SHEAVES OF MODULES

Contents

1. Introduction

In this chapter we work out basic notions of sheaves of modules. This in particular includes the case of abelian sheaves, since these may be viewed as sheaves of $\underline{\mathbf{Z}}$-modules. Basic references are [Ser55], [DG67] and [AGV71].

We work out what happens for sheaves of modules on ringed topoi in another chapter (see Modules on Sites, Section 1), although there we will mostly just duplicate the discussion from this chapter.

2. Pathology

A ringed space is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}$. We allow $\mathcal{O} = 0$ in the definition. In this case the category of modules has a single object (namely 0). It is still an abelian category etc, but it is a little degenerate. Similarly the sheaf $\mathcal{O}$ may be zero over open subsets of X, etc.

This doesn't happen when considering locally ringed spaces (as we will do later).

3. The abelian category of sheaves of modules

Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Definition 25.1. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules, see Sheaves, Definition 10.1. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}_X$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the map which on each open $U \subset X$ is the sum of the maps induced by φ, ψ. This is clearly again a map of sheaves of $\mathcal{O}_X$-modules. It is also clear that composition of maps of $\mathcal{O}_X$-modules is bilinear with respect to this addition. Thus $\textit{Mod}(\mathcal{O}_X)$ is a pre-additive category, see Homology, Definition 3.1.

We will denote 0 the sheaf of $\mathcal{O}_X$-modules which has constant value $\{0\}$ for all open $U \subset X$. Clearly this is both a final and an initial object of $\textit{Mod}(\mathcal{O}_X)$. Given a morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each open U, and (d) $\varphi_x = 0$ for all $x \in X$. See Sheaves, Lemma 16.1.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition 3.5. Thus $\textit{Mod}(\mathcal{O}_X)$ is an additive category, see Homology, Definition 3.8.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. We may define $\text{Ker}(\varphi)$ to be the subsheaf of $\mathcal{F}$ with sections

$$\text{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all open $U \subset X$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}_X$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\text{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$. Moreover, on the level of stalks we have $\text{Ker}(\varphi)_x = \text{Ker}(\varphi_x)$.

On the other hand, we define $\text{Coker}(\varphi)$ as the sheaf of $\mathcal{O}_X$-modules associated to the presheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \text{Coker}(\mathcal{G}(U) \to \mathcal{F}(U)) = \mathcal{F}(U)/\varphi(\mathcal{G}(U)).$$

Since taking stalks commutes with taking sheafification, see Sheaves, Lemma 17.2 we see that $\text{Coker}(\varphi)_x = \text{Coker}(\varphi_x)$. Thus the map $\mathcal{G} \to \text{Coker}(\varphi)$ is surjective (as a map of sheaves of sets), see Sheaves, Section 16. To show that this is a cokernel, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}_X$-modules such that $\beta \circ \varphi$ is zero, then you get for every open $U \subset X$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma 20.1) we obtain a canonical map $\text{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \text{Coker}(\varphi) \to \mathcal{H}$. The morphism $\text{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

Lemma 3.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is an abelian category. Moreover a complex*
$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$
is exact at $\mathcal{G}$ if and only if for all $x \in X$ the complex
$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$
is exact at $\mathcal{G}_x$.

Proof. By Homology, Definition 5.1 we have to show that image and coimage agree. By Sheaves, Lemma 16.1 it is enough to show that image and coimage have the same stalk at every $x \in X$. By the constructions of kernels and cokernels above these stalks are the coimage and image in the categories of $\mathcal{O}_{X,x}$-modules. Thus we get the result from the fact that the category of modules over a ring is abelian. □

Actually the category $Mod(\mathcal{O}_X)$ has many more properties. Here are two constructions we can do.

(1) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the product
$$\prod\nolimits_{i \in I} \mathcal{F}_i$$
which is the sheaf that associates to each open U the product of the modules $\mathcal{F}_i(U)$. This is also the categorical product, as in Categories, Definition 14.6.

(2) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the direct sum
$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i$$
which is the *sheafification* of the presheaf that associates to each open U the direct sum of the modules $\mathcal{F}_i(U)$. This is also the categorical coproduct, as in Categories, Definition 14.7. To see this you use the universal property of sheafification.

Using these we conclude that all limits and colimits exist in $Mod(\mathcal{O}_X)$.

Lemma 3.2. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *All limits exist in $Mod(\mathcal{O}_X)$. Limits are the same as the corresponding limits of presheaves of $\mathcal{O}_X$-modules (i.e., commute with taking sections over opens).*
(2) *All colimits exist in $Mod(\mathcal{O}_X)$. Colimits are the sheafification of the corresponding colimit in the category of presheaves. Taking colimits commutes with taking stalks.*
(3) *Filtered colimits are exact.*
(4) *Finite direct sums are the same as the corresponding finite direct sums of presheaves of $\mathcal{O}_X$-modules.*

Proof. As $Mod(\mathcal{O}_X)$ is abelian (Lemma 3.1) it has all finite limits and colimits (Homology, Lemma 5.5). Thus the existence of limits and colimits and their description follows from the existence of products and coproducts and their description (see discussion above) and Categories, Lemmas 14.11 and 14.12. Since sheafification commutes with taking stalks we see that colimits commute with taking stalks.

Part (3) signifies that given a system $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ of exact sequences of $\mathcal{O}_X$-modules over a directed set I the sequence $0 \to \operatorname{colim} \mathcal{F}_i \to \operatorname{colim} \mathcal{G}_i \to \operatorname{colim} \mathcal{H}_i \to 0$ is exact as well. Since we can check exactness on stalks (Lemma 3.1) this follows from the case of modules which is Algebra, Lemma 8.8. We omit the proof of (4). ☐

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section 23. See Homology, Lemma 7.2 for a description of exactness properties in terms of short exact sequences.

Lemma 3.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.*

(1) *The functor $f_* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ is left exact. In fact it commutes with all limits.*
(2) *The functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is right exact. In fact it commutes with all colimits.*
(3) *Pullback $f^{-1} : Ab(Y) \to Ab(X)$ on abelian sheaves is exact.*

Proof. Parts (1) and (2) hold because (f^*, f_*) is an adjoint pair of functors, see Sheaves, Lemma 26.2 and Categories, Section 24. Part (3) holds because exactness can be checked on stalks (Lemma 3.1) and the description of stalks of the pullback, see Sheaves, Lemma 22.1. ☐

Lemma 3.4. *Let $j : U \to X$ be an open immersion of topological spaces. The functor $j_! : Ab(U) \to Ab(X)$ is exact.*

Proof. Follows from the description of stalks given in Sheaves, Lemma 31.6. ☐

Lemma 3.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let I be a set. For $i \in I$, let $\mathcal{F}_i$ be a sheaf of $\mathcal{O}_X$-modules. For $U \subset X$ quasi-compact open the map*

$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i(U) \longrightarrow \left(\bigoplus\nolimits_{i \in I} \mathcal{F}_i\right)(U)$$

is bijective.

Proof. If s is an element of the right hand side, then there exists an open covering $U = \bigcup_{j \in J} U_j$ such that $s|_{U_j}$ is a finite sum $\sum_{i \in I_j} s_{ji}$ with $s_{ji} \in \mathcal{F}_i(U_j)$. Because U is quasi-compact we may assume that the covering is finite, i.e., that J is finite. Then $I' = \bigcup_{j \in J} I_j$ is a finite subset of I. Clearly, s is a section of the subsheaf $\bigoplus_{i \in I'} \mathcal{F}_i$. The result follows from the fact that for a finite direct sum sheafification is not needed, see Lemma 3.2 above. ☐

4. Sections of sheaves of modules

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $s \in \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ be a global section. There is a unique *map of $\mathcal{O}_X$-modules*

$$\mathcal{O}_X \longrightarrow \mathcal{F},\ f \longmapsto fs$$

associated to s. The notation above signifies that a local section f of $\mathcal{O}_X$, i.e., a section f over some open U, is mapped to the multiplication of f with the restriction of s to U. Conversely, any map $\varphi : \mathcal{O}_X \to \mathcal{F}$ gives rise to a section $s = \varphi(1)$ such that φ is the morphism associated to s.

Definition 4.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *generated by global sections* if there exist a set I, and global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that the map

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F}$$

which is the map associated to s_i on the summand corresponding to i, is surjective. In this case we say that the sections s_i *generate* $\mathcal{F}$.

We often use the abuse of notation introduced in Sheaves, Section 11 where, given a local section s of $\mathcal{F}$ defined in an open neighbourhood of a point $x \in X$, we denote s_x, or even s the image of s in the stalk $\mathcal{F}_x$.

Lemma 4.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let I be a set. Let $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ be global sections. The sections s_i generate $\mathcal{F}$ if and only if for all $x \in X$ the elements $s_{i,x} \in \mathcal{F}_x$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.*

Proof. Omitted. $\square$

Lemma 4.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{G}$ are generated by global sections then so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*

Proof. Omitted. $\square$

Lemma 4.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let I be a set. Let s_i, $i \in I$ be a collection of local sections of $\mathcal{F}$, i.e., $s_i \in \mathcal{F}(U_i)$ for some opens $U_i \subset X$. There exists a unique smallest subsheaf of $\mathcal{O}_X$-modules $\mathcal{G}$ such that each s_i corresponds to a local section of $\mathcal{G}$.*

Proof. Consider the subpresheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \{\text{sums } \sum\nolimits_{i \in J} f_i(s_i|_U) \text{ where } J \text{ is finite, } U \subset U_i \text{ for } i \in J, \text{ and } f_i \in \mathcal{O}_X(U)\}$$

Let $\mathcal{G}$ be the sheafification of this subpresheaf. This is a subsheaf of $\mathcal{F}$ by Sheaves, Lemma 16.3. Since all the finite sums clearly have to be in $\mathcal{G}$ this is the smallest subsheaf as desired. $\square$

Definition 4.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set I, and local sections s_i, $i \in I$ of $\mathcal{F}$ we say that the subsheaf $\mathcal{G}$ of Lemma 4.4 above is the *subsheaf generated by the s_i*.

Lemma 4.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set I, and local sections s_i, $i \in I$ of $\mathcal{F}$. Let $\mathcal{G}$ be the subsheaf generated by the s_i and let $x \in X$. Then $\mathcal{G}_x$ is the $\mathcal{O}_{X,x}$-submodule of $\mathcal{F}_x$ generated by the elements $s_{i,x}$ for those i such that s_i is defined at x.*

Proof. This is clear from the construction of $\mathcal{G}$ in the proof of Lemma 4.4. $\square$

5. Supports of modules and sections

Definition 5.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) The *support of $\mathcal{F}$* is the set of points $x \in X$ such that $\mathcal{F}_x \neq 0$.
(2) We denote $\text{Supp}(\mathcal{F})$ the support of $\mathcal{F}$.
(3) Let $s \in \Gamma(X, \mathcal{F})$ be a global section. The *support of s* is the set of points $x \in X$ such that the image $s_x \in \mathcal{F}_x$ of s is not zero.

Of course the support of a local section is then defined also since a local section is a global section of the restriction of $\mathcal{F}$.

Lemma 5.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ open.*

1. *The support of $s \in \mathcal{F}(U)$ is closed in U.*
2. *The support of fs is contained in the intersections of the supports of $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$.*
3. *The support of $s + s'$ is contained in the union of the supports of $s, s' \in \mathcal{F}(U)$.*
4. *The support of $\mathcal{F}$ is the union of the supports of all local sections of $\mathcal{F}$.*
5. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then the support of $\varphi(s)$ is contained in the support of $s \in \mathcal{F}(U)$.*

Proof. This is true because if $s_x = 0$, then s is zero in an open neighbourhood of x by definition of stalks. Similarly for f. Details omitted. □

In general the support of a sheaf of modules is not closed. Namely, the sheaf could be an abelian sheaf on $\mathbf{R}$ (with the usual archimedean topology) which is the direct sum of infinitely many nonzero skyscraper sheaves each supported at a single point p_i of $\mathbf{R}$. Then the support would be the set of points p_i which may not be closed.

Another example is to consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 31 the support of this sheaf is exactly U.

Lemma 5.3. *Let X be a topological space. The support of a sheaf of rings is closed.*

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

6. Closed immersions and abelian sheaves

Recall that we think of an abelian sheaf on a topological space X as a sheaf of $\underline{\mathbf{Z}}_X$-modules. Thus we may apply any results, definitions for sheaves of modules to abelian sheaves.

Lemma 6.1. *Let X be a topological space. Let $Z \subset X$ be a closed subset. Denote $i : Z \to X$ the inclusion map. The functor*

$$i_* : Ab(Z) \longrightarrow Ab(X)$$

is exact, fully faithful, with essential image exactly those abelian sheaves whose support is contained in Z. The functor i^{-1} is a left inverse to i_*.

Proof. Exactness follows from the description of stalks in Sheaves, Lemma 32.1 and Lemma 3.1. The rest was shown in Sheaves, Lemma 32.3. □

Let $\mathcal{F}$ be an abelian sheaf on the topological space X. Given a closed subset Z, there is a canonical abelian subsheaf of $\mathcal{F}$ which consists of exactly those sections whose support is contained in Z. Here is the exact statement.

Remark 6.2. Let X be a topological space. Let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be an abelian sheaf on X. For $U \subset X$ open set

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{ the support of } s \text{ is contained in } Z \cap U\}$$

Then $\mathcal{H}_Z(\mathcal{F})$ is an abelian subsheaf of $\mathcal{F}$. It is the largest abelian subsheaf of $\mathcal{F}$ whose support is contained in Z. By Lemma 6.1 we may (and we do) view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on Z. In this way we obtain a left exact functor

$$\textit{Ab}(X) \longrightarrow \textit{Ab}(Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as abelian sheaf on } Z$$

All of the statements made above follow directly from Lemma 5.2.

This seems like a good opportunity to show that the functor i_* has a right adjoint on abelian sheaves.

Lemma 6.3. *Let $i : Z \to X$ be the inclusion of a closed subset into the topological space X. The functor $\textit{Ab}(X) \to \textit{Ab}(Z)$, $\mathcal{F} \mapsto \mathcal{H}_Z(\mathcal{F})$ of Remark 6.2 is a right adjoint to $i_* : \textit{Ab}(Z) \to \textit{Ab}(X)$. In particular i_* commutes with arbitrary colimits.*

Proof. We have to show that for any abelian sheaf $\mathcal{F}$ on X and any abelian sheaf $\mathcal{G}$ on Z we have

$$\mathrm{Hom}_{\textit{Ab}(X)}(i_*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\textit{Ab}(Z)}(\mathcal{G}, \mathcal{H}_Z(\mathcal{F}))$$

This is clear because after all any section of $i_*\mathcal{G}$ has support in Z. Details omitted. $\square$

Remark 6.4. In Sheaves, Remark 32.5 we showed that i_* as a functor on the categories of sheaves of sets does not have a right adjoint simply because it is not exact. However, it is very close to being true, in fact, the functor i_* is exact on sheaves of pointed sets, sections with support in Z can be defined for sheaves of pointed sets, and $\mathcal{H}_Z$ makes sense and is a right adjoint to i_*.

7. A canonical exact sequence

We give this exact sequence its own section.

Lemma 7.1. *Let X be a topological space. Let $U \subset X$ be an open subset with complement $Z \subset X$. Denote $j : U \to X$ the open immersion and $i : Z \to X$ the closed immersion. For any sheaf of abelian groups $\mathcal{F}$ on X the adjunction mappings $j_!j^{-1}\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to i_*i^{-1}\mathcal{F}$ give a short exact sequence*

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

of sheaves of abelian groups. For any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of abelian sheaves on X we obtain a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & j_!j^{-1}\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & i_*i^{-1}\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & j_!j^{-1}\mathcal{G} \arrow[r] & \mathcal{G} \arrow[r] & i_*i^{-1}\mathcal{G} \arrow[r] & 0
\end{tikzcd}$$

Proof. The functoriality of the short exact sequence is immediate from the naturality of the adjunction mappings. We may check exactness on stalks (Lemma 3.1). For a description of the stalks in question see Sheaves, Lemmas 31.6 and 32.1. $\square$

8. Modules locally generated by sections

Let $(X, \mathcal{O}_X)$ be a ringed space. In this and the following section we will often restrict sheaves to open subspaces $U \subset X$, see Sheaves, Section 31. In particular, we will often denote the open subspace by $(U, \mathcal{O}_U)$ instead of the more correct notation $(U, \mathcal{O}_X|_U)$, see Sheaves, Definition 31.2.

Consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 31 the stalk of $j_!\underline{\mathbf{Z}}_U$ at $x = 0$ is 0. In fact the sections of this sheaf over any open interval containing 0 are 0. Thus there is no open neighbourhood of the point 0 over which the sheaf can be generated by sections.

Definition 8.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *locally generated by sections* if for every $x \in X$ there exists an open neighbourhood U of x such that $\mathcal{F}|_U$ is globally generated as a sheaf of $\mathcal{O}_U$-modules.

In other words there exists a set I and for each i a section $s_i \in \mathcal{F}(U)$ such that the associated map

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U$$

is surjective.

Lemma 8.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ is locally generated by sections if $\mathcal{G}$ is locally generated by sections.*

Proof. Given an open subspace V of Y we may consider the commutative diagram of ringed spaces

$$\begin{tikzcd}
(f^{-1}V, \mathcal{O}_{f^{-1}V}) \arrow[r, "j'"'] \arrow[d, "f'"'] & (X, \mathcal{O}_X) \arrow[d, "f"] \\
(V, \mathcal{O}_V) \arrow[r, "j"] & (Y, \mathcal{O}_Y)
\end{tikzcd}$$

We know that $f^*\mathcal{G}|_{f^{-1}V} \cong (f')^*(\mathcal{G}|_V)$, see Sheaves, Lemma 26.3. Thus we may assume that $\mathcal{G}$ is globally generated.

We have seen that f^* commutes with all colimits, and is right exact, see Lemma 3.3. Thus if we have a surjection

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the surjection

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. $\square$

9. Modules of finite type

Definition 9.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of *finite type* if for every $x \in X$ there exists an open neighbourhood U such that $\mathcal{F}|_U$ is generated by finitely many sections.

Lemma 9.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a finite type $\mathcal{O}_Y$-module is a finite type $\mathcal{O}_X$-module.*

Proof. Arguing as in the proof of Lemma 8.2 we may assume $\mathcal{G}$ is globally generated by finitely many sections. We have seen that f^* commutes with all colimits, and is right exact, see Lemma 3.3. Thus if we have a surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. □

Lemma 9.3. *Let X be a ringed space. The image of a morphism of $\mathcal{O}_X$-modules of finite type is of finite type. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}_1$ and $\mathcal{F}_3$ are of finite type, so is $\mathcal{F}_2$.*

Proof. The statement on images is trivial. The statement on short exact sequences comes from the fact that sections of $\mathcal{F}_3$ locally lift to sections of $\mathcal{F}_2$ and the corresponding result in the category of modules over a ring (applied to the stalks for example). □

Lemma 9.4. *Let X be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{F}$ of finite type and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ surjective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is surjective.*

Proof. Choose an open neighbourhood $U \subset X$ of x such that $\mathcal{F}$ is generated by $s_1, \ldots, s_n \in \mathcal{F}(U)$ over U. By assumption of surjectivity of φ_x, after shrinking U we may assume that $s_i = \varphi(t_i)$ for some $t_i \in \mathcal{G}(U)$. Then U works. □

Lemma 9.5. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $x \in X$. Assume $\mathcal{F}$ of finite type and $\mathcal{F}_x = 0$. Then there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is zero.*

Proof. This is a special case of Lemma 9.4 applied to the morphism $0 \to \mathcal{F}$. □

Lemma 9.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type then support of $\mathcal{F}$ is closed.*

Proof. This is a reformulation of Lemma 9.5. □

Lemma 9.7. *Let X be a ringed space. Let I be a preordered set and let $(\mathcal{F}_i, f_{ii'})$ be a system over I consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 21). Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ be the colimit. Assume (a) I is directed, (b) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and (c) X is quasi-compact. Then there exists an i such that $\mathcal{F}_i \to \mathcal{F}$ is surjective. If the transition maps $f_{ii'}$ are injective then we conclude that $\mathcal{F} = \mathcal{F}_i$ for some $i \in I$.*

Proof. Let $x \in X$. There exists an open neighbourhood $U \subset X$ of x and finitely many sections $s_j \in \mathcal{F}(U)$, $j = 1, \ldots, m$ such that $s_1, \ldots, s_m$ generate $\mathcal{F}$ as $\mathcal{O}_U$-module. After possibly shrinking U to a smaller open neighbourhood of x we may assume that each s_j comes from a section of $\mathcal{F}_i$ for some $i \in I$. Hence, since X is quasi-compact we can find a finite open covering $X = \bigcup_{j=1,\ldots,m} U_j$, and for each j an index i_j and finitely many sections $s_{jl} \in \mathcal{F}_{i_j}(U_j)$ whose images generate the restriction of $\mathcal{F}$ to U_j. Clearly, the lemma holds for any index $i \in I$ which is $\geq$ all i_j. □

Lemma 9.8. *Let X be a ringed space. There exists a set of $\mathcal{O}_X$-modules $\{\mathcal{F}_i\}_{i\in I}$ of finite type such that each finite type $\mathcal{O}_X$-module on X is isomorphic to exactly one of the $\mathcal{F}_i$.*

Proof. For each open covering $\mathcal{U} : X = \bigcup U_j$ consider the sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ such that each restriction $\mathcal{F}|_{U_j}$ is a quotient of $\mathcal{O}_{U_j}^{\oplus r_j}$ for some $r_j \geq 0$. These are parametrized by subsheaves $\mathcal{K}_j \subset \mathcal{O}_{U_j}^{\oplus r_j}$ and glueing data

$$\varphi_{jj'} : \mathcal{O}_{U_j\cap U_{j'}}^{\oplus r_j}/(\mathcal{K}_j|_{U_j\cap U_{j'}}) \longrightarrow \mathcal{O}_{U_j\cap U_{j'}}^{\oplus r_{j'}}/(\mathcal{K}_{j'}|_{U_j\cap U_{j'}})$$

see Sheaves, Section 33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j\in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. Hence the collection of all sheaves of $\mathcal{O}_X$-modules gotten from glueing quotients as above forms a set $\mathcal{I}$. By definition every finite type $\mathcal{O}_X$-module is isomorphic to an element of $\mathcal{I}$. Choosing an element out of each isomorphism class inside $\mathcal{I}$ gives the desired set of sheaves (uses axiom of choice). □

10. Quasi-coherent modules

In this section we introduce an abstract notion of quasi-coherent $\mathcal{O}_X$-module. This notion is very useful in algebraic geometry, since quasi-coherent modules on a scheme have a good description on any affine open. However, we warn the reader that in the general setting of (locally) ringed spaces this notion is not well behaved at all. The category of quasi-coherent sheaves is not abelian in general, infinite direct sums of quasi-coherent sheaves aren't quasi-coherent, etc, etc.

Definition 10.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a *quasi-coherent sheaf of $\mathcal{O}_X$-modules* if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_U$$

The category of quasi-coherent $\mathcal{O}_X$-modules is denoted $QCoh(\mathcal{O}_X)$.

The definition means that X is covered by open sets U such that $\mathcal{F}|_U$ has a *presentation* of the form

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

(1) for every point x of X there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by global sections, and
(2) for a suitable choice of these sections the kernel of the associated surjection is also generated by global sections.

Lemma 10.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. The direct sum of two quasi-coherent $\mathcal{O}_X$-modules is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. Omitted. □

Remark 10.3. Warning: It is not true in general that an infinite direct sum of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent. For more esoteric behaviour of quasi-coherent modules see Example 10.9.

Lemma 10.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a quasi-coherent $\mathcal{O}_Y$-module is quasi-coherent.*

Proof. Arguing as in the proof of Lemma 8.2 we may assume $\mathcal{G}$ has a global presentation by direct sums of copies of $\mathcal{O}_Y$. We have seen that f^* commutes with all colimits, and is right exact, see Lemma 3.3. Thus if we have an exact sequence

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_Y \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_Y \longrightarrow \mathcal{G} \longrightarrow 0$$

then upon applying f^* we obtain the exact sequence

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_X \longrightarrow f^*\mathcal{G} \longrightarrow 0.$$

This implies the lemma. $\square$

This gives plenty of examples of quasi-coherent sheaves.

Lemma 10.5. *Let $(X, \mathcal{O}_X)$ be ringed space. Let $\alpha : R \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism from a ring R into the ring of global sections on X. Let M be an R-module. The following three constructions give canonically isomorphic sheaves of $\mathcal{O}_X$-modules:*

(1) *Let $\pi : (X, \mathcal{O}_X) \longrightarrow (\{*\}, R)$ be the morphism of ringed spaces with $\pi : X \to \{*\}$ the unique map and with π-map $\pi^\sharp$ the given map $\alpha : R \to \Gamma(X, \mathcal{O}_X)$. Set $\mathcal{F}_1 = \pi^*M$.*
(2) *Choose a presentation $\bigoplus_{j\in J} R \to \bigoplus_{i\in I} R \to M \to 0$. Set*

$$\mathcal{F}_2 = \mathrm{Coker}\left(\bigoplus\nolimits_{j\in J} \mathcal{O}_X \to \bigoplus\nolimits_{i\in I} \mathcal{O}_X\right).$$

Here the map on the component $\mathcal{O}_X$ corresponding to $j \in J$ given by the section $\sum_i \alpha(r_{ij})$ where the r_{ij} are the matrix coefficients of the map in the presentation of M.
(3) *Set $\mathcal{F}_3$ equal to the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U) \otimes_R M$, where the map $R \to \mathcal{O}_X(U)$ is the composition of α and the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$.*

This construction has the following properties:

(1) *The resulting sheaf of $\mathcal{O}_X$-modules $\mathcal{F}_M = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$ is quasi-coherent.*
(2) *The construction gives a functor from the category of R-modules to the category of quasi-coherent sheaves on X which commutes with arbitrary colimits.*
(3) *For any $x \in X$ we have $\mathcal{F}_{M,x} = \mathcal{O}_{X,x} \otimes_R M$ functorial in M.*
(4) *Given any $\mathcal{O}_X$-module $\mathcal{G}$ we have*

$$\mathrm{Mor}_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{G}) = \mathrm{Hom}_R(M, \Gamma(X, \mathcal{G}))$$

where the R-module structure on $\Gamma(X, \mathcal{G})$ comes from the $\Gamma(X, \mathcal{O}_X)$-module structure via α.

Proof. The isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_3$ comes from the fact that π^* is defined as the sheafification of the presheaf in (3), see Sheaves, Section 26. The isomorphism between the constructions in (2) and (1) comes from the fact that the functor π^* is right exact, so $\pi^*(\bigoplus_{j\in J} R) \to \pi^*(\bigoplus_{i\in I} R) \to \pi^*M \to 0$ is exact, π^* commutes with arbitrary direct sums, see Lemma 3.3, and finally the fact that $\pi^*(R) = \mathcal{O}_X$.

Assertion (1) is clear from construction (2). Assertion (2) is clear since π^* has these properties. Assertion (3) follows from the description of stalks of pullback sheaves, see Sheaves, Lemma 26.4. Assertion (4) follows from adjointness of π_* and π^*. $\square$

Definition 10.6. In the situation of Lemma 10.5 we say $\mathcal{F}_M$ is the *sheaf associated to the module M and the ring map α*. If $R = \Gamma(X, \mathcal{O}_X)$ and $\alpha = \text{id}_R$ we simply say $\mathcal{F}_M$ is the *sheaf associated to the module M*.

Lemma 10.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. Let M be an R-module. Let $\mathcal{F}_M$ be the quasi-coherent sheaf of $\mathcal{O}_X$-modules associated to M. If $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces, then $g^*\mathcal{F}_M$ is the sheaf associated to the $\Gamma(Y, \mathcal{O}_Y)$-module $\Gamma(Y, \mathcal{O}_Y) \otimes_R M$.*

Proof. The assertion follows from the first description of $\mathcal{F}_M$ in Lemma 10.5 as π^*M, and the following commutative diagram of ringed spaces

$$\begin{tikzcd} (Y, \mathcal{O}_Y) \arrow[r, "\pi"'] \arrow[d, "g"'] & (\{*\}, \Gamma(Y, \mathcal{O}_Y)) \arrow[d, "\text{induced by } g^\sharp"] \\ (X, \mathcal{O}_X) \arrow[r, "\pi"] & (\{*\}, \Gamma(X, \mathcal{O}_X)) \end{tikzcd}$$

(Also use Sheaves, Lemma 26.3.) $\square$

Lemma 10.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$ be a point. Assume that x has a fundamental system of quasi-compact neighbourhoods. Consider any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Then there exists an open neighbourhood U of x such that $\mathcal{F}|_U$ is isomorphic to the sheaf of modules $\mathcal{F}_M$ on $(U, \mathcal{O}_U)$ associated to some $\Gamma(U, \mathcal{O}_U)$-module M.*

Proof. First we may replace X by an open neighbourhood of x and assume that $\mathcal{F}$ is isomorphic to the cokernel of a map

$$\Psi : \bigoplus\nolimits_{j \in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_X.$$

The problem is that this map may not be given by a "matrix", because the module of global sections of a direct sum is in general different from the direct sum of the modules of global sections.

Let $x \in E \subset X$ be a quasi-compact neighbourhood of x (note: E may not be open). Let $x \in U \subset E$ be an open neighbourhood of x contained in E. Next, we proceed as in the proof of Lemma 3.5. For each $j \in J$ denote $s_j \in \Gamma(X, \bigoplus_{i \in I} \mathcal{O}_X)$ the image of the section 1 in the summand $\mathcal{O}_X$ corresponding to j. There exists a finite collection of opens U_{jk}, $k \in K_j$ such that $E \subset \bigcup_{k \in K_j} U_{jk}$ and such that each restriction $s_j|_{U_{jk}}$ is a finite sum $\sum_{i \in I_{jk}} f_{jki}$ with $I_{jk} \subset I$, and f_{jki} in the summand $\mathcal{O}_X$ corresponding to $i \in I$. Set $I_j = \bigcup_{k \in K_j} I_{jk}$. This is a finite set. Since $U \subset E \subset \bigcup_{k \in K_j} U_{jk}$ the section $s_j|_U$ is a section of the finite direct sum $\bigoplus_{i \in I_j} \mathcal{O}_X$. By Lemma 3.2 we see that actually $s_j|_U$ is a sum $\sum_{i \in I_j} f_{ij}$ and $f_{ij} \in \mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_U)$.

At this point we can define a module M as the cokernel of the map

$$\bigoplus\nolimits_{j \in J} \Gamma(U, \mathcal{O}_U) \longrightarrow \bigoplus\nolimits_{i \in I} \Gamma(U, \mathcal{O}_U)$$

with matrix given by the (f_{ij}). By construction (2) of Lemma 10.5 we see that $\mathcal{F}_M$ has the same presentation as $\mathcal{F}|_U$ and therefore $\mathcal{F}_M \cong \mathcal{F}|_U$. $\square$

Example 10.9. Let X be countably many copies $L_1, L_2, L_3, \ldots$ of the real line all glued together at 0; a fundamental system of neighbourhoods of 0 being the collection $\{U_n\}_{n \in \mathbf{N}}$, with $U_n \cap L_i = (-1/n, 1/n)$. Let $\mathcal{O}_X$ be the sheaf of continuous real valued functions. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function which is identically zero on $(-1, 1)$ and identically 1 on $(-\infty, -2) \cup (2, \infty)$. Denote f_n the continuous function on X which is equal to $x \mapsto f(nx)$ on each $L_j = \mathbf{R}$. Let 1_{L_j} be the characteristic function of L_j. We consider the map

$$\bigoplus\nolimits_{j \in \mathbf{N}} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{j,i \in \mathbf{N}} \mathcal{O}_X, \quad e_j \longmapsto \sum\nolimits_{i \in \mathbf{N}} f_j 1_{L_i} e_{ij}$$

with obvious notation. This makes sense because this sum is locally finite as f_j is zero in a neighbourhood of 0. Over U_n the image of e_j, for $j > 2n$ is not a finite linear combination $\sum g_{ij} e_{ij}$ with g_{ij} continuous. Thus there is no neighbourhood of $0 \in X$ such that the displayed map is given by a "matrix" as in the proof of Lemma 10.8 above.

Note that $\bigoplus_{j \in \mathbf{N}} \mathcal{O}_X$ is the sheaf associated to the free module with basis e_j and similarly for the other direct sum. Thus we see that a morphism of sheaves associated to modules in general even locally on X does not come from a morphism of modules. Similarly there should be an example of a ringed space X and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}$ is not locally of the form $\mathcal{F}_M$. (Please email if you find one.) Moreover, there should be examples of locally compact spaces X and maps $\mathcal{F}_M \to \mathcal{F}_N$ which also do not locally come from maps of modules (the proof of Lemma 10.8 shows this cannot happen if N is free).

11. Modules of finite presentation

Here is the definition.

Definition 11.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of *finite presentation* if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$, and $n, m \in \mathbf{N}$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U$$

This means that X is covered by open sets U such that $\mathcal{F}|_U$ has a *presentation* of the form

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U \to 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

(1) for every point x of X there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by finitely many global sections, and
(2) for a suitable choice of these sections the kernel of the associated surjection is also generated by finitely many global sections.

Lemma 11.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Any $\mathcal{O}_X$-module of finite presentation is quasi-coherent.*

Proof. Immediate from definitions. ☐

Lemma 11.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation.*

(1) *If $\psi : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ is a surjection, then* $\mathrm{Ker}(\psi)$ *is of finite type.*

(2) *If $\theta : \mathcal{G} \to \mathcal{F}$ is surjective with $\mathcal{G}$ of finite type, then* $\text{Ker}(\theta)$ *is of finite type.*

Proof. Proof of (1). Let $x \in X$. Choose an open neighbourhood $U \subset X$ of x such that there exists a presentation

$$\mathcal{O}_U^{\oplus m} \xrightarrow{\chi} \mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} \mathcal{F}|_U \to 0.$$

Let e_k be the section generating the kth factor of $\mathcal{O}_X^{\oplus r}$. For every $k = 1, \ldots, r$ we can, after shrinking U to a small neighbourhood of x, lift $\psi(e_k)$ to a section $\tilde{e}_k$ of $\mathcal{O}_U^{\oplus n}$ over U. This gives a morphism of sheaves $\alpha : \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n}$ such that $\varphi \circ \alpha = \psi$. Similarly, after shrinking U, we can find a morphism $\beta : \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus r}$ such that $\psi \circ \beta = \varphi$. Then the map

$$\mathcal{O}_U^{\oplus m} \oplus \mathcal{O}_U^{\oplus r} \xrightarrow{\beta\circ\chi, 1-\beta\circ\alpha} \mathcal{O}_U^{\oplus r}$$

is a surjection onto the kernel of ψ.

To prove (2) we may locally choose a surjection $\eta : \mathcal{O}_X^{\oplus r} \to \mathcal{G}$. By part (1) we see $\text{Ker}(\theta \circ \eta)$ is of finite type. Since $\text{Ker}(\theta) = \eta(\text{Ker}(\theta \circ \eta))$ we win. $\square$

Lemma 11.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a module of finite presentation is of finite presentation.*

Proof. Exactly the same as the proof of Lemma 10.4 but with finite index sets. $\square$

Lemma 11.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. Let M be an R-module. The $\mathcal{O}_X$-module $\mathcal{F}_M$ associated to M is a directed colimit of finitely presented $\mathcal{O}_X$-modules.*

Proof. This follows immediately from Lemma 10.5 and the fact that any module is a directed colimit of finitely presented modules, see Algebra, Lemma 11.3. $\square$

Lemma 11.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $x \in X$ such that $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus r}$. Then there exists an open neighbourhood U of x such that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$.*

Proof. Choose $s_1, \ldots, s_r \in \mathcal{F}_x$ mapping to a basis of $\mathcal{O}_{X,x}^{\oplus r}$ by the isomorphism. Choose an open neighbourhood U of x such that s_i lifts to $s_i \in \mathcal{F}(U)$. After shrinking U we see that the induced map $\psi : \mathcal{O}_U^{\oplus r} \to \mathcal{F}|_U$ is surjective (Lemma 9.4). By Lemma 11.3 we see that $\text{Ker}(\psi)$ is of finite type. Then $\text{Ker}(\psi)_x = 0$ implies that $\text{Ker}(\psi)$ becomes zero after shrinking U once more (Lemma 9.5). $\square$

12. Coherent modules

A reference for this section is [Ser55].

The category of coherent sheaves on a ringed space X is a more reasonable object than the category of quasi-coherent sheaves, in the sense that it is at least an abelian subcategory of $\mathit{Mod}(\mathcal{O}_X)$ no matter what X is. On the other hand, the pullback of a coherent module is "almost never" coherent in the general setting of ringed spaces.

Definition 12.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a *coherent $\mathcal{O}_X$-module* if the following two conditions hold:

(1) $\mathcal{F}$ is of finite type, and

(2) for every open $U \subset X$ and every finite collection $s_i \in \mathcal{F}(U)$, $i = 1, \ldots, n$ the kernel of the associated map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type.

The category of coherent $\mathcal{O}_X$-modules is denoted $Coh(\mathcal{O}_X)$.

Lemma 12.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Any coherent $\mathcal{O}_X$-module is of finite presentation and hence quasi-coherent.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on X. Pick a point $x \in X$. By (1) of the definition of coherent, we may find an open neighbourhood U and sections s_i, $i = 1, \ldots, n$ of $\mathcal{F}$ over U such that $\Psi : \bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}$ is surjective. By (2) of the definition of coherent, we may find an open neighbourhood V, $x \in V \subset U$ and sections $t_1, \ldots, t_m$ of $\bigoplus_{i=1,\ldots,n} \mathcal{O}_V$ which generate the kernel of $\Psi|_V$. Then over V we get the presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \to \mathcal{F}|_V \to 0$$

as desired. $\square$

Example 12.3. Suppose that X is a point. In this case the definition above gives a notion for modules over rings. What does the definition of coherent mean? It is closely related to the notion of Noetherian, but it is not the same: Namely, the ring $R = \mathbf{C}[x_1, x_2, x_3, \ldots]$ is coherent as a module over itself but not Noetherian as a module over itself. See Algebra, Section 90 for more discussion.

Lemma 12.4. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *Any finite type subsheaf of a coherent sheaf is coherent.*
(2) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism from a finite type sheaf $\mathcal{F}$ to a coherent sheaf $\mathcal{G}$. Then $\mathrm{Ker}(\varphi)$ is of finite type.*
(3) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of coherent $\mathcal{O}_X$-modules. Then $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are coherent.*
(4) *Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are coherent so is the third.*
(5) *The category $Coh(\mathcal{O}_X)$ is a weak Serre subcategory of $Mod(\mathcal{O}_X)$. In particular, the category of coherent modules is abelian and the inclusion functor $Coh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ is exact.*

Proof. Condition (2) of Definition 12.1 holds for any subsheaf of a coherent sheaf. Thus we get (1).

Assume the hypotheses of (2). Let us show that $\mathrm{Ker}(\varphi)$ is of finite type. Pick $x \in X$. Choose an open neighbourhood U of x in X such that $\mathcal{F}|_U$ is generated by $s_1, \ldots, s_n$. By Definition 12.1 the kernel $\mathcal{K}$ of the induced map $\bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$, $e_i \mapsto \varphi(s_i)$ is of finite type. Hence $\mathrm{Ker}(\varphi)$ which is the image of the composition $\mathcal{K} \to \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{F}$ is of finite type.

Assume the hypotheses of (3). By (2) the kernel of φ is of finite type and hence by (1) it is coherent.

With the same hypotheses let us show that $\mathrm{Coker}(\varphi)$ is coherent. Since $\mathcal{G}$ is of finite type so is $\mathrm{Coker}(\varphi)$. Let $U \subset X$ be open and let $\overline{s}_i \in \mathrm{Coker}(\varphi)(U)$, $i = 1, \ldots, n$ be sections. We have to show that the kernel of the associated morphism $\overline{\Psi} : \bigoplus_{i=1}^n \mathcal{O}_U \to \mathrm{Coker}(\varphi)$ is of finite type. There exists an open covering of U such that on each open all the sections $\overline{s}_i$ lift to sections s_i of $\mathcal{G}$. Hence we may assume this is the case over U. We may in addition assume there are sections t_j, $j = 1, \ldots, m$ of $\text{Im}(\varphi)$ over U which generate $\text{Im}(\varphi)$ over U. Let $\Phi : \bigoplus_{j=1}^m \mathcal{O}_U \to \text{Im}(\varphi)$ be defined using t_j and $\Psi : \bigoplus_{j=1}^m \mathcal{O}_U \oplus \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$ using t_j and s_i. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{j=1}^m \mathcal{O}_U \arrow[r] \arrow[d, "\Phi"] & \bigoplus_{j=1}^m \mathcal{O}_U \oplus \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d, "\Psi"] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d, "\overline{\Psi}"] & 0 \\
0 \arrow[r] & \text{Im}(\varphi) \arrow[r] & \mathcal{G} \arrow[r] & \text{Coker}(\varphi) \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma we get an exact sequence $\text{Ker}(\Psi) \to \text{Ker}(\overline{\Psi}) \to 0$. Since $\text{Ker}(\Psi)$ is a finite type module, we see that $\text{Ker}(\overline{\Psi})$ has finite type.

Proof of part (4). Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. By part (3) it suffices to prove that if $\mathcal{F}_1$ and $\mathcal{F}_3$ are coherent so is $\mathcal{F}_2$. By Lemma 9.3 we see that $\mathcal{F}_2$ has finite type. Let $s_1, \ldots, s_n$ be finitely many local sections of $\mathcal{F}_2$ defined over a common open U of X. We have to show that the module of relations $\mathcal{K}$ between them is of finite type. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \arrow[r] & 0
\end{tikzcd}$$

with obvious notation. By the snake lemma we get a short exact sequence $0 \to \mathcal{K} \to \mathcal{K}_3 \to \mathcal{F}_1$ where $\mathcal{K}_3$ is the module of relations among the images of the sections s_i in $\mathcal{F}_3$. Since $\mathcal{F}_1$ is coherent we see that $\mathcal{K}$ is the kernel of a map from a finite type module to a coherent module and hence finite type by (2).

Proof of (5). This follows because (3) and (4) show that Homology, Lemma 10.3 applies. $\square$

Lemma 12.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is coherent if and only if it is of finite presentation.*

Proof. Omitted. $\square$

Lemma 12.6. *Let X be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{G}$ of finite type, $\mathcal{F}$ coherent and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ injective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is injective.*

Proof. Denote $\mathcal{K} \subset \mathcal{G}$ the kernel of φ. By Lemma 12.4 we see that $\mathcal{K}$ is a finite type $\mathcal{O}_X$-module. Our assumption is that $\mathcal{K}_x = 0$. By Lemma 9.5 there exists an open neighbourhood U of x such that $\mathcal{K}|_U = 0$. Then U works. $\square$

13. Closed immersions of ringed spaces

When do we declare a morphism of ringed spaces $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ to be a closed immersion?

Motivated by the example of a closed immersion of normal topological spaces (ringed with the sheaf of continuous functors), or differential manifolds (ringed with the sheaf of differentiable functions), it seems natural to assume at least:

(1) The map i is a closed immersion of topological spaces.
(2) The associated map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote the kernel by $\mathcal{I}$.

Already these conditions imply a number of pleasing results: For example we prove that the category of $\mathcal{O}_Z$-modules is equivalent to the category of $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ generalizing the result on abelian sheaves of Section 6

However, in the Stacks project we choose the definition that guarantees that if i is a closed immersion and $(X, \mathcal{O}_X)$ is a scheme, then also $(Z, \mathcal{O}_Z)$ is a scheme. Moreover, in this situation we want i_* and i^* to provide an equivalence between the category of quasi-coherent $\mathcal{O}_Z$-modules and the category of quasi-coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$. A minimal condition is that $i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules. A good way to guarantee that $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module is to assume that $\mathcal{I}$ is locally generated by sections. We can interpret this condition as saying "$(Z, \mathcal{O}_Z)$ is locally on $(X, \mathcal{O}_X)$ defined by setting some regular functions f_i, i.e., local sections of $\mathcal{O}_X$, equal to zero". This leads to the following definition.

Definition 13.1. A *closed immersion of ringed spaces*[1] is a morphism $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ with the following properties:

(1) The map i is a closed immersion of topological spaces.
(2) The associated map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote the kernel by $\mathcal{I}$.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

Actually, this definition still does not guarantee that i_* of a quasi-coherent $\mathcal{O}_Z$-module is a quasi-coherent $\mathcal{O}_X$-module. The problem is that it is not clear how to convert a local presentation of a quasi-coherent $\mathcal{O}_Z$-module into a local presentation for the pushforward. However, the following is trivial.

Lemma 13.2. *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a closed immersion of ringed spaces. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is locally on X the cokernel of a map of quasi-coherent $\mathcal{O}_X$-modules.*

Proof. This is true because $i_*\mathcal{O}_Z$ is quasi-coherent by definition. And locally on Z the sheaf $\mathcal{F}$ is a cokernel of a map between direct sums of copies of $\mathcal{O}_Z$. Moreover, any direct sum of copies of the *the same* quasi-coherent sheaf is quasi-coherent. And finally, i_* commutes with arbitrary colimits, see Lemma 6.3. Some details omitted. $\square$

Lemma 13.3. *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Assume i is a homeomorphism onto a closed subset of X and that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Let $\mathcal{F}$ be an $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is of finite type if and only if $\mathcal{F}$ is of finite type.*

Proof. Suppose that $\mathcal{F}$ is of finite type. Pick $x \in X$. If $x \notin Z$, then $i_*\mathcal{F}$ is zero in a neighbourhood of x and hence finitely generated in a neighbourhood of x. If $x = i(z)$, then choose an open neighbourhood $z \in V \subset Z$ and sections $s_1, \ldots, s_n \in \mathcal{F}(V)$ which generate $\mathcal{F}$ over V. Write $V = Z \cap U$ for some open

[1]This is nonstandard notation; see discussion above.

$U \subset X$. Note that U is a neighbourhood of x. Clearly the sections s_i give sections s_i of $i_*\mathcal{F}$ over U. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \longrightarrow i_*\mathcal{F}|_U$$

is surjective by inspection of what it does on stalks (here we use that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective). Hence $i_*\mathcal{F}$ is of finite type.

Conversely, suppose that $i_*\mathcal{F}$ is of finite type. Choose $z \in Z$. Set $x = i(z)$. By assumption there exists an open neighbourhood $U \subset X$ of x, and sections $s_1, \ldots, s_n \in (i_*\mathcal{F})(U)$ which generate $i_*\mathcal{F}$ over U. Set $V = Z \cap U$. By definition of i_* the sections s_i correspond to sections s_i of $\mathcal{F}$ over V. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \longrightarrow \mathcal{F}|_V$$

is surjective by inspection of what it does on stalks. Hence $\mathcal{F}$ is of finite type. $\square$

Lemma 13.4. *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Assume i is a homeomorphism onto a closed subset of X and $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote $\mathcal{I} \subset \mathcal{O}_X$ the kernel of $i^\sharp$. The functor*

$$i_* : \mathit{Mod}(\mathcal{O}_Z) \longrightarrow \mathit{Mod}(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. We claim that for an $\mathcal{O}_Z$-module $\mathcal{F}$ the canonical map

$$i^*i_*\mathcal{F} \longrightarrow \mathcal{F}$$

is an isomorphism. We check this on stalks. Say $z \in Z$ and $x = i(z)$. We have

$$(i^*i_*\mathcal{F})_z = (i_*\mathcal{F})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} = \mathcal{F}_z \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} = \mathcal{F}_z$$

by Sheaves, Lemma 26.4, the fact that $\mathcal{O}_{Z,z}$ is a quotient of $\mathcal{O}_{X,x}$, and Sheaves, Lemma 32.1. It follows that i_* is fully faithful.

Let $\mathcal{G}$ be a $\mathcal{O}_X$-module with $\mathcal{I}\mathcal{G} = 0$. We will prove the canonical map

$$\mathcal{G} \longrightarrow i_*i^*\mathcal{G}$$

is an isomorphism. This proves that $\mathcal{G} = i_*\mathcal{F}$ with $\mathcal{F} = i^*\mathcal{G}$ which finishes the proof. We check the displayed map induces an isomorphism on stalks. If $x \in X$, $x \notin i(Z)$, then $\mathcal{G}_x = 0$ because $\mathcal{I}_x = \mathcal{O}_{X,x}$ in this case. As above $(i_*i^*\mathcal{G})_x = 0$ by Sheaves, Lemma 32.1. On the other hand, if $x \in Z$, then we obtain the map

$$\mathcal{G}_x \longrightarrow \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,x}$$

by Sheaves, Lemmas 26.4 and 32.1. This map is an isomorphism because $\mathcal{O}_{Z,x} = \mathcal{O}_{X,x}/\mathcal{I}_x$ and because $\mathcal{G}_x$ is annihilated by $\mathcal{I}_x$ by assumption. $\square$

Remark 13.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. For an $\mathcal{O}_X$-module $\mathcal{F}$ we can consider the *submodule of sections with support in Z*, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset U \cap Z\}$$

Observe that $\mathcal{H}_Z(\mathcal{F})(U)$ is a module over $\mathcal{O}_X(U)$, i.e., $\mathcal{H}_Z(\mathcal{F})$ is an $\mathcal{O}_X$-module. By construction $\mathcal{H}_Z(\mathcal{F})$ is the largest $\mathcal{O}_X$-submodule of $\mathcal{F}$ whose support is contained in Z. Applying Lemma 13.4 to the morphism of ringed spaces $(Z, \mathcal{O}_X|_Z) \to (X, \mathcal{O}_X)$

we may (and we do) view $\mathcal{H}_Z(\mathcal{F})$ as an $\mathcal{O}_X|_Z$-module on Z. Thus we obtain a functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X|_Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as an } \mathcal{O}_X|_Z\text{-module on } Z$$

This functor is left exact, but in general not exact. All of the statements made above follow directly from Lemma 5.2. Clearly the construction is compatible with the construction in Remark 6.2.

Lemma 13.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor $\mathcal{H}_Z : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_X|_Z)$ of Remark 13.5 is right adjoint to $i_* : Mod(\mathcal{O}_X|_Z) \to Mod(\mathcal{O}_X)$.*

Proof. We have to show that for any $\mathcal{O}_X$-module $\mathcal{F}$ and any $\mathcal{O}_X|_Z$-module $\mathcal{G}$ we have

$$\mathrm{Hom}_{\mathcal{O}_X|_Z}(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{G}, \mathcal{F})$$

This is clear because after all any section of $i_*\mathcal{G}$ has support in Z. Details omitted. □

14. Locally free sheaves

Let $(X, \mathcal{O}_X)$ be a ringed space. Our conventions allow (some of) the stalks $\mathcal{O}_{X,x}$ to be the zero ring. This means we have to be a little careful when defining the rank of a locally free sheaf.

Definition 14.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We say $\mathcal{F}$ is *locally free* if for every point $x \in X$ there exist a set I and an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X|_U$ as an $\mathcal{O}_X|_U$-module.
(2) We say $\mathcal{F}$ is *finite locally free* if we may choose the index sets I to be finite.
(3) We say $\mathcal{F}$ is *finite locally free of rank r* if we may choose the index sets I to have cardinality r.

A finite direct sum of (finite) locally free sheaves is (finite) locally free. However, it may not be the case that an infinite direct sum of locally free sheaves is locally free.

Lemma 14.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is locally free then it is quasi-coherent.*

Proof. Omitted. □

Lemma 14.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a locally free $\mathcal{O}_X$-module.*

Proof. Omitted. □

Lemma 14.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose that the support of $\mathcal{O}_X$ is X, i.e., all stalks of $\mathcal{O}_X$ are nonzero rings. Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules. There exists a locally constant function*

$$rank_{\mathcal{F}} : X \longrightarrow \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any point $x \in X$ the cardinality of any set I such that $\mathcal{F}$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X$ in a neighbourhood of x is $rank_{\mathcal{F}}(x)$.

Proof. Under the assumption of the lemma the cardinality of I can be read off from the rank of the free module $\mathcal{F}_x$ over the nonzero ring $\mathcal{O}_{X,x}$, and it is constant in a neighbourhood of x. $\square$

Lemma 14.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $r \geq 0$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of finite locally free $\mathcal{O}_X$-modules of rank r. Then φ is an isomorphism if and only if φ is surjective.*

Proof. Assume φ is surjective. Pick $x \in X$. There exists an open neighbourhood U of x such that both $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are isomorphic to $\mathcal{O}_U^{\oplus r}$. Pick lifts of the free generators of $\mathcal{G}|_U$ to obtain a map $\psi : \mathcal{G}|_U \to \mathcal{F}|_U$ such that $\varphi|_U \circ \psi = \text{id}$. Hence we conclude that the map $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ induced by φ is surjective. Since both $\Gamma(U, \mathcal{F})$ and $\Gamma(U, \mathcal{G})$ are isomorphic to $\Gamma(U, \mathcal{O}_U)^{\oplus r}$ as an $\Gamma(U, \mathcal{O}_U)$-module we may apply Algebra, Lemma 16.4 to see that $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ is injective. This finishes the proof. $\square$

Lemma 14.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. If all stalks $\mathcal{O}_{X,x}$ are local rings, then any direct summand of a finite locally free $\mathcal{O}_X$-module is finite locally free.*

Proof. Assume $\mathcal{F}$ is a direct summand of the finite locally free $\mathcal{O}_X$-module $\mathcal{H}$. Let $x \in X$ be a point. Then $\mathcal{H}_x$ is a finite free $\mathcal{O}_{X,x}$-module. Because $\mathcal{O}_{X,x}$ is local, we see that $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus r}$ for some r, see Algebra, Lemma 78.2. By Lemma 11.6 we see that $\mathcal{F}$ is free of rank r in an open neighbourhood of x. (Note that $\mathcal{F}$ is of finite presentation as a summand of $\mathcal{H}$.) $\square$

15. Bilinear maps

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be $\mathcal{O}_X$-modules. A *bilinear map* $f : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ of sheaves of $\mathcal{O}_X$-modules is a map of sheaves of sets as indicated such that for every open $U \subset X$ the induced map

$$\mathcal{F}(U) \times \mathcal{G}(U) \to \mathcal{H}(U)$$

is an $\mathcal{O}_X(U)$-bilinear map of modules. Equivalently you can ask certain diagrams of maps of sheaves of sets commute, immitating the usual axioms for bilinear maps of modules. For example, the axiom $f(x + y, z) = f(x, z) + f(y, z)$ is represented by the commutativity of the diagram

$$\begin{tikzcd} \mathcal{F} \times \mathcal{F} \times \mathcal{G} \arrow[r, "(f \circ \mathrm{pr}_{13}, f \circ \mathrm{pr}_{23})"'] \arrow[d, "(+ \circ \mathrm{pr}_{12}, \mathrm{pr}_3)"'] & \mathcal{H} \times \mathcal{H} \arrow[d, "+"] \\ \mathcal{F} \times \mathcal{G} \arrow[r, "f"] & \mathcal{H} \end{tikzcd}$$

Another characterization is this: if $f : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ is a map of sheaves of sets and it induces a bilinar map of modules on stalks for all points of X, then f is a bilinear map of sheaves of modules. This is true as you can test whether local sections are equal by checking on stalks.

Let $\text{Mor}(-, -)$ denote morphisms in the category of sheaves of sets on X. Another characterization of a bilinear map is this: a map of sheaves of sets $f : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ is bilinear if given any sheaf of sets $\mathcal{S}$ the rule

$$\text{Mor}(\mathcal{S}, \mathcal{F}) \times \text{Mor}(\mathcal{S}, \mathcal{G}) \to \text{Mor}(\mathcal{S}, \mathcal{H}), \quad (a, b) \mapsto f \circ (a \times b)$$

is a bilinear map of modules over the ring $\mathrm{Mor}(\mathcal{S}, \mathcal{O}_X)$. We don't usually take this point of view as it is easier to think about sets of local sections and it is clearly equivalent.

Finally, here is yet another way to say the definition: $\mathcal{O}_X$ is a ring object in the category of sheaves of sets and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are module objects over this ring. Then a bilinear map can be defined for module objects over a ring object in any category. To formulate what is a ring object and what is a module object over a ring object, and what is a bilinear map of such in a category it is pleasant (but not strictly necessary) to assume the category has finite products; and this is true for the category of sheaves of sets.

16. Tensor product

We have already briefly discussed the tensor product in the setting of change of rings in Sheaves, Sections 6 and 20. Let us generalize this to tensor products of modules.

Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. We define first the *tensor product presheaf*

$$\mathcal{F} \otimes_{p, \mathcal{O}_X} \mathcal{G}$$

as the rule which assigns to $U \subset X$ open the $\mathcal{O}_X(U)$-module $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Having defined this we define the *tensor product sheaf* as the sheafification of the above:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F} \otimes_{p, \mathcal{O}_X} \mathcal{G})^{\#}$$

This can be characterized as the sheaf of $\mathcal{O}_X$-modules such that for any third sheaf of $\mathcal{O}_X$-modules $\mathcal{H}$ we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) = \mathrm{Bilin}_{\mathcal{O}_X}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

Here the right hand side indicates the set of bilinear maps of sheaves of $\mathcal{O}_X$-modules as defined in Section 15.

The tensor product of modules M, N over a ring R satisfies symmetry, namely $M \otimes_R N = N \otimes_R M$, hence the same holds for tensor products of sheaves of modules, i.e., we have

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

functorial in $\mathcal{F}$, $\mathcal{G}$. And since tensor product of modules satisfies associativity we also get canonical functorial isomorphisms

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H} = \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})$$

functorial in $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$.

Lemma 16.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $x \in X$. There is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules*

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$$

functorial in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Omitted. $\square$

Lemma 16.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}'$, $\mathcal{G}'$ be presheaves of $\mathcal{O}_X$-modules with sheafifications $\mathcal{F}$, $\mathcal{G}$. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F}' \otimes_{p, \mathcal{O}_X} \mathcal{G}')^{\#}$.*

Proof. Omitted. $\square$

Lemma 16.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of $\mathcal{O}_X$-modules then the induced sequence*

$$\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact.

Proof. This follows from the fact that exactness may be checked at stalks (Lemma 3.1), the description of stalks (Lemma 16.1) and the corresponding result for tensor products of modules (Algebra, Lemma 12.10). ☐

Lemma 16.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_Y$-modules. Then $f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ functorially in $\mathcal{F}$, $\mathcal{G}$.*

Proof. Omitted. ☐

Lemma 16.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any $\mathcal{O}_X$-module $\mathcal{F}$ the functor*

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$$

commutes with arbitrary colimits.

Proof. Let I be a preordered set and let $\{\mathcal{G}_i\}$ be a system over I. Set $\mathcal{G} = \operatorname{colim}_i \mathcal{G}_i$. Recall that $\mathcal{G}$ is the sheaf associated to the presheaf $\mathcal{G}' : U \mapsto \operatorname{colim}_i \mathcal{G}_i(U)$, see Sheaves, Section 29. By Lemma 16.2 the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \operatorname{colim}_i \mathcal{G}_i(U) = \operatorname{colim}_i \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}_i(U)$$

where the equality sign is Algebra, Lemma 12.9. Hence the lemma follows from the description of colimits in $Mod(\mathcal{O}_X)$, see Lemma 3.2. ☐

Lemma 16.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.*

1. *If $\mathcal{F}$, $\mathcal{G}$ are locally generated by sections, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
2. *If $\mathcal{F}$, $\mathcal{G}$ are of finite type, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
3. *If $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
4. *If $\mathcal{F}$, $\mathcal{G}$ are of finite presentation, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
5. *If $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is coherent.*
6. *If $\mathcal{F}$, $\mathcal{G}$ are coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
7. *If $\mathcal{F}$, $\mathcal{G}$ are locally free, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*

Proof. We first prove that the tensor product of locally free $\mathcal{O}_X$-modules is locally free. This follows if we show that $(\bigoplus_{i \in I} \mathcal{O}_X) \otimes_{\mathcal{O}_X} (\bigoplus_{j \in J} \mathcal{O}_X) \cong \bigoplus_{(i,j) \in I \times J} \mathcal{O}_X$. The sheaf $\bigoplus_{i \in I} \mathcal{O}_X$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{i \in I} \mathcal{O}_X(U)$. Hence the tensor product is the sheaf associated to the presheaf

$$U \longmapsto (\bigoplus\nolimits_{i \in I} \mathcal{O}_X(U)) \otimes_{\mathcal{O}_X(U)} (\bigoplus\nolimits_{j \in J} \mathcal{O}_X(U)).$$

We deduce what we want since for any ring R we have $(\bigoplus_{i \in I} R) \otimes_R (\bigoplus_{j \in J} R) = \bigoplus_{(i,j) \in I \times J} R$.

If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is exact, then by Lemma 16.3 the complex $\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$ is exact. Using this we can prove (5). Namely, in this case there exists locally such an exact sequence with $\mathcal{F}_i$, $i = 1, 2$ finite free. Hence the two terms $\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}$ are isomorphic to finite direct sums of $\mathcal{G}$ (for example by Lemma 16.5). Since finite direct sums are coherent sheaves, these are coherent and so is the cokernel of the map, see Lemma 12.4.

And if also $\mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G} \to 0$ is exact, then we see that

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \oplus \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_2 \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact. Using this we can for example prove (3). Namely, the assumption means that we can locally find presentations as above with $\mathcal{F}_i$ and $\mathcal{G}_i$ free $\mathcal{O}_X$-modules. Hence the displayed presentation is a presentation of the tensor product by free sheaves as well.

The proof of the other statements is omitted. □

17. Flat modules

We can define flat modules exactly as in the case of modules over rings.

Definition 17.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if the functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$$

is exact.

We can characterize flatness by looking at the stalks.

Lemma 17.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is flat if and only if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$.*

Proof. Assume $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$. In this case, if $\mathcal{G} \to \mathcal{H} \to \mathcal{K}$ is exact, then also $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{F}$ is exact because we can check exactness at stalks and because tensor product commutes with taking stalks, see Lemma 16.1. Conversely, suppose that $\mathcal{F}$ is flat, and let $x \in X$. Consider the skyscraper sheaves $i_{x,*}M$ where M is a $\mathcal{O}_{X,x}$-module. Note that

$$M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x = (i_{x,*}M \otimes_{\mathcal{O}_X} \mathcal{F})_x$$

again by Lemma 16.1. Since $i_{x,*}$ is exact, we see that the fact that $\mathcal{F}$ is flat implies that $M \mapsto M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x$ is exact. Hence $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module. □

Thus the following definition makes sense.

Definition 17.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$. An $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at x* if $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module.

Hence we see that $\mathcal{F}$ is a flat $\mathcal{O}_X$-module if and only if it is flat at every point.

Lemma 17.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. A filtered colimit of flat $\mathcal{O}_X$-modules is flat. A direct sum of flat $\mathcal{O}_X$-modules is flat.*

Proof. This follows from Lemma 16.5, Lemma 16.1, Algebra, Lemma 8.8, and the fact that we can check exactness at stalks. □

Lemma 17.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be open. The sheaf $j_{U!}\mathcal{O}_U$ is a flat sheaf of $\mathcal{O}_X$-modules.*

Proof. The stalks of $j_{U!}\mathcal{O}_U$ are either zero or equal to $\mathcal{O}_{X,x}$. Apply Lemma 17.2. □

Lemma 17.6. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

1. *Any sheaf of $\mathcal{O}_X$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
2. *Any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For every open $U \subset X$ and every $s \in \mathcal{F}(U)$ we get a morphism $j_{U!}\mathcal{O}_U \to \mathcal{F}$, namely the adjoint to the morphism $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$. Clearly the map

$$\bigoplus\nolimits_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective, and the source is flat by combining Lemmas 17.4 and 17.5. $\square$

Lemma 17.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$0 \to \mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is flat. Then for any $\mathcal{O}_X$-module $\mathcal{G}$ the sequence

$$0 \to \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0$$

is exact.

Proof. Using that $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for every $x \in X$ and that exactness can be checked on stalks, this follows from Algebra, Lemma 39.12. $\square$

Lemma 17.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules.

(1) *If $\mathcal{F}_2$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_1$.*
(2) *If $\mathcal{F}_1$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_2$.*

Proof. Since exactness and flatness may be checked at the level of stalks this follows from Algebra, Lemma 39.13. $\square$

Lemma 17.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{Q} \to 0$$

be an exact complex of $\mathcal{O}_X$-modules. If $\mathcal{Q}$ and all $\mathcal{F}_i$ are flat $\mathcal{O}_X$-modules, then for any $\mathcal{O}_X$-module $\mathcal{G}$ the complex

$$\ldots \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact also.

Proof. Follows from Lemma 17.7 by splitting the complex into short exact sequences and using Lemma 17.8 to prove inductively that $\mathrm{Im}(\mathcal{F}_{i+1} \to \mathcal{F}_i)$ is flat. $\square$

The following lemma gives one direction of the equational criterion of flatness (Algebra, Lemma 39.11).

Lemma 17.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a flat $\mathcal{O}_X$-module. Let $U \subset X$ be open and let*

$$\mathcal{O}_U \xrightarrow{(f_1, \ldots, f_n)} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} \mathcal{F}|_U$$

be a complex of $\mathcal{O}_U$-modules. For every $x \in U$ there exists an open neighbourhood $V \subset U$ of x and a factorization

$$\mathcal{O}_V^{\oplus n} \xrightarrow{A} \mathcal{O}_V^{\oplus m} \xrightarrow{(t_1, \ldots, t_m)} \mathcal{F}|_V$$

of $(s_1, \ldots, s_n)|_V$ such that $A \circ (f_1, \ldots, f_n)|_V = 0$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals generated by $f_1, \ldots, f_n$. Then $\sum f_i \otimes s_i$ is a section of $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ which maps to zero in $\mathcal{F}|_U$. As $\mathcal{F}|_U$ is flat the map $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{F}|_U$ is injective. Since $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is the sheaf associated to the presheaf tensor product, we see there exists an open neighbourhood $V \subset U$ of x such that $\sum f_i|_V \otimes s_i|_V$ is zero in $\mathcal{I}(V) \otimes_{\mathcal{O}(V)} \mathcal{F}(V)$. Unwinding the definitions using Algebra, Lemma 107.10 we find $t_1, \ldots, t_m \in \mathcal{F}(V)$ and $a_{ij} \in \mathcal{O}(V)$ such that $\sum a_{ij} f_i|_V = 0$ and $s_i|_V = \sum a_{ij} t_j$. $\square$

18. Duals

Let $(X, \mathcal{O}_X)$ be a ringed space. The category of $\mathcal{O}_X$-modules endowed with the tensor product constructed in Section 16 is a symmetric monoidal category. For an $\mathcal{O}_X$-module $\mathcal{F}$ the following are equivalent

(1) $\mathcal{F}$ has a left dual in the monoidal category of $\mathcal{O}_X$-modules,
(2) $\mathcal{F}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module, and
(3) $\mathcal{F}$ is of finite presentation and flat as an $\mathcal{O}_X$-module.

This is proved in Example 18.1 and Lemmas 18.2 and 18.3 of this section.

Example 18.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module which is locally a direct summand of a finite free $\mathcal{O}_X$-module. Then the map

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

is an isomorphism. Namely, this is a local question, it is true if $\mathcal{F}$ is finite free, and it holds for any summand of a module for which it is true. Denote

$$\eta : \mathcal{O}_X \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

the map sending 1 to the section corresponding to $\mathrm{id}_\mathcal{F}$ under the isomorphism above. Denote

$$\epsilon : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{O}_X$$

the evaluation map. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \eta, \epsilon$ is a left dual for $\mathcal{F}$ as in Categories, Definition 43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_\mathcal{F}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)}$.

Lemma 18.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{G}, \eta, \epsilon$ be a left dual of $\mathcal{F}$ in the monoidal category of $\mathcal{O}_X$-modules, see Categories, Definition 43.5. Then*

(1) *$\mathcal{F}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module,*
(2) *the map $e : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \to \mathcal{G}$ sending a local section λ to $(\lambda \otimes 1)(\eta)$ is an isomorphism,*
(3) *we have $\epsilon(f, g) = e^{-1}(g)(f)$ for local sections f and g of $\mathcal{F}$ and $\mathcal{G}$.*

Proof. The assumptions mean that

$$\mathcal{F} \xrightarrow{\eta \otimes 1} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{1 \otimes \epsilon} \mathcal{F} \quad \text{and} \quad \mathcal{G} \xrightarrow{1 \otimes \eta} \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \xrightarrow{\epsilon \otimes 1} \mathcal{G}$$

are the identity map. Let $x \in X$. We can find an open neighbourhood U of x, a finite number of sections $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ of $\mathcal{F}$ and $\mathcal{G}$ over U such that $\eta(1) = \sum f_i g_i$. Denote

$$\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$$

the map sending the ith basis vector to f_i. Then we can factor the map $\eta|_U$ over a map $\tilde\eta : \mathcal{O}_U \to \mathcal{O}_U^{\oplus n} \otimes_{\mathcal{O}_U} \mathcal{G}|_U$. We obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{F}|_U \arrow[r, "\eta\otimes 1"'] \arrow[dr, "\tilde\eta\otimes 1"'] & \mathcal{F}|_U \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U \arrow[r, "1\otimes\epsilon"'] & \mathcal{F}|_U \\
 & \mathcal{O}_U^{\oplus n} \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U \arrow[u] \arrow[r, "1\otimes\epsilon"] & \mathcal{O}_U^{\oplus n} \arrow[u]
\end{tikzcd}$$

This shows that the identity on $\mathcal{F}$ locally on X factors through a finite free module. This proves (1). Part (2) follows from Categories, Lemma 43.6 and its proof. Part (3) follows from the first equality of the proof. You can also deduce (2) and (3) from the uniqueness of left duals (Categories, Remark 43.7) and the construction of the left dual in Example 18.1. $\square$

Lemma 18.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a flat $\mathcal{O}_X$-module of finite presentation. Then $\mathcal{F}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module.*

Proof. After replacing X by the members of an open covering, we may assume there exists a presentation

$$\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{F} \to 0$$

Let $x \in X$. By Lemma 17.10 we can, after shrinking X to an open neighbourhood of x, assume there exists a factorization

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_1} \to \mathcal{F}$$

such that the composition $\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_1}$ annihilates the first summand of $\mathcal{O}_X^{\oplus r}$. Repeating this argument $r-1$ more times we obtain a factorization

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_r} \to \mathcal{F}$$

such that the composition $\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_r}$ is zero. This means that the surjection $\mathcal{O}_X^{\oplus n_r} \to \mathcal{F}$ has a section and we win. $\square$

19. Constructible sheaves of sets

Let X be a topological space. Given a set S recall that $\underline{S}$ or $\underline{S}_X$ denotes the constant sheaf with value S, see Sheaves, Definition 7.4. Let $U \subset X$ be an open of a topological space X. We will denote j_U the inclusion morphism and we will denote $j_{U!} : \mathit{Sh}(U) \to \mathit{Sh}(X)$ the extension by the empty set described in Sheaves, Section 31.

Lemma 19.1. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf of sets on X. There exists a set I and for each $i \in I$ an element $U_i \in \mathcal{B}$ and a finite set S_i such that there exists a surjection $\coprod_{i\in I} j_{U_i!}\underline{S_i} \to \mathcal{F}$.*

Proof. Let S be a singleton set. We will prove the result with $S_i = S$. For every $x \in X$ and element $s \in \mathcal{F}_x$ we can choose a $U(x,s) \in \mathcal{B}$ and $s(x,s) \in \mathcal{F}(U(x,s))$ which maps to s in $\mathcal{F}_x$. By Sheaves, Lemma 31.4 the section $s(x,s)$ corresponds to a map of sheaves $j_{U(x,s)!}\underline{S} \to \mathcal{F}$. Then

$$\coprod\nolimits_{(x,s)} j_{U(x,s)!}\underline{S} \to \mathcal{F}$$

is surjective on stalks and hence surjective. $\square$

Lemma 19.2. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X and assume that each $U \in \mathcal{B}$ is quasi-compact. Then every sheaf of sets on X is a filtered colimit of sheaves of the form*

$$\textit{Coequalizer}\left(\coprod\nolimits_{b=1,\ldots,m} j_{V_b!}\underline{S_b} \rightrightarrows \coprod\nolimits_{a=1,\ldots,n} j_{U_a!}\underline{S_a} \right) \tag{19.2.1}$$

with U_a and V_b in $\mathcal{B}$ and S_a and S_b finite sets.

Proof. By Lemma 19.1 every sheaf of sets $\mathcal{F}$ is the target of a surjection whose source $\mathcal{F}_0$ is a coproduct of sheaves the form $j_{U!}\underline{S}$ with $U \in \mathcal{B}$ and S finite. Applying this to $\mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ we find that $\mathcal{F}$ is a coequalizer of a pair of maps

$$\coprod\nolimits_{b\in B} j_{V_b!}\underline{S_b} \rightrightarrows \coprod\nolimits_{a\in A} j_{U_a!}\underline{S_a}$$

for some index sets A, B and V_b and U_a in $\mathcal{B}$ and S_a and S_b finite. For every finite subset $B' \subset B$ there is a finite subset $A' \subset A$ such that the coproduct over $b \in B'$ maps into the coproduct over $a \in A'$ via both maps. Namely, we can view the right hand side as a filtered colimit with injective transition maps. Hence taking sections over the quasi-compact opens V_b, $b \in B'$ commutes with this coproduct, see Sheaves, Lemma 29.1. Thus our sheaf is the colimit of the cokernels of these maps between finite coproducts. $\square$

Lemma 19.3. *Let X be a spectral topological space. Let $\mathcal{B}$ be the set of quasi-compact open subsets of X. Let $\mathcal{F}$ be a sheaf of sets as in Equation (19.2.1). Then there exists a continuous spectral map $f : X \to Y$ to a finite sober topological space Y and a sheaf of sets $\mathcal{G}$ on Y with finite stalks such that $f^{-1}\mathcal{G} \cong \mathcal{F}$.*

Proof. We can write $X = \lim X_i$ as a directed limit of finite sober spaces, see Topology, Lemma 23.14. Of course the transition maps $X_{i'} \to X_i$ are spectral and hence by Topology, Lemma 24.5 the maps $p_i : X \to X_i$ are spectral. For some i we can find opens $U_{a,i}$ and $V_{b,i}$ of X_i whose inverse images are U_a and V_b, see Topology, Lemma 24.6. The two maps

$$\beta, \gamma : \coprod\nolimits_{b\in B} j_{V_b!}\underline{S_b} \longrightarrow \coprod\nolimits_{a\in A} j_{U_a!}\underline{S_a}$$

whose coequalizer is $\mathcal{F}$ correspond by adjunction to two families

$$\beta_b, \gamma_b : S_b \longrightarrow \Gamma(V_b, \coprod\nolimits_{a\in A} j_{U_a!}\underline{S_a}), \quad b \in B$$

of maps of sets. Observe that $p_i^{-1}(j_{U_{a,i}!}\underline{S_a}) = j_{U_a!}\underline{S_a}$ and $(X_{i'} \to X_i)^{-1}(j_{U_{a,i}!}\underline{S_a}) = j_{U_{a,i'}!}\underline{S_a}$. It follows from Sheaves, Lemma 29.3 (and using that S_b and B are finite sets) that after increasing i we find maps

$$\beta_{b,i}, \gamma_{b,i} : S_b \longrightarrow \Gamma(V_{b,i}, \coprod\nolimits_{a\in A} j_{U_{a,i}!}\underline{S_a}), \quad b \in B$$

which give rise to the maps β_b and γ_b after pulling back by p_i. These maps correspond in turn to maps of sheaves

$$\beta_i, \gamma_i : \coprod\nolimits_{b\in B} j_{V_{b,i}!}\underline{S_b} \longrightarrow \coprod\nolimits_{a\in A} j_{U_{a,i}!}\underline{S_a}$$

on X_i. Then we can take $Y = X_i$ and

$$\mathcal{G} = \text{Coequalizer}\left(\coprod\nolimits_{b=1,\ldots,m} j_{V_{b,i}!}\underline{S_b} \rightrightarrows \coprod\nolimits_{a=1,\ldots,n} j_{U_{a,i}!}\underline{S_a} \right)$$

We omit some details. $\square$

Lemma 19.4. *Let X be a spectral topological space. Let $\mathcal{B}$ be the set of quasi-compact open subsets of X. Let $\mathcal{F}$ be a sheaf of sets as in Equation (19.2.1). Then there exist finitely many constructible closed subsets $Z_1, \ldots, Z_n \subset X$ and finite sets S_i such that $\mathcal{F}$ is isomorphic to a subsheaf of $\prod (Z_i \to X)_* \underline{S_i}$.*

Proof. By Lemma 19.3 we reduce to the case of a finite sober topological space and a sheaf with finite stalks. In this case $\mathcal{F} \subset \prod_{x \in X} i_{x,*}\mathcal{F}_x$ where $i_x : \overline{\{x\}} \to X$ is the embedding. We omit the proof that $i_{x,*}\mathcal{F}_x$ is a constant sheaf on $\overline{\{x\}}$. $\square$

20. Flat morphisms of ringed spaces

The pointwise definition is motivated by Lemma 17.2 and Definition 17.3 above.

Definition 20.1. Let $f : X \to Y$ be a morphism of ringed spaces. Let $x \in X$. We say f is *flat at x* if the map of rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. We say f is *flat* if f is flat at every $x \in X$.

Consider the map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We see that the stalk at x is the ring map $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Hence f is flat at x if and only if $\mathcal{O}_X$ is flat at x as an $f^{-1}\mathcal{O}_Y$-module. And f is flat if and only if $\mathcal{O}_X$ is flat as an $f^{-1}\mathcal{O}_Y$-module. A very special case of a flat morphism is an open immersion.

Lemma 20.2. *Let $f : X \to Y$ be a flat morphism of ringed spaces. Then the pullback functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is exact.*

Proof. The functor f^* is the composition of the exact functor $f^{-1} : Mod(\mathcal{O}_Y) \to Mod(f^{-1}\mathcal{O}_Y)$ and the change of rings functor

$$Mod(f^{-1}\mathcal{O}_Y) \to Mod(\mathcal{O}_X), \quad \mathcal{F} \longmapsto \mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Thus the result follows from the discussion following Definition 20.1. $\square$

Definition 20.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We say that $\mathcal{F}$ is *flat over Y at a point $x \in X$* if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module.
(2) We say that $\mathcal{F}$ is *flat over Y* if $\mathcal{F}$ is flat over Y at every point x of X.

With this definition we see that $\mathcal{F}$ is flat over Y at x if and only if $\mathcal{F}$ is flat at x as an $f^{-1}\mathcal{O}_Y$-module because $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,f(x)}$ by Sheaves, Lemma 21.5.

Lemma 20.4. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module flat over Y. Then the functor*

$$Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X), \quad \mathcal{G} \longmapsto f^*\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact.

Proof. This is true because $f^*\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{F}$, the functor f^{-1} is exact, and $\mathcal{F}$ is a flat $f^{-1}\mathcal{O}_Y$-module. $\square$

21. Symmetric and exterior powers

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the *tensor algebra* of $\mathcal{F}$ to be the sheaf of noncommutative $\mathcal{O}_X$-algebras

$$\mathrm{T}(\mathcal{F}) = \mathrm{T}_{\mathcal{O}_X}(\mathcal{F}) = \bigoplus\nolimits_{n \geq 0} \mathrm{T}^n(\mathcal{F}).$$

Here $\mathrm{T}^0(\mathcal{F}) = \mathcal{O}_X$, $\mathrm{T}^1(\mathcal{F}) = \mathcal{F}$ and for $n \geq 2$ we have

$$\mathrm{T}^n(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{F} \quad (n \text{ factors})$$

We define $\wedge(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections $s \otimes s$ of $\mathrm{T}^2(\mathcal{F})$ where s is a local section of $\mathcal{F}$. This is called the *exterior algebra* of $\mathcal{F}$. Similarly, we define $\mathrm{Sym}(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections of the form $s \otimes t - t \otimes s$ of $\mathrm{T}^2(\mathcal{F})$.

Both $\wedge(\mathcal{F})$ and $\mathrm{Sym}(\mathcal{F})$ are graded $\mathcal{O}_X$-algebras, with grading inherited from $\mathrm{T}(\mathcal{F})$. Moreover $\mathrm{Sym}(\mathcal{F})$ is commutative, and $\wedge(\mathcal{F})$ is graded commutative.

Lemma 21.1. *In the situation described above. The sheaf $\wedge^n\mathcal{F}$ is the sheafification of the presheaf*

$$U \longmapsto \wedge^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

See Algebra, Section 13. Similarly, the sheaf $Sym^n\mathcal{F}$ is the sheafification of the presheaf

$$U \longmapsto Sym^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

Proof. Omitted. It may be more efficient to define $\mathrm{Sym}(\mathcal{F})$ and $\wedge(\mathcal{F})$ in this way instead of the method given above. □

Lemma 21.2. *In the situation described above. Let $x \in X$. There are canonical isomorphisms of $\mathcal{O}_{X,x}$-modules $T(\mathcal{F})_x = T(\mathcal{F}_x)$, $Sym(\mathcal{F})_x = Sym(\mathcal{F}_x)$, and $\wedge(\mathcal{F})_x = \wedge(\mathcal{F}_x)$.*

Proof. Clear from Lemma 21.1 above, and Algebra, Lemma 13.5. □

Lemma 21.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^*T(\mathcal{F}) = T(f^*\mathcal{F})$, and similarly for the exterior and symmetric algebras associated to $\mathcal{F}$.*

Proof. Omitted. □

Lemma 21.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ be an exact sequence of sheaves of $\mathcal{O}_X$-modules. For each $n \geq 1$ there is an exact sequence*

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} Sym^{n-1}(\mathcal{F}_1) \to Sym^n(\mathcal{F}_1) \to Sym^n(\mathcal{F}) \to 0$$

and similarly an exact sequence

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \wedge^{n-1}(\mathcal{F}_1) \to \wedge^n(\mathcal{F}_1) \to \wedge^n(\mathcal{F}) \to 0$$

Proof. See Algebra, Lemma 13.2. □

Lemma 21.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

1. *If $\mathcal{F}$ is locally generated by sections, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
2. *If $\mathcal{F}$ is of finite type, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
3. *If $\mathcal{F}$ is of finite presentation, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*

(4) *If $\mathcal{F}$ is coherent, then for $n > 0$ each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$ is coherent.*
(5) *If $\mathcal{F}$ is quasi-coherent, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
(6) *If $\mathcal{F}$ is locally free, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*

Proof. These statements for $\mathrm{T}^n(\mathcal{F})$ follow from Lemma 16.6.

Statements (1) and (2) follow from the fact that $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$ are quotients of $\mathrm{T}^n(\mathcal{F})$.

Statement (6) follows from Algebra, Lemma 13.1.

For (3) and (5) we will use Lemma 21.4 above. By locally choosing a presentation $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ with $\mathcal{F}_i$ free, or finite free and applying the lemma we see that $\mathrm{Sym}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ has a similar presentation; here we use (6) and Lemma 16.6.

To prove (4) we will use Algebra, Lemma 13.3. We may localize on X and assume that $\mathcal{F}$ is generated by a finite set $(s_i)_{i\in I}$ of global sections. The lemma mentioned above combined with Lemma 21.1 above implies that for $n \geq 2$ there exists an exact sequence

$$\bigoplus\nolimits_{j\in J} \mathrm{T}^{n-2}(\mathcal{F}) \to \mathrm{T}^n(\mathcal{F}) \to \mathrm{Sym}^n(\mathcal{F}) \to 0$$

where the index set J is finite. Now we know that $\mathrm{T}^{n-2}(\mathcal{F})$ is finitely generated and hence the image of the first arrow is a coherent subsheaf of $\mathrm{T}^n(\mathcal{F})$, see Lemma 12.4. By that same lemma we conclude that $\mathrm{Sym}^n(\mathcal{F})$ is coherent. □

Lemma 21.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*
(1) *If $\mathcal{F}$ is quasi-coherent, then so is each $T(\mathcal{F})$, $\wedge(\mathcal{F})$, and $Sym(\mathcal{F})$.*
(2) *If $\mathcal{F}$ is locally free, then so is each $T(\mathcal{F})$, $\wedge(\mathcal{F})$, and $Sym(\mathcal{F})$.*

Proof. It is not true that an infinite direct sum $\bigoplus \mathcal{G}_i$ of locally free modules is locally free, or that an infinite direct sum of quasi-coherent modules is quasi-coherent. The problem is that given a point $x \in X$ the open neighbourhoods U_i of x on which $\mathcal{G}_i$ becomes free (resp. has a suitable presentation) may have an intersection which is not an open neighbourhood of x. However, in the proof of Lemma 21.5 we saw that once a suitable open neighbourhood for $\mathcal{F}$ has been chosen, then this open neighbourhood works for each of the sheaves $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$. The lemma follows. □

22. Internal Hom

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

It follows from the discussion in Sheaves, Section 33 that this is a sheaf of abelian groups. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}_X(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a sheaf of $\mathcal{O}_X$-modules. We will denote this sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

For every $x \in X$ there is also a canonical morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

which is rarely an isomorphism.

Lemma 22.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be $\mathcal{O}_X$-modules. There is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

which is functorial in all three entries (sheaf Hom in all three spots). In particular, to give a morphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{H}$ is the same as giving a morphism $\mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})$.

Proof. This is the analogue of Algebra, Lemma 12.8. The proof is the same, and is omitted. ☐

Lemma 22.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.*

(1) *If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, then*

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G})$$

is exact.

(2) *If $0 \to \mathcal{G} \to \mathcal{G}_1 \to \mathcal{G}_2$ is an exact sequence of $\mathcal{O}_X$-modules, then*

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_1) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_2)$$

is exact.

Proof. Let $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ be as in (1). For every $U \subset X$ open the sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_1|_U, \mathcal{G}|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_2|_U, \mathcal{G}|_U)$$

is exact by Homology, Lemma 5.8. This means that taking sections over U of the sequence of sheaves in (1) produces an exact sequence of abelian groups. Hence the sequence in (1) is exact by definition. The proof of (2) is exactly the same. ☐

Lemma 22.3. *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Then we have*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{F}_{\mathcal{O}_1}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G}))$$

bifunctorially in $\mathcal{F} \in Mod(\mathcal{O}_2)$ and $\mathcal{G} \in Mod(\mathcal{O}_1)$.

Proof. Omitted. This is the analogue of Algebra, Lemma 14.4 and is proved in exactly the same way. ☐

Lemma 22.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type then the canonical map*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

is injective. If $\mathcal{F}$ is finitely presented, this canonical morphism is an isomorphism.

Proof. The map sends the equivalence class of (U, φ) in $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$, where $x \in U \subset X$ is open and $\varphi \in \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$, to the the induced map on stalks at x, namely $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$.

Suppose $\mathcal{F}$ is of finite type. Pick a representative (U, φ) of an element σ in the kernel of the map, i.e., $\varphi_x = 0$. Shrinking U if necessary, choose sections $s^1, \ldots, s^n \in \mathcal{F}(U)$ generating $\mathcal{F}|_U$. Since $\varphi_x(s^i_x) = 0$ and we are dealing with a finite number of sections, we can find an open neighborhood $V \subset U$ of x such that $\varphi_V(s^i|_V) = 0$ for all $i = 1, \ldots, n$. Since $s^i|_V$, $i = 1, \ldots, n$ generate $\mathcal{F}|_V$ this means that $\varphi|_V = 0$. Since (U, φ) is equivalent to $(V, \varphi|_V)$ we conclude $\sigma = 0$ and injectivity of the map follows.

Next, assume $\mathcal{F}$ is finitely presented. By localizing on X we may assume that $\mathcal{F}$ has a presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to \mathcal{F} \to 0.$$

By Lemma 22.2 this gives an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \bigoplus_{i=1,\ldots,n} \mathcal{G} \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}$. Taking stalks we get an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ and the result follows since $\mathcal{F}_x$ sits in an exact sequence $\bigoplus_{j=1,\ldots,m} \mathcal{O}_{X,x} \longrightarrow \bigoplus_{i=1,\ldots,n} \mathcal{O}_{X,x} \to \mathcal{F}_x \to 0$ which induces the exact sequence $0 \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ which is the same as the one above. $\square$

Lemma 22.5. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_Y$-modules. If $\mathcal{F}$ is finitely presented and f is flat, then the canonical map*

$$f^* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G})$$

is an isomorphism.

Proof. Note that $f^*\mathcal{F}$ is also finitely presented (Lemma 11.4). Let $x \in X$ map to $y \in Y$. Looking at the stalks at x we get an isomorphism by Lemma 22.4 and More on Algebra, Lemma 65.4 to see that in this case Hom commutes with base change by $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Second proof: use the exact same argument as given in the proof of Lemma 22.4. $\square$

Lemma 22.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is finitely presented then the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is locally a kernel of a map between finite direct sums of copies of $\mathcal{G}$. In particular, if $\mathcal{G}$ is coherent then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent too.*

Proof. The first assertion we saw in the proof of Lemma 22.4. And the result for coherent sheaves then follows from Lemma 12.4. $\square$

Lemma 22.7. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Let $\mathcal{G} = \mathrm{colim}_{\lambda \in \Lambda} \mathcal{G}_\lambda$ be a filtered colimit of $\mathcal{O}_X$-modules. Then the canonical map*

$$\mathrm{colim}_\lambda \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_\lambda) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

is an isomorphism.

Proof. Taking colimits of sheaves of modules commutes with restriction to opens, see Sheaves, Section 29. Hence we may assume $\mathcal{F}$ has a global presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to \mathcal{F} \to 0$$

The functor $\mathcal{H}om_{\mathcal{O}_X}(-, -)$ commutes with finite direct sums in either variable and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, -)$ is the identity functor. By this and by Lemma 22.2 we obtain an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{G} \to \bigoplus\nolimits_{j=1,\ldots,m} \mathcal{G}$$

Since filtered colimits are exact in $Mod(\mathcal{O}_X)$ also the top row in the following commutative diagram is exact

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{colim}_\lambda \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_\lambda) \arrow[r] \arrow[d] & \mathrm{colim}_\lambda \bigoplus_{i=1,\ldots,n} \mathcal{G}_\lambda \arrow[r] \arrow[d] & \mathrm{colim}_\lambda \bigoplus_{j=1,\ldots,m} \mathcal{G}_\lambda \arrow[d] \\ 0 \arrow[r] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \arrow[r] & \bigoplus_{i=1,\ldots,n} \mathcal{G} \arrow[r] & \bigoplus_{j=1,\ldots,m} \mathcal{G} \end{tikzcd}$$

Since the right two vertical arrows are isomorphisms we conclude. $\square$

Lemma 22.8. *Let X be a ringed space. Let I be a preordered set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 21). Assume*

1. *I is directed,*
2. *$\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation, and*
3. *X has a cofinal system of open coverings $\mathcal{U} : X = \bigcup_{j \in J} U_j$ with J finite and $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$.*

Then we have

$$\mathrm{colim}_i \mathrm{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \mathrm{Hom}_X(\mathcal{G}, \mathrm{colim}_i \mathcal{F}_i).$$

Proof. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathrm{colim}\, \mathcal{F}_i)$ and $\mathcal{H}_i = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}_i)$. Recall that

$$\mathrm{Hom}_X(\mathcal{G}, \mathcal{F}) = \Gamma(X, \mathcal{H}) \quad \text{and} \quad \mathrm{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \Gamma(X, \mathcal{H}_i)$$

by construction. By Lemma 22.7 we have $\mathcal{H} = \mathrm{colim}\, \mathcal{H}_i$. Thus the lemma follows from Sheaves, Lemma 29.1. $\square$

Remark 22.9. In the lemma above some condition beyond the condition that X is quasi-compact is necessary. See Sheaves, Example 29.2.

23. The annihilator of a sheaf of modules

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. There is a canonical map of sheaves of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

which sends a local section $f \in \mathcal{O}_X(U)$ to the map $f : \mathcal{F}|_U \to \mathcal{F}|_U$ given by multiplication by f.

Definition 23.1. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The *annihilator* of $\mathcal{F}$, denoted $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})$ is the kernel of the map $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ discussed above.

For each $x \in X$, there is an inclusion of ideals of $\mathcal{O}_{X,x}$:

$$(\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F}))_x \subset \mathrm{Ann}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \tag{23.1.1}$$

since after all any section of $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})$ will annihilate the stalks of $\mathcal{F}$ at all points at which it is defined. Here is a simple situation in which (23.1.1) becomes an equality.

Lemma 23.2. *Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type, then $(Ann_{\mathcal{O}_X}(\mathcal{F}))_x = Ann_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$.*

Proof. By Lemma 22.4 the map

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_x \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{F}_x)$$

is injective. Thus any section f of $\mathcal{O}_X$ over an open neighbourhood U of x which acts as zero on $\mathcal{F}_x$ will act as zero on $\mathcal{F}|_V$ for some $U \supset V \ni x$ open. Hence the inclusion (23.1.1) is an equality. $\square$

Lemma 23.3. *Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf. If $\mathcal{I} \subset Ann_{\mathcal{O}_X}(\mathcal{F})$, then $\mathcal{F}$ has a natural $\mathcal{O}_X/\mathcal{I}$-module structure which agrees with the usual commutative algebra construction on stalks.*

Proof. Applying the universal property of the cokernel of the inclusion $\mathcal{I} \to \mathcal{O}_X$, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \\ \mathcal{O}_X/\mathcal{I} \arrow[ru, dashed] & \end{tikzcd}$$

of $\mathcal{O}_X$-modules. By Lemma 22.1 the resulting map $\mathcal{O}_X/\mathcal{I} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ corresponds to a map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X/\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F}$$

which means we have an $\mathcal{O}_X/\mathcal{I}$-module structure on $\mathcal{F}$ compatible with the given $\mathcal{O}_X$-module structure. We omit the verification of the statement on stalks. $\square$

Lemma 23.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{O}_X$ and $\mathcal{F}$ are coherent, then so is $Ann_{\mathcal{O}_X}(\mathcal{F})$.*

Proof. Since $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})$ is the kernel of $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ by Lemma 12.4 it suffices to show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ is coherent. This follows from Lemma 22.6 and the fact that $\mathcal{F}$ is coherent and a fortiori finitely presented (Lemma 12.2). $\square$

24. Koszul complexes

We suggest first reading the section on Koszul complexes in More on Algebra, Section 28. We define the Koszul complex in the category of $\mathcal{O}_X$-modules as follows.

Definition 24.1. Let X be a ringed space. Let $\varphi : \mathcal{E} \to \mathcal{O}_X$ be an $\mathcal{O}_X$-module map. The *Koszul complex* $K_\bullet(\varphi)$ associated to φ is the sheaf of commutative differential graded algebras defined as follows:

1. the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(\mathcal{E})$,
2. the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all local sections e of $\mathcal{E} = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is a wedge product of local sections of $\mathcal{E}$, then

$$d(e_1 \wedge \ldots \wedge e_n) = \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n.$$

It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \wedge e$ and hence factors through the exterior algebra.

Definition 24.2. Let X be a ringed space and let $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$. The *Koszul complex on* $f_1, \ldots, f_r$ is the Koszul complex associated to the map $(f_1, \ldots, f_n) : \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X$. Notation $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$, or $K_\bullet(\mathcal{O}_X, f_\bullet)$.

Of course, given an $\mathcal{O}_X$-module map $\varphi : \mathcal{E} \to \mathcal{O}_X$, if $\mathcal{E}$ is finite locally free, then $K_\bullet(\varphi)$ is locally on X isomorphic to a Koszul complex $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$.

25. Invertible modules

Similarly to the case of modules over rings (More on Algebra, Section 117) we have the following definition.

Definition 25.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An *invertible $\mathcal{O}_X$-module* is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that the functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{F} \longmapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an equivalence of categories. We say that $\mathcal{L}$ is *trivial* if it is isomorphic as an $\mathcal{O}_X$-module to $\mathcal{O}_X$.

Lemma 25.4 below explains the relationship with locally free modules of rank 1.

Lemma 25.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. Equivalent are*

1. *$\mathcal{L}$ is invertible, and*
2. *there exists an $\mathcal{O}_X$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \cong \mathcal{O}_X$.*

In this case $\mathcal{L}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module and the module $\mathcal{N}$ in (2) is isomorphic to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

Proof. Assume (1). Then the functor $- \otimes_{\mathcal{O}_X} \mathcal{L}$ is essentially surjective, hence there exists an $\mathcal{O}_X$-module $\mathcal{N}$ as in (2). If (2) holds, then the functor $- \otimes_{\mathcal{O}_X} \mathcal{N}$ is a quasi-inverse to the functor $- \otimes_{\mathcal{O}_X} \mathcal{L}$ and we see that (1) holds.

Assume (1) and (2) hold. Denote $\psi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \to \mathcal{O}_X$ the given isomorphism. Let $x \in X$. Choose an open neighbourhood U an integer $n \geq 1$ and sections $s_i \in \mathcal{L}(U)$, $t_i \in \mathcal{N}(U)$ such that $\psi(\sum s_i \otimes t_i) = 1$. Consider the isomorphisms

$$\mathcal{L}|_U \to \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{N}|_U \to \mathcal{L}|_U$$

where the first arrow sends s to $\sum s_i \otimes s \otimes t_i$ and the second arrow sends $s \otimes s' \otimes t$ to $\psi(s' \otimes t)s$. We conclude that $s \mapsto \sum \psi(s \otimes t_i)s_i$ is an automorphism of $\mathcal{L}|_U$. This automorphism factors as

$$\mathcal{L}|_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{L}|_U$$

where the first arrow is given by $s \mapsto (\psi(s \otimes t_1), \ldots, \psi(s \otimes t_n))$ and the second arrow by $(a_1, \ldots, a_n) \mapsto \sum a_i s_i$. In this way we conclude that $\mathcal{L}|_U$ is a direct summand of a finite free $\mathcal{O}_U$-module.

Assume (1) and (2) hold. Consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

To finish the proof of the lemma we will show this is an isomorphism by checking it induces isomorphisms on stalks. Let $x \in X$. Since we know (by the previous paragraph) that $\mathcal{L}$ is a finitely presented $\mathcal{O}_X$-module we can use Lemma 22.4 to see that it suffices to show that

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{L}_x, \mathcal{O}_{X,x}) \longrightarrow \mathcal{O}_{X,x}$$

is an isomorphism. Since $\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{N}_x = (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N})_x = \mathcal{O}_{X,x}$ (Lemma 16.1) the desired result follows from More on Algebra, Lemma 117.2. $\square$

Lemma 25.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{L}$ of an invertible $\mathcal{O}_Y$-module is invertible.*

Proof. By Lemma 25.2 there exists an $\mathcal{O}_Y$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{N} \cong \mathcal{O}_Y$. Pulling back we get $f^*\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N} \cong \mathcal{O}_X$ by Lemma 16.4. Thus $f^*\mathcal{L}$ is invertible by Lemma 25.2. □

Lemma 25.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Any locally free $\mathcal{O}_X$-module of rank 1 is invertible. If all stalks $\mathcal{O}_{X,x}$ are local rings, then the converse holds as well (but in general this is not the case).*

Proof. The parenthetical statement follows by considering a one point space X with sheaf of rings $\mathcal{O}_X$ given by a ring R. Then invertible $\mathcal{O}_X$-modules correspond to invertible R-modules, hence as soon as $\mathrm{Pic}(R)$ is not the trivial group, then we get an example.

Assume $\mathcal{L}$ is locally free of rank 1 and consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

Looking over an open covering trivialization $\mathcal{L}$, we see that this map is an isomorphism. Hence $\mathcal{L}$ is invertible by Lemma 25.2.

Assume all stalks $\mathcal{O}_{X,x}$ are local rings and $\mathcal{L}$ invertible. In the proof of Lemma 25.2 we have seen that $\mathcal{L}_x$ is an invertible $\mathcal{O}_{X,x}$-module for all $x \in X$. Since $\mathcal{O}_{X,x}$ is local, we see that $\mathcal{L}_x \cong \mathcal{O}_{X,x}$ (More on Algebra, Section 117). Since $\mathcal{L}$ is of finite presentation by Lemma 25.2 we conclude that $\mathcal{L}$ is locally free of rank 1 by Lemma 11.6. □

Lemma 25.5. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *If $\mathcal{L}$, $\mathcal{N}$ are invertible $\mathcal{O}_X$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$.*
(2) *If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then so is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ and the evaluation map $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \to \mathcal{O}_X$ is an isomorphism.*

Proof. Part (1) is clear from the definition and part (2) follows from Lemma 25.2 and its proof. □

Definition 25.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Given an invertible sheaf $\mathcal{L}$ on X and $n \in \mathbf{Z}$ we define the nth *tensor power* $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$ as the image of $\mathcal{O}_X$ under applying the equivalence $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ exactly n times.

This makes sense also for negative n as we've defined an invertible $\mathcal{O}_X$-module as one for which tensoring is an equivalence. More explicitly, we have

$$\mathcal{L}^{\otimes n} = \begin{cases} \mathcal{O}_X & \text{if} \quad n = 0 \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) & \text{if} \quad n = -1 \\ \mathcal{L} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L} & \text{if} \quad n > 0 \\ \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} & \text{if} \quad n < -1 \end{cases}$$

see Lemma 25.5. With this definition we have canonical isomorphisms $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes n+m}$, and these isomorphisms satisfy a commutativity and an associativity constraint (formulation omitted).

Let $(X, \mathcal{O}_X)$ be a ringed space. We can define a $\mathbf{Z}$-graded ring structure on $\bigoplus \Gamma(X, \mathcal{L}^{\otimes n})$ by mapping $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $t \in \Gamma(X, \mathcal{L}^{\otimes m})$ to the section corresponding to $s \otimes t$ in $\Gamma(X, \mathcal{L}^{\otimes n+m})$. We omit the verification that this defines a commutative and associative ring with 1. However, by our conventions in Algebra, Section 56 a graded ring has no nonzero elements in negative degrees. This leads to the following definition.

Definition 25.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Given an invertible sheaf $\mathcal{L}$ on X we define the *associated graded ring* to be

$$\Gamma_*(X, \mathcal{L}) = \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which we think of as a graded $\Gamma_*(X, \mathcal{L})$-module.

We often write simply $\Gamma_*(\mathcal{L})$ and $\Gamma_*(\mathcal{F})$ (although this is ambiguous if $\mathcal{F}$ is invertible). The multiplication of $\Gamma_*(\mathcal{L})$ on $\Gamma_*(\mathcal{F})$ is defined using the isomorphisms above. If $\gamma : \mathcal{F} \to \mathcal{G}$ is a $\mathcal{O}_X$-module map, then we get an $\Gamma_*(\mathcal{L})$-module homomorphism $\gamma : \Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$. If $\alpha : \mathcal{L} \to \mathcal{N}$ is an $\mathcal{O}_X$-module map between invertible $\mathcal{O}_X$-modules, then we obtain a graded ring homomorphism $\Gamma_*(\mathcal{L}) \to \Gamma_*(\mathcal{N})$. If $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces and if $\mathcal{L}$ is invertible on X, then we get an invertible sheaf $f^*\mathcal{L}$ on Y (Lemma 25.3) and an induced homomorphism of graded rings

$$f^* : \Gamma_*(X, \mathcal{L}) \longrightarrow \Gamma_*(Y, f^*\mathcal{L})$$

Furthermore, there are some compatibilities between the constructions above whose statements we omit.

Lemma 25.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. There exists a set of invertible modules $\{\mathcal{L}_i\}_{i \in I}$ such that each invertible module on X is isomorphic to exactly one of the $\mathcal{L}_i$.*

Proof. Recall that any invertible $\mathcal{O}_X$-module is locally a direct summand of a finite free $\mathcal{O}_X$-module, see Lemma 25.2. For each open covering $\mathcal{U} : X = \bigcup_{j \in J} U_j$ and map $r : J \to \mathbf{N}$ consider the sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ such that $\mathcal{F}_j = \mathcal{F}|_{U_j}$ is a direct summand of $\mathcal{O}_{U_j}^{\oplus r(j)}$. The collection of isomorphism classes of $\mathcal{F}_j$ is a set, because $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus r}, \mathcal{O}_U^{\oplus r})$ is a set. The sheaf $\mathcal{F}$ is gotten by glueing $\mathcal{F}_j$, see Sheaves, Section 33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j \in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. For each covering there is a set of maps $r : J \to \mathbf{N}$. Hence the collection of all $\mathcal{F}$ forms a set. $\square$

This lemma says roughly speaking that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 25.5 says that tensor product defines the structure of an abelian group on this set.

Definition 25.9. Let $(X, \mathcal{O}_X)$ be a ringed space. The *Picard group* $\mathrm{Pic}(X)$ of X is the abelian group whose elements are isomorphism classes of invertible $\mathcal{O}_X$-modules, with addition corresponding to tensor product.

Lemma 25.10. *Let X be a ringed space. Assume that each stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For any section $s \in \Gamma(X, \mathcal{L})$ the set*

$$X_s = \{x \in X \mid \textit{image } s \not\in \mathfrak{m}_x\mathcal{L}_x\}$$

is open in X. The map $s : \mathcal{O}_{X_s} \to \mathcal{L}|_{X_s}$ is an isomorphism, and there exists a section s' of $\mathcal{L}^{\otimes -1}$ over X_s such that $s'(s|_{X_s}) = 1$.

Proof. Suppose $x \in X_s$. We have an isomorphism

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{L}^{\otimes -1})_x \longrightarrow \mathcal{O}_{X,x}$$

by Lemma 25.5. Both $\mathcal{L}_x$ and $(\mathcal{L}^{\otimes -1})_x$ are free $\mathcal{O}_{X,x}$-modules of rank 1. We conclude from Algebra, Nakayama's Lemma 20.1 that s_x is a basis for $\mathcal{L}_x$. Hence there exists a basis element $t_x \in (\mathcal{L}^{\otimes -1})_x$ such that $s_x \otimes t_x$ maps to 1. Choose an open neighbourhood U of x such that t_x comes from a section t of $\mathcal{L}^{\otimes -1}$ over U and such that $s \otimes t$ maps to $1 \in \mathcal{O}_X(U)$. Clearly, for every $x' \in U$ we see that s generates the module $\mathcal{L}_{x'}$. Hence $U \subset X_s$. This proves that X_s is open. Moreover, the section t constructed over U above is unique, and hence these glue to give the section s' of the lemma. $\square$

It is also true that, given a morphism of locally ringed spaces $f : Y \to X$ (see Schemes, Definition 2.1) that the inverse image $f^{-1}(X_s)$ is equal to Y_{f^*s}, where $f^*s \in \Gamma(Y, f^*\mathcal{L})$ is the pullback of s.

26. Rank and determinant

Let $(X, \mathcal{O}_X)$ be a ringed space. Consider the category *Vect*(X) of finite locally free $\mathcal{O}_X$-modules. This is an exact category (see Injectives, Remark 9.6) whose admissible epimorphisms are surjections and whose admissible monomorphisms are kernels of surjections. Moreover, there is a set of isomorphism classes of objects of *Vect*(X) (proof omitted). Thus we can form the zeroth Grothendieck K-group $K_0(\mathit{Vect}(X))$. Explicitly, in this case $K_0(\mathit{Vect}(X))$ is the abelian group generated by $[\mathcal{E}]$ for $\mathcal{E}$ a finite locally free $\mathcal{O}_X$-module, subject to the relations

$$[\mathcal{E}'] = [\mathcal{E}] + [\mathcal{E}'']$$

whenever there is a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of finite locally free $\mathcal{O}_X$-modules.

Ranks. Assume all stalks $\mathcal{O}_{X,x}$ are nonzero rings. Given a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$, the *rank* is a locally constant function

$$\mathrm{rank}_\mathcal{E} : X \longrightarrow \mathbf{Z}_{\geq 0}, \quad x \longmapsto \mathrm{rank}_{\mathcal{O}_{X,x}} \mathcal{E}_x$$

See Lemma 14.4. By definition of locally free modules the function $\mathrm{rank}_\mathcal{E}$ is locally constant. If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a short exact sequence of finite locally free $\mathcal{O}_X$-modules, then $\mathrm{rank}_\mathcal{E} = \mathrm{rank}_{\mathcal{E}'} + \mathrm{rank}_{\mathcal{E}''}$, Thus the rank defines a homomorphism

$$K_0(\mathit{Vect}(X)) \longrightarrow \mathrm{Map}_{cont}(X, \mathbf{Z}), \quad [\mathcal{E}] \longmapsto \mathrm{rank}_\mathcal{E}$$

Determinants. Given a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ we obtain a disjoint union decomposition

$$X = X_0 \amalg X_1 \amalg X_2 \amalg \ldots$$

with X_i open and closed, such that $\mathcal{E}$ is finite locally free of rank i on X_i (this is exactly the same as saying the $\mathrm{rank}_\mathcal{E}$ is locally constant). In this case we define $\det(\mathcal{E})$ as the invertible sheaf on X which is equal to $\wedge^i(\mathcal{E}|_{X_i})$ on X_i for all $i \geq 0$. Since the decomposition above is disjoint, there are no glueing conditions to check. By Lemma 26.1 below this defines a homomorphism

$$\det : K_0(\mathit{Vect}(X)) \longrightarrow \mathrm{Pic}(X), \quad [\mathcal{E}] \longmapsto \det(\mathcal{E})$$

of abelian groups. The elements of $\mathrm{Pic}(X)$ we get in this manner are locally free of rank 1 (see below the lemma for a generalization).

Lemma 26.1. *Let X be a ringed space. Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be a short exact sequence of finite locally free $\mathcal{O}_X$-modules. Then there is a canonical isomorphism*

$$\det(\mathcal{E}') \otimes_{\mathcal{O}_X} \det(\mathcal{E}'') \longrightarrow \det(\mathcal{E})$$

of $\mathcal{O}_X$-modules.

Proof. We can decompose X into disjoint open and closed subsets such that both $\mathcal{E}'$ and $\mathcal{E}''$ have constant rank on them. Thus we reduce to the case where $\mathcal{E}'$ and $\mathcal{E}''$ have constant rank, say r' and r''. In this situation we define

$$\wedge^{r'}(\mathcal{E}') \otimes_{\mathcal{O}_X} \wedge^{r''}(\mathcal{E}'') \longrightarrow \wedge^{r'+r''}(\mathcal{E})$$

as follows. Given local sections $s'_1, \ldots, s'_{r'}$ of $\mathcal{E}'$ and local sections $s''_1, \ldots, s''_{r''}$ of $\mathcal{E}''$ we map

$$s'_1 \wedge \ldots \wedge s'_{r'} \otimes s''_1 \wedge \ldots \wedge s''_{r''} \quad \text{to} \quad s'_1 \wedge \ldots \wedge s'_{r'} \wedge \tilde{s}''_1 \wedge \ldots \wedge \tilde{s}''_{r''}$$

where $\tilde{s}''_i$ is a local lift of the section s''_i to a section of $\mathcal{E}$. We omit the details. $\square$

Let $(X, \mathcal{O}_X)$ be a ringed space. Instead of looking at finite locally free $\mathcal{O}_X$-modules we can look at those $\mathcal{O}_X$-modules $\mathcal{F}$ which are locally on X a direct summand of a finite free $\mathcal{O}_X$-module. This is the same thing as asking $\mathcal{F}$ to be a flat $\mathcal{O}_X$-module of finite presentation, see Lemma 18.3. If all the stalks $\mathcal{O}_{X,x}$ are local, then such a module $\mathcal{F}$ is finite locally free, see Lemma 14.6. In general however this will not be the case; for example X could be a point and $\Gamma(X, \mathcal{O}_X)$ could be the product $A \times B$ of two nonzero rings and $\mathcal{F}$ could correspond to $A \times 0$. Thus for such a module the rank function is undefined. However, it turns out we can still define $\det(\mathcal{F})$ and this will be an invertible $\mathcal{O}_X$-module in the sense of Definition 25.1 (not necessarily locally free of rank 1). Our construction will agree with the one above in the case that $\mathcal{F}$ is finite locally free. We urge the reader to skip the rest of this section.

Lemma 26.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a flat and finitely presented $\mathcal{O}_X$-module. Denote*

$$\det(\mathcal{F}) \subset \wedge^*_{\mathcal{O}_X}(\mathcal{F})$$

the annihilator of $\mathcal{F} \subset \wedge^*_{\mathcal{O}_X}(\mathcal{F})$. Then $\det(\mathcal{F})$ is an invertible $\mathcal{O}_X$-module.

Proof. To prove this we may work locally on X. Hence we may assume $\mathcal{F}$ is a direct summand of a finite free module, see Lemma 18.3. Say $\mathcal{F} \oplus \mathcal{G} = \mathcal{O}_X^{\oplus n}$. Set $R = \mathcal{O}_X(X)$. Then we see $\mathcal{F}(X) \oplus \mathcal{G}(X) = R^{\oplus n}$ and correspondingly $\mathcal{F}(U) \oplus \mathcal{G}(U) = \mathcal{O}_X(U)^{\oplus n}$ for all opens $U \subset X$. We conclude that $\mathcal{F} = \mathcal{F}_M$ as in Lemma 10.5 with $M = \mathcal{F}(X)$ a finite projective R-module. In other words, we have $\mathcal{F}(U) = M \otimes_R \mathcal{O}_X(U)$. This implies that $\det(M) \otimes_R \mathcal{O}_X(U) = \det(\mathcal{F}(U))$ for all open $U \subset X$ with det as in More on Algebra, Section 118. By More on Algebra, Remark 118.1 we see that

$$\det(M) \otimes_R \mathcal{O}_X(U) = \det(\mathcal{F}(U)) \subset \wedge^*_{\mathcal{O}_X(U)}(\mathcal{F}(U))$$

is the annihilator of $\mathcal{F}(U)$. We conclude that $\det(\mathcal{F})$ as defined in the statement of the lemma is equal to $\mathcal{F}_{\det(M)}$. Some details omitted; one has to be careful as annihilators cannot be defined as the sheafification of taking annihilators on sections over opens. Thus $\det(\mathcal{F})$ is the pullback of an invertible module and we conclude. $\square$

27. Localizing sheaves of rings

Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset, see Algebra, Definition 9.1. In this case we can consider the presheaf of rings

$$\mathcal{S}^{-1}\mathcal{O}_X : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U).$$

The restriction mapping sends the section f/s, $f \in \mathcal{O}_X(U)$, $s \in \mathcal{S}(U)$ to $(f|_V)/(s|_V)$ if $V \subset U$ are opens of X.

Lemma 27.1. *Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset.*

1. *There is a map of presheaves of rings $\mathcal{O}_X \to \mathcal{S}^{-1}\mathcal{O}_X$ such that every local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{O}_X$.*
2. *For any homomorphism of presheaves of rings $\mathcal{O}_X \to \mathcal{A}$ such that each local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{A}$ there exists a unique factorization $\mathcal{S}^{-1}\mathcal{O}_X \to \mathcal{A}$.*
3. *For any $x \in X$ we have*
$$(\mathcal{S}^{-1}\mathcal{O}_X)_x = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$
4. *The sheafification $(\mathcal{S}^{-1}\mathcal{O}_X)^\#$ is a sheaf of rings with a map of sheaves of rings $(\mathcal{O}_X)^\# \to (\mathcal{S}^{-1}\mathcal{O}_X)^\#$ which is universal for maps of $(\mathcal{O}_X)^\#$ into sheaves of rings such that each local section of $\mathcal{S}$ maps to an invertible section.*
5. *For any $x \in X$ we have*
$$(\mathcal{S}^{-1}\mathcal{O}_X)^\#_x = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$

Proof. Omitted. ☐

Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules In this case we can consider the presheaf of $\mathcal{S}^{-1}\mathcal{O}_X$-modules

$$\mathcal{S}^{-1}\mathcal{F} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{F}(U).$$

The restriction mapping sends the section t/s, $t \in \mathcal{F}(U)$, $s \in \mathcal{S}(U)$ to $(t|_V)/(s|_V)$ if $V \subset U$ are opens of X.

Lemma 27.2. *Let X be a topological space. Let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. For any presheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have*

$$\mathcal{S}^{-1}\mathcal{F} = \mathcal{S}^{-1}\mathcal{O}_X \otimes_{p,\mathcal{O}_X} \mathcal{F}$$

(see Sheaves, Section 6 for notation) and if $\mathcal{F}$ and $\mathcal{O}_X$ are sheaves then

$$(\mathcal{S}^{-1}\mathcal{F})^\# = (\mathcal{S}^{-1}\mathcal{O}_X)^\# \otimes_{\mathcal{O}_X} \mathcal{F}$$

(see Sheaves, Section 20 for notation).

Proof. Omitted. ☐

28. Modules of differentials

In this section we briefly explain how to define the module of relative differentials for a morphism of ringed spaces. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 131).

Definition 28.1. Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be an $\mathcal{O}_2$-module. An $\mathcal{O}_1$*-derivation* or more precisely a φ*-derivation* into $\mathcal{F}$ is a map $D : \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1 \to \mathcal{O}_2$, and satisfies the *Leibniz rule*

$$D(ab) = aD(b) + D(a)b$$

for all a, b local sections of $\mathcal{O}_2$ (wherever they are both defined). We denote $\text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ the set of φ-derivations into $\mathcal{F}$.

This is the sheaf theoretic analogue of Algebra, Definition 131.1. Given a derivation $D : \mathcal{O}_2 \to \mathcal{F}$ as in the definition the map on global sections

$$D : \Gamma(X, \mathcal{O}_2) \longrightarrow \Gamma(X, \mathcal{F})$$

is a $\Gamma(X, \mathcal{O}_1)$-derivation as in the algebra definition. Note that if $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_2$-modules, then there is an induced map

$$\text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \longrightarrow \text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G})$$

given by the rule $D \mapsto \alpha \circ D$. In other words we obtain a functor.

Lemma 28.2. *Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The functor*

$$\textit{Mod}(\mathcal{O}_2) \longrightarrow \textit{Ab}, \quad \mathcal{F} \longmapsto \textit{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$$

is representable.

Proof. This is proved in exactly the same way as the analogous statement in algebra. During this proof, for any sheaf of sets $\mathcal{F}$ on X, let us denote $\mathcal{O}_2[\mathcal{F}]$ the sheafification of the presheaf $U \mapsto \mathcal{O}_2(U)[\mathcal{F}(U)]$ where this denotes the free $\mathcal{O}_2(U)$-module on the set $\mathcal{F}(U)$. For $s \in \mathcal{F}(U)$ we denote $[s]$ the corresponding section of $\mathcal{O}_2[\mathcal{F}]$ over U. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules, then there is a canonical map

$$c : \mathcal{O}_2[\mathcal{F}] \longrightarrow \mathcal{F}$$

which on the presheaf level is given by the rule $\sum f_s[s] \mapsto \sum f_s s$. We will employ the short hand $[s] \mapsto s$ to describe this map and similarly for other maps below. Consider the map of $\mathcal{O}_2$-modules

$$\begin{matrix} \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] & \longrightarrow & \mathcal{O}_2[\mathcal{O}_2] \\ [(a, b)] \oplus [(f, g)] \oplus [h] & \longmapsto & [a + b] - [a] - [b] + \\ & & [fg] - g[f] - f[g] + \\ & & [\varphi(h)] \end{matrix} \tag{28.2.1}$$

with short hand notation as above. Set $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ equal to the cokernel of this map. Then it is clear that there exists a map of sheaves of sets

$$\text{d} : \mathcal{O}_2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

mapping a local section f to the image of $[f]$ in $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$. By construction d is a $\mathcal{O}_1$-derivation. Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules and let $D : \mathcal{O}_2 \to \mathcal{F}$ be a $\mathcal{O}_1$-derivation. Then we can consider the $\mathcal{O}_2$-linear map $\mathcal{O}_2[\mathcal{O}_2] \to \mathcal{F}$ which sends

$[g]$ to $D(g)$. It follows from the definition of a derivation that this map annihilates sections in the image of the map (28.2.1) and hence defines a map

$$\alpha_D : \Omega_{\mathcal{O}_2/\mathcal{O}_1} \longrightarrow \mathcal{F}$$

Since it is clear that $D = \alpha_D \circ \mathrm{d}$ the lemma is proved. $\square$

Definition 28.3. Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. The *module of differentials* of φ is the object representing the functor $\mathcal{F} \mapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ which exists by Lemma 28.2. It is denoted $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$, and the *universal φ-derivation* is denoted $\mathrm{d} : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$.

Note that $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (28.2.1) of $\mathcal{O}_2$-modules. Moreover the map d is described by the rule that $\mathrm{d}f$ is the image of the local section $[f]$.

Lemma 28.4. *Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Then $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the sheaf associated to the presheaf $U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$.*

Proof. Consider the map (28.2.1). There is a similar map of presheaves whose value on the open U is

$$\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_1(U)] \longrightarrow \mathcal{O}_2(U)[\mathcal{O}_2(U)]$$

The cokernel of this map has value $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$ over U by the construction of the module of differentials in Algebra, Definition 131.2. On the other hand, the sheaves in (28.2.1) are the sheafifications of the presheaves above. Thus the result follows as sheafification is exact. $\square$

Lemma 28.5. *Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. For $U \subset X$ open there is a canonical isomorphism*

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}|_U = \Omega_{(\mathcal{O}_2|_U)/(\mathcal{O}_1|_U)}$$

compatible with universal derivations.

Proof. Holds because $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (28.2.1). $\square$

Lemma 28.6. *Let $f : Y \to X$ be a continuous map of topological spaces. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Then there is a canonical identification $f^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} = \Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$ compatible with universal derivations.*

Proof. This holds because the sheaf $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (28.2.1) and a similar statement holds for $\Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$, because the functor f^{-1} is exact, and because $f^{-1}(\mathcal{O}_2[\mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2]$, $f^{-1}(\mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2 \times f^{-1}\mathcal{O}_2]$, and $f^{-1}(\mathcal{O}_2[\mathcal{O}_1]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_1]$. $\square$

Lemma 28.7. *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Let $x \in X$. Then we have $\Omega_{\mathcal{O}_2/\mathcal{O}_1,x} = \Omega_{\mathcal{O}_{2,x}/\mathcal{O}_{1,x}}$.*

Proof. This is a special case of Lemma 28.6 for the inclusion map $\{x\} \to X$. An alternative proof is to use Lemma 28.4, Sheaves, Lemma 17.2, and Algebra, Lemma 131.5 $\square$

Lemma 28.8. *Let X be a topological space. Let*

$$\begin{tikzcd} \mathcal{O}_2 \arrow[r, "\varphi"'] & \mathcal{O}'_2 \\ \mathcal{O}_1 \arrow[u] \arrow[r] & \mathcal{O}'_1 \arrow[u] \end{tikzcd}$$

be a commutative diagram of sheaves of rings on X. The map $\mathcal{O}_2 \to \mathcal{O}'_2$ composed with the map $d : \mathcal{O}'_2 \to \Omega_{\mathcal{O}'_2/\mathcal{O}'_1}$ is a $\mathcal{O}_1$-derivation. Hence we obtain a canonical map of $\mathcal{O}_2$-modules $\Omega_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega_{\mathcal{O}'_2/\mathcal{O}'_1}$. It is uniquely characterized by the property that $d(f) \mapsto d(\varphi(f))$ for any local section f of $\mathcal{O}_2$. In this way $\Omega_{-/-}$ becomes a functor on the category of arrows of sheaves of rings.

Proof. This lemma proves itself. $\square$

Lemma 28.9. *In Lemma 28.8 suppose that $\mathcal{O}_2 \to \mathcal{O}'_2$ is surjective with kernel $\mathcal{I} \subset \mathcal{O}_2$ and assume that $\mathcal{O}_1 = \mathcal{O}'_1$. Then there is a canonical exact sequence of $\mathcal{O}'_2$-modules*

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}'_2 \longrightarrow \Omega_{\mathcal{O}'_2/\mathcal{O}_1} \longrightarrow 0$$

The leftmost map is characterized by the rule that a local section f of $\mathcal{I}$ maps to $df \otimes 1$.

Proof. For a local section f of $\mathcal{I}$ denote $\overline{f}$ the image of f in $\mathcal{I}/\mathcal{I}^2$. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1f_2 \otimes 1 = 0$ if f_1, f_2 are local sections of $\mathcal{I}$. And this is clear from the Leibniz rule $\mathrm{d}f_1f_2 \otimes 1 = (f_1\mathrm{d}f_2 + f_2\mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_1 \otimes f_2 = 0$. A similar computation show this map is $\mathcal{O}'_2 = \mathcal{O}_2/\mathcal{I}$-linear. The map on the right is the one from Lemma 28.8. To see that the sequence is exact, we can check on stalks (Lemma 3.1). By Lemma 28.7 this follows from Algebra, Lemma 131.9. $\square$

Definition 28.10. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces.

(1) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. An *S-derivation* into $\mathcal{F}$ is a $f^{-1}\mathcal{O}_S$-derivation, or more precisely a $f^\sharp$-derivation in the sense of Definition 28.1. We denote $\mathrm{Der}_S(\mathcal{O}_X, \mathcal{F})$ the set of S-derivations into $\mathcal{F}$.
(2) The *sheaf of differentials* $\Omega_{X/S}$ *of X over S* is the module of differentials $\Omega_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}$ endowed with its universal S-derivation $\mathrm{d}_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$.

Here is a particular situation where derivations come up naturally.

Lemma 28.11. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

Here $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\pi : \mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I} = \mathrm{Ker}(\pi)$ is its kernel. Assume $\mathcal{I}$ an ideal sheaf with square zero in $\mathcal{A}$. So $\mathcal{I}$ has a natural structure of an $\mathcal{O}_X$-module. A section $s : \mathcal{O}_X \to \mathcal{A}$ of π is a $f^{-1}\mathcal{O}_S$-algebra map such that $\pi \circ s = id$. Given any section $s : \mathcal{O}_X \to \mathcal{A}$ of π and any S-derivation $D : \mathcal{O}_X \to \mathcal{I}$ the map

$$s + D : \mathcal{O}_X \to \mathcal{A}$$

is a section of π and every section s' is of the form $s + D$ for a unique S-derivation D.

Proof. Recall that the $\mathcal{O}_X$-module structure on $\mathcal{I}$ is given by $h\tau = \tilde{h}\tau$ (multiplication in $\mathcal{A}$) where h is a local section of $\mathcal{O}_X$, and $\tilde{h}$ is a local lift of h to a local section of $\mathcal{A}$, and τ is a local section of $\mathcal{I}$. In particular, given s, we may use $\tilde{h} = s(h)$. To verify that $s + D$ is a homomorphism of sheaves of rings we compute

$$\begin{aligned}(s+D)(ab) &= s(ab) + D(ab) \\ &= s(a)s(b) + aD(b) + D(a)b \\ &= s(a)s(b) + s(a)D(b) + D(a)s(b) \\ &= (s(a) + D(a))(s(b) + D(b))\end{aligned}$$

by the Leibniz rule. In the same manner one shows $s + D$ is a $f^{-1}\mathcal{O}_S$-algebra map because D is an S-derivation. Conversely, given s' we set $D = s' - s$. Details omitted. ☐

Lemma 28.12. *Let*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "h'"'] & X \arrow[d, "h"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a commutative diagram of ringed spaces.

1. *The canonical map $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with $f_*d_{X'/S'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/S'}$ is a S-derivation and we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/S} \to f_*\Omega_{X'/S'}$.*
2. *The commutative diagram*

$$\begin{tikzcd} f^{-1}\mathcal{O}_X \arrow[r] & \mathcal{O}_{X'} \\ f^{-1}h^{-1}\mathcal{O}_S \arrow[r] \arrow[u] & (h')^{-1}\mathcal{O}_{S'} \arrow[u] \end{tikzcd}$$

induces by Lemmas 28.6 and 28.8 a canonical map $f^{-1}\Omega_{X/S} \to \Omega_{X'/S'}$.

These two maps correspond (via adjointness of f_* and f^* and via $f^*\Omega_{X/S} = f^{-1}\Omega_{X/S} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'}$ and Sheaves, Lemma 20.2) to the same $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/S} \longrightarrow \Omega_{X'/S'}$$

which is uniquely characterized by the property that $f^*d_{X/S}(a)$ maps to $d_{X'/S'}(f^*a)$ for any local section a of $\mathcal{O}_X$.

Proof. Omitted. ☐

Lemma 28.13. *Let*

$$\begin{tikzcd} X'' \arrow[r, "g"] \arrow[d] & X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

be a commutative diagram of ringed spaces. With notation as in Lemma 28.12 we have

$$c_{f\circ g} = c_g \circ g^*c_f$$

as maps $(f \circ g)^*\Omega_{X/S} \to \Omega_{X''/S''}$.

Proof. Omitted. ☐

29. Finite order differential operators

In this section we introduce differential operators of finite order. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 133).

Definition 29.1. Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Let $k \geq 0$ be an integer. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. A *differential operator $D : \mathcal{F} \to \mathcal{G}$ of order k* is an is an $\mathcal{O}_1$-linear map such that for all local sections g of $\mathcal{O}_2$ the map $s \mapsto D(gs) - gD(s)$ is a differential operator of order $k-1$. For the base case $k = 0$ we define a differential operator of order 0 to be an $\mathcal{O}_2$-linear map.

If $D : \mathcal{F} \to \mathcal{G}$ is a differential operator of order k, then for all local sections g of $\mathcal{O}_2$ the map gD is a differential operator of order k. The sum of two differential operators of order k is another. Hence the set of all these

$$\text{Diff}^k(\mathcal{F}, \mathcal{G}) = \text{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G})$$

is a $\Gamma(X, \mathcal{O}_2)$-module. We have

$$\text{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

The rule which maps $U \subset X$ open to the module of differential operators $D : \mathcal{F}|_U \to \mathcal{G}|_U$ of order k is a sheaf of $\mathcal{O}_2$-modules on X. Thus we obtain a sheaf of differential operators (if we ever need this we will add a definition here).

Lemma 29.2. *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on X. Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. If $D : \mathcal{E} \to \mathcal{F}$ and $D' : \mathcal{F} \to \mathcal{G}$ are differential operators of order k and k', then $D' \circ D$ is a differential operator of order $k + k'$.*

Proof. Let g be a local section of $\mathcal{O}_2$. Then the map which sends a local section x of $\mathcal{E}$ to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on $k + k'$. $\square$

Lemma 29.3. *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. Let $k \geq 0$. There exists a sheaf of $\mathcal{O}_2$-modules $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ and a canonical isomorphism*

$$\mathit{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{O}_2}(\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}), \mathcal{G})$$

functorial in the $\mathcal{O}_2$-module $\mathcal{G}$.

Proof. The existence follows from general category theoretic arguments (insert future reference here), but we will also give a direct construction as this construction will be useful in the future proofs. We will freely use the notation introduced in the proof of Lemma 28.2. Given any differential operator $D : \mathcal{F} \to \mathcal{G}$ we obtain an $\mathcal{O}_2$-linear map $L_D : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ sending $[m]$ to $D(m)$. If D has order 0 then L_D annihilates the local sections

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0 m]$$

where g_0 is a local section of $\mathcal{O}_2$ and m, m' are local sections of $\mathcal{F}$. If D has order 1, then L_D annihilates the local sections

$$[m + m' - [m] - [m'], \quad f[m] - [fm], \quad g_0g_1[m] - g_0[g_1m] - g_1[g_0m] + [g_1g_0m]$$

where f is a local section of $\mathcal{O}_1$, g_0, g_1 are local sections of $\mathcal{O}_2$, and m, m' are local sections of $\mathcal{F}$. If D has order k, then L_D annihilates the local sections $[m + m'] - [m] - [m']$, $f[m] - [fm]$, and the local sections

$$g_0g_1 \ldots g_k[m] - \sum g_0 \ldots \hat{g}_i \ldots g_k[g_im] + \ldots + (-1)^{k+1}[g_0 \ldots g_km]$$

Conversely, if $L : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ is an $\mathcal{O}_2$-linear map annihilating all the local sections listed in the previous sentence, then $m \mapsto L([m])$ is a differential operator of order k. Thus we see that $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ is the quotient of $\mathcal{O}_2[\mathcal{F}]$ by the $\mathcal{O}_2$-submodule generated by these local sections. $\square$

Definition 29.4. Let X be a topoological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. The module $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ constructed in Lemma 29.3 is called the *module of principal parts of order* k of $\mathcal{F}$.

Note that the inclusions

$$\mathrm{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

correspond via Yoneda's lemma (Categories, Lemma 3.5) to surjections

$$\ldots \to \mathcal{P}^2_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^0_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) = \mathcal{F}$$

Lemma 29.5. *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_2$-modules. Then $\mathcal{P}^k_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}(\mathcal{F}^\#)$ is the sheaf associated to the presheaf $U \mapsto P^k_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}(\mathcal{F}(U))$.*

Proof. This can be proved in exactly the same way as is done for the sheaf of differentials in Lemma 28.4. Perhaps a more pleasing approach is to use the universal property of Lemma 29.3 directly to see the equality. We omit the details. $\square$

Lemma 29.6. *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. There is a canonical short exact sequence*

$$0 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{F} \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{F} \to 0$$

functorial in $\mathcal{F}$ called the sequence of principal parts.

Proof. Follows from the commutative algebra version (Algebra, Lemma 133.6) and Lemmas 28.4 and 29.5. $\square$

Remark 29.7. Let X be a topological space. Suppose given a commutative diagram of sheaves of rings

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array}$$

on X, a $\mathcal{B}$-module $\mathcal{F}$, a $\mathcal{B}'$-module $\mathcal{F}'$, and a $\mathcal{B}$-linear map $\mathcal{F} \to \mathcal{F}'$. Then we get a compatible system of module maps

$$\begin{tikzcd} \ldots \arrow[r] & \mathcal{P}^2_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \arrow[r] & \mathcal{P}^1_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \arrow[r] & \mathcal{P}^0_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \\ \ldots \arrow[r] & \mathcal{P}^2_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[r] \arrow[u] & \mathcal{P}^1_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[r] \arrow[u] & \mathcal{P}^0_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[u] \end{tikzcd}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules $\mathcal{P}^k_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$ in terms of (local) generators and relations in the proof of Lemma 29.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 29.6.

Next, we extend our definition to morphisms of ringed spaces.

Definition 29.8. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $k \geq 0$ be an integer. A *differential operator of order k on X/S* is a differential operator $D : \mathcal{F} \to \mathcal{G}$ with respect to $f^\sharp : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$ We denote $\text{Diff}^k_{X/S}(\mathcal{F}, \mathcal{G})$ the set of these differential operators.

30. The de Rham complex

The section is the analogue of Algebra, Section 132 for morphisms of ringed spaces. We urge the reader to read that section first.

Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Denote $\mathrm{d} : \mathcal{B} \to \Omega_{\mathcal{B}/\mathcal{A}}$ the module of differentials with its universal $\mathcal{A}$-derivation constructed in Section 28. Let

$$\Omega^i_{\mathcal{B}/\mathcal{A}} = \wedge^i_{\mathcal{B}}(\Omega_{\mathcal{B}/\mathcal{A}})$$

for $i \geq 0$ be the ith exterior power as in Section 21.

Definition 30.1. In the situation above, the *de Rham complex of $\mathcal{B}$ over $\mathcal{A}$* is the unique complex

$$\Omega^0_{\mathcal{B}/\mathcal{A}} \to \Omega^1_{\mathcal{B}/\mathcal{A}} \to \Omega^2_{\mathcal{B}/\mathcal{A}} \to \ldots$$

of sheaves of $\mathcal{A}$-modules whose differential in degree 0 is given by $\mathrm{d} : \mathcal{B} \to \Omega_{\mathcal{B}/\mathcal{A}}$ and whose differentials in higher degrees have the following property

$$\mathrm{d}\left(b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p\right) = \mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p \tag{30.1.1}$$

where $b_0, \ldots, b_p \in \mathcal{B}(U)$ are sections over a common open $U \subset X$.

We could construct this complex by repeating the cumbersome arguments given in Algebra, Section 132. Instead we recall that $\Omega_{\mathcal{B}/\mathcal{A}}$ is the sheafification of the presheaf $U \mapsto \Omega_{\mathcal{B}(U)/\mathcal{A}(U)}$, see Lemma 28.4. Thus $\Omega^i_{\mathcal{B}/\mathcal{A}}$ is the sheafification of the presheaf $U \mapsto \Omega^i_{\mathcal{B}(U)/\mathcal{A}(U)}$, see Lemma 21.1. Therefore we can define the de Rham complex as the sheafification of the rule

$$U \longmapsto \Omega^\bullet_{\mathcal{B}(U)/\mathcal{A}(U)}$$

Lemma 30.2. *Let $f : Y \to X$ be a continuous map of topological spaces. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. Then there is a canonical identification $f^{-1}\Omega^\bullet_{\mathcal{B}/\mathcal{A}} = \Omega^\bullet_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$ of de Rham complexes.*

Proof. Omitted. Hint: compare with Lemma 28.6. $\square$

Lemma 30.3. *Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. The differentials $d : \Omega^i_{\mathcal{B}/\mathcal{A}} \to \Omega^{i+1}_{\mathcal{B}/\mathcal{A}}$ are differential operators of order 1.*

Proof. Via our construction of the de Rham complex above as the sheafification of the rule $U \mapsto \Omega^\bullet_{\mathcal{B}(U)/\mathcal{A}(U)}$ this follows from Algebra, Lemma 133.8. $\square$

Let X be a topological space. Let

$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \end{tikzcd}$$

be a commutative diagram of sheaves of rings on X. There is a natural map of de Rham complexes

$$\Omega^\bullet_{\mathcal{B}/\mathcal{A}} \longrightarrow \Omega^\bullet_{\mathcal{B}'/\mathcal{A}'}$$

Namely, in degree 0 this is the map $\mathcal{B} \to \mathcal{B}'$, in degree 1 this is the map $\Omega_{\mathcal{B}/\mathcal{A}} \to \Omega_{\mathcal{B}'/\mathcal{A}'}$ constructed in Section 28, and for $p \geq 2$ it is the induced map $\Omega^p_{\mathcal{B}/\mathcal{A}} = \wedge^p_{\mathcal{B}}(\Omega_{\mathcal{B}/\mathcal{A}}) \to \wedge^p_{\mathcal{B}'}(\Omega_{\mathcal{B}'/\mathcal{A}'}) = \Omega^p_{\mathcal{B}'/\mathcal{A}'}$. The compatibility with differentials follows from the characterization of the differentials by the formula (30.1.1).

Definition 30.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The *de Rham complex* of f or of X over Y is the complex

$$\Omega^\bullet_{X/Y} = \Omega^\bullet_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$$

Consider a commutative diagram of ringed spaces

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d, "h'"'] & X \arrow[d, "h"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Then we obtain a canonical map

$$\Omega^\bullet_{X/S} \to f_*\Omega^\bullet_{X'/S'}$$

of de Rham complexes. Namely, the commutative diagram of sheaves of rings

$$\begin{tikzcd} f^{-1}\mathcal{O}_X \arrow[r] & \mathcal{O}_{X'} \\ f^{-1}h^{-1}\mathcal{O}_S \arrow[u] \arrow[r] & (h')^{-1}\mathcal{O}_{S'} \arrow[u] \end{tikzcd}$$

on X' produces a map of complexes (see above)

$$f^{-1}\Omega^\bullet_{X/S} = \Omega^\bullet_{f^{-1}\mathcal{O}_X/f^{-1}h^{-1}\mathcal{O}_S} \longrightarrow \Omega^\bullet_{\mathcal{O}_{X'}/(h')^{-1}\mathcal{O}_{S'}} = \Omega^\bullet_{X'/S'}$$

(using Lemma 30.2 for the first equality) and then we can use adjunction.

Lemma 30.5. *Let $f : X \to Y$ be a morphism of ringed spaces. The differentials $d : \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y}$ are differential operators of order 1 on X/Y.*

Proof. Immediate from Lemma 30.3 and the definition. $\square$

31. The naive cotangent complex

This section is the analogue of Algebra, Section 134 for morphisms of ringed spaces. We urge the reader to read that section first.

Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. In this section, for any sheaf of sets $\mathcal{E}$ on X we denote $\mathcal{A}[\mathcal{E}]$ the sheafification of the presheaf $U \mapsto \mathcal{A}(U)[\mathcal{E}(U)]$. Here $\mathcal{A}(U)[\mathcal{E}(U)]$ denotes the polynomial algebra over $\mathcal{A}(U)$ whose variables correspond to the elements of $\mathcal{E}(U)$. We denote $[e] \in \mathcal{A}(U)[\mathcal{E}(U)]$ the variable corresponding to $e \in \mathcal{E}(U)$. There is a canonical surjection of $\mathcal{A}$-algebras

$$\mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{B}, \quad [b] \longmapsto b \tag{31.0.1}$$

whose kernel we denote $\mathcal{I} \subset \mathcal{A}[\mathcal{B}]$. It is a simple observation that $\mathcal{I}$ is generated by the local sections $[b][b'] - [bb']$ and $[a] - a$. According to Lemma 28.9 there is a canonical map

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B} \tag{31.0.2}$$

whose cokernel is canonically isomorphic to $\Omega_{\mathcal{B}/\mathcal{A}}$.

Definition 31.1. Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. The *naive cotangent complex* $NL_{\mathcal{B}/\mathcal{A}}$ is the chain complex (31.0.2)

$$NL_{\mathcal{B}/\mathcal{A}} = \left(\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}\right)$$

with $\mathcal{I}/\mathcal{I}^2$ placed in degree -1 and $\Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$ placed in degree 0.

This construction satisfies a functoriality similar to that discussed in Lemma 28.8 for modules of differentials. Namely, given a commutative diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \end{tikzcd} \tag{31.1.1}$$

of sheaves of rings on X there is a canonical $\mathcal{B}$-linear map of complexes

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'}$$

Namely, the maps in the commutative diagram give rise to a canonical map $\mathcal{A}[\mathcal{B}] \to \mathcal{A}'[\mathcal{B}']$ which maps $\mathcal{I}$ into $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}'[\mathcal{B}'] \to \mathcal{B}')$. Thus a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{I}'/(\mathcal{I}')^2$ and a map between modules of differentials, which together give the desired map between the naive cotangent complexes. The map is compatible with compositions in the following sense: given a commutative diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \arrow[r] & \mathcal{B}'' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \arrow[r] & \mathcal{A}'' \arrow[u] \end{tikzcd}$$

of sheaves of rings then the composition

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'} \longrightarrow NL_{\mathcal{B}''/\mathcal{A}''}$$

is the map for the outer rectangle.

We can choose a different presentation of $\mathcal{B}$ as a quotient of a polynomial algebra over $\mathcal{A}$ and still obtain the same object of $D(\mathcal{B})$. To explain this, suppose that

$\mathcal{E}$ is a sheaves of sets on X and $\alpha : \mathcal{E} \to \mathcal{B}$ a map of sheaves of sets. Then we obtain an $\mathcal{A}$-algebra homomorphism $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. If this map is surjective, i.e., if $\alpha(\mathcal{E})$ generates $\mathcal{B}$ as an $\mathcal{A}$-algebra, then we set

$$NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}\right)$$

where $\mathcal{J} \subset \mathcal{A}[\mathcal{E}]$ is the kernel of the surjection $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. Here is the result.

Lemma 31.2. *In the situation above there is a canonical isomorphism $NL(\alpha) = NL_{\mathcal{B}/\mathcal{A}}$ in $D(\mathcal{B})$.*

Proof. Observe that $NL_{\mathcal{B}/\mathcal{A}} = NL(\mathrm{id}_{\mathcal{B}})$. Thus it suffices to show that given two maps $\alpha_i : \mathcal{E}_i \to \mathcal{B}$ as above, there is a canonical quasi-isomorphism $NL(\alpha_1) = NL(\alpha_2)$ in $D(\mathcal{B})$. To see this set $\mathcal{E} = \mathcal{E}_1 \amalg \mathcal{E}_2$ and $\alpha = \alpha_1 \amalg \alpha_2 : \mathcal{E} \to \mathcal{B}$. Set $\mathcal{J}_i = \mathrm{Ker}(\mathcal{A}[\mathcal{E}_i] \to \mathcal{B})$ and $\mathcal{J} = \mathrm{Ker}(\mathcal{A}[\mathcal{E}] \to \mathcal{B})$. We obtain maps $\mathcal{A}[\mathcal{E}_i] \to \mathcal{A}[\mathcal{E}]$ which send $\mathcal{J}_i$ into $\mathcal{J}$. Thus we obtain canonical maps of complexes

$$NL(\alpha_i) \longrightarrow NL(\alpha)$$

and it suffices to show these maps are quasi-isomorphism. To see this it suffices to check on stalks (Lemma 3.1). If $x \in X$ then the stalk of $NL(\alpha)$ is the complex $NL(\alpha_x)$ of Algebra, Section 134 associated to the presentation $\mathcal{A}_x[\mathcal{E}_x] \to \mathcal{B}_x$ coming from the map $\alpha_x : \mathcal{E}_x \to \mathcal{B}_x$. (Some details omitted; use Lemma 28.7 to see compatibility of forming differentials and taking stalks.) We conclude the result holds by Algebra, Lemma 134.2. $\square$

Lemma 31.3. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on Y. Then $f^{-1} NL_{\mathcal{B}/\mathcal{A}} = NL_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.*

Proof. Omitted. Hint: Use Lemma 28.6. $\square$

Lemma 31.4. *Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. Let $x \in X$. Then we have $NL_{\mathcal{B}/\mathcal{A},x} = NL_{\mathcal{B}_x/\mathcal{A}_x}$.*

Proof. This is a special case of Lemma 31.3 for the inclusion map $\{x\} \to X$. $\square$

Lemma 31.5. *Let X be a topological space. Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be maps of sheaves of rings. Let C be the cone (Derived Categories, Definition 9.1) of the map of complexes $NL_{\mathcal{C}/\mathcal{A}} \to NL_{\mathcal{C}/\mathcal{B}}$. There is a canonical map*

$$c : NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \longrightarrow C[-1]$$

of complexes of $\mathcal{C}$-modules which produces a canonical six term exact sequence

$$\begin{tikzcd}
H^0(NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C}) \arrow[r] & H^0(NL_{\mathcal{C}/\mathcal{A}}) \arrow[r] & H^0(NL_{\mathcal{C}/\mathcal{B}}) \arrow[r] & 0 \\
H^{-1}(NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C}) \arrow[r] & H^{-1}(NL_{\mathcal{C}/\mathcal{A}}) \arrow[r] & H^{-1}(NL_{\mathcal{C}/\mathcal{B}}) \arrow[ull] &
\end{tikzcd}$$

of cohomology sheaves.

Proof. To give the map c we have to give a map $c_1 : NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \to NL_{\mathcal{C}/\mathcal{A}}$ and an explicit homotopy between the composition

$$NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \to NL_{\mathcal{C}/\mathcal{A}} \to NL_{\mathcal{C}/\mathcal{B}}$$

and the zero map, see Derived Categories, Lemma 9.3. For c_1 we use the functoriality described above for the obvious diagram. For the homotopy we use the map

$$NL^0_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \longrightarrow NL^{-1}_{\mathcal{C}/\mathcal{B}}, \quad \mathrm{d}[b] \otimes 1 \longmapsto [\varphi(b)] - b[1]$$

where $\varphi : \mathcal{B} \to \mathcal{C}$ is the given map. Please compare with Algebra, Remark 134.5. To see the consequence for cohomology sheaves, it suffices to show that $H^0(c)$ is an isomorphism and $H^{-1}(c)$ surjective. To see this we can look at stalks, see Lemma 31.4, and then we can use the corresponding result in commutative algebra, see Algebra, Lemma 134.4. Some details omitted. □

The cotangent complex of a morphism of ringed spaces is defined in terms of the cotangent complex we defined above.

Definition 31.6. The *naive cotangent complex* $NL_f = NL_{X/Y}$ of a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is $NL_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$.

Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

of ringed spaces, there is a canonical map $c : g^* NL_{X/Y} \to NL_{X'/Y'}$. Namely, it is the map

$$g^* NL_{X/Y} = \mathcal{O}_{X'} \otimes_{g^{-1}\mathcal{O}_X} NL_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_Y} \longrightarrow NL_{\mathcal{O}_{X'}/(f')^{-1}\mathcal{O}_{Y'}} = NL_{X'/Y'}$$

where the arrow comes from the commutative diagram of sheaves of rings

$$\begin{tikzcd} g^{-1}\mathcal{O}_X \arrow[r, "g^\sharp"] & \mathcal{O}_{X'} \\ g^{-1}f^{-1}\mathcal{O}_Y \arrow[u, "g^{-1}f^\sharp"] \arrow[r, "g^{-1}h^\sharp"] & (f')^{-1}\mathcal{O}_{Y'} \arrow[u, "(f')^\sharp"'] \end{tikzcd}$$

as in (31.1.1) above. Given a second such diagram

$$\begin{tikzcd} X'' \arrow[r, "g'"] \arrow[d] & X' \arrow[d] \\ Y'' \arrow[r] & Y' \end{tikzcd}$$

the composition of $(g')^*c$ and the map $c' : (g')^* NL_{X'/Y'} \to NL_{X''/Y''}$ is the map $(g \circ g')^* NL_{X''/Y''} \to NL_{X/Y}$.

Lemma 31.7. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Let C be the cone of the map $NL_{X/Z} \to NL_{X/Y}$ of complexes of $\mathcal{O}_X$-modules. There is a canonical map*

$$f^* NL_{Y/Z} \to C[-1]$$

which produces a canonical six term exact sequence

$$\begin{tikzcd} H^0(f^* NL_{Y/Z}) \arrow[r] & H^0(NL_{X/Z}) \arrow[r] & H^0(NL_{X/Y}) \arrow[r] & 0 \\ H^{-1}(f^* NL_{Y/Z}) \arrow[r] & H^{-1}(NL_{X/Z}) \arrow[r] & H^{-1}(NL_{X/Y}) \arrow[ull] \end{tikzcd}$$

of cohomology sheaves.

Proof. Consider the maps of sheaves rings

$$(g \circ f)^{-1}\mathcal{O}_Z \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$$

and apply Lemma 31.5. □

32. Other chapters

References

[AGV71] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier, *Theorie de topos et cohomologie etale des schemas I, II, III*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer, 1971.

[DG67] Jean Dieudonné and Alexander Grothendieck, *Éléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1961–1967).

[Ser55] Jean-Pierre Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278.