COHOMOLOGY ON SITES

Contents

1. Introduction

In this document we work out some topics on cohomology of sheaves. We work out what happens for sheaves on sites, although often we will simply duplicate the discussion, the constructions, and the proofs from the topological case in the case. Basic references are [AGV71], [God73] and [Ive86].

2. Cohomology of sheaves

Let $\mathcal{C}$ be a site, see Sites, Definition 6.2. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. We know that the category of abelian sheaves on $\mathcal{C}$ has enough injectives, see Injectives, Theorem 7.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

$$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet)) \tag{2.0.1}$$

to be the *ith cohomology group of the abelian sheaf $\mathcal{F}$ over the object U*. In other words, these are the right derived functors of the functor $\mathcal{F} \mapsto \mathcal{F}(U)$. The family of functors $H^i(U, -)$ forms a universal δ-functor $Ab(\mathcal{C}) \to Ab$.

It sometimes happens that the site $\mathcal{C}$ does not have a final object. In this case we define the *global sections* of a presheaf of sets $\mathcal{F}$ over $\mathcal{C}$ to be the set

$$\Gamma(\mathcal{C}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(e, \mathcal{F}) \tag{2.0.2}$$

where e is a final object in the category of presheaves on $\mathcal{C}$. In this case, given an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$, we define the *ith cohomology group of $\mathcal{F}$ on $\mathcal{C}$* as follows

$$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet)) \tag{2.0.3}$$

in other words, it is the ith right derived functor of the global sections functor. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $Ab(\mathcal{C}) \to Ab$.

Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi, see Sites, Definition 15.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet) \tag{2.0.4}$$

to be the *ith higher direct image of* $\mathcal{F}$. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $Ab(\mathcal{C}) \to Ab(\mathcal{D})$.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site, see Modules on Sites, Definition 6.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. We know that the category of $\mathcal{O}$-modules has enough injectives, see Injectives, Theorem 8.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

$$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet)) \tag{2.0.5}$$

to be the *the ith cohomology group of* $\mathcal{F}$ *over* U. The family of functors $H^i(U, -)$ forms a universal δ-functor $Mod(\mathcal{O}) \to \text{Mod}_{\mathcal{O}(U)}$. Similarly

$$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet)) \tag{2.0.6}$$

it the *ith cohomology group of* $\mathcal{F}$ *on* $\mathcal{C}$. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $Mod(\mathcal{C}) \to \text{Mod}_{\Gamma(\mathcal{C}, \mathcal{O})}$.

Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi, see Modules on Sites, Definition 7.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet) \tag{2.0.7}$$

to be the *ith higher direct image of* $\mathcal{F}$. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $Mod(\mathcal{O}) \to Mod(\mathcal{O}')$.

3. Derived functors

We briefly explain an approach to right derived functors using resolution functors. Namely, suppose that $(\mathcal{C}, \mathcal{O})$ is a ringed site. In this chapter we will write

$$K(\mathcal{O}) = K(Mod(\mathcal{O})) \quad \text{and} \quad D(\mathcal{O}) = D(Mod(\mathcal{O}))$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 8.1 and Definition 11.3. By Derived Categories, Remark 24.3 there exists a resolution functor

$$j = j_{(\mathcal{C}, \mathcal{O})} : K^+(Mod(\mathcal{O})) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $Mod(\mathcal{O})$ which consists of injective $\mathcal{O}$-modules. For any left exact functor $F : Mod(\mathcal{O}) \to \mathcal{B}$ into any abelian category $\mathcal{B}$ we will denote RF the right derived functor of Derived Categories, Section 20 constructed using the resolution functor j just described:

$$RF = F \circ j' : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{B}) \tag{3.0.1}$$

see Derived Categories, Lemma 25.1 for notation. Note that we may think of RF as defined on $Mod(\mathcal{O})$, $\text{Comp}^+(Mod(\mathcal{O}))$, or $K^+(\mathcal{O})$ depending on the situation. According to Derived Categories, Definition 16.2 we obtain the $i$the right derived functor

$$R^i F = H^i \circ RF : Mod(\mathcal{O}) \longrightarrow \mathcal{B} \tag{3.0.2}$$

so that $R^0 F = F$ and $\{R^i F, \delta\}_{i \geq 0}$ is universal δ-functor, see Derived Categories, Lemma 20.4.

Here are two special cases of this construction. Given a ring R we write $K(R) = K(\text{Mod}_R)$ and $D(R) = D(\text{Mod}_R)$ and similarly for the bounded versions. For any object U of $\mathcal{C}$ have a left exact functor $\Gamma(U, -) : \textit{Mod}(\mathcal{O}) \longrightarrow \text{Mod}_{\mathcal{O}(U)}$ which gives rise to

$$R\Gamma(U, -) : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{O}(U))$$

by the discussion above. Note that $H^i(U, -) = R^i\Gamma(U, -)$ is compatible with (2.0.5) above. We similarly have

$$R\Gamma(\mathcal{C}, -) : D^+(\mathcal{O}) \longrightarrow D^+(\Gamma(\mathcal{C}, \mathcal{O}))$$

compatible with (2.0.6). If $f : (\textit{Sh}(\mathcal{C}), \mathcal{O}) \to (\textit{Sh}(\mathcal{D}), \mathcal{O}')$ is a morphism of ringed topoi then we get a left exact functor $f_* : \textit{Mod}(\mathcal{O}) \to \textit{Mod}(\mathcal{O}')$ which gives rise to *derived pushforward*

$$Rf_* : D^+(\mathcal{O}) \to D^+(\mathcal{O}')$$

The ith cohomology sheaf of $Rf_*\mathcal{F}^\bullet$ is denoted $R^if_*\mathcal{F}^\bullet$ and called the ith *higher direct image* in accordance with (2.0.7). The displayed functors above are exact functor of derived categories.

4. First cohomology and torsors

Definition 4.1. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A *pseudo torsor*, or more precisely a *pseudo $\mathcal{G}$-torsor*, is a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive.

A *morphism of pseudo $\mathcal{G}$-torsors* $\mathcal{F} \to \mathcal{F}'$ is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. A *torsor*, or more precisely a *$\mathcal{G}$-torsor*, is a pseudo $\mathcal{G}$-torsor such that in addition

(2) for every $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i\in I}$ of U such that $\mathcal{F}(U_i)$ is nonempty for all $i \in I$.

A *morphism of $\mathcal{G}$-torsors* is simply a morphism of pseudo $\mathcal{G}$-torsors. The *trivial $\mathcal{G}$-torsor* is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

Lemma 4.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\Gamma(\mathcal{C}, \mathcal{F}) \neq \emptyset$.*

Proof. Omitted. $\square$

Lemma 4.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{H}$ be an abelian sheaf on $\mathcal{C}$. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(\mathcal{C}, \mathcal{H})$.*

Proof. Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \in \text{Ob}(\mathcal{C})$ the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by sections of the form $[s] - [s']$. There is a canonical map $a : \text{Ker}(\sigma) \to \mathcal{H}$ which maps

$[s] - [s'] \mapsto h$ where h is the local section of $\mathcal{H}$ such that $h \cdot s = s'$. Consider the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\sigma) \arrow[r] \arrow[d, "a"] & \mathbf{Z}[\mathcal{F}] \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{E} \arrow[r] & \underline{\mathbf{Z}} \arrow[r] & 0 \end{tikzcd}$$

Here $\mathcal{E}$ is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_{\mathcal{F}} \in H^1(\mathcal{C}, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(\mathcal{C}, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(\mathcal{C}, \mathcal{H})$ we can associate to ξ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in H^0(\mathcal{C}, \mathcal{Q})$ because $H^1(\mathcal{C}, \mathcal{I}) = 0$ (see Derived Categories, Lemma 20.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a $\mathcal{H}$-torsor.

We omit the verification that the two constructions given above are mutually inverse. $\square$

5. First cohomology and extensions

Lemma 5.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. There is a canonical bijection*

$$\mathrm{Ext}^1_{Mod(\mathcal{O})}(\mathcal{O}, \mathcal{F}) \longrightarrow H^1(\mathcal{C}, \mathcal{F})$$

which associates to the extension

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$$

the image of $1 \in \Gamma(\mathcal{C}, \mathcal{O})$ *in* $H^1(\mathcal{C}, \mathcal{F})$.

Proof. Let us construct the inverse of the map given in the lemma. Let $\xi \in H^1(\mathcal{C}, \mathcal{F})$. Choose an injection $\mathcal{F} \subset \mathcal{I}$ with $\mathcal{I}$ injective in $Mod(\mathcal{O})$. Set $\mathcal{Q} = \mathcal{I}/\mathcal{F}$. By the long exact sequence of cohomology, we see that ξ is the image of a section $\tilde{\xi} \in \Gamma(\mathcal{C}, \mathcal{Q}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{Q})$. Now, we just form the pullback

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, "\tilde{\xi}"] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd}$$

see Homology, Section 6. $\square$

The following lemma will be superseded by the more general Lemma 12.4.

Lemma 5.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. Let $\mathcal{F}_{ab}$ denote the underlying sheaf of abelian groups. Then there is a functorial isomorphism*

$$H^1(\mathcal{C}, \mathcal{F}_{ab}) = H^1(\mathcal{C}, \mathcal{F})$$

where the left hand side is cohomology computed in $Ab(\mathcal{C})$ and the right hand side is cohomology computed in $Mod(\mathcal{O})$.

Proof. Let $\underline{\mathbf{Z}}$ denote the constant sheaf $\mathbf{Z}$. As $Ab(\mathcal{C}) = Mod(\underline{\mathbf{Z}})$ we may apply Lemma 5.1 twice, and it follows that we have to show

$$\mathrm{Ext}^1_{Mod(\mathcal{O})}(\mathcal{O}, \mathcal{F}) = \mathrm{Ext}^1_{Mod(\underline{\mathbf{Z}})}(\underline{\mathbf{Z}}, \mathcal{F}_{ab}).$$

Suppose that $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$ is an extension in $Mod(\mathcal{O})$. Then we can use the obvious map of abelian sheaves $1 : \underline{\mathbf{Z}} \to \mathcal{O}$ and pullback to obtain an extension $\mathcal{E}_{ab}$, like so:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{ab} \arrow[r] \arrow[d, equal] & \mathcal{E}_{ab} \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O} \arrow[r] & 0
\end{tikzcd}$$

The converse is a little more fun. Suppose that $0 \to \mathcal{F}_{ab} \to \mathcal{E}_{ab} \to \underline{\mathbf{Z}} \to 0$ is an extension in $Mod(\underline{\mathbf{Z}})$. Since $\underline{\mathbf{Z}}$ is a flat $\underline{\mathbf{Z}}$-module we see that the sequence

$$0 \to \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to 0$$

is exact, see Modules on Sites, Lemma 28.9. Of course $\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} = \mathcal{O}$. Hence we can form the pushout via the ($\mathcal{O}$-linear) multiplication map $\mu : \mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{F}$ to get an extension of $\mathcal{O}$ by $\mathcal{F}$, like this

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \arrow[r] \arrow[d, "\mu"] & \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O} \arrow[r] & 0
\end{tikzcd}$$

which is the desired extension. We omit the verification that these constructions are mutually inverse. □

6. First cohomology and invertible sheaves

The Picard group of a ringed site is defined in Modules on Sites, Section 32.

Lemma 6.1. *Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. There is a canonical isomorphism*

$$H^1(\mathcal{C}, \mathcal{O}^*) = \mathrm{Pic}(\mathcal{O}).$$

of abelian groups.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}$-module. Consider the presheaf $\mathcal{L}^*$ defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s\cdot -} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if $f \in \mathcal{O}^*(U)$ and $s \in \mathcal{L}^*(U)$, then clearly $fs \in \mathcal{L}^*(U)$. By the same token, if $s, s' \in \mathcal{L}^*(U)$ then there exists a unique $f \in \mathcal{O}^*(U)$ such that $fs = s'$. Moreover, the sheaf $\mathcal{L}^*$ has sections locally by Modules on Sites, Lemma 40.7. In other words we see that $\mathcal{L}^*$ is a $\mathcal{O}^*$-torsor. Thus we get a map

$$\begin{matrix}\text{set of invertible sheaves on } (\mathcal{C}, \mathcal{O}) \\ \text{up to isomorphism}\end{matrix} \longrightarrow \begin{matrix}\text{set of } \mathcal{O}^*\text{-torsors} \\ \text{up to isomorphism}\end{matrix}$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 4.3 the right hand side is canonically bijective to $H^1(\mathcal{C}, \mathcal{O}^*)$. Thus we have to show this map is injective and surjective.

Injective. If the torsor $\mathcal{L}^*$ is trivial, this means by Lemma 4.2 that $\mathcal{L}^*$ has a global section. Hence this means exactly that $\mathcal{L} \cong \mathcal{O}$ is the neutral element in $\mathrm{Pic}(\mathcal{O})$.

Surjective. Let $\mathcal{F}$ be an $\mathcal{O}^*$-torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}(U))/\mathcal{O}^*(U)$$

where the action of $f \in \mathcal{O}^*(U)$ on (s, g) is $(fs, f^{-1}g)$. Then $\mathcal{L}_1$ is a presheaf of $\mathcal{O}$-modules by setting $(s, g)+(s', g') = (s, g+(s'/s)g')$ where s'/s is the local section f of $\mathcal{O}^*$ such that $fs = s'$, and $h(s, g) = (s, hg)$ for h a local section of $\mathcal{O}$. We omit the verification that the sheafification $\mathcal{L} = \mathcal{L}_1^\#$ is an invertible $\mathcal{O}$-module whose associated $\mathcal{O}^*$-torsor $\mathcal{L}^*$ is isomorphic to $\mathcal{F}$. $\square$

7. Locality of cohomology

The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an object of the site.

Lemma 7.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$.*

1. *If $\mathcal{I}$ is an injective $\mathcal{O}$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.*
2. *For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(\mathcal{C}/U, \mathcal{F}|_U)$.*

Proof. Recall that the functor j_U^{-1} of restriction to U is a right adjoint to the functor $j_{U!}$ of extension by 0, see Modules on Sites, Section 19. Moreover, $j_{U!}$ is exact. Hence (1) follows from Homology, Lemma 29.1.

By definition $H^p(U, \mathcal{F}) = H^p(\mathcal{I}^\bullet(U))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Mod(\mathcal{O})$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in $Mod(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\mathcal{I}^\bullet|_U(U))$. Of course $\mathcal{F}(U) = \mathcal{F}|_U(U)$ for any sheaf $\mathcal{F}$ on $\mathcal{C}$. Hence the equality in (2). $\square$

The following lemma will be use to see what happens if we change a partial universe, or to compare cohomology of the small and big étale sites.

Lemma 7.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume u satisfies the hypotheses of Sites, Lemma 21.8. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the associated morphism of topoi. For any abelian sheaf $\mathcal{F}$ on $\mathcal{D}$ we have isomorphisms*

$$R\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = R\Gamma(\mathcal{D}, \mathcal{F}),$$

in particular $H^p(\mathcal{C}, g^{-1}\mathcal{F}) = H^p(\mathcal{D}, \mathcal{F})$ and for any $U \in \mathrm{Ob}(\mathcal{C})$ we have isomorphisms

$$R\Gamma(U, g^{-1}\mathcal{F}) = R\Gamma(u(U), \mathcal{F}),$$

in particular $H^p(U, g^{-1}\mathcal{F}) = H^p(u(U), \mathcal{F})$. All of these isomorphisms are functorial in $\mathcal{F}$.

Proof. Since it is clear that $\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = \Gamma(\mathcal{D}, \mathcal{F})$ by hypothesis (e), it suffices to show that g^{-1} transforms injective abelian sheaves into injective abelian sheaves. As usual we use Homology, Lemma 29.1 to see this. The left adjoint to g^{-1} is $g_! = f^{-1}$ with the notation of Sites, Lemma 21.8 which is an exact functor. Hence the lemma does indeed apply. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\varphi : U \to V$ be a morphism of $\mathcal{O}$. Then there is a canonical *restriction mapping*

$$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U \tag{7.2.1}$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor H^n. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 10.5.

The following lemma says that it is possible to kill higher cohomology classes by going to a covering.

Lemma 7.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let U be an object of $\mathcal{C}$. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists a covering $\{U_i \to U\}$ of $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$ is exact in degree n. Hence $\mathrm{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \mathrm{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since $\tilde{\xi}$ is a section of the kernel sheaf over U we conclude there exists a covering $\{U_i \to U\}$ of the site such that $\tilde{\xi}|_{U_i}$ is the image under d of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$. By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. $\square$

Lemma 7.4. *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. For any $\mathcal{F} \in \mathrm{Ob}(\mathit{Mod}(\mathcal{O}_\mathcal{C}))$ the sheaf $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf*

$$V \longmapsto H^i(u(V), \mathcal{F})$$

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^if_*\mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \ldots$$

By definition of the abelian category structure on $\mathcal{O}_\mathcal{D}$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(u(V)) \to \mathcal{I}^{i+1}(u(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(u(V)) \to \mathcal{I}^i(u(V)))}$$

which is equal to $H^i(u(V), \mathcal{F})$ and we win. $\square$

8. The Čech complex and Čech cohomology

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target, see Sites, Definition 6.1. Assume that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. Set

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in the factor $\mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p})$. We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}|_{U_{i_0} \times_U \ldots \times_U U_{i_{p+1}}} \tag{8.0.1}$$

where the restriction is via the projection map

$$U_{i_0} \times_U \ldots \times_U U_{i_{p+1}} \longrightarrow U_{i_0} \times_U \ldots \times_U \widehat{U_{i_j}} \times_U \ldots \times_U U_{i_{p+1}}.$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

Definition 8.1. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the *Čech complex* associated to $\mathcal{F}$ and the family $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the *Čech cohomology groups* of $\mathcal{F}$ with respect to $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

We observe that any covering $\{U_i \to U\}$ of a site $\mathcal{C}$ is a family of morphisms with fixed target to which the definition applies.

Lemma 8.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The following are equivalent*

(1) *$\mathcal{F}$ is an abelian sheaf on $\mathcal{C}$ and*
(2) *for every covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$ the natural map*

$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$

(see Sites, Section 10) is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every covering of $\mathcal{C}$. $\square$

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be another. Let $f : U \to V$, $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$ be a morphism of families of morphisms with fixed target, see Sites, Section 8. In this case we get a map of Čech complexes

$$\varphi : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \tag{8.2.1}$$

which in degree p is given by

$$\varphi(s)_{i_0 \ldots i_p} = (f_{i_0} \times \ldots \times f_{i_p})^* s_{\alpha(i_0) \ldots \alpha(i_p)}$$

9. Čech cohomology as a functor on presheaves

Warning: In this section we work exclusively with abelian presheaves on a category. The results are completely wrong in the setting of sheaves and categories of sheaves!

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in $\mathcal{F}$. In fact, it is a functor

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow \text{Comp}^+(Ab) \tag{9.0.1}$$

see Derived Categories, Definition 8.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 13.9.

Lemma 9.1. *The functor given by Equation (9.0.1) is an exact functor (see Homology, Lemma 7.2).*

Proof. For any object W of $\mathcal{C}$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $PAb(\mathcal{C})$ to Ab. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. □

Lemma 9.2. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category $PAb(\mathcal{C})$ to the category of $\mathbf{Z}$-modules (see Homology, Definition 12.1).*

Proof. By Lemma 9.1 a short exact sequence of abelian presheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathbf{Z}$-modules. Hence we can use Homology, Lemma 13.12 to get the boundary maps $\delta_{\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. □

Lemma 9.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Consider the chain complex $\mathbf{Z}_{\mathcal{U}, \bullet}$ of abelian presheaves*

$$\ldots \to \bigoplus_{i_0 i_1 i_2} \mathbf{Z}_{U_{i_0} \times_U U_{i_1} \times_U U_{i_2}} \to \bigoplus_{i_0 i_1} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \to \bigoplus_{i_0} \mathbf{Z}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$\mathbf{Z}_{U_{i_0} \times_U \ldots \times_u U_{i_{p+1}}} \longrightarrow \mathbf{Z}_{U_{i_0} \times_U \ldots \widehat{U_{i_j}} \ldots \times_U U_{i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map. Then there is an isomorphism

$$\text{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U}, \bullet}, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in \text{Ob}(PAb(\mathcal{C}))$.

Proof. This is a tautology based on the fact that

$$\begin{aligned}\mathrm{Hom}_{PAb(\mathcal{C})}(\bigoplus_{i_0\ldots i_p} \mathbf{Z}_{U_{i_0}\times_U \ldots \times_U U_{i_p}}, \mathcal{F}) &= \prod_{i_0\ldots i_p} \mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0}\times_U \ldots \times_U U_{i_p}}, \mathcal{F}) \\ &= \prod_{i_0\ldots i_p} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p})\end{aligned}$$

see Modules on Sites, Lemma 4.2. □

Lemma 9.4. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The chain complex $\mathbf{Z}_{\mathcal{U},\bullet}$ of presheaves of Lemma 9.3 above is exact in positive degrees, i.e., the homology presheaves $H_i(\mathbf{Z}_{\mathcal{U},\bullet})$ are zero for $i > 0$.*

Proof. Let V be an object of $\mathcal{C}$. We have to show that the chain complex of abelian groups $\mathbf{Z}_{\mathcal{U},\bullet}(V)$ is exact in degrees > 0. This is the complex

$$\begin{array}{c} \ldots \\ \downarrow \\ \bigoplus_{i_0 i_1 i_2} \mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \\ \downarrow \\ \bigoplus_{i_0 i_1} \mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0} \times_U U_{i_1})] \\ \downarrow \\ \bigoplus_{i_0} \mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0})] \\ \downarrow \\ 0 \end{array}$$

For any morphism $\varphi : V \to U$ denote $\mathrm{Mor}_\varphi(V, U_i) = \{\varphi_i : V \to U_i \mid f_i \circ \varphi_i = \varphi\}$. We will use a similar notation for $\mathrm{Mor}_\varphi(V, U_{i_0} \times_U \ldots \times_U U_{i_p})$. Note that composing with the various projection maps between the fibred products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ preserves these morphism sets. Hence we see that the complex above is the same as the complex

$$\begin{array}{c} \ldots \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0 i_1 i_2} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0 i_1} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0} \times_U U_{i_1})] \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0})] \\ \downarrow \\ 0 \end{array}$$

Next, we make the remark that we have

$$\mathrm{Mor}_\varphi(V, U_{i_0} \times_U \ldots \times_U U_{i_p}) = \mathrm{Mor}_\varphi(V, U_{i_0}) \times \ldots \times \mathrm{Mor}_\varphi(V, U_{i_p})$$

Using this and the fact that $\mathbf{Z}[A] \oplus \mathbf{Z}[B] = \mathbf{Z}[A \amalg B]$ we see that the complex becomes

$$\begin{array}{c} \ldots \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1 i_2} \mathrm{Mor}_\varphi(V, U_{i_0}) \times \mathrm{Mor}_\varphi(V, U_{i_1}) \times \mathrm{Mor}_\varphi(V, U_{i_2})\right] \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1} \mathrm{Mor}_\varphi(V, U_{i_0}) \times \mathrm{Mor}_\varphi(V, U_{i_1})\right] \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0} \mathrm{Mor}_\varphi(V, U_{i_0})\right] \\ \downarrow \\ 0 \end{array}$$

Finally, on setting $S_\varphi = \coprod_{i \in I} \mathrm{Mor}_\varphi(V, U_i)$ we see that we get

$$\bigoplus\nolimits_\varphi (\ldots \to \mathbf{Z}[S_\varphi \times S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi] \to 0 \to \ldots)$$

Thus we have simplified our task. Namely, it suffices to show that for any nonempty set S the (extended) complex of free abelian groups

$$\ldots \to \mathbf{Z}[S \times S \times S] \to \mathbf{Z}[S \times S] \to \mathbf{Z}[S] \xrightarrow{\Sigma} \mathbf{Z} \to 0 \to \ldots$$

is exact in all degrees. To see this fix an element $s \in S$, and use the homotopy

$$n_{(s_0, \ldots, s_p)} \longmapsto n_{(s, s_0, \ldots, s_p)}$$

with obvious notations. $\square$

Lemma 9.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. The chain complex*

$$\mathbf{Z}_{\mathcal{U},\bullet} \otimes_{p,\mathbf{Z}} \mathcal{O}$$

is exact in positive degrees. Here $\mathbf{Z}_{\mathcal{U},\bullet}$ is the chain complex of Lemma 9.3, and the tensor product is over the constant presheaf of rings with value $\mathbf{Z}$.

Proof. Let V be an object of $\mathcal{C}$. In the proof of Lemma 9.4 we saw that $\mathbf{Z}_{\mathcal{U},\bullet}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathbf{Z}$ placed in degree zero. Hence also $\mathbf{Z}_{\mathcal{U},\bullet}(V) \otimes_{\mathbf{Z}} \mathcal{O}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathcal{O}(V)$ placed in degree zero. Or you can use Modules on Sites, Lemma 28.11, which applies since the presheaves $\mathbf{Z}_{\mathcal{U},i}$ are flat, and the proof of Lemma 9.4 shows that $H_0(\mathbf{Z}_{\mathcal{U},\bullet})$ is a flat presheaf also. $\square$

Lemma 9.6. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The Čech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow Ab.$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(PAb(\mathcal{C})) \longrightarrow D^+(\mathbf{Z})$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of abelian presheaves has enough injectives, see Injectives, Proposition 6.1. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of abelian presheaves to the category of $\mathbf{Z}$-modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 20.

Let $\mathcal{I}$ be a injective abelian presheaf. In this case the functor $\mathrm{Hom}_{PAb(\mathcal{C})}(-, \mathcal{I})$ is exact on $PAb(\mathcal{C})$. By Lemma 9.3 we have

$$\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 9.4 we have that $\mathbf{Z}_{\mathcal{U},\bullet}$ is exact in positive degrees. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 9.2) satisfies the assumptions of Homology, Lemma 12.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 20.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 12.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any abelian presheaf on $\mathcal{C}$. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $PAb(\mathcal{C})$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Next, consider the total complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$ associated to this double complex, see Homology, Section 18. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 25.4. Namely, the columns of the double complex are exact in positive degrees because the Čech complex as a functor is exact (Lemma 9.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Čech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathbf{Z})$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. $\square$

10. Čech cohomology and cohomology

The relationship between cohomology and Čech cohomology comes from the fact that the Čech cohomology of an injective abelian sheaf is zero. To see this we note that an injective abelian sheaf is an injective abelian presheaf and then we apply results in Čech cohomology in the preceding section.

Lemma 10.1. *Let $\mathcal{C}$ be a site. An injective abelian sheaf is also injective as an object in the category $PAb(\mathcal{C})$.*

Proof. Apply Homology, Lemma 29.1 to the categories $\mathcal{A} = Ab(\mathcal{C})$, $\mathcal{B} = PAb(\mathcal{C})$, the inclusion functor and sheafification. (See Modules on Sites, Section 3 to see that all assumptions of the lemma are satisfied.) □

Lemma 10.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective abelian sheaf, i.e., an injective object of $Ab(\mathcal{C})$. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. By Lemma 10.1 we see that $\mathcal{I}$ is an injective object in $PAb(\mathcal{C})$. Hence we can apply Lemma 9.6 (or its proof) to see the vanishing of higher Čech cohomology group. For the zeroth see Lemma 8.2. □

Lemma 10.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. There is a transformation*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors $Ab(\mathcal{C}) \to D^+(\mathbf{Z})$. In particular this gives a transformation of functors $\check{H}^p(U, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $Ab(\mathcal{C})$.

Proof. Let $\mathcal{F}$ be an abelian sheaf. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Next, consider the associated total complex $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$, see Homology, Definition 18.3. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 25.4 to see that α is a quasi-isomorphism. Namely, Lemma 10.2 implies that the qth row of the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathbf{Z})$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. □

Lemma 10.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be an abelian sheaf on $\mathcal{C}$. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. The map*

$$\check{H}^1(\mathcal{U}, \mathcal{G}) \longrightarrow H^1(U, \mathcal{G})$$

is injective and identifies $\check{H}^1(\mathcal{U}, \mathcal{G})$ via the bijection of Lemma 4.3 with the set of isomorphism classes of $\mathcal{G}|_U$-torsors which restrict to trivial torsors over each U_i.

Proof. To see this we construct an inverse map. Namely, let $\mathcal{F}$ be a $\mathcal{G}|_U$-torsor on $\mathcal{C}/U$ whose restriction to $\mathcal{C}/U_i$ is trivial. By Lemma 4.2 this means there exists a section $s_i \in \mathcal{F}(U_i)$. On $U_{i_0} \times_U U_{i_1}$ there is a unique section $s_{i_0 i_1}$ of $\mathcal{G}$ such that $s_{i_0 i_1} \cdot s_{i_0}|_{U_{i_0} \times_U U_{i_1}} = s_{i_1}|_{U_{i_0} \times_U U_{i_1}}$. An easy computation shows that $s_{i_0 i_1}$ is a Čech cocycle and that its class is well defined (i.e., does not depend on the choice of the sections s_i). The inverse maps the isomorphism class of $\mathcal{F}$ to the cohomology class of the cocycle $(s_{i_0 i_1})$. We omit the verification that this map is indeed an inverse. □

Lemma 10.5. *Let $\mathcal{C}$ be a site. Consider the functor $i : Ab(\mathcal{C}) \to PAb(\mathcal{C})$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 7.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an object U of $\mathcal{C}$ are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

Lemma 10.6. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. For any abelian sheaf $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 22.2) for the functors

$$i : Ab(\mathcal{C}) \to PAb(\mathcal{C}) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \to Ab.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 8.2. We have that $i(\mathcal{I})$ is Čech acyclic by Lemma 10.2. And we have that $\check{H}^p(\mathcal{U}, -) = R^p \check{H}^0(\mathcal{U}, -)$ as functors on $PAb(\mathcal{C})$ by Lemma 9.6. Putting everything together gives the lemma. □

Lemma 10.7. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Assume that $H^i(U_{i_0} \times_U \ldots \times_U U_{i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$.*

Proof. We will use the spectral sequence of Lemma 10.6. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. □

Lemma 10.8. *Let $\mathcal{C}$ be a site. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let U be an object of $\mathcal{C}$. If there exists a cofinal system of coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0} \times_U U_{i_1}} - s_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathcal{F}(U_{i_0} \times_U U_{i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0} \times_U U_{i_1}} - t_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. $\square$

Lemma 10.9. *(Variant of Cohomology, Lemma 11.8.) Let $\mathcal{C}$ be a site. Let $Cov_{\mathcal{C}}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 6.2). Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$, and $Cov \subset Cov_{\mathcal{C}}$ be subsets. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Assume that*

1. *For every $\mathcal{U} \in Cov$, $\mathcal{U} = \{U_i \to U\}_{i \in I}$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.*
3. *For every $\mathcal{U} \in Cov$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective abelian sheaf. By Lemma 10.2 $\mathcal{I}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 10.8 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in$ Cov we get a short exact sequence of Čech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Čech cohomology groups for any covering $\mathcal{U} \in$ Cov. This implies that $\mathcal{Q}$ is also an abelian sheaf with vanishing higher Čech cohomology for all $\mathcal{U} \in$ Cov.

Next, we look at the long exact cohomology sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(U, \mathcal{F}) \arrow[r] & H^0(U, \mathcal{I}) \arrow[r] & H^0(U, \mathcal{Q}) \arrow[dll] \\
 & H^1(U, \mathcal{F}) \arrow[r] & H^1(U, \mathcal{I}) \arrow[r] & H^1(U, \mathcal{Q}) \arrow[dll] \\
 & \ldots & \ldots & \ldots
\end{tikzcd}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary abelian sheaf with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$ we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

11. Second cohomology and gerbes

Let $p : \mathcal{S} \to \mathcal{C}$ be a gerbe over a site all of whose automorphism groups are commutative. In this situation the first and second cohomology groups of the sheaf of automorphisms (Stacks, Lemma 11.8) controls the existence of objects.

The following lemma will be made obsolete by a more complete discussion of this relationship we will add in the future.

Lemma 11.1. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a gerbe over a site whose automorphism sheaves are abelian. Let $\mathcal{G}$ be the sheaf of abelian groups constructed in Stacks, Lemma 11.8. Let U be an object of $\mathcal{C}$ such that*

(1) *there exists a cofinal system of coverings $\{U_i \to U\}$ of U in $\mathcal{C}$ such that $H^1(U_i, \mathcal{G}) = 0$ and $H^1(U_i \times_U U_j, \mathcal{G}) = 0$ for all i, j, and*
(2) *$H^2(U, \mathcal{G}) = 0$.*

Then there exists an object of $\mathcal{S}$ lying over U.

Proof. By Stacks, Definition 11.1 there exists a covering $\mathcal{U} = \{U_i \to U\}$ and x_i in $\mathcal{S}$ lying over U_i. Write $U_{ij} = U_i \times_U U_j$. By (1) after refining the covering we may assume that $H^1(U_i, \mathcal{G}) = 0$ and $H^1(U_{ij}, \mathcal{G}) = 0$. Consider the sheaf

$$\mathcal{F}_{ij} = \mathit{Isom}(x_i|_{U_{ij}}, x_j|_{U_{ij}})$$

on $\mathcal{C}/U_{ij}$. Since $\mathcal{G}|_{U_{ij}} = \mathit{Aut}(x_i|_{U_{ij}})$ we see that there is an action

$$\mathcal{G}|_{U_{ij}} \times \mathcal{F}_{ij} \to \mathcal{F}_{ij}$$

by precomposition. It is clear that $\mathcal{F}_{ij}$ is a pseudo $\mathcal{G}|_{U_{ij}}$-torsor and in fact a torsor because any two objects of a gerbe are locally isomorphic. By our choice of the covering and by Lemma 4.3 these torsors are trivial (and hence have global sections by Lemma 4.2). In other words, we can choose isomorphisms

$$\varphi_{ij} : x_i|_{U_{ij}} \longrightarrow x_j|_{U_{ij}}$$

To find an object x over U we are going to massage our choice of these φ_{ij} to get a descent datum (which is necessarily effective as $p : \mathcal{S} \to \mathcal{C}$ is a stack). Namely, the obstruction to being a descent datum is that the cocycle condition may not hold. Namely, set $U_{ijk} = U_i \times_U U_j \times_U U_k$. Then we can consider

$$g_{ijk} = \varphi_{ik}^{-1}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}}$$

which is an automorphism of x_i over U_{ijk}. Thus we may and do consider g_{ijk} as a section of $\mathcal{G}$ over U_{ijk}. A computation (omitted) shows that $(g_{i_0 i_1 i_2})$ is a 2-cocycle in the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{G})$ of $\mathcal{G}$ with respect to the covering $\mathcal{U}$. By the spectral sequence of Lemma 10.6 and since $H^1(U_i, \mathcal{G}) = 0$ for all i we see that $\check{H}^2(\mathcal{U}, \mathcal{G}) \to H^2(U, \mathcal{G})$ is injective. Hence $(g_{i_0 i_1 i_2})$ is a coboundary by our assumption that $H^2(U, \mathcal{G}) = 0$. Thus we can find sections $g_{ij} \in \mathcal{G}(U_{ij})$ such that $g_{ik}^{-1}|_{U_{ijk}} g_{jk}|_{U_{ijk}} g_{ij}|_{U_{ijk}} = g_{ijk}$ for all i, j, k. After replacing φ_{ij} by $\varphi_{ij} g_{ij}^{-1}$ we see that φ_{ij} gives a descent datum on the objects x_i over U_i and the proof is complete. □

12. Cohomology of modules

Everything that was said for cohomology of abelian sheaves goes for cohomology of modules, since the two agree.

Lemma 12.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. An injective sheaf of modules is also injective as an object in the category $PMod(\mathcal{O})$.*

Proof. Apply Homology, Lemma 29.1 to the categories $\mathcal{A} = Mod(\mathcal{O})$, $\mathcal{B} = PMod(\mathcal{O})$, the inclusion functor and sheafification. (See Modules on Sites, Section 11 to see that all assumptions of the lemma are satisfied.) $\square$

Lemma 12.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the functor $i : Mod(\mathcal{C}) \to PMod(\mathcal{C})$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 7.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Mod(\mathcal{O})$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an object U of $\mathcal{C}$ are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. $\square$

Lemma 12.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective $\mathcal{O}$-module, i.e., an injective object of $Mod(\mathcal{O})$. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. Lemma 9.3 gives the first equality in the following sequence of equalities

$$\begin{aligned} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) &= \mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= \mathrm{Mor}_{PMod(\mathbf{Z})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= \mathrm{Mor}_{PMod(\mathcal{O})}(\mathbf{Z}_{\mathcal{U},\bullet} \otimes_{p,\mathbf{Z}} \mathcal{O}, \mathcal{I}) \end{aligned}$$

The third equality by Modules on Sites, Lemma 9.2. By Lemma 12.1 we see that $\mathcal{I}$ is an injective object in $PMod(\mathcal{O})$. Hence $\mathrm{Hom}_{PMod(\mathcal{O})}(-, \mathcal{I})$ is an exact functor. By Lemma 9.5 we see the vanishing of higher Čech cohomology groups. For the zeroth see Lemma 8.2. $\square$

Lemma 12.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module, and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then we have*

$$H^i(\mathcal{C}, \mathcal{F}_{ab}) = H^i(\mathcal{C}, \mathcal{F})$$

and for any object U of $\mathcal{C}$ we also have

$$H^i(U, \mathcal{F}_{ab}) = H^i(U, \mathcal{F}).$$

Here the left hand side is cohomology computed in $Ab(\mathcal{C})$ and the right hand side is cohomology computed in $Mod(\mathcal{O})$.

Proof. By Derived Categories, Lemma 20.4 the δ-functor $(\mathcal{F} \mapsto H^p(U, \mathcal{F}))_{p \geq 0}$ is universal. The functor $\textit{Mod}(\mathcal{O}) \to \textit{Ab}(\mathcal{C})$, $\mathcal{F} \mapsto \mathcal{F}_{ab}$ is exact. Hence $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is a δ-functor also. Suppose we show that $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is also universal. This will imply the second statement of the lemma by uniqueness of universal δ-functors, see Homology, Lemma 12.5. Since $\textit{Mod}(\mathcal{O})$ has enough injectives, it suffices to show that $H^i(U, \mathcal{I}_{ab}) = 0$ for any injective object $\mathcal{I}$ in $\textit{Mod}(\mathcal{O})$, see Homology, Lemma 12.4.

Let $\mathcal{I}$ be an injective object of $\textit{Mod}(\mathcal{O})$. Apply Lemma 10.9 with $\mathcal{F} = \mathcal{I}$, $\mathcal{B} = \mathcal{C}$ and $\text{Cov} = \text{Cov}_{\mathcal{C}}$. Assumption (3) of that lemma holds by Lemma 12.3. Hence we see that $H^i(U, \mathcal{I}_{ab}) = 0$ for every object U of $\mathcal{C}$.

If $\mathcal{C}$ has a final object then this also implies the first equality. If not, then according to Sites, Lemma 29.5 we see that the ringed topos $(\textit{Sh}(\mathcal{C}), \mathcal{O})$ is equivalent to a ringed topos where the underlying site does have a final object. Hence the lemma follows. $\square$

Lemma 12.5. *Let $\mathcal{C}$ be a site. Let I be a set. For $i \in I$ let $\mathcal{F}_i$ be an abelian sheaf on $\mathcal{C}$. Let $U \in \text{Ob}(\mathcal{C})$. The canonical map*

$$H^p(U, \prod\nolimits_{i \in I} \mathcal{F}_i) \longrightarrow \prod\nolimits_{i \in I} H^p(U, \mathcal{F}_i)$$

is an isomorphism for $p = 0$ and injective for $p = 1$.

Proof. The statement for $p = 0$ is true because the product of sheaves is equal to the product of the underlying presheaves, see Sites, Lemma 10.1. Proof for $p = 1$. Set $\mathcal{F} = \prod \mathcal{F}_i$. Let $\xi \in H^1(U, \mathcal{F})$ map to zero in $\prod H^1(U, \mathcal{F}_i)$. By locality of cohomology, see Lemma 7.3, there exists a covering $\mathcal{U} = \{U_j \to U\}$ such that $\xi|_{U_j} = 0$ for all j. By Lemma 10.4 this means ξ comes from an element $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F})$. Since the maps $\check{H}^1(\mathcal{U}, \mathcal{F}_i) \to H^1(U, \mathcal{F}_i)$ are injective for all i (by Lemma 10.4), and since the image of ξ is zero in $\prod H^1(U, \mathcal{F}_i)$ we see that the image $\check{\xi}_i = 0$ in $\check{H}^1(\mathcal{U}, \mathcal{F}_i)$. However, since $\mathcal{F} = \prod \mathcal{F}_i$ we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the product of the complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_i)$, hence by Homology, Lemma 32.1 we conclude that $\check{\xi} = 0$ as desired. $\square$

Lemma 12.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $a : U' \to U$ be a monomorphism in $\mathcal{C}$. Then for any injective $\mathcal{O}$-module $\mathcal{I}$ the restriction mapping $\mathcal{I}(U) \to \mathcal{I}(U')$ is surjective.*

Proof. Let $j : \mathcal{C}/U \to \mathcal{C}$ and $j' : \mathcal{C}/U' \to \mathcal{C}$ be the localization morphisms (Modules on Sites, Section 19). Since $j_!$ is a left adjoint to restriction we see that for any sheaf $\mathcal{F}$ of $\mathcal{O}$-modules

$$\text{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, \mathcal{F}) = \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

Similarly, the sheaf $j'_!\mathcal{O}_{U'}$ represents the functor $\mathcal{F} \mapsto \mathcal{F}(U')$. Moreover below we describe a canonical map of $\mathcal{O}$-modules

$$j'_!\mathcal{O}_{U'} \longrightarrow j_!\mathcal{O}_U$$

which corresponds to the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ via Yoneda's lemma (Categories, Lemma 3.5). It suffices to prove the displayed map of modules is injective, see Homology, Lemma 27.2.

To construct our map it suffices to construct a map between the presheaves which assign to an object V of $\mathcal{C}$ the $\mathcal{O}(V)$-module

$$\bigoplus\nolimits_{\varphi' \in \mathrm{Mor}_\mathcal{C}(V, U')} \mathcal{O}(V) \quad \text{and} \quad \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V, U)} \mathcal{O}(V)$$

see Modules on Sites, Lemma 19.2. We take the map which maps the summand corresponding to φ' to the summand corresponding to $\varphi = a \circ \varphi'$ by the identity map on $\mathcal{O}(V)$. As a is a monomorphism, this map is injective. As sheafification is exact, the result follows. $\square$

13. Totally acyclic sheaves

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a presheaf of sets on $\mathcal{C}$ (we intentionally use a roman capital here to distinguish from abelian sheaves). Given a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we set

$$\mathcal{F}(K) = \mathrm{Mor}_{PSh(\mathcal{C})}(K, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(K^\#, \mathcal{F})$$

The functor $\mathcal{F} \mapsto \mathcal{F}(K)$ is a left exact functor $\textit{Mod}(\mathcal{O}) \to \textit{Ab}$ hence we have its right derived functors. We will denote these $H^p(K, \mathcal{F})$ so that $H^0(K, \mathcal{F}) = \mathcal{F}(K)$.

Here are some observations:

(1) Since $\mathcal{F}(K) = \mathcal{F}(K^\#)$, we have $H^p(K, \mathcal{F}) = H^p(K^\#, \mathcal{F})$. Allowing K to be a presheaf in the definition above is a purely notational convenience.
(2) Suppose that $K = h_U$ or $K = h_U^\#$ for some object U of $\mathcal{C}$. Then $H^p(K, \mathcal{F}) = H^p(U, \mathcal{F})$, because $\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$, see Sites, Section 12.
(3) If $\mathcal{O} = \mathbf{Z}$ (the constant sheaf), then the cohomology groups are functors $H^p(K, -) : \textit{Ab}(\mathcal{C}) \to \textit{Ab}$ since $\textit{Mod}(\mathcal{O}) = \textit{Ab}(\mathcal{C})$ in this case.

We can translate some of our already proven results using this language.

Lemma 13.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a presheaf of sets on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then $H^p(K, \mathcal{F}) = H^p(K, \mathcal{F}_{ab})$.*

Proof. We may replace K by its sheafification and assume K is a sheaf. Note that both $H^p(K, \mathcal{F})$ and $H^p(K, \mathcal{F}_{ab})$ depend only on the topos, not on the underlying site. Hence by Sites, Lemma 29.5 we may replace $\mathcal{C}$ by a "larger" site such that $K = h_U$ for some object U of $\mathcal{C}$. In this case the result follows from Lemma 12.4. $\square$

Lemma 13.2. *Let $\mathcal{C}$ be a site. Let $K' \to K$ be a map of presheaves of sets on $\mathcal{C}$ whose sheafification is surjective. Set $K'_p = K' \times_K \ldots \times_K K'$ ($p+1$-factors). For every abelian sheaf $\mathcal{F}$ there is a spectral sequence with $E_1^{p,q} = H^q(K'_p, \mathcal{F})$ converging to $H^{p+q}(K, \mathcal{F})$.*

Proof. Since sheafification is exact, we see that $(K'_p)^\#$ is equal to $(K')^\# \times_{K^\#} \ldots \times_{K^\#} (K')^\#$ (p+1-factors). Thus we may replace K and K' by their sheafifications and assume $K \to K'$ is a surjective map of sheaves. After replacing $\mathcal{C}$ by a "larger" site as in Sites, Lemma 29.5 we may assume that K, K' are objects of $\mathcal{C}$ and that $\mathcal{U} = \{K' \to K\}$ is a covering. Then we have the Čech to cohomology spectral sequence of Lemma 10.6 whose E_1 page is as indicated in the statement of the lemma. $\square$

Lemma 13.3. *Let $\mathcal{C}$ be a site. Let K be a sheaf of sets on $\mathcal{C}$. Consider the morphism of topoi $j : Sh(\mathcal{C}/K) \to Sh(\mathcal{C})$, see Sites, Lemma 30.3. Then j^{-1} preserves injectives and $H^p(K, \mathcal{F}) = H^p(\mathcal{C}/K, j^{-1}\mathcal{F})$ for any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$.*

Proof. By Sites, Lemmas 30.1 and 30.3 the morphism of topoi j is equivalent to a localization. Hence this follows from Lemma 7.1. $\square$

Keeping in mind Lemma 13.1 we see that the following definition is the "correct one" also for sheaves of modules on ringed sites.

Definition 13.4. Let $\mathcal{C}$ be a site. We say an abelian sheaf $\mathcal{F}$ is *totally acyclic*[1] if for every sheaf of sets K we have $H^p(K, \mathcal{F}) = 0$ for all $p \geq 1$.

It is clear that being totally acyclic is an intrinsic property, i.e., preserved under equivalences of topoi. A totally acyclic sheaf has vanishing higher cohomology on all objects of the site, but in general the condition of being totally acyclic is strictly stronger. Here is a characterization of totally acyclic sheaves which is sometimes useful.

Lemma 13.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian sheaf. If*

(1) *$H^p(U, \mathcal{F}) = 0$ for $p > 0$ and $U \in \mathrm{Ob}(\mathcal{C})$, and*
(2) *for every surjection $K' \to K$ of sheaves of sets the extended Čech complex*
$$0 \to H^0(K, \mathcal{F}) \to H^0(K', \mathcal{F}) \to H^0(K' \times_K K', \mathcal{F}) \to \ldots$$
is exact,

then $\mathcal{F}$ is totally acyclic (and the converse holds too).

Proof. By assumption (1) we have $H^p(h_U^\#, g^{-1}\mathcal{I}) = 0$ for all $p > 0$ and all objects U of $\mathcal{C}$. Note that if $K = \coprod K_i$ is a coproduct of sheaves of sets on $\mathcal{C}$ then $H^p(K, g^{-1}\mathcal{I}) = \prod H^p(K_i, g^{-1}\mathcal{I})$. For any sheaf of sets K there exists a surjection
$$K' = \coprod h_{U_i}^\# \longrightarrow K$$
see Sites, Lemma 12.5. Thus we conclude that: (*) for every sheaf of sets K there exists a surjection $K' \to K$ of sheaves of sets such that $H^p(K', \mathcal{F}) = 0$ for $p > 0$. We claim that (*) and condition (2) imply that $\mathcal{F}$ is totally acyclic. Note that conditions (*) and (2) only depend on $\mathcal{F}$ as an object of the topos $Sh(\mathcal{C})$ and not on the underlying site. (We will not use property (1) in the rest of the proof.)

We are going to prove by induction on $n \geq 0$ that (*) and (2) imply the following induction hypothesis IH_n: $H^p(K, \mathcal{F}) = 0$ for all $0 < p \leq n$ and all sheaves of sets K. Note that IH_0 holds. Assume IH_n. Pick a sheaf of sets K. Pick a surjection $K' \to K$ such that $H^p(K', \mathcal{F}) = 0$ for all $p > 0$. We have a spectral sequence with
$$E_1^{p,q} = H^q(K'_p, \mathcal{F})$$
covering to $H^{p+q}(K, \mathcal{F})$, see Lemma 13.2. By IH_n we see that $E_1^{p,q} = 0$ for $0 < q \leq n$ and by assumption (2) we see that $E_2^{p,0} = 0$ for $p > 0$. Finally, we have $E_1^{0,q} = 0$ for $q > 0$ because $H^q(K', \mathcal{F}) = 0$ by choice of K'. Hence we conclude that $H^{n+1}(K, \mathcal{F}) = 0$ because all the terms $E_2^{p,q}$ with $p + q = n + 1$ are zero. $\square$

[1]Although this terminology is is used in [AGV71, Vbis, Proposition 1.3.10] this is probably nonstandard notation. In [AGV71, V, Definition 4.1] this property is dubbed "flasque", but we cannot use this because it would clash with our definition of flasque sheaves on topological spaces. Please email stacks.project@gmail.com if you have a better suggestion.

14. The Leray spectral sequence

The key to proving the existence of the Leray spectral sequence is the following lemma.

Lemma 14.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Then for any injective object $\mathcal{I}$ in $Mod(\mathcal{O}_\mathcal{C})$ the pushforward $f_*\mathcal{I}$ is totally acyclic.*

Proof. Let K be a sheaf of sets on $\mathcal{D}$. By Modules on Sites, Lemma 7.2 we may replace $\mathcal{C}$, $\mathcal{D}$ by "larger" sites such that f comes from a morphism of ringed sites induced by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that $K = h_V$ for some object V of $\mathcal{D}$.

Thus we have to show that $H^q(V, f_*\mathcal{I})$ is zero for $q > 0$ and all objects V of $\mathcal{D}$ when f is given by a morphism of ringed sites. Let $\mathcal{V} = \{V_j \to V\}$ be any covering of $\mathcal{D}$. Since u is continuous we see that $\mathcal{U} = \{u(V_j) \to u(V)\}$ is a covering of $\mathcal{C}$. Then we have an equality of Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I})$$

by the definition of f_*. By Lemma 12.3 we see that the cohomology of this complex is zero in positive degrees. We win by Lemma 10.9. □

For flat morphisms the functor f_* preserves injective modules. In particular the functor $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ always transforms injective abelian sheaves into injective abelian sheaves.

Lemma 14.2. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. If f is flat, then $f_*\mathcal{I}$ is an injective $\mathcal{O}_\mathcal{D}$-module for any injective $\mathcal{O}_\mathcal{C}$-module $\mathcal{I}$.*

Proof. In this case the functor f^* is exact, see Modules on Sites, Lemma 31.2. Hence the result follows from Homology, Lemma 29.1. □

Lemma 14.3. *Let $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ be a ringed topos. A totally acyclic sheaf is right acyclic for the following functors:*

1. *the functor $H^0(U, -)$ for any object U of $\mathcal{C}$,*
2. *the functor $\mathcal{F} \mapsto \mathcal{F}(K)$ for any presheaf of sets K,*
3. *the functor $\Gamma(\mathcal{C}, -)$ of global sections,*
4. *the functor f_* for any morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi.*

Proof. Part (2) is the definition of a totally acyclic sheaf. Part (1) is a consequence of (2) as pointed out in the discussion following the definition of totally acyclic sheaves. Part (3) is a special case of (2) where $K = e$ is the final object of $Sh(\mathcal{C})$.

To prove (4) we may assume, by Modules on Sites, Lemma 7.2 that f is given by a morphism of sites. In this case we see that R^if_*, $i > 0$ of a totally acyclic sheaf are zero by the description of higher direct images in Lemma 7.4. □

Remark 14.4. As a consequence of the results above we find that Derived Categories, Lemma 22.1 applies to a number of situations. For example, given a morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi we have

$$R\Gamma(\mathcal{D}, Rf_*\mathcal{F}) = R\Gamma(\mathcal{C}, \mathcal{F})$$

for any sheaf of $\mathcal{O}_\mathcal{C}$-modules $\mathcal{F}$. Namely, for an injective $\mathcal{O}_\mathcal{X}$-module $\mathcal{I}$ the $\mathcal{O}_\mathcal{D}$-module $f_*\mathcal{I}$ is totally acyclic by Lemma 14.1 and a totally acyclic sheaf is acyclic for $\Gamma(\mathcal{D}, -)$ by Lemma 14.3.

Lemma 14.5 (Leray spectral sequence). *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_\mathcal{C}$-modules. There is a spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{D}, R^qf_*(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{C}, \mathcal{F}^\bullet)$.

Proof. This is just the Grothendieck spectral sequence Derived Categories, Lemma 22.2 coming from the composition of functors $\Gamma(\mathcal{C}, -) = \Gamma(\mathcal{D}, -) \circ f_*$. To see that the assumptions of Derived Categories, Lemma 22.2 are satisfied, see Lemmas 14.1 and 14.3. $\square$

Lemma 14.6. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{C}$-module.*

1. *If $R^qf_*\mathcal{F} = 0$ for $q > 0$, then $H^p(\mathcal{C}, \mathcal{F}) = H^p(\mathcal{D}, f_*\mathcal{F})$ for all p.*
2. *If $H^p(\mathcal{D}, R^qf_*\mathcal{F}) = 0$ for all q and $p > 0$, then $H^q(\mathcal{C}, \mathcal{F}) = H^0(\mathcal{D}, R^qf_*\mathcal{F})$ for all q.*

Proof. These are two simple conditions that force the Leray spectral sequence to converge. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves. $\square$

Lemma 14.7 (Relative Leray spectral sequence). *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{E}), \mathcal{O}_\mathcal{E})$ be morphisms of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{C}$-module. There is a spectral sequence with*

$$E_2^{p,q} = R^pg_*(R^qf_*\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_*\mathcal{F}$. This spectral sequence is functorial in $\mathcal{F}$, and there is a version for bounded below complexes of $\mathcal{O}_\mathcal{C}$-modules.

Proof. This is a Grothendieck spectral sequence for composition of functors, see Derived Categories, Lemma 22.2 and Lemmas 14.1 and 14.3. $\square$

15. The base change map

In this section we construct the base change map in some cases; the general case is treated in Remark 19.3. The discussion in this section avoids using derived pullback by restricting to the case of a base change by a flat morphism of ringed sites. Before we state the result, let us discuss flat pullback on the derived category. Suppose $g : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is a flat morphism of ringed topoi. By Modules on Sites, Lemma 31.2 the functor $g^* : Mod(\mathcal{O}_\mathcal{D}) \to Mod(\mathcal{O}_\mathcal{C})$ is exact. Hence it has a derived functor

$$g^* : D(\mathcal{O}_\mathcal{D}) \to D(\mathcal{O}_\mathcal{C})$$

which is computed by simply pulling back an representative of a given object in $D(\mathcal{O}_\mathcal{D})$, see Derived Categories, Lemma 16.9. It preserved the bounded (above, below) subcategories. Hence as indicated we indicate this functor by g^* rather than Lg^*.

Lemma 15.1. *Let*

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g'"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "f"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{tikzcd}$$

be a commutative diagram of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_{\mathcal{C}}$-modules. Assume both g and g' are flat. Then there exists a canonical base change map

$$g^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*(g')^*\mathcal{F}^\bullet$$

in $D^+(\mathcal{O}_{\mathcal{D}'})$.

Proof. Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^*\mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 14.2 we see that $(g')_*\mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_*(g')^*\mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 18.6 and 18.7 the arrow β in the diagram

$$\begin{tikzcd} (g')_*(g')^*\mathcal{F}^\bullet \arrow[r] & (g')_*\mathcal{J}^\bullet \\ \mathcal{F}^\bullet \arrow[u, "adjunction"] \arrow[r] & \mathcal{I}^\bullet \arrow[u, "\beta"'] \end{tikzcd}$$

exists and is unique up to homotopy. Pushing down to $\mathcal{D}$ we get

$$f_*\beta : f_*\mathcal{I}^\bullet \longrightarrow f_*(g')_*\mathcal{J}^\bullet = g_*(f')_*\mathcal{J}^\bullet$$

By adjunction of g^* and g_* we get a map of complexes $g^*f_*\mathcal{I}^\bullet \to (f')_*\mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map β and everything was done on the level of complexes. □

16. Cohomology and colimits

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \to Mod(\mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram over the index category $\mathcal{I}$, see Categories, Section 14. For each i there is a canonical map $\mathcal{F}_i \to \mathrm{colim}_i \mathcal{F}_i$ which induces a map on cohomology. Hence we get a canonical map

$$\mathrm{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \mathrm{colim}_i \mathcal{F}_i)$$

for every $p \geq 0$ and every object U of $\mathcal{C}$. These maps are in general not isomorphisms, even for $p = 0$.

The following lemma is the analogue of Sites, Lemma 17.7 for cohomology.

Lemma 16.1. *Let $\mathcal{C}$ be a site. Let $Cov_{\mathcal{C}}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 6.2). Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$, and $Cov \subset Cov_{\mathcal{C}}$ be subsets. Assume that*

1. *For every $\mathcal{U} \in Cov$ we have $\mathcal{U} = \{U_i \to U\}_{i \in I}$ with I finite, $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.*

Then the map

$$\mathrm{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \mathrm{colim}_i \mathcal{F}_i)$$

is an isomorphism for every $p \geq 0$, every $U \in \mathcal{B}$, and every filtered diagram $\mathcal{I} \to Ab(\mathcal{C})$.

Proof. To prove the lemma we will argue by induction on p. Note that we require in (1) the coverings $\mathcal{U} \in \text{Cov}$ to be finite, so that all the elements of $\mathcal{B}$ are quasi-compact. Hence (2) and (1) imply that any $U \in \mathcal{B}$ satisfies the hypothesis of Sites, Lemma 17.7 (4). Thus we see that the result holds for $p = 0$. Now we assume the lemma holds for p and prove it for $p + 1$.

Choose a filtered diagram $\mathcal{F} : \mathcal{I} \to Ab(\mathcal{C})$, $i \mapsto \mathcal{F}_i$. Since $Ab(\mathcal{C})$ has functorial injective embeddings, see Injectives, Theorem 7.4, we can find a morphism of filtered diagrams $\mathcal{F} \to \mathcal{I}$ such that each $\mathcal{F}_i \to \mathcal{I}_i$ is an injective map of abelian sheaves into an injective abelian sheaf. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

Since colimits of sheaves are the sheafification of colimits on the level of presheaves, since sheafification is exact, and since filtered colimits of abelian groups are exact (see Algebra, Lemma 8.8), we see the sequence

$$0 \to \text{colim}_i\, \mathcal{F}_i \to \text{colim}_i\, \mathcal{I}_i \to \text{colim}_i\, \mathcal{Q}_i \to 0.$$

is also a short exact sequence. We claim that $H^q(U, \text{colim}_i\, \mathcal{I}_i) = 0$ for all $U \in \mathcal{B}$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$\begin{tikzcd}
\text{colim}_i\, H^p(U, \mathcal{I}_i) \arrow[r] \arrow[d] & \text{colim}_i\, H^p(U, \mathcal{Q}_i) \arrow[r] \arrow[d] & \text{colim}_i\, H^{p+1}(U, \mathcal{F}_i) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^p(U, \text{colim}_i\, \mathcal{I}_i) \arrow[r] & H^p(U, \text{colim}_i\, \mathcal{Q}_i) \arrow[r] & H^{p+1}(U, \text{colim}_i\, \mathcal{F}_i) \arrow[r] & 0
\end{tikzcd}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is exact by an application of Algebra, Lemma 8.8. Hence by the snake lemma we deduce the result for $p + 1$.

It remains to show that the claim is true. We will use Lemma 10.9. By the result for $p = 0$ we see that for $\mathcal{U} \in \text{Cov}$ we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{colim}_i\, \mathcal{I}_i) = \text{colim}_i\, \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the $U_{j_0} \times_U \ldots \times_U U_{j_p}$ are in $\mathcal{B}$. By Lemma 10.2 each of the complexes in the colimit of Čech complexes is acyclic in degree ≥ 1. Hence by Algebra, Lemma 8.8 we see that also the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{colim}_i\, \mathcal{I}_i)$ is acyclic in degrees ≥ 1. In other words we see that $\check{H}^p(\mathcal{U}, \text{colim}_i\, \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 10.9. are satisfied and the claim follows. $\square$

Lemma 16.2. *Let $\mathcal{C}$ be a site. Let $S \subset \text{Ob}(Sh(\mathcal{C}))$ be a subset. Denote $*$ the final object of $Sh(\mathcal{C})$. Assume*

(1) *for some $K \in S$ the map $K \to *$ is surjective,*
(2) *given a surjective map of sheaves $\mathcal{F} \to K$ with $K \in S$ there exists a $K' \in S$ and a map $K' \to \mathcal{F}$ such that the composition $K' \to K$ is surjective,*
(3) *given $K, K' \in S$ there is a surjection $K'' \to K \times K'$ with $K'' \in S$,*
(4) *given $a, b : K \to K'$ with $K, K' \in S$ there exists a surjection $K'' \to Equalizer(a, b)$ with $K'' \in S$, and*
(5) *every $K \in S$ is quasi-compact (Sites, Definition 17.4).*

Then for all $p \geq 0$ the map

$$\operatorname{colim}_\lambda H^p(\mathcal{C}, \mathcal{F}_\lambda) \longrightarrow H^p(\mathcal{C}, \operatorname{colim}_\lambda \mathcal{F}_\lambda)$$

is an isomorphism for every filtered diagram $\Lambda \to Ab(\mathcal{C})$, $\lambda \mapsto \mathcal{F}_\lambda$.

Proof. We will prove this by induction on p. The base case $p = 0$ follows from Sites, Lemma 17.8 part (4). We check the assumptions hold, but we urge the reader to skip this part. Suppose $\mathcal{F} \to *$ is surjective. Choose $K \in S$ and $K \to *$ surjective as in (1). Then $\mathcal{F} \times K \to K$ is surjective. Choose $K' \to \mathcal{F} \times K$ with $K' \in S$ and $K' \to K$ surjective as in (2). Then there is a map $K' \to \mathcal{F}$ and $K' \to *$ is surjective. Hence Sites, Lemma 17.8 assumption (4)(a) is satisfied. By Sites, Lemma 17.5, assumptions (3) and (5) we see that $K \times K$ is quasi-compact for all $K \in S$. Hence Sites, Lemma 17.8 assumption (4)(b) is satisfied. This finishes the proof of the base case.

Induction step. Assume the result holds for H^p for $p \leq p_0$ and for all topoi $Sh(\mathcal{C})$ such that a set $S \subset \text{Ob}(Sh(\mathcal{C}))$ can be found satisfying (1) – (5). Arguing exactly as in the proof of Lemma 16.1 we see that it suffices to show: given a filtered colimit $\mathcal{I} = \operatorname{colim} \mathcal{I}_\lambda$ with $\mathcal{I}_\lambda$ injective abelian sheaves, we have $H^{p_0+1}(\mathcal{C}, \mathcal{I}) = 0$. Choose $K \to *$ surjective with $K \in S$ as in (1). Denote K^n the n-fold self product of K. Consider the spectral sequence

$$E_1^{p,q} = H^q(K^{p+1}, \mathcal{I}) \Rightarrow H^{p+q}(*, \mathcal{I}) = H^{p+q}(\mathcal{C}, \mathcal{I})$$

of Lemma 13.2. Recall that $H^q(K^{p+1}, \mathcal{F}) = H^q(\mathcal{C}/K^{p+1}, j^{-1}\mathcal{F})$, for any abelian sheaf on $\mathcal{C}$, see Lemma 13.3. We have $j^{-1}\mathcal{I} = \operatorname{colim} j^{-1}\mathcal{I}_\lambda$ as j^{-1} commutes with colimits. The restrictions $j^{-1}\mathcal{I}_\lambda$ are injective abelian sheaves on $\mathcal{C}/K^{p+1}$ by Lemma 7.1. Below we will show that the induction hypothesis applies to $\mathcal{C}/K^{p+1}$ and hence we see that $H^q(K^{p+1}, \mathcal{I}) = \operatorname{colim} H^q(K^{p+1}, \mathcal{I}_\lambda) = 0$ for $q < p_0 + 1$ (vanishing as $\mathcal{I}_\lambda$ is injective). It follows that

$$H^{p_0+1}(\mathcal{C}, \mathcal{I}) = H^{p_0+1}\left(\ldots \to H^0(K^{p_0}, \mathcal{I}) \to H^0(K^{p_0+1}, \mathcal{I}) \to H^0(K^{p_0+2}, \mathcal{I}) \to \ldots\right)$$

Again using the induction hypothesis, the complex depicted on the right hand side is the colimit over Λ of the complexes

$$\ldots \to H^0(K^{p_0}, \mathcal{I}_\lambda) \to H^0(K^{p_0+1}, \mathcal{I}_\lambda) \to H^0(K^{p_0+2}, \mathcal{I}_\lambda) \to \ldots$$

These complexes are exact as $\mathcal{I}_\lambda$ is an injective abelian sheaf (follows from the spectral sequence for example). Since filtered colimits are exact in the category of abelian groups we obtain the desired vanishing.

We still have to show that the induction hypothesis applies to the site $\mathcal{C}/K^n$ for all $n \geq 1$. Recall that $Sh(\mathcal{C}/K^n) = Sh(\mathcal{C})/K^n$, see Sites, Lemma 30.3. Thus we may work in $Sh(\mathcal{C})/K^n$. Denote $S_n \subset \text{Ob}(Sh(\mathcal{C}/K^n)$ the set of objects of the form $K' \to K^n$. We check each property in turn:

(1) By (3) and induction there exists a surjection $K' \to K^n$ with $K' \in S$. Then $(K' \to K^n) \to (K^n \to K^n)$ is a surjection in $Sh(\mathcal{C})/K^n$ and $K^n \to K^n$ is the final object of $Sh(\mathcal{C})/K^n$. Hence (1) holds for S_n,
(2) Property (2) for S_n is an immediate consquence of (2) for S.
(3) Let $a : K_1 \to K^n$ and $b : K_2 \to K^n$ be in S_n. Then $(K_1 \to K^n) \times (K_2 \to K^n)$ is the object $K_1 \times_{K^n} K_2 \to K^n$ of $Sh(\mathcal{C})/K^n$. The subsheaf $K_1 \times_{K^n} K_2 \subset K_1 \times K_2$ is the equalizer of $a \circ \text{pr}_1$ and $b \circ \text{pr}_2$. Write

$a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. Pick $K_3 \to K_1 \times K_2$ surjective with $K_3 \in S$; this is possibly by assumption (3) for $\mathcal{C}$. Pick

$$K_4 \longrightarrow \text{Equalizer}(K_3 \to K_1 \times K_2 \xrightarrow{a_1, b_1} K)$$

surjective with $K_4 \in S$. This is possible by assumption (4) for $\mathcal{C}$. Pick

$$K_5 \longrightarrow \text{Equalizer}(K_4 \to K_1 \times K_2 \xrightarrow{a_2, b_2} K)$$

surjective with $K_5 \in S$. Again this is possible. Continue in this fashion until we get

$$K_{3+n} \longrightarrow \text{Equalizer}(K_{2+n} \to K_1 \times K_2 \xrightarrow{a_n, b_n} K)$$

surjective with $K_{3+n} \in S$. By construction $K_{3+n} \to K_1 \times_{K^n} K_2$ is surjective. Hence $(K_{3+n} \to K^n)$ is in S_n and surjects onto the product $(K_1 \to K^n) \times (K_2 \to K^n)$. Thus (3) holds for S_n.

(4) Property (4) for S_n is an immediate consequence of property (4) for S.
(5) Property (5) for S_n is a consequence of property (5) for S. Namely, an object $\mathcal{F} \to K^n$ of $\mathit{Sh}(\mathcal{C})/K^n$ corresponds to a quasi-compact object of $\mathit{Sh}(\mathcal{C}/K^n)$ if and only if $\mathcal{F}$ is a quasi-compact object of $\mathit{Sh}(\mathcal{C})$.

This finishes the proof of the lemma. □

Remark 16.3. Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Let $S \subset \text{Ob}(\mathit{Sh}(\mathcal{C}))$ be the set of sheaves K which have the form

$$K = \coprod\nolimits_{i=1,\ldots,n} h_{U_i}^\#$$

with $U_1, \ldots, U_n \in \mathcal{B}$. Then we can ask: when does this set satisfy the assumptions of Lemma 16.2? One answer is that it suffices if

(1) for some $n \geq 0$, $U_1, \ldots, U_n \in \mathcal{B}$ the map $\coprod_{i=1,\ldots,n} h_{U_i}^\# \to *$ is surjective,
(2) every covering of $U \in \mathcal{B}$ can be refined by a covering of the form $\{U_i \to U\}_{i=1,\ldots,n}$ with $U_i \in \mathcal{B}$,
(3) given $U, U' \in \mathcal{B}$ there exist $n \geq 0$, $U_1, \ldots, U_n \in \mathcal{B}$, maps $U_i \to U$ and $U_i \to U'$ such that $\coprod_{i=1,\ldots,n} h_{U_i}^\# \to h_U^\# \times h_{U'}^\#$ is surjective,
(4) given morphisms $a, b : U \to U'$ in $\mathcal{C}$ with $U, U' \in \mathcal{B}$, there exist $U_1, \ldots, U_n \in \mathcal{B}$, maps $U_i \to U$ equalizing a, b such that $\coprod_{i=1,\ldots,n} h_{U_i}^\# \to \text{Equalizer}(h_a^\#, h_b^\# : h_U^\# \to h_{U'}^\#)$ is surjective.

We omit the detailed verification, except to mention that part (2) above insures that every element of $\mathcal{B}$ is quasi-compact and hence every $K \in S$ is quasi-compact as well by Sites, Lemma 17.6.

Lemma 16.4. *Let $\mathcal{I}$ be a cofiltered index category and let $(\mathcal{C}_i, f_a)$ be an inverse system of sites over $\mathcal{I}$ as in Sites, Situation 18.1. Set $\mathcal{C} = \text{colim}\, \mathcal{C}_i$ as in Sites, Lemmas 18.2 and 18.3. Moreover, assume given*

(1) an abelian sheaf $\mathcal{F}_i$ on $\mathcal{C}_i$ for all $i \in \text{Ob}(\mathcal{I})$,
(2) for $a : j \to i$ a map $\varphi_a : f_a^{-1}\mathcal{F}_i \to \mathcal{F}_j$ of abelian sheaves on $\mathcal{C}_j$

such that $\varphi_c = \varphi_b \circ f_b^{-1}\varphi_a$ whenever $c = a \circ b$. Then there exists a map of systems $(\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ such that $\mathcal{F}_i \to \mathcal{G}_i$ is injective and $\mathcal{G}_i$ is an injective abelian sheaf.

Proof. For each i we pick an injection $\mathcal{F}_i \to \mathcal{A}_i$ where $\mathcal{A}_i$ is an injective abelian sheaf on $\mathcal{C}_i$. Then we can consider the family of maps

$$\gamma_i : \mathcal{F}_i \longrightarrow \prod\nolimits_{b:k\to i} f_{b,*}\mathcal{A}_k = \mathcal{G}_i$$

where the component maps are the maps adjoint to the maps $f_b^{-1}\mathcal{F}_i \to \mathcal{F}_k \to \mathcal{A}_k$. For $a : j \to i$ in $\mathcal{I}$ there is a canonical map

$$\psi_a : f_a^{-1}\mathcal{G}_i \to \mathcal{G}_j$$

whose components are the canonical maps $f_b^{-1}f_{a\circ b,*}\mathcal{A}_k \to f_{b,*}\mathcal{A}_k$ for $b : k \to j$. Thus we find an injection $(\gamma_i) : (\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ of systems of abelian sheaves. Note that $\mathcal{G}_i$ is an injective sheaf of abelian groups on $\mathcal{C}_i$, see Lemma 14.2 and Homology, Lemma 27.3. This finishes the construction. □

Lemma 16.5. *In the situation of Lemma 16.4 set $\mathcal{F} = \operatorname{colim} f_i^{-1}\mathcal{F}_i$. Let $i \in \mathrm{Ob}(\mathcal{I})$, $X_i \in Ob(\mathcal{C}_i)$. Then*

$$\operatorname{colim}_{a:j\to i} H^p(u_a(X_i), \mathcal{F}_j) = H^p(u_i(X_i), \mathcal{F})$$

for all $p \geq 0$.

Proof. The case $p = 0$ is Sites, Lemma 18.4.

Choose $(\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ as in Lemma 16.4. Arguing exactly as in the proof of Lemma 16.1 we see that it suffices to prove that $H^p(X, \operatorname{colim} f_i^{-1}\mathcal{G}_i) = 0$ for $p > 0$.

Set $\mathcal{G} = \operatorname{colim} f_i^{-1}\mathcal{G}_i$. To show vanishing of cohomology of $\mathcal{G}$ on every object of $\mathcal{C}$ we show that the Čech cohomology of $\mathcal{G}$ for any covering $\mathcal{U}$ of $\mathcal{C}$ is zero (Lemma 10.9). The covering $\mathcal{U}$ comes from a covering $\mathcal{U}_i$ of $\mathcal{C}_i$ for some i. We have

$$\check{C}^\bullet(\mathcal{U}, \mathcal{G}) = \operatorname{colim}_{a:j\to i} \check{C}^\bullet(u_a(\mathcal{U}_i), \mathcal{G}_j)$$

by the case $p = 0$. The right hand side is acyclic in positive degrees as a filtered colimit of acyclic complexes by Lemma 10.2. See Algebra, Lemma 8.8. □

17. Flat resolutions

In this section we redo the arguments of Cohomology, Section 26 in the setting of ringed sites and ringed topoi.

Lemma 17.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}$-modules. The functors*

$$K(Mod(\mathcal{O})) \longrightarrow K(Mod(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet)$$

and

$$K(Mod(\mathcal{O})) \longrightarrow K(Mod(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{G}^\bullet)$$

are exact functors of triangulated categories.

Proof. This follows from Derived Categories, Remark 10.9. □

Definition 17.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}$-modules is called *K-flat* if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the complex

$$\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

is acyclic.

Lemma 17.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor*

$$K(Mod(\mathcal{O})) \longrightarrow K(Mod(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{K}^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 17.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. □

Lemma 17.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. If $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}$-modules, then $\mathcal{K}^\bullet|_U$ is a K-flat complex of $\mathcal{O}_U$-modules.*

Proof. Let $\mathcal{G}^\bullet$ be an exact complex of $\mathcal{O}_U$-modules. Since $j_{U!}$ is exact (Modules on Sites, Lemma 19.3) and $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}$-modules we see that the complex

$$j_{U!}(\text{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_U} \mathcal{K}^\bullet|_U)) = \text{Tot}(j_{U!}\mathcal{G}^\bullet \otimes_\mathcal{O} \mathcal{K}^\bullet)$$

is exact. Here the equality comes from Modules on Sites, Lemma 27.9 and the fact that $j_{U!}$ commutes with direct sums (as a left adjoint). We conclude because $j_{U!}$ reflects exactness by Modules on Sites, Lemma 19.4. □

Lemma 17.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}$-modules, then $Tot(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}$-modules.*

Proof. Follows from the isomorphism

$$\text{Tot}(\mathcal{M}^\bullet \otimes_\mathcal{O} \text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet)) = \text{Tot}(\text{Tot}(\mathcal{M}^\bullet \otimes_\mathcal{O} \mathcal{K}^\bullet) \otimes_\mathcal{O} \mathcal{L}^\bullet)$$

and the definition. □

Lemma 17.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$ be a distinguished triangle in $K(Mod(\mathcal{O}))$. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 17.1 and the fact that in a distinguished triangle in $K(Mod(\mathcal{O}))$ if two out of three are acyclic, so is the third. □

Lemma 17.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $0 \to \mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \mathcal{K}_3^\bullet \to 0$ be a short exact sequence of complexes such that the terms of $\mathcal{K}_3^\bullet$ are flat $\mathcal{O}$-modules. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

Proof. By Modules on Sites, Lemma 28.9 for every complex $\mathcal{L}^\bullet$ we obtain a short exact sequence

$$0 \to \text{Tot}(\mathcal{L}^\bullet \otimes_\mathcal{O} \mathcal{K}_1^\bullet) \to \text{Tot}(\mathcal{L}^\bullet \otimes_\mathcal{O} \mathcal{K}_1^\bullet) \to \text{Tot}(\mathcal{L}^\bullet \otimes_\mathcal{O} \mathcal{K}_1^\bullet) \to 0$$

of complexes. Hence the lemma follows from the long exact sequence of cohomology sheaves and the definition of K-flat complexes. □

Lemma 17.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A bounded above complex of flat $\mathcal{O}$-modules is K-flat.*

Proof. Let $\mathcal{K}^\bullet$ be a bounded above complex of flat $\mathcal{O}$-modules. Let $\mathcal{L}^\bullet$ be an acyclic complex of $\mathcal{O}$-modules. Note that $\mathcal{L}^\bullet = \text{colim}_m \tau_{\leq m}\mathcal{L}^\bullet$ where we take termwise colimits. Hence also

$$\text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet) = \text{colim}_m \text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \tau_{\leq m}\mathcal{L}^\bullet)$$

termwise. Hence to prove the complex on the left is acyclic it suffices to show each of the complexes on the right is acyclic. Since $\tau_{\leq m}\mathcal{L}^\bullet$ is acyclic this reduces us to the case where $\mathcal{L}^\bullet$ is bounded above. In this case the spectral sequence of Homology, Lemma 25.3 has

$$'E_1^{p,q} = H^p(\mathcal{L}^\bullet \otimes_R \mathcal{K}^q)$$

which is zero as $\mathcal{K}^q$ is flat and $\mathcal{L}^\bullet$ acyclic. Hence we win. $\square$

Lemma 17.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\mathrm{colim}_i \mathcal{K}_i^\bullet$ is K-flat.*

Proof. Because we are taking termwise colimits it is clear that

$$\mathrm{colim}_i \mathrm{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{K}_i^\bullet) = \mathrm{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathrm{colim}_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. $\square$

Lemma 17.10. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}$-modules there exists a commutative diagram of complexes of $\mathcal{O}$-modules*

$$\begin{tikzcd}
\mathcal{K}_1^\bullet \arrow[r] \arrow[d] & \mathcal{K}_2^\bullet \arrow[r] \arrow[d] & \ldots \\
\tau_{\leq 1}\mathcal{G}^\bullet \arrow[r] & \tau_{\leq 2}\mathcal{G}^\bullet \arrow[r] & \ldots
\end{tikzcd}$$

with the following properties: (1) the vertical arrows are quasi-isomorphisms and termwise surjective, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $\mathrm{colim}\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Proof. The existence of the diagram and properties (1), (2), (3) follows immediately from Modules on Sites, Lemma 28.8 and Derived Categories, Lemma 29.1. The induced map $\mathrm{colim}\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. $\square$

Lemma 17.11. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ there exists a K-flat complex $\mathcal{K}^\bullet$ whose terms are flat $\mathcal{O}$-modules and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ which is termwise surjective.*

Proof. Choose a diagram as in Lemma 17.10. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules on Sites, Lemma 28.7. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 17.8. Thus $\mathrm{colim}\, \mathcal{K}_n^\bullet$ is K-flat by Lemma 17.9. The induced map $\mathrm{colim}\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism and termwise surjective by construction. Property (3) of Lemma 17.10 shows that $\mathrm{colim}\, \mathcal{K}_n^m$ is a direct sum of flat modules and hence flat which proves the final assertion. $\square$

Lemma 17.12. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the induced map*

$$Tot(id_{\mathcal{F}^\bullet} \otimes \alpha) : Tot(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{P}^\bullet) \longrightarrow Tot(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{Q}^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 17.11. Consider the commutative diagram

$$\begin{tikzcd}
\text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{P}^\bullet) \arrow[r] \arrow[d] & \text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{Q}^\bullet) \arrow[d] \\
\text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{P}^\bullet) \arrow[r] & \text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{Q}^\bullet)
\end{tikzcd}$$

The result follows as by Lemma 17.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 17.11. By Lemma 17.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}) \longrightarrow K(\mathcal{O}), \quad \mathcal{G}^\bullet \longmapsto \text{Tot}(\mathcal{G}^\bullet \otimes_\mathcal{O} \mathcal{K}^\bullet)$$

By Lemma 17.3 this functor induces a functor $D(\mathcal{O}) \to D(\mathcal{O})$ simply because $D(\mathcal{O})$ is the localization of $K(\mathcal{O})$ at quasi-isomorphisms. By Lemma 17.12 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

Definition 17.13. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. The *derived tensor product*

$$- \otimes_\mathcal{O}^\mathbf{L} \mathcal{F}^\bullet : D(\mathcal{O}) \longrightarrow D(\mathcal{O})$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes_\mathcal{O}^\mathbf{L} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes_\mathcal{O}^\mathbf{L} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O})$. Hence when we write $\mathcal{F}^\bullet \otimes_\mathcal{O}^\mathbf{L} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

Definition 17.14. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules. The *Tor*'s of $\mathcal{F}$ and $\mathcal{G}$ are defined by the formula

$$\text{Tor}_p^\mathcal{O}(\mathcal{F}, \mathcal{G}) = H^{-p}(\mathcal{F} \otimes_\mathcal{O}^\mathbf{L} \mathcal{G})$$

with derived tensor product as defined above.

This definition implies that for every short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we have a long exact cohomology sequence

$$\begin{tikzcd}
\mathcal{F}_1 \otimes_\mathcal{O} \mathcal{G} \arrow[r] & \mathcal{F}_2 \otimes_\mathcal{O} \mathcal{G} \arrow[r] & \mathcal{F}_3 \otimes_\mathcal{O} \mathcal{G} \arrow[r] & 0 \\
\text{Tor}_1^\mathcal{O}(\mathcal{F}_1, \mathcal{G}) \arrow[r] & \text{Tor}_1^\mathcal{O}(\mathcal{F}_2, \mathcal{G}) \arrow[r] & \text{Tor}_1^\mathcal{O}(\mathcal{F}_3, \mathcal{G}) \arrow[ull] &
\end{tikzcd}$$

for every $\mathcal{O}$-module $\mathcal{G}$. This will be called the long exact sequence of Tor associated to the situation.

Lemma 17.15. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module. The following are equivalent*

1. *$\mathcal{F}$ is a flat $\mathcal{O}$-module, and*
2. *$Tor_1^\mathcal{O}(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{O}$-module $\mathcal{G}$.*

Proof. If $\mathcal{F}$ is flat, then $\mathcal{F} \otimes_{\mathcal{O}} -$ is an exact functor and the satellites vanish. Conversely assume (2) holds. Then if $\mathcal{G} \to \mathcal{H}$ is injective with cokernel $\mathcal{Q}$, the long exact sequence of Tor shows that the kernel of $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}$ is a quotient of $\text{Tor}_1^{\mathcal{O}}(\mathcal{F}, \mathcal{Q})$ which is zero by assumption. Hence $\mathcal{F}$ is flat. $\square$

Lemma 17.16. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a K-flat, acyclic complex with flat terms. Then $\mathcal{F} = \text{Ker}(\mathcal{K}^n \to \mathcal{K}^{n+1})$ is a flat $\mathcal{O}$-module.*

Proof. Observe that
$$\ldots \to \mathcal{K}^{n-2} \to \mathcal{K}^{n-1} \to \mathcal{F} \to 0$$
is a flat resolution of our module $\mathcal{F}$. Since a bounded above complex of flat modules is K-flat (Lemma 17.8) we may use this resolution to compute $\text{Tor}_i(\mathcal{F}, \mathcal{G})$ for any $\mathcal{O}$-module $\mathcal{G}$. On the one hand $\mathcal{K}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G}$ is zero in $D(\mathcal{O})$ because $\mathcal{K}^\bullet$ is acyclic and on the other hand it is represented by $\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{G}$. Hence we see that
$$\mathcal{K}^{n-3} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{K}^{n-2} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{K}^{n-1} \otimes_{\mathcal{O}} \mathcal{G}$$
is exact. Thus $\text{Tor}_1(\mathcal{F}, \mathcal{G}) = 0$ and we conclude by Lemma 17.15. $\square$

Lemma 17.17. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. Let $a : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ be a map of complexes of $\mathcal{O}$-modules. If $\mathcal{K}^\bullet$ is K-flat, then there exist a complex $\mathcal{N}^\bullet$ and maps of complexes $b : \mathcal{K}^\bullet \to \mathcal{N}^\bullet$ and $c : \mathcal{N}^\bullet \to \mathcal{L}^\bullet$ such that*

1. *$\mathcal{N}^\bullet$ is K-flat,*
2. *c is a quasi-isomorphism,*
3. *a is homotopic to $c \circ b$.*

If the terms of $\mathcal{K}^\bullet$ are flat, then we may choose $\mathcal{N}^\bullet$, b, and c such that the same is true for $\mathcal{N}^\bullet$.

Proof. We will use that the homotopy category $K(Mod(\mathcal{O}))$ is a triangulated category, see Derived Categories, Proposition 10.3. Choose a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{L}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$. Choose a quasi-isomorphism $\mathcal{M}^\bullet \to \mathcal{C}^\bullet$ with $\mathcal{M}^\bullet$ K-flat with flat terms, see Lemma 17.11. By the axioms of triangulated categories, we may fit the composition $\mathcal{M}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$ into a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{N}^\bullet \to \mathcal{M}^\bullet \to \mathcal{K}^\bullet[1]$. By Lemma 17.6 we see that $\mathcal{N}^\bullet$ is K-flat. Again using the axioms of triangulated categories, we can choose a map $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ fitting into the following morphism of distinghuised triangles
$$\begin{tikzcd}
\mathcal{K}^\bullet \arrow[r] \arrow[d] & \mathcal{N}^\bullet \arrow[r] \arrow[d] & \mathcal{M}^\bullet \arrow[r] \arrow[d] & \mathcal{K}^\bullet[1] \arrow[d] \\
\mathcal{K}^\bullet \arrow[r] & \mathcal{L}^\bullet \arrow[r] & \mathcal{C}^\bullet \arrow[r] & \mathcal{K}^\bullet[1]
\end{tikzcd}$$
Since two out of three of the arrows are quasi-isomorphisms, so is the third arrow $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ by the long exact sequences of cohomology associated to these distinguished triangles (or you can look at the image of this diagram in $D(\mathcal{O})$ and use Derived Categories, Lemma 4.3 if you like). This finishes the proof of (1), (2), and (3). To prove the final assertion, we may choose $\mathcal{N}^\bullet$ such that $\mathcal{N}^n \cong \mathcal{M}^n \oplus \mathcal{K}^n$, see Derived Categories, Lemma 10.7. Hence we get the desired flatness if the terms of $\mathcal{K}^\bullet$ are flat. $\square$

18. Derived pullback

Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}') \to D(\mathcal{O})$$

Lemma 18.1. *Let $f : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{K}^\bullet$ be a K-flat complex of $\mathcal{O}$-modules whose terms are flat $\mathcal{O}$-modules. Then $f^*\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}'$-modules whose terms are flat $\mathcal{O}'$-modules.*

Proof. The terms $f^*\mathcal{K}^n$ are flat $\mathcal{O}'$-modules by Modules on Sites, Lemma 39.1. Choose a diagram

$$\begin{tikzcd} \mathcal{K}_1^\bullet \arrow[r] \arrow[d] & \mathcal{K}_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}\mathcal{K}^\bullet \arrow[r] & \tau_{\leq 2}\mathcal{K}^\bullet \arrow[r] & \ldots \end{tikzcd}$$

as in Lemma 17.10. We will use all of the properties stated in the lemma without further mention. Each $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules on Sites, Lemma 28.7. Consider the short exact sequence of complexes

$$0 \to \mathcal{M}^\bullet \to \operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{K}^\bullet \to 0$$

defining $\mathcal{M}^\bullet$. By Lemmas 17.8 and 17.9 the complex $\operatorname{colim} \mathcal{K}_n^\bullet$ is K-flat and by Modules on Sites, Lemma 28.5 it has flat terms. By Modules on Sites, Lemma 28.10 $\mathcal{M}^\bullet$ has flat terms, by Lemma 17.7 $\mathcal{M}^\bullet$ is K-flat, and by the long exact cohomology sequence $\mathcal{M}^\bullet$ is acyclic (because the second arrow is a quasi-isomorphism). The pullback $f^*(\operatorname{colim} \mathcal{K}_n^\bullet) = \operatorname{colim} f^*\mathcal{K}_n^\bullet$ is a colimit of bounded below complexes of flat $\mathcal{O}'$-modules and hence is K-flat (by the same lemmas as above). The pullback of our short exact sequence

$$0 \to f^*\mathcal{M}^\bullet \to f^*(\operatorname{colim} \mathcal{K}_n^\bullet) \to f^*\mathcal{K}^\bullet \to 0$$

is a short exact sequence of complexes by Modules on Sites, Lemma 39.4. Hence by Lemma 17.7 it suffices to show that $f^*\mathcal{M}^\bullet$ is K-flat. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{K}^\bullet$ is acyclic as well as K-flat and with flat terms. Then Lemma 17.16 guarantees that all terms of $\tau_{\leq n}\mathcal{K}^\bullet$ are flat $\mathcal{O}$-modules. We choose a diagram as above and we will use all the properties proven above for this diagram. Denote $\mathcal{M}_n^\bullet$ the kernel of the map of complexes $\mathcal{K}_n^\bullet \to \tau_{\leq n}\mathcal{K}^\bullet$ so that we have short exact sequences of complexes

$$0 \to \mathcal{M}_n^\bullet \to \mathcal{K}_n^\bullet \to \tau_{\leq n}\mathcal{K}^\bullet \to 0$$

By Modules on Sites, Lemma 28.10 we see that the terms of the complex $\mathcal{M}_n^\bullet$ are flat. Hence we see that $\mathcal{M} = \operatorname{colim} \mathcal{M}_n^\bullet$ is a filtered colimit of bounded below complexes of flat modules in this case. Thus $f^*\mathcal{M}^\bullet$ is K-flat (same argument as above) and we win. $\square$

Lemma 18.2. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. There exists an exact functor*

$$Lf^* : D(\mathcal{O}') \longrightarrow D(\mathcal{O})$$

of triangulated categories so that $Lf^*\mathcal{K}^\bullet = f^*\mathcal{K}^\bullet$ for any K-flat complex $\mathcal{K}^\bullet$ with flat terms and in particular for any bounded above complex of flat $\mathcal{O}'$-modules.

Proof. To see this we use the general theory developed in Derived Categories, Section 14. Set $\mathcal{D} = K(\mathcal{O}')$ and $\mathcal{D}' = D(\mathcal{O})$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}')$. This gives a situation as in Derived Categories, Situation 14.1 so that Derived Categories, Definition 14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 14.15 with $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ the collection of K-flat complexes $\mathcal{K}^\bullet$ with flat terms. Namely, (1) follows from Lemma 17.11 and to see (2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between elements of $\mathcal{P}$ the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O} \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O}$$

The functor f^{-1} is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. The complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}'$-modules by Lemma 18.1 because we can consider the morphism of ringed topoi $(Sh(\mathcal{C}), f^{-1}\mathcal{O}') \to (Sh(\mathcal{C}'), \mathcal{O}')$. Hence Lemma 17.12 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}') = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O})$$

see Derived Categories, Equation (14.9.1). Finally, Derived Categories, Lemma 14.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is in $\mathcal{P}$. The proof is finished by observing that bounded above complexes of flat modules are in $\mathcal{P}$ by Lemma 17.8. □

Lemma 18.3. *Consider morphisms of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{E}), \mathcal{O}_\mathcal{E})$. Then $Lf^* \circ Lg^* = L(g \circ f)^*$ as functors $D(\mathcal{O}_\mathcal{E}) \to D(\mathcal{O}_\mathcal{C})$.*

Proof. Let E be an object of $D(\mathcal{O}_\mathcal{E})$. We may represent E by a K-flat complex $\mathcal{K}^\bullet$ with flat terms, see Lemma 17.11. By construction Lg^*E is computed by $g^*\mathcal{K}^\bullet$, see Lemma 18.2. By Lemma 18.1 the complex $g^*\mathcal{K}^\bullet$ is K-flat with flat terms. Hence Lf^*Lg^*E is represented by $f^*g^*\mathcal{K}^\bullet$. Since also $L(g \circ f)^*E$ is represented by $(g \circ f)^*\mathcal{K}^\bullet = f^*g^*\mathcal{K}^\bullet$ we conclude. □

Lemma 18.4. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. There is a canonical bifunctorial isomorphism*

$$Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}'} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}'))$.

Proof. By our construction of derived pullback in Lemma 18.2. and the existence of resolutions in Lemma 17.11 we may replace $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ by complexes of $\mathcal{O}'$-modules which are K-flat and have flat terms. In this case $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}'} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet$. The complex $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$ is K-flat with flat terms by Lemma 17.5 and Modules on Sites, Lemma 28.12. Hence the isomorphism of the lemma comes from the isomorphism

$$\mathrm{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}} f^*\mathcal{G}^\bullet) \longrightarrow f^*\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_\mathcal{O} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}'} \mathcal{G}^q)$ of Modules on Sites, Lemma 26.2. □

Lemma 18.5. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. There is a canonical bifunctorial isomorphism*

$$\mathcal{F}^\bullet \otimes_\mathcal{O}^{\mathbf{L}} Lf^*\mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_Y}^{\mathbf{L}} f^{-1}\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet$ in $D(\mathcal{O})$ and $\mathcal{G}^\bullet$ in $D(\mathcal{O}')$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}$-module and let $\mathcal{G}$ be an $\mathcal{O}'$-module. Then $\mathcal{F} \otimes_\mathcal{O} f^*\mathcal{G} = \mathcal{F} \otimes_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{G}$ because $f^*\mathcal{G} = \mathcal{O} \otimes_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{G}$. The lemma follows from this and the definitions. □

Lemma 18.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}$-modules.*

1. *If $\mathcal{K}^\bullet$ is K-flat, then for every point p of the site $\mathcal{C}$ the complex of $\mathcal{O}_p$-modules $\mathcal{K}_p^\bullet$ is K-flat in the sense of More on Algebra, Definition 59.1*
2. *If $\mathcal{C}$ has enough points, then the converse is true.*

Proof. Proof of (2). If $\mathcal{C}$ has enough points and $\mathcal{K}_p^\bullet$ is K-flat for all points p of $\mathcal{C}$ then we see that $\mathcal{K}^\bullet$ is K-flat because $\otimes$ and direct sums commute with taking stalks and because we can check exactness at stalks, see Modules on Sites, Lemma 14.4.

Proof of (1). Assume $\mathcal{K}^\bullet$ is K-flat. Choose a quasi-isomorphism $a : \mathcal{L}^\bullet \to \mathcal{K}^\bullet$ such that $\mathcal{L}^\bullet$ is K-flat with flat terms, see Lemma 17.11. Any pullback of $\mathcal{L}^\bullet$ is K-flat, see Lemma 18.1. In particular the stalk $\mathcal{L}_p^\bullet$ is a K-flat complex of $\mathcal{O}_p$-modules. Thus the cone $C(a)$ on a is a K-flat (Lemma 17.6) acyclic complex of $\mathcal{O}$-modules and it suffuces to show the stalk of $C(a)$ is K-flat (by More on Algebra, Lemma 59.5). Thus we may assume that $\mathcal{K}^\bullet$ is K-flat and acyclic.

Assume $\mathcal{K}^\bullet$ is acyclic and K-flat. Before continuing we replace the site $\mathcal{C}$ by another one as in Sites, Lemma 29.5 to insure that $\mathcal{C}$ has all finite limits. This implies the category of neighbourhoods of p is filtered (Sites, Lemma 33.2) and the colimit defining the stalk of a sheaf is filtered. Let M be a finitely presented $\mathcal{O}_p$-module. It suffices to show that $\mathcal{K}^\bullet \otimes_{\mathcal{O}_p} M$ is acyclic, see More on Algebra, Lemma 59.9. Since $\mathcal{O}_p$ is the filtered colimit of $\mathcal{O}(U)$ where U runs over the neighbourhoods of p, we can find a neighbourhood (U, x) of p and a finitely presented $\mathcal{O}(U)$-module M' whose base change to $\mathcal{O}_p$ is M, see Algebra, Lemma 127.6. By Lemma 17.4 we may replace $\mathcal{C}, \mathcal{O}, \mathcal{K}^\bullet$ by $\mathcal{C}/U, \mathcal{O}_U, \mathcal{K}^\bullet|_U$. We conclude that we may assume there exists an $\mathcal{O}$-module $\mathcal{F}$ such that $M \cong \mathcal{F}_p$. Since $\mathcal{K}^\bullet$ is K-flat and acyclic, we see that $\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{F}$ is acyclic (as it computes the derived tensor product by definition). Taking stalks is an exact functor, hence we get that $\mathcal{K}^\bullet \otimes_{\mathcal{O}_p} M$ is acyclic as desired. □

Lemma 18.7. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. If $\mathcal{C}$ has enough points, then the pullback of a K-flat complex of $\mathcal{O}'$-modules is a K-flat complex of $\mathcal{O}$-modules.*

Proof. This follows from Lemma 18.6, Modules on Sites, Lemma 36.4, and More on Algebra, Lemma 59.3. □

Lemma 18.8. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_\mathcal{D}$-modules. The diagram*

$$\begin{tikzcd}
Lf^*(\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \arrow[r] \arrow[d] & Lf^* \mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \arrow[d] \\
Lf^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*\mathcal{M}^\bullet \arrow[d] & f^* \mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \arrow[d] \\
f^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} f^*\mathcal{M}^\bullet \arrow[r] & \mathit{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{M}^\bullet)
\end{tikzcd}$$

commutes.

Proof. We will use the existence of K-flat resolutions with flat terms (Lemma 17.11), we will use that derived pullback is computed by such complexes (Lemma 18.2), and that pullbacks preserve these properties (Lemma 18.1). If we choose such resolutions $\mathcal{P}^\bullet \to \mathcal{K}^\bullet$ and $\mathcal{Q}^\bullet \to \mathcal{M}^\bullet$, then we see that

$$\begin{tikzcd}
Lf^*\mathrm{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{Q}^\bullet) \arrow[r] \arrow[d] & Lf^*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \arrow[d] \\
f^*\mathrm{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{Q}^\bullet) \arrow[r] \arrow[d] & f^*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \arrow[d] \\
\mathrm{Tot}(f^*\mathcal{P}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{Q}^\bullet) \arrow[r] & \mathrm{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{M}^\bullet)
\end{tikzcd}$$

commutes. However, now the left hand side of the diagram is the left hand side of the diagram by our choice of $\mathcal{P}^\bullet$ and $\mathcal{Q}^\bullet$ and Lemma 17.5. $\square$

19. Cohomology of unbounded complexes

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The category $\mathit{Mod}(\mathcal{O})$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus\nolimits_{U \in \mathrm{Ob}(\mathcal{C})} j_{U!}\mathcal{O}_U,$$

see Modules on Sites, Section 14 and Lemmas 28.7 and 28.8. By Injectives, Theorem 12.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}$-modules and moreover this embedding can be chosen functorial in $\mathcal{F}^\bullet$. It follows from Derived Categories, Lemma 31.7 that

(1) any exact functor $F : K(\mathit{Mod}(\mathcal{O})) \to \mathcal{D}$ into a trianguated category $\mathcal{D}$ has a right derived functor $RF : D(\mathcal{O}) \to \mathcal{D}$,
(2) for any additive functor $F : \mathit{Mod}(\mathcal{O}) \to \mathcal{A}$ into an abelian category $\mathcal{A}$ we consider the exact functor $F : K(\mathit{Mod}(\mathcal{O})) \to D(\mathcal{A})$ induced by F and we obtain a right derived functor $RF : D(\mathcal{O}) \to K(\mathcal{A})$.

By construction we have $RF(\mathcal{F}^\bullet) = F(\mathcal{I}^\bullet)$ where $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is as above.

Here are some examples of the above:

(1) The functor $\Gamma(\mathcal{C}, -) : \mathit{Mod}(\mathcal{O}) \to \mathrm{Mod}_{\Gamma(\mathcal{C}, \mathcal{O})}$ gives rise to
$$R\Gamma(\mathcal{C}, -) : D(\mathcal{O}) \longrightarrow D(\Gamma(\mathcal{C}, \mathcal{O}))$$

We shall use the notation $H^i(\mathcal{C}, K) = H^i(R\Gamma(\mathcal{C}, K))$ for cohomology.

(2) For an object U of $\mathcal{C}$ we consider the functor $\Gamma(U, -) : \textit{Mod}(\mathcal{O}) \to \text{Mod}_{\Gamma(U,\mathcal{O})}$. This gives rise to

$$R\Gamma(U, -) : D(\mathcal{O}) \to D(\Gamma(U, \mathcal{O}))$$

We shall use the notation $H^i(U, K) = H^i(R\Gamma(U, K))$ for cohomology.

(3) For a morphism of ringed topoi $f : (\textit{Sh}(\mathcal{C}), \mathcal{O}) \to (\textit{Sh}(\mathcal{D}), \mathcal{O}')$ we consider the functor $f_* : \textit{Mod}(\mathcal{O}) \to \textit{Mod}(\mathcal{O}')$ which gives rise to the total direct image

$$Rf_* : D(\mathcal{O}) \longrightarrow D(\mathcal{O}')$$

on unbounded derived categories.

Lemma 19.1. *Let $f : (\textit{Sh}(\mathcal{C}), \mathcal{O}) \to (\textit{Sh}(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. The functor Rf_* defined above and the functor Lf^* defined in Lemma 18.2 are adjoint:*

$$\text{Hom}_{D(\mathcal{O})}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \text{Hom}_{D(\mathcal{O}')}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

bifunctorially in $\mathcal{F}^\bullet \in \text{Ob}(D(\mathcal{O}))$ and $\mathcal{G}^\bullet \in \text{Ob}(D(\mathcal{O}'))$.

Proof. This follows formally from the fact that Rf_* and Lf^* exist, see Derived Categories, Lemma 30.3. $\square$

Lemma 19.2. *Let $f : (\textit{Sh}(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (\textit{Sh}(\mathcal{D}), \mathcal{O}_\mathcal{D})$ and $g : (\textit{Sh}(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (\textit{Sh}(\mathcal{E}), \mathcal{O}_\mathcal{E})$ be morphisms of ringed topoi. Then $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors $D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_\mathcal{E})$.*

Proof. By Lemma 19.1 we see that $Rg_* \circ Rf_*$ is adjoint to $Lf^* \circ Lg^*$. We have $Lf^* \circ Lg^* = L(g \circ f)^*$ by Lemma 18.3 and hence by uniqueness of adjoint functors we have $Rg_* \circ Rf_* = R(g \circ f)_*$. $\square$

Remark 19.3. The construction of unbounded derived functor Lf^* and Rf_* allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{tikzcd}
(\textit{Sh}(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g'"'] \arrow[d, "f'"'] & (\textit{Sh}(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "f"] \\
(\textit{Sh}(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (\textit{Sh}(\mathcal{D}), \mathcal{O}_\mathcal{D})
\end{tikzcd}$$

is a commutative diagram of ringed topoi. Let K be an object of $D(\mathcal{O}_\mathcal{C})$. Then there exists a canonical base change map

$$Lg^*Rf_*K \longrightarrow R(f')_*L(g')^*K$$

in $D(\mathcal{O}_{\mathcal{D}'})$. Namely, this map is adjoint to a map $L(f')^*Lg^*Rf_*K \to L(g')^*K$. Since $L(f')^* \circ Lg^* = L(g')^* \circ Lf^*$ we see this is the same as a map $L(g')^*Lf^*Rf_*K \to L(g')^*K$ which we can take to be $L(g')^*$ of the adjunction map $Lf^*Rf_*K \to K$.

Remark 19.4. Consider a commutative diagram

$$\begin{tikzcd}
(Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'}) \arrow[r, "k"'] \arrow[d, "f'"'] & (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) \arrow[d, "f"] \\
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "l"] \arrow[d, "g'"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "g"] \\
(Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "m"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})
\end{tikzcd}$$

of ringed topoi. Then the base change maps of Remark 19.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned}
Lm^* \circ R(g \circ f)_* & = Lm^* \circ Rg_* \circ Rf_* \\
& \to Rg'_* \circ Ll^* \circ Rf_* \\
& \to Rg'_* \circ Rf'_* \circ Lk^* \\
& = R(g' \circ f')_* \circ Lk^*
\end{aligned}$$

is the base change map for the rectangle. We omit the verification.

Remark 19.5. Consider a commutative diagram

$$\begin{tikzcd}
(Sh(\mathcal{C}''), \mathcal{O}_{\mathcal{C}''}) \arrow[r, "g'"'] \arrow[d, "f''"'] & (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "f"] \\
(Sh(\mathcal{D}''), \mathcal{O}_{\mathcal{D}''}) \arrow[r, "h'"] & (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "h"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})
\end{tikzcd}$$

of ringed topoi. Then the base change maps of Remark 19.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned}
L(h \circ h')^* \circ Rf_* & = L(h')^* \circ Lh^* \circ Rf_* \\
& \to L(h')^* \circ Rf'_* \circ Lg^* \\
& \to Rf''_* \circ L(g')^* \circ Lg^* \\
& = Rf''_* \circ L(g \circ g')^*
\end{aligned}$$

is the base change map for the rectangle. We omit the verification.

Lemma 19.6. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_{\mathcal{C}}$-modules. The diagram*

$$\begin{tikzcd}
Lf^* f_* \mathcal{K}^\bullet \arrow[r] \arrow[d] & f^* f_* \mathcal{K}^\bullet \arrow[d] \\
Lf^* Rf_* \mathcal{K}^\bullet \arrow[r] & \mathcal{K}^\bullet
\end{tikzcd}$$

coming from $Lf^* \to f^*$ on complexes, $f_* \to Rf_*$ on complexes, and adjunction $Lf^* \circ Rf_* \to id$ commutes in $D(\mathcal{O}_{\mathcal{C}})$.

Proof. We will use the existence of K-flat resolutions and K-injective resolutions, see Lemmas 17.11, 18.2, and 18.1 and the discussion above. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_{\mathcal{C}}$-modules. Choose a quasi-isomorphism $\mathcal{Q}^\bullet \to f_*\mathcal{I}^\bullet$ where $\mathcal{Q}^\bullet$ is a K-flat complex of $\mathcal{O}_\mathcal{D}$-modules with flat terms. We can choose a K-flat complex of $\mathcal{O}_\mathcal{D}$-modules $\mathcal{P}^\bullet$ with flat terms and a diagram of morphisms of complexes

$$\begin{tikzcd} \mathcal{P}^\bullet \arrow[r] \arrow[d] & f_*\mathcal{K}^\bullet \arrow[d] \\ \mathcal{Q}^\bullet \arrow[r] & f_*\mathcal{I}^\bullet \end{tikzcd}$$

commutative up to homotopy where the top horizontal arrow is a quasi-isomorphism. Namely, we can first choose such a diagram for some complex $\mathcal{P}^\bullet$ because the quasi-isomorphisms form a multiplicative system in the homotopy category of complexes and then we can choose a resolution of $\mathcal{P}^\bullet$ by a K-flat complex with flat terms. Taking pullbacks we obtain a diagram of morphisms of complexes

$$\begin{tikzcd} f^*\mathcal{P}^\bullet \arrow[r] \arrow[d] & f^*f_*\mathcal{K}^\bullet \arrow[r] \arrow[d] & \mathcal{K}^\bullet \arrow[d] \\ f^*\mathcal{Q}^\bullet \arrow[r] & f^*f_*\mathcal{I}^\bullet \arrow[r] & \mathcal{I}^\bullet \end{tikzcd}$$

commutative up to homotopy. The outer rectangle witnesses the truth of the statement in the lemma. $\square$

Remark 19.7. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. The adjointness of Lf^* and Rf_* allows us to construct a relative cup product

$$Rf_*K \otimes_{\mathcal{O}_\mathcal{D}}^\mathbf{L} Rf_*L \longrightarrow Rf_*(K \otimes_{\mathcal{O}_\mathcal{C}}^\mathbf{L} L)$$

in $D(\mathcal{O}_\mathcal{D})$ for all K, L in $D(\mathcal{O}_\mathcal{C})$. Namely, this map is adjoint to a map $Lf^*(Rf_*K \otimes_{\mathcal{O}_\mathcal{D}}^\mathbf{L} Rf_*L) \to K \otimes_{\mathcal{O}_\mathcal{C}}^\mathbf{L} L$ for which we can take the composition of the isomorphism $Lf^*(Rf_*K \otimes_{\mathcal{O}_\mathcal{D}}^\mathbf{L} Rf_*L) = Lf^*Rf_*K \otimes_{\mathcal{O}_\mathcal{C}}^\mathbf{L} Lf^*Rf_*L$ (Lemma 18.4) with the map $Lf^*Rf_*K \otimes_{\mathcal{O}_\mathcal{C}}^\mathbf{L} Lf^*Rf_*L \to K \otimes_{\mathcal{O}_\mathcal{C}}^\mathbf{L} L$ coming from the counit $Lf^* \circ Rf_* \to \mathrm{id}$.

Lemma 19.8. *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \subset Ab(\mathcal{C})$ denote the Serre subcategory consisting of torsion abelian sheaves. Then the functor $D(\mathcal{A}) \to D_\mathcal{A}(\mathcal{C})$ is an equivalence.*

Proof. A key observation is that an injective abelian sheaf $\mathcal{I}$ is divisible. Namely, if $s \in \mathcal{I}(U)$ is a local section, then we interpret s as a map $s : j_{U!}\mathbf{Z} \to \mathcal{I}$ and we apply the defining property of an injective object to the injective map of sheaves $n : j_{U!}\mathbf{Z} \to j_{U!}\mathbf{Z}$ to see that there exists an $s' \in \mathcal{I}(U)$ with $ns' = s$.

For a sheaf $\mathcal{F}$ denote $\mathcal{F}_{tor}$ its torsion subsheaf. We claim that if $\mathcal{I}^\bullet$ is a complex of injective abelian sheaves whose cohomology sheaves are torsion, then

$$\mathcal{I}^\bullet_{tor} \to \mathcal{I}^\bullet$$

is a quasi-isomorphism. Namely, by flatness of $\mathbf{Q}$ over $\mathbf{Z}$ we have

$$H^p(\mathcal{I}^\bullet) \otimes_\mathbf{Z} \mathbf{Q} = H^p(\mathcal{I}^\bullet \otimes_\mathbf{Z} \mathbf{Q})$$

which is zero because the cohomology sheaves are torsion. By divisibility (shown above) we see that $\mathcal{I}^\bullet \to \mathcal{I}^\bullet \otimes_\mathbf{Z} \mathbf{Q}$ is surjective with kernel $\mathcal{I}^\bullet_{tor}$. The claim follows from the long exact sequence of cohomology sheaves associated to the short exact sequence you get.

To prove the lemma we will construct right adjoint $T : D(\mathcal{C}) \to D(\mathcal{A})$. Namely, given K in $D(\mathcal{C})$ we can represent K by a K-injective complex $\mathcal{I}^\bullet$ whose cohomology sheaves are injective, see Injectives, Theorem 12.6. Then we set $T(K) = \mathcal{I}^\bullet_{tor}$, in other words, T is the right derived functor of taking torsion. The functor T is a right adjoint to $i : D(\mathcal{A}) \to D_\mathcal{A}(\mathcal{C})$. This readily follows from the observation that if $\mathcal{F}^\bullet$ is a complex of torsion sheaves, then

$$\mathrm{Hom}_{K(\mathcal{A})}(\mathcal{F}^\bullet, I^\bullet_{tor}) = \mathrm{Hom}_{K(Ab(\mathcal{C}))}(\mathcal{F}^\bullet, I^\bullet)$$

in particular $\mathcal{I}^\bullet_{tor}$ is a K-injective complex of $\mathcal{A}$. Some details omitted; in case of doubt, it also follows from the more general Derived Categories, Lemma 30.3. Our claim above gives that $L = T(i(L))$ for L in $D(\mathcal{A})$ and $i(T(K)) = K$ if K is in $D_\mathcal{A}(\mathcal{C})$. Using Categories, Lemma 24.4 the result follows. $\square$

20. Some properties of K-injective complexes

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Denote $j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ the corresponding localization morphism. The pullback functor j^* is exact as it is just the restriction functor. Thus derived pullback Lj^* is computed on any complex by simply restricting the complex. We often simply denote the corresponding functor

$$D(\mathcal{O}) \to D(\mathcal{O}_U), \quad E \mapsto j^*E = E|_U$$

Similarly, extension by zero $j_! : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ (see Modules on Sites, Definition 19.1) is an exact functor (Modules on Sites, Lemma 19.3). Thus it induces a functor

$$j_! : D(\mathcal{O}_U) \to D(\mathcal{O}), \quad F \mapsto j_!F$$

by simply applying $j_!$ to any complex representing the object F.

Lemma 20.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. The restriction of a K-injective complex of $\mathcal{O}$-modules to $\mathcal{C}/U$ is a K-injective complex of $\mathcal{O}_U$-modules.*

Proof. Follows immediately from Derived Categories, Lemma 31.9 and the fact that the restriction functor has the exact left adjoint $j_!$. See discussion above. $\square$

Lemma 20.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. For K in $D(\mathcal{O})$ we have $H^p(U, K) = H^p(\mathcal{C}/U, K|_{\mathcal{C}/U})$.*

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing K. Then

$$H^q(U, K) = H^q(\Gamma(U, \mathcal{I}^\bullet)) = H^q(\Gamma(\mathcal{C}/U, \mathcal{I}^\bullet|_{\mathcal{C}/U}))$$

by construction of cohomology. By Lemma 20.1 the complex $\mathcal{I}^\bullet|_{\mathcal{C}/U}$ is a K-injective complex representing $K|_{\mathcal{C}/U}$ and the lemma follows. $\square$

Lemma 20.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be an object of $D(\mathcal{O})$. The sheafification of*

$$U \mapsto H^q(U, K) = H^q(\mathcal{C}/U, K|_{\mathcal{C}/U})$$

is the qth cohomology sheaf $H^q(K)$ of K.

Proof. The equality $H^q(U, K) = H^q(\mathcal{C}/U, K|_{\mathcal{C}/U})$ holds by Lemma 20.2. Choose a K-injective complex $\mathcal{I}^\bullet$ representing K. Then

$$H^q(U, K) = \frac{\mathrm{Ker}(\mathcal{I}^q(U) \to \mathcal{I}^{q+1}(U))}{\mathrm{Im}(\mathcal{I}^{q-1}(U) \to \mathcal{I}^q(U))}.$$

by our construction of cohomology. Since $H^q(K) = \mathrm{Ker}(\mathcal{I}^q \to \mathcal{I}^{q+1}) / \mathrm{Im}(\mathcal{I}^{q-1} \to \mathcal{I}^q)$ the result is clear. $\square$

Lemma 20.4. *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Given $V \in \mathcal{D}$, set $U = u(V)$ and denote $g : (\mathcal{C}/U, \mathcal{O}_U) \to (\mathcal{D}/V, \mathcal{O}_V)$ the induced morphism of ringed sites (Modules on Sites, Lemma 20.1). Then $(Rf_*E)|_{\mathcal{D}/V} = Rg_*(E|_{\mathcal{C}/U})$ for E in $D(\mathcal{O}_\mathcal{C})$.*

Proof. Represent E by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_\mathcal{C}$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_{\mathcal{C}/U}) = g_*(\mathcal{I}^\bullet|_{\mathcal{C}/U})$ by Lemma 20.1. Since it is clear that $(f_*\mathcal{F})|_{\mathcal{D}/V} = g_*(\mathcal{F}|_{\mathcal{C}/U})$ for any sheaf $\mathcal{F}$ on $\mathcal{C}$ (see Modules on Sites, Lemma 20.1 or the more basic Sites, Lemma 28.1) the result follows. ☐

Lemma 20.5. *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Then $R\Gamma(\mathcal{D}, -) \circ Rf_* = R\Gamma(\mathcal{C}, -)$ as functors $D(\mathcal{O}_\mathcal{C}) \to D(\Gamma(\mathcal{O}_\mathcal{D}))$. More generally, for $V \in \mathcal{D}$ with $U = u(V)$ we have $R\Gamma(U, -) = R\Gamma(V, -) \circ Rf_*$.*

Proof. Consider the punctual topos pt endowed with $\mathcal{O}_{pt}$ given by the ring $\Gamma(\mathcal{O}_\mathcal{D})$. There is a canonical morphism $(\mathcal{D}, \mathcal{O}_\mathcal{D}) \to (pt, \mathcal{O}_{pt})$ of ringed topoi inducing the identification on global sections of structure sheaves. Then $D(\mathcal{O}_{pt}) = D(\Gamma(\mathcal{O}_\mathcal{D}))$. The assertion $R\Gamma(\mathcal{D}, -) \circ Rf_* = R\Gamma(\mathcal{C}, -)$ follows from Lemma 19.2 applied to

$$(\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D}) \to (pt, \mathcal{O}_{pt})$$

The second (more general) statement follows from the first statement after applying Lemma 20.4. ☐

Lemma 20.6. *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let K be in $D(\mathcal{O}_\mathcal{C})$. Then $H^i(Rf_*K)$ is the sheaf associated to the presheaf*

$$V \mapsto H^i(u(V), K) = H^i(V, Rf_*K)$$

Proof. The equality $H^i(u(V), K) = H^i(V, Rf_*K)$ follows upon taking cohomology from the second statement in Lemma 20.5. Then the statement on sheafification follows from Lemma 20.3. ☐

Lemma 20.7. *Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Let K be an object of $D(\mathcal{O}_\mathcal{C})$ and denote K_{ab} its image in $D(\underline{\mathbf{Z}}_\mathcal{C})$.*

1. *There is a canonical map $R\Gamma(\mathcal{C}, K) \to R\Gamma(\mathcal{C}, K_{ab})$ which is an isomorphism in $D(Ab)$.*
2. *For any $U \in \mathcal{C}$ there is a canonical map $R\Gamma(U, K) \to R\Gamma(U, K_{ab})$ which is an isomorphism in $D(Ab)$.*
3. *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. There is a canonical map $Rf_*K \to Rf_*(K_{ab})$ which is an isomorphism in $D(\underline{\mathbf{Z}}_\mathcal{D})$.*

Proof. The map is constructed as follows. Choose a K-injective complex $\mathcal{I}^\bullet$ representing K. Choose a quasi-isomorpism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is a K-injective complex of abelian groups. Then the map in (1) is given by $\Gamma(\mathcal{C}, \mathcal{I}^\bullet) \to \Gamma(\mathcal{C}, \mathcal{J}^\bullet)$ (2) is given by $\Gamma(U, \mathcal{I}^\bullet) \to \Gamma(U, \mathcal{J}^\bullet)$ and the map in (3) is given by $f_*\mathcal{I}^\bullet \to f_*\mathcal{J}^\bullet$. To show that these maps are isomorphisms, it suffices to prove they induce isomorphisms on cohomology groups and cohomology sheaves. By Lemmas 20.2 and 20.6 it suffices to show that the map

$$H^0(\mathcal{C}, K) \longrightarrow H^0(\mathcal{C}, K_{ab})$$

is an isomorphism. Observe that

$$H^0(\mathcal{C}, K) = \mathrm{Hom}_{D(\mathcal{O}_\mathcal{C})}(\mathcal{O}_\mathcal{C}, K)$$

and similarly for the other group. Choose any complex $\mathcal{K}^\bullet$ of $\mathcal{O}_\mathcal{C}$-modules representing K. By construction of the derived category as a localization we have

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{C})}(\mathcal{O}_\mathcal{C}, K) = \mathrm{colim}_{s : \mathcal{F}^\bullet \to \mathcal{O}_\mathcal{C}} \mathrm{Hom}_{K(\mathcal{O}_\mathcal{C})}(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

where the colimit is over quasi-isomorphisms s of complexes of $\mathcal{O}_\mathcal{C}$-modules. Similarly, we have

$$\mathrm{Hom}_{D(\underline{\mathbf{Z}}_\mathcal{C})}(\underline{\mathbf{Z}}_\mathcal{C}, K) = \mathrm{colim}_{s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_\mathcal{C}} \mathrm{Hom}_{K(\underline{\mathbf{Z}}_\mathcal{C})}(\mathcal{G}^\bullet, \mathcal{K}^\bullet)$$

Next, we observe that the quasi-isomorphisms $s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_\mathcal{C}$ with $\mathcal{G}^\bullet$ bounded above complex of flat $\underline{\mathbf{Z}}_\mathcal{C}$-modules is cofinal in the system. (This follows from Modules on Sites, Lemma 28.8 and Derived Categories, Lemma 15.4; see discussion in Section 17.) Hence we can construct an inverse to the map $H^0(\mathcal{C}, K) \longrightarrow H^0(\mathcal{C}, K_{ab})$ by representing an element $\xi \in H^0(\mathcal{C}, K_{ab})$ by a pair

$$(s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_\mathcal{C}, a : \mathcal{G}^\bullet \to \mathcal{K}^\bullet)$$

with $\mathcal{G}^\bullet$ a bounded above complex of flat $\underline{\mathbf{Z}}_\mathcal{C}$-modules and sending this to

$$(\mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_\mathcal{C}} \mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{C}, \mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_\mathcal{C}} \mathcal{O}_\mathcal{C} \to \mathcal{K}^\bullet)$$

The only thing to note here is that the first arrow is a quasi-isomorphism by Lemmas 17.12 and 17.8. We omit the detailed verification that this construction is indeed an inverse. $\square$

Lemma 20.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Denote $j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ the corresponding localization morphism. The restriction functor $D(\mathcal{O}) \to D(\mathcal{O}_U)$ is a right adjoint to extension by zero $j_! : D(\mathcal{O}_U) \to D(\mathcal{O})$.*

Proof. We have to show that

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!E, F) = \mathrm{Hom}_{D(\mathcal{O}_U)}(E, F|_U)$$

Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_U$-modules representing E and choose a K-injective complex $\mathcal{I}^\bullet$ representing F. By Lemma 20.1 the complex $\mathcal{I}^\bullet|_U$ is K-injective as well. Hence we see that the formula above becomes

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!\mathcal{E}^\bullet, \mathcal{I}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{I}^\bullet|_U)$$

which holds as $|_U$ and $j_!$ are adjoint functors (Modules on Sites, Lemma 19.2) and Derived Categories, Lemma 31.2. $\square$

Lemma 20.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. For L in $D(\mathcal{O}_U)$ and K in $D(\mathcal{O})$ we have $j_!L \otimes^{\mathbf{L}}_\mathcal{O} K = j_!(L \otimes^{\mathbf{L}}_{\mathcal{O}_U} K|_U)$.*

Proof. Represent L by a complex of $\mathcal{O}_U$-modules and K by a K-flat complexe of $\mathcal{O}$-modules and apply Modules on Sites, Lemma 27.9. Details omitted. $\square$

Lemma 20.10. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a flat morphism of ringed topoi. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_\mathcal{C}$-modules, then $f_*\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_\mathcal{D}$-modules.*

Proof. This is true because

$$\mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(\mathcal{F}^\bullet, f_*\mathcal{I}^\bullet) = \mathrm{Hom}_{K(\mathcal{O}_\mathcal{C})}(f^*\mathcal{F}^\bullet, \mathcal{I}^\bullet)$$

by Modules on Sites, Lemma 13.2 and the fact that f^* is exact as f is assumed to be flat. $\square$

Lemma 20.11. *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a map of sheaves of rings. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules, then $\mathcal{H}om_\mathcal{O}(\mathcal{O}', \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}'$-modules.*

Proof. This is true because $\mathrm{Hom}_{K(\mathcal{O}')}(\mathcal{G}^\bullet, \mathrm{Hom}_\mathcal{O}(\mathcal{O}', \mathcal{I}^\bullet)) = \mathrm{Hom}_{K(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{I}^\bullet)$ by Modules on Sites, Lemma 27.8. $\square$

21. Localization and cohomology

Let $\mathcal{C}$ be a site. Let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then we obtain a morphism of topoi

$$j_{X/Y} : Sh(\mathcal{C}/X) \longrightarrow Sh(\mathcal{C}/Y)$$

See Sites, Sections 25 and 27. Some questions about cohomology are easier for this type of morphisms of topoi. Here is an example where we get a trivial type of base change theorem.

Lemma 21.1. *Let $\mathcal{C}$ be a site. Let*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a cartesian diagram of $\mathcal{C}$. Then we have $j_{Y'/Y}^{-1} \circ Rj_{X/Y,*} = Rj_{X'/Y',*} \circ j_{X'/X}^{-1}$ as functors $D(\mathcal{C}/X) \to D(\mathcal{C}/Y')$.

Proof. Let $E \in D(\mathcal{C}/X)$. Choose a K-injective complex $\mathcal{I}^\bullet$ of abelian sheaves on $\mathcal{C}/X$ representing E. By Lemma 20.1 we see that $j_{X'/X}^{-1}\mathcal{I}^\bullet$ is K-injective too. Hence we may compute $Rj_{X'/Y'}(j_{X'/X}^{-1}E)$ by $j_{X'/Y',*}j_{X'/X}^{-1}\mathcal{I}^\bullet$. Thus we see that the equality holds by Sites, Lemma 27.5. $\square$

If we have a ringed site $(\mathcal{C}, \mathcal{O})$ and a morphism $f : X \to Y$ of $\mathcal{C}$, then $j_{X/Y}$ becomes a morphism of ringed topoi

$$j_{X/Y} : (Sh(\mathcal{C}/X), \mathcal{O}_X) \longrightarrow (Sh(\mathcal{C}/Y), \mathcal{O}_Y)$$

See Modules on Sites, Lemma 19.5.

Lemma 21.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a cartesian diagram of $\mathcal{C}$. Then we have $j_{Y'/Y}^{*} \circ Rj_{X/Y,*} = Rj_{X'/Y',*} \circ j_{X'/X}^{*}$ as functors $D(\mathcal{O}_X) \to D(\mathcal{O}_{Y'})$.

Proof. Since $j_{Y'/Y}^{-1}\mathcal{O}_Y = \mathcal{O}_{Y'}$ we have $j_{Y'/Y}^* = Lj_{Y'/Y}^* = j_{Y'/Y}^{-1}$. Similarly we have $j_{X'/X}^* = Lj_{X'/X}^* = j_{X'/X}^{-1}$. Thus by Lemma 20.7 it suffices to prove the result on derived categories of abelian sheaves which we did in Lemma 21.1. $\square$

22. Inverse systems and cohomology

We prove some results on inverse systems of sheaves of modules.

Lemma 22.1. *Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume

$$\bigoplus\nolimits_{n \geq 0} H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-module. Then the inverse system $M_n = H^p(\mathcal{C}, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[2].

Proof. Set $N_n = H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and let $\delta_n : M_n \to N_n$ be the boundary map on cohomology coming from the short exact sequence $0 \to I^n\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$. Then $\bigoplus \text{Im}(\delta_n) \subset \bigoplus N_n$ is a graded submodule. Namely, if $s \in M_n$ and $f \in I^m$, then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I^n\mathcal{F}_{n+1} \arrow[r] \arrow[d, "f"'] & \mathcal{F}_{n+1} \arrow[r] \arrow[d, "f"'] & \mathcal{F}_n \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & I^{n+m}\mathcal{F}_{n+m+1} \arrow[r] & \mathcal{F}_{n+m+1} \arrow[r] & \mathcal{F}_{n+m} \arrow[r] & 0
\end{tikzcd}$$

The middle vertical map is given by lifting a local section of $\mathcal{F}_{n+1}$ to a section of $\mathcal{F}_{n+m+1}$ and then multiplying by f; similarly for the other vertical arrows. We conclude that $\delta_{n+m}(fs) = f\delta_n(s)$. By assumption we can find $s_j \in M_{n_j}$, $j = 1, \ldots, N$ such that $\delta_{n_j}(s_j)$ generate $\bigoplus \text{Im}(\delta_n)$ as a graded module. Let $n > c = \max(n_j)$. Let $s \in M_n$. Then we can find $f_j \in I^{n-n_j}$ such that $\delta_n(s) = \sum f_j\delta_{n_j}(s_j)$. We conclude that $\delta(s - \sum f_js_j) = 0$, i.e., we can find $s' \in M_{n+1}$ mapping to $s - \sum f_js_j$ in M_n. It follows that

$$\text{Im}(M_{n+1} \to M_{n-c}) = \text{Im}(M_n \to M_{n-c})$$

Namely, the elements f_js_j map to zero in M_{n-c}. This proves the lemma. $\square$

Lemma 22.2. *Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Given n define

$$N_n = \bigcap\nolimits_{m \geq n} \text{Im}\left(H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-module, then the inverse system $M_n = H^p(\mathcal{C}, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[3].

Proof. The proof is exactly the same as the proof of Lemma 22.1. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and that the boundary maps $\delta_n : M_n \to H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ have image contained in N_n.

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[2] In fact, there exists a $c \geq 0$ such that $\text{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.

[3] In fact, there exists a $c \geq 0$ such that $\text{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.

Suppose that $\xi \in N_n$ and $f \in I^k$. Choose $m \gg n+k$. Choose $\xi' \in H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{m+1})$ lifting ξ. We consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & I^n\mathcal{F}_{m+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_{m+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_n \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & I^{n+k}\mathcal{F}_{m+1} \arrow[r] & \mathcal{F}_{m+1} \arrow[r] & \mathcal{F}_{n+k} \arrow[r] & 0
\end{tikzcd}$$

constructed as in the proof of Lemma 22.1. We get an induced map on cohomology and we see that $f\xi' \in H^{p+1}(\mathcal{C}, I^{n+k}\mathcal{F}_{m+1})$ maps to $f\xi$. Since this is true for all $m \gg n+k$ we see that $f\xi$ is in N_{n+k} as desired.

To see the boundary maps δ_n have image contained in N_n we consider the diagrams

$$\begin{tikzcd}
0 \arrow[r] & I^n\mathcal{F}_{m+1} \arrow[r] \arrow[d] & \mathcal{F}_{m+1} \arrow[r] \arrow[d] & \mathcal{F}_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^n\mathcal{F}_{n+1} \arrow[r] & \mathcal{F}_{n+1} \arrow[r] & \mathcal{F}_n \arrow[r] & 0
\end{tikzcd}$$

for $m \geq n$. Looking at the induced maps on cohomology we conclude. $\square$

Lemma 22.3. *Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume

$$\bigoplus\nolimits_{n\geq 0} H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the limit topology on $M = \lim H^p(\mathcal{C}, \mathcal{F}_n)$ is the I-adic topology.

Proof. Set $F^n = \mathrm{Ker}(M \to H^p(\mathcal{C}, \mathcal{F}_n))$ for $n \geq 1$ and $F^0 = M$. Observe that $IF^n \subset F^{n+1}$. In particular $I^nM \subset F^n$. Hence the I-adic topology is finer than the limit topology. For the converse, we will show that given n there exists an $m \geq n$ such that $F^m \subset I^nM$[4]. We have injective maps

$$F^n/F^{n+1} \longrightarrow H^p(\mathcal{C}, \mathcal{F}_{n+1})$$

whose image is contained in the image of $H^p(\mathcal{C}, I^n\mathcal{F}_{n+1}) \to H^p(\mathcal{C}, \mathcal{F}_{n+1})$. Denote

$$E_n \subset H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$$

the inverse image of F^n/F^{n+1}. Then $\bigoplus E_n$ is a graded $\bigoplus I^n/I^{n+1}$-submodule of $\bigoplus H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and $\bigoplus E_n \to \bigoplus F^n/F^{n+1}$ is a homomorphism of graded modules; details omitted. By assumption $\bigoplus E_n$ is generated by finitely many homogeneous elements over $\bigoplus I^n/I^{n+1}$. Since $E_n \to F^n/F^{n+1}$ is surjective, we see that the same thing is true of $\bigoplus F^n/F^{n+1}$. Hence we can find r and $c_1, \ldots, c_r \geq 0$ and $a_i \in F^{c_i}$ whose images in $\bigoplus F^n/F^{n+1}$ generate. Set $c = \max(c_i)$.

For $n \geq c$ we claim that $IF^n = F^{n+1}$. The claim shows that $F^{n+c} = I^nF^c \subset I^nM$ as desired. To prove the claim suppose $a \in F^{n+1}$. The image of a in F^{n+1}/F^{n+2} is a linear combination of our a_i. Therefore $a - \sum f_ia_i \in F^{n+2}$ for some $f_i \in I^{n+1-c_i}$. Since $I^{n+1-c_i} = I \cdot I^{n-c_i}$ as $n \geq c_i$ we can write $f_i = \sum g_{i,j}h_{i,j}$ with $g_{i,j} \in I$ and $h_{i,j}a_i \in F^n$. Thus we see that $F^{n+1} = F^{n+2} + IF^n$. A simple induction argument

[4]In fact, there exist a $c \geq 0$ such that $F^{n+c} \subset I^nM$ for all n.

gives $F^{n+1} = F^{n+e} + IF^n$ for all $e > 0$. It follows that IF^n is dense in F^{n+1}. Choose generators $k_1, \ldots, k_r$ of I and consider the continuous map

$$u : (F^n)^{\oplus r} \longrightarrow F^{n+1}, \quad (x_1, \ldots, x_r) \mapsto \sum k_i x_i$$

(in the limit topology). By the above the image of $(F^m)^{\oplus r}$ under u is dense in F^{m+1} for all $m \geq n$. By the open mapping lemma (More on Algebra, Lemma 36.5) we find that u is open. Hence u is surjective. Hence $IF^n = F^{n+1}$ for $n \geq c$. This concludes the proof. □

Lemma 22.4. *Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Given n define

$$N_n = \bigcap\nolimits_{m \geq n} \mathrm{Im}\left(H^p(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module, then the limit topology on $M = \lim H^p(\mathcal{C}, \mathcal{F}_n)$ is the I-adic topology.

Proof. The proof is exactly the same as the proof of Lemma 22.3. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and that $F^n/F^{n+1} \subset H^p(\mathcal{C}, \mathcal{F}_{n+1})$ is contained in the image of $N_n \to H^p(\mathcal{C}, \mathcal{F}_{n+1})$. In the proof of Lemma 22.2 we have seen the statement on the module structure.

Let $t \in F^n$. Choose an element $s \in H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$ which maps to the image of t in $H^p(\mathcal{C}, \mathcal{F}_{n+1})$. We have to show that s is in N_n. Now F^n is the kernel of the map from $M \to H^p(\mathcal{C}, \mathcal{F}_n)$ hence for all $m \geq n$ we can map t to an element $t_m \in H^p(\mathcal{C}, \mathcal{F}_{m+1})$ which maps to zero in $H^p(\mathcal{C}, \mathcal{F}_n)$. Consider the cohomology sequence

$$H^{p-1}(\mathcal{C}, \mathcal{F}_n) \to H^p(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^p(\mathcal{C}, \mathcal{F}_{m+1}) \to H^p(\mathcal{C}, \mathcal{F}_n)$$

coming from the short exact sequence $0 \to I^n\mathcal{F}_{m+1} \to \mathcal{F}_{m+1} \to \mathcal{F}_n \to 0$. We can choose $s_m \in H^p(\mathcal{C}, I^n\mathcal{F}_{m+1})$ mapping to t_m. Comparing the sequence above with the one for $m = n$ we see that s_m maps to s up to an element in the image of $H^{p-1}(\mathcal{C}, \mathcal{F}_n) \to H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$. However, this map factors through the map $H^p(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and we see that s is in the image as desired. □

23. Derived and homotopy limits

Let $\mathcal{C}$ be a site. Consider the category $\mathcal{C} \times \mathbf{N}$ with $\mathrm{Mor}((U, n), (V, m)) = \emptyset$ if $n > m$ and $\mathrm{Mor}((U, n), (V, m)) = \mathrm{Mor}(U, V)$ else. We endow this with the structure of a site by letting coverings be families $\{(U_i, n) \to (U, n)\}$ such that $\{U_i \to U\}$ is a covering of $\mathcal{C}$. Then the reader verifies immediately that sheaves on $\mathcal{C} \times \mathbf{N}$ are the same thing as inverse systems of sheaves on $\mathcal{C}$. In particular $Ab(\mathcal{C} \times \mathbf{N})$ is inverse systems of abelian sheaves on $\mathcal{C}$. Consider now the functor

$$\lim : Ab(\mathcal{C} \times \mathbf{N}) \to Ab(\mathcal{C})$$

which takes an inverse system to its limit. This is nothing but g_* where $g : Sh(\mathcal{C} \times \mathbf{N}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to the continuous and cocontinuous functor $\mathcal{C} \times \mathbf{N} \to \mathcal{C}$. (Observe that g^{-1} assigns to a sheaf on $\mathcal{C}$ the corresponding constant inverse system.)

By the general machinery explained above we obtain a derived functor

$$R\lim = Rg_* : D(\mathcal{C} \times \mathbf{N}) \to D(\mathcal{C}).$$

As indicated this functor is often denoted $R\lim$.

On the other hand, the continuous and cocontinuous functors $\mathcal{C} \to \mathcal{C} \times \mathbf{N}$, $U \mapsto (U, n)$ define morphisms of topoi $i_n : Sh(\mathcal{C}) \to Sh(\mathcal{C} \times \mathbf{N})$. Of course i_n^{-1} is the functor which picks the nth term of the inverse system. Thus there are transformations of functors $i_{n+1}^{-1} \to i_n^{-1}$. Hence given $K \in D(\mathcal{C} \times \mathbf{N})$ we get $K_n = i_n^{-1}K \in D(\mathcal{C})$ and maps $K_{n+1} \to K_n$. In Derived Categories, Definition 34.1 we have defined the notion of a homotopy limit

$$R\lim K_n \in D(\mathcal{C})$$

We claim the two notions agree (as far as it makes sense).

Lemma 23.1. *Let $\mathcal{C}$ be a site. Let K be an object of $D(\mathcal{C} \times \mathbf{N})$. Set $K_n = i_n^{-1}K$ as above. Then*

$$R\lim K \cong R\lim K_n$$

in $D(\mathcal{C})$.

Proof. To calculate $R\lim$ on an object K of $D(\mathcal{C} \times \mathbf{N})$ we choose a K-injective representative $\mathcal{I}^\bullet$ whose terms are injective objects of $Ab(\mathcal{C} \times \mathbf{N})$, see Injectives, Theorem 12.6. We may and do think of $\mathcal{I}^\bullet$ as an inverse system of complexes $(\mathcal{I}_n^\bullet)$ and then we see that

$$R\lim K = \lim \mathcal{I}_n^\bullet$$

where the right hand side is the termwise inverse limit.

Let $\mathcal{J} = (\mathcal{J}_n)$ be an injective object of $Ab(\mathcal{C} \times \mathbf{N})$. The morphisms $(U, n) \to (U, n+1)$ are monomorphisms of $\mathcal{C} \times \mathbf{N}$, hence $\mathcal{J}(U, n+1) \to \mathcal{J}(U, n)$ is surjective (Lemma 12.6). It follows that $\mathcal{J}_{n+1} \to \mathcal{J}_n$ is surjective as a map of presheaves.

Note that the functor i_n^{-1} has an exact left adjoint $i_{n,!}$. Namely, $i_{n,!}\mathcal{F}$ is the inverse system $\ldots 0 \to 0 \to \mathcal{F} \to \ldots \to \mathcal{F}$. Thus the complexes $i_n^{-1}\mathcal{I}^\bullet = \mathcal{I}_n^\bullet$ are K-injective by Derived Categories, Lemma 31.9.

Because we chose our K-injective complex to have injective terms we conclude that

$$0 \to \lim \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to 0$$

is a short exact sequence of complexes of abelian sheaves as it is a short exact sequence of complexes of abelian presheaves. Moreover, the products in the middle and the right represent the products in $D(\mathcal{C})$, see Injectives, Lemma 13.4 and its proof (this is where we use that $\mathcal{I}_n^\bullet$ is K-injective). Thus $R\lim K$ is a homotopy limit of the inverse system (K_n) by definition of homotopy limits in triangulated categories. □

Lemma 23.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The functors $R\Gamma(\mathcal{C}, -)$ and $R\Gamma(U, -)$ for $U \in \mathrm{Ob}(\mathcal{C})$ commute with $R\lim$. Moreover, there are short exact sequences*

$$0 \to R^1\lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

for any inverse system (K_n) in $D(\mathcal{O})$ and $m \in \mathbf{Z}$. Similar for $H^m(\mathcal{C}, R\lim K_n)$.

Proof. The first statement follows from Injectives, Lemma 13.6. Then we may apply More on Algebra, Remark 86.10 to $R\lim R\Gamma(U, K_n) = R\Gamma(U, R\lim K_n)$ to get the short exact sequences. $\square$

Lemma 23.3. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Then Rf_* commutes with $R\lim$, i.e., Rf_* commutes with derived limits.*

Proof. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. By induction on n we may choose actual complexes $\mathcal{K}_n^\bullet$ of $\mathcal{O}$-modules and maps of complexes $\mathcal{K}_{n+1}^\bullet \to \mathcal{K}_n^\bullet$ representing the maps $K_{n+1} \to K_n$ in $D(\mathcal{O})$. In other words, there exists an object K in $D(\mathcal{C} \times \mathbf{N})$ whose associated inverse system is the given one. Next, consider the commutative diagram

$$\begin{tikzcd} Sh(\mathcal{C} \times \mathbf{N}) \arrow[r, "g"'] \arrow[d, "f \times 1"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{C}' \times \mathbf{N}) \arrow[r, "g'"] & Sh(\mathcal{C}') \end{tikzcd}$$

of morphisms of topoi. It follows that $R\lim R(f \times 1)_*K = Rf_*R\lim K$. Working through the definitions and using Lemma 23.1 we obtain that $R\lim(Rf_*K_n) = Rf_*(R\lim K_n)$.

Alternate proof in case $\mathcal{C}$ has enough points. Consider the defining distinguished triangle

$$R\lim K_n \to \prod K_n \to \prod K_n$$

in $D(\mathcal{O})$. Applying the exact functor Rf_* we obtain the distinguished triangle

$$Rf_*(R\lim K_n) \to Rf_*\left(\prod K_n\right) \to Rf_*\left(\prod K_n\right)$$

in $D(\mathcal{O}')$. Thus we see that it suffices to prove that Rf_* commutes with products in the derived category (which are not just given by products of complexes, see Injectives, Lemma 13.4). However, since Rf_* is a right adjoint by Lemma 19.1 this follows formally (see Categories, Lemma 24.5). Caution: Note that we cannot apply Categories, Lemma 24.5 directly as $R\lim K_n$ is not a limit in $D(\mathcal{O})$. $\square$

Remark 23.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system in $D(\mathcal{O})$. Set $K = R\lim K_n$. For each n and m let $\mathcal{H}_n^m = H^m(K_n)$ be the mth cohomology sheaf of K_n and similarly set $\mathcal{H}^m = H^m(K)$. Let us denote $\underline{\mathcal{H}}_n^m$ the presheaf

$$U \longmapsto \underline{\mathcal{H}}_n^m(U) = H^m(U, K_n)$$

Similarly we set $\underline{\mathcal{H}}^m(U) = H^m(U, K)$. By Lemma 20.3 we see that $\mathcal{H}_n^m$ is the sheafification of $\underline{\mathcal{H}}_n^m$ and $\mathcal{H}^m$ is the sheafification of $\underline{\mathcal{H}}^m$. Here is a diagram

$$\begin{tikzcd} K \arrow[d, equal] & \underline{\mathcal{H}}^m \arrow[r] \arrow[d] & \mathcal{H}^m \arrow[d] \\ R\lim K_n & \lim \underline{\mathcal{H}}_n^m \arrow[r] & \lim \mathcal{H}_n^m \end{tikzcd}$$

In general it may not be the case that $\lim \mathcal{H}_n^m$ is the sheafification of $\lim \underline{\mathcal{H}}_n^m$. If $U \in \mathcal{C}$, then we have short exact sequences

$$0 \to R^1\lim \underline{\mathcal{H}}_n^{m-1}(U) \to \underline{\mathcal{H}}^m(U) \to \lim \underline{\mathcal{H}}_n^m(U) \to 0 \tag{23.4.1}$$

by Lemma 23.2.

The following lemma applies to an inverse system of quasi-coherent modules with surjective transition maps on an algebraic space or an algebraic stack.

Lemma 23.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{F}_n)$ be an inverse system of $\mathcal{O}$-modules. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Assume*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *$H^p(U, \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$,*
(3) *the inverse system $\mathcal{F}_n(U)$ has vanishing $R^1 \lim$ for $U \in \mathcal{B}$.*

Then $R\lim \mathcal{F}_n = \lim \mathcal{F}_n$ and we have $H^p(U, \lim \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$.

Proof. Set $K_n = \mathcal{F}_n$ and $K = R\lim \mathcal{F}_n$. Using the notation of Remark 23.4 and assumption (2) we see that for $U \in \mathcal{B}$ we have $\underline{\mathcal{H}}_n^m(U) = 0$ when $m \neq 0$ and $\underline{\mathcal{H}}_n^0(U) = \mathcal{F}_n(U)$. From Equation (23.4.1) and assumption (3) we see that $\underline{\mathcal{H}}^m(U) = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n(U)$ when $m = 0$. Sheafifying using (1) we find that $\mathcal{H}^m = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n$ when $m = 0$. Hence $K = \lim \mathcal{F}_n$. Since $H^m(U, K) = \underline{\mathcal{H}}^m(U) = 0$ for $m > 0$ (see above) we see that the second assertion holds. □

Lemma 23.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system in $D(\mathcal{O})$. Let $V \in \text{Ob}(\mathcal{C})$ and $m \in \mathbf{Z}$. Assume there exist an integer $n(V)$ and a cofinal system Cov_V of coverings of V such that for $\{V_i \to V\} \in Cov_V$*

(1) *$R^1 \lim H^{m-1}(V_i, K_n) = 0$, and*
(2) *$H^m(V_i, K_n) \to H^m(V_i, K_{n(V)})$ is injective for $n \geq n(V)$.*

Then the map on sections $H^m(R\lim K_n)(V) \to H^m(K_{n(V)})(V)$ is injective.

Proof. Let $\gamma \in H^m(R\lim K_n)(V)$ map to zero in $H^m(K_{n(V)})(V)$. Since $H^m(R\lim K_n)$ is the sheafification of $U \mapsto H^m(U, R\lim K_n)$ (by Lemma 20.3) we can choose $\{V_i \to V\} \in \text{Cov}_V$ and elements $\tilde{\gamma}_i \in H^m(V_i, R\lim K_n)$ mapping to $\gamma|_{V_i}$. Then $\tilde{\gamma}_i$ maps to $\tilde{\gamma}_{i,n(V)} \in H^m(V_i, K_{n(V)})$. Using that $H^m(K_{n(V)})$ is the sheafification of $U \mapsto H^m(U, K_{n(V)})$ (by Lemma 20.3 again) we see that after replacing $\{V_i \to V\}$ by a refinement we may assume that $\tilde{\gamma}_{i,n(V)} = 0$ for all i. For this covering we consider the short exact sequences

$$0 \to R^1 \lim H^{m-1}(V_i, K_n) \to H^m(V_i, R\lim K_n) \to \lim H^m(V_i, K_n) \to 0$$

of Lemma 23.2. By assumption (1) the group on the left is zero and by assumption (2) the group on the right maps injectively into $H^m(V_i, K_{n(V)})$. We conclude $\tilde{\gamma}_i = 0$ and hence $\gamma = 0$ as desired. □

Lemma 23.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Assume*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$, and*
(2) *for every $V \in \mathcal{B}$ there exist a function $p(V, -) : \mathbf{Z} \to \mathbf{Z}$ and a cofinal system Cov_V of coverings of V such that*
$$H^p(V_i, H^{m-p}(E)) = 0$$
for all $\{V_i \to V\} \in Cov_V$ and all integers p, m satisfying $p > p(V, m)$.

Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O})$.

Proof. Set $K_n = \tau_{\geq -n} E$ and $K = R\lim K_n$. The canonical map $E \to K$ comes from the canonical maps $E \to K_n = \tau_{\geq -n} E$. We have to show that $E \to K$ induces an isomorphism $H^m(E) \to H^m(K)$ of cohomology sheaves. In the rest of the proof we fix m. If $n \geq -m$, then the map $E \to \tau_{\geq -n}E = K_n$ induces an isomorphism $H^m(E) \to H^m(K_n)$. To finish the proof it suffices to show that for every $V \in \mathcal{B}$ there exists an integer $n(V) \geq -m$ such that the map $H^m(K)(V) \to H^m(K_{n(V)})(V)$ is injective. Namely, then the composition

$$H^m(E)(V) \to H^m(K)(V) \to H^m(K_{n(V)})(V)$$

is a bijection and the second arrow is injective, hence the first arrow is bijective. By property (1) this will imply $H^m(E) \to H^m(K)$ is an isomorphism. Set

$$n(V) = 1 + \max\{-m, p(V, m-1) - m, -1 + p(V, m) - m, -2 + p(V, m+1) - m\}.$$

so that in any case $n(V) \geq -m$. Claim: the maps

$$H^{m-1}(V_i, K_{n+1}) \to H^{m-1}(V_i, K_n) \quad \text{and} \quad H^m(V_i, K_{n+1}) \to H^m(V_i, K_n)$$

are isomorphisms for $n \geq n(V)$ and $\{V_i \to V\} \in \mathrm{Cov}_V$. The claim implies conditions (1) and (2) of Lemma 23.6 are satisfied and hence implies the desired injectivity. Recall (Derived Categories, Remark 12.4) that we have distinguished triangles

$$H^{-n-1}(E)[n+1] \to K_{n+1} \to K_n \to H^{-n-1}(E)[n+2]$$

Looking at the asssociated long exact cohomology sequence the claim follows if

$$H^{m+n}(V_i, H^{-n-1}(E)), \quad H^{m+n+1}(V_i, H^{-n-1}(E)), \quad H^{m+n+2}(V_i, H^{-n-1}(E))$$

are zero for $n \geq n(V)$ and $\{V_i \to V\} \in \mathrm{Cov}_V$. This follows from our choice of $n(V)$ and the assumption in the lemma. $\square$

Lemma 23.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

1. *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$, and*
2. *for every $V \in \mathcal{B}$ there exist an integer $d_V \geq 0$ and a cofinal system Cov_V of coverings of V such that*

$$H^p(V_i, H^q(E)) = 0 \text{ for } \{V_i \to V\} \in Cov_V,\ p > d_V,\ \text{and } q < 0$$

Then the canonical map $E \to R\lim \tau_{\geq -n}E$ is an isomorphism in $D(\mathcal{O})$.

Proof. This follows from Lemma 23.7 with $p(V, m) = d_V + \max(0, m)$. $\square$

Lemma 23.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Assume there exists a function $p(-) : \mathbf{Z} \to \mathbf{Z}$ and a subset $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ such that*

1. *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
2. *$H^p(V, H^{m-p}(E)) = 0$ for $p > p(m)$ and $V \in \mathcal{B}$.*

Then the canonical map $E \to R\lim \tau_{\geq -n}E$ is an isomorphism in $D(\mathcal{O})$.

Proof. Apply Lemma 23.7 with $p(V, m) = p(m)$ and Cov_V equal to the set of coverings $\{V_i \to V\}$ with $V_i \in \mathcal{B}$ for all i. $\square$

Lemma 23.10. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Assume there exists an integer $d \geq 0$ and a subset $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ such that*

1. *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
2. *$H^p(V, H^q(E)) = 0$ for $p > d$, $q < 0$, and $V \in \mathcal{B}$.*

Then the canonical map $E \to R\lim \tau_{\geq -n}E$ is an isomorphism in $D(\mathcal{O})$.

Proof. Apply Lemma 23.8 with $d_V = d$ and Cov_V equal to the set of coverings $\{V_i \to V\}$ with $V_i \in \mathcal{B}$ for all i. $\square$

The lemmas above can be used to compute cohomology in certain situations.

Lemma 23.11. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be an object of $D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

1. *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
2. *$H^p(U, H^q(K)) = 0$ for all $p > 0$, $q \in \mathbf{Z}$, and $U \in \mathcal{B}$.*

Then $H^q(U, K) = H^0(U, H^q(K))$ for $q \in \mathbf{Z}$ and $U \in \mathcal{B}$.

Proof. Observe that $K = R\lim \tau_{\geq -n} K$ by Lemma 23.10 with $d = 0$. Let $U \in \mathcal{B}$. By Equation (23.4.1) we get a short exact sequence

$$0 \to R^1 \lim H^{q-1}(U, \tau_{\geq -n} K) \to H^q(U, K) \to \lim H^q(U, \tau_{\geq -n} K) \to 0$$

Condition (2) implies $H^q(U, \tau_{\geq -n} K) = H^0(U, H^q(\tau_{\geq -n} K))$ for all q by using the spectral sequence of Derived Categories, Lemma 21.3. The spectral sequence converges because $\tau_{\geq -n} K$ is bounded below. If $n > -q$ then we have $H^q(\tau_{\geq -n} K) = H^q(K)$. Thus the systems on the left and the right of the displayed short exact sequence are eventually constant with values $H^0(U, H^{q-1}(K))$ and $H^0(U, H^q(K))$ and the lemma follows. $\square$

Here is another case where we can describe the derived limit.

Lemma 23.12. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

1. *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
2. *for all $U \in \mathcal{B}$ and all $q \in \mathbf{Z}$ we have*
   (a) *$H^p(U, H^q(K_n)) = 0$ for $p > 0$,*
   (b) *the inverse system $H^0(U, H^q(K_n))$ has vanishing $R^1 \lim$.*

Then $H^q(R\lim K_n) = \lim H^q(K_n)$ for $q \in \mathbf{Z}$.

Proof. Set $K = R\lim K_n$. We will use notation as in Remark 23.4. Let $U \in \mathcal{B}$. By Lemma 23.11 and (2)(a) we have $H^q(U, K_n) = H^0(U, H^q(K_n))$. Using that the functor $R\Gamma(U, -)$ commutes with derived limits we have

$$H^q(U, K) = H^q(R\lim R\Gamma(U, K_n)) = \lim H^0(U, H^q(K_n))$$

where the final equality follows from More on Algebra, Remark 86.10 and assumption (2)(b). Thus $H^q(U, K)$ is the inverse limit the sections of the sheaves $H^q(K_n)$ over U. Since $\lim H^q(K_n)$ is a sheaf we find using assumption (1) that $H^q(K)$, which is the sheafification of the presheaf $U \mapsto H^q(U, K)$, is equal to $\lim H^q(K_n)$. This proves the lemma. $\square$

24. Producing K-injective resolutions

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules. The category $\mathit{Mod}(\mathcal{O})$ has enough injectives, hence we can use Derived Categories, Lemma 29.3 produce a diagram

$$\begin{tikzcd} \ldots \arrow[r] & \tau_{\geq -2}\mathcal{F}^\bullet \arrow[r] \arrow[d] & \tau_{\geq -1}\mathcal{F}^\bullet \arrow[d] \\ \ldots \arrow[r] & \mathcal{I}_2^\bullet \arrow[r] & \mathcal{I}_1^\bullet \end{tikzcd}$$

in the category of complexes of $\mathcal{O}$-modules such that

1. the vertical arrows are quasi-isomorphisms,

(2) $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,
(3) the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 31.4 and 31.8 this is a K-injective complex. In general the canonical map

$$\mathcal{F}^\bullet \to \mathcal{I}^\bullet \tag{24.0.1}$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

Lemma 24.1. *In the situation described above. Denote $\mathcal{H}^m = H^m(\mathcal{F}^\bullet)$ the mth cohomology sheaf. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $d \in \mathbf{N}$. Assume*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$ we have $H^p(U, \mathcal{H}^q) = 0$ for $p > d$ and $q < 0$*[5].

Then (24.0.1) is a quasi-isomorphism.

Proof. By Derived Categories, Lemma 34.4 it suffices to show that the canonical map $\mathcal{F}^\bullet \to R\lim \tau_{\geq -n}\mathcal{F}^\bullet$ is an isomorphism. This follows from Lemma 23.10. $\square$

Here is a technical lemma about cohomology sheaves of termwise limits of inverse systems of complexes of modules. We should avoid using this lemma as much as possible and instead use arguments with derived inverse limits.

Lemma 24.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{F}_n^\bullet)$ be an inverse system of complexes of $\mathcal{O}$-modules. Let $m \in \mathbf{Z}$. Suppose given $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ and an integer n_0 such that*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$*
 (a) *the systems of abelian groups $\mathcal{F}_n^{m-2}(U)$ and $\mathcal{F}_n^{m-1}(U)$ have vanishing $R^1\lim$ (for example these have the Mittag-Leffler property),*
 (b) *the system of abelian groups $H^{m-1}(\mathcal{F}_n^\bullet(U))$ has vanishing $R^1\lim$ (for example it has the Mittag-Leffler property), and*
 (c) *we have $H^m(\mathcal{F}_n^\bullet(U)) = H^m(\mathcal{F}_{n_0}^\bullet(U))$ for all $n \geq n_0$.*

Then the maps $H^m(\mathcal{F}^\bullet) \to \lim H^m(\mathcal{F}_n^\bullet) \to H^m(\mathcal{F}_{n_0}^\bullet)$ are isomorphisms of sheaves where $\mathcal{F}^\bullet = \lim \mathcal{F}_n^\bullet$ is the termwise inverse limit.

Proof. Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{F}^\bullet(U))$ is the cohomology of

$$\lim_n \mathcal{F}_n^{m-2}(U) \to \lim_n \mathcal{F}_n^{m-1}(U) \to \lim_n \mathcal{F}_n^m(U) \to \lim_n \mathcal{F}_n^{m+1}(U)$$

in the third spot from the left. By assumptions (2)(a) and (2)(b) we may apply More on Algebra, Lemma 86.3 to conclude that

$$H^m(\mathcal{F}^\bullet(U)) = \lim H^m(\mathcal{F}_n^\bullet(U))$$

By assumption (2)(c) we conclude

$$H^m(\mathcal{F}^\bullet(U)) = H^m(\mathcal{F}_n^\bullet(U))$$

for all $n \geq n_0$. By assumption (1) we conclude that the sheafification of $U \mapsto H^m(\mathcal{F}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^m(\mathcal{F}_n^\bullet(U))$ for all $n \geq n_0$. Thus the inverse system of sheaves $H^m(\mathcal{F}_n^\bullet)$ is constant for $n \geq n_0$ with value $H^m(\mathcal{F}^\bullet)$ which proves the lemma. $\square$

[5]It suffices if $\forall m$, $\exists p(m)$, $H^p(U.\mathcal{H}^{m-p}) = 0$ for $p > p(m)$, see Lemma 23.9.

25. Bounded cohomological dimension

In this section we ask when a functor Rf_* has bounded cohomological dimension. This is a rather subtle question when we consider unbounded complexes.

Situation 25.1. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{A} \subset \textit{Mod}(\mathcal{O})$ be a weak Serre subcategory. We assume the following is true: there exists a subset $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ such that

(1) every object of $\mathcal{C}$ has a covering whose members are in $\mathcal{B}$, and
(2) for every $V \in \mathcal{B}$ there exists an integer d_V and a cofinal system Cov_V of coverings of V such that
$$H^p(V_i, \mathcal{F}) = 0 \text{ for } \{V_i \to V\} \in \text{Cov}_V,\ p > d_V,\ \text{and } \mathcal{F} \in \text{Ob}(\mathcal{A})$$

Lemma 25.2. *In Situation 25.1 for any $E \in D_\mathcal{A}(\mathcal{O})$ the canonical map $E \to R\lim \tau_{\geq -n}E$ is an isomorphism in $D(\mathcal{O})$.*

Proof. Follows immediately from Lemma 23.8. $\square$

Lemma 25.3. *In Situation 25.1 let (K_n) be an inverse system in $D^+_\mathcal{A}(\mathcal{O})$. Assume that for every j the inverse system $(H^j(K_n))$ in $\mathcal{A}$ is eventually constant with value $\mathcal{H}^j$. Then $H^j(R\lim K_n) = \mathcal{H}^j$ for all j.*

Proof. Let $V \in \mathcal{B}$. Let $\{V_i \to V\}$ be in the set Cov_V of Situation 25.1. Because K_n is bounded below there is a spectral sequence
$$E_2^{p,q} = H^p(V_i, H^q(K_n))$$
converging to $H^{p+q}(V_i, K_n)$. See Derived Categories, Lemma 21.3. Observe that $E_2^{p,q} = 0$ for $p > d_V$ by assumption. Pick n_0 such that
$$\begin{matrix} \mathcal{H}^{j+1} & = & H^{j+1}(K_n), \\ \mathcal{H}^j & = & H^j(K_n), \\ \ldots, & & \\ \mathcal{H}^{j-d_V-2} & = & H^{j-d_V-2}(K_n) \end{matrix}$$
for all $n \geq n_0$. Comparing the spectral sequences above for K_n and K_{n_0}, we see that for $n \geq n_0$ the cohomology groups $H^{j-1}(V_i, K_n)$ and $H^j(V_i, K_n)$ are independent of n. It follows that the map on sections $H^j(R\lim K_n)(V) \to H^j(K_n)(V)$ is injective for n large enough (depending on V), see Lemma 23.6. Since every object of $\mathcal{C}$ can be covered by elements of $\mathcal{B}$, we conclude that the map $H^j(R\lim K_n) \to \mathcal{H}^j$ is injective.

Surjectivity is shown in a similar manner. Namely, pick $U \in \text{Ob}(\mathcal{C})$ and $\gamma \in \mathcal{H}^j(U)$. We want to lift γ to a section of $H^j(R\lim K_n)$ after replacing U by the members of a covering. Hence we may assume $U = V \in \mathcal{B}$ by property (1) of Situation 25.1. Pick n_0 such that
$$\begin{matrix} \mathcal{H}^{j+1} & = & H^{j+1}(K_n), \\ \mathcal{H}^j & = & H^j(K_n), \\ \ldots, & & \\ \mathcal{H}^{j-d_V-2} & = & H^{j-d_V-2}(K_n) \end{matrix}$$
for all $n \geq n_0$. Choose an element $\{V_i \to V\}$ of Cov_V such that $\gamma|_{V_i} \in \mathcal{H}^j(V_i) = H^j(K_{n_0})(V_i)$ lifts to an element $\gamma_{n_0,i} \in H^j(V_i, K_{n_0})$. This is possible because $H^j(K_{n_0})$ is the sheafification of $U \mapsto H^j(U, K_{n_0})$ by Lemma 20.3. By the discussion in the first paragraph of the proof we have that $H^{j-1}(V_i, K_n)$ and $H^j(V_i, K_n)$ are

independent of $n \geq n_0$. Hence $\gamma_{n_0,i}$ lifts to an element $\gamma_i \in H^j(V_i, R\lim K_n)$ by Lemma 23.2. This finishes the proof. $\square$

Lemma 25.4. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{A} \subset Mod(\mathcal{O})$ and $\mathcal{A}' \subset Mod(\mathcal{O}')$ be weak Serre subcategories. Assume there is an integer N such that*

1. *$\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 25.1,*
2. *$\mathcal{C}', \mathcal{O}', \mathcal{A}'$ satisfy the assumption of Situation 25.1,*
3. *$R^pf_*\mathcal{F} \in \mathrm{Ob}(\mathcal{A}')$ for $p \geq 0$ and $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$,*
4. *$R^pf_*\mathcal{F} = 0$ for $p > N$ and $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$,*

Then for K in $D_\mathcal{A}(\mathcal{O})$ we have

(a) *Rf_*K is in $D_{\mathcal{A}'}(\mathcal{O}')$,*
(b) *the map $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is an isomorphism for $j \geq N - n$.*

Proof. By Lemma 25.2 we have $K = R\lim \tau_{\geq -n}K$. By Lemma 23.3 we have $Rf_*K = R\lim Rf_*\tau_{\geq -n}K$. The complexes $Rf_*\tau_{\geq -n}K$ are bounded below. The spectral sequence

$$E_2^{p,q} = R^pf_*H^q(\tau_{\geq -n}K)$$

converging to $H^{p+q}(Rf_*\tau_{\geq -n}K)$ (Derived Categories, Lemma 21.3) and assumption (3) show that $Rf_*\tau_{\geq -n}K$ lies in $D^+_{\mathcal{A}'}(\mathcal{O}')$, see Homology, Lemma 24.11. Observe that for $m \geq n$ the map

$$Rf_*(\tau_{\geq -m}K) \longrightarrow Rf_*(\tau_{\geq -n}K)$$

induces an isomorphism on cohomology sheaves in degrees $j \geq -n + N$ by the spectral sequences above. Hence we may apply Lemma 25.3 to conclude. $\square$

It turns out that we sometimes need a variant of the lemma above where the assumptions are sligthly different.

Situation 25.5. Let $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ be a morphism of ringed sites. Let $u : \mathcal{C}' \to \mathcal{C}$ be the corresponding continuous functor of sites. Let $\mathcal{A} \subset Mod(\mathcal{O})$ be a weak Serre subcategory. We assume the following is true: there exists a subset $\mathcal{B}' \subset \mathrm{Ob}(\mathcal{C}')$ such that

1. every object of $\mathcal{C}'$ has a covering whose members are in $\mathcal{B}'$, and
2. for every $V' \in \mathcal{B}'$ there exists an integer $d_{V'}$ and a cofinal system $\mathrm{Cov}_{V'}$ of coverings of V' such that
$$H^p(u(V'_i), \mathcal{F}) = 0 \text{ for } \{V'_i \to V'\} \in \mathrm{Cov}_{V'},\ p > d_{V'},\ \text{and } \mathcal{F} \in \mathrm{Ob}(\mathcal{A})$$

Lemma 25.6. *Let $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ be a morphism of ringed sites. assume moreover there is an integer N such that*

1. *$\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 25.1,*
2. *$f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ and $\mathcal{A}$ satisfy the assumption of Situation 25.5,*
3. *$R^pf_*\mathcal{F} = 0$ for $p > N$ and $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$,*

Then for K in $D_\mathcal{A}(\mathcal{O})$ the map $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is an isomorphism for $j \geq N - n$.

Proof. Let K be in $D_\mathcal{A}(\mathcal{O})$. By Lemma 25.2 we have $K = R\lim \tau_{\geq -n}K$. By Lemma 23.3 we have $Rf_*K = R\lim Rf_*(\tau_{\geq -n}K)$. Let $V' \in \mathcal{B}'$ and let $\{V'_i \to V'\}$ be an element of $\mathrm{Cov}_{V'}$. Then we consider

$$H^j(V'_i, Rf_*K) = H^j(u(V'_i), K) \quad \text{and} \quad H^j(V'_i, Rf_*(\tau_{\geq -n}K)) = H^j(u(V'_i), \tau_{\geq -n}K)$$

The assumption in Situation 25.5 implies that the last group is independent of n for n large enough depending on j and $d_{V'}$. Some details omitted. We apply this for j and $j-1$ and via Lemma 23.2 this gives that

$$H^j(V_i', Rf_*K) = \lim H^j(V_i', Rf_*(\tau_{\geq -n}K))$$

and the system on the right is constant for n larger than a constant depending only on $d_{V'}$ and j. Thus Lemma 23.6 implies that

$$H^j(Rf_*K)(V') \longrightarrow \left(\lim H^j(Rf_*(\tau_{\geq -n}K))\right)(V')$$

is injective. Since the elements $V' \in \mathcal{B}'$ cover every object of $\mathcal{C}'$ we conclude that the map $H^j(Rf_*K) \to \lim H^j(Rf_*(\tau_{\geq -n}K))$ is injective. The spectral sequence

$$E_2^{p,q} = R^pf_*H^q(\tau_{\geq -n}K)$$

converging to $H^{p+q}(Rf_*(\tau_{\geq -n}K))$ (Derived Categories, Lemma 21.3) and assumption (3) show that $H^j(Rf_*(\tau_{\geq -n}K))$ is constant for $n \geq N-j$. Hence $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is injective for $j \geq N-n$.

Thus we proved the lemma with "isomorphism" in the last line of the lemma replaced by "injective". However, now choose j and n with $j \geq N-n$. Then consider the distinguished triangle

$$\tau_{\leq -n-1}K \to K \to \tau_{\geq -n}K \to (\tau_{\leq -n-1}K)[1]$$

See Derived Categories, Remark 12.4. Since $\tau_{\geq -n}\tau_{\leq -n-1}K = 0$, the injectivity already proven for $\tau_{-n-1}K$ implies

$$0 = H^j(Rf_*(\tau_{\leq -n-1}K)) = H^{j+1}(Rf_*(\tau_{\leq -n-1}K)) = H^{j+2}(Rf_*(\tau_{\leq -n-1}K)) = \ldots$$

By the long exact cohomology sequence associated to the distinguished triangle

$$Rf_*(\tau_{\leq -n-1}K) \to Rf_*K \to Rf_*(\tau_{\geq -n}K) \to Rf_*(\tau_{\leq -n-1}K)[1]$$

this implies that $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is an isomorphism. □

26. Mayer-Vietoris

For the usual statement and proof of Mayer-Vietoris, please see Cohomology, Section 8.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider a commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

in the category $\mathcal{C}$. In this situation, given an object K of $D(\mathcal{O})$ we get what looks like the beginning of a distinguished triangle

$$R\Gamma(X, K) \to R\Gamma(Z, K) \oplus R\Gamma(Y, K) \to R\Gamma(E, K)$$

In the following lemma we make this more precise.

Lemma 26.1. *In the situation above, choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}$-modules representing K. Using -1 times the canonical map for one of the four arrows we get maps of complexes*

$$\mathcal{I}^\bullet(X) \xrightarrow{\alpha} \mathcal{I}^\bullet(Z) \oplus \mathcal{I}^\bullet(Y) \xrightarrow{\beta} \mathcal{I}^\bullet(E)$$

with $\beta \circ \alpha = 0$. Thus a canonical map

$$c^K_{X,Z,Y,E} : \mathcal{I}^\bullet(X) \longrightarrow C(\beta)^\bullet[-1]$$

This map is canonical in the sense that a different choice of K-injective complex representing K determines an isomorphic arrow in the derived category of abelian groups. If $c^K_{X,Z,Y,E}$ is an isomorphism, then using its inverse we obtain a canonical distinguished triangle

$$R\Gamma(X, K) \to R\Gamma(Z, K) \oplus R\Gamma(Y, K) \to R\Gamma(E, K) \to R\Gamma(X, K)[1]$$

All of these constructions are functorial in K.

Proof. This lemma proves itself. For example, if $\mathcal{J}^\bullet$ is a second K-injective complex representing K, then we can choose a quasi-isomorphism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ which determines quasi-isomorphisms between all the complexes in sight. Details omitted. For the construction of cones and the relationship with distinguished triangles see Derived Categories, Sections 9 and 10. $\square$

Lemma 26.2. *In the situation above, let $K_1 \to K_2 \to K_3 \to K_1[1]$ be a distinguished triangle in $D(\mathcal{O})$. If $c^{K_i}_{X,Z,Y,E}$ is a quasi-isomorphism for two i out of $\{1, 2, 3\}$, then it is a quasi-isomorphism for the third i.*

Proof. By rotating the triangle we may assume $c^{K_1}_{X,Z,Y,E}$ and $c^{K_2}_{X,Z,Y,E}$ are quasi-isomorphisms. Choose a map $f : \mathcal{I}^\bullet_1 \to \mathcal{I}^\bullet_2$ of K-injective complexes of $\mathcal{O}$-modules representing $K_1 \to K_2$. Then K_3 is represented by the K-injective complex $C(f)^\bullet$, see Derived Categories, Lemma 31.3. Then the morphism $c^{K_3}_{X,Z,Y,E}$ is an isomorphism as it is the third leg in a map of distinguished triangles in $K(Ab)$ whose other two legs are quasi-isomorphisms. Some details omitted; use Derived Categories, Lemma 4.3. $\square$

Let us give a criterion for when this does produce a distinguished triangle.

Lemma 26.3. *In the situation above assume*

1. $h^\#_X = h^\#_Y \amalg_{h^\#_E} h^\#_Z$, *and*
2. $h^\#_E \to h^\#_Y$ *is injective.*

Then the construction of Lemma 26.1 produces a distinguished triangle

$$R\Gamma(X, K) \to R\Gamma(Z, K) \oplus R\Gamma(Y, K) \to R\Gamma(E, K) \to R\Gamma(X, K)[1]$$

functorial for K in $D(\mathcal{C})$.

Proof. We can represent K by a K-injective complex whose terms are injective abelian sheaves, see Section 19. Thus it suffices to show: if $\mathcal{I}$ is an injective abelian sheaf, then

$$0 \to \mathcal{I}(X) \to \mathcal{I}(Z) \oplus \mathcal{I}(Y) \to \mathcal{I}(E) \to 0$$

is a short exact sequence. The first arrow is injective because by condition (1) the map $h_Y \amalg h_Z \to h_X$ becomes surjective after sheafification, which means that $\{Y \to X, Z \to X\}$ can be refined by a covering of X. The last arrow is surjective because $\mathcal{I}(Y) \to \mathcal{I}(E)$ is surjective. Namely, we have $\mathcal{I}(E) = \mathrm{Hom}(\mathbf{Z}^\#_E, \mathcal{I})$, $\mathcal{I}(Y) = \mathrm{Hom}(\mathbf{Z}^\#_Y, \mathcal{I})$, the map $\mathbf{Z}^\#_E \to \mathbf{Z}^\#_Y$ is injective by (2), and $\mathcal{I}$ is an injective abelian sheaf. Please compare with Modules on Sites, Section 5. Finally, suppose we have

$s \in \mathcal{I}(Y)$ and $t \in \mathcal{F}(Z)$ mapping to the same element of $\mathcal{I}(E)$. Then s and t define a map

$$s \amalg t : h_Y^\# \amalg h_Z^\# \longrightarrow \mathcal{I}$$

which by assumption factors through $h_Y^\# \amalg_{h_E^\#} h_Z^\#$. Thus by assumption (1) we obtain a unique map $h_X^\# \to \mathcal{I}$ which corresponds to an element of $\mathcal{I}(X)$ restricting to s on Y and t on Z. $\square$

Lemma 26.4. *Let $\mathcal{C}$ be a site. Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{E} \arrow[r] & \mathcal{G} \end{tikzcd}$$

of presheaves of sets on $\mathcal{C}$ and assume that

(1) $\mathcal{G}^\# = \mathcal{E}^\# \amalg_{\mathcal{D}^\#} \mathcal{F}^\#$, *and*
(2) $\mathcal{D}^\# \to \mathcal{F}^\#$ *is injective.*

Then there is a canonical distinguished triangle

$$R\Gamma(\mathcal{G}, K) \to R\Gamma(\mathcal{E}, K) \oplus R\Gamma(\mathcal{F}, K) \to R\Gamma(\mathcal{D}, K) \to R\Gamma(\mathcal{G}, K)[1]$$

functorial in $K \in D(\mathcal{C})$ where $R\Gamma(\mathcal{G}, -)$ is the cohomology discussed in Section 13.

Proof. Since sheafification is exact and since $R\Gamma(\mathcal{G}, -) = R\Gamma(\mathcal{G}^\#, -)$ we may assume $\mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}$ are sheaves of sets. Moreover, the cohomology $R\Gamma(\mathcal{G}, -)$ only depends on the topos, not on the underlying site. Hence by Sites, Lemma 29.5 we may replace $\mathcal{C}$ by a "larger" site with a subcanonical topology such that $\mathcal{G} = h_X$, $\mathcal{F} = h_Y$, $\mathcal{E} = h_Z$, and $\mathcal{D} = h_E$ for some objects X, Y, Z, E of $\mathcal{C}$. In this case the result follows from Lemma 26.3. $\square$

27. Comparing two topologies

Let $\mathcal{C}$ be a category. Let $\text{Cov}(\mathcal{C}) \supset \text{Cov}'(\mathcal{C})$ be two ways to endow $\mathcal{C}$ with the structure of a site. Denote τ the topology corresponding to $\text{Cov}(\mathcal{C})$ and τ' the topology corresponding to $\text{Cov}'(\mathcal{C})$. Then the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

where ϵ_* is the identity functor on underlying presheaves and where ϵ^{-1} is the τ-sheafification of a τ'-sheaf. See Sites, Examples 14.3 and 22.3. In the situation above we have the following

(1) $\epsilon_* : Sh(\mathcal{C}_\tau) \to Sh(\mathcal{C}_{\tau'})$ is fully faithful and $\epsilon^{-1} \circ \epsilon_* = \text{id}$,
(2) $\epsilon_* : Ab(\mathcal{C}_\tau) \to Ab(\mathcal{C}_{\tau'})$ is fully faithful and $\epsilon^{-1} \circ \epsilon_* = \text{id}$,
(3) $R\epsilon_* : D(\mathcal{C}_\tau) \to D(\mathcal{C}_{\tau'})$ is fully faithful and $\epsilon^{-1} \circ R\epsilon_* = \text{id}$,
(4) if $\mathcal{O}$ is a sheaf of rings for the τ-topology, then $\mathcal{O}$ is also a sheaf for the τ'-topology and ϵ becomes a flat morphism of ringed sites

$$\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$$

(5) $\epsilon_* : Mod(\mathcal{O}_\tau) \to Mod(\mathcal{O}_{\tau'})$ is fully faithful and $\epsilon^* \circ \epsilon_* = \text{id}$
(6) $R\epsilon_* : D(\mathcal{O}_\tau) \to D(\mathcal{O}_{\tau'})$ is fully faithful and $\epsilon^* \circ R\epsilon_* = \text{id}$.

Here are some explanations.

Ad (1). Let $\mathcal{F}$ be a sheaf of sets in the τ-topology. Then $\epsilon_*\mathcal{F}$ is just $\mathcal{F}$ viewed as a sheaf in the τ'-topology. Applying ϵ^{-1} means taking the τ-sheafification of $\mathcal{F}$, which doesn't do anything as $\mathcal{F}$ is already a τ-sheaf. Thus $\epsilon^{-1}(\epsilon_*\mathcal{F})) = \mathcal{F}$. The fully faithfulness follows by Categories, Lemma 24.4.

Ad (2). This is a consequence of (1) since pullback and pushforward of abelian sheaves is the same as doing those operations on the underlying sheaves of sets.

Ad (3). Let K be an object of $D(\mathcal{C}_\tau)$. To compute $R\epsilon_*K$ we choose a K-injective complex $\mathcal{I}^\bullet$ representing K and we set $R\epsilon_*K = \epsilon_*\mathcal{I}^\bullet$. Since $\epsilon^{-1} : D(\mathcal{C}_{\tau'}) \to D(\mathcal{C}_\tau)$ is computed on an object L by applying the exact functor ϵ^{-1} to any complex of abelian sheaves representing L, we find that $\epsilon^{-1}R\epsilon_*K$ is represented by $\epsilon^{-1}\epsilon_*\mathcal{I}^\bullet$. By Part (1) we have $\mathcal{I}^\bullet = \epsilon^{-1}\epsilon_*\mathcal{I}^\bullet$. In other words, we have $\epsilon^{-1} \circ R\epsilon_* = \text{id}$ and we conclude as before.

Ad (4). Observe that $\epsilon^{-1}\mathcal{O}_{\tau'} = \mathcal{O}_\tau$, see discussion in part (1). Hence ϵ is a flat morphism of ringed sites, see Modules on Sites, Definition 31.1. Not only that, it is moreover clear that $\epsilon^* = \epsilon^{-1}$ on $\mathcal{O}_{\tau'}$-modules (the pullback as a module has the same underlying abelian sheaf as the pullback of the underlying abelian sheaf).

Ad (5). This is clear from (2) and what we said in (4).

Ad (6). This is analogous to (3). We omit the details.

28. Formalities on cohomological descent

In this section we discuss only to what extent a morphism of ringed topoi determines an embedding from the derived category downstairs to the derived category upstairs. Here is a typical result.

Lemma 28.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Consider the full subcategory $D' \subset D(\mathcal{O}_\mathcal{D})$ consisting of objects K such that*

$$K \longrightarrow Rf_*Lf^*K$$

is an isomorphism. Then D' is a saturated triangulated strictly full subcategory of $D(\mathcal{O}_\mathcal{D})$ and the functor $Lf^* : D' \to D(\mathcal{O}_\mathcal{C})$ is fully faithful.

Proof. See Derived Categories, Definition 6.1 for the definition of saturated in this setting. See Derived Categories, Lemma 4.16 for a discussion of triangulated subcategories. The canonical map of the lemma is the unit of the adjoint pair of functors (Lf^*, Rf_*), see Lemma 19.1. Having said this the proof that D' is a saturated triangulated subcategory is omitted; it follows formally from the fact that Lf^* and Rf_* are exact functors of triangulated categories. The final part follows formally from fact that Lf^* and Rf_* are adjoint; compare with Categories, Lemma 24.4. $\square$

Lemma 28.2. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Consider the full subcategory $D' \subset D(\mathcal{O}_\mathcal{C})$ consisting of objects K such that*

$$Lf^*Rf_*K \longrightarrow K$$

is an isomorphism. Then D' is a saturated triangulated strictly full subcategory of $D(\mathcal{O}_\mathcal{C})$ and the functor $Rf_* : D' \to D(\mathcal{O}_\mathcal{D})$ is fully faithful.

Proof. See Derived Categories, Definition 6.1 for the definition of saturated in this setting. See Derived Categories, Lemma 4.16 for a discussion of triangulated subcategories. The canonical map of the lemma is the counit of the adjoint pair of functors (Lf^*, Rf_*), see Lemma 19.1. Having said this the proof that D' is a saturated triangulated subcategory is omitted; it follows formally from the fact that Lf^* and Rf_* are exact functors of triangulated categories. The final part follows formally from fact that Lf^* and Rf_* are adjoint; compare with Categories, Lemma 24.4. $\square$

Lemma 28.3. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let K be an object of $D(\mathcal{O}_\mathcal{C})$. Assume*

1. *f is flat,*
2. *K is bounded below,*
3. *$f^*Rf_*H^q(K) \to H^q(K)$ is an isomorphism.*

Then $f^*Rf_*K \to K$ is an isomorphism.

Proof. Observe that $f^*Rf_*K \to K$ is an isomorphism if and only if it is an isomorphism on cohomology sheaves H^j. Observe that $H^j(f^*Rf_*K) = f^*H^j(Rf_*K) = f^*H^j(Rf_*\tau_{\leq j}K) = H^j(f^*Rf_*\tau_{\leq j}K)$. Hence we may assume that K is bounded. Then property (3) tells us the cohomology sheaves are in the triangulated subcategory $D' \subset D(\mathcal{O}_\mathcal{C})$ of Lemma 28.2. Hence K is in it too. $\square$

Lemma 28.4. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let K be an object of $D(\mathcal{O}_\mathcal{D})$. Assume*

1. *f is flat,*
2. *K is bounded below,*
3. *$H^q(K) \to Rf_*f^*H^q(K)$ is an isomorphism.*

Then $K \to Rf_*f^*K$ is an isomorphism.

Proof. Observe that $K \to Rf_*f^*K$ is an isomorphism if and only if it is an isomorphism on cohomology sheaves H^j. Observe that $H^j(Rf_*f^*K) = H^j(Rf_*\tau_{\leq j}f^*K) = H^j(Rf_*f^*\tau_{\leq j}K)$. Hence we may assume that K is bounded. Then property (3) tells us the cohomology sheaves are in the triangulated subcategory $D' \subset D(\mathcal{O}_\mathcal{D})$ of Lemma 28.1. Hence K is in it too. $\square$

Lemma 28.5. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{A} \subset Mod(\mathcal{O})$ and $\mathcal{A}' \subset Mod(\mathcal{O}')$ be weak Serre subcategories. Assume*

1. *f is flat,*
2. *f^* induces an equivalence of categories $\mathcal{A}' \to \mathcal{A}$,*
3. *$\mathcal{F}' \to Rf_*f^*\mathcal{F}'$ is an isomorphism for $\mathcal{F}' \in \mathrm{Ob}(\mathcal{A}')$.*

Then $f^* : D^+_{\mathcal{A}'}(\mathcal{O}') \to D^+_\mathcal{A}(\mathcal{O})$ is an equivalence of categories with quasi-inverse given by $Rf_* : D^+_\mathcal{A}(\mathcal{O}) \to D^+_{\mathcal{A}'}(\mathcal{O}')$.

Proof. By assumptions (2) and (3) and Lemmas 28.4 and 28.1 we see that $f^* : D^+_{\mathcal{A}'}(\mathcal{O}') \to D^+_\mathcal{A}(\mathcal{O})$ is fully faithful. Let $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$. Then we can write $\mathcal{F} = f^*\mathcal{F}'$. Then $Rf_*\mathcal{F} = Rf_*f^*\mathcal{F}' = \mathcal{F}'$. In particular, we have $R^pf_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F} \in \mathrm{Ob}(\mathcal{A}')$. Thus for any $K \in D^+_\mathcal{A}(\mathcal{O})$ we see, using the spectral sequence $E_2^{p,q} = R^pf_*H^q(K)$ converging to $R^{p+q}f_*K$, that Rf_*K is in $D^+_{\mathcal{A}'}(\mathcal{O}')$. Of course, it also follows from Lemmas 28.3 and 28.2 that $Rf_* : D^+_\mathcal{A}(\mathcal{O}) \to D^+_{\mathcal{A}'}(\mathcal{O}')$ is fully faithful. Since f^* and Rf_* are adjoint we then get the result of the lemma, for example by Categories, Lemma 24.4. $\square$

Lemma 28.6. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{A} \subset Mod(\mathcal{O})$ and $\mathcal{A}' \subset Mod(\mathcal{O}')$ be weak Serre subcategories. Assume*

1. *f is flat,*
2. *f^* induces an equivalence of categories $\mathcal{A}' \to \mathcal{A}$,*
3. *$\mathcal{F}' \to Rf_*f^*\mathcal{F}'$ is an isomorphism for $\mathcal{F}' \in \mathrm{Ob}(\mathcal{A}')$,*
4. *$\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 25.1,*
5. *$\mathcal{C}', \mathcal{O}', \mathcal{A}'$ satisfy the assumption of Situation 25.1.*

Then $f^* : D_{\mathcal{A}'}(\mathcal{O}') \to D_{\mathcal{A}}(\mathcal{O})$ is an equivalence of categories with quasi-inverse given by $Rf_* : D_{\mathcal{A}}(\mathcal{O}) \to D_{\mathcal{A}'}(\mathcal{O}')$.

Proof. Since f^* is exact, it is clear that f^* defines a functor $f^* : D_{\mathcal{A}'}(\mathcal{O}') \to D_{\mathcal{A}}(\mathcal{O})$ as in the statement of the lemma and that moreover this functor commutes with the truncation functors $\tau_{\geq -n}$. We already know that f^* and Rf_* are quasi-inverse equivalence on the corresponding bounded below categories, see Lemma 28.5. By Lemma 25.4 with $N = 0$ we see that Rf_* indeed defines a functor $Rf_* : D_{\mathcal{A}}(\mathcal{O}) \to D_{\mathcal{A}'}(\mathcal{O}')$ and that moreover this functor commutes with the truncation functors $\tau_{\geq -n}$. Thus for K in $D_{\mathcal{A}}(\mathcal{O})$ the map $f^*Rf_*K \to K$ is an isomorphism as this is true on trunctions. Similarly, for K' in $D_{\mathcal{A}'}(\mathcal{O}')$ the map $K' \to Rf_*f^*K'$ is an isomorphism as this is true on trunctions. This finishes the proof. $\square$

Lemma 28.7. *Let $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ be a morphism of ringed sites. Let $\mathcal{A} \subset Mod(\mathcal{O})$ and $\mathcal{A}' \subset Mod(\mathcal{O}')$ be weak Serre subcategories. Assume*

1. *f is flat,*
2. *f^* induces an equivalence of categories $\mathcal{A}' \to \mathcal{A}$,*
3. *$\mathcal{F}' \to Rf_*f^*\mathcal{F}'$ is an isomorphism for $\mathcal{F}' \in \mathrm{Ob}(\mathcal{A}')$,*
4. *$\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 25.1,*
5. *$f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ and $\mathcal{A}$ satisfy the assumption of Situation 25.5.*

Then $f^* : D_{\mathcal{A}'}(\mathcal{O}') \to D_{\mathcal{A}}(\mathcal{O})$ is an equivalence of categories with quasi-inverse given by $Rf_* : D_{\mathcal{A}}(\mathcal{O}) \to D_{\mathcal{A}'}(\mathcal{O}')$.

Proof. The proof of this lemma is exactly the same as the proof of Lemma 28.6 except the reference to Lemma 25.4 is replaced by a reference to Lemma 25.6. $\square$

29. Comparing two topologies, II

Let $\mathcal{C}$ be a category. Let $\mathrm{Cov}(\mathcal{C}) \supset \mathrm{Cov}'(\mathcal{C})$ be two ways to endow $\mathcal{C}$ with the structure of a site. Denote τ the topology corresponding to $\mathrm{Cov}(\mathcal{C})$ and τ' the topology corresponding to $\mathrm{Cov}'(\mathcal{C})$. Then the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

where ϵ_* is the identity functor on underlying presheaves and where ϵ^{-1} is the τ-sheafification of a τ'-sheaf (hence clearly exact). Let $\mathcal{O}$ be a sheaf of rings for the τ-topology. Then $\mathcal{O}$ is also a sheaf for the τ'-topology and ϵ becomes a morphism of ringed sites

$$\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$$

For more discussion, see Section 27.

Lemma 29.1. *With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $\mathcal{A} \subset PMod(\mathcal{O})$ be a full subcategory. Assume*

1. *every object of $\mathcal{A}$ is a sheaf for the τ-topology,*

(2) *$\mathcal{A}$ is a weak Serre subcategory of Mod($\mathcal{O}_\tau$),*
(3) *every object of $\mathcal{C}$ has a τ'-covering whose members are elements of $\mathcal{B}$, and*
(4) *for every $U \in \mathcal{B}$ we have $H^p_\tau(U, \mathcal{F}) = 0$, $p > 0$ for all $\mathcal{F} \in \mathcal{A}$.*

Then $\mathcal{A}$ is a weak Serre subcategory of Mod($\mathcal{O}_{\tau'}$) and there is an equivalence of triangulated categories $D_\mathcal{A}(\mathcal{O}_\tau) = D_\mathcal{A}(\mathcal{O}_{\tau'})$ given by ϵ^* and $R\epsilon_*$.

Proof. Since $\epsilon^{-1}\mathcal{O}_{\tau'} = \mathcal{O}_\tau$ we see that ϵ is a flat morphism of ringed sites and that in fact $\epsilon^{-1} = \epsilon^*$ on sheaves of modules. By property (1) we can think of every object of $\mathcal{A}$ as a sheaf of $\mathcal{O}_\tau$-modules and as a sheaf of $\mathcal{O}_{\tau'}$-modules. In other words, we have fully faithful inclusion functors

$$\mathcal{A} \to \mathit{Mod}(\mathcal{O}_\tau) \to \mathit{Mod}(\mathcal{O}_{\tau'})$$

To avoid confusion we will denote $\mathcal{A}' \subset \mathit{Mod}(\mathcal{O}_{\tau'})$ the image of $\mathcal{A}$. Then it is clear that $\epsilon_* : \mathcal{A} \to \mathcal{A}'$ and $\epsilon^* : \mathcal{A}' \to \mathcal{A}$ are quasi-inverse equivalences (see discussion preceding the lemma and use that objects of $\mathcal{A}'$ are sheaves in the τ topology).

Conditions (3) and (4) imply that $R^p\epsilon_*\mathcal{F} = 0$ for $p > 0$ and $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$. This is true because $R^p\epsilon_*$ is the sheaf associated to the presheave $U \mapsto H^p_\tau(U, \mathcal{F})$, see Lemma 7.4. Thus any exact complex in $\mathcal{A}$ (which is the same thing as an exact complex in $\mathit{Mod}(\mathcal{O}_\tau)$ whose terms are in $\mathcal{A}$, see Homology, Lemma 10.3) remains exact upon applying the functor ϵ_*.

Consider an exact sequence

$$\mathcal{F}'_0 \to \mathcal{F}'_1 \to \mathcal{F}'_2 \to \mathcal{F}'_3 \to \mathcal{F}'_4$$

in $\mathit{Mod}(\mathcal{O}_{\tau'})$ with $\mathcal{F}'_0, \mathcal{F}'_1, \mathcal{F}'_3, \mathcal{F}'_4$ in $\mathcal{A}'$. Apply the exact functor ϵ^* to get an exact sequence

$$\epsilon^*\mathcal{F}'_0 \to \epsilon^*\mathcal{F}'_1 \to \epsilon^*\mathcal{F}'_2 \to \epsilon^*\mathcal{F}'_3 \to \epsilon^*\mathcal{F}'_4$$

in $\mathit{Mod}(\mathcal{O}_\tau)$. Since $\mathcal{A}$ is a weak Serre subcategory and since $\epsilon^*\mathcal{F}'_0, \epsilon^*\mathcal{F}'_1, \epsilon^*\mathcal{F}'_3, \epsilon^*\mathcal{F}'_4$ are in $\mathcal{A}$, we conclude that $\epsilon^*\mathcal{F}_2$ is in $\mathcal{A}$ by Homology, Definition 10.1. Consider the map of sequences

$$\begin{tikzcd}
\mathcal{F}'_0 \arrow[r] \arrow[d] & \mathcal{F}'_1 \arrow[r] \arrow[d] & \mathcal{F}'_2 \arrow[r] \arrow[d] & \mathcal{F}'_3 \arrow[r] \arrow[d] & \mathcal{F}'_4 \arrow[d] \\
\epsilon_*\epsilon^*\mathcal{F}'_0 \arrow[r] & \epsilon_*\epsilon^*\mathcal{F}'_1 \arrow[r] & \epsilon_*\epsilon^*\mathcal{F}'_2 \arrow[r] & \epsilon_*\epsilon^*\mathcal{F}'_3 \arrow[r] & \epsilon_*\epsilon^*\mathcal{F}'_4
\end{tikzcd}$$

The lower row is exact by the discussion in the preceding paragraph. The vertical arrows with index 0, 1, 3, 4 are isomorphisms by the discussion in the first paragraph. By the 5 lemma (Homology, Lemma 5.20) we find that $\mathcal{F}'_2 \cong \epsilon_*\epsilon^*\mathcal{F}'_2$ and hence $\mathcal{F}'_2$ is in $\mathcal{A}'$. In this way we see that $\mathcal{A}'$ is a weak Serre subcategory of $\mathit{Mod}(\mathcal{O}_{\tau'})$, see Homology, Definition 10.1.

At this point it makes sense to talk about the derived categories $D_\mathcal{A}(\mathcal{O}_\tau)$ and $D_{\mathcal{A}'}(\mathcal{O}_{\tau'})$, see Derived Categories, Section 17. To finish the proof we show that conditions (1) – (5) of Lemma 28.7 apply. We have already seen (1), (2), (3) above. Note that since every object has a τ'-covering by objects of $\mathcal{B}$, a fortiori every object has a τ-covering by objects of $\mathcal{B}$. Hence condition (4) of Lemma 28.7 is satisfied. Similarly, condition (5) is satisfied as well. □

Lemma 29.2. *With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let A be a set and for $\alpha \in A$ let*

$$\begin{tikzcd} E_\alpha \arrow[r] \arrow[d] & Y_\alpha \arrow[d] \\ Z_\alpha \arrow[r] & X_\alpha \end{tikzcd}$$

be a commutative diagram in the category $\mathcal{C}$. Assume that

(1) *a τ'-sheaf $\mathcal{F}'$ is a τ-sheaf if $\mathcal{F}'(X_\alpha) = \mathcal{F}'(Z_\alpha) \times_{\mathcal{F}'(E_\alpha)} \mathcal{F}'(Y_\alpha)$ for all α,*

(2) *for K' in $D(\mathcal{O}_{\tau'})$ in the essential image of $R\epsilon_*$ the maps $c^{K'}_{X_\alpha, Z_\alpha, Y_\alpha, E_\alpha}$ of Lemma 26.1 are isomorphisms for all α.*

Then $K' \in D^+(\mathcal{O}_{\tau'})$ is in the essential image of $R\epsilon_*$ if and only if the maps $c^{K'}_{X_\alpha, Z_\alpha, Y_\alpha, E_\alpha}$ are isomorphisms for all α.

Proof. The "only if" direction is implied by assumption (2). On the other hand, if K' has a unique nonzero cohomology sheaf, then the "if" direction follows from assumption (1). In general we will use an induction argument to prove the "if" direction. Let us say an object K' of $D^+(\mathcal{O}_{\tau'})$ satisfies (P) if the maps $c^{K'}_{X_\alpha, Z_\alpha, Y_\alpha, E_\alpha}$ are isomorphisms for all $\alpha \in A$.

Namely, let K' be an object of $D^+(\mathcal{O}_{\tau'})$ satisfying (P). Choose a distinguished triangle

$$K' \to R\epsilon_*\epsilon^{-1}K' \to M' \to K'[1]$$

in $D^+(\mathcal{O}_{\tau'})$ where the first arrow is the adjuntion map. By (2) and Lemma 26.2 we see that M' has (P). On the other hand, applying ϵ^{-1} and using that $\epsilon^{-1}R\epsilon_* = \text{id}$ by Section 27 we find that $\epsilon^{-1}M' = 0$. In the next paragraph we will show $M' = 0$ which finishes the proof.

Let K' be an object of $D^+(\mathcal{O}_{\tau'})$ satisfying (P) with $\epsilon^{-1}K' = 0$. We will show $K' = 0$. Namely, given $n \in \mathbf{Z}$ such that $H^i(K') = 0$ for $i < n$ we will show that $H^n(K') = 0$. For $\alpha \in A$ we have a distinguished triangle

$$R\Gamma_{\tau'}(X_\alpha, K') \to R\Gamma_{\tau'}(Z_\alpha, K') \oplus R\Gamma_{\tau'}(Y_\alpha, K') \to R\Gamma_{\tau'}(E_\alpha, K') \to R\Gamma_{\tau'}(X_\alpha, K')[1]$$

by Lemma 26.1. Taking cohomology in degree n and using the assumed vanishing of cohomology sheaves of K' we obtain an exact sequence

$$0 \to H^n_{\tau'}(X_\alpha, K') \to H^n_{\tau'}(Z_\alpha, K') \oplus H^n_{\tau'}(Y_\alpha, K') \to H^n_{\tau'}(E_\alpha, K')$$

which is the same as the exact sequence

$$0 \to \Gamma(X_\alpha, H^n(K')) \to \Gamma(Z_\alpha, H^n(K')) \oplus \Gamma(Y_\alpha, H^n(K')) \to \Gamma(E_\alpha, H^n(K'))$$

We conclude that $H^n(K')$ is a a τ-sheaf by assumption (1). However, since the τ-sheafification $\epsilon^{-1}H^n(K') = H^n(\epsilon^{-1}K')$ is 0 as $\epsilon^{-1}K' = 0$ we conclude that $H^n(K') = 0$ as desired. $\square$

Lemma 29.3. *With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let*

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

be a commutative diagram in the category $\mathcal{C}$ such that

(1) $h_X^\# = h_Y^\# \amalg_{h_E^\#} h_Z^\#$, *and*
(2) $h_E^\# \to h_Y^\#$ *is injective*

where $^\#$ *denotes* τ*-sheafification. Then for* $K' \in D(\mathcal{O}_{\tau'})$ *in the essential image of* $R\epsilon_*$ *the map* $c^{K'}_{X,Z,Y,E}$ *of Lemma 26.1 (using the* τ'*-topology) is an isomorphism.*

Proof. This helper lemma is an almost immediate consequence of Lemma 26.3 and we strongly urge the reader skip the proof. Say $K' = R\epsilon_* K$. Choose a K-injective complex of $\mathcal{O}_\tau$-modules $\mathcal{J}^\bullet$ representing K. Then $\epsilon_*\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}_{\tau'}$-modules representing K', see Lemma 20.10. Next,

$$0 \to \mathcal{J}^\bullet(X) \xrightarrow{\alpha} \mathcal{J}^\bullet(Z) \oplus \mathcal{J}^\bullet(Y) \xrightarrow{\beta} \mathcal{J}^\bullet(E) \to 0$$

is a short exact sequence of complexes of abelian groups, see Lemma 26.3 and its proof. Since this is the same as the sequence of complexes of abelian groups which is used to define $c^{K'}_{X,Z,Y,E}$, we conclude. $\square$

30. Comparing cohomology

We develop some general theory which will help us compare cohomology in different topologies. Given $\mathcal{C}$, τ, and τ' as in Section 27 and a morphism $f : X \to Y$ in $\mathcal{C}$ we obtain a commutative diagram of morphisms of topoi

(30.0.1)
$$\begin{tikzcd} Sh(\mathcal{C}_\tau/X) \arrow[r, "f_\tau"'] \arrow[d, "\epsilon_X"'] & Sh(\mathcal{C}_\tau/Y) \arrow[d, "\epsilon_Y"] \\ Sh(\mathcal{C}_{\tau'}/X) \arrow[r, "f_{\tau'}"] & Sh(\mathcal{C}_{\tau'}/X) \end{tikzcd}$$

Here the morphism ϵ_X, resp. ϵ_Y is the comparison morphism of Section 27 for the category $\mathcal{C}/X$ endowed with the two topologies τ and τ'. The morphisms f_τ and $f_{\tau'}$ are "relocalization" morphisms (Sites, Lemma 25.8). The commutativity of the diagram is a special case of Sites, Lemma 28.1 (applied with $\mathcal{C} = \mathcal{C}_\tau/Y$, $\mathcal{D} = \mathcal{C}_{\tau'}/Y$, $u = \text{id}$, $U = X$, and $V = X$). We also get $\epsilon_{X,*} \circ f_\tau^{-1} = f_{\tau'}^{-1} \circ \epsilon_{Y,*}$ either from the lemma or because it is obvious.

Situation 30.1. With $\mathcal{C}$, τ, and τ' as in Section 27. Assume we are given a subset $\mathcal{P} \subset \text{Arrows}(\mathcal{C})$ and for every object X of $\mathcal{C}$ we are given a weak Serre subcategory $\mathcal{A}'_X \subset Ab(\mathcal{C}_{\tau'}/X)$. We make the following assumption:

(1) given $f : X \to Y$ in $\mathcal{P}$ and $Y' \to Y$ general, then $X \times_Y Y'$ exists and $X \times_Y Y' \to Y'$ is in $\mathcal{P}$,
(2) $f_{\tau'}^{-1}$ sends $\mathcal{A}'_Y$ into $\mathcal{A}'_X$ for any morphism $f : X \to Y$ of $\mathcal{C}$,
(3) given X in $\mathcal{C}$ and $\mathcal{F}'$ in $\mathcal{A}'_X$, then $\mathcal{F}'$ satisfies the sheaf condition for τ-coverings, i.e., $\mathcal{F}' = \epsilon_{X,*}\epsilon_X^{-1}\mathcal{F}'$,
(4) if $f : X \to Y$ in $\mathcal{P}$ and $\mathcal{F}' \in \text{Ob}(\mathcal{A}'_X)$, then $R^i f_{\tau',*}\mathcal{F}' \in \text{Ob}(\mathcal{A}'_Y)$ for $i \geq 0$.
(5) if $\{U_i \to U\}_{i \in I}$ is a τ-covering, then there exist
    (a) a τ'-covering $\{V_j \to U\}_{j \in J}$,
    (b) a τ-covering $\{f_j : W_j \to V_j\}$ consisting of a single $f_j \in \mathcal{P}$, and
    (c) a τ'-covering $\{W_{jk} \to W_j\}_{k \in K_j}$

    such that $\{W_{jk} \to U\}_{j \in J, k \in K_j}$ is a refinement of $\{U_i \to U\}_{i \in I}$.

Lemma 30.2. *In Situation 30.1 for* X *in* $\mathcal{C}$ *denote* $\mathcal{A}_X$ *the objects of* $Ab(\mathcal{C}_\tau/X)$ *of the form* $\epsilon_X^{-1}\mathcal{F}'$ *with* $\mathcal{F}'$ *in* $\mathcal{A}'_X$*. Then*

(1) *for $\mathcal{F}$ in $Ab(\mathcal{C}_\tau/X)$ we have $\mathcal{F} \in \mathcal{A}_X \Leftrightarrow \epsilon_{X,*}\mathcal{F} \in \mathcal{A}'_X$, and*
(2) *f_τ^{-1} sends $\mathcal{A}_Y$ into $\mathcal{A}_X$ for any morphism $f : X \to Y$ of $\mathcal{C}$.*

Proof. Part (1) follows from (3) and part (2) follows from (2) and the commutativity of (30.0.1) which gives $\epsilon_X^{-1} \circ f_{\tau'}^{-1} = f_\tau^{-1} \circ \epsilon_Y^{-1}$. □

Our next goal is to prove Lemmas 30.10 and 30.9. We will do this by an induction argument using the following induction hypothesis.

(V_n) For X in $\mathcal{C}$ and $\mathcal{F}$ in $\mathcal{A}_X$ we have $R^i\epsilon_{X,*}\mathcal{F} = 0$ for $1 \leq i \leq n$.

Lemma 30.3. *In Situation 30.1 assume (V_n) holds. For $f : X \to Y$ in $\mathcal{P}$ and $\mathcal{F}$ in $\mathcal{A}_X$ we have $R^i f_{\tau',*}\epsilon_{X,*}\mathcal{F} = \epsilon_{Y,*}R^i f_{\tau,*}\mathcal{F}$ for $i \leq n$.*

Proof. We will use the commutative diagram (30.0.1) without further mention. In particular have

$$Rf_{\tau',*}R\epsilon_{X,*}\mathcal{F} = R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F}$$

Assumption (V_n) tells us that $\epsilon_{X,*}\mathcal{F} \to R\epsilon_{X,*}\mathcal{F}$ is an isomorphism in degrees $\leq n$. Hence $Rf_{\tau',*}\epsilon_{X,*}\mathcal{F} \to Rf_{\tau',*}R\epsilon_{X,*}\mathcal{F}$ is an isomorphism in degrees $\leq n$. We conclude that

$$R^i f_{\tau',*}\epsilon_{X,*}\mathcal{F} \to H^i(R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F})$$

is an isomorphism for $i \leq n$. We will prove the lemma by looking at the second page of the spectral sequence of Lemma 14.7 for $R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F}$. Here is a picture:

$$\begin{matrix}
\ldots & \ldots & \ldots & \ldots \\
\epsilon_{Y,*}R^2f_{\tau,*}\mathcal{F} & R^1\epsilon_{Y,*}R^2f_{\tau,*}\mathcal{F} & R^2\epsilon_{Y,*}R^2f_{\tau,*}\mathcal{F} & \ldots \\
\epsilon_{Y,*}R^1f_{\tau,*}\mathcal{F} & R^1\epsilon_{Y,*}R^1f_{\tau,*}\mathcal{F} & R^2\epsilon_{Y,*}R^1f_{\tau,*}\mathcal{F} & \ldots \\
\epsilon_{Y,*}f_{\tau,*}\mathcal{F} & R^1\epsilon_{Y,*}f_{\tau,*}\mathcal{F} & R^2\epsilon_{Y,*}f_{\tau,*}\mathcal{F} & \ldots
\end{matrix}$$

Let (C_m) be the hypothesis: $R^i f_{\tau',*}\epsilon_{X,*}\mathcal{F} = \epsilon_{Y,*}R^i f_{\tau,*}\mathcal{F}$ for $i \leq m$. Observe that (C_0) holds. We will show that (C_{m-1}) $\Rightarrow$ (C_m) for $m < n$. Namely, if (C_{m-1}) holds, then for $n \geq p > 0$ and $q \leq m-1$ we have

$$\begin{aligned}
R^p\epsilon_{Y,*}R^qf_{\tau,*}\mathcal{F} &= R^p\epsilon_{Y,*}\epsilon_Y^{-1}\epsilon_{Y,*}R^qf_{\tau,*}\mathcal{F} \\
&= R^p\epsilon_{Y,*}\epsilon_Y^{-1}R^qf_{\tau',*}\epsilon_{X,*}\mathcal{F} = 0
\end{aligned}$$

First equality as $\epsilon_Y^{-1}\epsilon_{Y,*} = \text{id}$, the second by ($C_{m-1}$), and the final by by ($V_n$) because $\epsilon_Y^{-1}R^qf_{\tau',*}\epsilon_{X,*}\mathcal{F}$ is in $\mathcal{A}_Y$ by (4). Looking at the spectral sequence we see that $E_2^{0,m} = \epsilon_{Y,*}R^mf_{\tau,*}\mathcal{F}$ is the only nonzero term $E_2^{p,q}$ with $p+q = m$. Recall that $\mathrm{d}_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$. Hence there are no nonzero differentials $\mathrm{d}_r^{p,q}$, $r \geq 2$ either emanating or entering this spot. We conclude that $H^m(R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F}) = \epsilon_{Y,*}R^mf_{\tau,*}\mathcal{F}$ which implies (C_m) by the discussion above.

Finally, assume (C_{n-1}). The same analysis shows that $E_2^{0,n} = \epsilon_{Y,*}R^nf_{\tau,*}\mathcal{F}$ is the only nonzero term $E_2^{p,q}$ with $p+q = n$. We do still have no nonzero differentials entering this spot, but there can be a nonzero differential emanating it. Namely, the map $d_{n+1}^{0,n} : \epsilon_{Y,*}R^nf_{\tau,*}\mathcal{F} \to R^{n+1}\epsilon_{Y,*}f_{\tau,*}\mathcal{F}$. We conclude that there is an exact sequence

$$0 \to R^nf_{\tau',*}\epsilon_{X,*}\mathcal{F} \to \epsilon_{Y,*}R^nf_{\tau,*}\mathcal{F} \to R^{n+1}\epsilon_{Y,*}f_{\tau,*}\mathcal{F}$$

By (4) and (3) the sheaf $R^nf_{\tau',*}\epsilon_{X,*}\mathcal{F}$ satisfies the sheaf property for τ-coverings as does $\epsilon_{Y,*}R^nf_{\tau,*}\mathcal{F}$ (use the description of ϵ_* in Section 27). However, the τ-sheafification of the τ'-sheaf $R^{n+1}\epsilon_{Y,*}f_{\tau,*}\mathcal{F}$ is zero (by locality of cohomology; use

Lemmas 7.3 and 7.4). Thus $R^n f_{\tau',*}\epsilon_{X,*}\mathcal{F} \to \epsilon_{Y,*}R^n f_{\tau,*}\mathcal{F}$ has to be an isomorphism and the proof is complete. $\square$

If E', resp. E is an object of $D(\mathcal{C}_{\tau'}/X)$, resp. $D(\mathcal{C}_\tau/X)$ then we will write $H^n_{\tau'}(U, E')$, resp. $H^n_\tau(U, E)$ for the cohomology of E', resp. E over an object U of $\mathcal{C}/X$.

Lemma 30.4. *In Situation 30.1 if (V_n) holds, then for X in $\mathcal{C}$ and $L \in D(\mathcal{C}_{\tau'}/X)$ with $H^i(L) = 0$ for $i < 0$ and $H^i(L)$ in $\mathcal{A}'_X$ for $0 \leq i \leq n$ we have $H^n_{\tau'}(X, L) = H^n_\tau(X, \epsilon_X^{-1}L)$.*

Proof. By Lemma 20.5 we have $H^n_\tau(X, \epsilon_X^{-1}L) = H^n_{\tau'}(X, R\epsilon_{X,*}\epsilon_X^{-1}L)$. There is a spectral sequence

$$E_2^{p,q} = R^p\epsilon_{X,*}\epsilon_X^{-1}H^q(L)$$

converging to $H^{p+q}(R\epsilon_{X,*}\epsilon_X^{-1}L)$. By (V_n) we have the vanishing of $E_2^{p,q}$ for $0 < p \leq n$ and $0 \leq q \leq n$. Thus $E_2^{0,q} = \epsilon_{X,*}\epsilon_X^{-1}H^q(L) = H^q(L)$ are the only nonzero terms $E_2^{p,q}$ with $p + q \leq n$. It follows that the map

$$L \longrightarrow R\epsilon_{X,*}\epsilon_X^{-1}L$$

is an isomorphism in degrees $\leq n$ (small detail omitted). Hence we find that $H^i_{\tau'}(X, L) = H^i_{\tau'}(X, R\epsilon_{X,*}\epsilon_X^{-1}L)$ for $i \leq n$. Thus the lemma is proved. $\square$

Lemma 30.5. *In Situation 30.1 if (V_n) holds, then for X in $\mathcal{C}$ and $\mathcal{F}$ in $\mathcal{A}_X$ the map $H^{n+1}_{\tau'}(X, \epsilon_{X,*}\mathcal{F}) \to H^{n+1}_\tau(X, \mathcal{F})$ is injective with image those classes which become trivial on a τ'-covering of X.*

Proof. Recall that $\epsilon_X^{-1}\epsilon_{X,*}\mathcal{F} = \mathcal{F}$ hence the map is given by pulling back cohomology classes by ϵ_X. The Leray spectral sequence (Lemma 14.5)

$$E_2^{p,q} = H^p_{\tau'}(X, R^q\epsilon_{X,*}\mathcal{F}) \Rightarrow H^{p+q}_\tau(X, \mathcal{F})$$

combined with the assumed vanishing gives an exact sequence

$$0 \to H^{n+1}_{\tau'}(X, \epsilon_{X,*}\mathcal{F}) \to H^{n+1}_\tau(X, \mathcal{F}) \to H^0_{\tau'}(X, R^{n+1}\epsilon_{X,*}\mathcal{F})$$

This is a restatement of the lemma. $\square$

Lemma 30.6. *In Situation 30.1 let $f : X \to Y$ be in $\mathcal{P}$ such that $\{X \to Y\}$ is a τ-covering. Let $\mathcal{F}'$ be in $\mathcal{A}'_Y$. If $n \geq 0$ and*

$$\theta \in \mathit{Equalizer}\left(H^{n+1}_{\tau'}(X, \mathcal{F}') \rightrightarrows H^{n+1}_{\tau'}(X \times_Y X, \mathcal{F}') \right)$$

then there exists a τ'-covering $\{Y_i \to Y\}$ such that θ restricts to zero in $H^{n+1}_{\tau'}(Y_i \times_Y X, \mathcal{F}')$.

Proof. Observe that $X \times_Y X$ exists by (1). For Z in $\mathcal{C}/Y$ denote $\mathcal{F}'|_Z$ the restriction of $\mathcal{F}'$ to $\mathcal{C}_{\tau'}/Z$. Recall that $H^{n+1}_{\tau'}(X, \mathcal{F}') = H^{n+1}(\mathcal{C}_{\tau'}/X, \mathcal{F}'|_X)$, see Lemma 7.1. The lemma asserts that the image $\overline{\theta} \in H^0(Y, R^{n+1}f_{\tau',*}\mathcal{F}'|_X)$ of θ is zero. Consider the cartesian diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

By trivial base change (Lemma 21.1) we have

$$f_{\tau'}^{-1}R^{n+1}f_{\tau',*}(\mathcal{F}'|_X) = R^{n+1}\mathrm{pr}_{1,\tau',*}(\mathcal{F}'|_{X \times_Y X})$$

If $\mathrm{pr}_1^{-1}\theta = \mathrm{pr}_2^{-1}\theta$, then the section $f_{\tau'}^{-1}\overline{\theta}$ of $f_{\tau'}^{-1}R^{n+1}f_{\tau',*}(\mathcal{F}'|_X)$ is zero, because it is clear that $\mathrm{pr}_1^{-1}\theta$ maps to the zero element in $H^0(X, R^{n+1}\mathrm{pr}_{1,\tau',*}(\mathcal{F}'|_{X\times_Y X}))$. By (2) we have $\mathcal{F}'|_X$ in $\mathcal{A}'_X$. Thus $\mathcal{G}' = R^{n+1}f_{\tau',*}(\mathcal{F}'|_X)$ is an object of $\mathcal{A}'_Y$ by (4). Thus $\mathcal{G}'$ satisfies the sheaf property for τ-coverings by (3). Since $\{X \to Y\}$ is a τ-covering we conclude that restriction $\mathcal{G}'(Y) \to \mathcal{G}'(X)$ is injective. It follows that $\overline{\theta}$ is zero. $\square$

Lemma 30.7. *In Situation 30.1 we have* $(V_n) \Rightarrow (V_{n+1})$.

Proof. Let X in $\mathcal{C}$ and $\mathcal{F}$ in $\mathcal{A}_X$. Let $\xi \in H^{n+1}_\tau(U, \mathcal{F})$ for some U/X. We have to show that ξ restricts to zero on the members of a τ'-covering of U. See Lemma 7.4. It follows from this that we may replace U by the members of a τ'-covering of U.

By locality of cohomology (Lemma 7.3) we can choose a τ-covering $\{U_i \to U\}$ such that ξ restricts to zero on U_i. Choose $\{V_j \to V\}$, $\{f_j : W_j \to V_j\}$, and $\{W_{jk} \to W_j\}$ as in (5). After replacing both U by V_j and $\mathcal{F}$ by its restriction to $\mathcal{C}_\tau/V_j$, which is allowed by (1), we reduce to the case discussed in the next paragraph.

Here $f : X \to Y$ is an element of $\mathcal{P}$ such that $\{X \to Y\}$ is a τ-covering, $\mathcal{F}$ is an object of $\mathcal{A}_Y$, and $\xi \in H^{n+1}_\tau(Y, \mathcal{F})$ is such that there exists a τ'-covering $\{X_i \to X\}_{i\in I}$ such that ξ restricts to zero on X_i for all $i \in I$. Problem: show that ξ restricts to zero on a τ'-covering of Y.

By Lemma 30.5 there exists a unique τ'-cohomology class $\theta \in H^{n+1}_{\tau'}(X, \epsilon_{X,*}\mathcal{F})$ whose image is $\xi|_X$. Since $\xi|_X$ pulls back to the same class on $X \times_Y X$ via the two projections, we find that the same is true for θ (by uniqueness). By Lemma 30.6 we see that after replacing Y by the members of a τ'-covering, we may assume that $\theta = 0$. Consequently, we may assume that $\xi|_X$ is zero.

Let $f : X \to Y$ be an element of $\mathcal{P}$ such that $\{X \to Y\}$ is a τ-covering, $\mathcal{F}$ is an object of $\mathcal{A}_Y$, and $\xi \in H^{n+1}_\tau(Y, \mathcal{F})$ maps to zero in $H^{n+1}_\tau(X, \mathcal{F})$. Problem: show that ξ restricts to zero on a τ'-covering of Y.

The assumptions tell us ξ maps to zero under the map

$$\mathcal{F} \longrightarrow Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

Use Lemma 20.5. A simple argument using the distinguished triangle of truncations (Derived Categories, Remark 12.4) shows that ξ maps to zero under the map

$$\mathcal{F} \longrightarrow \tau_{\leq n}Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

We will compare this with the map $\epsilon_{Y,*}\mathcal{F} \to K$ where

$$K = \tau_{\leq n}Rf_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F} = \tau_{\leq n}Rf_{\tau',*}\epsilon_{X,*}f_\tau^{-1}\mathcal{F}$$

The equality $\epsilon_{X,*}f_\tau^{-1} = f_{\tau'}^{-1}\epsilon_{Y,*}$ is a property of (30.0.1). Consider the map

$$Rf_{\tau',*}\epsilon_{X,*}f_\tau^{-1}\mathcal{F} \longrightarrow Rf_{\tau',*}R\epsilon_{X,*}f_\tau^{-1}\mathcal{F} = R\epsilon_{Y,*}Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

used in the proof of Lemma 30.3 which induces by adjunction a map

$$\epsilon_Y^{-1}Rf_{\tau',*}\epsilon_{X,*}f_\tau^{-1}\mathcal{F} \to Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

Taking trunctions we find a map

$$\epsilon_Y^{-1}K \longrightarrow \tau_{\leq n}Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

which is an isomorphism by Lemma 30.3; the lemma applies because $f_\tau^{-1}\mathcal{F}$ is in $\mathcal{A}_X$ by Lemma 30.2. Choose a distinguished triangle

$$\epsilon_{Y,*}\mathcal{F} \to K \to L \to \epsilon_{Y,*}\mathcal{F}[1]$$

The map $\mathcal{F} \to f_{\tau,*}f_\tau^{-1}\mathcal{F}$ is injective as $\{X \to Y\}$ is a τ-covering. Thus $\epsilon_{Y,*}\mathcal{F} \to \epsilon_{Y,*}f_{\tau,*}f_\tau^{-1}\mathcal{F} = f_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F}$ is injective too. Hence L only has nonzero cohomology sheaves in degrees $0, \ldots, n$. As $f_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F}$ is in $\mathcal{A}'_Y$ by (2) and (4) we conclude that

$$H^0(L) = \text{Coker}(\epsilon_{Y,*}\mathcal{F} \to f_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F})$$

is in the weak Serre subcategory $\mathcal{A}'_Y$. For $1 \leq i \leq n$ we see that $H^i(L) = R^if_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F}$ is in $\mathcal{A}'_Y$ by (2) and (4). Pulling back the distinguished triangle above by ϵ_Y we get the distinguished triangle

$$\mathcal{F} \to \tau_{\leq n}Rf_{\tau,*}f_\tau^{-1}\mathcal{F} \to \epsilon_Y^{-1}L \to \mathcal{F}[1]$$

Since ξ maps to zero in the middle term we find that ξ is the image of an element $\xi' \in H^n_\tau(Y, \epsilon_Y^{-1}L)$. By Lemma 30.4 we have

$$H^n_{\tau'}(Y, L) = H^n_\tau(Y, \epsilon_Y^{-1}L),$$

Thus we may lift ξ' to an element of $H^n_{\tau'}(Y, L)$ and take the boundary into $H^{n+1}_{\tau'}(Y, \epsilon_{Y,*}\mathcal{F})$ to see that ξ is in the image of the canonical map $H^{n+1}_{\tau'}(Y, \epsilon_{Y,*}\mathcal{F}) \to H^{n+1}_\tau(Y, \mathcal{F})$. By locality of cohomology for $H^{n+1}_{\tau'}(Y, \epsilon_{Y,*}\mathcal{F})$, see Lemma 7.3, we conclude. ☐

Lemma 30.8. *In Situation 30.1 we have that (V_n) is true for all n. Moreover:*

1. *For X in $\mathcal{C}$ and $K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$ the map $K' \to R\epsilon_{X,*}(\epsilon_X^{-1}K')$ is an isomorphism.*
2. *For $f : X \to Y$ in $\mathcal{P}$ and $K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$ we have $Rf_{\tau',*}K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/Y)$ and $\epsilon_Y^{-1}(Rf_{\tau',*}K') = Rf_{\tau,*}(\epsilon_X^{-1}K')$.*

Proof. Observe that (V_0) holds as it is the empty condition. Then we get (V_n) for all n by Lemma 30.7.

Proof of (1). The object $K = \epsilon_X^{-1}K'$ has cohomology sheaves $H^i(K) = \epsilon_X^{-1}H^i(K')$ in $\mathcal{A}_X$. Hence the spectral sequence

$$E_2^{p,q} = R^p\epsilon_{X,*}H^q(K) \Rightarrow H^{p+q}(R\epsilon_{X,*}K)$$

degenerates by (V_n) for all n and we find

$$H^n(R\epsilon_{X,*}K) = \epsilon_{X,*}H^n(K) = \epsilon_{X,*}\epsilon_X^{-1}H^i(K') = H^i(K').$$

again because $H^i(K')$ is in $\mathcal{A}'_X$. Thus the canonical map $K' \to R\epsilon_{X,*}(\epsilon_X^{-1}K')$ is an isomorphism.

Proof of (2). Using the spectral sequence

$$E_2^{p,q} = R^pf_{\tau',*}H^q(K') \Rightarrow R^{p+q}f_{\tau',*}K'$$

the fact that $R^pf_{\tau',*}H^q(K')$ is in $\mathcal{A}'_Y$ by (4), the fact that $\mathcal{A}'_Y$ is a weak Serre subcategory of $Ab(\mathcal{C}_{\tau'}/Y)$, and Homology, Lemma 24.11 we conclude that $Rf_{\tau',*}K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$. To finish the proof we have to show the base change map

$$\epsilon_Y^{-1}(Rf_{\tau',*}K') \longrightarrow Rf_{\tau,*}(\epsilon_X^{-1}K')$$

is an isomorphism. Comparing the spectral sequence above to the spectral sequence

$$E_2^{p,q} = R^pf_{\tau,*}H^q(\epsilon_X^{-1}K') \Rightarrow R^{p+q}f_{\tau,*}\epsilon_X^{-1}K'$$

we reduce this to the case where K' has a single nonzero cohomology sheaf $\mathcal{F}'$ in $\mathcal{A}'_X$; details omitted. Then Lemma 30.3 gives $\epsilon_Y^{-1}R^if_{\tau',*}\mathcal{F}' = R^if_{\tau,*}\epsilon_X^{-1}\mathcal{F}'$ for all i and the proof is complete. □

Lemma 30.9. *In Situation 30.1. For any X in $\mathcal{C}$ the category $\mathcal{A}_X \subset Ab(\mathcal{C}_\tau/X)$ is a weak Serre subcategory and the functor*

$$R\epsilon_{X,*} : D^+_{\mathcal{A}_X}(\mathcal{C}_\tau/X) \longrightarrow D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$$

is an equivalence with quasi-inverse given by ϵ_X^{-1}.

Proof. We need to check the conditions listed in Homology, Lemma 10.3 for $\mathcal{A}_X$. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map in $\mathcal{A}_X$, then $\epsilon_{X,*}\varphi : \epsilon_{X,*}\mathcal{F} \to \epsilon_{X,*}\mathcal{G}$ is a map in $\mathcal{A}'_X$. Hence $\mathrm{Ker}(\epsilon_{X,*}\varphi)$ and $\mathrm{Coker}(\epsilon_{X,*}\varphi)$ are objects of $\mathcal{A}'_X$ as this is a weak Serre subcategory of $Ab(\mathcal{C}_{\tau'}/X)$. Applying ϵ_X^{-1} we obtain an exact sequence

$$0 \to \epsilon_X^{-1}\mathrm{Ker}(\epsilon_{X,*}\varphi) \to \mathcal{F} \to \mathcal{G} \to \epsilon_X^{-1}\mathrm{Coker}(\epsilon_{X,*}\varphi) \to 0$$

and we see that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are in $\mathcal{A}_X$. Finally, suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence in $Ab(\mathcal{C}_\tau/X)$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ in $\mathcal{A}_X$. Then applying $\epsilon_{X,*}$ we obtain an exact sequence

$$0 \to \epsilon_{X,*}\mathcal{F}_1 \to \epsilon_{X,*}\mathcal{F}_2 \to \epsilon_{X,*}\mathcal{F}_3 \to R^1\epsilon_{X,*}\mathcal{F}_1 = 0$$

Vanishing by Lemma 30.8. Hence $\epsilon_{X,*}\mathcal{F}_2$ is in $\mathcal{A}'_X$ as this is a weak Serre subcategory of $Ab(\mathcal{C}_{\tau'}/X)$. Pulling back by ϵ_X we conclude that $\mathcal{F}_2$ is in $\mathcal{A}_X$.

Thus $\mathcal{A}_X$ is a weak Serre subcategory of $Ab(\mathcal{C}_\tau/X)$ and it makes sense to consider the category $D^+_{\mathcal{A}_X}(\mathcal{C}_\tau/X)$. Observe that $\epsilon_X^{-1} : \mathcal{A}'_X \to \mathcal{A}_X$ is an equivalence and that $\mathcal{F}' \to R\epsilon_{X,*}\epsilon_X^{-1}\mathcal{F}'$ is an isomorphism for $\mathcal{F}'$ in $\mathcal{A}'_X$ since we have (V_n) for all n by Lemma 30.8. Thus we conclude by Lemma 28.5. □

Lemma 30.10. *In Situation 30.1. Let X be in $\mathcal{C}$.*

1. *for $\mathcal{F}'$ in $\mathcal{A}'_X$ we have $H^n_{\tau'}(X, \mathcal{F}') = H^n_\tau(X, \epsilon_X^{-1}\mathcal{F}')$,*
2. *for $K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$ we have $H^n_{\tau'}(X, K') = H^n_\tau(X, \epsilon_X^{-1}K')$.*

Proof. This follows from Lemma 30.8 by Remark 14.4. □

31. Cohomology on Hausdorff and locally quasi-compact spaces

We continue our convention to say "Hausdorff and locally quasi-compact" instead of saying "locally compact" as is often done in the literature. Let LC denote the category whose objects are Hausdorff and locally quasi-compact topological spaces and whose morphisms are continuous maps.

Lemma 31.1. *The category LC has fibre products and a final object and hence has arbitrary finite limits. Given morphisms $X \to Z$ and $Y \to Z$ in LC with X and Y quasi-compact, then $X \times_Z Y$ is quasi-compact.*

Proof. The final object is the singleton space. Given morphisms $X \to Z$ and $Y \to Z$ of LC the fibre product $X \times_Z Y$ is a subspace of $X \times Y$. Hence $X \times_Z Y$ is Hausdorff as $X \times Y$ is Hausdorff by Topology, Section 3.

If X and Y are quasi-compact, then $X \times Y$ is quasi-compact by Topology, Theorem 14.4. Since $X \times_Z Y$ is a closed subset of $X \times Y$ (Topology, Lemma 3.4) we find that $X \times_Z Y$ is quasi-compact by Topology, Lemma 12.3.

Finally, returning to the general case, if $x \in X$ and $y \in Y$ we can pick quasi-compact neighbourhoods $x \in E \subset X$ and $y \in F \subset Y$ and we find that $E \times_Z F$ is a quasi-compact neighbourhood of (x, y) by the result above. Thus $X \times_Z Y$ is an object of LC by Topology, Lemma 13.2. $\square$

We can endow LC with a stronger topology than the usual one.

Definition 31.2. Let $\{f_i : X_i \to X\}$ be a family of morphisms with fixed target in the category LC. We say this family is a *qc covering*[6] if for every $x \in X$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x.

Observe that an open covering $X = \bigcup U_i$ of an object of LC gives a qc covering $\{U_i \to X\}$ because X is locally quasi-compact. We start with the obligatory lemma.

Lemma 31.3. *Let X be a Hausdorff and locally quasi-compact space, in other words, an object of LC.*

1. *If $X' \to X$ is an isomorphism in LC then $\{X' \to X\}$ is a qc covering.*
2. *If $\{f_i : X_i \to X\}_{i \in I}$ is a qc covering and for each i we have a qc covering $\{g_{ij} : X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a qc covering.*
3. *If $\{X_i \to X\}_{i \in I}$ is a qc covering and $X' \to X$ is a morphism of LC then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a qc covering.*

Proof. Part (1) holds by the remark above that open coverings are qc coverings.

Proof of (2). Let $x \in X$. Choose $i_1, \ldots, i_n \in I$ and $E_a \subset X_{i_a}$ quasi-compact such that $\bigcup f_{i_a}(E_a)$ is a neighbourhood of x. For every $e \in E_a$ we can find a finite subset $J_e \subset J_{i_a}$ and quasi-compact $F_{e,j} \subset X_{ij}$, $j \in J_e$ such that $\bigcup g_{ij}(F_{e,j})$ is a neighbourhood of e. Since E_a is quasi-compact we find a finite collection $e_1, \ldots, e_{m_a}$ such that

$$E_a \subset \bigcup\nolimits_{k=1,\ldots,m_a} \bigcup\nolimits_{j \in J_{e_k}} g_{ij}(F_{e_k,j})$$

Then we find that

$$\bigcup\nolimits_{a=1,\ldots,n} \bigcup\nolimits_{k=1,\ldots,m_a} \bigcup\nolimits_{j \in J_{e_k}} f_i(g_{ij}(F_{e_k,j}))$$

is a neighbourhood of x.

Proof of (3). Let $x' \in X'$ be a point. Let $x \in X$ be its image. Choose $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x. Choose a quasi-compact neighbourhood $F \subset X'$ of x' which maps into the quasi-compact neighbourhood $\bigcup f_{i_j}(E_j)$ of x. Then $F \times_X E_j \subset X' \times_X X_{i_j}$ is a quasi-compact subset and F is the image of the map $\coprod F \times_X E_j \to F$. Hence the base change is a qc covering and the proof is finished. $\square$

Since all objects of LC are Hausdorff any morphism $f : X \to Y$ of LC is a separated continuous map of topological spaces. Hence f is a proper map of topological spaces if and only if f is universally closed. See discussion in Topology, Section 17.

[6] This is nonstandard notation. We chose it to remind the reader of fpqc coverings of schemes.

Lemma 31.4. *Let $f : X \to Y$ be a morphism of LC. If f is proper and surjective, then $\{f : X \to Y\}$ is a qc covering.*

Proof. Let $y \in Y$ be a point. For each $x \in X_y$ choose a quasi-compact neighbourhood $E_x \subset X$. Choose $x \in U_x \subset E_x$ open. Since f is proper the fibre X_y is quasi-compact and we find $x_1, \ldots, x_n \in X_y$ such that $X_y \subset U_{x_1} \cup \ldots \cup U_{x_n}$. We claim that $f(E_{x_1}) \cup \ldots \cup f(E_{x_n})$ is a neighbourhood of y. Namely, as f is closed (Topology, Theorem 17.5) we see that $Z = f(X \setminus U_{x_1} \cup \ldots \cup U_{x_n})$ is a closed subset of Y not containing y. As f is surjective we see that $Y \setminus Z$ is contained in $f(E_{x_1}) \cup \ldots \cup f(E_{x_n})$ as desired. □

Besides some set theoretic issues Lemma 31.3 shows that LC with the collection of qc coverings forms a site. We will denote this site (suitably modified to overcome the set theoretical issues) LC_{qc}.

Remark 31.5 (Set theoretic issues)**.** The category LC is a "big" category as its objects form a proper class. Similarly, the coverings form a proper class. Let us define the *size* of a topological space X to be the cardinality of the set of points of X. Choose a function *Bound* on cardinals, for example as in Sets, Equation (9.1.1). Finally, let S_0 be an initial set of objects of LC, for example $S_0 = \{(\mathbf{R}, \text{euclidean topology})\}$. Exactly as in Sets, Lemma 9.2 we can choose a limit ordinal α such that $LC_\alpha = LC \cap V_\alpha$ contains S_0 and is preserved under all countable limits and colimits which exist in LC. Moreover, if $X \in LC_\alpha$ and if $Y \in LC$ and $\text{size}(Y) \leq Bound(\text{size}(X))$, then Y is isomorphic to an object of LC_α. Next, we apply Sets, Lemma 11.1 to choose set Cov of qc covering on LC_α such that every qc covering in LC_α is combinatorially equivalent to a covering this set. In this way we obtain a site (LC_α, Cov) which we will denote LC_{qc}.

There is a second topology on the site LC_{qc} of Remark 31.5. Namely, given an object X we can consider all coverings $\{X_i \to X\}$ of LC_{qc} such that $X_i \to X$ is an open immersion. We denote this site LC_{Zar}. The identity functor $LC_{Zar} \to LC_{qc}$ is continuous and defines a morphism of sites

$$\epsilon : LC_{qc} \longrightarrow LC_{Zar}$$

See Section 27. For a Hausdorff and locally quasi-compact topological space X, more precisely for $X \in \text{Ob}(LC_{qc})$, we denote the induced morphism

$$\epsilon_X : LC_{qc}/X \longrightarrow LC_{Zar}/X$$

(see Sites, Lemma 28.1). Let X_{Zar} be the site whose objects are opens of X, see Sites, Example 6.4. There is a morphism of sites

$$\pi_X : LC_{Zar}/X \longrightarrow X_{Zar}$$

given by the continuous functor $X_{Zar} \to LC_{Zar}/X$, $U \mapsto U$. Namely, X_{Zar} has fibre products and a final object and the functor above commutes with these and Sites, Proposition 14.7 applies. We often think of π as a morphism of topoi

$$\pi_X : Sh(LC_{Zar}/X) \longrightarrow Sh(X)$$

using the equality $Sh(X_{Zar}) = Sh(X)$.

Lemma 31.6. *Let X be an object of LC_{qc}. Let $\mathcal{F}$ be a sheaf on X. The rule*

$$LC_{qc}/X \longrightarrow Sets, \quad (f : Y \to X) \longmapsto \Gamma(Y, f^{-1}\mathcal{F})$$

is a sheaf and a fortiori also a sheaf on LC_{Zar}/X. This sheaf is equal to $\pi_X^{-1}\mathcal{F}$ on LC_{Zar}/X and $\epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$ on LC_{qc}/X.

Proof. Denote $\mathcal{G}$ the presheaf given by the formula in the lemma. Of course the pullback f^{-1} in the formula denotes usual pullback of sheaves on topological spaces. It is immediate from the definitions that $\mathcal{G}$ is a sheaf for the Zar topology.

Let $Y \to X$ be a morphism in LC_{qc}. Let $\mathcal{V} = \{g_i : Y_i \to Y\}_{i \in I}$ be a qc covering. To prove $\mathcal{G}$ is a sheaf for the qc topology it suffices to show that $\mathcal{G}(Y) \to H^0(\mathcal{V}, \mathcal{G})$ is an isomorphism, see Sites, Section 10. We first point out that the map is injective as a qc covering is surjective and we can detect equality of sections at stalks (use Sheaves, Lemmas 11.1 and 21.4). Thus $\mathcal{G}$ is a separated presheaf on LC_{qc} hence it suffices to show that any element $(s_i) \in H^0(\mathcal{V}, \mathcal{G})$ maps to an element in the image of $\mathcal{G}(Y)$ after replacing $\mathcal{V}$ by a refinement (Sites, Theorem 10.10).

Identifying sheaves on $Y_{i,Zar}$ and sheaves on Y_i we find that $\mathcal{G}|_{Y_{i,Zar}}$ is the pullback of $f^{-1}\mathcal{F}$ under the continuous map $g_i : Y_i \to Y$. Thus we can choose an open covering $Y_i = \bigcup V_{ij}$ such that for each j there is an open $W_{ij} \subset Y$ and a section $t_{ij} \in \mathcal{G}(W_{ij})$ such that V_{ij} maps into W_{ij} and such that $s|_{V_{ij}}$ is the pullback of t_{ij}. In other words, after refining the covering $\{Y_i \to Y\}$ we may assume there are opens $W_i \subset Y$ such that $Y_i \to Y$ factors through W_i and sections t_i of $\mathcal{G}$ over W_i which restrict to the given sections s_i. Moreover, if $y \in Y$ is in the image of both $Y_i \to Y$ and $Y_j \to Y$, then the images $t_{i,y}$ and $t_{j,y}$ in the stalk $f^{-1}\mathcal{F}_y$ agree (because s_i and s_j agree over $Y_i \times_Y Y_j$). Thus for $y \in Y$ there is a well defined element t_y of $f^{-1}\mathcal{F}_y$ agreeing with $t_{i,y}$ whenever y is in the image of $Y_i \to Y$. We will show that the element (t_y) comes from a global section of $f^{-1}\mathcal{F}$ over Y which will finish the proof of the lemma.

It suffices to show that this is true locally on Y, see Sheaves, Section 17. Let $y_0 \in Y$. Pick $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset Y_{i_j}$ such that $\bigcup g_{i_j}(E_j)$ is a neighbourhood of y_0. Let $V \subset Y$ be an open neighbourhood of y_0 contained in $\bigcup g_{i_j}(E_j)$ and contained in $W_{i_1} \cap \ldots \cap W_{i_n}$. Since $t_{i_1,y_0} = \ldots = t_{i_n,y_0}$, after shrinking V we may assume the sections $t_{i_j}|_V$, $j = 1, \ldots, n$ of $f^{-1}\mathcal{F}$ agree. As $V \subset \bigcup g_{i_j}(E_j)$ we see that $(t_y)_{y \in V}$ comes from this section.

We still have to show that $\mathcal{G}$ is equal to $\epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$ on LC_{qc}, resp. $\pi_X^{-1}\mathcal{F}$ on LC_{Zar}. In both cases the pullback is defined by taking the presheaf

$$(f : Y \to X) \longmapsto \operatorname{colim}_{f(Y) \subset U \subset X} \mathcal{F}(U)$$

and then sheafifying. Sheafifying in the Zar topology exactly produces our sheaf $\mathcal{G}$ and the fact that $\mathcal{G}$ is a qc sheaf, shows that it works as well in the qc topology. $\square$

Let $X \in \mathrm{Ob}(LC_{Zar})$ and let $\mathcal{H}$ be an abelian sheaf on LC_{Zar}/X. Then we will write $H^n_{Zar}(U, \mathcal{H})$ for the cohomology of $\mathcal{H}$ over an object U of LC_{Zar}/X.

Lemma 31.7. *Let X be an object of LC_{Zar}. Then*

1. *for $\mathcal{F} \in Ab(X)$ we have $H^n_{Zar}(X, \pi_X^{-1}\mathcal{F}) = H^n(X, \mathcal{F})$,*
2. *$\pi_{X,*} : Ab(LC_{Zar}/X) \to Ab(X)$ is exact,*
3. *the unit $id \to \pi_{X,*} \circ \pi_X^{-1}$ of the adjunction is an isomorphism, and*
4. *for $K \in D(X)$ the canonical map $K \to R\pi_{X,*}\pi_X^{-1}K$ is an isomorphism.*

Let $f : X \to Y$ be a morphism of LC_{Zar}. Then

(5) *there is a commutative diagram*

$$\begin{tikzcd} Sh(LC_{Zar}/X) \arrow[r, "f_{Zar}"'] \arrow[d, "\pi_X"'] & Sh(LC_{Zar}/Y) \arrow[d, "\pi_Y"] \\ Sh(X_{Zar}) \arrow[r, "f"] & Sh(Y_{Zar}) \end{tikzcd}$$

of topoi,

(6) *for* $L \in D^+(Y)$ *we have* $H^n_{Zar}(X, \pi_Y^{-1}L) = H^n(X, f^{-1}L)$,

(7) *if* f *is proper, then we have*
   (a) $\pi_Y^{-1} \circ f_* = f_{Zar,*} \circ \pi_X^{-1}$ *as functors* $Sh(X) \to Sh(LC_{Zar}/Y)$,
   (b) $\pi_Y^{-1} \circ Rf_* = Rf_{Zar,*} \circ \pi_X^{-1}$ *as functors* $D^+(X) \to D^+(LC_{Zar}/Y)$.

Proof. Proof of (1). The equality $H^n_{Zar}(X, \pi_X^{-1}\mathcal{F}) = H^n(X, \mathcal{F})$ is a general fact coming from the trivial observation that coverings of X in LC_{Zar} are the same thing as open coverings of X. The reader who wishes to see a detailed proof should apply Lemma 7.2 to the functor $X_{Zar} \to LC_{Zar}$.

Proof of (2). This is true because $\pi_{X,*} = \tau_X^{-1}$ for some morphism of topoi $\tau_X : Sh(X_{Zar}) \to Sh(LC_{Zar})$ as follows from Sites, Lemma 21.8 applied to the functor $X_{Zar} \to LC_{Zar}/X$ used to define π_X.

Proof of (3). This is true because $\tau_X^{-1} \circ \pi_X^{-1}$ is the identity functor by Sites, Lemma 21.8. Or you can deduce it from the explicit description of π_X^{-1} in Lemma 31.6.

Proof of (4). Apply (3) to an complex of abelian sheaves representing K.

Proof of (5). The morphism of topoi f_{Zar} comes from an application of Sites, Lemma 25.8 and in our case comes from the continuous functor $Z/Y \mapsto Z \times_Y X/X$ by Sites, Lemma 27.3. The diagram commutes simply because the corresponding continuous functors compose correctly (see Sites, Lemma 14.4).

Proof of (6). We have $H^n_{Zar}(X, \pi_Y^{-1}\mathcal{G}) = H^n_{Zar}(X, f_{Zar}^{-1}\pi_Y^{-1}\mathcal{G})$ for $\mathcal{G}$ in $Ab(Y)$, see Lemma 7.1. This is equal to $H^n_{Zar}(X, \pi_X^{-1}f^{-1}\mathcal{G})$ by the commutativity of the diagram in (5). Hence we conclude by (1) in the case L consists of a single sheaf in degree 0. The general case follows by representing L by a bounded below complex of abelian sheaves.

Proof of (7a). Let $\mathcal{F}$ be a sheaf on X. Let $g : Z \to Y$ be an object of LC_{Zar}/Y. Consider the fibre product

$$\begin{tikzcd} Z' \arrow[r, "f'"'] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then we have

$$(f_{Zar,*}\pi_X^{-1}\mathcal{F})(Z/Y) = (\pi_X^{-1}\mathcal{F})(Z'/X) = \Gamma(Z', (g')^{-1}\mathcal{F}) = \Gamma(Z, f'_*(g')^{-1}\mathcal{F})$$

the second equality by Lemma 31.6. On the other hand

$$(\pi_Y^{-1}f_*\mathcal{F})(Z/Y) = \Gamma(Z, g^{-1}f_*\mathcal{F})$$

again by Lemma 31.6. Hence by proper base change for sheaves of sets (Cohomology, Lemma 18.3) we conclude the two sets are canonically isomorphic. The isomorphism is compatible with restriction mappings and defines an isomorphism $\pi_Y^{-1} f_* \mathcal{F} = f_{Zar,*} \pi_X^{-1} \mathcal{F}$. Thus an isomorphism of functors $\pi_Y^{-1} \circ f_* = f_{Zar,*} \circ \pi_X^{-1}$.

Proof of (7b). Let $K \in D^+(X)$. By Lemma 20.6 the nth cohomology sheaf of $Rf_{Zar,*}\pi_X^{-1}K$ is the sheaf associated to the presheaf

$$(g : Z \to Y) \longmapsto H^n_{Zar}(Z', \pi_X^{-1}K)$$

with notation as above. Observe that

$$\begin{aligned} H^n_{Zar}(Z', \pi_X^{-1}K) &= H^n(Z', (g')^{-1}K) \\ &= H^n(Z, Rf'_*(g')^{-1}K) \\ &= H^n(Z, g^{-1}Rf_*K) \\ &= H^n_{Zar}(Z, \pi_Y^{-1}Rf_*K) \end{aligned}$$

The first equality is (6) applied to K and $g' : Z' \to X$. The second equality is Leray for $f' : Z' \to Z$ (Cohomology, Lemma 13.1). The third equality is the proper base change theorem (Cohomology, Theorem 18.2). The fourth equality is (6) applied to $g : Z \to Y$ and Rf_*K. Thus $Rf_{Zar,*}\pi_X^{-1}K$ and $\pi_Y^{-1}Rf_*K$ have the same cohomology sheaves. We omit the verification that the canonical base change map $\pi_Y^{-1}Rf_*K \to Rf_{Zar,*}\pi_X^{-1}K$ induces this isomorphism. □

In the situation of Lemma 31.6 the composition of ϵ and π and the equality $Sh(X) = Sh(X_{Zar})$ determine a morphism of topoi

$$a_X : Sh(LC_{qc}/X) \longrightarrow Sh(X)$$

Lemma 31.8. *Let $f : X \to Y$ be a morphism of LC_{qc}. Then there are commutative diagrams of topoi*

$$\begin{tikzcd} Sh(LC_{qc}/X) \arrow[r, "f_{qc}"'] \arrow[d, "\epsilon_X"'] & Sh(LC_{qc}/Y) \arrow[d, "\epsilon_Y"] \\ Sh(LC_{Zar}/X) \arrow[r, "f_{Zar}"] & Sh(LC_{Zar}/Y) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Sh(LC_{qc}/X) \arrow[r, "f_{qc}"'] \arrow[d, "a_X"'] & Sh(LC_{qc}/Y) \arrow[d, "a_Y"] \\ Sh(X) \arrow[r, "f"] & Sh(Y) \end{tikzcd}$$

with $a_X = \pi_X \circ \epsilon_X$, $a_Y = \pi_X \circ \epsilon_X$. If f is proper, then $a_Y^{-1} \circ f_* = f_{qc,*} \circ a_X^{-1}$.

Proof. The morphism of topoi f_{qc} is the one from Sites, Lemma 25.8 which in our case comes from the continuous functor $Z/Y \mapsto Z \times_Y X/X$, see Sites, Lemma 27.3. The diagram on the left commutes because the corresponding continuous functors compose correctly (see Sites, Lemma 14.4). The diagram on the right commutes because the one on the left does and because of part (5) of Lemma 31.7.

Proof of the final assertion. The reader may repeat the proof of part (7a) of Lemma 31.7; we will instead deduce this from it. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 31.6 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1} f_* \mathcal{F} \to f_{qc,*} a_X^{-1} \mathcal{F}$ is an isomorphism, it suffices to show that

$$\pi_Y^{-1} f_* \mathcal{F} = \epsilon_{Y,*} a_Y^{-1} f_* \mathcal{F} \to \epsilon_{Y,*} f_{qc,*} a_X^{-1} \mathcal{F} = f_{Zar,*} \epsilon_{X,*} a_X^{-1} \mathcal{F} = f_{Zar,*} \pi_X^{-1} \mathcal{F}$$

is an isomorphism. This is part (7a) of Lemma 31.7. □

Lemma 31.9. *Consider the comparison morphism $\epsilon : LC_{qc} \to LC_{Zar}$. Let $\mathcal{P}$ denote the class of proper maps of topological spaces. For X in LC_{Zar} denote $\mathcal{A}'_X \subset Ab(LC_{Zar}/X)$ the full subcategory consisting of sheaves of the form $\pi_X^{-1}\mathcal{F}$ with $\mathcal{F}$ in $Ab(X)$. Then (1), (2), (3), (4), and (5) of Situation 30.1 hold.*

Proof. We first show that $\mathcal{A}'_X \subset Ab(LC_{Zar}/X)$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful by Lemma 31.7 part (3). If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $Ab(LC_{Zar}/X)$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 31.7 part (2). Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Property (1) holds by Lemma 31.1 and the fact that the base change of a proper map is a proper map (see Topology, Theorem 17.5 and Lemma 4.4).

Property (2) follows from the commutative diagram (5) in Lemma 31.7.

Property (3) is Lemma 31.6.

Property (4) is Lemma 31.7 part (7)(b).

Proof of (5). Suppose given a qc covering $\{U_i \to U\}$. For $u \in U$ pick $i_1, \ldots, i_m \in I$ and quasi-compact subsets $E_j \subset U_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of u. Observe that $Y = \coprod_{j=1,\ldots,m} E_j \to U$ is proper as a continuous map between Hausdorff quasi-compact spaces (Topology, Lemma 17.7). Choose an open neighbourhood $u \in V$ contained in $\bigcup f_{i_j}(E_j)$. Then $Y \times_U V \to V$ is a surjective proper morphism and hence a *qc* covering by Lemma 31.4. Since we can do this for every $u \in U$ we see that (5) holds. $\square$

Lemma 31.10. *With notation as above.*

1. *For $X \in \mathrm{Ob}(LC_{qc})$ and an abelian sheaf $\mathcal{F}$ on X we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.*
2. *For a proper morphism $f : X \to Y$ in LC_{qc} and abelian sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^if_*\mathcal{F}) = R^if_{qc,*}(a_X^{-1}\mathcal{F})$ for all i.*
3. *For $X \in \mathrm{Ob}(LC_{qc})$ and K in $D^+(X)$ the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.*
4. *For a proper morphism $f : X \to Y$ in LC_{qc} and K in $D^+(X)$ we have $a_Y^{-1}(Rf_*K) = Rf_{qc,*}(a_X^{-1}K)$.*

Proof. By Lemma 31.9 the lemmas in Section 30 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Lemma 30.2 is the essential image of $a_X^{-1} : Ab(X) \to Ab(LC_{qc}/X)$.

Part (1) is equivalent to (V_n) for all n which holds by Lemma 30.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Lemma 30.3.

Part (3) follows from Lemma 30.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}(LC_{Zar}/X)$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Lemma 30.8 part (2) for the same reason. $\square$

Lemma 31.11. *Let X be an object of LC_{qc}. For $K \in D^+(X)$ the map*

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh(LC_{qc}/X) \to Sh(X)$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$.

By Lemma 31.6 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence Lemma 14.7. By Lemma 31.10 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 31.7 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. □

Lemma 31.12. *With $X \in \mathrm{Ob}(LC_{qc})$ and $a_X : Sh(LC_{qc}/X) \to Sh(X)$ as above:*

(1) *for an abelian sheaf $\mathcal{F}$ on X we have $H^n(X, \mathcal{F}) = H^n_{qc}(X, a_X^{-1}\mathcal{F})$,*
(2) *for $K \in D^+(X)$ we have $H^n(X, K) = H^n_{qc}(X, a_X^{-1}K)$.*

For example, if A is an abelian group, then we have $H^n(X, \underline{A}) = H^n_{qc}(X, \underline{A})$.

Proof. This follows from Lemma 31.11 by Remark 14.4. □

32. Spectral sequences for Ext

In this section we collect various spectral sequences that come up when considering the Ext functors. For any pair of complexes $\mathcal{G}^\bullet, \mathcal{F}^\bullet$ of complexes of modules on a ringed site $(\mathcal{C}, \mathcal{O})$ we denote

$$\mathrm{Ext}^n_{\mathcal{O}}(\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{F}^\bullet[n])$$

according to our general conventions in Derived Categories, Section 27.

Example 32.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a bounded above complex of $\mathcal{O}$-modules. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Then there is a spectral sequence with E_2-page

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathcal{O}}(H^{-j}(\mathcal{K}^\bullet), \mathcal{F}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{F})$$

and another spectral sequence with E_1-page

$$E_1^{i,j} = \mathrm{Ext}^j_{\mathcal{O}}(\mathcal{K}^{-i}, \mathcal{F}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{F}).$$

To construct these spectral sequences choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and consider the two spectral sequences coming from the double complex $\mathrm{Hom}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$, see Homology, Section 25.

33. Cup product

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, M be objects of $D(\mathcal{O})$. Set $A = \Gamma(\mathcal{C}, \mathcal{O})$. The (global) cup product in this setting is a map

$$R\Gamma(\mathcal{C}, K) \otimes_A^{\mathbf{L}} R\Gamma(\mathcal{C}, M) \longrightarrow R\Gamma(\mathcal{C}, K \otimes_{\mathcal{O}}^{\mathbf{L}} M)$$

in $D(A)$. We define it as the relative cup product for the morphism of ringed topoi $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(pt), A)$ as in Remark 19.7.

Let us formulate and prove a natural compatibility of the relative cup product. Namely, suppose that we have a morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_\mathcal{C}$-modules. There is a naive cup product

$$\mathrm{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \longrightarrow f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)$$

We claim that this is related to the relative cup product.

Lemma 33.1. *In the situation above the following diagram commutes*

$$\begin{tikzcd}
f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet \arrow[r] \arrow[d] & Rf_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_*\mathcal{M}^\bullet \arrow[d, "\text{Remark 19.7}"] \\
Tot(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \arrow[d, "\text{naive cup product}"'] & Rf_*(\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \arrow[d] \\
f_* Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \arrow[r] & Rf_* Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{tikzcd}$$

Proof. By the construction in Remark 19.7 we see that going around the diagram clockwise the map

$$f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet \longrightarrow Rf_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)$$

is adjoint to the map

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &= Lf^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Lf^*f_*\mathcal{M}^\bullet \\
&\to Lf^*Rf_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Lf^*Rf_*\mathcal{M}^\bullet \\
&\to \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet \\
&\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

By Lemma 19.6 this is also equal to

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &= Lf^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Lf^*f_*\mathcal{M}^\bullet \\
&\to f^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f^*f_*\mathcal{M}^\bullet \\
&\to \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet \\
&\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

Going around anti-clockwise we obtain the map adjoint to the map

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &\to Lf^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \\
&\to Lf^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to Lf^*Rf_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

By Lemma 19.6 this is also equal to

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &\to Lf^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \\
&\to Lf^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to f^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

Now the proof is finished by a contemplation of the diagram

$$\begin{tikzcd}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \arrow[rr] \arrow[d] & & Lf^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*f_*\mathcal{M}^\bullet \arrow[d] \\
Lf^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \arrow[r] \arrow[d, "naive"'] & f^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \arrow[ddl, "naive"] \arrow[dd] & f^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} f^*f_*\mathcal{M}^\bullet \arrow[ddl] \arrow[dd] \\
Lf^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \arrow[d] & & \\
f^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \arrow[dr] & \text{Tot}(f^*f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} f^*f_*\mathcal{M}^\bullet) \arrow[d] & \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet \arrow[dl] \\
& \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) &
\end{tikzcd}$$

All of the polygons in this diagram commute. The top one commutes by Lemma 18.8. The square with the two naive cup products commutes because $Lf^* \to f^*$ is functorial in the complex of modules. Similarly with the square involving the two maps $\mathcal{A}^\bullet \otimes^{\mathbf{L}} \mathcal{B}^\bullet \to \text{Tot}(\mathcal{A}^\bullet \otimes \mathcal{B}^\bullet)$. Finally, the commutativity of the remaining square is true on the level of complexes and may be viewed as the definiton of the naive cup product (by the adjointness of f^* and f_*). The proof is finished because going around the diagram on the outside are the two maps given above. $\square$

Lemma 33.2. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. The relative cup product of Remark 19.7 is associative in the sense that the diagram*

$$\begin{tikzcd}
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*M \arrow[r] \arrow[d] & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*M \arrow[d] \\
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*(L \otimes^{\mathbf{L}}_{\mathcal{O}} M) \arrow[r] & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L \otimes^{\mathbf{L}}_{\mathcal{O}} M)
\end{tikzcd}$$

is commutative in $D(\mathcal{O}')$ for all K, L, M in $D(\mathcal{O})$.

Proof. Going around either side we obtain the map adjoint to the obvious map

$$\begin{aligned} Lf^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*M) = Lf^*(Rf_*K) \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*(Rf_*L) \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*(Rf_*M) \\ \to K \otimes^{\mathbf{L}}_{\mathcal{O}} L \otimes^{\mathbf{L}}_{\mathcal{O}} M \end{aligned}$$

in $D(\mathcal{O})$. $\square$

Lemma 33.3. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. The relative cup product of Remark 19.7 is commutative in the sense that the diagram*

$$\begin{tikzcd}
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L \arrow[r] \arrow[d, "\psi"'] & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) \arrow[d, "Rf_*\psi"] \\
Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*K \arrow[r] & Rf_*(L \otimes^{\mathbf{L}}_{\mathcal{O}} K)
\end{tikzcd}$$

is commutative in $D(\mathcal{O}')$ for all K, L in $D(\mathcal{O})$. Here ψ is the commutativity constraint on the derived category (Lemma 48.5).

Proof. Omitted. ☐

Lemma 33.4. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and $f' : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}''), \mathcal{O}'')$ be morphisms of ringed topoi. The relative cup product of Remark 19.7 is compatible with compositions in the sense that the diagram*

$$\begin{tikzcd}
R(f' \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}''} R(f' \circ f)_*L \arrow[rr, equal] \arrow[d] & & Rf'_*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}''} Rf'_*Rf_*L \arrow[d] \\
R(f' \circ f)_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) \arrow[r, equal] & Rf'_*Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) & Rf'_*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L) \arrow[l]
\end{tikzcd}$$

is commutative in $D(\mathcal{O}'')$ for all K, L in $D(\mathcal{O})$.

Proof. This is true because going around the diagram either way we obtain the map adjoint to the map

$$\begin{aligned}
& L(f' \circ f)^* \left(R(f' \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}''} R(f' \circ f)_*L\right) \\
& = L(f' \circ f)^*R(f' \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}} L(f' \circ f)^*R(f' \circ f)_*L) \\
& \to K \otimes^{\mathbf{L}}_{\mathcal{O}} L
\end{aligned}$$

in $D(\mathcal{O})$. To see this one uses that the composition of the counits like so

$$L(f' \circ f)^*R(f' \circ f)_* = Lf^*L(f')^*Rf'_*Rf_* \to Lf^*Rf_* \to \text{id}$$

is the counit for $L(f' \circ f)^*$ and $R(f' \circ f)_*$. See Categories, Lemma 24.9. ☐

34. Hom complexes

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ be two complexes of $\mathcal{O}$-modules. We construct a complex of $\mathcal{O}$-modules $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. Namely, for each n we set

$$\mathcal{H}om^n(\mathcal{L}^\bullet, \mathcal{M}^\bullet) = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

It is a good idea to think of $\mathcal{H}om^n$ as the sheaf of $\mathcal{O}$-modules of all $\mathcal{O}$-linear maps from $\mathcal{L}^\bullet$ to $\mathcal{M}^\bullet$ (viewed as graded $\mathcal{O}$-modules) which are homogenous of degree n. In this terminology, we define the differential by the rule

$$\text{d}(f) = \text{d}_{\mathcal{M}} \circ f - (-1)^n f \circ \text{d}_{\mathcal{L}}$$

for $f \in \mathcal{H}om^n_{\mathcal{O}}(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. We omit the verification that $\text{d}^2 = 0$. This construction is a special case of Differential Graded Algebra, Example 26.6. It follows immediately from the construction that we have

$$H^n(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \text{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet|_U, \mathcal{M}^\bullet[n]|_U) \tag{34.0.1}$$

for all $n \in \mathbf{Z}$ and every $U \in \text{Ob}(\mathcal{C})$. Similarly, we have

$$H^n(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \text{Hom}_{K(\mathcal{O})}(\mathcal{L}^\bullet, \mathcal{M}^\bullet[n]) \tag{34.0.2}$$

for the complex of global sections.

Lemma 34.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is an isomorphism*

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)) = \mathcal{H}om^\bullet(\mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules functorial in $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 71.1. ☐

Lemma 34.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism*

$$Tot(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_\mathcal{O} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 71.3. □

Lemma 34.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism*

$$Tot(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{M}^\bullet, Tot(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 71.4. □

Lemma 34.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism*

$$\mathcal{K}^\bullet \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, Tot(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}$-modules functorial in both complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 71.5. □

Lemma 34.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism*

$$Tot(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_\mathcal{O} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 71.6. □

Lemma 34.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L and M be objects of $D(\mathcal{O})$. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing M. Let $\mathcal{L}^\bullet$ be a complex of $\mathcal{O}$-modules representing L. Then*

$$H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

for all $U \in \mathrm{Ob}(\mathcal{C})$. Similarly, $H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O})}(L, M)$.

Proof. We have

$$\begin{aligned} H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) &= \mathrm{Hom}_{K(\mathcal{O}_U)}(L|_U, M|_U) \\ &= \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U) \end{aligned}$$

The first equality is (34.0.1). The second equality is true because $\mathcal{I}^\bullet|_U$ is K-injective by Lemma 20.1. The proof of the last equation is similar except that it uses (34.0.2). □

Lemma 34.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{I}')^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism of K-injective complexes of $\mathcal{O}$-modules. Let $(\mathcal{L}')^\bullet \to \mathcal{L}^\bullet$ be a quasi-isomorphism of complexes of $\mathcal{O}$-modules. Then*

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet) \longrightarrow \mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet)$$

is a quasi-isomorphism.

Proof. Let M be the object of $D(\mathcal{O})$ represented by $\mathcal{I}^\bullet$ and $(\mathcal{I}')^\bullet$. Let L be the object of $D(\mathcal{O})$ represented by $\mathcal{L}^\bullet$ and $(\mathcal{L}')^\bullet$. By Lemma 34.6 we see that the sheaves

$$H^0(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet)) \quad \text{and} \quad H^0(\mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet))$$

are both equal to the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

Thus the map is a quasi-isomorphism. □

Lemma 34.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules. Let $\mathcal{L}^\bullet$ be a K-flat complex of $\mathcal{O}$-modules. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}$-modules.*

Proof. Namely, if $\mathcal{K}^\bullet$ is an acyclic complex of $\mathcal{O}$-modules, then

$$\begin{aligned}
\mathrm{Hom}_{K(\mathcal{O})}(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) &= H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)))) \\
&= H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet))) \\
&= \mathrm{Hom}_{K(\mathcal{O})}(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet) \\
&= 0
\end{aligned}$$

The first equality by (34.0.2). The second equality by Lemma 34.1. The third equality by (34.0.2). The final equality because $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$ is acyclic because $\mathcal{L}^\bullet$ is K-flat (Definition 17.2) and because $\mathcal{I}^\bullet$ is K-injective. □

35. Internal hom in the derived category

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L, M be objects of $D(\mathcal{O})$. We would like to construct an object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O})$ such that for every third object K of $D(\mathcal{O})$ there exists a canonical bijection

$$\mathrm{Hom}_{D(\mathcal{O})}(K, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O})}(K \otimes_{\mathcal{O}}^{\mathbf{L}} L, M) \tag{35.0.1}$$

Observe that this formula defines $R\mathcal{H}om(L, M)$ up to unique isomorphism by the Yoneda lemma (Categories, Lemma 3.5).

To construct such an object, choose a K-injective complex of $\mathcal{O}$-modules $\mathcal{I}^\bullet$ representing M and any complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. Then we set Then we set

$$R\mathcal{H}om(L, M) = \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

where the right hand side is the complex of $\mathcal{O}$-modules constructed in Section 34. This is well defined by Lemma 34.7. We get a functor

$$D(\mathcal{O})^{opp} \times D(\mathcal{O}) \longrightarrow D(\mathcal{O}), \quad (K, L) \longmapsto R\mathcal{H}om(K, L)$$

As a prelude to proving (35.0.1) we compute the cohomology groups of $R\mathcal{H}om(K, L)$.

Lemma 35.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. For every object U of $\mathcal{C}$ we have*

$$H^0(U, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

and we have $H^0(\mathcal{C}, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O})}(L, M)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}$-modules representing M and a K-flat complex $\mathcal{L}^\bullet$ representing L. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is K-injective by Lemma 34.8. Hence we can compute cohomology over U by simply taking sections over U and the result follows from Lemma 34.6. □

Lemma 35.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L, M be objects of $D(\mathcal{O})$. With the construction as described above there is a canonical isomorphism*

$$R\mathcal{H}om(K, R\mathcal{H}om(L, M)) = R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}} L, M)$$

in $D(\mathcal{O})$ functorial in K, L, M which recovers (35.0.1) on taking $H^0(\mathcal{C}, -)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M and a K-flat complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. For any complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ we have

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) = \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet)$$

by Lemma 34.1. Note that the left hand side represents $R\mathcal{H}om(K, R\mathcal{H}om(L, M))$ (use Lemma 34.8) and that the right hand side represents $R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}} L, M)$. This proves the displayed formula of the lemma. Taking global sections and using Lemma 35.1 we obtain (35.0.1). □

Lemma 35.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. The construction of $R\mathcal{H}om(K, L)$ commutes with restrictions, i.e., for every object U of $\mathcal{C}$ we have $R\mathcal{H}om(K|_U, L|_U) = R\mathcal{H}om(K, L)|_U$.*

Proof. This is clear from the construction and Lemma 20.1. □

Lemma 35.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The bifunctor $R\mathcal{H}om(-, -)$ transforms distinguished triangles into distinguished triangles in both variables.*

Proof. This follows from the observation that the assignment

$$(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \longmapsto \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$$

transforms a termwise split short exact sequences of complexes in either variable into a termwise split short exact sequence. Details omitted. □

Lemma 35.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L, M be objects of $D(\mathcal{O})$. There is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

in $D(\mathcal{O})$ functorial in K, L, M.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and a K-flat complex $\mathcal{K}^\bullet$ representing K. The map is defined using the map

$$\mathrm{Tot}(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet), \mathcal{I}^\bullet)$$

of Lemma 34.5. By our particular choice of complexes the left hand side represents $R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}} K$ and the right hand side represents $R\mathcal{H}om(R\mathcal{H}om(K, L), M)$. We omit the proof that this is functorial in all three objects of $D(\mathcal{O})$. □

Lemma 35.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given K, L, M in $D(\mathcal{O})$ there is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, M)$$

in $D(\mathcal{O})$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and any complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ representing K. By Lemma 34.2 there is a map of complexes

$$\mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)\right) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

The complexes of $\mathcal{O}$-modules $\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet)$, $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, and $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$ represent $R\mathcal{H}om(L, M)$, $R\mathcal{H}om(K, L)$, and $R\mathcal{H}om(K, M)$. If we choose a K-flat complex $\mathcal{H}^\bullet$ and a quasi-isomorphism $\mathcal{H}^\bullet \to \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, then there is a map

$$\mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}^\bullet\right) \longrightarrow \mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)\right)$$

whose source represents $R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(K, L)$. Composing the two displayed arrows gives the desired map. We omit the proof that the construction is functorial. ☐

Lemma 35.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given K, L, M in $D(\mathcal{O})$ there is a canonical morphism*

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes_{\mathcal{O}}^{\mathbf{L}} L)$$

in $D(\mathcal{O})$ functorial in K, L, M.

Proof. Choose a K-flat complex $\mathcal{K}^\bullet$ representing K, and a K-injective complex $\mathcal{I}^\bullet$ representing L, and choose any complex of $\mathcal{O}$-modules $\mathcal{M}^\bullet$ representing M. Choose a quasi-isomorphism $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet) \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is K-injective. Then we use the map

$$\mathrm{Tot}\left(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{I}^\bullet)\right) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{I}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{J}^\bullet)$$

where the first map is the map from Lemma 34.3. ☐

Lemma 35.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given K, L in $D(\mathcal{O})$ there is a canonical morphism*

$$K \longrightarrow R\mathcal{H}om(L, K \otimes_{\mathcal{O}}^{\mathbf{L}} L)$$

in $D(\mathcal{O})$ functorial in both K and L.

Proof. Choose a K-flat complex $\mathcal{K}^\bullet$ representing K and any complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. Choose a K-injective complex $\mathcal{J}^\bullet$ and a quasi-isomorphism $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) \to \mathcal{J}^\bullet$. Then we use

$$\mathcal{K}^\bullet \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{J}^\bullet)$$

where the first map comes from Lemma 34.4. ☐

Lemma 35.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L be an object of $D(\mathcal{O})$. Set $L^\vee = R\mathcal{H}om(L, \mathcal{O})$. For M in $D(\mathcal{O})$ there is a canonical map*

$$M \otimes_{\mathcal{O}}^{\mathbf{L}} L^\vee \longrightarrow R\mathcal{H}om(L, M) \tag{35.9.1}$$

which induces a canonical map

$$H^0(\mathcal{C}, M \otimes_{\mathcal{O}}^{\mathbf{L}} L^\vee) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(L, M)$$

functorial in M in $D(\mathcal{O})$.

Proof. The map (35.9.1) is a special case of Lemma 35.6 using the identification $M = R\mathcal{H}om(\mathcal{O}, M)$. $\square$

Remark 35.10. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let K, L be objects of $D(\mathcal{O}_\mathcal{C})$. We claim there is a canonical map

$$Rf_*R\mathcal{H}om(L, K) \longrightarrow R\mathcal{H}om(Rf_*L, Rf_*K)$$

Namely, by (35.0.1) this is the same thing as a map $Rf_*R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_*L \to Rf_*K$. For this we can use the composition

$$Rf_*R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_*L \to Rf_*(R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} L) \to Rf_*K$$

where the first arrow is the relative cup product (Remark 19.7) and the second arrow is Rf_* applied to the canonical map $R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} L \to K$ coming from Lemma 35.6 (with $\mathcal{O}_\mathcal{C}$ in one of the spots).

Remark 35.11. Let $h : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let K, L be objects of $D(\mathcal{O}')$. We claim there is a canonical map

$$Lh^*R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O})$. Namely, by (35.0.1) proved in Lemma 35.2 such a map is the same thing as a map

$$Lh^*R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} Lh^*K \longrightarrow Lh^*L$$

The source of this arrow is $Lh^*(\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K)$ by Lemma 18.4 hence it suffices to construct a canonical map

$$R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K \longrightarrow L.$$

For this we take the arrow corresponding to

$$\mathrm{id} : R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, L)$$

via (35.0.1).

Remark 35.12. Suppose that

$$\begin{tikzcd}
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "h"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "f"] \\
(Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})
\end{tikzcd}$$

is a commutative diagram of ringed topoi. Let K, L be objects of $D(\mathcal{O}_\mathcal{C})$. We claim there exists a canonical base change map

$$Lg^*Rf_*R\mathcal{H}om(K, L) \longrightarrow R(f')_*R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_{\mathcal{D}'})$. Namely, we take the map adjoint to the composition

$$\begin{aligned} L(f')^*Lg^*Rf_*R\mathcal{H}om(K, L) &= Lh^*Lf^*Rf_*R\mathcal{H}om(K, L) \\ &\to Lh^*R\mathcal{H}om(K, L) \\ &\to R\mathcal{H}om(Lh^*K, Lh^*L) \end{aligned}$$

where the first arrow uses the adjunction mapping $Lf^*Rf_* \to \mathrm{id}$ and the second arrow is the canonical map constructed in Remark 35.11.

36. Global derived hom

Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $K, L \in D(\mathcal{O})$. Using the construction of the internal hom in the derived category we obtain a well defined object

$$R\operatorname{Hom}_{\mathcal{O}}(K, L) = R\Gamma(\mathcal{C}, R\mathcal{H}om(K, L))$$

in $D(\Gamma(\mathcal{C}, \mathcal{O}))$. By Lemma 35.1 we have

$$H^0(R\operatorname{Hom}_{\mathcal{O}}(K, L)) = \operatorname{Hom}_{D(\mathcal{O})}(K, L)$$

and

$$H^p(R\operatorname{Hom}_{\mathcal{O}}(K, L)) = \operatorname{Ext}^p_{D(\mathcal{O})}(K, L)$$

If $f : (\mathcal{C}', \mathcal{O}') \to (\mathcal{C}, \mathcal{O})$ is a morphism of ringed topoi, then there is a canonical map

$$R\operatorname{Hom}_{\mathcal{O}}(K, L) \longrightarrow R\operatorname{Hom}_{\mathcal{O}'}(Lf^*K, Lf^*L)$$

in $D(\Gamma(\mathcal{O}))$ by taking global sections of the map defined in Remark 35.11.

37. Derived lower shriek

In this section we study morphisms g of ringed topoi where besides Lg^* and Rg_* there also exists a derived functor $Lg_!$.

Lemma 37.1. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and let $\mathcal{I}$ be an injective $\mathcal{O}_\mathcal{D}$-module. Then $H^p(U, g^{-1}\mathcal{I}) = 0$ for all $p > 0$ and $U \in \operatorname{Ob}(\mathcal{C})$.*

Proof. The vanishing of the lemma follows from Lemma 10.9 if we can prove vanishing of all higher Čech cohomology groups $\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I})$ for any covering $\mathcal{U} = \{U_i \to U\}$ of $\mathcal{C}$. Since u is continuous, $u(\mathcal{U}) = \{u(U_i) \to u(U)\}$ is a covering of $\mathcal{D}$, and $u(U_{i_0} \times_U \ldots \times_U U_{i_n}) = u(U_{i_0}) \times_{u(U)} \ldots \times_{u(U)} u(U_{i_n})$. Thus we have

$$\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I}) = \check{H}^p(u(\mathcal{U}), \mathcal{I})$$

because $g^{-1} = u^p$ by Sites, Lemma 21.5. Since $\mathcal{I}$ is an injective $\mathcal{O}_\mathcal{D}$-module these Čech cohomology groups vanish, see Lemma 12.3. □

Lemma 37.2. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and set $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$. The functor $g_! : Mod(\mathcal{O}_\mathcal{C}) \to Mod(\mathcal{O}_\mathcal{D})$ (see Modules on Sites, Lemma 41.1) has a left derived functor*

$$Lg_! : D(\mathcal{O}_\mathcal{C}) \longrightarrow D(\mathcal{O}_\mathcal{D})$$

which is left adjoint to g^*. Moreover, for $U \in \operatorname{Ob}(\mathcal{C})$ we have

$$Lg_!(j_{U!}\mathcal{O}_U) = g_!j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}.$$

where $j_{U!}$ and $j_{u(U)!}$ are extension by zero associated to the localization morphism $j_U : \mathcal{C}/U \to \mathcal{C}$ and $j_{u(U)} : \mathcal{D}/u(U) \to \mathcal{D}$.

Proof. We are going to use Derived Categories, Proposition 29.2 to construct $Lg_!$. To do this we have to verify assumptions (1), (2), (3), (4), and (5) of that proposition. First, since $g_!$ is a left adjoint we see that it is right exact and commutes with all colimits, so (5) holds. Conditions (3) and (4) hold because the category of modules on a ringed site is a Grothendieck abelian category. Let $\mathcal{P} \subset \operatorname{Ob}(Mod(\mathcal{O}_\mathcal{C}))$ be the collection of $\mathcal{O}_\mathcal{C}$-modules which are direct sums of modules of the form

$j_{U!}\mathcal{O}_U$. Note that $g_! j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}$, see proof of Modules on Sites, Lemma 41.1. Every $\mathcal{O}_\mathcal{C}$-module is a quotient of an object of $\mathcal{P}$, see Modules on Sites, Lemma 28.8. Thus (1) holds. Finally, we have to prove (2). Let $\mathcal{K}^\bullet$ be a bounded above acyclic complex of $\mathcal{O}_\mathcal{C}$-modules with $\mathcal{K}^n \in \mathcal{P}$ for all n. We have to show that $g_!\mathcal{K}^\bullet$ is exact. To do this it suffices to show, for every injective $\mathcal{O}_\mathcal{D}$-module $\mathcal{I}$ that

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) = 0$$

for all $n \in \mathbf{Z}$. Since $\mathcal{I}$ is injective we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) &= \mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(g_!\mathcal{K}^\bullet, \mathcal{I})) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I})) \end{aligned}$$

the last equality by the adjointness of $g_!$ and g^{-1}.

The vanishing of this group would be clear if $g^{-1}\mathcal{I}$ were an injective $\mathcal{O}_\mathcal{C}$-module. But $g^{-1}\mathcal{I}$ isn't necessarily an injective $\mathcal{O}_\mathcal{C}$-module as $g_!$ isn't exact in general. We do know that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{I}) = H^p(U, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1$$

Here the first equality follows from $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, \mathcal{H}) = \mathcal{H}(U)$ and taking derived functors and the vanishing of $H^p(U, g^{-1}\mathcal{I})$ for $p > 0$ and $U \in \mathrm{Ob}(\mathcal{C})$ follows from Lemma 37.1. Since each $\mathcal{K}^{-q}$ is a direct sum of modules of the form $j_{U!}\mathcal{O}_U$ we see that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1 \text{ and all } q$$

Let us use the spectral sequence (see Example 32.1)

$$E_1^{p,q} = \mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}) = 0.$$

Note that the spectral sequence abuts to zero as $\mathcal{K}^\bullet$ is acyclic (hence vanishes in the derived category, hence produces vanishing ext groups). By the vanishing of higher exts proved above the only nonzero terms on the E_1 page are the terms $E_1^{0,q} = \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I})$. We conclude that the complex $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I})$ is acyclic as desired.

Thus the left derived functor $Lg_!$ exists. It is left adjoint to $g^{-1} = g^* = Rg^* = Lg^*$, i.e., we have

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{C})}(K, g^*L) = \mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(Lg_!K, L) \tag{37.2.1}$$

by Derived Categories, Lemma 30.3. This finishes the proof. □

Remark 37.3. Warning! Let $u : \mathcal{C} \to \mathcal{D}$, g, $\mathcal{O}_\mathcal{D}$, and $\mathcal{O}_\mathcal{C}$ be as in Lemma 37.2. In general it is **not** the case that the diagram

$$\begin{tikzcd} D(\mathcal{O}_\mathcal{C}) \arrow[r, "Lg_!"'] \arrow[d, "forget"'] & D(\mathcal{O}_\mathcal{D}) \arrow[d, "forget"] \\ D(\mathcal{C}) \arrow[r, "Lg_!^{Ab}"] & D(\mathcal{D}) \end{tikzcd}$$

commutes where the functor $Lg_!^{Ab}$ is the one constructed in Lemma 37.2 but using the constant sheaf $\mathbf{Z}$ as the structure sheaf on both $\mathcal{C}$ and $\mathcal{D}$. In general it isn't even the case that $g_! = g_!^{Ab}$ (see Modules on Sites, Remark 41.2), but this phenomenon

can occur even if $g_! = g_!^{Ab}$! Namely, the construction of $Lg_!$ in the proof of Lemma 37.2 shows that $Lg_!$ agrees with $Lg_!^{Ab}$ if and only if the canonical maps

$$Lg_!^{Ab} j_{U!}\mathcal{O}_U \longrightarrow j_{u(U)!}\mathcal{O}_{u(U)}$$

are isomorphisms in $D(\mathcal{D})$ for all objects U in $\mathcal{C}$. In general all we can say is that there exists a natural transformation

$$Lg_!^{Ab} \circ forget \longrightarrow forget \circ Lg_!$$

Lemma 37.4. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and let $\mathcal{I}$ be an injective $\mathcal{O}_\mathcal{D}$-module. If $g_!^{Sh} : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ commutes with fibre products*[7]*, then $g^{-1}\mathcal{I}$ is totally acyclic.*

Proof. We will use the criterion of Lemma 13.5. Condition (1) holds by Lemma 37.1. Let $K' \to K$ be a surjective map of sheaves of sets on $\mathcal{C}$. Since $g_!^{Sh}$ is a left adjoint, we see that $g_!^{Sh}K' \to g_!^{Sh}K$ is surjective. Observe that

$$\begin{aligned} H^0(K' \times_K \ldots \times_K K', g^{-1}\mathcal{I}) &= H^0(g_!^{Sh}(K' \times_K \ldots \times_K K'), \mathcal{I}) \\ &= H^0(g_!^{Sh}K' \times_{g_!^{Sh}K} \ldots \times_{g_!^{Sh}K} g_!^{Sh}K', \mathcal{I}) \end{aligned}$$

by our assumption on $g_!^{Sh}$. Since $\mathcal{I}$ is an injective module it is totally acyclic by Lemma 14.1 (applied to the identity). Hence we can use the converse of Lemma 13.5 to see that the complex

$$0 \to H^0(K, g^{-1}\mathcal{I}) \to H^0(K', g^{-1}\mathcal{I}) \to H^0(K' \times_K K', g^{-1}\mathcal{I}) \to \ldots$$

is exact as desired. $\square$

Lemma 37.5. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $U \in \mathrm{Ob}(\mathcal{C})$.*

1. *For M in $D(\mathcal{D})$ we have $R\Gamma(U, g^{-1}M) = R\Gamma(u(U), M)$.*
2. *If $\mathcal{O}_\mathcal{D}$ is a sheaf of rings and $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$, then for M in $D(\mathcal{O}_\mathcal{D})$ we have $R\Gamma(U, g^*M) = R\Gamma(u(U), M)$.*

Proof. In the bounded below case (1) and (2) can be seen by representing K by a bounded below complex of injectives and using Lemma 37.1 as well as Leray's acyclicity lemma. In the unbounded case, first note that (1) is a special case of (2). For (2) we can use

$$R\Gamma(U, g^*M) = R\operatorname{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^*M) = R\operatorname{Hom}_{\mathcal{O}_\mathcal{D}}(j_{u(U)!}\mathcal{O}_{u(U)}, M) = R\Gamma(u(U), M)$$

where the middle equality is a consequence of Lemma 37.2. $\square$

Lemma 37.6. *Assume given a commutative diagram*

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "(g', (g')^\sharp)"'] \arrow[d, "(f', (f')^\sharp)"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "(f, f^\sharp)"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "(g, g^\sharp)"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{tikzcd}$$

of ringed topoi. Assume

1. *f, f', g, and g' correspond to cocontinuous functors u, u', v, and v' as in Sites, Lemma 21.1,*

[7]Holds if $\mathcal{C}$ has finite connected limits and u commutes with them, see Sites, Lemma 21.6.

(2) $v \circ u' = u \circ v'$,
(3) *v and v' are continuous as well as cocontinuous,*
(4) *for any object V' of $\mathcal{D}'$ the functor ${}_{V'}^{u'}\mathcal{I} \to {}_{v(V')}^{u}\mathcal{I}$ given by v is cofinal,*
(5) $g^{-1}\mathcal{O}_\mathcal{D} = \mathcal{O}_{\mathcal{D}'}$ *and* $(g')^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$, *and*
(6) $g'_! : Ab(\mathcal{C}') \to Ab(\mathcal{C})$ *is exact*[8].

Then we have $Rf'_* \circ (g')^* = g^* \circ Rf_*$ as functors $D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_{\mathcal{D}'})$.

Proof. We have $g^* = Lg^* = g^{-1}$ and $(g')^* = L(g')^* = (g')^{-1}$ by condition (5). By Lemma 20.7 it suffices to prove the result on the derived category $D(\mathcal{C})$ of abelian sheaves. Choose an object $K \in D(\mathcal{C})$. Let $\mathcal{I}^\bullet$ be a K-injective complex of abelian sheaves on $\mathcal{C}$ representing K. By Derived Categories, Lemma 31.9 and assumption (6) we find that $(g')^{-1}\mathcal{I}^\bullet$ is a K-injective complex of abelian sheaves on $\mathcal{C}'$. By Modules on Sites, Lemma 41.3 we find that $f'_*(g')^{-1}\mathcal{I}^\bullet = g^{-1}f_*\mathcal{I}^\bullet$. Since $f_*\mathcal{I}^\bullet$ represents Rf_*K and since $f'_*(g')^{-1}\mathcal{I}^\bullet$ represents $Rf'_*(g')^{-1}K$ we conclude. $\square$

Lemma 37.7. *Consider a commutative diagram*

$$\begin{tikzcd}
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "{(g', (g')^\sharp)}"'] \arrow[d, "{(f', (f')^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "{(f, f^\sharp)}"] \\
(Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "{(g, g^\sharp)}"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})
\end{tikzcd}$$

of ringed topoi and suppose we have functors

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "v'"'] & \mathcal{C} \\
\mathcal{D}' \arrow[r, "v"] \arrow[u, "u'"] & \mathcal{D} \arrow[u, "u"']
\end{tikzcd}$$

such that (with notation as in Sites, Sections 14 and 21) we have

(1) *u and u' are continuous and give rise to the morphisms f and f',*
(2) *v and v' are cocontinuous giving rise to the morphisms g and g',*
(3) $u \circ v = v' \circ u'$,
(4) *v and v' are continuous as well as cocontinuous, and*
(5) $g^{-1}\mathcal{O}_\mathcal{D} = \mathcal{O}_{\mathcal{D}'}$ *and* $(g')^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$.

Then $Rf'_* \circ (g')^* = g^* \circ Rf_*$ as functors $D^+(\mathcal{O}_\mathcal{C}) \to D^+(\mathcal{O}_{\mathcal{D}'})$. If in addition

(6) $g'_! : Ab(\mathcal{C}') \to Ab(\mathcal{C})$ *is exact*[9],

then $Rf'_* \circ (g')^* = g^* \circ Rf_*$ as functors $D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_{\mathcal{D}'})$.

Proof. We have $g^* = Lg^* = g^{-1}$ and $(g')^* = L(g')^* = (g')^{-1}$ by condition (5). By Lemma 20.7 it suffices to prove the result on the derived category $D^+(\mathcal{C})$ or $D(\mathcal{C})$ of abelian sheaves.

Choose an object $K \in D^+(\mathcal{C})$. Let $\mathcal{I}^\bullet$ be a bounded below complex of injective abelian sheaves on $\mathcal{C}$ representing K. By Lemma 37.1 we see that $H^p(U', (g')^{-1}\mathcal{I}^q) = 0$ for all $p > 0$ and any q and any $U' \in \text{Ob}(\mathcal{C}')$. Recall that $R^pf'_*(g')^{-1}\mathcal{I}^q$ is the sheaf associated to the presheaf $V' \mapsto H^p(u'(V'), (g')^{-1}\mathcal{I}^q)$, see Lemma 7.4. Thus we see

[8]Holds if fibre products and equalizers exist in $\mathcal{C}'$ and v' commutes with them, see Modules on Sites, Lemma 16.3.

[9]Holds if fibre products and equalizers exist in $\mathcal{C}'$ and v' commutes with them, see Modules on Sites, Lemma 16.3.

that $(g')^{-1}\mathcal{I}^q$ is right acyclic for the functor f'_*. By Leray's acyclicity lemma (Derived Categories, Lemma 16.7) we find that $f'_*(g')^*\mathcal{I}^\bullet$ represents $Rf'_*(g')^{-1}K$. By Modules on Sites, Lemma 41.4 we find that $f'_*(g')^{-1}\mathcal{I}^\bullet = g^{-1}f_*\mathcal{I}^\bullet$. Since $g^{-1}f_*\mathcal{I}^\bullet$ represents $g^{-1}Rf_*K$ we conclude.

Choose an object $K \in D(\mathcal{C})$. Let $\mathcal{I}^\bullet$ be a K-injective complex of abelian sheaves on $\mathcal{C}$ representing K. By Derived Categories, Lemma 31.9 and assumption (6) we find that $(g')^{-1}\mathcal{I}^\bullet$ is a K-injective complex of abelian sheaves on $\mathcal{C}'$. By Modules on Sites, Lemma 41.4 we find that $f'_*(g')^{-1}\mathcal{I}^\bullet = g^{-1}f_*\mathcal{I}^\bullet$. Since $f_*\mathcal{I}^\bullet$ represents Rf_*K and since $f'_*(g')^{-1}\mathcal{I}^\bullet$ represents $Rf'_*(g')^{-1}K$ we conclude. □

38. Derived lower shriek for fibred categories

In this section we work out some special cases of the situation discussed in Section 37. We make sure that we have equality between lower shriek on modules and sheaves of abelian groups. We encourage the reader to skip this section on a first reading.

Situation 38.1. Here $(\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a ringed site and $p : \mathcal{C} \to \mathcal{D}$ is a fibred category. We endow $\mathcal{C}$ with the topology inherited from $\mathcal{D}$ (Stacks, Section 10). We denote $\pi : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ the morphism of topoi associated to p (Stacks, Lemma 10.3). We set $\mathcal{O}_\mathcal{C} = \pi^{-1}\mathcal{O}_\mathcal{D}$ so that we obtain a morphism of ringed topoi

$$\pi : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$$

Lemma 38.2. *Assumptions and notation as in Situation 38.1. For $U \in \mathrm{Ob}(\mathcal{C})$ consider the induced morphism of topoi*

$$\pi_U : Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{D}/p(U))$$

Then there exists a morphism of topoi

$$\sigma : Sh(\mathcal{D}/p(U)) \to Sh(\mathcal{C}/U)$$

such that $\pi_U \circ \sigma = id$ and $\sigma^{-1} = \pi_{U,*}$.

Proof. Observe that π_U is the restriction of π to the localizations, see Sites, Lemma 28.4. For an object $V \to p(U)$ of $\mathcal{D}/p(U)$ denote $V \times_{p(U)} U \to U$ the strongly cartesian morphism of $\mathcal{C}$ over $\mathcal{D}$ which exists as p is a fibred category. The functor

$$v : \mathcal{D}/p(U) \to \mathcal{C}/U, \quad V/p(U) \mapsto V \times_{p(U)} U/U$$

is continuous by the definition of the topology on $\mathcal{C}$. Moreover, it is a right adjoint to p by the definition of strongly cartesian morphisms. Hence we are in the situation discussed in Sites, Section 22 and we see that the sheaf $\pi_{U,*}\mathcal{F}$ is equal to $V \mapsto \mathcal{F}(V \times_{p(U)} U)$ (see especially Sites, Lemma 22.2).

But here we have more. Namely, the functor v is also cocontinuous (as all morphisms in coverings of $\mathcal{C}$ are strongly cartesian). Hence v defines a morphism σ as indicated in the lemma. The equality $\sigma^{-1} = \pi_{U,*}$ is immediate from the definition. Since $\pi_U^{-1}\mathcal{G}$ is given by the rule $U'/U \mapsto \mathcal{G}(p(U')/p(U))$ it follows that $\sigma^{-1} \circ \pi_U^{-1} = \mathrm{id}$ which proves the equality $\pi_U \circ \sigma = \mathrm{id}$. □

Situation 38.3. Let $(\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a ringed site. Let $u : \mathcal{C}' \to \mathcal{C}$ be a 1-morphism of fibred categories over $\mathcal{D}$ (Categories, Definition 33.9). Endow $\mathcal{C}$ and $\mathcal{C}'$ with their inherited topologies (Stacks, Definition 10.2) and let $\pi : Sh(\mathcal{C}) \to Sh(\mathcal{D})$, $\pi' : Sh(\mathcal{C}') \to Sh(\mathcal{D})$, and $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ be the corresponding morphisms of topoi (Stacks, Lemma 10.3). Set $\mathcal{O}_\mathcal{C} = \pi^{-1}\mathcal{O}_\mathcal{D}$ and $\mathcal{O}_{\mathcal{C}'} = (\pi')^{-1}\mathcal{O}_\mathcal{D}$. Observe that $g^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$ so that

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[rr, "g"'] \arrow[dr, "\pi'"'] & & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[dl, "\pi"] \\ & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) & \end{tikzcd}$$

is a commutative diagram of morphisms of ringed topoi.

Lemma 38.4. *Assumptions and notation as in Situation 38.3. For $U' \in \mathrm{Ob}(\mathcal{C}')$ set $U = u(U')$ and $V = p'(U')$ and consider the induced morphisms of ringed topoi*

$$\begin{tikzcd} (Sh(\mathcal{C}'/U'), \mathcal{O}_{U'}) \arrow[rr, "g'"'] \arrow[dr, "\pi'_{U'}"'] & & (Sh(\mathcal{C}), \mathcal{O}_U) \arrow[dl, "\pi_U"] \\ & (Sh(\mathcal{D}/V), \mathcal{O}_V) & \end{tikzcd}$$

Then there exists a morphism of topoi

$$\sigma' : Sh(\mathcal{D}/V) \to Sh(\mathcal{C}'/U'),$$

such that setting $\sigma = g' \circ \sigma'$ we have $\pi'_{U'} \circ \sigma' = id$, $\pi_U \circ \sigma = id$, $(\sigma')^{-1} = \pi'_{U',*}$, and $\sigma^{-1} = \pi_{U,*}$.

Proof. Let $v' : \mathcal{D}/V \to \mathcal{C}'/U'$ be the functor constructed in the proof of Lemma 38.2 starting with $p' : \mathcal{C}' \to \mathcal{D}'$ and the object U'. Since u is a 1-morphism of fibred categories over $\mathcal{D}$ it transforms strongly cartesian morphisms into strongly cartesian morphisms, hence the functor $v = u \circ v'$ is the functor of the proof of Lemma 38.2 relative to $p : \mathcal{C} \to \mathcal{D}$ and U. Thus our lemma follows from that lemma. $\square$

Lemma 38.5. *Assumption and notation as in Situation 38.3.*

(1) *There are left adjoints $g_! : Mod(\mathcal{O}_{\mathcal{C}'}) \to Mod(\mathcal{O}_\mathcal{C})$ and $g_!^{Ab} : Ab(\mathcal{C}') \to Ab(\mathcal{C})$ to $g^* = g^{-1}$ on modules and on abelian sheaves.*

(2) *The diagram*

$$\begin{tikzcd} Mod(\mathcal{O}_{\mathcal{C}'}) \arrow[r, "g_!"'] \arrow[d] & Mod(\mathcal{O}_\mathcal{C}) \arrow[d] \\ Ab(\mathcal{C}') \arrow[r, "g_!^{Ab}"] & Ab(\mathcal{C}) \end{tikzcd}$$

commutes.

(3) *There are left adjoints $Lg_! : D(\mathcal{O}_{\mathcal{C}'}) \to D(\mathcal{O}_\mathcal{C})$ and $Lg_!^{Ab} : D(\mathcal{C}') \to D(\mathcal{C})$ to $g^* = g^{-1}$ on derived categories of modules and abelian sheaves.*

(4) *The diagram*

$$\begin{tikzcd} D(\mathcal{O}_{\mathcal{C}'}) \arrow[r, "Lg_!"'] \arrow[d] & D(\mathcal{O}_\mathcal{C}) \arrow[d] \\ D(\mathcal{C}') \arrow[r, "Lg_!^{Ab}"] & D(\mathcal{C}) \end{tikzcd}$$

commutes.

Proof. The functor u is continuous and cocontinuous Stacks, Lemma 10.3. Hence the existence of the functors $g_!$, $g_!^{Ab}$, $Lg_!$, and $Lg_!^{Ab}$ can be found in Modules on Sites, Sections 16 and 41 and Section 37.

To prove (2) it suffices to show that the canonical map

$$g_!^{Ab} j_{U'!}\mathcal{O}_{U'} \to j_{u(U')!}\mathcal{O}_{u(U')}$$

is an isomorphism for all objects U' of $\mathcal{C}'$, see Modules on Sites, Remark 41.2. Similarly, to prove (4) it suffices to show that the canonical map

$$Lg_!^{Ab} j_{U'!}\mathcal{O}_{U'} \to j_{u(U')!}\mathcal{O}_{u(U')}$$

is an isomorphism in $D(\mathcal{C})$ for all objects U' of $\mathcal{C}'$, see Remark 37.3. This will also imply the previous formula hence this is what we will show.

We will use that for a localization morphism j the functors $j_!$ and $j_!^{Ab}$ agree (see Modules on Sites, Remark 19.6) and that $j_!$ is exact (Modules on Sites, Lemma 19.3). Let us adopt the notation of Lemma 38.4. Since $Lg_!^{Ab} \circ j_{U'!} = j_{U!} \circ L(g')_!^{Ab}$ (by commutativity of Sites, Lemma 28.4 and uniqueness of adjoint functors) it suffices to prove that $L(g')_!^{Ab}\mathcal{O}_{U'} = \mathcal{O}_U$. Using the results of Lemma 38.4 we have for any object E of $D(\mathcal{C}/u(U'))$ the following sequence of equalities

$$\begin{aligned}
\mathrm{Hom}_{D(\mathcal{C}/U)}(L(g')_!^{Ab}\mathcal{O}_{U'}, E) & = \mathrm{Hom}_{D(\mathcal{C}'/U')}(\mathcal{O}_{U'}, (g')^{-1}E) \\
& = \mathrm{Hom}_{D(\mathcal{C}'/U')}((\pi'_{U'})^{-1}\mathcal{O}_V, (g')^{-1}E) \\
& = \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, R\pi'_{U',*}(g')^{-1}E) \\
& = \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, (\sigma')^{-1}(g')^{-1}E) \\
& = \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \sigma^{-1}E) \\
& = \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \pi_{U,*}E) \\
& = \mathrm{Hom}_{D(\mathcal{C}/U)}(\pi_U^{-1}\mathcal{O}_V, E) \\
& = \mathrm{Hom}_{D(\mathcal{C}/U)}(\mathcal{O}_U, E)
\end{aligned}$$

By Yoneda's lemma we conclude. $\square$

Remark 38.6. Assumptions and notation as in Situation 38.1. Note that setting $\mathcal{C}' = \mathcal{D}$ and u equal to the structure functor of $\mathcal{C}$ gives a situation as in Situation 38.3. Hence Lemma 38.5 tells us we have functors $\pi_!$, $\pi_!^{Ab}$, $L\pi_!$, and $L\pi_!^{Ab}$ such that $forget \circ \pi_! = \pi_!^{Ab} \circ forget$ and $forget \circ L\pi_! = L\pi_!^{Ab} \circ forget$.

Remark 38.7. Assumptions and notation as in Situation 38.3. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$, let $\mathcal{F}'$ be an abelian sheaf on $\mathcal{C}'$, and let $t : \mathcal{F}' \to g^{-1}\mathcal{F}$ be a map. Then we obtain a canonical map

$$L\pi'_!(\mathcal{F}') \longrightarrow L\pi_!(\mathcal{F})$$

by using the adjoint $g_!\mathcal{F}' \to \mathcal{F}$ of t, the map $Lg_!(\mathcal{F}') \to g_!\mathcal{F}'$, and the equality $L\pi'_! = L\pi_! \circ Lg_!$.

Lemma 38.8. *Assumptions and notation as in Situation 38.1. For $\mathcal{F}$ in $Ab(\mathcal{C})$ the sheaf $\pi_!\mathcal{F}$ is the sheaf associated to the presheaf*

$$V \longmapsto \mathrm{colim}_{\mathcal{C}_V^{opp}} \mathcal{F}|_{\mathcal{C}_V}$$

with restriction maps as indicated in the proof.

Proof. Denote $\mathcal{H}$ be the rule of the lemma. For a morphism $h : V' \to V$ of $\mathcal{D}$ there is a pullback functor $h^* : \mathcal{C}_V \to \mathcal{C}_{V'}$ of fibre categories (Categories, Definition 33.6). Moreover for $U \in \mathrm{Ob}(\mathcal{C}_V)$ there is a strongly cartesian morphism $h^*U \to U$ covering h. Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \longrightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

Hence a map $\mathcal{H}(V) \to \mathcal{H}(V')$ between colimits, see Categories, Lemma 14.8.

To prove the lemma we show that

$$\mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{H}, \mathcal{G}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \pi^{-1}\mathcal{G})$$

for every sheaf $\mathcal{G}$ on $\mathcal{C}$. An element of the left hand side is a compatible system of maps $\mathcal{F}(U) \to \mathcal{G}(p(U))$ for all U in $\mathcal{C}$. Since $\pi^{-1}\mathcal{G}(U) = \mathcal{G}(p(U))$ by our choice of topology on $\mathcal{C}$ we see the same thing is true for the right hand side and we win. $\square$

39. Homology on a category

In the case of a category over a point we will baptize the left derived lower shriek functors the homology functors.

Example 39.1 (Category over point)**.** Let $\mathcal{C}$ be a category. Endow $\mathcal{C}$ with the chaotic topology (Sites, Example 6.6). Thus presheaves and sheaves agree on $\mathcal{C}$. The functor $p : \mathcal{C} \to *$ where $*$ is the category with a single object and a single morphism is cocontinuous and continuous. Let $\pi : Sh(\mathcal{C}) \to Sh(*)$ be the corresponding morphism of topoi. Let B be a ring. We endow $*$ with the sheaf of rings B and $\mathcal{C}$ with $\mathcal{O}_\mathcal{C} = \pi^{-1}B$ which we will denote $\underline{B}$. In this way

$$\pi : (Sh(\mathcal{C}), \underline{B}) \to (Sh(*), B)$$

is an example of Situation 38.1. By Remark 38.6 we do not need to distinguish between $\pi_!$ on modules or abelian sheaves. By Lemma 38.8 we see that $\pi_!\mathcal{F} = \mathrm{colim}_{\mathcal{C}^{opp}} \mathcal{F}$. Thus $L_n\pi_!$ is the nth left derived functor of taking colimits. In the following, we write

$$H_n(\mathcal{C}, \mathcal{F}) = L_n\pi_!(\mathcal{F})$$

and we will name this the *nth homology group of* $\mathcal{F}$ on $\mathcal{C}$.

Example 39.2 (Computing homology)**.** In Example 39.1 we can compute the functors $H_n(\mathcal{C}, -)$ as follows. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Consider the chain complex

$$K_\bullet(\mathcal{F}) : \ \ldots \to \bigoplus\nolimits_{U_2 \to U_1 \to U_0} \mathcal{F}(U_0) \to \bigoplus\nolimits_{U_1 \to U_0} \mathcal{F}(U_0) \to \bigoplus\nolimits_{U_0} \mathcal{F}(U_0)$$

where the transition maps are given by

$$(U_2 \to U_1 \to U_0, s) \longmapsto (U_1 \to U_0, s) - (U_2 \to U_0, s) + (U_2 \to U_1, s|_{U_1})$$

and similarly in other degrees. By construction

$$H_0(\mathcal{C}, \mathcal{F}) = \mathrm{colim}_{\mathcal{C}^{opp}} \mathcal{F} = H_0(K_\bullet(\mathcal{F})),$$

see Categories, Lemma 14.12. The construction of $K_\bullet(\mathcal{F})$ is functorial in $\mathcal{F}$ and transforms short exact sequences of $Ab(\mathcal{C})$ into short exact sequences of complexes. Thus the sequence of functors $\mathcal{F} \mapsto H_n(K_\bullet(\mathcal{F}))$ forms a δ-functor, see Homology, Definition 12.1 and Lemma 13.12. For $\mathcal{F} = j_{U!}\mathbf{Z}_U$ the complex $K_\bullet(\mathcal{F})$ is the complex associated to the free $\mathbf{Z}$-module on the simplicial set $X_\bullet$ with terms

$$X_n = \coprod\nolimits_{U_n \to \ldots \to U_1 \to U_0} \mathrm{Mor}_\mathcal{C}(U_0, U)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$. Namely, the map $X_\bullet \to \{*\}$ is obvious, the map $\{*\} \to X_n$ is given by mapping $*$ to $(U \to \ldots \to U, \mathrm{id}_U)$, and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 26.2) defining the homotopy between the two maps $X_\bullet \to X_\bullet$ are given by the rule

$$h_{n,i} : (U_n \to \ldots \to U_0, f) \longmapsto (U_n \to \ldots \to U_i \to U \to \ldots \to U, \mathrm{id})$$

for $i > 0$ and $h_{n,0} = \mathrm{id}$. Verifications omitted. This implies that $K_\bullet(j_{U!}\mathbf{Z}_U)$ has trivial cohomology in negative degrees (by the functoriality of Simplicial, Remark 26.4 and the result of Simplicial, Lemma 27.1). Thus $K_\bullet(\mathcal{F})$ computes the left derived functors $H_n(\mathcal{C}, -)$ of $H_0(\mathcal{C}, -)$ for example by (the duals of) Homology, Lemma 12.4 and Derived Categories, Lemma 16.6.

Example 39.3. Let $u : \mathcal{C}' \to \mathcal{C}$ be a functor. Endow $\mathcal{C}'$ and $\mathcal{C}$ with the chaotic topology as in Example 39.1. The functors u, $\mathcal{C}' \to *$, and $\mathcal{C} \to *$ where $*$ is the category with a single object and a single morphism are cocontinuous and continuous. Let $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$, $\pi' : Sh(\mathcal{C}') \to Sh(*)$, and $\pi : Sh(\mathcal{C}) \to Sh(*)$, be the corresponding morphisms of topoi. Let B be a ring. We endow $*$ with the sheaf of rings B and $\mathcal{C}'$, $\mathcal{C}$ with the constant sheaf $\underline{B}$. In this way

$$\begin{tikzcd} (Sh(\mathcal{C}'), \underline{B}) \arrow[rr, "g"'] \arrow[dr, "\pi'"'] & & (Sh(\mathcal{C}), \underline{B}) \arrow[dl, "\pi"] \\ & (Sh(*), B) & \end{tikzcd}$$

is an example of Situation 38.3. Thus Lemma 38.5 applies to g so we do not need to distinguish between $g_!$ on modules or abelian sheaves. In particular Remark 38.7 produces canonical maps

$$H_n(\mathcal{C}', \mathcal{F}') \longrightarrow H_n(\mathcal{C}, \mathcal{F})$$

whenever we have $\mathcal{F}$ in $Ab(\mathcal{C})$, $\mathcal{F}'$ in $Ab(\mathcal{C}')$, and a map $t : \mathcal{F}' \to g^{-1}\mathcal{F}$. In terms of the computation of homology given in Example 39.2 we see that these maps come from a map of complexes

$$K_\bullet(\mathcal{F}') \longrightarrow K_\bullet(\mathcal{F})$$

given by the rule

$$(U'_n \to \ldots \to U'_0, s') \longmapsto (u(U'_n) \to \ldots \to u(U'_0), t(s'))$$

with obvious notation.

Remark 39.4. Notation and assumptions as in Example 39.1. Let $\mathcal{F}^\bullet$ be a bounded complex of abelian sheaves on $\mathcal{C}$. For any object U of $\mathcal{C}$ there is a canonical map

$$\mathcal{F}^\bullet(U) \longrightarrow L\pi_!(\mathcal{F}^\bullet)$$

in $D(Ab)$. If $\mathcal{F}^\bullet$ is a complex of $\underline{B}$-modules then this map is in $D(B)$. To prove this, note that we compute $L\pi_!(\mathcal{F}^\bullet)$ by taking a quasi-isomorphism $\mathcal{P}^\bullet \to \mathcal{F}^\bullet$ where $\mathcal{P}^\bullet$ is a complex of projectives. However, since the topology is chaotic this means that $\mathcal{P}^\bullet(U) \to \mathcal{F}^\bullet(U)$ is a quasi-isomorphism hence can be inverted in $D(Ab)$, resp. $D(B)$. Composing with the canonical map $\mathcal{P}^\bullet(U) \to \pi_!(\mathcal{P}^\bullet)$ coming from the computation of $\pi_!$ as a colimit we obtain the desired arrow.

Lemma 39.5. *Notation and assumptions as in Example 39.1. If $\mathcal{C}$ has either an initial or a final object, then $L\pi_! \circ \pi^{-1} = id$ on $D(Ab)$, resp. $D(B)$.*

Proof. If $\mathcal{C}$ has an initial object, then $\pi_!$ is computed by evaluating on this object and the statement is clear. If $\mathcal{C}$ has a final object, then $R\pi_*$ is computed by evaluating on this object, hence $R\pi_* \circ \pi^{-1} \cong \text{id}$ on $D(Ab)$, resp. $D(B)$. This implies that $\pi^{-1} : D(Ab) \to D(\mathcal{C})$, resp. $\pi^{-1} : D(B) \to D(\underline{B})$ is fully faithful, see Categories, Lemma 24.4. Then the same lemma implies that $L\pi_! \circ \pi^{-1} = \text{id}$ as desired. $\square$

Lemma 39.6. *Notation and assumptions as in Example 39.1. Let $B \to B'$ be a ring map. Consider the commutative diagram of ringed topoi*

$$\begin{tikzcd} (Sh(\mathcal{C}), \underline{B}) \arrow[d, "\pi"'] & (Sh(\mathcal{C}), \underline{B'}) \arrow[l, "h"'] \arrow[d, "\pi'"] \\ (*, B) & (*, B') \arrow[l, "f"] \end{tikzcd}$$

Then $L\pi_! \circ Lh^* = Lf^* \circ L\pi'_!$.

Proof. Both functors are right adjoint to the obvious functor $D(B') \to D(\underline{B})$. $\square$

Lemma 39.7. *Notation and assumptions as in Example 39.1. Let $U_\bullet$ be a cosimplicial object in $\mathcal{C}$ such that for every $U \in \text{Ob}(\mathcal{C})$ the simplicial set $\text{Mor}_\mathcal{C}(U_\bullet, U)$ is homotopy equivalent to the constant simplicial set on a singleton. Then*

$$L\pi_!(\mathcal{F}) = \mathcal{F}(U_\bullet)$$

in $D(Ab)$, resp. $D(B)$ functorially in $\mathcal{F}$ in $Ab(\mathcal{C})$, resp. $Mod(\underline{B})$.

Proof. As $L\pi_!$ agrees for modules and abelian sheaves by Lemma 38.5 it suffices to prove this when $\mathcal{F}$ is an abelian sheaf. For $U \in \text{Ob}(\mathcal{C})$ the abelian sheaf $j_{U!}\mathbf{Z}_U$ is a projective object of $Ab(\mathcal{C})$ since $\text{Hom}(j_{U!}\mathbf{Z}_U, \mathcal{F}) = \mathcal{F}(U)$ and taking sections is an exact functor as the topology is chaotic. Every abelian sheaf is a quotient of a direct sum of $j_{U!}\mathbf{Z}_U$ by Modules on Sites, Lemma 28.8. Thus we can compute $L\pi_!(\mathcal{F})$ by choosing a resolution

$$\ldots \to \mathcal{G}^{-1} \to \mathcal{G}^0 \to \mathcal{F} \to 0$$

whose terms are direct sums of sheaves of the form above and taking $L\pi_!(\mathcal{F}) = \pi_!(\mathcal{G}^\bullet)$. Consider the double complex $A^{\bullet, \bullet} = \mathcal{G}^\bullet(U_\bullet)$. The map $\mathcal{G}^0 \to \mathcal{F}$ gives a map of complexes $A^{0,\bullet} \to \mathcal{F}(U_\bullet)$. Since $\pi_!$ is computed by taking the colimit over $\mathcal{C}^{opp}$ (Lemma 38.8) we see that the two compositions $\mathcal{G}^m(U_1) \to \mathcal{G}^m(U_0) \to \pi_!\mathcal{G}^m$ are equal. Thus we obtain a canonical map of complexes

$$\text{Tot}(A^{\bullet,\bullet}) \longrightarrow \pi_!(\mathcal{G}^\bullet) = L\pi_!(\mathcal{F})$$

To prove the lemma it suffices to show that the complexes

$$\ldots \to \mathcal{G}^m(U_1) \to \mathcal{G}^m(U_0) \to \pi_!\mathcal{G}^m \to 0$$

are exact, see Homology, Lemma 25.4. Since the sheaves $\mathcal{G}^m$ are direct sums of the sheaves $j_{U!}\mathbf{Z}_U$ we reduce to $\mathcal{G} = j_{U!}\mathbf{Z}_U$. The complex $j_{U!}\mathbf{Z}_U(U_\bullet)$ is the complex of abelian groups associated to the free $\mathbf{Z}$-module on the simplicial set $\text{Mor}_\mathcal{C}(U_\bullet, U)$ which we assumed to be homotopy equivalent to a singleton. We conclude that

$$j_{U!}\mathbf{Z}_U(U_\bullet) \to \mathbf{Z}$$

is a homotopy equivalence of abelian groups hence a quasi-isomorphism (Simplicial, Remark 26.4 and Lemma 27.1). This finishes the proof since $\pi_! j_{U!}\mathbf{Z}_U = \mathbf{Z}$ as was shown in the proof of Lemma 38.5. $\square$

Lemma 39.8. *Notation and assumptions as in Example 39.3. If there exists a cosimplicial object $U'_\bullet$ of $\mathcal{C}'$ such that Lemma 39.7 applies to both $U'_\bullet$ in $\mathcal{C}'$ and $u(U'_\bullet)$ in $\mathcal{C}$, then we have $L\pi'_! \circ g^{-1} = L\pi_!$ as functors $D(\mathcal{C}) \to D(Ab)$, resp. $D(\mathcal{C}, \underline{B}) \to D(B)$.*

Proof. Follows immediately from Lemma 39.7 and the fact that g^{-1} is given by precomposing with u. $\square$

Lemma 39.9. *Let $\mathcal{C}_i$, $i = 1, 2$ be categories. Let $u_i : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_i$ be the projection functors. Let B be a ring. Let $g_i : (Sh(\mathcal{C}_1 \times \mathcal{C}_2), \underline{B}) \to (Sh(\mathcal{C}_i), \underline{B})$ be the corresponding morphisms of ringed topoi, see Example 39.3. For $K_i \in D(\mathcal{C}_i, B)$ we have*

$$L(\pi_1 \times \pi_2)_!(g_1^{-1}K_1 \otimes_{\underline{B}}^{\mathbf{L}} g_2^{-1}K_2) = L\pi_{1,!}(K_1) \otimes_B^{\mathbf{L}} L\pi_{2,!}(K_2)$$

in $D(B)$ with obvious notation.

Proof. As both sides commute with colimits, it suffices to prove this for $K_1 = j_{U!}\underline{B}_U$ and $K_2 = j_{V!}\underline{B}_V$ for $U \in \mathrm{Ob}(\mathcal{C}_1)$ and $V \in \mathrm{Ob}(\mathcal{C}_2)$. See construction of $L\pi_!$ in Lemma 37.2. In this case

$$g_1^{-1}K_1 \otimes_{\underline{B}}^{\mathbf{L}} g_2^{-1}K_2 = g_1^{-1}K_1 \otimes_{\underline{B}} g_2^{-1}K_2 = j_{(U,V)!}\underline{B}_{(U,V)}$$

Verification omitted. Hence the result follows as both the left and the right hand side of the formula of the lemma evaluate to B, see construction of $L\pi_!$ in Lemma 37.2. $\square$

Lemma 39.10. *Notation and assumptions as in Example 39.1. If there exists a cosimplicial object $U_\bullet$ of $\mathcal{C}$ such that Lemma 39.7 applies, then*

$$L\pi_!(K_1 \otimes_{\underline{B}}^{\mathbf{L}} K_2) = L\pi_!(K_1) \otimes_B^{\mathbf{L}} L\pi_!(K_2)$$

for all $K_i \in D(\underline{B})$.

Proof. Consider the diagram of categories and functors

$$\begin{tikzcd} & & \mathcal{C} \\ \mathcal{C} \arrow[r, "u"] & \mathcal{C} \times \mathcal{C} \arrow[ur, "u_1"] \arrow[dr, "u_2"'] & \\ & & \mathcal{C} \end{tikzcd}$$

where u is the diagonal functor and u_i are the projection functors. This gives morphisms of ringed topoi g, g_1, g_2. For any object (U_1, U_2) of $\mathcal{C}$ we have

$$\mathrm{Mor}_{\mathcal{C}\times\mathcal{C}}(u(U_\bullet), (U_1, U_2)) = \mathrm{Mor}_\mathcal{C}(U_\bullet, U_1) \times \mathrm{Mor}_\mathcal{C}(U_\bullet, U_2)$$

which is homotopy equivalent to a point by Simplicial, Lemma 26.10. Thus Lemma 39.8 gives $L\pi_!(g^{-1}K) = L(\pi \times \pi)_!(K)$ for any K in $D(\mathcal{C} \times \mathcal{C}, B)$. Take $K = g_1^{-1}K_1 \otimes_B^{\mathbf{L}} g_2^{-1}K_2$. Then $g^{-1}K = K_1 \otimes_{\underline{B}}^{\mathbf{L}} K_2$ because $g^{-1} = g^* = Lg^*$ commutes with derived tensor product (Lemma 18.4). To finish we apply Lemma 39.9. $\square$

Remark 39.11 (Simplicial modules)**.** Let $\mathcal{C} = \Delta$ and let B be any ring. This is a special case of Example 39.1 where the assumptions of Lemma 39.7 hold. Namely, let $U_\bullet$ be the cosimplicial object of Δ given by the identity functor. To verify the condition we have to show that for $[m] \in \text{Ob}(\Delta)$ the simplicial set $\Delta[m] : n \mapsto \text{Mor}_\Delta([n], [m])$ is homotopy equivalent to a point. This is explained in Simplicial, Example 26.7.

In this situation the category $\textit{Mod}(\underline{B})$ is just the category of simplicial B-modules and the functor $L\pi_!$ sends a simplicial B-module $M_\bullet$ to its associated complex $s(M_\bullet)$ of B-modules. Thus the results above can be reinterpreted in terms of results on simplicial modules. For example a special case of Lemma 39.10 is: if $M_\bullet$, $M'_\bullet$ are flat simplicial B-modules, then the complex $s(M_\bullet \otimes_B M'_\bullet)$ is quasi-isomorphic to the total complex associated to the double complex $s(M_\bullet) \otimes_B s(M'_\bullet)$. (Hint: use flatness to convert from derived tensor products to usual tensor products.) This is a special case of the Eilenberg-Zilber theorem which can be found in [EZ53].

Lemma 39.12. *Let $\mathcal{C}$ be a category (endowed with chaotic topology). Let $\mathcal{O} \to \mathcal{O}'$ be a map of sheaves of rings on $\mathcal{C}$. Assume*

1. *there exists a cosimplicial object $U_\bullet$ in $\mathcal{C}$ as in Lemma 39.7, and*
2. *$L\pi_!\mathcal{O} \to L\pi_!\mathcal{O}'$ is an isomorphism.*

For K in $D(\mathcal{O})$ we have

$$L\pi_!(K) = L\pi_!(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}')$$

in $D(Ab)$.

Proof. Note: in this proof $L\pi_!$ denotes the left derived functor of $\pi_!$ on abelian sheaves. Since $L\pi_!$ commutes with colimits, it suffices to prove this for bounded above complexes of $\mathcal{O}$-modules (compare with argument of Derived Categories, Proposition 29.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are direct sums of $j_{U!}\mathcal{O}_U$ with $U \in \text{Ob}(\mathcal{C})$, see Modules on Sites, Lemma 28.8. Thus it suffices to prove the lemma for $j_{U!}\mathcal{O}_U$. By assumption

$$S_\bullet = \text{Mor}_\mathcal{C}(U_\bullet, U)$$

is a simplicial set homotopy equivalent to the constant simplicial set on a singleton. Set $P_n = \mathcal{O}(U_n)$ and $P'_n = \mathcal{O}'(U_n)$. Observe that the complex associated to the simplicial abelian group

$$X_\bullet : n \longmapsto \bigoplus\nolimits_{s \in S_n} P_n$$

computes $L\pi_!(j_{U!}\mathcal{O}_U)$ by Lemma 39.7. Since $j_{U!}\mathcal{O}_U$ is a flat $\mathcal{O}$-module we have $j_{U!}\mathcal{O}_U \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}' = j_{U!}\mathcal{O}'_U$ and $L\pi_!$ of this is computed by the complex associated to the simplicial abelian group

$$X'_\bullet : n \longmapsto \bigoplus\nolimits_{s \in S_n} P'_n$$

As the rule which to a simplicial set $T_\bullet$ associated the simplicial abelian group with terms $\bigoplus_{t \in T_n} P_n$ is a functor, we see that $X_\bullet \to P_\bullet$ is a homotopy equivalence of simplicial abelian groups. Similarly, the rule which to a simplicial set $T_\bullet$ associates the simplicial abelian group with terms $\bigoplus_{t \in T_n} P'_n$ is a functor. Hence $X'_\bullet \to P'_\bullet$ is a homotopy equivalence of simplicial abelian groups. By assumption $P_\bullet \to P'_\bullet$ is a quasi-isomorphism (since $P_\bullet$, resp. $P'_\bullet$ computes $L\pi_!\mathcal{O}$, resp. $L\pi_!\mathcal{O}'$ by Lemma 39.7). We conclude that $X_\bullet$ and $X'_\bullet$ are quasi-isomorphic as desired. $\square$

Remark 39.13. Let $\mathcal{C}$ and B be as in Example 39.1. Assume there exists a cosimplicial object as in Lemma 39.7. Let $\mathcal{O} \to \underline{B}$ be a map sheaf of rings on $\mathcal{C}$ which induces an isomorphism $L\pi_!\mathcal{O} \to L\pi_!\underline{B}$. In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{O}) \longrightarrow D(B)$$

Namely, for any object K of $D(\mathcal{O})$ we have $L\pi_!^{Ab}(K) = L\pi_!^{Ab}(K \otimes_{\mathcal{O}}^{\mathbf{L}} \underline{B})$ by Lemma 39.12. Thus we can define the displayed functor as the composition of $- \otimes_{\mathcal{O}}^{\mathbf{L}} \underline{B}$ with the functor $L\pi_! : D(\underline{B}) \to D(B)$. In other words, we obtain a B-module structure on $L\pi_!(K)$ coming from the (canonical, functorial) identification of $L\pi_!(K)$ with $L\pi_!(K \otimes_{\mathcal{O}}^{\mathbf{L}} \underline{B})$ of the lemma.

40. Calculating derived lower shriek

In this section we apply the results from Section 39 to compute $L\pi_!$ in Situation 38.1 and $Lg_!$ in Situation 38.3.

Lemma 40.1. *Assumptions and notation as in Situation 38.1. For $\mathcal{F}$ in $PAb(\mathcal{C})$ and $n \geq 0$ consider the abelian sheaf $L_n(\mathcal{F})$ on $\mathcal{D}$ which is the sheaf associated to the presheaf*

$$V \longmapsto H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V})$$

with restriction maps as indicated in the proof. Then $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$.

Proof. For a morphism $h : V' \to V$ of $\mathcal{D}$ there is a pullback functor $h^* : \mathcal{C}_V \to \mathcal{C}_{V'}$ of fibre categories (Categories, Definition 33.6). Moreover for $U \in \mathrm{Ob}(\mathcal{C}_V)$ there is a strongly cartesian morphism $h^*U \to U$ covering h. Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \longrightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

By Example 39.3 we obtain the desired restriction map

$$H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V}) \longrightarrow H_n(\mathcal{C}_{V'}, \mathcal{F}|_{\mathcal{C}_{V'}})$$

Let us denote $L_{n,p}(\mathcal{F})$ this presheaf, so that $L_n(\mathcal{F}) = L_{n,p}(\mathcal{F})^\#$. The canonical map $\gamma : \mathcal{F} \to \mathcal{F}^+$ (Sites, Theorem 10.10) defines a canonical map $L_{n,p}(\mathcal{F}) \to L_{n,p}(\mathcal{F}^+)$. We have to prove this map becomes an isomorphism after sheafification.

Let us use the computation of homology given in Example 39.2. Denote $K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$ the complex associated to the restriction of $\mathcal{F}$ to the fibre category $\mathcal{C}_V$. By the remarks above we obtain a presheaf $K_\bullet(\mathcal{F})$ of complexes

$$V \longmapsto K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$$

whose cohomology presheaves are the presheaves $L_{n,p}(\mathcal{F})$. Thus it suffices to show that

$$K_\bullet(\mathcal{F}) \longrightarrow K_\bullet(\mathcal{F}^+)$$

becomes an isomorphism on sheafification.

Injectivity. Let V be an object of $\mathcal{D}$ and let $\xi \in K_n(\mathcal{F})(V)$ be an element which maps to zero in $K_n(\mathcal{F}^+)(V)$. We have to show there exists a covering $\{V_j \to V\}$ such that $\xi|_{V_j}$ is zero in $K_n(\mathcal{F})(V_j)$. We write

$$\xi = \sum (U_{i,n+1} \to \ldots \to U_{i,0}, \sigma_i)$$

with $\sigma_i \in \mathcal{F}(U_{i,0})$. We arrange it so that each sequence of morphisms $U_n \to \ldots \to U_0$ of $\mathcal{C}_V$ occurs are most once. Since the sums in the definition of the complex $K_\bullet$ are direct sums, the only way this can map to zero in $K_\bullet(\mathcal{F}^+)(V)$ is if all σ_i map to zero in $\mathcal{F}^+(U_{i,0})$. By construction of $\mathcal{F}^+$ there exist coverings $\{U_{i,0,j} \to U_{i,0}\}$ such that $\sigma_i|_{U_{i,0,j}}$ is zero. By our construction of the topology on $\mathcal{C}$ we can write $U_{i,0,j} \to U_{i,0}$ as the pullback (Categories, Definition 33.6) of some morphisms $V_{i,j} \to V$ and moreover each $\{V_{i,j} \to V\}$ is a covering. Choose a covering $\{V_j \to V\}$ dominating each of the coverings $\{V_{i,j} \to V\}$. Then it is clear that $\xi|_{V_j} = 0$.

Surjectivity. Proof omitted. Hint: Argue as in the proof of injectivity. □

Lemma 40.2. *Assumptions and notation as in Situation 38.1. For $\mathcal{F}$ in $Ab(\mathcal{C})$ and $n \geq 0$ the sheaf $L_n\pi_!(\mathcal{F})$ is equal to the sheaf $L_n(\mathcal{F})$ constructed in Lemma 40.1.*

Proof. Consider the sequence of functors $\mathcal{F} \mapsto L_n(\mathcal{F})$ from $PAb(\mathcal{C}) \to Ab(\mathcal{C})$. Since for each $V \in \mathrm{Ob}(\mathcal{D})$ the sequence of functors $H_n(\mathcal{C}_V, -)$ forms a δ-functor so do the functors $\mathcal{F} \mapsto L_n(\mathcal{F})$. Our goal is to show these form a universal δ-functor. In order to do this we construct some abelian presheaves on which these functors vanish.

For $U' \in \mathrm{Ob}(\mathcal{C})$ consider the abelian presheaf $\mathcal{F}_{U'} = j_{U'!}^{PAb}\mathbf{Z}_{U'}$ (Modules on Sites, Remark 19.7). Recall that

$$\mathcal{F}_{U'}(U) = \bigoplus\nolimits_{\mathrm{Mor}_\mathcal{C}(U,U')} \mathbf{Z}$$

If U lies over $V = p(U)$ in $\mathcal{D}$) and U' lies over $V' = p(U')$ then any morphism $a : U \to U'$ factors uniquely as $U \to h^*U' \to U'$ where $h = p(a) : V \to V'$ (see Categories, Definition 33.6). Hence we see that

$$\mathcal{F}_{U'}|_{\mathcal{C}_V} = \bigoplus\nolimits_{h \in \mathrm{Mor}_\mathcal{D}(V,V')} j_{h^*U'!}\mathbf{Z}_{h^*U'}$$

where $j_{h^*U'} : Sh(\mathcal{C}_V/h^*U') \to Sh(\mathcal{C}_V)$ is the localization morphism. The sheaves $j_{h^*U'!}\mathbf{Z}_{h^*U'}$ have vanishing higher homology groups (see Example 39.2). We conclude that $L_n(\mathcal{F}_{U'}) = 0$ for all $n > 0$ and all U'. It follows that any abelian presheaf $\mathcal{F}$ is a quotient of an abelian presheaf $\mathcal{G}$ with $L_n(\mathcal{G}) = 0$ for all $n > 0$ (Modules on Sites, Lemma 28.8). Since $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$ we see that the same thing is true for abelian sheaves. Thus the sequence of functors $L_n(-)$ is a universal delta functor on $Ab(\mathcal{C})$ (Homology, Lemma 12.4). Since we have agreement with $H^{-n}(L\pi_!(-))$ for $n = 0$ by Lemma 38.8 we conclude by uniqueness of universal δ-functors (Homology, Lemma 12.5) and Derived Categories, Lemma 16.6. □

Lemma 40.3. *Assumptions and notation as in Situation 38.3. For an abelian sheaf $\mathcal{F}'$ on $\mathcal{C}'$ the sheaf $L_ng_!(\mathcal{F}')$ is the sheaf associated to the presheaf*

$$U \longmapsto H_n(\mathcal{I}_U, \mathcal{F}'_U)$$

For notation and restriction maps see proof.

Proof. Say $p(U) = V$. The category $\mathcal{I}_U$ is the category of pairs (U', φ) where $\varphi : U \to u(U')$ is a morphism of $\mathcal{C}$ with $p(\varphi) = \mathrm{id}_V$, i.e., φ is a morphism of the fibre category $\mathcal{C}_V$. Morphisms $(U'_1, \varphi_1) \to (U'_2, \varphi_2)$ are given by morphisms $a : U'_1 \to U'_2$ of the fibre category $\mathcal{C}'_V$ such that $\varphi_2 = u(a) \circ \varphi_1$. The presheaf $\mathcal{F}'_U$ sends (U', φ) to $\mathcal{F}'(U')$. We will construct the restriction mappings below.

Choose a factorization

$$\mathcal{C}' \underset{w}{\overset{u'}{\rightleftarrows}} \mathcal{C}'' \xrightarrow{u''} \mathcal{C}$$

of u as in Categories, Lemma 33.14. Then $g_! = g''_! \circ g'_!$ and similarly for derived functors. On the other hand, the functor $g'_!$ is exact, see Modules on Sites, Lemma 16.6. Thus we get $Lg_!(\mathcal{F}') = Lg''_!(\mathcal{F}'')$ where $\mathcal{F}'' = g'_!\mathcal{F}'$. Note that $\mathcal{F}'' = h^{-1}\mathcal{F}'$ where $h : Sh(\mathcal{C}'') \to Sh(\mathcal{C}')$ is the morphism of topoi associated to w, see Sites, Lemma 23.1. The functor u'' turns $\mathcal{C}''$ into a fibred category over $\mathcal{C}$, hence Lemma 40.2 applies to the computation of $L_n g''_!$. The result follows as the construction of $\mathcal{C}''$ in the proof of Categories, Lemma 33.14 shows that the fibre category $\mathcal{C}''_U$ is equal to $\mathcal{I}_U$. Moreover, $h^{-1}\mathcal{F}'|_{\mathcal{C}''_U}$ is given by the rule described above (as w is continuous and cocontinuous by Stacks, Lemma 10.3 so we may apply Sites, Lemma 21.5). $\square$

41. Simplicial modules

Let $A_\bullet$ be a simplicial ring. Recall that we may think of $A_\bullet$ as a sheaf on Δ (endowed with the chaotic topology), see Simplicial, Section 4. Then a simplicial module $M_\bullet$ over $A_\bullet$ is just a sheaf of $A_\bullet$-modules on Δ. In other words, for every $n \geq 0$ we have an A_n-module M_n and for every map $\varphi : [n] \to [m]$ we have a corresponding map

$$M_\bullet(\varphi) : M_m \longrightarrow M_n$$

which is $A_\bullet(\varphi)$-linear such that these maps compose in the usual manner.

Let $\mathcal{C}$ be a site. A *simplicial sheaf of rings* $\mathcal{A}_\bullet$ on $\mathcal{C}$ is a simplicial object in the category of sheaves of rings on $\mathcal{C}$. In this case the assignment $U \mapsto \mathcal{A}_\bullet(U)$ is a sheaf of simplicial rings and in fact the two notions are equivalent. A similar discussion holds for simplicial abelian sheaves, simplicial sheaves of Lie algebras, and so on.

However, as in the case of simplicial rings above, there is another way to think about simplicial sheaves. Namely, consider the projection

$$p : \Delta \times \mathcal{C} \longrightarrow \mathcal{C}$$

This defines a fibred category with strongly cartesian morphisms exactly the morphisms of the form $([n], U) \to ([n], V)$. We endow the category $\Delta \times \mathcal{C}$ with the topology inherited from $\mathcal{C}$ (see Stacks, Section 10). The simple description of the coverings in $\Delta \times \mathcal{C}$ (Stacks, Lemma 10.1) immediately implies that a simplicial sheaf of rings on $\mathcal{C}$ is the same thing as a sheaf of rings on $\Delta \times \mathcal{C}$.

By analogy with the case of simplicial modules over a simplicial ring, we define simplicial modules over simplicial sheaves of rings as follows.

Definition 41.1. Let $\mathcal{C}$ be a site. Let $\mathcal{A}_\bullet$ be a simplicial sheaf of rings on $\mathcal{C}$. A *simplicial $\mathcal{A}_\bullet$-module* $\mathcal{F}_\bullet$ (sometimes called a *simplicial sheaf of $\mathcal{A}_\bullet$-modules*) is a sheaf of modules over the sheaf of rings on $\Delta \times \mathcal{C}$ associated to $\mathcal{A}_\bullet$.

We obtain a category $Mod(\mathcal{A}_\bullet)$ of simplicial modules and a corresponding derived category $D(\mathcal{A}_\bullet)$. Given a map $\mathcal{A}_\bullet \to \mathcal{B}_\bullet$ of simplicial sheaves of rings we obtain a functor

$$- \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{B}_\bullet : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{B}_\bullet)$$

Moreover, the material of the preceding sections determines a functor

$$L\pi_! : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{C})$$

Given a simplicial module $\mathcal{F}_\bullet$ the object $L\pi_!(\mathcal{F}_\bullet)$ is represented by the associated chain complex $s(\mathcal{F}_\bullet)$ (Simplicial, Section 23). This follows from Lemmas 40.2 and 39.7.

Lemma 41.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{A}_\bullet \to \mathcal{B}_\bullet$ be a homomorphism of simplicial sheaves of rings on $\mathcal{C}$. If $L\pi_!\mathcal{A}_\bullet \to L\pi_!\mathcal{B}_\bullet$ is an isomorphism in $D(\mathcal{C})$, then we have*

$$L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{B}_\bullet)$$

for all K in $D(\mathcal{A}_\bullet)$.

Proof. Let $([n], U)$ be an object of $\Delta \times \mathcal{C}$. Since $L\pi_!$ commutes with colimits, it suffices to prove this for bounded above complexes of $\mathcal{O}$-modules (compare with argument of Derived Categories, Proposition 29.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are flat modules, see Modules on Sites, Lemma 28.8. Thus it suffices to prove the lemma for a flat $\mathcal{A}_\bullet$-module $\mathcal{F}$. In this case the derived tensor product is the usual tensor product and is a sheaf also. Hence by Lemma 40.2 we can compute the cohomology sheaves of both sides of the equation by the procedure of Lemma 40.1. Thus it suffices to prove the result for the restriction of $\mathcal{F}$ to the fibre categories (i.e., to $\Delta \times U$). In this case the result follows from Lemma 39.12. $\square$

Remark 41.3. Let $\mathcal{C}$ be a site. Let $\epsilon : \mathcal{A}_\bullet \to \mathcal{O}$ be an augmentation (Simplicial, Definition 20.1) in the category of sheaves of rings. Assume ϵ induces a quasi-isomorphism $s(\mathcal{A}_\bullet) \to \mathcal{O}$. In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{O})$$

Namely, for any object K of $D(\mathcal{A}_\bullet)$ we have $L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O})$ by Lemma 41.2. Thus we can define the displayed functor as the composition of $- \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O}$ with the functor $L\pi_! : D(\Delta \times \mathcal{C}, \pi^{-1}\mathcal{O}) \to D(\mathcal{O})$ of Remark 38.6. In other words, we obtain a $\mathcal{O}$-module structure on $L\pi_!(K)$ coming from the (canonical, functorial) identification of $L\pi_!(K)$ with $L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O})$ of the lemma.

42. Cohomology on a category

In the situation of Example 39.1 in addition to the derived functor $L\pi_!$, we also have the functor $R\pi_*$. For an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we have $H_n(\mathcal{C}, \mathcal{F}) = H^{-n}(L\pi_!\mathcal{F})$ and $H^n(\mathcal{C}, \mathcal{F}) = H^n(R\pi_*\mathcal{F})$.

Example 42.1 (Computing cohomology)**.** In Example 39.1 we can compute the functors $H^n(\mathcal{C}, -)$ as follows. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Consider the cochain complex

$$K^\bullet(\mathcal{F}) : \prod\nolimits_{U_0} \mathcal{F}(U_0) \to \prod\nolimits_{U_0 \to U_1} \mathcal{F}(U_0) \to \prod\nolimits_{U_0 \to U_1 \to U_2} \mathcal{F}(U_0) \to \ldots$$

where the transition maps are given by

$$(s_{U_0 \to U_1}) \longmapsto ((U_0 \to U_1 \to U_2) \mapsto s_{U_0 \to U_1} - s_{U_0 \to U_2} + s_{U_1 \to U_2}|_{U_0})$$

and similarly in other degrees. By construction

$$H^0(\mathcal{C}, \mathcal{F}) = \lim\nolimits_{\mathcal{C}^{opp}} \mathcal{F} = H^0(K^\bullet(\mathcal{F})),$$

see Categories, Lemma 14.11. The construction of $K^\bullet(\mathcal{F})$ is functorial in $\mathcal{F}$ and transforms short exact sequences of $Ab(\mathcal{C})$ into short exact sequences of complexes. Thus the sequence of functors $\mathcal{F} \mapsto H^n(K^\bullet(\mathcal{F}))$ forms a δ-functor, see Homology, Definition 12.1 and Lemma 13.12. For an object U of $\mathcal{C}$ denote $p_U : Sh(*) \to Sh(\mathcal{C})$ the corresponding point with p_U^{-1} equal to evaluation at U, see Sites, Example 33.8. Let A be an abelian group and set $\mathcal{F} = p_{U,*}A$. In this case the complex $K^\bullet(\mathcal{F})$ is the complex with terms $\text{Map}(X_n, A)$ where

$$X_n = \coprod\nolimits_{U_0 \to \ldots \to U_{n-1} \to U_n} \text{Mor}_\mathcal{C}(U, U_0)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$. Namely, the map $X_\bullet \to \{*\}$ is obvious, the map $\{*\} \to X_n$ is given by mapping $*$ to $(U \to \ldots \to U, \text{id}_U)$, and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 26.2) defining the homotopy between the two maps $X_\bullet \to X_\bullet$ are given by the rule

$$h_{n,i} : (U_0 \to \ldots \to U_n, f) \longmapsto (U \to \ldots \to U \to U_i \to \ldots \to U_n, \text{id})$$

for $i > 0$ and $h_{n,0} = \text{id}$. Verifications omitted. Since $\text{Map}(-, A)$ is a contravariant functor, implies that $K^\bullet(p_{U,*}A)$ has trivial cohomology in positive degrees (by the functoriality of Simplicial, Remark 26.4 and the result of Simplicial, Lemma 28.6). This implies that $K^\bullet(\mathcal{F})$ is acyclic in positive degrees also if $\mathcal{F}$ is a product of sheaves of the form $p_{U,*}A$. As every abelian sheaf on $\mathcal{C}$ embeds into such a product we conclude that $K^\bullet(\mathcal{F})$ computes the left derived functors $H^n(\mathcal{C}, -)$ of $H^0(\mathcal{C}, -)$ for example by Homology, Lemma 12.4 and Derived Categories, Lemma 16.6.

Example 42.2 (Computing Exts)**.** In Example 39.1 assume we are moreover given a sheaf of rings $\mathcal{O}$ on $\mathcal{C}$. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules. Consider the complex $K^\bullet(\mathcal{G}, \mathcal{F})$ with degree n term

$$\prod\nolimits_{U_0 \to U_1 \to \ldots \to U_n} \text{Hom}_{\mathcal{O}(U_n)}(\mathcal{G}(U_n), \mathcal{F}(U_0))$$

and transition map given by

$$(\varphi_{U_0 \to U_1}) \longmapsto ((U_0 \to U_1 \to U_2) \mapsto \varphi_{U_0 \to U_1} \circ \rho_{U_1}^{U_2} - \varphi_{U_0 \to U_2} + \rho_{U_0}^{U_1} \circ \varphi_{U_1 \to U_2}$$

and similarly in other degrees. Here the ρ's indicate restriction maps. By construction

$$\text{Hom}_\mathcal{O}(\mathcal{G}, \mathcal{F}) = H^0(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for all pairs of $\mathcal{O}$-modules $\mathcal{F}, \mathcal{G}$. The assignment $(\mathcal{G}, \mathcal{F}) \mapsto K^\bullet(\mathcal{G}, \mathcal{F})$ is a bifunctor which transforms direct sums in the first variable into products and commutes with products in the second variable. We claim that

$$\text{Ext}^i_\mathcal{O}(\mathcal{G}, \mathcal{F}) = H^i(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for $i \geq 0$ provided either

(1) $\mathcal{G}(U)$ is a projective $\mathcal{O}(U)$-module for all $U \in \text{Ob}(\mathcal{C})$, or
(2) $\mathcal{F}(U)$ is an injective $\mathcal{O}(U)$-module for all $U \in \text{Ob}(\mathcal{C})$.

Namely, case (1) the functor $K^\bullet(\mathcal{G}, -)$ is an exact functor from the category of $\mathcal{O}$-modules to the category of cochain complexes of abelian groups. Thus, arguing as in Example 42.1, it suffices to show that $K^\bullet(\mathcal{G}, \mathcal{F})$ is acyclic in positive degrees when $\mathcal{F}$ is $p_{U,*}A$ for an $\mathcal{O}(U)$-module A. Choose a short exact sequence

$$0 \to \mathcal{G}' \to \bigoplus j_{U_i!}\mathcal{O}_{U_i} \to \mathcal{G} \to 0 \tag{42.2.1}$$

see Modules on Sites, Lemma 28.8. Since (1) holds for the middle and right sheaves, it also holds for $\mathcal{G}'$ and evaluating (42.2.1) on an object of $\mathcal{C}$ gives a split exact sequence of modules. We obtain a short exact sequence of complexes

$$0 \to K^\bullet(\mathcal{G}, \mathcal{F}) \to \prod K^\bullet(j_{U_i!}\mathcal{O}_{U_i}, \mathcal{F}) \to K^\bullet(\mathcal{G}', \mathcal{F}) \to 0$$

for any $\mathcal{F}$, in particular $\mathcal{F} = p_{U,*}A$. On H^0 we obtain

$$0 \to \mathrm{Hom}(\mathcal{G}, p_{U,*}A) \to \mathrm{Hom}(\prod j_{U_i!}\mathcal{O}_{U_i}, p_{U,*}A) \to \mathrm{Hom}(\mathcal{G}', p_{U,*}A) \to 0$$

which is exact as $\mathrm{Hom}(\mathcal{H}, p_{U,*}A) = \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{H}(U), A)$ and the sequence of sections of (42.2.1) over U is split exact. Thus we can use dimension shifting to see that it suffices to prove $K^\bullet(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is acyclic in positive degrees for all $U, U' \in \mathrm{Ob}(\mathcal{C})$. In this case $K^n(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is equal to

$$\prod\nolimits_{U \to U_0 \to U_1 \to \ldots \to U_n \to U'} A$$

In other words, $K^\bullet(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is the complex with terms $\mathrm{Map}(X_\bullet, A)$ where

$$X_n = \coprod\nolimits_{U_0 \to \ldots \to U_{n-1} \to U_n} \mathrm{Mor}_\mathcal{C}(U, U_0) \times \mathrm{Mor}_\mathcal{C}(U_n, U')$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$ as can be proved in exactly the same way as the corresponding statement in Example 42.1. This finishes the proof of the claim.

The argument in case (2) is similar (but dual).

43. Modules on a category

The material in this section will be used to define a variant of the derived category of quasi-coherent modules on a stack in groupoids over the category of schemes. See Sheaves on Stacks, Section 26.

Let $\mathcal{C}$ be a category. We think of $\mathcal{C}$ as a site with the chaotic topology. As in Example 42.2 we let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. In other words, $\mathcal{O}$ is a presheaf of rings on the category $\mathcal{C}$, see Categories, Definition 3.3.

Definition 43.1. In the situation above, we denote $QC(\mathcal{C}, \mathcal{O})$ or simply $QC(\mathcal{O})$ the full subcategory of $D(\mathcal{O}) = D(\mathcal{C}, \mathcal{O})$ consisting of objects K such that for all $U \to V$ in $\mathcal{C}$ the canonical map

$$R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \longrightarrow R\Gamma(U, K)$$

is an isomorphism in $D(\mathcal{O}(U))$.

Lemma 43.2. *In the situation above, the subcategory $QC(\mathcal{O})$ is a strictly full, saturated, triangulated subcategory of $D(\mathcal{O})$ preserved by arbitrary direct sums.*

Proof. Let U be an object of $\mathcal{C}$. Since the topology on $\mathcal{C}$ is chaotic, the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is exact and commutes with direct sums. Hence the exact functor $K \mapsto R\Gamma(U, K)$ is computed by representing K by any complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules and taking $\mathcal{F}^\bullet(U)$. Thus $R\Gamma(U, -)$ commutes with direct sums, see Injectives, Lemma 13.4. Similarly, given a morphism $U \to V$ of $\mathcal{C}$ the derived tensor product functor $- \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) : D(\mathcal{O}(V)) \to D(\mathcal{O}(U))$ is exact and commutes with direct sums. The lemma follows from these observations in a straightforward manner; details omitted. $\square$

Lemma 43.3. *In the situation above, suppose that M is an object of $QC(\mathcal{O})$ and $b \in \mathbf{Z}$ such that $H^i(M) = 0$ for all $i > b$. Then $H^b(M)$ is a quasi-coherent module on $(\mathcal{C}, \mathcal{O})$ in the sense of Modules on Sites, Definition 23.1.*

Proof. By Modules on Sites, Lemma 24.2 it suffices to show that for every morphism $U \to V$ of $\mathcal{C}$ the map

$$H^p(M)(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \to H^b(M)(U)$$

is an isomorphism. We are given that the map

$$R\Gamma(V, M) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \to R\Gamma(U, M)$$

is an isomorphism. Thus the result by the Tor spectral sequence for example. Details omitted. $\square$

Lemma 43.4. *In the situation above, suppose that $\mathcal{C}$ has a final object X. Set $R = \mathcal{O}(X)$ and denote $f : (\mathcal{C}, \mathcal{O}) \to (pt, R)$ the obvious morphism of sites. Then $QC(\mathcal{O}) = D(R)$ given by Lf^* and Rf_*.*

Proof. Omitted. $\square$

Lemma 43.5. *In the situation above, suppose that K is an object of $QC(\mathcal{O})$ and M arbitrary in $D(\mathcal{O})$. For every object U of $\mathcal{C}$ we have*

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U, M|_U) = R\,\mathrm{Hom}_{\mathcal{O}(U)}(R\Gamma(U, K), R\Gamma(U, M))$$

Proof. We may replace $\mathcal{C}$ by $\mathcal{C}/U$. Thus we may assume $U = X$ is a final object of $\mathcal{C}$. By Lemma 43.4 we see that $K = Lf^*P$ where $P = R\Gamma(U, K) = R\Gamma(X, K) = Rf_*K$. Thus the result because Lf^* is the left adjoint to $Rf_*(-) = R\Gamma(U, -)$. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be as above. For a complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules we define the *size* $|\mathcal{F}^\bullet|$ of $\mathcal{F}^\bullet$ as

$$|\mathcal{F}^\bullet| = \left|\coprod\nolimits_{i \in \mathbf{Z},\ U \in \mathrm{Ob}(\mathcal{C})} \mathcal{F}^i(U)\right|$$

For an object K of $D(\mathcal{O})$ we define the *size* $|K|$ of K to be the cardinal

$$|K| = \min\{|\mathcal{F}^\bullet| \text{ where } \mathcal{F}^\bullet \text{ represents } K\}$$

By properties of cardinals the minimum exists.

Lemma 43.6. *In the situation above, there exists a cardinal κ with the following property: given a complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules and subsets $\Omega^i_U \subset \mathcal{F}^i(U)$ there exists a subcomplex $\mathcal{H}^\bullet \subset \mathcal{F}^\bullet$ with $\Omega^i_U \subset \mathcal{H}^i(U)$ and $|\mathcal{H}^\bullet| \leq \max(\kappa, |\bigcup \Omega^i_U|)$.*

Proof. Define $\mathcal{H}^i(U)$ to be the $\mathcal{O}(U)$-submodule of $\mathcal{F}^i(U)$ generated by the images of Ω^i_V and $\mathrm{d}(\Omega^{i-1}_U)$ by restriction along any morphism $f : U \to V$. The cardinality of $\mathcal{H}^i(U)$ is bounded by the maximum of $\aleph_0$, the cardinality of the $\mathcal{O}(U)$, the cardinality of $\mathrm{Arrows}(\mathcal{C})$, and $|\bigcup \Omega^i_U|$. Details omitted. $\square$

Lemma 43.7. *In the situation above, there exists a cardinal κ with the following property: given a complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules representing an object K of $D(\mathcal{O})$ there exists a subcomplex $\mathcal{H}^\bullet \subset \mathcal{F}^\bullet$ such that $\mathcal{H}^\bullet$ represents K and such that $|\mathcal{H}^\bullet| \leq \max(\kappa, |K|)$.*

Proof. First, for every i and U we choose a subset $\Omega_U^i \subset \mathrm{Ker}(\mathrm{d} : \mathcal{F}^i(U) \to \mathcal{F}^{i+1}(U))$ mapping bijectively onto $H^i(K)(U) = H^i(\mathcal{F}^\bullet(U))$. Hence $|\Omega_U^i| \leq |K|$ as we may represent K by a complex whose size is $|K|$. Applying Lemma 43.6 we find a subcomplex $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ of size at most $\max(\kappa, |K|)$ containing Ω_U^i and hence such that $H^i(\mathcal{S}^\bullet) \to H^i(\mathcal{F}^\bullet)$ is a surjection of sheaves.

We are going to inductively construct subcomplexes

$$\mathcal{S}^\bullet = \mathcal{S}_0^\bullet \subset \mathcal{S}_1^\bullet \subset \mathcal{S}_2^\bullet \subset \ldots \subset \mathcal{F}^\bullet$$

of size $\leq \max(\kappa, |K|)$ such that the kernel of $H^i(\mathcal{S}_n^\bullet) \to H^i(\mathcal{F}^\bullet)$ is the same as the kernel of $H^i(\mathcal{S}_n^\bullet) \to H^i(\mathcal{S}_{n+1}^\bullet)$. Once this is done we can take $\mathcal{H}^\bullet = \bigcup \mathcal{S}_n^\bullet$ as our solution.

Construction of $\mathcal{S}_{n+1}^\bullet$ given $\mathcal{S}_n^\bullet$. For ever U and i let $\Omega_U^{i-1} \subset \mathcal{F}^{i-1}(U)$ be a subset such that $\mathrm{d} : \mathcal{F}^{i-1}(U) \to \mathcal{F}^i(U)$ maps Ω_U^{i-1} bijectively onto

$$\mathcal{S}_n^i(U) \cap \mathrm{Im}(\mathrm{d} : \mathcal{F}^{i-1}(U) \to \mathcal{F}^i(U))$$

Observe that $|\Omega_U^i| \leq |K|$ because $\mathcal{S}_n^i(U)$ is so bounded. Then we get $\mathcal{S}_{n+1}^\bullet$ by an application of Lemma 43.6 to the subsets

$$\mathcal{S}^i(U) \cup \Omega_U^i \subset \mathcal{F}^i(U)$$

and everything is clear. $\square$

Lemma 43.8. *In the situation above, there exists a cardinal κ with the following properties:*

1. *for every nonzero object K of $QC(\mathcal{O})$ there exists a nonzero morphism $E \to K$ of $QC(\mathcal{O})$ such that $|E| \leq \kappa$,*
2. *for every morphism $\alpha : E \to \bigoplus_n K_n$ of $QC(\mathcal{O})$ such that $|E| \leq \kappa$, there exist morphisms $E_n \to K_n$ in $QC(\mathcal{O})$ with $|E_n| \leq \kappa$ such that α factors through $\bigoplus E_n \to \bigoplus K_n$.*

Proof. Let κ be an upper bound for the following set of cardinals:

1. $|\coprod_V j_{U!}\mathcal{O}_U(V)|$ for all $U \in \mathrm{Ob}(\mathcal{C})$,
2. the cardinals $\kappa(\mathcal{O}(V) \to \mathcal{O}(U))$ found in More on Algebra, Lemma 102.5 for all morphisms $U \to V$ in $\mathcal{C}$,
3. the cardinal found in Lemma 43.7.

We claim that for any complex $\mathcal{F}^\bullet$ representing an object of $QC(\mathcal{O})$ and any subcomplex $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ with $|\mathcal{S}^\bullet| \leq \kappa$ there exists a subcomplex $\mathcal{H}^\bullet$ of $\mathcal{F}^\bullet$ containing $\mathcal{S}^\bullet$ such that $\mathcal{H}^\bullet$ represents an object of $QC(\mathcal{O})$ and such that $|\mathcal{H}^\bullet| \leq \kappa$. In the next two paragraphs we show that the claim implies the lemma.

As in (1) let K be a nonzero object of $QC(\mathcal{O})$. Say K is represented by the complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$. Then $H^i(\mathcal{F}^\bullet)$ is nonzero for some i. Hence there exists an object U of $\mathcal{C}$ and a section $s \in \mathcal{F}^i(U)$ with $d(s) = 0$ which determines a nonzero section of $H^i(\mathcal{F}^\bullet)$ over U. Then the image of $s : j_{U!}\mathcal{O}_U[-i] \to \mathcal{F}^\bullet$ is a subcomplex $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ with $|\mathcal{S}^\bullet| \leq \kappa$. Applying the claim we get $\mathcal{H}^\bullet \to \mathcal{F}^\bullet$ in $QC(\mathcal{O})$ nonzero with $|\mathcal{H}^\bullet| \leq \kappa$. Thus (1) holds.

Let $\alpha : E \to \bigoplus K_n$ be as in (2). Choose any complexes $\mathcal{K}_n^\bullet$ representing K_n. Then $\bigoplus \mathcal{K}_n^\bullet$ represents $\bigoplus K_n$. By the construction of the derived category we can represent E by a complex $\mathcal{E}^\bullet$ such that α is represented by a morphism $a : \mathcal{E}^\bullet \to \bigoplus \mathcal{K}_n^\bullet$ of complexes. By Lemma 43.7 and our choice of κ above we may assume

$|\mathcal{E}^\bullet| \leq \kappa$. By the claim we get subcomplexes $\mathcal{E}_n^\bullet \subset \mathcal{K}_n^\bullet$ representing objects E_n of $QC(\mathcal{O})$ with $|E_n| \leq \kappa$ containing the image of $a_n : \mathcal{E}^\bullet \to \mathcal{K}_n^\bullet$ as desired.

Proof of the claim. Let $\mathcal{F}^\bullet$ be a complex representing an object of $QC(\mathcal{O})$ and let $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ be a subcomplex of size $\leq \kappa$. We are going to inductively construct subcomplexes

$$\mathcal{S}^\bullet = \mathcal{S}_0^\bullet \subset \mathcal{S}_1^\bullet \subset \mathcal{S}_2^\bullet \subset \ldots \subset \mathcal{F}^\bullet$$

of size $\leq \kappa$ such that for every morphism $f : U \to V$ of $\mathcal{C}$ and every $i \in \mathbf{Z}$

(1) the kernel of the arrow $H^i(\mathcal{S}_n^\bullet(V) \otimes_{\mathcal{O}(V)}^{\mathbf{L}} \mathcal{O}(U)) \to H^i(\mathcal{S}_n^\bullet(U))$ maps to zero in $H^i(\mathcal{S}_{n+1}^\bullet(V) \otimes_{\mathcal{O}(V)}^{\mathbf{L}} \mathcal{O}(U))$,
(2) the image of the arrow $H^i(\mathcal{S}_n^\bullet(U)) \to H^i(\mathcal{S}_{n+1}^\bullet(U))$ is contained in the image of $H^i(\mathcal{S}_{n+1}^\bullet(V) \otimes_{\mathcal{O}(V)}^{\mathbf{L}} \mathcal{O}(U)) \to H^i(\mathcal{S}_{n+1}^\bullet(U))$,

Once this is done we can set $\mathcal{H}^\bullet = \bigcup \mathcal{S}_n^\bullet$. Namely, since derived tensor product and taking cohomology of complexes of modules over rings commute with filtered colimits, the conditions (1) and (2) together will guarantee that

$$\mathcal{H}^\bullet(V) \otimes_{\mathcal{O}(V)}^{\mathbf{L}} \mathcal{O}(U) \longrightarrow \mathcal{H}^\bullet(U)$$

is an isomorphism on cohomology in all degrees and hence an isomorphism in $D(\mathcal{O}(U))$ for all $f : U \to V$ in $\mathcal{C}$. Hence $\mathcal{H}^\bullet$ represents an object of $QC(\mathcal{O})$ as desired.

Construction of $\mathcal{S}_{n+1}$ given $\mathcal{S}_n$. For every morphism $f : U \to V$ of $\mathcal{C}$ we consider the commutative diagram

$$\begin{tikzcd} \mathcal{S}_n^\bullet(V) \arrow[r] \arrow[d] & \mathcal{S}_n^\bullet(U) \arrow[d] \\ \mathcal{F}^\bullet(V) \arrow[r] & \mathcal{F}^\bullet(U) \end{tikzcd}$$

This is a diagram as in More on Algebra, Lemma 102.5 for the ring map $\mathcal{O}(V) \to \mathcal{O}(U)$, i.e., the bottom row induces an isomorphism

$$\mathcal{F}^\bullet(V) \otimes_{\mathcal{O}(V)}^{\mathbf{L}} \mathcal{O}(U) \longrightarrow \mathcal{F}^\bullet(U)$$

in $D(\mathcal{O}(U))$. Thus we may choose subcomplexes

$$\mathcal{S}_n^\bullet(V) \subset M_f^\bullet \subset \mathcal{F}^\bullet(V) \quad \text{and} \quad \mathcal{S}_n^\bullet(U) \subset N_f^\bullet \subset \mathcal{F}^\bullet(U)$$

as in More on Algebra, Lemma 102.5 and in particular we see that $|N_f^i|, |M_f^i| \leq \kappa$. Next, we apply Lemma 43.6 using the subsets

$$\mathcal{S}_n^i(U) \amalg \coprod\nolimits_{f:U\to V} N_f^i \amalg \coprod\nolimits_{g:W\to U} M_g^i \subset \mathcal{F}^i(U)$$

to find a subcomplex

$$\mathcal{S}_n^\bullet \subset \mathcal{S}_{n+1}^\bullet \subset \mathcal{F}^\bullet$$

with containing those subsets and such that $|\mathcal{S}_{n+1}^\bullet| \leq \kappa$. Conditions (1) and (2) hold because the corresponding statements hold for $\mathcal{S}_n^\bullet(V) \subset M_f^\bullet$ and $\mathcal{S}_n^\bullet(U) \subset N_f^\bullet$ by the construction in More on Algebra, Lemma 102.5. Thus the proof is complete. $\square$

Proposition 43.9. *Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. With $QC(\mathcal{O})$ as in Definition 43.1 we have*

(1) *$QC(\mathcal{O})$ is a strictly full, saturated, triangulated subcategory of $D(\mathcal{O})$ preserved by arbitrary direct sums,*

(2) *any contravariant cohomological functor $H : QC(\mathcal{O}) \to Ab$ which transforms direct sums into products is representable,*
(3) *any exact functor $F : QC(\mathcal{O}) \to \mathcal{D}$ of triangulated categories which transforms direct sums into direct sums has an exact right adjoint, and*
(4) *the inclusion functor $QC(\mathcal{O}) \to D(\mathcal{O})$ has an exact right adjoint.*

Proof. Part (1) is Lemma 43.2. Part (2) follows from Lemma 43.8 and Derived Categories, Lemma 39.1. Part (3) follows from Lemma 43.8 and Derived Categories, Proposition 39.2. Part (4) is a special case of (3). $\square$

Let $u : \mathcal{C}' \to \mathcal{C}$ be a functor between categories. If we view $\mathcal{C}$ and $\mathcal{C}'$ as sites with the chaotic topology, then u is a continuous and cocontinuous functor. Hence we obtain a morphism $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ of topoi, see Sites, Lemma 21.1. Additionally, suppose given sheaves of rings $\mathcal{O}$ on $\mathcal{C}$ and $\mathcal{O}'$ on $\mathcal{C}'$ and a map $g^\sharp : g^{-1}\mathcal{O} \to \mathcal{O}'$. We denote the corresponding morphism of ringed topoi simply $g : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$, see Modules on Sites, Section 7.

Lemma 43.10. *Let $g : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be as above. Then the functor $Lg^* : D(\mathcal{O}) \to D(\mathcal{O}')$ maps $QC(\mathcal{O})$ into $QC(\mathcal{O}')$.*

Proof. Let $U' \in \text{Ob}(\mathcal{C}')$ with image $U = u(U')$ in $\mathcal{C}$. Let pt denote the category with a single object and a single morphism. Denote $(Sh(pt), \mathcal{O}'(U'))$ and $(Sh(pt), \mathcal{O}(U))$ the ringed topoi as indicated. Of course we identify the derived category of modules on these ringed topoi with $D(\mathcal{O}'(U'))$ and $D(\mathcal{O}(U))$. Then we have a commutative diagram of ringed topoi

$$\begin{tikzcd}
(Sh(pt), \mathcal{O}'(U')) \arrow[r, "U'"'] \arrow[d] & (Sh(\mathcal{C}'), \mathcal{O}') \arrow[d, "g"] \\
(Sh(pt), \mathcal{O}(U)) \arrow[r, "U"] & (Sh(\mathcal{C}), \mathcal{O})
\end{tikzcd}$$

Pullback along the lower horizontal morphism sends K in $D(\mathcal{O})$ to $R\Gamma(U, K)$. Pullback by the left vertical arrow sends M to $M \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U')$. Going around the diagram either direction produces the same result (Lemma 18.3) and hence we conclude

$$R\Gamma(U', Lg^*K) = R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U')$$

Finally, let $f' : U' \to V'$ be a morphism in $\mathcal{C}'$ and denote $f = u(f') : U = u(U') \to V = u(V')$ the image in $\mathcal{C}$. If K is in $QC(\mathcal{O})$ then we have

$$\begin{aligned}
R\Gamma(V', Lg^*K) \otimes^{\mathbf{L}}_{\mathcal{O}'(V')} \mathcal{O}'(U') &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}'(V') \otimes^{\mathbf{L}}_{\mathcal{O}'(V')} \mathcal{O}'(U') \\
&= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}'(U') \\
&= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U') \\
&= R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U') \\
&= R\Gamma(U', Lg^*K)
\end{aligned}$$

as desired. Here we have used the observation above both for U' and V'. $\square$

Lemma 43.11. *Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Assume for all $U \to V$ in $\mathcal{C}$ the restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$ is a flat ring map. Then $QC(\mathcal{O})$ agrees with the subcategory $D_{QCoh}(\mathcal{O}) \subset D(\mathcal{O})$ of complexes whose cohomology sheaves are quasi-coherent.*

Proof. Recall that $QCoh(\mathcal{O}) \subset Mod(\mathcal{O})$ is a weak Serre subcategory under our assumptions, see Modules on Sites, Lemma 24.3. Thus taking the full subcategory

$$D_{QCoh}(\mathcal{O}) = D_{QCoh(\mathcal{O})}(Mod(\mathcal{O}))$$

of $D(\mathcal{O})$ makes sense, see Derived Categories, Section 17. (Strictly speaking we don't need this in the proof of the lemma.)

Let M be an object of $QC(\mathcal{O})$. Since for every morphism $U \to V$ in $\mathcal{C}$ the restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$ is flat, we see that

$$\begin{aligned} H^i(M)(U) &= H^i(R\Gamma(U, M)) \\ &= H^i(R\Gamma(V, M) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U)) \\ &= H^i(R\Gamma(V, M)) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \\ &= H^i(M)(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \end{aligned}$$

and hence $H^i(M)$ is quasi-coherent by Modules on Sites, Lemma 24.2. The first and last equality above follow from the fact that taking sections over an object of $\mathcal{C}$ is an exact functor due to the fact that the topology on $\mathcal{C}$ is chaotic.

Conversely, if M is an object of $D_{QCoh}(\mathcal{O})$, then due to Modules on Sites, Lemma 24.2 we see that the map $R\Gamma(V, M) \to R\Gamma(U, M)$ induces isomorphisms $H^i(M)(U) \to H^i(M)(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$. Whence $R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \to R\Gamma(U, K)$ is an isomorphism in $D(\mathcal{O}(U))$ by the flatness of $\mathcal{O}(V) \to \mathcal{O}(U)$ and we conclude that M is in $QC(\mathcal{O})$. $\square$

Lemma 43.12. *Let $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ be as in Section 27. Assume*

(1) *τ' is the chaotic topology on the category $\mathcal{C}$,*
(2) *for all $U \in \mathrm{Ob}(\mathcal{C})$ and all K-flat complexes of $\mathcal{O}(U)$-modules $M^\bullet$ the map*

$$M^\bullet \longrightarrow R\Gamma((\mathcal{C}/U)_\tau, (M^\bullet \otimes_{\mathcal{O}(U)} \mathcal{O}_U)^\#)$$

is a quasi-isomorphism (see proof for an explanation).

Then ϵ^* and $R\epsilon_*$ define mutually quasi-inverse equivalences between $QC(\mathcal{O})$ and the full subcategory of $D(\mathcal{C}_\tau, \mathcal{O}_\tau)$ consisting of objects K such that $R\epsilon_* K$ is in $QC(\mathcal{O})$[10].

Proof. We will use the observations made in Section 27 without further mention. Since $R\epsilon_*$ is fully faithful and $\epsilon^* \circ R\epsilon_* = \mathrm{id}$, to prove the lemma it suffices to show that for M in $QC(\mathcal{O})$ we have $R\epsilon_*(\epsilon^* M) = M$. Condition (2) is exactly the condition needed to see this. Namely, we choose a K-flat complex $\mathcal{M}^\bullet$ of $\mathcal{O}$-modules with flat terms representing M. Then we see that $\epsilon^* M$ is represented by the τ-sheafification $(\mathcal{M}^\bullet)^\#$ of $\mathcal{M}^\bullet$. Let $U \in \mathrm{Ob}(\mathcal{C})$. By Leray we get

$$R\Gamma(U, R\epsilon_*(\epsilon^* M)) = R\Gamma((\mathcal{C}/U)_\tau, (\mathcal{M}^\bullet)^\#|_{\mathcal{C}/U}) = R\Gamma((\mathcal{C}/U)_\tau, (\mathcal{M}^\bullet|_{\mathcal{C}/U})^\#)$$

The last equality since sheafification commutes with restriction to $\mathcal{C}/U$. As usual, denote $\mathcal{O}_U$ the restriction of $\mathcal{O}$ to $\mathcal{C}/U$. Consider the map

$$\mathcal{M}^\bullet(U) \otimes_{\mathcal{O}(U)} \mathcal{O}_U \longrightarrow \mathcal{M}^\bullet|_{\mathcal{C}/U}$$

of complexes of $\mathcal{O}_U$-modules (in τ'-topology). By our choice of $\mathcal{M}^\bullet$ the complex $\mathcal{M}^\bullet(U)$ is a K-flat complex of $\mathcal{O}(U)$-modules; see Lemma 18.1 and use that the inclusion of U into $\mathcal{C}$ defines a morphism of ringed topoi $(Sh(pt), \mathcal{O}(U)) \to$

[10] This means that $R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \to R\Gamma(U, K)$ is an isomorphism for all $U \to V$ in $\mathcal{C}$.

$(Sh(\mathcal{C}_{\tau'}), \mathcal{O})$. Since M is in $QC(\mathcal{O})$ we conclude that the displayed arrow is a quasi-isomorphism. Since sheafification is exact, we see that the same remains true after sheafification. Hence

$$R\Gamma(U, R\epsilon_*(\epsilon^* M)) = R\Gamma((\mathcal{C}/U)_\tau, (M^\bullet \otimes_{\mathcal{O}(U)} \mathcal{O}_U)^\#)$$

and assumption (2) tells us this is equal to $R\Gamma(U, M) = \mathcal{M}^\bullet(U)$ as desired. $\square$

Lemma 43.13. *Notation and assumptions as in Lemma 43.12. Suppose that K is an object of $QC(\mathcal{O})$ and M arbitrary in $D(\mathcal{O}_\tau)$. For every object U of $\mathcal{C}$ we have*

$$\mathrm{Hom}_{D((\mathcal{O}_U)_\tau)}(\epsilon^* K|_U, M|_U) = R\,\mathrm{Hom}_{\mathcal{O}(U)}(R\Gamma(U, K), R\Gamma(U, M))$$

Proof. We have

$$\mathrm{Hom}_{D((\mathcal{O}_U)_\tau)}(\epsilon^* K|_U, M|_U) = \mathrm{Hom}_{D((\mathcal{O}_U)_{\tau'})}(K|_U, R\epsilon_* M|_U)$$

by adjunction. Hence the result by Lemma 43.5 and the fact that

$$R\Gamma(U, M) = R\Gamma(U, R\epsilon_* M)$$

by Leray. $\square$

44. Strictly perfect complexes

This section is the analogue of Cohomology, Section 46.

Definition 44.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. We say $\mathcal{E}^\bullet$ is *strictly perfect* if $\mathcal{E}^i$ is zero for all but finitely many i and $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{O}$-module for all i.

Let U be an object of $\mathcal{C}$. We will often say "Let $\mathcal{E}^\bullet$ be a strictly perfect complex of $\mathcal{O}_U$-modules" to mean $\mathcal{E}^\bullet$ is a strictly perfect complex of modules on the ringed site $(\mathcal{C}/U, \mathcal{O}_U)$, see Modules on Sites, Definition 19.1.

Lemma 44.2. *The cone on a morphism of strictly perfect complexes is strictly perfect.*

Proof. This is immediate from the definitions. $\square$

Lemma 44.3. *The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.*

Proof. Omitted. $\square$

Lemma 44.4. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. If $\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_\mathcal{D}$-modules, then $f^*\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_\mathcal{C}$-modules.*

Proof. We have seen in Modules on Sites, Lemma 17.2 that the pullback of a finite free module is finite free. The functor f^* is additive functor hence preserves direct summands. The lemma follows. $\square$

Lemma 44.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Given a solid diagram of $\mathcal{O}_U$-modules*

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[dr, dotted] & \mathcal{F} \\ & \mathcal{G} \arrow[u, "p"'] \end{tikzcd}$$

with $\mathcal{E}$ a direct summand of a finite free $\mathcal{O}_U$-module and p surjective, then there exists a covering $\{U_i \to U\}$ such that a dotted arrow making the diagram commute exists over each U_i.

Proof. We may assume $\mathcal{E} = \mathcal{O}_U^{\oplus n}$ for some n. In this case finding the dotted arrow is equivalent to lifting the images of the basis elements in $\Gamma(U, \mathcal{F})$. This is locally possible by the characterization of surjective maps of sheaves (Sites, Section 11). $\square$

Lemma 44.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$.*

1. *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect and $\mathcal{F}^\bullet$ acyclic. Then there exists a covering $\{U_i \to U\}$ such that each $\alpha|_{U_i}$ is homotopic to zero.*
2. *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^i = 0$ for $i < a$, and $H^i(\mathcal{F}^\bullet) = 0$ for $i \geq a$. Then there exists a covering $\{U_i \to U\}$ such that each $\alpha|_{U_i}$ is homotopic to zero.*

Proof. The first statement follows from the second, hence we only prove (2). We will prove this by induction on the length of the complex $\mathcal{E}^\bullet$. If $\mathcal{E}^\bullet \cong \mathcal{E}[-n]$ for some direct summand $\mathcal{E}$ of a finite free $\mathcal{O}$-module and integer $n \geq a$, then the result follows from Lemma 44.5 and the fact that $\mathcal{F}^{n-1} \to \mathrm{Ker}(\mathcal{F}^n \to \mathcal{F}^{n+1})$ is surjective by the assumed vanishing of $H^n(\mathcal{F}^\bullet)$. If $\mathcal{E}^i$ is zero except for $i \in [a, b]$, then we have a split exact sequence of complexes

$$0 \to \mathcal{E}^b[-b] \to \mathcal{E}^\bullet \to \sigma_{\leq b-1}\mathcal{E}^\bullet \to 0$$

which determines a distinguished triangle in $K(\mathcal{O}_U)$. Hence an exact sequence

$$\mathrm{Hom}_{K(\mathcal{O}_U)}(\sigma_{\leq b-1}\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^b[-b], \mathcal{F}^\bullet)$$

by the axioms of triangulated categories. The composition $\mathcal{E}^b[-b] \to \mathcal{F}^\bullet$ is homotopic to zero on the members of a covering of U by the above, whence we may assume our map comes from an element in the left hand side of the displayed exact sequence above. This element is zero on the members of a covering of U by induction hypothesis. $\square$

Lemma 44.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Given a solid diagram of complexes of $\mathcal{O}_U$-modules*

$$\begin{tikzcd} \mathcal{E}^\bullet \arrow[r, "\alpha"'] \arrow[dr, dotted] & \mathcal{F}^\bullet \\ & \mathcal{G}^\bullet \arrow[u, "f"'] \end{tikzcd}$$

with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^j = 0$ for $j < a$ and $H^j(f)$ an isomorphism for $j > a$ and surjective for $j = a$, then there exists a covering $\{U_i \to U\}$ and for each i a dotted arrow over U_i making the diagram commute up to homotopy.

Proof. Our assumptions on f imply the cone $C(f)^\bullet$ has vanishing cohomology sheaves in degrees $\geq a$. Hence Lemma 44.6 guarantees there is a covering $\{U_i \to U\}$ such that the composition $\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet$ is homotopic to zero over U_i. Since

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet \to \mathcal{G}^\bullet[1]$$

restricts to a distinguished triangle in $K(\mathcal{O}_{U_i})$ we see that we can lift $\alpha|_{U_i}$ up to homotopy to a map $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{G}^\bullet|_{U_i}$ as desired. $\square$

Lemma 44.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect.*

1. *For any element $\alpha \in \text{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ there exists a covering $\{U_i \to U\}$ such that $\alpha|_{U_i}$ is given by a morphism of complexes $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{F}^\bullet|_{U_i}$.*
2. *Given a morphism of complexes $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ whose image in the group $\text{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is zero, there exists a covering $\{U_i \to U\}$ such that $\alpha|_{U_i}$ is homotopic to zero.*

Proof. Proof of (1). By the construction of the derived category we can find a quasi-isomorphism $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and a map of complexes $\beta : \mathcal{E}^\bullet \to \mathcal{G}^\bullet$ such that $\alpha = f^{-1}\beta$. Thus the result follows from Lemma 44.7. We omit the proof of (2). $\square$

Lemma 44.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}$-modules with $\mathcal{E}^\bullet$ strictly perfect. Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms*

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 35.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex. Let $(\mathcal{H}')^\bullet$ be the complex with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{I}^p)$$

which represents $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ by the construction in Section 35. It suffices to show that the map

$$\mathcal{H}^\bullet \longrightarrow (\mathcal{H}')^\bullet$$

is a quasi-isomorphism. Given an object U of $\mathcal{C}$ we have by inspection

$$H^0(\mathcal{H}^\bullet(U)) = \text{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U) \to H^0((\mathcal{H}')^\bullet(U)) = \text{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U)$$

By Lemma 44.8 the sheafification of $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^0((\mathcal{H}')^\bullet(U))$. A similar argument can be given for the other cohomology sheaves. Thus $\mathcal{H}^\bullet$ is quasi-isomorphic to $(\mathcal{H}')^\bullet$ which proves the lemma. $\square$

Lemma 44.10. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}$-modules with*

1. *$\mathcal{F}^n = 0$ for $n \ll 0$,*
2. *$\mathcal{E}^n = 0$ for $n \gg 0$, and*
3. *$\mathcal{E}^n$ isomorphic to a direct summand of a finite free $\mathcal{O}$-module.*

Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 35.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injectives. Note that $\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 31.4). Hence the construction in Section 35 shows that $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $(\mathcal{H}')^\bullet$ with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{I}^p) = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

(equality because there are only finitely many nonzero terms). Note that $\mathcal{H}^\bullet$ is the total complex associated to the double complex with terms $\mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^p)$ and similarly for $(\mathcal{H}')^\bullet$. The natural map $(\mathcal{H}')^\bullet \to \mathcal{H}^\bullet$ comes from a map of double complexes. Thus to show this map is a quasi-isomorphism, we may use the spectral sequence of a double complex (Homology, Lemma 25.3)

$$'E_1^{p,q} = H^p(\mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{H}^\bullet)$ and similarly for $(\mathcal{H}')^\bullet$. To finish the proof of the lemma it suffices to show that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism

$$H^p(\mathcal{H}om_\mathcal{O}(\mathcal{E}, \mathcal{F}^\bullet)) \longrightarrow H^p(\mathcal{H}om_\mathcal{O}(\mathcal{E}, \mathcal{I}^\bullet))$$

on cohomology sheaves whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}$-module. Since this is clear when $\mathcal{E}$ is finite free the result follows. □

45. Pseudo-coherent modules

In this section we discuss pseudo-coherent complexes.

Definition 45.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. Let $m \in \mathbf{Z}$.

(1) We say $\mathcal{E}^\bullet$ is *m-pseudo-coherent* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ and for each i a morphism of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ where $\mathcal{E}_i$ is a strictly perfect complex of $\mathcal{O}_{U_i}$-modules and $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective.
(2) We say $\mathcal{E}^\bullet$ is *pseudo-coherent* if it is m-pseudo-coherent for all m.
(3) We say an object E of $D(\mathcal{O})$ is *m-pseudo-coherent* (resp. *pseudo-coherent*) if and only if it can be represented by a m-pseudo-coherent (resp. pseudo-coherent) complex of $\mathcal{O}$-modules.

If $\mathcal{C}$ has a final object X which is quasi-compact (for example if every covering of X can be refined by a finite covering), then an m-pseudo-coherent object of $D(\mathcal{O})$ is in $D^-(\mathcal{O})$. But this need not be the case in general.

Lemma 45.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$.*

(1) *If $\mathcal{C}$ has a final object X and if there exist a covering $\{U_i \to X\}$, strictly perfect complexes $\mathcal{E}_i^\bullet$ of $\mathcal{O}_{U_i}$-modules, and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $H^j(\alpha_i)$ an isomorphism for $j > m$ and $H^m(\alpha_i)$ surjective, then E is m-pseudo-coherent.*
(2) *If E is m-pseudo-coherent, then any complex of $\mathcal{O}$-modules representing E is m-pseudo-coherent.*
(3) *If for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ is m-pseudo-coherent, then E is m-pseudo-coherent.*

Proof. Let $\mathcal{F}^\bullet$ be any complex representing E and let X, $\{U_i \to X\}$, and $\alpha_i : \mathcal{E}_i \to E|_{U_i}$ be as in (1). We will show that $\mathcal{F}^\bullet$ is m-pseudo-coherent as a complex, which will prove (1) and (2) in case $\mathcal{C}$ has a final object. By Lemma 44.8 we can after refining the covering $\{U_i \to X\}$ represent the maps α_i by maps of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{F}^\bullet|_{U_i}$. By assumption $H^j(\alpha_i)$ are isomorphisms for $j > m$, and $H^m(\alpha_i)$ is surjective whence $\mathcal{F}^\bullet$ is m-pseudo-coherent.

Proof of (2). By the above we see that $\mathcal{F}^\bullet|_U$ is m-pseudo-coherent as a complex of $\mathcal{O}_U$-modules for all objects U of $\mathcal{C}$. It is a formal consequence of the definitions that $\mathcal{F}^\bullet$ is m-pseudo-coherent.

Proof of (3). Follows from the definitions and Sites, Definition 6.2 part (2). ☐

Lemma 45.3. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let E be an object of $D(\mathcal{O}_\mathcal{C})$. If E is m-pseudo-coherent, then Lf^*E is m-pseudo-coherent.*

Proof. Say f is given by the functor $u : \mathcal{D} \to \mathcal{C}$. Let U be an object of $\mathcal{C}$. By Sites, Lemma 14.10 we can find a covering $\{U_i \to U\}$ and for each i a morphism $U_i \to u(V_i)$ for some object V_i of $\mathcal{D}$. By Lemma 45.2 it suffices to show that $Lf^*E|_{U_i}$ is m-pseudo-coherent. To do this it is enough to show that $Lf^*E|_{u(V_i)}$ is m-pseudo-coherent, since $Lf^*E|_{U_i}$ is the restriction of $Lf^*E|_{u(V_i)}$ to $\mathcal{C}/U_i$ (via Modules on Sites, Lemma 19.5). By the commutative diagram of Modules on Sites, Lemma 20.1 it suffices to prove the lemma for the morphism of ringed sites $(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \to (\mathcal{D}/V_i, \mathcal{O}_{V_i})$. Thus we may assume $\mathcal{D}$ has a final object Y such that $X = u(Y)$ is a final object of $\mathcal{C}$.

Let $\{V_i \to Y\}$ be a covering such that for each i there exists a strictly perfect complex $\mathcal{F}_i^\bullet$ of $\mathcal{O}_{V_i}$-modules and a morphism $\alpha_i : \mathcal{F}_i^\bullet \to E|_{V_i}$ of $D(\mathcal{O}_{V_i})$ such that $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective. Arguing as above it suffices to prove the result for $(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \to (\mathcal{D}/V_i, \mathcal{O}_{V_i})$. Hence we may assume that there exists a strictly perfect complex $\mathcal{F}^\bullet$ of $\mathcal{O}_\mathcal{D}$-modules and a morphism $\alpha : \mathcal{F}^\bullet \to E$ of $D(\mathcal{O}_\mathcal{D})$ such that $H^j(\alpha)$ is an isomorphism for $j > m$ and $H^m(\alpha)$ is surjective. In this case, choose a distinguished triangle

$$\mathcal{F}^\bullet \to E \to C \to \mathcal{F}^\bullet[1]$$

The assumption on α means exactly that the cohomology sheaves $H^j(C)$ are zero for all $j \geq m$. Applying Lf^* we obtain the distinguished triangle

$$Lf^*\mathcal{F}^\bullet \to Lf^*E \to Lf^*C \to Lf^*\mathcal{F}^\bullet[1]$$

By the construction of Lf^* as a left derived functor we see that $H^j(Lf^*C) = 0$ for $j \geq m$ (by the dual of Derived Categories, Lemma 16.1). Hence $H^j(Lf^*\alpha)$ is an isomorphism for $j > m$ and $H^m(Lf^*\alpha)$ is surjective. On the other hand, since $\mathcal{F}^\bullet$ is a bounded above complex of flat $\mathcal{O}_\mathcal{D}$-modules we see that $Lf^*\mathcal{F}^\bullet = f^*\mathcal{F}^\bullet$. Applying Lemma 44.4 we conclude. ☐

Lemma 45.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site and $m \in \mathbf{Z}$. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$.*

1. *If K is $(m+1)$-pseudo-coherent and L is m-pseudo-coherent then M is m-pseudo-coherent.*
2. *If K and M are m-pseudo-coherent, then L is m-pseudo-coherent.*
3. *If L is $(m+1)$-pseudo-coherent and M is m-pseudo-coherent, then K is $(m+1)$-pseudo-coherent.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. Choose a covering $\{U_i \to U\}$ and maps $\alpha_i : \mathcal{K}_i^\bullet \to K|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{K}_i^\bullet$ strictly perfect and $H^j(\alpha_i)$ isomorphisms for $j > m+1$ and surjective for $j = m+1$. We may replace $\mathcal{K}_i^\bullet$ by $\sigma_{\geq m+1}\mathcal{K}_i^\bullet$ and hence we may assume that $\mathcal{K}_i^j = 0$ for $j < m+1$. After refining the covering we may choose maps $\beta_i : \mathcal{L}_i^\bullet \to L|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{L}_i^\bullet$ strictly perfect such that $H^j(\beta)$ is an isomorphism for $j > m$ and surjective for $j = m$. By Lemma 44.7 we can, after refining the covering, find maps of complexes $\gamma_i : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ such that the diagrams

$$\begin{tikzcd} K|_{U_i} \arrow[r] & L|_{U_i} \\ \mathcal{K}_i^\bullet \arrow[u, "\alpha_i"] \arrow[r, "\gamma_i"] & \mathcal{L}_i^\bullet \arrow[u, "\beta_i"'] \end{tikzcd}$$

are commutative in $D(\mathcal{O}_{U_i})$ (this requires representing the maps α_i, β_i and $K|_{U_i} \to L|_{U_i}$ by actual maps of complexes; some details omitted). The cone $C(\gamma_i)^\bullet$ is strictly perfect (Lemma 44.2). The commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(\mathcal{K}_i^\bullet, \mathcal{L}_i^\bullet, C(\gamma_i)^\bullet) \longrightarrow (K|_{U_i}, L|_{U_i}, M|_{U_i}).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 5.19 and 5.20 that $C(\gamma_i)^\bullet \to M|_{U_i}$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence M is m-pseudo-coherent by Lemma 45.2.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. $\square$

Lemma 45.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$.*

1. *If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes_{\mathcal{O}}^{\mathbf{L}} L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.*
2. *If K and L are pseudo-coherent, then $K \otimes_{\mathcal{O}}^{\mathbf{L}} L$ is pseudo-coherent.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. By replacing U by the members of a covering and replacing $\mathcal{C}$ by the localization $\mathcal{C}/U$ we may assume there exist strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ and maps $\alpha : \mathcal{K}^\bullet \to K$ and $\beta : \mathcal{L}^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) \to K \otimes_{\mathcal{O}}^{\mathbf{L}} L$$

induces isomorphisms on cohomology sheaves in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted).

Proof of (2). Let U be an object of $\mathcal{C}$. We may first replace U by the members of a covering and $\mathcal{C}$ by the localization $\mathcal{C}/U$ to reduce to the case that K and L are bounded above. Then the statement follows immediately from case (1). $\square$

Lemma 45.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $m \in \mathbf{Z}$. If $K \oplus L$ is m-pseudo-coherent (resp. pseudo-coherent) in $D(\mathcal{O})$ so are K and L.*

Proof. Assume that $K \oplus L$ is m-pseudo-coherent. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume $K \oplus L \in D^-(\mathcal{O}_U)$, hence $L \in D^-(\mathcal{O}_U)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 4.10. By Lemma 45.4 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. Since L is bounded above we see that $L[n]$ is

m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n-1], L[n-2], \ldots, L$ are m-pseudo-coherent as desired. □

Lemma 45.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be an object of $D(\mathcal{O})$. Let $m \in \mathbf{Z}$.*

1. *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m$, then $H^m(K)$ is a finite type $\mathcal{O}$-module.*
2. *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m+1$, then $H^{m+1}(K)$ is a finitely presented $\mathcal{O}$-module.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. We have to show that $H^m(K)$ is can be generated by finitely many sections over the members of a covering of U (see Modules on Sites, Definition 23.1). Thus during the proof we may (finitely often) choose a covering $\{U_i \to U\}$ and replace $\mathcal{C}$ by $\mathcal{C}/U_i$ and U by U_i. In particular, by our definitions we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It suffices to prove the result for $\mathcal{E}^\bullet$. Let n be the largest integer such that $\mathcal{E}^n \neq 0$. If $n = m$, then $H^m(\mathcal{E}^\bullet)$ is a quotient of $\mathcal{E}^n$ and the result is clear. If $n > m$, then $\mathcal{E}^{n-1} \to \mathcal{E}^n$ is surjective as $H^n(E^\bullet) = 0$. By Lemma 44.5 we can (after replacing U by the members of a covering) find a section of this surjection and write $\mathcal{E}^{n-1} = \mathcal{E}' \oplus \mathcal{E}^n$. Hence it suffices to prove the result for the complex $(\mathcal{E}')^\bullet$ which is the same as $\mathcal{E}^\bullet$ except has $\mathcal{E}'$ in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). Pick an object U of $\mathcal{C}$. As in the proof of (1) we may work locally on U. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $\mathcal{E}^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K) \cong H^{m+1}(\mathcal{E}^\bullet) = \text{Coker}(\mathcal{E}^m \to \mathcal{E}^{m+1})$ which is of finite presentation. □

46. Tor dimension

In this section we take a closer look at resolutions by flat modules.

Definition 46.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$ with $a \leq b$.

1. We say E has *tor-amplitude in* $[a, b]$ if $H^i(E \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}) = 0$ for all $\mathcal{O}$-modules $\mathcal{F}$ and all $i \not\in [a, b]$.
2. We say E has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some a, b.
3. We say E *locally has finite tor dimension* if for any object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ has finite tor dimension for all i.

An $\mathcal{O}$-module $\mathcal{F}$ has *tor dimension* $\leq d$ if $\mathcal{F}[0]$ viewed as an object of $D(\mathcal{O})$ has tor-amplitude in $[-d, 0]$.

Note that if E as in the definition has finite tor dimension, then E is an object of $D^b(\mathcal{O})$ as can be seen by taking $\mathcal{F} = \mathcal{O}$ in the definition above.

Lemma 46.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a bounded above complex of flat $\mathcal{O}$-modules with tor-amplitude in $[a, b]$. Then $\mathrm{Coker}(d_{\mathcal{E}^\bullet}^{a-1})$ is a flat $\mathcal{O}$-module.*

Proof. As $\mathcal{E}^\bullet$ is a bounded above complex of flat modules we see that $\mathcal{E}^\bullet \otimes_\mathcal{O} \mathcal{F} = \mathcal{E}^\bullet \otimes_\mathcal{O}^\mathbf{L} \mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$. Hence for every $\mathcal{O}$-module $\mathcal{F}$ the sequence

$$\mathcal{E}^{a-2} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{E}^a \otimes_\mathcal{O} \mathcal{F}$$

is exact in the middle. Since $\mathcal{E}^{a-2} \to \mathcal{E}^{a-1} \to \mathcal{E}^a \to \mathrm{Coker}(d^{a-1}) \to 0$ is a flat resolution this implies that $\mathrm{Tor}_1^\mathcal{O}(\mathrm{Coker}(d^{a-1}), \mathcal{F}) = 0$ for all $\mathcal{O}$-modules $\mathcal{F}$. This means that $\mathrm{Coker}(d^{a-1})$ is flat, see Lemma 17.15. $\square$

Lemma 46.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent*

1. *E has tor-amplitude in $[a, b]$.*
2. *E is represented by a complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$.*

Proof. If (2) holds, then we may compute $E \otimes_\mathcal{O}^\mathbf{L} \mathcal{F} = \mathcal{E}^\bullet \otimes_\mathcal{O} \mathcal{F}$ and it is clear that (1) holds.

Assume that (1) holds. We may represent E by a bounded above complex of flat $\mathcal{O}$-modules $\mathcal{K}^\bullet$, see Section 17. Let n be the largest integer such that $\mathcal{K}^n \neq 0$. If $n > b$, then $\mathcal{K}^{n-1} \to \mathcal{K}^n$ is surjective as $H^n(\mathcal{K}^\bullet) = 0$. As $\mathcal{K}^n$ is flat we see that $\mathrm{Ker}(\mathcal{K}^{n-1} \to \mathcal{K}^n)$ is flat (Modules on Sites, Lemma 28.10). Hence we may replace $\mathcal{K}^\bullet$ by $\tau_{\leq n-1}\mathcal{K}^\bullet$. Thus, by induction on n, we reduce to the case that $K^\bullet$ is a complex of flat $\mathcal{O}$-modules with $\mathcal{K}^i = 0$ for $i > b$.

Set $\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet$. Everything is clear except that $\mathcal{E}^a$ is flat which follows immediately from Lemma 46.2 and the definitions. $\square$

Lemma 46.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a \in \mathbf{Z}$. The following are equivalent*

1. *E has tor-amplitude in $[a, \infty]$.*
2. *E can be represented by a K-flat complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, \infty]$.*

Moreover, we can choose $\mathcal{E}^\bullet$ such that any pullback by a morphism of ringed sites is a K-flat complex with flat terms.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1) holds. First we choose a K-flat complex $\mathcal{K}^\bullet$ with flat terms representing E, see Lemma 17.11. For any $\mathcal{O}$-module $\mathcal{M}$ the cohomology of

$$\mathcal{K}^{n-1} \otimes_\mathcal{O} \mathcal{M} \to \mathcal{K}^n \otimes_\mathcal{O} \mathcal{M} \to \mathcal{K}^{n+1} \otimes_\mathcal{O} \mathcal{M}$$

computes $H^n(E \otimes_\mathcal{O}^\mathbf{L} \mathcal{M})$. This is always zero for $n < a$. Hence if we apply Lemma 46.2 to the complex $\ldots \to \mathcal{K}^{a-1} \to \mathcal{K}^a \to \mathcal{K}^{a+1}$ we conclude that $\mathcal{N} = \mathrm{Coker}(\mathcal{K}^{a-1} \to \mathcal{K}^a)$ is a flat $\mathcal{O}$-module. We set

$$\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet = (\ldots \to 0 \to \mathcal{N} \to \mathcal{K}^{a+1} \to \ldots)$$

The kernel $\mathcal{L}^\bullet$ of $\mathcal{K}^\bullet \to \mathcal{E}^\bullet$ is the complex

$$\mathcal{L}^\bullet = (\ldots \to \mathcal{K}^{a-1} \to \mathcal{I} \to 0 \to \ldots)$$

where $\mathcal{I} \subset \mathcal{K}^a$ is the image of $\mathcal{K}^{a-1} \to \mathcal{K}^a$. Since we have the short exact sequence $0 \to \mathcal{I} \to \mathcal{K}^a \to \mathcal{N} \to 0$ we see that $\mathcal{I}$ is a flat $\mathcal{O}$-module. Thus $\mathcal{L}^\bullet$ is a bounded above complex of flat modules, hence K-flat by Lemma 17.8. It follows that $\mathcal{E}^\bullet$ is K-flat by Lemma 17.7.

Proof of the final assertion. Let $f : (\mathcal{C}', \mathcal{O}') \to (\mathcal{C}, \mathcal{O})$ be a morphism of ringed sites. By Lemma 18.1 the complex $f^*\mathcal{K}^\bullet$ is K-flat with flat terms. The complex $f^*\mathcal{L}^\bullet$ is K-flat as it is a bounded above complex of flat $\mathcal{O}'$-modules. We have a short exact sequence of complexes of $\mathcal{O}'$-modules

$$0 \to f^*\mathcal{L}^\bullet \to f^*\mathcal{K}^\bullet \to f^*\mathcal{E}^\bullet \to 0$$

because the short exact sequence $0 \to \mathcal{I} \to \mathcal{K}^a \to \mathcal{N} \to 0$ of flat modules pulls back to a short exact sequence. By Lemma 17.7. the complex $f^*\mathcal{E}^\bullet$ is K-flat and the proof is complete. □

Lemma 46.5. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let E be an object of $D(\mathcal{O}_\mathcal{D})$. If E has tor amplitude in $[a, b]$, then Lf^*E has tor amplitude in $[a, b]$.*

Proof. Assume E has tor amplitude in $[a, b]$. By Lemma 46.3 we can represent E by a complex of $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \not\in [a, b]$. Then Lf^*E is represented by $f^*\mathcal{E}^\bullet$. By Modules on Sites, Lemma 39.1 the module $f^*\mathcal{E}^i$ are flat. Thus by Lemma 46.3 we conclude that Lf^*E has tor amplitude in $[a, b]$. □

Lemma 46.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$.*

1. *If K has tor-amplitude in $[a + 1, b + 1]$ and L has tor-amplitude in $[a, b]$ then M has tor-amplitude in $[a, b]$.*
2. *If K and M have tor-amplitude in $[a, b]$, then L has tor-amplitude in $[a, b]$.*
3. *If L has tor-amplitude in $[a + 1, b + 1]$ and M has tor-amplitude in $[a, b]$, then K has tor-amplitude in $[a + 1, b + 1]$.*

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes_\mathcal{O}^\mathbf{L} \mathcal{F}$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. □

Lemma 46.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. If K has tor-amplitude in $[a, b]$ and L has tor-amplitude in $[c, d]$ then $K \otimes_\mathcal{O}^\mathbf{L} L$ has tor amplitude in $[a + c, b + d]$.*

Proof. Omitted. Hint: use the spectral sequence for tors. □

Lemma 46.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $a, b \in \mathbf{Z}$. For K, L objects of $D(\mathcal{O})$ if $K \oplus L$ has tor amplitude in $[a, b]$ so do K and L.*

Proof. Clear from the fact that the Tor functors are additive. □

Lemma 46.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. Let K be an object of $D(\mathcal{O})$.*

1. *If $K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}$ is bounded above, then $K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n$ is uniformly bounded above for all n.*
2. *If $K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}$ as an object of $D(\mathcal{O}/\mathcal{I})$ has tor amplitude in $[a, b]$, then $K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n$ as an object of $D(\mathcal{O}/\mathcal{I}^n)$ has tor amplitude in $[a, b]$ for all n.*

Proof. Proof of (1). Assume that $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}$ is bounded above, say $H^i(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}) = 0$ for $i > b$. Note that we have distinguished triangles

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1} \to K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^{n+1} \to K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n \to K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1}[1]$$

and that

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1} = \left(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}\right) \otimes_{\mathcal{O}/\mathcal{I}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1}$$

By induction we conclude that $H^i(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n) = 0$ for $i > b$ for all n.

Proof of (2). Assume $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}$ as an object of $D(\mathcal{O}/\mathcal{I})$ has tor amplitude in $[a, b]$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}/\mathcal{I}^n$-modules. Then we have a finite filtration

$$0 \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

whose successive quotients are sheaves of $\mathcal{O}/\mathcal{I}$-modules. Thus to prove that $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n$ has tor amplitude in $[a, b]$ it suffices to show $H^i(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n \otimes_{\mathcal{O}/\mathcal{I}^n}^{\mathbf{L}} \mathcal{G})$ is zero for $i \notin [a, b]$ for all $\mathcal{O}/\mathcal{I}$-modules $\mathcal{G}$. Since

$$\left(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n\right) \otimes_{\mathcal{O}/\mathcal{I}^n}^{\mathbf{L}} \mathcal{G} = \left(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}\right) \otimes_{\mathcal{O}/\mathcal{I}}^{\mathbf{L}} \mathcal{G}$$

for every sheaf of $\mathcal{O}/\mathcal{I}$-modules $\mathcal{G}$ the result follows. ☐

Lemma 46.10. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$.*

1. *If E has tor amplitude in $[a, b]$, then for every point p of the site $\mathcal{C}$ the object E_p of $D(\mathcal{O}_p)$ has tor amplitude in $[a, b]$.*
2. *If $\mathcal{C}$ has enough points, then the converse is true.*

Proof. Proof of (1). This follows because taking stalks at p is the same as pulling back by the morphism of ringed sites $(p, \mathcal{O}_p) \to (\mathcal{C}, \mathcal{O})$ and hence we can apply Lemma 46.5.

Proof of (2). If $\mathcal{C}$ has enough points, then we can check vanishing of $H^i(E \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F})$ at stalks, see Modules on Sites, Lemma 14.4. Since $H^i(E \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F})_p = H^i(E_p \otimes_{\mathcal{O}_p}^{\mathbf{L}} \mathcal{F}_p)$ we conclude. ☐

47. Perfect complexes

In this section we discuss properties of perfect complexes on ringed sites.

Definition 47.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. We say $\mathcal{E}^\bullet$ is *perfect* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that for each i there exists a morphism of complexes $\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ which is a quasi-isomorphism with $\mathcal{E}_i^\bullet$ strictly perfect. An object E of $D(\mathcal{O})$ is *perfect* if it can be represented by a perfect complex of $\mathcal{O}$-modules.

Lemma 47.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$.*

1. *If $\mathcal{C}$ has a final object X and there exist a covering $\{U_i \to X\}$, strictly perfect complexes $\mathcal{E}_i^\bullet$ of $\mathcal{O}_{U_i}$-modules, and isomorphisms $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$, then E is perfect.*
2. *If E is perfect, then any complex representing E is perfect.*

Proof. Identical to the proof of Lemma 45.2. ☐

Lemma 47.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a \leq b$ be integers. If E has tor amplitude in $[a, b]$ and is $(a-1)$-pseudo-coherent, then E is perfect.*

Proof. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering and $\mathcal{C}$ by the localization $\mathcal{C}/U$ we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to E$ such that $H^i(\alpha)$ is an isomorphism for $i \geq a$. We may and do replace $\mathcal{E}^\bullet$ by $\sigma_{\geq a-1}\mathcal{E}^\bullet$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to E \to C \to \mathcal{E}^\bullet[1]$$

From the vanishing of cohomology sheaves of E and $\mathcal{E}^\bullet$ and the assumption on α we obtain $C \cong \mathcal{K}[a-2]$ with $\mathcal{K} = \text{Ker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Applying $- \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}$ the assumption that E has tor amplitude in $[a, b]$ implies $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F}$ has image $\text{Ker}(\mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}} \mathcal{F})$. It follows that $\text{Tor}_1^{\mathcal{O}}(\mathcal{E}', \mathcal{F}) = 0$ where $\mathcal{E}' = \text{Coker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Hence $\mathcal{E}'$ is flat (Lemma 17.15). Thus there exists a covering $\{U_i \to U\}$ such that $\mathcal{E}'|_{U_i}$ is a direct summand of a finite free module by Modules on Sites, Lemma 29.3. Thus the complex

$$\mathcal{E}'|_{U_i} \to \mathcal{E}^{a-1}|_{U_i} \to \ldots \to \mathcal{E}^b|_{U_i}$$

is quasi-isomorphic to $E|_{U_i}$ and E is perfect. $\square$

Lemma 47.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. The following are equivalent*

(1) *E is perfect, and*
(2) *E is pseudo-coherent and locally has finite tor dimension.*

Proof. Assume (1). Let U be an object of $\mathcal{C}$. By definition there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ is represented by a strictly perfect complex. Thus E is pseudo-coherent (i.e., m-pseudo-coherent for all m) by Lemma 45.2. Moreover, a direct summand of a finite free module is flat, hence $E|_{U_i}$ has finite Tor dimension by Lemma 46.3. Thus (2) holds.

Assume (2). Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume there exist integers $a \leq b$ such that $E|_U$ has tor amplitude in $[a, b]$. Since $E|_U$ is m-pseudo-coherent for all m we conclude using Lemma 47.3. $\square$

Lemma 47.5. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let E be an object of $D(\mathcal{O}_\mathcal{D})$. If E is perfect in $D(\mathcal{O}_\mathcal{D})$, then Lf^*E is perfect in $D(\mathcal{O}_\mathcal{C})$.*

Proof. This follows from Lemma 47.4, 46.5, and 45.3. $\square$

Lemma 47.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$. If two out of three of K, L, M are perfect then the third is also perfect.*

Proof. First proof: Combine Lemmas 47.4, 45.4, and 46.6. Second proof (sketch): Say K and L are perfect. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume that $K|_U$ and $L|_U$ are represented by strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$. After replacing U by the members of a covering we may assume the map $K|_U \to L|_U$ is given by a map of complexes $\alpha : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, see Lemma 44.8. Then $M|_U$ is isomorphic to the cone of α which is strictly perfect by Lemma 44.2. $\square$

Lemma 47.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If K, L are perfect objects of $D(\mathcal{O})$, then so is $K \otimes_{\mathcal{O}}^{\mathbf{L}} L$.*

Proof. Follows from Lemmas 47.4, 45.5, and 46.7. $\square$

Lemma 47.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $K \oplus L$ is a perfect object of $D(\mathcal{O})$, then so are K and L.*

Proof. Follows from Lemmas 47.4, 45.6, and 46.8. $\square$

48. Duals

In this section we characterize the dualizable objects of the category of complexes and of the derived category. In particular, we will see that an object of $D(\mathcal{O})$ has a dual if and only if it is perfect (this follows from Example 48.6 and Lemma 48.7).

Lemma 48.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. The category of complexes of $\mathcal{O}$-modules with tensor product defined by $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet = Tot(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{G}^\bullet)$ is a symmetric monoidal category.*

Proof. Omitted. Hints: as unit $\mathbf{1}$ we take the complex having $\mathcal{O}$ in degree 0 and zero in other degrees with obvious isomorphisms $\text{Tot}(\mathbf{1} \otimes_\mathcal{O} \mathcal{G}^\bullet) = \mathcal{G}^\bullet$ and $\text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathbf{1}) = \mathcal{F}^\bullet$. to prove the lemma you have to check the commutativity of various diagrams, see Categories, Definitions 43.1 and 43.9. The verifications are straightforward in each case. $\square$

Example 48.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules such that for every $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}^\bullet|_{U_i}$ is strictly perfect. Consider the complex

$$\mathcal{G}^\bullet = \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{O})$$

as in Section 34. Let

$$\eta : \mathcal{O} \to \text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{G}^\bullet) \quad \text{and} \quad \epsilon : \text{Tot}(\mathcal{G}^\bullet \otimes_\mathcal{O} \mathcal{F}^\bullet) \to \mathcal{O}$$

be $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O} \to \mathcal{F}^n \otimes_\mathcal{O} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_\mathcal{O} \mathcal{F}^n \to \mathcal{O}$ are as in Modules on Sites, Example 29.1. Then $\mathcal{G}^\bullet, \eta, \epsilon$ is a left dual for $\mathcal{F}^\bullet$ as in Categories, Definition 43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \text{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \text{id}_{\mathcal{G}^\bullet}$. Please compare with More on Algebra, Lemma 72.2.

Lemma 48.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules. If $\mathcal{F}^\bullet$ has a left dual in the monoidal category of complexes of $\mathcal{O}$-modules (Categories, Definition 43.5) then for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}^\bullet|_{U_i}$ is strictly perfect and the left dual is as constructed in Example 48.2.*

Proof. By uniqueness of left duals (Categories, Remark 43.7) we get the final statement provided we show that $\mathcal{F}^\bullet$ is as stated. Let $\mathcal{G}^\bullet, \eta, \epsilon$ be a left dual. Write $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O} \to \mathcal{F}^n \otimes_\mathcal{O} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_\mathcal{O} \mathcal{F}^n \to \mathcal{O}$. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \text{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \text{id}_{\mathcal{G}^\bullet}$ by Categories, Definition 43.5 we see immediately that we have $(1 \otimes \epsilon_n) \circ (\eta_n \otimes 1) = \text{id}_{\mathcal{F}^n}$ and $(\epsilon_n \otimes 1) \circ (1 \otimes \eta_n) = \text{id}_{\mathcal{G}^{-n}}$. In other words, we see that $\mathcal{G}^{-n}$ is a left dual of $\mathcal{F}^n$ and we see that Modules on Sites, Lemma 29.2 applies to each $\mathcal{F}^n$. Let U be an object of $\mathcal{C}$. There exists a covering $\{U_i \to U\}$ such that for every i only a finite number of $\eta_n|_{U_i}$ are nonzero. Thus after replacing U by U_i we may assume only a finite number of $\eta_n|_U$ are nonzero and by the lemma cited this implies only a finite number of $\mathcal{F}^n|_U$ are nonzero. Using the lemma again we can then find a covering $\{U_i \to U\}$ such that each $\mathcal{F}^n|_{U_i}$ is a direct summand of a finite free $\mathcal{O}$-module and the proof is complete. $\square$

Lemma 48.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a perfect object of $D(\mathcal{O})$. Then $K^\vee = R\mathcal{H}om(K, \mathcal{O})$ is a perfect object too and $(K^\vee)^\vee \cong K$. There are functorial isomorphisms*
$$M \otimes^{\mathbf{L}}_{\mathcal{O}} K^\vee = R\mathcal{H}om_{\mathcal{O}}(K, M)$$
and
$$H^0(\mathcal{C}, M \otimes^{\mathbf{L}}_{\mathcal{O}} K^\vee) = \mathrm{Hom}_{D(\mathcal{O})}(K, M)$$
for M in $D(\mathcal{O})$.

Proof. We will us without further mention that formation of internal hom commutes with restriction (Lemma 35.3). Let U be an arbitrary object of $\mathcal{C}$. To check that $K^\vee$ is perfect, it suffices to show that there exists a covering $\{U_i \to U\}$ such that $K^\vee|_{U_i}$ is perfect for all i. There is a canonical map
$$K = R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, \mathcal{O}_X), \mathcal{O}_X) = (K^\vee)^\vee$$
see Lemma 35.5. It suffices to prove there is a covering $\{U_i \to U\}$ such that the restriction of this map to $\mathcal{C}/U_i$ is an isomorphism for all i. By Lemma 35.9 to see the final statement it suffices to check that the map (35.9.1)
$$M \otimes^{\mathbf{L}}_{\mathcal{O}} K^\vee \longrightarrow R\mathcal{H}om(K, M)$$
is an isomorphism. This is a local question as well (in the sense above). Hence it suffices to prove the lemma when K is represented by a strictly perfect complex.

Assume K is represented by the strictly perfect complex $\mathcal{E}^\bullet$. Then it follows from Lemma 44.9 that $K^\vee$ is represented by the complex whose terms are $(\mathcal{E}^n)^\vee = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^n, \mathcal{O})$ in degree $-n$. Since $\mathcal{E}^n$ is a direct summand of a finite free $\mathcal{O}$-module, so is $(\mathcal{E}^n)^\vee$. Hence $K^\vee$ is represented by a strictly perfect complex too and we see that $K^\vee$ is perfect. The map $K \to (K^\vee)^\vee$ is an isomorphism as it is given up to sign by the evaluation maps $\mathcal{E}^n \to ((\mathcal{E}^n)^\vee)^\vee$ which are isomorphisms. To see that (35.9.1) is an isomorphism, represent M by a K-flat complex $\mathcal{F}^\bullet$. By Lemma 44.9 the complex $R\mathcal{H}om(K, M)$ is represented by the complex with terms
$$\bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^p)$$
On the other hand, the object $M \otimes^{\mathbf{L}}_{\mathcal{O}} K^\vee$ is represented by the complex with terms
$$\bigoplus\nolimits_{n=p+q} \mathcal{F}^p \otimes_{\mathcal{O}} (\mathcal{E}^{-q})^\vee$$
Thus the assertion that (35.9.1) is an isomorphism reduces to the assertion that the canonical map
$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{F})$$
is an isomorphism when $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}$-module and $\mathcal{F}$ is any $\mathcal{O}$-module. This follows immediately from the corresponding statement when $\mathcal{E}$ is finite free. ☐

Lemma 48.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The derived category $D(\mathcal{O})$ is a symmetric monoidal category with tensor product given by derived tensor product with usual associativity and commutativity constraints (for sign rules, see More on Algebra, Section 72).*

Proof. Omitted. Compare with Lemma 48.1. ☐

Example 48.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a perfect object of $D(\mathcal{O})$. Set $K^\vee = R\mathcal{H}om(K, \mathcal{O})$ as in Lemma 48.4. Then the map

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} K^\vee \longrightarrow R\mathcal{H}om(K, K)$$

is an isomorphism (by the lemma). Denote

$$\eta : \mathcal{O} \longrightarrow K \otimes_{\mathcal{O}}^{\mathbf{L}} K^\vee$$

the map sending 1 to the section corresponding to id_K under the isomorphism above. Denote

$$\epsilon : K^\vee \otimes_{\mathcal{O}}^{\mathbf{L}} K \longrightarrow \mathcal{O}$$

the evaluation map (to construct it you can use Lemma 35.6 for example). Then $K^\vee, \eta, \epsilon$ is a left dual for K as in Categories, Definition 43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_K$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{K^\vee}$.

Lemma 48.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let M be an object of $D(\mathcal{O})$. If M has a left dual in the monoidal category $D(\mathcal{O})$ (Categories, Definition 43.5) then M is perfect and the left dual is as constructed in Example 48.6.*

Proof. Let N, η, ϵ be a left dual. Observe that for any object U of $\mathcal{C}$ the restriction $N|_U, \eta|_U, \epsilon|_U$ is a left dual for $M|_U$.

Let U be an object of $\mathcal{C}$. It suffices to find a covering $\{U_i \to U\}_{i \in I}$ fo $\mathcal{C}$ such that $M|_{U_i}$ is a perfect object of $D(\mathcal{O}_{U_i})$. Hence we may replace $\mathcal{C}, \mathcal{O}, M, N, \eta, \epsilon$ by $\mathcal{C}/U, \mathcal{O}_U, M|_U, N|_U, \eta|_U, \epsilon|_U$ and assume $\mathcal{C}$ has a final object X. Moreover, during the proof we can (finitely often) replace X by the members of a covering $\{U_i \to X\}$ of X.

We are going to use the following argument several times. Choose any complex $\mathcal{M}^\bullet$ of $\mathcal{O}$-modules representing M. Choose a K-flat complex $\mathcal{N}^\bullet$ representing N whose terms are flat $\mathcal{O}$-modules, see Lemma 17.11. Consider the map

$$\eta : \mathcal{O} \to \mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{N}^\bullet)$$

After replacing X by the members of a covering, we can find an integer N and for $i = 1, \ldots, N$ integers $n_i \in \mathbf{Z}$ and sections f_i and g_i of $\mathcal{M}^{n_i}$ and $\mathcal{N}^{-n_i}$ such that

$$\eta(1) = \sum\nolimits_i f_i \otimes g_i$$

Let $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ be any subcomplex of $\mathcal{O}$-modules containing the sections f_i for $i = 1, \ldots, N$. Since $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{N}^\bullet) \subset \mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{N}^\bullet)$ by flatness of the modules $\mathcal{N}^n$, we see that η factors through

$$\tilde{\eta} : \mathcal{O} \to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{N}^\bullet)$$

Denoting K the object of $D(\mathcal{O})$ represented by $\mathcal{K}^\bullet$ we find a commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\eta\otimes 1"'] \arrow[dr, "\tilde{\eta}\otimes 1"'] & M \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M \arrow[r, "1\otimes\epsilon"'] & M \\
& K \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M \arrow[u] \arrow[r, "1\otimes\epsilon"] & K \arrow[u]
\end{tikzcd}$$

Since the composition of the upper row is the identity on M we conclude that M is a direct summand of K in $D(\mathcal{O})$.

As a first use of the argument above, we can choose the subcomplex $\mathcal{K}^\bullet = \sigma_{\geq a}\tau_{\leq b}\mathcal{M}^\bullet$ with $a < n_i < b$ for $i = 1, \ldots, N$. Thus M is a direct summand in $D(\mathcal{O})$ of a bounded complex and we conclude we may assume M is in $D^b(\mathcal{O})$. (Recall that the process above involves replacing X by the members of a covering.)

Since M is in $D^b(\mathcal{O})$ we may choose $\mathcal{M}^\bullet$ to be a bounded above complex of flat modules (by Modules, Lemma 17.6 and Derived Categories, Lemma 15.4). Then we can choose $\mathcal{K}^\bullet = \sigma_{\geq a}\mathcal{M}^\bullet$ with $a < n_i$ for $i = 1, \ldots, N$ in the argument above. Thus we find that we may assume M is a direct summand in $D(\mathcal{O})$ of a bounded complex of flat modules. In particular, we find M has finite tor amplitude.

Say M has tor amplitude in $[a, b]$. Assuming M is m-pseudo-coherent we are going to show that (after replacing X by the members of a covering) we may assume M is $(m-1)$-pseudo-coherent. This will finish the proof by Lemma 47.3 and the fact that M is $(b+1)$-pseudo-coherent in any case. After replacing X by the members of a covering we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to M$ in $D(\mathcal{O})$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and surjective for $i = m$. We may and do assume that $\mathcal{E}^i = 0$ for $i < m$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to M \to L \to \mathcal{E}^\bullet[1]$$

Observe that $H^i(L) = 0$ for $i \geq m$. Thus we may represent L by a complex $\mathcal{L}^\bullet$ with $\mathcal{L}^i = 0$ for $i \geq m$. The map $L \to \mathcal{E}^\bullet[1]$ is given by a map of complexes $\mathcal{L}^\bullet \to \mathcal{E}^\bullet[1]$ which is zero in all degrees except in degree $m-1$ where we obtain a map $\mathcal{L}^{m-1} \to \mathcal{E}^m$, see Derived Categories, Lemma 27.3. Then M is represented by the complex

$$\mathcal{M}^\bullet : \ldots \to \mathcal{L}^{m-2} \to \mathcal{L}^{m-1} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots$$

Apply the discussion in the second paragraph to this complex to get sections f_i of $\mathcal{M}^{n_i}$ for $i = 1, \ldots, N$. For $n < m$ let $\mathcal{K}^n \subset \mathcal{L}^n$ be the $\mathcal{O}$-submodule generated by the sections f_i for $n_i = n$ and $d(f_i)$ for $n_i = n - 1$. For $n \geq m$ set $\mathcal{K}^n = \mathcal{E}^n$. Clearly, we have a morphism of distinguished triangles

$$\begin{tikzcd}
\mathcal{E}^\bullet \arrow[r] & \mathcal{M}^\bullet \arrow[r] & \mathcal{L}^\bullet \arrow[r] & \mathcal{E}^\bullet[1] \\
\mathcal{E}^\bullet \arrow[r] \arrow[u] & \mathcal{K}^\bullet \arrow[r] \arrow[u] & \sigma_{\leq m-1}\mathcal{K}^\bullet \arrow[r] \arrow[u] & \mathcal{E}^\bullet[1] \arrow[u]
\end{tikzcd}$$

where all the morphisms are as indicated above. Denote K the object of $D(\mathcal{O})$ corresponding to the complex $\mathcal{K}^\bullet$. By the arguments in the second paragraph of the proof we obtain a morphism $s : M \to K$ in $D(\mathcal{O})$ such that the composition $M \to K \to M$ is the identity on M. We don't know that the diagram

$$\begin{tikzcd}
\mathcal{E}^\bullet \arrow[r] & \mathcal{K}^\bullet \arrow[r, equal] & K \\
\mathcal{E}^\bullet \arrow[r, "i"] \arrow[u, "\mathrm{id}"] & \mathcal{M}^\bullet \arrow[r, equal] & M \arrow[u, "s"']
\end{tikzcd}$$

commutes, but we do know it commutes after composing with the map $K \to M$. By Lemma 44.8 after replacing X by the members of a covering, we may assume that $s \circ i$ is given by a map of complexes $\sigma : \mathcal{E}^\bullet \to \mathcal{K}^\bullet$. By the same lemma we may assume the composition of σ with the inclusion $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ is homotopic to zero by some homotopy $\{h^i : \mathcal{E}^i \to \mathcal{M}^{i-1}\}$. Thus, after replacing $\mathcal{K}^{m-1}$ by $\mathcal{K}^{m-1} + \mathrm{Im}(h^m)$ (note that after doing this it is still the case that $\mathcal{K}^{m-1}$ is generated by finitely many global sections), we see that σ itself is homotopic to zero! This means that we have a commutative solid diagram

$$\begin{tikzcd}
\mathcal{E}^\bullet \arrow[r] & M \arrow[r] & \mathcal{L}^\bullet \arrow[r] & \mathcal{E}^\bullet[1] \\
\mathcal{E}^\bullet \arrow[r] \arrow[u] & K \arrow[r] \arrow[u] & \sigma_{\leq m-1}\mathcal{K}^\bullet \arrow[r] \arrow[u] & \mathcal{E}^\bullet[1] \arrow[u] \\
\mathcal{E}^\bullet \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "s"] & \mathcal{L}^\bullet \arrow[r] \arrow[u, dotted] & \mathcal{E}^\bullet[1] \arrow[u]
\end{tikzcd}$$

By the axioms of triangulated categories we obtain a dotted arrow fitting into the diagram. Looking at cohomology sheaves in degree $m-1$ we see that we obtain

$$\begin{tikzcd}
H^{m-1}(M) \arrow[r] & H^{m-1}(\mathcal{L}^\bullet) \arrow[r] & H^m(\mathcal{E}^\bullet) \\
H^{m-1}(K) \arrow[r] \arrow[u] & H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \arrow[r] \arrow[u] & H^m(\mathcal{E}^\bullet) \arrow[u] \\
H^{m-1}(M) \arrow[r] \arrow[u] & H^{m-1}(\mathcal{L}^\bullet) \arrow[r] \arrow[u] & H^m(\mathcal{E}^\bullet) \arrow[u]
\end{tikzcd}$$

Since the vertical compositions are the identity in both the left and right column, we conclude the vertical composition $H^{m-1}(\mathcal{L}^\bullet) \to H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ in the middle is surjective! In particular $H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ is surjective. Using the induced map of long exact sequences of cohomology sheaves from the morphism of triangles above, a diagram chase shows this implies $H^i(K) \to H^i(M)$ is an isomorphism for $i \geq m$ and surjective for $i = m-1$. By construction we can choose an $r \geq 0$ and a surjection $\mathcal{O}^{\oplus r} \to \mathcal{K}^{m-1}$. Then the composition

$$(\mathcal{O}^{\oplus r} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots) \longrightarrow K \longrightarrow M$$

induces an isomorphism on cohomology sheaves in degrees $\geq m$ and a surjection in degree $m-1$ and the proof is complete. $\square$

Lemma 48.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(K_n)_{n \in \mathbf{N}}$ be a system of perfect objects of $D(\mathcal{O})$. Let $K = hocolim K_n$ be the derived colimit (Derived Categories, Definition 33.1). Then for any object E of $D(\mathcal{O})$ we have*

$$R\mathcal{H}om(K, E) = R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}} K_n^\vee$$

where $(K_n^\vee)$ is the inverse system of dual perfect complexes.

Proof. By Lemma 48.4 we have $R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}} K_n^\vee = R\lim R\mathcal{H}om(K_n, E)$ which fits into the distinguished triangle

$$R\lim R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E)$$

Because K similarly fits into the distinguished triangle $\bigoplus K_n \to \bigoplus K_n \to K$ it suffices to show that $\prod R\mathcal{H}om(K_n, E) = R\mathcal{H}om(\bigoplus K_n, E)$. This is a formal consequence of (35.0.1) and the fact that derived tensor product commutes with direct sums. $\square$

49. Invertible objects in the derived category

We characterize invertible objects in the derived category of a ringed space (both in the case of a locally ringed topos and in the general case).

Lemma 49.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. Set $R = \Gamma(\mathcal{C}, \mathcal{O})$. The category of $\mathcal{O}$-modules which are summands of finite free $\mathcal{O}$-modules is equivalent to the category of finite projective R-modules.*

Proof. Observe that a finite projective R-module is the same thing as a summand of a finite free R-module. The equivalence is given by the functor $\mathcal{E} \mapsto \Gamma(\mathcal{C}, \mathcal{E})$. The inverse functor is given by the following construction. Consider the morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathrm{pt})$ with f_* given by taking global sections and f^{-1} by sending a set S, i.e., an object of $Sh(\mathrm{pt})$, to the constant sheaf with value S. We obtain a morphism $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathrm{pt}), R)$ of ringed topoi by using the identity map $R \to f_*\mathcal{O}$. Then the inverse functor is given by f^*. $\square$

Lemma 49.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let M be an object of $D(\mathcal{O})$. The following are equivalent*

(1) *M is invertible in $D(\mathcal{O})$, see Categories, Definition 43.4, and*
(2) *there is a locally finite[11] direct product decomposition*
$$\mathcal{O} = \prod\nolimits_{n \in \mathbf{Z}} \mathcal{O}_n$$
and for each n there is an invertible $\mathcal{O}_n$-module $\mathcal{H}^n$ (Modules on Sites, Definition 32.1) and $M = \bigoplus \mathcal{H}^n[-n]$ in $D(\mathcal{O})$.

If (1) and (2) hold, then M is a perfect object of $D(\mathcal{O})$. If $(\mathcal{C}, \mathcal{O})$ is a locally ringed site these condition are also equivalent to

(3) *for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ and for each i an integer n_i such that $M|_{U_i}$ is represented by an invertible $\mathcal{O}_{U_i}$-module placed in degree n_i.*

Proof. Assume (2). Consider the object $R\mathcal{H}om(M, \mathcal{O})$ and the composition map
$$R\mathcal{H}om(M, \mathcal{O}) \otimes_{\mathcal{O}}^{\mathbf{L}} M \to \mathcal{O}$$
To prove this is an isomorphism, we may work locally. Thus we may assume $\mathcal{O} = \prod_{a \leq n \leq b} \mathcal{O}_n$ and $M = \bigoplus_{a \leq n \leq b} \mathcal{H}^n[-n]$. Then it suffices to show that
$$R\mathcal{H}om(\mathcal{H}^m, \mathcal{O}) \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{H}^n$$
is zero if $n \neq m$ and equal to $\mathcal{O}_n$ if $n = m$. The case $n \neq m$ follows from the fact that $\mathcal{O}_n$ and $\mathcal{O}_m$ are flat $\mathcal{O}$-algebras with $\mathcal{O}_n \otimes_{\mathcal{O}} \mathcal{O}_m = 0$. Using the local structure of invertible $\mathcal{O}$-modules (Modules on Sites, Lemma 32.2) and working locally the isomorphism in case $n = m$ follows in a straightforward manner; we omit the details. Because $D(\mathcal{O})$ is symmetric monoidal, we conclude that M is invertible.

Assume (1). The description in (2) shows that we have a candidate for $\mathcal{O}_n$, namely, $\mathcal{H}om_{\mathcal{O}}(H^n(M), H^n(M))$. If this is a locally finite family of sheaves of rings and if $\mathcal{O} = \prod \mathcal{O}_n$, then we immediately obtain the direct sum decomposition $M = \bigoplus H^n(M)[-n]$ using the idempotents in $\mathcal{O}$ coming from the product decomposition.

[11] This means that for every object U of $\mathcal{C}$ there is a covering $\{U_i \to U\}$ such that for every i the sheaf $\mathcal{O}_n|_{U_i}$ is nonzero for only a finite number of n.

This shows that in order to prove (2) we may work locally in the following sense. Let U be an object of $\mathcal{C}$. We have to show there exists a covering $\{U_i \to U\}$ of U such that with $\mathcal{O}_n$ as above we have the statements above and those of (2) after restriction to $\mathcal{C}/U_i$. Thus we may assume $\mathcal{C}$ has a final object X and during the proof of (2) we may finitely many times replace X by the members of a covering of X.

Choose an object N of $D(\mathcal{O})$ and an isomorphism $M \otimes_{\mathcal{O}}^{\mathbf{L}} N \cong \mathcal{O}$. Then N is a left dual for M in the monoidal category $D(\mathcal{O})$ and we conclude that M is perfect by Lemma 48.7. By symmetry we see that N is perfect. After replacing X by the members of a covering, we may assume M and N are represented by a strictly perfect complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$. Then $M \otimes_{\mathcal{O}}^{\mathbf{L}} N$ is represented by $\text{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet)$. After replacing X by the members of a covering of X we may assume the mutually inverse isomorphisms $\mathcal{O} \to M \otimes_{\mathcal{O}}^{\mathbf{L}} N$ and $M \otimes_{\mathcal{O}}^{\mathbf{L}} N \to \mathcal{O}$ are given by maps of complexes

$$\alpha : \mathcal{O} \to \text{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet) \quad \text{and} \quad \beta : \text{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet) \to \mathcal{O}$$

See Lemma 44.8. Then $\beta \circ \alpha = 1$ as maps of complexes and $\alpha \circ \beta = 1$ as a morphism in $D(\mathcal{O})$. After replacing X by the members of a covering of X we may assume the composition $\alpha \circ \beta$ is homotopic to 1 by some homotopy θ with components

$$\theta^n : \text{Tot}^n(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet) \to \text{Tot}^{n-1}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet)$$

by the same lemma as before. Set $R = \Gamma(\mathcal{C}, \mathcal{O})$. By Lemma 49.1 we find that we obtain

1. $M^\bullet = \Gamma(X, \mathcal{E}^\bullet)$ is a bounded complex of finite projective R-modules,
2. $N^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ is a bounded complex of finite projective R-modules,
3. α and β correspond to maps of complexes $a : R \to \text{Tot}(M^\bullet \otimes_R N^\bullet)$ and $b : \text{Tot}(M^\bullet \otimes_R N^\bullet) \to R$,
4. θ^n corresponds to a map $h^n : \text{Tot}^n(M^\bullet \otimes_R N^\bullet) \to \text{Tot}^{n-1}(M^\bullet \otimes_R N^\bullet)$, and
5. $b \circ a = 1$ and $b \circ a - 1 = dh + hd$,

It follows that $M^\bullet$ and $N^\bullet$ define mutually inverse objects of $D(R)$. By More on Algebra, Lemma 126.4 we find a product decomposition $R = \prod_{a \leq n \leq b} R_n$ and invertible R_n-modules H^n such that $M^\bullet \cong \bigoplus_{a \leq n \leq b} H^n[-n]$. This isomorphism in $D(R)$ can be lifted to an morphism

$$\bigoplus H^n[-n] \longrightarrow M^\bullet$$

of complexes because each H^n is projective as an R-module. Correspondingly, using Lemma 49.1 again, we obtain an morphism

$$\bigoplus H^n \otimes_R \mathcal{O}[-n] \to \mathcal{E}^\bullet$$

which is an isomorphism in $D(\mathcal{O})$. Here $M \otimes_R \mathcal{O}$ denotes the functor from finite projective R-modules to $\mathcal{O}$-modules constructed in the proof of Lemma 49.1. Setting $\mathcal{O}_n = R_n \otimes_R \mathcal{O}$ we conclude (2) is true.

If $(\mathcal{C}, \mathcal{O})$ is a locally ringed site, then given an object U and a finite product decomposition $\mathcal{O}|_U = \prod_{a \leq n \leq b} \mathcal{O}_n|_U$ we can find a covering $\{U_i \to U\}$ such that for every i there is at most one n with $\mathcal{O}_n|_{U_i}$ nonzero. This follows readily from part (2) of Modules on Sites, Lemma 40.1 and the definition of locally ringed sites as given in Modules on Sites, Definition 40.4. From this the implication (2) $\Rightarrow$ (3) is easily seen. The implication (3) $\Rightarrow$ (2) holds without any assumptions on the ringed site. We omit the details. $\square$

50. Projection formula

Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $E \in D(\mathcal{O}_\mathcal{C})$ and $K \in D(\mathcal{O}_\mathcal{D})$. Without any further assumptions there is a map

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K) \tag{50.0.1}$$

Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K) = Lf^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K \longrightarrow E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$ and Lemmas 18.4 and 19.1. A reasonably general version of the projection formula is the following.

Lemma 50.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $E \in D(\mathcal{O}_\mathcal{C})$ and $K \in D(\mathcal{O}_\mathcal{D})$. If K is perfect, then*

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K)$$

in $D(\mathcal{O}_\mathcal{D})$.

Proof. To check (50.0.1) is an isomorphism we may work locally on $\mathcal{D}$, i.e., for any object V of $\mathcal{D}$ we have to find a covering $\{V_j \to V\}$ such that the map restricts to an isomorphism on V_j. By definition of perfect objects, this means we may assume K is represented by a strictly perfect complex of $\mathcal{O}_\mathcal{D}$-modules. Note that, completely generally, the statement is true for $K = K_1 \oplus K_2$, if and only if the statement is true for K_1 and K_2. Hence we may assume K is a finite complex of finite free $\mathcal{O}_\mathcal{D}$-modules. In this case a simple argument involving stupid truncations reduces the statement to the case where K is represented by a finite free $\mathcal{O}_\mathcal{D}$-module. Since the statement is invariant under finite direct summands in the K variable, we conclude it suffices to prove it for $K = \mathcal{O}_\mathcal{D}[n]$ in which case it is trivial. $\square$

Remark 50.2. The map (50.0.1) is compatible with the base change map of Remark 19.3 in the following sense. Namely, suppose that

$$\begin{tikzcd}
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g'"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "f"] \\
(Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})
\end{tikzcd}$$

is a commutative diagram of ringed topoi. Let $E \in D(\mathcal{O}_\mathcal{C})$ and $K \in D(\mathcal{O}_\mathcal{D})$. Then the diagram

$$\begin{tikzcd}
Lg^*(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K) \arrow[r, "p"'] \arrow[d, "t"'] & Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K) \arrow[d, "b"'] \\
Lg^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} Lg^*K \arrow[d, "b"'] & Rf'_*L(g')^*(E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K) \arrow[d, "t"'] \\
Rf'_*L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} Lg^*K \arrow[rd, "p"'] & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} L(g')^*Lf^*K) \arrow[d, "c"'] \\
 & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} L(f')^*Lg^*K)
\end{tikzcd}$$

is commutative. Here arrows labeled t are gotten by an application of Lemma 18.4, arrows labeled b by an application of Remark 19.3, arrows labeled p by an application of (50.0.1), and c comes from $L(g')^* \circ Lf^* = L(f')^* \circ Lg^*$. We omit the verification.

51. Weakly contractible objects

An object U of a site is *weakly contractible* if every surjection $\mathcal{F} \to \mathcal{G}$ of sheaves of sets gives rise to a surjection $\mathcal{F}(U) \to \mathcal{G}(U)$, see Sites, Definition 40.2.

Lemma 51.1. *Let $\mathcal{C}$ be a site. Let U be a weakly contractible object of $\mathcal{C}$. Then*

1. *the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor $Ab(\mathcal{C}) \to Ab$,*
2. *$H^p(U, \mathcal{F}) = 0$ for every abelian sheaf $\mathcal{F}$ and all $p \geq 1$, and*
3. *for any sheaf of groups $\mathcal{G}$ any $\mathcal{G}$-torsor has a section over U.*

Proof. The first statement follows immediately from the definition (see also Homology, Section 7). The higher derived functors vanish by Derived Categories, Lemma 16.9. Let $\mathcal{F}$ be a $\mathcal{G}$-torsor. Then $\mathcal{F} \to *$ is a surjective map of sheaves. Hence (3) follows from the definition as well. $\square$

It is convenient to list some consequences of having enough weakly contractible objects here.

Proposition 51.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ such that every $U \in \mathcal{B}$ is weakly contractible and every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Then*

1. *A complex $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ of $\mathcal{O}$-modules is exact, if and only if $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact for all $U \in \mathcal{B}$.*
2. *Every object K of $D(\mathcal{O})$ is a derived limit of its canonical truncations: $K = R\lim \tau_{\geq -n} K$.*
3. *Given an inverse system $\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$ with surjective transition maps, the projection $\lim \mathcal{F}_n \to \mathcal{F}_1$ is surjective.*
4. *Products are exact on $Mod(\mathcal{O})$.*
5. *Products on $D(\mathcal{O})$ can be computed by taking products of any representative complexes.*

(6) *If* $(\mathcal{F}_n)$ *is an inverse system of* $\mathcal{O}$*-modules, then* $R^p \lim \mathcal{F}_n = 0$ *for all* $p > 1$ *and*

$$R^1 \lim \mathcal{F}_n = \mathrm{Coker}(\prod \mathcal{F}_n \to \prod \mathcal{F}_n)$$

where the map is $(x_n) \mapsto (x_n - f(x_{n+1}))$.

(7) *If* (K_n) *is an inverse system of objects of* $D(\mathcal{O})$*, then there are short exact sequences*

$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(R\lim K_n) \to \lim H^p(K_n) \to 0$$

Proof. Proof of (1). If the sequence is exact, then evaluating at any weakly contractible element of $\mathcal{C}$ gives an exact sequence by Lemma 51.1. Conversely, assume that $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact for all $U \in \mathcal{B}$. Let V be an object of $\mathcal{C}$ and let $s \in \mathcal{F}_2(V)$ be an element of the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$. By assumption there exists a covering $\{U_i \to V\}$ with $U_i \in \mathcal{B}$. Then $s|_{U_i}$ lifts to a section $s_i \in \mathcal{F}_1(U_i)$. Thus s is a section of the image sheaf $\mathrm{Im}(\mathcal{F}_1 \to \mathcal{F}_2)$. In other words, the sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact.

Proof of (2). This holds by Lemma 23.10 with $d = 0$.

Proof of (3). Let $(\mathcal{F}_n)$ be a system as in (2) and set $\mathcal{F} = \lim \mathcal{F}_n$. If $U \in \mathcal{B}$, then $\mathcal{F}(U) = \lim \mathcal{F}_n(U)$ surjects onto $\mathcal{F}_1(U)$ as all the transition maps $\mathcal{F}_{n+1}(U) \to \mathcal{F}_n(U)$ are surjective. Thus $\mathcal{F} \to \mathcal{F}_1$ is surjective by Sites, Definition 11.1 and the assumption that every object has a covering by elements of $\mathcal{B}$.

Proof of (4). Let $\mathcal{F}_{i,1} \to \mathcal{F}_{i,2} \to \mathcal{F}_{i,3}$ be a family of exact sequences of $\mathcal{O}$-modules. We want to show that $\prod \mathcal{F}_{i,1} \to \prod \mathcal{F}_{i,2} \to \prod \mathcal{F}_{i,3}$ is exact. We use the criterion of (1). Let $U \in \mathcal{B}$. Then

$$(\prod \mathcal{F}_{i,1})(U) \to (\prod \mathcal{F}_{i,2})(U) \to (\prod \mathcal{F}_{i,3})(U)$$

is the same as

$$\prod \mathcal{F}_{i,1}(U) \to \prod \mathcal{F}_{i,2}(U) \to \prod \mathcal{F}_{i,3}(U)$$

Each of the sequences $\mathcal{F}_{i,1}(U) \to \mathcal{F}_{i,2}(U) \to \mathcal{F}_{i,3}(U)$ are exact by (1). Thus the displayed sequences are exact by Homology, Lemma 32.1. We conclude by (1) again.

Proof of (5). Follows from (4) and (slightly generalized) Derived Categories, Lemma 34.2.

Proof of (6) and (7). We refer to Section 23 for a discussion of derived and homotopy limits and their relationship. By Derived Categories, Definition 34.1 we have a distinguished triangle

$$R\lim K_n \to \prod K_n \to \prod K_n \to R\lim K_n[1]$$

Taking the long exact sequence of cohomology sheaves we obtain

$$H^{p-1}(\prod K_n) \to H^{p-1}(\prod K_n) \to H^p(R\lim K_n) \to H^p(\prod K_n) \to H^p(\prod K_n)$$

Since products are exact by (4) this becomes

$$\prod H^{p-1}(K_n) \to \prod H^{p-1}(K_n) \to H^p(R\lim K_n) \to \prod H^p(K_n) \to \prod H^p(K_n)$$

Now we first apply this to the case $K_n = \mathcal{F}_n[0]$ where $(\mathcal{F}_n)$ is as in (6). We conclude that (6) holds. Next we apply it to (K_n) as in (7) and we conclude (7) holds. $\square$

52. Compact objects

In this section we study compact objects in the derived category of modules on a ringed site. We recall that compact objects are defined in Derived Categories, Definition 37.1.

Lemma 52.1. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $S \subset \mathrm{Ob}(\mathcal{A})$ be a set of objects such that*

1. *any object of $\mathcal{A}$ is a quotient of a direct sum of elements of S, and*
2. *for any $E \in S$ the functor $\mathrm{Hom}_\mathcal{A}(E, -)$ commutes with direct sums.*

Then every compact object of $D(\mathcal{A})$ is a direct summand in $D(\mathcal{A})$ of a finite complex of finite direct sums of elements of S.

Proof. Assume $K \in D(\mathcal{A})$ is a compact object. Represent K by a complex $K^\bullet$ and consider the map

$$K^\bullet \longrightarrow \bigoplus\nolimits_{n \geq 0} \tau_{\geq n} K^\bullet$$

where we have used the canonical truncations, see Homology, Section 15. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that $K \to \tau_{\geq n} K$ is zero for at least one n, i.e., K is in $D^-(R)$.

We may represent K by a bounded above complex $K^\bullet$ each of whose terms is a direct sum of objects from S, see Derived Categories, Lemma 15.4. Note that we have

$$K^\bullet = \bigcup\nolimits_{n \leq 0} \sigma_{\geq n} K^\bullet$$

where we have used the stupid truncations, see Homology, Section 15. Hence by Derived Categories, Lemmas 33.7 and 33.9 we see that $1 : K^\bullet \to K^\bullet$ factors through $\sigma_{\geq n} K^\bullet \to K^\bullet$ in $D(R)$. Thus we see that $1 : K^\bullet \to K^\bullet$ factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in $D(\mathcal{A})$ for some complex $L^\bullet$ which is bounded and whose terms are direct sums of elements of S. Say L^i is zero for $i \not\in [a, b]$. Let c be the largest integer $\leq b + 1$ such that L^i a finite direct sum of elements of S for $i < c$. Claim: if $c < b + 1$, then we can modify $L^\bullet$ to increase c. By induction this claim will show we have a factorization of 1_K as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in $D(\mathcal{A})$ where L can be represented by a finite complex of finite direct sums of elements of S. Note that $e = \varphi \circ \psi \in \mathrm{End}_{D(\mathcal{A})}(L)$ is an idempotent. By Derived Categories, Lemma 4.14 we see that $L = \mathrm{Ker}(e) \oplus \mathrm{Ker}(1 - e)$. The map $\varphi : K \to L$ induces an isomorphism with $\mathrm{Ker}(1 - e)$ in $D(R)$ and we conclude.

Proof of the claim. Write $L^c = \bigoplus_{\lambda \in \Lambda} E_\lambda$. Since L^{c-1} is a finite direct sum of elements of S we can by assumption (2) find a finite subset $\Lambda' \subset \Lambda$ such that $L^{c-1} \to L^c$ factors through $\bigoplus_{\lambda \in \Lambda'} E_\lambda \subset L^c$. Consider the map of complexes

$$\pi : L^\bullet \longrightarrow (\bigoplus\nolimits_{\lambda \in \Lambda \setminus \Lambda'} E_\lambda)[-i]$$

given by the projection onto the factors corresponding to $\Lambda \setminus \Lambda'$ in degree i. By our assumption on K we see that, after possibly replacing Λ' by a larger finite subset, we may assume that $\pi \circ \varphi = 0$ in $D(\mathcal{A})$. Let $(L')^\bullet \subset L^\bullet$ be the kernel of π. Since π

is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in $D(\mathcal{A})$ (see Derived Categories, Lemma 12.1). Since $\mathrm{Hom}_{D(\mathcal{A})}(K, -)$ is homological (see Derived Categories, Lemma 4.2) and $\pi \circ \varphi = 0$, we can find a morphism $\varphi' : K^\bullet \to (L')^\bullet$ in $D(\mathcal{A})$ whose composition with $(L')^\bullet \to L^\bullet$ gives φ. Setting ψ' equal to the composition of ψ with $(L')^\bullet \to L^\bullet$ we obtain a new factorization. Since $(L')^\bullet$ agrees with $L^\bullet$ except in degree c and since $(L')^c = \bigoplus_{\lambda \in \Lambda'} E_\lambda$ the claim is proved. $\square$

Lemma 52.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Assume every object of $\mathcal{C}$ has a covering by quasi-compact objects. Then every compact object of $D(\mathcal{O})$ is a direct summand in $D(\mathcal{O})$ of a finite complex whose terms are finite direct sums of $\mathcal{O}$-modules of the form $j_!\mathcal{O}_U$ where U is a quasi-compact object of $\mathcal{C}$.*

Proof. Apply Lemma 52.1 where $S \subset \mathrm{Ob}(\textit{Mod}(\mathcal{O}))$ is the set of modules of the form $j_!\mathcal{O}_U$ with $U \in \mathrm{Ob}(\mathcal{C})$ quasi-compact. Assumption (1) holds by Modules on Sites, Lemma 28.8 and the assumption that every U can be covered by quasi-compact objects. Assumption (2) follows as

$$\mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$$

which commutes with direct sums by Sites, Lemma 17.7. $\square$

In the situation of the lemma above it is not always true that the modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O})$ (even if U is a quasi-compact object of $\mathcal{C}$). Here are two lemmas addressing this issue.

Lemma 52.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Assume the functors $\mathcal{F} \mapsto H^p(U, \mathcal{F})$ commute with direct sums. Then $\mathcal{O}$-module $j_!\mathcal{O}_U$ is a compact object of $D^+(\mathcal{O})$ in the following sense: if $M = \bigoplus_{i \in I} M_i$ in $D(\mathcal{O})$ is bounded below, then $\mathrm{Hom}(j_{U!}\mathcal{O}_U, M) = \bigoplus_{i \in I} \mathrm{Hom}(j_{U!}\mathcal{O}_U, M_i)$.*

Proof. Since $\mathrm{Hom}(j_{U!}\mathcal{O}_U, -)$ is the same as the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ by Modules on Sites, Equation (19.2.1) it suffices to prove that $H^p(U, M) = \bigoplus H^p(U, M_i)$. Let $\mathcal{I}_i$, $i \in I$ be a collection of injective $\mathcal{O}$-modules. By assumption we have

$$H^p(U, \bigoplus\nolimits_{i \in I} \mathcal{I}_i) = \bigoplus\nolimits_{i \in I} H^p(U, \mathcal{I}_i) = 0$$

for all p. Since $M = \bigoplus M_i$ is bounded below, we see that there exists an $a \in \mathbf{Z}$ such that $H^n(M_i) = 0$ for $n < a$. Thus we can choose complexes of injective $\mathcal{O}$-modues $\mathcal{I}_i^\bullet$ representing M_i with $\mathcal{I}_i^n = 0$ for $n < a$, see Derived Categories, Lemma 18.3. By Injectives, Lemma 13.4 we see that the direct sum complex $\bigoplus \mathcal{I}_i^\bullet$ represents M. By Leray acyclicity (Derived Categories, Lemma 16.7) we see that

$$R\Gamma(U, M) = \Gamma(U, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus \Gamma(U, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus R\Gamma(U, M_i)$$

as desired. $\square$

Lemma 52.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site with set of coverings $\mathit{Cov}_{\mathcal{C}}$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$, and $\mathit{Cov} \subset \mathit{Cov}_{\mathcal{C}}$ be subsets. Assume that*

1. *For every $\mathcal{U} \in \mathit{Cov}$ we have $\mathcal{U} = \{U_i \to U\}_{i \in I}$ with I finite, $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.*

Then for $U \in \mathcal{B}$ the object $j_{U!}\mathcal{O}_U$ is a compact object of $D^+(\mathcal{O})$ in the following sense: if $M = \bigoplus_{i \in I} M_i$ in $D(\mathcal{O})$ is bounded below, then $\text{Hom}(j_{U!}\mathcal{O}_U, M) = \bigoplus_{i \in I} \text{Hom}(j_{U!}\mathcal{O}_U, M_i)$.

Proof. This follows from Lemma 52.3 and Lemma 16.1. $\square$

Lemma 52.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. The $\mathcal{O}$-module $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O})$ if there exists an integer d such that*

1. *$H^p(U, \mathcal{F}) = 0$ for all $p > d$, and*
2. *the functors $\mathcal{F} \mapsto H^p(U, \mathcal{F})$ commute with direct sums.*

Proof. Assume (1) and (2). Recall that $\text{Hom}(j_!\mathcal{O}_U, K) = R\Gamma(U, K)$ for K in $D(\mathcal{O})$. Thus we have to show that $R\Gamma(U, -)$ commutes with direct sums. The first assumption means that the functor $F = H^0(U, -)$ has finite cohomological dimension. Moreover, the second assumption implies any direct sum of injective modules is acyclic for F. Let K_i be a family of objects of $D(\mathcal{O})$. Choose K-injective representatives $I_i^\bullet$ with injective terms representing K_i, see Injectives, Theorem 12.6. Since we may compute RF by applying F to any complex of acyclics (Derived Categories, Lemma 32.2) and since $\bigoplus K_i$ is represented by $\bigoplus I_i^\bullet$ (Injectives, Lemma 13.4) we conclude that $R\Gamma(U, \bigoplus K_i)$ is represented by $\bigoplus H^0(U, I_i^\bullet)$. Hence $R\Gamma(U, -)$ commutes with direct sums as desired. $\square$

Lemma 52.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$ which is quasi-compact and weakly contractible. Then $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O})$.*

Proof. Combine Lemmas 52.5 and 51.1 with Modules on Sites, Lemma 30.3. $\square$

Lemma 52.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Assume $\mathcal{C}$ has the following properties*

1. *$\mathcal{C}$ has a quasi-compact final object X,*
2. *every quasi-compact object of $\mathcal{C}$ has a cofinal system of coverings which are finite and consist of quasi-compact objects,*
3. *for a finite covering $\{U_i \to U\}_{i \in I}$ with U, U_i quasi-compact the fibre products $U_i \times_U U_j$ are quasi-compact.*

Let K be a perfect object of $D(\mathcal{O})$. Then

(a) *K is a compact object of $D^+(\mathcal{O})$ in the following sense: if $M = \bigoplus_{i \in I} M_i$ is bounded below, then $\text{Hom}(K, M) = \bigoplus_{i \in I} \text{Hom}(K, M_i)$.*
(b) *If $(\mathcal{C}, \mathcal{O})$ has finite cohomological dimension, i.e., if there exists a d such that $H^i(X, \mathcal{F}) = 0$ for $i > d$ for any $\mathcal{O}$-module $\mathcal{F}$, then K is a compact object of $D(\mathcal{O})$.*

Proof. Let $K^\vee$ be the dual of K, see Lemma 48.4. Then we have

$$\text{Hom}_{D(\mathcal{O})}(K, M) = H^0(X, K^\vee \otimes_{\mathcal{O}}^{\mathbf{L}} M)$$

functorially in M in $D(\mathcal{O})$. Since $K^\vee \otimes_{\mathcal{O}}^{\mathbf{L}} -$ commutes with direct sums it suffices to show that $R\Gamma(X, -)$ commutes with the relevant direct sums.

Proof of (a). After reformulation as above this is a special case of Lemma 52.4 with $U = X$.

Proof of (b). Since $R\Gamma(X, K) = R\,\text{Hom}(\mathcal{O}, K)$ and since $H^p(X, -)$ commutes with direct sums by Lemma 16.1 this is a special case of Lemma 52.5. $\square$

53. Complexes with locally constant cohomology sheaves

Locally constant sheaves are introduced in Modules on Sites, Section 43. Let $\mathcal{C}$ be a site. Let Λ be a ring. We denote $D(\mathcal{C}, \Lambda)$ the derived category of the abelian category of $\underline{\Lambda}$-modules on $\mathcal{C}$.

Lemma 53.1. *Let $\mathcal{C}$ be a site with final object X. Let Λ be a Noetherian ring. Let $K \in D^b(\mathcal{C}, \Lambda)$ with $H^i(K)$ locally constant sheaves of Λ-modules of finite type. Then there exists a covering $\{U_i \to X\}$ such that each $K|_{U_i}$ is represented by a complex of locally constant sheaves of Λ-modules of finite type.*

Proof. Let $a \leq b$ be such that $H^i(K) = 0$ for $i \not\in [a, b]$. By induction on $b - a$ we will prove there exists a covering $\{U_i \to X\}$ such that $K|_{U_i}$ can be represented by a complex $\underline{M^\bullet}_{U_i}$ with M^p a finite type Λ-module and $M^p = 0$ for $p \not\in [a, b]$. If $b = a$, then this is clear. In general, we may replace X by the members of a covering and assume that $H^b(K)$ is constant, say $H^b(K) = \underline{M}$. By Modules on Sites, Lemma 42.5 the module M is a finite Λ-module. Choose a surjection $\Lambda^{\oplus r} \to M$ given by generators $x_1, \ldots, x_r$ of M.

By a slight generalization of Lemma 7.3 (details omitted) there exists a covering $\{U_i \to X\}$ such that $x_i \in H^0(X, H^b(K))$ lifts to an element of $H^b(U_i, K)$. Thus, after replacing X by the U_i we reach the situation where there is a map $\underline{\Lambda^{\oplus r}}[-b] \to K$ inducing a surjection on cohomology sheaves in degree b. Choose a distinguished triangle

$$\underline{\Lambda^{\oplus r}}[-b] \to K \to L \to \underline{\Lambda^{\oplus r}}[-b + 1]$$

Now the cohomology sheaves of L are nonzero only in the interval $[a, b - 1]$, agree with the cohomology sheaves of K in the interval $[a, b - 2]$ and there is a short exact sequence

$$0 \to H^{b-1}(K) \to H^{b-1}(L) \to \underline{\mathrm{Ker}(\Lambda^{\oplus r} \to M)} \to 0$$

in degree $b - 1$. By Modules on Sites, Lemma 43.5 we see that $H^{b-1}(L)$ is locally constant of finite type. By induction hypothesis we obtain an isomorphism $\underline{M^\bullet} \to L$ in $D(\mathcal{C}, \underline{\Lambda})$ with M^p a finite Λ-module and $M^p = 0$ for $p \not\in [a, b - 1]$. The map $L \to \Lambda^{\oplus r}[-b + 1]$ gives a map $\underline{M^{b-1}} \to \underline{\Lambda^{\oplus r}}$ which locally is constant (Modules on Sites, Lemma 43.3). Thus we may assume it is given by a map $M^{b-1} \to \Lambda^{\oplus r}$. The distinguished triangle shows that the composition $M^{b-2} \to M^{b-1} \to \Lambda^{\oplus r}$ is zero and the axioms of triangulated categories produce an isomorphism

$$\underline{M^a \to \ldots \to M^{b-1} \to \Lambda^{\oplus r}} \longrightarrow K$$

in $D(\mathcal{C}, \Lambda)$. □

Let $\mathcal{C}$ be a site. Let Λ be a ring. Using the morphism $Sh(\mathcal{C}) \to Sh(pt)$ we see that there is a functor $D(\Lambda) \to D(\mathcal{C}, \Lambda)$, $K \mapsto \underline{K}$.

Lemma 53.2. *Let $\mathcal{C}$ be a site with final object X. Let Λ be a ring. Let*

1. *K a perfect object of $D(\Lambda)$,*
2. *a finite complex $K^\bullet$ of finite projective Λ-modules representing K,*
3. *$\mathcal{L}^\bullet$ a complex of sheaves of Λ-modules, and*
4. *$\varphi : \underline{K} \to \mathcal{L}^\bullet$ a map in $D(\mathcal{C}, \Lambda)$.*

Then there exists a covering $\{U_i \to X\}$ and maps of complexes $\alpha_i : \underline{K}^\bullet|_{U_i} \to \mathcal{L}^\bullet|_{U_i}$ representing $\varphi|_{U_i}$.

Proof. Follows immediately from Lemma 44.8. □

Lemma 53.3. *Let $\mathcal{C}$ be a site with final object X. Let Λ be a ring. Let K, L be objects of $D(\Lambda)$ with K perfect. Let $\varphi : \underline{K} \to \underline{L}$ be map in $D(\mathcal{C}, \Lambda)$. There exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is equal to $\underline{\alpha_i}$ for some map $\alpha_i : K \to L$ in $D(\Lambda)$.*

Proof. Follows from Lemma 53.2 and Modules on Sites, Lemma 43.3. □

Lemma 53.4. *Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $K, L \in D^-(\mathcal{C}, \Lambda)$. If the cohomology sheaves of K and L are locally constant sheaves of Λ-modules of finite type, then the cohomology sheaves of $K \otimes_\Lambda^{\mathbf{L}} L$ are locally constant sheaves of Λ-modules of finite type.*

Proof. We'll prove this as an application of Lemma 53.1. Note that $H^i(K \otimes_\Lambda^{\mathbf{L}} L)$ is the same as $H^i(\tau_{\geq i-1} K \otimes_\Lambda^{\mathbf{L}} \tau_{\geq i-1} L)$. Thus we may assume K and L are bounded. By Lemma 53.1 we may assume that K and L are represented by complexes of locally constant sheaves of Λ-modules of finite type. Then we can replace these complexes by bounded above complexes of finite free Λ-modules. In this case the result is clear. □

Lemma 53.5. *Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $K \in D^-(\mathcal{C}, \Lambda)$. If the cohomology sheaves of $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ are locally constant sheaves of Λ/I-modules of finite type, then the cohomology sheaves of $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$ are locally constant sheaves of Λ/I^n-modules of finite type for all $n \geq 1$.*

Proof. Recall that the locally constant sheaves of Λ-modules of finite type form a weak Serre subcategory of all $\underline{\Lambda}$-modules, see Modules on Sites, Lemma 43.5. Thus the subcategory of $D(\mathcal{C}, \Lambda)$ consisting of complexes whose cohomology sheaves are locally constant sheaves of Λ-modules of finite type forms a strictly full, saturated triangulated subcategory of $D(\mathcal{C}, \Lambda)$, see Derived Categories, Lemma 17.1. Next, consider the distinguished triangles

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} \to K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^{n+1}} \to K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n} \to K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}}[1]$$

and the isomorphisms

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} = \left(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}\right) \otimes_{\Lambda/I}^{\mathbf{L}} \underline{I^n/I^{n+1}}$$

Combined with Lemma 53.4 we obtain the result. □

54. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra

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