SHEAVES ON ALGEBRAIC STACKS

Contents

1. Introduction

There is a myriad of ways to think about sheaves on algebraic stacks. In this chapter we discuss one approach, which is particularly well adapted to our foundations for algebraic stacks. Whenever we introduce a type of sheaves we will indicate the precise relationship with similar notions in the literature. The goal of this chapter is to state those results that are either obviously true or straightforward to prove and leave more intricate constructions till later.

In fact, it turns out that to develop a fully fledged theory of constructible étale sheaves and/or an adequate discussion of derived categories of complexes $\mathcal{O}$-modules whose cohomology sheaves are quasi-coherent takes a significant amount of work, see [Ols07]. We will return to this in Cohomology of Stacks, Section 1.

In the literature and in research papers on sheaves on algebraic stacks the lisse-étale site of an algebraic stack often plays a prominent role. However, it is a problematic beast, because it turns out that a morphism of algebraic stacks does not induce a morphism of lisse-étale topoi. We have therefore made the design decision to avoid any mention of the lisse-étale site as long as possible. Arguments that traditionally use the lisse-étale site will be replaced by an argument using a Čech covering in the site $\mathcal{X}_{smooth}$ defined below.

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 2 for an explanation.

2. Conventions

The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 2. For convenience we repeat them here.

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We record what changes if you change the big fppf site elsewhere (insert future reference here).

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

3. Presheaves

In this section we define presheaves on categories fibred in groupoids over $(Sch/S)_{fppf}$, but most of the discussion works for categories over any base category. This section also serves to introduce the notation we will use later on.

Definition 3.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) A *presheaf on* $\mathcal{X}$ is a presheaf on the underlying category of $\mathcal{X}$.
(2) A *morphism of presheaves on* $\mathcal{X}$ is a morphism of presheaves on the underlying category of $\mathcal{X}$.

We denote $PSh(\mathcal{X})$ the category of presheaves on $\mathcal{X}$.

This defines presheaves of sets. Of course we can also talk about presheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of *abelian presheaves*, i.e., presheaves of abelian groups, is denoted $PAb(\mathcal{X})$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Recall that this means just that f is a functor over $(Sch/S)_{fppf}$. The material in Sites, Section 19 provides us with a pair of adjoint functors[1]

$$f^p : PSh(\mathcal{Y}) \longrightarrow PSh(\mathcal{X}) \quad \text{and} \quad {}_pf : PSh(\mathcal{X}) \longrightarrow PSh(\mathcal{Y}). \tag{3.1.1}$$

[1]These functors will be denoted f^{-1} and f_* after Lemma 4.4 has been proved.

The adjointness is

$$\mathrm{Mor}_{PSh(\mathcal{X})}(f^p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pf\mathcal{F})$$

where $\mathcal{F} \in \mathrm{Ob}(PSh(\mathcal{X}))$ and $\mathcal{G} \in \mathrm{Ob}(PSh(\mathcal{Y}))$. We call $f^p\mathcal{G}$ the *pullback* of $\mathcal{G}$. It follows from the definitions that

$$f^p\mathcal{G}(x) = \mathcal{G}(f(x))$$

for any $x \in \mathrm{Ob}(\mathcal{X})$. The presheaf ${}_pf\mathcal{F}$ is called the *pushforward* of $\mathcal{F}$. It is described by the formula

$$({}_pf\mathcal{F})(y) = \lim_{f(x)\to y} \mathcal{F}(x).$$

The rest of this section should probably be moved to the chapter on sites and in any case should be skipped on a first reading.

Lemma 3.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Then $(g \circ f)^p = f^p \circ g^p$ and there is a canonical isomorphism ${}_p(g \circ f) \to {}_pg \circ {}_pf$ compatible with adjointness of $(f^p, {}_pf)$, $(g^p, {}_pg)$, and $((g \circ f)^p, {}_p(g \circ f))$.*

Proof. Let $\mathcal{H}$ be a presheaf on $\mathcal{Z}$. Then $(g \circ f)^p\mathcal{H} = f^p(g^p\mathcal{H})$ is given by the equalities

$$(g \circ f)^p\mathcal{H}(x) = \mathcal{H}((g \circ f)(x)) = \mathcal{H}(g(f(x))) = f^p(g^p\mathcal{H})(x).$$

We omit the verification that this is compatible with restriction maps.

Next, we define the transformation ${}_p(g\circ f) \to {}_pg\circ{}_pf$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. If z is an object of $\mathcal{Z}$ then we get a category $\mathcal{J}$ of quadruples $(x, f(x) \to y, y, g(y) \to z)$ and a category $\mathcal{I}$ of pairs $(x, g(f(x)) \to z)$. There is a canonical functor $\mathcal{J} \to \mathcal{I}$ sending the object $(x, \alpha : f(x) \to y, y, \beta : g(y) \to z)$ to $(x, \beta \circ f(\alpha) : g(f(x)) \to z)$. This gives the arrow in

$$\begin{aligned} ({}_p(g \circ f)\mathcal{F})(z) &= \lim_{g(f(x))\to z} \mathcal{F}(x) \\ &= \lim_{\mathcal{I}} \mathcal{F} \\ &\to \lim_{\mathcal{J}} \mathcal{F} \\ &= \lim_{g(y)\to z} \Big(\lim_{f(x)\to y} \mathcal{F}(x) \Big) \\ &= ({}_pg \circ {}_pf\mathcal{F})(x) \end{aligned}$$

by Categories, Lemma 14.9. We omit the verification that this is compatible with restriction maps. An alternative to this direct construction is to define ${}_p(g \circ f) \cong {}_pg \circ {}_pf$ as the unique map compatible with the adjointness properties. This also has the advantage that one does not need to prove the compatibility.

Compatibility with adjointness of $(f^p, {}_pf)$, $(g^p, {}_pg)$, and $((g \circ f)^p, {}_p(g \circ f))$ means that given presheaves $\mathcal{H}$ and $\mathcal{F}$ as above we have a commutative diagram

$$\begin{tikzcd} \mathrm{Mor}_{PSh(\mathcal{X})}(f^pg^p\mathcal{H}, \mathcal{F}) \arrow[r, equal] \arrow[d, equal] & \mathrm{Mor}_{PSh(\mathcal{Y})}(g^p\mathcal{H}, {}_pf\mathcal{F}) \arrow[r, equal] & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{H}, {}_pg{}_pf\mathcal{F}) \\ \mathrm{Mor}_{PSh(\mathcal{X})}((g \circ f)^p\mathcal{G}, \mathcal{F}) \arrow[rr, equal] & & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_p(g \circ f)\mathcal{F}) \arrow[u] \end{tikzcd}$$

Proof omitted. $\square$

Lemma 3.3. *Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $t : f \to g$ be a 2-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assigned to t there are canonical isomorphisms of functors*

$$t^p : g^p \longrightarrow f^p \quad \text{and} \quad {}_pt : {}_pf \longrightarrow {}_pg$$

which compatible with adjointness of $(f^p, {}_pf)$ and $(g^p, {}_pg)$ and with vertical and horizontal composition of 2-morphisms.

Proof. Let $\mathcal{G}$ be a presheaf on $\mathcal{Y}$. Then $t^p : g^p\mathcal{G} \to f^p\mathcal{G}$ is given by the family of maps

$$g^p\mathcal{G}(x) = \mathcal{G}(g(x)) \xrightarrow{\mathcal{G}(t_x)} \mathcal{G}(f(x)) = f^p\mathcal{G}(x)$$

parametrized by $x \in \mathrm{Ob}(\mathcal{X})$. This makes sense as $t_x : f(x) \to g(x)$ and $\mathcal{G}$ is a contravariant functor. We omit the verification that this is compatible with restriction mappings.

To define the transformation ${}_pt$ for $y \in \mathrm{Ob}(\mathcal{Y})$ define ${}^f_y\mathcal{I}$, resp. ${}^g_y\mathcal{I}$ to be the category of pairs $(x, \psi : f(x) \to y)$, resp. $(x, \psi : g(x) \to y)$, see Sites, Section 19. Note that t defines a functor ${}_yt : {}^g_y\mathcal{I} \to {}^f_y\mathcal{I}$ given by the rule

$$(x, g(x) \to y) \longmapsto (x, f(x) \xrightarrow{t_x} g(x) \to y).$$

Note that for $\mathcal{F}$ a presheaf on $\mathcal{X}$ the composition of ${}_yt$ with $\mathcal{F} : {}^f_y\mathcal{I}^{opp} \to \textit{Sets}$, $(x, f(x) \to y) \mapsto \mathcal{F}(x)$ is equal to $\mathcal{F} : {}^g_y\mathcal{I}^{opp} \to \textit{Sets}$. Hence by Categories, Lemma 14.9 we get for every $y \in \mathrm{Ob}(\mathcal{Y})$ a canonical map

$$({}_pf\mathcal{F})(y) = \lim_{{}^f_y\mathcal{I}} \mathcal{F} \longrightarrow \lim_{{}^g_y\mathcal{I}} \mathcal{F} = ({}_pg\mathcal{F})(y)$$

We omit the verification that this is compatible with restriction mappings. An alternative to this direct construction is to define ${}_pt$ as the unique map compatible with the adjointness properties of the pairs $(f^p, {}_pf)$ and $(g^p, {}_pg)$ (see below). This also has the advantage that one does not need to prove the compatibility.

Compatibility with adjointness of $(f^p, {}_pf)$ and $(g^p, {}_pg)$ means that given presheaves $\mathcal{G}$ and $\mathcal{F}$ as above we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Mor}_{PSh(\mathcal{X})}(f^p\mathcal{G}, \mathcal{F}) \arrow[r, equal] \arrow[d, "-\circ t^p"'] & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pf\mathcal{F}) \arrow[d, "{}_pt\circ -"] \\
\mathrm{Mor}_{PSh(\mathcal{X})}(g^p\mathcal{G}, \mathcal{F}) \arrow[r, equal] & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pg\mathcal{F})
\end{tikzcd}$$

Proof omitted. Hint: Work through the proof of Sites, Lemma 19.2 and observe the compatibility from the explicit description of the horizontal and vertical maps in the diagram.

We omit the verification that this is compatible with vertical and horizontal compositions. Hint: The proof of this for t^p is straightforward and one can conclude that this holds for the ${}_pt$ maps using compatibility with adjointness. $\square$

4. Sheaves

We first make an observation that is important and trivial (especially for those readers who do not worry about set theoretical issues).

Consider a big fppf site Sch_{fppf} as in Topologies, Definition 7.6 and denote its underlying category Sch_α. Besides being the underlying category of a fppf site, the category Sch_α can also can serve as the underlying category for a big Zariski site, a big étale site, a big smooth site, and a big syntomic site, see Topologies, Remark 11.1. We denote these sites Sch_{Zar}, $Sch_{étale}$, Sch_{smooth}, and $Sch_{syntomic}$. In this situation, since we have defined the big Zariski site $(Sch/S)_{Zar}$ of S, the big étale site $(Sch/S)_{étale}$ of S, the big smooth site $(Sch/S)_{smooth}$ of S, the big syntomic site $(Sch/S)_{syntomic}$ of S, and the big fppf site $(Sch/S)_{fppf}$ of S as the localizations (see Sites, Section 25) Sch_{Zar}/S, $Sch_{étale}/S$, Sch_{smooth}/S, $Sch_{syntomic}/S$, and Sch_{fppf}/S of these (absolute) big sites we see that all of these have the same underlying category, namely Sch_α/S.

It follows that if we have a category $p : \mathcal{X} \to (Sch/S)_{fppf}$ fibred in groupoids, then $\mathcal{X}$ inherits a Zariski, étale, smooth, syntomic, and fppf topology, see Stacks, Definition 10.2.

Definition 4.1. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$.

(1) The *associated Zariski site*, denoted $\mathcal{X}_{Zar}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{Zar}$.
(2) The *associated étale site*, denoted $\mathcal{X}_{étale}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{étale}$.
(3) The *associated smooth site*, denoted $\mathcal{X}_{smooth}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{smooth}$.
(4) The *associated syntomic site*, denoted $\mathcal{X}_{syntomic}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{syntomic}$.
(5) The *associated fppf site*, denoted $\mathcal{X}_{fppf}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{fppf}$.

This definition makes sense by the discussion above. If $\mathcal{X}$ is an algebraic stack, the literature calls $\mathcal{X}_{fppf}$ (or a site equivalent to it) the *big fppf site* of $\mathcal{X}$ and similarly for the other ones. We may occasionally use this terminology to distinguish this construction from others.

Remark 4.2. We only use this notation when the symbol $\mathcal{X}$ refers to a category fibred in groupoids, and not a scheme, an algebraic space, etc. In this way we will avoid confusion with the small étale site of a scheme, or algebraic space which is denoted $X_{étale}$ (in which case we use a roman capital instead of a calligraphic one).

Now that we have these topologies defined we can say what it means to have a sheaf on $\mathcal{X}$, i.e., define the corresponding topoi.

Definition 4.3. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$.

(1) We say $\mathcal{F}$ is a *Zariski sheaf*, or a *sheaf for the Zariski topology* if $\mathcal{F}$ is a sheaf on the associated Zariski site $\mathcal{X}_{Zar}$.
(2) We say $\mathcal{F}$ is an *étale sheaf*, or a *sheaf for the étale topology* if $\mathcal{F}$ is a sheaf on the associated étale site $\mathcal{X}_{étale}$.

(3) We say $\mathcal{F}$ is a *smooth sheaf*, or a *sheaf for the smooth topology* if $\mathcal{F}$ is a sheaf on the associated smooth site $\mathcal{X}_{smooth}$.
(4) We say $\mathcal{F}$ is a *syntomic sheaf*, or a *sheaf for the syntomic topology* if $\mathcal{F}$ is a sheaf on the associated syntomic site $\mathcal{X}_{syntomic}$.
(5) We say $\mathcal{F}$ is an *fppf sheaf*, or a *sheaf*, or a *sheaf for the fppf topology* if $\mathcal{F}$ is a sheaf on the associated fppf site $\mathcal{X}_{fppf}$.

A morphism of sheaves is just a morphism of presheaves. We denote these categories of sheaves $Sh(\mathcal{X}_{Zar})$, $Sh(\mathcal{X}_{étale})$, $Sh(\mathcal{X}_{smooth})$, $Sh(\mathcal{X}_{syntomic})$, and $Sh(\mathcal{X}_{fppf})$.

Of course we can also talk about sheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of *abelian sheaves*, i.e., sheaves of abelian groups, is denoted $Ab(\mathcal{X}_{fppf})$ and similarly for the other topologies. If $\mathcal{X}$ is an algebraic stack, then $Sh(\mathcal{X}_{fppf})$ is equivalent (modulo set theoretical problems) to what in the literature would be termed the *category of sheaves on the big fppf site of* $\mathcal{X}$. Similar for other topologies. We may occasionally use this terminology to distinguish this construction from others.

Since the topologies are listed in increasing order of strength we have the following strictly full inclusions

$$Sh(\mathcal{X}_{fppf}) \subset Sh(\mathcal{X}_{syntomic}) \subset Sh(\mathcal{X}_{smooth}) \subset Sh(\mathcal{X}_{étale}) \subset Sh(\mathcal{X}_{Zar}) \subset PSh(\mathcal{X})$$

We sometimes write $Sh(\mathcal{X}_{fppf}) = Sh(\mathcal{X})$ and $Ab(\mathcal{X}_{fppf}) = Ab(\mathcal{X})$ in accordance with our terminology that a sheaf on $\mathcal{X}$ is an fppf sheaf on $\mathcal{X}$.

With this setup functoriality of these topoi is straightforward, and moreover, is compatible with the inclusion functors above.

Lemma 4.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functors ${}_pf$ and f^p of (3.1.1) transform τ sheaves into τ sheaves and define a morphism of topoi $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$.*

Proof. This follows immediately from Stacks, Lemma 10.3. $\square$

In other words, pushforward and pullback of presheaves as defined in Section 3 also produces *pushforward* and *pullback* of τ-sheaves. Having said all of the above we see that we can write $f^p = f^{-1}$ and ${}_pf = f_*$ without any possibility of confusion.

Definition 4.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We denote

$$f = (f^{-1}, f_*) : Sh(\mathcal{X}_{fppf}) \longrightarrow Sh(\mathcal{Y}_{fppf})$$

the *associated morphism of fppf topoi* constructed above. Similarly for the associated Zariski, étale, smooth, and syntomic topoi.

As discussed in Sites, Section 44 the same formula (on the underlying sheaf of sets) defines pushforward and pullback for sheaves (for one of our topologies) of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc.

5. Computing pushforward

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. Let $y \in \mathrm{Ob}(\mathcal{Y})$. We can compute $f_*\mathcal{F}(y)$ in the following way. Suppose that y lies over the scheme V and using the 2-Yoneda lemma think of y as a 1-morphism. Consider the projection

$$\mathrm{pr} : (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X}$$

Then we have a canonical identification

$$f_*\mathcal{F}(y) = \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big) \tag{5.0.1}$$

Namely, objects of the 2-fibre product are triples $(h : U \to V, x, f(x) \to h^*y)$. Dropping the h from the notation we see that this is equivalent to the data of an object x of $\mathcal{X}$ and a morphism $\alpha : f(x) \to y$ of $\mathcal{Y}$. Since $f_*\mathcal{F}(y) = \lim_{f(x)\to y} \mathcal{F}(x)$ by definition the equality follows.

As a consequence we have the following "base change" result for pushforwards. This result is trivial and hinges on the fact that we are using "big" sites.

Lemma 5.1. *Let S be a scheme. Let*

$$\begin{tikzcd} \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

be a 2-cartesian diagram of categories fibred in groupoids over S. Then we have a canonical isomorphism

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

functorial in the presheaf $\mathcal{F}$ on $\mathcal{X}$.

Proof. Given an object y' of $\mathcal{Y}'$ over V there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'),\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y',\mathcal{Y}'} (\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X})$$

Hence by (5.0.1) a bijection $g^{-1}f_*\mathcal{F}(y') \to f'_*(g')^{-1}\mathcal{F}(y')$. We omit the verification that this is compatible with restriction mappings. $\square$

In the case of a representable morphism of categories fibred in groupoids this formula (5.0.1) simplifies. We suggest the reader skip the rest of this section.

Lemma 5.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent*

1. *f is representable, and*
2. *for every $y \in \mathrm{Ob}(\mathcal{Y})$ the functor $\mathcal{X}^{opp} \to Sets$, $x \mapsto \mathrm{Mor}_{\mathcal{Y}}(f(x), y)$ is representable.*

Proof. According to the discussion in Algebraic Stacks, Section 6 we see that f is representable if and only if for every $y \in \mathrm{Ob}(\mathcal{Y})$ lying over U the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable, i.e., of the form $(Sch/V_y)_{fppf}$ for some scheme V_y over U. Objects in this 2-fibre products are triples $(h : V \to U, x, \alpha : f(x) \to h^*y)$ where α lies over id_V. Dropping the h from the notation we see that this is equivalent to the data of an object x of $\mathcal{X}$ and a morphism $f(x) \to y$. Hence the 2-fibre product is representable by V_y and $f(x_y) \to y$ where x_y is an object of $\mathcal{X}$

over V_y if and only if the functor in (2) is representable by x_y with universal object a map $f(x_y) \to y$. □

Let

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{Y} \arrow[dl, "q"] \\ & (Sch/S)_{fppf} & \end{tikzcd}$$

be a 1-morphism of categories fibred in groupoids. Assume f is representable. For every $y \in \mathrm{Ob}(\mathcal{Y})$ we choose an object $u(y) \in \mathrm{Ob}(\mathcal{X})$ representing the functor $x \mapsto \mathrm{Mor}_{\mathcal{Y}}(f(x), y)$ of Lemma 5.2 (this is possible by the axiom of choice). The objects come with canonical morphisms $f(u(y)) \to y$ by construction. For every morphism $\beta : y' \to y$ in $\mathcal{Y}$ we obtain a unique morphism $u(\beta) : u(y') \to u(y)$ in $\mathcal{X}$ such that the diagram

$$\begin{tikzcd} f(u(y')) \arrow[r, "f(u(\beta))"] \arrow[d] & f(u(y)) \arrow[d] \\ y' \arrow[r] & y \end{tikzcd}$$

commutes. In other words, $u : \mathcal{Y} \to \mathcal{X}$ is a functor. In fact, we can say a little bit more. Namely, suppose that $V' = q(y')$, $V = q(y)$, $U' = p(u(y'))$ and $U = p(u(y))$. Then

$$\begin{tikzcd} U' \arrow[r, "p(u(\beta))"] \arrow[d] & U \arrow[d] \\ V' \arrow[r, "q(\beta)"] & V \end{tikzcd}$$

is a fibre product square. This is true because $U' \to U$ represents the base change $(Sch/V')_{fppf} \times_{y', \mathcal{Y}} \mathcal{X} \to (Sch/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}$ of $V' \to V$.

Lemma 5.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Then the functor $u : \mathcal{Y}_\tau \to \mathcal{X}_\tau$ is continuous and defines a morphism of sites $\mathcal{X}_\tau \to \mathcal{Y}_\tau$ which induces the same morphism of topoi $Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ as the morphism f constructed in Lemma 4.4. Moreover, $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$ for any presheaf $\mathcal{F}$ on $\mathcal{X}$.*

Proof. Let $\{y_i \to y\}$ be a τ-covering in $\mathcal{Y}$. By definition this simply means that $\{q(y_i) \to q(y)\}$ is a τ-covering of schemes. By the final remark above the lemma we see that $\{p(u(y_i)) \to p(u(y))\}$ is the base change of the τ-covering $\{q(y_i) \to q(y)\}$ by $p(u(y)) \to q(y)$, hence is itself a τ-covering by the axioms of a site. Hence $\{u(y_i) \to u(y)\}$ is a τ-covering of $\mathcal{X}$. This proves that u is continuous.

Let's use the notation u_p, u_s, u^p, u^s of Sites, Sections 5 and 13. If we can show the final assertion of the lemma, then we see that $f_* = u^p = u^s$ (by continuity of u seen above) and hence by adjointness $f^{-1} = u_s$ which will prove u_s is exact, hence that u determines a morphism of sites, and the equality will be clear as well. To see that $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$ note that by definition

$$f_*\mathcal{F}(y) = ({}_pf\mathcal{F})(y) = \lim_{f(x) \to y} \mathcal{F}(x).$$

Since $u(y)$ is a final object in the category the limit is taken over we conclude. □

6. The structure sheaf

Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$ has a final object, namely, $\text{id} : (Sch/S)_{fppf} \to (Sch/S)_{fppf}$ and p is a 1-morphism from $\mathcal{X}$ to this final object. Hence any presheaf $\mathcal{G}$ on $(Sch/S)_{fppf}$ gives a presheaf $p^{-1}\mathcal{G}$ on $\mathcal{X}$ defined by the rule $p^{-1}\mathcal{G}(x) = \mathcal{G}(p(x))$. Moreover, the discussion in Section 4 shows that $p^{-1}\mathcal{G}$ is a τ sheaf whenever $\mathcal{G}$ is a τ-sheaf.

Recall that the site $(Sch/S)_{fppf}$ is a ringed site with structure sheaf $\mathcal{O}$ defined by the rule

$$(Sch/S)^{opp} \longrightarrow Rings, \quad U/S \longmapsto \Gamma(U, \mathcal{O}_U)$$

see Descent, Definition 8.2.

Definition 6.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The *structure sheaf of* $\mathcal{X}$ is the sheaf of rings $\mathcal{O}_\mathcal{X} = p^{-1}\mathcal{O}$.

For an object x of $\mathcal{X}$ lying over U we have $\mathcal{O}_\mathcal{X}(x) = \mathcal{O}(U) = \Gamma(U, \mathcal{O}_U)$. Needless to say $\mathcal{O}_\mathcal{X}$ is also a Zariski, étale, smooth, and syntomic sheaf, and hence each of the sites $\mathcal{X}_{Zar}$, $\mathcal{X}_{étale}$, $\mathcal{X}_{smooth}$, $\mathcal{X}_{syntomic}$, and $\mathcal{X}_{fppf}$ is a ringed site. This construction is functorial as well.

Lemma 6.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. There is a canonical identification $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ which turns $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ into a morphism of ringed topoi.*

Proof. Denote $p : \mathcal{X} \to (Sch/S)_{fppf}$ and $q : \mathcal{Y} \to (Sch/S)_{fppf}$ the structural functors. Then $p = q \circ f$, hence $p^{-1} = f^{-1} \circ q^{-1}$ by Lemma 3.2. Since $\mathcal{O}_\mathcal{X} = p^{-1}\mathcal{O}$ and $\mathcal{O}_\mathcal{Y} = q^{-1}\mathcal{O}$ the result follows. $\square$

Remark 6.3. In the situation of Lemma 6.2 the morphism of ringed topoi $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ is flat as is clear from the equality $f^{-1}\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$. This is a bit counter intuitive, for example because a closed immersion of algebraic stacks is typically not flat (as a morphism of algebraic stacks). However, exactly the same thing happens when taking a closed immersion $i : X \to Y$ of schemes: in this case the associated morphism of big τ-sites $i : (Sch/X)_\tau \to (Sch/Y)_\tau$ also is flat.

7. Sheaves of modules

Since we have a structure sheaf we have modules.

Definition 7.1. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$.

(1) A *presheaf of modules on* $\mathcal{X}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules. The category of presheaves of modules is denoted $PMod(\mathcal{O}_\mathcal{X})$.
(2) We say a presheaf of modules $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$*-module*, or more precisely a *sheaf of* $\mathcal{O}_\mathcal{X}$*-modules* if $\mathcal{F}$ is an fppf sheaf. The category of $\mathcal{O}_\mathcal{X}$-modules is denoted $Mod(\mathcal{O}_\mathcal{X})$.

These (pre)sheaves of modules occur in the literature as *(pre)sheaves of* $\mathcal{O}_\mathcal{X}$*-modules on the big fppf site of* $\mathcal{X}$. We will occasionally use this terminology if we want to distinguish these categories from others. We will also encounter presheaves of modules which are sheaves in the Zariski, étale, smooth, or syntomic topologies (without necessarily being sheaves). If need be these will be denoted $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and similarly for the other topologies.

Next, we address functoriality – first for presheaves of modules. Let

$$
\begin{tikzcd}
\mathcal{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{Y} \arrow[dl, "q"] \\
& (Sch/S)_{fppf} &
\end{tikzcd}
$$

be a 1-morphism of categories fibred in groupoids. The functors f^{-1}, f_* on abelian presheaves extend to functors

$$f^{-1} : PMod(\mathcal{O}_\mathcal{Y}) \longrightarrow PMod(\mathcal{O}_\mathcal{X}) \quad \text{and} \quad f_* : PMod(\mathcal{O}_\mathcal{X}) \longrightarrow PMod(\mathcal{O}_\mathcal{Y}) \tag{7.1.1}$$

This is immediate for f^{-1} because $f^{-1}\mathcal{G}(x) = \mathcal{G}(f(x))$ which is a module over $\mathcal{O}_\mathcal{Y}(f(x)) = \mathcal{O}(q(f(x))) = \mathcal{O}(p(x)) = \mathcal{O}_\mathcal{X}(x)$. Alternatively it follows because $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ and because f^{-1} commutes with limits (on presheaves). Since f_* is a right adjoint it commutes with all limits (on presheaves) in particular products. Hence we can extend f_* to a functor on presheaves of modules as in the proof of Modules on Sites, Lemma 12.1. We claim that the functors (7.1.1) form an adjoint pair of functors:

$$\mathrm{Mor}_{PMod(\mathcal{O}_\mathcal{X})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PMod(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}).$$

As $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ this follows from Modules on Sites, Lemma 12.3 by endowing $\mathcal{X}$ and $\mathcal{Y}$ with the chaotic topology.

Next, we discuss functoriality for modules, i.e., for sheaves of modules in the fppf topology. Denote by f also the induced morphism of ringed topoi, see Lemma 6.2 (for the fppf topologies right now). Note that the functors f^{-1} and f_* of (7.1.1) preserve the subcategories of sheaves of modules, see Lemma 4.4. Hence it follows immediately that

$$f^{-1} : Mod(\mathcal{O}_\mathcal{Y}) \longrightarrow Mod(\mathcal{O}_\mathcal{X}) \quad \text{and} \quad f_* : Mod(\mathcal{O}_\mathcal{X}) \longrightarrow Mod(\mathcal{O}_\mathcal{Y}) \tag{7.1.2}$$

form an adjoint pair of functors:

$$\mathrm{Mor}_{Mod(\mathcal{O}_\mathcal{X})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}).$$

By uniqueness of adjoints we conclude that $f^* = f^{-1}$ where f^* is as defined in Modules on Sites, Section 13 for the morphism of ringed topoi f above. Of course we could have seen this directly because $f^*(-) = f^{-1}(-) \otimes_{f^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}$ and because $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$.

Similarly for sheaves of modules in the Zariski, étale, smooth, syntomic topology.

8. Representable categories

In this short section we compare our definitions with what happens in case the algebraic stacks in question are representable.

Lemma 8.1. *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over (Sch/S). Assume $\mathcal{X}$ is representable by a scheme X. For $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ there is a canonical equivalence*

$$(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X}) = ((Sch/X)_\tau, \mathcal{O}_X)$$

of ringed sites.

Proof. This follows by choosing an equivalence $(Sch/X)_\tau \to \mathcal{X}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ and using the functoriality of the construction $\mathcal{X} \rightsquigarrow \mathcal{X}_\tau$. $\square$

Lemma 8.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over S. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by schemes X, Y. Let $f : X \to Y$ be the morphism of schemes corresponding to f. For $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ the morphism of ringed topoi $f : (Sh(\mathcal{X}_\tau), \mathcal{O}_\mathcal{X}) \to (Sh(\mathcal{Y}_\tau), \mathcal{O}_\mathcal{Y})$ agrees with the morphism of ringed topoi $f : (Sh((Sch/X)_\tau), \mathcal{O}_X) \to (Sh((Sch/Y)_\tau), \mathcal{O}_Y)$ via the identifications of Lemma 8.1.*

Proof. Follows by unwinding the definitions. $\square$

9. Restriction

A trivial but useful observation is that the localization of a category fibred in groupoids at an object is equivalent to the big site of the scheme it lies over.

Lemma 9.1. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. The functor p induces an equivalence of sites $\mathcal{X}_\tau/x \to (Sch/U)_\tau$.*

Proof. Special case of Stacks, Lemma 10.4. $\square$

We use the lemma above to talk about the pullback and the restriction of a (pre)sheaf to a scheme.

Definition 9.2. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$.

(1) The *pullback $x^{-1}\mathcal{F}$ of $\mathcal{F}$* is the restriction $\mathcal{F}|_{(\mathcal{X}/x)}$ viewed as a presheaf on $(Sch/U)_{fppf}$ via the equivalence $\mathcal{X}/x \to (Sch/U)_{fppf}$ of Lemma 9.1.
(2) The *restriction of $\mathcal{F}$ to $U_{étale}$* is $x^{-1}\mathcal{F}|_{U_{étale}}$, abusively written $\mathcal{F}|_{U_{étale}}$.

This notation makes sense because to the object x the 2-Yoneda lemma, see Algebraic Stacks, Section 5 associates a 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}/x$ which is quasi-inverse to $p : \mathcal{X}/x \to (Sch/U)_{fppf}$. Hence $x^{-1}\mathcal{F}$ truly is the pullback of $\mathcal{F}$ via this 1-morphism. In particular, by the material above, if $\mathcal{F}$ is a sheaf (or a Zariski, étale, smooth, syntomic sheaf), then $x^{-1}\mathcal{F}$ is a sheaf on $(Sch/U)_{fppf}$ (or on $(Sch/U)_{Zar}$, $(Sch/U)_{étale}$, $(Sch/U)_{smooth}$, $(Sch/U)_{syntomic}$).

Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\varphi : x \to y$ be a morphism of $\mathcal{X}$ lying over the morphism of schemes $a : U \to V$. Recall that a induces a morphism of small étale sites $a_{small} : U_{étale} \to V_{étale}$, see Étale Cohomology, Section 34. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. Let $\mathcal{F}|_{U_{étale}}$ and $\mathcal{F}|_{V_{étale}}$ be the restrictions of $\mathcal{F}$ via x and y. There is a natural *comparison* map

$$c_\varphi : \mathcal{F}|_{V_{étale}} \longrightarrow a_{small,*}(\mathcal{F}|_{U_{étale}}) \tag{9.2.1}$$

of presheaves on $U_{étale}$. Namely, if $V' \to V$ is étale, set $U' = V' \times_V U$ and define c_φ on sections over V' via

$$\begin{tikzcd}
a_{small,*}(\mathcal{F}|_{U_{étale}})(V') \arrow[r, equal] & \mathcal{F}|_{U_{étale}}(U') \arrow[r, equal] & \mathcal{F}(x') \\
\mathcal{F}|_{V_{étale}}(V') \arrow[u, "c_\varphi"] \arrow[rr, equal] & & \mathcal{F}(y') \arrow[u, "\mathcal{F}(\varphi')"']
\end{tikzcd}$$

Here $\varphi' : x' \to y'$ is a morphism of $\mathcal{X}$ fitting into a commutative diagram

$$\begin{tikzcd} x' \arrow[r] \arrow[d, "\varphi'"'] & x \arrow[d, "\varphi"] \\ y' \arrow[r] & y \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d, "a"] \\ V' \arrow[r] & V \end{tikzcd}$$

The existence and uniqueness of φ' follow from the axioms of a category fibred in groupoids. We omit the verification that c_φ so defined is indeed a map of presheaves (i.e., compatible with restriction mappings) and that it is functorial in $\mathcal{F}$. In case $\mathcal{F}$ is a sheaf for the étale topology we obtain a *comparison* map

$$c_\varphi : a_{small}^{-1}(\mathcal{F}|_{V_{étale}}) \longrightarrow \mathcal{F}|_{U_{étale}} \tag{9.2.2}$$

which is also denoted c_φ as indicated (this is the customary abuse of notation in not distinguishing between adjoint maps).

Lemma 9.3. *Let $\mathcal{F}$ be an étale sheaf on $\mathcal{X} \to (Sch/S)_{fppf}$.*

(1) *If $\varphi : x \to y$ and $\psi : y \to z$ are morphisms of $\mathcal{X}$ lying over $a : U \to V$ and $b : V \to W$, then the composition*
$$a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{étale}})) \xrightarrow{a_{small}^{-1}c_\psi} a_{small}^{-1}(\mathcal{F}|_{V_{étale}}) \xrightarrow{c_\varphi} \mathcal{F}|_{U_{étale}}$$
is equal to $c_{\psi\circ\varphi}$ via the identification
$$(b \circ a)_{small}^{-1}(\mathcal{F}|_{W_{étale}}) = a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{étale}})).$$
(2) *If $\varphi : x \to y$ lies over an étale morphism of schemes $a : U \to V$, then (9.2.2) is an isomorphism.*
(3) *Suppose $f : \mathcal{Y} \to \mathcal{X}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ and y is an object of $\mathcal{Y}$ lying over the scheme U with image $x = f(y)$. Then there is a canonical identification $f^{-1}\mathcal{F}|_{U_{étale}} = \mathcal{F}|_{U_{étale}}$.*
(4) *Moreover, given $\psi : y' \to y$ in $\mathcal{Y}$ lying over $a : U' \to U$ the comparison map $c_\psi : a_{small}^{-1}(f^{-1}\mathcal{F}|_{U_{étale}}) \to f^{-1}\mathcal{F}|_{U'_{étale}}$ is equal to the comparison map $c_{f(\psi)} : a_{small}^{-1}\mathcal{F}|_{U_{étale}} \to \mathcal{F}|_{U'_{étale}}$ via the identifications in (3).*

Proof. The verification of these properties is omitted. □

Next, we turn to the restriction of (pre)sheaves of modules.

Lemma 9.4. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. The equivalence of Lemma 9.1 extends to an equivalence of ringed sites $(\mathcal{X}_\tau/x, \mathcal{O}_\mathcal{X}|_x) \to ((Sch/U)_\tau, \mathcal{O})$.*

Proof. This is immediate from the construction of the structure sheaves. □

Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a (pre)sheaf of modules on $\mathcal{X}$ as in Definition 7.1. Let x be an object of $\mathcal{X}$ lying over U. Then Lemma 9.4 guarantees that the restriction $x^{-1}\mathcal{F}$ is a (pre)sheaf of modules on $(Sch/U)_{fppf}$. We will sometimes write $x^*\mathcal{F} = x^{-1}\mathcal{F}$ in this case. Similarly, if $\mathcal{F}$ is a sheaf for the Zariski, étale, smooth, or syntomic topology, then $x^{-1}\mathcal{F}$ is as well. Moreover, the restriction $\mathcal{F}|_{U_{étale}} = x^{-1}\mathcal{F}|_{U_{étale}}$ to U is a presheaf of $\mathcal{O}_{U_{étale}}$-modules. If $\mathcal{F}$ is a sheaf for the étale topology, then $\mathcal{F}|_{U_{étale}}$ is a sheaf of modules. Moreover, if $\varphi : x \to y$ is a morphism of $\mathcal{X}$ lying over $a : U \to V$ then the comparison map (9.2.2) is compatible with $a^\sharp_{small}$ (see Descent, Remark 8.4) and induces a *comparison* map

$$c_\varphi : a_{small}^*(\mathcal{F}|_{V_{étale}}) \longrightarrow \mathcal{F}|_{U_{étale}} \tag{9.4.1}$$

of $\mathcal{O}_{U_{étale}}$-modules. Note that the properties (1), (2), (3), and (4) of Lemma 9.3 hold in the setting of étale sheaves of modules as well. We will use this in the following without further mention.

Lemma 9.5. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The site $\mathcal{X}_\tau$ has enough points.*

Proof. By Sites, Lemma 38.5 we have to show that there exists a family of objects x of $\mathcal{X}$ such that $\mathcal{X}_\tau/x$ has enough points and such that the sheaves $h_x^\#$ cover the final object of the category of sheaves. By Lemma 9.1 and Étale Cohomology, Lemma 30.1 we see that $\mathcal{X}_\tau/x$ has enough points for every object x and we win. $\square$

10. Restriction to algebraic spaces

In this section we consider sheaves on categories representable by algebraic spaces. The following lemma is the analogue of Topologies, Lemma 4.14 for algebraic spaces.

Lemma 10.1. *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume $\mathcal{X}$ is representable by an algebraic space F. Then there exists a continuous and cocontinuous functor $F_{étale} \to \mathcal{X}_{étale}$ which induces a morphism of ringed sites*

$$\pi_F : (\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \longrightarrow (F_{étale}, \mathcal{O}_F)$$

and a morphism of ringed topoi

$$i_F : (Sh(F_{étale}), \mathcal{O}_F) \longrightarrow (Sh(\mathcal{X}_{étale}), \mathcal{O}_\mathcal{X})$$

such that $\pi_F \circ i_F = id$. Moreover $\pi_{F,*} = i_F^{-1}$.

Proof. Choose an equivalence $j : \mathcal{S}_F \to \mathcal{X}$, see Algebraic Stacks, Sections 7 and 8. An object of $F_{étale}$ is a scheme U together with an étale morphism $\varphi : U \to F$. Then φ is an object of $\mathcal{S}_F$ over U. Hence $j(\varphi)$ is an object of $\mathcal{X}$ over U. In this way j induces a functor $u : F_{étale} \to \mathcal{X}$. It is clear that u is continuous and cocontinuous for the étale topology on $\mathcal{X}$. Since j is an equivalence, the functor u is fully faithful. Also, fibre products and equalizers exist in $F_{étale}$ and u commutes with them because these are computed on the level of underlying schemes in $F_{étale}$. Thus Sites, Lemmas 21.5, 21.6, and 21.7 apply. In particular u defines a morphism of topoi $i_F : Sh(F_{étale}) \to Sh(\mathcal{X}_{étale})$ and there exists a left adjoint $i_{F,!}$ of i_F^{-1} which commutes with fibre products and equalizers.

We claim that $i_{F,!}$ is exact. If this is true, then we can define π_F by the rules $\pi_F^{-1} = i_{F,!}$ and $\pi_{F,*} = i_F^{-1}$ and everything is clear. To prove the claim, note that we already know that $i_{F,!}$ is right exact and preserves fibre products. Hence it suffices to show that $i_{F,!}* = *$ where $*$ indicates the final object in the category of sheaves of sets. Let U be a scheme and let $\varphi : U \to F$ be surjective and étale. Set $R = U \times_F U$. Then

$$h_R \rightrightarrows h_U \longrightarrow *$$

is a coequalizer diagram in $Sh(F_{étale})$. Using the right exactness of $i_{F,!}$, using $i_{F,!} = (u_p\ )^\#$, and using Sites, Lemma 5.6 we see that

$$h_{u(R)} \rightrightarrows h_{u(U)} \longrightarrow i_{F,!}*$$

is a coequalizer diagram in $Sh(F_{étale})$. Using that j is an equivalence and that $F = U/R$ it follows that the coequalizer in $Sh(\mathcal{X}_{étale})$ of the two maps $h_{u(R)} \to h_{u(U)}$ is $*$. We omit the proof that these morphisms are compatible with structure sheaves. $\square$

Remark 10.2. The constructions in Lemma 10.1 are compatible with étale localization. Here is a precise formulation. Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G, and that the induced morphism $f : F \to G$ of algebraic spaces is étale. Denote $f_{small} : F_{étale} \to G_{étale}$ the corresponding morphism of ringed topoi. Then

$$\begin{tikzcd}
(Sh(F_{étale}), \mathcal{O}_F) \arrow[r, "f_{small}"'] \arrow[d, "i_F"'] & (Sh(G_{étale}), \mathcal{O}_G) \arrow[d, "i_G"] \\
(Sh(\mathcal{X}_{étale}), \mathcal{O}_\mathcal{X}) \arrow[r, "f"'] \arrow[d, "\pi_F"'] & (Sh(\mathcal{Y}_{étale}), \mathcal{O}_\mathcal{Y}) \arrow[d, "\pi_G"] \\
(Sh(F_{étale}), \mathcal{O}_F) \arrow[r, "f_{small}"] & (Sh(G_{étale}), \mathcal{O}_G)
\end{tikzcd}$$

is a commutative diagram of ringed topoi. We omit the details.

Assume $\mathcal{X}$ is an algebraic stack represented by the algebraic space F. Let $j : \mathcal{S}_F \to \mathcal{X}$ be an equivalence and denote $u : F_{étale} \to \mathcal{X}_{étale}$ the functor of the proof of Lemma 10.1 above. Given a sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$ we have

$$\pi_{F,*}\mathcal{F}(U) = i_F^{-1}\mathcal{F}(U) = \mathcal{F}(u(U)).$$

This is why we often think of i_F^{-1} as a *restriction functor* similarly to Definition 9.2 and to the restriction of a sheaf on the big étale site of a scheme to the small étale site of a scheme. We often use the notation

$$\mathcal{F}|_{F_{étale}} = i_F^{-1}\mathcal{F} = \pi_{F,*}\mathcal{F} \tag{10.2.1}$$

in this situation.

Lemma 10.3. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G. Denote $f : F \to G$ the induced morphism of algebraic spaces, and $f_{small} : F_{étale} \to G_{étale}$ the corresponding morphism of ringed topoi. Then*

$$\begin{tikzcd}
(Sh(\mathcal{X}_{étale}), \mathcal{O}_\mathcal{X}) \arrow[r, "f"'] \arrow[d, "\pi_F"'] & (Sh(\mathcal{Y}_{étale}), \mathcal{O}_\mathcal{Y}) \arrow[d, "\pi_G"] \\
(Sh(F_{étale}), \mathcal{O}_F) \arrow[r, "f_{small}"] & (Sh(G_{étale}), \mathcal{O}_G)
\end{tikzcd}$$

is a commutative diagram of ringed topoi.

Proof. This is similar to Topologies, Lemma 4.17 (3) but there is a small snag due to the fact that $F \to G$ may not be representable by schemes. In particular we don't get a commutative diagram of ringed sites, but only a commutative diagram of ringed topoi.

Before we start the proof proper, we choose equivalences $j : \mathcal{S}_F \to \mathcal{X}$ and $j' : \mathcal{S}_G \to \mathcal{Y}$ which induce functors $u : F_{étale} \to \mathcal{X}$ and $u' : G_{étale} \to \mathcal{Y}$ as in the proof of Lemma 10.1. Because of the 2-functoriality of sheaves on categories fibred in groupoids over Sch_{fppf} (see discussion in Section 3) we may assume that $\mathcal{X} = \mathcal{S}_F$ and $\mathcal{Y} = \mathcal{S}_G$ and that $f : \mathcal{S}_F \to \mathcal{S}_G$ is the functor associated to the morphism $f : F \to G$. Correspondingly we will omit u and u' from the notation, i.e., given an object $U \to F$ of $F_{étale}$ we denote U/F the corresponding object of $\mathcal{X}$. Similarly for G.

Let $\mathcal{G}$ be a sheaf on $\mathcal{X}_{étale}$. To prove (2) we compute $\pi_{G,*}f_*\mathcal{G}$ and $f_{small,*}\pi_{F,*}\mathcal{G}$. To do this let $V \to G$ be an object of $G_{étale}$. Then

$$\pi_{G,*}f_*\mathcal{G}(V) = f_*\mathcal{G}(V/G) = \Gamma\Big((Sch/V)_{fppf} \times_{\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{G}\Big)$$

see (5.0.1). The fibre product in the formula is

$$(Sch/V)_{fppf} \times_{\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{\mathcal{S}_G} \mathcal{S}_F = \mathcal{S}_{V \times_G F}$$

i.e., it is the split category fibred in groupoids associated to the algebraic space $V \times_G F$. And $\mathrm{pr}^{-1}\mathcal{G}$ is a sheaf on $\mathcal{S}_{V\times_G F}$ for the étale topology.

In particular, if $V \times_G F$ is representable, i.e., if it is a scheme, then $\pi_{G,*}f_*\mathcal{G}(V) = \mathcal{G}(V \times_G F/F)$ and also

$$f_{small,*}\pi_{F,*}\mathcal{G}(V) = \pi_{F,*}\mathcal{G}(V \times_G F) = \mathcal{G}(V \times_G F/F)$$

which proves the desired equality in this special case.

In general, choose a scheme U and a surjective étale morphism $U \to V \times_G F$. Set $R = U \times_{V\times_G F} U$. Then $U/V \times_G F$ and $R/V \times_G F$ are objects of the fibre product category above. Since $\mathrm{pr}^{-1}\mathcal{G}$ is a sheaf for the étale topology on $\mathcal{S}_{V\times_G F}$ the diagram

$$\Gamma\Big((Sch/V)_{fppf} \times_{\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{G}\Big) \longrightarrow \mathrm{pr}^{-1}\mathcal{G}(U/V \times_G F) \rightrightarrows \mathrm{pr}^{-1}\mathcal{G}(R/V \times_G F)$$

is an equalizer diagram. Note that $\mathrm{pr}^{-1}\mathcal{G}(U/V\times_G F) = \mathcal{G}(U/F)$ and $\mathrm{pr}^{-1}\mathcal{G}(R/V\times_G F) = \mathcal{G}(R/F)$ by the definition of pullbacks. Moreover, by the material in Properties of Spaces, Section 18 (especially, Properties of Spaces, Remark 18.4 and Lemma 18.8) we see that there is an equalizer diagram

$$f_{small,*}\pi_{F,*}\mathcal{G}(V) \longrightarrow \pi_{F,*}\mathcal{G}(U/F) \rightrightarrows \pi_{F,*}\mathcal{G}(R/F)$$

Since we also have $\pi_{F,*}\mathcal{G}(U/F) = \mathcal{G}(U/F)$ and $\pi_{F,*}\mathcal{G}(U/F) = \mathcal{G}(U/F)$ we obtain a canonical identification $f_{small,*}\pi_{F,*}\mathcal{G}(V) = \pi_{G,*}f_*\mathcal{G}(V)$. We omit the proof that this is compatible with restriction mappings and that it is functorial in $\mathcal{G}$. $\square$

Let $f : \mathcal{X} \to \mathcal{Y}$ and $f : F \to G$ be as in the second part of the lemma above. A consequence of the lemma, using (10.2.1), is that

$$(f_*\mathcal{F})|_{G_{étale}} = f_{small,*}(\mathcal{F}|_{F_{étale}}) \tag{10.3.1}$$

for any sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$. Moreover, if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (10.3.1) is an isomorphism of $\mathcal{O}_G$-modules on $G_{étale}$.

Finally, suppose that we have a 2-commutative diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "a"] \arrow[dr, "f"'] & \mathcal{V} \arrow[d, "g"] \arrow[dl, phantom, "\Swarrow\varphi"] \\ & \mathcal{X} \end{tikzcd}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$, that $\mathcal{F}$ is a sheaf on $\mathcal{X}_{étale}$, and that $\mathcal{U}, \mathcal{V}$ are representable by algebraic spaces U, V. Then we obtain a comparison map

$$c_\varphi : a_{small}^{-1}(g^{-1}\mathcal{F}|_{V_{étale}}) \longrightarrow f^{-1}\mathcal{F}|_{U_{étale}} \tag{10.3.2}$$

where $a : U \to V$ denotes the morphism of algebraic spaces corresponding to a. This is the analogue of (9.2.2). We define c_φ as the adjoint to the map

$$g^{-1}\mathcal{F}|_{V_{étale}} \longrightarrow a_{small,*}(f^{-1}\mathcal{F}|_{U_{étale}}) = (a_*f^{-1}\mathcal{F})|_{V_{étale}}$$

(equality by (10.3.1)) which is the restriction to V (10.2.1) of the map

$$g^{-1}\mathcal{F} \to a_*a^{-1}g^{-1}\mathcal{F} = a_*f^{-1}\mathcal{F}$$

where the last equality uses the 2-commutativity of the diagram above. In case $\mathcal{F}$ is a sheaf of $\mathcal{O}_\mathcal{X}$-modules c_φ induces a *comparison* map

$$c_\varphi : a_{small}^{*}(g^{*}\mathcal{F}|_{V_{étale}}) \longrightarrow f^{*}\mathcal{F}|_{U_{étale}} \tag{10.3.3}$$

of $\mathcal{O}_{U_{étale}}$-modules. This is the analogue of (9.4.1). Note that the properties (1), (2), (3), and (4) of Lemma 9.3 hold in this setting as well.

11. Quasi-coherent modules

At this point we can apply the general definition of a quasi-coherent module to the situation discussed in this chapter.

Definition 11.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. A *quasi-coherent module on* $\mathcal{X}$, or a *quasi-coherent* $\mathcal{O}_\mathcal{X}$*-module* is a quasi-coherent module on the ringed site $(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ as in Modules on Sites, Definition 23.1. The category of quasi-coherent sheaves on $\mathcal{X}$ is denoted $QCoh(\mathcal{O}_\mathcal{X})$.

If $\mathcal{X}$ is an algebraic stack, then this definition agrees with all definitions in the literature in the sense that $QCoh(\mathcal{O}_\mathcal{X})$ is equivalent (modulo set theoretic issues) to any variant of this category defined in the literature. For example, we will match our definition with the definition in [Ols07, Definition 6.1] in Cohomology on Stacks, Lemma 12.2. We will also see alternative constructions of this category later on.

In general (as is the case for morphisms of schemes) the pushforward of quasi-coherent sheaf along a 1-morphism is not quasi-coherent. Pullback does preserve quasi-coherence.

Lemma 11.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The pullback functor $f^* = f^{-1} : Mod(\mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{O}_\mathcal{X})$ preserves quasi-coherent sheaves.*

Proof. This is a general fact, see Modules on Sites, Lemma 23.4. □

It turns out that quasi-coherent sheaves have a very simple characterization in terms of their pullbacks. See also Lemma 12.2 for a characterization in terms of restrictions.

Lemma 11.3. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if $x^*\mathcal{F}$ is a quasi-coherent sheaf on $(Sch/U)_{fppf}$ for every object x of $\mathcal{X}$ with $U = p(x)$.*

Proof. By Lemma 11.2 the condition is necessary. Conversely, since $x^*\mathcal{F}$ is just the restriction to $\mathcal{X}_{fppf}/x$ we see that it is sufficient directly from the definition of a quasi-coherent sheaf (and the fact that the notion of being quasi-coherent is an intrinsic property of sheaves of modules, see Modules on Sites, Section 18). □

Lemma 11.4. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of modules on $\mathcal{X}$. The following are equivalent*

1. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$ and $\mathcal{F}$ is a quasi-coherent module on $(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$ in the sense of Modules on Sites, Definition 23.1,*
2. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and $\mathcal{F}$ is a quasi-coherent module on $(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ in the sense of Modules on Sites, Definition 23.1, and*
3. *$\mathcal{F}$ is a quasi-coherent module on $\mathcal{X}$ in the sense of Definition 11.1.*

Proof. Assume either (1), (2), or (3) holds. Let x be an object of $\mathcal{X}$ lying over the scheme U. Recall that $x^*\mathcal{F} = x^{-1}\mathcal{F}$ is just the restriction to $\mathcal{X}/x = (Sch/U)_\tau$ where $\tau = fppf$, $\tau = étale$, or $\tau = Zar$, see Section 9. By the definition of quasi-coherent modules on a ringed site this restriction is quasi-coherent provided $\mathcal{F}$ is. By Descent, Proposition 8.9 we see that $x^*\mathcal{F}$ is the sheaf associated to a quasi-coherent $\mathcal{O}_U$-module and is therefore a quasi-coherent module in the fppf, étale, and Zariski topology; here we also use Descent, Lemma 8.1 and Definition 8.2. Since this holds for every object x of $\mathcal{X}$, we see that $\mathcal{F}$ is a sheaf in any of the three topologies. Moreover, we find that $\mathcal{F}$ is quasi-coherent in any of the three topologies directly from the definition of being quasi-coherent and the fact that x is an arbitrary object of $\mathcal{X}$. □

12. Locally quasi-coherent modules

Although there is a variant for the Zariski topology, it seems that the étale topology is the natural topology to use in the following definition.

Definition 12.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. We say $\mathcal{F}$ is *locally quasi-coherent*[2] if $\mathcal{F}$ is a sheaf for the étale topology and for every object x of $\mathcal{X}$ the restriction $x^*\mathcal{F}|_{U_{étale}}$ is a quasi-coherent sheaf. Here $U = p(x)$.

We use $LQCoh(\mathcal{O}_\mathcal{X})$ to indicate the category of locally quasi-coherent modules. We now have the following diagram of categories of modules

$$
\begin{tikzcd}
QCoh(\mathcal{O}_\mathcal{X}) \arrow[r] \arrow[d] & Mod(\mathcal{O}_\mathcal{X}) \arrow[d] \\
LQCoh(\mathcal{O}_\mathcal{X}) \arrow[r] & Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})
\end{tikzcd}
$$

where the arrows are strictly full embeddings. It turns out that many results for quasi-coherent sheaves have a counter part for locally quasi-coherent modules. Moreover, from many points of view (as we shall see later) this is a natural category to consider. For example the quasi-coherent sheaves are exactly those locally quasi-coherent modules that are "cartesian", i.e., satisfy the second condition of the lemma below.

[2]This is nonstandard notation.

Lemma 12.2. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if the following two conditions hold*

1. *$\mathcal{F}$ is locally quasi-coherent, and*
2. *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ lying over $f : U \to V$ the comparison map $c_\varphi : f_{small}^*\mathcal{F}|_{V_{\acute{e}tale}} \to \mathcal{F}|_{U_{\acute{e}tale}}$ of (9.4.1) is an isomorphism.*

Proof. Assume $\mathcal{F}$ is quasi-coherent. Then $\mathcal{F}$ is a sheaf for the fppf topology, hence a sheaf for the étale topology. Moreover, any pullback of $\mathcal{F}$ to a ringed topos is quasi-coherent, hence the restrictions $x^*\mathcal{F}|_{U_{\acute{e}tale}}$ are quasi-coherent. This proves $\mathcal{F}$ is locally quasi-coherent. Let y be an object of $\mathcal{X}$ with $V = p(y)$. We have seen that $\mathcal{X}/y = (Sch/V)_{fppf}$. By Descent, Proposition 8.9 it follows that $y^*\mathcal{F}$ is the quasi-coherent module associated to a (usual) quasi-coherent module $\mathcal{F}_V$ on the scheme V. Hence certainly the comparison maps (9.4.1) are isomorphisms.

Conversely, suppose that $\mathcal{F}$ satisfies (1) and (2). Let y be an object of $\mathcal{X}$ with $V = p(y)$. Denote $\mathcal{F}_V$ the quasi-coherent module on the scheme V corresponding to the restriction $y^*\mathcal{F}|_{V_{\acute{e}tale}}$ which is quasi-coherent by assumption (1), see Descent, Proposition 8.9. Condition (2) now signifies that the restrictions $x^*\mathcal{F}|_{U_{\acute{e}tale}}$ for x over y are each isomorphic to the (étale sheaf associated to the) pullback of $\mathcal{F}_V$ via the corresponding morphism of schemes $U \to V$. Hence $y^*\mathcal{F}$ is the sheaf on $(Sch/V)_{fppf}$ associated to $\mathcal{F}_V$. Hence it is quasi-coherent (by Descent, Proposition 8.9 again) and we see that $\mathcal{F}$ is quasi-coherent on $\mathcal{X}$ by Lemma 11.3. □

Lemma 12.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The pullback functor $f^* = f^{-1} : Mod(\mathcal{Y}_{\acute{e}tale}, \mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ preserves locally quasi-coherent sheaves.*

Proof. Let $\mathcal{G}$ be locally quasi-coherent on $\mathcal{Y}$. Choose an object x of $\mathcal{X}$ lying over the scheme U. The restriction $x^*f^*\mathcal{G}|_{U_{\acute{e}tale}}$ equals $(f \circ x)^*\mathcal{G}|_{U_{\acute{e}tale}}$ hence is a quasi-coherent sheaf by assumption on $\mathcal{G}$. □

Lemma 12.4. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.*

1. *The category $LQCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$.*
2. *The category $LQCoh(\mathcal{O}_\mathcal{X})$ is abelian with kernels and cokernels computed in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$, in other words the inclusion functor is exact.*
3. *Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ if two out of three are locally quasi-coherent so is the third.*
4. *Given $\mathcal{F}, \mathcal{G}$ in $LQCoh(\mathcal{O}_\mathcal{X})$ the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ is an object of $LQCoh(\mathcal{O}_\mathcal{X})$.*
5. *Given $\mathcal{F}, \mathcal{G}$ in $LQCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ of finite presentation on $\mathcal{X}_{\acute{e}tale}$ the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ is an object of $LQCoh(\mathcal{O}_\mathcal{X})$.*

Proof. In the arguments below x denotes an arbitrary object of $\mathcal{X}$ lying over the scheme U. To show that an object $\mathcal{H}$ of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ is in $LQCoh(\mathcal{O}_\mathcal{X})$ we will show that the restriction $x^*\mathcal{H}|_{U_{\acute{e}tale}} = \mathcal{H}|_{U_{\acute{e}tale}}$ is a quasi-coherent object of $Mod(U_{\acute{e}tale}, \mathcal{O}_U)$.

Proof of (1). Let $\mathcal{I} \to LQCoh(\mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ be a diagram. Consider the object $\mathcal{F} = \mathrm{colim}_i\, \mathcal{F}_i$ of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$. The pullback functor x^* commutes with all colimits as it is a left adjoint. Hence $x^*\mathcal{F} = \mathrm{colim}_i\, x^*\mathcal{F}_i$. Similarly we have $x^*\mathcal{F}|_{U_{\acute{e}tale}} =$

$\mathrm{colim}_i\, x^*\mathcal{F}_i|_{U_{\acute{e}tale}}$. Now by assumption each $x^*\mathcal{F}_i|_{U_{\acute{e}tale}}$ is quasi-coherent. Hence $\mathrm{colim}_i\, x^*\mathcal{F}_i|_{U_{\acute{e}tale}}$ is quasi-coherent by Descent, Lemma 10.3. Thus $x^*\mathcal{F}|_{U_{\acute{e}tale}}$ is quasi-coherent as desired.

Proof of (2). It follows from (1) that cokernels exist in $LQCoh(\mathcal{O}_\mathcal{X})$ and agree with the cokernels computed in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $LQCoh(\mathcal{O}_\mathcal{X})$ and let $\mathcal{K} = \mathrm{Ker}(\varphi)$ computed in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$. If we can show that $\mathcal{K}$ is a locally quasi-coherent module, then the proof of (2) is complete. To see this, note that kernels are computed in the category of presheaves (no sheafification necessary). Hence $\mathcal{K}|_{U_{\acute{e}tale}}$ is the kernel of the map $\mathcal{F}|_{U_{\acute{e}tale}} \to \mathcal{G}|_{U_{\acute{e}tale}}$, i.e., is the kernel of a map of quasi-coherent sheaves on $U_{\acute{e}tale}$ whence quasi-coherent by Descent, Lemma 10.3. This proves (2).

Proof of (3). Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$. Since we are using the étale topology, the restriction $0 \to \mathcal{F}_1|_{U_{\acute{e}tale}} \to \mathcal{F}_2|_{U_{\acute{e}tale}} \to \mathcal{F}_3|_{U_{\acute{e}tale}} \to 0$ is a short exact sequence too. Hence (3) follows from the corresponding statement in Descent, Lemma 10.3.

Proof of (4). Let $\mathcal{F}$ and $\mathcal{G}$ be in $LQCoh(\mathcal{O}_\mathcal{X})$. Since restriction to $U_{\acute{e}tale}$ is given by pullback along the morphism of ringed topoi $U_{\acute{e}tale} \to (Sch/U)_{\acute{e}tale} \to \mathcal{X}_{\acute{e}tale}$ we see that the restriction of the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ to $U_{\acute{e}tale}$ is equal to $\mathcal{F}|_{U_{\acute{e}tale}} \otimes_{\mathcal{O}_U} \mathcal{G}|_{U_{\acute{e}tale}}$, see Modules on Sites, Lemma 26.2. Since $\mathcal{F}|_{U_{\acute{e}tale}}$ and $\mathcal{G}|_{U_{\acute{e}tale}}$ are quasi-coherent, so is their tensor product, see Descent, Lemma 10.3.

Proof of (5). Let $\mathcal{F}$ and $\mathcal{G}$ be in $LQCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ of finite presentation. Since $(Sch/U)_{\acute{e}tale} = \mathcal{X}_{\acute{e}tale}/x$ is a localization of $\mathcal{X}_{\acute{e}tale}$ at an object we see that the restriction of $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ to $(Sch/U)_{\acute{e}tale}$ is equal to

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}|_{(Sch/U)_{\acute{e}tale}}}(\mathcal{F}|_{(Sch/U)_{\acute{e}tale}}, \mathcal{G}|_{(Sch/U)_{\acute{e}tale}})$$

by Modules on Sites, Lemma 27.2. The morphism of ringed topoi $(U_{\acute{e}tale}, \mathcal{O}_U) \to ((Sch/U)_{\acute{e}tale}, \mathcal{O})$ is flat as the pullback of $\mathcal{O}$ is $\mathcal{O}_U$. Hence the pullback of $\mathcal{H}$ by this morphism is equal to $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{\acute{e}tale}}, \mathcal{G}|_{U_{\acute{e}tale}})$ by Modules on Sites, Lemma 31.4. In other words, the restriction of $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ to $U_{\acute{e}tale}$ is $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{\acute{e}tale}}, \mathcal{G}|_{U_{\acute{e}tale}})$. Since $\mathcal{F}|_{U_{\acute{e}tale}}$ and $\mathcal{G}|_{U_{\acute{e}tale}}$ are quasi-coherent, so is $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{\acute{e}tale}}, \mathcal{G}|_{U_{\acute{e}tale}})$, see Descent, Lemma 10.3. We conclude as before. □

In the generality discussed here the category of quasi-coherent sheaves is not abelian. See Examples, Section 13. Here is what we can prove without any further work.

Lemma 12.5. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.*

(1) *The category $QCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the categories $Mod(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$, $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$, $Mod(\mathcal{O}_\mathcal{X})$, and $LQCoh(\mathcal{O}_\mathcal{X})$.*
(2) *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ the tensor products $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ computed in $Mod(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$, $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$, or $Mod(\mathcal{O}_\mathcal{X})$ agree and the common value is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*
(3) *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ finite locally free (in fppf, or equivalently étale, or equivalently Zariski topology) the internal homs $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ computed in $Mod(\mathcal{X}_{Zar}, \mathcal{O}_\mathcal{X})$, $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$, or $Mod(\mathcal{O}_\mathcal{X})$ agree and the common value is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*

Proof. Let x be an arbitrary object of $\mathcal{X}$ lying over the scheme U. Let $\tau \in \{Zariski, \acute{e}tale, fppf\}$. To show that an object $\mathcal{H}$ of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ is in $QCoh(\mathcal{O}_\mathcal{X})$ it suffices show that the restriction $x^*\mathcal{H}$ (Section 9) is a quasi-coherent object of

$Mod((Sch/U)_\tau, \mathcal{O})$. See Lemmas 11.3 and 11.4. Similarly for being finite locally free. Recall that $(Sch/U)_\tau = \mathcal{X}_\tau/x$ is a localization of $\mathcal{X}_\tau$ at an object. Hence restriction commutes with colimits, tensor products, and forming internal hom (see Modules on Sites, Lemmas 14.3, 26.2, and 27.2). This reduces the lemma to Descent, Lemma 10.6. □

13. Stackification and sheaves

It turns out that the category of sheaves on a category fibred in groupoids only "knows about" the stackification.

Lemma 13.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f induces an equivalence of stackifications, then the morphism of topoi $f : Sh(\mathcal{X}_{fppf}) \to Sh(\mathcal{Y}_{fppf})$ is an equivalence.*

Proof. We may assume $\mathcal{Y}$ is the stackification of $\mathcal{X}$. We claim that $f : \mathcal{X} \to \mathcal{Y}$ is a special cocontinuous functor, see Sites, Definition 29.2 which will prove the lemma. By Stacks, Lemma 10.3 the functor f is continuous and cocontinuous. By Stacks, Lemma 8.1 we see that conditions (3), (4), and (5) of Sites, Lemma 29.1 hold. □

Lemma 13.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f induces an equivalence of stackifications, then f^* induces equivalences $Mod(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{Y})$ and $QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{Y})$.*

Proof. We may assume $\mathcal{Y}$ is the stackification of $\mathcal{X}$. The first assertion is clear from Lemma 13.1 and $\mathcal{O}_\mathcal{X} = f^{-1}\mathcal{O}_\mathcal{Y}$. Pullback of quasi-coherent sheaves are quasi-coherent, see Lemma 11.2. Hence it suffices to show that if $f^*\mathcal{G}$ is quasi-coherent, then $\mathcal{G}$ is. To see this, let y be an object of $\mathcal{Y}$. Translating the condition that $\mathcal{Y}$ is the stackification of $\mathcal{X}$ we see there exists an fppf covering $\{y_i \to y\}$ in $\mathcal{Y}$ such that $y_i \cong f(x_i)$ for some x_i object of $\mathcal{X}$. Say x_i and y_i lie over the scheme U_i. Then $f^*\mathcal{G}$ being quasi-coherent, means that $x_i^*f^*\mathcal{G}$ is quasi-coherent. Since $x_i^*f^*\mathcal{G}$ is isomorphic to $y_i^*\mathcal{G}$ (as sheaves on $(Sch/U_i)_{fppf}$ we see that $y_i^*\mathcal{G}$ is quasi-coherent. It follows from Modules on Sites, Lemma 23.3 that the restriction of $\mathcal{G}$ to $\mathcal{Y}/y$ is quasi-coherent. Hence $\mathcal{G}$ is quasi-coherent by Lemma 11.3. □

14. Quasi-coherent sheaves and presentations

Let us first match quasi-coherent sheaves with our previously defined notions for schemes and algebraic spaces.

Lemma 14.1. *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids wich is representable by an algebraic space F. If $\mathcal{F}$ is in $LQCoh(\mathcal{O}_\mathcal{X})$ then the restriction $\mathcal{F}|_{F_{\acute{e}tale}}$ (10.2.1) is quasi-coherent.*

Proof. Let U be a scheme étale over F. Then $\mathcal{F}|_{U_{\acute{e}tale}} = (\mathcal{F}|_{F_{\acute{e}tale}})|_{U_{\acute{e}tale}}$. This is clear but see also Remark 10.2. Thus the assertion follows from the definitions. □

Lemma 14.2. *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids wich is representable by an algebraic space F. The functor (10.2.1) defines an equivalence*

$$QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_F), \quad \mathcal{F} \longmapsto \mathcal{F}|_{F_{\acute{e}tale}}$$

with quasi-inverse given by $\mathcal{G} \mapsto \pi_F^*\mathcal{G}$. This equivalence is compatible with pullback for morphisms between categories fibred in groupoids representable by algebraic spaces.

Proof. By Lemma 11.4 we may work with the étale topology. We will use the notation and results of Lemma 10.1 without further mention. Recall that the restriction functor $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \to Mod(F_{étale}, \mathcal{O}_F)$, $\mathcal{F} \mapsto \mathcal{F}|_{F_{étale}}$ is given by i_F^*. By Lemma 14.1 or by Modules on Sites, Lemma 23.4 we see that $\mathcal{F}|_{F_{étale}}$ is quasi-coherent if $\mathcal{F}$ is quasi-coherent. Hence we get a functor as indicated in the statement of the lemma and we get a functor π_F^* in the opposite direction. Since $\pi_F \circ i_F = \mathrm{id}$ we see that $i_F^*\pi_F^*\mathcal{G} = \mathcal{G}$.

For $\mathcal{F}$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ there is a canonical map $\pi_F^*(\mathcal{F}|_{F_{étale}}) \to \mathcal{F}$, namely the map adjoint to the identification $\mathcal{F}|_{F_{étale}} = \pi_{F,*}\mathcal{F}$. We will show that this map is an isomorphism if $\mathcal{F}$ is a quasi-coherent module on $\mathcal{X}$. Choose a scheme U and a surjective étale morphism $U \to F$. Denote $x : U \to \mathcal{X}$ the corresponding object of $\mathcal{X}$ over U. It suffices to show that $\pi_F^*(\mathcal{F}|_{F_{étale}}) \to \mathcal{F}$ is an isomorphism after restricting to $\mathcal{X}_{étale}/x = (Sch/U)_{étale}$. Since $U \to F$ is étale, it follows from Remark 10.2 that

$$\pi_F^*(\mathcal{F}|_{F_{étale}})|_{\mathcal{X}_{étale}/x} = \pi_U^*(\mathcal{F}|_{U_{étale}})$$

and that the restriction of the map $\pi_F^*(\mathcal{F}|_{F_{étale}}) \to \mathcal{F}$ to $\mathcal{X}_{étale}/x = (Sch/U)_{étale}$ is equal to the corresponding map $\pi_U^*(\mathcal{F}|_{U_{étale}}) \to \mathcal{F}|_{(Sch/U)_{étale}}$. Since we have seen the result is true for schemes in Descent, Section 8[3] we conclude.

Compatibility with pullbacks follows from the fact that the quasi-inverse is given by π_F^* and the commutative diagram of ringed topoi in Lemma 10.3. □

In Groupoids in Spaces, Definition 12.1 we have the defined the notion of a quasi-coherent module on an arbitrary groupoid. The following (formal) proposition tells us that we can study quasi-coherent sheaves on quotient stacks in terms of quasi-coherent modules on presentations.

Proposition 14.3. *Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Let $\mathcal{X} = [U/R]$ be the quotient stack. The category of quasi-coherent modules on $\mathcal{X}$ is equivalent to the category of quasi-coherent modules on (U, R, s, t, c).*

Proof. We will construct quasi-inverse functors

$$QCoh(\mathcal{O}_\mathcal{X}) \longleftrightarrow QCoh(U, R, s, t, c).$$

where $QCoh(U, R, s, t, c)$ denotes the category of quasi-coherent modules on the groupoid (U, R, s, t, c).

Let $\mathcal{F}$ be an object of $QCoh(\mathcal{O}_\mathcal{X})$. Denote $\mathcal{U}$, $\mathcal{R}$ the categories fibred in groupoids corresponding to U and R. Denote x the (defining) object of $\mathcal{X}$ over U. Recall that we have a 2-commutative diagram

$$\begin{tikzcd} \mathcal{R} \arrow[r, "s"'] \arrow[d, "t"'] & \mathcal{U} \arrow[d, "x"] \\ \mathcal{U} \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

[3]Namely, if U is a scheme and $\mathcal{F}$ is quasi-coherent on $(Sch/U)_{étale}$, then $\mathcal{F} = \mathcal{H}^a$ for some quasi-coherent module $\mathcal{H}$ on the scheme U by Descent, Proposition 8.9. In other words, $\mathcal{F} = (\mathrm{id}_{étale,Zar})^*\mathcal{H}$ by Descent, Remark 8.6 with notation as in Descent, Lemma 8.5. Then we have $\mathrm{id}_{étale,Zar} = \pi_U \circ \mathrm{id}_{small,étale,Zar}$ and hence we see that $\mathcal{F} = \pi_U^*\mathcal{G}$ where $\mathcal{G} = (\mathrm{id}_{small,étale,Zar})^*\mathcal{H}$ is quasi-coherent. Then $\pi_U^* i_U^* \mathcal{F} = \pi_U^* i_U^* \pi_U^* \mathcal{G} = \pi_U^*\mathcal{G} = \mathcal{F}$ as desired.

See Groupoids in Spaces, Lemma 20.3. By Lemma 3.3 the 2-arrow inherent in the diagram induces an isomorphism $\alpha : t^*x^*\mathcal{F} \to s^*x^*\mathcal{F}$ which satisfies the cocycle condition over $\mathcal{R} \times_{s,\mathcal{U},t} \mathcal{R}$; this is a consequence of Groupoids in Spaces, Lemma 23.1. Thus if we set $\mathcal{G} = x^*\mathcal{F}|_{U_{étale}}$ then the equivalence of categories in Lemma 14.2 (used several times compatibly with pullbacks) gives an isomorphism $\alpha : t_{small}^*\mathcal{G} \to s_{small}^*\mathcal{G}$ satisfying the cocycle condition on $R \times_{s,U,t} R$, i.e., $(\mathcal{G}, \alpha)$ is an object of $QCoh(U, R, s, t, c)$. The rule $\mathcal{F} \mapsto (\mathcal{G}, \alpha)$ is our functor from left to right.

Construction of the functor in the other direction. Let $(\mathcal{G}, \alpha)$ be an object of $QCoh(U, R, s, t, c)$. According to Lemma 13.2 the stackification map $[U/_pR] \to [U/R]$ (see Groupoids in Spaces, Definition 20.1) induces an equivalence of categories of quasi-coherent sheaves. Thus it suffices to construct a quasi-coherent module $\mathcal{F}$ on $[U/_pR]$.

Recall that an object $x = (T, u)$ of $[U/_pR]$ is given by a scheme T and a morphism $u : T \to U$. A morphism $(T, u) \to (T', u')$ is given by a pair (f, r) where $f : T \to T'$ and $r : T \to R$ with $s \circ r = u$ and $t \circ r = u' \circ f$. Let us call a *special morphism* any morphism of the form $(f, e \circ u' \circ f) : (T, u' \circ f) \to (T', u')$. The category of (T, u) with special morphisms is just the category of schemes over U.

With this notation in place, given an object (T, u) of $[U/_pR]$, we set

$$\mathcal{F}(T, u) := \Gamma(T, u_{small}^*\mathcal{G}).$$

Given a morphism $(f, r) : (T, u) \to (T', u')$ we get a map

$$\begin{aligned}
\mathcal{F}(T', u') &= \Gamma(T', (u')_{small}^*\mathcal{G}) \\
&\to \Gamma(T, f_{small}^*(u')_{small}^*\mathcal{G}) = \Gamma(T, (u' \circ f)_{small}^*\mathcal{G}) \\
&= \Gamma(T, (t \circ r)_{small}^*\mathcal{G}) = \Gamma(T, r_{small}^* t_{small}^*\mathcal{G}) \\
&\to \Gamma(T, r_{small}^* s_{small}^*\mathcal{G}) = \Gamma(T, (s \circ r)_{small}^*\mathcal{G}) \\
&= \Gamma(T, u_{small}^*\mathcal{G}) \\
&= \mathcal{F}(T, u)
\end{aligned}$$

where the first arrow is pullback along f and the second arrow is α. Note that if (T, r) is a special morphism, then this map is just pullback along f as $e_{small}^*\alpha = \text{id}$ by the axioms of a sheaf of quasi-coherent modules on a groupoid. The cocycle condition implies that $\mathcal{F}$ is a presheaf of modules (details omitted). We see that the restriction of $\mathcal{F}$ to $(Sch/T)_{fppf}$ is quasi-coherent by the simple description of the restriction maps of $\mathcal{F}$ in case of a special morphism. Hence $\mathcal{F}$ is a sheaf on $[U/_pR]$ and quasi-coherent (Lemma 11.3).

We omit the verification that the functors constructed above are quasi-inverse to each other. □

We finish this section with a technical lemma on maps out of quasi-coherent sheaves. It is an analogue of Schemes, Lemma 7.1. We will see later (Criteria for Representability, Theorem 17.2) that the assumptions on the groupoid imply that $\mathcal{X}$ is an algebraic stack.

Lemma 14.4. *Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Let $\mathcal{X} = [U/R]$ be the quotient stack.*

Denote x the object of $\mathcal{X}$ over U. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module, and let $\mathcal{H}$ be any object of $Mod(\mathcal{O}_\mathcal{X})$. The map

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{H}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(x^*\mathcal{F}|_{U_{étale}}, x^*\mathcal{H}|_{U_{étale}}), \quad \phi \longmapsto x^*\phi|_{U_{étale}}$$

is injective and its image consists of exactly those $\varphi : x^*\mathcal{F}|_{U_{étale}} \to x^*\mathcal{H}|_{U_{étale}}$ which give rise to a commutative diagram

$$\begin{tikzcd}
s_{small}^*(x^*\mathcal{F}|_{U_{étale}}) \arrow[r] \arrow[d, "s_{small}^*\varphi"] & (x \circ s)^*\mathcal{F}|_{R_{étale}} = (x \circ t)^*\mathcal{F}|_{R_{étale}} & t_{small}^*(x^*\mathcal{F}|_{U_{étale}}) \arrow[l] \arrow[d, "t_{small}^*\varphi"'] \\
s_{small}^*(x^*\mathcal{H}|_{U_{étale}}) \arrow[r] & (x \circ s)^*\mathcal{H}|_{R_{étale}} = (x \circ t)^*\mathcal{H}|_{R_{étale}} & t_{small}^*(x^*\mathcal{H}|_{U_{étale}}) \arrow[l]
\end{tikzcd}$$

of modules on $R_{étale}$ where the horizontal arrows are the comparison maps (10.3.3).

Proof. According to Lemma 13.2 the stackification map $[U/_pR] \to [U/R]$ (see Groupoids in Spaces, Definition 20.1) induces an equivalence of categories of quasi-coherent sheaves and of fppf $\mathcal{O}$-modules. Thus it suffices to prove the lemma with $\mathcal{X} = [U/_pR]$. By Proposition 14.3 and its proof there exists a quasi-coherent module $(\mathcal{G}, \alpha)$ on (U, R, s, t, c) such that $\mathcal{F}$ is given by the rule $\mathcal{F}(T, u) = \Gamma(T, u^*\mathcal{G})$. In particular $x^*\mathcal{F}|_{U_{étale}} = \mathcal{G}$ and it is clear that the map of the statement of the lemma is injective. Moreover, given a map $\varphi : \mathcal{G} \to x^*\mathcal{H}|_{U_{étale}}$ and given any object $y = (T, u)$ of $[U/_pR]$ we can consider the map

$$\mathcal{F}(y) = \Gamma(T, u^*\mathcal{G}) \xrightarrow{u_{small}^*\varphi} \Gamma(T, u_{small}^* x^*\mathcal{H}|_{U_{étale}}) \to \Gamma(T, y^*\mathcal{H}|_{T_{étale}}) = \mathcal{H}(y)$$

where the second arrow is the comparison map (9.4.1) for the sheaf $\mathcal{H}$. This assignment is compatible with the restriction mappings of the sheaves $\mathcal{F}$ and $\mathcal{G}$ for morphisms of $[U/_pR]$ if the cocycle condition of the lemma is satisfied. Proof omitted. Hint: the restriction maps of $\mathcal{F}$ are made explicit in terms of $(\mathcal{G}, \alpha)$ in the proof of Proposition 14.3. □

15. Quasi-coherent sheaves on algebraic stacks

Let $\mathcal{X}$ be an algebraic stack over S. By Algebraic Stacks, Lemma 16.2 we can find an equivalence $[U/R] \to \mathcal{X}$ where (U, R, s, t, c) is a smooth groupoid in algebraic spaces. Then

$$QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(\mathcal{O}_{[U/R]}) \cong QCoh(U, R, s, t, c)$$

where the second equivalence is Proposition 14.3. Hence the category of quasi-coherent sheaves on an algebraic stack is equivalent to the category of quasi-coherent modules on a smooth groupoid in algebraic spaces. In particular, by Groupoids in Spaces, Lemma 12.6 we see that $QCoh(\mathcal{O}_\mathcal{X})$ is abelian!

There is something slightly disconcerting about our current setup. It is that the fully faithful embedding

$$QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow Mod(\mathcal{O}_\mathcal{X})$$

is in general **not** exact. However, exactly the same thing happens for schemes: for most schemes X the embedding

$$QCoh(\mathcal{O}_X) \cong QCoh((Sch/X)_{fppf}, \mathcal{O}_X) \longrightarrow Mod((Sch/X)_{fppf}, \mathcal{O}_X)$$

isn't exact, see Descent, Lemma 10.2. Parenthetically, the example in the proof of Descent, Lemma 10.2 shows that in general the strictly full embedding $QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh(\mathcal{O}_\mathcal{X})$ isn't exact either.

We collect all the results obtained so far in a single statement.

Lemma 15.1. *Let $\mathcal{X}$ be an algebraic stack over S.*

1. *If $[U/R] \to \mathcal{X}$ is a presentation of $\mathcal{X}$ then there is a canonical equivalence $QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(U, R, s, t, c)$.*
2. *The category $QCoh(\mathcal{O}_\mathcal{X})$ is abelian.*
3. *The inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ is right exact but* **not** *exact in general.*
4. *The category $QCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $Mod(\mathcal{O}_\mathcal{X})$.*
5. *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ in $Mod(\mathcal{O}_\mathcal{X})$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*
6. *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ finite locally free the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in $Mod(\mathcal{O}_\mathcal{X})$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*
7. *Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ in $Mod(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ quasi-coherent, then $\mathcal{F}_2$ is quasi-coherent.*

Proof. Properties (4), (5), and (6) were proven in Lemma 12.5. Part (1) is Proposition 14.3. Part (2) follows from part (1) and Groupoids in Spaces, Lemma 12.6 as discussed above. Right exactness of the inclusion functor in (3) follows from (4); please compare with Homology, Lemma 7.2. For the nonexactness of the inclusion functor in part (3) see Descent, Lemma 10.2. To see (7) observe that it suffices to check the restriction of $\mathcal{F}_2$ to the big site of a scheme is quasi-coherent (Lemma 11.3), hence this follows from the corresponding part of Descent, Lemma 10.2. ☐

Next we construct the coherator for modules on an algebraic stack.

Proposition 15.2. *Let $\mathcal{X}$ be an algebraic stack over S.*

1. *The category $QCoh(\mathcal{O}_\mathcal{X})$ is a Grothendieck abelian category. Consequently, $QCoh(\mathcal{O}_\mathcal{X})$ has enough injectives and all limits.*
2. *The inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ has a right adjoint*[4]

$$Q : Mod(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$

such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

Proof. This proof is a repeat of the proof in the case of schemes, see Properties, Proposition 23.4 and the case of algebraic spaces, see Properties of Spaces, Proposition 32.2. We advise the reader to read either of those proofs first.

Part (1) means $QCoh(\mathcal{O}_\mathcal{X})$ (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 10. By Lemma 15.1 colimits in $QCoh(\mathcal{O}_X)$ exist and agree with colimits in $Mod(\mathcal{O}_X)$. By Modules on Sites, Lemma 14.2 filtered colimits are exact. Hence (a) and (b) hold.

Choose a presentation $\mathcal{X} = [U/R]$ so that (U, R, s, t, c) is a smooth groupoid in algebraic spaces and in particular s and t are flat morphisms of algebraic spaces.

[4] This functor is sometimes called the *coherator*.

By Lemma 15.1 above we have $QCoh(\mathcal{O}_\mathcal{X}) = QCoh(U, R, s, t, c)$. By Groupoids in Spaces, Lemma 14.2 there exists a set T and a family $(\mathcal{F}_t)_{t\in T}$ of quasi-coherent sheaves on $\mathcal{X}$ such that every quasi-coherent sheaf on $\mathcal{X}$ is the directed colimit of its subsheaves which are isomorphic to one of the $\mathcal{F}_t$. Thus $\bigoplus_t \mathcal{F}_t$ is a generator of $QCoh(\mathcal{O}_X)$ and we conclude that (c) holds. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 11.7 and Lemma 13.2.

Proof of (2). To construct Q we use the following general procedure. Given an object $\mathcal{F}$ of $Mod(\mathcal{O}_\mathcal{X})$ we consider the functor

$$QCoh(\mathcal{O}_\mathcal{X})^{opp} \longrightarrow Sets, \quad \mathcal{G} \longmapsto \mathrm{Hom}_\mathcal{X}(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 13.1. Thus there exists a quasi-coherent sheaf $Q(\mathcal{F})$ and a functorial isomorphism $\mathrm{Hom}_\mathcal{X}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_\mathcal{X}(\mathcal{G}, Q(\mathcal{F}))$ for $\mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$. By the Yoneda lemma (Categories, Lemma 3.5) the construction $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$ is functorial in $\mathcal{F}$. By construction Q is a right adjoint to the inclusion functor. The fact that $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism when $\mathcal{F}$ is quasi-coherent is a formal consequence of the fact that the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ is fully faithful. □

16. Cohomology

Let S be a scheme and let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. For any $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ the categories $Ab(\mathcal{X}_\tau)$ and $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ have enough injectives, see Injectives, Theorems 7.4 and 8.4. Thus we can use the machinery of Cohomology on Sites, Section 2 to define the cohomology groups

$$H^p(\mathcal{X}_\tau, \mathcal{F}) = H^p_\tau(\mathcal{X}, \mathcal{F}) \quad \text{and} \quad H^p(x, \mathcal{F}) = H^p_\tau(x, \mathcal{F})$$

for any $x \in \mathrm{Ob}(\mathcal{X})$ and any object $\mathcal{F}$ of $Ab(\mathcal{X}_\tau)$ or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Moreover, if $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, then we obtain the higher direct images $R^if_*\mathcal{F}$ in $Ab(\mathcal{Y}_\tau)$ or $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$. Of course, as explained in Cohomology on Sites, Section 3 there are also derived versions of $H^p(-)$ and R^if_*.

Lemma 16.1. *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ be an object lying over the scheme U. Let $\mathcal{F}$ be an object of $Ab(\mathcal{X}_\tau)$ or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Then*

$$H^p_\tau(x, \mathcal{F}) = H^p((Sch/U)_\tau, x^{-1}\mathcal{F})$$

and if $\tau = étale$, then we also have

$$H^p_{étale}(x, \mathcal{F}) = H^p(U_{étale}, \mathcal{F}|_{U_{étale}}).$$

Proof. The first statement follows from Cohomology on Sites, Lemma 7.1 and the equivalence of Lemma 9.4. The second statement follows from the first combined with Étale Cohomology, Lemma 20.3. □

17. Injective sheaves

The pushforward of an injective abelian sheaf or module is injective.

Lemma 17.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$.*

(1) *$f_*\mathcal{I}$ is injective in $Ab(\mathcal{Y}_\tau)$ for $\mathcal{I}$ injective in $Ab(\mathcal{X}_\tau)$, and*
(2) *$f_*\mathcal{I}$ is injective in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$ for $\mathcal{I}$ injective in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.*

Proof. This follows formally from the fact that f^{-1} is an exact left adjoint of f_*, see Homology, Lemma 29.1. □

In the rest of this section we prove that pullback f^{-1} has a left adjoint $f_!$ on abelian sheaves and modules. If f is representable (by schemes or by algebraic spaces), then it will turn out that $f_!$ is exact and f^{-1} will preserve injectives. We first prove a few preliminary lemmas about fibre products and equalizers in categories fibred in groupoids and their behaviour with respect to morphisms.

Lemma 17.2. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.*
(1) *The category $\mathcal{X}$ has fibre products.*
(2) *If the Isom-presheaves of $\mathcal{X}$ are representable by algebraic spaces, then $\mathcal{X}$ has equalizers.*
(3) *If $\mathcal{X}$ is an algebraic stack (or more generally a quotient stack), then $\mathcal{X}$ has equalizers.*

Proof. Part (1) follows Categories, Lemma 35.15 as $(Sch/S)_{fppf}$ has fibre products.

Let $a, b : x \to y$ be morphisms of $\mathcal{X}$. Set $U = p(x)$ and $V = p(y)$. The category of schemes has equalizers hence we can let $W \to U$ be the equalizer of $p(a)$ and $p(b)$. Denote $c : z \to x$ a morphism of $\mathcal{X}$ lying over $W \to U$. The equalizer of a and b, if it exists, is the equalizer of $a \circ c$ and $b \circ c$. Thus we may assume that $p(a) = p(b) = f : U \to V$. As $\mathcal{X}$ is fibred in groupoids, there exists a unique automorphism $i : x \to x$ in the fibre category of $\mathcal{X}$ over U such that $a \circ i = b$. Again the equalizer of a and b is the equalizer of id_x and i. Recall that the $Isom_\mathcal{X}(x)$ is the presheaf on $(Sch/U)_{fppf}$ which to T/U associates the set of automorphisms of $x|_T$ in the fibre category of $\mathcal{X}$ over T, see Stacks, Definition 2.2. If $Isom_\mathcal{X}(x)$ is representable by an algebraic space $G \to U$, then we see that id_x and i define morphisms $e, i : U \to G$ over U. Set $M = U \times_{e,G,i} U$, which by Morphisms of Spaces, Lemma 4.7 is a scheme. Then it is clear that $x|_M \to x$ is the equalizer of the maps id_x and i in $\mathcal{X}$. This proves (2).

If $\mathcal{X} = [U/R]$ for some groupoid in algebraic spaces (U, R, s, t, c) over S, then the hypothesis of (2) holds by Bootstrap, Lemma 11.5. If $\mathcal{X}$ is an algebraic stack, then we can choose a presentation $[U/R] \cong \mathcal{X}$ by Algebraic Stacks, Lemma 16.2. □

Lemma 17.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.*
(1) *The functor f transforms fibre products into fibre products.*
(2) *If f is faithful, then f transforms equalizers into equalizers.*

Proof. By Categories, Lemma 35.15 we see that a fibre product in $\mathcal{X}$ is any commutative square lying over a fibre product diagram in $(Sch/S)_{fppf}$. Similarly for $\mathcal{Y}$. Hence (1) is clear.

Let $x \to x'$ be the equalizer of two morphisms $a, b : x' \to x''$ in $\mathcal{X}$. We will show that $f(x) \to f(x')$ is the equalizer of $f(a)$ and $f(b)$. Let $y \to f(x)$ be a morphism of $\mathcal{Y}$ equalizing $f(a)$ and $f(b)$. Say x, x', x'' lie over the schemes U, U', U'' and y lies over V. Denote $h : V \to U'$ the image of $y \to f(x)$ in the category of schemes.

The morphism $y \to f(x)$ is isomorphic to $f(h^*x') \to f(x')$ by the axioms of fibred categories. Hence, as f is faithful, we see that $h^*x' \to x'$ equalizes a and b. Thus we obtain a unique morphism $h^*x' \to x$ whose image $y = f(h^*x') \to f(x)$ is the desired morphism in $\mathcal{Y}$. $\square$

Lemma 17.4. *Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Z} \to \mathcal{Y}$ be faithful 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

1. *the functor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y}$ is faithful, and*
2. *if $\mathcal{X}, \mathcal{Z}$ have equalizers, so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. We think of objects in $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ as quadruples (U, x, z, α) where $\alpha : f(x) \to g(z)$ is an isomorphism over U, see Categories, Lemma 32.3. A morphism $(U, x, z, \alpha) \to (U', x', z', \alpha')$ is a pair of morphisms $a : x \to x'$ and $b : z \to z'$ compatible with α and α'. Thus it is clear that if f and g are faithful, so is the functor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y}$. Now, suppose that $(a, b), (a', b') : (U, x, z, \alpha) \to (U', x', z', \alpha')$ are two morphisms of the 2-fibre product. Then consider the equalizer $x'' \to x$ of a and a' and the equalizer $z'' \to z$ of b and b'. Since f commutes with equalizers (by Lemma 17.3) we see that $f(x'') \to f(x)$ is the equalizer of $f(a)$ and $f(a')$. Similarly, $g(z'') \to g(z)$ is the equalizer of $g(b)$ and $g(b')$. Picture

$$
\begin{tikzcd}
f(x'') \arrow[r] \arrow[d, dotted, "\alpha''"'] & f(x) \arrow[r, shift left, "f(a)"] \arrow[r, shift right, "f(a')"'] \arrow[d, "\alpha"'] & f(x') \arrow[d, "\alpha'"] \\
g(z'') \arrow[r] & g(z) \arrow[r, shift left, "g(b)"] \arrow[r, shift right, "g(b')"'] & g(z')
\end{tikzcd}
$$

It is clear that the dotted arrow exists and is an isomorphism. However, it is not a priori the case that the image of α'' in the category of schemes is the identity of its source. On the other hand, the existence of α'' means that we can assume that x'' and z'' are defined over the same scheme and that the morphisms $x'' \to x$ and $z'' \to z$ have the same image in the category of schemes. Redoing the diagram above we see that the dotted arrow now does project to an identity morphism and we win. Some details omitted. $\square$

As we are working with big sites we have the following somewhat counter intuitive result (which also holds for morphisms of big sites of schemes). Warning: This result isn't true if we drop the hypothesis that f is faithful.

Lemma 17.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functor $f^{-1} : Ab(\mathcal{Y}_\tau) \to Ab(\mathcal{X}_\tau)$ has a left adjoint $f_! : Ab(\mathcal{X}_\tau) \to Ab(\mathcal{Y}_\tau)$. If f is faithful and $\mathcal{X}$ has equalizers, then*

1. *$f_!$ is exact, and*
2. *$f^{-1}\mathcal{I}$ is injective in $Ab(\mathcal{X}_\tau)$ for $\mathcal{I}$ injective in $Ab(\mathcal{Y}_\tau)$.*

Proof. By Stacks, Lemma 10.3 the functor f is continuous and cocontinuous. Hence by Modules on Sites, Lemma 16.2 the functor $f^{-1} : Ab(\mathcal{Y}_\tau) \to Ab(\mathcal{X}_\tau)$ has a left adjoint $f_! : Ab(\mathcal{X}_\tau) \to Ab(\mathcal{Y}_\tau)$. To see (1) we apply Modules on Sites, Lemma 16.3 and to see that the hypotheses of that lemma are satisfied use Lemmas 17.2 and 17.3 above. Part (2) follows from this formally, see Homology, Lemma 29.1. $\square$

Lemma 17.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functor $f^* : Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ has a left adjoint $f_! : Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X}) \to Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$ which agrees with the functor $f_!$ of Lemma 17.5 on underlying abelian sheaves. If f is faithful and $\mathcal{X}$ has equalizers, then*

1. *$f_!$ is exact, and*
2. *$f^{-1}\mathcal{I}$ is injective in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ for $\mathcal{I}$ injective in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{X})$.*

Proof. Recall that f is a continuous and cocontinuous functor of sites and that $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$. Hence Modules on Sites, Lemma 41.1 implies f^* has a left adjoint $f_!^{Mod}$. Let x be an object of $\mathcal{X}$ lying over the scheme U. Then f induces an equivalence of ringed sites

$$\mathcal{X}/x \longrightarrow \mathcal{Y}/f(x)$$

as both sides are equivalent to $(Sch/U)_\tau$, see Lemma 9.4. Modules on Sites, Remark 41.2 shows that $f_!$ agrees with the functor on abelian sheaves.

Assume now that $\mathcal{X}$ has equalizers and that f is faithful. Lemma 17.5 tells us that $f_!$ is exact. Finally, Homology, Lemma 29.1 implies the statement on pullbacks of injective modules. $\square$

18. The Čech complex

To compute the cohomology of a sheaf on an algebraic stack we compare it to the cohomology of the sheaf restricted to coverings of the given algebraic stack.

Throughout this section the situation will be as follows. We are given a 1-morphism of categories fibred in groupoids

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "f"] \arrow[dr, "q"'] & & \mathcal{X} \arrow[dl, "p"] \\ & (Sch/S)_{fppf} & \end{tikzcd} \tag{18.0.1}$$

We are going to think about $\mathcal{U}$ as a "covering" of $\mathcal{X}$. Hence we want to consider the simplicial object

$$\mathcal{U} \times_\mathcal{X} \mathcal{U} \times_\mathcal{X} \mathcal{U} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{U} \times_\mathcal{X} \mathcal{U} \substack{\longrightarrow\\\longrightarrow} \mathcal{U}$$

in the category of categories fibred in groupoids over $(Sch/S)_{fppf}$. However, since this is a $(2,1)$-category and not a category, we should say explicitly what we mean. Namely, we let $\mathcal{U}_n$ be the category with objects $(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n)$ where $\alpha_i : f(u_i) \to x$ is an isomorphism in $\mathcal{X}$. We denote $f_n : \mathcal{U}_n \to \mathcal{X}$ the 1-morphism which assigns to $(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n)$ the object x. Note that $\mathcal{U}_0 = \mathcal{U}$ and $f_0 = f$. Given a map $\varphi : [m] \to [n]$ we consider the 1-morphism $\mathcal{U}_\varphi : \mathcal{U}_n \longrightarrow \mathcal{U}_n$ given by

$$(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n) \longmapsto (u_{\varphi(0)}, \ldots, u_{\varphi(m)}, x, \alpha_{\varphi(0)}, \ldots, \alpha_{\varphi(m)})$$

on objects. All of these 1-morphisms compose correctly on the nose (no 2-morphisms required) and all of these 1-morphisms are 1-morphisms over $\mathcal{X}$. We denote $\mathcal{U}_\bullet$ this simplicial object. If $\mathcal{F}$ is a presheaf of sets on $\mathcal{X}$, then we obtain a cosimplicial set

$$\Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \substack{\longrightarrow\\\longrightarrow} \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \Gamma(\mathcal{U}_2, f_2^{-1}\mathcal{F})$$

Here the arrows are the pullback maps along the given morphisms of the simplicial object. If $\mathcal{F}$ is a presheaf of abelian groups, this is a cosimplicial abelian group.

Let $\mathcal{U} \to \mathcal{X}$ be as above and let $\mathcal{F}$ be an abelian presheaf on $\mathcal{X}$. The *Čech complex* associated to the situation is denoted $\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F})$. It is the cochain complex associated to the cosimplicial abelian group above, see Simplicial, Section 25. It has terms

$$\check{\mathcal{C}}^n(\mathcal{U} \to \mathcal{X}, \mathcal{F}) = \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}).$$

The boundary maps are the maps

$$d^n = \sum\nolimits_{i=0}^{n+1} (-1)^i \delta^{n+1}_i : \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}) \longrightarrow \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F})$$

where δ^{n+1}_i corresponds to the map $[n] \to [n+1]$ omitting the index i. Note that the map $\Gamma(\mathcal{X}, \mathcal{F}) \to \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0)$ is in the kernel of the differential d^0. Hence we define the *extended Čech complex* to be the complex

$$\ldots \to 0 \to \Gamma(\mathcal{X}, \mathcal{F}) \to \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0) \to \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}_1) \to \ldots$$

with $\Gamma(\mathcal{X}, \mathcal{F})$ placed in degree -1. The extended Čech complex is acyclic if and only if the canonical map

$$\Gamma(\mathcal{X}, \mathcal{F})[0] \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F})$$

is a quasi-isomorphism of complexes.

Lemma 18.1. *Generalities on Čech complexes.*

(1) *If*

$$\begin{tikzcd} \mathcal{V} \arrow[r, "h"] \arrow[d, "g"'] & \mathcal{U} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, "e"] & \mathcal{X} \end{tikzcd}$$

is 2*-commutative diagram of categories fibred in groupoids over* $(Sch/S)_{fppf}$*, then there is a morphism of Čech complexes*

$$\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{V} \to \mathcal{Y}, e^{-1}\mathcal{F})$$

(2) *if* h *and* e *are equivalences, then the map of (1) is an isomorphism,*

(3) *if* $f, f' : \mathcal{U} \to \mathcal{X}$ *are* 2*-isomorphic, then the associated Čech complexes are isomorphic.*

Proof. In the situation of (1) let $t : f \circ h \to e \circ g$ be a 2-morphism. The map on complexes is given in degree n by pullback along the 1-morphisms $\mathcal{V}_n \to \mathcal{U}_n$ given by the rule

$$(v_0, \ldots, v_n, y, \beta_0, \ldots, \beta_n) \longmapsto (h(v_0), \ldots, h(v_n), e(y), e(\beta_0) \circ t_{v_0}, \ldots, e(\beta_n) \circ t_{v_n}).$$

For (2), note that pullback on global sections is an isomorphism for any presheaf of sets when the pullback is along an equivalence of categories. Part (3) follows on combining (1) and (2). □

Lemma 18.2. *If there exists a* 1*-morphism* $s : \mathcal{X} \to \mathcal{U}$ *such that* $f \circ s$ *is* 2*-isomorphic to* $id_\mathcal{X}$ *then the extended Čech complex is homotopic to zero.*

Proof. Set $\mathcal{U}' = \mathcal{U} \times_\mathcal{X} \mathcal{X}$ equal to the fibre product as described in Categories, Lemma 32.3. Set $f' : \mathcal{U}' \to \mathcal{X}$ equal to the second projection. Then $\mathcal{U} \to \mathcal{U}'$, $u \mapsto (u, f(x), 1)$ is an equivalence over $\mathcal{X}$, hence we may replace $(\mathcal{U}, f)$ by $(\mathcal{U}', f')$ by Lemma 18.1. The advantage of this is that now f' has a section s' such that $f' \circ s' = \mathrm{id}_\mathcal{X}$ on the nose. Namely, if $t : s \circ f \to \mathrm{id}_\mathcal{X}$ is a 2-isomorphism then we can set $s'(x) = (s(x), x, t_x)$. Thus we may assume that $f \circ s = \mathrm{id}_\mathcal{X}$.

In the case that $f \circ s = \mathrm{id}_\mathcal{X}$ the result follows from general principles. We give the homotopy explicitly. Namely, for $n \geq 0$ define $s_n : \mathcal{U}_n \to \mathcal{U}_{n+1}$ to be the 1-morphism defined by the rule on objects

$$(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n) \longmapsto (u_0, \ldots, u_n, s(x), x, \alpha_0, \ldots, \alpha_n, \mathrm{id}_x).$$

Define

$$h^{n+1} : \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F}) \longrightarrow \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F})$$

as pullback along s_n. We also set $s_{-1} = s$ and $h^0 : \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \to \Gamma(\mathcal{X}, \mathcal{F})$ equal to pullback along s_{-1}. Then the family of maps $\{h^n\}_{n \geq 0}$ is a homotopy between 1 and 0 on the extended Čech complex. $\square$

19. The relative Čech complex

Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ as in (18.0.1). Consider the associated simplicial object $\mathcal{U}_\bullet$ and the maps $f_n : \mathcal{U}_n \to \mathcal{X}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Finally, suppose that $\mathcal{F}$ is a sheaf (of sets) on $\mathcal{X}_\tau$. Then

$$f_{0,*}f_0^{-1}\mathcal{F} \rightrightarrows f_{1,*}f_1^{-1}\mathcal{F} \Rrightarrow f_{2,*}f_2^{-1}\mathcal{F}$$

is a cosimplicial sheaf on $\mathcal{X}_\tau$ where we use the pullback maps introduced in Sites, Section 45. If $\mathcal{F}$ is an abelian sheaf, then $f_{n,*}f_n^{-1}\mathcal{F}$ form a cosimplicial abelian sheaf on $\mathcal{X}_\tau$. The associated complex (see Simplicial, Section 25)

$$\ldots \to 0 \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

is called the *relative Čech complex* associated to the situation. We will denote this complex $\mathcal{K}^\bullet(f, \mathcal{F})$. The *extended relative Čech complex* is the complex

$$\ldots \to 0 \to \mathcal{F} \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

with $\mathcal{F}$ in degree -1. The extended relative Čech complex is acyclic if and only if the map $\mathcal{F}[0] \to \mathcal{K}^\bullet(f, \mathcal{F})$ is a quasi-isomorphism of complexes of sheaves.

Remark 19.1. We can define the complex $\mathcal{K}^\bullet(f, \mathcal{F})$ also if $\mathcal{F}$ is a presheaf, only we cannot use the reference to Sites, Section 45 to define the pullback maps. To explain the pullback maps, suppose given a commutative diagram

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "h"] \arrow[dr, "g"'] & & \mathcal{U} \arrow[dl, "f"] \\ & \mathcal{X} & \end{tikzcd}$$

of categories fibred in groupoids over $(Sch/S)_{fppf}$ and a presheaf $\mathcal{G}$ on $\mathcal{U}$ we can define the pullback map $f_*\mathcal{G} \to g_*h^{-1}\mathcal{G}$ as the composition

$$f_*\mathcal{G} \longrightarrow f_*h_*h^{-1}\mathcal{G} = g_*h^{-1}\mathcal{G}$$

where the map comes from the adjunction map $\mathcal{G} \to h_* h^{-1}\mathcal{G}$. This works because in our situation the functors h_* and h^{-1} are adjoint in presheaves (and agree with their counter parts on sheaves). See Sections 3 and 4.

Lemma 19.2. *Generalities on relative Čech complexes.*

(1) *If*

$$\begin{tikzcd} \mathcal{V} \arrow[r, "h"'] \arrow[d, "g"'] & \mathcal{U} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, "e"] & \mathcal{X} \end{tikzcd}$$

is 2*-commutative diagram of categories fibred in groupoids over* $(Sch/S)_{fppf}$*, then there is a morphism* $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \to \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$.

(2) *if* h *and* e *are equivalences, then the map of (1) is an isomorphism,*

(3) *if* $f, f' : \mathcal{U} \to \mathcal{X}$ *are* 2*-isomorphic, then the associated relative Čech complexes are isomorphic,*

Proof. Literally the same as the proof of Lemma 18.1 using the pullback maps of Remark 19.1. $\square$

Lemma 19.3. *If there exists a* 1*-morphism* $s : \mathcal{X} \to \mathcal{U}$ *such that* $f \circ s$ *is* 2*-isomorphic to* $id_{\mathcal{X}}$ *then the extended relative Čech complex is homotopic to zero.*

Proof. Literally the same as the proof of Lemma 18.2. $\square$

Remark 19.4. Let us "compute" the value of the relative Čech complex on an object x of $\mathcal{X}$. Say $p(x) = U$. Consider the 2-fibre product diagram (which serves to introduce the notation $g : \mathcal{V} \to \mathcal{Y}$)

$$\begin{tikzcd} \mathcal{V} \arrow[r, equal] \arrow[d, "g"'] & (Sch/U)_{fppf} \times_{x, \mathcal{X}} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, equal] & (Sch/U)_{fppf} \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

Note that the morphism $\mathcal{V}_n \to \mathcal{U}_n$ of the proof of Lemma 18.1 induces an equivalence $\mathcal{V}_n = (Sch/U)_{fppf} \times_{x,\mathcal{X}} \mathcal{U}_n$. Hence we see from (5.0.1) that

$$\Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{\mathcal{C}}^\bullet(\mathcal{V} \to \mathcal{Y}, x^{-1}\mathcal{F})$$

In words: The value of the relative Čech complex on an object x of $\mathcal{X}$ is the Čech complex of the base change of f to $\mathcal{X}/x \cong (Sch/U)_{fppf}$. This implies for example that Lemma 18.2 implies Lemma 19.3 and more generally that results on the (usual) Čech complex imply results for the relative Čech complex.

Lemma 19.5. *Let*

$$\begin{tikzcd} \mathcal{V} \arrow[r, "h"'] \arrow[d, "g"'] & \mathcal{U} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, "e"] & \mathcal{X} \end{tikzcd}$$

be a 2*-fibre product of categories fibred in groupoids over* $(Sch/S)_{fppf}$ *and let* $\mathcal{F}$ *be an abelian presheaf on* $\mathcal{X}$*. Then the map* $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \to \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$ *of Lemma 19.2 is an isomorphism of complexes of abelian presheaves.*

Proof. Let y be an object of $\mathcal{Y}$ lying over the scheme T. Set $x = e(y)$. We are going to show that the map induces an isomorphism on sections over y. Note that

$$\Gamma(y, e^{-1}\mathcal{K}^\bullet(f, \mathcal{F})) = \Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{\mathcal{C}}^\bullet((Sch/T)_{fppf} \times_{x,\mathcal{X}} \mathcal{U} \to (Sch/T)_{fppf}, x^{-1}\mathcal{F})$$

by Remark 19.4. On the other hand,

$$\Gamma(y, \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})) = \check{\mathcal{C}}^\bullet((Sch/T)_{fppf} \times_{y,\mathcal{Y}} \mathcal{V} \to (Sch/T)_{fppf}, y^{-1}e^{-1}\mathcal{F})$$

also by Remark 19.4. Note that $y^{-1}e^{-1}\mathcal{F} = x^{-1}\mathcal{F}$ and since the diagram is 2-cartesian the 1-morphism

$$(Sch/T)_{fppf} \times_{y,\mathcal{Y}} \mathcal{V} \to (Sch/T)_{fppf} \times_{x,\mathcal{X}} \mathcal{U}$$

is an equivalence. Hence the map on sections over y is an isomorphism by Lemma 18.1. $\square$

Exactness can be checked on a "covering".

Lemma 19.6. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

be a complex in $Ab(\mathcal{X}_\tau)$. Assume that

1. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$, and*
2. *$f^{-1}\mathcal{F} \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H}$ is exact.*

Then the sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact.

Proof. Let x be an object of $\mathcal{X}$ lying over the scheme T. Consider the sequence $x^{-1}\mathcal{F} \to x^{-1}\mathcal{G} \to x^{-1}\mathcal{H}$ of abelian sheaves on $(Sch/T)_\tau$. It suffices to show this sequence is exact. By assumption there exists a τ-covering $\{T_i \to T\}$ such that $x|_{T_i}$ is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$ over T_i and moreover the sequence $u_i^{-1}f^{-1}\mathcal{F} \to u_i^{-1}f^{-1}\mathcal{G} \to u_i^{-1}f^{-1}\mathcal{H}$ of abelian sheaves on $(Sch/T_i)_\tau$ is exact. Since $u_i^{-1}f^{-1}\mathcal{F} = x^{-1}\mathcal{F}|_{(Sch/T_i)_\tau}$ we conclude that the sequence $x^{-1}\mathcal{F} \to x^{-1}\mathcal{G} \to x^{-1}\mathcal{H}$ become exact after localizing at each of the members of a covering, hence the sequence is exact. $\square$

Proposition 19.7. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. If*

1. *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$, and*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*

then the extended relative Čech complex

$$\ldots \to 0 \to \mathcal{F} \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

is exact in $Ab(\mathcal{X}_\tau)$.

Proof. By Lemma 19.6 it suffices to check exactness after pulling back to $\mathcal{U}$. By Lemma 19.5 the pullback of the extended relative Čech complex is isomorphic to the extend relative Čech complex for the morphism $\mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{U}$ and an abelian sheaf on $\mathcal{U}_\tau$. Since there is a section $\Delta_{\mathcal{U}/\mathcal{X}} : \mathcal{U} \to \mathcal{U} \times_\mathcal{X} \mathcal{U}$ exactness follows from Lemma 19.3. $\square$

Using this we can construct the Čech-to-cohomology spectral sequence as follows. We first give a technical, precise version. In the next section we give a version that applies only to algebraic stacks.

Lemma 19.8. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence of abelian groups

$$E_1^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of $\mathcal{F}$ in the τ-topology.

Proof. Before we start the proof we make some remarks. By Lemma 17.4 (and induction) all of the categories fibred in groupoids $\mathcal{U}_p$ have equalizers and all of the morphisms $f_p : \mathcal{U}_p \to \mathcal{X}$ are faithful. Let $\mathcal{I}$ be an injective object of $Ab(\mathcal{X}_\tau)$. By Lemma 17.5 we see $f_p^{-1}\mathcal{I}$ is an injective object of $Ab((\mathcal{U}_p)_\tau)$. Hence $f_{p,*}f_p^{-1}\mathcal{I}$ is an injective object of $Ab(\mathcal{X}_\tau)$ by Lemma 17.1. Hence Proposition 19.7 shows that the extended relative Čech complex

$$\ldots \to 0 \to \mathcal{I} \to f_{0,*}f_0^{-1}\mathcal{I} \to f_{1,*}f_1^{-1}\mathcal{I} \to f_{2,*}f_2^{-1}\mathcal{I} \to \ldots$$

is an exact complex in $Ab(\mathcal{X}_\tau)$ all of whose terms are injective. Taking global sections of this complex is exact and we see that the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{I})$ is quasi-isomorphic to $\Gamma(\mathcal{X}_\tau, \mathcal{I})[0]$.

With these preliminaries out of the way consider the two spectral sequences associated to the double complex (see Homology, Section 25)

$$\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{I}^\bullet)$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The discussion above shows that Homology, Lemma 25.4 applies which shows that $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$ is quasi-isomorphic to the total complex associated to the double complex. By our remarks above the complex $f_p^{-1}\mathcal{I}^\bullet$ is an injective resolution of $f_p^{-1}\mathcal{F}$. Hence the other spectral sequence is as indicated in the lemma. □

To be sure there is a version for modules as well.

Lemma 19.9. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence of $\Gamma(\mathcal{O}_\mathcal{X})$-modules

$$E_1^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^*\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of $\mathcal{F}$ in the τ-topology.

Proof. The proof of this lemma is identical to the proof of Lemma 19.8 except that it uses an injective resolution in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ and it uses Lemma 17.6 instead of Lemma 17.5. ☐

Here is a lemma that translates a more usual kind of covering in the kinds of coverings we have encountered above.

Lemma 19.10. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

(1) *Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then for any object y of $\mathcal{Y}$ there exists an fppf covering $\{y_i \to y\}$ and objects x_i of $\mathcal{X}$ such that $f(x_i) \cong y_i$ in $\mathcal{Y}$.*
(2) *Assume that f is representable by algebraic spaces, surjective, and smooth. Then for any object y of $\mathcal{Y}$ there exists an étale covering $\{y_i \to y\}$ and objects x_i of $\mathcal{X}$ such that $f(x_i) \cong y_i$ in $\mathcal{Y}$.*

Proof. Proof of (1). Suppose that y lies over the scheme V. We may think of y as a morphism $(Sch/V)_{fppf} \to \mathcal{Y}$. By definition the 2-fibre product $\mathcal{X} \times_\mathcal{Y} (Sch/V)_{fppf}$ is representable by an algebraic space W and the morphism $W \to V$ is surjective, flat, and locally of finite presentation. Choose a scheme U and a surjective étale morphism $U \to W$. Then $U \to V$ is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 39.7, 39.8, 5.4, 28.2, and 30.3). Hence $\{U \to V\}$ is an fppf covering. Denote x the object of $\mathcal{X}$ over U corresponding to the 1-morphism $(Sch/U)_{fppf} \to \mathcal{X}$. Then $\{f(x) \to y\}$ is the desired fppf covering of $\mathcal{Y}$.

Proof of (2). Suppose that y lies over the scheme V. We may think of y as a morphism $(Sch/V)_{fppf} \to \mathcal{Y}$. By definition the 2-fibre product $\mathcal{X} \times_\mathcal{Y} (Sch/V)_{fppf}$ is representable by an algebraic space W and the morphism $W \to V$ is surjective and smooth. Choose a scheme U and a surjective étale morphism $U \to W$. Then $U \to V$ is also surjective and smooth (see Morphisms of Spaces, Lemmas 39.6, 5.4, and 37.2). Hence $\{U \to V\}$ is a smooth covering. By More on Morphisms, Lemma 38.7 there exists an étale covering $\{V_i \to V\}$ such that each $V_i \to V$ factors through U. Denote x_i the object of $\mathcal{X}$ over V_i corresponding to the 1-morphism

$$(Sch/V_i)_{fppf} \to (Sch/U)_{fppf} \to \mathcal{X}.$$

Then $\{f(x_i) \to y\}$ is the desired étale covering of $\mathcal{Y}$. ☐

Lemma 19.11. *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Assume*

(1) *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$,*
(2) *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
(3) *the category $\mathcal{U}$ has equalizers, and*
(4) *the functor f is faithful.*

Then there is a first quadrant spectral sequence of abelian sheaves on $\mathcal{Y}_\tau$

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$

where all higher direct images are computed in the τ-topology.

Proof. Note that the assumptions on $f : \mathcal{U} \to \mathcal{X}$ and $\mathcal{F}$ are identical to those in Lemma 19.8. Hence the preliminary remarks made in the proof of that lemma hold here also. These remarks imply in particular that

$$0 \to g_*\mathcal{I} \to (g \circ f_0)_* f_0^{-1}\mathcal{I} \to (g \circ f_1)_* f_1^{-1}\mathcal{I} \to \ldots$$

is exact if $\mathcal{I}$ is an injective object of $Ab(\mathcal{X}_\tau)$. Having said this, consider the two spectral sequences of Homology, Section 25 associated to the double complex $\mathcal{C}^{\bullet,\bullet}$ with terms

$$\mathcal{C}^{p,q} = (g \circ f_p)_* \mathcal{I}^q$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The first spectral sequence implies, via Homology, Lemma 25.4, that $g_*\mathcal{I}^\bullet$ is quasi-isomorphic to the total complex associated to $\mathcal{C}^{\bullet,\bullet}$. Since $f_p^{-1}\mathcal{I}^\bullet$ is an injective resolution of $f_p^{-1}\mathcal{F}$ (see Lemma 17.5) the second spectral sequence has terms $E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F}$ as in the statement of the lemma. □

Lemma 19.12. *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$

where all higher direct images are computed in the τ-topology.

Proof. The proof is identical to the proof of Lemma 19.11 except that it uses an injective resolution in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ and it uses Lemma 17.6 instead of Lemma 17.5. □

20. Cohomology on algebraic stacks

Let $\mathcal{X}$ be an algebraic stack over S. In the sections above we have seen how to define sheaves for the étale, ..., fppf topologies on $\mathcal{X}$. In fact, we have constructed a site $\mathcal{X}_\tau$ for each $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. There is a notion of an abelian sheaf $\mathcal{F}$ on these sites. In the chapter on cohomology of sites we have explained how to define cohomology. Putting all of this together, let's define the *derived global sections* or *total cohomology*

$$R\Gamma_{Zar}(\mathcal{X}, \mathcal{F}), R\Gamma_{\acute{e}tale}(\mathcal{X}, \mathcal{F}), \ldots, R\Gamma_{fppf}(\mathcal{X}, \mathcal{F})$$

as $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The ith cohomology group of $\mathcal{F}$ is the ith cohomology of the total cohomology. We will denote this

$$H^i_{Zar}(\mathcal{X}, \mathcal{F}), H^i_{\acute{e}tale}(\mathcal{X}, \mathcal{F}), \ldots, H^i_{fppf}(\mathcal{X}, \mathcal{F}).$$

It will turn out that $H^i_{\acute{e}tale} = H^i_{smooth}$ because of More on Morphisms, Lemma 38.7.

If $\mathcal{F}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules which is a sheaf in the τ-topology, then we use injective resolutions in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ to compute its total cohomology, resp.

cohomology groups; the end result is quasi-isomorphic, resp. isomorphic to the cohomology of $\mathcal{F}$ viewed as a sheaf of abelian groups by the very general Cohomology on Sites, Lemma 12.4.

So far our only tool to compute cohomology groups is the result on Čech complexes proved above. We rephrase it here in the language of algebraic stacks for the étale and the fppf topology. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of algebraic stacks. Recall that

$$f_p : \mathcal{U}_p = \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \longrightarrow \mathcal{X}$$

is the structure morphism where there are $(p+1)$-factors. Also, recall that a sheaf on $\mathcal{X}$ is a sheaf for the fppf topology. Note that if $\mathcal{U}$ is an algebraic space, then $f : \mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, see Algebraic Stacks, Lemma 10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack $\mathcal{X}$ by a scheme.

Proposition 20.1. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of algebraic stacks.*

(1) *Let $\mathcal{F}$ be an abelian étale sheaf on $\mathcal{X}$. Assume that f is representable by algebraic spaces, surjective, and smooth. Then there is a spectral sequence*

$$E_1^{p,q} = H^q_{\acute{e}tale}(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}_{\acute{e}tale}(\mathcal{X}, \mathcal{F})$$

(2) *Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{X}$. Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then there is a spectral sequence*

$$E_1^{p,q} = H^q_{fppf}(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}_{fppf}(\mathcal{X}, \mathcal{F})$$

Proof. To see this we will check the hypotheses (1) – (4) of Lemma 19.8. The 1-morphism f is faithful by Algebraic Stacks, Lemma 15.2. This proves (4). Hypothesis (3) follows from the fact that $\mathcal{U}$ is an algebraic stack, see Lemma 17.2. To see (2) apply Lemma 19.10. Condition (1) is satisfied by fiat. ☐

21. Higher direct images and algebraic stacks

Let $g : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. In the sections above we have constructed a morphism of ringed topoi $g : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ for each $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. In the chapter on cohomology of sites we have explained how to define higher direct images. Hence the *total direct image* $Rg_*\mathcal{F}$ is defined as $g_*\mathcal{I}^\bullet$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The ith higher direct image $R^ig_*\mathcal{F}$ is the ith cohomology of the total direct image. Important: it matters which topology τ is used here!

If $\mathcal{F}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules which is a sheaf in the τ-topology, then we use injective resolutions in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ to compute total direct image and higher direct images.

So far our only tool to compute the higher direct images of g_* is the result on Čech complexes proved above. This requires the choice of a "covering" $f : \mathcal{U} \to \mathcal{X}$. If $\mathcal{U}$ is an algebraic space, then $f : \mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, see Algebraic Stacks, Lemma 10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack $\mathcal{X}$ by a scheme.

Proposition 21.1. *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of algebraic stacks.*

(1) *Assume that f is representable by algebraic spaces, surjective and smooth.*
 (a) *If $\mathcal{F}$ is in $Ab(\mathcal{X}_{étale})$ then there is a spectral sequence*
$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
in $Ab(\mathcal{Y}_{étale})$ with higher direct images computed in the étale topology.
 (b) *If $\mathcal{F}$ is in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ then there is a spectral sequence*
$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
in $Mod(\mathcal{Y}_{étale}, \mathcal{O}_\mathcal{Y})$.

(2) *Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation.*
 (a) *If $\mathcal{F}$ is in $Ab(\mathcal{X})$ then there is a spectral sequence*
$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
in $Ab(\mathcal{Y})$ with higher direct images computed in the fppf topology.
 (b) *If $\mathcal{F}$ is in $Mod(\mathcal{O}_\mathcal{X})$ then there is a spectral sequence*
$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
in $Mod(\mathcal{O}_\mathcal{Y})$.

Proof. To see this we will check the hypotheses (1) – (4) of Lemma 19.11 and Lemma 19.12. The 1-morphism f is faithful by Algebraic Stacks, Lemma 15.2. This proves (4). Hypothesis (3) follows from the fact that $\mathcal{U}$ is an algebraic stack, see Lemma 17.2. To see (2) apply Lemma 19.10. Condition (1) is satisfied by fiat in all four cases. □

Here is a description of higher direct images for a morphism of algebraic stacks.

Lemma 21.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks*[5] *over S. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $\mathcal{F}$ be an object of $Ab(\mathcal{X}_\tau)$ or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Then the sheaf $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf*
$$y \longmapsto H^i_\tau\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ pr^{-1}\mathcal{F}\Big)$$
Here y is an object of $\mathcal{Y}$ lying over the scheme V.

Proof. Choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. By the formula for pushforward (5.0.1) we see that $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf which associates to y the cohomology of the complex
$$\begin{matrix} \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i-1}\Big) \\ \downarrow \\ \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i}\Big) \\ \downarrow \\ \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i+1}\Big) \end{matrix}$$
Since pr^{-1} is exact, it suffices to show that pr^{-1} preserves injectives. This follows from Lemmas 17.5 and 17.6 as well as the fact that pr is a representable morphism of algebraic stacks (so that pr is faithful by Algebraic Stacks, Lemma 15.2 and that $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ has equalizers by Lemma 17.2). □

[5]This result should hold for any 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.

Here is a trivial base change result.

Lemma 21.3. *Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let*

$$\begin{tikzcd} \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

be a 2-cartesian diagram of algebraic stacks over S. Then the base change map is an isomorphism

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

functorial for $\mathcal{F}$ in $Ab(\mathcal{X}_\tau)$ or $\mathcal{F}$ in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.

Proof. The isomorphism $g^{-1}f_*\mathcal{F} = f'_*(g')^{-1}\mathcal{F}$ is Lemma 5.1 (and it holds for arbitrary presheaves). For the total direct images, there is a base change map because the morphisms g and g' are flat, see Cohomology on Sites, Section 15. To see that this map is a quasi-isomorphism we can use that for an object y' of $\mathcal{Y}'$ over a scheme V there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'),\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y',\mathcal{Y}'} (\mathcal{Y}' \times_\mathcal{Y} \mathcal{X})$$

We conclude that the induced map $g^{-1}R^if_*\mathcal{F} \to R^if'_*(g')^{-1}\mathcal{F}$ is an isomorphism by Lemma 21.2. □

22. Comparison

In this section we collect some results on comparing cohomology defined using stacks and using algebraic spaces.

Lemma 22.1. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S representable by the algebraic space F.*

1. *If $\mathcal{I}$ injective in $Ab(\mathcal{X}_{étale})$, then $\mathcal{I}|_{F_{étale}}$ is injective in $Ab(F_{étale})$,*
2. *If $\mathcal{I}^\bullet$ is a K-injective complex in $Ab(\mathcal{X}_{étale})$, then $\mathcal{I}^\bullet|_{F_{étale}}$ is a K-injective complex in $Ab(F_{étale})$.*

The same does not hold for modules.

Proof. This follows formally from the fact that the restriction functor $\pi_{F,*} = i_F^{-1}$ (see Lemma 10.1) is right adjoint to the exact functor π_F^{-1}, see Homology, Lemma 29.1 and Derived Categories, Lemma 31.9. To see that the lemma does not hold for modules, we refer the reader to Étale Cohomology, Lemma 99.1. □

Lemma 22.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over S. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G. Denote $f : F \to G$ the induced morphism of algebraic spaces.*

1. *For any $\mathcal{F} \in Ab(\mathcal{X}_{étale})$ we have*
$$(Rf_*\mathcal{F})|_{G_{étale}} = Rf_{small,*}(\mathcal{F}|_{F_{étale}})$$
in $D(G_{étale})$.
2. *For any object $\mathcal{F}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ we have*
$$(Rf_*\mathcal{F})|_{G_{étale}} = Rf_{small,*}(\mathcal{F}|_{F_{étale}})$$
in $D(\mathcal{O}_G)$.

Proof. Part (1) follows immediately from Lemma 22.1 and (10.3.1) on choosing an injective resolution of $\mathcal{F}$.

Part (2) can be proved as follows. In Lemma 10.3 we have seen that $\pi_G \circ f = f_{small} \circ \pi_F$ as morphisms of ringed sites. Hence we obtain $R\pi_{G,*} \circ Rf_* = Rf_{small,*} \circ R\pi_{F,*}$ by Cohomology on Sites, Lemma 19.2. Since the restriction functors $\pi_{F,*}$ and $\pi_{G,*}$ are exact, we conclude. □

Lemma 22.3. *Let S be a scheme. Consider a 2-fibre product square*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

of algebraic stacks over S. Assume that f is representable by algebraic spaces and that $\mathcal{Y}'$ is representable by an algebraic space G'. Then $\mathcal{X}'$ is representable by an algebraic space F' and denoting $f' : F' \to G'$ the induced morphism of algebraic spaces we have

$$g^{-1}(Rf_*\mathcal{F})|_{G'_{\acute{e}tale}} = Rf'_{small,*}((g')^{-1}\mathcal{F}|_{F'_{\acute{e}tale}})$$

for any $\mathcal{F}$ in $Ab(\mathcal{X}_{\acute{e}tale})$ or in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$

Proof. Follows formally on combining Lemmas 21.3 and 22.2. □

23. Change of topology

Here is a technical lemma which tells us that the fppf cohomology of a locally quasi-coherent sheaf is equal to its étale cohomology provided the comparison maps are isomorphisms for morphisms of $\mathcal{X}$ lying over flat morphisms.

Lemma 23.1. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. Assume*

(a) *$\mathcal{F}$ is locally quasi-coherent, and*
(b) *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ which lies over a morphism of schemes $f : U \to V$ which is flat and locally of finite presentation the comparison map $c_\varphi : f_{small}^*\mathcal{F}|_{V_{\acute{e}tale}} \to \mathcal{F}|_{U_{\acute{e}tale}}$ of (9.4.1) is an isomorphism.*

Then $\mathcal{F}$ is a sheaf for the fppf topology.

Proof. Let $\{x_i \to x\}$ be an fppf covering of $\mathcal{X}$ lying over the fppf covering $\{f_i : U_i \to U\}$ of schemes over S. By assumption the restriction $\mathcal{G} = \mathcal{F}|_{U_{\acute{e}tale}}$ is quasi-coherent and the comparison maps $f_{i,small}^*\mathcal{G} \to \mathcal{F}|_{U_{i,\acute{e}tale}}$ are isomorphisms. Hence the sheaf condition for $\mathcal{F}$ and the covering $\{x_i \to x\}$ is equivalent to the sheaf condition for $\mathcal{G}^a$ on $(Sch/U)_{fppf}$ and the covering $\{U_i \to U\}$ which holds by Descent, Lemma 8.1. □

Lemma 23.2. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{F}$ be a presheaf $\mathcal{O}_\mathcal{X}$-module such that*

(a) *$\mathcal{F}$ is locally quasi-coherent, and*
(b) *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ which lies over a morphism of schemes $f : U \to V$ which is flat and locally of finite presentation, the comparison map $c_\varphi : f_{small}^*\mathcal{F}|_{V_{\acute{e}tale}} \to \mathcal{F}|_{U_{\acute{e}tale}}$ of (9.4.1) is an isomorphism.*

Then $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module and we have the following

(1) *If* $\epsilon : \mathcal{X}_{fppf} \to \mathcal{X}_{étale}$ *is the comparison morphism, then* $R\epsilon_*\mathcal{F} = \epsilon_*\mathcal{F}$.
(2) *The cohomology groups* $H^p_{fppf}(\mathcal{X}, \mathcal{F})$ *are equal to the cohomology groups computed in the étale topology on* $\mathcal{X}$. *Similarly for the cohomology groups* $H^p_{fppf}(x, \mathcal{F})$ *and the derived versions* $R\Gamma(\mathcal{X}, \mathcal{F})$ *and* $R\Gamma(x, \mathcal{F})$.
(3) *If* $f : \mathcal{X} \to \mathcal{Y}$ *is a* 1*-morphism of categories fibred in groupoids over* $(Sch/S)_{fppf}$ *then* $R^if_*\mathcal{F}$ *is equal to the fppf-sheafification of the higher direct image computed in the étale cohomology. Similarly for derived pull-back.*

Proof. The assertion that $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module follows from Lemma 23.1. Note that ϵ is a morphism of sites given by the identity functor on $\mathcal{X}$. The sheaf $R^p\epsilon_*\mathcal{F}$ is therefore the sheaf associated to the presheaf $x \mapsto H^p_{fppf}(x, \mathcal{F})$, see Cohomology on Sites, Lemma 7.4. To prove (1) it suffices to show that $H^p_{fppf}(x, \mathcal{F}) = 0$ for $p > 0$ whenever x lies over an affine scheme U. By Lemma 16.1 we have $H^p_{fppf}(x, \mathcal{F}) = H^p((Sch/U)_{fppf}, x^{-1}\mathcal{F})$. Combining Descent, Lemma 12.4 with Cohomology of Schemes, Lemma 2.2 we see that these cohomology groups are zero.

We have seen above that $\epsilon_*\mathcal{F}$ and $\mathcal{F}$ are the sheaves on $\mathcal{X}_{étale}$ and $\mathcal{X}_{fppf}$ corresponding to the same presheaf on $\mathcal{X}$ (and this is true more generally for any sheaf in the fppf topology on $\mathcal{X}$). We often abusively identify $\mathcal{F}$ and $\epsilon_*\mathcal{F}$ and this is the sense in which parts (2) and (3) of the lemma should be understood. Thus part (2) follows formally from (1) and the Leray spectral sequence, see Cohomology on Sites, Lemma 14.6.

Finally we prove (3). The sheaf $R^if_*\mathcal{F}$ (resp. $Rf_{étale,*}\mathcal{F}$) is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big)$$

where τ is $fppf$ (resp. *étale*), see Lemma 21.2. Note that $\mathrm{pr}^{-1}\mathcal{F}$ satisfies properties (a) and (b) also (by Lemmas 12.3 and 9.3), hence these two presheaves are equal by (2). This immediately implies (3). □

We will use the following lemma to compare étale cohomology of sheaves on algebraic stacks with cohomology on the lisse-étale topos.

Lemma 23.3. *Let* S *be a scheme. Let* $\mathcal{X}$ *be an algebraic stack over* S. *Let* $\tau =$ *étale (resp.* $\tau = fppf$*). Let* $\mathcal{X}' \subset \mathcal{X}$ *be a full subcategory with the following properties*

(1) *if* $x \to x'$ *is a morphism of* $\mathcal{X}$ *which lies over a smooth (resp. flat and locally finitely presented) morphism of schemes and* $x' \in \mathrm{Ob}(\mathcal{X}')$, *then* $x \in \mathrm{Ob}(\mathcal{X}')$, *and*
(2) *there exists an object* $x \in \mathrm{Ob}(\mathcal{X}')$ *lying over a scheme* U *such that the associated* 1*-morphism* $x : (Sch/U)_{fppf} \to \mathcal{X}$ *is smooth and surjective.*

We get a site $\mathcal{X}'_\tau$ *by declaring a covering of* $\mathcal{X}'$ *to be any family of morphisms* $\{x_i \to x\}$ *in* $\mathcal{X}'$ *which is a covering in* $\mathcal{X}_\tau$. *Then the inclusion functor* $\mathcal{X}' \to \mathcal{X}_\tau$ *is fully faithful, cocontinuous, and continuous, whence defines a morphism of topoi*

$$g : Sh(\mathcal{X}'_\tau) \longrightarrow Sh(\mathcal{X}_\tau)$$

and $H^p(\mathcal{X}'_\tau, g^{-1}\mathcal{F}) = H^p(\mathcal{X}_\tau, \mathcal{F})$ *for all* $p \geq 0$ *and all* $\mathcal{F} \in Ab(\mathcal{X}_\tau)$.

Proof. Note that assumption (1) implies that if $\{x_i \to x\}$ is a covering of $\mathcal{X}_\tau$ and $x \in \mathrm{Ob}(\mathcal{X}')$, then we have $x_i \in \mathrm{Ob}(\mathcal{X}')$. Hence we see that $\mathcal{X}' \to \mathcal{X}$ is continuous and cocontinuous as the coverings of objects of $\mathcal{X}'_\tau$ agree with their coverings seen as objects of $\mathcal{X}_\tau$. We obtain the morphism g and the functor g^{-1} is identified with the restriction functor, see Sites, Lemma 21.5.

In particular, if $\{x_i \to x\}$ is a covering in $\mathcal{X}'_\tau$, then for any abelian sheaf $\mathcal{F}$ on $\mathcal{X}$ then

$$\check{H}^p(\{x_i \to x\}, g^{-1}\mathcal{F}) = \check{H}^p(\{x_i \to x\}, \mathcal{F})$$

Thus if $\mathcal{I}$ is an injective abelian sheaf on $\mathcal{X}_\tau$ then we see that the higher Čech cohomology groups are zero (Cohomology on Sites, Lemma 10.2). Hence $H^p(x, g^{-1}\mathcal{I}) = 0$ for all objects x of $\mathcal{X}'$ (Cohomology on Sites, Lemma 10.9). In other words injective abelian sheaves on $\mathcal{X}_\tau$ are right acyclic for the functor $H^0(x, g^{-1}-)$. It follows that $H^p(x, g^{-1}\mathcal{F}) = H^p(x, \mathcal{F})$ for all $\mathcal{F} \in Ab(\mathcal{X})$ and all $x \in \mathrm{Ob}(\mathcal{X}')$.

Choose an object $x \in \mathcal{X}'$ lying over a scheme U as in assumption (2). In particular $\mathcal{X}/x \to \mathcal{X}$ is a morphism of algebraic stacks which representable by algebraic spaces, surjective, and smooth. (Note that $\mathcal{X}/x$ is equivalent to $(Sch/U)_{fppf}$, see Lemma 9.1.) The map of sheaves

$$h_x \longrightarrow *$$

in $Sh(\mathcal{X}_\tau)$ is surjective. Namely, for any object x' of $\mathcal{X}$ there exists a τ-covering $\{x'_i \to x'\}$ such that there exist morphisms $x'_i \to x$, see Lemma 19.10. Since g is exact, the map of sheaves

$$g^{-1}h_x \longrightarrow * = g^{-1}*$$

in $Sh(\mathcal{X}'_\tau)$ is surjective also. Let $h_{x,n}$ be the $(n+1)$-fold product $h_x \times \ldots \times h_x$. Then we have spectral sequences

$$E_1^{p,q} = H^q(h_{x,p}, \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F}) \tag{23.3.1}$$

and

$$E_1^{p,q} = H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}'_\tau, g^{-1}\mathcal{F}) \tag{23.3.2}$$

see Cohomology on Sites, Lemma 13.2.

Case I: $\mathcal{X}$ has a final object x which is also an object of $\mathcal{X}'$. This case follows immediately from the discussion in the second paragraph above.

Case II: $\mathcal{X}$ is representable by an algebraic space F. In this case the sheaves $h_{x,n}$ are representable by an object x_n in $\mathcal{X}$. (Namely, if $\mathcal{S}_F = \mathcal{X}$ and $x : U \to F$ is the given object, then $h_{x,n}$ is representable by the object $U \times_F \ldots \times_F U \to F$ of $\mathcal{S}_F$.) It follows that $H^q(h_{x,p}, \mathcal{F}) = H^q(x_p, \mathcal{F})$. The morphisms $x_n \to x$ lie over smooth morphisms of schemes, hence $x_n \in \mathcal{X}'$ for all n. Hence $H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) = H^q(x_p, g^{-1}\mathcal{F})$. Thus in the two spectral sequences (23.3.1) and (23.3.2) above the $E_1^{p,q}$ terms agree by the discussion in the second paragraph. The lemma follows in Case II as well.

Case III: $\mathcal{X}$ is an algebraic stack. We claim that in this case the cohomology groups $H^q(h_{x,p}, \mathcal{F})$ and $H^q(g^{-1}h_{x,n}, g^{-1}\mathcal{F})$ agree by Case II above. Once we have proved this the result will follow as before.

Namely, consider the category $\mathcal{X}/h_{x,n}$, see Sites, Lemma 30.3. Since $h_{x,n}$ is the $(n+1)$-fold product of h_x an object of this category is an $(n+2)$-tuple $(y, s_0, \ldots, s_n)$ where y is an object of $\mathcal{X}$ and each $s_i : y \to x$ is a morphism of $\mathcal{X}$. This is a category over $(Sch/S)_{fppf}$. There is an equivalence

$$\mathcal{X}/h_{x,n} \longrightarrow (Sch/U)_{fppf} \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} (Sch/U)_{fppf} =: \mathcal{U}_n$$

over $(Sch/S)_{fppf}$. Namely, if $x : (Sch/U)_{fppf} \to \mathcal{X}$ also denotes the 1-morphism associated with x and $p : \mathcal{X} \to (Sch/S)_{fppf}$ the structure functor, then we can think of $(y, s_0, \ldots, s_n)$ as $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ where y is an object of $\mathcal{X}$, $f_i : p(y) \to p(x)$ is a morphism of schemes, and $\alpha_i : y \to x(f_i)$ an isomorphism. The category of $2n+3$-tuples $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ is an incarnation of the $(n+1)$-fold fibred product $\mathcal{U}_n$ of algebraic stacks displayed above, as we discussed in Section 18. By Cohomology on Sites, Lemma 13.3 we have

$$H^p(\mathcal{U}_n, \mathcal{F}|_{\mathcal{U}_n}) = H^p(\mathcal{X}/h_{x,n}, \mathcal{F}|_{\mathcal{X}/h_{x,n}}) = H^p(h_{x,n}, \mathcal{F}).$$

Finally, we discuss the "primed" analogue of this. Namely, $\mathcal{X}'/h_{x,n}$ corresponds, via the equivalence above to the full subcategory $\mathcal{U}'_n \subset \mathcal{U}_n$ consisting of those tuples $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ with $y \in \mathcal{X}'$. Hence certainly property (1) of the statement of the lemma holds for the inclusion $\mathcal{U}'_n \subset \mathcal{U}_n$. To see property (2) choose an object $\xi = (y, s_0, \ldots, s_n)$ which lies over a scheme W such that $(Sch/W)_{fppf} \to \mathcal{U}_n$ is smooth and surjective (this is possible as $\mathcal{U}_n$ is an algebraic stack). Then $(Sch/W)_{fppf} \to \mathcal{U}_n \to (Sch/U)_{fppf}$ is smooth as a composition of base changes of the morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$, see Algebraic Stacks, Lemmas 10.6 and 10.5. Thus axiom (1) for $\mathcal{X}$ implies that y is an object of $\mathcal{X}'$ whence ξ is an object of $\mathcal{U}'_n$. Using again

$$H^p(\mathcal{U}'_n, \mathcal{F}|_{\mathcal{U}'_n}) = H^p(\mathcal{X}'/h_{x,n}, \mathcal{F}|_{\mathcal{X}'/h_{x,n}}) = H^p(g^{-1}h_{x,n}, g^{-1}\mathcal{F}).$$

we now can use Case II for $\mathcal{U}'_n \subset \mathcal{U}_n$ to conclude. $\square$

24. Restricting to affines

In this section, given a category $\mathcal{X}$ fibred in groupoids over $(Sch/S)_{fppf}$ we will consider the full subcategory $\mathcal{X}_{affine}$ of $\mathcal{X}$ consisting of objects x lying over affine schemes U. We will see how, for any topology τ finer than the Zariski topology, the category of sheaves on $\mathcal{X}$ and $\mathcal{X}_{affine,\tau}$ agree.

Definition 24.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The *associated affine site* is the full subcategory $\mathcal{X}_{affine}$ of $\mathcal{X}$ whose objects are those $x \in \mathrm{Ob}(\mathcal{X})$ lying over a scheme U such that U is affine. The topology on $\mathcal{X}_{affine}$ will be the chaotic one, i.e., such that sheaves on $\mathcal{X}_{affine}$ are the same as presheaves.

Thus the functor $p : \mathcal{X} \to (Sch/S)_{fppf}$ restricts to a functor

$$p : \mathcal{X}_{affine} \longrightarrow (Aff/S)_{fppf}$$

where the notation on the right hand side is the one introduced in Topologies, Definition 7.8. It is clear that $\mathcal{X}_{affine}$ is fibred in groupoids over $(Aff/S)_{fppf}$. It follows that $\mathcal{X}_{affine}$ inherits a Zariski, étale, smooth, syntomic, and fppf topology from $(Aff/S)_{Zar}$, $(Aff/S)_{étale}$, $(Aff/S)_{smooth}$, $(Aff/S)_{syntomic}$, and $(Aff/S)_{fppf}$, see Stacks, Definition 10.2.

Definition 24.2. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) The *associated affine Zariski site* $\mathcal{X}_{affine,Zar}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(Aff/S)_{Zar}$.
(2) The *associated affine étale site* $\mathcal{X}_{affine,étale}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(Aff/S)_{étale}$.
(3) The *associated affine smooth site* $\mathcal{X}_{affine,smooth}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(Aff/S)_{smooth}$.
(4) The *associated affine syntomic site* $\mathcal{X}_{affine,syntomic}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(Aff/S)_{syntomic}$.

(5) The *associated affine fppf site* $\mathcal{X}_{affine,fppf}$ is the structure of site on $\mathcal{X}_{affine}$ inherited from $(Aff/S)_{fppf}$.

This definition makes sense by the discussion above. For each $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ a family of morphisms $\{x_i \to x\}_{i\in I}$ with fixed target in $\mathcal{X}_{affine}$ is a covering in $\mathcal{X}_{affine,\tau}$ if and only if the family of morphisms $\{p(x_i) \to p(x)\}_{i\in I}$ of affine schemes is a standard τ-covering as defined in Topologies, Definitions 3.4, 4.5, 5.5, 6.5, and 7.5.

Lemma 24.3. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. The functor $\mathcal{X}_{affine,\tau} \to \mathcal{X}_\tau$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh(\mathcal{X}_{affine,\tau})$ to $Sh(\mathcal{X}_\tau)$.*

Proof. Omitted. Hint: the proof is exactly the same as the proof of Topologies, Lemmas 3.10, 4.11, 5.9, 6.9, and 7.11. □

Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let us denote $\mathcal{O}$ the restriction of $\mathcal{O}_\mathcal{X}$ to $\mathcal{X}_{affine}$. Then $\mathcal{O}$ is a sheaf in the Zariski, étale, smooth, syntomic, and fppf topologies on $\mathcal{X}_{affine}$. Furthermore, the equivalence of topoi of Lemma 24.3 extends to an equivalence

$$(Sh(\mathcal{X}_{affine,\tau}), \mathcal{O}) \longrightarrow (Sh(\mathcal{X}_\tau), \mathcal{O}_\mathcal{X}) \tag{24.3.1}$$

of ringed topoi for $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$.

25. Quasi-coherent modules and affines

Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. In Section 24 we have associated to this a ringed site $(\mathcal{X}_{affine}, \mathcal{O})$.

Lemma 25.1. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be an $\mathcal{O}$-module on $\mathcal{X}_{affine}$. The following are equivalent*

(1) *for every morphism $x \to x'$ of $\mathcal{X}_{affine}$ the map $\mathcal{F}(x') \otimes_{\mathcal{O}(x')} \mathcal{O}(x) \to \mathcal{F}(x)$ is an isomorphism,*
(2) *$\mathcal{F}$ is a quasi-coherent module on $(\mathcal{X}_{affine}, \mathcal{O})$ in the sense of Modules on Sites, Definition 23.1,*
(3) *$\mathcal{F}$ is a sheaf for the Zariski topology on $\mathcal{X}_{affine}$ and a quasi-coherent module on $(\mathcal{X}_{affine,Zar}, \mathcal{O})$ in the sense of Modules on Sites, Definition 23.1,*
(4) *same as in (3) for the étale topology,*
(5) *same as in (3) for the smooth topology,*
(6) *same as in (3) for the syntomic topology,*
(7) *same as in (3) for the fppf topology, and*
(8) *$\mathcal{F}$ corresponds to a quasi-coherent module on $\mathcal{X}$ via the equivalence (24.3.1).*

Proof. To make sense out of part (2), recall that $\mathcal{X}_{affine}$ is a site gotten by endowing the category $\mathcal{X}_{affine}$ with the chaotic topology (Definition 24.1) and hence a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ is the same thing as a presheaf of $\mathcal{O}$-modules. Conditions (1) and (2) are equivalent by Modules on Sites, Lemma 24.2. Observe that for $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ the presheaf $\mathcal{F}$ is a τ-sheaf if and only if for all $x \in \mathrm{Ob}(\mathcal{X}_{affine})$ the restriction to $\mathcal{X}_{affine}/x$ is a τ-sheaf. Set $U = p(x)$. Similarly to the discussion in Section 9 the object x of $\mathcal{X}_{affine}$ induces an equivalence $\mathcal{X}_{affine,étale}/x \to (Aff/U)_{étale}$ of sites. In this way we see that the equivalence of (1) with (3) – (7) follows from Descent, Lemma 11.1 applied to each of these sites. The equivalence of (8) and (7) is immediate from the fact that "being quasi-coherent" is an intrinsic property of sheaves of modules, see Modules on Sites, Section 18 $\square$

Lemma 25.2. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be an $\mathcal{O}$-module on $\mathcal{X}_{affine}$. The following are equivalent*

(1) *for every morphism $x \to x'$ of $\mathcal{X}_{affine}$ such that $p(x) \to p(x')$ is an étale morphism (of affine schemes), the map $\mathcal{F}(x') \otimes_{\mathcal{O}(x')} \mathcal{O}(x) \to \mathcal{F}(x)$ is an isomorphism,*
(2) *$\mathcal{F}$ is a sheaf for the étale topology on $\mathcal{X}_{affine}$ and for every object x of $\mathcal{X}_{affine}$ the restriction $x^*\mathcal{F}|_{U_{affine,étale}}$ is quasi-coherent where $U = p(x)$,*
(3) *$\mathcal{F}$ corresponds to a locally quasi-coherent module on $\mathcal{X}$ via the equivalence (24.3.1) for the étale topology.*

Proof. To make sense out of condition (2), recall that $U_{affine,étale}$ is the full subcategory of $U_{étale}$ consisting of affine objects, see Topologies, Definition 4.8. Similarly to the discussion in Section 9 the object x of $\mathcal{X}_{affine}$ induces an equivalence $\mathcal{X}_{affine,étale}/x \to (Aff/U)_{étale}$ of sites. Then $x^*\mathcal{F}$ is the sheaf of modules on $(Aff/U)_{étale}$ corresponding to the restriction $\mathcal{F}|_{\mathcal{X}_{affine,étale}/x}$. Finally, using the continuous and cocontinuous inclusion functor $U_{affine,étale} \to (Aff/U)_{étale}$ we can further restrict and obtain $x^*\mathcal{F}|_{U_{affine,étale}}$.

The equivalence of (1) and (2) follows from the remarks above and Descent, Lemma 11.2 applied to the restriction of $\mathcal{F}$ to $U_{affine,étale}$ for every object x of $\mathcal{X}$ lying over an affine scheme U. The equivalence of (2) and (3) is immediate from the definitions and the fact that quasi-coherent modules on $U_{affine,étale}$ and $U_{étale}$ correspond (again by Descent, Lemma 11.2 for example). $\square$

26. Quasi-coherent objects in the derived category

Algebraic geometers have contemplated invariants for non-representable functors X (valued in sets or groupoids) on Sch/S for decades. For instance, before the notion of a stack was invented, Mumford defined [Mum65] the Picard groupoid $\mathrm{Pic}(X)$ for the moduli functor X of elliptic curves as the 2-limit $Pic(U)$ over the category of all schemes U equipped with a map to X (i.e., with a family of elliptic curves). Similarly, Beilinson-Drinfeld defined [BD] the category $QCoh(X)$ for an ind-scheme $X = \mathrm{colim}\, X_i$ as the 2-limit $\lim QCoh(X_i)$. This strategy is sufficient for defining 1-categorical invariants like $QCoh(-)$, but inadequate for derived categorical ones (such as the quasi-coherent derived category) as 2-limits of triangulated categories are poorly behaved. With the advent of higher categorical technology and derived algebraic geometry, this problem can be resolved gracefully: one can define the quasi-coherent derived ∞-category $\mathcal{D}_{qc}(X)$ of the functor X as the limit $\lim \mathcal{D}_{qc}(U)$, where U ranges over all derived affines over X (see [Lur04]).

The goal of this section is to attach a triangulated category $QC(X)$ to a functor X (valued in sets or groupoids) as above. In fact, the construction works for any category $p : \mathcal{X} \to (Sch/S)_{fppf}$ fibred in groupoids (not just split ones). In good cases, the category $QC(\mathcal{X})$ can be shown to agree with the homotopy category of $\mathcal{D}_{qc}(\mathcal{X})$, though it is outside the scope of this document to explain this comparison. The salient features of the construction are:

(a) $QC(\mathcal{X})$ is a full subcategory of $D(\mathcal{X}_{affine}, \mathcal{O})$ by construction,

(b) $QC(\mathcal{X})$ agrees with $D_{QCoh}(\mathcal{O}_X)$ when $\mathcal{X}$ is representable by the algebraic space X,
(c) $QC(\mathcal{X})$ agrees with $D_{QCoh}(\mathcal{O}_\mathcal{X})$ when $\mathcal{X}$ is an algebraic stack,
(d) when $X = \text{Spf}(A)$ is an affine formal algebraic space attached to a noetherian ring A equipped with the I-adic topology for an ideal I, the triangulated category $QC(X)$ agrees with the full subcategory $D_{comp}(A, I) \subset D(A)$ of derived complete objects.

These results are proven in Proposition 26.4, Derived Categories of Stacks, Proposition 8.4, and Proposition 26.5.

As a motivation for the precise definition of $QC(\mathcal{X})$ we point the reader to the characterization, in Lemma 25.1, of quasi-coherent modules on $\mathcal{X}$ as presheaves of $\mathcal{O}$-modules on $\mathcal{X}_{affine}$ which satisfy a kind of base change property.

Definition 26.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{O}$ be the sheaf of rings on $\mathcal{X}_{affine}$ introduced in Section 24. We define the *triangulated category of quasi-coherent objects in the derived category* by the formula

$$QC(\mathcal{X}) = QC(\mathcal{X}_{affine}, \mathcal{O})$$

where the right hand side is as defined in Cohomology on Sites, Definition 43.1.

Note that this makes sense as $\mathcal{X}_{affine}$ is a category and is viewed as a site by endowing it with the chaotic topology and $\mathcal{O}$ is a sheaf of rings on this category, exactly as required in Cohomology on Sites, Definition 43.1.

The relationship of this definition with the category of quasi-coherent modules on $\mathcal{X}$ is not so clear in general! For example, suppose that M is an object of $QC(\mathcal{X})$. Then the cohomology sheaves $H^i(M)$ of M are (pre)sheaves of $\mathcal{O}$-modules on $\mathcal{X}_{affine}$, but in general they are not quasi-coherent. The last nonvanishing cohomology sheaf is quasi-coherent however.

Lemma 26.2. *In the situation of Definition 26.1 suppose that M is an object of $QC(\mathcal{X})$ and $b \in \mathbf{Z}$ such that $H^i(M) = 0$ for all $i > b$. Then $H^b(M)$ is a quasi-coherent module on $(\mathcal{X}_{affine}, \mathcal{O})$, see Lemma 25.1.*

Proof. Special case of Cohomology on Sites, Lemma 43.3. $\square$

Lemma 26.3. *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The comparision morphism $\epsilon : \mathcal{X}_{affine,étale} \to \mathcal{X}_{affine}$ satisfies the assumptions and conclusions of Cohomology on Sites, Lemma 43.12.*

Proof. Assumption (1) holds by definition of $\mathcal{X}_{affine}$. For condition (2) we use that for $x \in \text{Ob}(\mathcal{X})$ lying over the affine scheme $U = p(x)$ we have an equivalence $\mathcal{X}_{affine,étale}/x = (Aff/U)_{étale}$ compatible with structure sheaves; see discussion in Section 9. Thus it suffices to show: given an affine scheme $U = \text{Spec}(R)$ and a complex of R-modules $M^\bullet$ the total cohomology of the complex of modules on $(Aff/U)_{étale}$ associated to $M^\bullet$ is quasi-isomorphic to $M^\bullet$. This follows from a combination of: Derived Categories of Schemes, Lemma 3.5 (total cohomology of complexes of modules over affines in the Zariski topology), Derived Categories of Spaces, Remark 6.3 (agreement between total cohomology in small Zariski and étale topologies for quasi-coherent complexes of modules), and Étale Cohomology, Lemma 99.3 (to see that the étale cohomology of a complex of modules on the big étale site of a scheme may be computed after restricting to the small étale site). $\square$

If we apply the definition in case our category fibred in groupoids $\mathcal{X}$ is representable by an algebraic space X, then we recover $D_{QCoh}(\mathcal{O}_X)$. We will later state and prove the analogous result for algebraic stacks (insert future reference here).

Proposition 26.4. *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume $\mathcal{X}$ is representable by an algebraic space X. Then $QC(\mathcal{X})$ is canonically equivalent to $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Denote X_{affine} the category of affine schemes étale over X endowed with the chaotic topology and its structure sheaf $\mathcal{O}_X$, see Derived Categories of Spaces, Section 30. The functor $u : X_{étale} \to \mathcal{X}_{étale}$ of Lemma 10.1 gives rise to a functor $X_{affine} \to \mathcal{X}_{affine}$. This is compatible with structure sheaves and produces a functor

$$G : QC(\mathcal{X}) = QC(\mathcal{X}_{affine}, \mathcal{O}) \longrightarrow QC(X_{affine}, \mathcal{O}_X)$$

See Cohomology on Sites, Lemma 43.10. By Derived Categories of Spaces, Lemma 30.1 the triangulated category $QC(X_{affine}, \mathcal{O}_X)$ is equivalent to $D_{QCoh}(\mathcal{O}_X)$. Hence it suffices to prove that G is an equivalence.

Consider the flat comparision morphisms $\epsilon_{\mathcal{X}} : \mathcal{X}_{affine,étale} \to \mathcal{X}_{affine}$ and $\epsilon_X : X_{affine,étale} \to X_{affine}$ of ringed sites. Lemma 26.3 and (the proof of) Derived Categories of Spaces, Lemma 30.1 show that the functors $\epsilon_{\mathcal{X}}^*$ and ϵ_X^* identify $QC(\mathcal{X}_{affine}, \mathcal{O})$ and $QC(X_{affine}, \mathcal{O}_X)$ with subcategories $Q_{\mathcal{X}} \subset D(\mathcal{X}_{affine,étale}, \mathcal{O})$ and $Q_X \subset D(X_{affine,étale}, \mathcal{O}_X)$. With these identifications the functor G in the first paragraph is induced by the functor

$$Li_X^* = R\pi_{X,*} : D(\mathcal{X}_{affine,étale}, \mathcal{O}) \longrightarrow D(X_{affine,étale}, \mathcal{O}_X)$$

where i_X and π_X are the morphisms from Lemma 10.1 but with the étale sites replaced by the corresponding affine ones. The reader can show that this replacement is permissible either by reproving the lemma for the affine sites directly or by using the equivalences of topoi $Sh(\mathcal{X}_{affine,étale}) = Sh(\mathcal{X}_{étale})$ and $Sh(X_{affine,étale}) = Sh(X_{étale})$. The lemma also tells us Li_X^* has a left adjoint

$$L\pi_X^* : D(X_{affine,étale}, \mathcal{O}_X) \longrightarrow D(\mathcal{X}_{affine,étale}, \mathcal{O})$$

and moreover we have $Li_X^* \circ L\pi_X^* = \mathrm{id}$ since $\pi_X \circ i_X$ is the identity. Thus it suffices to show that (a) $L\pi_X^*$ sends Q_X into $Q_{\mathcal{X}}$ and (b) the kernel of Li_X^* is 0. See Derived Categories, Lemma 7.2.

Proof of (a). By Derived Categories of Spaces, Lemma 30.1 we have $Q_X = D_{QCoh}(X_{affine,étale}, \mathcal{O}_X)$. Let K be an object of Q_X. Let x be an object of $\mathcal{X}_{affine,étale}$ lying over the affine scheme $U = p(x)$. Denote $f : U \to X$ the morphism corresponding to x. Then we see that

$$R\Gamma(x, L\pi_X^* K) = R\Gamma(U, Lf^* K)$$

This follows from transitivity of pullbacks; see discussion in Section 10. Next, suppose that $x \to x'$ is a morphism of $\mathcal{X}_{affine,étale}$ lying over the morphism $h : U \to U'$ of affine schemes. As before denote $f : U \to X$ and $f' : U' \to X$ the morphisms corresponding to x and x' so that we have $f = f' \circ h$. Then

$$\begin{aligned} R\Gamma(x, L\pi_X^*K) &= R\Gamma(U, Lf^*K) \\ &= R\Gamma(U, Lh^*L(f')^*K) \\ &= R\Gamma(U', L(f')^*K) \otimes^{\mathbf{L}}_{\mathcal{O}(U')} \mathcal{O}(U) \\ &= R\Gamma(x', L\pi_X^*K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \end{aligned}$$

and hence we have (a) by the footnote in the statement of Cohomology on Sites, Lemma 43.12. The third equality is Derived Categories of Schemes, Lemma 3.8.

Proof of (b). Let M be an object of $Q_{\mathcal{X}}$ such that $Li_X^*M = 0$. Let x' be an object of $\mathcal{X}_{affine,étale}$ lying over the affine scheme $U' = p(x')$ and assume that the corresponding morphism $f' : U' \to X$ is étale. Then $f' : U' \to X$ is an object of $X_{affine,étale}$ and the condition $Li_X^*M = 0$ implies that $M|_{U'_{étale}} = 0$. In particular, we see that $R\Gamma(x', M) = 0$. However, for an arbitrary object x of the site $\mathcal{X}_{affine,étale}$ there exists a covering $\{x_i \to x\}$ such that for each i there is a morphism $x_i \to x'_i$ with x'_i corresponding to an object of $X_{affine,étale}$. Now since M is in $Q_{\mathcal{X}}$ we have

$$R\Gamma(x_i, M) = R\Gamma(x'_i, M) \otimes^{\mathbf{L}}_{\mathcal{O}(x'_i)} \mathcal{O}(x_i) = 0$$

and we conclude that M is zero as desired. □

To show that the construction produces an interesting category in another case, let us state and prove a characterization of $QC(\text{Spf}(A))$ for the formal spectrum of a Noetherian adic ring A.

Proposition 26.5. *Let S be a scheme. Let $X = Spf(A)$ where A is an an adic Noetherian topological S-algebra with ideal of definition I, see More on Algebra, Definition 36.1 and Formal Spaces, Definition 9.9. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ the be category fibred in sets associated to the functor X, see Categories, Example 38.5. Then $QC(\mathcal{X})$ is canonically equivalent to the category $D_{comp}(A, I)$ of objects of $D(A)$ which are derived complete with respect to I.*

Proof. Recall that $X = \text{colim}\,\text{Spec}(A/I^n)$ as an fppf sheaf. An object of $\mathcal{X}_{affine}$ is the same thing as an affine scheme $U = \text{Spec}(R)$ with a given morphism $f : U \to X$. By Formal Spaces, Lemma 9.4 there exists an $n \geq 1$ such that f factors through the monomorphism $\text{Spec}(A/I^n) \to X$. Consider the full subcategory $\mathcal{C} \subset \mathcal{X}_{affine}$ consisting of the objects $\text{Spec}(A/I^n) \to X$. By the remarks just made and Differential Graded Sheaves, Lemma 34.1 restriction to $\mathcal{C}$ is an exact equivalence $QC(\mathcal{X}) \to QC(\mathcal{C}, \mathcal{O}|_{\mathcal{C}})$. For simplicity, let us assume that $I^n \neq I^{n+1}$ for all $n \geq 1$. Then $(\mathcal{C}, \mathcal{O}|_{\mathcal{C}})$ is isomorphic as a ringed site to the ringed site $(\mathbf{N}, (A/I^n))$, see Differential Graded Sheaves, Section 35. Hence we conclude by Differential Graded Sheaves, Proposition 35.4. □

The following lemma will be used in comparing $QC(\mathcal{X})$ to $D_{QCoh}(\mathcal{O}_{\mathcal{X}})$ when $\mathcal{X}$ is an algebraic stack.

Lemma 26.6. *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The comparision morphism $\epsilon : \mathcal{X}_{affine,fppf} \to \mathcal{X}_{affine}$ satisfies the assumptions and conclusions of Cohomology on Sites, Lemma 43.12.*

Proof. The proof is exactly the same as the proof of Lemma 26.3. Assumption (1) holds by definition of $\mathcal{X}_{affine}$. For condition (2) we use that for $x \in \text{Ob}(\mathcal{X})$ lying over the affine scheme $U = p(x)$ we have an equivalence $\mathcal{X}_{affine,étale}/x = (\textit{Aff}/U)_{étale}$ compatible with structure sheaves; see discussion in Section 9. Thus it suffices to show: given an affine scheme $U = \text{Spec}(R)$ and a complex of R-modules $M^\bullet$ the total cohomology of the complex of modules on $(\textit{Aff}/U)_{fppf}$ associated to $M^\bullet$ is quasi-isomorphic to $M^\bullet$. This is Étale Cohomology, Lemma 101.3. $\square$

27. Other chapters

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