TOPOLOGIES ON SCHEMES

Contents

1. Introduction

In this document we explain what the different topologies on the category of schemes are. Some references are [Gro71] and [BLR90]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

2. The general procedure

In this section we explain a general procedure for producing the sites we will be working with. Suppose we want to study sheaves over schemes with respect to some topology τ. In order to get a site, as in Sites, Definition 6.2, of schemes with that topology we have to do some work. Namely, we cannot simply say "consider all schemes with the Zariski topology" since that would give a "big" category. Instead, in each section of this chapter we will proceed as follows:

(1) We define a class Cov_τ of coverings of schemes satisfying the axioms of Sites, Definition 6.2. It will always be the case that a Zariski open covering of a scheme is a covering for τ.
(2) We single out a notion of standard τ-covering within the category of affine schemes.
(3) We define what is an "absolute" big τ-site Sch_τ. These are the sites one gets by appropriately choosing a set of schemes and a set of coverings.

(4) For any object S of Sch_τ we define the big τ-site $(Sch/S)_\tau$ and for suitable τ the small[1] τ-site S_τ.
(5) In addition there is a site $(Aff/S)_\tau$ using the notion of standard τ-covering of affines[2] whose category of sheaves is equivalent to the category of sheaves on $(Sch/S)_\tau$.

The above is a little clumsy in that we do not end up with a canonical choice for the big τ-site of a scheme, or even the small τ-site of a scheme. If you are willing to ignore set theoretic difficulties, then you can work with classes and end up with canonical big and small sites...

3. The Zariski topology

Definition 3.1. Let T be a scheme. A *Zariski covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is an open immersion and such that $T = \bigcup f_i(T_i)$.

This defines a (proper) class of coverings. Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 3.2. *Let T be a scheme.*
(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a Zariski covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering and for each i we have a Zariski covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a Zariski covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a Zariski covering.*

Proof. Omitted. $\square$

Lemma 3.3. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be a Zariski covering of T. Then there exists a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is a standard open of T, see Schemes, Definition 5.2. Moreover, we may choose each U_j to be an open of one of the T_i.*

Proof. Follows as T is quasi-compact and standard opens form a basis for its topology. This is also proved in Schemes, Lemma 5.1. $\square$

Thus we define the corresponding standard coverings of affines as follows.

Definition 3.4. Compare Schemes, Definition 5.2. Let T be an affine scheme. A *standard Zariski covering* of T is a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ with each $U_j \to T$ inducing an isomorphism with a standard affine open of T.

Definition 3.5. A *big Zariski site* is any site Sch_{Zar} as in Sites, Definition 6.2 constructed as follows:
(1) Choose any set of schemes S_0, and any set of Zariski coverings Cov_0 among these schemes.
(2) As underlying category of Sch_{Zar} take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.

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[1] The words big and small here do not relate to bigness/smallness of the corresponding categories.

[2] In the case of the ph topology we deviate very slightly from this approach, see Definition 8.11 and the surrounding discussion.

(3) As coverings of Sch_{Zar} choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of Zariski coverings, and the set Cov_0 chosen above.

It is shown in Sites, Lemma 8.8 that, after having chosen the category Sch_α, the category of sheaves on Sch_α does not depend on the choice of coverings chosen in (3) above. In other words, the topos $Sh(Sch_{Zar})$ only depends on the choice of the category Sch_α. It is shown in Sets, Lemma 9.9 that these categories are closed under many constructions of algebraic geometry, e.g., fibre products and taking open and closed subschemes. We can also show that the exact choice of Sch_α does not matter too much, see Section 12.

Another approach would be to assume the existence of a strongly inaccessible cardinal and to define Sch_{Zar} to be the category of schemes contained in a chosen universe with set of coverings the Zariski coverings contained in that same universe.

Before we continue with the introduction of the big Zariski site of a scheme S, let us point out that the topology on a big Zariski site Sch_{Zar} is in some sense induced from the Zariski topology on the category of all schemes.

Lemma 3.6. *Let Sch_{Zar} be a big Zariski site as in Definition 3.5. Let $T \in \mathrm{Ob}(Sch_{Zar})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary Zariski covering of T. There exists a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{Zar} which is tautologically equivalent (see Sites, Definition 8.2) to $\{T_i \to T\}_{i \in I}$.*

Proof. Since each $T_i \to T$ is an open immersion, we see by Sets, Lemma 9.9 that each T_i is isomorphic to an object V_i of Sch_{Zar}. The covering $\{V_i \to T\}_{i \in I}$ is tautologically equivalent to $\{T_i \to T\}_{i \in I}$ (using the identity map on I both ways). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{Zar} by Sets, Lemma 11.1. $\square$

Definition 3.7. Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S.
   (1) The *big Zariski site of S*, denoted $(Sch/S)_{Zar}$, is the site Sch_{Zar}/S introduced in Sites, Section 25.
   (2) The *small Zariski site of S*, which we denote S_{Zar}, is the full subcategory of $(Sch/S)_{Zar}$ whose objects are those U/S such that $U \to S$ is an open immersion. A covering of S_{Zar} is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}(S_{Zar})$.
   (3) The *big affine Zariski site of S*, denoted $(Aff/S)_{Zar}$, is the full subcategory of $(Sch/S)_{Zar}$ consisting of objects U/S such that U is an affine scheme. A covering of $(Aff/S)_{Zar}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}((Aff/S)_{Zar})$ which is a standard Zariski covering.
   (4) The *small affine Zariski site of S*, denoted $S_{affine,Zar}$, is the full subcategory of S_{Zar} whose objects are those U/S such that U is an affine scheme. A covering of $S_{affine,Zar}$ is any covering $\{U_i \to U\}$ of S_{Zar} with $U \in \mathrm{Ob}(S_{affine,Zar})$ which is a standard Zariski covering.

It is not completely clear that the small Zariski site, the big affine Zariski site, and the small affine Zariski site are sites. We check this now.

Lemma 3.8. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The structures S_{Zar}, $(Aff/S)_{Zar}$, and $S_{affine,Zar}$ defined above are sites.*

Proof. Let us show that S_{Zar} is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 6.2. Since $(Sch/S)_{Zar}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}(S_{Zar})$ we also have $U_i \in \mathrm{Ob}(S_{Zar})$. This follows from the definitions as the composition of open immersions is an open immersion.

Let us show that $(Aff/S)_{Zar}$ is a site. Reasoning as above, it suffices to show that the collection of standard Zariski coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 6.2. Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. For each $i \in \{1, \ldots, n\}$ let $g_{i1}, \ldots, g_{in_i} \in R_{f_i}$ be elements generating the unit ideal of R_{f_i}. Write $g_{ij} = f_{ij}/f_i^{e_{ij}}$ which is possible. After replacing f_{ij} by $f_i f_{ij}$ if necessary, we have that $D(f_{ij}) \subset D(f_i) \cong \mathrm{Spec}(R_{f_i})$ is equal to $D(g_{ij}) \subset \mathrm{Spec}(R_{f_i})$. Hence we see that the family of morphisms $\{D(g_{ij}) \to \mathrm{Spec}(R)\}$ is a standard Zariski covering. From these considerations it follows that (2) holds for standard Zariski coverings. We omit the verification of (1) and (3).

We omit the proof that $S_{affine,Zar}$ is a site. $\square$

Lemma 3.9. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The underlying categories of the sites Sch_{Zar}, $(Sch/S)_{Zar}$, S_{Zar}, $(Aff/S)_{Zar}$, and $S_{affine,Zar}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{Zar}$, and S_{Zar} both have a final object, namely S/S.*

Proof. For Sch_{Zar} it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{Zar})$. The fibre product $V \times_U W$ in Sch_{Zar} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{Zar}$. This proves the result for $(Sch/S)_{Zar}$. If $U \to S$, $V \to U$ and $W \to U$ are open immersions then so is $V \times_U W \to S$ and hence we get the result for S_{Zar}. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{Zar}$ and $S_{affine,Zar}$. $\square$

Next, we check that the big, resp. small affine site defines the same topos as the big, resp. small site.

Lemma 3.10. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The functor $(Aff/S)_{Zar} \to (Sch/S)_{Zar}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh((Aff/S)_{Zar})$ to $Sh((Sch/S)_{Zar})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 29.1. Denote the inclusion functor $u : (Aff/S)_{Zar} \to (Sch/S)_{Zar}$. Being cocontinuous just means that any Zariski covering of T/S, T affine, can be refined by a standard Zariski covering of T. This is the content of Lemma 3.3. Hence (1) holds. We see u is continuous simply because a standard Zariski covering is a Zariski covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. $\square$

Lemma 3.11. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The functor $S_{affine,Zar} \to S_{Zar}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh(S_{affine,Zar})$ to $Sh(S_{Zar})$.*

Proof. Omitted. Hint: compare with the proof of Lemma 3.10. $\square$

Let us check that the notion of a sheaf on the small Zariski site corresponds to notion of a sheaf on S.

Lemma 3.12. *The category of sheaves on S_{Zar} is equivalent to the category of sheaves on the underlying topological space of S.*

Proof. We will use repeatedly that for any object U/S of S_{Zar} the morphism $U \to S$ is an isomorphism onto an open subscheme. Let $\mathcal{F}$ be a sheaf on S. Then we define a sheaf on S_{Zar} by the rule $\mathcal{F}'(U/S) = \mathcal{F}(\mathrm{Im}(U \to S))$. For the converse, we choose for every open subscheme $U \subset S$ an object $U'/S \in \mathrm{Ob}(S_{Zar})$ with $\mathrm{Im}(U' \to S) = U$ (here you have to use Sets, Lemma 9.9). Given a sheaf $\mathcal{G}$ on S_{Zar} we define a sheaf on S by setting $\mathcal{G}'(U) = \mathcal{G}(U'/S)$. To see that $\mathcal{G}'$ is a sheaf we use that for any open covering $U = \bigcup_{i \in I} U_i$ the covering $\{U_i \to U\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j' \to U'\}_{j \in J}$ in S_{Zar} by Sets, Lemma 11.1, and we use Sites, Lemma 8.4. Details omitted. $\square$

From now on we will not make any distinction between a sheaf on S_{Zar} or a sheaf on S. We will always use the procedures of the proof of the lemma to go between the two notions. Next, we establish some relationships between the topoi associated to these sites.

Lemma 3.13. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor $T_{Zar} \to (Sch/S)_{Zar}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{Zar}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{Zar} \to (Sch/S)_{Zar}$. In other words, given and open immersion $j : U \to T$ corresponding to an object of T_{Zar} we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 3.9. Moreover, T_{Zar} has equalizers (as any two morphisms with the same source and target are the same) and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 21.5 and 21.6. $\square$

Lemma 3.14. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The inclusion functor $S_{Zar} \to (Sch/S)_{Zar}$ satisfies the hypotheses of Sites, Lemma 21.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{Zar} \longrightarrow S_{Zar}$$

and a morphism of topoi

$$i_S : Sh(S_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 3.13. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{Zar} \to (Sch/S)_{Zar}$, in addition to the properties seen in the proof of Lemma 3.13 above, also is fully faithful and transforms the final object into the final object. The lemma follows. $\square$

Definition 3.15. In the situation of Lemma 3.14 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small Zariski site*, and for a sheaf $\mathcal{F}$ on the big Zariski site we denote $\mathcal{F}|_{S_{Zar}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\mathrm{Mor}_{Sh(S_{Zar})}(\mathcal{F}|_{S_{Zar}}, \mathcal{G}) = \mathrm{Mor}_{Sh((Sch/S)_{Zar})}(\mathcal{F}, i_{S,*}\mathcal{G})$$

$$\mathrm{Mor}_{Sh(S_{Zar})}(\mathcal{G}, \mathcal{F}|_{S_{Zar}}) = \mathrm{Mor}_{Sh((Sch/S)_{Zar})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$.

Lemma 3.16. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor*

$$u : (Sch/T)_{Zar} \longrightarrow (Sch/S)_{Zar}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{Zar} \longrightarrow (Sch/T)_{Zar}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 3.13). Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for $f_{big,*}$. $\square$

Lemma 3.17. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}.*

  (1) *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 3.13 and i_T as in Lemma 3.14.*
  (2) *The functor $S_{Zar} \to T_{Zar}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of topoi*

$$f_{small} : Sh(T_{Zar}) \longrightarrow Sh(S_{Zar}).$$

  The functors f_{small}^{-1} and $f_{small,*}$ agree with the usual notions f^{-1} and f_* is we identify sheaves on T_{Zar}, resp. S_{Zar} with sheaves on T, resp. S via Lemma 3.12.
  (3) *We have a commutative diagram of morphisms of sites*

$$
\begin{array}{ccc}
T_{Zar} & \xleftarrow{\ \pi_T\ } & (Sch/T)_{Zar} \\
{\scriptstyle f_{small}}\downarrow & & \downarrow{\scriptstyle f_{big}} \\
S_{Zar} & \xleftarrow{\ \pi_S\ } & (Sch/S)_{Zar}
\end{array}
$$

so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.

(4) *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

Statement (2): See Sites, Example 14.2.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functor $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. $\qquad\qquad\square$

In the situation of the lemma, using the terminology of Definition 3.15 we have: for $\mathcal{F}$ a sheaf on the big Zariski site of T

$$(f_{big,*}\mathcal{F})|_{S_{Zar}} = f_{small,*}(\mathcal{F}|_{T_{Zar}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small Zariski site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

Lemma 3.18. *Given schemes X, Y, Z in $(Sch/S)_{Zar}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 3.16. For the functors on the small sites this is Sheaves, Lemma 21.2 via the identification of Lemma 3.12. $\qquad\square$

Lemma 3.19. *Let Sch_{Zar} be a big Zariski site. Consider a cartesian diagram*

$$
\begin{array}{ccc}
T' & \xrightarrow{\,g'\,} & T \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
S' & \xrightarrow{\,g\,} & S
\end{array}
$$

in Sch_{Zar}. Then $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.

Proof. Since the diagram is cartesian, we have for U'/S' that $U' \times_{S'} T' = U' \times_S T$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(Sch/T)_{Zar}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{S'} T')$ on S'_{Zar} (use Lemmas 3.13 and 3.17). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 28.1. $\qquad\square$

We can think about a sheaf on the big Zariski site of S as a collection of "usual" sheaves on all schemes over S.

Lemma 3.20. *Let S be a scheme contained in a big Zariski site Sch_{Zar}. A sheaf $\mathcal{F}$ on the big Zariski site $(Sch/S)_{Zar}$ is given by the following data:*

(1) *for every $T/S \in \mathrm{Ob}((Sch/S)_{Zar})$ a sheaf $\mathcal{F}_T$ on T,*

(2) *for every $f : T' \to T$ in $(Sch/S)_{Zar}$ a map $c_f : f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

(a) *given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{Zar}$ the composition $c_g \circ g^{-1}c_f$ is equal to $c_{f \circ g}$, and*

(b) *if $f : T' \to T$ in $(Sch/S)_{Zar}$ is an open immersion then c_f is an isomorphism.*

Proof. This lemma follows from a purely sheaf theoretic statement discussed in Sites, Remark 26.7. We also give a direct proof in this case.

Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U'/S)$ for any open $U \subset T$, and $U' \to T$ an open immersion in $(Sch/T)_{Zar}$ with image U, see Lemmas 3.12 and 3.13. Hence given $f : T' \to T$ over S and $U, U' \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U'/S) \to \mathcal{F}(U' \times_T T'/S) = \mathcal{F}_{T'}(f^{-1}(U))$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U' \times_T T' \to U'$ over S. The collection of these maps are compatible with restrictions, and hence define an f-map c_f from $\mathcal{F}_T$ to $\mathcal{F}_{T'}$, see Sheaves, Definition 21.7 and the discussion surrounding it. It is clear that $c_{f \circ g}$ is the composition of c_f and c_g, since composition of restriction maps of $\mathcal{F}$ gives restriction maps.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ (where we think of c_f as an f-map again). The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. $\square$

4. The étale topology

Let S be a scheme. We would like to define the étale-topology on the category of schemes over S. According to our general principle we first introduce the notion of an étale covering.

Definition 4.1. Let T be a scheme. An *étale covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is étale and such that $T = \bigcup f_i(T_i)$.

Lemma 4.2. *Any Zariski covering is an étale covering.*

Proof. This is clear from the definitions and the fact that an open immersion is an étale morphism, see Morphisms, Lemma 36.9. $\square$

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 4.3. *Let T be a scheme.*
  (1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an étale covering of T.*
  (2) *If $\{T_i \to T\}_{i \in I}$ is an étale covering and for each i we have an étale covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an étale covering.*
  (3) *If $\{T_i \to T\}_{i \in I}$ is an étale covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an étale covering.*

Proof. Omitted. $\square$

Lemma 4.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an étale covering of T. Then there exists an étale covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted. $\square$

Thus we define the corresponding standard coverings of affines as follows.

Definition 4.5. Let T be an affine scheme. A *standard étale covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\dots,m}$ with each U_j is affine and étale over T and $T = \bigcup f_j(U_j)$.

In the definition above we do **not** assume the morphisms f_j are standard étale. The reason is that if we did then the standard étale coverings would not define a site on *Aff/S*, for example because of Algebra, Lemma 144.2 part (4). On the other hand, an étale morphism of affines is automatically standard smooth, see Algebra, Lemma 143.2. Hence a standard étale covering is a standard smooth covering and a standard syntomic covering.

Definition 4.6. A *big étale site* is any site $Sch_{\acute{e}tale}$ as in Sites, Definition 6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of étale coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of étale coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big étale site of a scheme S, let us point out that the topology on a big étale site $Sch_{\acute{e}tale}$ is in some sense induced from the étale topology on the category of all schemes.

Lemma 4.7. *Let $Sch_{\acute{e}tale}$ be a big étale site as in Definition 4.6. Let $T \in \mathrm{Ob}(Sch_{\acute{e}tale})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary étale covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j \in J}$ of T in the site $Sch_{\acute{e}tale}$ which refines $\{T_i \to T\}_{i \in I}$.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a standard étale covering, then it is tautologically equivalent to a covering in $Sch_{\acute{e}tale}$.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{\acute{e}tale}$.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 4.3 the refinement $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an étale covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of $Sch_{\acute{e}tale}$. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of $Sch_{\acute{e}tale}$ by a second application of Sets, Lemma 9.9. The covering $\{V_i \to T\}_{i \in I}$ refines $\{T_i \to T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site $Sch_{\acute{e}tale}$ by Sets, Lemma 9.9. The covering $\{U_j \to T\}_{j \in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of $Sch_{\acute{e}tale}$ by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. $\square$

Definition 4.8. Let S be a scheme. Let $Sch_{\acute{e}tale}$ be a big étale site containing S.

(1) The *big étale site of S*, denoted $(Sch/S)_{\acute{e}tale}$, is the site $Sch_{\acute{e}tale}/S$ introduced in Sites, Section 25.

(2) The *small étale site of S*, which we denote $S_{étale}$, is the full subcategory
of $(Sch/S)_{étale}$ whose objects are those U/S such that $U \to S$ is étale.
A covering of $S_{étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with $U \in$
$\mathrm{Ob}(S_{étale})$.
(3) The *big affine étale site of S*, denoted $(Aff/S)_{étale}$, is the full subcategory of
$(Sch/S)_{étale}$ whose objects are those U/S such that U is an affine scheme.
A covering of $(Aff/S)_{étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with
$U \in \mathrm{Ob}((Aff/S)_{étale})$ which is a standard étale covering.
(4) The *small affine étale site of S*, denoted $S_{affine,étale}$, is the full subcategory
of $S_{étale}$ whose objects are those U/S such that U is an affine scheme.
A covering of $S_{affine,étale}$ is any covering $\{U_i \to U\}$ of $S_{étale}$ with $U \in$
$\mathrm{Ob}(S_{affine,étale})$ which is a standard étale covering.

It is not completely clear that the big affine étale site, the small étale site, and the
small affine étale site are sites. We check this now.

Lemma 4.9. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The
structures $S_{étale}$, $(Aff/S)_{étale}$, and $S_{affine,étale}$ are sites.*

Proof. Let us show that $S_{étale}$ is a site. It is a category with a given set of
families of morphisms with fixed target. Thus we have to show properties (1), (2)
and (3) of Sites, Definition 6.2. Since $(Sch/S)_{étale}$ is a site, it suffices to prove
that given any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with $U \in \mathrm{Ob}(S_{étale})$ we also
have $U_i \in \mathrm{Ob}(S_{étale})$. This follows from the definitions as the composition of étale
morphisms is an étale morphism.

Let us show that $(Aff/S)_{étale}$ is a site. Reasoning as above, it suffices to show that
the collection of standard étale coverings of affines satisfies properties (1), (2) and
(3) of Sites, Definition 6.2. This is clear since for example, given a standard étale
covering $\{T_i \to T\}_{i \in I}$ and for each i we have a standard étale covering $\{T_{ij} \to
T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard étale covering because $\bigcup_{i \in I} J_i$ is
finite and each T_{ij} is affine.

We omit the proof that $S_{affine,étale}$ is a site. $\square$

Lemma 4.10. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S.
The underlying categories of the sites $Sch_{étale}$, $(Sch/S)_{étale}$, $S_{étale}$, $(Aff/S)_{étale}$,
and $S_{affine,étale}$ have fibre products. In each case the obvious functor into the
category Sch of all schemes commutes with taking fibre products. The categories
$(Sch/S)_{étale}$, and $S_{étale}$ both have a final object, namely S/S.*

Proof. For $Sch_{étale}$ it is true by construction, see Sets, Lemma 9.9. Suppose we
have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{étale})$.
The fibre product $V \times_U W$ in $Sch_{étale}$ is a fibre product in Sch and is the fibre
product of V/S with W/S over U/S in the category of all schemes over S, and
hence also a fibre product in $(Sch/S)_{étale}$. This proves the result for $(Sch/S)_{étale}$.
If $U \to S$, $V \to U$ and $W \to U$ are étale then so is $V \times_U W \to S$ and hence we get
the result for $S_{étale}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for
$(Aff/S)_{étale}$ and $S_{affine,étale}$. $\square$

Next, we check that the big, resp. small affine site defines the same topos as the
big, resp. small site.

Lemma 4.11. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The functor $(Aff/S)_{étale} \to (Sch/S)_{étale}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{étale})$ to $Sh((Sch/S)_{étale})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 29.1. Denote the inclusion functor $u : (Aff/S)_{étale} \to (Sch/S)_{étale}$. Being cocontinuous just means that any étale covering of T/S, T affine, can be refined by a standard étale covering of T. This is the content of Lemma 4.4. Hence (1) holds. We see u is continuous simply because a standard étale covering is a étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. $\square$

Lemma 4.12. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The functor $S_{affine,étale} \to S_{étale}$ is special cocontinuous and induces an equivalence of topoi from $Sh(S_{affine,étale})$ to $Sh(S_{étale})$.*

Proof. Omitted. Hint: compare with the proof of Lemma 4.11. $\square$

Next, we establish some relationships between the topoi associated to these sites.

Lemma 4.13. *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$. The functor $T_{étale} \to (Sch/S)_{étale}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{étale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{étale} \to (Sch/S)_{étale}$. In other words, given an étale morphism $j : U \to T$ corresponding to an object of $T_{étale}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 4.10. Let $a, b : U \to V$ be two morphisms in $T_{étale}$. In this case the equalizer of a and b (in the category of schemes) is

$$V \times_{\Delta_{V/T}, V \times_T V, (a,b)} U \times_T U$$

which is a fibre product of schemes étale over T, hence étale over T. Thus $T_{étale}$ has equalizers and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 21.5 and 21.6. $\square$

Lemma 4.14. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The inclusion functor $S_{étale} \to (Sch/S)_{étale}$ satisfies the hypotheses of Sites, Lemma 21.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

and a morphism of topoi

$$i_S : Sh(S_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 4.13. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{\acute{e}tale} \to (Sch/S)_{\acute{e}tale}$, in addition to the properties seen in the proof of Lemma 4.13 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 21.8. $\square$

Definition 4.15. In the situation of Lemma 4.14 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small étale site*, and for a sheaf $\mathcal{F}$ on the big étale site we denote $\mathcal{F}|_{S_{\acute{e}tale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the small site that

$$\mathrm{Mor}_{Sh(S_{\acute{e}tale})}(\mathcal{F}|_{S_{\acute{e}tale}}, \mathcal{G}) = \mathrm{Mor}_{Sh((Sch/S)_{\acute{e}tale})}(\mathcal{F}, i_{S,*}\mathcal{G})$$

$$\mathrm{Mor}_{Sh(S_{\acute{e}tale})}(\mathcal{G}, \mathcal{F}|_{S_{\acute{e}tale}}) = \mathrm{Mor}_{Sh((Sch/S)_{\acute{e}tale})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{\acute{e}tale}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{\acute{e}tale}} = \mathcal{G}$.

Lemma 4.16. *Let $Sch_{\acute{e}tale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{\acute{e}tale}$. The functor*

$$u : (Sch/T)_{\acute{e}tale} \longrightarrow (Sch/S)_{\acute{e}tale}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{\acute{e}tale} \longrightarrow (Sch/T)_{\acute{e}tale}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{\acute{e}tale}) \longrightarrow Sh((Sch/S)_{\acute{e}tale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 4.13). Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for $f_{big,*}$. $\square$

Lemma 4.17. *Let $Sch_{\acute{e}tale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{\acute{e}tale}$.*

(1) *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 4.13 and i_T as in Lemma 4.14.*
(2) *The functor $S_{\acute{e}tale} \to T_{\acute{e}tale}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of sites*

$$f_{small} : T_{\acute{e}tale} \longrightarrow S_{\acute{e}tale}$$

We have $f_{small,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$.
(3) *We have a commutative diagram of morphisms of sites*

$$
\begin{array}{ccc}
T_{\acute{e}tale} & \xleftarrow{\;\pi_T\;} & (Sch/T)_{\acute{e}tale} \\
{\scriptstyle f_{small}} \downarrow & & \downarrow {\scriptstyle f_{big}} \\
S_{\acute{e}tale} & \xleftarrow{\;\pi_S\;} & (Sch/S)_{\acute{e}tale}
\end{array}
$$

so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.
(4) *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : S_{étale} \to T_{étale}$, $u(U \to S) = (U \times_S T \to T)$ transforms coverings into coverings and commutes with fibre products, see Lemma 4.3 (3) and 4.10. Moreover, both $S_{étale}, T_{étale}$ have final objects, namely S/S and T/T and $u(S/S) = T/T$. Hence by Sites, Proposition 14.7 the functor u corresponds to a morphism of sites $T_{étale} \to S_{étale}$. This in turn gives rise to the morphism of topoi, see Sites, Lemma 15.2. The description of the pushforward is clear from these references.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functors $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. $\square$

In the situation of the lemma, using the terminology of Definition 4.15 we have: for $\mathcal{F}$ a sheaf on the big étale site of T

$$(f_{big,*}\mathcal{F})|_{S_{étale}} = f_{small,*}(\mathcal{F}|_{T_{étale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

Lemma 4.18. *Given schemes X, Y, Y in $Sch_{étale}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 4.16. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 4.17. $\square$

Lemma 4.19. *Let $Sch_{étale}$ be a big étale site. Consider a cartesian diagram*

$$
\begin{array}{ccc}
T' & \xrightarrow{g'} & T \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
S' & \xrightarrow{g} & S
\end{array}
$$

in $Sch_{étale}$. Then $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.

Proof. Since the diagram is cartesian, we have for U'/S' that $U' \times_{S'} T' = U' \times_S T$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(Sch/T)_{étale}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{S'} T')$ on $S'_{étale}$ (use Lemmas 4.13 and 4.16). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 28.1. $\square$

We can think about a sheaf on the big étale site of S as a collection of "usual" sheaves on all schemes over S.

Lemma 4.20. *Let S be a scheme contained in a big étale site $Sch_{étale}$. A sheaf $\mathcal{F}$ on the big étale site $(Sch/S)_{étale}$ is given by the following data:*

    (1) *for every $T/S \in \mathrm{Ob}((Sch/S)_{étale})$ a sheaf $\mathcal{F}_T$ on $T_{étale}$,*
    (2) *for every $f : T' \to T$ in $(Sch/S)_{étale}$ a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

    (a) *given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{étale}$ the composition $c_g \circ g_{small}^{-1} c_f$ is equal to $c_{f \circ g}$, and*

(b) *if $f : T' \to T$ in $(Sch/S)_{\acute{e}tale}$ is étale then c_f is an isomorphism.*

Proof. This lemma follows from a purely sheaf theoretic statement discussed in Sites, Remark 26.7. We also give a direct proof in this case.

Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{\acute{e}tale})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U/S)$ for any $U \to T$ in $T_{\acute{e}tale}$ see Lemma 4.13. Hence given $f : T' \to T$ over S and $U \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U/S) \to \mathcal{F}(U \times_T T'/S) = \mathcal{F}_{T'}(U \times_T T')$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U \times_T T' \to U$ over S. The collection of these maps are compatible with restrictions, and hence define a map $c'_f : \mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ where $u : T_{\acute{e}tale} \to T'_{\acute{e}tale}$ is the base change functor associated to f. By adjunction of $f_{small,*}$ (see Sites, Section 13) with f_{small}^{-1} this is the same as a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$. It is clear that $c'_{f \circ g}$ is the composition of c'_f and $f_{small,*}c'_g$, since composition of restriction maps of $\mathcal{F}$ gives restriction maps, and this gives the desired relationship among c_f, c_g and $c_{f \circ g}$.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{\acute{e}tale})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ where we think of c_f as a map $\mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ again. The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. $\square$

5. The smooth topology

In this section we define the smooth topology. This is a bit pointless as it will turn out later (see More on Morphisms, Section 38) that this topology defines the same topos as the étale topology. But still it makes sense and it is used occasionally.

Definition 5.1. Let T be a scheme. A *smooth covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is smooth and such that $T = \bigcup f_i(T_i)$.

Lemma 5.2. *Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.*

Proof. This is clear from the definitions, the fact that an étale morphism is smooth see Morphisms, Definition 36.1 and Lemma 4.2. $\square$

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 5.3. *Let T be a scheme.*
  (1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a smooth covering of T.*
  (2) *If $\{T_i \to T\}_{i \in I}$ is a smooth covering and for each i we have a smooth covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a smooth covering.*
  (3) *If $\{T_i \to T\}_{i \in I}$ is a smooth covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a smooth covering.*

Proof. Omitted. $\square$

Lemma 5.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be a smooth covering of T. Then there exists a smooth covering $\{U_j \to T\}_{j=1,\dots,m}$ which is a refinement of*

$\{T_i \to T\}_{i \in I}$ *such that each U_j is an affine scheme, and such that each morphism $U_j \to T$ is standard smooth, see Morphisms, Definition 34.1. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted, but see Algebra, Lemma 137.10. $\qquad\qquad\qquad\qquad\qquad\square$

Thus we define the corresponding standard coverings of affines as follows.

Definition 5.5. Let T be an affine scheme. A *standard smooth covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\dots,m}$ with each U_j is affine, $U_j \to T$ standard smooth and $T = \bigcup f_j(U_j)$.

Definition 5.6. A *big smooth site* is any site Sch_{smooth} as in Sites, Definition 6.2 constructed as follows:

  (1) Choose any set of schemes S_0, and any set of smooth coverings Cov_0 among these schemes.
  (2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.
  (3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of smooth coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big smooth site of a scheme S, let us point out that the topology on a big smooth site Sch_{smooth} is in some sense induced from the smooth topology on the category of all schemes.

Lemma 5.7. *Let Sch_{smooth} be a big smooth site as in Definition 5.6. Let $T \in \text{Ob}(Sch_{smooth})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary smooth covering of T.*

  (1) *There exists a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{smooth} which refines $\{T_i \to T\}_{i \in I}$.*
  (2) *If $\{T_i \to T\}_{i \in I}$ is a standard smooth covering, then it is tautologically equivalent to a covering of Sch_{smooth}.*
  (3) *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{smooth}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 5.3 the refinement $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a smooth covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{smooth}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{smooth} by a second application of Sets, Lemma 9.9. The covering $\{V_i \to T\}_{i \in I}$ refines $\{T_i \to T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{smooth} by Sets, Lemma 9.9. The covering $\{U_j \to T\}_{j \in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{smooth} by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. $\qquad\qquad\square$

Definition 5.8. Let S be a scheme. Let Sch_{smooth} be a big smooth site containing S.

(1) The *big smooth site of S*, denoted $(Sch/S)_{smooth}$, is the site Sch_{smooth}/S introduced in Sites, Section 25.
(2) The *big affine smooth site of S*, denoted $(Aff/S)_{smooth}$, is the full subcategory of $(Sch/S)_{smooth}$ whose objects are affine U/S. A covering of $(Aff/S)_{smooth}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{smooth}$ which is a standard smooth covering.

Next, we check that the big affine site defines the same topos as the big site.

Lemma 5.9. *Let S be a scheme. Let Sch_{smooth} be a big smooth site containing S. The functor $(Aff/S)_{smooth} \to (Sch/S)_{smooth}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{smooth})$ to $Sh((Sch/S)_{smooth})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 29.1. Denote the inclusion functor $u : (Aff/S)_{smooth} \to (Sch/S)_{smooth}$. Being cocontinuous just means that any smooth covering of T/S, T affine, can be refined by a standard smooth covering of T. This is the content of Lemma 5.4. Hence (1) holds. We see u is continuous simply because a standard smooth covering is a smooth covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. $\square$

To be continued...

Lemma 5.10. *Let Sch_{smooth} be a big smooth site. Let $f : T \to S$ be a morphism in Sch_{smooth}. The functor*

$$u : (Sch/T)_{smooth} \longrightarrow (Sch/S)_{smooth}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{smooth} \longrightarrow (Sch/T)_{smooth}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{smooth}) \longrightarrow Sh((Sch/S)_{smooth})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for $f_{big,*}$. $\square$

6. The syntomic topology

In this section we define the syntomic topology. This topology is quite interesting in that it often has the same cohomology groups as the fppf topology but is technically easier to deal with.

Definition 6.1. Let T be a scheme. An *syntomic covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is syntomic and such that $T = \bigcup f_i(T_i)$.

Lemma 6.2. *Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.*

Proof. This is clear from the definitions and the fact that a smooth morphism is syntomic, see Morphisms, Lemma 34.7 and Lemma 5.2. $\square$

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 6.3. *Let T be a scheme.*
  (1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a syntomic covering of T.*
  (2) *If $\{T_i \to T\}_{i \in I}$ is a syntomic covering and for each i we have a syntomic covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a syntomic covering.*
  (3) *If $\{T_i \to T\}_{i \in I}$ is a syntomic covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a syntomic covering.*

Proof. Omitted. $\square$

Lemma 6.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be a syntomic covering of T. Then there exists a syntomic covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme, and such that each morphism $U_j \to T$ is standard syntomic, see Morphisms, Definition 30.1. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted, but see Algebra, Lemma 136.15. $\square$

Thus we define the corresponding standard coverings of affines as follows.

Definition 6.5. Let T be an affine scheme. A *standard syntomic covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, $U_j \to T$ standard syntomic and $T = \bigcup f_j(U_j)$.

Definition 6.6. A *big syntomic site* is any site $Sch_{syntomic}$ as in Sites, Definition 6.2 constructed as follows:
  (1) Choose any set of schemes S_0, and any set of syntomic coverings Cov_0 among these schemes.
  (2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.
  (3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of syntomic coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big syntomic site of a scheme S, let us point out that the topology on a big syntomic site $Sch_{syntomic}$ is in some sense induced from the syntomic topology on the category of all schemes.

Lemma 6.7. *Let $Sch_{syntomic}$ be a big syntomic site as in Definition 6.6. Let $T \in \mathrm{Ob}(Sch_{syntomic})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary syntomic covering of T.*
  (1) *There exists a covering $\{U_j \to T\}_{j \in J}$ of T in the site $Sch_{syntomic}$ which refines $\{T_i \to T\}_{i \in I}$.*
  (2) *If $\{T_i \to T\}_{i \in I}$ is a standard syntomic covering, then it is tautologically equivalent to a covering in $Sch_{syntomic}$.*

(3) If $\{T_i \to T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{syntomic}$.

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 6.3 the refinement $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a syntomic covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of $Sch_{syntomic}$. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of $Sch_{syntomic}$ by a second application of Sets, Lemma 9.9. The covering $\{V_i \to T\}_{i \in I}$ refines $\{T_i \to T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site $Sch_{syntomic}$ by Sets, Lemma 9.9. The covering $\{U_j \to T\}_{j \in J}$ is a covering as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of $Sch_{syntomic}$ by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. $\square$

Definition 6.8. Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S.

(1) The *big syntomic site of S*, denoted $(Sch/S)_{syntomic}$, is the site $Sch_{syntomic}/S$ introduced in Sites, Section 25.
(2) The *big affine syntomic site of S*, denoted $(Aff/S)_{syntomic}$, is the full subcategory of $(Sch/S)_{syntomic}$ whose objects are affine U/S. A covering of $(Aff/S)_{syntomic}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{syntomic}$ which is a standard syntomic covering.

Next, we check that the big affine site defines the same topos as the big site.

Lemma 6.9. *Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S. The functor $(Aff/S)_{syntomic} \to (Sch/S)_{syntomic}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{syntomic})$ to $Sh((Sch/S)_{syntomic})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 29.1. Denote the inclusion functor $u : (Aff/S)_{syntomic} \to (Sch/S)_{syntomic}$. Being cocontinuous just means that any syntomic covering of T/S, T affine, can be refined by a standard syntomic covering of T. This is the content of Lemma 6.4. Hence (1) holds. We see u is continuous simply because a standard syntomic covering is a syntomic covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. $\square$

To be continued...

Lemma 6.10. *Let $Sch_{syntomic}$ be a big syntomic site. Let $f : T \to S$ be a morphism in $Sch_{syntomic}$. The functor*

$$u : (Sch/T)_{syntomic} \longrightarrow (Sch/S)_{syntomic}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{syntomic} \longrightarrow (Sch/T)_{syntomic}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{syntomic}) \longrightarrow Sh((Sch/S)_{syntomic})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for $f_{big,*}$. $\square$

7. The fppf topology

Let S be a scheme. We would like to define the fppf-topology[3] on the category of schemes over S. According to our general principle we first introduce the notion of an fppf-covering.

Definition 7.1. Let T be a scheme. An *fppf covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is flat, locally of finite presentation and such that $T = \bigcup f_i(T_i)$.

Lemma 7.2. *Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.*

Proof. This is clear from the definitions, the fact that a syntomic morphism is flat and locally of finite presentation, see Morphisms, Lemmas 30.6 and 30.7, and Lemma 6.2. $\square$

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 7.3. *Let T be a scheme.*
  (1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fppf covering of T.*
  (2) *If $\{T_i \to T\}_{i \in I}$ is an fppf covering and for each i we have an fppf covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fppf covering.*
  (3) *If $\{T_i \to T\}_{i \in I}$ is an fppf covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fppf covering.*

Proof. The first assertion is clear. The second follows as the composition of flat morphisms is flat (see Morphisms, Lemma 25.6) and the composition of morphisms of finite presentation is of finite presentation (see Morphisms, Lemma 21.3). The third follows as the base change of a flat morphism is flat (see Morphisms, Lemma 25.8) and the base change of a morphism of finite presentation is of finite presentation (see Morphisms, Lemma 21.4). Moreover, the base change of a surjective family of morphisms is surjective (proof omitted). $\square$

Lemma 7.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fppf covering of T. Then there exists an fppf covering $\{U_j \to T\}_{j=1,\dots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definitions using that a morphism which is flat and locally of finite presentation is open, see Morphisms, Lemma 25.10. $\square$

Thus we define the corresponding standard coverings of affines as follows.

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[3]The letters fppf stand for "fidèlement plat de présentation finie".

Definition 7.5. Let T be an affine scheme. A *standard fppf covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, flat and of finite presentation over T and $T = \bigcup f_j(U_j)$.

Definition 7.6. A *big fppf site* is any site Sch_{fppf} as in Sites, Definition 6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of fppf coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of fppf coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big fppf site of a scheme S, let us point out that the topology on a big fppf site Sch_{fppf} is in some sense induced from the fppf topology on the category of all schemes.

Lemma 7.7. *Let Sch_{fppf} be a big fppf site as in Definition 7.6. Let $T \in \mathrm{Ob}(Sch_{fppf})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary fppf covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{fppf} which refines $\{T_i \to T\}_{i \in I}$.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a standard fppf covering, then it is tautologically equivalent to a covering of Sch_{fppf}.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{fppf}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 7.3 the refinement $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fppf covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{fppf}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{fppf} by a second application of Sets, Lemma 9.9. The covering $\{V_i \to T\}_{i \in I}$ refines $\{T_i \to T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{fppf} by Sets, Lemma 9.9. The covering $\{U_j \to T\}_{j \in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{fppf} by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. $\square$

Definition 7.8. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S.

(1) The *big fppf site of S*, denoted $(Sch/S)_{fppf}$, is the site Sch_{fppf}/S introduced in Sites, Section 25.
(2) The *big affine fppf site of S*, denoted $(Aff/S)_{fppf}$, is the full subcategory of $(Sch/S)_{fppf}$ whose objects are affine U/S. A covering of $(Aff/S)_{fppf}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{fppf}$ which is a standard fppf covering.

It is not completely clear that the big affine fppf site is a site. We check this now.

Lemma 7.9. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. Then $(Aff/S)_{fppf}$ is a site.*

Proof. Let us show that $(Aff/S)_{fppf}$ is a site. Reasoning as in the proof of Lemma 4.9 it suffices to show that the collection of standard fppf coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 6.2. This is clear since for example, given a standard fppf covering $\{T_i \to T\}_{i \in I}$ and for each i we have a standard fppf covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard fppf covering because $\bigcup_{i \in I} J_i$ is finite and each T_{ij} is affine. $\square$

Lemma 7.10. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The underlying categories of the sites Sch_{fppf}, $(Sch/S)_{fppf}$, and $(Aff/S)_{fppf}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_{fppf}$ has a final object, namely S/S.*

Proof. For Sch_{fppf} it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{fppf})$. The fibre product $V \times_U W$ in Sch_{fppf} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{fppf}$. This proves the result for $(Sch/S)_{fppf}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{fppf}$. $\square$

Next, we check that the big affine site defines the same topos as the big site.

Lemma 7.11. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The functor $(Aff/S)_{fppf} \to (Sch/S)_{fppf}$ is cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{fppf})$ to $Sh((Sch/S)_{fppf})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 29.1. Denote the inclusion functor $u : (Aff/S)_{fppf} \to (Sch/S)_{fppf}$. Being cocontinuous just means that any fppf covering of T/S, T affine, can be refined by a standard fppf covering of T. This is the content of Lemma 7.4. Hence (1) holds. We see u is continuous simply because a standard fppf covering is a fppf covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. $\square$

Next, we establish some relationships between the topoi associated to these sites.

Lemma 7.12. *Let Sch_{fppf} be a big fppf site. Let $f : T \to S$ be a morphism in Sch_{fppf}. The functor*

$$u : (Sch/T)_{fppf} \longrightarrow (Sch/S)_{fppf}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{fppf} \longrightarrow (Sch/T)_{fppf}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{fppf}) \longrightarrow Sh((Sch/S)_{fppf})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for $f_{big,*}$. $\square$

Lemma 7.13. *Given schemes X, Y, Y in $(Sch/S)_{fppf}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 7.12. $\square$

8. The ph topology

In this section we define the ph topology. This is the topology generated by Zariski coverings and proper surjective morphisms, see Lemma 8.15.

We borrow our notation/terminology from the paper [GL01] by Goodwillie and Lichtenbaum. These authors show that if we restrict to the subcategory of Noetherian schemes, then the ph topology is the same as the "h topology" as originally defined by Voevodsky: this is the topology generated by Zariski open coverings and finite type morphisms which are universally submersive. They also show that the two topologies do not agree on non-Noetherian schemes, see [GL01, Example 4.5]. We return to (our version of) the h topology in More on Flatness, Section 34.

Before we can define the coverings in our topology we need to do a bit of work.

Definition 8.1. Let T be an affine scheme. A *standard ph covering* is a family $\{f_j : U_j \to T\}_{j=1,\dots,m}$ constructed from a proper surjective morphism $f : U \to T$ and an affine open covering $U = \bigcup_{j=1,\dots,m} U_j$ by setting $f_j = f|_{U_j}$.

It follows immediately from Chow's lemma that we can refine a standard ph covering by a standard ph covering corresponding to a surjective projective morphism.

Lemma 8.2. *Let $\{f_j : U_j \to T\}_{j=1,\dots,m}$ be a standard ph covering. Let $T' \to T$ be a morphism of affine schemes. Then $\{U_j \times_T T' \to T'\}_{j=1,\dots,m}$ is a standard ph covering.*

Proof. Let $f : U \to T$ be proper surjective and let an affine open covering $U = \bigcup_{j=1,\dots,m} U_j$ be given as in Definition 8.1. Then $U \times_T T' \to T'$ is proper surjective (Morphisms, Lemmas 9.4 and 41.5). Also, $U \times_T T' = \bigcup_{j=1,\dots,m} U_j \times_T T'$ is an affine open covering. This concludes the proof. $\square$

Lemma 8.3. *Let T be an affine scheme. Each of the following types of families of maps with target T has a refinement by a standard ph covering:*
   (1) *any Zariski open covering of T,*
   (2) *$\{W_{ji} \to T\}_{j=1,\dots,m, i=1,\dots n_j}$ where $\{W_{ji} \to U_j\}_{i=1,\dots,n_j}$ and $\{U_j \to T\}_{j=1,\dots,m}$ are standard ph coverings.*

Proof. Part (1) follows from the fact that any Zariski open covering of T can be refined by a finite affine open covering.

Proof of (3). Choose $U \to T$ proper surjective and $U = \bigcup_{j=1,\dots,m} U_j$ as in Definition 8.1. Choose $W_j \to U_j$ proper surjective and $W_j = \bigcup W_{ji}$ as in Definition 8.1. By Chow's lemma (Limits, Lemma 12.1) we can find $W_j' \to W_j$ proper surjective and

closed immersions $W_j' \to \mathbf{P}_{U_j}^{e_j}$. Thus, after replacing W_j by W_j' and $W_j = \bigcup W_{ji}$ by a suitable affine open covering of W_j', we may assume there is a closed immersion $W_j \subset \mathbf{P}_{U_j}^{e_j}$ for all $j = 1, \ldots, m$.

Let $\overline{W}_j \subset \mathbf{P}_U^{e_j}$ be the scheme theoretic closure of W_j. Then $W_j \subset \overline{W}_j$ is an open subscheme; in fact W_j is the inverse image of $U_j \subset U$ under the morphism $\overline{W}_j \to U$. (To see this use that $W_j \to \mathbf{P}_U^{e_j}$ is quasi-compact and hence formation of the scheme theoretic image commutes with restriction to opens, see Morphisms, Section 6.) Let $Z_j = U \setminus U_j$ with reduced induced closed subscheme structure. Then

$$V_j = \overline{W}_j \amalg Z_j \to U$$

is proper surjective and the open subscheme $W_j \subset V_j$ is the inverse image of U_j. Hence for $v \in V_j$, $v \notin W_j$ we can pick an affine open neighbourhood $v \in V_{j,v} \subset V_j$ which maps into $U_{j'}$ for some $1 \le j' \le m$.

To finish the proof we consider the proper surjective morphism

$$V = V_1 \times_U V_2 \times_U \ldots \times_U V_m \longrightarrow U \longrightarrow T$$

and the covering of V by the affine opens

$$V_{1,v_1} \times_U \ldots \times_U V_{j-1,v_{j-1}} \times_U W_{ji} \times_U V_{j+1,v_{j+1}} \times_U \ldots \times_U V_{m,v_m}$$

These do indeed form a covering, because each point of U is in some U_j and the inverse image of U_j in V is equal to $V_1 \times \ldots \times V_{j-1} \times W_j \times V_{j+1} \times \ldots \times V_m$. Observe that the morphism from the affine open displayed above to T factors through W_{ji} thus we obtain a refinement. Finally, we only need a finite number of these affine opens as V is quasi-compact (as a scheme proper over the affine scheme T). $\qquad\square$

Definition 8.4. Let T be a scheme. A *ph covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that f_i is locally of finite type and such that for every affine open $U \subset T$ there exists a standard ph covering $\{U_j \to U\}_{j=1,\ldots,m}$ refining the family $\{T_i \times_T U \to U\}_{i \in I}$.

A standard ph covering is a ph covering by Lemma 8.2.

Lemma 8.5. *A Zariski covering is a ph covering*[4].

Proof. This is true because a Zariski covering of an affine scheme can be refined by a standard ph covering by Lemma 8.3. $\qquad\square$

Lemma 8.6. *Let $f : Y \to X$ be a surjective proper morphism of schemes. Then $\{Y \to X\}$ is a ph covering.*

Proof. Omitted. $\qquad\square$

Lemma 8.7. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms such that f_i is locally of finite type for all i. The following are equivalent*

(1) *$\{T_i \to T\}_{i \in I}$ is a ph covering,*
(2) *there is a ph covering which refines $\{T_i \to T\}_{i \in I}$, and*
(3) *$\{\coprod_{i \in I} T_i \to T\}$ is a ph covering.*

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[4]We will see in More on Morphisms, Lemma 48.7 that fppf coverings (and hence syntomic, smooth, or étale coverings) are ph coverings as well.

Proof. The equivalence of (1) and (2) follows immediately from Definition 8.4 and the fact that a refinement of a refinement is a refinement. Because of the equivalence of (1) and (2) and since $\{T_i \to T\}_{i \in I}$ refines $\{\coprod_{i \in I} T_i \to T\}$ we see that (1) implies (3). Finally, assume (3) holds. Let $U \subset T$ be an affine open and let $\{U_j \to U\}_{j=1,\ldots,m}$ be a standard ph covering which refines $\{U \times_T \coprod_{i \in I} T_i \to U\}$. This means that for each j we have a morphism

$$h_j : U_j \longrightarrow U \times_T \coprod_{i \in I} T_i = \coprod_{i \in I} U \times_T T_i$$

over U. Since U_j is quasi-compact, we get disjoint union decompositions $U_j = \coprod_{i \in I} U_{j,i}$ by open and closed subschemes almost all of which are empty such that $h_j|_{U_{j,i}}$ maps $U_{j,i}$ into $U \times_T T_i$. It follows that

$$\{U_{j,i} \to U\}_{j=1,\ldots,m, \; i \in I, \; U_{j,i} \neq \emptyset}$$

is a standard ph covering (small detail omitted) refining $\{U \times_T T_i \to U\}_{i \in I}$. Thus (1) holds. $\square$

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 8.8. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a ph covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a ph covering and for each i we have a ph covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a ph covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a ph covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a ph covering.*

Proof. Assertion (1) is clear.

Proof of (3). The base change $T_i \times_T T' \to T'$ is locally of finite type by Morphisms, Lemma 15.4. hence we only need to check the condition on affine opens. Let $U' \subset T'$ be an affine open subscheme. Since U' is quasi-compact we can find a finite affine open covering $U' = U'_1 \cup \ldots \cup U'$ such that $U'_j \to T$ maps into an affine open $U_j \subset T$. Choose a standard ph covering $\{U_{jl} \to U_j\}_{l=1,\ldots,n_j}$ refining $\{T_i \times_T U_j \to U_j\}$. By Lemma 8.2 the base change $\{U_{jl} \times_{U_j} U'_j \to U'_j\}$ is a standard ph covering. Note that $\{U'_j \to U'\}$ is a standard ph covering as well. By Lemma 8.3 the family $\{U_{jl} \times_{U_j} U'_j \to U'\}$ can be refined by a standard ph covering. Since $\{U_{jl} \times_{U_j} U'_j \to U'\}$ refines $\{T_i \times_T U' \to U'\}$ we conclude.

Proof of (2). Composition preserves being locally of finite type, see Morphisms, Lemma 15.3. Hence we only need to check the condition on affine opens. Let $U \subset T$ be affine open. First we pick a standard ph covering $\{U_k \to U\}_{k=1,\ldots,m}$ refining $\{T_i \times_T U \to U\}$. Say the refinement is given by morphisms $U_k \to T_{i_k}$ over T. Then

$$\{T_{i_k j} \times_{T_{i_k}} U_k \to U_k\}_{j \in J_{i_k}}$$

is a ph covering by part (3). As U_k is affine, we can find a standard ph covering $\{U_{ka} \to U_k\}_{a=1,\ldots,b_k}$ refining this family. Then we apply Lemma 8.3 to see that $\{U_{ka} \to U\}$ can be refined by a standard ph covering. Since $\{U_{ka} \to U\}$ refines $\{T_{ij} \times_T U \to U\}$ this finishes the proof. $\square$

Definition 8.9. A *big ph site* is any site Sch_{ph} as in Sites, Definition 6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of ph coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of ph coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big ph site of a scheme S, let us point out that the topology on a big ph site Sch_{ph} is in some sense induced from the ph topology on the category of all schemes.

Lemma 8.10. *Let Sch_{ph} be a big ph site as in Definition 8.9. Let $T \in \mathrm{Ob}(Sch_{ph})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary ph covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{ph} which refines $\{T_i \to T\}_{i \in I}$.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a standard ph covering, then it is tautologically equivalent to a covering of Sch_{ph}.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{ph}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemmas 8.5 and 8.8 the refinement $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a ph covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{ph}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{ph} by a second application of Sets, Lemma 9.9. The covering $\{V_i \to T\}_{i \in I}$ refines $\{T_i \to T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{ph} by Sets, Lemma 9.9. The covering $\{U_j \to T\}_{j \in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{ph} by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. $\square$

Definition 8.11. Let S be a scheme. Let Sch_{ph} be a big ph site containing S.

(1) The *big ph site of S*, denoted $(Sch/S)_{ph}$, is the site Sch_{ph}/S introduced in Sites, Section 25.
(2) The *big affine ph site of S*, denoted $(Aff/S)_{ph}$, is the full subcategory of $(Sch/S)_{ph}$ whose objects are affine U/S. A covering of $(Aff/S)_{ph}$ is any finite covering $\{U_i \to U\}$ of $(Sch/S)_{ph}$ with U_i and U affine.

Observe that the coverings in $(Aff/S)_{ph}$ are *not* given by standard ph coverings. The reason is simply that this would fail the second axiom of Sites, Definition 6.2. Rather, the coverings in $(Aff/S)_{ph}$ are those finite families $\{U_i \to U\}$ of finite type morphisms between affine objects of $(Sch/S)_{ph}$ which can be refined by a standard ph covering. We explicitly state and prove that the big affine ph site is a site.

Lemma 8.12. *Let S be a scheme. Let Sch_{ph} be a big ph site containing S. Then $(Aff/S)_{ph}$ is a site.*

Proof. Reasoning as in the proof of Lemma 4.9 it suffices to show that the collection of finite ph coverings $\{U_i \to U\}$ with U, U_i affine satisfies properties (1), (2) and (3) of Sites, Definition 6.2. This is clear since for example, given a finite ph covering $\{T_i \to T\}_{i \in I}$ with T_i, T affine, and for each i a finite ph covering $\{T_{ij} \to T_i\}_{j \in J_i}$ with T_{ij} affine , then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a ph covering (Lemma 8.8), $\bigcup_{i \in I} J_i$ is finite and each T_{ij} is affine. $\square$

Lemma 8.13. *Let S be a scheme. Let Sch_{ph} be a big ph site containing S. The underlying categories of the sites Sch_{ph}, $(Sch/S)_{ph}$, and $(Aff/S)_{ph}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_{ph}$ has a final object, namely S/S.*

Proof. For Sch_{ph} it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{ph})$. The fibre product $V \times_U W$ in Sch_{ph} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{ph}$. This proves the result for $(Sch/S)_{ph}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{ph}$. $\square$

Next, we check that the big affine site defines the same topos as the big site.

Lemma 8.14. *Let S be a scheme. Let Sch_{ph} be a big ph site containing S. The functor $(Aff/S)_{ph} \to (Sch/S)_{ph}$ is cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{ph})$ to $Sh((Sch/S)_{ph})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 29.1. Denote the inclusion functor $u : (Aff/S)_{ph} \to (Sch/S)_{ph}$. Being cocontinuous follows because any ph covering of T/S, T affine, can be refined by a standard ph covering of T by definition. Hence (1) holds. We see u is continuous simply because a finite ph covering of an affine by affines is a ph covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering (which is a ph covering). $\square$

Lemma 8.15. *Let $\mathcal{F}$ be a presheaf on $(Sch/S)_{ph}$. Then $\mathcal{F}$ is a sheaf if and only if*

(1) *$\mathcal{F}$ satisfies the sheaf condition for Zariski coverings, and*
(2) *if $f : V \to U$ is proper surjective, then $\mathcal{F}(U)$ maps bijectively to the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$.*

Moreover, in the presence of (1) property (2) is equivalent to property

(2') *the sheaf property for $\{V \to U\}$ as in (2) with U affine.*

Proof. We will show that if (1) and (2) hold, then $\mathcal{F}$ is sheaf. Let $\{T_i \to T\}$ be a ph covering, i.e., a covering in $(Sch/S)_{ph}$. We will verify the sheaf condition for this covering. Let $s_i \in \mathcal{F}(T_i)$ be sections which restrict to the same section over $T_i \times_T T_{i'}$. We will show that there exists a unique section $s \in \mathcal{F}(T)$ restricting to s_i over T_i. Let $T = \bigcup U_j$ be an affine open covering. By property (1) it suffices to produce sections $s_j \in \mathcal{F}(U_j)$ which agree on $U_j \cap U_{j'}$ in order to produce s. Consider the ph coverings $\{T_i \times_T U_j \to U_j\}$. Then $s_{ji} = s_i|_{T_i \times_T U_j}$ are sections agreeing over $(T_i \times_T U_j) \times_{U_j} (T_{i'} \times_T U_j)$. Choose a proper surjective morphism $V_j \to U_j$ and a finite affine open covering $V_j = \bigcup V_{jk}$ such that the standard ph covering $\{V_{jk} \to U_j\}$ refines $\{T_i \times_T U_j \to U_j\}$. If $s_{jk} \in \mathcal{F}(V_{jk})$ denotes the

pullback of s_{ji} to V_{jk} by the implied morphisms, then we find that s_{jk} glue to a section $s'_j \in \mathcal{F}(V_j)$. Using the agreement on overlaps once more, we find that s'_j is in the equalizer of the two maps $\mathcal{F}(V_j) \to \mathcal{F}(V_j \times_{U_j} V_j)$. Hence by (2) we find that s'_j comes from a unique section $s_j \in \mathcal{F}(U_j)$. We omit the verification that these sections s_j have all the desired properties.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $V \to U$ is a morphism of $(Sch/S)_{ph}$ which is proper and surjective. Choose an affine open covering $U = \bigcup U_i$ and set $V_i = V \times_U U_i$. Then we see that $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective because we know $\mathcal{F}(U_i) \to \mathcal{F}(V_i)$ is injective by (2') and we know $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective by (1). Finally, suppose that we are given an $t \in \mathcal{F}(V)$ in the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$. Then $t|_{V_i}$ is in the equalizer of the two maps $\mathcal{F}(V_i) \to \mathcal{F}(V_i \times_{U_i} V_i)$ for all i. Hence we obtain a unique section $s_i \in \mathcal{F}(U_i)$ mapping to $t|_{V_i}$ for all i by (2'). We omit the verification that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j; this uses the uniqueness property just shown. By the sheaf property for the covering $U = \bigcup U_i$ we obtain a section $s \in \mathcal{F}(U)$. We omit the proof that s maps to t in $\mathcal{F}(V)$. $\square$

Next, we establish some relationships between the topoi associated to these sites.

Lemma 8.16. *Let Sch_{ph} be a big ph site. Let $f : T \to S$ be a morphism in Sch_{ph}. The functor*

$$u : (Sch/T)_{ph} \longrightarrow (Sch/S)_{ph}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{ph} \longrightarrow (Sch/T)_{ph}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{ph}) \longrightarrow Sh((Sch/S)_{ph})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\text{Mor}_S(u(U), V) = \text{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for $f_{big,*}$. $\square$

Lemma 8.17. *Given schemes X, Y, Y in $(Sch/S)_{ph}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 8.16. $\square$

9. The fpqc topology

Definition 9.1. Let T be a scheme. An *fpqc covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is flat and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \dots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}$, $j = 1, \dots, n$ with $\bigcup_{j=1}^n f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. It is slightly harder to recognize an fpqc covering, hence we provide some lemmas to do so.

Lemma 9.2. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

(1) *$\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering,*
(2) *each f_i is flat and for every affine open $U \subset T$ there exist quasi-compact opens $U_i \subset T_i$ which are almost all empty, such that $U = \bigcup f_i(U_i)$,*
(3) *each f_i is flat and there exists an affine open covering $T = \bigcup_{\alpha \in A} U_\alpha$ and for each $\alpha \in A$ there exist $i_{\alpha,1}, \ldots, i_{\alpha,n(\alpha)} \in I$ and quasi-compact opens $U_{\alpha,j} \subset T_{i_{\alpha,j}}$ such that $U_\alpha = \bigcup_{j=1,\ldots,n(\alpha)} f_{i_{\alpha,j}}(U_{\alpha,j})$.*

If T is quasi-separated, these are also equivalent to

(4) *each f_i is flat, and for every $t \in T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of t in T.*

Proof. We omit the proof of the equivalence of (1), (2), and (3). From now on assume T is quasi-separated. We prove (4) implies (2). Let $U \subset T$ be an affine open. To prove (2) it suffices to show that for every $t \in U$ there exist finitely many quasi-compact opens $U_j \subset T_{i_j}$ such that $f_{i_j}(U_j) \subset U$ and such that $\bigcup f_{i_j}(U_j)$ is a neighbourhood of t in U. By assumption there do exist finitely many quasi-compact opens $U'_j \subset T_{i_j}$ such that such that $\bigcup f_{i_j}(U'_j)$ is a neighbourhood of t in T. Since T is quasi-separated we see that $U_j = U'_j \cap f_j^{-1}(U)$ is quasi-compact open as desired. Since it is clear that (2) implies (4) the proof is finished. $\square$

Lemma 9.3. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

(1) *$\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering, and*
(2) *setting $T' = \coprod_{i \in I} T_i$, and $f = \coprod_{i \in I} f_i$ the family $\{f : T' \to T\}$ is an fpqc covering.*

Proof. Suppose that $U \subset T$ is an affine open. If (1) holds, then we find $i_1, \ldots, i_n \in I$ and affine opens $U_j \subset T_{i_j}$ such that $U = \bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$. Then $U_1 \amalg \ldots \amalg U_n \subset T'$ is a quasi-compact open surjecting onto U. Thus $\{f : T' \to T\}$ is an fpqc covering by Lemma 9.2. Conversely, if (2) holds then there exists a quasi-compact open $U' \subset T'$ with $U = f(U')$. Then $U_j = U' \cap T_j$ is quasi-compact open in T_j and empty for almost all j. By Lemma 9.2 we see that (1) holds. $\square$

Lemma 9.4. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*
(2) *the family $\{f_i : T_i \to T\}_{i \in I}$ can be refined by an fpqc covering of T.*

Then $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering of T.

Proof. Let $\{g_j : X_j \to T\}_{j \in J}$ be an fpqc covering refining $\{f_i : T_i \to T\}$. Suppose that $U \subset T$ is affine open. Choose $j_1, \ldots, j_m \in J$ and $V_k \subset X_{j_k}$ affine open such that $U = \bigcup g_{j_k}(V_k)$. For each j pick $i_j \in I$ and a morphism $h_j : X_j \to T_{i_j}$ such that $g_j = f_{i_j} \circ h_j$. Since $h_{j_k}(V_k)$ is quasi-compact we can find a quasi-compact open $h_{j_k}(V_k) \subset U_k \subset f_{i_{j_k}}^{-1}(U)$. Then $U = \bigcup f_{i_{j_k}}(U_k)$. We conclude that $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering by Lemma 9.2. $\square$

Lemma 9.5. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that*

  (1) *each f_i is flat, and*
  (2) *there exists an fpqc covering $\{g_j : S_j \to T\}_{j \in J}$ such that each $\{S_j \times_T T_i \to S_j\}_{i \in I}$ is an fpqc covering.*

Then $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering of T.

Proof. We will use Lemma 9.2 without further mention. Let $U \subset T$ be an affine open. By (2) we can find quasi-compact opens $V_j \subset S_j$ for $j \in J$, almost all empty, such that $U = \bigcup g_j(V_j)$. Then for each j we can choose quasi-compact opens $W_{ij} \subset S_j \times_T T_i$ for $i \in I$, almost all empty, with $V_j = \bigcup_i \mathrm{pr}_1(W_{ij})$. Thus $\{S_j \times_T T_i \to T\}$ is an fpqc covering. Since this covering refines $\{f_i : T_i \to T\}$ we conclude by Lemma 9.4. $\qquad\square$

Lemma 9.6. *Any fppf covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.*

Proof. We will show that an fppf covering is an fpqc covering, and then the rest follows from Lemma 7.2. Let $\{f_i : U_i \to U\}_{i \in I}$ be an fppf covering. By definition this means that the f_i are flat which checks the first condition of Definition 9.1. To check the second, let $V \subset U$ be an affine open subset. Write $f_i^{-1}(V) = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since each f_i is open (Morphisms, Lemma 25.10), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} f_i(V_{ij})$ is an open covering of V. Since V is quasi-compact, this covering has a finite refinement. This finishes the proof. $\qquad\square$

The fpqc[5] topology cannot be treated in the same way as the fppf topology[6]. Namely, suppose that R is a nonzero ring. We will see in Lemma 9.14 that there does not exist a set A of fpqc-coverings of $\mathrm{Spec}(R)$ such that every fpqc-covering can be refined by an element of A. If $R = k$ is a field, then the reason for this unboundedness is that there does not exist a field extension of k such that every field extension of k is contained in it.

If you ignore set theoretic difficulties, then you run into presheaves which do not have a sheafification, see [Wat75, Theorem 5.5]. A mildly interesting option is to consider only those faithfully flat ring extensions $R \to R'$ where the cardinality of R' is suitably bounded. (And if you consider all schemes in a fixed universe as in SGA4 then you are bounding the cardinality by a strongly inaccessible cardinal.) However, it is not so clear what happens if you change the cardinal to a bigger one.

For these reasons we do not introduce fpqc sites and we will not consider cohomology with respect to the fpqc-topology.

On the other hand, given a contravariant functor $F : Sch^{opp} \to Sets$ it does make sense to ask whether F satisfies the sheaf property for the fpqc topology, see below. Moreover, we can wonder about descent of object in the fpqc topology, etc. Simply put, for certain results the correct generality is to work with fpqc coverings.

Lemma 9.7. *Let T be a scheme.*
  (1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fpqc covering of T.*
  (2) *If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fpqc covering.*

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[5]The letters fpqc stand for "fidèlement plat quasi-compacte".

[6]A more precise statement would be that the analogue of Lemma 7.7 for the fpqc topology does not hold.

(3) If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fpqc covering.

Proof. Part (1) is immediate. Recall that the composition of flat morphisms is flat and that the base change of a flat morphism is flat (Morphisms, Lemmas 25.8 and 25.6). Thus we can apply Lemma 9.2 in each case to check that our families of morphisms are fpqc coverings.

Proof of (2). Assume $\{T_i \to T\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{f_{ij} : T_{ij} \to T_i\}_{j \in J_i}$. Let $U \subset T$ be an affine open. We can find quasi-compact opens $U_i \subset T_i$ for $i \in I$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then for each i we can choose quasi-compact opens $W_{ij} \subset T_{ij}$ for $j \in J_i$, almost all empty, with $U_i = \bigcup_j f_{ij}(U_{ij})$. Thus $\{T_{ij} \to T\}$ is an fpqc covering.

Proof of (3). Assume $\{T_i \to T\}_{i \in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes. Let $U' \subset T'$ be an affine open which maps into the affine open $U \subset T$. Choose quasi-compact opens $U_i \subset T_i$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then $U' \times_U U_i$ is a quasi-compact open of $T' \times_T T_i$ and $U' = \bigcup \mathrm{pr}_1(U' \times_U U_i)$. Since T' can be covered by such affine opens $U' \subset T'$ we see that $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fpqc covering by Lemma 9.2. $\qquad \square$

Lemma 9.8. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of T. Then there exists an fpqc covering $\{U_j \to T\}_{j=1,\ldots,n}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definition. $\qquad \square$

Definition 9.9. Let T be an affine scheme. A *standard fpqc covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,n}$ with each U_j is affine, flat over T and $T = \bigcup f_j(U_j)$.

Since we do not introduce the affine site we have to show directly that the collection of all standard fpqc coverings satisfies the axioms.

Lemma 9.10. *Let T be an affine scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a standard fpqc covering of T.*

(2) *If $\{T_i \to T\}_{i \in I}$ is a standard fpqc covering and for each i we have a standard fpqc covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard fpqc covering.*

(3) *If $\{T_i \to T\}_{i \in I}$ is a standard fpqc covering and $T' \to T$ is a morphism of affine schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a standard fpqc covering.*

Proof. This follows formally from the fact that compositions and base changes of flat morphisms are flat (Morphisms, Lemmas 25.8 and 25.6) and that fibre products of affine schemes are affine (Schemes, Lemma 17.2). $\qquad \square$

Lemma 9.11. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*

(2) *every affine scheme Z and morphism $h : Z \to T$ there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ which refines the family $\{T_i \times_T Z \to Z\}_{i \in I}$.*

Then $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering of T.

Proof. Let $T = \bigcup U_\alpha$ be an affine open covering. For each α the pullback family $\{T_i \times_T U_\alpha \to U_\alpha\}$ can be refined by a standard fpqc covering, hence is an fpqc covering by Lemma 9.4. As $\{U_\alpha \to T\}$ is an fpqc covering we conclude that $\{T_i \to T\}$ is an fpqc covering by Lemma 9.5. $\qquad\square$

Definition 9.12. Let F be a contravariant functor on the category of schemes with values in sets.

(1) Let $\{U_i \to T\}_{i \in I}$ be a family of morphisms of schemes with fixed target. We say that F *satisfies the sheaf property for the given family* if for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \times_T U_j} = \xi_j|_{U_i \times_T U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
(2) We say that F *satisfies the sheaf property for the fpqc topology* if it satisfies the sheaf property for any fpqc covering.

We try to avoid using the terminology "F is a sheaf" in this situation since we are not defining a category of fpqc sheaves as we explained above.

Lemma 9.13. *Let F be a contravariant functor on the category of schemes with values in sets. Then F satisfies the sheaf property for the fpqc topology if and only if it satisfies*

(1) *the sheaf property for every Zariski covering, and*
(2) *the sheaf property for any standard fpqc covering.*

Moreover, in the presence of (1) property (2) is equivalent to property

(2') *the sheaf property for $\{V \to U\}$ with V, U affine and $V \to U$ faithfully flat.*

Proof. Assume (1) and (2) hold. Let $\{f_i : T_i \to T\}_{i \in I}$ be an fpqc covering. Let $s_i \in F(T_i)$ be a family of elements such that s_i and s_j map to the same element of $F(T_i \times_T T_j)$. Let $W \subset T$ be the maximal open subset such that there exists a unique $s \in F(W)$ with $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$ for all i. Such a maximal open exists because F satisfies the sheaf property for Zariski coverings; in fact W is the union of all opens with this property. Let $t \in T$. We will show $t \in W$. To do this we pick an affine open $t \in U \subset T$ and we will show there is a unique $s \in F(U)$ with $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$ for all i.

By Lemma 9.8 we can find a standard fpqc covering $\{U_j \to U\}_{j=1,\dots,n}$ refining $\{U \times_T T_i \to U\}$, say by morphisms $h_j : U_j \to T_{i_j}$. By (2) we obtain a unique element $s \in F(U)$ such that $s|_{U_j} = F(h_j)(s_{i_j})$. Note that for any scheme $V \to U$ over U there is a unique section $s_V \in F(V)$ which restricts to $F(h_j \circ \mathrm{pr}_2)(s_{i_j})$ on $V \times_U U_j$ for $j = 1, \dots, n$. Namely, this is true if V is affine by (2) as $\{V \times_U U_j \to V\}$ is a standard fpqc covering and in general this follows from (1) and the affine case by choosing an affine open covering of V. In particular, $s_V = s|_V$. Now, taking $V = U \times_T T_i$ and using that $s_{i_j}|_{T_{i_j} \times_T T_i} = s_i|_{T_{i_j} \times_T T_i}$ we conclude that $s|_{U \times_T T_i} = s_V = s_i|_{U \times_T T_i}$ which is what we had to show.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $\{T_i \to T\}$ is a standard fpqc covering, then $\coprod T_i \to T$ is a faithfully flat morphism of affine schemes. In the presence of (1) we have $F(\coprod T_i) = \prod F(T_i)$ and similarly $F((\coprod T_i) \times_T (\coprod T_i)) = \prod F(T_i \times_T T_{i'})$. Thus the sheaf condition for $\{T_i \to T\}$ and $\{\coprod T_i \to T\}$ is the same. $\qquad\square$

The following lemma is here just to point out set theoretical difficulties do indeed arise and should be ignored by most readers.

Lemma 9.14. *Let R be a nonzero ring. There does not exist a set A of fpqc-coverings of $\mathrm{Spec}(R)$ such that every fpqc-covering can be refined by an element of A.*

Proof. Let us first explain this when $R = k$ is a field. For any set I consider the purely transcendental field extension $k_I = k(\{t_i\}_{i\in I})/k$. Since $k \to k_I$ is faithfully flat we see that $\{\mathrm{Spec}(k_I) \to \mathrm{Spec}(k)\}$ is an fpqc covering. Let A be a set and for each $\alpha \in A$ let $\mathcal{U}_\alpha = \{S_{\alpha,j} \to \mathrm{Spec}(k)\}_{j\in J_\alpha}$ be an fpqc covering. If $\mathcal{U}_\alpha$ refines $\{\mathrm{Spec}(k_I) \to \mathrm{Spec}(k)\}$ then the morphisms $S_{\alpha,j} \to \mathrm{Spec}(k)$ factor through $\mathrm{Spec}(k_I)$. Since $\mathcal{U}_\alpha$ is a covering, at least some $S_{\alpha,j}$ is nonempty. Pick a point $s \in S_{\alpha,j}$. Since we have the factorization $S_{\alpha,j} \to \mathrm{Spec}(k_I) \to \mathrm{Spec}(k)$ we obtain a homomorphism of fields $k_I \to \kappa(s)$. In particular, we see that the cardinality of $\kappa(s)$ is at least the cardinality of I. Thus if we take I to be a set of cardinality bigger than the cardinalities of the residue fields of all the schemes $S_{\alpha,j}$, then such a factorization does not exist and the lemma holds for $R = k$.

General case. Since R is nonzero it has a maximal prime ideal $\mathfrak{m}$ with residue field κ. Let I be a set and consider $R_I = S_I^{-1}R[\{t_i\}_{i\in I}]$ where $S_I \subset R[\{t_i\}_{i\in I}]$ is the multiplicative subset of $f \in R[\{t_i\}_{i\in I}]$ such that f maps to a nonzero element of $R/\mathfrak{p}[\{t_i\}_{i\in I}]$ for all primes $\mathfrak{p}$ of R. Then R_I is a faithfully flat R-algebra and $\{\mathrm{Spec}(R_I) \to \mathrm{Spec}(R)\}$ is an fpqc covering. We leave it as an exercise to the reader to show that $R_I \otimes_R \kappa \cong \kappa(\{t_i\}_{i\in I}) = \kappa_I$ with notation as above (hint: use that $R \to \kappa$ is surjective and that any $f \in R[\{t_i\}_{i\in I}]$ one of whose monomials occurs with coefficient 1 is an element of S_I). Let A be a set and for each $\alpha \in A$ let $\mathcal{U}_\alpha = \{S_{\alpha,j} \to \mathrm{Spec}(R)\}_{j\in J_\alpha}$ be an fpqc covering. If $\mathcal{U}_\alpha$ refines $\{\mathrm{Spec}(R_I) \to \mathrm{Spec}(R)\}$, then by base change we conclude that $\{S_{\alpha,j} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa) \to \mathrm{Spec}(\kappa)\}$ refines $\{\mathrm{Spec}(\kappa_I) \to \mathrm{Spec}(\kappa)\}$. Hence by the result of the previous paragraph, there exists an I such that this is not the case and the lemma is proved. $\square$

10. The V topology

The V topology is stronger than all other topologies in this chapter. Roughly speaking it is generated by Zariski coverings and by quasi-compact morphisms satisfying a lifting property for specializations (Lemma 10.13). However, the procedure we will use to define V coverings is a bit different. We will first define standard V coverings of affines and then use these to define V coverings in general. Typographical point: in the literature sometimes "v-covering" is used instead of "V covering".

Definition 10.1. Let T be an affine scheme. A *standard V covering* is a finite family $\{T_j \to T\}_{j=1,\ldots,m}$ with T_j affine such that for every morphism $g : \mathrm{Spec}(V) \to T$ where V is a valuation ring, there is an extension $V \subset W$ of valuation rings (More on Algebra, Definition 123.1), an index $1 \le j \le m$, and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & T_j \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \xrightarrow{\ g\ } & T \end{array}$$

We first prove a few basic lemmas about this notion.

Lemma 10.2. *A standard fpqc covering is a standard V covering.*

Proof. Let $\{X_i \to X\}_{i=1,\ldots,n}$ be a standard fpqc covering (Definition 9.9). Let $g : \mathrm{Spec}(V) \to X$ be a morphism where V is a valuation ring. Let $x \in X$ be the image of the closed point of $\mathrm{Spec}(V)$. Choose an i and a point $x_i \in X_i$ mapping to x. Then $\mathrm{Spec}(V) \times_X X_i$ has a point x_i' mapping to the closed point of $\mathrm{Spec}(V)$. Since $\mathrm{Spec}(V) \times_X X_i \to \mathrm{Spec}(V)$ is flat we can find a specialization $x_i'' \rightsquigarrow x_i'$ of points of $\mathrm{Spec}(V) \times_X X_i$ with x_i'' mapping to the generic point of $\mathrm{Spec}(V)$, see Morphisms, Lemma 25.9. By Schemes, Lemma 20.4 we can choose a valuation ring W and a morphism $h : \mathrm{Spec}(W) \to \mathrm{Spec}(V) \times_X X_i$ such that h maps the generic point of $\mathrm{Spec}(W)$ to x_i'' and the closed point of $\mathrm{Spec}(W)$ to x_i'. We obtain a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(W) & \longrightarrow & X_i \\
\downarrow & & \downarrow \\
\mathrm{Spec}(V) & \longrightarrow & X
\end{array}
$$

where $V \to W$ is an extension of valuation rings. This proves the lemma. $\qquad\square$

Lemma 10.3. *A standard ph covering is a standard V covering.*

Proof. Let T be an affine scheme. Let $f : U \to T$ be a proper surjective morphism. Let $U = \bigcup_{j=1,\ldots,m} U_j$ be a finite affine open covering. We have to show that $\{U_j \to T\}$ is a standard V covering, see Definition 8.1. Let $g : \mathrm{Spec}(V) \to T$ be a morphism where V is a valuation ring with fraction field K. Since $U \to T$ is surjective, we may choose a field extension L/K and a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(L) & \longrightarrow & U \\
\downarrow & & \downarrow \\
\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(V) & \xrightarrow{\ g\ } & T
\end{array}
$$

By Algebra, Lemma 50.2 we can choose a valuation ring $W \subset L$ dominating V. By the valuative criterion of properness (Morphisms, Lemma 42.1) we can then find the morphism h in the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(W) & \xrightarrow{\ h\ } & U \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(V) & \xrightarrow{\ g\ } & X
\end{array}
$$

Since $\mathrm{Spec}(W)$ has a unique closed point, we see that $\mathrm{Im}(h)$ is contained in U_j for some j. Thus $h : \mathrm{Spec}(W) \to U_j$ is the desired lift and we conclude $\{U_j \to T\}$ is a standard V covering. $\qquad\square$

Lemma 10.4. *Let $\{T_j \to T\}_{j=1,\ldots,m}$ be a standard V covering. Let $T' \to T$ be a morphism of affine schemes. Then $\{T_j \times_T T' \to T'\}_{j=1,\ldots,m}$ is a standard V covering.*

Proof. Let $\mathrm{Spec}(V) \to T'$ be a morphism where V is a valuation ring. By assumption we can find an extension of valuation rings $V \subset W$, an i, and a commutative

diagram

$$\begin{CD}
\mathrm{Spec}(W) @>>> T_i \\
@VVV @VVV \\
\mathrm{Spec}(V) @>>> T
\end{CD}$$

By the universal property of fibre products we obtain a morphism $\mathrm{Spec}(W) \to T' \times_T T_i$ as desired. $\qquad\square$

Lemma 10.5. *Let T be an affine scheme. Let $\{T_j \to T\}_{j=1,\dots,m}$ be a standard V covering. Let $\{T_{ji} \to T_j\}_{i=1,\dots n_j}$ be a standard V covering. Then $\{T_{ji} \to T\}_{i,j}$ is a standard V covering.*

Proof. This follows formally from the observation that if $V \subset W$ and $W \subset \Omega$ are extensions of valuation rings, then $V \subset \Omega$ is an extension of valuation rings. $\qquad\square$

Lemma 10.6. *Let T be an affine scheme. Let $\{T_j \to T\}_{j=1,\dots,m}$ be a family of morphisms with T_j affine for all j. The following are equivalent*

(1) $\{T_j \to T\}_{j=1,\dots,m}$ *is a standard V covering,*
(2) *there is a standard V covering which refines $\{T_j \to T\}_{j=1,\dots,m}$, and*
(3) $\{\coprod_{j=1,\dots,m} T_j \to T\}$ *is a standard V covering.*

Proof. Omitted. Hints: This follows almost immediately from the definition. The only slightly interesting point is that a morphism from the spectrum of a local ring into $\coprod_{j=1,\dots,m} T_j$ must factor through some T_j. $\qquad\square$

Definition 10.7. Let T be a scheme. A *V covering of T* is a family of morphisms $\{T_i \to T\}_{i \in I}$ of schemes such that for every affine open $U \subset T$ there exists a standard V covering $\{U_j \to U\}_{j=1,\dots,m}$ refining the family $\{T_i \times_T U \to U\}_{i \in I}$.

The V topology has the same set theoretical problems as the fpqc topology. Thus we refrain from defining V sites and we will not consider cohomology with respect to the V topology. On the other hand, given a $F : Sch^{opp} \to Sets$ it does make sense to ask whether F satisfies the sheaf property for the V topology, see below. Moreover, we can wonder about descent of object in the V topology, etc.

Lemma 10.8. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms. The following are equivalent*

(1) $\{T_i \to T\}_{i \in I}$ *is a V covering,*
(2) *there is a V covering which refines $\{T_i \to T\}_{i \in I}$, and*
(3) $\{\coprod_{i \in I} T_i \to T\}$ *is a V covering.*

Proof. Omitted. Hint: compare with the proof of Lemma 8.7. $\qquad\square$

Lemma 10.9. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a V covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a V covering and for each i we have a V covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a V covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a V covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a V covering.*

Proof. Assertion (1) is clear.

Proof of (3). Let $U' \subset T'$ be an affine open subscheme. Since U' is quasi-compact we can find a finite affine open covering $U' = U'_1 \cup \ldots \cup U'$ such that $U'_j \to T$ maps into an affine open $U_j \subset T$. Choose a standard V covering $\{U_{jl} \to U_j\}_{l=1,\ldots,n_j}$ refining $\{T_i \times_T U_j \to U_j\}$. By Lemma 10.4 the base change $\{U_{jl} \times_{U_j} U'_j \to U'_j\}$ is a standard V covering. Note that $\{U'_j \to U'\}$ is a standard V covering (for example by Lemma 10.2). By Lemma 10.5 the family $\{U_{jl} \times_{U_j} U'_j \to U'\}$ is a standard V covering. Since $\{U_{jl} \times_{U_j} U'_j \to U'\}$ refines $\{T_i \times_T U' \to U'\}$ we conclude.

Proof of (2). Let $U \subset T$ be affine open. First we pick a standard V covering $\{U_k \to U\}_{k=1,\ldots,m}$ refining $\{T_i \times_T U \to U\}$. Say the refinement is given by morphisms $U_k \to T_{i_k}$ over T. Then

$$\{T_{i_k j} \times_{T_{i_k}} U_k \to U_k\}_{j \in J_{i_k}}$$

is a V covering by part (3). As U_k is affine, we can find a standard V covering $\{U_{ka} \to U_k\}_{a=1,\ldots,b_k}$ refining this family. Then we apply Lemma 10.5 to see that $\{U_{ka} \to U\}$ is a standard V covering which refines $\{T_{ij} \times_T U \to U\}$. This finishes the proof. $\square$

Lemma 10.10. *Any fpqc covering is a V covering. A fortiori, any fppf, syntomic, smooth, étale or Zariski covering is a V covering. Also, a ph covering is a V covering.*

Proof. An fpqc covering can affine locally be refined by a standard fpqc covering, see Lemmas 9.8. A standard fpqc covering is a standard V covering, see Lemma 10.2. Hence the first statement follows from our definition of V covers in terms of standard V coverings. The conclusion for fppf, syntomic, smooth, étale or Zariski coverings follows as these are fpqc coverings, see Lemma 9.6.

The statement on ph coverings follows from Lemma 10.3 in the same manner. $\square$

Definition 10.11. Let F be a contravariant functor on the category of schemes with values in sets. We say that F *satisfies the sheaf property for the V topology* if it satisfies the sheaf property for any V covering (see Definition 9.12).

We try to avoid using the terminology "F is a sheaf" in this situation since we are not defining a category of V sheaves as we explained above.

Lemma 10.12. *Let F be a contravariant functor on the category of schemes with values in sets. Then F satisfies the sheaf property for the V topology if and only if it satisfies*

(1) *the sheaf property for every Zariski covering, and*
(2) *the sheaf property for any standard V covering.*

Moreover, in the presence of (1) property (2) is equivalent to property

(2') *the sheaf property for a standard V covering of the form $\{V \to U\}$, i.e., consisting of a single arrow.*

Proof. Assume (1) and (2) hold. Let $\{f_i : T_i \to T\}_{i \in I}$ be a V covering. Let $s_i \in F(T_i)$ be a family of elements such that s_i and s_j map to the same element of $F(T_i \times_T T_j)$. Let $W \subset T$ be the maximal open subset such that there exists a unique $s \in F(W)$ with $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$ for all i. Such a maximal open exists because F satisfies the sheaf property for Zariski coverings; in fact W is the union

of all opens with this property. Let $t \in T$. We will show $t \in W$. To do this we pick an affine open $t \in U \subset T$ and we will show there is a unique $s \in F(U)$ with $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$ for all i.

We can find a standard V covering $\{U_j \to U\}_{j=1,\ldots,n}$ refining $\{U \times_T T_i \to U\}$, say by morphisms $h_j : U_j \to T_{i_j}$. By (2) we obtain a unique element $s \in F(U)$ such that $s|_{U_j} = F(h_j)(s_{i_j})$. Note that for any scheme $V \to U$ over U there is a unique section $s_V \in F(V)$ which restricts to $F(h_j \circ \mathrm{pr}_2)(s_{i_j})$ on $V \times_U U_j$ for $j = 1, \ldots, n$. Namely, this is true if V is affine by (2) as $\{V \times_U U_j \to V\}$ is a standard V covering (Lemma 10.4) and in general this follows from (1) and the affine case by choosing an affine open covering of V. In particular, $s_V = s|_V$. Now, taking $V = U \times_T T_i$ and using that $s_{i_j}|_{T_{i_j} \times_T T_i} = s_i|_{T_{i_j} \times_T T_i}$ we conclude that $s|_{U \times_T T_i} = s_V = s_i|_{U \times_T T_i}$ which is what we had to show.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $\{T_i \to T\}_{i=1,\ldots,n}$ is a standard V covering, then $\coprod_{i=1,\ldots,n} T_i \to T$ is a morphism of affine schemes which is clearly also a standard V covering. In the presence of (1) we have $F(\coprod T_i) = \prod F(T_i)$ and similarly $F((\coprod T_i) \times_T (\coprod T_i)) = \prod F(T_i \times_T T_{i'})$. Thus the sheaf condition for $\{T_i \to T\}$ and $\{\coprod T_i \to T\}$ is the same. $\square$

The following lemma shows that being a V covering is related to the possibility of lifting specializations.

Lemma 10.13. *Let $X \to Y$ be a quasi-compact morphism of schemes. The following are equivalent*

(1) *$\{X \to Y\}$ is a V covering,*
(2) *for any valuation ring V and morphism $g : \mathrm{Spec}(V) \to Y$ there exists an extension of valuation rings $V \subset W$ and a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & Y \end{array}$$

(3) *for any morphism $Z \to Y$ and specialization $z' \rightsquigarrow z$ of points in Z, there is a specialization $w' \rightsquigarrow w$ of points in $Z \times_Y X$ mapping to $z' \rightsquigarrow z$.*

Proof. Assume (1) and let $g : \mathrm{Spec}(V) \to Y$ be as in (2). Since V is a local ring there is an affine open $U \subset Y$ such that g factors through U. By Definition 10.7 we can find a standard V covering $\{U_j \to U\}$ refining $\{X \times_Y U \to U\}$. By Definition 10.1 we can find a j, an extension of valuation rings $V \subset W$ and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow U_j & \dashrightarrow X \\ \downarrow & \downarrow & \swarrow \\ \mathrm{Spec}(V) & \longrightarrow Y & \end{array}$$

We have the dotted arrow making the diagram commute by the refinement property of the covering and we see that (2) holds.

Assume (2) and let $Z \to Y$ and $z' \rightsquigarrow z$ be as in (3). By Schemes, Lemma 20.4 we can find a valuation ring V and a morphism $\mathrm{Spec}(V) \to Z$ such that the closed

point of $\mathrm{Spec}(V)$ maps to z and the generic point of $\mathrm{Spec}(V)$ maps to z'. By (2) we can find an extension of valuation rings $V \subset W$ and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) \longrightarrow Z & \longrightarrow & Y \end{array}$$

The generic and closed points of $\mathrm{Spec}(W)$ map to points $w' \rightsquigarrow w$ in $Z \times_Y X$ via the induced morphism $\mathrm{Spec}(W) \to Z \times_Y X$. This shows that (3) holds.

Assume (3) holds and let $U \subset Y$ be an affine open. Choose a finite affine open covering $U \times_Y X = \bigcup_{j=1,\dots,m} U_j$. This is possible as $X \to Y$ is quasi-compact. We claim that $\{U_j \to U\}$ is a standard V covering. The claim implies (1) is true and finishes the proof of the lemma. In order to prove the claim, let V be a valuation ring and let $g : \mathrm{Spec}(V) \to U$ be a morphism. By (3) we find a specialization $w' \rightsquigarrow w$ of points of

$$T = \mathrm{Spec}(V) \times_X Y = \mathrm{Spec}(V) \times_U (U \times_X Y)$$

such that w' maps to the generic point of $\mathrm{Spec}(V)$ and w maps to the closed point of $\mathrm{Spec}(V)$. By Schemes, Lemma 20.4 we can find a valuation ring W and a morphism $\mathrm{Spec}(W) \to T$ such that the generic point of $\mathrm{Spec}(W)$ maps to w' and the closed point of $\mathrm{Spec}(W)$ maps to w. The composition $\mathrm{Spec}(W) \to T \to \mathrm{Spec}(V)$ corresponds to an inclusion $V \subset W$ which presents W as an extension of the valuation ring V. Since $T = \bigcup \mathrm{Spec}(V) \times_U U_j$ is an open covering, we see that $\mathrm{Spec}(W) \to T$ factors through $\mathrm{Spec}(V) \times_U U_j$ for some j. Thus we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & U_j \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & U \end{array}$$

and the proof of the claim is complete. $\qquad\square$

A V covering gives a universally submersive family of maps. The converse of this lemma is false, see Examples, Section 78.

Lemma 10.14. *Let $\{f_i : X_i \to X\}_{i \in I}$ be a V covering. Then*

$$\coprod_{i \in I} f_i : \coprod_{i \in I} X_i \longrightarrow X$$

is a universally submersive morphism of schemes (Morphisms, Definition 24.1).

Proof. We will use without further mention that the base change of a V covering is a V covering (Lemma 10.9). In particular it suffices to show that the morphism is submersive. Being submersive is clearly Zariski local on the base. Thus we may assume X is affine. Then $\{X_i \to X\}$ can be refined by a standard V covering $\{Y_j \to X\}$. If we can show that $\coprod Y_j \to X$ is submersive, then since there is a factorization $\coprod Y_j \to \coprod X_i \to X$ we conclude that $\coprod X_i \to X$ is submersive. Set $Y = \coprod Y_j$ and consider the morphism of affines $f : Y \to X$. By Lemma 10.13 we know that we can lift any specialization $x' \rightsquigarrow x$ in X to some specialization $y' \rightsquigarrow y$ in Y. Thus if $T \subset X$ is a subset such that $f^{-1}(T)$ is closed in Y, then $T \subset X$ is closed under specialization. Since $f^{-1}(T) \subset Y$ with the reduced induced

closed subscheme structure is an affine scheme, we conclude that $T \subset X$ is closed by Algebra, Lemma 41.5. Hence f is submersive. $\qquad\qquad\qquad\qquad\qquad\square$

11. Change of topologies

Let $f : X \to Y$ be a morphism of schemes over a base scheme S. In this case we have the following morphisms of sites[7] (with suitable choices of sites as in Remark 11.1 below):

(1) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{fppf}$,
(2) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{syntomic}$,
(3) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{smooth}$,
(4) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{étale}$,
(5) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{Zar}$,
(6) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{syntomic}$,
(7) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{smooth}$,
(8) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{étale}$,
(9) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{Zar}$,
(10) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{smooth}$,
(11) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{étale}$,
(12) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{Zar}$,
(13) $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{étale}$,
(14) $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{Zar}$,
(15) $(Sch/X)_{Zar} \longrightarrow (Sch/Y)_{Zar}$,
(16) $(Sch/X)_{fppf} \longrightarrow Y_{étale}$,
(17) $(Sch/X)_{syntomic} \longrightarrow Y_{étale}$,
(18) $(Sch/X)_{smooth} \longrightarrow Y_{étale}$,
(19) $(Sch/X)_{étale} \longrightarrow Y_{étale}$,
(20) $(Sch/X)_{fppf} \longrightarrow Y_{Zar}$,
(21) $(Sch/X)_{syntomic} \longrightarrow Y_{Zar}$,
(22) $(Sch/X)_{smooth} \longrightarrow Y_{Zar}$,
(23) $(Sch/X)_{étale} \longrightarrow Y_{Zar}$,
(24) $(Sch/X)_{Zar} \longrightarrow Y_{Zar}$,
(25) $X_{étale} \longrightarrow Y_{étale}$,
(26) $X_{étale} \longrightarrow Y_{Zar}$,
(27) $X_{Zar} \longrightarrow Y_{Zar}$,

In each case the underlying continuous functor $Sch/Y \to Sch/X$, or $Y_\tau \to Sch/X$ is the functor $Y'/Y \mapsto X \times_Y Y'/X$. Namely, in the sections above we have seen the morphisms $f_{big} : (Sch/X)_\tau \to (Sch/Y)_\tau$ and $f_{small} : X_\tau \to Y_\tau$ for τ as above. We also have seen the morphisms of sites $\pi_Y : (Sch/Y)_\tau \to Y_\tau$ for $\tau \in \{étale, Zariski\}$. On the other hand, it is clear that the identity functor $(Sch/X)_\tau \to (Sch/X)_{\tau'}$ defines a morphism of sites when τ is a stronger topology than τ'. Hence composing these gives the list of possible morphisms above.

Because of the simple description of the underlying functor it is clear that given morphisms of schemes $X \to Y \to Z$ the composition of two of the morphisms of sites above, e.g.,

$$(Sch/X)_{\tau_0} \longrightarrow (Sch/Y)_{\tau_1} \longrightarrow (Sch/Z)_{\tau_2}$$

---

[7]We have not included the comparison between the ph topology and the others; for this see More on Morphisms, Remark 48.8.

is the corresponding morphism of sites associated to the morphism of schemes $X \to Z$.

Remark 11.1. Take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set of schemes $\{X, Y, S\}$. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained. Next, for $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic\}$ let Sch_τ have the same underlying category as Sch_{fppf} with coverings $\mathrm{Cov}_\tau \subset \mathrm{Cov}_{fppf}$ simply the subset of τ-coverings. It is straightforward to check that this gives rise to a big site Sch_τ.

12. Change of big sites

In this section we explain what happens on changing the big Zariski/fppf/étale sites.

Let $\tau, \tau' \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Given two big sites Sch_τ and $Sch'_{\tau'}$ we say that Sch_τ *is contained in* $Sch'_{\tau'}$ if $\mathrm{Ob}(Sch_\tau) \subset \mathrm{Ob}(Sch'_{\tau'})$ and $\mathrm{Cov}(Sch_\tau) \subset \mathrm{Cov}(Sch'_{\tau'})$. In this case τ is stronger than τ', for example, no fppf site can be contained in an étale site.

Lemma 12.1. *Any set of big Zariski sites is contained in a common big Zariski site. The same is true, mutatis mutandis, for big fppf and big étale sites.*

Proof. This is true because the union of a set of sets is a set, and the constructions in Sets, Lemmas 9.2 and 11.1 allow one to start with any initially given set of schemes and coverings. $\square$

Lemma 12.2. *Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Suppose given big sites Sch_τ and Sch'_τ. Assume that Sch_τ is contained in Sch'_τ. The inclusion functor $Sch_\tau \to Sch'_\tau$ satisfies the assumptions of Sites, Lemma 21.8. There are morphisms of topoi*

$$
\begin{aligned}
g : Sh(Sch_\tau) &\longrightarrow Sh(Sch'_\tau) \\
f : Sh(Sch'_\tau) &\longrightarrow Sh(Sch_\tau)
\end{aligned}
$$

such that $f \circ g \cong id$. For any object S of Sch_τ the inclusion functor $(Sch/S)_\tau \to (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 21.8 also. Hence similarly we obtain morphisms

$$
\begin{aligned}
g : Sh((Sch/S)_\tau) &\longrightarrow Sh((Sch'/S)_\tau) \\
f : Sh((Sch'/S)_\tau) &\longrightarrow Sh((Sch/S)_\tau)
\end{aligned}
$$

with $f \circ g \cong id$.

Proof. Assumptions (b), (c), and (e) of Sites, Lemma 21.8 are immediate for the functors $Sch_\tau \to Sch'_\tau$ and $(Sch/S)_\tau \to (Sch'/S)_\tau$. Property (a) holds by Lemma 3.6, 4.7, 5.7, 6.7, or 7.7. Property (d) holds because fibre products in the categories Sch_τ, Sch'_τ exist and are compatible with fibre products in the category of schemes. $\square$

Discussion: The functor $g^{-1} = f_*$ is simply the restriction functor which associates to a sheaf $\mathcal{G}$ on Sch'_τ the restriction $\mathcal{G}|_{Sch_\tau}$. Hence this lemma simply says that given any sheaf of sets $\mathcal{F}$ on Sch_τ there exists a canonical sheaf $\mathcal{F}'$ on Sch'_τ such

that $\mathcal{F}|_{Sch'_\tau} = \mathcal{F}'$. In fact the sheaf $\mathcal{F}'$ has the following description: it is the sheafification of the presheaf

$$Sch'_\tau \longrightarrow Sets, \quad V \longmapsto \mathrm{colim}_{V \to U} \, \mathcal{F}(U)$$

where U is an object of Sch_τ. This is true because $\mathcal{F}' = f^{-1}\mathcal{F} = (u_p\mathcal{F})^\#$ according to Sites, Lemmas 21.5 and 21.8.

13. Extending functors

Let us start with a simple example which explains what we are doing. Let R be a ring. Suppose F is a functor defined on the category $\mathcal{C}$ of R-algebras of the form

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

for $n, m \geq 0$ integers and $f_1, \ldots, f_m \in R[x_1, \ldots, x_m]$ elements. Then for any R-algebra B we can define

$$F'(B) = \mathrm{colim}_{A \to B, \ A \in \mathcal{C}} \, F(A)$$

It turns out F' is the unique functor on the category of all R-algebras which extends F and commutes with filtered colimits. The same procedure works in the category of schemes if we impose that our functor is a Zariski sheaf.

Lemma 13.1. *Let S be a scheme. Let $\mathcal{C}$ be a full subcategory of the category Sch/S of all schemes over S. Assume*

    (1) *if $X \to S$ is an object of $\mathcal{C}$ and $U \subset X$ is an affine open, then $U \to S$ is isomorphic to an object of $\mathcal{C}$,*

    (2) *if V is an affine scheme lying over an affine open $U \subset S$ such that $V \to U$ is of finite presentation, then $V \to S$ is isomorphic to an object of $\mathcal{C}$.*

Let $F : \mathcal{C}^{opp} \to Sets$ be a functor. Assume

    (a) *for any Zariski covering $\{f_i : X_i \to X\}_{i \in I}$ with X, X_i objects of $\mathcal{C}$ we have the sheaf condition for F and this family[8],*

    (b) *if $X = \lim X_i$ is a directed limit of affine schemes over S with X, X_i objects of $\mathcal{C}$, then $F(X) = \mathrm{colim}\, F(X_i)$.*

Then there is a unique way to extend F to a functor $F' : (Sch/S)^{opp} \to Sets$ satisfying the analogues of (a) and (b), i.e., F' satisfies the sheaf condition for any Zariski covering and $F'(X) = \mathrm{colim}\, F'(X_i)$ whenever $X = \lim X_i$ is a directed limit of affine schemes over S.

Proof. The idea will be to first extend F to a sufficiently large collection of affine schemes over S and then use the Zariski sheaf property to extend to all schemes.

Suppose that V is an affine scheme over S whose structure morphism $V \to S$ factors through some affine open $U \subset S$. In this case we can write

$$V = \lim V_i$$

as a cofiltered limit with $V_i \to U$ of finite presentation and V_i affine. See Algebra, Lemma 127.2. By conditions (1) and (2) we may replace our V_i by objects of $\mathcal{C}$.

---

[8]As we do not know that $X_i \times_X X_j$ is in $\mathcal{C}$ this has to be interpreted as follows: by property (1) there exist Zariski coverings $\{U_{ijk} \to X_i \times_X X_j\}_{k \in K_{ij}}$ with U_{ijk} an object of $\mathcal{C}$. Then the sheaf condition says that $F(X)$ is the equalizer of the two maps from $\prod F(X_i)$ to $\prod F(U_{ijk})$.

Observe that $V_i \to S$ is locally of finite presentation (if S is quasi-separated, then these morphisms are actually of finite presentation). Then we set

$$F'(V) = \operatorname{colim} F(V_i)$$

Actually, we can give a more canonical expression, namely

$$F'(V) = \operatorname{colim}_{V \to V'} F(V')$$

where the colimit is over the category of morphisms $V \to V'$ over S where V' is an object of $\mathcal{C}$ whose structure morphism $V' \to S$ is locally of finite presentation. The reason this is the same as the first formula is that by Limits, Proposition 6.1 our inverse system V_i is cofinal in this category! Finally, note that if V were an object of $\mathcal{C}$, then $F'(V) = F(V)$ by assumption (b).

The second formula turns F' into a contravariant functor on the category formed by affine schemes V over S whose structure morphism factors through an affine open of S. Let V be such an affine scheme over S and suppose that $V = \bigcup_{k=1,\ldots,n} V_k$ is a finite open covering by affines. Then it makes sense to ask if the sheaf condition holds for F' and this open covering. This is true and easy to show: write $V = \lim V_i$ as in the previous paragraph. By Limits, Lemma 4.11 for all sufficiently large i we can find affine opens $V_{i,k} \subset V_i$ compatible with transition maps pulling back to V_k in V. Thus

$$F'(V_k) = \operatorname{colim} F(V_{i,k}) \quad \text{and} \quad F'(V_k \cap V_l) = \operatorname{colim} F(V_{i,k} \cap V_{i,l})$$

Strictly speaking in these formulas we need to replace $V_{i,k}$ and $V_{i,k} \cap V_{i,l}$ by isomorphic affine objects of $\mathcal{C}$ before applying the functor F. Since I is directed the colimits pass through equalizers. Hence the sheaf condition (b) for F and the Zariski coverings $\{V_{i,k} \to V_i\}$ implies the sheaf condition for F' and this covering.

Let X be a general scheme over S. Let $\mathcal{B}_X$ denote the collection of affine opens of X whose structure morphism to S maps into an affine open of S. It is clear that $\mathcal{B}_X$ is a basis for the topology of X. By the result of the previous paragraph and Sheaves, Lemma 30.4 we see that F' is a sheaf on $\mathcal{B}_X$. Hence F' restricted to $\mathcal{B}_X$ extends uniquely to a sheaf F'_X on X, see Sheaves, Lemma 30.6. If X is an object of $\mathcal{C}$ then we have a canonical identification $F'_X(X) = F(X)$ by the agreement of F' and F on the objects for which they are both defined and the fact that F satisfies the sheaf condition for Zariski coverings.

Let $f : X \to Y$ be a morphism of schemes over S. We get a unique f-map from F'_Y to F'_X compatible with the maps $F'(V) \to F'(U)$ for all $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ with $f(U) \subset V$, see Sheaves, Lemma 30.16. We omit the verification that these maps compose correctly given morphisms $X \to Y \to Z$ of schemes over S. We also omit the verification that if f is a morphism of $\mathcal{C}$, then the induced map $F'_Y(Y) \to F'_X(X)$ is the same as the map $F(Y) \to F(X)$ via the identifications $F'_X(X) = F(X)$ and $F'_Y(Y) = F(Y)$ above. In this way we see that the desired extension of F is the functor which sends X/S to $F'_X(X)$.

Property (a) for the functor $X \mapsto F'_X(X)$ is almost immediate from the construction; we omit the details. Suppose that $X = \lim_{i \in I} X_i$ is a directed limit of affine schemes over S. We have to show that

$$F'_X(X) = \operatorname{colim}_{i \in I} F'_{X_i}(X_i)$$

First assume that there is some $i \in I$ such that $X_i \to S$ factors through an affine open $U \subset S$. Then F' is defined on X and on $X_{i'}$ for $i' \geq i$ and we see that $F'_{X_{i'}}(X_{i'}) = F'(X_{i'})$ for $i' \geq i$ and $F'_X(X) = F'(X)$. In this case every arrow $X \to V$ with V locally of finite presentation over S factors as $X \to X_{i'} \to V$ for some $i' \geq i$, see Limits, Proposition 6.1. Thus we have

$$
\begin{aligned}
F'_X(X) &= F'(X) \\
&= \operatorname{colim}_{X \to V} F(V) \\
&= \operatorname{colim}_{i' \geq i} \operatorname{colim}_{X_{i'} \to V} F(V) \\
&= \operatorname{colim}_{i' \geq i} F'(X_{i'}) \\
&= \operatorname{colim}_{i' \geq i} F'_{X_{i'}}(X_{i'}) \\
&= \operatorname{colim}_{i' \in I} F'_{X_{i'}}(X_{i'})
\end{aligned}
$$

as desired. Finally, in general we pick any $i \in I$ and we choose a finite affine open covering $V_i = V_{i,1} \cup \ldots \cup V_{i,n}$ such that $V_{i,k} \to S$ factors through an affine open of S. Let $V_k \subset V$ and $V_{i',k}$ for $i' \geq i$ be the inverse images of $V_{i,k}$. By the previous case we see that

$$
F'_{V_k}(V_k) = \operatorname{colim}_{i' \geq i} F'_{V_{i',k}}(V_{i',k})
$$

and

$$
F'_{V_k \cap V_l}(V_k \cap V_l) = \operatorname{colim}_{i' \geq i} F'_{V_{i',k} \cap V_{i',l}}(V_{i',k} \cap V_{i',l})
$$

By the sheaf property and exactness of filtered colimits we find that $F'_X(X) = \operatorname{colim}_{i \in I} F'_{X_i}(X_i)$ also in this case. This finishes the proof of property (b) and hence finishes the proof of the lemma. $\qquad\square$

Lemma 13.2. *Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Let T be an affine scheme which is written as a limit $T = \lim_{i \in I} T_i$ of a directed inverse system of affine schemes.*

(1) *Let $\mathcal{V} = \{V_j \to T\}_{j=1,\ldots,m}$ be a standard τ-covering of T, see Definitions 3.4, 4.5, 5.5, 6.5, and 7.5. Then there exists an index i and a standard τ-covering $\mathcal{V}_i = \{V_{i,j} \to T_i\}_{j=1,\ldots,m}$ whose base change $T \times_{T_i} \mathcal{V}_i$ to T is isomorphic to $\mathcal{V}$.*

(2) *Let $\mathcal{V}_i$, $\mathcal{V}'_i$ be a pair of standard τ-coverings of T_i. If $f : T \times_{T_i} \mathcal{V}_i \to T \times_{T_i} \mathcal{V}'_i$ is a morphism of coverings of T, then there exists an index $i' \geq i$ and a morphism $f_{i'} : T_{i'} \times_{T_i} \mathcal{V} \to T_{i'} \times_{T_i} \mathcal{V}'_i$ whose base change to T is f.*

(3) *If $f, g : \mathcal{V} \to \mathcal{V}'_i$ are morphisms of standard τ-coverings of T_i whose base changes f_T, g_T to T are equal then there exists an index $i' \geq i$ such that $f_{T_{i'}} = g_{T_{i'}}$.*

In other words, the category of standard τ-coverings of T is the colimit over I of the categories of standard τ-coverings of T_i.

Proof. Let us prove this for $\tau = fppf$. By Limits, Lemma 10.1 the category of schemes of finite presentation over T is the colimit over I of the categories of finite presentation over T_i. By Limits, Lemmas 8.2 and 8.7 the same is true for category of schemes which are affine, flat and of finite presentation over T. To finish the proof of the lemma it suffices to show that if $\{V_{j,i} \to T_i\}_{j=1,\ldots,m}$ is a finite family of flat finitely presented morphisms with $V_{j,i}$ affine, and the base change $\coprod_j T \times_{T_i} V_{j,i} \to T$ is surjective, then for some $i' \geq i$ the morphism $\coprod T_{i'} \times_{T_i} V_{j,i} \to T_{i'}$ is surjective. Denote $W_{i'} \subset T_{i'}$, resp. $W \subset T$ the image. Of course $W = T$ by assumption. Since

the morphisms are flat and of finite presentation we see that W_i is a quasi-compact open of T_i, see Morphisms, Lemma 25.10. Moreover, $W = T \times_{T_i} W_i$ (formation of image commutes with base change). Hence by Limits, Lemma 4.11 we conclude that $W_{i'} = T_{i'}$ for some large enough i' and we win.

For $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic\}$ a standard τ-covering is a standard fppf covering. Hence the fully faithfulness of the functor holds. The only issue is to show that given a standard fppf covering $\mathcal{V}_i$ for some i such that $\mathcal{V}_i \times_{T_i} T$ is a standard τ-covering, then $\mathcal{V}_i \times_{T_i} T_{i'}$ is a standard τ-covering for all $i' \gg i$. This follows immediately from Limits, Lemmas 8.12, 8.10, 8.9, and 8.16. $\square$

Lemma 13.3. *Let S, $\mathcal{C}$, F satisfy conditions (1), (2), (a), and (b) of Lemma 13.1 and denote $F' : (Sch/S)^{opp} \to Sets$ the unique extension constructed in the lemma. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Assume*

(c) *for any standard τ-covering $\{V_i \to V\}_{i=1,\dots,n}$ of affines in Sch/S such that $V \to S$ factors through an affine open $U \subset S$ and $V \to U$ is of finite presentation, the sheaf condition hold for F and $\{V_i \to V\}_{i=1,\dots,n}$[9].*

Then F' satisfies the sheaf condition for all τ-coverings.

Proof. Let X be a scheme over S and let $\{X_i \to X\}_{i \in I}$ be a τ-covering. Let $s_i \in F'(X_i)$ be elements such that s_i and s_j map to the same element of $F'(X_i \times_X X_j)$ for all $i, j \in I$. We have to show that there is a unique element $s \in F'(X)$ restricting to $s_i \in F'(X_i)$ for all $i \in I$.

Special case: X is an affine such that the structure morphism maps into an affine open U of S and the covering $\{X_i \to X\}_{i \in I}$ is a standard τ-covering. In this case we can write

$$X = \lim V_k$$

as a cofiltered limit with $V_k \to U$ of finite presentation and V_k affine. See Algebra, Lemma 127.2. By Lemma 13.2 there exists a k and a standard τ-covering $\{V_{k,i} \to V_k\}_{i \in I}$ whose base change to X is the given covering. For $k' \geq k$ denote $\{V_{k',i} \to V_{k'}\}_{i \in I}$ the base change to $V_{k'}$ of our covering. Then we see that

$F'(X) = \mathrm{colim}_{k' \geq k} F(V_k)$

$\qquad = \mathrm{colim}_{k' \geq k} \mathrm{Equalizer}(\prod F(V_{k',i}) \rightrightarrows \prod F(V_{k',i} \times_{V_{k'}} V_{k',j}))$

$\qquad = \mathrm{Equalizer}(\mathrm{colim}_{k' \geq k} \prod F(V_{k',i}) \rightrightarrows \mathrm{colim}_{k' \geq k} \prod F(V_{k',i} \times_{V_{k'}} V_{k',j}))$

$\qquad = \mathrm{Equalizer}(\prod F'(X_i) \rightrightarrows \prod F'(X_i \times_X X_j))$

The first equality holds by construction of F'. The second holds by assumption (c). The third holds because filtered colimits are exact. The fourth again holds by construction of F'. In this way we find that the sheaf property holds for F' with respect to $\{X_i \to X\}_{i \in I}$.

General case. Choose an affine open covering $X = \bigcup U_k$ such that each U_k maps into an affine open of S. For every k we may choose a standard τ-covering $\{V_{k,j} \to U_k\}_{j=1,\dots,m_k}$ which refines $\{X_i \times_X U_k \to U_k\}_{i \in I}$. For each $j \in \{1, \dots, m_k\}$ choose an index $i_{k,j} \in I$ and a morphism $g_{k,j} : V_{k,j} \to X_{i_{k,j}}$ over X. Let $s_{k,j}$ be the element of $F'(V_{k,j})$ we get by restricting $s_{i_{k,j}}$ via $g_{k,j}$. Observe that $s_{k,j}$ and $s_{k',j'}$ restrict to

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[9]This makes sense as V, V_i, and $V_i \times_V V_j$ are isomorphic to objects of $\mathcal{C}$ by (2).

the same element of $F'(V_{k,j} \times_X V_{k',j'})$ for all k and k' and all $j \in \{1, \ldots, m_k\}$ and $j' \in \{1, \ldots, m_{k'}\}$; verification omitted. In particular, by the result of the previous paragraph there is a unique element $s_k \in F'(U_k)$ restricting to $s_{k,j}$ for all j. With this notation we are ready to finish the proof.

Proof of uniqueness of s: this is true because F' satisfies the sheaf property for Zariski coverings and $s|_{U_k}$ must be equal to s_k because both restrict to $s_{k,j}$ for all j. This uniqueness then shows that s_k and $s_{k'}$ must restrict to the same section of F' over (the non-affine scheme) $U_k \cap U_{k'}$ because these sections restrict to the same section over the τ-covering $\{V_{k,j} \times_X V_{k',j'} \to U_k \cap U_{k'}\}$. Thus by the sheaf property for Zariski coverings, there is a unique section s of F' over X whose restriction to U_k is s_k. We omit the verification (similar to the above) that s restricts to s_i over X_i. $\square$

Lemma 13.4. *Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Let S be a scheme contained in a big site Sch_τ. Let $F : (Sch/S)^{opp}_\tau \to Sets$ be a τ-sheaf satisfying property (b) of Lemma 13.1 with $\mathcal{C} = (Sch/S)_\tau$. Then the extension F' of F to the category of all schemes over S satisfies the sheaf condition for all τ-coverings.*

Proof. This follows from Lemma 13.3 applied with $\mathcal{C} = (Sch/S)_\tau$. Conditions (1), (2), (a), and (b) of Lemma 13.1 hold; we omit the details. Thus we get our unique extension F' to the category of all schemes over S. Finally, observe that any standard τ-covering is tautologically equivalent to a covering in $(Sch/S)_\tau$, see Sets, Lemma 9.9 as well as Lemmas 3.6, 4.7, 5.7, 6.7, and 7.7. By Sites, Lemma 8.4 the sheaf property passes through tautological equivalence of coverings. Hence the fact that F is a τ-sheaf implies that property (c) of Lemma 13.3 holds and we conclude. $\square$

14. Other chapters

Preliminaries

Schemes

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