DERIVED CATEGORIES OF STACKS

Contents

1. Introduction

In this chapter we write about derived categories associated to algebraic stacks. This means in particular derived categories of quasi-coherent sheaves, i.e., we prove analogues of the results on schemes (see Derived Categories of Schemes, Section 1) and algebraic spaces (see Derived Categories of Spaces, Section 1). The results in this chapter are different from those in [LMB00] mainly because we consistently use the "big sites". Before reading this chapter please take a quick look at the chapters "Sheaves on Algebraic Stacks" and "Cohomology of Algebraic Stacks" where the terminology we use here is introduced.

2. Conventions, notation, and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2. We use notation as explained in Cohomology of Stacks, Section 3.

3. The lisse-étale and the flat-fppf sites

The section is the analogue of Cohomology of Stacks, Section 14 for derived categories.

Lemma 3.1. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Cohomology of Stacks, Lemmas 14.2 and 14.4.*

(1) *The functor $g_! : Ab(\mathcal{X}_{lisse,étale}) \to Ab(\mathcal{X}_{étale})$ has a left derived functor*

$$Lg_! : D(\mathcal{X}_{lisse,étale}) \longrightarrow D(\mathcal{X}_{étale})$$

which is left adjoint to g^{-1} and such that $g^{-1}Lg_! = id$.

(2) *The functor* $g_! : Mod(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ *has a left derived functor*

$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{lisse,étale}}) \longrightarrow D(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$

which is left adjoint to g^* *and such that* $g^*Lg_! = id$.

(3) *The functor* $g_! : Ab(\mathcal{X}_{flat,fppf}) \to Ab(\mathcal{X}_{fppf})$ *has a left derived functor*

$$Lg_! : D(\mathcal{X}_{flat,fppf}) \longrightarrow D(\mathcal{X}_{fppf})$$

which is left adjoint to g^{-1} *and such that* $g^{-1}Lg_! = id$.

(4) *The functor* $g_! : Mod(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}}) \to Mod(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ *has a left derived functor*

$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \longrightarrow D(\mathcal{O}_\mathcal{X})$$

which is left adjoint to g^* *and such that* $g^*Lg_! = id$.

Warning: It is not clear (a priori) that $Lg_!$ *on modules agrees with* $Lg_!$ *on abelian sheaves, see Cohomology on Sites, Remark 37.3.*

Proof. The existence of the functor $Lg_!$ and adjointness to g^* is Cohomology on Sites, Lemma 37.2. (For the case of abelian sheaves use the constant sheaf $\mathbf{Z}$ as the structure sheaves.) Moreover, it is computed on a complex $\mathcal{H}^\bullet$ by taking a suitable left resolution $\mathcal{K}^\bullet \to \mathcal{H}^\bullet$ and applying the functor $g_!$ to $\mathcal{K}^\bullet$. Since $g^{-1}g_!\mathcal{K}^\bullet = \mathcal{K}^\bullet$ by Cohomology of Stacks, Lemmas 14.4 and 14.2 we see that the final assertion holds in each case. ☐

Lemma 3.2. *With assumptions and notation as in Cohomology of Stacks, Lemma 15.1. We have*

$$g^{-1} \circ Rf_* = Rf'_* \circ (g')^{-1} \quad \text{and} \quad L(g')_! \circ (f')^{-1} = f^{-1} \circ Lg_!$$

on unbounded derived categories (both for the case of modules and for the case of abelian sheaves).

Proof. Let $\tau = étale$ (resp. $\tau = fppf$). Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{X}_\tau$. By Cohomology of Stacks, Lemma 15.3 the canonical (base change) map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism. The rest of the proof is formal. Since cohomology of abelian groups and sheaves of modules agree we also conclude that $g^{-1}Rf_*\mathcal{F} = Rf'_*(g')^{-1}\mathcal{F}$ when $\mathcal{F}$ is a sheaf of modules on $\mathcal{X}_\tau$.

Next we show that for $\mathcal{G}$ (either sheaf of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$) the canonical map

$$L(g')_!(f')^{-1}\mathcal{G} \to f^{-1}Lg_!\mathcal{G}$$

is an isomorphism. To see this it is enough to prove for any injective sheaf $\mathcal{I}$ on $\mathcal{X}_\tau$ the induced map

$$\mathrm{Hom}(L(g')_!(f')^{-1}\mathcal{G}, \mathcal{I}[n]) \leftarrow \mathrm{Hom}(f^{-1}Lg_!\mathcal{G}, \mathcal{I}[n])$$

is an isomorphism for all $n \in \mathbf{Z}$. (Hom's taken in suitable derived categories.) By the adjointness of f^{-1} and Rf_*, the adjointness of $Lg_!$ and g^{-1}, and their "primed" versions this follows from the isomorphism $g^{-1}Rf_*\mathcal{I} \to Rf'_*(g')^{-1}\mathcal{I}$ proved above.

In the case of a bounded complex $\mathcal{G}^\bullet$ (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$) the canonical map

$$L(g')_!(f')^{-1}\mathcal{G}^\bullet \to f^{-1}Lg_!\mathcal{G}^\bullet \tag{3.2.1}$$

is an isomorphism as follows from the case of a sheaf by the usual arguments involving truncations and the fact that the functors $L(g')_!(f')^{-1}$ and $f^{-1}Lg_!$ are exact functors of triangulated categories.

Suppose that $\mathcal{G}^\bullet$ is a bounded above complex (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$). The canonical map (3.2.1) is an isomorphism because we can use the stupid truncations $\sigma_{\geq -n}$ (see Homology, Section 15) to write $\mathcal{G}^\bullet$ as a colimit $\mathcal{G}^\bullet = \operatorname{colim} \mathcal{G}_n^\bullet$ of bounded complexes. This gives a distinguished triangle

$$\bigoplus\nolimits_{n\geq 1} \mathcal{G}_n^\bullet \to \bigoplus\nolimits_{n\geq 1} \mathcal{G}_n^\bullet \to \mathcal{G}^\bullet \to \ldots$$

and each of the functors $L(g')_!$, $(f')^{-1}$, f^{-1}, $Lg_!$ commutes with direct sums (of complexes).

If $\mathcal{G}^\bullet$ is an arbitrary complex (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$) then we use the canonical truncations $\tau_{\leq n}$ (see Homology, Section 15) to write $\mathcal{G}^\bullet$ as a colimit of bounded above complexes and we repeat the argument of the paragraph above.

Finally, by the adjointness of f^{-1} and Rf_*, the adjointness of $Lg_!$ and g^{-1}, and their "primed" versions we conclude that the first identity of the lemma follows from the second in full generality. $\square$

Lemma 3.3. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Cohomology of Stacks, Lemma 14.2.*

1. *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-module on the lisse-étale site of $\mathcal{X}$. For all $p \in \mathbf{Z}$ the sheaf $H^p(Lg_!\mathcal{H})$ is a locally quasi-coherent module with the flat base change property on $\mathcal{X}$.*
2. *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-module on the flat-fppf site of $\mathcal{X}$. For all $p \in \mathbf{Z}$ the sheaf $H^p(Lg_!\mathcal{H})$ is a locally quasi-coherent module with the flat base change property on $\mathcal{X}$.*

Proof. Pick a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By Modules on Sites, Definition 23.1 there exists an étale (resp. fppf) covering $\{U_i \to U\}_{i\in I}$ such that each pullback $f_i^{-1}\mathcal{H}$ has a global presentation (see Modules on Sites, Definition 17.1). Here $f_i : U_i \to \mathcal{X}$ is the composition $U_i \to U \to \mathcal{X}$ which is a morphism of algebraic stacks. (Recall that the pullback "is" the restriction to $\mathcal{X}/f_i$, see Sheaves on Stacks, Definition 9.2 and the discussion following.) After refining the covering we may assume each U_i is an affine scheme. Since each f_i is smooth (resp. flat) by Lemma 3.2 we see that $f_i^{-1}Lg_!\mathcal{H} = Lg_{i,!}(f_i')^{-1}\mathcal{H}$. Using Cohomology of Stacks, Lemma 8.2 we reduce the statement of the lemma to the case where $\mathcal{H}$ has a global presentation and where $\mathcal{X} = (Sch/X)_{fppf}$ for some affine scheme $X = \operatorname{Spec}(A)$.

Say our presentation looks like

$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{H} \longrightarrow 0$$

where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). Note that the site $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) has a final object, namely X/X which is quasi-compact (see Cohomology on Sites, Section 16). Hence we have

$$\Gamma(\bigoplus\nolimits_{i \in I} \mathcal{O}) = \bigoplus\nolimits_{i \in I} A$$

by Sites, Lemma 17.7. Hence the map in the presentation corresponds to a similar presentation

$$\bigoplus\nolimits_{j \in J} A \longrightarrow \bigoplus\nolimits_{i \in I} A \longrightarrow M \longrightarrow 0$$

of an A-module M. Moreover, $\mathcal{H}$ is equal to the restriction to the lisse-étale (resp. flat-fppf) site of the quasi-coherent sheaf M^a associated to M. Choose a resolution

$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$

by free A-modules. The complex

$$\ldots \mathcal{O} \otimes_A F_2 \to \mathcal{O} \otimes_A F_1 \to \mathcal{O} \otimes_A F_0 \to \mathcal{H} \to 0$$

is a resolution of $\mathcal{H}$ by free $\mathcal{O}$-modules because for each object U/X of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) the structure morphism $U \to X$ is flat. Hence by construction the value of $Lg_!\mathcal{H}$ is

$$\ldots \to \mathcal{O}_\mathcal{X} \otimes_A F_2 \to \mathcal{O}_\mathcal{X} \otimes_A F_1 \to \mathcal{O}_\mathcal{X} \otimes_A F_0 \to 0 \to \ldots$$

Since this is a complex of quasi-coherent modules on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$) it follows from Cohomology of Stacks, Proposition 8.1 that $H^p(Lg_!\mathcal{H})$ is quasi-coherent. □

4. Cohomology and the lisse-étale and flat-fppf sites

We have already seen that cohomology of a sheaf on an algebraic stack $\mathcal{X}$ can be computed on flat-fppf site. In this section we prove the same is true for (possibly) unbounded objects of the direct category of $\mathcal{X}$.

Lemma 4.1. *Let $\mathcal{X}$ be an algebraic stack. We have $Lg_!\mathbf{Z} = \mathbf{Z}$ for either $Lg_!$ as in Lemma 3.1 part (1) or $Lg_!$ as in Lemma 3.1 part (3).*

Proof. We prove this for the comparison between the flat-fppf site with the fppf site; the case of the lisse-étale site is exactly the same. We have to show that $H^i(Lg_!\mathbf{Z})$ is 0 for $i \neq 0$ and that the canonical map $H^0(Lg_!\mathbf{Z}) \to \mathbf{Z}$ is an isomorphism. Let $f : \mathcal{U} \to \mathcal{X}$ be a surjective, flat morphism where $\mathcal{U}$ is a scheme such that f is also locally of finite presentation. (For example, pick a presentation $U \to \mathcal{X}$ and let $\mathcal{U}$ be the algebraic stack corresponding to U.) By Sheaves on Stacks, Lemmas 19.6 and 19.10 it suffices to show that the pullback $f^{-1}H^i(Lg_!\mathbf{Z})$ is 0 for $i \neq 0$ and that the pullback $H^0(Lg_!\mathbf{Z}) \to f^{-1}\mathbf{Z}$ is an isomorphism. By Lemma 3.2 we find $f^{-1}Lg_!\mathbf{Z} = L(g')_!\mathbf{Z}$ where $g' : Sh(\mathcal{U}_{flat,fppf}) \to Sh(\mathcal{U}_{fppf})$ is the corresponding comparision morphism for $\mathcal{U}$. This reduces us to the case studied in the next paragraph.

Assume $\mathcal{X} = (Sch/X)_{fppf}$ for some scheme X. In this case the category $\mathcal{X}_{flat,fppf}$ has a final object e, namely X/X, and moreover the functor $u : \mathcal{X}_{flat,fppf} \to \mathcal{X}_{fppf}$ sends e to the final object. Since $\mathbf{Z}$ is the free abelian sheaf on the final object (provided the final object exists) we find that $Lg_!\mathbf{Z} = \mathbf{Z}$ by the very construction of $Lg_!$ in Cohomology on Sites, Lemma 37.2. □

Lemma 4.2. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 3.1.*

(1) *For K in $D(\mathcal{X}_{étale})$ we have*
   (a) $R\Gamma(\mathcal{X}_{étale}, K) = R\Gamma(\mathcal{X}_{lisse,étale}, g^{-1}K)$, *and*
   (b) $R\Gamma(x, K) = R\Gamma(\mathcal{X}_{lisse,étale}/x, g^{-1}K)$ *for any object x of $\mathcal{X}_{lisse,étale}$.*
(2) *For K in $D(\mathcal{X}_{fppf})$ we have*
   (a) $R\Gamma(\mathcal{X}_{fppf}, K) = R\Gamma(\mathcal{X}_{flat,fppf}, g^{-1}K)$, *and*
   (b) $H^p(x, K) = R\Gamma(\mathcal{X}_{flat,fppf}/x, g^{-1}K)$ *for any object x of $\mathcal{X}_{flat,fppf}$.*

In both cases, the same holds for modules, since we have $g^{-1} = g^*$ and there is no difference in computing cohomology by Cohomology on Sites, Lemma 20.7.

Proof. We prove this for the comparison between the flat-fppf site with the fppf site; the case of the lisse-étale site is exactly the same. By Lemma 4.1 we have $Lg_!\mathbf{Z} = \mathbf{Z}$. Then we obtain

$$\begin{aligned} R\Gamma(\mathcal{X}_{fppf}, K) &= R\operatorname{Hom}(\mathbf{Z}, K) \\ &= R\operatorname{Hom}(Lg_!\mathbf{Z}, K) \\ &= R\operatorname{Hom}(\mathbf{Z}, g^{-1}K) \\ &= R\Gamma(\mathcal{X}_{lisse,étale}, g^{-1}K) \end{aligned}$$

This proves (1)(a). Part (1)(b) follows from part (1)(a). Namely, if x lies over the scheme U, then the site $\mathcal{X}_{étale}/x$ is equivalent to $(Sch/U)_{étale}$ and $\mathcal{X}_{lisse,étale}$ is equivalent to $U_{lisse,étale}$. □

5. Derived categories of quasi-coherent modules

Let $\mathcal{X}$ be an algebraic stack. As the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ isn't exact, we cannot define $D_{QCoh}(\mathcal{O}_\mathcal{X})$ as the full subcategory of $D(\mathcal{O}_\mathcal{X})$ consisting of complexes with quasi-coherent cohomology sheaves. Instead we define the derived category of quasi-coherent modules as a quotient by analogy with Cohomology of Stacks, Remark 10.7.

Recall that $LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{O}_\mathcal{X})$ denotes the full subcategory of locally quasi-coherent $\mathcal{O}_\mathcal{X}$-modules with the flat base change property, see Cohomology of Stacks, Section 8. We will abbreviate

$$D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) = D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{O}_\mathcal{X})$$

From Derived Categories, Lemma 17.1 and Cohomology of Stacks, Proposition 8.1 part (2) we deduce that $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$.

Let $Parasitic(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{O}_\mathcal{X})$ denote the full subcategory of parasitic $\mathcal{O}_\mathcal{X}$-modules, see Cohomology of Stacks, Section 9. Let us abbreviate

$$D_{Parasitic}(\mathcal{O}_\mathcal{X}) = D_{Parasitic(\mathcal{O}_\mathcal{X})}(\mathcal{O}_\mathcal{X})$$

As before this is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$ since $Parasitic(\mathcal{O}_\mathcal{X})$ is a Serre subcategory of $Mod(\mathcal{O}_\mathcal{X})$, see Cohomology of Stacks, Lemma 9.2.

The intersection of the weak Serre subcategories $Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ of $Mod(\mathcal{O}_\mathcal{X})$ is another one. Let us similarly abbreviate

$$\begin{aligned} D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) &= D_{Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{O}_\mathcal{X}) \\ &= D_{Parasitic}(\mathcal{O}_\mathcal{X}) \cap D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \end{aligned}$$

As before this is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$. Hence a fortiori it is a strictly full, saturated triangulated subcategory of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$.

Definition 5.1. Let $\mathcal{X}$ be an algebraic stack. With notation as above we define the *derived category of $\mathcal{O}_\mathcal{X}$-modules with quasi-coherent cohomology sheaves* as the Verdier quotient[1]

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) = D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})/D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$$

The Verdier quotient is defined in Derived Categories, Section 6. A morphism $a : E \to E'$ of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ becomes an isomorphism in $D_{QCoh}(\mathcal{O}_\mathcal{X})$ if and only if the cone $C(a)$ has parasitic cohomology sheaves, see Derived Categories, Lemma 6.10.

Consider the functors

$$D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \xrightarrow{H^i} LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{X})$$

Note that Q annihilates the subcategory $Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, see Cohomology of Stacks, Lemma 10.2. By Derived Categories, Lemma 6.8 we obtain a cohomological functor

$$H^i : D_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X}) \tag{5.1.1}$$

Moreover, note that $E \in D_{QCoh}(\mathcal{O}_\mathcal{X})$ is zero if and only if $H^i(E) = 0$ for all $i \in \mathbf{Z}$ since the kernel of Q is exactly equal to $Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ by Cohomology of Stacks, Lemma 10.2.

Note that the categories $Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ and $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ are also weak Serre subcategories of the abelian category $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ of modules in the étale topology, see Cohomology of Stacks, Proposition 8.1 and Lemma 9.2. Hence the statement of the following lemma makes sense.

Lemma 5.2. *Let $\mathcal{X}$ be an algebraic stack. Abbreviate $\mathcal{P}_\mathcal{X} = Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. The comparison morphism $\epsilon : \mathcal{X}_{fppf} \to \mathcal{X}_{étale}$ induces a commutative diagram*

$$\begin{tikzcd}
D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \arrow[r] & D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \arrow[r] & D(\mathcal{O}_\mathcal{X}) \\
D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \arrow[r] \arrow[u, "\epsilon^*"] & D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \arrow[r] \arrow[u, "\epsilon^*"] & D(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \arrow[u, "\epsilon^*"]
\end{tikzcd}$$

Moreover, the left two vertical arrows are equivalences of triangulated categories, hence we also obtain an equivalence

$$D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})/D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

Proof. Since ϵ^* is exact it is clear that we obtain a diagram as in the statement of the lemma. We will show the middle vertical arrow is an equivalence by applying Cohomology on Sites, Lemma 29.1 to the following situation: $\mathcal{C} = \mathcal{X}$, $\tau = fppf$, $\tau' = étale$, $\mathcal{O} = \mathcal{O}_\mathcal{X}$, $\mathcal{A} = LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, and $\mathcal{B}$ is the set of objects of $\mathcal{X}$ lying over affine schemes. To see the lemma applies we have to check conditions (1), (2), (3), (4). Conditions (1) and (2) are clear from the discussion above (explicitly this

[1] This definition is different from the one in the literature, see [Ols07, 6.3], but it agrees with that definition by Lemma 5.3.

follows from Cohomology of Stacks, Proposition 8.1). Condition (3) holds because every scheme has a Zariski open covering by affines. Condition (4) follows from Descent, Lemma 12.4.

We omit the verification that the equivalence of categories $\epsilon^* : D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X}) \to D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ induces an equivalence of the subcategories of complexes with parasitic cohomology sheaves. $\square$

Let $\mathcal{X}$ be an algebraic stack. By Cohomology of Stacks, Lemma 16.4 the category of quasi-coherent modules $QCoh(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}})$ forms a weak Serre subcategory of $Mod(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}})$ and the category of quasi-coherent modules $QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$ forms a weak Serre subcategory of $Mod(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Thus we can consider

$$D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}}) = D_{QCoh(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}})}(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}}) \subset D(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}})$$

and similarly

$$D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) = D_{QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \subset D(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

As above these are strictly full, saturated triangulated subcategories. It turns out that $D_{QCoh}(\mathcal{O}_\mathcal{X})$ is equivalent to either of these.

Lemma 5.3. *Let $\mathcal{X}$ be an algebraic stack. Set $\mathcal{P}_\mathcal{X} = Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*

1. *Let $\mathcal{F}^\bullet$ be an object of $D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$. With g as in Cohomology of Stacks, Lemma 14.2 for the lisse-étale site we have*
   (a) *$g^*\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}})$,*
   (b) *$g^*\mathcal{F}^\bullet = 0$ if and only if $\mathcal{F}^\bullet$ is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$,*
   (c) *$Lg_!\mathcal{H}^\bullet$ is in $D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ for $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}})$, and*
   (d) *the functors g^* and $Lg_!$ define mutually inverse functors*

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) \underset{Lg_!}{\overset{g^*}{\rightleftarrows}} D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}})$$

2. *Let $\mathcal{F}^\bullet$ be an object of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. With g as in Cohomology of Stacks, Lemma 14.2 for the flat-fppf site we have*
   (a) *$g^*\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$,*
   (b) *$g^*\mathcal{F}^\bullet = 0$ if and only if $\mathcal{F}^\bullet$ is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$,*
   (c) *$Lg_!\mathcal{H}^\bullet$ is in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ for $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$, and*
   (d) *the functors g^* and $Lg_!$ define mutually inverse functors*

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) \underset{Lg_!}{\overset{g^*}{\rightleftarrows}} D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

Proof. The functor $g^* = g^{-1}$ is exact, hence (1)(a), (2)(a), (1)(b), and (2)(b) follow from Cohomology of Stacks, Lemmas 16.3 and 14.6.

Proof of (1)(c) and (2)(c). The construction of $Lg_!$ in Lemma 3.1 (via Cohomology on Sites, Lemma 37.2 which in turn uses Derived Categories, Proposition 29.2) shows that $Lg_!$ on any object $\mathcal{H}^\bullet$ of $D(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}})$ is computed as

$$Lg_!\mathcal{H}^\bullet = \operatorname{colim} g_!\mathcal{K}_n^\bullet = g_! \operatorname{colim} \mathcal{K}_n^\bullet$$

(termwise colimits) where the quasi-isomorphism $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{H}^\bullet$ induces quasi-isomorphisms $\mathcal{K}_n^\bullet \to \tau_{\leq n}\mathcal{H}^\bullet$. Since the inclusion functors

$$LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X}) \quad \text{and} \quad LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{O}_\mathcal{X})$$

are compatible with filtered colimits we see that it suffices to prove (c) on bounded above complexes $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}})$ and in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. In this case to show that $H^n(Lg_!\mathcal{H}^\bullet)$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ we can argue by induction on the integer m such that $\mathcal{H}^i = 0$ for $i > m$. If $m < n$, then $H^n(Lg_!\mathcal{H}^\bullet) = 0$ and the result holds. In general consider the distinguished triangle

$$\tau_{\leq m-1}\mathcal{H}^\bullet \to \mathcal{H}^\bullet \to H^m(\mathcal{H}^\bullet)[-m] \to \ldots$$

(Derived Categories, Remark 12.4) and apply the functor $Lg_!$. Since $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ is a weak Serre subcategory of the module category it suffices to prove (c) for two out of three. We have the result for $Lg_!\tau_{\leq m-1}\mathcal{H}^\bullet$ by induction and we have the result for $Lg_!H^m(\mathcal{H}^\bullet)[-m]$ by Lemma 3.3. Whence (c) holds.

Let us prove (2)(d). By (2)(a) and (2)(b) the functor $g^{-1} = g^*$ induces a functor

$$c : D_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

see Derived Categories, Lemma 6.8. Thus we have the following diagram of triangulated categories

$$\begin{tikzcd} D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \arrow[rr, "q"] \arrow[dr, shift left, "g^{-1}"] & & D_{QCoh}(\mathcal{O}_\mathcal{X}) \arrow[dl, "c"] \\ & D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \arrow[ul, shift left, "Lg_!"] & \end{tikzcd}$$

where q is the quotient functor, the inner triangle is commutative, and $g^{-1}Lg_! = \mathrm{id}$. For any object of E of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ the map $a : Lg_!g^{-1}E \to E$ maps to a quasi-isomorphism in $D(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Hence the cone on a maps to zero under g^{-1} and by (2)(b) we see that $q(a)$ is an isomorphism. Thus $q \circ Lg_!$ is a quasi-inverse to c.

In the case of the lisse-étale site exactly the same argument as above proves that

$$D_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})/D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$$

is equivalent to $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,\acute{e}tale}})$. Applying the last equivalence of Lemma 5.2 finishes the proof. □

The following lemma tells us that the quotient functor $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{X})$ has a left adjoint. See Remark 5.5.

Lemma 5.4. *Let $\mathcal{X}$ be an algebraic stack. Let E be an object of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. There exists a canonical distinguished triangle*

$$E' \to E \to P \to E'[1]$$

in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ such that P is in $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ and

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{X})}(E', P') = 0$$

for all P' in $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$.

Proof. Consider the morphism of ringed topoi $g : Sh(\mathcal{X}_{flat,fppf}) \longrightarrow Sh(\mathcal{X}_{fppf})$ studied in Cohomology of Stacks, Section 14. Set $E' = Lg_! g^* E$ and let P be the cone on the adjunction map $E' \to E$, see Lemma 3.1 part (4). By Lemma 5.3 parts (2)(a) and (2)(c) we have that E' is in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. Hence also P is in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. The map $g^*E' \to g^*E$ is an isomorphism as $g^*Lg_! = \text{id}$ by Lemma 3.1 part (4). Hence $g^*P = 0$ and whence P is an object of $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ by Lemma 5.3 part (2)(b). Finally, for P' in $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ we have

$$\text{Hom}(E', P') = \text{Hom}(Lg_! g^* E, P') = \text{Hom}(g^*E, g^*P') = 0$$

as $g^*P' = 0$ by Lemma 5.3 part (2)(b). The distinguished triangle $E' \to E \to P \to E'[1]$ is canonical (more precisely unique up to isomorphism of triangles induces the identity on E) by the discussion in Derived Categories, Section 40. $\square$

Remark 5.5. The result of Lemma 5.4 tells us that

$$D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \subset D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$$

is a left admissible subcategory, see Derived Categories, Section 40. In particular, if $\mathcal{A} \subset D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ denotes its left orthogonal, then Derived Categories, Proposition 40.10 implies that $\mathcal{A}$ is right admissible in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ and that the composition

$$\mathcal{A} \longrightarrow D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

is an equivalence. This means that we can view $D_{QCoh}(\mathcal{O}_\mathcal{X})$ as a strictly full saturated triangulated subcategory of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ and also of $D(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$.

6. Derived pushforward of quasi-coherent modules

As a first application of the material above we construct the derived pushforward. In Examples, Section 60 the reader can find an example of a quasi-compact and quasi-separated morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks such that the direct image functor Rf_* does not induce a functor $D_{QCoh}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$. Thus restricting to bounded below complexes is necessary.

Proposition 6.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor Rf_* induces a commutative diagram*

$$\begin{tikzcd}
D^+_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \arrow[r] \arrow[d, "Rf_*"] & D^+_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \arrow[r] \arrow[d, "Rf_*"] & D(\mathcal{O}_\mathcal{X}) \arrow[d, "Rf_*"] \\
D^+_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y}) \arrow[r] & D^+_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y}) \arrow[r] & D(\mathcal{O}_\mathcal{Y})
\end{tikzcd}$$

and hence induces a functor

$$Rf_{QCoh,*} : D^+_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D^+_{QCoh}(\mathcal{O}_\mathcal{Y})$$

on quotient categories. Moreover, the functor $R^i f_{QCoh}$ of Cohomology of Stacks, Proposition 11.1 are equal to $H^i \circ Rf_{QCoh,*}$ with H^i as in (5.1.1).

Proof. We have to show that Rf_*E is an object of $D^+_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$ for E in $D^+_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. This follows from Cohomology of Stacks, Proposition 8.1 and the spectral sequence $R^i f_* H^j(E) \Rightarrow R^{i+j} f_* E$. The case of parasitic modules works the same way using Cohomology of Stacks, Lemma 9.3. The final statement is clear from the definition of H^i in (5.1.1). $\square$

7. Derived pullback of quasi-coherent modules

Derived pullback of complexes with quasi-coherent cohomology sheaves exists in general.

Proposition 7.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The exact functor f^* induces a commutative diagram*

$$\begin{tikzcd} D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \arrow[r] & D(\mathcal{O}_\mathcal{X}) \\ D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y}) \arrow[r] \arrow[u, "f^*"] & D(\mathcal{O}_\mathcal{Y}) \arrow[u, "f^*"'] \end{tikzcd}$$

The composition

$$D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y}) \xrightarrow{f^*} D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X}) \xrightarrow{q_\mathcal{X}} D_{QCoh}(\mathcal{O}_\mathcal{X})$$

is left derivable with respect to the localization $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$ and we may define Lf^*_{QCoh} as its left derived functor

$$Lf^*_{QCoh} : D_{QCoh}(\mathcal{O}_\mathcal{Y}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

(see Derived Categories, Definitions 14.2 and 14.9). If f is quasi-compact and quasi-separated, then Lf^*_{QCoh} and $Rf_{QCoh,*}$ satisfy the following adjointness:

$$\mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(Lf^*_{QCoh}A, B) = \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{Y})}(A, Rf_{QCoh,*}B)$$

for $A \in D_{QCoh}(\mathcal{O}_\mathcal{Y})$ and $B \in D^+_{QCoh}(\mathcal{O}_\mathcal{X})$.

Proof. To prove the first statement, we have to show that f^*E is an object of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ for E in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$. Since $f^* = f^{-1}$ is exact this follows immediately from the fact that f^* maps $LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})$ into $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ by Cohomology of Stacks, Proposition 8.1.

Set $\mathcal{D} = D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$. Let S be the collection of morphisms in $\mathcal{D}$ whose cone is an object of $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$. Set $\mathcal{D}' = D_{QCoh}(\mathcal{O}_\mathcal{X})$. Set $F = q_\mathcal{X} \circ f^* : \mathcal{D} \to \mathcal{D}'$. Then $\mathcal{D}, S, \mathcal{D}', F$ are as in Derived Categories, Situation 14.1 and Definition 14.2. Let us prove that $LF(E)$ is defined for any object E of $\mathcal{D}$. Namely, consider the triangle

$$E' \to E \to P \to E'[1]$$

constructed in Lemma 5.4. Note that $s : E' \to E$ is an element of S. We claim that E' computes LF. Namely, suppose that $s' : E'' \to E$ is another element of S, i.e., fits into a triangle $E'' \to E \to P' \to E''[1]$ with P' in $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$. By Lemma 5.4 (and its proof) we see that $E' \to E$ factors through $E'' \to E$. Thus we see that $E' \to E$ is cofinal in the system S/E. Hence it is clear that E' computes LF.

To see the final statement, write $B = q_\mathcal{X}(H)$ and $A = q_\mathcal{Y}(E)$. Choose $E' \to E$ as above. We will use on the one hand that $Rf_{QCoh,*}(B) = q_\mathcal{Y}(Rf_*H)$ and on the other that $Lf^*_{QCoh}(A) = q_\mathcal{X}(f^*E')$.

$$\begin{aligned} \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(Lf^*_{QCoh}A, B) &= \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(q_\mathcal{X}(f^*E'), q_\mathcal{X}(H)) \\ &= \mathrm{colim}_{H \to H'} \mathrm{Hom}_{D(\mathcal{O}_\mathcal{X})}(f^*E', H') \\ &= \mathrm{colim}_{H \to H'} \mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*H') \\ &= \mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*H) \\ &= \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{Y})}(A, Rf_{QCoh,*}B) \end{aligned}$$

Here the colimit is over morphisms $s : H \to H'$ in $D^+_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ whose cone $P(s)$ is an object of $D^+_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. The first equality we've seen above. The second equality holds by construction of the Verdier quotient. The third equality holds by Cohomology on Sites, Lemma 19.1. Since $Rf_*P(s)$ is an object of $D^+_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{Y})$ by Proposition 6.1 we see that $\mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*P(s)) = 0$. Thus the fourth equality holds. The final equality holds by construction of E'. □

8. Quasi-coherent objects in the derived category

This section is the continuation of Sheaves on Stacks, Section 26. Let $\mathcal{X}$ be an algebraic stack. In that section we defined a triangulated category

$$QC(\mathcal{X}) = QC(\mathcal{X}_{affine}, \mathcal{O})$$

and we proved that if $\mathcal{X}$ is representable by an algebraic space X then $QC(\mathcal{X})$ is equivalent to $D_{QCoh}(\mathcal{O}_X)$. It turns out that we have developed just enough theory to prove the same thing is true for any algebraic stack.

Lemma 8.1. *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D(\mathcal{X}_{fppf})$ whose cohomology sheaves are parasitic. Then $R\Gamma(x, K) = 0$ for all objects x of $\mathcal{X}$ lying over a scheme U such that $U \to \mathcal{X}$ is flat.*

Proof. Denote $g : Sh(\mathcal{X}_{flat,fppf}) \to Sh(\mathcal{X}_{fppf})$ the morphism of topoi discussed in Section 3. Let x be an object of $\mathcal{X}$ lying over a scheme U such that $U \to \mathcal{X}$ is flat, i.e., x is an object of $\mathcal{X}_{flat,fppf}$. By Lemma 4.2 part (2)(b) we have $R\Gamma(x, K) = R\Gamma(\mathcal{X}_{flat,fppf}/x, g^{-1}K)$. However, our assumption means that the cohomology sheaves of the object $g^{-1}K$ of $D(\mathcal{X}_{flat,fppf})$ are zero, see Cohomology of Stacks, Definition 9.1. Hence $g^{-1}K = 0$ and we win. □

Lemma 8.2. *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D(\mathcal{X}_{fppf})$ such that $R\Gamma(x, K) = 0$ for all objects x of $\mathcal{X}$ lying over an affine scheme U such that $U \to \mathcal{X}$ is flat. Then $H^i(\mathcal{X}, K) = 0$ for all i.*

Proof. Denote $g : Sh(\mathcal{X}_{flat,fppf}) \to Sh(\mathcal{X}_{fppf})$ the morphism of topoi discussed in Section 3. By Lemma 4.2 part (2)(b) our assumption means that $g^{-1}K$ has vanishing cohomology over every object of $\mathcal{X}_{flat,fppf}$ which lies over an affine scheme. Since every object x of $\mathcal{X}_{flat,fppf}$ has a covering by such objects, we conclude that $g^{-1}K$ has vanishing cohomology sheaves, i.e., we conclude $g^{-1}K = 0$. Then of course $R\Gamma(\mathcal{X}_{flat,fppf}, g^{-1}K) = 0$ which in turn implies what we want by Lemma 4.2 part (2)(a). □

Lemma 8.3. *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Then $Lg_!K$ satisfies the following property: for any morphism $x \to x'$ of $\mathcal{X}_{affine}$ the map*

$$R\Gamma(x', Lg_!K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \longrightarrow R\Gamma(x, Lg_!K)$$

is a quasi-isomorphism.

Proof. By Lemma 5.3 part (2)(c) the object $Lg_!K$ is in $D_{LQCoh^{fbc}}(\mathcal{O}_{\mathcal{X}})$. It follows readily from this that the map displayed in the lemma is an isomorphism if $\mathcal{O}(x') \to \mathcal{O}(x)$ is a flat ring map; we omit the details.

In this paragraph we argue that the question is local for the étale topology. Let $x \to x'$ be a general morphism of $\mathcal{X}_{affine}$. Let $\{x'_i \to x'\}$ be a covering in $\mathcal{X}_{affine,étale}$. Set $x_i = x \times_{x'} x'_i$ so that $\{x_i \to x\}$ is a covering of $\mathcal{X}_{affine,étale}$ too. Then $\mathcal{O}(x') \to \prod \mathcal{O}(x'_i)$ is a faithfully flat étale ring map and

$$\prod \mathcal{O}(x_i) = \mathcal{O}(x) \otimes_{\mathcal{O}(x')} \left(\prod \mathcal{O}(x'_i)\right)$$

Thus a simple algebra argument we omit shows that it suffices to prove the result in the statement of the lemma holds for each of the morphisms $x_i \to x'_i$ in $\mathcal{X}_{affine}$. In other words, the problem is local in the étale topology.

Choose a scheme X and a surjective smooth morphism $f : X \to \mathcal{X}$. We may view f as an object of $\mathcal{X}$ (by our abuse of notation) and then $(Sch/X)_{fppf} = \mathcal{X}/f$, see Sheaves on Stacks, Section 9. By Sheaves on Stacks, Lemma 19.10 for example, there exist an étale covering $\{x'_i \to x'\}$ such that $x'_i : U'_i = p(x'_i) \to \mathcal{X}$ factors through f. By the result of the previous paragraph, we may assume that $x \to x'$ is a morphism which is the image of a morphism $U \to U'$ of $(Aff/X)_{fppf}$ by the functor $(Sch/X)_{fppf} \to \mathcal{X}$. At this point we see use that the restriction to $(Sch/X)_{fppf}$ of $Lg_!K$ is equal to $f^*Lg_!K = L(g')_!(f')^*K$ by Lemma 3.2. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{X} = (Sch/X)_{fppf}$ and $x \to x'$ corresponds to the morphism of affine schemes $U \to U'$. We may still work étale (or Zariski) locally on U' and hence we may assume $U' \to X$ factors through some affine open of X. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{X} = (Sch/X)_{fppf}$ where $X = \text{Spec}(R)$ is an affine scheme and $x \to x'$ corresponds to the morphism of affine schemes $U \to U'$. Let $M^\bullet$ be a complex of R-modules representing $R\Gamma(X, K)$. By the construction in More on Algebra, Lemma 59.10 we may assume $M^\bullet = \text{colim}\, P_n^\bullet$ where each $P_n^\bullet$ is a bounded above complex of free R-modules. Details omitted; see also More on Algebra, Remark 59.11. Consider the complex of modules $M^\bullet_{flat,fppf}$ on $X_{flat,fppf} = (Sch/X)_{flat,fppf}$ given by the rule

$$U \longmapsto \Gamma(U, M^\bullet \otimes_R \mathcal{O}_U)$$

This is a complex of sheaves by the discussion in Descent, Section 8. There is a canonical map $M^\bullet_{flat,fppf} \to K$ which by our initial remarks of the proof produces an isomorphism on sections over the affine objects of $X_{flat,fppf}$. Since every object of $X_{flat,fppf}$ has a covering by affine objects we see that $M^\bullet_{flat,fppf}$ agrees with K.

Let $M^\bullet_{fppf}$ be the complex of modules on X_{fppf} given by the same formula as displayed above. Recall that $Lg_!\mathcal{O} = g_!\mathcal{O} = \mathcal{O}$. Since $Lg_!$ is the left derived functor of $g_!$ we conclude that $Lg_!P^\bullet_{n,flat,fppf} = P^\bullet_{n,fppf}$. Since the functor $Lg_!$ commutes with homotopy colimits (or by its construction in Cohomology on Sites, Lemma 37.2) and since $M^\bullet = \operatorname{colim} P_n^\bullet$ we conclude that $Lg_! M^\bullet_{flat,fppf} = M^\bullet_{fppf}$. Say $U = \operatorname{Spec}(A)$, $U' = \operatorname{Spec}(A')$ and $U \to U'$ corresponds to the ring map $A' \to A$. From the above we see that

$$R\Gamma(U, Lg_! K) = M^\bullet \otimes_R A \quad \text{and} \quad R\Gamma(U', Lg_! K) = M^\bullet \otimes_R A'$$

Since $M^\bullet$ is a K-flat complex of R-modules, by transitivity of tensor product it follows that

$$R\Gamma(U', Lg_! K) \otimes^{\mathbf{L}}_{A'} A \longrightarrow R\Gamma(U, Lg_! K)$$

is a quasi-isomorphism as desired. □

Proposition 8.4. *Let $\mathcal{X}$ be an algebraic stack. Then $QC(\mathcal{X})$ is canonically equivalent to $D_{QCoh}(\mathcal{O}_\mathcal{X})$.*

Proof. By Sheaves on Stacks, Lemma 26.6 pullback by the comparison morphism $\epsilon : \mathcal{X}_{affine,fppf} \to \mathcal{X}_{affine}$ identifies $QC(\mathcal{X})$ with a full subcategory $Q_\mathcal{X} \subset D(\mathcal{X}_{affine,fppf}, \mathcal{O})$. Using the equivalence of ringed topoi in Sheaves on Stacks, Equation (24.3.1) we may and do view $Q_\mathcal{X}$ as a full subcategory of $D(\mathcal{X}_{fppf}, \mathcal{O})$.

Similarly by Lemma 5.4 and Remark 5.5 we find that $D_{QCoh}(\mathcal{O}_\mathcal{X})$ may be viewed as the left orthogonal $\mathcal{A}$ of the left admissible subcategory $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$ of $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$.

To finish we will show that $Q_\mathcal{X}$ is equal to $\mathcal{A}$ as subcategories of $D(\mathcal{X}_{fppf}, \mathcal{O})$.

Step 1: $Q_\mathcal{X}$ is contained in $D_{LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. An object K of $Q_\mathcal{X}$ is characterized by the property that K, viewed as an object of $D(\mathcal{X}_{affine,fppf}, \mathcal{O})$ satisfies $R\epsilon_* K$ is an object of $QC(\mathcal{X}_{affine}, \mathcal{O})$. This in turn means exactly that for all morphisms $x \to x'$ of $\mathcal{X}_{affine}$ the map

$$R\Gamma(x', K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \longrightarrow R\Gamma(x, K)$$

is an isomorphism, see footnote in statement of Cohomology on Sites, Lemma 43.12. Now, if $x' \to x$ lies over a flat morphism of affine schemes, then this means that

$$H^i(x', K) \otimes_{\mathcal{O}(x')} \mathcal{O}(x) \cong H^i(x, K)$$

This clearly means that $H^i(K)$ is a sheaf for the étale topology (Sheaves on Stacks, Lemma 25.1) and that it has the flat base change property (small detail omitted).

Step 2: $Q_\mathcal{X}$ is contained in $\mathcal{A}$. To see this it suffices to show that for K in $Q_\mathcal{X}$ we have $\operatorname{Hom}(K, P) = 0$ for all P in $D_{Parasitic \cap LQCoh^{fbc}}(\mathcal{O}_\mathcal{X})$. Consider the object

$$H = R\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(K, P)$$

Let x be an object of $\mathcal{X}$ which lies over an affine scheme $U = p(x)$. By Cohomology on Sites, Lemma 35.1 we have the first equality in

$$R\Gamma(x, H) = R\operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(K|_{\mathcal{X}/x}, P|_{\mathcal{X}/x}) = R\operatorname{Hom}_{\mathcal{O}}(K|_{\mathcal{X}_{affine}/x}, P|_{\mathcal{X}_{affine}/x})$$

The second equality stems from the fact that the topos of the site $\mathcal{X}/x$ is equivalent to the topos of the site $\mathcal{X}_{affine}/x$, see Sheaves on Stacks, Equation (24.3.1). We may write $K = \epsilon^* N$ for some N in $QC(\mathcal{O})$. Then by Cohomology on Sites, Lemma 43.13 we see that

$$R\Gamma(x, H) = R\operatorname{Hom}_{D(\mathcal{O}(x))}(R\Gamma(x, N), R\Gamma(x, P))$$

By Lemma 8.1 we see that $R\Gamma(x, P) = 0$ if $U \to \mathcal{X}$ is flat and hence $R\Gamma(x, H) = 0$ under the same hypothesis. By Lemma 8.2 we conclude that $R\Gamma(\mathcal{X}, H) = 0$ and therefore $\mathrm{Hom}(K, P) = 0$.

Step 3: $\mathcal{A}$ is contained in $Q_{\mathcal{X}}$. Let K be an object of $\mathcal{A}$ and let $x \to x'$ be a morphism of $\mathcal{X}_{affine}$. We have to show that

$$R\Gamma(x', K) \otimes^{\mathbf{L}}_{\mathcal{O}(x')} \mathcal{O}(x) \longrightarrow R\Gamma(x, K)$$

is a quasi-isomorphism, see footnote in statement of Cohomology on Sites, Lemma 43.12. By the proof of Lemma 5.4 and the discussion in Remark 5.5 we see that $\mathcal{A}$ is the image of the restriction of $Lg_!$ to $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Thus we may assume $K = Lg_! M$ for some M in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Then the desired equality follow from Lemma 8.3. □

9. Other chapters

References

[LMB00] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 39, Springer-Verlag, 2000.

[Ols07] Martin Christian Olsson, *Sheaves on Artin stacks*, J. Reine Angew. Math. **603** (2007), 55–112.