

DESIRABLES

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1. Introduction

This is basically just a list of things that we want to put in the stacks project. As we add material to the Stacks project continuously this is always somewhat behind the current state of the Stacks project. In fact, it may have been a mistake to try and list things we should add, because it seems impossible to keep it up to date.

Last updated: Thursday, August 31, 2017.

2. Conventions

We should have a chapter with a short list of conventions used in the document. This chapter already exists, see Conventions, Section 1, but a lot more could be added there. Especially useful would be to find “hidden” conventions and tacit assumptions and put those there.

3. Sites and Topoi

We have a chapter on sites and sheaves, see Sites, Section 1. We have a chapter on ringed sites (and topoi) and modules on them, see Modules on Sites, Section 1. We have a chapter on cohomology in this setting, see Cohomology on Sites, Section 1. But a lot more could be added, especially in the chapter on cohomology.

4. Stacks

We have a chapter on (abstract) stacks, see Stacks, Section 1. It would be nice if

- (1) improve the discussion on “stackyfication”,
- (2) give examples of stackyfication,
- (3) more examples in general,
- (4) improve the discussion of gerbes.

Example result which has not been added yet: Given a sheaf of abelian groups \mathcal{F} over \mathcal{C} the set of equivalence classes of gerbes banded by \mathcal{F} is bijective to $H^2(\mathcal{C}, \mathcal{F})$.

5. Simplicial methods

We have a chapter on simplicial methods, see Simplicial, Section 1. This has to be reviewed and improved. The discussion of the relationship between simplicial homotopy (also known as combinatorial homotopy) and Kan complexes should be improved upon. There is a chapter on simplicial spaces, see Simplicial Spaces, Section 1. This chapter briefly discusses simplicial topological spaces, simplicial sites, and simplicial topoi. We can further develop “simplicial algebraic geometry” to discuss simplicial schemes (or simplicial algebraic spaces, or simplicial algebraic stacks) and treat geometric questions, their cohomology, etc.

6. Cohomology of schemes

There is already a chapter on cohomology of quasi-coherent sheaves, see Cohomology of Schemes, Section 1. We have a chapter discussing the derived category of quasi-coherent sheaves on a scheme, see Derived Categories of Schemes, Section 1. We have a chapter discussing duality for Noetherian schemes and relative duality for morphisms of schemes, see Duality for Schemes, Section 1. We also have chapters on étale cohomology of schemes and on crystalline cohomology of schemes. But most of the material in these chapters is very basic and a lot more could/should be added there.

7. Deformation theory à la Schlessinger

We have a chapter on this material, see Formal Deformation Theory, Section 1. We have a chapter discussing examples of the general theory, see Deformation Problems, Section 1. We have a chapter, see Deformation Theory, Section 1 which discusses deformations of rings (and modules), deformations of ringed spaces (and sheaves of modules), deformations of ringed topoi (and sheaves of modules). In this chapter we use the naive cotangent complex to describe obstructions, first order deformations, and infinitesimal automorphisms. This material has found some applications to algebraicity of moduli stacks in later chapters. There is also a chapter discussing the full cotangent complex, see Cotangent, Section 1.

8. Definition of algebraic stacks

An algebraic stack is a stack in groupoids over the category of schemes with the fppf topology that has a diagonal representable by algebraic spaces and is the target of a surjective smooth morphism from a scheme. See Algebraic Stacks, Section 12. A “Deligne-Mumford stack” is an algebraic stack for which there exists a scheme and a surjective étale morphism from that scheme to it as in the paper [DM69] of Deligne and Mumford, see Algebraic Stacks, Definition 12.2. We will reserve the term “Artin stack” for a stack such as in the papers by Artin, see [Art69], [Art70], and [Art74]. A possible definition is that an Artin stack is an algebraic stack \mathcal{X} over a locally Noetherian scheme S such that $\mathcal{X} \rightarrow S$ is locally of finite type¹.

9. Examples of schemes, algebraic spaces, algebraic stacks

The Stacks project currently contains two chapters discussing moduli stacks and their properties, see Moduli Stacks, Section 1 and Moduli of Curves, Section 1. Over time we intend to add more, for example:

- (1) \mathcal{A}_g , i.e., principally polarized abelian schemes of genus g ,
- (2) $\mathcal{A}_1 = \mathcal{M}_{1,1}$, i.e., 1-pointed smooth projective genus 1 curves,
- (3) $\mathcal{M}_{g,n}$, i.e., smooth projective genus g -curves with n pairwise distinct labeled points,
- (4) $\overline{\mathcal{M}}_{g,n}$, i.e., stable n -pointed nodal projective genus g -curves,
- (5) $\mathcal{H}om_S(\mathcal{X}, \mathcal{Y})$, moduli of morphisms (with suitable conditions on the stacks \mathcal{X} , \mathcal{Y} and the base scheme S),
- (6) $Bun_G(X) = \mathcal{H}om_S(X, BG)$, the stack of G -bundles of the geometric Langlands programme (with suitable conditions on the scheme X , the group scheme G , and the base scheme S),
- (7) $\mathcal{P}ic_{\mathcal{X}/S}$, i.e., the Picard stack associated to an algebraic stack over a base scheme (or space).

More generally, the Stacks project is somewhat lacking in geometrically meaningful examples.

10. Properties of algebraic stacks

This is perhaps one of the easier projects to work on, as most of the basic theory is there now. Of course these things are really properties of morphisms of stacks. We can define singularities (up to smooth factors) etc. Prove that a connected normal stack is irreducible, etc.

11. Lisse étale site of an algebraic stack

This has been introduced in Cohomology of Stacks, Section 14. An example to show that it is not functorial with respect to 1-morphisms of algebraic stacks is discussed in Examples, Section 58. Of course a lot more could be said about this, but it turns out to be very useful to prove things using the “big” étale site as much as possible.

¹Namely, these are exactly the algebraic stacks over S satisfying Artin’s axioms [-1], [0], [1], [2], [3], [4], [5] of Artin’s Axioms, Section 14.

12. Things you always wanted to know but were afraid to ask

There are going to be lots of lemmas that you use over and over again that are useful but aren't really mentioned specifically in the literature, or it isn't easy to find references for. Bag of tricks.

Example: Given two groupoids in schemes $R \rightrightarrows U$ and $R' \rightrightarrows U'$ what does it mean to have a 1-morphism $[U/R] \rightarrow [U'/R']$ purely in terms of groupoids in schemes.

13. Quasi-coherent sheaves on stacks

These are defined and discussed in the chapter Cohomology of Stacks, Section 1. Derived categories of modules are discussed in the chapter Derived Categories of Stacks, Section 1. A lot more could be added to these chapters.

14. Flat and smooth

Artin's theorem that having a flat surjection from a scheme is a replacement for the smooth surjective condition. This is now available as Criteria for Representability, Theorem 16.1.

15. Artin's representability theorem

This is discussed in the chapter Artin's Axioms, Section 1. We also have an application, see Quot, Theorem 5.12. There should be a lot more applications and the chapter itself has to be cleaned up as well.

16. DM stacks are finitely covered by schemes

We already have the corresponding result for algebraic spaces, see Limits of Spaces, Section 16. What is missing is the result for DM and quasi-DM stacks.

17. Martin Olsson's paper on properness

This proves two notions of proper are the same. The first part of this is now available in the form of Chow's lemma for algebraic stacks, see More on Morphisms of Stacks, Theorem 10.3. As a consequence we show that it suffices to use DVR's in checking the valuative criterion for properness for algebraic stacks in certain cases, see More on Morphisms of Stacks, Section 11.

18. Proper pushforward of coherent sheaves

We can start working on this now that we have Chow's lemma for algebraic stacks, see previous section.

19. Keel and Mori

See [KM97]. Their result has been added in More on Morphisms of Stacks, Section 13.

20. Add more here

Actually, no we should never have started this list as part of the Stacks project itself! There is a todo list somewhere else which is much easier to update.

21. Other chapters

Preliminaries

- (1) Introduction
- (2) Conventions
- (3) Set Theory
- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces
- (7) Sites and Sheaves
- (8) Stacks
- (9) Fields
- (10) Commutative Algebra
- (11) Brauer Groups
- (12) Homological Algebra
- (13) Derived Categories
- (14) Simplicial Methods
- (15) More on Algebra
- (16) Smoothing Ring Maps
- (17) Sheaves of Modules
- (18) Modules on Sites
- (19) Injectives
- (20) Cohomology of Sheaves
- (21) Cohomology on Sites
- (22) Differential Graded Algebra
- (23) Divided Power Algebra
- (24) Differential Graded Sheaves
- (25) Hypercoverings

Schemes

- (26) Schemes
- (27) Constructions of Schemes
- (28) Properties of Schemes
- (29) Morphisms of Schemes
- (30) Cohomology of Schemes
- (31) Divisors
- (32) Limits of Schemes
- (33) Varieties
- (34) Topologies on Schemes
- (35) Descent
- (36) Derived Categories of Schemes
- (37) More on Morphisms
- (38) More on Flatness
- (39) Groupoid Schemes
- (40) More on Groupoid Schemes
- (41) Étale Morphisms of Schemes

Topics in Scheme Theory

- (42) Chow Homology

- (43) Intersection Theory
- (44) Picard Schemes of Curves
- (45) Weil Cohomology Theories
- (46) Adequate Modules
- (47) Dualizing Complexes
- (48) Duality for Schemes
- (49) Discriminants and Differents
- (50) de Rham Cohomology
- (51) Local Cohomology
- (52) Algebraic and Formal Geometry
- (53) Algebraic Curves
- (54) Resolution of Surfaces
- (55) Semistable Reduction
- (56) Functors and Morphisms
- (57) Derived Categories of Varieties
- (58) Fundamental Groups of Schemes
- (59) Étale Cohomology
- (60) Crystalline Cohomology
- (61) Pro-étale Cohomology
- (62) Relative Cycles
- (63) More Étale Cohomology
- (64) The Trace Formula

Algebraic Spaces

- (65) Algebraic Spaces
- (66) Properties of Algebraic Spaces
- (67) Morphisms of Algebraic Spaces
- (68) Decent Algebraic Spaces
- (69) Cohomology of Algebraic Spaces
- (70) Limits of Algebraic Spaces
- (71) Divisors on Algebraic Spaces
- (72) Algebraic Spaces over Fields
- (73) Topologies on Algebraic Spaces
- (74) Descent and Algebraic Spaces
- (75) Derived Categories of Spaces
- (76) More on Morphisms of Spaces
- (77) Flatness on Algebraic Spaces
- (78) Groupoids in Algebraic Spaces
- (79) More on Groupoids in Spaces
- (80) Bootstrap
- (81) Pushouts of Algebraic Spaces

Topics in Geometry

- (82) Chow Groups of Spaces
- (83) Quotients of Groupoids
- (84) More on Cohomology of Spaces
- (85) Simplicial Spaces
- (86) Duality for Spaces

(87) Formal Algebraic Spaces	(103) Cohomology of Algebraic Stacks
(88) Algebraization of Formal Spaces	(104) Derived Categories of Stacks
(89) Resolution of Surfaces Revisited	(105) Introducing Algebraic Stacks
Deformation Theory	(106) More on Morphisms of Stacks
(90) Formal Deformation Theory	(107) The Geometry of Stacks
(91) Deformation Theory	Topics in Moduli Theory
(92) The Cotangent Complex	(108) Moduli Stacks
(93) Deformation Problems	(109) Moduli of Curves
Algebraic Stacks	Miscellany
(94) Algebraic Stacks	(110) Examples
(95) Examples of Stacks	(111) Exercises
(96) Sheaves on Algebraic Stacks	(112) Guide to Literature
(97) Criteria for Representability	(113) Desirables
(98) Artin's Axioms	(114) Coding Style
(99) Quot and Hilbert Spaces	(115) Obsolete
(100) Properties of Algebraic Stacks	(116) GNU Free Documentation License
(101) Morphisms of Algebraic Stacks	(117) Auto Generated Index
(102) Limits of Algebraic Stacks	

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- [DM69] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHES **36** (1969), 75–110.
- [KM97] Sean Keel and Shigefumi Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), 193–213.