ALGEBRAIC STACKS

Contents

1. Introduction

This is where we define algebraic stacks and make some very elementary observations. The general philosophy will be to have no separation conditions whatsoever and add those conditions necessary to make lemmas, propositions, theorems true/provable. Thus the notions discussed here differ slightly from those in other places in the literature, e.g., [LMB00].

This chapter is not an introduction to algebraic stacks. For an informal discussion of algebraic stacks, please take a look at Introducing Algebraic Stacks, Section 1.

2. Conventions

The conventions we use in this chapter are the same as those in the chapter on algebraic spaces. For convenience we repeat them here.

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We discuss what changes if you change the big fppf site in Section 18.

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

If U, T are schemes over S, then we denote $U(T)$ for the set of T-valued points *over* S. In a formula: $U(T) = \mathrm{Mor}_S(T, U)$.

Note that any fpqc covering is a universal effective epimorphism, see Descent, Lemma 13.7. Hence the topology on Sch_{fppf} is weaker than the canonical topology and all representable presheaves are sheaves.

3. Notation

We use the letters S, T, U, V, X, Y to indicate schemes. We use the letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ to indicate categories (fibred, fibred in groupoids, stacks, ...) over $(Sch/S)_{fppf}$. We use small case letters f, g for functors such as $f : \mathcal{X} \to \mathcal{Y}$ over $(Sch/S)_{fppf}$. We use capital F, G, H for algebraic spaces over S, and more generally for presheaves of sets on $(Sch/S)_{fppf}$. (In future chapters we will revert to using also X, Y, etc for algebraic spaces.)

The reason for these choices is that we want to clearly distinguish between the different types of objects in this chapter, to build the foundations.

4. Representable categories fibred in groupoids

Let S be a scheme contained in Sch_{fppf}. The basic object of study in this chapter will be a category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$, see Categories, Definition 35.1. We will often simply say "let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$" to indicate this situation. A 1-morphism $\mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ will be a 1-morphism in the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$, see Categories, Definition 35.6. It is simply a functor $\mathcal{X} \to \mathcal{Y}$ over $(Sch/S)_{fppf}$. We recall this is really a $(2,1)$-category and that all 2-fibre products exist.

Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Recall that $\mathcal{X}$ is said to be *representable* if there exists a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and an equivalence

$$j : \mathcal{X} \longrightarrow (Sch/U)_{fppf}$$

of categories over $(Sch/S)_{fppf}$, see Categories, Definition 40.1. We will sometimes say that $\mathcal{X}$ is *representable by a scheme* to distinguish from the case where $\mathcal{X}$ is representable by an algebraic space (see below).

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by U, V, then we have

$$(4.0.1) \qquad \mathrm{Mor}_{Cat/(Sch/S)_{fppf}}(\mathcal{X}, \mathcal{Y}) \Big/ \text{2-isomorphism} = \mathrm{Mor}_{Sch/S}(U, V)$$

see Categories, Lemma 40.3. More precisely, any 1-morphism $\mathcal{X} \to \mathcal{Y}$ gives rise to a morphism $U \to V$. Conversely, given a morphism of schemes $U \to V$ over S there exists a 1-morphism $\phi : \mathcal{X} \to \mathcal{Y}$ which gives rise to $U \to V$ and which is unique up to unique 2-isomorphism.

5. The 2-Yoneda lemma

Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$, and let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We will frequently use the 2-Yoneda lemma, see Categories, Lemma 41.2. Technically it says that there is an equivalence of categories

$$\mathrm{Mor}_{Cat/(Sch/S)_{fppf}}((Sch/U)_{fppf}, \mathcal{X}) \longrightarrow \mathcal{X}_U, \quad f \longmapsto f(U/U).$$

It says that 1-morphisms $(Sch/U)_{fppf} \to \mathcal{X}$ correspond to objects x of the fibre category $\mathcal{X}_U$. Namely, given a 1-morphism $f : (Sch/U)_{fppf} \to \mathcal{X}$ we obtain the object $x = f(U/U) \in \mathrm{Ob}(\mathcal{X}_U)$. Conversely, given a choice of pullbacks for $\mathcal{X}$ as in Categories, Definition 33.6, and an object x of $\mathcal{X}_U$, we obtain a functor $(Sch/U)_{fppf} \to \mathcal{X}$ defined by the rule

$$(\varphi : V \to U) \longmapsto \varphi^* x$$

on objects. By abuse of notation we use $x : (Sch/U)_{fppf} \to \mathcal{X}$ to indicate this functor. It indeed has the property that $x(U/U) = x$ and moreover, given any other functor f with $f(U/U) = x$ there exists a unique 2-isomorphism $x \to f$. In other words the functor x is well determined by the object x up to unique 2-isomorphism.

We will use this without further mention in the following.

6. Representable morphisms of categories fibred in groupoids

Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a *representable* 1-*morphism*, see Categories, Definition 42.3. This means that for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y \in \mathrm{Ob}(\mathcal{Y}_U)$ the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable. Choose a representing object V_y and an equivalence

$$(Sch/V_y)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}.$$

The projection $(Sch/V_y)_{fppf} \to (Sch/U)_{fppf} \times_{\mathcal{Y}} \mathcal{Y} \to (Sch/U)_{fppf}$ comes from a morphism of schemes $f_y : V_y \to U$, see Section 4. We represent this by the diagram

(6.0.1)
$$
\begin{array}{ccccc}
V_y & \rightsquigarrow & (Sch/V_y)_{fppf} & \longrightarrow & \mathcal{X} \\
\downarrow {\scriptstyle f_y} & & \downarrow & & \downarrow {\scriptstyle f} \\
U & \rightsquigarrow & (Sch/U)_{fppf} & \xrightarrow{\ y\ } & \mathcal{Y}
\end{array}
$$

where the squiggly arrows represent the 2-Yoneda embedding. Here are some lemmas about this notion that work in great generality (namely, they work for categories fibred in groupoids over any base category which has fibre products).

Lemma 6.1. *Let $f : X \to Y$ be a morphism of $(Sch/S)_{fppf}$. Then the 1-morphism induced by f*

$$(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{fppf}$$

is a representable 1-morphism.

Proof. This is formal and relies only on the fact that the category $(Sch/S)_{fppf}$ has fibre products. $\square$

Lemma 6.2. *Let S be an object of Sch_{fppf}. Consider a 2-commutative diagram*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \scriptstyle{f'} \downarrow & & \downarrow \scriptstyle{f} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences. Then f is representable if and only if f' is representable.

Proof. Omitted. $\square$

Lemma 6.3. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be representable 1-morphisms. Then*

$$g \circ f : \mathcal{X} \longrightarrow \mathcal{Z}$$

is a representable 1-morphism.

Proof. This is entirely formal and works in any category. $\square$

Lemma 6.4. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$\begin{array}{ccc} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \overset{g'}{\longrightarrow} & \mathcal{X} \\ \scriptstyle{f'} \downarrow & & \downarrow \scriptstyle{f} \\ \mathcal{Z} & \overset{g}{\longrightarrow} & \mathcal{Y} \end{array}$$

Then the base change f' is a representable 1-morphism.

Proof. This is entirely formal and works in any category. $\square$

Lemma 6.5. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be representable 1-morphisms. Then*

$$f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is a representable 1-morphism.

Proof. Write $f_1 \times f_2$ as the composition $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. The first arrow is the base change of f_1 by the map $\mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1$, and the second arrow is the base change of f_2 by the map $\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_2$. Hence this lemma is a formal consequence of Lemmas 6.3 and 6.4. $\square$

7. Split categories fibred in groupoids

Let S be a scheme contained in Sch_{fppf}. Recall that given a "presheaf of groupoids"

$$F : (Sch/S)_{fppf}^{opp} \longrightarrow Groupoids$$

we get a category fibred in groupoids $\mathcal{S}_F$ over $(Sch/S)_{fppf}$, see Categories, Example 37.1. Any category fibred in groupoids isomorphic (!) to one of these is called a *split category fibred in groupoids*. Any category fibred in groupoids is equivalent to a split one.

If F is a presheaf of sets then $\mathcal{S}_F$ is fibred in sets, see Categories, Definition 38.2, and Categories, Example 38.5. The rule $F \mapsto \mathcal{S}_F$ is in some sense fully faithful on presheaves, see Categories, Lemma 38.6. If F, G are presheaves, then

$$\mathcal{S}_{F \times G} = \mathcal{S}_F \times_{(Sch/S)_{fppf}} \mathcal{S}_G$$

and if $F \to H$ and $G \to H$ are maps of presheaves of sets, then

$$\mathcal{S}_{F \times_H G} = \mathcal{S}_F \times_{\mathcal{S}_H} \mathcal{S}_G$$

where the right hand sides are 2-fibre products. This is immediate from the definitions as the fibre categories of $\mathcal{S}_F, \mathcal{S}_G, \mathcal{S}_H$ have only identity morphisms.

An even more special case is where $F = h_X$ is a representable presheaf. In this case we have $\mathcal{S}_{h_X} = (Sch/X)_{fppf}$, see Categories, Example 38.7.

We will use the notation $\mathcal{S}_F$ without further mention in the following.

8. Categories fibred in groupoids representable by algebraic spaces

A slightly weaker notion than being representable is the notion of being representable by algebraic spaces which we discuss in this section. This discussion might have been avoided had we worked with some category $Spaces_{fppf}$ of algebraic spaces instead of the category Sch_{fppf}. However, it seems to us natural to consider the category of schemes as the natural collection of "test objects" over which the fibre categories of an algebraic stack are defined.

In analogy with Categories, Definitions 40.1 we make the following definition.

Definition 8.1. Let S be a scheme contained in Sch_{fppf}. A category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ is called *representable by an algebraic space over S* if there exists an algebraic space F over S and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ of categories over $(Sch/S)_{fppf}$.

We continue our abuse of notation in suppressing the equivalence j whenever we encounter such a situation. It follows formally from the above that if $\mathcal{X}$ is representable (by a scheme), then it is representable by an algebraic space. Here is the analogue of Categories, Lemma 40.2.

Lemma 8.2. *Let S be a scheme contained in Sch_{fppf}. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Then $\mathcal{X}$ is representable by an algebraic space over S if and only if the following conditions are satisfied:*

(1) *$\mathcal{X}$ is fibred in setoids[1], and*
(2) *the presheaf $U \mapsto \mathrm{Ob}(\mathcal{X}_U)/\cong$ is an algebraic space.*

Proof. Omitted, but see Categories, Lemma 40.2. $\qquad\square$

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by algebraic spaces F, G over S, then we have

$$(8.2.1) \qquad \mathrm{Mor}_{Cat/(Sch/S)_{fppf}}(\mathcal{X}, \mathcal{Y}) \Big/ \text{2-isomorphism} = \mathrm{Mor}_{Sch/S}(F, G)$$

see Categories, Lemma 39.6. More precisely, any 1-morphism $\mathcal{X} \to \mathcal{Y}$ gives rise to a morphism $F \to G$. Conversely, given a morphism of sheaves $F \to G$ over S there exists a 1-morphism $\phi : \mathcal{X} \to \mathcal{Y}$ which gives rise to $F \to G$ and which is unique up to unique 2-isomorphism.

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[1]This means that it is fibred in groupoids and objects in the fibre categories have no nontrivial automorphisms, see Categories, Definition 38.2.

9. Morphisms representable by algebraic spaces

In analogy with Categories, Definition 42.3 we make the following definition.

Definition 9.1. Let S be a scheme contained in Sch_{fppf}. A 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ is called *representable by algebraic spaces* if for any $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y : (Sch/U)_{fppf} \to \mathcal{Y}$ the category fibred in groupoids

$$(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$$

over $(Sch/U)_{fppf}$ is representable by an algebraic space over U.

Choose an algebraic space F_y over U which represents $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$. We may think of F_y as an algebraic space over S which comes equipped with a canonical morphism $f_y : F_y \to U$ over S, see Spaces, Section 16. Here is the diagram

(9.1.1)
$$
\begin{array}{ccccc}
F_y & \leftsquigarrow & (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X} & \xrightarrow{\ \mathrm{pr}_1\ } & \mathcal{X} \\
f_y \downarrow & & \mathrm{pr}_0 \downarrow & & \downarrow f \\
U & \leftsquigarrow & (Sch/U)_{fppf} & \xrightarrow{\ y\ } & \mathcal{Y}
\end{array}
$$

where the squiggly arrows represent the construction which associates to a stack fibred in setoids its associated sheaf of isomorphism classes of objects. The right square is 2-commutative, and is a 2-fibre product square.

Here is the analogue of Categories, Lemma 42.5.

Lemma 9.2. *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are necessary and sufficient conditions for f to be representable by algebraic spaces:*

   (1) *for each scheme U/S the functor $f_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$ between fibre categories is faithful, and*

   (2) *for each U and each $y \in \mathrm{Ob}(\mathcal{Y}_U)$ the presheaf*

$$(h : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathrm{Ob}(\mathcal{X}_V), \phi : h^*y \to f(x)\}/\cong$$

   is an algebraic space over U.

Here we have made a choice of pullbacks for $\mathcal{Y}$.

Proof. This follows from the description of fibre categories of the 2-fibre products $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ in Categories, Lemma 42.1 combined with Lemma 8.2. $\square$

Here are some lemmas about this notion that work in great generality.

Lemma 9.3. *Let S be an object of Sch_{fppf}. Consider a 2-commutative diagram*

$$
\begin{array}{ccc}
\mathcal{X}' & \longrightarrow & \mathcal{X} \\
f' \downarrow & & \downarrow f \\
\mathcal{Y}' & \longrightarrow & \mathcal{Y}
\end{array}
$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences. Then f is representable by algebraic spaces if and only if f' is representable by algebraic spaces.

Proof. Omitted. $\square$

Lemma 9.4. *Let S be an object of Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over S. If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces over S, then the 1-morphism f is representable by algebraic spaces.*

Proof. Omitted. This relies only on the fact that the category of algebraic spaces over S has fibre products, see Spaces, Lemma 7.3. $\square$

Lemma 9.5. *Let S be an object of Sch_{fppf}. Let $a : F \to G$ be a map of presheaves of sets on $(Sch/S)_{fppf}$. Denote $a' : \mathcal{S}_F \to \mathcal{S}_G$ the associated map of categories fibred in sets. Then a is representable by algebraic spaces (see Bootstrap, Definition 3.1) if and only if a' is representable by algebraic spaces.*

Proof. Omitted. $\square$

Lemma 9.6. *Let S be an object of Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in setoids over $(Sch/S)_{fppf}$. Let F, resp. G be the presheaf which to T associates the set of isomorphism classes of objects of $\mathcal{X}_T$, resp. $\mathcal{Y}_T$. Let $a : F \to G$ be the map of presheaves corresponding to f. Then a is representable by algebraic spaces (see Bootstrap, Definition 3.1) if and only if f is representable by algebraic spaces.*

Proof. Omitted. Hint: Combine Lemmas 9.3 and 9.5. $\square$

Lemma 9.7. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$
\begin{array}{ccc}
\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \xrightarrow{\ g'\ } & \mathcal{X} \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{Z} & \xrightarrow{\ g\ } & \mathcal{Y}
\end{array}
$$

Then the base change f' is a 1-morphism representable by algebraic spaces.

Proof. This is formal. $\square$

Lemma 9.8. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms. Assume*

    (1) *f is representable by algebraic spaces, and*
    (2) *$\mathcal{Z}$ is representable by an algebraic space over S.*

Then the 2-fibre product $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ is representable by an algebraic space.

Proof. This is a reformulation of Bootstrap, Lemma 3.6. First note that $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ is fibred in setoids over $(Sch/S)_{fppf}$. Hence it is equivalent to $\mathcal{S}_F$ for some presheaf F on $(Sch/S)_{fppf}$, see Categories, Lemma 39.5. Moreover, let G be an algebraic space which represents $\mathcal{Z}$. The 1-morphism $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ is representable by algebraic spaces by Lemma 9.7. And $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ corresponds to a morphism $F \to G$ by Categories, Lemma 39.6. Then $F \to G$ is representable by algebraic spaces by Lemma 9.6. Hence Bootstrap, Lemma 3.6 implies that F is an algebraic space as desired. $\square$

Let S, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, f, g be as in Lemma 9.8. Let F and G be algebraic spaces over S such that F represents $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ and G represents $\mathcal{Z}$. The 1-morphism $f' : \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ corresponds to a morphism $f' : F \to G$ of algebraic spaces by (8.2.1). Thus we have the following diagram

$$(9.8.1) \qquad \begin{array}{ccccc} F & \rightsquigarrow & \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ G & \rightsquigarrow & \mathcal{Z} & \xrightarrow{\ g\ } & \mathcal{Y} \end{array}$$

where the squiggly arrows represent the construction which associates to a stack fibred in setoids its associated sheaf of isomorphism classes of objects.

Lemma 9.9. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. If $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ are 1-morphisms representable by algebraic spaces, then*

$$g \circ f : \mathcal{X} \longrightarrow \mathcal{Z}$$

is a 1-morphism representable by algebraic spaces.

Proof. This follows from Lemma 9.8. Details omitted. $\qquad\square$

Lemma 9.10. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. Then*

$$f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is a 1-morphism representable by algebraic spaces.

Proof. Write $f_1 \times f_2$ as the composition $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. The first arrow is the base change of f_1 by the map $\mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1$, and the second arrow is the base change of f_2 by the map $\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_2$. Hence this lemma is a formal consequence of Lemmas 9.9 and 9.7. $\qquad\square$

Lemma 9.11. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X} \to \mathcal{Z}$ is representable by algebraic spaces and $\mathcal{Y}$ is a stack in groupoids, then $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a stack in groupoids.*

Proof. The property of a morphism being representable by algebraic spaces is preserved under base-change (Lemma 9.8), and so, passing to the base-change $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ over $\mathcal{Y}$, we may reduce to the case of a morphism of categories fibred in groupoids $\mathcal{X} \to \mathcal{Y}$ which is representable by algebraic spaces, and whose target is a stack in groupoids; our goal is then to prove that $\mathcal{X}$ is also a stack in groupoids. This follows from Stacks, Lemma 6.11 whose assumptions are satisfied as a result of Lemma 9.2. $\qquad\square$

10. Properties of morphisms representable by algebraic spaces

Here is the definition that makes this work.

Definition 10.1. Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume f is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

(1) is preserved under any base change, and

(2) is fppf local on the base, see Descent on Spaces, Definition 10.1.

In this case we say that f has *property* $\mathcal{P}$ if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y \in \mathcal{Y}_U$ the resulting morphism of algebraic spaces $f_y : F_y \to U$, see diagram (9.1.1), has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the target. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

Lemma 10.2. *Let S be an object of Sch_{fppf}. Let $\mathcal{P}$ be as in Definition 10.1. Consider a 2-commutative diagram*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \scriptstyle f' \downarrow & & \downarrow \scriptstyle f \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences and f (or equivalently f') is representable by algebraic spaces. Then f has $\mathcal{P}$ if and only if f' has $\mathcal{P}$.

Proof. Note that this makes sense by Lemma 9.3. Proof omitted. $\square$

Here is a sanity check.

Lemma 10.3. *Let S be a scheme contained in Sch_{fppf}. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be as in Definition 10.1. Assume a is representable by algebraic spaces. Then $a : F \to G$ has property $\mathcal{P}$ (see Bootstrap, Definition 4.1) if and only if the corresponding morphism $\mathcal{S}_F \to \mathcal{S}_G$ of categories fibred in groupoids has property $\mathcal{P}$.*

Proof. Note that the lemma makes sense by Lemma 9.5. Proof omitted. $\square$

Lemma 10.4. *Let S be an object of Sch_{fppf}. Let $\mathcal{P}$ be as in Definition 10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in setoids over $(Sch/S)_{fppf}$. Let F, resp. G be the presheaf which to T associates the set of isomorphism classes of objects of $\mathcal{X}_T$, resp. $\mathcal{Y}_T$. Let $a : F \to G$ be the map of presheaves corresponding to f. Then a has $\mathcal{P}$ if and only if f has $\mathcal{P}$.*

Proof. The lemma makes sense by Lemma 9.6. The lemma follows on combining Lemmas 10.2 and 10.3. $\square$

Lemma 10.5. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 10.1 which is stable under composition. Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms which are representable by algebraic spaces. If f and g have property $\mathcal{P}$ so does $g \circ f : \mathcal{X} \to \mathcal{Z}$.*

Proof. Note that the lemma makes sense by Lemma 9.9. Proof omitted. $\square$

Lemma 10.6. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the 2-fibre product diagram*

$$
\begin{array}{ccc}
\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \xrightarrow{\;g'\;} & \mathcal{X} \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\mathcal{Z} & \xrightarrow{\;g\;} & \mathcal{Y}
\end{array}
$$

If f has $\mathcal{P}$, then the base change f' has $\mathcal{P}$.

Proof. The lemma makes sense by Lemma 9.7. Proof omitted. $\square$

Lemma 10.7. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$
\begin{array}{ccc}
\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \xrightarrow{\;g'\;} & \mathcal{X} \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\mathcal{Z} & \xrightarrow{\;g\;} & \mathcal{Y}
\end{array}
$$

Assume that for every scheme U and object x of $\mathcal{Y}_U$, there exists an fppf covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of the functor $g : \mathcal{Z}_{U_i} \to \mathcal{Y}_{U_i}$. In this case, if f' has $\mathcal{P}$, then f has $\mathcal{P}$.

Proof. Proof omitted. Hint: Compare with the proof of Spaces, Lemma 5.6. $\square$

Lemma 10.8. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{P}$ be a property as in Definition 10.1 which is stable under composition. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. If f_1 and f_2 have property $\mathcal{P}$ so does $f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$.*

Proof. The lemma makes sense by Lemma 9.10. Proof omitted. $\square$

Lemma 10.9. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $\mathcal{P}, \mathcal{P}'$ be properties as in Definition 10.1. Suppose that for any morphism of algebraic spaces $a : F \to G$ we have $\mathcal{P}(a) \Rightarrow \mathcal{P}'(a)$. If f has property $\mathcal{P}$ then f has property $\mathcal{P}'$.*

Proof. Formal. $\square$

Lemma 10.10. *Let S be a scheme contained in Sch_{fppf}. Let $j : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume j is representable by algebraic spaces and a monomorphism (see Definition 10.1 and Descent on Spaces, Lemma 11.30). Then j is fully faithful on fibre categories.*

Proof. We have seen in Lemma 9.2 that j is faithful on fibre categories. Consider a scheme U, two objects u, v of $\mathcal{X}_U$, and an isomorphism $t : j(u) \to j(v)$ in $\mathcal{Y}_U$. We have to construct an isomorphism in $\mathcal{X}_U$ between u and v. By the 2-Yoneda lemma

(see Section 5) we think of u, v as 1-morphisms $u, v : (Sch/U)_{fppf} \to \mathcal{X}$ and we consider the 2-fibre product

$$(Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X}.$$

By assumption this is representable by an algebraic space $F_{j \circ v}$, over U and the morphism $F_{j \circ v} \to U$ is a monomorphism. But since $(1_U, v, 1_{j(v)})$ gives a 1-morphism of $(Sch/U)_{fppf}$ into the displayed 2-fibre product, we see that $F_{j \circ v} = U$ (here we use that if $V \to U$ is a monomorphism of algebraic spaces which has a section, then $V = U$). Therefore the 1-morphism projecting to the first coordinate

$$(Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X} \to (Sch/U)_{fppf}$$

is an equivalence of fibre categories. Since $(1_U, u, t)$ and $(1_U, v, 1_{j(v)})$ give two objects in $((Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X})_U$ which have the same first coordinate, there must be a 2-morphism between them in the 2-fibre product. This is by definition a morphism $\tilde{t} : u \to v$ such that $j(\tilde{t}) = t$. $\square$

Here is a characterization of those categories fibred in groupoids for which the diagonal is representable by algebraic spaces.

Lemma 10.11. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent:*

1. *the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
2. *for every scheme U over S, and any $x, y \in \mathrm{Ob}(\mathcal{X}_U)$ the sheaf $Isom(x, y)$ is an algebraic space over U,*
3. *for every scheme U over S, and any $x \in \mathrm{Ob}(\mathcal{X}_U)$ the associated 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ is representable by algebraic spaces,*
4. *for every pair of schemes T_1, T_2 over S, and any $x_i \in \mathrm{Ob}(\mathcal{X}_{T_i})$, $i = 1, 2$ the 2-fibre product $(Sch/T_1)_{fppf} \times_{x_1, \mathcal{X}, x_2} (Sch/T_2)_{fppf}$ is representable by an algebraic space,*
5. *for every representable category fibred in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ every 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces,*
6. *for every pair $\mathcal{T}_1, \mathcal{T}_2$ of representable categories fibred in groupoids over $(Sch/S)_{fppf}$ and any 1-morphisms $x_i : \mathcal{T}_i \to \mathcal{X}$, $i = 1, 2$ the 2-fibre product $\mathcal{T}_1 \times_{x_1, \mathcal{X}, x_2} \mathcal{T}_2$ is representable by an algebraic space,*
7. *for every category fibred in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ which is representable by an algebraic space every 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces,*
8. *for every pair $\mathcal{T}_1, \mathcal{T}_2$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ which are representable by algebraic spaces, and any 1-morphisms $x_i : \mathcal{T}_i \to \mathcal{X}$ the 2-fibre product $\mathcal{T}_1 \times_{x_1, \mathcal{X}, x_2} \mathcal{T}_2$ is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 2.5 and the definitions. Let us prove the equivalence of (1) and (3). Write $\mathcal{C} = (Sch/S)_{fppf}$ for the base category. We will use some of the observations of the proof of the similar Categories, Lemma 42.6. We will use the symbol $\cong$ to mean "equivalence of categories fibred in groupoids over $\mathcal{C} = (Sch/S)_{fppf}$". Assume (1). Suppose given U and x as in (3). For any scheme V and $y \in \mathrm{Ob}(\mathcal{X}_V)$ we see (compare reference above) that

$$\mathcal{C}/U \times_{x, \mathcal{X}, y} \mathcal{C}/V \cong (\mathcal{C}/U \times_S V) \times_{(x, y), \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$$

which is representable by an algebraic space by assumption. Conversely, assume (3). Consider any scheme U over S and a pair (x, x') of objects of $\mathcal{X}$ over U. We have to show that $\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (x, x')} U$ is representable by an algebraic space. This is clear because (compare reference above)

$$\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (x, x')} \mathcal{C}/U \cong (\mathcal{C}/U \times_{x, \mathcal{X}, x'} \mathcal{C}/U) \times_{\mathcal{C}/U \times_S U, \Delta} \mathcal{C}/U$$

and the right hand side is representable by an algebraic space by assumption and the fact that the category of algebraic spaces over S has fibre products and contains U and S.

The equivalences $(3) \Leftrightarrow (4)$, $(5) \Leftrightarrow (6)$, and $(7) \Leftrightarrow (8)$ are formal. The equivalences $(3) \Leftrightarrow (5)$ and $(4) \Leftrightarrow (6)$ follow from Lemma 9.3. Assume (3), and let $\mathcal{U} \to \mathcal{X}$ be as in (7). To prove (7) we have to show that for every scheme V and 1-morphism $y : (Sch/V)_{fppf} \to \mathcal{X}$ the 2-fibre product $(Sch/V)_{fppf} \times_{y, \mathcal{X}} \mathcal{U}$ is representable by an algebraic space. Property (3) tells us that y is representable by algebraic spaces hence Lemma 9.8 implies what we want. Finally, (7) directly implies (3). $\qquad\square$

In the situation of the lemma, for any 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ as in the lemma, it makes sense to say that x has property $\mathcal{P}$, for any property as in Definition 10.1. In particular this holds for $\mathcal{P} = $ "surjective", $\mathcal{P} = $ "smooth", and $\mathcal{P} = $ "étale", see Descent on Spaces, Lemmas 11.6, 11.26, and 11.28. We will use these three cases in the definitions of algebraic stacks below.

11. Stacks in groupoids

Let S be a scheme contained in Sch_{fppf}. Recall that a category $p : \mathcal{X} \to (Sch/S)_{fppf}$ over $(Sch/S)_{fppf}$ is said to be a *stack in groupoids* (see Stacks, Definition 5.1) if and only if

1. $p : \mathcal{X} \to (Sch/S)_{fppf}$ is fibred in groupoids over $(Sch/S)_{fppf}$,
2. for all $U \in \mathrm{Ob}((Sch/S)_{fppf})$, for all $x, y \in \mathrm{Ob}(\mathcal{X}_U)$ the presheaf $Isom(x, y)$ is a sheaf on the site $(Sch/U)_{fppf}$, and
3. for all coverings $\mathcal{U} = \{U_i \to U\}$ in $(Sch/S)_{fppf}$, all descent data (x_i, ϕ_{ij}) for $\mathcal{U}$ are effective.

For examples see Examples of Stacks, Section 9 ff.

12. Algebraic stacks

Here is the definition of an algebraic stack. We remark that condition (2) implies we can make sense out of the condition in part (3) that $(Sch/U)_{fppf} \to \mathcal{X}$ is smooth and surjective, see discussion following Lemma 10.11.

Definition 12.1. Let S be a base scheme contained in Sch_{fppf}. An *algebraic stack over S* is a category

$$p : \mathcal{X} \to (Sch/S)_{fppf}$$

over $(Sch/S)_{fppf}$ with the following properties:

1. The category $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$.
2. The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.

(3) There exists a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a 1-morphism $(Sch/U)_{fppf} \to \mathcal{X}$ which is surjective and smooth[2].

There are some differences with other definitions found in the literature.

The first is that we require $\mathcal{X}$ to be a stack in groupoids in the fppf topology, whereas in many references the étale topology is used. It somehow seems to us that the fppf topology is the natural topology to work with. In the end the resulting 2-category of algebraic stacks ends up being the same. This is explained in Criteria for Representability, Section 19.

The second is that we only require the diagonal map of $\mathcal{X}$ to be representable by algebraic spaces, whereas in most references some other conditions are imposed. Our point of view is to try to prove a certain number of the results that follow only assuming that the diagonal of $\mathcal{X}$ be representable by algebraic spaces, and simply add an additional hypothesis wherever this is necessary. It has the added benefit that any algebraic space (as defined in Spaces, Definition 6.1) gives rise to an algebraic stack.

The third is that in some papers it is required that there exists a scheme U and a surjective and étale morphism $U \to \mathcal{X}$. In the groundbreaking paper [DM69] where algebraic stacks were first introduced Deligne and Mumford used this definition and showed that the moduli stack of stable genus $g > 1$ curves is an algebraic stack which has an étale covering by a scheme. Michael Artin, see [Art74], realized that many natural results on algebraic stacks generalize to the case where one only assume a smooth covering by a scheme. Hence our choice above. To distinguish the two cases one sees the terms "Deligne-Mumford stack" and "Artin stack" used in the literature. We will reserve the term "Artin stack" for later use (insert future reference here), and continue to use "algebraic stack", but we will use "Deligne-Mumford stack" to indicate those algebraic stacks which have an étale covering by a scheme.

Definition 12.2. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. We say $\mathcal{X}$ is a *Deligne-Mumford stack* if there exists a scheme U and a surjective étale morphism $(Sch/U)_{fppf} \to \mathcal{X}$.

We will compare our notion of a Deligne-Mumford stack with the notion as defined in the paper by Deligne and Mumford later (see insert future reference here).

The category of algebraic stacks over S forms a 2-category. Here is the precise definition.

Definition 12.3. Let S be a scheme contained in Sch_{fppf}. The 2-*category of algebraic stacks over S* is the sub 2-category of the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$ (see Categories, Definition 35.6) defined as follows:
  (1) Its objects are those categories fibred in groupoids over $(Sch/S)_{fppf}$ which are algebraic stacks over S.
  (2) Its 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ are any functors of categories over $(Sch/S)_{fppf}$, as in Categories, Definition 32.1.

---

[2]In future chapters we will denote this simply $U \to \mathcal{X}$ as is customary in the literature. Another good alternative would be to formulate this condition as the existence of a representable category fibred in groupoids $\mathcal{U}$ and a surjective smooth 1-morphism $\mathcal{U} \to \mathcal{X}$.

(3) Its 2-morphisms are transformations between functors over $(Sch/S)_{fppf}$, as in Categories, Definition 32.1.

In other words this 2-category is the full sub 2-category of $Cat/(Sch/S)_{fppf}$ whose objects are algebraic stacks. Note that every 2-morphism is automatically an isomorphism. Hence this is actually a $(2, 1)$-category and not just a 2-category.

We will see later (insert future reference here) that this 2-category has 2-fibre products.

Similar to the remark above the 2-category of algebraic stacks over S is a full sub 2-category of the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$. It turns out that it is closed under equivalences. Here is the precise statement.

Lemma 12.4. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be categories over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are equivalent as categories over $(Sch/S)_{fppf}$. Then $\mathcal{X}$ is an algebraic stack if and only if $\mathcal{Y}$ is an algebraic stack. Similarly, $\mathcal{X}$ is a Deligne-Mumford stack if and only if $\mathcal{Y}$ is a Deligne-Mumford stack.*

Proof. Assume $\mathcal{X}$ is an algebraic stack (resp. a Deligne-Mumford stack). By Stacks, Lemma 5.4 this implies that $\mathcal{Y}$ is a stack in groupoids over Sch_{fppf}. Choose an equivalence $f : \mathcal{X} \to \mathcal{Y}$ over Sch_{fppf}. This gives a 2-commutative diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\ \ f\ \ } & \mathcal{Y} \\
\Delta_{\mathcal{X}} \downarrow & & \downarrow \Delta_{\mathcal{Y}} \\
\mathcal{X} \times \mathcal{X} & \xrightarrow{f \times f} & \mathcal{Y} \times \mathcal{Y}
\end{array}
$$

whose horizontal arrows are equivalences. This implies that $\Delta_{\mathcal{Y}}$ is representable by algebraic spaces according to Lemma 9.3. Finally, let U be a scheme over S, and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a 1-morphism which is surjective and smooth (resp. étale). Considering the diagram

$$
\begin{array}{ccc}
(Sch/U)_{fppf} & \xrightarrow{\ \ \mathrm{id}\ \ } & (Sch/U)_{fppf} \\
x \downarrow & & \downarrow f \circ x \\
\mathcal{X} & \xrightarrow{\ \ f\ \ } & \mathcal{Y}
\end{array}
$$

and applying Lemma 10.2 we conclude that $f \circ x$ is surjective and smooth (resp. étale) as desired. $\square$

13. Algebraic stacks and algebraic spaces

In this section we discuss some simple criteria which imply that an algebraic stack is an algebraic space. The main result is that this happens exactly when objects of fibre categories have no nontrivial automorphisms. This is not a triviality! Before we come to this we first do a sanity check.

Lemma 13.1. *Let S be a scheme contained in Sch_{fppf}.*

(1) *A category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ which is representable by an algebraic space is a Deligne-Mumford stack.*

(2) *If F is an algebraic space over S, then the associated category fibred in groupoids $p : \mathcal{S}_F \to (Sch/S)_{fppf}$ is a Deligne-Mumford stack.*

(3) *If $X \in \mathrm{Ob}((Sch/S)_{fppf})$, then $(Sch/X)_{fppf} \to (Sch/S)_{fppf}$ is a Deligne-Mumford stack.*

Proof. It is clear that (2) implies (3). Parts (1) and (2) are equivalent by Lemma 12.4. Hence it suffices to prove (2). First, we note that $\mathcal{S}_F$ is stack in sets since F is a sheaf (Stacks, Lemma 6.3). A fortiori it is a stack in groupoids. Second the diagonal morphism $\mathcal{S}_F \to \mathcal{S}_F \times \mathcal{S}_F$ is the same as the morphism $\mathcal{S}_F \to \mathcal{S}_{F \times F}$ which comes from the diagonal of F. Hence this is representable by algebraic spaces according to Lemma 9.4. Actually it is even representable (by schemes), as the diagonal of an algebraic space is representable, but we do not need this. Let U be a scheme and let $h_U \to F$ be a surjective étale morphism. We may think of this as a surjective étale morphism of algebraic spaces. Hence by Lemma 10.3 the corresponding 1-morphism $(Sch/U)_{fppf} \to \mathcal{S}_F$ is surjective and étale. $\qquad\square$

The following result says that a Deligne-Mumford stack whose inertia is trivial "is" an algebraic space. This lemma will be obsoleted by the stronger Proposition 13.3 below which says that this holds more generally for algebraic stacks...

Lemma 13.2. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. The following are equivalent*

(1) *$\mathcal{X}$ is a Deligne-Mumford stack and is a stack in setoids,*
(2) *$\mathcal{X}$ is a Deligne-Mumford stack such that the canonical 1-morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is an equivalence, and*
(3) *$\mathcal{X}$ is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 7.2. The implication (3) $\Rightarrow$ (1) follows from Lemma 13.1. Finally, assume (1). By Stacks, Lemma 6.3 there exists a sheaf F on $(Sch/S)_{fppf}$ and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$. By Lemma 9.5 the fact that $\Delta_\mathcal{X}$ is representable by algebraic spaces, means that $\Delta_F : F \to F \times F$ is representable by algebraic spaces. Let U be a scheme, and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a surjective étale morphism. The composition $j \circ x : (Sch/U)_{fppf} \to \mathcal{S}_F$ corresponds to a morphism $h_U \to F$ of sheaves. By Bootstrap, Lemma 5.1 this morphism is representable by algebraic spaces. Hence by Lemma 10.4 we conclude that $h_U \to F$ is surjective and étale. Finally, we apply Bootstrap, Theorem 6.1 to see that F is an algebraic space. $\qquad\square$

Proposition 13.3. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. The following are equivalent*

(1) *$\mathcal{X}$ is a stack in setoids,*
(2) *the canonical 1-morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is an equivalence, and*
(3) *$\mathcal{X}$ is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 7.2. The implication (3) $\Rightarrow$ (1) follows from Lemma 13.2. Finally, assume (1). By Stacks, Lemma 6.3 there exists an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ where F is a sheaf on $(Sch/S)_{fppf}$. By Lemma 9.5 the fact that $\Delta_\mathcal{X}$ is representable by algebraic spaces, means that $\Delta_F : F \to F \times F$ is representable by algebraic spaces. Let U be a scheme and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a surjective smooth morphism. The composition $j \circ x : (Sch/U)_{fppf} \to \mathcal{S}_F$ corresponds to a morphism $h_U \to F$ of sheaves. By Bootstrap, Lemma 5.1 this morphism is representable by algebraic spaces. Hence by Lemma 10.4 we conclude that $h_U \to F$ is surjective and smooth. In particular it is

surjective, flat and locally of finite presentation (by Lemma 10.9 and the fact that a smooth morphism of algebraic spaces is flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 37.5 and 37.7). Finally, we apply Bootstrap, Theorem 10.1 to see that F is an algebraic space. $\square$

14. 2-Fibre products of algebraic stacks

The 2-category of algebraic stacks has products and 2-fibre products. The first lemma is really a special case of Lemma 14.3 but its proof is slightly easier.

Lemma 14.1. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks over S. Then $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is an algebraic stack, and is a product in the 2-category of algebraic stacks over S.*

Proof. An object of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over T is just a pair (x, y) where x is an object of $\mathcal{X}_T$ and y is an object of $\mathcal{Y}_T$. Hence it is immediate from the definitions that $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is a stack in groupoids. If (x, y) and (x', y') are two objects of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over T, then

$$Isom((x, y), (x', y')) = Isom(x, x') \times Isom(y, y').$$

Hence it follows from the equivalences in Lemma 10.11 and the fact that the category of algebraic spaces has products that the diagonal of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is representable by algebraic spaces. Finally, suppose that $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, and let x, y be surjective smooth morphisms $x : (Sch/U)_{fppf} \to \mathcal{X}$, $y : (Sch/V)_{fppf} \to \mathcal{Y}$. Note that

$$(Sch/U \times_S V)_{fppf} = (Sch/U)_{fppf} \times_{(Sch/S)_{fppf}} (Sch/V)_{fppf}.$$

The object $(\mathrm{pr}_U^* x, \mathrm{pr}_V^* y)$ of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over $(Sch/U \times_S V)_{fppf}$ thus defines a 1-morphism

$$(Sch/U \times_S V)_{fppf} \longrightarrow \mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$$

which is the composition of base changes of x and y, hence is surjective and smooth, see Lemmas 10.6 and 10.5. We conclude that $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is indeed an algebraic stack. We omit the verification that it really is a product. $\square$

Lemma 14.2. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{Z}$ be a stack in groupoids over $(Sch/S)_{fppf}$ whose diagonal is representable by algebraic spaces. Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks over S. Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids. Then the 2-fibre product $\mathcal{X} \times_{f, \mathcal{Z}, g} \mathcal{Y}$ is an algebraic stack.*

Proof. We have to check conditions (1), (2), and (3) of Definition 12.1. The first condition follows from Stacks, Lemma 5.6.

The second condition we have to check is that the *Isom*-sheaves are representable by algebraic spaces. To do this, suppose that T is a scheme over S, and u, v are objects of $(\mathcal{X} \times_{f, \mathcal{Z}, g} \mathcal{Y})_T$. By our construction of 2-fibre products (which goes all the way back to Categories, Lemma 32.3) we may write $u = (x, y, \alpha)$ and $v = (x', y', \alpha')$. Here $\alpha : f(x) \to g(y)$ and similarly for α'. Then it is clear that

$$
\begin{array}{ccc}
Isom(u, v) & \longrightarrow & Isom(y, y') \\
\downarrow & & \downarrow{\scriptstyle \phi \mapsto g(\phi) \circ \alpha} \\
Isom(x, x') & \xrightarrow{\psi \mapsto \alpha' \circ f(\psi)} & Isom(f(x), g(y'))
\end{array}
$$

is a cartesian diagram of sheaves on $(Sch/T)_{fppf}$. Since by assumption the sheaves $Isom(y, y')$, $Isom(x, x')$, $Isom(f(x), g(y'))$ are algebraic spaces (see Lemma 10.11) we see that $Isom(u, v)$ is an algebraic space.

Let $U, V \in \text{Ob}((Sch/S)_{fppf})$, and let x, y be surjective smooth morphisms $x : (Sch/U)_{fppf} \to \mathcal{X}$, $y : (Sch/V)_{fppf} \to \mathcal{Y}$. Consider the morphism

$$(Sch/U)_{fppf} \times_{f \circ x, \mathcal{Z}, g \circ y} (Sch/V)_{fppf} \longrightarrow \mathcal{X} \times_{f, \mathcal{Z}, g} \mathcal{Y}.$$

As the diagonal of $\mathcal{Z}$ is representable by algebraic spaces the source of this arrow is representable by an algebraic space F, see Lemma 10.11. Moreover, the morphism is the composition of base changes of x and y, hence surjective and smooth, see Lemmas 10.6 and 10.5. Choosing a scheme W and a surjective étale morphism $W \to F$ we see that the composition of the displayed 1-morphism with the corresponding 1-morphism

$$(Sch/W)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{f \circ x, \mathcal{Z}, g \circ y} (Sch/V)_{fppf}$$

is surjective and smooth which proves the last condition. $\square$

Lemma 14.3. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be algebraic stacks over S. Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of algebraic stacks. Then the 2-fibre product $\mathcal{X} \times_{f, \mathcal{Z}, g} \mathcal{Y}$ is an algebraic stack. It is also the 2-fibre product in the 2-category of algebraic stacks over $(Sch/S)_{fppf}$.*

Proof. The fact that $\mathcal{X} \times_{f, \mathcal{Z}, g} \mathcal{Y}$ is an algebraic stack follows from the stronger Lemma 14.2. The fact that $\mathcal{X} \times_{f, \mathcal{Z}, g} \mathcal{Y}$ is a 2-fibre product in the 2-category of algebraic stacks over S follows formally from the fact that the 2-category of algebraic stacks over S is a full sub 2-category of the 2-category of stacks in groupoids over $(Sch/S)_{fppf}$. $\square$

15. Algebraic stacks, overhauled

Some basic results on algebraic stacks.

Lemma 15.1. *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. Let $V \in \text{Ob}((Sch/S)_{fppf})$. Let $y : (Sch/V)_{fppf} \to \mathcal{Y}$ be surjective and smooth. Then there exists an object $U \in \text{Ob}((Sch/S)_{fppf})$ and a 2-commutative diagram*

$$
\begin{array}{ccc}
(Sch/U)_{fppf} & \xrightarrow{\ a\ } & (Sch/V)_{fppf} \\
{\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\
\mathcal{X} & \xrightarrow{\ f\ } & \mathcal{Y}
\end{array}
$$

with x surjective and smooth.

Proof. First choose $W \in \text{Ob}((Sch/S)_{fppf})$ and a surjective smooth 1-morphism $z : (Sch/W)_{fppf} \to \mathcal{X}$. As $\mathcal{Y}$ is an algebraic stack we may choose an equivalence

$$j : \mathcal{S}_F \longrightarrow (Sch/W)_{fppf} \times_{f \circ z, \mathcal{Y}, y} (Sch/V)_{fppf}$$

where F is an algebraic space. By Lemma 10.6 the morphism $\mathcal{S}_F \to (Sch/W)_{fppf}$ is surjective and smooth as a base change of y. Hence by Lemma 10.5 we see that $\mathcal{S}_F \to \mathcal{X}$ is surjective and smooth. Choose an object $U \in \text{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to F$. Then applying Lemma 10.5 once more we obtain the desired properties. $\square$

This lemma is a generalization of Proposition 13.3.

Lemma 15.2. *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. The following are equivalent:*

(1) *for $U \in \mathrm{Ob}((Sch/S)_{fppf})$ the functor $f : \mathcal{X}_U \to \mathcal{Y}_U$ is faithful,*
(2) *the functor f is faithful, and*
(3) *f is representable by algebraic spaces.*

Proof. Parts (1) and (2) are equivalent by general properties of 1-morphisms of categories fibred in groupoids, see Categories, Lemma 35.9. We see that (3) implies (2) by Lemma 9.2. Finally, assume (2). Let U be a scheme. Let $y \in \mathrm{Ob}(\mathcal{Y}_U)$. We have to prove that

$$\mathcal{W} = (Sch/U)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}$$

is representable by an algebraic space over U. Since $(Sch/U)_{fppf}$ is an algebraic stack we see from Lemma 14.3 that $\mathcal{W}$ is an algebraic stack. On the other hand the explicit description of objects of $\mathcal{W}$ as triples $(V, x, \alpha : y(V) \to f(x))$ and the fact that f is faithful, shows that the fibre categories of $\mathcal{W}$ are setoids. Hence Proposition 13.3 guarantees that $\mathcal{W}$ is representable by an algebraic space. $\square$

Lemma 15.3. *Let S be a scheme contained in Sch_{fppf}. Let $u : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

(1) *$\mathcal{U}$ is representable by an algebraic space, and*
(2) *u is representable by algebraic spaces, surjective and smooth,*

then $\mathcal{X}$ is an algebraic stack over S.

Proof. We have to show that $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces, see Definition 12.1. Given two schemes T_1, T_2 over S denote $\mathcal{T}_i = (Sch/T_i)_{fppf}$ the associated representable fibre categories. Suppose given 1-morphisms $f_i : \mathcal{T}_i \to \mathcal{X}$. According to Lemma 10.11 it suffices to prove that the 2-fibered product $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is representable by an algebraic space. By Stacks, Lemma 6.8 this is in any case a stack in setoids. Thus $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ corresponds to some sheaf F on $(Sch/S)_{fppf}$, see Stacks, Lemma 6.3. Let U be the algebraic space which represents $\mathcal{U}$. By assumption

$$\mathcal{T}_i' = \mathcal{U} \times_{u, \mathcal{X}, f_i} \mathcal{T}_i$$

is representable by an algebraic space T_i' over S. Hence $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2'$ is representable by the algebraic space $T_1' \times_U T_2'$. Consider the commutative diagram

In this diagram the bottom square, the right square, the back square, and the front square are 2-fibre products. A formal argument then shows that $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \to \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is the "base change" of $\mathcal{U} \to \mathcal{X}$, more precisely the diagram

$$
\begin{array}{ccc}
\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' & \longrightarrow & \mathcal{U} \\
\downarrow & & \downarrow \\
\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 & \longrightarrow & \mathcal{X}
\end{array}
$$

is a 2-fibre square. Hence $T_1' \times_U T_2' \to F$ is representable by algebraic spaces, smooth, and surjective, see Lemmas 9.6, 9.7, 10.4, and 10.6. Therefore F is an algebraic space by Bootstrap, Theorem 10.1 and we win. $\square$

An application of Lemma 15.3 is that something which is an algebraic space over an algebraic stack is an algebraic stack. This is the analogue of Bootstrap, Lemma 3.6. Actually, it suffices to assume the morphism $\mathcal{X} \to \mathcal{Y}$ is "algebraic", as we will see in Criteria for Representability, Lemma 8.2.

Lemma 15.4. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that*

(1) *$\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, and*
(2) *$\mathcal{Y}$ is an algebraic stack over S.*

Then $\mathcal{X}$ is an algebraic stack over S.

Proof. Let $\mathcal{V} \to \mathcal{Y}$ be a surjective smooth 1-morphism from a representable stack in groupoids to $\mathcal{Y}$. This exists by Definition 12.1. Then the 2-fibre product $\mathcal{U} = \mathcal{V} \times_{\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space by Lemma 9.8. The 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, smooth, and surjective, see Lemmas 9.7 and 10.6. By Lemma 15.3 we conclude that $\mathcal{X}$ is an algebraic stack. $\square$

Lemma 15.5. *Let S be a scheme contained in Sch_{fppf}. Let $j : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume j is representable by algebraic spaces. Then, if $\mathcal{Y}$ is a stack in groupoids (resp. an algebraic stack), so is $\mathcal{X}$.*

Proof. The statement on algebraic stacks will follow from the statement on stacks in groupoids by Lemma 15.4. If j is representable by algebraic spaces, then j is faithful on fibre categories and for each U and each $y \in \mathrm{Ob}(\mathcal{Y}_U)$ the presheaf

$$(h : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathrm{Ob}(\mathcal{X}_V), \phi : h^*y \to f(x)\}/\cong$$

is an algebraic space over U. See Lemma 9.2. In particular this presheaf is a sheaf and the conclusion follows from Stacks, Lemma 6.11. $\square$

16. From an algebraic stack to a presentation

Given an algebraic stack over S we obtain a groupoid in algebraic spaces over S whose associated quotient stack is the algebraic stack.

Recall that if (U, R, s, t, c) is a groupoid in algebraic spaces over S then $[U/R]$ denotes the quotient stack associated to this datum, see Groupoids in Spaces, Definition 20.1. In general $[U/R]$ is **not** an algebraic stack. In particular the stack $[U/R]$ occurring in the following lemma is in general not algebraic.

Lemma 16.1. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{U}$ be an algebraic stack over S which is representable by an algebraic space. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism. Then*

(1) *the 2-fibre product $\mathcal{R} = \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U}$ is representable by an algebraic space,*

(2) *there is a canonical equivalence*

$$\mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} = \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R},$$

(3) *the projection pr_{02} induces via (2) a 1-morphism*

$$pr_{02} : \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R} \longrightarrow \mathcal{R}$$

(4) *let U, R be the algebraic spaces representing $\mathcal{U}, \mathcal{R}$ and $t, s : R \to U$ and $c : R \times_{s,U,t} R \to R$ are the morphisms corresponding to the 1-morphisms $pr_0, pr_1 : \mathcal{R} \to \mathcal{U}$ and $pr_{02} : \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R} \to \mathcal{R}$ above, then the quintuple (U, R, s, t, c) is a groupoid in algebraic spaces over S,*

(5) *the morphism f induces a canonical 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ of stacks in groupoids over $(Sch/S)_{fppf}$, and*

(6) *the 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ is fully faithful.*

Proof. Proof of (1). By definition $\Delta_{\mathcal{X}}$ is representable by algebraic spaces so Lemma 10.11 applies to show that $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces. Hence the result follows from Lemma 9.8.

Let T be a scheme over S. By construction of the 2-fibre product (see Categories, Lemma 32.3) we see that the objects of the fibre category $\mathcal{R}_T$ are triples (a, b, α) where $a, b \in \mathrm{Ob}(\mathcal{U}_T)$ and $\alpha : f(a) \to f(b)$ is a morphism in the fibre category $\mathcal{X}_T$.

Proof of (2). The equivalence comes from repeatedly applying Categories, Lemmas 31.8 and 31.10. Let us identify $\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$ with $(\mathcal{U} \times_{\mathcal{X}} \mathcal{U}) \times_{\mathcal{X}} \mathcal{U}$. If T is a scheme over S, then on fibre categories over T this equivalence maps the object $((a, b, \alpha), c, \beta)$ on the left hand side to the object $((a, b, \alpha), (b, c, \beta))$ of the right hand side.

Proof of (3). The 1-morphism pr_{02} is constructed in the proof of Categories, Lemma 31.9. In terms of the description of objects of the fibre category above we see that $((a, b, \alpha), (b, c, \beta))$ maps to $(a, c, \beta \circ \alpha)$.

Unfortunately, this is *not compatible* with our conventions on groupoids where we always have $j = (t, s) : R \to U$, and we "think" of a T-valued point r of R as a morphism $r : s(r) \to t(r)$. However, this does not affect the proof of (4), since the opposite of a groupoid is a groupoid. But in the proof of (5) it is responsible for the inverses in the displayed formula below.

Proof of (4). Recall that the sheaf U is isomorphic to the sheaf $T \mapsto \mathrm{Ob}(\mathcal{U}_T)/\cong$, and similarly for R, see Lemma 8.2. It follows from Categories, Lemma 39.8 that this description is compatible with 2-fibre products so we get a similar matching of $\mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R}$ and $R \times_{s,U,t} R$. The morphisms $t, s : R \to U$ and $c : R \times_{s,U,t} R \to R$ we get from the general equality (8.2.1). Explicitly these maps are the transformations of functors that come from letting pr_0, pr_0, pr_{02} act on isomorphism classes of objects of fibre categories. Hence to show that we obtain a groupoid in algebraic spaces it suffices to show that for every scheme T over S the structure

$$(\mathrm{Ob}(\mathcal{U}_T)/\cong, \mathrm{Ob}(\mathcal{R}_T)/\cong, pr_1, pr_0, pr_{02})$$

is a groupoid which is clear from our description of objects of $\mathcal{R}_T$ above.

Proof of (5). We will eventually apply Groupoids in Spaces, Lemma 23.2 to obtain the functor $[U/R] \to \mathcal{X}$. Consider the 1-morphism $f : \mathcal{U} \to \mathcal{X}$. We have a 2-arrow $\tau : f \circ \mathrm{pr}_1 \to f \circ \mathrm{pr}_0$ by definition of $\mathcal{R}$ as the 2-fibre product. Namely, on an object (a, b, α) of $\mathcal{R}$ over T it is the map $\alpha^{-1} : b \to a$. We claim that

$$\tau \circ \mathrm{id}_{\mathrm{pr}_{02}} = (\tau \star \mathrm{id}_{\mathrm{pr}_0}) \circ (\tau \star \mathrm{id}_{\mathrm{pr}_1}).$$

This identity says that given an object $((a, b, \alpha), (b, c, \beta))$ of $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R}$ over T, then the composition of

$$c \xrightarrow{\beta^{-1}} b \xrightarrow{\alpha^{-1}} a$$

is the same as the arrow $(\beta \circ \alpha)^{-1} : a \to c$. This is clearly true, hence the claim holds. In this way we see that all the assumption of Groupoids in Spaces, Lemma 23.2 are satisfied for the structure $(\mathcal{U}, \mathcal{R}, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02})$ and the 1-morphism f and the 2-morphism τ. Except, to apply the lemma we need to prove this holds for the structure $(\mathcal{S}_U, \mathcal{S}_R, s, t, c)$ with suitable morphisms.

Now there should be some general abstract nonsense argument which transfer these data between the two, but it seems to be quite long. Instead, we use the following trick. Pick a quasi-inverse $j^{-1} : \mathcal{S}_U \to \mathcal{U}$ of the canonical equivalence $j : \mathcal{U} \to \mathcal{S}_U$ which comes from $U(T) = \mathrm{Ob}(\mathcal{U}_T)/\cong$. This just means that for every scheme T/S and every object $a \in \mathcal{U}_T$ we have picked out a particular element of its isomorphism class, namely $j^{-1}(j(a))$. Using j^{-1} we may therefore see $\mathcal{S}_U$ as a subcategory of $\mathcal{U}$. Having chosen this subcategory we can consider those objects (a, b, α) of $\mathcal{R}_T$ such that a, b are objects of $(\mathcal{S}_U)_T$, i.e., such that $j^{-1}(j(a)) = a$ and $j^{-1}(j(b)) = b$. Then it is clear that this forms a subcategory of $\mathcal{R}$ which maps isomorphically to $\mathcal{S}_R$ via the canonical equivalence $\mathcal{R} \to \mathcal{S}_R$. Moreover, this is clearly compatible with forming the 2-fibre product $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R}$. Hence we see that we may simply restrict f to $\mathcal{S}_U$ and restrict τ to a transformation between functors $\mathcal{S}_R \to \mathcal{X}$. Hence it is clear that the displayed equality of Groupoids in Spaces, Lemma 23.2 holds since it holds even as an equality of transformations of functors $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R} \to \mathcal{X}$ before restricting to the subcategory $\mathcal{S}_{R \times_{s, U, t} R}$.

This proves that Groupoids in Spaces, Lemma 23.2 applies and we get our desired morphism of stacks $f_{can} : [U/R] \to \mathcal{X}$. We briefly spell out how f_{can} is defined in this special case. On an object a of $\mathcal{S}_U$ over T we have $f_{can}(a) = f(a)$, where we think of $\mathcal{S}_U \subset \mathcal{U}$ by the chosen embedding above. If a, b are objects of $\mathcal{S}_U$ over T, then a morphism $\varphi : a \to b$ in $[U/R]$ is by definition an object of the form $\varphi = (b, a, \alpha)$ of $\mathcal{R}$ over T. (Note switch.) And the rule in the proof of Groupoids in Spaces, Lemma 23.2 is that

(16.1.1) $$f_{can}(\varphi) = \left(f(a) \xrightarrow{\alpha^{-1}} f(b) \right).$$

Proof of (6). Both $[U/R]$ and $\mathcal{X}$ are stacks. Hence given a scheme T/S and objects a, b of $[U/R]$ over T we obtain a transformation of fppf sheaves

$$\mathit{Isom}(a, b) \longrightarrow \mathit{Isom}(f_{can}(a), f_{can}(b))$$

on $(Sch/T)_{fppf}$. We have to show that this is an isomorphism. We may work fppf locally on T, hence we may assume that a, b come from morphisms $a, b : T \to U$. By the embedding $\mathcal{S}_U \subset \mathcal{U}$ above we may also think of a, b as objects of $\mathcal{U}$ over

T. In Groupoids in Spaces, Lemma 22.1 we have seen that the left hand sheaf is represented by the algebraic space

$$R \times_{(t,s), U \times_S U, (b,a)} T$$

over T. On the other hand, the right hand side is by Stacks, Lemma 2.5 equal to the sheaf associated to the following stack in setoids:

$$\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}, (f \circ b, f \circ a)} T = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}, (f,f)} (\mathcal{U} \times \mathcal{U}) \times_{\mathcal{U} \times \mathcal{U}, (b,a)} T = \mathcal{R} \times_{(\mathrm{pr}_0, \mathrm{pr}_1), \mathcal{U} \times \mathcal{U}, (b,a)} T$$

which is representable by the fibre product displayed above. At this point we have shown that the two *Isom*-sheaves are isomorphic. Our 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ induces this isomorphism on *Isom*-sheaves by Equation (16.1.1). $\square$

We can use the previous very abstract lemma to produce presentations.

Lemma 16.2. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. Let U be an algebraic space over S. Let $f : \mathcal{S}_U \to \mathcal{X}$ be a surjective smooth morphism. Let (U, R, s, t, c) be the groupoid in algebraic spaces and $f_{can} : [U/R] \to \mathcal{X}$ be the result of applying Lemma 16.1 to U and f. Then*

(1) *the morphisms s, t are smooth, and*
(2) *the 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ is an equivalence.*

Proof. The morphisms s, t are smooth by Lemmas 10.2 and 10.3. As the 1-morphism f is smooth and surjective it is clear that given any scheme T and any object $a \in \mathrm{Ob}(\mathcal{X}_T)$ there exists a smooth and surjective morphism $T' \to T$ such that $a|'_T$ comes from an object of $[U/R]_{T'}$. Since $f_{can} : [U/R] \to \mathcal{X}$ is fully faithful, we deduce that $[U/R] \to \mathcal{X}$ is essentially surjective as descent data on objects are effective on both sides, see Stacks, Lemma 4.8. $\square$

Remark 16.3. If the morphism $f : \mathcal{S}_U \to \mathcal{X}$ of Lemma 16.2 is only assumed surjective, flat and locally of finite presentation, then it will still be the case that $f_{can} : [U/R] \to \mathcal{X}$ is an equivalence. In this case the morphisms s, t will be flat and locally of finite presentation, but of course not smooth in general.

Lemma 16.2 suggests the following definitions.

Definition 16.4. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. We say (U, R, s, t, c) is a *smooth groupoid*[3] if $s, t : R \to U$ are smooth morphisms of algebraic spaces.

Definition 16.5. Let $\mathcal{X}$ be an algebraic stack over S. A *presentation* of $\mathcal{X}$ is given by a smooth groupoid (U, R, s, t, c) in algebraic spaces over S, and an equivalence $f : [U/R] \to \mathcal{X}$.

We have seen above that every algebraic stack has a presentation. Our next task is to show that every smooth groupoid in algebraic spaces over S gives rise to an algebraic stack.

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[3]This terminology might be a bit confusing: it does not imply that $[U/R]$ is smooth over anything.

17. The algebraic stack associated to a smooth groupoid

In this section we start with a smooth groupoid in algebraic spaces and we show that the associated quotient stack is an algebraic stack.

Lemma 17.1. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Then the diagonal of $[U/R]$ is representable by algebraic spaces.*

Proof. It suffices to show that the *Isom*-sheaves are algebraic spaces, see Lemma 10.11. This follows from Bootstrap, Lemma 11.5. $\square$

Lemma 17.2. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces over S. Then the morphism $\mathcal{S}_U \to [U/R]$ is smooth and surjective.*

Proof. Let T be a scheme and let $x : (Sch/T)_{fppf} \to [U/R]$ be a 1-morphism. We have to show that the projection

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} \longrightarrow (Sch/T)_{fppf}$$

is surjective and smooth. We already know that the left hand side is representable by an algebraic space F, see Lemmas 17.1 and 10.11. Hence we have to show the corresponding morphism $F \to T$ of algebraic spaces is surjective and smooth. Since we are working with properties of morphisms of algebraic spaces which are local on the target in the fppf topology we may check this fppf locally on T. By construction, there exists an fppf covering $\{T_i \to T\}$ of T such that $x|_{(Sch/T_i)_{fppf}}$ comes from a morphism $x_i : T_i \to U$. (Note that $F \times_T T_i$ represents the 2-fibre product $\mathcal{S}_U \times_{[U/R]} (Sch/T_i)_{fppf}$ so everything is compatible with the base change via $T_i \to T$.) Hence we may assume that x comes from $x : T \to U$. In this case we see that

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} = (\mathcal{S}_U \times_{[U/R]} \mathcal{S}_U) \times_{\mathcal{S}_U} (Sch/T)_{fppf} = \mathcal{S}_R \times_{\mathcal{S}_U} (Sch/T)_{fppf}$$

The first equality by Categories, Lemma 31.10 and the second equality by Groupoids in Spaces, Lemma 22.2. Clearly the last 2-fibre product is represented by the algebraic space $F = R \times_{s,U,x} T$ and the projection $R \times_{s,U,x} T \to T$ is smooth as the base change of the smooth morphism of algebraic spaces $s : R \to U$. It is also surjective as s has a section (namely the identity $e : U \to R$ of the groupoid). This proves the lemma. $\square$

Here is the main result of this section.

Theorem 17.3. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces over S. Then the quotient stack $[U/R]$ is an algebraic stack over S.*

Proof. We check the three conditions of Definition 12.1. By construction we have that $[U/R]$ is a stack in groupoids which is the first condition.

The second condition follows from the stronger Lemma 17.1.

Finally, we have to show there exists a scheme W over S and a surjective smooth 1-morphism $(Sch/W)_{fppf} \longrightarrow \mathcal{X}$. First choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $W \to U$. Note that this gives a surjective étale morphism $\mathcal{S}_W \to \mathcal{S}_U$ of categories fibred in sets, see Lemma 10.3. Of course then $\mathcal{S}_W \to \mathcal{S}_U$

is also surjective and smooth, see Lemma 10.9. Hence $\mathcal{S}_W \to \mathcal{S}_U \to [U/R]$ is surjective and smooth by a combination of Lemmas 17.2 and 10.5. $\qquad\square$

18. Change of big site

In this section we briefly discuss what happens when we change big sites. The upshot is that we can always enlarge the big site at will, hence we may assume any set of schemes we want to consider is contained in the big fppf site over which we consider our algebraic space. We encourage the reader to skip this section.

Pullbacks of stacks is defined in Stacks, Section 12.

Lemma 18.1. *Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 12. Let S be an object of Sch_{fppf}. Let $f : (Sch'/S)_{fppf} \to (Sch/S)_{fppf}$ the morphism of sites corresponding to the inclusion functor $u : (Sch/S)_{fppf} \to (Sch'/S)_{fppf}$. Let $\mathcal{X}$ be a stack in groupoids over $(Sch/S)_{fppf}$.*

    (1) *if $\mathcal{X}$ is representable by some $X \in \mathrm{Ob}((Sch/S)_{fppf})$, then $f^{-1}\mathcal{X}$ is representable too, in fact it is representable by the same scheme X, now viewed as an object of $(Sch'/S)_{fppf}$,*

    (2) *if $\mathcal{X}$ is representable by $F \in Sh((Sch/S)_{fppf})$ which is an algebraic space, then $f^{-1}\mathcal{X}$ is representable by the algebraic space $f^{-1}F$,*

    (3) *if $\mathcal{X}$ is an algebraic stack, then $f^{-1}\mathcal{X}$ is an algebraic stack, and*

    (4) *if $\mathcal{X}$ is a Deligne-Mumford stack, then $f^{-1}\mathcal{X}$ is a Deligne-Mumford stack too.*

Proof. Let us prove (3). By Lemma 16.2 we may write $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces (U, R, s, t, c). By Groupoids in Spaces, Lemma 28.1 we see that $f^{-1}[U/R] = [f^{-1}U/f^{-1}R]$. Of course $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a smooth groupoid in algebraic spaces too. Hence (3) is proved.

Now the other cases (1), (2), (4) each mean that $\mathcal{X}$ has a presentation $[U/R]$ of a particular kind, and hence translate into the same kind of presentation for $f^{-1}\mathcal{X} = [f^{-1}U/f^{-1}R]$. Whence the lemma is proved. $\qquad\square$

It is not true (in general) that the restriction of an algebraic space over the bigger site is an algebraic space over the smaller site (simply by reasons of cardinality). Hence we can only ever use a simple lemma of this kind to enlarge the base category and never to shrink it.

Lemma 18.2. *Suppose Sch_{fppf} is contained in Sch'_{fppf}. Let S be an object of Sch_{fppf}. Denote Algebraic-Stacks$/S$ the 2-category of algebraic stacks over S defined using Sch_{fppf}. Similarly, denote Algebraic-Stacks$'/S$ the 2-category of algebraic stacks over S defined using Sch'_{fppf}. The rule $\mathcal{X} \mapsto f^{-1}\mathcal{X}$ of Lemma 18.1 defines a functor of 2-categories*

$$Algebraic\text{-}Stacks/S \longrightarrow Algebraic\text{-}Stacks'/S$$

which defines equivalences of morphism categories

$$\mathrm{Mor}_{Algebraic\text{-}Stacks/S}(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathrm{Mor}_{Algebraic\text{-}Stacks'/S}(f^{-1}\mathcal{X}, f^{-1}\mathcal{Y})$$

for every objects $\mathcal{X}, \mathcal{Y}$ of Algebraic-Stacks$/S$. An object $\mathcal{X}'$ of Algebraic-Stacks$'/S$ is equivalence to $f^{-1}\mathcal{X}$ for some $\mathcal{X}$ in Algebraic-Stacks$/S$ if and only if it has a

presentation $\mathcal{X} = [U'/R']$ with U', R' isomorphic to $f^{-1}U$, $f^{-1}R$ for some $U, R \in$ Spaces/S.

Proof. The statement on morphism categories is a consequence of the more general Stacks, Lemma 12.12. The characterization of the "essential image" follows from the description of f^{-1} in the proof of Lemma 18.1. $\square$

19. Change of base scheme

In this section we briefly discuss what happens when we change base schemes. The upshot is that given a morphism $S \to S'$ of base schemes, any algebraic stack over S can be viewed as an algebraic stack over S'.

Lemma 19.1. *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. The constructions A and B of Stacks, Section 13 above give isomorphisms of 2-categories*

$$\left\{ \begin{array}{c} \text{2-category of algebraic} \\ \text{stacks } \mathcal{X} \text{ over } S \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{2-category of pairs } (\mathcal{X}', f) \text{ consisting of an} \\ \text{algebraic stack } \mathcal{X}' \text{ over } S' \text{ and a morphism} \\ f : \mathcal{X}' \to (Sch/S)_{fppf} \text{ of algebraic stacks over } S' \end{array} \right\}$$

Proof. The statement makes sense as the functor $j : (Sch/S)_{fppf} \to (Sch/S')_{fppf}$ is the localization functor associated to the object S/S' of $(Sch/S')_{fppf}$. By Stacks, Lemma 13.2 the only thing to show is that the constructions A and B preserve the subcategories of algebraic stacks. For example, if $\mathcal{X} = [U/R]$ then construction A applied to $\mathcal{X}$ just produces $\mathcal{X}' = \mathcal{X}$. Conversely, if $\mathcal{X}' = [U'/R']$ the morphism p induces morphisms of algebraic spaces $U' \to S$ and $R' \to S$, and then $\mathcal{X} = [U'/R']$ but now viewed as a stack over S. Hence the lemma is clear. $\square$

Definition 19.2. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. If $p : \mathcal{X} \to (Sch/S)_{fppf}$ is an algebraic stack over S, then $\mathcal{X}$ *viewed as an algebraic stack over S'* is the algebraic stack

$$\mathcal{X} \longrightarrow (Sch/S')_{fppf}$$

gotten by applying construction A of Lemma 19.1 to $\mathcal{X}$.

Conversely, what if we start with an algebraic stack $\mathcal{X}'$ over S' and we want to get an algebraic stack over S? Well, then we consider the 2-fibre product

$$\mathcal{X}'_S = (Sch/S)_{fppf} \times_{(Sch/S')_{fppf}} \mathcal{X}'$$

which is an algebraic stack over S' according to Lemma 14.3. Moreover, it comes equipped with a natural 1-morphism $p : \mathcal{X}'_S \to (Sch/S)_{fppf}$ and hence by Lemma 19.1 it corresponds in a canonical way to an algebraic stack over S.

Definition 19.3. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let $\mathcal{X}'$ be an algebraic stack over S'. The *change of base of $\mathcal{X}'$* is the algebraic stack $\mathcal{X}'_S$ over S described above.

20. Other chapters

References

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