EXERCISES

Contents

1. Algebra

This first section just contains some assorted questions.

Exercise 1.1. Let A be a ring, and $\mathfrak{m}$ a maximal ideal. In $A[X]$ let $\tilde{\mathfrak{m}}_1 = (\mathfrak{m}, X)$ and $\tilde{\mathfrak{m}}_2 = (\mathfrak{m}, X-1)$. Show that

$$A[X]_{\tilde{\mathfrak{m}}_1} \cong A[X]_{\tilde{\mathfrak{m}}_2}.$$

Exercise 1.2. Find an example of a non Noetherian ring R such that every finitely generated ideal of R is finitely presented as an R-module. (A ring is said to be *coherent* if the last property holds.)

Exercise 1.3. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring. For any finite A-module M define $r(M)$ to be the minimum number of generators of M as an A-module. This number equals $\dim_k M/\mathfrak{m}M = \dim_k M \otimes_A k$ by NAK.

(1) Show that $r(M \otimes_A N) = r(M)r(N)$.
(2) Let $I \subset A$ be an ideal with $r(I) > 1$. Show that $r(I^2) < r(I)^2$.
(3) Conclude that if every ideal in A is a flat module, then A is a PID (or a field).

Exercise 1.4. Let k be a field. Show that the following pairs of k-algebras are not isomorphic:

(1) $k[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_{n+1}]$ for any $n \geq 1$.
(2) $k[a,b,c,d,e,f]/(ab+cd+ef)$ and $k[x_1, \ldots, x_n]$ for $n = 5$.
(3) $k[a,b,c,d,e,f]/(ab+cd+ef)$ and $k[x_1, \ldots, x_n]$ for $n = 6$.

Remark 1.5. Of course the idea of this exercise is to find a simple argument in each case rather than applying a "big" theorem. Nonetheless it is good to be guided by general principles.

Exercise 1.6. Algebra. (Silly and should be easy.)

(1) Give an example of a ring A and a nonsplit short exact sequence of A-modules
$$0 \to M_1 \to M_2 \to M_3 \to 0.$$
(2) Give an example of a nonsplit sequence of A-modules as above and a faithfully flat $A \to B$ such that
$$0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0.$$
is split as a sequence of B-modules.

Exercise 1.7. Suppose that k is a field having a primitive nth root of unity ζ. This means that $\zeta^n = 1$, but $\zeta^m \neq 1$ for $0 < m < n$.

(1) Show that the characteristic of k is prime to n.
(2) Suppose that $a \in k$ is an element of k which is not an dth power in k for any divisor d of n for $n \geq d > 1$. Show that $k[x]/(x^n - a)$ is a field. (Hint: Consider a splitting field for $x^n - a$ and use Galois theory.)

Exercise 1.8. Let $\nu : k[x] \setminus \{0\} \to \mathbf{Z}$ be a map with the following properties: $\nu(fg) = \nu(f) + \nu(g)$ whenever f, g not zero, and $\nu(f+g) \geq min(\nu(f), \nu(g))$ whenever f, g, $f+g$ are not zero, and $\nu(c) = 0$ for all $c \in k^*$.

(1) Show that if f, g, and $f+g$ are nonzero and $\nu(f) \neq \nu(g)$ then we have equality $\nu(f+g) = min(\nu(f), \nu(g))$.
(2) Show that if $f = \sum a_i x^i$, $f \neq 0$, then $\nu(f) \geq min(\{i\nu(x)\}_{a_i \neq 0})$. When does equality hold?
(3) Show that if ν attains a negative value then $\nu(f) = -n \deg(f)$ for some $n \in \mathbf{N}$.
(4) Suppose $\nu(x) \geq 0$. Show that $\{f \mid f = 0,\ or\ \nu(f) > 0\}$ is a prime ideal of $k[x]$.
(5) Describe all possible ν.

Let A be a ring. An *idempotent* is an element $e \in A$ such that $e^2 = e$. The elements 1 and 0 are always idempotent. A *nontrivial idempotent* is an idempotent which is not equal to zero. Two idempotents $e, e' \in A$ are called *orthogonal* if $ee' = 0$.

Exercise 1.9. Let A be a ring. Show that A is a product of two nonzero rings if and only if A has a nontrivial idempotent.

Exercise 1.10. Let A be a ring and let $I \subset A$ be a locally nilpotent ideal. Show that the map $A \to A/I$ induces a bijection on idempotents. (Hint: It may be easier to prove this when I is nilpotent. Do this first. Then use "absolute Noetherian reduction" to reduce to the nilpotent case.)

2. Colimits

Definition 2.1. A *directed set* is a nonempty set I endowed with a preorder $\leq$ such that given any pair $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. A *system of rings* over I is given by a ring A_i for each $i \in I$ and a map of rings $\varphi_{ij} : A_i \to A_j$ whenever $i \leq j$ such that the composition $A_i \to A_j \to A_k$ is equal to $A_i \to A_k$ whenever $i \leq j \leq k$.

One similarly defines systems of groups, modules over a fixed ring, vector spaces over a field, etc.

Exercise 2.2. Let I be a directed set and let (A_i, φ_{ij}) be a system of rings over I. Show that there exists a ring A and maps $\varphi_i : A_i \to A$ such that $\varphi_j \circ \varphi_{ij} = \varphi_i$ for all $i \leq j$ with the following universal property: Given any ring B and maps $\psi_i : A_i \to B$ such that $\psi_j \circ \varphi_{ij} = \psi_i$ for all $i \leq j$, then there exists a unique ring map $\psi : A \to B$ such that $\psi_i = \psi \circ \varphi_i$.

Definition 2.3. The ring A constructed in Exercise 2.2 is called the *colimit* of the system. Notation $\operatorname{colim} A_i$.

Exercise 2.4. Let $(I, \geq)$ be a directed set and let (A_i, φ_{ij}) be a system of rings over I with colimit A. Prove that there is a bijection

$$\operatorname{Spec}(A) = \{(\mathfrak{p}_i)_{i \in I} \mid \mathfrak{p}_i \subset A_i \text{ and } \mathfrak{p}_i = \varphi_{ij}^{-1}(\mathfrak{p}_j)\ \forall i \leq j\} \subset \prod\nolimits_{i \in I} \operatorname{Spec}(A_i)$$

The set on the right hand side of the equality is the limit of the sets $\operatorname{Spec}(A_i)$. Notation $\lim \operatorname{Spec}(A_i)$.

Exercise 2.5. Let $(I, \geq)$ be a directed set and let (A_i, φ_{ij}) be a system of rings over I with colimit A. Suppose that $\operatorname{Spec}(A_j) \to \operatorname{Spec}(A_i)$ is surjective for all $i \leq j$. Show that $\operatorname{Spec}(A) \to \operatorname{Spec}(A_i)$ is surjective for all i. (Hint: You can try to use Tychonoff, but there is also a basically trivial direct algebraic proof based on Algebra, Lemma 17.9.)

Exercise 2.6. Let $A \subset B$ be an integral ring extension. Prove that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective. Use the exercises above, the fact that this holds for a finite ring extension (proved in the lectures), and by proving that $B = \operatorname{colim} B_i$ is a directed colimit of finite extensions $A \subset B_i$.

Exercise 2.7. Let $(I, \geq)$ be a directed set. Let A be a ring and let $(N_i, \varphi_{i,i'})$ be a directed system of A-modules indexed by I. Suppose that M is another A-module. Prove that

$$\operatorname{colim}_{i \in I} M \otimes_A N_i \cong M \otimes_A \Big(\operatorname{colim}_{i \in I} N_i\Big).$$

Definition 2.8. A module M over R is said to be of *finite presentation* over R if it is isomorphic to the cokernel of a map of finite free modules $R^{\oplus n} \to R^{\oplus m}$.

Exercise 2.9. Prove that any module over any ring is
(1) the colimit of its finitely generated submodules, and
(2) in some way a colimit of finitely presented modules.

3. Additive and abelian categories

Exercise 3.1. Let k be a field. Let $\mathcal{C}$ be the category of filtered vector spaces over k, see Homology, Definition 19.1 for the definition of a filtered object of any category.

(1) Show that this is an additive category (explain carefuly what the direct sum of two objects is).
(2) Let $f : (V, F) \to (W, F)$ be a morphism of $\mathcal{C}$. Show that f has a kernel and cokernel (explain precisely what the kernel and cokernel of f are).
(3) Give an example of a map of $\mathcal{C}$ such that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is not an isomorphism.

Exercise 3.2. Let R be a Noetherian domain. Let $\mathcal{C}$ be the category of finitely generated torsion free R-modules.

(1) Show that this is an additive category.
(2) Let $f : N \to M$ be a morphism of $\mathcal{C}$. Show that f has a kernel and cokernel (make sure you define precisely what the kernel and cokernel of f are).
(3) Give an example of a Noetherian domain R and a map of $\mathcal{C}$ such that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is not an isomorphism.

Exercise 3.3. Give an example of a category which is additive and has kernels and cokernels but which is not as in Exercises 3.1 and 3.2.

4. Tensor product

Tensor products are introduced in Algebra, Section 12. Let R be a ring. Let Mod_R be the category of R-modules. We will say that a functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$

(1) is additive if $F : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(F(M), F(N))$ is a homomorphism of abelian groups for any R-modules M, N, see Homology, Definition 3.1.
(2) R-linear if $F : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(F(M), F(N))$ is R-linear for any R-modules M, N,
(3) right exact if for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ the sequence $F(M_1) \to F(M_2) \to F(M_3) \to 0$ is exact,
(4) left exact if for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ the sequence $0 \to F(M_1) \to F(M_2) \to F(M_3)$ is exact,
(5) commutes with direct sums, if given a set I and R-modules M_i the maps $F(M_i) \to F(\bigoplus M_i)$ induce an isomorphism $\bigoplus F(M_i) = F(\bigoplus M_i)$.

Exercise 4.1. Let R be a ring. With notation as above.

(1) Give an example of a ring R and an additive functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ which is not R-linear.
(2) Let N be an R-module. Show that the functor $F(M) = M \otimes_R N$ is R-linear, right exact, and commutes with direct sums,
(3) Conversely, show that any functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ which is R-linear, right exact, and commutes with direct sums is of the form $F(M) = M \otimes_R N$ for some R-module N.
(4) Show that if in (3) we drop the assumption that F commutes with direct sums, then the conclusion no longer holds.

5. Flat ring maps

Exercise 5.1. Let S be a multiplicative subset of the ring A.

(1) For an A-module M show that $S^{-1}M = S^{-1}A \otimes_A M$.
(2) Show that $S^{-1}A$ is flat over A.

Exercise 5.2. Find an injection $M_1 \to M_2$ of A-modules such that $M_1 \otimes N \to M_2 \otimes N$ is not injective in the following cases:

(1) $A = k[x,y]$ and $N = (x,y) \subset A$. (Here and below k is a field.)
(2) $A = k[x,y]$ and $N = A/(x,y)$.

Exercise 5.3. Give an example of a ring A and a finite A-module M which is a flat but not a projective A-module.

Remark 5.4. If M is of finite presentation and flat over A, then M is projective over A. Thus your example will have to involve a ring A which is not Noetherian. I know of an example where A is the ring of $\mathcal{C}^\infty$-functions on $\mathbf{R}$.

Exercise 5.5. Find a flat but not free module over $\mathbf{Z}_{(2)}$.

Exercise 5.6. Flat deformations.

(1) Suppose that k is a field and $k[\epsilon]$ is the ring of dual numbers $k[\epsilon] = k[x]/(x^2)$ and $\epsilon = \bar{x}$. Show that for any k-algebra A there is a flat $k[\epsilon]$-algebra B such that A is isomorphic to $B/\epsilon B$.
(2) Suppose that $k = \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ and

$$A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^p, x_2^p, x_3^p, x_4^p, x_5^p, x_6^p).$$

Show that there exists a flat $\mathbf{Z}/p^2\mathbf{Z}$-algebra B such that B/pB is isomorphic to A. (So here p plays the role of ϵ.)
(3) Now let $p = 2$ and consider the same question for $k = \mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$ and

$$A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_1x_2 + x_3x_4 + x_5x_6).$$

However, in this case show that there does *not* exist a flat $\mathbf{Z}/4\mathbf{Z}$-algebra B such that $B/2B$ is isomorphic to A. (Find the trick! The same example works in arbitrary characteristic $p > 0$, except that the computation is more difficult.)

Exercise 5.7. Let $(A, \mathfrak{m}, k)$ be a local ring and let k'/k be a finite field extension. Show there exists a flat, local map of local rings $A \to B$ such that $\mathfrak{m}_B = \mathfrak{m}B$ and $B/\mathfrak{m}B$ is isomorphic to k' as k-algebra. (Hint: first do the case where $k \subset k'$ is generated by a single element.)

Remark 5.8. The same result holds for arbitrary field extensions K/k.

6. The Spectrum of a ring

Exercise 6.1. Compute $\mathrm{Spec}(\mathbf{Z})$ as a set and describe its topology.

Exercise 6.2. Let A be any ring. For $f \in A$ we define $D(f) := \{\mathfrak{p} \subset A \mid f \notin \mathfrak{p}\}$. Prove that the open subsets $D(f)$ form a basis of the topology of $\mathrm{Spec}(A)$.

Exercise 6.3. Prove that the map $I \mapsto V(I)$ defines a natural bijection

$$\{I \subset A \text{ with } I = \sqrt{I}\} \longrightarrow \{T \subset \mathrm{Spec}(A) \text{ closed}\}$$

Definition 6.4. A topological space X is called *quasi-compact* if for any open covering $X = \bigcup_{i \in I} U_i$ there is a finite subset $\{i_1, \ldots, i_n\} \subset I$ such that $X = U_{i_1} \cup \ldots U_{i_n}$.

Exercise 6.5. Prove that $\mathrm{Spec}(A)$ is quasi-compact for any ring A.

Definition 6.6. A topological space X is said to verify the separation axiom T_0 if for any pair of points $x, y \in X$, $x \neq y$ there is an open subset of X containing one but not the other. We say that X is *Hausdorff* if for any pair $x, y \in X$, $x \neq y$ there are disjoint open subsets U, V such that $x \in U$ and $y \in V$.

Exercise 6.7. Show that Spec(A) is **not** Hausdorff in general. Prove that Spec(A) is T_0. Give an example of a topological space X that is not T_0.

Remark 6.8. Usually the word compact is reserved for quasi-compact and Hausdorff spaces.

Definition 6.9. A topological space X is called *irreducible* if X is not empty and if $X = Z_1 \cup Z_2$ with $Z_1, Z_2 \subset X$ closed, then either $Z_1 = X$ or $Z_2 = X$. A subset $T \subset X$ of a topological space is called *irreducible* if it is an irreducible topological space with the topology induced from X. This definition implies T is irreducible if and only if the closure $\bar{T}$ of T in X is irreducible.

Exercise 6.10. Prove that Spec(A) is irreducible if and only if $Nil(A)$ is a prime ideal and that in this case it is the unique minimal prime ideal of A.

Exercise 6.11. Prove that a closed subset $T \subset \mathrm{Spec}(A)$ is irreducible if and only if it is of the form $T = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$.

Definition 6.12. A point x of an irreducible topological space X is called a *generic point* of X if X is equal to the closure of the subset $\{x\}$.

Exercise 6.13. Show that in a T_0 space X every irreducible closed subset has at most one generic point.

Exercise 6.14. Prove that in Spec(A) every irreducible closed subset *does* have a generic point. In fact show that the map $\mathfrak{p} \mapsto \overline{\{\mathfrak{p}\}}$ is a bijection of Spec(A) with the set of irreducible closed subsets of X.

Exercise 6.15. Give an example to show that an irreducible subset of Spec($\mathbf{Z}$) does not necessarily have a generic point.

Definition 6.16. A topological space X is called *Noetherian* if any decreasing sequence $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ of closed subsets of X stabilizes. (It is called *Artinian* if any increasing sequence of closed subsets stabilizes.)

Exercise 6.17. Show that if the ring A is Noetherian then the topological space Spec(A) is Noetherian. Give an example to show that the converse is false. (The same for Artinian if you like.)

Definition 6.18. A maximal irreducible subset $T \subset X$ is called an *irreducible component* of the space X. Such an irreducible component of X is automatically a closed subset of X.

Exercise 6.19. Prove that any irreducible subset of X is contained in an irreducible component of X.

Exercise 6.20. Prove that a Noetherian topological space X has only finitely many irreducible components, say $X_1, \ldots, X_n$, and that $X = X_1 \cup X_2 \cup \ldots \cup X_n$. (Note that any X is always the union of its irreducible components, but that if $X = \mathbf{R}$ with its usual topology for instance then the irreducible components of X are the one point subsets. This is not terribly interesting.)

Exercise 6.21. Show that irreducible components of Spec(A) correspond to minimal primes of A.

Definition 6.22. A point $x \in X$ is called *closed* if $\overline{\{x\}} = \{x\}$. Let x, y be points of X. We say that x is a *specialization* of y, or that y is a *generalization* of x if $x \in \overline{\{y\}}$.

Exercise 6.23. Show that closed points of $\mathrm{Spec}(A)$ correspond to maximal ideals of A.

Exercise 6.24. Show that $\mathfrak{p}$ is a generalization of $\mathfrak{q}$ in $\mathrm{Spec}(A)$ if and only if $\mathfrak{p} \subset \mathfrak{q}$. Characterize closed points, maximal ideals, generic points and minimal prime ideals in terms of generalization and specialization. (Here we use the terminology that a point of a possibly reducible topological space X is called a generic point if it is a generic points of one of the irreducible components of X.)

Exercise 6.25. Let I and J be ideals of A. What is the condition for $V(I)$ and $V(J)$ to be disjoint?

Definition 6.26. A topological space X is called *connected* if it is nonempty and not the union of two nonempty disjoint open subsets. A *connected component* of X is a maximal connected subset. Any point of X is contained in a connected component of X and any connected component of X is closed in X. (But in general a connected component need not be open in X.)

Exercise 6.27. Let A be a nonzero ring. Show that $\mathrm{Spec}(A)$ is disconnected iff $A \cong B \times C$ for certain nonzero rings B, C.

Exercise 6.28. Let T be a connected component of $\mathrm{Spec}(A)$. Prove that T is stable under generalization. Prove that T is an open subset of $\mathrm{Spec}(A)$ if A is Noetherian. (Remark: This is wrong when A is an infinite product of copies of $\mathbf{F}_2$ for example. The spectrum of this ring consists of infinitely many closed points.)

Exercise 6.29. Compute $\mathrm{Spec}(k[x])$, i.e., describe the prime ideals in this ring, describe the possible specializations, and describe the topology. (Work this out when k is algebraically closed but also when k is not.)

Exercise 6.30. Compute $\mathrm{Spec}(k[x, y])$, where k is algebraically closed. [Hint: use the morphism $\varphi : \mathrm{Spec}(k[x, y]) \to \mathrm{Spec}(k[x])$; if $\varphi(\mathfrak{p}) = (0)$ then localize with respect to $S = \{f \in k[x] \mid f \neq 0\}$ and use result of lecture on localization and Spec.] (Why do you think algebraic geometers call this affine 2-space?)

Exercise 6.31. Compute $\mathrm{Spec}(\mathbf{Z}[y])$. [Hint: as above.] (Affine 1-space over $\mathbf{Z}$.)

7. Localization

Exercise 7.1. Let A be a ring. Let $S \subset A$ be a multiplicative subset. Let M be an A-module. Let $N \subset S^{-1}M$ be an $S^{-1}A$-submodule. Show that there exists an A-submodule $N' \subset M$ such that $N = S^{-1}N'$. (This useful result applies in particular to ideals of $S^{-1}A$.)

Exercise 7.2. Let A be a ring. Let M be an A-module. Let $m \in M$.

(1) Show that $I = \{a \in A \mid am = 0\}$ is an ideal of A.
(2) For a prime $\mathfrak{p}$ of A show that the image of m in $M_{\mathfrak{p}}$ is zero if and only if $I \not\subset \mathfrak{p}$.
(3) Show that m is zero if and only if the image of m is zero in $M_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of A.
(4) Show that m is zero if and only if the image of m is zero in $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of A.
(5) Show that $M = 0$ if and only if $M_{\mathfrak{m}}$ is zero for all maximal ideals $\mathfrak{m}$.

Exercise 7.3. Find a pair (A, f) where A is a domain with three or more pairwise distinct primes and $f \in A$ is an element such that the principal localization $A_f = \{1, f, f^2, \ldots\}^{-1}A$ is a field.

Exercise 7.4. Let A be a ring. Let M be a finite A-module. Let $S \subset A$ be a multiplicative set. Assume that $S^{-1}M = 0$. Show that there exists an $f \in S$ such that the principal localization $M_f = \{1, f, f^2, \ldots\}^{-1}M$ is zero.

Exercise 7.5. Give an example of a triple (A, I, S) where A is a ring, $0 \neq I \neq A$ is a proper nonzero ideal, and $S \subset A$ is a multiplicative subset such that $A/I \cong S^{-1}A$ as A-algebras.

8. Nakayama's Lemma

Exercise 8.1. Let A be a ring. Let I be an ideal of A. Let M be an A-module. Let $x_1, \ldots, x_n \in M$. Assume that

(1) M/IM is generated by $x_1, \ldots, x_n$,
(2) M is a finite A-module,
(3) I is contained in every maximal ideal of A.

Show that $x_1, \ldots, x_n$ generate M. (Suggested solution: Reduce to a localization at a maximal ideal of A using Exercise 7.2 and exactness of localization. Then reduce to the statement of Nakayama's lemma in the lectures by looking at the quotient of M by the submodule generated by $x_1, \ldots, x_n$.)

9. Length

Definition 9.1. Let A be a ring. Let M be an A-module. The *length* of M as an R-module is

$$\text{length}_A(M) = \sup\{n \mid \exists\ 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

In other words, the supremum of the lengths of chains of submodules.

Exercise 9.2. Show that a module M over a ring A has length 1 if and only if it is isomorphic to $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in A.

Exercise 9.3. Compute the length of the following modules over the following rings. Briefly(!) explain your answer. (Please feel free to use additivity of the length function in short exact sequences, see Algebra, Lemma 52.3).

(1) The length of $\mathbf{Z}/120\mathbf{Z}$ over $\mathbf{Z}$.
(2) The length of $\mathbf{C}[x]/(x^{100} + x + 1)$ over $\mathbf{C}[x]$.
(3) The length of $\mathbf{R}[x]/(x^4 + 2x^2 + 1)$ over $\mathbf{R}[x]$.

Exercise 9.4. Let $A = k[x, y]_{(x,y)}$ be the local ring of the affine plane at the origin. Make any assumption you like about the field k. Suppose that $f = x^3 + x^2y^2 + y^{100}$ and $g = y^3 - x^{999}$. What is the length of $A/(f, g)$ as an A-module? (Possible way to proceed: think about the ideal that f and g generate in quotients of the form $A/\mathfrak{m}_A^n = k[x, y]/(x, y)^n$ for varying n. Try to find n such that $A/(f, g) + \mathfrak{m}_A^n \cong A/(f, g) + \mathfrak{m}_A^{n+1}$ and use NAK.)

10. Associated primes

Associated primes are discussed in Algebra, Section 63

Exercise 10.1. Compute the set of associated primes for each of the following modules.

1. $R = k[x, y]$ and $M = R/(xy(x + y))$,
2. $R = \mathbf{Z}[x]$ and $M = R/(300x + 75)$, and
3. $R = k[x, y, z]$ and $M = R/(x^3, x^2y, xz)$.

Here as usual k is a field.

Exercise 10.2. Give an example of a Noetherian ring R and a prime ideal $\mathfrak{p}$ such that $\mathfrak{p}$ is not the only associated prime of $R/\mathfrak{p}^2$.

Exercise 10.3. Let R be a Noetherian ring with incomparable prime ideals $\mathfrak{p}$, $\mathfrak{q}$, i.e., $\mathfrak{p} \not\subset \mathfrak{q}$ and $\mathfrak{q} \not\subset \mathfrak{p}$.

1. Show that for $N = R/(\mathfrak{p} \cap \mathfrak{q})$ we have $\text{Ass}(N) = \{\mathfrak{p}, \mathfrak{q}\}$.
2. Show by an example that the module $M = R/\mathfrak{p}\mathfrak{q}$ can have an associated prime not equal to $\mathfrak{p}$ or $\mathfrak{q}$.

11. Ext groups

Ext groups are defined in Algebra, Section 71.

Exercise 11.1. Compute all the Ext groups $\text{Ext}^i(M, N)$ of the given modules in the category of $\mathbf{Z}$-modules (also known as the category of abelian groups).

1. $M = \mathbf{Z}$ and $N = \mathbf{Z}$,
2. $M = \mathbf{Z}/4\mathbf{Z}$ and $N = \mathbf{Z}/8\mathbf{Z}$,
3. $M = \mathbf{Q}$ and $N = \mathbf{Z}/2\mathbf{Z}$, and
4. $M = \mathbf{Z}/2\mathbf{Z}$ and $N = \mathbf{Q}/\mathbf{Z}$.

Exercise 11.2. Let $R = k[x, y]$ where k is a field.

1. Show by hand that the Koszul complex
$$0 \to R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^{\oplus 2} \xrightarrow{(x,y)} R \xrightarrow{f \mapsto f(0,0)} k \to 0$$
is exact.
2. Compute $\text{Ext}^i_R(k, k)$ where $k = R/(x, y)$ as an R-module.

Exercise 11.3. Give an example of a Noetherian ring R and finite modules M, N such that $\text{Ext}^i_R(M, N)$ is nonzero for all $i \geq 0$.

Exercise 11.4. Give an example of a ring R and ideal I such that $\text{Ext}^1_R(R/I, R/I)$ is not a finite R-module. (We know this cannot happen if R is Noetherian by Algebra, Lemma 71.9.)

12. Depth

Depth is defined in Algebra, Section 72 and further studied in Dualizing Complexes, Section 11.

Exercise 12.1. Let R be a ring, $I \subset R$ an ideal, and M an R-module. Compute $\text{depth}_I(M)$ in the following cases.

(1) $R = \mathbf{Z}$, $I = (30)$, $M = \mathbf{Z}$,
(2) $R = \mathbf{Z}$, $I = (30)$, $M = \mathbf{Z}/(300)$,
(3) $R = \mathbf{Z}$, $I = (30)$, $M = \mathbf{Z}/(7)$,
(4) $R = k[x, y, z]/(x^2 + y^2 + z^2)$, $I = (x, y, z)$, $M = R$,
(5) $R = k[x, y, z, w]/(xz, xw, yz, yw)$, $I = (x, y, z, w)$, $M = R$.

Here k is a field. In the last two cases feel free to localize at the maximal ideal I.

Exercise 12.2. Give an example of a Noetherian local ring $(R, \mathfrak{m}, \kappa)$ of depth ≥ 1 and a prime ideal $\mathfrak{p}$ such that

(1) $\text{depth}_{\mathfrak{m}}(R) \geq 1$,
(2) $\text{depth}_{\mathfrak{p}}(R_{\mathfrak{p}}) = 0$, and
(3) $\dim(R_{\mathfrak{p}}) \geq 1$.

If we don't ask for (3) then the exercise is too easy. Why?

Exercise 12.3. Let $(R, \mathfrak{m})$ be a local Noetherian domain. Let M be a finite R-module.

(1) If M is torsion free, show that M has depth at least 1 over R.
(2) Give an example with depth equal to 1.

Exercise 12.4. For every $m \geq n \geq 0$ give an example of a Noetherian local ring R with $\dim(R) = m$ and $\text{depth}(R) = n$.

Exercise 12.5. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a finite R-module. Show that there exists a canonical short exact sequence

$$0 \to K \to M \to Q \to 0$$

such that the following are true

(1) $\text{depth}(Q) \geq 1$,
(2) K is zero or $\text{Supp}(K) = \{\mathfrak{m}\}$, and
(3) $\text{length}_R(K) < \infty$.

Hint: using the Noetherian property show that there exists a maximal submodule K as in (2) and then show that $Q = M/K$ satisfies (1) and K satisfies (3).

Exercise 12.6. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a finite R-module of depth ≥ 2. Let $N \subset M$ be a nonzero submodule.

(1) Show that $\text{depth}(N) \geq 1$.
(2) Show that $\text{depth}(N) = 1$ if and only if the quotient module M/N has $\text{depth}(M/N) = 0$.
(3) Show there exists a submodule $N' \subset M$ with $N \subset N'$ of finite colength, i.e., $\text{length}_R(N'/N) < \infty$, such that N' has depth ≥ 2. Hint: Apply Exercise 12.5 to M/N and choose N' to be the inverse image of K.

Exercise 12.7. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Assume that R is reduced, i.e., R has no nonzero nilpotent elements. Assume moreover that R has two distinct minimal primes $\mathfrak{p}$ and $\mathfrak{q}$.

(1) Show that the sequence of R-modules

$$0 \to R \to R/\mathfrak{p} \oplus R/\mathfrak{q} \to R/\mathfrak{p} + \mathfrak{q} \to 0$$

is exact (check at all the spots). The maps are $x \mapsto (x \bmod \mathfrak{p}, x \bmod \mathfrak{q})$ and $(y \bmod \mathfrak{p}, z \bmod \mathfrak{q}) \mapsto (y - z \bmod \mathfrak{p} + \mathfrak{q})$.
(2) Show that if $\text{depth}(R) \geq 2$, then $\dim(R/\mathfrak{p} + \mathfrak{q}) \geq 1$.

(3) Show that if $\text{depth}(R) \geq 2$, then $U = \text{Spec}(R) \setminus \{\mathfrak{m}\}$ is a connected topological space.

This proves a very special case of Hartshorne's connectedness theorem which says that the punctured spectrum U of a local Noetherian ring of depth ≥ 2 is connected.

Exercise 12.8. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $x, y \in \mathfrak{m}$ be a regular sequence of length 2. For any $n \geq 2$ show that there do not exist $a, b \in R$ with

$$x^{n-1}y^{n-1} = ax^n + by^n$$

Suggestion: First try for $n = 2$ to see how to argue. Remark: There is a vast generalization of this result called the monomial conjecture.

13. Cohen-Macaulay modules and rings

Cohen-Macaulay modules are studied in Algebra, Section 103 and Cohen-Macaulay rings are studied in Algebra, Section 104.

Exercise 13.1. In the following cases, please answer yes or no. No explanation or proof necessary.

(1) Let p be a prime number. Is the local ring $\mathbf{Z}_{(p)}$ a Cohen-Macaulay local ring?
(2) Let p be a prime number. Is the local ring $\mathbf{Z}_{(p)}$ a regular local ring?
(3) Let k be a field. Is the local ring $k[x]_{(x)}$ a Cohen-Macaulay local ring?
(4) Let k be a field. Is the local ring $k[x]_{(x)}$ a regular local ring?
(5) Let k be a field. Is the local ring $(k[x,y]/(y^2 - x^3))_{(x,y)} = k[x,y]_{(x,y)}/(y^2 - x^3)$ a Cohen-Macaulay local ring?
(6) Let k be a field. Is the local ring $(k[x,y]/(y^2, xy))_{(x,y)} = k[x,y]_{(x,y)}/(y^2, xy)$ a Cohen-Macaulay local ring?

14. Singularities

Exercise 14.1. Let k be any field. Suppose that $A = k[[x,y]]/(f)$ and $B = k[[u,v]]/(g)$, where $f = xy$ and $g = uv + \delta$ with $\delta \in (u,v)^3$. Show that A and B are isomorphic rings.

Remark 14.2. A singularity on a curve over a field k is called an ordinary double point if the complete local ring of the curve at the point is of the form $k'[[x,y]]/(f)$, where (a) k' is a finite separable extension of k, (b) the initial term of f has degree two, i.e., it looks like $q = ax^2 + bxy + cy^2$ for some $a, b, c \in k'$ not all zero, and (c) q is a nondegenerate quadratic form over k' (in char 2 this means that b is not zero). In general there is one isomorphism class of such rings for each isomorphism class of pairs (k', q).

Exercise 14.3. Let R be a ring. Let $n \geq 1$. Let A, B be $n \times n$ matrices with coefficients in R such that $AB = f1_{n\times n}$ for some nonzerodivisor f in R. Set $S = R/(f)$. Show that

$$\ldots \to S^{\oplus n} \xrightarrow{B} S^{\oplus n} \xrightarrow{A} S^{\oplus n} \xrightarrow{B} S^{\oplus n} \to \ldots$$

is exact.

15. Constructible sets

Let k be an algebraically closed field, for example the field $\mathbf{C}$ of complex numbers. Let $n \geq 0$. A polynomial $f \in k[x_1, \ldots, x_n]$ gives a function $f : k^n \to k$ by evaluation. A subset $Z \subset k^n$ is called an *algebraic set* if it is the common vanishing set of a collection of polynomials.

Exercise 15.1. Prove that an algebraic set can always be written as the zero locus of finitely many polynomials.

With notation as above a subset $E \subset k^n$ is called *constructible* if it is a finite union of sets of the form $Z \cap \{f \neq 0\}$ where f is a polynomial.

Exercise 15.2. Show the following

1. the complement of a constructible set is a constructible set,
2. a finite union of constructible sets is a constructible set,
3. a finite intersection of constructible sets is a constructible set, and
4. any constructible set E can be written as a finite disjoint union $E = \coprod E_i$ with each E_i of the form $Z \cap \{f \neq 0\}$ where Z is an algebraic set and f is a polynomial.

Exercise 15.3. Let R be a ring. Let $f = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 \in R[x]$. (As usual this notation means $a_0, \ldots, a_d \in R$.) Let $g \in R[x]$. Prove that we can find $N \geq 0$ and $r, q \in R[x]$ such that

$$a_d^N g = qf + r$$

with $\deg(r) < d$, i.e., for some $c_i \in R$ we have $r = c_0 + c_1 x + \ldots + c_{d-1} x^{d-1}$.

16. Hilbert Nullstellensatz

Exercise 16.1. *A silly argument using the complex numbers!* Let $\mathbf{C}$ be the complex number field. Let V be a vector space over $\mathbf{C}$. The spectrum of a linear operator $T : V \to V$ is the set of complex numbers $\lambda \in \mathbf{C}$ such that the operator $T - \lambda \mathrm{id}_V$ is not invertible.

1. Show that $\mathbf{C}(X)$ has uncountable dimension over $\mathbf{C}$.
2. Show that any linear operator on V has a nonempty spectrum if the dimension of V is finite or countable.
3. Show that if a finitely generated $\mathbf{C}$-algebra R is a field, then the map $\mathbf{C} \to R$ is an isomorphism.
4. Show that any maximal ideal $\mathfrak{m}$ of $\mathbf{C}[x_1, \ldots, x_n]$ is of the form $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ for some $\alpha_i \in \mathbf{C}$.

Remark 16.2. Let k be a field. Then for every integer $n \in \mathbf{N}$ and every maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ the quotient $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension of k. This will be shown later in the course. Of course (please check this) it implies a similar statement for maximal ideals of finitely generated k-algebras. The exercise above proves it in the case $k = \mathbf{C}$.

Exercise 16.3. Let k be a field. Please use Remark 16.2.

1. Let R be a k-algebra. Suppose that $\dim_k R < \infty$ and that R is a domain. Show that R is a field.
2. Suppose that R is a finitely generated k-algebra, and $f \in R$ not nilpotent. Show that there exists a maximal ideal $\mathfrak{m} \subset R$ with $f \notin \mathfrak{m}$.

(3) Show by an example that this statement fails when R is not of finite type over a field.
(4) Show that any radical ideal $I \subset \mathbf{C}[x_1, \ldots, x_n]$ is the intersection of the maximal ideals containing it.

Remark 16.4. This is the Hilbert Nullstellensatz. Namely it says that the closed subsets of $\mathrm{Spec}(k[x_1, \ldots, x_n])$ (which correspond to radical ideals by a previous exercise) are determined by the closed points contained in them.

Exercise 16.5. Let $A = \mathbf{C}[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$. Let I be the ideal of A generated by the entries of the matrix XY, with

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Find the irreducible components of the closed subset $V(I)$ of $\mathrm{Spec}(A)$. (I mean describe them and give equations for each of them. You do not have to prove that the equations you write down define prime ideals.) Hints:

(1) You may use the Hilbert Nullstellensatz, and it suffices to find irreducible locally closed subsets which cover the set of closed points of $V(I)$.
(2) There are two easy components.
(3) An image of an irreducible set under a continuous map is irreducible.

17. Dimension

Exercise 17.1. Construct a ring A with finitely many prime ideals having dimension > 1.

Exercise 17.2. Let $f \in \mathbf{C}[x, y]$ be a nonconstant polynomial. Show that $\mathbf{C}[x, y]/(f)$ has dimension 1.

Exercise 17.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $n \geq 1$. Let $\mathfrak{m}' = (\mathfrak{m}, x_1, \ldots, x_n)$ in the polynomial ring $R[x_1, \ldots, x_n]$. Show that

$$\dim(R[x_1, \ldots, x_n]_{\mathfrak{m}'}) = \dim(R) + n.$$

18. Catenary rings

Definition 18.1. A Noetherian ring A is said to be *catenary* if for any triple of prime ideals $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \mathfrak{p}_3$ we have

$$ht(\mathfrak{p}_3/\mathfrak{p}_1) = ht(\mathfrak{p}_3/\mathfrak{p}_2) + ht(\mathfrak{p}_2/\mathfrak{p}_1).$$

Here $ht(\mathfrak{p}/\mathfrak{q})$ means the height of $\mathfrak{p}/\mathfrak{q}$ in the ring $A/\mathfrak{q}$. In a formula

$$ht(\mathfrak{p}/\mathfrak{q}) = \dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}) = \dim((A/\mathfrak{q})_\mathfrak{p}) = \dim((A/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}})$$

A topological space X is *catenary*, if given $T \subset T' \subset X$ with T and T' closed and irreducible, then there exists a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_n = T'$$

and every such chain has the same (finite) length.

Exercise 18.2. Show that the notion of catenary defined in Algebra, Definition 105.1 agrees with the notion of Definition 18.1 for Noetherian rings.

Exercise 18.3. Show that a Noetherian local domain of dimension 2 is catenary.

Exercise 18.4. Let k be a field. Show that a finite type k-algebra is catenary.

Exercise 18.5. Give an example of a finite, sober, catenary topological space X which does not have a dimension function $\delta : X \to \mathbf{Z}$. Here $\delta : X \to \mathbf{Z}$ is a dimension function if for $x, y \in X$ we have

1. $x \rightsquigarrow y$ and $x \neq y$ implies $\delta(x) > \delta(y)$,
2. $x \rightsquigarrow y$ and $\delta(x) \geq \delta(y) + 2$ implies there exists a $z \in X$ with $x \rightsquigarrow z \rightsquigarrow y$ and $\delta(x) > \delta(z) > \delta(y)$.

Describe your space clearly and succintly explain why there cannot be a dimension function.

19. Fraction fields

Exercise 19.1. Consider the domain

$$\mathbf{Q}[r, s, t]/(s^2 - (r-1)(r-2)(r-3), t^2 - (r+1)(r+2)(r+3)).$$

Find a domain of the form $\mathbf{Q}[x, y]/(f)$ with isomorphic field of fractions.

20. Transcendence degree

Exercise 20.1. Let $K'/K/k$ be field extensions with K' algebraic over K. Prove that $\text{trdeg}_k(K) = \text{trdeg}_k(K')$. (Hint: Show that if $x_1, \ldots, x_d \in K$ are algebraically independent over k and $d < \text{trdeg}_k(K')$ then $k(x_1, \ldots, x_d) \subset K$ cannot be algebraic.)

Exercise 20.2. Let k be a field. Let K/k be a finitely generated extension of transcendence degree d. If $V, W \subset K$ are finite dimensional k-subvector spaces denote

$$VW = \{f \in K \mid f = \sum\nolimits_{i=1,\ldots,n} v_i w_i \text{ for some } n \text{ and } v_i \in V, w_i \in W\}$$

This is a finite dimensional k-subvector space. Set $V^2 = VV$, $V^3 = VV^2$, etc.

1. Show you can find $V \subset K$ and $\epsilon > 0$ such that $\dim V^n \geq \epsilon n^d$ for all $n \geq 1$.
2. Conversely, show that for every finite dimensional $V \subset K$ there exists a $C > 0$ such that $\dim V^n \leq C n^d$ for all $n \geq 1$. (One possible way to proceed: First do this for subvector spaces of $k[x_1, \ldots, x_d]$. Then do this for subvector spaces of $k(x_1, \ldots, x_d)$. Finally, if $K/k(x_1, \ldots, x_d)$ is a finite extension choose a basis of K over $k(x_1, \ldots, x_d)$ and argue using expansion in terms of this basis.)
3. Conclude that you can redefine the transcendence degree in terms of growth of powers of finite dimensional subvector spaces of K.

This is related to Gelfand-Kirillov dimension of (noncommutative) algebras over k.

21. Dimension of fibres

Some questions related to the dimension formula, see Algebra, Section 113.

Exercise 21.1. Let k be your favorite algebraically closed field. Below $k[x]$ and $k[x, y]$ denote the polynomial rings.

1. For every integer $n \geq 0$ find a finite type extension $k[x] \subset A$ of domains such that the spectrum of A/xA has exactly n irreducible components.
2. Make an example of a finite type extension $k[x] \subset A$ of domains such that the spectrum of $A/(x - \alpha)A$ is nonempty and reducible for every $\alpha \in k$.

(3) Make an example of a finite type extension $k[x, y] \subset A$ of domains such that the spectrum of $A/(x-\alpha, y-\beta)A$ is irreducible[1] for all $(\alpha, \beta) \in k^2 \setminus \{(0, 0)\}$ and the spectrum of $A/(x, y)A$ is nonempty and reducible.

Exercise 21.2. Let k be your favorite algebraically closed field. Let $n \geq 1$. Let $k[x_1, \ldots, x_n]$ be the polynomial ring. Set $\mathfrak{m} = (x_1, \ldots, x_n)$. Let $k[x_1, \ldots, x_n] \subset A$ be a finite type extension of domains. Set $d = \dim(A)$.

(1) Show that $d - 1 \geq \dim(A/\mathfrak{m}A) \geq d - n$ if $A/\mathfrak{m}A \neq 0$.
(2) Show by example that every value can occur.
(3) Show by example that $\mathrm{Spec}(A/\mathfrak{m}A)$ can have irreducible components of different dimensions.

22. Finite locally free modules

Definition 22.1. Let A be a ring. Recall that a *finite locally free* A-module M is a module such that for every $\mathfrak{p} \in \mathrm{Spec}(A)$ there exists an $f \in A$, $f \notin \mathfrak{p}$ such that M_f is a finite free A_f-module. We say M is an *invertible module* if M is finite locally free of rank 1, i.e., for every $\mathfrak{p} \in \mathrm{Spec}(A)$ there exists an $f \in A$, $f \notin \mathfrak{p}$ such that $M_f \cong A_f$ as an A_f-module.

Exercise 22.2. Prove that the tensor product of finite locally free modules is finite locally free. Prove that the tensor product of two invertible modules is invertible.

Definition 22.3. Let A be a ring. The *class group of* A, sometimes called the *Picard group of* A is the set $\mathrm{Pic}(A)$ of isomorphism classes of invertible A-modules endowed with a group operation defined by tensor product (see Exercise 22.2).

Note that the class group of A is trivial exactly when every invertible module is isomorphic to a free module of rank 1.

Exercise 22.4. Show that the class groups of the following rings are trivial

(1) a polynomial ring $A = k[x]$ where k is a field,
(2) the integers $A = \mathbf{Z}$,
(3) a polynomial ring $A = k[x, y]$ where k is a field, and
(4) the quotient $k[x, y]/(xy)$ where k is a field.

Exercise 22.5. Show that the class group of the ring $A = k[x, y]/(y^2 - f(x))$ where k is a field of characteristic not 2 and where $f(x) = (x - t_1) \ldots (x - t_n)$ with $t_1, \ldots, t_n \in k$ distinct and $n \geq 3$ an odd integer is not trivial. (Hint: Show that the ideal $(y, x - t_1)$ defines a nontrivial element of $\mathrm{Pic}(A)$.)

Exercise 22.6. Let A be a ring.

(1) Suppose that M is a finite locally free A-module, and suppose that $\varphi : M \to M$ is an endomorphism. Define/construct the *trace* and *determinant* of φ and prove that your construction is "functorial in the triple (A, M, φ)".
(2) Show that if M, N are finite locally free A-modules, and if $\varphi : M \to N$ and $\psi : N \to M$ then $\mathrm{Trace}(\varphi \circ \psi) = \mathrm{Trace}(\psi \circ \varphi)$ and $\det(\varphi \circ \psi) = \det(\psi \circ \varphi)$.
(3) In case M is finite locally free show that Trace defines an A-linear map $\mathrm{End}_A(M) \to A$ and det defines a multiplicative map $\mathrm{End}_A(M) \to A$.

Exercise 22.7. Now suppose that B is an A-algebra which is finite locally free as an A-module, in other words B is a finite locally free A-algebra.

[1]Recall that irreducible implies nonempty.

(1) Define $\mathrm{Trace}_{B/A}$ and $\mathrm{Norm}_{B/A}$ using Trace and det from Exercise 22.6.
(2) Let $b \in B$ and let $\pi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the induced morphism. Show that $\pi(V(b)) = V(\mathrm{Norm}_{B/A}(b))$. (Recall that $V(f) = \{\mathfrak{p} \mid f \in \mathfrak{p}\}$.)
(3) (Base change.) Suppose that $i : A \to A'$ is a ring map. Set $B' = B \otimes_A A'$. Indicate why $i(\mathrm{Norm}_{B/A}(b))$ equals $\mathrm{Norm}_{B'/A'}(b \otimes 1)$.
(4) Compute $\mathrm{Norm}_{B/A}(b)$ when $B = A \times A \times A \times \ldots \times A$ and $b = (a_1, \ldots, a_n)$.
(5) Compute the norm of $y - y^3$ under the finite flat map $\mathbf{Q}[x] \to \mathbf{Q}[y]$, $x \to y^n$. (Hint: use the "base change" $A = \mathbf{Q}[x] \subset A' = \mathbf{Q}(\zeta_n)(x^{1/n})$.)

23. Glueing

Exercise 23.1. Suppose that A is a ring and M is an A-module. Let f_i, $i \in I$ be a collection of elements of A such that

$$\mathrm{Spec}(A) = \bigcup D(f_i).$$

(1) Show that if M_{f_i} is a finite A_{f_i}-module, then M is a finite A-module.
(2) Show that if M_{f_i} is a flat A_{f_i}-module, then M is a flat A-module. (This is kind of silly if you think about it right.)

Remark 23.2. In algebraic geometric language this means that the property of "being finitely generated" or "being flat" is local for the Zariski topology (in a suitable sense). You can also show this for the property "being of finite presentation".

Exercise 23.3. Suppose that $A \to B$ is a ring map. Let $f_i \in A$, $i \in I$ and $g_j \in B$, $j \in J$ be collections of elements such that

$$\mathrm{Spec}(A) = \bigcup D(f_i) \quad \text{and} \quad \mathrm{Spec}(B) = \bigcup D(g_j).$$

Show that if $A_{f_i} \to B_{f_i g_j}$ is of finite type for all i, j then $A \to B$ is of finite type.

24. Going up and going down

Definition 24.1. Let $\phi : A \to B$ be a homomorphism of rings. We say that the *going-up theorem* holds for ϕ if the following condition is satisfied:

(GU) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$, there exists $P' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$ such that $P \subset P'$.

Similarly, we say that the *going-down theorem* holds for ϕ if the following condition is satisfied:

(GD) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$, there exists $P \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$ such that $P \subset P'$.

Exercise 24.2. In each of the following cases determine whether (GU), (GD) holds, and explain why. (Use any Prop/Thm/Lemma you can find, but check the hypotheses in each case.)

(1) k is a field, $A = k$, $B = k[x]$.
(2) k is a field, $A = k[x]$, $B = k[x, y]$.
(3) $A = \mathbf{Z}$, $B = \mathbf{Z}[1/11]$.
(4) k is an algebraically closed field, $A = k[x, y]$, $B = k[x, y, z]/(x^2 - y, z^2 - x)$.
(5) $A = \mathbf{Z}$, $B = \mathbf{Z}[i, 1/(2 + i)]$.
(6) $A = \mathbf{Z}$, $B = \mathbf{Z}[i, 1/(14 + 7i)]$.
(7) k is an algebraically closed field, $A = k[x]$, $B = k[x, y, 1/(xy - 1)]/(y^2 - y)$.

Exercise 24.3. Let A be a ring. Let $B = A[x]$ be the polynomial algebra in one variable over A. Let $f = a_0 + a_1 x + \ldots + a_r x^r \in B = A[x]$. Prove carefully that the image of $D(f)$ in $\mathrm{Spec}(A)$ is equal to $D(a_0) \cup \ldots \cup D(a_r)$.

Exercise 24.4. Let k be an algebraically closed field. Compute the image in $\mathrm{Spec}(k[x,y])$ of the following maps:

(1) $\mathrm{Spec}(k[x, yx^{-1}]) \to \mathrm{Spec}(k[x,y])$, where $k[x,y] \subset k[x, yx^{-1}] \subset k[x,y,x^{-1}]$. (Hint: To avoid confusion, give the element yx^{-1} another name.)
(2) $\mathrm{Spec}(k[x,y,a,b]/(ax - by - 1)) \to \mathrm{Spec}(k[x,y])$.
(3) $\mathrm{Spec}(k[t, 1/(t-1)]) \to \mathrm{Spec}(k[x,y])$, induced by $x \mapsto t^2$, and $y \mapsto t^3$.
(4) $k = \mathbf{C}$ (complex numbers), $\mathrm{Spec}(k[s,t]/(s^3 + t^3 - 1)) \to \mathrm{Spec}(k[x,y])$, where $x \mapsto s^2$, $y \mapsto t^2$.

Remark 24.5. Finding the image as above usually is done by using elimination theory.

25. Fitting ideals

Exercise 25.1. Let R be a ring and let M be a finite R-module. Choose a presentation

$$\bigoplus\nolimits_{j \in J} R \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0.$$

of M. Note that the map $R^{\oplus n} \to M$ is given by a sequence of elements $x_1, \ldots, x_n$ of M. The elements x_i are *generators* of M. The map $\bigoplus_{j \in J} R \to R^{\oplus n}$ is given by a $n \times J$ matrix A with coefficients in R. In other words, $A = (a_{ij})_{i=1,\ldots,n, j \in J}$. The columns $(a_{1j}, \ldots, a_{nj})$, $j \in J$ of A are said to be the *relations*. Any vector $(r_i) \in R^{\oplus n}$ such that $\sum r_i x_i = 0$ is a linear combination of the columns of A. Of course any finite R-module has a lot of different presentations.

(1) Show that the ideal generated by the $(n-k) \times (n-k)$ minors of A is independent of the choice of the presentation. This ideal is the *kth Fitting ideal of M*. Notation $Fit_k(M)$.
(2) Show that $Fit_0(M) \subset Fit_1(M) \subset Fit_2(M) \subset \ldots$. (Hint: Use that a determinant can be computed by expanding along a column.)
(3) Show that the following are equivalent:
    (a) $Fit_{r-1}(M) = (0)$ and $Fit_r(M) = R$, and
    (b) M is locally free of rank r.

26. Hilbert functions

Definition 26.1. A *numerical polynomial* is a polynomial $f(x) \in \mathbf{Q}[x]$ such that $f(n) \in \mathbf{Z}$ for every integer n.

Definition 26.2. A *graded module* M over a ring A is an A-module M endowed with a direct sum decomposition $\bigoplus_{n \in \mathbf{Z}} M_n$ into A-submodules. We will say that M is *locally finite* if all of the M_n are finite A-modules. Suppose that A is a Noetherian ring and that φ is a *Euler-Poincaré function* on finite A-modules. This means that for every finitely generated A-module M we are given an integer $\varphi(M) \in \mathbf{Z}$ and for every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we have $\varphi(M) = \varphi(M') + \varphi(M'')$. The *Hilbert function* of a locally finite graded module M (with respect to φ) is the function $\chi_\varphi(M, n) = \varphi(M_n)$. We say that

M has a *Hilbert polynomial* if there is some numerical polynomial P_φ such that $\chi_\varphi(M, n) = P_\varphi(n)$ for all sufficiently large integers n.

Definition 26.3. A *graded A-algebra* is a graded A-module $B = \bigoplus_{n \geq 0} B_n$ together with an A-bilinear map

$$B \times B \longrightarrow B, \ (b, b') \longmapsto bb'$$

that turns B into an A-algebra so that $B_n \cdot B_m \subset B_{n+m}$. Finally, a *graded module M over a graded A-algebra B* is given by a graded A-module M together with a (compatible) B-module structure such that $B_n \cdot M_d \subset M_{n+d}$. Now you can define *homomorphisms of graded modules/rings*, *graded submodules*, *graded ideals*, *exact sequences of graded modules*, etc, etc.

Exercise 26.4. Let $A = k$ a field. What are all possible Euler-Poincaré functions on finite A-modules in this case?

Exercise 26.5. Let $A = \mathbf{Z}$. What are all possible Euler-Poincaré functions on finite A-modules in this case?

Exercise 26.6. Let $A = k[x, y]/(xy)$ with k algebraically closed. What are all possible Euler-Poincaré functions on finite A-modules in this case?

Exercise 26.7. Suppose that A is Noetherian. Show that the kernel of a map of locally finite graded A-modules is locally finite.

Exercise 26.8. Let k be a field and let $A = k$ and $B = k[x, y]$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

Exercise 26.9. Let k be a field and let $A = k$ and $B = k[x, y]/(x^2, xy)$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

Exercise 26.10. Let k be a field and let $A = k$. Let $\varphi(M) = \dim_k(M)$. Fix $d \in \mathbf{N}$. Consider the graded A-algebra $B = k[x, y, z]/(x^d + y^d + z^d)$, where x, y, z each have degree 1. Compute the Hilbert function of B. Is there a Hilbert polynomial in this case?

27. Proj of a ring

Definition 27.1. Let R be a graded ring. A *homogeneous* ideal is simply an ideal $I \subset R$ which is also a graded submodule of R. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of f into homogeneous pieces in R then $f_i \in I$ for each i.

Definition 27.2. We define the *homogeneous spectrum Proj*(R) of the graded ring R to be the set of homogeneous, prime ideals $\mathfrak{p}$ of R such that $R_+ \not\subset \mathfrak{p}$. Note that $\mathrm{Proj}(R)$ is a subset of $\mathrm{Spec}(R)$ and hence has a natural induced topology.

Definition 27.3. Let $R = \oplus_{d\geq 0} R_d$ be a graded ring, let $f \in R_d$ and assume that $d \geq 1$. We define $R_{(f)}$ to be the subring of R_f consisting of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$. Furthermore, we define

$$D_+(f) = \{\mathfrak{p} \in \text{Proj}(R) | f \notin \mathfrak{p}\}.$$

Finally, for a homogeneous ideal $I \subset R$ we define $V_+(I) = V(I) \cap \text{Proj}(R)$.

Exercise 27.4. On the topology on $\text{Proj}(R)$. With definitions and notation as above prove the following statements.

(1) Show that $D_+(f)$ is open in $\text{Proj}(R)$.
(2) Show that $D_+(ff') = D_+(f) \cap D_+(f')$.
(3) Let $g = g_0 + \ldots + g_m$ be an element of R with $g_i \in R_i$. Express $D(g) \cap \text{Proj}(R)$ in terms of $D_+(g_i)$, $i \geq 1$ and $D(g_0) \cap \text{Proj}(R)$. No proof necessary.
(4) Let $g \in R_0$ be a homogeneous element of degree 0. Express $D(g) \cap \text{Proj}(R)$ in terms of $D_+(f_\alpha)$ for a suitable family $f_\alpha \in R$ of homogeneous elements of positive degree.
(5) Show that the collection $\{D_+(f)\}$ of opens forms a basis for the topology of $\text{Proj}(R)$.
(6) Show that there is a canonical bijection $D_+(f) \to \text{Spec}(R_{(f)})$. (Hint: Imitate the proof for Spec but at some point thrown in the radical of an ideal.)
(7) Show that the map from (6) is a homeomorphism.
(8) Give an example of an R such that $\text{Proj}(R)$ is not quasi-compact. No proof necessary.
(9) Show that any closed subset $T \subset \text{Proj}(R)$ is of the form $V_+(I)$ for some homogeneous ideal $I \subset R$.

Remark 27.5. There is a continuous map $\text{Proj}(R) \longrightarrow \text{Spec}(R_0)$.

Exercise 27.6. If $R = A[X]$ with $\deg(X) = 1$, show that the natural map $\text{Proj}(R) \to \text{Spec}(A)$ is a bijection and in fact a homeomorphism.

Exercise 27.7. Blowing up: part I. In this exercise $R = Bl_I(A) = A \oplus I \oplus I^2 \oplus \ldots$. Consider the natural map $b : \text{Proj}(R) \to \text{Spec}(A)$. Set $U = \text{Spec}(A) - V(I)$. Show that

$$b : b^{-1}(U) \longrightarrow U$$

is a homeomorphism. Thus we may think of U as an open subset of $\text{Proj}(R)$. Let $Z \subset \text{Spec}(A)$ be an irreducible closed subscheme with generic point $\xi \in Z$. Assume that $\xi \notin V(I)$, in other words $Z \not\subset V(I)$, in other words $\xi \in U$, in other words $Z \cap U \neq \emptyset$. We define the *strict transform* Z' of Z to be the closure of the unique point ξ' lying above ξ. Another way to say this is that Z' is the closure in $\text{Proj}(R)$ of the locally closed subset $Z \cap U \subset U \subset \text{Proj}(R)$.

Exercise 27.8. Blowing up: Part II. Let $A = k[x, y]$ where k is a field, and let $I = (x, y)$. Let R be the blowup algebra for A and I.

(1) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{y\})$ are disjoint.
(2) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ are not disjoint.
(3) Find an ideal $J \subset A$ such that $V(J) = V(I)$ and such that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ in the blowup along J are disjoint.

Exercise 27.9. Let R be a graded ring.

(1) Show that $\mathrm{Proj}(R)$ is empty if $R_n = (0)$ for all $n >> 0$.
(2) Show that $\mathrm{Proj}(R)$ is an irreducible topological space if R is a domain and R_+ is not zero. (Recall that the empty topological space is not irreducible.)

Exercise 27.10. Blowing up: Part III. Consider A, I and U, Z as in the definition of strict transform. Let $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. Let $\bar{A} = A/\mathfrak{p}$ and let $\bar{I}$ be the image of I in $\bar{A}$.

(1) Show that there exists a surjective ring map $R := Bl_I(A) \to \bar{R} := Bl_{\bar{I}}(\bar{A})$.
(2) Show that the ring map above induces a bijective map from $\mathrm{Proj}(\bar{R})$ onto the strict transform Z' of Z. (This is not so easy. Hint: Use 5(b) above.)
(3) Conclude that the strict transform $Z' = V_+(P)$ where $P \subset R$ is the homogeneous ideal defined by $P_d = I^d \cap \mathfrak{p}$.
(4) Suppose that $Z_1 = V(\mathfrak{p})$ and $Z_2 = V(\mathfrak{q})$ are irreducible closed subsets defined by prime ideals such that $Z_1 \not\subset Z_2$, and $Z_2 \not\subset Z_1$. Show that blowing up the ideal $I = \mathfrak{p} + \mathfrak{q}$ separates the strict transforms of Z_1 and Z_2, i.e., $Z_1' \cap Z_2' = \emptyset$. (Hint: Consider the homogeneous ideal P and Q from part (c) and consider $V(P + Q)$.)

28. Cohen-Macaulay rings of dimension 1

Definition 28.1. A Noetherian local ring A is said to be *Cohen-Macaulay* of dimension d if it has dimension d and there exists a system of parameters $x_1, \ldots, x_d$ for A such that x_i is a nonzerodivisor in $A/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, d$.

Exercise 28.2. Cohen-Macaulay rings of dimension 1. Part I: Theory.

(1) Let $(A, \mathfrak{m})$ be a local Noetherian with $\dim A = 1$. Show that if $x \in \mathfrak{m}$ is not a zerodivisor then
    (a) $\dim A/xA = 0$, in other words A/xA is Artinian, in other words $\{x\}$ is a system of parameters for A.
    (b) A is has no embedded prime.
(2) Conversely, let $(A, \mathfrak{m})$ be a local Noetherian ring of dimension 1. Show that if A has no embedded prime then there exists a nonzerodivisor in $\mathfrak{m}$.

Exercise 28.3. Cohen-Macaulay rings of dimension 1. Part II: Examples.

(1) Let A be the local ring at (x, y) of $k[x, y]/(x^2, xy)$.
    (a) Show that A has dimension 1.
    (b) Prove that every element of $\mathfrak{m} \subset A$ is a zerodivisor.
    (c) Find $z \in \mathfrak{m}$ such that $\dim A/zA = 0$ (no proof required).
(2) Let A be the local ring at (x, y) of $k[x, y]/(x^2)$. Find a nonzerodivisor in $\mathfrak{m}$ (no proof required).

Exercise 28.4. Local rings of embedding dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Show that the function $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is either constant with value 1, or its values are

$$1, 1, \ldots, 1, 0, 0, 0, 0, 0, \ldots$$

Exercise 28.5. Regular local rings of dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension 1. Recall that this means that A has dimension 1 and embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Let $x \in \mathfrak{m}$ be any element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is not zero.

(1) Show that for every element y of $\mathfrak{m}$ there exists an integer n such that y can be written as $y = ux^n$ with $u \in A^*$ a unit.
(2) Show that x is a nonzerodivisor in A.
(3) Conclude that A is a domain.

Exercise 28.6. Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring with associated graded $Gr_{\mathfrak{m}}(A)$.

(1) Suppose that $x \in \mathfrak{m}^d$ maps to a nonzerodivisor $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree d of $Gr_{\mathfrak{m}}(A)$. Show that x is a nonzerodivisor.
(2) Suppose the depth of A is at least 1. Namely, suppose that there exists a nonzerodivisor $y \in \mathfrak{m}$. In this case we can do better: assume just that $x \in \mathfrak{m}^d$ maps to the element $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree d of $Gr_{\mathfrak{m}}(A)$ which is a nonzerodivisor on sufficiently high degrees: $\exists N$ such that for all $n \geq N$ the map of multiplication by $\bar{x}$

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow \mathfrak{m}^{n+d}/\mathfrak{m}^{n+d+1}$$

is injective. Then show that x is a nonzerodivisor.

Exercise 28.7. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of dimension 1. Assume also that the embedding dimension of A is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notation: $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Pick generators $x, y \in \mathfrak{m}$ and write $Gr_{\mathfrak{m}}(A) = k[\bar{x}, \bar{y}]/I$ for some homogeneous ideal I.

(1) Show that there exists a homogeneous element $F \in k[\bar{x}, \bar{y}]$ such that $I \subset (F)$ with equality in all sufficiently high degrees.
(2) Show that $f(n) \leq n + 1$.
(3) Show that if $f(n) < n + 1$ then $n \geq \deg(F)$.
(4) Show that if $f(n) < n + 1$, then $f(n + 1) \leq f(n)$.
(5) Show that $f(n) = \deg(F)$ for all $n >> 0$.

Exercise 28.8. Cohen-Macaulay rings of dimension 1 and embedding dimension 2. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring which is Cohen-Macaulay of dimension 1. Assume also that the embedding dimension of A is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notations: f, F, $x, y \in \mathfrak{m}$, I as in Ex. 6 above. Please use any results from the problems above.

(1) Suppose that $z \in \mathfrak{m}$ is an element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is a linear form $\alpha\bar{x} + \beta\bar{y} \in k[\bar{x}, \bar{y}]$ which is coprime with f.
  (a) Show that z is a nonzerodivisor on A.
  (b) Let $d = \deg(F)$. Show that $\mathfrak{m}^n = z^{n+1-d}\mathfrak{m}^{d-1}$ for all sufficiently large n. (Hint: First show $z^{n+1-d}\mathfrak{m}^{d-1} \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is surjective by what you know about $Gr_{\mathfrak{m}}(A)$. Then use NAK.)

(2) What condition on k guarantees the existence of such a z? (No proof required; it's too easy.)
Now we are going to assume there exists a z as above. This turns out to be a harmless assumption (in the sense that you can reduce to the situation where it holds in order to obtain the results in parts (d) and (e) below).
(3) Now show that $\mathfrak{m}^\ell = z^{\ell-d+1}\mathfrak{m}^{d-1}$ for all $\ell \geq d$.
(4) Conclude that $I = (F)$.
(5) Conclude that the function f has values
$$2, 3, 4, \ldots, d-1, d, d, d, d, d, d, d, \ldots$$

Remark 28.9. This suggests that a local Noetherian Cohen-Macaulay ring of dimension 1 and embedding dimension 2 is of the form B/FB, where B is a 2-dimensional regular local ring. This is more or less true (under suitable "niceness" properties of the ring).

29. Infinitely many primes

A section with a collection of strange questions on rings where infinitely many primes are not invertible.

Exercise 29.1. Give an example of a finite type $\mathbf{Z}$-algebra R with the following two properties:

(1) There is no ring map $R \to \mathbf{Q}$.
(2) For every prime p there exists a maximal ideal $\mathfrak{m} \subset R$ such that $R/\mathfrak{m} \cong \mathbf{F}_p$.

Exercise 29.2. For $f \in \mathbf{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbf{F}_p[x]$. Give an example of an $f \in \mathbf{Z}[x, u]$ such that the following two properties hold:

(1) There exist infinitely many p such that f_p does not have a zero in $\mathbf{F}_p$.
(2) For all $p >> 0$ the polynomial f_p either has a linear or a quadratic factor.

Exercise 29.3. For $f \in \mathbf{Z}[x, y, u, v]$ we define $f_p(x, y) = f(x, y, x^p, y^p) \bmod p \in \mathbf{F}_p[x, y]$. Give an "interesting" example of an f such that f_p is reducible for all $p >> 0$. For example, $f = xv - yu$ with $f_p = xy^p - x^py = xy(x^{p-1} - y^{p-1})$ is "uninteresting"; any f depending only on x, u is "uninteresting", etc.

Remark 29.4. Let $h \in \mathbf{Z}[y]$ be a monic polynomial of degree d. Then:

(1) The map $A = \mathbf{Z}[x] \to B = \mathbf{Z}[y]$, $x \mapsto h$ is finite locally free of rank d.
(2) For all primes p the map $A_p = \mathbf{F}_p[x] \to B_p = \mathbf{F}_p[y]$, $y \mapsto h(y) \bmod p$ is finite locally free of rank d.

Exercise 29.5. Let h, A, B, A_p, B_p be as in the remark. For $f \in \mathbf{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbf{F}_p[x]$. For $g \in \mathbf{Z}[y, v]$ we define $g_p(y) = g(y, y^p) \bmod p \in \mathbf{F}_p[y]$.

(1) Give an example of a h and g such that there does not exist a f with the property
$$f_p = Norm_{B_p/A_p}(g_p).$$
(2) Show that for any choice of h and g as above there exists a nonzero f such that for all p we have
$$Norm_{B_p/A_p}(g_p) \quad \text{divides} \quad f_p.$$
If you want you can restrict to the case $h = y^n$, even with $n = 2$, but it is true in general.

(3) Discuss the relevance of this to Exercises 6 and 7 of the previous set.

Exercise 29.6. Unsolved problems. They may be really hard or they may be easy. I don't know.

(1) Is there any $f \in \mathbf{Z}[x, u]$ such that f_p is irreducible for an infinite number of p? (Hint: Yes, this happens for $f(x, u) = u - x - 1$ and also for $f(x, u) = u^2 - x^2 + 1$.)
(2) Let $f \in \mathbf{Z}[x, u]$ nonzero, and suppose $\deg_x(f_p) = dp + d'$ for all large p. (In other words $\deg_u(f) = d$ and the coefficient c of u^d in f has $\deg_x(c) = d'$.) Suppose we can write $d = d_1 + d_2$ and $d' = d'_1 + d'_2$ with $d_1, d_2 > 0$ and $d'_1, d'_2 \geq 0$ such that for all sufficiently large p there exists a factorization

$$f_p = f_{1,p} f_{2,p}$$

with $\deg_x(f_{1,p}) = d_1 p + d'_1$. Is it true that f comes about via a norm construction as in Exercise 4? (More precisely, are there a h and g such that $Norm_{B_p/A_p}(g_p)$ divides f_p for all $p >> 0$.)
(3) Analogous question to the one in (b) but now with $f \in \mathbf{Z}[x_1, x_2, u_1, u_2]$ irreducible and just assuming that $f_p(x_1, x_2) = f(x_1, x_2, x_1^p, x_2^p) \bmod p$ factors for all $p >> 0$.

30. Filtered derived category

In order to do the exercises in this section, please read the material in Homology, Section 19. We will say A is a filtered object of $\mathcal{A}$, to mean that A comes endowed with a filtration F which we omit from the notation.

Exercise 30.1. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume that the filtration on I is finite and that each $\mathrm{gr}^p(I)$ is an injective object of $\mathcal{A}$. Show that there exists an isomorphism $I \cong \bigoplus \mathrm{gr}^p(I)$ with filtration $F^p(I)$ corresponding to $\bigoplus_{p' \geq p} \mathrm{gr}^p(I)$.

Exercise 30.2. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume that the filtration on I is finite. Show the following are equivalent:

(1) For any solid diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & B \arrow[dl, dashed] \\ I & \end{tikzcd}$$

of filtered objects with (i) the filtrations on A and B are finite, and (ii) $\mathrm{gr}(\alpha)$ injective the dotted arrow exists making the diagram commute.
(2) Each $\mathrm{gr}^p I$ is injective.

Note that given a morphism $\alpha : A \to B$ of filtered objects with finite filtrations to say that $\mathrm{gr}(\alpha)$ injective is the same thing as saying that α is a *strict monomorphism* in the category $\mathrm{Fil}(\mathcal{A})$. Namely, being a monomorphism means $\mathrm{Ker}(\alpha) = 0$ and strict means that this also implies $\mathrm{Ker}(\mathrm{gr}(\alpha)) = 0$. See Homology, Lemma 19.13. (We only use the term "injective" for a morphism in an abelian category, although it makes sense in any additive category having kernels.) The exercises above justifies the following definition.

Definition 30.3. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume the filtration on I is finite. We say I is *filtered injective* if each $\text{gr}^p(I)$ is an injective object of $\mathcal{A}$.

We make the following definition to avoid having to keep saying "with a finite filtration" everywhere.

Definition 30.4. Let $\mathcal{A}$ be an abelian category. We denote $\mathit{Fil}^f(\mathcal{A})$ the full subcategory of $\text{Fil}(\mathcal{A})$ whose objects consist of those $A \in \text{Ob}(\text{Fil}(\mathcal{A}))$ whose filtration is finite.

Exercise 30.5. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let A be an object of $\text{Fil}^f(\mathcal{A})$. Show that there exists a strict monomorphism $\alpha : A \to I$ of A into a filtered injective object I of $\text{Fil}^f(\mathcal{A})$.

Definition 30.6. Let $\mathcal{A}$ be an abelian category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\text{Fil}(\mathcal{A})$. We say that α is a *filtered quasi-isomorphism* if for each $p \in \mathbf{Z}$ the morphism $\text{gr}^p(K^\bullet) \to \text{gr}^p(L^\bullet)$ is a quasi-isomorphism.

Definition 30.7. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\text{Fil}^f(\mathcal{A})$. We say that $K^\bullet$ is *filtered acyclic* if for each $p \in \mathbf{Z}$ the complex $\text{gr}^p(K^\bullet)$ is acyclic.

Exercise 30.8. Let $\mathcal{A}$ be an abelian category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of bounded below complexes of $\text{Fil}^f(\mathcal{A})$. (Note the superscript f.) Show that the following are equivalent:

1. α is a filtered quasi-isomorphism,
2. for each $p \in \mathbf{Z}$ the map $\alpha : F^pK^\bullet \to F^pL^\bullet$ is a quasi-isomorphism,
3. for each $p \in \mathbf{Z}$ the map $\alpha : K^\bullet/F^pK^\bullet \to L^\bullet/F^pL^\bullet$ is a quasi-isomorphism, and
4. the cone of α (see Derived Categories, Definition 9.1) is a filtered acyclic complex.

Moreover, show that if α is a filtered quasi-isomorphism then α is also a usual quasi-isomorphism.

Exercise 30.9. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let A be an object of $\text{Fil}^f(\mathcal{A})$. Show there exists a complex $I^\bullet$ of $\text{Fil}^f(\mathcal{A})$, and a morphism $A[0] \to I^\bullet$ such that

1. each I^p is filtered injective,
2. $I^p = 0$ for $p < 0$, and
3. $A[0] \to I^\bullet$ is a filtered quasi-isomorphism.

Exercise 30.10. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let $K^\bullet$ be a bounded below complex of objects of $\text{Fil}^f(\mathcal{A})$. Show there exists a filtered quasi-isomorphism $\alpha : K^\bullet \to I^\bullet$ with $I^\bullet$ a complex of $\text{Fil}^f(\mathcal{A})$ having filtered injective terms I^n, and bounded below. In fact, we may choose α such that each α^n is a strict monomorphism.

Exercise 30.11. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & L^\bullet \arrow[dl, dashed, "\beta"] \\ I^\bullet & \end{tikzcd}$$

of complexes of $\text{Fil}^f(\mathcal{A})$. Assume $K^\bullet$, $L^\bullet$ and $I^\bullet$ are bounded below and assume each I^n is a filtered injective object. Also assume that α is a filtered quasi-isomorphism.

(1) There exists a map of complexes β making the diagram commute up to homotopy.
(2) If α is a strict monomorphism in every degree then we can find a β which makes the diagram commute.

Exercise 30.12. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$, $K^\bullet$ be complexes of $\text{Fil}^f(\mathcal{A})$. Assume

(1) $K^\bullet$ bounded below and filtered acyclic, and
(2) $I^\bullet$ bounded below and consisting of filtered injective objects.

Then any morphism $K^\bullet \to I^\bullet$ is homotopic to zero.

Exercise 30.13. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"'] \arrow[d, "\gamma"'] & L^\bullet \arrow[dl, dashed, "\beta_i"] \\ I^\bullet & \end{tikzcd}$$

of complexes of $\text{Fil}^f(\mathcal{A})$. Assume $K^\bullet$, $L^\bullet$ and $I^\bullet$ bounded below and each I^n a filtered injective object. Also assume α a filtered quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

31. Regular functions

Exercise 31.1. Consider the affine curve X given by the equation $t^2 = s^5 + 8$ in $\mathbf{C}^2$ with coordinates s, t. Let $x \in X$ be the point with coordinates $(1, 3)$. Let $U = X \setminus \{x\}$. Prove that there is a regular function on U which is not the restriction of a regular function on $\mathbf{C}^2$, i.e., is not the restriction of a polynomial in s and t to U.

Exercise 31.2. Let $n \geq 2$. Let $E \subset \mathbf{C}^n$ be a finite subset. Show that any regular function on $\mathbf{C}^n \setminus E$ is a polynomial.

Exercise 31.3. Let $X \subset \mathbf{C}^n$ be an affine variety. Let us say X is a *cone* if $x = (a_1, \ldots, a_n) \in X$ and $\lambda \in \mathbf{C}$ implies $(\lambda a_1, \ldots, \lambda a_n) \in X$. Of course, if $\mathfrak{p} \subset \mathbf{C}[x_1, \ldots, x_n]$ is a prime ideal generated by homogeneous polynomials in $x_1, \ldots, x_n$, then the affine variety $X = V(\mathfrak{p}) \subset \mathbf{C}^n$ is a cone. Show that conversely the prime ideal $\mathfrak{p} \subset \mathbf{C}[x_1, \ldots, x_n]$ corresponding to a cone can be generated by homogeneous polynomials in $x_1, \ldots, x_n$.

Exercise 31.4. Give an example of an affine variety $X \subset \mathbf{C}^n$ which is a cone (see Exercise 31.3) and a regular function f on $U = X \setminus \{(0, \ldots, 0)\}$ which is not the restriction of a polynomial function on $\mathbf{C}^n$.

Exercise 31.5. In this exercise we try to see what happens with regular functions over non-algebraically closed fields. Let k be a field. Let $Z \subset k^n$ be a Zariski locally closed subset, i.e., there exist ideals $I \subset J \subset k[x_1, \ldots, x_n]$ such that

$$Z = \{a \in k^n \mid f(a) = 0 \ \forall \ f \in I, \ \exists \ g \in J, \ g(a) \neq 0\}.$$

A function $\varphi : Z \to k$ is said to be *regular* if for every $z \in Z$ there exists a Zariski open neighbourhood $z \in U \subset Z$ and polynomials $f, g \in k[x_1, \ldots, x_n]$ such that $g(u) \neq 0$ for all $u \in U$ and such that $\varphi(u) = f(u)/g(u)$ for all $u \in U$.

(1) If $k = \bar{k}$ and $Z = k^n$ show that regular functions are given by polynomials. (Only do this if you haven't seen this argument before.)
(2) If k is finite show that (a) every function φ is regular, (b) the ring of regular functions is finite dimensional over k. (If you like you can take $Z = k^n$ and even $n = 1$.)
(3) If $k = \mathbf{R}$ give an example of a regular function on $Z = \mathbf{R}$ which is not given by a polynomial.
(4) If $k = \mathbf{Q}_p$ give an example of a regular function on $Z = \mathbf{Q}_p$ which is not given by a polynomial.

32. Sheaves

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *monomorphism* if for every pair of morphisms $a, b : W \to X$ we have $f \circ a = f \circ b \Rightarrow a = b$. A monomorphism in the category of sets is an injective map of sets.

Exercise 32.1. Carefully prove that a map of sheaves of sets is an monomorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are injective.

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *isomorphism* if there exists a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. An isomorphism in the category of sets is a bijective map of sets.

Exercise 32.2. Carefully prove that a map of sheaves of sets is an isomorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are bijective.

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *epimorphism* if for every pair of morphisms $a, b : Y \to Z$ we have $a \circ f = b \circ f \Rightarrow a = b$. An epimorphism in the category of sets is a surjective map of sets.

Exercise 32.3. Carefully prove that a map of sheaves of sets is an epimorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are surjective.

Exercise 32.4. Let $f : X \to Y$ be a map of topological spaces. Prove pushforward f_* and pullback f^{-1} for sheaves of **sets** form an adjoint pair of functors.

Exercise 32.5. Let $j : U \to X$ be an open immersion. Show that

(1) Pullback $j^{-1} : \mathit{Sh}(X) \to \mathit{Sh}(U)$ has a left adjoint $j_! : \mathit{Sh}(U) \to \mathit{Sh}(X)$ called *extension by the empty set*.
(2) Characterize the stalks of $j_!(\mathcal{G})$ for $\mathcal{G} \in \mathit{Sh}(U)$.
(3) Pullback $j^{-1} : \mathit{Ab}(X) \to \mathit{Ab}(U)$ has a left adjoint $j_! : \mathit{Ab}(U) \to \mathit{Ab}(X)$ called *extension by zero*.
(4) Characterize the stalks of $j_!(\mathcal{G})$ for $\mathcal{G} \in \mathit{Ab}(U)$.

Observe that extension by zero differs from extension by the empty set!

Exercise 32.6. Let $X = \mathbf{R}$ with the usual topology. Let $\mathcal{O}_X = \underline{\mathbf{Z}/2\mathbf{Z}}_X$. Let $i : Z = \{0\} \to X$ be the inclusion and let $\mathcal{O}_Z = \underline{\mathbf{Z}/2\mathbf{Z}}_Z$. Prove the following (the first three follow from the definitions but if you are not clear on the definitions you should elucidate them):

(1) $i_*\mathcal{O}_Z$ is a skyscraper sheaf.

(2) There is a canonical surjective map from $\underline{\mathbf{Z}/2\mathbf{Z}}_X \to i_*\underline{\mathbf{Z}/2\mathbf{Z}}_Z$. Denote the kernel $\mathcal{I} \subset \mathcal{O}_X$.
(3) $\mathcal{I}$ is an ideal sheaf of $\mathcal{O}_X$.
(4) The sheaf $\mathcal{I}$ on X cannot be locally generated by sections (as in Modules, Definition 8.1.)

Exercise 32.7. Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Show that $\mathcal{F}$ is the quotient of a (possibly very large) direct sum of sheaves all of whose terms are of the form

$$j_!(\underline{\mathbf{Z}}_U)$$

where $U \subset X$ is open and $\underline{\mathbf{Z}}_U$ denotes the constant sheaf with value $\mathbf{Z}$ on U.

Remark 32.8. Let X be a topological space. In the category of abelian sheaves the direct sum of a family of sheaves $\{\mathcal{F}_i\}_{i\in I}$ is the sheaf associated to the presheaf $U \mapsto \oplus \mathcal{F}_i(U)$. Consequently the stalk of the direct sum at a point x is the direct sum of the stalks of the $\mathcal{F}_i$ at x.

Exercise 32.9. Let X be a topological space. Suppose we are given a collection of abelian groups A_x indexed by $x \in X$. Show that the rule $U \mapsto \prod_{x\in U} A_x$ with obvious restriction mappings defines a sheaf $\mathcal{G}$ of abelian groups. Show, by an example, that usually it is not the case that $\mathcal{G}_x = A_x$ for $x \in X$.

Exercise 32.10. Let X, A_x, $\mathcal{G}$ be as in Exercise 32.9. Let $\mathcal{B}$ be a basis for the topology of X, see Topology, Definition 5.1. For $U \in \mathcal{B}$ let A_U be a subgroup $A_U \subset \mathcal{G}(U) = \prod_{x\in U} A_x$. Assume that for $U \subset V$ with $U, V \in \mathcal{B}$ the restriction maps A_V into A_U. For $U \subset X$ open set

$$\mathcal{F}(U) = \left\{ (s_x)_{x\in U} \middle| \begin{array}{l} \text{for every } x \text{ in } U \text{ there exists } V \in \mathcal{B} \\ x \in V \subset U \text{ such that } (s_y)_{y\in V} \in A_V \end{array} \right\}$$

Show that $\mathcal{F}$ defines a sheaf of abelian groups on X. Show, by an example, that it is usually not the case that $\mathcal{F}(U) = A_U$ for $U \in \mathcal{B}$.

Exercise 32.11. Give an example of a topological space X and a functor

$$F : \mathit{Sh}(X) \longrightarrow \mathit{Sets}$$

which is exact (commutes with finite products and equalizers and commutes with finite coproducts and coequalizers, see Categories, Section 23), but there is no point $x \in X$ such that F is isomorphic to the stalk functor $\mathcal{F} \mapsto \mathcal{F}_x$.

33. Schemes

Let *LRS* be the category of locally ringed spaces. An affine scheme is an object in *LRS* isomorphic in *LRS* to a pair of the form $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$. A scheme is an object $(X, \mathcal{O}_X)$ of *LRS* such that every point $x \in X$ has an open neighbourhood $U \subset X$ such that the pair $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Exercise 33.1. Find a 1-point locally ringed space which is not a scheme.

Exercise 33.2. Suppose that X is a scheme whose underlying topological space has 2 points. Show that X is an affine scheme.

Exercise 33.3. Suppose that X is a scheme whose underlying topological space is a finite discrete set. Show that X is an affine scheme.

Exercise 33.4. Show that there exists a non-affine scheme having three points.

Exercise 33.5. Suppose that X is a nonempty quasi-compact scheme. Show that X has a closed point.

Remark 33.6. When $(X, \mathcal{O}_X)$ is a ringed space and $U \subset X$ is an open subset then $(U, \mathcal{O}_X|_U)$ is a ringed space. Notation: $\mathcal{O}_U = \mathcal{O}_X|_U$. There is a canonical morphisms of ringed spaces

$$j : (U, \mathcal{O}_U) \longrightarrow (X, \mathcal{O}_X).$$

If $(X, \mathcal{O}_X)$ is a locally ringed space, so is $(U, \mathcal{O}_U)$ and j is a morphism of locally ringed spaces. If $(X, \mathcal{O}_X)$ is a scheme so is $(U, \mathcal{O}_U)$ and j is a morphism of schemes. We say that $(U, \mathcal{O}_U)$ is an *open subscheme* of $(X, \mathcal{O}_X)$ and that j is an *open immersion*. More generally, any morphism $j' : (V, \mathcal{O}_V) \to (X, \mathcal{O}_X)$ that is *isomorphic* to a morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ as above is called an open immersion.

Exercise 33.7. Give an example of an affine scheme $(X, \mathcal{O}_X)$ and an open $U \subset X$ such that $(U, \mathcal{O}_X|U)$ is not an affine scheme.

Exercise 33.8. Given an example of a pair of affine schemes $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, an open subscheme $(U, \mathcal{O}_X|_U)$ of X and a morphism of schemes $(U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$ that does not extend to a morphism of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

Exercise 33.9. (This is pretty hard.) Given an example of a scheme X, and open subscheme $U \subset X$ and a closed subscheme $Z \subset U$ such that Z does not extend to a closed subscheme of X.

Exercise 33.10. Give an example of a scheme X, a field K, and a morphism of ringed spaces $\mathrm{Spec}(K) \to X$ which is NOT a morphism of schemes.

Exercise 33.11. Do all the exercises in [Har77, Chapter II], Sections 1 and 2... Just kidding!

Definition 33.12. A scheme X is called *integral* if X is nonempty and for every nonempty affine open $U \subset X$ the ring $\Gamma(U, \mathcal{O}_X) = \mathcal{O}_X(U)$ is a domain.

Exercise 33.13. Give an example of a morphism of *integral* schemes $f : X \to Y$ such that the induced maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are surjective for all $x \in X$, but f is not a closed immersion.

Exercise 33.14. Give an example of a fibre product $X \times_S Y$ such that X and Y are affine but $X \times_S Y$ is not.

Remark 33.15. It turns out this cannot happen with S separated. Do you know why?

Exercise 33.16. Give an example of a scheme V which is integral 1-dimensional scheme of finite type over $\mathbf{Q}$ such that $\mathrm{Spec}(\mathbf{C}) \times_{\mathrm{Spec}(\mathbf{Q})} V$ is not integral.

Exercise 33.17. Give an example of a scheme V which is integral 1-dimensional scheme of finite type over a field k such that $\mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} V$ is not reduced for some finite field extension k'/k.

Remark 33.18. If your scheme is affine then dimension is the same as the Krull dimension of the underlying ring. So you can use last semesters results to compute dimension.

34. Morphisms

An important question is, given a morphism $\pi : X \to S$, whether the morphism has a section or a rational section. Here are some example exercises.

Exercise 34.1. Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, t, 1/xt]) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t]).$$

(1) Show there does not exist a morphism $\sigma : S \to X$ such that $\pi \circ \sigma = \mathrm{id}_S$.
(2) Show there does exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 34.2. Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, t]/(x^2 + t)) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t]).$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 34.3. Let $A, B, C \in \mathbf{C}[t]$ be nonzero polynomials. Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, y, t]/(A + Bx^2 + Cy^2)) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t]).$$

Show there does exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$. (Hint: Symbolically, write $x = X/Z$, $y = Y/Z$ for some $X, Y, Z \in \mathbf{C}[t]$ of degree $\leq d$ for some d, and work out the condition that this solves the equation. Then show, using dimension theory, that if $d >> 0$ you can find nonzero X, Y, Z solving the equation.)

Remark 34.4. Exercise 34.3 is a special case of "Tsen's theorem". Exercise 34.5 shows that the method is limited to low degree equations (conics when the base and fibre have dimension 1).

Exercise 34.5. Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, y, t]/(1 + tx^3 + t^2y^3)) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t])$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 34.6. Consider the schemes

$$X = \mathrm{Spec}(\mathbf{C}[\{x_i\}_{i=1}^8, s, t]/(1 + sx_1^3 + s^2x_2^3 + tx_3^3 + stx_4^3 + s^2tx_5^3 + t^2x_6^3 + st^2x_7^3 + s^2t^2x_8^3))$$

and

$$S = \mathrm{Spec}(\mathbf{C}[s, t])$$

and the morphism of schemes

$$\pi : X \longrightarrow S$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 34.7. (For the number theorists.) Give an example of a closed subscheme

$$Z \subset \mathrm{Spec}\left(\mathbf{Z}[x, \frac{1}{x(x-1)(2x-1)}]\right)$$

such that the morphism $Z \to \mathrm{Spec}(\mathbf{Z})$ is finite and surjective.

Exercise 34.8. If you do not like number theory, you can try the variant where you look at
$$\mathrm{Spec}\left(\mathbf{F}_p[t, x, \frac{1}{x(x-t)(tx-1)}]\right) \longrightarrow \mathrm{Spec}(\mathbf{F}_p[t])$$
and you try to find a closed subscheme of the top scheme which maps finite surjectively to the bottom one. (There is a theoretical reason for having a finite ground field here; although it may not be necessary in this particular case.)

Remark 34.9. The interpretation of the results of Exercise 34.7 and 34.8 is that given the morphism $X \to S$ all of whose fibres are nonempty, there exists a finite surjective morphism $S' \to S$ such that the base change $X_{S'} \to S'$ does have a section. This is not a general fact, but it holds if the base is the spectrum of a dedekind ring with finite residue fields at closed points, and the morphism $X \to S$ is flat with geometrically irreducible generic fibre. See Exercise 34.10 below for an example where it doesn't work.

Exercise 34.10. Prove there exist a $f \in \mathbf{C}[x, t]$ which is not divisible by $t - \alpha$ for any $\alpha \in \mathbf{C}$ such that there does not exist any $Z \subset \mathrm{Spec}(\mathbf{C}[x, t, 1/f])$ which maps finite surjectively to $\mathrm{Spec}(\mathbf{C}[t])$. (I think that $f(x, t) = (xt - 2)(x - t + 3)$ works. To show any candidate has the required property is not so easy I think.)

Exercise 34.11. Let $A \to B$ be a finite type ring map. Suppose that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ factors through a closed immersion $\mathrm{Spec}(B) \to \mathbf{P}^n_A$ for some n. Prove that $A \to B$ is a finite ring map, i.e., that B is finite as an A-module. Hint: if A is Noetherian (please just assume this) you can argue using that $H^i(Z, \mathcal{O}_Z)$ for $i \in \mathbf{Z}$ is a finite A-module for every closed subscheme $Z \subset \mathbf{P}^n_A$.

Exercise 34.12. Let k be an algebraically closed field. Let $f : X \to Y$ be a morphism of projective varieties such that $f^{-1}(\{y\})$ is finite for every closed point $y \in Y$. Prove that f is finite as a morphism of schemes. Hints: (a) being finite is a local property, (b) try to reduce to Exercise 34.11, and (c) use a closed immersion $X \to \mathbf{P}^n_k$ to get a closed immersion $X \to \mathbf{P}^n_Y$ over Y.

35. Tangent Spaces

Definition 35.1. For any ring R we denote $R[\epsilon]$ the ring of *dual numbers*. As an R-module it is free with basis 1, ϵ. The ring structure comes from setting $\epsilon^2 = 0$.

Exercise 35.2. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point, let $s = f(x)$. Consider the solid commutative diagram
$$\begin{tikzcd}
\mathrm{Spec}(\kappa(x)) \arrow[r] \arrow[dr] \arrow[rr, bend left] & \mathrm{Spec}(\kappa(x)[\epsilon]) \arrow[r, dotted] \arrow[d] & X \arrow[d] \\
 & \mathrm{Spec}(\kappa(s)) \arrow[r] & S
\end{tikzcd}$$
with the curved arrow being the canonical morphism of $\mathrm{Spec}(\kappa(x))$ into X. If $\kappa(x) = \kappa(s)$ show that the set of dotted arrows which make the diagram commute are in one to one correspondence with the set of linear maps
$$\mathrm{Hom}_{\kappa(x)}(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}}, \kappa(x))$$
In other words: describe such a bijection. (This works more generally if $\kappa(x) \supset \kappa(s)$ is a separable algebraic extension.)

Definition 35.3. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. We dub the set of dotted arrows of Exercise 35.2 the *tangent space of X over S* and we denote it $T_{X/S,x}$. An element of this space is called a *tangent vector* of X/S at x.

Exercise 35.4. For any field K prove that the diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Spec}(K[\epsilon_1]) \arrow[d] \\ \mathrm{Spec}(K[\epsilon_2]) \arrow[r] & \mathrm{Spec}(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)) \end{tikzcd}$$

is a pushout diagram in the category of schemes. (Here $\epsilon_i^2 = 0$ as before.)

Exercise 35.5. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Define addition of tangent vectors, using Exercise 35.4 and a suitable morphism

$$\mathrm{Spec}(K[\epsilon]) \longrightarrow \mathrm{Spec}(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)).$$

Similarly, define scalar multiplication of tangent vectors (this is easier). Show that $T_{X/S,x}$ becomes a $\kappa(x)$-vector space with your constructions.

Exercise 35.6. Let k be a field. Consider the structure morphism $f : X = \mathbf{A}^1_k \to \mathrm{Spec}(k) = S$.

(1) Let $x \in X$ be a closed point. What is the dimension of $T_{X/S,x}$?
(2) Let $\eta \in X$ be the generic point. What is the dimension of $T_{X/S,\eta}$?
(3) Consider now X as a scheme over $\mathrm{Spec}(\mathbf{Z})$. What are the dimensions of $T_{X/\mathbf{Z},x}$ and $T_{X/\mathbf{Z},\eta}$?

Remark 35.7. Exercise 35.6 explains why it is necessary to consider the tangent space of X over S to get a good notion.

Exercise 35.8. Consider the morphism of schemes

$$f : X = \mathrm{Spec}(\mathbf{F}_p(t)) \longrightarrow \mathrm{Spec}(\mathbf{F}_p(t^p)) = S$$

Compute the tangent space of X/S at the unique point of X. Isn't that weird? What do you think happens if you take the morphism of schemes corresponding to $\mathbf{F}_p[t^p] \to \mathbf{F}_p[t]$?

Exercise 35.9. Let k be a field. Compute the tangent space of X/k at the point $x = (0, 0)$ where $X = \mathrm{Spec}(k[x, y]/(x^2 - y^3))$.

Exercise 35.10. Let $f : X \to Y$ be a morphism of schemes over S. Let $x \in X$ be a point. Set $y = f(x)$. Assume that the natural map $\kappa(y) \to \kappa(x)$ is bijective. Show, using the definition, that f induces a natural linear map

$$\mathrm{d}f : T_{X/S,x} \longrightarrow T_{Y/S,y}.$$

Match it with what happens on local rings via Exercise 35.2 in case $\kappa(x) = \kappa(s)$.

Exercise 35.11. Let k be an algebraically closed field. Let

$$\begin{array}{rcl} f : \mathbf{A}^n_k & \longrightarrow & \mathbf{A}^m_k \\ (x_1, \ldots, x_n) & \longmapsto & (f_1(x_i), \ldots, f_m(x_i)) \end{array}$$

be a morphism of schemes over k. This is given by m polynomials $f_1, \ldots, f_m$ in n variables. Consider the matrix

$$A = \left(\frac{\partial f_j}{\partial x_i}\right)$$

Let $x \in \mathbf{A}^n_k$ be a closed point. Set $y = f(x)$. Show that the map on tangent spaces $T_{\mathbf{A}^n_k/k,x} \to T_{\mathbf{A}^m_k/k,y}$ is given by the value of the matrix A at the point x.

36. Quasi-coherent Sheaves

Definition 36.1. Let X be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is *quasi-coherent* if for every affine open $\text{Spec}(R) = U \subset X$ the restriction $\mathcal{F}|_U$ is of the form $\widetilde{M}$ for some R-module M.

It is enough to check this conditions on the members of an affine open covering of X. See Schemes, Section 24 for more results.

Definition 36.2. Let X be a topological space. Let $x, x' \in X$. We say x is a *specialization* of x' if and only if $x \in \overline{\{x'\}}$.

Exercise 36.3. Let X be a scheme. Let $x, x' \in X$. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Suppose that (a) x is a specialization of x' and (b) $\mathcal{F}_{x'} \neq 0$. Show that $\mathcal{F}_x \neq 0$.

Exercise 36.4. Find an example of a scheme X, points $x, x' \in X$, a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that (a) x is a specialization of x' and (b) $\mathcal{F}_{x'} \neq 0$ and $\mathcal{F}_x = 0$.

Definition 36.5. A scheme X is called *locally Noetherian* if and only if for every point $x \in X$ there exists an affine open $\text{Spec}(R) = U \subset X$ such that R is Noetherian. A scheme is *Noetherian* if it is locally Noetherian and quasi-compact.

If X is locally Noetherian then any affine open of X is the spectrum of a Noetherian ring, see Properties, Lemma 5.2.

Definition 36.6. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is *coherent* if for every point $x \in X$ there exists an affine open $\text{Spec}(R) = U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\widetilde{M}$ for some finite R-module M.

Exercise 36.7. Let $X = \text{Spec}(R)$ be an affine scheme.

(1) Let $f \in R$. Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathcal{G} = \mathcal{F}|_{D(f)}$ for some quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.
(2) Let $I \subset R$ be an ideal. Let $i : Z \to X$ be the closed subscheme of X corresponding to I. Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_Z$-modules on the closed subscheme Z. Show that $\mathcal{G} = i^*\mathcal{F}$ for some quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. (Why is this silly?)
(3) Assume that R is Noetherian. Let $f \in R$. Let $\mathcal{G}$ be a coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathcal{G} = \mathcal{F}|_{D(f)}$ for some coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.

Remark 36.8. If $U \to X$ is a quasi-compact immersion then any quasi-coherent sheaf on U is the restriction of a quasi-coherent sheaf on X. If X is a Noetherian scheme, and $U \subset X$ is open, then any coherent sheaf on U is the restriction of a coherent sheaf on X. Of course the exercise above is easier, and shouldn't use these general facts.

37. Proj and projective schemes

Exercise 37.1. Give examples of graded rings S such that

(1) $\text{Proj}(S)$ is affine and nonempty, and
(2) $\text{Proj}(S)$ is integral, nonempty but not isomorphic to $\mathbf{P}^n_A$ for any $n \geq 0$, any ring A.

Exercise 37.2. Give an example of a nonconstant morphism of schemes $\mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^5_{\mathbf{C}}$ over $\text{Spec}(\mathbf{C})$.

Exercise 37.3. Give an example of an isomorphism of schemes

$$\mathbf{P}^1_{\mathbf{C}} \to \text{Proj}(\mathbf{C}[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2))$$

Exercise 37.4. Give an example of a morphism of schemes $f : X \to \mathbf{A}^1_{\mathbf{C}} = \text{Spec}(\mathbf{C}[T])$ such that the (scheme theoretic) fibre X_t of f over $t \in \mathbf{A}^1_{\mathbf{C}}$ is (a) isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ when t is a closed point not equal to 0, and (b) not isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ when $t = 0$. We will call X_0 the *special fibre* of the morphism. This can be done in many, many ways. Try to give examples that satisfy (each of) the following additional restraints (unless it isn't possible):

(1) Can you do it with special fibre projective?
(2) Can you do it with special fibre irreducible and projective?
(3) Can you do it with special fibre integral and projective?
(4) Can you do it with special fibre smooth and projective?
(5) Can you do it with f a flat morphism? This just means that for every affine open $\text{Spec}(A) \subset X$ the induced ring map $\mathbf{C}[t] \to A$ is flat, which in this case means that any nonzero polynomial in t is a nonzerodivisor on A.
(6) Can you do it with f a flat and projective morphism?
(7) Can you do it with f flat, projective and special fibre reduced?
(8) Can you do it with f flat, projective and special fibre irreducible?
(9) Can you do it with f flat, projective and special fibre integral?

What do you think happens when you replace $\mathbf{P}^1_{\mathbf{C}}$ with another variety over $\mathbf{C}$? (This can get very hard depending on which of the variants above you ask for.)

Exercise 37.5. Let $n \geq 1$ be any positive integer. Give an example of a surjective morphism $X \to \mathbf{P}^n_{\mathbf{C}}$ with X affine.

Exercise 37.6. Maps of Proj. Let R and S be graded rings. Suppose we have a ring map

$$\psi : R \to S$$

and an integer $e \geq 1$ such that $\psi(R_d) \subset S_{de}$ for all $d \geq 0$. (By our conventions this is not a homomorphism of graded rings, unless $e = 1$.)

(1) For which elements $\mathfrak{p} \in \text{Proj}(S)$ is there a well-defined corresponding point in $\text{Proj}(R)$? In other words, find a suitable open $U \subset \text{Proj}(S)$ such that ψ defines a continuous map $r_\psi : U \to \text{Proj}(R)$.
(2) Give an example where $U \neq \text{Proj}(S)$.
(3) Give an example where $U = \text{Proj}(S)$.
(4) (Do not write this down.) Convince yourself that the continuous map $U \to \text{Proj}(R)$ comes canonically with a map on sheaves so that r_ψ is a morphism of schemes:

$$\text{Proj}(S) \supset U \longrightarrow \text{Proj}(R).$$

(5) What can you say about this map if $R = \bigoplus_{d \geq 0} S_{de}$ (as a graded ring with S_e, S_{2e}, etc in degree 1, 2, etc) and ψ is the inclusion mapping?

Notation. Let R be a graded ring as above and let $n \geq 0$ be an integer. Let $X = \text{Proj}(R)$. Then there is a unique quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X(n)$ on X such that for every homogeneous element $f \in R$ of positive degree we have $\mathcal{O}_X|_{D_+(f)}$ is the quasi-coherent sheaf associated to the $R_{(f)} = (R_f)_0$-module $(R_f)_n$ (=elements homogeneous of degree n in $R_f = R[1/f]$). See [Har77, page 116+]. Note that there are natural maps

$$\mathcal{O}_X(n_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n_2) \longrightarrow \mathcal{O}_X(n_1 + n_2)$$

Exercise 37.7. Pathologies in Proj. Give examples of R as above such that

(1) $\mathcal{O}_X(1)$ is not an invertible $\mathcal{O}_X$-module.
(2) $\mathcal{O}_X(1)$ is invertible, but the natural map $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \to \mathcal{O}_X(2)$ is NOT an isomorphism.

Exercise 37.8. Let S be a graded ring. Let $X = \text{Proj}(S)$. Show that any finite set of points of X is contained in a standard affine open.

Exercise 37.9. Let S be a graded ring. Let $X = \text{Proj}(S)$. Let $Z, Z' \subset X$ be two closed subschemes. Let $\varphi : Z \to Z'$ be an isomorphism. Assume $Z \cap Z' = \emptyset$. Show that for any $z \in Z$ there exists an affine open $U \subset X$ such that $z \in U$, $\varphi(z) \in U$ and $\varphi(Z \cap U) = Z' \cap U$. (Hint: Use Exercise 37.8 and something akin to Schemes, Lemma 11.5.)

38. Morphisms from the projective line

In this section we study morphisms from $\mathbf{P}^1$ to projective schemes.

Exercise 38.1. Let k be a field. Let $k[t] \subset k(t)$ be the inclusion of the polynomial ring into its fraction field. Let X be a finite type scheme over k. Show that for any morphism

$$\varphi : \text{Spec}(k(t)) \longrightarrow X$$

over k, there exist a nonzero $f \in k[t]$ and a morphism $\psi : \text{Spec}(k[t, 1/f]) \to X$ over k such that φ is the composition

$$\text{Spec}(k(t)) \longrightarrow \text{Spec}(k[t, 1/f]) \longrightarrow X$$

Exercise 38.2. Let k be a field. Let $k[t] \subset k(t)$ be the inclusion of the polynomial ring into its fraction field. Show that for any morphism

$$\varphi : \text{Spec}(k(t)) \longrightarrow \mathbf{P}^n_k$$

over k, there exists a morphism $\psi : \text{Spec}(k[t]) \to \mathbf{P}^n_k$ over k such that φ is the composition

$$\text{Spec}(k(t)) \longrightarrow \text{Spec}(k[t]) \longrightarrow \mathbf{P}^n_k$$

Hint: the image of φ is in a standard open $D_+(T_i)$ for some i; then show that you can “clear denominators”.

Exercise 38.3. Let k be a field. Let $k[t] \subset k(t)$ be the inclusion of the polynomial ring into its fraction field. Let X be a projective scheme over k. Show that for any morphism

$$\varphi : \text{Spec}(k(t)) \longrightarrow X$$

over k, there exists a morphism $\psi : \mathrm{Spec}(k[t]) \to X$ over k such that φ is the composition

$$\mathrm{Spec}(k(t)) \longrightarrow \mathrm{Spec}(k[t]) \longrightarrow X$$

Hint: use Exercise 38.2.

Exercise 38.4. Let k be a field. Let X be a projective scheme over k. Let K be the function field of $\mathbf{P}^1_k$ (see hint below). Show that for any morphism

$$\varphi : \mathrm{Spec}(K) \longrightarrow X$$

over k, there exists a morphism $\psi : \mathbf{P}^1_k \to X$ over k such that φ is the composition

$$\mathrm{Spec}(k(t)) \longrightarrow \mathbf{P}^1_k \longrightarrow X$$

Hint: use Exercise 38.3 for each of the two pieces of the affine open covering $\mathbf{P}^1_k = D_+(T_0) \cup D_+(T_1)$, use that $D_+(T_0)$ is the spectrum of a polynomial ring and that K is the fraction field of this polynomial ring.

39. Morphisms from surfaces to curves

Exercise 39.1. Let R be a ring. Let $R \to k$ be a map from R to a field. Let $n \geq 0$. Show that

$$\mathrm{Mor}_{\mathrm{Spec}(R)}(\mathrm{Spec}(k), \mathbf{P}^n_R) = (k^{n+1} \setminus \{0\})/k^*$$

where k^* acts via scalar multiplication on k^{n+1}. From now on we denote $(x_0 : \ldots : x_n)$ the morphism $\mathrm{Spec}(k) \to \mathbf{P}^n_k$ corresponding to the equivalence class of the element $(x_0, \ldots, x_n) \in k^{n+1} \setminus \{0\}$.

Exercise 39.2. Let k be a field. Let $Z \subset \mathbf{P}^2_k$ be an irreducible and reduced closed subscheme. Show that either (a) Z is a closed point, or (b) there exists an homogeneous irreducible $F \in k[X_0, X_1, X_2]$ of degree > 0 such that $Z = V_+(F)$, or (c) $Z = \mathbf{P}^2_k$. (Hint: Look on a standard affine open.)

Exercise 39.3. Let k be a field. Let $Z_1, Z_2 \subset \mathbf{P}^2_k$ be irreducible closed subschemes of the form $V_+(F)$ for some homogeneous irreducible $F_i \in k[X_0, X_1, X_2]$ of degree > 0. Show that $Z_1 \cap Z_2$ is not empty. (Hint: Use dimension theory to estimate the dimension of the local ring of $k[X_0, X_1, X_2]/(F_1, F_2)$ at 0.)

Exercise 39.4. Show there does not exist a nonconstant morphism of schemes $\mathbf{P}^2_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ over $\mathrm{Spec}(\mathbf{C})$. Here a *constant morphism* is one whose image is a single point. (Hint: If the morphism is not constant consider the fibres over 0 and ∞ and argue that they have to meet to get a contradiction.)

Exercise 39.5. Let k be a field. Suppose that $X \subset \mathbf{P}^3_k$ is a closed subscheme given by a single homogeneous equation $F \in k[X_0, X_1, X_2, X_3]$. In other words,

$$X = \mathrm{Proj}(k[X_0, X_1, X_2, X_3]/(F)) \subset \mathbf{P}^3_k$$

as explained in the course. Assume that

$$F = X_0 G + X_1 H$$

for some homogeneous polynomials $G, H \in k[X_0, X_1, X_2, X_3]$ of positive degree. Show that if X_0, X_1, G, H have no common zeros then there exists a nonconstant morphism

$$X \longrightarrow \mathbf{P}^1_k$$

of schemes over $\mathrm{Spec}(k)$ which on field points (see Exercise 39.1) looks like $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1)$ whenever x_0 or x_1 is not zero.

40. Invertible sheaves

Definition 40.1. Let X be a locally ringed space. An *invertible $\mathcal{O}_X$-module* on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that every point has an open neighbourhood $U \subset X$ such that $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_U$ as $\mathcal{O}_U$-module. We say that $\mathcal{L}$ is trivial if it is isomorphic to $\mathcal{O}_X$ as a $\mathcal{O}_X$-module.

Exercise 40.2. General facts.

(1) Show that an invertible $\mathcal{O}_X$-module on a scheme X is quasi-coherent.
(2) Suppose $X \to Y$ is a morphism of locally ringed spaces, and $\mathcal{L}$ an invertible $\mathcal{O}_Y$-module. Show that $f^*\mathcal{L}$ is an invertible $\mathcal{O}_X$ module.

Exercise 40.3. Algebra.

(1) Show that an invertible $\mathcal{O}_X$-module on an affine scheme $\mathrm{Spec}(A)$ corresponds to an A-module M which is (i) finite, (ii) projective, (iii) locally free of rank 1, and hence (iv) flat, and (v) finitely presented. (Feel free to quote things from last semesters course; or from algebra books.)
(2) Suppose that A is a domain and that M is a module as in (a). Show that M is isomorphic as an A-module to an ideal $I \subset A$ such that $IA_{\mathfrak{p}}$ is principal for every prime $\mathfrak{p}$.

Definition 40.4. Let R be a ring. An *invertible module* M is an R-module M such that $\widetilde{M}$ is an invertible sheaf on the spectrum of R. We say M is *trivial* if $M \cong R$ as an R-module.

In other words, M is invertible if and only if it satisfies all of the following conditions: it is flat, of finite presentation, projective, and locally free of rank 1. (Of course it suffices for it to be locally free of rank 1).

Exercise 40.5. Simple examples.

(1) Let k be a field. Let $A = k[x]$. Show that $X = \mathrm{Spec}(A)$ has only trivial invertible $\mathcal{O}_X$-modules. In other words, show that every invertible A-module is free of rank 1.
(2) Let A be the ring

$$A = \{f \in k[x] \mid f(0) = f(1)\}.$$

Show there exists a nontrivial invertible A-module, unless $k = \mathbf{F}_2$. (Hint: Think about $\mathrm{Spec}(A)$ as identifying 0 and 1 in $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$.)
(3) Same question as in (2) for the ring $A = k[x^2, x^3] \subset k[x]$ (except now $k = \mathbf{F}_2$ works as well).

Exercise 40.6. Higher dimensions.

(1) Prove that every invertible sheaf on two dimensional affine space is trivial. More precisely, let $\mathbf{A}^2_k = \mathrm{Spec}(k[x, y])$ where k is a field. Show that every invertible sheaf on $\mathbf{A}^2_k$ is trivial. (Hint: One way to do this is to consider the corresponding module M, to look at $M \otimes_{k[x,y]} k(x)[y]$, and then use Exercise 40.5 (1) to find a generator for this; then you still have to think. Another way to is to use Exercise 40.3 and use what we know about ideals of the polynomial ring: primes of height one are generated by an irreducible polynomial; then you still have to think.)

(2) Prove that every invertible sheaf on any open subscheme of two dimensional affine space is trivial. More precisely, let $U \subset \mathbf{A}^2_k$ be an open subscheme where k is a field. Show that every invertible sheaf on U is trivial. Hint: Show that every invertible sheaf on U extends to one on $\mathbf{A}^2_k$. Not easy; but you can find it in [Har77].

(3) Find an example of a nontrivial invertible sheaf on a punctured cone over a field. More precisely, let k be a field and let $C = \text{Spec}(k[x, y, z]/(xy - z^2))$. Let $U = C \setminus \{(x, y, z)\}$. Find a nontrivial invertible sheaf on U. Hint: It may be easier to compute the group of isomorphism classes of invertible sheaves on U than to just find one. Note that U is covered by the opens $\text{Spec}(k[x, y, z, 1/x]/(xy - z^2))$ and $\text{Spec}(k[x, y, z, 1/y]/(xy - z^2))$ which are "easy" to deal with.

Definition 40.7. Let X be a locally ringed space. The *Picard group of X* is the set $\text{Pic}(X)$ of isomorphism classes of invertible $\mathcal{O}_X$-modules with addition given by tensor product. See Modules, Definition 25.9. For a ring R we set $\text{Pic}(R) = \text{Pic}(\text{Spec}(R))$.

Exercise 40.8. Let R be a ring.

(1) Show that if R is a Noetherian normal domain, then $\text{Pic}(R) = \text{Pic}(R[t])$. [Hint: There is a map $R[t] \to R$, $t \mapsto 0$ which is a left inverse to the map $R \to R[t]$. Hence it suffices to show that any invertible $R[t]$-module M such that $M/tM \cong R$ is free of rank 1. Let K be the fraction field of R. Pick a trivialization $K[t] \to M \otimes_{R[t]} K[t]$ which is possible by Exercise 40.5 (1). Adjust it so it agrees with the trivialization of M/tM above. Show that it is in fact a trivialization of M over $R[t]$ (this is where normality comes in).]

(2) Let k be a field. Show that $\text{Pic}(k[x^2, x^3, t]) \neq \text{Pic}(k[x^2, x^3])$.

41. Čech Cohomology

Exercise 41.1. Čech cohomology. Here k is a field.

(1) Let X be a scheme with an open covering $\mathcal{U} : X = U_1 \cup U_2$, with $U_1 = \text{Spec}(k[x])$, $U_2 = \text{Spec}(k[y])$ with $U_1 \cap U_2 = \text{Spec}(k[z, 1/z])$ and with open immersions $U_1 \cap U_2 \to U_1$ resp. $U_1 \cap U_2 \to U_2$ determined by $x \mapsto z$ resp. $y \mapsto z$ (and I really mean this). (We've seen in the lectures that such an X exists; it is the affine line with zero doubled.) Compute $\check{H}^1(\mathcal{U}, \mathcal{O})$; eg. give a basis for it as a k-vectorspace.

(2) For each element in $\check{H}^1(\mathcal{U}, \mathcal{O})$ construct an exact sequence of sheaves of $\mathcal{O}_X$-modules
$$0 \to \mathcal{O}_X \to E \to \mathcal{O}_X \to 0$$
such that the boundary $\delta(1) \in \check{H}^1(\mathcal{U}, \mathcal{O})$ equals the given element. (Part of the problem is to make sense of this. See also below. It is also OK to show abstractly such a thing has to exist.)

Definition 41.2. (Definition of delta.) Suppose that
$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
is a short exact sequence of abelian sheaves on any topological space X. The boundary map $\delta : H^0(X, \mathcal{F}_3) \to \check{H}^1(X, \mathcal{F}_1)$ is defined as follows. Take an element $\tau \in H^0(X, \mathcal{F}_3)$. Choose an open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ such that for each i

there exists a section $\tilde{\tau}_i \in \mathcal{F}_2$ lifting the restriction of τ to U_i. Then consider the assignment

$$(i_0, i_1) \longmapsto \tilde{\tau}_{i_0}|_{U_{i_0 i_1}} - \tilde{\tau}_{i_1}|_{U_{i_0 i_1}}.$$

This is clearly a 1-coboundary in the Čech complex $\check{C}^*(\mathcal{U}, \mathcal{F}_2)$. But we observe that (thinking of $\mathcal{F}_1$ as a subsheaf of $\mathcal{F}_2$) the RHS always is a section of $\mathcal{F}_1$ over $U_{i_0 i_1}$. Hence we see that the assignment defines a 1-cochain in the complex $\check{C}^*(\mathcal{U}, \mathcal{F}_2)$. The cohomology class of this 1-cochain is by definition $\delta(\tau)$.

42. Cohomology

Exercise 42.1. Let $X = \mathbf{R}$ with the usual Euclidean topology. Using only formal properties of cohomology (functoriality and the long exact cohomology sequence) show that there exists a sheaf $\mathcal{F}$ on X with nonzero $H^1(X, \mathcal{F})$.

Exercise 42.2. Let $X = U \cup V$ be a topological space written as the union of two opens. Then we have a long exact Mayer-Vietoris sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

What property of injective sheaves is essential for the construction of the Mayer-Vietoris long exact sequence? Why does it hold?

Exercise 42.3. Let X be a topological space.

(1) Show that $H^i(X, \mathcal{F})$ is zero for $i > 0$ if X has 2 or fewer points.
(2) What if X has 3 points?

Exercise 42.4. Let X be the spectrum of a local ring. Show that $H^i(X, \mathcal{F})$ is zero for $i > 0$ and any sheaf of abelian groups $\mathcal{F}$.

Exercise 42.5. Let $f : X \to Y$ be an affine morphism of schemes. Prove that $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$ for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Feel free to impose some further conditions on X and Y and use the agreement of Čech cohomology with cohomology for quasi-coherent sheaves and affine open coverings of separated schemes.

Exercise 42.6. Let A be a ring. Let $\mathbf{P}^n_A = \text{Proj}(A[T_0, \ldots, T_n])$ be projective space over A. Let $\mathbf{A}^{n+1}_A = \text{Spec}(A[T_0, \ldots, T_n])$ and let

$$U = \bigcup\nolimits_{i=0,\ldots,n} D(T_i) \subset \mathbf{A}^{n+1}_A$$

be the complement of the image of the closed immersion $0 : \text{Spec}(A) \to \mathbf{A}^{n+1}_A$. Construct an affine surjective morphism

$$f : U \longrightarrow \mathbf{P}^n_A$$

and prove that $f_*\mathcal{O}_U = \bigoplus_{d \in \mathbf{Z}} \mathcal{O}_{\mathbf{P}^n_A}(d)$. More generally, show that for a graded $A[T_0, \ldots, T_n]$-module M one has

$$f_*(\widetilde{M}|_U) = \bigoplus\nolimits_{d \in \mathbf{Z}} \widetilde{M(d)}$$

where on the left hand side we have the quasi-coherent sheaf $\widetilde{M}$ associated to M on $\mathbf{A}^{n+1}_A$ and on the right we have the quasi-coherent sheaves $\widetilde{M(d)}$ associated to the graded module $M(d)$.

Exercise 42.7. Let A be a ring and let $\mathbf{P}^n_A = \text{Proj}(A[T_0, \ldots, T_n])$ be projective space over A. Carefully compute the cohomology of the Serre twists $\mathcal{O}_{\mathbf{P}^n_A}(d)$ of the structure sheaf on $\mathbf{P}^n_A$. Feel free to use Čech cohomology and the agreement of Čech cohomology with cohomology for quasi-coherent sheaves and affine open coverings of separated schemes.

Exercise 42.8. Let A be a ring and let $\mathbf{P}^n_A = \text{Proj}(A[T_0, \ldots, T_n])$ be projective space over A. Let $F \in A[T_0, \ldots, T_n]$ be homogeneous of degree d. Let $X \subset \mathbf{P}^n_A$ be the closed subscheme corresponding to the graded ideal (F) of $A[T_0, \ldots, T_n]$. What can you say about $H^i(X, \mathcal{O}_X)$?

Exercise 42.9. Let R be a ring such that for any left exact functor $F : \text{Mod}_R \to Ab$ we have $R^iF = 0$ for $i > 0$. Show that R is a finite product of fields.

43. More cohomology

Exercise 43.1. Let k be a field. Let $X \subset \mathbf{P}^n_k$ be the "coordinate cross". Namely, let X be defined by the homogeneous equations

$$T_iT_j = 0 \text{ for } i > j > 0$$

where as usual we write $\mathbf{P}^n_k = \text{Proj}(k[T_0, \ldots, T_n])$. In other words, X is the closed subscheme corresponding to the quotient $k[T_0, \ldots, T_n]/(T_iT_j; i > j > 0)$ of the polynomial ring. Compute $H^i(X, \mathcal{O}_X)$ for all i. Hint: use Čech cohomology.

Exercise 43.2. Let A be a ring. Let $I = (f_1, \ldots, f_t)$ be a finitely generated ideal of A. Let $U \subset \text{Spec}(A)$ be the complement of $V(I)$. For any A-module M write down a complex of A-modules (in terms of A, $f_1, \ldots, f_t$, M) whose cohomology groups give $H^n(U, \widetilde{M})$.

Exercise 43.3. Let k be a field. Let $U \subset \mathbf{A}^d_k$ be the complement of the closed point 0 of $\mathbf{A}^d_k$. Compute $H^n(U, \mathcal{O}_U)$ for all n.

Exercise 43.4. Let k be a field. Find explicitly a scheme X projective over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ and $\dim_k H^1(X, \mathcal{O}_X) = 100$.

Exercise 43.5. Let $f : X \to Y$ be a finite locally free morphism of degree 2. Assume that X and Y are integral schemes and that 2 is invertible in the structure sheaf of Y, i.e., $2 \in \Gamma(Y, \mathcal{O}_Y)$ is invertible. Show that the $\mathcal{O}_Y$-module map

$$f^\sharp : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$$

has a left inverse, i.e., there is an $\mathcal{O}_Y$-module map $\tau : f_*\mathcal{O}_X \to \mathcal{O}_Y$ with $\tau \circ f^\sharp = \text{id}$. Conclude that $H^n(Y, \mathcal{O}_Y) \to H^n(X, \mathcal{O}_X)$ is injective[2].

Exercise 43.6. Let X be a scheme (or a locally ringed space). The rule $U \mapsto \mathcal{O}_X(U)^*$ defines a sheaf of groups denoted $\mathcal{O}_X^*$. Briefly explain why the Picard group of X (Definition 40.7) is equal to $H^1(X, \mathcal{O}_X^*)$.

Exercise 43.7. Give an example of an affine scheme X with nontrivial $\text{Pic}(X)$. Conclude using Exercise 43.6 that $H^1(X, -)$ is not the zero functor for any such X.

[2]There does exist a finite locally free morphism $X \to Y$ between integral schemes of degree 2 where the map $H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)$ is not injective.

Exercise 43.8. Let A be a ring. Let $I = (f_1, \ldots, f_t)$ be a finitely generated ideal of A. Let $U \subset \mathrm{Spec}(A)$ be the complement of $V(I)$. Given a quasi-coherent $\mathcal{O}_{\mathrm{Spec}(A)}$-module $\mathcal{F}$ and $\xi \in H^p(U, \mathcal{F})$ with $p > 0$, show that there exists $n > 0$ such that $f_i^n \xi = 0$ for $i = 1, \ldots, t$. Hint: One possible way to proceed is to use the complex you found in Exercise 43.2.

Exercise 43.9. Let A be a ring. Let $I = (f_1, \ldots, f_t)$ be a finitely generated ideal of A. Let $U \subset \mathrm{Spec}(A)$ be the complement of $V(I)$. Let M be an A-module whose I-torsion is zero, i.e., $0 = \mathrm{Ker}((f_1, \ldots, f_t) : M \to M^{\oplus t})$. Show that there is a canonical isomorphism

$$H^0(U, \widetilde{M}) = \mathrm{colim}\, \mathrm{Hom}_A(I^n, M).$$

Warning: this is not trivial.

Exercise 43.10. Let A be a Noetherian ring. Let I be an ideal of A. Let M be an A-module. Let $M[I^\infty]$ be the set of I-power torsion elements defined by

$$M[I^\infty] = \{x \in M \mid \text{ there exists an } n \geq 1 \text{ such that } I^n x = 0\}$$

Set $M' = M/M[I^\infty]$. Then the I-power torsion of M' is zero. Show that

$$\mathrm{colim}\, \mathrm{Hom}_A(I^n, M) = \mathrm{colim}\, \mathrm{Hom}_A(I^n, M').$$

Warning: this is not trivial. Hints: (1) try to reduce to M finite, (2) show any element of $\mathrm{Ext}^1_A(I^n, N)$ maps to zero in $\mathrm{Ext}^1_A(I^{n+m}, N)$ for some $m > 0$ if $N = M[I^\infty]$ and M finite, (3) show the same thing as in (2) for $\mathrm{Hom}_A(I^n, N)$, (3) consider the long exact sequence

$$0 \to \mathrm{Hom}_A(I^n, M[I^\infty]) \to \mathrm{Hom}_A(I^n, M) \to \mathrm{Hom}_A(I^n, M') \to \mathrm{Ext}^1_A(I^n, M[I^\infty])$$

for M finite and compare with the sequence for I^{n+m} to conclude.

44. Cohomology revisited

Exercise 44.1. Make an example of a field k, a curve X over k, an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a cohomology class $\xi \in H^1(X, \mathcal{L})$ with the following property: for every surjective finite morphism $\pi : Y \to X$ of schemes the element ξ pulls back to a nonzero element of $H^1(Y, \pi^*\mathcal{L})$. Hint: construct X, k, $\mathcal{L}$ such that there is a short exact sequence $0 \to \mathcal{L} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$ where $Z \subset X$ is a closed subscheme consisting of more than 1 closed point. Then look at what happens when you pullback.

Exercise 44.2. Let k be an algebraically closed field. Let X be a projective 1-dimensional scheme. Suppose that X contains a cycle of curves, i.e., suppose there exist an $n \geq 2$ and pairwise distinct 1-dimensional integral closed subschemes $C_1, \ldots, C_n$ and pairwise distinct closed points $x_1, \ldots, x_n \in X$ such that $x_n \in C_n \cap C_1$ and $x_i \in C_i \cap C_{i+1}$ for $i = 1, \ldots, n-1$. Prove that $H^1(X, \mathcal{O}_X)$ is nonzero. Hint: Let $\mathcal{F}$ be the image of the map $\mathcal{O}_X \to \bigoplus \mathcal{O}_{C_i}$, and show $H^1(X, \mathcal{F})$ is nonzero using that $\kappa(x_i) = k$ and $H^0(C_i, \mathcal{O}_{C_i}) = k$. Also use that $H^2(X, -) = 0$ by Grothendieck's theorem.

Exercise 44.3. Let X be a projective surface over an algebraically closed field k. Prove there exists a proper closed subscheme $Z \subset X$ such that $H^1(Z, \mathcal{O}_Z)$ is nonzero. Hint: Use Exercise 44.2.

Exercise 44.4. Let X be a projective surface over an algebraically closed field k. Show that for every $n \geq 0$ there exists a proper closed subscheme $Z \subset X$ such that $\dim_k H^1(Z, \mathcal{O}_Z) > n$. Only explain how to do this by modifying the arguments in Exercise 44.3 and 44.2; don't give all the details.

Exercise 44.5. Let X be a projective surface over an algebraically closed field k. Prove there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $H^2(X, \mathcal{F})$ is nonzero. Hint: Use the result of Exercise 44.4 and a cleverly chosen exact sequence.

Exercise 44.6. Let X and Y be schemes over a field k (feel free to assume X and Y are nice, for example qcqs or projective over k). Set $X \times Y = X \times_{\mathrm{Spec}(k)} Y$ with projections $p : X \times Y \to X$ and $q : X \times Y \to Y$. For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$ prove that

$$H^n(X \times Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G}) = \bigoplus\nolimits_{a+b=n} H^a(X, \mathcal{F}) \otimes_k H^b(Y, \mathcal{G})$$

or just show that this holds when one takes dimensions. Extra points for "clean" solutions.

Exercise 44.7. Let k be a field. Let $X = \mathbf{P}|^1 \times \mathbf{P}^1$ be the product of the projective line over k with itself with projections $p : X \to \mathbf{P}^1_k$ and $q : X \to \mathbf{P}^1_k$. Let

$$\mathcal{O}(a, b) = p^*\mathcal{O}_{\mathbf{P}^1_k}(a) \otimes_{\mathcal{O}_X} q^*\mathcal{O}_{\mathbf{P}^1_k}(b)$$

Compute the dimensions of $H^i(X, \mathcal{O}(a, b))$ for all i, a, b. Hint: Use Exercise 44.6.

45. Cohomology and Hilbert polynomials

Situation 45.1. Let k be a field. Let $X = \mathbf{P}^n_k$ be n-dimensional projective space. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Recall that

$$\chi(X, \mathcal{F}) = \sum\nolimits_{i=0}^{n} (-1)^i \dim_k H^i(X, \mathcal{F})$$

Recall that the *Hilbert polynomial* of $\mathcal{F}$ is the function

$$t \longmapsto \chi(X, \mathcal{F}(t))$$

We also recall that $\mathcal{F}(t) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(t)$ where $\mathcal{O}_X(t)$ is the tth twist of the structure sheaf as in Constructions, Definition 10.1. In Varieties, Subsection 35.13 we have proved the Hilbert polynomial is a polynomial in t.

Exercise 45.2. In Situation 45.1.

(1) If $P(t)$ is the Hilbert polynomial of $\mathcal{F}$, what is the Hilbert polynomial of $\mathcal{F}(-13)$.
(2) If P_i is the Hilbert polynomial of $\mathcal{F}_i$, what is the Hilbert polynomial of $\mathcal{F}_1 \oplus \mathcal{F}_2$.
(3) If P_i is the Hilbert polynomial of $\mathcal{F}_i$ and $\mathcal{F}$ is the kernel of a surjective map $\mathcal{F}_1 \to \mathcal{F}_2$, what is the Hilbert polynomial of $\mathcal{F}$?

Exercise 45.3. In Situation 45.1 assume $n \geq 1$. Find a coherent sheaf whose Hilbert polynomial is $t - 101$.

Exercise 45.4. In Situation 45.1 assume $n \geq 2$. Find a coherent sheaf whose Hilbert polynomial is $t^2/2 + t/2 - 1$. (This is a bit tricky; it suffices if you just show there is such a coherent sheaf.)

Exercise 45.5. In Situation 45.1 assume $n \geq 2$ and k algebraically closed. Let $C \subset X$ be an integral closed subscheme of dimension 1. In other words, C is a projective curve. Let $dt + e$ be the Hilbert polynomial of $\mathcal{O}_C$ viewed as a coherent sheaf on X.

(1) Give an upper bound on e. (Hints: Use that $\mathcal{O}_C(t)$ only has cohomology in degrees 0 and 1 and study $H^0(C, \mathcal{O}_C)$.)
(2) Pick a global section s of $\mathcal{O}_X(1)$ which intersects C transversally, i.e., such that there are pairwise distinct closed points $c_1, \ldots, c_r \in C$ and a short exact sequence
$$0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{O}_C(1) \to \bigoplus\nolimits_{i=1,\ldots,r} k_{c_i} \to 0$$
where k_{c_i} is the skyscraper sheaf with value k in c_i. (Such an s exists; please just use this.) Show that $r = d$. (Hint: twist the sequence and see what you get.)
(3) Twisting the short exact sequence gives a k-linear map $\varphi_t : \Gamma(C, \mathcal{O}_C(t)) \to \bigoplus_{i=1,\ldots,d} k$ for any t. Show that if this map is surjective for $t \geq d - 1$.
(4) Give a lower bound on e in terms of d. (Hint: show that $H^1(C, \mathcal{O}_C(d-2)) = 0$ using the result of (3) and use vanishing.)

Exercise 45.6. In Situation 45.1 assume $n = 2$. Let $s_1, s_2, s_3 \in \Gamma(X, \mathcal{O}_X(2))$ be three quadric equations. Consider the coherent sheaf
$$\mathcal{F} = \text{Coker}\left(\mathcal{O}_X(-2)^{\oplus 3} \xrightarrow{s_1, s_2, s_3} \mathcal{O}_X\right)$$
List the possible Hilbert polynomials of such $\mathcal{F}$. (Try to visualize intersections of quadrics in the projective plane.)

46. Curves

Exercise 46.1. Let k be an algebraically closed field. Let X be a projective curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ be global sections of $\mathcal{L}$. Prove there is a natural closed subscheme
$$Z \subset \mathbf{P}^n \times X$$
such that the closed point $((\lambda_0 : \ldots : \lambda_n), x)$ is in Z if and only if the section $\lambda_0 s_0 + \ldots + \lambda_n s_n$ vanishes at x. (Hint: describe Z affine locally.)

Exercise 46.2. Let k be an algebraically closed field. Let X be a smooth curve over k. Let $r \geq 1$. Show that the closed subset
$$D \subset X \times X^r = X^{r+1}$$
whose closed points are the tuples $(x, x_1, \ldots, x_r)$ with $x = x_i$ for some i, has an invertible ideal sheaf. In other words, show that D is an effective Cartier divisor. Hints: reduce to $r = 1$ and use that X is a smooth curves to say something about the diagonal (look in books for this).

Exercise 46.3. Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let T be a scheme of finite type over k and let
$$D_1 \subset X \times T \quad \text{and} \quad D_2 \subset X \times T$$
be two effective Cartier divisors such that for $t \in T$ the fibres $D_{i,t} \subset X_t$ are not dense (i.e., do not contain the generic point of the curve X_t). Prove that there is

a canonical closed subscheme $Z \subset T$ such that a closed point $t \in T$ is in Z if and only if for the scheme theoretic fibres $D_{1,t}$, $D_{2,t}$ of D_1, D_2 we have

$$D_{1,t} \subset D_{2,t}$$

as closed subschemes of X_t. Hints: Show that, possibly after shrinking T, you may assume $T = \mathrm{Spec}(A)$ is affine and there is an affine open $U \subset X$ such that $D_i \subset U \times T$. Then show that $M_1 = \Gamma(D_1, \mathcal{O}_{D_1})$ is a finite locally free A-module (here you will need some nontrivial algebra — ask your friends). After shrinking T you may assume M_1 is a free A-module. Then look at

$$\Gamma(U \times T, \mathcal{I}_{D_2}) \to M_1 = A^{\oplus N}$$

and you define Z as the closed subscheme cut out by the ideal generated by coefficients of vectors in the image of this map. Explain why this works (this will require perhaps a bit more commutative algebra).

Exercise 46.4. Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ be global sections of $\mathcal{L}$. Let $r \geq 1$. Prove there is a natural closed subscheme

$$Z \subset \mathbf{P}^n \times X \times \ldots \times X = \mathbf{P}^n \times X^r$$

such that the closed point $((\lambda_0 : \ldots : \lambda_n), x_1, \ldots, x_r)$ is in Z if and only if the section $s_\lambda = \lambda_0 s_0 + \ldots + \lambda_n s_n$ vanishes on the divisor $D = x_1 + \ldots + x_r$, i.e., the section s_λ is in $\mathcal{L}(-D)$. Hint: explain how this follows by combining then results of Exercises 46.2 and 46.3.

Exercise 46.5. Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Show that there is a natural closed subset

$$Z \subset X^r$$

such that a closed point $(x_1, \ldots, x_r)$ of X^r is in Z if and only if $\mathcal{L}(-x_1 - \ldots - x_r)$ has a nonzero global section. Hint: use Exercise 46.4.

Exercise 46.6. Let k be an algebraically closed field. Let X be a smooth projective curve over k. Let $r \geq s$ be integers. Show that there is a natural closed subset

$$Z \subset X^r \times X^s$$

such that a closed point $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ of $X^r \times X^s$ is in Z if and only if $x_1 + \ldots + x_r - y_1 - \ldots - y_s$ is linearly equivalent to an effective divisor. Hint: Choose an auxilliary invertible module $\mathcal{L}$ of very high degree so that $\mathcal{L}(-D)$ has a nonvanshing section for any effective divisor D of degree r. Then use the result of Exercise 46.5 twice.

Exercise 46.7. Choose your favorite algebraically closed field k. As best as you can determine all possible $\mathfrak{g}^r_d$ that can exist on some curve of genus 7. While doing this also try to

1. determine in which cases the $\mathfrak{g}^r_d$ is base point free, and
2. determine in which cases the $\mathfrak{g}^r_d$ gives a closed embedding in $\mathbf{P}^r$.

Do the same thing if you assume your curve is "general" (make up your own notion of general – this may be easier than the question above). Do the same thing if you assume your curve is hyperelliptic. Do the same thing if you assume your curve is trigonal (and not hyperelliptic). Etc.

47. Moduli

In this section we consider some naive approaches to moduli of algebraic geometric objects.

Let k be an algebraically closed field. Suppose that M is a moduli problem over k. We won't define exactly what this means here, but in each exercise it should be clear what we mean. To understand the following it suffices to know what the objects of M over k are, what the isomorphisms between objects of M over k are, and what the families of object of M over a variety are. Then we say the *number of moduli of* M is $d \geq 0$ if the following are true

(1) there is a finite number of families $X_i \to V_i$, $i = 1, \ldots, n$ such that every object of M over k is isomorphic to a fibre of one of these and such that $\max \dim(V_i) = d$, and
(2) there is no way to do this with a smaller d.

This is really just a very rough approximation of better notions in the literature.

Exercise 47.1. Let k be an algebraically closed field. Let $d \geq 1$ and $n \geq 1$. Let us say the moduli of hypersurfaces of degree d in P^n is given by

(1) an object is a hypersurface $X \subset \mathbf{P}^n_k$ of degree d,
(2) an isomorphism between two objects $X \subset \mathbf{P}^n_k$ and $Y \subset \mathbf{P}^n_k$ is an element $g \in \mathrm{PGL}_n(k)$ such that $g(X) = Y$, and
(3) a family of hypersurfaces over a variety V is a closed subscheme $X \subset \mathbf{P}^n_V$ such that for all $v \in V$ the scheme theoretic fibre X_v of $X \to V$ is a hypersurfaces in $\mathbf{P}^n_v$.

Compute (if you can – these get progressively harder)

(1) the number of moduli when $n = 1$ and d arbitrary,
(2) the number of moduli when $n = 2$ and $d = 1$,
(3) the number of moduli when $n = 2$ and $d = 2$,
(4) the number of moduli when $n \geq 1$ and $d = 2$,
(5) the number of moduli when $n = 2$ and $d = 3$,
(6) the number of moduli when $n = 3$ and $d = 3$, and
(7) the number of moduli when $n = 2$ and $d = 4$.

Exercise 47.2. Let k be an algebraically closed field. Let $g \geq 2$. Let us say the moduli of hyperelliptic curves of genus g is given by

(1) an object is a smooth projective hyperelliptic curve X of genus g,
(2) an isomorphism between two objects X and Y is an isomorphism $X \to Y$ of schemes over k, and
(3) a family of hyperelliptic curves of genus g over a variety V is a proper flat[3] morphism $X \to Y$ such that all scheme theoretic fibres of $X \to V$ are smooth projective hyperelliptic curves of genus g.

Show that the number of moduli is $2g - 1$.

48. Global Exts

Exercise 48.1. Let k be a field. Let $X = \mathbf{P}^3_k$. Let $L \subset X$ and $P \subset X$ be a line and a plane, viewed as closed subschemes cut out by 1, resp., 2 linear equations.

[3]You can drop this assumption without changing the answer to the question.

Compute the dimensions of

$$\mathrm{Ext}^i_X(\mathcal{O}_L, \mathcal{O}_P)$$

for all i. Make sure to do both the case where L is contained in P and the case where L is not contained in P.

Exercise 48.2. Let k be a field. Let $X = \mathbf{P}^n_k$. Let $Z \subset X$ be a closed k-rational point viewed as a closed subscheme. For example the point with homogeneous coordinates $(1 : 0 : \ldots : 0)$. Compute the dimensions of

$$\mathrm{Ext}^i_X(\mathcal{O}_Z, \mathcal{O}_Z)$$

for all i.

Exercise 48.3. Let X be a ringed space. Define cup-product maps

$$\mathrm{Ext}^i_X(\mathcal{G}, \mathcal{H}) \times \mathrm{Ext}^j_X(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^{i+j}_X(\mathcal{F}, \mathcal{H})$$

for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}, \mathcal{H}$. (Hint: this is a super general thing.)

Exercise 48.4. Let X be a ringed space. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$. Prove the following statements

(1) $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{G}) = \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}) = \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$, and
(2) $\mathrm{Ext}^i_X(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{G}) = \mathrm{Ext}^i_X(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})$.

Here $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. Conclude that

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{G}) = H^i(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})$$

Exercise 48.5. Let X be a ringed space. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Construct a trace map

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{E}) \to H^i(X, \mathcal{O}_X)$$

for all i. Generalize to a trace map

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \to H^i(X, \mathcal{F})$$

for any $\mathcal{O}_X$-module $\mathcal{F}$.

Exercise 48.6. Let k be a field. Let $X = \mathbf{P}^d_k$. Set $\omega_{X/k} = \mathcal{O}_X(-d-1)$. Prove that for finite locally free modules $\mathcal{E}$, $\mathcal{F}$ the cup product on Ext combined with the trace map on Ext

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{F} \otimes_{\mathcal{O}_X} \omega_{X/k}) \times \mathrm{Ext}^{d-i}_X(\mathcal{F}, \mathcal{E}) \to \mathrm{Ext}^d_X(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_X} \omega_{X/k}) \to H^d(X, \omega_{X/k}) = k$$

produces a nondegenerate pairing. Hint: you can either reprove duality in this setting or you can reduce to cohomology of sheaves and apply the Serre duality theorem as proved in the lectures.

49. Divisors

We collect all relevant definitions here in one spot for convenience.

Definition 49.1. Throughout, let S be any scheme and let X be a Noetherian, integral scheme.

(1) A *Weil divisor* on X is a formal linear combination $\Sigma n_i [Z_i]$ of prime divisors Z_i with integer coefficients.

(2) A *prime divisor* is a closed subscheme $Z \subset X$, which is integral with generic point $\xi \in Z$ such that $\mathcal{O}_{X,\xi}$ has dimension 1. We will use the notation $\mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ when $\xi \in Z \subset X$ is as above. Note that $\mathcal{O}_{X,Z} \subset K(X)$ is a subring of the function field of X.

(3) The *Weil divisor associated to a rational function* $f \in K(X)^*$ is the sum $\Sigma v_Z(f)[Z]$. Here $v_Z(f)$ is defined as follows
  (a) If $f \in \mathcal{O}_{X,Z}^*$ then $v_Z(f) = 0$.
  (b) If $f \in \mathcal{O}_{X,Z}$ then
$$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f)).$$
  (c) If $f = \frac{a}{b}$ with $a, b \in \mathcal{O}_{X,Z}$ then
$$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$

(4) An *effective Cartier divisor* on a scheme S is a closed subscheme $D \subset S$ such that every point $d \in D$ has an affine open neighbourhood $\text{Spec}(A) = U \subset S$ in S so that $D \cap U = \text{Spec}(A/(f))$ with $f \in A$ a nonzerodivisor.

(5) The *Weil divisor* $[D]$ *associated to an effective Cartier divisor* $D \subset X$ of our Noetherian integral scheme X is defined as the sum $\Sigma v_Z(D)[Z]$ where $v_Z(D)$ is defined as follows
  (a) If the generic point ξ of Z is not in D then $v_Z(D) = 0$.
  (b) If the generic point ξ of Z is in D then
$$v_Z(D) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f))$$
  where $f \in \mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ is the nonzerodivisor which defines D in an affine neighbourhood of ξ (as in (4) above).

(6) Let S be a scheme. The *sheaf of total quotient rings* $\mathcal{K}_S$ is the sheaf of $\mathcal{O}_S$-algebras which is the sheafification of the pre-sheaf $\mathcal{K}'$ defined as follows. For $U \subset S$ open we set $\mathcal{K}'(U) = S_U^{-1}\mathcal{O}_S(U)$ where $S_U \subset \mathcal{O}_S(U)$ is the multiplicative subset consisting of sections $f \in \mathcal{O}_S(U)$ such that the germ of f in $\mathcal{O}_{S,u}$ is a nonzerodivisor for every $u \in U$. In particular the elements of S_U are all nonzerodivisors. Thus $\mathcal{O}_S$ is a subsheaf of $\mathcal{K}_S$, and we get a short exact sequence
$$0 \to \mathcal{O}_S^* \to \mathcal{K}_S^* \to \mathcal{K}_S^*/\mathcal{O}_S^* \to 0.$$

(7) A *Cartier divisor* on a scheme S is a global section of the quotient sheaf $\mathcal{K}_S^*/\mathcal{O}_S^*$.

(8) The *Weil divisor associated to a Cartier divisor* $\tau \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ over our Noetherian integral scheme X is the sum $\Sigma v_Z(\tau)[Z]$ where $v_Z(\tau)$ is defined as by the following recipe
  (a) If the germ of τ at the generic point ξ of Z is zero – in other words the image of τ in the stalk $(\mathcal{K}^*/\mathcal{O}^*)_\xi$ is "zero" – then $v_Z(\tau) = 0$.
  (b) Find an affine open neighbourhood $\text{Spec}(A) = U \subset X$ so that $\tau|_U$ is the image of a section $f \in \mathcal{K}(U)$ and moreover $f = a/b$ with $a, b \in A$. Then we set
$$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$

Remarks 49.2. Here are some trivial remarks.

(1) On a Noetherian integral scheme X the sheaf $\mathcal{K}_X$ is constant with value the function field $K(X)$.

(2) To make sense out of the definitions above one needs to show that
$$\text{length}_{\mathcal{O}}(\mathcal{O}/(ab)) = \text{length}_{\mathcal{O}}(\mathcal{O}/(a)) + \text{length}_{\mathcal{O}}(\mathcal{O}/(b))$$
for any pair (a, b) of nonzero elements of a Noetherian 1-dimensional local domain $\mathcal{O}$. This will be done in the lectures.

Exercise 49.3. (On any scheme.) Describe how to assign a Cartier divisor to an effective Cartier divisor.

Exercise 49.4. (On an integral scheme.) Describe how to assign a Cartier divisor D to a rational function f such that the Weil divisor associated to D and to f agree. (This is silly.)

Exercise 49.5. Give an example of a Weil divisor on a variety which is not the Weil divisor associated to any Cartier divisor.

Exercise 49.6. Give an example of a Weil divisor D on a variety which is not the Weil divisor associated to any Cartier divisor but such that nD is the Weil divisor associated to a Cartier divisor for some $n > 1$.

Exercise 49.7. Give an example of a Weil divisor D on a variety which is not the Weil divisor associated to any Cartier divisor and such that nD is NOT the Weil divisor associated to a Cartier divisor for any $n > 1$. (Hint: Consider a cone, for example $X : xy - zw = 0$ in $\mathbf{A}^4_k$. Try to show that $D = [x = 0, z = 0]$ works.)

Exercise 49.8. On a separated scheme X of finite type over a field: Give an example of a Cartier divisor which is not the difference of two effective Cartier divisors. Hint: Find some X which does not have any nonempty effective Cartier divisors for example the scheme constructed in [Har77, III Exercise 5.9]. There is even an example with X a variety – namely the variety of Exercise 49.9.

Exercise 49.9. Example of a nonprojective proper variety. Let k be a field. Let $L \subset \mathbf{P}^3_k$ be a line and let $C \subset \mathbf{P}^3_k$ be a nonsingular conic. Assume that $C \cap L = \emptyset$. Choose an isomorphism $\varphi : L \to C$. Let X be the k-variety obtained by glueing C to L via φ. In other words there is a surjective proper birational morphism
$$\pi : \mathbf{P}^3_k \longrightarrow X$$
and an open $U \subset X$ such that $\pi : \pi^{-1}(U) \to U$ is an isomorphism, $\pi^{-1}(U) = \mathbf{P}^3_k \setminus (L \cup C)$ and such that $\pi|_L = \pi|_C \circ \varphi$. (These conditions do not yet uniquely define X. In order to do this you need to specify the structure sheaf of X along points of $Z = X \setminus U$.) Show X exists, is a proper variety, but is not projective. (Hint: For existence use the result of Exercise 37.9. For non-projectivity use that $\text{Pic}(\mathbf{P}^3_k) = \mathbf{Z}$ to show that X cannot have an ample invertible sheaf.)

50. Differentials

Definitions and results. Kähler differentials.

(1) Let $R \to A$ be a ring map. The *module of Kähler differentials of A over R* is denoted $\Omega_{A/R}$. It is generated by the elements $\mathrm{d}a$, $a \in A$ subject to the relations:
$$\mathrm{d}(a_1 + a_2) = \mathrm{d}a_1 + \mathrm{d}a_2, \quad \mathrm{d}(a_1 a_2) = a_1 \mathrm{d}a_2 + a_2 \mathrm{d}a_1, \quad \mathrm{d}r = 0$$
The canonical universal R-derivation $\mathrm{d} : A \to \Omega_{A/R}$ maps $a \mapsto \mathrm{d}a$.

(2) Consider the short exact sequence

$$0 \to I \to A \otimes_R A \to A \to 0$$

which defines the ideal I. There is a canonical derivation $\mathrm{d} : A \to I/I^2$ which maps a to the class of $a \otimes 1 - 1 \otimes a$. This is another presentation of the module of derivations of A over R, in other words

$$(I/I^2, \mathrm{d}) \cong (\Omega_{A/R}, \mathrm{d}).$$

(3) For multiplicative subsets $S_R \subset R$ and $S_A \subset A$ such that S_R maps into S_A we have

$$\Omega_{S_A^{-1}A/S_R^{-1}R} = S_A^{-1}\Omega_{A/R}.$$

(4) If A is a finitely presented R-algebra then $\Omega_{A/R}$ is a finitely presented A-module. Hence in this case the *fitting* ideals of $\Omega_{A/R}$ are defined.

(5) Let $f : X \to S$ be a morphism of schemes. There is a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\Omega_{X/S}$ and a $\mathcal{O}_S$-linear derivation

$$\mathrm{d} : \mathcal{O}_X \longrightarrow \Omega_{X/S}$$

such that for any affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ we have

$$\Gamma(\mathrm{Spec}(A), \Omega_{X/S}) = \Omega_{A/R}$$

compatibly with d.

Exercise 50.1. Let $k[\epsilon]$ be the ring of dual numbers over the field k, i.e., $\epsilon^2 = 0$.

(1) Consider the ring map

$$R = k[\epsilon] \to A = k[x, \epsilon]/(\epsilon x)$$

Show that the Fitting ideals of $\Omega_{A/R}$ are (starting with the zeroth Fitting ideal)

$$(\epsilon), A, A, \ldots$$

(2) Consider the map $R = k[t] \to A = k[x, y, t]/(x(y-t)(y-1), x(x-t))$. Show that the Fitting ideals of $\Omega_{A/R}$ in A are (assume characteristic k is zero for simplicity)

$$x(2x-t)(2y-t-1)A,\ (x, y, t) \cap (x, y-1, t),\ A,\ A, \ldots$$

So the 0-the Fitting ideal is cut out by a single element of A, the 1st Fitting ideal defines two closed points of $\mathrm{Spec}(A)$, and the others are all trivial.

(3) Consider the map $R = k[t] \to A = k[x, y, t]/(xy - t^n)$. Compute the Fitting ideals of $\Omega_{A/R}$.

Remark 50.2. The kth Fitting ideal of $\Omega_{X/S}$ is commonly used to define the singular scheme of the morphism $X \to S$ when X has relative dimension k over S. But as part (a) shows, you have to be careful doing this when your family does not have "constant" fibre dimension, e.g., when it is not flat. As part (b) shows, flatness doesn't guarantee it works either (and yes this is a flat family). In "good cases" – such as in (c) – for families of curves you expect the 0-th Fitting ideal to be zero and the 1st Fitting ideal to define (scheme-theoretically) the singular locus.

Exercise 50.3. Suppose that R is a ring and

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n).$$

Note that we are assuming that A is presented by the same number of equations as variables. Thus the matrix of partial derivatives

$$(\partial f_i/\partial x_j)$$

is $n \times n$, i.e., a square matrix. Assume that its determinant is invertible as an element in A. Note that this is exactly the condition that says that $\Omega_{A/R} = (0)$ in this case of n-generators and n relations. Let $\pi : B' \to B$ be a surjection of R-algebras whose kernel J has square zero (as an ideal in B'). Let $\varphi : A \to B$ be a homomorphism of R-algebras. Show there exists a unique homomorphism of R-algebras $\varphi' : A \to B'$ such that $\varphi = \pi \circ \varphi'$.

Exercise 50.4. Find a generalization of the result of Exercise 50.3 to the case where $A = R[x, y]/(f)$.

Exercise 50.5. Let k be a field, let $f_1, \ldots, f_c \in k[x_1, \ldots, x_n]$, and let $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Assume that $f_j(0, \ldots, 0) = 0$. This means that $\mathfrak{m} = (x_1, \ldots, x_n)A$ is a maximal ideal. Prove that the local ring $A_\mathfrak{m}$ is regular if the rank of the matrix

$$(\partial f_j/\partial x_i)|_{(x_1,\ldots,x_n)=(0,\ldots,0)}$$

is c. What is the dimension of $A_\mathfrak{m}$ in this case? Show that the converse is false by giving an example where $A_\mathfrak{m}$ is regular but the rank is less than c; what is the dimension of $A_\mathfrak{m}$ in your example?

51. Schemes, Final Exam, Fall 2007

These were the questions in the final exam of a course on Schemes, in the Spring of 2007 at Columbia University.

Exercise 51.1 (Definitions)**.** Provide definitions of the following concepts.

(1) X is a *scheme*
(2) the morphism of schemes $f : X \to Y$ is *finite*
(3) the morphisms of schemes $f : X \to Y$ is *of finite type*
(4) the scheme X is *Noetherian*
(5) the $\mathcal{O}_X$-module $\mathcal{L}$ on the scheme X is *invertible*
(6) the *genus* of a nonsingular projective curve over an algebraically closed field

Exercise 51.2. Let $X = \text{Spec}(\mathbf{Z}[x, y])$, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Suppose that $\mathcal{F}$ is zero when restricted to the standard affine open $D(x)$.

(1) Show that every global section s of $\mathcal{F}$ is killed by some power of x, i.e., $x^n s = 0$ for some $n \in \mathbf{N}$.
(2) Do you think the same is true if we do not assume that $\mathcal{F}$ is quasi-coherent?

Exercise 51.3. Suppose that $X \to \text{Spec}(R)$ is a proper morphism and that R is a discrete valuation ring with residue field k. Suppose that $X \times_{\text{Spec}(R)} \text{Spec}(k)$ is the empty scheme. Show that X is the empty scheme.

Exercise 51.4. Consider the projective[4] variety

$$\mathbf{P}^1 \times \mathbf{P}^1 = \mathbf{P}^1_{\mathbf{C}} \times_{\mathrm{Spec}(\mathbf{C})} \mathbf{P}^1_{\mathbf{C}}$$

over the field of complex numbers $\mathbf{C}$. It is covered by four affine pieces, corresponding to pairs of standard affine pieces of $\mathbf{P}^1_{\mathbf{C}}$. For example, suppose we use homogeneous coordinates X_0, X_1 on the first factor and Y_0, Y_1 on the second. Set $x = X_1/X_0$, and $y = Y_1/Y_0$. Then the 4 affine open pieces are the spectra of the rings

$$\mathbf{C}[x, y], \quad \mathbf{C}[x^{-1}, y], \quad \mathbf{C}[x, y^{-1}], \quad \mathbf{C}[x^{-1}, y^{-1}].$$

Let $X \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the closed subscheme which is the closure of the closed subset of the first affine piece given by the equation

$$y^3(x^4 + 1) = x^4 - 1.$$

(1) Show that X is contained in the union of the first and the last of the 4 affine open pieces.
(2) Show that X is a nonsingular projective curve.
(3) Consider the morphism $pr_2 : X \to \mathbf{P}^1$ (projection onto the first factor). On the first affine piece it is the map $(x, y) \mapsto x$. Briefly explain why it has degree 3.
(4) Compute the ramification points and ramification indices for the map $pr_2 : X \to \mathbf{P}^1$.
(5) Compute the genus of X.

Exercise 51.5. Let $X \to \mathrm{Spec}(\mathbf{Z})$ be a morphism of finite type. Suppose that there is an infinite number of primes p such that $X \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(\mathbf{F}_p)$ is not empty.

(1) Show that $X \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(\mathbf{Q})$ is not empty.
(2) Do you think the same is true if we replace the condition "finite type" by the condition "locally of finite type"?

52. Schemes, Final Exam, Spring 2009

These were the questions in the final exam of a course on Schemes, in the Spring of 2009 at Columbia University.

Exercise 52.1. Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $x \in X$ be a point. Assume that $\mathrm{Supp}(\mathcal{F}) = \{x\}$.

(1) Show that x is a closed point of X.
(2) Show that $H^0(X, \mathcal{F})$ is not zero.
(3) Show that $\mathcal{F}$ is generated by global sections.
(4) Show that $H^p(X, \mathcal{F}) = 0$ for $p > 0$.

Remark 52.2. Let k be a field. Let $\mathbf{P}^2_k = \mathrm{Proj}(k[X_0, X_1, X_2])$. Any invertible sheaf on $\mathbf{P}^2_k$ is isomorphic to $\mathcal{O}_{\mathbf{P}^2_k}(n)$ for some $n \in \mathbf{Z}$. Recall that

$$\Gamma(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(n)) = k[X_0, X_1, X_2]_n$$

is the degree n part of the polynomial ring. For a quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}^2_k$ set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^2_k}} \mathcal{O}_{\mathbf{P}^2_k}(n)$ as usual.

[4]The projective embedding is $((X_0, X_1), (Y_0, Y_1)) \mapsto (X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1)$ in other words $(x, y) \mapsto (1, y, x, xy)$.

Exercise 52.3. Let k be a field. Let $\mathcal{E}$ be a vector bundle on $\mathbf{P}^2_k$, i.e., a finite locally free $\mathcal{O}_{\mathbf{P}^2_k}$-module. We say $\mathcal{E}$ is *split* if $\mathcal{E}$ is isomorphic to a direct sum invertible $\mathcal{O}_{\mathbf{P}^2_k}$-modules.

(1) Show that $\mathcal{E}$ is split if and only if $\mathcal{E}(n)$ is split.
(2) Show that if $\mathcal{E}$ is split then $H^1(\mathbf{P}^2_k, \mathcal{E}(n)) = 0$ for all $n \in \mathbf{Z}$.
(3) Let
$$\varphi : \mathcal{O}_{\mathbf{P}^2_k} \longrightarrow \mathcal{O}_{\mathbf{P}^2_k}(1) \oplus \mathcal{O}_{\mathbf{P}^2_k}(1) \oplus \mathcal{O}_{\mathbf{P}^2_k}(1)$$
be given by linear forms $L_0, L_1, L_2 \in \Gamma(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(1))$. Assume $L_i \neq 0$ for some i. What is the condition on L_0, L_1, L_2 such that the cokernel of φ is a vector bundle? Why?
(4) Given an example of such a φ.
(5) Show that $\mathrm{Coker}(\varphi)$ is not split (if it is a vector bundle).

Remark 52.4. Freely use the following facts on dimension theory (and add more if you need more).

(1) The dimension of a scheme is the supremum of the length of chains of irreducible closed subsets.
(2) The dimension of a finite type scheme over a field is the maximum of the dimensions of its affine opens.
(3) The dimension of a Noetherian scheme is the maximum of the dimensions of its irreducible components.
(4) The dimension of an affine scheme coincides with the dimension of the corresponding ring.
(5) Let k be a field and let A be a finite type k-algebra. If A is a domain, and $x \neq 0$, then $\dim(A) = \dim(A/xA) + 1$.

Exercise 52.5. Let k be a field. Let X be a projective, reduced scheme over k. Let $f : X \to \mathbf{P}^1_k$ be a morphism of schemes over k. Assume there exists an integer $d \geq 0$ such that for every point $t \in \mathbf{P}^1_k$ the fibre $X_t = f^{-1}(t)$ is irreducible of dimension d. (Recall that an irreducible space is not empty.)

(1) Show that $\dim(X) = d + 1$.
(2) Let $X_0 \subset X$ be an irreducible component of X of dimension $d + 1$. Prove that for every $t \in \mathbf{P}^1_k$ the fibre $X_{0,t}$ has dimension d.
(3) What can you conclude about X_t and $X_{0,t}$ from the above?
(4) Show that X is irreducible.

Remark 52.6. Given a projective scheme X over a field k and a coherent sheaf $\mathcal{F}$ on X we set
$$\chi(X, \mathcal{F}) = \sum\nolimits_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Exercise 52.7. Let k be a field. Write $\mathbf{P}^3_k = \mathrm{Proj}(k[X_0, X_1, X_2, X_3])$. Let $C \subset \mathbf{P}^3_k$ be a *type* $(5, 6)$ *complete intersection curve*. This means that there exist $F \in k[X_0, X_1, X_2, X_3]_5$ and $G \in k[X_0, X_1, X_2, X_3]_6$ such that
$$C = \mathrm{Proj}(k[X_0, X_1, X_2, X_3]/(F, G))$$
is a variety of dimension 1. (Variety implies reduced and irreducible, but feel free to assume C is nonsingular if you like.) Let $i : C \to \mathbf{P}^3_k$ be the corresponding closed

immersion. Being a complete intersection also implies that

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3_k}(-11) \xrightarrow{\begin{pmatrix} -G \\ F \end{pmatrix}} \mathcal{O}_{\mathbf{P}^3_k}(-5) \oplus \mathcal{O}_{\mathbf{P}^3_k}(-6) \xrightarrow{(F,G)} \mathcal{O}_{\mathbf{P}^3_k} \longrightarrow i_*\mathcal{O}_C \longrightarrow 0$$

is an exact sequence of sheaves. Please use these facts to:

1. compute $\chi(C, i^*\mathcal{O}_{\mathbf{P}^3_k}(n))$ for any $n \in \mathbf{Z}$, and
2. compute the dimension of $H^1(C, \mathcal{O}_C)$.

Exercise 52.8. Let k be a field. Consider the rings

$$A = k[x, y]/(xy)$$
$$B = k[u, v]/(uv)$$
$$C = k[t, t^{-1}] \times k[s, s^{-1}]$$

and the k-algebra maps

$$\begin{array}{lll} A \longrightarrow C, & x \mapsto (t, 0), & y \mapsto (0, s) \\ B \longrightarrow C, & u \mapsto (t^{-1}, 0), & v \mapsto (0, s^{-1}) \end{array}$$

It is a true fact that these maps induce isomorphisms $A_{x+y} \to C$ and $B_{u+v} \to C$. Hence the maps $A \to C$ and $B \to C$ identify $\mathrm{Spec}(C)$ with open subsets of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$. Let X be the scheme obtained by glueing $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ along $\mathrm{Spec}(C)$:

$$X = \mathrm{Spec}(A) \amalg_{\mathrm{Spec}(C)} \mathrm{Spec}(B).$$

As we saw in the course such a scheme exists and there are affine opens $\mathrm{Spec}(A) \subset X$ and $\mathrm{Spec}(B) \subset X$ whose overlap is exactly $\mathrm{Spec}(C)$ identified with an open of each of these using the maps above.

1. Why is X separated?
2. Why is X of finite type over k?
3. Compute $H^1(X, \mathcal{O}_X)$, or what is its dimension?
4. What is a more geometric way to describe X?

53. Schemes, Final Exam, Fall 2010

These were the questions in the final exam of a course on Schemes, in the Fall of 2010 at Columbia University.

Exercise 53.1 (Definitions)**.** Provide definitions of the following concepts.

1. a separated scheme,
2. a quasi-compact morphism of schemes,
3. an affine morphism of schemes,
4. a multiplicative subset of a ring,
5. a Noetherian scheme,
6. a variety.

Exercise 53.2. Prime avoidance.

1. Let A be a ring. Let $I \subset A$ be an ideal and let $\mathfrak{q}_1$, $\mathfrak{q}_2$ be prime ideals such that $I \not\subset \mathfrak{q}_i$. Show that $I \not\subset \mathfrak{q}_1 \cup \mathfrak{q}_2$.
2. What is a geometric interpretation of (1)?
3. Let $X = \mathrm{Proj}(S)$ for some graded ring S. Let $x_1, x_2 \in X$. Show that there exists a standard open $D_+(F)$ which contains both x_1 and x_2.

Exercise 53.3. Why is a composition of affine morphisms affine?

Exercise 53.4 (Examples)**.** Give examples of the following:

(1) A reducible projective scheme over a field k.
(2) A scheme with 100 points.
(3) A non-affine morphism of schemes.

Exercise 53.5. Chevalley's theorem and the Hilbert Nullstellensatz.

(1) Let $\mathfrak{p} \subset \mathbf{Z}[x_1, \ldots, x_n]$ be a maximal ideal. What does Chevalley's theorem imply about $\mathfrak{p} \cap \mathbf{Z}$?
(2) In turn, what does the Hilbert Nullstellensatz imply about $\kappa(\mathfrak{p})$?

Exercise 53.6. Let A be a ring. Let $S = A[X]$ as a graded A-algebra where X has degree 1. Show that $\text{Proj}(S) \cong \text{Spec}(A)$ as schemes over A.

Exercise 53.7. Let $A \to B$ be a finite ring map. Show that $\text{Spec}(B)$ is a H-projective scheme over $\text{Spec}(A)$.

Exercise 53.8. Give an example of a scheme X over a field k such that X is irreducible and such that for some finite extension k'/k the base change $X_{k'} = X \times_{\text{Spec}(k)} \text{Spec}(k')$ is connected but reducible.

54. Schemes, Final Exam, Spring 2011

These were the questions in the final exam of a course on Schemes, in the Spring of 2011 at Columbia University.

Exercise 54.1 (Definitions)**.** Provide definitions of the italicized concepts.

(1) a *separated* scheme,
(2) a *universally closed* morphism of schemes,
(3) *A dominates B* for local rings A, B contained in a common field,
(4) the *dimension* of a scheme X,
(5) the *codimension* of an irreducible closed subscheme Y of a scheme X,

Exercise 54.2 (Results)**.** State something formally equivalent to the fact discussed in the course.

(1) The valuative criterion of properness for a morphism $X \to Y$ of varieties for example.
(2) The relationship between $\dim(X)$ and the function field $k(X)$ of X for a variety X over a field k.
(3) Fill in the blank: The category of nonsingular projective curves over k and nonconstant morphisms is anti-equivalent to ....
(4) Noether normalization.
(5) Jacobian criterion.

Exercise 54.3. Let k be a field. Let $F \in k[X_0, X_1, X_2]$ be a homogeneous form of degree d. Assume that $C = V_+(F) \subset \mathbf{P}^2_k$ is a smooth curve over k. Denote $i : C \to \mathbf{P}^2_k$ the corresponding closed immersion.

(1) Show that there is a short exact sequence
$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \to \mathcal{O}_{\mathbf{P}^2_k} \to i_*\mathcal{O}_C \to 0$$
of coherent sheaves on $\mathbf{P}^2_k$: tell me what the maps are and briefly why it is exact.

(2) Conclude that $H^0(C, \mathcal{O}_C) = k$.
(3) Compute the genus of C.
(4) Assume now that $P = (0 : 0 : 1)$ is not on C. Prove that $\pi : C \to \mathbf{P}^1_k$ given by $(a_0 : a_1 : a_2) \mapsto (a_0 : a_1)$ has degree d.
(5) Assume k is algebraically closed, assume all ramification indices (the "e_i") are 1 or 2, and assume the characteristic of k is not equal to 2. How many ramification points does $\pi : C \to \mathbf{P}^1_k$ have?
(6) In terms of F, what do you think is a set of equations of the set of ramification points of π?
(7) Can you guess K_C?

Exercise 54.4. Let k be a field. Let X be a "triangle" over k, i.e., you get X by glueing three copies of $\mathbf{A}^1_k$ to each other by identifying 0 on the first copy to 1 on the second copy, 0 on the second copy to 1 on the third copy, and 0 on the third copy to 1 on the first copy. It turns out that X is isomorphic to $\mathrm{Spec}(k[x,y]/(xy(x+y+1)))$; feel free to use this. Compute the Picard group of X.

Exercise 54.5. Let k be a field. Let $\pi : X \to Y$ be a finite birational morphism of curves with X a projective nonsingular curve over k. It follows from the material in the course that Y is a proper curve and that π is the normalization morphism of Y. We have also seen in the course that there exists a dense open $V \subset Y$ such that $U = \pi^{-1}(V)$ is a dense open in X and $\pi : U \to V$ is an isomorphism.

(1) Show that there exists an effective Cartier divisor $D \subset X$ such that $D \subset U$ and such that $\mathcal{O}_X(D)$ is ample on X.
(2) Let D be as in (1). Show that $E = \pi(D)$ is an effective Cartier divisor on Y.
(3) Briefly indicate why
  (a) the map $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ has a coherent cokernel Q which is supported in $Y \setminus V$, and
  (b) for every n there is a corresponding map $\mathcal{O}_Y(nE) \to \pi_*\mathcal{O}_X(nD)$ whose cokernel is isomorphic to Q.
(4) Show that $\dim_k H^0(X, \mathcal{O}_X(nD)) - \dim_k H^0(Y, \mathcal{O}_Y(nE))$ is bounded (by what?) and conclude that the invertible sheaf $\mathcal{O}_Y(nE)$ has lots of sections for large n (why?).

55. Schemes, Final Exam, Fall 2011

These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2011 at Columbia University.

Exercise 55.1 (Definitions)**.** Provide definitions of the italicized concepts.

(1) a *Noetherian* ring,
(2) a *Noetherian* scheme,
(3) a *finite* ring homomorphism,
(4) a *finite* morphism of schemes,
(5) the *dimension* of a ring.

Exercise 55.2 (Results)**.** State something formally equivalent to the fact discussed in the course.

(1) Zariski's Main Theorem.
(2) Noether normalization.

(3) Chinese remainder theorem.
(4) Going up for finite ring maps.

Exercise 55.3. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring whose residue field has characteristic not 2. Suppose that $\mathfrak{m}$ is generated by three elements x, y, z and that $x^2 + y^2 + z^2 = 0$ in A.

(1) What are the possible values of $\dim(A)$?
(2) Give an example to show that each value is possible.
(3) Show that A is a domain if $\dim(A) = 2$. (Hint: look at $\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.)

Exercise 55.4. Let A be a ring. Let $S \subset T \subset A$ be multiplicative subsets. Assume that

$$\{\mathfrak{q} \mid \mathfrak{q} \cap S = \emptyset\} = \{\mathfrak{q} \mid \mathfrak{q} \cap T = \emptyset\}.$$

Show that $S^{-1}A \to T^{-1}A$ is an isomorphism.

Exercise 55.5. Let k be an algebraically closed field. Let

$$V_0 = \{A \in \text{Mat}(3 \times 3, k) \mid \text{rank}(A) = 1\} \subset \text{Mat}(3 \times 3, k) = k^9.$$

(1) Show that V_0 is the set of closed points of a (Zariski) locally closed subset $V \subset \mathbf{A}^9_k$.
(2) Is V irreducible?
(3) What is $\dim(V)$?

Exercise 55.6. Prove that the ideal (x^2, xy, y^2) in $\mathbf{C}[x, y]$ cannot be generated by 2 elements.

Exercise 55.7. Let $f \in \mathbf{C}[x, y]$ be a nonconstant polynomial. Show that for some $\alpha, \beta \in \mathbf{C}$ the $\mathbf{C}$-algebra map

$$\mathbf{C}[t] \longrightarrow \mathbf{C}[x, y]/(f), \quad t \longmapsto \alpha x + \beta y$$

is finite.

Exercise 55.8. Show that given finitely many points $p_1, \ldots, p_n \in \mathbf{C}^2$ the scheme $\mathbf{A}^2_{\mathbf{C}} \setminus \{p_1, \ldots, p_n\}$ is a union of two affine opens.

Exercise 55.9. Show that there exists a surjective morphism of schemes $\mathbf{A}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$. (Surjective just means surjective on underlying sets of points.)

Exercise 55.10. Let k be an algebraically closed field. Let $A \subset B$ be an extension of domains which are both finite type k-algebras. Prove that the image of $\text{Spec}(B) \to \text{Spec}(A)$ contains a nonempty open subset of $\text{Spec}(A)$ using the following steps:

(1) Prove it if $A \to B$ is also finite.
(2) Prove it in case the fraction field of B is a finite extension of the fraction field of A.
(3) Reduce the statement to the previous case.

56. Schemes, Final Exam, Fall 2013

These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2013 at Columbia University.

Exercise 56.1 (Definitions)**.** Provide definitions of the italicized concepts.

(1) a *radical ideal* of a ring,

(2) a *finite type* ring homomorphism,
(3) a *differential a la Weil*,
(4) a *scheme*.

Exercise 56.2 (Results)**.** State something formally equivalent to the fact discussed in the course.

(1) result on hilbert polynomials of graded modules.
(2) dimension of a Noetherian local ring $(R, \mathfrak{m})$ and $\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.
(3) Riemann-Roch.
(4) Clifford's theorem.
(5) Chevalley's theorem.

Exercise 56.3. Let $A \to B$ be a ring map. Let $S \subset A$ be a multiplicative subset. Assume that $A \to B$ is of finite type and $S^{-1}A \to S^{-1}B$ is surjective. Show that there exists an $f \in S$ such that $A_f \to B_f$ is surjective.

Exercise 56.4. Give an example of an injective local homomorphism $A \to B$ of local rings, such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not surjective.

Situation 56.5 (Notation plane curve)**.** Let k be an algebraically closed field. Let $F(X_0, X_1, X_2) \in k[X_0, X_1, X_2]$ be an irreducible polynomial homogeneous of degree d. We let

$$D = V(F) \subset \mathbf{P}^2$$

be the projective plane curve given by the vanishing of F. Set $x = X_1/X_0$ and $y = X_2/X_0$ and $f(x, y) = X_0^{-d} F(X_0, X_1, X_2) = F(1, x, y)$. We denote K the fraction field of the domain $k[x, y]/(f)$. We let C be the abstract curve corresponding to K. Recall (from the lectures) that there is a surjective map $C \to D$ which is bijective over the nonsingular locus of D and an isomorphism if D is nonsingular. Set $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$. Finally, we denote $\omega = \mathrm{d}x/f_y = -\mathrm{d}y/f_x$ the element of $\Omega_{K/k}$ discussed in the lectures. Denote K_C the divisor of zeros and poles of ω.

Exercise 56.6. In Situation 56.5 assume $d \geq 3$ and that the curve D has exactly one singular point, namely $P = (1 : 0 : 0)$. Assume further that we have the expansion

$$f(x, y) = xy + h.o.t$$

around $P = (0, 0)$. Then C has two points v and w lying over P characterized by

$$v(x) = 1, v(y) > 1 \quad \text{and} \quad w(x) > 1, w(y) = 1$$

(1) Show that the element $\omega = \mathrm{d}x/f_y = -\mathrm{d}y/f_x$ of $\Omega_{K/k}$ has a first order pole at both v and w. (The behaviour of ω at nonsingular points is as discussed in the lectures.)
(2) In the lectures we have shown that ω vanishes to order $d-3$ at the divisor $X_0 = 0$ pulled back to C under the map $C \to D$. Combined with the information of (1) what is the degree of the divisor of zeros and poles of ω on C?
(3) What is the genus of the curve C?

Exercise 56.7. In Situation 56.5 assume $d = 5$ and that the curve $C = D$ is nonsingular. In the lectures we have shown that the genus of C is 6 and that the linear system K_C is given by

$$L(K_C) = \{h\omega \mid h \in k[x, y],\ \deg(h) \leq 2\}$$

where deg indicates total degree[5]. Let $P_1, P_2, P_3, P_4, P_5 \in D$ be pairwise distinct points lying in the affine open $X_0 \neq 0$. We denote $\sum P_i = P_1 + P_2 + P_3 + P_4 + P_5$ the corresponding divisor of C.

(1) Describe $L(K_C - \sum P_i)$ in terms of polynomials.
(2) What are the possibilities for $l(\sum P_i)$?

Exercise 56.8. Write down an F as in Situation 56.5 with $d = 100$ such that the genus of C is 0.

Exercise 56.9. Let k be an algebraically closed field. Let K/k be finitely generated field extension of transcendence degree 1. Let C be the abstract curve corresponding to K. Let $V \subset K$ be a g^r_d and let $\Phi : C \to \mathbf{P}^r$ be the corresponding morphism. Show that the image of C is contained in a quadric[6] if V is a complete linear system and d is large enough relative to the genus of C. (Extra credit: good bound on the degree needed.)

Exercise 56.10. Notation as in Situation 56.5. Let $U \subset \mathbf{P}^2_k$ be the open subscheme whose complement is D. Describe the k-algebra $A = \mathcal{O}_{\mathbf{P}^2_k}(U)$. Give an upper bound for the number of generators of A as a k-algebra.

57. Schemes, Final Exam, Spring 2014

These were the questions in the final exam of a course on Schemes, in the Spring of 2014 at Columbia University.

Exercise 57.1 (Definitions)**.** Let $(X, \mathcal{O}_X)$ be a scheme. Provide definitions of the italicized concepts.

(1) the *local ring of X at a point x*,
(2) a *quasi-coherent* sheaf of $\mathcal{O}_X$-modules,
(3) a *coherent* sheaf of $\mathcal{O}_X$-modules (please assume X is locally Noetherian,
(4) an *affine open* of X,
(5) a *finite morphism of schemes $X \to Y$*.

Exercise 57.2 (Theorems)**.** Precisely state a nontrivial fact discussed in the lectures related to each item.

(1) on birational invariance of pluri-genera of varieties,
(2) being an affine morphism is a local property,
(3) the topology of a scheme theoretic fibre of a morphism, and
(4) valuative criterion of properness.

Exercise 57.3. Let $X = \mathbf{A}^2_{\mathbf{C}}$ where $\mathbf{C}$ is the field of complex numbers. A *line* will mean a closed subscheme of X defined by one linear equation $ax + by + c = 0$ for some $a, b, c \in \mathbf{C}$ with $(a, b) \neq (0, 0)$. A *curve* will mean an irreducible (so nonempty) closed subscheme $C \subset X$ of dimension 1. A *quadric* will mean a curve defined by one quadratic equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ for some $a, b, c, d, e, f \in \mathbf{C}$ and $(a, b, c) \neq (0, 0, 0)$.

(1) Find a curve C such that every line has nonempty intersection with C.
(2) Find a curve C such that every line and every quadric has nonempty intersection with C.

[5]We get ≤ 2 because $d - 3 = 5 - 3 = 2$.

[6]A quadric is a degree 2 hypersurface, i.e., the zero set in $\mathbf{P}^r$ of a degree 2 homogeneous polynomial.

(3) Show that for every curve C there exists another curve such that $C \cap C' = \emptyset$.

Exercise 57.4. Let k be a field. Let $b : X \to \mathbf{A}^2_k$ be the blow up of the affine plane in the origin. In other words, if $\mathbf{A}^2_k = \mathrm{Spec}(k[x, y])$, then $X = \mathrm{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n)$ where $\mathfrak{m} = (x, y) \subset k[x, y]$. Prove the following statements

1. the scheme theoretic fibre E of b over the origin is isomorphic to $\mathbf{P}^1_k$,
2. E is an effective Cartier divisor on X,
3. the restriction of $\mathcal{O}_X(-E)$ to E is a line bundle of degree 1.

(Recall that $\mathcal{O}_X(-E)$ is the ideal sheaf of E in X.)

Exercise 57.5. Let k be a field. Let X be a projective variety over k. Show there exists an affine variety U over k and a surjective morphism of varieties $U \to X$.

Exercise 57.6. Let k be a field of characteristic $p > 0$ different from $2, 3$. Consider the closed subscheme X of $\mathbf{P}^n_k$ defined by

$$\sum\nolimits_{i=0,\ldots,n} X_i = 0, \quad \sum\nolimits_{i=0,\ldots,n} X_i^2 = 0, \quad \sum\nolimits_{i=0,\ldots,n} X_i^3 = 0$$

For which pairs (n, p) is this variety singular?

58. Commutative Algebra, Final Exam, Fall 2016

These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2016 at Columbia University.

Exercise 58.1 (Definitions)**.** Let R be a ring. Provide definitions of the italicized concepts.

1. the *local ring of R at a prime* $\mathfrak{p}$,
2. a *finite* R-module,
3. a *finitely presented* R-module,
4. R is *regular*,
5. R is *catenary*,
6. R is *Cohen-Macaulay*.

Exercise 58.2 (Theorems)**.** Precisely state a nontrivial fact discussed in the lectures related to each item.

1. regular rings,
2. associated primes of Cohen-Macaulay modules,
3. dimension of a finite type domain over a field, and
4. Chevalley's theorem.

Exercise 58.3. Let $A \to B$ be a ring map such that

1. A is local with maximal ideal $\mathfrak{m}$,
2. $A \to B$ is a finite[7] ring map,
3. $A \to B$ is injective (we think of A as a subring of B).

Show that there is a prime ideal $\mathfrak{q} \subset B$ with $\mathfrak{m} = A \cap \mathfrak{q}$.

Exercise 58.4. Let k be a field. Let $R = k[x, y, z, w]$. Consider the ideal $I = (xy, xz, xw)$. What are the irreducible components of $V(I) \subset \mathrm{Spec}(R)$ and what are their dimensions?

[7] Recall that this means B is finite as an A-module.

Exercise 58.5. Let k be a field. Let $A = k[x, x^{-1}]$ and $B = k[y]$. Show that any k-algebra map $\varphi : A \to B$ maps x to a constant.

Exercise 58.6. Consider the ring $R = \mathbf{Z}[x, y]/(xy - 7)$. Prove that R is regular.

Given a Noetherian local ring $(R, \mathfrak{m}, \kappa)$ for $n \geq 0$ we let $\varphi_R(n) = \dim_\kappa(\mathfrak{m}^n/\mathfrak{m}^{n+1})$.

Exercise 58.7. Does there exist a Noetherian local ring R with $\varphi_R(n) = n+1$ for all $n \geq 0$?

Exercise 58.8. Let R be a Noetherian local ring. Suppose that $\varphi_R(0) = 1$, $\varphi_R(1) = 3$, $\varphi_R(2) = 5$. Show that $\varphi_R(3) \leq 7$.

59. Schemes, Final Exam, Spring 2017

These were the questions in the final exam of a course on schemes, in the Spring of 2017 at Columbia University.

Exercise 59.1 (Definitions)**.** Let $f : X \to Y$ be a morphism of schemes. Provide brief definitions of the italicized concepts.

(1) the *scheme theoretic fibre* of f at $y \in Y$,
(2) f is a *finite morphism*,
(3) a *quasi-coherent* $\mathcal{O}_X$-module,
(4) X is *variety*,
(5) f is a *smooth morphism*,
(6) f is a *proper morphism*.

Exercise 59.2 (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item.

(1) pushforward of quasi-coherent sheaves,
(2) cohomology of coherent sheaves on projective varieties,
(3) Serre duality for a projective scheme over a field, and
(4) Riemann-Hurwitz.

Exercise 59.3. Let k be an algebraically closed field. Let $\ell > 100$ be a prime number different from the characteristic of k. Let X be the nonsingular projective model of the affine curve given by the equation

$$y^\ell = x(x-1)^3$$

in $\mathbf{A}^2_k$. Answer the following questions:

(1) What is the genus of X?
(2) Give an upper bound for the gonality[8] of X.

Exercise 59.4. Let k be an algebraically closed field. Let X be a reduced, projective scheme over k all of whose irreducible components have the same dimension 1. Let $\omega_{X/k}$ be the relative dualizing module. Show that if $\dim_k H^1(X, \omega_{X/k}) > 1$, then X is disconnected.

Exercise 59.5. Give an example of a scheme X and a nontrivial invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that both $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{\otimes -1})$ are nonzero.

[8]The gonality is the smallest degree of a nonconstant morphism from X to $\mathbf{P}^1_k$.

Exercise 59.6. Let k be an algebraically closed field. Let $g \geq 3$. Let X and X' be smooth projective curves over k of genus g and $g+1$. Let $Y \subset X \times X'$ be a curve such that the projections $Y \to X$ and $Y \to X'$ are nonconstant. Prove that the nonsingular projective model of Y has genus $\geq 2g+1$.

Exercise 59.7. Let k be a finite field. Let $g > 1$. Sketch a proof of the following: there are only a finite number of isomorphism classes of smooth projective curves over k of genus g. (You will get credit for even just trying to answer this.)

60. Commutative Algebra, Final Exam, Fall 2017

These were the questions in the final exam of a course on commutative algebra, in the Fall of 2017 at Columbia University.

Exercise 60.1 (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) the *left adjoint* of a functor $F : \mathcal{A} \to \mathcal{B}$,
(2) the *transcendence degree* of an extension L/K of fields,
(3) a *regular function* on a classical affine variety $X \subset k^n$,
(4) a *sheaf* on a topological space,
(5) a *local ring*, and
(6) a morphism of schemes $f : X \to Y$ being *affine*.

Exercise 60.2 (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) Yoneda lemma,
(2) Mayer-Vietoris,
(3) dimension and cohomology,
(4) Hilbert polynomial, and
(5) duality for projective space.

Exercise 60.3. Let k be an algebraically closed field. Consider the closed subset X of k^5 with Zariski topology and coordinates x_1, x_2, x_3, x_4, x_5 given by the equations

$$x_1^2 - x_4 = 0, \quad x_2^5 - x_5 = 0, \quad x_3^2 + x_3 + x_4 + x_5 = 0$$

What is the dimension of X and why?

Exercise 60.4. Let k be a field. Let $X = \mathbf{P}^1_k$ be the projective space of dimension 1 over k. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. For $d \in \mathbf{Z}$ denote $\mathcal{E}(d) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ the dth Serre twist of $\mathcal{E}$ and $h^i(X, \mathcal{E}(d)) = \dim_k H^i(X, \mathcal{E}(d))$.

(1) Why is there no $\mathcal{E}$ with $h^0(X, \mathcal{E}) = 5$ and $h^0(X, \mathcal{E}(1)) = 4$?
(2) Why is there no $\mathcal{E}$ with $h^1(X, \mathcal{E}(1)) = 5$ and $h^1(X, \mathcal{E}) = 4$?
(3) For which $a \in \mathbf{Z}$ can there exist a vector bundle $\mathcal{E}$ on X with

$$\begin{array}{ll} h^0(X, \mathcal{E}) \quad\; = 1 & h^1(X, \mathcal{E}) \quad\; = 1 \\ h^0(X, \mathcal{E}(1)) = 2 & h^1(X, \mathcal{E}(1)) = 0 \\ h^0(X, \mathcal{E}(2)) = 4 & h^1(X, \mathcal{E}(2)) = a \end{array}$$

Partial answers are welcomed and encouraged.

Exercise 60.5. Let X be a topological space which is the union $X = Y \cup Z$ of two closed subsets Y and Z whose intersection is denoted $W = Y \cap Z$. Denote $i : Y \to X$, $j : Z \to X$, and $k : W \to X$ the inclusion maps.

(1) Show that there is a short exact sequence of sheaves
$$0 \to \underline{\mathbf{Z}}_X \to i_*(\underline{\mathbf{Z}}_Y) \oplus j_*(\underline{\mathbf{Z}}_Z) \to k_*(\underline{\mathbf{Z}}_W) \to 0$$
where $\underline{\mathbf{Z}}_X$ denotes the constant sheaf with value $\mathbf{Z}$ on X, etc.

(2) What can you conclude about the relationship between the cohomology groups of X, Y, Z, W with $\mathbf{Z}$-coefficients?

Exercise 60.6. Let k be a field. Let $A = k[x_1, x_2, x_3, \ldots]$ be the polynomial ring in infinitely many variables. Denote $\mathfrak{m}$ the maximal ideal of A generated by all the variables. Let $X = \mathrm{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$.

(1) Show $H^1(U, \mathcal{O}_U) = 0$. Hint: Čech cohomology computation.
(2) What is your guess for $H^i(U, \mathcal{O}_U)$ for $i \geq 1$?

Exercise 60.7. Let A be a local ring. Let $a \in A$ be a nonzerodivisor. Let $I, J \subset A$ be ideals such that $IJ = (a)$. Show that the ideal I is principal, i.e., generated by one element (which will turn out to be a nonzerodivisor).

61. Schemes, Final Exam, Spring 2018

These were the questions in the final exam of a course on schemes, in the Spring of 2018 at Columbia University.

Exercise 61.1 (Definitions)**.** Provide brief definitions of the italicized concepts. Let k be an algebraically closed field. Let X be a projective curve over k.

(1) a *smooth* algebra over k,
(2) the *degree* of an invertible $\mathcal{O}_X$-module on X,
(3) the *genus* of X,
(4) the *Weil divisor class group* of X,
(5) X is *hyperelliptic*, and
(6) the *intersection number* of two curves on a smooth projective surface over k.

Exercise 61.2 (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) Riemann-Hurwitz theorem,
(2) Clifford's theorem,
(3) factorization of maps between smooth projective surfaces,
(4) Hodge index theorem, and
(5) Riemann hypothesis for curves over finite fields.

Exercise 61.3. Let k be an algebraically closed field. Let $X \subset \mathbf{P}^3_k$ be a smooth curve of degree d and genus ≥ 2. Assume X is not contained in a plane and that there is a line ℓ in $\mathbf{P}^3_k$ meeting X in $d - 2$ points. Show that X is hyperelliptic.

Exercise 61.4. Let k be an algebraically closed field. Let X be a projective curve with pairwise distinct singular points $p_1, \ldots, p_n$. Explain why the genus of the normalization of X is at most $-n + \dim_k H^1(X, \mathcal{O}_X)$.

Exercise 61.5. Let k be a field. Let $X = \mathrm{Spec}(k[x, y])$ be affine 2 space. Let
$$I = (x^3, x^2y, xy^2, y^3) \subset k[x, y].$$
Let $Y \subset X$ be the closed subscheme corresponding to I. Let $b : X' \to X$ be the blowing up of the ideal (x, y), i.e., the blow up of affine space at the origin.

(1) Show that the scheme theoretic inverse image $b^{-1}Y \subset X'$ is an effective Cartier divisor.
(2) Given an example of an ideal $J \subset k[x,y]$ with $I \subset J \subset (x,y)$ such that if $Z \subset X$ is the closed subscheme corresponding to J, then the scheme theoretic inverse image $b^{-1}Z$ is not an effective Cartier divisor.

Exercise 61.6. Let k be an algebraically closed field. Consider the following types of surfaces

(1) $S = C_1 \times C_2$ where C_1 and C_2 are smooth projective curves,
(2) $S = C_1 \times C_2$ where C_1 and C_2 are smooth projective curves and the genus of C_1 is > 0,
(3) $S \subset \mathbf{P}^3_k$ is a hypersurface of degree 4, and
(4) $S \subset \mathbf{P}^3_k$ is a smooth hypersurface of degree 4.

For each type briefly indicate why or why not the class of surfaces of this type contains rational surfaces.

Exercise 61.7. Let k be an algebraically closed field. Let $S \subset \mathbf{P}^3_k$ be a smooth hypersurface of degree d. Assume that S contains a line ℓ. What is the self square of ℓ viewed as a divisor on S?

62. Commutative Algebra, Final Exam, Fall 2019

These were the questions in the final exam of a course on commutative algebra, in the Fall of 2019 at Columbia University.

Exercise 62.1 (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) a *constructible subset* of a Noetherian topological space,
(2) the *localization* of an R-module M at a prime $\mathfrak{p}$,
(3) the *length* of a module over a Noetherian local ring $(A, \mathfrak{m}, \kappa)$,
(4) a *projective module* over a ring R, and
(5) a *Cohen-Macaulay* module over a Noetherian local ring $(A, \mathfrak{m}, \kappa)$.

Exercise 62.2 (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) images of constructible sets,
(2) Hilbert Nullstellensatz,
(3) dimension of finite type algebras over fields,
(4) Noether normalization, and
(5) regular local rings.

For a ring R and an ideal $I \subset R$ recall that $V(I)$ denotes the set of $\mathfrak{p} \in \mathrm{Spec}(R)$ with $I \subset \mathfrak{p}$.

Exercise 62.3 (Making primes)**.** Construct infinitely many distinct prime ideals $\mathfrak{p} \subset \mathbf{C}[x,y]$ such that $V(\mathfrak{p})$ contains (x,y) and $(x-1, y-1)$.

Exercise 62.4 (No prime)**.** Let $R = \mathbf{C}[x,y,z]/(xy)$. Argue briefly there does not exist a prime ideal $\mathfrak{p} \subset R$ such that $V(\mathfrak{p})$ contains $(x, y-1, z-5)$ and $(x-1, y, z-7)$.

Exercise 62.5 (Frobenius)**.** Let p be a prime number (you may assume $p = 2$ to simplify the formulas). Let R be a ring such that $p = 0$ in R.

(1) Show that the map $F : R \to R$, $x \mapsto x^p$ is a ring homomorphism.

(2) Show that $\text{Spec}(F) : \text{Spec}(R) \to \text{Spec}(R)$ is the identity map.

Recall that a *specialization* $x \rightsquigarrow y$ of points of a topological space simply means y is in the closure of x. We say $x \rightsquigarrow y$ is an *immediate specialization* if there does not exist a z different from x and y such that $x \rightsquigarrow z$ and $z \rightsquigarrow y$.

Exercise 62.6 (Dimension)**.** Suppose we have a sober topological space X containing 5 distinct points x, y, z, u, v having the following specializations

$$\begin{tikzcd} x \arrow[r] \arrow[d] & u & v \arrow[l] \arrow[dl] \\ y \arrow[r] & z & \end{tikzcd}$$

What is the minimal dimension such an X can have? If X is the spectrum of a finite type algebra over a field and $x \rightsquigarrow u$ is an immediate specialization, what can you say about the specialization $v \rightsquigarrow z$?

Exercise 62.7 (Tor computation)**.** Let $R = \mathbf{C}[x, y, z]$. Let $M = R/(x, z)$ and $N = R/(y, z)$. For which $i \in \mathbf{Z}$ is $\text{Tor}_i^R(M, N)$ nonzero?

Exercise 62.8. Let $A \to B$ be a flat local homomorphism of local Noetherian rings. Show that if A has depth k, then B has depth at least k.

63. Algebraic Geometry, Final Exam, Spring 2020

These were the questions in the final exam of a course on Algebraic Geometry, in the Spring of 2020 at Columbia University.

Exercise 63.1 (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) a *scheme*,
(2) a *morphism of schemes*,
(3) a *quasi-coherent module* on a scheme,
(4) a *variety* over a field k,
(5) a *curve* over a field k,
(6) a *finite morphism* of schemes,
(7) the *cohomology* of a sheaf of abelian groups $\mathcal{F}$ over a topological space X,
(8) a *dualizing sheaf* on a scheme X of dimension d proper over a field k, and
(9) a *rational map* from a variety X to a variety Y.

Exercise 63.2 (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) cohomology of abelian sheaves on a Noetherian topological space X of dimension d,
(2) sheaf of differentials $\Omega^1_{X/k}$ of a smooth variety over a field k,
(3) dualizing sheaf ω_X of a smooth projective variety X over the field k,
(4) a smooth proper genus 0 curve over an algebraically closed field k, and
(5) the genus of a plane curve of degree d.

Exercise 63.3. Let k be a field. Let X be a scheme over k. Assume $X = X_1 \cup X_2$ is an open covering with X_1, X_2 both isomorphic to $\mathbf{P}^1_k$ and $X_1 \cap X_2$ isomorphic to $\mathbf{A}^1_k$. (Such a scheme exists, for example you can take $\mathbf{P}^1_k$ with ∞ doubled.) Show that $\dim_k H^1(X, \mathcal{O}_X)$ is infinite.

Exercise 63.4. Let k be an algebraically closed field. Let Y be a smooth projective curve of genus 10. Find a good lower bound for the genus of a smooth projective curve X such that there exists a nonconstant morphism $f : X \to Y$ which is not an isomorphism.

Exercise 63.5. Let k be an algebraically closed field of characteristic 0. Let

$$X : T_0^d + T_1^d - T_2^d = 0 \subset \mathbf{P}^2_k$$

be the Fermat curve of degree $d \geq 3$. Consider the closed points $p = [1 : 0 : 1]$ and $q = [0 : 1 : 1]$ on X. Set $D = [p] - [q]$.

(1) Show that D is nontrivial in the Weil divisor class group.
(2) Show that dD is trivial in the Weil divisor class group. (Hint: try to show that both $d[p]$ and $d[q]$ are the intersection of X with a line in the plane.)

Exercise 63.6. Let k be an algebraically closed field. Consider the 2-uple embedding

$$\varphi : \mathbf{P}^2 \longrightarrow \mathbf{P}^5$$

In terms of the material/notation in the lectures this is the morphism

$$\varphi = \varphi_{\mathcal{O}_{\mathbf{P}^2}(2)} : \mathbf{P}^2 \longrightarrow \mathbf{P}(\Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2)))$$

In terms of homogeneous coordinates it is given by

$$[a_0 : a_1 : a_2] \longmapsto [a_0^2 : a_0a_1 : a_0a_2 : a_1^2 : a_1a_2 : a_2^2]$$

It is a closed immersion (please just use this). Let $I \subset k[T_0, \ldots, T_5]$ be the homogeneous ideal of $\varphi(\mathbf{P}^2)$, i.e., the elements of the homogeneous part I_d are the homogeneous polynomials $F(T_0, \ldots, T_5)$ of degree d which restrict to zero on the closed subscheme $\varphi(\mathbf{P}^2)$. Compute $\dim_k I_d$ as a function of d.

Exercise 63.7. Let k be an algebraically closed field. Let X be a proper scheme of dimension d over k with dualizing module ω_X. You are given the following information:

(1) $\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \times H^{d-i}(X, \mathcal{F}) \to H^d(X, \omega_X) \xrightarrow{t} k$ is nondegenerate for all i and for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$, and
(2) ω_X is finite locally free of some rank r.

Show that $r = 1$. (Hint: see what happens if you take $\mathcal{F}$ a suitable module supported at a closed point.)

64. Commutative Algebra, Final Exam, Fall 2021

These were the questions in the final exam of a course on commutative algebra, in the Fall of 2021 at Columbia University.

Exercise 64.1 (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) a *multiplicative subset* of a ring A,
(2) an *Artinian ring* A,
(3) the *spectrum of a ring* A as a topological space,
(4) a *flat ring map* $A \to B$,
(5) the *height* of a prime ideal $\mathfrak{p}$ in A, and
(6) the functors $Tor_i^A(-,-)$ over a ring A.

Exercise 64.2 (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) Artinian rings,
(2) flatness and prime ideals,
(3) lengths of $A/\mathfrak{m}^n$ for $(A, \mathfrak{m})$ Noetherian local,
(4) the dimension formula for universally catenary Noetherian rings,
(5) completion of a Noetherian local ring, and
(6) Matlis duality for Artinian local rings.

Exercise 64.3 (Units)**.** What is the structure of the group of units of $\mathbf{Z}[x, 1/x]$ as an abelian group? No explanation necessary.

Exercise 64.4 (Ideals)**.** Let $A = \mathbf{F}_2[x, y]/(x^2, xy, y^2)$ and denote $\overline{x}$ and $\overline{y}$ the images of x and y in A. List the ideals of A. No explanation necessary.

Exercise 64.5 (Tor and Ext)**.** Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Set $\varphi(n) = \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$.

(1) Show that $\text{Tor}_1^A(A/\mathfrak{m}^n, \kappa)$ has dimension $\varphi(n)$ as a κ-vector space.
(2) Show that $\text{Ext}_A^1(A/\mathfrak{m}^n, \kappa)$ has dimension $\varphi(n)$ as a κ-vector space.

Exercise 64.6 (Two vectors)**.** Let $A = \mathbf{Z}[a_1, a_2, a_3, b_1, b_2, b3]$. Set $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $A^{\oplus 3}$. Consider the set

$$Z = \{\mathfrak{p} \in \text{Spec}(A) \mid a, b \text{ map to linearly dependent vectors of } \kappa(\mathfrak{p})^{\oplus 3}\}$$

(1) Prove the Z is a closed subset of $\text{Spec}(A)$.
(2) What is the dimension $\dim(Z)$ of Z?
(3) What would happen to $\dim(Z)$ if we replaced $\mathbf{Z}$ by a field?

Exercise 64.7 (Injectives)**.** Let $(A, \mathfrak{m}, \kappa)$ be an Artinian local ring. Assume A is injective as an A-module. Show that $\text{Hom}_A(\kappa, A)$ has dimension 1 has a κ-vector space.

65. Algebraic Geometry, Final Exam, Spring 2022

These were the questions in the final exam of a course on Algebraic Geometry, in the Spring of 2022 at Columbia University.

Exercise 65.1 (Definitions)**.** Provide brief definitions of the italicized concepts.

(1) a *scheme*,
(2) a *quasi-coherent module* on a scheme X,
(3) a *flat* morphism of schemes $X \to Y$,
(4) a *finite* morphism of schemes $X \to Y$,
(5) a *group scheme* G over a base scheme S,
(6) a *family of varieties* over a base scheme S,
(7) the *degree* of a closed point x on a variety X over the field k,
(8) the usual *logarithmic height* of a point $p = (a_0 : \ldots : a_n)$ in $\mathbf{P}^n(\mathbf{Q})$, and
(9) a C_i *field*.

Exercise 65.2 (Theorems)**.** Precisely but briefly state a nontrivial fact discussed in the lectures related to each item (if there is more than one then just pick one of them).

(1) morphisms from a scheme X to the affine scheme $\mathrm{Spec}(A)$,
(2) cohomology of a quasi-coherent module $\mathcal{F}$ on an affine scheme X,
(3) the Picard group of $\mathbf{P}^1_k$ where k is a field,
(4) the dimensions of fibres of a flat proper morphism $X \to S$ for S Noetherian,
(5) $\mathbf{G}_m$-equivariant modules on a scheme S, and
(6) Bezout's theorem on intersections (restrict to a special case if you like).

Exercise 65.3 (Cubic hypersurfaces)**.** Let $F \in \mathbf{C}[T_0, \ldots, T_n]$ be homogeneous of degree 3. Given 3 vectors $x, y, z \in \mathbf{C}^{n+1}$ consider the condition

$$(*) \quad F(\lambda x + \mu y + \nu z) = 0 \text{ in } \mathbf{C}[\lambda, \mu, \nu]$$

(1) What is the dimension of the space of all choices of x, y, z?
(2) How many equations on the coordinates of x, y, and z is condition (*)?
(3) What is the expected dimension of the space of all triples x, y, z such that (*) is true?
(4) What is the dimension of the space of all triples such that x, y, z are linearly dependent?
(5) Conclude that on a hypersurface of degree 3 in $\mathbf{P}^n$ we expect to find a linear subspace of dimension 2 provided $n \geq a$ where it is up to you to find a.

Exercise 65.4 (Heights)**.** Let K be a field. Let $h_n : \mathbf{P}^n(K) \to \mathbf{R}$, $n \geq 0$ be a collection of functions satisfying the 2 axioms we discussed in the lectures. Let X be a projective variety over K. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and recall that we have constructed in the lectures an associated height function $h_{\mathcal{L}} : X(K) \to \mathbf{R}$. Let $\alpha : X \to X$ be an automorphism of X over K.

(1) Prove that $P \mapsto h_{\mathcal{L}}(\alpha(P))$ differs from the function $h_{\alpha^*\mathcal{L}}$ by a bounded amount. (Hint: recall that if there is a morphism $\varphi : X \to \mathbf{P}^n$ with $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbf{P}^n}(1)$, then by construction $h_{\mathcal{L}}(P) = h_n(\varphi(P))$ and play around with that. In general write $\mathcal{L}$ as a difference of two of these.)
(2) Assume that $h_{\mathcal{L}}(P) - h_{\mathcal{L}}(\alpha(P))$ is unbounded on $X(K)$. Show that $h_{\mathcal{N}}$ with $\mathcal{N} = \mathcal{L} \otimes \alpha^*\mathcal{L}^{\otimes -1}$ is unbounded on $X(K)$.
(3) Assume X is an elliptic curve and that $\mathcal{L}$ is a symmetric ample invertible module on X such that $h_{\mathcal{L}}$ is unbounded on $X(K)$. Show that there exists an invertible module $\mathcal{N}$ of degree 0 such that $h_{\mathcal{N}}$ is unbounded. (Hints: Recall that X is an abelian variety of dimension 1. Thus $h_{\mathcal{L}}$ is quadratic up to a constant by results in the lectures. Choose a suitable point $P_0 \in X(K)$. Let $\alpha : X \to X$ be translation by P_0. Consider $P \mapsto h_{\mathcal{L}}(P) - h_{\mathcal{L}}(P + P_0)$. Apply the results you proved above.)

Exercise 65.5 (Monomorphisms)**.** Let $f : X \to Y$ be a monomorphism in the category of schemes: for any pair of morphisms $a, b : T \to X$ of schemes if $f \circ a = f \circ b$, then $a = b$. Show that f is injective on points. Does you argument say anything else?

Exercise 65.6 (Fixed points)**.** Let k be an algebraically closed field.

(1) If $G = \mathbf{G}_{m,k}$ show that if G acts on a projective variety X over k, then the action has a fixed point, i.e., prove there exists a point $x \in X(k)$ such that $a(g, x) = x$ for all $g \in G(k)$.
(2) Same with $G = (\mathbf{G}_{m,k})^n$ equal to the product of $n \geq 1$ copies of the multiplicative group.

(3) Give an example of an action of a connected group scheme G on a smooth projective variety X which does not have a fixed point.

66. Other chapters

References

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, 1977.