MORE ON GROUPOID SCHEMES

Contents

1. Introduction

This chapter is devoted to advanced topics on groupoid schemes. Even though the results are stated in terms of groupoid schemes, the reader should keep in mind the 2-cartesian diagram

(1.0.1)
$$
\begin{array}{ccc}
R & \longrightarrow & U \\
\downarrow & & \downarrow \\
U & \longrightarrow & [U/R]
\end{array}
$$

where $[U/R]$ is the quotient stack, see Groupoids in Spaces, Remark 20.4. Many of the results are motivated by thinking about this diagram. See for example the beautiful paper [KM97] by Keel and Mori.

2. Notation

We continue to abide by the conventions and notation introduced in Groupoids, Section 2.

3. Useful diagrams

We briefly restate the results of Groupoids, Lemmas 13.4 and 13.5 for easy reference in this chapter. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. In the commutative diagram

(3.0.1)

$$
\begin{array}{ccc}
 & U & \\
 & \nearrow^{t} \quad \nwarrow^{t} & \\
R \xleftarrow{\mathrm{pr}_0} R \times_{s,U,t} R \xrightarrow{c} R \\
\downarrow s \quad\quad \downarrow \mathrm{pr}_1 \quad\quad \downarrow s \\
U \xleftarrow{t} R \xrightarrow{s} U
\end{array}
$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

The diagram

(3.0.2)

$$
\begin{array}{ccc}
R \times_{t,U,t} R \xrightarrow[\mathrm{pr}_0]{\mathrm{pr}_1} R \xrightarrow{t} U \\
\downarrow{\scriptstyle \mathrm{pr}_0 \times c\circ(i,1)} \quad\quad \downarrow{\scriptstyle \mathrm{id}_R} \quad \downarrow{\scriptstyle \mathrm{id}_U} \\
R \times_{s,U,t} R \xrightarrow[\mathrm{pr}_0]{c} R \xrightarrow{t} U \\
\downarrow{\scriptstyle \mathrm{pr}_1} \quad\quad\quad \downarrow{\scriptstyle s} \\
R \xrightarrow[t]{s} U
\end{array}
$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

4. Sheaf of differentials

The following lemma is the analogue of Groupoids, Lemma 6.3.

Lemma 4.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. The sheaf of differentials of R seen as a scheme over U via t is a quotient of the pullback via t of the conormal sheaf of the immersion $e : U \to R$. In a formula: there is a canonical surjection $t^*\mathcal{C}_{U/R} \to \Omega_{R/U}$. If s is flat, then this map is an isomorphism.*

Proof. Note that $e : U \to R$ is an immersion as it is a section of the morphism s, see Schemes, Lemma 21.11. Consider the following diagram

$$
\begin{array}{ccc}
R \xrightarrow{(1,i)} R \times_{s,U,t} R \xrightarrow{(\mathrm{pr}_0, i\circ\mathrm{pr}_1)} R \times_{t,U,t} R \\
\downarrow t \quad\quad\quad \downarrow c \\
U \xrightarrow{e} R
\end{array}
$$

The square on the left is cartesian, because if $a \circ b = e$, then $b = i(a)$. The composition of the horizontal maps is the diagonal morphism of $t : R \to U$. The right top horizontal arrow is an isomorphism. Hence since $\Omega_{R/U}$ is the conormal sheaf of the composition it is isomorphic to the conormal sheaf of $(1, i)$. By Morphisms, Lemma 31.4 we get the surjection $t^*\mathcal{C}_{U/R} \to \Omega_{R/U}$ and if c is flat, then this is an
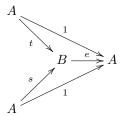
isomorphism. Since c is a base change of s by the properties of Diagram (3.0.2) we conclude that if s is flat, then c is flat, see Morphisms, Lemma 25.8. $\qquad\square$

5. Local structure

Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $u \in U$ be a point. In this section we explain what kind of structure we obtain on the local rings

$$A = \mathcal{O}_{U,u} \quad \text{and} \quad B = \mathcal{O}_{R,e(u)}$$

The convention we will use is to denote the local ring homomorphisms induced by the morphisms s, t, c, e, i by the corresponding letters. In particular we have a commutative diagram

of local rings. Thus if $I \subset B$ denotes the kernel of $e : B \to A$, then $B = s(A) \oplus I = t(A) \oplus I$. Let us denote

$$C = \mathcal{O}_{R \times_{s,U,t} R, (e(u), e(u))}$$

Then we have

$$C = (B \otimes_{s,A,t} B)_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B}$$

Let $J \subset C$ be the ideal of C generated by $I \otimes B + B \otimes I$. Then J is also the kernel of the local ring homomorphism

$$(e, e) : C \longrightarrow A$$

The composition law $c : R \times_{s,U,t} R \to R$ corresponds to a ring map

$$c : B \longrightarrow C$$

sending I into J.

Lemma 5.1. *The map $I/I^2 \to J/J^2$ induced by c is the composition*

$$I/I^2 \xrightarrow{(1,1)} I/I^2 \oplus I/I^2 \to J/J^2$$

where the second arrow comes from the equality $J = (I \otimes B + B \otimes I)C$. The map $i : B \to B$ induces the map $-1 : I/I^2 \to I/I^2$.

Proof. To describe a local homomorphism from C to another local ring it is enough to say what happens to elements of the form $b_1 \otimes b_2$. Keeping this in mind we have the two canonical maps

$$e_2 : C \to B, \ b_1 \otimes b_2 \mapsto b_1 s(e(b_2)), \quad e_1 : C \to B, \ b_1 \otimes b_2 \mapsto t(e(b_1))b_2$$

corresponding to the embeddings $R \to R \times_{s,U,t} R$ given by $r \mapsto (r, e(s(r)))$ and $r \mapsto (e(t(r)), r)$. These maps define maps $J/J^2 \to I/I^2$ which jointly give an inverse to the map $I/I^2 \oplus I/I^2 \to J/J^2$ of the lemma. Thus to prove statement we only have to show that $e_1 \circ c : B \to B$ and $e_2 \circ c : B \to B$ are the identity maps. This follows from the fact that both compositions $R \to R \times_{s,U,t} R \to R$ are identities.

The statement on i follows from the statement on c and the fact that $c \circ (1, i) = e \circ t$. Some details omitted. $\square$

6. Properties of groupoids

Let (U, R, s, t, c) be a groupoid scheme. The idea behind the results in this section is that $s : R \to U$ is a base change of the morphism $U \to [U/R]$ (see Diagram (1.0.1). Hence the local properties of $s : R \to U$ should reflect local properties of the morphism $U \to [U/R]$. This doesn't work, because $[U/R]$ is not always an algebraic stack, and hence we cannot speak of geometric or algebraic properties of $U \to [U/R]$. But it turns out that we can make some of it work without even referring to the quotient stack at all.

Here is a first example of such a result. The open $W \subset U'$ found in the lemma is roughly speaking the locus where the morphism $U' \to [U/R]$ has property $\mathcal{P}$.

Lemma 6.1. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid over S. Let $g : U' \to U$ be a morphism of schemes. Denote h the composition*

$$h : U' \times_{g,U,t} R \xrightarrow[pr_1]{} R \xrightarrow{s} U.$$

Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be properties of morphisms of schemes. Assume
  (1) *$\mathcal{R} \Rightarrow \mathcal{Q}$,*
  (2) *$\mathcal{Q}$ is preserved under base change and composition,*
  (3) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$ there exists a largest open $W(\mathcal{P}, f) \subset X$ such that $f|_{W(\mathcal{P},f)}$ has $\mathcal{P}$, and*
  (4) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$, and any morphism $Y' \to Y$ which has $\mathcal{R}$ we have $Y' \times_Y W(\mathcal{P}, f) = W(\mathcal{P}, f')$, where $f' : X_{Y'} \to Y'$ is the base change of f.*

If s, t have $\mathcal{R}$ and g has $\mathcal{Q}$, then there exists an open subscheme $W \subset U'$ such that $W \times_{g,U,t} R = W(\mathcal{P}, h)$.

Proof. Note that the following diagram is commutative

$$
\begin{array}{ccc}
U' \times_{g,U,t} R \times_{t,U,t} R & \xrightarrow{\quad pr_{12} \quad} & R \times_{t,U,t} R \\
\scriptstyle pr_{01} \downarrow \; \downarrow \scriptstyle pr_{02} & & \scriptstyle pr_{0} \downarrow \; \downarrow \scriptstyle pr_{1} \\
U' \times_{g,U,t} R & \xrightarrow{\quad pr_1 \quad} & R
\end{array}
$$

with both squares cartesian (this uses that the two maps $t \circ pr_i : R \times_{t,U,t} R \to U$ are equal). Combining this with the properties of diagram (3.0.2) we get a commutative diagram

$$
\begin{array}{ccc}
U' \times_{g,U,t} R \times_{t,U,t} R & \xrightarrow{\quad c\circ(i,1) \quad} & R \\
\scriptstyle pr_{01} \downarrow \; \downarrow \scriptstyle pr_{02} & & \scriptstyle t \downarrow \; \downarrow \scriptstyle s \\
U' \times_{g,U,t} R & \xrightarrow{\quad h \quad} & U
\end{array}
$$

where both squares are cartesian.

Assume s, t have $\mathcal{R}$ and g has $\mathcal{Q}$. Then h has $\mathcal{Q}$ as a composition of s (which has $\mathcal{R}$ hence $\mathcal{Q}$) and a base change of g (which has $\mathcal{Q}$). Thus $W(\mathcal{P}, h) \subset U' \times_{g,U,t} R$ exists. By our assumptions we have $pr_{01}^{-1}(W(\mathcal{P}, h)) = pr_{02}^{-1}(W(\mathcal{P}, h))$ since both are the largest open on which $c \circ (i, 1)$ has $\mathcal{P}$. Note that the projection $U' \times_{g,U,t} R \to U'$

has a section, namely $\sigma : U' \to U' \times_{g,U,t} R$, $u' \mapsto (u', e(g(u')))$. Also via the isomorphism

$$(U' \times_{g,U,t} R) \times_{U'} (U' \times_{g,U,t} R) = U' \times_{g,U,t} R \times_{t,U,t} R$$

the two projections of the left hand side to $U' \times_{g,U,t} R$ agree with the morphisms pr_{01} and pr_{02} on the right hand side. Since $\mathrm{pr}_{01}^{-1}(W(\mathcal{P}, h)) = \mathrm{pr}_{02}^{-1}(W(\mathcal{P}, h))$ we conclude that $W(\mathcal{P}, h)$ is the inverse image of a subset of U, which is necessarily the open set $W = \sigma^{-1}(W(\mathcal{P}, h))$. $\square$

Remark 6.2. Warning: Lemma 6.1 should be used with care. For example, it applies to $\mathcal{P} =$"flat", $\mathcal{Q} =$"empty", and $\mathcal{R} =$"flat and locally of finite presentation". But given a morphism of schemes $f : X \to Y$ the largest open $W \subset X$ such that $f|_W$ is flat is *not* the set of points where f is flat!

Remark 6.3. Notwithstanding the warning in Remark 6.2 there are some cases where Lemma 6.1 can be used without causing too much ambiguity. We give a list. In each case we omit the verification of assumptions (1) and (2) and we give references which imply (3) and (4). Here is the list:

(1) $\mathcal{Q} = \mathcal{R} =$"locally of finite type", and $\mathcal{P} =$"relative dimension $\leq d$". See Morphisms, Definition 29.1 and Morphisms, Lemmas 28.4 and 28.3.
(2) $\mathcal{Q} = \mathcal{R} =$"locally of finite type", and $\mathcal{P} =$"locally quasi-finite". This is the case $d = 0$ of the previous item, see Morphisms, Lemma 29.5.
(3) $\mathcal{Q} = \mathcal{R} =$"locally of finite type", and $\mathcal{P} =$"unramified". See Morphisms, Lemmas 35.3 and 35.15.

What is interesting about the cases listed above is that we do not need to assume that s, t are flat to get a conclusion about the locus where the morphism h has property $\mathcal{P}$. We continue the list:

(4) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"flat". See More on Morphisms, Theorem 15.1 and Lemma 15.2.
(5) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"Cohen-Macaulay". See More on Morphisms, Definition 22.1 and More on Morphisms, Lemmas 22.6 and 22.7.
(6) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"syntomic" use Morphisms, Lemma 30.12 (the locus is automatically open).
(7) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"smooth". See Morphisms, Lemma 34.15 (the locus is automatically open).
(8) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"étale". See Morphisms, Lemma 36.17 (the locus is automatically open).

Here is the second result. The R-invariant open $W \subset U$ should be thought of as the inverse image of the largest open of $[U/R]$ over which the morphism $U \to [U/R]$ has property $\mathcal{P}$.

Lemma 6.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$[1]. Let $\mathcal{P}$ be a property of morphisms*

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[1]The fact that $fpqc$ is missing is not a typo.

of schemes which is τ-local on the target (Descent, Definition 22.1). Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subscheme such that $s|_{s^{-1}(W)} : s^{-1}(W) \to W$ has property $\mathcal{P}$. Then W is R-invariant, see Groupoids, Definition 19.1.

Proof. The existence and properties of the open $W \subset U$ are described in Descent, Lemma 22.3. In Diagram (3.0.1) let $W_1 \subset R$ be the maximal open subscheme over which the morphism $\mathrm{pr}_1 : R \times_{s,U,t} R \to R$ has property $\mathcal{P}$. It follows from the aforementioned Descent, Lemma 22.3 and the assumption that $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology that $t^{-1}(W) = W_1 = s^{-1}(W)$ as desired. $\qquad\square$

Lemma 6.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be its stabilizer group scheme. Let $\tau \in \{fppf, \acute{e}tale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms which is τ-local on the target. Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subscheme such that $G_W \to W$ has property $\mathcal{P}$. Then W is R-invariant (see Groupoids, Definition 19.1).*

Proof. The existence and properties of the open $W \subset U$ are described in Descent, Lemma 22.3. The morphism

$$G \times_{U,t} R \longrightarrow R \times_{s,U} G, \quad (g,r) \longmapsto (r, r^{-1} \circ g \circ r)$$

is an isomorphism over R (where $\circ$ denotes composition in the groupoid). Hence $s^{-1}(W) = t^{-1}(W)$ by the properties of W proved in the aforementioned Descent, Lemma 22.3. $\qquad\square$

7. Comparing fibres

Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Diagram (3.0.1) gives us a way to compare the fibres of the map $s : R \to U$ in a groupoid. For a point $u \in U$ we will denote $F_u = s^{-1}(u)$ the scheme theoretic fibre of $s : R \to U$ over u. For example the diagram implies that if $u, u' \in U$ are points such that $s(r) = u$ and $t(r) = u'$, then $(F_u)_{\kappa(r)} \cong (F_{u'})_{\kappa(r)}$. This is a special case of the more general and more precise Lemma 7.1 below. To see this take $r' = i(r)$.

A pair (X, x) consisting of a scheme X and a point $x \in X$ is sometimes called the *germ of X at x*. A *morphism of germs* $f : (X, x) \to (S, s)$ is a morphism $f : U \to S$ defined on an open neighbourhood of x with $f(x) = s$. Two such f, f' are said to give the same morphism of germs if and only if f and f' agree in some open neighbourhood of x. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. We temporarily introduce the following concept: We say that two morphisms of germs $f : (X, x) \to (S, s)$ and $f' : (X', x') \to (S', s')$ are *isomorphic locally on the base in the τ-topology*, if there exists a pointed scheme (S'', s'') and morphisms of germs $g : (S'', s'') \to (S, s)$, and $g' : (S'', s'') \to (S', s')$ such that

(1) g and g' are an open immersion (resp. étale, smooth, syntomic, flat and locally of finite presentation) at s'',
(2) there exists an isomorphism

$$(S'' \times_{g,S,f} X, \tilde{x}) \cong (S'' \times_{g',S',f'} X', \tilde{x}')$$

of germs over the germ (S'', s'') for some choice of points $\tilde{x}$ and $\tilde{x}'$ lying over (s'', x) and (s'', x').

Finally, we simply say that the maps of germs $f : (X, x) \to (S, s)$ and $f' : (X', x') \to (S', s')$ are *flat locally on the base isomorphic* if there exist S'', s'', g, g' as above but with (1) replaced by the condition that g and g' are flat at s'' (this is much weaker than any of the τ conditions above as a flat morphism need not be open).

Lemma 7.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $r, r' \in R$ with $t(r) = t(r')$ in U. Set $u = s(r)$, $u' = s(r')$. Denote $F_u = s^{-1}(u)$ and $F_{u'} = s^{-1}(u')$ the scheme theoretic fibres.*

(1) *There exists a common field extension $\kappa(u) \subset k$, $\kappa(u') \subset k$ and an isomorphism $(F_u)_k \cong (F_{u'})_k$.*

(2) *We may choose the isomorphism of (1) such that a point lying over r maps to a point lying over r'.*

(3) *If the morphisms s, t are flat then the morphisms of germs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are flat locally on the base isomorphic.*

(4) *If the morphisms s, t are étale (resp. smooth, syntomic, or flat and locally of finite presentation) then the morphisms of germs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are locally on the base isomorphic in the étale (resp. smooth, syntomic, or fppf) topology.*

Proof. We repeatedly use the properties and the existence of diagram (3.0.1). By the properties of the diagram (and Schemes, Lemma 17.5) there exists a point ξ of $R \times_{s, U, t} R$ with $\mathrm{pr}_0(\xi) = r$ and $c(\xi) = r'$. Let $\tilde{r} = \mathrm{pr}_1(\xi) \in R$.

Proof of (1). Set $k = \kappa(\tilde{r})$. Since $t(\tilde{r}) = u$ and $s(\tilde{r}) = u'$ we see that k is a common extension of both $\kappa(u)$ and $\kappa(u')$ and in fact that both $(F_u)_k$ and $(F_{u'})_k$ are isomorphic to the fibre of $\mathrm{pr}_1 : R \times_{s, U, t} R \to R$ over $\tilde{r}$. Hence (1) is proved.

Part (2) follows since the point ξ maps to r, resp. r'.

Part (3) is clear from the above (using the point ξ for $\tilde{u}$ and $\tilde{u}'$) and the definitions.

If s and t are flat and of finite presentation, then they are open morphisms (Morphisms, Lemma 25.10). Hence the image of some affine open neighbourhood V'' of $\tilde{r}$ will cover an open neighbourhood V of u, resp. V' of u'. These can be used to show that properties (1) and (2) of the definition of "locally on the base isomorphic in the τ-topology". $\square$

8. Cohen-Macaulay presentations

Given any groupoid (U, R, s, t, c) with s, t flat and locally of finite presentation there exists an "equivalent" groupoid (U', R', s', t', c') such that s' and t' are Cohen-Macaulay morphisms (and locally of finite presentation). See More on Morphisms, Section 22 for more information on Cohen-Macaulay morphisms. Here "equivalent" can be taken to mean that the quotient stacks $[U/R]$ and $[U'/R']$ are equivalent stacks, see Groupoids in Spaces, Section 20 and Section 25.

Lemma 8.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Assume s and t are flat and locally of finite presentation. Then there exists an open $U' \subset U$ such that*
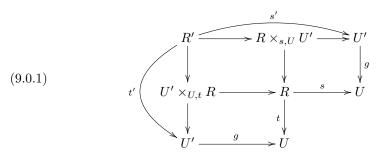
(1) *$t^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism s is Cohen-Macaulay,*

(2) *$s^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism t is Cohen-Macaulay,*

(3) *the morphism* $t|_{s^{-1}(U')} : s^{-1}(U') \to U$ *is surjective,*

(4) *the morphism* $s|_{t^{-1}(U')} : t^{-1}(U') \to U$ *is surjective, and*

(5) *the restriction* $R' = s^{-1}(U') \cap t^{-1}(U')$ *of* R *to* U' *defines a groupoid* (U', R', s', t', c') *which has the property that the morphisms* s' *and* t' *are Cohen-Macaulay and locally of finite presentation.*

Proof. Apply Lemma 6.1 with $g = \mathrm{id}$ and $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"Cohen-Macaulay", see Remark 6.3. This gives us an open $U' \subset U$ such that Let $t^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism s is Cohen-Macaulay. This proves (1). Let $i : R \to R$ be the inverse of the groupoid. Since i is an isomorphism, and $s \circ i = t$ and $t \circ i = s$ we see that $s^{-1}(U')$ is also the largest open of R on which t is Cohen-Macaulay. This proves (2). By More on Morphisms, Lemma 22.7 the open subset $t^{-1}(U')$ is dense in every fibre of $s : R \to U$. This proves (3). Same argument for (4). Part (5) is a formal consequence of (1) and (2) and the discussion of restrictions in Groupoids, Section 18. $\qquad \square$

9. Restricting groupoids

In this section we collect a bunch of lemmas on properties of groupoids which are inherited by restrictions. Most of these lemmas can be proved by contemplating the defining diagram

$$(9.0.1)$$

of a restriction. See Groupoids, Lemma 18.1.

Lemma 9.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g.*

(1) *If s, t are locally of finite type and g is locally of finite type, then s', t' are locally of finite type.*

(2) *If s, t are locally of finite presentation and g is locally of finite presentation, then s', t' are locally of finite presentation.*

(3) *If s, t are flat and g is flat, then s', t' are flat.*

(4) *Add more here.*

Proof. The property of being locally of finite type is stable under composition and arbitrary base change, see Morphisms, Lemmas 15.3 and 15.4. Hence (1) is clear from Diagram (9.0.1). For the other cases, see Morphisms, Lemmas 21.3, 21.4, 25.6, and 25.8. $\qquad \square$

The following lemma could have been used to prove the results of the preceding lemma in a more uniform way.

Lemma 9.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g, and let $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$. If $\mathcal{P}$ is a property of morphisms of schemes such that*

(1) *h has property $\mathcal{P}$, and*
(2) *$\mathcal{P}$ is preserved under base change,*

then s', t' have property $\mathcal{P}$.

Proof. This is clear as s' is the base change of h by Diagram (9.0.1) and t' is isomorphic to s' as a morphism of schemes. $\square$

Lemma 9.3. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ and $g' : U'' \to U'$ be morphisms of schemes. Set $g'' = g \circ g'$. Let (U', R', s', t', c') be the restriction of R to U'. Let $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$, let $h' = s' \circ pr_1 : U'' \times_{g',U',t} R \to U'$, and let $h'' = s \circ pr_1 : U'' \times_{g'',U,t} R \to U$. The following diagram is commutative*

$$
\begin{array}{ccccc}
U'' \times_{g',U',t} R' & \longleftarrow & (U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R) & \longrightarrow & U'' \times_{g'',U,t} R \\
\downarrow{\scriptstyle h'} & & \downarrow & & \downarrow{\scriptstyle h''} \\
U' & \xleftarrow{\ pr_0\ } & U' \times_{g,U,t} R & \xrightarrow{\ h\ } & U
\end{array}
$$

with both squares cartesian where the left upper horizontal arrow is given by the rule

$$
\begin{array}{ccc}
(U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R) & \longrightarrow & U'' \times_{g',U',t} R' \\
((u', r_0), (u'', r_1)) & \longmapsto & (u'', (c(r_1, i(r_0)), (g'(u''), u')))
\end{array}
$$

with notation as explained in the proof.

Proof. We work this out by exploiting the functorial point of view and reducing the lemma to a statement on arrows in restrictions of a groupoid category. In the last formula of the lemma the notation $((u', r_0), (u'', r_1))$ indicates a T-valued point of $(U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R)$. This means that u', u'', r_0, r_1 are T-valued points of U', U'', R, R and that $g(u') = t(r_0)$, $g(g'(u'')) = g''(u'') = t(r_1)$, and $s(r_0) = s(r_1)$. It would be more correct here to write $g \circ u' = t \circ r_0$ and so on but this makes the notation even more unreadable. If we think of r_1 and r_0 as arrows in a groupoid category then we can represent this by the picture

$$
t(r_0) = g(u') \xleftarrow{\ r_0\ } s(r_0) = s(r_1) \xrightarrow{\ r_1\ } t(r_1) = g(g'(u''))
$$

This diagram in particular demonstrates that the composition $c(r_1, i(r_0))$ makes sense. Recall that

$$
R' = R \times_{(t,s),U \times_S U, g \times g} U' \times_S U'
$$

hence a T-valued point of R' looks like $(r, (u'_0, u'_1))$ with $t(r) = g(u'_0)$ and $s(r) = g(u'_1)$. In particular given $((u', r_0), (u'', r_1))$ as above we get the T-valued point $(c(r_1, i(r_0)), (g'(u''), u'))$ of R' because we have $t(c(r_1, i(r_0))) = t(r_1) = g(g'(u''))$ and $s(c(r_1, i(r_0))) = s(i(r_0)) = t(r_0) = g(u')$. We leave it to the reader to show that the left square commutes with this definition.

To show that the left square is cartesian, suppose we are given (v'', p') and (v', p) which are T-valued points of $U'' \times_{g',U',t} R'$ and $U' \times_{g,U,t} R$ with $v' = s'(p')$. This also means that $g'(v'') = t'(p')$ and $g(v') = t(p)$. By the discussion above we know

that we can write $p' = (r, (u'_0, u'_1))$ with $t(r) = g(u'_0)$ and $s(r) = g(u'_1)$. Using this notation we see that $v' = s'(p') = u'_1$ and $g'(v'') = t'(p') = u'_0$. Here is a picture

$$s(p) \xrightarrow{\ p\ } g(v') = g(u'_1) \xrightarrow{\ r\ } g(u'_0) = g(g'(v''))$$

What we have to show is that there exists a unique T-valued point $((u', r_0), (u'', r_1))$ as above such that $v' = u'$, $p = r_0$, $v'' = u''$ and $p' = (c(r_1, i(r_0)), (g'(u''), u'))$. Comparing the two diagrams above it is clear that we have no choice but to take

$$((u', r_0), (u'', r_1)) = ((v', p), (v'', c(r, p)))$$

Some details omitted. $\qquad\square$

Lemma 9.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ and $g' : U'' \to U'$ be morphisms of schemes. Set $g'' = g \circ g'$. Let (U', R', s', t', c') be the restriction of R to U'. Let $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$, let $h' = s' \circ pr_1 : U'' \times_{g',U',t} R \to U'$, and let $h'' = s \circ pr_1 : U'' \times_{g'',U,t} R \to U$. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf, fpqc\}$. Let $\mathcal{P}$ be a property of morphisms of schemes which is preserved under base change, and which is local on the target for the τ-topology. If*

(1) *$h(U' \times_U R)$ is open in U,*
(2) *$\{h : U' \times_U R \to h(U' \times_U R)\}$ is a τ-covering,*
(3) *h' has property $\mathcal{P}$,*

then h'' has property $\mathcal{P}$. Conversely, if

(a) *$\{t : R \to U\}$ is a τ-covering,*
(d) *h'' has property $\mathcal{P}$,*

then h' has property $\mathcal{P}$.

Proof. This follows formally from the properties of the diagram of Lemma 9.3. In the first case, note that the image of the morphism h'' is contained in the image of h, as $g'' = g \circ g'$. Hence we may replace the U in the lower right corner of the diagram by $h(U' \times_U R)$. This explains the significance of conditions (1) and (2) in the lemma. In the second case, note that $\{pr_0 : U' \times_{g,U,t} R \to U'\}$ is a τ-covering as a base change of τ and condition (a). $\qquad\square$

10. Properties of groupoids on fields

A "groupoid on a field" indicates a groupoid scheme (U, R, s, t, c) where U is the spectrum of a field. It does **not** mean that (U, R, s, t, c) is defined over a field, more precisely, it does **not** mean that the morphisms $s, t : R \to U$ are equal. Given any field k, an abstract group G and a group homomorphism $\varphi : G \to \mathrm{Aut}(k)$ we obtain a groupoid scheme (U, R, s, t, c) over $\mathbf{Z}$ by setting

$$U = \mathrm{Spec}(k)$$
$$R = \coprod_{g \in G} \mathrm{Spec}(k)$$
$$s = \coprod_{g \in G} \mathrm{Spec}(\mathrm{id}_k)$$
$$t = \coprod_{g \in G} \mathrm{Spec}(\varphi(g))$$
$$c = \text{composition in } G$$

This example still is a groupoid scheme over $\mathrm{Spec}(k^G)$. Hence, if G is finite, then $U = \mathrm{Spec}(k)$ is finite over $\mathrm{Spec}(k^G)$. In some sense our goal in this section is to show that suitable finiteness conditions on s, t force any groupoid on a field to be defined over a finite index subfield $k' \subset k$.

If k is a field and (G, m) is a group scheme over k with structure morphism $p : G \to \mathrm{Spec}(k)$, then $(\mathrm{Spec}(k), G, p, p, m)$ is an example of a groupoid on a field (and in this case of course the whole structure is defined over a field). Hence this section can be viewed as the analogue of Groupoids, Section 7.

Lemma 10.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, then the composition morphism $c : R \times_{s,U,t} R \to R$ is open.*

Proof. The composition is isomorphic to the projection map $\mathrm{pr}_1 : R \times_{t,U,t} R \to R$ by Diagram (3.0.2). The projection is open by Morphisms, Lemma 23.4. $\square$

Lemma 10.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, then R is a separated scheme.*

Proof. By Groupoids, Lemma 7.3 the stabilizer group scheme $G \to U$ is separated. By Groupoids, Lemma 22.2 the morphism $j = (t, s) : R \to U \times_S U$ is separated. As U is the spectrum of a field the scheme $U \times_S U$ is affine (by the construction of fibre products in Schemes, Section 17). Hence R is a separated scheme, see Schemes, Lemma 21.12. $\square$
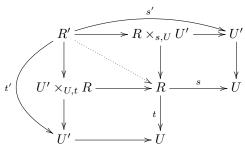
Lemma 10.3. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. For any points $r, r' \in R$ there exists a field extension k'/k and points $r_1, r_2 \in R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k')$ and a diagram*

$$R \xleftarrow{\mathrm{pr}_0} R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k') \xrightarrow{\varphi} R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k') \xrightarrow{\mathrm{pr}_0} R$$

such that φ is an isomorphism of schemes over $\mathrm{Spec}(k')$, we have $\varphi(r_1) = r_2$, $\mathrm{pr}_0(r_1) = r$, and $\mathrm{pr}_0(r_2) = r'$.

Proof. This is a special case of Lemma 7.1 parts (1) and (2). $\square$

Lemma 10.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. Let k'/k be a field extension, $U' = \mathrm{Spec}(k')$ and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $U' \to U$. In the defining diagram*

all the morphisms are surjective, flat, and universally open. The dotted arrow $R' \to R$ is in addition affine.

Proof. The morphism $U' \to U$ equals $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, hence is affine, surjective and flat. The morphisms $s, t : R \to U$ and the morphism $U' \to U$ are universally open by Morphisms, Lemma 23.4. Since R is not empty and U is the spectrum of a field the morphisms $s, t : R \to U$ are surjective and flat. Then you conclude by using Morphisms, Lemmas 9.4, 9.2, 23.3, 11.8, 11.7, 25.8, and 25.6. $\square$
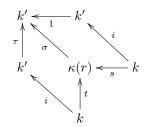
Lemma 10.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. For any point $r \in R$ there exist*

(1) *a field extension k'/k with k' algebraically closed,*
(2) *a point $r' \in R'$ where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$*

such that

(1) *the point r' maps to r under the morphism $R' \to R$, and*
(2) *the maps $s', t' : R' \to \mathrm{Spec}(k')$ induce isomorphisms $k' \to \kappa(r')$.*

Proof. Translating the geometric statement into a statement on fields, this means that we can find a diagram

where $i : k \to k'$ is the embedding of k into k', the maps $s, t : k \to \kappa(r)$ are induced by $s, t : R \to U$, and the map $\tau : k' \to k'$ is an automorphism. To produce such a diagram we may proceed in the following way:

(1) Pick $i : k \to k'$ a field map with k' algebraically closed of very large transcendence degree over k.
(2) Pick an embedding $\sigma : \kappa(r) \to k'$ such that $\sigma \circ s = i$. Such a σ exists because we can just choose a transcendence basis $\{x_\alpha\}_{\alpha \in A}$ of $\kappa(r)$ over k and find $y_\alpha \in k'$, $\alpha \in A$ which are algebraically independent over $i(k)$, and map $s(k)(\{x_\alpha\})$ into k' by the rules $s(\lambda) \mapsto i(\lambda)$ for $\lambda \in k$ and $x_\alpha \mapsto y_\alpha$ for $\alpha \in A$. Then extend to $\tau : \kappa(\alpha) \to k'$ using that k' is algebraically closed.
(3) Pick an automorphism $\tau : k' \to k'$ such that $\tau \circ i = \sigma \circ t$. To do this pick a transcendence basis $\{x_\alpha\}_{\alpha \in A}$ of k over its prime field. On the one hand, extend $\{i(x_\alpha)\}$ to a transcendence basis of k' by adding $\{y_\beta\}_{\beta \in B}$ and extend $\{\sigma(t(x_\alpha))\}$ to a transcendence basis of k' by adding $\{z_\gamma\}_{\gamma \in C}$. As k' is algebraically closed we can extend the isomorphism $\sigma \circ t \circ i^{-1} : i(k) \to \sigma(t(k))$ to an isomorphism $\tau' : \overline{i(k)} \to \overline{\sigma(t(k))}$ of their algebraic closures in k'. As k' has large transcendence degree we see that the sets B and C have the same cardinality. Thus we can use a bijection $B \to C$ to extend τ' to an isomorphism

$$\overline{i(k)}(\{y_\beta\}) \longrightarrow \overline{\sigma(t(k))}(\{z_\gamma\})$$

and then since k' is the algebraic closure of both sides we see that this extends to an automorphism $\tau : k' \to k'$ as desired.

This proves the lemma. $\square$

Lemma 10.6. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. If $r \in R$ is a point such that s, t induce isomorphisms $k \to \kappa(r)$, then the map*

$$R \longrightarrow R, \quad x \longmapsto c(r, x)$$

(see proof for precise notation) is an automorphism $R \to R$ which maps e to r.

Proof. This is completely obvious if you think about groupoids in a functorial way. But we will also spell it out completely. Denote $a : U \to R$ the morphism with image r such that $s \circ a = \mathrm{id}_U$ which exists by the hypothesis that $s : k \to \kappa(r)$ is an isomorphism. Similarly, denote $b : U \to R$ the morphism with image r such that $t \circ b = \mathrm{id}_U$. Note that $b = a \circ (t \circ a)^{-1}$, in particular $a \circ s \circ b = b$.

Consider the morphism $\Psi : R \to R$ given on T-valued points by

$$(f : T \to R) \longmapsto (c(a \circ t \circ f, f) : T \to R)$$

To see this is defined we have to check that $s \circ a \circ t \circ f = t \circ f$ which is obvious as $s \circ a = 1$. Note that $\Phi(e) = a$, so that in order to prove the lemma it suffices to show that Φ is an automorphism of R. Let $\Phi : R \to R$ be the morphism given on T-valued points by

$$(g : T \to R) \longmapsto (c(i \circ b \circ t \circ g, g) : T \to R).$$

This is defined because $s \circ i \circ b \circ t \circ g = t \circ b \circ t \circ g = t \circ g$. We claim that Φ and Ψ are inverse to each other. To see this we compute

$$\begin{aligned}
&c(a \circ t \circ c(i \circ b \circ t \circ g, g), c(i \circ b \circ t \circ g, g)) \\
&= c(a \circ t \circ i \circ b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
&= c(a \circ s \circ b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
&= c(b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
&= c(c(b \circ t \circ g, i \circ b \circ t \circ g), g)) \\
&= c(e, g) \\
&= g
\end{aligned}$$

where we have used the relation $a \circ s \circ b = b$ shown above. In the other direction we have

$$\begin{aligned}
&c(i \circ b \circ t \circ c(a \circ t \circ f, f), c(a \circ t \circ f, f)) \\
&= c(i \circ b \circ t \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
&= c(i \circ a \circ (t \circ a)^{-1} \circ t \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
&= c(i \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
&= c(c(i \circ a \circ t \circ f, a \circ t \circ f), f) \\
&= c(e, f) \\
&= f
\end{aligned}$$

The lemma is proved. $\square$

Lemma 10.7. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, $W \subset R$ is open, and $Z \to R$ is a morphism of schemes, then the image of the composition $Z \times_{s,U,t} W \to R \times_{s,U,t} R \to R$ is open.*

Proof. Write $U = \operatorname{Spec}(k)$. Consider a field extension k'/k. Denote $U' = \operatorname{Spec}(k')$. Let R' be the restriction of R via $U' \to U$. Set $Z' = Z \times_R R'$ and $W' = R' \times_R W$. Consider a point $\xi = (z, w)$ of $Z \times_{s,U,t} W$. Let $r \in R$ be the image of z under $Z \to R$. Pick $k' \supset k$ and $r' \in R'$ as in Lemma 10.5. We can choose $z' \in Z'$ mapping to z and r'. Then we can find $\xi' \in Z' \times_{s',U',t'} W'$ mapping to z' and ξ. The open $c(r', W')$ (Lemma 10.6) is contained in the image of $Z' \times_{s',U',t'} W' \to R'$. Observe that $Z' \times_{s',U',t'} W' = (Z \times_{s,U,t} W) \times_{R \times_{s,U,t} R} (R' \times_{s',U',t'} R')$. Hence the image of $Z' \times_{s',U',t'} W' \to R' \to R$ is contained in the image of $Z \times_{s,U,t} W \to R$. As $R' \to R$ is open (Lemma 10.4) we conclude the image contains an open neighbourhood of the image of ξ as desired. $\square$

Lemma 10.8. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \operatorname{Spec}(k)$ with k a field. By abuse of notation denote $e \in R$ the image of the identity morphism $e : U \to R$. Then*

(1) *every local ring $\mathcal{O}_{R,r}$ of R has a unique minimal prime ideal,*
(2) *there is exactly one irreducible component Z of R passing through e, and*
(3) *Z is geometrically irreducible over k via either s or t.*

Proof. Let $r \in R$ be a point. In this proof we will use the correspondence between irreducible components of R passing through a point r and minimal primes of the local ring $\mathcal{O}_{R,r}$ without further mention. Choose $k \subset k'$ and $r' \in R'$ as in Lemma 10.5. Note that $\mathcal{O}_{R,r} \to \mathcal{O}_{R',r'}$ is faithfully flat and local, see Lemma 10.4. Hence the result for $r' \in R'$ implies the result for $r \in R$. In other words we may assume that $s, t : k \to \kappa(r)$ are isomorphisms. By Lemma 10.6 there exists an automorphism moving e to r. Hence we may assume $r = e$, i.e., part (1) follows from part (2).

We first prove (2) in case k is separably algebraically closed. Namely, let $X, Y \subset R$ be irreducible components passing through e. Then by Varieties, Lemma 8.4 and 8.3 the scheme $X \times_{s,U,t} Y$ is irreducible as well. Hence $c(X \times_{s,U,t} Y) \subset R$ is an irreducible subset. We claim it contains both X and Y (as subsets of R). Namely, let T be the spectrum of a field. If $x : T \to X$ is a T-valued point of X, then $c(x, e \circ s \circ x) = x$ and $e \circ s \circ x$ factors through Y as $e \in Y$. Similarly for points of Y. This clearly implies that $X = Y$, i.e., there is a unique irreducible component of R passing through e.

Proof of (2) and (3) in general. Let $k \subset k'$ be a separable algebraic closure, and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $\operatorname{Spec}(k') \to \operatorname{Spec}(k)$. By the previous paragraph there is exactly one irreducible component Z' of R' passing through e'. Denote $e'' \in R \times_{s,U} U'$ the base change of e. As $R' \to R \times_{s,U} U'$ is faithfully flat, see Lemma 10.4, and $e' \mapsto e''$ we see that there is exactly one irreducible component Z'' of $R \times_{s,k} k'$ passing through e''. This implies, as $R \times_k k' \to R$ is faithfully flat, that there is exactly one irreducible component Z of R passing through e. This proves (2).

To prove (3) let $Z''' \subset R \times_k k'$ be an arbitrary irreducible component of $Z \times_k k'$. By Varieties, Lemma 8.13 we see that $Z''' = \sigma(Z'')$ for some $\sigma \in \operatorname{Gal}(k'/k)$. Since $\sigma(e'') = e''$ we see that $e'' \in Z'''$ and hence $Z''' = Z''$. This means that Z is geometrically irreducible over $\operatorname{Spec}(k)$ via the morphism s. The same argument implies that Z is geometrically irreducible over $\operatorname{Spec}(k)$ via the morphism t. $\square$

Lemma 10.9. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \operatorname{Spec}(k)$ with k a field. Assume s, t are locally of finite type. Then*

(1) *R is equidimensional,*
(2) *$\dim(R) = \dim_r(R)$ for all $r \in R$,*
(3) *for any $r \in R$ we have $\operatorname{trdeg}_{s(k)}(\kappa(r)) = \operatorname{trdeg}_{t(k)}(\kappa(r))$, and*
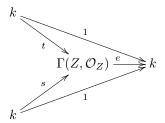(4) *for any closed point $r \in R$ we have $\dim(R) = \dim(\mathcal{O}_{R,r})$.*

Proof. Let $r, r' \in R$. Then $\dim_r(R) = \dim_{r'}(R)$ by Lemma 10.3 and Morphisms, Lemma 28.3. By Morphisms, Lemma 28.1 we have

$$\dim_r(R) = \dim(\mathcal{O}_{R,r}) + \operatorname{trdeg}_{s(k)}(\kappa(r)) = \dim(\mathcal{O}_{R,r}) + \operatorname{trdeg}_{t(k)}(\kappa(r)).$$

On the other hand, the dimension of R (or any open subset of R) is the supremum of the dimensions of the local rings of R, see Properties, Lemma 10.3. Clearly this is maximal for closed points r in which case $\operatorname{trdeg}_k(\kappa(r)) = 0$ (by the Hilbert Nullstellensatz, see Morphisms, Section 16). Hence the lemma follows. $\square$

Lemma 10.10. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \operatorname{Spec}(k)$ with k a field. Assume s, t are locally of finite type. Then $\dim(R) = \dim(G)$ where G is the stabilizer group scheme of R.*

Proof. Let $Z \subset R$ be the irreducible component passing through e (see Lemma 10.8) thought of as an integral closed subscheme of R. Let k'_s, resp. k'_t be the integral closure of $s(k)$, resp. $t(k)$ in $\Gamma(Z, \mathcal{O}_Z)$. Recall that k'_s and k'_t are fields, see Varieties, Lemma 28.4. By Varieties, Proposition 31.1 we have $k'_s = k'_t$ as subrings of $\Gamma(Z, \mathcal{O}_Z)$. As e factors through Z we obtain a commutative diagram

This on the one hand shows that $k'_s = s(k)$, $k'_t = t(k)$, so $s(k) = t(k)$, which combined with the diagram above implies that $s = t$! In other words, we conclude that Z is a closed subscheme of $G = R \times_{(t,s),U \times_S U, \Delta} U$. The lemma follows as both G and R are equidimensional, see Lemma 10.9 and Groupoids, Lemma 8.1. $\square$

Remark 10.11. Warning: Lemma 10.10 is wrong without the condition that s and t are locally of finite type. An easy example is to start with the action

$$\mathbf{G}_{m,\mathbf{Q}} \times_\mathbf{Q} \mathbf{A}^1_\mathbf{Q} \to \mathbf{A}^1_\mathbf{Q}$$

and restrict the corresponding groupoid scheme to the generic point of $\mathbf{A}^1_\mathbf{Q}$. In other words restrict via the morphism $\operatorname{Spec}(\mathbf{Q}(x)) \to \operatorname{Spec}(\mathbf{Q}[x]) = \mathbf{A}^1_\mathbf{Q}$. Then you get a groupoid scheme (U, R, s, t, c) with $U = \operatorname{Spec}(\mathbf{Q}(x))$ and

$$R = \operatorname{Spec}\left(\mathbf{Q}(x)[y]\left[\frac{1}{P(xy)}, P \in \mathbf{Q}[T], P \neq 0\right]\right)$$

In this case $\dim(R) = 1$ and $\dim(G) = 0$.

Lemma 10.12. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) $U = \mathrm{Spec}(k)$ *with k a field,*
(2) s, t *are locally of finite type, and*
(3) *the characteristic of k is zero.*

Then $s, t : R \to U$ are smooth.

Proof. By Lemma 4.1 the sheaf of differentials of $R \to U$ is free. Hence smoothness follows from Varieties, Lemma 25.1. $\square$

Lemma 10.13. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) $U = \mathrm{Spec}(k)$ *with k a field,*
(2) s, t *are locally of finite type,*
(3) R *is reduced, and*
(4) k *is perfect.*

Then $s, t : R \to U$ are smooth.

Proof. By Lemma 4.1 the sheaf $\Omega_{R/U}$ is free. Hence the lemma follows from Varieties, Lemma 25.2. $\square$

11. Morphisms of groupoids on fields

This section studies morphisms between groupoids on fields. This is slightly more general, but very akin to, studying morphisms of groupschemes over a field.

Situation 11.1. Let S be a scheme. Let $U = \mathrm{Spec}(k)$ be a scheme over S with k a field. Let (U, R_1, s_1, t_1, c_1), (U, R_2, s_2, t_2, c_2) be groupoid schemes over S with identical first component. Let $a : R_1 \to R_2$ be a morphism such that (id_U, a) defines a morphism of groupoid schemes over S, see Groupoids, Definition 13.1. In particular, the following diagrams commute

The following lemma is a generalization of Groupoids, Lemma 7.7.

Lemma 11.2. *Notation and assumptions as in Situation 11.1. If $a(R_1)$ is open in R_2, then $a(R_1)$ is closed in R_2.*

Proof. Let $r_2 \in R_2$ be a point in the closure of $a(R_1)$. We want to show $r_2 \in a(R_1)$. Pick $k \subset k'$ and $r_2' \in R_2'$ adapted to (U, R_2, s_2, t_2, c_2) and r_2 as in Lemma 10.5. Let R_i' be the restriction of R_i via the morphism $U' = \mathrm{Spec}(k') \to U = \mathrm{Spec}(k)$. Let $a' : R_1' \to R_2'$ be the base change of a. The diagram

$$
\begin{array}{ccc}
R_1' & \xrightarrow{\ a'\ } & R_2' \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\
R_1 & \xrightarrow{\ a\ } & R_2
\end{array}
$$

is a fibre square. Hence the image of a' is the inverse image of the image of a via the morphism $p_2 : R'_2 \to R_2$. By Lemma 10.4 the map p_2 is surjective and open. Hence by Topology, Lemma 6.4 we see that r'_2 is in the closure of $a'(R'_1)$. This means that we may assume that $r_2 \in R_2$ has the property that the maps $k \to \kappa(r_2)$ induced by s_2 and t_2 are isomorphisms.

In this case we can use Lemma 10.6. This lemma implies $c(r_2, a(R_1))$ is an open neighbourhood of r_2. Hence $a(R_1) \cap c(r_2, a(R_1)) \neq \emptyset$ as we assumed that r_2 was a point of the closure of $a(R_1)$. Using the inverse of R_2 and R_1 we see this means $c_2(a(R_1), a(R_1))$ contains r_2. As $c_2(a(R_1), a(R_1)) \subset a(c_1(R_1, R_1)) = a(R_1)$ we conclude $r_2 \in a(R_1)$ as desired. $\square$

Lemma 11.3. *Notation and assumptions as in Situation 11.1. Let $Z \subset R_2$ be the reduced closed subscheme (see Schemes, Definition 12.5) whose underlying topological space is the closure of the image of $a : R_1 \to R_2$. Then $c_2(Z \times_{s_2,U,t_2} Z) \subset Z$ set theoretically.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
R_1 \times_{s_1,U,t_1} R_1 & \longrightarrow & R_1 \\
\downarrow & & \downarrow \\
R_2 \times_{s_2,U,t_2} R_2 & \longrightarrow & R_2
\end{array}
$$

By Varieties, Lemma 24.2 the closure of the image of the left vertical arrow is (set theoretically) $Z \times_{s_2,U,t_2} Z$. Hence the result follows. $\square$

Lemma 11.4. *Notation and assumptions as in Situation 11.1. Assume that k is perfect. Let $Z \subset R_2$ be the reduced closed subscheme (see Schemes, Definition 12.5) whose underlying topological space is the closure of the image of $a : R_1 \to R_2$. Then*

$$(U, Z, s_2|_Z, t_2|_Z, c_2|_Z)$$

is a groupoid scheme over S.

Proof. We first explain why the statement makes sense. Since U is the spectrum of a perfect field k, the scheme Z is geometrically reduced over k (via either projection), see Varieties, Lemma 6.3. Hence the scheme $Z \times_{s_2,U,t_2} Z \subset Z$ is reduced, see Varieties, Lemma 6.7. Hence by Lemma 11.3 we see that c induces a morphism $Z \times_{s_2,U,t_2} Z \to Z$. Finally, it is clear that e_2 factors through Z and that the map $i_2 : R_2 \to R_2$ preserves Z. Since the morphisms of the septuple $(U, R_2, s_2, t_2, c_2, e_2, i_2)$ satisfies the axioms of a groupoid, it follows that after restricting to Z they satisfy the axioms. $\square$

Lemma 11.5. *Notation and assumptions as in Situation 11.1. If the image $a(R_1)$ is a locally closed subset of R_2 then it is a closed subset.*

Proof. Let $k \subset k'$ be a perfect closure of the field k. Let R'_i be the restriction of R_i via the morphism $U' = \mathrm{Spec}(k') \to \mathrm{Spec}(k)$. Note that the morphisms $R'_i \to R_i$ are universal homeomorphisms as compositions of base changes of the universal homeomorphism $U' \to U$ (see diagram in statement of Lemma 10.4). Hence it suffices to prove that $a'(R'_1)$ is closed in R'_2. In other words, we may assume that k is perfect.

If k is perfect, then the closure of the image is a groupoid scheme $Z \subset R_2$, by Lemma 11.4. By the same lemma applied to $\mathrm{id}_{R_1} : R_1 \to R_1$ we see that $(R_2)_{red}$ is a groupoid scheme. Thus we may apply Lemma 11.2 to the morphism $a|_{(R_2)_{red}} : (R_2)_{red} \to Z$ to conclude that Z equals the image of a. $\square$

Lemma 11.6. *Notation and assumptions as in Situation 11.1. Assume that $a : R_1 \to R_2$ is a quasi-compact morphism. Let $Z \subset R_2$ be the scheme theoretic image (see Morphisms, Definition 6.2) of $a : R_1 \to R_2$. Then*

$$(U, Z, s_2|_Z, t_2|_Z, c_2|_Z)$$

is a groupoid scheme over S.

Proof. The main difficulty is to show that $c_2|_{Z \times_{s_2, U, t_2} Z}$ maps into Z. Consider the commutative diagram

$$
\begin{array}{ccc}
R_1 \times_{s_1, U, t_1} R_1 & \longrightarrow & R_1 \\
\downarrow{\scriptstyle a \times a} & & \downarrow \\
R_2 \times_{s_2, U, t_2} R_2 & \longrightarrow & R_2
\end{array}
$$

By Varieties, Lemma 24.3 we see that the scheme theoretic image of $a \times a$ is $Z \times_{s_2, U, t_2} Z$. By the commutativity of the diagram we conclude that $Z \times_{s_2, U, t_2} Z$ maps into Z by the bottom horizontal arrow. As in the proof of Lemma 11.4 it is also true that $i_2(Z) \subset Z$ and that e_2 factors through Z. Hence we conclude as in the proof of that lemma. $\square$

Lemma 11.7. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a field. Let $Z \subset U \times_S U$ be the reduced closed subscheme (see Schemes, Definition 12.5) whose underlying topological space is the closure of the image of $j = (t, s) : R \to U \times_S U$. Then $pr_{02}(Z \times_{pr_1, U, pr_0} Z) \subset Z$ set theoretically.*

Proof. As $(U, U \times_S U, pr_1, pr_0, pr_{02})$ is a groupoid scheme over S this is a special case of Lemma 11.3. But we can also prove it directly as follows.

Write $U = \mathrm{Spec}(k)$. Denote R_s (resp. Z_s, resp. U_s^2) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via s (resp. $pr_1|_Z$, resp. pr_1). Similarly, denote ${}_tR$ (resp. ${}_tZ$, resp. ${}_tU^2$) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via t (resp. $pr_0|_Z$, resp. pr_0). The morphism j induces morphisms of schemes $j_s : R_s \to U_s^2$ and ${}_tj : {}_tR \to {}_tU^2$ over k. Consider the commutative diagram

$$
\begin{array}{ccc}
R_s \times_k {}_tR & \xrightarrow{\ c\ } & R \\
\downarrow{\scriptstyle j_s \times_t j} & & \downarrow{\scriptstyle j} \\
U_s^2 \times_k {}_tU^2 & \longrightarrow & U \times_S U
\end{array}
$$

By Varieties, Lemma 24.2 we see that the closure of the image of $j_s \times_t j$ is $Z_s \times_k {}_tZ$. By the commutativity of the diagram we conclude that $Z_s \times_k {}_tZ$ maps into Z by the bottom horizontal arrow. $\square$

Lemma 11.8. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a perfect field. Let $Z \subset U \times_S U$ be the reduced closed*

subscheme (see Schemes, Definition 12.5) whose underlying topological space is the closure of the image of $j = (t, s) : R \to U \times_S U$. Then

$$(U, Z, pr_0|_Z, pr_1|_Z, pr_{02}|_{Z \times_{pr_1, U, pr_0} Z})$$

is a groupoid scheme over S.

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 11.4. But we can also prove it directly as follows.

We first explain why the statement makes sense. Since U is the spectrum of a perfect field k, the scheme Z is geometrically reduced over k (via either projection), see Varieties, Lemma 6.3. Hence the scheme $Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z \subset Z$ is reduced, see Varieties, Lemma 6.7. Hence by Lemma 11.7 we see that pr_{02} induces a morphism $Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z \to Z$. Finally, it is clear that $\Delta_{U/S}$ factors through Z and that the map $\sigma : U \times_S U \to U \times_S U$, $(x, y) \mapsto (y, x)$ preserves Z. Since $(U, U \times_S U, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02}, \Delta_{U/S}, \sigma)$ satisfies the axioms of a groupoid, it follows that after restricting to Z they satisfy the axioms. $\square$

Lemma 11.9. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a field and assume R is quasi-compact (equivalently s, t are quasi-compact). Let $Z \subset U \times_S U$ be the scheme theoretic image (see Morphisms, Definition 6.2) of $j = (t, s) : R \to U \times_S U$. Then*

$$(U, Z, pr_0|_Z, pr_1|_Z, pr_{02}|_{Z \times_{pr_1, U, pr_0} Z})$$

is a groupoid scheme over S.

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 11.6. But we can also prove it directly as follows.

The main difficulty is to show that $\mathrm{pr}_{02}|_{Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z}$ maps into Z. Write $U = \mathrm{Spec}(k)$. Denote R_s (resp. Z_s, resp. U_s^2) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via s (resp. $\mathrm{pr}_1|_Z$, resp. pr_1). Similarly, denote $_tR$ (resp. $_tZ$, resp. $_tU^2$) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via t (resp. $\mathrm{pr}_0|_Z$, resp. pr_0). The morphism j induces morphisms of schemes $j_s : R_s \to U_s^2$ and $_tj : {}_tR \to {}_tU^2$ over k. Consider the commutative diagram

$$
\begin{array}{ccc}
R_s \times_{k} {}_tR & \xrightarrow{\ c\ } & R \\
{\scriptstyle j_s \times {}_tj} \downarrow & & \downarrow {\scriptstyle j} \\
U_s^2 \times_{k} {}_tU^2 & \longrightarrow & U \times_S U
\end{array}
$$

By Varieties, Lemma 24.3 we see that the scheme theoretic image of $j_s \times {}_tj$ is $Z_s \times_{k} {}_tZ$. By the commutativity of the diagram we conclude that $Z_s \times_{k} {}_tZ$ maps into Z by the bottom horizontal arrow. As in the proof of Lemma 11.8 it is also true that $\sigma(Z) \subset Z$ and that $\Delta_{U/S}$ factors through Z. Hence we conclude as in the proof of that lemma. $\square$

12. Slicing groupoids

The following lemma shows that we may slice a Cohen-Macaulay groupoid scheme in order to reduce the dimension of the fibres, provided that the dimension of the stabilizer is small. This is an essential step in the process of improving a given presentation of a quotient stack.

Situation 12.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let $u \in U$ be a point, and let $u' \in U'$ be a point such that $g(u') = u$. Given these data, denote (U', R', s', t', c') the restriction of (U, R, s, t, c) via the morphism g. Denote $G \to U$ the stabilizer group scheme of R, which is a locally closed subscheme of R. Denote h the composition

$$h = s \circ \mathrm{pr}_1 : U' \times_{g, U, t} R \longrightarrow U.$$

Denote $F_u = s^{-1}(u)$ (scheme theoretic fibre), and G_u the scheme theoretic fibre of G over u. Similarly for R' we denote $F'_{u'} = (s')^{-1}(u')$. Because $g(u') = u$ we have

$$F'_{u'} = h^{-1}(u) \times_{\mathrm{Spec}(\kappa(u))} \mathrm{Spec}(\kappa(u')).$$

The point $e(u) \in R$ may be viewed as a point on G_u and F_u also, and $e'(u')$ is a point of R' (resp. $G'_{u'}$, resp. $F'_{u'}$) which maps to $e(u)$ in R (resp. G_u, resp. F_u).

Lemma 12.2. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 16.3. With notation as in Situation 12.1, set*

$$d_1 = \dim(G_u), \quad d_2 = \dim_{e(u)}(F_u).$$

If $d_2 > d_1$, then there exist an affine scheme U' and a morphism $g : U' \to U$ such that (with notation as in Situation 12.1)

(1) *g is an immersion*
(2) *$u \in U'$,*
(3) *g is locally of finite presentation,*
(4) *the morphism $h : U' \times_{g, U, t} R \longrightarrow U$ is Cohen-Macaulay at $(u, e(u))$, and*
(5) *we have $\dim_{e'(u)}(F'_u) = d_2 - 1$.*

Proof. Let $\mathrm{Spec}(A) \subset U$ be an affine neighbourhood of u such that u corresponds to a closed point of U, see Morphisms, Lemma 16.4. Let $\mathrm{Spec}(B) \subset R$ be an affine neighbourhood of $e(u)$ which maps via j into the open $\mathrm{Spec}(A) \times_S \mathrm{Spec}(A) \subset U \times_S U$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to u. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to $e(u)$. Pictures:

$$
\begin{array}{ccc}
B \xleftarrow{\;s\;} A & & B_{\mathfrak{q}} \xleftarrow{\;s\;} A_{\mathfrak{m}} \\
{\scriptstyle t}\big\uparrow & \text{and} & {\scriptstyle t}\big\uparrow \\
A & & A_{\mathfrak{m}}
\end{array}
$$

Note that the two induced maps $s, t : \kappa(\mathfrak{m}) \to \kappa(\mathfrak{q})$ are equal and isomorphisms as $s \circ e = t \circ e = \mathrm{id}_U$. In particular we see that $\mathfrak{q}$ is a maximal ideal as well. The ring maps $s, t : A \to B$ are of finite presentation and flat. By assumption the ring

$$\mathcal{O}_{F_u, e(u)} = B_{\mathfrak{q}} / s(\mathfrak{m}) B_{\mathfrak{q}}$$

is Cohen-Macaulay of dimension d_2. The equality of dimension holds by Morphisms, Lemma 28.1.

Let R'' be the restriction of R to $u = \mathrm{Spec}(\kappa(u))$ via the morphism $\mathrm{Spec}(\kappa(u)) \to U$. As $u \to U$ is locally of finite type, we see that $(\mathrm{Spec}(\kappa(u)), R'', s'', t'', c'')$ is a groupoid scheme with s'', t'' locally of finite type, see Lemma 9.1. By Lemma 10.10 this implies that $\dim(G'') = \dim(R'')$. We also have $\dim(R'') = \dim_{e''}(R'') =$

$\dim(\mathcal{O}_{R'',e''})$, see Lemma 10.9. By Groupoids, Lemma 18.4 we have $G'' = G_u$. Hence we conclude that $\dim(\mathcal{O}_{R'',e''}) = d_1$.

As a scheme R'' is

$$R'' = R \times_{(U \times_S U)} \Big(\operatorname{Spec}(\kappa(\mathfrak{m})) \times_S \operatorname{Spec}(\kappa(\mathfrak{m})) \Big)$$

Hence an affine open neighbourhood of e'' is the spectrum of the ring

$$B \otimes_{(A \otimes A)} (\kappa(\mathfrak{m}) \otimes \kappa(\mathfrak{m})) = B/s(\mathfrak{m})B + t(\mathfrak{m})B$$

We conclude that

$$\mathcal{O}_{R'',e''} = B_\mathfrak{q}/s(\mathfrak{m})B_\mathfrak{q} + t(\mathfrak{m})B_\mathfrak{q}$$

and so now we know that this ring has dimension d_1.

We claim this implies we can find an element $f \in \mathfrak{m}$ such that

$$\dim(B_\mathfrak{q}/(s(\mathfrak{m})B_\mathfrak{q} + fB_\mathfrak{q}) < d_2$$

Namely, suppose $\mathfrak{n}_j \supset s(\mathfrak{m})B_\mathfrak{q}$, $j = 1, \ldots, m$ correspond to the minimal primes of the local ring $B_\mathfrak{q}/s(\mathfrak{m})B_\mathfrak{q}$. There are finitely many as this ring is Noetherian (since it is essentially of finite type over a field – but also because a Cohen-Macaulay ring is Noetherian). By the Cohen-Macaulay condition we have $\dim(B_\mathfrak{q}/\mathfrak{n}_j) = d_2$, for example by Algebra, Lemma 104.4. Note that $\dim(B_\mathfrak{q}/(\mathfrak{n}_j + t(\mathfrak{m})B_\mathfrak{q})) \leq d_1$ as it is a quotient of the ring $\mathcal{O}_{R'',e''} = B_\mathfrak{q}/s(\mathfrak{m})B_\mathfrak{q} + t(\mathfrak{m})B_\mathfrak{q}$ which has dimension d_1. As $d_1 < d_2$ this implies that $\mathfrak{m} \not\subset t^{-1}(\mathfrak{n}_i)$. By prime avoidance, see Algebra, Lemma 15.2, we can find $f \in \mathfrak{m}$ with $t(f) \notin \mathfrak{n}_j$ for $j = 1, \ldots, m$. For this choice of f we have the displayed inequality above, see Algebra, Lemma 60.13.

Set $A' = A/fA$ and $U' = \operatorname{Spec}(A')$. Then it is clear that $U' \to U$ is an immersion, locally of finite presentation and that $u \in U'$. Thus (1), (2) and (3) of the lemma hold. The morphism

$$U' \times_{g,U,t} R \longrightarrow U$$

factors through $\operatorname{Spec}(A)$ and corresponds to the ring map

$$B/t(f)B = A/(f) \otimes_{A,t} B \xleftarrow{\ s\ } A$$

Now, we see $t(f)$ is not a zerodivisor on $B_\mathfrak{q}/s(\mathfrak{m})B_\mathfrak{q}$ as this is a Cohen-Macaulay ring of positive dimension and f is not contained in any minimal prime, see for example Algebra, Lemma 104.2. Hence by Algebra, Lemma 128.5 we conclude that $s : A_\mathfrak{m} \to B_\mathfrak{q}/t(f)B_\mathfrak{q}$ is flat with fibre ring $B_\mathfrak{q}/(s(\mathfrak{m})B_\mathfrak{q} + t(f)B_\mathfrak{q})$ which is Cohen-Macaulay by Algebra, Lemma 104.2 again. This implies part (4) of the lemma. To see part (5) note that by Diagram (9.0.1) the fibre F'_u is equal to the fibre of h over u. Hence $\dim_{e'(u)}(F'_u) = \dim(B_\mathfrak{q}/(s(\mathfrak{m})B_\mathfrak{q} + t(f)B_\mathfrak{q}))$ by Morphisms, Lemma 28.1 and the dimension of this ring is $d_2 - 1$ by Algebra, Lemma 104.2 once more. This proves the final assertion of the lemma and we win. $\square$

Now that we know how to slice we can combine it with the preceding material to get the following "optimal" result. It is optimal in the sense that since G_u is a locally closed subscheme of F_u one always has the inequality $\dim(G_u) = \dim_{e(u)}(G_u) \leq \dim_{e(u)}(F_u)$ so it is not possible to slice more than in the lemma.

Lemma 12.3. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 16.3. With notation as in Situation 12.1 there exist an affine scheme U' and a morphism $g : U' \to U$ such that*

- (1) *g is an immersion,*
- (2) *$u \in U'$,*
- (3) *g is locally of finite presentation,*
- (4) *the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is Cohen-Macaulay and locally of finite presentation,*
- (5) *the morphisms $s', t' : R' \to U'$ are Cohen-Macaulay and locally of finite presentation, and*
- (6) *$\dim_{e(u)}(F'_u) = \dim(G'_u)$.*

Proof. As s is locally of finite presentation the scheme F_u is locally of finite type over $\kappa(u)$. Hence $\dim_{e(u)}(F_u) < \infty$ and we may argue by induction on $\dim_{e(u)}(F_u)$.

If $\dim_{e(u)}(F_u) = \dim(G_u)$ there is nothing to prove. Assume $\dim_{e(u)}(F_u) > \dim(G_u)$. This means that Lemma 12.2 applies and we find a morphism $g : U' \to U$ which has properties (1), (2), (3), instead of (6) we have $\dim_{e(u)}(F'_u) < \dim_{e(u)}(F_u)$, and instead of (4) and (5) we have that the composition

$$h = s \circ \mathrm{pr}_1 : U' \times_{g,U,t} R \longrightarrow U$$

is Cohen-Macaulay at the point $(u, e(u))$. We apply Remark 6.3 and we obtain an open subscheme $U'' \subset U'$ such that $U'' \times_{g,U,t} R \subset U' \times_{g,U,t} R$ is the largest open subscheme on which h is Cohen-Macaulay. Since $(u, e(u)) \in U'' \times_{g,U,t} R$ we see that $u \in U''$. Hence we may replace U' by U'' and assume that in fact h is Cohen-Macaulay everywhere! By Lemma 9.2 we conclude that s', t' are locally of finite presentation and Cohen-Macaulay (use Morphisms, Lemma 21.4 and More on Morphisms, Lemma 22.6).

By construction $\dim_{e'(u)}(F'_u) < \dim_{e(u)}(F_u)$, so we may apply the induction hypothesis to (U', R', s', t', c') and the point $u \in U'$. Note that u is also a finite type point of U' (for example you can see this using the characterization of finite type points from Morphisms, Lemma 16.4). Let $g' : U'' \to U'$ and $(U'', R'', s'', t'', c'')$ be the solution of the corresponding problem starting with (U', R', s', t', c') and the point $u \in U'$. We claim that the composition

$$g'' = g \circ g' : U'' \longrightarrow U$$

is a solution for the original problem. Properties (1), (2), (3), (5), and (6) are immediate. To see (4) note that the morphism

$$h'' = s \circ \mathrm{pr}_1 : U'' \times_{g'',U,t} R \longrightarrow U$$

is locally of finite presentation and Cohen-Macaulay by an application of Lemma 9.4 (use More on Morphisms, Lemma 22.11 to see that Cohen-Macaulay morphisms are fppf local on the target). $\square$

In case the stabilizer group scheme has fibres of dimension 0 this leads to the following slicing lemma.

Lemma 12.4. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 16.3. Assume that $G \to U$ is locally quasi-finite. With notation as in Situation 12.1 there exist an affine scheme U' and a morphism $g : U' \to U$ such that*

(1) *g is an immersion,*
(2) *$u \in U'$,*
(3) *g is locally of finite presentation,*
(4) *the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is flat, locally of finite presentation, and locally quasi-finite, and*
(5) *the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation, and locally quasi-finite.*

Proof. Take $g : U' \to U$ as in Lemma 12.3. Since $h^{-1}(u) = F'_u$ we see that h has relative dimension ≤ 0 at $(u, e(u))$. Hence, by Remark 6.3, we obtain an open subscheme $U'' \subset U'$ such that $u \in U''$ and $U'' \times_{g,U,t} R$ is the maximal open subscheme of $U' \times_{g,U,t} R$ on which h has relative dimension ≤ 0. After replacing U' by U'' we see that h has relative dimension ≤ 0. This implies that h is locally quasi-finite by Morphisms, Lemma 29.5. Since it is still locally of finite presentation and Cohen-Macaulay we see that it is flat, locally of finite presentation and locally quasi-finite, i.e., (4) above holds. This implies that s' is flat, locally of finite presentation and locally quasi-finite as a base change of h, see Lemma 9.2. $\qquad\square$

13. Étale localization of groupoids

In this section we begin applying the étale localization techniques of More on Morphisms, Section 41 to groupoid schemes. More advanced material of this kind can be found in More on Groupoids in Spaces, Section 15. Lemma 13.2 will be used to prove results on algebraic spaces separated and quasi-finite over a scheme, namely Morphisms of Spaces, Proposition 50.2 and its corollary Morphisms of Spaces, Lemma 51.1.

Lemma 13.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in S$ be a point, and let $u \in U$ be a point lying over p. Assume that*

(1) *$U \to S$ is locally of finite type,*
(2) *$U \to S$ is quasi-finite at u,*
(3) *$U \to S$ is separated,*
(4) *$R \to S$ is separated,*
(5) *s, t are flat and locally of finite presentation, and*
(6) *$s^{-1}(\{u\})$ is finite.*

Then there exists an étale neighbourhood $(S', p') \to (S, p)$ *with* $\kappa(p) = \kappa(p')$ *and a base change diagram*

$$
\begin{array}{ccccc}
R' \amalg W' & == & S' \times_S R & \longrightarrow & R \\
& & & & \\
\Big\downarrow{\scriptstyle t'} & & \Big\downarrow{\scriptstyle s'} & {\scriptstyle t}\Big\downarrow\ \Big\downarrow{\scriptstyle s} & \\
& & & & \\
U' \amalg W & == & S' \times_S U & \longrightarrow & U \\
& & \Big\downarrow & & \Big\downarrow \\
& & S' & \longrightarrow & S
\end{array}
$$

where the equal signs are decompositions into open and closed subschemes such that

   (a) *there exists a point* u' *of* U' *mapping to* u *in* U,
   (b) *the fibre* $(U')_{p'}$ *equals* $t'\big((s')^{-1}(\{u'\})\big)$ *set theoretically,*
   (c) *the fibre* $(R')_{p'}$ *equals* $(s')^{-1}\big((U')_{p'}\big)$ *set theoretically,*
   (d) *the schemes* U' *and* R' *are finite over* S',
   (e) *we have* $s'(R') \subset U'$ *and* $t'(R') \subset U'$,
   (f) *we have* $c'(R' \times_{s', U', t'} R') \subset R'$ *where* c' *is the base change of* c, *and*
   (g) *the morphisms* s', t', c' *determine a groupoid structure by taking the system* $(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{s', U', t'} R'})$.

Proof. Let us denote $f : U \to S$ the structure morphism of U. By assumption (6) we can write $s^{-1}(\{u\}) = \{r_1, \ldots, r_n\}$. Since this set is finite, we see that s is quasi-finite at each of these finitely many inverse images, see Morphisms, Lemma 20.7. Hence we see that $f \circ s : R \to S$ is quasi-finite at each r_i (Morphisms, Lemma 20.12). Hence r_i is isolated in the fibre R_p, see Morphisms, Lemma 20.6. Write $t(\{r_1, \ldots, r_n\}) = \{u_1, \ldots, u_m\}$. Note that it may happen that $m < n$ and note that $u \in \{u_1, \ldots, u_m\}$. Since t is flat and locally of finite presentation, the morphism of fibres $t_p : R_p \to U_p$ is flat and locally of finite presentation (Morphisms, Lemmas 25.8 and 21.4), hence open (Morphisms, Lemma 25.10). The fact that each r_i is isolated in R_p implies that each $u_j = t(r_i)$ is isolated in U_p. Using Morphisms, Lemma 20.6 again, we see that f is quasi-finite at $u_1, \ldots, u_m$.

Denote $F_u = s^{-1}(u)$ and $F_{u_j} = s^{-1}(u_j)$ the scheme theoretic fibres. Note that F_u is finite over $\kappa(u)$ as it is locally of finite type over $\kappa(u)$ with finitely many points (for example it follows from the much more general Morphisms, Lemma 57.9). By Lemma 7.1 we see that F_u and F_{u_j} become isomorphic over a common field extension of $\kappa(u)$ and $\kappa(u_j)$. Hence we see that F_{u_j} is finite over $\kappa(u_j)$. In particular we see $s^{-1}(\{u_j\})$ is a finite set for each $j = 1, \ldots, m$. Thus we see that assumptions (2) and (6) hold for each u_j also (above we saw that $U \to S$ is quasi-finite at u_j). Hence the argument of the first paragraph applies to each u_j and we see that $R \to U$ is quasi-finite at each of the points of

$$\{r_1, \ldots, r_N\} = s^{-1}(\{u_1, \ldots, u_m\})$$

Note that $t(\{r_1, \ldots, r_N\}) = \{u_1, \ldots, u_m\}$ and $t^{-1}(\{u_1, \ldots, u_m\}) = \{r_1, \ldots, r_N\}$ since R is a groupoid[2]. Moreover, we have $\mathrm{pr}_0(c^{-1}(\{r_1, \ldots, r_N\})) = \{r_1, \ldots, r_N\}$

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    [2]Explanation in groupoid language: The original set $\{r_1, \ldots, r_n\}$ was the set of arrows with source u. The set $\{u_1, \ldots, u_m\}$ was the set of objects isomorphic to u. And $\{r_1, \ldots, r_N\}$ is the set of all arrows between all the objects equivalent to u.

and $\mathrm{pr}_1(c^{-1}(\{r_1,\ldots,r_N\})) = \{r_1,\ldots,r_N\}$. Similarly we get $e(\{u_1,\ldots,u_m\}) \subset \{r_1,\ldots,r_N\}$ and $i(\{r_1,\ldots,r_N\}) = \{r_1,\ldots,r_N\}$.

We may apply More on Morphisms, Lemma 41.4 to the pairs $(U \to S, \{u_1,\ldots,u_m\})$ and $(R \to S, \{r_1,\ldots,r_N\})$ to get an étale neighbourhood $(S',p') \to (S,p)$ which induces an identification $\kappa(p) = \kappa(p')$ such that $S' \times_S U$ and $S' \times_S R$ decompose as

$$S' \times_S U = U' \amalg W, \quad S' \times_S R = R' \amalg W'$$

with $U' \to S'$ finite and $(U')_{p'}$ mapping bijectively to $\{u_1,\ldots,u_m\}$, and $R' \to S'$ finite and $(R')_{p'}$ mapping bijectively to $\{r_1,\ldots,r_N\}$. Moreover, no point of $W_{p'}$ (resp. $(W')_{p'}$) maps to any of the points u_j (resp. r_i). At this point (a), (b), (c), and (d) of the lemma are satisfied. Moreover, the inclusions of (e) and (f) hold on fibres over p', i.e., $s'((R')_{p'}) \subset (U')_{p'}$, $t'((R')_{p'}) \subset (U')_{p'}$, and $c'((R' \times_{s',U',t'} R')_{p'}) \subset (R')_{p'}$.

We claim that we can replace S' by a Zariski open neighbourhood of p' so that the inclusions of (e) and (f) hold. For example, consider the set $E = (s'|_{R'})^{-1}(W)$. This is open and closed in R' and does not contain any points of R' lying over p'. Since $R' \to S'$ is closed, after replacing S' by $S' \setminus (R' \to S')(E)$ we reach a situation where E is empty. In other words s' maps R' into U'. Note that this property is preserved under further shrinking S'. Similarly, we can arrange it so that t' maps R' into U'. At this point (e) holds. In the same manner, consider the set $E = (c'|_{R' \times_{s',U',t'} R'})^{-1}(W')$. It is open and closed in the scheme $R' \times_{s',U',t'} R'$ which is finite over S', and does not contain any points lying over p'. Hence after replacing S' by $S' \setminus (R' \times_{s',U',t'} R' \to S')(E)$ we reach a situation where E is empty. In other words we obtain the inclusion in (f). We may repeat the argument also with the identity $e' : S' \times_S U \to S' \times_S R$ and the inverse $i' : S' \times_S R \to S' \times_S R$ so that we may assume (after shrinking S' some more) that $(e'|_{U'})^{-1}(W') = \emptyset$ and $(i'|_{R'})^{-1}(W') = \emptyset$.

At this point we see that we may consider the structure

$$(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{t',U',s'} R'}, e'|_{U'}, i'|_{R'}).$$

The axioms of a groupoid scheme over S' hold because they hold for the groupoid scheme $(S' \times_S U, S' \times_S R, s', t', c', e', i')$. $\square$

Lemma 13.2. *Let S be a scheme. Let (U,R,s,t,c) be a groupoid scheme over S. Let $p \in S$ be a point, and let $u \in U$ be a point lying over p. Assume assumptions (1) – (6) of Lemma 13.1 hold as well as*

   (7) $j : R \to U \times_S U$ *is universally closed*[3].

Then we can choose $(S',p') \to (S,p)$ and decompositions $S' \times_S U = U' \amalg W$ and $S' \times_S R = R' \amalg W'$ and $u' \in U'$ such that (a) – (g) of Lemma 13.1 hold as well as

   (h) R' *is the restriction of $S' \times_S R$ to U'.*

Proof. We apply Lemma 13.1 for the groupoid (U,R,s,t,c) over the scheme S with points p and u. Hence we get an étale neighbourhood $(S',p') \to (S,p)$ and disjoint union decompositions

$$S' \times_S U = U' \amalg W, \quad S' \times_S R = R' \amalg W'$$

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[3] In view of the other conditions this is equivalent to requiring j to be proper.

and $u' \in U'$ satisfying conclusions (a), (b), (c), (d), (e), (f), and (g). We may shrink S' to a smaller neighbourhood of p' without affecting the conclusions (a) – (g). We will show that for a suitable shrinking conclusion (h) holds as well. Let us denote j' the base change of j to S'. By conclusion (e) it is clear that

$$j'^{-1}(U' \times_{S'} U') = R' \amalg Rest$$

for some open and closed $Rest$ piece. Since $U' \to S'$ is finite by conclusion (d) we see that $U' \times_{S'} U'$ is finite over S'. Since j is universally closed, also j' is universally closed, and hence $j'|_{Rest}$ is universally closed too. By conclusions (b) and (c) we see that the fibre of

$$(U' \times_{S'} U' \to S') \circ j'|_{Rest} : Rest \longrightarrow S'$$

over p' is empty. Hence, since $Rest \to S'$ is closed as a composition of closed morphisms, after replacing S' by $S' \backslash \mathrm{Im}(Rest \to S')$, we may assume that $Rest = \emptyset$. And this is exactly the condition that R' is the restriction of $S' \times_S R$ to the open subscheme $U' \subset S' \times_S U$, see Groupoids, Lemma 18.3 and its proof. $\qquad \square$

14. Finite groupoids

A groupoid scheme (U, R, s, t, c) is sometimes called *finite* if the morphisms s and t are finite. This is potentially confusing as it doesn't imply that U or R or the quotient sheaf U/R are finite over anything.

Lemma 14.1. *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume s, t are finite. There exists a sequence of R-invariant closed subschemes*

$$U = Z_0 \supset Z_1 \supset Z_2 \supset \ldots$$

such that $\bigcap Z_r = \emptyset$ and such that $s^{-1}(Z_{r-1}) \setminus s^{-1}(Z_r) \to Z_{r-1} \setminus Z_r$ is finite locally free of rank r.

Proof. Let $\{Z_r\}$ be the stratification of U given by the Fitting ideals of the finite type quasi-coherent modules $s_* \mathcal{O}_R$. See Divisors, Lemma 9.6. Since the identity $e : U \to R$ is a section to s we see that $s_* \mathcal{O}_R$ contains $\mathcal{O}_S$ as a direct summand. Hence $U = Z_{-1} = Z_0$ (details omitted). Since formation of Fitting ideals commutes with base change (More on Algebra, Lemma 8.4) we find that $s^{-1}(Z_r)$ corresponds to the rth Fitting ideal of $\mathrm{pr}_{1,*} \mathcal{O}_{R \times_{s,U,t} R}$ because the lower right square of diagram (3.0.1) is cartesian. Using the fact that the lower left square is also cartesian we conclude that $s^{-1}(Z_r) = t^{-1}(Z_r)$, in other words Z_r is R-invariant. The morphism $s^{-1}(Z_{r-1}) \setminus s^{-1}(Z_r) \to Z_{r-1} \setminus Z_r$ is finite locally free of rank r because the module $s_* \mathcal{O}_R$ pulls back to a finite locally free module of rank r on $Z_{r-1} \setminus Z_r$ by Divisors, Lemma 9.6. $\qquad \square$

Lemma 14.2. *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume s, t are finite. There exists an open subscheme $W \subset U$ and a closed subscheme $W' \subset W$ such that*

(1) *W and W' are R-invariant,*
(2) *$U = t(s^{-1}(\overline{W}))$ set theoretically,*
(3) *W is a thickening of W', and*
(4) *the maps s', t' of the restriction (W', R', s', t', c') are finite locally free.*

Proof. Consider the stratification $U = Z_0 \supset Z_1 \supset Z_2 \supset \ldots$ of Lemma 14.1.

We will construct disjoint unions $W = \coprod_{r \geq 1} W_r$ and $W' = \coprod_{r \geq 1} W'_r$ with each $W'_r \to W_r$ a thickening of R-invariant subschemes of U such that the morphisms s'_r, t'_r of the restrictions $(W'_r, R'_r, s'_r, t'_r, c'_r)$ are finite locally free of rank r. To begin we set $W_1 = W'_1 = U \backslash Z_1$. This is an R-invariant open subscheme of U, it is true that W_0 is a thickening of W'_0, and the maps s'_1, t'_1 of the restriction $(W'_1, R'_1, s'_1, t'_1, c'_1)$ are isomorphisms, i.e., finite locally free of rank 1. Moreover, every point of $U \setminus Z_1$ is in $t(s^{-1}(\overline{W_1}))$.

Assume we have found subschemes $W'_r \subset W_r \subset U$ for $r \leq n$ such that

- (1) $W_1, \ldots, W_n$ are disjoint,
- (2) W_r and W'_r are R-invariant,
- (3) $U \setminus Z_n \subset \bigcup_{r \leq n} t(s^{-1}(\overline{W_r}))$ set theoretically,
- (4) W_r is a thickening of W'_r,
- (5) the maps s'_r, t'_r of the restriction $(W'_r, R'_r, s'_r, t'_r, c'_r)$ are finite locally free of rank r.

Then we set

$$W_{n+1} = Z_n \setminus \left(Z_{n+1} \cup \bigcup_{r \leq n} t(s^{-1}(\overline{W_r})) \right)$$

set theoretically and

$$W'_{n+1} = Z_n \setminus \left(Z_{n+1} \cup \bigcup_{r \leq n} t(s^{-1}(\overline{W_r})) \right)$$

scheme theoretically. Then W_{n+1} is an R-invariant open subscheme of U because $Z_{n+1} \setminus \overline{U \setminus Z_{n+1}}$ is open in U and $\overline{U \setminus Z_{n+1}}$ is contained in the closed subset $\bigcup_{r \leq n} t(s^{-1}(\overline{W_r}))$ we are removing by property (3) and the fact that t is a closed morphism. It is clear that W'_{n+1} is a closed subscheme of W_{n+1} with the same underlying topological space. Finally, properties (1), (2) and (3) are clear and property (5) follows from Lemma 14.1.

By Lemma 14.1 we have $\bigcap Z_r = \emptyset$. Hence every point of U is contained in $U \setminus Z_n$ for some n. Thus we see that $U = \bigcup_{r \geq 1} t(s^{-1}(\overline{W_r}))$ set theoretically and we see that (2) holds. Thus $W' \subset W$ satisfy (1), (2), (3), and (4). □

Let (U, R, s, t, c) be a groupoid scheme. Given a point $u \in U$ the R-orbit of u is the subset $t(s^{-1}(\{u\}))$ of U.

Lemma 14.3. *In Lemma 14.2 assume in addition that s and t are of finite presentation. Then*

- (1) *the morphism $W' \to W$ is of finite presentation, and*
- (2) *if $u \in U$ is a point whose R-orbit consists of generic points of irreducible components of U, then $u \in W$.*

Proof. In this case the stratification $U = Z_0 \supset Z_1 \supset Z_2 \supset \ldots$ of Lemma 14.1 is given by closed immersions $Z_k \to U$ of finite presentation, see Divisors, Lemma 9.6. Part (1) follows immediately from this as $W' \to W$ is locally given by intersecting the open W by Z_r. To see part (2) let $\{u_1, \ldots, u_n\}$ be the orbit of u. Since the closed subschemes Z_k are R-invariant and $\bigcap Z_k = \emptyset$, we find an k such that $u_i \in Z_k$ and $u_i \notin Z_{k+1}$ for all i. The image of $Z_k \to U$ and $Z_{k+1} \to U$ is locally constructible (Morphisms, Theorem 22.3). Since $u_i \in U$ is a generic point of an irreducible component of U, there exists an open neighbourhood U_i of u_i which is contained

in $Z_k \setminus Z_{k+1}$ set theoretically (Properties, Lemma 2.2). In the proof of Lemma
14.2 we have constructed W as a disjoint union $\coprod W_r$ with $W_r \subset Z_{r-1} \setminus Z_r$ such
that $U = \bigcup t(s^{-1}(\overline{W_r}))$. As $\{u_1, \ldots, u_n\}$ is an R-orbit we see that $u \in t(s^{-1}(\overline{W_r}))$
implies $u_i \in \overline{W_r}$ for some i which implies $U_i \cap W_r \neq \emptyset$ which implies $r = k$. Thus
we conclude that u is in

$$W_{k+1} = Z_k \setminus \left(Z_{k+1} \cup \bigcup_{r \leq k} t(s^{-1}(\overline{W_r})) \right)$$

as desired. $\square$

Lemma 14.4. *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume s, t
are finite and of finite presentation and U quasi-separated. Let $u_1, \ldots, u_m \in U$ be
points whose orbits consist of generic points of irreducible components of U. Then
there exist R-invariant subschemes $V' \subset V \subset U$ such that*

> (1) $u_1, \ldots, u_m \in V'$,
> (2) V *is open in U,*
> (3) V' *and V are affine,*
> (4) $V' \subset V$ *is a thickening of finite presentation,*
> (5) *the morphisms s', t' of the restriction (V', R', s', t', c') are finite locally free.*

Proof. Let $W' \subset W \subset U$ be as in Lemma 14.2. By Lemma 14.3 we get $u_j \in W$
and that $W' \to W$ is a thickening of finite presentation. By Limits, Lemma 11.3
it suffices to find an R-invariant affine open subscheme V' of W' containing u_j
(because then we can let $V \subset W$ be the corresponding open subscheme which will
be affine). Thus we may replace (U, R, s, t, c) by the restriction (W', R', s', t', c')
to W'. In other words, we may assume we have a groupoid scheme (U, R, s, t, c)
whose morphisms s and t are finite locally free. By Properties, Lemma 29.1 we can
find an affine open containing the union of the orbits of $u_1, \ldots, u_m$. Finally, we can
apply Groupoids, Lemma 24.1 to conclude. $\square$

The following lemma is a special case of Lemma 14.4 but we redo the argument as
it is slightly easier in this case (it avoids using Lemma 14.3).

Lemma 14.5. *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Assume
s, t finite, U is locally Noetherian, and $u_1, \ldots, u_m \in U$ points whose orbits consist
of generic points of irreducible components of U. Then there exist R-invariant
subschemes $V' \subset V \subset U$ such that*

> (1) $u_1, \ldots, u_m \in V'$,
> (2) V *is open in U,*
> (3) V' *and V are affine,*
> (4) $V' \subset V$ *is a thickening,*
> (5) *the morphisms s', t' of the restriction (V', R', s', t', c') are finite locally free.*

Proof. Let $\{u_{j1}, \ldots, u_{jn_j}\}$ be the orbit of u_j. Let $W' \subset W \subset U$ be as in Lemma
14.2. Since $U = t(s^{-1}(\overline{W}))$ we see that at least one $u_{ji} \in \overline{W}$. Since u_{ji} is a
generic point of an irreducible component and U locally Noetherian, this implies
that $u_{ji} \in W$. Since W is R-invariant, we conclude that $u_j \in W$ and in fact
the whole orbit is contained in W. By Cohomology of Schemes, Lemma 13.3 it
suffices to find an R-invariant affine open subscheme V' of W' containing $u_1, \ldots, u_m$
(because then we can let $V \subset W$ be the corresponding open subscheme which will
be affine). Thus we may replace (U, R, s, t, c) by the restriction (W', R', s', t', c')
to W'. In other words, we may assume we have a groupoid scheme (U, R, s, t, c)

whose morphisms s and t are finite locally free. By Properties, Lemma 29.1 we can find an affine open containing $\{u_{ij}\}$ (a locally Noetherian scheme is quasi-separated by Properties, Lemma 5.4). Finally, we can apply Groupoids, Lemma 24.1 to conclude. $\square$

Lemma 14.6. *Let (U, R, s, t, c) be a groupoid scheme over a scheme S with s, t integral. Let $g : U' \to U$ be an integral morphism such that every R-orbit in U meets $g(U')$. Let (U', R', s', t', c') be the restriction of R to U'. If $u' \in U'$ is contained in an R'-invariant affine open, then the image $u \in U$ is contained in an R-invariant affine open of U.*

Proof. Let $W' \subset U'$ be an R'-invariant affine open. Set $\tilde{R} = U' \times_{g, U, t} R$ with maps $\mathrm{pr}_0 : \tilde{R} \to U'$ and $h = s \circ \mathrm{pr}_1 : \tilde{R} \to U$. Observe that pr_0 and h are integral. It follows that $\tilde{W} = \mathrm{pr}_0^{-1}(W')$ is affine. Since W' is R'-invariant, the image $W = h(\tilde{W})$ is set theoretically R-invariant and $\tilde{W} = h^{-1}(W)$ set theoretically (details omitted). Thus, if we can show that W is open, then W is a scheme and the morphism $\tilde{W} \to W$ is integral surjective which implies that W is affine by Limits, Proposition 11.2. However, our assumption on orbits meeting U' implies that $h : \tilde{R} \to U$ is surjective. Since an integral surjective morphism is submersive (Topology, Lemma 6.5 and Morphisms, Lemma 44.7) it follows that W is open. $\square$

The following technical lemma produces "almost" invariant functions in the situation of a finite groupoid on a quasi-affine scheme.

Lemma 14.7. *Let (U, R, s, t, c) be a groupoid scheme with s, t finite and of finite presentation. Let $u_1, \ldots, u_m \in U$ be points whose R-orbits consist of generic points of irreducible components of U. Let $j : U \to \mathrm{Spec}(A)$ be an immersion. Let $I \subset A$ be an ideal such that $j(U) \cap V(I) = \emptyset$ and $V(I) \cup j(U)$ is closed in $\mathrm{Spec}(A)$. Then there exists an $h \in I$ such that $j^{-1}D(h)$ is an R-invariant affine open subscheme of U containing $u_1, \ldots, u_m$.*

Proof. Let $u_1, \ldots, u_m \in V' \subset V \subset U$ be as in Lemma 14.4. Since $U \setminus V$ is closed in U, j an immersion, and $V(I) \cup j(U)$ is closed in $\mathrm{Spec}(A)$, we can find an ideal $J \subset I$ such that $V(J) = V(I) \cup j(U \setminus V)$. For example we can take the ideal of elements of I which vanish on $j(U \setminus V)$. Thus we can replace (U, R, s, t, c), $j : U \to \mathrm{Spec}(A)$, and I by (V', R', s', t', c'), $j|_{V'} : V' \to \mathrm{Spec}(A)$, and J. In other words, we may assume that U is affine and that s and t are finite locally free. Take any $f \in I$ which does not vanish at all the points in the R-orbits of $u_1, \ldots, u_m$ (Algebra, Lemma 15.2). Consider

$$g = \mathrm{Norm}_s(t^\sharp(j^\sharp(f))) \in \Gamma(U, \mathcal{O}_U)$$

Since $f \in I$ and since $V(I) \cup j(U)$ is closed we see that $U \cap D(f) \to D(f)$ is a closed immersion. Hence $f^n g$ is the image of an element $h \in I$ for some $n > 0$. We claim that h works. Namely, we have seen in Groupoids, Lemma 23.2 that g is an R-invariant function, hence $D(g) \subset U$ is R-invariant. Since f does not vanish on the orbit of u_j, the function g does not vanish at u_j. Moreover, we have $V(g) \supset V(j^\sharp(f))$ and hence $j^{-1}D(h) = D(g)$. $\square$

Lemma 14.8. *Let (U, R, s, t, c) be a groupoid scheme. If s, t are finite, and $u, u' \in R$ are distinct points in the same orbit, then u' is not a specialization of u.*

Proof. Let $r \in R$ with $s(r) = u$ and $t(r) = u'$. If $u \rightsquigarrow u'$ then we can find a nontrivial specialization $r \rightsquigarrow r'$ with $s(r') = u'$, see Schemes, Lemma 19.8. Set

$u'' = t(r')$. Note that $u'' \neq u'$ as there are no specializations in the fibres of a finite morphism. Hence we can continue and find a nontrivial specialization $r' \rightsquigarrow r''$ with $s(r'') = u''$, etc. This shows that the orbit of u contains an infinite sequence $u \rightsquigarrow u' \rightsquigarrow u'' \rightsquigarrow \ldots$ of specializations which is nonsense as the orbit $t(s^{-1}(\{u\}))$ is finite. $\qquad\square$

Lemma 14.9. *Let $j : V \to \mathrm{Spec}(A)$ be a quasi-compact immersion of schemes. Let $f \in A$ be such that $j^{-1}D(f)$ is affine and $j(V) \cap V(f)$ is closed. Then V is affine.*

Proof. This follows from Morphisms, Lemma 11.14 but we will also give a direct proof. Let $A' = \Gamma(V, \mathcal{O}_V)$. Then $j' : V \to \mathrm{Spec}(A')$ is a quasi-compact open immersion, see Properties, Lemma 18.4. Let $f' \in A'$ be the image of f. Then $(j')^{-1}D(f') = j^{-1}D(f)$ is affine. On the other hand, $j'(V) \cap V(f')$ is a subscheme of $\mathrm{Spec}(A')$ which maps isomorphically to the closed subscheme $j(V) \cap V(f)$ of $\mathrm{Spec}(A)$. Hence it is closed in $\mathrm{Spec}(A')$ for example by Schemes, Lemma 21.11. Thus we may replace A by A' and assume that j is an open immersion and $A = \Gamma(V, \mathcal{O}_V)$.

In this case we claim that $j(V) = \mathrm{Spec}(A)$ which finishes the proof. If not, then we can find a principal affine open $D(g) \subset \mathrm{Spec}(A)$ which meets the complement and avoids the closed subset $j(V) \cap V(f)$. Note that j maps $j^{-1}D(f)$ isomorphically onto $D(f)$, see Properties, Lemma 18.3. Hence $D(g)$ meets $V(f)$. On the other hand, $j^{-1}D(g)$ is a principal open of the affine open $j^{-1}D(f)$ hence affine. Hence by Properties, Lemma 18.3 again we see that $D(g)$ is isomorphic to $j^{-1}D(g) \subset j^{-1}D(f)$ which implies that $D(g) \subset D(f)$. This contradiction finishes the proof. $\qquad\square$

Lemma 14.10. *Let (U, R, s, t, c) be a groupoid scheme. Let $u \in U$. Assume*

(1) *s, t are finite morphisms,*
(2) *U is separated and locally Noetherian,*
(3) *$\dim(\mathcal{O}_{U,u'}) \leq 1$ for every point u' in the orbit of u.*

Then u is contained in an R-invariant affine open of U.

Proof. The R-orbit of u is finite. By conditions (2) and (3) it is contained in an affine open U' of U, see Varieties, Proposition 42.7. Then $t(s^{-1}(U \setminus U'))$ is an R-invariant closed subset of U which does not contain u. Thus $U \setminus t(s^{-1}(U \setminus U'))$ is an R-invariant open of U' containing u. Replacing U by this open we may assume U is quasi-affine.

By Lemma 14.6 we may replace U by its reduction and assume U is reduced. This means R-invariant subschemes $W' \subset W \subset U$ of Lemma 14.2 are equal $W' = W$. As $U = t(s^{-1}(\overline{W}))$ some point u' of the R-orbit of u is contained in $\overline{W}$ and by Lemma 14.6 we may replace U by $\overline{W}$ and u by u'. Hence we may assume there is a dense open R-invariant subscheme $W \subset U$ such that the morphisms s_W, t_W of the restriction (W, R_W, s_W, t_W, c_W) are finite locally free.

If $u \in W$ then we are done by Groupoids, Lemma 24.1 (because W is quasi-affine so any finite set of points of W is contained in an affine open, see Properties, Lemma 29.5). Thus we assume $u \notin W$ and hence none of the points of the orbit of u is in W. Let $\xi \in U$ be a point with a nontrivial specialization to a point u' in the orbit of u. Since there are no specializations among the points in the orbit of u (Lemma 14.8) we see that ξ is not in the orbit. By assumption (3) we see that ξ is a generic

point of U and hence $\xi \in W$. As U is Noetherian there are finitely many of these points $\xi_1, \ldots, \xi_m \in W$. Because s_W, t_W are flat the orbit of each ξ_j consists of generic points of irreducible components of W (and hence U).

Let $j : U \to \mathrm{Spec}(A)$ be an immersion of U into an affine scheme (this is possible as U is quasi-affine). Let $J \subset A$ be an ideal such that $V(J) \cap j(W) = \emptyset$ and $V(J) \cup j(W)$ is closed. Apply Lemma 14.7 to the groupoid scheme (W, R_W, s_W, t_W, c_W), the morphism $j|_W : W \to \mathrm{Spec}(A)$, the points ξ_j, and the ideal J to find an $f \in J$ such that $(j|_W)^{-1}D(f)$ is an R_W-invariant affine open containing ξ_j for all j. Since $f \in J$ we see that $j^{-1}D(f) \subset W$, i.e., $j^{-1}D(f)$ is an R-invariant affine open of U contained in W containing all ξ_j.

Let Z be the reduced induced closed subscheme structure on

$$U \setminus j^{-1}D(f) = j^{-1}V(f).$$

Then Z is set theoretically R-invariant (but it may not be scheme theoretically R-invariant). Let (Z, R_Z, s_Z, t_Z, c_Z) be the restriction of R to Z. Since $Z \to U$ is finite, it follows that s_Z and t_Z are finite. Since $u \in Z$ the orbit of u is in Z and agrees with the R_Z-orbit of u viewed as a point of Z. Since $\dim(\mathcal{O}_{U,u'}) \leq 1$ and since $\xi_j \notin Z$ for all j, we see that $\dim(\mathcal{O}_{Z,u'}) \leq 0$ for all u' in the orbit of u. In other words, the R_Z-orbit of u consists of generic points of irreducible components of Z.

Let $I \subset A$ be an ideal such that $V(I) \cap j(U) = \emptyset$ and $V(I) \cup j(U)$ is closed. Apply Lemma 14.7 to the groupoid scheme (Z, R_Z, s_Z, t_Z, c_Z), the restriction $j|_Z$, the ideal I, and the point $u \in Z$ to obtain $h \in I$ such that $j^{-1}D(h) \cap Z$ is an R_Z-invariant open affine containing u.

Consider the R_W-invariant (Groupoids, Lemma 23.2) function

$$g = \mathrm{Norm}_{s_W}(t_W^\sharp(j^\sharp(h)|_W)) \in \Gamma(W, \mathcal{O}_W)$$

(In the following we only need the restriction of g to $j^{-1}D(f)$ and in this case the norm is along a finite locally free morphism of affines.) We claim that

$$V = (W_g \cap j^{-1}D(f)) \cup (j^{-1}D(h) \cap Z)$$

is an R-invariant affine open of U which finishes the proof of the lemma. It is set theoretically R-invariant by construction. As V is a constuctible set, to see that it is open it suffices to show it is closed under generalization in U (Topology, Lemma 19.10 or the more general Topology, Lemma 23.6). Since $W_g \cap j^{-1}D(f)$ is open in U, it suffices to consider a specialization $u_1 \rightsquigarrow u_2$ of U with $u_2 \in j^{-1}D(h) \cap Z$. This means that h is nonzero in $j(u_2)$ and $u_2 \in Z$. If $u_1 \in Z$, then $j(u_1) \rightsquigarrow j(u_2)$ and since h is nonzero in $j(u_2)$ it is nonzero in $j(u_1)$ which implies $u_1 \in V$. If $u_1 \notin Z$ and also not in $W_g \cap j^{-1}D(f)$, then $u_1 \in W$, $u_1 \notin W_g$ because the complement of $Z = j^{-1}V(f)$ is contained in $W \cap j^{-1}D(f)$. Hence there exists a point $r_1 \in R$ with $s(r_1) = u_1$ such that h is zero in $t(r_1)$. Since s is finite we can find a specialization $r_1 \rightsquigarrow r_2$ with $s(r_2) = u_2$. However, then we conclude that h is zero in $u_2' = t(r_2)$ which contradicts the fact that $j^{-1}D(h) \cap Z$ is R-invariant and u_2 is in it. Thus V is open.

Observe that $V \subset j^{-1}D(h)$ for our function $h \in I$. Thus we obtain an immersion

$$j' : V \longrightarrow \mathrm{Spec}(A_h)$$

Let $f' \in A_h$ be the image of f. Then $(j')^{-1}D(f')$ is the principal open determined by g in the affine open $j^{-1}D(f)$ of U. Hence $(j')^{-1}D(f)$ is affine. Finally, $j'(V) \cap V(f') = j'(j^{-1}D(h) \cap Z)$ is closed in $\mathrm{Spec}(A_h/(f')) = \mathrm{Spec}((A/f)_h) = D(h) \cap V(f)$ by our choice of $h \in I$ and the ideal I. Hence we can apply Lemma 14.9 to conclude that V is affine as claimed above. $\qquad\square$

15. Descending ind-quasi-affine morphisms

Ind-quasi-affine morphisms were defined in More on Morphisms, Section 66. This section is the analogue of Descent, Section 38 for ind-quasi-affine-morphisms.

Let X be a quasi-separated scheme. Let $E \subset X$ be a subset which is an intersection of a nonempty family of quasi-compact opens of X. Say $E = \bigcap_{i \in I} U_i$ with $U_i \subset X$ quasi-compact open and I nonempty. By adding finite intersections we may assume that for $i, j \in I$ there exists a $k \in I$ with $U_k \subset U_i \cap U_j$. In this situation we have

$$(15.0.1) \qquad \Gamma(E, \mathcal{F}|_E) = \mathrm{colim}\, \Gamma(U_i, \mathcal{F}|_{U_i})$$

for any sheaf $\mathcal{F}$ defined on X. Namely, fix $i_0 \in I$ and replace X by U_{i_0} and I by $\{i \in I \mid U_i \subset U_{i_0}\}$. Then X is quasi-compact and quasi-separated, hence a spectral space, see Properties, Lemma 2.4. Then we see the equality holds by Topology, Lemma 24.7 and Sheaves, Lemma 29.4. (In fact, the formula holds for higher cohomology groups as well if $\mathcal{F}$ is abelian, see Cohomology, Lemma 19.2.)

Lemma 15.1. *Let X be an ind-quasi-affine scheme. Let $E \subset X$ be an intersection of a nonempty family of quasi-compact opens of X. Set $A = \Gamma(E, \mathcal{O}_X|_E)$ and $Y = \mathrm{Spec}(A)$. Then the canonical morphism*

$$j : (E, \mathcal{O}_X|_E) \longrightarrow (Y, \mathcal{O}_Y)$$

of Schemes, Lemma 6.4 determines an isomorphism $(E, \mathcal{O}_X|_E) \to (E', \mathcal{O}_Y|_{E'})$ where $E' \subset Y$ is an intersection of quasi-compact opens. If $W \subset E$ is open in X, then $j(W)$ is open in Y.

Proof. Note that $(E, \mathcal{O}_X|_E)$ is a locally ringed space so that Schemes, Lemma 6.4 applies to $A \to \Gamma(E, \mathcal{O}_X|_E)$. Write $E = \bigcap_{i \in I} U_i$ with $I \neq \emptyset$ and $U_i \subset X$ quasi-compact open. We may and do assume that for $i, j \in I$ there exists a $k \in I$ with $U_k \subset U_i \cap U_j$. Set $A_i = \Gamma(U_i, \mathcal{O}_{U_i})$. We obtain commutative diagrams

$$
\begin{array}{ccc}
(E, \mathcal{O}_X|_E) & \longrightarrow & (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \\
\downarrow & & \downarrow \\
(U_i, \mathcal{O}_{U_i}) & \longrightarrow & (\mathrm{Spec}(A_i), \mathcal{O}_{\mathrm{Spec}(A_i)})
\end{array}
$$

Since U_i is quasi-affine, we see that $U_i \to \mathrm{Spec}(A_i)$ is a quasi-compact open immersion. On the other hand $A = \mathrm{colim}\, A_i$. Hence $\mathrm{Spec}(A) = \lim \mathrm{Spec}(A_i)$ as topological spaces (Limits, Lemma 4.6). Since $E = \lim U_i$ (by Topology, Lemma 24.7) we see that $E \to \mathrm{Spec}(A)$ is a homeomorphism onto its image E' and that E' is the intersection of the inverse images of the opens $U_i \subset \mathrm{Spec}(A_i)$ in $\mathrm{Spec}(A)$. For any $e \in E$ the local ring $\mathcal{O}_{X,e}$ is the value of $\mathcal{O}_{U_i,e}$ which is the same as the value on $\mathrm{Spec}(A)$.

To prove the final assertion of the lemma we argue as follows. Pick $i, j \in I$ with $U_i \subset U_j$. Consider the following commutative diagrams

$$
\begin{array}{ccc}
U_i \longrightarrow \mathrm{Spec}(A_i) & \quad W \longrightarrow \mathrm{Spec}(A_i) & \quad W \longrightarrow \mathrm{Spec}(A) \\
\downarrow \qquad \qquad \downarrow & \quad \downarrow \qquad \qquad \downarrow & \quad \downarrow \qquad \qquad \downarrow \\
U_i \longrightarrow \mathrm{Spec}(A_j) & \quad W \longrightarrow \mathrm{Spec}(A_j) & \quad W \longrightarrow \mathrm{Spec}(A_j)
\end{array}
$$

By Properties, Lemma 18.5 the first diagram is cartesian. Hence the second is cartesian as well. Passing to the limit we find that the third diagram is cartesian, so the top horizontal arrow of this diagram is an open immersion. $\square$

Lemma 15.2. *Suppose given a cartesian diagram*

$$
\begin{array}{ccc}
X & \longrightarrow & \mathrm{Spec}(B) \\
{\scriptstyle f}\downarrow & & \downarrow \\
Y & \longrightarrow & \mathrm{Spec}(A)
\end{array}
$$

of schemes. Let $E \subset Y$ be an intersection of a nonempty family of quasi-compact opens of Y. Then

$$
\Gamma(f^{-1}(E), \mathcal{O}_X|_{f^{-1}(E)}) = \Gamma(E, \mathcal{O}_Y|_E) \otimes_A B
$$

provided Y is quasi-separated and $A \to B$ is flat.

Proof. Write $E = \bigcap_{i \in I} V_i$ with $V_i \subset Y$ quasi-compact open. We may and do assume that for $i, j \in I$ there exists a $k \in I$ with $V_k \subset V_i \cap V_j$. Then we have similarly that $f^{-1}(E) = \bigcap_{i \in I} f^{-1}(V_i)$ in X. Thus the result follows from equation (15.0.1) and the corresponding result for V_i and $f^{-1}(V_i)$ which is Cohomology of Schemes, Lemma 5.2. $\square$

Lemma 15.3 (Gabber)**.** *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$, see Descent, Definition 34.3. If each morphism $V_i \to X_i$ is ind-quasi-affine, then the descent datum is effective.*

Proof. Being ind-quasi-affine is a property of morphisms of schemes which is preserved under any base change, see More on Morphisms, Lemma 66.6. Hence Descent, Lemma 36.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $V \to X$ is ind-quasi-affine, in other words, V is ind-quasi-affine.

Let (U, R, s, t, c) be the groupoid scheme over S with $U = X$ and $R = X \times_S X$ and s, t, c as usual. By Groupoids, Lemma 21.3 the pair (V, φ) corresponds to a cartesian morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoid schemes. Let $u' \in U'$ be any point. By Groupoids, Lemmas 19.2, 19.3, and 19.4 we can choose $u' \in W \subset E \subset U'$ where W is open and R'-invariant, and E is set-theoretically R'-invariant and an intersection of a nonempty family of quasi-compact opens.

Translating back to (V, φ), for any $v \in V$ we can find $v \in W \subset E \subset V$ with the following properties: (a) W is open and $\varphi(W \times_S X) = X \times_S W$ and (b) E an

intersection of quasi-compact opens and $\varphi(E \times_S X) = X \times_S E$ set-theoretically. Here we use the notation $E \times_S X$ to mean the inverse image of E in $V \times_S X$ by the projection morphism and similarly for $X \times_S E$. By Lemma 15.2 this implies that φ defines an isomorphism

$$\Gamma(E, \mathcal{O}_V|_E) \otimes_R A = \Gamma(E \times_S X, \mathcal{O}_{V \times_S X}|_{E \times_S X})$$
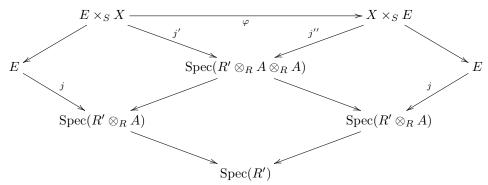$$\to \Gamma(X \times_S E, \mathcal{O}_{X \times_S V}|_{X \times_S E})$$
$$= A \otimes_R \Gamma(E, \mathcal{O}_V|_E)$$

of $A \otimes_R A$-algebras which we will call ψ. The cocycle condition for φ translates into the cocycle condition for ψ as in Descent, Definition 3.1 (details omitted). By Descent, Proposition 3.9 we find an R-algebra R' and an isomorphism $\chi : R' \otimes_R A \to \Gamma(E, \mathcal{O}_V|_E)$ of A-algebras, compatible with ψ and the canonical descent datum on $R' \otimes_R A$.

By Lemma 15.1 we obtain a canonical "embedding"

$$j : (E, \mathcal{O}_V|_E) \longrightarrow \mathrm{Spec}(\Gamma(E, \mathcal{O}_V|_E)) = \mathrm{Spec}(R' \otimes_R A)$$

of locally ringed spaces. The construction of this map is canonical and we get a commutative diagram

where j' and j'' come from the same construction applied to $E \times_S X \subset V \times_S X$ and $X \times_S E \subset X \times_S V$ via χ and the identifications used to construct ψ. It follows that $j(W)$ is an open subscheme of $\mathrm{Spec}(R' \otimes_R A)$ whose inverse image under the two projections $\mathrm{Spec}(R' \otimes_R A \otimes_R A) \to \mathrm{Spec}(R' \otimes_R A)$ are equal. By Descent, Lemma 13.6 we find an open $W_0 \subset \mathrm{Spec}(R')$ whose base change to $\mathrm{Spec}(A)$ is $j(W)$. Contemplating the diagram above we see that the descent datum $(W, \varphi|_{W \times_S X})$ is effective. By Descent, Lemma 35.13 we see that our descent datum is effective. $\square$

16. Other chapters

Preliminaries

References

[KM97] Sean Keel and Shigefumi Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), 193–213.