

# FLATNESS ON ALGEBRAIC SPACES

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## 1. Introduction

In this chapter, we discuss some advanced results on flat modules and flat morphisms in the setting of algebraic spaces. We strongly encourage the reader to take a look at the corresponding chapter in the setting of schemes first, see More on Flatness, Section 1. A reference is the paper [GR71] by Raynaud and Gruson.

## 2. Impurities

The section is the analogue of More on Flatness, Section 15.

**Situation 2.1.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a finite type, decent<sup>1</sup> morphism of algebraic spaces over  $S$ . Also,  $\mathcal{F}$  is a finite type quasi-coherent  $\mathcal{O}_X$ -module. Finally  $y \in |Y|$  is a point of  $Y$ .

In this situation consider a scheme  $T$ , a morphism  $g : T \rightarrow Y$ , a point  $t \in T$  with  $g(t) = y$ , a specialization  $t' \rightsquigarrow t$  in  $T$ , and a point  $\xi \in |X_T|$  lying over  $t'$ . Here

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<sup>1</sup>Quasi-separated morphisms are decent, see Decent Spaces, Lemma 17.2. For any morphism  $\mathrm{Spec}(k) \rightarrow Y$  where  $k$  is a field, the algebraic space  $X_k$  is of finite presentation over  $k$  because it is of finite type over  $k$  and quasi-separated by Decent Spaces, Lemma 14.1.

$X_T = T \times_Y X$ . Picture

$$(2.1.1) \quad \begin{array}{ccc} \xi & & X_T \longrightarrow X \\ \downarrow & & \downarrow f_T \quad \downarrow f \\ t' \rightsquigarrow t \longrightarrow y & & T \xrightarrow{g} Y \end{array}$$

Moreover, denote  $\mathcal{F}_T$  the pullback of  $\mathcal{F}$  to  $X_T$ .

**Definition 2.2.** In Situation 2.1 we say a diagram (2.1.1) defines an *impurity* of  $\mathcal{F}$  above  $y$  if  $\xi \in \text{Ass}_{X_T/T}(\mathcal{F}_T)$  and  $t \notin f_T(\overline{\{\xi\}})$ . We will indicate this by saying “let  $(g : T \rightarrow Y, t' \rightsquigarrow t, \xi)$  be an impurity of  $\mathcal{F}$  above  $y$ ”.

Another way to say this is:  $(g : T \rightarrow Y, t' \rightsquigarrow t, \xi)$  is an impurity of  $\mathcal{F}$  above  $y$  if there exists no specialization  $\xi \rightsquigarrow \theta$  in the topological space  $|X_T|$  with  $f_T(\theta) = t$ . Specializations in non-decent algebraic spaces do not behave well. If the morphism  $f$  is decent, then  $X_T$  is a decent algebraic space for all morphisms  $g : T \rightarrow Y$  as above, see Decent Spaces, Definition 17.1.

**Lemma 2.3.** In Situation 2.1. Let  $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$  be an impurity of  $\mathcal{F}$  above  $y$ . Assume  $T = \lim_{i \in I} T_i$  is a directed limit of affine schemes over  $Y$ . Then for some  $i$  the triple  $(T_i \rightarrow Y, t'_i \rightsquigarrow t_i, \xi_i)$  is an impurity of  $\mathcal{F}$  above  $y$ .

**Proof.** The notation in the statement means this: Let  $p_i : T \rightarrow T_i$  be the projection morphisms, let  $t_i = p_i(t)$  and  $t'_i = p_i(t')$ . Finally  $\xi_i \in |X_{T_i}|$  is the image of  $\xi$ . By Divisors on Spaces, Lemma 4.7 we have  $\xi_i \in \text{Ass}_{X_{T_i}/T_i}(\mathcal{F}_{T_i})$ . Thus the only point is to show that  $t_i \notin f_{T_i}(\overline{\{\xi_i\}})$  for some  $i$ .

Let  $Z_i \subset X_{T_i}$  be the reduced induced scheme structure on  $\overline{\{\xi_i\}} \subset |X_{T_i}|$  and let  $Z \subset X_T$  be the reduced induced scheme structure on  $\overline{\{\xi\}} \subset |X_T|$ . Then  $Z = \lim Z_i$  by Limits of Spaces, Lemma 5.4 (the lemma applies because each  $X_{T_i}$  is decent). Choose a field  $k$  and a morphism  $\text{Spec}(k) \rightarrow T$  whose image is  $t$ . Then

$$\emptyset = Z \times_T \text{Spec}(k) = (\lim Z_i) \times_{(\lim T_i)} \text{Spec}(k) = \lim Z_i \times_{T_i} \text{Spec}(k)$$

because limits commute with fibred products (limits commute with limits). Each  $Z_i \times_{T_i} \text{Spec}(k)$  is quasi-compact because  $X_{T_i} \rightarrow T_i$  is of finite type and hence  $Z_i \rightarrow T_i$  is of finite type. Hence  $Z_i \times_{T_i} \text{Spec}(k)$  is empty for some  $i$  by Limits of Spaces, Lemma 5.3. Since the image of the composition  $\text{Spec}(k) \rightarrow T \rightarrow T_i$  is  $t_i$  we obtain what we want.  $\square$

Impurities go up along flat base change.

**Lemma 2.4.** In Situation 2.1. Let  $(Y_1, y_1) \rightarrow (Y, y)$  be a morphism of pointed algebraic spaces over  $S$ . Assume  $Y_1 \rightarrow Y$  is flat at  $y_1$ . If  $(T \rightarrow Y, t' \rightsquigarrow t, \xi)$  is an impurity of  $\mathcal{F}$  above  $y$ , then there exists an impurity  $(T_1 \rightarrow Y_1, t'_1 \rightsquigarrow t_1, \xi_1)$  of the pullback  $\mathcal{F}_1$  of  $\mathcal{F}$  to  $X_1 = Y_1 \times_Y X$  over  $y_1$  such that  $T_1$  is étale over  $Y_1 \times_Y T$ .

**Proof.** Choose an étale morphism  $T_1 \rightarrow Y_1 \times_Y T$  where  $T_1$  is a scheme and let  $t_1 \in T_1$  be a point mapping to  $y_1$  and  $t$ . It is possible to find a pair  $(T_1, t_1)$  like this by Properties of Spaces, Lemma 4.3. The morphism of schemes  $T_1 \rightarrow T$  is flat at  $t_1$  (use Morphisms of Spaces, Lemma 30.4 and the definition of flat morphisms of algebraic spaces) there exists a specialization  $t'_1 \rightsquigarrow t_1$  lying over  $t' \rightsquigarrow t$ , see Morphisms, Lemma 25.9. Choose a point  $\xi_1 \in |X_{T_1}|$  mapping to  $t'_1$  and  $\xi$  with

$\xi_1 \in \text{Ass}_{X_{T_1}/T_1}(\mathcal{F}_{T_1})$ . point of  $\text{Spec}(\kappa(t'_1) \otimes_{\kappa(t')} \kappa(\xi))$ . This is possible by Divisors on Spaces, Lemma 4.7. As the closure  $Z_1$  of  $\{\xi_1\}$  in  $|X_{T_1}|$  maps into the closure of  $\{\xi\}$  in  $|X_T|$  we conclude that the image of  $Z_1$  in  $|T_1|$  cannot contain  $t_1$ . Hence  $(T_1 \rightarrow Y_1, t'_1 \rightsquigarrow t_1, \xi_1)$  is an impurity of  $\mathcal{F}_1$  above  $Y_1$ .  $\square$

**Lemma 2.5.** *In Situation 2.1. Let  $\bar{y}$  be a geometric point lying over  $y$ . Let  $\mathcal{O} = \mathcal{O}_{Y, \bar{y}}$  be the étale local ring of  $Y$  at  $\bar{y}$ . Denote  $Y^{sh} = \text{Spec}(\mathcal{O})$ ,  $X^{sh} = X \times_Y Y^{sh}$ , and  $\mathcal{F}^{sh}$  the pullback of  $\mathcal{F}$  to  $X^{sh}$ . The following are equivalent*

- (1) *there exists an impurity  $(Y^{sh} \rightarrow Y, y' \rightsquigarrow \bar{y}, \xi)$  of  $\mathcal{F}$  above  $y$ ,*
- (2) *every point of  $\text{Ass}_{X^{sh}/Y^{sh}}(\mathcal{F}^{sh})$  specializes to a point of the closed fibre  $X_{\bar{y}}$ ,*
- (3) *there exists an impurity  $(T \rightarrow Y, t' \rightsquigarrow t, \xi)$  of  $\mathcal{F}$  above  $y$  such that  $(T, t) \rightarrow (Y, y)$  is an étale neighbourhood, and*
- (4) *there exists an impurity  $(T \rightarrow Y, t' \rightsquigarrow t, \xi)$  of  $\mathcal{F}$  above  $y$  such that  $T \rightarrow Y$  is quasi-finite at  $t$ .*

**Proof.** That parts (1) and (2) are equivalent is immediate from the definition.

Recall that  $\mathcal{O} = \mathcal{O}_{Y, \bar{y}}$  is the filtered colimit of  $\mathcal{O}(V)$  over the category of étale neighbourhoods  $(V, \bar{v}) \rightarrow (Y, \bar{y})$  (Properties of Spaces, Lemma 19.3). Moreover, it suffices to consider affine étale neighbourhoods  $V$ . Hence  $Y^{sh} = \text{Spec}(\mathcal{O}) = \lim \text{Spec}(\mathcal{O}(V)) = \lim V$ . Thus we see that (1) implies (3) by Lemma 2.3.

Since an étale morphism is locally quasi-finite (Morphisms of Spaces, Lemma 39.5) we see that (3) implies (4).

Finally, assume (4). After replacing  $T$  by an open neighbourhood of  $t$  we may assume  $T \rightarrow Y$  is locally quasi-finite. By Lemma 2.4 we find an impurity  $(T_1 \rightarrow Y^{sh}, t'_1 \rightsquigarrow t_1, \xi_1)$  with  $T_1 \rightarrow T \times_Y Y^{sh}$  étale. Since an étale morphism is locally quasi-finite and using Morphisms of Spaces, Lemma 27.4 and Morphisms, Lemma 20.12 we see that  $T_1 \rightarrow Y^{sh}$  is locally quasi-finite. As  $\mathcal{O}$  is strictly henselian, we can apply More on Morphisms, Lemma 41.1 to see that after replacing  $T_1$  by an open and closed neighbourhood of  $t_1$  we may assume that  $T_1 \rightarrow Y^{sh} = \text{Spec}(\mathcal{O})$  is finite. Let  $\theta \in |X^{sh}|$  be the image of  $\xi_1$  and let  $y' \in \text{Spec}(\mathcal{O})$  be the image of  $t'_1$ . By Divisors on Spaces, Lemma 4.7 we see that  $\theta \in \text{Ass}_{X^{sh}/Y^{sh}}(\mathcal{F}^{sh})$ . Since  $\pi : X_{T_1} \rightarrow X^{sh}$  is finite, it induces a closed map  $|X_{T_1}| \rightarrow |X^{sh}|$ . Hence the image of  $\{\xi_1\}$  is  $\{\theta\}$ . It follows that  $(Y^{sh} \rightarrow Y, y' \rightsquigarrow \bar{y}, \theta)$  is an impurity of  $\mathcal{F}$  above  $y$  and the proof is complete.  $\square$

### 3. Relatively pure modules

This section is the analogue of More on Flatness, Section 16.

**Definition 3.1.** In Situation 2.1.

- (1) We say  $\mathcal{F}$  is *pure above  $y$*  if **none** of the equivalent conditions of Lemma 2.5 hold.
- (2) We say  $\mathcal{F}$  is *universally pure above  $y$*  if there does not exist any impurity of  $\mathcal{F}$  above  $y$ .
- (3) We say that  $X$  is *pure above  $y$*  if  $\mathcal{O}_X$  is pure above  $y$ .
- (4) We say  $\mathcal{F}$  is *universally  $Y$ -pure*, or *universally pure relative to  $Y$*  if  $\mathcal{F}$  is universally pure above  $y$  for every  $y \in |Y|$ .
- (5) We say  $\mathcal{F}$  is  *$Y$ -pure*, or *pure relative to  $Y$*  if  $\mathcal{F}$  is pure above  $y$  for every  $y \in |Y|$ .

(6) We say that  $X$  is  $Y$ -pure or *pure relative to  $Y$*  if  $\mathcal{O}_X$  is pure relative to  $Y$ .

The obligatory lemmas follow.

**Lemma 3.2.** *In Situation 2.1.*

- (1)  $\mathcal{F}$  is universally pure above  $y$ , and
- (2) for every morphism  $(Y', y') \rightarrow (Y, y)$  of pointed algebraic spaces the pullback  $\mathcal{F}_{Y'}$  is pure above  $y'$ .

*In particular,  $\mathcal{F}$  is universally pure relative to  $Y$  if and only if every base change  $\mathcal{F}_{Y'}$  of  $\mathcal{F}$  is pure relative to  $Y'$ .*

**Proof.** This is formal.  $\square$

**Lemma 3.3.** *In Situation 2.1. Let  $(Y', y') \rightarrow (Y, y)$  be a morphism of pointed algebraic spaces. If  $Y' \rightarrow Y$  is quasi-finite at  $y'$  and  $\mathcal{F}$  is pure above  $y$ , then  $\mathcal{F}_{Y'}$  is pure above  $y'$ .*

**Proof.** It  $(T \rightarrow Y', t' \rightsquigarrow t, \xi)$  is an impurity of  $\mathcal{F}_{Y'}$  above  $y'$  with  $T \rightarrow Y'$  quasi-finite at  $t$ , then  $(T \rightarrow Y, t' \rightarrow t, \xi)$  is an impurity of  $\mathcal{F}$  above  $y$  with  $T \rightarrow Y$  quasi-finite at  $t$ , see Morphisms of Spaces, Lemma 27.3. Hence the lemma follows immediately from the definition of purity.  $\square$

Purity satisfies flat descent.

**Lemma 3.4.** *In Situation 2.1. Let  $(Y_1, y_1) \rightarrow (Y, y)$  be a morphism of pointed algebraic spaces. Assume  $Y_1 \rightarrow Y$  is flat at  $y_1$ .*

- (1) If  $\mathcal{F}_{Y_1}$  is pure above  $y_1$ , then  $\mathcal{F}$  is pure above  $y$ .
- (2) If  $\mathcal{F}_{Y_1}$  is universally pure above  $y_1$ , then  $\mathcal{F}$  is universally pure above  $y$ .

**Proof.** This is true because impurities go up along a flat base change, see Lemma 2.4. For example part (1) follows because by any impurity  $(T \rightarrow Y, t' \rightsquigarrow t, \xi)$  of  $\mathcal{F}$  above  $y$  with  $T \rightarrow Y$  quasi-finite at  $t$  by the lemma leads to an impurity  $(T_1 \rightarrow Y_1, t'_1 \rightsquigarrow t_1, \xi_1)$  of the pullback  $\mathcal{F}_1$  of  $\mathcal{F}$  to  $X_1 = Y_1 \times_Y X$  over  $y_1$  such that  $T_1$  is étale over  $Y_1 \times_Y T$ . Hence  $T_1 \rightarrow Y_1$  is quasi-finite at  $t_1$  because étale morphisms are locally quasi-finite and compositions of locally quasi-finite morphisms are locally quasi-finite (Morphisms of Spaces, Lemmas 39.5 and 27.3). Similarly for part (2).  $\square$

**Lemma 3.5.** *In Situation 2.1. Let  $i : Z \rightarrow X$  be a closed immersion and assume that  $\mathcal{F} = i_* \mathcal{G}$  for some finite type, quasi-coherent sheaf  $\mathcal{G}$  on  $Z$ . Then  $\mathcal{G}$  is (universally) pure above  $y$  if and only if  $\mathcal{F}$  is (universally) pure above  $y$ .*

**Proof.** This follows from Divisors on Spaces, Lemma 4.9.  $\square$

**Lemma 3.6.** *In Situation 2.1.*

- (1) If the support of  $\mathcal{F}$  is proper over  $Y$ , then  $\mathcal{F}$  is universally pure relative to  $Y$ .
- (2) If  $f$  is proper, then  $\mathcal{F}$  is universally pure relative to  $Y$ .
- (3) If  $f$  is proper, then  $X$  is universally pure relative to  $Y$ .

**Proof.** First we reduce (1) to (2). Namely, let  $Z \subset X$  be the scheme theoretic support of  $\mathcal{F}$  (Morphisms of Spaces, Definition 15.4). Let  $i : Z \rightarrow X$  be the corresponding closed immersion and write  $\mathcal{F} = i_* \mathcal{G}$  for some finite type quasi-coherent  $\mathcal{O}_Z$ -module  $\mathcal{G}$ . In case (1)  $Z \rightarrow Y$  is proper by assumption. Thus by Lemma 3.5 case (1) reduces to case (2).

Assume  $f$  is proper. Let  $(g : T \rightarrow Y, t' \rightsquigarrow t, \xi)$  be an impurity of  $\mathcal{F}$  above  $y$ . Since  $f$  is proper, it is universally closed. Hence  $f_T : X_T \rightarrow T$  is closed. Since  $f_T(\xi) = t'$  this implies that  $t \in f(\overline{\{\xi\}})$  which is a contradiction.  $\square$

#### 4. Flat finite type modules

Please compare with More on Flatness, Sections 10, 13, and 26. Most of these results have immediate consequences of algebraic spaces by étale localization.

**Lemma 4.1.** *Let  $S$  be a scheme. Let  $X \rightarrow Y$  be a finite type morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a finite type quasi-coherent  $\mathcal{O}_X$ -module. Let  $y \in |Y|$  be a point. There exists an étale morphism  $(Y', y') \rightarrow (Y, y)$  with  $Y'$  an affine scheme and étale morphisms  $h_i : W_i \rightarrow X_{Y'}$ ,  $i = 1, \dots, n$  such that for each  $i$  there exists a complete dévissage of  $\mathcal{F}_i/W_i/Y'$  over  $y'$ , where  $\mathcal{F}_i$  is the pullback of  $\mathcal{F}$  to  $W_i$  and such that  $|(X_{Y'})_{y'}| \subset \bigcup h_i(W_i)$ .*

**Proof.** The question is étale local on  $Y$  hence we may assume  $Y$  is an affine scheme. Then  $X$  is quasi-compact, hence we can choose an affine scheme  $X'$  and a surjective étale morphism  $X' \rightarrow X$ . Then we may apply More on Flatness, Lemma 5.8 to  $X' \rightarrow Y$ ,  $(X' \rightarrow Y)^*\mathcal{F}$ , and  $y$  to get what we want.  $\square$

**Lemma 4.2.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module of finite type. Let  $y \in |Y|$  and  $F = f^{-1}(\{y\}) \subset |X|$ . Then the set*

$$\{x \in F \mid \mathcal{F} \text{ flat over } Y \text{ at } x\}$$

*is open in  $F$ .*

**Proof.** Choose a scheme  $V$ , a point  $v \in V$ , and an étale morphism  $V \rightarrow Y$  mapping  $v$  to  $y$ . Choose a scheme  $U$  and a surjective étale morphism  $U \rightarrow V \times_Y X$ . Then  $|U_v| \rightarrow F$  is an open continuous map of topological spaces as  $|U| \rightarrow |X|$  is continuous and open. Hence the result follows from the case of schemes which is More on Flatness, Lemma 10.4.  $\square$

**Lemma 4.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $x \in |X|$  with image  $y \in |Y|$ . Let  $\mathcal{F}$  be a finite type quasi-coherent sheaf on  $X$ . Let  $\mathcal{G}$  be a quasi-coherent sheaf on  $Y$ . If  $\mathcal{F}$  is flat at  $x$  over  $Y$ , then*

$$x \in \text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \Leftrightarrow y \in \text{WeakAss}_Y(\mathcal{G}) \text{ and } x \in \text{Ass}_{X/Y}(\mathcal{F}).$$

**Proof.** Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where  $U$  and  $V$  are schemes and the vertical arrows are surjective étale. Choose  $u \in U$  mapping to  $x$ . Let  $\mathcal{E} = \mathcal{F}|_U$  and  $\mathcal{H} = \mathcal{G}|_V$ . Let  $v \in V$  be the image of  $u$ . Then  $x \in \text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$  if and only if  $u \in \text{WeakAss}_X(\mathcal{E} \otimes_{\mathcal{O}_X} g^*\mathcal{H})$  by Divisors on Spaces, Definition 2.2. Similarly,  $y \in \text{WeakAss}_Y(\mathcal{G})$  if and only if  $v \in \text{WeakAss}_V(\mathcal{H})$ . Finally, we have  $x \in \text{Ass}_{X/Y}(\mathcal{F})$  if and only if  $u \in \text{Ass}_{U_v}(\mathcal{E}|_{U_v})$  by Divisors on Spaces, Definition 4.5. Observe that flatness of  $\mathcal{F}$  at  $x$  is equivalent

to flatness of  $\mathcal{E}$  at  $u$ , see Morphisms of Spaces, Definition 31.2. The equivalence for  $g : U \rightarrow V$ ,  $\mathcal{E}$ ,  $\mathcal{H}$ ,  $u$ , and  $v$  is More on Flatness, Lemma 13.3.  $\square$

**Lemma 4.4.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $\mathcal{F}$  be a finite type quasi-coherent sheaf on  $X$  which is flat over  $Y$ . Let  $\mathcal{G}$  be a quasi-coherent sheaf on  $Y$ . Then we have*

$$\text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G}) = \text{Ass}_{X/Y}(\mathcal{F}) \cap |f|^{-1}(\text{WeakAss}_Y(\mathcal{G}))$$

**Proof.** Immediate consequence of Lemma 4.3.  $\square$

**Theorem 4.5.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module. Assume*

- (1)  $X \rightarrow Y$  is locally of finite presentation,
- (2)  $\mathcal{F}$  is an  $\mathcal{O}_X$ -module of finite type, and
- (3) the set of weakly associated points of  $Y$  is locally finite in  $Y$ .

*Then  $U = \{x \in |X| : \mathcal{F} \text{ flat at } x \text{ over } Y\}$  is open in  $X$  and  $\mathcal{F}|_U$  is an  $\mathcal{O}_U$ -module of finite presentation and flat over  $Y$ .*

**Proof.** Condition (3) means that if  $V \rightarrow Y$  is a surjective étale morphism where  $V$  is a scheme, then the weakly associated points of  $V$  are locally finite on the scheme  $V$ . (Recall that the weakly associated points of  $V$  are exactly the inverse image of the weakly associated points of  $Y$  by Divisors on Spaces, Definition 2.2.) Having said this the question is étale local on  $X$  and  $Y$ , hence we may assume  $X$  and  $Y$  are schemes. Thus the result follows from More on Flatness, Theorem 13.6.  $\square$

**Lemma 4.6.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent sheaf on  $X$ . Let  $y \in |Y|$ . Set  $F = f^{-1}(\{y\}) \subset |X|$ . Assume that*

- (1)  $f$  is of finite type,
- (2)  $\mathcal{F}$  is of finite type, and
- (3)  $\mathcal{F}$  is flat over  $Y$  at all  $x \in F$ .

*Then there exists an étale morphism  $(Y', y') \rightarrow (Y, y)$  where  $Y'$  is a scheme and a commutative diagram of algebraic spaces*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ Y & \xleftarrow{\quad} & \text{Spec}(\mathcal{O}_{Y', y'}) \end{array}$$

*such that  $X' \rightarrow X \times_Y \text{Spec}(\mathcal{O}_{Y', y'})$  is étale,  $|X'_{y'}| \rightarrow F$  is surjective,  $X'$  is affine, and  $\Gamma(X', g^* \mathcal{F})$  is a free  $\mathcal{O}_{Y', y'}$ -module.*

**Proof.** Choose an étale morphism  $(Y', y') \rightarrow (Y, y)$  where  $Y'$  is an affine scheme. Then  $X \times_Y Y'$  is quasi-compact. Choose an affine scheme  $X'$  and a surjective étale morphism  $X' \rightarrow X \times_Y Y'$ . Picture

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ Y & \xleftarrow{\quad} & Y' \end{array}$$

Then  $\mathcal{F}' = g^*\mathcal{F}$  is flat over  $Y'$  at all points of  $X'_{y'}$ , see Morphisms of Spaces, Lemma 31.3. Hence we can apply the lemma in the case of schemes (More on Flatness, Lemma 12.11) to the morphism  $X' \rightarrow Y'$ , the quasi-coherent sheaf  $g^*\mathcal{F}$ , and the point  $y'$ . This gives an étale morphism  $(Y'', y'') \rightarrow (Y', y')$  and a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{g} & X' & \xleftarrow{g'} & X'' \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xleftarrow{\quad} & Y' & \xleftarrow{\quad} & \mathrm{Spec}(\mathcal{O}_{Y'', y''}) \end{array}$$

To get what we want we take  $(Y'', y'') \rightarrow (Y, y)$  and  $g \circ g' : X'' \rightarrow X$ .  $\square$

**Theorem 4.7.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module of finite type. Let  $x \in |X|$  with image  $y \in |Y|$ . Set  $F = f^{-1}(\{y\}) \subset |X|$ . Consider the conditions*

- (1)  $\mathcal{F}$  is flat at  $x$  over  $Y$ , and
- (2) for every  $x' \in F \cap \mathrm{Ass}_{X/Y}(\mathcal{F})$  which specializes to  $x$  we have that  $\mathcal{F}$  is flat at  $x'$  over  $Y$ .

Then we always have (2)  $\Rightarrow$  (1). If  $X$  and  $Y$  are decent, then (1)  $\Rightarrow$  (2).

**Proof.** Assume (2). Choose a scheme  $V$  and a surjective étale morphism  $V \rightarrow Y$ . Choose a scheme  $U$  and a surjective étale morphism  $U \rightarrow V \times_Y X$ . Choose a point  $u \in U$  mapping to  $x$ . Let  $v \in V$  be the image of  $u$ . We will deduce the result from the corresponding result for  $\mathcal{F}|_U = (U \rightarrow X)^*\mathcal{F}$  and the point  $u$ .  $U_v$ . This works because  $\mathrm{Ass}_{U/V}(\mathcal{F}|_U) \cap |U_v|$  is equal to  $\mathrm{Ass}_{U_v}(\mathcal{F}|_{U_v})$  and equal to the inverse image of  $F \cap \mathrm{Ass}_{X/Y}(\mathcal{F})$ . Since the map  $|U_v| \rightarrow F$  is continuous we see that specializations in  $|U_v|$  map to specializations in  $F$ , hence condition (2) is inherited by  $U \rightarrow V$ ,  $\mathcal{F}|_U$ , and the point  $u$ . Thus More on Flatness, Theorem 26.1 applies and we conclude that (1) holds.

If  $Y$  is decent, then we can represent  $y$  by a quasi-compact monomorphism  $\mathrm{Spec}(k) \rightarrow Y$  (by definition of decent spaces, see Decent Spaces, Definition 6.1). Then  $F = |X_k|$ , see Decent Spaces, Lemma 18.6. If in addition  $X$  is decent (or more generally if  $f$  is decent, see Decent Spaces, Definition 17.1 and Decent Spaces, Lemma 17.3), then  $X_y$  is a decent space too. Furthermore, specializations in  $F$  can be lifted to specializations in  $U_v \rightarrow X_y$ , see Decent Spaces, Lemma 12.2. Having said this it is clear that the reverse implication holds, because it holds in the case of schemes.  $\square$

**Lemma 4.8.** *Let  $S$  be a local scheme with closed point  $s$ . Let  $f : X \rightarrow S$  be a morphism from an algebraic space  $X$  to  $S$  which is locally of finite type. Let  $\mathcal{F}$  be a finite type quasi-coherent  $\mathcal{O}_X$ -module. Assume that*

- (1) every point of  $\mathrm{Ass}_{X/S}(\mathcal{F})$  specializes to a point of the closed fibre  $X_s^2$ ,
- (2)  $\mathcal{F}$  is flat over  $S$  at every point of  $X_s$ .

Then  $\mathcal{F}$  is flat over  $S$ .

**Proof.** This is immediate from the fact that it suffices to check for flatness at points of the relative assassin of  $\mathcal{F}$  over  $S$  by Theorem 4.7.  $\square$

<sup>2</sup>For example this holds if  $f$  is finite type and  $\mathcal{F}$  is pure along  $X_s$ , or if  $f$  is proper.

### 5. Flat finitely presented modules

This is the analogue of More on Flatness, Section 12.

**Proposition 5.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent sheaf on  $X$ . Let  $x \in |X|$  with image  $y \in |Y|$ . Assume that*

- (1)  *$f$  is locally of finite presentation,*
- (2)  *$\mathcal{F}$  is of finite presentation, and*
- (3)  *$\mathcal{F}$  is flat at  $x$  over  $Y$ .*

*Then there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (Y, y) & \xleftarrow{\quad} & (Y', y') \end{array}$$

*whose horizontal arrows are étale such that  $X', Y'$  are affine and such that  $\Gamma(X', g^*\mathcal{F})$  is a projective  $\Gamma(Y', \mathcal{O}_{Y'})$ -module.*

**Proof.** As formulated this proposition immediately reduces to the case of schemes, which is More on Flatness, Proposition 12.4.  $\square$

**Lemma 5.2.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent sheaf on  $X$ . Let  $y \in |Y|$ . Set  $F = f^{-1}(\{y\}) \subset |X|$ . Assume that*

- (1)  *$f$  is of finite presentation,*
- (2)  *$\mathcal{F}$  is of finite presentation, and*
- (3)  *$\mathcal{F}$  is flat over  $Y$  at all  $x \in F$ .*

*Then there exists a commutative diagram of algebraic spaces*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ Y & \xleftarrow{h} & Y' \end{array}$$

*such that  $h$  and  $g$  are étale, there is a point  $y' \in |Y'|$  mapping to  $y$ , we have  $F \subset g(|X'|)$ , the algebraic spaces  $X', Y'$  are affine, and  $\Gamma(X', g^*\mathcal{F})$  is a projective  $\Gamma(Y', \mathcal{O}_{Y'})$ -module.*

**Proof.** As formulated this lemma immediately reduces to the case of schemes, which is More on Flatness, Lemma 12.5.  $\square$

### 6. A criterion for purity

This section is the analogue of More on Flatness, Section 18.

**Lemma 6.1.** *Let  $S$  be a scheme. Let  $X$  be a decent algebraic space locally of finite type over  $S$ . Let  $\mathcal{F}$  be a finite type, quasi-coherent  $\mathcal{O}_X$ -module. Let  $s \in S$  such that  $\mathcal{F}$  is flat over  $S$  at all points of  $X_s$ . Let  $x' \in \text{Ass}_{X/S}(\mathcal{F})$ . If the closure of  $\{x'\}$  in  $|X|$  meets  $|X_s|$ , then the closure meets  $\text{Ass}_{X/S}(\mathcal{F}) \cap |X_s|$ .*



**Proof.** Observe that  $|X_s| \subset |X|$  is the set of points of  $|X|$  lying over  $s \in S$ , see Decent Spaces, Lemma 18.6. Let  $t \in |X_s|$  be a specialization of  $x'$  in  $|X|$ . Choose an affine scheme  $U$  and a point  $u \in U$  and an étale morphism  $\varphi : U \rightarrow X$  mapping  $u$  to  $t$ . By Decent Spaces, Lemma 12.2 we can choose a specialization  $u' \rightsquigarrow u$  with  $u'$  mapping to  $x'$ . Set  $g = f \circ \varphi$ . Observe that  $s' = g(u') = f(x')$  specializes to  $s$ . By our definition of  $\text{Ass}_{X/S}(\mathcal{F})$  we have  $u' \in \text{Ass}_{U/S}(\varphi^*\mathcal{F})$ . By the schemes version of this lemma (More on Flatness, Lemma 18.1) we see that there is a specialization  $u' \rightsquigarrow u$  with  $u \in \text{Ass}_{U_s}(\varphi^*\mathcal{F}_s) = \text{Ass}_{U/S}(\varphi^*\mathcal{F}) \cap U_s$ . Hence  $x = \varphi(u) \in \text{Ass}_{X/S}(\mathcal{F})$  lies over  $s$  and the lemma is proved.  $\square$

**Lemma 6.2.** *Let  $Y$  be an algebraic space over a scheme  $S$ . Let  $g : X' \rightarrow X$  be a morphism of algebraic spaces over  $Y$  with  $X$  locally of finite type over  $Y$ . Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module. If  $\text{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$ , then for any morphism  $Z \rightarrow Y$  we have  $\text{Ass}_{X_Z/Z}(\mathcal{F}_Z) \subset g_Z(|X'_Z|)$ .*

**Proof.** By Properties of Spaces, Lemma 4.3 the map  $|X'_Z| \rightarrow |X_Z| \times_{|X|} |X'|$  is surjective as  $X'_Z$  is equal to  $X_Z \times_X X'$ . By Divisors on Spaces, Lemma 4.7 the map  $|X_Z| \rightarrow |X|$  sends  $\text{Ass}_{X_Z/Z}(\mathcal{F}_Z)$  into  $\text{Ass}_{X/Y}(\mathcal{F})$ . The lemma follows.  $\square$

**Lemma 6.3.** *Let  $Y$  be an algebraic space over a scheme  $S$ . Let  $g : X' \rightarrow X$  be an étale morphism of algebraic spaces over  $Y$ . Assume the structure morphisms  $X' \rightarrow Y$  and  $X \rightarrow Y$  are decent and of finite type. Let  $\mathcal{F}$  be a finite type, quasi-coherent  $\mathcal{O}_X$ -module. Let  $y \in |Y|$ . Set  $F = f^{-1}(\{y\}) \subset |X|$ .*

- (1) *If  $\text{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$  and  $g^*\mathcal{F}$  is (universally) pure above  $y$ , then  $\mathcal{F}$  is (universally) pure above  $y$ .*
- (2) *If  $\mathcal{F}$  is pure above  $y$ ,  $g(|X'|)$  contains  $F$ , and  $Y$  is affine local with closed point  $y$ , then  $\text{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$ .*
- (3) *If  $\mathcal{F}$  is pure above  $y$ ,  $\mathcal{F}$  is flat at all points of  $F$ ,  $g(|X'|)$  contains  $\text{Ass}_{X/Y}(\mathcal{F}) \cap F$ , and  $Y$  is affine local with closed point  $y$ , then  $\text{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$ .*
- (4) *Add more here.*

**Proof.** The assumptions on  $X \rightarrow Y$  and  $X' \rightarrow Y$  guarantee that we may apply the material in Sections 2 and 3 to these morphisms and the sheaves  $\mathcal{F}$  and  $g^*\mathcal{F}$ . Since  $g$  is étale we see that  $\text{Ass}_{X'/Y}(g^*\mathcal{F})$  is the inverse image of  $\text{Ass}_{X/Y}(\mathcal{F})$  and the same remains true after base change.

Proof of (1). Assume  $\text{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$ . Suppose that  $(T \rightarrow Y, t' \rightsquigarrow t, \xi)$  is an impurity of  $\mathcal{F}$  above  $y$ . Since  $\text{Ass}_{X_T/T}(\mathcal{F}_T) \subset g_T(|X'_T|)$  by Lemma 6.2 we can choose a point  $\xi' \in |X'_T|$  mapping to  $\xi$ . By the above we see that  $(T \rightarrow Y, t' \rightsquigarrow t, \xi')$  is an impurity of  $g^*\mathcal{F}$  above  $y'$ . This implies (1) is true.

Proof of (2). This follows from the fact that  $g(|X'|)$  is open in  $|X|$  and the fact that by purity every point of  $\text{Ass}_{X/Y}(\mathcal{F})$  specializes to a point of  $F$ .

Proof of (3). This follows from the fact that  $g(|X'|)$  is open in  $|X|$  and the fact that by purity combined with Lemma 6.1 every point of  $\text{Ass}_{X/Y}(\mathcal{F})$  specializes to a point of  $\text{Ass}_{X/Y}(\mathcal{F}) \cap F$ .  $\square$

**Lemma 6.4.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module. Let  $y \in |Y|$ . Assume*

- (1)  *$f$  is decent and of finite type,*
- (2)  *$\mathcal{F}$  is of finite type,*

- (3)  $\mathcal{F}$  is flat over  $Y$  at all points lying over  $y$ , and
- (4)  $\mathcal{F}$  is pure above  $y$ .

Then  $\mathcal{F}$  is universally pure above  $y$ .

**Proof.** Consider the morphism  $\mathrm{Spec}(\mathcal{O}_{Y,\bar{y}}) \rightarrow Y$ . This is a flat morphism from the spectrum of a strictly henselian local ring which maps the closed point to  $y$ . By Lemma 3.4 we reduce to the case described in the next paragraph.

Assume  $Y$  is the spectrum of a strictly henselian local ring  $R$  with closed point  $y$ . By Lemma 4.6 there exists an étale morphism  $g : X' \rightarrow X$  with  $g(|X'|) \supset |X_y|$ , with  $X'$  affine, and with  $\Gamma(X', g^*\mathcal{F})$  a free  $R$ -module. Then  $g^*\mathcal{F}$  is universally pure relative to  $Y$ , see More on Flatness, Lemma 17.4. Hence it suffices to prove that  $g(|X'|)$  contains  $\mathrm{Ass}_{X/Y}(\mathcal{F})$  by Lemma 6.3 part (1). This in turn follows from Lemma 6.3 part (2).  $\square$

**Lemma 6.5.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a decent, finite type morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a finite type quasi-coherent  $\mathcal{O}_X$ -module. Assume  $\mathcal{F}$  is flat over  $Y$ . In this case  $\mathcal{F}$  is pure relative to  $Y$  if and only if  $\mathcal{F}$  is universally pure relative to  $Y$ .*

**Proof.** Immediate consequence of Lemma 6.4 and the definitions.  $\square$

**Lemma 6.6.** *Let  $Y$  be an algebraic space over a scheme  $S$ . Let  $g : X' \rightarrow X$  be a flat morphism of algebraic spaces over  $Y$  with  $X$  locally of finite type over  $Y$ . Let  $\mathcal{F}$  be a finite type quasi-coherent  $\mathcal{O}_X$ -module which is flat over  $Y$ . If  $\mathrm{Ass}_{X/Y}(\mathcal{F}) \subset g(|X'|)$  then the canonical map*

$$\mathcal{F} \longrightarrow g_*g^*\mathcal{F}$$

*is injective, and remains injective after any base change.*

**Proof.** The final assertion means that  $\mathcal{F}_Z \rightarrow (g_Z)_*g_Z^*\mathcal{F}_Z$  is injective for any morphism  $Z \rightarrow Y$ . Since the assumption on the relative assassin is preserved by base change (Lemma 6.2) it suffices to prove the injectivity of the displayed arrow.

Let  $\mathcal{K} = \mathrm{Ker}(\mathcal{F} \rightarrow g_*g^*\mathcal{F})$ . Our goal is to prove that  $\mathcal{K} = 0$ . In order to do this it suffices to prove that  $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$ , see Divisors on Spaces, Lemma 2.5. We have  $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F})$ , see Divisors on Spaces, Lemma 2.4. As  $\mathcal{F}$  is flat we see from Lemma 4.4 that  $\mathrm{WeakAss}_X(\mathcal{F}) \subset \mathrm{Ass}_{X/Y}(\mathcal{F})$ . By assumption any point  $x$  of  $\mathrm{Ass}_{X/Y}(\mathcal{F})$  is the image of some  $x' \in |X'|$ . Since  $g$  is flat the local ring map  $\mathcal{O}_{X,\bar{x}} \rightarrow \mathcal{O}_{X',\bar{x}'}$  is faithfully flat, hence the map

$$\mathcal{F}_{\bar{x}} \longrightarrow (g^*\mathcal{F})_{\bar{x}'} = \mathcal{F}_{\bar{x}} \otimes_{\mathcal{O}_{X,\bar{x}}} \mathcal{O}_{X',\bar{x}'}$$

is injective (see Algebra, Lemma 82.11). Since the displayed arrow factors through  $\mathcal{F}_{\bar{x}} \rightarrow (g_*g^*\mathcal{F})_{\bar{x}}$ , we conclude that  $\mathcal{K}_{\bar{x}} = 0$ . Hence  $x$  cannot be a weakly associated point of  $\mathcal{K}$  and we win.  $\square$

## 7. Flattening functors

This section is the analogue of More on Flatness, Section 20. We urge the reader to skip this section on a first reading.

**Situation 7.1.** Let  $S$  be a scheme. Let  $f : X \rightarrow B$  be a morphism of algebraic spaces over  $S$ . Let  $u : \mathcal{F} \rightarrow \mathcal{G}$  be a homomorphism of quasi-coherent  $\mathcal{O}_X$ -modules. For any scheme  $T$  over  $B$  we will denote  $u_T : \mathcal{F}_T \rightarrow \mathcal{G}_T$  the base change of  $u$  to  $T$ , in

other words,  $u_T$  is the pullback of  $u$  via the projection morphism  $X_T = X \times_B T \rightarrow X$ . In this situation we can consider the functor

$$(7.1.1) \quad F_{iso} : (Sch/B)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

There are variants  $F_{inj}$ ,  $F_{surj}$ ,  $F_{zero}$  where we ask that  $u_T$  is injective, surjective, or zero.

In Situation 7.1 we sometimes think of the functors  $F_{iso}$ ,  $F_{inj}$ ,  $F_{surj}$ , and  $F_{zero}$  as functors  $(Sch/S)^{opp} \rightarrow Sets$  endowed with a morphism  $F_{iso} \rightarrow B$ ,  $F_{inj} \rightarrow B$ ,  $F_{surj} \rightarrow B$ , and  $F_{zero} \rightarrow B$ . Namely, if  $T$  is a scheme over  $S$ , then an element  $h \in F_{iso}(T)$  is a morphism  $h : T \rightarrow B$  such that the base change of  $u$  via  $h$  is an isomorphism. In particular, when we say that  $F_{iso}$  is an algebraic space, we mean that the corresponding functor  $(Sch/S)^{opp} \rightarrow Sets$  is an algebraic space.

**Lemma 7.2.** *In Situation 7.1. Each of the functors  $F_{iso}$ ,  $F_{inj}$ ,  $F_{surj}$ ,  $F_{zero}$  satisfies the sheaf property for the fpqc topology.*

**Proof.** Let  $\{T_i \rightarrow T\}_{i \in I}$  be an fpqc covering of schemes over  $B$ . Set  $X_i = X_{T_i} = X \times_S T_i$  and  $u_i = u_{T_i}$ . Note that  $\{X_i \rightarrow X_T\}_{i \in I}$  is an fpqc covering of  $X_T$ , see Topologies on Spaces, Lemma 9.3. In particular, for every  $x \in |X_T|$  there exists an  $i \in I$  and an  $x_i \in |X_i|$  mapping to  $x$ . Since  $\mathcal{O}_{X_T, \bar{x}} \rightarrow \mathcal{O}_{X_i, \bar{x}_i}$  is flat, hence faithfully flat (see Morphisms of Spaces, Section 30). we conclude that  $(u_i)_{x_i}$  is injective, surjective, bijective, or zero if and only if  $(u_T)_x$  is injective, surjective, bijective, or zero. The lemma follows.  $\square$

**Lemma 7.3.** *In Situation 7.1 let  $X' \rightarrow X$  be a flat morphism of algebraic spaces. Denote  $u' : \mathcal{F}' \rightarrow \mathcal{G}'$  the pullback of  $u$  to  $X'$ . Denote  $F'_{iso}$ ,  $F'_{inj}$ ,  $F'_{surj}$ ,  $F'_{zero}$  the functors on  $Sch/B$  associated to  $u'$ .*

- (1) *If  $\mathcal{G}$  is of finite type and the image of  $|X'| \rightarrow |X|$  contains the support of  $\mathcal{G}$ , then  $F_{surj} = F'_{surj}$  and  $F_{zero} = F'_{zero}$ .*
- (2) *If  $\mathcal{F}$  is of finite type and the image of  $|X'| \rightarrow |X|$  contains the support of  $\mathcal{F}$ , then  $F_{inj} = F'_{inj}$  and  $F_{zero} = F'_{zero}$ .*
- (3) *If  $\mathcal{F}$  and  $\mathcal{G}$  are of finite type and the image of  $|X'| \rightarrow |X|$  contains the supports of  $\mathcal{F}$  and  $\mathcal{G}$ , then  $F_{iso} = F'_{iso}$ .*

**Proof.** let  $v : \mathcal{H} \rightarrow \mathcal{E}$  be a map of quasi-coherent modules on an algebraic space  $Y$  and let  $\varphi : Y' \rightarrow Y$  be a surjective flat morphism of algebraic spaces, then  $v$  is an isomorphism, injective, surjective, or zero if and only if  $\varphi^*v$  is an isomorphism, injective, surjective, or zero. Namely, for every  $y \in |Y|$  there exists a  $y' \in |Y'|$  and the map of local rings  $\mathcal{O}_{Y, \bar{y}} \rightarrow \mathcal{O}_{Y', \bar{y}'}$  is faithfully flat (see Morphisms of Spaces, Section 30). Of course, to check for injectivity or being zero it suffices to look at the points in the support of  $\mathcal{H}$ , and to check for surjectivity it suffices to look at points in the support of  $\mathcal{E}$ . Moreover, under the finite type assumptions as in the statement of the lemma, taking the supports commutes with base change, see Morphisms of Spaces, Lemma 15.2. Thus the lemma is clear.  $\square$

Recall that we've defined the scheme theoretic support of a finite type quasi-coherent module in Morphisms of Spaces, Definition 15.4.

**Lemma 7.4.** *In Situation 7.1.*

- (1) If  $\mathcal{G}$  is of finite type and the scheme theoretic support of  $\mathcal{G}$  is quasi-compact over  $B$ , then  $F_{\text{surj}}$  is limit preserving.
- (2) If  $\mathcal{F}$  of finite type and the scheme theoretic support of  $\mathcal{F}$  is quasi-compact over  $B$ , then  $F_{\text{zero}}$  is limit preserving.
- (3) If  $\mathcal{F}$  is of finite type,  $\mathcal{G}$  is of finite presentation, and the scheme theoretic supports of  $\mathcal{F}$  and  $\mathcal{G}$  are quasi-compact over  $B$ , then  $F_{\text{iso}}$  is limit preserving.

**Proof.** Proof of (1). Let  $i : Z \rightarrow X$  be the scheme theoretic support of  $\mathcal{G}$  and think of  $\mathcal{G}$  as a finite type quasi-coherent module on  $Z$ . We may replace  $X$  by  $Z$  and  $u$  by the map  $i^*\mathcal{F} \rightarrow \mathcal{G}$  (details omitted). Hence we may assume  $f$  is quasi-compact and  $\mathcal{G}$  of finite type. Let  $T = \lim_{i \in I} T_i$  be a directed limit of affine  $B$ -schemes and assume that  $u_T$  is surjective. Set  $X_i = X_{T_i} = X \times_S T_i$  and  $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \rightarrow \mathcal{G}_i = \mathcal{G}_{T_i}$ . To prove (1) we have to show that  $u_i$  is surjective for some  $i$ . Pick  $0 \in I$  and replace  $I$  by  $\{i \mid i \geq 0\}$ . Since  $f$  is quasi-compact we see  $X_0$  is quasi-compact. Hence we may choose a surjective étale morphism  $\varphi_0 : W_0 \rightarrow X_0$  where  $W_0$  is an affine scheme. Set  $W = W_0 \times_{T_0} T$  and  $W_i = W_0 \times_{T_0} T_i$  for  $i \geq 0$ . These are affine schemes endowed with a surjective étale morphisms  $\varphi : W \rightarrow X_T$  and  $\varphi_i : W_i \rightarrow X_i$ . Note that  $W = \lim W_i$ . Hence  $\varphi^*u_T$  is surjective and it suffices to prove that  $\varphi_i^*u_i$  is surjective for some  $i$ . Thus we have reduced the problem to the affine case which is Algebra, Lemma 127.5 part (2).

Proof of (2). Assume  $\mathcal{F}$  is of finite type with scheme theoretic support  $Z \subset B$  quasi-compact over  $B$ . Let  $T = \lim_{i \in I} T_i$  be a directed limit of affine  $B$ -schemes and assume that  $u_T$  is zero. Set  $X_i = T_i \times_B X$  and denote  $u_i : \mathcal{F}_i \rightarrow \mathcal{G}_i$  the pullback. Choose  $0 \in I$  and replace  $I$  by  $\{i \mid i \geq 0\}$ . Set  $Z_0 = Z \times_X X_0$ . By Morphisms of Spaces, Lemma 15.2 the support of  $\mathcal{F}_i$  is  $|Z_0|$ . Since  $|Z_0|$  is quasi-compact we can find an affine scheme  $W_0$  and an étale morphism  $W_0 \rightarrow X_0$  such that  $|Z_0| \subset \text{Im}(|W_0| \rightarrow |X_0|)$ . Set  $W = W_0 \times_{T_0} T$  and  $W_i = W_0 \times_{T_0} T_i$  for  $i \geq 0$ . These are affine schemes endowed with étale morphisms  $\varphi : W \rightarrow X_T$  and  $\varphi_i : W_i \rightarrow X_i$ . Note that  $W = \lim W_i$  and that the support of  $\mathcal{F}_T$  and  $\mathcal{F}_i$  is contained in the image of  $|W| \rightarrow |X_T|$  and  $|W_i| \rightarrow |X_i|$ . Now  $\varphi^*u_T$  is injective and it suffices to prove that  $\varphi_i^*u_i$  is injective for some  $i$ . Thus we have reduced the problem to the affine case which is Algebra, Lemma 127.5 part (1).

Proof of (3). This can be proven in exactly the same manner as in the previous two paragraphs using Algebra, Lemma 127.5 part (3). We can also deduce it from (1) and (2) as follows. Let  $T = \lim_{i \in I} T_i$  be a directed limit of affine  $B$ -schemes and assume that  $u_T$  is an isomorphism. By part (1) there exists an  $0 \in I$  such that  $u_{T_0}$  is surjective. Set  $\mathcal{K} = \text{Ker}(u_{T_0})$  and consider the map of quasi-coherent modules  $v : \mathcal{K} \rightarrow \mathcal{F}_{T_0}$ . For  $i \geq 0$  the base change  $v_{T_i}$  is zero if and only if  $u_i$  is an isomorphism. Moreover,  $v_T$  is zero. Since  $\mathcal{G}_{T_0}$  is of finite presentation,  $\mathcal{F}_{T_0}$  is of finite type, and  $u_{T_0}$  is surjective we conclude that  $\mathcal{K}$  is of finite type (Modules on Sites, Lemma 24.1). It is clear that the support of  $\mathcal{K}$  is contained in the support of  $\mathcal{F}_{T_0}$  which is quasi-compact over  $T_0$ . Hence we can apply part (2) to see that  $v_{T_i}$  is zero for some  $i$ .  $\square$

**Lemma 7.5.** *In Situation 7.1 suppose given an exact sequence*

$$\mathcal{F} \xrightarrow{u} \mathcal{G} \xrightarrow{v} \mathcal{H} \rightarrow 0$$

*Then we have  $F_{v, \text{iso}} = F_{u, \text{zero}}$  with obvious notation.*

**Proof.** Since pullback is right exact we see that  $\mathcal{F}_T \rightarrow \mathcal{G}_T \rightarrow \mathcal{H}_T \rightarrow 0$  is exact for every scheme  $T$  over  $B$ . Hence  $u_T$  is surjective if and only if  $v_T$  is an isomorphism.  $\square$

**Lemma 7.6.** *In Situation 7.1 suppose given an affine morphism  $i : Z \rightarrow X$  and a quasi-coherent  $\mathcal{O}_Z$ -module  $\mathcal{H}$  such that  $\mathcal{G} = i_*\mathcal{H}$ . Let  $v : i^*\mathcal{F} \rightarrow \mathcal{H}$  be the map adjoint to  $u$ . Then*

- (1)  $F_{v,zero} = F_{u,zero}$ , and
- (2) *if  $i$  is a closed immersion, then  $F_{v,surj} = F_{u,surj}$ .*

**Proof.** Let  $T$  be a scheme over  $B$ . Denote  $i_T : Z_T \rightarrow X_T$  the base change of  $i$  and  $\mathcal{H}_T$  the pullback of  $\mathcal{H}$  to  $Z_T$ . Observe that  $(i^*\mathcal{F})_T = i_T^*\mathcal{F}_T$  and  $i_{T,*}\mathcal{H}_T = (i_*\mathcal{H})_T$ . The first statement follows from commutativity of pullbacks and the second from Cohomology of Spaces, Lemma 11.1. Hence we see that  $u_T$  and  $v_T$  are adjoint maps as well. Thus  $u_T = 0$  if and only if  $v_T = 0$ . This proves (1). In case (2) we see that  $u_T$  is surjective if and only if  $v_T$  is surjective because  $u_T$  factors as

$$\mathcal{F}_T \rightarrow i_{T,*}i_T^*\mathcal{F}_T \xrightarrow{i_{T,*}v_T} i_{T,*}\mathcal{H}_T$$

and the fact that  $i_{T,*}$  is an exact functor fully faithfully embedding the category of quasi-coherent modules on  $Z_T$  into the category of quasi-coherent  $\mathcal{O}_{X_T}$ -modules. See Morphisms of Spaces, Lemma 14.1.  $\square$

**Lemma 7.7.** *In Situation 7.1 suppose given an affine morphism  $g : X \rightarrow X'$ . Set  $u' = f_*u : f_*\mathcal{F} \rightarrow f_*\mathcal{G}$ . Then  $F_{u,iso} = F_{u',iso}$ ,  $F_{u,inj} = F_{u',inj}$ ,  $F_{u,surj} = F_{u',surj}$ , and  $F_{u,zero} = F_{u',zero}$ .*

**Proof.** By Cohomology of Spaces, Lemma 11.1 we have  $g_{T,*}u_T = u'_T$ . Moreover,  $g_{T,*} : QCoh(\mathcal{O}_{X_T}) \rightarrow QCoh(\mathcal{O}_{X'})$  is a faithful, exact functor reflecting isomorphisms, injective maps, and surjective maps.  $\square$

**Situation 7.8.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module. For any scheme  $T$  over  $Y$  we will denote  $\mathcal{F}_T$  the base change of  $\mathcal{F}$  to  $T$ , in other words,  $\mathcal{F}_T$  is the pullback of  $\mathcal{F}$  via the projection morphism  $X_T = X \times_Y T \rightarrow X$ . Since the base change of a flat module is flat we obtain a functor

$$(7.8.1) \quad F_{flat} : (Sch/Y)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T, \\ \emptyset & \text{else.} \end{cases}$$

In Situation 7.8 we sometimes think of  $F_{flat}$  as a functor  $(Sch/S)^{opp} \rightarrow Sets$  endowed with a morphism  $F_{flat} \rightarrow Y$ . Namely, if  $T$  is a scheme over  $S$ , then an element  $h \in F_{flat}(T)$  is a morphism  $h : T \rightarrow Y$  such that the base change of  $\mathcal{F}$  via  $h$  is flat over  $T$ . In particular, when we say that  $F_{flat}$  is an algebraic space, we mean that the corresponding functor  $(Sch/S)^{opp} \rightarrow Sets$  is an algebraic space.

**Lemma 7.9.** *In Situation 7.8.*

- (1) *The functor  $F_{flat}$  satisfies the sheaf property for the fpqc topology.*
- (2) *If  $f$  is quasi-compact and locally of finite presentation and  $\mathcal{F}$  is of finite presentation, then the functor  $F_{flat}$  is limit preserving.*

**Proof.** Part (1) follows from the following statement: If  $T' \rightarrow T$  is a surjective flat morphism of algebraic spaces over  $Y$ , then  $\mathcal{F}_{T'}$  is flat over  $T'$  if and only if  $\mathcal{F}_T$  is flat over  $T$ , see Morphisms of Spaces, Lemma 31.3. Part (2) follows from Limits of

Spaces, Lemma 6.12 if  $f$  is also quasi-separated (i.e.,  $f$  is of finite presentation). For the general case, first reduce to the case where the base is affine and then cover  $X$  by finitely many affines to reduce to the quasi-separated case. Details omitted.  $\square$

## 8. Making a map zero

This section has no analogue in the corresponding chapter on schemes.

**Situation 8.1.** Let  $S = \operatorname{Spec}(R)$  be an affine scheme. Let  $X$  be an algebraic space over  $S$ . Let  $u : \mathcal{F} \rightarrow \mathcal{G}$  be a map of quasi-coherent  $\mathcal{O}_X$ -modules. Assume  $\mathcal{G}$  flat over  $S$ .

**Lemma 8.2.** *In Situation 8.1. Let  $T \rightarrow S$  be a quasi-compact morphism of schemes such that the base change  $u_T$  is zero. Then exists a closed subscheme  $Z \subset S$  such that (a)  $T \rightarrow S$  factors through  $Z$  and (b) the base change  $u_Z$  is zero. If  $\mathcal{F}$  is a finite type  $\mathcal{O}_X$ -module and the scheme theoretic support of  $\mathcal{F}$  is quasi-compact, then we can take  $Z \rightarrow S$  of finite presentation.*

**Proof.** Let  $U \rightarrow X$  be a surjective étale morphism of algebraic spaces where  $U = \coprod U_i$  is a disjoint union of affine schemes (see Properties of Spaces, Lemma 6.1). By Lemma 7.3 we see that we may replace  $X$  by  $U$ . In other words, we may assume that  $X = \coprod X_i$  is a disjoint union of affine schemes  $X_i$ . Suppose that we can prove the lemma for  $u_i = u|_{X_i}$ . Then we find a closed subscheme  $Z_i \subset S$  such that  $T \rightarrow S$  factors through  $Z_i$  and  $u_{i,Z_i}$  is zero. If  $Z_i = \operatorname{Spec}(R/I_i) \subset \operatorname{Spec}(R) = S$ , then taking  $Z = \operatorname{Spec}(R/\sum I_i)$  works. Thus we may assume that  $X = \operatorname{Spec}(A)$  is affine.

Choose a finite affine open covering  $T = T_1 \cup \dots \cup T_m$ . It is clear that we may replace  $T$  by  $\coprod_{j=1, \dots, m} T_j$ . Hence we may assume  $T$  is affine. Say  $T = \operatorname{Spec}(R')$ . Let  $u : M \rightarrow N$  be the homomorphisms of  $A$ -modules corresponding to  $u : \mathcal{F} \rightarrow \mathcal{G}$ . Then  $N$  is a flat  $R$ -module as  $\mathcal{G}$  is flat over  $S$ . The assumption of the lemma means that the composition

$$M \otimes_R R' \rightarrow N \otimes_R R'$$

is zero. Let  $z \in M$ . By Lazard's theorem (Algebra, Theorem 81.4) and the fact that  $\otimes$  commutes with colimits we can find free  $R$ -module  $F_z$ , an element  $\tilde{z} \in F_z$ , and a map  $F_z \rightarrow N$  such that  $u(z)$  is the image of  $\tilde{z}$  and  $\tilde{z}$  maps to zero in  $F_z \otimes_R R'$ . Choose a basis  $\{e_{z,\alpha}\}$  of  $F_z$  and write  $\tilde{z} = \sum f_{z,\alpha} e_{z,\alpha}$  with  $f_{z,\alpha} \in R$ . Let  $I \subset R$  be the ideal generated by the elements  $f_{z,\alpha}$  with  $z$  ranging over all elements of  $M$ . By construction  $I$  maps to zero in  $R'$  and the elements  $\tilde{z}$  map to zero in  $F_z/IF_z$  whence in  $N/IN$ . Thus  $Z = \operatorname{Spec}(R/I)$  is a solution to the problem in this case.

Assume  $\mathcal{F}$  is of finite type with quasi-compact scheme theoretic support. Write  $Z = \operatorname{Spec}(R/I)$ . Write  $I = \bigcup I_\lambda$  as a filtered union of finitely generated ideals. Set  $Z_\lambda = \operatorname{Spec}(R/I_\lambda)$ , so  $Z = \operatorname{colim} Z_\lambda$ . Since  $u_Z$  is zero, we see that  $u_{Z_\lambda}$  is zero for some  $\lambda$  by Lemma 7.4. This finishes the proof of the lemma.  $\square$

**Lemma 8.3.** *Let  $A$  be a ring. Let  $u : M \rightarrow N$  be a map of  $A$ -modules. If  $N$  is projective as an  $A$ -module, then there exists an ideal  $I \subset A$  such that for any ring map  $\varphi : A \rightarrow B$  the following are equivalent*

- (1)  $u \otimes 1 : M \otimes_A B \rightarrow N \otimes_A B$  is zero, and
- (2)  $\varphi(I) = 0$ .

**Proof.** As  $N$  is projective we can find a projective  $A$ -module  $C$  such that  $F = N \oplus C$  is a free  $R$ -module. By replacing  $u$  by  $u \oplus 1 : F = M \oplus C \rightarrow N \oplus C$  we see that we may assume  $N$  is free. In this case let  $I$  be the ideal of  $A$  generated by coefficients of all the elements of  $\text{Im}(u)$  with respect to some (fixed) basis of  $N$ .  $\square$

**Lemma 8.4.** *In Situation 8.1. Let  $T \subset S$  be a subset. Let  $s \in S$  be in the closure of  $T$ . For  $t \in T$ , let  $u_t$  be the pullback of  $u$  to  $X_t$  and let  $u_s$  be the pullback of  $u$  to  $X_s$ . If  $X$  is locally of finite presentation over  $S$ ,  $\mathcal{G}$  is of finite presentation<sup>3</sup>, and  $u_t = 0$  for all  $t \in T$ , then  $u_s = 0$ .*

**Proof.** To check whether  $u_s$  is zero, is étale local on the fibre  $X_s$ . Hence we may pick a point  $x \in |X_s| \subset |X|$  and check in an étale neighbourhood. Choose

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (S', s') \end{array}$$

as in Proposition 5.1. Let  $T' \subset S'$  be the inverse image of  $T$ . Observe that  $s'$  is in the closure of  $T'$  because  $S' \rightarrow S$  is open. Hence we reduce to the algebra problem described in the next paragraph.

We have an  $R$ -module map  $u : M \rightarrow N$  such that  $N$  is projective as an  $R$ -module and such that  $u_t : M \otimes_R \kappa(t) \rightarrow N \otimes_R \kappa(t)$  is zero for each  $t \in T$ . Problem: show that  $u_s = 0$ . Let  $I \subset R$  be the ideal defined in Lemma 8.3. Then  $I$  maps to zero in  $\kappa(t)$  for all  $t \in T$ . Hence  $T \subset V(I)$ . Since  $s$  is in the closure of  $T$  we see that  $s \in V(I)$ . Hence  $u_s = 0$ .  $\square$

It would be interesting to find a “simple” direct proof of either Lemma 8.5 or Lemma 8.6 using arguments like those used in Lemmas 8.2 and 8.4. A “classical” proof of this lemma when  $f : X \rightarrow B$  is a projective morphism and  $B$  a Noetherian scheme would be: (a) choose a relatively ample invertible sheaf  $\mathcal{O}_X(1)$ , (b) set  $u_n : f_*\mathcal{F}(n) \rightarrow f_*\mathcal{G}(n)$ , (c) observe that  $f_*\mathcal{G}(n)$  is a finite locally free sheaf for all  $n \gg 0$ , and (d)  $F_{\text{zero}}$  is represented by the vanishing locus of  $u_n$  for some  $n \gg 0$ .

**Lemma 8.5.** *In Situation 7.1. Assume*

- (1)  *$f$  is of finite presentation, and*
- (2)  *$\mathcal{G}$  is of finite presentation, flat over  $B$ , and pure relative to  $B$ .*

*Then  $F_{\text{zero}}$  is an algebraic space and  $F_{\text{zero}} \rightarrow B$  is a closed immersion. If  $\mathcal{F}$  is of finite type, then  $F_{\text{zero}} \rightarrow B$  is of finite presentation.*

**Proof.** By Lemma 6.5 the module  $\mathcal{G}$  is universally pure relative to  $B$ . In order to prove that  $F_{\text{zero}}$  is an algebraic space, it suffices to show that  $F_{\text{zero}} \rightarrow B$  is representable, see Spaces, Lemma 11.3. Let  $B' \rightarrow B$  be a morphism where  $B'$  is a scheme and let  $u' : \mathcal{F}' \rightarrow \mathcal{G}'$  be the pullback of  $u$  to  $X' = X_{B'}$ . Then the associated functor  $F'_{\text{zero}}$  equals  $F_{\text{zero}} \times_B B'$ . This reduces us to the case that  $B$  is a scheme.

Assume  $B$  is a scheme. We will show that  $F_{\text{zero}}$  is representable by a closed subscheme of  $B$ . By Lemma 7.2 and Descent, Lemmas 37.2 and 39.1 the question

<sup>3</sup>It would suffice if  $X$  is locally of finite type over  $S$  and  $\mathcal{G}$  is finitely presented relative to  $S$ , but this notion hasn't yet been defined in the setting of algebraic spaces. The definition for schemes is given in More on Morphisms, Section 58.

is local for the étale topology on  $B$ . Let  $b \in B$ . We first replace  $B$  by an affine neighbourhood of  $b$ . Choose a diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ B & \xleftarrow{\quad} & B' \end{array}$$

and  $b' \in B'$  mapping to  $b \in B$  as in Lemma 5.2. As we are working étale locally, we may replace  $B$  by  $B'$  and assume that we have a diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow & \swarrow \\ & B & \end{array}$$

with  $B$  and  $X'$  affine such that  $\Gamma(X', g^*\mathcal{G})$  is a projective  $\Gamma(B, \mathcal{O}_B)$ -module and  $g(|X'|) \supset |X_b|$ . Let  $U \subset X$  be the open subspace with  $|U| = g(|X'|)$ . By Divisors on Spaces, Lemma 4.10 the set

$$E = \{t \in B : \text{Ass}_{X_t}(\mathcal{G}_t) \subset |U_t|\} = \{t \in B : \text{Ass}_{X/B}(\mathcal{G}) \cap |X_t| \subset |U_t|\}$$

is constructible in  $B$ . By Lemma 6.3 part (2) we see that  $E$  contains  $\text{Spec}(\mathcal{O}_{B,b})$ . By Morphisms, Lemma 22.4 we see that  $E$  contains an open neighbourhood of  $b$ . Hence after replacing  $B$  by a smaller affine neighbourhood of  $b$  we may assume that  $\text{Ass}_{X/B}(\mathcal{G}) \subset g(|X'|)$ .

From Lemma 6.6 it follows that  $u : \mathcal{F} \rightarrow \mathcal{G}$  is injective if and only if  $g^*u : g^*\mathcal{F} \rightarrow g^*\mathcal{G}$  is injective, and the same remains true after any base change. Hence we have reduced to the case where, in addition to the assumptions in the theorem,  $X \rightarrow B$  is a morphism of affine schemes and  $\Gamma(X, \mathcal{G})$  is a projective  $\Gamma(B, \mathcal{O}_B)$ -module. This case follows immediately from Lemma 8.3.

We still have to show that  $F_{\text{zero}} \rightarrow B$  is of finite presentation if  $\mathcal{F}$  is of finite type. This follows from Lemma 7.4 combined with Limits of Spaces, Proposition 3.10.  $\square$

**Lemma 8.6.** *In Situation 7.1. Assume*

- (1)  *$f$  is locally of finite presentation,*
- (2)  *$\mathcal{G}$  is an  $\mathcal{O}_X$ -module of finite presentation flat over  $B$ ,*
- (3) *the support of  $\mathcal{G}$  is proper over  $B$ .*

*Then the functor  $F_{\text{zero}}$  is an algebraic space and  $F_{\text{zero}} \rightarrow B$  is a closed immersion. If  $\mathcal{F}$  is of finite type, then  $F_{\text{zero}} \rightarrow B$  is of finite presentation.*

**Proof.** If  $f$  is of finite presentation, then this follows immediately from Lemmas 8.5 and 3.6. This is the only case of interest and we urge the reader to skip the rest of the proof, which deals with the possibility (allowed by the assumptions in this lemma) that  $f$  is not quasi-separated or quasi-compact.

Let  $i : Z \rightarrow X$  be the closed subspace cut out by the zeroth fitting ideal of  $\mathcal{G}$  (Divisors on Spaces, Section 5). Then  $Z \rightarrow B$  is proper by assumption (see Derived Categories of Spaces, Section 7). On the other hand  $i$  is of finite presentation (Divisors on Spaces, Lemma 5.2 and Morphisms of Spaces, Lemma 28.12). There exists a quasi-coherent  $\mathcal{O}_Z$ -module  $\mathcal{H}$  of finite type with  $i_*\mathcal{H} = \mathcal{G}$  (Divisors on Spaces, Lemma 5.3). In fact  $\mathcal{H}$  is of finite presentation as an  $\mathcal{O}_Z$ -module by Algebra,



Lemma 6.4 (details omitted). Then  $F_{zero}$  is the same as the functor  $F_{zero}$  for the map  $i^*\mathcal{F} \rightarrow \mathcal{H}$  adjoint to  $u$ , see Lemma 7.6. The sheaf  $\mathcal{H}$  is flat relative to  $B$  because the same is true for  $\mathcal{G}$  (check on stalks; details omitted). Moreover, note that if  $\mathcal{F}$  is of finite type, then  $i^*\mathcal{F}$  is of finite type. Hence we have reduced the lemma to the case discussed in the first paragraph of the proof.  $\square$

## 9. Flattening a map

This section is the analogue of More on Flatness, Section 23. In particular the following result is a variant of More on Flatness, Theorem 23.3.

**Theorem 9.1.** *In Situation 7.1 assume*

- (1)  *$f$  is of finite presentation,*
- (2)  *$\mathcal{F}$  is of finite presentation, flat over  $B$ , and pure relative to  $B$ , and*
- (3)  *$u$  is surjective.*

*Then  $F_{iso}$  is representable by a closed immersion  $Z \rightarrow B$ . Moreover  $Z \rightarrow S$  is of finite presentation if  $\mathcal{G}$  is of finite presentation.*

**Proof.** Let  $\mathcal{K} = \text{Ker}(u)$  and denote  $v : \mathcal{K} \rightarrow \mathcal{F}$  the inclusion. By Lemma 7.5 we see that  $F_{u,iso} = F_{v,zero}$ . By Lemma 8.5 applied to  $v$  we see that  $F_{u,iso} = F_{v,zero}$  is representable by a closed subspace of  $B$ . Note that  $\mathcal{K}$  is of finite type if  $\mathcal{G}$  is of finite presentation, see Modules on Sites, Lemma 24.1. Hence we also obtain the final statement of the lemma.  $\square$

**Lemma 9.2.** *In Situation 7.1. Assume*

- (1)  *$f$  is locally of finite presentation,*
- (2)  *$\mathcal{F}$  is locally of finite presentation and flat over  $B$ ,*
- (3) *the support of  $\mathcal{F}$  is proper over  $B$ , and*
- (4)  *$u$  is surjective.*

*Then the functor  $F_{iso}$  is an algebraic space and  $F_{iso} \rightarrow B$  is a closed immersion. If  $\mathcal{G}$  is of finite presentation, then  $F_{iso} \rightarrow B$  is of finite presentation.*

**Proof.** Let  $\mathcal{K} = \text{Ker}(u)$  and denote  $v : \mathcal{K} \rightarrow \mathcal{F}$  the inclusion. By Lemma 7.5 we see that  $F_{u,iso} = F_{v,zero}$ . By Lemma 8.6 applied to  $v$  we see that  $F_{u,iso} = F_{v,zero}$  is representable by a closed subspace of  $B$ . Note that  $\mathcal{K}$  is of finite type if  $\mathcal{G}$  is of finite presentation, see Modules on Sites, Lemma 24.1. Hence we also obtain the final statement of the lemma.  $\square$

We will use the following (easy) result when discussing the Quot functor.

**Lemma 9.3.** *In Situation 7.1. Assume*

- (1)  *$f$  is locally of finite presentation,*
- (2)  *$\mathcal{G}$  is of finite type,*
- (3) *the support of  $\mathcal{G}$  is proper over  $B$ .*

*Then  $F_{surj}$  is an algebraic space and  $F_{surj} \rightarrow B$  is an open immersion.*

**Proof.** Consider  $\text{Coker}(u)$ . Observe that  $\text{Coker}(u_T) = \text{Coker}(u)_T$  for any  $T/B$ . Note that formation of the support of a finite type quasi-coherent module commutes with pullback (Morphisms of Spaces, Lemma 15.1). Hence  $F_{surj}$  is representable by the open subspace of  $B$  corresponding to the open set

$$|B| \setminus |f|(\text{Supp}(\text{Coker}(u)))$$

see Properties of Spaces, Lemma 4.8. This is an open because  $|f|$  is closed on  $\text{Supp}(\mathcal{G})$  and  $\text{Supp}(\text{Coker}(u))$  is a closed subset of  $\text{Supp}(\mathcal{G})$ .  $\square$

### 10. Flattening in the local case

This section is the analogue of More on Flatness, Section 24.

**Lemma 10.1.** *Let  $S$  be the spectrum of a henselian local ring with closed point  $s$ . Let  $X \rightarrow S$  be a morphism of algebraic spaces which is locally of finite type. Let  $\mathcal{F}$  be a finite type quasi-coherent  $\mathcal{O}_X$ -module. Let  $E \subset |X_s|$  be a subset. There exists a closed subscheme  $Z \subset S$  with the following property: for any morphism of pointed schemes  $(T, t) \rightarrow (S, s)$  the following are equivalent*

- (1)  $\mathcal{F}_T$  is flat over  $T$  at all points of  $|X_t|$  which map to a point of  $E \subset |X_s|$ ,  
and
- (2)  $\text{Spec}(\mathcal{O}_{T,t}) \rightarrow S$  factors through  $Z$ .

Moreover, if  $X \rightarrow S$  is locally of finite presentation,  $\mathcal{F}$  is of finite presentation, and  $E \subset |X_s|$  is closed and quasi-compact, then  $Z \rightarrow S$  is of finite presentation.

**Proof.** Choose a scheme  $U$  and an étale morphism  $\varphi : U \rightarrow X$ . Let  $E' \subset |U_s|$  be the inverse image of  $E$ . If  $E' \rightarrow E$  is surjective, then condition (1) is equivalent to:  $(\varphi^*\mathcal{F})_T$  is flat over  $T$  at all points of  $|U_t|$  which map to a point of  $E' \subset |U_t|$ . Choosing  $\varphi$  to be surjective, we reduced to the case of schemes which is More on Flatness, Lemma 24.3. If  $E$  is closed and quasi-compact, then we may choose  $U$  to be affine such that  $E' \rightarrow E$  is surjective. Then  $E'$  is closed and quasi-compact and the final statement follows from the final statement of More on Flatness, Lemma 24.3.  $\square$

### 11. Universal flattening

This section is the analogue of More on Flatness, Section 27. Our main aim is to prove Lemma 11.8. However, we do not see a way to deduce this result from the corresponding result for schemes directly. Hence we have to redevelop some of the material here. But first a definition.

**Definition 11.1.** Let  $S$  be a scheme. Let  $X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module. We say that the *universal flattening of  $\mathcal{F}$  exists* if the functor  $F_{flat}$  defined in Situation 7.8 is an algebraic space. We say that the *universal flattening of  $X$  exists* if the universal flattening of  $\mathcal{O}_X$  exists.

This is a bit unsatisfactory, because here the definition of universal flattening does not agree with the one used in the case of schemes, as we don't know whether every monomorphism of algebraic spaces is representable (More on Morphisms of Spaces, Section 4). Hopefully no confusion will ever result from this.

**Lemma 11.2.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces which is locally of finite type. Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module of finite type. Let  $n \geq 0$ . The following are equivalent*

(1) for some commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \varphi \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with surjective, étale vertical arrows where  $U$  and  $V$  are schemes, the sheaf  $\varphi^*\mathcal{F}$  is flat over  $V$  in dimensions  $\geq n$  (More on Flatness, Definition 20.10),

(2) for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \varphi \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with étale vertical arrows where  $U$  and  $V$  are schemes, the sheaf  $\varphi^*\mathcal{F}$  is flat over  $V$  in dimensions  $\geq n$ , and

(3) for  $x \in |X|$  such that  $\mathcal{F}$  is not flat at  $x$  over  $Y$  the transcendence degree of  $x/f(x)$  is  $< n$  (Morphisms of Spaces, Definition 33.1).

If this is true, then it remains true after any base change  $Y' \rightarrow Y$ .

**Proof.** Suppose that we have a diagram as in (1). Then the equivalence of the conditions in More on Flatness, Lemma 20.9 shows that (1) and (3) are equivalent. But condition (3) is inherited by  $\varphi^*\mathcal{F}$  for any  $U \rightarrow V$  as in (2). Whence we see that (3) implies (2) by the result for schemes again. The result for schemes also implies the statement on base change.  $\square$

**Definition 11.3.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module of finite type. Let  $n \geq 0$ . We say  $\mathcal{F}$  is flat over  $Y$  in dimensions  $\geq n$  if the equivalent conditions of Lemma 11.2 are satisfied.

**Situation 11.4.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module of finite type. For any scheme  $T$  over  $Y$  we will denote  $\mathcal{F}_T$  the base change of  $\mathcal{F}$  to  $T$ , in other words,  $\mathcal{F}_T$  is the pullback of  $\mathcal{F}$  via the projection morphism  $X_T = X \times_Y T \rightarrow X$ . Note that  $f_T : X_T \rightarrow T$  is of finite type and that  $\mathcal{F}_T$  is an  $\mathcal{O}_{X_T}$ -module of finite type (Morphisms of Spaces, Lemma 23.3 and Modules on Sites, Lemma 23.4). Let  $n \geq 0$ . By Definition 11.3 and Lemma 11.2 we obtain a functor

(11.4.1)

$$F_n : (Sch/Y)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T \text{ in } \dim \geq n, \\ \emptyset & \text{else.} \end{cases}$$

In Situation 11.4 we sometimes think of  $F_n$  as a functor  $(Sch/S)^{opp} \rightarrow Sets$  endowed with a morphism  $F_n \rightarrow Y$ . Namely, if  $T$  is a scheme over  $S$ , then an element  $h \in F_n(T)$  is a morphism  $h : T \rightarrow Y$  such that the base change of  $\mathcal{F}$  via  $h$  is flat over  $T$  in  $\dim \geq n$ . In particular, when we say that  $F_n$  is an algebraic space, we mean that the corresponding functor  $(Sch/S)^{opp} \rightarrow Sets$  is an algebraic space.

**Lemma 11.5.** In Situation 11.4.

(1) The functor  $F_n$  satisfies the sheaf property for the fpqc topology.

- (2) *If  $f$  is quasi-compact and locally of finite presentation and  $\mathcal{F}$  is of finite presentation, then the functor  $F_n$  is limit preserving.*

**Proof.** Proof of (1). Suppose that  $\{T_i \rightarrow T\}$  is an fpqc covering of a scheme  $T$  over  $Y$ . We have to show that if  $F_n(T_i)$  is nonempty for all  $i$ , then  $F_n(T)$  is nonempty. Choose a diagram as in part (1) of Lemma 11.2. Denote  $F'_n$  the corresponding functor for  $\varphi^*\mathcal{F}$  and the morphism  $U \rightarrow V$ . By More on Flatness, Lemma 20.12 we have the sheaf property for  $F'_n$ . Thus we get the sheaf property for  $F_n$  because for  $T \rightarrow Y$  we have  $F_n(T) = F'_n(V \times_Y T)$  by Lemma 11.2 and because  $\{V \times_Y T_i \rightarrow V \times_Y T\}$  is an fpqc covering.

Proof of (2). Suppose that  $T = \lim_{i \in I} T_i$  is a filtered limit of affine schemes  $T_i$  over  $Y$  and assume that  $F_n(T)$  is nonempty. We have to show that  $F_n(T_i)$  is nonempty for some  $i$ . Choose a diagram as in part (1) of Lemma 11.2. Fix  $i \in I$  and choose an affine open  $W_i \subset V \times_Y T_i$  mapping surjectively onto  $T_i$ . For  $i' \geq i$  let  $W_{i'}$  be the inverse image of  $W_i$  in  $V \times_Y T_{i'}$  and let  $W \subset V \times_Y T$  be the inverse image of  $W_i$ . Then  $W = \lim_{i' \geq i} W_{i'}$  is a filtered limit of affine schemes over  $V$ . By Lemma 11.2 again it suffices to show that  $F'_n(W_{i'})$  is nonempty for some  $i' \geq i$ . But we know that  $F'_n(W)$  is nonempty because of our assumption that  $F_n(T) = F'_n(V \times_Y T)$  is nonempty. Thus we can apply More on Flatness, Lemma 20.12 to conclude.  $\square$

**Lemma 11.6.** *In Situation 11.4. Let  $h : X' \rightarrow X$  be an étale morphism. Set  $\mathcal{F}' = h^*\mathcal{F}$  and  $f' = f \circ h$ . Let  $F'_n$  be (11.4.1) associated to  $(f' : X' \rightarrow Y, \mathcal{F}')$ . Then  $F_n$  is a subfunctor of  $F'_n$  and if  $h(X') \supset \text{Ass}_{X/Y}(\mathcal{F})$ , then  $F_n = F'_n$ .*

**Proof.** Choose  $U \rightarrow X$ ,  $V \rightarrow Y$ ,  $U \rightarrow V$  as in part (1) of Lemma 11.2. Choose a surjective étale morphism  $U' \rightarrow U \times_X X'$  where  $U'$  is a scheme. Then we have the lemma for the two functors  $F_{U,n}$  and  $F_{U',n}$  determined by  $U' \rightarrow U$  and  $\mathcal{F}|_U$  over  $V$ , see More on Flatness, Lemma 27.2. On the other hand, Lemma 11.2 tells us that given  $T \rightarrow Y$  we have  $F_n(T) = F_{U,n}(V \times_Y T)$  and  $F'_n(T) = F_{U',n}(V \times_Y T)$ . This proves the lemma.  $\square$

**Theorem 11.7.** *In Situation 11.4. Assume moreover that  $f$  is of finite presentation, that  $\mathcal{F}$  is an  $\mathcal{O}_X$ -module of finite presentation, and that  $\mathcal{F}$  is pure relative to  $Y$ . Then  $F_n$  is an algebraic space and  $F_n \rightarrow Y$  is a monomorphism of finite presentation.*

**Proof.** The functor  $F_n$  is a sheaf for the fppf topology by Lemma 11.5. Since  $F_n \rightarrow Y$  is a monomorphism of sheaves on  $(\text{Sch}/S)_{\text{fppf}}$  we see that  $\Delta : F_n \rightarrow F_n \times F_n$  is the pullback of the diagonal  $\Delta_Y : Y \rightarrow Y \times_S Y$ . Hence the representability of  $\Delta_Y$  implies the same thing for  $F_n$ . Therefore it suffices to prove that there exists a scheme  $W$  over  $S$  and a surjective étale morphism  $W \rightarrow F_n$ .

To construct  $W \rightarrow F_n$  choose an étale covering  $\{Y_i \rightarrow Y\}$  with  $Y_i$  a scheme. Let  $X_i = X \times_Y Y_i$  and let  $\mathcal{F}_i$  be the pullback of  $\mathcal{F}$  to  $X_i$ . Then  $\mathcal{F}_i$  is pure relative to  $Y_i$  either by definition or by Lemma 3.3. The other assumptions of the theorem are preserved as well. Finally, the restriction of  $F_n$  to  $Y_i$  is the functor  $F_n$  corresponding to  $X_i \rightarrow Y_i$  and  $\mathcal{F}_i$ . Hence it suffices to show: Given  $\mathcal{F}$  and  $f : X \rightarrow Y$  as in the statement of the theorem where  $Y$  is a scheme, the functor  $F_n$  is representable by a scheme  $Z_n$  and  $Z_n \rightarrow Y$  is a monomorphism of finite presentation.

Observe that a monomorphism of finite presentation is separated and quasi-finite (Morphisms, Lemma 20.15). Hence combining Descent, Lemma 39.1, More on

Morphisms, Lemma 57.1, and Descent, Lemmas 23.31 and 23.13 we see that the question is local for the étale topology on  $Y$ .

In particular the situation is local for the Zariski topology on  $Y$  and we may assume that  $Y$  is affine. In this case the dimension of the fibres of  $f$  is bounded above, hence we see that  $F_n$  is representable for  $n$  large enough. Thus we may use descending induction on  $n$ . Suppose that we know  $F_{n+1}$  is representable by a monomorphism  $Z_{n+1} \rightarrow Y$  of finite presentation. Consider the base change  $X_{n+1} = Z_{n+1} \times_Y X$  and the pullback  $\mathcal{F}_{n+1}$  of  $\mathcal{F}$  to  $X_{n+1}$ . The morphism  $Z_{n+1} \rightarrow Y$  is quasi-finite as it is a monomorphism of finite presentation, hence Lemma 3.3 implies that  $\mathcal{F}_{n+1}$  is pure relative to  $Z_{n+1}$ . Since  $F_n$  is a subfunctor of  $F_{n+1}$  we conclude that in order to prove the result for  $F_n$  it suffices to prove the result for the corresponding functor for the situation  $\mathcal{F}_{n+1}/X_{n+1}/Z_{n+1}$ . In this way we reduce to proving the result for  $F_n$  in case  $Y_{n+1} = Y$ , i.e., we may assume that  $\mathcal{F}$  is flat in dimensions  $\geq n+1$  over  $Y$ .

Fix  $n$  and assume  $\mathcal{F}$  is flat in dimensions  $\geq n+1$  over the affine scheme  $Y$ . To finish the proof we have to show that  $F_n$  is representable by a monomorphism  $Z_n \rightarrow S$  of finite presentation. Since the question is local in the étale topology on  $Y$  it suffices to show that for every  $y \in Y$  there exists an étale neighbourhood  $(Y', y') \rightarrow (Y, y)$  such that the result holds after base change to  $Y'$ . Thus by Lemma 4.1 we may assume there exist étale morphisms  $h_j : W_j \rightarrow X$ ,  $j = 1, \dots, m$  such that for each  $j$  there exists a complete dévissage of  $\mathcal{F}_j/W_j/Y$  over  $y$ , where  $\mathcal{F}_j$  is the pullback of  $\mathcal{F}$  to  $W_j$  and such that  $|X_y| \subset \bigcup h_j(W_j)$ . Since  $h_j$  is étale, by Lemma 11.2 the sheaves  $\mathcal{F}_j$  are still flat over in dimensions  $\geq n+1$  over  $Y$ . Set  $W = \bigcup h_j(W_j)$ , which is a quasi-compact open of  $X$ . As  $\mathcal{F}$  is pure along  $X_y$  we see that

$$E = \{t \in |Y| : \text{Ass}_{X_t}(\mathcal{F}_t) \subset W\}.$$

contains all generalizations of  $y$ . By Divisors on Spaces, Lemma 4.10  $E$  is a constructible subset of  $Y$ . We have seen that  $\text{Spec}(\mathcal{O}_{Y,y}) \subset E$ . By Morphisms, Lemma 22.4 we see that  $E$  contains an open neighbourhood of  $y$ . Hence after shrinking  $Y$  we may assume that  $E = Y$ . It follows from Lemma 11.6 that it suffices to prove the lemma for the functor  $F_n$  associated to  $X = \coprod W_j$  and  $\mathcal{F} = \coprod \mathcal{F}_j$ . If  $F_{j,n}$  denotes the functor for  $W_j \rightarrow Y$  and the sheaf  $\mathcal{F}_j$  we see that  $F_n = \coprod F_{j,n}$ . Hence it suffices to prove each  $F_{j,n}$  is representable by some monomorphism  $Z_{j,n} \rightarrow Y$  of finite presentation, since then

$$Z_n = Z_{1,n} \times_Y \dots \times_Y Z_{m,n}$$

Thus we have reduced the theorem to the special case handled in More on Flatness, Lemma 27.4.  $\square$

Thus we finally obtain the desired result.

**Lemma 11.8.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module.*

- (1) *If  $f$  is of finite presentation,  $\mathcal{F}$  is an  $\mathcal{O}_X$ -module of finite presentation, and  $\mathcal{F}$  is pure relative to  $Y$ , then there exists a universal flattening  $Y' \rightarrow Y$  of  $\mathcal{F}$ . Moreover  $Y' \rightarrow Y$  is a monomorphism of finite presentation.*
- (2) *If  $f$  is of finite presentation and  $X$  is pure relative to  $Y$ , then there exists a universal flattening  $Y' \rightarrow Y$  of  $X$ . Moreover  $Y' \rightarrow Y$  is a monomorphism of finite presentation.*

- (3) If  $f$  is proper and of finite presentation and  $\mathcal{F}$  is an  $\mathcal{O}_X$ -module of finite presentation, then there exists a universal flattening  $Y' \rightarrow Y$  of  $\mathcal{F}$ . Moreover  $Y' \rightarrow Y$  is a monomorphism of finite presentation.
- (4) If  $f$  is proper and of finite presentation then there exists a universal flattening  $Y' \rightarrow Y$  of  $X$ .

**Proof.** These statements follow immediately from Theorem 11.7 applied to  $F_0 = F_{flat}$  and the fact that if  $f$  is proper then  $\mathcal{F}$  is automatically pure over the base, see Lemma 3.6.  $\square$

## 12. Grothendieck's Existence Theorem

This section is the analogue of More on Flatness, Section 28 and continues the discussion in More on Morphisms of Spaces, Section 42. We will work in the following situation.

**Situation 12.1.** Here we have an inverse system of rings  $(A_n)$  with surjective transition maps whose kernels are locally nilpotent. Set  $A = \lim A_n$ . We have an algebraic space  $X$  separated and of finite presentation over  $A$ . We set  $X_n = X \times_{\text{Spec}(A)} \text{Spec}(A_n)$  and we view it as a closed subspace of  $X$ . We assume further given a system  $(\mathcal{F}_n, \varphi_n)$  where  $\mathcal{F}_n$  is a finitely presented  $\mathcal{O}_{X_n}$ -module, flat over  $A_n$ , with support proper over  $A_n$ , and

$$\varphi_n : \mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \longrightarrow \mathcal{F}_{n-1}$$

is an isomorphism (notation using the equivalence of Morphisms of Spaces, Lemma 14.1).

Our goal is to see if we can find a quasi-coherent sheaf  $\mathcal{F}$  on  $X$  such that  $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$  for all  $n$ .

**Lemma 12.2.** *In Situation 12.1 consider*

$$K = R \lim_{D_{QCoh}(\mathcal{O}_X)} (\mathcal{F}_n) = DQ_X(R \lim_{D(\mathcal{O}_X)} \mathcal{F}_n)$$

*Then  $K$  is in  $D_{QCoh}^b(\mathcal{O}_X)$  and in fact  $K$  has nonzero cohomology sheaves only in degrees  $\geq 0$ .*

**Proof.** Special case of Derived Categories of Spaces, Example 19.5.  $\square$

**Lemma 12.3.** *In Situation 12.1 let  $K$  be as in Lemma 12.2. For any perfect object  $E$  of  $D(\mathcal{O}_X)$  we have*

- (1)  $M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$  is a perfect object of  $D(A)$  and there is a canonical isomorphism  $R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes_A^{\mathbf{L}} A_n$  in  $D(A_n)$ ,
- (2)  $N = R\text{Hom}_X(E, K)$  is a perfect object of  $D(A)$  and there is a canonical isomorphism  $R\text{Hom}_{X_n}(E|_{X_n}, \mathcal{F}_n) = N \otimes_A^{\mathbf{L}} A_n$  in  $D(A_n)$ .

*In both statements  $E|_{X_n}$  denotes the derived pullback of  $E$  to  $X_n$ .*

**Proof.** Proof of (2). Write  $E_n = E|_{X_n}$  and  $N_n = R\text{Hom}_{X_n}(E_n, \mathcal{F}_n)$ . Recall that  $R\text{Hom}_{X_n}(-, -)$  is equal to  $R\Gamma(X_n, R\text{Hom}(-, -))$ , see Cohomology on Sites, Section 36. Hence by Derived Categories of Spaces, Lemma 25.8 we see that  $N_n$  is a perfect object of  $D(A_n)$  whose formation commutes with base change. Thus the maps  $N_n \otimes_{A_n}^{\mathbf{L}} A_{n-1} \rightarrow N_{n-1}$  coming from  $\varphi_n$  are isomorphisms. By More on Algebra, Lemma 97.3 we find that  $R \lim N_n$  is perfect and that its base change back to  $A_n$  recovers  $N_n$ . On the other hand, the exact functor  $R\text{Hom}_X(E, -) :$

$D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D(A)$  of triangulated categories commutes with products and hence with derived limits, whence

$$R\text{Hom}_X(E, K) = R\lim R\text{Hom}_X(E, \mathcal{F}_n) = R\lim R\text{Hom}_X(E_n, \mathcal{F}_n) = R\lim N_n$$

This proves (2). To see that (1) holds, translate it into (2) using Cohomology on Sites, Lemma 48.4.  $\square$

**Lemma 12.4.** *In Situation 12.1 let  $K$  be as in Lemma 12.2. Then  $K$  is pseudo-coherent relative to  $A$ .*

**Proof.** Combining Lemma 12.3 and Derived Categories of Spaces, Lemma 25.7 we see that  $R\Gamma(X, K \otimes^{\mathbf{L}} E)$  is pseudo-coherent in  $D(A)$  for all pseudo-coherent  $E$  in  $D(\mathcal{O}_X)$ . Thus the lemma follows from More on Morphisms of Spaces, Lemma 51.4.  $\square$

**Lemma 12.5.** *In Situation 12.1 let  $K$  be as in Lemma 12.2. For any étale morphism  $U \rightarrow X$  with  $U$  quasi-compact and quasi-separated we have*

$$R\Gamma(U, K) \otimes_A^{\mathbf{L}} A_n = R\Gamma(U_n, \mathcal{F}_n)$$

in  $D(A_n)$  where  $U_n = U \times_X X_n$ .

**Proof.** Fix  $n$ . By Derived Categories of Spaces, Lemma 27.3 there exists a system of perfect complexes  $E_m$  on  $X$  such that  $R\Gamma(U, K) = \text{hocolim}_m R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$ . In fact, this formula holds not just for  $K$  but for every object of  $D_{Q\text{Coh}}(\mathcal{O}_X)$ . Applying this to  $\mathcal{F}_n$  we obtain

$$\begin{aligned} R\Gamma(U_n, \mathcal{F}_n) &= R\Gamma(U, \mathcal{F}_n) \\ &= \text{hocolim}_m R\Gamma(X, \mathcal{F}_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 12.3 and the fact that  $-\otimes_A^{\mathbf{L}} A_n$  commutes with homotopy colimits we obtain the result.  $\square$

**Lemma 12.6.** *In Situation 12.1 let  $K$  be as in Lemma 12.2. Denote  $X_0 \subset |X|$  the closed subset consisting of points lying over the closed subset  $\text{Spec}(A_1) = \text{Spec}(A_2) = \dots$  of  $\text{Spec}(A)$ . There exists an open subspace  $W \subset X$  containing  $X_0$  such that*

- (1)  $H^i(K)|_W$  is zero unless  $i = 0$ ,
- (2)  $\mathcal{F} = H^0(K)|_W$  is of finite presentation, and
- (3)  $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ .

**Proof.** Fix  $n \geq 1$ . By construction there is a canonical map  $K \rightarrow \mathcal{F}_n$  in  $D_{Q\text{Coh}}(\mathcal{O}_X)$  and hence a canonical map  $H^0(K) \rightarrow \mathcal{F}_n$  of quasi-coherent sheaves. This explains the meaning of part (3).

Let  $x \in X_0$  be a point. We will find an open neighbourhood  $W$  of  $x$  such that (1), (2), and (3) are true. Since  $X_0$  is quasi-compact this will prove the lemma. Let  $U \rightarrow X$  be an étale morphism with  $U$  affine and  $u \in U$  a point mapping to  $x$ . Since  $|U| \rightarrow |X|$  is open it suffices to find an open neighbourhood of  $u$  in  $U$  where (1), (2), and (3) are true. Say  $U = \text{Spec}(B)$ . Choose a surjection  $P \rightarrow B$  with  $P$  smooth over  $A$ . By Lemma 12.4 and the definition of relative pseudo-coherence there exists a bounded above complex  $F^\bullet$  of finite free  $P$ -modules representing  $Ri_* K$  where

$i : U \rightarrow \operatorname{Spec}(P)$  is the closed immersion induced by the presentation. Let  $M_n$  be the  $B$ -module corresponding to  $\mathcal{F}_n|_U$ . By Lemma 12.5

$$H^i(F^\bullet \otimes_A A_n) = \begin{cases} 0 & \text{if } i \neq 0 \\ M_n & \text{if } i = 0 \end{cases}$$

Let  $i$  be the maximal index such that  $F^i$  is nonzero. If  $i \leq 0$ , then (1), (2), and (3) are true. If not, then  $i > 0$  and we see that the rank of the map

$$F^{i-1} \rightarrow F^i$$

in the point  $u$  is maximal. Hence in an open neighbourhood of  $u$  inside  $\operatorname{Spec}(P)$  the rank is maximal. Thus after replacing  $P$  by a principal localization we may assume that the displayed map is surjective. Since  $F^i$  is finite free we may choose a splitting  $F^{i-1} = F' \oplus F^i$ . Then we may replace  $F^\bullet$  by the complex

$$\dots \rightarrow F^{i-2} \rightarrow F' \rightarrow 0 \rightarrow \dots$$

and we win by induction on  $i$ .  $\square$

**Lemma 12.7.** *In Situation 12.1 let  $K$  be as in Lemma 12.2. Let  $W \subset X$  be as in Lemma 12.6. Set  $\mathcal{F} = H^0(K)|_W$ . Then, after possibly shrinking the open  $W$ , the support of  $\mathcal{F}$  is proper over  $A$ .*

**Proof.** Fix  $n \geq 1$ . Let  $I_n = \operatorname{Ker}(A \rightarrow A_n)$ . By More on Algebra, Lemma 11.3 the pair  $(A, I_n)$  is henselian. Let  $Z \subset W$  be the scheme theoretic support of  $\mathcal{F}$ . This is a closed subspace as  $\mathcal{F}$  is of finite presentation. By part (3) of Lemma 12.6 we see that  $Z \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A_n)$  is equal to the support of  $\mathcal{F}_n$  and hence proper over  $\operatorname{Spec}(A/I)$ . By More on Morphisms of Spaces, Lemma 36.10 we can write  $Z = Z_1 \amalg Z_2$  with  $Z_1, Z_2$  open and closed in  $Z$ , with  $Z_1$  proper over  $A$ , and with  $Z_1 \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A/I_n)$  equal to the support of  $\mathcal{F}_n$ . In other words,  $|Z_2|$  does not meet  $X_0$ . Hence after replacing  $W$  by  $W \setminus Z_2$  we obtain the lemma.  $\square$

**Theorem 12.8** (Grothendieck Existence Theorem). *In Situation 12.1 there exists a finitely presented  $\mathcal{O}_X$ -module  $\mathcal{F}$ , flat over  $A$ , with support proper over  $A$ , such that  $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$  for all  $n$  compatibly with the maps  $\varphi_n$ .*

**Proof.** Apply Lemmas 12.2, 12.3, 12.4, 12.5, 12.6, and 12.7 to get an open subspace  $W \subset X$  containing all points lying over  $\operatorname{Spec}(A_n)$  and a finitely presented  $\mathcal{O}_W$ -module  $\mathcal{F}$  whose support is proper over  $A$  with  $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{O}_{X_n}$  for all  $n \geq 1$ . (This makes sense as  $X_n \subset W$ .) By Lemma 3.6 we see that  $\mathcal{F}$  is universally pure relative to  $\operatorname{Spec}(A)$ . By Theorem 11.7 (for explanation, see Lemma 11.8) there exists a universal flattening  $S' \rightarrow \operatorname{Spec}(A)$  of  $\mathcal{F}$  and moreover the morphism  $S' \rightarrow \operatorname{Spec}(A)$  is a monomorphism of finite presentation. In particular  $S'$  is a scheme (this follows from the proof of the theorem but it also follows a posteriori by Morphisms of Spaces, Proposition 50.2). Since the base change of  $\mathcal{F}$  to  $\operatorname{Spec}(A_n)$  is  $\mathcal{F}_n$  we find that  $\operatorname{Spec}(A_n) \rightarrow \operatorname{Spec}(A)$  factors (uniquely) through  $S'$  for each  $n$ . By More on Flatness, Lemma 28.8 we see that  $S' = \operatorname{Spec}(A)$ . This means that  $\mathcal{F}$  is flat over  $A$ . Finally, since the scheme theoretic support  $Z$  of  $\mathcal{F}$  is proper over  $\operatorname{Spec}(A)$ , the morphism  $Z \rightarrow X$  is closed. Hence the pushforward  $(W \rightarrow X)_* \mathcal{F}$  is supported on  $W$  and has all the desired properties.  $\square$



### 13. Grothendieck's Existence Theorem, bis

In this section we prove an analogue for Grothendieck's existence theorem in the derived category, following the method used in Section 12 for quasi-coherent modules. This section is the analogue of More on Flatness, Section 29 for algebraic spaces. The classical case (for algebraic spaces) is discussed in More on Morphisms of Spaces, Section 42. We will work in the following situation.

**Situation 13.1.** Here we have an inverse system of rings  $(A_n)$  with surjective transition maps whose kernels are locally nilpotent. Set  $A = \lim A_n$ . We have an algebraic space  $X$  proper, flat, and of finite presentation over  $A$ . We set  $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$  and we view it as a closed subspace of  $X$ . We assume further given a system  $(K_n, \varphi_n)$  where  $K_n$  is a pseudo-coherent object of  $D(\mathcal{O}_{X_n})$  and

$$\varphi_n : K_n \longrightarrow K_{n-1}$$

is a map in  $D(\mathcal{O}_{X_n})$  which induces an isomorphism  $K_n \otimes_{\mathcal{O}_{X_n}}^{\mathbf{L}} \mathcal{O}_{X_{n-1}} \rightarrow K_{n-1}$  in  $D(\mathcal{O}_{X_{n-1}})$ .

More precisely, we should write  $\varphi_n : K_n \rightarrow Ri_{n-1,*}K_{n-1}$  where  $i_{n-1} : X_{n-1} \rightarrow X_n$  is the inclusion morphism and in this notation the condition is that the adjoint map  $Li_{n-1}^*K_n \rightarrow K_{n-1}$  is an isomorphism. Our goal is to find a pseudo-coherent  $K \in D(\mathcal{O}_X)$  such that  $K_n = K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_{X_n}$  for all  $n$  (with the same abuse of notation).

**Lemma 13.2.** *In Situation 13.1 consider*

$$K = R\lim_{D_{Qcoh}(\mathcal{O}_X)}(K_n) = DQ_X(R\lim_{D(\mathcal{O}_X)} K_n)$$

*Then  $K$  is in  $D_{Qcoh}^-(\mathcal{O}_X)$ .*

**Proof.** The functor  $DQ_X$  exists because  $X$  is quasi-compact and quasi-separated, see Derived Categories of Spaces, Lemma 19.1. Since  $DQ_X$  is a right adjoint it commutes with products and therefore with derived limits. Hence the equality in the statement of the lemma.

By Derived Categories of Spaces, Lemma 19.4 the functor  $DQ_X$  has bounded cohomological dimension. Hence it suffices to show that  $R\lim K_n \in D^-(\mathcal{O}_X)$ . To see this, let  $U \rightarrow X$  be étale with  $U$  affine. Then there is a canonical exact sequence

$$0 \rightarrow R^1 \lim H^{m-1}(U, K_n) \rightarrow H^m(U, R\lim K_n) \rightarrow \lim H^m(U, K_n) \rightarrow 0$$

by Cohomology on Sites, Lemma 23.2. Since  $U$  is affine and  $K_n$  is pseudo-coherent (and hence has quasi-coherent cohomology sheaves by Derived Categories of Spaces, Lemma 13.6) we see that  $H^m(U, K_n) = H^m(K_n)(U)$  by Derived Categories of Schemes, Lemma 3.5. Thus we conclude that it suffices to show that  $K_n$  is bounded above independent of  $n$ .

Since  $K_n$  is pseudo-coherent we have  $K_n \in D^-(\mathcal{O}_{X_n})$ . Suppose that  $a_n$  is maximal such that  $H^{a_n}(K_n)$  is nonzero. Of course  $a_1 \leq a_2 \leq a_3 \leq \dots$ . Note that  $H^{a_n}(K_n)$  is an  $\mathcal{O}_{X_n}$ -module of finite presentation (Cohomology on Sites, Lemma 45.7). We have  $H^{a_n}(K_{n-1}) = H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$ . Since  $X_{n-1} \rightarrow X_n$  is a thickening, it follows from Nakayama's lemma (Algebra, Lemma 20.1) that if  $H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$  is zero, then  $H^{a_n}(K_n)$  is zero too (argue by checking on stalks for example; small detail omitted). Thus  $a_{n-1} = a_n$  for all  $n$  and we conclude.  $\square$

**Lemma 13.3.** *In Situation 13.1 let  $K$  be as in Lemma 13.2. For any perfect object  $E$  of  $D(\mathcal{O}_X)$  the cohomology*

$$M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$$

*is a pseudo-coherent object of  $D(A)$  and there is a canonical isomorphism*

$$R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes_A^{\mathbf{L}} A_n$$

*in  $D(A_n)$ . Here  $E|_{X_n}$  denotes the derived pullback of  $E$  to  $X_n$ .*

**Proof.** Write  $E_n = E|_{X_n}$  and  $M_n = R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n})$ . By Derived Categories of Spaces, Lemma 25.5 we see that  $M_n$  is a pseudo-coherent object of  $D(A_n)$  whose formation commutes with base change. Thus the maps  $M_n \otimes_{A_n}^{\mathbf{L}} A_{n-1} \rightarrow M_{n-1}$  coming from  $\varphi_n$  are isomorphisms. By More on Algebra, Lemma 97.1 we find that  $R\lim M_n$  is pseudo-coherent and that its base change back to  $A_n$  recovers  $M_n$ . On the other hand, the exact functor  $R\Gamma(X, -) : D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D(A)$  of triangulated categories commutes with products and hence with derived limits, whence

$$R\Gamma(X, E \otimes^{\mathbf{L}} K) = R\lim R\Gamma(X, E \otimes^{\mathbf{L}} K_n) = R\lim R\Gamma(X_n, E_n \otimes^{\mathbf{L}} K_n) = R\lim M_n$$

as desired.  $\square$

**Lemma 13.4.** *In Situation 13.1 let  $K$  be as in Lemma 13.2. Then  $K$  is pseudo-coherent on  $X$ .*

**Proof.** Combining Lemma 13.3 and Derived Categories of Spaces, Lemma 25.7 we see that  $R\Gamma(X, K \otimes^{\mathbf{L}} E)$  is pseudo-coherent in  $D(A)$  for all pseudo-coherent  $E$  in  $D(\mathcal{O}_X)$ . Thus it follows from More on Morphisms of Spaces, Lemma 51.4 that  $K$  is pseudo-coherent relative to  $A$ . Since  $X$  is of flat and of finite presentation over  $A$ , this is the same as being pseudo-coherent on  $X$ , see More on Morphisms of Spaces, Lemma 45.4.  $\square$

**Lemma 13.5.** *In Situation 13.1 let  $K$  be as in Lemma 13.2. For any étale morphism  $U \rightarrow X$  with  $U$  quasi-compact and quasi-separated we have*

$$R\Gamma(U, K) \otimes_A^{\mathbf{L}} A_n = R\Gamma(U_n, K_n)$$

*in  $D(A_n)$  where  $U_n = U \times_X X_n$ .*

**Proof.** Fix  $n$ . By Derived Categories of Spaces, Lemma 27.3 there exists a system of perfect complexes  $E_m$  on  $X$  such that  $R\Gamma(U, K) = \text{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$ . In fact, this formula holds not just for  $K$  but for every object of  $D_{Q\text{Coh}}(\mathcal{O}_X)$ . Applying this to  $K_n$  we obtain

$$\begin{aligned} R\Gamma(U_n, K_n) &= R\Gamma(U, K_n) \\ &= \text{hocolim}_m R\Gamma(X, K_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 13.3 and the fact that  $-\otimes_A^{\mathbf{L}} A_n$  commutes with homotopy colimits we obtain the result.  $\square$

**Theorem 13.6** (Derived Grothendieck Existence Theorem). *In Situation 13.1 there exists a pseudo-coherent  $K$  in  $D(\mathcal{O}_X)$  such that  $K_n = K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_{X_n}$  for all  $n$  compatibly with the maps  $\varphi_n$ .*

**Proof.** Apply Lemmas 13.2, 13.3, 13.4 to get a pseudo-coherent object  $K$  of  $D(\mathcal{O}_X)$ . Choosing affine  $U$  in Lemma 13.5 it follows immediately that  $K$  restricts to  $K_n$  over  $X_n$ .  $\square$

**Remark 13.7.** The result in this section can be generalized. It is probably correct if we only assume  $X \rightarrow \operatorname{Spec}(A)$  to be separated, of finite presentation, and  $K_n$  pseudo-coherent relative to  $A_n$  supported on a closed subset of  $X_n$  proper over  $A_n$ . The outcome will be a  $K$  which is pseudo-coherent relative to  $A$  supported on a closed subset proper over  $A$ . If we ever need this, we will formulate a precise statement and prove it here.

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## References

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