MORE ON GROUPOIDS IN SPACES

Contents

1. Introduction

This chapter is devoted to advanced topics on groupoids in algebraic spaces. Even though the results are stated in terms of groupoids in algebraic spaces, the reader should keep in mind the 2-cartesian diagram

$$(1.0.1) \qquad \begin{array}{ccc} R & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & [U/R] \end{array}$$

where $[U/R]$ is the quotient stack, see Groupoids in Spaces, Remark 20.4. Many of the results are motivated by thinking about this diagram. See for example the beautiful paper [KM97] by Keel and Mori.

2. Notation

We continue to abide by the conventions and notation introduced in Groupoids in Spaces, Section 3.

3. Useful diagrams

We briefly restate the results of Groupoids in Spaces, Lemmas 11.4 and 11.5 for easy reference in this chapter. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. In the commutative diagram

$$(3.0.1)$$

$$
\begin{array}{ccc}
 & U & \\
 & {}^{t}\nearrow \quad \nwarrow^{t} & \\
R \xleftarrow{\mathrm{pr}_0} R \times_{s,U,t} R \xrightarrow{c} R & & \\
{}^{s}\downarrow \qquad \downarrow^{\mathrm{pr}_1} \qquad \downarrow^{s} & & \\
U \xleftarrow{t} R \xrightarrow{s} U & &
\end{array}
$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

The diagram

$$(3.0.2)$$

$$
\begin{array}{ccc}
R \times_{t,U,t} R \underset{\mathrm{pr}_0}{\overset{\mathrm{pr}_1}{\rightrightarrows}} R \xrightarrow{t} U & & \\
{}^{\mathrm{pr}_0 \times c \circ (i,1)}\downarrow \qquad \downarrow^{\mathrm{id}_R} \qquad \downarrow^{\mathrm{id}_U} & & \\
R \times_{s,U,t} R \underset{\mathrm{pr}_0}{\overset{c}{\rightrightarrows}} R \xrightarrow{t} U & & \\
{}^{\mathrm{pr}_1}\downarrow \qquad \downarrow^{s} & & \\
R \underset{t}{\overset{s}{\rightrightarrows}} U & &
\end{array}
$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

4. Local structure

Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over S. Let $\overline{u}$ be a geometric point of U. In this section we explain what kind of structure we obtain on the local rings (Properties of Spaces, Definition 22.2)

$$A = \mathcal{O}_{U,\overline{u}} \quad \text{and} \quad B = \mathcal{O}_{R,e(\overline{u})}$$

The convention we will use is to denote the local ring homomorphisms induced by the morphisms s, t, c, e, i by the corresponding letters. In particular we have a commutative diagram

$$
\begin{array}{ccc}
A & & \\
{}^{t}\searrow \quad {}^{1}\searrow & & \\
 & B \xrightarrow{e} A & \\
{}^{s}\nearrow \quad {}^{1}\nearrow & & \\
A & &
\end{array}
$$

of local rings. Thus if $I \subset B$ denotes the kernel of $e : B \to A$, then $B = s(A) \oplus I = t(A) \oplus I$. Let us denote

$$C = \mathcal{O}_{R \times_{s,U,t} R, (e,e)(\overline{u})}$$

Then we have

$$C = (B \otimes_{s,A,t} B)^h_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B}$$

because the localization $(B \otimes_{s,A,t} B)_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B}$ has separably closed residue field. Let $J \subset C$ be the ideal of C generated by $I \otimes B + B \otimes I$. Then J is also the kernel of the local ring homomorphism

$$(e,e) : C \longrightarrow A$$

The composition law $c : R \times_{s,U,t} R \to R$ corresponds to a ring map

$$c : B \longrightarrow C$$

sending I into J.

Lemma 4.1. *The map $I/I^2 \to J/J^2$ induced by c is the composition*

$$I/I^2 \xrightarrow{(1,1)} I/I^2 \oplus I/I^2 \to J/J^2$$

where the second arrow comes from the equality $J = (I \otimes B + B \otimes I)C$. The map $i : B \to B$ induces the map $-1 : I/I^2 \to I/I^2$.

Proof. To describe a local homomorphism from C to another henselian local ring it is enough to say what happens to elements of the form $b_1 \otimes b_2$ by Algebra, Lemma 155.6 for example. Keeping this in mind we have the two canonical maps

$$e_2 : C \to B, \ b_1 \otimes b_2 \mapsto b_1 s(e(b_2)), \quad e_1 : C \to B, \ b_1 \otimes b_2 \mapsto t(e(b_1))b_2$$

corresponding to the embeddings $R \to R \times_{s,U,t} R$ given by $r \mapsto (r, e(s(r)))$ and $r \mapsto (e(t(r)), r)$. These maps define maps $J/J^2 \to I/I^2$ which jointly give an inverse to the map $I/I^2 \oplus I/I^2 \to J/J^2$ of the lemma. Thus to prove statement we only have to show that $e_1 \circ c : B \to B$ and $e_2 \circ c : B \to B$ are the identity maps. This follows from the fact that both compositions $R \to R \times_{s,U,t} R \to R$ are identities.

The statement on i follows from the statement on c and the fact that $c \circ (1,i) = e \circ t$. Some details omitted. $\square$

5. Groupoid of sections

Suppose we have a groupoid $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c, e, i)$. Then we can construct a monoid Γ whose elements are maps $\delta : \mathrm{Ob} \to \mathrm{Arrows}$ with $s \circ \delta = \mathrm{id}_{\mathrm{Ob}}$ and composition given by

$$\delta_1 \circ \delta_2 = c(\delta_1 \circ t \circ \delta_2, \delta_2)$$

In other words, an element of Γ is a rule δ which prescribes an arrow emanating from every object and composition is the natural thing. For example
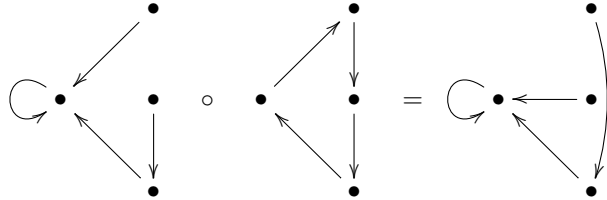
with obvious notation

The same procedure can be applied to a groupoid in algebraic spaces (U, R, s, t, c, e, i) over a scheme S. Namely, as elements of Γ we take the set

$$\Gamma = \{\delta : U \to R \mid s \circ \delta = \mathrm{id}_U\}$$

and composition $\circ : \Gamma \times \Gamma \to \Gamma$ is given by the rule above

(5.0.1) $$\delta_1 \circ \delta_2 = c(\delta_1 \circ t \circ \delta_2, \delta_2)$$

The identity is given by $e \in \Gamma$. The groupoid Γ is not a group in general because there may be elements $\delta \in \Gamma$ which do not have an inverse. Namely, it is clear that $\delta \in \Gamma$ will have an inverse if and only if $t \circ \delta$ is an automorphism of U and in this case $\delta^{-1} = i \circ \delta \circ (t \circ \delta)^{-1}$.

For later use we discuss what happens with the subgroupoid Γ_0 of Γ of sections which are infinitesimally close to the identity e. More precisely, suppose given an R-invariant closed subspace $U_0 \subset U$ such that U is a first order thickening of U_0. Denote $R_0 = s^{-1}(U_0) = t^{-1}(U_0)$ and let $(U_0, R_0, s_0, t_0, c_0, e_0, i_0)$ be the corresponding groupoid in algebraic spaces. Set

$$\Gamma_0 = \{\delta \in \Gamma \mid \delta|_{U_0} = e_0\}$$

If s and t are flat, then every element in Γ_0 is invertible. This follows because $t \circ \delta$ will be a morphism $U \to U$ inducing the identity on $\mathcal{O}_{U_0}$ and on $\mathcal{C}_{U_0/U}$ (Lemma 5.1) and we conclude because we have a short exact sequence $0 \to \mathcal{C}_{U_0/U} \to \mathcal{O}_U \to \mathcal{O}_{U_0} \to 0$.

Lemma 5.1. *In the situation discussed in this section, let $\delta \in \Gamma_0$ and $f = t \circ \delta :$ $U \to U$. If s, t are flat, then the canonical map $\mathcal{C}_{U_0/U} \to \mathcal{C}_{U_0/U}$ induced by f (More on Morphisms of Spaces, Lemma 5.3) is the identity map.*

Proof. To see this we extend the bottom of the diagram (3.0.2) as follows

$$
\begin{array}{ccccccc}
Y & \longrightarrow & R \times_{s,U,t} R & \overset{c}{\underset{\mathrm{pr}_0}{\rightrightarrows}} & R & \overset{t}{\longrightarrow} & U \\
\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle s} & & \\
U & \underset{\delta}{\longrightarrow} & R & \overset{s}{\underset{t}{\rightrightarrows}} & U & &
\end{array}
$$

where the left square is cartesian and this is our definition of Y; we will not need to know more about Y. There is a similar diagram with similar properties obtained by base change to U_0 everywhere. We are trying to show that $\mathrm{id}_U = s \circ \delta$ and $f = t \circ \delta$ induce the same maps on conormal sheaves. Since s is flat and surjective, it suffices to prove the same thing for the two compositions $a, b : Y \to R$ along the top row. Observe that $a_0 = b_0$ and that one of a and b is an isomorphism as we know that $s \circ \delta$ is an isomorphism. Therefore the two morphisms $a, b : Y \to R$ are morphisms between algebraic spaces flat over U (via the morphism $t : R \to U$ and the morphism $t \circ a = t \circ b : Y \to U$). This implies what we want. Namely, by the compatibility with compositions in More on Morphisms of Spaces, Lemma 5.4 we conclude that both maps $a_0^* \mathcal{C}_{R_0/R} \to \mathcal{C}_{Y_0/Y}$ fit into a commutative diagram

$$
\begin{array}{ccc}
a_0^* \mathcal{C}_{R_0/R} & \longrightarrow & \mathcal{C}_{Y_0/Y} \\
\uparrow & & \uparrow \\
a_0^* t_0^* \mathcal{C}_{U_0/U} & =\!=\!=\!= & (t_0 \circ a_0)^* \mathcal{C}_{U_0/U}
\end{array}
$$

whose vertical arrows are isomorphisms by More on Morphisms of Spaces, Lemma 18.1. Thus the lemma holds. $\square$

Let us identify the group Γ_0. Applying the discussion in More on Morphisms of Spaces, Remarks 17.3 and 17.7 to the diagram

$$
\begin{array}{ccc}
(U_0 \subset U) & \xdashrightarrow{(e_0,\delta)} & (R_0 \subset R) \\
& \searrow^{(\mathrm{id}_{U_0},\mathrm{id}_U)} \qquad \swarrow_{(s_0,s)} & \\
& (U_0 \subset U) &
\end{array}
$$

we see that $\delta = \theta \cdot e$ for a unique $\mathcal{O}_{U_0}$-linear map $\theta : e_0^* \Omega_{R_0/U_0} \to \mathcal{C}_{U_0/U}$. Thus we get a bijection

(5.1.1) $$\mathrm{Hom}_{\mathcal{O}_{U_0}}(e_0^* \Omega_{R_0/U_0}, \mathcal{C}_{U_0/U}) \longrightarrow \Gamma_0$$

by applying More on Morphisms of Spaces, Lemma 17.5.

Lemma 5.2. *The bijection (5.1.1) is an isomorphism of groups.*

Proof. Let $\delta_1, \delta_2 \in \Gamma_0$ correspond to θ_1, θ_2 as above and the composition $\delta = \delta_1 \circ \delta_2$ in Γ_0 correspond to θ. We have to show that $\theta = \theta_1 + \theta_2$. Recall (More on Morphisms of Spaces, Lemma 17.2) that $\theta_1, \theta_2, \theta$ correspond to derivations $D_1, D_2, D : e_0^{-1}\mathcal{O}_{R_0} \to \mathcal{C}_{U_0/U}$ given by $D_1 = \theta_1 \circ \mathrm{d}_{R_0/U_0}$ and so on. It suffices to check that $D = D_1 + D_2$.

We may check equality on stalks. Let $\overline{u}$ be a geometric point of U and let us use the local rings A, B, C introduced in Section 4. The morphisms δ_i correspond to ring maps $\delta_i : B \to A$. Let $K \subset A$ be the ideal of square zero such that $A/K = \mathcal{O}_{U_0,\overline{u}}$. In other words, K is the stalk of $\mathcal{C}_{U_0/U}$ at $\overline{u}$. The fact that $\delta_i \in \Gamma_0$ means exactly that $\delta_i(I) \subset K$. The derivation D_i is just the map $\delta_i - e : B \to A$. Since $B = s(A) \oplus I$ we see that D_i is determined by its restriction to I and that this is just given by $\delta_i|_I$. Moreover D_i and hence δ_i annihilates I^2 because $I = \mathrm{Ker}(I)$.

To finish the proof we observe that δ corresponds to the composition

$$B \to C = (B \otimes_{s,A,t} B)^h_{\mathfrak{m}_B \otimes B + B \otimes \mathfrak{m}_B} \to A$$

where the first arrow is c and the second arrow is determined by the rule $b_1 \otimes b_2 \mapsto \delta_2(t(\delta_1(b_1)))\delta_2(b_2)$ as follows from (5.0.1). By Lemma 4.1 we see that an element ζ of I maps to $\zeta \otimes 1 + 1 \otimes \zeta$ plus higher order terms. Hence we conclude that

$$D(\zeta) = (\delta_2 \circ t)(D_1(\zeta)) + D_2(\zeta)$$

However, by Lemma 5.1 the action of $\delta_2 \circ t$ on $K = \mathcal{C}_{U_0/U,\overline{u}}$ is the identity and we win. $\square$

6. Properties of groupoids

This section is the analogue of More on Groupoids, Section 6. The reader is strongly encouraged to read that section first.

The following lemma is the analogue of More on Groupoids, Lemma 6.4.

Lemma 6.1. *Let $B \to S$ be as in Section 2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $\tau \in \{fppf, \acute{e}tale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is τ-local on the target (Descent on Spaces, Definition 10.1). Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for*

the τ-topology. Let $W \subset U$ be the maximal open subspace such that $s^{-1}(W) \to W$ has property $\mathcal{P}$. Then W is R-invariant (Groupoids in Spaces, Definition 18.1).

Proof. The existence and properties of the open $W \subset U$ are described in Descent on Spaces, Lemma 10.3. In Diagram (3.0.1) let $W_1 \subset R$ be the maximal open subscheme over which the morphism $\mathrm{pr}_1 : R \times_{s,U,t} R \to R$ has property $\mathcal{P}$. It follows from the aforementioned Descent on Spaces, Lemma 10.3 and the assumption that $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology that $t^{-1}(W) = W_1 = s^{-1}(W)$ as desired. $\square$

Lemma 6.2. *Let $B \to S$ be as in Section 2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be its stabilizer group algebraic space. Let $\tau \in \{fppf, \acute{e}tale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is τ-local on the target. Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subspace such that $G_W \to W$ has property $\mathcal{P}$. Then W is R-invariant (see Groupoids in Spaces, Definition 18.1).*

Proof. The existence and properties of the open $W \subset U$ are described in Descent on Spaces, Lemma 10.3. The morphism

$$G \times_{U,t} R \longrightarrow R \times_{s,U} G, \quad (g, r) \longmapsto (r, r^{-1} \circ g \circ r)$$

is an isomorphism of algebraic spaces over R (where $\circ$ denotes composition in the groupoid). Hence $s^{-1}(W) = t^{-1}(W)$ by the properties of W proved in the aforementioned Descent on Spaces, Lemma 10.3. $\square$

7. Comparing fibres

This section is the analogue of More on Groupoids, Section 7. The reader is strongly encouraged to read that section first.

Lemma 7.1. *Let $B \to S$ be as in Section 2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let K be a field and let $r, r' : \mathrm{Spec}(K) \to R$ be morphisms such that $t \circ r = t \circ r' : \mathrm{Spec}(K) \to U$. Set $u = s \circ r$, $u' = s \circ r'$ and denote $F_u = \mathrm{Spec}(K) \times_{u,U,s} R$ and $F_{u'} = \mathrm{Spec}(K) \times_{u',U,s} R$ the fibre products. Then $F_u \cong F_{u'}$ as algebraic spaces over K.*

Proof. We use the properties and the existence of Diagram (3.0.1). There exists a morphism $\xi : \mathrm{Spec}(K) \to R \times_{s,U,t} R$ with $\mathrm{pr}_0 \circ \xi = r$ and $c \circ \xi = r'$. Let $\tilde{r} = \mathrm{pr}_1 \circ \xi : \mathrm{Spec}(K) \to R$. Then looking at the bottom two squares of Diagram (3.0.1) we see that both F_u and $F_{u'}$ are identified with the algebraic space $\mathrm{Spec}(K) \times_{\tilde{r},R,\mathrm{pr}_1} (R \times_{s,U,t} R)$. $\square$

Actually, in the situation of the lemma the morphisms of pairs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are locally isomorphic in the τ-topology, provided $\{s : R \to U\}$ is a τ-covering. We will insert a precise statement here if needed.

8. Restricting groupoids

In this section we collect a bunch of lemmas on properties of groupoids which are inherited by restrictions. Most of these lemmas can be proved by contemplating

the defining diagram

(8.0.1)

$$
\xymatrix{
R' \ar@<0.5ex>[r] \ar@<-0.5ex>[r] \ar[d]_{t'} & R \times_{s,U} U' \ar[r]^{s'} \ar[d] & U' \ar[d]^{g} \\
U' \times_{U,t} R \ar[r] \ar[d] & R \ar[r]^{s} \ar[d]^{t} & U \\
U' \ar[r]_{g} & U
}
$$

of a restriction. See Groupoids in Spaces, Lemma 17.1.

Lemma 8.1. *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces over B. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g.*

(1) *If s, t are locally of finite type and g is locally of finite type, then s', t' are locally of finite type.*

(2) *If s, t are locally of finite presentation and g is locally of finite presentation, then s', t' are locally of finite presentation.*

(3) *If s, t are flat and g is flat, then s', t' are flat.*

(4) *Add more here.*

Proof. The property of being locally of finite type is stable under composition and arbitrary base change, see Morphisms of Spaces, Lemmas 23.2 and 23.3. Hence (1) is clear from Diagram (8.0.1). For the other cases, see Morphisms of Spaces, Lemmas 28.2, 28.3, 30.3, and 30.4. $\square$

9. Properties of groups over fields and groupoids on fields

The reader is advised to first look at the corresponding sections for groupoid schemes, see Groupoids, Section 7 and More on Groupoids, Section 10.

Situation 9.1. Here S is a scheme, k is a field over S, and (G, m) is a group algebraic space over $\operatorname{Spec}(k)$.

Situation 9.2. Here S is a scheme, B is an algebraic space, and (U, R, s, t, c) is a groupoid in algebraic spaces over B with $U = \operatorname{Spec}(k)$ for some field k.

Note that in Situation 9.1 we obtain a groupoid in algebraic spaces

(9.2.1) $(\operatorname{Spec}(k), G, p, p, m)$

where $p : G \to \operatorname{Spec}(k)$ is the structure morphism of G, see Groupoids in Spaces, Lemma 15.1. This is a situation as in Situation 9.2. We will use this without further mention in the rest of this section.

Lemma 9.3. *In Situation 9.2 the composition morphism $c : R \times_{s,U,t} R \to R$ is flat and universally open. In Situation 9.1 the group law $m : G \times_k G \to G$ is flat and universally open.*

Proof. The composition is isomorphic to the projection map $\operatorname{pr}_1 : R \times_{t,U,t} R \to R$ by Diagram (3.0.2). The projection is flat as a base change of the flat morphism

t and open by Morphisms of Spaces, Lemma 6.6. The second assertion follows immediately from the first because m matches c in (9.2.1). $\square$

Note that the following lemma applies in particular when working with either quasi-separated or locally separated algebraic spaces (Decent Spaces, Lemma 15.2).

Lemma 9.4. *In Situation 9.2 assume R is a decent space. Then R is a separated algebraic space. In Situation 9.1 assume that G is a decent algebraic space. Then G is separated algebraic space.*

Proof. We first prove the second assertion. By Groupoids in Spaces, Lemma 6.1 we have to show that $e : S \to G$ is a closed immersion. This follows from Decent Spaces, Lemma 14.5.

Next, we prove the first assertion. To do this we may replace B by S. By the paragraph above the stabilizer group scheme $G \to U$ is separated. By Groupoids in Spaces, Lemma 29.2 the morphism $j = (t, s) : R \to U \times_S U$ is separated. As U is the spectrum of a field the scheme $U \times_S U$ is affine (by the construction of fibre products in Schemes, Section 17). Hence R is separated, see Morphisms of Spaces, Lemma 4.9. $\square$

Lemma 9.5. *In Situation 9.2. Let k'/k be a field extension, $U' = \mathrm{Spec}(k')$ and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $U' \to U$. In the defining diagram*

all the morphisms are surjective, flat, and universally open. The dotted arrow $R' \to R$ is in addition affine.

Proof. The morphism $U' \to U$ equals $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, hence is affine, surjective and flat. The morphisms $s, t : R \to U$ and the morphism $U' \to U$ are universally open by Morphisms, Lemma 23.4. Since R is not empty and U is the spectrum of a field the morphisms $s, t : R \to U$ are surjective and flat. Then you conclude by using Morphisms of Spaces, Lemmas 5.5, 5.4, 6.4, 20.5, 20.4, 30.4, and 30.3. $\square$

Lemma 9.6. *In Situation 9.2. For any point $r \in |R|$ there exist*

(1) *a field extension k'/k with k' algebraically closed,*
(2) *a point $r' : \mathrm{Spec}(k') \to R'$ where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$*

such that

(1) *the point r' maps to r under the morphism $R' \to R$, and*
(2) *the maps $s' \circ r', t' \circ r' : \mathrm{Spec}(k') \to \mathrm{Spec}(k')$ are automorphisms.*

Proof. Let's represent r by a morphism $r : \mathrm{Spec}(K) \to R$ for some field K. To prove the lemma we have to find an algebraically closed field k' and a commutative diagram

$$
\begin{array}{ccc}
k' & \xleftarrow{\ 1\ } & k' \\
{\scriptstyle \tau}\uparrow & {\scriptstyle \sigma}\nwarrow & \nwarrow{\scriptstyle i} \\
k' & K & \xleftarrow{s} k \\
& {\scriptstyle i}\nwarrow & \uparrow{\scriptstyle t} \\
& & k
\end{array}
$$

where $s, t : k \to K$ are the field maps coming from $s \circ r$ and $t \circ r$. In the proof of More on Groupoids, Lemma 10.5 it is shown how to construct such a diagram. $\square$

Lemma 9.7. *In Situation 9.2. If $r : \mathrm{Spec}(k) \to R$ is a morphism such that $s \circ r, t \circ r$ are automorphisms of $\mathrm{Spec}(k)$, then the map*

$$ R \longrightarrow R, \quad x \longmapsto c(r, x) $$

is an automorphism $R \to R$ which maps e to r.

Proof. Proof is identical to the proof of More on Groupoids, Lemma 10.6. $\square$

Lemma 9.8. *In Situation 9.2 the algebraic space R is geometrically unibranch. In Situation 9.1 the algebraic space G is geometrically unibranch.*

Proof. Let $r \in |R|$. We have to show that R is geometrically unibranch at r. Combining Lemma 9.5 with Descent on Spaces, Lemma 9.1 we see that it suffices to prove this in case k is algebraically closed and r comes from a morphism $r : \mathrm{Spec}(k) \to R$ such that $s \circ r$ and $t \circ r$ are automorphisms of $\mathrm{Spec}(k)$. By Lemma 9.7 we reduce to the case that $r = e$ is the identity of R and k is algebraically closed.

Assume $r = e$ and k is algebraically closed. Let $A = \mathcal{O}_{R,e}$ be the étale local ring of R at e and let $C = \mathcal{O}_{R \times_{s,U,t} R,(e,e)}$ be the étale local ring of $R \times_{s,U,t} R$ at (e,e). By More on Algebra, Lemma 107.4 the minimal prime ideals $\mathfrak{q}$ of C correspond 1-to-1 to pairs of minimal primes $\mathfrak{p}, \mathfrak{p}' \subset A$. On the other hand, the composition law induces a flat ring map

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ c^\sharp\ \ } & C \\
& & \uparrow \\
& & A \otimes_{s^\sharp,k,t^\sharp} A
\end{array}
\qquad
\begin{array}{c}
\mathfrak{q} \\
\mid \\
\mathfrak{p} \otimes A + A \otimes \mathfrak{p}'
\end{array}
$$

Note that $(c^\sharp)^{-1}(\mathfrak{q})$ contains both $\mathfrak{p}$ and $\mathfrak{p}'$ as the diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\ c^\sharp\ } & C \\
\uparrow & & \uparrow \\
A \otimes_{s^\sharp,k} k & \xleftarrow{1 \otimes e^\sharp} & A \otimes_{s^\sharp,k,t^\sharp} A
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\ c^\sharp\ } & C \\
\uparrow & & \uparrow \\
k \otimes_{k,t^\sharp} A & \xleftarrow{e^\sharp \otimes 1} & A \otimes_{s^\sharp,k,t^\sharp} A
\end{array}
$$

commute by (3.0.1). Since $c^\sharp$ is flat (as c is a flat morphism by Lemma 9.3), we see that $(c^\sharp)^{-1}(\mathfrak{q})$ is a minimal prime of A. Hence $\mathfrak{p} = (c^\sharp)^{-1}(\mathfrak{q}) = \mathfrak{p}'$. $\square$

In the following lemma we use dimension of algebraic spaces (at a point) as defined in Properties of Spaces, Section 9. We also use the dimension of the local ring defined in Properties of Spaces, Section 10 and transcendence degree of points, see Morphisms of Spaces, Section 33.

Lemma 9.9. *In Situation 9.2 assume s, t are locally of finite type. For all $r \in |R|$*

    (1) $\dim(R) = \dim_r(R)$,

    (2) *the transcendence degree of r over $\mathrm{Spec}(k)$ via s equals the transcendence degree of r over $\mathrm{Spec}(k)$ via t, and*

    (3) *if the transcendence degree mentioned in (2) is 0, then $\dim(R) = \dim(\mathcal{O}_{R,\overline{r}})$.*

Proof. Let $r \in |R|$. Denote $\mathrm{trdeg}(r/_s k)$ the transcendence degree of r over $\mathrm{Spec}(k)$ via s. Choose an étale morphism $\varphi : V \to R$ where V is a scheme and $v \in V$ mapping to r. Using the definitions mentioned above the lemma we see that

$$\dim_r(R) = \dim_v(V) = \dim(\mathcal{O}_{V,v}) + \mathrm{trdeg}_{s(k)}(\kappa(v)) = \dim(\mathcal{O}_{R,\overline{r}}) + \mathrm{trdeg}(r/_s k)$$

and similarly for t (the second equality by Morphisms, Lemma 28.1). Hence we see that $\mathrm{trdeg}(r/_s k) = \mathrm{trdeg}(r/_t k)$, i.e., (2) holds.

Let k'/k be a field extension. Note that the restriction R' of R to $\mathrm{Spec}(k')$ (see Lemma 9.5) is obtained from R by two base changes by morphisms of fields. Thus Morphisms of Spaces, Lemma 34.3 shows the dimension of R at a point is unchanged by this operation. Hence in order to prove (1) we may assume, by Lemma 9.6, that r is represented by a morphism $r : \mathrm{Spec}(k) \to R$ such that both $s \circ r$ and $t \circ r$ are automorphisms of $\mathrm{Spec}(k)$. In this case there exists an automorphism $R \to R$ which maps r to e (Lemma 9.7). Hence we see that $\dim_r(R) = \dim_e(R)$ for any r. By definition this means that $\dim_r(R) = \dim(R)$.

Part (3) is a formal consequence of the results obtained in the discussion above. $\square$

Lemma 9.10. *In Situation 9.1 assume G locally of finite type. For all $g \in |G|$*

    (1) $\dim(G) = \dim_g(G)$,

    (2) *if the transcendence degree of g over k is 0, then $\dim(G) = \dim(\mathcal{O}_{G,\overline{g}})$.*

Proof. Immediate from Lemma 9.9 via (9.2.1). $\square$

Lemma 9.11. *In Situation 9.2 assume s, t are locally of finite type. Let $G = \mathrm{Spec}(k) \times_{\Delta, \mathrm{Spec}(k) \times_B \mathrm{Spec}(k), t \times s} R$ be the stabilizer group algebraic space. Then we have $\dim(R) = \dim(G)$.*

Proof. Since G and R are equidimensional (see Lemmas 9.9 and 9.10) it suffices to prove that $\dim_e(R) = \dim_e(G)$. Let V be an affine scheme, $v \in V$, and let $\varphi : V \to R$ be an étale morphism of schemes such that $\varphi(v) = e$. Note that V is a Noetherian scheme as $s \circ \varphi$ is locally of finite type as a composition of morphisms locally of finite type and as V is quasi-compact (use Morphisms of Spaces, Lemmas 23.2, 39.8, and 28.5 and Morphisms, Lemma 15.6). Hence V is locally connected (see Properties, Lemma 5.5 and Topology, Lemma 9.6). Thus we may replace V by the connected component containing v (it is still affine as it is an open and closed subscheme of V). Set $T = V_{red}$ equal to the reduction of V. Consider the two morphisms $a, b : T \to \mathrm{Spec}(k)$ given by $a = s \circ \varphi|_T$ and $b = t \circ \varphi|_T$. Note that a, b induce the same field map $k \to \kappa(v)$ because $\varphi(v) = e$! Let $k_a \subset \Gamma(T, \mathcal{O}_T)$ be the integral closure of $a^\sharp(k) \subset \Gamma(T, \mathcal{O}_T)$. Similarly, let $k_b \subset \Gamma(T, \mathcal{O}_T)$ be the integral

closure of $b^\sharp(k) \subset \Gamma(T, \mathcal{O}_T)$. By Varieties, Proposition 31.1 we see that $k_a = k_b$. Thus we obtain the following commutative diagram

$$
\begin{array}{ccc}
k & & \\
 & \searrow^{a} & \\
 & & k_a = k_b \longrightarrow \Gamma(T, \mathcal{O}_T) \rightrightarrows \kappa(v) \\
 & \nearrow^{b} & \\
k & &
\end{array}
$$

As discussed above the long arrows are equal. Since $k_a = k_b \to \kappa(v)$ is injective we conclude that the two morphisms a and b agree. Hence $T \to R$ factors through G. It follows that $R_{red} = G_{red}$ in an open neighbourhood of e which certainly implies that $\dim_e(R) = \dim_e(G)$. $\qquad\square$

10. Group algebraic spaces over fields

There exists a nonseparated group algebraic space over a field, namely $\mathbf{G}_a/\mathbf{Z}$ over a field of characteristic zero, see Examples, Section 49. In fact any group scheme over a field is separated (Lemma 9.4) hence every nonseparated group algebraic space over a field is nonrepresentable. On the other hand, a group algebraic space over a field is separated as soon as it is decent, see Lemma 9.4. In this section we will show that a separated group algebraic space over a field is representable, i.e., a scheme.

Lemma 10.1. *Let k be a field with algebraic closure $\overline{k}$. Let G be a group algebraic space over k which is separated[1]. Then $G_{\overline{k}}$ is a scheme.*

Proof. By Spaces over Fields, Lemma 10.2 it suffices to show that G_K is a scheme for some field extension K/k. Denote $G'_K \subset G_K$ the schematic locus of G_K as in Properties of Spaces, Lemma 13.1. By Properties of Spaces, Proposition 13.3 we see that $G'_K \subset G_K$ is dense open, in particular not empty. Choose a scheme U and a surjective étale morphism $U \to G$. By Varieties, Lemma 14.2 if K is an algebraically closed field of large enough transcendence degree, then U_K is a Jacobson scheme and every closed point of U_K is K-rational. Hence G'_K has a K-rational point and it suffices to show that every K-rational point of G_K is in G'_K. If $g \in G_K(K)$ is a K-rational point and $g' \in G'_K(K)$ a K-rational point in the schematic locus, then we see that g is in the image of G'_K under the automorphism

$$
G_K \longrightarrow G_K, \quad h \longmapsto g(g')^{-1}h
$$

of G_K. Since automorphisms of G_K as an algebraic space preserve G'_K, we conclude that $g \in G'_K$ as desired. $\qquad\square$

Lemma 10.2. *Let k be a field. Let G be a group algebraic space over k. If G is separated and locally of finite type over k, then G is a scheme.*

Proof. This follows from Lemma 10.1, Groupoids, Lemma 8.6, and Spaces over Fields, Lemma 10.7. $\qquad\square$

Proposition 10.3. *Let k be a field. Let G be a group algebraic space over k. If G is separated, then G is a scheme.*

---

[1] It is enough to assume G is decent, e.g., locally separated or quasi-separated by Lemma 9.4.

Proof. This lemma generalizes Lemma 10.2 (which covers all cases one cares about in practice). The proof is very similar to the proof of Spaces over Fields, Lemma 10.7 used in the proof of Lemma 10.2 and we encourage the reader to read that proof first.

By Lemma 10.1 the base change $G_{\overline{k}}$ is a scheme. Let K/k be a purely transcendental extension of very large transcendence degree. By Spaces over Fields, Lemma 10.5 it suffices to show that G_K is a scheme. Let K^{perf} be the perfect closure of K. By Spaces over Fields, Lemma 10.1 it suffices to show that $G_{K^{perf}}$ is a scheme. Let $K \subset K^{perf} \subset \overline{K}$ be the algebraic closure of K. We may choose an embedding $\overline{k} \to \overline{K}$ over k, so that $G_{\overline{K}}$ is the base change of the scheme $G_{\overline{k}}$ by $\overline{k} \to \overline{K}$. By Varieties, Lemma 14.2 we see that $G_{\overline{K}}$ is a Jacobson scheme all of whose closed points have residue field $\overline{K}$.

Since $G_{\overline{K}} \to G_{K^{perf}}$ is surjective, it suffices to show that the image $g \in |G_{K^{perf}}|$ of an arbitrary closed point of $G_{\overline{K}}$ is in the schematic locus of G_K. In particular, we may represent g by a morphism $g : \mathrm{Spec}(L) \to G_{K^{perf}}$ where L/K^{perf} is separable algebraic (for example we can take $L = \overline{K}$). Thus the scheme

$$
\begin{aligned}
T &= \mathrm{Spec}(L) \times_{G_{K^{perf}}} G_{\overline{K}} \\
&= \mathrm{Spec}(L) \times_{\mathrm{Spec}(K^{perf})} \mathrm{Spec}(\overline{K}) \\
&= \mathrm{Spec}(L \otimes_{K^{perf}} \overline{K})
\end{aligned}
$$

is the spectrum of a $\overline{K}$-algebra which is a filtered colimit of algebras which are finite products of copies of $\overline{K}$. Thus by Groupoids, Lemma 7.13 we can find an affine open $W \subset G_{\overline{K}}$ containing the image of $g_{\overline{K}} : T \to G_{\overline{K}}$.

Choose a quasi-compact open $V \subset G_{K^{perf}}$ containing the image of W. By Spaces over Fields, Lemma 10.2 we see that $V_{K'}$ is a scheme for some finite extension K'/K^{perf}. After enlarging K' we may assume that there exists an affine open $U' \subset V_{K'} \subset G_{K'}$ whose base change to $\overline{K}$ recovers W (use that $V_{\overline{K}}$ is the limit of the schemes $V_{K''}$ for $K' \subset K'' \subset \overline{K}$ finite and use Limits, Lemmas 4.11 and 4.13). We may assume that K'/K^{perf} is a Galois extension (take the normal closure Fields, Lemma 16.3 and use that K^{perf} is perfect). Set $H = \mathrm{Gal}(K'/K^{perf})$. By construction the H-invariant closed subscheme $\mathrm{Spec}(L) \times_{G_{K^{perf}}} G_{K'}$ is contained in U'. By Spaces over Fields, Lemmas 10.3 and 10.4 we conclude. $\square$

11. No rational curves on groups

In this section we prove that there are no nonconstant morphisms from $\mathbf{P}^1$ to a group algebraic space locally of finite type over a field.

Lemma 11.1. *Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ and $g : X \to Z$ be morphisms of algebraic spaces over B. Assume*

(1) *$Y \to B$ is separated,*
(2) *g is surjective, flat, and locally of finite presentation,*
(3) *there is a scheme theoretically dense open $V \subset Z$ such that $f|_{g^{-1}(V)} : g^{-1}(V) \to Y$ factors through V.*

Then f factors through g.

Proof. Set $R = X \times_Z X$. By (2) we see that $Z = X/R$ as sheaves. Also (2) implies that the inverse image of V in R is scheme theoretically dense in R (Morphisms of Spaces, Lemma 30.11). The we see that the two compositions $R \to X \to Y$ are equal by Morphisms of Spaces, Lemma 17.8. The lemma follows. $\square$

Lemma 11.2. *Let k be a field. Let $n \geq 1$ and let $(\mathbf{P}_k^1)^n$ be the n-fold self product over $\mathrm{Spec}(k)$. Let $f : (\mathbf{P}_k^1)^n \to Z$ be a morphism of algebraic spaces over k. If Z is separated of finite type over k, then f factors as*

$$(\mathbf{P}_k^1)^n \xrightarrow{projection} (\mathbf{P}_k^1)^m \xrightarrow{finite} Z.$$

Proof. We may assume k is algebraically closed (details omitted); we only do this so we may argue using rational points, but the reader can work around this if she/he so desires. In the proof products are over k. The automorphism group algebraic space of $(\mathbf{P}_k^1)^n$ contains $G = (\mathrm{GL}_{2,k})^n$. If $C \subset (\mathbf{P}_k^1)^n$ is a closed subvariety (in particular irreducible over k) which is mapped to a point, then we can apply More on Morphisms of Spaces, Lemma 35.3 to the morphism

$$G \times C \to G \times Z, \quad (g, c) \mapsto (g, f(g \cdot c))$$

over G. Hence $g(C)$ is mapped to a point for $g \in G(k)$ lying in a Zariski open $U \subset G$. Suppose $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are k-valued points of $(\mathbf{P}_k^1)^n$. Let $I \subset \{1, \ldots, n\}$ be the set of indices i such that $x_i = y_i$. Then

$$\{g(x) \mid g(y) = y, \ g \in U(k)\}$$

is Zariski dense in the fibre of the projection $\pi_I : (\mathbf{P}_k^1)^n \to \prod_{i \in I} \mathbf{P}_k^1$ (exercise). Hence if $x, y \in C(k)$ are distinct, we conclude that f maps the whole fibre of π_I containing x, y to a single point. Moreover, the $U(k)$-orbit of C meets a Zariski open set of fibres of π_I. By Lemma 11.1 the morphism f factors through π_I. After repeating this process finitely many times we reach the stage where all fibres of f over k points are finite. In this case f is finite by More on Morphisms of Spaces, Lemma 35.2 and the fact that k points are dense in Z (Spaces over Fields, Lemma 16.2). $\square$

Lemma 11.3. *Let k be a field. Let G be a separated group algebraic space locally of finite type over k. There does not exist a nonconstant morphism $f : \mathbf{P}_k^1 \to G$ over $\mathrm{Spec}(k)$.*

Proof. Assume f is nonconstant. Consider the morphisms

$$\mathbf{P}_k^1 \times_{\mathrm{Spec}(k)} \ldots \times_{\mathrm{Spec}(k)} \mathbf{P}_k^1 \longrightarrow G, \quad (t_1, \ldots, t_n) \longmapsto f(g_1) \ldots f(g_n)$$

where on the right hand side we use multiplication in the group. By Lemma 11.2 and the assumption that f is nonconstant this morphism is finite onto its image. Hence $\dim(G) \geq n$ for all n, which is impossible by Lemma 9.10 and the fact that G is locally of finite type over k. $\square$

12. The finite part of a morphism

Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. For an algebraic space or a scheme T over S consider pairs (a, Z) where

(12.0.1)
$$\begin{aligned}
&a : T \to Y \text{ is a morphism over } S, \\
&Z \subset T \times_Y X \text{ is an open subspace} \\
&\text{such that } \mathrm{pr}_0|_Z : Z \to T \text{ is finite.}
\end{aligned}$$

Suppose $h : T' \to T$ is a morphism of algebraic spaces over S and (a, Z) is a pair as in (12.0.1) over T. Set $a' = a \circ h$ and $Z' = (h \times \mathrm{id}_X)^{-1}(Z) = T' \times_T Z$. Then (a', Z') is a pair as in (12.0.1) over T'. This follows as finite morphisms are preserved under base change, see Morphisms of Spaces, Lemma 45.5. Thus we obtain a functor

$$(12.0.2) \qquad \begin{array}{rccc} (X/Y)_{fin} : & (Sch/S)^{opp} & \longrightarrow & Sets \\ & T & \longmapsto & \{(a, Z) \text{ as above}\} \end{array}$$

For applications we are mainly interested in this functor $(X/Y)_{fin}$ when f is separated and locally of finite type. To get an idea of what this is all about, take a look at Remark 12.6.

Lemma 12.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then we have*

  (1) *The presheaf $(X/Y)_{fin}$ satisfies the sheaf condition for the fppf topology.*
  (2) *If T is an algebraic space over S, then there is a canonical bijection*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, (X/Y)_{fin}) = \{(a, Z) \text{ satisfying } 12.0.1\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering (by algebraic spaces). Let $s_i = (a_i, Z_i)$ be pairs over T_i satisfying 12.0.1 such that we have $s_i|_{T_i \times_T T_j} = s_j|_{T_i \times_T T_j}$. First, this implies in particular that a_i and a_j define the same morphism $T_i \times_T T_j \to Y$. By Descent on Spaces, Lemma 7.2 we deduce that there exists a unique morphism $a : T \to Y$ such that a_i equals the composition $T_i \to T \to Y$. Second, this implies that $Z_i \subset T_i \times_Y X$ are open subspaces whose inverse images in $(T_i \times_T T_j) \times_Y X$ are equal. Since $\{T_i \times_Y X \to T \times_Y X\}$ is an fppf covering we deduce that there exists a unique open subspace $Z \subset T \times_Y X$ which restricts back to Z_i over T_i, see Descent on Spaces, Lemma 7.1. We claim that the projection $Z \to T$ is finite. This follows as being finite is local for the fpqc topology, see Descent on Spaces, Lemma 11.23.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say "pair" we mean a pair satisfying conditions 12.0.1.

Let $v : T \to (X/Y)_{fin}$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in (X/Y)_{fin}(U)$ corresponds to a pair (a_U, Z_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, Z_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, Z) over T.

Conversely, let (a, Z) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, Z)|_U$ gives rise to a transformation of functors $v : h_U \to (X/Y)_{fin}$ by the Yoneda lemma (Categories, Lemma 3.5). As the two pullbacks $s^*(a, Z)|_U$ and $t^*(a, Z)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 9.1 and since $(X/Y)_{fin}$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to (X/Y)_{fin}$.

We omit the verification that the two constructions above are mutually inverse. $\square$

Lemma 12.2. *Let S be a scheme. Consider a commutative diagram*

of algebraic spaces over S. If j is an open immersion, then there is a canonical injective map of sheaves $j : (X'/Y)_{fin} \to (X/Y)_{fin}$.

Proof. If (a, Z) is a pair over T for X'/Y, then $(a, j(Z))$ is a pair over T for X/Y. $\qquad \square$

Lemma 12.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $X' \subset X$ be the maximal open subspace over which f is locally quasi-finite, see Morphisms of Spaces, Lemma 34.7. Then $(X/Y)_{fin} = (X'/Y)_{fin}$.*

Proof. Lemma 12.2 gives us an injective map $(X'/Y)_{fin} \to (X/Y)_{fin}$. Morphisms of Spaces, Lemma 34.7 assures us that formation of X' commutes with base change. Hence everything comes down to proving that if $Z \subset X$ is an open subspace such that $f|_Z : Z \to Y$ is finite, then $Z \subset X'$. This is true because a finite morphism is locally quasi-finite, see Morphisms of Spaces, Lemma 45.8. $\qquad \square$

Lemma 12.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let T be an algebraic space over S, and let (a, Z) be a pair as in 12.0.1. If f is separated, then Z is closed in $T \times_Y X$.*

Proof. A finite morphism of algebraic spaces is universally closed by Morphisms of Spaces, Lemma 45.9. Since f is separated so is the morphism $T \times_Y X \to T$, see Morphisms of Spaces, Lemma 4.4. Thus the closedness of Z follows from Morphisms of Spaces, Lemma 40.6. $\qquad \square$

Remark 12.5. Let $f : X \to Y$ be a separated morphism of algebraic spaces. The sheaf $(X/Y)_{fin}$ comes with a natural map $(X/Y)_{fin} \to Y$ by mapping the pair $(a, Z) \in (X/Y)_{fin}(T)$ to the element $a \in Y(T)$. We can use Lemma 12.4 to define operations

$$\star_i : (X/Y)_{fin} \times_Y (X/Y)_{fin} \longrightarrow (X/Y)_{fin}$$

by the rules

$$\star_1 : ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_1 \cup Z_2)$$
$$\star_2 : ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_1 \cap Z_2)$$
$$\star_3 : ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_1 \setminus Z_2)$$
$$\star_4 : ((a, Z_1), (a, Z_2)) \longmapsto (a, Z_2 \setminus Z_1).$$

The reason this works is that $Z_1 \cap Z_2$ is both open and closed inside Z_1 and Z_2 (which also implies that $Z_1 \cup Z_2$ is the disjoint union of the other three pieces). Thus we can think of $(X/Y)_{fin}$ as an $\mathbf{F}_2$-algebra (without unit) over Y with multiplication given by $ss' = \star_2(s, s')$, and addition given by

$$s + s' = \star_1(\star_3(s, s'), \star_4(s, s'))$$

which boils down to taking the symmetric difference. Note that in this sheaf of algebras $0 = (1_Y, \emptyset)$ and that indeed $s + s = 0$ for any local section s. If $f : X \to Y$

is finite, then this algebra has a unit namely $1 = (1_Y, X)$ and $\star_3(s, s') = s(1 + s')$, and $\star_4(s, s') = (1 + s)s'$.

Remark 12.6. Let $f : X \to Y$ be a separated, locally quasi-finite morphism of schemes. In this case the sheaf $(X/Y)_{fin}$ is closely related to the sheaf $f_! \mathbf{F}_2$ (insert future reference here) on $Y_{étale}$. Namely, if $V \to Y$ is étale, and $s \in \Gamma(V, f_! \mathbf{F}_2)$, then $s \in \Gamma(V \times_Y X, \mathbf{F}_2)$ is a section with proper support $Z = \mathrm{Supp}(s)$ over V. Since f is also locally quasi-finite we see that the projection $Z \to V$ is actually finite. Since the support of a section of a constant abelian sheaf is open we see that the pair $(V \to Y, \mathrm{Supp}(s))$ satisfies 12.0.1. In fact, $f_! \mathbf{F}_2 \cong (X/Y)_{fin}|_{Y_{étale}}$ in this case which also explains the $\mathbf{F}_2$-algebra structure introduced in Remark 12.5.

Lemma 12.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The diagonal of $(X/Y)_{fin} \to Y$*

$$(X/Y)_{fin} \longrightarrow (X/Y)_{fin} \times_Y (X/Y)_{fin}$$

is representable (by schemes) and an open immersion and the "absolute" diagonal

$$(X/Y)_{fin} \longrightarrow (X/Y)_{fin} \times (X/Y)_{fin}$$

is representable (by schemes).

Proof. The second statement follows from the first as the absolute diagonal is the composition of the relative diagonal and a base change of the diagonal of Y (which is representable by schemes), see Spaces, Section 3. To prove the first assertion we have to show the following: Given a scheme T and two pairs (a, Z_1) and (a, Z_2) over T with identical first component satisfying 12.0.1 there is an open subscheme $V \subset T$ with the following property: For any morphism of schemes $h : T' \to T$ we have

$$h(T') \subset V \Leftrightarrow \left(T' \times_T Z_1 = T' \times_T Z_2 \text{ as subspaces of } T' \times_Y X \right)$$

Let us construct V. Note that $Z_1 \cap Z_2$ is open in Z_1 and in Z_2. Since $\mathrm{pr}_0|_{Z_i} : Z_i \to T$ is finite, hence proper (see Morphisms of Spaces, Lemma 45.9) we see that

$$E = \mathrm{pr}_0|_{Z_1} (Z_1 \setminus Z_1 \cap Z_2)) \cup \mathrm{pr}_0|_{Z_2} (Z_2 \setminus Z_1 \cap Z_2))$$

is closed in T. Now it is clear that $V = T \setminus E$ works. $\square$

Lemma 12.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Suppose that U is a scheme, $U \to Y$ is an étale morphism and $Z \subset U \times_Y X$ is an open subspace finite over U. Then the induced morphism $U \to (X/Y)_{fin}$ is étale.*

Proof. This is formal from the description of the diagonal in Lemma 12.7 but we write it out since it is an important step in the development of the theory. We have to check that for any scheme T over S and a morphism $T \to (X/Y)_{fin}$ the projection map

$$T \times_{(X/Y)_{fin}} U \longrightarrow T$$

is étale. Note that

$$T \times_{(X/Y)_{fin}} U = (X/Y)_{fin} \times_{((X/Y)_{fin} \times_Y (X/Y)_{fin})} (T \times_Y U)$$

Applying the result of Lemma 12.7 we see that $T \times_{(X/Y)_{fin}} U$ is represented by an open subscheme of $T \times_Y U$. As the projection $T \times_Y U \to T$ is étale by Morphisms of Spaces, Lemma 39.4 we conclude. $\square$

Lemma 12.9. *Let S be a scheme. Let*

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
$$

be a fibre product square of algebraic spaces over S. Then

$$
\begin{array}{ccc}
(X'/Y')_{fin} & \longrightarrow & (X/Y)_{fin} \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
$$

is a fibre product square of sheaves on $(Sch/S)_{fppf}$.

Proof. It follows immediately from the definitions that the sheaf $(X'/Y')_{fin}$ is equal to the sheaf $Y' \times_Y (X/Y)_{fin}$. $\square$

Lemma 12.10. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated and locally quasi-finite, then there exists a scheme U étale over Y and a surjective étale morphism $U \to (X/Y)_{fin}$ over Y.*

Proof. Note that the assertion makes sense by the result of Lemma 12.7 on the diagonal of $(X/Y)_{fin}$, see Spaces, Lemma 5.10. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. By Lemma 12.9 the morphism $(V \times_Y X/V)_{fin} \to (X/Y)_{fin}$ is a base change of the map $V \to Y$ and hence is surjective and étale, see Spaces, Lemma 5.5. Hence it suffices to prove the lemma for $(V \times_Y X/V)_{fin}$. (Here we implicitly use that the composition of representable, surjective, and étale transformations of functors is again representable, surjective, and étale, see Spaces, Lemmas 3.2 and 5.4, and Morphisms, Lemmas 9.2 and 36.3.) Note that the properties of being separated and locally quasi-finite are preserved under base change, see Morphisms of Spaces, Lemmas 4.4 and 27.4. Hence $V \times_Y X \to V$ is separated and locally quasi-finite as well, and by Morphisms of Spaces, Proposition 50.2 we see that $V \times_Y X$ is a scheme as well. Thus we may assume that $f : X \to Y$ is a separated and locally quasi-finite morphism of schemes.

Pick a point $y \in Y$. Pick $x_1, \ldots, x_n \in X$ points lying over y. Pick an étale neighbourhood $a : (U, u) \to (Y, y)$ and a decomposition

$$
U \times_S X = W \amalg \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j}
$$

as in More on Morphisms, Lemma 41.5. Pick any subset

$$
I \subset \{(i,j) \mid 1 \le i \le n, \ 1 \le j \le m_i\}.
$$

Given these choices we obtain a pair (a, Z) with $Z = \bigcup_{(i,j) \in I} V_{i,j}$ which satisfies conditions 12.0.1. In other words we obtain a morphism $U \to (X/Y)_{fin}$. The construction of this morphism depends on all the things we picked above, so we should really write

$$
U(y, n, x_1, \ldots, x_n, a, I) \longrightarrow (X/Y)_{fin}
$$

This morphism is étale by Lemma 12.8.

Claim: The disjoint union of all of these is surjective onto $(X/Y)_{fin}$. It is clear that if the claim holds, then the lemma is true.

To show surjectivity we have to show the following (see Spaces, Remark 5.2): Given a scheme T over S, a point $t \in T$, and a map $T \to (X/Y)_{fin}$ we can find a datum $(y, n, x_1, \ldots, x_n, a, I)$ as above such that t is in the image of the projection map

$$U(y, n, x_1, \ldots, x_n, a, I) \times_{(X/Y)_{fin}} T \longrightarrow T.$$

To prove this we may clearly replace T by $\mathrm{Spec}(\overline{\kappa(t)})$ and $T \to (X/Y)_{fin}$ by the composition $\mathrm{Spec}(\overline{\kappa(t)}) \to T \to (X/Y)_{fin}$. In other words, we may assume that T is the spectrum of an algebraically closed field.

Let $T = \mathrm{Spec}(k)$ be the spectrum of an algebraically closed field k. The morphism $T \to (X/Y)_{fin}$ is given by a pair $(T \to Y, Z)$ satisfying conditions 12.0.1. Here is a picture:

$$
\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}(k) = T & \longrightarrow & Y
\end{array}
$$

Let $y \in Y$ be the image point of $T \to Y$. Since Z is finite over k it has finitely many points. Thus there exist finitely many points $x_1, \ldots, x_n \in X$ such that the image of Z in X is contained in $\{x_1, \ldots, x_n\}$. Choose $a : (U, u) \to (Y, y)$ adapted to y and $x_1, \ldots, x_n$ as above, which gives the diagram

$$
\begin{array}{ccc}
W \amalg \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j} & \longrightarrow & X \\
\downarrow & & \downarrow \\
U & \longrightarrow & Y.
\end{array}
$$

Since k is algebraically closed and $\kappa(y) \subset \kappa(u)$ is finite separable we may factor the morphism $T = \mathrm{Spec}(k) \to Y$ through the morphism $u = \mathrm{Spec}(\kappa(u)) \to \mathrm{Spec}(\kappa(y)) = y \subset Y$. With this choice we obtain the commutative diagram:

$$
\begin{array}{ccccc}
Z & \longrightarrow & W \amalg \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j} & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(k) & \longrightarrow & U & \longrightarrow & Y
\end{array}
$$

We know that the image of the left upper arrow ends up in $\coprod V_{i,j}$. Recall also that Z is an open subscheme of $\mathrm{Spec}(k) \times_Y X$ by definition of $(X/Y)_{fin}$ and that the right hand square is a fibre product square. Thus we see that

$$Z \subset \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} \mathrm{Spec}(k) \times_U V_{i,j}$$

is an open subscheme. By construction (see More on Morphisms, Lemma 41.5) each $V_{i,j}$ has a unique point $v_{i,j}$ lying over u with purely inseparable residue field extension $\kappa(v_{i,j})/\kappa(u)$. Hence each scheme $\mathrm{Spec}(k) \times_U V_{i,j}$ has exactly one point. Thus we see that

$$Z = \coprod_{(i,j) \in I} \mathrm{Spec}(k) \times_U V_{i,j}$$

for a unique subset $I \subset \{(i,j) \mid 1 \le i \le n, \ 1 \le j \le m_i\}$. Unwinding the definitions this shows that

$$U(y, n, x_1, \ldots, x_n, a, I) \times_{(X/Y)_{fin}} T$$

with I as found above is nonempty as desired. $\qquad\square$

Proposition 12.11. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is separated and locally of finite type. Then $(X/Y)_{fin}$ is an algebraic space. Moreover, the morphism $(X/Y)_{fin} \to Y$ is étale.*

Proof. By Lemma 12.3 we may replace X by the open subscheme which is locally quasi-finite over Y. Hence we may assume that f is separated and locally quasi-finite. We will check the three conditions of Spaces, Definition 6.1. Condition (1) follows from Lemma 12.1. Condition (2) follows from Lemma 12.7. Finally, condition (3) follows from Lemma 12.10. Thus $(X/Y)_{fin}$ is an algebraic space. Moreover, that lemma shows that there exists a commutative diagram

$$
\begin{array}{ccc}
U & \longrightarrow & (X/Y)_{fin} \\
 & \searrow \quad \swarrow & \\
 & Y &
\end{array}
$$

with horizontal arrow surjective and étale and south-east arrow étale. By Properties of Spaces, Lemma 16.3 this implies that the south-west arrow is étale as well. $\square$

Remark 12.12. The condition that f be separated cannot be dropped from Proposition 12.11. An example is to take X the affine line with zero doubled, see Schemes, Example 14.3, $Y = \mathbf{A}^1_k$ the affine line, and $X \to Y$ the obvious map. Recall that over $0 \in Y$ there are two points 0_1 and 0_2 in X. Thus $(X/Y)_{fin}$ has four points over 0, namely $\emptyset, \{0_1\}, \{0_2\}, \{0_1, 0_2\}$. Of these four points only three can be lifted to an open subscheme of $U \times_Y X$ finite over U for $U \to Y$ étale, namely $\emptyset, \{0_1\}, \{0_2\}$. This shows that $(X/Y)_{fin}$ if representable by an algebraic space is not étale over Y. Similar arguments show that $(X/Y)_{fin}$ is really not an algebraic space. Details omitted.

Remark 12.13. Let $Y = \mathbf{A}^1_{\mathbf{R}}$ be the affine line over the real numbers, and let $X = \mathrm{Spec}(\mathbf{C})$ mapping to the $\mathbf{R}$-rational point 0 in Y. In this case the morphism $f : X \to Y$ is finite, but it is not the case that $(X/Y)_{fin}$ is a scheme. Namely, one can show that in this case the algebraic space $(X/Y)_{fin}$ is isomorphic to the algebraic space of Spaces, Example 14.2 associated to the extension $\mathbf{R} \subset \mathbf{C}$. Thus it is really necessary to leave the category of schemes in order to represent the sheaf $(X/Y)_{fin}$, even when f is a finite morphism.

Lemma 12.14. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is separated, flat, and locally of finite presentation. In this case*

1. *$(X/Y)_{fin} \to Y$ is separated, representable, and étale, and*
2. *if Y is a scheme, then $(X/Y)_{fin}$ is (representable by) a scheme.*

Proof. Since f is in particular separated and locally of finite type (see Morphisms of Spaces, Lemma 28.5) we see that $(X/Y)_{fin}$ is an algebraic space by Proposition 12.11. To prove that $(X/Y)_{fin} \to Y$ is separated we have to show the following: Given a scheme T and two pairs (a, Z_1) and (a, Z_2) over T with identical first component satisfying 12.0.1 there is a closed subscheme $V \subset T$ with the following property: For any morphism of schemes $h : T' \to T$ we have

$$
h \text{ factors through } V \Leftrightarrow \left(T' \times_T Z_1 = T' \times_T Z_2 \text{ as subspaces of } T' \times_Y X \right)
$$

In the proof of Lemma 12.7 we have seen that $V = T' \setminus E$ is an open subscheme of T' with closed complement

$$E = \mathrm{pr}_0|_{Z_1} (Z_1 \setminus Z_1 \cap Z_2)) \cup \mathrm{pr}_0|_{Z_2} (Z_2 \setminus Z_1 \cap Z_2)) .$$

Thus everything comes down to showing that E is also open. By Lemma 12.4 we see that Z_1 and Z_2 are closed in $T' \times_Y X$. Hence $Z_1 \setminus Z_1 \cap Z_2$ is open in Z_1. As f is flat and locally of finite presentation, so is $\mathrm{pr}_0|_{Z_1}$. This is true as Z_1 is an open subspace of the base change $T' \times_Y X$, and Morphisms of Spaces, Lemmas 28.3 and Lemmas 30.4. Hence $\mathrm{pr}_0|_{Z_1}$ is open, see Morphisms of Spaces, Lemma 30.6. Thus $\mathrm{pr}_0|_{Z_1} (Z_1 \setminus Z_1 \cap Z_2))$ is open and it follows that E is open as desired.

We have already seen that $(X/Y)_{fin} \to Y$ is étale, see Proposition 12.11. Hence now we know it is locally quasi-finite (see Morphisms of Spaces, Lemma 39.5) and separated, hence representable by Morphisms of Spaces, Lemma 51.1. The final assertion is clear (if you like you can use Morphisms of Spaces, Proposition 50.2). $\qquad\square$

Variant: Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\sigma : Y \to X$ be a section of f. For an algebraic space or a scheme T over S consider pairs (a, Z) where

(12.14.1)
$$
\begin{aligned}
&a : T \to Y \text{ is a morphism over } S, \\
&Z \subset T \times_Y X \text{ is an open subspace} \\
&\text{such that } \mathrm{pr}_0|_Z : Z \to T \text{ is finite and} \\
&(1_T, \sigma \circ a) : T \to T \times_Y X \text{ factors through } Z.
\end{aligned}
$$

We will denote $(X/Y, \sigma)_{fin}$ the subfunctor of $(X/Y)_{fin}$ parametrizing these pairs.

Lemma 12.15. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\sigma : Y \to X$ be a section of f. Consider the transformation of functors*

$$t : (X/Y, \sigma)_{fin} \longrightarrow (X/Y)_{fin}.$$

defined above. Then

(1) *t is representable by open immersions,*
(2) *if f is separated, then t is representable by open and closed immersions,*
(3) *if $(X/Y)_{fin}$ is an algebraic space, then $(X/Y, \sigma)_{fin}$ is an algebraic space and an open subspace of $(X/Y)_{fin}$, and*
(4) *if $(X/Y)_{fin}$ is a scheme, then $(X/Y, \sigma)_{fin}$ is an open subscheme of it.*

Proof. Omitted. Hint: Given a pair (a, Z) over T as in (12.0.1) the inverse image of Z by $(1_T, \sigma \circ a) : T \to T \times_Y X$ is the open subscheme of T we are looking for. $\quad\square$

13. Finite collections of arrows

Let $\mathcal{C}$ be a groupoid, see Categories, Definition 2.5. As discussed in Groupoids, Section 13 this corresponds to a septuple $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c, e, i)$.

Using this data we can make another groupoid $\mathcal{C}_{fin}$ as follows:
(1) An object of $\mathcal{C}_{fin}$ consists of a finite subset $Z \subset \mathrm{Arrows}$ with the following properties:
   (a) $s(Z) = \{u\}$ is a singleton, and
   (b) $e(u) \in Z$.

(2) A morphism of $\mathcal{C}_{fin}$ consists of a pair (Z, z), where Z is an object of $\mathcal{C}_{fin}$ and $z \in Z$.
(3) The source of (Z, z) is Z.
(4) The target of (Z, z) is $t(Z, z) = \{z' \circ z^{-1}; z' \in Z\}$.
(5) Given (Z_1, z_1), (Z_2, z_2) such that $s(Z_1, z_1) = t(Z_2, z_2)$ the composition $(Z_1, z_1) \circ (Z_2, z_2)$ is $(Z_2, z_1 \circ z_2)$.

We omit the verification that this defines a groupoid. Pictorially an object of $\mathcal{C}_{fin}$ can be viewed as a diagram
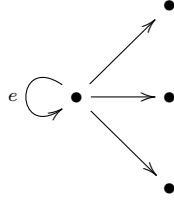
To make a morphism of $\mathcal{C}_{fin}$ you pick one of the arrows and you precompose the other arrows by its inverse. For example if we pick the middle horizontal arrow then the target is the picture
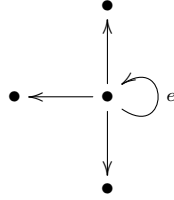
Note that the cardinalities of $s(Z, z)$ and $t(Z, z)$ are equal. So $\mathcal{C}_{fin}$ is really a countable disjoint union of groupoids.

14. The finite part of a groupoid

In this section we are going to use the idea explained in Section 13 to take the finite part of a groupoid in algebraic spaces.

Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. Assumption: The morphisms s, t are separated and locally of finite type. This notation and assumption will we be fixed throughout this section.

Denote R_s the algebraic space R seen as an algebraic space over U via s. Let $U' = (R_s/U, e)_{fin}$. Since s is separated and locally of finite type, by Proposition 12.11 and Lemma 12.15, we see that U' is an algebraic space endowed with an étale morphism $g : U' \to U$. Moreover, by Lemma 12.1 there exists a universal open subspace $Z_{univ} \subset R \times_{s,U,g} U'$ which is finite over U' and such that $(1_{U'}, e \circ g) : U' \to R \times_{s,U,g} U'$ factors through Z_{univ}. Moreover, by Lemma 12.4 the open

subspace Z_{univ} is also closed in $R \times_{s,U',g} U$. Picture so far:

$$
\begin{array}{ccc}
Z_{univ} & & \\
\downarrow & \searrow & \\
R \times_{s,U,g} U' & \longrightarrow & U' \\
\downarrow & & \downarrow g \\
R & \xrightarrow{\ \ s\ \ } & U
\end{array}
$$

Let T be a scheme over B. We see that a T-valued point of Z_{univ} may be viewed as a triple (u, Z, z) where

(1) $u : T \to U$ is a T-valued point of U,
(2) $Z \subset R \times_{s,U,u} T$ is an open and closed subspace finite over T such that $(e \circ u, 1_T)$ factors through it, and
(3) $z : T \to R$ is a T-valued point of R with $s \circ z = u$ and such that $(z, 1_T)$ factors through Z.

Having said this, it is morally clear from the discussion in Section 13 that we can turn (Z_{univ}, U') into a groupoid in algebraic spaces over B. To make sure will define the morphisms s', t', c', e', i' one by one using the functorial point of view. (Please don't read this before reading and understanding the simple construction in Section 13.)

The morphism $s' : Z_{univ} \to U'$ corresponds to the rule

$$s' : (u, Z, z) \mapsto (u, Z).$$

The morphism $t' : Z_{univ} \to U'$ is given by the rule

$$t' : (u, Z, z) \mapsto (t \circ z, c(Z, i \circ z)).$$

The entry $c(Z, i \circ z)$ makes sense as the map $c(-, i \circ z) : R \times_{s,U,u} T \to R \times_{s,U,t\circ z} T$ is an isomorphism with inverse $c(-, z)$. The morphism $e' : U' \to Z_{univ}$ is given by the rule

$$e' : (u, Z) \mapsto (u, Z, (e \circ u, 1_T)).$$

Note that this makes sense by the requirement that $(e \circ u, 1_T)$ factors through Z. The morphism $i' : Z_{univ} \to Z_{univ}$ is given by the rule

$$i' : (u, Z, z) \mapsto (t \circ z, c(Z, i \circ z), i \circ z).$$

Finally, composition is defined by the rule

$$c' : ((u_1, Z_1, z_1), (u_2, Z_2, z_2)) \mapsto (u_2, Z_2, z_1 \circ z_2).$$

We omit the verification that the axioms of a groupoid in algebraic spaces hold for $(U', Z_{univ}, s', t', c', e', i')$.

A final piece of information is that there is a canonical morphism of groupoids

$$(U', Z_{univ}, s', t', c', e', i') \longrightarrow (U, R, s, t, c, e, i)$$

Namely, the morphism $U' \to U$ is the morphism $g : U' \to U$ which is defined by the rule $(u, Z) \mapsto u$. The morphism $Z_{univ} \to R$ is defined by the rule $(u, Z, z) \mapsto z$. This finishes the construction. Let us summarize our findings as follows.

Lemma 14.1. *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. Assume the morphisms s, t are separated and locally of finite type. There exists a canonical morphism*

$$(U', Z_{univ}, s', t', c', e', i') \longrightarrow (U, R, s, t, c, e, i)$$

of groupoids in algebraic spaces over B where

(1) *$g : U' \to U$ is identified with $(R_s/U, e)_{fin} \to U$, and*
(2) *$Z_{univ} \subset R \times_{s,U,g} U'$ is the universal open (and closed) subspace finite over U' which contains the base change of the unit e.*

Proof. See discussion above. □

15. Étale localization of groupoid schemes

In this section we prove results similar to [KM97, Proposition 4.2]. We try to be a bit more general, and we try to avoid using Hilbert schemes by using the finite part of a morphism instead. The goal is to "split" a groupoid in algebraic spaces over a point after étale localization. Here is the definition (very similar to [KM97, Definition 4.1]).

Definition 15.1. Let S be a scheme. Let B be an algebraic space over S Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $u \in |U|$ be a point.

(1) We say R is *strongly split over u* if there exists an open subspace $P \subset R$ such that
    (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
    (b) $s|_P$, $t|_P$ are finite, and
    (c) $\{r \in |R| : s(r) = u, t(r) = u\} \subset |P|$.
    The choice of such a P will be called a *strong splitting of R over u*.
(2) We say R is *split over u* if there exists an open subspace $P \subset R$ such that
    (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
    (b) $s|_P$, $t|_P$ are finite, and
    (c) $\{g \in |G| : g \text{ maps to } u\} \subset |P|$ where $G \to U$ is the stabilizer.
    The choice of such a P will be called a *splitting of R over u*.
(3) We say R is *quasi-split over u* if there exists an open subspace $P \subset R$ such that
    (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
    (b) $s|_P$, $t|_P$ are finite, and
    (c) $e(u) \in |P|^2$.
    The choice of such a P will be called a *quasi-splitting of R over u*.

Note the similarity of the conditions on P to the conditions on pairs in (12.0.1). In particular, if s, t are separated, then P is also closed in R (see Lemma 12.4).

Suppose we start with a groupoid in algebraic spaces (U, R, s, t, c) over B and a point $u \in |U|$. Since the goal is to split the groupoid after étale localization we may as well replace U by an affine scheme (what we mean is that this is harmless for any possible application). Moreover, the additional hypotheses we are going to have to impose will force R to be a scheme at least in a neighbourhood of $\{r \in |R| : s(r) = u, t(r) = u\}$ or $e(u)$. This is why we start with a groupoid scheme as described below. However, our technique of proof leads us outside of the category

---

[2]This condition is implied by (a).

of schemes, which is why we have formulated a splitting for the case of groupoids in algebraic spaces above. On the other hand, we know of no applications but the case where the morphisms s, t are also flat and of finite presentation, in which case we end up back in the category of schemes.

Situation 15.2 (Strong splitting)**.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type,
(3) the set $\{r \in R : s(r) = u, t(r) = u\}$ is finite, and
(4) s is quasi-finite at each point of the set in (3).

Note that assumptions (3) and (4) are implied by the assumption that the fibre $s^{-1}(\{u\})$ is finite, see Morphisms, Lemma 20.7.

Situation 15.3 (Splitting)**.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type,
(3) the set $\{g \in G : g$ maps to $u\}$ is finite where $G \to U$ is the stabilizer, and
(4) s is quasi-finite at each point of the set in (3).

Situation 15.4 (Quasi-splitting)**.** Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type, and
(3) s is quasi-finite at $e(u)$.

For our application to the existence theorems for algebraic spaces the case of quasi-splittings is sufficient. Moreover, the quasi-splitting case will allow us to prove an étale local structure theorem for quasi-DM stacks. The splitting case will be used to prove a version of the Keel-Mori theorem. The strong splitting case applies to give an étale local structure theorem for quasi-DM algebraic stacks with quasi-compact diagonal.

Lemma 15.5 (Existence of strong splitting)**.** *In Situation 15.2 there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : \mathrm{Spec}(\kappa(u)) \to U'$ lying over $u : \mathrm{Spec}(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' is strongly split over u'.*

Proof. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 14.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z_{univ} \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $\mathrm{Spec}(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} \mathrm{Spec}(\kappa(u))$. The set $\{r \in R : s(r) = u, t(r) = u\}$ is finite by assumption

and $F_u \to \mathrm{Spec}(\kappa(u))$ is quasi-finite at each of its elements by assumption. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \amalg Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is $\{r \in R : s(r) = u, t(r) = u\}$. Note that $e(u) \in Z_u$. Hence by the construction of U' in Section 14 (u, Z_u) defines a $\mathrm{Spec}(\kappa(u))$-valued point u' of U'.

We still have to show that the set $\{r' \in |R'| : s'(r') = u', t'(r') = u'\}$ is contained in $|Z_{univ}|$. Pick any point r' in this set and represent it by a morphism $z' : \mathrm{Spec}(k) \to R'$. Denote $z : \mathrm{Spec}(k) \to R$ the composition of z' with the map $R' \to R$. Clearly, z defines an element of the set $\{r \in R : s(r) = u, t(r) = u\}$. Also, the compositions $s \circ z, t \circ z : \mathrm{Spec}(k) \to U$ factor through u, so we may think of $s \circ z, t \circ z$ as a morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(\kappa(u))$. Then $z' = (z, u' \circ t \circ z, u' \circ s \circ u)$ as morphisms into $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Consider the triple

$$(s \circ z, Z_u \times_{\mathrm{Spec}(\kappa(u)), s \circ z} \mathrm{Spec}(k), z)$$

where Z_u is as above. This defines a $\mathrm{Spec}(k)$-valued point of Z_{univ} whose image via s', t' in U' is u' and whose image via $Z_{univ} \to R'$ is the point r' by the relationship between z and z' mentioned above. This finishes the proof. $\qquad\square$

Lemma 15.6 (Existence of splitting). *In Situation 15.3 there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : \mathrm{Spec}(\kappa(u)) \to U'$ lying over $u : \mathrm{Spec}(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' is split over u'.*

Proof. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 14.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z_{univ} \subset R \times_{s, U, g} U'$ is open and $R' \to R \times_{s, U, g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $\mathrm{Spec}(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s, U} \mathrm{Spec}(\kappa(u))$. Let $G_u \subset F_u$ be the scheme theoretic fibre of $G \to U$ over u. By assumption G_u is finite and $F_u \to \mathrm{Spec}(\kappa(u))$ is quasi-finite at each point of G_u by assumption. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \amalg Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is G_u. Note that $e(u) \in Z_u$. Hence by the construction of U' in Section 14 (u, Z_u) defines a $\mathrm{Spec}(\kappa(u))$-valued point u' of U'.

We still have to show that the set $\{g' \in |G'| : g' \text{ maps to } u'\}$ is contained in $|Z_{univ}|$. Pick any point g' in this set and represent it by a morphism $z' : \mathrm{Spec}(k) \to G'$. Denote $z : \mathrm{Spec}(k) \to G$ the composition of z' with the map $G' \to G$. Clearly, z defines a point of G_u. In fact, let us write $\tilde{u} : \mathrm{Spec}(k) \to u \to U$ for the corresponding map to u or U. Consider the triple

$$(\tilde{u}, Z_u \times_{u, \tilde{u}} \mathrm{Spec}(k), z)$$

where Z_u is as above. This defines a $\text{Spec}(k)$-valued point of Z_{univ} whose image via s', t' in U' is u' and whose image via $Z_{univ} \to R'$ is the point z' (because the image in R is z). This finishes the proof. $\square$

Lemma 15.7 (Existence of quasi-splitting). *In Situation 15.4 there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : \text{Spec}(\kappa(u)) \to U'$ lying over $u : \text{Spec}(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.*

Proof. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 14.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z_{univ} \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $\text{Spec}(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} \text{Spec}(\kappa(u))$. The morphism $F_u \to \text{Spec}(\kappa(u))$ is quasi-finite at $e(u)$ by assumption. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \amalg Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is $e(u)$. Hence by the construction of U' in Section 14 (u, Z_u) defines a $\text{Spec}(\kappa(u))$-valued point u' of U'. To finish the proof we have to show that $e'(u') \in Z_{univ}$ which is clear. $\square$

Finally, when we add additional assumptions we obtain schemes.

Lemma 15.8. *In Situation 15.2 assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is strongly split over u'.*

Proof. This follows from the construction of U' in the proof of Lemma 15.5 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 12.14 and 12.15. $\square$

Lemma 15.9. *In Situation 15.3 assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is split over u'.*

Proof. This follows from the construction of U' in the proof of Lemma 15.6 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 12.14 and 12.15. $\square$

Lemma 15.10. *In Situation 15.4 assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.*

Proof. This follows from the construction of U' in the proof of Lemma 15.7 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 12.14 and 12.15. $\square$

In fact we can obtain affine schemes by applying an earlier result on finite locally free groupoids.

Lemma 15.11. *In Situation 15.2 assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is strongly split over u'.*

Proof. Let $U' \to U$ and $u' \in U'$ be the separated étale morphism of schemes we found in Lemma 15.8. Let $P \subset R'$ be the strong splitting of R' over u'. By More on Groupoids, Lemma 9.1 the morphisms $s', t' : R' \to U'$ are flat and locally of finite presentation. They are finite by assumption. Hence s', t' are finite locally free, see Morphisms, Lemma 48.2. In particular $t(s^{-1}(u'))$ is a finite set of points $\{u'_1, u'_2, \ldots, u'_n\}$ of U'. Choose a quasi-compact open $W \subset U'$ containing each u'_i. As U is affine the morphism $W \to U$ is quasi-compact (see Schemes, Lemma 19.2). The morphism $W \to U$ is also locally quasi-finite (see Morphisms, Lemma 36.6) and separated. Hence by More on Morphisms, Lemma 43.2 (a version of Zariski's Main Theorem) we conclude that W is quasi-affine. By Properties, Lemma 29.5 we see that $\{u'_1, \ldots, u'_n\}$ are contained in an affine open of U'. Thus we may apply Groupoids, Lemma 24.1 to conclude that there exists an affine P-invariant open $U'' \subset U'$ which contains u'.

To finish the proof denote $R'' = R|_{U''}$ the restriction of R to U''. This is the same as the restriction of R' to U''. As $P \subset R'$ is an open and closed subscheme, so is $P|_{U''} \subset R''$. By construction the open subscheme $U'' \subset U'$ is P-invariant which means that $P|_{U''} = (s'|_P)^{-1}(U'') = (t'|_P)^{-1}(U'')$ (see discussion in Groupoids, Section 19) so the restrictions of s'' and t'' to $P|_{U''}$ are still finite. The sub groupoid scheme $P|_{U''}$ is still a strong splitting of R'' over u''; above we verified (a), (b) and (c) holds as $\{r' \in R' : t'(r') = u', s'(r') = u'\} = \{r'' \in R'' : t''(r'') = u', s''(r'') = u'\}$ trivially. The lemma is proved. $\square$

Lemma 15.12. *In Situation 15.3 assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is split over u'.*

Proof. The proof of this lemma is literally the same as the proof of Lemma 15.11 except that "strong splitting" needs to be replaced by "splitting" (2 times) and that the reference to Lemma 15.8 needs to be replaced by a reference to Lemma 15.9. $\square$

Lemma 15.13. *In Situation 15.4 assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.*

Proof. The proof of this lemma is literally the same as the proof of Lemma 15.11 except that "strong splitting" needs to be replaced by "quasi-splitting" (2 times)

and that the reference to Lemma 15.8 needs to be replaced by a reference to Lemma 15.10. $\square$

16. Other chapters

References

[KM97] Sean Keel and Shigefumi Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), 193–213.