

LIMITS OF ALGEBRAIC SPACES

Contents

1. Introduction	1
2. Conventions	2
3. Morphisms of finite presentation	2
4. Limits of algebraic spaces	8
5. Descending properties	10
6. Descending properties of morphisms	16
7. Descending relative objects	21
8. Absolute Noetherian approximation	23
9. Applications	25
10. Relative approximation	28
11. Finite type closed in finite presentation	29
12. Approximating proper morphisms	31
13. Embedding into affine space	32
14. Sections with support in a closed subset	33
15. Characterizing affine spaces	35
16. Finite cover by a scheme	37
17. Obtaining schemes	39
18. Glueing in closed fibres	41
19. Application to modifications	42
20. Universally closed morphisms	44
21. Noetherian valuative criterion	47
22. Refined Noetherian valuative criteria	51
23. Descending finite type spaces	53
24. Other chapters	56
References	57

1. Introduction

In this chapter we put material related to limits of algebraic spaces. A first topic is the characterization of algebraic spaces F locally of finite presentation over the base S as limit preserving functors. We continue with a study of limits of inverse systems over directed sets (Categories, Definition 21.1) with affine transition maps. We discuss absolute Noetherian approximation for quasi-compact and quasi-separated algebraic spaces following [CLO12]. Another approach is due to David Rydh (see [Ryd08]) whose results also cover absolute Noetherian approximation for certain algebraic stacks.

2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf} . And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S . In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Morphisms of finite presentation

In this section we generalize Limits, Proposition 6.1 to morphisms of algebraic spaces. The motivation for the following definition comes from the proposition just cited.

Definition 3.1. Let S be a scheme.

- (1) A functor $F : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ is said to be *limit preserving* or *locally of finite presentation* if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S , we have

$$F(T) = \text{colim } F(T_i).$$

We sometimes say that F is *locally of finite presentation over S* .

- (2) Let $F, G : (Sch/S)_{fppf}^{opp} \rightarrow Sets$. A transformation of functors $a : F \rightarrow G$ is *limit preserving* or *locally of finite presentation* if for every scheme T over S and every $y \in G(T)$ the functor

$$F_y : (Sch/T)_{fppf}^{opp} \longrightarrow Sets, \quad T'/T \longmapsto \{x \in F(T') \mid a(x) = y|_{T'}\}$$

is locally of finite presentation over T^1 . We sometimes say that F is *relatively limit preserving* over G .

The functor F_y is in some sense the fiber of $a : F \rightarrow G$ over y , except that it is a presheaf on the big fppf site of T . A formula for this functor is:

$$(3.1.1) \quad F_y = F|_{(Sch/T)_{fppf}} \times_{G|_{(Sch/T)_{fppf}}} *$$

Here $*$ is the final object in the category of (pre)sheaves on $(Sch/T)_{fppf}$ (see Sites, Example 10.2) and the map $* \rightarrow G|_{(Sch/T)_{fppf}}$ is given by y . Note that if $j : (Sch/T)_{fppf} \rightarrow (Sch/S)_{fppf}$ is the localization functor, then the formula above becomes $F_y = j^{-1}F \times_{j^{-1}G} *$ and $j_!F_y$ is just the fiber product $F \times_{G,y} T$. (See Sites, Section 25, for information on localization, and especially Sites, Remark 25.10 for information on $j_!$ for presheaves.)

At this point we temporarily have two definitions of what it means for a morphism $X \rightarrow Y$ of algebraic spaces over S to be locally of finite presentation. Namely, one by Morphisms of Spaces, Definition 28.1 and one using that $X \rightarrow Y$ is a transformation of functors so that Definition 3.1 applies (we will use the terminology “limit preserving” for this notion as much as possible). We will show in Proposition 3.10 that these two definitions agree.

Lemma 3.2. *Let S be a scheme. Let $a : F \rightarrow G$ be a transformation of functors $(Sch/S)_{fppf}^{opp} \rightarrow Sets$. The following are equivalent*

¹The characterization (2) in Lemma 3.2 may be easier to parse.

- (1) $a : F \rightarrow G$ is limit preserving, and
 (2) for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S the diagram of sets

$$\begin{array}{ccc} \operatorname{colim}_i F(T_i) & \longrightarrow & F(T) \\ a \downarrow & & \downarrow a \\ \operatorname{colim}_i G(T_i) & \longrightarrow & G(T) \end{array}$$

is a fibre product diagram.

Proof. Assume (1). Consider $T = \lim_{i \in I} T_i$ as in (2). Let (y, x_T) be an element of the fibre product $\operatorname{colim}_i G(T_i) \times_{G(T)} F(T)$. Then y comes from $y_i \in G(T_i)$ for some i . Consider the functor F_{y_i} on $(\operatorname{Sch}/T_i)_{fppf}$ as in Definition 3.1. We see that $x_T \in F_{y_i}(T)$. Moreover $T = \lim_{i' \geq i} T_{i'}$ is a directed system of affine schemes over T_i . Hence (1) implies that x_T the image of a unique element x of $\operatorname{colim}_{i' \geq i} F_{y_i}(T_{i'})$. Thus x is the unique element of $\operatorname{colim} F(T_i)$ which maps to the pair (y, x_T) . This proves that (2) holds.

Assume (2). Let T be a scheme and $y_T \in G(T)$. We have to show that F_{y_T} is limit preserving. Let $T' = \lim_{i \in I} T'_i$ be an affine scheme over T which is the directed limit of affine scheme T'_i over T . Let $x_{T'} \in F_{y_T}$. Pick $i \in I$ which is possible as I is a directed set. Denote $y_i \in F(T'_i)$ the image of $y_{T'}$. Then we see that $(y_i, x_{T'})$ is an element of the fibre product $\operatorname{colim}_i G(T'_i) \times_{G(T')} F(T')$. Hence by (2) we get a unique element x of $\operatorname{colim}_i F(T'_i)$ mapping to $(y_i, x_{T'})$. It is clear that x defines an element of $\operatorname{colim}_i F_y(T'_i)$ mapping to $x_{T'}$ and we win. \square

Lemma 3.3. Let S be a scheme contained in $\operatorname{Sch}_{fppf}$. Let $F, G, H : (\operatorname{Sch}/S)_{fppf}^{opp} \rightarrow \operatorname{Sets}$. Let $a : F \rightarrow G$, $b : G \rightarrow H$ be transformations of functors. If a and b are limit preserving, then

$$b \circ a : F \longrightarrow H$$

is limit preserving.

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 3.2. Consider the diagram of sets

$$\begin{array}{ccc} \operatorname{colim}_i F(T_i) & \longrightarrow & F(T) \\ a \downarrow & & \downarrow a \\ \operatorname{colim}_i G(T_i) & \longrightarrow & G(T) \\ b \downarrow & & \downarrow b \\ \operatorname{colim}_i H(T_i) & \longrightarrow & H(T) \end{array}$$

By assumption the two squares are fibre product squares. Hence the outer rectangle is a fibre product diagram too which proves the lemma. \square

Lemma 3.4. Let S be a scheme contained in $\operatorname{Sch}_{fppf}$. Let $F, G, H : (\operatorname{Sch}/S)_{fppf}^{opp} \rightarrow \operatorname{Sets}$. Let $a : F \rightarrow G$, $b : G \rightarrow H$ be transformations of functors. If $b \circ a$ and b are limit preserving, then a is limit preserving.

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 3.2. Consider the diagram of sets

$$\begin{array}{ccc} \operatorname{colim}_i F(T_i) & \longrightarrow & F(T) \\ a \downarrow & & \downarrow a \\ \operatorname{colim}_i G(T_i) & \longrightarrow & G(T) \\ b \downarrow & & \downarrow b \\ \operatorname{colim}_i H(T_i) & \longrightarrow & H(T) \end{array}$$

By assumption the lower square and the outer rectangle are fibre products of sets. Hence the upper square is a fibre product square too which proves the lemma. \square

Lemma 3.5. *Let S be a scheme contained in $\operatorname{Sch}_{fppf}$. Let $F, G, H : (\operatorname{Sch}/S)_{fppf}^{opp} \rightarrow \operatorname{Sets}$. Let $a : F \rightarrow G$, $b : H \rightarrow G$ be transformations of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b, G, a} F & \xrightarrow{b'} & F \\ a' \downarrow & & \downarrow a \\ H & \xrightarrow{b} & G \end{array}$$

If a is limit preserving, then the base change a' is limit preserving.

Proof. Omitted. Hint: This is formal. \square

Lemma 3.6. *Let S be a scheme contained in $\operatorname{Sch}_{fppf}$. Let $E, F, G, H : (\operatorname{Sch}/S)_{fppf}^{opp} \rightarrow \operatorname{Sets}$. Let $a : F \rightarrow G$, $b : H \rightarrow G$, and $c : G \rightarrow E$ be transformations of functors. If c , $c \circ a$, and $c \circ b$ are limit preserving, then $F \times_G H \rightarrow E$ is too.*

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 3.2. Then we have

$$\operatorname{colim}(F \times_G H)(T_i) = \operatorname{colim} F(T_i) \times_{\operatorname{colim} G(T_i)} \operatorname{colim} H(T_i)$$

as filtered colimits commute with finite products. Our goal is thus to show that

$$\begin{array}{ccc} \operatorname{colim} F(T_i) \times_{\operatorname{colim} G(T_i)} \operatorname{colim} H(T_i) & \longrightarrow & F(T) \times_{G(T)} H(T) \\ \downarrow & & \downarrow \\ \operatorname{colim}_i E(T_i) & \longrightarrow & E(T) \end{array}$$

is a fibre product diagram. This follows from the observation that given maps of sets $E' \rightarrow E$, $F \rightarrow G$, $H \rightarrow G$, and $G \rightarrow E$ we have

$$E' \times_E (F \times_G H) = (E' \times_E F) \times_{(E' \times_E G)} (E' \times_E H)$$

Some details omitted. \square

Lemma 3.7. *Let S be a scheme contained in $\operatorname{Sch}_{fppf}$. Let $F : (\operatorname{Sch}/S)_{fppf}^{opp} \rightarrow \operatorname{Sets}$ be a functor. If F is limit preserving then its sheafification $F^\#$ is limit preserving.*

Proof. Assume F is limit preserving. It suffices to show that F^+ is limit preserving, since $F^\# = (F^+)^+$, see Sites, Theorem 10.10. Let T be an affine scheme over S , and let $T = \lim T_i$ be written as the directed limit of an inverse system of affine S schemes. Recall that $F^+(T)$ is the colimit of $\check{H}^0(\mathcal{V}, F)$ where the limit is over all

coverings of T in $(Sch/S)_{fppf}$. Any fppf covering of an affine scheme can be refined by a standard fppf covering, see Topologies, Lemma 7.4. Hence we can write

$$F^+(T) = \operatorname{colim}_{\mathcal{V} \text{ standard covering } T} \check{H}^0(\mathcal{V}, F).$$

Any $\mathcal{V} = \{T_k \rightarrow T\}_{k=1, \dots, n}$ in the colimit may be written as $V_i \times_{T_i} T$ for some i and some standard fppf covering $\mathcal{V}_i = \{T_{i,k} \rightarrow T_i\}_{k=1, \dots, n}$ of T_i . Denote $\mathcal{V}_{i'} = \{T_{i',k} \rightarrow T_{i'}\}_{k=1, \dots, n}$ the base change for $i' \geq i$. Then we see that

$$\begin{aligned} \operatorname{colim}_{i' \geq i} \check{H}^0(\mathcal{V}_i, F) &= \operatorname{colim}_{i' \geq i} \operatorname{Equalizer}(\prod F(T_{i',k}) \rightrightarrows \prod F(T_{i',k} \times_{T_{i'}} T_{i',l})) \\ &= \operatorname{Equalizer}(\operatorname{colim}_{i' \geq i} \prod F(T_{i',k}) \rightrightarrows \operatorname{colim}_{k' \geq k} \prod F(T_{i',k} \times_{T_{i'}} T_{i',l})) \\ &= \operatorname{Equalizer}(\prod F(T_k) \rightrightarrows \prod F(T_k \times_T T_l)) \\ &= \check{H}^0(\mathcal{V}, F) \end{aligned}$$

Here the second equality holds because filtered colimits are exact. The third equality holds because F is limit preserving and because $\lim_{i' \geq i} T_{i',k} = T_k$ and $\lim_{i' \geq i} T_{i',k} \times_{T_{i'}} T_{i',l} = T_k \times_T T_l$ by Limits, Lemma 2.3. If we use this for all coverings at the same time we obtain

$$\begin{aligned} F^+(T) &= \operatorname{colim}_{\mathcal{V} \text{ standard covering } T} \check{H}^0(\mathcal{V}, F) \\ &= \operatorname{colim}_{i \in I} \operatorname{colim}_{\mathcal{V}_i \text{ standard covering } T_i} \check{H}^0(T \times_{T_i} \mathcal{V}_i, F) \\ &= \operatorname{colim}_{i \in I} F^+(T_i) \end{aligned}$$

The switch of the order of the colimits is allowed by Categories, Lemma 14.10. \square

Lemma 3.8. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \rightarrow \mathbf{Sets}$ be a functor. Assume that*

- (1) *F is a sheaf, and*
- (2) *there exists an fppf covering $\{U_j \rightarrow S\}_{j \in J}$ such that $F|_{(Sch/U_j)_{fppf}}$ is limit preserving.*

Then F is limit preserving.

Proof. Let T be an affine scheme over S . Let I be a directed set, and let T_i be an inverse system of affine schemes over S such that $T = \lim T_i$. We have to show that the canonical map $\operatorname{colim} F(T_i) \rightarrow F(T)$ is bijective.

Choose some $0 \in I$ and choose a standard fppf covering $\{V_{0,k} \rightarrow T_0\}_{k=1, \dots, m}$ which refines the pullback $\{U_j \times_S T_0 \rightarrow T_0\}$ of the given fppf covering of S . For each $i \geq 0$ we set $V_{i,k} = T_i \times_{T_0} V_{0,k}$, and we set $V_k = T \times_{T_0} V_{0,k}$. Note that $V_k = \lim_{i \geq 0} V_{i,k}$, see Limits, Lemma 2.3.

Suppose that $x, x' \in \operatorname{colim} F(T_i)$ map to the same element of $F(T)$. Say x, x' are given by elements $x_i, x'_i \in F(T_i)$ for some $i \in I$ (we may choose the same i for both as I is directed). By assumption (2) and the fact that x_i, x'_i map to the same element of $F(T)$ this implies that

$$x_i|_{V_{i',k}} = x'_i|_{V_{i',k}}$$

for some suitably large $i' \in I$. We can choose the same i' for each k as $k \in \{1, \dots, m\}$ ranges over a finite set. Since $\{V_{i',k} \rightarrow T_{i'}\}$ is an fppf covering and F is a sheaf this implies that $x_i|_{T_{i'}} = x'_i|_{T_{i'}}$ as desired. This proves that the map $\operatorname{colim} F(T_i) \rightarrow F(T)$ is injective.

To show surjectivity we argue in a similar fashion. Let $x \in F(T)$. By assumption (2) for each k we can choose a i such that $x|_{V_k}$ comes from an element $x_{i,k} \in F(V_{i,k})$. As before we may choose a single i which works for all k . By the injectivity proved above we see that

$$x_{i,k}|_{V_{i',k} \times_{T_{i'}} V_{i',l}} = x_{i,l}|_{V_{i',k} \times_{T_{i'}} V_{i',l}}$$

for some large enough i' . Hence by the sheaf condition of F the elements $x_{i,k}|_{V_{i',k}}$ glue to an element $x_{i'} \in F(T_{i'})$ as desired. \square

Lemma 3.9. *Let S be a scheme contained in Sch_{fppf} . Let $F, G : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ be functors. If $a : F \rightarrow G$ is a transformation which is limit preserving, then the induced transformation of sheaves $F^\# \rightarrow G^\#$ is limit preserving.*

Proof. Suppose that T is a scheme and $y \in G^\#(T)$. We have to show the functor $F_y^\# : (Sch/T)_{fppf}^{opp} \rightarrow Sets$ constructed from $F^\# \rightarrow G^\#$ and y as in Definition 3.1 is limit preserving. By Equation (3.1.1) we see that $F_y^\#$ is a sheaf. Choose an fppf covering $\{V_j \rightarrow T\}_{j \in J}$ such that $y|_{V_j}$ comes from an element $y_j \in F(V_j)$. Note that the restriction of $F^\#$ to $(Sch/V_j)_{fppf}$ is just $F_{y_j}^\#$. If we can show that $F_{y_j}^\#$ is limit preserving then Lemma 3.8 guarantees that $F_y^\#$ is limit preserving and we win. This reduces us to the case $y \in G(T)$.

Let $y \in G(T)$. In this case we claim that $F_y^\# = (F_y)^\#$. This follows from Equation (3.1.1). Thus this case follows from Lemma 3.7. \square

Proposition 3.10. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . The following are equivalent:*

- (1) *The morphism f is a morphism of algebraic spaces which is locally of finite presentation, see Morphisms of Spaces, Definition 28.1.*
- (2) *The morphism $f : X \rightarrow Y$ is limit preserving as a transformation of functors, see Definition 3.1.*

Proof. Assume (1). Let T be a scheme and let $y \in Y(T)$. We have to show that $T \times_Y X$ is limit preserving over T in the sense of Definition 3.1. Hence we are reduced to proving that if X is an algebraic space which is locally of finite presentation over S as an algebraic space, then it is limit preserving as a functor $X : (Sch/S)_{fppf}^{opp} \rightarrow Sets$. To see this choose a presentation $X = U/R$, see Spaces, Definition 9.3. It follows from Morphisms of Spaces, Definition 28.1 that both U and R are schemes which are locally of finite presentation over S . Hence by Limits, Proposition 6.1 we have

$$U(T) = \operatorname{colim} U(T_i), \quad R(T) = \operatorname{colim} R(T_i)$$

whenever $T = \lim_i T_i$ in $(Sch/S)_{fppf}$. It follows that the presheaf

$$(Sch/S)_{fppf}^{opp} \longrightarrow Sets, \quad W \longmapsto U(W)/R(W)$$

is limit preserving. Hence by Lemma 3.7 its sheafification $X = U/R$ is limit preserving too.

Assume (2). Choose a scheme V and a surjective étale morphism $V \rightarrow Y$. Next, choose a scheme U and a surjective étale morphism $U \rightarrow V \times_Y X$. By Lemma 3.5 the transformation of functors $V \times_Y X \rightarrow V$ is limit preserving. By Morphisms of Spaces, Lemma 39.8 the morphism of algebraic spaces $U \rightarrow V \times_Y X$ is locally of finite presentation, hence limit preserving as a transformation of functors by the

first part of the proof. By Lemma 3.3 the composition $U \rightarrow V \times_Y X \rightarrow V$ is limit preserving as a transformation of functors. Hence the morphism of schemes $U \rightarrow V$ is locally of finite presentation by Limits, Proposition 6.1 (modulo a set theoretic remark, see last paragraph of the proof). This means, by definition, that (1) holds.

Set theoretic remark. Let $U \rightarrow V$ be a morphism of $(Sch/S)_{fppf}$. In the statement of Limits, Proposition 6.1 we characterize $U \rightarrow V$ as being locally of finite presentation if for *all* directed inverse systems $(T_i, f_{ii'})$ of affine schemes over V we have $U(T) = \text{colim } U(T_i)$, but in the current setting we may only consider affine schemes T_i over V which are (isomorphic to) an object of $(Sch/S)_{fppf}$. So we have to make sure that there are enough affines in $(Sch/S)_{fppf}$ to make the proof work. Inspecting the proof of (2) \Rightarrow (1) of Limits, Proposition 6.1 we see that the question reduces to the case that U and V are affine. Say $U = \text{Spec}(A)$ and $V = \text{Spec}(B)$. By construction of $(Sch/S)_{fppf}$ the spectrum of any ring of cardinality $\leq |B|$ is isomorphic to an object of $(Sch/S)_{fppf}$. Hence it suffices to observe that in the "only if" part of the proof of Algebra, Lemma 127.3 only A -algebras of cardinality $\leq |B|$ are used. \square

Remark 3.11. Here is an important special case of Proposition 3.10. Let S be a scheme. Let X be an algebraic space over S . Then X is locally of finite presentation over S if and only if X , as a functor $(Sch/S)^{opp} \rightarrow Sets$, is limit preserving. Compare with Limits, Remark 6.2. In fact, we will see in Lemma 3.12 below that it suffices if the map

$$\text{colim } X(T_i) \longrightarrow X(T)$$

is surjective whenever $T = \lim T_i$ is a directed limit of affine schemes over S .

Lemma 3.12. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . If for every directed limit $T = \lim_{i \in I} T_i$ of affine schemes over S the map*

$$\text{colim } X(T_i) \longrightarrow X(T) \times_{Y(T)} \text{colim } Y(T_i)$$

is surjective, then f is locally of finite presentation. In other words, in Proposition 3.10 part (2) it suffices to check surjectivity in the criterion of Lemma 3.2.

Proof. Choose a scheme V and a surjective étale morphism $g : V \rightarrow Y$. Next, choose a scheme U and a surjective étale morphism $h : U \rightarrow V \times_Y X$. It suffices to show for $T = \lim T_i$ as in the lemma that the map

$$\text{colim } U(T_i) \longrightarrow U(T) \times_{V(T)} \text{colim } V(T_i)$$

is surjective, because then $U \rightarrow V$ will be locally of finite presentation by Limits, Lemma 6.3 (modulo a set theoretic remark exactly as in the proof of Proposition 3.10). Thus we take $a : T \rightarrow U$ and $b_i : T_i \rightarrow V$ which determine the same morphism $T \rightarrow V$. Picture

$$\begin{array}{ccccc} T & \xrightarrow{\quad p_i \quad} & T_i & & \\ a \downarrow & & \swarrow \text{dotted} & \searrow b_i & \\ U & \xrightarrow{\quad h \quad} & X \times_Y V & \longrightarrow & V \\ & & \downarrow & & \downarrow g \\ & & X & \xrightarrow{\quad f \quad} & Y \end{array}$$

By the assumption of the lemma after increasing i we can find a morphism $c_i : T_i \rightarrow X$ such that $h \circ a = (b_i, c_i) \circ p_i : T_i \rightarrow V \times_Y X$ and such that $f \circ c_i = g \circ b_i$. Since h is an étale morphism of algebraic spaces (and hence locally of finite presentation), we have the surjectivity of

$$\operatorname{colim} U(T_i) \longrightarrow U(T) \times_{(X \times_Y V)(T)} \operatorname{colim} (X \times_Y V)(T_i)$$

by Proposition 3.10. Hence after increasing i again we can find the desired morphism $a_i : T_i \rightarrow U$ with $a = a_i \circ p_i$ and $b_i = (U \rightarrow V) \circ a_i$. \square

4. Limits of algebraic spaces

The following lemma explains how we think of limits of algebraic spaces in this chapter. We will use (without further mention) that the base change of an affine morphism of algebraic spaces is affine (see Morphisms of Spaces, Lemma 20.5).

Lemma 4.1. *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I in the category of algebraic spaces over S . If the morphisms $f_{ii'} : X_i \rightarrow X_{i'}$ are affine, then the limit $X = \lim_i X_i$ (as an fppf sheaf) is an algebraic space. Moreover,*

- (1) *each of the morphisms $f_i : X \rightarrow X_i$ is affine,*
- (2) *for any $i \in I$ and any morphism of algebraic spaces $T \rightarrow X_i$ we have*

$$X \times_{X_i} T = \lim_{i' \geq i} X_{i'} \times_{X_i} T.$$

as algebraic spaces over S .

Proof. Part (2) is a formal consequence of the existence of the limit $X = \lim X_i$ as an algebraic space over S . Choose an element $0 \in I$ (this is possible as a directed set is nonempty). Choose a scheme U_0 and a surjective étale morphism $U_0 \rightarrow X_0$. Set $R_0 = U_0 \times_{X_0} U_0$ so that $X_0 = U_0/R_0$. For $i \geq 0$ set $U_i = X_i \times_{X_0} U_0$ and $R_i = X_i \times_{X_0} R_0 = U_i \times_{X_i} U_i$. By Limits, Lemma 2.2 we see that $U = \lim_{i \geq 0} U_i$ and $R = \lim_{i \geq 0} R_i$ are schemes. Moreover, the two morphisms $s, t : R \rightarrow U$ are the base change of the two projections $R_0 \rightarrow U_0$ by the morphism $U \rightarrow U_0$, in particular étale. The morphism $R \rightarrow U \times_S U$ defines an equivalence relation as directed a limit of equivalence relations is an equivalence relation. Hence the morphism $R \rightarrow U \times_S U$ is an étale equivalence relation. We claim that the natural map

$$(4.1.1) \quad U/R \longrightarrow \lim X_i$$

is an isomorphism of fppf sheaves on the category of schemes over S . The claim implies $X = \lim X_i$ is an algebraic space by Spaces, Theorem 10.5.

Let Z be a scheme and let $a : Z \rightarrow \lim X_i$ be a morphism. Then $a = (a_i)$ where $a_i : Z \rightarrow X_i$. Set $W_0 = Z \times_{a_0, X_0} U_0$. Note that $W_0 = Z \times_{a_i, X_i} U_i$ for all $i \geq 0$ by our choice of $U_i \rightarrow X_i$ above. Hence we obtain a morphism $W_0 \rightarrow \lim_{i \geq 0} U_i = U$. Since $W_0 \rightarrow Z$ is surjective and étale, we conclude that (4.1.1) is a surjective map of sheaves. Finally, suppose that Z is a scheme and that $a, b : Z \rightarrow U/R$ are two morphisms which are equalized by (4.1.1). We have to show that $a = b$. After replacing Z by the members of an fppf covering we may assume there exist morphisms $a', b' : Z \rightarrow U$ which give rise to a and b . The condition that a, b are equalized by (4.1.1) means that for each $i \geq 0$ the compositions $a'_i, b'_i : Z \rightarrow U \rightarrow U_i$ are equal as morphisms into $U_i/R_i = X_i$. Hence $(a'_i, b'_i) : Z \rightarrow U_i \times_S U_i$ factors through R_i , say by some morphism $c_i : Z \rightarrow R_i$. Since $R = \lim_{i \geq 0} R_i$ we see that

$c = \lim c_i : Z \rightarrow R$ is a morphism which shows that a, b are equal as morphisms of Z into U/R .

Part (1) follows as we have seen above that $U_i \times_{X_i} X = U$ and $U \rightarrow U_i$ is affine by construction. \square

Lemma 4.2. *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S with affine transition maps. Let $X = \lim_i X_i$. Let $0 \in I$. Suppose that $T \rightarrow X_0$ is a morphism of algebraic spaces. Then*

$$T \times_{X_0} X = \lim_{i \geq 0} T \times_{X_0} X_i$$

as algebraic spaces over S .

Proof. The limit X is an algebraic space by Lemma 4.1. The equality is formal, see Categories, Lemma 14.10. \square

Lemma 4.3. *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{i'i}) \rightarrow (Y_i, g_{i'i})$ be a morphism of inverse systems over I of algebraic spaces over S . Assume*

- (1) *the morphisms $f_{i'i} : X_{i'} \rightarrow X_i$ are affine,*
- (2) *the morphisms $g_{i'i} : Y_{i'} \rightarrow Y_i$ are affine,*
- (3) *the morphisms $X_i \rightarrow Y_i$ are closed immersions.*

Then $\lim X_i \rightarrow \lim Y_i$ is a closed immersion.

Proof. Observe that $\lim X_i$ and $\lim Y_i$ exist by Lemma 4.1. Pick $0 \in I$ and choose an affine scheme V_0 and an étale morphism $V_0 \rightarrow Y_0$. Then the morphisms $V_i = Y_i \times_{Y_0} V_0 \rightarrow U_i = X_i \times_{X_0} V_0$ are closed immersions of affine schemes. Hence the morphism $V = Y \times_{Y_0} V_0 \rightarrow U = X \times_{X_0} V_0$ is a closed immersion because $V = \lim V_i$, $U = \lim U_i$ and because a limit of closed immersions of affine schemes is a closed immersion: a filtered colimit of surjective ring maps is surjective. Since the étale morphisms $V \rightarrow Y$ form an étale covering of Y as we vary our choice of $V_0 \rightarrow Y_0$ we see that the lemma is true. \square

Lemma 4.4. *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{i'i})$ be an inverse system over I of algebraic spaces over S . If X_i is reduced for all i , then X is reduced.*

Proof. Observe that $\lim X_i$ exists by Lemma 4.1. Pick $0 \in I$ and choose an affine scheme V_0 and an étale morphism $U_0 \rightarrow X_0$. Then the affine schemes $U_i = X_i \times_{X_0} U_0$ are reduced. Hence $U = X \times_{X_0} U_0$ is a reduced affine scheme as a limit of reduced affine schemes: a filtered colimit of reduced rings is reduced. Since the étale morphisms $U \rightarrow X$ form an étale covering of X as we vary our choice of $U_0 \rightarrow X_0$ we see that the lemma is true. \square

Lemma 4.5. *Let S be a scheme. Let $X \rightarrow Y$ be a morphism of algebraic spaces over S . The equivalent conditions (1) and (2) of Proposition 3.10 are also equivalent to*

- (3) *for every directed limit $T = \lim T_i$ of quasi-compact and quasi-separated algebraic spaces T_i over S with affine transition morphisms the diagram of*

sets

$$\begin{array}{ccc} \operatorname{colim}_i \operatorname{Mor}(T_i, X) & \longrightarrow & \operatorname{Mor}(T, X) \\ \downarrow & & \downarrow \\ \operatorname{colim}_i \operatorname{Mor}(T_i, Y) & \longrightarrow & \operatorname{Mor}(T, Y) \end{array}$$

is a fibre product diagram.

Proof. It is clear that (3) implies (2). We will assume (2) and prove (3). The proof is rather formal and we encourage the reader to find their own proof.

Let us first prove that (3) holds when T_i is in addition assumed separated for all i . Choose $i \in I$ and choose a surjective étale morphism $U_i \rightarrow T_i$ where U_i is affine. Using Lemma 4.2 we see that with $U = U_i \times_{T_i} T$ and $U_{i'} = U_i \times_{T_i} T_{i'}$ we have $U = \lim_{i' \geq i} U_{i'}$. Of course U and $U_{i'}$ are affine (see Lemma 4.1). Since T_i is separated, the fibre product $V_i = U_i \times_{T_i} U_i$ is an affine scheme as well and we obtain affine schemes $V = V_i \times_{T_i} T$ and $V_{i'} = V_i \times_{T_i} T_{i'}$ with $V = \lim_{i' \geq i} V_{i'}$. Observe that $U \rightarrow T$ and $U_i \rightarrow T_i$ are surjective étale and that $V = U \times_T U$ and $V_{i'} = U_{i'} \times_{T_{i'}} U_{i'}$. Note that $\operatorname{Mor}(T, X)$ is the equalizer of the two maps $\operatorname{Mor}(U, X) \rightarrow \operatorname{Mor}(V, X)$; this is true for example because X as a sheaf on $(\operatorname{Sch}/S)_{fppf}$ is the coequalizer of the two maps $h_V \rightarrow h_U$. Similarly $\operatorname{Mor}(T_{i'}, X)$ is the equalizer of the two maps $\operatorname{Mor}(U_{i'}, X) \rightarrow \operatorname{Mor}(V_{i'}, X)$. And of course the same thing is true with X replaced with Y . Condition (2) says that the diagrams of in (3) are fibre products in the case of $U = \lim U_i$ and $V = \lim V_i$. It follows formally that the same thing is true for $T = \lim T_i$.

In the general case, choose an affine scheme U , an $i \in I$, and a surjective étale morphism $U \rightarrow T_i$. Repeating the argument of the previous paragraph we still achieve the proof: the schemes $V_{i'}$, V are no longer affine, but they are still quasi-compact and separated and the result of the preceding paragraph applies. \square

5. Descending properties

This section is the analogue of Limits, Section 4.

Lemma 5.1. *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). If each X_i is decent (for example quasi-separated or locally separated) then $|X| = \lim_i |X_i|$ as sets.*

Proof. There is a canonical map $|X| \rightarrow \lim |X_i|$. Choose $0 \in I$. If $W_0 \subset X_0$ is an open subspace, then we have $f_0^{-1}W_0 = \lim_{i \geq 0} f_{i0}^{-1}W_0$, see Lemma 4.1. Hence, if we can prove the lemma for inverse systems where X_0 is quasi-compact, then the lemma follows in general. Thus we may and do assume X_0 is quasi-compact.

Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$. Recall that $U = \lim U_i$ and $R = \lim R_i$, see proof of Lemma 4.1. Recall that $|X| = |U|/|R|$ and $|X_i| = |U_i|/|R_i|$. By Limits, Lemma 4.6 we have $|U| = \lim |U_i|$ and $|R| = \lim |R_i|$.

Surjectivity of $|X| \rightarrow \lim |X_i|$. Let $(x_i) \in \lim |X_i|$. Denote $S_i \subset |U_i|$ the inverse image of x_i . This is a finite nonempty set by the definition of decent spaces (Decent Spaces, Definition 6.1). Hence $\lim S_i$ is nonempty, see Categories, Lemma 21.7. Let

$(u_i) \in \lim S_i \subset \lim |U_i|$. By the above this determines a point $u \in |U|$ which maps to an $x \in |X|$ mapping to the given element (x_i) of $\lim |X_i|$.

Injectivity of $|X| \rightarrow \lim |X_i|$. Suppose that $x, x' \in |X|$ map to the same point of $\lim |X_i|$. Choose lifts $u, u' \in |U|$ and denote $u_i, u'_i \in |U_i|$ the images. For each i let $T_i \subset |R_i|$ be the set of points mapping to $(u_i, u'_i) \in |U_i| \times |U_i|$. This is a finite set by the definition of decent spaces (Decent Spaces, Definition 6.1). Moreover T_i is nonempty as we've assumed that x and x' map to the same point of X_i . Hence $\lim T_i$ is nonempty, see Categories, Lemma 21.7. As before let $r \in |R| = \lim |R_i|$ be a point corresponding to an element of $\lim T_i$. Then r maps to (u, u') in $|U| \times |U|$ by construction and we see that $x = x'$ in $|X|$ as desired.

Parentetical statement: A quasi-separated algebraic space is decent, see Decent Spaces, Section 6 (the key observation to this is Properties of Spaces, Lemma 6.7). A locally separated algebraic space is decent by Decent Spaces, Lemma 15.2. \square

Lemma 5.2. *With same notation and assumptions as in Lemma 5.1 we have $|X| = \lim_i |X_i|$ as topological spaces.*

Proof. We will use the criterion of Topology, Lemma 14.3. We have seen that $|X| = \lim_i |X_i|$ as sets in Lemma 5.1. The maps $f_i : X \rightarrow X_i$ are morphisms of algebraic spaces hence determine continuous maps $|X| \rightarrow |X_i|$. Thus $f_i^{-1}(U_i)$ is open for each open $U_i \subset |X_i|$. Finally, let $x \in |X|$ and let $x \in V \subset |X|$ be an open neighbourhood. We have to find an i and an open neighbourhood $W_i \subset |X_i|$ of the image x with $f_i^{-1}(W_i) \subset V$. Choose $0 \in I$. Choose a scheme U_0 and a surjective étale morphism $U_0 \rightarrow X_0$. Set $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$ for $i \geq 0$. Then $U = \lim_{i \geq 0} U_i$ in the category of schemes by Lemma 4.1. Choose $u \in U$ mapping to x . By the result for schemes (Limits, Lemma 4.2) we can find an $i \geq 0$ and an open neighbourhood $E_i \subset U_i$ of the image of u whose inverse image in U is contained in the inverse image of V in U . Then we can set $W_i \subset |X_i|$ equal to the image of E_i . This works because $|U_i| \rightarrow |X_i|$ is open. \square

Lemma 5.3. *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). If each X_i is quasi-compact and nonempty, then $|X|$ is nonempty.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Then each U_i is a nonempty affine scheme. Hence $U = \lim U_i$ is nonempty (Limits, Lemma 4.3) and thus X is nonempty. \square

Lemma 5.4. *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). Let $x \in |X|$ with images $x_i \in |X_i|$. If each X_i is decent, then $\overline{\{x\}} = \lim_i \overline{\{x_i\}}$ as sets and as algebraic spaces if endowed with reduced induced scheme structure.*

Proof. Set $Z = \overline{\{x\}} \subset |X|$ and $Z_i = \overline{\{x_i\}} \subset |X_i|$. Since $|X| \rightarrow |X_i|$ is continuous we see that Z maps into Z_i for each i . Hence we obtain an injective map $Z \rightarrow \lim Z_i$ because $|X| = \lim |X_i|$ as sets (Lemma 5.1). Suppose that $x' \in |X|$ is not in Z . Then there is an open subset $U \subset |X|$ with $x' \in U$ and $x \notin U$. Since $|X| = \lim |X_i|$ as topological spaces (Lemma 5.2) we can write $U = \bigcup_{j \in J} f_j^{-1}(U_j)$ for some subset $J \subset I$ and opens $U_j \subset |X_j|$, see Topology, Lemma 14.2. Then we see that for some

$j \in J$ we have $f_j(x') \in U_j$ and $f_j(x) \notin U_j$. In other words, we see that $f_j(x') \notin Z_j$. Thus $Z = \lim Z_i$ as sets.

Next, endow Z and Z_i with their reduced induced scheme structures, see Properties of Spaces, Definition 12.5. The transition morphisms $X_{i'} \rightarrow X_i$ induce affine morphisms $Z_{i'} \rightarrow Z_i$ and the projections $X \rightarrow X_i$ induce compatible morphisms $Z \rightarrow Z_i$. Hence we obtain morphisms $Z \rightarrow \lim Z_i \rightarrow X$ of algebraic spaces. By Lemma 4.3 we see that $\lim Z_i \rightarrow X$ is a closed immersion. By Lemma 4.4 the algebraic space $\lim Z_i$ is reduced. By the above $Z \rightarrow \lim Z_i$ is bijective on points. By uniqueness of the reduced induced closed subscheme structure we find that this morphism is an isomorphism of algebraic spaces. \square

Situation 5.5. Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be the limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

Lemma 5.6. *Notation and assumptions as in Situation 5.5. Suppose that \mathcal{F}_0 is a quasi-coherent sheaf on X_0 . Set $\mathcal{F}_i = f_{0i}^* \mathcal{F}_0$ for $i \geq 0$ and set $\mathcal{F} = f_0^* \mathcal{F}_0$. Then*

$$\Gamma(X, \mathcal{F}) = \operatorname{colim}_{i \geq 0} \Gamma(X_i, \mathcal{F}_i)$$

Proof. Choose a surjective étale morphism $U_0 \rightarrow X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. In the proof of Lemma 4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Hence Limits, Lemma 4.7 implies that

$$\mathcal{F}(U) = \operatorname{colim} \mathcal{F}_i(U_i) \quad \text{and} \quad \mathcal{F}(R) = \operatorname{colim} \mathcal{F}_i(R_i).$$

The lemma follows as $\Gamma(X, \mathcal{F}) = \operatorname{Ker}(\mathcal{F}(U) \rightarrow \mathcal{F}(R))$ and similarly $\Gamma(X_i, \mathcal{F}_i) = \operatorname{Ker}(\mathcal{F}_i(U_i) \rightarrow \mathcal{F}_i(R_i))$. \square

Lemma 5.7. *Notation and assumptions as in Situation 5.5. For any quasi-compact open subspace $U \subset X$ there exists an i and a quasi-compact open $U_i \subset X_i$ whose inverse image in X is U .*

Proof. Follows formally from the construction of limits in Lemma 4.1 and the corresponding result for schemes: Limits, Lemma 4.11. \square

The following lemma will be superseded by the stronger Lemma 6.10.

Lemma 5.8. *Notation and assumptions as in Situation 5.5. Let $f_0 : Y_0 \rightarrow Z_0$ be a morphism of algebraic spaces over X_0 . Assume (a) $Y_0 \rightarrow X_0$ and $Z_0 \rightarrow X_0$ are representable, (b) Y_0, Z_0 quasi-compact and quasi-separated, (c) f_0 locally of finite presentation, and (d) $Y_0 \times_{X_0} X \rightarrow Z_0 \times_{X_0} X$ an isomorphism. Then there exists an $i \geq 0$ such that $Y_0 \times_{X_0} X_i \rightarrow Z_0 \times_{X_0} X_i$ is an isomorphism.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow X_0$. Set $U_i = U_0 \times_{X_0} X_i$ and $U = U_0 \times_{X_0} X$. Apply Limits, Lemma 8.11 to see that $Y_0 \times_{X_0} U_i \rightarrow Z_0 \times_{X_0} U_i$ is an isomorphism of schemes for some $i \geq 0$ (details omitted). As $U_i \rightarrow X_i$ is surjective étale, it follows that $Y_0 \times_{X_0} X_i \rightarrow Z_0 \times_{X_0} X_i$ is an isomorphism (details omitted). \square

Lemma 5.9. *Notation and assumptions as in Situation 5.5. If X is separated, then X_i is separated for some $i \in I$.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow X_0$. For $i \geq 0$ set $U_i = U_0 \times_{X_0} X_i$ and set $U = U_0 \times_{X_0} X$. Note that U_i and U are affine schemes which come equipped with surjective étale morphisms $U_i \rightarrow X_i$ and $U \rightarrow X$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$ with projections $s_i, t_i : R_i \rightarrow U_i$ and $s, t : R \rightarrow U$. Note that R_i and R are quasi-compact separated schemes (as the algebraic spaces X_i and X are quasi-separated). The maps $s_i : R_i \rightarrow U_i$ and $s : R \rightarrow U$ are of finite type. By definition X_i is separated if and only if $(t_i, s_i) : R_i \rightarrow U_i \times U_i$ is a closed immersion, and since X is separated by assumption, the morphism $(t, s) : R \rightarrow U \times U$ is a closed immersion. Since $R \rightarrow U$ is of finite type, there exists an i such that the morphism $R \rightarrow U_i \times U$ is a closed immersion (Limits, Lemma 4.16). Fix such an $i \in I$. Apply Limits, Lemma 8.5 to the system of morphisms $R_{i'} \rightarrow U_i \times U_{i'}$ for $i' \geq i$ (this is permissible as indeed $R_{i'} = R_i \times_{U_i \times U_i} U_i \times U_{i'}$) to see that $R_{i'} \rightarrow U_i \times U_{i'}$ is a closed immersion for i' sufficiently large. This implies immediately that $R_{i'} \rightarrow U_{i'} \times U_{i'}$ is a closed immersion finishing the proof of the lemma. \square

Lemma 5.10. *Notation and assumptions as in Situation 5.5. If X is affine, then there exists an i such that X_i is affine.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow X_0$. Set $U = U_0 \times_{X_0} X$ and $U_i = U_0 \times_{X_0} X_i$ for $i \geq 0$. Since the transition morphisms are affine, the algebraic spaces U_i and U are affine. Thus $U \rightarrow X$ is an étale morphism of affine schemes. Hence we can write $X = \text{Spec}(A)$, $U = \text{Spec}(B)$ and

$$B = A[x_1, \dots, x_n]/(g_1, \dots, g_n)$$

such that $\Delta = \det(\partial g_\lambda / \partial x_\mu)$ is invertible in B , see Algebra, Lemma 143.2. Set $A_i = \mathcal{O}_{X_i}(X_i)$. We have $A = \text{colim } A_i$ by Lemma 5.6. After increasing 0 we may assume we have $g_{1,i}, \dots, g_{n,i} \in A_i[x_1, \dots, x_n]$ mapping to g_1, \dots, g_n . Set

$$B_i = A_i[x_1, \dots, x_n]/(g_{1,i}, \dots, g_{n,i})$$

for all $i \geq 0$. Increasing 0 if necessary we may assume that $\Delta_i = \det(\partial g_{\lambda,i} / \partial x_\mu)$ is invertible in B_i for all $i \geq 0$. Thus $A_i \rightarrow B_i$ is an étale ring map. After increasing 0 we may assume also that $\text{Spec}(B_i) \rightarrow \text{Spec}(A_i)$ is surjective, see Limits, Lemma 8.15. Increasing 0 yet again we may choose elements $h_{1,i}, \dots, h_{n,i} \in \mathcal{O}_{U_i}(U_i)$ which map to the classes of x_1, \dots, x_n in $B = \mathcal{O}_U(U)$ and such that $g_{\lambda,i}(h_{\nu,i}) = 0$ in $\mathcal{O}_{U_i}(U_i)$. Thus we obtain a commutative diagram

$$(5.10.1) \quad \begin{array}{ccc} X_i & \longleftarrow & U_i \\ \downarrow & & \downarrow \\ \text{Spec}(A_i) & \longleftarrow & \text{Spec}(B_i) \end{array}$$

By construction $B_i = B_0 \otimes_{A_0} A_i$ and $B = B_0 \otimes_{A_0} A$. Consider the morphism

$$f_0 : U_0 \longrightarrow X_0 \times_{\text{Spec}(A_0)} \text{Spec}(B_0)$$

This is a morphism of quasi-compact and quasi-separated algebraic spaces representable, separated and étale over X_0 . The base change of f_0 to X is an isomorphism by our choices. Hence Lemma 5.8 guarantees that there exists an i

such that the base change of f_0 to X_i is an isomorphism, in other words the diagram (5.10.1) is cartesian. Thus Descent, Lemma 39.1 applied to the fppf covering $\{\mathrm{Spec}(B_i) \rightarrow \mathrm{Spec}(A_i)\}$ combined with Descent, Lemma 37.1 give that $X_i \rightarrow \mathrm{Spec}(A_i)$ is representable by a scheme affine over $\mathrm{Spec}(A_i)$ as desired. (Of course it then also follows that $X_i = \mathrm{Spec}(A_i)$ but we don't need this.) \square

Lemma 5.11. *Notation and assumptions as in Situation 5.5. If X is a scheme, then there exists an i such that X_i is a scheme.*

Proof. Choose a finite affine open covering $X = \bigcup W_j$. By Lemma 5.7 we can find an $i \in I$ and open subspaces $W_{j,i} \subset X_i$ whose base change to X is $W_j \rightarrow X$. By Lemma 5.10 we may assume that each $W_{j,i}$ is an affine scheme. This means that X_i is a scheme (see for example Properties of Spaces, Section 13). \square

Lemma 5.12. *Let S be a scheme. Let B be an algebraic space over S . Let $X = \lim X_i$ be a directed limit of algebraic spaces over B with affine transition morphisms. Let $Y \rightarrow X$ be a morphism of algebraic spaces over B .*

- (1) *If $Y \rightarrow X$ is a closed immersion, X_i quasi-compact, and $Y \rightarrow B$ locally of finite type, then $Y \rightarrow X_i$ is a closed immersion for i large enough.*
- (2) *If $Y \rightarrow X$ is an immersion, X_i quasi-separated, $Y \rightarrow B$ locally of finite type, and Y quasi-compact, then $Y \rightarrow X_i$ is an immersion for i large enough.*
- (3) *If $Y \rightarrow X$ is an isomorphism, X_i quasi-compact, $X_i \rightarrow B$ locally of finite type, the transition morphisms $X_{i'} \rightarrow X_i$ are closed immersions, and $Y \rightarrow B$ is locally of finite presentation, then $Y \rightarrow X_i$ is an isomorphism for i large enough.*
- (4) *If $Y \rightarrow X$ is a monomorphism, X_i quasi-separated, $Y \rightarrow B$ locally of finite type, and Y quasi-compact, then $Y \rightarrow X_i$ is a monomorphism for i large enough.*

Proof. Proof of (1). Choose $0 \in I$. As X_0 is quasi-compact, we can choose an affine scheme W and an étale morphism $W \rightarrow B$ such that the image of $|X_0| \rightarrow |B|$ is contained in $|W| \rightarrow |B|$. Choose an affine scheme U_0 and an étale morphism $U_0 \rightarrow X_0 \times_B W$ such that $U_0 \rightarrow X_0$ is surjective. (This is possible by our choice of W and the fact that X_0 is quasi-compact; details omitted.) Let $V \rightarrow Y$, resp. $U \rightarrow X$, resp. $U_i \rightarrow X_i$ be the base change of $U_0 \rightarrow X_0$ (for $i \geq 0$). It suffices to prove that $V \rightarrow U_i$ is a closed immersion for i sufficiently large. Thus we reduce to proving the result for $V \rightarrow U = \lim U_i$ over W . This follows from the case of schemes, which is Limits, Lemma 4.16.

Proof of (2). Choose $0 \in I$. Choose a quasi-compact open subspace $X'_0 \subset X_0$ such that $Y \rightarrow X_0$ factors through X'_0 . After replacing X_i by the inverse image of X'_0 for $i \geq 0$ we may assume all X'_i are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \rightarrow X$ factors through a closed immersion $Y \rightarrow U$ (U exists as Y is quasi-compact). By Lemma 5.7 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \rightarrow U_i$ is a closed immersion for some i . Thus (2) holds.

Proof of (3). Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow X_0$. Set $U_i = X_i \times_{X_0} U_0$, $U = X \times_{X_0} U_0 = Y \times_{X_0} U_0$. Then $U = \lim U_i$ is a limit of affine schemes, the transition maps of the system are closed immersions, and $U \rightarrow U_0$ is of finite presentation (because $U \rightarrow B$ is locally of finite presentation and $U_0 \rightarrow B$ is locally of finite type and Morphisms of Spaces, Lemma

28.9). Thus we've reduced to the following algebra fact: If $A = \lim A_i$ is a directed colimit of R -algebras with surjective transition maps and A of finite presentation over A_0 , then $A = A_i$ for some i . Namely, write $A = A_0/(f_1, \dots, f_n)$. Pick i such that f_1, \dots, f_n map to zero under the surjective map $A_0 \rightarrow A_i$.

Proof of (4). Set $Z_i = Y \times_{X_i} Y$. As the transition morphisms $X_{i'} \rightarrow X_i$ are affine hence separated, the transition morphisms $Z_{i'} \rightarrow Z_i$ are closed immersions, see Morphisms of Spaces, Lemma 4.5. We have $\lim Z_i = Y \times_X Y = Y$ as $Y \rightarrow X$ is a monomorphism. Choose $0 \in I$. Since $Y \rightarrow X_0$ is locally of finite type (Morphisms of Spaces, Lemma 23.6) the morphism $Y \rightarrow Z_0$ is locally of finite presentation (Morphisms of Spaces, Lemma 28.10). The morphisms $Z_i \rightarrow Z_0$ are locally of finite type (they are closed immersions). Finally, $Z_i = Y \times_{X_i} Y$ is quasi-compact as X_i is quasi-separated and Y is quasi-compact. Thus part (3) applies to $Y = \lim_{i \geq 0} Z_i$ over Z_0 and we conclude $Y = Z_i$ for some i . This proves (4) and the lemma. \square

Lemma 5.13. *Let S be a scheme. Let Y be an algebraic space over S . Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

- (1) Y is quasi-separated,
- (2) X_i is quasi-compact and quasi-separated,
- (3) the morphism $X \rightarrow Y$ is separated.

Then $X_i \rightarrow Y$ is separated for all i large enough.

Proof. Let $0 \in I$. Choose an affine scheme W and an étale morphism $W \rightarrow Y$ such that the image of $|W| \rightarrow |Y|$ contains the image of $|X_0| \rightarrow |Y|$. This is possible as X_0 is quasi-compact. It suffices to check that $W \times_Y X_i \rightarrow W$ is separated for some $i \geq 0$ because the diagonal of $W \times_Y X_i$ over W is the base change of $X_i \rightarrow X_i \times_Y X_i$ by the surjective étale morphism $(X_i \times_Y X_i) \times_Y W \rightarrow X_i \times_Y X_i$. Since Y is quasi-separated the algebraic spaces $W \times_Y X_i$ are quasi-compact (as well as quasi-separated). Thus we may base change to W and assume Y is an affine scheme. When Y is an affine scheme, we have to show that X_i is a separated algebraic space for i large enough and we are given that X is a separated algebraic space. Thus this case follows from Lemma 5.9. \square

Lemma 5.14. *Let S be a scheme. Let Y be an algebraic space over S . Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

- (1) Y quasi-compact and quasi-separated,
- (2) X_i quasi-compact and quasi-separated,
- (3) $X \rightarrow Y$ affine.

Then $X_i \rightarrow Y$ is affine for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \rightarrow Y$. Then $X \times_Y W$ is affine and it suffices to check that $X_i \times_Y W$ is affine for some i (Morphisms of Spaces, Lemma 20.3). This follows from Lemma 5.10. \square

Lemma 5.15. *Let S be a scheme. Let Y be an algebraic space over S . Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

- (1) Y quasi-compact and quasi-separated,
- (2) X_i quasi-compact and quasi-separated,

- (3) the transition morphisms $X_{i'} \rightarrow X_i$ are finite,
- (4) $X_i \rightarrow Y$ locally of finite type
- (5) $X \rightarrow Y$ integral.

Then $X_i \rightarrow Y$ is finite for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \rightarrow Y$. Then $X \times_Y W$ is finite over W and it suffices to check that $X_i \times_Y W$ is finite over W for some i (Morphisms of Spaces, Lemma 45.3). By Lemma 5.11 this reduces us to the case of schemes. In the case of schemes it follows from Limits, Lemma 4.19. \square

Lemma 5.16. *Let S be a scheme. Let Y be an algebraic space over S . Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

- (1) Y quasi-compact and quasi-separated,
- (2) X_i quasi-compact and quasi-separated,
- (3) the transition morphisms $X_{i'} \rightarrow X_i$ are closed immersions,
- (4) $X_i \rightarrow Y$ locally of finite type
- (5) $X \rightarrow Y$ is a closed immersion.

Then $X_i \rightarrow Y$ is a closed immersion for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \rightarrow Y$. Then $X \times_Y W$ is a closed subspace of W and it suffices to check that $X_i \times_Y W$ is a closed subspace W for some i (Morphisms of Spaces, Lemma 12.1). By Lemma 5.11 this reduces us to the case of schemes. In the case of schemes it follows from Limits, Lemma 4.20. \square

6. Descending properties of morphisms

This section is the analogue of Section 5 for properties of morphisms. We will work in the following situation.

Situation 6.1. Let S be a scheme. Let $B = \lim B_i$ be a limit of a directed inverse system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). Let $0 \in I$ and let $f_0 : X_0 \rightarrow Y_0$ be a morphism of algebraic spaces over B_0 . Assume B_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \rightarrow Y_i$ be the base change of f_0 to B_i and let $f : X \rightarrow Y$ be the base change of f_0 to B .

Lemma 6.2. *With notation and assumptions as in Situation 6.1. If*

- (1) f is étale,
- (2) f_0 is locally of finite presentation,

then f_i is étale for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \rightarrow Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow V_0 \times_{Y_0} X_0$. Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \rightarrow V_i$ is $U \rightarrow V$. Recall that $X_i \rightarrow Y_i$ is étale if and only if $U_i \rightarrow V_i$ is étale and similarly $X \rightarrow Y$ is étale if and only if $U \rightarrow V$ is étale (Morphisms of Spaces, Lemma 39.2). Since f_0 is locally of finite presentation, so is the morphism $U_0 \rightarrow V_0$. Hence the lemma follows from Limits, Lemma 8.10. \square

Lemma 6.3. *With notation and assumptions as in Situation 6.1. If*

- (1) *f is smooth,*
- (2) *f_0 is locally of finite presentation,*

then f_i is smooth for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \rightarrow Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow V_0 \times_{Y_0} X_0$. Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \rightarrow V_i$ is $U \rightarrow V$. Recall that $X_i \rightarrow Y_i$ is smooth if and only if $U_i \rightarrow V_i$ is smooth and similarly $X \rightarrow Y$ is smooth if and only if $U \rightarrow V$ is smooth (Morphisms of Spaces, Definition 37.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \rightarrow V_0$. Hence the lemma follows from Limits, Lemma 8.9. \square

Lemma 6.4. *With notation and assumptions as in Situation 6.1. If*

- (1) *f is surjective,*
- (2) *f_0 is locally of finite presentation,*

then f_i is surjective for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \rightarrow Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow V_0 \times_{Y_0} X_0$.

Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, the limit of the morphisms $U_i \rightarrow V_i$ is $U \rightarrow V$, and the morphisms $U_i \rightarrow X_i \times_{Y_i} V_i$ and $U \rightarrow X \times_Y V$ are surjective (as base changes of $U_0 \rightarrow X_0 \times_{Y_0} V_0$). In particular, we see that $X_i \rightarrow Y_i$ is surjective if and only if $U_i \rightarrow V_i$ is surjective and similarly $X \rightarrow Y$ is surjective if and only if $U \rightarrow V$ is surjective. Since f_0 is locally of finite presentation, so is the morphism $U_0 \rightarrow V_0$. Hence the lemma follows from the case of schemes (Limits, Lemma 8.15). \square

Lemma 6.5. *Notation and assumptions as in Situation 6.1. If*

- (1) *f is universally injective,*
- (2) *f_0 is locally of finite type,*

then f_i is universally injective for some $i \geq 0$.

Proof. Recall that a morphism $X \rightarrow Y$ is universally injective if and only if the diagonal $X \rightarrow X \times_Y X$ is surjective (Morphisms of Spaces, Definition 19.3 and Lemma 19.2). Observe that $X_0 \rightarrow X_0 \times_{Y_0} X_0$ is of locally of finite presentation (Morphisms of Spaces, Lemma 28.10). Hence the lemma follows from Lemma 6.4 by considering the morphism $X_0 \rightarrow X_0 \times_{Y_0} X_0$. \square

Lemma 6.6. *Notation and assumptions as in Situation 6.1. If f is affine, then f_i is affine for some $i \geq 0$.*

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \rightarrow Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is affine we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is affine. By Lemma 5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. For this i the morphism f_i is affine (Morphisms of Spaces, Lemma 20.3). \square

Lemma 6.7. *Notation and assumptions as in Situation 6.1. If*

- (1) *f is finite,*
- (2) *f_0 is locally of finite type,*

then f_i is finite for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \rightarrow Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is finite we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a scheme finite over V . By Lemma 5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \rightarrow V_i$ is finite by Limits, Lemma 8.3. For this i the morphism f_i is finite (Morphisms of Spaces, Lemma 45.3). \square

Lemma 6.8. *Notation and assumptions as in Situation 6.1. If*

- (1) *f is a closed immersion,*
- (2) *f_0 is locally of finite type,*

then f_i is a closed immersion for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \rightarrow Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is a closed immersion we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a closed subscheme of the affine scheme V . By Lemma 5.10 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \rightarrow V_i$ is a closed immersion by Limits, Lemma 8.5. For this i the morphism f_i is a closed immersion (Morphisms of Spaces, Lemma 45.3). \square

Lemma 6.9. *Notation and assumptions as in Situation 6.1. If f is separated, then f_i is separated for some $i \geq 0$.*

Proof. Apply Lemma 6.8 to the diagonal morphism $\Delta_{X_0/Y_0} : X_0 \rightarrow X_0 \times_{Y_0} X_0$. (Diagonal morphisms are locally of finite type and the fibre product $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated. Some details omitted.) \square

Lemma 6.10. *Notation and assumptions as in Situation 6.1. If*

- (1) *f is an isomorphism,*
- (2) *f_0 is locally of finite presentation,*

then f_i is an isomorphism for some $i \geq 0$.

Proof. Being an isomorphism is equivalent to being étale, universally injective, and surjective, see Morphisms of Spaces, Lemma 51.2. Thus the lemma follows from Lemmas 6.2, 6.4, and 6.5. \square

Lemma 6.11. *Notation and assumptions as in Situation 6.1. If*

- (1) *f is a monomorphism,*
- (2) *f_0 is locally of finite type,*

then f_i is a monomorphism for some $i \geq 0$.

Proof. Recall that a morphism is a monomorphism if and only if the diagonal is an isomorphism. The morphism $X_0 \rightarrow X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms of Spaces, Lemma 28.10. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated we conclude from Lemma 6.10 that $\Delta_i : X_i \rightarrow X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this i the morphism f_i is a monomorphism. \square

Lemma 6.12. *Notation and assumptions as in Situation 6.1. Let \mathcal{F}_0 be a quasi-coherent \mathcal{O}_{X_0} -module and denote \mathcal{F}_i the pullback to X_i and \mathcal{F} the pullback to X . If*

- (1) *\mathcal{F} is flat over Y ,*
- (2) *\mathcal{F}_0 is of finite presentation, and*
- (3) *f_0 is locally of finite presentation,*

then \mathcal{F}_i is flat over Y_i for some $i \geq 0$. In particular, if f_0 is locally of finite presentation and f is flat, then f_i is flat for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \rightarrow Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow V_0 \times_{Y_0} X_0$.

Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \rightarrow V_i$ is $U \rightarrow V$. Recall that \mathcal{F}_i is flat over Y_i if and only if $\mathcal{F}_i|_{U_i}$ is flat over V_i and similarly \mathcal{F} is flat over Y if and only if $\mathcal{F}|_U$ is flat over V (Morphisms of Spaces, Definition 30.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \rightarrow V_0$. Hence the lemma follows from Limits, Lemma 10.4. \square

Lemma 6.13. *Assumptions and notation as in Situation 6.1. If*

- (1) *f is proper, and*
- (2) *f_0 is locally of finite type,*

then there exists an i such that f_i is proper.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \rightarrow Y_0$. Set $V_i = Y_i \times_{Y_0} V_0$ and $V = Y \times_{Y_0} V_0$. It suffices to prove that the base change of f_i to V_i is proper, see Morphisms of Spaces, Lemma 40.2. Thus we may assume Y_0 is affine.

By Lemma 6.9 we see that f_i is separated for some $i \geq 0$. Replacing 0 by i we may assume that f_0 is separated. Observe that f_0 is quasi-compact. Thus f_0 is separated and of finite type. By Cohomology of Spaces, Lemma 18.1 we can choose a diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{\pi} & X'_0 \longrightarrow \mathbf{P}_{Y_0}^n \\ & \searrow & \downarrow \swarrow \\ & & Y_0 \end{array}$$

where $X'_0 \rightarrow \mathbf{P}_{Y_0}^n$ is an immersion, and $\pi : X'_0 \rightarrow X_0$ is proper and surjective. Introduce $X' = X'_0 \times_{Y_0} Y$ and $X'_i = X'_0 \times_{Y_0} Y_i$. By Morphisms of Spaces, Lemmas 40.4 and 40.3 we see that $X' \rightarrow Y$ is proper. Hence $X' \rightarrow \mathbf{P}_Y^n$ is a closed immersion (Morphisms of Spaces, Lemma 40.6). By Morphisms of Spaces, Lemma 40.7 it suffices to prove that $X'_i \rightarrow Y_i$ is proper for some i . By Lemma 6.8 we find that $X'_i \rightarrow \mathbf{P}_{Y_i}^n$ is a closed immersion for i large enough. Then $X'_i \rightarrow Y_i$ is proper and we win. \square

Lemma 6.14. *Assumptions and notation as in Situation 6.1. Let $d \geq 0$. If*

- (1) *f has relative dimension $\leq d$ (Morphisms of Spaces, Definition 33.2), and*
- (2) *f_0 is locally of finite type,*

then there exists an i such that f_i has relative dimension $\leq d$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \rightarrow Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow V_0 \times_{Y_0} X_0$. Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \rightarrow V_i$ is $U \rightarrow V$. In this situation $X_i \rightarrow Y_i$ has relative dimension $\leq d$ if and only if $U_i \rightarrow V_i$ has relative dimension $\leq d$ (as defined in Morphisms, Definition 29.1). To see the equivalence, use that the definition for morphisms of algebraic spaces involves Morphisms of Spaces, Definition 33.1 which uses étale localization. The same is true for $X \rightarrow Y$ and $U \rightarrow V$. Since f_0 is locally of finite type, so is the morphism $U_0 \rightarrow V_0$. Hence the lemma follows from the more general Limits, Lemma 18.1. \square

7. Descending relative objects

The following lemma is typical of the type of results in this section.

Lemma 7.1. *Let S be a scheme. Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S . Assume*

- (1) *the morphisms $f_{ii'} : X_i \rightarrow X_{i'}$ are affine,*
- (2) *the spaces X_i are quasi-compact and quasi-separated.*

Let $X = \lim_i X_i$. Then the category of algebraic spaces of finite presentation over X is the colimit over I of the categories of algebraic spaces of finite presentation over X_i .

Proof. Pick $0 \in I$. Choose a surjective étale morphism $U_0 \rightarrow X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \rightarrow U_i$ and $s, t : R \rightarrow U$ the two projections. In the proof of Lemma 4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Let Y be an algebraic space over S and let $Y \rightarrow X$ be a morphism of finite presentation. Set $V = U \times_X Y$. This is an algebraic space of finite presentation over U . Choose an affine scheme W and a surjective étale morphism $W \rightarrow V$. Then $W \rightarrow Y$ is surjective étale as well. Set $R' = W \times_Y W$ so that $Y = W/R'$ (see Spaces, Section 9). Note that W is a scheme of finite presentation over U and that R' is a scheme of finite presentation over R (details omitted). By Limits, Lemma 10.1 we can find an index i and a morphism of schemes $W_i \rightarrow U_i$ of finite presentation whose base change to U gives $W \rightarrow U$. Similarly we can find,

after possibly increasing i , a scheme R'_i of finite presentation over R_i whose base change to R is R' . The projection morphisms $s', t' : R' \rightarrow W$ are morphisms over the projection morphisms $s, t : R \rightarrow U$. Hence we can view s' , resp. t' as a morphism between schemes of finite presentation over U (with structure morphism $R' \rightarrow U$ given by $R' \rightarrow R$ followed by s , resp. t). Hence we can apply Limits, Lemma 10.1 again to see that, after possibly increasing i , there exist morphisms $s'_i, t'_i : R'_i \rightarrow W_i$, whose base change to U is S', t' . By Limits, Lemmas 8.10 and 8.14 we may assume that s'_i, t'_i are étale and that $j'_i : R'_i \rightarrow W_i \times_{X_i} W_i$ is a monomorphism (here we view j'_i as a morphism of schemes of finite presentation over U_i via one of the projections – it doesn't matter which one). Setting $Y_i = W_i/R'_i$ (see Spaces, Theorem 10.5) we obtain an algebraic space of finite presentation over X_i whose base change to X is isomorphic to Y .

This shows that every algebraic space of finite presentation over X comes from an algebraic space of finite presentation over some X_i , i.e., it shows that the functor of the lemma is essentially surjective. To show that it is fully faithful, consider an index $0 \in I$ and two algebraic spaces Y_0, Z_0 of finite presentation over X_0 . Set $Y_i = X_i \times_{X_0} Y_0$, $Y = X \times_{X_0} Y_0$, $Z_i = X_i \times_{X_0} Z_0$, and $Z = X \times_{X_0} Z_0$. Let $\alpha : Y \rightarrow Z$ be a morphism of algebraic spaces over X . Choose a surjective étale morphism $V_0 \rightarrow Y_0$ where V_0 is an affine scheme. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$ which are affine schemes endowed with surjective étale morphisms to Y_i and Y . The composition $V \rightarrow Y \rightarrow Z \rightarrow Z_0$ comes from a (essentially unique) morphism $V_i \rightarrow Z_0$ for some $i \geq 0$ by Proposition 3.10 (applied to $Z_0 \rightarrow X_0$ which is of finite presentation by assumption). After increasing i the two compositions

$$V_i \times_{Y_i} V_i \rightarrow V_i \rightarrow Z_0$$

are equal as this is true in the limit. Hence we obtain a (essentially unique) morphism $Y_i \rightarrow Z_0$. Since this is a morphism over X_0 it induces a morphism into $Z_i = Z_0 \times_{X_0} X_i$ as desired. \square

Lemma 7.2. *With notation and assumptions as in Lemma 7.1. The category of \mathcal{O}_X -modules of finite presentation is the colimit over I of the categories \mathcal{O}_{X_i} -modules of finite presentation.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \rightarrow X_0$. Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \rightarrow U_i$ and $s, t : R \rightarrow U$ the two projections. In the proof of Lemma 4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Moreover, it is also true that $R \times_{s, U, t} R = \text{colim } R_i \times_{s_i, U_i, t_i} R_i$. Thus we know that $QCoh(\mathcal{O}_U) = \text{colim } QCoh(\mathcal{O}_{U_i})$, $QCoh(\mathcal{O}_R) = \text{colim } QCoh(\mathcal{O}_{R_i})$, and $QCoh(\mathcal{O}_{R \times_{s, U, t} R}) = \text{colim } QCoh(\mathcal{O}_{R_i \times_{s_i, U_i, t_i} R_i})$ by Limits, Lemma 10.2. We have $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$ and $QCoh(\mathcal{O}_{X_i}) = QCoh(U_i, R_i, s_i, t_i, c_i)$, see Properties of Spaces, Proposition 32.1. Thus the result follows formally. \square

Lemma 7.3. *With notation and assumptions as in Lemma 7.1. Then*

- (1) *any finite locally free \mathcal{O}_X -module is the pullback of a finite locally free \mathcal{O}_{X_i} -module for some i ,*
- (2) *any invertible \mathcal{O}_X -module is the pullback of an invertible \mathcal{O}_{X_i} -module for some i .*

Proof. Proof of (2). Let \mathcal{L} be an invertible \mathcal{O}_X -module. Since invertible modules are of finite presentation we can find an i and modules \mathcal{L}_i and \mathcal{N}_i of finite presentation over X_i such that $f_i^* \mathcal{L}_i \cong \mathcal{L}$ and $f_i^* \mathcal{N}_i \cong \mathcal{L}^{\otimes -1}$, see Lemma 7.2. Since pullback commutes with tensor product we see that $f_i^*(\mathcal{L}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{N}_i)$ is isomorphic to \mathcal{O}_X . Since the tensor product of finitely presented modules is finitely presented, the same lemma implies that $f_{i'}^* \mathcal{L}_i \otimes_{\mathcal{O}_{X_{i'}}} f_{i'}^* \mathcal{N}_i$ is isomorphic to $\mathcal{O}_{X_{i'}}$ for some $i' \geq i$. It follows that $f_{i'}^* \mathcal{L}_i$ is invertible (Modules on Sites, Lemma 32.2) and the proof is complete.

Proof of (1). Omitted. Hint: argue as in the proof of (2) using that a module (on a locally ringed site) is finite locally free if and only if it has a dual, see Modules on Sites, Section 29. Alternatively, argue as in the proof for schemes, see Limits, Lemma 10.3. \square

8. Absolute Noetherian approximation

The following result is [CLO12, Theorem 1.2.2]. A key ingredient in the proof is Decent Spaces, Lemma 8.6.

Proposition 8.1. *Let X be a quasi-compact and quasi-separated algebraic space over $\mathrm{Spec}(\mathbf{Z})$. There exist a directed set I and an inverse system of algebraic spaces $(X_i, f_{ii'})$ over I such that*

- (1) *the transition morphisms $f_{ii'}$ are affine*
- (2) *each X_i is quasi-separated and of finite type over \mathbf{Z} , and*
- (3) *$X = \lim X_i$.*

Proof. We apply Decent Spaces, Lemma 8.6 to get open subspaces $U_p \subset X$, schemes V_p , and morphisms $f_p : V_p \rightarrow U_p$ with properties as stated. Note that $f_n : V_n \rightarrow U_n$ is an étale morphism of algebraic spaces whose restriction to the inverse image of $T_n = (V_n)_{red}$ is an isomorphism. Hence f_n is an isomorphism, for example by Morphisms of Spaces, Lemma 51.2. In particular U_n is a quasi-compact and separated scheme. Thus we can write $U_n = \lim U_{n,i}$ as a directed limit of schemes of finite type over \mathbf{Z} with affine transition morphisms, see Limits, Proposition 5.4. Thus, applying descending induction on p , we see that we have reduced to the problem posed in the following paragraph.

Here we have $U \subset X$, $U = \lim U_i$, $Z \subset X$, and $f : V \rightarrow X$ with the following properties

- (1) X is a quasi-compact and quasi-separated algebraic space,
- (2) V is a quasi-compact and separated scheme,
- (3) $U \subset X$ is a quasi-compact open subspace,
- (4) $(U_i, g_{ii'})$ is a directed inverse system of quasi-separated algebraic spaces of finite type over \mathbf{Z} with affine transition morphisms whose limit is U ,
- (5) $Z \subset X$ is a closed subspace such that $|X| = |U| \amalg |Z|$,
- (6) $f : V \rightarrow X$ is a surjective étale morphism such that $f^{-1}(Z) \rightarrow Z$ is an isomorphism.

Problem: Show that the conclusion of the proposition holds for X .

Note that $W = f^{-1}(U) \subset V$ is a quasi-compact open subscheme étale over U . Hence we may apply Lemmas 7.1 and 6.2 to find an index $0 \in I$ and an étale morphism $W_0 \rightarrow U_0$ of finite presentation whose base change to U produces W . Setting $W_i = W_0 \times_{U_0} U_i$ we see that $W = \lim_{i \geq 0} W_i$. After increasing 0 we may

assume the W_i are schemes, see Lemma 5.11. Moreover, W_i is of finite type over \mathbf{Z} .

Apply Limits, Lemma 5.3 to $W = \lim_{i \geq 0} W_i$ and the inclusion $W \subset V$. Replace I by the directed set J found in that lemma. This allows us to write V as a directed limit $V = \lim V_i$ of finite type schemes over \mathbf{Z} with affine transition maps such that each V_i contains W_i as an open subscheme (compatible with transition morphisms). For each i we can form the push out

$$\begin{array}{ccc} W_i & \longrightarrow & V_i \\ \Delta \downarrow & & \downarrow \\ W_i \times_{U_i} W_i & \longrightarrow & R_i \end{array}$$

in the category of schemes. Namely, the left vertical and upper horizontal arrows are open immersions of schemes. In other words, we can construct R_i as the glueing of V_i and $W_i \times_{U_i} W_i$ along the common open W_i (see Schemes, Section 14). Note that the étale projection maps $W_i \times_{U_i} W_i \rightarrow W_i$ extend to étale morphisms $s_i, t_i : R_i \rightarrow V_i$. It is clear that the morphism $j_i = (t_i, s_i) : R_i \rightarrow V_i \times V_i$ is an étale equivalence relation on V_i . Note that $W_i \times_{U_i} W_i$ is quasi-compact (as U_i is quasi-separated and W_i quasi-compact) and V_i is quasi-compact, hence R_i is quasi-compact. For $i \geq i'$ the diagram

$$(8.1.1) \quad \begin{array}{ccc} R_i & \longrightarrow & R_{i'} \\ s_i \downarrow & & \downarrow s_{i'} \\ V_i & \longrightarrow & V_{i'} \end{array}$$

is cartesian because

$$(W_{i'} \times_{U_{i'}} W_{i'}) \times_{U_{i'}} U_i = W_{i'} \times_{U_{i'}} U_i \times_{U_i} U_i \times_{U_{i'}} W_{i'} = W_i \times_{U_i} W_i.$$

Consider the algebraic space $X_i = V_i/R_i$ (see Spaces, Theorem 10.5). As V_i is of finite type over \mathbf{Z} and R_i is quasi-compact we see that X_i is quasi-separated and of finite type over \mathbf{Z} (see Properties of Spaces, Lemma 6.5 and Morphisms of Spaces, Lemmas 8.6 and 23.4). As the construction of R_i above is compatible with transition morphisms, we obtain morphisms of algebraic spaces $X_i \rightarrow X_{i'}$ for $i \geq i'$. The commutative diagrams

$$\begin{array}{ccc} V_i & \longrightarrow & V_{i'} \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X_{i'} \end{array}$$

are cartesian as (8.1.1) is cartesian, see Groupoids, Lemma 20.7. Since $V_i \rightarrow V_{i'}$ is affine, this implies that $X_i \rightarrow X_{i'}$ is affine, see Morphisms of Spaces, Lemma 20.3. Thus we can form the limit $X' = \lim X_i$ by Lemma 4.1. We claim that $X \cong X'$ which finishes the proof of the proposition.

Proof of the claim. Set $R = \lim R_i$. By construction the algebraic space X' comes equipped with a surjective étale morphism $V \rightarrow X'$ such that

$$V \times_{X'} V \cong R$$

(use Lemma 4.1). By construction $\lim W_i \times_{U_i} W_i = W \times_U W$ and $V = \lim V_i$ so that R is the union of $W \times_U W$ and V glued along W . Property (6) implies the

projections $V \times_X V \rightarrow V$ are isomorphisms over $f^{-1}(Z) \subset V$. Hence the scheme $V \times_X V$ is the union of the opens $\Delta_{V/X}(V)$ and $W \times_U W$ which intersect along $\Delta_{W/X}(W)$. We conclude that there exists a unique isomorphism $R \cong V \times_X V$ compatible with the projections to V . Since $V \rightarrow X$ and $V \rightarrow X'$ are surjective étale we see that

$$X = V/V \times_X V = V/R = V/V \times_{X'} V = X'$$

by Spaces, Lemma 9.1 and we win. \square

9. Applications

The following lemma can also be deduced directly from Decent Spaces, Lemma 8.6 without passing through absolute Noetherian approximation.

Lemma 9.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Every quasi-coherent \mathcal{O}_X -module is a filtered colimit of finitely presented \mathcal{O}_X -modules.*

Proof. We may view X as an algebraic space over $\text{Spec}(\mathbf{Z})$, see Spaces, Definition 16.2 and Properties of Spaces, Definition 3.1. Thus we may apply Proposition 8.1 and write $X = \lim X_i$ with X_i of finite presentation over \mathbf{Z} . Thus X_i is a Noetherian algebraic space, see Morphisms of Spaces, Lemma 28.6. The morphism $X \rightarrow X_i$ is affine, see Lemma 4.1. Conclusion by Cohomology of Spaces, Lemma 15.2. \square

The rest of this section consists of straightforward applications of Lemma 9.1.

Lemma 9.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then \mathcal{F} is the directed colimit of its finite type quasi-coherent submodules.*

Proof. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are finite type quasi-coherent \mathcal{O}_X -submodules then the image of $\mathcal{G} \oplus \mathcal{H} \rightarrow \mathcal{F}$ is another finite type quasi-coherent \mathcal{O}_X -submodule which contains both of them. In this way we see that the system is directed. To show that \mathcal{F} is the colimit of this system, write $\mathcal{F} = \text{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 9.1. Then the images $\mathcal{G}_i = \text{Im}(\mathcal{F}_i \rightarrow \mathcal{F})$ are finite type quasi-coherent subsheaves of \mathcal{F} . Since \mathcal{F} is the colimit of these the result follows. \square

Lemma 9.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Then we can write $\mathcal{F} = \lim \mathcal{F}_i$ where each \mathcal{F}_i is an \mathcal{O}_X -module of finite presentation and all transition maps $\mathcal{F}_i \rightarrow \mathcal{F}_{i'}$ are surjective.*

Proof. Write $\mathcal{F} = \text{colim} \mathcal{G}_i$ as a filtered colimit of finitely presented \mathcal{O}_X -modules (Lemma 9.1). We claim that $\mathcal{G}_i \rightarrow \mathcal{F}$ is surjective for some i . Namely, choose an étale surjection $U \rightarrow X$ where U is an affine scheme. Choose finitely many sections $s_k \in \mathcal{F}(U)$ generating $\mathcal{F}|_U$. Since U is affine we see that s_k is in the image of $\mathcal{G}_i \rightarrow \mathcal{F}$ for i large enough. Hence $\mathcal{G}_i \rightarrow \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{G}_i$ be the kernel of the map $\mathcal{G}_i \rightarrow \mathcal{F}$. Write $\mathcal{K} = \text{colim} \mathcal{K}_a$ as the filtered colimit of its finite type quasi-coherent submodules (Lemma 9.2). Then $\mathcal{F} = \text{colim} \mathcal{G}_i / \mathcal{K}_a$ is a solution to the problem posed by the lemma. \square

Let X be an algebraic space. In the following lemma we use the notion of a *finitely presented quasi-coherent \mathcal{O}_X -algebra* \mathcal{A} . This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \tilde{A}$ where A is a (commutative) R -algebra which is of finite presentation as an R -algebra.

Lemma 9.4. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let \mathcal{A} be a quasi-coherent \mathcal{O}_X -algebra. Then \mathcal{A} is a directed colimit of finitely presented quasi-coherent \mathcal{O}_X -algebras.*

Proof. First we write $\mathcal{A} = \mathrm{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 9.1. For each i let $\mathcal{B}_i = \mathrm{Sym}(\mathcal{F}_i)$ be the symmetric algebra on \mathcal{F}_i over \mathcal{O}_X . Write $\mathcal{I}_i = \mathrm{Ker}(\mathcal{B}_i \rightarrow \mathcal{A})$. Write $\mathcal{I}_i = \mathrm{colim}_j \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of \mathcal{I}_i , see Lemma 9.2. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the \mathcal{B}_i -ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i / \mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented \mathcal{O}_X -algebra. Define $(i, j) \leq (i', j')$ if $i \leq i'$ and the map $\mathcal{B}_i \rightarrow \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = \mathrm{colim}_{i,j} \mathcal{A}_{i,j}$. \square

Let X be an algebraic space. In the following lemma we use the notion of a *quasi-coherent \mathcal{O}_X -algebra of finite type*. This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \tilde{A}$ where A is a (commutative) R -algebra which is of finite type as an R -algebra.

Lemma 9.5. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let \mathcal{A} be a quasi-coherent \mathcal{O}_X -algebra. Then \mathcal{A} is the directed colimit of its finite type quasi-coherent \mathcal{O}_X -subalgebras.*

Proof. Omitted. Hint: Compare with the proof of Lemma 9.2. \square

Let X be an algebraic space. In the following lemma we use the notion of a *finite (resp. integral) quasi-coherent \mathcal{O}_X -algebra* \mathcal{A} . This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \tilde{A}$ where A is a (commutative) R -algebra which is finite (resp. integral) as an R -algebra.

Lemma 9.6. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let \mathcal{A} be a finite quasi-coherent \mathcal{O}_X -algebra. Then $\mathcal{A} = \mathrm{colim} \mathcal{A}_i$ is a directed colimit of finite and finitely presented quasi-coherent \mathcal{O}_X -algebras with surjective transition maps.*

Proof. By Lemma 9.3 there exists a finitely presented \mathcal{O}_X -module \mathcal{F} and a surjection $\mathcal{F} \rightarrow \mathcal{A}$. Using the algebra structure we obtain a surjection

$$\mathrm{Sym}_{\mathcal{O}_X}^*(\mathcal{F}) \longrightarrow \mathcal{A}$$

Denote \mathcal{J} the kernel. Write $\mathcal{J} = \mathrm{colim} \mathcal{E}_i$ as a filtered colimit of finite type \mathcal{O}_X -submodules \mathcal{E}_i (Lemma 9.2). Set

$$\mathcal{A}_i = \mathrm{Sym}_{\mathcal{O}_X}^*(\mathcal{F}) / (\mathcal{E}_i)$$

where (\mathcal{E}_i) indicates the ideal sheaf generated by the image of $\mathcal{E}_i \rightarrow \mathrm{Sym}_{\mathcal{O}_X}^*(\mathcal{F})$. Then each \mathcal{A}_i is a finitely presented \mathcal{O}_X -algebra, the transition maps are surjective, and $\mathcal{A} = \mathrm{colim} \mathcal{A}_i$. To finish the proof we still have to show that \mathcal{A}_i is a finite \mathcal{O}_X -algebra for i sufficiently large. To do this we choose an étale surjective map $U \rightarrow X$ where U is an affine scheme. Take generators $f_1, \dots, f_m \in \Gamma(U, \mathcal{F})$. As $\mathcal{A}(U)$ is a finite $\mathcal{O}_X(U)$ -algebra we see that for each j there exists a monic polynomial

$P_j \in \mathcal{O}(U)[T]$ such that $P_j(f_j)$ is zero in $\mathcal{A}(U)$. Since $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ by construction, we have $P_j(f_j) = 0$ in $\mathcal{A}_i(U)$ for all sufficiently large i . For such i the algebras \mathcal{A}_i are finite. \square

Lemma 9.7. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let \mathcal{A} be an integral quasi-coherent \mathcal{O}_X -algebra. Then*

- (1) \mathcal{A} is the directed colimit of its finite quasi-coherent \mathcal{O}_X -subalgebras, and
- (2) \mathcal{A} is a directed colimit of finite and finitely presented \mathcal{O}_X -algebras.

Proof. By Lemma 9.5 we have $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ where $\mathcal{A}_i \subset \mathcal{A}$ runs through the quasi-coherent \mathcal{O}_X -sub algebras of finite type. Any finite type quasi-coherent \mathcal{O}_X -subalgebra of \mathcal{A} is finite (use Algebra, Lemma 36.5 on affine schemes étale over X). This proves (1).

To prove (2), write $\mathcal{A} = \operatorname{colim} \mathcal{F}_i$ as a colimit of finitely presented \mathcal{O}_X -modules using Lemma 9.1. For each i , let \mathcal{J}_i be the kernel of the map

$$\operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \rightarrow \mathcal{J}_{i'}$ and we have $\mathcal{A} = \operatorname{colim} \operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent \mathcal{O}_X -algebras $\operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_i)/\mathcal{J}_i$ are finite (see above). Write $\mathcal{J}_i = \operatorname{colim} \mathcal{E}_{ik}$ as a colimit of finitely presented \mathcal{O}_X -modules. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \rightarrow \mathcal{E}_{i'k'}$ making

$$\begin{array}{ccc} \mathcal{J}_i & \longrightarrow & \mathcal{J}_{i'} \\ \uparrow & & \uparrow \\ \mathcal{E}_{ik} & \longrightarrow & \mathcal{E}_{i'k'} \end{array}$$

commute. This follows from Cohomology of Spaces, Lemma 5.3. This induces a map

$$\mathcal{A}_{ik} = \operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where (\mathcal{E}_{ik}) denotes the ideal generated by \mathcal{E}_{ik} . The quasi-coherent \mathcal{O}_X -algebras \mathcal{A}_{ki} are of finite presentation and finite for k large enough (see proof of Lemma 9.6). Finally, we have

$$\operatorname{colim} \mathcal{A}_{ik} = \operatorname{colim} \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 9.6 and the second equality because \mathcal{A} is the colimit of the modules \mathcal{F}_i . \square

Lemma 9.8. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let $U \subset X$ be a quasi-compact open. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent \mathcal{O}_U -submodule which is of finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. Denote $j : U \rightarrow X$ the inclusion morphism. As X is quasi-separated and U quasi-compact, the morphism j is quasi-compact. Hence $j_*\mathcal{G} \subset j_*\mathcal{F}|_U$ are quasi-coherent modules on X (Morphisms of Spaces, Lemma 11.2). Let $\mathcal{H} = \operatorname{Ker}(j_*\mathcal{G} \oplus \mathcal{F} \rightarrow j_*\mathcal{F}|_U)$. Then $\mathcal{H}|_U = \mathcal{G}$. By Lemma 9.2 we can find a finite type quasi-coherent submodule $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'|_U = \mathcal{H}|_U = \mathcal{G}$. Set $\mathcal{G}' = \operatorname{Im}(\mathcal{H}' \rightarrow \mathcal{F})$ to conclude. \square

10. Relative approximation

We discuss variants of Proposition 8.1 over a base.

Lemma 10.1. *Let $f : X \rightarrow Y$ be a morphism of quasi-compact and quasi-separated algebraic spaces over \mathbf{Z} . Then there exists a direct set I and an inverse system $(f_i : X_i \rightarrow Y_i)$ of morphisms algebraic spaces over I , such that the transition morphisms $X_i \rightarrow X_{i'}$ and $Y_i \rightarrow Y_{i'}$ are affine, such that X_i and Y_i are quasi-separated and of finite type over \mathbf{Z} , and such that $(X \rightarrow Y) = \lim(X_i \rightarrow Y_i)$.*

Proof. Write $X = \lim_{a \in A} X_a$ and $Y = \lim_{b \in B} Y_b$ as in Proposition 8.1, i.e., with X_a and Y_b quasi-separated and of finite type over \mathbf{Z} and with affine transition morphisms.

Fix $b \in B$. By Lemma 4.5 applied to Y_b and $X = \lim X_a$ over \mathbf{Z} we find there exists an $a \in A$ and a morphism $f_{a,b} : X_a \rightarrow Y_b$ making the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_a & \longrightarrow & Y_b \end{array}$$

commute. Let I be the set of triples $(a, b, f_{a,b})$ we obtain in this manner.

Let $(a, b, f_{a,b})$ and $(a', b', f_{a',b'})$ be in I . Let $b'' \leq \min(b, b')$. By Lemma 4.5 again, there exists an $a'' \geq \max(a, a')$ such that the compositions $X_{a''} \rightarrow X_a \rightarrow Y_b \rightarrow Y_{b''}$ and $X_{a''} \rightarrow X_{a'} \rightarrow Y_{b'} \rightarrow Y_{b''}$ are equal. We endow I with the preorder

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a', \quad b \geq b', \quad \text{and} \quad g_{b,b'} \circ f_{a,b} = f_{a',b'} \circ h_{a,a'}$$

where $h_{a,a'} : X_a \rightarrow X_{a'}$ and $g_{b,b'} : Y_b \rightarrow Y_{b'}$ are the transition morphisms. The remarks above show that I is directed and that the maps $I \rightarrow A$, $(a, b, f_{a,b}) \mapsto a$ and $I \rightarrow B$, $(a, b, f_{a,b}) \mapsto b$ are cofinal. If for $i = (a, b, f_{a,b})$ we set $X_i = X_a$, $Y_i = Y_b$, and $f_i = f_{a,b}$, then we get an inverse system of morphisms over I and we have

$$\lim_{i \in I} X_i = \lim_{a \in A} X_a = X \quad \text{and} \quad \lim_{i \in I} Y_i = \lim_{b \in B} Y_b = Y$$

by Categories, Lemma 17.4 (recall that limits over I are really limits over the opposite category associated to I and hence cofinal turns into initial). This finishes the proof. \square

Lemma 10.2. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume that*

- (1) X is quasi-compact and quasi-separated, and
- (2) Y is quasi-separated.

Then $X = \lim X_i$ is a limit of a directed inverse system of algebraic spaces X_i of finite presentation over Y with affine transition morphisms over Y .

Proof. Since $|f|(|X|)$ is quasi-compact we may replace Y by a quasi-compact open subspace whose set of points contains $|f|(|X|)$. Hence we may assume Y is quasi-compact as well. By Lemma 10.1 we can write $(X \rightarrow Y) = \lim(X_i \rightarrow Y_i)$ for some directed inverse system of morphisms of finite type schemes over \mathbf{Z} with affine transition morphisms. Since limits commute with limits (Categories, Lemma 14.10)

we have $X = \lim X_i \times_{Y_i} Y$. For $i \geq i'$ the transition morphism $X_i \times_{Y_i} Y \rightarrow X_{i'} \times_{Y_{i'}} Y$ is affine as the composition

$$X_i \times_{Y_i} Y \rightarrow X_i \times_{Y_{i'}} Y \rightarrow X_{i'} \times_{Y_{i'}} Y$$

where the first morphism is a closed immersion (by Morphisms of Spaces, Lemma 4.5) and the second is a base change of an affine morphism (Morphisms of Spaces, Lemma 20.5) and the composition of affine morphisms is affine (Morphisms of Spaces, Lemma 20.4). The morphisms f_i are of finite presentation (Morphisms of Spaces, Lemmas 28.7 and 28.9) and hence the base changes $X_i \times_{f_i, Y_i} Y \rightarrow Y$ are of finite presentation (Morphisms of Spaces, Lemma 28.3). \square

11. Finite type closed in finite presentation

This section is the analogue of Limits, Section 9.

Lemma 11.1. *Let S be a scheme. Let $f : X \rightarrow Y$ be an affine morphism of algebraic spaces over S . If Y quasi-compact and quasi-separated, then X is a directed limit $X = \lim X_i$ with each X_i affine and of finite presentation over Y .*

Proof. Consider the quasi-coherent \mathcal{O}_Y -module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 9.4 we can write $\mathcal{A} = \text{colim } \mathcal{A}_i$ as a directed colimit of finitely presented \mathcal{O}_Y -algebras \mathcal{A}_i . Set $X_i = \text{Spec}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 20.8. By construction $X_i \rightarrow Y$ is affine and of finite presentation and $X = \lim X_i$. \square

Lemma 11.2. *Let S be a scheme. Let $f : X \rightarrow Y$ be an integral morphism of algebraic spaces over S . Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where X_i are finite and of finite presentation over Y .*

Proof. Consider the quasi-coherent \mathcal{O}_Y -module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 9.7 we can write $\mathcal{A} = \text{colim } \mathcal{A}_i$ as a directed colimit of finite and finitely presented \mathcal{O}_Y -algebras \mathcal{A}_i . Set $X_i = \text{Spec}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 20.8. By construction $X_i \rightarrow Y$ is finite and of finite presentation and $X = \lim X_i$. \square

Lemma 11.3. *Let S be a scheme. Let $f : X \rightarrow Y$ be a finite morphism of algebraic spaces over S . Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the objects X_i are finite and of finite presentation over Y .*

Proof. Consider the finite quasi-coherent \mathcal{O}_Y -module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 9.6 we can write $\mathcal{A} = \text{colim } \mathcal{A}_i$ as a directed colimit of finite and finitely presented \mathcal{O}_Y -algebras \mathcal{A}_i with surjective transition maps. Set $X_i = \text{Spec}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 20.8. By construction $X_i \rightarrow Y$ is finite and of finite presentation, the transition maps are closed immersions, and $X = \lim X_i$. \square

Lemma 11.4. *Let S be a scheme. Let $f : X \rightarrow Y$ be a closed immersion of algebraic spaces over S . Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the morphisms $X_i \rightarrow Y$ are closed immersions of finite presentation.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subspace of Y . By Lemma 9.2 we can write $\mathcal{I} = \text{colim } \mathcal{I}_i$ as the filtered colimit of its finite type quasi-coherent submodules. Let X_i be the closed subspace of X cut out

by \mathcal{I}_i . Then $X_i \rightarrow Y$ is a closed immersion of finite presentation, and $X = \lim X_i$. Some details omitted. \square

Lemma 11.5. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume*

- (1) *f is locally of finite type and quasi-affine, and*
- (2) *Y is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \rightarrow Y$ and a closed immersion $X \rightarrow X'$ over Y .

Proof. By Morphisms of Spaces, Lemma 21.6 we can find a factorization $X \rightarrow Z \rightarrow Y$ where $X \rightarrow Z$ is a quasi-compact open immersion and $Z \rightarrow Y$ is affine. Write $Z = \lim Z_i$ with Z_i affine and of finite presentation over Y (Lemma 11.1). For some $0 \in I$ we can find a quasi-compact open $U_0 \subset Z_0$ such that X is isomorphic to the inverse image of U_0 in Z (Lemma 5.7). Let U_i be the inverse image of U_0 in Z_i , so $U = \lim U_i$. By Lemma 5.12 we see that $X \rightarrow U_i$ is a closed immersion for some i large enough. Setting $X' = U_i$ finishes the proof. \square

Lemma 11.6. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume:*

- (1) *f is of locally of finite type.*
- (2) *X is quasi-compact and quasi-separated, and*
- (3) *Y is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \rightarrow Y$ and a closed immersion $X \rightarrow X'$ of algebraic spaces over Y .

Proof. By Proposition 8.1 we can write $X = \lim_i X_i$ with X_i quasi-separated of finite type over \mathbf{Z} and with transition morphisms $f_{ii'} : X_i \rightarrow X_{i'}$ affine. Consider the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_{i,Y} & \longrightarrow & X_i \\ & \searrow & \downarrow & & \downarrow \\ & & Y & \longrightarrow & \mathrm{Spec}(\mathbf{Z}) \end{array}$$

Note that X_i is of finite presentation over $\mathrm{Spec}(\mathbf{Z})$, see Morphisms of Spaces, Lemma 28.7. Hence the base change $X_{i,Y} \rightarrow Y$ is of finite presentation by Morphisms of Spaces, Lemma 28.3. Observe that $\lim X_{i,Y} = X \times Y$ and that $X \rightarrow X \times Y$ is a monomorphism. By Lemma 5.12 we see that $X \rightarrow X_{i,Y}$ is a monomorphism for i large enough. Fix such an i . Note that $X \rightarrow X_{i,Y}$ is locally of finite type (Morphisms of Spaces, Lemma 23.6) and a monomorphism, hence separated and locally quasi-finite (Morphisms of Spaces, Lemma 27.10). Hence $X \rightarrow X_{i,Y}$ is representable. Hence $X \rightarrow X_{i,Y}$ is quasi-affine because we can use the principle Spaces, Lemma 5.8 and the result for morphisms of schemes More on Morphisms, Lemma 43.2. Thus Lemma 11.5 gives a factorization $X \rightarrow X' \rightarrow X_{i,Y}$ with $X \rightarrow X'$ a closed immersion and $X' \rightarrow X_{i,Y}$ of finite presentation. Finally, $X' \rightarrow Y$ is of finite presentation as a composition of morphisms of finite presentation (Morphisms of Spaces, Lemma 28.2). \square

Proposition 11.7. *Let S be a scheme. $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume*

- (1) f is of finite type and separated, and
- (2) Y is quasi-compact and quasi-separated.

Then there exists a separated morphism of finite presentation $f' : X' \rightarrow Y$ and a closed immersion $X \rightarrow X'$ over Y .

Proof. By Lemma 11.6 there is a closed immersion $X \rightarrow Z$ with Z/Y of finite presentation. Let $\mathcal{I} \subset \mathcal{O}_Z$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Z . By Lemma 9.2 we can write \mathcal{I} as a directed colimit $\mathcal{I} = \text{colim}_{a \in A} \mathcal{I}_a$ of its quasi-coherent sheaves of ideals of finite type. Let $X_a \subset Z$ be the closed subspace defined by \mathcal{I}_a . These form an inverse system indexed by A . The transition morphisms $X_a \rightarrow X_{a'}$ are affine because they are closed immersions. Each X_a is quasi-compact and quasi-separated since it is a closed subspace of Z and Z is quasi-compact and quasi-separated by our assumptions. We have $X = \lim_a X_a$ as follows directly from the fact that $\mathcal{I} = \text{colim}_{a \in A} \mathcal{I}_a$. Each of the morphisms $X_a \rightarrow Z$ is of finite presentation, see Morphisms, Lemma 21.7. Hence the morphisms $X_a \rightarrow Y$ are of finite presentation. Thus it suffices to show that $X_a \rightarrow Y$ is separated for some $a \in A$. This follows from Lemma 5.13 as we have assumed that $X \rightarrow Y$ is separated. \square

12. Approximating proper morphisms

Lemma 12.1. *Let S be a scheme. Let $f : X \rightarrow Y$ be a proper morphism of algebraic spaces over S with Y quasi-compact and quasi-separated. Then $X = \lim X_i$ is a directed limit of algebraic spaces X_i proper and of finite presentation over Y and with transition morphisms and morphisms $X \rightarrow X_i$ closed immersions.*

Proof. By Proposition 11.7 we can find a closed immersion $X \rightarrow X'$ with X' separated and of finite presentation over Y . By Lemma 11.4 we can write $X = \lim X_i$ with $X_i \rightarrow X'$ a closed immersion of finite presentation. We claim that for all i large enough the morphism $X_i \rightarrow Y$ is proper which finishes the proof.

To prove this we may assume that Y is an affine scheme, see Morphisms of Spaces, Lemma 40.2. Next, we use the weak version of Chow's lemma, see Cohomology of Spaces, Lemma 18.1, to find a diagram

$$\begin{array}{ccccc} X' & \xleftarrow{\pi} & X'' & \longrightarrow & \mathbf{P}_Y^n \\ & \searrow & \downarrow & \swarrow & \\ & & Y & & \end{array}$$

where $X'' \rightarrow \mathbf{P}_Y^n$ is an immersion, and $\pi : X'' \rightarrow X'$ is proper and surjective. Denote $X'_i \subset X''$, resp. $\pi^{-1}(X)$ the scheme theoretic inverse image of $X_i \subset X'$, resp. $X \subset X'$. Then $\lim X'_i = \pi^{-1}(X)$. Since $\pi^{-1}(X) \rightarrow Y$ is proper (Morphisms of Spaces, Lemmas 40.4), we see that $\pi^{-1}(X) \rightarrow \mathbf{P}_Y^n$ is a closed immersion (Morphisms of Spaces, Lemmas 40.6 and 12.3). Hence for i large enough we find that $X'_i \rightarrow \mathbf{P}_Y^n$ is a closed immersion by Lemma 5.16. Thus X'_i is proper over Y . For such i the morphism $X_i \rightarrow Y$ is proper by Morphisms of Spaces, Lemma 40.7. \square

Lemma 12.2. *Let $f : X \rightarrow Y$ be a proper morphism of algebraic spaces over \mathbf{Z} with Y quasi-compact and quasi-separated. Then there exists a directed set I , an inverse system $(f_i : X_i \rightarrow Y_i)$ of morphisms of algebraic spaces over I , such that the transition morphisms $X_i \rightarrow X_{i'}$ and $Y_i \rightarrow Y_{i'}$ are affine, such that f_i is proper*

and of finite presentation, such that Y_i is of finite presentation over \mathbf{Z} , and such that $(X \rightarrow Y) = \lim(X_i \rightarrow Y_i)$.

Proof. By Lemma 12.1 we can write $X = \lim_{k \in K} X_k$ with $X_k \rightarrow Y$ proper and of finite presentation. Next, by absolute Noetherian approximation (Proposition 8.1) we can write $Y = \lim_{j \in J} Y_j$ with Y_j of finite presentation over \mathbf{Z} . For each k there exists a j and a morphism $X_{k,j} \rightarrow Y_j$ of finite presentation with $X_k \cong Y \times_{Y_j} X_{k,j}$ as algebraic spaces over Y , see Lemma 7.1. After increasing j we may assume $X_{k,j} \rightarrow Y_j$ is proper, see Lemma 6.13. The set I will consist of these pairs (k, j) and the corresponding morphism is $X_{k,j} \rightarrow Y_j$. For every $k' \geq k$ we can find a $j' \geq j$ and a morphism $X_{j',k'} \rightarrow X_{j,k}$ over $Y_{j'} \rightarrow Y_j$ whose base change to Y gives the morphism $X_{k'} \rightarrow X_k$ (follows again from Lemma 7.1). These morphisms form the transition morphisms of the system. Some details omitted. \square

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms of Spaces, Definition 15.4.

Lemma 12.3. *Assumptions and notation as in Situation 6.1. Let \mathcal{F}_0 be a quasi-coherent \mathcal{O}_{X_0} -module. Denote \mathcal{F} and \mathcal{F}_i the pullbacks of \mathcal{F}_0 to X and X_i . Assume*

- (1) f_0 is locally of finite type,
- (2) \mathcal{F}_0 is of finite type,
- (3) the scheme theoretic support of \mathcal{F} is proper over Y .

Then the scheme theoretic support of \mathcal{F}_i is proper over Y_i for some i .

Proof. We may replace X_0 by the scheme theoretic support of \mathcal{F}_0 . By Morphisms of Spaces, Lemma 15.2 this guarantees that X_i is the support of \mathcal{F}_i and X is the support of \mathcal{F} . Then, if $Z \subset X$ denotes the scheme theoretic support of \mathcal{F} , we see that $Z \rightarrow X$ is a universal homeomorphism. We conclude that $X \rightarrow Y$ is proper as this is true for $Z \rightarrow Y$ by assumption, see Morphisms, Lemma 41.9. By Lemma 6.13 we see that $X_i \rightarrow Y$ is proper for some i . Then it follows that the scheme theoretic support Z_i of \mathcal{F}_i is proper over Y by Morphisms of Spaces, Lemmas 40.5 and 40.4. \square

13. Embedding into affine space

Some technical lemmas to be used in the proof of Chow's lemma later.

Lemma 13.1. *Let S be a scheme. Let $f : U \rightarrow X$ be a morphism of algebraic spaces over S . Assume U is an affine scheme, f is locally of finite type, and X quasi-separated and locally separated. Then there exists an immersion $U \rightarrow \mathbf{A}_X^n$ over X .*

Proof. Say $U = \text{Spec}(A)$. Write $A = \text{colim } A_i$ as a filtered colimit of finite type \mathbf{Z} -subalgebras. For each i the morphism $U \rightarrow U_i = \text{Spec}(A_i)$ induces a morphism

$$U \longrightarrow X \times U_i$$

over X . In the limit the morphism $U \rightarrow X \times U$ is an immersion as X is locally separated, see Morphisms of Spaces, Lemma 4.6. By Lemma 5.12 we see that $U \rightarrow X \times U_i$ is an immersion for some i . Since U_i is isomorphic to a closed subscheme of $\mathbf{A}_{\mathbf{Z}}^n$ the lemma follows. \square

Remark 13.2. We have seen in Examples, Section 28 that Lemma 13.1 does not hold if we drop the assumption that X be locally separated. This raises the question: Does Lemma 13.1 hold if we drop the assumption that X be quasi-separated? If you know the answer, please email stacks.project@gmail.com.

Lemma 13.3. *Let S be a scheme. Let $f : Y \rightarrow X$ be a morphism of algebraic spaces over S . Assume X Noetherian and f of finite presentation. Then there exists a dense open $V \subset Y$ and an immersion $V \rightarrow \mathbf{A}_X^n$.*

Proof. The assumptions imply that Y is Noetherian (Morphisms of Spaces, Lemma 28.6). Then Y is quasi-separated, hence has a dense open subscheme (Properties of Spaces, Proposition 13.3). Thus we may assume that Y is a Noetherian scheme. By removing intersections of irreducible components of Y (use Topology, Lemma 9.2 and Properties, Lemma 5.5) we may assume that Y is a disjoint union of irreducible Noetherian schemes. Since there is an immersion

$$\mathbf{A}_X^n \amalg \mathbf{A}_X^m \longrightarrow \mathbf{A}_X^{\max(n,m)+1}$$

(details omitted) we see that it suffices to prove the result in case Y is irreducible.

Assume Y is an irreducible scheme. Let $T \subset |X|$ be the closure of the image of $f : Y \rightarrow X$. Note that since $|Y|$ and $|X|$ are sober topological spaces (Properties of Spaces, Lemma 15.1) T is irreducible with a unique generic point ξ which is the image of the generic point η of Y . Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals cutting out the reduced induced space structure on T (Properties of Spaces, Definition 12.5). Since $\mathcal{O}_{Y,\eta}$ is an Artinian local ring we see that for some $n > 0$ we have $f^{-1}\mathcal{I}^n\mathcal{O}_{Y,\eta} = 0$. As $f^{-1}\mathcal{I}\mathcal{O}_Y$ is a finite type quasi-coherent ideal we conclude that $f^{-1}\mathcal{I}^n\mathcal{O}_V = 0$ for some nonempty open $V \subset Y$. Let $Z \subset X$ be the closed subspace cut out by \mathcal{I}^n . By construction $V \rightarrow Y \rightarrow X$ factors through Z . Because $\mathbf{A}_Z^n \rightarrow \mathbf{A}_X^n$ is an immersion, we may replace X by Z and Y by V . Hence we reach the situation where Y and X are irreducible and $Y \rightarrow X$ maps the generic point of Y onto the generic point of X .

Assume Y and X are irreducible, Y is a scheme, and $Y \rightarrow X$ maps the generic point of Y onto the generic point of X . By Properties of Spaces, Proposition 13.3 X has a dense open subscheme $U \subset X$. Choose a nonempty affine open $V \subset Y$ whose image in X is contained in U . By Morphisms, Lemma 39.2 we may factor $V \rightarrow U$ as $V \rightarrow \mathbf{A}_U^n \rightarrow U$. Composing with $\mathbf{A}_U^n \rightarrow \mathbf{A}_X^n$ we obtain the desired immersion. \square

14. Sections with support in a closed subset

This section is the analogue of Properties, Section 24.

Lemma 14.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $U \subset X$ be an open subspace. The following are equivalent:*

- (1) $U \rightarrow X$ is quasi-compact,
- (2) U is quasi-compact, and
- (3) there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $|X| \setminus |U| = |V(\mathcal{I})|$.

Proof. Let W be an affine scheme and let $\varphi : W \rightarrow X$ be a surjective étale morphism, see Properties of Spaces, Lemma 6.3. If (1) holds, then $\varphi^{-1}(U) \rightarrow W$ is quasi-compact, hence $\varphi^{-1}(U)$ is quasi-compact, hence U is quasi-compact (as

$|\varphi^{-1}(U)| \rightarrow |U|$ is surjective). If (2) holds, then $\varphi^{-1}(U)$ is quasi-compact because φ is quasi-compact since X is quasi-separated (Morphisms of Spaces, Lemma 8.10). Hence $\varphi^{-1}(U) \rightarrow W$ is a quasi-compact morphism of schemes by Properties, Lemma 24.1. It follows that $U \rightarrow X$ is quasi-compact by Morphisms of Spaces, Lemma 8.8. Thus (1) and (2) are equivalent.

Assume (1) and (2). By Properties of Spaces, Lemma 12.3 there exists a unique quasi-coherent sheaf of ideals \mathcal{J} cutting out the reduced induced closed subspace structure on $|X| \setminus |U|$. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an \mathcal{O}_U -modules of finite type. As U is quasi-compact it follows from Lemma 9.2 that there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. Then $|X| \setminus |U| = |V(\mathcal{I})|$ and we obtain (3). Conversely, if \mathcal{I} is as in (3), then $\varphi^{-1}(U) \subset W$ is a quasi-compact open by the lemma for schemes (Properties, Lemma 24.1) applied to $\varphi^{-1}\mathcal{I}$ on W . Thus (2) holds. \square

Lemma 14.2. *Let S be a scheme. Let X be an algebraic space over S . Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Consider the sheaf of \mathcal{O}_X -modules \mathcal{F}' which associates to every object U of $X_{\text{étale}}$ the module*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

Assume \mathcal{I} is of finite type. Then

- (1) \mathcal{F}' is a quasi-coherent sheaf of \mathcal{O}_X -modules,
- (2) for affine U in $X_{\text{étale}}$ we have $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, and
- (3) $\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.

Proof. It is clear that the rule defining \mathcal{F}' gives a subsheaf of \mathcal{F} . Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 24.2. \square

Definition 14.3. Let S be a scheme. Let X be an algebraic space over S . Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 14.2 above is called the *subsheaf of sections annihilated by \mathcal{I}* .

Lemma 14.4. *Let S be a scheme. Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S . Let $\mathcal{I} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals of finite type. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections annihilated by $f^{-1}\mathcal{I}\mathcal{O}_X$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections annihilated by \mathcal{I} .*

Proof. Omitted. Hint: The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent (Morphisms of Spaces, Lemma 11.2) so that Lemma 14.2 applies to \mathcal{I} and $f_*\mathcal{F}$. \square

Next we come to the sheaf of sections supported in a closed subset. Again this isn't always a quasi-coherent sheaf, but if the complement of the closed is "retrocompact" in the given algebraic space, then it is.

Lemma 14.5. *Let S be a scheme. Let X be an algebraic space over S . Let $T \subset |X|$ be a closed subset and let $U \subset X$ be the open subspace such that $T \amalg |U| = |X|$. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Consider the sheaf of \mathcal{O}_X -modules \mathcal{F}' which associates to every object $\varphi : W \rightarrow X$ of $X_{\text{étale}}$ the module*

$$\mathcal{F}'(W) = \{s \in \mathcal{F}(W) \mid \text{the support of } s \text{ is contained in } |\varphi|^{-1}(T)\}$$

If $U \rightarrow X$ is quasi-compact, then

- (1) for W affine there exist a finitely generated ideal $I \subset \mathcal{O}_X(W)$ such that $|\varphi|^{-1}(T) = V(I)$,
- (2) for W and I as in (1) we have $\mathcal{F}'(W) = \{x \in \mathcal{F}(W) \mid I^n x = 0 \text{ for some } n\}$,
- (3) \mathcal{F}' is a quasi-coherent sheaf of \mathcal{O}_X -modules.

Proof. It is clear that the rule defining \mathcal{F}' gives a subsheaf of \mathcal{F} . Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 24.5. \square

Definition 14.6. Let S be a scheme. Let X be an algebraic space over S . Let $T \subset |X|$ be a closed subset whose complement corresponds to an open subspace $U \subset X$ with quasi-compact inclusion morphism $U \rightarrow X$. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. The quasi-coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 14.5 above is called the *subsheaf of sections supported on T* .

Lemma 14.7. Let S be a scheme. Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S . Let $T \subset |Y|$ be a closed subset. Assume $|Y| \setminus T$ corresponds to an open subspace $V \subset Y$ such that $V \rightarrow Y$ is quasi-compact. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections supported on $|f|^{-1}T$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections supported on T .

Proof. Omitted. Hints: $|X| \setminus |f|^{-1}T$ is the support of the open subspace $U = f^{-1}V \subset X$. Since $V \rightarrow Y$ is quasi-compact, so is $U \rightarrow X$ (by base change). The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent. Hence Lemma 14.5 applies to T and $f_*\mathcal{F}$ as well as to $|f|^{-1}T$ and \mathcal{F} . The equality of the given quasi-coherent modules is immediate from the definitions. \square

15. Characterizing affine spaces

This section is the analogue of Limits, Section 11.

Lemma 15.1. Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume that f is surjective and finite, and assume that X is affine. Then Y is affine.

Proof. We may and do view $f : X \rightarrow Y$ as a morphism of algebraic space over $\text{Spec}(\mathbf{Z})$ (see Spaces, Definition 16.2). Note that a finite morphism is affine and universally closed, see Morphisms of Spaces, Lemma 45.7. By Morphisms of Spaces, Lemma 9.8 we see that Y is a separated algebraic space. As f is surjective and X is quasi-compact we see that Y is quasi-compact.

By Lemma 11.3 we can write $X = \lim X_a$ with each $X_a \rightarrow Y$ finite and of finite presentation. By Lemma 5.10 we see that X_a is affine for a large enough. Hence we may and do assume that $f : X \rightarrow Y$ is finite, surjective, and of finite presentation.

By Proposition 8.1 we may write $Y = \lim Y_i$ as a directed limit of algebraic spaces of finite presentation over \mathbf{Z} . By Lemma 7.1 we can find $0 \in I$ and a morphism $X_0 \rightarrow Y_0$ of finite presentation such that $X_i = X_0 \times_{Y_0} Y_i$ for $i \geq 0$ and such that $X = \lim_i X_i$. By Lemma 6.7 we see that $X_i \rightarrow Y_i$ is finite for i large enough. By Lemma 6.4 we see that $X_i \rightarrow Y_i$ is surjective for i large enough. By Lemma 5.10 we see that X_i is affine for i large enough. Hence for i large enough we can apply

Cohomology of Spaces, Lemma 17.3 to conclude that Y_i is affine. This implies that Y is affine and we conclude. \square

Proposition 15.2. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume that X is affine and f is surjective and universally closed². Then Y is affine.*

Proof. We may and do view $f : X \rightarrow Y$ as a morphism of algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 16.2). By Morphisms of Spaces, Lemma 9.8 we see that Y is a separated algebraic space. Then by Morphisms of Spaces, Lemma 20.11 we find that f is affine. Whereupon by Morphisms of Spaces, Lemma 45.7 we see that f is integral.

By the preceding paragraph, we may assume $f : X \rightarrow Y$ is surjective and integral, X is affine, and Y is separated. Since f is surjective and X is quasi-compact we also deduce that Y is quasi-compact.

Consider the sheaf $\mathcal{A} = f_*\mathcal{O}_X$. This is a quasi-coherent sheaf of \mathcal{O}_Y -algebras, see Morphisms of Spaces, Lemma 11.2. By Lemma 9.1 we can write $\mathcal{A} = \mathrm{colim}_i \mathcal{F}_i$ as a filtered colimit of finite type \mathcal{O}_Y -modules. Let $\mathcal{A}_i \subset \mathcal{A}$ be the \mathcal{O}_Y -subalgebra generated by \mathcal{F}_i . Since the map of algebras $\mathcal{O}_Y \rightarrow \mathcal{A}$ is integral, we see that each \mathcal{A}_i is a finite quasi-coherent \mathcal{O}_Y -algebra. Hence

$$X_i = \mathrm{Spec}_Y(\mathcal{A}_i) \rightarrow Y$$

is a finite morphism of algebraic spaces. Here Spec is the construction of Morphisms of Spaces, Lemma 20.7. It is clear that $X = \varinjlim_i X_i$. Hence by Lemma 5.10 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \rightarrow Y$ factors through each X_i we see that $X_i \rightarrow Y$ is surjective. Hence we conclude that Y is affine by Lemma 15.1. \square

The following corollary of the result above can be found in [CLO12].

Lemma 15.3. *Let S be a scheme. Let X be an algebraic space over S . If X_{red} is a scheme, then X is a scheme.*

Proof. Let $U' \subset X_{\mathrm{red}}$ be an open affine subscheme. Let $U \subset X$ be the open subspace corresponding to the open $|U'| \subset |X_{\mathrm{red}}| = |X|$. Then $U' \rightarrow U$ is surjective and integral. Hence U is affine by Proposition 15.2. Thus every point is contained in an open subscheme of X , i.e., X is a scheme. \square

Lemma 15.4. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f is integral and induces a bijection $|X| \rightarrow |Y|$. Then X is a scheme if and only if Y is a scheme.*

Proof. An integral morphism is representable by definition, hence if Y is a scheme, so is X . Conversely, assume that X is a scheme. Let $U \subset X$ be an affine open. An integral morphism is closed and $|f|$ is bijective, hence $|f|(|U|) \subset |Y|$ is open as the complement of $|f|(|X| \setminus |U|)$. Let $V \subset Y$ be the open subspace with $|V| = |f|(|U|)$, see Properties of Spaces, Lemma 4.8. Then $U \rightarrow V$ is integral and surjective, hence V is an affine scheme by Proposition 15.2. This concludes the proof. \square

Lemma 15.5. *Let S be a scheme. Let $f : X \rightarrow B$ and $B' \rightarrow B$ be morphisms of algebraic spaces over S . Assume*

²An integral morphism is universally closed, see Morphisms of Spaces, Lemma 45.7.

- (1) $B' \rightarrow B$ is a closed immersion,
- (2) $|B'| \rightarrow |B|$ is bijective,
- (3) $X \times_B B' \rightarrow B'$ is a closed immersion, and
- (4) $X \rightarrow B$ is of finite type or $B' \rightarrow B$ is of finite presentation.

Then $f : X \rightarrow B$ is a closed immersion.

Proof. Assumptions (1) and (2) imply that $B_{red} = B'_{red}$. Set $X' = X \times_B B'$. Then $X' \rightarrow X$ is closed immersion and $X'_{red} = X_{red}$. Let $U \rightarrow B$ be an étale morphism with U affine. Then $X' \times_B U \rightarrow X \times_B U$ is a closed immersion of algebraic spaces inducing an isomorphism on underlying reduced spaces. Since $X' \times_B U$ is a scheme (as $B' \rightarrow B$ and $X' \rightarrow B'$ are representable) so is $X \times_B U$ by Lemma 15.3. Hence $X \rightarrow B$ is representable too. Thus we reduce to the case of schemes, see Morphisms, Lemma 45.7. \square

16. Finite cover by a scheme

As an application of the limit results of this chapter, we prove that given any quasi-compact and quasi-separated algebraic space X , there is a scheme Y and a surjective, finite morphism $Y \rightarrow X$. We will rely on the already proven result that we can find a finite integral cover by a scheme, which was proved in Decent Spaces, Section 9.

Proposition 16.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S .*

- (1) *There exists a surjective finite morphism $Y \rightarrow X$ of finite presentation where Y is a scheme,*
- (2) *given a surjective étale morphism $U \rightarrow X$ we may choose $Y \rightarrow X$ such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \rightarrow X$ factors through U .*

Proof. Part (1) is the special case of (2) with $U = X$. Let $Y \rightarrow X$ be as in Decent Spaces, Lemma 9.2. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \rightarrow X$ factors through U . We can write $Y = \lim Y_i$ with $Y_i \rightarrow X$ finite and of finite presentation, see Lemma 11.2. For large enough i the algebraic space Y_i is a scheme, see Lemma 5.11. For large enough i we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in Y recovers V_j , see Lemma 5.7. For even larger i the morphisms $V_j \rightarrow U$ over X come from morphisms $V_{i,j} \rightarrow U$ over X , see Proposition 3.10. This finishes the proof. \square

Lemma 16.2. *Let S be a scheme. Let $f : X \rightarrow Y$ be an integral morphism of algebraic spaces over S . Assume Y quasi-compact and quasi-separated. Let $V \subset Y$ be a quasi-compact open subspace such that $f^{-1}(V) \rightarrow V$ is finite and of finite presentation. Then X can be written as a directed limit $X = \lim X_i$ where $f_i : X_i \rightarrow Y$ are finite and of finite presentation such that $f^{-1}(V) \rightarrow f_i^{-1}(V)$ is an isomorphism for all i .*

Proof. This lemma is a slight refinement of Proposition 16.1. Consider the integral quasi-coherent \mathcal{O}_Y -algebra $\mathcal{A} = f_* \mathcal{O}_X$. In the next paragraph, we will write $\mathcal{A} = \text{colim } \mathcal{A}_i$ as a directed colimit of finite and finitely presented \mathcal{O}_Y -algebras \mathcal{A}_i such that $\mathcal{A}_i|_V = \mathcal{A}|_V$. Having done this we set $X_i = \text{Spec}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 20.8. By construction $X_i \rightarrow Y$ is finite and of finite presentation, $X = \lim X_i$, and $f_i^{-1}(V) = f^{-1}(V)$.

The proof of the assertion on algebras is similar to the proof of part (2) of Lemma 9.7. First, write $\mathcal{A} = \operatorname{colim} \mathcal{F}_i$ as a colimit of finitely presented \mathcal{O}_Y -modules using Lemma 9.1. Since $\mathcal{A}|_V$ is a finite type \mathcal{O}_V -module we may and do assume that $\mathcal{F}_i|_V \rightarrow \mathcal{A}|_V$ is surjective for all i . For each i , let \mathcal{J}_i be the kernel of the map

$$\operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \rightarrow \mathcal{J}_{i'}$. We have $\mathcal{A} = \operatorname{colim} \operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent \mathcal{O}_X -algebras $\operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_i)/\mathcal{J}_i$ are finite (as finite type quasi-coherent subalgebras of the integral quasi-coherent \mathcal{O}_Y -algebra \mathcal{A} over \mathcal{O}_X). The restriction of $\operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_i)/\mathcal{J}_i$ to V is $\mathcal{A}|_V$ by the surjectivity above. Hence $\mathcal{J}_i|_V$ is finitely generated as an ideal sheaf of $\operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_i)|_V$ due to the fact that $\mathcal{A}|_V$ is finitely presented as an \mathcal{O}_V -algebra. Write $\mathcal{J}_i = \operatorname{colim} \mathcal{E}_{ik}$ as a colimit of finitely presented \mathcal{O}_X -modules. We may and do assume that $\mathcal{E}_{ik}|_V$ generates $\mathcal{J}_i|_V$ as a sheaf of ideal of $\operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_i)|_V$ by the statement on finite generation above. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \rightarrow \mathcal{E}_{i'k'}$ making

$$\begin{array}{ccc} \mathcal{J}_i & \longrightarrow & \mathcal{J}_{i'} \\ \uparrow & & \uparrow \\ \mathcal{E}_{ik} & \longrightarrow & \mathcal{E}_{i'k'} \end{array}$$

commute. This follows from Cohomology of Spaces, Lemma 5.3. This induces a map

$$\mathcal{A}_{ik} = \operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \operatorname{Sym}_{\mathcal{O}_X}^*(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where (\mathcal{E}_{ik}) denotes the ideal generated by \mathcal{E}_{ik} . The quasi-coherent \mathcal{O}_X -algebras \mathcal{A}_{ki} are of finite presentation and finite for k large enough (see proof of Lemma 9.6). Moreover we have $\mathcal{A}_{ik}|_V = \mathcal{A}|_V$ by construction. Finally, we have

$$\operatorname{colim} \mathcal{A}_{ik} = \operatorname{colim} \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 9.6 and the second equality because \mathcal{A} is the colimit of the modules \mathcal{F}_i . \square

Lemma 16.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S such that $|X|$ has finitely many irreducible components.*

- (1) *There exists a surjective finite morphism $f : Y \rightarrow X$ of finite presentation where Y is a scheme such that f is finite étale over a quasi-compact dense open $U \subset X$,*
- (2) *given a surjective étale morphism $V \rightarrow X$ we may choose $Y \rightarrow X$ such that for every $y \in Y$ there is an open neighbourhood $W \subset Y$ such that $W \rightarrow X$ factors through V .*

Proof. Part (1) is the special case of (2) with $V = X$.

Proof of (2). Let $\pi : Y \rightarrow X$ be as in Decent Spaces, Lemma 9.3 and let $U \subset X$ be a quasi-compact dense open such that $\pi^{-1}(U) \rightarrow U$ is finite étale. Choose a finite affine open covering $Y = \bigcup W_j$ such that $W_j \rightarrow X$ factors through V . We can write $Y = \lim Y_i$ with $\pi_i : Y_i \rightarrow X$ finite and of finite presentation such that $\pi^{-1}(U) \rightarrow \pi_i^{-1}(U)$ is an isomorphism, see Lemma 16.2. For large enough i the algebraic space Y_i is a scheme, see Lemma 5.11. For large enough i we can find affine opens $W_{i,j} \subset Y_i$ whose inverse image in Y recovers W_j , see Lemma 5.7. For

even larger i the morphisms $W_j \rightarrow V$ over X come from morphisms $W_{i,j} \rightarrow U$ over X , see Proposition 3.10. This finishes the proof. \square

Lemma 16.4. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . There exists a $t \geq 0$ and closed subspaces*

$$X \supset Z_0 \supset Z_1 \supset \dots \supset Z_t = \emptyset$$

such that $Z_i \rightarrow X$ is of finite presentation, $Z_0 \subset X$ is a thickening, and for each $i = 0, \dots, t-1$ there exists a scheme Y_i , a surjective, finite, and finitely presented morphism $Y_i \rightarrow Z_i$ which is finite étale over $Z_i \setminus Z_{i+1}$.

Proof. We may view X as an algebraic space over $\text{Spec}(\mathbf{Z})$, see Spaces, Definition 16.2 and Properties of Spaces, Definition 3.1. Thus we may apply Proposition 8.1. It follows that we can find an affine morphism $X \rightarrow X_0$ with X_0 of finite presentation over \mathbf{Z} . If we can prove the lemma for X_0 , then we can pull back the stratification and the morphisms to X and get the result for X ; some details omitted. This reduces us to the case discussed in the next paragraph.

Assume X is of finite presentation over \mathbf{Z} . Then X is Noetherian and $|X|$ is a Noetherian topological space (with finitely many irreducible components) of finite dimension. Hence we may use induction on $\dim(|X|)$. Any finite morphism towards X is of finite presentation, so we can ignore that requirement in the rest of the proof. By Lemma 16.3 there exists a surjective finite morphism $Y \rightarrow X$ which is finite étale over a dense open $U \subset X$. Set $Z_0 = X$ and let $Z_1 \subset X$ be the reduced closed subspace with $|Z_1| = |X| \setminus |U|$. By induction we find an integer $t \geq 0$ and a filtration

$$Z_1 \supset Z_{1,0} \supset Z_{1,1} \supset \dots \supset Z_{1,t} = \emptyset$$

by closed subspaces, where $Z_{1,0} \rightarrow Z_1$ is a thickening and there exist finite surjective morphisms $Y_{1,i} \rightarrow Z_{1,i}$ which are finite étale over $Z_{1,i} \setminus Z_{1,i+1}$. Since Z_1 is reduced, we have $Z_1 = Z_{1,0}$. Hence we can set $Z_i = Z_{1,i-1}$ and $Y_i = Y_{1,i-1}$ for $i \geq 1$ and the lemma is proved. \square

17. Obtaining schemes

A few more techniques to show an algebraic space is a scheme. The first is that we can show there is a minimal closed subspace which is not a scheme.

Lemma 17.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . If X is not a scheme, then there exists a closed subspace $Z \subset X$ such that Z is not a scheme, but every proper closed subspace $Z' \subset Z$ is a scheme.*

Proof. We prove this by Zorn's lemma. Let \mathcal{Z} be the set of closed subspaces Z which are not schemes ordered by inclusion. By assumption \mathcal{Z} contains X , hence is nonempty. If Z_α is a totally ordered subset of \mathcal{Z} , then $Z = \bigcap Z_\alpha$ is in \mathcal{Z} . Namely,

$$Z = \lim Z_\alpha$$

and the transition morphisms are affine. Thus we may apply Lemma 5.11 to see that if Z were a scheme, then so would one of the Z_α . (This works even if $Z = \emptyset$, but note that by Lemma 5.3 this cannot happen.) Thus \mathcal{Z} has minimal elements by Zorn's lemma. \square

Now we can prove a little bit about these minimal non-schemes.

Lemma 17.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Assume that every proper closed subspace $Z \subset X$ is a scheme, but X is not a scheme. Then X is reduced and irreducible.*

Proof. We see that X is reduced by Lemma 15.3. Choose closed subsets $T_1 \subset |X|$ and $T_2 \subset |X|$ such that $|X| = T_1 \cup T_2$. If T_1 and T_2 are proper closed subsets, then the corresponding reduced induced closed subspaces $Z_1, Z_2 \subset X$ (Properties of Spaces, Definition 12.5) are schemes and so is $Z = Z_1 \times_X Z_2 = Z_1 \cap Z_2$ as a closed subscheme of either Z_1 or Z_2 . Observe that the coproduct $Z_1 \amalg_Z Z_2$ exists in the category of schemes, see More on Morphisms, Lemma 67.8. One way to proceed, is to show that $Z_1 \amalg_Z Z_2$ is isomorphic to X , but we cannot use this here as the material on pushouts of algebraic spaces comes later in the theory. Instead we will use Lemma 15.1 to find an affine neighbourhood of every point. Namely, let $x \in |X|$. If $x \notin Z_1$, then x has a neighbourhood which is a scheme, namely, $X \setminus Z_1$. Similarly if $x \notin Z_2$. If $x \in Z = Z_1 \cap Z_2$, then we choose an affine open $U \subset Z_1 \amalg_Z Z_2$ containing x . Then $U_1 = Z_1 \cap U$ and $U_2 = Z_2 \cap U$ are affine opens whose intersections with Z agree. Since $|Z_1| = T_1$ and $|Z_2| = T_2$ are closed subsets of $|X|$ which intersect in $|Z|$, we find an open $W \subset |X|$ with $W \cap T_1 = |U_1|$ and $W \cap T_2 = |U_2|$. Let W denote the corresponding open subspace of X . Then $x \in |W|$ and the morphism $U_1 \amalg U_2 \rightarrow W$ is a surjective finite morphism whose source is an affine scheme. Thus W is an affine scheme by Lemma 15.1. \square

A key point in the following lemma is that we only need to check the condition in the images of points of X .

Lemma 17.3. *Let $f : X \rightarrow S$ be a quasi-compact and quasi-separated morphism from an algebraic space to a scheme S . If for every $x \in |X|$ with image $s = f(x) \in S$ the algebraic space $X \times_S \text{Spec}(\mathcal{O}_{S,s})$ is a scheme, then X is a scheme.*

Proof. Let $x \in |X|$. It suffices to find an open neighbourhood U of $s = f(x)$ such that $X \times_S U$ is a scheme. As $X \times_S \text{Spec}(\mathcal{O}_{S,s})$ is a scheme, then, since $\mathcal{O}_{S,s} = \text{colim } \mathcal{O}_S(U)$ where the colimit is over affine open neighbourhoods of s in S we see that

$$X \times_S \text{Spec}(\mathcal{O}_{S,s}) = \lim X \times_S U$$

By Lemma 5.11 we see that $X \times_S U$ is a scheme for some U . \square

Instead of restricting to local rings as in Lemma 17.3, we can restrict to closed subschemes of the base.

Lemma 17.4. *Let $\varphi : X \rightarrow \text{Spec}(A)$ be a quasi-compact and quasi-separated morphism from an algebraic space to an affine scheme. If X is not a scheme, then there exists an ideal $I \subset A$ such that the base change $X_{A/I}$ is not a scheme, but for every $I \subset I'$, $I \neq I'$ the base change $X_{A/I'}$ is a scheme.*

Proof. We prove this by Zorn's lemma. Let \mathcal{I} be the set of ideals I such that $X_{A/I}$ is not a scheme. By assumption \mathcal{I} contains (0) . If I_α is a chain of ideals in \mathcal{I} , then $I = \bigcup I_\alpha$ is in \mathcal{I} . Namely, $A/I = \text{colim } A/I_\alpha$, hence

$$X_{A/I} = \lim X_{A/I_\alpha}$$

Thus we may apply Lemma 5.11 to see that if $X_{A/I}$ were a scheme, then so would be one of the X_{A/I_α} . Thus \mathcal{I} has maximal elements by Zorn's lemma. \square

18. Glueing in closed fibres

Applying our theory above to the spectrum of a local ring we obtain a few pleasing glueing results for relative algebraic spaces. We first prove a helper lemma (which will be vastly generalized in Bootstrap, Section 11).

Lemma 18.1. *Let $S = U \cup W$ be an open covering of a scheme. Then the functor*

$$FP_S \longrightarrow FP_U \times_{FP_{U \cap W}} FP_W$$

given by base change is an equivalence where FP_T is the category of algebraic spaces of finite presentation over the scheme T .

Proof. First, since $S = U \cup W$ is a Zariski covering, we see that the category of sheaves on $(Sch/S)_{fppf}$ is equivalent to the category of triples $(\mathcal{F}_U, \mathcal{F}_W, \varphi)$ where \mathcal{F}_U is a sheaf on $(Sch/U)_{fppf}$, \mathcal{F}_W is a sheaf on $(Sch/W)_{fppf}$, and

$$\varphi : \mathcal{F}_U|_{(Sch/U \cap W)_{fppf}} \longrightarrow \mathcal{F}_W|_{(Sch/U \cap W)_{fppf}}$$

is an isomorphism. See Sites, Lemma 26.5 (note that no other gluing data are necessary because $U \times_S U = U$, $W \times_S W = W$ and that the cocycle condition is automatic for the same reason). Now, if the sheaf \mathcal{F} on $(Sch/S)_{fppf}$ maps to $(\mathcal{F}_U, \mathcal{F}_W, \varphi)$ via this equivalence, then \mathcal{F} is an algebraic space if and only if \mathcal{F}_U and \mathcal{F}_W are algebraic spaces. This follows immediately from Algebraic Spaces, Lemma 8.5 as $\mathcal{F}_U \rightarrow \mathcal{F}$ and $\mathcal{F}_W \rightarrow \mathcal{F}$ are representable by open immersions and cover \mathcal{F} . Finally, in this case the algebraic space \mathcal{F} is of finite presentation over S if and only if \mathcal{F}_U is of finite presentation over U and \mathcal{F}_W is of finite presentation over W by Morphisms of Spaces, Lemmas 8.8, 4.12, and 28.4. \square

Lemma 18.2. *Let S be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \rightarrow S$ is quasi-compact. With $V = \text{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ there is an equivalence of categories*

$$FP_S \longrightarrow FP_U \times_{FP_V} FP_{\text{Spec}(\mathcal{O}_{S,s})}$$

where FP_T is the category of algebraic spaces of finite presentation over T .

Proof. Let $W \subset S$ be an open neighbourhood of s . The functor

$$FP_S \rightarrow FP_U \times_{FP_{W \setminus \{s\}}} FP_W$$

is an equivalence of categories by Lemma 18.1. We have $\mathcal{O}_{S,s} = \text{colim } \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s . Hence $\text{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s . Thus the category of algebraic spaces of finite presentation over $\text{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of algebraic spaces of finite presentation over W where W runs over the affine open neighbourhoods of s , see Lemma 7.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \rightarrow S$ is quasi-compact. Hence $V = \lim W \cap U = \lim W \setminus \{s\}$ is a limit of quasi-compact and quasi-separated schemes (see Limits, Lemma 2.2). Thus also the category of algebraic spaces of finite presentation over V is the limit of the categories of algebraic spaces of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s . The lemma follows formally from a combination of these results. \square

Lemma 18.3. *Let S be a scheme. Let $U \subset S$ be a retrocompact open. Let $s \in S$ be a point in the complement of U . With $V = \operatorname{Spec}(\mathcal{O}_{S,s}) \cap U$ there is an equivalence of categories*

$$\operatorname{colim}_{s \in U' \supset U \text{ open}} FP_{U'} \longrightarrow FP_U \times_{FP_V} FP_{\operatorname{Spec}(\mathcal{O}_{S,s})}$$

where FP_T is the category of algebraic spaces of finite presentation over T .

Proof. Let $W \subset S$ be an open neighbourhood of s . By Lemma 18.1 the functor

$$FP_{U \cup W} \longrightarrow FP_U \times_{FP_{U \cap W}} FP_W$$

is an equivalence of categories. We have $\mathcal{O}_{S,s} = \operatorname{colim} \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s . Hence $\operatorname{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s . Thus the category of algebraic spaces of finite presentation over $\operatorname{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of algebraic spaces of finite presentation over W where W runs over the affine open neighbourhoods of s , see Lemma 7.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \rightarrow S$ is quasi-compact. Hence $V = \lim W \cap U$ is a limit of quasi-compact and quasi-separated schemes (see Limits, Lemma 2.2). Thus also the category of algebraic spaces of finite presentation over V is the limit of the categories of algebraic spaces of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s . The lemma follows formally from a combination of these results. \square

Lemma 18.4. *Let S be a scheme. Let $s_1, \dots, s_n \in S$ be pairwise distinct closed points such that $U = S \setminus \{s_1, \dots, s_n\} \rightarrow S$ is quasi-compact. With $S_i = \operatorname{Spec}(\mathcal{O}_{S,s_i})$ and $U_i = S_i \setminus \{s_i\}$ there is an equivalence of categories*

$$FP_S \longrightarrow FP_U \times_{(FP_{U_1} \times \dots \times FP_{U_n})} (FP_{S_1} \times \dots \times FP_{S_n})$$

where FP_T is the category of algebraic spaces of finite presentation over T .

Proof. For $n = 1$ this is Lemma 18.2. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $f_i : X_i \rightarrow S_i$ are objects of FP_{S_i} and $f : X \rightarrow U$ is an object of FP_U and we're given isomorphisms $X_i \times_{S_i} U_i = X \times_U U_i$. By Lemma 18.2 we can find a morphism $f' : X' \rightarrow U' = S \setminus \{s_1, \dots, s_{n-1}\}$ which is of finite presentation, which is isomorphic to X_i over S_i , which is isomorphic to X over U , and these isomorphisms are compatible with the given isomorphism $X_i \times_{S_i} U_i = X \times_U U_i$. Then we can apply induction to $f_i : X_i \rightarrow S_i$, $i \leq n-1$, $f' : X' \rightarrow U'$, and the induced isomorphisms $X_i \times_{S_i} U_i = X' \times_{U'} U_i$, $i \leq n-1$. This shows essential surjectivity. We omit the proof of fully faithfulness. \square

19. Application to modifications

Using limits we can describe the category of modifications of a decent algebraic space over a closed point in terms of the henselian local ring.

Lemma 19.1. *Let S be a scheme. Consider a separated étale morphism $f : V \rightarrow W$ of algebraic spaces over S . Assume there exists a closed subspace $T \subset W$ such that $f^{-1}T \rightarrow T$ is an isomorphism. Then, with $W^0 = W \setminus T$ and $V^0 = f^{-1}W^0$ the base change functor*

$$\left\{ \begin{array}{l} g : X \rightarrow W \text{ morphism of algebraic spaces} \\ g^{-1}(W^0) \rightarrow W^0 \text{ is an isomorphism} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} h : Y \rightarrow V \text{ morphism of algebraic spaces} \\ h^{-1}(V^0) \rightarrow V^0 \text{ is an isomorphism} \end{array} \right\}$$

is an equivalence of categories.

Proof. Since $V \rightarrow W$ is separated we see that $V \times_W V = \Delta(V) \amalg U$ for some open and closed subspace U of $V \times_W V$. By the assumption that $f^{-1}T \rightarrow T$ is an isomorphism we see that $U \times_W T = \emptyset$, i.e., the two projections $U \rightarrow V$ maps into V^0 .

Given $h : Y \rightarrow V$ in the right hand category, consider the contravariant functor X on $(Sch/S)_{fppf}$ defined by the rule

$$X(T) = \{(w, y) \mid w : T \rightarrow W, y : T \times_{w, W} V \rightarrow Y \text{ morphism over } V\}$$

Denote $g : X \rightarrow W$ the map sending $(w, y) \in X(T)$ to $w \in W(T)$. Since $h^{-1}V^0 \rightarrow V^0$ is an isomorphism, we see that if $w : T \rightarrow W$ maps into W^0 , then there is a unique choice for h . In other words $X \times_{g, W} W^0 = W^0$. On the other hand, consider a T -valued point (w, y, v) of $X \times_{g, W, f} V$. Then $w = f \circ v$ and

$$y : T \times_{f \circ v, W} V \longrightarrow V$$

is a morphism over V . Consider the morphism

$$T \times_{f \circ v, W} V \xrightarrow{(v, \text{id}_V)} V \times_W V = V \amalg U$$

The inverse image of V is T embedded via $(\text{id}_T, v) : T \rightarrow T \times_{f \circ v, W} V$. The composition $y' = y \circ (\text{id}_T, v) : T \rightarrow Y$ is a morphism with $v = h \circ y'$ which determines y because the restriction of y to the other part is uniquely determined as U maps into V^0 by the second projection. It follows that $X \times_{g, W, f} V \rightarrow Y$, $(w, y, v) \mapsto y'$ is an isomorphism.

Thus if we can show that X is an algebraic space, then we are done. Since $V \rightarrow W$ is separated and étale it is representable by Morphisms of Spaces, Lemma 51.1 (and Morphisms of Spaces, Lemma 39.5). Of course $W^0 \rightarrow W$ is representable and étale as it is an open immersion. Thus

$$W^0 \amalg Y = X \times_{g, W} W^0 \amalg X \times_{g, W, f} V = X \times_{g, W} (W^0 \amalg V) \longrightarrow X$$

is representable, surjective, and étale by Spaces, Lemmas 3.3 and 5.5. Thus X is an algebraic space by Spaces, Lemma 11.2. \square

Lemma 19.2. *Notation and assumptions as in Lemma 19.1. Let $g : X \rightarrow W$ correspond to $h : Y \rightarrow V$ via the equivalence. Then g is quasi-compact, quasi-separated, separated, locally of finite presentation, of finite presentation, locally of finite type, of finite type, proper, integral, finite, and add more here if and only if h is so.*

Proof. If g is quasi-compact, quasi-separated, separated, locally of finite presentation, of finite presentation, locally of finite type, of finite type, proper, finite, so is h as a base change of g by Morphisms of Spaces, Lemmas 8.4, 4.4, 28.3, 23.3, 40.3, 45.5. Conversely, let P be a property of morphisms of algebraic spaces which is étale local on the base and which holds for the identity morphism of any algebraic space. Since $\{W^0 \rightarrow W, V \rightarrow W\}$ is an étale covering, to prove that g has P it suffices to show that h has P . Thus we conclude using Morphisms of Spaces, Lemmas 8.8, 4.12, 28.4, 23.4, 40.2, 45.3. \square

Lemma 19.3. *Let S be a scheme. Let X be a decent algebraic space over S . Let $x \in |X|$ be a closed point such that $U = X \setminus \{x\} \rightarrow X$ is quasi-compact. With $V = \operatorname{Spec}(\mathcal{O}_{X,x}^h) \setminus \{\mathfrak{m}_x^h\}$ the base change functor*

$$\left\{ \begin{array}{l} f : Y \rightarrow X \text{ of finite presentation} \\ f^{-1}(U) \rightarrow U \text{ is an isomorphism} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} g : Y \rightarrow \operatorname{Spec}(\mathcal{O}_{X,x}^h) \text{ of finite presentation} \\ g^{-1}(V) \rightarrow V \text{ is an isomorphism} \end{array} \right\}$$

is an equivalence of categories.

Proof. Let $a : (W, w) \rightarrow (X, x)$ be an elementary étale neighbourhood of x with W affine as in Decent Spaces, Lemma 11.4. Since x is a closed point of X and w is the unique point of W lying over x , we see that w is a closed point of W . Since a is étale and identifies residue fields at x and w , it follows that a induces an isomorphism $a^{-1}x \rightarrow x$ (as closed subspaces of X and W). Thus we may apply Lemma 19.1 and 19.2 to reduce the problem to the case where X is an affine scheme.

Assume X is an affine scheme. Recall that $\mathcal{O}_{X,x}^h$ is the colimit of $\Gamma(U, \mathcal{O}_U)$ over affine elementary étale neighbourhoods $(U, u) \rightarrow (X, x)$. Recall that the category of these neighbourhoods is cofiltered, see Decent Spaces, Lemma 11.6 or More on Morphisms, Lemma 35.4. Then $\operatorname{Spec}(\mathcal{O}_{X,x}^h) = \lim U$ and $V = \lim U \setminus \{u\}$ (Lemma 4.1) where the limits are taken over the same category. Thus by Lemma 7.1 The category on the right is the colimit of the categories for the pairs (U, u) . And by the material in the first paragraph, each of these categories is equivalent to the category for the pair (X, x) . This finishes the proof. \square

20. Universally closed morphisms

In this section we discuss when a quasi-compact (but not necessarily separated) morphism is universally closed. We first prove a lemma which will allow us to check universal closedness after a base change which is locally of finite presentation.

Lemma 20.1. *Let S be a scheme. Let $f : X \rightarrow Y$ and $g : Z \rightarrow Y$ be morphisms of algebraic spaces over S . Let $z \in |Z|$ and let $T \subset |X \times_Y Z|$ be a closed subset with $z \notin \operatorname{Im}(T \rightarrow |Z|)$. If f is quasi-compact, then there exists an étale neighbourhood $(V, v) \rightarrow (Z, z)$, a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{a} & Z' \\ \downarrow & & \downarrow b \\ Z & \xrightarrow{g} & Y, \end{array}$$

and a closed subset $T' \subset |X \times_Y Z'|$ such that

- (1) *the morphism $b : Z' \rightarrow Y$ is locally of finite presentation,*
- (2) *with $z' = a(v)$ we have $z' \notin \operatorname{Im}(T' \rightarrow |Z'|)$, and*
- (3) *the inverse image of T in $|X \times_Y V|$ maps into T' via $|X \times_Y V| \rightarrow |X \times_Y Z'|$.*

Moreover, we may assume V and Z' are affine schemes and if Z is a scheme we may assume V is an affine open neighbourhood of z .

Proof. We will deduce this from the corresponding result for morphisms of schemes. Let $y \in |Y|$ be the image of z . First we choose an affine étale neighbourhood $(U, u) \rightarrow (Y, y)$ and then we choose an affine étale neighbourhood $(V, v) \rightarrow (Z, z)$ such that the morphism $V \rightarrow Y$ factors through U . Then we may replace

- (1) $X \rightarrow Y$ by $X \times_Y U \rightarrow U$,

- (2) $Z \rightarrow Y$ by $V \rightarrow U$,
- (3) z by v , and
- (4) T by its inverse image in $|(X \times_Y U) \times_U V| = |X \times_Y V|$.

In fact, below we will show that after replacing V by an affine open neighbourhood of v there will be a morphism $a : V \rightarrow Z'$ for some $Z' \rightarrow U$ of finite presentation and a closed subset T' of $|(X \times_Y U) \times_U Z'| = |X \times_Y Z'|$ such that T maps into T' and $a(v) \notin \text{Im}(T' \rightarrow |Z'|)$. Thus we may and do assume that Z and Y are affine schemes with the proviso that we need to find a solution where V is an open neighbourhood of z .

Since f is quasi-compact and Y is affine, the algebraic space X is quasi-compact. Choose an affine scheme W and a surjective étale morphism $W \rightarrow X$. Let $T_W \subset |W \times_Y Z|$ be the inverse image of T . Then z is not in the image of T_W . By the schemes case (Limits, Lemma 14.1) we can find an open neighbourhood $V \subset Z$ of z a commutative diagram of schemes

$$\begin{array}{ccc} V & \xrightarrow{a} & Z' \\ \downarrow & & \downarrow b \\ Z & \xrightarrow{g} & Y, \end{array}$$

and a closed subset $T' \subset |W \times_Y Z'|$ such that

- (1) the morphism $b : Z' \rightarrow Y$ is locally of finite presentation,
- (2) with $z' = a(z)$ we have $z' \notin \text{Im}(T' \rightarrow |Z'|)$, and
- (3) $T_1 = T_W \cap |W \times_Y V|$ maps into T' via $|W \times_Y V| \rightarrow |W \times_Y Z'|$.

The commutative diagram

$$\begin{array}{ccccc} W \times_Y Z & \longleftarrow & W \times_Y V & \xrightarrow{a_1} & W \times_Y Z' \\ \downarrow & & \downarrow c & & \downarrow q \\ X \times_Y Z & \longleftarrow & X \times_Y V & \xrightarrow{a_2} & X \times_Y Z' \end{array}$$

has cartesian squares and the vertical maps are, surjective, étale and a fortiori open. Looking at the left hand square we see that $T_1 = T_W \cap |W \times_Y V|$ is the inverse image of $T_2 = T \cap |X \times_Y V|$ by c . By Properties of Spaces, Lemma 4.3 we get $a_1(T_1) = q^{-1}(a_2(T_2))$. By Topology, Lemma 6.4 we get

$$q^{-1}(\overline{a_2(T_2)}) = \overline{q^{-1}(a_2(T_2))} = \overline{a_1(T_1)} \subset T'$$

As q is surjective the image of $\overline{a_2(T_2)} \rightarrow |Z'|$ does not contain z' since the same is true for T' . Thus we can take the diagram with Z', V, a, b above and the closed subset $\overline{a_2(T_2)} \subset |X \times_Y Z'|$ as a solution to the problem posed by the lemma. \square

Lemma 20.2. *Let S be a scheme. Let $f : X \rightarrow Y$ be a quasi-compact morphism of algebraic spaces over S . The following are equivalent*

- (1) f is universally closed,
- (2) for every morphism $Z \rightarrow Y$ which is locally of finite presentation the map $|X \times_Y Z| \rightarrow |Z|$ is closed, and
- (3) there exists a scheme V and a surjective étale morphism $V \rightarrow Y$ such that $|\mathbf{A}^n \times (X \times_Y V)| \rightarrow |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$.

Proof. It is clear that (1) implies (2). Suppose that $|X \times_Y Z| \rightarrow |Z|$ is not closed for some morphism of algebraic spaces $Z \rightarrow Y$ over S . This means that there exists some closed subset $T \subset |X \times_Y Z|$ such that $\text{Im}(T \rightarrow |Z|)$ is not closed. Pick $z \in |Z|$ in the closure of the image of T but not in the image. Apply Lemma 20.1. We find an étale neighbourhood $(V, v) \rightarrow (Z, z)$, a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & Z' \\ \downarrow & & \downarrow b \\ Z & \xrightarrow{g} & Y, \end{array}$$

and a closed subset $T' \subset |X \times_Y Z'|$ such that

- (1) the morphism $b : Z' \rightarrow Y$ is locally of finite presentation,
- (2) with $z' = a(v)$ we have $z' \notin \text{Im}(T' \rightarrow |Z'|)$, and
- (3) the inverse image of T in $|X \times_Y V|$ maps into T' via $|X \times_Y V| \rightarrow |X \times_Y Z'|$.

We claim that z' is in the closure of $\text{Im}(T' \rightarrow |Z'|)$ which implies that $|X \times_Y Z'| \rightarrow |Z'|$ is not closed. The claim shows that (2) implies (1). To see the claim is true we contemplate following commutative diagram

$$\begin{array}{ccccc} X \times_Y Z & \longleftarrow & X \times_Y V & \longrightarrow & X \times_Y Z' \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longleftarrow & V & \xrightarrow{a} & Z' \end{array}$$

Let $T_V \subset |X \times_Y V|$ be the inverse image of T . By Properties of Spaces, Lemma 4.3 the image of T_V in $|V|$ is the inverse image of the image of T in $|Z|$. Then since z is in the closure of the image of $T \rightarrow |Z|$ and since $|V| \rightarrow |Z|$ is open, we see that v is in the closure of the image of $T_V \rightarrow |V|$. Since the image of T_V in $|X \times_Y Z'|$ is contained in $|T'|$ it follows immediately that $z' = a(v)$ is in the closure of the image of T' .

It is clear that (1) implies (3). Let $V \rightarrow Y$ be as in (3). If we can show that $X \times_Y V \rightarrow V$ is universally closed, then f is universally closed by Morphisms of Spaces, Lemma 9.5. Thus it suffices to show that $f : X \rightarrow Y$ satisfies (2) if f is a quasi-compact morphism of algebraic spaces, Y is a scheme, and $|\mathbf{A}^n \times X| \rightarrow |\mathbf{A}^n \times Y|$ is closed for all n . Let $Z \rightarrow Y$ be locally of finite presentation. We have to show the map $|X \times_Y Z| \rightarrow |Z|$ is closed. This question is étale local on Z hence we may assume Z is affine (some details omitted). Since Y is a scheme, Z is affine, and $Z \rightarrow Y$ is locally of finite presentation we can find an immersion $Z \rightarrow \mathbf{A}^n \times Y$, see Morphisms, Lemma 39.2. Consider the cartesian diagram

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & \mathbf{A}^n \times X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathbf{A}^n \times Y \end{array} \quad \begin{array}{c} \text{inducing the} \\ \text{cartesian square} \end{array} \quad \begin{array}{ccc} |X \times_Y Z| & \longrightarrow & |\mathbf{A}^n \times X| \\ \downarrow & & \downarrow \\ |Z| & \longrightarrow & |\mathbf{A}^n \times Y| \end{array}$$

of topological spaces whose horizontal arrows are homeomorphisms onto locally closed subsets (Properties of Spaces, Lemma 12.1). Thus every closed subset T of $|X \times_Y Z|$ is the pullback of a closed subset T' of $|\mathbf{A}^n \times Y|$. Since the assumption is that the image of T' in $|\mathbf{A}^n \times X|$ is closed we conclude that the image of T in $|Z|$ is closed as desired. \square

Lemma 20.3. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f separated and of finite type. The following are equivalent*

- (1) *The morphism f is proper.*
- (2) *For any morphism $Y \rightarrow Z$ which is locally of finite presentation the map $|X \times_Y Z| \rightarrow |Z|$ is closed, and*
- (3) *there exists a scheme V and a surjective étale morphism $V \rightarrow Y$ such that $|\mathbf{A}^n \times (X \times_Y V)| \rightarrow |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$.*

Proof. In view of the fact that a proper morphism is the same thing as a separated, finite type, and universally closed morphism, this lemma is a special case of Lemma 20.2. \square

21. Noetherian valuative criterion

We have already proved some results in Cohomology of Spaces, Section 19. The corresponding section for schemes is Limits, Section 15.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

Lemma 21.1. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f finite type and Y locally Noetherian. Let $y \in |Y|$ be a point in the closure of the image of $|f|$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y . Moreover, we can assume that the point $x \in |X|$ corresponding to $\mathrm{Spec}(K) \rightarrow X$ is a codimension 0 point³ and that K is the residue field of a point on a scheme étale over X .

Proof. Choose an affine scheme V , a point $v \in V$ and an étale morphism $V \rightarrow Y$ mapping v to y . The map $|V| \rightarrow |Y|$ is open and by Properties of Spaces, Lemma 4.3 the image of $|X \times_Y V| \rightarrow |V|$ is the inverse image of the image of $|f|$. We conclude that the point v is in the closure of the image of $|X \times_Y V| \rightarrow |V|$. If we prove the lemma for $X \times_Y V \rightarrow V$ and the point v , then the lemma follows for f and y . In this way we reduce to the situation described in the next paragraph.

Assume we have $f : X \rightarrow Y$ and $y \in |Y|$ as in the lemma where Y is an affine scheme. Since f is quasi-compact, we conclude that X is quasi-compact. Hence we can choose an affine scheme W and a surjective étale morphism $W \rightarrow X$. Then the image of $|f|$ is the same as the image of $W \rightarrow Y$. In this way we reduce to the case of schemes which is Limits, Lemma 15.1. \square

First we state the result concerning separation. We will often use solid commutative diagrams of morphisms of algebraic spaces over a base scheme S having the following

³See discussion in Properties of Spaces, Section 11.

shape

$$(21.1.1) \quad \begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

with A a valuation ring and K its field of fractions.

Lemma 21.2. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f is quasi-separated and locally of finite type and Y is locally Noetherian. The following are equivalent:*

- (1) *The morphism f is separated.*
- (2) *For any diagram (21.1.1) there is at most one dotted arrow.*
- (3) *For all diagrams (21.1.1) with A a discrete valuation ring there is at most one dotted arrow.*
- (4) *For all diagrams (21.1.1) where A is a discrete valuation ring and where the image of $\mathrm{Spec}(K) \rightarrow X$ is a point of codimension 0 on X there is at most one dotted arrow.*

Proof. We have (1) \Rightarrow (2) by Morphisms of Spaces, Lemma 43.1. The implications (2) \Rightarrow (3) and (3) \Rightarrow (4) are immediate. It remains to show (4) implies (1).

Assume (4). We have to show that the diagonal $\Delta : X \rightarrow X \times_Y X$ is a closed immersion. We already know Δ is representable, separated, a monomorphism, and locally of finite type, see Morphisms of Spaces, Lemma 4.1. Choose an affine scheme U and an étale morphism $U \rightarrow X \times_Y X$. Set $V = X \times_{\Delta, X \times_Y X} U$. It suffices to show that $V \rightarrow U$ is a closed immersion (Morphisms of Spaces, Lemma 12.1). Since $X \times_Y X$ is locally of finite type over Y we see that U is Noetherian (use Morphisms of Spaces, Lemmas 23.2, 23.3, and 23.5). Note that V is a scheme as Δ is representable. Also, V is quasi-compact because f is quasi-separated. Hence $V \rightarrow U$ is separated and of finite type. Consider a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & V \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & U \end{array}$$

of morphisms of schemes where A is a discrete valuation ring with fraction field K and where K is the residue field of a generic point of the Noetherian scheme V . Since $V \rightarrow X$ is étale (as a base change of the étale morphism $U \rightarrow X \times_Y X$) we see that the image of $\mathrm{Spec}(K) \rightarrow V \rightarrow X$ is a point of codimension 0, see Properties of Spaces, Section 10. We can interpret the composition $\mathrm{Spec}(A) \rightarrow U \rightarrow X \times_Y X$ as a pair of morphisms $a, b : \mathrm{Spec}(A) \rightarrow X$ agreeing as morphisms into Y and equal when restricted to $\mathrm{Spec}(K)$ and that this restriction maps to a point of codimension 0. Hence our assumption (4) guarantees $a = b$ and we find the dotted arrow in the diagram. By Limits, Lemma 15.3 we conclude that $V \rightarrow U$ is proper. In other words, Δ is proper. Since Δ is a monomorphism, we find that Δ is a closed immersion (Étale Morphisms, Lemma 7.2) as desired. \square

Lemma 21.3. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f is quasi-separated and of finite type and Y is locally Noetherian. The following are equivalent:*

- (1) *f is proper,*
- (2) *f satisfies the valuative criterion, see Morphisms of Spaces, Definition 41.1,*
- (3) *for any diagram (21.1.1) there exists exactly one dotted arrow,*
- (4) *for all diagrams (21.1.1) with A a discrete valuation ring there exists exactly one dotted arrow, and*
- (5) *for all diagrams (21.1.1) where A is a discrete valuation ring and where the image of $\mathrm{Spec}(K) \rightarrow X$ is a point of codimension 0 on X there exists exactly one dotted arrow⁴.*

Proof. We have (1) \Leftrightarrow (2) \Leftrightarrow (3) by Morphisms of Spaces, Lemma 44.1. It is clear that (3) \Rightarrow (4) \Rightarrow (5). To finish the proof we will now show (5) implies (1).

Assume (5). By Lemma 21.2 we see that f is separated. To finish the proof it suffices to show that f is universally closed. Let $V \rightarrow Y$ be an étale morphism where V is an affine scheme. It suffices to show that the base change $V \times_Y X \rightarrow V$ is universally closed, see Morphisms of Spaces, Lemma 9.5. Let

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & V \times_Y X & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow & \nearrow \text{dotted} & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & V & \longrightarrow & Y \end{array}$$

of algebraic spaces over S be a commutative diagram where A is a discrete valuation ring with fraction field K and where $\mathrm{Spec}(K) \rightarrow V \times_Y X$ maps to a point of codimension 0 of the algebraic space $V \times_Y X$. Since $V \times_Y X \rightarrow X$ is étale it follows that the image of $\mathrm{Spec}(K) \rightarrow X$ is a point of codimension 0 of X . Thus by (5) we obtain the longer of the two dotted arrows fitting into the diagram. Then of course we obtain the shorter one as well. It follows that our assumptions hold for the morphism $V \times_Y X \rightarrow V$ and we reduce to the case discussed in the next paragraph.

Assume Y is a Noetherian affine scheme. In this case X is a separated Noetherian algebraic space (we already know f is separated) of finite type over Y . (In particular, the algebraic space X has a dense open subspace which is a scheme by Properties of Spaces, Proposition 13.3 although strictly speaking we will not need this.) Choose a quasi-projective scheme X' over Y and a proper surjective morphism $X' \rightarrow X$ as in the weak form of Chow's lemma (Cohomology of Spaces, Lemma 18.1). We may replace X' by the disjoint union of the irreducible components which dominate an irreducible component of X ; details omitted. In particular, we may assume that generic points of the scheme X' map to points of codimension 0 of X (in this case these are exactly the generic points of X). We claim that $X' \rightarrow Y$ is proper. The claim implies X is proper over Y by Morphisms of Spaces, Lemma 40.7. To prove this, according to Limits, Lemma 15.3 it suffices to prove

⁴There is a sharper formulation where in the existence part one only requires the dotted arrow exists after an extension of discrete valuation rings.

that in every solid commutative diagram

$$\begin{array}{ccccc}
 \mathrm{Spec}(K) & \longrightarrow & X' & \longrightarrow & X \\
 \downarrow & & \nearrow a & \nearrow b & \downarrow \\
 \mathrm{Spec}(A) & \longrightarrow & & & Y
 \end{array}$$

where A is a dvr with fraction field K and where K is the residue field of a generic point of X' we can find the dotted arrow a (we already know uniqueness as X' is separated). By assumption (5) we can find the dotted arrow b . Then the morphism $X' \times_{X,b} \mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A)$ is a proper morphism of schemes and by the valuative criterion for morphisms of schemes we can lift b to the desired morphism a . \square

Lemma 21.4. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume Y is locally Noetherian and f is of finite type. Then the following are equivalent*

- (1) f is universally closed,
- (2) f satisfies the existence part of the valuative criterion,
- (3) there exists a scheme V and a surjective étale morphism $V \rightarrow Y$ such that $|\mathbf{A}^n \times X \times_Y V| \rightarrow |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$,
- (4) for all diagrams (21.1.1) with A a discrete valuation ring there there exists a finite separable extension K'/K of fields, a discrete valuation ring $A' \subset K'$ dominating A , and a morphism $\mathrm{Spec}(A') \rightarrow X$ such that the following diagram commutes

$$\begin{array}{ccccc}
 \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\
 \downarrow & & \nearrow & & \downarrow \\
 \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y
 \end{array}$$

- (5) for all diagrams (21.1.1) with A a discrete valuation ring there there exists a field extension K'/K , a valuation ring $A' \subset K'$ dominating A , and a morphism $\mathrm{Spec}(A') \rightarrow X$ such that the following diagram commutes

$$\begin{array}{ccccc}
 \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\
 \downarrow & & \nearrow & & \downarrow \\
 \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y
 \end{array}$$

Proof. Parts (1), (2), and (3) are equivalent by Lemma 20.2 and Morphisms of Spaces, Lemma 42.1. These equivalent conditions imply part (4) as Morphisms of Spaces, Lemma 41.3 tells us that we may always choose K'/K finite separable in the existence part of the valuative criterion and this automatically forces A' to be a discrete valuation ring by Krull-Akizuki (Algebra, Lemma 119.12). The implication (4) \Rightarrow (5) is immediate. In the rest of the proof we show that (5) implies (1).

Assume (5). Chose an affine scheme V and an étale morphism $V \rightarrow Y$. It suffices to show that the base change of f to V is universally closed, see Morphisms of Spaces, Lemma 9.5. Exactly as in the proof of Lemma 21.3 we see that assumption (5) is inherited by this base change; details omitted. This reduces us to the case discussed in the next paragraph.

Assume Y is a Noetherian affine scheme and we have (5). To prove that f is universally closed it suffices to show that $|X \times \mathbf{A}^n| \rightarrow |Y \times \mathbf{A}^n|$ is closed for all n (by the discussion above). Since assumption (5) is inherited by the product morphism $X \times \mathbf{A}^n \rightarrow Y \times \mathbf{A}^n$ (details omitted) we reduce to proving that $|X| \rightarrow |Y|$ is closed.

Assume Y is a Noetherian affine scheme and we have (5). Let $T \subset |X|$ be a closed subset. We have to show that the image of T in $|Y|$ is closed. We may replace X by the reduced induced closed subspace structure on T ; we omit the verification that property (5) is preserved by this replacement. Thus we reduce to proving that the image of $|X| \rightarrow |Y|$ is closed.

Let $y \in |Y|$ be a point in the closure of the image of $|X| \rightarrow |Y|$. By Lemma 21.1 we may choose a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y . It follows immediately from property (5) that y is in the image of $|X| \rightarrow |Y|$ and the proof is complete. \square

22. Refined Noetherian valuative criteria

This section is the analogue of Limits, Section 16. One usually does not have to consider all possible diagrams with valuation rings when checking valuative criteria.

Lemma 22.1. *Let S be a scheme. Let $f : X \rightarrow Y$ and $h : U \rightarrow X$ be morphisms of algebraic spaces over S . Assume that Y is locally Noetherian, that f and h are of finite type, that f is separated, and that the image of $|h| : |U| \rightarrow |X|$ is dense in $|X|$. If given any commutative solid diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow \text{dotted} & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & & & Y \end{array}$$

where A is a discrete valuation ring with field of fractions K , there exists a dotted arrow making the diagram commute, then f is proper.

Proof. It suffices to prove that f is universally closed. Let $V \rightarrow Y$ be an étale morphism where V is an affine scheme. By Morphisms of Spaces, Lemma 9.5 it suffices to prove that the base change $X \times_Y V \rightarrow V$ is universally closed. By Properties of Spaces, Lemma 4.3 the image I of $|U \times_Y V| \rightarrow |X \times_Y V|$ is the inverse image of the image of $|h|$. Since $|X \times_Y V| \rightarrow |X|$ is open (Properties of Spaces, Lemma 16.7) we conclude that I is dense in $|X \times_Y V|$. Therefore the assumptions of the lemma are satisfied for the morphisms $U \times_Y V \rightarrow X \times_Y V \rightarrow V$. Hence we may assume Y is an affine scheme.

Assume Y is an affine scheme. Then U is quasi-compact. Choose an affine scheme and a surjective étale morphism $W \rightarrow U$. Then we may and do replace U by W and assume that U is affine. By the weak version of Chow's lemma (Cohomology of

(Spaces, Lemma 18.1) we can choose a surjective proper morphism $X' \rightarrow X$ where X' is a scheme. Then $U' = X' \times_X U$ is a scheme and $U' \rightarrow X'$ is of finite type. We may replace X' by the scheme theoretic image of $h' : U' \rightarrow X'$ and hence $h'(U')$ is dense in X' . We claim that for every diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U' & \xrightarrow{h} & X' \\ \downarrow & & & \nearrow \text{dotted} & \downarrow f' \\ \mathrm{Spec}(A) & \longrightarrow & & & Y \end{array}$$

where A is a discrete valuation ring with field of fractions K , there exists a dotted arrow making the diagram commute. Namely, we first get an arrow $\mathrm{Spec}(A) \rightarrow X$ by the assumption of the lemma and then we lift this to an arrow $\mathrm{Spec}(A) \rightarrow X'$ using the valuative criterion for properness (Morphisms of Spaces, Lemma 44.1). The morphism $X' \rightarrow Y$ is separated as a composition of a proper and a separated morphism. Thus by the case of schemes the morphism $X' \rightarrow Y$ is proper (Limits, Lemma 16.1). By Morphisms of Spaces, Lemma 40.7 we conclude that $X \rightarrow Y$ is proper. \square

Lemma 22.2. *Let S be a scheme. Let $f : X \rightarrow Y$ and $h : U \rightarrow X$ be morphisms of algebraic spaces over S . Assume that Y is locally Noetherian, that f is locally of finite type and quasi-separated, that h is of finite type, and that the image of $|h| : |U| \rightarrow |X|$ is dense in $|X|$. If given any commutative solid diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow \text{dotted} & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & & & Y \end{array}$$

where A is a discrete valuation ring with field of fractions K , there exists at most one dotted arrow making the diagram commute, then f is separated.

Proof. We will apply Lemma 22.1 to the morphisms $U \rightarrow X$ and $\Delta : X \rightarrow X \times_Y X$. We check the conditions. Observe that Δ is quasi-compact because f is quasi-separated. Of course Δ is locally of finite type and separated (true for any diagonal morphism). Finally, suppose given a commutative solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow \text{dotted} & \downarrow \Delta \\ \mathrm{Spec}(A) & \xrightarrow{(a,b)} & & & X \times_Y X \end{array}$$

where A is a discrete valuation ring with field of fractions K . Then a and b give two dotted arrows in the diagram of the lemma and have to be equal. Hence as dotted arrow we can use $a = b$ which gives existence. This finishes the proof. \square

Lemma 22.3. *Let S be a scheme. Let $f : X \rightarrow Y$ and $h : U \rightarrow X$ be morphisms of algebraic spaces over S . Assume that Y is locally Noetherian, that f and h are of finite type, that f is quasi-separated, and that $h(U)$ is dense in X . If given any*

commutative solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow \text{dotted} & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & & & Y \end{array}$$

where A is a discrete valuation ring with field of fractions K , there exists a unique dotted arrow making the diagram commute, then f is proper.

Proof. Combine Lemmas 22.2 and 22.1. \square

23. Descending finite type spaces

This section continues the theme of Section 11 in the spirit of the results discussed in Section 7. It is also the analogue of Limits, Section 22 for algebraic spaces.

Situation 23.1. Let S be a scheme, for example $\mathrm{Spec}(\mathbf{Z})$. Let $B = \lim_{i \in I} B_i$ be the limit of a directed inverse system of Noetherian spaces over S with affine transition morphisms $B_{i'} \rightarrow B_i$ for $i' \geq i$.

Lemma 23.2. *In Situation 23.1. Let $X \rightarrow B$ be a quasi-separated and finite type morphism of algebraic spaces. Then there exists an $i \in I$ and a diagram*

$$(23.2.1) \quad \begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_i \end{array}$$

such that $W \rightarrow B_i$ is of finite type and such that the induced morphism $X \rightarrow B \times_{B_i} W$ is a closed immersion.

Proof. By Lemma 11.6 we can find a closed immersion $X \rightarrow X'$ over B where X' is an algebraic space of finite presentation over B . By Lemma 7.1 we can find an i and a morphism of finite presentation $X'_i \rightarrow B_i$ whose pull back is X' . Set $W = X'_i$. \square

Lemma 23.3. *In Situation 23.1. Let $X \rightarrow B$ be a quasi-separated and finite type morphism of algebraic spaces. Given $i \in I$ and a diagram*

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_i \end{array}$$

as in (23.2.1) for $i' \geq i$ let $X_{i'}$ be the scheme theoretic image of $X \rightarrow B_{i'} \times_{B_i} W$. Then $X = \lim_{i' \geq i} X_{i'}$.

Proof. Since X is quasi-compact and quasi-separated formation of the scheme theoretic image of $X \rightarrow B_{i'} \times_{B_i} W$ commutes with étale localization (Morphisms of Spaces, Lemma 16.3). Hence we may and do assume W is affine and maps into an affine U_i étale over B_i . Then

$$B_{i'} \times_{B_i} W = B_{i'} \times_{B_i} U_i \times_{U_i} W = U_{i'} \times_{U_i} W$$

where $U_{i'} = B_{i'} \times_{B_i} U_i$ is affine as the transition morphisms are affine. Thus the lemma follows from the case of schemes which is Limits, Lemma 22.3. \square

Lemma 23.4. *In Situation 23.1. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces quasi-separated and of finite type over B . Let*

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_{i_1} \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \longrightarrow & V \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_{i_2} \end{array}$$

be diagrams as in (23.2.1). Let $X = \lim_{i \geq i_1} X_i$ and $Y = \lim_{i \geq i_2} Y_i$ be the corresponding limit descriptions as in Lemma 23.3. Then there exists an $i_0 \geq \max(i_1, i_2)$ and a morphism

$$(f_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \rightarrow (Y_i)_{i \geq i_0}$$

of inverse systems over $(B_i)_{i \geq i_0}$ such that $f = \lim_{i \geq i_0} f_i$. If $(g_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \rightarrow (Y_i)_{i \geq i_0}$ is a second morphism of inverse systems over $(B_i)_{i \geq i_0}$ such that $f = \lim_{i \geq i_0} g_i$ then $f_i = g_i$ for all $i \geq i_0$.

Proof. Since $V \rightarrow B_{i_2}$ is of finite presentation and $X = \lim_{i \geq i_1} X_i$ we can appeal to Proposition 3.10 as improved by Lemma 4.5 to find an $i_0 \geq \max(i_1, i_2)$ and a morphism $h : X_{i_0} \rightarrow V$ over B_{i_2} such that $X \rightarrow X_{i_0} \rightarrow V$ is equal to $X \rightarrow Y \rightarrow V$. For $i \geq i_0$ we get a commutative solid diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_i & \longrightarrow & X_{i_0} \\ \downarrow & & \downarrow & & \downarrow h \\ Y & \longrightarrow & Y_i & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & B_i & \longrightarrow & B_{i_0} \end{array}$$

Since $X \rightarrow X_i$ has scheme theoretically dense image and since Y_i is the scheme theoretic image of $Y \rightarrow B_i \times_{B_{i_2}} V$ we find that the morphism $X_i \rightarrow B_i \times_{B_{i_2}} V$ induced by the diagram factors through Y_i (Morphisms of Spaces, Lemma 16.6). This proves existence.

Uniqueness. Let $E_i \rightarrow X_i$ be the equalizer of f_i and g_i for $i \geq i_0$. We have $E_i = Y_i \times_{\Delta, Y_i \times_{B_i} Y_i, (f_i, g_i)} X_i$. Hence $E_i \rightarrow X_i$ is a monomorphism of finite presentation as a base change of the diagonal of Y_i over B_i , see Morphisms of Spaces, Lemmas 4.1 and 28.10. Since X_i is a closed subspace of $B_i \times_{B_{i_0}} X_{i_0}$ and similarly for Y_i we see that

$$E_i = X_i \times_{(B_i \times_{B_{i_0}} X_{i_0})} (B_i \times_{B_{i_0}} E_{i_0}) = X_i \times_{X_{i_0}} E_{i_0}$$

Similarly, we have $X = X \times_{X_{i_0}} E_{i_0}$. Hence we conclude that $E_i = X_i$ for i large enough by Lemma 6.10. \square

Remark 23.5. In Situation 23.1 Lemmas 23.2, 23.3, and 23.4 tell us that the category of algebraic spaces quasi-separated and of finite type over B is equivalent to certain types of inverse systems of algebraic spaces over $(B_i)_{i \in I}$, namely the ones produced by applying Lemma 23.3 to a diagram of the form (23.2.1). For example, given $X \rightarrow B$ finite type and quasi-separated if we choose two different diagrams $X \rightarrow V_1 \rightarrow B_{i_1}$ and $X \rightarrow V_2 \rightarrow B_{i_2}$ as in (23.2.1), then applying Lemma 23.4 to id_X (in two directions) we see that the corresponding limit descriptions of X are canonically isomorphic (up to shrinking the directed set I). And so on and so forth.

Lemma 23.6. *Notation and assumptions as in Lemma 23.4. If f is flat and of finite presentation, then there exists an $i_3 > i_0$ such that for $i \geq i_3$ we have f_i is flat, $X_i = Y_i \times_{Y_{i_3}} X_{i_3}$, and $X = Y \times_{Y_{i_3}} X_{i_3}$.*

Proof. By Lemma 7.1 we can choose an $i \geq i_2$ and a morphism $U \rightarrow Y_i$ of finite presentation such that $X = Y \times_{Y_i} U$ (this is where we use that f is of finite presentation). After increasing i we may assume that $U \rightarrow Y_i$ is flat, see Lemma 6.12. As discussed in Remark 23.5 we may and do replace the initial diagram used to define the system $(X_i)_{i \geq i_1}$ by the system corresponding to $X \rightarrow U \rightarrow B_i$. Thus $X_{i'}$ for $i' \geq i$ is defined as the scheme theoretic image of $X \rightarrow B_{i'} \times_{B_i} U$.

Because $U \rightarrow Y_i$ is flat (this is where we use that f is flat), because $X = Y \times_{Y_i} U$, and because the scheme theoretic image of $Y \rightarrow Y_i$ is Y_i , we see that the scheme theoretic image of $X \rightarrow U$ is U (Morphisms of Spaces, Lemma 30.12). Observe that $Y_{i'} \rightarrow B_{i'} \times_{B_i} Y_i$ is a closed immersion for $i' \geq i$ by construction of the system of Y_j . Then the same argument as above shows that the scheme theoretic image of $X \rightarrow B_{i'} \times_{B_i} U$ is equal to the closed subspace $Y_{i'} \times_{Y_i} U$. Thus we see that $X_{i'} = Y_{i'} \times_{Y_i} U$ for all $i' \geq i$ and hence the lemma holds with $i_3 = i$. \square

Lemma 23.7. *Notation and assumptions as in Lemma 23.4. If f is smooth, then there exists an $i_3 > i_0$ such that for $i \geq i_3$ we have f_i is smooth.*

Proof. Combine Lemmas 23.6 and 6.3. \square

Lemma 23.8. *Notation and assumptions as in Lemma 23.4. If f is proper, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is proper.*

Proof. By the discussion in Remark 23.5 the choice of i_1 and W fitting into a diagram as in (23.2.1) is immaterial for the truth of the lemma. Thus we choose W as follows. First we choose a closed immersion $X \rightarrow X'$ with $X' \rightarrow Y$ proper and of finite presentation, see Lemma 12.1. Then we choose an $i_3 \geq i_2$ and a proper morphism $W \rightarrow Y_{i_3}$ such that $X' = Y \times_{Y_{i_3}} W$. This is possible because $Y = \lim_{i \geq i_2} Y_i$ and Lemmas 10.2 and 6.13. With this choice of W it is immediate from the construction that for $i \geq i_3$ the algebraic space X_i is a closed subspace of $Y_i \times_{Y_{i_3}} W \subset B_i \times_{B_{i_3}} W$ and hence proper over Y_i . \square

Lemma 23.9. *In Situation 23.1 suppose that we have a cartesian diagram*

$$\begin{array}{ccc} X^1 & \xrightarrow{p} & X^3 \\ q \downarrow & & \downarrow a \\ X^2 & \xrightarrow{b} & X^4 \end{array}$$

of algebraic spaces quasi-separated and of finite type over B . For each $j = 1, 2, 3, 4$ choose $i_j \in I$ and a diagram

$$\begin{array}{ccc} X^j & \longrightarrow & W^j \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_{i_j} \end{array}$$

as in (23.2.1). Let $X^j = \lim_{i \geq i_j} X_i^j$ be the corresponding limit descriptions as in Lemma 23.4. Let $(a_i)_{i \geq i_5}$, $(b_i)_{i \geq i_6}$, $(p_i)_{i \geq i_7}$, and $(q_i)_{i \geq i_8}$ be the corresponding

morphisms of inverse systems constructed in Lemma 23.4. Then there exists an $i_9 \geq \max(i_5, i_6, i_7, i_8)$ such that for $i \geq i_9$ we have $a_i \circ p_i = b_i \circ q_i$ and such that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion. If a and b are flat and of finite presentation, then there exists an $i_{10} \geq \max(i_5, i_6, i_7, i_8, i_9)$ such that for $i \geq i_{10}$ the last displayed morphism is an isomorphism.

Proof. According to the discussion in Remark 23.5 the choice of W^1 fitting into a diagram as in (23.2.1) is immaterial for the truth of the lemma. Thus we may choose $W^1 = W^2 \times_{W^4} W^3$. Then it is immediate from the construction of X_i^1 that $a_i \circ p_i = b_i \circ q_i$ and that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion.

If a and b are flat and of finite presentation, then so are p and q as base changes of a and b . Thus we can apply Lemma 23.6 to each of a , b , p , q , and $a \circ p = b \circ q$. It follows that there exists an $i_9 \in I$ such that

$$(q_i, p_i) : X_i^1 \rightarrow X_i^2 \times_{X_i^4} X_i^3$$

is the base change of (q_{i_9}, p_{i_9}) by the morphism by the morphism $X_i^4 \rightarrow X_{i_9}^4$ for all $i \geq i_9$. We conclude that (q_i, p_i) is an isomorphism for all sufficiently large i by Lemma 6.10. \square

24. Other chapters

Preliminaries

- (1) Introduction
- (2) Conventions
- (3) Set Theory
- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces
- (7) Sites and Sheaves
- (8) Stacks
- (9) Fields
- (10) Commutative Algebra
- (11) Brauer Groups
- (12) Homological Algebra
- (13) Derived Categories
- (14) Simplicial Methods
- (15) More on Algebra
- (16) Smoothing Ring Maps
- (17) Sheaves of Modules
- (18) Modules on Sites
- (19) Injectives
- (20) Cohomology of Sheaves
- (21) Cohomology on Sites
- (22) Differential Graded Algebra

- (23) Divided Power Algebra
- (24) Differential Graded Sheaves
- (25) Hypercoverings

Schemes

- (26) Schemes
- (27) Constructions of Schemes
- (28) Properties of Schemes
- (29) Morphisms of Schemes
- (30) Cohomology of Schemes
- (31) Divisors
- (32) Limits of Schemes
- (33) Varieties
- (34) Topologies on Schemes
- (35) Descent
- (36) Derived Categories of Schemes
- (37) More on Morphisms
- (38) More on Flatness
- (39) Groupoid Schemes
- (40) More on Groupoid Schemes
- (41) Étale Morphisms of Schemes

Topics in Scheme Theory

- (42) Chow Homology
- (43) Intersection Theory

(44) Picard Schemes of Curves	(83) Quotients of Groupoids
(45) Weil Cohomology Theories	(84) More on Cohomology of Spaces
(46) Adequate Modules	(85) Simplicial Spaces
(47) Dualizing Complexes	(86) Duality for Spaces
(48) Duality for Schemes	(87) Formal Algebraic Spaces
(49) Discriminants and Differents	(88) Algebraization of Formal Spaces
(50) de Rham Cohomology	(89) Resolution of Surfaces Revisited
(51) Local Cohomology	Deformation Theory
(52) Algebraic and Formal Geometry	(90) Formal Deformation Theory
(53) Algebraic Curves	(91) Deformation Theory
(54) Resolution of Surfaces	(92) The Cotangent Complex
(55) Semistable Reduction	(93) Deformation Problems
(56) Functors and Morphisms	Algebraic Stacks
(57) Derived Categories of Varieties	(94) Algebraic Stacks
(58) Fundamental Groups of Schemes	(95) Examples of Stacks
(59) Étale Cohomology	(96) Sheaves on Algebraic Stacks
(60) Crystalline Cohomology	(97) Criteria for Representability
(61) Pro-étale Cohomology	(98) Artin's Axioms
(62) Relative Cycles	(99) Quot and Hilbert Spaces
(63) More Étale Cohomology	(100) Properties of Algebraic Stacks
(64) The Trace Formula	(101) Morphisms of Algebraic Stacks
Algebraic Spaces	(102) Limits of Algebraic Stacks
(65) Algebraic Spaces	(103) Cohomology of Algebraic Stacks
(66) Properties of Algebraic Spaces	(104) Derived Categories of Stacks
(67) Morphisms of Algebraic Spaces	(105) Introducing Algebraic Stacks
(68) Decent Algebraic Spaces	(106) More on Morphisms of Stacks
(69) Cohomology of Algebraic Spaces	(107) The Geometry of Stacks
(70) Limits of Algebraic Spaces	Topics in Moduli Theory
(71) Divisors on Algebraic Spaces	(108) Moduli Stacks
(72) Algebraic Spaces over Fields	(109) Moduli of Curves
(73) Topologies on Algebraic Spaces	Miscellany
(74) Descent and Algebraic Spaces	(110) Examples
(75) Derived Categories of Spaces	(111) Exercises
(76) More on Morphisms of Spaces	(112) Guide to Literature
(77) Flatness on Algebraic Spaces	(113) Desirables
(78) Groupoids in Algebraic Spaces	(114) Coding Style
(79) More on Groupoids in Spaces	(115) Obsolete
(80) Bootstrap	(116) GNU Free Documentation License
(81) Pushouts of Algebraic Spaces	(117) Auto Generated Index
Topics in Geometry	
(82) Chow Groups of Spaces	

References

- [CLO12] Brian Conrad, Max Lieblich, and Martin Olsson, *Nagata compactification for algebraic spaces*, J. Inst. Math. Jussieu **11** (2012), no. 4, 747–814.
- [Ryd08] David Rydh, *Noetherian approximation of algebraic spaces and stacks*, math.AG/0904.0227 (2008).