WEIL COHOMOLOGY THEORIES

Contents

1. Introduction

In this chapter we discuss Weil cohomology theories for smooth projective schemes over a base field. Briefly, for us such a cohomology theory H^* is one which has Künneth, Poincaré duality, and cycle classes (with suitable compatibilities). We warn the reader that there is no universal agreement in the literature as to what constitutes a "Weil cohomology theory".

Before reading this chapter the reader should take a look at Categories, Section 43 and Homology, Section 17 where we define (symmetric) monoidal categories and we develop just enough basic language concerning these categories for the needs of this chapter. Equipped with this language we construct in Section 3 the symmetric monoidal graded category whose objects are smooth projective schemes and whose morphisms are correspondences. In Section 4 we add images of projectors and invert the Lefschetz motive in order to obtain the symmetric monoidal Karoubian category M_k of Chow motives. This category comes equipped with a contravariant functor

$$h : \{\text{smooth projective schemes over } k\} \longrightarrow M_k$$

As we will see below, a key property of a Weil cohomology theory is that it factors over h.

First, in the case of an algebraically closed base field, we define what we call a "classical Weil cohomology theory", see Section 7. This notion is the same as the

notion introduced in [Kle68, Section 1.2] and agrees with the notion introduced in [Kle72, page 65]. However, our notion does not a priori agree with the notion introduced in [Kle94, page 10] because there the author adds two Lefschetz type axioms and it isn't known whether any classical Weil cohomology theory as defined in this chapter satisfies those axioms. At the end of Section 7 we show that a classical Weil cohomology theory is of the form $H^* = G \circ h$ where G is a symmetric monoidal functor from M_k to the category of graded vector spaces over the coefficient field of H^*.

In Section 8 we prove a couple of lemmas on cycle groups over non-closed fields which will be used in discussing Weil cohomology theories on smooth projective schemes over arbitrary fields.

Our motivation for our axioms of a Weil cohomology theory H^* over a general base field k are the following

(1) $H^* = G \circ h$ for a symmetric monoidal functor G from M_k to the category of graded vector spaces over the coefficient field F of H^*,
(2) G should send the Tate motive (inverse of the Lefschetz motive) to a 1-dimensional vector space $F(1)$ sitting in degree -2,
(3) when k is algebraically closed we should recover the notion discussion in Section 7 up to choosing a basis element of $F(1)$.

First, in Section 9 we analyze the first two conditions. After developing a few more results in Section 10 in Section 11 we add the necessary axioms to obtain property (3).

In the final Section 14 we detail an alternative approach to Weil cohomology theories, using a first Chern class map instead of cycle classes. It is this approach that will be most suited for proving that certain cohomology theories are Weil cohomology theories in later chapters, see de Rham Cohomology, Section 22.

2. Conventions and notation

Let F be a field. In this chapter we view the category of F-graded vector spaces as an F-linear symmetric monoidal category with associativity constraint as usual and with commutativity constraint involving signs. See Homology, Example 17.4.

Let R be a ring. In this chapter a *graded commutative R-algebra A* is a commutative differential graded R-algebra (Differential Graded Algebra, Definitions 3.1 and 3.3) whose differential is zero. Thus A is an R-module endowed with a grading $A = \bigoplus_{n \in \mathbf{Z}} A^n$ by R-submodules. The R-bilinear multiplication

$$A^n \times A^m \longrightarrow A^{n+m}, \quad \alpha \times \beta \longmapsto \alpha \cup \beta$$

will be called the *cup product* in this chapter. The commutativity constraint is $\alpha \cup \beta = (-1)^{nm} \beta \cup \alpha$ if $\alpha \in A^n$ and $\beta \in A^m$. Finally, there is a multiplicative unit $1 \in A^0$, or equivalently, there is an additive and multiplicative map $R \to A^0$ which is compatible the R-module structure on A.

Let k be a field. Let X be a scheme of finite type over k. The Chow groups $\mathrm{CH}_k(X)$ of X of cycles of dimension k modulo rational equivalence have been defined in Chow Homology, Definition 19.1. If X is normal or Cohen-Macaulay, then we can also consider the Chow groups $\mathrm{CH}^p(X)$ of cycles of codimension p (Chow Homology, Section 42) and then $[X] \in \mathrm{CH}^0(X)$ denotes the "fundamental class" of X, see

Chow Homology, Remark 42.2. If X is smooth and α and β are cycles on X, then $\alpha \cdot \beta$ denotes the intersection product of α and β, see Chow Homology, Section 62.

3. Correspondences

Let k be a field. For schemes X and Y over k we denote $X \times Y$ the product of X and Y in the category of schemes over k. In this section we construct the graded category over $\mathbf{Q}$ whose objects are smooth projective schemes over k and whose morphisms are correspondences.

Let X and Y be smooth projective schemes over k. Let $X = \coprod X_d$ be the decomposition of X into the open and closed subschemes which are equidimensional with $\dim(X_d) = d$. We define the $\mathbf{Q}$-vector space *of correspondences of degree r from X to Y* by the formula:

$$\operatorname{Corr}^r(X, Y) = \bigoplus_d \operatorname{CH}^{d+r}(X_d \times Y) \otimes \mathbf{Q} \subset \operatorname{CH}^*(X \times Y) \otimes \mathbf{Q}$$

Given $c \in \operatorname{Corr}^r(X, Y)$ and $\beta \in \operatorname{CH}_k(Y) \otimes \mathbf{Q}$ we can define the *pullback* of β by c using the formula

$$c^*(\beta) = \operatorname{pr}_{1,*}(c \cdot \operatorname{pr}_2^* \beta) \quad \text{in} \quad \operatorname{CH}_{k-r}(X) \otimes \mathbf{Q}$$

This makes sense because pr_2 is flat of relative dimension d on $X_d \times Y$, hence $\operatorname{pr}_2^* \beta$ is a cycle of dimension $d + k$ on $X_d \times Y$, hence $c \cdot \operatorname{pr}_2^* \alpha$ is a cycle of dimension $k - r$ on $X_d \times Y$ whose pushforward by the proper morphism pr_1 is a cycle of the same dimension. Similarly, switching to grading by codimension, given $\alpha \in \operatorname{CH}^i(X) \otimes \mathbf{Q}$ we can define the *pushforward* of α by c using the formula

$$c_*(\alpha) = \operatorname{pr}_{2,*}(c \cdot \operatorname{pr}_1^* \alpha) \quad \text{in} \quad \operatorname{CH}^{i+r}(Y) \otimes \mathbf{Q}$$

This makes sense because $\operatorname{pr}_1^* \alpha$ is a cycle of codimension i on $X \times Y$, hence $c \cdot \operatorname{pr}_1^* \alpha$ is a cycle of codimension $i + d + r$ on $X_d \times Y$, which pushes forward to a cycle of codimension $i + r$ on Y.

Given a three smooth projective schemes X, Y, Z over k we define a *composition of correspondences*

$$\operatorname{Corr}^s(Y, Z) \times \operatorname{Corr}^r(X, Y) \longrightarrow \operatorname{Corr}^{r+s}(X, Z)$$

by the rule

$$(c', c) \longmapsto c' \circ c = \operatorname{pr}_{13,*}(\operatorname{pr}_{12}^* c \cdot \operatorname{pr}_{23}^* c')$$

where $\operatorname{pr}_{12} : X \times Y \times Z \to X \times Y$ is the projection and similarly for pr_{13} and pr_{23}.

Lemma 3.1. *We have the following for correspondences:*

(1) *composition of correspondences is $\mathbf{Q}$-bilinear and associative,*
(2) *there is a canonical isomorphism*

$$\operatorname{CH}_{-r}(X) \otimes \mathbf{Q} = \operatorname{Corr}^r(X, \operatorname{Spec}(k))$$

such that pullback by correspondences corresponds to composition,
(3) *there is a canonical isomorphism*

$$\operatorname{CH}^r(X) \otimes \mathbf{Q} = \operatorname{Corr}^r(\operatorname{Spec}(k), X)$$

such that pushforward by correspondences corresponds to composition,
(4) *composition of correspondences is compatible with pushforward and pullback of cycles.*

Proof. Bilinearity follows immediately from the linearity of pushforward and pull-back and the bilinearity of the intersection product. To prove associativity, say we have X, Y, Z, W and $c \in \mathrm{Corr}(X, Y)$, $c' \in \mathrm{Corr}(Y, Z)$, and $c'' \in \mathrm{Corr}(Z, W)$. Then we have

$$
\begin{aligned}
c'' \circ (c' \circ c) &= \mathrm{pr}_{14,*}^{134}(\mathrm{pr}_{13}^{134,*}\mathrm{pr}_{13,*}^{123}(\mathrm{pr}_{12}^{123,*}c \cdot \mathrm{pr}_{23}^{123,*}c') \cdot \mathrm{pr}_{34}^{134,*}c'') \\
&= \mathrm{pr}_{14,*}^{134}(\mathrm{pr}_{134,*}^{1234}\mathrm{pr}_{123}^{1234,*}(\mathrm{pr}_{12}^{123,*}c \cdot \mathrm{pr}_{23}^{123,*}c') \cdot \mathrm{pr}_{34}^{134,*}c'') \\
&= \mathrm{pr}_{14,*}^{134}(\mathrm{pr}_{134,*}^{1234}(\mathrm{pr}_{12}^{1234,*}c \cdot \mathrm{pr}_{23}^{1234,*}c') \cdot \mathrm{pr}_{34}^{134,*}c'') \\
&= \mathrm{pr}_{14,*}^{134}\mathrm{pr}_{134,*}^{1234}((\mathrm{pr}_{12}^{1234,*}c \cdot \mathrm{pr}_{23}^{1234,*}c') \cdot \mathrm{pr}_{34}^{1234,*}c'') \\
&= \mathrm{pr}_{14,*}^{1234}((\mathrm{pr}_{12}^{1234,*}c \cdot \mathrm{pr}_{23}^{1234,*}c') \cdot \mathrm{pr}_{34}^{1234,*}c'')
\end{aligned}
$$

Here we use the notation

$$
p_{134}^{1234} : X \times Y \times Z \times W \to X \times Z \times W \quad \text{and} \quad p_{14}^{134} : X \times Z \times W \to X \times W
$$

the projections and similarly for other indices. The first equality is the definition of the composition. The second equality holds because $\mathrm{pr}_{13}^{134,*}\mathrm{pr}_{13,*}^{123} = \mathrm{pr}_{134,*}^{1234}\mathrm{pr}_{123}^{1234,*}$ by Chow Homology, Lemma 15.1. The third equality holds because intersection product commutes with the gysin map for p_{123}^{1234} (which is given by flat pullback), see Chow Homology, Lemma 62.3. The fourth equality follows from the projection formula for p_{134}^{1234}, see Chow Homology, Lemma 62.4. The fourth equality is that proper pushforward is compatible with composition, see Chow Homology, Lemma 12.2. Since intersection product is associative by Chow Homology, Lemma 62.1 this concludes the proof of associativity of composition of correspondences.

We omit the proofs of (2) and (3) as these are essentially proved by carefully bookkeeping where various cycles live and in what (co)dimension.

The statement on pushforward and pullback of cycles means that $(c' \circ c)^*(\alpha) = c^*((c')^*(\alpha))$ and $(c' \circ c)_*(\alpha) = (c')_*(c_*(\alpha))$. This follows on combining (1), (2), and (3). $\qquad\square$

Example 3.2. Let $f : Y \to X$ be a morphism of smooth projective schemes over k. Denote $\Gamma_f \subset X \times Y$ the graph of f. More precisely, Γ_f is the image of the closed immersion

$$
(f, \mathrm{id}_Y) : Y \longrightarrow X \times Y
$$

Let $X = \coprod X_d$ be the decomposition of X into its open and closed parts X_d which are equidimensional of dimension d. Then $\Gamma_f \cap (X_d \times Y)$ has pure codimension d. Hence $[\Gamma_f] \in \mathrm{CH}^*(X \times Y) \otimes \mathbf{Q}$ is contained in $\mathrm{Corr}^0(X \times Y)$, i.e., $[\Gamma_f]$ is a correspondence of degree 0 from X to Y.

Lemma 3.3. *Smooth projective schemes over k with correspondences and composition of correspondences as defined above form a graded category over* $\mathbf{Q}$ *(Differential Graded Algebra, Definition 25.1).*

Proof. Everything is clear from the construction and Lemma 3.1 except for the existence of identity morphisms. Given a smooth projective scheme X consider the class $[\Delta]$ of the diagonal $\Delta \subset X \times X$ in $\mathrm{Corr}^0(X, X)$. We note that Δ is equal to the graph of the identity $\mathrm{id}_X : X \to X$ which is a fact we will use below.

To prove that $[\Delta]$ can serve as an identity we have to show that $[\Delta] \circ c = c$ and $c' \circ [\Delta] = c'$ for any correspondences $c \in \mathrm{Corr}^r(Y, X)$ and $c' \in \mathrm{Corr}^s(X, Y)$. For

the second case we have to show that

$$c' = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*[\Delta] \cdot \mathrm{pr}_{23}^* c')$$

where $\mathrm{pr}_{12} : X \times X \times Y \to X \times X$ is the projection and simlarly for pr_{13} and pr_{23}. We may write $c' = \sum a_i[Z_i]$ for some integral closed subschemes $Z_i \subset X \times Y$ and rational numers a_i. Thus it clearly suffices to show that

$$[Z] = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*[\Delta] \cdot \mathrm{pr}_{23}^*[Z])$$

in the chow group of $X \times Y$ for any integral closed subscheme Z of $X \times Y$. After replacing X and Y by the irreducible component containing the image of Z under the two projections we may assume X and Y are integral as well. Then we have to show

$$[Z] = \mathrm{pr}_{13,*}([\Delta \times Y] \cdot [X \times Z])$$

Denote $Z' \subset X \times X \times Y$ the image of Z by the morphism $(\Delta, 1) : X \times Y \to X \times X \times Y$. Then Z' is a closed subscheme of $X \times X \times Y$ isomorphic to Z and $Z' = \Delta \times Y \cap X \times Z$ scheme theoretically. By Chow Homology, Lemma 62.5[1] we conclude that

$$[Z'] = [\Delta \times Y] \cdot [X \times Z]$$

Since Z' maps isomorphically to Z by pr_{13} also we conclude. The verification that $[\Delta] \circ c = c$ is similar and we omit it. $\square$

Lemma 3.4. *There is a contravariant functor from the category of smooth projective schemes over k to the category of correspondences which is the identity on objects and sends $f : Y \to X$ to the element $[\Gamma_f] \in Corr^0(X, Y)$.*

Proof. In the proof of Lemma 3.3 we have seen that this construction sends identities to identities. To finish the proof we have to show if $g : Z \to Y$ is another morphism of smooth projective schemes over k, then we have $[\Gamma_g] \circ [\Gamma_f] = [\Gamma_{f \circ g}]$ in $\mathrm{Corr}^0(X, Z)$. Arguing as in the proof of Lemma 3.3 we see that it suffices to show

$$[\Gamma_{f \circ g}] = \mathrm{pr}_{13,*}([\Gamma_f \times Z] \cdot [X \times \Gamma_g])$$

in $\mathrm{CH}^*(X \times Z)$ when X, Y, Z are integral. Denote $Z' \subset X \times Y \times Z$ the image of the closed immersion $(f \circ g, g, 1) : Z \to X \times Y \times Z$. Then $Z' = \Gamma_f \times Z \cap X \times \Gamma_g$ scheme theoretically and we conclude using Chow Homology, Lemma 62.5 that

$$[Z'] = [\Gamma_f \times Z] \cdot [X \times \Gamma_g]$$

Since it is clear that $\mathrm{pr}_{13,*}([Z']) = [\Gamma_{f \circ g}]$ the proof is complete. $\square$

Remark 3.5. Let X and Y be smooth projective schemes over k. Assume X is equidimensional of dimension d and Y is equidimensional of dimension e. Then the isomorphism $X \times Y \to Y \times X$ switching the factors determines an isomorphism

$$\mathrm{Corr}^r(X, Y) \longrightarrow \mathrm{Corr}^{d-e+r}(Y, X), \quad c \longmapsto c^t$$

called the *transpose*. It acts on cycles as well as cycle classes. An example which is sometimes useful, is the transpose $[\Gamma_f]^t = [\Gamma_f^t]$ of the graph of a morphism $f : Y \to X$.

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[1]The reader verifies that $\dim(Z') = \dim(\Delta \times Y) + \dim(X \times Z) - \dim(X \times X \times Y)$ and that Z' has a unique generic point mapping to the generic point of Z (where the local ring is CM) and to some point of X (where the local ring is CM). Thus all the hypothese of the lemma are indeed verified.

Lemma 3.6. *Let $f : Y \to X$ be a morphism of smooth projective schemes over k. Let $[\Gamma_f] \in Corr^0(X, Y)$ be as in Example 3.2. Then*

    (1) *pushforward of cycles by the correspondence $[\Gamma_f]$ agrees with the gysin map $f^! : CH^*(X) \to CH^*(Y)$,*

    (2) *pullback of cycles by the correspondence $[\Gamma_f]$ agrees with the pushforward map $f_* : CH_*(Y) \to CH_*(X)$,*

    (3) *if X and Y are equidimensional of dimensions d and e, then*

        (a) *pushforward of cycles by the correspondence $[\Gamma_f^t]$ of Remark 3.5 corresponds to pushforward of cycles by f, and*

        (b) *pullback of cycles by the correspondence $[\Gamma_f^t]$ of Remark 3.5 corresponds to the gysin map $f^!$.*

Proof. Proof of (1). Recall that $[\Gamma_f]_*(\alpha) = \mathrm{pr}_{2,*}([\Gamma_f] \cdot \mathrm{pr}_1^* \alpha)$. We have

$$[\Gamma_f] \cdot \mathrm{pr}_1^* \alpha = (f,1)_*((f,1)^! \mathrm{pr}_1^* \alpha) = (f,1)_*((f,1)^! \mathrm{pr}_1^! \alpha) = (f,1)_*(f^! \alpha)$$

The first equality by Chow Homology, Lemma 62.6. The second by Chow Homology, Lemma 59.5. The third because $\mathrm{pr}_1 \circ (f,1) = f$ and Chow Homology, Lemma 59.6. Then we coclude because $\mathrm{pr}_{2,*} \circ (f,1)_* = 1_*$ by Chow Homology, Lemma 12.2.

Proof of (2). Recall that $[\Gamma_f]_*(\beta) = \mathrm{pr}_{1,*}([\Gamma_f] \cdot \mathrm{pr}_2^* \beta)$. Arguing exactly as above we have

$$[\Gamma_f] \cdot \mathrm{pr}_2^* \beta = (f,1)_* \beta$$

Thus the result follows as before.

Proof of (3). Proved in exactly the same manner as above. $\square$

Example 3.7. Let $X = \mathbf{P}_k^1$. Then we have

$$\mathrm{Corr}^0(X, X) = \mathrm{CH}^1(X \times X) \otimes \mathbf{Q} = \mathrm{CH}_1(X \times X) \otimes \mathbf{Q}$$

Choose a k-rational point $x \in X$ and consider the cycles $c_0 = [x \times X]$ and $c_2 = [X \times x]$. A computation shows that $1 = [\Delta] = c_0 + c_2$ in $\mathrm{Corr}^0(X, X)$ and that we have the following rules for composition $c_0 \circ c_0 = c_0$, $c_0 \circ c_2 = 0$, $c_2 \circ c_0 = 0$, and $c_2 \circ c_2 = c_2$. In other words, c_0 and c_2 are orthogonal idempotents in the algebra $\mathrm{Corr}^0(X, X)$ and in fact we get

$$\mathrm{Corr}^0(X, X) = \mathbf{Q} \times \mathbf{Q}$$

as a $\mathbf{Q}$-algebra.

The category of correspondences is a symmetric monoidal category. Given smooth projective schemes X and Y over k, we define $X \otimes Y = X \times Y$. Given four smooth projective schemes X, X', Y, Y' over k we define a tensor product

$$\otimes : \mathrm{Corr}^r(X, Y) \times \mathrm{Corr}^{r'}(X', Y') \longrightarrow \mathrm{Corr}^{r+r'}(X \times X', Y \times Y')$$

by the rule

$$(c, c') \longmapsto c \otimes c' = \mathrm{pr}_{13}^* c \cdot \mathrm{pr}_{24}^* c'$$

where $\mathrm{pr}_{13} : X \times X' \times Y \times Y' \to X \times Y$ and $\mathrm{pr}_{24} : X \times X' \times Y \times Y' \to X' \times Y'$ are the projections. As associativity constraint

$$X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$$

we use the usual associativity constraint on products of schemes. The commutativity constraint will be given by the isomorphism $X \times Y \to Y \times X$ switching the factors.

Lemma 3.8. *The tensor product of correspondences defined above turns the category of correspondences into a symmetric monoidal category with unit* $\mathrm{Spec}(k)$*.*

Proof. Omitted. $\square$

Lemma 3.9. *Let* $f : Y \to X$ *be a morphism of smooth projective schemes over* k*. Assume* X *and* Y *equidimensional of dimensions* d *and* e*. Denote* $a = [\Gamma_f] \in Corr^0(X, Y)$ *and* $a^t = [\Gamma_f^t] \in Corr^{d-e}(Y, X)$*. Set* $\eta_X = [\Gamma_{X \to X \times X}] \in Corr^0(X \times X, X)$*,* $\eta_Y = [\Gamma_{Y \to Y \times Y}] \in Corr^0(Y \times Y, Y)$*,* $[X] \in Corr^{-d}(X, \mathrm{Spec}(k))$*, and* $[Y] \in Corr^{-e}(Y, \mathrm{Spec}(k))$*. The diagram*

$$
\begin{array}{ccccc}
X \otimes Y & \xrightarrow{\ a \otimes id\ } & Y \otimes Y & \xrightarrow{\ \eta_Y\ } & Y \\
{\scriptstyle id \otimes a^t} \downarrow & & & & \downarrow {\scriptstyle [Y]} \\
X \otimes X & \xrightarrow{\ \eta_X\ } & X & \xrightarrow{\ [X]\ } & \mathrm{Spec}(k)
\end{array}
$$

is commutative in the category of correspondences.

Proof. Recall that $Corr^r(W, \mathrm{Spec}(k)) = \mathrm{CH}_{-r}(W)$ for any smooth projective scheme W over k and given $c \in Corr^s(W', W)$ the composition with c agrees with pullback by c as a map $\mathrm{CH}_{-r}(W) \to \mathrm{CH}_{-r-s}(W')$ (Lemma 3.1). Finally, we have Lemma 3.6 which tells us how to convert this into usual pushforward and pullback of cycles. We have

$$(a \otimes \mathrm{id})^* \eta_Y^*[Y] = (a \otimes \mathrm{id})^*[\Delta_Y] = (f \times \mathrm{id})_*\Delta_Y = [\Gamma_f]$$

and the other way around we get

$$(\mathrm{id} \otimes a^t)^* \eta_X^*[X] = (\mathrm{id} \otimes a^t)^*[\Delta_X] = (\mathrm{id} \times f)^![\Delta_X] = [\Gamma_f]$$

The last equality follows from Chow Homology, Lemma 59.8. In other words, going either way around the diagram we obtain the element of $Corr^d(X \times Y, \mathrm{Spec}(k))$ corresponding to the cycle $\Gamma_f \subset X \times Y$. $\square$

4. Chow motives

We fix a base field k. In this section we construct an additive Karoubian $\mathbf{Q}$-linear category M_k endowed with a symmetric monoidal structure and a contravariant functor

$$h : \{\text{smooth projective schemes over } k\} \longrightarrow M_k$$

which maps products to tensor products and disjoint unions to direct sums. Our construction will be characterized by the fact that h factors through the symmetric monoidal category whose objects are smooth projective varieties and whose morphisms are correspondences of degree 0 such that the image of the projector c_2 on $h(\mathbf{P}_k^1)$ from Example 3.7 is invertible in M_k, see Lemma 4.8. At the end of the section we will show that every motive, i.e., every object of M_k to has a (left) dual, see Lemma 4.10.

A *motive* or a *Chow motive* over k will be a triple (X, p, m) where

    (1) X is a smooth projective scheme over k,
    (2) $p \in \mathrm{Corr}^0(X, X)$ satisfies $p \circ p = p$,
    (3) $m \in \mathbf{Z}$.

Given a second motive (Y, q, n) we define a *morphism of motives* or a *morphism of Chow motives* to be an element of

$$\mathrm{Hom}((X, p, m), (Y, q, n)) = q \circ \mathrm{Corr}^{n-m}(X, Y) \circ p \subset \mathrm{Corr}^{n-m}(X, Y)$$

Composition of morphisms of motives is defined using the composition of correspondences defined above.

Lemma 4.1. *The category M_k whose objects are motives over k and morphisms are morphisms of motives over k is a $\mathbf{Q}$-linear category. There is a contravariant functor*

$$h : \{\textit{smooth projective schemes over } k\} \longrightarrow M_k$$

defined by $h(X) = (X, 1, 0)$ and $h(f) = [\Gamma_f]$.

Proof. Follows immediately from Lemma 3.4. $\square$

Lemma 4.2. *The category M_k is Karoubian.*

Proof. Let $M = (X, p, m)$ be a motive and let $a \in \mathrm{Mor}(M, M)$ be a projector. Then $a = a \circ a$ both in $\mathrm{Mor}(M, M)$ as well as in $\mathrm{Corr}^0(X, X)$. Set $N = (X, a, m)$. Since we have $a = p \circ a \circ a$ in $\mathrm{Corr}^0(X, X)$ we see that $a : N \to M$ is a morphism of M_k. Next, suppose that $b : (Y, q, n) \to M$ is a morphism such that $(1 - a) \circ b = 0$. Then $b = a \circ b$ as well as $b = b \circ q$. Hence b is a morphism $b : (Y, q, n) \to N$. Thus we see that the projector $1 - a$ has a kernel, namely N and we find that M_k is Karoubian, see Homology, Definition 4.1. $\square$

We define a functor

$$\otimes : M_k \times M_k \longrightarrow M_k$$

On objects we use the formula

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \otimes q, m + n)$$

On morphisms, we use

$$\mathrm{Mor}((X, p, m), (Y, q, n)) \times \mathrm{Mor}((X', p', m'), (Y', q', n'))$$

$$\downarrow$$

$$\mathrm{Mor}((X \times X', p \otimes p', m + m'), (Y \times Y', q \otimes q', n + n'))$$

given by the rule $(a, a') \longmapsto a \otimes a'$ where $\otimes$ on correspondences is as in Section 3. This makes sense: by definition of morphisms of motives we can write $a = q \circ c \circ p$ and $a' = q' \circ c' \circ p'$ with $c \in \mathrm{Corr}^{n-m}(X, Y)$ and $c' \in \mathrm{Corr}^{n'-m'}(X', Y')$ and then we obtain

$$a \otimes a' = (q \circ c \circ p) \otimes (q' \circ c' \circ p') = (q \otimes q') \circ (c \otimes c') \circ (p \otimes p')$$

which is indeed a morphism of motives from $(X \times X', p \otimes p', m + m')$ to $(Y \times Y', q \otimes q', n + n')$.

Lemma 4.3. *The category M_k with tensor product defined as above is symmetric monoidal with the obvious associativity and commutativity constraints and with unit $\mathbf{1} = (\mathrm{Spec}(k), 1, 0)$.*

Proof. Follows readily from Lemma 3.8. Details omitted. $\square$

The motives $\mathbf{1}(n) = (\mathrm{Spec}(k), 1, n)$ are useful. Observe that

$$\mathbf{1} = \mathbf{1}(0) \quad \text{and} \quad \mathbf{1}(n + m) = \mathbf{1}(n) \otimes \mathbf{1}(m)$$

Thus tensoring with $\mathbf{1}(1)$ is an autoequivalence of the category of motives. Given a motive M we sometimes write $M(n) = M \otimes \mathbf{1}(n)$. Observe that if $M = (X, p, m)$, then $M(n) = (X, p, m + n)$.

Lemma 4.4. *With notation as in Example 3.7*
   (1) *the motive $(X, c_0, 0)$ is isomorphic to the motive $\mathbf{1} = (\mathrm{Spec}(k), 1, 0)$.*
   (2) *the motive $(X, c_2, 0)$ is isomorphic to the motive $\mathbf{1}(-1) = (\mathrm{Spec}(k), 1, -1)$.*

Proof. We will use Lemma 3.4 without further mention. The structure morphism $X \to \mathrm{Spec}(k)$ gives a correspondence $a \in \mathrm{Corr}^0(\mathrm{Spec}(k), X)$. On the other hand, the rational point x is a morphism $\mathrm{Spec}(k) \to X$ which gives a correspondence $b \in \mathrm{Corr}^0(X, \mathrm{Spec}(k))$. We have $b \circ a = 1$ as a correspondence on $\mathrm{Spec}(k)$. The composition $a \circ b$ corresponds to the graph of the composition $X \to x \to X$ which is $c_0 = [x \times X]$. Thus $a = a \circ b \circ a = c_0 \circ a$ and $b = a \circ b \circ a = b \circ c_0$. Hence, unwinding the definitions, we see that a and b are mutually inverse morphisms $a : (\mathrm{Spec}(k), 1, 0) \to (X, c_0, 0)$ and $b : (X, c_0, 0) \to (\mathrm{Spec}(k), 1, 0)$.

We will proceed exactly as above to prove the second statement. Denote

$$a' \in \mathrm{Corr}^1(\mathrm{Spec}(k), X) = \mathrm{CH}^1(X)$$

the class of the point x. Denote

$$b' \in \mathrm{Corr}^{-1}(X, \mathrm{Spec}(k)) = \mathrm{CH}_1(X)$$

the class of $[X]$. We have $b' \circ a' = 1$ as a correspondence on $\mathrm{Spec}(k)$ because $[x] \cdot [X] = [x]$ on $X = \mathrm{Spec}(k) \times X \times \mathrm{Spec}(k)$. Computing the intersection product $\mathrm{pr}_{12}^* b' \cdot \mathrm{pr}_{23}^* a'$ on $X \times \mathrm{Spec}(k) \times X$ gives the cycle $X \times \mathrm{Spec}(k) \times x$. Hence the composition $a' \circ b'$ is equal to c_2 as a correspondence on X. Thus $a' = a' \circ b \circ a' = c_2 \circ a'$ and $b' = b' \circ a' \circ b' = b' \circ c_2$. Recall that

$$\mathrm{Mor}((\mathrm{Spec}(k), 1, -1), (X, c_2, 0)) = c_2 \circ \mathrm{Corr}^1(\mathrm{Spec}(k), X) \subset \mathrm{Corr}^1(\mathrm{Spec}(k), X)$$

and

$$\mathrm{Mor}((X, c_2, 0), (\mathrm{Spec}(k), 1, -1)) = \mathrm{Corr}^{-1}(X, \mathrm{Spec}(k)) \circ c_2 \subset \mathrm{Corr}^{-1}(X, \mathrm{Spec}(k))$$

Hence, we see that a' and b' are mutually inverse morphisms $a' : (\mathrm{Spec}(k), 1, -1) \to (X, c_0, 0)$ and $b' : (X, c_0, 0) \to (\mathrm{Spec}(k), 1, -1)$. $\qquad\square$

Remark 4.5 (Lefschetz and Tate motive)**.** Let $X = \mathbf{P}_k^1$ and c_2 be as in Example 3.7. In the literature the motive $(X, c_2, 0)$ is sometimes called the *Lefschetz motive* and depending on the reference the notation L, $\mathbf{L}$, $\mathbf{Q}(-1)$, or $h^2(\mathbf{P}_k^1)$ may be used to denote it. By Lemma 4.4 the Lefschetz motive is isomorphic to $\mathbf{1}(-1)$. Hence the Lefschetz motive is invertible (Categories, Definition 43.4) with inverse $\mathbf{1}(1)$. The motive $\mathbf{1}(1)$ is sometimes called the *Tate motive* and depending on the reference the notation L^{-1}, $\mathbf{L}^{-1}$, $\mathbf{T}$, or $\mathbf{Q}(1)$ may be used to denote it.

Lemma 4.6. *The category M_k is additive.*

Proof. Let (Y, p, m) and (Z, q, n) be motives. If $n = m$, then a direct sum is given by $(Y \amalg Z, p + q, m)$, with obvious notation. Details omitted.

Suppose that $n < m$. Let X, c_2 be as in Example 3.7. Then we consider

$$\begin{aligned}
(Z, q, n) &= (Z, q, m) \otimes (\mathrm{Spec}(k), 1, -1) \otimes \ldots \otimes (\mathrm{Spec}(k), 1, -1) \\
&\cong (Z, q, m) \otimes (X, c_2, 0) \otimes \ldots \otimes (X, c_2, 0) \\
&\cong (Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m)
\end{aligned}$$

where we have used Lemma 4.4. This reduces us to the case discussed in the first paragraph. $\square$

Lemma 4.7. *In M_k we have $h(\mathbf{P}_k^1) \cong \mathbf{1} \oplus \mathbf{1}(-1)$.*

Proof. This follows from Example 3.7 and Lemma 4.4. $\square$

Lemma 4.8. *Let X, c_2 be as in Example 3.7. Let $\mathcal{C}$ be a $\mathbf{Q}$-linear Karoubian symmetric monoidal category. Any $\mathbf{Q}$-linear functor*

$$F : \left\{ \begin{array}{c} \text{smooth projective schemes over } k \\ \text{morphisms are correspondences of degree } 0 \end{array} \right\} \longrightarrow \mathcal{C}$$

of symmetric monoidal categories such that the image of $F(c_2)$ on $F(X)$ is an invertible object, factors uniquely through a functor $F : M_k \to \mathcal{C}$ of symmetric monoidal categories.

Proof. Denote U in $\mathcal{C}$ the invertible object which is assumed to exist in the statement of the lemma. We extend F to motives by setting

$$F(X, p, m) = \text{(the image of the projector } F(p) \text{ in } F(X)) \otimes U^{\otimes -m}$$

which makes sense because U is invertible and because $\mathcal{C}$ is Karoubian. An important feature of this choice is that $F(X, c_2, 0) = U$. Observe that

$$\begin{aligned}
F((X, p, m) \otimes (Y, q, n)) &= F(X \times Y, p \otimes q, m + n) \\
&= \text{(the image of } F(p \otimes q) \text{ in } F(X \times Y)) \otimes U^{\otimes -m-n} \\
&= F(X, p, m) \otimes F(Y, q, n)
\end{aligned}$$

Thus we see that our rule is compatible with tensor products on the level of objects (details omitted).

Next, we extend F to morphisms of motives. Suppose that

$$a \in \mathrm{Hom}((Y, p, m), (Z, q, n)) = q \circ \mathrm{Corr}^{n-m}(Y, Z) \circ p \subset \mathrm{Corr}^{n-m}(Y, Z)$$

is a morphism. If $n = m$, then a is a correspondence of degree 0 and we can use $F(a) : F(Y) \to F(Z)$ to get the desired map $F(Y, p, m) \to F(Z, q, n)$. If $n < m$ we get canonical identifications

$$\begin{aligned}
s : F((Z, q, n)) &\to F(Z, q, m) \otimes U^{m-n} \\
&\to F(Z, q, m) \otimes F(X, c_2, 0) \otimes \ldots \otimes F(X, c_2, 0) \\
&\to F((Z, q, m) \otimes (X, c_2, 0) \otimes \ldots \otimes (X, c_2, 0)) \\
&\to F((Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m))
\end{aligned}$$

Namely, for the first isomorphism we use the definition of F on motives above. For the second, we use the choice of U. For the third we use the compatibility of F on tensor products of motives. The fourth is the definition of tensor products on motives. On the other hand, since we similarly have an isomorphism

$$\sigma : (Z, q, n) \to (Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m)$$

(see proof of Lemma 4.6). Composing a with this isomorphism gives

$$\sigma \circ a \in \mathrm{Hom}((Y, p, m), (Z \times X^{m-n}, q \otimes c_2 \otimes \ldots \otimes c_2, m))$$

Putting everything together we obtain

$$s^{-1} \circ F(\sigma \circ a) : F(Y, p, m) \to F(Z, q, n)$$

If $n > m$ we similarly define isomorphisms

$$t : F((Y, p, m)) \to F((Y \times X^{n-m}, p \otimes c_2 \otimes \ldots \otimes c_2, n))$$

and

$$\tau : (Y, p, m)) \to (Y \times X^{n-m}, p \otimes c_2 \otimes \ldots \otimes c_2, n)$$

and we set $F(a) = F(a \circ \tau^{-1}) \circ t$. We omit the verification that this construction defines a functor of symmetric monoidal categories. $\square$

Lemma 4.9. *Let X be a smooth projective scheme over k which is equidimensional of dimension d. Then $h(X)(d)$ is a left dual to $h(X)$ in M_k.*

Proof. We will use Lemma 3.1 without further mention. We compute

$$\mathrm{Hom}(\mathbf{1}, h(X) \otimes h(X)(d)) = \mathrm{Corr}^d(\mathrm{Spec}(k), X \times X) = \mathrm{CH}^d(X \times X)$$

Here we have $\eta = [\Delta]$. On the other hand, we have

$$\mathrm{Hom}(h(X)(d) \otimes h(X), \mathbf{1}) = \mathrm{Corr}^{-d}(X \times X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X \times X)$$

and here we have the class $\epsilon = [\Delta]$ of the diagonal as well. The composition of the correspondence $[\Delta] \otimes 1$ with $1 \otimes [\Delta]$ either way is the correspondence $[\Delta] = 1$ in $\mathrm{Corr}^0(X, X)$ which proves the required diagrams of Categories, Definition 43.5 commute. Namely, observe that

$$[\Delta] \otimes 1 \in \mathrm{Corr}^d(X, X \times X \times X) = \mathrm{CH}^{2d}(X \times X \times X \times X)$$

is given by the class of the cycle $\mathrm{pr}_{23}^{1234,-1}(\Delta) \cap \mathrm{pr}_{14}^{1234,-1}(\Delta)$ with obvious notation. Similarly, the class

$$1 \otimes [\Delta] \in \mathrm{Corr}^{-d}(X \times X \times X, X) = \mathrm{CH}^{2d}(X \times X \times X \times X)$$

is given by the class of the cycle $\mathrm{pr}_{23}^{1234,-1}(\Delta) \cap \mathrm{pr}_{14}^{1234,-1}(\Delta)$. The composition $(1 \otimes [\Delta]) \circ ([\Delta] \otimes 1)$ is by definition the pushforward $\mathrm{pr}_{15,*}^{12345}$ of the intersection product

$$[\mathrm{pr}_{23}^{12345,-1}(\Delta) \cap \mathrm{pr}_{14}^{12345,-1}(\Delta)] \cdot [\mathrm{pr}_{34}^{12345,-1}(\Delta) \cap \mathrm{pr}_{15}^{12345,-1}(\Delta)] = [\text{small diagonal in } X^5]$$

which is equal to Δ as desired. We omit the proof of the formula for the composition in the other order. $\square$

Lemma 4.10. *Every object of M_k has a left dual.*

Proof. Let $M = (X, p, m)$ be an object of M_k. Then M is a summand of $(X, 0, m) = h(X)(m)$. By Homology, Lemma 17.3 it suffices to show that $h(X)(m) = h(X) \otimes \mathbf{1}(m)$ has a dual. By construction $\mathbf{1}(-m)$ is a left dual of $\mathbf{1}(m)$. Hence it suffices to show that $h(X)$ has a left dual, see Categories, Lemma 43.8. Let $X = \coprod X_i$ be the decomposition of X into irreducible components. Then $h(X) = \bigoplus h(X_i)$ and it suffices to show that $h(X_i)$ has a left dual, see Homology, Lemma 17.2. This follows from Lemma 4.9. $\square$

5. Chow groups of motives

We define the Chow groups of a motive as follows.

Definition 5.1. Let k be a base field. Let $M = (X, p, m)$ be a Chow motive over k. For $i \in \mathbf{Z}$ we define the *ith Chow group of M* by the formula

$$\mathrm{CH}^i(M) = p\left(\mathrm{CH}^{i+m}(X) \otimes \mathbf{Q}\right)$$

We have $\mathrm{CH}^i(h(X)) = \mathrm{CH}^i(X) \otimes \mathbf{Q}$ if X is a smooth projective scheme over k.

Observe that $\mathrm{CH}^i(-)$ is a functor from M_k to $\mathbf{Q}$-vector spaces. Indeed, if $c : M \to N$ is a morphism of motives $M = (X, p, m)$ and $N = (Y, q, n)$, then c is a correspondence of degree $n - m$ from X to Y and hence pushforward along c (Section 3) is a family of maps

$$c_* : \mathrm{CH}^{i+m}(X) \otimes \mathbf{Q} \longrightarrow \mathrm{CH}^{i+n}(Y) \otimes \mathbf{Q}$$

Since $c = q \circ c \circ p$ by definition of morphisms of motives, we see that indeed we obtain

$$c_* : \mathrm{CH}^i(M) \to \mathrm{CH}^i(N)$$

for all $i \in \mathbf{Z}$. This is compatible with compositions of morphisms of motives by Lemma 3.1. This functoriality of Chow groups can also be deduced from the following lemma.

Lemma 5.2. *Let k be a base field. The functor $\mathrm{CH}^i(-)$ on the category of motives M_k is representable by $\mathbf{1}(-i)$, i.e., we have*

$$\mathrm{CH}^i(M) = \mathrm{Hom}_{M_k}(\mathbf{1}(-i), M)$$

functorially in M in M_k.

Proof. Immediate from the definitions and Lemma 3.1. $\qquad\square$

The reader can imagine that we can use Lemma 5.2, the Yoneda lemma, and the duality in Lemma 4.9 to obtain the following.

Lemma 5.3 (Manin)**.** *Let k be a base field. Let $c : M \to N$ be a morphism of motives. If for every smooth projective scheme X over k the map $c \otimes 1 : M \otimes h(X) \to N \otimes h(X)$ induces an isomorphism on Chow groups, then c is an isomorphism.*

Proof. Any object L of M_k is a summand of $h(X)(m)$ for some smooth projective scheme X over k and some $m \in \mathbf{Z}$. Observe that the Chow groups of $M \otimes h(X)(m)$ are the same as the Chow groups of of $M \otimes h(X)$ up to a shift in degrees. Hence our assumption implies that $c \otimes 1 : M \otimes L \to N \otimes L$ induces an isomorphism on Chow groups for every object L of M_k. By Lemma 5.2 we see that

$$\mathrm{Hom}_{M_k}(\mathbf{1}, M \otimes L) \to \mathrm{Hom}_{M_k}(\mathbf{1}, N \otimes L)$$

is an isomorphism for every L. Since every object of M_k has a left dual (Lemma 4.10) we conclude that

$$\mathrm{Hom}_{M_k}(K, M) \to \mathrm{Hom}_{M_k}(K, N)$$

is an isomorphism for every object K of M_k, see Categories, Lemma 43.6. We conclude by the Yoneda lemma (Categories, Lemma 3.5). $\qquad\square$

6. Projective space bundle formula

Let k be a base field. Let X be a smooth projective scheme over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Our convention is that the *projective bundle associated to* $\mathcal{E}$ is the morphism

$$P = \mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{E})) \xrightarrow{\ p\ } X$$

over X with $\mathcal{O}_P(1)$ normalized so that $p_*(\mathcal{O}_P(1)) = \mathcal{E}$. Recall that

$$[\Gamma_p] \in \mathrm{Corr}^0(X, P) \subset \mathrm{CH}^*(X \times P) \otimes \mathbf{Q}$$

See Example 3.2. For $i = 0, \ldots, r-1$ consider the correspondences

$$c_i = c_1(\mathrm{pr}_2^*\mathcal{O}_P(1))^i \cap [\Gamma_p] \in \mathrm{Corr}^i(X, P)$$

We may and do think of c_i as a morphism $h(X)(-i) \to h(P)$.

Lemma 6.1 (Projective space bundle formula). *In the situation above, the map*

$$\sum_{i=0,\ldots,r-1} c_i : \bigoplus_{i=0,\ldots,r-1} h(X)(-i) \longrightarrow h(P)$$

is an isomorphism in the category of motives.

Proof. By Lemma 5.3 it suffices to show that our map defines an isomorphism on Chow groups of motives after taking the product with any smooth projective scheme Z. Observe that $P \times Z \to X \times Z$ is the projective bundle associated to the pullback of $\mathcal{E}$ to $X \times Z$. Hence the statement on Chow groups is true by the projective space bundle formula given in Chow Homology, Lemma 36.2. Namely, pushforward of cycles along $[\Gamma_p]$ is given by pullback of cycles by p according to Lemma 3.6 and Chow Homology, Lemma 59.5. Hence pushforward along c_i sends α to $c_1(\mathcal{O}_P(1))^i \cap p^*\alpha$. Some details omitted. $\square$

In the situation above, for $j = 0, \ldots, r-1$ consider the correspondences

$$c_j' = c_1(\mathrm{pr}_1^*\mathcal{O}_P(1))^{r-1-j} \cap [\Gamma_p^t] \in \mathrm{Corr}^{-j}(P, X)$$

For $i, j \in \{0, \ldots, r-1\}$ we have

$$c_j' \circ c_i = \mathrm{pr}_{13,*}\left(c_1(\mathrm{pr}_2^*\mathcal{O}_P(1))^{i+r-1-j} \cap (\mathrm{pr}_{12}^*[\Gamma_p] \cdot \mathrm{pr}_{23}^*[\Gamma_p^t])\right)$$

The cycles $\mathrm{pr}_{12}^{-1}\Gamma_p$ and $\mathrm{pr}_{23}^{-1}\Gamma_p^t$ intersect transversally and with intersection equal to the image of $(p, 1, p) : P \to X \times P \times X$. Observe that the fibres of $(p, p) = \mathrm{pr}_{13} \circ (p, 1, p) : P \to X \times X$ have dimension $r - 1$. We immediately conclude $c_j' \circ c_i = 0$ for $i + r - 1 - j < r - 1$, in other words when $i < j$. On the other hand, by the projective space bundle formula (Chow Homology, Lemma 36.2) the cycle $c_1(\mathcal{O}_P(1))^{r-1} \cap [P]$ maps to $[X]$ in X. Hence for $i = j$ the pushforward above gives the class of the diagonal and hence we see that

$$c_i' \circ c_i = 1 \in \mathrm{Corr}^0(X, X)$$

for all $i \in \{0, \ldots, r-1\}$. Thus we see that the matrix of the composition

$$\bigoplus h(X)(-i) \xrightarrow{\ \oplus c_i\ } h(P) \xrightarrow{\ \oplus c_j'\ } \bigoplus h(X)(-j)$$

is invertible (upper triangular with 1s on the diagonal). We conclude from the projective space bundle formula (Lemma 6.1) that also the composition the other way around is invertible, but it seems a bit harder to prove this directly.

Lemma 6.2. *Let* $p : P \to X$ *be as in Lemma 6.1. The class* $[\Delta_P]$ *of the diagonal of* P *in* $\mathrm{CH}^*(P \times P)$ *can be written as*

$$[\Delta_P] = \left(\sum_{i=0,\ldots,r-1} \binom{r-1}{i} c_{r-1-i}(pr_1^* \mathcal{S}^\vee) \cap c_1(pr_2^* \mathcal{O}_P(1))^i \right) \cap (p \times p)^*[\Delta_X]$$

where $\mathcal{S}$ *is the kernel of the canonical surjection* $p^* \mathcal{E} \to \mathcal{O}_P(1)$.

Proof. Observe that $(p \times p)^*[\Delta_X] = [P \times_X P]$. Since $\Delta_P \subset P \times_X P \subset P \times P$ and since capping with Chern classes commutes with proper pushforward (Chow Homology, Lemma 38.4) it suffices to show that the class of $\Delta_P \subset P \times_X P$ in $\mathrm{CH}^*(P \times_X P)$ is equal to

$$\left(\sum_{i=0,\ldots,r-1} \binom{r-1}{i} c_{r-1-i}(q_1^* \mathcal{S}^\vee) \cap c_1(q_2^* \mathcal{O}_P(1))^i \right) \cap [P \times_X P]$$

where $q_i : P \times_X P \to P$, $i = 1, 2$ are the projections. Set $q = p \circ q_1 = p \circ q_2 :$ $P \times_X P \to X$. Consider the maps

$$q_1^* \mathcal{S} \otimes q_2^* \mathcal{O}_P(-1) \to q^* \mathcal{E} \otimes q^* \mathcal{E}^\vee \to \mathcal{O}_{P \times_X P}$$

where the final arrow is the pullback by q of the evaluation map $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee \to \mathcal{O}_X$. The source of the composition is a module locally free of rank $r - 1$ and a local calculation shows that this map vanishes exactly along Δ_P. By Chow Homology, Lemma 44.1 the class $[\Delta_P]$ is the top Chern class of the dual

$$q_1^* \mathcal{S}^\vee \otimes q_2^* \mathcal{O}_P(1)$$

The desired result follows from Chow Homology, Lemma 39.1. $\square$

7. Classical Weil cohomology theories

In this section we define what we will call a classical Weil cohomology theory. This is exactly what is called a Weil cohomology theory in [Kle68, Section 1.2].

We fix an algebraically closed field k (the base field). In this section *variety* will mean a variety over k, see Varieties, Section 3. We fix a field F of characteristic 0 (the coefficient field). A Weil cohomology theory is given by data (D1), (D2), and (D3) subject to axioms (A), (B), and (C).

The data is given by:

(D1) A contravariant functor H^* from the category of smooth projective varieties to the category of graded commutative F-algebras.

(D2) For every smooth projective variety X a group homomorphism $\gamma : \mathrm{CH}^i(X) \to H^{2i}(X)$.

(D3) For every smooth projective variety X of dimension d a map $\int_X : H^{2d}(X) \to F$.

We make some remarks to explain what this means and to introduce some terminology associated with this.

Remarks on (D1). Given a smooth projective variety X we say that $H^*(X)$ is the *cohomology* of X. Given a morphism $f : X \to Y$ of smooth projective varieties we denote $f^* : H^*(Y) \to H^*(X)$ the map $H^*(f)$ and we call it the *pullback map*.

Remarks on (D2). The map γ is called the *cycle class map*. We say that $\gamma(\alpha)$ is the *cohomology class* of α. If $Z \subset Y \subset X$ are closed subschemes with Y and X smooth projective varieties and Z integral, then $[Z]$ could mean the class of the cycle $[Z]$

in $\mathrm{CH}^*(Y)$ or in $\mathrm{CH}^*(X)$. In this case the notation $\gamma([Z])$ is ambiguous and the intended meaning has to be deduced from context.

Remarks on (D3). The map $\int_X$ is sometimes called the *trace map* and is sometimes denoted Tr_X.

The first axiom is often called *Poincaré duality*

(A) Let X be a smooth projective variety of dimension d. Then
   (a) $\dim_F H^i(X) < \infty$ for all i,
   (b) $H^i(X) \times H^{2d-i}(X) \to H^{2d}(X) \to F$ is a perfect pairing for all i where the final map is the trace map $\int_X$,
   (c) $H^i(X) = 0$ unless $i \in [0, 2d]$, and
   (d) $\int_X : H^{2d}(X) \to F$ is an isomorphism.

Let $f : X \to Y$ be a morphism of smooth projective varieties with $\dim(X) = d$ and $\dim(Y) = e$. Using Poincaré duality we can define a *pushforward*

$$f_* : H^{2d-i}(X) \longrightarrow H^{2e-i}(Y)$$

as the contragredient of the linear map $f^* : H^i(Y) \to H^i(X)$. In a formula, for $a \in H^{2d-i}(X)$, the element $f_* a \in H^{2e-i}(Y)$ is characterized by

$$\int_X f^* b \cup a = \int_Y b \cup f_* a$$

for all $b \in H^i(Y)$.

Lemma 7.1. *Assume given (D1) and (D3) satisfying (A). For $f : X \to Y$ a morphism of smooth projective varieties we have $f_*(f^* b \cup a) = b \cup f_* a$. If $g : Y \to Z$ is a second morphism of smooth projective varieties, then $g_* \circ f_* = (g \circ f)_*$.*

Proof. The first equality holds because

$$\int_Y c \cup b \cup f_* a = \int_X f^* c \cup f^* b \cup a = \int_Y c \cup f_*(f^* b \cup a).$$

The second equality holds because

$$\int_Z c \cup (g \circ f)_* a = \int_X (g \circ f)^* c \cup a = \int_X f^* g^* c \cup a = \int_Y g^* c \cup f_* a = \int_Z c \cup g_* f_* a$$

This ends the proof. $\square$

The second axiom says that H^* respects the monoidal structure given by products via the *Künneth formula*

(B) Let X and Y be smooth projective varieties. The map

$$H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y), \quad a \otimes b \mapsto \mathrm{pr}_1^* a \cup \mathrm{pr}_2^* b$$

   is an isomorphism.

The third axiom concerns the cycle class maps

(C) The cycle class maps satisfy the following rules
   (a) for a morphism $f : X \to Y$ of smooth projective varieties we have $\gamma(f^! \beta) = f^* \gamma(\beta)$ for $\beta \in \mathrm{CH}^*(Y)$,
   (b) for a morphism $f : X \to Y$ of smooth projective varieties we have $\gamma(f_* \alpha) = f_* \gamma(\alpha)$ for $\alpha \in \mathrm{CH}^*(X)$,
   (c) for any smooth projective variety X we have $\gamma(\alpha \cdot \beta) = \gamma(\alpha) \cup \gamma(\beta)$ for $\alpha, \beta \in \mathrm{CH}^*(X)$, and

(d) $\int_{\mathrm{Spec}(k)} \gamma([\mathrm{Spec}(k)]) = 1$.

Remark 7.2. Let X be a smooth projective variety. We obtain maps

$$H^*(X) \otimes_F H^*(X) \longrightarrow H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$$

where the first arrow is as in axiom (B) and Δ^* is pullback along the diagonal morphism $\Delta : X \to X \times X$. The composition is the cup product as pullback is an algebra homomorphism and $\mathrm{pr}_i \circ \Delta = \mathrm{id}$. On the other hand, given cycles α, β on X the intersection product is defined by the formula

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta)$$

In other words, $\alpha \cdot \beta$ is the pullback of the exterior product $\alpha \times \beta$ on $X \times X$ by the diagonal. Note also that $\alpha \times \beta = \mathrm{pr}_1^* \alpha \cdot \mathrm{pr}_2^* \beta$ in $\mathrm{CH}^*(X \times X)$ (we omit the proof). Hence, given axiom (C)(a), axiom (C)(c) is equivalent to the statement that γ is compatible with exterior product in the sense that $\gamma(\alpha \times \beta)$ is equal to $\mathrm{pr}_1^* \gamma(\alpha) \cup \mathrm{pr}_2^* \gamma(\beta)$. This is how axiom (C)(c) is formulated in [Kle68].

Definition 7.3. Let k be an algebraically closed field. Let F be a field of characteristic 0. A *classical Weil cohomology theory* over k with coefficients in F is given by data (D1), (D2), and (D3) satisfying Poincaré duality, the Künneth formula, and compatibility with cycle classes, more precisely, satisfying (A), (B), and (C).

We do a tiny bit of work.

Lemma 7.4. *Let H^* be a classical Weil cohomology theory (Definition 7.3). Let X be a smooth projective variety of dimension d. The diagram*

$$
\begin{array}{ccc}
\mathrm{CH}^d(X) & \xrightarrow{\ \gamma\ } & H^{2d}(X) \\
\| & & \downarrow{\scriptstyle \int_X} \\
\mathrm{CH}_0(X) & \xrightarrow{\ \deg\ } & F
\end{array}
$$

commutes where $\deg : \mathrm{CH}_0(X) \to \mathbf{Z}$ is the degree of zero cycles discussed in Chow Homology, Section 41.

Proof. The result holds for $\mathrm{Spec}(k)$ by axiom (C)(d). Let $x : \mathrm{Spec}(k) \to X$ be a closed point of X. Then we have $\gamma([x]) = x_* \gamma([\mathrm{Spec}(k)])$ in $H^{2d}(X)$ by axiom (C)(b). Hence $\int_X \gamma([x]) = 1$ by the definition of x_*. $\qquad\square$

Lemma 7.5. *Let H^* be a classical Weil cohomology theory (Definition 7.3). Let X and Y be smooth projective varieties. Then $\int_{X \times Y} = \int_X \otimes \int_Y$.*

Proof. Say $\dim(X) = d$ and $\dim(Y) = e$. By axiom (B) we have $H^{2d+2e}(X \times Y) = H^{2d}(X) \otimes H^{2e}(Y)$ and by axiom (A)(d) this is 1-dimensional. By Lemma 7.4 this 1-dimensional vector space generated by the class $\gamma([x \times y])$ of a closed point (x, y) and $\int_{X \times Y} \gamma([x \times y]) = 1$. Since $\gamma([x \times y]) = \gamma([x]) \otimes \gamma([y])$ by axioms (C)(a) and (C)(c) and since $\int_X \gamma([x]) = 1$ and $\int_Y \gamma([y]) = 1$ we conclude. $\qquad\square$

Lemma 7.6. *Let H^* be a classical Weil cohomology theory (Definition 7.3). Let X and Y be smooth projective varieties. Then $\mathrm{pr}_{2,*} : H^*(X \times Y) \to H^*(Y)$ sends $a \otimes b$ to $(\int_X a)b$.*

Proof. This is equivalent to the result of Lemma 7.5. $\qquad\square$

Lemma 7.7. *Let H^* be a classical Weil cohomology theory (Definition 7.3). Let X be a smooth projective variety of dimension d. Choose a basis $e_{i,j}, j = 1, \ldots, \beta_i$ of $H^i(X)$ over F. Using Künneth write*

$$\gamma([\Delta]) = \sum_{i=0,\ldots,2d} \sum_j e_{i,j} \otimes e'_{2d-i,j} \quad in \quad \bigoplus_i H^i(X) \otimes_F H^{2d-i}(X)$$

with $e'_{2d-i,j} \in H^{2d-i}(X)$. Then $\int_X e_{i,j} \cup e'_{2d-i,j'} = (-1)^i \delta_{jj'}$.

Proof. Recall that $\Delta^* : H^*(X \times X) \to H^*(X)$ is equal to the cup product map $H^*(X) \otimes_F H^*(X) \to H^*(X)$, see Remark 7.2. On the other hand we have $\gamma([\Delta]) = \Delta_* \gamma([X]) = \Delta_* 1$ by axiom (C)(b) and the fact that $\gamma([X]) = 1$. Namely, $[X] \cdot [X] = [X]$ hence by axiom (C)(c) the cohomology class $\gamma([X])$ is 0 or 1 in the 1-dimensional F-algebra $H^0(X)$; here we have also used axioms (A)(d) and (A)(b). But $\gamma([X])$ cannot be zero as $[X] \cdot [x] = [x]$ for a closed point x of X and we have the nonvanishing of $\gamma([x])$ by Lemma 7.4. Hence

$$\int_{X \times X} \gamma([\Delta]) \cup a \otimes b = \int_{X \times X} \Delta_* 1 \cup a \otimes b = \int_X a \cup b$$

by the definition of Δ_*. On the other hand, we have

$$\int_{X \times X} (\sum e_{i,j} \otimes e'_{2d-i,j}) \cup a \otimes b = \sum (\int_X a \cup e_{i,j})(\int_X e'_{2d-i,j} \cup b)$$

by Lemma 7.5; note that we made two switches of order so that the sign is 1. Thus if we choose a such that $\int_X a \cup e_{i,j} = 1$ and all other pairings equal to zero, then we conclude that $\int_X e'_{2d-i,j} \cup b = \int_X a \cup b$ for all b, i.e., $e'_{2d-i,j} = a$. This proves the lemma. $\qquad\square$

Lemma 7.8. *Let H^* be a classical Weil cohomology theory (Definition 7.3). Let X be a smooth projective variety. We have*

$$\sum_{i=0,\ldots,2\dim(X)} (-1)^i \dim_F H^i(X) = \deg([\Delta] \cdot [\Delta]) = \deg(c_d(\mathcal{T}_X) \cap [X])$$

Proof. Equality on the right. We have $[\Delta] \cdot [\Delta] = \Delta_*(\Delta^![\Delta])$ (Chow Homology, Lemma 62.6). Since Δ_* preserves degrees of 0-cycles it suffices to compute the degree of $\Delta^![\Delta]$. The class $\Delta^![\Delta]$ is given by capping $[\Delta]$ with the top Chern class of the normal sheaf of $\Delta \subset X \times X$ (Chow Homology, Lemma 54.5). Since the conormal sheaf of Δ is $\Omega_{X/k}$ (Morphisms, Lemma 32.7) we see that the normal sheaf is equal to the tangent sheaf $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ as desired.

Equality on the left. By Lemma 7.4 we have

$$\deg([\Delta] \cdot [\Delta]) = \int_{X \times X} \gamma([\Delta]) \cup \gamma([\Delta])$$

$$= \int_{X \times X} \Delta_* 1 \cup \gamma([\Delta])$$

$$= \int_{X \times X} \Delta_*(\Delta^* \gamma([\Delta]))$$

$$= \int_X \Delta^* \gamma([\Delta])$$

Write $\gamma([\Delta]) = \sum e_{i,j} \otimes e'_{2d-i,j}$ as in Lemma 7.7. Recalling that Δ^* is given by cup product we obtain

$$\int_X \sum_{i,j} e_{i,j} \cup e'_{2d-i,j} = \sum_{i,j} \int_X e_{i,j} \cup e'_{2d-i,j} = \sum_{i,j} (-1)^i = \sum (-1)^i \beta_i$$

as desired. $\square$

We will now tie classical Weil cohomology theories in with motives as follows.

Lemma 7.9. *Let k be an algebraically closed field. Let F be a field of characteristic 0. Consider a $\mathbf{Q}$-linear functor*

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories such that $G(\mathbf{1}(1))$ is nonzero only in degree -2. Then we obtain data (D1), (D2), (D3) satisfying all of (A), (B), (C) except for possibly (A)(c) and (A)(d).

Proof. We obtain a contravariant functor from the category of smooth projective varieties to the category of graded F-vector spaces by setting $H^*(X) = G(h(X))$. By assumption we have a canonical isomorphism

$$H^*(X \times Y) = G(h(X \times Y)) = G(h(X) \otimes h(Y)) = G(h(X)) \otimes G(h(Y)) = H^*(X) \otimes H^*(Y)$$

compatible with pullbacks. Using pullback along the diagonal $\Delta : X \to X \times X$ we obtain a canonical map

$$H^*(X) \otimes H^*(X) = H^*(X \times X) \to H^*(X)$$

of graded vector spaces compatible with pullbacks. This defines a functorial graded F-algebra structure on $H^*(X)$. Since Δ commutes with the commutativity constraint $h(X) \otimes h(X) \to h(X) \otimes h(X)$ (switching the factors) and since G is a functor of symmetric monoidal categories (so compatible with commutativity constraints), and by our convention in Homology, Example 17.4 we conclude that $H^*(X)$ is a graded commutative algebra! Hence we get our datum (D1).

Since $\mathbf{1}(1)$ is invertible in the category of motives we see that $G(\mathbf{1}(1))$ is invertible in the category of graded F-vector spaces. Thus $\sum_i \dim_F G^i(\mathbf{1}(1)) = 1$. By assumption we only get something nonzero in degree -2 and we may choose an isomorphism $F[2] \to G(\mathbf{1}(1))$ of graded F-vector spaces. Here and below $F[n]$ means the graded F-vector space which has F in degree $-n$ and zero elsewhere. Using compatibility with tensor products, we find for all $n \in \mathbf{Z}$ an isomorphism $F[2n] \to G(\mathbf{1}(n))$ compatible with tensor products.

Let X be a smooth projective variety. By Lemma 3.1 we have

$$\text{CH}^r(X) \otimes \mathbf{Q} = \text{Corr}^r(\text{Spec}(k), X) = \text{Hom}(\mathbf{1}(-r), h(X))$$

Applying the functor G we obtain

$$\gamma : \text{CH}^r(X) \otimes \mathbf{Q} \longrightarrow \text{Hom}(G(\mathbf{1}(-r)), H^*(X)) = H^{2r}(X)$$

This is the datum (D2).

Let X be a smooth projective variety of dimension d. By Lemma 3.1 we have

$$\text{Mor}(h(X)(d), \mathbf{1}) = \text{Mor}((X, 1, d), (\text{Spec}(k), 1, 0)) = \text{Corr}^{-d}(X, \text{Spec}(k)) = \text{CH}_d(X)$$

Thus the class of the cycle $[X]$ in $\mathrm{CH}_d(X)$ defines a morphism $h(X)(d) \to \mathbf{1}$. Applying G we obtain

$$H^*(X) \otimes F[-2d] = G(h(X)(d)) \longrightarrow G(\mathbf{1}) = F$$

This map is zero except in degree 0 where we obtain $\int_X : H^{2d}(X) \to F$. This is the datum (D3).

Let X be a smooth projective variety of dimension d. By Lemma 4.9 we know that $h(X)(d)$ is a left dual to $h(X)$. Hence $G(h(X)(d)) = H^*(X) \otimes F[-2d]$ is a left dual to $H^*(X)$ in the category of graded F-vector spaces. By Homology, Lemma 17.5 we find that $\sum_i \dim_F H^i(X) < \infty$ and that $\epsilon : h(X)(d) \otimes h(X) \to \mathbf{1}$ produces nondegenerate pairings $H^{2d-i}(X) \otimes_F H^i(X) \to F$. In the proof of Lemma 4.9 we have seen that ϵ is given by $[\Delta]$ via the identifications

$$\mathrm{Hom}(h(X)(d) \otimes h(X), \mathbf{1}) = \mathrm{Corr}^{-d}(X \times X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X \times X)$$

Thus ϵ is the composition of $[X] : h(X)(d) \to \mathbf{1}$ and $h(\Delta)(d) : h(X)(d) \otimes h(X) \to h(X)(d)$. It follows that the pairings above are given by cup product followed by $\int_X$. This proves axiom (A) parts (a) and (b).

Axiom (B) follows from the assumption that G is compatible with tensor structures and our construction of the cup product above.

Axiom (C). Our construction of γ takes a cycle α on X, interprets it as a correspondence a from $\mathrm{Spec}(k)$ to X of some degree, and then applies G. If $f : Y \to X$ is a morphism of smooth projective varieties, then $f^! \alpha$ is the pushforward (!) of α by the correspondence $[\Gamma_f]$ from X to Y, see Lemma 3.6. Hence $f^! \alpha$ viewed as a correspondence from $\mathrm{Spec}(k)$ to Y is equal to $a \circ [\Gamma_f]$, see Lemma 3.1. Since G is a functor, we conclude γ is compatible with pullbacks, i.e., axiom (C)(a) holds.

Let $f : Y \to X$ be a morphism of smooth projective varieties and let $\beta \in \mathrm{CH}^r(Y)$ be a cycle on Y. We have to show that

$$\int_Y \gamma(\beta) \cup f^*c = \int_X \gamma(f_*\beta) \cup c$$

for all $c \in H^*(X)$. Let $a, a^t, \eta_X, \eta_Y, [X], [Y]$ be as in Lemma 3.9. Let b be β viewed as a correspondence from $\mathrm{Spec}(k)$ to Y of degree r. Then $f_*\beta$ viewed as a correspondence from $\mathrm{Spec}(k)$ to X is equal to $a^t \circ b$, see Lemmas 3.6 and 3.1. The displayed equality above holds if we can show that

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{b \otimes 1} h(Y)(r) \otimes h(X) \xrightarrow{1 \otimes a} h(Y)(r) \otimes h(Y) \xrightarrow{\eta_Y} h(Y)(r) \xrightarrow{[Y]} \mathbf{1}(r-e)$$

is equal to

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{a^t \circ b \otimes 1} h(X)(r+d-e) \otimes h(X) \xrightarrow{\eta_X} h(X)(r+d-e) \xrightarrow{[X]} \mathbf{1}(r-e)$$

This follows immediately from Lemma 3.9. Thus we have axiom (C)(b).

To prove axiom (C)(c) we use the discussion in Remark 7.2. Hence it suffices to prove that γ is compatible with exterior products. Let X, Y be smooth projective varieties and let α, β be cycles on them. Denote a, b the corresponding correspondences from $\mathrm{Spec}(k)$ to X, Y. Then $\alpha \times \beta$ corresponds to the correspondence $a \otimes b$ from $\mathrm{Spec}(k)$ to $X \otimes Y = X \times Y$. Hence the requirement follows from the fact that G is compatible with the tensor structures on both sides.

Axiom (C)(d) follows because the cycle $[\mathrm{Spec}(k)]$ corresponds to the identity morphism on $h(\mathrm{Spec}(k))$. This finishes the proof of the lemma. $\square$

Lemma 7.10. *Let k be an algebraically closed field. Let F be a field of characteristic 0. Let H^* be a classical Weil cohomology theory. Then we can construct a $\mathbf{Q}$-linear functor*

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories such that $H^*(X) = G(h(X))$.

Proof. By Lemma 4.8 it suffices to construct a functor G on the category of smooth projective schemes over k with morphisms given by correspondences of degree 0 such that the image of $G(c_2)$ on $G(\mathbf{P}^1)$ is an invertible graded F-vector space. Since every smooth projective scheme is canonically a disjoint union of smooth projective varieties, it suffices to construct G on the category whose objects are smooth projective varieties and whose morphisms are correspondences of degree 0. (Some details omitted.)

Given a smooth projective variety X we set $G(X) = H^*(X)$.

Given a correspondence $c \in \mathrm{Corr}^0(X, Y)$ between smooth projective varieties we consider the map $G(c) : G(X) = H^*(X) \to G(Y) = H^*(Y)$ given by the rule

$$a \longmapsto G(c)(a) = \mathrm{pr}_{2,*}(\gamma(c) \cup \mathrm{pr}_1^* a)$$

It is clear that $G(c)$ is additive in c and hence $\mathbf{Q}$-linear. Compatibility of γ with pullbacks, pushforwards, and intersection products given by axioms (C)(a), (C)(b), and (C)(c) shows that we have $G(c' \circ c) = G(c') \circ G(c)$ if $c' \in \mathrm{Corr}^0(Y, Z)$. Namely, for $a \in H^*(X)$ we have

$$
\begin{aligned}
(G(c') \circ G(c))(a) &= \mathrm{pr}_{3,*}^{23}(\gamma(c') \cup \mathrm{pr}_2^{23,*}(\mathrm{pr}_{2,*}^{12}(\gamma(c) \cup \mathrm{pr}_1^{12,*}a))) \\
&= \mathrm{pr}_{3,*}^{23}(\gamma(c') \cup \mathrm{pr}_{23,*}^{123}(\mathrm{pr}_{12}^{123,*}(\gamma(c) \cup \mathrm{pr}_1^{12,*}a))) \\
&= \mathrm{pr}_{3,*}^{23}\mathrm{pr}_{23,*}^{123}(\mathrm{pr}_{23}^{123,*}\gamma(c') \cup \mathrm{pr}_{12}^{123,*}\gamma(c) \cup \mathrm{pr}_1^{123,*}a) \\
&= \mathrm{pr}_{3,*}^{23}\mathrm{pr}_{23,*}^{123}(\gamma(\mathrm{pr}_{23}^{123,*}c') \cup \gamma(\mathrm{pr}_{12}^{123,*}c) \cup \mathrm{pr}_1^{123,*}a) \\
&= \mathrm{pr}_{3,*}^{13}\mathrm{pr}_{13,*}^{123}(\gamma(\mathrm{pr}_{23}^{123,*}c' \cdot \mathrm{pr}_{12}^{123,*}c) \cup \mathrm{pr}_1^{123,*}a) \\
&= \mathrm{pr}_{3,*}^{13}(\gamma(\mathrm{pr}_{13,*}^{123}(\mathrm{pr}_{23}^{123,*}c' \cdot \mathrm{pr}_{12}^{123,*}c)) \cup \mathrm{pr}_1^{13,*}a) \\
&= G(c' \circ c)(a)
\end{aligned}
$$

with obvious notation. The first equality follows from the definitions. The second equality holds because $\mathrm{pr}_2^{23,*} \circ \mathrm{pr}_{2,*}^{12} = \mathrm{pr}_{23,*}^{123} \circ \mathrm{pr}_{12}^{123,*}$ as follows immediately from the description of pushforward along projections given in Lemma 7.6. The third equality holds by Lemma 7.1 and the fact that H^* is a functor. The fourth equalith holds by axiom (C)(a) and the fact that the gysin map agrees with flat pullback for flat morphisms (Chow Homology, Lemma 59.5). The fifth equality uses axiom (C)(c) as well as Lemma 7.1 to see that $\mathrm{pr}_{3,*}^{23} \circ \mathrm{pr}_{23,*}^{123} = \mathrm{pr}_{3,*}^{13} \circ \mathrm{pr}_{13,*}^{123}$. The sixth equality uses the projection formula from Lemma 7.1 as well as axiom (C)(b) to see that $\mathrm{pr}_{13,*}^{123}\gamma(\mathrm{pr}_{23}^{123,*}c' \cdot \mathrm{pr}_{12}^{123,*}c) = \gamma(\mathrm{pr}_{13,*}^{123}(\mathrm{pr}_{23}^{123,*}c' \cdot \mathrm{pr}_{12}^{123,*}c))$. Finally, the last equality is the definition.

To finish the proof that G is a functor, we have to show identities are preserved. In other words, if $1 = [\Delta] \in \mathrm{Corr}^0(X, X)$ is the identity in the category of correspondences (see Lemma 3.3 and its proof), then we have to show that $G([\Delta]) = \mathrm{id}$.

This follows from the determination of $\gamma([\Delta])$ in Lemma 7.7 and Lemma 7.6. This finishes the construction of G as a functor on smooth projective varieties and correspondences of degree 0.

It follows from axioms (A)(c) and (A)(d) that $G(\mathrm{Spec}(k)) = H^*(\mathrm{Spec}(k))$ is canonically isomorphic to F as an F-algebra. The Künneth axiom (B) shows our functor is compatible with tensor products. Thus our functor is a functor of symmetric monoidal categories.

We still have to check that the image of $G(c_2)$ on $G(\mathbf{P}^1)$ is an invertible graded F-vector space (in particular we don't know yet that G extends to M_k). By axiom (A)(d) the map $\int_{\mathbf{P}^1} : H^2(\mathbf{P}^1) \to F$ is an isomorphism. By axiom (A)(b) we see that $\dim_F H^0(\mathbf{P}^1) = 1$. By Lemma 7.8 and axiom (A)(c) we obtain $2 - \dim_F H^1(\mathbf{P}^1) = c_1(T_{\mathbf{P}^1}) = 2$. Hence $H^1(\mathbf{P}^1) = 0$. Thus

$$G(\mathbf{P}^1) = H^0(\mathbf{P}^1) \oplus H^2(\mathbf{P}^1)$$

Recall that $1 = c_0 + c_2$ is a decomposition of the identity into a sum of orthogonal idempotents in $\mathrm{Corr}^0(\mathbf{P}^1, \mathbf{P}^1)$, see Example 3.7. We have $c_0 = a \circ b$ where $a \in \mathrm{Corr}^0(\mathrm{Spec}(k), \mathbf{P}^1)$ and $b \in \mathrm{Corr}^0(\mathbf{P}^1, \mathrm{Spec}(k))$ and where $b \circ a = 1$ in $\mathrm{Corr}^0(\mathrm{Spec}(k), \mathrm{Spec}(k))$, see proof of Lemma 4.4. Since $F = G(\mathrm{Spec}(k))$, it follows from functoriality that $G(c_0)$ is the projector onto the summand $H^0(\mathbf{P}^1) \subset G(\mathbf{P}^1)$. Hence $G(c_2)$ must necessarily be the projection onto $H^2(\mathbf{P}^1)$ and the proof is complete. $\qquad\square$

Proposition 7.11. *Let k be an algebraically closed field. Let F be a field of characteristic 0. A classical Weil cohomology theory is the same thing as a $\mathbf{Q}$-linear functor*

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories together with an isomorphism $F[2] \to G(\mathbf{1}(1))$ of graded F-vector spaces such that in addition

(1) *$G(h(X))$ lives in nonnegative degrees, and*
(2) *$\dim_F G^0(h(X)) = 1$*

for any smooth projective variety X.

Proof. Given G and $F[2] \to G(\mathbf{1}(1))$ by setting $H^*(X) = G(h(X))$ we obtain data (D1), (D2), and (D3) satisfying all of (A), (B), and (C) except for possibly (A)(c) and (A)(d), see Lemma 7.9 and its proof. Observe that assumptions (1) and (2) imply axioms (A)(c) and (A)(d) in the presence of the known axioms (A)(a) and (A)(b).

Conversely, given H^* we get a functor G by the construction of Lemma 7.10. Let $X = \mathbf{P}^1, c_0, c_2$ be as in Example 3.7. We have constructed an isomorphism $1(-1) \to (X, c_2, 0)$ of motives in Lemma 4.4. In the proof of Lemma 7.10 we have seen that $G(1(-1)) = G(X, c_2, 0) = H^2(\mathbf{P}^1)[-2]$. Hence the isomorphism $\int_{\mathbf{P}^1} : H^2(\mathbf{P}^1) \to F$ of axiom (A)(d) gives an isomorphism $G(1(-1)) \to F[-2]$ which determines an isomorphism $F[2] \to G(\mathbf{1}(1))$. Finally, since $G(h(X)) = H^*(X)$ assumptions (1) and (2) follow from axiom (A). $\qquad\square$

8. Cycles over non-closed fields

Some lemmas which will help us in our study of motives over base fields which are not algebraically closed.

Lemma 8.1. *Let k be a field. Let X be a smooth projective scheme over k. Then $\mathrm{CH}_0(X)$ is generated by classes of closed points whose residue fields are separable over k.*

Proof. The lemma is immediate if k has characteristic 0 or is perfect. Thus we may assume k is an infinite field of characteristic $p > 0$.

We may assume X is irreducible of dimension d. Then $k' = H^0(X, \mathcal{O}_X)$ is a finite separable field extension of k and that X is geometrically integral over k'. See Varieties, Lemmas 25.4, 9.3, and 9.4. We may and do replace k by k' and assume that X is geometrically integral.

Let $x \in X$ be a closed point. To prove the lemma we are going to show that $[x] \in \mathrm{CH}_0(X)$ is rationally equivalent to an integer linear combination of classes of closed points whose residue fields are separable over k. Choose an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$. Set

$$V = \{s \in H^0(X, \mathcal{L}) \mid s(x) = 0\}$$

After replacing $\mathcal{L}$ by a power we may assume (a) $\mathcal{L}$ is very ample, (b) V generates $\mathcal{L}$ over $X \setminus x$, (c) the morphism $X \setminus x \to \mathbf{P}(V)$ is an immersion, (d) the map $V \to \mathfrak{m}_x \mathcal{L}_x / \mathfrak{m}_x^2 \mathcal{L}_x$ is surjective, see Morphisms, Lemma 39.5, Varieties, Lemma 47.1, and Properties, Proposition 26.13. Consider the set

$$V^d \supset U = \{(s_1, \ldots, s_d) \in V^d \mid s_1, \ldots, s_d \text{ generate } \mathfrak{m}_x \mathcal{L}_x / \mathfrak{m}_x^2 \mathcal{L}_x \text{ over } \kappa(x)\}$$

Since $\mathcal{O}_{X,x}$ is a regular local ring of dimension d we have $\dim_{\kappa(x)}(\mathfrak{m}_x / \mathfrak{m}_x^2) = d$ and hence we see that U is a nonempty (Zariski) open of V^d. For $(s_1, \ldots, s_d) \in U$ set $H_i = Z(s_i)$. Since $s_1, \ldots, s_d$ generate $\mathfrak{m}_x \mathcal{L}_x$ we see that

$$H_1 \cap \ldots \cap H_d = x \amalg Z$$

scheme theoretically for some closed subscheme $Z \subset X$. By Bertini (in the form of Varieties, Lemma 47.3) for a general element $s_1 \in V$ the scheme $H_1 \cap (X \setminus x)$ is smooth over k of dimension $d - 1$. Having chosen s_1, for a general element $s_2 \in V$ the scheme $H_1 \cap H_2 \cap (X \setminus x)$ is smooth over k of dimension $d - 2$. And so on. We conclude that for sufficiently general $(s_1, \ldots, s_d) \in U$ the scheme Z is étale over $\mathrm{Spec}(k)$. In particular $H_1 \cap \ldots \cap H_d$ has dimension 0 and hence

$$[H_1] \cdot \ldots \cdot [H_d] = [x] + [Z]$$

in $\mathrm{CH}_0(X)$ by repeated application of Chow Homology, Lemma 62.5 (details omitted). This finishes the proof as it shows that $[x] \sim_{rat} -[Z] + [Z']$ where $Z' = H_1' \cap \ldots \cap H_d'$ is a general complete intersection of vanishing loci of sufficiently general sections of $\mathcal{L}$ which will be étale over k by the same argument as before. □

Lemma 8.2. *Let K/k be an algebraic field extension. Let X be a finite type scheme over k. Then $\mathrm{CH}_i(X_K) = \mathrm{colim}\, \mathrm{CH}_i(X_{k'})$ where the colimit is over the subextensions $K/k'/k$ with k'/k finite.*

Proof. This is a special case of Chow Homology, Lemma 67.10. □

Lemma 8.3. *Let k be a field. Let X be a geometrically irreducible smooth projective scheme over k. Let $x, x' \in X$ be k-rational points. Let n be an integer invertible in k. Then there exists a finite separable extension k'/k such that the pullback of $[x] - [x']$ to $X_{k'}$ is divisible by n in $\mathrm{CH}_0(X_{k'})$.*

Proof. Let k' be a separable algebraic closure of k. Suppose that we can show the the pullback of $[x] - [x']$ to $X_{k'}$ is divisible by n in $\mathrm{CH}_0(X_{k'})$. Then we conclude by Lemma 8.2. Thus we may and do assume k is separably algebraically closed.

Suppose $\dim(X) > 1$. Let $\mathcal{L}$ be an ample invertible sheaf on X. Set

$$V = \{s \in H^0(X, \mathcal{L}) \mid s(x) = 0 \text{ and } s(x') = 0\}$$

After replacing $\mathcal{L}$ by a power we see that for a general $v \in V$ the corresponding divisor $H_v \subset X$ is smooth away from x and x', see Varieties, Lemmas 47.1 and 47.3. To find v we use that k is infinite (being separably algebraically closed). If we choose s general, then the image of s in $\mathfrak{m}_x \mathcal{L}_x / \mathfrak{m}_x^2 \mathcal{L}_x$ will be nonzero, which implies that H_v is smooth at x (details omitted). Similarly for x'. Thus H_v is smooth. By Varieties, Lemma 48.3 (applied to the base change of everything to the algebraic closure of k) we see that H_v is geometrically connected. It suffices to prove the result for $[x] - [x']$ seen as an element of $\mathrm{CH}_0(H_v)$. In this way we reduce to the case of a curve.

Assume X is a curve. Then we see that $\mathcal{O}_X(x - x')$ defines a k-rational point g of $J = \underline{\mathrm{Pic}}^0_{X/k}$, see Picard Schemes of Curves, Lemma 6.7. Recall that J is a proper smooth variety over k which is also a group scheme over k (same reference). Hence J is geometrically integral (see Varieties, Lemma 7.13 and 25.4). In other words, J is an abelian variety, see Groupoids, Definition 9.1. Thus $[n] : J \to J$ is finite étale by Groupoids, Proposition 9.11 (this is where we use n is invertible in k). Since k is separably closed we conclude that $g = [n](g')$ for some $g' \in J(k)$. If $\mathcal{L}$ is the degree 0 invertible module on X corresponding to g', then we conclude that $\mathcal{O}_X(x - x') \cong \mathcal{L}^{\otimes n}$ as desired. $\square$

Lemma 8.4. *Let K/k be an algebraic extension of fields. Let X be a finite type scheme over k. The kernel of the map $\mathrm{CH}_i(X) \to \mathrm{CH}_i(X_K)$ constructed in Lemma 8.2 is torsion.*

Proof. It clearly suffices to show that the kernel of flat pullback $\mathrm{CH}_i(X) \to \mathrm{CH}_i(X_{k'})$ by $\pi : X_{k'} \to X$ is torsion for any finite extension k'/k. This is clear because $\pi_* \pi^* \alpha = [k' : k]\alpha$ by Chow Homology, Lemma 15.2. $\square$

Lemma 8.5 (Voevodsky). *Let k be a field. Let X be a geometrically irreducible smooth projective scheme over k. Let $x, x' \in X$ be k-rational points. For n large enough the class of the zero cycle*

$$([x] - [x']) \times \ldots \times ([x] - [x']) \in \mathrm{CH}_0(X^n)$$

is torsion.

Proof. If we can show this after base change to the algebraic closure of k, then the result follows over k because the kernel of pullback is torsion by Lemma 8.4. Hence we may and do assume k is algebraically closed.

Using Bertini we can choose a smooth curve $C \subset X$ passing through x and x'. See proof of Lemma 8.3. Hence we may assume X is a curve.

Assume X is a curve and k is algebraically closed. Write $S^n(X) = \underline{\mathrm{Hilb}}^n_{X/k}$ with notation as in Picard Schemes of Curves, Sections 2 and 3. There is a canonical morphism

$$\pi : X^n \longrightarrow S^n(X)$$

which sends the k-rational point $(x_1, \ldots, x_n)$ to the k-rational point corresponding to the divisor $[x_1] + \ldots + [x_n]$ on X. There is a faithful action of the symmetric group S_n on X^n. The morphism π is S_n-invariant and the fibres of π are S_n-orbits (set theoretically). Finally, π is finite flat of degree $n!$, see Picard Schemes of Curves, Lemma 3.4.

Let α_n be the zero cycle on X^n given by the formula in the statement of the lemma. Let $\mathcal{L} = \mathcal{O}_X(x - x')$. Then $c_1(\mathcal{L}) \cap [X] = [x] - [x']$. Thus

$$\alpha_n = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_n) \cap [X^n]$$

where $\mathcal{L}_i = \mathrm{pr}_i^* \mathcal{L}$ and $\mathrm{pr}_i : X^n \to X$ is the ith projection. By either Divisors, Lemma 17.6 or Divisors, Lemma 17.7 there is a norm for π. Set $\mathcal{N} = \mathrm{Norm}_\pi(\mathcal{L}_1)$, see Divisors, Lemma 17.2. We have

$$\pi^* \mathcal{N} = (\mathcal{L}_1 \otimes \ldots \otimes \mathcal{L}_n)^{\otimes(n-1)!}$$

in $\mathrm{Pic}(X^n)$ by a calculation. Deails omitted; hint: this follows from the fact that $\mathrm{Norm}_\pi : \pi_* \mathcal{O}_{X^n} \to \mathcal{O}_{S^n(X)}$ composed with the natural map $\pi_* \mathcal{O}_{S^n(X)} \to \mathcal{O}_{X^n}$ is equal to the product over all $\sigma \in S_n$ of the action of σ on $\pi_* \mathcal{O}_{X^n}$. Consider

$$\beta_n = c_1(\mathcal{N})^n \cap [S^n(X)]$$

in $\mathrm{CH}_0(S^n(X))$. Observe that $c_1(\mathcal{L}_i) \cap c_1(\mathcal{L}_i) = 0$ because $\mathcal{L}_i$ is pulled back from a curve, see Chow Homology, Lemma 34.6. Thus we see that

$$\begin{aligned}
\pi^* \beta_n &= ((n-1)!)^n \big(\sum_{i=1,\ldots,n} c_1(\mathcal{L}_i) \big)^n \cap [X^n] \\
&= ((n-1)!)^n n^n c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_n) \cap [X^n] \\
&= (n!)^n \alpha_n
\end{aligned}$$

Thus it suffices to show that β_n is torsion.

There is a canonical morphism

$$f : S^n(X) \longrightarrow \underline{\mathrm{Pic}}^n_{X/k}$$

See Picard Schemes of Curves, Lemma 6.7. For $n \geq 2g - 1$ this morphism is a projective space bundle (details omitted; compare with the proof of Picard Schemes of Curves, Lemma 6.7). The invertible sheaf $\mathcal{N}$ is trivial on the fibres of f, see below. Thus by the projective space bundle formula (Chow Homology, Lemma 36.2) we see that $\mathcal{N} = f^* \mathcal{M}$ for some invertible module $\mathcal{M}$ on $\underline{\mathrm{Pic}}^n_{X/k}$. Of course, then we see that

$$c_1(\mathcal{N})^n = f^*(c_1(\mathcal{M})^n)$$

is zero because $n > g = \dim(\underline{\mathrm{Pic}}^n_{X/k})$ and we can use Chow Homology, Lemma 34.6 as before.

We still have to show that $\mathcal{N}$ is trivial on a fibre F of f. Since the fibres of f are projective spaces and since $\mathrm{Pic}(\mathbf{P}^m_k) = \mathbf{Z}$ (Divisors, Lemma 28.5), this can be shown by computing the degree of $\mathcal{N}$ on a line contained in the fibre. Instead we will prove it by proving that $\mathcal{N}$ is algebraically equivalent to zero. First we claim there is a connected finite type scheme T over k, an invertible module $\mathcal{L}'$

on $T \times X$ and k-rational points $p, q \in T$ such that $\mathcal{M}_p \cong \mathcal{O}_X$ and $\mathcal{M}_q = \mathcal{L}$. Namely, since $\mathcal{L} = \mathcal{O}_X(x - x')$ we can take $T = X$, $p = x'$, $q = x$, and $\mathcal{L}' = \mathcal{O}_{X \times X}(\Delta) \otimes \mathrm{pr}_2^* \mathcal{O}_X(-x')$. Then we let $\mathcal{L}'_i$ on $T \times X^n$ for $i = 1, \ldots, n$ be the pullback of $\mathcal{L}'$ by $\mathrm{id}_T \times \mathrm{pr}_i : T \times X^n \to T \times X$. Finally, we let $\mathcal{N}' = \mathrm{Norm}_{\mathrm{id}_T \times \pi}(\mathcal{L}'_1)$ on $T \times S^n(X)$. By construction we have $\mathcal{N}'_p = \mathcal{O}_{S^n(X)}$ and $\mathcal{N}'_q = \mathcal{N}$. We conclude that

$$\mathcal{N}'|_{T \times F}$$

is an invertible module on $T \times F \cong T \times \mathbf{P}_k^m$ whose fibre over p is the trivial invertible module and whose fibre over q is $\mathcal{N}|_F$. Since the euler characteristic of the trivial bundle is 1 and since this euler characteristic is locally constant in families (Derived Categories of Schemes, Lemma 32.2) we conclude $\chi(F, \mathcal{N}^{\otimes s}|_F) = 1$ for all $s \in \mathbf{Z}$. This can happen only if $\mathcal{N}|_F \cong \mathcal{O}_F$ (see Cohomology of Schemes, Lemma 8.1) and the proof is complete. Some details omitted. $\square$

9. Weil cohomology theories, I

This section is the analogue of Section 7 over arbitrary fields. In other words, we work out what data and axioms correspond to functors G of symmetric monoidal categories from the category of motives to the category of graded vector spaces such that $G(\mathbf{1}(1))$ sits in degree -2. In Section 11 we will define a Weil cohomology theory by adding a single suplementary condition.

We fix a field k (the base field). We fix a field F of characteristic 0 (the coefficient field). The data is given by:

- (D0) A 1-dimensional F-vector space $F(1)$.
- (D1) A contravariant functor H^* from the category of smooth projective schemes over k to the category of graded commutative F-algebras.
- (D2) For every smooth projective scheme X over k a group homomorphism $\gamma : \mathrm{CH}^i(X) \to H^{2i}(X)(i)$.
- (D3) For every nonempty smooth projective scheme X over k which is equidimensional of dimension d a map $\int_X : H^{2d}(X)(d) \to F$.

We make some remarks to explain what this means and to introduce some terminology associated with this.

Remarks on (D0). The vector space $F(1)$ gives rise to *Tate twists* on the category of F-vector spaces. Namely, for $n \in \mathbf{Z}$ we set $F(n) = F(1)^{\otimes n}$ if $n \geq 0$, we set $F(-1) = \mathrm{Hom}_F(F(1), F)$, and we set $F(n) = F(-1)^{\otimes -n}$ if $n < 0$. Please compare with More on Algebra, Section 117. For an F-vector space V we define the *nth Tate twist*

$$V(n) = V \otimes_F F(n)$$

We will use obvious notation, e.g., given F-vector spaces U, V and W and a linear map $U \otimes_F V \to W$ we obtain a linear map $U(n) \otimes_F V(m) \to W(n+m)$ for $n, m \in \mathbf{Z}$.

Remarks on (D1). Given a smooth projective scheme X over k we say that $H^*(X)$ is the *cohomology* of X. Given a morphism $f : X \to Y$ of smooth projective schemes over k we denote $f^* : H^*(Y) \to H^*(X)$ the map $H^*(f)$ and we call it the *pullback map*.

Remarks on (D2). The map γ is called the *cycle class map*. We say that $\gamma(\alpha)$ is the *cohomology class* of α. If $Z \subset Y \subset X$ are closed subschemes with Y and X smooth projective over k and Z integral, then $[Z]$ could mean the class of the cycle

$[Z]$ in $\mathrm{CH}^*(Y)$ or in $\mathrm{CH}^*(X)$. In this case the notation $\gamma([Z])$ is ambiguous and the intended meaning has to be deduced from context.

Remarks on (D3). The map $\int_X$ is sometimes called the *trace map* and is sometimes denoted Tr_X.

The first axiom is often called *Poincaré duality*

  (A) Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. Then
    (a) $\dim_F H^i(X) < \infty$ for all i,
    (b) $H^i(X) \times H^{2d-i}(X)(d) \to H^{2d}(X)(d) \to F$ is a perfect pairing for all i where the final map is the trace map $\int_X$.

Let $f : X \to Y$ be a morphism of nonempty smooth projective schemes with X equidimensional of dimension d and Y is equidimensional of dimension e. Using Poincaré duality we can define a *pushforward*

$$f_* : H^{2d-i}(X)(d) \longrightarrow H^{2e-i}(Y)(e)$$

as the contragredient of the linear map $f^* : H^i(Y) \to H^i(X)$. In a formula, for $a \in H^{2d-i}(X)(d)$, the element $f_*a \in H^{2e-i}(Y)(e)$ is characterized by

$$\int_X f^*b \cup a = \int_Y b \cup f_*a$$

for all $b \in H^i(Y)$.

Lemma 9.1. *Assume given (D0), (D1), and (D3) satisfying (A). For $f : X \to Y$ a morphism of nonempty equidimensional smooth projective schemes over k we have $f_*(f^*b \cup a) = b \cup f_*a$. If $g : Y \to Z$ is a second morphism with Z nonempty smooth projective and equidimensional, then $g_* \circ f_* = (g \circ f)_*$.*

Proof. The first equality holds because

$$\int_Y c \cup b \cup f_*a = \int_X f^*c \cup f^*b \cup a = \int_Y c \cup f_*(f^*b \cup a).$$

The second equality holds because

$$\int_Z c \cup (g \circ f)_*a = \int_X (g \circ f)^*c \cup a = \int_X f^*g^*c \cup a = \int_Y g^*c \cup f_*a = \int_Z c \cup g_*f_*a$$

This ends the proof. $\qquad\square$

The second axiom says that H^* respects the monoidal structure given by products via the *Künneth formula*

  (B) Let X and Y be smooth projective schemes over k.
    (a) $H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y)$, $\alpha \otimes \beta \mapsto \mathrm{pr}_1^*\alpha \cup \mathrm{pr}_2^*\beta$ is an isomorphism,
    (b) if X and Y are nonempty and equidimensional, then $\int_{X \times Y} = \int_X \otimes \int_Y$ via (a).

Using axiom (B)(b) we can compute pushforwards along projections.

Lemma 9.2. *Assume given (D0), (D1), and (D3) satisfying (A) and (B). Let X and Y be nonempty smooth projective schemes over k equidimensional of dimensions d and e. Then $\mathrm{pr}_{2,*} : H^*(X \times Y)(d + e) \to H^*(Y)(e)$ sends $a \otimes b$ to $(\int_X a)b$.*

Proof. This follows from axioms (B)(a) and (B)(b). $\qquad\square$

The third axiom concerns the cycle class maps

(C) The cycle class maps satisfy the following rules
- (a) for a morphism $f : X \to Y$ of smooth projective schemes over k we have $\gamma(f^! \beta) = f^* \gamma(\beta)$ for $\beta \in \mathrm{CH}^*(Y)$,
- (b) for a morphism $f : X \to Y$ of nonempty equidimensional smooth projective schemes over k we have $\gamma(f_* \alpha) = f_* \gamma(\alpha)$ for $\alpha \in \mathrm{CH}^*(X)$,
- (c) for any smooth projective scheme X over k we have $\gamma(\alpha \cdot \beta) = \gamma(\alpha) \cup \gamma(\beta)$ for $\alpha, \beta \in \mathrm{CH}^*(X)$, and
- (d) $\int_{\mathrm{Spec}(k)} \gamma([\mathrm{Spec}(k)]) = 1$.

Let us elucidate axiom (C)(b). Namely, say $f : X \to Y$ is as in (C)(b) with $\dim(X) = d$ and $\dim(Y) = e$. Then we see that pushforward on Chow groups gives

$$f_* : \mathrm{CH}^{d-i}(X) = \mathrm{CH}_i(X) \to \mathrm{CH}_i(Y) = \mathrm{CH}^{e-i}(Y)$$

Say $\alpha \in \mathrm{CH}^{d-i}(X)$. On the one hand, we have $f_* \alpha \in \mathrm{CH}^{e-i}(Y)$ and hence $\gamma(f_* \alpha) \in H^{2e-2i}(Y)(e-i)$. On the other hand, we have $\gamma(\alpha) \in H^{2d-2i}(X)(d-i)$ and hence $f_* \gamma(\alpha) \in H^{2e-2i}(Y)(e-i)$ as well. Thus the condition $\gamma(f_* \alpha) = f_* \gamma(\alpha)$ makes sense.

Remark 9.3. Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C)(a). Let X be a smooth projective scheme over k. We obtain maps

$$H^*(X) \otimes_F H^*(X) \longrightarrow H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$$

where the first arrow is as in axiom (B) and Δ^* is pullback along the diagonal morphism $\Delta : X \to X \times X$. The composition is the cup product as pullback is an algebra homomorphism and $\mathrm{pr}_i \circ \Delta = \mathrm{id}$. On the other hand, given cycles α, β on X the intersection product is defined by the formula

$$\alpha \cdot \beta = \Delta^!(\alpha \times \beta)$$

In other words, $\alpha \cdot \beta$ is the pullback of the exterior product $\alpha \times \beta$ on $X \times X$ by the diagonal. Note also that $\alpha \times \beta = \mathrm{pr}_1^* \alpha \cdot \mathrm{pr}_2^* \beta$ in $\mathrm{CH}^*(X \times X)$ (we omit the proof). Hence, given axiom (C)(a), axiom (C)(c) is equivalent to the statement that γ is compatible with exterior product in the sense that $\gamma(\alpha \times \beta)$ is equal to $\mathrm{pr}_1^* \gamma(\alpha) \cup \mathrm{pr}_2^* \gamma(\beta)$.

Lemma 9.4. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Then $H^i(\mathrm{Spec}(k)) = 0$ for $i \neq 0$ and there is a unique F-algebra isomorphism $F = H^0(\mathrm{Spec}(k))$. We have $\gamma([\mathrm{Spec}(k)]) = 1$ and $\int_{\mathrm{Spec}(k)} 1 = 1$.*

Proof. By axiom (C)(d) we see that $H^0(\mathrm{Spec}(k))$ is nonzero and even $\gamma([\mathrm{Spec}(k)])$ is nonzero. Since $\mathrm{Spec}(k) \times \mathrm{Spec}(k) = \mathrm{Spec}(k)$ we get

$$H^*(\mathrm{Spec}(k)) \otimes_F H^*(\mathrm{Spec}(k)) = H^*(\mathrm{Spec}(k))$$

by axiom (B)(a) which implies (look at dimensions) that only H^0 is nonzero and moreover has dimension 1. Thus $F = H^0(\mathrm{Spec}(k))$ via the unique F-algebra isomorphism given by mapping $1 \in F$ to $1 \in H^0(\mathrm{Spec}(k))$. Since $[\mathrm{Spec}(k)] \cdot [\mathrm{Spec}(k)] = [\mathrm{Spec}(k)]$ in the Chow ring of $\mathrm{Spec}(k)$ we conclude that $\gamma([\mathrm{Spec}(k)]) \cup \gamma([\mathrm{Spec}(k)]) = \gamma([\mathrm{Spec}(k)])$ by axiom (C)(c). Since we already know that $\gamma([\mathrm{Spec}(k)])$ is nonzero we conclude that it has to be equal to 1. Finally, we have $\int_{\mathrm{Spec}(k)} 1 = 1$ since $\int_{\mathrm{Spec}(k)} \gamma([\mathrm{Spec}(k)]) = 1$ by axiom (C)(d). $\square$

Lemma 9.5. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a smooth projective scheme over k. If $X = \emptyset$, then $H^*(X) = 0$. If X is nonempty, then $\gamma([X]) = 1$ and $1 \neq 0$ in $H^0(X)$.*

Proof. First assume X is nonempty. Observe that $[X]$ is the pullback of $[\mathrm{Spec}(k)]$ by the structure morphism $p : X \to \mathrm{Spec}(k)$. Hence we get $\gamma([X]) = 1$ by axiom (C)(a) and Lemma 9.4. Let $X' \subset X$ be an irreducible component. By functoriality it suffices to show $1 \neq 0$ in $H^0(X')$. Thus we may and do assume X is irreducible, and in particular nonempty and equidimensional, say of dimension d. To see that $1 \neq 0$ it suffices to show that $H^*(X)$ is nonzero.

Let $x \in X$ be a closed point whose residue field k' is separable over k, see Varieties, Lemma 25.6. Let $i : \mathrm{Spec}(k') \to X$ be the inclusion morphism. Denote $p : X \to \mathrm{Spec}(k)$ is the structure morphism. Observe that $p_* i_* [\mathrm{Spec}(k')] = [k' : k][\mathrm{Spec}(k)]$ in $\mathrm{CH}_0(\mathrm{Spec}(k))$. Using axiom (C)(b) twice and Lemma 9.4 we conclude that

$$p_* i_* \gamma([\mathrm{Spec}(k')]) = \gamma([k' : k][\mathrm{Spec}(k)]) = [k' : k] \in F = H^0(\mathrm{Spec}(k))$$

is nonzero. Thus $i_* \gamma([\mathrm{Spec}(k)]) \in H^{2d}(X)(d)$ is nonzero (because it maps to something nonzero via p_*). This concludes the proof in case X is nonempty.

Finally, we consider the case of the empty scheme. Axiom (B)(a) gives $H^*(\emptyset) \otimes H^*(\emptyset) = H^*(\emptyset)$ and we get that $H^*(\emptyset)$ is either zero or 1-dimensional in degree 0. Then axiom (B)(a) again shows that $H^*(\emptyset) \otimes H^*(X) = H^*(\emptyset)$ for all smooth projective schemes X over k. Using axiom (A)(b) and the nonvanishing of $H^0(X)$ we've seen above we find that $H^*(X)$ is nonzero in at least two degrees if $\dim(X) > 0$. This then forces $H^*(\emptyset)$ to be zero. $\square$

Lemma 9.6. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let $i : X \to Y$ be a closed immersion of nonempty smooth projective equidimensional schemes over k. Then $\gamma([X]) = i_* 1$ in $H^{2c}(Y)(c)$ where $c = \dim(Y) - \dim(X)$.*

Proof. This is true because $1 = \gamma([X])$ in $H^0(X)$ by Lemma 9.5 and then we can apply axiom (C)(b). $\square$

Lemma 9.7. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a nonempty smooth projective scheme over k equidimensional of dimension d. Choose a basis $e_{i,j}, j = 1, \ldots, \beta_i$ of $H^i(X)$ over F. Using Künneth write*

$$\gamma([\Delta]) = \sum_i \sum_j e_{i,j} \otimes e'_{2d-i,j} \quad in \quad \bigoplus_i H^i(X) \otimes_F H^{2d-i}(X)(d)$$

with $e'_{2d-i,j} \in H^{2d-i}(X)(d)$. Then $\int_X e_{i,j} \cup e'_{2d-i,j'} = (-1)^i \delta_{jj'}$.

Proof. Recall that $\Delta^* : H^*(X \times X) \to H^*(X)$ is equal to the cup product map $H^*(X) \otimes_F H^*(X) \to H^*(X)$, see Remark 9.3. On the other hand, recall that $\gamma([\Delta]) = \Delta_* 1$ (Lemma 9.6) and hence

$$\int_{X \times X} \gamma([\Delta]) \cup a \otimes b = \int_{X \times X} \Delta_* 1 \cup a \otimes b = \int_X a \cup b$$

by Lemma 9.1. On the other hand, we have

$$\int_{X \times X} (\sum e_{i,j} \otimes e'_{2d-i,j}) \cup a \otimes b = \sum (\int_X a \cup e_{i,j})(\int_X e'_{2d-i,j} \cup b)$$

by axiom (B)(b); note that we made two switches of order so that the sign for each term is 1. Thus if we choose a such that $\int_X a \cup e_{i,j} = 1$ and all other pairings equal to zero, then we conclude that $\int_X e'_{2d-i,j} \cup b = \int_X a \cup b$ for all b, i.e., $e'_{2d-i,j} = a$. This proves the lemma. $\square$

Lemma 9.8. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Then $H^*(\mathbf{P}^1_k)$ is 1-dimensional in dimensions 0 and 2 and zero in other degrees.*

Proof. Let $x \in \mathbf{P}^1_k$ be a k-rational point. Observe that $\Delta = \mathrm{pr}_1^* x + \mathrm{pr}_2^* x$ as divisors on $\mathbf{P}^1_k \times \mathbf{P}^1_k$. Using axiom (C)(a) and additivity of γ we see that

$$\gamma([\Delta]) = \mathrm{pr}_1^* \gamma([x]) + \mathrm{pr}_2^* \gamma([x]) = \gamma([x]) \otimes 1 + 1 \otimes \gamma([x])$$

in $H^*(\mathbf{P}^1_k \times \mathbf{P}^1_k) = H^*(\mathbf{P}^1_k) \otimes_F H^*(\mathbf{P}^1_k)$. However, by Lemma 9.7 we know that $\gamma([\Delta])$ cannot be written as a sum of fewer than $\sum \beta_i$ pure tensors where $\beta_i = \dim_F H^i(\mathbf{P}^1_k)$. Thus we see that $\sum \beta_i \leq 2$. By Lemma 9.5 we have $H^0(\mathbf{P}^1_k) \neq 0$. By Poincaré duality, more precisely axiom (A)(b), we have $\beta_0 = \beta_2$. Therefore the lemma holds. $\square$

Lemma 9.9. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). If X and Y are smooth projective schemes over k, then $H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$, $a \mapsto (i^*a, j^*a)$ is an isomorphism where i, j are the coprojections.*

Proof. If X or Y is empty, then this is true because $H^*(\emptyset) = 0$ by Lemma 9.5. Thus we may assume both X and Y are nonempty.

We first show that the map is injective. First, observe that we can find morphisms $X' \to X$ and $Y' \to Y$ of smooth projective schemes so that X' and Y' are equidimensional of the same dimension and such that $X' \to X$ and $Y' \to Y$ each have a section. Namely, decompose $X = \coprod X_d$ and $Y = \coprod Y_e$ into open and closed subschemes equidimensional of dimension d and e. Then take $X' = \coprod X_d \times \mathbf{P}^{n-d}$ and $Y' = \coprod Y_e \times \mathbf{P}^{n-e}$ for some n sufficiently large. Thus pullback by $X' \amalg Y' \to X \amalg Y$ is injective (because there is a section) and it suffices to show the injectivity for X', Y' as we do in the next parapgrah.

Let us show the map is injective when X and Y are equidimensional of the same dimension d. Observe that $[X \amalg Y] = [X] + [Y]$ in $\mathrm{CH}^0(X \amalg Y)$ and that $[X]$ and $[Y]$ are orthogonal idempotents in $\mathrm{CH}^0(X \amalg Y)$. Thus

$$1 = \gamma([X \amalg Y] = \gamma([X]) + \gamma([Y]) = i_* 1 + j_* 1$$

is a decomposition into orthogonal idempotents. Here we have used Lemmas 9.5 and 9.6 and axiom (C)(c). Then we see that

$$a = a \cup 1 = a \cup i_* 1 + a \cup j_* 1 = i_*(i^*a) + j_*(j^*a)$$

by the projection formula (Lemma 9.1) and hence the map is injective.

We show the map is surjective. Write $e = \gamma([X])$ and $f = \gamma([Y])$ viewed as elements in $H^0(X \amalg Y)$. We have $i^*e = 1$, $i^*f = 0$, $j^*e = 0$, and $j^*f = 1$ by axiom (C)(a). Hence if $i^* : H^*(X \amalg Y) \to H^*(X)$ and $j^* : H^*(X \amalg Y) \to H^*(Y)$ are surjective, then so is (i^*, j^*). Namely, for $a, a' \in H^*(X \amalg Y)$ we have

$$(i^*a, j^*a') = (i^*(a \cup e + a' \cup f), j^*(a \cup e + a' \cup f))$$

By symmetry it suffices to show $i^* : H^*(X \amalg Y) \to H^*(X)$ is surjective. If there is a morphism $Y \to X$, then there is a morphism $g : X \amalg Y \to X$ with $g \circ i = \mathrm{id}_X$

and we conclude. To finish the proof, observe that in order to prove i^* is surjective, it suffices to do so after tensoring by a nonzero graded F-vector space. Hence by axiom (B)(b) and nonvanishing of cohomology (Lemma 9.5) it suffices to prove i^* is surjective after replacing X and Y by $X \times \mathrm{Spec}(k')$ and $Y \times \mathrm{Spec}(k')$ for some finite separable extension k'/k. If we choose k' such that there exists a closed point $x \in X$ with $\kappa(x) = k'$ (and this is possible by Varieties, Lemma 25.6) then there is a morphism $Y \times \mathrm{Spec}(k') \to X \times \mathrm{Spec}(k')$ and we find that the proof is complete. $\square$

Lemma 9.10. *Let k be a field. Let F be a field of characteristic 0. Assume given a $\mathbf{Q}$-linear functor*

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories such that $G(\mathbf{1}(1))$ is nonzero only in degree -2. Then we obtain data (D0), (D1), (D2), and (D3) satisfying all of (A), (B), and (C) above.

Proof. This proof is the same as the proof of Lemma 7.9; we urge the reader to read the proof of that lemma instead.

We obtain a contravariant functor from the category of smooth projective schemes over k to the category of graded F-vector spaces by setting $H^*(X) = G(h(X))$. By assumption we have a canonical isomorphism

$$H^*(X \times Y) = G(h(X \times Y)) = G(h(X) \otimes h(Y)) = G(h(X)) \otimes G(h(Y)) = H^*(X) \otimes H^*(Y)$$

compatible with pullbacks. Using pullback along the diagonal $\Delta : X \to X \times X$ we obtain a canonical map

$$H^*(X) \otimes H^*(X) = H^*(X \times X) \to H^*(X)$$

of graded vector spaces compatible with pullbacks. This defines a functorial graded F-algebra structure on $H^*(X)$. Since Δ commutes with the commutativity constraint $h(X) \otimes h(X) \to h(X) \otimes h(X)$ (switching the factors) and since G is a functor of symmetric monoidal categories (so compatible with commutativity constraints), and by our convention in Homology, Example 17.4 we conclude that $H^*(X)$ is a graded commutative algebra! Hence we get our datum (D1).

Since $\mathbf{1}(1)$ is invertible in the category of motives we see that $G(\mathbf{1}(1))$ is invertible in the category of graded F-vector spaces. Thus $\sum_i \dim_F G^i(\mathbf{1}(1)) = 1$. By assumption we only get something nonzero in degree -2. Our datum (D0) is the vector space $F(1) = G^{-2}(\mathbf{1}(1))$. Since G is a symmetric monoidal functor we see that $F(n) = G^{-2n}(\mathbf{1}(n))$ for all $n \in \mathbf{Z}$. It follows that

$$H^{2r}(X)(r) = G^{2r}(h(X)) \otimes G^{-2r}(\mathbf{1}(r)) = G^0(h(X)(r))$$

a formula we will frequently use below.

Let X be a smooth projective scheme over k. By Lemma 3.1 we have

$$\mathrm{CH}^r(X) \otimes \mathbf{Q} = \mathrm{Corr}^r(\mathrm{Spec}(k), X) = \mathrm{Hom}(\mathbf{1}(-r), h(X)) = \mathrm{Hom}(\mathbf{1}, h(X)(r))$$

Applying the functor G this maps into $\mathrm{Hom}(G(\mathbf{1}), G(h(X)(r)))$. By taking the image of 1 in $G^0(\mathbf{1}) = F$ into $G^0(h(X)(r)) = H^{2r}(X)(r)$ we obtain

$$\gamma : \mathrm{CH}^r(X) \otimes \mathbf{Q} \longrightarrow H^{2r}(X)(r)$$

This is the datum (D2).

Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. By Lemma 3.1 we have

$$\mathrm{Mor}(h(X)(d), \mathbf{1}) = \mathrm{Mor}((X, 1, d), (\mathrm{Spec}(k), 1, 0)) = \mathrm{Corr}^{-d}(X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X)$$

Thus the class of the cycle $[X]$ in $\mathrm{CH}_d(X)$ defines a morphism $h(X)(d) \to \mathbf{1}$. Applying G and taking degree 0 parts we obtain

$$H^{2d}(X)(d) = G^0(h(X)(d)) \longrightarrow G^0(\mathbf{1}) = F$$

This map $\int_X : H^{2d}(X)(d) \to F$ is the datum (D3).

Let X be a smooth projective scheme over k which is nonempty and equidimensional of dimension d. By Lemma 4.9 we know that $h(X)(d)$ is a left dual to $h(X)$. Hence $G(h(X)(d)) = H^*(X) \otimes_F F(d)[2d]$ is a left dual to $H^*(X)$ in the category of graded F-vector spaces. Here $[n]$ is the shift functor on graded vector spaces. By Homology, Lemma 17.5 we find that $\sum_i \dim_F H^i(X) < \infty$ and that $\epsilon : h(X)(d) \otimes h(X) \to \mathbf{1}$ produces nondegenerate pairings $H^{2d-i}(X)(d) \otimes_F H^i(X) \to F$. In the proof of Lemma 4.9 we have seen that ϵ is given by $[\Delta]$ via the identifications

$$\mathrm{Hom}(h(X)(d) \otimes h(X), \mathbf{1}) = \mathrm{Corr}^{-d}(X \times X, \mathrm{Spec}(k)) = \mathrm{CH}_d(X \times X)$$

Thus ϵ is the composition of $[X] : h(X)(d) \to \mathbf{1}$ and $h(\Delta)(d) : h(X)(d) \otimes h(X) \to h(X)(d)$. It follows that the pairings above are given by cup product followed by $\int_X$. This proves axiom (A).

Axiom (B) follows from the assumption that G is compatible with tensor structures and our construction of the cup product above.

Axiom (C). Our construction of γ takes a cycle α on X, interprets it a correspondence a from $\mathrm{Spec}(k)$ to X of some degree, and then applies G. If $f : Y \to X$ is a morphism of nonempty equidimensional smooth projective schemes over k, then $f^!\alpha$ is the pushforward (!) of α by the correspondence $[\Gamma_f]$ from X to Y, see Lemma 3.6. Hence $f^!\alpha$ viewed as a correspondence from $\mathrm{Spec}(k)$ to Y is equal to $a \circ [\Gamma_f]$, see Lemma 3.1. Since G is a functor, we conclude γ is compatible with pullbacks, i.e., axiom (C)(a) holds.

Let $f : Y \to X$ be a morphism of nonempty equidimensional smooth projective schemes over k and let $\beta \in \mathrm{CH}^r(Y)$ be a cycle on Y. We have to show that

$$\int_Y \gamma(\beta) \cup f^*c = \int_X \gamma(f_*\beta) \cup c$$

for all $c \in H^*(X)$. Let $a, a^t, \eta_X, \eta_Y, [X], [Y]$ be as in Lemma 3.9. Let b be β viewed as a correspondence from $\mathrm{Spec}(k)$ to Y of degree r. Then $f_*\beta$ viewed as a correspondence from $\mathrm{Spec}(k)$ to X is equal to $a^t \circ b$, see Lemmas 3.6 and 3.1. The displayed equality above holds if we can show that

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{b \otimes 1} h(Y)(r) \otimes h(X) \xrightarrow{1 \otimes a} h(Y)(r) \otimes h(Y) \xrightarrow{\eta_Y} h(Y)(r) \xrightarrow{[Y]} \mathbf{1}(r-e)$$

is equal to

$$h(X) = \mathbf{1} \otimes h(X) \xrightarrow{a^t \circ b \otimes 1} h(X)(r+d-e) \otimes h(X) \xrightarrow{\eta_X} h(X)(r+d-e) \xrightarrow{[X]} \mathbf{1}(r-e)$$

This follows immediately from Lemma 3.9. Thus we have axiom (C)(b).

To prove axiom (C)(c) we use the discussion in Remark 7.2. Hence it suffices to prove that γ is compatible with exterior products. Let X, Y be nonempty smooth projective schemes over k and let α, β be cycles on them. Denote a, b the

corresponding correspondences from $\operatorname{Spec}(k)$ to X, Y. Then $\alpha \times \beta$ corresponds to the correspondence $a \otimes b$ from $\operatorname{Spec}(k)$ to $X \otimes Y = X \times Y$. Hence the requirement follows from the fact that G is compatible with the tensor structures on both sides.

Axiom (C)(d) follows because the cycle $[\operatorname{Spec}(k)]$ corresponds to the identity morphism on $h(\operatorname{Spec}(k))$. This finishes the proof of the lemma. $\qquad\square$

Lemma 9.11. *Let k be a field. Let F be a field of characteristic 0. Given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C) we can construct a $\mathbf{Q}$-linear functor*

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

of symmetric monoidal categories such that $H^*(X) = G(h(X))$.

Proof. The proof of this lemma is the same as the proof of Lemma 7.10; we urge the reader to read the proof of that lemma instead.

By Lemma 4.8 it suffices to construct a functor G on the category of smooth projective schemes over k with morphisms given by correspondences of degree 0 such that the image of $G(c_2)$ on $G(\mathbf{P}^1_k)$ is an invertible graded F-vector space.

Let X be a smooth projective scheme over k. There is a canonical decomposition

$$X = \coprod_{0 \le d \le \dim(X)} X_d$$

into open and closed subschemes such that X_d is equidimensional of dimension d. By Lemma 9.9 we have correspondingly

$$H^*(X) \longrightarrow \prod_{0 \le d \le \dim(X)} H^*(X_d)$$

If Y is a second smooth projective scheme over k and we similarly decompose $Y = \coprod Y_e$, then

$$\operatorname{Corr}^0(X, Y) = \bigoplus \operatorname{Corr}^0(X_d, Y_e)$$

As well we have $X \otimes Y = \coprod X_d \otimes Y_e$ in the category of correspondences. From these observations it follows that it suffices to construct G on the category whose objects are equidimensional smooth projective schemes over k and whose morphisms are correspondences of degree 0. (Some details omitted.)

Given an equdimensional smooth projective scheme X over k we set $G(X) = H^*(X)$. Observe that $G(X) = 0$ if $X = \emptyset$ (Lemma 9.5). Thus maps from and to $G(\emptyset)$ are zero and we may and do assume our schemes are nonempty in the discussions below.

Given a correspondence $c \in \operatorname{Corr}^0(X, Y)$ between nonempty equdimensional smooth projective schemes over k we consider the map $G(c) : G(X) = H^*(X) \to G(Y) = H^*(Y)$ given by the rule

$$a \longmapsto G(c)(a) = \operatorname{pr}_{2,*}(\gamma(c) \cup \operatorname{pr}_1^* a)$$

It is clear that $G(c)$ is additive in c and hence $\mathbf{Q}$-linear. Compatibility of γ with pullbacks, pushforwards, and intersection products given by axioms (C)(a), (C)(b),

and (C)(c) shows that we have $G(c' \circ c) = G(c') \circ G(c)$ if $c' \in \mathrm{Corr}^0(Y, Z)$. Namely, for $a \in H^*(X)$ we have

$$
\begin{aligned}
(G(c') \circ G(c))(a) &= \mathrm{pr}^{23}_{3,*}(\gamma(c') \cup \mathrm{pr}^{23,*}_2(\mathrm{pr}^{12}_{2,*}(\gamma(c) \cup \mathrm{pr}^{12,*}_1 a))) \\
&= \mathrm{pr}^{23}_{3,*}(\gamma(c') \cup \mathrm{pr}^{123}_{23,*}(\mathrm{pr}^{123,*}_{12}(\gamma(c) \cup \mathrm{pr}^{12,*}_1 a))) \\
&= \mathrm{pr}^{23}_{3,*}\mathrm{pr}^{123}_{23,*}(\mathrm{pr}^{123,*}_{23}\gamma(c') \cup \mathrm{pr}^{123,*}_{12}\gamma(c) \cup \mathrm{pr}^{123,*}_1 a) \\
&= \mathrm{pr}^{23}_{3,*}\mathrm{pr}^{123}_{23,*}(\gamma(\mathrm{pr}^{123,*}_{23}c') \cup \gamma(\mathrm{pr}^{123,*}_{12}c) \cup \mathrm{pr}^{123,*}_1 a) \\
&= \mathrm{pr}^{13}_{3,*}\mathrm{pr}^{123}_{13,*}(\gamma(\mathrm{pr}^{123,*}_{23}c' \cdot \mathrm{pr}^{123,*}_{12}c) \cup \mathrm{pr}^{123,*}_1 a) \\
&= \mathrm{pr}^{13}_{3,*}(\gamma(\mathrm{pr}^{123}_{13,*}(\mathrm{pr}^{123,*}_{23}c' \cdot \mathrm{pr}^{123,*}_{12}c)) \cup \mathrm{pr}^{13,*}_1 a) \\
&= G(c' \circ c)(a)
\end{aligned}
$$

with obvious notation. The first equality follows from the definitions. The second equality holds because $\mathrm{pr}^{23,*}_2 \circ \mathrm{pr}^{12}_{2,*} = \mathrm{pr}^{123}_{23,*} \circ \mathrm{pr}^{123,*}_{12}$ as follows immediately from the description of pushforward along projections given in Lemma 9.2. The third equality holds by Lemma 9.1 and the fact that H^* is a functor. The fourth equalith holds by axiom (C)(a) and the fact that the gysin map agrees with flat pullback for flat morphisms (Chow Homology, Lemma 59.5). The fifth equality uses axiom (C)(c) as well as Lemma 9.1 to see that $\mathrm{pr}^{23}_{3,*} \circ \mathrm{pr}^{123}_{23,*} = \mathrm{pr}^{13}_{3,*} \circ \mathrm{pr}^{123}_{13,*}$. The sixth equality uses the projection formula from Lemma 9.1 as well as axiom (C)(b) to see that $\mathrm{pr}^{123}_{13,*}\gamma(\mathrm{pr}^{123,*}_{23}c' \cdot \mathrm{pr}^{123,*}_{12}c) = \gamma(\mathrm{pr}^{123}_{13,*}(\mathrm{pr}^{123,*}_{23}c' \cdot \mathrm{pr}^{123,*}_{12}c))$. Finally, the last equality is the definition.

To finish the proof that G is a functor, we have to show identities are preserved. In other words, if $1 = [\Delta] \in \mathrm{Corr}^0(X, X)$ is the identity in the category of correspondences (Lemma 3.3), then we have to show that $G([\Delta]) = \mathrm{id}$. This follows from the determination of $\gamma([\Delta])$ in Lemma 9.7 and Lemma 9.2. This finishes the construction of G as a functor on smooth projective schemes over k and correspondences of degree 0.

By Lemma 9.4 we have that $G(\mathrm{Spec}(k)) = H^*(\mathrm{Spec}(k))$ is canonically isomorphic to F as an F-algebra. The Künneth axiom (B)(a) shows our functor is compatible with tensor products. Thus our functor is a functor of symmetric monoidal categories.

We still have to check that the image of $G(c_2)$ on $G(\mathbf{P}^1_k) = H^*(\mathbf{P}^1_k)$ is an invertible graded F-vector space (in particular we don't know yet that G extends to M_k). By Lemma 9.8 we only have nonzero cohomology in degrees 0 and 2 both of dimension 1. We have $1 = c_0 + c_2$ is a decomposition of the identity into a sum of orthogonal idempotents in $\mathrm{Corr}^0(\mathbf{P}^1_k, \mathbf{P}^1_k)$, see Example 3.7. Further we have $c_0 = a \circ b$ where $a \in \mathrm{Corr}^0(\mathrm{Spec}(k), \mathbf{P}^1_k)$ and $b \in \mathrm{Corr}^0(\mathbf{P}^1_k, \mathrm{Spec}(k))$ and where $b \circ a = 1$ in $\mathrm{Corr}^0(\mathrm{Spec}(k), \mathrm{Spec}(k))$, see proof of Lemma 4.4. Thus $G(c_0)$ is the projector onto the degree 0 part. It follows that $G(c_2)$ must be the projector onto the degree 2 part and the proof is complete. $\qquad\square$

Proposition 9.12. *Let k be a field. Let F be a field of characteristic 0. There is a 1-to-1 correspondence between the following*

(1) *data (D0), (D1), (D2), and (D3) satisfying (A), (B), and(C), and*
(2) $\mathbf{Q}$*-linear symmetric monoidal functors*

$$
G : M_k \longrightarrow \text{graded } F\text{-vector spaces}
$$

such that $G(\mathbf{1}(1))$ *is nonzero only in degree* -2.

Proof. Given G as in (2) by setting $H^*(X) = G(h(X))$ we obtain data (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C), see Lemma 9.10 and its proof.

Conversely, given data (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C) we get a functor G as in (2) by the construction of the proof of Lemma 9.11.

We omit the detailed proof that these constructions are inverse to each other. $\square$

10. Further properties

In this section we prove a few more results one obtains if given data (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C) as in Section 9.

Lemma 10.1. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let* X, Y *be nonempty smooth projective schemes both equidimensional of dimension d over k. Then* $\int_{X \amalg Y} = \int_X + \int_Y$.

Proof. Denote $i : X \to X \amalg Y$ and $j : Y \to X \amalg Y$ be the coprojections. By Lemma 9.9 the map $(i^*, j^*) : H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$ is an isomorphism. The statement of the lemma means that under the isomorphism $(i^*, j^*) : H^{2d}(X \amalg Y)(d) \to H^{2d}(X)(d) \oplus H^{2d}(Y)(d)$ the map $\int_X + \int_Y$ is tranformed into $\int_{X \amalg Y}$. This is true because

$$\int_{X \amalg Y} a = \int_{X \amalg Y} i_*(i^*a) + j_*(j^*a) = \int_X i^*a + \int_Y j^*a$$

where the equality $a = i_*(i^*a) + j_*(j^*a)$ was shown in the proof of Lemma 9.9. $\square$

Lemma 10.2. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a smooth projective scheme of dimension zero over k. Then*

  (1) $H^i(X) = 0$ *for* $i \neq 0$,
  (2) $H^0(X)$ *is a finite separable algebra over* F,
  (3) $\dim_F H^0(X) = \deg(X \to \mathrm{Spec}(F))$,
  (4) $\int_X : H^0(X) \to F$ *is the trace map*,
  (5) $\gamma([X]) = 1$, *and*
  (6) $\int_X \gamma([X]) = \deg(X \to \mathrm{Spec}(k))$.

Proof. We can write $X = \mathrm{Spec}(k')$ where k' is a finite separable algebra over k. Observe that $\deg(X \to \mathrm{Spec}(k)) = [k' : k]$. Choose a finite Galois extension k''/k containing each of the factors of k'. (Recall that a finite separable k-algebra is a product of finite separable field extension of k.) Set $\Sigma = \mathrm{Hom}_k(k', k'')$. Then we get

$$k' \otimes_k k'' = \prod\nolimits_{\sigma \in \Sigma} k''$$

Setting $Y = \mathrm{Spec}(k'')$ axioms (B)(a) and Lemma 9.9 give

$$H^*(X) \otimes_F H^*(Y) = \prod\nolimits_{\sigma \in \Sigma} H^*(Y)$$

as graded commutative F-algebras. By Lemma 9.5 the F-algebra $H^*(Y)$ is nonzero. Comparing dimensions on either side of the displayed equation we conclude that $H^*(X)$ sits only in degree 0 and $\dim_F H^0(X) = [k' : k]$. Applying this to Y we get $H^*(Y) = H^0(Y)$. Since

$$H^0(X) \otimes_F H^0(Y) = H^0(Y) \times \ldots \times H^0(Y)$$

as F-algebras, it follows that $H^0(X)$ is a separable F-algebra because we may check this after the faithfully flat base change $F \to H^0(Y)$.

The displayed isomorphism above is given by the map

$$H^0(X) \otimes_F H^0(Y) \longrightarrow \prod_{\sigma \in \Sigma} H^0(Y), \quad a \otimes b \longmapsto \prod_\sigma \mathrm{Spec}(\sigma)^* a \cup b$$

Via this isomorphism we have $\int_{X \times Y} = \sum_\sigma \int_Y$ by Lemma 10.1. Thus

$$\int_X a = \mathrm{pr}_{1,*}(a \otimes 1) = \sum \mathrm{Spec}(\sigma)^* a$$

in $H^0(Y)$; the first equality by Lemma 9.2 and the second by the observation we just made. Choose an algebraic closure $\overline{F}$ and a F-algebra map $\tau : H^0(Y) \to \overline{F}$. The isomorphism above base changes to the isomorphism

$$H^0(X) \otimes_F \overline{F} \longrightarrow \prod_{\sigma \in \Sigma} \overline{F}, \quad a \otimes b \longmapsto \prod_\sigma \tau(\mathrm{Spec}(\sigma)^* a) b$$

It follows that $a \mapsto \tau(\mathrm{Spec}(\sigma)^* a)$ is a full set of embeddings of $H^0(X)$ into $\overline{F}$. Applying τ to the formula for $\int_X a$ obtained above we conclude that $\int_X$ is the trace map. By Lemma 9.5 we have $\gamma([X]) = 1$. Finally, we have $\int_X \gamma([X]) = \deg(X \to \mathrm{Spec}(k))$ because $\gamma([X]) = 1$ and the trace of 1 is equal to $[k' : k]$ $\qquad\square$

Lemma 10.3. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a nonempty smooth projective scheme equidimensional of dimension d over k. The diagram*

$$
\begin{array}{ccc}
\mathrm{CH}^d(X) & \xrightarrow{\ \gamma\ } & H^{2d}(X)(d) \\
\| & & \downarrow{\scriptstyle \int_X} \\
\mathrm{CH}_0(X) & \xrightarrow{\ \mathrm{deg}\ } & F
\end{array}
$$

commutes where $\deg : \mathrm{CH}_0(X) \to \mathbf{Z}$ *is the degree of zero cycles discussed in Chow Homology, Section 41.*

Proof. Let x be a closed point of X whose residue field is separable over k. View x as a scheme and denote $i : x \to X$ the inclusion morphism. To avoid confusion denote $\gamma' : \mathrm{CH}_0(x) \to H^0(x)$ the cycle class map for x. Then we have

$$\int_X \gamma([x]) = \int_X \gamma(i_*[x]) = \int_X i_* \gamma'([x]) = \int_x \gamma'([x]) = \deg(x \to \mathrm{Spec}(k))$$

The second equality is axiom (C)(b) and the third equality is the definition of i_* on cohomology. The final equality is Lemma 10.2. This proves the lemma because $\mathrm{CH}_0(X)$ is generated by the classes of points x as above by Lemma 8.1. $\qquad\square$

Lemma 10.4. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. We have*

$$\sum_i (-1)^i \dim_F H^i(X) = \deg(\Delta \cdot \Delta) = \deg(c_d(\mathcal{T}_{X/k}))$$

Proof. Equality on the right. We have $[\Delta] \cdot [\Delta] = \Delta_*(\Delta^![\Delta])$ (Chow Homology, Lemma 62.6). Since Δ_* preserves degrees of 0-cycles it suffices to compute the degree of $\Delta^![\Delta]$. The class $\Delta^![\Delta]$ is given by capping $[\Delta]$ with the top Chern class of the normal sheaf of $\Delta \subset X \times X$ (Chow Homology, Lemma 54.5). Since the

conormal sheaf of Δ is $\Omega_{X/k}$ (Morphisms, Lemma 32.7) we see that the normal sheaf is equal to the tangent sheaf $\mathcal{T}_{X/k} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ as desired.

Equality on the left. By Lemma 10.3 we have

$$
\begin{aligned}
\deg([\Delta] \cdot [\Delta]) &= \int_{X \times X} \gamma([\Delta]) \cup \gamma([\Delta]) \\
&= \int_{X \times X} \Delta_* 1 \cup \gamma([\Delta]) \\
&= \int_{X \times X} \Delta_*(\Delta^* \gamma([\Delta])) \\
&= \int_X \Delta^* \gamma([\Delta])
\end{aligned}
$$

We have used Lemmas 9.6 and 9.1. Write $\gamma([\Delta]) = \sum e_{i,j} \otimes e'_{2d-i,j}$ as in Lemma 9.7. Recalling that Δ^* is given by cup product (Remark 9.3) we obtain

$$
\int_X \sum_{i,j} e_{i,j} \cup e'_{2d-i,j} = \sum_{i,j} \int_X e_{i,j} \cup e'_{2d-i,j} = \sum_{i,j}(-1)^i = \sum(-1)^i \beta_i
$$

as desired. $\qquad\square$

Lemma 10.5. *Let F be a field of characteristic 0. Let F' and F_i, $i = 1, \ldots, r$ be finite separable F-algebras. Let A be a finite F-algebra. Let $\sigma, \sigma' : A \to F'$ and $\sigma_i : A \to F_i$ be F-algebra maps. Assume σ and σ' surjective. If there is a relation*

$$
Tr_{F'/F} \circ \sigma - Tr_{F'/F} \circ \sigma' = n(\sum m_i \, Tr_{F_i/F} \circ \sigma_i)
$$

where $n > 1$ and m_i are integers, then $\sigma = \sigma'$.

Proof. We may write $A = \prod A_j$ as a finite product of local Artinian F-algebras $(A_j, \mathfrak{m}_j, \kappa_j)$, see Algebra, Lemma 53.2 and Proposition 60.7. Denote $A' = \prod \kappa_j$ where the product is over those j such that κ_j/k is separable. Then each of the maps $\sigma, \sigma', \sigma_i$ factors over the map $A \to A'$. After replacing A by A' we may assume A is a finite separable F-algebra.

Choose an algebraic closure $\overline{F}$. Set $\overline{A} = A \otimes_F \overline{F}$, $\overline{F}' = F' \otimes_F \overline{F}$, and $\overline{F}_i = F_i \otimes_F \overline{F}$. We can base change σ, σ', σ_i to get $\overline{F}$ algebra maps $\overline{A} \to \overline{F}'$ and $\overline{A} \to \overline{F}_i$. Moreover $\mathrm{Tr}_{\overline{F}'/\overline{F}}$ is the base change of $\mathrm{Tr}_{F'/F}$ and similarly for $\mathrm{Tr}_{F_i/F}$. Thus we may replace F by $\overline{F}$ and we reduce to the case discussed in the next paragraph.

Assume F is algebraically closed and A a finite separable F-algebra. Then each of A, F', F_i is a product of copies of F. Let us say an element e of a product $F \times \ldots \times F$ of copies of F is a minimal idempotent if it generates one of the factors, i.e., if $e = (0, \ldots, 0, 1, 0, \ldots, 0)$. Let $e \in A$ be a minimal idempotent. Since σ and σ' are surjective, we see that $\sigma(e)$ and $\sigma'(e)$ are minimal idempotents or zero. If $\sigma \neq \sigma'$, then we can choose a minimal idempotent $e \in A$ such that $\sigma(e) = 0$ and $\sigma'(e) \neq 0$ or vice versa. Then $\mathrm{Tr}_{F'/F}(\sigma(e)) = 0$ and $\mathrm{Tr}_{F'/F}(\sigma'(e)) = 1$ or vice versa. On the other hand, $\sigma_i(e)$ is an idempotent and hence $\mathrm{Tr}_{F_i/F}(\sigma_i(e)) = r_i$ is an integer. We conclude that

$$
-1 = \sum n m_i r_i = n(\sum m_i r_i) \quad \text{or} \quad 1 = \sum n m_i r_i = n(\sum m_i r_i)
$$

which is impossible. $\qquad\square$

Lemma 10.6. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let k'/k be a finite separable extension. Let X be a smooth projective scheme over k'. Let $x, x' \in X$ be k'-rational points. If $\gamma(x) \neq \gamma(x')$, then $[x] - [x']$ is not divisible by any integer $n > 1$ in $\mathrm{CH}_0(X)$.*

Proof. If x and x' lie on distinct irreducible components of X, then the result is obvious. Thus we may X irreducible of dimension d. Say $[x] - [x']$ is divisible by $n > 1$ in $\mathrm{CH}_0(X)$. We may write $[x] - [x'] = n(\sum m_i[x_i])$ in $\mathrm{CH}_0(X)$ for some $x_i \in X$ closed points whose residue fields are separable over k by Lemma 8.1. Then

$$\gamma([x]) - \gamma([x']) = n(\sum m_i \gamma([x_i]))$$

in $H^{2d}(X)(d)$. Denote $i^*, (i')^*, i_i^*$ the pullback maps $H^0(X) \to H^0(x)$, $H^0(X) \to H^0(x')$, $H^0(X) \to H^0(x_i)$. Recall that $H^0(x)$ is a finite separable F-algebra and that $\int_x : H^0(x) \to F$ is the trace map (Lemma 10.2) which we will denote Tr_x. Similarly for x' and x_i. Then by Poincaré duality in the form of axiom (A)(b) the equation above is dual to

$$\mathrm{Tr}_x \circ i^* - \mathrm{Tr}_{x'} \circ (i')^* = n(\sum m_i \mathrm{Tr}_{x_i} \circ i_i^*)$$

which takes place in $\mathrm{Hom}_F(H^0(X), F)$. Finally, observe that i^* and $(i')^*$ are surjective as x and x' are k'-rational points and hence the compositions $H^0(\mathrm{Spec}(k')) \to H^0(X) \to H^0(x)$ and $H^0(\mathrm{Spec}(k')) \to H^0(X) \to H^0(x')$ are isomorphisms. By Lemma 10.5 we conclude that $i^* = (i')^*$ which contradicts the assumption that $\gamma([x]) \neq \gamma([x'])$. $\square$

Lemma 10.7. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let k'/k be a finite separable extension. Let X be a geometrically irreducible smooth projective scheme over k' of dimension d. Then $\gamma : \mathrm{CH}_0(X) \to H^{2d}(X)(d)$ factors through $\deg : \mathrm{CH}_0(X) \to \mathbf{Z}$.*

Proof. By Lemma 8.1 it suffices to show: given closed points $x, x' \in X$ whose residue fields are separable over k we have $\deg(x')\gamma([x]) = \deg(x)\gamma([x'])$.

We first reduce to the case of k'-rational points. Let k''/k' be a Galois extension such that $\kappa(x)$ and $\kappa(x')$ embed into k'' over k. Set $Y = X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k'')$ and denote $p : Y \to X$ the projection. By our choice of k''/k' there exists a k''-rational point y, resp. y' on Y mapping to x, resp. x'. Then $p_*[y] = [k'' : \kappa(x)][x]$ and $p_*[y'] = [k'' : \kappa(x')][x']$ in $\mathrm{CH}_0(X)$. By compatibility with pushforwards given in axiom (C)(b) it suffices to prove $\gamma([y]) = \gamma([y'])$ in $\mathrm{CH}^{2d}(Y)(d)$. This reduces us to the discussion in the next paragraph.

Assume x and x' are k'-rational points. By Lemma 8.3 there exists a finite separable extension k''/k' of fields such that the pullback $[y] - [y']$ of the difference $[x] - [x']$ becomes divisible by an integer $n > 1$ on $Y = X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k'')$. (Note that $y, y' \in Y$ are k''-rational points.) By Lemma 10.6 we have $\gamma([y]) = \gamma([y'])$ in $H^{2d}(Y)(d)$. By compatibility with pushforward in axiom (C)(b) we conclude the same for x and x'. $\square$

Lemma 10.8. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let $f : X \to Y$ be a dominant morphism of irreducible smooth projective schemes over k. Then $H^*(Y) \to H^*(X)$ is injective.*

Proof. There exists an integral closed subscheme $Z \subset X$ of the same dimension as Y mapping onto Y. Thus $f_*[Z] = m[Y]$ for some $m > 0$. Then $f_*\gamma([Z]) = m\gamma([Y]) = m$ in $H^*(Y)$ because of Lemma 9.5. Hence by the projection formula (Lemma 9.1) we have $f_*(f^*a \cup \gamma([Z])) = ma$ and we conclude. $\square$

Lemma 10.9. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let $k''/k'/k$ be finite separable algebras and let X be a smooth projective scheme over k'. Then*

$$H^*(X) \otimes_{H^0(\mathrm{Spec}(k'))} H^0(\mathrm{Spec}(k'')) = H^*(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k''))$$

Proof. We will use the results of Lemma 10.2 without further mention. Write

$$k' \otimes_k k'' = k'' \times l$$

for some finite separable k'-algebra l. Write $F' = H^0(\mathrm{Spec}(k'))$, $F'' = H^0(\mathrm{Spec}(k''))$, and $G = H^0(\mathrm{Spec}(l))$. Since $\mathrm{Spec}(k') \times \mathrm{Spec}(k'') = \mathrm{Spec}(k'') \amalg \mathrm{Spec}(l)$ we deduce from axiom (B)(a) and Lemma 9.9 that we have

$$F' \otimes_F F'' = F'' \times G$$

The map from left to right identifies F'' with $F' \otimes_{F'} F''$. By the same token we have

$$H^*(X) \otimes_F F'' = H^*(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k'')) \times H^*(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(l))$$

as modules over $F' \otimes_F F'' = F'' \times G$. This proves the lemma. $\square$

11. Weil cohomology theories, II

For us a Weil cohomology theory will be the analogue of a classical Weil cohomology theory (Section 7) when the ground field k is not algebraically closed. In Section 9 we listed axioms which guarantee our cohomology theory comes from a symmetric monoidal functor on the category of motives over k. Missing from our axioms so far are the condition $H^i(X) = 0$ for $i < 0$ and a condition on $H^{2d}(X)(d)$ for X equidimensional of dimension d corresponding to the classical axioms (A)(c) and (A)(d). Let us first convince the reader that it is necessary to impose such conditions.

Example 11.1. Let $k = \mathbf{C}$ and $F = \mathbf{C}$ both be equal to the field of complex numbers. For X smooth projective over k denote $H^{p,q}(X) = H^q(X, \Omega^p_{X/k})$. Let $(H')^*$ be the functor which sends X to $(H')^*(X) = \bigoplus H^{p,q}(X)$ with the usual cup product. This is a classical Weil cohomology theory (insert future reference here). By Proposition 7.11 we obtain a $\mathbf{Q}$-linear symmetric monoidal functor G' from M_k to the category of graded F-vector spaces. Of course, in this case for every M in M_k the value $G'(M)$ is naturally bigraded, i.e., we have

$$(G')(M) = \bigoplus (G')^{p,q}(M), \quad (G')^n = \bigoplus_{n=p+q} (G')^{p,q}(M)$$

with $(G')^{p,q}$ sitting in total degree $p+q$ as indicated. Now we are going to construct a $\mathbf{Q}$-linear symmetric monoidal functor G to the category of graded F-vector spaces by setting

$$G^n(M) = \bigoplus_{n=3p-q} (G')^{p,q}(M)$$

We omit the verification that this defines a symmetric monoidal functor (a technical point is that because we chose odd numbers 3 and -1 above the functor G

is compatible with the commutativity constraints). Observe that $G(\mathbf{1}(1))$ is still sitting in degree -2! Hence by Lemma 7.9 we obtain a functor H^*, cycle classes γ, and trace maps satisfying all classical axioms (A), (B), (C), except for possibly the classical axioms (A)(a) and (A)(d). However, if E is an elliptic curve over k, then we find $\dim H^{-1}(E) = 1$, i.e., axiom (A)(a) is indeed violated.

Lemma 11.2. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). Let X be a smooth projective scheme over k. Set $k' = \Gamma(X, \mathcal{O}_X)$. The following are equivalent*

(1) *there exist finitely many closed points $x_1, \ldots, x_r \in X$ whose residue fields are separable over k such that $H^0(X) \to H^0(x_1) \oplus \ldots \oplus H^0(x_r)$ is injective,*
(2) *the map $H^0(\mathrm{Spec}(k')) \to H^0(X)$ is an isomorphism.*

If this is true, then $H^0(X)$ is a finite separable algebra over F. If X is equidimensional of dimension d, then (1) and (2) are also equivalent to

(3) *the classes of closed points generate $H^{2d}(X)(d)$ as a module over $H^0(X)$.*

Proof. We observe that the statement makes sense because k' is a finite separable algebra over k (Varieties, Lemma 9.3) and hence $\mathrm{Spec}(k')$ is smooth and projective over k. The compatibility of H^* with direct sums (Lemmas 9.9 and 10.1) shows that it suffices to prove the lemma when X is connected. Hence we may assume X is irreducible and we have to show the equivalence of (1), (2), and (3). Set $d = \dim(X)$. This implies that k' is a field finite separable over k and that X is geometrically irreducible over k', see Varieties, Lemmas 9.3 and 9.4.

By Lemma 8.1 we see that the closed points in (3) may be assumed to have separable residue fields over k. By axioms (A)(a) and (A)(b) we see that conditions (1) and (3) are equivalent.

If (2) holds, then pick any closed point $x \in X$ whose residue field is finite separable over k'. Then $H^0(\mathrm{Spec}(k')) = H^0(X) \to H^0(x)$ is injective for example by Lemma 10.8.

Assume the equivalent conditions (1) and (3) hold. Choose $x_1, \ldots, x_r \in X$ as in (1). Choose a finite separable extension k''/k'. By Lemma 10.9 we have

$$H^0(X) \otimes_{H^0(\mathrm{Spec}(k'))} H^0(\mathrm{Spec}(k'')) = H^0(X \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k''))$$

Thus in order to show that $H^0(\mathrm{Spec}(k')) \to H^0(X)$ is an isomorphism we may replace k' by k''. Thus we may assume $x_1, \ldots, x_r$ are k'-rational points (this replaces each x_i with multiple points, so r is increased in this step). By Lemma 10.7 $\gamma(x_1) = \gamma(x_2) = \ldots = \gamma(x_r)$. By axiom (A)(b) all the maps $H^0(X) \to H^0(x_i)$ are the same. This means (2) holds.

Finally, Lemma 10.2 implies $H^0(X)$ is a separable F-algebra if (1) holds.   □

Lemma 11.3. *Assume given (D0), (D1), (D2), and (D3) satisfying (A), (B), and (C). If there exists a smooth projective scheme Y over k such that $H^i(Y)$ is nonzero for some $i < 0$, then there exists an equidimensional smooth projective scheme X over k such that the equivalent conditions of Lemma 11.2 fail for X.*

Proof. By Lemma 9.9 we may assume Y is irreducible and a fortiori equidimensional. If i is odd, then after replacing Y by $Y \times Y$ we find an example where Y

is equidimensional and $i = -2l$ for some $l > 0$. Set $X = Y \times (\mathbf{P}_k^1)^l$. Using axiom (B)(a) we obtain

$$H^0(X) \supset H^0(Y) \oplus H^i(Y) \otimes_F H^2(\mathbf{P}_k^1)^{\otimes_F l}$$

with both summands nonzero. Thus it is clear that $H^0(X)$ cannot be isomorphic to H^0 of the spectrum of $\Gamma(X, \mathcal{O}_X) = \Gamma(Y, \mathcal{O}_Y)$ as this falls into the first summand. $\square$

Thus it makes sense to finally make the following definition.

Definition 11.4. Let k be a field. Let F be a field of characteristic 0. A *Weil cohomology theory* over k with coefficients in F is given by data (D0), (D1), (D2), and (D3) satisfying Poincaré duality, the Künneth formula, and compatibility with cycle classes, more precisely, satisfying axioms (A), (B), and (C) of Section 9 and in addition such that the equivalent conditions (1) and (2) of Lemma 11.2 hold for every smooth projective X over k.

By Lemma 11.3 this means also that there are no nonzero negative cohomology groups. In particular, if k is algebraically closed, then a Weil cohomology theory as above together with an isomorphism $F \to F(1)$ is the same thing as a classical Weil cohomology theory.

Remark 11.5. Let H^* be a Weil cohomology theory (Definition 11.4). Let X be a geometrically irreducible smooth projective scheme of dimension d over k' with k'/k a finite separable extension of fields. Suppose that

$$H^0(\mathrm{Spec}(k')) = F_1 \times \ldots \times F_r$$

for some fields F_i. Then we accordingly can write

$$H^*(X) = \prod\nolimits_{i=1,\ldots,r} H^*(X) \otimes_{H^0(\mathrm{Spec}(k'))} F_i$$

Now, our final assumption in Definition 11.4 tells us that $H^0(X)$ is free of rank 1 over $\prod F_i$. In other words, each of the factors $H^0(X) \otimes_{H^0(\mathrm{Spec}(k'))} F_i$ has dimension 1 over F_i. Poincaré duality then tells us that the same is true for cohomology in degree $2d$. What isn't clear however is that the same holds in other degrees. Namely, we don't know that given $0 < n < \dim(X)$ the integers

$$\dim_{F_i} H^n(X) \otimes_{H^0(\mathrm{Spec}(k'))} F_i$$

are independent of i! This question is closely related to the following open question: given an algebraically closed base field $\overline{k}$, a field of characteristic zero F, a classical Weil cohomology theory H^* over $\overline{k}$ with coefficient field F, and a smooth projective variety X over $\overline{k}$ is it true that the betti numbers of X

$$\beta_i = \dim_F H^i(X)$$

are independent of F and the Weil cohomology theory H^*?

Proposition 11.6. *Let k be a field. Let F be a field of characteristic* 0. *A Weil cohomology theory is the same thing as a* **Q***-linear symmetric monoidal functor*

$$G : M_k \longrightarrow \text{graded } F\text{-vector spaces}$$

such that

  (1) $G(\mathbf{1}(1))$ *is nonzero only in degree* -2, *and*

(2) *for every smooth projective scheme X over k with $k' = \Gamma(X, \mathcal{O}_X)$ the homomorphism $G(h(\mathrm{Spec}(k'))) \to G(h(X))$ of graded F-vector spaces is an isomorphism in degree 0.*

Proof. Immediate consequence of Proposition 9.12 and Definition 11.4. Of course we could replace (2) by the condition that $G(h(X)) \to \bigoplus G(h(x_i))$ is injective in degree 0 for some choice of closed points $x_1, \dots, x_r \in X$ whose residue fields are separable over k. $\qquad\square$

12. Chern classes

In this section we discuss how given a first Chern class and a projective space bundle formula we can get all Chern classes. A reference for this section is [Gro58] although our axioms are slightly different.

Let $\mathcal{C}$ be a category of schemes with the following properties

(1) Every $X \in \mathrm{Ob}(\mathcal{C})$ is quasi-compact and quasi-separated.
(2) If $X \in \mathrm{Ob}(\mathcal{C})$ and $U \subset X$ is open and closed, then $U \to X$ is a morphism of $\mathcal{C}$. If $X' \to X$ is a morphism of $\mathcal{C}$ factoring through U, then $X' \to U$ is a morphism of $\mathcal{C}$.
(3) If $X \in \mathrm{Ob}(\mathcal{C})$ and if $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module, then
    (a) $p : \mathbf{P}(\mathcal{E}) \to X$ is a morphism of $\mathcal{C}$,
    (b) for a morphism $f : X' \to X$ in $\mathcal{C}$ the induced morphism $\mathbf{P}(f^*\mathcal{E}) \to \mathbf{P}(\mathcal{E})$ is a morphism of $\mathcal{C}$,
    (c) if $\mathcal{E} \to \mathcal{F}$ is a surjection onto another finite locally free $\mathcal{O}_X$-module then the closed immersion $\mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E})$ is a morphism of $\mathcal{C}$.

Next, assume given a contravariant functor A from the category $\mathcal{C}$ to the category of graded algebras. Here a graded algebra A is a unital, associative, not necessarily commutative $\mathbf{Z}$-algebra A endowed with a grading $A = \bigoplus_{i \geq 0} A^i$. Given a morphism $f : X' \to X$ of $\mathcal{C}$ we denote $f^* : A(X) \to A(X')$ the induced algebra map. We will denote the product of $a, b \in A(X)$ by $a \cup b$. Finally, we assume given for every object X of $\mathcal{C}$ an additive map

$$c_1^A : \mathrm{Pic}(X) \longrightarrow A^1(X)$$

We assume the following axioms are satisfied

(1) Given $X \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{L} \in \mathrm{Pic}(X)$ the element $c_1^A(\mathcal{L})$ is in the center of the algebra $A(X)$.
(2) If $X \in \mathrm{Ob}(\mathcal{C})$ and $X = U \amalg V$ with U and V open and closed, then $A(X) = A(U) \times A(V)$ via the induced maps $A(X) \to A(U)$ and $A(X) \to A(V)$.
(3) If $f : X' \to X$ is a morphism of $\mathcal{C}$ and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then $f^* c_1^A(\mathcal{L}) = c_1^A(f^*\mathcal{L})$.
(4) Given $X \in \mathrm{Ob}(\mathcal{C})$ and locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank r consider the morphism $p : P = \mathbf{P}(\mathcal{E}) \to X$ of $\mathcal{C}$. Then the map

$$\bigoplus_{i=0,\dots,r-1} A(X) \longrightarrow A(P), \quad (a_0, \dots, a_{r-1}) \longmapsto \sum c_1^A(\mathcal{O}_P(1))^i \cup p^*(a_i)$$

is bijective.
(5) Let $X \in \mathrm{Ob}(\mathcal{C})$ and let $\mathcal{E} \to \mathcal{F}$ be a surjection of finite locally free $\mathcal{O}_X$-modules of ranks $r + 1$ and r. Denote $i : P' = \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E}) = P$ the corresponding incusion morphism. This is a morphism of $\mathcal{C}$ which exhibits

P' as an effective Cartier divisor on P. Then for $a \in A(P)$ with $i^*a = 0$ we have $a \cup c_1^A(\mathcal{O}_P(P')) = 0$.

To formulate our result recall that $Vect(X)$ denotes the (exact) category of finite locally free $\mathcal{O}_X$-modules. In Derived Categories of Schemes, Section 38 we have defined the zeroth K-group $K_0(Vect(X))$ of this category. Moreover, we have seen that $K_0(Vect(X))$ is a ring, see Derived Categories of Schemes, Remark 38.6.

Proposition 12.1. *In the situation above there is a unique rule which assigns to every $X \in \mathrm{Ob}(\mathcal{C})$ a "total Chern class"*

$$c^A : K_0(Vect(X)) \longrightarrow \prod_{i \geq 0} A^i(X)$$

with the following properties

(1) *For $X \in \mathrm{Ob}(\mathcal{C})$ we have $c^A(\alpha + \beta) = c^A(\alpha)c^A(\beta)$ and $c^A(0) = 1$.*
(2) *If $f : X' \to X$ is a morphism of $\mathcal{C}$, then $f^* \circ c^A = c^A \circ f^*$.*
(3) *Given $X \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{L} \in \mathrm{Pic}(X)$ we have $c^A([\mathcal{L}]) = 1 + c_1^A(\mathcal{L})$.*

Proof. Let $X \in \mathrm{Ob}(\mathcal{C})$ and let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. We first show how to define an element $c^A(\mathcal{E}) \in A(X)$.

As a first step, let $X = \bigcup X_r$ be the decomposition into open and closed subschemes such that $\mathcal{E}|_{X_r}$ has constant rank r. Since X is quasi-compact, this decomposition is finite. Hence $A(X) = \prod A(X_r)$. Thus it suffices to define $c^A(\mathcal{E})$ when $\mathcal{E}$ has constant rank r. In this case let $p : P \to X$ be the projective bundle of $\mathcal{E}$. We can uniquely define elements $c_i^A(\mathcal{E}) \in A^i(X)$ for $i \geq 0$ such that $c_0^A(\mathcal{E}) = 1$ and the equation

$$(12.1.1) \qquad \sum_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cup p^* c_{r-i}^A(\mathcal{E}) = 0$$

is true. As usual we set $c^A(\mathcal{E}) = c_0^A(\mathcal{E}) + c_1^A(\mathcal{E}) + \ldots + c_r^A(\mathcal{E})$ in $A(X)$.

If $\mathcal{E}$ is invertible, then $c^A(\mathcal{E}) = 1 + c_1^A(\mathcal{L})$. This follows immediately from the construction above.

The elements $c_i^A(\mathcal{E})$ are in the center of $A(X)$. Namely, to prove this we may assume $\mathcal{E}$ has constant rank r. Let $p : P \to X$ be the corresponding projective bundle. if $a \in A(X)$ then $p^*a \cup (-1)^r c_1(\mathcal{O}_P(1))^r = (-1)^r c_1(\mathcal{O}_P(1))^r \cup p^*a$ and hence we must have the same for all the other terms in the expression defining $c_i^A(\mathcal{E})$ as well and we conclude.

If $f : X' \to X$ is a morphism of $\mathcal{C}$, then $f^* c_i^A(\mathcal{E}) = c_i^A(f^*\mathcal{E})$. Namely, to prove this we may assume $\mathcal{E}$ has constant rank r. Let $p : P \to X$ and $p' : P' \to X'$ be the projective bundles corresponding to $\mathcal{E}$ and $f^*\mathcal{E}$. The induced morphism $g : P' \to P$ is a morphism of $\mathcal{C}$. The pullback by g of the equality defining $c_i^A(\mathcal{E})$ is the corresponding equation for $f^*\mathcal{E}$ and we conclude.

Let $X \in \mathrm{Ob}(\mathcal{C})$. Consider a short exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

of finite locally free $\mathcal{O}_X$-modules with $\mathcal{L}$ invertible. Then

$$c^A(\mathcal{E}) = c^A(\mathcal{L})c^A(\mathcal{F})$$

Namely, by the construction of c_i^A we may assume $\mathcal{E}$ has constant rank $r + 1$ and $\mathcal{F}$ has constant rank r. The inclusion

$$i : P' = \mathbf{P}(\mathcal{F}) \longrightarrow \mathbf{P}(\mathcal{E}) = P$$

is a morphism of $\mathcal{C}$ and it is the zero scheme of a regular section of the invertible module $\mathcal{L}^{\otimes -1} \otimes \mathcal{O}_P(1)$. The element

$$\sum_{i=0}^{r} (-1)^i c_1^A(\mathcal{O}_P(1))^i \cup p^* c_i^A(\mathcal{F})$$

pulls back to zero on P' by definition. Hence we see that

$$\left(c_1^A(\mathcal{O}_P(1)) - c_1^A(\mathcal{L})\right) \cup \left(\sum_{i=0}^{r} (-1)^i c_1^A(\mathcal{O}_P(1))^i \cup p^* c_i^A(\mathcal{F})\right) = 0$$

in $A^*(P)$ by assumption (5) on our cohomology A. By definition of $c_1^A(\mathcal{E})$ this gives the desired equality.

Let $X \in \mathrm{Ob}(\mathcal{C})$. Consider a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

of finite locally free $\mathcal{O}_X$-modules. Then

$$c^A(\mathcal{F}) = c^A(\mathcal{E}) c^A(\mathcal{G})$$

Namely, by the construction of c_i^A we may assume $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ have constant ranks r, s, and t. We prove it by induction on r. The case $r = 1$ was done above. If $r > 1$, then it suffices to check this after pulling back by the morphism $\mathbf{P}(\mathcal{E}^\vee) \to X$. Thus we may assume we have an invertible submodule $\mathcal{L} \subset \mathcal{E}$ such that both $\mathcal{E}' = \mathcal{E}/\mathcal{L}$ and $\mathcal{F}' = \mathcal{E}/\mathcal{L}$ are finite locally free (of ranks $s - 1$ and $t - 1$). Then we have

$$c^A(\mathcal{E}) = c^A(\mathcal{L}) c^A(\mathcal{E}') \quad \text{and} \quad c^A(\mathcal{F}) = c^A(\mathcal{L}) c^A(\mathcal{F}')$$

Since we have the short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{F}' \to \mathcal{G} \to 0$$

we see by induction hypothesis that

$$c^A(\mathcal{F}') = c^A(\mathcal{E}') c^A(\mathcal{G})$$

Thus the result follows from a formal calculation.

At this point for $X \in \mathrm{Ob}(\mathcal{C})$ we can define $c^A : K_0(\mathit{Vect}(X)) \to A(X)$. Namely, we send a generator $[\mathcal{E}]$ to $c^A(\mathcal{E})$ and we extend multiplicatively. Thus for example $c^A(-[\mathcal{E}]) = c^A(\mathcal{E})^{-1}$ is the formal inverse of $a^A([\mathcal{E}])$. The multiplicativity in short exact sequences shown above guarantees that this works.

Uniqueness. Suppose $X \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module. We want to show that conditions (1), (2), and (3) of the lemma uniquely determine $c^A([\mathcal{E}])$. To prove this we may assume $\mathcal{E}$ has constant rank r; this already uses (2). Then we may use induction on r. If $r = 1$, then uniqueness follows from (3). If $r > 1$ we pullback using (2) to the projective bundle $p : P \to X$ and we see that we may assume we have a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ with $\mathcal{E}'$ and $\mathcal{E}''$ having lower rank. By induction hypothesis $c^A(\mathcal{E}')$ and $c^A(\mathcal{E}'')$ are uniquely determined. Thus uniqueness for $\mathcal{E}$ by the axiom (1). $\qquad\square$

Lemma 12.2. *In the situation above. Let $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{E}_i$ be a finite collection of locally free $\mathcal{O}_X$-modules of rank r_i. There exists a morphism $p : P \to X$ in $\mathcal{C}$ such that*

(1) $p^* : A(X) \to A(P)$ is injective,
(2) each $p^*\mathcal{E}_i$ has a filtration whose successive quotients $\mathcal{L}_{i,1}, \ldots, \mathcal{L}_{i,r_i}$ are invertible $\mathcal{O}_P$-modules.

Proof. We may assume $r_i \geq 1$ for all i. We will prove the lemma by induction on $\sum (r_i - 1)$. If this integer is 0, then $\mathcal{E}_i$ is invertible for all i and we conclude by taking $\pi = \mathrm{id}_X$. If not, then we can pick an i such that $r_i > 1$ and consider the projective bundle $p : P \to X$ associated to $\mathcal{E}_i$. We have a short exact sequence

$$0 \to \mathcal{F} \to p^*\mathcal{E}_i \to \mathcal{O}_P(1) \to 0$$

of finite locally free $\mathcal{O}_P$-modules of ranks $r_i - 1$, r_i, and 1. Observe that $p^* : A(X) \to A(P)$ is injective by assumption. By the induction hypothesis applied to the finite locally free $\mathcal{O}_P$-modules $\mathcal{F}$ and $p^*\mathcal{E}_{i'}$ for $i' \neq i$, we find a morphism $p' : P' \to P$ with properties stated as in the lemma. Then the composition $p \circ p' : P' \to X$ does the job. $\square$

Lemma 12.3. Let $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then

$$c_i^A(\mathcal{E} \otimes \mathcal{L}) = \sum_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}^A(\mathcal{E}) \cup c_1^A(\mathcal{L})^j$$

Proof. By the construction of c_i^A we may assume $\mathcal{E}$ has constant rank r. Let $p : P \to X$ and $p' : P' \to X$ be the projective bundle associated to $\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{L}$. Then there is an isomorphism $g : P \to P'$ such that $g^*\mathcal{O}_{P'}(1) = \mathcal{O}_P(1) \otimes p^*\mathcal{L}$. See Constructions, Lemma 20.1. Thus

$$g^* c_1^A(\mathcal{O}_{P'}(1)) = c_1^A(\mathcal{O}_P(1)) + p^* c_1^A(\mathcal{L})$$

The desired equality follows formally from this and the definition of Chern classes using equation (12.1.1). $\square$

Proposition 12.4. In the situation above assume $A(X)$ is a **Q**-algebra for all $X \in \mathrm{Ob}(\mathcal{C})$. Then there is a unique rule which assigns to every $X \in \mathrm{Ob}(\mathcal{C})$ a "chern character"

$$ch^A : K_0(\mathit{Vect}(X)) \longrightarrow \prod_{i \geq 0} A^i(X)$$

with the following properties

(1) ch^A is a ring map for all $X \in \mathrm{Ob}(\mathcal{C})$.
(2) If $f : X' \to X$ is a morphism of $\mathcal{C}$, then $f^* \circ ch^A = ch^A \circ f^*$.
(3) Given $X \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{L} \in \mathrm{Pic}(X)$ we have $ch^A([\mathcal{L}]) = \exp(c_1^A(\mathcal{L}))$.

Proof. Let $X \in \mathrm{Ob}(\mathcal{C})$ and let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. We first show how to define the rank $r^A(\mathcal{E}) \in A^0(X)$. Namely, let $X = \bigcup X_r$ be the decomposition into open and closed subschemes such that $\mathcal{E}|_{X_r}$ has constant rank r. Since X is quasi-compact, this decomposition is finite, say $X = X_0 \amalg X_1 \amalg \ldots \amalg X_n$. Then $A(X) = A(X_0) \times A(X_1) \times \ldots \times A(X_n)$. Thus we can define $r^A(\mathcal{E}) = (0, 1, \ldots, n) \in A^0(X)$.

Let $P_p(c_1, \ldots, c_p)$ be the polynomials constructed in Chow Homology, Example 43.6. Then we can define

$$ch^A(\mathcal{E}) = r^A(\mathcal{E}) + \sum_{i \geq 1} (1/i!) P_i(c_1^A(\mathcal{E}), \ldots, c_i^A(\mathcal{E})) \in \prod_{i \geq 0} A^i(X)$$

where ci^A are the Chern classes of Proposition 12.1. It follows immediately that we have property (2) and (3) of the lemma.

We still have to show the following three statements

(1) If $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ is a short exact sequence of finite locally free $\mathcal{O}_X$-modules on $X \in \mathrm{Ob}(\mathcal{C})$, then $ch^A(\mathcal{E}) = ch^A(\mathcal{E}_1) + ch^A(\mathcal{E}_2)$.

(2) If $\mathcal{E}_1$ and $\mathcal{E}_2 \to 0$ are finite locally free $\mathcal{O}_X$-modules on $X \in \mathrm{Ob}(\mathcal{C})$, then $ch^A(\mathcal{E}_1 \otimes \mathcal{E}_2) = ch^A(\mathcal{E}_1)ch^A(\mathcal{E}_2)$.

Namely, the first will prove that ch^A factors through $K_0(Vect(X))$ and the first and the second will combined show that ch^A is a ring map.

To prove these statements we can reduce to the case where $\mathcal{E}_1$ and $\mathcal{E}_2$ have constant ranks r_1 and r_2. In this case the equalities in $A^0(X)$ are immediate. To prove the equalities in higher degrees, by Lemma 12.2 we may assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ have filtrations whose graded pieces are invertible modules $\mathcal{L}_{1,j}$, $j = 1, \ldots, r_1$ and $\mathcal{L}_{2,j}$, $j = 1, \ldots, r_2$. Using the multiplicativity of Chern classes we get

$$c_i^A(\mathcal{E}_1) = s_i(c_1^A(\mathcal{L}_{1,1}), \ldots, c_1^A(\mathcal{L}_{1,r_1}))$$

where s_i is the ith elementary symmetric function as in Chow Homology, Example 43.6. Similarly for $c_i^A(\mathcal{E}_2)$. In case (1) we get

$$c_i^A(\mathcal{E}) = s_i(c_1^A(\mathcal{L}_{1,1}), \ldots, c_1^A(\mathcal{L}_{1,r_1}), c_1^A(\mathcal{L}_{2,1}), \ldots, c_1^A(\mathcal{L}_{2,r_2}))$$

and for case (2) we get

$$c_i^A(\mathcal{E}_1 \otimes \mathcal{E}_2) = s_i(c_1^A(\mathcal{L}_{1,1}) + c_1^A(\mathcal{L}_{2,1}), \ldots, c_1^A(\mathcal{L}_{1,r_1}) + c_1^A(\mathcal{L}_{2,r_2}))$$

By the definition of the polynomials P_i we see that this means

$$P_i(c_1^A(\mathcal{E}_1), \ldots, c_i^A(\mathcal{E}_1)) = \sum\nolimits_{j=1,\ldots,r_1} c_1^A(\mathcal{L}_{1,j})^i$$

and similarly for $\mathcal{E}_2$. In case (1) we have also

$$P_i(c_1^A(\mathcal{E}), \ldots, c_i^A(\mathcal{E})) = \sum\nolimits_{j=1,\ldots,r_1} c_1^A(\mathcal{L}_{1,j})^i + \sum\nolimits_{j=1,\ldots,r_2} c_1^A(\mathcal{L}_{2,j})^i$$

In case (2) we get accordingly

$$P_i(c_1^A(\mathcal{E}_1 \otimes \mathcal{E}_2), \ldots, c_i^A(\mathcal{E}_1 \otimes \mathcal{E}_2)) = \sum\nolimits_{j=1,\ldots,r_1} \sum\nolimits_{j'=1,\ldots,r_2} (c_1^A(\mathcal{L}_{1,j}) + c_1^A(\mathcal{L}_{2,j'}))^i$$

Thus the desired equalities are now consequences of elementary identities between symmetric polynomials.

We omit the proof of uniqueness. $\qquad\square$

Lemma 12.5. *In the situation above let $X \in \mathrm{Ob}(\mathcal{C})$. If ψ^2 is as in Chow Homology, Lemma 56.1 and c^A and ch^A are as in Propositions 12.1 and 12.4 then we have $c_i^A(\psi^2(\alpha)) = 2^i c_i^A(\alpha)$ and $ch_i^A(\psi^2(\alpha)) = 2^i ch_i^A(\alpha)$ for all $\alpha \in K_0(Vect(X))$.*

Proof. Observe that the map $\prod_{i \geq 0} A^i(X) \to \prod_{i \geq 0} A^i(X)$ multiplying by 2^i on $A^i(X)$ is a ring map. Hence, since ψ^2 is also a ring map, it suffices to prove the formulas for additive generators of $K_0(Vect(X))$. Thus we may assume $\alpha = [\mathcal{E}]$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. By construction of the Chern classes of $\mathcal{E}$ we immediately reduce to the case where $\mathcal{E}$ has constant rank r. In this case, we can choose a projective smooth morphism $p : P \to X$ such that restriction $A^*(X) \to A^*(P)$ is injective and such that $p^*\mathcal{E}$ has a finite filtration whose graded

parts are invertible $\mathcal{O}_P$-modules $\mathcal{L}_j$, see Lemma 12.2. Then $[p^*\mathcal{E}] = \sum[\mathcal{L}_j]$ and hence $\psi^2([p^*\mathcal{E}]) = \sum[\mathcal{L}_j^{\otimes 2}]$ by definition of ψ^2. Setting $x_j = c_1^A(\mathcal{L}_j)$ we have

$$c^A(\alpha) = \prod(1+x_j) \quad \text{and} \quad c^A(\psi^2(\alpha)) = \prod(1+2x_j)$$

in $\prod A^i(P)$ and we have

$$ch^A(\alpha) = \sum \exp(x_j) \quad \text{and} \quad ch^A(\psi^2(\alpha)) = \sum \exp(2x_j)$$

in $\prod A^i(P)$. From these formulas the desired result follows. $\square$

13. Exterior powers and K-groups

We do the minimal amount of work to define the lambda operators. Let X be a scheme. Recall that $Vect(X)$ denotes the category of finite locally free $\mathcal{O}_X$-modules. Moreover, recall that we have constructed a zeroth K-group $K_0(Vect(X))$ associated to this category in Derived Categories of Schemes, Section 38. Finally, $K_0(Vect(X))$ is a ring, see Derived Categories of Schemes, Remark 38.6.

Lemma 13.1. *Let X be a scheme. There are maps*

$$\lambda^r : K_0(Vect(X)) \longrightarrow K_0(Vect(X))$$

which sends $[\mathcal{E}]$ to $[\wedge^r(\mathcal{E})]$ when $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module and which are compatible with pullbacks.

Proof. Consider the ring $R = K_0(Vect(X))[[t]]$ where t is a variable. For a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ we set

$$c(\mathcal{E}) = \sum_{i=0}^{\infty} [\wedge^i(\mathcal{E})]t^i$$

in R. We claim that given a short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

of finite locally free $\mathcal{O}_X$-modules we have $c(\mathcal{E}) = c(\mathcal{E}')c(\mathcal{E}'')$. The claim implies that c extends to a map

$$c : K_0(Vect(X)) \longrightarrow R$$

which converts addition in $K_0(Vect(X))$ to multiplication in R. Writing $c(\alpha) = \sum \lambda^i(\alpha)t^i$ we obtain the desired operators λ^i.

To see the claim, we consider the short exact sequence as a filtration on $\mathcal{E}$ with 2 steps. We obtain an induced filtration on $\wedge^r(\mathcal{E})$ with $r+1$ steps and subquotients

$$\wedge^r(\mathcal{E}'), \wedge^{r-1}(\mathcal{E}') \otimes \mathcal{E}'', \wedge^{r-2}(\mathcal{E}') \otimes \wedge^2(\mathcal{E}''), \dots \wedge^r(\mathcal{E}'')$$

Thus we see that $[\wedge^r(\mathcal{E})]$ is equal to

$$\sum_{i=0}^{r} [\wedge^{r-i}(\mathcal{E}')][\wedge^i(\mathcal{E}'')]$$

and the result follows easily from this and elementary algebra. $\square$

14. Weil cohomology theories, III

Let k be a field. Let F be a field of characteristic zero. Suppose we are given the following data

- (D0) A 1-dimensional F-vector space $F(1)$.
- (D1) A contravariant functor $H^*(-)$ from the category of smooth projective schemes over k to the category of graded commutative F-algebras.
- (D2') For every smooth projective scheme X over k a homomorphism $c_1^H : \mathrm{Pic}(X) \to H^2(X)(1)$ of abelian groups.

We will use the terminology, notation, and conventions regarding (D0) and (D1) as discussed in Section 9. Given a smooth projective scheme X over k and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ the cohomology class $c_1^H(\mathcal{L}) \in H^2(X)(1)$ of (D2') is sometimes called the *first Chern class of $\mathcal{L}$ in cohomology.*

Here is the list of axioms.

- (A1) H^* is compatible with finite coproducts
- (A2) c_1^H is compatible with pullbacks
- (A3) Let X be a smooth projective scheme over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $r \geq 1$. Consider the morphism $p : P = \mathbf{P}(\mathcal{E}) \to X$. Then the map

$$\bigoplus_{i=0,\ldots,r-1} H^*(X)(-i) \longrightarrow H^*(P), \quad (a_0, \ldots, a_{r-1}) \longmapsto \sum c_1^H(\mathcal{O}_P(1))^i \cup p^*(a_i)$$

    is an isomorphism of F-vector spaces.
- (A4) Let $i : Y \to X$ be the inclusion of an effective Cartier divisor over k with both X and Y smooth and projective over k. For $a \in H^*(X)$ with $i^*a = 0$ we have $a \cup c_1^H(\mathcal{O}_X(Y)) = 0$.
- (A5) H^* is compatible with finite products
- (A6) Let X be a nonempty smooth, projective scheme over k equidimensional of dimension d. Then there exists an F-linear map $\lambda : H^{2d}(X)(d) \to F$ such that $(\mathrm{id} \otimes \lambda)\gamma([\Delta]) = 1$ in $H^*(X)$.
- (A7) If $b : X' \to X$ is the blowing up of a smooth center in a smooth projective scheme X over k[2], then $b^* : H^*(X) \to H^*(X')$ is injective.
- (A8) If X is a smooth projective scheme over k and $k' = \Gamma(X, \mathcal{O}_X)$, then the map $H^0(\mathrm{Spec}(k')) \to H^0(X)$ is an isomorphism.
- (A9) Let X be a nonempty smooth projective scheme over k equidimensional of dimension d. Let $i : Y \to X$ be a nonempty effective Cartier divisor smooth over k. For $a \in H^{2d-2}(X)(d-1)$ we have $\lambda_Y(i^*(a)) = \lambda_X(a \cup c_1^H(\mathcal{O}_X(Y)))$ where λ_Y and λ_X are as in axiom (A6) for X and Y.

Let us explain more precisely what we mean by each of these axioms. Axioms (A3), (A4), and (A7) are clear as stated.

Ad (A1). This means that $H^*(\emptyset) = 0$ and that $(i^*, j^*) : H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$ is an isomorphism where i and j are the coprojections.

Ad (A2). This means that given a morphism $f : X \to Y$ of smooth projective schemes over k and an invertible $\mathcal{O}_Y$-module $\mathcal{N}$ we have $f^* c_1^H(\mathcal{L}) = c_1^H(f^*\mathcal{L})$.

---

[2]Then X' is smooth and projective over k as well, see More on Morphisms, Lemma 17.3.

Ad (A5). This means that $H^*(\mathrm{Spec}(k)) = F$ and that for X and Y smooth projective over k the map $H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y)$, $a \otimes b \mapsto p^*(a) \cup q^*(b)$ is an isomorphism where p and q are the projections.

Ad (A6). Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. By Lemma 14.2 if we have axioms (A1) – (A4) we can consider the class of the diagonal

$$\gamma([\Delta]) \in H^{2d}(X \times X)(d) = \bigoplus_i H^i(X) \otimes_F H^{2d-i}(X)(d)$$

where the tensor decomposition comes from axiom (A5). Given an F-linear map $\lambda : H^{2d}(X)(d) \to F$ we may also view λ as an F-linear map $\lambda : H^*(X)(d) \to F$ by precomposing with the projection onto $H^{2d}(X)(d)$. Having said this axiom (A6) makes sense.

Ad (A8). Let X be a smooth projective scheme over k. Then $k' = \Gamma(X, \mathcal{O}_X)$ is a finite separable k-algebra (Varieties, Lemma 9.3) and hence $\mathrm{Spec}(k')$ is smooth and projective over k. Thus we may apply H^* to $\mathrm{Spec}(k')$ and axiom (A8) makes sense.

Ad (A9). We will see in Remark 14.6 that if we have axioms (A1) – (A7) then the map λ of axiom (A6) is unique.

Lemma 14.1. *Assume given (D0), (D1), and (D2') satisfying axioms (A1), (A2), (A3), and (A4). There is a unique rule which assigns to every smooth projective X over k a ring homomorphism*

$$ch^H : K_0(\mathit{Vect}(X)) \longrightarrow \prod_{i \geq 0} H^{2i}(X)(i)$$

compatible with pullbacks such that $ch^H(\mathcal{L}) = \exp(c_1^H(\mathcal{L}))$ for any invertible $\mathcal{O}_X$-module $\mathcal{L}$.

Proof. Immediate from Proposition 12.4 applied to the category of smooth projective schemes over k, the functor $A : X \mapsto \bigoplus_{i \geq 0} H^{2i}(X)(i)$, and the map c_1^H. $\square$

Lemma 14.2. *Assume given (D0), (D1), and (D2') satisfying axioms (A1), (A2), (A3), and (A4). There is a unique rule which assigns to every smooth projective X over k a graded ring homomorphism*

$$\gamma : \mathrm{CH}^*(X) \longrightarrow \bigoplus_{i \geq 0} H^{2i}(X)(i)$$

compatible with pullbacks such that $ch^H(\alpha) = \gamma(ch(\alpha))$ for α in $K_0(\mathit{Vect}(X))$.

Proof. Recall that we have an isomorphism

$$K_0(\mathit{Vect}(X)) \otimes \mathbf{Q} \longrightarrow \mathrm{CH}^*(X) \otimes \mathbf{Q}, \quad \alpha \longmapsto ch(\alpha) \cap [X]$$

see Chow Homology, Lemma 58.1. It is an isomorphism of rings by Chow Homology, Remark 56.5. We define γ by the formula $\gamma(\alpha) = ch^H(\alpha')$ where ch^H is as in Lemma 14.1 and $\alpha' \in K_0(\mathit{Vect}(X))$ is such that $ch(\alpha') \cap [X] = \alpha$ in $\mathrm{CH}^*(X) \otimes \mathbf{Q}$.

The construction $\alpha \mapsto \gamma(\alpha)$ is compatible with pullbacks because both ch^H and taking Chern classes is compatible with pullbacks, see Lemma 14.1 and Chow Homology, Remark 59.9.

We still have to see that γ is graded. Let $\psi^2 : K_0(\mathit{Vect}(X)) \to K_0(\mathit{Vect}(X))$ be the second Adams operator, see Chow Homology, Lemma 56.1. If $\alpha \in \mathrm{CH}^i(X)$ and

$\alpha' \in K_0(Vect(X)) \otimes \mathbf{Q}$ is the unique element with $ch(\alpha') \cap [X] = \alpha$, then we have seen in Chow Homology, Section 58 that $\psi^2(\alpha') = 2^i \alpha'$. Hence we conclude that $ch^H(\alpha') \in H^{2i}(X)(i)$ by Lemma 12.5 as desired. $\qquad\square$

Lemma 14.3. *Let $b : X' \to X$ be the blowing up of a smooth projective scheme over k in a smooth closed subscheme $Z \subset X$. Picture*

$$
\begin{array}{ccc}
E & \xrightarrow{\ j\ } & X' \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle b} \\
Z & \xrightarrow{\ i\ } & X
\end{array}
$$

Assume there exists an element of $K_0(X)$ whose restriction to Z is equal to the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$. Assume every irreducible component of Z has codimension r in X. Then there exists a cycle $\theta \in \mathrm{CH}^{r-1}(X')$ such that $b^![Z] = [E] \cdot \theta$ in $\mathrm{CH}^r(X')$ and $\pi_* j^!(\theta) = [Z]$ in $\mathrm{CH}^r(Z)$.

Proof. The scheme X is smooth and projective over k and hence we have $K_0(X) = K_0(Vect(X))$. See Derived Categories of Schemes, Lemmas 36.2 and 38.5. Let $\alpha \in K_0(\mathrm{Vect}(X))$ be an element whose restriction to Z is $\mathcal{C}_{Z/X}$. By Chow Homology, Lemma 56.3 there exists an element $\alpha^\vee$ which restricts to $\mathcal{C}_{Z/X}^\vee$. By the blow up formula (Chow Homology, Lemma 59.11) we have

$$b^![Z] = b^! i_*[Z] = j_* res(b^!)([Z]) = j_*(c_{r-1}(\mathcal{F}^\vee) \cap \pi^*[Z]) = j_*(c_{r-1}(\mathcal{F}^\vee) \cap [E])$$

where $\mathcal{F}$ is the kernel of the surjection $\pi^* \mathcal{C}_{Z/X} \to \mathcal{C}_{E/X'}$. Observe that $b^* \alpha^\vee - [\mathcal{O}_{X'}(E)]$ is an element of $K_0(\mathrm{Vect}(X'))$ which restricts to $[\pi^* \mathcal{C}_{Z/X}^\vee] - [\mathcal{C}_{E/X'}^\vee] = [\mathcal{F}^\vee]$ on E. Since capping with Chern classes commutes with j_* we conclude that the above is equal to

$$c_{r-1}(b^* \alpha^\vee - [\mathcal{O}_{X'}(E)]) \cap [E]$$

in the chow group of X'. Hence we see that setting

$$\theta = c_{r-1}(b^* \alpha^\vee - [\mathcal{O}_{X'}(E)]) \cap [X']$$

we get the first relation $\theta \cdot [E] = b^![Z]$ for example by Chow Homology, Lemma 62.2. For the second relation observe that

$$j^! \theta = j^!(c_{r-1}(b^* \alpha^\vee - [\mathcal{O}_{X'}(E)]) \cap [X']) = c_{r-1}(\mathcal{F}^\vee) \cap j^![X'] = c_{r-1}(\mathcal{F}^\vee) \cap [E]$$

in the chow groups of E. To prove that π_* of this is equal to $[Z]$ it suffices to prove that the degree of the codimension $r-1$ cycle $(-1)^{r-1} c_{r-1}(\mathcal{F}) \cap [E]$ on the fibres of π is 1. This is a computation we omit. $\qquad\square$

Lemma 14.4. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A4) and (A7). Let X be a smooth projective scheme over k. Let $Z \subset X$ be a smooth closed subscheme such that every irreducible component of Z has codimension r in X. Assume the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$ is the restriction of an element of $K_0(X)$. If $a \in H^*(X)$ and $a|_Z = 0$ in $H^*(Z)$, then $\gamma([Z]) \cup a = 0$.*

Proof. Let $b : X' \to X$ be the blowing up. By (A7) it suffices to show that

$$b^*(\gamma([Z]) \cup a) = b^* \gamma([Z]) \cup b^* a = 0$$

By Lemma 14.3 we have

$$b^* \gamma([Z]) = \gamma(b^*[Z]) = \gamma([E] \cdot \theta) = \gamma([E]) \cup \gamma(\theta)$$

Hence because b^*a restricts to zero on E and since $\gamma([E]) = c_1^H(\mathcal{O}_{X'}(E))$ we get what we want from (A4). $\square$

Lemma 14.5. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then axiom (A) of Section 9 holds with $\int_X = \lambda$ as in axiom (A6).*

Proof. Let X be a nonempty smooth projective scheme over k which is equidimensional of dimension d. We will show that the graded F-vector space $H^*(X)(d)[2d]$ is a left dual to $H^*(X)$. This will prove what we want by Homology, Lemma 17.5. We are going to use axiom (A5) which in particular says that

$$H^*(X \times X)(d) = \bigoplus H^i(X) \otimes H^j(X)(d) = \bigoplus H^i(X)(d) \otimes H^j(X)$$

Define a map

$$\eta : F \longrightarrow H^*(X \times X)(d)$$

by multiplying by $\gamma([\Delta]) \in H^{2d}(X \times X)(d)$. On the other hand, define a map

$$\epsilon : H^*(X \times X)(d) \longrightarrow H^*(X)(d) \xrightarrow{\lambda} F$$

by first using pullback Δ^* by the diagonal morphism $\Delta : X \to X \times X$ and then using the F-linear map $\lambda : H^{2d}(X)(d) \to F$ of axiom (A6) precomposed by the projection $H^*(X)(d) \to H^{2d}(X)(d)$. In order to show that $H^*(X)(d)$ is a left dual to $H^*(X)$ we have to show that the composition of the maps

$$\eta \otimes 1 : H^*(X) \longrightarrow H^*(X \times X \times X)(d)$$

and

$$1 \otimes \epsilon : H^*(X \times X \times X)(d) \longrightarrow H^*(X)$$

is the identity. If $a \in H^*(X)$ then we see that the composition maps a to

$$(1 \otimes \lambda)(\Delta_{23}^*(q_{12}^*\gamma([\Delta]) \cup q_3^*a)) = (1 \otimes \lambda)(\gamma([\Delta]) \cup p_2^*a)$$

where $q_i : X \times X \times X \to X$ and $q_{ij} : X \times X \times X \to X \times X$ are the projections, $\Delta_{23} : X \times X \to X \times X \times X$ is the diagonal, and $p_i : X \times X \to X$ are the projections. The equality holds because $\Delta_{23}^*(q_{12}^*\gamma([\Delta]) = \Delta_{23}^*\gamma([\Delta \times X]) = \gamma([\Delta])$ and because $\Delta_{23}^*q_3^*a = p_2^*a$. Since $\gamma([\Delta]) \cup p_1^*a = \gamma([\Delta]) \cup p_2^*a$ (see below) the above simplifies to

$$(1 \otimes \lambda)(\gamma([\Delta]) \cup p_1^*a) = a$$

by our choice of λ as desired. The second condition $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \text{id}$ of Categories, Definition 43.5 is proved in exactly the same manner.

Note that p_1^*a and pr_2^*a restrict to the same cohomology class on $\Delta \subset X \times X$. Moreover we have $\mathcal{C}_{\Delta/X \times X} = \Omega_\Delta^1$ which is the restriction of $p_1^*\Omega_X^1$. Hence Lemma 14.4 implies $\gamma([\Delta]) \cup p_1^*a = \gamma([\Delta]) \cup p_2^*a$ and the proof is complete. $\square$

Remark 14.6 (Uniqueness of trace maps). Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Let X be a smooth projective scheme over k which is nonempty and equidimensional of dimension d. Combining what was said in the proofs of Lemma 14.5 and Homology, Lemma 17.5 we see that

$$\gamma([\Delta]) \in \bigoplus_i H^i(X) \otimes H^{2d-i}(X)(d)$$

defines a perfect duality between $H^i(X)$ and $H^{2d-i}(X)(d)$ for all i. In particular, the linear map $\int_X = \lambda : H^{2d}(X)(d) \to F$ of axiom (A6) is unique! We will call the linear map $\int_X$ the trace map of X from now on.

Lemma 14.7. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then axiom (B) of Section 9 holds.*

Proof. Axiom (B)(a) is immediate from axiom (A5). Let X and Y be nonempty smooth projective schemes over k equidimensional of dimensions d and e. To see that axiom (B)(b) holds, observe that the diagonal $\Delta_{X \times Y}$ of $X \times Y$ is the intersection product of the pullbacks of the diagonals Δ_X of X and Δ_Y of Y by the projections $p : X \times Y \times X \times Y \to X \times X$ and $q : X \times Y \times X \times Y \to Y \times Y$. Compatibility of γ with intersection products then gives that

$$\gamma([\Delta_{X \times Y}]) \in H^{2d+2e}(X \times Y \times X \times Y)(d+e)$$

is the cup product of the pullbacks of $\gamma([\Delta_X])$ and $\gamma([\Delta_Y])$ by p and q. Write

$$\gamma([\Delta_{X \times Y}]) = \sum \eta_{X \times Y, i} \text{ with } \eta_{X \times Y, i} \in H^i(X \times Y) \otimes H^{2d+2e-i}(X \times Y)(d+e)$$

and similarly $\gamma([\Delta_X]) = \sum \eta_{X,i}$ and $\gamma([\Delta_Y]) = \sum \eta_{Y,i}$. The observation above implies we have

$$\eta_{X \times Y, 0} = \sum\nolimits_{i \in \mathbf{Z}} p^* \eta_{X,i} \cup q^* \eta_{Y,-i}$$

(If our cohomology theory vanishes in negative degrees, which will be true in almost all cases, then only the term for $i = 0$ contributes and $\eta_{X \times Y, 0}$ lies in $H^0(X) \otimes H^0(Y) \otimes H^{2d}(X)(d) \otimes H^{2e}(Y)(e)$ as expected, but we don't need this.) Since $\lambda_X : H^{2d}(X)(d) \to F$ and $\lambda_Y : H^{2e}(Y)(e) \to F$ send $\eta_{X,0}$ and $\eta_{Y,0}$ to 1 in $H^*(X)$ and $H^*(Y)$, we see that $\lambda_X \otimes \lambda_Y$ sends $\eta_{X \times Y, 0}$ to 1 in $H^*(X) \otimes H^*(Y) = H^*(X \times Y)$ and the proof is complete. $\square$

Lemma 14.8. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then axiom (C)(d) of Section 9 holds.*

Proof. We have $\gamma([\mathrm{Spec}(k)]) = 1 \in H^*(\mathrm{Spec}(k))$ by construction. Since

$$H^0(\mathrm{Spec}(k)) = F, \quad H^0(\mathrm{Spec}(k) \times \mathrm{Spec}(k)) = H^0(\mathrm{Spec}(k)) \otimes_F H^0(\mathrm{Spec}(k))$$

the map $\int_{\mathrm{Spec}(k)} = \lambda$ of axiom (A6) must send 1 to 1 because we have seen that $\int_{\mathrm{Spec}(k) \times \mathrm{Spec}(k)} = \int_{\mathrm{Spec}(k)} \int_{\mathrm{Spec}(k)}$ in Lemma 14.7. $\square$

Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Then we obtain data (D0), (D1), (D2), and (D3) of Section 9 satisfying axioms (A), (B) and (C)(a), (C)(c), and (C)(d) of Section 9, see Lemmas 14.5, 14.7, and 14.8. Moreover, we have the pushforwards $f_* : H^*(X) \to H^*(Y)$ as constructed in Section 9. The only axiom of Section 9 which isn't clear yet is axiom (C)(b).

Lemma 14.9. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Let $p : P \to X$ be as in axiom (A3) with X nonempty equidimensional. Then γ commutes with pushforward along p.*

Proof. It suffices to prove this on generators for $\mathrm{CH}_*(P)$. Thus it suffices to prove this for a cycle class of the form $\xi^i \cdot p^* \alpha$ where $0 \le i \le r - 1$ and $\alpha \in \mathrm{CH}_*(X)$. Note that $p_*(\xi^i \cdot p^* \alpha) = 0$ if $i < r - 1$ and $p_*(\xi^{r-1} \cdot p^* \alpha) = \alpha$. On the other hand, we have $\gamma(\xi^i \cdot p^* \alpha) = c^i \cup p^* \gamma(\alpha)$ and by the projection formula (Lemma 9.1) we have

$$p_* \gamma(\xi^i \cdot p^* \alpha) = p_*(c^i) \cup \gamma(\alpha)$$

Thus it suffices to show that $p_* c^i = 0$ for $i < r - 1$ and $p_* c^{r-1} = 1$. Equivalently, it suffices to prove that $\lambda_P : H^{2d+2r-2}(P)(d+r-1) \to F$ defined by the rules

$$\lambda_P(c^i \cup p^*(a)) = \begin{cases} 0 & \text{if} \quad i < r - 1 \\ \int_X(a) & \text{if} \quad i = r - 1 \end{cases}$$

satisfies the condition of axiom (A5). This follows from the computation of the class of the diagonal of P in Lemma 6.2. $\square$

Lemma 14.10. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). If k'/k is a Galois extension, then we have $\int_{\mathrm{Spec}(k')} 1 = [k' : k]$.*

Proof. We observe that

$$\mathrm{Spec}(k') \times \mathrm{Spec}(k') = \coprod_{\sigma \in \mathrm{Gal}(k'/k)} (\mathrm{Spec}(\sigma) \times \mathrm{id})^{-1} \Delta$$

as cycles on $\mathrm{Spec}(k') \times \mathrm{Spec}(k')$. Our construction of γ always sends $[X]$ to 1 in $H^0(X)$. Thus $1 \otimes 1 = 1 = \sum (\mathrm{Spec}(\sigma) \times \mathrm{id})^* \gamma([\Delta])$. Denote $\lambda : H^0(\mathrm{Spec}(k')) \to F$ the map from axiom (A6), in other words $(\mathrm{id} \otimes \lambda)(\gamma(\Delta)) = 1$ in $H^0(\mathrm{Spec}(k'))$. We obtain

$$\begin{aligned} \lambda(1)1 &= (\mathrm{id} \otimes \lambda)(1 \otimes 1) \\ &= (\mathrm{id} \otimes \lambda)(\sum (\mathrm{Spec}(\sigma) \times \mathrm{id})^* \gamma([\Delta])) \\ &= \sum (\mathrm{Spec}(\sigma) \times \mathrm{id})^*((\mathrm{id} \otimes \lambda)(\gamma([\Delta])) \\ &= \sum (\mathrm{Spec}(\sigma) \times \mathrm{id})^*(1) \\ &= [k' : k] \end{aligned}$$

Since λ is another name for $\int_{\mathrm{Spec}(k')}$ (Remark 14.6) the proof is complete. $\square$

Lemma 14.11. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). In order to show that γ commutes with pushforward it suffices to show that $i_*(1) = \gamma([Z])$ if $i : Z \to X$ is a closed immersion of nonempty smooth projective equidimensional schemes over k.*

Proof. We will use without further mention that we've constructed our cycle class map γ in Lemma 14.2 compatible with intersection products and pullbacks and that we've already shown axioms (A), (B), (C)(a), (C)(c), and (C)(d) of Section 9, see Lemma 14.5, Remark 14.6, and Lemmas 14.7 and 14.8. In particular, we may use (for example) Lemma 9.1 to see that pushforward on H^* is compatible with composition and satisfies the projection formula.

Let $f : X \to Y$ be a morphism of nonempty equidimensional smooth projective schemes over k. We are trying to show $f_* \gamma(\alpha) = \gamma(f_* \alpha)$ for any cycle class α on X. We can write α as a **Q**-linear combination of products of Chern classes of locally free $\mathcal{O}_X$-modules (Chow Homology, Lemma 58.1). Thus we may assume α is a product of Chern classes of finite locally free $\mathcal{O}_X$-modules $\mathcal{E}_1, \ldots, \mathcal{E}_r$. Pick $p : P \to X$ as in the splitting principle (Chow Homology, Lemma 43.1). By Chow Homology, Remark 43.2 we see that p is a composition of projective space bundles and that $\alpha = p_*(\xi_1 \cap \ldots \cap \xi_d \cap \cdot p^* \alpha)$ where ξ_i are first Chern classes of invertible modules. By Lemma 14.9 we know that p_* commutes with cycle classes. Thus it suffices to prove the property for the composition $f \circ p$. Since $p^* \mathcal{E}_1, \ldots, p^* \mathcal{E}_r$ have filtrations whose successive quotients are invertible modules, this reduces us to the

case where α is of the form $\xi_1 \cap \ldots \cap \xi_t \cap [X]$ for some first Chern classes ξ_i of invertible modules $\mathcal{L}_i$.

Assume $\alpha = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_t) \cap [X]$ for some invertible modules $\mathcal{L}_i$ on X. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For $n \gg 0$ the invertible $\mathcal{O}_X$-modules $\mathcal{L}^{\otimes n}$ and $\mathcal{L}_1 \otimes \mathcal{L}^{\otimes n}$ are both very ample on X over k, see Morphisms, Lemma 39.8. Since $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_1 \otimes \mathcal{L}^{\otimes n}) - c_1(\mathcal{L}^{\otimes n})$ this reduces us to the case where $\mathcal{L}_1$ is very ample. Repeating this with $\mathcal{L}_i$ for $i = 2, \ldots, t$ we reduce to the case where $\mathcal{L}_i$ is very ample on X over k for all $i = 1, \ldots, t$.

Assume k is infinite and $\alpha = c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_t) \cap [X]$ for some very ample invertible modules $\mathcal{L}_i$ on X over k. By Bertini in the form of Varieties, Lemma 47.3 we can successively find regular sections s_i of $\mathcal{L}_i$ such that the schemes $Z(s_1) \cap \ldots \cap Z(s_i)$ are smooth over k and of codimension i in X. By the construction of capping with the first class of an invertible module (going back to Chow Homology, Definition 24.1), this reduces us to the case where $\alpha = [Z]$ for some nonempty smooth closed subscheme $Z \subset X$ which is equidimensional.

Assume $\alpha = [Z]$ where $Z \subset X$ is a smooth closed subscheme. Choose a closed embedding $X \to \mathbf{P}^n$. We can factor f as

$$X \to Y \times \mathbf{P}^n \to Y$$

Since we know the result for the second morphism by Lemma 14.9 it suffices to prove the result when $\alpha = [Z]$ where $i : Z \to X$ is a closed immersion and f is a closed immersion. Then $j = f \circ i$ is a closed embedding too. Using the hypothesis for i and j we win.

We still have to prove the lemma in case k is finite. We urge the reader to skip the rest of the proof. Everything we said above continues to work, except that we do not know we can choose the sections s_i cutting out our Z over k as k is finite. However, we do know that we can find s_i over the algebraic closure $\bar{k}$ of k (by the same lemma). This means that we can find a finite extension k'/k such that our sections s_i are defined over k'. Denote $\pi : X_{k'} \to X$ the projection. The arguments above shows that we get the desired conclusion (from the assumption in the lemma) for the cycle $\pi^* \alpha$ and the morphism $f \circ \pi : X_{k'} \to Y$. We have $\pi_* \pi^* \alpha = [k' : k]\alpha$, see Chow Homology, Lemma 15.2. On the other hand, we have

$$\pi_* \gamma(\pi^* \alpha) = \pi_* \pi^* \gamma(\alpha) = \gamma(\alpha) \pi_* 1$$

by the projection formula for our cohomology theory. Observe that π is a projection (!) and hence we have $\pi_*(1) = \int_{\mathrm{Spec}(k')}(1)1$ by Lemma 9.2. Thus to finish the proof in the finite field case, it suffices to prove that $\int_{\mathrm{Spec}(k')}(1) = [k' : k]$ which we do in Lemma 14.10. $\qquad\square$

In the lemmas below we use the Grassmannians defined and constructed in Constructions, Section 22.

Lemma 14.12. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). Given integers $0 < l < n$ and a nonempty equidimensional smooth projective scheme X over k consider the projection morphism $p : X \times \mathbf{G}(l, n) \to X$. Then γ commutes with pushforward along p.*

Proof. If $l = 1$ or $l = n - 1$ then p is a projective bundle and the result follows from Lemma 14.9. In general there exists a morphism

$$h : Y \to X \times \mathbf{G}(l, n)$$

such that both h and $p \circ h$ are compositions of projective space bundles. Namely, denote $\mathbf{G}(1, 2, \ldots, l; n)$ the partial flag variety. Then the morphism

$$\mathbf{G}(1, 2, \ldots, l; n) \to \mathbf{G}(l, n)$$

is a compostion of projective space bundles and similarly the structure morphism $\mathbf{G}(1, 2, \ldots, l; n) \to \mathrm{Spec}(k)$ is of this form. Thus we may set $Y = X \times \mathbf{G}(1, 2, \ldots, l; n)$. Since every cycle on $X \times \mathbf{G}(l, n)$ is the pushforward of a cycle on Y, the result for $Y \to X$ and the result for $Y \to X \times \mathbf{G}(l, n)$ imply the result for p. $\qquad\square$

Lemma 14.13. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). In order to show that γ commutes with pushforward it suffices to show that $i_*(1) = \gamma([Z])$ if $i : Z \to X$ is a closed immersion of nonempty smooth projective equidimensional schemes over k such that the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$ is the pullback of a class in $K_0(X)$.*

Proof. By Lemma 14.11 it suffices to show that $i_*(1) = \gamma([Z])$ if $i : Z \to X$ is a closed immersion of nonempty smooth projective equidimensional schemes over k. Say Z has codimension r in X. Let $\mathcal{L}$ be a sufficiently ample invertible module on X. Choose $n > 0$ and a surjection

$$\mathcal{O}_Z^{\oplus n} \to \mathcal{C}_{Z/X} \otimes \mathcal{L}|_Z$$

This gives a morphism $g : Z \to \mathbf{G}(n - r, n)$ to the Grassmannian over k, see Constructions, Section 22. Consider the composition

$$Z \to X \times \mathbf{G}(n - r, n) \to X$$

Pushforward along the second morphism is compatible with classes of cycles by Lemma 14.12. The conormal sheaf $\mathcal{C}$ of the closed immersion $Z \to X \times \mathbf{G}(n - r, n)$ sits in a short exact sequence

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{C} \to g^* \Omega_{\mathbf{G}(n-r,n)} \to 0$$

See More on Morphisms, Lemma 11.13. Since $\mathcal{C}_{Z/X} \otimes \mathcal{L}|_Z$ is the pull back of a finite locally free sheaf on $\mathbf{G}(n - r, n)$ we conclude that the class of $\mathcal{C}$ in $K_0(Z)$ is the pullback of a class in $K_0(X \times \mathbf{G}(n - r, n))$. Hence we have the property for $Z \to X \times \mathbf{G}(n - r, n)$ and we conclude. $\qquad\square$

Lemma 14.14. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A7). If $k''/k'/k$ are finite separable field extensions, then $H^0(\mathrm{Spec}(k')) \to H^0(\mathrm{Spec}(k''))$ is injective.*

Proof. We may replace k'' by its normal closure over k which is Galois over k, see Fields, Lemma 21.5. Then k'' is Galois over k' as well, see Fields, Lemma 21.4. We deduce we have an isomorphism

$$k' \otimes_k k'' \longrightarrow \prod_{\sigma \in \mathrm{Gal}(k''/k')} k'', \quad \eta \otimes \zeta \longmapsto (\eta \sigma(\zeta))_\sigma$$

This produces an isomorphism $\coprod_\sigma \mathrm{Spec}(k'') \to \mathrm{Spec}(k') \times \mathrm{Spec}(k'')$ which on cohomology produces the isomorphism

$$H^*(\mathrm{Spec}(k')) \otimes_F H^*(\mathrm{Spec}(k'')) \to \prod_\sigma H^*(\mathrm{Spec}(k'')), \quad a' \otimes a'' \longmapsto (\pi^* a' \cup \mathrm{Spec}(\sigma)^* a'')_\sigma$$

where $\pi : \mathrm{Spec}(k'') \to \mathrm{Spec}(k')$ is the morphism corresponding to the inclusion of k' in k''. We conclude the lemma is true by taking $a'' = 1$. $\qquad\square$

Lemma 14.15. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A8). Let $b : X' \to X$ be a blowing up of a smooth projective scheme X over k which is nonempty equidimensional of dimension d in a nonwhere dense smooth center Z. Then $b_*(1) = 1$.*

Proof. We may replace X by a connected component of X (some details omitted). Thus we may assume X is connected and hence irreducible. Set $k' = \Gamma(X, \mathcal{O}_X) = \Gamma(X', \mathcal{O}_{X'})$; we omit the proof of the equality. Choose a closed point $x' \in X'$ which isn't contained in the exceptional divisor and whose residue field k'' is separable over k; this is possible by Varieties, Lemma 25.6. Denote $x \in X$ the image (whose residue field is equal to k'' as well of course). Consider the diagram

$$
\begin{array}{ccc}
x' \times X' & \longrightarrow & X' \times X' \\
\downarrow & & \downarrow \\
x \times X & \longrightarrow & X \times X
\end{array}
$$

The class of the diagonal $\Delta = \Delta_X$ pulls back to the class of the "diagonal point" $\delta_x : x \to x \times X$ and similarly for the class of the diagonal Δ'. On the other hand, the diagonal point δ_x pulls back to the diagonal point $\delta_{x'}$ by the left vertical arrow. Write $\gamma([\Delta]) = \sum \eta_i$ with $\eta_i \in H^i(X) \otimes H^{2d-i}(X)(d)$ and $\gamma([\Delta']) = \sum \eta_i'$ with $\eta_i' \in H^i(X') \otimes H^{2d-i}(X')(d)$. The arguments above show that η_0 and η_0' map to the same class in

$$ H^0(x') \otimes_F H^{2d}(X')(d) $$

We have $H^0(\mathrm{Spec}(k')) = H^0(X) = H^0(X')$ by axiom (A8). By Lemma 14.14 this common value maps injectively into $H^0(x')$. We conclude that η_0 maps to η_0' by the map

$$ H^0(X) \otimes_F H^{2d}(X)(d) \longrightarrow H^0(X') \otimes_F H^{2d}(X')(d) $$

This means that $\int_X$ is equal to $\int_{X'}$ composed with the pullback map. This proves the lemma. $\qquad\square$

Lemma 14.16. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A8). Then the cycle class map γ commutes with pushforward.*

Proof. Let $i : Z \to X$ be as in Lemma 14.13. Consider the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\ j\ } & X' \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle b} \\
Z & \xrightarrow{\ i\ } & X
\end{array}
$$

Let $\theta \in \mathrm{CH}^{r-1}(X')$ be as in Lemma 14.3. Then $\pi_* j^! \theta = [Z]$ in $\mathrm{CH}_*(Z)$ implies that $\pi_* \gamma(j^! \theta) = 1$ by Lemma 14.9 because π is a projective space bundle. Hence we see that

$$ i_*(1) = i_*(\pi_*(\gamma(j^! \theta))) = b_* j_*(j^* \gamma(\theta)) = b_*(j_*(1) \cup \gamma(\theta)) $$

We have $j_*(1) = \gamma([E])$ by (A9). Thus this is equal to

$$ b_*(\gamma([E]) \cup \gamma(\theta)) = b_*(\gamma([E] \cdot \theta)) = b_*(\gamma(b^*[Z])) = b_* b^* \gamma([Z]) = b_*(1) \cup \gamma([Z]) $$

Since $b_*(1) = 1$ by Lemma 14.15 the proof is complete. $\qquad\square$

Proposition 14.17. *Assume given data (D0), (D1), and (D2') satisfying axioms (A1) – (A8). Then we have a Weil cohomology theory.*

Proof. We have axioms (A), (B) and (C)(a), (C)(c), and (C)(d) of Section 9 by Lemmas 14.5, 14.7, and 14.8. We have axiom (C)(b) by Lemma 14.16. Finally, the additional condition of Definition 11.4 holds because it is the same as our axiom (A8). □

The following lemma is sometimes useful to show that we get a Weil cohomology theory over a nonclosed field by reducing to a closed one.

Lemma 14.18. *Let k'/k be an extension of fields. Let F'/F be an extension of fields of characteristic 0. Assume given*

(1) *data (D0), (D1), (D2') for k and F denoted $F(1), H^*, c_1^H$,*
(2) *data (D0), (D1), (D2') for k' and F' denoted $F'(1), (H')^*, c_1^{H'}$, and*
(3) *an isomorphism $F(1) \otimes_F F' \to F'(1)$, functorial isomorphisms $H^*(X) \otimes_F F' \to (H')^*(X_{k'})$ on the category of smooth projective schemes X over k such that the diagrams*

$$
\begin{array}{ccc}
\mathrm{Pic}(X) & \xrightarrow{\ c_1^H\ } & H^2(X)(1) \\
\downarrow & & \downarrow \\
\mathrm{Pic}(X_{k'}) & \xrightarrow{\ c_1^{H'}\ } & (H')^2(X_{k'})(1)
\end{array}
$$

commute.

In this case, if $F'(1), (H')^*, c_1^{H'}$ satisfy axioms (A1) – (A9), then the same is true for $F(1), H^*, c_1^H$.

Proof. We go by the axioms one by one.

Axiom (A1). We have to show $H^*(\emptyset) = 0$ and that $(i^*, j^*) : H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$ is an isomorphism where i and j are the coprojections. By the functorial nature of the isomorphisms $H^*(X) \otimes_F F' \to (H')^*(X_{k'})$ this follows from linear algebra: if $\varphi : V \to W$ is an F-linear map of F-vector spaces, then φ is an isomorphism if and only if $\varphi_{F'} : V \otimes_F F' \to W \otimes_F F'$ is an isomorphism.

Axiom (A2). This means that given a morphism $f : X \to Y$ of smooth projective schemes over k and an invertible $\mathcal{O}_Y$-module $\mathcal{N}$ we have $f^* c_1^H(\mathcal{L}) = c_1^H(f^*\mathcal{L})$. This is immediately clear from the corresponding property for $c_1^{H'}$, the commutative diagrams in the lemma, and the fact that the canonical map $V \to V \otimes_F F'$ is injective for any F-vector space V.

Axiom (A3). This follows from the principle stated in the proof of axiom (A1) and compatibility of c_1^H and $c_1^{H'}$.

Axiom (A4). Let $i : Y \to X$ be the inclusion of an effective Cartier divisor over k with both X and Y smooth and projective over k. For $a \in H^*(X)$ with $i^*a = 0$ we have to show $a \cup c_1^H(\mathcal{O}_X(Y)) = 0$. Denote $a' \in (H')^*(X_{k'})$ the image of a. The assumption implies that $(i')^*a' = 0$ where $i' : Y_{k'} \to X_{k'}$ is the base change of i. Hence we get $a' \cup c_1^{H'}(\mathcal{O}_{X_{k'}}(Y_{k'})) = 0$ by the axiom for $(H')^*$. Since $a' \cup c_1^{H'}(\mathcal{O}_{X_{k'}}(Y_{k'}))$ is the image of $a \cup c_1^H(\mathcal{O}_X(Y))$ we conclude by the princple stated in the proof of axiom (A2).

Axiom (A5). This means that $H^*(\mathrm{Spec}(k)) = F$ and that for X and Y smooth projective over k the map $H^*(X) \otimes_F H^*(Y) \to H^*(X \times Y)$, $a \otimes b \mapsto p^*(a) \cup q^*(b)$ is an isomorphism where p and q are the projections. This follows from the principle stated in the proof of axiom (A1).

We interrupt the flow of the arguments to show that for every smooth projective scheme X over k the diagram

$$
\begin{array}{ccc}
\mathrm{CH}^*(X) & \xrightarrow{\ \gamma\ } & \bigoplus H^{2i}(X)(i) \\
{\scriptstyle g^*}\big\downarrow & & \big\downarrow \\
\mathrm{CH}^*(X_{k'}) & \xrightarrow{\ \gamma'\ } & \bigoplus (H')^{2i}(X_{k'})(i)
\end{array}
$$

commutes. Observe that we have γ as we know axioms (A1) – (A4) already; see Lemma 14.2. Also, the left vertical arrow is the one discussed in Chow Homology, Section 67 for the morphism of base schemes $g : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$. More precisely, it is the map given in Chow Homology, Lemma 67.4. Pick $\alpha \in \mathrm{CH}^*(X)$. Write $\alpha = ch(\beta) \cap [X]$ in $\mathrm{CH}^*(X) \otimes \mathbf{Q}$ for some $\beta \in K_0(\mathit{Vect}(X)) \otimes \mathbf{Q}$ so that $\gamma(\alpha) = ch^H(\beta)$; this is our construction of γ. Since the map of Chow Homology, Lemma 67.4 is compatible with capping with Chern classes by Chow Homology, Lemma 67.8 we see that $g^*\alpha = ch((X_{k'} \to X)^*\beta) \cap [X_{k'}]$. Hence $\gamma'(g^*\alpha) = ch^{H'}((X_{k'} \to X)^*\beta)$. Thus commutativity of the diagram will hold if for any locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r and $0 \le i \le r$ the element $c_i^H(\mathcal{E})$ of $H^{2i}(X)(i)$ maps to the element $c_i^{H'}(\mathcal{E}_{k'})$ in $(H')^{2i}(X_{k'})(i)$. Because we have the projective space bundle formula for both X and X' we may replace X by a projective space bundle over X finitely many times to show this. Thus we may assume $\mathcal{E}$ has a filtration whose graded pieces are invertible $\mathcal{O}_X$-modules $\mathcal{L}_1, \ldots, \mathcal{L}_r$. See Chow Homology, Lemma 43.1 and Remark 43.2. Then $c_i^H(\mathcal{E}$ is the ith elementary symmetric polynomial in $c_1^H(\mathcal{L}_1), \ldots, c_1^H(\mathcal{L}_r)$ and we conclude by our assumption that we have agreement for first Chern classes.

Axiom (A6). Suppose given F-vector spaces V, W, an element $v \in V$, and a tensor $\xi \in V \otimes_F W$. Denote $V' = V \otimes_F F'$, $W' = W \otimes_F F'$ and v', ξ' the images of v, ξ in V', $V' \otimes_{F'} W'$. The linear algebra principle we will use in the proof of axiom (A6) is the following: there exists an F-linear map $\lambda : W \to F$ such that $(1 \otimes \lambda)\xi = v$ if and only if there exists an F'-linear map $\lambda' : W \otimes_F F' \to F'$ such that $(1 \otimes \lambda')\xi' = v'$.

Let X be a nonempty equidimensional smooth projective scheme over k of dimension d. Denote $\gamma = \gamma([\Delta])$ in $H^{2d}(X \times X)(d)$ (unadorned fibre products will be over k). Observe/recall that this makes sense as we know axioms (A1) – (A4) already; see Lemma 14.2. We may decompose

$$\gamma = \sum \gamma_i, \quad \gamma_i \in H^i(X) \otimes_F H^{2d-i}(X)(d)$$

in the Künneth decomposition. Similarly, denote $\gamma' = \gamma([\Delta']) = \sum \gamma_i'$ in $(H')^{2d}(X_{k'} \times_{k'} X_{k'})(d)$. By the linear algebra princple mentioned above, it suffices to show that γ_0 maps to γ_0' in $(H')^0(X) \otimes_{F'} (H')^{2d}(X')(d)$. By the compatibility of Künneth decompositions we see that it suffice to show that γ maps to γ' in

$$(H')^{2d}(X_{k'} \times_{k'} X_{k'})(d) = (H')^{2d}((X \times X)_{k'})(d)$$

Since $\Delta_{k'} = \Delta'$ this follows from the discussion above.

Axiom (A7). This follows from the linear algebra fact: a linear map $V \to W$ of F-vector spaces is injective if and only if $V \otimes_F F' \to W \otimes_F F'$ is injective.

Axiom (A8). Follows from the linear algebra fact used in the proof of axiom (A1).

Axiom (A9). Let X be a nonempty smooth projective scheme over k equidimensional of dimension d. Let $i : Y \to X$ be a nonempty effective Cartier divisor smooth over k. Let λ_Y and λ_X be as in axiom (A6) for X and Y. We have to show: for $a \in H^{2d-2}(X)(d-1)$ we have $\lambda_Y(i^*(a)) = \lambda_X(a \cup c_1^H(\mathcal{O}_X(Y)))$. By Remark 14.6 we know that $\lambda_X : H^{2d}(X)(d) \to F$ and $\lambda_Y : H^{2d-2}(Y)(d-1)$ are uniquely determined by the requirement in axiom (A6). Having said this, it follows from our proof of axiom (A6) for H^* above that $\lambda_X \otimes \mathrm{id}_{F'}$ corresponds to $\lambda_{X_{k'}}$ via the given identification $H^{2d}(X)(d) \otimes_F F' = H^{2d}(X_{k'})(d)$. Thus the fact that we know axiom (A9) for $F'(1), (H')^*, c_1^{H'}$ implies the axiom for $F(1), H^*, c_1^H$ by a diagram chase. This completes the proof of the lemma. □

15. Other chapters

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