

OBSOLETE

Contents

1. Introduction	1
2. Preliminaries	1
3. Homological algebra	2
4. Obsolete algebra lemmas	2
5. Lemmas related to ZMT	7
6. Formally smooth ring maps	9
7. Sites and sheaves	10
8. Cohomology	11
9. Differential graded algebra	15
10. Simplicial methods	15
11. Results on schemes	16
12. Derived categories of varieties	18
13. Representability in the regular proper case	20
14. Functor of quotients	21
15. Spaces and fpqc coverings	21
16. Very reasonable algebraic spaces	22
17. Obsolete lemmas on algebraic spaces	23
18. Obsolete lemmas on algebraic stacks	23
19. Variants of cotangent complexes for schemes	24
20. Deformations and obstructions of flat modules	26
21. The stack of coherent sheaves in the non-flat case	28
22. Modifications	31
23. Intersection theory	32
23.4. Blowing up lemmas	33
24. Commutativity of intersecting divisors	39
25. Dualizing modules on regular proper models	43
26. Duplicate and split out references	44
27. Other chapters	46
References	47

1. Introduction

In this chapter we put some lemmas that have become “obsolete” (see [Mil17]).

2. Preliminaries

Remark 2.1. The information which used to be contained in this remark is now subsumed in the combination of Categories, Lemmas 24.4 and 24.3.

3. Homological algebra

Remark 3.1. The following remarks are obsolete as they are subsumed in Homology, Lemmas 24.11 and 25.3. Let \mathcal{A} be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a weak Serre subcategory (see Homology, Definition 10.1). Suppose that $K^{\bullet, \bullet}$ is a double complex to which Homology, Lemma 25.3 applies such that for some $r \geq 0$ all the objects $'E_r^{p, q}$ belong to \mathcal{C} . Then all the cohomology groups $H^n(sK^\bullet)$ belong to \mathcal{C} . Namely, the assumptions imply that the kernels and images of $'d_r^{p, q}$ are in \mathcal{C} . Whereupon we see that each $'E_{r+1}^{p, q}$ is in \mathcal{C} . By induction we see that each $'E_\infty^{p, q}$ is in \mathcal{C} . Hence each $H^n(sK^\bullet)$ has a finite filtration whose subquotients are in \mathcal{C} . Using that \mathcal{C} is closed under extensions we conclude that $H^n(sK^\bullet)$ is in \mathcal{C} as claimed. The same result holds for the second spectral sequence associated to $K^{\bullet, \bullet}$. Similarly, if (K^\bullet, F) is a filtered complex to which Homology, Lemma 24.11 applies and for some $r \geq 0$ all the objects $E_r^{p, q}$ belong to \mathcal{C} , then each $H^n(K^\bullet)$ is an object of \mathcal{C} .

4. Obsolete algebra lemmas

Lemma 4.1. *Let M be an R -module of finite presentation. For any surjection $\alpha : R^{\oplus n} \rightarrow M$ the kernel of α is a finite R -module.*

Proof. This is a special case of Algebra, Lemma 5.3. □

Lemma 4.2. *Let $\varphi : R \rightarrow S$ be a ring map. If*

- (1) *for any $x \in S$ there exists $n > 0$ such that x^n is in the image of φ , and*
- (2) *for any $x \in \text{Ker}(\varphi)$ there exists $n > 0$ such that $x^n = 0$,*

then φ induces a homeomorphism on spectra. Given a prime number p such that

- (a) *S is generated as an R -algebra by elements x such that there exists an $n > 0$ with $x^{p^n} \in \varphi(R)$ and $p^n x \in \varphi(R)$, and*
- (b) *the kernel of φ is generated by nilpotent elements,*

then (1) and (2) hold, and for any ring map $R \rightarrow R'$ the ring map $R' \rightarrow R' \otimes_R S$ also satisfies (a), (b), (1), and (2) and in particular induces a homeomorphism on spectra.

Proof. This is a combination of Algebra, Lemmas 46.3 and 46.7. □

The following technical lemma says that you can lift any sequence of relations from a fibre to the whole space of a ring map which is essentially of finite type, in a suitable sense.

Lemma 4.3. *Let $R \rightarrow S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Let $\mathfrak{q} \subset S$ be a prime lying over \mathfrak{p} . Assume $S_{\mathfrak{q}}$ is essentially of finite type over $R_{\mathfrak{p}}$. Assume given*

- (1) *an integer $n \geq 0$,*
- (2) *a prime $\mathfrak{a} \subset \kappa(\mathfrak{p})[x_1, \dots, x_n]$,*
- (3) *a surjective $\kappa(\mathfrak{p})$ -homomorphism*

$$\psi : (\kappa(\mathfrak{p})[x_1, \dots, x_n])_{\mathfrak{a}} \longrightarrow S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}},$$

and

- (4) *elements $\bar{f}_1, \dots, \bar{f}_e$ in $\text{Ker}(\psi)$.*

Then there exist

- (1) *an integer $m \geq 0$,*
- (2) *and element $g \in S$, $g \notin \mathfrak{q}$,*

(3) a map

$$\Psi : R[x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}] \longrightarrow S_g,$$

and

(4) elements $f_1, \dots, f_e, f_{e+1}, \dots, f_{e+m}$ of $\text{Ker}(\Psi)$

such that

(1) the following diagram commutes

$$\begin{array}{ccc} R[x_1, \dots, x_{n+m}] & \xrightarrow{x_{n+j} \mapsto 0} & (\kappa(\mathfrak{p})[x_1, \dots, x_n])_{\mathfrak{a}}, \\ \Psi \downarrow & & \downarrow \psi \\ S_g & \xrightarrow{\quad} & S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \end{array}$$

(2) the element f_i , $i \leq n$ maps to a unit times \bar{f}_i in the local ring

$$(\kappa(\mathfrak{p})[x_1, \dots, x_{n+m}])_{(\mathfrak{a}, x_{n+1}, \dots, x_{n+m})},$$

(3) the element f_{e+j} maps to a unit times x_{n+j} in the same local ring, and

(4) the induced map $R[x_1, \dots, x_{n+m}]_{\mathfrak{b}} \rightarrow S_{\mathfrak{q}}$ is surjective, where $\mathfrak{b} = \Psi^{-1}(\mathfrak{q}S_g)$.

Proof. We claim that it suffices to prove the lemma in case R and S are local with maximal ideals \mathfrak{p} and \mathfrak{q} . Namely, suppose we have constructed

$$\Psi' : R_{\mathfrak{p}}[x_1, \dots, x_{n+m}] \longrightarrow S_{\mathfrak{q}}$$

and $f'_1, \dots, f'_{e+m} \in R_{\mathfrak{p}}[x_1, \dots, x_{n+m}]$ with all the required properties. Then there exists an element $f \in R$, $f \notin \mathfrak{p}$ such that each ff'_k comes from an element $f_k \in R[x_1, \dots, x_{n+m}]$. Moreover, for a suitable $g \in S$, $g \notin \mathfrak{q}$ the elements $\Psi'(x_i)$ are the image of elements $y_i \in S_g$. Let Ψ be the R -algebra map defined by the rule $\Psi(x_i) = y_i$. Since $\Psi(f_i)$ is zero in the localization $S_{\mathfrak{q}}$ we may after possibly replacing g assume that $\Psi(f_i) = 0$. This proves the claim.

Thus we may assume R and S are local with maximal ideals \mathfrak{p} and \mathfrak{q} . Pick $y_1, \dots, y_n \in S$ such that $y_i \bmod \mathfrak{p}S = \psi(x_i)$. Let $y_{n+1}, \dots, y_{n+m} \in S$ be elements which generate an R -subalgebra of which S is the localization. These exist by the assumption that S is essentially of finite type over R . Since ψ is surjective we may write $y_{n+j} \bmod \mathfrak{p}S = \psi(h_j)$ for some $h_j \in \kappa(\mathfrak{p})[x_1, \dots, x_n]_{\mathfrak{a}}$. Write $h_j = g_j/d$, $g_j \in \kappa(\mathfrak{p})[x_1, \dots, x_n]$ for some common denominator $d \in \kappa(\mathfrak{p})[x_1, \dots, x_n]$, $d \notin \mathfrak{a}$. Choose lifts $G_j, D \in R[x_1, \dots, x_n]$ of g_j and d . Set $y'_{n+j} = D(y_1, \dots, y_n)y_{n+j} - G_j(y_1, \dots, y_n)$. By construction $y'_{n+j} \in \mathfrak{p}S$. It is clear that $y_1, \dots, y_n, y'_n, \dots, y'_{n+m}$ generate an R -subalgebra of S whose localization is S . We define

$$\Psi : R[x_1, \dots, x_{n+m}] \rightarrow S$$

to be the map that sends x_i to y_i for $i = 1, \dots, n$ and x_{n+j} to y'_{n+j} for $j = 1, \dots, m$. Properties (1) and (4) are clear by construction. Moreover the ideal \mathfrak{b} maps onto the ideal $(\mathfrak{a}, x_{n+1}, \dots, x_{n+m})$ in the polynomial ring $\kappa(\mathfrak{p})[x_1, \dots, x_{n+m}]$.

Denote $J = \text{Ker}(\Psi)$. We have a short exact sequence

$$0 \rightarrow J_{\mathfrak{b}} \rightarrow R[x_1, \dots, x_{n+m}]_{\mathfrak{b}} \rightarrow S_{\mathfrak{q}} \rightarrow 0.$$

The surjectivity comes from our choice of $y_1, \dots, y_n, y'_n, \dots, y'_{n+m}$ above. This implies that

$$J_{\mathfrak{b}}/\mathfrak{p}J_{\mathfrak{b}} \rightarrow \kappa(\mathfrak{p})[x_1, \dots, x_{n+m}]_{(\mathfrak{a}, x_{n+1}, \dots, x_{n+m})} \rightarrow S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \rightarrow 0$$

is exact. By construction x_i maps to $\psi(x_i)$ and x_{n+j} maps to zero under the last map. Thus it is easy to choose f_i as in (2) and (3) of the lemma. \square

Remark 4.4 (Projective resolutions). Let R be a ring. For any set S we let $F(S)$ denote the free R -module on S . Then any left R -module has the following two step resolution

$$F(M \times M) \oplus F(R \times M) \rightarrow F(M) \rightarrow M \rightarrow 0.$$

The first map is given by the rule

$$[m_1, m_2] \oplus [r, m] \mapsto [m_1 + m_2] - [m_1] - [m_2] + [rm] - r[m].$$

Lemma 4.5. *Let S be a multiplicative set of A . Then the map*

$$f : \operatorname{Spec}(S^{-1}A) \longrightarrow \operatorname{Spec}(A)$$

induced by the canonical ring map $A \rightarrow S^{-1}A$ is a homeomorphism onto its image and $\operatorname{Im}(f) = \{\mathfrak{p} \in \operatorname{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}$.

Proof. This is a duplicate of Algebra, Lemma 17.5. \square

Lemma 4.6. *Let $A \rightarrow B$ be a finite type, flat ring map with A an integral domain. Then B is a finitely presented A -algebra.*

Proof. Special case of More on Flatness, Proposition 13.10. \square

Lemma 4.7. *Let R be a domain with fraction field K . Let $S = R[x_1, \dots, x_n]$ be a polynomial ring over R . Let M be a finite S -module. Assume that M is flat over R . If for every subring $R \subset R' \subset K$, $R \neq R'$ the module $M \otimes_R R'$ is finitely presented over $S \otimes_R R'$, then M is finitely presented over S .*

Proof. This lemma is true because M is finitely presented even without the assumption that $M \otimes_R R'$ is finitely presented for every R' as in the statement of the lemma. This follows from More on Flatness, Proposition 13.10. Originally this lemma had an erroneous proof (thanks to Ofer Gabber for finding the gap) and was used in an alternative proof of the proposition cited. To reinstate this lemma, we need a correct argument in case R is a local normal domain using only results from the chapters on commutative algebra; please email stacks.project@gmail.com if you have an argument. \square

Lemma 4.8. *Let $A \rightarrow B$ be a ring map. Let $f \in B$. Assume that*

- (1) $A \rightarrow B$ is flat,
- (2) f is a nonzerodivisor, and
- (3) $A \rightarrow B/fB$ is flat.

Then for every ideal $I \subset A$ the map $f : B/IB \rightarrow B/IB$ is injective.

Proof. Note that $IB = I \otimes_A B$ and $I(B/fB) = I \otimes_A B/fB$ by the flatness of B and B/fB over A . In particular $IB/fIB \cong I \otimes_A B/fB$ maps injectively into B/fB . Hence the result follows from the snake lemma applied to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I \otimes_A B & \longrightarrow & B & \longrightarrow & B/IB \longrightarrow 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f \\ 0 & \longrightarrow & I \otimes_A B & \longrightarrow & B & \longrightarrow & B/IB \longrightarrow 0 \end{array}$$

with exact rows. \square

Lemma 4.9. *If $R \rightarrow S$ is a faithfully flat ring map then for every R -module M the map $M \rightarrow S \otimes_R M$, $x \mapsto 1 \otimes x$ is injective.*

Proof. This lemma is a duplicate of Algebra, Lemma 82.11. \square

Remark 4.10. This reference/tag used to refer to a Section in the chapter Smoothing Ring Maps, but the material has since been subsumed in Algebra, Section 127.

Lemma 4.11. *Let (R, \mathfrak{m}) be a reduced Noetherian local ring of dimension 1 and let $x \in \mathfrak{m}$ be a nonzerodivisor. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be the minimal primes of R . Then*

$$\text{length}_R(R/(x)) = \sum_i \text{ord}_{R/\mathfrak{q}_i}(x)$$

Proof. Special (very easy) case of Chow Homology, Lemma 3.2. \square

Lemma 4.12. *Let A be a Noetherian local normal domain of dimension 2. For $f \in \mathfrak{m}$ nonzero denote $\text{div}(f) = \sum n_i(\mathfrak{p}_i)$ the divisor associated to f on the punctured spectrum of A . We set $|f| = \sum n_i$. There exist integers N and M such that $|f + g| \leq M$ for all $g \in \mathfrak{m}^N$.*

Proof. Pick $h \in \mathfrak{m}$ such that f, h is a regular sequence in A (this follows from Algebra, Lemmas 157.4 and 72.7). We will prove the lemma with $M = \text{length}_A(A/(f, h))$ and with N any integer such that $\mathfrak{m}^N \subset (f, h)$. Such an integer N exists because $\sqrt{(f, h)} = \mathfrak{m}$. Note that $M = \text{length}_A(A/(f + g, h))$ for all $g \in \mathfrak{m}^N$ because $(f, h) = (f + g, h)$. This moreover implies that $f + g, h$ is a regular sequence in A too, see Algebra, Lemma 104.2. Now suppose that $\text{div}(f + g) = \sum m_j(\mathfrak{q}_j)$. Then consider the map

$$c : A/(f + g) \longrightarrow \prod A/\mathfrak{q}_j^{(m_j)}$$

where $\mathfrak{q}_j^{(m_j)}$ is the symbolic power, see Algebra, Section 64. Since A is normal, we see that $A_{\mathfrak{q}_i}$ is a discrete valuation ring and hence

$$A_{\mathfrak{q}_i}/(f + g) = A_{\mathfrak{q}_i}/\mathfrak{q}_i^{m_i} A_{\mathfrak{q}_i} = (A/\mathfrak{q}_i^{(m_i)})_{\mathfrak{q}_i}$$

Since $V(f + g, h) = \{\mathfrak{m}\}$ this implies that c becomes an isomorphism on inverting h (small detail omitted). Since h is a nonzerodivisor on $A/(f + g)$ we see that the length of $A/(f + g, h)$ equals the Herbrand quotient $e_A(A/(f + g), 0, h)$ as defined in Chow Homology, Section 2. Similarly the length of $A/(h, \mathfrak{q}_j^{(m_j)})$ equals $e_A(A/\mathfrak{q}_j^{(m_j)}, 0, h)$. Then we have

$$\begin{aligned} M &= \text{length}_A(A/(f + g, h)) \\ &= e_A(A/(f + g), 0, h) \\ &= \sum_i e_A(A/\mathfrak{q}_i^{(m_i)}, 0, h) \\ &= \sum_i \sum_{m=0, \dots, m_i-1} e_A(\mathfrak{q}_i^{(m)}/\mathfrak{q}_i^{(m+1)}, 0, h) \end{aligned}$$

The equalities follow from Chow Homology, Lemmas 2.3 and 2.4 using in particular that the cokernel of c has finite length as discussed above. It is straightforward to prove that $e_A(\mathfrak{q}_i^{(m)}/\mathfrak{q}_i^{(m+1)}, 0, h)$ is at least 1 by Nakayama's lemma. This finishes the proof of the lemma. \square

Lemma 4.13. *Let $A \rightarrow B$ be a flat local homomorphism of Noetherian local rings. If A and $B/\mathfrak{m}_A B$ are Gorenstein, then B is Gorenstein.*

Proof. Follows immediately from Dualizing Complexes, Lemma 21.8. \square

Lemma 4.14. *Let (A, \mathfrak{m}) be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A -module. Let s be an integer. Assume*

- (1) *A has a dualizing complex,*
- (2) *if $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$, then $\text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) > s$.*

Then there exists an $n > 0$ and an ideal $J \subset A$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$ such that JI^n annihilates $H_{\mathfrak{m}}^i(M)$ for $i \leq s$.

Proof. According to Local Cohomology, Lemma 9.4 we have to show this for the finite A -module $E^i = \text{Ext}_A^{-i}(M, \omega_A^\bullet)$ for $i \leq s$. The support Z of $E^0 \oplus \dots \oplus E^s$ is closed in $\text{Spec}(A)$ and does not contain any prime as in (2). Hence it is contained in $V(JI^n)$ for some J as in the statement of the lemma. \square

Lemma 4.15. *Let (A, \mathfrak{m}) be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A -module. Let s and d be integers. Assume*

- (a) *A has a dualizing complex,*
- (b) *$cd(A, I) \leq d$,*
- (c) *if $\mathfrak{p} \notin V(I)$ then $\text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) > s$ or $\text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) > d + s$.*

Then the assumptions of Algebraic and Formal Geometry, Lemma 10.4 hold for A, I, \mathfrak{m}, M and $H_{\mathfrak{m}}^i(M) \rightarrow \lim H_{\mathfrak{m}}^i(M/I^n M)$ is an isomorphism for $i \leq s$ and these modules are annihilated by a power of I .

Proof. The assumptions of Algebraic and Formal Geometry, Lemma 10.4 by the more general Algebraic and Formal Geometry, Lemma 10.5. Then the conclusion of Algebraic and Formal Geometry, Lemma 10.4 gives the second statement. \square

Lemma 4.16. *In Algebraic and Formal Geometry, Situation 10.1 we have $H_{\mathfrak{a}}^s(M) = \lim H_{\mathfrak{a}}^s(M/I^n M)$.*

Proof. This is immediate from Algebraic and Formal Geometry, Theorem 10.8. The original version of this lemma, which had additional assumptions, was superseded by the this theorem. \square

Lemma 4.17. *Let A be a Noetherian ring. Let $f \in \mathfrak{a}$ be an element of an ideal of A . Let $U = \text{Spec}(A) \setminus V(\mathfrak{a})$. Assume*

- (1) *A has a dualizing complex and is complete with respect to f ,*
- (2) *A_f is (S_2) and for every minimal prime $\mathfrak{p} \subset A$, $f \notin \mathfrak{p}$ and $\mathfrak{q} \in V(\mathfrak{p}) \cap V(\mathfrak{a})$ we have $\dim((A/\mathfrak{p})_{\mathfrak{q}}) \geq 3$.*

Then the completion functor

$$\text{Coh}(\mathcal{O}_U) \longrightarrow \text{Coh}(U, I\mathcal{O}_U), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects.

Proof. This lemma is a special case of Algebraic and Formal Geometry, Lemma 15.6. \square

5. Lemmas related to ZMT

The lemmas in this section were originally used in the proof of the (algebraic version of) Zariski's Main Theorem, Algebra, Theorem 123.12.

Lemma 5.1. *Let R be a ring and let $\varphi : R[x] \rightarrow S$ be a ring map. Let $t \in S$. If t is integral over $R[x]$, then there exists an $\ell \geq 0$ such that for every $a \in R$ the element $\varphi(a)^\ell t$ is integral over $\varphi_a : R[y] \rightarrow S$, defined by $y \mapsto \varphi(ax)$ and $r \mapsto \varphi(r)$ for $r \in R$.*

Proof. Say $t^d + \sum_{i < d} \varphi(f_i)t^i = 0$ with $f_i \in R[x]$. Let ℓ be the maximum degree in x of all the f_i . Multiply the equation by $\varphi(a)^\ell$ to get $\varphi(a)^\ell t^d + \sum_{i < d} \varphi(a^\ell f_i)t^i = 0$. Note that each $\varphi(a^\ell f_i)$ is in the image of φ_a . The result follows from Algebra, Lemma 123.1. \square

Lemma 5.2. *Let $\varphi : R \rightarrow S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \dots + \varphi(a_n)t^n = 0$. Set $u_n = \varphi(a_n)$, $u_{n-1} = u_n t + \varphi(a_{n-1})$, and so on till $u_1 = u_2 t + \varphi(a_1)$. Then all of u_n, u_{n-1}, \dots, u_1 and $u_n t, u_{n-1} t, \dots, u_1 t$ are integral over R , and the ideals $(\varphi(a_0), \dots, \varphi(a_n))$ and (u_n, \dots, u_1) of S are equal.*

Proof. We prove this by induction on n . As $u_n = \varphi(a_n)$ we conclude from Algebra, Lemma 123.1 that $u_n t$ is integral over R . Of course $u_n = \varphi(a_n)$ is integral over R . Then $u_{n-1} = u_n t + \varphi(a_{n-1})$ is integral over R (see Algebra, Lemma 36.7) and we have

$$\varphi(a_0) + \varphi(a_1)t + \dots + \varphi(a_{n-1})t^{n-1} + u_{n-1}t^{n-1} = 0.$$

Hence by the induction hypothesis applied to the map $S' \rightarrow S$ where S' is the integral closure of R in S and the displayed equation we see that u_{n-1}, \dots, u_1 and $u_{n-1}t, \dots, u_1t$ are all in S' too. The statement on the ideals is immediate from the shape of the elements and the fact that $u_1 t + \varphi(a_0) = 0$. \square

Lemma 5.3. *Let $\varphi : R \rightarrow S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \dots + \varphi(a_n)t^n = 0$. Let $J \subset S$ be an ideal such that for at least one i we have $\varphi(a_i) \notin J$. Then there exists a $u \in S$, $u \notin J$ such that both u and ut are integral over R .*

Proof. This is immediate from Lemma 5.2 since one of the elements u_i will not be in J . \square

The following two lemmas are a way of describing closed subschemes of \mathbf{P}_R^1 cut out by one (nondegenerate) equation.

Lemma 5.4. *Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogeneous of degree d . Assume that for every prime \mathfrak{p} of R at least one coefficient of F is not in \mathfrak{p} . Let $S = R[X, Y]/(F)$ as a graded ring. Then for all $n \geq d$ the R -module S_n is finite locally free of rank d .*

Proof. The R -module S_n has a presentation

$$R[X, Y]_{n-d} \rightarrow R[X, Y]_n \rightarrow S_n \rightarrow 0.$$

Thus by Algebra, Lemma 79.4 it is enough to show that multiplication by F induces an injective map $\kappa(\mathfrak{p})[X, Y] \rightarrow \kappa(\mathfrak{p})[X, Y]$ for all primes \mathfrak{p} . This is clear from the assumption that F does not map to the zero polynomial mod \mathfrak{p} . The assertion on ranks is clear from this as well. \square

Lemma 5.5. *Let k be a field. Let $F, G \in k[X, Y]$ be homogeneous of degrees d, e . Assume F, G relatively prime. Then multiplication by G is injective on $S = k[X, Y]/(F)$.*

Proof. This is one way to define “relatively prime”. If you have another definition, then you can show it is equivalent to this one. \square

Lemma 5.6. *Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogeneous of degree d . Let $S = R[X, Y]/(F)$ as a graded ring. Let $\mathfrak{p} \subset R$ be a prime such that some coefficient of F is not in \mathfrak{p} . There exists an $f \in R$ $f \notin \mathfrak{p}$, an integer e , and a $G \in R[X, Y]_e$ such that multiplication by G induces isomorphisms $(S_n)_f \rightarrow (S_{n+e})_f$ for all $n \geq d$.*

Proof. During the course of the proof we may replace R by R_f for $f \in R$, $f \notin \mathfrak{p}$ (finitely often). As a first step we do such a replacement such that some coefficient of F is invertible in R . In particular the modules S_n are now locally free of rank d for $n \geq d$ by Lemma 5.4. Pick any $G \in R[X, Y]_e$ such that the image of G in $\kappa(\mathfrak{p})[X, Y]$ is relatively prime to the image of $F(X, Y)$ (this is possible for some e). Apply Algebra, Lemma 79.4 to the map induced by multiplication by G from $S_d \rightarrow S_{d+e}$. By our choice of G and Lemma 5.5 we see $S_d \otimes \kappa(\mathfrak{p}) \rightarrow S_{d+e} \otimes \kappa(\mathfrak{p})$ is bijective. Thus, after replacing R by R_f for a suitable f we may assume that $G : S_d \rightarrow S_{d+e}$ is bijective. This in turn implies that the image of G in $\kappa(\mathfrak{p}')[X, Y]$ is relatively prime to the image of F for all primes \mathfrak{p}' of R . And then by Algebra, Lemma 79.4 again we see that all the maps $G : S_d \rightarrow S_{d+e}$, $n \geq d$ are isomorphisms. \square

Remark 5.7. Let R be a ring. Suppose that we have $F \in R[X, Y]_d$ and $G \in R[X, Y]_e$ such that, setting $S = R[X, Y]/(F)$ we have (1) S_n is finite locally free of rank d for all $n \geq d$, and (2) multiplication by G defines isomorphisms $S_n \rightarrow S_{n+e}$ for all $n \geq d$. In this case we may define a finite, locally free R -algebra A as follows:

- (1) as an R -module $A = S_{ed}$, and
- (2) multiplication $A \times A \rightarrow A$ is given by the rule that $H_1 H_2 = H_3$ if and only if $G^d H_3 = H_1 H_2$ in S_{2ed} .

This makes sense because multiplication by G^d induces a bijective map $S_{de} \rightarrow S_{2de}$. It is easy to see that this defines a ring structure. Note the confusing fact that the element G^d defines the unit element of the ring A .

Lemma 5.8. *Let R be a ring, let $f \in R$. Suppose we have S, S' and the solid arrows forming the following commutative diagram of rings*

$$\begin{array}{ccccc}
 & & S'' & & \\
 & \nearrow & \vdots & \nwarrow & \\
 R & \xrightarrow{\quad} & & \xrightarrow{\quad} & S \\
 \downarrow & & \vdots & & \downarrow \\
 R_f & \xrightarrow{\quad} & S' & \xrightarrow{\quad} & S_f
 \end{array}$$

Assume that $R_f \rightarrow S'$ is finite. Then we can find a finite ring map $R \rightarrow S''$ and dotted arrows as in the diagram such that $S' = (S'')_f$.

Proof. Namely, suppose that S' is generated by x_i over R_f , $i = 1, \dots, w$. Let $P_i(t) \in R_f[t]$ be a monic polynomial such that $P_i(x_i) = 0$. Say P_i has degree

$d_i > 0$. Write $P_i(t) = t^{d_i} + \sum_{j < d_i} (a_{ij}/f^n)t^j$ for some uniform n . Also write the image of x_i in S_f as g_i/f^n for suitable $g_i \in S$. Then we know that the element $\xi_i = f^{nd_i}g_i^{d_i} + \sum_{j < d_i} f^{n(d_i-j)}a_{ij}g_i^j$ of S is killed by a power of f . Hence upon increasing n to n' , which replaces g_i by $f^{n'-n}g_i$ we may assume $\xi_i = 0$. Then S' is generated by the elements $f^n x_i$, each of which is a zero of the monic polynomial $Q_i(t) = t^{d_i} + \sum_{j < d_i} f^{n(d_i-j)}a_{ij}t^j$ with coefficients in R . Also, by construction $Q_i(f^n g_i) = 0$ in S . Thus we get a finite R -algebra $S'' = R[z_1, \dots, z_w]/(Q_1(z_1), \dots, Q_w(z_w))$ which fits into a commutative diagram as above. The map $\alpha : S'' \rightarrow S$ maps z_i to $f^n g_i$ and the map $\beta : S'' \rightarrow S'$ maps z_i to $f^n x_i$. It may not yet be the case that β induces an isomorphism $(S'')_f \cong S'$. For the moment we only know that this map is surjective. The problem is that there could be elements $h/f^n \in (S'')_f$ which map to zero in S' but are not zero. In this case $\beta(h)$ is an element of S such that $f^N \beta(h) = 0$ for some N . Thus $f^N h$ is an element of the ideal $J = \{h \in S'' \mid \alpha(h) = 0 \text{ and } \beta(h) = 0\}$ of S'' . OK, and it is easy to see that S''/J does the job. \square

6. Formally smooth ring maps

Lemma 6.1. *Let R be a ring. Let S be a R -algebra. If S is of finite presentation and formally smooth over R then S is smooth over R .*

Proof. See Algebra, Proposition 138.13. \square

Remark 6.2. This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 6.3 which became unused because of a rearrangement of the material.

Remark 6.3. This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 6.3 which became unused because of a rearrangement of the material.

Remark 6.4. This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 6.3 which became unused because of a rearrangement of the material.

Remark 6.5. This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Proposition 6.3 which became unused because of a rearrangement of the material.

Remark 6.6. This tag used to refer to an equation in the proof of Algebraization of Formal Spaces, Lemma 9.1 which became unused because of a rearrangement of the material.

Lemma 6.7. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let t be the minimal number of generators for I . Let C be a Noetherian I -adically complete A -algebra. There exists an integer $d \geq 0$ depending only on $I \subset A \rightarrow C$ with the following property: given*

- (1) $c \geq 0$ and B in Algebraization of Formal Spaces, Equation (2.0.2) such that for $a \in I^c$ multiplication by a on $NL_{B/A}^\wedge$ is zero in $D(B)$,
- (2) an integer $n > 2t \max(c, d)$,
- (3) an A/I^n -algebra map $\psi_n : B/I^n B \rightarrow C/I^n C$,

there exists a map $\varphi : B \rightarrow C$ of A -algebras such that $\psi_n \bmod I^{m-c} = \varphi \bmod I^{m-c}$ with $m = \lfloor \frac{n}{t} \rfloor$.

Proof. This lemma has been obsoleted by the stronger Algebraization of Formal Spaces, Lemma 5.3. In fact, we will deduce the lemma from it.

Let $I \subset A \rightarrow C$ be given as in the statement above. Denote $d(\mathrm{Gr}_I(C))$ and $q(\mathrm{Gr}_I(C))$ the integers found in Local Cohomology, Section 22. Observe that t is an upper bound for the minimal number of generators of IC and hence we have $d(\mathrm{Gr}_I(C)) + 1 \leq t$, see discussion in Local Cohomology, Section 22. We may and do assume $t \geq 1$ since otherwise the lemma does not say anything. We claim that the lemma is true with

$$d = q(\mathrm{Gr}_I(C))$$

Namely, suppose that c, B, n, ψ_n are as in the statement above. Then we see that

$$n > 2t \max(c, d) \Rightarrow n \geq 2tc + 1 \Rightarrow n \geq 2(d(\mathrm{Gr}_I(C)) + 1)c + 1$$

On the other hand, we have

$$n > 2t \max(c, d) \Rightarrow n > t(c + d) \Rightarrow n \geq q(C) + tc \geq q(\mathrm{Gr}_I(C)) + (d(\mathrm{Gr}_I(C)) + 1)c$$

Hence the assumptions of Algebraization of Formal Spaces, Lemma 5.3 are satisfied and we obtain an A -algebra homomorphism $\varphi : B \rightarrow C$ which is congruent with ψ_n module $I^{n-(d(\mathrm{Gr}_I(C))+1)c}C$. Since

$$\begin{aligned} n - (d(\mathrm{Gr}_I(C)) + 1)c &= \frac{n}{t} + \frac{(t-1)n}{t} - (d(\mathrm{Gr}_I(C)) + 1)c \\ &\geq \frac{n}{t} + \frac{(d(\mathrm{Gr}_I(C))n)}{t} - (d(\mathrm{Gr}_I(C)) + 1)c \\ &> \frac{n}{t} + \frac{d(\mathrm{Gr}_I(C))2tc}{t} - (d(\mathrm{Gr}_I(C)) + 1)c \\ &= \frac{n}{t} + 2d(\mathrm{Gr}_I(C))c - (d(\mathrm{Gr}_I(C)) + 1)c \\ &= \frac{n}{t} + d(\mathrm{Gr}_I(C))c - c \\ &\geq m - c \end{aligned}$$

we see that we have the congruence of φ and ψ_n module $I^{m-c}C$ as desired. \square

7. Sites and sheaves

Remark 7.1 (No map from lower shriek to pushforward). Let U be an object of a site \mathcal{C} . For any abelian sheaf \mathcal{G} on \mathcal{C}/U one may wonder whether there is a canonical map

$$c : j_{U!}\mathcal{G} \longrightarrow j_{U*}\mathcal{G}$$

To construct such a thing is the same as constructing a map $j_U^{-1}j_{U!}\mathcal{G} \rightarrow \mathcal{G}$. Note that restriction commutes with sheafification. Thus we can use the presheaf of Modules on Sites, Lemma 19.2. Hence it suffices to define for V/U a map

$$\bigoplus_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V, U)} \mathcal{G}(V \xrightarrow{\varphi} U) \longrightarrow \mathcal{G}(V/U)$$

compatible with restrictions. It looks like we can take the which is zero on all summands except for the one where φ is the structure morphism $\varphi_0 : V \rightarrow U$ where we take 1. However, this isn't compatible with restriction mappings: namely,

if $\alpha : V' \rightarrow V$ is a morphism of \mathcal{C} , then denote V'/U the object of \mathcal{C}/U with structure morphism $\varphi'_0 = \varphi_0 \circ \alpha$. We need to check that the diagram

$$\begin{array}{ccc} \bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V, U)} \mathcal{G}(V \xrightarrow{\varphi} U) & \longrightarrow & \mathcal{G}(V/U) \\ \downarrow & & \downarrow \\ \bigoplus_{\varphi' \in \text{Mor}_{\mathcal{C}}(V', U)} \mathcal{G}(V' \xrightarrow{\varphi'} U) & \longrightarrow & \mathcal{G}(V'/U) \end{array}$$

commutes. The problem here is that there may be a morphism $\varphi : V \rightarrow U$ different from φ_0 such that $\varphi \circ \alpha = \varphi'_0$. Thus the left vertical arrow will send the summand corresponding to φ into the summand on which the lower horizontal arrow is equal to 1 and almost surely the diagram doesn't commute.

8. Cohomology

Obsolete lemmas about cohomology.

Lemma 8.1. *Let I be an ideal of a ring A . Let X be a scheme over $\text{Spec}(A)$. Let*

$$\dots \rightarrow \mathcal{F}_3 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1$$

be an inverse system of \mathcal{O}_X -modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n \mathcal{F}_{n+1}$. Assume

$$\bigoplus_{n \geq 0} H^1(X, I^n \mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$ -module. Then the inverse system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. This is a special case of the more general Cohomology, Lemma 35.1. \square

Lemma 8.2. *Let I be an ideal of a ring A . Let X be a scheme over $\text{Spec}(A)$. Let*

$$\dots \rightarrow \mathcal{F}_3 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1$$

be an inverse system of \mathcal{O}_X -modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n \mathcal{F}_{n+1}$. Given n define

$$H_n^1 = \bigcap_{m \geq n} \text{Im} (H^1(X, I^m \mathcal{F}_{m+1}) \rightarrow H^1(X, I^n \mathcal{F}_{n+1}))$$

If $\bigoplus H_n^1$ satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$ -module, then the inverse system $M_n = \Gamma(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. This is a special case of the more general Cohomology, Lemma 35.2. \square

Lemma 8.3. *Let I be a finitely generated ideal of a ring A . Let X be a scheme over $\text{Spec}(A)$. Let*

$$\dots \rightarrow \mathcal{F}_3 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1$$

be an inverse system of \mathcal{O}_X -modules such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n \mathcal{F}_{n+1}$. Assume

$$\bigoplus_{n \geq 0} H^0(X, I^n \mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$ -module. Then the limit topology on $M = \lim \Gamma(X, \mathcal{F}_n)$ is the I -adic topology.

Proof. This is a special case of the more general Cohomology, Lemma 35.3. \square

Lemma 8.4. *Let $(Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}})$ be a ringed topos. For any complex of $\mathcal{O}_{\mathcal{C}}$ -modules \mathcal{G}^\bullet there exists a quasi-isomorphism $K^\bullet \rightarrow \mathcal{G}^\bullet$ such that f^*K^\bullet is a K -flat complex of $\mathcal{O}_{\mathcal{D}}$ -modules for any morphism $f : (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \rightarrow (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}})$ of ringed topoi.*

Proof. This follows from Cohomology on Sites, Lemmas 17.11 and 18.1. \square

Remark 8.5. This remark used to discuss what we know about pullbacks of K -flat complexes being K -flat or not, but is now obsoleted by Cohomology on Sites, Lemma 18.1.

The following lemma computes the cohomology sheaves of the derived limit in a special case.

Lemma 8.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Let $d \in \mathbf{N}$. Assume*

- (1) K_n is an object of $D^+(\mathcal{O})$ for all n ,
- (2) for $q \in \mathbf{Z}$ there exists $n(q)$ such that $H^q(K_{n+1}) \rightarrow H^q(K_n)$ is an isomorphism for $n \geq n(q)$,
- (3) every object of \mathcal{C} has a covering whose members are elements of \mathcal{B} ,
- (4) for every $U \in \mathcal{B}$ we have $H^p(U, H^q(K_n)) = 0$ for $p > d$ and all q .

Then we have $H^m(R\lim K_n) = \lim H^m(K_n)$ for all $m \in \mathbf{Z}$.

Proof. Set $K = R\lim K_n$. Let $U \in \mathcal{B}$. For each n there is a spectral sequence

$$H^p(U, H^q(K_n)) \Rightarrow H^{p+q}(U, K_n)$$

which converges as K_n is bounded below, see Derived Categories, Lemma 21.3. If we fix $m \in \mathbf{Z}$, then we see from our assumption (4) that only $H^p(U, H^q(K_n))$ contribute to $H^m(U, K_n)$ for $0 \leq p \leq d$ and $m - d \leq q \leq m$. By assumption (2) this implies that $H^m(U, K_{n+1}) \rightarrow H^m(U, K_n)$ is an isomorphism as soon as $n \geq \max n(m), \dots, n(m-d)$. The functor $R\Gamma(U, -)$ commutes with derived limits by Injectives, Lemma 13.6. Thus we have

$$H^m(U, K) = H^m(R\lim R\Gamma(U, K_n))$$

On the other hand we have just seen that the complexes $R\Gamma(U, K_n)$ have eventually constant cohomology groups. Thus by More on Algebra, Remark 86.10 we find that $H^m(U, K)$ is equal to $H^m(U, K_n)$ for all $n \gg 0$ for some bound independent of $U \in \mathcal{B}$. Pick such an n . Finally, recall that $H^m(K)$ is the sheafification of the presheaf $U \mapsto H^m(U, K)$ and $H^m(K_n)$ is the sheafification of the presheaf $U \mapsto H^m(U, K_n)$. On the elements of \mathcal{B} these presheaves have the same values. Therefore assumption (3) guarantees that the sheafifications are the same too. The lemma follows. \square

Lemma 8.7. *In Simplicial Spaces, Situation 3.3 let a_0 be an augmentation towards a site \mathcal{D} as in Simplicial Spaces, Remark 4.1. Suppose given strictly full weak Serre subcategories*

$$\mathcal{A} \subset \text{Ab}(\mathcal{D}), \quad \mathcal{A}_n \subset \text{Ab}(\mathcal{C}_n)$$

Then

- (1) *the collection of abelian sheaves \mathcal{F} on \mathcal{C}_{total} whose restriction to \mathcal{C}_n is in \mathcal{A}_n for all n is a strictly full weak Serre subcategory $\mathcal{A}_{total} \subset \text{Ab}(\mathcal{C}_{total})$.*

If a_n^{-1} sends \mathcal{A} into \mathcal{A}_n for all n , then

- (2) *a^{-1} sends \mathcal{A} into \mathcal{A}_{total} and*

- (3) a^{-1} sends $D_{\mathcal{A}}(\mathcal{D})$ into $D_{\mathcal{A}_{total}}(\mathcal{C}_{total})$.
 If $R^q a_{n,*}$ sends \mathcal{A}_n into \mathcal{A} for all n, q , then
 (4) $R^q a_*$ sends \mathcal{A}_{total} into \mathcal{A} for all q , and
 (5) Ra_* sends $D_{\mathcal{A}_{total}}^+(\mathcal{C}_{total})$ into $D_{\mathcal{A}}^+(\mathcal{D})$.

Proof. The only interesting assertions are (4) and (5). Part (4) follows from the spectral sequence in Simplicial Spaces, Lemma 9.3 and Homology, Lemma 24.11. Then part (5) follows by considering the spectral sequence associated to the canonical filtration on an object K of $D_{\mathcal{A}_{total}}^+(\mathcal{C}_{total})$ given by truncations. We omit the details. \square

Remark 8.8. This tag used to refer to a section of the chapter on cohomology listing topics to be treated.

Remark 8.9. This tag used to refer to a section of the chapter on cohomology listing topics to be treated.

Remark 8.10. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.3 pertaining to the situation described in Cohomology on Sites, Lemma 31.9.

Remark 8.11. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.4 pertaining to the situation described in Cohomology on Sites, Lemma 31.9.

Remark 8.12. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.7 pertaining to the situation described in Cohomology on Sites, Lemma 31.9.

Remark 8.13. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.3 pertaining to the situation described in Étale Cohomology, Lemma 100.5.

Remark 8.14. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.4 pertaining to the situation described in Étale Cohomology, Lemma 100.5.

Remark 8.15. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.7 pertaining to the situation described in Étale Cohomology, Lemma 100.5.

Remark 8.16. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.3 pertaining to the situation described in Étale Cohomology, Lemma 102.4.

Remark 8.17. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.4 pertaining to the situation described in Étale Cohomology, Lemma 102.4.

Remark 8.18. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.5 pertaining to the situation described in Étale Cohomology, Lemma 102.4.

Remark 8.19. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.6 pertaining to the situation described in Étale Cohomology, Lemma 102.4.

Remark 8.20. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.7 pertaining to the situation described in Étale Cohomology, Lemma 102.4.

Remark 8.21. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.3 pertaining to the situation described in Étale Cohomology, Lemma 103.4.

Remark 8.22. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.4 pertaining to the situation described in Étale Cohomology, Lemma 103.4.

Remark 8.23. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.5 pertaining to the situation described in Étale Cohomology, Lemma 103.4.

Remark 8.24. This tag used to refer to the special case of Cohomology on Sites, Lemma 30.7 pertaining to the situation described in Étale Cohomology, Lemma 103.4.

Remark 8.25. This tag used to be in the chapter on étale cohomology, but is no longer suitable there because of a reorganization. The content of the tag was the following: Étale Cohomology, Lemma 77.3 can be used to prove that if $f : X \rightarrow Y$ is a separated, finite type morphism of schemes and Y is Noetherian, then $Rf_!$ induces a functor $D_{ctf}(X_{\text{étale}}, \Lambda) \rightarrow D_{ctf}(Y_{\text{étale}}, \Lambda)$. An example of this argument, when Y is the spectrum of a field and X is a curve is given in The Trace Formula, Proposition 13.1.

Lemma 8.26. *Let $f : X \rightarrow Y$ be a locally quasi-finite morphism of schemes. There exists a unique functor $f^! : \text{Ab}(Y_{\text{étale}}) \rightarrow \text{Ab}(X_{\text{étale}})$ such that*

- (1) *for any open $j : U \rightarrow X$ with $f \circ j$ separated there is a canonical isomorphism $j^! \circ f^! = (f \circ j)^!$, and*
- (2) *these isomorphisms for $U \subset U' \subset X$ are compatible with the isomorphisms in More Étale Cohomology, Lemma 6.3.*

Proof. Immediate consequence of More Étale Cohomology, Lemmas 6.1 and 6.3. \square

Proposition 8.27. *Let $f : X \rightarrow Y$ be a locally quasi-finite morphism. There exist adjoint functors $f_! : \text{Ab}(X_{\text{étale}}) \rightarrow \text{Ab}(Y_{\text{étale}})$ and $f^! : \text{Ab}(Y_{\text{étale}}) \rightarrow \text{Ab}(X_{\text{étale}})$ with the following properties*

- (1) *the functor $f^!$ is the one constructed in More Étale Cohomology, Lemma 6.1,*
- (2) *for any open $j : U \rightarrow X$ with $f \circ j$ separated there is a canonical isomorphism $f_! \circ j_! = (f \circ j)_!$, and*
- (3) *these isomorphisms for $U \subset U' \subset X$ are compatible with the isomorphisms in More Étale Cohomology, Lemma 3.13.*

Proof. See More Étale Cohomology, Sections 4 and 6. \square

Lemma 8.28. *Let $f : X \rightarrow Y$ be a morphism of schemes which is locally quasi-finite. For an abelian group A and a geometric point $\bar{y} : \text{Spec}(k) \rightarrow Y$ we have $f^!(\bar{y}_* A) = \prod_{f(\bar{x})=\bar{y}} \bar{x}_* A$.*

Proof. Follows from the corresponding statement in More Étale Cohomology, Lemma 6.1. \square

Lemma 8.29. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be composable locally quasi-finite morphisms of schemes. Then $g_! \circ f_! = (g \circ f)_!$ and $f^! \circ g^! = (g \circ f)^!$.*

Proof. Combination of More Étale Cohomology, Lemmas 4.12 and 6.3. \square

9. Differential graded algebra

Lemma 9.1. *Let (A, d) and (B, d) be differential graded algebras. Let N be a differential graded (A, B) -bimodule with property (P). Let M be a differential graded A -module with property (P). Then $Q = M \otimes_A N$ is a differential graded B -module which represents $M \otimes_A^L N$ in $D(B)$ and which has a filtration*

$$0 = F_{-1}Q \subset F_0Q \subset F_1Q \subset \dots \subset Q$$

by differential graded submodules such that $Q = \bigcup F_p Q$, the inclusions $F_i Q \rightarrow F_{i+1} Q$ are admissible monomorphisms, the quotients $F_{i+1} Q / F_i Q$ are isomorphic as differential graded B -modules to a direct sum of $(A \otimes_R B)[k]$.

Proof. Choose filtrations F_\bullet on M and N . Then consider the filtration on $Q = M \otimes_A N$ given by

$$F_n(Q) = \sum_{i+j=n} F_i(M) \otimes_A F_j(N)$$

This is clearly a differential graded B -submodule. We see that

$$F_n(Q)/F_{n-1}(Q) = \bigoplus_{i+j=n} F_i(M)/F_{i-1}(M) \otimes_A F_j(N)/F_{j-1}(N)$$

for example because the filtration of M is split in the category of graded A -modules. Since by assumption the quotients on the right hand side are isomorphic to direct sums of shifts of A and $A \otimes_R B$ and since $A \otimes_A (A \otimes_R B) = A \otimes_R B$, we conclude that the left hand side is a direct sum of shifts of $A \otimes_R B$ as a differential graded B -module. (Warning: Q does not have a structure of (A, B) -bimodule.) This proves the first statement of the lemma. The second statement is immediate from the definition of the functor in Differential Graded Algebra, Lemma 33.2. \square

10. Simplicial methods

Lemma 10.1. *Assumptions and notation as in Simplicial, Lemma 32.1. There exists a section $g : U \rightarrow V$ to the morphism f and the composition $g \circ f$ is homotopy equivalent to the identity on V . In particular, the morphism f is a homotopy equivalence.*

Proof. Immediate from Simplicial, Lemmas 32.1 and 30.8. \square

Lemma 10.2. *Let \mathcal{C} be a category with finite coproducts and finite limits. Let X be an object of \mathcal{C} . Let $k \geq 0$. The canonical map*

$$\mathrm{Hom}(\Delta[k], X) \longrightarrow \mathrm{cosk}_1 \mathrm{sk}_1 \mathrm{Hom}(\Delta[k], X)$$

is an isomorphism.

Proof. For any simplicial object V we have

$$\begin{aligned} \text{Mor}(V, \text{cosk}_1 \text{sk}_1 \text{Hom}(\Delta[k], X)) &= \text{Mor}(\text{sk}_1 V, \text{sk}_1 \text{Hom}(\Delta[k], X)) \\ &= \text{Mor}(i_{1!} \text{sk}_1 V, \text{Hom}(\Delta[k], X)) \\ &= \text{Mor}(i_{1!} \text{sk}_1 V \times \Delta[k], X) \end{aligned}$$

The first equality by the adjointness of sk and cosk , the second equality by the adjointness of $i_{1!}$ and sk_1 , and the first equality by Simplicial, Definition 17.1 where the last X denotes the constant simplicial object with value X . By Simplicial, Lemma 20.2 an element in this set depends only on the terms of degree 0 and 1 of $i_{1!} \text{sk}_1 V \times \Delta[k]$. These agree with the degree 0 and 1 terms of $V \times \Delta[k]$, see Simplicial, Lemma 21.3. Thus the set above is equal to $\text{Mor}(V \times \Delta[k], X) = \text{Mor}(V, \text{Hom}(\Delta[k], X))$. \square

Lemma 10.3. *Let \mathcal{C} be a category. Let X be an object of \mathcal{C} such that the self products $X \times \dots \times X$ exist. Let $k \geq 0$ and let $C[k]$ be as in Simplicial, Example 5.6. With notation as in Simplicial, Lemma 15.2 the canonical map*

$$\text{Hom}(C[k], X)_1 \longrightarrow (\text{cosk}_0 \text{sk}_0 \text{Hom}(C[k], X))_1$$

is identified with the map

$$\prod_{\alpha: [k] \rightarrow [1]} X \longrightarrow X \times X$$

which is the projection onto the factors where α is a constant map.

Proof. This is shown in the proof of Hypercoverings, Lemma 7.3. \square

11. Results on schemes

Lemmas that seem superfluous.

Lemma 11.1. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $X \subset \mathbf{P}_R^n$ be a closed subscheme. Assume that $R = \Gamma(X, \mathcal{O}_X)$. Then the special fibre X_k is geometrically connected.*

Proof. This is a special case of More on Morphisms, Theorem 53.5. \square

Lemma 11.2. *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ . Let \mathcal{P} be a property of coherent sheaves on X such that*

- (1) *For any short exact sequence of coherent sheaves if two out of three of them have property \mathcal{P} then so does the third.*
- (2) *If \mathcal{P} holds for a direct sum of coherent sheaves then it holds for both.*
- (3) *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have \mathcal{P} for $(Z \rightarrow X)_* \mathcal{I}$.*
- (4) *There exists some coherent sheaf \mathcal{G} on X such that*
 - (a) *$\text{Supp}(\mathcal{G}) = Z_0$,*
 - (b) *\mathcal{G}_ξ is annihilated by \mathfrak{m}_ξ , and*
 - (c) *property \mathcal{P} holds for \mathcal{G} .*

Then property \mathcal{P} holds for every coherent sheaf \mathcal{F} on X whose support is contained in Z_0 .

Proof. The proof is a variant on the proof of Cohomology of Schemes, Lemma 12.5. In exactly the same manner as in that proof we see that any coherent sheaf whose support is strictly contained in Z_0 has property \mathcal{P} .

Consider a coherent sheaf \mathcal{G} as in (3). By Cohomology of Schemes, Lemma 12.2 there exists a sheaf of ideals \mathcal{I} on Z_0 and a short exact sequence

$$0 \rightarrow ((Z_0 \rightarrow X)_*\mathcal{I})^{\oplus r} \rightarrow \mathcal{G} \rightarrow \mathcal{Q} \rightarrow 0$$

where the support of \mathcal{Q} is strictly contained in Z_0 . In particular $r > 0$ and \mathcal{I} is nonzero because the support of \mathcal{G} is equal to Z . Since \mathcal{Q} has property \mathcal{P} we conclude that also $((Z_0 \rightarrow X)_*\mathcal{I})^{\oplus r}$ has property \mathcal{P} . By (2) we deduce property \mathcal{P} for $(Z_0 \rightarrow X)_*\mathcal{I}$. Slotting this into the proof of Cohomology of Schemes, Lemma 12.5 at the appropriate point gives the lemma. Some details omitted. \square

Lemma 11.3. *Let X be a Noetherian scheme. Let \mathcal{P} be a property of coherent sheaves on X such that*

- (1) *For any short exact sequence of coherent sheaves if two out of three of them have property \mathcal{P} then so does the third.*
- (2) *If \mathcal{P} holds for a direct sum of coherent sheaves then it holds for both.*
- (3) *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf \mathcal{G} such that*
 - (a) *$\text{Supp}(\mathcal{G}) = Z$,*
 - (b) *\mathcal{G}_ξ is annihilated by \mathfrak{m}_ξ , and*
 - (c) *property \mathcal{P} holds for \mathcal{G} .*

Then property \mathcal{P} holds for every coherent sheaf on X .

Proof. This follows from Lemma 11.2 in exactly the same way that Cohomology of Schemes, Lemma 12.6 follows from Cohomology of Schemes, Lemma 12.5. \square

Lemma 11.4. *Let X be a scheme. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}' \subset \mathcal{F}$ be quasi-coherent \mathcal{O}_X -modules. Assume that*

- (1) *X is quasi-compact,*
- (2) *\mathcal{F} is of finite type, and*
- (3) *$\mathcal{F}'|_{X_s} = \mathcal{F}|_{X_s}$.*

Then there exists an $n \geq 0$ such that multiplication by s^n on \mathcal{F} factors through \mathcal{F}' .

Proof. In other words we claim that $s^n \mathcal{F} \subset \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ for some $n \geq 0$. In other words, we claim that the quotient map $\mathcal{F} \rightarrow \mathcal{F}/\mathcal{F}'$ becomes zero after multiplying by a power of s . This follows from Properties, Lemma 17.3. \square

Lemma 11.5. *Let $f : X \rightarrow Y$ be a morphism schemes. Assume*

- (1) *X and Y are integral schemes,*
- (2) *f is locally of finite type and dominant,*
- (3) *f is either quasi-compact or separated,*
- (4) *f is generically finite, i.e., one of (1) – (5) of Morphisms, Lemma 51.7 holds.*

Then there is a nonempty open $V \subset Y$ such that $f^{-1}(V) \rightarrow V$ is finite locally free of degree $\deg(X/Y)$. In particular, the degrees of the fibres of $f^{-1}(V) \rightarrow V$ are bounded by $\deg(X/Y)$.

Proof. We may choose V such that $f^{-1}(V) \rightarrow V$ is finite. Then we may shrink V and assume that $f^{-1}(V) \rightarrow V$ is flat and of finite presentation by generic flatness (Morphisms, Proposition 27.1). Then the morphism is finite locally free by Morphisms, Lemma 48.2. Since V is irreducible the morphism has a fixed degree. The final statement follows from this and Morphisms, Lemma 57.3. \square

12. Derived categories of varieties

Some lemma which were originally part of the chapter on derived categories of varieties but are no longer needed.

Lemma 12.1. *Let k be a field. Let X be a separated scheme of finite type over k which is regular. Let $F : D_{\text{perf}}(\mathcal{O}_X) \rightarrow D_{\text{perf}}(\mathcal{O}_X)$ be a k -linear exact functor. Assume for every coherent \mathcal{O}_X -module \mathcal{F} with $\dim(\text{Supp}(\mathcal{F})) = 0$ there is an isomorphism of k -vector spaces*

$$\text{Hom}_X(\mathcal{F}, M) = \text{Hom}_X(\mathcal{F}, F(M))$$

*functorial in M in $D_{\text{perf}}(\mathcal{O}_X)$. Then there exists an automorphism $f : X \rightarrow X$ over k which induces the identity on the underlying topological space¹ and an invertible \mathcal{O}_X -module \mathcal{L} such that F and $F'(M) = f^*M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{L}$ are siblings.*

Proof. By Derived Categories of Varieties, Lemma 11.2 we conclude that for every coherent \mathcal{O}_X -module \mathcal{F} whose support is a closed point there are isomorphisms

$$H^0(X, M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}) = H^0(X, F(M) \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F})$$

functorial in M .

Let $x \in X$ be a closed point and apply the above with $\mathcal{F} = \mathcal{O}_x$ the skyscraper sheaf with value $\kappa(x)$ at x . We find

$$\dim_{\kappa(x)} \text{Tor}_p^{\mathcal{O}_{X,x}}(M_x, \kappa(x)) = \dim_{\kappa(x)} \text{Tor}_p^{\mathcal{O}_{X,x}}(F(M)_x, \kappa(x))$$

for all $p \in \mathbf{Z}$. In particular, if $H^i(M) = 0$ for $i > 0$, then $H^i(F(M)) = 0$ for $i > 0$ by Derived Categories of Varieties, Lemma 11.3.

If \mathcal{E} is locally free of rank r , then $F(\mathcal{E})$ is locally free of rank r . This is true because a perfect complex K over $\mathcal{O}_{X,x}$ with

$$\dim_{\kappa(x)} \text{Tor}_i^{\mathcal{O}_{X,x}}(K, \kappa(x)) = \begin{cases} r & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

is equal to a free module of rank r placed in degree 0. See for example More on Algebra, Lemma 75.6.

If M is supported on a closed subscheme $Z \subset X$, then $F(M)$ is also supported on Z . This is clear because we will have $M \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_x = 0$ for $x \notin Z$ and hence the same will be true for $F(M)$ and hence we get the conclusion from Derived Categories of Varieties, Lemma 11.3.

In particular $F(\mathcal{O}_x)$ is supported at $\{x\}$. Let $i \in \mathbf{Z}$ be the minimal integer such that $H^i(\mathcal{O}_x) \neq 0$. We know that $i \leq 0$. If $i < 0$, then there is a morphism $\mathcal{O}_x[-i] \rightarrow F(\mathcal{O}_x)$ which contradicts the fact that all morphisms $\mathcal{O}_x[-i] \rightarrow \mathcal{O}_x$ are zero. Thus $F(\mathcal{O}_x) = \mathcal{H}[0]$ where \mathcal{H} is a skyscraper sheaf at x .

Let \mathcal{G} be a coherent \mathcal{O}_X -module with $\dim(\text{Supp}(\mathcal{G})) = 0$. Then there exists a filtration

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \dots \subset \mathcal{G}_n = \mathcal{G}$$

such that for $n \geq i \geq 1$ the quotient $\mathcal{G}_i/\mathcal{G}_{i-1}$ is isomorphic to \mathcal{O}_{x_i} for some closed point $x_i \in X$. Then we get distinguished triangles

$$F(\mathcal{G}_{i-1}) \rightarrow F(\mathcal{G}_i) \rightarrow F(\mathcal{O}_{x_i})$$

¹This often forces f to be the identity, see Varieties, Lemma 32.1.

and using induction we find that $F(\mathcal{G}_i)$ is a coherent sheaf placed in degree 0.

Let \mathcal{G} be a coherent \mathcal{O}_X -module. We know that $H^i(F(\mathcal{G})) = 0$ for $i > 0$. To get a contradiction assume that $H^i(F(\mathcal{G}))$ is nonzero for some $i < 0$. We choose i minimal with this property so that we have a morphism $H^i(F(\mathcal{G}))[-i] \rightarrow F(\mathcal{G})$ in $D_{\text{perf}}(\mathcal{O}_X)$. Choose a closed point $x \in X$ in the support of $H^i(F(\mathcal{G}))$. By More on Algebra, Lemma 100.2 there exists an $n > 0$ such that

$$H^i(F(\mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n \longrightarrow \text{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n)$$

is nonzero. Next, we take $m \geq 1$ and we consider the short exact sequence

$$0 \rightarrow \mathfrak{m}_x^m \mathcal{G} \rightarrow \mathcal{G} \rightarrow \mathcal{G}/\mathfrak{m}_x^m \mathcal{G} \rightarrow 0$$

By the above we know that $F(\mathcal{G}/\mathfrak{m}_x^m \mathcal{G})$ is a sheaf placed in degree 0. Hence $H^i(F(\mathfrak{m}_x^m \mathcal{G})) \rightarrow H^i(F(\mathcal{G}))$ is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} H^i(F(\mathfrak{m}_x^m \mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n & \longrightarrow & \text{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathfrak{m}_x^m \mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \\ \downarrow & & \downarrow \\ H^i(F(\mathcal{G}))_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^n & \longrightarrow & \text{Tor}_{-i}^{\mathcal{O}_{X,x}}(F(\mathcal{G})_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \end{array}$$

Since the left vertical arrow is an isomorphism and the bottom arrow is nonzero, we conclude that the right vertical arrow is nonzero for all $m \geq 1$. On the other hand, by the first paragraph of the proof, we know this arrow is isomorphic to the arrow

$$\text{Tor}_{-i}^{\mathcal{O}_{X,x}}(\mathfrak{m}_x^m \mathcal{G}_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \longrightarrow \text{Tor}_{-i}^{\mathcal{O}_{X,x}}(\mathcal{G}_x, \mathcal{O}_{X,x}/\mathfrak{m}_x^n)$$

However, this arrow is zero for $m \gg n$ by More on Algebra, Lemma 102.2 which is the contradiction we're looking for.

Thus we know that F preserves coherent modules. By Derived Categories of Varieties, Lemma 12.2 we find F is a sibling to the Fourier-Mukai functor F' given by a coherent $\mathcal{O}_{X \times X}$ -module \mathcal{K} flat over X via pr_1 and finite over X via pr_2 . Since $F(\mathcal{O}_X)$ is an invertible \mathcal{O}_X -module \mathcal{L} placed in degree 0 we see that

$$\mathcal{L} \cong F(\mathcal{O}_X) \cong F'(\mathcal{O}_X) \cong \text{pr}_{2,*} \mathcal{K}$$

Thus by Functors and Morphisms, Lemma 7.6 there is a morphism $s : X \rightarrow X \times X$ with $\text{pr}_2 \circ s = \text{id}_X$ such that $\mathcal{K} = s_* \mathcal{L}$. Set $f = \text{pr}_1 \circ s$. Then we have

$$\begin{aligned} F'(M) &= R\text{pr}_{2,*}(L\text{pr}_1^* K \otimes \mathcal{K}) \\ &= R\text{pr}_{2,*}(L\text{pr}_1^* M \otimes s_* \mathcal{L}) \\ &= R\text{pr}_{2,*}(Rs_*(Lf^* M \otimes \mathcal{L})) \\ &= Lf^* M \otimes \mathcal{L} \end{aligned}$$

where we have used Derived Categories of Schemes, Lemma 22.1 in the third step. Since for all closed points $x \in X$ the module $F(\mathcal{O}_x)$ is supported at x , we see that f induces the identity on the underlying topological space of X . We still have to show that f is an isomorphism which we will do in the next paragraph.

Let $x \in X$ be a closed point. For $n \geq 1$ denote $\mathcal{O}_{x,n}$ the skyscraper sheaf at x with value $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$. We have

$$\text{Hom}_X(\mathcal{O}_{x,m}, \mathcal{O}_{x,n}) \cong \text{Hom}_X(\mathcal{O}_{x,m}, F(\mathcal{O}_{x,n})) \cong \text{Hom}_X(\mathcal{O}_{x,m}, f^* \mathcal{O}_{x,n} \otimes \mathcal{L})$$

functorially with respect to \mathcal{O}_X -module homomorphisms between the $\mathcal{O}_{x,n}$. (The first isomorphism exists by assumption and the second isomorphism because F and F' are siblings.) For $m \geq n$ we have $\mathcal{O}_{X,x}/\mathfrak{m}_x^n = \text{Hom}_X(\mathcal{O}_{x,m}, \mathcal{O}_{x,n})$ via the action on $\mathcal{O}_{x,n}$ we conclude that $f^\# : \mathcal{O}_{X,x}/\mathfrak{m}_x^n \rightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is bijective for all n . Thus f induces isomorphisms on complete local rings at closed points and hence is étale (Étale Morphisms, Lemma 11.3). Looking at closed points we see that $\Delta_f : X \rightarrow X \times_{f,X,f} X$ (which is an open immersion as f is étale) is bijective hence an isomorphism. Hence f is a monomorphism. Finally, we conclude f is an isomorphism as Descent, Lemma 25.1 tells us it is an open immersion. \square

13. Representability in the regular proper case

This section is obsolete because we improved Derived Categories of Varieties, Theorem 6.3 to apply to all proper schemes over a field (whereas before we only proved it for projective schemes over a field).

Lemma 13.1. *Let $f : X' \rightarrow X$ be a proper birational morphism of integral Noetherian schemes with X regular. The map $\mathcal{O}_X \rightarrow Rf_*\mathcal{O}_{X'}$ canonically splits in $D(\mathcal{O}_X)$.*

Proof. Set $E = Rf_*\mathcal{O}_{X'}$ in $D(\mathcal{O}_X)$. Observe that E is in $D_{Coh}^b(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 11.3. By Derived Categories of Schemes, Lemma 11.8 we find that E is a perfect object of $D(\mathcal{O}_X)$. Since $\mathcal{O}_{X'}$ is a sheaf of algebras, we have the relative cup product $\mu : E \otimes_{\mathcal{O}_X}^L E \rightarrow E$ by Cohomology, Remark 28.7. Let $\sigma : E \otimes E^\vee \rightarrow E^\vee \otimes E$ be the commutativity constraint on the symmetric monoidal category $D(\mathcal{O}_X)$ (Cohomology, Lemma 50.6). Denote $\eta : \mathcal{O}_X \rightarrow E \otimes E^\vee$ and $\epsilon : E^\vee \otimes E \rightarrow \mathcal{O}_X$ the maps constructed in Cohomology, Example 50.7. Then we can consider the map

$$E \xrightarrow{\eta \otimes 1} E \otimes E^\vee \otimes E \xrightarrow{\sigma \otimes 1} E^\vee \otimes E \otimes E \xrightarrow{1 \otimes \mu} E^\vee \otimes E \xrightarrow{\epsilon} \mathcal{O}_X$$

We claim that this map is a one sided inverse to the map in the statement of the lemma. To see this it suffices to show that the composition $\mathcal{O}_X \rightarrow \mathcal{O}_X$ is the identity map. This we may do in the generic point of X (or on an open subscheme of X over which f is an isomorphism). In this case $E = \mathcal{O}_X$ and μ is the usual multiplication map and the result is clear. \square

Lemma 13.2. *Let X be a proper scheme over a field k which is regular. Let $K \in \text{Ob}(D_{QCoh}(\mathcal{O}_X))$. The following are equivalent*

- (1) $K \in D_{Coh}^b(\mathcal{O}_X) = D_{perf}(\mathcal{O}_X)$, and
- (2) $\sum_{i \in \mathbf{Z}} \dim_k \text{Ext}_X^i(E, K) < \infty$ for all perfect E in $D(\mathcal{O}_X)$.

Proof. The equality in (1) holds by Derived Categories of Schemes, Lemma 11.8. The implication (1) \Rightarrow (2) follows from Derived Categories of Varieties, Lemma 5.3. The implication (2) \Rightarrow (1) follows from More on Morphisms, Lemma 69.6. \square

Lemma 13.3. *Let X be a proper scheme over a field k which is regular.*

- (1) *Let $F : D_{perf}(\mathcal{O}_X)^{opp} \rightarrow \text{Vect}_k$ be a k -linear cohomological functor such that*

$$\sum_{n \in \mathbf{Z}} \dim_k F(E[n]) < \infty$$

for all $E \in D_{perf}(\mathcal{O}_X)$. Then F is isomorphic to a functor of the form $E \mapsto \text{Hom}_X(E, K)$ for some $K \in D_{perf}(\mathcal{O}_X)$.

(2) Let $G : D_{\text{perf}}(\mathcal{O}_X) \rightarrow \text{Vect}_k$ be a k -linear homological functor such that

$$\sum_{n \in \mathbf{Z}} \dim_k G(E[n]) < \infty$$

for all $E \in D_{\text{perf}}(\mathcal{O}_X)$. Then G is isomorphic to a functor of the form $E \mapsto \text{Hom}_X(K, E)$ for some $K \in D_{\text{perf}}(\mathcal{O}_X)$.

Proof. This follows from Derived Categories of Varieties, Theorem 6.3 and Lemma 6.4. We also give another proof below.

Proof of (1). The derived category $D_{Q\text{Coh}}(\mathcal{O}_X)$ has direct sums, is compactly generated, and $D_{\text{perf}}(\mathcal{O}_X)$ is the full subcategory of compact objects, see Derived Categories of Schemes, Lemma 3.1, Theorem 15.3, and Proposition 17.1. By Derived Categories of Varieties, Lemma 6.2 we may assume $F(E) = \text{Hom}_X(E, K)$ for some $K \in \text{Ob}(D_{Q\text{Coh}}(\mathcal{O}_X))$. Then it follows that K is in $D_{\text{Coh}}^b(\mathcal{O}_X)$ by Lemma 13.2.

Proof of (2). Consider the contravariant functor $E \mapsto E^\vee$ on $D_{\text{perf}}(\mathcal{O}_X)$, see Cohomology, Lemma 50.5. This functor is an exact anti-self-equivalence of $D_{\text{perf}}(\mathcal{O}_X)$. Hence we may apply part (1) to the functor $F(E) = G(E^\vee)$ to find $K \in D_{\text{perf}}(\mathcal{O}_X)$ such that $G(E^\vee) = \text{Hom}_X(E, K)$. It follows that $G(E) = \text{Hom}_X(E^\vee, K) = \text{Hom}_X(K^\vee, E)$ and we conclude that taking K^\vee works. \square

14. Functor of quotients

Lemma 14.1. Let $S = \text{Spec}(R)$ be an affine scheme. Let X be an algebraic space over S . Let $q_i : \mathcal{F} \rightarrow \mathcal{Q}_i$, $i = 1, 2$ be surjective maps of quasi-coherent \mathcal{O}_X -modules. Assume \mathcal{Q}_1 flat over S . Let $T \rightarrow S$ be a quasi-compact morphism of schemes such that there exists a factorization

$$\begin{array}{ccc} & \mathcal{F}_T & \\ q_{1,T} \swarrow & & \searrow q_{2,T} \\ \mathcal{Q}_{1,T} & \xleftarrow{\dots\dots\dots} & \mathcal{Q}_{2,T} \end{array}$$

Then exists a closed subscheme $Z \subset S$ such that (a) $T \rightarrow S$ factors through Z and (b) $q_{1,Z}$ factors through $q_{2,Z}$. If $\text{Ker}(q_2)$ is a finite type \mathcal{O}_X -module and X quasi-compact, then we can take $Z \rightarrow S$ of finite presentation.

Proof. Apply Flatness on Spaces, Lemma 8.2 to the map $\text{Ker}(q_2) \rightarrow \mathcal{Q}_1$. \square

15. Spaces and fpqc coverings

The material here was made obsolete by Gabber's argument showing that algebraic spaces satisfy the sheaf condition with respect to fpqc coverings. Please visit Properties of Spaces, Section 17.

Lemma 15.1. Let S be a scheme. Let X be an algebraic space over S . Let $\{f_i : T_i \rightarrow T\}_{i \in I}$ be a fpqc covering of schemes over S . Then the map

$$\text{Mor}_S(T, X) \longrightarrow \prod_{i \in I} \text{Mor}_S(T_i, X)$$

is injective.

Proof. Immediate consequence of Properties of Spaces, Proposition 17.1. \square

Lemma 15.2. *Let S be a scheme. Let X be an algebraic space over S . Let $X = \bigcup_{j \in J} X_j$ be a Zariski covering, see Spaces, Definition 12.5. If each X_j satisfies the sheaf property for the fpqc topology then X satisfies the sheaf property for the fpqc topology.*

Proof. This is true because all algebraic spaces satisfy the sheaf property for the fpqc topology, see Properties of Spaces, Proposition 17.1. \square

Lemma 15.3. *Let S be a scheme. Let X be an algebraic space over S . If X is Zariski locally quasi-separated over S , then X satisfies the sheaf condition for the fpqc topology.*

Proof. Immediate consequence of the general Properties of Spaces, Proposition 17.1. \square

Remark 15.4. This remark used to discuss to what extent the original proof of Lemma 15.3 (of December 18, 2009) generalizes.

16. Very reasonable algebraic spaces

Material that is somewhat obsolete.

Lemma 16.1. *Let S be a scheme. Let X be a reasonable algebraic space over S . Then $|X|$ is Kolmogorov (see Topology, Definition 8.6).*

Proof. Follows from the definitions and Decent Spaces, Lemma 12.3. \square

In the rest of this section we make some remarks about very reasonable algebraic spaces. If there exists a scheme U and a surjective, étale, quasi-compact morphism $U \rightarrow X$, then X is very reasonable, see Decent Spaces, Lemma 4.7.

Lemma 16.2. *A scheme is very reasonable.*

Proof. This is true because the identity map is a quasi-compact, surjective étale morphism. \square

Lemma 16.3. *Let S be a scheme. Let X be an algebraic space over S . If there exists a Zariski open covering $X = \bigcup X_i$ such that each X_i is very reasonable, then X is very reasonable.*

Proof. This is case (ϵ) of Decent Spaces, Lemma 5.2. \square

Lemma 16.4. *An algebraic space which is Zariski locally quasi-separated is very reasonable. In particular any quasi-separated algebraic space is very reasonable.*

Proof. This is one of the implications of Decent Spaces, Lemma 5.1. \square

Lemma 16.5. *Let S be a scheme. Let X, Y be algebraic spaces over S . Let $Y \rightarrow X$ be a representable morphism. If X is very reasonable, so is Y .*

Proof. This is case (ϵ) of Decent Spaces, Lemma 5.3. \square

Remark 16.6. Very reasonable algebraic spaces form a strictly larger collection than Zariski locally quasi-separated algebraic spaces. Consider an algebraic space of the form $X = [U/G]$ (see Spaces, Definition 14.4) where G is a finite group acting without fixed points on a non-quasi-separated scheme U . Namely, in this case $U \times_X U = U \times G$ and clearly both projections to U are quasi-compact, hence

X is very reasonable. On the other hand, the diagonal $U \times_X U \rightarrow U \times U$ is not quasi-compact, hence this algebraic space is not quasi-separated. Now, take U the infinite affine space over a field k of characteristic $\neq 2$ with zero doubled, see Schemes, Example 21.4. Let $0_1, 0_2$ be the two zeros of U . Let $G = \{+1, -1\}$, and let -1 act by -1 on all coordinates, and by switching 0_1 and 0_2 . Then $[U/G]$ is very reasonable but not Zariski locally quasi-separated (details omitted).

Warning: The following lemma should be used with caution, as the schemes U_i in it are not necessarily separated or even quasi-separated.

Lemma 16.7. *Let S be a scheme. Let X be a very reasonable algebraic space over S . There exists a set of schemes U_i and morphisms $U_i \rightarrow X$ such that*

- (1) *each U_i is a quasi-compact scheme,*
- (2) *each $U_i \rightarrow X$ is étale,*
- (3) *both projections $U_i \times_X U_i \rightarrow U_i$ are quasi-compact, and*
- (4) *the morphism $\coprod U_i \rightarrow X$ is surjective (and étale).*

Proof. Decent Spaces, Definition 6.1 says that there exist $U_i \rightarrow X$ such that (2), (3) and (4) hold. Fix i , and set $R_i = U_i \times_X U_i$, and denote $s, t : R_i \rightarrow U_i$ the projections. For any affine open $W \subset U_i$ the open $W' = t(s^{-1}(W)) \subset U_i$ is a quasi-compact R_i -invariant open (see Groupoids, Lemma 19.2). Hence W' is a quasi-compact scheme, $W' \rightarrow X$ is étale, and $W' \times_X W' = s^{-1}(W') = t^{-1}(W')$ so both projections $W' \times_X W' \rightarrow W'$ are quasi-compact. This means the family of $W' \rightarrow X$, where $W \subset U_i$ runs through the members of affine open coverings of the U_i gives what we want. \square

17. Obsolete lemmas on algebraic spaces

Lemmas that seem superfluous or are no longer used in the text.

Lemma 17.1. *In Cohomology of Spaces, Situation 16.1 the morphism $p : X \rightarrow \text{Spec}(A)$ is surjective.*

Proof. This lemma was originally used in the proof of Cohomology of Spaces, Proposition 16.7 but now is a consequence of it. \square

Lemma 17.2. *In Cohomology of Spaces, Situation 16.1 the morphism $p : X \rightarrow \text{Spec}(A)$ is universally closed.*

Proof. This lemma was originally used in the proof of Cohomology of Spaces, Proposition 16.7 but now is a consequence of it. \square

Remark 17.3. This tag used to refer to an equation in the proof of Formal Spaces, Lemma 20.4.

Remark 17.4. This tag used to refer to an equation in the proof of Formal Spaces, Lemma 20.4.

18. Obsolete lemmas on algebraic stacks

Lemmas that seem superfluous or are no longer used in the text.

Lemma 18.1. *Let S be a locally Noetherian scheme. Let \mathcal{X} be a category fibred in groupoids over $(\text{Sch}/S)_{fppf}$ having (RS^*) . Let x be an object of \mathcal{X} over an affine scheme U of finite type over S . Let $u_n \in U$, $n \geq 1$ be pairwise distinct finite type*

points such that x is not versal at u_n for all n . After replacing u_n by a subsequence, there exist morphisms

$$x \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \quad \text{in } \mathcal{X} \text{ lying over } U \rightarrow U_1 \rightarrow U_2 \rightarrow \dots$$

over S such that

- (1) for each n the morphism $U \rightarrow U_n$ is a first order thickening,
- (2) for each n we have a short exact sequence

$$0 \rightarrow \kappa(u_n) \rightarrow \mathcal{O}_{U_n} \rightarrow \mathcal{O}_{U_{n-1}} \rightarrow 0$$

with $U_0 = U$ for $n = 1$,

- (3) for each n there does **not** exist a pair (W, α) consisting of an open neighbourhood $W \subset U_n$ of u_n and a morphism $\alpha : x_n|_W \rightarrow x$ such that the composition

$$x|_{U \cap W} \xrightarrow{\text{restriction of } x \rightarrow x_n} x_n|_W \xrightarrow{\alpha} x$$

is the canonical morphism $x|_{U \cap W} \rightarrow x$.

Proof. This lemma was originally used in the proof of a criterion for openness of versality (Artin's Axioms, Lemma 20.3) but it got replaced by Artin's Axioms, Lemma 20.1 from which it readily follows. Namely, after replacing u_n , $n \geq 1$ by a subsequence we may and do assume that there are no specializations among these points, see Properties, Lemma 5.11. Then we can apply Artin's Axioms, Lemma 20.1 to finish the proof. \square

19. Variants of cotangent complexes for schemes

This section gives an alternative construction of the cotangent complex of a morphism of schemes. This section is currently in the obsolete chapter as we can get by with the easier version discussed in Cotangent, Section 25 for applications.

Let $f : X \rightarrow Y$ be a morphism of schemes. Let $\mathcal{C}_{X/Y}$ be the category whose objects are commutative diagrams

$$(19.0.1) \quad \begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow{\quad} & A \\ \downarrow & & \downarrow & \searrow i & \\ Y & \longleftarrow & V & & \end{array}$$

of schemes where

- (1) U is an open subscheme of X ,
- (2) V is an open subscheme of Y , and
- (3) there exists an isomorphism $A = V \times \text{Spec}(P)$ over V where P is a polynomial algebra over \mathbf{Z} (on some set of variables).

In other words, A is an (infinite dimensional) affine space over V . Morphisms are given by commutative diagrams.

Notation. An object of $\mathcal{C}_{X/Y}$, i.e., a diagram (19.0.1), is often denoted $U \rightarrow A$ where it is understood that (a) U is an open subscheme of X , (b) $U \rightarrow A$ is a morphism over Y , (c) the image of the structure morphism $A \rightarrow Y$ is an open $V \subset Y$, and (d) $A \rightarrow V$ is an affine space. We'll write $U \rightarrow A/V$ to indicate $V \subset Y$

is the image of $A \rightarrow Y$. Recall that X_{Zar} denotes the small Zariski site X . There are forgetful functors

$$\mathcal{C}_{X/Y} \rightarrow X_{Zar}, (U \rightarrow A) \mapsto U \quad \text{and} \quad \mathcal{C}_{X/Y} \mapsto Y_{Zar}, (U \rightarrow A/V) \mapsto V.$$

Lemma 19.1. *Let $X \rightarrow Y$ be a morphism of schemes.*

- (1) *The category $\mathcal{C}_{X/Y}$ is fibred over X_{Zar} .*
- (2) *The category $\mathcal{C}_{X/Y}$ is fibred over Y_{Zar} .*
- (3) *The category $\mathcal{C}_{X/Y}$ is fibred over the category of pairs (U, V) where $U \subset X$, $V \subset Y$ are open and $f(U) \subset V$.*

Proof. Ad (1). Given an object $U \rightarrow A$ of $\mathcal{C}_{X/Y}$ and a morphism $U' \rightarrow U$ of X_{Zar} consider the object $i' : U' \rightarrow A$ of $\mathcal{C}_{X/Y}$ where i' is the composition of i and $U' \rightarrow U$. The morphism $(U' \rightarrow A) \rightarrow (U \rightarrow A)$ of $\mathcal{C}_{X/Y}$ is strongly cartesian over X_{Zar} .

Ad (2). Given an object $U \rightarrow A/V$ and $V' \rightarrow V$ we can set $U' = U \cap f^{-1}(V')$ and $A' = V' \times_V A$ to obtain a strongly cartesian morphism $(U' \rightarrow A') \rightarrow (U \rightarrow A)$ over $V' \rightarrow V$.

Ad (3). Denote $(X/Y)_{Zar}$ the category in (3). Given $U \rightarrow A/V$ and a morphism $(U', V') \rightarrow (U, V)$ in $(X/Y)_{Zar}$ we can consider $A' = V' \times_V A$. Then the morphism $(U' \rightarrow A'/V') \rightarrow (U \rightarrow A/V)$ is strongly cartesian in $\mathcal{C}_{X/Y}$ over $(X/Y)_{Zar}$. \square

We obtain a topology τ_X on $\mathcal{C}_{X/Y}$ by using the topology inherited from X_{Zar} (see Stacks, Section 10). If not otherwise stated this is the topology on $\mathcal{C}_{X/Y}$ we will consider. To be precise, a family of morphisms $\{(U_i \rightarrow A_i) \rightarrow (U \rightarrow A)\}$ is a covering of $\mathcal{C}_{X/Y}$ if and only if

- (1) $U = \bigcup U_i$, and
- (2) $A_i = A$ for all i .

We obtain the same collection of sheaves if we allow $A_i \cong A$ in (2). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/Y}) \rightarrow Sh(X_{Zar})$.

The site $\mathcal{C}_{X/Y}$ comes with several sheaves of rings.

- (1) The sheaf \mathcal{O} given by the rule $(U \rightarrow A) \mapsto \mathcal{O}(A)$.
- (2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \rightarrow A) \mapsto \mathcal{O}(U)$.
- (3) The sheaf $\underline{\mathcal{O}}_Y$ given by the rule $(U \rightarrow A/V) \mapsto \mathcal{O}(V)$.

We obtain morphisms of ringed topoi

$$(19.1.1) \quad \begin{array}{ccc} (Sh(\mathcal{C}_{X/Y}), \underline{\mathcal{O}}_X) & \xrightarrow{i} & (Sh(\mathcal{C}_{X/Y}), \mathcal{O}) \\ \pi \downarrow & & \\ (Sh(X_{Zar}), \mathcal{O}_X) & & \end{array}$$

The morphism i is the identity on underlying topoi and $i^\# : \mathcal{O} \rightarrow \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 38.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \rightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \rightarrow D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \rightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \rightarrow D(\underline{\mathcal{O}}_X)$.

Remark 19.2. We obtain a second topology τ_Y on $\mathcal{C}_{X/Y}$ by taking the topology inherited from Y_{Zar} . There is a third topology $\tau_{X \rightarrow Y}$ where a family of morphisms $\{(U_i \rightarrow A_i) \rightarrow (U \rightarrow A)\}$ is a covering if and only if $U = \bigcup U_i$, $V = \bigcup V_i$

and $A_i \cong V_i \times_V A$. This is the topology inherited from the topology on the site $(X/Y)_{Zar}$ whose underlying category is the category of pairs (U, V) as in Lemma 19.1 part (3). The coverings of $(X/Y)_{Zar}$ are families $\{(U_i, V_i) \rightarrow (U, V)\}$ such that $U = \bigcup U_i$ and $V = \bigcup V_i$. There are morphisms of topoi

$$Sh(\mathcal{C}_{X/Y}) = Sh(\mathcal{C}_{X/Y}, \tau_X) \longleftarrow Sh(\mathcal{C}_{X/Y}, \tau_{X \rightarrow Y}) \longrightarrow Sh(\mathcal{C}_{X/Y}, \tau_Y)$$

(recall that τ_X is our “default” topology). The pullback functors for these arrows are sheafification and pushforward is the identity on underlying presheaves. The diagram of topoi

$$\begin{array}{ccccc} Sh(X_{Zar}) & \xleftarrow{\pi} & Sh(\mathcal{C}_{X/Y}) & \xleftarrow{\quad} & Sh(\mathcal{C}_{X/Y}, \tau_{X \rightarrow Y}) \\ \downarrow f & & & & \downarrow \\ Sh(Y_{Zar}) & \xleftarrow{\quad} & & & Sh(\mathcal{C}_{X/Y}, \tau_Y) \end{array}$$

is **not** commutative. Namely, the pullback of a nonzero abelian sheaf on Y is a nonzero abelian sheaf on $(\mathcal{C}_{X/Y}, \tau_{X \rightarrow Y})$, but we can certainly find examples where such a sheaf pulls back to zero on X . Note that any presheaf \mathcal{F} on Y_{Zar} gives a sheaf $\underline{\mathcal{F}}$ on $\mathcal{C}_{Y/X}$ by the rule which assigns to $(U \rightarrow A/V)$ the set $\mathcal{F}(V)$. Even if \mathcal{F} happens to be a sheaf it isn't true in general that $\underline{\mathcal{F}} = \pi^{-1}f^{-1}\mathcal{F}$. This is related to the noncommutativity of the diagram above, as we can describe $\underline{\mathcal{F}}$ as the pushforward of the pullback of \mathcal{F} to $Sh(\mathcal{C}_{X/Y}, \tau_{X \rightarrow Y})$ via the lower horizontal and right vertical arrows. An example is the sheaf $\underline{\mathcal{O}}_Y$. But what is true is that there is a map $\underline{\mathcal{F}} \rightarrow \pi^{-1}f^{-1}\mathcal{F}$ which is transformed (as we shall see later) into an isomorphism after applying $\pi_!$.

20. Deformations and obstructions of flat modules

In this section we sketch a construction of a deformation theory for the stack of coherent sheaves for any algebraic space X over a ring Λ . This material is obsolete due to the improved discussion in Quot, Section 6.

Our setup will be the following. We assume given

- (1) a ring Λ ,
- (2) an algebraic space X over Λ ,
- (3) a Λ -algebra A , set $X_A = X \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(A)$, and
- (4) a finitely presented \mathcal{O}_{X_A} -module \mathcal{F} flat over A .

In this situation we will consider all possible surjections

$$0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0$$

where A' is a Λ -algebra whose kernel I is an ideal of square zero in A' . Given A' we obtain a first order thickening $X_A \rightarrow X_{A'}$ of algebraic spaces over $\mathrm{Spec}(\Lambda)$. For each of these we consider the problem of lifting \mathcal{F} to a finitely presented module \mathcal{F}' on $X_{A'}$ flat over A' . We would like to replicate the results of Deformation Theory, Lemma 12.1 in this setting.

To be more precise let $Lift(\mathcal{F}, A')$ denote the category of pairs (\mathcal{F}', α) where \mathcal{F}' is a finitely presented module on $X_{A'}$ flat over A' and $\alpha : \mathcal{F}'|_{X_A} \rightarrow \mathcal{F}$ is an isomorphism. Morphisms $(\mathcal{F}'_1, \alpha_1) \rightarrow (\mathcal{F}'_2, \alpha_2)$ are isomorphisms $\mathcal{F}'_1 \rightarrow \mathcal{F}'_2$ which are compatible with α_1 and α_2 . The set of isomorphism classes of $Lift(\mathcal{F}, A')$ is denoted $Lift(\mathcal{F}, A')$.

Let \mathcal{G} be a sheaf of $\mathcal{O}_X \otimes_\Lambda A$ -modules on $X_{\acute{e}tale}$ flat over A . We introduce the category $Lift(\mathcal{G}, A')$ of pairs (\mathcal{G}', β) where \mathcal{G}' is a sheaf of $\mathcal{O}_X \otimes_\Lambda A'$ -modules flat over A' and β is an isomorphism $\mathcal{G}' \otimes_{A'} A \rightarrow \mathcal{G}$.

Lemma 20.1. *Notation and assumptions as above. Let $p : X_A \rightarrow X$ denote the projection. Given A' denote $p' : X_{A'} \rightarrow X$ the projection. The functor p'_* induces an equivalence of categories between*

- (1) *the category $Lift(\mathcal{F}, A')$, and*
- (2) *the category $Lift(p_*\mathcal{F}, A')$.*

Proof. FIXME. □

Let \mathcal{H} be a sheaf of $\mathcal{O} \otimes_\Lambda A$ -modules on $\mathcal{C}_{X/\Lambda}$ flat over A . We introduce the category $Lift_{\mathcal{O}}(\mathcal{H}, A')$ whose objects are pairs (\mathcal{H}', γ) where \mathcal{H}' is a sheaf of $\mathcal{O} \otimes_\Lambda A'$ -modules flat over A' and $\gamma : \mathcal{H}' \otimes_{A'} A \rightarrow \mathcal{H}$ is an isomorphism of $\mathcal{O} \otimes_\Lambda A$ -modules.

Let \mathcal{G} be a sheaf of $\mathcal{O}_X \otimes_\Lambda A$ -modules on $X_{\acute{e}tale}$ flat over A . Consider the morphisms i and π of Cotangent, Equation (27.1.1). Denote $\underline{\mathcal{G}} = \pi^{-1}(\mathcal{G})$. It is simply given by the rule $(U \rightarrow \mathbf{A}) \mapsto \mathcal{G}(U)$ hence it is a sheaf of $\underline{\mathcal{O}}_X \otimes_\Lambda A$ -modules. Denote $i_*\underline{\mathcal{G}}$ the same sheaf but viewed as a sheaf of $\mathcal{O} \otimes_\Lambda A$ -modules.

Lemma 20.2. *Notation and assumptions as above. The functor $\pi_!$ induces an equivalence of categories between*

- (1) *the category $Lift_{\mathcal{O}}(i_*\underline{\mathcal{G}}, A')$, and*
- (2) *the category $Lift(\mathcal{G}, A')$.*

Proof. FIXME. □

Lemma 20.3. *Notation and assumptions as in Lemma 20.2. Consider the object*

$$L = L(\Lambda, X, A, \mathcal{G}) = L\pi_!(Li^*(i_*(\underline{\mathcal{G}})))$$

of $D(\mathcal{O}_X \otimes_\Lambda A)$. Given a surjection $A' \rightarrow A$ of Λ -algebras with square zero kernel I we have

- (1) *The category $Lift(\mathcal{G}, A')$ is nonempty if and only if a certain class $\xi \in \text{Ext}_{\mathcal{O}_X \otimes_\Lambda A}^2(L, \mathcal{G} \otimes_A I)$ is zero.*
- (2) *If $Lift(\mathcal{G}, A')$ is nonempty, then $Lift(\mathcal{G}, A')$ is principal homogeneous under $\text{Ext}_{\mathcal{O}_X \otimes_\Lambda A}^1(L, \mathcal{G} \otimes_A I)$.*
- (3) *Given a lift \mathcal{G}' , the set of automorphisms of \mathcal{G}' which pull back to $id_{\mathcal{G}}$ is canonically isomorphic to $\text{Ext}_{\mathcal{O}_X \otimes_\Lambda A}^0(L, \mathcal{G} \otimes_A I)$.*

Proof. FIXME. □

Finally, we put everything together as follows.

Proposition 20.4. *With $\Lambda, X, A, \mathcal{F}$ as above. There exists a canonical object $L = L(\Lambda, X, A, \mathcal{F})$ of $D(X_A)$ such that given a surjection $A' \rightarrow A$ of Λ -algebras with square zero kernel I we have*

- (1) *The category $Lift(\mathcal{F}, A')$ is nonempty if and only if a certain class $\xi \in \text{Ext}_{X_A}^2(L, \mathcal{F} \otimes_A I)$ is zero.*
- (2) *If $Lift(\mathcal{F}, A')$ is nonempty, then $Lift(\mathcal{F}, A')$ is principal homogeneous under $\text{Ext}_{X_A}^1(L, \mathcal{F} \otimes_A I)$.*
- (3) *Given a lift \mathcal{F}' , the set of automorphisms of \mathcal{F}' which pull back to $id_{\mathcal{F}}$ is canonically isomorphic to $\text{Ext}_{X_A}^0(L, \mathcal{F} \otimes_A I)$.*

Proof. FIXME. □

Lemma 20.5. *In the situation of Proposition 20.4, if $X \rightarrow \mathrm{Spec}(\Lambda)$ is locally of finite type and Λ is Noetherian, then L is pseudo-coherent.*

Proof. FIXME. □

21. The stack of coherent sheaves in the non-flat case

In Quot, Theorem 5.12 the assumption that $f : X \rightarrow B$ is flat is not necessary. In this section we modify the method of proof based on ideas from derived algebraic geometry to get around the flatness hypothesis. An entirely different method is used in Quot, Section 6 to get exactly the same result; this is why the method from this section is obsolete.

The only step in the proof of Quot, Theorem 5.12 which uses flatness is in the application of Quot, Lemma 5.11. The lemma is used to construct an obstruction theory as in Artin's Axioms, Section 24. The proof of the lemma relies on Deformation Theory, Lemmas 12.1 and 12.5 from Deformation Theory, Section 12. This is how the assumption that f is flat comes about. Before we go on, note that results (2) and (3) of Deformation Theory, Lemmas 12.1 do hold without the assumption that f is flat as they rely on Deformation Theory, Lemmas 11.7 and 11.4 which do not have any flatness assumptions.

Before we give the details we give some motivation for the construction from derived algebraic geometry, since we think it will clarify what follows. Let A be a finite type algebra over the locally Noetherian base S . Denote $X \otimes^{\mathbf{L}} A$ a “derived base change” of X to A and denote $i : X_A \rightarrow X \otimes^{\mathbf{L}} A$ the canonical inclusion morphism. The object $X \otimes^{\mathbf{L}} A$ does not (yet) have a definition in the Stacks project; we may think of it as the algebraic space X_A endowed with a simplicial sheaf of rings $\mathcal{O}_{X \otimes^{\mathbf{L}} A}$ whose homology sheaves are

$$H_i(\mathcal{O}_{X \otimes^{\mathbf{L}} A}) = \mathrm{Tor}_i^{\mathcal{O}_S}(\mathcal{O}_X, A).$$

The morphism $X \otimes^{\mathbf{L}} A \rightarrow \mathrm{Spec}(A)$ is flat (the terms of the simplicial sheaf of rings being A -flat), so the usual material for deformations of flat modules applies to it. Thus we see that we get an obstruction theory using the groups

$$\mathrm{Ext}_{X \otimes^{\mathbf{L}} A}^i(i_*\mathcal{F}, i_*\mathcal{F} \otimes_A M)$$

where $i = 0, 1, 2$ for inf auts, inf defs, obstructions. Note that a flat deformation of $i_*\mathcal{F}$ to $X \otimes^{\mathbf{L}} A'$ is automatically of the form $i'_*\mathcal{F}'$ where \mathcal{F}' is a flat deformation of \mathcal{F} . By adjunction of the functors Li^* and $i_* = Ri_*$ these ext groups are equal to

$$\mathrm{Ext}_{X_A}^i(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$$

Thus we obtain obstruction groups of exactly the same form as in the proof of Quot, Lemma 5.11 with the only change being that one replaces the first occurrence of \mathcal{F} by the complex $Li^*(i_*\mathcal{F})$.

Below we prove the non-flat version of the lemma by a “direct” construction of $E(\mathcal{F}) = Li^*(i_*\mathcal{F})$ and direct proof of its relationship to the deformation theory of \mathcal{F} . In fact, it suffices to construct $\tau_{\geq -2}E(\mathcal{F})$, as we are only interested in the

ext groups $\text{Ext}_{X_A}^i(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i = 0, 1, 2$. We can even identify the cohomology sheaves

$$H^i(E(\mathcal{F})) = \begin{cases} 0 & \text{if } i > 0 \\ \mathcal{F} & \text{if } i = 0 \\ 0 & \text{if } i = -1 \\ \text{Tor}_1^{\mathcal{O}_S}(\mathcal{O}_X, A) \otimes_{\mathcal{O}_X} \mathcal{F} & \text{if } i = -2 \end{cases}$$

This observation will guide our construction of $E(\mathcal{F})$ in the remarks below.

Remark 21.1 (Direct construction). Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . Let U be another algebraic space over B . Denote $q : X \times_B U \rightarrow U$ the second projection. Consider the distinguished triangle

$$Lq^*L_{U/B} \rightarrow L_{X \times_B U/B} \rightarrow E \rightarrow Lq^*L_{U/B}[1]$$

of Cotangent, Section 28. For any sheaf \mathcal{F} of $\mathcal{O}_{X \times_B U}$ -modules we have the Atiyah class

$$\mathcal{F} \rightarrow L_{X \times_B U/B} \otimes_{\mathcal{O}_{X \times_B U}}^{\mathbf{L}} \mathcal{F}[1]$$

see Cotangent, Section 19. We can compose this with the map to E and choose a distinguished triangle

$$E(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}}^{\mathbf{L}} E[1] \rightarrow E(\mathcal{F})[1]$$

in $D(\mathcal{O}_{X \times_B U})$. By construction the Atiyah class lifts to a map

$$e_{\mathcal{F}} : E(\mathcal{F}) \rightarrow Lq^*L_{U/B} \otimes_{\mathcal{O}_{X \times_B U}}^{\mathbf{L}} \mathcal{F}[1]$$

fitting into a morphism of distinguished triangles

$$\begin{array}{ccccc} \mathcal{F} \otimes^{\mathbf{L}} Lq^*L_{U/B}[1] & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} L_{X \times_B U/B}[1] & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} E[1] \\ e_{\mathcal{F}} \uparrow & & \text{Atiyah} \uparrow & & \uparrow = \\ E(\mathcal{F}) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} E[1] \end{array}$$

Given $S, B, X, f, U, \mathcal{F}$ we fix a choice of $E(\mathcal{F})$ and $e_{\mathcal{F}}$.

Remark 21.2 (Construction of obstruction class). With notation as in Remark 21.1 let $i : U \rightarrow U'$ be a first order thickening of U over B . Let $\mathcal{I} \subset \mathcal{O}_{U'}$ be the quasi-coherent sheaf of ideals cutting out B in B' . The fundamental triangle

$$Li^*L_{U'/B} \rightarrow L_{U/B} \rightarrow L_{U/U'} \rightarrow Li^*L_{U'/B}[1]$$

together with the map $L_{U/U'} \rightarrow \mathcal{I}[1]$ determine a map $e_{U'} : L_{U/B} \rightarrow \mathcal{I}[1]$. Combined with the map $e_{\mathcal{F}}$ of the previous remark we obtain

$$(\text{id}_{\mathcal{F}} \otimes Lq^*e_{U'}) \cup e_{\mathcal{F}} : E(\mathcal{F}) \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I}[2]$$

(we have also composed with the map from the derived tensor product to the usual tensor product). In other words, we obtain an element

$$\xi_{U'} \in \text{Ext}_{\mathcal{O}_{X \times_B U}}^2(E(\mathcal{F}), \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I})$$

Lemma 21.3. *In the situation of Remark 21.2 assume that \mathcal{F} is flat over U . Then the vanishing of the class $\xi_{U'}$ is a necessary and sufficient condition for the existence of a $\mathcal{O}_{X \times_B U'}$ -module \mathcal{F}' flat over U' with $i^*\mathcal{F}' \cong \mathcal{F}$.*

Proof (sketch). We will use the criterion of Deformation Theory, Lemma 11.8. We will abbreviate $\mathcal{O} = \mathcal{O}_{X \times_B U}$ and $\mathcal{O}' = \mathcal{O}_{X \times_B U'}$. Consider the short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{U'} \rightarrow \mathcal{O}_U \rightarrow 0.$$

Let $\mathcal{J} \subset \mathcal{O}'$ be the quasi-coherent sheaf of ideals cutting out $X \times_B U$. By the above we obtain an exact sequence

$$\mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \rightarrow q^* \mathcal{I} \rightarrow \mathcal{J} \rightarrow 0$$

where the $\mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U)$ is an abbreviation for

$$\mathrm{Tor}_1^{h^{-1}\mathcal{O}_B}(p^{-1}\mathcal{O}_X, q^{-1}\mathcal{O}_U) \otimes_{(p^{-1}\mathcal{O}_X \otimes_{h^{-1}\mathcal{O}_B} q^{-1}\mathcal{O}_U)} \mathcal{O}.$$

Tensoring with \mathcal{F} we obtain the exact sequence

$$\mathcal{F} \otimes_{\mathcal{O}} \mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \rightarrow \mathcal{F} \otimes_{\mathcal{O}} q^* \mathcal{I} \rightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{J} \rightarrow 0$$

(Note that the roles of the letters \mathcal{I} and \mathcal{J} are reversed relative to the notation in Deformation Theory, Lemma 11.8.) Condition (1) of the lemma is that the last map above is an isomorphism, i.e., that the first map is zero. The vanishing of this map may be checked on stalks at geometric points $\bar{z} = (\bar{x}, \bar{u}) : \mathrm{Spec}(k) \rightarrow X \times_B U$. Set $R = \mathcal{O}_{B, \bar{b}}$, $A = \mathcal{O}_{X, \bar{x}}$, $B = \mathcal{O}_{U, \bar{u}}$, and $C = \mathcal{O}_{\bar{z}}$. By Cotangent, Lemma 28.2 and the defining triangle for $E(\mathcal{F})$ we see that

$$H^{-2}(E(\mathcal{F}))_{\bar{z}} = \mathcal{F}_{\bar{z}} \otimes \mathrm{Tor}_1^R(A, B)$$

The map $\xi_{U'}$ therefore induces a map

$$\mathcal{F}_{\bar{z}} \otimes \mathrm{Tor}_1^R(A, B) \longrightarrow \mathcal{F}_{\bar{z}} \otimes_B \mathcal{I}_{\bar{u}}$$

We claim this map is the same as the stalk of the map described above (proof omitted; this is a purely ring theoretic statement). Thus we see that condition (1) of Deformation Theory, Lemma 11.8 is equivalent to the vanishing $H^{-2}(\xi_{U'}) : H^{-2}(E(\mathcal{F})) \rightarrow \mathcal{F} \otimes \mathcal{I}$.

To finish the proof we show that, assuming that condition (1) is satisfied, condition (2) is equivalent to the vanishing of $\xi_{U'}$. In the rest of the proof we write $\mathcal{F} \otimes \mathcal{I}$ to denote $\mathcal{F} \otimes_{\mathcal{O}} q^* \mathcal{I} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{J}$. A consideration of the spectral sequence

$$\mathrm{Ext}^i(H^{-j}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I}) \Rightarrow \mathrm{Ext}^{i+j}(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I})$$

using that $H^0(E(\mathcal{F})) = \mathcal{F}$ and $H^{-1}(E(\mathcal{F})) = 0$ shows that there is an exact sequence

$$0 \rightarrow \mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I}) \rightarrow \mathrm{Ext}^2(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I}) \rightarrow \mathrm{Hom}(H^{-2}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I})$$

Thus our element $\xi_{U'}$ is an element of $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I})$. The proof is finished by showing this element agrees with the element of Deformation Theory, Lemma 11.8 a verification we omit. \square

Lemma 21.4. *In Quot, Situation 5.1 assume that S is a locally Noetherian scheme and $S = B$. Let $\mathcal{X} = \mathrm{Coh}_{X/B}$. Then we have openness of versality for \mathcal{X} (see Artin's Axioms, Definition 13.1).*

Proof (sketch). Let $U \rightarrow S$ be of finite type morphism of schemes, x an object of \mathcal{X} over U and $u_0 \in U$ a finite type point such that x is versal at u_0 . After shrinking U we may assume that u_0 is a closed point (Morphisms, Lemma 16.1) and $U = \mathrm{Spec}(A)$ with $U \rightarrow S$ mapping into an affine open $\mathrm{Spec}(\Lambda)$ of S . We

will use Artin's Axioms, Lemma 24.4 to prove the lemma. Let \mathcal{F} be the coherent module on $X_A = \text{Spec}(A) \times_S X$ flat over A corresponding to the given object x .

Choose $E(\mathcal{F})$ and $e_{\mathcal{F}}$ as in Remark 21.1. The description of the cohomology sheaves of $E(\mathcal{F})$ shows that

$$\text{Ext}^1(E(\mathcal{F}), \mathcal{F} \otimes_A M) = \text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for any A -module M . Using this and using Deformation Theory, Lemma 11.7 we have an isomorphism of functors

$$T_x(M) = \text{Ext}_{X_A}^1(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

By Lemma 21.3 given any surjection $A' \rightarrow A$ of Λ -algebras with square zero kernel I we have an obstruction class

$$\xi_{A'} \in \text{Ext}_{X_A}^2(E(\mathcal{F}), \mathcal{F} \otimes_A I)$$

Apply Derived Categories of Spaces, Lemma 23.3 to the computation of the Ext groups $\text{Ext}_{X_A}^i(E(\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We omit the verification that $E(\mathcal{F})$ is in D_{Coh}^- ; hint: use Cotangent, Lemma 5.4. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes_A^{\mathbf{L}} M) \longrightarrow \text{Ext}_{X_A}^i(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

for $i \leq m$ compatible with boundary maps. This object K , together with the displayed identifications above gives us a datum as in Artin's Axioms, Situation 24.2. Finally, condition (iv) of Artin's Axioms, Lemma 24.3 holds by a variant of Deformation Theory, Lemma 12.5 whose formulation and proof we omit. Thus Artin's Axioms, Lemma 24.4 applies and the lemma is proved. \square

Theorem 21.5. *Let S be a scheme. Let $f : X \rightarrow B$ be morphism of algebraic spaces over S . Assume that f is of finite presentation and separated. Then $\text{Coh}_{X/B}$ is an algebraic stack over S .*

Proof. This theorem is a copy of Quot, Theorem 6.1. The reason we have this copy here is that with the material in this section we get a second proof (as discussed at the beginning of this section). Namely, we argue exactly as in the proof of Quot, Theorem 5.12 except that we substitute Lemma 21.4 for Quot, Lemma 5.11. \square

22. Modifications

Here is an obsolete result on the category of Algebraization of Formal Spaces, Equation (30.0.1). Please visit Algebraization of Formal Spaces, Section 30 for the current material.

Lemma 22.1. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. The category of Algebraization of Formal Spaces, Equation (30.0.1) for A is equivalent to the category Algebraization of Formal Spaces, Equation (30.0.1) for the henselization A^h of A .*

Proof. This is a special case of Algebraization of Formal Spaces, Lemma 30.3. \square

The following lemma on rational singularities is no longer needed in the chapter on resolving surface singularities.

Lemma 22.2. *In Resolution of Surfaces, Situation 9.1. Let M be a finite reflexive A -module. Let $M \otimes_A \mathcal{O}_X$ denote the pullback of the associated \mathcal{O}_S -module. Then $M \otimes_A \mathcal{O}_X$ maps onto its double dual.*

Proof. Let $\mathcal{F} = (M \otimes_A \mathcal{O}_X)^{**}$ be the double dual and let $\mathcal{F}' \subset \mathcal{F}$ be the image of the evaluation map $M \otimes_A \mathcal{O}_X \rightarrow \mathcal{F}$. Then we have a short exact sequence

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{Q} \rightarrow 0$$

Since X is normal, the local rings $\mathcal{O}_{X,x}$ are discrete valuation rings for points of codimension 1 (see Properties, Lemma 12.5). Hence $\mathcal{Q}_x = 0$ for such points by More on Algebra, Lemma 23.3. Thus \mathcal{Q} is supported in finitely many closed points and is globally generated by Cohomology of Schemes, Lemma 9.10. We obtain the exact sequence

$$0 \rightarrow H^0(X, \mathcal{F}') \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{Q}) \rightarrow 0$$

because \mathcal{F}' is generated by global sections (Resolution of Surfaces, Lemma 9.2). Since $X \rightarrow \operatorname{Spec}(A)$ is an isomorphism over the complement of the closed point, and since M is reflexive, we see that the maps

$$M \rightarrow H^0(X, \mathcal{F}') \rightarrow H^0(X, \mathcal{F})$$

induce isomorphisms after localization at any nonmaximal prime of A . Hence these maps are isomorphisms by More on Algebra, Lemma 23.13 and the fact that reflexive modules over normal rings have property (S_2) (More on Algebra, Lemma 23.18). Thus we conclude that $\mathcal{Q} = 0$ as desired. \square

23. Intersection theory

Lemma 23.1. *Let $b : X' \rightarrow X$ be the blowing up of a smooth projective scheme over a field k in a smooth closed subscheme $Z \subset X$. Picture*

$$\begin{array}{ccc} E & \xrightarrow{j} & X' \\ \pi \downarrow & & \downarrow b \\ Z & \xrightarrow{i} & X \end{array}$$

Assume there exists an element of $K_0(X)$ whose restriction to Z is equal to the class of $\mathcal{C}_{Z/X}$ in $K_0(Z)$. Then $[Lb^\mathcal{O}_Z] = [\mathcal{O}_E] \cdot \alpha''$ in $K_0(X')$ for some $\alpha'' \in K_0(X')$.*

Proof. The schemes X, X', E, Z are smooth and projective over k and hence we have $K'_0(X) = K_0(X) = K_0(\operatorname{Vect}(X)) = K_0(D_{\operatorname{Coh}}^b(X))$ and similarly for the other 3. See Derived Categories of Schemes, Lemmas 38.1, 38.4, and 38.5. We will switch between these versions at will in this proof. Consider the short exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \pi^*\mathcal{C}_{Z/X} \rightarrow \mathcal{C}_{E/X'} \rightarrow 0$$

of finite locally free \mathcal{O}_E -modules defining \mathcal{F} . Observe that $\mathcal{C}_{E/X'} = \mathcal{O}_{X'}(-E)|_E$ is the restriction of the invertible \mathcal{O}_X -module $\mathcal{O}_{X'}(-E)$. Let $\alpha \in K_0(X)$ be an element such that $i^*\alpha = [\mathcal{C}_{Z/X}]$ in $K_0(Z)$. Let $\alpha' = b^*\alpha - [\mathcal{O}_{X'}(-E)]$. Then $j^*\alpha' = [\mathcal{F}]$. We deduce that $j^*\lambda^i(\alpha') = [\wedge^i(\mathcal{F})]$ by Weil Cohomology Theories, Lemma 13.1. This means that $[\mathcal{O}_E] \cdot \alpha' = [\wedge^i \mathcal{F}]$ in $K_0(X)$, see Derived Categories of Schemes, Lemma 38.8. Let r be the maximum codimension of an irreducible component of Z in X . A computation which we omit shows that $H^{-i}(Lb^*\mathcal{O}_Z) = \wedge^i \mathcal{F}$ for $i \geq 0, 1, \dots, r-1$ and zero in other degrees. It follows that in $K_0(X)$ we

have

$$\begin{aligned} [Lb^*\mathcal{O}_Z] &= \sum_{i=0,\dots,r-1} (-1)^i [\wedge^i \mathcal{F}] \\ &= \sum_{i=0,\dots,r-1} (-1)^i [\mathcal{O}_E] \lambda^i(\alpha') \\ &= [\mathcal{O}_E] \left(\sum_{i=0,\dots,r-1} (-1)^i \lambda^i(\alpha') \right) \end{aligned}$$

This proves the lemma with $\alpha'' = \sum_{i=0,\dots,r-1} (-1)^i \lambda^i(\alpha')$. \square

Lemma 23.2. *Let (S, δ) be as in Chow Homology, Situation 7.1. Let X be locally of finite type over S . Let X be integral and $n = \dim_\delta(X)$. Let $a \in \Gamma(X, \mathcal{O}_X)$ be a nonzero function. Let $i : D = Z(a) \rightarrow X$ be the closed immersion of the zero scheme of a . Let $f \in R(X)^*$. In this case $i^* \operatorname{div}_X(f) = 0$ in $A_{n-2}(D)$.*

Proof. Special case of Chow Homology, Lemma 30.1. \square

Remark 23.3. This remark used to say that it wasn't clear whether the arrows of Chow Homology, Lemma 23.2 were isomorphisms in general. However, we've now found a proof of this fact.

23.4. Blowing up lemmas. In this section we prove some lemmas on representing Cartier divisors by suitable effective Cartier divisors on blowups. These lemmas can be found in [Ful98, Section 2.4]. We have adapted the formulation so they also work in the non-finite type setting. It may happen that the morphism b of Lemma 23.11 is a composition of infinitely many blowups, but over any given quasi-compact open $W \subset X$ one needs only finitely many blowups (and this is the result of loc. cit.).

Lemma 23.5. *Let (S, δ) be as in Chow Homology, Situation 7.1. Let X, Y be locally of finite type over S . Let $f : X \rightarrow Y$ be a proper morphism. Let $D \subset Y$ be an effective Cartier divisor. Assume X, Y integral, $n = \dim_\delta(X) = \dim_\delta(Y)$ and f dominant. Then*

$$f_*[f^{-1}(D)]_{n-1} = [R(X) : R(Y)][D]_{n-1}.$$

In particular if f is birational then $f_[f^{-1}(D)]_{n-1} = [D]_{n-1}$.*

Proof. Immediate from Chow Homology, Lemma 26.3 and the fact that D is the zero scheme of the canonical section 1_D of $\mathcal{O}_X(D)$. \square

Lemma 23.6. *Let (S, δ) be as in Chow Homology, Situation 7.1. Let X be locally of finite type over S . Assume X integral with $\dim_\delta(X) = n$. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let s be a nonzero meromorphic section of \mathcal{L} . Let $U \subset X$ be the maximal open subscheme such that s corresponds to a section of \mathcal{L} over U . There exists a projective morphism*

$$\pi : X' \longrightarrow X$$

such that

- (1) X' is integral,
- (2) $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \rightarrow U$ is an isomorphism,
- (3) there exist effective Cartier divisors $D, E \subset X'$ such that

$$\pi^* \mathcal{L} = \mathcal{O}_{X'}(D - E),$$

- (4) the meromorphic section s corresponds, via the isomorphism above, to the meromorphic section $1_D \otimes (1_E)^{-1}$ (see Divisors, Definition 14.1),

(5) we have

$$\pi_*([D]_{n-1} - [E]_{n-1}) = \operatorname{div}_{\mathcal{L}}(s)$$

in $Z_{n-1}(X)$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf of denominators of s , see Divisors, Definition 23.10. By Divisors, Lemma 34.6 we get (2), (3), and (4). By Divisors, Lemma 32.9 we get (1). By Divisors, Lemma 32.13 the morphism π is projective. We still have to prove (5). By Chow Homology, Lemma 26.3 we have

$$\pi_*(\operatorname{div}_{\mathcal{L}'}(s')) = \operatorname{div}_{\mathcal{L}}(s).$$

Hence it suffices to show that $\operatorname{div}_{\mathcal{L}'}(s') = [D]_{n-1} - [E]_{n-1}$. This follows from the equality $s' = 1_D \otimes 1_E^{-1}$ and additivity, see Divisors, Lemma 27.5. \square

Definition 23.7. Let (S, δ) be as in Chow Homology, Situation 7.1. Let X be locally of finite type over S . Assume X integral and $\dim_{\delta}(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X . Let $Z \subset X$ be an integral closed subscheme with $\dim_{\delta}(Z) = n - 1$. The ϵ -invariant of this situation is

$$\epsilon_Z(D_1, D_2) = n_Z \cdot m_Z$$

where n_Z , resp. m_Z is the coefficient of Z in the $(n-1)$ -cycle $[D_1]_{n-1}$, resp. $[D_2]_{n-1}$.

Lemma 23.8. Let (S, δ) be as in Chow Homology, Situation 7.1. Let X be locally of finite type over S . Assume X integral and $\dim_{\delta}(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X . Let Z be an open and closed subscheme of the scheme $D_1 \cap D_2$. Assume $\dim_{\delta}(D_1 \cap D_2 \setminus Z) \leq n - 2$. Then there exists a morphism $b : X' \rightarrow X$, and Cartier divisors D'_1, D'_2, E on X' with the following properties

- (1) X' is integral,
- (2) b is projective,
- (3) b is the blowup of X in the closed subscheme Z ,
- (4) $E = b^{-1}(Z)$,
- (5) $b^{-1}(D_1) = D'_1 + E$, and $b^{-1}D_2 = D'_2 + E$,
- (6) $\dim_{\delta}(D'_1 \cap D'_2) \leq n - 2$, and if $Z = D_1 \cap D_2$ then $D'_1 \cap D'_2 = \emptyset$,
- (7) for every integral closed subscheme W' with $\dim_{\delta}(W') = n - 1$ we have
 - (a) if $\epsilon_{W'}(D'_1, E) > 0$, then setting $W = b(W')$ we have $\dim_{\delta}(W) = n - 1$ and

$$\epsilon_{W'}(D'_1, E) < \epsilon_W(D_1, D_2),$$

- (b) if $\epsilon_{W'}(D'_2, E) > 0$, then setting $W = b(W')$ we have $\dim_{\delta}(W) = n - 1$ and

$$\epsilon_{W'}(D'_2, E) < \epsilon_W(D_1, D_2),$$

Proof. Note that the quasi-coherent ideal sheaf $\mathcal{I} = \mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ defines the scheme theoretic intersection $D_1 \cap D_2 \subset X$. Since Z is a union of connected components of $D_1 \cap D_2$ we see that for every $z \in Z$ the kernel of $\mathcal{O}_{X,z} \rightarrow \mathcal{O}_{Z,z}$ is equal to \mathcal{I}_z . Let $b : X' \rightarrow X$ be the blowup of X in Z . (So Zariski locally around Z it is the blowup of X in \mathcal{I} .) Denote $E = b^{-1}(Z)$ the corresponding effective Cartier divisor, see Divisors, Lemma 32.4. Since $Z \subset D_1$ we have $E \subset f^{-1}(D_1)$ and hence $D_1 = D'_1 + E$ for some effective Cartier divisor $D'_1 \subset X'$, see Divisors, Lemma 13.8. Similarly $D_2 = D'_2 + E$. This takes care of assertions (1) – (5).

Note that if W' is as in (7) (a) or (7) (b), then the image W of W' is contained in $D_1 \cap D_2$. If W is not contained in Z , then b is an isomorphism at the generic

point of W and we see that $\dim_\delta(W) = \dim_\delta(W') = n - 1$ which contradicts the assumption that $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n - 2$. Hence $W \subset Z$. This means that to prove (6) and (7) we may work locally around Z on X .

Thus we may assume that $X = \text{Spec}(A)$ with A a Noetherian domain, and $D_1 = \text{Spec}(A/a)$, $D_2 = \text{Spec}(A/b)$ and $Z = D_1 \cap D_2$. Set $I = (a, b)$. Since A is a domain and $a, b \neq 0$ we can cover the blowup by two patches, namely $U = \text{Spec}(A[s]/(as - b))$ and $V = \text{Spec}(A[t]/(bt - a))$. These patches are glued using the isomorphism $A[s, s^{-1}]/(as - b) \cong A[t, t^{-1}]/(bt - a)$ which maps s to t^{-1} . The effective Cartier divisor E is described by $\text{Spec}(A[s]/(as - b, a)) \subset U$ and $\text{Spec}(A[t]/(bt - a, b)) \subset V$. The closed subscheme D'_1 corresponds to $\text{Spec}(A[t]/(bt - a, t)) \subset U$. The closed subscheme D'_2 corresponds to $\text{Spec}(A[s]/(as - b, s)) \subset V$. Since “ $ts = 1$ ” we see that $D'_1 \cap D'_2 = \emptyset$.

Suppose we have a prime $\mathfrak{q} \subset A[s]/(as - b)$ of height one with $s, a \in \mathfrak{q}$. Let $\mathfrak{p} \subset A$ be the corresponding prime of A . Observe that $a, b \in \mathfrak{p}$. By the dimension formula we see that $\dim(A_{\mathfrak{p}}) = 1$ as well. The final assertion to be shown is that

$$\text{ord}_{A_{\mathfrak{p}}}(a)\text{ord}_{A_{\mathfrak{p}}}(b) > \text{ord}_{B_{\mathfrak{q}}}(a)\text{ord}_{B_{\mathfrak{q}}}(s)$$

where $B = A[s]/(as - b)$. By Algebra, Lemma 124.1 we have $\text{ord}_{A_{\mathfrak{p}}}(x) \geq \text{ord}_{B_{\mathfrak{q}}}(x)$ for $x = a, b$. Since $\text{ord}_{B_{\mathfrak{q}}}(s) > 0$ we win by additivity of the ord function and the fact that $as = b$. \square

Definition 23.9. Let X be a scheme. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X . Suppose given a function $I \rightarrow \mathbf{Z}_{\geq 0}$, $i \mapsto n_i$. The *sum of the effective Cartier divisors* $D = \sum n_i D_i$, is the unique effective Cartier divisor $D \subset X$ such that on any quasi-compact open $U \subset X$ we have $D|_U = \sum_{D_i \cap U \neq \emptyset} n_i D_i|_U$ is the sum as in Divisors, Definition 13.6.

Lemma 23.10. Let (S, δ) be as in Chow Homology, Situation 7.1. Let X be locally of finite type over S . Assume X integral and $\dim_\delta(X) = n$. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X . Suppose given $n_i \geq 0$ for $i \in I$. Then

$$[D]_{n-1} = \sum_i n_i [D_i]_{n-1}$$

in $Z_{n-1}(X)$.

Proof. Since we are proving an equality of cycles we may work locally on X . Hence this reduces to a finite sum, and by induction to a sum of two effective Cartier divisors $D = D_1 + D_2$. By Chow Homology, Lemma 24.2 we see that $D_1 = \text{div}_{\mathcal{O}_X(D_1)}(1_{D_1})$ where 1_{D_1} denotes the canonical section of $\mathcal{O}_X(D_1)$. Of course we have the same statement for D_2 and D . Since $1_D = 1_{D_1} \otimes 1_{D_2}$ via the identification $\mathcal{O}_X(D) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ we win by Divisors, Lemma 27.5. \square

Lemma 23.11. Let (S, δ) be as in Chow Homology, Situation 7.1. Let X be locally of finite type over S . Assume X integral and $\dim_\delta(X) = d$. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X . Assume that for all $\{i, j, k\} \subset I$, $\#\{i, j, k\} = 3$ we have $D_i \cap D_j \cap D_k = \emptyset$. Then there exist

- (1) an open subscheme $U \subset X$ with $\dim_\delta(X \setminus U) \leq d - 3$,
- (2) a morphism $b : U' \rightarrow U$, and
- (3) effective Cartier divisors $\{D'_j\}_{j \in J}$ on U'

with the following properties:

- (1) b is proper morphism $b : U' \rightarrow U$,
- (2) U' is integral,
- (3) b is an isomorphism over the complement of the union of the pairwise intersections of the $D_i|_U$,
- (4) $\{D'_j\}_{j \in J}$ is a locally finite collection of effective Cartier divisors on U' ,
- (5) $\dim_\delta(D'_j \cap D'_{j'}) \leq d - 2$ if $j \neq j'$, and
- (6) $b^{-1}(D_i|_U) = \sum n_{ij} D'_j$ for certain $n_{ij} \geq 0$.

Moreover, if X is quasi-compact, then we may assume $U = X$ in the above.

Proof. Let us first prove this in the quasi-compact case, since it is perhaps the most interesting case. In this case we produce inductively a sequence of blowups

$$X = X_0 \xleftarrow{b_0} X_1 \xleftarrow{b_1} X_2 \leftarrow \dots$$

and finite sets of effective Cartier divisors $\{D_{n,i}\}_{i \in I_n}$. At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \amalg P(I_n)$ where $P(I_n)$ denotes the set of pairs of elements of I_n . Finally, we will have

$$b_n^{-1}(D_{n,i}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$$

We conclude that for each $n \geq 0$ we have $(b_0 \circ \dots \circ b_n)^{-1}(D_i)$ is a nonnegative integer combination of the divisors $D_{n+1,j}$, $j \in I_{n+1}$.

To start the induction we set $X_0 = X$ and $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\}_{i \in I_n})$ let X_{n+1} be the blowup of X_n in the closed subscheme $Z_n = \bigcup_{\{i,i'\} \in P(I_n)} D_{n,i} \cap D_{n,i'}$. Note that the closed subschemes $D_{n,i} \cap D_{n,i'}$ are pairwise disjoint by our assumption on triple intersections. In other words we may write $Z_n = \coprod_{\{i,i'\} \in P(I_n)} D_{n,i} \cap D_{n,i'}$. Moreover, in a Zariski neighbourhood of $D_{n,i} \cap D_{n,i'}$ the morphism b_n is equal to the blowup of the scheme X_n in the closed subscheme $D_{n,i} \cap D_{n,i'}$, and the results of Lemma 23.8 apply. Hence setting $D_{n+1,\{i,i'\}} = b_n^{-1}(D_i \cap D_{i'})$ we get an effective Cartier divisor. The Cartier divisors $D_{n+1,\{i,i'\}}$ are pairwise disjoint. Clearly we have $b_n^{-1}(D_{n,i}) \supset D_{n+1,\{i,i'\}}$ for every $i' \in I_n$, $i' \neq i$. Hence, applying Divisors, Lemma 13.8 we see that indeed $b^{-1}(D_{n,i}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$ for some effective Cartier divisor $D_{n+1,i}$ on X_{n+1} . In a neighbourhood of $D_{n+1,\{i,i'\}}$ these divisors $D_{n+1,i}$ play the role of the primed divisors of Lemma 23.8. In particular we conclude that $D_{n+1,i} \cap D_{n+1,i'} = \emptyset$ if $i \neq i'$, $i, i' \in I_n$ by part (6) of Lemma 23.8. This already implies that triple intersections of the divisors $D_{n+1,i}$ are zero.

OK, and at this point we can use the quasi-compactness of X to conclude that the invariant

(23.11.1)

$$\epsilon(X, \{D_i\}_{i \in I}) = \max\{\epsilon_Z(D_i, D_{i'}) \mid Z \subset X, \dim_\delta(Z) = d - 1, \{i, i'\} \in P(I)\}$$

is finite, since after all each D_i has at most finitely many irreducible components. We claim that for some n the invariant $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$ is zero. Namely, if not then by Lemma 23.8 we have a strictly decreasing sequence

$$\epsilon(X, \{D_i\}_{i \in I}) = \epsilon(X_0, \{D_{0,i}\}_{i \in I_0}) > \epsilon(X_1, \{D_{1,i}\}_{i \in I_1}) > \dots$$

of positive integers which is a contradiction. Take n with invariant $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$ equal to zero. This means that there is no integral closed subscheme $Z \subset X_n$

and no pair of indices $i, i' \in I_n$ such that $\epsilon_Z(D_{n,i}, D_{n,i'}) > 0$. In other words, $\dim_\delta(D_{n,i}, D_{n,i'}) \leq d-2$ for all pairs $\{i, i'\} \in P(I_n)$ as desired.

Next, we come to the general case where we no longer assume that the scheme X is quasi-compact. The problem with the idea from the first part of the proof is that we may get an infinite sequence of blowups with centers dominating a fixed point of X . In order to avoid this we cut out suitable closed subsets of codimension ≥ 3 at each stage. Namely, we will construct by induction a sequence of morphisms having the following shape

$$\begin{array}{c}
 X = X_0 \\
 \uparrow j_0 \\
 U_0 \xleftarrow{b_0} X_1 \\
 \uparrow j_1 \\
 U_1 \xleftarrow{b_1} X_2 \\
 \uparrow j_2 \\
 U_2 \xleftarrow{b_2} X_3
 \end{array}$$

Each of the morphisms $j_n : U_n \rightarrow X_n$ will be an open immersion. Each of the morphisms $b_n : X_{n+1} \rightarrow U_n$ will be a proper birational morphism of integral schemes. As in the quasi-compact case we will have effective Cartier divisors $\{D_{n,i}\}_{i \in I_n}$ on X_n . At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \amalg P(I_n)$ where $P(I_n)$ denotes the set of pairs of elements of I_n . Finally, we will arrange it so that

$$b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1, \{i, i'\}}$$

We start the induction by setting $X_0 = X$, $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\})$ we construct the open subscheme U_n as follows. For each pair $\{i, i'\} \in P(I_n)$ consider the closed subscheme $D_{n,i} \cap D_{n,i'}$. This has “good” irreducible components which have δ -dimension $d-2$ and “bad” irreducible components which have δ -dimension $d-1$. Let us set

$$\text{Bad}(i, i') = \bigcup_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W)=d-1} W$$

and similarly

$$\text{Good}(i, i') = \bigcup_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W)=d-2} W.$$

Then $D_{n,i} \cap D_{n,i'} = \text{Bad}(i, i') \cup \text{Good}(i, i')$ and moreover we have $\dim_\delta(\text{Bad}(i, i') \cap \text{Good}(i, i')) \leq d-3$. Here is our choice of U_n :

$$U_n = X_n \setminus \bigcup_{\{i, i'\} \in P(I_n)} \text{Bad}(i, i') \cap \text{Good}(i, i').$$

By our condition on triple intersections of the divisors $D_{n,i}$ we see that the union is actually a disjoint union. Moreover, we see that (as a scheme)

$$D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n} = Z_{n,i,i'} \amalg G_{n,i,i'}$$

where $Z_{n,i,i'}$ is δ -equidimensional of dimension $d-1$ and $G_{n,i,i'}$ is δ -equidimensional of dimension $d-2$. (So topologically $Z_{n,i,i'}$ is the union of the bad components but throw out intersections with good components.) Finally we set

$$Z_n = \bigcup_{\{i,i'\} \in P(I_n)} Z_{n,i,i'} = \coprod_{\{i,i'\} \in P(I_n)} Z_{n,i,i'},$$

and we let $b_n : X_{n+1} \rightarrow X_n$ be the blowup in Z_n . Note that Lemma 23.8 applies to the morphism $b_n : X_{n+1} \rightarrow X_n$ locally around each of the loci $D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n}$. Hence, exactly as in the first part of the proof we obtain effective Cartier divisors $D_{n+1,\{i,i'\}}$ for $\{i,i'\} \in P(I_n)$ and effective Cartier divisors $D_{n+1,i}$ for $i \in I_n$ such that $b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$. For each n denote $\pi_n : X_n \rightarrow X$ the morphism obtained as the composition $j_0 \circ \dots \circ j_{n-1} \circ b_{n-1}$.

Claim: given any quasi-compact open $V \subset X$ for all sufficiently large n the maps

$$\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V) \leftarrow \dots$$

are all isomorphisms. Namely, if the map $\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V)$ is not an isomorphism, then $Z_{n,i,i'} \cap \pi_n^{-1}(V) \neq \emptyset$ for some $\{i,i'\} \in P(I_n)$. Hence there exists an irreducible component $W \subset D_{n,i} \cap D_{n,i'}$ with $\dim_\delta(W) = d-1$. In particular we see that $\epsilon_W(D_{n,i}, D_{n,i'}) > 0$. Applying Lemma 23.8 repeatedly we see that

$$\epsilon_W(D_{n,i}, D_{n,i'}) < \epsilon(V, \{D_i|_V\}) - n$$

with $\epsilon(V, \{D_i|_V\})$ as in (23.11.1). Since V is quasi-compact, we have $\epsilon(V, \{D_i|_V\}) < \infty$ and taking $n > \epsilon(V, \{D_i|_V\})$ we see the result.

Note that by construction the difference $X_n \setminus U_n$ has $\dim_\delta(X_n \setminus U_n) \leq d-3$. Let $T_n = \pi_n(X_n \setminus U_n)$ be its image in X . Traversing in the diagram of maps above using each b_n is closed it follows that $T_0 \cup \dots \cup T_n$ is a closed subset of X for each n . Any $t \in T_n$ satisfies $\delta(t) \leq d-3$ by construction. Hence $\overline{T_n} \subset X$ is a closed subset with $\dim_\delta(T_n) \leq d-3$. By the claim above we see that for any quasi-compact open $V \subset X$ we have $T_n \cap V \neq \emptyset$ for at most finitely many n . Hence $\{\overline{T_n}\}_{n \geq 0}$ is a locally finite collection of closed subsets, and we may set $U = X \setminus \bigcup \overline{T_n}$. This will be U as in the lemma.

Note that $U_n \cap \pi_n^{-1}(U) = \pi_n^{-1}(U)$ by construction of U . Hence all the morphisms

$$b_n : \pi_{n+1}^{-1}(U) \longrightarrow \pi_n^{-1}(U)$$

are proper. Moreover, by the claim they eventually become isomorphisms over each quasi-compact open of X . Hence we can define

$$U' = \lim_n \pi_n^{-1}(U).$$

The induced morphism $b : U' \rightarrow U$ is proper since this is local on U , and over each compact open the limit stabilizes. Similarly we set $J = \bigcup_{n \geq 0} I_n$ using the inclusions $I_n \rightarrow I_{n+1}$ from the construction. For $j \in J$ choose an n_0 such that j corresponds to $i \in I_{n_0}$ and define $D'_j = \lim_{n \geq n_0} D_{n,i}$. Again this makes sense as locally over X the morphisms stabilize. The other claims of the lemma are verified as in the case of a quasi-compact X . \square

24. Commutativity of intersecting divisors

The results of this section were originally used to provide an alternative proof of the lemmas of Chow Homology, Section 28 and a weak version of Chow Homology, Lemma 30.5.

Lemma 24.1. *Let (S, δ) be as in Chow Homology, Situation 7.1. Let X be locally of finite type over S . Let $\{i_j : D_j \rightarrow X\}_{j \in J}$ be a locally finite collection of effective Cartier divisors on X . Let $n_j > 0$, $j \in J$. Set $D = \sum_{j \in J} n_j D_j$, and denote $i : D \rightarrow X$ the inclusion morphism. Let $\alpha \in Z_{k+1}(X)$. Then*

$$p : \coprod_{j \in J} D_j \longrightarrow D$$

is proper and

$$i^* \alpha = p_* \left(\sum n_j i_j^* \alpha \right)$$

in $\text{CH}_k(D)$.

Proof. The proof of this lemma is made a bit longer than expected by a subtlety concerning infinite sums of rational equivalences. In the quasi-compact case the family D_j is finite and the result is altogether easy and a straightforward consequence of Chow Homology, Lemma 24.2 and Divisors, Lemma 27.5 and the definitions.

The morphism p is proper since the family $\{D_j\}_{j \in J}$ is locally finite. Write $\alpha = \sum_{a \in A} m_a [W_a]$ with $W_a \subset X$ an integral closed subscheme of δ -dimension $k+1$. Denote $i_a : W_a \rightarrow X$ the closed immersion. We assume that $m_a \neq 0$ for all $a \in A$ such that $\{W_a\}_{a \in A}$ is locally finite on X .

Observe that by Chow Homology, Definition 29.1 the class $i^* \alpha$ is the class of a cycle $\sum m_a \beta_a$ for certain $\beta_a \in Z_k(W_a \cap D)$. Namely, if $W_a \not\subset D$ then $\beta_a = [D \cap W_a]_k$ and if $W_a \subset D$, then β_a is a cycle representing $c_1(\mathcal{O}_X(D)) \cap [W_a]$.

For each $a \in A$ write $J = J_{a,1} \amalg J_{a,2} \amalg J_{a,3}$ where

- (1) $j \in J_{a,1}$ if and only if $W_a \cap D_j = \emptyset$,
- (2) $j \in J_{a,2}$ if and only if $W_a \neq W_a \cap D_1 \neq \emptyset$, and
- (3) $j \in J_{a,3}$ if and only if $W_a \subset D_j$.

Since the family $\{D_j\}$ is locally finite we see that $J_{a,3}$ is a finite set. For every $a \in A$ and $j \in J$ we choose a cycle $\beta_{a,j} \in Z_k(W_a \cap D_j)$ as follows

- (1) if $j \in J_{a,1}$ we set $\beta_{a,j} = 0$,
- (2) if $j \in J_{a,2}$ we set $\beta_{a,j} = [D_j \cap W_a]_k$, and
- (3) if $j \in J_{a,3}$ we choose $\beta_{a,j} \in Z_k(W_a)$ representing $c_1(i_a^* \mathcal{O}_X(D_j)) \cap [W_j]$.

We claim that

$$\beta_a \sim_{\text{rat}} \sum_{j \in J} n_j \beta_{a,j}$$

in $\text{CH}_k(W_a \cap D)$.

Case I: $W_a \not\subset D$. In this case $J_{a,3} = \emptyset$. Thus it suffices to show that $[D \cap W_a]_k = \sum n_j [D_j \cap W_a]_k$ as cycles. This is Lemma 23.10.

Case II: $W_a \subset D$. In this case β_a is a cycle representing $c_1(i_a^* \mathcal{O}_X(D)) \cap [W_a]$. Write $D = D_{a,1} + D_{a,2} + D_{a,3}$ with $D_{a,s} = \sum_{j \in J_{a,s}} n_j D_j$. By Divisors, Lemma 27.5 we

have

$$\begin{aligned} c_1(i_a^* \mathcal{O}_X(D)) \cap [W_a] &= c_1(i_a^* \mathcal{O}_X(D_{a,1})) \cap [W_a] + c_1(i_a^* \mathcal{O}_X(D_{a,2})) \cap [W_a] \\ &\quad + c_1(i_a^* \mathcal{O}_X(D_{a,3})) \cap [W_a]. \end{aligned}$$

It is clear that the first term of the sum is zero. Since $J_{a,3}$ is finite we see that the last term agrees with $\sum_{j \in J_{a,3}} n_j c_1(i_a^* \mathcal{L}_j) \cap [W_a]$, see Divisors, Lemma 27.5. This is represented by $\sum_{j \in J_{a,3}} n_j \beta_{a,j}$. Finally, by Case I we see that the middle term is represented by the cycle $\sum_{j \in J_{a,2}} n_j [D_j \cap W_a]_k = \sum_{j \in J_{a,2}} n_j \beta_{a,j}$. Whence the claim in this case.

At this point we are ready to finish the proof of the lemma. Namely, we have $i^* D \sim_{\text{rat}} \sum m_a \beta_a$ by our choice of β_a . For each a we have $\beta_a \sim_{\text{rat}} \sum_j \beta_{a,j}$ with the rational equivalence taking place on $D \cap W_a$. Since the collection of closed subschemes $D \cap W_a$ is locally finite on D , we see that also $\sum m_a \beta_a \sim_{\text{rat}} \sum_{a,j} m_a \beta_{a,j}$ on D ! (See Chow Homology, Remark 19.6.) Ok, and now it is clear that $\sum_a m_a \beta_{a,j}$ (viewed as a cycle on D_j) represents $i_j^* \alpha$ and hence $\sum_{a,j} m_a \beta_{a,j}$ represents $p_* \sum_j i_j^* \alpha$ and we win. \square

Lemma 24.2. *Let (S, δ) be as in Chow Homology, Situation 7.1. Let X be locally of finite type over S . Assume X integral and $\dim_\delta(X) = n$. Let D, D' be effective Cartier divisors on X . Assume $\dim_\delta(D \cap D') = n - 2$. Let $i : D \rightarrow X$, resp. $i' : D' \rightarrow X$ be the corresponding closed immersions. Then*

- (1) *there exists a cycle $\alpha \in Z_{n-2}(D \cap D')$ whose pushforward to D represents $i^*[D']_{n-1} \in \text{CH}_{n-2}(D)$ and whose pushforward to D' represents $(i')^*[D]_{n-1} \in \text{CH}_{n-2}(D')$, and*
- (2) *we have*

$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$

in $\text{CH}_{n-2}(X)$.

Proof. Part (2) is a trivial consequence of part (1). Let us write $[D]_{n-1} = \sum n_a [Z_a]$ and $[D']_{n-1} = \sum m_b [Z_b]$ with Z_a the irreducible components of D and $[Z_b]$ the irreducible components of D' . According to Chow Homology, Definition 29.1, we have $i^* D' = \sum m_b i^* [Z_b]$ and $(i')^* D = \sum n_a (i')^* [Z_a]$. By assumption, none of the irreducible components Z_b is contained in D , and hence $i^* [Z_b] = [Z_b \cap D]_{n-2}$ by definition. Similarly $(i')^* [Z_a] = [Z_a \cap D']_{n-2}$. Hence we are trying to prove the equality of cycles

$$\sum n_a [Z_a \cap D']_{n-2} = \sum m_b [Z_b \cap D]_{n-2}$$

which are indeed supported on $D \cap D'$. Let $W \subset X$ be an integral closed subscheme with $\dim_\delta(W) = n - 2$. Let $\xi \in W$ be its generic point. Set $R = \mathcal{O}_{X,\xi}$. It is a Noetherian local domain. Note that $\dim(R) = 2$. Let $f \in R$, resp. $f' \in R$ be an element defining the ideal of D , resp. D' . By assumption $\dim(R/(f, f')) = 0$. Let $\mathfrak{q}'_1, \dots, \mathfrak{q}'_t \subset R$ be the minimal primes over (f') , let $\mathfrak{q}_1, \dots, \mathfrak{q}_s \subset R$ be the minimal primes over (f) . The equality above comes down to the equality

$$\sum_{i=1, \dots, s} \text{length}_{R_{\mathfrak{q}_i}}(R_{\mathfrak{q}_i}/(f)) \text{ord}_{R/\mathfrak{q}_i}(f') = \sum_{j=1, \dots, t} \text{length}_{R_{\mathfrak{q}'_j}}(R_{\mathfrak{q}'_j}/(f')) \text{ord}_{R/\mathfrak{q}'_j}(f).$$

By Chow Homology, Lemma 3.1 applied with $M = R/(f)$ the left hand side of this equation is equal to

$$\text{length}_R(R/(f, f')) - \text{length}_R(\text{Ker}(f' : R/(f) \rightarrow R/(f)))$$

OK, and now we note that $\text{Ker}(f' : R/(f) \rightarrow R/(f))$ is canonically isomorphic to $((f) \cap (f'))/(ff')$ via the map $x \bmod (f) \mapsto f'x \bmod (ff')$. Hence the left hand side is

$$\text{length}_R(R/(f, f')) - \text{length}_R((f) \cap (f'))/(ff')$$

Since this is symmetric in f and f' we win. \square

Lemma 24.3. *Let (S, δ) be as in Chow Homology, Situation 7.1. Let X be locally of finite type over S . Assume X integral and $\dim_\delta(X) = n$. Let $\{D_j\}_{j \in J}$ be a locally finite collection of effective Cartier divisors on X . Let $n_j, m_j \geq 0$ be collections of nonnegative integers. Set $D = \sum n_j D_j$ and $D' = \sum m_j D_j$. Assume that $\dim_\delta(D_j \cap D_{j'}) = n - 2$ for every $j \neq j'$. Then $D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$ in $\text{CH}_{n-2}(X)$.*

Proof. This lemma is a trivial consequence of Lemmas 23.10 and 24.2 in case the sums are finite, e.g., if X is quasi-compact. Hence we suggest the reader skip the proof.

Here is the proof in the general case. Let $i_j : D_j \rightarrow X$ be the closed immersions. Let $p : \coprod D_j \rightarrow X$ denote coproduct of the morphisms i_j . Let $\{Z_a\}_{a \in A}$ be the collection of irreducible components of $\bigcup D_j$. For each j we write

$$[D_j]_{n-1} = \sum d_{j,a} [Z_a].$$

By Lemma 23.10 we have

$$[D]_{n-1} = \sum n_j d_{j,a} [Z_a], \quad [D']_{n-1} = \sum m_j d_{j,a} [Z_a].$$

By Lemma 24.1 we have

$$D \cdot [D']_{n-1} = p_* \left(\sum n_j i_j^* [D']_{n-1} \right), \quad D' \cdot [D]_{n-1} = p_* \left(\sum m_{j'} i_{j'}^* [D]_{n-1} \right).$$

As in the definition of the Gysin homomorphisms (see Chow Homology, Definition 29.1) we choose cycles $\beta_{a,j}$ on $D_j \cap Z_a$ representing $i_j^* [Z_a]$. (Note that in fact $\beta_{a,j} = [D_j \cap Z_a]_{n-2}$ if Z_a is not contained in D_j , i.e., there is no choice in that case.) Now since p is a closed immersion when restricted to each of the D_j we can (and we will) view $\beta_{a,j}$ as a cycle on X . Plugging in the formulas for $[D]_{n-1}$ and $[D']_{n-1}$ obtained above we see that

$$D \cdot [D']_{n-1} = \sum_{j,j',a} n_j m_{j'} d_{j',a} \beta_{a,j}, \quad D' \cdot [D]_{n-1} = \sum_{j,j',a} m_{j'} n_j d_{j,a} \beta_{a,j'}.$$

Moreover, with the same conventions we also have

$$D_j \cdot [D_{j'}]_{n-1} = \sum d_{j',a} \beta_{a,j}.$$

In these terms Lemma 24.2 (see also its proof) says that for $j \neq j'$ the cycles $\sum d_{j',a}\beta_{a,j}$ and $\sum d_{j,a}\beta_{a,j'}$ are equal as cycles! Hence we see that

$$\begin{aligned}
D \cdot [D']_{n-1} &= \sum_{j,j',a} n_j m_{j'} d_{j',a} \beta_{a,j} \\
&= \sum_{j \neq j'} n_j m_{j'} \left(\sum_a d_{j',a} \beta_{a,j} \right) + \sum_{j,a} n_j m_j d_{j,a} \beta_{a,j} \\
&= \sum_{j \neq j'} n_j m_{j'} \left(\sum_a d_{j,a} \beta_{a,j'} \right) + \sum_{j,a} n_j m_j d_{j,a} \beta_{a,j} \\
&= \sum_{j,j',a} m_{j'} n_j d_{j,a} \beta_{a,j'} \\
&= D' \cdot [D]_{n-1}
\end{aligned}$$

and we win. \square

Lemma 24.4. *Let (S, δ) be as in Chow Homology, Situation 7.1. Let X be locally of finite type over S . Assume X integral and $\dim_\delta(X) = n$. Let D, D' be effective Cartier divisors on X . Then*

$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$

in $\text{CH}_{n-2}(X)$.

First proof of Lemma 24.4. First, let us prove this in case X is quasi-compact. In this case, apply Lemma 23.11 to X and the two element set $\{D, D'\}$ of effective Cartier divisors. Thus we get a proper morphism $b : X' \rightarrow X$, a finite collection of effective Cartier divisors $D'_j \subset X'$ intersecting pairwise in codimension ≥ 2 , with $b^{-1}(D) = \sum n_j D'_j$, and $b^{-1}(D') = \sum m_j D'_j$. Note that $b_*[b^{-1}(D)]_{n-1} = [D]_{n-1}$ in $Z_{n-1}(X)$ and similarly for D' , see Lemma 23.5. Hence, by Chow Homology, Lemma 26.4 we have

$$D \cdot [D']_{n-1} = b_* (b^{-1}(D) \cdot [b^{-1}(D')]_{n-1})$$

in $\text{CH}_{n-2}(X)$ and similarly for the other term. Hence the lemma follows from the equality $b^{-1}(D) \cdot [b^{-1}(D')]_{n-1} = b^{-1}(D') \cdot [b^{-1}(D)]_{n-1}$ in $\text{CH}_{n-2}(X')$ of Lemma 24.3.

Note that in the proof above, each referenced lemma works also in the general case (when X is not assumed quasi-compact). The only minor change in the general case is that the morphism $b : U' \rightarrow U$ we get from applying Lemma 23.11 has as its target an open $U \subset X$ whose complement has codimension ≥ 3 . Hence by Chow Homology, Lemma 19.3 we see that $\text{CH}_{n-2}(U) = \text{CH}_{n-2}(X)$ and after replacing X by U the rest of the proof goes through unchanged. \square

Second proof of Lemma 24.4. Let $\mathcal{I} = \mathcal{O}_X(-D)$ and $\mathcal{I}' = \mathcal{O}_X(-D')$ be the invertible ideal sheaves of D and D' . We denote $\mathcal{I}_{D'} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_{D'}$ and $\mathcal{I}'_D = \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{O}_D$. We can restrict the inclusion map $\mathcal{I} \rightarrow \mathcal{O}_X$ to D' to get a map

$$\varphi : \mathcal{I}_{D'} \longrightarrow \mathcal{O}_{D'}$$

and similarly

$$\psi : \mathcal{I}'_D \longrightarrow \mathcal{O}_D$$

It is clear that

$$\text{Coker}(\varphi) \cong \mathcal{O}_{D \cap D'} \cong \text{Coker}(\psi)$$

and

$$\mathrm{Ker}(\varphi) \cong \frac{\mathcal{I} \cap \mathcal{I}'}{\mathcal{I}\mathcal{I}'} \cong \mathrm{Ker}(\psi).$$

Hence we see that

$$\gamma = [\mathcal{I}_{D'}] - [\mathcal{O}_{D'}] = [\mathcal{I}'_D] - [\mathcal{O}_D]$$

in $K_0(\mathrm{Coh}_{\leq n-1}(X))$. On the other hand it is clear that

$$[\mathcal{I}'_D]_{n-1} = [D]_{n-1}, \quad [\mathcal{I}_{D'}]_{n-1} = [D']_{n-1}.$$

and that

$$\mathcal{O}_X(D') \otimes \mathcal{I}'_D = \mathcal{O}_D, \quad \mathcal{O}_X(D) \otimes \mathcal{I}_{D'} = \mathcal{O}_{D'}.$$

By Chow Homology, Lemma 69.7 (applied two times) this means that the element γ is an element of $B_{n-2}(X)$, and maps to both $c_1(\mathcal{O}_X(D')) \cap [D]_{n-1}$ and to $c_1(\mathcal{O}_X(D)) \cap [D']_{n-1}$ and we win (since the map $B_{n-2}(X) \rightarrow \mathrm{CH}_{n-2}(X)$ is well defined – which is the key to this proof). \square

25. Dualizing modules on regular proper models

In Semistable Reduction, Situation 9.3 we let $\omega_{X/R}^\bullet = f^! \mathcal{O}_{\mathrm{Spec}(R)}$ be the relative dualizing complex of $f : X \rightarrow \mathrm{Spec}(R)$ as introduced in Duality for Schemes, Remark 12.5. Since f is Gorenstein of relative dimension 1 by Semistable Reduction, Lemma 9.2 we can use Duality for Schemes, Lemmas 25.10, 21.7, and 25.4 to see that

$$\omega_{X/R}^\bullet = \omega_X[1]$$

for some invertible \mathcal{O}_X -module ω_X . This invertible module is often called the *relative dualizing module of X over R* . Since R is regular (hence Gorenstein) of dimension 1 we see that $\omega_R^\bullet = R[1]$ is a normalized dualizing complex for R . Hence $\omega_X = H^{-2}(f^! \omega_R^\bullet)$ and we see that ω_X is not just a relative dualizing module but also a dualizing module, see Duality for Schemes, Example 22.1. Thus ω_X represents the functor

$$\mathrm{Coh}(\mathcal{O}_X) \rightarrow \mathrm{Sets}, \quad \mathcal{F} \mapsto \mathrm{Hom}_R(H^1(X, \mathcal{F}), R)$$

by Duality for Schemes, Lemma 22.5. This gives an alternative definition of the relative dualizing module in Semistable Reduction, Situation 9.3. The formation of ω_X commutes with arbitrary base change (for any proper Gorenstein morphism of given relative dimension); this follows from the corresponding fact for the relative dualizing complex discussed in Duality for Schemes, Remark 12.5 which goes back to Duality for Schemes, Lemma 12.4. Thus ω_X pulls back to the dualizing module ω_C of C over K discussed in Algebraic Curves, Lemma 4.2. Note that ω_C is isomorphic to $\Omega_{C/K}$ by Algebraic Curves, Lemma 4.1. Similarly $\omega_X|_{X_k}$ is the dualizing module ω_{X_k} of X_k over k .

Lemma 25.1. *In Semistable Reduction, Situation 9.3 the dualizing module of C_i over k is*

$$\omega_{C_i} = \omega_X(C_i)|_{C_i}$$

where ω_X is as above.

Proof. Let $t : C_i \rightarrow X$ be the closed immersion. Since t is the inclusion of an effective Cartier divisor we conclude from Duality for Schemes, Lemmas 9.7 and

14.2 that we have $t^!(\mathcal{L}) = \mathcal{L}(C_i)|_{C_i}$ for every invertible \mathcal{O}_X -module \mathcal{L} . Consider the commutative diagram

$$\begin{array}{ccc} C_i & \xrightarrow{t} & X \\ g \downarrow & & \downarrow f \\ \mathrm{Spec}(k) & \xrightarrow{s} & \mathrm{Spec}(R) \end{array}$$

Observe that C_i is a Gorenstein curve (Semistable Reduction, Lemma 9.2) with invertible dualizing module ω_{C_i} characterized by the property $\omega_{C_i}[0] = g^!\mathcal{O}_{\mathrm{Spec}(k)}$. See Algebraic Curves, Lemma 4.1, its proof, and Algebraic Curves, Lemmas 4.2 and 5.2. On the other hand, $s^!(R[1]) = k$ and hence

$$\omega_{C_i}[0] = g^!s^!(R[1]) = t^!f^!(R[1]) = t^!\omega_X$$

Combining the above we obtain the statement of the lemma. \square

26. Duplicate and split out references

This section is a place where we collect duplicates and references which used to say several things at the same time but are now split into their constituent parts.

Lemma 26.1. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then \mathcal{F} is the directed colimit of its finite type quasi-coherent submodules.*

Proof. This is a duplicate of Properties, Lemma 22.3. \square

Lemma 26.2. *Let S be a scheme. Let X be an algebraic space over S . The map $\{\mathrm{Spec}(k) \rightarrow X \text{ monomorphism}\} \rightarrow |X|$ is injective.*

Proof. This is a duplicate of Properties of Spaces, Lemma 4.12. \square

Theorem 26.3. *Let $S = \mathrm{Spec}(K)$ with K a field. Let \bar{s} be a geometric point of S . Let $G = \mathrm{Gal}_{\kappa(\bar{s})}$ denote the absolute Galois group. Then there is an equivalence of categories $\mathrm{Sh}(S_{\text{étale}}) \rightarrow G\text{-Sets}$, $\mathcal{F} \mapsto \mathcal{F}_{\bar{s}}$.*

Proof. This is a duplicate of Étale Cohomology, Theorem 56.3. \square

Remark 26.4. You got here because of a duplicate tag. Please see Formal Deformation Theory, Section 12 for the actual content.

Lemma 26.5. *Let X be a locally ringed space. A direct summand of a finite free \mathcal{O}_X -module is finite locally free.*

Proof. This is a duplicate of Modules, Lemma 14.6. \square

Lemma 26.6. *Let R be a ring. Let E be an R -module. The following are equivalent*

- (1) *E is an injective R -module, and*
- (2) *given an ideal $I \subset R$ and a module map $\varphi : I \rightarrow E$ there exists an extension of φ to an R -module map $R \rightarrow E$.*

Proof. This is Baer's criterion, see Injectives, Lemma 2.6. \square

Lemma 26.7. *Let R be a local ring.*

- (1) *If (M, N, φ, ψ) is a 2-periodic complex such that M, N have finite length. Then $e_R(M, N, \varphi, \psi) = \mathrm{length}_R(M) - \mathrm{length}_R(N)$.*

- (2) If (M, φ, ψ) is a $(2, 1)$ -periodic complex such that M has finite length. Then $e_R(M, \varphi, \psi) = 0$.
- (3) Suppose that we have a short exact sequence of 2-periodic complexes

$$0 \rightarrow (M_1, N_1, \varphi_1, \psi_1) \rightarrow (M_2, N_2, \varphi_2, \psi_2) \rightarrow (M_3, N_3, \varphi_3, \psi_3) \rightarrow 0$$

If two out of three have cohomology modules of finite length so does the third and we have

$$e_R(M_2, N_2, \varphi_2, \psi_2) = e_R(M_1, N_1, \varphi_1, \psi_1) + e_R(M_3, N_3, \varphi_3, \psi_3).$$

Proof. This follows from Chow Homology, Lemmas 2.3 and 2.4. \square

Lemma 26.8. Let A be a ring and let I be an A -module.

- (1) The set of extensions of rings $0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0$ where I is an ideal of square zero is canonically bijective to $\text{Ext}_A^1(NL_{A/\mathbf{Z}}, I)$.
- (2) Given a ring map $A \rightarrow B$, a B -module N , an A -module map $c : I \rightarrow N$, and given extensions of rings with square zero kernels:
- (a) $0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0$ corresponding to $\alpha \in \text{Ext}_A^1(NL_{A/\mathbf{Z}}, I)$, and
 - (b) $0 \rightarrow N \rightarrow B' \rightarrow B \rightarrow 0$ corresponding to $\beta \in \text{Ext}_B^1(NL_{B/\mathbf{Z}}, N)$
- then there is a map $A' \rightarrow B'$ fitting into Deformation Theory, Equation (2.0.1) if and only if β and α map to the same element of $\text{Ext}_A^1(NL_{A/\mathbf{Z}}, N)$.

Proof. This follows from Deformation Theory, Lemmas 2.3 and 2.5. \square

Lemma 26.9. Let (S, \mathcal{O}_S) be a ringed space and let \mathcal{J} be an \mathcal{O}_S -module.

- (1) The set of extensions of sheaves of rings $0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_{S'} \rightarrow \mathcal{O}_S \rightarrow 0$ where \mathcal{J} is an ideal of square zero is canonically bijective to $\text{Ext}_{\mathcal{O}_S}^1(NL_{S/\mathbf{Z}}, \mathcal{J})$.
- (2) Given a morphism of ringed spaces $f : (X, \mathcal{O}_X) \rightarrow (S, \mathcal{O}_S)$, an \mathcal{O}_X -module \mathcal{G} , an f -map $c : \mathcal{J} \rightarrow \mathcal{G}$, and given extensions of sheaves of rings with square zero kernels:
- (a) $0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_{S'} \rightarrow \mathcal{O}_S \rightarrow 0$ corresponding to $\alpha \in \text{Ext}_{\mathcal{O}_S}^1(NL_{S/\mathbf{Z}}, \mathcal{J})$,
 - (b) $0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}_{X'} \rightarrow \mathcal{O}_X \rightarrow 0$ corresponding to $\beta \in \text{Ext}_{\mathcal{O}_X}^1(NL_{X/\mathbf{Z}}, \mathcal{G})$
- then there is a morphism $X' \rightarrow S'$ fitting into Deformation Theory, Equation (7.0.1) if and only if β and α map to the same element of $\text{Ext}_{\mathcal{O}_X}^1(Lf^* NL_{S/\mathbf{Z}}, \mathcal{G})$.

Proof. This follows from Deformation Theory, Lemmas 7.4 and 7.6. \square

Lemma 26.10. Let $(Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$ be a ringed topos and let \mathcal{J} be an $\mathcal{O}_{\mathcal{B}}$ -module.

- (1) The set of extensions of sheaves of rings $0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_{\mathcal{B}'} \rightarrow \mathcal{O}_{\mathcal{B}} \rightarrow 0$ where \mathcal{J} is an ideal of square zero is canonically bijective to $\text{Ext}_{\mathcal{O}_{\mathcal{B}}}^1(NL_{\mathcal{O}_{\mathcal{B}}/\mathbf{Z}}, \mathcal{J})$.
- (2) Given a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, an \mathcal{O} -module \mathcal{G} , an $f^{-1}\mathcal{O}_{\mathcal{B}}$ -module map $c : f^{-1}\mathcal{J} \rightarrow \mathcal{G}$, and given extensions of sheaves of rings with square zero kernels:
- (a) $0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_{\mathcal{B}'} \rightarrow \mathcal{O}_{\mathcal{B}} \rightarrow 0$ corresponding to $\alpha \in \text{Ext}_{\mathcal{O}_{\mathcal{B}}}^1(NL_{\mathcal{O}_{\mathcal{B}}/\mathbf{Z}}, \mathcal{J})$,
 - (b) $0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}' \rightarrow \mathcal{O} \rightarrow 0$ corresponding to $\beta \in \text{Ext}_{\mathcal{O}}^1(NL_{\mathcal{O}/\mathbf{Z}}, \mathcal{G})$
- then there is a morphism $(Sh(\mathcal{C}), \mathcal{O}') \rightarrow (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}'})$ fitting into Deformation Theory, Equation (13.0.1) if and only if β and α map to the same element of $\text{Ext}_{\mathcal{O}}^1(Lf^* NL_{\mathcal{O}_{\mathcal{B}}/\mathbf{Z}}, \mathcal{G})$.

Proof. This follows from Deformation Theory, Lemmas 13.4 and 13.6. \square

Remark 26.11. This tag used to point to a section describing several examples of deformation problems. These now each have their own section. See Deformation Problems, Sections 4, 5, 6, and 7.

Lemma 26.12. *Deformation Problems, Examples 4.1, 5.1, 6.1, and 7.1 satisfy the Rim-Schlessinger condition (RS).*

Proof. This follows from Deformation Problems, Lemmas 4.2, 5.2, 6.2, and 7.2. \square

Lemma 26.13. *We have the following canonical k -vector space identifications:*

- (1) *In Deformation Problems, Example 4.1 if $x_0 = (k, V)$, then $T_{x_0}\mathcal{F} = (0)$ and $\text{Inf}_{x_0}(\mathcal{F}) = \text{End}_k(V)$ are finite dimensional.*
- (2) *In Deformation Problems, Example 5.1 if $x_0 = (k, V, \rho_0)$, then $T_{x_0}\mathcal{F} = \text{Ext}_{k[\Gamma]}^1(V, V) = H^1(\Gamma, \text{End}_k(V))$ and $\text{Inf}_{x_0}(\mathcal{F}) = H^0(\Gamma, \text{End}_k(V))$ are finite dimensional if Γ is finitely generated.*
- (3) *In Deformation Problems, Example 6.1 if $x_0 = (k, V, \rho_0)$, then $T_{x_0}\mathcal{F} = H_{\text{cont}}^1(\Gamma, \text{End}_k(V))$ and $\text{Inf}_{x_0}(\mathcal{F}) = H_{\text{cont}}^0(\Gamma, \text{End}_k(V))$ are finite dimensional if Γ is topologically finitely generated.*
- (4) *In Deformation Problems, Example 7.1 if $x_0 = (k, P)$, then $T_{x_0}\mathcal{F}$ and $\text{Inf}_{x_0}(\mathcal{F}) = \text{Der}_k(P, P)$ are finite dimensional if P is finitely generated over k .*

Proof. This follows from Deformation Problems, Lemmas 4.3, 5.3, 6.3, and 7.3. \square

27. Other chapters

Preliminaries	(24) Differential Graded Sheaves
	(25) Hypercoverings
(1) Introduction	Schemes
(2) Conventions	(26) Schemes
(3) Set Theory	(27) Constructions of Schemes
(4) Categories	(28) Properties of Schemes
(5) Topology	(29) Morphisms of Schemes
(6) Sheaves on Spaces	(30) Cohomology of Schemes
(7) Sites and Sheaves	(31) Divisors
(8) Stacks	(32) Limits of Schemes
(9) Fields	(33) Varieties
(10) Commutative Algebra	(34) Topologies on Schemes
(11) Brauer Groups	(35) Descent
(12) Homological Algebra	(36) Derived Categories of Schemes
(13) Derived Categories	(37) More on Morphisms
(14) Simplicial Methods	(38) More on Flatness
(15) More on Algebra	(39) Groupoid Schemes
(16) Smoothing Ring Maps	(40) More on Groupoid Schemes
(17) Sheaves of Modules	(41) Étale Morphisms of Schemes
(18) Modules on Sites	Topics in Scheme Theory
(19) Injectives	(42) Chow Homology
(20) Cohomology of Sheaves	(43) Intersection Theory
(21) Cohomology on Sites	(44) Picard Schemes of Curves
(22) Differential Graded Algebra	(45) Weil Cohomology Theories
(23) Divided Power Algebra	

(46) Adequate Modules	(84) More on Cohomology of Spaces
(47) Dualizing Complexes	(85) Simplicial Spaces
(48) Duality for Schemes	(86) Duality for Spaces
(49) Discriminants and Differents	(87) Formal Algebraic Spaces
(50) de Rham Cohomology	(88) Algebraization of Formal Spaces
(51) Local Cohomology	(89) Resolution of Surfaces Revisited
(52) Algebraic and Formal Geometry	Deformation Theory
(53) Algebraic Curves	(90) Formal Deformation Theory
(54) Resolution of Surfaces	(91) Deformation Theory
(55) Semistable Reduction	(92) The Cotangent Complex
(56) Functors and Morphisms	(93) Deformation Problems
(57) Derived Categories of Varieties	Algebraic Stacks
(58) Fundamental Groups of Schemes	(94) Algebraic Stacks
(59) Étale Cohomology	(95) Examples of Stacks
(60) Crystalline Cohomology	(96) Sheaves on Algebraic Stacks
(61) Pro-étale Cohomology	(97) Criteria for Representability
(62) Relative Cycles	(98) Artin's Axioms
(63) More Étale Cohomology	(99) Quot and Hilbert Spaces
(64) The Trace Formula	(100) Properties of Algebraic Stacks
Algebraic Spaces	(101) Morphisms of Algebraic Stacks
(65) Algebraic Spaces	(102) Limits of Algebraic Stacks
(66) Properties of Algebraic Spaces	(103) Cohomology of Algebraic Stacks
(67) Morphisms of Algebraic Spaces	(104) Derived Categories of Stacks
(68) Decent Algebraic Spaces	(105) Introducing Algebraic Stacks
(69) Cohomology of Algebraic Spaces	(106) More on Morphisms of Stacks
(70) Limits of Algebraic Spaces	(107) The Geometry of Stacks
(71) Divisors on Algebraic Spaces	Topics in Moduli Theory
(72) Algebraic Spaces over Fields	(108) Moduli Stacks
(73) Topologies on Algebraic Spaces	(109) Moduli of Curves
(74) Descent and Algebraic Spaces	Miscellany
(75) Derived Categories of Spaces	(110) Examples
(76) More on Morphisms of Spaces	(111) Exercises
(77) Flatness on Algebraic Spaces	(112) Guide to Literature
(78) Groupoids in Algebraic Spaces	(113) Desirables
(79) More on Groupoids in Spaces	(114) Coding Style
(80) Bootstrap	(115) Obsolete
(81) Pushouts of Algebraic Spaces	(116) GNU Free Documentation License
Topics in Geometry	(117) Auto Generated Index
(82) Chow Groups of Spaces	
(83) Quotients of Groupoids	

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