# PROPERTIES OF SCHEMES

## Contents



## 1. Introduction

In this chapter we introduce some absolute properties of schemes. A foundational reference is [DG67].

## 2. Constructible sets

Constructible and locally constructible sets are introduced in Topology, Section 15. We may characterize locally constructible subsets of schemes as follows.

**Lemma 2.1.** *Let $X$ be a scheme. A subset $E$ of $X$ is locally constructible in $X$ if and only if $E \cap U$ is constructible in $U$ for every affine open $U$ of $X$.*

**Proof.** Assume $E$ is locally constructible. Then there exists an open covering $X = \bigcup U_i$ such that $E \cap U_i$ is constructible in $U_i$ for each $i$. Let $V \subset X$ be any affine open. We can find a finite open affine covering $V = V_1 \cup \ldots \cup V_m$ such that for each $j$ we have $V_j \subset U_i$ for some $i = i(j)$. By Topology, Lemma 15.4 we see that each $E \cap V_j$ is constructible in $V_j$. Since the inclusions $V_j \to V$ are quasi-compact (see Schemes, Lemma 19.2) we conclude that $E \cap V$ is constructible in $V$ by Topology, Lemma 15.6. The converse implication is immediate. $\square$

**Lemma 2.2.** *Let $X$ be a scheme and let $E \subset X$ be a locally constructible subset. Let $\xi \in X$ be a generic point of an irreducible component of $X$.*

1. *If $\xi \in E$, then an open neighbourhood of $\xi$ is contained in $E$.*
2. *If $\xi \notin E$, then an open neighbourhood of $\xi$ is disjoint from $E$.*

**Proof.** As the complement of a locally constructible subset is locally constructible it suffices to show (2). We may assume $X$ is affine and hence $E$ constructible (Lemma 2.1). In this case $X$ is a spectral space (Algebra, Lemma 26.2). Then $\xi \notin E$ implies $\xi \notin \overline{E}$ by Topology, Lemma 23.6 and the fact that there are no points of $X$ different from $\xi$ which specialize to $\xi$. $\square$

**Lemma 2.3.** *Let $X$ be a quasi-separated scheme. The intersection of any two quasi-compact opens of $X$ is a quasi-compact open of $X$. Every quasi-compact open of $X$ is retrocompact in $X$.*

**Proof.** If $U$ and $V$ are quasi-compact open then $U \cap V = \Delta^{-1}(U \times V)$, where $\Delta : X \to X \times X$ is the diagonal. As $X$ is quasi-separated we see that $\Delta$ is quasi-compact. Hence we see that $U \cap V$ is quasi-compact as $U \times V$ is quasi-compact (details omitted; use Schemes, Lemma 17.4 to see $U \times V$ is a finite union of affines). The other assertions follow from the first and Topology, Lemma 27.1. $\square$

**Lemma 2.4.** *Let $X$ be a quasi-compact and quasi-separated scheme. Then the underlying topological space of $X$ is a spectral space.*

**Proof.** By Topology, Definition 23.1 we have to check that $X$ is sober, quasi-compact, has a basis of quasi-compact opens, and the intersection of any two quasi-compact opens is quasi-compact. This follows from Schemes, Lemma 11.1 and 11.2 and Lemma 2.3 above. $\square$

**Lemma 2.5.** *Let $X$ be a quasi-compact and quasi-separated scheme. Any locally constructible subset of $X$ is constructible.*

**Proof.** As $X$ is quasi-compact we can choose a finite affine open covering $X = V_1 \cup \ldots \cup V_m$. As $X$ is quasi-separated each $V_i$ is retrocompact in $X$ by Lemma 2.3. Hence by Topology, Lemma 15.6 we see that $E \subset X$ is constructible in $X$ if and only if $E \cap V_j$ is constructible in $V_j$. Thus we win by Lemma 2.1. $\square$

**Lemma 2.6.** *Let $X$ be a scheme. A subset $E$ of $X$ is retrocompact in $X$ if and only if $E \cap U$ is quasi-compact for every affine open $U$ of $X$.*

**Proof.** Immediate from the fact that every quasi-compact open of $X$ is a finite union of affine opens. $\square$

**Lemma 2.7.** *A partition $X = \coprod_{i \in I} X_i$ of a scheme $X$ with retrocompact parts is locally finite if and only if the parts are locally constructible.*

**Proof.** See Topology, Definitions 12.1, 28.1, and 28.4 for the definitions of retrocompact, partition, and locally finite.

If the partition is locally finite and $U \subset X$ is an affine open, then we see that $U = \coprod_{i \in I} U \cap X_i$ is a finite partition (more precisely, all but a finite number of its parts are empty). Hence $U \cap X_i$ is quasi-compact and its complement is retrocompact in $U$ as a finite union of retrocompact parts. Thus $U \cap X_i$ is constructible by Topology, Lemma 15.13. It follows that $X_i$ is locally constructible by Lemma 2.1.

Assume the parts are locally constructible. Then for any affine open $U \subset X$ we obtain a covering $U = \coprod X_i \cap U$ by constructible subsets. Since the constructible topology is quasi-compact, see Topology, Lemma 23.2, this covering has a finite refinement, i.e., the partition is locally finite. $\square$

## 3. Integral, irreducible, and reduced schemes

**Definition 3.1.** Let $X$ be a scheme. We say $X$ is *integral* if it is nonempty and for every nonempty affine open $\mathrm{Spec}(R) = U \subset X$ the ring $R$ is an integral domain.

**Lemma 3.2.** *Let $X$ be a scheme. The following are equivalent.*

1. *The scheme $X$ is reduced, see Schemes, Definition 12.1.*
2. *There exists an affine open covering $X = \bigcup U_i$ such that each $\Gamma(U_i, \mathcal{O}_X)$ is reduced.*
3. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.*
4. *For every open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.*

**Proof.** See Schemes, Lemmas 12.2 and 12.3. $\square$

**Lemma 3.3.** *Let $X$ be a scheme. The following are equivalent.*

1. *The scheme $X$ is irreducible.*
2. *There exists an affine open covering $X = \bigcup_{i \in I} U_i$ such that $I$ is not empty, $U_i$ is irreducible for all $i \in I$, and $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$.*
3. *The scheme $X$ is nonempty and every nonempty affine open $U \subset X$ is irreducible.*

**Proof.** Assume (1). By Schemes, Lemma 11.1 we see that $X$ has a unique generic point $\eta$. Then $X = \overline{\{\eta\}}$. Hence $\eta$ is an element of every nonempty affine open $U \subset X$. This implies that $\eta \in U$ is dense hence $U$ is irreducible. It also implies any two nonempty affines meet. Thus (1) implies both (2) and (3).

Assume (2). Suppose $X = Z_1 \cup Z_2$ is a union of two closed subsets. For every $i$ we see that either $U_i \subset Z_1$ or $U_i \subset Z_2$. Pick some $i \in I$ and assume $U_i \subset Z_1$ (possibly after renumbering $Z_1$, $Z_2$). For any $j \in I$ the open subset $U_i \cap U_j$ is dense in $U_j$ and contained in the closed subset $Z_1 \cap U_j$. We conclude that also $U_j \subset Z_1$. Thus $X = Z_1$ as desired.

Assume (3). Choose an affine open covering $X = \bigcup_{i \in I} U_i$. We may assume that each $U_i$ is nonempty. Since $X$ is nonempty we see that $I$ is not empty. By assumption each $U_i$ is irreducible. Suppose $U_i \cap U_j = \emptyset$ for some pair $i, j \in I$. Then the open $U_i \amalg U_j = U_i \cup U_j$ is affine, see Schemes, Lemma 6.8. Hence it is irreducible by assumption which is absurd. We conclude that (3) implies (2). The lemma is proved. $\square$

**Lemma 3.4.** *A scheme $X$ is integral if and only if it is reduced and irreducible.*

**Proof.** If $X$ is irreducible, then every affine open $\operatorname{Spec}(R) = U \subset X$ is irreducible. If $X$ is reduced, then $R$ is reduced, by Lemma 3.2 above. Hence $R$ is reduced and $(0)$ is a prime ideal, i.e., $R$ is an integral domain.

If $X$ is integral, then for every nonempty affine open $\operatorname{Spec}(R) = U \subset X$ the ring $R$ is reduced and hence $X$ is reduced by Lemma 3.2. Moreover, every nonempty affine open is irreducible. Hence $X$ is irreducible, see Lemma 3.3. $\square$

In Examples, Section 6 we construct a connected affine scheme all of whose local rings are domains, but which is not integral.

## 4. Types of schemes defined by properties of rings

In this section we study what properties of rings allow one to define local properties of schemes.

**Definition 4.1.** Let $P$ be a property of rings. We say that $P$ is *local* if the following hold:

(1) For any ring $R$, and any $f \in R$ we have $P(R) \Rightarrow P(R_f)$.
(2) For any ring $R$, and $f_i \in R$ such that $(f_1, \ldots, f_n) = R$ then $\forall i, P(R_{f_i}) \Rightarrow P(R)$.

**Definition 4.2.** Let $P$ be a property of rings. Let $X$ be a scheme. We say $X$ is *locally $P$* if for any $x \in X$ there exists an affine open neighbourhood $U$ of $x$ in $X$ such that $\mathcal{O}_X(U)$ has property $P$.

This is only a good notion if the property is local. Even if $P$ is a local property we will not automatically use this definition to say that a scheme is "locally $P$" unless we also explicitly state the definition elsewhere.

**Lemma 4.3.** *Let $X$ be a scheme. Let $P$ be a local property of rings. The following are equivalent:*

(1) *The scheme $X$ is locally $P$.*
(2) *For every affine open $U \subset X$ the property $P(\mathcal{O}_X(U))$ holds.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies $P$.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme $X_j$ is locally $P$.*

*Moreover, if $X$ is locally $P$ then every open subscheme is locally $P$.*

**Proof.** Of course $(1) \Leftrightarrow (3)$ and $(2) \Rightarrow (1)$. If $(3) \Rightarrow (2)$, then the final statement of the lemma holds and it follows easily that (4) is also equivalent to (1). Thus we show $(3) \Rightarrow (2)$.

Let $X = \bigcup U_i$ be an affine open covering, say $U_i = \operatorname{Spec}(R_i)$. Assume $P(R_i)$. Let $\operatorname{Spec}(R) = U \subset X$ be an arbitrary affine open. By Schemes, Lemma 11.6 there exists a standard covering of $U = \operatorname{Spec}(R)$ by standard opens $D(f_j)$ such that each ring $R_{f_j}$ is a principal localization of one of the rings $R_i$. By Definition 4.1 (1) we get $P(R_{f_j})$. Whereupon $P(R)$ by Definition 4.1 (2). $\square$

Here is a sample application.

**Lemma 4.4.** *Let $X$ be a scheme. Then $X$ is reduced if and only if $X$ is "locally reduced" in the sense of Definition 4.2.*

**Proof.** This is clear from Lemma 3.2. □

**Lemma 4.5.** *The following properties of a ring $R$ are local.*
(1) *(Cohen-Macaulay.) The ring $R$ is Noetherian and CM, see Algebra, Definition 104.6.*
(2) *(Regular.) The ring $R$ is Noetherian and regular, see Algebra, Definition 110.7.*
(3) *(Absolutely Noetherian.) The ring $R$ is of finite type over $\mathbf{Z}$.*
(4) *Add more here as needed.*[1]

**Proof.** Omitted. □

## 5. Noetherian schemes

Recall that a ring $R$ is *Noetherian* if it satisfies the ascending chain condition of ideals. Equivalently every ideal of $R$ is finitely generated.

**Definition 5.1.** Let $X$ be a scheme.
(1) We say $X$ is *locally Noetherian* if every $x \in X$ has an affine open neighbourhood $\text{Spec}(R) = U \subset X$ such that the ring $R$ is Noetherian.
(2) We say $X$ is *Noetherian* if $X$ is locally Noetherian and quasi-compact.

Here is the standard result characterizing locally Noetherian schemes.

**Lemma 5.2.** *Let $X$ be a scheme. The following are equivalent:*
(1) *The scheme $X$ is locally Noetherian.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme $X_j$ is locally Noetherian.*

*Moreover, if $X$ is locally Noetherian then every open subscheme is locally Noetherian.*

**Proof.** To show this it suffices to show that being Noetherian is a local property of rings, see Lemma 4.3. Any localization of a Noetherian ring is Noetherian, see Algebra, Lemma 31.1. By Algebra, Lemma 23.2 we see the second property to Definition 4.1. □

**Lemma 5.3.** *Any immersion $Z \to X$ with $X$ locally Noetherian is quasi-compact.*

**Proof.** A closed immersion is clearly quasi-compact. A composition of quasi-compact morphisms is quasi-compact, see Topology, Lemma 12.2. Hence it suffices to show that an open immersion into a locally Noetherian scheme is quasi-compact. Using Schemes, Lemma 19.2 we reduce to the case where $X$ is affine. Any open subset of the spectrum of a Noetherian ring is quasi-compact (for example combine Algebra, Lemma 31.5 and Topology, Lemmas 9.2 and 12.13). □

[1] But we only list those properties here which we have not already dealt with separately somewhere else.

**Lemma 5.4.** *A locally Noetherian scheme is quasi-separated.*

**Proof.** By Schemes, Lemma 21.6 we have to show that the intersection $U \cap V$ of two affine opens of $X$ is quasi-compact. This follows from Lemma 5.3 above on considering the open immersion $U \cap V \to U$ for example. (But really it is just because any open of the spectrum of a Noetherian ring is quasi-compact.) □

**Lemma 5.5.** *A (locally) Noetherian scheme has a (locally) Noetherian underlying topological space, see Topology, Definition 9.1.*

**Proof.** This is because a Noetherian scheme is a finite union of spectra of Noetherian rings and Algebra, Lemma 31.5 and Topology, Lemma 9.4. □

**Lemma 5.6.** *Any locally closed subscheme of a (locally) Noetherian scheme is (locally) Noetherian.*

**Proof.** Omitted. Hint: Any quotient, and any localization of a Noetherian ring is Noetherian. For the Noetherian case use again that any subset of a Noetherian space is a Noetherian space (with induced topology). □

**Lemma 5.7.** *A Noetherian scheme has a finite number of irreducible components.*

**Proof.** The underlying topological space of a Noetherian scheme is Noetherian (Lemma 5.5) and we conclude because a Noetherian topological space has only finitely many irreducible components (Topology, Lemma 9.2). □

**Lemma 5.8.** *Any morphism of schemes $f : X \to Y$ with $X$ Noetherian is quasi-compact.*

**Proof.** Use Lemma 5.5 and use that any subset of a Noetherian topological space is quasi-compact (see Topology, Lemmas 9.2 and 12.13). □

Here is a fun lemma. It says that every locally Noetherian scheme has plenty of closed points (at least one in every closed subset).

**Lemma 5.9.** *Any nonempty locally Noetherian scheme has a closed point. Any nonempty closed subset of a locally Noetherian scheme has a closed point. Equivalently, any point of a locally Noetherian scheme specializes to a closed point.*

**Proof.** The second assertion follows from the first (using Schemes, Lemma 12.4 and Lemma 5.6). Consider any nonempty affine open $U \subset X$. Let $x \in U$ be a closed point. If $x$ is a closed point of $X$ then we are done. If not, let $X_0 \subset X$ be the reduced induced closed subscheme structure on $\overline{\{x\}}$. Then $U_0 = U \cap X_0$ is an affine open of $X_0$ by Schemes, Lemma 10.1 and $U_0 = \{x\}$. Let $y \in X_0$, $y \neq x$ be a specialization of $x$. Consider the local ring $R = \mathcal{O}_{X_0,y}$. This is a Noetherian local ring as $X_0$ is Noetherian by Lemma 5.6. Denote $V \subset \text{Spec}(R)$ the inverse image of $U_0$ in $\text{Spec}(R)$ by the canonical morphism $\text{Spec}(R) \to X_0$ (see Schemes, Section 13.) By construction $V$ is a singleton with unique point corresponding to $x$ (use Schemes, Lemma 13.2). By Algebra, Lemma 61.1 we see that $\dim(R) = 1$. In other words, we see that $y$ is an immediate specialization of $x$ (see Topology, Definition 20.1). In other words, any point $y \neq x$ such that $x \rightsquigarrow y$ is an immediate specialization of $x$. Clearly each of these points is a closed point as desired. □

**Lemma 5.10.** *Let $X$ be a locally Noetherian scheme. Let $x' \rightsquigarrow x$ be a specialization of points of $X$. Then*

(1) *there exists a discrete valuation ring $R$ and a morphism $f : \mathrm{Spec}(R) \to X$ such that the generic point $\eta$ of $\mathrm{Spec}(R)$ maps to $x'$ and the special point maps to $x$, and*
(2) *given a finitely generated field extension $K/\kappa(x')$ we may arrange it so that the extension $\kappa(\eta)/\kappa(x')$ induced by $f$ is isomorphic to the given one.*

**Proof.** Let $x' \rightsquigarrow x$ be a specialization in $X$, and let $K/\kappa(x')$ be a finitely generated extension of fields. By Schemes, Lemma 13.2 and the discussion following Schemes, Lemma 13.3 this leads to ring maps $\mathcal{O}_{X,x} \to \kappa(x') \to K$. Let $R \subset K$ be any discrete valuation ring whose field of fractions is $K$ and which dominates the image of $\mathcal{O}_{X,x} \to K$, see Algebra, Lemma 119.13. The ring map $\mathcal{O}_{X,x} \to R$ induces the morphism $f : \mathrm{Spec}(R) \to X$, see Schemes, Lemma 13.1. This morphism has all the desired properties by construction. □

**Lemma 5.11.** *Let $S$ be a Noetherian scheme. Let $T \subset S$ be an infinite subset. Then there exists an infinite subset $T' \subset T$ such that there are no nontrivial specializations among the points $T'$.*

**Proof.** Let $T_0 \subset T$ be the set of $t \in T$ which do not specialize to another point of $T$. If $T_0$ is infinite, then $T' = T_0$ works. Hence we may and do assume $T_0$ is finite. Inductively, for $i > 0$, consider the set $T_i \subset T$ of $t \in T$ such that

(1) $t \notin T_{i-1} \cup T_{i-2} \cup \ldots \cup T_0$,
(2) there exist a nontrivial specialization $t \rightsquigarrow t'$ with $t' \in T_{i-1}$, and
(3) for any nontrivial specialization $t \rightsquigarrow t'$ with $t' \in T$ we have $t' \in T_{i-1} \cup T_{i-2} \cup \ldots \cup T_0$.

Again, if $T_i$ is infinite, then $T' = T_i$ works. Let $d$ be the maximum of the dimensions of the local rings $\mathcal{O}_{S,t}$ for $t \in T_0$; then $d$ is an integer because $T_0$ is finite and the dimensions of the local rings are finite by Algebra, Proposition 60.9. Then $T_i = \emptyset$ for $i > d$. Namely, if $t \in T_i$ then we can find a sequence of nontrivial specializations $t = t_i \rightsquigarrow t_{i-1} \rightsquigarrow \ldots \rightsquigarrow t_0$ with $t_0 \in T_0$. As the points $t = t_i, t_{i-1}, \ldots, t_0$ are in $\mathrm{Spec}(\mathcal{O}_{S,t_0})$ (Schemes, Lemma 13.2), we see that $i \leq d$. Thus $\bigcup T_i = T_d \cup \ldots \cup T_0$ is a finite subset of $T$.

Suppose $t \in T$ is not in $\bigcup T_i$. Then there must be a specialization $t \rightsquigarrow t'$ with $t' \in T$ and $t' \notin \bigcup T_i$. (Namely, if every specialization of $t$ is in the finite set $T_d \cup \ldots \cup T_0$, then there is a maximum $i$ such that there is some specialization $t \rightsquigarrow t'$ with $t' \in T_i$ and then $t \in T_{i+1}$ by construction.) Hence we get an infinite sequence

$$t \rightsquigarrow t' \rightsquigarrow t'' \rightsquigarrow \ldots$$

of nontrivial specializations between points of $T \setminus \bigcup T_i$. This is impossible because the underlying topological space of $S$ is Noetherian by Lemma 5.4. □

**Lemma 5.12.** *Let $S$ be a Noetherian scheme. Let $T \subset S$ be a subset. Let $T_0 \subset T$ be the set of $t \in T$ such that there is no nontrivial specialization $t' \rightsquigarrow t$ with $t' \in T'$. Then (a) there are no specializations among the points of $T_0$, (b) every point of $T$ is a specialization of a point of $T_0$, and (c) the closures of $T$ and $T_0$ are the same.*

**Proof.** Recall that $\dim(\mathcal{O}_{S,s}) < \infty$ for any $s \in S$, see Algebra, Proposition 60.9. Let $t \in T$. If $t' \rightsquigarrow t$, then by dimension theory $\dim(\mathcal{O}_{S,t'}) \leq \dim(\mathcal{O}_{S,t})$ with equality if and only if $t' = t$. Thus if we pick $t' \rightsquigarrow t$ with $\dim(\mathcal{O}_{T,t'})$ minimal, then $t' \in T_0$. In other words, every $t \in T$ is the specialization of an element of $T_0$. □

**Lemma 5.13.** *Let $S$ be a Noetherian scheme. Let $T \subset S$ be an infinite dense subset. Then there exist a countable subset $E \subset T$ which is dense in $S$.*

**Proof.** Let $T'$ be the set of points $s \in S$ such that $\overline{\{s\}} \cap T$ contains a countable subset whose closure is $\overline{\{s\}}$. Since a finite set is countable we have $T \subset T'$. For $s \in T'$ choose such a countable subset $E_s \subset \overline{\{s\}} \cap T$. Let $E' = \{s_1, s_2, s_3, \ldots\} \subset T'$ be a countable subset. Then the closure of $E'$ in $S$ is the closure of the countable subset $\bigcup_n E_{s_n}$ of $T$. It follows that if $Z$ is an irreducible component of the closure of $E'$, then the generic point of $Z$ is in $T'$.

Denote $T'_0 \subset T'$ the subset of $t \in T'$ such that there is no nontrivial specialization $t' \rightsquigarrow t$ with $t' \in T'$ as in Lemma 5.12 whose results we will use without further mention. If $T'_0$ is infinite, then we choose a countable subset $E' \subset T'_0$. By the argument in the first paragraph, the generic points of the irreducible components of the closure of $E'$ are in $T'$. However, since one of these points specializes to infinitely many distinct elements of $E' \subset T'_0$ this is a contradiction. Thus $T'_0$ is finite, say $T'_0 = \{s_1, \ldots, s_m\}$. Then it follows that $S$, which is the closure of $T$, is contained in the closure of $\{s_1, \ldots, s_m\}$, which in turn is contained in the closure of the countable subset $E_{s_1} \cup \ldots \cup E_{s_m} \subset T$ as desired. □

## 6. Jacobson schemes

Recall that a space is said to be *Jacobson* if the closed points are dense in every closed subset, see Topology, Section 18.

**Definition 6.1.** A scheme $S$ is said to be *Jacobson* if its underlying topological space is Jacobson.

Recall that a ring $R$ is Jacobson if every radical ideal of $R$ is the intersection of maximal ideals, see Algebra, Definition 35.1.

**Lemma 6.2.** *An affine scheme* $\mathrm{Spec}(R)$ *is Jacobson if and only if the ring $R$ is Jacobson.*

**Proof.** This is Algebra, Lemma 35.4. □

Here is the standard result characterizing Jacobson schemes. Intuitively it claims that Jacobson $\Leftrightarrow$ locally Jacobson.

**Lemma 6.3.** *Let $X$ be a scheme. The following are equivalent:*

(1) *The scheme $X$ is Jacobson.*
(2) *The scheme $X$ is "locally Jacobson" in the sense of Definition 4.2.*
(3) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Jacobson.*
(4) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Jacobson.*
(5) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme $X_j$ is Jacobson.*

*Moreover, if $X$ is Jacobson then every open subscheme is Jacobson.*

**Proof.** The final assertion of the lemma holds by Topology, Lemma 18.5. The equivalence of (5) and (1) is Topology, Lemma 18.4. Hence, using Lemma 6.2, we see that (1) $\Leftrightarrow$ (2). To finish proving the lemma it suffices to show that "Jacobson" is a local property of rings, see Lemma 4.3. Any localization of a Jacobson ring at an element is Jacobson, see Algebra, Lemma 35.14. Suppose $R$ is a ring, $f_1, \ldots, f_n \in R$

generate the unit ideal and each $R_{f_i}$ is Jacobson. Then we see that $\mathrm{Spec}(R) = \bigcup D(f_i)$ is a union of open subsets which are all Jacobson, and hence $\mathrm{Spec}(R)$ is Jacobson by Topology, Lemma 18.4 again. This proves the second property of Definition 4.1. $\square$

Many schemes used commonly in algebraic geometry are Jacobson, see Morphisms, Lemma 16.10. We mention here the following interesting case.

**Lemma 6.4.** *Examples of Noetherian Jacobson schemes.*

1. *If $(R, \mathfrak{m})$ is a Noetherian local ring, then the punctured spectrum $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ is a Jacobson scheme.*
2. *If $R$ is a Noetherian ring with Jacobson radical $rad(R)$ then $\mathrm{Spec}(R) \setminus V(rad(R))$ is a Jacobson scheme.*
3. *If $(R, I)$ is a Zariski pair (More on Algebra, Definition 10.1) with $R$ Noetherian, then $\mathrm{Spec}(R) \setminus V(I)$ is a Jacobson scheme.*

**Proof.** Proof of (3). Observe that $\mathrm{Spec}(R) - V(I)$ has a covering by the affine opens $\mathrm{Spec}(R_f)$ for $f \in I$. The rings $R_f$ are Jacobson by More on Algebra, Lemma 10.5. Hence $\mathrm{Spec}(R) \setminus V(I)$ is Jacobson by Lemma 6.3. Parts (1) and (2) are special cases of (3).

Direct proof of case (1). Since $\mathrm{Spec}(R)$ is a Noetherian scheme, $S$ is a Noetherian scheme (Lemma 5.6). Hence $S$ is a sober, Noetherian topological space (use Schemes, Lemma 11.1). Assume $S$ is not Jacobson to get a contradiction. By Topology, Lemma 18.3 there exists some non-closed point $\xi \in S$ such that $\{\xi\}$ is locally closed. This corresponds to a prime $\mathfrak{p} \subset R$ such that (1) there exists a prime $\mathfrak{q}$, $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$ with both inclusions strict, and (2) $\{\mathfrak{p}\}$ is open in $\mathrm{Spec}(R/\mathfrak{p})$. This is impossible by Algebra, Lemma 61.1. $\square$

## 7. Normal schemes

Recall that a ring $R$ is said to be normal if all its local rings are normal domains, see Algebra, Definition 37.11. A normal domain is a domain which is integrally closed in its field of fractions, see Algebra, Definition 37.1. Thus it makes sense to define a normal scheme as follows.

**Definition 7.1.** A scheme $X$ is *normal* if and only if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a normal domain.

This seems to be the definition used in EGA, see [DG67, 0, 4.1.4]. Suppose $X = \mathrm{Spec}(A)$, and $A$ is reduced. Then saying that $X$ is normal is not equivalent to saying that $A$ is integrally closed in its total ring of fractions. However, if $A$ is Noetherian then this is the case (see Algebra, Lemma 37.16).

**Lemma 7.2.** *Let $X$ be a scheme. The following are equivalent:*

1. *The scheme $X$ is normal.*
2. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is normal.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is normal.*
4. *There exists an open covering $X = \bigcup X_j$ such that each open subscheme $X_j$ is normal.*

*Moreover, if $X$ is normal then every open subscheme is normal.*

**Proof.** This is clear from the definitions. □

**Lemma 7.3.** *A normal scheme is reduced.*

**Proof.** Immediate from the definitions. □

**Lemma 7.4.** *Let $X$ be an integral scheme. Then $X$ is normal if and only if for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is a normal domain.*

**Proof.** This follows from Algebra, Lemma 37.10. □

**Lemma 7.5.** *Let $X$ be a scheme such that any quasi-compact open has a finite number of irreducible components. The following are equivalent:*

1. *$X$ is normal, and*
2. *$X$ is a disjoint union of normal integral schemes.*

**Proof.** It is immediate from the definitions that (2) implies (1). Let $X$ be a normal scheme such that every quasi-compact open has a finite number of irreducible components. If $X$ is affine then $X$ satisfies (2) by Algebra, Lemma 37.16. For a general $X$, let $X = \bigcup X_i$ be an affine open covering. Note that also each $X_i$ has but a finite number of irreducible components, and the lemma holds for each $X_i$. Let $T \subset X$ be an irreducible component. By the affine case each intersection $T \cap X_i$ is open in $X_i$ and an integral normal scheme. Hence $T \subset X$ is open, and an integral normal scheme. This proves that $X$ is the disjoint union of its irreducible components, which are integral normal schemes. □

**Lemma 7.6.** *Let $X$ be a Noetherian scheme. The following are equivalent:*

1. *$X$ is normal, and*
2. *$X$ is a finite disjoint union of normal integral schemes.*

**Proof.** This is a special case of Lemma 7.5 because a Noetherian scheme has a Noetherian underlying topological space (Lemma 5.5 and Topology, Lemma 9.2). □

**Lemma 7.7.** *Let $X$ be a locally Noetherian scheme. The following are equivalent:*

1. *$X$ is normal, and*
2. *$X$ is a disjoint union of integral normal schemes.*

**Proof.** Omitted. Hint: This is purely topological from Lemma 7.6. □

**Remark 7.8.** Let $X$ be a normal scheme. If $X$ is locally Noetherian then we see that $X$ is integral if and only if $X$ is connected, see Lemma 7.7. But there exists a connected affine scheme $X$ such that $\mathcal{O}_{X,x}$ is a domain for all $x \in X$, but $X$ is not irreducible, see Examples, Section 6. This example is even a normal scheme (proof omitted), so beware!

**Lemma 7.9.** *Let $X$ be an integral normal scheme. Then $\Gamma(X, \mathcal{O}_X)$ is a normal domain.*

**Proof.** Set $R = \Gamma(X, \mathcal{O}_X)$. It is clear that $R$ is a domain. Suppose $f = a/b$ is an element of its fraction field which is integral over $R$. Say we have $f^d + \sum_{i=0,\ldots,d-1} a_i f^i = 0$ with $a_i \in R$. Let $U \subset X$ be a nonempty affine open. Since $b \in R$ is not zero and since $X$ is integral we see that also $b|_U \in \mathcal{O}_X(U)$ is not zero. Hence $a/b$ is an element of the fraction field of $\mathcal{O}_X(U)$ which is integral over $\mathcal{O}_X(U)$

(because we can use the same polynomial $f^d + \sum_{i=0,\ldots,d-1} a_i|_U f^i = 0$ on $U$). Since $\mathcal{O}_X(U)$ is a normal domain (Lemma 7.2), we see that $f_U = (a|_U)/(b|_U) \in \mathcal{O}_X(U)$. It is clear that $f_U|_V = f_V$ whenever $V \subset U \subset X$ are nonempty affine open. Hence the local sections $f_U$ glue to an element $g \in R = \Gamma(X, \mathcal{O}_X)$. Then $bg$ and $a$ restrict to the same element of $\mathcal{O}_X(U)$ for all $U$ as above, hence $bg = a$, in other words, $g$ maps to $f$ in the fraction field of $R$. □

## 8. Cohen-Macaulay schemes

Recall, see Algebra, Definition 104.1, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be Cohen-Macaulay if $\text{depth}_{\mathfrak{m}}(R) = \dim(R)$. Recall that a Noetherian ring $R$ is said to be Cohen-Macaulay if every local ring $R_{\mathfrak{p}}$ of $R$ is Cohen-Macaulay, see Algebra, Definition 104.6.

**Definition 8.1.** Let $X$ be a scheme. We say $X$ is *Cohen-Macaulay* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of $x$ such that the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.

**Lemma 8.2.** *Let $X$ be a scheme. The following are equivalent:*

1. *$X$ is Cohen-Macaulay,*
2. *$X$ is locally Noetherian and all of its local rings are Cohen-Macaulay, and*
3. *$X$ is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay.*

**Proof.** Algebra, Lemma 104.5 says that the localization of a Cohen-Macaulay local ring is Cohen-Macaulay. The lemma follows by combining this with Lemma 5.2, with the existence of closed points on locally Noetherian schemes (Lemma 5.9), and the definitions. □

**Lemma 8.3.** *Let $X$ be a scheme. The following are equivalent:*

1. *The scheme $X$ is Cohen-Macaulay.*
2. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and Cohen-Macaulay.*
4. *There exists an open covering $X = \bigcup X_j$ such that each open subscheme $X_j$ is Cohen-Macaulay.*

*Moreover, if $X$ is Cohen-Macaulay then every open subscheme is Cohen-Macaulay.*

**Proof.** Combine Lemmas 5.2 and 8.2. □

More information on Cohen-Macaulay schemes and depth can be found in Cohomology of Schemes, Section 11.

## 9. Regular schemes

Recall, see Algebra, Definition 60.10, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be *regular* if $\mathfrak{m}$ can be generated by $\dim(R)$ elements. Recall that a Noetherian ring $R$ is said to be *regular* if every local ring $R_{\mathfrak{p}}$ of $R$ is regular, see Algebra, Definition 110.7.

**Definition 9.1.** Let $X$ be a scheme. We say $X$ is *regular*, or *nonsingular* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of $x$ such that the ring $\mathcal{O}_X(U)$ is Noetherian and regular.

**Lemma 9.2.** *Let $X$ be a scheme. The following are equivalent:*

(1) *$X$ is regular,*
(2) *$X$ is locally Noetherian and all of its local rings are regular, and*
(3) *$X$ is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is regular.*

**Proof.** By the discussion in Algebra preceding Algebra, Definition 110.7 we know that the localization of a regular local ring is regular. The lemma follows by combining this with Lemma 5.2, with the existence of closed points on locally Noetherian schemes (Lemma 5.9), and the definitions. □

**Lemma 9.3.** *Let $X$ be a scheme. The following are equivalent:*

(1) *The scheme $X$ is regular.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and regular.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and regular.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme $X_j$ is regular.*

*Moreover, if $X$ is regular then every open subscheme is regular.*

**Proof.** Combine Lemmas 5.2 and 9.2. □

**Lemma 9.4.** *A regular scheme is normal.*

**Proof.** See Algebra, Lemma 157.5. □

## 10. Dimension

The dimension of a scheme is just the dimension of its underlying topological space.

**Definition 10.1.** Let $X$ be a scheme.

(1) The *dimension* of $X$ is just the dimension of $X$ as a topological spaces, see Topology, Definition 10.1.
(2) For $x \in X$ we denote $\dim_x(X)$ the dimension of the underlying topological space of $X$ at $x$ as in Topology, Definition 10.1. We say $\dim_x(X)$ is the *dimension of $X$ at $x$*.

As a scheme has a sober underlying topological space (Schemes, Lemma 11.1) we may compute the dimension of $X$ as the supremum of the lengths $n$ of chains

$$T_0 \subset T_1 \subset \ldots \subset T_n$$

of irreducible closed subsets of $X$, or as the supremum of the lengths $n$ of chains of specializations

$$\xi_n \leadsto \xi_{n-1} \leadsto \ldots \leadsto \xi_0$$

of points of $X$.

**Lemma 10.2.** *Let $X$ be a scheme. The following are equal*

(1) *The dimension of $X$.*
(2) *The supremum of the dimensions of the local rings of $X$.*

(3) *The supremum of* $\dim_x(X)$ *for* $x \in X$.

**Proof.** Note that given a chain of specializations

$$\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

of points of $X$ all of the points $\xi_i$ correspond to prime ideals of the local ring of $X$ at $\xi_0$ by Schemes, Lemma 13.2. Hence we see that the dimension of $X$ is the supremum of the dimensions of its local rings. In particular $\dim_x(X) \geq \dim(\mathcal{O}_{X,x})$ as $\dim_x(X)$ is the minimum of the dimensions of open neighbourhoods of $x$. Thus $\sup_{x \in X} \dim_x(X) \geq \dim(X)$. On the other hand, it is clear that $\sup_{x \in X} \dim_x(X) \leq \dim(X)$ as $\dim(U) \leq \dim(X)$ for any open subset of $X$. $\square$

**Lemma 10.3.** *Let $X$ be a scheme. Let $Y \subset X$ be an irreducible closed subset. Let $\xi \in Y$ be the generic point. Then*

$$\mathit{codim}(Y, X) = \dim(\mathcal{O}_{X,\xi})$$

*where the codimension is as defined in Topology, Definition 11.1.*

**Proof.** By Topology, Lemma 11.2 we may replace $X$ by an affine open neighbourhood of $\xi$. In this case the result follows easily from Algebra, Lemma 26.3. $\square$

**Lemma 10.4.** *Let $X$ be a scheme. Let $x \in X$. Then $x$ is a generic point of an irreducible component of $X$ if and only if* $\dim(\mathcal{O}_{X,x}) = 0$.

**Proof.** This follows from Lemma 10.3 for example. $\square$

**Lemma 10.5.** *A locally Noetherian scheme of dimension* 0 *is a disjoint union of spectra of Artinian local rings.*

**Proof.** A Noetherian ring of dimension 0 is a finite product of Artinian local rings, see Algebra, Proposition 60.7. Hence an affine open of a locally Noetherian scheme $X$ of dimension 0 has discrete underlying topological space. This implies that the topology on $X$ is discrete. The lemma follows easily from these remarks. $\square$

**Lemma 10.6.** *Let $X$ be a scheme of dimension zero. The following are equivalent*

(1) *$X$ is quasi-separated,*
(2) *$X$ is separated,*
(3) *$X$ is Hausdorff,*
(4) *every affine open is closed.*

*In this case the connected components of $X$ are points and every quasi-compact open of $X$ is affine. In particular, if $X$ is quasi-compact, then $X$ is affine.*

**Proof.** As the dimension of $X$ is zero, we see that for any affine open $U \subset X$ the space $U$ is profinite and satisfies a bunch of other properties which we will use freely below, see Algebra, Lemma 26.5. We choose an affine open covering $X = \bigcup U_i$.

If (4) holds, then $U_i \cap U_j$ is a closed subset of $U_i$, hence quasi-compact, hence $X$ is quasi-separated, by Schemes, Lemma 21.6, hence (1) holds.

If (1) holds, then $U_i \cap U_j$ is a quasi-compact open of $U_i$ hence closed in $U_i$. Then $U_i \cap U_j \to U_i$ is an open immersion whose image is closed, hence it is a closed immersion. In particular $U_i \cap U_j$ is affine and $\mathcal{O}(U_i) \to \mathcal{O}_X(U_i \cap U_j)$ is surjective. Thus $X$ is separated by Schemes, Lemma 21.7, hence (2) holds.

Assume (2) and let $x, y \in X$. Say $x \in U_i$. If $y \in U_i$ too, then we can find disjoint open neighbourhoods of $x$ and $y$ because $U_i$ is Hausdorff. Say $y \notin U_i$ and $y \in U_j$. Then $y \notin U_i \cap U_j$ which is an affine open of $U_j$ and hence closed in $U_j$. Thus we can find an open neighbourhood of $y$ not meeting $U_i$ and we conclude that $X$ is Hausdorff, hence (3) holds.

Assume (3). Let $U \subset X$ be affine open. Then $U$ is closed in $X$ by Topology, Lemma 12.4. This proves (4) holds.

Assume $X$ satisfies the equivalent conditions (1) – (4). We prove the final statements of the lemma. Say $x, y \in X$ with $x \neq y$. Since $y$ does not specialize to $x$ we can choose $U \subset X$ affine open with $x \in U$ and $y \notin U$. Then we see that $X = U \amalg (X \setminus U)$ is a decomposistion into open and closed subsets which shows that $x$ and $y$ do not belong to the same connected component of $X$. Next, assume $U \subset X$ is a quasi-compact open. Write $U = U_1 \cup \ldots \cup U_n$ as a union of affine opens. We will prove by induction on $n$ that $U$ is affine. This immediately reduces us to the case $n = 2$. In this case we have $U = (U_1 \setminus U_2) \amalg (U_1 \cap U_2) \amalg (U_2 \setminus U_1)$ and the arguments above show that each of the pieces is affine. □

## 11. Catenary schemes

Recall that a topological space $X$ is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 11.4.

**Definition 11.1.** Let $S$ be a scheme. We say $S$ is *catenary* if the underlying topological space of $S$ is catenary.

Recall that a ring $A$ is called *catenary* if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ there exists a maximal chain of primes

$$\mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$$

and all of these have the same length. See Algebra, Definition 105.1.

**Lemma 11.2.** *Let $S$ be a scheme. The following are equivalent*

1. *$S$ is catenary,*
2. *there exists an open covering of $S$ all of whose members are catenary schemes,*
3. *for every affine open* $\mathrm{Spec}(R) = U \subset S$ *the ring $R$ is catenary, and*
4. *there exists an affine open covering $S = \bigcup U_i$ such that each $U_i$ is the spectrum of a catenary ring.*

*Moreover, in this case any locally closed subscheme of $S$ is catenary as well.*

**Proof.** Combine Topology, Lemma 11.5, and Algebra, Lemma 105.2. □

**Lemma 11.3.** *Let $S$ be a locally Noetherian scheme. The following are equivalent:*

1. *$S$ is catenary, and*
2. *locally in the Zariski topology there exists a dimension function on $S$ (see Topology, Definition 20.1).*

**Proof.** This follows from Topology, Lemmas 11.5, 20.2, and 20.4, Schemes, Lemma 11.1 and finally Lemma 5.5. □

It turns out that a scheme is catenary if and only if its local rings are catenary.

**Lemma 11.4.** *Let $X$ be a scheme. The following are equivalent*

(1) *$X$ is catenary, and*
(2) *for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ is catenary.*

**Proof.** Assume $X$ is catenary. Let $x \in X$. By Lemma 11.2 we may replace $X$ by an affine open neighbourhood of $x$, and then $\Gamma(X, \mathcal{O}_X)$ is a catenary ring. By Algebra, Lemma 105.4 any localization of a catenary ring is catenary. Whence $\mathcal{O}_{X,x}$ is catenary.

Conversely assume all local rings of $X$ are catenary. Let $Y \subset Y'$ be an inclusion of irreducible closed subsets of $X$. Let $\xi \in Y$ be the generic point. Let $\mathfrak{p} \subset \mathcal{O}_{X,\xi}$ be the prime corresponding to the generic point of $Y'$, see Schemes, Lemma 13.2. By that same lemma the irreducible closed subsets of $X$ in between $Y$ and $Y'$ correspond to primes $\mathfrak{q} \subset \mathcal{O}_{X,\xi}$ with $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}_\xi$. Hence we see all maximal chains of these are finite and have the same length as $\mathcal{O}_{X,\xi}$ is a catenary ring. $\square$

## 12. Serre's conditions

Here are two technical notions that are often useful. See also Cohomology of Schemes, Section 11.

**Definition 12.1.** Let $X$ be a locally Noetherian scheme. Let $k \geq 0$.

(1) We say $X$ is *regular in codimension $k$*, or we say $X$ has property $(R_k)$ if for every $x \in X$ we have
$$\dim(\mathcal{O}_{X,x}) \leq k \Rightarrow \mathcal{O}_{X,x} \text{ is regular}$$
(2) We say $X$ has property $(S_k)$ if for every $x \in X$ we have $\text{depth}(\mathcal{O}_{X,x}) \geq \min(k, \dim(\mathcal{O}_{X,x}))$.

The phrase "regular in codimension $k$" makes sense since we have seen in Section 11 that if $Y \subset X$ is irreducible closed with generic point $x$, then $\dim(\mathcal{O}_{X,x}) = \text{codim}(Y, X)$. For example condition $(R_0)$ means that for every generic point $\eta \in X$ of an irreducible component of $X$ the local ring $\mathcal{O}_{X,\eta}$ is a field. But for general Noetherian schemes it can happen that the regular locus of $X$ is badly behaved, so care has to be taken.

**Lemma 12.2.** *Let $X$ be a locally Noetherian scheme. Then $X$ is regular if and only if $X$ has $(R_k)$ for all $k \geq 0$.*

**Proof.** Follows from Lemma 9.2 and the definitions. $\square$

**Lemma 12.3.** *Let $X$ be a locally Noetherian scheme. Then $X$ is Cohen-Macaulay if and only if $X$ has $(S_k)$ for all $k \geq 0$.*

**Proof.** By Lemma 8.2 we reduce to looking at local rings. Hence the lemma is true because a Noetherian local ring is Cohen-Macaulay if and only if it has depth equal to its dimension. $\square$

**Lemma 12.4.** *Let $X$ be a locally Noetherian scheme. Then $X$ is reduced if and only if $X$ has properties $(S_1)$ and $(R_0)$.*

**Proof.** This is Algebra, Lemma 157.3. $\square$

**Lemma 12.5.** *Let $X$ be a locally Noetherian scheme. Then $X$ is normal if and only if $X$ has properties $(S_2)$ and $(R_1)$.*

**Proof.** This is Algebra, Lemma 157.4. □

**Lemma 12.6.** *Let $X$ be a locally Noetherian scheme which is normal and has dimension $\leq 1$. Then $X$ is regular.*

**Proof.** This follows from Lemma 12.5 and the definitions. □

**Lemma 12.7.** *Let $X$ be a locally Noetherian scheme which is normal and has dimension $\leq 2$. Then $X$ is Cohen-Macaulay.*

**Proof.** This follows from Lemma 12.5 and the definitions. □

## 13. Japanese and Nagata schemes

The notions considered in this section are not prominently defined in EGA. A "universally Japanese scheme" is mentioned and defined in [DG67, IV Corollary 5.11.4]. A "Japanese scheme" is mentioned in [DG67, IV Remark 10.4.14 (ii)] but no definition is given. A Nagata scheme (as given below) occurs in a few places in the literature (see for example [Liu02, Definition 8.2.30] and [Gre76, Page 142]).

We briefly recall that a domain $R$ is called *Japanese* if the integral closure of $R$ in any finite extension of its fraction field is finite over $R$. A ring $R$ is called *universally Japanese* if for any finite type ring map $R \to S$ with $S$ a domain $S$ is Japanese. A ring $R$ is called *Nagata* if it is Noetherian and $R/\mathfrak{p}$ is Japanese for every prime $\mathfrak{p}$ of $R$.

**Definition 13.1.** Let $X$ be a scheme.

(1) Assume $X$ integral. We say $X$ is *Japanese* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Japanese (see Algebra, Definition 161.1).
(2) We say $X$ is *universally Japanese* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is universally Japanese (see Algebra, Definition 162.1).
(3) We say $X$ is *Nagata* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Nagata (see Algebra, Definition 162.1).

Being Nagata is the same thing as being locally Noetherian and universally Japanese, see Lemma 13.8.

**Remark 13.2.** In [Hoo72] a (locally Noetherian) scheme $X$ is called Japanese if for every $x \in X$ and every associated prime $\mathfrak{p}$ of $\mathcal{O}_{X,x}$ the ring $\mathcal{O}_{X,x}/\mathfrak{p}$ is Japanese. We do not use this definition since there exists a one dimensional Noetherian domain with excellent (in particular Japanese) local rings whose normalization is not finite. See [Hoc73, Example 1] or [HL07] or [ILO14, Exposé XIX]. On the other hand, we could circumvent this problem by calling a scheme $X$ Japanese if for every affine open $\text{Spec}(A) \subset X$ the ring $A/\mathfrak{p}$ is Japanese for every associated prime $\mathfrak{p}$ of $A$.

**Lemma 13.3.** *A Nagata scheme is locally Noetherian.*

**Proof.** This is true because a Nagata ring is Noetherian by definition. □

**Lemma 13.4.** *Let $X$ be an integral scheme. The following are equivalent:*

(1) *The scheme $X$ is Japanese.*
(2) *For every affine open $U \subset X$ the domain $\mathcal{O}_X(U)$ is Japanese.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Japanese.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme $X_j$ is Japanese.*

*Moreover, if $X$ is Japanese then every open subscheme is Japanese.*

**Proof.** This follows from Lemma 4.3 and Algebra, Lemmas 161.3 and 161.4. □

**Lemma 13.5.** *Let $X$ be a scheme. The following are equivalent:*

(1) *The scheme $X$ is universally Japanese.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is universally Japanese.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is universally Japanese.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme $X_j$ is universally Japanese.*

*Moreover, if $X$ is universally Japanese then every open subscheme is universally Japanese.*

**Proof.** This follows from Lemma 4.3 and Algebra, Lemmas 162.4 and 162.7. □

**Lemma 13.6.** *Let $X$ be a scheme. The following are equivalent:*

(1) *The scheme $X$ is Nagata.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Nagata.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Nagata.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme $X_j$ is Nagata.*

*Moreover, if $X$ is Nagata then every open subscheme is Nagata.*

**Proof.** This follows from Lemma 4.3 and Algebra, Lemmas 162.6 and 162.7. □

**Lemma 13.7.** *Let $X$ be a locally Noetherian scheme. Then $X$ is Nagata if and only if every integral closed subscheme $Z \subset X$ is Japanese.*

**Proof.** Assume $X$ is Nagata. Let $Z \subset X$ be an integral closed subscheme. Let $z \in Z$. Let $\text{Spec}(A) = U \subset X$ be an affine open containing $z$ such that $A$ is Nagata. Then $Z \cap U \cong \text{Spec}(A/\mathfrak{p})$ for some prime $\mathfrak{p}$, see Schemes, Lemma 10.1 (and Definition 3.1). By Algebra, Definition 162.1 we see that $A/\mathfrak{p}$ is Japanese. Hence $Z$ is Japanese by definition.

Assume every integral closed subscheme of $X$ is Japanese. Let $\text{Spec}(A) = U \subset X$ be any affine open. As $X$ is locally Noetherian we see that $A$ is Noetherian (Lemma 5.2). Let $\mathfrak{p} \subset A$ be a prime ideal. We have to show that $A/\mathfrak{p}$ is Japanese. Let $T \subset U$ be the closed subset $V(\mathfrak{p}) \subset \text{Spec}(A)$. Let $\overline{T} \subset X$ be the closure. Then $\overline{T}$ is irreducible as the closure of an irreducible subset. Hence the reduced closed subscheme defined by $\overline{T}$ is an integral closed subscheme (called $\overline{T}$ again), see Schemes, Lemma 12.4. In other words, $\text{Spec}(A/\mathfrak{p})$ is an affine open of an integral closed subscheme of $X$. This subscheme is Japanese by assumption and by Lemma 13.4 we see that $A/\mathfrak{p}$ is Japanese. □

**Lemma 13.8.** *Let $X$ be a scheme. The following are equivalent:*

(1) *$X$ is Nagata, and*
(2) *$X$ is locally Noetherian and universally Japanese.*

**Proof.** This is Algebra, Proposition 162.15. $\square$

This discussion will be continued in Morphisms, Section 18.

## 14. The singular locus

Here is the definition.

**Definition 14.1.** Let $X$ be a locally Noetherian scheme. The *regular locus* $\text{Reg}(X)$ of $X$ is the set of $x \in X$ such that $\mathcal{O}_{X,x}$ is a regular local ring. The *singular locus* $\text{Sing}(X)$ is the complement $X \setminus \text{Reg}(X)$, i.e., the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is not a regular local ring.

The regular locus of a locally Noetherian scheme is stable under generalizations, see the discussion preceding Algebra, Definition 110.7. However, for general locally Noetherian schemes the regular locus need not be open. In More on Algebra, Section 47 the reader can find some criteria for when this is the case. We will discuss this further in Morphisms, Section 19.

## 15. Local irreducibility

Recall that in More on Algebra, Section 106 we introduced the notion of a (geometrically) unibranch local ring.

**Definition 15.1.** Let $X$ be a scheme. Let $x \in X$. We say $X$ is *unibranch at $x$* if the local ring $\mathcal{O}_{X,x}$ is unibranch. We say $X$ is *geometrically unibranch at $x$* if the local ring $\mathcal{O}_{X,x}$ is geometrically unibranch. We say $X$ is *unibranch* if $X$ is unibranch at all of its points. We say $X$ is *geometrically unibranch* if $X$ is geometrically unibranch at all of its points.

To be sure, it can happen that a local ring $A$ is geometrically unibranch (in the sense of More on Algebra, Definition 106.1) but the scheme $\text{Spec}(A)$ is not geometrically unibranch in the sense of Definition 15.1. For example this happens if $A$ is the local ring at the vertex of the cone over an irreducible plane curve which has ordinary double point singularity (a node).

**Lemma 15.2.** *A normal scheme is geometrically unibranch.*

**Proof.** This follows from the definitions. Namely, a scheme is normal if the local rings are normal domains. It is immediate from the More on Algebra, Definition 106.1 that a local normal domain is geometrically unibranch. $\square$

**Lemma 15.3.** *Let $X$ be a Noetherian scheme. The following are equivalent*

(1) *$X$ is geometrically unibranch (Definition 15.1),*
(2) *for every point $x \in X$ which is not the generic point of an irreducible component of $X$, the punctured spectrum of the strict henselization $\mathcal{O}_{X,x}^{sh}$ is connected.*

**Proof.** More on Algebra, Lemma 106.5 shows that (1) implies that the punctured spectra in (2) are irreducible and in particular connected.

Assume (2). Let $x \in X$. We have to show that $\mathcal{O}_{X,x}$ is geometrically unibranch. By induction on $\dim(\mathcal{O}_{X,x})$ we may assume that the result holds for every nontrivial generalization of $x$. We may replace $X$ by $\text{Spec}(\mathcal{O}_{X,x})$. In other words, we may assume that $X = \text{Spec}(A)$ with $A$ local and that $A_\mathfrak{p}$ is geometrically unibranch for each nonmaximal prime $\mathfrak{p} \subset A$.

Let $A^{sh}$ be the strict henselization of $A$. If $\mathfrak{q} \subset A^{sh}$ is a prime lying over $\mathfrak{p} \subset A$, then $A_\mathfrak{p} \to A^{sh}_\mathfrak{q}$ is a filtered colimit of étale algebras. Hence the strict henselizations of $A_\mathfrak{p}$ and $A^{sh}_\mathfrak{q}$ are isomorphic. Thus by More on Algebra, Lemma 106.5 we conclude that $A^{sh}_\mathfrak{q}$ has a unique minimal prime ideal for every nonmaximal prime $\mathfrak{q}$ of $A^{sh}$.

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal primes of $A^{sh}$. We have to show that $r = 1$. By the above we see that $V(\mathfrak{q}_1) \cap V(\mathfrak{q}_j) = \{\mathfrak{m}^{sh}\}$ for $j = 2, \ldots, r$. Hence $V(\mathfrak{q}_1) \setminus \{\mathfrak{m}^{sh}\}$ is an open and closed subset of the punctured spectrum of $A^{sh}$ which is a contradiction with the assumption that this punctured spectrum is connected unless $r = 1$. $\square$

**Definition 15.4.** Let $X$ be a scheme. Let $x \in X$. The *number of branches of $X$ at $x$* is the number of branches of the local ring $\mathcal{O}_{X,x}$ as defined in More on Algebra, Definition 106.6. The *number of geometric branches of $X$ at $x$* is the number of geometric branches of the local ring $\mathcal{O}_{X,x}$ as defined in More on Algebra, Definition 106.6.

Often we want to compare this with the branches of the complete local ring, but the comparison is not straightforward in general; some information on this topic can be found in More on Algebra, Section 108.

**Lemma 15.5.** *Let $X$ be a scheme and $x \in X$. Let $X_i$, $i \in I$ be the irreducible components of $X$ passing through $x$. Then the number of (geometric) branches of $X$ at $x$ is the sum over $i \in I$ of the number of (geometric) branches of $X_i$ at $x$.*

**Proof.** We view the $X_i$ as integral closed subschemes of $X$, see Schemes, Definition 12.5 and Lemma 3.4. Observe that the number of (geometric) branches of $X_i$ at $x$ is at least 1 for all $i$ (essentially by definition). Recall that the $X_i$ correspond 1-to-1 with the minimal prime ideals $\mathfrak{p}_i \subset \mathcal{O}_{X,x}$, see Algebra, Lemma 26.3. Thus, if $I$ is infinite, then $\mathcal{O}_{X,x}$ has infinitely many minimal primes, whence both $\mathcal{O}^h_{X,x}$ and $\mathcal{O}^{sh}_{X,x}$ have infinitely many minimal primes (combine Algebra, Lemmas 30.5 and 30.7 and the injectivity of the maps $\mathcal{O}_{X,x} \to \mathcal{O}^h_{X,x} \to \mathcal{O}^{sh}_{X,x}$). In this case the number of (geometric) branches of $X$ at $x$ is defined to be $\infty$ which is also true for the sum. Thus we may assume $I$ is finite. Let $A'$ be the integral closure of $\mathcal{O}_{X,x}$ in the total ring of fractions $Q$ of $(\mathcal{O}_{X,x})_{red}$. Let $A'_i$ be the integral closure of $\mathcal{O}_{X,x}/\mathfrak{p}_i$ in the total ring of fractions $Q_i$ of $\mathcal{O}_{X,x}/\mathfrak{p}_i$. By Algebra, Lemma 25.4 we have $Q = \prod_{i \in I} Q_i$. Thus $A' = \prod A'_i$. Then the equality of the lemma follows from More on Algebra, Lemma 106.7 which expresses the number of (geometric) branches in terms of the maximal ideals of $A'$. $\square$

**Lemma 15.6.** *Let $X$ be a scheme. Let $x \in X$.*

1. *The number of branches of $X$ at $x$ is 1 if and only if $X$ is unibranch at $x$.*
2. *The number of geometric branches of $X$ at $x$ is 1 if and only if $X$ is geometrically unibranch at $x$.*

**Proof.** This lemma follows immediately from the definitions and the corresponding result for rings, see More on Algebra, Lemma 106.7. $\square$

## 16. Characterizing modules of finite type and finite presentation

Let $X$ be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following lemma implies that $\mathcal{F}$ is of finite type (see Modules, Definition 9.1) if and only if $\mathcal{F}$ is on each open affine $\text{Spec}(A) = U \subset X$ of the form $\widetilde{M}$ for some finite type $A$-module $M$. Similarly, $\mathcal{F}$ is of finite presentation (see Modules, Definition 11.1) if and only if $\mathcal{F}$ is on each open affine $\text{Spec}(A) = U \subset X$ of the form $\widetilde{M}$ for some finitely presented $A$-module $M$.

**Lemma 16.1.** *Let $X = \text{Spec}(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module if and only if $M$ is a finite $R$-module.*

**Proof.** Assume $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module. This means there exists an open covering of $X$ such that $\widetilde{M}$ restricted to the members of this covering is globally generated by finitely many sections. Thus there also exists a standard open covering $X = \bigcup_{i=1,\ldots,n} D(f_i)$ such that $\widetilde{M}|_{D(f_i)}$ is generated by finitely many sections. Thus $M_{f_i}$ is finitely generated for each $i$. Hence we conclude by Algebra, Lemma 23.2. $\square$

**Lemma 16.2.** *Let $X = \text{Spec}(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation if and only if $M$ is an $R$-module of finite presentation.*

**Proof.** Assume $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation. By Lemma 16.1 we see that $M$ is a finite $R$-module. Choose a surjection $R^n \to M$ with kernel $K$. By Schemes, Lemma 5.4 there is a short exact sequence

$$0 \to \widetilde{K} \to \bigoplus \mathcal{O}_X^{\oplus n} \to \widetilde{M} \to 0$$

By Modules, Lemma 11.3 we see that $\widetilde{K}$ is a finite type $\mathcal{O}_X$-module. Hence by Lemma 16.1 again we see that $K$ is a finite $R$-module. Hence $M$ is an $R$-module of finite presentation. $\square$

## 17. Sections over principal opens

Here is a typical result of this kind. We will use a more naive but more direct method of proof in later lemmas.

**Lemma 17.1.** *Let $X$ be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Denote $X_f \subset X$ the open where $f$ is invertible, see Schemes, Lemma 6.2. If $X$ is quasi-compact and quasi-separated, the canonical map*

$$\Gamma(X, \mathcal{O}_X)_f \longrightarrow \Gamma(X_f, \mathcal{O}_X)$$

*is an isomorphism. Moreover, if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules the map*

$$\Gamma(X, \mathcal{F})_f \longrightarrow \Gamma(X_f, \mathcal{F})$$

*is an isomorphism.*

**Proof.** Write $R = \Gamma(X, \mathcal{O}_X)$. Consider the canonical morphism

$$\varphi : X \longrightarrow \mathrm{Spec}(R)$$

of schemes, see Schemes, Lemma 6.4. Then the inverse image of the standard open $D(f)$ on the right hand side is $X_f$ on the left hand side. Moreover, since $X$ is assumed quasi-compact and quasi-separated the morphism $\varphi$ is quasi-compact and quasi-separated, see Schemes, Lemma 19.2 and 21.13. Hence by Schemes, Lemma 24.1 we see that $\varphi_*\mathcal{F}$ is quasi-coherent. Hence we see that $\varphi_*\mathcal{F} = \widetilde{M}$ with $M = \Gamma(X, \mathcal{F})$ as an $R$-module. Thus we see that

$$\Gamma(X_f, \mathcal{F}) = \Gamma(D(f), \varphi_*\mathcal{F}) = \Gamma(D(f), \widetilde{M}) = M_f$$

which is exactly the content of the lemma. The first displayed isomorphism of the lemma follows by taking $\mathcal{F} = \mathcal{O}_X$. $\square$

Recall that given a scheme $X$, an invertible sheaf $\mathcal{L}$ on $X$, and a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we get a graded ring $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ and a graded $\Gamma_*(X, \mathcal{L})$-module $\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ see Modules, Definition 25.7. If we have moreover a section $s \in \Gamma(X, \mathcal{L})$, then we obtain a map

$$\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{F}|_{X_s}) \tag{17.1.1}$$

which sends $t/s^n$ where $t \in \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ to $t|_{X_s} \otimes s|_{X_s}^{-n}$. This makes sense because $X_s \subset X$ is by definition the open over which $s$ has an inverse, see Modules, Lemma 25.10.

**Lemma 17.2.** *Let $X$ be a scheme. Let $\mathcal{L}$ be an invertible sheaf on $X$. Let $s \in \Gamma(X, \mathcal{L})$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

1. *If $X$ is quasi-compact, then (17.1.1) is injective, and*
2. *if $X$ is quasi-compact and quasi-separated, then (17.1.1) is an isomorphism.*

*In particular, the canonical map*

$$\Gamma_*(X, \mathcal{L})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{O}_X), \quad a/s^n \longmapsto a \otimes s^{-n}$$

*is an isomorphism if $X$ is quasi-compact and quasi-separated.*

**Proof.** Assume $X$ is quasi-compact. Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ with $U_j$ affine and $\mathcal{L}|_{U_j} \cong \mathcal{O}_{U_j}$. Via this isomorphism, the image $s|_{U_j}$ corresponds to some $f_j \in \Gamma(U_j, \mathcal{O}_{U_j})$. Then $X_s \cap U_j = D(f_j)$.

Proof of (1). Let $t/s^n$ be an element in the kernel of (17.1.1). Then $t|_{X_s} = 0$. Hence $(t|_{U_j})|_{D(f_j)} = 0$. By Lemma 17.1 we conclude that $f_j^{e_j} t|_{U_j} = 0$ for some $e_j \geq 0$. Let $e = \max(e_j)$. Then we see that $t \otimes s^e$ restricts to zero on $U_j$ for all $j$, hence is zero. Since $t/s^n$ is equal to $t \otimes s^e/s^{n+e}$ in $\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)}$ we conclude that $t/s^n = 0$ as desired.

Proof of (2). Assume $X$ is quasi-compact and quasi-separated. Then $U_j \cap U_{j'}$ is quasi-compact for all pairs $j, j'$, see Schemes, Lemma 21.6. By part (1) we know (17.1.1) is injective. Let $t' \in \Gamma(X_s, \mathcal{F}|_{X_s})$. For every $j$, there exist an integer $e_j \geq 0$ and $t'_j \in \Gamma(U_j, \mathcal{F}|_{U_j})$ such that $t'|_{D(f_j)}$ corresponds to $t'_j/f_j^{e_j}$ via the isomorphism of Lemma 17.1. Set $e = \max(e_j)$ and

$$t_j = f_j^{e - e_j} t'_j \otimes q_j^e \in \Gamma(U_j, (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes e})|_{U_j})$$

where $q_j \in \Gamma(U_j, \mathcal{L}|_{U_j})$ is the trivializing section coming from the isomorphism $\mathcal{L}|_{U_j} \cong \mathcal{O}_{U_j}$. In particular we have $s|_{U_j} = f_j q_j$. Using this a calculation shows that $t_j|_{U_j \cap U_{j'}}$ and $t_{j'}|_{U_j \cap U_{j'}}$ map to the same section of $\mathcal{F}$ over $U_j \cap U_{j'} \cap X_s$. By quasi-compactness of $U_j \cap U_{j'}$ and part (1) there exists an integer $e' \geq 0$ such that

$$t_j|_{U_j \cap U_{j'}} \otimes s^{e'}|_{U_j \cap U_{j'}} = t_{j'}|_{U_j \cap U_{j'}} \otimes s^{e'}|_{U_j \cap U_{j'}}$$

as sections of $\mathcal{F} \otimes \mathcal{L}^{\otimes e+e'}$ over $U_j \cap U_{j'}$. We may choose the same $e'$ to work for all pairs $j, j'$. Then the sheaf conditions implies there is a section $t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes e+e'})$ whose restriction to $U_j$ is $t_j \otimes s^{e'}|_{U_j}$. A simple computation shows that $t/s^{e+e'}$ maps to $t'$ as desired. $\square$

Let $X$ be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. Consider the graded $\Gamma_*(X, \mathcal{L})$-module

$$M = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

Next, let $s \in \Gamma(X, \mathcal{L})$ be a section. Then there is a canonical map

$$M_{(s)} \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_s}}(\mathcal{F}|_{X_s}, \mathcal{G}|_{X_s}) \tag{17.2.1}$$

which sends $\alpha/s^n$ to the map $\alpha|_{X_s} \otimes s|_{X_s}^{-n}$. The following lemma, combined with Lemma 22.4, says roughly that, if $X$ is quasi-compact and quasi-separated, the category of finitely presented $\mathcal{O}_{X_s}$-modules is the category of finitely presented $\mathcal{O}_X$-modules with the multiplicative system of maps $s^n : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ inverted.

**Lemma 17.3.** *Let $X$ be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules.*

1. *If $X$ is quasi-compact and $\mathcal{F}$ is of finite type, then (17.2.1) is injective, and*
2. *if $X$ is quasi-compact and quasi-separated and $\mathcal{F}$ is of finite presentation, then (17.2.1) is bijective.*

**Proof.** We first prove the lemma in case $X = \mathrm{Spec}(A)$ is affine and $\mathcal{L} = \mathcal{O}_X$. In this case $s$ corresponds to an element $f \in A$. Say $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$ for some $A$-modules $M$ and $N$. Then the lemma translates (via Lemmas 16.1 and 16.2) into the following algebra statements

1. If $M$ is a finite $A$-module and $\varphi : M \to N$ is an $A$-module map such that the induced map $M_f \to N_f$ is zero, then $f^n \varphi = 0$ for some $n$.
2. If $M$ is a finitely presented $A$-module, then $\mathrm{Hom}_A(M, N)_f = \mathrm{Hom}_{A_f}(M_f, N_f)$.

The second statement is Algebra, Lemma 10.2 and we omit the proof of the first statement.

Next, we prove (1) for general $X$. Assume $X$ is quasi-compact and hoose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ with $U_j$ affine and $\mathcal{L}|_{U_j} \cong \mathcal{O}_{U_j}$. Via this isomorphism, the image $s|_{U_j}$ corresponds to some $f_j \in \Gamma(U_j, \mathcal{O}_{U_j})$. Then $X_s \cap U_j = D(f_j)$. Let $\alpha/s^n$ be an element in the kernel of (17.2.1). Then $\alpha|_{X_s} = 0$. Hence $(\alpha|_{U_j})|_{D(f_j)} = 0$. By the affine case treated above we conclude that $f_j^{e_j} \alpha|_{U_j} = 0$ for some $e_j \geq 0$. Let $e = \max(e_j)$. Then we see that $\alpha \otimes s^e$ restricts to zero on $U_j$ for all $j$, hence is zero. Since $\alpha/s^n$ is equal to $\alpha \otimes s^e/s^{n+e}$ in $M_{(s)}$ we conclude that $\alpha/s^n = 0$ as desired.

Proof of (2). Since $\mathcal{F}$ is of finite presentation, the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, see Schemes, Section 24. Moreover, it is clear that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$$

for all $n$. Hence in this case the statement follows from Lemma 17.2 applied to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. $\square$

## 18. Quasi-affine schemes

**Definition 18.1.** A scheme $X$ is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme.

**Lemma 18.2.** *Let $A$ be a ring and let $U \subset \text{Spec}(A)$ be a quasi-compact open subscheme. For $\mathcal{F}$ quasi-coherent on $U$ the canonical map*

$$\widetilde{H^0(U, \mathcal{F})}|_U \to \mathcal{F}$$

*is an isomorphism.*

**Proof.** Denote $j : U \to \text{Spec}(A)$ the inclusion morphism. Then $H^0(U, \mathcal{F}) = H^0(\text{Spec}(A), j_*\mathcal{F})$ and $j_*\mathcal{F}$ is quasi-coherent by Schemes, Lemma 24.1. Hence $j_*\mathcal{F} = \widetilde{H^0(U, \mathcal{F})}$ by Schemes, Lemma 7.5. Restricting back to $U$ we get the lemma. $\square$

**Lemma 18.3.** *Let $X$ be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Assume $X$ is quasi-compact and quasi-separated and assume that $X_f$ is affine. Then the canonical morphism*

$$j : X \longrightarrow \text{Spec}(\Gamma(X, \mathcal{O}_X))$$

*from Schemes, Lemma 6.4 induces an isomorphism of $X_f = j^{-1}(D(f))$ onto the standard affine open $D(f) \subset \text{Spec}(\Gamma(X, \mathcal{O}_X))$.*

**Proof.** This is clear as $j$ induces an isomorphism of rings $\Gamma(X, \mathcal{O}_X)_f \to \mathcal{O}_X(X_f)$ by Lemma 17.1 above. $\square$

**Lemma 18.4.** *Let $X$ be a scheme. Then $X$ is quasi-affine if and only if the canonical morphism*

$$X \longrightarrow \text{Spec}(\Gamma(X, \mathcal{O}_X))$$

*from Schemes, Lemma 6.4 is a quasi-compact open immersion.*

**Proof.** If the displayed morphism is a quasi-compact open immersion then $X$ is isomorphic to a quasi-compact open subscheme of $\text{Spec}(\Gamma(X, \mathcal{O}_X))$ and clearly $X$ is quasi-affine.

Assume $X$ is quasi-affine, say $X \subset \text{Spec}(R)$ is quasi-compact open. This in particular implies that $X$ is separated, see Schemes, Lemma 23.9. Let $A = \Gamma(X, \mathcal{O}_X)$. Consider the ring map $R \to A$ coming from $R = \Gamma(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ and the restriction mapping of the sheaf $\mathcal{O}_{\text{Spec}(R)}$. By Schemes, Lemma 6.4 we obtain a factorization:

$$X \longrightarrow \text{Spec}(A) \longrightarrow \text{Spec}(R)$$

of the inclusion morphism. Let $x \in X$. Choose $r \in R$ such that $x \in D(r)$ and $D(r) \subset X$. Denote $f \in A$ the image of $r$ in $A$. The open $X_f$ of Lemma 17.1 above is equal to $D(r) \subset X$ and hence $A_f \cong R_r$ by the conclusion of that lemma. Hence $D(r) \to \text{Spec}(A)$ is an isomorphism onto the standard affine open $D(f)$ of $\text{Spec}(A)$. Since $X$ can be covered by such affine opens $D(f)$ we win. $\square$

**Lemma 18.5.** *Let $U \to V$ be an open immersion of quasi-affine schemes. Then*

$$\begin{tikzcd} U \arrow[rr, "j"'] \arrow[d] & & \mathrm{Spec}(\Gamma(U, \mathcal{O}_U)) \arrow[d] \\ U \arrow[r] & V \arrow[r, "j'"] & \mathrm{Spec}(\Gamma(V, \mathcal{O}_V)) \end{tikzcd}$$

*is cartesian.*

**Proof.** The diagram is commutative by Schemes, Lemma 6.4. Write $A = \Gamma(U, \mathcal{O}_U)$ and $B = \Gamma(V, \mathcal{O}_V)$. Let $g \in B$ be such that $V_g$ is affine and contained in $U$. This means that if $f$ is the image of $g$ in $A$, then $U_f = V_g$. By Lemma 18.3 we see that $j'$ induces an isomorphism of $V_g$ with the standard open $D(g)$ of $\mathrm{Spec}(B)$. Thus $V_g \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is an isomorphism onto $D(f) \subset \mathrm{Spec}(A)$. By Lemma 18.3 again $j$ maps $U_f$ isomorphically to $D(f)$. Thus we see that $U_f = U_f \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$. Since by Lemma 18.4 we can cover $U$ by $V_g = U_f$ as above, we see that $U \to U \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$ is an isomorphism. $\square$

**Lemma 18.6.** *Let $X$ be a quasi-affine scheme. There exists an integer $n \geq 0$, an affine scheme $T$, and a morphism $T \to X$ such that for every morphism $X' \to X$ with $X'$ affine the fibre product $X' \times_X T$ is isomorphic to $\mathbf{A}^n_{X'}$ over $X'$.*

**Proof.** By definition, there exists a ring $A$ such that $X$ is isomorphic to a quasi-compact open subscheme $U \subset \mathrm{Spec}(A)$. Recall that the standard opens $D(f) \subset \mathrm{Spec}(A)$ form a basis for the topology, see Algebra, Section 17. Since $U$ is quasi-compact we can choose $f_1, \ldots, f_n \in A$ such that $U = D(f_1) \cup \ldots \cup D(f_n)$. Thus we may assume $X = \mathrm{Spec}(A) \setminus V(I)$ where $I = (f_1, \ldots, f_n)$. We set

$$T = \mathrm{Spec}(A[t, x_1, \ldots, x_n]/(f_1x_1 + \ldots + f_nx_n - 1))$$

The structure morphism $T \to \mathrm{Spec}(A)$ factors through the open $X$ to give the morphism $T \to X$. If $X' = \mathrm{Spec}(A')$ and the morphism $X' \to X$ corresponds to the ring map $A \to A'$, then the images $f'_1, \ldots, f'_n \in A'$ of $f_1, \ldots, f_n$ generate the unit ideal in $A'$. Say $1 = f'_1a'_1 + \ldots + f'_na'_n$. The base change $X' \times_X T$ is the spectrum of $A'[t, x_1, \ldots, x_n]/(f'_1x_1 + \ldots + f'_nx_n - 1)$. We claim the $A'$-algebra homomorphism

$$\varphi : A'[y_1, \ldots, y_n] \longrightarrow A'[t, x_1, \ldots, x_n, x_{n+1}]/(f'_1x_1 + \ldots + f'_nx_n - 1)$$

sending $y_i$ to $a'_it + x_i$ is an isomorphism. The claim finishes the proof of the lemma. The inverse of $\varphi$ is given by the $A'$-algebra homomorphism

$$\psi : A'[t, x_1, \ldots, x_n, x_{n+1}]/(f'_1x_1 + \ldots + f'_nx_n - 1) \longrightarrow A'[y_1, \ldots, y_n]$$

sending $t$ to $-1 + f'_1y_1 + \ldots + f'_ny_n$ and $x_i$ to $y_i + a'_i - a'_i(f'_1y_1 + \ldots + f'_ny_n)$ for $i = 1, \ldots, n$. This makes sense because $\sum f'_ix_i$ is mapped to

$$\sum f'_i(y_i + a'_i - a'_i(\sum f'_jy_j)) = (\sum f'_iy_i) + 1 - (\sum f'_jy_j) = 1$$

To see the maps are mutually inverse one computes as follows:

$$\begin{gathered} \varphi(\psi(t) = \varphi(-1 + \sum f'_iy_i) = -1 + \sum f'_i(a'_it + x_i) = t \\ \varphi(\psi(x_i)) = \varphi(y_i + a'_i - a'_i(\sum f'_jy_j)) = a'_it + x_i + a'_i - a'_i(\sum f'_ja'_jt + f'_jx_j) = x_i \\ \psi(\varphi(y_i)) = \psi(a'_it + x_i) = a'_i(-1 + \sum f'_jy_j) + y_i + a'_i - a'_i(\sum f'_jy_j) = y_i \end{gathered}$$

This finishes the proof. $\square$

## 19. Flat modules

On any ringed space $(X, \mathcal{O}_X)$ we know what it means for an $\mathcal{O}_X$-module to be flat (at a point), see Modules, Definition 17.1 (Definition 17.3). For quasi-coherent sheaves on an affine scheme this matches the notion defined in the algebra chapter.

**Lemma 19.1.** *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some $R$-module $M$. The quasi-coherent sheaf $\mathcal{F}$ is a flat $\mathcal{O}_X$-module if and only if $M$ is a flat $R$-module.*

**Proof.** Flatness of $\mathcal{F}$ may be checked on the stalks, see Modules, Lemma 17.2. The same is true in the case of modules over a ring, see Algebra, Lemma 39.18. And since $\mathcal{F}_x = M_\mathfrak{p}$ if $x$ corresponds to $\mathfrak{p}$ the lemma is true. □

## 20. Locally free modules

On any ringed space we know what it means for an $\mathcal{O}_X$-module to be (finite) locally free. On an affine scheme this matches the notion defined in the algebra chapter.

**Lemma 20.1.** *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some $R$-module $M$. The quasi-coherent sheaf $\mathcal{F}$ is a (finite) locally free $\mathcal{O}_X$-module of if and only if $M$ is a (finite) locally free $R$-module.*

**Proof.** Follows from the definitions, see Modules, Definition 14.1 and Algebra, Definition 78.1. □

We can characterize finite locally free modules in many different ways.

**Lemma 20.2.** *Let $X$ be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:*

1. *$\mathcal{F}$ is a flat $\mathcal{O}_X$-module of finite presentation,*
2. *$\mathcal{F}$ is $\mathcal{O}_X$-module of finite presentation and for all $x \in X$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module,*
3. *$\mathcal{F}$ is a locally free, finite type $\mathcal{O}_X$-module,*
4. *$\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module, and*
5. *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, for every $x \in X$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, and the function*
$$\rho_\mathcal{F} : X \to \mathbf{Z}, \quad x \longmapsto \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
*is locally constant in the Zariski topology on $X$.*

**Proof.** This lemma immediately reduces to the affine case. In this case the lemma is a reformulation of Algebra, Lemma 78.2. The translation uses Lemmas 16.1, 16.2, 19.1, and 20.1. □

**Lemma 20.3.** *Let $X$ be a reduced scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the equivalent conditions of Lemma 20.2 are also equivalent to*

(6) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type and the function*
$$\rho_\mathcal{F} : X \to \mathbf{Z}, \quad x \longmapsto \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
*is locally constant in the Zariski topology on $X$.*

**Proof.** This lemma immediately reduces to the affine case. In this case the lemma is a reformulation of Algebra, Lemma 78.3. □

## 21. Locally projective modules

A consequence of the work done in the algebra chapter is that it makes sense to define a locally projective module as follows.

**Definition 21.1.** Let $X$ be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *locally projective* if for every affine open $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is projective.

**Lemma 21.2.** *Let $X$ be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is locally projective, and*
(2) *there exists an affine open covering $X = \bigcup U_i$ such that the $\mathcal{O}_X(U_i)$-module $\mathcal{F}(U_i)$ is projective for every $i$.*

*In particular, if $X = \mathrm{Spec}(A)$ and $\mathcal{F} = \widetilde{M}$ then $\mathcal{F}$ is locally projective if and only if $M$ is a projective $A$-module.*

**Proof.** First, note that if $M$ is a projective $A$-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a projective $B$-module, see Algebra, Lemma 94.1. Hence if $U$ is an affine open such that $\mathcal{F}(U)$ is a projective $\mathcal{O}_X(U)$-module, then the standard open $D(f)$ is an affine open such that $\mathcal{F}(D(f))$ is a projective $\mathcal{O}_X(D(f))$-module for all $f \in \mathcal{O}_X(U)$. Assume (2) holds. Let $U \subset X$ be an arbitrary affine open. We can find an open covering $U = \bigcup_{j=1,\ldots,m} D(f_j)$ by finitely many standard opens $D(f_j)$ such that for each $j$ the open $D(f_j)$ is a standard open of some $U_i$, see Schemes, Lemma 11.5. Hence, if we set $A = \mathcal{O}_X(U)$ and if $M$ is an $A$-module such that $\mathcal{F}|_U$ corresponds to $M$, then we see that $M_{f_j}$ is a projective $A_{f_j}$-module. It follows that $A \to B = \prod A_{f_j}$ is a faithfully flat ring map such that $M \otimes_A B$ is a projective $B$-module. Hence $M$ is projective by Algebra, Theorem 95.6. $\square$

**Lemma 21.3.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $\mathcal{G}$ is locally projective on $Y$, then $f^*\mathcal{G}$ is locally projective on $X$.*

**Proof.** See Algebra, Lemma 94.1. $\square$

## 22. Extending quasi-coherent sheaves

It is sometimes useful to be able to show that a given quasi-coherent sheaf on an open subscheme extends to the whole scheme.

**Lemma 22.1.** *Let $j : U \to X$ be a quasi-compact open immersion of schemes.*

(1) *Any quasi-coherent sheaf on $U$ extends to a quasi-coherent sheaf on $X$.*
(2) *Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent subsheaf. There exists a quasi-coherent subsheaf $\mathcal{H}$ of $\mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ as subsheaves of $\mathcal{F}|_U$.*
(3) *Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Let $\mathcal{G}$ be a quasi-coherent sheaf on $U$. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. There exists a quasi-coherent sheaf $\mathcal{H}$ of $\mathcal{O}_X$-modules and a map $\psi : \mathcal{H} \to \mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ and that $\psi|_U = \varphi$.*

**Proof.** An immersion is separated (see Schemes, Lemma 23.8) and $j$ is quasi-compact by assumption. Hence for any quasi-coherent sheaf $\mathcal{G}$ on $U$ the sheaf $j_*\mathcal{G}$ is an extension to $X$. See Schemes, Lemma 24.1 and Sheaves, Section 31.

Assume $\mathcal{F}$, $\mathcal{G}$ are as in (2). Then $j_*\mathcal{G}$ is a quasi-coherent sheaf on $X$ (see above). It is a subsheaf of $j_*j^*\mathcal{F}$. Hence the kernel

$$\mathcal{H} = \operatorname{Ker}(\mathcal{F} \oplus j_*\mathcal{G} \longrightarrow j_*j^*\mathcal{F})$$

is quasi-coherent as well, see Schemes, Section 24. It is formal to check that $\mathcal{H} \subset \mathcal{F}$ and that $\mathcal{H}|_U = \mathcal{G}$ (using the material in Sheaves, Section 31 again).

Part (3) is proved in the same manner as (2). Just take $\mathcal{H} = \operatorname{Ker}(\mathcal{F} \oplus j_*\mathcal{G} \to j_*j^*\mathcal{F})$ with its obvious map to $\mathcal{F}$ and its obvious identification with $\mathcal{G}$ over $U$. □

**Lemma 22.2.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent $\mathcal{O}_U$-submodule which is of finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

**Proof.** Let $n$ be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that $X$ is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend $\mathcal{G}$ to a $\mathcal{G}_1$ over $U \cup U_1$ to a $\mathcal{G}_2$ over $U \cup U_1 \cup U_2$ to a $\mathcal{G}_3$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with $V$ affine. Since $X$ is quasi-separated and $U$, $V$ are quasi-compact open, we see that $U \cap V$ is a quasi-compact open. It suffices to prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U\cap V})$ since we can glue the resulting sheaf $\mathcal{G}'$ over $V$ to the given sheaf $\mathcal{G}$ over $U$ along the common value over $U \cap V$. Thus we reduce to the case where $X$ is affine.

Assume $X = \operatorname{Spec}(R)$. Write $\mathcal{F} = \widetilde{M}$ for some $R$-module $M$. By Lemma 22.1 above we may find a quasi-coherent subsheaf $\mathcal{H} \subset \mathcal{F}$ which restricts to $\mathcal{G}$ over $U$. Write $\mathcal{H} = \widetilde{N}$ for some $R$-module $N$. For every $u \in U$ there exists an $f \in R$ such that $u \in D(f) \subset U$ and such that $N_f$ is finitely generated, see Lemma 16.1. Since $U$ is quasi-compact we can cover it by finitely many $D(f_i)$ such that $N_{f_i}$ is generated by finitely many elements, say $x_{i,1}/f_i^N, \ldots, x_{i,r_i}/f_i^N$. Let $N' \subset N$ be the submodule generated by the elements $x_{i,j}$. Then the subsheaf $\mathcal{G}' = \widetilde{N'} \subset \mathcal{H} \subset \mathcal{F}$ works. □

**Lemma 22.3.** *Let $X$ be a quasi-compact and quasi-separated scheme. Any quasi-coherent sheaf of $\mathcal{O}_X$-modules is the directed colimit of its quasi-coherent $\mathcal{O}_X$-submodules which are of finite type.*

**Proof.** The colimit is directed because if $\mathcal{G}_1$, $\mathcal{G}_2$ are quasi-coherent subsheaves of finite type, then the image of $\mathcal{G}_1 \oplus \mathcal{G}_2 \to \mathcal{F}$ is a quasi-coherent submodule of finite type. Let $U \subset X$ be any affine open, and let $s \in \Gamma(U, \mathcal{F})$ be any section. Let $\mathcal{G} \subset \mathcal{F}|_U$ be the subsheaf generated by $s$. Then clearly $\mathcal{G}$ is quasi-coherent and has finite type as an $\mathcal{O}_U$-module. By Lemma 22.2 we see that $\mathcal{G}$ is the restriction of a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{F}$ which has finite type. Since $X$ has a basis for the topology consisting of affine opens we conclude that every local section of $\mathcal{F}$ is locally contained in a quasi-coherent submodule of finite type. Thus we win. □

**Lemma 22.4.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{G}$ be an $\mathcal{O}_U$-module which is of finite presentation. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. Then there exists an $\mathcal{O}_X$-module $\mathcal{G}'$ of finite presentation, and a morphism of $\mathcal{O}_X$-modules $\varphi' : \mathcal{G}' \to \mathcal{F}$ such that $\mathcal{G}'|_U = \mathcal{G}$ and such that $\varphi'|_U = \varphi$.*

**Proof.** The beginning of the proof is a repeat of the beginning of the proof of Lemma 22.2. We write it out carefuly anyway.

Let $n$ be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that $X$ is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend the pair $(\mathcal{G}, \varphi)$ to a pair $(\mathcal{G}_1, \varphi_1)$ over $U \cup U_1$ to a pair $(\mathcal{G}_2, \varphi_2)$ over $U \cup U_1 \cup U_2$ to a pair $(\mathcal{G}_3, \varphi_3)$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with $V$ affine. Since $X$ is quasi-separated and $U$ quasi-compact, we see that $U \cap V \subset V$ is quasi-compact. Suppose we prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U \cap V}, \varphi|_{U \cap V})$ thereby producing $(\mathcal{G}', \varphi')$ over $V$. Then we can glue $\mathcal{G}'$ over $V$ to the given sheaf $\mathcal{G}$ over $U$ along the common value over $U \cap V$, and similarly we can glue the map $\varphi'$ to the map $\varphi$ along the common value over $U \cap V$. Thus we reduce to the case where $X$ is affine.

Assume $X = \mathrm{Spec}(R)$. By Lemma 22.1 above we may find a quasi-coherent sheaf $\mathcal{H}$ with a map $\psi : \mathcal{H} \to \mathcal{F}$ over $X$ which restricts to $\mathcal{G}$ and $\varphi$ over $U$. By Lemma 22.2 we can find a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'|_U = \mathcal{G}$. Thus after replacing $\mathcal{H}$ by $\mathcal{H}'$ and $\psi$ by the restriction of $\psi$ to $\mathcal{H}'$ we may assume that $\mathcal{H}$ is of finite type. By Lemma 16.2 we conclude that $\mathcal{H} = \widetilde{N}$ with $N$ a finitely generated $R$-module. Hence there exists a surjection as in the following short exact sequence of quasi-coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{K} \to \mathcal{O}_X^{\oplus n} \to \mathcal{H} \to 0$$

where $\mathcal{K}$ is defined as the kernel. Since $\mathcal{G}$ is of finite presentation and $\mathcal{H}|_U = \mathcal{G}$ by Modules, Lemma 11.3 the restriction $\mathcal{K}|_U$ is an $\mathcal{O}_U$-module of finite type. Hence by Lemma 22.2 again we see that there exists a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{K}' \subset \mathcal{K}$ such that $\mathcal{K}'|_U = \mathcal{K}|_U$. The solution to the problem posed in the lemma is to set

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}'$$

which is clearly of finite presentation and restricts to give $\mathcal{G}$ on $U$ with $\varphi'$ equal to the composition

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}' \to \mathcal{O}_X^{\oplus n}/\mathcal{K} = \mathcal{H} \xrightarrow{\psi} \mathcal{F}.$$

This finishes the proof of the lemma. $\square$

**Lemma 22.5.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{G}$ be an $\mathcal{O}_U$-module.*

1. *If $\mathcal{G}$ is quasi-coherent and of finite type, then there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}'$ of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*
2. *If $\mathcal{G}$ is of finite presentation, then there exists an $\mathcal{O}_X$-module $\mathcal{G}'$ of finite presentation such that $\mathcal{G}'|_U = \mathcal{G}$.*

**Proof.** Part (2) is the special case of Lemma 22.4 where $\mathcal{F} = 0$. For part (1) we first write $\mathcal{G} = \mathcal{F}|_U$ for some quasi-coherent $\mathcal{O}_X$-module by Lemma 22.1 and then we apply Lemma 22.2 with $\mathcal{G} = \mathcal{F}|_U$. $\square$

The following lemma says that every quasi-coherent sheaf on a quasi-compact and quasi-separated scheme is a filtered colimit of $\mathcal{O}$-modules of finite presentation. Actually, we reformulate this in (perhaps more familiar) terms of directed colimits over directed sets in the next lemma.

**Lemma 22.6.** *Let $X$ be a scheme. Assume $X$ is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist*

1. *a filtered index category $\mathcal{I}$ (see Categories, Definition 19.1),*
2. *a diagram $\mathcal{I} \to Mod(\mathcal{O}_X)$ (see Categories, Section 14), $i \mapsto \mathcal{F}_i$,*
3. *morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$*

*such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms $\varphi_i$ induce an isomorphism*

$$\operatorname{colim}_i \mathcal{F}_i = \mathcal{F}.$$

**Proof.** Choose a set $I$ and for each $i \in I$ an $\mathcal{O}_X$-module of finite presentation and a homomorphism of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$ with the following property: For any $\psi : \mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ of finite presentation there is an $i \in I$ such that there exists an isomorphism $\alpha : \mathcal{F}_i \to \mathcal{G}$ with $\varphi_i = \psi \circ \alpha$. It is clear from Modules, Lemma 9.8 that such a set exists (see also its proof). We denote $\mathcal{I}$ the category with $\mathrm{Ob}(\mathcal{I}) = I$ and given $i, i' \in I$ we set

$$\mathrm{Mor}_{\mathcal{I}}(i, i') = \{\alpha : \mathcal{F}_i \to \mathcal{F}_{i'} \mid \alpha \circ \varphi_{i'} = \varphi_i\}.$$

We claim that $\mathcal{I}$ is a filtered category and that $\mathcal{F} = \operatorname{colim}_i \mathcal{F}_i$.

Let $i, i' \in I$. Then we can consider the morphism

$$\mathcal{F}_i \oplus \mathcal{F}_{i'} \longrightarrow \mathcal{F}$$

which is the direct sum of $\varphi_i$ and $\varphi_{i'}$. Since a direct sum of finitely presented $\mathcal{O}_X$-modules is finitely presented we see that there exists some $i'' \in I$ such that $\varphi_{i''} : \mathcal{F}_{i''} \to \mathcal{F}$ is isomorphic to the displayed arrow towards $\mathcal{F}$ above. Since there are commutative diagrams

$$\begin{tikzcd} \mathcal{F}_i \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, equal] \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} \arrow[r] & \mathcal{F} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{F}_{i'} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, equal] \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} \arrow[r] & \mathcal{F} \end{tikzcd}$$

we see that there are morphisms $i \to i''$ and $i' \to i''$ in $\mathcal{I}$. Next, suppose that we have $i, i' \in I$ and morphisms $\alpha, \beta : i \to i'$ (corresponding to $\mathcal{O}_X$-module maps $\alpha, \beta : \mathcal{F}_i \to \mathcal{F}_{i'}$). In this case consider the coequalizer

$$\mathcal{G} = \mathrm{Coker}(\mathcal{F}_i \xrightarrow{\alpha - \beta} \mathcal{F}_{i'})$$

Note that $\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation. Since by definition of morphisms in the category $\mathcal{I}$ we have $\varphi_{i'} \circ \alpha = \varphi_{i'} \circ \beta$ we see that we get an induced map $\psi : \mathcal{G} \to \mathcal{F}$. Hence again the pair $(\mathcal{G}, \psi)$ is isomorphic to the pair $(\mathcal{F}_{i''}, \varphi_{i''})$ for some $i''$. Hence we see that there exists a morphism $i' \to i''$ in $\mathcal{I}$ which equalizes $\alpha$ and $\beta$. Thus we have shown that the category $\mathcal{I}$ is filtered.

We still have to show that the colimit of the diagram is $\mathcal{F}$. By definition of the colimit, and by our definition of the category $\mathcal{I}$ there is a canonical map

$$\varphi : \operatorname{colim}_i \mathcal{F}_i \longrightarrow \mathcal{F}.$$

Pick $x \in X$. Let us show that $\varphi_x$ is an isomorphism. Recall that

$$(\operatorname{colim}_i \mathcal{F}_i)_x = \operatorname{colim}_i \mathcal{F}_{i,x},$$

see Sheaves, Section 29. First we show that the map $\varphi_x$ is injective. Suppose that $s \in \mathcal{F}_{i,x}$ is an element such that $s$ maps to zero in $\mathcal{F}_x$. Then there exists a quasi-compact open $U$ such that $s$ comes from $s \in \mathcal{F}_i(U)$ and such that $\varphi_i(s) = 0$ in $\mathcal{F}(U)$. By Lemma 22.2 we can find a finite type quasi-coherent subsheaf $\mathcal{K} \subset \text{Ker}(\varphi_i)$ which restricts to the quasi-coherent $\mathcal{O}_U$-submodule of $\mathcal{F}_i$ generated by $s$: $\mathcal{K}|_U = \mathcal{O}_U \cdot s \subset \mathcal{F}_i|_U$. Clearly, $\mathcal{F}_i/\mathcal{K}$ is of finite presentation and the map $\varphi_i$ factors through the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Hence we can find an $i' \in I$ and a morphism $\alpha : \mathcal{F}_i \to \mathcal{F}_{i'}$ in $\mathcal{I}$ which can be identified with the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Then it follows that the section $s$ maps to zero in $\mathcal{F}_{i'}(U)$ and in particular in $(\text{colim}_i \mathcal{F}_i)_x = \text{colim}_i \mathcal{F}_{i,x}$. The injectivity follows. Finally, we show that the map $\varphi_x$ is surjective. Pick $s \in \mathcal{F}_x$. Choose a quasi-compact open neighbourhood $U \subset X$ of $x$ such that $s$ corresponds to a section $s \in \mathcal{F}(U)$. Consider the map $s : \mathcal{O}_U \to \mathcal{F}$ (multiplication by $s$). By Lemma 22.4 there exists an $\mathcal{O}_X$-module $\mathcal{G}$ of finite presentation and an $\mathcal{O}_X$-module map $\mathcal{G} \to \mathcal{F}$ such that $\mathcal{G}|_U \to \mathcal{F}|_U$ is identified with $s : \mathcal{O}_U \to \mathcal{F}$. Again by definition of $\mathcal{I}$ there exists an $i \in I$ such that $\mathcal{G} \to \mathcal{F}$ is isomorphic to $\varphi_i : \mathcal{F}_i \to \mathcal{F}$. Clearly there exists a section $s' \in \mathcal{F}_i(U)$ mapping to $s \in \mathcal{F}(U)$. This proves surjectivity and the proof of the lemma is complete. $\square$

**Lemma 22.7.** *Let $X$ be a scheme. Assume $X$ is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist*

1. *a directed set $I$ (see Categories, Definition 21.1),*
2. *a system $(\mathcal{F}_i, \varphi_{ii'})$ over $I$ in $Mod(\mathcal{O}_X)$ (see Categories, Definition 21.2)*
3. *morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$*

*such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms $\varphi_i$ induce an isomorphism*

$$\text{colim}_i \mathcal{F}_i = \mathcal{F}.$$

**Proof.** This is a direct consequence of Lemma 22.6 and Categories, Lemma 21.5 (combined with the fact that colimits exist in the category of sheaves of $\mathcal{O}_X$-modules, see Sheaves, Section 29). $\square$

**Lemma 22.8.** *Let $X$ be a scheme. Assume $X$ is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then we can write $\mathcal{F} = \text{colim}\, \mathcal{F}_i$ with $\mathcal{F}_i$ of finite presentation and all transition maps $\mathcal{F}_i \to \mathcal{F}_{i'}$ surjective.*

**Proof.** Write $\mathcal{F} = \text{colim}\, \mathcal{G}_i$ as a filtered colimit of finitely presented $\mathcal{O}_X$-modules (Lemma 22.7). We claim that $\mathcal{G}_i \to \mathcal{F}$ is surjective for some $i$. Namely, choose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$. Choose sections $s_{jl} \in \mathcal{F}(U_j)$ generating $\mathcal{F}|_{U_j}$, see Lemma 16.1. By Sheaves, Lemma 29.1 we see that $s_{jl}$ is in the image of $\mathcal{G}_i \to \mathcal{F}$ for $i$ large enough. Hence $\mathcal{G}_i \to \mathcal{F}$ is surjective for $i$ large enough. Choose such an $i$ and let $\mathcal{K} \subset \mathcal{G}_i$ be the kernel of the map $\mathcal{G}_i \to \mathcal{F}$. Write $\mathcal{K} = \text{colim}\, \mathcal{K}_a$ as the filtered colimit of its finite type quasi-coherent submodules (Lemma 22.3). Then $\mathcal{F} = \text{colim}\, \mathcal{G}_i/\mathcal{K}_a$ is a solution to the problem posed by the lemma. $\square$

**Lemma 22.9.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be a quasi-compact open such that $\mathcal{F}|_U$ is of finite presentation. Then there exists a map of $\mathcal{O}_X$-modules $\varphi : \mathcal{G} \to \mathcal{F}$ with (a) $\mathcal{G}$ of finite presentation, (b) $\varphi$ is surjective, and (c) $\varphi|_U$ is an isomorphism.*

**Proof.** Write $\mathcal{F} = \text{colim}\, \mathcal{F}_i$ as a directed colimit with each $\mathcal{F}_i$ of finite presentation, see Lemma 22.7. Choose a finite affine open covering $X = \bigcup V_j$ and choose finitely many sections $s_{jl} \in \mathcal{F}(V_j)$ generating $\mathcal{F}|_{V_j}$, see Lemma 16.1. By Sheaves, Lemma 29.1 we see that $s_{jl}$ is in the image of $\mathcal{F}_i \to \mathcal{F}$ for $i$ large enough. Hence $\mathcal{F}_i \to \mathcal{F}$ is surjective for $i$ large enough. Choose such an $i$ and let $\mathcal{K} \subset \mathcal{F}_i$ be the kernel of the map $\mathcal{F}_i \to \mathcal{F}$. Since $\mathcal{F}_U$ is of finite presentation, we see that $\mathcal{K}|_U$ is of finite type, see Modules, Lemma 11.3. Hence we can find a finite type quasi-coherent submodule $\mathcal{K}' \subset \mathcal{K}$ with $\mathcal{K}'|_U = \mathcal{K}|_U$, see Lemma 22.2. Then $\mathcal{G} = \mathcal{F}_i/\mathcal{K}'$ with the given map $\mathcal{G} \to \mathcal{F}$ is a solution. $\square$

Let $X$ be a scheme. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite presentation*. This means that for every affine open $\text{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where $A$ is a (commutative) $R$-algebra which is of finite presentation as an $R$-algebra.

**Lemma 22.10.** *Let $X$ be a scheme. Assume $X$ is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. There exist*

1. *a directed set $I$ (see Categories, Definition 21.1),*
2. *a system $(\mathcal{A}_i, \varphi_{ii'})$ over $I$ in the category of $\mathcal{O}_X$-algebras,*
3. *morphisms of $\mathcal{O}_X$-algebras $\varphi_i : \mathcal{A}_i \to \mathcal{A}$*

*such that each $\mathcal{A}_i$ is a quasi-coherent $\mathcal{O}_X$-algebra of finite presentation and such that the morphisms $\varphi_i$ induce an isomorphism*

$$\text{colim}_i\, \mathcal{A}_i = \mathcal{A}.$$

**Proof.** First we write $\mathcal{A} = \text{colim}_i\, \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 22.7. For each $i$ let $\mathcal{B}_i = \text{Sym}(\mathcal{F}_i)$ be the symmetric algebra on $\mathcal{F}_i$ over $\mathcal{O}_X$. Write $\mathcal{I}_i = \text{Ker}(\mathcal{B}_i \to \mathcal{A})$. Write $\mathcal{I}_i = \text{colim}_j\, \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of $\mathcal{I}_i$, see Lemma 22.3. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the $\mathcal{B}_i$-ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i/\mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented $\mathcal{O}_X$-algebra. Define $(i,j) \leq (i',j')$ if $i \leq i'$ and the map $\mathcal{B}_i \to \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = \text{colim}_{i,j}\, \mathcal{A}_{i,j}$. $\square$

Let $X$ be a scheme. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type*. This means that for every affine open $\text{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where $A$ is a (commutative) $R$-algebra which is of finite type as an $R$-algebra.

**Lemma 22.11.** *Let $X$ be a scheme. Assume $X$ is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is the directed colimit of its finite type quasi-coherent $\mathcal{O}_X$-subalgebras.*

**Proof.** If $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ are quasi-coherent $\mathcal{O}_X$-subalgebras of finite type, then the image of $\mathcal{A}_1 \otimes_{\mathcal{O}_X} \mathcal{A}_2 \to \mathcal{A}$ is also a quasi-coherent $\mathcal{O}_X$-subalgebra of finite type (some details omitted) which contains both $\mathcal{A}_1$ and $\mathcal{A}_2$. In this way we see that the system is directed. To show that $\mathcal{A}$ is the colimit of this system, write $\mathcal{A} = \text{colim}_i\, \mathcal{A}_i$ as a directed colimit of finitely presented quasi-coherent $\mathcal{O}_X$-algebras as in Lemma 22.10. Then the images $\mathcal{A}'_i = \text{Im}(\mathcal{A}_i \to \mathcal{A})$ are quasi-coherent subalgebras of $\mathcal{A}$ of finite type. Since $\mathcal{A}$ is the colimit of these the result follows. $\square$

Let $X$ be a scheme. In the following lemma we use the notion of a *finite (resp. integral) quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$*. This means that for every affine open $\mathrm{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where $A$ is a (commutative) $R$-algebra which is finite (resp. integral) as an $R$-algebra.

**Lemma 22.12.** *Let $X$ be a scheme. Assume $X$ is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a finite quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ is a directed colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras such that all transition maps $\mathcal{A}_{i'} \to \mathcal{A}_i$ are surjective.*

**Proof.** By Lemma 22.8 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ and a surjection $\mathcal{F} \to \mathcal{A}$. Using the algebra structure we obtain a surjection

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow \mathcal{A}$$

Denote $\mathcal{J}$ the kernel. Write $\mathcal{J} = \mathrm{colim}\, \mathcal{E}_i$ as a filtered colimit of finite type $\mathcal{O}_X$-submodules $\mathcal{E}_i$ (Lemma 22.3). Set

$$\mathcal{A}_i = \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F})/(\mathcal{E}_i)$$

where $(\mathcal{E}_i)$ indicates the ideal sheaf generated by the image of $\mathcal{E}_i \to \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F})$. Then each $\mathcal{A}_i$ is a finitely presented $\mathcal{O}_X$-algebra, the transition maps are surjections, and $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$. To finish the proof we still have to show that $\mathcal{A}_i$ is a finite $\mathcal{O}_X$-algebra for $i$ sufficiently large. To do this we choose an affine open covering $X = U_1 \cup \ldots \cup U_m$. Take generators $f_{j,1}, \ldots, f_{j,N_j} \in \Gamma(U_i, \mathcal{F})$. As $\mathcal{A}(U_j)$ is a finite $\mathcal{O}_X(U_j)$-algebra we see that for each $k$ there exists a monic polynomial $P_{j,k} \in \mathcal{O}(U_j)[T]$ such that $P_{j,k}(f_{j,k})$ is zero in $\mathcal{A}(U_j)$. Since $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ by construction, we have $P_{j,k}(f_{j,k}) = 0$ in $\mathcal{A}_i(U_j)$ for all sufficiently large $i$. For such $i$ the algebras $\mathcal{A}_i$ are finite. $\square$

**Lemma 22.13.** *Let $X$ be a scheme. Assume $X$ is quasi-compact and quasi-separated. Let $\mathcal{A}$ be an integral quasi-coherent $\mathcal{O}_X$-algebra. Then*

1. *$\mathcal{A}$ is the directed colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras, and*
2. *$\mathcal{A}$ is a direct colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras.*

**Proof.** By Lemma 22.11 we have $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ where $\mathcal{A}_i \subset \mathcal{A}$ runs through the quasi-coherent $\mathcal{O}_X$-algebras of finite type. Any finite type quasi-coherent $\mathcal{O}_X$-subalgebra of $\mathcal{A}$ is finite (apply Algebra, Lemma 36.5 to $\mathcal{A}_i(U) \subset \mathcal{A}(U)$ for affine opens $U$ in $X$). This proves (1).

To prove (2), write $\mathcal{A} = \mathrm{colim}\, \mathcal{F}_i$ as a colimit of finitely presented $\mathcal{O}_X$-modules using Lemma 22.7. For each $i$, let $\mathcal{J}_i$ be the kernel of the map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \to \mathcal{J}_{i'}$ and we have $\mathcal{A} = \mathrm{colim}\, \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent $\mathcal{O}_X$-algebras $\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ are finite (see above). Write $\mathcal{J}_i = \mathrm{colim}\, \mathcal{E}_{ik}$ as a colimit of finitely presented $\mathcal{O}_X$-modules. Given $i' \geq i$ and $k$ there exists a $k'$ such that we have a map $\mathcal{E}_{ik} \to \mathcal{E}_{i'k'}$ making

$$\begin{tikzcd}
\mathcal{J}_i \arrow[r] & \mathcal{J}_{i'} \\
\mathcal{E}_{ik} \arrow[r] \arrow[u] & \mathcal{E}_{i'k'} \arrow[u]
\end{tikzcd}$$

commute. This follows from Modules, Lemma 22.8. This induces a map

$$\mathcal{A}_{ik} = \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where $(\mathcal{E}_{ik})$ denotes the ideal generated by $\mathcal{E}_{ik}$. The quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_{ki}$ are of finite presentation and finite for $k$ large enough (see proof of Lemma 22.12). Finally, we have

$$\mathrm{colim}\, \mathcal{A}_{ik} = \mathrm{colim}\, \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 22.12 and the second equality because $\mathcal{A}$ is the colimit of the modules $\mathcal{F}_i$. □

## 23. Gabber's result

In this section we prove a result of Gabber which guarantees that on every scheme there exists a cardinal $\kappa$ such that every quasi-coherent module $\mathcal{F}$ is the union of its quasi-coherent $\kappa$-generated subsheaves. It follows that the category of quasi-coherent sheaves on a scheme is a Grothendieck abelian category having limits and enough injectives[2].

**Definition 23.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\kappa$ be an infinite cardinal. We say a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *$\kappa$-generated* if there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is generated by a subset $R_i \subset \mathcal{F}(U_i)$ whose cardinality is at most $\kappa$.

Note that a direct sum of at most $\kappa$ $\kappa$-generated modules is again $\kappa$-generated because $\kappa \otimes \kappa = \kappa$, see Sets, Section 6. In particular this holds for the direct sum of two $\kappa$-generated modules. Moreover, a quotient of a $\kappa$-generated sheaf is $\kappa$-generated. (But the same needn't be true for submodules.)

**Lemma 23.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\kappa$ be a cardinal. There exists a set $T$ and a family $(\mathcal{F}_t)_{t \in T}$ of $\kappa$-generated $\mathcal{O}_X$-modules such that every $\kappa$-generated $\mathcal{O}_X$-module is isomorphic to one of the $\mathcal{F}_t$.*

**Proof.** There is a set of coverings of $X$ (provided we disallow repeats). Suppose $X = \bigcup U_i$ is a covering and suppose $\mathcal{F}_i$ is an $\mathcal{O}_{U_i}$-module. Then there is a set of isomorphism classes of $\mathcal{O}_X$-modules $\mathcal{F}$ with the property that $\mathcal{F}|_{U_i} \cong \mathcal{F}_i$ since there is a set of glueing maps. This reduces us to proving there is a set of (isomorphism classes of) quotients $\oplus_{k \in \kappa} \mathcal{O}_X \to \mathcal{F}$ for any ringed space $X$. This is clear. □

Here is the result the title of this section refers to.

**Lemma 23.3.** *Let $X$ be a scheme. There exists a cardinal $\kappa$ such that every quasi-coherent module $\mathcal{F}$ is the directed colimit of its quasi-coherent $\kappa$-generated submodules.*

**Proof.** Choose an affine open covering $X = \bigcup_{i \in I} U_i$. For each pair $i, j$ choose an affine open covering $U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk}$. Write $U_i = \mathrm{Spec}(A_i)$ and $U_{ijk} = \mathrm{Spec}(A_{ijk})$. Let $\kappa$ be any infinite cardinal $\geq$ than the cardinality of any of the sets $I$, $I_{ij}$.

Let $\mathcal{F}$ be a quasi-coherent sheaf. Set $M_i = \mathcal{F}(U_i)$ and $M_{ijk} = \mathcal{F}(U_{ijk})$. Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}.$$

[2]Nicely explained in a blog post by Akhil Mathew.

see Schemes, Lemma 7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every $i, j \in I$, $k \in I_{ij}$ and $m \in M_i$ a finite subset $S(i, j, k, m) \subset M_j$ such that we have

$$m \otimes 1 = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes a_{m'}$$

in $M_{ijk}$ for some $a_{m'} \in A_{ijk}$. Moreover, let's agree that $S(i, i, k, m) = \{m\}$ for all $i, j = i, k, m$ as above. Fix such a map.

Given a family $\mathcal{S} = (S_i)_{i \in I}$ of subsets $S_i \subset M_i$ of cardinality at most $\kappa$ we set $\mathcal{S}' = (S'_i)$ where

$$S'_j = \bigcup\nolimits_{(i,k,m) \text{ such that } m \in S_i} S(i, j, k, m)$$

Note that $S_i \subset S'_i$. Note that $S'_i$ has cardinality at most $\kappa$ because it is a union over a set of cardinality at most $\kappa$ of finite sets. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $\mathcal{S}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{S}^{(n)}$. Writing $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$ we see that for any element $m \in S_i^{(\infty)}$ the image of $m$ in $M_{ijk}$ can be written as a finite sum $\sum m' \otimes a_{m'}$ with $m' \in S_j^{(\infty)}$. In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk}.$$

as submodules of $M_{ijk}$. Thus there exists a quasi-coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{G}(U_i) = N_i$. Moreover, by construction the sheaf $\mathcal{G}$ is $\kappa$-generated.

Let $\{\mathcal{G}_t\}_{t \in T}$ be the set of $\kappa$-generated quasi-coherent subsheaves. If $t, t' \in T$ then $\mathcal{G}_t + \mathcal{G}_{t'}$ is also a $\kappa$-generated quasi-coherent subsheaf as it is the image of the map $\mathcal{G}_t \oplus \mathcal{G}_{t'} \to \mathcal{F}$. Hence the system (ordered by inclusion) is directed. The arguments above show that every section of $\mathcal{F}$ over $U_i$ is in one of the $\mathcal{G}_t$ (because we can start with $\mathcal{S}$ such that the given section is an element of $S_i$). Hence $\operatorname{colim}_t \mathcal{G}_t \to \mathcal{F}$ is both injective and surjective as desired. $\square$

**Proposition 23.4.** *Let $X$ be a scheme.*

1. *The category $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category. Consequently, $QCoh(\mathcal{O}_X)$ has enough injectives and all limits.*
2. *The inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ has a right adjoint*[3]

$$Q : Mod(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

*such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.*

**Proof.** Part (1) means $QCoh(\mathcal{O}_X)$ (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 10. By Schemes, Section 24 colimits in $QCoh(\mathcal{O}_X)$ exist and agree with colimits in $Mod(\mathcal{O}_X)$. By Modules, Lemma 3.2 filtered colimits are exact. Hence (a) and (b) hold. To construct a generator $U$, pick a cardinal $\kappa$ as in Lemma 23.3. Pick a collection $(\mathcal{F}_t)_{t \in T}$ of $\kappa$-generated quasi-coherent sheaves as in Lemma 23.2. Set $U = \bigoplus_{t \in T} \mathcal{F}_t$. Since every object of $QCoh(\mathcal{O}_X)$ is a filtered colimit of $\kappa$-generated quasi-coherent modules, i.e.,

[3] This functor is sometimes called the *coherator*.

of objects isomorphic to $\mathcal{F}_t$, it is clear that $U$ is a generator. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 11.7 and Lemma 13.2.

Proof of (2). To construct $Q$ we use the following general procedure. Given an object $\mathcal{F}$ of $\mathit{Mod}(\mathcal{O}_X)$ we consider the functor

$$QCoh(\mathcal{O}_X)^{opp} \longrightarrow Sets, \quad \mathcal{G} \longmapsto \mathrm{Hom}_X(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 13.1. Thus there exists a quasi-coherent sheaf $Q(\mathcal{F})$ and a functorial isomorphism $\mathrm{Hom}_X(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_X(\mathcal{G}, Q(\mathcal{F}))$ for $\mathcal{G}$ in $QCoh(\mathcal{O}_X)$. By the Yoneda lemma (Categories, Lemma 3.5) the construction $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$ is functorial in $\mathcal{F}$. By construction $Q$ is a right adjoint to the inclusion functor. The fact that $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism when $\mathcal{F}$ is quasi-coherent is a formal consequence of the fact that the inclusion functor $QCoh(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$ is fully faithful. □

## 24. Sections with support in a closed subset

Given any topological space $X$, a closed subset $Z \subset X$, and an abelian sheaf $\mathcal{F}$ you can take the subsheaf of sections whose support is contained in $Z$. If $X$ is a scheme, $Z$ a closed subscheme, and $\mathcal{F}$ a quasi-coherent module there is a variant where you take sections which are scheme theoretically supported on $Z$. However, in the scheme setting you have to be careful because the resulting $\mathcal{O}_X$-module may not be quasi-coherent.

**Lemma 24.1.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be an open subscheme. The following are equivalent:*

1. *$U$ is retrocompact in $X$,*
2. *$U$ is quasi-compact,*
3. *$U$ is a finite union of affine opens, and*
4. *there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $X \setminus U = V(\mathcal{I})$ (set theoretically).*

**Proof.** The equivalence of (1), (2), and (3) follows from Lemma 2.3. Assume (1), (2), (3). Let $T = X \setminus U$. By Schemes, Lemma 12.4 there exists a unique quasi-coherent sheaf of ideals $\mathcal{J}$ cutting out the reduced induced closed subscheme structure on $T$. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an $\mathcal{O}_U$-modules of finite type. By Lemma 22.2 there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. Then $X \setminus U = V(\mathcal{I})$ and we obtain (4). Conversely, if $\mathcal{I}$ is as in (4) and $W = \mathrm{Spec}(R) \subset X$ is an affine open, then $\mathcal{I}|_W = \widetilde{I}$ for some finitely generated ideal $I \subset R$, see Lemma 16.1. It follows that $U \cap W = \mathrm{Spec}(R) \setminus V(I)$ is quasi-compact, see Algebra, Lemma 29.1. Hence $U \subset X$ is retrocompact by Lemma 2.6. □

**Lemma 24.2.** *Let $X$ be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every open $U \subset X$*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

*Assume $\mathcal{I}$ is of finite type. Then*

1. *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules,*
2. *on any affine open $U \subset X$ we have $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, and*

(3) $\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.

**Proof.** It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$ (the sheaf condition is easy to verify). Hence we may work locally on $X$ to verify the other statements. In other words we may assume that $X = \mathrm{Spec}(A)$, $\mathcal{F} = \widetilde{M}$ and $\mathcal{I} = \widetilde{I}$. It is clear that in this case $\mathcal{F}'(U) = \{x \in M \mid Ix = 0\} =: M'$ because $\widetilde{I}$ is generated by its global sections $I$ which proves (2). To show $\mathcal{F}'$ is quasi-coherent it suffices to show that for every $f \in A$ we have $\{x \in M_f \mid I_f x = 0\} = (M')_f$. Write $I = (g_1, \ldots, g_t)$, which is possible because $\mathcal{I}$ is of finite type, see Lemma 16.1. If $x = y/f^n$ and $I_f x = 0$, then that means that for every $i$ there exists an $m \geq 0$ such that $f^m g_i x = 0$. We may choose one $m$ which works for all $i$ (and this is where we use that $I$ is finitely generated). Then we see that $f^m x \in M'$ and $x/f^n = f^m x/f^{n+m}$ in $(M')_f$ as desired. The proof of (3) is similar and omitted. □

**Definition 24.3.** Let $X$ be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 24.2 above is called the *subsheaf of sections annihilated by* $\mathcal{I}$.

**Lemma 24.4.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections annihilated by $f^{-1}\mathcal{I}\mathcal{O}_X$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections annihilated by $\mathcal{I}$.*

**Proof.** Omitted. (Hint: The assumption that $f$ is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent so that Lemma 24.2 applies to $\mathcal{I}$ and $f_*\mathcal{F}$.) □

For an abelian sheaf on a topological space we have discussed the subsheaf of sections with support in a closed subset in Modules, Remark 6.2. For quasi-coherent modules this submodule isn't always a quasi-coherent module, but if the closed subset has a retrocompact complement, then it is.

**Lemma 24.5.** *Let $X$ be a scheme. Let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every open $U \subset X$*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \textit{the support of } s \textit{ is contained in } Z \cap U\}$$

*If $X \setminus Z$ is a retrocompact open of $X$, then*

1. *for an affine open $U \subset X$ there exist a finitely generated ideal $I \subset \mathcal{O}_X(U)$ such that $Z \cap U = V(I)$,*
2. *for $U$ and $I$ as in (1) we have $\mathcal{F}'(U) = \{x \in \mathcal{F}(U) \mid I^n x = 0 \textit{ for some } n\}$,*
3. *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

**Proof.** Part (1) is Algebra, Lemma 29.1. Let $U = \mathrm{Spec}(A)$ and $I$ be as in (1). Then $\mathcal{F}|_U$ is the quasi-coherent sheaf associated to some $A$-module $M$. We have

$$\mathcal{F}'(U) = \{x \in M \mid x = 0 \text{ in } M_\mathfrak{p} \text{ for all } \mathfrak{p} \notin Z\}.$$

by Modules, Definition 5.1. Thus $x \in \mathcal{F}'(U)$ if and only if $V(\mathrm{Ann}(x)) \subset V(I)$, see Algebra, Lemma 40.7. Since $I$ is finitely generated this is equivalent to $I^n x = 0$ for some $n$. This proves (2).

Proof of (3). Observe that given $U \subset X$ open there is an exact sequence

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}(U \setminus Z)$$

If we denote $j : X \setminus Z \to X$ the inclusion morphism, then we observe that $\mathcal{F}(U \setminus Z)$ is the sections of the module $j_*(\mathcal{F}|_{X\setminus Z})$ over $U$. Thus we have an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to j_*(\mathcal{F}|_{X\setminus Z})$$

The restriction $\mathcal{F}|_{X\setminus Z}$ is quasi-coherent. Hence $j_*(\mathcal{F}|_{X\setminus Z})$ is quasi-coherent by Schemes, Lemma 24.1 and our assumption that $j$ is quasi-compact (any open immersion is separated). Hence $\mathcal{F}'$ is quasi-coherent as a kernel of a map of quasi-coherent modules, see Schemes, Section 24. □

**Definition 24.6.** Let $X$ be a scheme. Let $T \subset X$ be a closed subset whose complement is retrocompact in $X$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The quasi-coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 24.5 is called the *subsheaf of sections supported on $T$*.

**Lemma 24.7.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. Let $Z \subset Y$ be a closed subset such that $Y \setminus Z$ is retrocompact in $Y$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections supported in $f^{-1}Z$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections supported in $Z$.*

**Proof.** Omitted. (Hint: First show that $X \setminus f^{-1}Z$ is retrocompact in $X$ as $Y \setminus Z$ is retrocompact in $Y$. Hence Lemma 24.5 applies to $f^{-1}Z$ and $\mathcal{F}$. As $f$ is quasi-compact and quasi-separated we see that $f_*\mathcal{F}$ is quasi-coherent. Hence Lemma 24.5 applies to $Z$ and $f_*\mathcal{F}$. Finally, match the sheaves directly.) □

## 25. Sections of quasi-coherent sheaves

Here is a computation of sections of a quasi-coherent sheaf on a quasi-compact open of an affine spectrum.

**Lemma 25.1.** *Let $A$ be a ring. Let $I \subset A$ be a finitely generated ideal. Let $M$ be an $A$-module. Then there is a canonical map*

$$\mathrm{colim}_n \mathrm{Hom}_A(I^n, M) \longrightarrow \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}).$$

*This map is always injective. If for all $x \in M$ we have $Ix = 0 \Rightarrow x = 0$ then this map is an isomorphism. In general, set $M_n = \{x \in M \mid I^n x = 0\}$, then there is an isomorphism*

$$\mathrm{colim}_n \mathrm{Hom}_A(I^n, M/M_n) \longrightarrow \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}).$$

**Proof.** Since $I^{n+1} \subset I^n$ and $M_n \subset M_{n+1}$ we can use composition via these maps to get canonical maps of $A$-modules

$$\mathrm{Hom}_A(I^n, M) \longrightarrow \mathrm{Hom}_A(I^{n+1}, M)$$

and

$$\mathrm{Hom}_A(I^n, M/M_n) \longrightarrow \mathrm{Hom}_A(I^{n+1}, M/M_{n+1})$$

which we will use as the transition maps in the systems. Given an $A$-module map $\varphi : I^n \to M$, then we get a map of sheaves $\widetilde{\varphi} : \widetilde{I^n} \to \widetilde{M}$ which we can restrict to the open $\mathrm{Spec}(A) \setminus V(I)$. Since $\widetilde{I^n}$ restricted to this open gives the structure sheaf we get an element of $\Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. We omit the verification that this is compatible with the transition maps in the system $\mathrm{Hom}_A(I^n, M)$. This gives the first arrow. To get the second arrow we note that $\widetilde{M}$ and $\widetilde{M/M_n}$ agree over the open $\mathrm{Spec}(A) \setminus V(I)$ since the sheaf $\widetilde{M_n}$ is clearly supported on $V(I)$. Hence we can use the same mechanism as before.

Next, we work out how to define this arrow in terms of algebra. Say $I = (f_1, \ldots, f_t)$. Then $\mathrm{Spec}(A) \setminus V(I) = \bigcup_{i=1,\ldots,t} D(f_i)$. Hence

$$0 \to \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}) \to \bigoplus\nolimits_i M_{f_i} \to \bigoplus\nolimits_{i,j} M_{f_i f_j}$$

is exact. Suppose that $\varphi : I^n \to M$ is an $A$-module map. Consider the vector of elements $\varphi(f_i^n)/f_i^n \in M_{f_i}$. It is easy to see that this vector maps to zero in the second direct sum of the exact sequence above. Whence an element of $\Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. We omit the verification that this description agrees with the one given above.

Let us show that the first arrow is injective using this description. Namely, if $\varphi$ maps to zero, then for each $i$ the element $\varphi(f_i^n)/f_i^n$ is zero in $M_{f_i}$. In other words we see that for each $i$ we have $f_i^m \varphi(f_i^n) = 0$ for some $m \geq 0$. We may choose a single $m$ which works for all $i$. Then we see that $\varphi(f_i^{n+m}) = 0$ for all $i$. It is easy to see that this means that $\varphi|_{I^{t(n+m-1)+1}} = 0$ in other words that $\varphi$ maps to zero in the $t(n+m-1)+1$st term of the colimit. Hence injectivity follows.

Note that each $M_n = 0$ in case we have $Ix = 0 \Rightarrow x = 0$ for $x \in M$. Thus to finish the proof of the lemma it suffices to show that the second arrow is an isomorphism.

Let us attempt to construct an inverse of the second map of the lemma. Let $s \in \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. This corresponds to a vector $x_i/f_i^n$ with $x_i \in M$ of the first direct sum of the exact sequence above. Hence for each $i, j$ there exists $m \geq 0$ such that $f_i^m f_j^m (f_j^n x_i - f_i^n x_j) = 0$ in $M$. We may choose a single $m$ which works for all pairs $i, j$. After replacing $x_i$ by $f_i^m x_i$ and $n$ by $n + m$ we see that we get $f_j^n x_i = f_i^n x_j$ in $M$ for all $i, j$. Let us introduce

$$K_n = \{x \in M \mid f_1^n x = \ldots = f_t^n x = 0\}$$

We claim there is an $A$-module map

$$\varphi : I^{t(n-1)+1} \longrightarrow M/K_n$$

which maps the monomial $f_1^{e_1} \ldots f_t^{e_t}$ with $\sum e_i = t(n-1)+1$ to the class modulo $K_n$ of the expression $f_1^{e_1} \ldots f_i^{e_i - n} \ldots f_t^{e_t} x_i$ where $i$ is chosen such that $e_i \geq n$ (note that there is at least one such $i$). To see that this is indeed the case suppose that

$$\sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} = 0$$

is a relation between the monomials with coefficients $a_E$ in $A$. Then we would map this to

$$z = \sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_{i(E)}^{e_{i(E)}-n} \ldots f_t^{e_t} x_{i(E)}$$

where for each multiindex $E$ we have chosen a particular $i(E)$ such that $e_{i(E)} \geq n$. Note that if we multiply this by $f_j^n$ for any $j$, then we get zero, since by the relations $f_j^n x_i = f_i^n x_j$ above we get

$$\begin{aligned} f_j^n z &= \sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_j^{e_j+n} \ldots f_{i(E)}^{e_{i(E)}-n} \ldots f_t^{e_t} x_{i(E)} \\ &= \sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} x_j = 0. \end{aligned}$$

Hence $z \in K_n$ and we see that every relation gets mapped to zero in $M/K_n$. This proves the claim.

Note that $K_n \subset M_{t(n-1)+1}$. Hence the map $\varphi$ in particular gives rise to an $A$-module map $I^{t(n-1)+1} \to M/M_{t(n-1)+1}$. This proves the second arrow of the lemma is surjective. We omit the proof of injectivity. $\square$

**Example 25.2.** We will give two examples showing that the first displayed map of Lemma 25.1 is not an isomorphism.

Let $k$ be a field. Consider the ring

$$A = k[x, y, z_1, z_2, \ldots]/(x^n z_n).$$

Set $I = (x)$ and let $M = A$. Then the element $y/x$ defines a section of the structure sheaf of $\text{Spec}(A)$ over $D(x) = \text{Spec}(A) \setminus V(I)$. We claim that $y/x$ is not in the image of the canonical map $\text{colim}\,\text{Hom}_A(I^n, A) \to A_x = \mathcal{O}(D(x))$. Namely, if so it would come from a homomorphism $\varphi : I^n \to A$ for some $n$. Set $a = \varphi(x^n)$. Then we would have $x^m(xa - x^n y) = 0$ for some $m > 0$. This would mean that $x^{m+1}a = x^{m+n}y$. This would mean that $\varphi(x^{n+m+1}) = x^{m+n}y$. This leads to a contradiction because it would imply that

$$0 = \varphi(0) = \varphi(z_{n+m+1}x^{n+m+1}) = x^{m+n}yz_{n+m+1}$$

which is not true in the ring $A$.

Let $k$ be a field. Consider the ring

$$A = k[f, g, x, y, \{a_n, b_n\}_{n \geq 1}]/(fy - gx, \{a_n f^n + b_n g^n\}_{n \geq 1}).$$

Set $I = (f, g)$ and let $M = A$. Then $x/f \in A_f$ and $y/g \in A_g$ map to the same element of $A_{fg}$. Hence these define a section $s$ of the structure sheaf of $\text{Spec}(A)$ over $D(f) \cup D(g) = \text{Spec}(A) \setminus V(I)$. However, there is no $n \geq 0$ such that $s$ comes from an $A$-module map $\varphi : I^n \to A$ as in the source of the first displayed arrow of Lemma 25.1. Namely, given such a module map set $x_n = \varphi(f^n)$ and $y_n = \varphi(g^n)$. Then $f^m x_n = f^{n+m-1}x$ and $g^m y_n = g^{n+m-1}y$ for some $m \geq 0$ (see proof of the lemma). But then we would have $0 = \varphi(0) = \varphi(a_{n+m}f^{n+m} + b_{n+m}g^{n+m}) = a_{n+m}f^{n+m-1}x + b_{n+m}g^{n+m-1}y$ which is not the case in the ring $A$.

We will improve on the following lemma in the Noetherian case, see Cohomology of Schemes, Lemma 10.5.

**Lemma 25.3.** *Let $X$ be a quasi-compact scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $Z \subset X$ be the closed subscheme defined by $\mathcal{I}$ and set $U = X \setminus Z$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The canonical map*

$$\text{colim}_n \text{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F})$$

*is injective. Assume further that $X$ is quasi-separated. Let $\mathcal{F}_n \subset \mathcal{F}$ be subsheaf of sections annihilated by $\mathcal{I}^n$. The canonical map*

$$\text{colim}_n \text{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}/\mathcal{F}_n) \longrightarrow \Gamma(U, \mathcal{F})$$

*is an isomorphism.*

**Proof.** Let $\text{Spec}(A) = W \subset X$ be an affine open. Write $\mathcal{F}|_W = \widetilde{M}$ for some $A$-module $M$ and $\mathcal{I}|_W = \widetilde{I}$ for some finite type ideal $I \subset A$. Restricting the first displayed map of the lemma to $W$ we obtain the first displayed map of Lemma

25.1. Since we can cover $X$ by a finite number of affine opens this proves the first displayed map of the lemma is injective.

We have $\mathcal{F}_n|_W = \widetilde{M_n}$ where $M_n \subset M$ is defined as in Lemma 25.1 (details omitted). The lemma guarantees that we have a bijection

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_W}(\mathcal{I}^n|_W, (\mathcal{F}/\mathcal{F}_n)|_W) \longrightarrow \Gamma(U \cap W, \mathcal{F})$$

for any such affine open $W$.

To see the second displayed arrow of the lemma is bijective, we choose a finite affine open covering $X = \bigcup_{j=1,\ldots,m} W_j$. The injectivity follows immediately from the above and the finiteness of the covering. If $X$ is quasi-separated, then for each pair $j, j'$ we choose a finite affine open covering

$$W_j \cap W_{j'} = \bigcup\nolimits_{k=1,\ldots,m_{jj'}} W_{jj'k}.$$

Let $s \in \Gamma(U, \mathcal{F})$. As seen above for each $j$ there exists an $n_j$ and a map $\varphi_j : \mathcal{I}^{n_j}|_{W_j} \to (\mathcal{F}/\mathcal{F}_{n_j})|_{W_j}$ which corresponds to $s|_{U\cap W_j}$. By the same token for each triple $(j, j', k)$ there exists an integer $n_{jj'k}$ such that the restriction of $\varphi_j$ and $\varphi_{j'}$ as maps $\mathcal{I}^{n_{jj'k}} \to \mathcal{F}/\mathcal{F}_{n_{jj'k}}$ agree over $W_{jj'k}$. Let $n = \max\{n_j, n_{jj'k}\}$ and we see that the $\varphi_j$ glue as maps $\mathcal{I}^n \to \mathcal{F}/\mathcal{F}_n$ over $X$. This proves surjectivity of the map. □

## 26. Ample invertible sheaves

Recall from Modules, Lemma 25.10 that given an invertible sheaf $\mathcal{L}$ on a locally ringed space $X$, and given a global section $s$ of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x\mathcal{L}_x\}$ is open. A general remark is that $X_s \cap X_{s'} = X_{ss'}$, where $ss'$ denote the section $s \otimes s' \in \Gamma(X, \mathcal{L} \otimes \mathcal{L}')$.

**Definition 26.1.** Let $X$ be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *ample* if

(1) $X$ is quasi-compact, and
(2) for every $x \in X$ there exists an $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and $X_s$ is affine.

**Lemma 26.2.** *Let $X$ be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $n \geq 1$. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ is ample.*

**Proof.** This follows from the fact that $X_{s^n} = X_s$. □

**Lemma 26.3.** *Let $X$ be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For any closed subscheme $Z \subset X$ the restriction of $\mathcal{L}$ to $Z$ is ample.*

**Proof.** This is clear since a closed subset of a quasi-compact space is quasi-compact and a closed subscheme of an affine scheme is affine (see Schemes, Lemma 8.2). □

**Lemma 26.4.** *Let $X$ be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. For any affine $U \subset X$ the intersection $U \cap X_s$ is affine.*

**Proof.** This translates into the following algebra problem. Let $R$ be a ring. Let $N$ be an invertible $R$-module (i.e., locally free of rank 1). Let $s \in N$ be an element. Then $U = \{\mathfrak{p} \mid s \notin \mathfrak{p}N\}$ is an affine open subset of $\mathrm{Spec}(R)$.

Let $A = \bigoplus_{n\geq 0} A_n$ be the symmetric algebra of $N$ (which is commutative) and view $s$ as an element of $A_1$. Set $B = A/(s-1)A$. This is an $R$-algebra whose construction commutes with any base change $R \to R'$. Thus $B' = B \otimes_R R'$ is the zero ring if $s$ maps to zero in $N' = N \otimes_R R'$. It follows that if $x \in \mathrm{Spec}(R) \setminus U$, then $B \otimes_R \kappa(x) = 0$. We conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ factors through $U$ as the fibres over $x \notin U$ are empty. On the other hand, if $\mathrm{Spec}(R') \subset U$ is an affine open, then $s$ maps to a basis element of $N'$ and we see that $B' = R'[s]/(s-1) \cong R'$. It follows that $\mathrm{Spec}(B) \to U$ is an isomorphism and $U$ is indeed affine. $\square$

**Lemma 26.5.** *Let $X$ be a scheme. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible $\mathcal{O}_X$-modules. If*

1. *$\mathcal{L}$ is ample, and*
2. *the open sets $X_t$ where $t \in \Gamma(X, \mathcal{M}^{\otimes m})$ for $m > 0$ cover $X$,*

*then $\mathcal{L} \otimes \mathcal{M}$ is ample.*

**Proof.** We check the conditions of Definition 26.1. As $\mathcal{L}$ is ample we see that $X$ is quasi-compact. Let $x \in X$. Choose $n \geq 1$, $m \geq 1$, $s \in \Gamma(X, \mathcal{L}^{\otimes n})$, and $t \in \Gamma(X, \mathcal{M}^{\otimes m})$ such that $x \in X_s$, $x \in X_t$ and $X_s$ is affine. Then $s^m t^n \in \Gamma(X, (\mathcal{L} \otimes \mathcal{M})^{\otimes nm})$, $x \in X_{s^m t^n}$, and $X_{s^m t^n}$ is affine by Lemma 26.4. $\square$

**Lemma 26.6.** *Let $X$ be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume the open sets $X_s$, where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, form a basis for the topology on $X$. Then among those opens, the open sets $X_s$ which are affine form a basis for the topology on $X$.*

**Proof.** Let $x \in X$. Choose an affine open neighbourhood $\mathrm{Spec}(R) = U \subset X$ of $x$. By assumption, there exists a $n \geq 1$ and a $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $X_s \subset U$. By Lemma 26.4 above the intersection $X_s = U \cap X_s$ is affine. Since $U$ can be chosen arbitrarily small we win. $\square$

**Lemma 26.7.** *Let $X$ be a scheme and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume for every point $x$ of $X$ there exists $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and $X_s$ is affine. Then $X$ is separated.*

**Proof.** We show first that $X$ is quasi-separated. By assumption we can find a covering of $X$ by affine opens of the form $X_s$. By Lemma 26.4, the intersection of any two such sets is affine, so Schemes, Lemma 21.6 implies that $X$ is quasi-separated.

To show that $X$ is separated, we can use the valuative criterion, Schemes, Lemma 22.2. Thus, let $A$ be a valuation ring with fraction field $K$ and consider two morphisms $f, g : \mathrm{Spec}(A) \to X$ such that the two compositions $\mathrm{Spec}(K) \to \mathrm{Spec}(A) \to X$ agree. As $A$ is local, there exists $p, q \geq 1$, $s \in \Gamma(X, \mathcal{L}^{\otimes p})$, and $t \in \Gamma(X, \mathcal{L}^{\otimes q})$ such that $X_s$ and $X_t$ are affine, $f(\mathrm{Spec}\, A) \subseteq X_s$, and $g(\mathrm{Spec}\, A) \subseteq X_t$. We now replace $s$ by $s^q$, $t$ by $t^p$, and $\mathcal{L}$ by $\mathcal{L}^{\otimes pq}$. This is harmless as $X_s = X_{s^q}$ and $X_t = X_{t^p}$, and now $s$ and $t$ are both sections of the same sheaf $\mathcal{L}$.

The quasi-coherent module $f^*\mathcal{L}$ corresponds to an $A$-module $M$ and $g^*\mathcal{L}$ corresponds to an $A$-module $N$ by our classification of quasi-coherent modules over affine schemes (Schemes, Lemma 7.4). The $A$-modules $M$ and $N$ are locally free of rank 1 (Lemma 20.1) and as $A$ is local they are free (Algebra, Lemma 55.8). Therefore we may identify $M$ and $N$ with $A$-submodules of $M \otimes_A K$ and $N \otimes_A K$. The equality $f|_{\mathrm{Spec}(K)} = g|_{\mathrm{Spec}(K)}$ determines an isomorphism $\phi \colon M \otimes_A K \to N \otimes_A K$.

Let $x \in M$ and $y \in N$ be the elements corresponding to the pullback of $s$ along $f$ and $g$, respectively. These satisfy $\phi(x \otimes 1) = y \otimes 1$. The image of $f$ is contained in

$X_s$, so $x \notin \mathfrak{m}_A M$, that is, $x$ generates $M$. Hence $\phi$ determines an isomorphism of $M$ with the submodule of $N$ generated by $y$. Arguing symmetrically using $t$, $\phi^{-1}$ determines an isomorphism of $N$ with a submodule of $M$. Consequently $\phi$ restricts to an isomorphism of $M$ and $N$. Since $x$ generates $M$, its image $y$ generates $N$, implying $y \notin \mathfrak{m}_A N$. Therefore $g(\mathrm{Spec}(A)) \subseteq X_s$. Because $X_s$ is affine, it is separated by Schemes, Lemma 21.15, and we conclude $f = g$. □

**Lemma 26.8.** *Let $X$ be a scheme. If there exists an ample invertible sheaf on $X$ then $X$ is separated.*

**Proof.** Follows immediately from Lemma 26.7 and Definition 26.1. □

**Lemma 26.9.** *Let $X$ be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. If every point of $X$ is contained in one of the open subschemes $X_s$, for some $s \in S_+$ homogeneous, then there is a canonical morphism of schemes*

$$f : X \longrightarrow Y = Proj(S),$$

*to the homogeneous spectrum of $S$ (see Constructions, Section 8). This morphism has the following properties*

1. *$f^{-1}(D_+(s)) = X_s$ for any $s \in S_+$ homogeneous,*
2. *there are $\mathcal{O}_X$-module maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ compatible with multiplication maps, see Constructions, Equation (10.1.1),*
3. *the composition $S_n \to \Gamma(Y, \mathcal{O}_Y(n)) \to \Gamma(X, \mathcal{L}^{\otimes n})$ is the identity map, and*
4. *for every $x \in X$ there is an integer $d \geq 1$ and an open neighbourhood $U \subset X$ of $x$ such that $f^*\mathcal{O}_Y(dn)|_U \to \mathcal{L}^{\otimes dn}|_U$ is an isomorphism for all $n \in \mathbf{Z}$.*

**Proof.** Denote $\psi : S \to \Gamma_*(X, \mathcal{L})$ the identity map. We are going to use the triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ of Constructions, Lemma 14.1. By assumption the open subscheme $U(\psi)$ of equals $X$. Hence $r_{\mathcal{L},\psi} : U(\psi) \to Y$ is defined on all of $X$. We set $f = r_{\mathcal{L},\psi}$. The maps in part (2) are the components of $\theta$. Part (3) follows from condition (2) in the lemma cited above. Part (1) follows from (3) combined with condition (1) in the lemma cited above. Part (4) follows from the last statement in Constructions, Lemma 14.1 since the map $\alpha$ mentioned there is an isomorphism. □

**Lemma 26.10.** *Let $X$ be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume (a) every point of $X$ is contained in one of the open subschemes $X_s$, for some $s \in S_+$ homogeneous, and (b) $X$ is quasi-compact. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 26.9 above is quasi-compact with dense image.*

**Proof.** To prove $f$ is quasi-compact it suffices to show that $f^{-1}(D_+(s))$ is quasi-compact for any $s \in S_+$ homogeneous. Write $X = \bigcup_{i=1,\ldots,n} X_i$ as a finite union of affine opens. By Lemma 26.4 each intersection $X_s \cap X_i$ is affine. Hence $X_s = \bigcup_{i=1,\ldots,n} X_s \cap X_i$ is quasi-compact. Assume that the image of $f$ is not dense to get a contradiction. Then, since the opens $D_+(s)$ with $s \in S_+$ homogeneous form a basis for the topology on $\mathrm{Proj}(S)$, we can find such an $s$ with $D_+(s) \neq \emptyset$ and $f(X) \cap D_+(s) = \emptyset$. By Lemma 26.9 this means $X_s = \emptyset$. By Lemma 17.2 this means that a power $s^n$ is the zero section of $\mathcal{L}^{\otimes n \deg(s)}$. This in turn means that $D_+(s) = \emptyset$ which is the desired contradiction. □

**Lemma 26.11.** *Let $X$ be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume $\mathcal{L}$ is ample. Then the canonical morphism of schemes $f : X \longrightarrow$ Proj$(S)$ of Lemma 26.9 is an open immersion with dense image.*

**Proof.** By Lemma 26.7 we see that $X$ is quasi-separated. Choose finitely many $s_1, \ldots, s_n \in S_+$ homogeneous such that $X_{s_i}$ are affine, and $X = \bigcup X_{s_i}$. Say $s_i$ has degree $d_i$. The inverse image of $D_+(s_i)$ under $f$ is $X_{s_i}$, see Lemma 26.9. By Lemma 17.2 the ring map

$$(S^{(d_i)})_{(s_i)} = \Gamma(D_+(s_i), \mathcal{O}_{\text{Proj}(S)}) \longrightarrow \Gamma(X_{s_i}, \mathcal{O}_X)$$

is an isomorphism. Hence $f$ induces an isomorphism $X_{s_i} \to D_+(s_i)$. Thus $f$ is an isomorphism of $X$ onto the open subscheme $\bigcup_{i=1,\ldots,n} D_+(s_i)$ of Proj$(S)$. The image is dense by Lemma 26.10. □

**Lemma 26.12.** *Let $X$ be a scheme. Let $S$ be a graded ring. Assume $X$ is quasi-compact, and assume there exists an open immersion*

$$j : X \longrightarrow Y = \mathit{Proj}(S).$$

*Then $j^*\mathcal{O}_Y(d)$ is an invertible ample sheaf for some $d > 0$.*

**Proof.** This is Constructions, Lemma 10.6. □

**Proposition 26.13.** *Let $X$ be a quasi-compact scheme. Let $\mathcal{L}$ be an invertible sheaf on $X$. Set $S = \Gamma_*(X, \mathcal{L})$. The following are equivalent:*

1. *$\mathcal{L}$ is ample,*
2. *the open sets $X_s$, with $s \in S_+$ homogeneous, cover $X$ and the associated morphism $X \to$ Proj$(S)$ is an open immersion,*
3. *the open sets $X_s$, with $s \in S_+$ homogeneous, form a basis for the topology of $X$,*
4. *the open sets $X_s$, with $s \in S_+$ homogeneous, which are affine form a basis for the topology of $X$,*
5. *for every quasi-coherent sheaf $\mathcal{F}$ on $X$ the sum of the images of the canonical maps*
$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$
*with $n \geq 1$ equals $\mathcal{F}$,*
6. *same property as (5) with $\mathcal{F}$ ranging over all quasi-coherent sheaves of ideals,*
7. *$X$ is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on $X$ there exists an integer $n_0$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$,*
8. *$X$ is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on $X$ there exist integers $n > 0$, $k \geq 0$ such that $\mathcal{F}$ is a quotient of a direct sum of $k$ copies of $\mathcal{L}^{\otimes -n}$, and*
9. *same as in (8) with $\mathcal{F}$ ranging over all sheaves of ideals of finite type on $X$.*

**Proof.** Lemma 26.11 is (1) ⇒ (2). Lemmas 26.2 and 26.12 provide the implication (1) ⇐ (2). The implications (2) ⇒ (4) ⇒ (3) are clear from Constructions, Section 8. Lemma 26.6 is (3) ⇒ (1). Thus we see that the first 4 conditions are all equivalent.

Assume the equivalent conditions (1) – (4). Note that in particular $X$ is separated (as an open subscheme of the separated scheme Proj$(S)$). Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Choose $s \in S_+$ homogeneous such that $X_s$ is affine. We claim that any section $m \in \Gamma(X_s, \mathcal{F})$ is in the image of one of the maps displayed in (5) above. This will imply (5) since these affines $X_s$ cover $X$. Namely, by Lemma 17.2 we may write $m$ as the image of $m' \otimes s^{-n}$ for some $n \geq 1$, some $m' \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This proves the claim.

Clearly (5) $\Rightarrow$ (6). Let us assume (6) and prove $\mathcal{L}$ is ample. Pick $x \in X$. Let $U \subset X$ be an affine open which contains $x$. Set $Z = X \setminus U$. We may think of $Z$ as a reduced closed subscheme, see Schemes, Section 12. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to the closed subscheme $Z$. By assumption (6), there exists an $n \geq 1$ and a section $s \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n})$ such that $s$ does not vanish at $x$ (more precisely such that $s \notin \mathfrak{m}_x \mathcal{I}_x \otimes \mathcal{L}_x^{\otimes n}$). We may think of $s$ as a section of $\mathcal{L}^{\otimes n}$. Since it clearly vanishes along $Z$ we see that $X_s \subset U$. Hence $X_s$ is affine, see Lemma 26.4. This proves that $\mathcal{L}$ is ample. At this point we have proved that (1) – (6) are equivalent.

Assume the equivalent conditions (1) – (6). In the following we will use the fact that the tensor product of two sheaves of modules which are globally generated is globally generated without further mention (see Modules, Lemma 4.3). By (1) we can find elements $s_i \in S_{d_i}$ with $d_i \geq 1$ such that $X = \bigcup_{i=1,\ldots,n} X_{s_i}$. Set $d = d_1 \ldots d_n$. It follows that $\mathcal{L}^{\otimes d}$ is globally generated by

$$s_1^{d/d_1}, \ldots, s_n^{d/d_n}.$$

This means that if $\mathcal{L}^{\otimes j}$ is globally generated then so is $\mathcal{L}^{\otimes j+dn}$ for all $n \geq 0$. Fix a $j \in \{0, \ldots, d-1\}$. For any point $x \in X$ there exists an $n \geq 1$ and a global section $s$ of $\mathcal{L}^{j+dn}$ which does not vanish at $x$, as follows from (5) applied to $\mathcal{F} = \mathcal{L}^{\otimes j}$ and ample invertible sheaf $\mathcal{L}^{\otimes d}$. Since $X$ is quasi-compact there we may find a finite list of integers $n_i$ and global sections $s_i$ of $\mathcal{L}^{\otimes j+dn_i}$ which do not vanish at any point of $X$. Since $\mathcal{L}^{\otimes d}$ is globally generated this means that $\mathcal{L}^{\otimes j+dn}$ is globally generated where $n = \max\{n_i\}$. Since we proved this for every congruence class mod $d$ we conclude that there exists an $n_0 = n_0(\mathcal{L})$ such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. At this point we see that if $\mathcal{F}$ is globally generated then so is $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for all $n \geq n_0$.

We continue to assume the equivalent conditions (1) – (6). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. Denote $\mathcal{F}_n \subset \mathcal{F}$ the image of the canonical map of (5). By construction $\mathcal{F}_n \otimes \mathcal{L}^{\otimes n}$ is globally generated. By (5) we see $\mathcal{F}$ is the sum of the subsheaves $\mathcal{F}_n$, $n \geq 1$. By Modules, Lemma 9.7 we see that $\mathcal{F} = \sum_{n=1,\ldots,N} \mathcal{F}_n$ for some $N \geq 1$. It follows that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated whenever $n \geq N + n_0(\mathcal{L})$ with $n_0(\mathcal{L})$ as above. We conclude that (1) – (6) implies (7).

Assume (7). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. By (7) there exists an integer $n \geq 1$ such that the canonical map

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

is surjective. Let $I$ be the set of finite subsets of $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ partially ordered by inclusion. Then $I$ is a directed partially ordered set. For $i = \{s_1, \ldots, s_{r(i)}\}$ let $\mathcal{F}_i \subset \mathcal{F}$ be the image of the map

$$\bigoplus\nolimits_{j=1,\ldots,r(i)} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

which is multiplication by $s_j$ on the $j$th factor. The surjectivity above implies that $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$. Hence Modules, Lemma 9.7 applies and we conclude that $\mathcal{F} = \mathcal{F}_i$ for some $i$. Hence we have proved (8). In other words, (7) $\Rightarrow$ (8).

The implication (8) $\Rightarrow$ (9) is trivial.

Finally, assume (9). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. By Lemma 22.3 (this is where we use the condition that $X$ be quasi-separated) we see that $\mathcal{I} = \operatorname{colim}_\alpha I_\alpha$ with each $I_\alpha$ quasi-coherent of finite type. Since by assumption each of the $I_\alpha$ is a quotient of negative tensor powers of $\mathcal{L}$ we conclude the same for $\mathcal{I}$ (but of course without the finiteness or boundedness of the powers). Hence we conclude that (9) implies (6). This ends the proof of the proposition. □

**Lemma 26.14.** *Let $X$ be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $i : X' \to X$ be a morphism of schemes. Assume at least one of the following conditions holds*

1. *$i$ is a quasi-compact immersion,*
2. *$X'$ is quasi-compact and $i$ is an immersion,*
3. *$i$ is quasi-compact and induces a homeomorphism between $X'$ and $i(X')$,*
4. *$X'$ is quasi-compact and $i$ induces a homeomorphism between $X'$ and $i(X')$.*

*Then $i^*\mathcal{L}$ is ample on $X'$.*

**Proof.** Observe that in cases (1) and (3) the scheme $X'$ is quasi-compact as $X$ is quasi-compact by Definition 26.1. Thus it suffices to prove (2) and (4). Since (2) is a special case of (4) it suffices to prove (4).

Assume condition (4) holds. For $s \in \Gamma(X, \mathcal{L}^{\otimes d})$ denote $s' = i^*s$ the pullback of $s$ to $X'$. Note that $s'$ is a section of $(i^*\mathcal{L})^{\otimes d}$. By Proposition 26.13 the opens $X_s$, for $s \in \Gamma(X, \mathcal{L}^{\otimes d})$, form a basis for the topology on $X$. Since $X'_{s'} = i^{-1}(X_s)$ and since $X' \to i(X')$ is a homeomorphism, we conclude the opens $X'_{s'}$ form a basis for the topology of $X'$. Hence $i^*\mathcal{L}$ is ample by Proposition 26.13. □

**Lemma 26.15.** *Let $S$ be a quasi-separated scheme. Let $X$, $Y$ be schemes over $S$. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module and let $\mathcal{N}$ be an ample invertible $\mathcal{O}_Y$-module. Then $\mathcal{M} = pr_1^*\mathcal{L} \otimes_{\mathcal{O}_{X \times_S Y}} pr_2^*\mathcal{N}$ is an ample invertible sheaf on $X \times_S Y$.*

**Proof.** The morphism $i : X \times_S Y \to X \times Y$ is a quasi-compact immersion, see Schemes, Lemma 21.9. On the other hand, $\mathcal{M}$ is the pullback by $i$ of the corresponding invertible module on $X \times Y$. By Lemma 26.14 it suffices to prove the lemma for $X \times Y$. We check (1) and (2) of Definition 26.1 for $\mathcal{M}$ on $X \times Y$.

Since $X$ and $Y$ are quasi-compact, so is $X \times Y$. Let $z \in X \times Y$ be a point. Let $x \in X$ and $y \in Y$ be the projections. Choose $n > 0$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $X_s$ is an affine open neighbourhood of $x$. Choose $m > 0$ and $t \in \Gamma(Y, \mathcal{N}^{\otimes m})$ such that $Y_t$ is an affine open neighbourhood of $y$. Then $r = \mathrm{pr}_1^*s \otimes \mathrm{pr}_2^*t$ is a section of $\mathcal{M}$ with $(X \times Y)_r = X_s \times Y_t$. This is an affine open neighbourhood of $z$ and the proof is complete. □

## 27. Affine and quasi-affine schemes

**Lemma 27.1.** *Let $X$ be a scheme. Then $X$ is quasi-affine if and only if $\mathcal{O}_X$ is ample.*

**Proof.** Suppose that $X$ is quasi-affine. Set $A = \Gamma(X, \mathcal{O}_X)$. Consider the open immersion

$$j : X \longrightarrow \operatorname{Spec}(A)$$

from Lemma 18.4. Note that $\operatorname{Spec}(A) = \operatorname{Proj}(A[T])$, see Constructions, Example 8.14. Hence we can apply Lemma 26.12 to deduce that $\mathcal{O}_X$ is ample.

Suppose that $\mathcal{O}_X$ is ample. Note that $\Gamma_*(X, \mathcal{O}_X) \cong A[T]$ as graded rings. Hence the result follows from Lemmas 26.11 and 18.4 taking into account that $\operatorname{Spec}(A) = \operatorname{Proj}(A[T])$ for any ring $A$ as seen above. $\square$

**Lemma 27.2.** *Let $X$ be a quasi-affine scheme. For any quasi-compact immersion $i : X' \to X$ the scheme $X'$ is quasi-affine.*

**Proof.** This can be proved directly without making use of the material on ample invertible sheaves; we urge the reader to do this on a napkin. Since $X$ is quasi-affine, we have that $\mathcal{O}_X$ is ample by Lemma 27.1. Then $\mathcal{O}_{X'}$ is ample by Lemma 26.14. Then $X'$ is quasi-affine by Lemma 27.1. $\square$

**Lemma 27.3.** *Let $X$ be a scheme. Suppose that there exist finitely many elements $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that*

1. *each $X_{f_i}$ is an affine open of $X$, and*
2. *the ideal generated by $f_1, \ldots, f_n$ in $\Gamma(X, \mathcal{O}_X)$ is equal to the unit ideal.*

*Then $X$ is affine.*

**Proof.** Assume we have $f_1, \ldots, f_n$ as in the lemma. We may write $1 = \sum g_i f_i$ for some $g_j \in \Gamma(X, \mathcal{O}_X)$ and hence it is clear that $X = \bigcup X_{f_i}$. (The $f_i$'s cannot all vanish at a point.) Since each $X_{f_i}$ is quasi-compact (being affine) it follows that $X$ is quasi-compact. Hence we see that $X$ is quasi-affine by Lemma 27.1 above. Consider the open immersion

$$j : X \to \operatorname{Spec}(\Gamma(X, \mathcal{O}_X)),$$

see Lemma 18.4. The inverse image of the standard open $D(f_i)$ on the right hand side is equal to $X_{f_i}$ on the left hand side and the morphism $j$ induces an isomorphism $X_{f_i} \cong D(f_i)$, see Lemma 18.3. Since the $f_i$ generate the unit ideal we see that $\operatorname{Spec}(\Gamma(X, \mathcal{O}_X)) = \bigcup_{i=1,\ldots,n} D(f_i)$. Thus $j$ is an isomorphism. $\square$

## 28. Quasi-coherent sheaves and ample invertible sheaves

Theme of this section: in the presence of an ample invertible sheaf every quasi-coherent sheaf comes from a graded module.

**Situation 28.1.** Let $X$ be a scheme. Let $\mathcal{L}$ be an ample invertible sheaf on $X$. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. Set $Y = \operatorname{Proj}(S)$. Let $f : X \to Y$ be the canonical morphism of Lemma 26.9. It comes equipped with a $\mathbf{Z}$-graded $\mathcal{O}_X$-algebra map $\bigoplus f^*\mathcal{O}_Y(n) \to \bigoplus \mathcal{L}^{\otimes n}$.

The following lemma is really a special case of the next lemma but it seems like a good idea to point out its validity first.

**Lemma 28.2.** *In Situation 28.1. The canonical morphism $f : X \to Y$ maps $X$ into the open subscheme $W = W_1 \subset Y$ where $\mathcal{O}_Y(1)$ is invertible and where all multiplication maps $\mathcal{O}_Y(n) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \to \mathcal{O}_Y(n+m)$ are isomorphisms (see Constructions, Lemma 10.4). Moreover, the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are all isomorphisms.*

**Proof.** By Proposition 26.13 there exists an integer $n_0$ such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. Let $x \in X$ be a point. By the above we can find $a \in S_{n_0}$ and $b \in S_{n_0+1}$ such that $a$ and $b$ do not vanish at $x$. Hence $f(x) \in D_+(a) \cap D_+(b) = D_+(ab)$. By Constructions, Lemma 10.4 we see that $f(x) \in W_1$ as desired. By Constructions, Lemma 14.1 which was used in the construction of the map $f$ the maps $f^*\mathcal{O}_Y(n_0) \to \mathcal{L}^{\otimes n_0}$ and $f^*\mathcal{O}_Y(n_0+1) \to \mathcal{L}^{\otimes n_0+1}$ are isomorphisms in a neighbourhood of $x$. By compatibility with the algebra structure and the fact that $f$ maps into $W$ we conclude all the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are isomorphisms in a neighbourhood of $x$. Hence we win. $\square$

Recall from Modules, Definition 25.7 that given a locally ringed space $X$, an invertible sheaf $\mathcal{L}$, and a $\mathcal{O}_X$-module $\mathcal{F}$ we have the graded $\Gamma_*(X, \mathcal{L})$-module

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}).$$

The following lemma says that, in Situation 28.1, we can recover a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ from this graded module. Take a look also at Constructions, Lemma 13.8 where we prove this lemma in the special case $X = \mathbf{P}^n_R$.

**Lemma 28.3.** *In Situation 28.1. Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Set $M = \Gamma_*(X, \mathcal{L}, \mathcal{F})$ as a graded $S$-module. There are isomorphisms*

$$f^*\widetilde{M} \longrightarrow \mathcal{F}$$

*functorial in $\mathcal{F}$ such that $M_0 \to \Gamma(Proj(S), \widetilde{M}) \to \Gamma(X, \mathcal{F})$ is the identity map.*

**Proof.** Let $s \in S_+$ be homogeneous such that $X_s$ is affine open in $X$. Recall that $\widetilde{M}|_{D_+(s)}$ corresponds to the $S_{(s)}$-module $M_{(s)}$, see Constructions, Lemma 8.4. Recall that $f^{-1}(D_+(s)) = X_s$. As $X$ carries an ample invertible sheaf it is quasi-compact and quasi-separated, see Section 26. By Lemma 17.2 there is a canonical isomorphism $M_{(s)} = \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \to \Gamma(X_s, \mathcal{F})$. Since $\mathcal{F}$ is quasi-coherent this leads to a canonical isomorphism

$$f^*\widetilde{M}|_{X_s} \to \mathcal{F}|_{X_s}$$

Since $\mathcal{L}$ is ample on $X$ we know that $X$ is covered by the affine opens of the form $X_s$. Hence it suffices to prove that the displayed maps glue on overlaps. Proof of this is omitted. $\square$

**Remark 28.4.** With assumptions and notation of Lemma 28.3. Denote the displayed map of the lemma by $\theta_\mathcal{F}$. Note that the isomorphism $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ of Lemma 28.2 is just $\theta_{\mathcal{L}^{\otimes n}}$. Consider the multiplication maps

$$\widetilde{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n) \longrightarrow \widetilde{M(n)}$$

see Constructions, Equation (10.1.5). Pull this back to $X$ and consider

$$\begin{tikzcd} f^*\widetilde{M} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y(n) \arrow[r] \arrow[d, "\theta_\mathcal{F} \otimes \theta_{\mathcal{L}^{\otimes n}}"'] & f^*\widetilde{M(n)} \arrow[d, "\theta_{\mathcal{F} \otimes \mathcal{L}^{\otimes n}}"] \\ \mathcal{F} \otimes \mathcal{L}^{\otimes n} \arrow[r, "\mathrm{id}"] & \mathcal{F} \otimes \mathcal{L}^{\otimes n} \end{tikzcd}$$

Here we have used the obvious identification $M(n) = \Gamma_*(X, \mathcal{L}, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This diagram commutes. Proof omitted.

It should be possible to deduce the following lemma from Lemma 28.3 (or conversely) but it seems simpler to just repeat the proof.

**Lemma 28.5.** *Let $S$ be a graded ring such that $X = Proj(S)$ is quasi-compact. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Set $M = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ as a graded $S$-module, see Constructions, Section 10. The map*

$$\widetilde{M} \longrightarrow \mathcal{F}$$

*of Constructions, Lemma 10.7 is an isomorphism. If $X$ is covered by standard opens $D_+(f)$ where $f$ has degree 1, then the induced maps $M_n \to \Gamma(X, \mathcal{F}(n))$ are the identity maps.*

**Proof.** Since $X$ is quasi-compact we can find homogeneous elements $f_1, \ldots, f_n \in S$ of positive degrees such that $X = D_+(f_1) \cup \ldots \cup D_+(f_n)$. Let $d$ be the least common multiple of the degrees of $f_1, \ldots, f_n$. After replacing $f_i$ by a power we may assume that each $f_i$ has degree $d$. Then we see that $\mathcal{L} = \mathcal{O}_X(d)$ is invertible, the multiplication maps $\mathcal{O}_X(ad) \otimes \mathcal{O}_X(bd) \to \mathcal{O}_X((a+b)d)$ are isomorphisms, and each $f_i$ determines a global section $s_i$ of $\mathcal{L}$ such that $X_{s_i} = D_+(f_i)$, see Constructions, Lemmas 10.4 and 10.5. Thus $\Gamma(X, \mathcal{F}(ad)) = \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$. Recall that $\widetilde{M}|_{D_+(f_i)}$ corresponds to the $S_{(f_i)}$-module $M_{(f_i)}$, see Constructions, Lemma 8.4. Since the degree of $f_i$ is $d$, the isomorphism class of $M_{(f_i)}$ depends only on the homogeneous summands of $M$ of degree divisible by $d$. More precisely, the isomorphism class of $M_{(f_i)}$ depends only on the graded $\Gamma_*(X, \mathcal{L})$-module $\Gamma_*(X, \mathcal{L}, \mathcal{F})$ and the image $s_i$ of $f_i$ in $\Gamma_*(X, \mathcal{L})$. The scheme $X$ is quasi-compact by assumption and separated by Constructions, Lemma 8.8. By Lemma 17.2 there is a canonical isomorphism

$$M_{(f_i)} = \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s_i)} \to \Gamma(X_{s_i}, \mathcal{F}).$$

The construction of the map in Constructions, Lemma 10.7 then shows that it is an isomorphism over $D_+(f_i)$ hence an isomorphism as $X$ is covered by these opens. We omit the proof of the final statement. $\square$

## 29. Finding suitable affine opens

In this section we collect some results on the existence of affine opens in more and less general situations.

**Lemma 29.1.** *Let $X$ be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of $X$, see Topology, Section 8. Let $\eta_i \in Z_i$ be their generic points, see Schemes, Lemma 11.1. There exist affine open neighbourhoods $\eta_i \in U_i$ such that $U_i \cap U_j = \emptyset$ for all $i \neq j$. In particular, $U = U_1 \cup \ldots \cup U_n$ is an affine open containing all of the points $\eta_1, \ldots, \eta_n$.*

**Proof.** Let $V_i$ be any affine open containing $\eta_i$ and disjoint from the closed set $Z_1 \cup \ldots \hat{Z}_i \ldots \cup Z_n$. Since $X$ is quasi-separated for each $i$ the union $W_i = \bigcup_{j, j \neq i} V_i \cap V_j$ is a quasi-compact open of $V_i$ not containing $\eta_i$. We can find open neighbourhoods $U_i \subset V_i$ containing $\eta_i$ and disjoint from $W_i$ by Algebra, Lemma 26.4. Finally, $U$ is affine since it is the spectrum of the ring $R_1 \times \ldots \times R_n$ where $R_i = \mathcal{O}_X(U_i)$, see Schemes, Lemma 6.8. $\square$

**Remark 29.2.** Lemma 29.1 above is false if $X$ is not quasi-separated. Here is an example. Take $R = \mathbf{Q}[x, y_1, y_2, \ldots]/((x-i)y_i)$. Consider the minimal prime ideal $\mathfrak{p} = (y_1, y_2, \ldots)$ of $R$. Glue two copies of $\mathrm{Spec}(R)$ along the (not quasi-compact) open $\text{Spec}(R)\setminus V(\mathfrak{p})$ to get a scheme $X$ (glueing as in Schemes, Example 14.3). Then the two maximal points of $X$ corresponding to $\mathfrak{p}$ are not contained in a common affine open. The reason is that any open of $\text{Spec}(R)$ containing $\mathfrak{p}$ contains infinitely many of the "lines" $x = i$, $y_j = 0$, $j \neq i$ with parameter $y_i$. Details omitted.

Notwithstanding the example above, for "most" finite sets of irreducible closed subsets one can apply Lemma 29.1 above, at least if $X$ is quasi-compact. This is true because $X$ contains a dense open which is separated.

**Lemma 29.3.** *Let $X$ be a quasi-compact scheme. There exists a dense open $V \subset X$ which is separated.*

**Proof.** Say $X = \bigcup_{i=1,\ldots,n} U_i$ is a union of $n$ affine open subschemes. We will prove the lemma by induction on $n$. It is trivial for $n = 1$. Let $V' \subset \bigcup_{i=1,\ldots,n-1} U_i$ be a separated dense open subscheme, which exists by induction hypothesis. Consider

$$V = V' \amalg (U_n \setminus \overline{V'}).$$

It is clear that $V$ is separated and a dense open subscheme of $X$. □

It turns out that, even if $X$ is quasi-separated as well as quasi-compact, there does not exist a separated, quasi-compact dense open, see Examples, Lemma 26.2. Here is a slight refinement of Lemma 29.1 above.

**Lemma 29.4.** *Let $X$ be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of $X$. Let $\eta_i \in Z_i$ be their generic points. Let $x \in X$ be arbitrary. There exists an affine open $U \subset X$ containing $x$ and all the $\eta_i$.*

**Proof.** Suppose that $x \in Z_1 \cap \ldots \cap Z_r$ and $x \notin Z_{r+1}, \ldots, Z_n$. Then we may choose an affine open $W \subset X$ such that $x \in W$ and $W \cap Z_i = \emptyset$ for $i = r+1, \ldots, n$. Note that clearly $\eta_i \in W$ for $i = 1, \ldots, r$. By Lemma 29.1 we may choose affine opens $U_i \subset X$ which are pairwise disjoint such that $\eta_i \in U_i$ for $i = r+1, \ldots, n$. Since $X$ is quasi-separated the opens $W \cap U_i$ are quasi-compact and do not contain $\eta_i$ for $i = r+1, \ldots, n$. Hence by Algebra, Lemma 26.4 we may shrink $U_i$ such that $W \cap U_i = \emptyset$ for $i = r+1, \ldots, n$. Then the union $U = W \cup \bigcup_{i=r+1,\ldots,n} U_i$ is disjoint and hence (by Schemes, Lemma 6.8) a suitable affine open. □

**Lemma 29.5.** *Let $X$ be a scheme. Assume either*

1. *The scheme $X$ is quasi-affine.*
2. *The scheme $X$ is isomorphic to a locally closed subscheme of an affine scheme.*
3. *There exists an ample invertible sheaf on $X$.*
4. *The scheme $X$ is isomorphic to a locally closed subscheme of Proj$(S)$ for some graded ring $S$.*

*Then for any finite subset $E \subset X$ there exists an affine open $U \subset X$ with $E \subset U$.*

**Proof.** By Properties, Definition 18.1 a quasi-affine scheme is a quasi-compact open subscheme of an affine scheme. Any affine scheme $\text{Spec}(R)$ is isomorphic to $\text{Proj}(R[X])$ where $R[X]$ is graded by setting $\deg(X) = 1$. By Proposition 26.13 if $X$ has an ample invertible sheaf then $X$ is isomorphic to an open subscheme of $\text{Proj}(S)$ for some graded ring $S$. Hence, it suffices to prove the lemma in case (4). (We urge the reader to prove case (2) directly for themselves.)

Thus assume $X \subset \text{Proj}(S)$ is a locally closed subscheme where $S$ is some graded ring. Let $T = \overline{X} \setminus X$. Recall that the standard opens $D_+(f)$ form a basis of the topology on $\text{Proj}(S)$. Since $E$ is finite we may choose finitely many homogeneous elements $f_i \in S_+$ such that

$$E \subset D_+(f_1) \cup \ldots \cup D_+(f_n) \subset \text{Proj}(S) \setminus T$$

Suppose that $E = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ as a subset of $\text{Proj}(S)$. Consider the ideal $I = (f_1, \ldots, f_n) \subset S$. Since $I \not\subset \mathfrak{p}_j$ for all $j = 1, \ldots, m$ we see from Algebra, Lemma 57.6 that there exists a homogeneous element $f \in I$, $f \notin \mathfrak{p}_j$ for all $j = 1, \ldots, m$. Then $E \subset D_+(f) \subset D_+(f_1) \cup \ldots \cup D_+(f_n)$. Since $D_+(f)$ does not meet $T$ we see that $X \cap D_+(f)$ is a closed subscheme of the affine scheme $D_+(f)$, hence is an affine open of $X$ as desired. $\square$

**Lemma 29.6.** *Let $X$ be a scheme. Let $\mathcal{L}$ be an ample invertible sheaf on $X$. Let*

$$E \subset W \subset X$$

*with $E$ finite and $W$ open in $X$. Then there exists an $n > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $X_s$ is affine and $E \subset X_s \subset W$.*

**Proof.** The reader can modify the proof of Lemma 29.5 to prove this lemma; we will instead deduce the lemma from it. By Lemma 29.5 we can choose an affine open $U \subset W$ such that $E \subset U$. Consider the graded ring $S = \Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$. For each $x \in E$ let $\mathfrak{p}_x \subset S$ be the graded ideal of sections vanishing at $x$. It is clear that $\mathfrak{p}_x$ is a prime ideal and since some power of $\mathcal{L}$ is globally generated, it is clear that $S_+ \not\subset \mathfrak{p}_x$. Let $I \subset S$ be the graded ideal of sections vanishing on all points of $X \setminus U$. Since the sets $X_s$ form a basis for the topology we see that $I \not\subset \mathfrak{p}_x$ for all $x \in E$. By (graded) prime avoidance (Algebra, Lemma 57.6) we can find $s \in I$ homogeneous with $s \notin \mathfrak{p}_x$ for all $x \in E$. Then $E \subset X_s \subset U$ and $X_s$ is affine by Lemma 26.4. $\square$

**Lemma 29.7.** *Let $X$ be a quasi-affine scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $E \subset W \subset X$ with $E$ finite and $W$ open. Then there exists an $s \in \Gamma(X, \mathcal{L})$ such that $X_s$ is affine and $E \subset X_s \subset W$.*

**Proof.** The proof of this lemma has a lot in common with the proof of Algebra, Lemma 15.2. Say $E = \{x_1, \ldots, x_n\}$. If $E = W = \emptyset$, then $s = 0$ works. If $W \neq \emptyset$, then we may assume $E \neq \emptyset$ by adding a point if necessary. Thus we may assume $n \geq 1$. We will prove the lemma by induction on $n$.

Base case: $n = 1$. After replacing $W$ by an affine open neighbourhood of $x_1$ in $W$, we may assume $W$ is affine. Combining Lemmas 27.1 and Proposition 26.13 we see that every quasi-coherent $\mathcal{O}_X$-module is globally generated. Hence there exists a global section $s$ of $\mathcal{L}$ which does not vanish at $x_1$. On the other hand, let $Z \subset X$ be the reduced induced closed subscheme on $X \setminus W$. Applying global generation to the quasi-coherent ideal sheaf $\mathcal{I}$ of $Z$ we find a global section $f$ of $\mathcal{I}$ which does not vanish at $x_1$. Then $s' = fs$ is a global section of $\mathcal{L}$ which does not vanish at $x_1$ such that $X_{s'} \subset W$. Then $X_{s'}$ is affine by Lemma 26.4.

Induction step for $n > 1$. If there is a specialization $x_i \rightsquigarrow x_j$ for $i \neq j$, then it suffices to prove the lemma for $\{x_1, \ldots, x_n\} \setminus \{x_i\}$ and we are done by induction. Thus we may assume there are no specializations among the $x_i$. By either Lemma 29.5 or Lemma 29.6 we may assume $W$ is affine. By induction we can find a global section $s$ of $\mathcal{L}$ such that $X_s \subset W$ is affine and contains $x_1, \ldots, x_{n-1}$. If $x_n \in X_s$ then we are done. Assume $s$ is zero at $x_n$. By the case $n = 1$ we can find a global section $s'$ of $\mathcal{L}$ with $\{x_n\} \subset X_{s'} \subset W \setminus \overline{\{x_1, \ldots, x_{n-1}\}}$. Here we use that $x_n$ is not a specialization of $x_1, \ldots, x_{n-1}$. Then $s + s'$ is a global section of $\mathcal{L}$ which is nonvanishing at $x_1, \ldots, x_n$ with $X_{s+s'} \subset W$ and we conclude as before. $\square$

**Lemma 29.8.** *Let $X$ be a scheme and $x \in X$ a point. There exists an affine open neighbourhood $U \subset X$ of $x$ such that the canonical map $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is injective in each of the following cases:*
1. *$X$ is integral,*
2. *$X$ is locally Noetherian,*
3. *$X$ is reduced and has a finite number of irreducible components.*

**Proof.** After translation into algebra, this follows from Algebra, Lemma 31.9. $\square$

## 30. Other chapters

## References


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