FORMAL ALGEBRAIC SPACES

Contents

1. Introduction

Formal schemes were introduced in [DG67]. A more general version of formal schemes was introduced in [McQ02] and another in [Yas09]. Formal algebraic spaces were introduced in [Knu71]. Related material and much besides can be found in [Abb10] and [FK]. This chapter introduces the notion of formal algebraic spaces we will work with. Our definition is general enough to allow most classes of formal schemes/spaces in the literature as full subcategories.

Although we do discuss the comparison of some of these alternative theories with ours, we do not always give full details when it is not necessary for the logical development of the theory.

Besides introducing formal algebraic spaces, we also prove a few very basic properties and we discuss a few types of morphisms.

2. Formal schemes à la EGA

In this section we review the construction of formal schemes in [DG67]. This notion, although very useful in algebraic geometry, may not always be the correct one to consider. Perhaps it is better to say that in the setup of the theory a number of choices are made, where for different purposes others might work better. And indeed in the literature one can find many different closely related theories adapted to the problem the authors may want to consider. Still, one of the major advantages of the theory as sketched here is that one gets to work with definite geometric objects.

Before we start we should point out an issue with the sheaf condition for sheaves of topological rings or more generally sheaves of topological spaces. Namely, the big categories

(1) category of topological spaces,
(2) category of topological groups,
(3) category of topological rings,
(4) category of topological modules over a given topological ring,

endowed with their natural forgetful functors to *Sets* are not examples of types of algebraic structures as defined in Sheaves, Section 15. Thus we cannot blithely apply to them the machinery developed in that chapter. On the other hand, each of the categories listed above has limits and equalizers and the forgetful functor to sets, groups, rings, modules commutes with them (see Topology, Lemmas 14.1, 30.3, 30.8, and 30.11). Thus we can define the notion of a sheaf as in Sheaves, Definition 9.1 and the underlying presheaf of sets, groups, rings, or modules is a sheaf. The key difference is that for an open covering $U = \bigcup_{i \in I} U_i$ the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

has to be an equalizer diagram in the category of topological spaces, topological groups, topological rings, topological modules, i.e., that the first map identifies $\mathcal{F}(U)$ with a subspace of $\prod_{i \in I} \mathcal{F}(U_i)$ which is endowed with the product topology.

The stalk $\mathcal{F}_x$ of a sheaf $\mathcal{F}$ of topological spaces, topological groups, topological rings, or topological modules at a point $x \in X$ is defined as the colimit over open neighbourhoods

$$\mathcal{F}_x = \text{colim}_{x \in U} \mathcal{F}(U)$$

in the corresponding category. This is the same as taking the colimit on the level of sets, groups, rings, or modules (see Topology, Lemmas 29.1, 30.6, 30.9, and 30.12) but comes equipped with a topology. Warning: the topology one gets depends on which category one is working with, see Examples, Section 77. One can sheafify presheaves of topological spaces, topological groups, topological rings, or topological modules and taking stalks commutes with this operation, see Remark 2.4.

Let $f : X \to Y$ be a continuous map of topological spaces. There is a functor f_* from the category of sheaves of topological spaces, topological groups, topological rings, topological modules, to the corresponding category of sheaves on Y which is defined by setting $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$ as usual. (We delay discussing the pullback in this setting till later.) We define the notion of an f-map $\xi : \mathcal{G} \to \mathcal{F}$ between a sheaf of topological spaces $\mathcal{G}$ on Y and a sheaf of topological spaces $\mathcal{F}$ on X in exactly the same manner as in Sheaves, Definition 21.7 with the additional constraint that $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$ be continuous for every open $V \subset Y$. We have

$$\{f\text{-maps from } \mathcal{G} \text{ to } \mathcal{F}\} = \mathrm{Mor}_{Sh(Y,Top)}(\mathcal{G}, f_*\mathcal{F})$$

as in Sheaves, Lemma 21.8. Similarly for sheaves of topological groups, topological rings, topological modules. Finally, let $\xi : \mathcal{G} \to \mathcal{F}$ be an f-map as above. Then given $x \in X$ with image $y = f(x)$ there is a continuous map

$$\xi_x : \mathcal{G}_y \longrightarrow \mathcal{F}_x$$

of stalks defined in exactly the same manner as in the discussion following Sheaves, Definition 21.9.

Using the discussion above, we can define a category $LTRS$ of "locally topologically ringed spaces". An object is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of topological rings $\mathcal{O}_X$ whose stalks $\mathcal{O}_{X,x}$ are local rings (if one forgets about the topology). A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of $LTRS$ is a pair $(f, f^\sharp)$ where $f : X \to Y$ is a continuous map of topological spaces and $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$ is an f-map such that for every $x \in X$ the induced map

$$f^\sharp_x : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a local homomorphism of local rings (forgetting about the topologies). The composition works in exactly the same manner as composition of morphisms of locally ringed spaces.

Assume now that the topological space X has a basis consisting of quasi-compact opens. Given a sheaf $\mathcal{F}$ of sets, groups, rings, modules over a ring, one can endow $\mathcal{F}$ with the structure of a sheaf of topological spaces, topological groups, topological rings, topological modules. Namely, if $U \subset X$ is quasi-compact open, we endow $\mathcal{F}(U)$ with the discrete topology. If $U \subset X$ is arbitrary, then we choose an open covering $U = \bigcup_{i \in I} U_i$ by quasi-compact opens and we endow $\mathcal{F}(U)$ with the induced topology from $\prod_{i \in I} \mathcal{F}(U_i)$ (as we should do according to our discussion above). The reader may verify (omitted) that we obtain a sheaf of topological spaces, topological groups, topological rings, topological modules in this fashion. Let us say that a sheaf of topological spaces, topological groups, topological rings, topological modules is *pseudo-discrete* if the topology on $\mathcal{F}(U)$ is discrete for every quasi-compact open $U \subset X$. Then the construction given above is an adjoint to the forgetful functor and induces an equivalence between the category of sheaves of sets and the category of pseudo-discrete sheaves of topological spaces (similarly for groups, rings, modules).

Grothendieck and Dieudonné first define formal affine schemes. These correspond to admissible topological rings A, see More on Algebra, Definition 36.1. Namely, given A one considers a fundamental system I_λ of ideals of definition for the ring A. (In any admissible topological ring the family of all ideals of definition forms a fundamental system.) For each λ we can consider the scheme $\mathrm{Spec}(A/I_\lambda)$. For $I_\lambda \subset I_\mu$ the induced morphism

$$\mathrm{Spec}(A/I_\mu) \to \mathrm{Spec}(A/I_\lambda)$$

is a thickening because $I_\mu^n \subset I_\lambda$ for some n. Another way to see this, is to notice that the image of each of the maps

$$\mathrm{Spec}(A/I_\lambda) \to \mathrm{Spec}(A)$$

is a homeomorphism onto the set of open prime ideals of A. This motivates the definition

$$\mathrm{Spf}(A) = \{\text{open prime ideals } \mathfrak{p} \subset A\}$$

endowed with the topology coming from $\mathrm{Spec}(A)$. For each λ we can consider the structure sheaf $\mathcal{O}_{\mathrm{Spec}(A/I_\lambda)}$ as a sheaf on $\mathrm{Spf}(A)$. Let $\mathcal{O}_\lambda$ be the corresponding pseudo-discrete sheaf of topological rings, see above. Then we set

$$\mathcal{O}_{\mathrm{Spf}(A)} = \lim \mathcal{O}_\lambda$$

where the limit is taken in the category of sheaves of topological rings. The pair $(\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})$ is called the *formal spectrum* of A.

At this point one should check several things. The first is that the stalks $\mathcal{O}_{\mathrm{Spf}(A),x}$ are local rings (forgetting about the topology). The second is that given $f \in A$, for the corresponding open $D(f) \cap \mathrm{Spf}(A)$ we have

$$\Gamma(D(f) \cap \mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}) = A_{\{f\}} = \lim (A/I_\lambda)_f$$

as topological rings where I_λ is a fundamental system of ideals of definition as above. Moreover, the ring $A_{\{f\}}$ is admissible too and $(\mathrm{Spf}(A_f), \mathcal{O}_{\mathrm{Spf}(A_{\{f\}})})$ is isomorphic to $(D(f) \cap \mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}|_{D(f)\cap\mathrm{Spf}(A)})$. Finally, given a pair of admissible topological rings A, B we have

$$\mathrm{Mor}_{LTRS}((\mathrm{Spf}(B), \mathcal{O}_{\mathrm{Spf}(B)}), (\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})) = \mathrm{Hom}_{cont}(A, B) \tag{2.0.1}$$

where $LTRS$ is the category of "locally topologically ringed spaces" as defined above.

Having said this, in [DG67] a *formal scheme* is defined as a pair $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ where $\mathfrak{X}$ is a topological space and $\mathcal{O}_\mathfrak{X}$ is a sheaf of topological rings such that every point has an open neighbourhood isomorphic (in $LTRS$) to an affine formal scheme. A *morphism of formal schemes* $f : (\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \to (\mathfrak{Y}, \mathcal{O}_\mathfrak{Y})$ is a morphism in the category $LTRS$.

Let A be a ring endowed with the discrete topology. Then A is admissible and the formal scheme $\mathrm{Spf}(A)$ is equal to $\mathrm{Spec}(A)$. The structure sheaf $\mathcal{O}_{\mathrm{Spf}(A)}$ is the pseudo-discrete sheaf of topological rings associated to $\mathcal{O}_{\mathrm{Spec}(A)}$, in other words, its underlying sheaf of rings is equal to $\mathcal{O}_{\mathrm{Spec}(A)}$ and the ring $\mathcal{O}_{\mathrm{Spf}(A)}(U) = \mathcal{O}_{\mathrm{Spec}(A)}(U)$ over a quasi-compact open U has the discrete topology, but not in general. Thus we can associate to every affine scheme a formal affine scheme. In exactly the same manner we can start with a general scheme $(X, \mathcal{O}_X)$ and associate to it $(X, \mathcal{O}'_X)$

where $\mathcal{O}'_X$ is the pseudo-discrete sheaf of topological rings whose underlying sheaf of rings is $\mathcal{O}_X$. This construction is compatible with morphisms and defines a functor

$$\textit{Schemes} \longrightarrow \textit{Formal Schemes} \tag{2.0.2}$$

It follows in a straightforward manner from (2.0.1) that this functor is fully faithful.

Let $\mathfrak{X}$ be a formal scheme. Let us define the *size* of the formal scheme by the formula $\text{size}(\mathfrak{X}) = \max(\aleph_0, \kappa_1, \kappa_2)$ where κ_1 is the cardinality of the formal affine opens of $\mathfrak{X}$ and κ_2 is the supremum of the cardinalities of $\mathcal{O}_\mathfrak{X}(\mathfrak{U})$ where $\mathfrak{U} \subset \mathfrak{X}$ is such a formal affine open.

Lemma 2.1. *Choose a category of schemes Sch_α as in Sets, Lemma 9.2. Given a formal scheme $\mathfrak{X}$ let*

$$h_\mathfrak{X} : (Sch_\alpha)^{opp} \longrightarrow Sets, \quad h_\mathfrak{X}(S) = \text{Mor}_{\textit{Formal Schemes}}(S, \mathfrak{X})$$

be its functor of points. Then we have

$$\text{Mor}_{\textit{Formal Schemes}}(\mathfrak{X}, \mathfrak{Y}) = \text{Mor}_{PSh(Sch_\alpha)}(h_\mathfrak{X}, h_\mathfrak{Y})$$

provided the size of $\mathfrak{X}$ is not too large.

Proof. First we observe that $h_\mathfrak{X}$ satisfies the sheaf property for the Zariski topology for any formal scheme $\mathfrak{X}$ (see Schemes, Definition 15.3). This follows from the local nature of morphisms in the category of formal schemes. Also, for an open immersion $\mathfrak{V} \to \mathfrak{W}$ of formal schemes, the corresponding transformation of functors $h_\mathfrak{V} \to h_\mathfrak{W}$ is injective and representable by open immersions (see Schemes, Definition 15.3). Choose an open covering $\mathfrak{X} = \bigcup \mathfrak{U}_i$ of a formal scheme by affine formal schemes $\mathfrak{U}_i$. Then the collection of functors $h_{\mathfrak{U}_i}$ covers $h_\mathfrak{X}$ (see Schemes, Definition 15.3). Finally, note that

$$h_{\mathfrak{U}_i} \times_{h_\mathfrak{X}} h_{\mathfrak{U}_j} = h_{\mathfrak{U}_i \cap \mathfrak{U}_j}$$

Hence in order to give a map $h_\mathfrak{X} \to h_\mathfrak{Y}$ is equivalent to giving a family of maps $h_{\mathfrak{U}_i} \to h_\mathfrak{Y}$ which agree on overlaps. Thus we can reduce the bijectivity (resp. injectivity) of the map of the lemma to bijectivity (resp. injectivity) for the pairs $(\mathfrak{U}_i, \mathfrak{Y})$ and injectivity (resp. nothing) for $(\mathfrak{U}_i \cap \mathfrak{U}_j, \mathfrak{Y})$. In this way we reduce to the case where $\mathfrak{X}$ is an affine formal scheme. Say $\mathfrak{X} = \text{Spf}(A)$ for some admissible topological ring A. Also, choose a fundamental system of ideals of definition $I_\lambda \subset A$.

We can also localize on $\mathfrak{Y}$. Namely, suppose that $\mathfrak{V} \subset \mathfrak{Y}$ is an open formal subscheme and $\varphi : h_\mathfrak{X} \to h_\mathfrak{Y}$. Then

$$h_\mathfrak{V} \times_{h_\mathfrak{Y}, \varphi} h_\mathfrak{X} \to h_\mathfrak{X}$$

is representable by open immersions. Pulling back to $\text{Spec}(A/I_\lambda)$ for all λ we find an open subscheme $U_\lambda \subset \text{Spec}(A/I_\lambda)$. However, for $I_\lambda \subset I_\mu$ the morphism $\text{Spec}(A/I_\lambda) \to \text{Spec}(A/I_\mu)$ pulls back U_μ to U_λ. Thus these glue to give an open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$. A straightforward argument (omitted) shows that

$$h_\mathfrak{U} = h_\mathfrak{V} \times_{h_\mathfrak{Y}} h_\mathfrak{X}$$

In this way we see that given an open covering $\mathfrak{Y} = \bigcup \mathfrak{V}_j$ and a transformation of functors $\varphi : h_\mathfrak{X} \to h_\mathfrak{Y}$ we obtain a corresponding open covering of $\mathfrak{X}$. Since $\mathfrak{X}$ is affine, we can refine this covering by a finite open covering $\mathfrak{X} = \mathfrak{U}_1 \cup \ldots \cup \mathfrak{U}_n$

by affine formal subschemes. In other words, for each i there is a j and a map $\varphi_i : h_{\mathfrak{U}_i} \to h_{\mathfrak{V}_j}$ such that

$$\begin{tikzcd} h_{\mathfrak{U}_i} \arrow[r, "\varphi_i"'] \arrow[d] & h_{\mathfrak{V}_j} \arrow[d] \\ h_{\mathfrak{X}} \arrow[r, "\varphi"] & h_{\mathfrak{Y}} \end{tikzcd}$$

commutes. With a few additional arguments (which we omit) this implies that it suffices to prove the bijectivity of the lemma in case both $\mathfrak{X}$ and $\mathfrak{Y}$ are affine formal schemes.

Assume $\mathfrak{X}$ and $\mathfrak{Y}$ are affine formal schemes. Say $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$. Let $\varphi : h_{\mathfrak{X}} \to h_{\mathfrak{Y}}$ be a transformation of functors. Let $I_\lambda \subset A$ be a fundamental system of ideals of definition. The canonical inclusion morphism $i_\lambda : \mathrm{Spec}(A/I_\lambda) \to \mathfrak{X}$ maps to a morphism $\varphi(i_\lambda) : \mathrm{Spec}(A/I_\lambda) \to \mathfrak{Y}$. By (2.0.1) this corresponds to a continuous map $\chi_\lambda : B \to A/I_\lambda$. Since φ is a transformation of functors it follows that for $I_\lambda \subset I_\mu$ the composition $B \to A/I_\lambda \to A/I_\mu$ is equal to χ_μ. In other words we obtain a ring map

$$\chi = \lim \chi_\lambda : B \longrightarrow \lim A/I_\lambda = A$$

This is a continuous homomorphism because the inverse image of I_λ is open for all λ (as A/I_λ has the discrete topology and χ_λ is continuous). Thus we obtain a morphism $\mathrm{Spf}(\chi) : \mathfrak{X} \to \mathfrak{Y}$ by (2.0.1). We omit the verification that this construction is the inverse to the map of the lemma in this case.

Set theoretic remarks. To make this work on the given category of schemes Sch_α we just have to make sure all the schemes used in the proof above are isomorphic to objects of Sch_α. In fact, a careful analysis shows that it suffices if the schemes $\mathrm{Spec}(A/I_\lambda)$ occurring above are isomorphic to objects of Sch_α. For this it certainly suffices to assume the size of $\mathfrak{X}$ is at most the size of a scheme contained in Sch_α. $\square$

Lemma 2.2. *Let $\mathfrak{X}$ be a formal scheme. The functor of points $h_{\mathfrak{X}}$ (see Lemma 2.1) satisfies the sheaf condition for fpqc coverings.*

Proof. Topologies, Lemma 9.13 reduces us to the case of a Zariski covering and a covering $\{\mathrm{Spec}(S) \to \mathrm{Spec}(R)\}$ with $R \to S$ faithfully flat. We observed in the proof of Lemma 2.1 that $h_{\mathfrak{X}}$ satisfies the sheaf condition for Zariski coverings.

Suppose that $R \to S$ is a faithfully flat ring map. Denote $\pi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ the corresponding morphism of schemes. It is surjective and flat. Let $f : \mathrm{Spec}(S) \to \mathfrak{X}$ be a morphism such that $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$ as maps $\mathrm{Spec}(S \otimes_R S) \to \mathfrak{X}$. By Descent, Lemma 13.1 we see that as a map on the underlying sets f is of the form $f = g \circ \pi$ for some (set theoretic) map $g : \mathrm{Spec}(R) \to \mathfrak{X}$. By Morphisms, Lemma 25.12 and the fact that f is continuous we see that g is continuous.

Pick $y \in \mathrm{Spec}(R)$. Choose $\mathfrak{U} \subset \mathfrak{X}$ an affine formal open subscheme containing $g(y)$. Say $\mathfrak{U} = \mathrm{Spf}(A)$ for some admissible topological ring A. By the above we may choose an $r \in R$ such that $y \in D(r) \subset g^{-1}(\mathfrak{U})$. The restriction of f to $\pi^{-1}(D(r))$ into $\mathfrak{U}$ corresponds to a continuous ring map $A \to S_r$ by (2.0.1). The two induced ring maps $A \to S_r \otimes_{R_r} S_r = (S \otimes_R S)_r$ are equal by assumption on f. Note that $R_r \to S_r$ is faithfully flat. By Descent, Lemma 3.6 the equalizer of the two arrows $S_r \to S_r \otimes_{R_r} S_r$ is R_r. We conclude that $A \to S_r$ factors uniquely through a map

$A \to R_r$ which is also continuous as it has the same (open) kernel as the map $A \to S_r$. This map in turn gives a morphism $D(r) \to \mathfrak{U}$ by (2.0.1).

What have we proved so far? We have shown that for any $y \in \text{Spec}(R)$ there exists a standard affine open $y \in D(r) \subset \text{Spec}(R)$ such that the morphism $f|_{\pi^{-1}(D(r))} : \pi^{-1}(D(r)) \to \mathfrak{X}$ factors uniquely though some morphism $D(r) \to \mathfrak{X}$. We omit the verification that these morphisms glue to the desired morphism $\text{Spec}(R) \to \mathfrak{X}$. $\square$

Remark 2.3 (McQuillan's variant)**.** There is a variant of the construction of formal schemes due to McQuillan, see [McQ02]. He suggests a slight weakening of the condition of admissibility. Namely, recall that an admissible topological ring is a complete (and separated by our conventions) topological ring A which is linearly topologized such that there exists an ideal of definition: an open ideal I such that any neighbourhood of 0 contains I^n for some $n \geq 1$. McQuillan works with what we will call *weakly admissible* topological rings. A weakly admissible topological ring A is a complete (and separated by our conventions) topological ring which is linearly topologized such that there exists an *weak ideal of definition*: an open ideal I such that for all $f \in I$ we have $f^n \to 0$ for $n \to \infty$. Similarly to the admissible case, if I is a weak ideal of definition and $J \subset A$ is an open ideal, then $I \cap J$ is a weak ideal of definition. Thus the weak ideals of definition form a fundamental system of open neighbourhoods of 0 and one can proceed along much the same route as above to define a larger category of formal schemes based on this notion. The analogues of Lemmas 2.1 and 2.2 still hold in this setting (with the same proof).

Remark 2.4 (Sheafification of presheaves of topological spaces)**.** In this remark we briefly discuss sheafification of presheaves of topological spaces. The exact same arguments work for presheaves of topological abelian groups, topological rings, and topological modules (over a given topological ring). In order to do this in the correct generality let us work over a site $\mathcal{C}$. The reader who is interested in the case of (pre)sheaves over a topological space X should think of objects of $\mathcal{C}$ as the opens of X, of morphisms of $\mathcal{C}$ as inclusions of opens, and of coverings in $\mathcal{C}$ as coverings in X, see Sites, Example 6.4. Denote $\mathit{Sh}(\mathcal{C}, \mathit{Top})$ the category of sheaves of topological spaces on $\mathcal{C}$ and denote $\mathit{PSh}(\mathcal{C}, \mathit{Top})$ the category of presheaves of topological spaces on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of topological spaces on $\mathcal{C}$. The sheafification $\mathcal{F}^\#$ should satisfy the formula

$$\text{Mor}_{\mathit{PSh}(\mathcal{C}, \mathit{Top})}(\mathcal{F}, \mathcal{G}) = \text{Mor}_{\mathit{Sh}(\mathcal{C}, \mathit{Top})}(\mathcal{F}^\#, \mathcal{G})$$

functorially in $\mathcal{G}$ from $\mathit{Sh}(\mathcal{C}, \mathit{Top})$. In other words, we are trying to construct the left adjoint to the inclusion functor $\mathit{Sh}(\mathcal{C}, \mathit{Top}) \to \mathit{PSh}(\mathcal{C}, \mathit{Top})$. We first claim that $\mathit{Sh}(\mathcal{C}, \mathit{Top})$ has limits and that the inclusion functor commutes with them. Namely, given a category $\mathcal{I}$ and a functor $i \mapsto \mathcal{G}_i$ into $\mathit{Sh}(\mathcal{C}, \mathit{Top})$ we simply define

$$(\lim \mathcal{G}_i)(U) = \lim \mathcal{G}_i(U)$$

where we take the limit in the category of topological spaces (Topology, Lemma 14.1). This defines a sheaf because limits commute with limits (Categories, Lemma 14.10) and in particular products and equalizers (which are the operations used in the sheaf axiom). Finally, a morphism of presheaves from $\mathcal{F} \to \lim \mathcal{G}_i$ is clearly the same thing as a compatible system of morphisms $\mathcal{F} \to \mathcal{G}_i$. In other words, the object $\lim \mathcal{G}_i$ is the limit in the category of presheaves of topological spaces and a fortiori in the category of sheaves of topological spaces. Our second claim is that any morphism of presheaves $\mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ an object of $\mathit{Sh}(\mathcal{C}, \mathit{Top})$ factors through a subsheaf $\mathcal{G}' \subset \mathcal{G}$ whose size is bounded. Here we define the *size* $|\mathcal{H}|$ of a sheaf of topological spaces $\mathcal{H}$ to be the cardinal $\sup_{U \in \mathrm{Ob}(\mathcal{C})} |\mathcal{H}(U)|$. To prove our claim we let

$$\mathcal{G}'(U) = \left\{ \quad s \in \mathcal{G}(U) \quad \middle| \quad \begin{array}{c} \text{there exists a covering } \{U_i \to U\}_{i \in I} \\ \text{such that } s|_{U_i} \in \mathrm{Im}(\mathcal{F}(U_i) \to \mathcal{G}(U_i)) \end{array} \right\}$$

We endow $\mathcal{G}'(U)$ with the induced topology. Then $\mathcal{G}'$ is a sheaf of topological spaces (details omitted) and $\mathcal{G}' \to \mathcal{G}$ is a morphism through which the given map $\mathcal{F} \to \mathcal{G}$ factors. Moreover, the size of $\mathcal{G}'$ is bounded by some cardinal κ depending only on $\mathcal{C}$ and the presheaf $\mathcal{F}$ (hint: use that coverings in $\mathcal{C}$ form a set by our conventions). Putting everything together we see that the assumptions of Categories, Theorem 25.3 are satisfied and we obtain sheafification as the left adjoint of the inclusion functor from sheaves to presheaves. Finally, let p be a point of the site $\mathcal{C}$ given by a functor $u : \mathcal{C} \to \mathit{Sets}$, see Sites, Definition 32.2. For a topological space M the presheaf defined by the rule

$$U \mapsto \mathrm{Map}(u(U), M) = \prod\nolimits_{x \in u(U)} M$$

endowed with the product topology is a sheaf of topological spaces. Hence the exact same argument as given in the proof of Sites, Lemma 32.5 shows that $\mathcal{F}_p = \mathcal{F}_p^\#$, in other words, sheafification commutes with taking stalks at a point.

3. Conventions and notation

The conventions from now on will be similar to the conventions in Properties of Spaces, Section 2. Thus from now on the standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site. For topological rings A we assume only that all discrete quotients have this property (but usually we assume more, compare with Remark 11.5).

Let S be a scheme and let X be a "space" over S, i.e., a sheaf on $(\mathit{Sch}/S)_{fppf}$. In this chapter we will write $X \times_S X$ for the product of X with itself in the category of sheaves on $(\mathit{Sch}/S)_{fppf}$ instead of $X \times X$. Moreover, if X and Y are "spaces" then we say "let $f : X \to Y$ be a morphism" to indicate that f is a natural transformation of functors, i.e., a map of sheaves on $(\mathit{Sch}/S)_{fppf}$. Similarly, if U is a scheme over S and X is a "space" over S, then we say "let $f : U \to X$ be a morphism" or "let $g : X \to U$ be a morphism" to indicate that f or g is a map of sheaves $h_U \to X$ or $X \to h_U$ where h_U is as in Categories, Example 3.4.

4. Topological rings and modules

This section is a continuation of More on Algebra, Section 36. Let R be a topological ring and let M be a linearly topologized R-module. When we say "*let M_λ be a fundamental system of open submodules*" we will mean that each M_λ is an open submodule and that any neighbourhood of 0 contains one of the M_λ. In other words, this means that M_λ is a fundamental system of neighbourhoods of 0 in M consisting of submodules. Similarly, if R is a linearly topologized ring, then we say "*let I_λ be a fundamental system of open ideals*" to mean that I_λ is a fundamental system of neighbourhoods of 0 in R consisting of ideals.

Example 4.1. Let R be a linearly topologized ring and let M be a linearly topologized R-module. Let I_λ be a fundamental system of open ideals in R and let M_μ be a fundamental system of open submodules of M. The continuity of $+ : M \times M \to M$ is automatic and the continuity of $R \times M \to M$ signifies

$$\forall f, x, \mu\ \exists \lambda, \nu,\ (f + I_\lambda)(x + M_\nu) \subset fx + M_\mu$$

Since $fM_\nu + I_\lambda M_\nu \subset M_\mu$ if $M_\nu \subset M_\mu$ we see that the condition is equivalent to

$$\forall x, \mu\ \exists \lambda\ I_\lambda x \subset M_\mu$$

However, it need not be the case that given μ there is a λ such that $I_\lambda M \subset M_\mu$. For example, consider $R = k[[t]]$ with the t-adic topology and $M = \bigoplus_{n \in \mathbf{N}} R$ with fundamental system of open submodules given by

$$M_m = \bigoplus\nolimits_{n \in \mathbf{N}} t^{nm} R$$

Since every $x \in M$ has finitely many nonzero coordinates we see that, given m and x there exists a k such that $t^k x \in M_m$. Thus M is a linearly topologized R-module, but it isn't true that given m there is a k such that $t^k M \subset M_m$. On the other hand, if $R \to S$ is a continuous map of linearly topologized rings, then the corresponding statement does hold, i.e., for every open ideal $J \subset S$ there exists an open ideal $I \subset R$ such that $IS \subset J$ (as the reader can easily deduce from continuity of the map $R \to S$).

Lemma 4.2. *Let R be a topological ring. Let M be a linearly topologized R-module and let M_λ, $\lambda \in \Lambda$ be a fundamental system of open submodules. Let $N \subset M$ be a submodule. The closure of N is $\bigcap_{\lambda \in \Lambda}(N + M_\lambda)$.*

Proof. Since each $N + M_\lambda$ is open, it is also closed. Hence the intersection is closed. If $x \in M$ is not in the closure of N, then $(x + M_\lambda) \cap N = 0$ for some λ. Hence $x \notin N + M_\lambda$. This proves the lemma. $\square$

Unless otherwise mentioned we endow submodules and quotient modules with the induced topology. Let M be a linearly topologized module over a topological ring R, and let $0 \to N \to M \to Q \to 0$ be a short exact sequence of R-modules. If M_λ is a fundamental system of open submodules of M, then $N \cap M_\lambda$ is a fundamental system of open submodules of N. If $\pi : M \to Q$ is the quotient map, then $\pi(M_\lambda)$ is a fundamental system of open submodules of Q. In particular these induced topologies are linear topologies.

Lemma 4.3. *Let R be a topological ring. Let M be a linearly topologized R-module. Let $N \subset M$ be a submodule. Then*

(1) $0 \to N^\wedge \to M^\wedge \to (M/N)^\wedge$ *is exact, and*
(2) $N^\wedge$ *is the closure of the image of* $N \to M^\wedge$.

Proof. Let M_λ, $\lambda \in \Lambda$ be a fundamental system of open submodules. Then $N \cap M_\lambda$ is a fundamental system of open submodules of N and $M_\lambda + N/N$ is a fundamental system of open submodules of M/N. Thus we see that (1) follows from the exactness of the sequences

$$0 \to N/N \cap M_\lambda \to M/M_\lambda \to M/(M_\lambda + N) \to 0$$

and the fact that taking limits commutes with limits. The second statement follows from this and the fact that $N \to N^\wedge$ has dense image and that the kernel of $M^\wedge \to (M/N)^\wedge$ is closed. $\square$

Lemma 4.4. *Let R be a topological ring. Let M be a complete, linearly topologized R-module. Let $N \subset M$ be a closed submodule. If M has a countable fundamental system of neighbourhoods of 0, then M/N is complete and the map $M \to M/N$ is open.*

Proof. Let M_n, $n \in \mathbf{N}$ be a fundamental system of open submodules of M. We may assume $M_{n+1} \subset M_n$ for all n. The system $(M_n + N)/N$ is a fundamental system in M/N. Hence we have to show that $M/N = \lim M/(M_n + N)$. Consider the short exact sequences

$$0 \to N/N \cap M_n \to M/M_n \to M/(M_n + N) \to 0$$

Since the transition maps of the system $\{N/N \cap M_n\}$ are surjective we see that $M = \lim M/M_n$ (by completeness of M) surjects onto $\lim M/(M_n + N)$ by Algebra, Lemma 86.4. As N is closed we see that the kernel of $M \to \lim M/(M_n + N)$ is N (see Lemma 4.2). Finally, $M \to M/N$ is open by definition of the quotient topology. □

Lemma 4.5. *Let R be a topological ring. Let M be a linearly topologized R-module. Let $N \subset M$ be a submodule. Assume M has a countable fundamental system of neighbourhoods of 0. Then*
1. *$0 \to N^\wedge \to M^\wedge \to (M/N)^\wedge \to 0$ is exact,*
2. *$N^\wedge$ is the closure of the image of $N \to M^\wedge$,*
3. *$M^\wedge \to (M/N)^\wedge$ is open.*

Proof. We have $0 \to N^\wedge \to M^\wedge \to (M/N)^\wedge$ is exact and statement (2) by Lemma 4.3. This produces a canonical map $c : M^\wedge/N^\wedge \to (M/N)^\wedge$. The module $M^\wedge/N^\wedge$ is complete and $M^\wedge \to M^\wedge/N^\wedge$ is open by Lemma 4.4. By the universal property of completion we obtain a canonical map $b : (M/N)^\wedge \to M^\wedge/N^\wedge$. Then b and c are mutually inverse as they are on a dense subset. □

Lemma 4.6. *Let R be a topological ring. Let M be a topological R-module. Let $I \subset R$ be a finitely generated ideal. Assume M has an open submodule whose topology is I-adic. Then $M^\wedge$ has an open submodule whose topology is I-adic and we have $M^\wedge/I^nM^\wedge = M/I^nM$ for all $n \geq 1$.*

Proof. Let $M' \subset M$ be an open submodule whose topology is I-adic. Then $\{I^nM'\}_{n\geq 1}$ is a fundamental system of open submodules of M. Thus $M^\wedge = \lim M/I^nM'$ contains $(M')^\wedge = \lim M'/I^nM'$ as an open submodule and the topology on $(M')^\wedge$ is I-adic by Algebra, Lemma 96.3. Since I is finitely generated, I^n is finitely generated, say by $f_1, \ldots, f_r$. Observe that the surjection $(f_1, \ldots, f_r) : M^{\oplus r} \to I^nM$ is continuous and open by our description of the topology on M above. By Lemma 4.5 applied to this surjection and to the short exact sequence $0 \to I^nM \to M \to M/I^nM \to 0$ we conclude that

$$(f_1, \ldots, f_r) : (M^\wedge)^{\oplus r} \longrightarrow M^\wedge$$

surjects onto the kernel of the surjection $M^\wedge \to M/I^nM$. Since $f_1, \ldots, f_r$ generate I^n we conclude. □

Definition 4.7. Let R be a topological ring. Let M and N be linearly topologized R-modules. The *tensor product* of M and N is the (usual) tensor product $M \otimes_R N$ endowed with the linear topology defined by declaring

$$\mathrm{Im}(M_\mu \otimes_R N + M \otimes_R N_\nu \longrightarrow M \otimes_R N)$$

to be a fundamental system of open submodules, where $M_\mu \subset M$ and $N_\nu \subset N$ run through fundamental systems of open submodules in M and N. The *completed tensor product*

$$M \widehat{\otimes}_R N = \lim M \otimes_R N/(M_\mu \otimes_R N + M \otimes_R N_\nu) = \lim M/M_\mu \otimes_R N/N_\nu$$

is the completion of the tensor product.

Observe that the topology on R is immaterial for the construction of the tensor product or the completed tensor product. If $R \to A$ and $R \to B$ are continuous maps of linearly topologized rings, then the construction above gives a tensor product $A \otimes_R B$ and a completed tensor product $A \widehat{\otimes}_R B$.

We record here the notions introduced in Remark 2.3.

Definition 4.8. Let A be a linearly topologized ring.

(1) An element $f \in A$ is called *topologically nilpotent* if $f^n \to 0$ as $n \to \infty$.
(2) A *weak ideal of definition* for A is an open ideal $I \subset A$ consisting entirely of topologically nilpotent elements.
(3) We say A is *weakly pre-admissible* if A has a weak ideal of definition.
(4) We say A is *weakly admissible* if A is weakly pre-admissible and complete[1].

Given a weak ideal of definition I in a linearly topologized ring A and an open ideal J the intersection $I \cap J$ is a weak ideal of definition. Hence if there is one weak ideal of definition, then there is a fundamental system of open ideals consisting of weak ideals of definition. In particular, given a weakly admissible topological ring A then $A = \lim A/I_\lambda$ where $\{I_\lambda\}$ is a fundamental system of weak ideals of definition.

Lemma 4.9. *Let A be a weakly admissible topological ring. Let $I \subset A$ be a weak ideal of definition. Then (A, I) is a henselian pair.*

Proof. Let $A \to A'$ be an étale ring map and let $\sigma : A' \to A/I$ be an A-algebra map. By More on Algebra, Lemma 11.6 it suffices to lift σ to an A-algebra map $A' \to A$. To do this, as A is complete, it suffices to find, for every open ideal $J \subset I$, a unique A-algebra map $A' \to A/J$ lifting σ. Since I is a weak ideal of definition, the ideal I/J is locally nilpotent. We conclude by More on Algebra, Lemma 11.2. □

Lemma 4.10. *Let B be a linearly topologized ring. The set of topologically nilpotent elements of B is a closed, radical ideal of B. Let $\varphi : A \to B$ be a continuous map of linearly topologized rings.*

(1) *If $f \in A$ is topologically nilpotent, then $\varphi(f)$ is topologically nilpotent.*
(2) *If $I \subset A$ consists of topologically nilpotent elements, then the closure of $\varphi(I)B$ consists of topologically nilpotent elements.*

Proof. Let $\mathfrak{b} \subset B$ be the set of topologically nilpotent elements. We omit the proof of the fact that $\mathfrak{b}$ is a radical ideal (good exercise in the definitions). Let g be an element of the closure of $\mathfrak{b}$. Our goal is to show that g is topologically nilpotent. Let $J \subset B$ be an open ideal. We have to show $g^e \in J$ for some $e \geq 1$. We have $g \in \mathfrak{b} + J$ by Lemma 4.2. Hence $g = f + h$ for some $f \in \mathfrak{b}$ and $h \in J$. Pick $m \geq 1$ such that $f^m \in J$. Then $g^{m+1} \in J$ as desired.

Let $\varphi : A \to B$ be as in the statement of the lemma. Assertion (1) is clear and assertion (2) follows from this and the fact that $\mathfrak{b}$ is a closed ideal. □

[1] By our conventions this includes separated.

Lemma 4.11. *Let $A \to B$ be a continuous map of linearly topologized rings. Let $I \subset A$ be an ideal. The closure of IB is the kernel of $B \to B\widehat{\otimes}_A A/I$.*

Proof. Let J_μ be a fundamental system of open ideals of B. The closure of IB is $\bigcap(IB + J_\lambda)$ by Lemma 4.2. Let I_μ be a fundamental system of open ideals in A. Then

$$B\widehat{\otimes}_A A/I = \lim(B/J_\lambda \otimes_A A/(I_\mu + I)) = \lim B/(J_\lambda + I_\mu B + IB)$$

Since $A \to B$ is continuous, for every λ there is a μ such that $I_\mu B \subset J_\lambda$, see discussion in Example 4.1. Hence the limit can be written as $\lim B/(J_\lambda + IB)$ and the result is clear. □

Lemma 4.12. *Let $B \to A$ and $B \to C$ be continuous homomorphisms of linearly topologized rings.*

1. *If A and C are weakly pre-admissible, then $A\widehat{\otimes}_B C$ is weakly admissible.*
2. *If A and C are pre-admissible, then $A\widehat{\otimes}_B C$ is admissible.*
3. *If A and C have a countable fundamental system of open ideals, then $A\widehat{\otimes}_B C$ has a countable fundamental system of open ideals.*
4. *If A and C are pre-adic and have finitely generated ideals of definition, then $A\widehat{\otimes}_B C$ is adic and has a finitely generated ideal of definition.*
5. *If A and C are pre-adic Noetherian rings and $B/\mathfrak{b} \to A/\mathfrak{a}$ is of finite type where $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ are the ideals of topologically nilpotent elements, then $A\widehat{\otimes}_B C$ is adic Noetherian.*

Proof. Let $I_\lambda \subset A$, $\lambda \in \Lambda$ and $J_\mu \subset C$, $\mu \in M$ be fundamental systems of open ideals, then by definition

$$A\widehat{\otimes}_B C = \lim_{\lambda,\mu} A/I_\lambda \otimes_B C/J_\mu$$

with the limit topology. Thus a fundamental system of open ideals is given by the kernels $K_{\lambda,\mu}$ of the maps $A\widehat{\otimes}_B C \to A/I_\lambda \otimes_B C/J_\mu$. Note that $K_{\lambda,\mu}$ is the closure of the ideal $I_\lambda(A\widehat{\otimes}_B C)+J_\mu(A\widehat{\otimes}_B C)$. Finally, we have a ring homomorphism $\tau : A \otimes_B C \to A\widehat{\otimes}_B C$ with dense image.

Proof of (1). If I_λ and J_μ consist of topologically nilpotent elements, then so does $K_{\lambda,\mu}$ by Lemma 4.10. Hence $A\widehat{\otimes}_B C$ is weakly admissible by definition.

Proof of (2). Assume for some λ_0 and μ_0 the ideals $I = I_{\lambda_0} \subset A$ and $J_{\mu_0} \subset C$ are ideals of definition. Thus for every λ there exists an n such that $I^n \subset I_\lambda$. For every μ there exists an m such that $J^m \subset J_\mu$. Then

$$\left(I(A\widehat{\otimes}_B C) + J(A\widehat{\otimes}_B C)\right)^{n+m} \subset I_\lambda(A\widehat{\otimes}_B C) + J_\mu(A\widehat{\otimes}_B C)$$

It follows that the open ideal $K = K_{\lambda_0,\mu_0}$ satisfies $K^{n+m} \subset K_{\lambda,\mu}$. Hence K is an ideal of definition of $A\widehat{\otimes}_B C$ and $A\widehat{\otimes}_B C$ is admissible by definition.

Proof of (3). If Λ and M are countable, so is $\Lambda \times M$.

Proof of (4). Assume $\Lambda = \mathbf{N}$ and $M = \mathbf{N}$ and we have finitely generated ideals $I \subset A$ and $J \subset C$ such that $I_n = I^n$ and $J_n = J^n$. Then

$$I(A\widehat{\otimes}_B C) + J(A\widehat{\otimes}_B C)$$

is a finitely generated ideal and it is easily seen that $A\widehat{\otimes}_B C$ is the completion of $A \otimes_B C$ with respect to this ideal. Hence (4) follows from Algebra, Lemma 96.3.

Proof of (5). Let $\mathfrak{c} \subset C$ be the ideal of topologically nilpotent elements. Since A and C are adic Noetherian, we see that $\mathfrak{a}$ and $\mathfrak{c}$ are ideals of definition (details omitted). From part (4) we already know that $A\widehat{\otimes}_B C$ is adic and that $\mathfrak{a}(A\widehat{\otimes}_B C) + \mathfrak{c}(A\widehat{\otimes}_B C)$ is a finitely generated ideal of definition. Since

$$A\widehat{\otimes}_B C/\left(\mathfrak{a}(A\widehat{\otimes}_B C) + \mathfrak{c}(A\widehat{\otimes}_B C)\right) = A/\mathfrak{a} \otimes_{B/\mathfrak{b}} C/\mathfrak{c}$$

is Noetherian as a finite type algebra over the Noetherian ring $C/\mathfrak{c}$ we conclude by Algebra, Lemma 97.5. □

5. Taut ring maps

It turns out to be convenient to have a name for the following property of continuous maps between linearly topologized rings.

Definition 5.1. Let $\varphi : A \to B$ be a continuous map of linearly topologized rings. We say φ is *taut*[2] if for every open ideal $I \subset A$ the closure of the ideal $\varphi(I)B$ is open and these closures form a fundamental system of open ideals.

If $\varphi : A \to B$ is a continuous map of linearly topologized rings and I_λ a fundamental system of open ideals of A, then φ is taut if and only if the closures of $I_\lambda B$ are open and form a fundamental system of open ideals in A.

Lemma 5.2. *Let $\varphi : A \to B$ be a continuous map of weakly admissible topological rings. The following are equivalent*

1. *φ is taut,*
2. *for every weak ideal of definition $I \subset A$ the closure of $\varphi(I)B$ is a weak ideal of definition of B and these form a fundamental system of weak ideals of definition of B.*

Proof. The remarks following Definition 5.1 show that (2) implies (1). Conversely, assume φ is taut. If $I \subset A$ is a weak ideal of definition, then the closure of $\varphi(I)B$ is open by definition of tautness and consists of topologically nilpotent elements by Lemma 4.10. Hence the closure of $\varphi(I)B$ is a weak ideal of definition. Furthermore, by definition of tautness these ideals form a fundamental system of open ideals and we see that (2) is true. □

Lemma 5.3. *Let A be a linearly topologized ring. The map $A \to A^\wedge$ from A to its completion is taut.*

Proof. Let I_λ be a fundamental system of open ideals of A. Recall that $A^\wedge = \lim A/I_\lambda$ with the limit topology, which means that the kernels $J_\lambda = \text{Ker}(A^\wedge \to A/I_\lambda)$ form a fundamental system of open ideals of $A^\wedge$. Since J_λ is the closure of $I_\lambda A^\wedge$ (compare with Lemma 4.11) we conclude. □

Lemma 5.4. *Let $A \to B$ and $B \to C$ be continuous homomorphisms of linearly topologized rings. If $A \to B$ and $B \to C$ are taut, then $A \to C$ is taut.*

Proof. Omitted. Hint: if $I \subset A$ is an ideal and J is the closure of IB, then the closure of JC is equal to the closure of IC. □

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[2]This is nonstandard notation. The definition generalizes to modules, by saying a linearly topologized A-module M is A-taut if for every open ideal $I \subset A$ the closure of IM in M is open and these closures form a fundamental system of neighbourhoods of 0 in M.

Lemma 5.5. *Let $A \to B$ and $B \to C$ be continuous homomorphisms of linearly topologized rings. If $A \to C$ is taut, then $B \to C$ is taut.*

Proof. Let $J \subset B$ be an open ideal with inverse image $I \subset A$. Then the closure of JC contains the closure of IC. Hence this closure is open as $A \to C$ is taut. Let I_λ be a fundamental system of open ideals of A. Let K_λ be the closure of $I_\lambda C$. Since $A \to C$ is taut, these form a fundamental system of open ideals of C. Denote $J_\lambda \subset B$ the inverse image of K_λ. Then the closure of $J_\lambda C$ is K_λ. Hence we see that the closures of the ideals JC, where J runs over the open ideals of B form a fundamental system of open ideals of C. □

Lemma 5.6. *Let $A \to B$ and $A \to C$ be continuous homomorphisms of linearly topologized rings. If $A \to B$ is taut, then $C \to B\widehat{\otimes}_A C$ is taut.*

Proof. Let $K \subset C$ be an open ideal. Choose any open ideal $I \subset A$ whose image in C is contained in J. By assumption the closure J of IB is open. Since $A \to B$ is taut we see that $B\widehat{\otimes}_A C$ is the limit of the rings $B/J \otimes_{A/I} C/K$ over all choices of K and I, i.e, the ideals $J(B\widehat{\otimes}_A C) + K(B\widehat{\otimes}_A C)$ form a fundamental system of open ideals. Now, since $B \to B\widehat{\otimes}_A C$ is continuous we see that J maps into the closure of $K(B\widehat{\otimes}_A C)$ (as I maps into K). Hence this closure is equal to $J(B\widehat{\otimes}_A C) + K(B\widehat{\otimes}_A C)$ and the proof is complete. □

Lemma 5.7. *Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A has a countable fundamental system of open ideals, then B has a countable fundamental system of open ideals.*

Proof. Immediate from the definitions. □

Lemma 5.8. *Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A is weakly pre-admissible, then B is weakly pre-admissible.*

Proof. Let $I \subset A$ be a weak ideal of definition. Then the closure J of IB is open and consists of topologically nilpotent elements by Lemma 4.10. Hence J is a weak ideal of definition of B. □

Lemma 5.9. *Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A is pre-admissible, then B is pre-admissible.*

Proof. Let $I \subset A$ be an ideal of definition. Let $I_\lambda \subset A$ be a fundamental system of open ideals. Then the closure J of IB is open and the closures J_λ of $I_\lambda B$ are open and form a fundamental system of open ideals of B. For every λ there is an n such that $I^n \subset I_\lambda$. Observe that J^n is contained in the closure of $I^n B$. Thus $J^n \subset J_\lambda$ and we conclude J is an ideal of definition. □

Lemma 5.10. *Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. Assume*

(1) *φ is taut and has dense image,*
(2) *A is complete and has a countable fundamental system of open ideals, and*
(3) *B is separated.*

Then φ is surjective and open, B is complete, and $B = A/K$ for some closed ideal $K \subset A$.

Proof. By the open mapping lemma (More on Algebra, Lemma 36.5) combined with tautness of φ, we see the map φ is open. Since the image of φ is dense, we see that φ is surjective. The kernel K of φ is closed as φ is continuous. It follows that $B = A/K$ is complete, see for example Lemma 4.4. $\square$

6. Adic ring maps

Let us make the following definition.

Definition 6.1. Let A and B be pre-adic topological rings. A ring homomorphism $\varphi : A \to B$ is *adic*[3] if there exists an ideal of definition $I \subset A$ such that the topology on B is the I-adic topology.

If $\varphi : A \to B$ is an adic homomorphism of pre-adic rings, then φ is continuous and the topology on B is the I-adic topology for every ideal of definition I of A.

Lemma 6.2. *Let $A \to B$ and $B \to C$ be continuous homomorphisms of pre-adic rings. If $A \to B$ and $B \to C$ are adic, then $A \to C$ is adic.*

Proof. Omitted. $\square$

Lemma 6.3. *Let $A \to B$ and $B \to C$ be continuous homomorphisms of pre-adic rings. If $A \to C$ is adic, then $B \to C$ is adic.*

Proof. Choose an ideal of definition I of A. As $A \to C$ is adic, we see that IC is an ideal of definition of C. As $B \to C$ is continuous, we can find an ideal of definition $J \subset B$ mapping into IC. As $A \to B$ is continuous the inverse image $I' \subset I$ of J in I is an ideal of definition of A too. Hence $I'C \subset JC \subset IC$ is sandwiched between two ideals of definition, hence is an ideal of definition itself. $\square$

Lemma 6.4. *Let $\varphi : A \to B$ be a continuous homomorphism between pre-adic topological rings. If φ is adic, then φ is taut.*

Proof. Immediate from the definitions. $\square$

The next lemma says two things

(1) the property of being adic ascents along taut maps of complete linearly topologized rings, and
(2) the properties "φ is taut" and "φ is adic" are equivalent for continuous maps $\varphi : A \to B$ between adic rings if A has a finitely generated ideal of definition.

Because of (2) we can say that "tautness" generalizes "adicness" to continuous ring maps between arbitrary linearly topologized rings. See also Section 23.

Lemma 6.5. *Let $\varphi : A \to B$ be a continuous map of linearly topologized rings. If φ is taut, A is pre-adic and has a finitely generated ideal of definition, and B is complete, then B is adic and has a finitely generated ideal of definition and the ring map φ is adic.*

Proof. Choose a finitely generated ideal of definition I of A. Let J_n be the closure of $\varphi(I^n)B$ in B. Since B is complete we have $B = \lim B/J_n$. Let $B' = \lim B/I^nB$ be the I-adic completion of B. By Algebra, Lemma 96.3, the I-adic topology on B' is complete and $B'/I^nB' = B/I^nB$. Thus the ring map $B' \to B$ is continuous

[3]This may be nonstandard terminology.

and has dense image as $B' \to B/I^nB \to B/J_n$ is surjective for all n. Finally, the map $B' \to B$ is taut because $(I^nB')B = I^nB$ and $A \to B$ is taut. By Lemma 5.10 we see that $B' \to B$ is open and surjective. Thus the topology on B is the I-adic topology and the proof is complete. $\square$

7. Weakly adic rings

We suggest the reader skip this section. The following is a natural generalization of adic rings.

Definition 7.1. Let A be a linearly topologized ring.

(1) We say A is *weakly pre-adic*[4] if there exists an ideal $I \subset A$ such that the closure of I^n is open for all $n \geq 0$ and these closures form a fundamental system of open ideals.
(2) We say A is *weakly adic* if A is weakly pre-adic and complete[5].

For complete linearly topologized rings we have the following implications

$$\begin{tikzcd}
\text{adic + Noetherian} \arrow[d, Rightarrow] & \\
\text{adic + finitely generated ideal of definition} \arrow[d, Rightarrow] & \\
\text{adic} \arrow[d, Rightarrow] & \\
\text{weakly adic} \arrow[d, Rightarrow] & \\
\text{admissible + first countable} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{admissible} \arrow[d, Rightarrow] \\
\text{weakly admissible + first countable} \arrow[r, Rightarrow] & \text{weakly admissible}
\end{tikzcd}$$

where "first countable" means that our topological ring has a countable fundamental system of open ideals. There is a similar diagram of implications for noncomplete linearly topologized rings (i.e., using the notions of pre-adic, weakly pre-adic, pre-admissible, and weakly pre-admissible). Contrary to what happens with pre-adic rings the completion of a weakly pre-adic ring is weakly adic as the following lemma characterizing weakly pre-adic rings shows.

Lemma 7.2. *Let A be a linearly topologized ring. The following are equivalent*

(1) *A is weakly pre-adic,*
(2) *there exists a taut continuous ring map $A' \to A$ where A' is a pre-adic topological ring, and*
(3) *A is pre-admissible and there exists an ideal of definition I such that the closure of I^n is open for all $n \geq 1$, and*

[4] In [GR04] the authors say A is *c-adic*.
[5] By our conventions this includes separated.

(4) *A is pre-admissible and for every ideal of definition I the closure of I^n is open for all $n \geq 1$.*

The completion of a weakly pre-adic ring is weakly adic. If A is weakly adic, then A is admissible and has a countable fundamental system of open ideals.

Proof. Assume (1). Choose an ideal I such that the closure of I^n is open for all n and such that these closures form a fundamental system of open ideals. Denote $A' = A$ endowed with the I-adic topology. Then $A' \to A$ is taut by definition and we see that (2) holds.

Assume (2). Let $I' \subset A'$ be an ideal of definition. Denote I the closure of $I'A$. Tautness of $A' \to A$ means that the closures I_n of $(I')^n A$ are open and form a fundamental system of open ideals. Thus $I = I_1$ is open and the closures of I^n are equal to I_n and hence open and form a fundamental system of open ideals. Thus certainly I is an ideal of definition such that the closure of I^n is open for all n. Hence (3) holds.

If $I \subset A$ is as in (3), then I is an ideal as in Definition 7.1 and we see that (1) holds. Also, if $I' \subset A$ is any other ideal of definition, then I' is open (see More on Algebra, Definition 36.1) and hence contains I^n for some $n \geq 1$. Thus $(I')^m$ contains I^{nm} for all $m \geq 1$ and we conclude that the closures of $(I')^m$ are open for all m. In this way we see that (3) implies (4). The implication (4) $\Rightarrow$ (3) is trivial.

Let A be weakly pre-adic. Choose $A' \to A$ as in (2). By Lemmas 5.3 and 5.4 the composition $A' \to A^\wedge$ is taut. Hence $A^\wedge$ is weakly pre-adic by the equivalence of (2) and (1). Since the completion of a linearly topologized ring A is complete (More on Algebra, Section 36) we see that $A^\wedge$ is weakly adic.

Let A be weakly adic. Then A is complete and and pre-admissible by (1) $\Rightarrow$ (3) and hence A is admissible. Of course by definition A has a countable fundamental system of open ideals. □

We give two criteria that guarantee that a weakly adic ring is adic and has a finitely generated ideal of definition.

Lemma 7.3. *Let A be a complete linearly topologized ring. Let $I \subset A$ be a finitely generated ideal such that the closure of I^n is open for all $n \geq 0$ and these closures form a fundamental system of open ideals. Then A is adic and has a finitely generated ideal of definition.*

Proof. Denote A' the ring A endowed with the I-adic topology. The assumptions tells us that $A' \to A$ is taut. We conclude by Lemma 6.5 (to be sure, this lemma also tells us that I is an ideal of definition). □

Lemma 7.4. *Let A be a weakly adic topological ring. Let I be an ideal of definition such that I/I_2 is a finitely generated module where I_2 is the closure of I^2. Then A is adic and has a finitely generated ideal of definition.*

Proof. We use the characterization of Lemma 7.2 without further mention. Choose $f_1, \ldots, f_r \in I$ which map to generators of I/I_2. Set $I' = (f_1, \ldots, f_r)$. We have $I' + I_2 = I$. Then I_2 is the closure of $I^2 = (I' + I_2)^2 \subset I' + I_3$ where I_3 is the closure of I^3. Hence $I' + I_3 = I$. Continuing in this fashion we see that $I' + I_n = I$ for all $n \geq 2$ where I_n is the closure of I^n. In other words, the closure of I' in A

is I. Hence the closure of $(I')^n$ is I_n. Thus the closures of $(I')^n$ are a fundamental system of open ideals of A. We conclude by Lemma 7.3. □

A key feature of the property "weakly pre-adic" is that it ascents along taut ring homomorphisms of linearly topologized rings.

Lemma 7.5. *Let $\varphi : A \to B$ be a continuous homomorphism of linearly topologized rings. If φ is taut and A is weakly pre-adic, then B is weakly pre-adic.*

Proof. Let $I \subset A$ be an ideal such that the closure I_n of I^n is open and these closures define a fundamental system of open ideals. Then the closure of I^nB is equal to the closure of I_nB. Since φ is taut, these closures are open and form a fundamental system of open ideals of B. Hence B is weakly pre-adic. □

Lemma 7.6. *Let $B \to A$ and $B \to C$ be continuous homomorphisms of linearly topologized rings. If A and C are weakly pre-adic, then $A\widehat{\otimes}_B C$ is weakly adic.*

Proof. We will use the characterization of Lemma 7.2 without further mention. By Lemma 4.12 we know that $A\widehat{\otimes}_B C$ is admissible. Moreover, the proof of that lemma shows that the closure $K \subset A\widehat{\otimes}_B C$ is an ideal of definition, when $I \subset A$ and $J \subset C$ of $I(A\widehat{\otimes}_B C) + J(A\widehat{\otimes}_B C)$ are ideals of definition. Then it suffices to show that the closure of K^n is open for all $n \geq 1$. Since the ideal K^n contains $I^n(A\widehat{\otimes}_B C) + J^n(A\widehat{\otimes}_B C)$, since the closure of I^n in A is open, and since the closure of J^n in C is open, we see that the closure of K^n is open in $A\widehat{\otimes}_B C$. □

8. Descending properties

In this section we consider the following situation

(1) $\varphi : A \to B$ is a continuous map of linearly topologized topological rings,
(2) φ is taut, and
(3) for every open ideal $I \subset A$ if $J \subset B$ denotes the closure of IB, then the map $A/I \to B/J$ is faithfully flat.

We are going to show that properties of B are inherited by A in this situation.

Lemma 8.1. *In the situation above, if B has a countable fundamental system of open ideals, then A has a countable fundamental system of open ideals.*

Proof. Choose a fundamental system $B \supset J_1 \supset J_2 \supset \ldots$ of open ideals. By tautness of φ, for every n we can find an open ideal I_n such that $J_n \supset I_nB$. We claim that I_n is a fundamental system of open ideals of A. Namely, suppose that $I \subset A$ is open. As φ is taut, the closure of IB is open and hence contains J_n for some n large enough. Hence $I_nB \subset IB$. Let J be the closure of IB in B. Since $A/I \to B/J$ is faithfully flat, it is injective. Hence, since $I_n \to A/I \to B/J$ is zero as $I_nB \subset IB \subset J$, we conclude that $I_n \to A/I$ is zero. Hence $I_n \subset I$ and we win. □

Lemma 8.2. *In the situation above, if B is weakly pre-admissible, then A is weakly pre-admissible.*

Proof. Let $J \subset B$ be a weak ideal of definition. Let $I \subset A$ be an open ideal such that $IB \subset J$. To show that I is a weak ideal of definition we have to show that any $f \in I$ is topologically nilpotent. Let $I' \subset A$ be an open ideal. Denote $J' \subset B$ the closure of $I'B$. Then $A/I' \to B/J'$ is faithfully flat, hence injective. Thus in order to show that $f^n \in I'$ it suffices to show that $\varphi(f)^n \in J'$. This holds for $n \gg 0$ since $\varphi(f) \in J$, the ideal J is a weak ideal of defintion of B, and J' is open in B. □

Lemma 8.3. *In the situation above, if B is pre-admissible, then A is pre-admissible.*

Proof. Let $J \subset B$ be a weak ideal of definition. Let $I \subset A$ be an open ideal such that $IB \subset J$. Let $I' \subset A$ be an open ideal. To show that I is an ideal of definition we have to show that $I^n \subset I'$ for $n \gg 0$. Denote $J' \subset B$ the closure of $I'B$. Then $A/I' \to B/J'$ is faithfully flat, hence injective. Thus in order to show that $I^n \subset I'$ it suffices to show that $\varphi(I)^n \subset J'$. This holds for $n \gg 0$ since $\varphi(I) \subset J$, the ideal J is an ideal of defintion of B, and J' is open in B. □

Lemma 8.4. *In the situation above, if B is weakly pre-adic, then A is weakly pre-adic.*

Proof. We will use the characterization of weakly pre-adic rings given in Lemma 7.2 without further mention. By Lemma 8.3 the topological ring A is pre-admissible. Let $I \subset A$ be an ideal of definition. Fix $n \geq 1$. To prove the lemma we have to show that the closure of I^n is open. Let $I_\lambda \subset A$ be a fundamental system of open ideals. Denote $J \subset B$, resp. $J_\lambda \subset B$ the closure of IB, resp. $I_\lambda B$. Since B is weakly pre-adic, the closure of J^n is open. Hence there exists a λ such that

$$J_\lambda \subset \bigcap\nolimits_\mu (J^n + J_\mu)$$

because the right hand side is the closure of J^n by Lemma 4.2. This means that the image of J_λ in B/J_μ is contained in the image of J^n in B/J_μ. Observe that the image of J^n in B/J_μ is equal to the image of I^nB in B/J_μ (since every element of J is congruent to an element of IB modulo J_μ). Since $A/I_\mu \to B/J_\mu$ is faithfully flat and since $I_\lambda B \subset J_\lambda$, we conclude that the image of I_λ in A/I_μ is contained in the image of I^n. We conclude that I_λ is contained in the closure of I^n and the proof is complete. □

Lemma 8.5. *In the situation above, if B is adic and has a finitely generated ideal of definition and A is complete, then A is adic and has a finitely generated ideal of definition.*

Proof. We already know that A is weakly adic and a fortiori admissible by Lemma 8.4 (and Lemma 7.2 to see that adic rings are weakly adic). Let $I \subset A$ be an ideal of definition. Let $J \subset B$ be a finitely generated ideal of definition. Since the closure of IB is open, we can find an $n > 0$ such that J^n is contained in the closure of IB. Thus after replacing J by J^n we may assume J is a finitely generated ideal of definition contained in the closure of IB. By Lemma 4.2 this certainly implies that

$$J \subset IB + J^2$$

Consider the finitely generated A-module $M = (J + IB)/IB$. The displayed equation shows that $JM = M$. By Lemma 4.9 (for example) we see that J is contained in the Jacobson radical of B. Hence by Nakayama's lemma, more precisely part (2) of Algebra, Lemma 20.1, we conclude $M = 0$. Thus $J \subset IB$.

Since J is finitely generated, we can find a finitely generated ideal $I' \subset I$ such that $J \subset I'B$. Since $A \to B$ is continuous, $J \subset B$ is open, and I is an ideal of definition, we can find an $n > 0$ such that $I^nB \subset J$. Let $J_{n+1} \subset B$ be the closure of $I^{n+1}B$. We have

$$I^n \cdot (B/J_{n+1}) \subset J \cdot (B/J_{n+1}) \subset I' \cdot (B/J_{n+1})$$

Since $A/I^{n+1} \to B/J_{n+1}$ is faithfully flat, this implies $I^n \cdot (A/I^{n+1}) \subset I' \cdot (A/I^{n+1})$ which in turn means

$$I^n \subset I' + I^{n+1}$$

This implies $I^n \subset I' + I^{n+k}$ for all $k \geq 1$ which in turn implies that $I^{nm} \subset (I')^m + I^{nm+k}$ for all $k, m \geq 1$. This implies that the closure of $(I')^m$ contains I^{nm}. Since the closure of I^{nm} is open as A is weakly adic, we conclude that the closure $(I')^m$ is open for all m. Since these closures form a fundamental system of open ideals of A (as the same thing is true for the closures of I^n) we conclude by Lemma 7.3. $\square$

9. Affine formal algebraic spaces

In this section we introduce affine formal algebraic spaces. These will in fact be the same as what are called affine formal schemes in [BD]. However, we will call them affine formal algebraic spaces, in order to prevent confusion with the notion of an affine formal scheme as defined in [DG67].

Recall that a thickening of schemes is a closed immersion which induces a surjection on underlying topological spaces, see More on Morphisms, Definition 2.1.

Definition 9.1. Let S be a scheme. We say a sheaf X on $(Sch/S)_{fppf}$ is an *affine formal algebraic space* if there exist

(1) a directed set Λ,
(2) a system $(X_\lambda, f_{\lambda\mu})$ over Λ in $(Sch/S)_{fppf}$ where
    (a) each X_λ is affine,
    (b) each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a thickening,

such that

$$X \cong \operatorname{colim}_{\lambda \in \Lambda} X_\lambda$$

as fppf sheaves and X satisfies a set theoretic condition (see Remark 11.5). A *morphism of affine formal algebraic spaces* over S is a map of sheaves.

Observe that the system $(X_\lambda, f_{\lambda\mu})$ is not part of the data. Suppose that U is a quasi-compact scheme over S. Since the transition maps are monomorphisms, we see that

$$X(U) = \operatorname{colim} X_\lambda(U)$$

by Sites, Lemma 17.7. Thus the fppf sheafification inherent in the colimit of the definition is a Zariski sheafification which does not do anything for quasi-compact schemes.

Lemma 9.2. *Let S be a scheme. If X is an affine formal algebraic space over S, then the diagonal morphism $\Delta : X \to X \times_S X$ is representable and a closed immersion.*

Proof. Suppose given $U \to X$ and $V \to X$ where U, V are schemes over S. Let us show that $U \times_X V$ is representable. Write $X = \operatorname{colim} X_\lambda$ as in Definition 9.1. The discussion above shows that Zariski locally on U and V the morphisms factors through some X_λ. In this case $U \times_X V = U \times_{X_\lambda} V$ which is a scheme. Thus the diagonal is representable, see Spaces, Lemma 5.10. Given $(a, b) : W \to X \times_S X$ where W is a scheme over S consider the map $X \times_{\Delta, X \times_S X, (a,b)} W \to W$. As before locally on W the morphisms a and b map into the affine scheme X_λ for some λ and then we get the morphism $X_\lambda \times_{\Delta_\lambda, X_\lambda \times_S X_\lambda, (a,b)} W \to W$. This is the base change of $\Delta_\lambda : X_\lambda \to X_\lambda \times_S X_\lambda$ which is a closed immersion as $X_\lambda \to S$ is separated (because X_λ is affine). Thus $X \to X \times_S X$ is a closed immersion. $\square$

A morphism of schemes $X \to X'$ is a thickening if it is a closed immersion and induces a surjection on underlying sets of points, see (More on Morphisms, Definition 2.1). Hence the property of being a thickening is preserved under arbitrary base change and fpqc local on the target, see Spaces, Section 4. Thus Spaces, Definition 5.1 applies to "thickening" and we know what it means for a representable transformation $F \to G$ of presheaves on $(Sch/S)_{fppf}$ to be a thickening. We observe that this does not clash with our definition (More on Morphisms of Spaces, Definition 9.1) of thickenings in case F and G are algebraic spaces.

Lemma 9.3. *Let $X_\lambda, \lambda \in \Lambda$ and $X = \operatorname{colim} X_\lambda$ be as in Definition 9.1. Then $X_\lambda \to X$ is representable and a thickening.*

Proof. The statement makes sense by the discussion in Spaces, Section 3 and 5. By Lemma 9.2 the morphisms $X_\lambda \to X$ are representable. Given $U \to X$ where U is a scheme, then the discussion following Definition 9.1 shows that Zariski locally on U the morphism factors through some X_μ with $\lambda \leq \mu$. In this case $U \times_X X_\lambda = U \times_{X_\mu} X_\lambda$ so that $U \times_X X_\lambda \to U$ is a base change of the thickening $X_\lambda \to X_\mu$. $\square$

Lemma 9.4. *Let $X_\lambda, \lambda \in \Lambda$ and $X = \operatorname{colim} X_\lambda$ be as in Definition 9.1. If Y is a quasi-compact algebraic space over S, then any morphism $Y \to X$ factors through an X_λ.*

Proof. Choose an affine scheme V and a surjective étale morphism $V \to Y$. The composition $V \to Y \to X$ factors through X_λ for some λ by the discussion following Definition 9.1. Since $V \to Y$ is a surjection of sheaves, we conclude. $\square$

Lemma 9.5. *Let S be a scheme. Let X be a sheaf on $(Sch/S)_{fppf}$. Then X is an affine formal algebraic space if and only if the following hold*

(1) *any morphism $U \to X$ where U is an affine scheme over S factors through a morphism $T \to X$ which is representable and a thickening with T an affine scheme over S, and*
(2) *a set theoretic condition as in Remark 11.5.*

Proof. It follows from Lemmas 9.3 and 9.4 that an affine formal algebraic space satisfies (1) and (2). In order to prove the converse we may assume X is not empty. Let Λ be the category of representable morphisms $T \to X$ which are thickenings where T is an affine scheme over S. This category is directed. Since X is not empty, Λ contains at least one object. If $T \to X$ and $T' \to X$ are in Λ, then we can factor $T \amalg T' \to X$ through $T'' \to X$ in Λ. Between any two objects of Λ there is a unique arrow or none. Thus Λ is a directed set and by assumption $X = \operatorname{colim}_{T \to X \text{ in } \Lambda} T$. To finish the proof we need to show that any arrow $T \to T'$ in Λ is a thickening. This is true because $T' \to X$ is a monomorphism of sheaves, so that $T = T \times_{T'} T' = T \times_X T'$ and hence the morphism $T \to T'$ equals the projection $T \times_X T' \to T'$ which is a thickening because $T \to X$ is a thickening. $\square$

For a general affine formal algebraic space X there is no guarantee that X has enough functions to separate points (for example). See Examples, Section 74. To characterize those that do we offer the following lemma.

Lemma 9.6. *Let S be a scheme. Let X be an fppf sheaf on $(Sch/S)_{fppf}$ which satisfies the set theoretic condition of Remark 11.5. The following are equivalent:*

1. *there exists a weakly admissible topological ring A over S (see Remark 2.3) such that $X = \text{colim}_{I \subset A \text{ weak ideal of definition}} \text{Spec}(A/I)$,*
2. *X is an affine formal algebraic space and there exists an S-algebra A and a map $X \to \text{Spec}(A)$ such that for a closed immersion $T \to X$ with T an affine scheme the composition $T \to \text{Spec}(A)$ is a closed immersion,*
3. *X is an affine formal algebraic space and there exists an S-algebra A and a map $X \to \text{Spec}(A)$ such that for a closed immersion $T \to X$ with T a scheme the composition $T \to \text{Spec}(A)$ is a closed immersion,*
4. *X is an affine formal algebraic space and for some choice of $X = \text{colim}\, X_\lambda$ as in Definition 9.1 the projections $\lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda}) \to \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ are surjective,*
5. *X is an affine formal algebraic space and for any choice of $X = \text{colim}\, X_\lambda$ as in Definition 9.1 the projections $\lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda}) \to \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ are surjective.*

Moreover, the weakly admissible topological ring is $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ endowed with its limit topology and the weak ideals of definition classify exactly the morphisms $T \to X$ which are representable and thickenings.

Proof. It is clear that (5) implies (4).

Assume (4) for $X = \text{colim}\, X_\lambda$ as in Definition 9.1. Set $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$. Let $T \to X$ be a closed immersion with T a scheme (note that $T \to X$ is representable by Lemma 9.2). Since $X_\lambda \to X$ is a thickening, so is $X_\lambda \times_X T \to T$. On the other hand, $X_\lambda \times_X T \to X_\lambda$ is a closed immersion, hence $X_\lambda \times_X T$ is affine. Hence T is affine by Limits, Proposition 11.2. Then $T \to X$ factors through X_λ for some λ by Lemma 9.4. Thus $A \to \Gamma(X_\lambda, \mathcal{O}) \to \Gamma(T, \mathcal{O})$ is surjective. In this way we see that (3) holds.

It is clear that (3) implies (2).

Assume (2) for A and $X \to \text{Spec}(A)$. Write $X = \text{colim}\, X_\lambda$ as in Definition 9.1. Then $A_\lambda = \Gamma(X_\lambda, \mathcal{O})$ is a quotient of A by assumption (2). Hence $A^\wedge = \lim A_\lambda$ is a complete topological ring, see discussion in More on Algebra, Section 36. The maps $A^\wedge \to A_\lambda$ are surjective as $A \to A_\lambda$ is. We claim that for any λ the kernel $I_\lambda \subset A^\wedge$ of $A^\wedge \to A_\lambda$ is a weak ideal of definition. Namely, it is open by definition of the limit topology. If $f \in I_\lambda$, then for any $\mu \in \Lambda$ the image of f in A_μ is zero in all the residue fields of the points of X_μ. Hence it is a nilpotent element of A_μ. Hence some power $f^n \in I_\mu$. Thus $f^n \to 0$ as $n \to 0$. Thus $A^\wedge$ is weakly admissible. Finally, suppose that $I \subset A^\wedge$ is a weak ideal of definition. Then $I \subset A^\wedge$ is open and hence there exists some λ such that $I \supset I_\lambda$. Thus we obtain a morphism $\text{Spec}(A^\wedge/I) \to \text{Spec}(A_\lambda) \to X$. Then it follows that $X = \text{colim}\, \text{Spec}(A^\wedge/I)$ where now the colimit is over all weak ideals of definition. Thus (1) holds.

Assume (1). In this case it is clear that X is an affine formal algebraic space. Let $X = \text{colim}\, X_\lambda$ be any presentation as in Definition 9.1. For each λ we can find a weak ideal of definition $I \subset A$ such that $X_\lambda \to X$ factors through $\text{Spec}(A/I) \to X$, see Lemma 9.4. Then $X_\lambda = \text{Spec}(A/I_\lambda)$ with $I \subset I_\lambda$. Conversely, for any weak ideal of definition $I \subset A$ the morphism $\text{Spec}(A/I) \to X$ factors through X_λ for some λ, i.e., $I_\lambda \subset I$. It follows that each I_λ is a weak ideal of definition and that they form a cofinal subset of the set of weak ideals of definition. Hence $A = \lim A/I = \lim A/I_\lambda$ and we see that (5) is true and moreover that $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$. $\square$

With this lemma in hand we can make the following definition.

Definition 9.7. Let S be a scheme. Let X be an affine formal algebraic space over S. We say X is *McQuillan* if X satisfies the equivalent conditions of Lemma 9.6. Let A be the weakly admissible topological ring associated to X. We say

(1) X is *classical* if X is McQuillan and A is admissible (More on Algebra, Definition 36.1),
(2) X is *weakly adic* if X is McQuillan and A is weakly adic (Definition 7.1),
(3) X is *adic* if X is McQuillan and A is adic (More on Algebra, Definition 36.1),
(4) X is *adic** if X is McQuillan, A is adic, and A has a finitely generated ideal of definition, and
(5) X is *Noetherian* if X is McQuillan and A is both Noetherian and adic.

In [FK] they use the terminology "of finite ideal type" for the property that an adic topological ring A contains a finitely generated ideal of definition. Given an affine formal algebraic space X here are the implications among the notions introduced in the definition:

$$\begin{tikzcd}
X \text{ Noetherian} \arrow[r, Rightarrow] & X \text{ adic*} \arrow[r, Rightarrow] & X \text{ adic} \arrow[dll, Rightarrow] \\
X \text{ weakly adic} \arrow[r, Rightarrow] & X \text{ classical} \arrow[r, Rightarrow] & X \text{ McQuillan}
\end{tikzcd}$$

See discussion in Section 7 and for a precise statement see Lemma 10.3.

Remark 9.8. The classical affine formal algebraic spaces correspond to the affine formal schemes considered in EGA ([DG67]). To explain this we assume our base scheme is $\text{Spec}(\mathbf{Z})$. Let $\mathfrak{X} = \text{Spf}(A)$ be an affine formal scheme. Let $h_\mathfrak{X}$ be its functor of points as in Lemma 2.1. Then $h_\mathfrak{X} = \text{colim}\, h_{\text{Spec}(A/I)}$ where the colimit is over the collection of ideals of definition of the admissible topological ring A. This follows from (2.0.1) when evaluating on affine schemes and it suffices to check on affine schemes as both sides are fppf sheaves, see Lemma 2.2. Thus $h_\mathfrak{X}$ is an affine formal algebraic space. In fact, it is a classical affine formal algebraic space by Definition 9.7. Thus Lemma 2.1 tells us the category of affine formal schemes is equivalent to the category of classical affine formal algebraic spaces.

Having made the connection with affine formal schemes above, it seems natural to make the following definition.

Definition 9.9. Let S be a scheme. Let A be a weakly admissible topological ring over S, see Definition 4.8[6]. The *formal spectrum* of A is the affine formal algebraic space

$$\text{Spf}(A) = \text{colim}\, \text{Spec}(A/I)$$

where the colimit is over the set of weak ideals of definition of A and taken in the category $Sh((Sch/S)_{fppf})$.

[6] See More on Algebra, Definition 36.1 for the classical case and see Remark 2.3 for a discussion of differences.

Such a formal spectrum is McQuillan by construction and conversely every McQuillan affine formal algebraic space is isomorphic to a formal spectrum. To be sure, in our theory there exist affine formal algebraic spaces which are not the formal spectrum of any weakly admissible topological ring. Following [Yas09] we could introduce S-pro-rings to be pro-objects in the category of S-algebras, see Categories, Remark 22.5. Then every affine formal algebraic space over S would be the formal spectrum of such an S-pro-ring. We will not do this and instead we will work directly with the corresponding affine formal algebraic spaces.

The construction of the formal spectrum is functorial. To explain this let $\varphi : B \to A$ be a continuous map of weakly admissible topological rings over S. Then

$$\mathrm{Spf}(\varphi) : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$$

is the unique morphism of affine formal algebraic spaces such that the diagrams

$$\begin{tikzcd} \mathrm{Spec}(B/J) \arrow[r] \arrow[d] & \mathrm{Spec}(A/I) \arrow[d] \\ \mathrm{Spf}(B) \arrow[r] & \mathrm{Spf}(A) \end{tikzcd}$$

commute for all weak ideals of definition $I \subset A$ and $J \subset B$ with $\varphi(I) \subset J$. Since continuity of φ implies that for every weak ideal of definition $J \subset B$ there is a weak ideal of definition $I \subset A$ with the required property, we see that the required commutativities uniquely determine and define $\mathrm{Spf}(\varphi)$.

Lemma 9.10. *Let S be a scheme. Let A, B be weakly admissible topological rings over S. Any morphism $f : Spf(B) \to Spf(A)$ of affine formal algebraic spaces over S is equal to $Spf(f^\sharp)$ for a unique continuous S-algebra map $f^\sharp : A \to B$.*

Proof. Let $f : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$ be as in the lemma. Let $J \subset B$ be a weak ideal of definition. By Lemma 9.4 there exists a weak ideal of definition $I \subset A$ such that $\mathrm{Spec}(B/J) \to \mathrm{Spf}(B) \to \mathrm{Spf}(A)$ factors through $\mathrm{Spec}(A/I)$. By Schemes, Lemma 6.4 we obtain an S-algebra map $A/I \to B/J$. These maps are compatible for varying J and define the map $f^\sharp : A \to B$. This map is continuous because for every weak ideal of definition $J \subset B$ there is a weak ideal of definition $I \subset A$ such that $f^\sharp(I) \subset J$. The equality $f = \mathrm{Spf}(f^\sharp)$ holds by our choice of the ring maps $A/I \to B/J$ which make up $f^\sharp$. □

Lemma 9.11. *Let S be a scheme. Let $f : X \to Y$ be a map of presheaves on $(Sch/S)_{fppf}$. If X is an affine formal algebraic space and f is representable by algebraic spaces and locally quasi-finite, then f is representable (by schemes).*

Proof. Let T be a scheme over S and $T \to Y$ a map. We have to show that the algebraic space $X \times_Y T$ is a scheme. Write $X = \mathrm{colim}\, X_\lambda$ as in Definition 9.1. Let $W \subset X \times_Y T$ be a quasi-compact open subspace. The restriction of the projection $X \times_Y T \to X$ to W factors through X_λ for some λ. Then

$$W \to X_\lambda \times_S T$$

is a monomorphism (hence separated) and locally quasi-finite (because $W \to X \times_Y T \to T$ is locally quasi-finite by our assumption on $X \to Y$, see Morphisms of Spaces, Lemma 27.8). Hence W is a scheme by Morphisms of Spaces, Proposition 50.2. Thus $X \times_Y T$ is a scheme by Properties of Spaces, Lemma 13.1. □

10. Countably indexed affine formal algebraic spaces

These are the affine formal algebraic spaces as in the following lemma.

Lemma 10.1. *Let S be a scheme. Let X be an affine formal algebraic space over S. The following are equivalent*

1. *there exists a system $X_1 \to X_2 \to X_3 \to \ldots$ of thickenings of affine schemes over S such that $X = \operatorname{colim} X_n$,*
2. *there exists a choice $X = \operatorname{colim} X_\lambda$ as in Definition 9.1 such that Λ is countable.*

Proof. This follows from the observation that a countable directed set has a cofinal subset isomorphic to $(\mathbf{N}, \geq)$. See proof of Algebra, Lemma 86.3. □

Definition 10.2. Let S be a scheme. Let X be an affine formal algebraic space over S. We say X is *countably indexed* if the equivalent conditions of Lemma 10.1 are satisfied.

In the language of [BD] this is expressed by saying that X is an $\aleph_0$-ind scheme.

Lemma 10.3. *Let X be an affine formal algebraic space over a scheme S.*

1. *If X is Noetherian, then X is adic*.*
2. *If X is adic*, then X is adic.*
3. *If X is adic, then X is weakly adic.*
4. *If X is weakly adic, then X is classical.*
5. *If X is weakly adic, then X is countably indexed.*
6. *If X is countably indexed, then X is McQuillan.*

Proof. Statements (1), (2), (3), and (4) follow by writing $X = \operatorname{Spf}(A)$ and where A is a weakly admissible (hence complete) linearly topologized ring and using the implications between the various types of such rings discussed in Section 7.

Proof of (5). By definition there exists a weakly adic topological ring A such that $X = \operatorname{colim} \operatorname{Spec}(A/I)$ where the colimit is over the ideals of definition of A. As A is weakly adic, there exits in particular a countable fundamental system I_λ of open ideals, see Definition 7.1. Then $X = \operatorname{colim} \operatorname{Spec}(A/I_n)$ by definition of $\operatorname{Spf}(A)$. Thus X is countably indexed.

Proof of (6). Write $X = \operatorname{colim} X_n$ for some system $X_1 \to X_2 \to X_3 \to \ldots$ of thickenings of affine schemes over S. Then

$$A = \lim \Gamma(X_n, \mathcal{O}_{X_n})$$

surjects onto each $\Gamma(X_n, \mathcal{O}_{X_n})$ because the transition maps are surjections as the morphisms $X_n \to X_{n+1}$ are closed immersions. Hence X is McQuillan. □

Lemma 10.4. *Let S be a scheme. Let X be a presheaf on $(Sch/S)_{fppf}$. The following are equivalent*

1. *X is a countably indexed affine formal algebraic space,*
2. *$X = Spf(A)$ where A is a weakly admissible topological S-algebra which has a countable fundamental system of neighbourhoods of 0,*
3. *$X = Spf(A)$ where A is a weakly admissible topological S-algebra which has a fundamental system $A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$ of weak ideals of definition,*

(4) *$X = Spf(A)$ where A is a complete topological S-algebra with a fundamental system of open neighbourhoods of 0 given by a countable sequence $A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$ of ideals such that I_n/I_{n+1} is locally nilpotent, and*
(5) *$X = Spf(A)$ where $A = \lim B/J_n$ with the limit topology where $B \supset J_1 \supset J_2 \supset J_3 \supset \ldots$ is a sequence of ideals in an S-algebra B with J_n/J_{n+1} locally nilpotent.*

Proof. Assume (1). By Lemma 10.3 we can write $X = \mathrm{Spf}(A)$ where A is a weakly admissible topological S-algebra. For any presentation $X = \mathrm{colim}\, X_n$ as in Lemma 10.1 part (1) we see that $A = \lim A_n$ with $X_n = \mathrm{Spec}(A_n)$ and $A_n = A/I_n$ for some weak ideal of definition $I_n \subset A$. This follows from the final statement of Lemma 9.6 which moreover implies that $\{I_n\}$ is a fundamental system of open neighbourhoods of 0. Thus we have a sequence

$$A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$$

of weak ideals of definition with $A = \lim A/I_n$. In this way we see that condition (1) implies each of the conditions (2) – (5).

Assume (5). First note that the limit topology on $A = \lim B/J_n$ is a linearly topologized, complete topology, see More on Algebra, Section 36. If $f \in A$ maps to zero in B/J_1, then some power maps to zero in B/J_2 as its image in J_1/J_2 is nilpotent, then a further power maps to zero in J_2/J_3, etc, etc. In this way we see the open ideal $\mathrm{Ker}(A \to B/J_1)$ is a weak ideal of definition. Thus A is weakly admissible. In this way we see that (5) implies (2).

It is clear that (4) is a special case of (5) by taking $B = A$. It is clear that (3) is a special case of (2).

Assume A is as in (2). Let E_n be a countable fundamental system of neighbourhoods of 0 in A. Since A is a weakly admissible topological ring we can find open ideals $I_n \subset E_n$. We can also choose a weak ideal of definition $J \subset A$. Then $J \cap I_n$ is a fundamental system of weak ideals of definition of A and we get $X = \mathrm{Spf}(A) = \mathrm{colim}\,\mathrm{Spec}(A/(J \cap I_n))$ which shows that X is a countably indexed affine formal algebraic space. □

Lemma 10.5. *Let S be a scheme. Let X be an affine formal algebraic space. The following are equivalent*

(1) *X is Noetherian,*
(2) *X is adic* and for every closed immersion $T \to X$ with T a scheme, T is Noetherian,*
(3) *X is adic* and for some choice of $X = \mathrm{colim}\, X_\lambda$ as in Definition 9.1 the schemes X_λ are Noetherian, and*
(4) *X is weakly adic and for some choice $X = \mathrm{colim}\, X_\lambda$ as in Definition 9.1 the schemes X_λ are Noetherian.*

Proof. Assume X is Noetherian. Then $X = \mathrm{Spf}(A)$ where A is a Noetherian adic ring. Let $T \to X$ be a closed immersion where T is a scheme. By Lemma 9.6 we see that T is affine and that $T \to \mathrm{Spec}(A)$ is a closed immersion. Since A is Noetherian, we see that T is Noetherian. In this way we see that (1) $\Rightarrow$ (2).

The implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are immediate (see Lemma 10.3).

To prove (3) $\Rightarrow$ (1) write $X = \text{Spf}(A)$ for some adic ring A with finitely generated ideal of definition I. We are also given that the rings A/I_λ are Noetherian for some fundamental system of open ideals I_λ. Since I is open, we can find a λ such that $I_\lambda \subset I$. Then A/I is Noetherian and we conclude that A is Noetherian by Algebra, Lemma 97.5.

To prove (4) $\Rightarrow$ (3) write $X = \text{Spf}(A)$ for some weakly adic ring A. Then A is admissible and has an ideal of definition I and the closure I_2 of I^2 is open, see Lemma 7.2. We are also given that the rings A/I_λ are Noetherian for some fundamental system of open ideals I_λ. Choose a λ such that $I_\lambda \subset I_2$. Then A/I_2 is Noetherian as a quotient of A/I_λ. Hence I/I_2 is a finite A-module. Hence A is an adic ring with a finitely generated ideal of definition by Lemma 7.4. Thus X is adic* and (3) holds. $\square$

11. Formal algebraic spaces

We take a break from our habit of introducing new concepts first for rings, then for schemes, and then for algebraic spaces, by introducing formal algebraic spaces without first introducing formal schemes. The general idea will be that a formal algebraic space is a sheaf in the fppf topology which étale locally is an affine formal scheme in the sense of [BD]. Related material can be found in [Yas09].

In the definition of a formal algebraic space we are going to borrow some terminology from Bootstrap, Sections 3 and 4.

Definition 11.1. Let S be a scheme. We say a sheaf X on $(Sch/S)_{fppf}$ is a *formal algebraic space* if there exist a family of maps $\{X_i \to X\}_{i \in I}$ of sheaves such that

(1) X_i is an affine formal algebraic space,
(2) $X_i \to X$ is representable by algebraic spaces and étale,
(3) $\coprod X_i \to X$ is surjective as a map of sheaves

and X satisfies a set theoretic condition (see Remark 11.5). A *morphism of formal algebraic spaces* over S is a map of sheaves.

Discussion. Sanity check: an affine formal algebraic space is a formal algebraic space. In the situation of the definition the morphisms $X_i \to X$ are representable (by schemes), see Lemma 9.11. By Bootstrap, Lemma 4.6 we could instead of asking $\coprod X_i \to X$ to be surjective as a map of sheaves, require that it be surjective (which makes sense because it is representable).

Our notion of a formal algebraic space is **very general**. In fact, even affine formal algebraic spaces as defined above are very nasty objects.

Lemma 11.2. *Let S be a scheme. If X is a formal algebraic space over S, then the diagonal morphism $\Delta : X \to X \times_S X$ is representable, a monomorphism, locally quasi-finite, locally of finite type, and separated.*

Proof. Suppose given $U \to X$ and $V \to X$ with U, V schemes over S. Then $U \times_X V$ is a sheaf. Choose $\{X_i \to X\}$ as in Definition 11.1. For every i the morphism

$$(U \times_X X_i) \times_{X_i} (V \times_X X_i) = (U \times_X V) \times_X X_i \to U \times_X V$$

is representable and étale as a base change of $X_i \to X$ and its source is a scheme (use Lemmas 9.2 and 9.11). These maps are jointly surjective hence $U \times_X V$ is an algebraic space by Bootstrap, Theorem 10.1. The morphism $U \times_X V \to U \times_S V$ is a monomorphism. It is also locally quasi-finite, because on precomposing with the morphism displayed above we obtain the composition

$$(U \times_X X_i) \times_{X_i} (V \times_X X_i) \to (U \times_X X_i) \times_S (V \times_X X_i) \to U \times_S V$$

which is locally quasi-finite as a composition of a closed immersion (Lemma 9.2) and an étale morphism, see Descent on Spaces, Lemma 19.2. Hence we conclude that $U \times_X V$ is a scheme by Morphisms of Spaces, Proposition 50.2. Thus Δ is representable, see Spaces, Lemma 5.10.

In fact, since we've shown above that the morphisms of schemes $U \times_X V \to U \times_S V$ are aways monomorphisms and locally quasi-finite we conclude that $\Delta : X \to X \times_S X$ is a monomorphism and locally quasi-finite, see Spaces, Lemma 5.11. Then we can use the principle of Spaces, Lemma 5.8 to see that Δ is separated and locally of finite type. Namely, a monomorphism of schemes is separated (Schemes, Lemma 23.3) and a locally quasi-finite morphism of schemes is locally of finite type (follows from the definition in Morphisms, Section 20). □

Lemma 11.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism from an algebraic space over S to a formal algebraic space over S. Then f is representable by algebraic spaces.*

Proof. Let $Z \to Y$ be a morphism where Z is a scheme over S. We have to show that $X \times_Y Z$ is an algebraic space. Choose a scheme U and a surjective étale morphism $U \to X$. Then $U \times_Y Z \to X \times_Y Z$ is representable surjective étale (Spaces, Lemma 5.5) and $U \times_Y Z$ is a scheme by Lemma 11.2. Hence the result by Bootstrap, Theorem 10.1. □

Remark 11.4. Modulo set theoretic issues the category of formal schemes à la EGA (see Section 2) is equivalent to a full subcategory of the category of formal algebraic spaces. To explain this we assume our base scheme is Spec($\mathbf{Z}$). By Lemma 2.2 the functor of points $h_{\mathfrak{X}}$ associated to a formal scheme $\mathfrak{X}$ is a sheaf in the fppf topology. By Lemma 2.1 the assignment $\mathfrak{X} \mapsto h_{\mathfrak{X}}$ is a fully faithful embedding of the category of formal schemes into the category of fppf sheaves. Given a formal scheme $\mathfrak{X}$ we choose an open covering $\mathfrak{X} = \bigcup \mathfrak{X}_i$ with $\mathfrak{X}_i$ affine formal schemes. Then $h_{\mathfrak{X}_i}$ is an affine formal algebraic space by Remark 9.8. The morphisms $h_{\mathfrak{X}_i} \to h_{\mathfrak{X}}$ are representable and open immersions. Thus $\{h_{\mathfrak{X}_i} \to h_{\mathfrak{X}}\}$ is a family as in Definition 11.1 and we see that $h_{\mathfrak{X}}$ is a formal algebraic space.

Remark 11.5. Let S be a scheme and let $(Sch/S)_{fppf}$ be a big fppf site as in Topologies, Definition 7.8. As our set theoretic condition on X in Definitions 9.1 and 11.1 we take: there exist objects U, R of $(Sch/S)_{fppf}$, a morphism $U \to X$ which is a surjection of fppf sheaves, and a morphism $R \to U \times_X U$ which is a surjection of fppf sheaves. In other words, we require our sheaf to be a coequalizer of two maps between representable sheaves. Here are some observations which imply this notion behaves reasonably well:

(1) Suppose $X = \text{colim}_{\lambda \in \Lambda} X_\lambda$ and the system satisfies conditions (1) and (2) of Definition 9.1. Then $U = \coprod_{\lambda \in \Lambda} X_\lambda \to X$ is a surjection of fppf sheaves. Moreover, $U \times_X U$ is a closed subscheme of $U \times_S U$ by Lemma 9.2. Hence if U is representable by an object of $(Sch/S)_{fppf}$ then $U \times_S U$ is too (see Sets, Lemma 9.9) and the set theoretic condition is satisfied. This is always the case if Λ is countable, see Sets, Lemma 9.9.

(2) Sanity check. Let $\{X_i \to X\}_{i \in I}$ be as in Definition 11.1 (with the set theoretic condition as formulated above) and assume that each X_i is actually an affine scheme. Then X is an algebraic space. Namely, if we choose a larger big fppf site $(Sch'/S)_{fppf}$ such that $U' = \coprod X_i$ and $R' = \coprod X_i \times_X X_j$ are representable by objects in it, then $X' = U'/R'$ will be an object of the category of algebraic spaces for this choice. Then an application of Spaces, Lemma 15.2 shows that X is an algebraic space for $(Sch/S)_{fppf}$.
(3) Let $\{X_i \to X\}_{i \in I}$ be a family of maps of sheaves satisfying conditions (1), (2), (3) of Definition 11.1. For each i we can pick $U_i \in \text{Ob}((Sch/S)_{fppf})$ and $U_i \to X_i$ which is a surjection of sheaves. Thus if I is not too large (for example countable) then $U = \coprod U_i \to X$ is a surjection of sheaves and U is representable by an object of $(Sch/S)_{fppf}$. To get $R \in \text{Ob}((Sch/S)_{fppf})$ surjecting onto $U \times_X U$ it suffices to assume the diagonal $\Delta : X \to X \times_S X$ is not too wild, for example this always works if the diagonal of X is quasi-compact, i.e., X is quasi-separated.

12. The reduction

All formal algebraic spaces have an underlying reduced algebraic space as the following lemma demonstrates.

Lemma 12.1. *Let S be a scheme. Let X be a formal algebraic space over S. There exists a reduced algebraic space X_{red} and a representable morphism $X_{red} \to X$ which is a thickening. A morphism $U \to X$ with U a reduced algebraic space factors uniquely through X_{red}.*

Proof. First assume that X is an affine formal algebraic space. Say $X = \text{colim}\, X_\lambda$ as in Definition 9.1. Since the transition morphisms are thickenings, the affine schemes X_λ all have isomorphic reductions X_{red}. The morphism $X_{red} \to X$ is representable and a thickening by Lemma 9.3 and the fact that compositions of thickenings are thickenings. We omit the verification of the universal property (use Schemes, Definition 12.5, Schemes, Lemma 12.7, Properties of Spaces, Definition 12.5, and Properties of Spaces, Lemma 12.4).

Let X and $\{X_i \to X\}_{i \in I}$ be as in Definition 11.1. For each i let $X_{i,red} \to X_i$ be the reduction as constructed above. For $i, j \in I$ the projection $X_{i,red} \times_X X_j \to X_{i,red}$ is an étale (by assumption) morphism of schemes (by Lemma 9.11). Hence $X_{i,red} \times_X X_j$ is reduced (see Descent, Lemma 18.1). Thus the projection $X_{i,red} \times_X X_j \to X_j$ factors through $X_{j,red}$ by the universal property. We conclude that

$$R_{ij} = X_{i,red} \times_X X_j = X_{i,red} \times_X X_{j,red} = X_i \times_X X_{j,red}$$

because the morphisms $X_{i,red} \to X_i$ are injections of sheaves. Set $U = \coprod X_{i,red}$, set $R = \coprod R_{ij}$, and denote $s, t : R \to U$ the two projections. As a sheaf $R = U \times_X U$ and s and t are étale. Then $(t, s) : R \to U$ defines an étale equivalence relation by our observations above. Thus $X_{red} = U/R$ is an algebraic space by Spaces, Theorem 10.5. By construction the diagram

$$\begin{tikzcd} \coprod X_{i,red} \arrow[r] \arrow[d] & \coprod X_i \arrow[d] \\ X_{red} \arrow[r] & X \end{tikzcd}$$

is cartesian. Since the right vertical arrow is étale surjective and the top horizontal arrow is representable and a thickening we conclude that $X_{red} \to X$ is representable by Bootstrap, Lemma 5.2 (to verify the assumptions of the lemma use that a surjective étale morphism is surjective, flat, and locally of finite presentation and use that thickenings are separated and locally quasi-finite). Then we can use Spaces, Lemma 5.6 to conclude that $X_{red} \to X$ is a thickening (use that being a thickening is equivalent to being a surjective closed immersion).

Finally, suppose that $U \to X$ is a morphism with U a reduced algebraic space over S. Then each $X_i \times_X U$ is étale over U and therefore reduced (by our definition of reduced algebraic spaces in Properties of Spaces, Section 7). Then $X_i \times_X U \to X_i$ factors through $X_{i,red}$. Hence $U \to X$ factors through X_{red} because $\{X_i \times_X U \to U\}$ is an étale covering. □

Example 12.2. Let A be a weakly admissible topological ring. In this case we have

$$\mathrm{Spf}(A)_{red} = \mathrm{Spec}(A/\mathfrak{a})$$

where $\mathfrak{a} \subset A$ is the ideal of topologically nilpotent elements. Namely, $\mathfrak{a}$ is a radical ideal (Lemma 4.10) which is open because A is weakly admissible.

Lemma 12.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S which is representable by algebraic spaces and smooth (for example étale). Then $X_{red} = X \times_Y Y_{red}$.*

Proof. (The étale case follows directly from the construction of the underlying reduced algebraic space in the proof of Lemma 12.1.) Assume f is smooth. Observe that $X \times_Y Y_{red} \to Y_{red}$ is a smooth morphism of algebraic spaces. Hence $X \times_Y Y_{red}$ is a reduced algebraic space by Descent on Spaces, Lemma 9.5. Then the univeral property of reduction shows that the canonical morphism $X_{red} \to X \times_Y Y_{red}$ is an isomorphism. □

Lemma 12.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S which is representable by algebraic spaces. Then f is surjective in the sense of Bootstrap, Definition 4.1 if and only if $f_{red} : X_{red} \to Y_{red}$ is a surjective morphism of algebraic spaces.*

Proof. Omitted. □

13. Colimits of algebraic spaces along thickenings

A special type of formal algebraic space is one which can globally be written as a cofiltered colimit of algebraic spaces along thickenings as in the following lemma. We will see later (in Section 18) that any quasi-compact and quasi-separated formal algebraic space is such a global colimit.

Lemma 13.1. *Let S be a scheme. Suppose given a directed set Λ and a system of algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a thickening. Then $X = \mathrm{colim}_{\lambda \in \Lambda} X_\lambda$ is a formal algebraic space over S.*

Proof. Since we take the colimit in the category of fppf sheaves, we see that X is a sheaf. Choose and fix $\lambda \in \Lambda$. Choose an étale covering $\{X_{i,\lambda} \to X_\lambda\}$ where X_i

is an affine scheme over S, see Properties of Spaces, Lemma 6.1. For each $\mu \geq \lambda$ there exists a cartesian diagram

$$\begin{tikzcd} X_{i,\lambda} \arrow[r] \arrow[d] & X_{i,\mu} \arrow[d] \\ X_\lambda \arrow[r] & X_\mu \end{tikzcd}$$

with étale vertical arrows, see More on Morphisms of Spaces, Theorem 8.1 (this also uses that a thickening is a surjective closed immersion which satisfies the conditions of the theorem). Moreover, these diagrams are unique up to unique isomorphism and hence $X_{i,\mu} = X_\mu \times_{X_{\mu'}} X_{i,\mu'}$ for $\mu' \geq \mu$. The morphisms $X_{i,\mu} \to X_{i,\mu'}$ is a thickening as a base change of a thickening. Each $X_{i,\mu}$ is an affine scheme by Limits of Spaces, Proposition 15.2 and the fact that $X_{i,\lambda}$ is affine. Set $X_i = \text{colim}_{\mu \geq \lambda} X_{i,\mu}$. Then X_i is an affine formal algebraic space. The morphism $X_i \to X$ is étale because given an affine scheme U any $U \to X$ factors through X_μ for some $\mu \geq \lambda$ (details omitted). In this way we see that X is a formal algebraic space. $\square$

Let S be a scheme. Let X be a formal algebraic space over S. How does one prove or check that X is a global colimit as in Lemma 13.1? To do this we look for maps $i : Z \to X$ where Z is an algebraic space over S and i is surjective and a closed immersion, in other words, i is a thickening. This makes sense as i is representable by algebraic spaces (Lemma 11.3) and we can use Bootstrap, Definition 4.1 as before.

Example 13.2. Let $(A, \mathfrak{m}, \kappa)$ be a valuation ring, which is (π)-adically complete for some nonzero $\pi \in \mathfrak{m}$. Assume also that $\mathfrak{m}$ is not finitely generated. An example is $A = \mathcal{O}_{\mathbf{C}_p}$ and $\pi = p$ where $\mathcal{O}_{\mathbf{C}_p}$ is the ring of integers of the field of p-adic complex numbers $\mathbf{C}_p$ (this is the completion of the algebraic closure of $\mathbf{Q}_p$). Another example is

$$A = \left\{ \sum\nolimits_{\alpha \in \mathbf{Q},\ \alpha \geq 0} a_\alpha t^\alpha \middle| \begin{matrix} a_\alpha \in \kappa \text{ and for all } n \text{ there are only a} \\ \text{finite number of nonzero } a_\alpha \text{ with } \alpha \leq n \end{matrix} \right\}$$

and $\pi = t$. Then $X = \text{Spf}(A)$ is an affine formal algebraic space and $\text{Spec}(\kappa) \to X$ is a thickening which corresponds to the weak ideal of definition $\mathfrak{m} \subset A$ which is however not an ideal of definition.

Remark 13.3 (Weak ideals of definition)**.** Let $\mathfrak{X}$ be a formal scheme in the sense of McQuillan, see Remark 2.3. An *weak ideal of definition* for $\mathfrak{X}$ is an ideal sheaf $\mathcal{I} \subset \mathcal{O}_\mathfrak{X}$ such that for all $\mathfrak{U} \subset \mathfrak{X}$ affine formal open subscheme the ideal $\mathcal{I}(\mathfrak{U}) \subset \mathcal{O}_\mathfrak{X}(\mathfrak{U})$ is a weak ideal of definition of the weakly admissible topological ring $\mathcal{O}_\mathfrak{X}(\mathfrak{U})$. It suffices to check the condition on the members of an affine open covering. There is a one-to-one correspondence

$$\{\text{weak ideals of definition for } \mathfrak{X}\} \leftrightarrow \{\text{thickenings } i : Z \to h_\mathfrak{X} \text{ as above}\}$$

This correspondence associates to $\mathcal{I}$ the scheme $Z = (\mathfrak{X}, \mathcal{O}_\mathfrak{X}/\mathcal{I})$ together with the obvious morphism to $\mathfrak{X}$. A *fundamental system of weak ideals of definition* is a collection of weak ideals of definition $\mathcal{I}_\lambda$ such that on every affine open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$ the ideals

$$I_\lambda = \mathcal{I}_\lambda(\mathfrak{U}) \subset A = \Gamma(\mathfrak{U}, \mathcal{O}_\mathfrak{X})$$

form a fundamental system of weak ideals of definition of the weakly admissible topological ring A. It suffices to check on the members of an affine open covering. We conclude that the formal algebraic space $h_{\mathfrak{X}}$ associated to the McQuillan formal scheme $\mathfrak{X}$ is a colimit of schemes as in Lemma 13.1 if and only if there exists a fundamental system of weak ideals of definition for $\mathfrak{X}$.

Remark 13.4 (Ideals of definition)**.** Let $\mathfrak{X}$ be a formal scheme à la EGA. An *ideal of definition* for $\mathfrak{X}$ is an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$ such that for all $\mathfrak{U} \subset \mathfrak{X}$ affine formal open subscheme the ideal $\mathcal{I}(\mathfrak{U}) \subset \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ is an ideal of definition of the admissible topological ring $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$. It suffices to check the condition on the members of an affine open covering. We do **not** get the same correspondence between ideals of definition and thickenings $Z \to h_{\mathfrak{X}}$ as in Remark 13.3; an example is given in Example 13.2. A *fundamental system of ideals of definition* is a collection of ideals of definition $\mathcal{I}_\lambda$ such that on every affine open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$ the ideals

$$I_\lambda = \mathcal{I}_\lambda(\mathfrak{U}) \subset A = \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$$

form a fundamental system of ideals of definition of the admissible topological ring A. It suffices to check on the members of an affine open covering. Suppose that $\mathfrak{X}$ is quasi-compact and that $\{\mathcal{I}_\lambda\}_{\lambda \in \Lambda}$ is a fundamental system of weak ideals of definition. If A is an admissible topological ring then all sufficiently small open ideals are ideals of definition (namely any open ideal contained in an ideal of definition is an ideal of definition). Thus since we only need to check on the finitely many members of an affine open covering we see that $\mathcal{I}_\lambda$ is an ideal of definition for λ sufficiently large. Using the discussion in Remark 13.3 we conclude that the formal algebraic space $h_{\mathfrak{X}}$ associated to the quasi-compact formal scheme $\mathfrak{X}$ à la EGA is a colimit of schemes as in Lemma 13.1 if and only if there exists a fundamental system of ideals of definition for $\mathfrak{X}$.

14. Completion along a closed subset

Our notion of a formal algebraic space is well adapted to taking the completion along a closed subset.

Lemma 14.1. *Let S be a scheme. Let X be an affine scheme over S. Let $T \subset |X|$ be a closed subset. Then the functor*

$$(Sch/S)_{fppf} \longrightarrow Sets, \quad U \longmapsto \{f : U \to X \mid f(|U|) \subset T\}$$

is a McQuillan affine formal algebraic space.

Proof. Say $X = \text{Spec}(A)$ and T corresponds to the radical ideal $I \subset A$. Let $U = \text{Spec}(B)$ be an affine scheme over S and let $f : U \to X$ be an element of $F(U)$. Then f corresponds to a ring map $\varphi : A \to B$ such that every prime of B contains $\varphi(I)B$. Thus every element of $\varphi(I)$ is nilpotent in B, see Algebra, Lemma 17.2. Setting $J = \text{Ker}(\varphi)$ we conclude that I/J is a locally nilpotent ideal in A/J. Equivalently, $V(J) = V(I) = T$. In other words, the functor of the lemma equals $\text{colim}\, \text{Spec}(A/J)$ where the colimit is over the collection of ideals J with $V(J) = T$. Thus our functor is an affine formal algebraic space. It is McQuillan (Definition 9.7) because the maps $A \to A/J$ are surjective and hence $A^\wedge = \lim A/J \to A/J$ is surjective, see Lemma 9.6. $\square$

Lemma 14.2. *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. Then the functor*

$$(Sch/S)_{fppf} \longrightarrow Sets, \quad U \longmapsto \{f : U \to X \mid f(|U|) \subset T\}$$

is a formal algebraic space.

Proof. Denote F the functor. Let $\{U_i \to U\}$ be an fppf covering. Then $\coprod |U_i| \to |U|$ is surjective. Since X is an fppf sheaf, it follows that F is an fppf sheaf.

Let $\{g_i : X_i \to X\}$ be an étale covering such that X_i is affine for all i, see Properties of Spaces, Lemma 6.1. The morphisms $F \times_X X_i \to F$ are étale (see Spaces, Lemma 5.5) and the map $\coprod F \times_X X_i \to F$ is a surjection of sheaves. Thus it suffices to prove that $F \times_X X_i$ is an affine formal algebraic space. A U-valued point of $F \times_X X_i$ is a morphism $U \to X_i$ whose image is contained in the closed subset $g_i^{-1}(T) \subset |X_i|$. Thus this follows from Lemma 14.1. $\square$

Definition 14.3. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. The formal algebraic space of Lemma 14.2 is called the *completion of X along T*.

In [DG67, Chapter I, Section 10.8] the notation $X_{/T}$ is used to denote the completion and we will occasionally use this notation as well. Let $f : X \to X'$ be a morphism of algebraic spaces over a scheme S. Suppose that $T \subset |X|$ and $T' \subset |X'|$ are closed subsets such that $|f|(T) \subset T'$. Then it is clear that f defines a morphism of formal algebraic spaces

$$X_{/T} \longrightarrow X'_{/T'}$$

between the completions.

Lemma 14.4. *Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $T \subset |X|$ be a closed subset and let $T' = |f|^{-1}(T) \subset |X'|$. Then*

$$\begin{tikzcd} X'_{/T'} \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ X_{/T} \arrow[r] & X \end{tikzcd}$$

is a cartesian diagram of sheaves. In particular, the morphism $X'_{/T'} \to X_{/T}$ is representable by algebraic spaces.

Proof. Namely, suppose that $Y \to X$ is a morphism from a scheme into X such that $|Y|$ maps into T. Then $Y \times_X X' \to X$ is a morphism of algebraic spaces such that $|Y \times_X X'|$ maps into T'. Hence the functor $Y \times_{X_{/T}} X'_{/T'}$ is represented by $Y \times_X X'$ and we see that the lemma holds. $\square$

Lemma 14.5. *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. The reduction $(X_{/T})_{red}$ of the completion $X_{/T}$ of X along T is the reduced induced closed subspace Z of X corresponding to T.*

Proof. It follows from Lemma 12.1, Properties of Spaces, Definition 12.5 (which uses Properties of Spaces, Lemma 12.3 to construct Z), and the definition of $X_{/T}$ that Z and $(X_{/T})_{red}$ are reduced algebraic spaces characterized the same mapping property: a morphism $g : Y \to X$ whose source is a reduced algebraic space factors through them if and only if $|Y|$ maps into $T \subset |X|$. $\square$

Lemma 14.6. *Let S be a scheme. Let $X = \mathrm{Spec}(A)$ be an affine scheme over S. Let $T \subset X$ be a closed subset. Let $X_{/T}$ be the formal completion of X along T.*

1. *If $X \setminus T$ is quasi-compact, i.e., T is constructible, then $X_{/T}$ is adic*.*
2. *If $T = V(I)$ for some finitely generated ideal $I \subset A$, then $X_{/T} = Spf(A^\wedge)$ where $A^\wedge$ is the I-adic completion of A.*
3. *If X is Noetherian, then $X_{/T}$ is Noetherian.*

Proof. By Algebra, Lemma 29.1 if (1) holds, then we can find an ideal $I \subset A$ as in (2). If (3) holds then we can find an ideal $I \subset A$ as in (2). Moreover, completions of Noetherian rings are Noetherian by Algebra, Lemma 97.6. All in all we see that it suffices to prove (2).

Proof of (2). Let $I = (f_1, \ldots, f_r) \subset A$ cut out T. If $Z = \mathrm{Spec}(B)$ is an affine scheme and $g : Z \to X$ is a morphism with $g(Z) \subset T$ (set theoretically), then $g^\sharp(f_i)$ is nilpotent in B for each i. Thus I^n maps to zero in B for some n. Hence we see that $X_{/T} = \mathrm{colim}\,\mathrm{Spec}(A/I^n) = \mathrm{Spf}(A^\wedge)$. $\square$

The following lemma is due to Ofer Gabber.

Lemma 14.7. *Let S be a scheme. Let $X = \mathrm{Spec}(A)$ be an affine scheme over S. Let $T \subset X$ be a closed subscheme.*

1. *If the formal completion $X_{/T}$ is countably indexed and there exist countably many $f_1, f_2, f_3, \ldots \in A$ such that $T = V(f_1, f_2, f_3, \ldots)$, then $X_{/T}$ is adic*.*
2. *The conclusion of (1) is wrong if we omit the assumption that T can be cut out by countably many functions in X.*

Proof. The assumption that $X_{/T}$ is countably indexed means that there exists a sequence of ideals
$$A \supset J_1 \supset J_2 \supset J_3 \supset \ldots$$
with $V(J_n) = T$ such that every ideal $J \subset A$ with $V(J) = T$ there exists an n such that $J \supset J_n$.

To construct an example for (2) let ω_1 be the first uncountable ordinal. Let k be a field and let A be the k-algebra generated by x_α, $\alpha \in \omega_1$ and $y_{\alpha\beta}$ with $\alpha \in \beta \in \omega_1$ subject to the relations $x_\alpha = y_{\alpha\beta}x_\beta$. Let $T = V(x_\alpha)$. Let $J_n = (x_\alpha^n)$. If $J \subset A$ is an ideal such that $V(J) = T$, then $x_\alpha^{n_\alpha} \in J$ for some $n_\alpha \geq 1$. One of the sets $\{\alpha \mid n_\alpha = n\}$ must be unbounded in ω_1. Then the relations imply that $J_n \subset J$.

To see that (2) holds it now suffices to show that $A^\wedge = \lim A/J_n$ is not a ring complete with respect to a finitely generated ideal. For $\gamma \in \omega_1$ let A_γ be the quotient of A by the ideal generated by x_α, $\alpha \in \gamma$ and $y_{\alpha\beta}$, $\alpha \in \gamma$. As A/J_1 is reduced, every topologically nilpotent element f of $\lim A/J_n$ is in $J_1^\wedge = \lim J_1/J_n$. This means f is an infinite series involving only a countable number of generators. Hence f dies in $A_\gamma^\wedge = \lim A_\gamma/J_nA_\gamma$ for some γ. Note that $A^\wedge \to A_\gamma^\wedge$ is continuous and open by Lemma 4.5. If the topology on $A^\wedge$ was I-adic for some finitely generated ideal $I \subset A^\wedge$, then I would go to zero in some $A_\gamma^\wedge$. This would mean that $A_\gamma^\wedge$ is discrete, which is not the case as there is a surjective continuous and open (by Lemma 4.5) map $A_\gamma^\wedge \to k[[t]]$ given by $x_\alpha \mapsto t$, $y_{\alpha\beta} \mapsto 1$ for $\gamma = \alpha$ or $\gamma \in \alpha$.

Before we prove (1) we first prove the following: If $I \subset A^\wedge$ is a finitely generated ideal whose closure $\bar{I}$ is open, then $I = \bar{I}$. Since $V(J_n^2) = T$ there exists an m such that $J_n^2 \supset J_m$. Thus, we may assume that $J_n^2 \supset J_{n+1}$ for all n by passing to a subsequence. Set $J_n^\wedge = \lim_{k \geq n} J_n/J_k \subset A^\wedge$. Since the closure $\bar{I} = \bigcap (I + J_n^\wedge)$ (Lemma 4.2) is open we see that there exists an m such that $I + J_n^\wedge \supset J_m^\wedge$ for all $n \geq m$. Fix such an m. We have

$$J_{n-1}^\wedge I + J_{n+1}^\wedge \supset J_{n-1}^\wedge (I + J_{n+1}^\wedge) \supset J_{n-1}^\wedge J_m^\wedge$$

for all $n \geq m+1$. Namely, the first inclusion is trivial and the second was shown above. Because $J_{n-1}J_m \supset J_{n-1}^2 \supset J_n$ these inclusions show that the image of J_n in $A^\wedge$ is contained in the ideal $J_{n-1}^\wedge I + J_{n+1}^\wedge$. Because this ideal is open we conclude that

$$J_{n-1}^\wedge I + J_{n+1}^\wedge \supset J_n^\wedge.$$

Say $I = (g_1, \ldots, g_t)$. Pick $f \in J_{m+1}^\wedge$. Using the last displayed inclusion, valid for all $n \geq m+1$, we can write by induction on $c \geq 0$

$$f = \sum f_{i,c} g_i \mod J_{m+1+c}^\wedge$$

with $f_{i,c} \in J_m^\wedge$ and $f_{i,c} \equiv f_{i,c-1} \bmod J_{m+c}^\wedge$. It follows that $IJ_m^\wedge \supset J_{m+1}^\wedge$. Combined with $I + J_{m+1}^\wedge \supset J_m^\wedge$ we conclude that I is open.

Proof of (1). Assume $T = V(f_1, f_2, f_3, \ldots)$. Let $I_m \subset A^\wedge$ be the ideal generated by $f_1, \ldots, f_m$. We distinguish two cases.

Case I: For some m the closure of I_m is open. Then I_m is open by the result of the previous paragraph. For any n we have $(J_n)^2 \supset J_{n+1}$ by design, so the closure of $(J_n^\wedge)^2$ contains $J_{n+1}^\wedge$ and thus is open. Taking n large, it follows that the closure of the product of any two open ideals in $A^\wedge$ is open. Let us prove I_m^k is open for $k \geq 1$ by induction on k. The case $k = 1$ is our hypothesis on m in Case I. For $k > 1$, suppose I_m^{k-1} is open. Then $I_m^k = I_m^{k-1} \cdot I_m$ is the product of two open ideals and hence has open closure. But then since I_m^k is finitely generated it follows that I_m^k is open by the previous paragraph (applied to $I = I_m^k$), so we can continue the induction on k. As each element of I_m is topologically nilpotent, we conclude that I_m is an ideal of definition which proves that $A^\wedge$ is adic with a finitely generated ideal of definition, i.e., $X_{/T}$ is adic*.

Case II. For all m the closure $\bar{I}_m$ of I_m is not open. Then the topology on $A^\wedge/\bar{I}_m$ is not discrete. This means we can pick $\phi(m) \geq m$ such that

$$\mathrm{Im}(J_{\phi(m)} \to A/(f_1, \ldots, f_m)) \neq \mathrm{Im}(J_{\phi(m)+1} \to A/(f_1, \ldots, f_m))$$

To see this we have used that $A^\wedge/(\bar{I}_m + J_n^\wedge) = A/((f_1, \ldots, f_m) + J_n)$. Choose exponents $e_i > 0$ such that $f_i^{e_i} \in J_{\phi(m)+1}$ for $0 < m < i$. Let $J = (f_1^{e_1}, f_2^{e_2}, f_3^{e_3}, \ldots)$. Then $V(J) = T$. We claim that $J \not\supset J_n$ for all n which is a contradiction proving Case II does not occur. Namely, the image of J in $A/(f_1, \ldots, f_m)$ is contained in the image of $J_{\phi(m)+1}$ which is properly contained in the image of J_m. □

15. Fibre products

Obligatory section about fibre products of formal algebraic spaces.

Lemma 15.1. *Let S be a scheme. Let $\{X_i \to X\}_{i \in I}$ be a family of maps of sheaves on $(Sch/S)_{fppf}$. Assume (a) X_i is a formal algebraic space over S, (b) $X_i \to X$ is representable by algebraic spaces and étale, and (c) $\coprod X_i \to X$ is a surjection of sheaves. Then X is a formal algebraic space over S.*

Proof. For each i pick $\{X_{ij} \to X_i\}_{j \in J_i}$ as in Definition 11.1. Then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a family as in Definition 11.1 for X. $\square$

Lemma 15.2. *Let S be a scheme. Let X, Y be formal algebraic spaces over S and let Z be a sheaf whose diagonal is representable by algebraic spaces. Let $X \to Z$ and $Y \to Z$ be maps of sheaves. Then $X \times_Z Y$ is a formal algebraic space.*

Proof. Choose $\{X_i \to X\}$ and $\{Y_j \to Y\}$ as in Definition 11.1. Then $\{X_i \times_Z Y_j \to X \times_Z Y\}$ is a family of maps which are representable by algebraic spaces and étale. Thus Lemma 15.1 tells us it suffices to show that $X \times_Z Y$ is a formal algebraic space when X and Y are affine formal algebraic spaces.

Assume X and Y are affine formal algebraic spaces. Write $X = \operatorname{colim} X_\lambda$ and $Y = \operatorname{colim} Y_\mu$ as in Definition 9.1. Then $X \times_Z Y = \operatorname{colim} X_\lambda \times_Z Y_\mu$. Each $X_\lambda \times_Z Y_\mu$ is an algebraic space. For $\lambda \leq \lambda'$ and $\mu \leq \mu'$ the morphism

$$X_\lambda \times_Z Y_\mu \to X_\lambda \times_Z Y_{\mu'} \to X_{\lambda'} \times_Z Y_{\mu'}$$

is a thickening as a composition of base changes of thickenings. Thus we conclude by applying Lemma 13.1. $\square$

Lemma 15.3. *Let S be a scheme. The category of formal algebraic spaces over S has fibre products.*

Proof. Special case of Lemma 15.2 because formal algebraic spaces have representable diagonals, see Lemma 11.2. $\square$

Lemma 15.4. *Let S be a scheme. Let $X \to Z$ and $Y \to Z$ be morphisms of formal algebraic spaces over S. Then $(X \times_Z Y)_{red} = (X_{red} \times_{Z_{red}} Y_{red})_{red}$.*

Proof. This follows from the universal property of the reduction in Lemma 12.1. $\square$

We have already proved the following lemma (without knowing that fibre products exist).

Lemma 15.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The diagonal morphism $\Delta : X \to X \times_Y X$ is representable (by schemes), a monomorphism, locally quasi-finite, locally of finite type, and separated.*

Proof. Let T be a scheme and let $T \to X \times_Y X$ be a morphism. Then

$$T \times_{(X \times_Y X)} X = T \times_{(X \times_S X)} X$$

Hence the result follows immediately from Lemma 11.2. $\square$

16. Separation axioms for formal algebraic spaces

This section is about "absolute" separation conditions on formal algebraic spaces. We will discuss separation conditions for morphisms of formal algebraic spaces later.

Lemma 16.1. *Let S be a scheme. Let X be a formal algebraic space over S. The following are equivalent*

1. *the reduction of X (Lemma 12.1) is a quasi-separated algebraic space,*
2. *for $U \to X$, $V \to X$ with U, V quasi-compact schemes the fibre product $U \times_X V$ is quasi-compact,*
3. *for $U \to X$, $V \to X$ with U, V affine the fibre product $U \times_X V$ is quasi-compact.*

Proof. Observe that $U \times_X V$ is a scheme by Lemma 11.2. Let $U_{red}, V_{red}, X_{red}$ be the reduction of U, V, X. Then

$$U_{red} \times_{X_{red}} V_{red} = U_{red} \times_X V_{red} \to U \times_X V$$

is a thickening of schemes. From this the equivalence of (1) and (2) is clear, keeping in mind the analogous lemma for algebraic spaces, see Properties of Spaces, Lemma 3.3. We omit the proof of the equivalence of (2) and (3). □

Lemma 16.2. *Let S be a scheme. Let X be a formal algebraic space over S. The following are equivalent*

1. *the reduction of X (Lemma 12.1) is a separated algebraic space,*
2. *for $U \to X$, $V \to X$ with U, V affine the fibre product $U \times_X V$ is affine and*
$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(U \times_X V)$$
is surjective.

Proof. If (2) holds, then X_{red} is a separated algebraic space by applying Properties of Spaces, Lemma 3.3 to morphisms $U \to X_{red}$ and $V \to X_{red}$ with U, V affine and using that $U \times_{X_{red}} V = U \times_X V$.

Assume (1). Let $U \to X$ and $V \to X$ be as in (2). Observe that $U \times_X V$ is a scheme by Lemma 11.2. Let $U_{red}, V_{red}, X_{red}$ be the reduction of U, V, X. Then

$$U_{red} \times_{X_{red}} V_{red} = U_{red} \times_X V_{red} \to U \times_X V$$

is a thickening of schemes. It follows that $(U \times_X V)_{red} = (U_{red} \times_{X_{red}} V_{red})_{red}$. In particular, we see that $(U \times_X V)_{red}$ is an affine scheme and that

$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}((U \times_X V)_{red})$$

is surjective, see Properties of Spaces, Lemma 3.3. Then $U \times_X V$ is affine by Limits of Spaces, Proposition 15.2. On the other hand, the morphism $U \times_X V \to U \times V$ of affine schemes is the composition

$$U \times_X V = X \times_{(X \times_S X)} (U \times_S V) \to U \times_S V \to U \times V$$

The first morphism is a monomorphism and locally of finite type (Lemma 11.2). The second morphism is an immersion (Schemes, Lemma 21.9). Hence the composition is a monomorphism which is locally of finite type. On the other hand, the composition is integral as the map on underlying reduced affine schemes is a closed immersion by the above and hence universally closed (use Morphisms, Lemma 44.7). Thus the ring map

$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(U \times_X V)$$

is an epimorphism which is integral of finite type hence finite hence surjective (use Morphisms, Lemma 44.4 and Algebra, Lemma 107.6). □

Definition 16.3. Let S be a scheme. Let X be a formal algebraic space over S. We say

1. X is *quasi-separated* if the equivalent conditions of Lemma 16.1 are satisfied.
2. X is *separated* if the equivalent conditions of Lemma 16.2 are satisfied.

The following lemma implies in particular that the completed tensor product of weakly admissible topological rings is a weakly admissible topological ring.

Lemma 16.4. *Let S be a scheme. Let $X \to Z$ and $Y \to Z$ be morphisms of formal algebraic spaces over S. Assume Z separated.*

1. *If X and Y are affine formal algebraic spaces, then so is $X \times_Z Y$.*
2. *If X and Y are McQuillan affine formal algebraic spaces, then so is $X \times_Z Y$.*
3. *If X, Y, and Z are McQuillan affine formal algebraic spaces corresponding to the weakly admissible topological S-algebras A, B, and C, then $X \times_Z Y$ corresponds to $A \widehat{\otimes}_C B$.*

Proof. Write $X = \text{colim}\, X_\lambda$ and $Y = \text{colim}\, Y_\mu$ as in Definition 9.1. Then $X \times_Z Y = \text{colim}\, X_\lambda \times_Z Y_\mu$. Since Z is separated the fibre products are affine, hence we see that (1) holds. Assume X and Y corresponds to the weakly admissible topological S-algebras A and B and $X_\lambda = \text{Spec}(A/I_\lambda)$ and $Y_\mu = \text{Spec}(B/J_\mu)$. Then

$$X_\lambda \times_Z Y_\mu \to X_\lambda \times Y_\mu \to \text{Spec}(A \otimes B)$$

is a closed immersion. Thus one of the conditions of Lemma 9.6 holds and we conclude that $X \times_Z Y$ is McQuillan. If also Z is McQuillan corresponding to C, then

$$X_\lambda \times_Z Y_\mu = \text{Spec}(A/I_\lambda \otimes_C B/J_\mu)$$

hence we see that the weakly admissible topological ring corresponding to $X \times_Z Y$ is the completed tensor product (see Definition 4.7). $\square$

Lemma 16.5. *Let S be a scheme. Let X be a formal algebraic space over S. Let $U \to X$ be a morphism where U is a separated algebraic space over S. Then $U \to X$ is separated.*

Proof. The statement makes sense because $U \to X$ is representable by algebraic spaces (Lemma 11.3). Let T be a scheme and $T \to X$ a morphism. We have to show that $U \times_X T \to T$ is separated. Since $U \times_X T \to U \times_S T$ is a monomorphism, it suffices to show that $U \times_S T \to T$ is separated. As this is the base change of $U \to S$ this follows. We used in the argument above: Morphisms of Spaces, Lemmas 4.4, 4.8, 10.3, and 4.11. $\square$

17. Quasi-compact formal algebraic spaces

Here is the characterization of quasi-compact formal algebraic spaces.

Lemma 17.1. *Let S be a scheme. Let X be a formal algebraic space over S. The following are equivalent*

1. *the reduction of X (Lemma 12.1) is a quasi-compact algebraic space,*
2. *we can find $\{X_i \to X\}_{i \in I}$ as in Definition 11.1 with I finite,*
3. *there exists a morphism $Y \to X$ representable by algebraic spaces which is étale and surjective and where Y is an affine formal algebraic space.*

Proof. Omitted. $\square$

Definition 17.2. Let S be a scheme. Let X be a formal algebraic space over S. We say X is *quasi-compact* if the equivalent conditions of Lemma 17.1 are satisfied.

Lemma 17.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

1. *the induced map $f_{red} : X_{red} \to Y_{red}$ between reductions (Lemma 12.1) is a quasi-compact morphism of algebraic spaces,*

(2) *for every quasi-compact scheme T and morphism $T \to Y$ the fibre product $X \times_Y T$ is a quasi-compact formal algebraic space,*
(3) *for every affine scheme T and morphism $T \to Y$ the fibre product $X \times_Y T$ is a quasi-compact formal algebraic space, and*
(4) *there exists a covering $\{Y_j \to Y\}$ as in Definition 11.1 such that each $X \times_Y Y_j$ is a quasi-compact formal algebraic space.*

Proof. Omitted. $\square$

Definition 17.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. We say f is *quasi-compact* if the equivalent conditions of Lemma 17.3 are satisfied.

This agrees with the already existing notion when the morphism is representable by algebraic spaces (and in particular when it is representable).

Lemma 17.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S which is representable by algebraic spaces. Then f is quasi-compact in the sense of Definition 17.4 if and only if f is quasi-compact in the sense of Bootstrap, Definition 4.1.*

Proof. This is immediate from the definitions and Lemma 17.3. $\square$

18. Quasi-compact and quasi-separated formal algebraic spaces

The following result is due to Yasuda, see [Yas09, Proposition 3.32].

Lemma 18.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated formal algebraic space over S. Then $X = \operatorname{colim} X_\lambda$ for a system of algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over a directed set Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a thickening.*

Proof. By Lemma 17.1 we may choose an affine formal algebraic space Y and a representable surjective étale morphism $Y \to X$. Write $Y = \operatorname{colim} Y_\lambda$ as in Definition 9.1.

Pick $\lambda \in \Lambda$. Then $Y_\lambda \times_X Y$ is a scheme by Lemma 9.11. The reduction (Lemma 12.1) of $Y_\lambda \times_X Y$ is equal to the reduction of $Y_{red} \times_{X_{red}} Y_{red}$ which is quasi-compact as X is quasi-separated and Y_{red} is affine. Therefore $Y_\lambda \times_X Y$ is a quasi-compact scheme. Hence there exists a $\mu \geq \lambda$ such that $\text{pr}_2 : Y_\lambda \times_X Y \to Y$ factors through Y_μ, see Lemma 9.4. Let Z_λ be the scheme theoretic image of the morphism $\text{pr}_2 : Y_\lambda \times_X Y \to Y_\mu$. This is independent of the choice of μ and we can and will think of $Z_\lambda \subset Y$ as the scheme theoretic image of the morphism $\text{pr}_2 : Y_\lambda \times_X Y \to Y$. Observe that Z_λ is also equal to the scheme theoretic image of the morphism $\text{pr}_1 : Y \times_X Y_\lambda \to Y$ since this is isomorphic to the morphism used to define Z_λ. We claim that $Z_\lambda \times_X Y = Y \times_X Z_\lambda$ as subfunctors of $Y \times_X Y$. Namely, since $Y \to X$ is étale we see that $Z_\lambda \times_X Y$ is the scheme theoretic image of the morphism

$$\text{pr}_{13} = \text{pr}_1 \times \text{id}_Y : Y \times_X Y_\lambda \times_X Y \longrightarrow Y \times_X Y$$

by Morphisms of Spaces, Lemma 16.3. By the same token, $Y \times_X Z_\lambda$ is the scheme theoretic image of the morphism

$$\text{pr}_{13} = \text{id}_Y \times \text{pr}_2 : Y \times_X Y_\lambda \times_X Y \longrightarrow Y \times_X Y$$

The claim follows. Then $R_\lambda = Z_\lambda \times_X Y = Y \times_X Z_\lambda$ together with the morphism $R_\lambda \to Z_\lambda \times_S Z_\lambda$ defines an étale equivalence relation. In this way we obtain an algebraic space $X_\lambda = Z_\lambda / R_\lambda$. By construction the diagram

$$\begin{tikzcd} Z_\lambda \arrow[r] \arrow[d] & Y \arrow[d] \\ X_\lambda \arrow[r] & X \end{tikzcd}$$

is cartesian (because X is the coequalizer of the two projections $R = Y \times_X Y \to Y$, because $Z_\lambda \subset Y$ is R-invariant, and because R_λ is the restriction of R to Z_λ). Hence $X_\lambda \to X$ is representable and a closed immersion, see Spaces, Lemma 11.5. On the other hand, since $Y_\lambda \subset Z_\lambda$ we see that $(X_\lambda)_{red} = X_{red}$, in other words, $X_\lambda \to X$ is a thickening. Finally, we claim that

$$X = \operatorname{colim} X_\lambda$$

We have $Y \times_X X_\lambda = Z_\lambda \supset Y_\lambda$. Every morphism $T \to X$ where T is a scheme over S lifts étale locally to a morphism into Y which lifts étale locally into a morphism into some Y_λ. Hence $T \to X$ lifts étale locally on T to a morphism into X_λ. This finishes the proof. $\square$

Remark 18.2. In this remark we translate the statement and proof of Lemma 18.1 into the language of formal schemes à la EGA. Looking at Remark 13.4 we see that the lemma can be translated as follows

(*) Every quasi-compact and quasi-separated formal scheme has a fundamental system of ideals of definition.

To prove this we first use the induction principle (reformulated for quasi-compact and quasi-separated formal schemes) of Cohomology of Schemes, Lemma 4.1 to reduce to the following situation: $\mathfrak{X} = \mathfrak{U} \cup \mathfrak{V}$ with $\mathfrak{U}$, $\mathfrak{V}$ open formal subschemes, with $\mathfrak{V}$ affine, and the result is true for $\mathfrak{U}$, $\mathfrak{V}$, and $\mathfrak{U}\cap\mathfrak{V}$. Pick any ideals of definition $\mathcal{I} \subset \mathcal{O}_\mathfrak{U}$ and $\mathcal{J} \subset \mathcal{O}_\mathfrak{V}$. By our assumption that we have a fundamental system of ideals of definition on $\mathfrak{U}$ and $\mathfrak{V}$ and because $\mathfrak{U} \cap \mathfrak{V}$ is quasi-compact, we can find ideals of definition $\mathcal{I}' \subset \mathcal{I}$ and $\mathcal{J}' \subset \mathcal{J}$ such that

$$\mathcal{I}'|_{\mathfrak{U}\cap\mathfrak{V}} \subset \mathcal{J}|_{\mathfrak{U}\cap\mathfrak{V}} \quad \text{and} \quad \mathcal{J}'|_{\mathfrak{U}\cap\mathfrak{V}} \subset \mathcal{I}|_{\mathfrak{U}\cap\mathfrak{V}}$$

Let $U \to U' \to \mathfrak{U}$ and $V \to V' \to \mathfrak{V}$ be the closed immersions determined by the ideals of definition $\mathcal{I}' \subset \mathcal{I} \subset \mathcal{O}_\mathfrak{U}$ and $\mathcal{J}' \subset \mathcal{J} \subset \mathcal{O}_\mathfrak{V}$. Let $\mathfrak{U} \cap V$ denote the open subscheme of V whose underlying topological space is that of $\mathfrak{U}\cap\mathfrak{V}$. By our choice of $\mathcal{I}'$ there is a factorization $\mathfrak{U} \cap V \to U'$. We define similarly $U \cap \mathfrak{V}$ which factors through V'. Then we consider

$$Z_U = \text{scheme theoretic image of } U \amalg (\mathfrak{U} \cap V) \longrightarrow U'$$

and

$$Z_V = \text{scheme theoretic image of } (U \cap \mathfrak{V}) \amalg V \longrightarrow V'$$

Since taking scheme theoretic images of quasi-compact morphisms commutes with restriction to opens (Morphisms, Lemma 6.3) we see that $Z_U \cap \mathfrak{V} = \mathfrak{U} \cap Z_V$. Thus Z_U and Z_V glue to a scheme Z which comes equipped with a morphism $Z \to \mathfrak{X}$. Analogous to the discussion in Remark 13.3 we see that Z corresponds to a weak ideal of definition $\mathcal{I}_Z \subset \mathcal{O}_\mathfrak{X}$. Note that $Z_U \subset U'$ and that $Z_V \subset V'$. Thus the collection of all $\mathcal{I}_Z$ constructed in this manner forms a fundamental system of weak ideals of definition. Hence a subfamily gives a fundamental system of ideals of definition, see Remark 13.4.

Lemma 18.3. *Let S be a scheme. Let X be a formal algebraic space over S. Then X is an affine formal algebraic space if and only if its reduction X_{red} (Lemma 12.1) is affine.*

Proof. By Lemmas 16.1 and 17.1 and Definitions 16.3 and 17.2 we see that X is quasi-compact and quasi-separated. By Yasuda's lemma (Lemma 18.1) we can write $X = \mathrm{colim}\, X_\lambda$ as a filtered colimit of thickenings of algebraic spaces. However, each X_λ is affine by Limits of Spaces, Lemma 15.3 because $(X_\lambda)_{red} = X_{red}$. Hence X is an affine formal algebraic space by definition. □

19. Morphisms representable by algebraic spaces

Let $f : X \to Y$ be a morphism of formal algebraic spaces which is representable by algebraic spaces. For these types of morphisms we have a lot of theory at our disposal, thanks to the work done in the chapters on algebraic spaces.

Lemma 19.1. *The composition of morphisms representable by algebraic spaces is representable by algebraic spaces. The same holds for representable (by schemes).*

Proof. See Bootstrap, Lemma 3.8. □

Lemma 19.2. *A base change of a morphism representable by algebraic spaces is representable by algebraic spaces. The same holds for representable (by schemes).*

Proof. See Bootstrap, Lemma 3.3. □

Lemma 19.3. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. If $g \circ f : X \to Z$ is representable by algebraic spaces, then $f : X \to Y$ is representable by algebraic spaces.*

Proof. Note that the diagonal of $Y \to Z$ is representable by Lemma 15.5. Thus $X \to Y$ is representable by algebraic spaces by Bootstrap, Lemma 3.10. □

The property of being representable by algebraic spaces is local on the source and the target.

Lemma 19.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent:*

(1) *the morphism f is representable by algebraic spaces,*
(2) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are formal algebraic spaces, the vertical arrows are representable by algebraic spaces, $U \to X$ is surjective étale, and $U \to V$ is representable by algebraic spaces,

(3) *for any commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are formal algebraic spaces and the vertical arrows are representable by algebraic spaces, the morphism $U \to V$ is representable by algebraic spaces,

(4) *there exists a covering $\{Y_j \to Y\}$ as in Definition 11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 11.1 such that $X_{ji} \to Y_j$ is representable by algebraic spaces for each j and i,*

(5) *there exist a covering $\{X_i \to X\}$ as in Definition 11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces, such that $X_i \to Y_i$ is representable by algebraic spaces, and*

(6) *add more here.*

Proof. It is clear that (1) implies (2) because we can take $U = X$ and $V = Y$. Conversely, (2) implies (1) by Bootstrap, Lemma 11.4 applied to $U \to X \to Y$.

Assume (1) is true and consider a diagram as in (3). Then $U \to Y$ is representable by algebraic spaces (as the composition $U \to X \to Y$, see Bootstrap, Lemma 3.8) and factors through V. Thus $U \to V$ is representable by algebraic spaces by Lemma 19.3.

It is clear that (3) implies (2). Thus now (1) – (3) are equivalent.

Observe that the condition in (4) makes sense as the fibre product $Y_j \times_Y X$ is a formal algebraic space by Lemma 15.3. It is clear that (4) implies (5).

Assume $X_i \to Y_i \to Y$ as in (5). Then we set $V = \coprod Y_i$ and $U = \coprod X_i$ to see that (5) implies (2).

Finally, assume (1) – (3) are true. Thus we can choose any covering $\{Y_j \to Y\}$ as in Definition 11.1 and for each j any covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 11.1. Then $X_{ij} \to Y_j$ is representable by algebraic spaces by (3) and we see that (4) is true. This concludes the proof. □

Lemma 19.5. *Let S be a scheme. Let Y be an affine formal algebraic space over S. Let $f : X \to Y$ be a map of sheaves on $(Sch/S)_{fppf}$ which is representable by algebraic spaces. Then X is a formal algebraic space.*

Proof. Write $Y = \operatorname{colim} Y_\lambda$ as in Definition 9.1. For each λ the fibre product $X \times_Y Y_\lambda$ is an algebraic space. Hence $X = \operatorname{colim} X \times_Y Y_\lambda$ is a formal algebraic space by Lemma 13.1. □

Lemma 19.6. *Let S be a scheme. Let Y be a formal algebraic space over S. Let $f : X \to Y$ be a map of sheaves on $(Sch/S)_{fppf}$ which is representable by algebraic spaces. Then X is a formal algebraic space.*

Proof. Let $\{Y_i \to Y\}$ be as in Definition 11.1. Then $X \times_Y Y_i \to X$ is a family of morphisms representable by algebraic spaces, étale, and jointly surjective. Thus it suffices to show that $X \times_Y Y_i$ is a formal algebraic space, see Lemma 15.1. This follows from Lemma 19.5. □

Lemma 19.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces. Then f is representable (by schemes) and affine.*

Proof. We will show that f is affine; it will then follow that f is representable and affine by Morphisms of Spaces, Lemma 20.3. Write $Y = \mathrm{colim}\, Y_\mu$ and $X = \mathrm{colim}\, X_\lambda$ as in Definition 9.1. Let $T \to Y$ be a morphism where T is a scheme over S. We have to show that $X \times_Y T \to T$ is affine, see Bootstrap, Definition 4.1. To do this we may assume that T is affine and we have to prove that $X \times_Y T$ is affine. In this case $T \to Y$ factors through $Y_\mu \to Y$ for some μ, see Lemma 9.4. Since f is quasi-compact we see that $X \times_Y T$ is quasi-compact (Lemma 17.3). Hence $X \times_Y T \to X$ factors through X_λ for some λ. Similarly $X_\lambda \to Y$ factors through Y_μ after increasing μ. Then $X \times_Y T = X_\lambda \times_{Y_\mu} T$. We conclude as fibre products of affine schemes are affine. □

Lemma 19.8. *Let S be a scheme. Let $\varphi : A \to B$ be a continuous map of weakly admissible topological rings over S. The following are equivalent*

1. *$Spf(\varphi) : Spf(B) \to Spf(A)$ is representable by algebraic spaces,*
2. *$Spf(\varphi) : Spf(B) \to Spf(A)$ is representable (by schemes),*
3. *φ is taut, see Definition 5.1.*

Proof. Parts (1) and (2) are equivalent by Lemma 19.7.

Assume the equivalent conditions (1) and (2) hold. If $I \subset A$ is a weak ideal of definition, then $\mathrm{Spec}(A/I) \to \mathrm{Spf}(A)$ is representable and a thickening (this is clear from the construction of the formal spectrum but it also follows from Lemma 9.6). Then $\mathrm{Spec}(A/I) \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B) \to \mathrm{Spf}(B)$ is representable and a thickening as a base change. Hence by Lemma 9.6 there is a weak ideal of definition $J(I) \subset B$ such that $\mathrm{Spec}(A/I) \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B) = \mathrm{Spec}(B/J(I))$ as subfunctors of $\mathrm{Spf}(B)$. We obtain a cartesian diagram

$$\begin{tikzcd} \mathrm{Spec}(B/J(I)) \arrow[r] \arrow[d] & \mathrm{Spec}(A/I) \arrow[d] \\ \mathrm{Spf}(B) \arrow[r] & \mathrm{Spf}(A) \end{tikzcd}$$

By Lemma 16.4 we see that $B/J(I) = B \widehat{\otimes}_A A/I$. It follows that $J(I)$ is the closure of the ideal $\varphi(I)B$, see Lemma 4.11. Since $\mathrm{Spf}(A) = \mathrm{colim}\, \mathrm{Spec}(A/I)$ with I as above, we find that $\mathrm{Spf}(B) = \mathrm{colim}\, \mathrm{Spec}(B/J(I))$. Thus the ideals $J(I)$ form a fundamental system of weak ideals of definition (see Lemma 9.6). Hence (3) holds.

Assume (3) holds. We are essentially just going to reverse the arguments given in the previous paragraph. Let $I \subset A$ be a weak ideal of definition. By Lemma 16.4 we get a cartesian diagram

$$\begin{tikzcd} \mathrm{Spf}(B \widehat{\otimes}_A A/I) \arrow[r] \arrow[d] & \mathrm{Spec}(A/I) \arrow[d] \\ \mathrm{Spf}(B) \arrow[r] & \mathrm{Spf}(A) \end{tikzcd}$$

If $J(I)$ is the closure of IB, then $J(I)$ is open in B by tautness of φ. Hence if J is open in B and $J \subset J(B)$, then $B/J \otimes_A A/I = B/(IB + J) = B/J(I)$ because

$J(I) = \bigcap_{J \subset B \text{ open}}(IB + J)$ by Lemma 4.2. Hence the limit defining the completed tensor product collapses to give $B\widehat{\otimes}_A A/I = B/J(I)$. Thus $\text{Spf}(B\widehat{\otimes}_A A/I) = \text{Spec}(B/J(I))$. This proves that $\text{Spf}(B) \times_{\text{Spf}(A)} \text{Spec}(A/I)$ is representable for every weak ideal of definition $I \subset A$. Since every morphism $T \to \text{Spf}(A)$ with T quasi-compact factors through $\text{Spec}(A/I)$ for some weak ideal of definition I (Lemma 9.4) we conclude that $\text{Spf}(\varphi)$ is representable, i.e., (2) holds. This finishes the proof. □

Lemma 19.9. *Let S be a scheme. Let Y be an affine formal algebraic space. Let $f : X \to Y$ be a map of sheaves on $(Sch/S)_{fppf}$ which is representable and affine. Then*

1. *X is an affine formal algebraic space,*
2. *if Y is countably indexed, then X is countably indexed,*
3. *if Y is countably indexed and classical, then X is countably indexed and classical,*
4. *if Y is weakly adic, then X is weakly adic,*
5. *if Y is adic*, then X is adic*, and*
6. *if Y is Noetherian and f is (locally) of finite type, then X is Noetherian.*

Proof. Proof of (1). Write $Y = \text{colim}_{\lambda \in \Lambda} Y_\lambda$ as in Definition 9.1. Since f is representable and affine, the fibre products $X_\lambda = Y_\lambda \times_Y X$ are affine. And $X = \text{colim}\, Y_\lambda \times_Y X$. Thus X is an affine formal algebraic space.

Proof of (2). If Y is countably indexed, then in the argument above we may assume Λ is countable. Then we immediately see that X is countably indexed too.

Proof of (3), (4), and (5). In each of these cases the assumptions imply that Y is a countably indexed affine formal algebraic space (Lemma 10.3) and hence X is too by (2). Thus we may write $X = \text{Spf}(A)$ and $Y = \text{Spf}(B)$ for some weakly admissible topological S-algebras A and B, see Lemma 10.4. By Lemma 9.10 the morphism f corresponds to a continuous S-algebra homomorphism $\varphi : B \to A$. We see from Lemma 19.8 that φ is taut. We conclude that (3) follows from Lemma 5.9, (4) follows from Lemma 7.5, and (5) follows from Lemma 6.5.

Proof of (6). Combining (3) with Lemma 10.3 we see that X is adic*. Thus we can use the criterion of Lemma 10.5. First, it tells us the affine schemes Y_λ are Noetherian. Then $X_\lambda \to Y_\lambda$ is of finite type, hence X_λ is Noetherian too (Morphisms, Lemma 15.6). Then the criterion tells us X is Noetherian and the proof is complete. □

Lemma 19.10. *Let S be a scheme. Let $f : X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces. Then*

1. *if Y is countably indexed, then X is countably indexed,*
2. *if Y is countably indexed and classical, then X is countably indexed and classical,*
3. *if Y is weakly adic, then X is weakly adic,*
4. *if Y is adic*, then X is adic*, and*
5. *if Y is Noetherian and f is (locally) of finite type, then X is Noetherian.*

Proof. Combine Lemmas 19.7 and 19.9. □

Example 19.11. Let B be a weakly admissible topological ring. Let $B \to A$ be a ring map (no topology). Then we can consider

$$A^\wedge = \lim A/JA$$

where the limit is over all weak ideals of definition J of B. Then $A^\wedge$ (endowed with the limit topology) is a complete linearly topologized ring. The (open) kernel I of the surjection $A^\wedge \to A/JA$ is the closure of $JA^\wedge$, see Lemma 4.2. By Lemma 4.10 we see that I consists of topologically nilpotent elements. Thus I is a weak ideal of definition of $A^\wedge$ and we conclude $A^\wedge$ is a weakly admissible topological ring. Thus $\varphi : B \to A^\wedge$ is taut map of weakly admissible topological rings and

$$\mathrm{Spf}(A^\wedge) \longrightarrow \mathrm{Spf}(B)$$

is a special case of the phenomenon studied in Lemma 19.8.

Remark 19.12 (Warning)**.** The discussion in Lemmas 19.8, 19.9, and 19.10 is sharp in the following two senses:

(1) If A and B are weakly admissible rings and $\varphi : A \to B$ is a continuous map, then $\mathrm{Spf}(\varphi) : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is in general not representable.
(2) If $f : Y \to X$ is a representable morphism of affine formal algebraic spaces and $X = \mathrm{Spf}(A)$ is McQuillan, then it does not follow that Y is McQuillan.

An example for (1) is to take $A = k$ a field (with discrete topology) and $B = k[[t]]$ with the t-adic topology. An example for (2) is given in Examples, Section 74.

The warning above notwithstanding, we do have the following result.

Lemma 19.13. *Let S be a scheme. Let Y be a McQuillan affine formal algebraic space over S, i.e., $Y = Spf(B)$ for some weakly admissible topological S-algebra B. Then there is an equivalence of categories between*

(1) *the category of morphisms $f : X \to Y$ of affine formal algebraic spaces which are representable by algebraic spaces and étale, and*
(2) *the category of topological B-algebras of the form $A^\wedge$ where A is an étale B-algebra and $A^\wedge = \lim A/JA$ with $J \subset B$ running over the weak ideals of definition of B.*

The equivalence is given by sending $A^\wedge$ to $X = Spf(A^\wedge)$. In particular, any X as in (1) is McQuillan.

Proof. Let A be an étale B-algebra. Then $B/J \to A/JA$ is étale for every open ideal $J \subset B$. Hence the morphism $\mathrm{Spf}(A^\wedge) \to Y$ is representable and étale. The functor Spf is fully faithful by Lemma 9.10. To finish the proof we will show in the next paragraph that any $X \to Y$ as in (1) is in the essential image.

Choose a weak ideal of definition $J_0 \subset B$. Set $Y_0 = \mathrm{Spec}(B/J_0)$ and $X_0 = Y_0 \times_Y X$. Then $X_0 \to Y_0$ is an étale morphism of affine schemes (see Lemma 19.7). Say $X_0 = \mathrm{Spec}(A_0)$. By Algebra, Lemma 143.10 we can find an étale algebra map $B \to A$ such that $A_0 \cong A/J_0A$. Consider an ideal of definition $J \subset J_0$. As above we may write $\mathrm{Spec}(B/J) \times_Y X = \mathrm{Spec}(\bar{A})$ for some étale ring map $B/J \to \bar{A}$. Then both $B/J \to \bar{A}$ and $B/J \to A/JA$ are étale ring maps lifting the étale ring map $B/J_0 \to A_0$. By More on Algebra, Lemma 11.2 there is a unique B/J-algebra isomorphism $\varphi_J : A/JA \to \bar{A}$ lifting the identification modulo J_0. Since the maps φ_J are unique they are compatible for varying J. Thus

$$X = \mathrm{colim}\, \mathrm{Spec}(B/J) \times_Y X = \mathrm{colim}\, \mathrm{Spec}(A/JA) = \mathrm{Spf}(A)$$

and we see that the lemma holds. ☐

Lemma 19.14. *With notation and assumptions as in Lemma 19.13 let $f : X \to Y$ correspond to $B \to A^\wedge$. The following are equivalent*

1. *$f : X \to Y$ is surjective,*
2. *$B \to A$ is faithfully flat,*
3. *for every weak ideal of definition $J \subset B$ the ring map $B/J \to A/JA$ is faithfully flat, and*
4. *for some weak ideal of definition $J \subset B$ the ring map $B/J \to A/JA$ is faithfully flat.*

Proof. Let $J \subset B$ be a weak ideal of definition. As every element of J is topologically nilpotent, we see that every element of $1 + J$ is a unit. It follows that J is contained in the Jacobson radical of B (Algebra, Lemma 19.1). Hence a flat ring map $B \to A$ is faithfully flat if and only if $B/J \to A/JA$ is faithfully flat (Algebra, Lemma 39.16). In this way we see that (2) – (4) are equivalent. If (1) holds, then for every weak ideal of definition $J \subset B$ the morphism $\mathrm{Spec}(A/JA) = \mathrm{Spec}(B/J) \times_Y X \to \mathrm{Spec}(B/J)$ is surjective which implies (3). Conversely, assume (3). A morphism $T \to Y$ with T quasi-compact factors through $\mathrm{Spec}(B/J)$ for some ideal of definition J of B (Lemma 9.4). Hence $X \times_Y T = \mathrm{Spec}(A/JA) \times_{\mathrm{Spec}(B/J)} T \to T$ is surjective as a base change of the surjective morphism $\mathrm{Spec}(A/JA) \to \mathrm{Spec}(B/J)$. Thus (1) holds. ☐

20. Types of formal algebraic spaces

In this section we define "locally Noetherian", "locally adic*", "locally weakly adic", "locally countably indexed and classical", and "locally countably indexed" formal algebraic spaces. The types "locally adic", "locally classical", and "locally McQuillan" are missing as we do not know how to prove the analogue of the following lemmas for those cases (it would suffice to prove the analogue of these lemmas for étale coverings between affine formal algebraic spaces).

Lemma 20.1. *Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is countably indexed if and only if Y is countably indexed.*

Proof. Assume X is countably indexed. We write $X = \mathrm{colim}\, X_n$ as in Lemma 10.1. Write $Y = \mathrm{colim}\, Y_\lambda$ as in Definition 9.1. For every n we can pick a λ_n such that $X_n \to Y$ factors through Y_{λ_n}, see Lemma 9.4. On the other hand, for every λ the scheme $Y_\lambda \times_Y X$ is affine (Lemma 19.7) and hence $Y_\lambda \times_Y X \to X$ factors through X_n for some n (Lemma 9.4). Picture

$$\begin{tikzcd}
Y_\lambda \times_Y X \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & X \arrow[d] \\
Y_\lambda \arrow[r, dotted] \arrow[rr, bend right] & Y_{\lambda_n} \arrow[r] & Y
\end{tikzcd}$$

If we can show the dotted arrow exists, then we conclude that $Y = \mathrm{colim}\, Y_{\lambda_n}$ and Y is countably indexed. To do this we pick a μ with $\mu \geq \lambda$ and $\mu \geq \lambda_n$. Thus both $Y_\lambda \to Y$ and $Y_{\lambda_n} \to Y$ factor through $Y_\mu \to Y$. Say $Y_\mu = \mathrm{Spec}(B_\mu)$, the closed subscheme Y_λ corresponds to $J \subset B_\mu$, and the closed subscheme Y_{λ_n} corresponds to $J' \subset B_\mu$. We are trying to show that $J' \subset J$. By the diagram above we know

$J'A_\mu \subset JA_\mu$ where $Y_\mu \times_Y X = \mathrm{Spec}(A_\mu)$. Since $X \to Y$ is surjective and flat the morphism $Y_\lambda \times_Y X \to Y_\lambda$ is a faithfully flat morphism of affine schemes, hence $B_\mu \to A_\mu$ is faithfully flat. Thus $J' \subset J$ as desired.

Assume Y is countably indexed. Then X is countably indexed by Lemma 19.10. □

Lemma 20.2. *Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is countably indexed and classical if and only if Y is countably indexed and classical.*

Proof. We have already seen the implication in one direction in Lemma 19.10. For the other direction, note that by Lemma 20.1 we may assume both X and Y are countably indexed. Thus $X = \mathrm{Spf}(A)$ and $Y = \mathrm{Spf}(B)$ for some weakly admissible topological S-algebras A and B, see Lemma 10.4. By Lemma 9.10 the morphism $X \to Y$ corresponds to a continuous S-algebra homomorphism $\varphi : B \to A$. We see from Lemma 19.8 that φ is taut. Let $J \subset B$ be an open ideal and let $I \subset A$ be the closure of JA. By Lemmas 16.4 and 4.11 we see that $\mathrm{Spec}(B/J) \times_Y X = \mathrm{Spec}(A/I)$. Hence $B/J \to A/I$ is faithfully flat (since $X \to Y$ is surjective and flat). This means that $\varphi : B \to A$ is as in Section 8 (with the roles of A and B swapped). We conclude that the lemma holds by Lemma 8.2. □

Lemma 20.3. *Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is weakly adic if and only if Y is weakly adic.*

Proof. The proof is exactly the same as the proof of Lemma 20.2 except that at the end we use Lemma 8.4. □

Lemma 20.4. *Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, and flat. Then X is adic* if and only if Y is adic*.*

Proof. The proof is exactly the same as the proof of Lemma 20.2 except that at the end we use Lemma 8.5. □

Lemma 20.5. *Let S be a scheme. Let $X \to Y$ be a morphism of affine formal algebraic spaces which is representable by algebraic spaces, surjective, flat, and (locally) of finite type. Then X is Noetherian if and only if Y is Noetherian.*

Proof. Observe that a Noetherian affine formal algebraic space is adic*, see Lemma 10.3. Thus by Lemma 20.4 we may assume that both X and Y are adic*. We will use the criterion of Lemma 10.5 to see that the lemma holds. Namely, write $Y = \mathrm{colim}\, Y_n$ as in Lemma 10.1. For each n set $X_n = Y_n \times_Y X$. Then X_n is an affine scheme (Lemma 19.7) and $X = \mathrm{colim}\, X_n$. Each of the morphisms $X_n \to Y_n$ is faithfully flat and of finite type. Thus the lemma follows from the fact that in this situation X_n is Noetherian if and only if Y_n is Noetherian, see Algebra, Lemma 164.1 (to go down) and Algebra, Lemma 31.1 (to go up). □

Lemma 20.6. *Let S be a scheme. Let*

$$P \in \left\{ \begin{matrix} \textit{countably indexed}, \\ \textit{countably indexed and classical}, \\ \textit{weakly adic},\ \textit{adic}*,\ \textit{Noetherian} \end{matrix} \right\}$$

Let X be a formal algebraic space over S. The following are equivalent

(1) *if Y is an affine formal algebraic space and $f : Y \to X$ is representable by algebraic spaces and étale, then Y has property P,*
(2) *for some $\{X_i \to X\}_{i \in I}$ as in Definition 11.1 each X_i has property P.*

Proof. It is clear that (1) implies (2). Assume (2) and let $Y \to X$ be as in (1). Since the fibre products $X_i \times_X Y$ are formal algebraic spaces (Lemma 15.2) we can pick coverings $\{X_{ij} \to X_i \times_X Y\}$ as in Definition 11.1. Since Y is quasi-compact, there exist $(i_1, j_1), \ldots, (i_n, j_n)$ such that

$$X_{i_1 j_1} \amalg \ldots \amalg X_{i_n j_n} \longrightarrow Y$$

is surjective and étale. Then $X_{i_k j_k} \to X_{i_k}$ is representable by algebraic spaces and étale hence $X_{i_k j_k}$ has property P by Lemma 19.10. Then $X_{i_1 j_1} \amalg \ldots \amalg X_{i_n j_n}$ is an affine formal algebraic space with property P (small detail omitted on finite disjoint unions of affine formal algebraic spaces). Hence we conclude by applying one of Lemmas 20.1, 20.2, 20.3, 20.4, and 20.5. $\square$

The previous lemma clears the way for the following definition.

Definition 20.7. Let S be a scheme. Let X be a formal algebraic space over S. We say X is *locally countably indexed*, *locally countably indexed and classical*, *locally weakly adic*, *locally adic**, or *locally Noetherian* if the equivalent conditions of Lemma 20.6 hold for the corresponding property.

The formal completion of a locally Noetherian algebraic space along a closed subset is a locally Noetherian formal algebraic space.

Lemma 20.8. *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $X_{/T}$ be the formal completion of X along T.*

(1) *If $X \setminus T \to X$ is quasi-compact, then $X_{/T}$ is locally adic*.*
(2) *If X is locally Noetherian, then $X_{/T}$ is locally Noetherian.*

Proof. Choose a surjective étale morphism $U \to X$ with $U = \coprod U_i$ a disjoint union of affine schemes, see Properties of Spaces, Lemma 6.1. Let $T_i \subset U_i$ be the inverse image of T. We have $X_{/T} \times_X U_i = (U_i)_{/T_i}$ (Lemma 14.4). Hence $\{(U_i)_{/T_i} \to X_{/T}\}$ is a covering as in Definition 11.1. Moreover, if $X \setminus T \to X$ is quasi-compact, so is $U_i \setminus T_i \to U_i$ and if X is locally Noetherian, so is U_i. Thus the lemma follows from the affine case which is Lemma 14.6. $\square$

Remark 20.9 (Warning)**.** Suppose $X = \text{Spec}(A)$ and $T \subset X$ is the zero locus of a finitely generated ideal $I \subset A$. Let $J = \sqrt{I}$ be the radical of I. Then from the definitions we see that $X_{/T} = \text{Spf}(A^\wedge)$ where $A^\wedge = \lim A/I^n$ is the I-adic completion of A. On the other hand, the map $A^\wedge \to \lim A/J^n$ from the I-adic completion to the J-adic completion can fail to be a ring isomorphisms. As an example let

$$A = \bigcup\nolimits_{n \geq 1} \mathbf{C}[t^{1/n}]$$

and $I = (t)$. Then $J = \mathfrak{m}$ is the maximal ideal of the valuation ring A and $J^2 = J$. Thus the J-adic completion of A is $\mathbf{C}$ whereas the I-adic completion is the valuation ring described in Example 13.2 (but in particular it is easy to see that $A \subset A^\wedge$).

Lemma 20.10. *Let S be a scheme. Let $X \to Y$ and $Z \to Y$ be morphisms of formal algebraic space over S. Then*

(1) *If X and Z are locally countably indexed, then $X \times_Y Z$ is locally countably indexed.*
(2) *If X and Z are locally countably indexed and classical, then $X \times_Y Z$ is locally countably indexed and classical.*
(3) *If X and Z are weakly adic, then $X \times_Y Z$ is weakly adic.*
(4) *If X and Z are locally adic*, then $X \times_Y Z$ is locally adic*.*
(5) *If X and Z are locally Noetherian and $X_{red} \to Y_{red}$ is locally of finite type, then $X \times_Y Z$ is locally Noetherian.*

Proof. Choose a covering $\{Y_j \to Y\}$ as in Definition 11.1. For each j choose a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 11.1. For each j choose a covering $\{Z_{jk} \to Y_j \times_Y Z\}$ as in Definition 11.1. Observe that $X_{ji} \times_{Y_j} Z_{jk}$ is an affine formal algebraic space by Lemma 16.4. Hence

$$\{X_{ji} \times_{Y_j} Z_{jk} \to X \times_Y Z\}$$

is a covering as in Definition 11.1. Thus it suffices to prove (1), (2), (3), and (4) in case X, Y, and Z are affine formal algebraic spaces.

Assume X and Z are countably indexed. Say $X = \operatorname{colim} X_n$ and $Z = \operatorname{colim} Z_m$ as in Lemma 10.1. Write $Y = \operatorname{colim}_{\lambda \in \Lambda} Y_\lambda$ as in Definition 9.1. For each n and m we can find $\lambda_{n,m} \in \Lambda$ such that $X_n \to Y$ and $Z_m \to Y$ factor through $Y_{\lambda_{n,m}}$ (for example see Lemma 9.4). Pick $\lambda_0 \in \Lambda$. By induction for $t \geq 1$ pick an element $\lambda_t \in \Lambda$ such that $\lambda_t \geq \lambda_{n,m}$ for all $1 \leq n, m \leq t$ and $\lambda_t \geq \lambda_{t-1}$. Set $Y' = \operatorname{colim} Y_{\lambda_t}$. Then $Y' \to Y$ is a monomorphism such that $X \to Y$ and $Z \to Y$ factor through Y'. Hence we may replace Y by Y', i.e., we may assume that Y is countably indexed.

Assume X, Y, and Z are countably indexed. By Lemma 10.4 we can write $X = \operatorname{Spf}(A)$, $Y = \operatorname{Spf}(B)$, $Z = \operatorname{Spf}(C)$ for some weakly admissible topological rings A, B, and C. The morphsms $X \to Y$ and $Z \to Y$ are given by continuous ring maps $B \to A$ and $B \to C$, see Lemma 9.10. By Lemma 16.4 we see that $X \times_Y Z = \operatorname{Spf}(A \widehat{\otimes}_B C)$ and that $A \widehat{\otimes}_B C$ is a weakly admissible topological ring. In particular, we see that $X \times_Y Z$ is countably indexed by Lemma 4.12 part (3). This proves (1).

Proof of (2). In this case X and Z are countably indexed and hence the arguments above show that $X \times_Y Z$ is the formal spectrum of $A \widehat{\otimes}_B C$ where A and C are admissible. Then $A \widehat{\otimes}_B C$ is admissible by Lemma 4.12 part (2).

Proof of (3). As before we conclude that $X \times_Y Z$ is the formal spectrum of $A \widehat{\otimes}_B C$ where A and C are weakly adic. Then $A \widehat{\otimes}_B C$ is weakly adic by Lemma 7.6.

Proof of (4). Arguing as above, this follows from Lemma 4.12 part (4).

Proof of (5). To deduce case (5) from Lemma 4.12 part (5) we need to show the hypotheses match. Namely, with notation as in the first parapgrah of the proof, if $X_{red} \to Y_{red}$ is locally of finite type, then $(X_{ji})_{red} \to (Y_j)_{red}$ is locally of finite type. This follows from Morphisms of Spaces, Lemma 23.4 and the fact that in the commutative diagram

$$\begin{tikzcd} (X_{ji})_{red} \arrow[r] \arrow[d] & (Y_j)_{red} \arrow[d] \\ X_{red} \arrow[r] & Y_{red} \end{tikzcd}$$

the vertical morphisms are étale. Namely, we have $(X_{ji})_{red} = X_{ij} \times_X X_{red}$ and $(Y_j)_{red} = Y_j \times_Y Y_{red}$ by Lemma 12.3. Thus as above we reduce to the case where X, Y, Z are affine formal algebraic spaces, X, Z are Noetherian, and $X_{red} \to Y_{red}$ is of finite type. Next, in the second paragraph of the proof we replaced Y by Y' but by construction $Y_{red} = Y'_{red}$, hence the finite type assumption is preserved by this replacement. Then we see that X, Y, Z correspond to A, B, C and $X \times_Y Z$ to $A \widehat{\otimes}_B C$ with A, C Noetherian adic. Finally, taking the reduction corresponds to dividing by the ideal of topologically nilpotent elements (Example 12.2) hence the fact that $X_{red} \to Y_{red}$ is of finite type does indeed mean that $B/\mathfrak{b} \to A/\mathfrak{a}$ is of finite type and the proof is complete. $\square$

Lemma 20.11. *Let S be a scheme. Let X be a locally Noetherian formal algebraic space over S. Then $X = \operatorname{colim} X_n$ for a system $X_1 \to X_2 \to X_3 \to \ldots$ of finite order thickenings of locally Noetherian algebraic spaces over S where $X_1 = X_{red}$ and X_n is the nth infinitesimal neighbourhood of X_1 in X_m for all $m \geq n$.*

Proof. We only sketch the proof and omit some of the details. Set $X_1 = X_{red}$. Define $X_n \subset X$ as the subfunctor defined by the rule: a morphism $f : T \to X$ where T is a scheme factors through X_n if and only if the nth power of the ideal sheaf of the closed immersion $X_1 \times_X T \to T$ is zero. Then $X_n \subset X$ is a subsheaf as vanishing of quasi-coherent modules can be checked fppf locally. We claim that $X_n \to X$ is representable by schemes, a closed immersion, and that $X = \operatorname{colim} X_n$ (as fppf sheaves). To check this we may work étale locally on X. Hence we may assume $X = \operatorname{Spf}(A)$ is a Noetherian affine formal algebraic space. Then $X_1 = \operatorname{Spec}(A/\mathfrak{a})$ where $\mathfrak{a} \subset A$ is the ideal of topologically nilpotent elements of the Noetherian adic topological ring A. Then $X_n = \operatorname{Spec}(A/\mathfrak{a}^n)$ and we obtain what we want. $\square$

21. Morphisms and continuous ring maps

In this section we denote *WAdm* the category of weakly admissible topological rings and continuous ring homomorphisms. We define full subcategories

$$\textit{WAdm} \supset \textit{WAdm}^{count} \supset \textit{WAdm}^{cic} \supset \textit{WAdm}^{weakly\ adic} \supset \textit{WAdm}^{adic*} \supset \textit{WAdm}^{Noeth}$$

whose objects are

1. (1) $\textit{WAdm}^{count}$: those weakly admissible topological rings A which have a countable fundamental system of open ideals,
2. (2) $\textit{WAdm}^{cic}$: the admissible topological rings A which have a countable fundamental system of open ideals,
3. (3) $\textit{WAdm}^{weakly\ adic}$: the weakly adic topological rings (Section 7),
4. (4) $\textit{WAdm}^{adic*}$: the adic topological rings which have a finitely generated ideal of definition, and
5. (5) $\textit{WAdm}^{Noeth}$: the adic topological rings which are Noetherian.

Clearly, the formal spectra of these types of rings are the basic building blocks of locally countably indexed, locally countably indexed and classical, locally weakly adic, locally adic*, and locally Noetherian formal algebraic spaces.

We briefly review the relationship between morphisms of countably indexed, affine formal algebraic spaces and morphisms of $\textit{WAdm}^{count}$. Let S be a scheme. Let X and Y be countably indexed, affine formal algebraic spaces. Write $X = \operatorname{Spf}(A)$ and $Y = \operatorname{Spf}(B)$ topological S-algebras A and B in $\textit{WAdm}^{count}$, see Lemma 10.4. By

Lemma 9.10 there is a 1-to-1 correspondence between morphisms $f : X \to Y$ and continuous maps

$$\varphi : B \longrightarrow A$$

of topological S-algebras. The relationship is given by $f \mapsto f^\sharp$ and $\varphi \mapsto \mathrm{Spf}(\varphi)$.

Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces and $U \to X$ and $V \to Y$ representable by algebraic spaces and étale. By Definition 20.7 (and hence via Lemma 20.6) we see that U and V are countably indexed affine formal algebraic spaces. By the discussion in the previous paragraph we see that $U \to V$ is isomorphic to $\mathrm{Spf}(\varphi)$ for some continuous map

$$\varphi : B \longrightarrow A$$

of topological S-algebras in $\textit{WAdm}^{count}$.

Lemma 21.1. *Let $A \in \mathrm{Ob}(\textit{WAdm})$. Let $A \to A'$ be a ring map (no topology). Let $(A')^\wedge = \lim_{I \subset A \text{ w.i.d}} A'/IA'$ be the object of WAdm constructed in Example 19.11.*

1. *If A is in $\textit{WAdm}^{count}$, so is $(A')^\wedge$.*
2. *If A is in $\textit{WAdm}^{cic}$, so is $(A')^\wedge$.*
3. *If A is in $\textit{WAdm}^{weakly\ adic}$, so is $(A')^\wedge$.*
4. *If A is in $\textit{WAdm}^{adic*}$, so is $(A')^\wedge$.*
5. *If A is in $\textit{WAdm}^{Noeth}$ and A' is Noetherian, then $(A')^\wedge$ is in $\textit{WAdm}^{Noeth}$.*

Proof. Recall that $A \to (A')^\wedge$ is taut, see discussion in Example 19.11. Hence statements (1), (2), (3), and (4) follow from Lemmas 5.7, 5.9, 7.5, and 6.5. Finally, assume that A is Noetherian and adic. By (4) we know that $(A')^\wedge$ is adic. By Algebra, Lemma 97.6 we see that $(A')^\wedge$ is Noetherian. Hence (5) holds. $\square$

Situation 21.2. Let P be a property of morphisms of $\textit{WAdm}^{count}$. Consider commutative diagrams

$$\begin{tikzcd} A \arrow[r] & (A')^\wedge \\ B \arrow[u, "\varphi"] \arrow[r] & (B')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd} \tag{21.2.1}$$

satisfying the following conditions

1. A and B are objects of $\textit{WAdm}^{count}$,
2. $A \to A'$ and $B \to B'$ are étale ring maps,
3. $(A')^\wedge = \lim A'/IA'$, resp. $(B')^\wedge = \lim B'/JB'$ where $I \subset A$, resp. $J \subset B$ runs through the weakly admissible ideals of definition of A, resp. B,
4. $\varphi : B \to A$ and $\varphi' : (B')^\wedge \to (A')^\wedge$ are continuous.

By Lemma 21.1 the topological rings $(A')^\wedge$ and $(B')^\wedge$ are objects of $\textit{WAdm}^{count}$. We say P is a *local property* if the following axioms hold:

1. for any diagram (21.2.1) we have $P(\varphi) \Rightarrow P(\varphi')$,

(2) for any diagram (21.2.1) with $A \to A'$ faithfully flat we have $P(\varphi') \Rightarrow P(\varphi)$,
(3) if $P(B \to A_i)$ for $i = 1, \ldots, n$, then $P(B \to \prod_{i=1,\ldots,n} A_i)$.

Axiom (3) makes sense as *WAdm*^{count} has finite products.

Lemma 21.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. Let P be a local property of morphisms of WAdm^{count}. The following are equivalent*

(1) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a morphism of WAdm^{count} with property P,

(2) *there exists a covering $\{Y_j \to Y\}$ as in Definition 11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 11.1 such that each $X_{ji} \to Y_j$ corresponds to a morphism of WAdm^{count} with property P, and*

(3) *there exist a covering $\{X_i \to X\}$ as in Definition 11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a morphism of WAdm^{count} with property P.*

Proof. It is clear that (1) implies (2) and that (2) implies (3). Assume $\{X_i \to X\}$ and $X_i \to Y_i \to Y$ as in (3) and let a diagram as in (1) be given. Since $Y_i \times_Y V$ is a formal algebraic space (Lemma 15.2) we may pick coverings $\{Y_{ij} \to Y_i \times_Y V\}$ as in Definition 11.1. For each (i, j) we may similarly choose coverings $\{X_{ijk} \to Y_{ij} \times_{Y_i} X_i \times_X U\}$ as in Definition 11.1. Since U is quasi-compact we can choose $(i_1, j_1, k_1), \ldots, (i_n, j_n, k_n)$ such that

$$X_{i_1 j_1 k_1} \amalg \ldots \amalg X_{i_n j_n k_n} \longrightarrow U$$

is surjective. For $s = 1, \ldots, n$ consider the commutative diagram

$$\begin{tikzcd}[column sep=small] & & & X_{i_s j_s k_s} \arrow[dl] \arrow[d] \arrow[dr] & & & \\ X \arrow[d] & X_{i_s} \arrow[l] \arrow[d] & X_{i_s} \times_X U \arrow[l] \arrow[d] & Y_{i_s j_s} \arrow[dl] \arrow[dr] & X_{i_s} \times_X U \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & X \arrow[d] \\ Y & Y_{i_s} \arrow[l] & Y_{i_s} \times_Y V \arrow[l] & & Y_{i_s} \times_Y V \arrow[r] & V \arrow[r] & Y \end{tikzcd}$$

Let us say that P holds for a morphism of countably indexed affine formal algebraic spaces if it holds for the corresponding morphism of *WAdm*^{count}. Observe that the maps $X_{i_s j_s k_s} \to X_{i_s}$, $Y_{i_s j_s} \to Y_{i_s}$ are given by completions of étale ring maps, see Lemma 19.13. Hence we see that $P(X_{i_s} \to Y_{i_s})$ implies $P(X_{i_s j_s k_s} \to Y_{i_s j_s})$ by axiom (1). Observe that the maps $Y_{i_s j_s} \to V$ are given by completions of étale rings maps (same lemma as before). By axiom (2) applied to the diagram

$$\begin{tikzcd} X_{i_s j_s k_s} \arrow[r, equal] \arrow[d] & X_{i_s j_s k_s} \arrow[d] \\ Y_{i_s j_s} \arrow[r] & V \end{tikzcd}$$

(this is permissible as identities are faithfully flat ring maps) we conclude that $P(X_{i_s j_s k_s} \to V)$ holds. By axiom (3) we find that $P(\coprod_{s=1,\ldots,n} X_{i_s j_s k_s} \to V)$ holds. Since the morphism $\coprod X_{i_s j_s k_s} \to U$ is surjective by construction, the corresponding morphism of WAdm^{count} is the completion of a faithfully flat étale ring map, see Lemma 19.14. One more application of axiom (2) (with $B' = B$) implies that $P(U \to V)$ is true as desired. $\square$

Remark 21.4 (Variant for adic-star)**.** Let P be a property of morphisms of WAdm^{adic*}. We say P is a *local property* if axioms (1), (2), (3) of Situation 21.2 hold for morphisms of WAdm^{adic*}. In exactly the same way we obtain a variant of Lemma 21.3 for morphisms between locally adic* formal algebraic spaces over S.

Remark 21.5 (Variant for Noetherian)**.** Let P be a property of morphisms of WAdm^{Noeth}. We say P is a *local property* if axioms (1), (2), (3), of Situation 21.2 hold for morphisms of WAdm^{Noeth}. In exactly the same way we obtain a variant of Lemma 21.3 for morphisms between locally Noetherian formal algebraic spaces over S.

Situation 21.6. Let P be a local property of morphisms of WAdm^{count}, see Situation 21.2. We say P is *stable under base change* if given $B \to A$ and $B \to C$ in WAdm^{count} we have $P(B \to A) \Rightarrow P(C \to A\widehat{\otimes}_B C)$. This makes sense as $A\widehat{\otimes}_B C$ is an object of WAdm^{count} by Lemma 4.12.

Lemma 21.7. *Let S be a scheme. Let P be a local property of morphisms of WAdm^{count} which is stable under base change. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally countably indexed formal algebraic spaces over S. If f satisfies the equivalent conditions of Lemma 21.3 then so does $pr_2 : X \times_Y Z \to Z$.*

Proof. Choose a covering $\{Y_j \to Y\}$ as in Definition 11.1. For each j choose a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 11.1. For each j choose a covering $\{Z_{jk} \to Y_j \times_Y Z\}$ as in Definition 11.1. Observe that $X_{ji} \times_{Y_j} Z_{jk}$ is an affine formal algebraic space which is countably indexed, see Lemma 20.10. Then we see that

$$\{X_{ji} \times_{Y_j} Z_{jk} \to X \times_Y Z\}$$

is a covering as in Definition 11.1. Moreover, the morphisms $X_{ji} \times_{Y_j} Z_{jk} \to Z$ factor through Z_{jk}. By assumption we know that $X_{ji} \to Y_j$ corresponds to a morphism $B_j \to A_{ji}$ of WAdm^{count} having property P. The morphisms $Z_{jk} \to Y_j$ correspond to morphisms $B_j \to C_{jk}$ in WAdm^{count}. Since $X_{ji} \times_{Y_j} Z_{jk} = \mathrm{Spf}(A_{ji}\widehat{\otimes}_{B_j} C_{jk})$ by Lemma 16.4 we see that it suffices to show that $C_{jk} \to A_{ji}\widehat{\otimes}_{B_j} C_{jk}$ has property P which is exactly what the condition that P is stable under base change guarantees. $\square$

Remark 21.8 (Variant for adic-star)**.** Let P be a local property of morphisms of WAdm^{adic*}, see Remark 21.4. We say P is *stable under base change* if given $B \to A$

and $B \to C$ in WAdm^{adic*} we have $P(B \to A) \Rightarrow P(C \to A\widehat{\otimes}_B C)$. This makes sense as $A\widehat{\otimes}_B C$ is an object of WAdm^{adic*} by Lemma 4.12. In exactly the same way we obtain a variant of Lemma 21.7 for morphisms between locally adic* formal algebraic spaces over S.

Remark 21.9 (Variant for Noetherian)**.** Let P be a local property of morphisms of WAdm^{Noeth}, see Remark 21.5. We say P is *stable under base change* if given $B \to A$ and $B \to C$ in WAdm^{Noeth} the property $P(B \to A)$ implies both that $A\widehat{\otimes}_B C$ is adic Noetherian[7] and that $P(C \to A\widehat{\otimes}_B C)$. In exactly the same way we obtain a variant of Lemma 21.7 for morphisms between locally Noetherian formal algebraic spaces over S.

Remark 21.10 (Another variant for Noetherian)**.** Let P and Q be local properties of morphisms of WAdm^{Noeth}, see Remark 21.5. We say *P is stable under base change by Q* if given $B \to A$ and $B \to C$ in WAdm^{Noeth} satisfying $P(B \to A)$ and $Q(B \to C)$, then $A\widehat{\otimes}_B C$ is adic Noetherian and $P(C \to A\widehat{\otimes}_B C)$ holds. Arguing exactly as in the proof of Lemma 21.7 we obtain the following statement: given morphisms $f : X \to Y$ and $g : Y \to Z$ of locally Noetherian formal algebraic spaces over S such that

(1) the equivalent conditions of Lemma 21.3 hold for f and P,
(2) the equivalent conditions of Lemma 21.3 hold for g and Q,

then the equivalent conditions of Lemma 21.3 hold for $\mathrm{pr}_2 : X \times_Y Z \to Z$ and P.

Situation 21.11. Let P be a local property of morphisms of WAdm^{count}, see Situation 21.2. We say P is *stable under composition* if given $B \to A$ and $C \to B$ in WAdm^{count} we have $P(B \to A) \wedge P(C \to B) \Rightarrow P(C \to A)$.

Lemma 21.12. *Let S be a scheme. Let P be a local property of morphisms of WAdm^{count} which is stable under composition. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally countably indexed formal algebraic spaces over S. If f and g satisfies the equivalent conditions of Lemma 21.3 then so does $g \circ f : X \to Z$.*

Proof. Choose a covering $\{Z_k \to Z\}$ as in Definition 11.1. For each k choose a covering $\{Y_{kj} \to Z_k \times_Z Y\}$ as in Definition 11.1. For each k and j choose a covering $\{X_{kji} \to Y_{kj} \times_Y X\}$ as in Definition 11.1. If f and g satisfies the equivalent conditions of Lemma 21.3 then $X_{kji} \to Y_{jk}$ and $Y_{jk} \to Z_k$ correspond to arrows $B_{kj} \to A_{kji}$ and $C_k \to B_{kj}$ of WAdm^{count} having property P. Hence the compositions do too and we conclude. $\square$

Remark 21.13 (Variant for adic-star)**.** Let P be a local property of morphisms of WAdm^{adic*}, see Remark 21.4. We say P is *stable under composition* if given $B \to A$ and $C \to B$ in WAdm^{adic*} we have $P(B \to A) \wedge P(C \to B) \Rightarrow P(C \to A)$. In exactly the same way we obtain a variant of Lemma 21.12 for morphisms between locally adic* formal algebraic spaces over S.

Remark 21.14 (Variant for Noetherian)**.** Let P be a local property of morphisms of WAdm^{Noeth}, see Remark 21.5. We say P is *stable under composition* if given $B \to A$ and $C \to B$ in WAdm^{Noeth} we have $P(B \to A) \wedge P(C \to B) \Rightarrow P(C \to A)$. In exactly the same way we obtain a variant of Lemma 21.12 for morphisms between locally Noetherian formal algebraic spaces over S.

[7]See Lemma 4.12 for a criterion.

Situation 21.15. Let P be a local property of morphisms of WAdm^{count}, see Situation 21.2. We say P *has the cancellation property* if given $B \to A$ and $C \to B$ in WAdm^{count} we have $P(C \to B) \wedge P(C \to A) \Rightarrow P(B \to A)$.

Lemma 21.16. *Let S be a scheme. Let P be a local property of morphisms of WAdm^{count} which has the cancellation property. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally countably indexed formal algebraic spaces over S. If $g \circ f$ and g satisfies the equivalent conditions of Lemma 21.3 then so does $f : X \to Y$.*

Proof. Choose a covering $\{Z_k \to Z\}$ as in Definition 11.1. For each k choose a covering $\{Y_{kj} \to Z_k \times_Z Y\}$ as in Definition 11.1. For each k and j choose a covering $\{X_{kji} \to Y_{kj} \times_Y X\}$ as in Definition 11.1. Let $X_{kji} \to Y_{jk}$ and $Y_{jk} \to Z_k$ correspond to arrows $B_{kj} \to A_{kji}$ and $C_k \to B_{kj}$ of WAdm^{count}. If $g \circ f$ and g satisfies the equivalent conditions of Lemma 21.3 then $C_k \to B_{kj}$ and $C_k \to A_{kji}$ satisfy P. Hence $B_{kj} \to A_{kji}$ does too and we conclude. □

Remark 21.17 (Variant for adic-star)**.** Let P be a local property of morphisms of WAdm^{adic*}, see Remark 21.4. We say P *has the cancellation property* if given $B \to A$ and $C \to B$ in WAdm^{adic*} we have $P(C \to A) \wedge P(C \to B) \Rightarrow P(B \to A)$. In exactly the same way we obtain a variant of Lemma 21.12 for morphisms between locally adic* formal algebraic spaces over S.

Remark 21.18 (Variant for Noetherian)**.** Let P be a local property of morphisms of WAdm^{Noeth}, see Remark 21.5. We say P *has the cancellation property* if given $B \to A$ and $C \to B$ in WAdm^{Noeth} we have $P(C \to B) \wedge P(C \to A) \Rightarrow P(C \to B)$. In exactly the same way we obtain a variant of Lemma 21.12 for morphisms between locally Noetherian formal algebraic spaces over S.

22. Taut ring maps and representability by algebraic spaces

In this section we briefly show that morphisms between locally countably index formal algebraic spaces correspond étale locally to taut continuous ring homomorphisms between weakly admissible topological rings having countable fundamental systems of open ideals. In fact, this is rather clear from Lemma 19.8 and we encourage the reader to skip this section.

Lemma 22.1. *Let $B \to A$ be an arrow of WAdm^{count}. The following are equivalent*

(a) *$B \to A$ is taut (Definition 5.1),*
(b) *for $B \supset J_1 \supset J_2 \supset J_3 \supset \ldots$ a fundamental system of weak ideals of definitions there exist a commutative diagram*

$$\begin{tikzcd} A \arrow[r] & \ldots \arrow[r] & A_3 \arrow[r] & A_2 \arrow[r] & A_1 \\ B \arrow[r] \arrow[u] & \ldots \arrow[r] & B/J_3 \arrow[r] \arrow[u] & B/J_2 \arrow[r] \arrow[u] & B/J_1 \arrow[u] \end{tikzcd}$$

such that $A_{n+1}/J_n A_{n+1} = A_n$ and $A = \lim A_n$ as topological ring.

Moreover, these equivalent conditions define a local property, i.e., they satisfy axioms (1), (2), (3).

Proof. The equivalence of (a) and (b) is immediate. Below we will give an algebraic proof of the axioms, but it turns out we've already proven them. Namely, using Lemma 19.8 the equivalent conditions (a) and (b) translate to saying the corresponding morphism of affine formal algebraic spaces is representable by algebraic spaces. Since this condition is "étale local on the source and target" by Lemma 19.4 we immediately get axioms (1), (2), and (3).

Direct algebraic proof of (1), (2), (3). Let a diagram (21.2.1) as in Situation 21.2 be given. By Example 19.11 the maps $A \to (A')^\wedge$ and $B \to (B')^\wedge$ satisfy (a) and (b).

Assume (a) and (b) hold for φ. Let $J \subset B$ be a weak ideal of definition. Then the closure of JA, resp. $J(B')^\wedge$ is a weak ideal of definition $I \subset A$, resp. $J' \subset (B')^\wedge$. Then the closure of $I(A')^\wedge$ is a weak ideal of definition $I' \subset (A')^\wedge$. A topological argument shows that I' is also the closure of $J(A')^\wedge$ and of $J'(A')^\wedge$. Finally, as J runs over a fundamental system of weak ideals of definition of B so do the ideals I and I' in A and $(A')^\wedge$. It follows that (a) holds for φ'. This proves (1).

Assume $A \to A'$ is faithfully flat and that (a) and (b) hold for φ'. Let $J \subset B$ be a weak ideal of definition. Using (a) and (b) for the maps $B \to (B')^\wedge \to (A')^\wedge$ we find that the closure I' of $J(A')^\wedge$ is a weak ideal of definition. In particular, I' is open and hence the inverse image of I' in A is open. Now we have (explanation below)

$$\begin{aligned} A \cap I' &= A \cap \bigcap (J(A')^\wedge + \mathrm{Ker}((A')^\wedge \to A'/I_0A')) \\ &= A \cap \bigcap \mathrm{Ker}((A')^\wedge \to A'/JA' + I_0A') \\ &= \bigcap (JA + I_0) \end{aligned}$$

which is the closure of JA by Lemma 4.2. The intersections are over weak ideals of definition $I_0 \subset A$. The first equality because a fundamental system of neighbourhoods of 0 in $(A')^\wedge$ are the kernels of the maps $(A')^\wedge \to A'/I_0A'$. The second equality is trivial. The third equality because $A \to A'$ is faithfully flat, see Algebra, Lemma 82.11. Thus the closure of JA is open. By Lemma 4.10 the closure of JA is a weak ideal of definition of A. Finally, given a weak ideal of definition $I \subset A$ we can find J such that $J(A')^\wedge$ is contained in the closure of $I(A')^\wedge$ by property (a) for $B \to (B')^\wedge$ and φ'. Thus we see that (a) holds for φ. This proves (2).

We omit the proof of (3). □

Lemma 22.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. The following are equivalent*

(1) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a taut map $B \to A$ of $\textit{WAdm}^{count}$,

(2) *there exists a covering $\{Y_j \to Y\}$ as in Definition 11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 11.1 such that each $X_{ji} \to Y_j$ corresponds to a taut ring map in $\textit{WAdm}^{count}$,*

(3) *there exist a covering $\{X_i \to X\}$ as in Definition 11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a taut ring map in $\textit{WAdm}^{count}$, and*
(4) *f is representable by algebraic spaces.*

Proof. The property of a map in $\textit{WAdm}^{count}$ being "taut" is a local property by Lemma 22.1. Thus Lemma 21.3 exactly tells us that (1), (2), and (3) are equivalent. On the other hand, by Lemma 19.8 being "taut" on maps in $\textit{WAdm}^{count}$ corresponds exactly to being "representable by algebraic spaces" for the corresponding morphisms of countably indexed affine formal algebraic spaces. Thus the implication (1) $\Rightarrow$ (2) of Lemma 19.4 shows that (4) implies (1) of the current lemma. Similarly, the implication (4) $\Rightarrow$ (1) of Lemma 19.4 shows that (2) implies (4) of the current lemma. $\square$

23. Adic morphisms

This section matches the occasionally used notion of an "adic morphism" $f : X \to Y$ of locally adic* formal algebraic spaces X and Y on the one hand with representability of f by algebraic spaces and on the other hand with our notion of taut continuous ring homomorphisms. First we recall that tautness is equivalent to adicness for adic rings with finitely generated ideal of definition.

Lemma 23.1. *Let A and B be pre-adic topological rings. Let $\varphi : A \to B$ be a continuous ring homomorphism.*

(1) *If φ is adic, then φ is taut.*
(2) *If B is complete, A has a finitely generated ideal of definition, and φ is taut, then φ is adic.*

In particular the conditions "φ is adic" and "φ is taut" are equivalent on the category $\textit{WAdm}^{adic*}$.

Proof. Part (1) is Lemma 6.4. Part (2) is Lemma 6.5. The final statement is a consequence of (1) and (2). $\square$

Let S be a scheme. Let $f : X \to Y$ be a morphism of locally adic* formal algebraic spaces over S. By Lemma 22.2 the following are equivalent

(1) f is representable by algebraic spaces (in other words, the equivalent conditions of Lemma 19.4 hold),
(2) for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to an adic[8] map in $\textit{WAdm}^{adic*}$.

[8]Equivalently taut by Lemma 23.1.

In this situation we will say that f *is an adic morphism* (the formal definition is below). This notion/terminology will only be defined/used for morphisms between formal algebraic spaces which are locally adic* since otherwise we don't have the equivalence between (1) and (2) above.

Definition 23.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Assume X and Y are locally adic*. We say f is an *adic morphism* if f is representable by algebraic spaces. See discussion above.

24. Morphisms of finite type

Due to how things are setup in the Stacks project, the following is really the correct thing to do and stronger notions should have a different name.

Definition 24.1. Let S be a scheme. Let $f : Y \to X$ be a morphism of formal algebraic spaces over S.

(1) We say f is *locally of finite type* if f is representable by algebraic spaces and is locally of finite type in the sense of Bootstrap, Definition 4.1.
(2) We say f is of *finite type* if f is locally of finite type and quasi-compact (Definition 17.4).

We will discuss the relationship between finite type morphisms of certain formal algebraic spaces and continuous ring maps $A \to B$ which are topologically of finite type in Section 29.

Lemma 24.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

(1) *f is of finite type,*
(2) *f is representable by algebraic spaces and is of finite type in the sense of Bootstrap, Definition 4.1.*

Proof. This follows from Bootstrap, Lemma 4.5, the implication "quasi-compact + locally of finite type $\Rightarrow$ finite type" for morphisms of algebraic spaces, and Lemma 17.5. $\square$

Lemma 24.3. *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. See Bootstrap, Lemma 4.3 and use Morphisms of Spaces, Lemma 23.2. $\square$

Lemma 24.4. *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. See Bootstrap, Lemma 4.2 and use Morphisms of Spaces, Lemma 23.3. $\square$

Lemma 24.5. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. If $g \circ f : X \to Z$ is locally of finite type, then $f : X \to Y$ is locally of finite type.*

Proof. By Lemma 19.3 we see that f is representable by algebraic spaces. Let T be a scheme and let $T \to Z$ be a morphism. Then we can apply Morphisms of Spaces, Lemma 23.6 to the morphisms $T \times_Z X \to T \times_Z Y \to T$ of algebraic spaces to conclude. $\square$

Being locally of finite type is local on the source and the target.

Lemma 24.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent:*

(1) *the morphism f is locally of finite type,*
(2) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are formal algebraic spaces, the vertical arrows are representable by algebraic spaces and étale, $U \to X$ is surjective, and $U \to V$ is locally of finite type,
(3) *for any commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are formal algebraic spaces and vertical arrows representable by algebraic spaces and étale, the morphism $U \to V$ is locally of finite type,
(4) *there exists a covering $\{Y_j \to Y\}$ as in Definition 11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 11.1 such that $X_{ji} \to Y_j$ is locally of finite type for each j and i,*
(5) *there exist a covering $\{X_i \to X\}$ as in Definition 11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, such that $X_i \to Y_i$ is locally of finite type, and*
(6) *add more here.*

Proof. In each of the 5 cases the morphism $f : X \to Y$ is representable by algebraic spaces, see Lemma 19.4. We will use this below without further mention.

It is clear that (1) implies (2) because we can take $U = X$ and $V = Y$. Conversely, assume given a diagram as in (2). Let T be a scheme and let $T \to Y$ be a morphism. Then we can consider

$$\begin{tikzcd} U \times_Y T \arrow[r] \arrow[d] & V \times_Y T \arrow[d] \\ X \times_Y T \arrow[r] & T \end{tikzcd}$$

The vertical arrows are étale and the top horizontal arrow is locally of finite type as base changes of such morphisms. Hence by Morphisms of Spaces, Lemma 23.4 we conclude that $X \times_Y T \to T$ is locally of finite type. In other words (1) holds.

Assume (1) is true and consider a diagram as in (3). Then $U \to Y$ is locally of finite type (as the composition $U \to X \to Y$, see Bootstrap, Lemma 4.3). Let T be a scheme and let $T \to V$ be a morphism. Then the projection $T \times_V U \to T$ factors as

$$T \times_V U = (T \times_Y U) \times_{(V \times_Y V)} V \to T \times_Y U \to T$$

The second arrow is locally of finite type (as a base change of the composition $U \to X \to Y$) and the first is the base change of the diagonal $V \to V \times_Y V$ which is locally of finite type by Lemma 15.5.

It is clear that (3) implies (2). Thus now (1) – (3) are equivalent.

Observe that the condition in (4) makes sense as the fibre product $Y_j \times_Y X$ is a formal algebraic space by Lemma 15.3. It is clear that (4) implies (5).

Assume $X_i \to Y_i \to Y$ as in (5). Then we set $V = \coprod Y_i$ and $U = \coprod X_i$ to see that (5) implies (2).

Finally, assume (1) – (3) are true. Thus we can choose any covering $\{Y_j \to Y\}$ as in Definition 11.1 and for each j any covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 11.1. Then $X_{ij} \to Y_j$ is locally of finite type by (3) and we see that (4) is true. This concludes the proof. □

Example 24.7. Let S be a scheme. Let A be a weakly admissible topological ring over S. Let $A \to A'$ be a finite type ring map. Then

$$(A')^\wedge = \lim_{I \subset A \ w.i.d.} A'/IA'$$

is a weakly admissible ring and the corresponding morphism $\text{Spf}((A')^\wedge) \to \text{Spf}(A)$ is representable, see Example 19.11. If $T \to \text{Spf}(A)$ is a morphism where T is a quasi-compact scheme, then this factors through $\text{Spec}(A/I)$ for some weak ideal of definition $I \subset A$ (Lemma 9.4). Then $T \times_{\text{Spf}(A)} \text{Spf}((A')^\wedge)$ is equal to $T \times_{\text{Spec}(A/I)} \text{Spec}(A'/IA')$ and we see that $\text{Spf}((A')^\wedge) \to \text{Spf}(A)$ is of finite type.

Lemma 24.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. If Y is locally Noetherian and f locally of finite type, then X is locally Noetherian.*

Proof. Pick $\{Y_j \to Y\}$ and $\{X_{ij} \to Y_j \times_Y X\}$ as in Lemma 24.6. Then it follows from Lemma 19.9 that each X_{ij} is Noetherian. This proves the lemma. □

Lemma 24.9. *Let S be a scheme. Let $f : X \to Y$ and $Z \to Y$ be morphisms of formal algebraic spaces over S. If Z is locally Noetherian and f locally of finite type, then $Z \times_Y X$ is locally Noetherian.*

Proof. The morphism $Z \times_Y X \to Z$ is locally of finite type by Lemma 24.4. Hence this follows from Lemma 24.8. □

25. Surjective morphisms

By Lemma 12.4 the following definition does not clash with the already existing definitions for morphisms of algebraic spaces or morphisms of formal algebraic spaces which are representable by algebraic spaces.

Definition 25.1. Let S be a scheme. A morphism $f : X \to Y$ of formal algebraic spaces over S is said to be *surjective* if it induces a surjective morphism $X_{red} \to Y_{red}$ on underlying reduced algebraic spaces.

Lemma 25.2. *The composition of two surjective morphisms is a surjective morphism.*

Proof. Omitted. □

Lemma 25.3. *A base change of a surjective morphism is a surjective morphism.*

Proof. Omitted. □

Lemma 25.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

1. *f is surjective,*
2. *for every scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a surjective morphism of formal algebraic spaces,*
3. *for every affine scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a surjective morphism of formal algebraic spaces,*
4. *there exists a covering $\{Y_j \to Y\}$ as in Definition 11.1 such that each $X \times_Y Y_j \to Y_j$ is a surjective morphism of formal algebraic spaces,*
5. *there exists a surjective morphism $Z \to Y$ of formal algebraic spaces such that $X \times_Y Z \to Z$ is surjective, and*
6. *add more here.*

Proof. Omitted. □

26. Monomorphisms

Here is the definition.

Definition 26.1. Let S be a scheme. A morphism of formal algebraic spaces over S is called a *monomorphism* if it is an injective map of sheaves.

An example is the following. Let X be an algebraic space and let $T \subset |X|$ be a closed subset. Then the morphism $X_{/T} \to X$ from the formal completion of X along T to X is a monomorphism. In particular, monomorphisms of formal algebraic spaces are in general not representable.

Lemma 26.2. *The composition of two monomorphisms is a monomorphism.*

Proof. Omitted. □

Lemma 26.3. *A base change of a monomorphism is a monomorphism.*

Proof. Omitted. □

Lemma 26.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

1. *f is a monomorphism,*
2. *for every scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a monomorphism of formal algebraic spaces,*
3. *for every affine scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a monomorphism of formal algebraic spaces,*
4. *there exists a covering $\{Y_j \to Y\}$ as in Definition 11.1 such that each $X \times_Y Y_j \to Y_j$ is a monomorphism of formal algebraic spaces, and*
5. *there exists a family of morphisms $\{Y_j \to Y\}$ such that $\coprod Y_j \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$ such that each $X \times_Y Y_j \to Y_j$ is a monomorphism for all j,*
6. *there exists a morphism $Z \to Y$ of formal algebraic spaces which is representable by algebraic spaces, surjective, flat, and locally of finite presentation such that $X \times_Y Z \to X$ is a monomorphism, and*
7. *add more here.*

Proof. Omitted. □

27. Closed immersions

Here is the definition.

Definition 27.1. Let S be a scheme. Let $f : Y \to X$ be a morphism of formal algebraic spaces over S. We say f is a *closed immersion* if f is representable by algebraic spaces and a closed immersion in the sense of Bootstrap, Definition 4.1.

Please skip the initial the obligatory lemmas when reading this section.

Lemma 27.2. *The composition of two closed immersions is a closed immersion.*

Proof. Omitted. □

Lemma 27.3. *A base change of a closed immersion is a closed immersion.*

Proof. Omitted. □

Lemma 27.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

1. *f is a closed immersion,*
2. *for every scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a closed immersion,*
3. *for every affine scheme T and morphism $T \to Y$ the projection $X \times_Y T \to T$ is a closed immersion,*
4. *there exists a covering $\{Y_j \to Y\}$ as in Definition 11.1 such that each $X \times_Y Y_j \to Y_j$ is a closed immersion, and*
5. *there exists a morphism $Z \to Y$ of formal algebraic spaces which is representable by algebraic spaces, surjective, flat, and locally of finite presentation such that $X \times_Y Z \to X$ is a closed immersion, and*
6. *add more here.*

Proof. Omitted. □

Lemma 27.5. *Let S be a scheme. Let X be a McQuillan affine formal algebraic space over S. Let $f : Y \to X$ be a closed immersion of formal algebraic spaces over S. Then Y is a McQuillan affine formal algebraic space and f corresponds to a continuous homomorphism $A \to B$ of weakly admissible topological S-algebras which is taut, has closed kernel, and has dense image.*

Proof. Write $X = \text{Spf}(A)$ where A is a weakly admissible topological ring. Let I_λ be a fundamental system of weakly admissible ideals of definition in A. Then $Y \times_X \text{Spec}(A/I_\lambda)$ is a closed subscheme of $\text{Spec}(A/I_\lambda)$ and hence affine (Definition 27.1). Say $Y \times_X \text{Spec}(A/I_\lambda) = \text{Spec}(B_\lambda)$. The ring map $A/I_\lambda \to B_\lambda$ is surjective. Hence the projections

$$B = \lim B_\lambda \longrightarrow B_\lambda$$

are surjective as the compositions $A \to B \to B_\lambda$ are surjective. It follows that Y is McQuillan by Lemma 9.6. The ring map $A \to B$ is taut by Lemma 19.8. The kernel is closed because B is complete and $A \to B$ is continuous. Finally, as $A \to B_\lambda$ is surjective for all λ we see that the image of A in B is dense. □

Even though we have the result above, in general we do not know how closed immersions behave when the target is a McQuillan affine formal algebraic space, see Remark 29.4.

Example 27.6. Let S be a scheme. Let A be a weakly admissible topological ring over S. Let $K \subset A$ be a closed ideal. Setting

$$B = (A/K)^\wedge = \lim_{I \subset A \ w.i.d.} A/(I + K)$$

the morphism $\text{Spf}(B) \to \text{Spf}(A)$ is representable, see Example 19.11. If $T \to \text{Spf}(A)$ is a morphism where T is a quasi-compact scheme, then this factors through $\text{Spec}(A/I)$ for some weak ideal of definition $I \subset A$ (Lemma 9.4). Then $T \times_{\text{Spf}(A)} \text{Spf}(B)$ is equal to $T \times_{\text{Spec}(A/I)} \text{Spec}(A/(K + I))$ and we see that $\text{Spf}(B) \to \text{Spf}(A)$ is a closed immersion. The kernel of $A \to B$ is K as K is closed, but beware that in general the ring map $A \to B = (A/K)^\wedge$ need not be surjective.

Lemma 27.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume*

1. *f is representable by algebraic spaces,*
2. *f is a monomorphism,*
3. *the inclusion $Y_{red} \to Y$ factors through f, and*
4. *f is locally of finite type or Y is locally Noetherian.*

Then f is a closed immersion.

Proof. Assumptions (2) and (3) imply that $X_{red} = X \times_Y Y_{red} = Y_{red}$. We will use this without further mention.

If $Y' \to Y$ is an étale morphism of formal algebraic spaces over S, then the base change $f' : X \times_Y Y' \to Y'$ satisfies conditions (1) – (4). Hence by Lemma 27.4 we may assume Y is an affine formal algebraic space.

Say $Y = \text{colim}_{\lambda \in \Lambda} Y_\lambda$ as in Definition 9.1. Then $X_\lambda = X \times_Y Y_\lambda$ is an algebraic space endowed with a monomorphism $f_\lambda : X_\lambda \to Y_\lambda$ which induces an isomorphism $X_{\lambda,red} \to Y_{\lambda,red}$. Thus X_λ is an affine scheme by Limits of Spaces, Proposition 15.2 (as $X_{\lambda,red} \to X_\lambda$ is surjective and integral). To finish the proof it suffices to show that $X_\lambda \to Y_\lambda$ is a closed immersion which we will do in the next paragraph.

Let $X \to Y$ be a monomorphism of affine schemes such that $X_{red} = X \times_Y Y_{red} = Y_{red}$. In general, this does not imply that $X \to Y$ is a closed immersion, see Examples, Section 35. However, under our assumption (4) we know that in the previous parapgrah either $X_\lambda \to Y_\lambda$ is of finite type or Y_λ is Noetherian. This means that $X \to Y$ corresponds to a ring map $R \to A$ such that $R/I \to A/IA$ is an isomorphism where $I \subset R$ is the nil radical (ie., the maximal locally nilpotent ideal of R) and either $R \to A$ is of finite type or R is Noetherian. In the first case $R \to A$ is surjective by Algebra, Lemma 126.9 and in the second case I is finitely generated, hence nilpotent, hence $R \to A$ is surjective by Nakayama's lemma, see Algebra, Lemma 20.1 part (11). ☐

28. Restricted power series

Let A be a topological ring complete with respect to a linear topology (More on Algebra, Definition 36.1). Let I_λ be a fundamental system of open ideals. Let $r \geq 0$ be an integer. In this setting one often denotes

$$A\{x_1, \ldots, x_r\} = \lim_\lambda A/I_\lambda[x_1, \ldots, x_r] = \lim_\lambda (A[x_1, \ldots, x_r]/I_\lambda A[x_1, \ldots, x_r])$$

endowed with the limit topology. In other words, this is the completion of the polynomial ring with respect to the ideals I_λ. We can think of elements of $A\{x_1, \ldots, x_r\}$

as power series

$$f = \sum\nolimits_{E=(e_1,\ldots,e_r)} a_E x_1^{e_1} \ldots x_r^{e_r}$$

in $x_1, \ldots, x_r$ with coefficients $a_E \in A$ which tend to zero in the topology of A. In other words, for any λ all but a finite number of a_E are in I_λ. For this reason elements of $A\{x_1, \ldots, x_r\}$ are sometimes called *restricted power series*. Sometimes this ring is denoted $A\langle x_1, \ldots, x_r\rangle$; we will refrain from using this notation.

Remark 28.1 (Universal property restricted power series)**.** Let $A \to C$ be a continuous map of complete linearly topologized rings. Then any A-algebra map $A[x_1, \ldots x_r] \to C$ extends uniquely to a continuous map $A\{x_1, \ldots, x_r\} \to C$ on restricted power series.

Remark 28.2. Let A be a ring and let $I \subset A$ be an ideal. If A is I-adically complete, then the I-adic completion $A[x_1, \ldots, x_r]^\wedge$ of $A[x_1, \ldots, x_r]$ is the restricted power series ring over A as a ring. However, it is not clear that $A[x_1, \ldots, x_r]^\wedge$ is I-adically complete. We think of the topology on $A\{x_1, \ldots, x_r\}$ as the limit topology (which is always complete) whereas we often think of the topology on $A[x_1, \ldots, x_r]^\wedge$ as the I-adic topology (not always complete). If I is finitely generated, then $A\{x_1, \ldots, x_r\} = A[x_1, \ldots, x_r]^\wedge$ as topological rings, see Algebra, Lemma 96.3.

29. Algebras topologically of finite type

Here is our definition. This definition is not generally agreed upon. Many authors impose further conditions, often because they are only interested in specific types of rings and not the most general case.

Definition 29.1. Let $A \to B$ be a continuous map of topological rings (More on Algebra, Definition 36.1). We say B is *topologically of finite type over* A if there exists an A-algebra map $A[x_1, \ldots, x_n] \to B$ whose image is dense in B.

If A is a complete, linearly topologized ring, then the restricted power series ring $A\{x_1, \ldots, x_r\}$ is topologically of finite type over A. If k is a field, then the power series ring $k[[x_1, \ldots, x_r]]$ is topologically of finite type over k.

For continuous taut maps of weakly admissible topological rings, being topologically of finite type corresponds exactly to morphisms of finite type between the associated affine formal algebraic spaces.

Lemma 29.2. *Let S be a scheme. Let $\varphi : A \to B$ be a continuous map of weakly admissible topological rings over S. The following are equivalent*

1. *$Spf(\varphi) : Y = Spf(B) \to Spf(A) = X$ is of finite type,*
2. *φ is taut and B is topologically of finite type over A.*

Proof. We can use Lemma 19.8 to relate tautness of φ to representability of $\text{Spf}(\varphi)$. We will use this without further mention below. It follows that $X = \operatorname{colim} \text{Spec}(A/I)$ and $Y = \operatorname{colim} \text{Spec}(B/J(I))$ where $I \subset A$ runs over the weak ideals of definition of A and $J(I)$ is the closure of IB in B.

Assume (2). Choose a ring map $A[x_1, \ldots, x_r] \to B$ whose image is dense. Then $A[x_1, \ldots, x_r] \to B \to B/J(I)$ has dense image too which means that it is surjective. Therefore $B/J(I)$ is of finite type over A/I. Let $T \to X$ be a morphism with T a quasi-compact scheme. Then $T \to X$ factors through $\text{Spec}(A/I)$ for some I

(Lemma 9.4). Then $T \times_X Y = T \times_{\mathrm{Spec}(A/I)} \mathrm{Spec}(B/J(I))$, see proof of Lemma 19.8. Hence $T \times_Y X \to T$ is of finite type as the base change of the morphism $\mathrm{Spec}(B/J(I)) \to \mathrm{Spec}(A/I)$ which is of finite type. Thus (1) is true.

Assume (1). Pick any $I \subset A$ as above. Since $\mathrm{Spec}(A/I) \times_X Y = \mathrm{Spec}(B/J(I))$ we see that $A/I \to B/J(I)$ is of finite type. Choose $b_1, \ldots, b_r \in B$ mapping to generators of $B/J(I)$ over A/I. We claim that the image of the ring map $A[x_1, \ldots, x_r] \to B$ sending x_i to b_i is dense. To prove this, let $I' \subset I$ be a second weak ideal of definition. Then we have

$$B/(J(I') + IB) = B/J(I)$$

because $J(I)$ is the closure of IB and because $J(I')$ is open. Hence we may apply Algebra, Lemma 126.9 to see that $(A/I')[x_1, \ldots, x_r] \to B/J(I')$ is surjective. Thus (2) is true, concluding the proof. $\square$

Let A be a topological ring complete with respect to a linear topology. Let (I_λ) be a fundamental system of open ideals. Let $\mathcal{C}$ be the category of inverse systems (B_λ) where

(1) B_λ is a finite type A/I_λ-algebra, and
(2) $B_\mu \to B_\lambda$ is an A/I_μ-algebra homomorphism which induces an isomorphism $B_\mu/I_\lambda B_\mu \to B_\lambda$.

Morphisms in $\mathcal{C}$ are given by compatible systems of homomorphisms.

Lemma 29.3. *Let S be a scheme. Let X be an affine formal algebraic space over S. Assume X is McQuillan and let A be the weakly admissible topological ring associated to X. Then there is an anti-equivalence of categories between*

(1) *the category $\mathcal{C}$ introduced above, and*
(2) *the category of maps $Y \to X$ of finite type of affine formal algebraic spaces.*

Proof. Let (I_λ) be a fundamental system of weakly admissible ideals of definition in A. Consider Y as in (2). Then $Y \times_X \mathrm{Spec}(A/I_\lambda)$ is affine (Definition 24.1 and Lemma 19.7). Say $Y \times_X \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$. The ring map $A/I_\lambda \to B_\lambda$ is of finite type because $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A/I_\lambda)$ is of finite type (by Definition 24.1). Then (B_λ) is an object of $\mathcal{C}$.

Conversely, given an object (B_λ) of $\mathcal{C}$ we can set $Y = \mathrm{colim}\, \mathrm{Spec}(B_\lambda)$. This is an affine formal algebraic space. We claim that

$$Y \times_X \mathrm{Spec}(A/I_\lambda) = (\mathrm{colim}_\mu \mathrm{Spec}(B_\mu)) \times_X \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$$

To show this it suffices we get the same values if we evaluate on a quasi-compact scheme U. A morphism $U \to (\mathrm{colim}_\mu \mathrm{Spec}(B_\mu)) \times_X \mathrm{Spec}(A/I_\lambda)$ comes from a morphism $U \to \mathrm{Spec}(B_\mu) \times_{\mathrm{Spec}(A/I_\mu)} \mathrm{Spec}(A/I_\lambda)$ for some $\mu \geq \lambda$ (use Lemma 9.4 two times). Since $\mathrm{Spec}(B_\mu) \times_{\mathrm{Spec}(A/I_\mu)} \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$ by our second assumption on objects of $\mathcal{C}$ this proves what we want. Using this we can show the morphism $Y \to X$ is of finite type. Namely, we note that for any morphism $U \to X$ with U a quasi-compact scheme, we get a factorization $U \to \mathrm{Spec}(A/I_\lambda) \to X$ for some λ (see lemma cited above). Hence

$$Y \times_X U = Y \times_X \mathrm{Spec}(A/I_\lambda)) \times_{\mathrm{Spec}(A/I_\lambda)} U = \mathrm{Spec}(B_\lambda) \times_{\mathrm{Spec}(A/I_\lambda)} U$$

is a scheme of finite type over U as desired. Thus the construction $(B_\lambda) \mapsto \mathrm{colim}\, \mathrm{Spec}(B_\lambda)$ does give a functor from category (1) to category (2).

To finish the proof we show that the above constructions define quasi-inverse functors between the categories (1) and (2). In one direction you have to show that

$$(\mathrm{colim}_\mu \operatorname{Spec}(B_\mu)) \times_X \operatorname{Spec}(A/I_\lambda) = \operatorname{Spec}(B_\lambda)$$

for any object (B_λ) in the category $\mathcal{C}$. This we proved above. For the other direction you have to show that

$$Y = \mathrm{colim}(Y \times_X \operatorname{Spec}(A/I_\lambda))$$

given Y in the category (2). Again this is true by evaluating on quasi-compact test objects and because $X = \mathrm{colim} \operatorname{Spec}(A/I_\lambda)$. □

Remark 29.4. Let A be a weakly admissible topological ring and let (I_λ) be a fundamental system of weak ideals of definition. Let $X = \operatorname{Spf}(A)$, in other words, X is a McQuillan affine formal algebraic space. Let $f : Y \to X$ be a morphism of affine formal algebraic spaces. In general it will not be true that Y is McQuillan. More specifically, we can ask the following questions:

(1) Assume that $f : Y \to X$ is a closed immersion. Then Y is McQuillan and f corresponds to a continuous map $\varphi : A \to B$ of weakly admissible topological rings which is taut, whose kernel $K \subset A$ is a closed ideal, and whose image $\varphi(A)$ is dense in B, see Lemma 27.5. What conditions on A guarantee that $B = (A/K)^\wedge$ as in Example 27.6?
(2) What conditions on A guarantee that closed immersions $f : Y \to X$ correspond to quotients A/K of A by closed ideals, in other words, the corresponding continuous map φ is surjective and open?
(3) Suppose that $f : Y \to X$ is of finite type. Then we get $Y = \mathrm{colim} \operatorname{Spec}(B_\lambda)$ where (B_λ) is an object of $\mathcal{C}$ by Lemma 29.3. In this case it is true that there exists a fixed integer r such that B_λ is generated by r elements over A/I_λ for all λ (the argument is essentially already given in the proof of (1) $\Rightarrow$ (2) in Lemma 29.2). However, it is not clear that the projections $\lim B_\lambda \to B_\lambda$ are surjective, i.e., it is not clear that Y is McQuillan. Is there an example where Y is not McQuillan?
(4) Suppose that $f : Y \to X$ is of finite type and Y is McQuillan. Then f corresponds to a continuous map $\varphi : A \to B$ of weakly admissible topological rings. In fact φ is taut and B is topologically of finite type over A, see Lemma 29.2. In other words, f factors as
$$Y \longrightarrow \mathbf{A}^r_X \longrightarrow X$$
where the first arrow is a closed immersion of McQuillan affine formal algebraic spaces. However, then questions (1) and (2) are in force for $Y \to \mathbf{A}^r_X$.

Below we will answer these questions when X is countably indexed, i.e., when A has a countable fundamental system of open ideals. If you have answers to these questions in greater generality, or if you have counter examples, please email stacks.project@gmail.com.

Lemma 29.5. *Let S be a scheme. Let X be a countably indexed affine formal algebraic space over S. Let $f : Y \to X$ be a closed immersion of formal algebraic spaces over S. Then Y is a countably indexed affine formal algebraic space and f corresponds to $A \to A/K$ where A is an object of $\textit{WAdm}^{count}$ (Section 21) and $K \subset A$ is a closed ideal.*

Proof. By Lemma 10.4 we see that $X = \mathrm{Spf}(A)$ where A is an object of $\textit{WAdm}^{count}$. Since a closed immersion is representable and affine, we conclude by Lemma 19.9 that Y is an affine formal algebraic space and countably index. Thus applying Lemma 10.4 again we see that $Y = \mathrm{Spf}(B)$ with B an object of $\textit{WAdm}^{count}$. By Lemma 27.5 we conclude that f is given by a morphism $A \to B$ of $\textit{WAdm}^{count}$ which is taut and has dense image. To finish the proof we apply Lemma 5.10. $\square$

Lemma 29.6. *Let $B \to A$ be an arrow of $\textit{WAdm}^{count}$, see Section 21. The following are equivalent*

(a) *$B \to A$ is taut and $B/J \to A/I$ is of finite type for every weak ideal of definition $J \subset B$ where $I \subset A$ is the closure of JA,*
(b) *$B \to A$ is taut and $B/J_\lambda \to A/I_\lambda$ is of finite type for a cofinal system (J_λ) of weak ideals of definition of B where $I_\lambda \subset A$ is the closure of $J_\lambda A$,*
(c) *$B \to A$ is taut and A is topologically of finite type over B,*
(d) *A is isomorphic as a topological B-algebra to a quotient of $B\{x_1, \ldots, x_n\}$ by a closed ideal.*

Moreover, these equivalent conditions define a local property, i.e., they satisfy Axioms (1), (2), (3).

Proof. The implications (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (a), (d) $\Rightarrow$ (c) are straightforward from the definitions. Assume (b) holds and let $J \subset B$ and $I \subset A$ be as in (a). Choose a commutative diagram

$$\begin{tikzcd} A \arrow[r] & \ldots \arrow[r] & A_3 \arrow[r] & A_2 \arrow[r] & A_1 \\ B \arrow[r] \arrow[u] & \ldots \arrow[r] & B/J_3 \arrow[r] \arrow[u] & B/J_2 \arrow[r] \arrow[u] & B/J_1 \arrow[u] \end{tikzcd}$$

such that $A_{n+1}/J_n A_{n+1} = A_n$ and such that $A = \lim A_n$ as in Lemma 22.1. For every m there exists a λ such that $J_\lambda \subset J_m$. Since $B/J_\lambda \to A/I_\lambda$ is of finite type, this implies that $B/J_m \to A/I_m$ is of finite type. Let $\alpha_1, \ldots, \alpha_n \in A_1$ be generators of A_1 over B/J_1. Since A is a countable limit of a system with surjective transition maps, we can find $a_1, \ldots, a_n \in A$ mapping to $\alpha_1, \ldots, \alpha_n$ in A_1. By Remark 28.1 we find a continuous map $B\{x_1, \ldots, x_n\} \to A$ mapping x_i to a_i. This map induces surjections $(B/J_m)[x_1, \ldots, x_n] \to A_m$ by Algebra, Lemma 126.9. For $m \geq 1$ we obtain a short exact sequence

$$0 \to K_m \to (B/J_m)[x_1, \ldots, x_n] \to A_m \to 0$$

The induced transition maps $K_{m+1} \to K_m$ are surjective because $A_{m+1}/J_m A_{m+1} = A_m$. Hence the inverse limit of these short exact sequences is exact, see Algebra, Lemma 86.4. Since $B\{x_1, \ldots, x_n\} = \lim (B/J_m)[x_1, \ldots, x_n]$ and $A = \lim A_m$ we conclude that $B\{x_1, \ldots, x_n\} \to A$ is surjective and open. As A is complete the kernel is a closed ideal. In this way we see that (a), (b), (c), and (d) are equivalent.

Let a diagram (21.2.1) as in Situation 21.2 be given. By Example 24.7 the maps $A \to (A')^\wedge$ and $B \to (B')^\wedge$ satisfy (a), (b), (c), and (d). Moreover, by Lemma 22.1 in order to prove Axioms (1) and (2) we may assume both $B \to A$ and $(B')^\wedge \to (A')^\wedge$ are taut. Now pick a weak ideal of definition $J \subset B$. Let $J' \subset (B')^\wedge$, $I \subset A$, $I' \subset (A')^\wedge$ be the closure of $J(B')^\wedge$, JA, $J(A')^\wedge$. By what was said above, it suffices to consider the commutative diagram

$$\begin{tikzcd} A/I \arrow[r] & (A')^\wedge/I' \\ B/J \arrow[r] \arrow[u, "\overline{\varphi}"] & (B')^\wedge/J' \arrow[u, "\overline{\varphi}'"'] \end{tikzcd}$$

and to show (1) $\overline{\varphi}$ finite type $\Rightarrow$ $\overline{\varphi}'$ finite type, and (2) if $A \to A'$ is faithfully flat, then $\overline{\varphi}'$ finite type $\Rightarrow$ $\overline{\varphi}$ finite type. Note that $(B')^\wedge/J' = B'/JB'$ and $(A')^\wedge/I' = A'/IA'$ by the construction of the topologies on $(B')^\wedge$ and $(A')^\wedge$. In particular the horizontal maps in the diagram are étale. Part (1) now follows from Algebra, Lemma 6.2 and part (2) from Descent, Lemma 14.2 as the ring map $A/I \to (A')^\wedge/I' = A'/IA'$ is faithfully flat and étale.

We omit the proof of Axiom (3). $\square$

Lemma 29.7. *In Lemma 29.6 if B is admissible (for example adic), then the equivalent conditions (a) – (d) are also equivalent to*

(e) *$B \to A$ is taut and $B/J \to A/I$ is of finite type for some ideal of definition $J \subset B$ where $I \subset A$ is the closure of JA.*

Proof. It is enough to show that (e) implies (a). Let $J' \subset B$ be a weak ideal of definition and let $I' \subset A$ be the closure of $J'A$. We have to show that $B/J' \to A/I'$ is of finite type. If the corresponding statement holds for the smaller weak ideal of definition $J'' = J' \cap J$, then it holds for J'. Thus we may assume $J' \subset J$. As J is an ideal of definition (and not just a weak ideal of definition), we get $J^n \subset J'$ for some $n \geq 1$. Thus we can consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & I/I' \arrow[r] & A/I' \arrow[r] & A/I \arrow[r] & 0 \\ 0 \arrow[r] & J/J' \arrow[r] \arrow[u] & B/J' \arrow[r] \arrow[u] & B/J \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. Since $I' \subset A$ is open and since I is the closure of JA we see that $I/I' = (J/J') \cdot A/I'$. Because J/J' is a nilpotent ideal and as $B/J \to A/I$ is of finite type, we conclude from Algebra, Lemma 126.8 that A/I' is of finite type over B/J' as desired. $\square$

Lemma 29.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of affine formal algebraic spaces. Assume Y countably indexed. The following are equivalent*

(1) *f is locally of finite type,*
(2) *f is of finite type,*
(3) *f corresponds to a morphism $B \to A$ of $\textit{WAdm}^{count}$ (Section 21) satisfying the equivalent conditions of Lemma 29.6.*

Proof. Since X and Y are affine it is clear that conditions (1) and (2) are equivalent. In cases (1) and (2) the morphism f is representable by algebraic spaces by definition, hence affine by Lemma 19.7. Thus if (1) or (2) holds we see that X is countably indexed by Lemma 19.9. Write $X = \text{Spf}(A)$ and $Y = \text{Spf}(B)$ for topological S-algebras A and B in $\textit{WAdm}^{count}$, see Lemma 10.4. By Lemma 9.10 we see that f corresponds to a continuous map $B \to A$. Hence now the result follows from Lemma 29.2. $\square$

Lemma 29.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. The following are equivalent*

1. *for every commutative diagram*
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a morphism of $WAdm^{count}$ which is taut and topologically of finite type,
2. *there exists a covering $\{Y_j \to Y\}$ as in Definition 11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Definition 11.1 such that each $X_{ji} \to Y_j$ corresponds to a morphism of $WAdm^{count}$ which is taut and topologically of finite type,*
3. *there exist a covering $\{X_i \to X\}$ as in Definition 11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a morphism of $WAdm^{count}$ which is, taut and topologically of finite type, and*
4. *f is locally of finite type.*

Proof. By Lemma 29.6 the property $P(\varphi) =$"φ is taut and topologically of finite type" is local on WAdm^{count}. Hence by Lemma 21.3 we see that conditions (1), (2), and (3) are equivalent. On the other hand, by Lemma 29.8 the condition P on morphisms of $WAdm^{count}$ corresponds exactly to morphisms of countably indexed, affine formal algebraic spaces being locally of finite type. Thus the implication (1) $\Rightarrow$ (3) of Lemma 24.6 shows that (4) implies (1) of the current lemma. Similarly, the implication (4) $\Rightarrow$ (1) of Lemma 24.6 shows that (2) implies (4) of the current lemma. □

30. Separation axioms for morphisms

This section is the analogue of Morphisms of Spaces, Section 4 for morphisms of formal algebraic spaces.

Definition 30.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism.

1. We say f is *separated* if $\Delta_{X/Y}$ is a closed immersion.
2. We say f is *quasi-separated* if $\Delta_{X/Y}$ is quasi-compact.

Since $\Delta_{X/Y}$ is representable (by schemes) by Lemma 15.5 we can test this by considering morphisms $T \to X \times_Y X$ from affine schemes T and checking whether

$$E = T \times_{X \times_Y X} X \longrightarrow T$$

is quasi-compact or a closed immersion, see Lemma 17.5 or Definition 27.1. Note that the scheme E is the equalizer of two morphisms $a, b : T \to X$ which agree as morphisms into Y and that $E \to T$ is a monomorphism and locally of finite type.

Lemma 30.2. *All of the separation axioms listed in Definition 30.1 are stable under base change.*

Proof. Let $f : X \to Y$ and $Y' \to Y$ be morphisms of formal algebraic spaces. Let $f' : X' \to Y'$ be the base change of f by $Y' \to Y$. Then $\Delta_{X'/Y'}$ is the base change of $\Delta_{X/Y}$ by the morphism $X' \times_{Y'} X' \to X \times_Y X$. Each of the properties of the diagonal used in Definition 30.1 is stable under base change. Hence the lemma is true. $\square$

Lemma 30.3. *Let S be a scheme. Let $f : X \to Z$, $g : Y \to Z$ and $Z \to T$ be morphisms of formal algebraic spaces over S. Consider the induced morphism $i : X \times_Z Y \to X \times_T Y$. Then*

1. *i is representable (by schemes), locally of finite type, locally quasi-finite, separated, and a monomorphism,*
2. *if $Z \to T$ is separated, then i is a closed immersion, and*
3. *if $Z \to T$ is quasi-separated, then i is quasi-compact.*

Proof. By general category theory the following diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "i"'] \arrow[d] & X \times_T Y \arrow[d] \\ Z \arrow[r, "\Delta_{Z/T}"] & Z \times_T Z \end{tikzcd}$$

is a fibre product diagram. Hence i is the base change of the diagonal morphism $\Delta_{Z/T}$. Thus the lemma follows from Lemma 15.5. $\square$

Lemma 30.4. *All of the separation axioms listed in Definition 30.1 are stable under composition of morphisms.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces to which the axiom in question applies. The diagonal $\Delta_{X/Z}$ is the composition

$$X \longrightarrow X \times_Y X \longrightarrow X \times_Z X.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 30.3 above we see that the second arrow also has this property. Hence the lemma follows since the composition of (representable) morphisms with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$. $\square$

Lemma 30.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Let $\mathcal{P}$ be any of the separation axioms of Definition 30.1. The following are equivalent*

1. *f is $\mathcal{P}$,*
2. *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
3. *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
4. *for every affine scheme Z and every morphism $Z \to Y$ the formal algebraic space $Z \times_Y X$ is $\mathcal{P}$ (see Definition 16.3),*
5. *there exists a covering $\{Y_j \to Y\}$ as in Definition 11.1 such that the base change $Y_j \times_Y X \to Y_j$ has $\mathcal{P}$ for all j.*

Proof. We will repeatedly use Lemma 30.2 without further mention. In particular, it is clear that (1) implies (2) and (2) implies (3).

Assume (3) and let $Z \to Y$ be a morphism where Z is an affine scheme. Let U, V be affine schemes and let $a : U \to Z \times_Y X$ and $b : V \to Z \times_Y X$ be morphisms. Then

$$U \times_{Z \times_Y X} V = (Z \times_Y X) \times_{\Delta, (Z \times_Y X) \times_Z (Z \times_Y X)} (U \times_Z V)$$

and we see that this is quasi-compact if $\mathcal{P} =$"quasi-separated" or an affine scheme equipped with a closed immersion into $U \times_Z V$ if $\mathcal{P} =$"separated". Thus (4) holds.

Assume (4) and let $Z \to Y$ be a morphism where Z is an affine scheme. Let U, V be affine schemes and let $a : U \to Z \times_Y X$ and $b : V \to Z \times_Y X$ be morphisms. Reading the argument above backwards, we see that $U \times_{Z \times_Y X} V \to U \times_Z V$ is quasi-compact if $\mathcal{P} =$"quasi-separated" or a closed immersion if $\mathcal{P} =$"separated". Since we can choose U and V as above such that U varies through an étale covering of $Z \times_Y X$, we find that the corresponding morphisms

$$U \times_Z V \to (Z \times_Y X) \times_Z (Z \times_Y X)$$

form an étale covering by affines. Hence we conclude that $\Delta : (Z \times_Y X) \to (Z \times_Y X) \times_Z (Z \times_Y X)$ is quasi-compact, resp. a closed immersion. Thus (3) holds.

Let us prove that (3) implies (5). Assume (3) and let $\{Y_j \to Y\}$ be as in Definition 11.1. We have to show that the morphisms

$$\Delta_j : Y_j \times_Y X \longrightarrow (Y_j \times_Y X) \times_{Y_j} (Y_j \times_Y X) = Y_j \times_Y X \times_Y X$$

has the corresponding property (i.e., is quasi-compact or a closed immersion). Write $Y_j = \mathrm{colim}\, Y_{j,\lambda}$ as in Definition 9.1. Replacing Y_j by $Y_{j,\lambda}$ in the formula above, we have the property by our assumption that (3) holds. Since the displayed arrow is the colimit of the arrows $\Delta_{j,\lambda}$ and since we can test whether Δ_j has the corresponding property by testing after base change by affine schemes mapping into $Y_j \times_Y X \times_Y X$, we conclude by Lemma 9.4.

Let us prove that (5) implies (1). Let $\{Y_j \to Y\}$ be as in (5). Then we have the fibre product diagram

$$\begin{tikzcd}
\coprod Y_j \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
\coprod Y_j \times_Y X \times_Y X \arrow[r] & X \times_Y X
\end{tikzcd}$$

By assumption the left vertical arrow is quasi-compact or a closed immersion. It follows from Spaces, Lemma 5.6 that also the right vertical arrow is quasi-compact or a closed immersion. ☐

31. Proper morphisms

Here is the definition we will use.

Definition 31.1. Let S be a scheme. Let $f : Y \to X$ be a morphism of formal algebraic spaces over S. We say f is *proper* if f is representable by algebraic spaces and is proper in the sense of Bootstrap, Definition 4.1.

It follows from the definitions that a proper morphism is of finite type.

Lemma 31.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. The following are equivalent*

(1) *f is proper,*

(2) *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is proper,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is proper,*
(4) *for every affine scheme Z and every morphism $Z \to Y$ the formal algebraic space $Z \times_Y X$ is an algebraic space proper over Z,*
(5) *there exists a covering $\{Y_j \to Y\}$ as in Definition 11.1 such that the base change $Y_j \times_Y X \to Y_j$ is proper for all j.*

Proof. Omitted. ☐

Lemma 31.3. *Proper morphisms of formal algebraic spaces are preserved by base change.*

Proof. This is an immediate consequence of Lemma 31.2 and transitivity of base change. ☐

32. Formal algebraic spaces and fpqc coverings

This section is the analogue of Properties of Spaces, Section 17. Please read that section first.

Lemma 32.1. *Let S be a scheme. Let X be a formal algebraic space over S. Then X satisfies the sheaf property for the fpqc topology.*

Proof. The proof is **identical** to the proof of Properties of Spaces, Proposition 17.1. Since X is a sheaf for the Zariski topology it suffices to show the following. Given a surjective flat morphism of affines $f : T' \to T$ we have: $X(T)$ is the equalizer of the two maps $X(T') \to X(T' \times_T T')$. See Topologies, Lemma 9.13.

Let $a, b : T \to X$ be two morphisms such that $a \circ f = b \circ f$. We have to show $a = b$. Consider the fibre product

$$E = X \times_{\Delta_{X/S}, X \times_S X, (a,b)} T.$$

By Lemma 11.2 the morphism $\Delta_{X/S}$ is a representable monomorphism. Hence $E \to T$ is a monomorphism of schemes. Our assumption that $a \circ f = b \circ f$ implies that $T' \to T$ factors (uniquely) through E. Consider the commutative diagram

$$\begin{tikzcd}
T' \times_T E \arrow[r] \arrow[d] & E \arrow[d] \\
T' \arrow[r] \arrow[ru] \arrow[u, bend left] & T
\end{tikzcd}$$

Since the projection $T' \times_T E \to T'$ is a monomorphism with a section we conclude it is an isomorphism. Hence we conclude that $E \to T$ is an isomorphism by Descent, Lemma 23.17. This means $a = b$ as desired.

Next, let $c : T' \to X$ be a morphism such that the two compositions $T' \times_T T' \to T' \to X$ are the same. We have to find a morphism $a : T \to X$ whose composition with $T' \to T$ is c. Choose a formal affine scheme U and an étale morphism $U \to X$ such that the image of $|U| \to |X_{red}|$ contains the image of $|c| : |T'| \to |X_{red}|$. This is possible by Definition 11.1, Properties of Spaces, Lemma 4.6, the fact that a finite union of formal affine algebraic spaces is a formal affine algebraic space, and the fact that $|T'|$ is quasi-compact (small argument omitted). The morphism $U \to X$ is representable by schemes (Lemma 9.11) and separated (Lemma 16.5). Thus

$$V = U \times_{X,c} T' \longrightarrow T'$$

is an étale and separated morphism of schemes. It is also surjective by our choice of $U \to X$ (if you do not want to argue this you can replace U by a disjoint union of formal affine algebraic spaces so that $U \to X$ is surjective everything else still works as well). The fact that $c \circ \mathrm{pr}_0 = c \circ \mathrm{pr}_1$ means that we obtain a descent datum on $V/T'/T$ (Descent, Definition 34.1) because

$$\begin{aligned} V \times_{T'} (T' \times_T T') &= U \times_{X, c\circ \mathrm{pr}_0} (T' \times_T T') \\ &= (T' \times_T T') \times_{c \circ \mathrm{pr}_1, X} U \\ &= (T' \times_T T') \times_{T'} V \end{aligned}$$

The morphism $V \to T'$ is ind-quasi-affine by More on Morphisms, Lemma 66.8 (because étale morphisms are locally quasi-finite, see Morphisms, Lemma 36.6). By More on Groupoids, Lemma 15.3 the descent datum is effective. Say $W \to T$ is a morphism such that there is an isomorphism $\alpha : T' \times_T W \to V$ compatible with the given descent datum on V and the canonical descent datum on $T' \times_T W$. Then $W \to T$ is surjective and étale (Descent, Lemmas 23.7 and 23.29). Consider the composition

$$b' : T' \times_T W \longrightarrow V = U \times_{X,c} T' \longrightarrow U$$

The two compositions $b' \circ (\mathrm{pr}_0, 1), b' \circ (\mathrm{pr}_1, 1) : (T' \times_T T') \times_T W \to T' \times_T W \to U$ agree by our choice of α and the corresponding property of c (computation omitted). Hence b' descends to a morphism $b : W \to U$ by Descent, Lemma 13.7. The diagram

$$\begin{tikzcd} T' \times_T W \arrow[r] \arrow[d] & W \arrow[r, "b"'] & U \arrow[d] \\ T' \arrow[rr, "c"] & & X \end{tikzcd}$$

is commutative. What this means is that we have proved the existence of a étale locally on T, i.e., we have an $a' : W \to X$. However, since we have proved uniqueness in the first paragraph, we find that this étale local solution satisfies the glueing condition, i.e., we have $\mathrm{pr}_0^* a' = \mathrm{pr}_1^* a'$ as elements of $X(W \times_T W)$. Since X is an étale sheaf we find an unique $a \in X(T)$ restricting to a' on W. □

33. Maps out of affine formal schemes

We prove a few results that will be useful later. In the paper [Bha16] the reader can find very general results of a similar nature.

Lemma 33.1. *Let S be a scheme. Let A be a weakly admissible topological S-algebra. Let X be an affine scheme over S. Then the natural map*

$$\mathrm{Mor}_S(\mathrm{Spec}(A), X) \longrightarrow \mathrm{Mor}_S(\mathit{Spf}(A), X)$$

is bijective.

Proof. If X is affine, say $X = \mathrm{Spec}(B)$, then we see from Lemma 9.10 that morphisms $\mathrm{Spf}(A) \to \mathrm{Spec}(B)$ correspond to continuous S-algebra maps $B \to A$ where B has the discrete topology. These are just S-algebra maps, which correspond to morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$. □

Lemma 33.2. *Let S be a scheme. Let A be a weakly admissible topological S-algebra such that A/I is a local ring for some weak ideal of definition $I \subset A$. Let X be a scheme over S. Then the natural map*

$$\mathrm{Mor}_S(\mathrm{Spec}(A), X) \longrightarrow \mathrm{Mor}_S(\mathit{Spf}(A), X)$$

is bijective.

Proof. Let $\varphi : \mathrm{Spf}(A) \to X$ be a morphism. Since $\mathrm{Spec}(A/I)$ is local we see that φ maps $\mathrm{Spec}(A/I)$ into an affine open $U \subset X$. However, this then implies that $\mathrm{Spec}(A/J)$ maps into U for every ideal of definition J. Hence we may apply Lemma 33.1 to see that φ comes from a morphism $\mathrm{Spec}(A) \to X$. This proves surjectivity of the map. We omit the proof of injectivity. $\square$

Lemma 33.3. *Let S be a scheme. Let R be a complete local Noetherian S-algebra. Let X be an algebraic space over S. Then the natural map*

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \mathrm{Mor}_S(\mathit{Spf}(R), X)$$

is bijective.

Proof. Let $\mathfrak m$ be the maximal ideal of R. We have to show that

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak m^n), X)$$

is bijective for R as above.

Injectivity: Let $x, x' : \mathrm{Spec}(R) \to X$ be two morphisms mapping to the same element in the right hand side. Consider the fibre product

$$T = \mathrm{Spec}(R) \times_{(x,x'), X \times_S X, \Delta} X$$

Then T is a scheme and $T \to \mathrm{Spec}(R)$ is locally of finite type, monomorphism, separated, and locally quasi-finite, see Morphisms of Spaces, Lemma 4.1. In particular T is locally Noetherian, see Morphisms, Lemma 15.6. Let $t \in T$ be the unique point mapping to the closed point of $\mathrm{Spec}(R)$ which exists as x and x' agree over $R/\mathfrak m$. Then $R \to \mathcal{O}_{T,t}$ is a local ring map of Noetherian rings such that $R/\mathfrak m^n \to \mathcal{O}_{T,t}/\mathfrak m^n\mathcal{O}_{T,t}$ is an isomorphism for all n (because x and x' agree over $\mathrm{Spec}(R/\mathfrak m^n)$ for all n). Since $\mathcal{O}_{T,t}$ maps injectively into its completion (see Algebra, Lemma 51.4) we conclude that $R = \mathcal{O}_{T,t}$. Hence x and x' agree over R.

Surjectivity: Let (x_n) be an element of the right hand side. Choose a scheme U and a surjective étale morphism $U \to X$. Denote $x_0 : \mathrm{Spec}(k) \to X$ the morphism induced on the residue field $k = R/\mathfrak m$. The morphism of schemes $U \times_{X,x_0} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is surjective étale. Thus $U \times_{X,x_0} \mathrm{Spec}(k)$ is a nonempty disjoint union of spectra of finite separable field extensions of k, see Morphisms, Lemma 36.7. Hence we can find a finite separable field extension k'/k and a k'-point $u_0 : \mathrm{Spec}(k') \to U$ such that

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r, "u_0"'] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "x_0"] & X \end{tikzcd}$$

commutes. Let $R \subset R'$ be the finite étale extension of Noetherian complete local rings which induces k'/k on residue fields (see Algebra, Lemmas 153.7 and 153.9). Denote x'_n the restriction of x_n to $\mathrm{Spec}(R'/\mathfrak m^n R')$. By More on Morphisms of Spaces, Lemma 16.8 we can find an element $(u'_n) \in \lim \mathrm{Mor}_S(\mathrm{Spec}(R'/\mathfrak m^n R'), U)$

mapping to (x'_n). By Lemma 33.2 the family (u'_n) comes from a unique morphism $u' : \mathrm{Spec}(R') \to U$. Denote $x' : \mathrm{Spec}(R') \to X$ the composition. Note that $R' \otimes_R R'$ is a finite product of spectra of Noetherian complete local rings to which our current discussion applies. Hence the diagram

$$\begin{tikzcd} \mathrm{Spec}(R' \otimes_R R') \arrow[r] \arrow[d] & \mathrm{Spec}(R') \arrow[d, "x'"] \\ \mathrm{Spec}(R') \arrow[r, "x'"] & X \end{tikzcd}$$

is commutative by the injectivity shown above and the fact that x'_n is the restriction of x_n which is defined over $R/\mathfrak{m}^n$. Since $\{\mathrm{Spec}(R') \to \mathrm{Spec}(R)\}$ is an fppf covering we conclude that x' descends to a morphism $x : \mathrm{Spec}(R) \to X$. We omit the proof that x_n is the restriction of x to $\mathrm{Spec}(R/\mathfrak{m}^n)$. ☐

Lemma 33.4. *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset such that $X \setminus T \to X$ is quasi-compact. Let R be a complete local Noetherian S-algebra. Then an adic morphism $p : Spf(R) \to X_{/T}$ corresponds to a unique morphism $g : \mathrm{Spec}(R) \to X$ such that $g^{-1}(T) = \{\mathfrak{m}_R\}$.*

Proof. The statement makes sense because $X_{/T}$ is adic* by Lemma 20.8 (and hence we're allowed to use the terminology adic for morphisms, see Definition 23.2). Let p be given. By Lemma 33.3 we get a unique morphism $g : \mathrm{Spec}(R) \to X$ corresponding to the composition $\mathrm{Spf}(R) \to X_{/T} \to X$. Let $Z \subset X$ be the reduced induced closed subspace structure on T. The incusion morphism $Z \to X$ corresponds to a morphism $Z \to X_{/T}$. Since p is adic it is representable by algebraic spaces and we find

$$\mathrm{Spf}(R) \times_{X_{/T}} Z = \mathrm{Spf}(R) \times_X Z$$

is an algebraic space endowed with a closed immersion to $\mathrm{Spf}(R)$. (Equality holds because $X_{/T} \to X$ is a monomorphism.) Thus this fibre product is equal to $\mathrm{Spec}(R/J)$ for some ideal $J \subset R$ wich contains $\mathfrak{m}_R^{n_0}$ for some $n_0 \geq 1$. This implies that $\mathrm{Spec}(R) \times_X Z$ is a closed subscheme of $\mathrm{Spec}(R)$, say $\mathrm{Spec}(R) \times_X Z = \mathrm{Spec}(R/I)$, whose intersection with $\mathrm{Spec}(R/\mathfrak{m}_R^n)$ for $n \geq n_0$ is equal to $\mathrm{Spec}(R/J)$. In algebraic terms this says $I + \mathfrak{m}_R^n = J + \mathfrak{m}_R^n = J$ for all $n \geq n_0$. By Krull's intersection theorem this implies $I = J$ and we conclude. ☐

34. The small étale site of a formal algebraic space

The motivation for the following definition comes from classical formal schemes: the underlying topological space of a formal scheme $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is the underlying topological space of the reduction $\mathfrak{X}_{red}$.

An important remark is the following. Suppose that X is an algebraic space with reduction X_{red} (Properties of Spaces, Definition 12.5). Then we have

$$X_{spaces,étale} = X_{red,spaces,étale}, \quad X_{étale} = X_{red,étale}, \quad X_{affine,étale} = X_{red,affine,étale}$$

by More on Morphisms of Spaces, Theorem 8.1 and Lemma 8.2. Therefore the following definition does not conflict with the already existing notion in case our formal algebraic space happens to be an algebraic space.

Definition 34.1. Let S be a scheme. Let X be a formal algebraic space with reduction X_{red} (Lemma 12.1).

(1) The *small étale site* $X_{étale}$ of X is the site $X_{red,étale}$ of Properties of Spaces, Definition 18.1.
(2) The site $X_{spaces,étale}$ is the site $X_{red,spaces,étale}$ of Properties of Spaces, Definition 18.2.
(3) The site $X_{affine,étale}$ is the site $X_{red,affine,étale}$ of Properties of Spaces, Lemma 18.6.

In Lemma 34.6 we will see that $X_{spaces,étale}$ can be described by in terms of morphisms of formal algebraic spaces which are representable by algebraic spaces and étale. By Properties of Spaces, Lemmas 18.3 and 18.6 we have identifications

$$Sh(X_{étale}) = Sh(X_{spaces,étale}) = Sh(X_{affine,étale}) \tag{34.1.1}$$

We will call this the *(small) étale topos* of X.

Lemma 34.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S.*

(1) *There is a continuous functor $Y_{spaces,étale} \to X_{spaces,étale}$ which induces a morphism of sites*
$$f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}.$$
(2) *The rule $f \mapsto f_{spaces,étale}$ is compatible with compositions, in other words $(f \circ g)_{spaces,étale} = f_{spaces,étale} \circ g_{spaces,étale}$ (see Sites, Definition 14.5).*
(3) *The morphism of topoi associated to $f_{spaces,étale}$ induces, via (34.1.1), a morphism of topoi $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ whose construction is compatible with compositions.*

Proof. The only point here is that f induces a morphism of reductions $X_{red} \to Y_{red}$ by Lemma 12.1. Hence this lemma is immediate from the corresponding lemma for morphisms of algebraic spaces (Properties of Spaces, Lemma 18.8). □

If the morphism of formal algebraic spaces $X \to Y$ is étale, then the morphism of topoi $Sh(X_{étale}) \to Sh(Y_{étale})$ is a localization. Here is a statement.

Lemma 34.3. *Let S be a scheme, and let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Assume f is representable by algebraic spaces and étale. In this case there is a cocontinuous functor $j : X_{étale} \to Y_{étale}$. The morphism of topoi f_{small} is the morphism of topoi associated to j, see Sites, Lemma 21.1. Moreover, j is continuous as well, hence Sites, Lemma 21.5 applies.*

Proof. This will follow immediately from the case of algebraic spaces (Properties of Spaces, Lemma 18.11) if we can show that the induced morphism $X_{red} \to Y_{red}$ is étale. Observe that $X \times_Y Y_{red}$ is an algebraic space, étale over the reduced algebraic space Y_{red}, and hence reduced itself (by our definition of reduced algebraic spaces in Properties of Spaces, Section 7. Hence $X_{red} = X \times_Y Y_{red}$ as desired. □

Lemma 34.4. *Let S be a scheme. Let X be an affine formal algebraic space over S. Then $X_{affine,étale}$ is equivalent to the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that*

(1) *U is an affine formal algebraic space,*
(2) *φ is representable by algebraic spaces and étale.*

Proof. Denote $\mathcal{C}$ the category introduced in the lemma. Observe that for $\varphi : U \to X$ in $\mathcal{C}$ the morphism φ is representable (by schemes) and affine, see Lemma 19.7. Recall that $X_{affine,étale} = X_{red,affine,étale}$. Hence we can define a functor

$$\mathcal{C} \longrightarrow X_{affine,étale}, \quad (U \to X) \longmapsto U \times_X X_{red}$$

because $U \times_X X_{red}$ is an affine scheme.

To finish the proof we will construct a quasi-inverse. Namely, write $X = \operatorname{colim} X_\lambda$ as in Definition 9.1. For each λ we have $X_{red} \subset X_\lambda$ is a thickening. Thus for every λ we have an equivalence

$$X_{red,affine,étale} = X_{\lambda,affine,étale}$$

for example by More on Algebra, Lemma 11.2. Hence if $U_{red} \to X_{red}$ is an étale morphism with U_{red} affine, then we obtain a system of étale morphisms $U_\lambda \to X_\lambda$ of affine schemes compatible with the transition morphisms in the system defining X. Hence we can take

$$U = \operatorname{colim} U_\lambda$$

as our affine formal algebraic space over X. The construction gives that $U \times_X X_\lambda = U_\lambda$. This shows that $U \to X$ is representable and étale. We omit the verification that the constructions are mutually inverse to each other. □

Lemma 34.5. *Let S be a scheme. Let X be an affine formal algebraic space over S. Assume X is McQuillan, i.e., equal to Spf(A) for some weakly admissible topological S-algebra A. Then $(X_{affine,étale})^{opp}$ is equivalent to the category whose*

1. *objects are A-algebras of the form $B^\wedge = \lim B/JB$ where $A \to B$ is an étale ring map and J runs over the weak ideals of definition of A, and*
2. *morphisms are continuous A-algebra homomorphisms.*

Proof. Combine Lemmas 34.4 and 19.13. □

Lemma 34.6. *Let S be a scheme. Let X be a formal algebraic space over S. Then $X_{spaces,étale}$ is equivalent to the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that φ is representable by algebraic spaces and étale.*

Proof. Denote $\mathcal{C}$ the category introduced in the lemma. Recall that $X_{spaces,étale} = X_{red,spaces,étale}$. Hence we can define a functor

$$\mathcal{C} \longrightarrow X_{spaces,étale}, \quad (U \to X) \longmapsto U \times_X X_{red}$$

because $U \times_X X_{red}$ is an algebraic space étale over X_{red}.

To finish the proof we will construct a quasi-inverse. Choose an object $\psi : V \to X_{red}$ of $X_{red,spaces,étale}$. Consider the functor $U_{V,\psi} : (Sch/S)_{fppf} \to Sets$ given by

$$U_{V,\psi}(T) = \{(a,b) \mid a : T \to X,\ b : T \times_{a,X} X_{red} \to V,\ \psi \circ b = a|_{T \times_{a,X} X_{red}}\}$$

We claim that the transformation $U_{V,\psi} \to X$, $(a,b) \mapsto a$ defines an object of the category $\mathcal{C}$. First, let's prove that $U_{V,\psi}$ is a formal algebraic space. Observe that $U_{V,\psi}$ is a sheaf for the fppf topology (some details omitted). Next, suppose that $X_i \to X$ is an étale covering by affine formal algebraic spaces as in Definition 11.1. Set $V_i = V \times_{X_{red}} X_{i,red}$ and denote $\psi_i : V_i \to X_{i,red}$ the projection. Then we have

$$U_{V,\psi} \times_X X_i = U_{V_i,\psi_i}$$

by a formal argument because $X_{i,red} = X_i \times_X X_{red}$ (as $X_i \to X$ is representable by algebraic spaces and étale). Hence it suffices to show that U_{V_i,ψ_i} is an affine formal algebraic space, because then we will have a covering $U_{V_i,\psi_i} \to U_{V,\psi}$ as in Definition 11.1. On the other hand, we have seen in the proof of Lemma 34.3 that $\psi_i : V_i \to X_i$ is the base change of a representable and étale morphism $U_i \to X_i$ of affine formal algebraic spaces. Then it is not hard to see that $U_i = U_{V_i,\psi_i}$ as desired.

We omit the verification that $U_{V,\psi} \to X$ is representable by algebraic spaces and étale. Thus we obtain our functor $(V,\psi) \mapsto (U_{V,\psi} \to X)$ in the other direction. We omit the verification that the constructions are mutually inverse to each other. $\square$

Lemma 34.7. *Let S be a scheme. Let X be a formal algebraic space over S. Then $X_{affine,étale}$ is equivalent to the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that*

1. *U is an affine formal algebraic space,*
2. *φ is representable by algebraic spaces and étale.*

Proof. This follows by combining Lemmas 34.6 and 18.3. $\square$

35. The structure sheaf

Let X be a formal algebraic space. A structure sheaf for X is a sheaf of topological rings $\mathcal{O}_X$ on the étale site $X_{étale}$ (which we defined in Section 34) such that

$$\mathcal{O}_X(U_{red}) = \lim \Gamma(U_\lambda, \mathcal{O}_{U_\lambda})$$

as topological rings whenever

1. $\varphi : U \to X$ is a morphism of formal algebraic spaces,
2. U is an affine formal algebraic space,
3. φ is representable by algebraic spaces and étale,
4. $U_{red} \to X_{red}$ is the corresponding affine object of $X_{étale}$, see Lemma 34.7,
5. $U = \operatorname{colim} U_\lambda$ is a colimit representation for U as in Definition 9.1.

Structure sheaves exist but may behave in unexpected manner.

Lemma 35.1. *Every formal algebraic space has a structure sheaf.*

Proof. Let S be a scheme. Let X be a formal algebraic space over S. By (34.1.1) it suffices to construct $\mathcal{O}_X$ as a sheaf of topological rings on $X_{affine,étale}$. Denote $\mathcal{C}$ the category whose objects are morphisms $\varphi : U \to X$ of formal algebraic spaces such that U is an affine formal algebraic space and φ is representable by algebraic spaces and étale. By Lemma 34.7 the functor $U \mapsto U_{red}$ is an equivalence of categories $\mathcal{C} \to X_{affine,étale}$. Hence by the rule given above the lemma, we already have $\mathcal{O}_X$ as a presheaf of topological rings on $X_{affine,étale}$. Thus it suffices to check the sheaf condition.

By definition of $X_{affine,étale}$ a covering corresponds to a finite family $\{g_i : U_i \to U\}_{i=1,\ldots,n}$ of morphisms of $\mathcal{C}$ such that $\{U_{i,red} \to U_{red}\}$ is an étale covering. The morphisms g_i are representably by algebraic spaces (Lemma 19.3) hence affine (Lemma 19.7). Then g_i is étale (follows formally from Properties of Spaces, Lemma 16.6 as U_i and U are étale over X in the sense of Bootstrap, Section 4). Finally, write $U = \operatorname{colim} U_\lambda$ as in Definition 9.1.

With these preparations out of the way, we can prove the sheaf property as follows. For each λ we set $U_{i,\lambda} = U_i \times_U U_\lambda$ and $U_{ij,\lambda} = (U_i \times_U U_j) \times_U U_\lambda$. By the above, these are affine schemes, $\{U_{i,\lambda} \to U_\lambda\}$ is an étale covering, and $U_{ij,\lambda} = U_{i,\lambda} \times_{U_\lambda} U_{j,\lambda}$.

Also we have $U_i = \operatorname{colim} U_{i,\lambda}$ and $U_i \times_U U_j = \operatorname{colim} U_{ij,\lambda}$. For each λ we have an exact sequence

$$0 \to \Gamma(U_\lambda, \mathcal{O}_{U_\lambda}) \to \prod\nolimits_i \Gamma(U_{i,\lambda}, \mathcal{O}_{U_{i,\lambda}}) \to \prod\nolimits_{i,j} \Gamma(U_{ij,\lambda}, \mathcal{O}_{U_{ij,\lambda}})$$

as we have the sheaf condition for the structure sheaf on U_λ and the étale topology (see Étale Cohomology, Proposition 17.1). Since limits commute with limits, the inverse limit of these exact sequences is an exact sequence

$$0 \to \lim \Gamma(U_\lambda, \mathcal{O}_{U_\lambda}) \to \prod\nolimits_i \lim \Gamma(U_{i,\lambda}, \mathcal{O}_{U_{i,\lambda}}) \to \prod\nolimits_{i,j} \lim \Gamma(U_{ij,\lambda}, \mathcal{O}_{U_{ij,\lambda}})$$

which exactly means that

$$0 \to \mathcal{O}_X(U_{red}) \to \prod\nolimits_i \mathcal{O}_X(U_{i,red}) \to \prod\nolimits_{i,j} \mathcal{O}_X((U_i \times_U U_j)_{red})$$

is exact and hence the sheaf propery holds as desired. □

Remark 35.2. The structure sheaf does not always have "enough sections". In Examples, Section 74 we have seen that there exist affine formal algebraic spaces which aren't McQuillan and there are even examples whose points are not separated by regular functions.

In the next lemma we prove that the structure sheaf on a countably indexed affine formal scheme has vanishing higher cohomology. For non-countably indexed ones, presumably this generally doesn't hold.

Lemma 35.3. *If X is a countably indexed affine formal algebraic space, then we have $H^n(X_{étale}, \mathcal{O}_X) = 0$ for $n > 0$.*

Proof. We may work with $X_{affine,étale}$ as this gives the same topos. We will apply Cohomology on Sites, Lemma 10.9 to show we have vanishing. Since $X_{affine,étale}$ has finite disjoint unions, this reduces us to the Čech complex of a covering given by a single arrow $\{U_{red} \to V_{red}\}$ in $X_{affine,étale} = X_{red,affine,étale}$ (see Étale Cohomology, Lemma 22.1). Thus we have to show that

$$0 \to \mathcal{O}_X(V_{red}) \to \mathcal{O}_X(U_{red}) \to \mathcal{O}_X(U_{red} \times_{V_{red}} U_{red}) \to \ldots$$

is exact. We will do this below in the case $V_{red} = X_{red}$. The general case is proven in exactly the same way.

Recall that $X = \text{Spf}(A)$ where A is a weakly admissible topological ring having a countable fundamental system of weak ideals of definition. We have seen in Lemmas 34.4 and 34.5 that the object U_{red} in $X_{affine,étale}$ corresponds to a morphism $U \to X$ of affine formal algebraic spaces which is representable by algebraic space and étale and $U = \text{Spf}(B^\wedge)$ where B is an étale A-algebra. By our rule for the structure sheaf we see

$$\mathcal{O}_X(U_{red}) = B^\wedge$$

We recall that $B^\wedge = \lim B/JB$ where the limit is over weak ideals of definition $J \subset A$. Working through the definitions we obtain

$$\mathcal{O}_X(U_{red} \times_{X_{red}} U_{red}) = (B \otimes_A B)^\wedge$$

and so on. Since $U \to X$ is a covering the map $A \to B$ is faithfully flat, see Lemma 19.14. Hence the complex

$$0 \to A \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \ldots$$

is **universally** exact, see Descent, Lemma 3.6. Our goal is to show that

$$H^n(0 \to A^\wedge \to B^\wedge \to (B \otimes_A B)^\wedge \to (B \otimes_A B \otimes_A B)^\wedge \to \ldots)$$

is zero for $n > 0$. To see what is going on, let's split our exact complex (before completion) into short exact sequences

$$0 \to A \to B \to M_1 \to 0, \quad 0 \to M_i \to B^{\otimes_A i+1} \to M_{i+1} \to 0$$

By what we said above, these are universally exact short exact sequences. Hence $JM_i = M_i \cap J(B^{\otimes_A i+1})$ for every ideal J of A. In particular, the topology on M_i as a submodule of $B^{\otimes_A i+1}$ is the same as the topology on M_i as a quotient module of $B^{\otimes_A i}$. Therefore, since there exists a countable fundamental system of weak ideals of definition in A, the sequences

$$0 \to A^\wedge \to B^\wedge \to M_1^\wedge \to 0, \quad 0 \to M_i^\wedge \to (B^{\otimes_A i+1})^\wedge \to M_{i+1}^\wedge \to 0$$

remain exact by Lemma 4.5. This proves the lemma. $\square$

Remark 35.4. Even if the structure sheaf has good properties, this does not mean there is a good theory of quasi-coherent modules. For example, in Examples, Section 13 we have seen that for almost any Noetherian affine formal algebraic spaces the most natural notion of a quasi-coherent module leads to a category of modules which is not abelian.

36. Colimits of formal algebraic spaces

In this section we generalize the result of Section 13 to the case of systems of morphisms of formal algebraic spaces. We remark that in the lemmas below the condition "$f_{\lambda\mu} : X_\lambda \to X_\mu$ is a closed immersion inducing an isomorphism $X_{\lambda,red} \to X_{\mu,red}$" can be reformulated as "$f_{\lambda\mu}$ is representable and a thickening".

Lemma 36.1. *Let S be a scheme. Suppose given a directed set Λ and a system of affine formal algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a closed immersion inducing an isomorphism $X_{\lambda,red} \to X_{\mu,red}$. Then $X = \mathrm{colim}_{\lambda\in\Lambda} X_\lambda$ is an affine formal algebraic space over S.*

Proof. We may write $X_\lambda = \mathrm{colim}_{\omega\in\Omega_\lambda} X_{\lambda,\omega}$ as the colimit of affine schemes over a directed set Ω_λ such that the transition morphisms $X_{\lambda,\omega} \to X_{\lambda,\omega'}$ are thickenings. For each $\lambda, \mu \in \Lambda$ and $\omega \in \Omega_\lambda$, with $\mu \geq \lambda$ there exists an $\omega' \in \Omega_\mu$ such that the morphism $X_{\lambda,\omega} \to X_\mu$ factors through $X_{\mu,\omega'}$, see Lemma 9.4. Then the morphism $X_{\lambda,\omega} \to X_{\mu,\omega'}$ is a closed immersion inducing an isomorphism on reductions and hence a thickening. Set $\Omega = \coprod_{\lambda\in\Lambda} \Omega_\lambda$ and say $(\lambda, \omega) \leq (\mu, \omega')$ if and only if $\lambda \leq \mu$ and $X_{\lambda,\omega} \to X_\mu$ factors through $X_{\mu,\omega'}$. It follows from the above that Ω is a directed set and that $X = \mathrm{colim}_{\lambda\in\Lambda} X_\lambda = \mathrm{colim}_{(\lambda,\omega)\in\Omega} X_{\lambda,\omega}$. This finishes the proof. $\square$

Lemma 36.2. *Let S be a scheme. Suppose given a directed set Λ and a system of formal algebraic spaces $(X_\lambda, f_{\lambda\mu})$ over Λ where each $f_{\lambda\mu} : X_\lambda \to X_\mu$ is a closed immersion inducing an isomorphism $X_{\lambda,red} \to X_{\mu,red}$. Then $X = \mathrm{colim}_{\lambda\in\Lambda} X_\lambda$ is a formal algebraic space over S.*

Proof. Since we take the colimit in the category of fppf sheaves, we see that X is a sheaf. Choose and fix $\lambda \in \Lambda$. Choose a covering $\{X_{i,\lambda} \to X_\lambda\}$ as in Definition

11.1. In particular, we see that $\{X_{i,\lambda,red} \to X_{\lambda,red}\}$ is an étale covering by affine schemes. For each $\mu \geq \lambda$ there exists a cartesian diagram

$$\begin{tikzcd} X_{i,\lambda} \arrow[r] \arrow[d] & X_{i,\mu} \arrow[d] \\ X_\lambda \arrow[r] & X_\mu \end{tikzcd}$$

with étale vertical arrows. Namely, the étale morphism $X_{i,\lambda,red} \to X_{\lambda,red} = X_{\mu,red}$ corresponds to an étale morphism $X_{i,\mu} \to X_\mu$ of formal algebraic spaces with $X_{i,\mu}$ an affine formal algebraic space, see Lemma 34.4. The same lemma implies the base change of $X_{i,\mu}$ to X_λ agrees with $X_{i,\lambda}$. It also follows that $X_{i,\mu} = X_\mu \times_{X_{\mu'}} X_{i,\mu'}$ for $\mu' \geq \mu \geq \lambda$. Set $X_i = \text{colim}\, X_{i,\mu}$. Then $X_{i,\mu} = X_i \times_X X_\mu$ (as functors). Since any morphism $T \to X = \text{colim}\, X_\mu$ from an affine (or quasi-compact) scheme T maps into X_μ for some μ, we see conclude that $\text{colim}\, X_{i,\mu} \to \text{colim}\, X_\mu$ is étale. Thus, if we can show that $\text{colim}\, X_{i,\mu}$ is an affine formal algebraic space, then the lemma holds. Note that the morphisms $X_{i,\mu} \to X_{i,\mu'}$ are closed immersions as a base change of the closed immersion $X_\mu \to X_{\mu'}$. Finally, the morphism $X_{i,\mu,red} \to X_{i,\mu',red}$ is an isomorphism as $X_{\mu,red} \to X_{\mu',red}$ is an isomorphism. Hence this reduces us to the case discussed in Lemma 36.1. $\square$

37. Recompletion

In this section we define the completion of a formal algebraic space along a closed subset of its reduction. It is the natural generalization of Section 14.

Lemma 37.1. *Let S be a scheme. Let X be an affine formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. Then the functor*

$$X_{/T} : (Sch/S)_{fppf} \longrightarrow Sets, \quad U \longmapsto \{f : U \to X : f(|U|) \subset T\}$$

is an affine formal algebraic space.

Proof. Write $X = \text{colim}\, X_\lambda$ as in Definition 9.1. Then $X_{\lambda,red} = X_{red}$ and we may and do view T as a closed subset of $|X_\lambda| = |X_{\lambda,red}|$. By Lemma 14.1 for each λ the completion $(X_\lambda)_{/T}$ is an affine formal algebraic space. The transition morphisms $(X_\lambda)_{/T} \to (X_\mu)_{/T}$ are closed immersions as base changes of the transition morphisms $X_\lambda \to X_\mu$, see Lemma 14.4. Also the morphisms $((X_\lambda)_{/T})_{red} \to ((X_\mu)_/T)_{red}$ are isomorphisms by Lemma 14.5. Since $X_{/T} = \text{colim}(X_\lambda)_{/T}$ we conclude by Lemma 36.1. $\square$

Lemma 37.2. *Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. Then the functor*

$$X_{/T} : (Sch/S)_{fppf} \longrightarrow Sets, \quad U \longmapsto \{f : U \to X \mid f(|U|) \subset T\}$$

is a formal algebraic space.

Proof. The functor $X_{/T}$ is an fppf sheaf since if $\{U_i \to U\}$ is an fppf covering, then $\coprod |U_i| \to |U|$ is surjective.

Choose a covering $\{g_i : X_i \to X\}_{i \in I}$ as in Definition 11.1. The morphisms $X_i \times_X X_{/T} \to X_{/T}$ are étale (see Spaces, Lemma 5.5) and the map $\coprod X_i \times_X X_{/T} \to X_{/T}$ is a surjection of sheaves. Thus it suffices to prove that $X_{/T} \times_X X_i$ is an affine formal algebraic space. A U-valued point of $X_i \times_X X_{/T}$ is a morphism $U \to X_i$

whose image is contained in the closed subset $|g_{i,red}|^{-1}(T) \subset |X_{i,red}|$. Thus this follows from Lemma 37.1. $\square$

Definition 37.3. Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. The formal algebraic space $X_{/T}$ of Lemma 14.2 is called the *completion of X along T*.

Let $f : X \to X'$ be a morphism of formal algebraic spaces over a scheme S. Suppose that $T \subset |X_{red}|$ and $T' \subset |X'_{red}|$ are closed subsets such that $|f_{red}|(T) \subset T'$. Then it is clear that f defines a morphism of formal algebraic spaces

$$X_{/T} \longrightarrow X'_{/T'}$$

between the completions.

Lemma 37.4. *Let S be a scheme. Let $f : X' \to X$ be a morphism of formal algebraic spaces over S. Let $T \subset |X_{red}|$ be a closed subset and let $T' = |f_{red}|^{-1}(T) \subset |X'_{red}|$. Then*

$$\begin{tikzcd} X'_{/T'} \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ X_{/T} \arrow[r] & X \end{tikzcd}$$

is a cartesian diagram of formal algebraic spaces over S.

Proof. Namely, observe that the horizontal arrows are monomorphisms by construction. Thus it suffices to show that a morphism $g : U \to X'$ from a scheme U defines a point of $X'_{/T}$ if and only if $f \circ g$ defines a point of $X_{/T}$. In other words, we have to show that $g(U)$ is contained in $T' \subset |X'_{red}|$ if and only if $(f \circ g)(U)$ is contained in $T \subset |X_{red}|$. This follows immediately from our choice of T' as the inverse image of T. $\square$

Lemma 37.5. *Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset |X_{red}|$ be a closed subset. The reduction $(X_{/T})_{red}$ of the completion $X_{/T}$ of X along T is the reduced induced closed subspace Z of X_{red} corresponding to T.*

Proof. It follows from Lemma 12.1, Properties of Spaces, Definition 12.5 (which uses Properties of Spaces, Lemma 12.3 to construct Z), and the definition of $X_{/T}$ that Z and $(X_{/T})_{red}$ are reduced algebraic spaces characterized the same mapping property: a morphism $g : Y \to X$ whose source is a reduced algebraic space factors through them if and only if $|Y|$ maps into $T \subset |X|$. $\square$

Lemma 37.6. *Let S be a scheme. Let X be an affine formal algebraic space over S. Let $T \subset X_{red}$ be a closed subset and let $X_{/T}$ be the formal completion of X along T. Then*

1. *$X_{/T}$ is an affine formal algebraic space,*
2. *if X is McQuillan, then $X_{/T}$ is McQuillan,*
3. *if $|X_{red}| \setminus T$ is quasi-compact and X is countably indexed, then $X_{/T}$ is countably indexed,*
4. *if $|X_{red}| \setminus T$ is quasi-compact and X is adic*, then $X_{/T}$ is adic*,*
5. *if X is Noetherian, then $X_{/T}$ is Noetherian.*

Proof. Part (1) is Lemma 37.1. If X is McQuillan, then $X = \text{Spf}(A)$ for some weakly admissible topological ring A. Then $X_{/T} \to X \to \text{Spec}(A)$ satisfies property (2) of Lemma 9.6 and hence $X_{/T}$ is McQuillan, see Definition 9.7.

Assume X and T are as in (3). Then $X = \text{Spf}(A)$ where A has a fundamental system $A \supset I_1 \supset I_2 \supset I_3 \supset \ldots$ of weak ideals of definition, see Lemma 10.4. By Algebra, Lemma 29.1 we can find a finitely generated ideal $\overline{J} = (\overline{f}_1, \ldots, \overline{f}_r) \subset A/I_1$ such that T is cut out by $\overline{J}$ inside $\text{Spec}(A/I_1) = |X_{red}|$. Choose $f_i \in A$ lifting $\overline{f}_i$. If $Z = \text{Spec}(B)$ is an affine scheme and $g : Z \to X$ is a morphism with $g(Z) \subset T$ (set theoretically), then $g^\sharp : A \to B$ factors through A/I_n for some n and $g^\sharp(f_i)$ is nilpotent in B for each i. Thus $J_{m,n} = (f_1, \ldots, f_r)^m + I_n$ maps to zero in B for some $n, m \geq 1$. It follows that $X_{/T}$ is the formal spectrum of $\lim_{n,m} A/J_{m,n}$ and hence countably indexed. This proves (3).

Proof of (4). Here the argument is the same as in (3). However, here we may choose $I_n = I^n$ for some finitely generated ideal $I \subset A$. Then it is clear that $X_{/T}$ is the formal spectrum of $\lim A/J^n$ where $J = (f_1, \ldots, f_r) + I$. Some details omitted.

Proof of (5). In this case X_{red} is the spectrum of a Noetherian ring and hence the assumption that $|X_{red}| \setminus T$ is quasi-compact is satisfied. Thus as in the proof of (4) we see that $X_{/T}$ is the spectrum of $\lim A/J^n$ which is a Noetherian adic topological ring, see Algebra, Lemma 97.6. □

Lemma 37.7. *Let S be a scheme. Let X be a formal algebraic space over S. Let $T \subset X_{red}$ be a closed subset and let $X_{/T}$ be the formal completion of X along T. Then*

1. *if $X_{red} \setminus T \to X_{red}$ is quasi-compact and X is locally countably indexed, then $X_{/T}$ is locally countably indexed,*
2. *if $X_{red} \setminus T \to X_{red}$ is quasi-compact and X is locally adic*, then $X_{/T}$ is locally adic*, and*
3. *if X is locally Noetherian, then $X_{/T}$ is locally Noetherian.*

Proof. Choose a covering $\{X_i \to X\}$ as in Definition 11.1. Let $T_i \subset X_{i,red}$ be the inverse image of T. We have $X_i \times_X X_{/T} = (X_i)_{/T_i}$ (Lemma 37.4). Hence $\{(X_i)_{/T_i} \to X_{/T}\}$ is a covering as in Definition 11.1. Moreover, if $X_{red} \setminus T \to X_{red}$ is quasi-compact, so is $X_{i,red} \setminus T_i \to X_{i,red}$ and if X is locally countably indexed, or locally adic*, pr locally Noetherian, the is X_i is countably index, or adic*, or Noetherian. Thus the lemma follows from the affine case which is Lemma 37.6. □

38. Completion along a closed subspace

This section is the analgue of Section 14 for completions with respect to a closed subspace.

Definition 38.1. Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace and denote $Z_n \subset X$ the nth order infinitesimal neighbourhood. The formal algebraic space

$$X_Z^\wedge = \text{colim}\, Z_n$$

(see Lemma 36.2) is called the *completion of X along Z*.

Observe that if $T = |Z|$ then there is a canonical morphism $X_Z^\wedge \to X_{/T}$ comparing the completions along Z and T (Section 14) which need not be an isomorphism.

Let $f : X \to X'$ be a morphism of algebraic spaces over a scheme S. Suppose that $Z \subset X$ and $Z' \subset X'$ are closed subspaces such that $f|_Z$ maps Z into Z' inducing a morphism $Z \to Z'$. Then it is clear that f defines a morphism of formal algebraic spaces

$$X^\wedge_Z \longrightarrow (X')^\wedge_{Z'}$$

between the completions.

Lemma 38.2. *Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a closed subspace and let $Z' = f^{-1}(Z) = X' \times_X Z$. Then*

$$\begin{tikzcd} (X')^\wedge_{Z'} \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ X^\wedge_Z \arrow[r] & X \end{tikzcd}$$

is a cartesian diagram of sheaves. In particular, the morphism $(X')^\wedge_{Z'} \to X^\wedge_Z$ is representable by algebraic spaces.

Proof. Namely, suppose that $Y \to X$ is a morphism from a scheme into X such that $Y \to X$ factors through Z. Then $Y \times_X X' \to X$ is a morphism of algebraic spaces such that $Y \times_X X' \to X'$ factors through Z'. Since $Z'_n = X' \times_X Z_n$ for all $n \geq 1$ the same is true for the infinitesimal neighbourhoods. Hence the cartesian square of functors follows from the formulas $X^\wedge_Z = \operatorname{colim} Z_n$ and $(X')^\wedge_{Z'} = \operatorname{colim} Z'_n$. ☐

Lemma 38.3. *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. The reduction $(X^\wedge_Z)_{red}$ of the completion $X^\wedge_Z$ of X along Z is Z_{red}.*

Proof. Omitted. ☐

Lemma 38.4. *Let S be a scheme. Let $X = \operatorname{Spec}(A)$ be an affine scheme over S. Let $Z \subset X$ be a closed subscheme. Let $X^\wedge_Z$ be the formal completion of X along Z.*

1. *The affine formal algebraic space $X^\wedge_Z$ is weakly adic.*
2. *If $Z \to X$ is of finite presentation, then $X^\wedge_Z$ is adic*.*
3. *If $Z = V(I)$ for some finitely generated ideal $I \subset A$, then $X^\wedge_Z = Spf(A^\wedge)$ where $A^\wedge$ is the I-adic completion of A.*
4. *If X is Noetherian, then $X^\wedge_Z$ is Noetherian.*

Proof. Omitted. ☐

Lemma 38.5. *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. Let $X^\wedge_Z$ be the formal completion of X along Z.*

1. *The formal algebraic space $X^\wedge_Z$ is locally weakly adic.*
2. *If $Z \to X$ is of finite presentation, then $X^\wedge_Z$ is locally adic*.*
3. *If X is locally Noetherian, then X_Z is locally Noetherian.*

Proof. Omitted. ☐

39. Other chapters

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