of a Galois category $(\mathcal{C}, F)$, the automorphism group $\text{Aut}(X)$ has order at most $|F(X)|$. Namely, given $s \in F(X)$ and $g \in \text{Aut}(X)$ we see that $g(s) = s$ if and only if $g = \text{id}_X$ by (7). We say X is *Galois* if equality holds. Equivalently, X is Galois if it is connected and $\text{Aut}(X)$ acts transitively on $F(X)$.

Lemma 3.8. *Let $(\mathcal{C}, F)$ be a Galois category. For any connected object X of $\mathcal{C}$ there exists a Galois object Y and a morphism $Y \to X$.*

Proof. We will use the results of Lemma 3.7 without further mention. Let $n = |F(X)|$. Consider X^n endowed with its natural action of S_n. Let

$$X^n = \coprod\nolimits_{t \in T} Z_t$$

be the decomposition into connected objects. Pick a t such that $F(Z_t)$ contains $(s_1, \ldots, s_n)$ with s_i pairwise distinct. If $(s'_1, \ldots, s'_n) \in F(Z_t)$ is another element, then we claim s'_i are pairwise distinct as well. Namely, if not, say $s'_i = s'_j$, then Z_t is the image of an connected component of X^{n-1} under the diagonal morphism

$$\Delta_{ij} : X^{n-1} \longrightarrow X^n$$

Since morphisms of connected objects are epimorphisms and induce surjections after applying F it would follow that $s_i = s_j$ which is not the case.

Let $G \subset S_n$ be the subgroup of elements with $g(Z_t) = Z_t$. Looking at the action of S_n on

$$F(X)^n = F(X^n) = \coprod\nolimits_{t' \in T} F(Z_{t'})$$

we see that $G = \{g \in S_n \mid g(s_1, \ldots, s_n) \in F(Z_t)\}$. Now pick a second element $(s'_1, \ldots, s'_n) \in F(Z_t)$. Above we have seen that s'_i are pairwise distinct. Thus we can find a $g \in S_n$ with $g(s_1, \ldots, s_n) = (s'_1, \ldots, s'_n)$. In other words, the action of G on $F(Z_t)$ is transitive and the proof is complete. □

Here is a key lemma.

Lemma 3.9. *Let $(\mathcal{C}, F)$ be a Galois category. Let $G = Aut(F)$ be as in Example 3.5. For any connected X in $\mathcal{C}$ the action of G on $F(X)$ is transitive.*

Proof. We will use the results of Lemma 3.7 without further mention. Let I be the set of isomorphism classes of Galois objects in $\mathcal{C}$. For each $i \in I$ let X_i be a representative of the isomorphism class. Choose $\gamma_i \in F(X_i)$ for each $i \in I$. We define a partial ordering on I by setting $i \geq i'$ if and only if there is a morphism $f_{ii'} : X_i \to X_{i'}$. Given such a morphism we can post-compose by an automorphism $X_{i'} \to X_{i'}$ to assure that $F(f_{ii'})(\gamma_i) = \gamma_{i'}$. With this normalization the morphism $f_{ii'}$ is unique. Observe that I is a directed partially ordered set: (Categories, Definition 21.1) if $i_1, i_2 \in I$ there exists a Galois object Y and a morphism $Y \to X_{i_1} \times X_{i_2}$ by Lemma 3.8 applied to a connected component of $X_{i_1} \times X_{i_2}$. Then $Y \cong X_i$ for some $i \in I$ and $i \geq i_1$, $i \geq I_2$.

We claim that the functor F is isomorphic to the functor F' which sends X to

$$F'(X) = \text{colim}_I \, \text{Mor}_{\mathcal{C}}(X_i, X)$$

via the transformation of functors $t : F' \to F$ defined as follows: given $f : X_i \to X$ we set $t_X(f) = F(f)(\gamma_i)$. Using (7) we find that t_X is injective. To show surjectivity, let $\gamma \in F(X)$. Then we can immediately reduce to the case where X is connected by the definition of a Galois category. Then we may assume X is Galois by Lemma 3.8. In this case X is isomorphic to X_i for some i and we can choose the isomorphism $X_i \to X$ such that γ_i maps to γ (by definition of Galois objects). We conclude that t is an isomorphism.

Set $A_i = \text{Aut}(X_i)$. We claim that for $i \geq i'$ there is a canonical map $h_{ii'} : A_i \to A_{i'}$ such that for all $a \in A_i$ the diagram

$$\begin{tikzcd} X_i \arrow[r, "f_{ii'}"'] \arrow[d, "a"'] & X_{i'} \arrow[d, "h_{ii'}(a)"] \\ X_i \arrow[r, "f_{ii'}"] & X_{i'} \end{tikzcd}$$

commutes. Namely, just let $h_{ii'}(a) = a' : X_{i'} \to X_{i'}$ be the unique automorphism such that $F(a')(\gamma_{i'}) = F(f_{ii'} \circ a)(\gamma_i)$. As before this makes the diagram commute and moreover the choice is unique. It follows that $h_{i'i''} \circ h_{ii'} = h_{ii''}$ if $i \geq i' \geq i''$. Since $F(X_i) \to F(X_{i'})$ is surjective we see that $A_i \to A_{i'}$ is surjective. Taking the inverse limit we obtain a group

$$A = \lim_I A_i$$

This is a profinite group since the automorphism groups are finite. The map $A \to A_i$ is surjective for all i by Categories, Lemma 21.7.

Since elements of A act on the inverse system X_i we get an action of A (on the right) on F' by pre-composing. In other words, we get a homomorphism $A^{opp} \to G$. Since $A \to A_i$ is surjective we conclude that G acts transitively on $F(X_i)$ for all i. Since every connected object is dominated by one of the X_i we conclude the lemma is true. ☐

Proposition 3.10. *Let $(\mathcal{C}, F)$ be a Galois category. Let $G = Aut(F)$ be as in Example 3.5. The functor $F : \mathcal{C} \to$ Finite-G-Sets (3.5.1) an equivalence.*

Proof. We will use the results of Lemma 3.7 without further mention. In particular we know the functor is faithful. By Lemma 3.9 we know that for any connected X the action of G on $F(X)$ is transitive. Hence F preserves the decomposition into connected components (existence of which is an axiom of a Galois category). Let X and Y be objects and let $s : F(X) \to F(Y)$ be a map. Then the graph $\Gamma_s \subset F(X) \times F(Y)$ of s is a union of connected components. Hence there exists a union of connected components Z of $X \times Y$, which comes equipped with a monomorphism $Z \to X \times Y$, with $F(Z) = \Gamma_s$. Since $F(Z) \to F(X)$ is bijective we see that $Z \to X$ is an isomorphism and we conclude that $s = F(f)$ where $f : X \cong Z \to Y$ is the composition. Hence F is fully faithful.

To finish the proof we show that F is essentially surjective. It suffices to show that G/H is in the essential image for any open subgroup $H \subset G$ of finite index. By definition of the topology on G there exists a finite collection of objects X_i such that

$$\text{Ker}(G \longrightarrow \prod\nolimits_i \text{Aut}(F(X_i)))$$

is contained in H. We may assume X_i is connected for all i. We can choose a Galois object Y mapping to a connected component of $\prod X_i$ using Lemma 3.8. Choose an isomorphism $F(Y) = G/U$ in *G-sets* for some open subgroup $U \subset G$. As Y is Galois, the group $\mathrm{Aut}(Y) = \mathrm{Aut}_{G\text{-}Sets}(G/U)$ acts transitively on $F(Y) = G/U$. This implies that U is normal. Since $F(Y)$ surjects onto $F(X_i)$ for each i we see that $U \subset H$. Let $M \subset \mathrm{Aut}(Y)$ be the finite subgroup corresponding to

$$(H/U)^{opp} \subset (G/U)^{opp} = \mathrm{Aut}_{G\text{-}Sets}(G/U) = \mathrm{Aut}(Y).$$

Set $X = Y/M$, i.e., X is the coequalizer of the arrows $m : Y \to Y$, $m \in M$. Since F is exact we see that $F(X) = G/H$ and the proof is complete. ☐

Lemma 3.11. *Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be Galois categories. Let $H : \mathcal{C} \to \mathcal{C}'$ be an exact functor. There exists an isomorphism $t : F' \circ H \to F$. The choice of t determines a continuous homomorphism $h : G' = Aut(F') \to Aut(F) = G$ and a 2-commutative diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"'] \arrow[d] & \mathcal{C}' \arrow[d] \\ \textit{Finite-G-Sets} \arrow[r, "h"] & \textit{Finite-}G'\textit{-Sets} \end{tikzcd}$$

The map h is independent of t up to an inner automorphism of G. Conversely, given a continuous homomorphism $h : G' \to G$ there is an exact functor $H : \mathcal{C} \to \mathcal{C}'$ and an isomorphism t recovering h as above.

Proof. By Proposition 3.10 and Lemma 3.3 we may assume $\mathcal{C} = $ *Finite-G-Sets* and F is the forgetful functor and similarly for $\mathcal{C}'$. Thus the existence of t follows from Lemma 3.4. The map h comes from transport of structure via t. The commutativity of the diagram is obvious. Uniqueness of h up to inner conjugation by an element of G comes from the fact that the choice of t is unique up to an element of G. The final statement is straightforward. ☐

4. Functors and homomorphisms

Let $(\mathcal{C}, F)$, $(\mathcal{C}', F')$, $(\mathcal{C}'', F'')$ be Galois categories. Set $G = \mathrm{Aut}(F)$, $G' = \mathrm{Aut}(F')$, and $G'' = \mathrm{Aut}(F'')$. Let $H : \mathcal{C} \to \mathcal{C}'$ and $H' : \mathcal{C}' \to \mathcal{C}''$ be exact functors. Let $h : G' \to G$ and $h' : G'' \to G'$ be the corresponding continuous homomorphism as in Lemma 3.11. In this section we consider the corresponding 2-commutative diagram

(4.0.1)
$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"'] \arrow[d] & \mathcal{C}' \arrow[r, "H'"'] \arrow[d] & \mathcal{C}'' \arrow[d] \\ \textit{Finite-G-Sets} \arrow[r, "h"] & \textit{Finite-}G'\textit{-Sets} \arrow[r, "h'"] & \textit{Finite-}G''\textit{-Sets} \end{tikzcd}$$

and we relate exactness properties of the sequence $1 \to G'' \to G' \to G \to 1$ to properties of the functors H and H'.

Lemma 4.1. *In diagram (4.0.1) the following are equivalent*

(1) *$h : G' \to G$ is surjective,*
(2) *$H : \mathcal{C} \to \mathcal{C}'$ is fully faithful,*
(3) *if $X \in \mathrm{Ob}(\mathcal{C})$ is connected, then $H(X)$ is connected,*

(4) *if $X \in \mathrm{Ob}(\mathcal{C})$ is connected and there is a morphism $*' \to H(X)$ in $\mathcal{C}'$, then there is a morphism $* \to X$, and*
(5) *for any object X of $\mathcal{C}$ the map $\mathrm{Mor}_\mathcal{C}(*, X) \to \mathrm{Mor}_{\mathcal{C}'}(*', H(X))$ is bijective.*

Here $*$ and $*'$ are final objects of $\mathcal{C}$ and $\mathcal{C}'$.

Proof. The implications (5) ⇒ (4) and (2) ⇒ (5) are clear.

Assume (3). Let X be a connected object of $\mathcal{C}$ and let $*' \to H(X)$ be a morphism. Since $H(X)$ is connected by (3) we see that $*' \to H(X)$ is an isomorphism. Hence the G'-set corresponding to $H(X)$ has exactly one element, which means the G-set corresponding to X has one element which means X is isomorphic to the final object of $\mathcal{C}$, in particular there is a map $* \to X$. In this way we see that (3) ⇒ (4).

If (1) is true, then the functor *Finite-G-Sets* $\to$ *Finite-G'-Sets* is fully faithful: in this case a map of G-sets commutes with the action of G if and only if it commutes with the action of G'. Thus (1) ⇒ (2).

If (1) is true, then for a G-set X the G-orbits and G'-orbits agree. Thus (1) ⇒ (3).

To finish the proof it suffices to show that (4) implies (1). If (1) is false, i.e., if h is not surjective, then there is an open subgroup $U \subset G$ containing $h(G')$ which is not equal to G. Then the finite G-set $M = G/U$ has a transitive action but G' has a fixed point. The object X of $\mathcal{C}$ corresponding to M would contradict (3). In this way we see that (3) ⇒ (1) and the proof is complete. □

Lemma 4.2. *In diagram (4.0.1) the following are equivalent*

(1) *$h \circ h'$ is trivial, and*
(2) *the image of $H' \circ H$ consists of objects isomorphic to finite coproducts of final objects.*

Proof. We may replace H and H' by the canonical functors *Finite-G-Sets* $\to$ *Finite-G'-Sets* $\to$ *Finite-G''-Sets* determined by h and h'. Then we are saying that the action of G'' on every G-set is trivial if and only if the homomorphism $G'' \to G$ is trivial. This is clear. □

Lemma 4.3. *In diagram (4.0.1) the following are equivalent*

(1) *the sequence $G'' \xrightarrow{h'} G' \xrightarrow{h} G \to 1$ is exact in the following sense: h is surjective, $h \circ h'$ is trivial, and $\mathrm{Ker}(h)$ is the smallest closed normal subgroup containing $\mathrm{Im}(h')$,*
(2) *H is fully faithful and an object X' of $\mathcal{C}'$ is in the essential image of H if and only if $H'(X')$ is isomorphic to a finite coproduct of final objects, and*
(3) *H is fully faithful, $H \circ H'$ sends every object to a finite coproduct of final objects, and for an object X' of $\mathcal{C}'$ such that $H'(X')$ is a finite coproduct of final objects there exists an object X of $\mathcal{C}$ and an epimorphism $H(X) \to X'$.*

Proof. By Lemmas 4.1 and 4.2 we may assume that H is fully faithful, h is surjective, $H' \circ H$ maps objects to disjoint unions of the final object, and $h \circ h'$ is trivial. Let $N \subset G'$ be the smallest closed normal subgroup containing the image of h'. It is clear that $N \subset \mathrm{Ker}(h)$. We may assume the functors H and H' are the canonical functors *Finite-G-Sets* $\to$ *Finite-G'-Sets* $\to$ *Finite-G''-Sets* determined by h and h'.

Suppose that (2) holds. This means that for a finite G'-set X' such that G'' acts trivially, the action of G' factors through G. Apply this to $X' = G'/U'N$ where U'

is a small open subgroup of G'. Then we see that $\text{Ker}(h) \subset U'N$ for all U'. Since N is closed this implies $\text{Ker}(h) \subset N$, i.e., (1) holds.

Suppose that (1) holds. This means that $N = \text{Ker}(h)$. Let X' be a finite G'-set such that G'' acts trivially. This means that $\text{Ker}(G' \to \text{Aut}(X'))$ is a closed normal subgroup containing $\text{Im}(h')$. Hence $N = \text{Ker}(h)$ is contained in it and the G'-action on X' factors through G, i.e., (2) holds.

Suppose that (3) holds. This means that for a finite G'-set X' such that G'' acts trivially, there is a surjection of G'-sets $X \to X'$ where X is a G-set. Clearly this means the action of G' on X' factors through G, i.e., (2) holds.

The implication (2) $\Rightarrow$ (3) is immediate. This finishes the proof. $\square$

Lemma 4.4. *In diagram (4.0.1) the following are equivalent*

1. *h' is injective, and*
2. *for every connected object X'' of $\mathcal{C}''$ there exists an object X' of $\mathcal{C}'$ and a diagram*
$$X'' \leftarrow Y'' \to H(X')$$
in $\mathcal{C}''$ where $Y'' \to X''$ is an epimorphism and $Y'' \to H(X')$ is a monomorphism.

Proof. We may replace H' by the corresponding functor between the categories of finite G'-sets and finite G''-sets.

Assume $h' : G'' \to G'$ is injective. Let $H'' \subset G''$ be an open subgroup. Since the topology on G'' is the induced topology from G' there exists an open subgroup $H' \subset G'$ such that $(h')^{-1}(H') \subset H''$. Then the desired diagram is

$$G''/H'' \leftarrow G''/(h')^{-1}(H') \to G'/H'$$

Conversely, assume (2) holds for the functor *Finite-G'-Sets* $\to$ *Finite-G''-Sets*. Let $g'' \in \text{Ker}(h')$. Pick any open subgroup $H'' \subset G''$. By assumption there exists a finite G'-set X' and a diagram

$$G''/H'' \leftarrow Y'' \to X'$$

of G''-sets with the left arrow surjective and the right arrow injective. Since g'' is in the kernel of h' we see that g'' acts trivially on X'. Hence g'' acts trivially on Y'' and hence trivially on G''/H''. Thus $g'' \in H''$. As this holds for all open subgroups we conclude that g'' is the identity element as desired. $\square$

Lemma 4.5. *In diagram (4.0.1) the following are equivalent*

1. *the image of h' is normal, and*
2. *for every connected object X' of $\mathcal{C}'$ such that there is a morphism from the final object of $\mathcal{C}''$ to $H'(X')$ we have that $H'(X')$ is isomorphic to a finite coproduct of final objects.*

Proof. This translates into the following statement for the continuous group homomorphism $h' : G'' \to G'$: the image of h' is normal if and only if every open subgroup $U' \subset G'$ which contains $h'(G'')$ also contains every conjugate of $h'(G'')$. The result follows easily from this; some details omitted. $\square$

5. Finite étale morphisms

In this section we prove enough basic results on finite étale morphisms to be able to construct the étale fundamental group.

Let X be a scheme. We will use the notation $\textit{FÉt}_X$ to denote the category of schemes finite and étale over X. Thus

(1) an object of $\textit{FÉt}_X$ is a finite étale morphism $Y \to X$ with target X, and
(2) a morphism in $\textit{FÉt}_X$ from $Y \to X$ to $Y' \to X$ is a morphism $Y \to Y'$ making the diagram

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

commute.

We will often call an object of $\textit{FÉt}_X$ a *finite étale cover* of X (even if Y is empty). It turns out that there is a stack $p : \textit{FÉt} \to \textit{Sch}$ over the category of schemes whose fibre over X is the category $\textit{FÉt}_X$ just defined. See Examples of Stacks, Section 6.

Example 5.1. Let k be an algebraically closed field and $X = \mathrm{Spec}(k)$. In this case $\textit{FÉt}_X$ is equivalent to the category of finite sets. This works more generally when k is separably algebraically closed. The reason is that a scheme étale over k is the disjoint union of spectra of fields finite separable over k, see Morphisms, Lemma 36.7.

Lemma 5.2. *Let X be a scheme. The category $\textit{FÉt}_X$ has finite limits and finite colimits and for any morphism $X' \to X$ the base change functor $\textit{FÉt}_X \to \textit{FÉt}_{X'}$ is exact.*

Proof. Finite limits and left exactness. By Categories, Lemma 18.4 it suffices to show that $\textit{FÉt}_X$ has a final object and fibred products. This is clear because the category of all schemes over X has a final object (namely X) and fibred products. Also, fibred products of schemes finite étale over X are finite étale over X. Moreover, it is clear that base change commutes with these operations and hence base change is left exact (Categories, Lemma 23.2).

Finite colimits and right exactness. By Categories, Lemma 18.7 it suffices to show that $\textit{FÉt}_X$ has finite coproducts and coequalizers. Finite coproducts are given by disjoint unions (the empty coproduct is the empty scheme). Let $a, b : Z \to Y$ be two morphisms of $\textit{FÉt}_X$. Since $Z \to X$ and $Y \to X$ are finite étale we can write $Z = \underline{\mathrm{Spec}}(\mathcal{C})$ and $Y = \underline{\mathrm{Spec}}(\mathcal{B})$ for some finite locally free $\mathcal{O}_X$-algebras $\mathcal{C}$ and $\mathcal{B}$. The morphisms a, b induce two maps $a^\sharp, b^\sharp : \mathcal{B} \to \mathcal{C}$. Let $\mathcal{A} = \mathrm{Eq}(a^\sharp, b^\sharp)$ be their equalizer. If

$$\underline{\mathrm{Spec}}(\mathcal{A}) \longrightarrow X$$

is finite étale, then it is clear that this is the coequalizer (after all we can write any object of $\textit{FÉt}_X$ as the relative spectrum of a sheaf of $\mathcal{O}_X$-algebras). This we may do after replacing X by the members of an étale covering (Descent, Lemmas 23.23 and 23.29). Thus by Étale Morphisms, Lemma 18.3 we may assume that

$Y = \coprod_{i=1,\ldots,n} X$ and $Z = \coprod_{j=1,\ldots,m} X$. Then

$$\mathcal{C} = \prod\nolimits_{1 \leq j \leq m} \mathcal{O}_X \quad \text{and} \quad \mathcal{B} = \prod\nolimits_{1 \leq i \leq n} \mathcal{O}_X$$

After a further replacement by the members of an open covering we may assume that a, b correspond to maps $a_s, b_s : \{1, \ldots, m\} \to \{1, \ldots, n\}$, i.e., the summand X of Z corresponding to the index j maps into the summand X of Y corresponding to the index $a_s(j)$, resp. $b_s(j)$ under the morphism a, resp. b. Let $\{1, \ldots, n\} \to T$ be the coequalizer of a_s, b_s. Then we see that

$$\mathcal{A} = \prod\nolimits_{t \in T} \mathcal{O}_X$$

whose spectrum is certainly finite étale over X. We omit the verification that this is compatible with base change. Thus base change is a right exact functor. $\square$

Remark 5.3. Let X be a scheme. Consider the natural functors $F_1 : \textit{FÉt}_X \to \textit{Sch}$ and $F_2 : \textit{FÉt}_X \to \textit{Sch}/X$. Then

(1) The functors F_1 and F_2 commute with finite colimits.
(2) The functor F_2 commutes with finite limits,
(3) The functor F_1 commutes with connected finite limits, i.e., with equalizers and fibre products.

The results on limits are immediate from the discussion in the proof of Lemma 5.2 and Categories, Lemma 16.2. It is clear that F_1 and F_2 commute with finite coproducts. By the dual of Categories, Lemma 23.2 we need to show that F_1 and F_2 commute with coequalizers. In the proof of Lemma 5.2 we saw that coequalizers in $\textit{FÉt}_X$ look étale locally like this

$$\coprod\nolimits_{j \in J} U \overset{a}{\underset{b}{\rightrightarrows}} \coprod\nolimits_{i \in I} U \longrightarrow \coprod\nolimits_{t \in \mathrm{Coeq}(a,b)} U$$

which is certainly a coequalizer in the category of schemes. Hence the statement follows from the fact that being a coequalizer is fpqc local as formulated precisely in Descent, Lemma 13.8.

Lemma 5.4. *Let X be a scheme. Given U, V finite étale over X there exists a scheme W finite étale over X such that*

$$\mathrm{Mor}_X(X, W) = \mathrm{Mor}_X(U, V)$$

and such that the same remains true after any base change.

Proof. By More on Morphisms, Lemma 68.4 there exists a scheme W representing $\mathit{Mor}_X(U, V)$. (Use that an étale morphism is locally quasi-finite by Morphisms, Lemmas 36.6 and that a finite morphism is separated.) This scheme clearly satisfies the formula after any base change. To finish the proof we have to show that $W \to X$ is finite étale. This we may do after replacing X by the members of an étale covering (Descent, Lemmas 23.23 and 23.6). Thus by Étale Morphisms, Lemma 18.3 we may assume that $U = \coprod_{i=1,\ldots,n} X$ and $V = \coprod_{j=1,\ldots,m} X$. In this case $W = \coprod_{\alpha : \{1,\ldots,n\} \to \{1,\ldots,m\}} X$ by inspection (details omitted) and the proof is complete. $\square$

Let X be a scheme. A *geometric point* of X is a morphism $\mathrm{Spec}(k) \to X$ where k is algebraically closed. Such a point is usually denoted $\overline{x}$, i.e., by an overlined small case letter. We often use $\overline{x}$ to denote the scheme $\mathrm{Spec}(k)$ as well as the morphism, and we use $\kappa(\overline{x})$ to denote k. We say $\overline{x}$ *lies over* x to indicate that $x \in X$ is the image of $\overline{x}$. We will discuss this further in Étale Cohomology, Section 29. Given $\overline{x}$ and an étale morphism $U \to X$ we can consider

$$|U_{\overline{x}}| : \text{the underlying set of points of the scheme } U_{\overline{x}} = U \times_X \overline{x}$$

Since $U_{\overline{x}}$ as a scheme over $\overline{x}$ is a disjoint union of copies of $\overline{x}$ (Morphisms, Lemma 36.7) we can also describe this set as

$$|U_{\overline{x}}| = \left\{ \begin{matrix} \text{commutative} \\ \text{diagrams} \end{matrix} \quad \begin{tikzcd} \overline{x} \arrow[r, "\overline{u}"] \arrow[dr, "\overline{x}"'] & U \arrow[d] \\ & X \end{tikzcd} \right\}$$

The assignment $U \mapsto |U_{\overline{x}}|$ is a functor which is often denoted $F_{\overline{x}}$.

Lemma 5.5. *Let X be a connected scheme. Let $\overline{x}$ be a geometric point. The functor*

$$F_{\overline{x}} : \textit{FÉt}_X \longrightarrow \textit{Sets}, \quad Y \longmapsto |Y_{\overline{x}}|$$

defines a Galois category (Definition 3.6).

Proof. After identifying $\textit{FÉt}_{\overline{x}}$ with the category of finite sets (Example 5.1) we see that our functor $F_{\overline{x}}$ is nothing but the base change functor for the morphism $\overline{x} \to X$. Thus we see that $\textit{FÉt}_X$ has finite limits and finite colimits and that $F_{\overline{x}}$ is exact by Lemma 5.2. We will also use that finite limits in $\textit{FÉt}_X$ agree with the corresponding finite limits in the category of schemes over X, see Remark 5.3.

If $Y' \to Y$ is a monomorphism in $\textit{FÉt}_X$ then we see that $Y' \to Y' \times_Y Y'$ is an isomorphism, and hence $Y' \to Y$ is a monomorphism of schemes. It follows that $Y' \to Y$ is an open immersion (Étale Morphisms, Theorem 14.1). Since Y' is finite over X and Y separated over X, the morphism $Y' \to Y$ is finite (Morphisms, Lemma 44.14), hence closed (Morphisms, Lemma 44.11), hence it is the inclusion of an open and closed subscheme of Y. It follows that Y is a connected objects of the category $\textit{FÉt}_X$ (as in Definition 3.6) if and only if Y is connected as a scheme. Then it follows from Topology, Lemma 7.7 that Y is a finite coproduct of its connected components both as a scheme and in the sense of Definition 3.6.

Let $Y \to Z$ be a morphism in $\textit{FÉt}_X$ which induces a bijection $F_{\overline{x}}(Y) \to F_{\overline{x}}(Z)$. We have to show that $Y \to Z$ is an isomorphism. By the above we may assume Z is connected. Since $Y \to Z$ is finite étale and hence finite locally free it suffices to show that $Y \to Z$ is finite locally free of degree 1. This is true in a neighbourhood of any point of Z lying over $\overline{x}$ and since Z is connected and the degree is locally constant we conclude. □

6. Fundamental groups

In this section we define Grothendieck's algebraic fundamental group. The following definition makes sense thanks to Lemma 5.5.

Definition 6.1. Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X. The *fundamental group* of X with *base point* $\overline{x}$ is the group

$$\pi_1(X, \overline{x}) = \text{Aut}(F_{\overline{x}})$$

of automorphisms of the fibre functor $F_{\overline{x}} : \textit{FÉt}_X \to \textit{Sets}$ endowed with its canonical profinite topology from Lemma 3.1.

Combining the above with the material from Section 3 we obtain the following theorem.

Theorem 6.2. *Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X.*

1. *The fibre functor $F_{\overline{x}}$ defines an equivalence of categories*
$$F\acute{E}t_X \longrightarrow \textit{Finite-}\pi_1(X, \overline{x})\textit{-Sets}$$
2. *Given a second geometric point $\overline{x}'$ of X there exists an isomorphism $t : F_{\overline{x}} \to F_{\overline{x}'}$. This gives an isomorphism $\pi_1(X, \overline{x}) \to \pi_1(X, \overline{x}')$ compatible with the equivalences in (1). This isomorphism is independent of t up to inner conjugation.*
3. *Given a morphism $f : X \to Y$ of connected schemes denote $\overline{y} = f \circ \overline{x}$. There is a canonical continuous homomorphism*
$$f_* : \pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$$
such that the diagram
$$\begin{tikzcd} F\acute{E}t_Y \arrow[rr, "base\ change"'] \arrow[d, "F_{\overline{y}}"'] & & F\acute{E}t_X \arrow[d, "F_{\overline{x}}"] \\ \textit{Finite-}\pi_1(Y, \overline{y})\textit{-Sets} \arrow[rr, "f_*"] & & \textit{Finite-}\pi_1(X, \overline{x})\textit{-Sets} \end{tikzcd}$$
is commutative.

Proof. Part (1) follows from Lemma 5.5 and Proposition 3.10. Part (2) is a special case of Lemma 3.11. For part (3) observe that the diagram
$$\begin{tikzcd} F\acute{E}t_Y \arrow[r] \arrow[d, "F_{\overline{y}}"'] & F\acute{E}t_X \arrow[d, "F_{\overline{x}}"] \\ Sets \arrow[r, equal] & Sets \end{tikzcd}$$
is commutative (actually commutative, not just 2-commutative) because $\overline{y} = f \circ \overline{x}$. Hence we can apply Lemma 3.11 with the implied transformation of functors to get (3). □

Lemma 6.3. *Let K be a field and set $X = \mathrm{Spec}(K)$. Let $\overline{K}$ be an algebraic closure and denote $\overline{x} : \mathrm{Spec}(\overline{K}) \to X$ the corresponding geometric point. Let $K^{sep} \subset \overline{K}$ be the separable algebraic closure.*

1. *The functor of Lemma 2.2 induces an equivalence*
$$F\acute{E}t_X \longrightarrow \textit{Finite-Gal}(K^{sep}/K)\textit{-Sets}.$$
compatible with $F_{\overline{x}}$ and the functor Finite-Gal(K^{sep}/K)-Sets $\to$ Sets.
2. *This induces a canonical isomorphism*
$$\textit{Gal}(K^{sep}/K) \longrightarrow \pi_1(X, \overline{x})$$
of profinite topological groups.

Proof. The functor of Lemma 2.2 is the same as the functor $F_{\overline{x}}$ because for any Y étale over X we have
$$\mathrm{Mor}_X(\mathrm{Spec}(\overline{K}), Y) = \mathrm{Mor}_X(\mathrm{Spec}(K^{sep}), Y)$$

Namely, as seen in the proof of Lemma 2.2 we have $Y = \coprod_{i \in I} \mathrm{Spec}(L_i)$ with L_i/K finite separable over K. Hence any K-algebra homomorphism $L_i \to \overline{K}$ factors through K^{sep}. Also, note that $F_{\overline{x}}(Y)$ is finite if and only if I is finite if and only if $Y \to X$ is finite étale. This proves (1).

Part (2) is a formal consequence of (1), Lemma 3.11, and Lemma 3.3. (Please also see the remark below.) □

Remark 6.4. In the situation of Lemma 6.3 let us give a more explicit construction of the isomorphism $\mathrm{Gal}(K^{sep}/K) \to \pi_1(X, \overline{x}) = \mathrm{Aut}(F_{\overline{x}})$. Observe that $\mathrm{Gal}(K^{sep}/K) = \mathrm{Aut}(\overline{K}/K)$ as $\overline{K}$ is the perfection of K^{sep}. Since $F_{\overline{x}}(Y) = \mathrm{Mor}_X(\mathrm{Spec}(\overline{K}), Y)$ we may consider the map

$$\mathrm{Aut}(\overline{K}/K) \times F_{\overline{x}}(Y) \to F_{\overline{x}}(Y), \quad (\sigma, \overline{y}) \mapsto \sigma \cdot \overline{y} = \overline{y} \circ \mathrm{Spec}(\sigma)$$

This is an action because

$$\sigma\tau \cdot \overline{y} = \overline{y} \circ \mathrm{Spec}(\sigma\tau) = \overline{y} \circ \mathrm{Spec}(\tau) \circ \mathrm{Spec}(\sigma) = \sigma \cdot (\tau \cdot \overline{y})$$

The action is functorial in $Y \in \textit{FÉt}_X$ and we obtain the desired map.

7. Galois covers of connected schemes

Let X be a connected scheme with geometric point $\overline{x}$. Since $F_{\overline{x}} : \textit{FÉt}_X \to \textit{Sets}$ is a Galois category (Lemma 5.5) the material in Section 3 applies. In this section we explicity transfer some of the terminology and results to the setting of schemes and finite étale morphisms.

We will say a finite étale morphism $Y \to X$ is a *Galois cover* if Y defines a Galois object of $\textit{FÉt}_X$. For a finite étale morphism $Y \to X$ with $G = \mathrm{Aut}_X(Y)$ the following are equivalent

1. Y is a Galois cover of X,
2. Y is connected and $|G|$ is equal to the degree of $Y \to X$,
3. Y is connected and G acts transitively on $F_{\overline{x}}(Y)$, and
4. Y is connected and G acts simply transitively on $F_{\overline{x}}(Y)$.

This follows immediately from the discussion in Section 3.

For any finite étale morphism $f : Y \to X$ with Y connected, there is a finite étale Galois cover $Y' \to X$ which dominates Y (Lemma 3.8).

The Galois objects of $\textit{FÉt}_X$ correspond, via the equivalence

$$F_{\overline{x}} : \textit{FÉt}_X \to \textit{Finite-}\pi_1(X, \overline{x})\textit{-Sets}$$

of Theorem 6.2, with the finite $\pi_1(X, \overline{x})$-*Sets* of the form $G = \pi_1(X, \overline{x})/H$ where H is a normal open subgroup. Equivalently, if G is a finite group and $\pi_1(X, \overline{x}) \to G$ is a continuous surjection, then G viewed as a $\pi_1(X, \overline{x})$-set corresponds to a Galois covering.

If $Y_i \to X$, $i = 1, 2$ are finite étale Galois covers with Galois groups G_i, then there exists a finite étale Galois cover $Y \to X$ whose Galois group is a subgroup of $G_1 \times G_2$. Namely, take the corresponding continuous homomorphisms $\pi_1(X, \overline{x}) \to G_i$ and let G be the image of the induced continuous homomorphism $\pi_1(X, \overline{x}) \to G_1 \times G_2$.

8. Topological invariance of the fundamental group

The main result of this section is that a universal homeomorphism of connected schemes induces an isomorphism on fundamental groups. See Proposition 8.4.

Instead of directly proving two schemes have the same fundamental group, we often prove that their categories of finite étale coverings are the same. This of course implies that their fundamental groups are equal provided they are connected.

Lemma 8.1. *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes such that the base change functor $\textit{FÉt}_Y \to \textit{FÉt}_X$ is an equivalence of categories. In this case*

(1) *f induces a homeomorphism $\pi_0(X) \to \pi_0(Y)$,*
(2) *if X or equivalently Y is connected, then $\pi_1(X, \overline{x}) = \pi_1(Y, \overline{y})$.*

Proof. Let $Y = Y_0 \amalg Y_1$ be a decomposition into nonempty open and closed subschemes. We claim that $f(X)$ meets both Y_i. Namely, if not, say $f(X) \subset Y_1$, then we can consider the finite étale morphism $V = Y_1 \to Y$. This is not an isomorphism but $V \times_Y X \to X$ is an isomorphism, which is a contradiction.

Suppose that $X = X_0 \amalg X_1$ is a decomposition into open and closed subschemes. Consider the finite étale morphism $U = X_1 \to X$. Then $U = X \times_Y V$ for some finite étale morphism $V \to Y$. The degree of the morphism $V \to Y$ is locally constant, hence we obtain a decomposition $Y = \coprod_{d \geq 0} Y_d$ into open and closed subschemes such that $V \to Y$ has degree d over Y_d. Since $f^{-1}(Y_d) = \emptyset$ for $d > 1$ we conclude that $Y_d = \emptyset$ for $d > 1$ by the above. And we conclude that $f^{-1}(Y_i) = X_i$ for $i = 0, 1$.

It follows that f^{-1} induces a bijection between the set of open and closed subsets of Y and the set of open and closed subsets of X. Note that X and Y are spectral spaces, see Properties, Lemma 2.4. By Topology, Lemma 12.10 the lattice of open and closed subsets of a spectral space determines the set of connected components. Hence $\pi_0(X) \to \pi_0(Y)$ is bijective. Since $\pi_0(X)$ and $\pi_0(Y)$ are profinite spaces (Topology, Lemma 22.5) we conclude that $\pi_0(X) \to \pi_0(Y)$ is a homeomorphism by Topology, Lemma 17.8. This proves (1). Part (2) is immediate. $\square$

The following lemma tells us that the fundamental group of a henselian pair is the fundamental group of the closed subset.

Lemma 8.2. *Let (A, I) be a henselian pair. Set $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/I)$. The functor*

$$\textit{FÉt}_X \longrightarrow \textit{FÉt}_Z, \quad U \longmapsto U \times_X Z$$

is an equivalence of categories.

Proof. This is a translation of More on Algebra, Lemma 13.2. $\square$

The following lemma tells us that the fundamental group of a thickening is the same as the fundamental group of the original. We will use this in the proof of the strong proposition concerning universal homeomorphisms below.

Lemma 8.3. *Let $X \subset X'$ be a thickening of schemes. The functor*

$$\textit{FÉt}_{X'} \longrightarrow \textit{FÉt}_X, \quad U' \longmapsto U' \times_{X'} X$$

is an equivalence of categories.

Proof. For a discussion of thickenings see More on Morphisms, Section 2. Let $U' \to X'$ be an étale morphism such that $U = U' \times_{X'} X \to X$ is finite étale. Then $U' \to X'$ is finite étale as well. This follows for example from More on Morphisms, Lemma 3.4. Now, if $X \subset X'$ is a finite order thickening then this remark combined with Étale Morphisms, Theorem 15.2 proves the lemma. Below we will prove the lemma for general thickenings, but we suggest the reader skip the proof.

Let $X' = \bigcup X'_i$ be an affine open covering. Set $X_i = X \times_{X'} X'_i$, $X'_{ij} = X'_i \cap X'_j$, $X_{ij} = X \times_{X'} X'_{ij}$, $X'_{ijk} = X'_i \cap X'_j \cap X'_k$, $X_{ijk} = X \times_{X'} X'_{ijk}$. Suppose that we can prove the theorem for each of the thickenings $X_i \subset X'_i$, $X_{ij} \subset X'_{ij}$, and $X_{ijk} \subset X'_{ijk}$. Then the result follows for $X \subset X'$ by relative glueing of schemes, see Constructions, Section 2. Observe that the schemes X'_i, X'_{ij}, X'_{ijk} are each separated as open subschemes of affine schemes. Repeating the argument one more time we reduce to the case where the schemes X'_i, X'_{ij}, X'_{ijk} are affine.

In the affine case we have $X' = \mathrm{Spec}(A')$ and $X = \mathrm{Spec}(A'/I')$ where I' is a locally nilpotent ideal. Then (A', I') is a henselian pair (More on Algebra, Lemma 11.2) and the result follows from Lemma 8.2 (which is much easier in this case). $\square$

The "correct" way to prove the following proposition would be to deduce it from the invariance of the étale site, see Étale Cohomology, Theorem 45.2.

Proposition 8.4. *Let $f : X \to Y$ be a universal homeomorphism of schemes. Then*

$$\textit{FÉt}_Y \longrightarrow \textit{FÉt}_X, \quad V \longmapsto V \times_Y X$$

is an equivalence. Thus if X and Y are connected, then f induces an isomorphism $\pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$ of fundamental groups.

Proof. Recall that a universal homeomorphism is the same thing as an integral, universally injective, surjective morphism, see Morphisms, Lemma 45.5. In particular, the diagonal $\Delta : X \to X \times_Y X$ is a thickening by Morphisms, Lemma 10.2. Thus by Lemma 8.3 we see that given a finite étale morphism $U \to X$ there is a unique isomorphism

$$\varphi : U \times_Y X \to X \times_Y U$$

of schemes finite étale over $X \times_Y X$ which pulls back under Δ to $\mathrm{id} : U \to U$ over X. Since $X \to X \times_Y X \times_Y X$ is a thickening as well (it is bijective and a closed immersion) we conclude that (U, φ) is a descent datum relative to X/Y. By Étale Morphisms, Proposition 20.6 we conclude that $U = X \times_Y V$ for some $V \to Y$ quasi-compact, separated, and étale. We omit the proof that $V \to Y$ is finite (hints: the morphism $U \to V$ is surjective and $U \to Y$ is integral). We conclude that $\textit{FÉt}_Y \to \textit{FÉt}_X$ is essentially surjective.

Arguing in the same manner as above we see that given $V_1 \to Y$ and $V_2 \to Y$ in $\textit{FÉt}_Y$ any morphism $a : X \times_Y V_1 \to X \times_Y V_2$ over X is compatible with the canonical descent data. Thus a descends to a morphism $V_1 \to V_2$ over Y by Étale Morphisms, Lemma 20.3. $\square$

9. Finite étale covers of proper schemes

In this section we show that the fundamental group of a connected proper scheme over a henselian local ring is the same as the fundamental group of its special fibre. We also prove a variant of this result for a henselian pair.

We also show that the fundamental group of a connected proper scheme over an algebraically closed field k does not change if we replace k by an algebraically closed extension.

Instead of stating and proving the results in the connected case we prove the results in general and we leave it to the reader to deduce the result for fundamental groups using Lemma 8.1.

Lemma 9.1. *Let A be a henselian local ring. Let X be a proper scheme over A with closed fibre X_0. Then the functor*

$$F\acute{E}t_X \to F\acute{E}t_{X_0}, \quad U \longmapsto U_0 = U \times_X X_0$$

is an equivalence of categories.

Proof. The proof given here is an example of applying algebraization and approximation. We proceed in a number of stages.

Essential surjectivity when A is a complete local Noetherian ring. Let $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^{n+1})$. By Étale Morphisms, Theorem 15.2 the inclusions

$$X_0 \to X_1 \to X_2 \to \ldots$$

induce equivalence of categories between the category of schemes étale over X_0 and the category of schemes étale over X_n. Moreover, if $U_n \to X_n$ corresponds to a finite étale morphism $U_0 \to X_0$, then $U_n \to X_n$ is finite too, for example by More on Morphisms, Lemma 3.3. In this case the morphism $U_0 \to \mathrm{Spec}(A/\mathfrak{m})$ is proper as X_0 is proper over $A/\mathfrak{m}$. Thus we may apply Grothendieck's algebraization theorem (in the form of Cohomology of Schemes, Lemma 28.2) to see that there is a finite morphism $U \to X$ whose restriction to X_0 recovers U_0. By More on Morphisms, Lemma 12.3 we see that $U \to X$ is étale at every point of U_0. However, since every point of U specializes to a point of U_0 (as U is proper over A), we conclude that $U \to X$ is étale. In this way we conclude the functor is essentially surjective.

Fully faithfulness when A is a complete local Noetherian ring. Let $U \to X$ and $V \to X$ be finite étale morphisms and let $\varphi_0 : U_0 \to V_0$ be a morphism over X_0. Look at the morphism

$$\Gamma_{\varphi_0} : U_0 \longrightarrow U_0 \times_{X_0} V_0$$

This morphism is both finite étale and a closed immersion. By essential surjectivity applied to $X = U \times_X V$ we find a finite étale morphism $W \to U \times_X V$ whose special fibre is isomorphic to Γ_{φ_0}. Consider the projection $W \to U$. It is finite étale and an isomorphism over U_0 by construction. By Étale Morphisms, Lemma 14.2 $W \to U$ is an isomorphism in an open neighbourhood of U_0. Thus it is an isomorphism and the composition $\varphi : U \cong W \to V$ is the desired lift of φ_0.

Essential surjectivity when A is a henselian local Noetherian G-ring. Let $U_0 \to X_0$ be a finite étale morphism. Let $A^\wedge$ be the completion of A with respect to the maximal ideal. Let $X^\wedge$ be the base change of X to $A^\wedge$. By the result above there exists a finite étale morphism $V \to X^\wedge$ whose special fibre is U_0. Write $A^\wedge = \mathrm{colim}\, A_i$ with $A \to A_i$ of finite type. By Limits, Lemma 10.1 there exists an i and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X^\wedge$ is V. After increasing i we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 8.3 and 8.10). Writing

$$A_i = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

the ring map $A_i \to A^\wedge$ can be reinterpreted as a solution $(a_1, \ldots, a_n)$ in $A^\wedge$ for the system of equations $f_j = 0$. By Smoothing Ring Maps, Theorem 13.1 we can approximate this solution (to order 11 for example) by a solution $(b_1, \ldots, b_n)$ in A. Translating back we find an A-algebra map $A_i \to A$ which gives the same closed point as the original map $A_i \to A^\wedge$ (as $11 > 1$). The base change $U \to X$ of $V \to X_{A_i}$ by this ring map will therefore be a finite étale morphism whose special fibre is isomorphic to U_0.

Fully faithfulness when A is a henselian local Noetherian G-ring. This can be deduced from essential surjectivity in exactly the same manner as was done in the case that A is complete Noetherian.

General case. Let $(A, \mathfrak{m})$ be a henselian local ring. Set $S = \text{Spec}(A)$ and denote $s \in S$ the closed point. By Limits, Lemma 13.3 we can write $X \to \text{Spec}(A)$ as a cofiltered limit of proper morphisms $X_i \to S_i$ with S_i of finite type over $\mathbf{Z}$. For each i let $s_i \in S_i$ be the image of s. Since $S = \lim S_i$ and $A = \mathcal{O}_{S,s}$ we have $A = \text{colim}\, \mathcal{O}_{S_i,s_i}$. The ring $A_i = \mathcal{O}_{S_i,s_i}$ is a Noetherian local G-ring (More on Algebra, Proposition 50.12). By More on Algebra, Lemma 12.5 we see that $A = \text{colim}\, A_i^h$. By More on Algebra, Lemma 50.8 the rings A_i^h are G-rings. Thus we see that $A = \text{colim}\, A_i^h$ and

$$X = \lim(X_i \times_{S_i} \text{Spec}(A_i^h))$$

as schemes. The category of schemes finite étale over X is the limit of the category of schemes finite étale over $X_i \times_{S_i} \text{Spec}(A_i^h)$ (by Limits, Lemmas 10.1, 8.3, and 8.10) The same thing is true for schemes finite étale over $X_0 = \lim(X_i \times_{S_i} s_i)$. Thus we formally deduce the result for $X/\text{Spec}(A)$ from the result for the $(X_i \times_{S_i} \text{Spec}(A_i^h))/\text{Spec}(A_i^h)$ which we dealt with above. $\square$

Lemma 9.2. *Let (A, I) be a henselian pair. Let X be a proper scheme over A. Set $X_0 = X \times_{\text{Spec}(A)} \text{Spec}(A/I)$. Then the functor*

$$\textit{FÉt}_X \to \textit{FÉt}_{X_0}, \quad U \longmapsto U_0 = U \times_X X_0$$

is an equivalence of categories.

Proof. The proof of this lemma is exactly the same as the proof of Lemma 9.1.

Essential surjectivity when A is Noetherian and I-adically complete. Let $X_n = X \times_{\text{Spec}(A)} \text{Spec}(A/I^{n+1})$. By Étale Morphisms, Theorem 15.2 the inclusions

$$X_0 \to X_1 \to X_2 \to \ldots$$

induce equivalence of categories between the category of schemes étale over X_0 and the category of schemes étale over X_n. Moreover, if $U_n \to X_n$ corresponds to a finite étale morphism $U_0 \to X_0$, then $U_n \to X_n$ is finite too, for example by More on Morphisms, Lemma 3.3. In this case the morphism $U_0 \to \text{Spec}(A/I)$ is proper as X_0 is proper over A/I. Thus we may apply Grothendieck's algebraization theorem (in the form of Cohomology of Schemes, Lemma 28.2) to see that there is a finite morphism $U \to X$ whose restriction to X_0 recovers U_0. By More on Morphisms, Lemma 12.3 we see that $U \to X$ is étale at every point of U_0. However, since every point of U specializes to a point of U_0 (as U is proper over A), we conclude that $U \to X$ is étale. In this way we conclude the functor is essentially surjective.

Fully faithfulness when A is Noetherian and I-adically complete. Let $U \to X$ and $V \to X$ be finite étale morphisms and let $\varphi_0 : U_0 \to V_0$ be a morphism over X_0. Look at the morphism

$$\Gamma_{\varphi_0} : U_0 \longrightarrow U_0 \times_{X_0} V_0$$

This morphism is both finite étale and a closed immersion. By essential surjectivity applied to $X = U \times_X V$ we find a finite étale morphism $W \to U \times_X V$ whose special fibre is isomorphic to Γ_{φ_0}. Consider the projection $W \to U$. It is finite étale and an isomorphism over U_0 by construction. By Étale Morphisms, Lemma 14.2 $W \to U$ is an isomorphism in an open neighbourhood of U_0. Thus it is an isomorphism and the composition $\varphi : U \cong W \to V$ is the desired lift of φ_0.

Essential surjectivity when (A, I) is a henselian pair and A is a Noetherian G-ring. Let $U_0 \to X_0$ be a finite étale morphism. Let $A^\wedge$ be the completion of A with respect to I. Observe that $A^\wedge$ is a Noetherian ring which is $IA^\wedge$-adically complete, see Algebra, Lemmas 97.4 and 97.6. Let $X^\wedge$ be the base change of X to $A^\wedge$. By the result above there exists a finite étale morphism $V \to X^\wedge$ whose special fibre is U_0. Write $A^\wedge = \operatorname{colim} A_i$ with $A \to A_i$ of finite type. By Limits, Lemma 10.1 there exists an i and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X^\wedge$ is V. After increasing i we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 8.3 and 8.10). Writing

$$A_i = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

the ring map $A_i \to A^\wedge$ can be reinterpreted as a solution $(a_1, \ldots, a_n)$ in $A^\wedge$ for the system of equations $f_j = 0$. By Smoothing Ring Maps, Lemma 14.1 we can approximate this solution (to order 11 for example) by a solution $(b_1, \ldots, b_n)$ in A. Translating back we find an A-algebra map $A_i \to A$ which gives the same closed point as the original map $A_i \to A^\wedge$ (as $11 > 1$). The base change $U \to X$ of $V \to X_{A_i}$ by this ring map will therefore be a finite étale morphism whose special fibre is isomorphic to U_0.

Fully faithfulness when $(A, I$ is a henselian pair and A is a Noetherian G-ring. This can be deduced from essential surjectivity in exactly the same manner as was done in the case that A is complete Noetherian.

General case. Let (A, I) be a henselian pair. Set $S = \operatorname{Spec}(A)$ and denote $S_0 = \operatorname{Spec}(A/I)$. By Limits, Lemma 13.3 we can write $X \to \operatorname{Spec}(A)$ as a cofiltered limit of proper morphisms $X_i \to S_i$ with S_i affine and of finite type over $\mathbf{Z}$. Write $S_i = \operatorname{Spec}(A_i)$ and denote $I_i \subset A_i$ the inverse image of I by the map $A_i \to A$. Set $S_{i,0} = \operatorname{Spec}(A_i/I_i)$. Since $S = \lim S_i$ we have $A = \operatorname{colim} A_i$. Thus we also have $I = \operatorname{colim} I_i$ and $A/I = \operatorname{colim} A_i/I_i$. The ring A_i is a Noetherian G-ring (More on Algebra, Proposition 50.12). Denote (A_i^h, I_i^h) the henselization of the pair (A_i, I_i). By More on Algebra, Lemma 12.5 we see that $A = \operatorname{colim} A_i^h$. By More on Algebra, Lemma 50.15 the rings A_i^h are G-rings. Thus we see that $A = \operatorname{colim} A_i^h$ and

$$X = \lim(X_i \times_{S_i} \operatorname{Spec}(A_i^h))$$

as schemes. The category of schemes finite étale over X is the limit of the category of schemes finite étale over $X_i \times_{S_i} \operatorname{Spec}(A_i^h)$ (by Limits, Lemmas 10.1, 8.3, and 8.10) The same thing is true for schemes finite étale over $X_0 = \lim(X_i \times_{S_i} S_{i,0})$. Thus we formally deduce the result for $X/\operatorname{Spec}(A)$ from the result for the $(X_i \times_{S_i} \operatorname{Spec}(A_i^h))/\operatorname{Spec}(A_i^h)$ which we dealt with above. $\square$

Lemma 9.3. *Let k'/k be an extension of algebraically closed fields. Let X be a proper scheme over k. Then the functor*

$$U \longmapsto U_{k'}$$

is an equivalence of categories between schemes finite étale over X and schemes finite étale over $X_{k'}$.

Proof. Let us prove the functor is essentially surjective. Let $U' \to X_{k'}$ be a finite étale morphism. Write $k' = \operatorname{colim} A_i$ as a filtered colimit of finite type k-algebras. By Limits, Lemma 10.1 there exists an i and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X_{k'}$ is U'. After increasing i we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 8.3 and 8.10). Since k is algebraically closed we can find a k-valued point t in $\operatorname{Spec}(A_i)$. Let $U = (U_i)_t$ be the fibre of U_i over t. Let A_i^h be the henselization of $(A_i)_\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal corresponding to the point t. By Lemma 9.1 we see that $(U_i)_{A_i^h} = U \times \operatorname{Spec}(A_i^h)$ as schemes over $X_{A_i^h}$. Now since A_i^h is algebraic over A_i (see for example discussion in Smoothing Ring Maps, Example 13.3) and since k' is algebraically closed we can find a ring map $A_i^h \to k'$ extending the given inclusion $A_i \subset k'$. Hence we conclude that U' is isomorphic to the base change of U. The proof of fully faithfulness is exactly the same. $\square$

10. Local connectedness

In this section we ask when $\pi_1(U) \to \pi_1(X)$ is surjective for U a dense open of a scheme X. We will see that this is the case (roughly) when $U \cap B$ is connected for any small "ball" B around a point $x \in X \setminus U$.

Lemma 10.1. *Let $f : X \to Y$ be a morphism of schemes. If $f(X)$ is dense in Y then the base change functor $\textit{FÉt}_Y \to \textit{FÉt}_X$ is faithful.*

Proof. Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given W finite étale over Y and a morphism $s : X \to W$ over X there is at most one section $t : Y \to W$ such that $s = t \circ f$. Consider two sections $t_1, t_2 : Y \to W$ such that $s = t_1 \circ f = t_2 \circ f$. Since the equalizer of t_1 and t_2 is closed in Y (Schemes, Lemma 21.5) and since $f(X)$ is dense in Y we see that t_1 and t_2 agree on Y_{red}. Then it follows that t_1 and t_2 have the same image which is an open and closed subscheme of W mapping isomorphically to Y (Étale Morphisms, Proposition 6.1) hence they are equal. $\square$

The condition in the following lemma that the punctured spectrum of the strict henselization is connected follows for example from the assumption that the local ring is geometrically unibranch, see More on Algebra, Lemma 106.5. There is a partial converse in Properties, Lemma 15.3.

Lemma 10.2. *Let $(A, \mathfrak{m})$ be a local ring. Set $X = \operatorname{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. If the punctured spectrum of the strict henselization of A is connected, then*

$$\textit{FÉt}_X \longrightarrow \textit{FÉt}_U, \quad Y \longmapsto Y \times_X U$$

is a fully faithful functor.

Proof. Assume A is strictly henselian. In this case any finite étale cover Y of X is isomorphic to a finite disjoint union of copies of X. Thus it suffices to prove that any morphism $U \to U \amalg \ldots \amalg U$ over U, extends uniquely to a morphism

$X \to X \amalg \ldots \amalg X$ over X. If U is connected (in particular nonempty), then this is true.

The general case. Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given Y finite étale over X any morphism $s : U \to Y$ over X extends to a morphism $t : X \to Y$ over X. Let A^{sh} be the strict henselization of A and denote $X^{sh} = \text{Spec}(A^{sh})$, $U^{sh} = U \times_X X^{sh}$, $Y^{sh} = Y \times_X X^{sh}$. By the first paragraph and our assumption on A, we can extend the base change $s^{sh} : U^{sh} \to Y^{sh}$ of s to $t^{sh} : X^{sh} \to Y^{sh}$. Set $A' = A^{sh} \otimes_A A^{sh}$. Then the two pullbacks t'_1, t'_2 of t^{sh} to $X' = \text{Spec}(A')$ are extensions of the pullback s' of s to $U' = U \times_X X'$. As $A \to A'$ is flat we see that $U' \subset X'$ is (topologically) dense by going down for $A \to A'$ (Algebra, Lemma 39.19). Thus $t'_1 = t'_2$ by Lemma 10.1. Hence t^{sh} descends to a morphism $t : X \to Y$ for example by Descent, Lemma 13.7. □

In view of Lemma 10.2 it is interesting to know when the punctured spectrum of a ring (and of its strict henselization) is connected. There is a famous lemma due to Hartshorne which gives a sufficient condition, see Local Cohomology, Lemma 3.1.

Lemma 10.3. *Let X be a scheme. Let $U \subset X$ be a dense open. Assume*
1. *the underlying topological space of X is Noetherian, and*
2. *for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.*

Then $F\acute{E}t_X \to F\acute{e}t_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 10.1.

Let $x \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$. Set $V = U \times_X \text{Spec}(\mathcal{O}_{X,x})$. By our choice of x this is the punctured spectrum of $\text{Spec}(\mathcal{O}_{X,x})$. By Lemma 10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ uniquely to a morphism $(Y_1)_{\text{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\text{Spec}(\mathcal{O}_{X,x})}$. By Limits, Lemma 20.3 we find an open $U \subset U'$ containing x and an extension $\varphi' : (Y_1)_{U'} \to (Y_2)_{U'}$ of φ. Since the underlying topological space of X is Noetherian this finishes the proof by Noetherian induction on the complement of the open over which φ is defined. □

Lemma 10.4. *Let X be a scheme. Let $U \subset X$ be a dense open. Assume*
1. *$U \to X$ is quasi-compact,*
2. *every point of $X \setminus U$ is closed, and*
3. *for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.*

Then $F\acute{E}t_X \to F\acute{e}t_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 10.1.

Let $x \in X \setminus U$. Set $V = U \times_X \text{Spec}(\mathcal{O}_{X,x})$. Since every point of $X \setminus U$ is closed V is the punctured spectrum of $\text{Spec}(\mathcal{O}_{X,x})$. By Lemma 10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ uniquely to a morphism $(Y_1)_{\text{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\text{Spec}(\mathcal{O}_{X,x})}$. By Limits, Lemma 20.3 (this uses that U is retrocompact in X) we find an open

$U \subset U'_x$ containing x and an extension $\varphi'_x : (Y_1)_{U'_x} \to (Y_2)_{U'_x}$ of φ. Note that given two points $x, x' \in X \setminus U$ the morphisms φ'_x and $\varphi'_{x'}$ agree over $U'_x \cap U'_{x'}$ as U is dense in that open (Lemma 10.1). Thus we can extend φ to $\bigcup U'_x = X$ as desired. □

Lemma 10.5. *Let X be a scheme. Let $U \subset X$ be a dense open. Assume*

(1) *every quasi-compact open of X has finitely many irreducible components,*
(2) *for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.*

Then $F\acute{E}t_X \to F\acute{e}t_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 10.1. We will prove existence by showing that we can enlarge U if $U \neq X$ and using Zorn's lemma to finish the proof.

Let $x \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$. Set $V = U \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. By our choice of x this is the punctured spectrum of $\mathrm{Spec}(\mathcal{O}_{X,x})$. By Lemma 10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ (uniquely) to a morphism $(Y_1)_{\mathrm{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\mathrm{Spec}(\mathcal{O}_{X,x})}$. Choose an affine neighbourhood $W \subset X$ of x. Since $U \cap W$ is dense in W it contains the generic points $\eta_1, \ldots, \eta_n$ of W. Choose an affine open $W' \subset W \cap U$ containing $\eta_1, \ldots, \eta_n$. Set $V' = W' \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. By Limits, Lemma 20.3 applied to $x \in W \supset W'$ we find an open $W' \subset W'' \subset W$ with $x \in W''$ and a morphism $\varphi'' : (Y_1)_{W''} \to (Y_2)_{W''}$ agreeing with φ over W'. Since W' is dense in $W'' \cap U$, we see by Lemma 10.1 that φ and φ'' agree over $U \cap W'$. Thus φ and φ'' glue to a morphism φ' over $U' = U \cup W''$ agreeing with φ over U. Observe that $x \in U'$ so that we've extended φ to a strictly larger open.

Consider the set $\mathcal{S}$ of pairs (U', φ') where $U \subset U'$ and φ' is an extension of φ. We endow $\mathcal{S}$ with a partial ordering in the obvious manner. If (U'_i, φ'_i) is a totally ordered subset, then it has a maximum (U', φ'). Just take $U' = \bigcup U'_i$ and let $\varphi' : (Y_1)_{U'} \to (Y_2)_{U'}$ be the morphism agreeing with φ'_i over U'_i. Thus Zorn's lemma applies and $\mathcal{S}$ has a maximal element. By the argument above we see that this maximal element is an extension of φ over all of X. □

Lemma 10.6. *Let $(A, \mathfrak{m})$ be a local ring. Set $X = \mathrm{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$. Let U^{sh} be the punctured spectrum of the strict henselization A^{sh} of A. Assume U is quasi-compact and U^{sh} is connected. Then the sequence*

$$\pi_1(U^{sh}, \overline{u}) \to \pi_1(U, \overline{u}) \to \pi_1(X, \overline{u}) \to 1$$

is exact in the sense of Lemma 4.3 part (1).

Proof. The map $\pi_1(U) \to \pi_1(X)$ is surjective by Lemmas 10.2 and 4.1.

Write $X^{sh} = \mathrm{Spec}(A^{sh})$. Let $Y \to X$ be a finite étale morphism. Then $Y^{sh} = Y \times_X X^{sh} \to X^{sh}$ is a finite étale morphism. Since A^{sh} is strictly henselian we see that Y^{sh} is isomorphic to a disjoint union of copies of X^{sh}. Thus the same is true for $Y \times_X U^{sh}$. It follows that the composition $\pi_1(U^{sh}) \to \pi_1(U) \to \pi_1(X)$ is trivial, see Lemma 4.2.

To finish the proof, it suffices according to Lemma 4.3 to show the following: Given a finite étale morphism $V \to U$ such that $V \times_U U^{sh}$ is a disjoint union of copies of U^{sh}, we can find a finite étale morphism $Y \to X$ with $V \cong Y \times_X U$ over U. The assumption implies that there exists a finite étale morphism $Y^{sh} \to X^{sh}$ and an isomorphism $V \times_U U^{sh} \cong Y^{sh} \times_{X^{sh}} U^{sh}$. Consider the following diagram

$$\begin{tikzcd}
U \arrow[d] & U^{sh} \arrow[l] \arrow[d] & U^{sh} \times_U U^{sh} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & U^{sh} \times_U U^{sh} \times_U U^{sh} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] \\
X & X^{sh} \arrow[l] & X^{sh} \times_X X^{sh} \arrow[l, shift left] \arrow[l, shift right] & X^{sh} \times_X X^{sh} \times_X X^{sh} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

Since $U \subset X$ is quasi-compact by assumption, all the downward arrows are quasi-compact open immersions. Let $\xi \in X^{sh} \times_X X^{sh}$ be a point not in $U^{sh} \times_U U^{sh}$. Then ξ lies over the closed point x^{sh} of X^{sh}. Consider the local ring homomorphism

$$A^{sh} = \mathcal{O}_{X^{sh},x^{sh}} \to \mathcal{O}_{X^{sh} \times_X X^{sh},\xi}$$

determined by the first projection $X^{sh} \times_X X^{sh}$. This is a filtered colimit of local homomorphisms which are localizations étale ring maps. Since A^{sh} is strictly henselian, we conclude that it is an isomorphism. Since this holds for every ξ in the complement it follows there are no specializations among these points and hence every such ξ is a closed point (you can also prove this directly). As the local ring at ξ is isomorphic to A^{sh}, it is strictly henselian and has connected punctured spectrum. Similarly for points ξ of $X^{sh} \times_X X^{sh} \times_X X^{sh}$ not in $U^{sh} \times_U U^{sh} \times_U U^{sh}$. It follows from Lemma 10.4 that pullback along the vertical arrows induce fully faithful functors on the categories of finite étale schemes. Thus the canonical descent datum on $V \times_U U^{sh}$ relative to the fpqc covering $\{U^{sh} \to U\}$ translates into a descent datum for Y^{sh} relative to the fpqc covering $\{X^{sh} \to X\}$. Since $Y^{sh} \to X^{sh}$ is finite hence affine, this descent datum is effective (Descent, Lemma 37.1). Thus we get an affine morphism $Y \to X$ and an isomorphism $Y \times_X X^{sh} \to Y^{sh}$ compatible with descent data. By fully faithfulness of descent data (as in Descent, Lemma 35.11) we get an isomorphism $V \to U \times_X Y$. Finally, $Y \to X$ is finite étale as $Y^{sh} \to X^{sh}$ is, see Descent, Lemmas 23.29 and 23.23. □

Let X be an irreducible scheme. Let $\eta \in X$ be the generic point. The canonical morphism $\eta \to X$ induces a canonical map

$$\mathrm{Gal}(\kappa(\eta)^{sep}/\kappa(\eta)) = \pi_1(\eta, \overline{\eta}) \longrightarrow \pi_1(X, \overline{\eta}) \tag{10.6.1}$$

The identification on the left hand side is Lemma 6.3.

Lemma 10.7. *Let X be an irreducible, geometrically unibranch scheme. For any nonempty open $U \subset X$ the canonical map*

$$\pi_1(U, \overline{u}) \longrightarrow \pi_1(X, \overline{u})$$

is surjective. The map (10.6.1) $\pi_1(\eta, \overline{\eta}) \to \pi_1(X, \overline{\eta})$ is surjective as well.

Proof. By Lemma 8.3 we may replace X by its reduction. Thus we may assume that X is an integral scheme. By Lemma 4.1 the assertion of the lemma translates into the statement that the functors $F\acute{E}t_X \to F\acute{E}t_U$ and $F\acute{E}t_X \to F\acute{E}t_\eta$ are fully faithful.

The result for $F\acute{E}t_X \to F\acute{E}t_U$ follows from Lemma 10.5 and the fact that for a local ring A which is geometrically unibranch its strict henselization has an irreducible spectrum. See More on Algebra, Lemma 106.5.

Observe that the residue field $\kappa(\eta) = \mathcal{O}_{X,\eta}$ is the filtered colimit of $\mathcal{O}_X(U)$ over $U \subset X$ nonempty open affine. Hence $F\acute{E}t_\eta$ is the colimit of the categories $F\acute{E}t_U$

over such U, see Limits, Lemmas 10.1, 8.3, and 8.10. A formal argument then shows that fully faithfulness for $F\acute{E}t_X \to F\acute{E}t_\eta$ follows from the fully faithfulness of the functors $F\acute{E}t_X \to F\acute{E}t_U$. $\square$

Lemma 10.8. *Let X be a scheme. Let $x_1, \ldots, x_n \in X$ be a finite number of closed points such that*

1. *$U = X \setminus \{x_1, \ldots, x_n\}$ is connected and is a retrocompact open of X, and*
2. *for each i the punctured spectrum U_i^{sh} of the strict henselization of $\mathcal{O}_{X,x_i}$ is connected.*

Then the map $\pi_1(U) \to \pi_1(X)$ is surjective and the kernel is the smallest closed normal subgroup of $\pi_1(U)$ containing the image of $\pi_1(U_i^{sh}) \to \pi_1(U)$ for $i = 1, \ldots, n$.

Proof. Surjectivity follows from Lemmas 10.4 and 4.1. We can consider the sequence of maps

$$\pi_1(U) \to \ldots \to \pi_1(X \setminus \{x_1, x_2\}) \to \pi_1(X \setminus \{x_1\}) \to \pi_1(X)$$

A group theory argument then shows it suffices to prove the statement on the kernel in the case $n = 1$ (details omitted). Write $x = x_1$, $U^{sh} = U_1^{sh}$, set $A = \mathcal{O}_{X,x}$, and let A^{sh} be the strict henselization. Consider the diagram

$$\begin{tikzcd} U \arrow[d] & \mathrm{Spec}(A) \setminus \{\mathfrak{m}\} \arrow[l] \arrow[d] & U^{sh} \arrow[l] \arrow[d] \\ X & \mathrm{Spec}(A) \arrow[l] & \mathrm{Spec}(A^{sh}) \arrow[l] \end{tikzcd}$$

By Lemma 4.3 we have to show finite étale morphisms $V \to U$ which pull back to trivial coverings of U^{sh} extend to finite étale schemes over X. By Lemma 10.6 we know the corresponding statement for finite étale schemes over the punctured spectrum of A. However, by Limits, Lemma 20.1 schemes of finite presentation over X are the same thing as schemes of finite presentation over U and A glued over the punctured spectrum of A. This finishes the proof. $\square$

11. Fundamental groups of normal schemes

Let X be an integral, geometrically unibranch scheme. In the previous section we have seen that the fundamental group of X is a quotient of the Galois group of the function field K of X. Since the map is continuous the kernel is a normal closed subgroup of the Galois group. Hence this kernel corresponds to a Galois extension M/K by Galois theory (Fields, Theorem 22.4). In this section we will determine M when X is a normal integral scheme.

Let X be an integral normal scheme with function field K. Let L/K be a finite extension. Consider the normalization $Y \to X$ of X in the morphism $\mathrm{Spec}(L) \to X$ as defined in Morphisms, Section 53. We will say (in this setting) that X *is unramified in* L if $Y \to X$ is an unramified morphism of schemes. In Lemma 13.4 we will elucidate this condition. Observe that the scheme theoretic fibre of $Y \to X$ over $\mathrm{Spec}(K)$ is $\mathrm{Spec}(L)$. Hence the field extension L/K is separable if X is unramified in L, see Morphisms, Lemmas 35.11.

Lemma 11.1. *In the situation above the following are equivalent*

1. *X is unramified in L,*
2. *$Y \to X$ is étale, and*

(3) *$Y \to X$ is finite étale.*

Proof. Observe that $Y \to X$ is an integral morphism. In each case the morphism $Y \to X$ is locally of finite type by definition. Hence we find that in each case $Y \to X$ is finite by Morphisms, Lemma 44.4. In particular we see that (2) is equivalent to (3). An étale morphism is unramified, hence (2) implies (1).

Conversely, assume $Y \to X$ is unramified. Since a normal scheme is geometrically unibranch (Properties, Lemma 15.2), we see that the morphism $Y \to X$ is étale by More on Morphisms, Lemma 37.2. We also give a direct proof in the next paragraph.

Let $x \in X$. We can choose an étale neighbourhood $(U, u) \to (X, x)$ such that

$$Y \times_X U = \coprod V_j \longrightarrow U$$

is a disjoint union of closed immersions, see Étale Morphisms, Lemma 17.3. Shrinking we may assume U is quasi-compact. Then U has finitely many irreducible components (Descent, Lemma 16.3). Since U is normal (Descent, Lemma 18.2) the irreducible components of U are open and closed (Properties, Lemma 7.5) and we may assume U is irreducible. Then U is an integral scheme whose generic point ξ maps to the generic point of X. On the other hand, we know that $Y \times_X U$ is the normalization of U in $\text{Spec}(L) \times_X U$ by More on Morphisms, Lemma 19.2. Every point of $\text{Spec}(L) \times_X U$ maps to ξ. Thus every V_j contains a point mapping to ξ by Morphisms, Lemma 53.9. Thus $V_j \to U$ is an isomorphism as $U = \overline{\{\xi\}}$. Thus $Y \times_X U \to U$ is étale. By Descent, Lemma 23.29 we conclude that $Y \to X$ is étale over the image of $U \to X$ (an open neighbourhood of x). $\square$

Lemma 11.2. *Let X be a normal integral scheme with function field K. Let $Y \to X$ be a finite étale morphism. If Y is connected, then Y is an integral normal scheme and Y is the normalization of X in the function field of Y.*

Proof. The scheme Y is normal by Descent, Lemma 18.2. Since $Y \to X$ is flat every generic point of Y maps to the generic point of X by Morphisms, Lemma 25.9. Since $Y \to X$ is finite we see that Y has a finite number of irreducible components. Thus Y is the disjoint union of a finite number of integral normal schemes by Properties, Lemma 7.5. Thus if Y is connected, then Y is an integral normal scheme.

Let L be the function field of Y and let $Y' \to X$ be the normalization of X in L. By Morphisms, Lemma 53.4 we obtain a factorization $Y' \to Y \to X$ and $Y' \to Y$ is the normalization of Y in L. Since Y is normal it is clear that $Y' = Y$ (this can also be deduced from Morphisms, Lemma 54.8). $\square$

Proposition 11.3. *Let X be a normal integral scheme with function field K. Then the canonical map (10.6.1)*

$$Gal(K^{sep}/K) = \pi_1(\eta, \overline{\eta}) \longrightarrow \pi_1(X, \overline{\eta})$$

is identified with the quotient map $Gal(K^{sep}/K) \to Gal(M/K)$ where $M \subset K^{sep}$ is the union of the finite subextensions L such that X is unramified in L.

Proof. The normal scheme X is geometrically unibranch (Properties, Lemma 15.2). Hence Lemma 10.7 applies to X. Thus $\pi_1(\eta, \overline{\eta}) \to \pi_1(X, \overline{\eta})$ is surjective and top horizontal arrow of the commutative diagram

$$\begin{tikzcd} F\acute{E}t_X \arrow[r] \arrow[d] \arrow[dr, "c"] & F\acute{E}t_\eta \arrow[d] \\ \textit{Finite-}\pi_1(X, \overline{\eta})\textit{-sets} \arrow[r] & \textit{Finite-}\mathrm{Gal}(K^{sep}/K)\textit{-sets} \end{tikzcd}$$

is fully faithful. The left vertical arrow is the equivalence of Theorem 6.2 and the right vertical arrow is the equivalence of Lemma 6.3. The lower horizontal arrow is induced by the map of the proposition. By Lemmas 11.1 and 11.2 we see that the essential image of c consists of $\mathrm{Gal}(K^{sep}/K)$-*Sets* isomorphic to sets of the form

$$S = \mathrm{Hom}_K(\prod\nolimits_{i=1,\ldots,n} L_i, K^{sep}) = \coprod\nolimits_{i=1,\ldots,n} \mathrm{Hom}_K(L_i, K^{sep})$$

with L_i/K finite separable such that X is unramified in L_i. Thus if $M \subset K^{sep}$ is as in the statement of the lemma, then $\mathrm{Gal}(K^{sep}/M)$ is exactly the subgroup of $\mathrm{Gal}(K^{sep}/K)$ acting trivially on every object in the essential image of c. On the other hand, the essential image of c is exactly the category of S such that the $\mathrm{Gal}(K^{sep}/K)$-action factors through the surjection $\mathrm{Gal}(K^{sep}/K) \to \pi_1(X, \overline{\eta})$. We conclude that $\mathrm{Gal}(K^{sep}/M)$ is the kernel. Hence $\mathrm{Gal}(K^{sep}/M)$ is a normal subgroup, M/K is Galois, and we have a short exact sequence

$$1 \to \mathrm{Gal}(K^{sep}/M) \to \mathrm{Gal}(K^{sep}/K) \to \mathrm{Gal}(M/K) \to 1$$

by Galois theory (Fields, Theorem 22.4 and Lemma 22.5). The proof is done. □

Lemma 11.4. *Let $(A, \mathfrak{m})$ be a normal local ring. Set $X = \mathrm{Spec}(A)$. Let A^{sh} be the strict henselization of A. Let K and K^{sh} be the fraction fields of A and A^{sh}. Then the sequence*

$$\pi_1(\mathrm{Spec}(K^{sh})) \to \pi_1(\mathrm{Spec}(K)) \to \pi_1(X) \to 1$$

is exact in the sense of Lemma 4.3 part (1).

Proof. Note that A^{sh} is a normal domain, see More on Algebra, Lemma 45.6. The map $\pi_1(\mathrm{Spec}(K)) \to \pi_1(X)$ is surjective by Proposition 11.3.

Write $X^{sh} = \mathrm{Spec}(A^{sh})$. Let $Y \to X$ be a finite étale morphism. Then $Y^{sh} = Y \times_X X^{sh} \to X^{sh}$ is a finite étale morphism. Since A^{sh} is strictly henselian we see that Y^{sh} is isomorphic to a disjoint union of copies of X^{sh}. Thus the same is true for $Y \times_X \mathrm{Spec}(K^{sh})$. It follows that the composition $\pi_1(\mathrm{Spec}(K^{sh})) \to \pi_1(X)$ is trivial, see Lemma 4.2.

To finish the proof, it suffices according to Lemma 4.3 to show the following: Given a finite étale morphism $V \to \mathrm{Spec}(K)$ such that $V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^{sh})$ is a disjoint union of copies of $\mathrm{Spec}(K^{sh})$, we can find a finite étale morphism $Y \to X$ with $V \cong Y \times_X \mathrm{Spec}(K)$ over $\mathrm{Spec}(K)$. Write $V = \mathrm{Spec}(L)$, so L is a finite product of finite separable extensions of K. Let $B \subset L$ be the integral closure of A in L. If $A \to B$ is étale, then we can take $Y = \mathrm{Spec}(B)$ and the proof is complete. By Algebra, Lemma 147.4 (and a limit argument we omit) we see that $B \otimes_A A^{sh}$ is the integral closure of A^{sh} in $L^{sh} = L \otimes_K K^{sh}$. Our assumption is that L^{sh} is a product of copies of K^{sh} and hence B^{sh} is a product of copies of A^{sh}. Thus $A^{sh} \to B^{sh}$ is étale. As $A \to A^{sh}$ is faithfully flat it follows that $A \to B$ is étale (Descent, Lemma 23.29) as desired. ☐

12. Group actions and integral closure

In this section we continue the discussion of More on Algebra, Section 110. Recall that a normal local ring is a domain by definition.

Lemma 12.1. *Let A be a normal domain whose fraction field K is separably algebraically closed. Let $\mathfrak{p} \subset A$ be a nonzero prime ideal. Then the residue field $\kappa(\mathfrak{p})$ is algebraically closed.*

Proof. Assume the lemma is not true to get a contradiction. Then there exists a monic irreducible polynomial $P(T) \in \kappa(\mathfrak{p})[T]$ of degree $d > 1$. After replacing P by $a^d P(a^{-1}T)$ for suitable $a \in A$ (to clear denominators) we may assume that P is the image of a monic polynomial Q in $A[T]$. Observe that Q is irreducible in $K[T]$. Namely a factorization over K leads to a factorization over A by Algebra, Lemma 38.5 which we could reduce modulo $\mathfrak{p}$ to get a factorization of P. As K is separably closed, Q is not a separable polynomial (Fields, Definition 12.2). Then the characteristic of K is $p > 0$ and Q has vanishing linear term (Fields, Definition 12.2). However, then we can replace Q by $Q + aT$ where $a \in \mathfrak{p}$ is nonzero to get a contradiction. ☐

Lemma 12.2. *A normal local ring with separably closed fraction field is strictly henselian.*

Proof. Let $(A, \mathfrak{m}, \kappa)$ be normal local with separably closed fraction field K. If $A = K$, then we are done. If not, then the residue field κ is algebraically closed by Lemma 12.1 and it suffices to check that A is henselian. Let $f \in A[T]$ be monic and let $a_0 \in \kappa$ be a root of multiplicity 1 of the reduction $\overline{f} \in \kappa[T]$. Let $f = \prod f_i$ be the factorization in $K[T]$. By Algebra, Lemma 38.5 we have $f_i \in A[T]$. Thus a_0 is a root of f_i for some i. After replacing f by f_i we may assume f is irreducible. Then, since the derivative f' cannot be zero in $A[T]$ as a_0 is a single root, we conclude that f is linear due to the fact that K is separably algebraically closed. Thus A is henselian, see Algebra, Definition 153.1. ☐

Lemma 12.3. *Let G be a finite group acting on a ring R. Let $R^G \to A$ be a ring map. Let $\mathfrak{q}' \subset A \otimes_{R^G} R$ be a prime lying over the prime $\mathfrak{q} \subset R$. Then*

$$I_\mathfrak{q} = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q} \text{ and } \sigma \bmod \mathfrak{q} = id_{\kappa(\mathfrak{q})}\}$$

is equal to

$$I_{\mathfrak{q}'} = \{\sigma \in G \mid \sigma(\mathfrak{q}') = \mathfrak{q}' \text{ and } \sigma \bmod \mathfrak{q}' = id_{\kappa(\mathfrak{q}')}\}$$

Proof. Since $\mathfrak{q}$ is the inverse image of $\mathfrak{q}'$ and since $\kappa(\mathfrak{q}) \subset \kappa(\mathfrak{q}')$, we get $I_{\mathfrak{q}'} \subset I_\mathfrak{q}$. Conversely, if $\sigma \in I_\mathfrak{q}$, the σ acts trivially on the fibre ring $A \otimes_{R^G} \kappa(\mathfrak{q})$. Thus σ fixes all the primes lying over $\mathfrak{q}$ and induces the identity on their residue fields. ☐

Lemma 12.4. *Let G be a finite group acting on a ring R. Let $\mathfrak{q} \subset R$ be a prime. Set*

$$I = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q} \text{ and } \sigma \bmod \mathfrak{q} = id_\mathfrak{q}\}$$

Then $R^G \to R^I$ is étale at $R^I \cap \mathfrak{q}$.

Proof. The strategy of the proof is to use étale localization to reduce to the case where $R \to R^I$ is a local isomorphism at $R^I \cap \mathfrak{p}$. Let $R^G \to A$ be an étale ring map. We claim that if the result holds for the action of G on $A \otimes_{R^G} R$ and some prime $\mathfrak{q}'$ of $A \otimes_{R^G} R$ lying over $\mathfrak{q}$, then the result is true.

To check this, note that since $R^G \to A$ is flat we have $A = (A \otimes_{R^G} R)^G$, see More on Algebra, Lemma 110.7. By Lemma 12.3 the group I does not change. Then a second application of More on Algebra, Lemma 110.7 shows that $A \otimes_{R^G} R^I = (A \otimes_{R^G} R)^I$ (because $R^I \to A \otimes_{R^G} R^I$ is flat). Thus

$$\begin{tikzcd} \mathrm{Spec}((A \otimes_{R^G} R)^I) \arrow[r] \arrow[d] & \mathrm{Spec}(R^I) \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(R^G) \end{tikzcd}$$

is cartesian and the horizontal arrows are étale. Thus if the left vertical arrow is étale in some open neighbourhood W of $(A \otimes_{R^G} R)^I \cap \mathfrak{q}'$, then the right vertical arrow is étale at the points of the (open) image of W in $\mathrm{Spec}(R^I)$, see Descent, Lemma 14.5. In particular the morphism $\mathrm{Spec}(R^I) \to \mathrm{Spec}(R^G)$ is étale at $R^I \cap \mathfrak{q}$.

Let $\mathfrak{p} = R^G \cap \mathfrak{q}$. By More on Algebra, Lemma 110.8 the fibre of $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ over $\mathfrak{p}$ is finite. Moreover the residue field extensions at these points are algebraic, normal, with finite automorphism groups by More on Algebra, Lemma 110.9. Thus we may apply More on Morphisms, Lemma 42.1 to the integral ring map $R^G \to R$ and the prime $\mathfrak{p}$. Combined with the claim above we reduce to the case where $R = A_1 \times \ldots \times A_n$ with each A_i having a single prime $\mathfrak{q}_i$ lying over $\mathfrak{p}$ such that the residue field extensions $\kappa(\mathfrak{q}_i)/\kappa(\mathfrak{p})$ are purely inseparable. Of course $\mathfrak{q}$ is one of these primes, say $\mathfrak{q} = \mathfrak{q}_1$.

It may not be the case that G permutes the factors A_i (this would be true if the spectrum of A_i were connected, for example if R^G was local). This we can fix as follows; we suggest the reader think this through for themselves, perhaps using idempotents instead of topology. Recall that the product decomposition gives a corresponding disjoint union decomposition of $\mathrm{Spec}(R)$ by open and closed subsets U_i. Since G is finite, we can refine this covering by a finite disjoint union decomposition $\mathrm{Spec}(R) = \coprod_{j \in J} W_j$ by open and closed subsets W_j, such that for all $j \in J$ there exists a $j' \in J$ with $\sigma(W_j) = W_{j'}$. The union of the W_j not meeting $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ is a closed subset not meeting the fibre over $\mathfrak{p}$ hence maps to a closed subset of $\mathrm{Spec}(R^G)$ not meeting $\mathfrak{p}$ as $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ is closed. Hence after replacing R^G by a principal localization (permissible by the claim) we may assume each W_j meets one of the points $\mathfrak{q}_i$. Then we set $U_i = W_j$ if $\mathfrak{q}_i \in W_j$. The corresponding product decomposition $R = A_1 \times \ldots \times A_n$ is one where G permutes the factors A_i.

Thus we may assume we have a product decomposition $R = A_1 \times \ldots \times A_n$ compatible with G-action, where each A_i has a single prime $\mathfrak{q}_i$ lying over $\mathfrak{p}$ and the field extensions $\kappa(\mathfrak{q}_i)/\kappa(\mathfrak{p})$ are purely inseparable. Write $A' = A_2 \times \ldots \times A_n$ so that

$$R = A_1 \times A'$$

Since $\mathfrak{q} = \mathfrak{q}_1$ we find that every $\sigma \in I$ preserves the product decomposition above. Hence

$$R^I = (A_1)^I \times (A')^I$$

Observe that $I = D = \{\sigma \in G \mid \sigma(\mathfrak q) = \mathfrak q\}$ because $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is purely inseparable. Since the action of G on primes over $\mathfrak p$ is transitive (More on Algebra, Lemma 110.8) we conclude that, the index of I in G is n and we can write $G = eI \amalg \sigma_2 I \amalg \ldots \amalg \sigma_n I$ so that $A_i = \sigma_i(A_1)$ for $i = 2, \ldots, n$. It follows that

$$R^G = (A_1)^I.$$

Thus the map $R^G \to R^I$ is étale at $R^I \cap \mathfrak q$ and the proof is complete. $\square$

The following lemma generalizes More on Algebra, Lemma 112.8.

Lemma 12.5. *Let A be a normal domain with fraction field K. Let L/K be a (possibly infinite) Galois extension. Let $G = Gal(L/K)$ and let B be the integral closure of A in L. Let $\mathfrak q \subset B$. Set*

$$I = \{\sigma \in G \mid \sigma(\mathfrak q) = \mathfrak q \text{ and } \sigma \bmod \mathfrak q = id_{\kappa(\mathfrak q)}\}$$

Then $(B^I)_{B^I \cap \mathfrak q}$ is a filtered colimit of étale A-algebras.

Proof. We can write L as the filtered colimit of finite Galois extensions of K. Hence it suffices to prove this lemma in case L/K is a finite Galois extension, see Algebra, Lemma 154.3. Since $A = B^G$ as A is integrally closed in $K = L^G$ the result follows from Lemma 12.4. $\square$

13. Ramification theory

In this section we continue the discussion of More on Algebra, Section 112 and we relate it to our discussion of the fundamental groups of schemes.

Let $(A, \mathfrak m, \kappa)$ be a normal local ring with fraction field K. Choose a separable algebraic closure K^{sep}. Let A^{sep} be the integral closure of A in K^{sep}. Choose maximal ideal $\mathfrak m^{sep} \subset A^{sep}$. Let $A \subset A^h \subset A^{sh}$ be the henselization and strict henselization. Observe that A^h and A^{sh} are normal rings as well (More on Algebra, Lemma 45.6). Denote K^h and K^{sh} their fraction fields. Since $(A^{sep})_{\mathfrak m^{sep}}$ is strictly henselian by Lemma 12.2 we can choose an A-algebra map $A^{sh} \to (A^{sep})_{\mathfrak m^{sep}}$. Namely, first choose a κ-embedding[4] $\kappa(\mathfrak m^{sh}) \to \kappa(\mathfrak m^{sep})$ and then extend (uniquely) to an A-algebra homomorphism by Algebra, Lemma 155.10. We get the following diagram

$$\begin{tikzcd}
K^{sep} & K^{sh} \arrow[l] & K^h \arrow[l] & K \arrow[l] \\
(A^{sep})_{\mathfrak m^{sep}} \arrow[u] & A^{sh} \arrow[l] \arrow[u] & A^h \arrow[l] \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

We can take the fundamental groups of the spectra of these rings. Of course, since K^{sep}, $(A^{sep})_{\mathfrak m^{sep}}$, and A^{sh} are strictly henselian, for them we obtain trivial groups. Thus the interesting part is the following

(13.0.1)
$$\begin{tikzcd}
\pi_1(U^{sh}) \arrow[r] \arrow[dr, "1"'] & \pi_1(U^h) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \\
 & \pi_1(X^h) \arrow[r] & \pi_1(X)
\end{tikzcd}$$

[4] This is possible because $\kappa(\mathfrak m^{sh})$ is a separable algebraic closure of κ and $\kappa(\mathfrak m^{sep})$ is an algebraic closure of κ by Lemma 12.1.

Here X^h and X are the spectra of A^h and A and U^{sh}, U^h, U are the spectra of K^{sh}, K^h, and K. The label 1 means that the map is trivial; this follows as it factors through the trivial group $\pi_1(X^{sh})$. On the other hand, the profinite group $G = \text{Gal}(K^{sep}/K)$ acts on A^{sep} and we can make the following definitions

$$D = \{\sigma \in G \mid \sigma(\mathfrak{m}^{sep}) = \mathfrak{m}^{sep}\} \supset I = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}^{sep} = \text{id}_{\kappa(\mathfrak{m}^{sep})}\}$$

These groups are sometimes called the *decomposition group* and the *inertia group* especially when A is a discrete valuation ring.

Lemma 13.1. *In the situation described above, via the isomorphism $\pi_1(U) = Gal(K^{sep}/K)$ the diagram (13.0.1) translates into the diagram*

$$\begin{tikzcd}
I \arrow[r] \arrow[rd, "1"'] & D \arrow[r] \arrow[d] & Gal(K^{sep}/K) \arrow[d] \\
 & Gal(\kappa(\mathfrak{m}^{sh})/\kappa) \arrow[r] & Gal(M/K)
\end{tikzcd}$$

where $K^{sep}/M/K$ is the maximal subextension unramified with respect to A. Moreover, the vertical arrows are surjective, the kernel of the left vertical arrow is I and the kernel of the right vertical arrow is the smallest closed normal subgroup of $Gal(K^{sep}/K)$ containing I.

Proof. By construction the group D acts on $(A^{sep})_{\mathfrak{m}^{sep}}$ over A. By the uniqueness of $A^{sh} \to (A^{sep})_{\mathfrak{m}^{sep}}$ given the map on residue fields (Algebra, Lemma 155.10) we see that the image of $A^{sh} \to (A^{sep})_{\mathfrak{m}^{sep}}$ is contained in $((A^{sep})_{\mathfrak{m}^{sep}})^I$. On the other hand, Lemma 12.5 shows that $((A^{sep})_{\mathfrak{m}^{sep}})^I$ is a filtered colimit of étale extensions of A. Since A^{sh} is the maximal such extension, we conclude that $A^{sh} = ((A^{sep})_{\mathfrak{m}^{sep}})^I$. Hence $K^{sh} = (K^{sep})^I$.

Recall that I is the kernel of a surjective map $D \to \text{Aut}(\kappa(\mathfrak{m}^{sep})/\kappa)$, see More on Algebra, Lemma 110.10. We have $\text{Aut}(\kappa(\mathfrak{m}^{sep})/\kappa) = \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$ as we have seen above that these fields are the algebraic and separable algebraic closures of κ. On the other hand, any automorphism of A^{sh} over A is an automorphism of A^{sh} over A^h by the uniqueness in Algebra, Lemma 155.6. Furthermore, A^{sh} is the colimit of finite étale extensions $A^h \subset A'$ which correspond 1-to-1 with finite separable extension κ'/κ, see Algebra, Remark 155.4. Thus

$$\text{Aut}(A^{sh}/A) = \text{Aut}(A^{sh}/A^h) = \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$$

Let κ'/κ be a finite Galois extension with Galois group G. Let $A^h \subset A'$ be the finite étale extension corresponding to $\kappa \subset \kappa'$ by Algebra, Lemma 153.7. Then it follows that $(A')^G = A^h$ by looking at fraction fields and degrees (small detail omitted). Taking the colimit we conclude that $(A^{sh})^{\text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)} = A^h$. Combining all of the above, we find $A^h = ((A^{sep})_{\mathfrak{m}^{sep}})^D$. Hence $K^h = (K^{sep})^D$.

Since U, U^h, U^{sh} are the spectra of the fields K, K^h, K^{sh} we see that the top lines of the diagrams correspond via Lemma 6.3. By Lemma 8.2 we have $\pi_1(X^h) = \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$. The exactness of the sequence $1 \to I \to D \to \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa) \to 1$ was pointed out above. By Proposition 11.3 we see that $\pi_1(X) = \text{Gal}(M/K)$. Finally, the statement on the kernel of $\text{Gal}(K^{sep}/K) \to \text{Gal}(M/K) = \pi_1(X)$ follows from Lemma 11.4. This finishes the proof. $\square$

Let X be a normal integral scheme with function field K. Let K^{sep} be a separable algebraic closure of K. Let $X^{sep} \to X$ be the normalization of X in K^{sep}. Since $G = \text{Gal}(K^{sep}/K)$ acts on K^{sep} we obtain a right action of G on X^{sep}. For $y \in X^{sep}$ define

$$D_y = \{\sigma \in G \mid \sigma(y) = y\} \supset I_y = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}_y = \text{id}_{\kappa(y)}\}$$

similarly to the above. On the other hand, for $x \in X$ let $\mathcal{O}^{sh}_{X,x}$ be a strict henselization, let K^{sh}_x be the fraction field of $\mathcal{O}^{sh}_{X,x}$ and choose a K-embedding $K^{sh}_x \to K^{sep}$.

Lemma 13.2. *Let X be a normal integral scheme with function field K. With notation as above, the following three subgroups of $Gal(K^{sep}/K) = \pi_1(\text{Spec}(K))$ are equal*

1. *the kernel of the surjection $Gal(K^{sep}/K) \longrightarrow \pi_1(X)$,*
2. *the smallest normal closed subgroup containing I_y for all $y \in X^{sep}$, and*
3. *the smallest normal closed subgroup containing $Gal(K^{sep}/K^{sh}_x)$ for all $x \in X$.*

Proof. The equivalence of (2) and (3) follows from Lemma 13.1 which tells us that I_y is conjugate to $\text{Gal}(K^{sep}/K^{sh}_x)$ if y lies over x. By Lemma 11.4 we see that $\text{Gal}(K^{sep}/K^{sh}_x)$ maps trivially to $\pi_1(\text{Spec}(\mathcal{O}_{X,x}))$ and therefore the subgroup $N \subset G = \text{Gal}(K^{sep}/K)$ of (2) and (3) is contained in the kernel of $G \longrightarrow \pi_1(X)$.

To prove the other inclusion, since N is normal, it suffices to prove: given $N \subset U \subset G$ with U open normal, the quotient map $G \to G/U$ factors through $\pi_1(X)$. In other words, if L/K is the Galois extension corresponding to U, then we have to show that X is unramified in L (Section 11, especially Proposition 11.3). It suffices to do this when X is affine (we do this so we can refer to algebra results in the rest of the proof). Let $Y \to X$ be the normalization of X in L. The inclusion $L \subset K^{sep}$ induces a morphism $\pi : X^{sep} \to Y$. For $y \in X^{sep}$ the inertia group of $\pi(y)$ in $\text{Gal}(L/K)$ is the image of I_y in $\text{Gal}(L/K)$; this follows from More on Algebra, Lemma 110.11. Since $N \subset U$ all these inertia groups are trivial. We conclude that $Y \to X$ is étale by applying Lemma 12.4. (Alternative: you can use Lemma 11.4 to see that the pullback of Y to $\text{Spec}(\mathcal{O}_{X,x})$ is étale for all $x \in X$ and then conclude from there with a bit more work.) $\square$

Example 13.3. Let X be a normal integral Noetherian scheme with function field K. Purity of branch locus (see below) tells us that if X is regular, then it suffices in Lemma 13.2 to consider the inertia groups $I = \pi_1(\text{Spec}(K^{sh}_x))$ for points x of codimension 1 in X. In general this is not enough however. Namely, let $Y = \mathbf{A}^n_k = \text{Spec}(k[t_1, \ldots, t_n])$ where k is a field not of characteristic 2. Let $G = \{\pm 1\}$ be the group of order 2 acting on Y by multiplication on the coordinates. Set

$$X = \text{Spec}(k[t_it_j, i, j \in \{1, \ldots, n\}])$$

The embedding $k[t_it_j] \subset k[t_1, \ldots, t_n]$ defines a degree 2 morphism $Y \to X$ which is unramified everywhere except over the maximal ideal $\mathfrak{m} = (t_it_j)$ which is a point of codimension n in X.

Lemma 13.4. *Let X be an integral normal scheme with function field K. Let L/K be a finite extension. Let $Y \to X$ be the normalization of X in L. The following are equivalent*

1. *X is unramified in L as defined in Section 11,*

(2) *$Y \to X$ is an unramified morphism of schemes,*
(3) *$Y \to X$ is an étale morphism of schemes,*
(4) *$Y \to X$ is a finite étale morphism of schemes,*
(5) *for $x \in X$ the projection $Y \times_X \text{Spec}(\mathcal{O}_{X,x}) \to \text{Spec}(\mathcal{O}_{X,x})$ is unramified,*
(6) *same as in (5) but with $\mathcal{O}^h_{X,x}$,*
(7) *same as in (5) but with $\mathcal{O}^{sh}_{X,x}$,*
(8) *for $x \in X$ the scheme theoretic fibre Y_x is étale over x of degree $\geq [L : K]$.*

If L/K is Galois with Galois group G, then these are also equivalent to

(9) *for $y \in Y$ the group $I_y = \{g \in G \mid g(y) = y \text{ and } g \bmod \mathfrak{m}_y = id_{\kappa(y)}\}$ is trivial.*

Proof. The equivalence of (1) and (2) is the definition of (1). The equivalence of (2), (3), and (4) is Lemma 11.1. It is straightforward to prove that (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (6), (6) $\Rightarrow$ (7).

Assume (7). Observe that $\mathcal{O}^{sh}_{X,x}$ is a normal local domain (More on Algebra, Lemma 45.6). Let $L^{sh} = L \otimes_K K^{sh}_x$ where K^{sh}_x is the fraction field of $\mathcal{O}^{sh}_{X,x}$. Then $L^{sh} = \prod_{i=1,\ldots,n} L_i$ with L_i/K^{sh}_x finite separable. By Algebra, Lemma 147.4 (and a limit argument we omit) we see that $Y \times_X \text{Spec}(\mathcal{O}^{sh}_{X,x})$ is the integral closure of $\text{Spec}(\mathcal{O}^{sh}_{X,x})$ in L^{sh}. Hence by Lemma 11.1 (applied to the factors L_i of L^{sh}) we see that $Y \times_X \text{Spec}(\mathcal{O}^{sh}_{X,x}) \to \text{Spec}(\mathcal{O}^{sh}_{X,x})$ is finite étale. Looking at the generic point we see that the degree is equal to $[L : K]$ and hence we see that (8) is true.

Assume (8). Assume that $x \in X$ and that the scheme theoretic fibre Y_x is étale over x of degree $\geq [L : K]$. Observe that this means that Y has $\geq [L : K]$ geometric points lying over x. We will show that $Y \to X$ is finite étale over a neighbourhood of x. This will prove (1) holds. To prove this we may assume $X = \text{Spec}(R)$, the point x corresponds to the prime $\mathfrak{p} \subset R$, and $Y = \text{Spec}(S)$. We apply More on Morphisms, Lemma 42.1 and we find an étale neighbourhood $(U, u) \to (X, x)$ such that $Y \times_X U = V_1 \amalg \ldots \amalg V_m$ such that V_i has a unique point v_i lying over u with $\kappa(v_i)/\kappa(u)$ purely inseparable. Shrinking U if necessary we may assume U is a normal integral scheme with generic point ξ (use Descent, Lemmas 16.3 and 18.2 and Properties, Lemma 7.5). By our remark on geometric points we see that $m \geq [L : K]$. On the other hand, by More on Morphisms, Lemma 19.2 we see that $\coprod V_i \to U$ is the normalization of U in $\text{Spec}(L) \times_X U$. As $K \subset \kappa(\xi)$ is finite separable, we can write $\text{Spec}(L) \times_X U = \text{Spec}(\prod_{i=1,\ldots,n} L_i)$ with $L_i/\kappa(\xi)$ finite and $[L : K] = \sum [L_i : \kappa(\xi)]$. Since V_j is nonempty for each j and $m \geq [L : K]$ we conclude that $m = n$ and $[L_i : \kappa(\xi)] = 1$ for all i. Then $V_j \to U$ is an isomorphism in particular étale, hence $Y \times_X U \to U$ is étale. By Descent, Lemma 23.29 we conclude that $Y \to X$ is étale over the image of $U \to X$ (an open neighbourhood of x).

Assume L/K is Galois and (9) holds. Then $Y \to X$ is étale by Lemma 12.5. We omit the proof that (1) implies (9). $\square$

In the case of infinite Galois extensions of discrete valuation rings we can say a tiny bit more. To do so we introduce the following notation. A subset $S \subset \mathbf{N}$ of integers is *multiplicativity directed* if $1 \in S$ and for $n, m \in S$ there exists $k \in S$ with $n|k$ and $m|k$. Define a partial ordering on S by the rule $n \geq_S m$ if and only if $m|n$. Given a field κ we obtain an inverse system of finite groups $\{\mu_n(\kappa)\}_{n \in S}$ with transition maps

$$\mu_n(\kappa) \longrightarrow \mu_m(\kappa), \quad \zeta \longmapsto \zeta^{n/m}$$

for $n \geq_S m$. Then we can form the profinite group

$$\lim_{n \in S} \mu_n(\kappa)$$

Observe that the limit is cofiltered (as S is directed). The construction is functorial in κ. In particular $\text{Aut}(\kappa)$ acts on this profinite group. For example, if $S = \{1, n\}$, then this gives $\mu_n(\kappa)$. If $S = \{1, \ell, \ell^2, \ell^3, \ldots\}$ for some prime ℓ different from the characteristic of κ this produces $\lim_n \mu_{\ell^n}(\kappa)$ which is sometimes called the ℓ-adic Tate module of the multiplicative group of κ (compare with More on Algebra, Example 93.5).

Lemma 13.5. *Let A be a discrete valuation ring with fraction field K. Let L/K be a (possibly infinite) Galois extension. Let B be the integral closure of A in L. Let $\mathfrak{m}$ be a maximal ideal of B. Let $G = Gal(L/K)$, $D = \{\sigma \in G \mid \sigma(\mathfrak{m}) = \mathfrak{m}\}$, and $I = \{\sigma \in D \mid \sigma \bmod \mathfrak{m} = id_{\kappa(\mathfrak{m})}\}$. The decomposition group D fits into a canonical exact sequence*

$$1 \to I \to D \to Aut(\kappa(\mathfrak{m})/\kappa_A) \to 1$$

The inertia group I fits into a canonical exact sequence

$$1 \to P \to I \to I_t \to 1$$

such that

1. *P is a normal subgroup of D,*
2. *P is a pro-p-group if the characteristic of κ_A is $p > 1$ and $P = \{1\}$ if the characteristic of κ_A is zero,*
3. *there is a multiplicatively directed $S \subset \mathbf{N}$ such that $\kappa(\mathfrak{m})$ contains a primitive nth root of unity for each $n \in S$ (elements of S are prime to p),*
4. *there exists a canonical surjective map*

$$\theta_{can} : I \to \lim_{n \in S} \mu_n(\kappa(\mathfrak{m}))$$

whose kernel is P, which satisfies $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$, and which induces an isomorphism $I_t \to \lim_{n \in S} \mu_n(\kappa(\mathfrak{m}))$.

Proof. This is mostly a reformulation of the results on finite Galois extensions proved in More on Algebra, Section 112. The surjectivity of the map $D \to \text{Aut}(\kappa(\mathfrak{m})/\kappa)$ is More on Algebra, Lemma 110.10. This gives the first exact sequence.

To construct the second short exact sequence let Λ be the set of finite Galois subextensions, i.e., $\lambda \in \Lambda$ corresponds to $L/L_\lambda/K$. Set $G_\lambda = \text{Gal}(L_\lambda/K)$. Recall that G_λ is an inverse system of finite groups with surjective transition maps and that $G = \lim_{\lambda \in \Lambda} G_\lambda$, see Fields, Lemma 22.3. We let B_λ be the integral closure of A in L_λ. Then we set $\mathfrak{m}_\lambda = \mathfrak{m} \cap B_\lambda$ and we denote $P_\lambda, I_\lambda, D_\lambda$ the wild inertia, inertia, and decomposition group of $\mathfrak{m}_\lambda$, see More on Algebra, Lemma 112.5. For $\lambda \geq \lambda'$ the restriction defines a commutative diagram

$$\begin{tikzcd}
P_\lambda \arrow[r] \arrow[d] & I_\lambda \arrow[r] \arrow[d] & D_\lambda \arrow[r] \arrow[d] & G_\lambda \arrow[d] \\
P_{\lambda'} \arrow[r] & I_{\lambda'} \arrow[r] & D_{\lambda'} \arrow[r] & G_{\lambda'}
\end{tikzcd}$$

with surjective vertical maps, see More on Algebra, Lemma 112.10.

From the definitions it follows immediately that $I = \lim I_\lambda$ and $D = \lim D_\lambda$ under the isomorphism $G = \lim G_\lambda$ above. Since $L = \operatorname{colim} L_\lambda$ we have $B = \operatorname{colim} B_\lambda$ and $\kappa(\mathfrak m) = \operatorname{colim} \kappa(\mathfrak m_\lambda)$. Since the transition maps of the system D_λ are compatible with the maps $D_\lambda \to \text{Aut}(\kappa(\mathfrak m_\lambda)/\kappa)$ (see More on Algebra, Lemma 112.10) we see that the map $D \to \text{Aut}(\kappa(\mathfrak m)/\kappa)$ is the limit of the maps $D_\lambda \to \text{Aut}(\kappa(\mathfrak m_\lambda)/\kappa)$.

There exist canonical maps

$$\theta_{\lambda,can} : I_\lambda \longrightarrow \mu_{n_\lambda}(\kappa(\mathfrak m_\lambda))$$

where $n_\lambda = |I_\lambda|/|P_\lambda|$, where $\mu_{n_\lambda}(\kappa(\mathfrak m_\lambda))$ has order n_λ, such that $\theta_{\lambda,can}(\tau\sigma\tau^{-1}) = \tau(\theta_{\lambda,can}(\sigma))$ for $\tau \in D_\lambda$ and $\sigma \in I_\lambda$, and such that we get commutative diagrams

$$\begin{tikzcd} I_\lambda \arrow[r, "\theta_{\lambda,can}"'] \arrow[d] & \mu_{n_\lambda}(\kappa(\mathfrak m_\lambda)) \arrow[d, "(-)^{n_\lambda/n_{\lambda'}}"] \\ I_{\lambda'} \arrow[r, "\theta_{\lambda',can}"] & \mu_{n_{\lambda'}}(\kappa(\mathfrak m_{\lambda'})) \end{tikzcd}$$

see More on Algebra, Remark 112.11.

Let $S \subset \mathbf{N}$ be the collection of integers n_λ. Since Λ is directed, we see that S is multiplicatively directed. By the displayed commutative diagrams above we can take the limits of the maps $\theta_{\lambda,can}$ to obtain

$$\theta_{can} : I \to \lim_{n\in S} \mu_n(\kappa(\mathfrak m)).$$

This map is continuous (small detail omitted). Since the transition maps of the system of I_λ are surjective and Λ is directed, the projections $I \to I_\lambda$ are surjective. For every λ the diagram

$$\begin{tikzcd} I \arrow[r, "\theta_{can}"'] \arrow[d] & \lim_{n\in S} \mu_n(\kappa(\mathfrak m)) \arrow[d] \\ I_\lambda \arrow[r, "\theta_{\lambda,can}"] & \mu_{n_\lambda}(\kappa(\mathfrak m_\lambda)) \end{tikzcd}$$

commutes. Hence the image of θ_{can} surjects onto the finite group $\mu_{n_\lambda}(\kappa(\mathfrak m)) = \mu_{n_\lambda}(\kappa(\mathfrak m_\lambda))$ of order n_λ (see above). It follows that the image of θ_{can} is dense. On the other hand θ_{can} is continuous and the source is a profinite group. Hence θ_{can} is surjective by a topological argument.

The property $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$ follows from the corresponding properties of the maps $\theta_{\lambda,can}$ and the compatibility of the map $D \to \text{Aut}(\kappa(\mathfrak m))$ with the maps $D_\lambda \to \text{Aut}(\kappa(\mathfrak m_\lambda))$. Setting $P = \text{Ker}(\theta_{can})$ this implies that P is a normal subgroup of D. Setting $I_t = I/P$ we obtain the isomorphism $I_t \to \lim_{n\in S} \mu_n(\kappa(\mathfrak m))$ from the surjectivity of θ_{can}.

To finish the proof we show that $P = \lim P_\lambda$ which proves that P is a pro-p-group. Recall that the tame inertia group $I_{\lambda,t} = I_\lambda/P_\lambda$ has order n_λ. Since the transition maps $P_\lambda \to P_{\lambda'}$ are surjective and Λ is directed, we obtain a short exact sequence

$$1 \to \lim P_\lambda \to I \to \lim I_{\lambda,t} \to 1$$

(details omitted). Since for each λ the map $\theta_{\lambda,can}$ induces an isomorphism $I_{\lambda,t} \cong \mu_{n_\lambda}(\kappa(\mathfrak m))$ the desired result follows. $\square$

Lemma 13.6. *Let A be a discrete valuation ring with fraction field K. Let K^{sep} be a separable closure of K. Let A^{sep} be the integral closure of A in K^{sep}. Let $\mathfrak{m}^{sep}$ be a maximal ideal of A^{sep}. Let $\mathfrak{m} = \mathfrak{m}^{sep} \cap A$, let $\kappa = A/\mathfrak{m}$, and let $\overline{\kappa} = A^{sep}/\mathfrak{m}^{sep}$. Then $\overline{\kappa}$ is an algebraic closure of κ. Let $G = Gal(K^{sep}/K)$, $D = \{\sigma \in G \mid \sigma(\mathfrak{m}^{sep}) = \mathfrak{m}^{sep}\}$, and $I = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}^{sep} = id_{\kappa(\mathfrak{m}^{sep})}\}$. The decomposition group D fits into a canonical exact sequence*

$$1 \to I \to D \to Gal(\kappa^{sep}/\kappa) \to 1$$

where $\kappa^{sep} \subset \overline{\kappa}$ is the separable closure of κ. The inertia group I fits into a canonical exact sequence

$$1 \to P \to I \to I_t \to 1$$

such that

1. *P is a normal subgroup of D,*
2. *P is a pro-p-group if the characteristic of κ_A is $p > 1$ and $P = \{1\}$ if the characteristic of κ_A is zero,*
3. *there exists a canonical surjective map*

$$\theta_{can} : I \to \lim_{n \text{ prime to } p} \mu_n(\kappa^{sep})$$

whose kernel is P, which satisfies $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$, and which induces an isomorphism $I_t \to \lim_{n \text{ prime to } p} \mu_n(\kappa^{sep})$.

Proof. The field $\overline{\kappa}$ is the algebraic closure of κ by Lemma 12.1. Most of the statements immediately follow from the corresponding parts of Lemma 13.5. For example because $\text{Aut}(\overline{\kappa}/\kappa) = \text{Gal}(\kappa^{sep}/\kappa)$ we obtain the first sequence. Then the only other assertion that needs a proof is the fact that with S as in Lemma 13.5 the limit $\lim_{n \in S} \mu_n(\overline{\kappa})$ is equal to $\lim_{n \text{ prime to } p} \mu_n(\kappa^{sep})$. To see this it suffices to show that every integer n prime to p divides an element of S. Let $\pi \in A$ be a uniformizer and consider the splitting field L of the polynomial $X^n - \pi$. Since the polynomial is separable we see that L is a finite Galois extension of K. Choose an embedding $L \to K^{sep}$. Observe that if B is the integral closure of A in L, then the ramification index of $A \to B_{\mathfrak{m}^{sep} \cap B}$ is divisible by n (because π has an nth root in B; in fact the ramification index equals n but we do not need this). Then it follows from the construction of the S in the proof of Lemma 13.5 that n divides an element of S. □

14. Geometric and arithmetic fundamental groups

In this section we work out what happens when comparing the fundamental group of a scheme X over a field k with the fundamental group of $X_{\overline{k}}$ where $\overline{k}$ is the algebraic closure of k.

Lemma 14.1. *Let I be a directed set. Let X_i be an inverse system of quasi-compact and quasi-separated schemes over I with affine transition morphisms. Let $X = \lim X_i$ as in Limits, Section 2. Then there is an equivalence of categories*

$$\text{colim}\, F\acute{E}t_{X_i} = F\acute{E}t_X$$

If X_i is connected for all sufficiently large i and $\overline{x}$ is a geometric point of X, then

$$\pi_1(X, \overline{x}) = \lim \pi_1(X_i, \overline{x})$$

Proof. The equivalence of categories follows from Limits, Lemmas 10.1, 8.3, and 8.10. The second statement is formal given the statement on categories. □

Lemma 14.2. *Let k be a field with perfection k^{perf}. Let X be a connected scheme over k. Then $X_{k^{perf}}$ is connected and $\pi_1(X_{k^{perf}}) \to \pi_1(X)$ is an isomorphism.*

Proof. Special case of topological invariance of the fundamental group. See Proposition 8.4. To see that $\mathrm{Spec}(k^{perf}) \to \mathrm{Spec}(k)$ is a universal homeomorphism you can use Algebra, Lemma 46.10. □

Lemma 14.3. *Let k be a field with algebraic closure $\overline{k}$. Let X be a quasi-compact and quasi-separated scheme over k. If the base change $X_{\overline{k}}$ is connected, then there is a short exact sequence*

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \pi_1(\mathrm{Spec}(k)) \to 1$$

of profinite topological groups.

Proof. Connected objects of $F\acute{E}t_{\mathrm{Spec}(k)}$ are of the form $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ with k'/k a finite separable extension. Then $X_{\mathrm{Spec}\, k'}$ is connected, as the morphism $X_{\overline{k}} \to X_{\mathrm{Spec}(k')}$ is surjective and $X_{\overline{k}}$ is connected by assumption. Thus $\pi_1(X) \to \pi_1(\mathrm{Spec}(k))$ is surjective by Lemma 4.1.

Before we go on, note that we may assume that k is a perfect field. Namely, we have $\pi_1(X_{k^{perf}}) = \pi_1(X)$ and $\pi_1(\mathrm{Spec}(k^{perf})) = \pi_1(\mathrm{Spec}(k))$ by Lemma 14.2.

It is clear that the composition of the functors $F\acute{E}t_{\mathrm{Spec}(k)} \to F\acute{E}t_X \to F\acute{E}t_{X_{\overline{k}}}$ sends objects to disjoint unions of copies of $X_{\mathrm{Spec}(\overline{k})}$. Therefore the composition $\pi_1(X_{\overline{k}}) \to \pi_1(X) \to \pi_1(\mathrm{Spec}(k))$ is the trivial homomorphism by Lemma 4.2.

Let $U \to X$ be a finite étale morphism with U connected. Observe that $U \times_X X_{\overline{k}} = U_{\overline{k}}$. Suppose that $U_{\overline{k}} \to X_{\overline{k}}$ has a section $s : X_{\overline{k}} \to U_{\overline{k}}$. Then $s(X_{\overline{k}})$ is an open connected component of $U_{\overline{k}}$. For $\sigma \in \mathrm{Gal}(\overline{k}/k)$ denote s^σ the base change of s by $\mathrm{Spec}(\sigma)$. Since $U_{\overline{k}} \to X_{\overline{k}}$ is finite étale it has only a finite number of sections. Thus

$$\overline{T} = \bigcup s^\sigma(X_{\overline{k}})$$

is a finite union and we see that $\overline{T}$ is a $\mathrm{Gal}(\overline{k}/k)$-stable open and closed subset. By Varieties, Lemma 7.10 we see that $\overline{T}$ is the inverse image of a closed subset $T \subset U$. Since $U_{\overline{k}} \to U$ is open (Morphisms, Lemma 23.4) we conclude that T is open as well. As U is connected we see that $T = U$. Hence $U_{\overline{k}}$ is a (finite) disjoint union of copies of $X_{\overline{k}}$. By Lemma 4.5 we conclude that the image of $\pi_1(X_{\overline{k}}) \to \pi_1(X)$ is normal.

Let $V \to X_{\overline{k}}$ be a finite étale cover. Recall that $\overline{k}$ is the union of finite separable extensions of k. By Lemma 14.1 we find a finite separable extension k'/k and a finite étale morphism $U \to X_{k'}$ such that $V = X_{\overline{k}} \times_{X_{k'}} U = U \times_{\mathrm{Spec}(k')} \mathrm{Spec}(\overline{k})$. Then the composition $U \to X_{k'} \to X$ is finite étale and $U \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ contains $V = U \times_{\mathrm{Spec}(k')} \mathrm{Spec}(\overline{k})$ as an open and closed subscheme. (Because $\mathrm{Spec}(\overline{k})$ is an open and closed subscheme of $\mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ via the multiplication map $k' \otimes_k \overline{k} \to \overline{k}$.) By Lemma 4.4 we conclude that $\pi_1(X_{\overline{k}}) \to \pi_1(X)$ is injective.

Finally, we have to show that for any finite étale morphism $U \to X$ such that $U_{\overline{k}}$ is a disjoint union of copies of $X_{\overline{k}}$ there is a finite étale morphism $V \to \mathrm{Spec}(k)$ and a surjection $V \times_{\mathrm{Spec}(k)} X \to U$. See Lemma 4.3. Arguing as above using Lemma 14.1 we find a finite separable extension k'/k such that there is an isomorphism $U_{k'} \cong \coprod_{i=1,\ldots,n} X_{k'}$. Thus setting $V = \coprod_{i=1,\ldots,n} \mathrm{Spec}(k')$ we conclude. □

15. Homotopy exact sequence

In this section we discuss the following result. Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups. See Proposition 15.2.

Lemma 15.1. *Let $f : X \to S$ be a proper morphism of schemes. Let $X \to S' \to S$ be the Stein factorization of f, see More on Morphisms, Theorem 53.5. If f is of finite presentation, flat, with geometrically reduced fibres, then $S' \to S$ is finite étale.*

Proof. This follows from Derived Categories of Schemes, Lemma 32.8 and the information contained in More on Morphisms, Theorem 53.5. □

Proposition 15.2. *Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence*

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups.

Proof. Let $Y \to X$ be a finite étale morphism. Consider the Stein factorization

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

of $Y \to S$. By Lemma 15.1 the morphism $T \to S$ is finite étale. In this way we obtain a functor $F\acute{E}t_X \to F\acute{E}t_S$. For any finite étale morphism $U \to S$ a morphism $Y \to U \times_S X$ over X is the same thing as a morphism $Y \to U$ over S and such a morphism factors uniquely through the Stein factorization, i.e., corresponds to a unique morphism $T \to U$ (by the construction of the Stein factorization as a relative normalization in More on Morphisms, Lemma 53.1 and factorization by Morphisms, Lemma 53.4). Thus we see that the functors $F\acute{E}t_X \to F\acute{E}t_S$ and $F\acute{E}t_S \to F\acute{E}t_X$ are adjoints. Note that the Stein factorization of $U \times_S X \to S$ is U, because the fibres of $U \times_S X \to U$ are geometrically connected.

By the discussion above and Categories, Lemma 24.4 we conclude that $F\acute{E}t_S \to F\acute{E}t_X$ is fully faithful, i.e., $\pi_1(X) \to \pi_1(S)$ is surjective (Lemma 4.1).

It is immediate that the composition $F\acute{E}t_S \to F\acute{E}t_X \to F\acute{E}t_{X_{\overline{s}}}$ sends any U to a disjoint union of copies of $X_{\overline{s}}$. Hence $\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S)$ is trivial by Lemma 4.2.

Let $Y \to X$ be a finite étale morphism with Y connected such that $Y \times_X X_{\overline{s}}$ contains a connected component Z isomorphic to $X_{\overline{s}}$. Consider the Stein factorization T as above. Let $\overline{t} \in T_{\overline{s}}$ be the point corresponding to the fibre Z. Observe that T is connected (as the image of a connected scheme) and by the surjectivity above $T \times_S X$ is connected. Now consider the factorization

$$\pi : Y \longrightarrow T \times_S X$$

Let $\overline{x} \in X_{\overline{s}}$ be any closed point. Note that $\kappa(\overline{t}) = \kappa(\overline{s}) = \kappa(\overline{x})$ is an algebraically closed field. Then the fibre of π over $(\overline{t}, \overline{x})$ consists of a unique point, namely the unique point $\overline{z} \in Z$ corresponding to $\overline{x} \in X_{\overline{s}}$ via the isomorphism $Z \to X_{\overline{s}}$. We conclude that the finite étale morphism π has degree 1 in a neighbourhood of $(\overline{t}, \overline{x})$. Since $T \times_S X$ is connected it has degree 1 everywhere and we find that $Y \cong T \times_S X$. Thus $Y \times_X X_{\overline{s}}$ splits completely. Combining all of the above we see that Lemmas 4.3 and 4.5 both apply and the proof is complete. $\square$

16. Specialization maps

In this section we construct specialization maps. Let $f : X \to S$ be a proper morphism of schemes with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization of points in S. Let $\overline{s}$ and $\overline{s}'$ be geometric points lying over s and s'. Then there is a specialization map

$$sp : \pi_1(X_{\overline{s}'}) \longrightarrow \pi_1(X_{\overline{s}})$$

The construction of this map is as follows. Let A be the strict henselization of $\mathcal{O}_{S,s}$ with respect to $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$, see Algebra, Definition 155.3. Since $s' \rightsquigarrow s$ the point s' corresponds to a point of $\mathrm{Spec}(\mathcal{O}_{S,s})$ and hence there is at least one point (and potentially many points) of $\mathrm{Spec}(A)$ over s' whose residue field is a separable algebraic extension of $\kappa(s')$. Since $\kappa(\overline{s}')$ is algebraically closed we can choose a morphism $\varphi : \overline{s}' \to \mathrm{Spec}(A)$ giving rise to a commutative diagram

$$\begin{tikzcd} \overline{s}' \arrow[r, "\varphi"'] \arrow[dr] & \mathrm{Spec}(A) \arrow[d] & \overline{s} \arrow[l] \arrow[dl] \\ & S & \end{tikzcd}$$

The specialization map is the composition

$$\pi_1(X_{\overline{s}'}) \longrightarrow \pi_1(X_A) = \pi_1(X_{\kappa(s)^{sep}}) = \pi_1(X_{\overline{s}})$$

where the first equality is Lemma 9.1 and the second follows from Lemmas 14.2 and 9.3. By construction the specialization map fits into a commutative diagram

$$\begin{tikzcd} \pi_1(X_{\overline{s}'}) \arrow[rr, "sp"'] \arrow[dr] & & \pi_1(X_{\overline{s}}) \arrow[dl] \\ & \pi_1(X) & \end{tikzcd}$$

provided that X is connected. The specialization map depends on the choice of $\varphi : \overline{s}' \to \mathrm{Spec}(A)$ above and we will write sp_φ if we want to indicate this.

Lemma 16.1. *Consider a commutative diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r] & S \end{tikzcd}$$

of schemes where f and g are proper with geometrically connected fibres. Let $t' \rightsquigarrow t$ be a specialization of points in T and consider a specialization map $sp : \pi_1(Y_{\overline{t}'}) \to$

$\pi_1(Y_{\overline{t}})$ as above. Then there is a commutative diagram

$$\begin{tikzcd} \pi_1(Y_{\overline{t}'}) \arrow[r, "sp"'] \arrow[d] & \pi_1(Y_{\overline{t}}) \arrow[d] \\ \pi_1(X_{\overline{s}'}) \arrow[r, "sp"] & \pi_1(X_{\overline{s}}) \end{tikzcd}$$

of specialization maps where $\overline{s}$ and $\overline{s}'$ are the images of $\overline{t}$ and $\overline{t}'$.

Proof. Let B be the strict henselization of $\mathcal{O}_{T,t}$ with respect to $\kappa(t) \subset \kappa(t)^{sep} \subset \kappa(\overline{t})$. Pick $\psi : \overline{t}' \to \text{Spec}(B)$ lifting $\overline{t}' \to T$ as in the construction of the specialization map. Let s and s' denote the images of t and t' in S. Let A be the strict henselization of $\mathcal{O}_{S,s}$ with respect to $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$. Since $\kappa(\overline{s}) = \kappa(\overline{t})$, by the functoriality of strict henselization (Algebra, Lemma 155.10) we obtain a ring map $A \to B$ fitting into the commutative diagram

$$\begin{tikzcd} \overline{t}' \arrow[r, "\psi"'] \arrow[d] & \text{Spec}(B) \arrow[r] \arrow[d] & T \arrow[d] \\ \overline{s}' \arrow[r, "\varphi"] & \text{Spec}(A) \arrow[r] & S \end{tikzcd}$$

Here the morphism $\varphi : \overline{s}' \to \text{Spec}(A)$ is simply taken to be the composition $\overline{t}' \to \text{Spec}(B) \to \text{Spec}(A)$. Applying base change we obtain a commutative diagram

$$\begin{tikzcd} Y_{\overline{t}'} \arrow[r] \arrow[d] & Y_B \arrow[d] \\ X_{\overline{s}'} \arrow[r] & X_A \end{tikzcd}$$

and from the construction of the specialization map the commutativity of this diagram implies the commutativity of the diagram of the lemma. $\square$

Lemma 16.2. *Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s'' \rightsquigarrow s' \rightsquigarrow s$ be specializations of points of S. A composition of specialization maps $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ is a specialization map $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}})$.*

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct the first specialization map. Let $\mathcal{O}_{S,s'} \to A'$ be the strict henselization constructed using $\kappa(s') \subset \kappa(\overline{s}')$. By functoriality of strict henselization, there is a map $A \to A'$ such that the composition with $A' \to \kappa(\overline{s}')$ is the given map (Algebra, Lemma 154.6). Next, let $A' \to \kappa(\overline{s}'')$ be the map used to construct the second specialization map. Then it is clear that the composition of the first and second specialization maps is the specialization map $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}})$ constructed using $A \to A' \to \kappa(\overline{s}'')$. $\square$

Let $X \to S$ be a proper morphism with geometrically connected fibres. Let R be a strictly henselian valuation ring with algebraically closed fraction field and let $\text{Spec}(R) \to S$ be a morphism. Let $\eta, s \in \text{Spec}(R)$ be the generic and closed point. Then we can consider the specialization map

$$sp_R : \pi_1(X_\eta) \to \pi_1(X_s)$$

for the base change $X_R/\operatorname{Spec}(R)$. Note that this makes sense as both η and s have algebraically closed residue fields.

Lemma 16.3. *Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization of points of S and let $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ be a specialization map. Then there exists a strictly henselian valuation ring R over S with algebraically closed fraction field such that sp is isomorphic to sp_R defined above.*

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct sp. Let $R \subset \kappa(\overline{s}')$ be a valuation ring with fraction field $\kappa(\overline{s}')$ dominating the image of A. See Algebra, Lemma 50.2. Observe that R is strictly henselian for example by Lemma 12.2 and Algebra, Lemma 50.3. Then the lemma is clear. $\square$

Let $X \to S$ be a proper morphism with geometrically connected fibres. Let R be a strictly henselian discrete valuation ring and let $\operatorname{Spec}(R) \to S$ be a morphism. Let $\eta, s \in \operatorname{Spec}(R)$ be the generic and closed point. Then we can consider the specialization map

$$sp_R : \pi_1(X_{\overline{\eta}}) \to \pi_1(X_s)$$

for the base change $X_R/\operatorname{Spec}(R)$. Note that this makes sense as s has algebraically closed residue field.

Lemma 16.4. *Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization of points of S and let $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ be a specialization map. If S is Noetherian, then there exists a strictly henselian discrete valuation ring R over S such that sp is isomorphic to sp_R defined above.*

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct sp. Let $R \subset \kappa(\overline{s}')$ be a discrete valuation ring dominating the image of A, see Algebra, Lemma 119.13. Choose a diagram of fields

$$\begin{tikzcd} \kappa(\overline{s}) \arrow[r] & k \\ A/\mathfrak{m}_A \arrow[u] \arrow[r] & R/\mathfrak{m}_R \arrow[u] \end{tikzcd}$$

with k algebraically closed. Let R^{sh} be the strict henselization of R constructed using $R \to k$. Then R^{sh} is a discrete valuation ring by More on Algebra, Lemma 45.11. Denote η, o the generic and closed point of $\operatorname{Spec}(R^{sh})$. Since the diagram of schemes

$$\begin{tikzcd} \overline{\eta} \arrow[r] \arrow[d] & \operatorname{Spec}(R^{sh}) \arrow[d] & \operatorname{Spec}(k) \arrow[l] \arrow[d] \\ \overline{s}' \arrow[r] & \operatorname{Spec}(A) & \overline{s} \arrow[l] \end{tikzcd}$$

commutes, we obtain a commutative diagram

$$\begin{tikzcd} \pi_1(X_{\overline{\eta}}) \arrow[r, "sp_{R^{sh}}"'] \arrow[d] & \pi_1(X_o) \arrow[d] \\ \pi_1(X_{\overline{s}'}) \arrow[r, "sp"] & X_{\overline{s}} \end{tikzcd}$$

of specialization maps by the construction of these maps. Since the vertical arrows are isomorphisms (Lemma 9.3), this proves the lemma. ☐

17. Restriction to a closed subscheme

In this section we prove some results about the restriction functor

$$F\acute{E}t_X \longrightarrow F\acute{E}t_Y, \quad U \longmapsto V = U \times_X Y$$

where X is a scheme and Y is a closed subscheme. Using the topological invariance of the fundamental group, we can relate the study of this functor to the completion functor on finite locally free modules.

In the following lemmas we use the concept of coherent formal modules defined in Cohomology of Schemes, Section 23. Given a Noetherian scheme and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we will say an object $(\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$ is *finite locally free* if each $\mathcal{F}_n$ is a finite locally free $\mathcal{O}_X/\mathcal{I}^n$-module.

Lemma 17.1. *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Assume the completion functor*

$$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects (see above). Then the restriction functor $F\acute{E}t_X \to F\acute{E}t_Y$ is fully faithful.

Proof. Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given U finite étale over X and a morphism $t : Y \to U$ over X there exists a unique section $s : X \to U$ such that $t = s|_Y$. Picture

$$\begin{tikzcd} & U \arrow[d, "f"] \\ Y \arrow[ur] \arrow[r] & X \arrow[u, dotted, bend left] \end{tikzcd}$$

Finding the dotted arrow s is the same thing as finding an $\mathcal{O}_X$-algebra map

$$s^\sharp : f_*\mathcal{O}_U \longrightarrow \mathcal{O}_X$$

which reduces modulo the ideal sheaf of Y to the given algebra map $t^\sharp : f_*\mathcal{O}_U \to \mathcal{O}_Y$. By Lemma 8.3 we can lift t uniquely to a compatible system of maps $t_n : Y_n \to U$ and hence a map

$$\lim t_n^\sharp : f_*\mathcal{O}_U \longrightarrow \lim \mathcal{O}_{Y_n}$$

of sheaves of algebras on X. Since $f_*\mathcal{O}_U$ is a finite locally free $\mathcal{O}_X$-module, we conclude that we get a unique $\mathcal{O}_X$-module map $\sigma : f_*\mathcal{O}_U \to \mathcal{O}_X$ whose completion is $\lim t_n^\sharp$. To see that σ is an algebra homomorphism, we need to check that the diagram

$$\begin{tikzcd} f_*\mathcal{O}_U \otimes_{\mathcal{O}_X} f_*\mathcal{O}_U \arrow[r] \arrow[d, "\sigma\otimes\sigma"'] & f_*\mathcal{O}_U \arrow[d, "\sigma"] \\ \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

commutes. For every n we know this diagram commutes after restricting to Y_n, i.e., the diagram commutes after applying the completion functor. Hence by faithfulness of the completion functor we conclude. ☐

Lemma 17.2. *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Assume the completion functor*

$$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is an equivalence on full subcategories of finite locally free objects (see above). Then the restriction functor $F\acute{E}t_X \to F\acute{E}t_Y$ is an equivalence.

Proof. The restriction functor is fully faithful by Lemma 17.1.

Let $U_1 \to Y$ be a finite étale morphism. To finish the proof we will show that U_1 is in the essential image of the restriction functor.

For $n \geq 1$ let Y_n be the nth infinitesimal neighbourhood of Y. By Lemma 8.3 there is a unique finite étale morphism $\pi_n : U_n \to Y_n$ whose base change to $Y = Y_1$ recovers $U_1 \to Y_1$. Consider the sheaves $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$. We may and do view $\mathcal{F}_n$ as an $\mathcal{O}_X$-module on X wich is locally isomorphic to $(\mathcal{O}_X/f^{n+1}\mathcal{O}_X)^{\oplus r}$. This $(\mathcal{F}_n)$ is a finite locally free object of $Coh(X, \mathcal{I})$. By assumption there exists a finite locally free $\mathcal{O}_X$-module $\mathcal{F}$ and a compatible system of isomorphisms

$$\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$$

of $\mathcal{O}_X$-modules.

To construct an algebra structure on $\mathcal{F}$ consider the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_X} \mathcal{F}_n \to \mathcal{F}_n$ coming from the fact that $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$ are sheaves of algebras. These define a map

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F})^\wedge \longrightarrow \mathcal{F}^\wedge$$

in the category $Coh(X, \mathcal{I})$. Hence by assumption we may assume there is a map $\mu : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ whose restriction to Y_n gives the multiplication maps above. By faithfulness of the functor in the statement of the lemma, we conclude that μ defines a commutative $\mathcal{O}_X$-algebra structure on $\mathcal{F}$ compatible with the given algebra structures on $\mathcal{F}_n$. Setting

$$U = \underline{\mathrm{Spec}}_X((\mathcal{F}, \mu))$$

we obtain a finite locally free scheme $\pi : U \to X$ whose restriction to Y is isomorphic to U_1. The the discriminant of π is the zero set of the section

$$\det(Q_\pi) : \mathcal{O}_X \longrightarrow \wedge^{top}(\pi_*\mathcal{O}_U)^{\otimes -2}$$

constructed in Discriminants, Section 3. Since the restriction of this to Y_n is an isomorphism for all n by Discriminants, Lemma 3.1 we conclude that it is an isomorphism. Thus π is étale by Discriminants, Lemma 3.1. $\square$

Lemma 17.3. *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $\mathcal{V}$ be the set of open subschemes $V \subset X$ containing Y ordered by reverse inclusion. Assume the completion functor*

$$\mathrm{colim}_{\mathcal{V}}\, Coh(\mathcal{O}_V) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

defines is fully faithful on the full subcategory of finite locally free objects (see above). Then the restriction functor $\mathrm{colim}_{\mathcal{V}}\, F\acute{E}t_V \to F\acute{E}t_Y$ is fully faithful.

Proof. Observe that $\mathcal{V}$ is a directed set, so the colimits are as in Categories, Section 19. The rest of the argument is almost exactly the same as the argument in the proof of Lemma 17.1; we urge the reader to skip it.

Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given U finite étale over $V \in \mathcal{V}$ and a morphism $t : Y \to U$ over V there exists a $V' \geq V$ and a morphism $s : V' \to U$ over V such that $t = s|_Y$. Picture

$$\begin{tikzcd} & & U \arrow[d, "f"] \\ Y \arrow[r] \arrow[urr] & V' \arrow[r] \arrow[ur, dotted] & V \end{tikzcd}$$

Finding the dotted arrow s is the same thing as finding an $\mathcal{O}_{V'}$-algebra map

$$s^\sharp : f_*\mathcal{O}_U|_{V'} \longrightarrow \mathcal{O}_{V'}$$

which reduces modulo the ideal sheaf of Y to the given algebra map $t^\sharp : f_*\mathcal{O}_U \to \mathcal{O}_Y$. By Lemma 8.3 we can lift t uniquely to a compatible system of maps $t_n : Y_n \to U$ and hence a map

$$\lim t_n^\sharp : f_*\mathcal{O}_U \longrightarrow \lim \mathcal{O}_{Y_n}$$

of sheaves of algebras on V. Observe that $f_*\mathcal{O}_U$ is a finite locally free $\mathcal{O}_V$-module. Hence we get a $V' \geq V$ a map $\sigma : f_*\mathcal{O}_U|_{V'} \to \mathcal{O}_{V'}$ whose completion is $\lim t_n^\sharp$. To see that σ is an algebra homomorphism, we need to check that the diagram

$$\begin{tikzcd} (f_*\mathcal{O}_U \otimes_{\mathcal{O}_V} f_*\mathcal{O}_U)|_{V'} \arrow[r] \arrow[d, "\sigma\otimes\sigma"'] & f_*\mathcal{O}_U|_{V'} \arrow[d, "\sigma"] \\ \mathcal{O}_{V'} \otimes_{\mathcal{O}_{V'}} \mathcal{O}_{V'} \arrow[r] & \mathcal{O}_{V'} \end{tikzcd}$$

commutes. For every n we know this diagram commutes after restricting to Y_n, i.e., the diagram commutes after applying the completion functor. Hence by faithfulness of the completion functor we deduce that there exists a $V'' \geq V'$ such that $\sigma|_{V''}$ is an algebra homomorphism as desired. $\square$

Lemma 17.4. *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $\mathcal{V}$ be the set of open subschemes $V \subset X$ containing Y ordered by reverse inclusion. Assume the completion functor*

$$\mathrm{colim}_{\mathcal{V}}\, Coh(\mathcal{O}_V) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

defines an equivalence of the full subcategories of finite locally free objects (see explanation above). Then the restriction functor

$$\mathrm{colim}_{\mathcal{V}}\, F\acute{E}t_V \to F\acute{E}t_Y$$

is an equivalence.

Proof. Observe that $\mathcal{V}$ is a directed set, so the colimits are as in Categories, Section 19. The rest of the argument is almost exactly the same as the argument in the proof of Lemma 17.2; we urge the reader to skip it.

The restriction functor is fully faithful by Lemma 17.3.

Let $U_1 \to Y$ be a finite étale morphism. To finish the proof we will show that U_1 is in the essential image of the restriction functor.

For $n \geq 1$ let Y_n be the nth infinitesimal neighbourhood of Y. By Lemma 8.3 there is a unique finite étale morphism $\pi_n : U_n \to Y_n$ whose base change to $Y = Y_1$ recovers $U_1 \to Y_1$. Consider the sheaves $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$. We may and do view $\mathcal{F}_n$

as an $\mathcal{O}_X$-module on X wich is locally isomorphic to $(\mathcal{O}_X/f^{n+1}\mathcal{O}_X)^{\oplus r}$. This $(\mathcal{F}_n)$ is a finite locally free object of $Coh(X, \mathcal{I})$. By assumption there exists a $V \in \mathcal{V}$ and a finite locally free $\mathcal{O}_V$-module $\mathcal{F}$ and a compatible system of isomorphisms

$$\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$$

of $\mathcal{O}_V$-modules.

To construct an algebra structure on $\mathcal{F}$ consider the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_V} \mathcal{F}_n \to \mathcal{F}_n$ coming from the fact that $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$ are sheaves of algebras. These define a map

$$(\mathcal{F} \otimes_{\mathcal{O}_V} \mathcal{F})^\wedge \longrightarrow \mathcal{F}^\wedge$$

in the category $Coh(X, \mathcal{I})$. Hence by assumption after shrinking V we may assume there is a map $\mu : \mathcal{F} \otimes_{\mathcal{O}_V} \mathcal{F} \to \mathcal{F}$ whose restriction to Y_n gives the multiplication maps above. After possibly shrinking further we may assume μ defines a commutative $\mathcal{O}_V$-algebra structure on $\mathcal{F}$ compatible with the given algebra structures on $\mathcal{F}_n$. Setting

$$U = \underline{\mathrm{Spec}}_V((\mathcal{F}, \mu))$$

we obtain a finite locally free scheme over V whose restriction to Y is isomorphic to U_1. It follows that $U \to V$ is étale at all points lying over Y, see More on Morphisms, Lemma 12.3. Thus after shrinking V once more we may assume $U \to V$ is finite étale. This finishes the proof. □

Lemma 17.5. *Let X be a scheme and let $Y \subset X$ be a closed subscheme. If every connected component of X meets Y, then the restriction functor $\textit{FÉt}_X \to \textit{FÉt}_Y$ is faithful.*

Proof. Let $a, b : U \to U'$ be two morphisms of schemes finite étale over X whose restriction to Y are the same. The image of a connected component of U is an connected component of X; this follows from Topology, Lemma 7.7 applied to the restriction of $U \to X$ to a connected component of X. Hence the image of every connected component of U meets Y by assumption. We conclude that $a = b$ after restriction to each connected component of U by Étale Morphisms, Proposition 6.3. Since the equalizer of a and b is an open subscheme of U (as the diagonal of U' over X is open) we conclude. □

Lemma 17.6. *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Assume one of the following holds*

1. *X is quasi-affine and $\Gamma(X, \mathcal{O}_X) \to \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$ is an isomorphism, or*
2. *X has an ample invertible module $\mathcal{L}$ and $\Gamma(X, \mathcal{L}^{\otimes m}) \to \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$ is an isomorphism for all $m \gg 0$, or*
3. *for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ the map $\Gamma(X, \mathcal{E}) \to \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$ is an isomorphism.*

Then the restriction functor $\textit{FÉt}_X \to \textit{FÉt}_Y$ is fully faithful.

Proof. This lemma follows formally from Lemma 17.1 and Algebraic and Formal Geometry, Lemma 15.1. □

Lemma 17.7. *Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Let $Y_n \subset X$ be the nth infinitesimal neighbourhood of Y in X. Let $\mathcal{V}$ be the set of* *open subschemes $V \subset X$ containing Y ordered by reverse inclusion. Assume one of the following holds*

(1) *X is quasi-affine and*

$$\mathrm{colim}_{\mathcal{V}}\, \Gamma(V, \mathcal{O}_V) \longrightarrow \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$$

is an isomorphism, or

(2) *X has an ample invertible module $\mathcal{L}$ and*

$$\mathrm{colim}_{\mathcal{V}}\, \Gamma(V, \mathcal{L}^{\otimes m}) \longrightarrow \lim \Gamma(Y_n, \mathcal{L}^{\otimes m}|_{Y_n})$$

is an isomorphism for all $m \gg 0$, or

(3) *for every $V \in \mathcal{V}$ and every finite locally free $\mathcal{O}_V$-module $\mathcal{E}$ the map*

$$\mathrm{colim}_{V' \geq V}\, \Gamma(V', \mathcal{E}|_{V'}) \longrightarrow \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$$

is an isomorphism.

Then the functor

$$\mathrm{colim}_{\mathcal{V}}\, \textit{FÉt}_V \to \textit{FÉt}_Y$$

is fully faithful.

Proof. This lemma follows formally from Lemma 17.3 and Algebraic and Formal Geometry, Lemma 15.2. □

18. Pushouts and fundamental groups

Here is the main result.

Lemma 18.1. *In More on Morphisms, Situation 67.1, for example if $Z \to Y$ and $Z \to X$ are closed immersions of schemes, there is an equivalence of categories*

$$\textit{FÉt}_{Y \amalg_Z X} \longrightarrow \textit{FÉt}_Y \times_{\textit{FÉt}_Z} \textit{FÉt}_X$$

Proof. The pushout exists by More on Morphisms, Proposition 67.3. The functor is given by sending a scheme U finite étale over the pushout to the base changes $Y' = U \times_{Y \amalg_Z X} Y$ and $X' = U \times_{Y \amalg_Z X} X$ and the natural isomorphism $Y' \times_Y Z \to X' \times_X Z$ over Z. To prove this functor is an equivalence we use More on Morphisms, Lemma 67.7 to construct a quasi-inverse functor. The only thing left to prove is to show that given a morphism $U \to Y \amalg_Z X$ which is separated, quasi-finite and étale such that $X' \to X$ and $Y' \to Y$ are finite, then $U \to Y \amalg_Z X$ is finite. This can either be deduced from the corresponding algebra fact (More on Algebra, Lemma 6.7) or it can be seen because

$$X' \amalg Y' \to U$$

is surjective and X' and Y' are proper over $Y \amalg_Z X$ (this uses the description of the pushout in More on Morphisms, Proposition 67.3) and then we can apply Morphisms, Lemma 41.10 to conclude that U is proper over $Y \amalg_Z X$. Since a quasi-finite and proper morphism is finite (More on Morphisms, Lemma 44.1) we win. □

19. Finite étale covers of punctured spectra, I

We first prove some results á la Lefschetz.

Situation 19.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. We set $X = \mathrm{Spec}(A)$ and $X_0 = \mathrm{Spec}(A/fA)$ and we let $U = X \setminus \{\mathfrak{m}\}$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A and A/fA.

Recall that for a scheme X the category of schemes finite étale over X is denoted $\textit{FÉt}_X$, see Section 5. In Situation 19.1 we will study the base change functors

$$\begin{tikzcd} \textit{FÉt}_X \arrow[r] \arrow[d] & \textit{FÉt}_U \arrow[d] \\ \textit{FÉt}_{X_0} \arrow[r] & \textit{FÉt}_{U_0} \end{tikzcd}$$

In many case the right vertical arrow is faithful.

Lemma 19.2. *In Situation 19.1. Assume one of the following holds*

(1) $\dim(A/\mathfrak{p}) \geq 2$ *for every minimal prime* $\mathfrak{p} \subset A$ *with* $f \notin \mathfrak{p}$, *or*
(2) *every connected component of* U *meets* U_0.

Then

$$\textit{FÉt}_U \longrightarrow \textit{FÉt}_{U_0}, \quad V \longmapsto V_0 = V \times_U U_0$$

is a faithful functor.

Proof. Case (2) is immediate from Lemma 17.5. Assumption (1) implies every irreducible component of U meets U_0, see Algebra, Lemma 60.13. Hence (1) follows from (2). $\square$

Before we prove something more interesting, we need a couple of lemmas.

Lemma 19.3. *In Situation 19.1. Let* $V \to U$ *be a finite morphism. Let* $A^\wedge$ *be the* $\mathfrak{m}$*-adic completion of* A, *let* $X' = \mathrm{Spec}(A^\wedge)$ *and let* U' *and* V' *be the base changes of* U *and* V *to* X'. *If* $Y' \to X'$ *is a finite morphism such that* $V' = Y' \times_{X'} U'$, *then there exists a finite morphism* $Y \to X$ *such that* $V = Y \times_X U$ *and* $Y' = Y \times_X X'$.

Proof. This is a straightforward application of More on Algebra, Proposition 89.15. Namely, choose generators $f_1, \ldots, f_t$ of $\mathfrak{m}$. For each i write $V \times_U D(f_i) = \mathrm{Spec}(B_i)$. For $1 \leq i, j \leq n$ we obtain an isomorphism $\alpha_{ij} : (B_i)_{f_j} \to (B_j)_{f_i}$ of $A_{f_i f_j}$-algebras because the spectrum of both represent $V \times_U D(f_i f_j)$. Write $Y' = \mathrm{Spec}(B')$. Since $V \times_U U' = Y \times_{X'} U'$ we get isomorphisms $\alpha_i : B'_{f_i} \to B_i \otimes_A A^\wedge$. A straightforward argument shows that $(B', B_i, \alpha_i, \alpha_{ij})$ is an object of $\mathrm{Glue}(A \to A^\wedge, f_1, \ldots, f_t)$, see More on Algebra, Remark 89.10. Applying the proposition cited above (and using More on Algebra, Remark 89.19 to obtain the algebra structure) we find an A-algebra B such that $\mathrm{Can}(B)$ is isomorphic to $(B', B_i, \alpha_i, \alpha_{ij})$. Setting $Y = \mathrm{Spec}(B)$ we see that $Y \to X$ is a morphism which comes equipped with compatible isomorphisms $V \cong Y \times_X U$ and $Y' = Y \times_X X'$ as desired. $\square$

Lemma 19.4. *In Situation 19.1 assume* A *is henselian or more generally that* $(A, (f))$ *is a henselian pair. Let* $A^\wedge$ *be the* $\mathfrak{m}$*-adic completion of* A, *let* $X' = \mathrm{Spec}(A^\wedge)$ *and let* U' *and* U'_0 *be the base changes of* U *and* U_0 *to* X'. *If* $\textit{FÉt}_{U'} \to \textit{FÉt}_{U'_0}$ *is fully faithful, then* $\textit{FÉt}_U \to \textit{FÉt}_{U_0}$ *is fully faithful.*

Proof. Assume $F\acute{E}t_{U'} \longrightarrow F\acute{E}t_{U'_0}$ is a fully faithful. Since $X' \to X$ is faithfully flat, it is immediate that the functor $V \to V_0 = V \times_U U_0$ is faithful. Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given V finite étale over U we have

$$\mathrm{Mor}_U(U, V) = \mathrm{Mor}_{U_0}(U_0, V_0)$$

The we assume we have a morphism $s_0 : U_0 \to V_0$ over U_0 and we will produce a morphism $s : U \to V$ over U.

By our assumption there does exist a morphism $s' : U' \to V'$ whose restriction to V'_0 is the base change s'_0 of s_0. Since $V' \to U'$ is finite étale this means that $V' = s'(U') \amalg W'$ for some $W' \to U'$ finite and étale. Choose a finite morphism $Z' \to X'$ such that $W' = Z' \times_{X'} U'$. This is possible by Zariski's main theorem in the form stated in More on Morphisms, Lemma 43.3 (small detail omitted). Then

$$V' = s'(U') \amalg W' \longrightarrow X' \amalg Z' = Y'$$

is an open immersion such that $V' = Y' \times_{X'} U'$. By Lemma 19.3 we can find $Y \to X$ finite such that $V = Y \times_X U$ and $Y' = Y \times_X X'$. Write $Y = \mathrm{Spec}(B)$ so that $Y' = \mathrm{Spec}(B \otimes_A A^\wedge)$. Then $B \otimes_A A^\wedge$ has an idempotent e' corresponding to the open and closed subscheme X' of $Y' = X' \amalg Z'$.

The case A is henselian (slightly easier). The image $\overline{e}$ of e' in $B \otimes_A \kappa(\mathfrak{m}) = B/\mathfrak{m}B$ lifts to an idempotent e of B as A is henselian (because B is a product of local rings by Algebra, Lemma 153.3). Then we see that e maps to e' by uniqueness of lifts of idempotents (using that $B \otimes_A A^\wedge$ is a product of local rings). Let $Y_1 \subset Y$ be the open and closed subscheme corresponding to e. Then $Y_1 \times_X X' = s'(X')$ which implies that $Y_1 \to X$ is an isomorphism (by faithfully flat descent) and gives the desired section.

The case where $(A, (f))$ is a henselian pair. Here we use that s' is a lift of s'_0. Namely, let $Y_{0,1} \subset Y_0 = Y \times_X X_0$ be the closure of $s_0(U_0) \subset V_0 = Y_0 \times_{X_0} U_0$. As $X' \to X$ is flat, the base change $Y'_{0,1} \subset Y'_0$ is the closure of $s'_0(U'_0)$ which is equal to $X'_0 \subset Y'_0$ (see Morphisms, Lemma 25.16). Since $Y'_0 \to Y_0$ is submersive (Morphisms, Lemma 25.12) we conclude that $Y_{0,1}$ is open and closed in Y_0. Let $e_0 \in B/fB$ be the corresponding idempotent. By More on Algebra, Lemma 11.6 we can lift e_0 to an idempotent $e \in B$. Then we conclude as before. $\square$

In Situation 19.1 fully faithfulness of the restriction functor $F\acute{E}t_U \longrightarrow F\acute{E}t_{U_0}$ holds under fairly mild assumptions. In particular, the assumptions often do not imply U is a connected scheme, but the conclusion guarantees that U and U_0 have the same number of connected components.

Lemma 19.5. *In Situation 19.1. Assume*

(a) *A has a dualizing complex,*
(b) *the pair $(A, (f))$ is henselian,*
(c) *one of the following is true*
    (i) *A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 3, or*
    (ii) *for every prime $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ we have depth$(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 2$.*

Then the restriction functor $F\acute{E}t_U \longrightarrow F\acute{E}t_{U_0}$ is fully faithful.

Proof. Let A' be the $\mathfrak{m}$-adic completion of A. We will show that the hypotheses remain true for A'. This is clear for conditions (a) and (b). Condition (c)(ii) is preserved by Local Cohomology, Lemma 11.3. Next, assume (c)(i) holds. Since A is universally catenary (Dualizing Complexes, Lemma 17.4) we see that every irreducible component of $\text{Spec}(A')$ not contained in $V(f)$ has dimension ≥ 3, see More on Algebra, Proposition 109.5. Since $A \to A'$ is flat with Gorenstein fibres, the condition that A_f is (S_2) implies that A'_f is (S_2). References used: Dualizing Complexes, Section 23, More on Algebra, Section 51, and Algebra, Lemma 163.4. Thus by Lemma 19.4 we may assume that A is a Noetherian complete local ring.

Assume A is a complete local ring in addition to the other assumptions. By Lemma 17.1 the result follows from Algebraic and Formal Geometry, Lemma 15.6. $\square$

Lemma 19.6. *In Situation 19.1. Assume*

1. *$H^1_\mathfrak{m}(A)$ and $H^2_\mathfrak{m}(A)$ are annihilated by a power of f, and*
2. *A is henselian or more generally $(A,(f))$ is a henselian pair.*

Then the restriction functor $\textit{FÉt}_U \longrightarrow \textit{FÉt}_{U_0}$ is fully faithful.

Proof. By Lemma 19.4 we may assume that A is a Noetherian complete local ring. (The assumptions carry over; use Dualizing Complexes, Lemma 9.3.) By Lemma 17.1 the result follows from Algebraic and Formal Geometry, Lemma 15.5. $\square$

Lemma 19.7. *In Situation 19.1 assume A has depth ≥ 3 and A is henselian or more generally $(A,(f))$ is a henselian pair. Then the restriction functor $\textit{FÉt}_U \to \textit{FÉt}_{U_0}$ is fully faithful.*

Proof. The assumption of depth forces $H^1_\mathfrak{m}(A) = H^2_\mathfrak{m}(A) = 0$, see Dualizing Complexes, Lemma 11.1. Hence Lemma 19.6 applies. $\square$

20. Purity in local case, I

Let $(A,\mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$ be the punctured spectrum. We say *purity holds for* $(A,\mathfrak{m})$ if the restriction functor

$$\textit{FÉt}_X \longrightarrow \textit{FÉt}_U$$

is essentially surjective. In this section we try to understand how the question changes when one passes from X to a hypersurface X_0 in X, in other words, we study a kind of local Lefschetz property for the fundamental groups of punctured spectra. These results will be useful to proceed by induction on dimension in the proofs of our main results on local purity, namely, Lemma 21.3, Proposition 25.3, and Proposition 26.4.

Lemma 20.1. *Let $(A,\mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Let $\pi : Y \to X$ be a finite morphism such that depth$(\mathcal{O}_{Y,y}) \geq 2$ for all closed points $y \in Y$. Then Y is the spectrum of $B = \mathcal{O}_Y(\pi^{-1}(U))$.*

Proof. Set $V = \pi^{-1}(U)$ and denote $\pi' : V \to U$ the restriction of π. Consider the $\mathcal{O}_X$-module map

$$\pi_*\mathcal{O}_Y \longrightarrow j_*\pi'_*\mathcal{O}_V$$

where $j : U \to X$ is the inclusion morphism. We claim Divisors, Lemma 5.11 applies to this map. If so, then $B = \Gamma(Y, \mathcal{O}_Y)$ and we see that the lemma holds. Let $x \in X$ be the closed point. It suffices to show that $\text{depth}((\pi_*\mathcal{O}_Y)_x) \geq 2$. Let $y_1, \ldots, y_n \in Y$ be the points mapping to x. By Algebra, Lemma 72.11 it suffices to show that $\text{depth}(\mathcal{O}_{Y,y_i}) \geq 2$ for $i = 1, \ldots, n$. Since this is the assumption of the lemma the proof is complete. □

Lemma 20.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Let V be finite étale over U. Assume A has depth ≥ 2. The following are equivalent*

(1) *$V = Y \times_X U$ for some $Y \to X$ finite étale,*
(2) *$B = \Gamma(V, \mathcal{O}_V)$ is finite étale over A.*

Proof. Denote $\pi : V \to U$ the given finite étale morphism. Assume Y as in (1) exists. Let $x \in X$ be the point corresponding to $\mathfrak{m}$. Let $y \in Y$ be a point mapping to x. We claim that $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$. This is true because $Y \to X$ is étale and hence $A = \mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ have the same depth (Algebra, Lemma 163.2). Hence Lemma 20.1 applies and $Y = \text{Spec}(B)$.

The implication (2) ⇒ (1) is easier and the details are omitted. □

Lemma 20.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Assume A is normal of dimension ≥ 2. The functor*

$$\textit{FÉt}_U \longrightarrow \left\{ \begin{matrix} \textit{finite normal } A\textit{-algebras } B \textit{ such} \\ \textit{that } \text{Spec}(B) \to X \textit{ is étale over } U \end{matrix} \right\}, \quad V \longmapsto \Gamma(V, \mathcal{O}_V)$$

is an equivalence. Moreover, $V = Y \times_X U$ for some $Y \to X$ finite étale if and only if $B = \Gamma(V, \mathcal{O}_V)$ is finite étale over A.

Proof. Observe that $\text{depth}(A) \geq 2$ because A is normal (Serre's criterion for normality, Algebra, Lemma 157.4). Thus the final statement follows from Lemma 20.2. Given $\pi : V \to U$ finite étale, set $B = \Gamma(V, \mathcal{O}_V)$. If we can show that B is normal and finite over A, then we obtain the displayed functor. Since there is an obvious quasi-inverse functor, this is also all that we have to show.

Since A is normal, the scheme V is normal (Descent, Lemma 18.2). Hence V is a finite disjoint union of integral schemes (Properties, Lemma 7.6). Thus we may assume V is integral. In this case the function field L of V (Morphisms, Section 49) is a finite separable extension of the fraction field of A (because we get it by looking at the generic fibre of $V \to U$ and using Morphisms, Lemma 36.7). By Algebra, Lemma 161.8 the integral closure $B' \subset L$ of A in L is finite over A. By More on Algebra, Lemma 23.20 we see that B' is a reflexive A-module, which in turn implies that $\text{depth}_A(B') \geq 2$ by More on Algebra, Lemma 23.18.

Let $f \in \mathfrak{m}$. Then $B_f = \Gamma(V \times_U D(f), \mathcal{O}_V)$ (Properties, Lemma 17.1). Hence $B'_f = B_f$ because B_f is normal (see above), finite over A_f with fraction field L. It follows that $V = \text{Spec}(B') \times_X U$. Then we conclude that $B = B'$ from Lemma 20.1 applied to $\text{Spec}(B') \to X$. This lemma applies because the localizations $B'_{\mathfrak{m}'}$ of B' at maximal ideals $\mathfrak{m}' \subset B'$ lying over $\mathfrak{m}$ have depth ≥ 2 by Algebra, Lemma 72.11 and the remark on depth in the preceding paragraph. □

Lemma 20.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Let V be finite étale over U. Let $A^\wedge$ be the $\mathfrak{m}$-adic completion of A, let $X' = \text{Spec}(A^\wedge)$ and let U' and V' be the base changes of U and V to X'. The following are equivalent*

(1) *$V = Y \times_X U$ for some $Y \to X$ finite étale, and*
(2) *$V' = Y' \times_{X'} U'$ for some $Y' \to X'$ finite étale.*

Proof. The implication (1) $\Rightarrow$ (2) follows from taking the base change of a solution $Y \to X$. Let $Y' \to X'$ be as in (2). By Lemma 19.3 we can find $Y \to X$ finite such that $V = Y \times_X U$ and $Y' = Y \times_X X'$. By descent we see that $Y \to X$ is finite étale (Algebra, Lemmas 83.2 and 143.3). This finishes the proof. $\square$

The point of the following two lemmas is that the assumptions do not force A to have depth ≥ 3. For example if A is a complete normal local domain of dimension ≥ 3 and $f \in \mathfrak{m}$ is nonzero, then the assumptions are satisfied.

Lemma 20.5. *In Situation 19.1. Let V be finite étale over U. Assume*

(a) *A has a dualizing complex,*
(b) *the pair $(A, (f))$ is henselian,*
(c) *one of the following is true*
  (i) *A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 3, or*
  (ii) *for every prime $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ we have depth$(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 2$.*
(d) *$V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.*

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Proof. We reduce to the complete case using Lemma 20.4. (The assumptions carry over; see proof of Lemma 19.5.)

In the complete case we can lift $Y_0 \to X_0$ to a finite étale morphism $Y \to X$ by More on Algebra, Lemma 13.2; observe that (A, fA) is a henselian pair by More on Algebra, Lemma 11.4. Then we can use Lemma 19.5 to see that V is isomorphic to $Y \times_X U$ and the proof is complete. $\square$

Lemma 20.6. *In Situation 19.1. Let V be finite étale over U. Assume*

(1) *$H^1_\mathfrak{m}(A)$ and $H^2_\mathfrak{m}(A)$ are annihilated by a power of f,*
(2) *$V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.*

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Proof. We reduce to the complete case using Lemma 20.4. (The assumptions carry over; use Dualizing Complexes, Lemma 9.3.)

In the complete case we can lift $Y_0 \to X_0$ to a finite étale morphism $Y \to X$ by More on Algebra, Lemma 13.2; observe that (A, fA) is a henselian pair by More on Algebra, Lemma 11.4. Then we can use Lemma 19.6 to see that V is isomorphic to $Y \times_X U$ and the proof is complete. $\square$

Lemma 20.7. *In Situation 19.1. Let V be finite étale over U. Assume*

(1) *A has depth ≥ 3,*
(2) *$V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.*

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Proof. The assumption of depth forces $H^1_\mathfrak{m}(A) = H^2_\mathfrak{m}(A) = 0$, see Dualizing Complexes, Lemma 11.1. Hence Lemma 20.6 applies. $\square$

21. Purity of branch locus

We will use the discriminant of a finite locally free morphism. See Discriminants, Section 3.

Lemma 21.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\dim(A) \geq 1$. Let $f \in \mathfrak{m}$. Then there exist a $\mathfrak{p} \in V(f)$ with $\dim(A_\mathfrak{p}) = 1$.*

Proof. By induction on $\dim(A)$. If $\dim(A) = 1$, then $\mathfrak{p} = \mathfrak{m}$ works. If $\dim(A) > 1$, then let $Z \subset \text{Spec}(A)$ be an irreducible component of dimension > 1. Then $V(f) \cap Z$ has dimension > 0 (Algebra, Lemma 60.13). Pick a prime $\mathfrak{q} \in V(f) \cap Z$, $\mathfrak{q} \neq \mathfrak{m}$ corresponding to a closed point of the punctured spectrum of A; this is possible by Properties, Lemma 6.4. Then $\mathfrak{q}$ is not the generic point of Z. Hence $0 < \dim(A_\mathfrak{q}) < \dim(A)$ and $f \in \mathfrak{q}A_\mathfrak{q}$. By induction on the dimension we can find $f \in \mathfrak{p} \subset A_\mathfrak{q}$ with $\dim((A_\mathfrak{q})_\mathfrak{p}) = 1$. Then $\mathfrak{p} \cap A$ works. □

Lemma 21.2. *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$. Assume*

(1) *f is flat,*
(2) *f is quasi-finite at x,*
(3) *x is not a generic point of an irreducible component of X,*
(4) *for specializations $x' \rightsquigarrow x$ with $\dim(\mathcal{O}_{X,x'}) = 1$ our f is unramified at x'.*

Then f is étale at x.

Proof. Observe that the set of points where f is unramified is the same as the set of points where f is étale and that this set is open. See Morphisms, Definitions 35.1 and 36.1 and Lemma 36.16. To check f is étale at x we may work étale locally on the base and on the target (Descent, Lemmas 23.29 and 31.1). Thus we can apply More on Morphisms, Lemma 41.1 and assume that $f : X \to Y$ is finite and that x is the unique point of X lying over $y = f(x)$. Then it follows that f is finite locally free (Morphisms, Lemma 48.2).

Assume f is finite locally free and that x is the unique point of X lying over $y = f(x)$. By Discriminants, Lemma 3.1 we find a locally principal closed subscheme $D_\pi \subset Y$ such that $y' \in D_\pi$ if and only if there exists an $x' \in X$ with $f(x') = y'$ and f ramified at x'. Thus we have to prove that $y \notin D_\pi$. Assume $y \in D_\pi$ to get a contradiction.

By condition (3) we have $\dim(\mathcal{O}_{X,x}) \geq 1$. We have $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$ by Algebra, Lemma 112.7. By Lemma 21.1 we can find $y' \in D_\pi$ specializing to y with $\dim(\mathcal{O}_{Y,y'}) = 1$. Choose $x' \in X$ with $f(x') = y'$ where f is ramified. Since f is finite it is closed, and hence $x' \rightsquigarrow x$. We have $\dim(\mathcal{O}_{X,x'}) = \dim(\mathcal{O}_{Y,y'}) = 1$ as before. This contradicts property (4). □

Lemma 21.3. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$. Set $X = \text{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$. Then*

(1) *the functor $\textit{FÉt}_X \to \textit{FÉt}_U$ is essentially surjective, i.e., purity holds for A,*
(2) *any finite $A \to B$ with B normal which induces a finite étale morphism on punctured spectra is étale.*

Proof. Recall that a regular local ring is normal by Algebra, Lemma 157.5. Hence (1) and (2) are equivalent by Lemma 20.3. We prove the lemma by induction on d.

The case $d = 2$. In this case $A \to B$ is flat. Namely, we have going down for $A \to B$ by Algebra, Proposition 38.7. Then $\dim(B_{\mathfrak{m}'}) = 2$ for all maximal ideals $\mathfrak{m}' \subset B$ by Algebra, Lemma 112.7. Then $B_{\mathfrak{m}'}$ is Cohen-Macaulay by Algebra, Lemma 157.4. Hence and this is the important step Algebra, Lemma 128.1 applies to show $A \to B_{\mathfrak{m}'}$ is flat. Then Algebra, Lemma 39.18 shows $A \to B$ is flat. Thus we can apply Lemma 21.2 (or you can directly argue using the easier Discriminants, Lemma 3.1) to see that $A \to B$ is étale.

The case $d \geq 3$. Let $V \to U$ be finite étale. Let $f \in \mathfrak{m}_A$, $f \notin \mathfrak{m}_A^2$. Then A/fA is a regular local ring of dimension $d - 1 \geq 2$, see Algebra, Lemma 106.3. Let U_0 be the punctured spectrum of A/fA and let $V_0 = V \times_U U_0$. By Lemma 20.7 it suffices to show that V_0 is in the essential image of $\textit{FÉt}_{\mathrm{Spec}(A/fA)} \to \textit{FÉt}_{U_0}$. This follows from the induction hypothesis. □

Lemma 21.4 (Purity of branch locus)**.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$ and set $y = f(x)$. Assume*

1. *$\mathcal{O}_{X,x}$ is normal,*
2. *$\mathcal{O}_{Y,y}$ is regular,*
3. *f is quasi-finite at x,*
4. $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) \geq 1$
5. *for specializations $x' \rightsquigarrow x$ with $\dim(\mathcal{O}_{X,x'}) = 1$ our f is unramified at x'.*

Then f is étale at x.

Proof. We will prove the lemma by induction on $d = \dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$.

An uninteresting case is when $d = 1$. In that case we are assuming that f is unramified at x and that $\mathcal{O}_{Y,y}$ is a discrete valuation ring (Algebra, Lemma 119.7). Then $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$ (otherwise the map would not be quasi-finite at x) and we see that f is flat at x. Since flat + unramified is étale we conclude (some details omitted).

The case $d \geq 2$. We will use induction on d to reduce to the case discussed in Lemma 21.3. To check f is étale at x we may work étale locally on the base and on the target (Descent, Lemmas 23.29 and 31.1). Thus we can apply More on Morphisms, Lemma 41.1 and assume that $f : X \to Y$ is finite and that x is the unique point of X lying over y. Here we use that étale extensions of local rings do not change dimension, normality, and regularity, see More on Algebra, Section 44 and Étale Morphisms, Section 19.

Next, we can base change by $\mathrm{Spec}(\mathcal{O}_{Y,y})$ and assume that Y is the spectrum of a regular local ring. It follows that $X = \mathrm{Spec}(\mathcal{O}_{X,x})$ as every point of X necessarily specializes to x.

The ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is finite and necessarily injective (by equality of dimensions). We conclude we have going down for $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ by Algebra, Proposition 38.7 (and the fact that a regular ring is a normal ring by Algebra, Lemma 157.5). Pick $x' \in X$, $x' \neq x$ with image $y' = f(x')$. Then $\mathcal{O}_{X,x'}$ is normal as a localization of a normal domain. Similarly, $\mathcal{O}_{Y,y'}$ is regular (see Algebra, Lemma 110.6). We have $\dim(\mathcal{O}_{X,x'}) = \dim(\mathcal{O}_{Y,y'})$ by Algebra, Lemma 112.7 (we checked going down above). Of course these dimensions are strictly less than d as $x' \neq x$ and by induction on d we conclude that f is étale at x'.

Thus we arrive at the following situation: We have a finite local homomorphism $A \to B$ of Noetherian local rings of dimension $d \geq 2$, with A regular, B normal, which induces a finite étale morphism $V \to U$ on punctured spectra. Our goal is to show that $A \to B$ is étale. This follows from Lemma 21.3 and the proof is complete. □

The following lemma is sometimes useful to find the maximal open subset over which a finite étale morphism extends.

Lemma 21.5. *Let $j : U \to X$ be an open immersion of locally Noetherian schemes such that depth$(\mathcal{O}_{X,x}) \geq 2$ for $x \notin U$. Let $\pi : V \to U$ be finite étale. Then*

1. *$\mathcal{B} = j_*\pi_*\mathcal{O}_V$ is a reflexive coherent $\mathcal{O}_X$-algebra, set $Y = \underline{\mathrm{Spec}}_X(\mathcal{B})$,*
2. *$Y \to X$ is the unique finite morphism such that $V = Y \times_X U$ and depth$(\mathcal{O}_{Y,y}) \geq 2$ for $y \in Y \setminus V$,*
3. *$Y \to X$ is étale at y if and only if $Y \to X$ is flat at y, and*
4. *$Y \to X$ is étale if and only if $\mathcal{B}$ is finite locally free as an $\mathcal{O}_X$-module.*

Moreover, (a) the construction of $\mathcal{B}$ and $Y \to X$ commutes with base change by flat morphisms $X' \to X$ of locally Noetherian schemes, and (b) if $V' \to U'$ is a finite étale morphism with $U \subset U' \subset X$ open which restricts to $V \to U$ over U, then there is a unique isomorphism $Y' \times_X U' = V'$ over U'.

Proof. Observe that $\pi_*\mathcal{O}_V$ is a finite locally free $\mathcal{O}_U$-module, in particular reflexive. By Divisors, Lemma 12.12 the module $j_*\pi_*\mathcal{O}_V$ is the unique reflexive coherent module on X restricting to $\pi_*\mathcal{O}_V$ over U. This proves (1).

By construction $Y \times_X U = V$. Since $\mathcal{B}$ is coherent, we see that $Y \to X$ is finite. We have depth$(\mathcal{B}_x) \geq 2$ for $x \in X \setminus U$ by Divisors, Lemma 12.11. Hence depth$(\mathcal{O}_{Y,y}) \geq 2$ for $y \in Y \setminus V$ by Algebra, Lemma 72.11. Conversely, suppose that $\pi' : Y' \to X$ is a finite morphism such that $V = Y' \times_X U$ and depth$(\mathcal{O}_{Y',y'}) \geq 2$ for $y' \in Y' \setminus V$. Then $\pi'_*\mathcal{O}_{Y'}$ restricts to $\pi_*\mathcal{O}_V$ over U and satisfies depth$((\pi'_*\mathcal{O}_{Y'})_x) \geq 2$ for $x \in X \setminus U$ by Algebra, Lemma 72.11. Then $\pi'_*\mathcal{O}_{Y'}$ is canonically isomorphic to $j_*\pi_*\mathcal{O}_V$ for example by Divisors, Lemma 5.11. This proves (2).

If $Y \to X$ is étale at y, then $Y \to X$ is flat at y. Conversely, suppose that $Y \to X$ is flat at y. If $y \in V$, then $Y \to X$ is étale at y. If $y \notin V$, then we check (1), (2), (3), and (4) of Lemma 21.2 hold to see that $Y \to X$ is étale at y. Parts (1) and (2) are clear and so is (3) since depth$(\mathcal{O}_{Y,y}) \geq 2$. If $y' \rightsquigarrow y$ is a specialization and $\dim(\mathcal{O}_{Y,y'}) = 1$, then $y' \in V$ since otherwise the depth of this local ring would be 2 a contradiction by Algebra, Lemma 72.3. Hence $Y \to X$ is étale at y' and we conclude (4) of Lemma 21.2 holds too. This finishes the proof of (3).

Part (4) follows from (3) and the fact that $((Y \to X)_*\mathcal{O}_Y)_x$ is a flat $\mathcal{O}_{X,x}$-module if and only if $\mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,x}$-module for all $y \in Y$ mapping to x, see Algebra, Lemma 39.18. Here we also use that a finite flat module over a Noetherian ring is finite locally free, see Algebra, Lemma 78.2 (and Algebra, Lemma 31.4).

As to the final assertions of the lemma, part (a) follows from flat base change, see Cohomology of Schemes, Lemma 5.2 and part (b) follows from the uniqueness in (2) applied to the restriction $Y \times_X U'$. □

Lemma 21.6. *Let $j : U \to X$ be an open immersion of Noetherian schemes such that purity holds for $\mathcal{O}_{X,x}$ for all $x \notin U$. Then*

$$F\acute{E}t_X \longrightarrow F\acute{E}t_U$$

is essentially surjective.

Proof. Let $V \to U$ be a finite étale morphism. By Noetherian induction it suffices to extend $V \to U$ to a finite étale morphism to a strictly larger open subset of X. Let $x \in X \setminus U$ be the generic point of an irreducible component of $X \setminus U$.

Then the inverse image U_x of U in $\mathrm{Spec}(\mathcal{O}_{X,x})$ is the punctured spectrum of $\mathcal{O}_{X,x}$. By assumption $V_x = V \times_U U_x$ is the restriction of a finite étale morphism $Y_x \to \mathrm{Spec}(\mathcal{O}_{X,x})$ to U_x. By Limits, Lemma 20.3 we find an open subscheme $U \subset U' \subset X$ containing x and a morphism $V' \to U'$ of finite presentation whose restriction to U recovers $V \to U$ and whose restriction to $\mathrm{Spec}(\mathcal{O}_{X,x})$ recovering Y_x. Finally, the morphism $V' \to U'$ is finite étale after possible shrinking U' to a smaller open by Limits, Lemma 20.4. □

22. Finite étale covers of punctured spectra, II

In this section we prove some variants of the material discussed in Section 19. Suppose we have a Noetherian local ring $(A, \mathfrak{m})$ and $f \in \mathfrak{m}$. We set $X = \mathrm{Spec}(A)$ and $X_0 = \mathrm{Spec}(A/fA)$ and we let $U = X \setminus \{\mathfrak{m}\}$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A and A/fA. All of this is exactly as in Situation 19.1. The difference is that we will consider the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} \textit{FÉt}_{U'} \longrightarrow \textit{FÉt}_{U_0}$$

In other words, we will not try to lift finite étale coverings of U_0 to all of U, but just to some open neighbourhood U' of U_0 in U.

Lemma 22.1. *In Situation 19.1. Let $U' \subset U$ be open and contain U_0. Assume for $\mathfrak{p} \subset A$ minimal with $\mathfrak{p} \in U'$, $\mathfrak{p} \notin U_0$ we have $\dim(A/\mathfrak{p}) \geq 2$. Then*

$$\textit{FÉt}_{U'} \longrightarrow \textit{FÉt}_{U_0}, \quad V' \longmapsto V_0 = V' \times_{U'} U_0$$

is a faithful functor. Moreover, there exists a U' satisfying the assumption and any smaller open $U'' \subset U'$ containing U_0 also satisfies this assumption. In particular, the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} \textit{FÉt}_{U'} \longrightarrow \textit{FÉt}_{U_0}$$

is faithful.

Proof. By Algebra, Lemma 60.13 we see that $V(\mathfrak{p})$ meets U_0 for every prime $\mathfrak{p}$ of A with $\dim(A/\mathfrak{p}) \geq 2$. Thus the displayed functor is faithful for a U as in the statement by Lemma 17.5. To see the existence of such a U' note that for $\mathfrak{p} \subset A$ with $\mathfrak{p} \in U$, $\mathfrak{p} \notin U_0$ with $\dim(A/\mathfrak{p}) = 1$ then $\mathfrak{p}$ corresponds to a closed point of U and hence $V(\mathfrak{p}) \cap U_0 = \emptyset$. Thus we can take U' to be the complement of the irreducible components of X which do not meet U_0 and have dimension 1. □

Lemma 22.2. *In Situation 19.1 assume*

1. *A has a dualizing complex and is f-adically complete,*
2. *every irreducible component of X not contained in X_0 has dimension ≥ 3.*

Then the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} \textit{FÉt}_{U'} \longrightarrow \textit{FÉt}_{U_0}$$

is fully faithful.

Proof. To prove this we may replace A by its reduction by the topological invariance of the fundamental group, see Lemma 8.3. Then the result follows from Lemma 17.3 and Algebraic and Formal Geometry, Lemma 15.7. □

Lemma 22.3. *In Situation 19.1 assume*

1. *A is f-adically complete,*

(2) *f is a nonzerodivisor.*
(3) *$H^1_{\mathfrak m}(A/fA)$ is a finite A-module.*

Then the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} \textit{FÉt}_{U'} \longrightarrow \textit{FÉt}_{U_0}$$

is fully faithful.

Proof. Follows from Lemma 17.3 and Algebraic and Formal Geometry, Lemma 15.8. ☐

23. Finite étale covers of punctured spectra, III

In this section we study when in Situation 19.1. the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} \textit{FÉt}_{U'} \longrightarrow \textit{FÉt}_{U_0}$$

is an equivalence of categories.

Lemma 23.1. *In Situation 19.1 assume*

(1) *A has a dualizing complex and is f-adically complete,*
(2) *one of the following is true*
    (a) *A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 4, or*
    (b) *if $\mathfrak p \notin V(f)$ and $V(\mathfrak p) \cap V(f) \neq \{\mathfrak m\}$, then depth$(A_{\mathfrak p}) + \dim(A/\mathfrak p) > 3$.*

Then the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} \textit{FÉt}_{U'} \longrightarrow \textit{FÉt}_{U_0}$$

is an equivalence.

Proof. This follows from Lemma 17.4 and Algebraic and Formal Geometry, Lemma 24.1. ☐

Lemma 23.2. *In Situation 19.1 assume*

(1) *A is f-adically complete,*
(2) *f is a nonzerodivisor,*
(3) *$H^1_{\mathfrak m}(A/fA)$ and $H^2_{\mathfrak m}(A/fA)$ are finite A-modules.*

Then the restriction functor

$$\mathrm{colim}_{U_0 \subset U' \subset U \text{ open}} \textit{FÉt}_{U'} \longrightarrow \textit{FÉt}_{U_0}$$

is an equivalence.

Proof. This follows from Lemma 17.4 and Algebraic and Formal Geometry, Lemma 24.2. ☐

Remark 23.3. Let $(A, \mathfrak m)$ be a complete local Noetherian ring and $f \in \mathfrak m$ nonzero. Suppose that A_f is (S_2) and every irreducible component of $\mathrm{Spec}(A)$ has dimension ≥ 4. Then Lemma 23.1 tells us that the category

$$\mathrm{colim}_{U' \subset U \text{ open, } U_0 \subset U} \text{ category of schemes finite étale over } U'$$

is equivalent to the category of schemes finite étale over U_0. For example this holds if A is a normal domain of dimension ≥ 4!

24. Finite étale covers of punctured spectra, IV

Let X, X_0, U, U_0 be as in Situation 19.1. In this section we ask when the restriction functor

$$F\acute{E}t_U \longrightarrow F\acute{E}t_{U_0}$$

is essentially surjective. We will do this by taking results from Section 23 and then filling in the gaps using purity. Recall that we say *purity holds* for a Noetherian local ring $(A, \mathfrak{m})$ if the restriction functor $F\acute{E}t_X \to F\acute{E}t_U$ is essentially surjective where $X = \text{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$.

Lemma 24.1. *In Situation 19.1 assume*

1. *A has a dualizing complex and is f-adically complete,*
2. *one of the following is true*
   1. (a) *A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 4, or*
   2. (b) *if $\mathfrak{p} \not\in V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then depth$(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.*
3. *for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$.*

Then the restriction functor $F\acute{E}t_U \to F\acute{E}t_{U_0}$ is essentially surjective.

Proof. Let $V_0 \to U_0$ be a finite étale morphism. By Lemma 23.1 there exists an open $U' \subset U$ containing U_0 and a finite étale morphism $V' \to U$ whose base change to U_0 is isomorphic to $V_0 \to U_0$. Since $U' \supset U_0$ we see that $U \setminus U'$ consists of points corresponding to prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ as in (3). By assumption we can find finite étale morphisms $V'_i \to \text{Spec}(A_{\mathfrak{p}_i})$ agreeing with $V' \to U'$ over $U' \times_U \text{Spec}(A_{\mathfrak{p}_i})$. By Limits, Lemma 20.1 applied n times we see that $V' \to U'$ extends to a finite étale morphism $V \to U$. $\square$

Lemma 24.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume*

1. *A is f-adically complete,*
2. *f is a nonzerodivisor,*
3. *$H^1_\mathfrak{m}(A/fA)$ and $H^2_\mathfrak{m}(A/fA)$ are finite A-modules,*
4. *for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$.*

Then the restriction functor $F\acute{E}t_U \to F\acute{E}t_{U_0}$ is essentially surjective.

Proof. The proof is identical to the proof of Lemma 24.1 using Lemma 23.2 in stead of Lemma 23.1. $\square$

25. Purity in local case, II

This section is the continuation of Section 20. Recall that we say *purity holds* for a Noetherian local ring $(A, \mathfrak{m})$ if the restriction functor $F\acute{E}t_X \to F\acute{E}t_U$ is essentially surjective where $X = \text{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$.

Lemma 25.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume*

1. *A has a dualizing complex and is f-adically complete,*
2. *one of the following is true*
   1. (a) *A_f is (S_2) and every irreducible component of X not contained in X_0 has dimension ≥ 4, or*
   2. (b) *if $\mathfrak{p} \not\in V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then depth$(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.*
3. *for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$, and*
4. *purity holds for A.*

Then purity holds for A/fA.

Proof. Denote $X = \mathrm{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$ the punctured spectrum. Similarly we have $X_0 = \mathrm{Spec}(A/fA)$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$. Let $V_0 \to U_0$ be a finite étale morphism. By Lemma 24.1 we find a finite étale morphism $V \to U$ whose base change to U_0 is isomorphic to $V_0 \to U_0$. By assumption (5) we find that $V \to U$ extends to a finite étale morphism $Y \to X$. Then the restriction of Y to X_0 is the desired extension of $V_0 \to U_0$. □

Lemma 25.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $f \in \mathfrak{m}$. Assume*

1. *A is f-adically complete,*
2. *f is a nonzerodivisor,*
3. *$H^1_\mathfrak{m}(A/fA)$ and $H^2_\mathfrak{m}(A/fA)$ are finite A-modules,*
4. *for every maximal ideal $\mathfrak{p} \subset A_f$ purity holds for $(A_f)_\mathfrak{p}$,*
5. *purity holds for A.*

Then purity holds for A/fA.

Proof. The proof is identical to the proof of Lemma 25.1 using Lemma 24.2 in stead of Lemma 24.1. □

Now we can bootstrap the earlier results to prove that purity holds for complete intersections of dimension ≥ 3. Recall that a Noetherian local ring is called a complete intersection if its completion is the quotient of a regular local ring by the ideal generated by a regular sequence. See the discussion in Divided Power Algebra, Section 8.

Proposition 25.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. If A is a complete intersection of dimension ≥ 3, then purity holds for A in the sense that any finite étale cover of the punctured spectrum extends.*

Proof. By Lemma 20.4 we may assume that A is a complete local ring. By assumption we can write $A = B/(f_1, \ldots, f_r)$ where B is a complete regular local ring and $f_1, \ldots, f_r$ is a regular sequence. We will finish the proof by induction on r. The base case is $r = 0$ which follows from Lemma 21.3 which applies to regular rings of dimension ≥ 2.

Assume that $A = B/(f_1, \ldots, f_r)$ and that the proposition holds for $r - 1$. Set $A' = B/(f_1, \ldots, f_{r-1})$ and apply Lemma 25.2 to $f_r \in A'$. This is permissible: condition (1) holds as $f_1, \ldots, f_r$ is a regular sequence, condition (2) holds as B and hence A' is complete, condition (3) holds as $A = A'/f_rA'$ is Cohen-Macaulay of dimension $\dim(A) \geq 3$, see Dualizing Complexes, Lemma 11.1, condition (4) holds by induction hypothesis as $\dim((A'_{f_r})_\mathfrak{p}) \geq 3$ for a maximal prime $\mathfrak{p}$ of A'_{f_r} and as $(A'_{f_r})_\mathfrak{p} = B_\mathfrak{q}/(f_1, \ldots, f_{r-1})$ for some $\mathfrak{q} \subset B$, condition (5) holds by induction hypothesis. □

26. Purity in local case, III

In this section is a continuation of the discussion in Sections 20 and 25.

Lemma 26.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of depth ≥ 2. Let $B = A[[x_1, \ldots, x_d]]$ with $d \geq 1$. Set $Y = \mathrm{Spec}(B)$ and $Y_0 = V(x_1, \ldots, x_d)$. For any open subscheme $V \subset Y$ with $V_0 = V \cap Y_0$ equal to $Y_0 \setminus \{\mathfrak{m}_B\}$ the restriction functor*

$$F\acute{E}t_V \longrightarrow F\acute{E}t_{V_0}$$

is fully faithful.

Proof. Set $I = (x_1, \ldots, x_d)$. Set $X = \mathrm{Spec}(A)$. If we use the map $Y \to X$ to identify Y_0 with X, then V_0 is identified with the punctured spectrum U of A. Pushing forward modules by this affine morphism we get

$$\begin{aligned} \lim_n \Gamma(V_0, \mathcal{O}_V/I^n\mathcal{O}_V) &= \lim_n \Gamma(V_0, \mathcal{O}_Y/I^n\mathcal{O}_Y) \\ &= \lim_n \Gamma(U, \mathcal{O}_U[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n) \\ &= \lim_n A[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n \\ &= B \end{aligned}$$

Namely, as the depth of A is ≥ 2 we have $\Gamma(U, \mathcal{O}_U) = A$, see Local Cohomology, Lemma 8.2. Thus for any $V \subset Y$ open as in the lemma we get

$$B = \Gamma(Y, \mathcal{O}_Y) \to \Gamma(V, \mathcal{O}_V) \to \lim_n \Gamma(V_0, \mathcal{O}_Y/I^n\mathcal{O}_Y) = B$$

which implies both arrows are isomorphisms (small detail omitted). By Algebraic and Formal Geometry, Lemma 15.1 we conclude that $\mathit{Coh}(\mathcal{O}_V) \to \mathit{Coh}(V, I\mathcal{O}_V)$ is fully faithful on the full subcategory of finite locally free objects. Thus we conclude by Lemma 17.1. ☐

Lemma 26.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of depth ≥ 2. Let $B = A[[x_1, \ldots, x_d]]$ with $d \geq 1$. For any open $V \subset Y = \mathrm{Spec}(B)$ which contains*

1. *any prime $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap A \neq \mathfrak{m}$,*
2. *the prime $\mathfrak{m}B$*

the functor $F\acute{E}t_Y \to F\acute{E}t_V$ is an equivalence. In particular purity holds for B.

Proof. A prime $\mathfrak{q} \subset B$ which is not contained in V lies over $\mathfrak{m}$. In this case $A \to B_\mathfrak{q}$ is a flat local homomorphism and hence $\mathrm{depth}(B_\mathfrak{q}) \geq 2$ (Algebra, Lemma 163.2). Thus the functor is fully faithful by Lemma 10.3 combined with Local Cohomology, Lemma 3.1.

Let $W \to V$ be a finite étale morphism. Let $B \to C$ be the unique finite ring map such that $\mathrm{Spec}(C) \to Y$ is the finite morphism extending $W \to V$ constructed in Lemma 21.5. Observe that $C = \Gamma(W, \mathcal{O}_W)$.

Set $Y_0 = V(x_1, \ldots, x_d)$ and $V_0 = V \cap Y_0$. Set $X = \mathrm{Spec}(A)$. If we use the map $Y \to X$ to identify Y_0 with X, then V_0 is identified with the punctured spectrum U of A. Thus we may view $W_0 = W \times_Y Y_0$ as a finite étale scheme over U. Then

$$W_0 \times_U (U \times_X Y) \quad \text{and} \quad W \times_V (U \times_X Y)$$

are schemes finite étale over $U \times_X Y$ which restrict to isomorphic finite étale schemes over V_0. By Lemma 26.1 applied to the open $U \times_X Y$ we obtain an isomorphism

$$W_0 \times_U (U \times_X Y) \longrightarrow W \times_V (U \times_X Y)$$

over $U \times_X Y$.

Observe that $C_0 = \Gamma(W_0, \mathcal{O}_{W_0})$ is a finite A-algebra by Lemma 21.5 applied to $W_0 \to U \subset X$ (exactly as we did for $B \to C$ above). Since the construction in Lemma 21.5 is compatible with flat base change and with change of opens, the isomorphism above induces an isomorphism

$$\Psi : C \longrightarrow C_0 \otimes_A B$$

of finite B-algebras. However, we know that $\text{Spec}(C) \to Y$ is étale at all points above at least one point of Y lying over $\mathfrak{m} \in X$. Since Ψ is an isomorphism, we conclude that $\text{Spec}(C_0) \to X$ is étale above $\mathfrak{m}$ (small detail omitted). Of course this means that $A \to C_0$ is finite étale and hence $B \to C$ is finite étale. □

Lemma 26.3. *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ be an open subscheme. Assume*

(1) *f is smooth,*
(2) *S is Noetherian,*
(3) *for $s \in S$ with depth$(\mathcal{O}_{S,s}) \leq 1$ we have $X_s = U_s$,*
(4) *$U_s \subset X_s$ is dense for all $s \in S$.*

Then $F\acute{E}t_X \to F\acute{E}t_U$ is an equivalence.

Proof. The functor is fully faithful by Lemma 10.3 combined with Local Cohomology, Lemma 3.1 (plus an application of Algebra, Lemma 163.2 to check the depth condition).

Let $\pi : V \to U$ be a finite étale morphism. Let $Y \to X$ be the finite morphism constructed in Lemma 21.5. We have to show that $Y \to X$ is finite étale. To show that this is true for all points $x \in X$ mapping to a given point $s \in S$ we may perform a base change by a flat morphism $S' \to S$ of Noetherian schemes such that s is in the image. This follows from the compatibility of the construction in Lemma 21.5 with flat base change.

After enlarging U we may assume $U \subset X$ is the maximal open over which $Y \to X$ is finite étale. Let $Z \subset X$ be the complement of U. To get a contradiction, assume $Z \neq \emptyset$. Let $s \in S$ be a point in the image of $Z \to S$ such that no strict generalization of s is in the image. Then after base change to $\text{Spec}(\mathcal{O}_{S,s})$ we see that $S = \text{Spec}(A)$ with $(A, \mathfrak{m}, \kappa)$ a local Noetherian ring of depth ≥ 2 and Z contained in the closed fibre X_s and nowhere dense in X_s. Choose a closed point $z \in Z$. Then $\kappa(z)/\kappa$ is finite (by the Hilbert Nullstellensatz, see Algebra, Theorem 34.1). Choose a finite flat morphism $(S', s') \to (S, s)$ of local schemes realizing the residue field extension $\kappa(z)/\kappa$, see Algebra, Lemma 159.3. After doing a base change by $S' \to S$ we reduce to the case where $\kappa(z) = \kappa$.

By More on Morphisms, Lemma 38.5 there exists a locally closed subscheme $S' \subset X$ passing through z such that $S' \to S$ is étale at z. After performing the base change by $S' \to S$, we may assume there is a section $\sigma : S \to X$ such that $\sigma(s) = z$. Choose an affine neighbourhood $\text{Spec}(B) \subset X$ of s. Then $A \to B$ is a smooth ring map which has a section $\sigma : B \to A$. Denote $I = \text{Ker}(\sigma)$ and denote $B^\wedge$ the I-adic completion of B. Then $B^\wedge \cong A[[x_1, \ldots, x_d]]$ for some $d \geq 0$, see Algebra, Lemma 139.4. Observe that $d > 0$ since otherwise we see that $X \to S$ is étale at z which would imply that z is a generic point of X_s and hence $z \in U$ by assumption (4). Similarly, if $d > 0$, then $\mathfrak{m}B^\wedge$ maps into U via the morphism $\text{Spec}(B^\wedge) \to X$. It suffices prove $Y \to X$ is finite étale after base change to $\text{Spec}(B^\wedge)$. Since $B \to B^\wedge$ is flat (Algebra, Lemma 97.2) this follows from Lemma 26.2 and the uniqueness in the construction of $Y \to X$. □

Proposition 26.4. *Let $A \to B$ be a local homomorphism of local Noetherian rings. Assume A has depth ≥ 2, $A \to B$ is formally smooth for the $\mathfrak{m}_B$-adic topology, and* $\dim(B) > \dim(A)$. *For any open $V \subset Y = \text{Spec}(B)$ which contains*

(1) *any prime* $\mathfrak{q} \subset B$ *such that* $\mathfrak{q} \cap A \neq \mathfrak{m}_A$,
(2) *the prime* $\mathfrak{m}_A B$

the functor $\textit{FÉt}_Y \to \textit{FÉt}_V$ *is an equivalence. In particular purity holds for* B.

Proof. A prime $\mathfrak{q} \subset B$ which is not contained in V lies over $\mathfrak{m}_A$. In this case $A \to B_{\mathfrak{q}}$ is a flat local homomorphism and hence $\text{depth}(B_{\mathfrak{q}}) \geq 2$ (Algebra, Lemma 163.2). Thus the functor is fully faithful by Lemma 10.3 combined with Local Cohomology, Lemma 3.1.

Denote $A^\wedge$ and $B^\wedge$ the completions of A and B with respect to their maximal ideals. Observe that the assumptions of the proposition hold for $A^\wedge \to B^\wedge$, see More on Algebra, Lemmas 43.1, 43.2, and 37.4. By the uniqueness and compatibility with flat base change of the construction of Lemma 21.5 it suffices to prove the essential surjectivity for $A^\wedge \to B^\wedge$ and the inverse image of V (details omitted; compare with Lemma 20.4 for the case where V is the punctured spectrum). By More on Algebra, Proposition 49.2 this means we may assume $A \to B$ is regular.

Let $W \to V$ be a finite étale morphism. By Popescu's theorem (Smoothing Ring Maps, Theorem 12.1) we can write $B = \text{colim}\, B_i$ as a filtered colimit of smooth A-algebras. We can pick an i and an open $V_i \subset \text{Spec}(B_i)$ whose inverse image is V (Limits, Lemma 4.11). After increasing i we may assume there is a finite étale morphism $W_i \to V_i$ whose base change to V is $W \to V$, see Limits, Lemmas 10.1, 8.3, and 8.10. We may assume the complement of V_i is contained in the closed fibre of $\text{Spec}(B_i) \to \text{Spec}(A)$ as this is true for V (either choose V_i this way or use the lemma above to show this is true for i large enough). Let η be the generic point of the closed fibre of $\text{Spec}(B) \to \text{Spec}(A)$. Since $\eta \in V$, the image of η is in V_i. Hence after replacing V_i by an affine open neighbourhood of the image of the closed point of $\text{Spec}(B)$, we may assume that the closed fibre of $\text{Spec}(B_i) \to \text{Spec}(A)$ is irreducible and that its generic point is contained in V_i (details omitted; use that a scheme smooth over a field is a disjoint union of irreducible schemes). At this point we may apply Lemma 26.3 to see that $W_i \to V_i$ extends to a finite étale morphism $\text{Spec}(C_i) \to \text{Spec}(B_i)$ and pulling back to $\text{Spec}(B)$ we conclude that W is in the essential image of the functor $\textit{FÉt}_Y \to \textit{FÉt}_V$ as desired. □

27. Lefschetz for the fundamental group

Of course we have already proven a bunch of results of this type in the local case. In this section we discuss the projective case.

Proposition 27.1. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s. Assume that for all $x \in X \setminus Y$ we have*

$$\textit{depth}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 1$$

Then the restriction functor $\textit{FÉt}_X \to \textit{FÉt}_Y$ is fully faithful. In fact, for any open subscheme $V \subset X$ containing Y the restriction functor $\textit{FÉt}_V \to \textit{FÉt}_Y$ is fully faithful.

Proof. The first statement is a formal consequence of Lemma 17.6 and Algebraic and Formal Geometry, Proposition 28.1. The second statement follows from Lemma 17.6 and Algebraic and Formal Geometry, Lemma 28.2. □

Proposition 27.2. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s. Let $\mathcal{V}$ be the set of open subschemes of X containing Y ordered by reverse inclusion. Assume that for all $x \in X \setminus Y$ we have*

$$depth(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 2$$

Then the restriction functor

$$\mathrm{colim}_{\mathcal{V}}\, F\acute{E}t_V \to F\acute{E}t_Y$$

is an equivalence.

Proof. This is a formal consequence of Lemma 17.4 and Algebraic and Formal Geometry, Proposition 28.7. ☐

Proposition 27.3. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Let $Y = Z(s)$ be the zero scheme of s. Assume that for all $x \in X \setminus Y$ we have*

$$depth(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 2$$

and that for $x \in X \setminus Y$ closed purity holds for $\mathcal{O}_{X,x}$. Then the restriction functor $F\acute{E}t_X \to F\acute{E}t_Y$ is an equivalence. If X or equivalently Y is connected, then

$$\pi_1(Y, \overline{y}) \to \pi_1(X, \overline{y})$$

is an isomorphism for any geometric point $\overline{y}$ of Y.

Proof. Fully faithfulness holds by Proposition 27.1. By Proposition 27.2 any object of $F\acute{E}t_Y$ is isomorphic to the fibre product $U \times_V Y$ for some finite étale morphism $U \to V$ where $V \subset X$ is an open subscheme containing Y. The complement $T = X \setminus V$ is[5] a finite set of closed points of $X \setminus Y$. Say $T = \{x_1, \ldots, x_n\}$. By assumption we can find finite étale morphisms $V_i' \to \mathrm{Spec}(\mathcal{O}_{X,x_i})$ agreeing with $U \to V$ over $V \times_X \mathrm{Spec}(\mathcal{O}_{X,x_i})$. By Limits, Lemma 20.1 applied n times we see that $U \to V$ extends to a finite étale morphism $U' \to X$ as desired. See Lemma 8.1 for the final statement. ☐

28. Purity of ramification locus

In this section we discuss the analogue of purity of branch locus for generically finite morphisms. Apparently, this result is due to Gabber. A special case is van der Waerden's purity theorem for the locus where a birational morphism from a normal variety to a smooth variety is not an isomorphism.

Lemma 28.1. *Let A be a Noetherian normal local domain of dimension 2. Assume A is Nagata, has a dualizing module ω_A, and has a resolution of singularities $f : X \to \mathrm{Spec}(A)$. Let ω_X be as in Resolution of Surfaces, Remark 7.7. If $\omega_X \cong \mathcal{O}_X(E)$ for some effective Cartier divisor $E \subset X$ supported on the exceptional fibre, then A defines a rational singularity. If f is a minimal resolution, then $E = 0$.*

[5]Namely, T is proper over k (being closed in X) and affine (being closed in the affine scheme $X \setminus Y$, see Morphisms, Lemma 43.18) and hence finite over k (Morphisms, Lemma 44.11). Thus T is a finite set of closed points.

Proof. There is a trace map $Rf_*\omega_X \to \omega_A$, see Duality for Schemes, Section 7. By Grauert-Riemenschneider (Resolution of Surfaces, Proposition 7.8) we have $R^1f_*\omega_X = 0$. Thus the trace map is a map $f_*\omega_X \to \omega_A$. Then we can consider

$$\mathcal{O}_{\mathrm{Spec}(A)} = f_*\mathcal{O}_X \to f_*\omega_X \to \omega_A$$

where the first map comes from the map $\mathcal{O}_X \to \mathcal{O}_X(E) = \omega_X$ which is assumed to exist in the statement of the lemma. The composition is an isomorphism by Divisors, Lemma 2.11 as it is an isomorphism over the punctured spectrum of A (by the assumption in the lemma and the fact that f is an isomorphism over the punctured spectrum) and A and ω_A are A-modules of depth 2 (by Algebra, Lemma 157.4 and Dualizing Complexes, Lemma 17.5). Hence $f_*\omega_X \to \omega_A$ is surjective whence an isomorphism. Thus $Rf_*\omega_X = \omega_A$ which by duality implies $Rf_*\mathcal{O}_X = \mathcal{O}_{\mathrm{Spec}(A)}$. Whence $H^1(X, \mathcal{O}_X) = 0$ which implies that A defines a rational singularity (see discussion in Resolution of Surfaces, Section 8 in particular Lemmas 8.7 and 8.1). If f is minimal, then $E = 0$ because the map $f^*\omega_A \to \omega_X$ is surjective by a repeated application of Resolution of Surfaces, Lemma 9.7 and $\omega_A \cong A$ as we've seen above. ☐

Lemma 28.2. *Let $f : X \to \mathrm{Spec}(A)$ be a finite type morphism. Let $x \in X$ be a point. Assume*

1. *A is an excellent regular local ring,*
2. *$\mathcal{O}_{X,x}$ is normal of dimension 2,*
3. *f is étale outside of $\overline{\{x\}}$.*

Then f is étale at x.

Proof. We first replace X by an affine open neighbourhood of x. Observe that $\mathcal{O}_{X,x}$ is an excellent local ring (More on Algebra, Lemma 52.2). Thus we can choose a minimal resolution of singularities $W \to \mathrm{Spec}(\mathcal{O}_{X,x})$, see Resolution of Surfaces, Theorem 14.5. After possibly replacing X by an affine open neighbourhood of x we can find a proper morphism $b : X' \to X$ such that $X' \times_X \mathrm{Spec}(\mathcal{O}_{X,x}) = W$, see Limits, Lemma 20.1. After shrinking X further, we may assume X' is regular. Namely, we know W is regular and X' is excellent and the regular locus of the spectrum of an excellent ring is open. Since $W \to \mathrm{Spec}(\mathcal{O}_{X,x})$ is projective (as a sequence of normalized blowing ups), we may assume after shrinking X that b is projective (details omitted). Let $U = X \setminus \overline{\{x\}}$. Since $W \to \mathrm{Spec}(\mathcal{O}_{X,x})$ is an isomorphism over the punctured spectrum, we may assume $b : X' \to X$ is an isomorphism over U. Thus we may and will think of U as an open subscheme of X' as well. Set $f' = f \circ b : X' \to \mathrm{Spec}(A)$.

Since A is regular we see that $\mathcal{O}_Y$ is a dualizing complex for Y. Hence $f^!\mathcal{O}_Y$ is a dualzing complex on X (Duality for Schemes, Lemma 17.7). The Cohen-Macaulay locus of X is open by Duality for Schemes, Lemma 23.1 (this can also be proven using excellency). Since $\mathcal{O}_{X,x}$ is Cohen-Macaulay, after shrinking X we may assume X is Cohen-Macaulay. Observe that an étale morphism is a local complete intersection. Thus Duality for Schemes, Lemma 29.3 applies with $r = 0$ and we get a map

$$\mathcal{O}_X \longrightarrow \omega_{X/Y} = H^0(f^!\mathcal{O}_Y)$$

which is an isomorphism over $X \setminus \overline{\{x\}}$. Since $\omega_{X/Y}$ is (S_2) by Duality for Schemes, Lemma 21.5 we find this map is an isomorphism by Divisors, Lemma 2.11. This already shows that X and in particular $\mathcal{O}_{X,x}$ is Gorenstein.

Set $\omega_{X'/Y} = H^0((f')^!\mathcal{O}_Y)$. Arguing in exactly the same manner as above we find that $(f')^!\mathcal{O}_Y = \omega_{X'/Y}[0]$ is a dualizing complex for X'. Since X' is regular the morphism $X' \to Y$ is a local complete intersection morphism, see More on Morphisms, Lemma 62.11. By Duality for Schemes, Lemma 29.2 there exists a map

$$\mathcal{O}_{X'} \longrightarrow \omega_{X'/Y}$$

which is an isomorphism over U. We conclude $\omega_{X'/Y} = \mathcal{O}_{X'}(E)$ for some effective Cartier divisor $E \subset X'$ disjoint from U.

Since $\omega_{X/Y} = \mathcal{O}_Y$ we see that $\omega_{X'/Y} = b^! f^! \mathcal{O}_Y = b^! \mathcal{O}_X$. Returning to $W \to \mathrm{Spec}(\mathcal{O}_{X,x})$ we see that $\omega_W = \mathcal{O}_W(E|_W)$. By Lemma 28.1 we find $E|_W = 0$. This means that $f' : X' \to Y$ is étale by (the already used) Duality for Schemes, Lemma 29.2. This immediately finishes the proof, as étaleness of f' forces b to be an isomorphism. $\square$

Lemma 28.3 (Purity of ramification locus)**.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$ and set $y = f(x)$. Assume*

1. *$\mathcal{O}_{X,x}$ is normal of dimension ≥ 1,*
2. *$\mathcal{O}_{Y,y}$ is regular,*
3. *f is locally of finite type, and*
4. *for specializations $x' \rightsquigarrow x$ with $\dim(\mathcal{O}_{X,x'}) = 1$ our f is étale at x'.*

Then f is étale at x.

Proof. We will prove the lemma by induction on $d = \dim(\mathcal{O}_{X,x})$.

An uninteresting case is $d = 1$ since in that case the morphism f is étale at x by assumption. Assume $d \geq 2$.

We can base change by $\mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$ without affecting the conclusion of the lemma, see Morphisms, Lemma 36.17. Thus we may assume $Y = \mathrm{Spec}(A)$ where A is a regular local ring and y corresponds to the maximal ideal $\mathfrak{m}$ of A.

Let $x' \rightsquigarrow x$ be a specialization with $x' \neq x$. Then $\mathcal{O}_{X,x'}$ is normal as a localization of $\mathcal{O}_{X,x}$. If x' is not a generic point of X, then $1 \leq \dim(\mathcal{O}_{X,x'}) < d$ and we conclude that f is étale at x' by induction hypothesis. Thus we may assume that f is étale at all points specializing to x. Since the set of points where f is étale is open in X (by definition) we may after replacing X by an open neighbourhood of x assume that f is étale away from $\overline{\{x\}}$. In particular, we see that f is étale except at points lying over the closed point $y \in Y = \mathrm{Spec}(A)$.

Let $X' = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A^\wedge)$. Let $x' \in X'$ be the unique point lying over x. By the above we see that X' is étale over $\mathrm{Spec}(A^\wedge)$ away from the closed fibre and hence X' is normal away from the closed fibre. Since X is normal we conclude that X' is normal by Resolution of Surfaces, Lemma 11.6. Then if we can show $X' \to \mathrm{Spec}(A^\wedge)$ is étale at x', then f is étale at x (by the aforementioned Morphisms, Lemma 36.17). Thus we may and do assume A is a regular complete local ring.

The case $d = 2$ now follows from Lemma 28.2.

Assume $d > 2$. Let $t \in \mathfrak{m}$, $t \notin \mathfrak{m}^2$. Set $Y_0 = \mathrm{Spec}(A/tA)$ and $X_0 = X \times_Y Y_0$. Then $X_0 \to Y_0$ is étale away from the fibre over the closed point. Since $d > 2$ we have $\dim(\mathcal{O}_{X_0,x}) = d - 1$ is ≥ 2. The normalization $X_0' \to X_0$ is surjective and finite (as we're working over a complete local ring and such rings are Nagata). Let $x' \in X_0'$

be a point mapping to x. By induction hypothesis the morphism $X'_0 \to Y$ is étale at x'. From the inclusions $\kappa(y) \subset \kappa(x) \subset \kappa(x')$ we conclude that $\kappa(x)$ is finite over $\kappa(y)$. Hence x is a closed point of the fibre of $X \to Y$ over y. But since x is also a generic point of this fibre, we conclude that f is quasi-finite at x and we reduce to the case of purity of branch locus, see Lemma 21.4. □

29. Affineness of complement of ramification locus

Let $f : X \to Y$ be a finite type morphism of Noetherian schemes with X normal and Y regular. Let $V \subset X$ be the maximal open subscheme where f is étale. The discussion in [DG67, Chapter IV, Section 21.12] suggests that $V \to X$ might be an affine morphism. Observe that if $V \to X$ is affine, then we deduce purity of ramification locus (Lemma 28.3) by using Divisors, Lemma 16.4. Thus affineness of $V \to X$ is a "strong" form of purity for the ramification locus. In this section we prove $V \to X$ is affine when X and Y are equicharacteristic and excellent, see Theorem 29.3. It seems reasonable to guess the result remains true for X and Y of mixed characteristic (but still excellent).

Lemma 29.1. *Let $(A, \mathfrak{m})$ be a regular local ring which contains a field. Let $f : V \to \mathrm{Spec}(A)$ be étale and quasi-compact. Assume that $\mathfrak{m} \notin f(V)$ and assume that $g : V \to \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ is affine. Then $H^i(V, \mathcal{O}_V)$, $i > 0$ is isomorphic to a direct sum of copies of the injective hull of the residue field of A.*

Proof. Denote $U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ the punctured spectrum. Thus $g : V \to U$ is affine. We have $H^i(V, \mathcal{O}_V) = H^i(U, g_*\mathcal{O}_V)$ by Cohomology of Schemes, Lemma 2.4. The $\mathcal{O}_U$-module $g_*\mathcal{O}_V$ is quasi-coherent by Schemes, Lemma 24.1. For any quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$ the cohomology $H^i(U, \mathcal{F})$, $i > 0$ is $\mathfrak{m}$-power torsion, see for example Local Cohomology, Lemma 2.2. In particular, the A-modules $H^i(V, \mathcal{O}_V)$, $i > 0$ are $\mathfrak{m}$-power torsion. For any flat ring map $A \to A'$ we have $H^i(V, \mathcal{O}_V) \otimes_A A' = H^i(V', \mathcal{O}_{V'})$ where $V' = V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$ by flat base change Cohomology of Schemes, Lemma 5.2. If we take A' to be the completion of A (flat by More on Algebra, Section 43), then we see that

$$H^i(V, \mathcal{O}_V) = H^i(V, \mathcal{O}_V) \otimes_A A' = H^i(V', \mathcal{O}_{V'}), \quad \text{for } i > 0$$

The first equality by the torsion property we just proved and More on Algebra, Lemma 89.3. Moreover, the injective hull of the residue field k is the same for A and A', see Dualizing Complexes, Lemma 7.4. In this way we reduce to the case $A = k[[x_1, \ldots, x_d]]$, see Algebra, Section 160.

Assume the characteristic of k is $p > 0$. Since $F : A \to A$, $a \mapsto a^p$ is flat (Local Cohomology, Lemma 17.6) and since $V \times_{\mathrm{Spec}(A), \mathrm{Spec}(F)} \mathrm{Spec}(A) \cong V$ as schemes over $\mathrm{Spec}(A)$ by Étale Morphisms, Lemma 14.3 the above gives $H^i(V, \mathcal{O}_V) \otimes_{A,F} A \cong H^i(V, \mathcal{O}_V)$. Thus we get the result by Local Cohomology, Lemma 18.2.

Assume the characteristic of k is 0. By Local Cohomology, Lemma 19.3 there are additive operators D_j, $j = 1, \ldots, d$ on $H^i(V, \mathcal{O}_V)$ satisfying the Leibniz rule with respect to $\partial_j = \partial/\partial x_j$. Thus we get the result by Local Cohomology, Lemma 18.1. □

Lemma 29.2. *In the situation of Lemma 29.1 assume that $H^i(V, \mathcal{O}_V) = 0$ for $i \geq \dim(A) - 1$. Then V is affine.*

Proof. Let $k = A/\mathfrak{m}$. Since $V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = \emptyset$, by cohomology and base change we have

$$R\Gamma(V, \mathcal{O}_V) \otimes_A^{\mathbf{L}} k = 0$$

See Derived Categories of Schemes, Lemma 22.5. Thus there is a spectral sequence (More on Algebra, Example 62.4)

$$E_2^{p,q} = \mathrm{Tor}_{-p}(k, H^q(V, \mathcal{O}_V)), \quad d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$$

and $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ converging to zero. By Lemma 29.1, Dualizing Complexes, Lemma 21.9, and our assumption $H^i(V, \mathcal{O}_V) = 0$ for $i \geq \dim(A) - 1$ we conclude that there is no nonzero differential entering or leaving the $(p,q) = (0,0)$ spot. Thus $H^0(V, \mathcal{O}_V) \otimes_A k = 0$. This means that if $\mathfrak{m} = (x_1, \ldots, x_d)$ then we have an open covering $V = \bigcup V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_{x_i})$ by affine open subschemes $V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_{x_i})$ (because V is affine over the punctured spectrum of A) such that $x_1, \ldots, x_d$ generate the unit ideal in $\Gamma(V, \mathcal{O}_V)$. This implies V is affine by Properties, Lemma 27.3. ☐

Theorem 29.3. *Let Y be an excellent regular scheme over a field. Let $f : X \to Y$ be a finite type morphism of schemes with X normal. Let $V \subset X$ be the maximal open subscheme where f is étale. Then the inclusion morphism $V \to X$ is affine.*

Proof. Let $x \in X$ with image $y \in Y$. It suffices to prove that $V \cap W$ is affine for some affine open neighbourhood W of x. Since $\mathrm{Spec}(\mathcal{O}_{X,x})$ is the limit of the schemes W, this holds if and only if

$$V_x = V \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$$

is affine (Limits, Lemma 4.13). Thus, if the theorem holds for the morphism $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$, then the theorem holds. In particular, we may assume Y is regular of finite dimension, which allows us to do induction on the dimension $d = \dim(Y)$. Combining this with the same argument again, we may assume that Y is local with closed point y and that $V \cap (X \setminus f^{-1}(\{y\}) \to X \setminus f^{-1}(\{y\})$ is affine.

Let $x \in X$ be a point lying over y. If $x \in V$, then there is nothing to prove. Observe that $f^{-1}(\{y\}) \cap V$ is a finite set of closed points (the fibres of an étale morphism are discrete). Thus after replacing X by an affine open neighbourhood of x we may assume $y \notin f(V)$. We have to prove that V is affine.

Let $e(V)$ be the maximum i with $H^i(V, \mathcal{O}_V) \neq 0$. As X is affine the integer $e(V)$ is the maximum of the numbers $e(V_x)$ where $x \in X \setminus V$, see Local Cohomology, Lemma 4.6 and the characterization of cohomological dimension in Local Cohomology, Lemma 4.1. We have $e(V_x) \leq \dim(\mathcal{O}_{X,x}) - 1$ by Local Cohomology, Lemma 4.7. If $\dim(\mathcal{O}_{X,x}) \geq 2$ then purity of ramification locus (Lemma 28.3) shows that V_x is strictly smaller than the punctured spectrum of $\mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is normal and excellent, this implies $e(V_x) \leq \dim(\mathcal{O}_{X,x}) - 2$ by Hartshorne-Lichtenbaum vanishing (Local Cohomology, Lemma 16.7). On the other hand, since $X \to Y$ is of finite type and $V \subset X$ is dense (after possibly replacing X by the closure of V), we see that $\dim(\mathcal{O}_{X,x}) \leq d$ by the dimension formula (Morphisms, Lemma 52.1). Whence $e(V) \leq \max(0, d-2)$. Thus V is affine by Lemma 29.2 if $d \geq 2$. If $d = 1$ or $d = 0$, then the punctured spectrum of $\mathcal{O}_{Y,y}$ is affine and hence V is affine. ☐

30. Specialization maps in the smooth proper case

In this section we discuss the following result. Let $f : X \to S$ be a proper smooth morphism of schemes. Let $s \rightsquigarrow s'$ be a specialization of points in S. Then the specialization map

$$sp : \pi_1(X_{\overline{s}}) \longrightarrow \pi_1(X_{\overline{s}'})$$

of Section 16 is surjective and

(1) if the characteristic of $\kappa(s')$ is zero, then it is an isomorphism, or
(2) if the characteristic of $\kappa(s')$ is $p > 0$, then it induces an isomorphism on maximal prime-to-p quotients.

Lemma 30.1. *Let $f : X \to S$ be a flat proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization. If X_s is geometrically reduced, then the specialization map $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ is surjective.*

Proof. Since X_s is geometrically reduced, we may assume all fibres are geometrically reduced after possibly shrinking S, see More on Morphisms, Lemma 26.7. Let $\mathcal{O}_{S,s} \to A \to \kappa(\overline{s}')$ be as in the construction of the specialization map, see Section 16. Thus it suffices to show that

$$\pi_1(X_{\overline{s}'}) \to \pi_1(X_A)$$

is surjective. This follows from Proposition 15.2 and $\pi_1(\text{Spec}(A)) = \{1\}$. $\square$

Proposition 30.2. *Let $f : X \to S$ be a smooth proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization. If the characteristic to $\kappa(s)$ is zero, then the specialization map*

$$sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$$

is an isomorphism.

Proof. The map is surjective by Lemma 30.1. Thus we have to show it is injective.

We may assume S is affine. Then S is a cofiltered limit of affine schemes of finite type over $\mathbf{Z}$. Hence we can assume $X \to S$ is the base change of $X_0 \to S_0$ where S_0 is the spectrum of a finite type $\mathbf{Z}$-algebra and $X_0 \to S_0$ is smooth and proper. See Limits, Lemma 10.1, 8.9, and 13.1. By Lemma 16.1 we reduce to the case where the base is Noetherian.

Applying Lemma 16.4 we reduce to the case where the base S is the spectrum of a strictly henselian discrete valuation ring A and we are looking at the specialization map over A. Let K be the fraction field of A. Choose an algebraic closure $\overline{K}$ which corresponds to a geometric generic point $\overline{\eta}$ of $\text{Spec}(A)$. For $\overline{K}/L/K$ finite separable, let $B \subset L$ be the integral closure of A in L. This is a discrete valuation ring by More on Algebra, Remark 111.6.

Let $X \to \text{Spec}(A)$ be as in the previous paragraph. To show injectivity of the specialization map it suffices to prove that every finite étale cover V of $X_{\overline{\eta}}$ is the base change of a finite étale cover $Y \to X$. Namely, then $\pi_1(X_{\overline{\eta}}) \to \pi_1(X) = \pi_1(X_s)$ is injective by Lemma 4.4.

Given V we can first descend V to $V' \to X_{K^{sep}}$ by Lemma 14.2 and then to $V'' \to X_L$ by Lemma 14.1. Let $Z \to X_B$ be the normalization of X_B in V''. Observe that Z is normal and that $Z_L = V''$ as schemes over X_L. Hence $Z \to X_B$ is finite étale over the generic fibre. The problem is that we do not know that

$Z \to X_B$ is everywhere étale. Since $X \to \mathrm{Spec}(A)$ has geometrically connected smooth fibres, we see that the special fibre X_s is geometrically irreducible. Hence the special fibre of $X_B \to \mathrm{Spec}(B)$ is irreducible; let ξ_B be its generic point. Let $\xi_1, \ldots, \xi_r$ be the points of Z mapping to ξ_B. Our first (and it will turn out only) problem is now that the extensions

$$\mathcal{O}_{X_B,\xi_B} \subset \mathcal{O}_{Z,\xi_i}$$

of discrete valuation rings may be ramified. Let e_i be the ramification index of this extension. Note that since the characteristic of $\kappa(s)$ is zero, the ramification is tame!

To get rid of the ramification we are going to choose a further finite separable extension $K^{sep}/L'/L/K$ such that the ramification index e of the induced extensions B'/B is divisible by e_i. Consider the normalized base change Z' of Z with respect to $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$, see discussion in More on Morphisms, Section 65. Let $\xi_{i,j}$ be the points of Z' mapping to $\xi_{B'}$ and to ξ_i in Z. Then the local rings

$$\mathcal{O}_{Z',\xi_{i,j}}$$

are localizations of the integral closure of $\mathcal{O}_{Z,\xi_i}$ in $L' \otimes_L F_i$ where F_i is the fraction field of $\mathcal{O}_{Z,\xi_i}$; details omitted. Hence Abhyankar's lemma (More on Algebra, Lemma 114.4) tells us that

$$\mathcal{O}_{X_{B'},\xi_{B'}} \subset \mathcal{O}_{Z',\xi_{i,j}}$$

is unramified. We conclude that the morphism $Z' \to X_{B'}$ is étale away from codimension 1. Hence by purity of branch locus (Lemma 21.4) we see that $Z' \to X_{B'}$ is finite étale!

However, since the residue field extension induced by $A \to B'$ is trivial (as the residue field of A is algebraically closed being separably closed of characteristic zero) we conclude that Z' is the base change of a finite étale cover $Y \to X$ by applying Lemma 9.1 twice (first to get Y over A, then to prove that the pullback to B is isomorphic to Z'). This finishes the proof. □

Let G be a profinite group. Let p be a prime number. The *maximal prime-to-p quotient* is by definition

$$G' = \lim_{U \subset G \text{ open, normal, index prime to } p} G/U$$

If X is a connected scheme and p is given, then the maximal prime-to-p quotient of $\pi_1(X)$ is denoted $\pi_1'(X)$.

Theorem 30.3. *Let $f : X \to S$ be a smooth proper morphism with geometrically connected fibres. Let $s' \leadsto s$ be a specialization. If the characteristic of $\kappa(s)$ is p, then the specialization map*

$$sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$$

is surjective and induces an isomorphism

$$\pi_1'(X_{\overline{s}'}) \cong \pi_1'(X_{\overline{s}})$$

of the maximal prime-to-p quotients

Proof. This is proved in exactly the same manner as Proposition 30.2 with the following differences

(1) Given X/A we no longer show that the functor $F\acute{E}t_X \to F\acute{E}t_{X_{\overline{\eta}}}$ is essentially surjective. We show only that Galois objects whose Galois group has order prime to p are in the essential image. This will be enough to conclude the injectivity of $\pi_1'(X_{\overline{s}'}) \to \pi_1'(X_{\overline{s}})$ by exactly the same argument.
(2) The extensions $\mathcal{O}_{X_B,\xi_B} \subset \mathcal{O}_{Z,\xi_i}$ are tamely ramified as the associated extension of fraction fields is Galois with group of order prime to p. See More on Algebra, Lemma 112.2.
(3) The extension κ_B/κ_A is no longer necessarily trivial, but it is purely inseparable. Hence the morphism $X_{\kappa_B} \to X_{\kappa_A}$ is a universal homeomorphism and induces an isomorphism of fundamental groups by Proposition 8.4.

$\square$

31. Tame ramification

Let $X \to Y$ be a finite étale morphism of schemes of finite type over $\mathbf{Z}$. There are many ways to define what it means for f to be tamely ramified at ∞. The article [KS10] discusses to what extent these notions agree.

In this section we discuss a different more elementary question which precedes the notion of tameness at infinity. Please compare with the (slightly different) discussion in [GM71]. Assume we are given

(1) a locally Noetherian scheme X,
(2) a dense open $U \subset X$,
(3) a finite étale morphism $f : Y \to U$

such that for every prime divisor $Z \subset X$ with $Z \cap U = \emptyset$ the local ring $\mathcal{O}_{X,\xi}$ of X at the generic point ξ of Z is a discrete valuation ring. Setting K_ξ equal to the fraction field of $\mathcal{O}_{X,\xi}$ we obtain a cartesian square

$$\begin{array}{ccc} \mathrm{Spec}(K_\xi) & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{X,\xi}) & \longrightarrow & X \end{array}$$

of schemes. In particular, we see that $Y \times_U \mathrm{Spec}(K_\xi)$ is the spectrum of a finite separable algebra L_ξ/K_ξ. Then we say Y *is unramified over X in codimension* 1, resp. Y *is tamely ramified over X in codimension* 1 if L_ξ/K_ξ is unramified, resp. tamely ramified with respect to $\mathcal{O}_{X,\xi}$ for every (Z,ξ) as above, see More on Algebra, Definition 111.7. More precisely, we decompose L_ξ into a product of finite separable field extensions of K_ξ and we require each of these to be unramified, resp. tamely ramified with respect to $\mathcal{O}_{X,\xi}$.

Lemma 31.1. *Let $X' \to X$ be a morphism of locally Noetherian schemes. Let $U \subset X$ be a dense open. Assume*

(1) *$U' = f^{-1}(U)$ is dense open in X',*
(2) *for every prime divisor $Z \subset X$ with $Z \cap U = \emptyset$ the local ring $\mathcal{O}_{X,\xi}$ of X at the generic point ξ of Z is a discrete valuation ring,*
(3) *for every prime divisor $Z' \subset X'$ with $Z' \cap U' = \emptyset$ the local ring $\mathcal{O}_{X',\xi'}$ of X' at the generic point ξ' of Z' is a discrete valuation ring,*
(4) *if $\xi' \in X'$ is as in (3), then $\xi = f(\xi')$ is as in (2).*

Then if $f : Y \to U$ is finite étale and Y is unramified, resp. tamely ramified over X in codimension 1, then $Y' = Y \times_X X' \to U'$ is finite étale and Y' is unramified, resp. tamely ramified over X' in codimension 1.

Proof. The only interesting fact in this lemma is the commutative algebra result given in More on Algebra, Lemma 114.9. $\square$

Using the terminology introduced above, we can reformulate our purity results obtained earlier in the following pleasing manner.

Lemma 31.2. *Let X be a locally Noetherian scheme. Let $U \subset X$ be open and dense. Let $Y \to U$ be a finite étale morphism. Assume*

(1) *Y is unramified over X in codimension 1, and*
(2) *$\mathcal{O}_{X,x}$ is regular for all $x \in X \setminus U$.*

Then there exists a finite étale morphism $Y' \to X$ whose restriction to $X \setminus D$ is Y.

Proof. Let $\xi \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$ of codimension 1. Then $\mathcal{O}_{X,\xi}$ is a discrete valuation ring. As in the discussion above, write $Y \times_U \mathrm{Spec}(K_\xi) = \mathrm{Spec}(L_\xi)$. Denote B_ξ the integral closure of $\mathcal{O}_{X,\xi}$ in L_ξ. Our assumption that Y is unramified over X in codimension 1 signifies that $\mathcal{O}_{X,\xi} \to B_\xi$ is finite étale. Thus we get $Y_\xi \to \mathrm{Spec}(\mathcal{O}_{X,\xi})$ finite étale and an isomorphism

$$Y \times_U \mathrm{Spec}(K_\xi) \cong Y_\xi \times_{\mathrm{Spec}(\mathcal{O}_{X,\xi})} \mathrm{Spec}(K_\xi)$$

over $\mathrm{Spec}(K_\xi)$. By Limits, Lemma 20.3 we find an open subscheme $U \subset U' \subset X$ containing ξ and a morphism $Y' \to U'$ of finite presentation whose restriction to U recovers Y and whose restriction to $\mathrm{Spec}(\mathcal{O}_{X,\xi})$ recovers Y_ξ. Finally, the morphism $Y' \to U'$ is finite étale after possible shrinking U' to a smaller open by Limits, Lemma 20.4. Repeating the argument with the other generic points of $X \setminus U$ of codimension 1 we may assume that we have a finite étale morphism $Y' \to U'$ extending $Y \to U$ to an open subscheme containing $U' \subset X$ containing U and all codimension 1 points of $X \setminus U$. We finish by applying Lemma 21.6 to $Y' \to U'$. Namely, all local rings $\mathcal{O}_{X,x}$ for $x \in X \setminus U'$ are regular and have $\dim(\mathcal{O}_{X,x}) \geq 2$. Hence we have purity for $\mathcal{O}_{X,x}$ by Lemma 21.3. $\square$

Lemma 31.3. *Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor such that D is a regular scheme. Let $Y \to X \setminus D$ be a finite étale morphism. If Y is unramified over X in codimension 1, then there exists a finite étale morphism $Y' \to X$ whose restriction to $X \setminus D$ is Y.*

Proof. This is a special case of Lemma 31.2. First, D is nowhere dense in X (see discussion in Divisors, Section 13) and hence $X \setminus D$ is dense in X. Second, the ring $\mathcal{O}_{X,x}$ is a regular local ring for all $x \in D$ by Algebra, Lemma 106.7 and our assumption that $\mathcal{O}_{D,x}$ is regular. $\square$

Example 31.4 (Standard tamely ramified morphism)**.** Let A be a Noetherian ring. Let $f \in A$ be a nonzerodivisor such that A/fA is reduced. This implies that $A_\mathfrak{p}$ is a discrete valuation ring with uniformizer f for any minimal prime $\mathfrak{p}$ over f. Let $e \geq 1$ be an integer which is invertible in A. Set

$$C = A[x]/(x^e - f)$$

Then $\text{Spec}(C) \to \text{Spec}(A)$ is a finite locally free morphism which is étale over the spectrum of A_f. The finite étale morphism

$$\text{Spec}(C_f) \longrightarrow \text{Spec}(A_f)$$

is tamely ramified over $\text{Spec}(A)$ in codimension 1. The tameness follows immediately from the characterization of tamely ramified extensions in More on Algebra, Lemma 114.7.

Here is a version of Abhyankar's lemma for regular divisors.

Lemma 31.5 (Abhyankar's lemma for regular divisor)**.** *Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor such that D is a regular scheme. Let $Y \to X \setminus D$ be a finite étale morphism. If Y is tamely ramified over X in codimension 1, then étale locally on X the morphism $Y \to X$ is as given as a finite disjoint union of standard tamely ramified morphisms as described in Example 31.4.*

Proof. Before we start we note that $\mathcal{O}_{X,x}$ is a regular local ring for all $x \in D$. This follows from Algebra, Lemma 106.7 and our assumption that $\mathcal{O}_{D,x}$ is regular. Below we will also use that regular rings are normal, see Algebra, Lemma 157.5.

To prove the lemma we may work locally on X. Thus we may assume $X = \text{Spec}(A)$ and $D \subset X$ is given by a nonzerodivisor $f \in A$. Then $Y = \text{Spec}(B)$ as a finite étale scheme over A_f. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes of A over f. Then $A_i = A_{\mathfrak{p}_i}$ is a discrete valuation ring; denote its fraction field K_i. By assumption

$$K_i \otimes_{A_f} B = \prod L_{ij}$$

is a finite product of fields each tamely ramified with respect to A_i. Choose $e \geq 1$ sufficiently divisible (namely, divisible by all ramification indices for L_{ij} over A_i as in More on Algebra, Remark 111.6). Warning: at this point we do not know that e is invertible on A.

Consider the finite free A-algebra

$$A' = A[x]/(x^e - f)$$

Observe that $f' = x$ is a nonzerodivisor in A' and that $A'/f'A' \cong A/fA$ is a regular ring. Set $B' = B \otimes_A A' = B \otimes_{A_f} A'_{f'}$. By Abhyankar's lemma (More on Algebra, Lemma 114.4) we see that $\text{Spec}(B')$ is unramified over $\text{Spec}(A')$ in codimension 1. Namely, by Lemma 31.1 we see that $\text{Spec}(B')$ is still at least tamely ramified over $\text{Spec}(A')$ in codimension 1. But Abhyankar's lemma tells us that the ramification indices have all become equal to 1. By Lemma 31.3 we conclude that $\text{Spec}(B') \to \text{Spec}(A'_{f'})$ extends to a finite étale morphism $\text{Spec}(C) \to \text{Spec}(A')$.

For a point $x \in D$ corresponding to $\mathfrak{p} \in V(f)$ denote A^{sh} a strict henselization of $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$. Observe that A^{sh} and $A^{sh}/fA^{sh} = (A/fA)^{sh}$ (Algebra, Lemma 156.4) are regular local rings, see More on Algebra, Lemma 45.10. Observe that A' has a unique prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ with identical residue field. Thus

$$(A')^{sh} = A^{sh} \otimes_A A' = A^{sh}[x]/(x^e - f)$$

is a strictly henselian local ring finite over A^{sh} (Algebra, Lemma 156.3). Since f' is a nonzerodivisor in $(A')^{sh}$ and since $(A')^{sh}/f'(A')^{sh} = A^{sh}/fA^{sh}$ is regular, we conclude that $(A')^{sh}$ is a regular local ring (see above). Observe that the induced extension

$$Q(A^{sh}) \subset Q((A')^{sh}) = Q(A^{sh})[x]/(x^e - f)$$

of fraction fields has degree e (and not less). Since $A' \to C$ is finite étale we see that $A^{sh} \otimes_A C$ is a finite product of copies of $(A')^{sh}$ (Algebra, Lemma 153.6). We have the inclusions

$$A^{sh}_f \subset A^{sh} \otimes_A B \subset A^{sh} \otimes_A B' = A^{sh} \otimes_A C_{f'}$$

and each of these rings is Noetherian and normal; this follows from Algebra, Lemma 163.9 for the ring in the middle. Taking total quotient rings, using the product decomposition of $A^{sh} \otimes_A C$ and using Fields, Lemma 24.3 we conclude that there is an isomorphism

$$Q(A^{sh}) \otimes_A B \cong \prod\nolimits_{i \in I} F_i, \quad F_i \cong Q(A^{sh})[x]/(x^{e_i} - f)$$

of $Q(A^{sh})$-algebras for some finite set I and integers $e_i | e$. Since $A^{sh} \otimes_A B$ is a normal ring, it must be the integral closure of A^{sh} in its total quotient ring. We conclude that we have an isomorphism

$$A^{sh} \otimes_A B \cong \prod A^{sh}_f[x]/(x^{e_i} - f)$$

over A^{sh}_f because the algebras $A^{sh}[x]/(x^{e_i} - f)$ are regular and hence normal. The discriminant of $A^{sh}[x]/(x^{e_i} - f)$ over A^{sh} is $e_i^{e_i} f^{e_i - 1}$ (up to sign; calculation omitted). Since $A_f \to B$ is finite étale we see that e_i must be invertible in A^{sh}_f. On the other hand, since $A_f \to B$ is tamely ramified over $\text{Spec}(A)$ in codimension 1, by Lemma 31.1 the ring map $A^{sh}_f \to A^{sh} \otimes_A B$ is tamely ramified over $\text{Spec}(A^{sh})$ in codimension 1. This implies e_i is nonzero in A^{sh}/fA^{sh} (as it must map to an invertible element of the fraction field of this domain by definition of tamely ramified extensions). We conclude that $V(e_i) \subset \text{Spec}(A^{sh})$ has codimension ≥ 2 which is absurd unless it is empty. In other words, e_i is an invertible element of A^{sh}. We conclude that the pullback of Y to $\text{Spec}(A^{sh})$ is indeed a finite disjoint union of standard tamely ramified morphisms.

To finish the proof, we write $A^{sh} = \text{colim}\, A_\lambda$ as a filtered colimit of étale A-algebras A_λ. The isomorphism

$$A^{sh} \otimes_A B \cong \prod\nolimits_{i \in I} A^{sh}_f[x]/(x^{e_i} - f)$$

descends to an isomorphism

$$A_\lambda \otimes_A B \cong \prod\nolimits_{i \in I} (A_\lambda)_f[x]/(x^{e_i} - f)$$

for suitably large λ. After increasing λ a bit more we may assume e_i is invertible in A_λ. Then $\text{Spec}(A_\lambda) \to \text{Spec}(A)$ is the desired étale neighbourhood of x and the proof is complete. □

Lemma 31.6. *In the situation of Lemma 31.5 the normalization of X in Y is a finite locally free morphism $\pi : Y' \to X$ such that*

1. *the restriction of Y' to $X \setminus D$ is isomorphic to Y,*
2. *$D' = \pi^{-1}(D)_{red}$ is an effective Cartier divisor on Y', and*
3. *D' is a regular scheme.*

Moreover, étale locally on X the morphism $Y' \to X$ is a finite disjoint union of morphisms

$$\mathrm{Spec}(A[x]/(x^e - f)) \to \mathrm{Spec}(A)$$

where A is a Noetherian ring, $f \in A$ is a nonzerodivisor with A/fA regular, and $e \geq 1$ is invertible in A.

Proof. This is just an addendum to Lemma 31.5 and in fact the truth of this lemma follows almost immediately if you've read the proof of that lemma. But we can also deduce the lemma from the result of Lemma 31.5. Namely, taking the normalization of X in Y commutes with étale base change, see More on Morphisms, Lemma 19.2. Hence we see that we may prove the statements on the local structure of $Y' \to X$ étale locally on X. Thus, by Lemma 31.5 we may assume that $X = \mathrm{Spec}(A)$ where A is a Noetherian ring, that we have a nonzerodivisor $f \in A$ such that A/fA is regular, and that Y is a finite disjoint union of spectra of rings $A_f[x]/(x^e - f)$ where e is invertible in A. We omit the verification that the integral closure of A in $A_f[x]/(x^e - f)$ is equal to $A' = A[x]/(x^e - f)$. (To see this argue that the localizations of A' at primes lying over (f) are regular.) We omit the details. $\square$

Lemma 31.7. *In the situation of Lemma 31.5 let $Y' \to X$ be as in Lemma 31.6. Let R be a discrete valuation ring with fraction field K. Let*

$$t : \mathrm{Spec}(R) \to X$$

be a morphism such that the scheme theoretic inverse image $t^{-1}D$ is the reduced closed point of $\mathrm{Spec}(R)$.

(1) *If $t|_{\mathrm{Spec}(K)}$ lifts to a point of Y, then we get a lift $t' : \mathrm{Spec}(R) \to Y'$ such that $Y' \to X$ is étale along $t'(\mathrm{Spec}(R))$.*
(2) *If $\mathrm{Spec}(K) \times_X Y$ is isomorphic to a disjoint union of copies of $\mathrm{Spec}(K)$, then $Y' \to X$ is finite étale over an open neighbourhood of $t(\mathrm{Spec}(R))$.*

Proof. By the valuative criterion of properness applied to the finite morphism $Y' \to X$ we see that $\mathrm{Spec}(K)$-valued points of Y matching $t|_{\mathrm{Spec}(K)}$ as maps into X lift uniquely to morphisms $t' : \mathrm{Spec}(R) \to Y'$. Thus statement (1) make sense.

Choose an étale neighbourhood $(U, u) \to (X, t(\mathfrak{m}_R))$ such that $U = \mathrm{Spec}(A)$ and such that $Y' \times_X U \to U$ has a description as in Lemma 31.6 for some $f \in A$. Then $\mathrm{Spec}(R) \times_X U \to \mathrm{Spec}(R)$ is étale and surjective. If R' denotes the local ring of $\mathrm{Spec}(R) \times_X U$ lying over the closed point of $\mathrm{Spec}(R)$, then R' is a discrete valuation ring and $R \subset R'$ is an unramified extension of discrete valuation rings (More on Algebra, Lemma 44.4). The assumption on t signifies that the map $A \to R'$ corresponding to

$$\mathrm{Spec}(R') \to \mathrm{Spec}(R) \times_X U \to U$$

maps f to a uniformizer $\pi \in R'$. Now suppose that

$$Y' \times_X U = \coprod\nolimits_{i \in I} \mathrm{Spec}(A[x]/(x^{e_i} - f))$$

for some $e_i \geq 1$. Then we see that

$$\mathrm{Spec}(R') \times_U (Y' \times_X U) = \coprod\nolimits_{i \in I} \mathrm{Spec}(R'[x]/(x^{e_i} - \pi))$$

The rings $R'[x]/(x^{e_i} - f)$ are discrete valuation rings (More on Algebra, Lemma 114.2) and hence have no map into the fraction field of R' unless $e_i = 1$.

Proof of (1). In this case the map $t' : \mathrm{Spec}(R) \to Y'$ base changes to determine a corresponding map $t'' : \mathrm{Spec}(R') \to Y' \times_X U$ which must map into a summand corresponding to $i \in I$ with $e_i = 1$ by the discussion above. Thus clearly we see that $Y' \times_X U \to U$ is étale along the image of t''. Since being étale is a property one can check after étale base chamge, this proves (1).

Proof of (2). In this case the assumption implies that $e_i = 1$ for all $i \in I$. Thus $Y' \times_X U \to U$ is finite étale and we conclude as before. $\square$

Lemma 31.8. *Let S be an integral normal Noetherian scheme with generic point η. Let $f : X \to S$ be a smooth morphism with geometrically connected fibres. Let $\sigma : S \to X$ be a section of f. Let $Z \to X_\eta$ be a finite étale Galois cover (Section 7) with group G of order invertible on S such that Z has a $\kappa(\eta)$-rational point mapping to $\sigma(\eta)$. Then there exists a finite étale Galois cover $Y \to X$ with group G whose restriction to X_η is Z.*

Proof. First assume $S = \mathrm{Spec}(R)$ is the spectrum of a discrete valuation ring R with closed point $s \in S$. Then X_s is an effective Cartier divisor in X and X_s is regular as a scheme smooth over a field. Moreover the generic fibre X_η is the open subscheme $X \setminus X_s$. It follows from More on Algebra, Lemma 112.2 and the assumption on G that Z is tamely ramified over X in codimension 1. Let $Z' \to X$ be as in Lemma 31.6. Observe that the action of G on Z extends to an action of G on Z'. By Lemma 31.7 we see that $Z' \to X$ is finite étale over an open neighbourhood of $\sigma(y)$. Since X_s is irreducible, this implies $Z \to X_\eta$ is unramified over X in codimension 1. Then we get a finite étale morphism $Y \to X$ whose restriction to X_η is Z by Lemma 31.3. Of course $Y \cong Z'$ (details omitted; hint: compute étale locally) and hence Y is a Galois cover with group G.

General case. Let $U \subset S$ be a maximal open subscheme such that there exists a finite étale Galois cover $Y \to X \times_S U$ with group G whose restriction to X_η is isomorphic to Z. Assume $U \neq S$ to get a contradiction. Let $s \in S \setminus U$ be a generic point of an irreducible component of $S \setminus U$. Then the inverse image U_s of U in $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the punctured spectrum of $\mathcal{O}_{S,s}$. We claim $Y \times_S U_s \to X \times_S U_s$ is the restriction of a finite étale Galois cover $Y'_s \to X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ with group G.

Let us first prove the claim produces the desired contradiction. By Limits, Lemma 20.3 we find an open subscheme $U \subset U' \subset S$ containing s and a morphism $Y'' \to U'$ of finite presentation whose restriction to U recovers $Y' \to U$ and whose restriction to $\mathrm{Spec}(\mathcal{O}_{S,s})$ recovers Y'_s. Moreover, by the equivalence of categories given in the lemma, we may assume after shrinking U' there is a morphism $Y'' \to U' \times_S X$ and there is an action of G on Y'' over $U' \times_S X$ compatible with the given morphisms and actions after base change to U and $\mathrm{Spec}(\mathcal{O}_{S,s})$. After shrinking U' further if necessary, we may assume $Y'' \to U \times_S X$ is finite étale, see Limits, Lemma 20.4. This means we have found a strictly larger open of S over which Y extends to a finite étale Galois cover with group G which gives the contradiction we were looking for.

Proof of the claim. We may and do replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$. Then $S = \mathrm{Spec}(A)$ where $(A, \mathfrak{m})$ is a local normal domain. Also $U \subset S$ is the punctured spectrum and we have a finite étale Galois cover $Y \to X \times_S U$ with group G. If $\dim(A) = 1$, then we can construct the extension of Y to a Galois covering of X by the first paragraph of the proof. Thus we may assume $\dim(A) \geq 2$ and hence $\mathrm{depth}(A) \geq 2$

as S is normal, see Algebra, Lemma 157.4. Since $X \to S$ is flat, we conclude that $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ for every point $x \in X$ mapping to s, see Algebra, Lemma 163.2. Let

$$Y' \longrightarrow X$$

be the finite morphism constructed in Lemma 21.5 using $Y \to X \times_S U$. Observe that we obtain a canonical G-action on Y. Thus all that remains is to show that Y' is étale over X. In fact, by Lemma 26.3 (for example) it even suffices to show that $Y' \to X$ is étale over the (unique) generic point of the fibre X_s. This we do by a local calculation in a (formal) neighbourhood of $\sigma(s)$.

Choose an affine open $\text{Spec}(B) \subset X$ containing $\sigma(s)$. Then $A \to B$ is a smooth ring map which has a section $\sigma : B \to A$. Denote $I = \text{Ker}(\sigma)$ and denote $B^\wedge$ the I-adic completion of B. Then $B^\wedge \cong A[[x_1, \ldots, x_d]]$ for some $d \geq 0$, see Algebra, Lemma 139.4. Of course $B \to B^\wedge$ is flat (Algebra, Lemma 97.2) and the image of $\text{Spec}(B^\wedge) \to X$ contains the generic point of X_s. Let $V \subset \text{Spec}(B^\wedge)$ be the inverse image of U. Consider the finite étale morphism

$$W = Y \times_{(X \times_S U)} V \longrightarrow V$$

By the compatibility of the construction of Y' with flat base change in Lemma 21.5 we find that the base chang $Y' \times_X \text{Spec}(B^\wedge) \to \text{Spec}(B^\wedge)$ is constructed from $W \to V$ over $\text{Spec}(B^\wedge)$ by the procedure in Lemma 21.5. Set $V_0 = V \cap V(x_1, \ldots, x_d) \subset V$ and $W_0 = W \times_V V_0$. This is a normal integral scheme which maps into $\sigma(S)$ by the morphism $\text{Spec}(B^\wedge) \to X$ and in fact is identified with $\sigma(U)$. Hence we know that $W_0 \to V_0 = U$ completely decomposes as this is true for its generic fibre by our assumption on $Z \to X_\eta$ having a $\kappa(\eta)$-rational point lying over $\sigma(\eta)$ (and of course the G-action then implies the whole fibre $Z_{\sigma(\eta)}$ is a disjoint union of copies of the scheme $\eta = \text{Spec}(\kappa(\eta))$). Finally, by Lemma 26.1 we have

$$W_0 \times_U V \cong W$$

This shows that W is a disjoint union of copies of V and hence $Y' \times_X \text{Spec}(B^\wedge)$ is a disjoint union of copies of $\text{Spec}(B^\wedge)$ and the proof is complete. ☐

Lemma 31.9. *Let S be a quasi-compact and quasi-separated integral normal scheme with generic point η. Let $f : X \to S$ be a quasi-compact and quasi-separated smooth morphism with geometrically connected fibres. Let $\sigma : S \to X$ be a section of f. Let $Z \to X_\eta$ be a finite étale Galois cover (Section 7) with group G of order invertible on S such that Z has a $\kappa(\eta)$-rational point mapping to $\sigma(\eta)$. Then there exists a finite étale Galois cover $Y \to X$ with group G whose restriction to X_η is Z.*

Proof. If S is Noetherian, then this is the result of Lemma 31.8. The general case follows from this by a standard limit argument. We strongly urge the reader to skip the proof.

We can write $S = \lim S_i$ as a directed limit of a system of schemes with affine transition morphisms and with S_i of finite type over $\mathbf{Z}$, see Limits, Proposition 5.4. For each i let $S \to S_i' \to S_i$ be the normalization of S_i in S, see Morphisms, Section 53. Combining Algebra, Proposition 162.16 Morphisms, Lemmas 53.15 and 53.13 we conclude that S_i' is of finite type over $\mathbf{Z}$, finite over S_i, and that S_i' is an integral normal scheme such that $S \to S_i'$ is dominant. By Morphisms, Lemma 53.5 we obtain transition morphisms $S_{i'}' \to S_i'$ compatible with the transition morphisms $S_{i'} \to S_i$ and with the morphisms with source S. We claim that $S = \lim S_i'$. Proof of claim omitted (hint: look on affine opens over a chosen affine open in S_i for some i to translate this into a straightforward algebra problem). We conclude that we may write $S = \lim S_i$ as a directed limit of a system of normal integral schemes S_i with affine transition morphisms and with S_i of finite type over $\mathbf{Z}$.

For some i we can find a smooth morphism $X_i \to S_i$ of finite presentation whose base change to S is $X \to S$. See Limits, Lemmas 10.1 and 8.9. After increasing i we may assume the section σ lifts to a section $\sigma_i : S_i \to X_i$ (by the equivalence of categories in Limits, Lemma 10.1). We may replace X_i by the open subscheme X_i^0 of it studied in More on Morphisms, Section 29 since the image of $X \to X_i$ clearly maps into it (openness by More on Morphisms, Lemma 29.6). Thus we may assume the fibres of $X_i \to S_i$ are geometrically connected. After increasing i we may assume $|G|$ is invertible on S_i. Let $\eta_i \in S_i$ be the generic point. Since X_η is the limit of the schemes X_{i,η_i} we can use the exact same arguments to descent $Z \to X_\eta$ to some finite étale Galois cover $Z_i \to X_{i,\eta_i}$ after possibly increasing i. See Lemma 14.1. After possibly increasing i once more we may assume Z_i has a $\kappa(\eta_i)$-rational point mapping to $\sigma_i(\eta_i)$. Then we apply the lemma in the Noetherian case and we pullback to X to conclude. $\square$

32. Tricks in positive characteristic

In Piotr Achinger's paper [Ach17] it is shown that an affine scheme in positive characteristic is always a $K(\pi, 1)$. In this section we explain the more elementary parts of [Ach17]. Namely, we show that for a field k of positive characteristic an affine scheme étale over $\mathbf{A}^n_k$ is actually finite étale over $\mathbf{A}^n_k$ (by a different morphism). We also show that a closed immersion of connected affine schemes in positive characteristic induces an injective map on étale fundamental groups.

Let k be a field of characteristic $p > 0$. Let

$$k[x_1, \ldots, x_n] \longrightarrow A$$

be a surjection of finite type k-algebras whose source is the polynomial algebra on $x_1, \ldots, x_n$. Denote $I \subset k[x_1, \ldots, x_n]$ the kernel so that we have $A = k[x_1, \ldots, x_n]/I$. We do not assume A is nonzero (in other words, we allow the case where A is the zero ring and $I = k[x_1, \ldots, x_n]$). Finally, we assume given a finite étale ring map $\pi : A \to B$.

Suppose given $k, n, k[x_1, \ldots, x_n] \to A, I, \pi : A \to B$. Let C be a k-algebra. Consider commutative diagrams

$$\begin{tikzcd}
 & B \\
C \arrow[r] & C/\varphi(I)C \arrow[u, "\tau"] \\
k[x_1, \ldots, x_n] \arrow[u, "\varphi"] \arrow[r] & A \arrow[u] \arrow[uu, bend right, "\pi"']
\end{tikzcd}$$

where φ is an étale k-algebra map and τ is a surjective k-algebra map. Let C, φ, τ be given. For any $r \geq 0$ and $y_1, \ldots, y_r \in C$ which generate C as an algebra over $\mathrm{Im}(\varphi)$ let $s = s(r, y_1, \ldots, y_r) \in \{0, \ldots, r\}$ be the maximal element such that y_i is integral over $\mathrm{Im}(\varphi)$ for $1 \leq i \leq s$. We define $NF(C, \varphi, \tau)$ to be the minimum value of $r - s = r - s(r, y_1, \ldots, y_r)$ for all choices of r and $y_1, \ldots, y_r$ as above. Observe that $NF(C, \varphi, \tau)$ is 0 if and only if φ is finite.

Lemma 32.1. *In the situation above, if $NF(C, \varphi, \tau) > 0$, then there exist an étale k-algebra map φ' and a surjective k-algebra map τ' fitting into the commutative diagram*

$$\begin{tikzcd}
 & B \\
C \arrow[r] & C/\varphi'(I)C \arrow[u, "\tau'"'] \\
k[x_1, \ldots, x_n] \arrow[u, "\varphi'"] \arrow[r] & A \arrow[u] \arrow[uu, bend right, "\pi"']
\end{tikzcd}$$

with $NF(C, \varphi', \tau') < NF(C, \varphi, \tau)$.

Proof. Choose $r \geq 0$ and $y_1, \ldots, y_r \in C$ which generate C over $\mathrm{Im}(\varphi)$ and let $0 \leq s \leq r$ be such that $y_1, \ldots, y_s$ are integral over $\mathrm{Im}(\varphi)$ such that $r - s = NF(C, \varphi, \tau) > 0$. Since B is finite over A, the image of y_{s+1} in B satisfies a monic polynomial over A. Hence we can find $d \geq 1$ and $f_1, \ldots, f_d \in k[x_1, \ldots, x_n]$ such that

$$z = y_{s+1}^d + \varphi(f_1) y_{s+1}^{d-1} + \ldots + \varphi(f_d) \in J = \mathrm{Ker}(C \to C/\varphi(I)C \xrightarrow{\tau} B)$$

Since $\varphi : k[x_1, \ldots, x_n] \to C$ is étale, we can find a nonzero and nonconstant polynomial $g \in k[T_1, \ldots, T_{n+1}]$ such that

$$g(\varphi(x_1), \ldots, \varphi(x_n), z) = 0 \quad \text{in} \quad C$$

To see this you can use for example that $C \otimes_{\varphi, k[x_1, \ldots, x_n]} k(x_1, \ldots, x_n)$ is a finite product of finite separable field extensions of $k(x_1, \ldots, x_n)$ (see Algebra, Lemmas 143.4) and hence z satisfies a monic polynomial over $k(x_1, \ldots, x_n)$. Clearing denominators we obtain g.

The existence of g and Algebra, Lemma 115.2 produce integers $e_1, e_2, \ldots, e_n \geq 1$ such that z is integral over the subring C' of C generated by $t_1 = \varphi(x_1) + z^{pe_1}, \ldots, t_n = \varphi(x_n) + z^{pe_n}$. Of course, the elements $\varphi(x_1), \ldots, \varphi(x_n)$ are also integral over C' as are the elements $y_1, \ldots, y_s$. Finally, by our choice of z the element y_{s+1} is integral over C' too.

Consider the ring map

$$\varphi' : k[x_1, \ldots, x_n] \longrightarrow C, \quad x_i \longmapsto t_i$$

with image C'. Since $\mathrm{d}(\varphi(x_i)) = \mathrm{d}(t_i) = \mathrm{d}(\varphi'(x_i))$ in $\Omega_{C/k}$ (and this is where we use the characteristic of k is $p > 0$) we conclude that φ' is étale because φ is étale, see Algebra, Lemma 151.9. Observe that $\varphi'(x_i) - \varphi(x_i) = t_i - \varphi(x_i) = z^{pe_i}$ is in the kernel J of the map $C \to C/\varphi(I)C \to B$ by our choice of z as an element of J. Hence for $f \in I$ the element

$$\varphi'(f) = f(t_1, \ldots, t_n) = f(\varphi(x_1) + z^{pe_1}, \ldots, \varphi(x_n) + z^{pe_n}) = \varphi(f) + \text{element of } (z)$$

is in J as well. In other words, $\varphi'(I)C \subset J$ and we obtain a surjection

$$\tau' : C/\varphi'(I)C \longrightarrow C/J \cong B$$

of algebras étale over A. Finally, the algebra C is generated by the elements $\varphi(x_1), \ldots, \varphi(x_n), y_1, \ldots, y_r$ over $C' = \mathrm{Im}(\varphi')$ with $\varphi(x_1), \ldots, \varphi(x_n), y_1, \ldots, y_{s+1}$ integral over $C' = \mathrm{Im}(\varphi')$. Hence $NF(C, \varphi', \tau') < r - s = NF(C, \varphi, \tau)$. This finishes the proof. $\square$

Lemma 32.2. *Let k be a field of characteristic $p > 0$. Let $X \to \mathbf{A}^n_k$ be an étale morphism with X affine. Then there exists a finite étale morphism $X \to \mathbf{A}^n_k$.*

Proof. Write $X = \mathrm{Spec}(C)$. Set $A = 0$ and denote $I = k[x_1, \ldots, x_n]$. By assumption there exists some étale k-algebra map $\varphi : k[x_1, \ldots, x_n] \to C$. Denote $\tau : C/\varphi(I)C \to 0$ the unique surjection. We may choose φ and τ such that $N(C, \varphi, \tau)$ is minimal. By Lemma 32.1 we get $N(C, \varphi, \tau) = 0$. Hence φ is finite étale. $\square$

Lemma 32.3. *Let k be a field of characteristic $p > 0$. Let $Z \subset \mathbf{A}^n_k$ be a closed subscheme. Let $Y \to Z$ be finite étale. There exists a finite étale morphism $f : U \to \mathbf{A}^n_k$ such that there is an open and closed immersion $Y \to f^{-1}(Z)$ over Z.*

Proof. Let us turn the problem into algebra. Write $\mathbf{A}^n_k = \mathrm{Spec}(k[x_1, \ldots, x_n])$. Then $Z = \mathrm{Spec}(A)$ where $A = k[x_1, \ldots, x_n]/I$ for some ideal $I \subset k[x_1, \ldots, x_n]$. Write $Y = \mathrm{Spec}(B)$ so that $Y \to Z$ corresponds to the finite étale k-algebra map $A \to B$.

By Algebra, Lemma 143.10 there exists an étale ring map

$$\varphi : k[x_1, \ldots, x_n] \to C$$

and a surjective A-algebra map $\tau : C/\varphi(I)C \to B$. (We can even choose C, φ, τ such that τ is an isomorphism, but we won't use this). We may choose φ and τ such that $N(C, \varphi, \tau)$ is minimal. By Lemma 32.1 we get $N(C, \varphi, \tau) = 0$. Hence φ is finite étale.

Let $f : U = \mathrm{Spec}(C) \to \mathbf{A}^n_k$ be the finite étale morphism corresponding to φ. The morphism $Y \to f^{-1}(Z) = \mathrm{Spec}(C/\varphi(I)C)$ induced by τ is a closed immersion as τ is surjective and open as it is an étale morphism by Morphisms, Lemma 36.18. This finishes the proof. $\square$

Here is the main result.

Proposition 32.4. *Let p be a prime number. Let $i : Z \to X$ be a closed immersion of connected affine schemes over $\mathbf{F}_p$. For any geometric point $\overline{z}$ of Z the map*

$$\pi_1(Z, \overline{z}) \to \pi_1(X, \overline{z})$$

is injective.

Proof. Let $Y \to Z$ be a finite étale morphism. It suffices to construct a finite étale morphism $f : U \to X$ such that Y is isomorphic to an open and closed subscheme of $f^{-1}(Z)$, see Lemma 4.4. Write $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(R)$ so the closed immersion $Y \to X$ is given by a surjection $R \to A$. We may write $A = \mathrm{colim}\, A_i$ as the filtered colimit of its $\mathbf{F}_p$-subalgebras of finite type. By Lemma 14.1 we can find an i and a finite étale morphism $Y_i \to Z_i = \mathrm{Spec}(A_i)$ such that $Y = Z \times_{Z_i} Y_i$.

Choose a surjection $\mathbf{F}_p[x_1, \ldots, x_n] \to A_i$. This determines a closed immersion

$$Z_i = \mathrm{Spec}(A_i) \longrightarrow X_i = \mathbf{A}^n_{\mathbf{F}_p} = \mathrm{Spec}(\mathbf{F}_p[x_1, \ldots, x_n])$$

By the universal property of polynomial algebras and since $R \to A$ is surjective, we can find a commutative diagram

$$\begin{tikzcd} \mathbf{F}_p[x_1, \ldots, x_n] \arrow[r] \arrow[d] & A_i \arrow[d] \\ R \arrow[r] & A \end{tikzcd}$$

of $\mathbf{F}_p$-algebras. Thus we have a commutative diagram

$$\begin{tikzcd} Y_i \arrow[r] & Z_i \arrow[r] & X_i \\ Y \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u] & X \arrow[u] \end{tikzcd}$$

whose right square is cartesian. Clearly, if we can find $f_i : U_i \to X_i$ finite étale such that Y_i is isomorphic to an open and closed subscheme of $f_i^{-1}(Z_i)$, then the base change $f : U \to X$ of f_i by $X \to X_i$ is a solution to our problem. Thus we conclude by applying Lemma 32.3 to $Y_i \to Z_i \to X_i = \mathbf{A}^n_{\mathbf{F}_p}$. □

33. Other chapters

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(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
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Schemes

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(27) Constructions of Schemes
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(49) Discriminants and Differents
(50) de Rham Cohomology
(51) Local Cohomology
(52) Algebraic and Formal Geometry
(53) Algebraic Curves

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