

# GROUPOID SCHEMES

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## 1. Introduction

This chapter is devoted to generalities concerning groupoid schemes. See for example the beautiful paper [KM97] by Keel and Mori.

## 2. Notation

Let  $S$  be a scheme. If  $U, T$  are schemes over  $S$  we denote  $U(T)$  for the set of  $T$ -valued points of  $U$  over  $S$ . In a formula:  $U(T) = \text{Mor}_S(T, U)$ . We try to reserve the letter  $T$  to denote a “test scheme” over  $S$ , as in the discussion that follows.

Suppose we are given schemes  $X, Y$  over  $S$  and a morphism of schemes  $f : X \rightarrow Y$  over  $S$ . For any scheme  $T$  over  $S$  we get an induced map of sets

$$f : X(T) \longrightarrow Y(T)$$

which as indicated we denote by  $f$  also. In fact this construction is functorial in the scheme  $T/S$ . Yoneda's Lemma, see Categories, Lemma 3.5, says that  $f$  determines and is determined by this transformation of functors  $f : h_X \rightarrow h_Y$ . More generally, we use the same notation for maps between fibre products. For example, if  $X, Y, Z$  are schemes over  $S$ , and if  $m : X \times_S Y \rightarrow Z \times_S Z$  is a morphism of schemes over  $S$ , then we think of  $m$  as corresponding to a collection of maps between  $T$ -valued points

$$X(T) \times Y(T) \longrightarrow Z(T) \times Z(T).$$

And so on and so forth.

We continue our convention to label projection maps starting with index 0, so we have  $\text{pr}_0 : X \times_S Y \rightarrow X$  and  $\text{pr}_1 : X \times_S Y \rightarrow Y$ .

### 3. Equivalence relations

Recall that a *relation*  $R$  on a set  $A$  is just a subset of  $R \subset A \times A$ . We usually write  $aRb$  to indicate  $(a, b) \in R$ . We say the relation is *transitive* if  $aRb, bRc \Rightarrow aRc$ . We say the relation is *reflexive* if  $aRa$  for all  $a \in A$ . We say the relation is *symmetric* if  $aRb \Rightarrow bRa$ . A relation is called an *equivalence relation* if it is transitive, reflexive and symmetric.

In the setting of schemes we are going to relax the notion of a relation a little bit and just require  $R \rightarrow A \times A$  to be a map. Here is the definition.

**Definition 3.1.** Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ .

- (1) A *pre-relation* on  $U$  over  $S$  is any morphism of schemes  $j : R \rightarrow U \times_S U$ . In this case we set  $t = \text{pr}_0 \circ j$  and  $s = \text{pr}_1 \circ j$ , so that  $j = (t, s)$ .
- (2) A *relation* on  $U$  over  $S$  is a monomorphism of schemes  $j : R \rightarrow U \times_S U$ .
- (3) A *pre-equivalence relation* is a pre-relation  $j : R \rightarrow U \times_S U$  such that the image of  $j : R(T) \rightarrow U(T) \times U(T)$  is an equivalence relation for all  $T/S$ .
- (4) We say a morphism  $R \rightarrow U \times_S U$  of schemes is an *equivalence relation on  $U$  over  $S$*  if and only if for every scheme  $T$  over  $S$  the  $T$ -valued points of  $R$  define an equivalence relation on the set of  $T$ -valued points of  $U$ .

In other words, an equivalence relation is a pre-equivalence relation such that  $j$  is a relation.

**Lemma 3.2.** Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ . Let  $j : R \rightarrow U \times_S U$  be a pre-relation. Let  $g : U' \rightarrow U$  be a morphism of schemes. Finally, set

$$R' = (U' \times_S U') \times_{U \times_S U} R \xrightarrow{j'} U' \times_S U'$$

Then  $j'$  is a pre-relation on  $U'$  over  $S$ . If  $j$  is a relation, then  $j'$  is a relation. If  $j$  is a pre-equivalence relation, then  $j'$  is a pre-equivalence relation. If  $j$  is an equivalence relation, then  $j'$  is an equivalence relation.

**Proof.** Omitted. □

**Definition 3.3.** Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ . Let  $j : R \rightarrow U \times_S U$  be a pre-relation. Let  $g : U' \rightarrow U$  be a morphism of schemes. The pre-relation  $j' : R' \rightarrow U' \times_S U'$  is called the *restriction*, or *pullback* of the pre-relation  $j$  to  $U'$ . In this situation we sometimes write  $R' = R|_{U'}$ .

**Lemma 3.4.** Let  $j : R \rightarrow U \times_S U$  be a pre-relation. Consider the relation on points of the scheme  $U$  defined by the rule

$$x \sim y \Leftrightarrow \exists r \in R : t(r) = x, s(r) = y.$$

If  $j$  is a pre-equivalence relation then this is an equivalence relation.

**Proof.** Suppose that  $x \sim y$  and  $y \sim z$ . Pick  $r \in R$  with  $t(r) = x, s(r) = y$  and pick  $r' \in R$  with  $t(r') = y, s(r') = z$ . Pick a field  $K$  fitting into the following commutative diagram

$$\begin{array}{ccc} \kappa(r) & \longrightarrow & K \\ \uparrow & & \uparrow \\ \kappa(y) & \longrightarrow & \kappa(r') \end{array}$$

Denote  $x_K, y_K, z_K : \text{Spec}(K) \rightarrow U$  the morphisms

$$\begin{aligned} \text{Spec}(K) &\rightarrow \text{Spec}(\kappa(r)) \rightarrow \text{Spec}(\kappa(x)) \rightarrow U \\ \text{Spec}(K) &\rightarrow \text{Spec}(\kappa(r)) \rightarrow \text{Spec}(\kappa(y)) \rightarrow U \\ \text{Spec}(K) &\rightarrow \text{Spec}(\kappa(r')) \rightarrow \text{Spec}(\kappa(z)) \rightarrow U \end{aligned}$$

By construction  $(x_K, y_K) \in j(R(K))$  and  $(y_K, z_K) \in j(R(K))$ . Since  $j$  is a pre-equivalence relation we see that also  $(x_K, z_K) \in j(R(K))$ . This clearly implies that  $x \sim z$ .

The proof that  $\sim$  is reflexive and symmetric is omitted.  $\square$

**Lemma 3.5.** Let  $j : R \rightarrow U \times_S U$  be a pre-relation. Assume

- (1)  $s, t$  are unramified,
- (2) for any algebraically closed field  $k$  over  $S$  the map  $R(k) \rightarrow U(k) \times U(k)$  is an equivalence relation,
- (3) there are morphisms  $e : U \rightarrow R, i : R \rightarrow R, c : R \times_{s,U,t} R \rightarrow R$  such that

$$\begin{array}{ccccc} U & \xrightarrow{e} & R & & R & \xrightarrow{i} & R & & R \times_{s,U,t} R & \xrightarrow{c} & R \\ \Delta \downarrow & & \downarrow j & & \downarrow j & & \downarrow j & & \downarrow j \times j & & \downarrow j \\ U \times_S U & \xrightarrow{\quad} & U \times_S U & \xrightarrow{\text{flip}} & U \times_S U & & U \times_S U \times_S U & \xrightarrow{pr_{02}} & U \times_S U \end{array}$$

are commutative.

Then  $j$  is an equivalence relation.

**Proof.** By condition (1) and Morphisms, Lemma 35.16 we see that  $j$  is a unramified. Then  $\Delta_j : R \rightarrow R \times_{U \times_S U} R$  is an open immersion by Morphisms, Lemma 35.13. However, then condition (2) says  $\Delta_j$  is bijective on  $k$ -valued points, hence  $\Delta_j$  is an isomorphism, hence  $j$  is a monomorphism. Then it easily follows from the commutative diagrams that  $R(T) \subset U(T) \times U(T)$  is an equivalence relation for all schemes  $T$  over  $S$ .  $\square$

#### 4. Group schemes

Let us recall that a *group* is a pair  $(G, m)$  where  $G$  is a set, and  $m : G \times G \rightarrow G$  is a map of sets with the following properties:

- (1) (associativity)  $m(g, m(g', g'')) = m(m(g, g'), g'')$  for all  $g, g', g'' \in G$ ,
- (2) (identity) there exists a unique element  $e \in G$  (called the *identity*, *unit*, or 1 of  $G$ ) such that  $m(g, e) = m(e, g) = g$  for all  $g \in G$ , and
- (3) (inverse) for all  $g \in G$  there exists a  $i(g) \in G$  such that  $m(g, i(g)) = m(i(g), g) = e$ , where  $e$  is the identity.

Thus we obtain a map  $e : \{*\} \rightarrow G$  and a map  $i : G \rightarrow G$  so that the quadruple  $(G, m, e, i)$  satisfies the axioms listed above.

A *homomorphism of groups*  $\psi : (G, m) \rightarrow (G', m')$  is a map of sets  $\psi : G \rightarrow G'$  such that  $m'(\psi(g), \psi(g')) = \psi(m(g, g'))$ . This automatically insures that  $\psi(e) = e'$  and  $i'(\psi(g)) = \psi(i(g))$ . (Obvious notation.) We will use this below.

**Definition 4.1.** Let  $S$  be a scheme.

- (1) A *group scheme over  $S$*  is a pair  $(G, m)$ , where  $G$  is a scheme over  $S$  and  $m : G \times_S G \rightarrow G$  is a morphism of schemes over  $S$  with the following property: For every scheme  $T$  over  $S$  the pair  $(G(T), m)$  is a group.
- (2) A *morphism  $\psi : (G, m) \rightarrow (G', m')$  of group schemes over  $S$*  is a morphism  $\psi : G \rightarrow G'$  of schemes over  $S$  such that for every  $T/S$  the induced map  $\psi : G(T) \rightarrow G'(T)$  is a homomorphism of groups.

Let  $(G, m)$  be a group scheme over the scheme  $S$ . By the discussion above (and the discussion in Section 2) we obtain morphisms of schemes over  $S$ : (identity)  $e : S \rightarrow G$  and (inverse)  $i : G \rightarrow G$  such that for every  $T$  the quadruple  $(G(T), m, e, i)$  satisfies the axioms of a group listed above.

Let  $(G, m), (G', m')$  be group schemes over  $S$ . Let  $f : G \rightarrow G'$  be a morphism of schemes over  $S$ . It follows from the definition that  $f$  is a morphism of group schemes over  $S$  if and only if the following diagram is commutative:

$$\begin{array}{ccc} G \times_S G & \xrightarrow{f \times f} & G' \times_S G' \\ m \downarrow & & \downarrow m' \\ G & \xrightarrow{f} & G' \end{array}$$

**Lemma 4.2.** Let  $(G, m)$  be a group scheme over  $S$ . Let  $S' \rightarrow S$  be a morphism of schemes. The pullback  $(G_{S'}, m_{S'})$  is a group scheme over  $S'$ .

**Proof.** Omitted. □

**Definition 4.3.** Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ .

- (1) A *closed subgroup scheme* of  $G$  is a closed subscheme  $H \subset G$  such that  $m|_{H \times_S H}$  factors through  $H$  and induces a group scheme structure on  $H$  over  $S$ .
- (2) An *open subgroup scheme* of  $G$  is an open subscheme  $G' \subset G$  such that  $m|_{G' \times_S G'}$  factors through  $G'$  and induces a group scheme structure on  $G'$  over  $S$ .

Alternatively, we could say that  $H$  is a closed subgroup scheme of  $G$  if it is a group scheme over  $S$  endowed with a morphism of group schemes  $i : H \rightarrow G$  over  $S$  which identifies  $H$  with a closed subscheme of  $G$ .

**Lemma 4.4.** *Let  $S$  be a scheme. Let  $(G, m, e, i)$  be a group scheme over  $S$ .*

- (1) *A closed subscheme  $H \subset G$  is a closed subgroup scheme if and only if  $e : S \rightarrow G$ ,  $m|_{H \times_S H} : H \times_S H \rightarrow G$ , and  $i|_H : H \rightarrow G$  factor through  $H$ .*
- (2) *An open subscheme  $H \subset G$  is an open subgroup scheme if and only if  $e : S \rightarrow G$ ,  $m|_{H \times_S H} : H \times_S H \rightarrow G$ , and  $i|_H : H \rightarrow G$  factor through  $H$ .*

**Proof.** Looking at  $T$ -valued points this translates into the well known conditions characterizing subsets of groups as subgroups.  $\square$

**Definition 4.5.** Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ .

- (1) We say  $G$  is a *smooth group scheme* if the structure morphism  $G \rightarrow S$  is smooth.
- (2) We say  $G$  is a *flat group scheme* if the structure morphism  $G \rightarrow S$  is flat.
- (3) We say  $G$  is a *separated group scheme* if the structure morphism  $G \rightarrow S$  is separated.

Add more as needed.

## 5. Examples of group schemes

**Example 5.1** (Multiplicative group scheme). Consider the functor which associates to any scheme  $T$  the group  $\Gamma(T, \mathcal{O}_T^*)$  of units in the global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_m = \text{Spec}(\mathbf{Z}[x, x^{-1}])$$

The morphism giving the group structure is the morphism

$$\begin{aligned} \mathbf{G}_m \times \mathbf{G}_m &\rightarrow \mathbf{G}_m \\ \text{Spec}(\mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}]) &\rightarrow \text{Spec}(\mathbf{Z}[x, x^{-1}]) \\ \mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}] &\leftarrow \mathbf{Z}[x, x^{-1}] \\ x \otimes x &\leftarrow x \end{aligned}$$

Hence we see that  $\mathbf{G}_m$  is a group scheme over  $\mathbf{Z}$ . For any scheme  $S$  the base change  $\mathbf{G}_{m,S}$  is a group scheme over  $S$  whose functor of points is

$$T/S \mapsto \mathbf{G}_{m,S}(T) = \mathbf{G}_m(T) = \Gamma(T, \mathcal{O}_T^*)$$

as before.

**Example 5.2** (Roots of unity). Let  $n \in \mathbf{N}$ . Consider the functor which associates to any scheme  $T$  the subgroup of  $\Gamma(T, \mathcal{O}_T^*)$  consisting of  $n$ th roots of unity. This is representable by the scheme

$$\mu_n = \text{Spec}(\mathbf{Z}[x]/(x^n - 1)).$$

The morphism giving the group structure is the morphism

$$\begin{aligned} \mu_n \times \mu_n &\rightarrow \mu_n \\ \text{Spec}(\mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1)) &\rightarrow \text{Spec}(\mathbf{Z}[x]/(x^n - 1)) \\ \mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1) &\leftarrow \mathbf{Z}[x]/(x^n - 1) \\ x \otimes x &\leftarrow x \end{aligned}$$

Hence we see that  $\mu_n$  is a group scheme over  $\mathbf{Z}$ . For any scheme  $S$  the base change  $\mu_{n,S}$  is a group scheme over  $S$  whose functor of points is

$$T/S \mapsto \mu_{n,S}(T) = \mu_n(T) = \{f \in \Gamma(T, \mathcal{O}_T^*) \mid f^n = 1\}$$

as before.

**Example 5.3** (Additive group scheme). Consider the functor which associates to any scheme  $T$  the group  $\Gamma(T, \mathcal{O}_T)$  of global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_a = \operatorname{Spec}(\mathbf{Z}[x])$$

The morphism giving the group structure is the morphism

$$\begin{aligned} \mathbf{G}_a \times \mathbf{G}_a &\rightarrow \mathbf{G}_a \\ \operatorname{Spec}(\mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x]) &\rightarrow \operatorname{Spec}(\mathbf{Z}[x]) \\ \mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x] &\leftarrow \mathbf{Z}[x] \\ x \otimes 1 + 1 \otimes x &\leftarrow x \end{aligned}$$

Hence we see that  $\mathbf{G}_a$  is a group scheme over  $\mathbf{Z}$ . For any scheme  $S$  the base change  $\mathbf{G}_{a,S}$  is a group scheme over  $S$  whose functor of points is

$$T/S \mapsto \mathbf{G}_{a,S}(T) = \mathbf{G}_a(T) = \Gamma(T, \mathcal{O}_T)$$

as before.

**Example 5.4** (General linear group scheme). Let  $n \geq 1$ . Consider the functor which associates to any scheme  $T$  the group

$$\operatorname{GL}_n(\Gamma(T, \mathcal{O}_T))$$

of invertible  $n \times n$  matrices over the global sections of the structure sheaf. This is representable by the scheme

$$\operatorname{GL}_n = \operatorname{Spec}(\mathbf{Z}[\{x_{ij}\}_{1 \leq i,j \leq n}][1/d])$$

where  $d = \det((x_{ij}))$  with  $(x_{ij})$  the  $n \times n$  matrix with entry  $x_{ij}$  in the  $(i, j)$ -spot. The morphism giving the group structure is the morphism

$$\begin{aligned} \operatorname{GL}_n \times \operatorname{GL}_n &\rightarrow \operatorname{GL}_n \\ \operatorname{Spec}(\mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d]) &\rightarrow \operatorname{Spec}(\mathbf{Z}[x_{ij}, 1/d]) \\ \mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d] &\leftarrow \mathbf{Z}[x_{ij}, 1/d] \\ \sum x_{ik} \otimes x_{kj} &\leftarrow x_{ij} \end{aligned}$$

Hence we see that  $\operatorname{GL}_n$  is a group scheme over  $\mathbf{Z}$ . For any scheme  $S$  the base change  $\operatorname{GL}_{n,S}$  is a group scheme over  $S$  whose functor of points is

$$T/S \mapsto \operatorname{GL}_{n,S}(T) = \operatorname{GL}_n(T) = \operatorname{GL}_n(\Gamma(T, \mathcal{O}_T))$$

as before.

**Example 5.5.** The determinant defines a morphism of group schemes

$$\det : \operatorname{GL}_n \longrightarrow \mathbf{G}_m$$

over  $\mathbf{Z}$ . By base change it gives a morphism of group schemes  $\operatorname{GL}_{n,S} \rightarrow \mathbf{G}_{m,S}$  over any base scheme  $S$ .

**Example 5.6** (Constant group). Let  $G$  be an abstract group. Consider the functor which associates to any scheme  $T$  the group of locally constant maps  $T \rightarrow G$  (where  $T$  has the Zariski topology and  $G$  the discrete topology). This is representable by the scheme

$$G_{\operatorname{Spec}(\mathbf{Z})} = \coprod_{g \in G} \operatorname{Spec}(\mathbf{Z}).$$

The morphism giving the group structure is the morphism

$$G_{\mathrm{Spec}(\mathbf{Z})} \times_{\mathrm{Spec}(\mathbf{Z})} G_{\mathrm{Spec}(\mathbf{Z})} \longrightarrow G_{\mathrm{Spec}(\mathbf{Z})}$$

which maps the component corresponding to the pair  $(g, g')$  to the component corresponding to  $gg'$ . For any scheme  $S$  the base change  $G_S$  is a group scheme over  $S$  whose functor of points is

$$T/S \longmapsto G_S(T) = \{f : T \rightarrow G \text{ locally constant}\}$$

as before.

## 6. Properties of group schemes

In this section we collect some simple properties of group schemes which hold over any base.

**Lemma 6.1.** *Let  $S$  be a scheme. Let  $G$  be a group scheme over  $S$ . Then  $G \rightarrow S$  is separated (resp. quasi-separated) if and only if the identity morphism  $e : S \rightarrow G$  is a closed immersion (resp. quasi-compact).*

**Proof.** We recall that by Schemes, Lemma 21.11 we have that  $e$  is an immersion which is a closed immersion (resp. quasi-compact) if  $G \rightarrow S$  is separated (resp. quasi-separated). For the converse, consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\Delta_{G/S}} & G \times_S G \\ \downarrow & & \downarrow (g, g') \mapsto m(i(g), g') \\ S & \xrightarrow{e} & G \end{array}$$

It is an exercise in the functorial point of view in algebraic geometry to show that this diagram is cartesian. In other words, we see that  $\Delta_{G/S}$  is a base change of  $e$ . Hence if  $e$  is a closed immersion (resp. quasi-compact) so is  $\Delta_{G/S}$ , see Schemes, Lemma 18.2 (resp. Schemes, Lemma 19.3).  $\square$

**Lemma 6.2.** *Let  $S$  be a scheme. Let  $G$  be a group scheme over  $S$ . Let  $T$  be a scheme over  $S$  and let  $\psi : T \rightarrow G$  be a morphism over  $S$ . If  $T$  is flat over  $S$ , then the morphism*

$$T \times_S G \longrightarrow G, \quad (t, g) \longmapsto m(\psi(t), g)$$

*is flat. In particular, if  $G$  is flat over  $S$ , then  $m : G \times_S G \rightarrow G$  is flat.*

**Proof.** Consider the diagram

$$\begin{array}{ccccc} T \times_S G & \xrightarrow{(t, g) \mapsto (t, m(\psi(t), g))} & T \times_S G & \xrightarrow{\mathrm{pr}} & G \\ & & \downarrow & & \downarrow \\ & & T & \longrightarrow & S \end{array}$$

The left top horizontal arrow is an isomorphism and the square is cartesian. Hence the lemma follows from Morphisms, Lemma 25.8.  $\square$

**Lemma 6.3.** *Let  $(G, m, e, i)$  be a group scheme over the scheme  $S$ . Denote  $f : G \rightarrow S$  the structure morphism. Then there exist canonical isomorphisms*

$$\Omega_{G/S} \cong f^* \mathcal{C}_{S/G} \cong f^* e^* \Omega_{G/S}$$

*where  $\mathcal{C}_{S/G}$  denotes the conormal sheaf of the immersion  $e$ . In particular, if  $S$  is the spectrum of a field, then  $\Omega_{G/S}$  is a free  $\mathcal{O}_G$ -module.*

**Proof.** By Morphisms, Lemma 32.10 we have

$$\Omega_{G \times_S G/G} = \text{pr}_0^* \Omega_{G/S}$$

where on the left hand side we view  $G \times_S G$  as a scheme over  $G$  using  $\text{pr}_1$ . Let  $\tau : G \times_S G \rightarrow G \times_S G$  be the “shearing map” given by  $(g, h) \mapsto (m(g, h), h)$  on points. This map is an automorphism of  $G \times_S G$  viewed as a scheme over  $G$  via the projection  $\text{pr}_1$ . Combining these two remarks we obtain an isomorphism

$$\tau^* \text{pr}_0^* \Omega_{G/S} \rightarrow \text{pr}_0^* \Omega_{G/S}$$

Since  $\text{pr}_0 \circ \tau = m$  this can be rewritten as an isomorphism

$$m^* \Omega_{G/S} \rightarrow \text{pr}_0^* \Omega_{G/S}$$

Pulling back this isomorphism by  $(e \circ f, \text{id}_G) : G \rightarrow G \times_S G$  and using that  $m \circ (e \circ f, \text{id}_G) = \text{id}_G$  and  $\text{pr}_0 \circ (e \circ f, \text{id}_G) = e \circ f$  we obtain an isomorphism

$$\Omega_{G/S} \rightarrow f^* e^* \Omega_{G/S}$$

as desired. By Morphisms, Lemma 32.16 we have  $\mathcal{C}_{S/G} \cong e^* \Omega_{G/S}$ . If  $S$  is the spectrum of a field, then any  $\mathcal{O}_S$ -module on  $S$  is free and the final statement follows.  $\square$

**Lemma 6.4.** *Let  $S$  be a scheme. Let  $G$  be a group scheme over  $S$ . Let  $s \in S$ . Then the composition*

$$T_{G/S, e(s)} \oplus T_{G/S, e(s)} = T_{G \times_S G/S, (e(s), e(s))} \rightarrow T_{G/S, e(s)}$$

*is addition of tangent vectors. Here the  $=$  comes from Varieties, Lemma 16.7 and the right arrow is induced from  $m : G \times_S G \rightarrow G$  via Varieties, Lemma 16.6.*

**Proof.** We will use Varieties, Equation (16.3.1) and work with tangent vectors in fibres. An element  $\theta$  in the first factor  $T_{G_s/s, e(s)}$  is the image of  $\theta$  via the map  $T_{G_s/s, e(s)} \rightarrow T_{G_s \times G_s/s, (e(s), e(s))}$  coming from  $(1, e) : G_s \rightarrow G_s \times G_s$ . Since  $m \circ (1, e) = 1$  we see that  $\theta$  maps to  $\theta$  by functoriality. Since the map is linear we see that  $(\theta_1, \theta_2)$  maps to  $\theta_1 + \theta_2$ .  $\square$

## 7. Properties of group schemes over a field

In this section we collect some properties of group schemes over a field. In the case of group schemes which are (locally) algebraic over a field we can say a lot more, see Section 8.

**Lemma 7.1.** *If  $(G, m)$  is a group scheme over a field  $k$ , then the multiplication map  $m : G \times_k G \rightarrow G$  is open.*

**Proof.** The multiplication map is isomorphic to the projection map  $\text{pr}_0 : G \times_k G \rightarrow G$  because the diagram

$$\begin{array}{ccc} G \times_k G & \xrightarrow{(g, g') \mapsto (m(g, g'), g')} & G \times_k G \\ \downarrow m & & \downarrow (g, g') \mapsto g \\ G & \xrightarrow{\text{id}} & G \end{array}$$

is commutative with isomorphisms as horizontal arrows. The projection is open by Morphisms, Lemma 23.4.  $\square$



**Lemma 7.2.** *If  $(G, m)$  is a group scheme over a field  $k$ . Let  $U \subset G$  open and  $T \rightarrow G$  a morphism of schemes. Then the image of the composition  $T \times_k U \rightarrow G \times_k G \rightarrow G$  is open.*

**Proof.** For any field extension  $K/k$  the morphism  $G_K \rightarrow G$  is open (Morphisms, Lemma 23.4). Every point  $\xi$  of  $T \times_k U$  is the image of a morphism  $(t, u) : \text{Spec}(K) \rightarrow T \times_k U$  for some  $K$ . Then the image of  $T_K \times_K U_K = (T \times_k U)_K \rightarrow G_K$  contains the translate  $t \cdot U_K$  which is open. Combining these facts we see that the image of  $T \times_k U \rightarrow G$  contains an open neighbourhood of the image of  $\xi$ . Since  $\xi$  was arbitrary we win.  $\square$

**Lemma 7.3.** *Let  $G$  be a group scheme over a field. Then  $G$  is a separated scheme.*

**Proof.** Say  $S = \text{Spec}(k)$  with  $k$  a field, and let  $G$  be a group scheme over  $S$ . By Lemma 6.1 we have to show that  $e : S \rightarrow G$  is a closed immersion. By Morphisms, Lemma 20.2 the image of  $e : S \rightarrow G$  is a closed point of  $G$ . It is clear that  $\mathcal{O}_G \rightarrow e_* \mathcal{O}_S$  is surjective, since  $e_* \mathcal{O}_S$  is a skyscraper sheaf supported at the neutral element of  $G$  with value  $k$ . We conclude that  $e$  is a closed immersion by Schemes, Lemma 24.2.  $\square$

**Lemma 7.4.** *Let  $G$  be a group scheme over a field  $k$ . Then*

- (1) *every local ring  $\mathcal{O}_{G,g}$  of  $G$  has a unique minimal prime ideal,*
- (2) *there is exactly one irreducible component  $Z$  of  $G$  passing through  $e$ , and*
- (3)  *$Z$  is geometrically irreducible over  $k$ .*

**Proof.** For any point  $g \in G$  there exists a field extension  $K/k$  and a  $K$ -valued point  $g' \in G(K)$  mapping to  $g$ . If we think of  $g'$  as a  $K$ -rational point of the group scheme  $G_K$ , then we see that  $\mathcal{O}_{G,g} \rightarrow \mathcal{O}_{G_K,g'}$  is a faithfully flat local ring map (as  $G_K \rightarrow G$  is flat, and a local flat ring map is faithfully flat, see Algebra, Lemma 39.17). The result for  $\mathcal{O}_{G_K,g'}$  implies the result for  $\mathcal{O}_{G,g}$ , see Algebra, Lemma 30.5. Hence in order to prove (1) it suffices to prove it for  $k$ -rational points  $g$  of  $G$ . In this case translation by  $g$  defines an automorphism  $G \rightarrow G$  which maps  $e$  to  $g$ . Hence  $\mathcal{O}_{G,g} \cong \mathcal{O}_{G,e}$ . In this way we see that (2) implies (1), since irreducible components passing through  $e$  correspond one to one with minimal prime ideals of  $\mathcal{O}_{G,e}$ .

In order to prove (2) and (3) it suffices to prove (2) when  $k$  is algebraically closed. In this case, let  $Z_1, Z_2$  be two irreducible components of  $G$  passing through  $e$ . Since  $k$  is algebraically closed the closed subscheme  $Z_1 \times_k Z_2 \subset G \times_k G$  is irreducible too, see Varieties, Lemma 8.4. Hence  $m(Z_1 \times_k Z_2)$  is contained in an irreducible component of  $G$ . On the other hand it contains  $Z_1$  and  $Z_2$  since  $m|_{e \times G} = \text{id}_G$  and  $m|_{G \times e} = \text{id}_G$ . We conclude  $Z_1 = Z_2$  as desired.  $\square$

**Remark 7.5.** Warning: The result of Lemma 7.4 does not mean that every irreducible component of  $G/k$  is geometrically irreducible. For example the group scheme  $\mu_{3,\mathbf{Q}} = \text{Spec}(\mathbf{Q}[x]/(x^3 - 1))$  over  $\mathbf{Q}$  has two irreducible components corresponding to the factorization  $x^3 - 1 = (x - 1)(x^2 + x + 1)$ . The first factor corresponds to the irreducible component passing through the identity, and the second irreducible component is not geometrically irreducible over  $\text{Spec}(\mathbf{Q})$ .

**Lemma 7.6.** *Let  $G$  be a group scheme over a perfect field  $k$ . Then the reduction  $G_{\text{red}}$  of  $G$  is a closed subgroup scheme of  $G$ .*

**Proof.** Omitted. Hint: Use that  $G_{\text{red}} \times_k G_{\text{red}}$  is reduced by Varieties, Lemmas 6.3 and 6.7.  $\square$

**Lemma 7.7.** *Let  $k$  be a field. Let  $\psi : G' \rightarrow G$  be a morphism of group schemes over  $k$ . If  $\psi(G')$  is open in  $G$ , then  $\psi(G')$  is closed in  $G$ .*

**Proof.** Let  $U = \psi(G') \subset G$ . Let  $Z = G \setminus \psi(G') = G \setminus U$  with the reduced induced closed subscheme structure. By Lemma 7.2 the image of

$$Z \times_k G' \longrightarrow Z \times_k U \longrightarrow G$$

is open (the first arrow is surjective). On the other hand, since  $\psi$  is a homomorphism of group schemes, the image of  $Z \times_k G' \rightarrow G$  is contained in  $Z$  (because translation by  $\psi(g')$  preserves  $U$  for all points  $g'$  of  $G'$ ; small detail omitted). Hence  $Z \subset G$  is an open subset (although not necessarily an open subscheme). Thus  $U = \psi(G')$  is closed.  $\square$

**Lemma 7.8.** *Let  $i : G' \rightarrow G$  be an immersion of group schemes over a field  $k$ . Then  $i$  is a closed immersion, i.e.,  $i(G')$  is a closed subgroup scheme of  $G$ .*

**Proof.** To show that  $i$  is a closed immersion it suffices to show that  $i(G')$  is a closed subset of  $G$ . Let  $k \subset k'$  be a perfect extension of  $k$ . If  $i(G'_{k'}) \subset G_{k'}$  is closed, then  $i(G') \subset G$  is closed by Morphisms, Lemma 25.12 (as  $G_{k'} \rightarrow G$  is flat, quasi-compact and surjective). Hence we may and do assume  $k$  is perfect. We will use without further mention that products of reduced schemes over  $k$  are reduced. We may replace  $G'$  and  $G$  by their reductions, see Lemma 7.6. Let  $\overline{G'} \subset G$  be the closure of  $i(G')$  viewed as a reduced closed subscheme. By Varieties, Lemma 24.1 we conclude that  $\overline{G'} \times_k \overline{G'}$  is the closure of the image of  $G' \times_k G' \rightarrow G \times_k G$ . Hence

$$m(\overline{G'} \times_k \overline{G'}) \subset \overline{G'}$$

as  $m$  is continuous. It follows that  $\overline{G'} \subset G$  is a (reduced) closed subgroup scheme. By Lemma 7.7 we see that  $i(G') \subset \overline{G'}$  is also closed which implies that  $i(G') = \overline{G'}$  as desired.  $\square$

**Lemma 7.9.** *Let  $G$  be a group scheme over a field  $k$ . If  $G$  is irreducible, then  $G$  is quasi-compact.*

**Proof.** Suppose that  $K/k$  is a field extension. If  $G_K$  is quasi-compact, then  $G$  is too as  $G_K \rightarrow G$  is surjective. By Lemma 7.4 we see that  $G_K$  is irreducible. Hence it suffices to prove the lemma after replacing  $k$  by some extension. Choose  $K$  to be an algebraically closed field extension of very large cardinality. Then by Varieties, Lemma 14.2, we see that  $G_K$  is a Jacobson scheme all of whose closed points have residue field equal to  $K$ . In other words we may assume  $G$  is a Jacobson scheme all of whose closed points have residue field  $k$ .

Let  $U \subset G$  be a nonempty affine open. Let  $g \in G(k)$ . Then  $gU \cap U \neq \emptyset$ . Hence we see that  $g$  is in the image of the morphism

$$U \times_{\text{Spec}(k)} U \longrightarrow G, \quad (u_1, u_2) \longmapsto u_1 u_2^{-1}$$

Since the image of this morphism is open (Lemma 7.1) we see that the image is all of  $G$  (because  $G$  is Jacobson and closed points are  $k$ -rational). Since  $U$  is affine, so is  $U \times_{\text{Spec}(k)} U$ . Hence  $G$  is the image of a quasi-compact scheme, hence quasi-compact.  $\square$

**Lemma 7.10.** *Let  $G$  be a group scheme over a field  $k$ . If  $G$  is connected, then  $G$  is irreducible.*

**Proof.** By Varieties, Lemma 7.14 we see that  $G$  is geometrically connected. If we show that  $G_K$  is irreducible for some field extension  $K/k$ , then the lemma follows. Hence we may apply Varieties, Lemma 14.2 to reduce to the case where  $k$  is algebraically closed,  $G$  is a Jacobson scheme, and all the closed points are  $k$ -rational.

Let  $Z \subset G$  be the unique irreducible component of  $G$  passing through the neutral element, see Lemma 7.4. Endowing  $Z$  with the reduced induced closed subscheme structure, we see that  $Z \times_k Z$  is reduced and irreducible (Varieties, Lemmas 6.7 and 8.4). We conclude that  $m|_{Z \times_k Z} : Z \times_k Z \rightarrow G$  factors through  $Z$ . Hence  $Z$  becomes a closed subgroup scheme of  $G$ .

To get a contradiction, assume there exists another irreducible component  $Z' \subset G$ . Then  $Z \cap Z' = \emptyset$  by Lemma 7.4. By Lemma 7.9 we see that  $Z$  is quasi-compact. Thus we may choose a quasi-compact open  $U \subset G$  with  $Z \subset U$  and  $U \cap Z' = \emptyset$ . The image  $W$  of  $Z \times_k U \rightarrow G$  is open in  $G$  by Lemma 7.2. On the other hand,  $W$  is quasi-compact as the image of a quasi-compact space. We claim that  $W$  is closed.

Proof of the claim. Since  $W$  is quasi-compact, we see that points in the closure of  $W$  are specializations of points of  $W$  (Morphisms, Lemma 6.5). Thus we have to show that any irreducible component  $Z'' \subset G$  of  $G$  which meets  $W$  is contained in  $W$ . As  $G$  is Jacobson and closed points are rational,  $Z'' \cap W$  has a rational point  $g \in Z''(k) \cap W(k)$  and hence  $Z'' = Zg$ . But  $W = m(Z \times_k U)$  by construction, so  $Z'' \cap W \neq \emptyset$  implies  $Z'' \subset W$ .

By the claim  $W \subset G$  is an open and closed subset of  $G$ . Now  $W \cap Z' = \emptyset$  since otherwise by the argument given in the preceding paragraph we would get  $Z' = Zg$  for some  $g \in W(k)$ . Then as  $Z$  is a subgroup we could even pick  $g \in U(k)$  which would contradict  $Z' \cap U = \emptyset$ . Hence  $W \subset G$  is a proper open and closed subset which contradicts the assumption that  $G$  is connected.  $\square$

**Proposition 7.11.** *Let  $G$  be a group scheme over a field  $k$ . There exists a canonical closed subgroup scheme  $G^0 \subset G$  with the following properties*

- (1)  $G^0 \rightarrow G$  is a flat closed immersion,
- (2)  $G^0 \subset G$  is the connected component of the identity,
- (3)  $G^0$  is geometrically irreducible, and
- (4)  $G^0$  is quasi-compact.

**Proof.** Let  $G^0$  be the connected component of the identity with its canonical scheme structure (Morphisms, Definition 26.3). To show that  $G^0$  is a closed subgroup scheme we will use the criterion of Lemma 4.4. The morphism  $e : \text{Spec}(k) \rightarrow G$  factors through  $G^0$  as we chose  $G^0$  to be the connected component of  $G$  containing  $e$ . Since  $i : G \rightarrow G$  is an automorphism fixing  $e$ , we see that  $i$  sends  $G^0$  into itself. By Varieties, Lemma 7.13 the scheme  $G^0$  is geometrically connected over  $k$ . Thus  $G^0 \times_k G^0$  is connected (Varieties, Lemma 7.4). Thus  $m(G^0 \times_k G^0) \subset G^0$  set theoretically. Thus  $m|_{G^0 \times_k G^0} : G^0 \times_k G^0 \rightarrow G$  factors through  $G^0$  by Morphisms, Lemma 26.1. Hence  $G^0$  is a closed subgroup scheme of  $G$ . By Lemma 7.10 we see that  $G^0$  is irreducible. By Lemma 7.4 we see that  $G^0$  is geometrically irreducible. By Lemma 7.9 we see that  $G^0$  is quasi-compact.  $\square$

**Lemma 7.12.** *Let  $k$  be a field. Let  $T = \operatorname{Spec}(A)$  where  $A$  is a directed colimit of algebras which are finite products of copies of  $k$ . For any scheme  $X$  over  $k$  we have  $|T \times_k X| = |T| \times |X|$  as topological spaces.*

**Proof.** By taking an affine open covering we reduce to the case of an affine  $X$ . Say  $X = \operatorname{Spec}(B)$ . Write  $A = \operatorname{colim} A_i$  with  $A_i = \prod_{t \in T_i} k$  and  $T_i$  finite. Then  $T_i = |\operatorname{Spec}(A_i)|$  with the discrete topology and the transition morphisms  $A_i \rightarrow A_{i'}$  are given by set maps  $T_{i'} \rightarrow T_i$ . Thus  $|T| = \lim T_i$  as a topological space, see Limits, Lemma 4.6. Similarly we have

$$\begin{aligned} |T \times_k X| &= |\operatorname{Spec}(A \otimes_k B)| \\ &= |\operatorname{Spec}(\operatorname{colim} A_i \otimes_k B)| \\ &= \lim |\operatorname{Spec}(A_i \otimes_k B)| \\ &= \lim |\operatorname{Spec}(\prod_{t \in T_i} B)| \\ &= \lim T_i \times |X| \\ &= (\lim T_i) \times |X| \\ &= |T| \times |X| \end{aligned}$$

by the lemma above and the fact that limits commute with limits.  $\square$

The following lemma says that in fact we can put a “algebraic profinite family of points” in an affine open. We urge the reader to read Lemma 8.6 first.

**Lemma 7.13.** *Let  $k$  be an algebraically closed field. Let  $G$  be a group scheme over  $k$ . Assume that  $G$  is Jacobson and that all closed points are  $k$ -rational. Let  $T = \operatorname{Spec}(A)$  where  $A$  is a directed colimit of algebras which are finite products of copies of  $k$ . For any morphism  $f : T \rightarrow G$  there exists an affine open  $U \subset G$  containing  $f(T)$ .*

**Proof.** Let  $G^0 \subset G$  be the closed subgroup scheme found in Proposition 7.11. The first two paragraphs serve to reduce to the case  $G = G^0$ .

Observe that  $T$  is a directed inverse limit of finite topological spaces (Limits, Lemma 4.6), hence profinite as a topological space (Topology, Definition 22.1). Let  $W \subset G$  be a quasi-compact open containing the image of  $T \rightarrow G$ . After replacing  $W$  by the image of  $G^0 \times W \rightarrow G \times G \rightarrow G$  we may assume that  $W$  is invariant under the action of left translation by  $G^0$ , see Lemma 7.2. Consider the composition

$$\psi = \pi \circ f : T \xrightarrow{f} W \xrightarrow{\pi} \pi_0(W)$$

The space  $\pi_0(W)$  is profinite (Topology, Lemma 23.9 and Properties, Lemma 2.4). Let  $F_\xi \subset T$  be the fibre of  $T \rightarrow \pi_0(W)$  over  $\xi \in \pi_0(W)$ . Assume that for all  $\xi$  we can find an affine open  $U_\xi \subset W$  with  $F_\xi \subset U$ . Since  $\psi : T \rightarrow \pi_0(W)$  is universally closed as a map of topological spaces (Topology, Lemma 17.7), we can find a quasi-compact open  $V_\xi \subset \pi_0(W)$  such that  $\psi^{-1}(V_\xi) \subset f^{-1}(U_\xi)$  (easy topological argument omitted). After replacing  $U_\xi$  by  $U_\xi \cap \pi^{-1}(V_\xi)$ , which is open and closed in  $U_\xi$  hence affine, we see that  $U_\xi \subset \pi^{-1}(V_\xi)$  and  $U_\xi \cap T = \psi^{-1}(V_\xi)$ . By Topology, Lemma 22.4 we can find a finite disjoint union decomposition  $\pi_0(W) = \bigcup_{i=1, \dots, n} V_i$  by quasi-compact opens such that  $V_i \subset V_{\xi_i}$  for some  $i$ . Then we see that

$$f(T) \subset \bigcup_{i=1, \dots, n} U_{\xi_i} \cap \pi^{-1}(V_i)$$

the right hand side of which is a finite disjoint union of affines, therefore affine.

Let  $Z$  be a connected component of  $G$  which meets  $f(T)$ . Then  $Z$  has a  $k$ -rational point  $z$  (because all residue fields of the scheme  $T$  are isomorphic to  $k$ ). Hence  $Z = G^0 z$ . By our choice of  $W$ , we see that  $Z \subset W$ . The argument in the preceding paragraph reduces us to the problem of finding an affine open neighbourhood of  $f(T) \cap Z$  in  $W$ . After translation by a rational point we may assume that  $Z = G^0$  (details omitted). Observe that the scheme theoretic inverse image  $T' = f^{-1}(G^0) \subset T$  is a closed subscheme, which has the same type. After replacing  $T$  by  $T'$  we may assume that  $f(T) \subset G^0$ . Choose an affine open neighbourhood  $U \subset G$  of  $e \in G$ , so that in particular  $U \cap G^0$  is nonempty. We will show there exists a  $g \in G^0(k)$  such that  $f(T) \subset g^{-1}U$ . This will finish the proof as  $g^{-1}U \subset W$  by the left  $G^0$ -invariance of  $W$ .

The arguments in the preceding two paragraphs allow us to pass to  $G^0$  and reduce the problem to the following: Assume  $G$  is irreducible and  $U \subset G$  an affine open neighbourhood of  $e$ . Show that  $f(T) \subset g^{-1}U$  for some  $g \in G(k)$ . Consider the morphism

$$U \times_k T \longrightarrow G \times_k T, \quad (t, u) \longrightarrow (uf(t)^{-1}, t)$$

which is an open immersion (because the extension of this morphism to  $G \times_k T \rightarrow G \times_k T$  is an isomorphism). By our assumption on  $T$  we see that we have  $|U \times_k T| = |U| \times |T|$  and similarly for  $G \times_k T$ , see Lemma 7.12. Hence the image of the displayed open immersion is a finite union of boxes  $\bigcup_{i=1, \dots, n} U_i \times V_i$  with  $V_i \subset T$  and  $U_i \subset G$  quasi-compact open. This means that the possible opens  $Uf(t)^{-1}$ ,  $t \in T$  are finite in number, say  $Uf(t_1)^{-1}, \dots, Uf(t_r)^{-1}$ . Since  $G$  is irreducible the intersection

$$Uf(t_1)^{-1} \cap \dots \cap Uf(t_r)^{-1}$$

is nonempty and since  $G$  is Jacobson with closed points  $k$ -rational, we can choose a  $k$ -valued point  $g \in G(k)$  of this intersection. Then we see that  $g \in Uf(t)^{-1}$  for all  $t \in T$  which means that  $f(t) \in g^{-1}U$  as desired.  $\square$

**Remark 7.14.** If  $G$  is a group scheme over a field, is there always a quasi-compact open and closed subgroup scheme? By Proposition 7.11 this question is only interesting if  $G$  has infinitely many connected components (geometrically).

**Lemma 7.15.** *Let  $G$  be a group scheme over a field. There exists an open and closed subscheme  $G' \subset G$  which is a countable union of affines.*

**Proof.** Let  $e \in U(k)$  be a quasi-compact open neighbourhood of the identity element. By replacing  $U$  by  $U \cap i(U)$  we may assume that  $U$  is invariant under the inverse map. As  $G$  is separated this is still a quasi-compact set. Set

$$G' = \bigcup_{n \geq 1} m_n(U \times_k \dots \times_k U)$$

where  $m_n : G \times_k \dots \times_k G \rightarrow G$  is the  $n$ -slot multiplication map  $(g_1, \dots, g_n) \mapsto m(m(\dots(m(g_1, g_2), g_3), \dots), g_n)$ . Each of these maps are open (see Lemma 7.1) hence  $G'$  is an open subgroup scheme. By Lemma 7.7 it is also a closed subgroup scheme.  $\square$

### 8. Properties of algebraic group schemes

Recall that a scheme over a field  $k$  is (locally) algebraic if it is (locally) of finite type over  $\text{Spec}(k)$ , see Varieties, Definition 20.1. This is the sense of algebraic we are using in the title of this section.

**Lemma 8.1.** *Let  $k$  be a field. Let  $G$  be a locally algebraic group scheme over  $k$ . Then  $G$  is equidimensional and  $\dim(G) = \dim_g(G)$  for all  $g \in G$ . For any closed point  $g \in G$  we have  $\dim(G) = \dim(\mathcal{O}_{G,g})$ .*

**Proof.** Let us first prove that  $\dim_g(G) = \dim_{g'}(G)$  for any pair of points  $g, g' \in G$ . By Morphisms, Lemma 28.3 we may extend the ground field at will. Hence we may assume that both  $g$  and  $g'$  are defined over  $k$ . Hence there exists an automorphism of  $G$  mapping  $g$  to  $g'$ , whence the equality. By Morphisms, Lemma 28.1 we have  $\dim_g(G) = \dim(\mathcal{O}_{G,g}) + \text{trdeg}_k(\kappa(g))$ . On the other hand, the dimension of  $G$  (or any open subset of  $G$ ) is the supremum of the dimensions of the local rings of  $G$ , see Properties, Lemma 10.3. Clearly this is maximal for closed points  $g$  in which case  $\text{trdeg}_k(\kappa(g)) = 0$  (by the Hilbert Nullstellensatz, see Morphisms, Section 16). Hence the lemma follows.  $\square$

The following result is sometimes referred to as Cartier's theorem.

**Lemma 8.2.** *Let  $k$  be a field of characteristic 0. Let  $G$  be a locally algebraic group scheme over  $k$ . Then the structure morphism  $G \rightarrow \text{Spec}(k)$  is smooth, i.e.,  $G$  is a smooth group scheme.*

**Proof.** By Lemma 6.3 the module of differentials of  $G$  over  $k$  is free. Hence smoothness follows from Varieties, Lemma 25.1.  $\square$

**Remark 8.3.** Any group scheme over a field of characteristic 0 is reduced, see [Per75, I, Theorem 1.1 and I, Corollary 3.9, and II, Theorem 2.4] and also [Per76, Proposition 4.2.8]. This was a question raised in [Oor66, page 80]. We have seen in Lemma 8.2 that this holds when the group scheme is locally of finite type.

**Lemma 8.4.** *Let  $k$  be a perfect field of characteristic  $p > 0$  (see Lemma 8.2 for the characteristic zero case). Let  $G$  be a locally algebraic group scheme over  $k$ . If  $G$  is reduced then the structure morphism  $G \rightarrow \text{Spec}(k)$  is smooth, i.e.,  $G$  is a smooth group scheme.*

**Proof.** By Lemma 6.3 the sheaf  $\Omega_{G/k}$  is free. Hence the lemma follows from Varieties, Lemma 25.2.  $\square$

**Remark 8.5.** Let  $k$  be a field of characteristic  $p > 0$ . Let  $\alpha \in k$  be an element which is not a  $p$ th power. The closed subgroup scheme

$$G = V(x^p + \alpha y^p) \subset \mathbf{G}_{a,k}^2$$

is reduced and irreducible but not smooth (not even normal).

The following lemma is a special case of Lemma 7.13 with a somewhat easier proof.

**Lemma 8.6.** *Let  $k$  be an algebraically closed field. Let  $G$  be a locally algebraic group scheme over  $k$ . Let  $g_1, \dots, g_n \in G(k)$  be  $k$ -rational points. Then there exists an affine open  $U \subset G$  containing  $g_1, \dots, g_n$ .*

**Proof.** We first argue by induction on  $n$  that we may assume all  $g_i$  are on the same connected component of  $G$ . Namely, if not, then we can find a decomposition  $G = W_1 \amalg W_2$  with  $W_i$  open in  $G$  and (after possibly renumbering)  $g_1, \dots, g_r \in W_1$  and  $g_{r+1}, \dots, g_n \in W_2$  for some  $0 < r < n$ . By induction we can find affine opens  $U_1$  and  $U_2$  of  $G$  with  $g_1, \dots, g_r \in U_1$  and  $g_{r+1}, \dots, g_n \in U_2$ . Then

$$g_1, \dots, g_n \in (U_1 \cap W_1) \cup (U_2 \cap W_2)$$

is a solution to the problem. Thus we may assume  $g_1, \dots, g_n$  are all on the same connected component of  $G$ . Translating by  $g_1^{-1}$  we may assume  $g_1, \dots, g_n \in G^0$  where  $G^0 \subset G$  is as in Proposition 7.11. Choose an affine open neighbourhood  $U$  of  $e$ , in particular  $U \cap G^0$  is nonempty. Since  $G^0$  is irreducible we see that

$$G^0 \cap (Ug_1^{-1} \cap \dots \cap Ug_n^{-1})$$

is nonempty. Since  $G \rightarrow \operatorname{Spec}(k)$  is locally of finite type, also  $G^0 \rightarrow \operatorname{Spec}(k)$  is locally of finite type, hence any nonempty open has a  $k$ -rational point. Thus we can pick  $g \in G^0(k)$  with  $g \in Ug_i^{-1}$  for all  $i$ . Then  $g_i \in g^{-1}U$  for all  $i$  and  $g^{-1}U$  is the affine open we were looking for.  $\square$

**Lemma 8.7.** *Let  $k$  be a field. Let  $G$  be an algebraic group scheme over  $k$ . Then  $G$  is quasi-projective over  $k$ .*

**Proof.** By Varieties, Lemma 15.1 we may assume that  $k$  is algebraically closed. Let  $G^0 \subset G$  be the connected component of  $G$  as in Proposition 7.11. Then every other connected component of  $G$  has a  $k$ -rational point and hence is isomorphic to  $G^0$  as a scheme. Since  $G$  is quasi-compact and Noetherian, there are finitely many of these connected components. Thus we reduce to the case discussed in the next paragraph.

Let  $G$  be a connected algebraic group scheme over an algebraically closed field  $k$ . If the characteristic of  $k$  is zero, then  $G$  is smooth over  $k$  by Lemma 8.2. If the characteristic of  $k$  is  $p > 0$ , then we let  $H = G_{red}$  be the reduction of  $G$ . By Divisors, Proposition 17.9 it suffices to show that  $H$  has an ample invertible sheaf. (For an algebraic scheme over  $k$  having an ample invertible sheaf is equivalent to being quasi-projective over  $k$ , see for example the very general More on Morphisms, Lemma 49.1.) By Lemma 7.6 we see that  $H$  is a group scheme over  $k$ . By Lemma 8.4 we see that  $H$  is smooth over  $k$ . This reduces us to the situation discussed in the next paragraph.

Let  $G$  be a quasi-compact irreducible smooth group scheme over an algebraically closed field  $k$ . Observe that the local rings of  $G$  are regular and hence UFDs (Varieties, Lemma 25.3 and More on Algebra, Lemma 121.2). The complement of a nonempty affine open of  $G$  is the support of an effective Cartier divisor  $D$ . This follows from Divisors, Lemma 16.6. (Observe that  $G$  is separated by Lemma 7.3.) We conclude there exists an effective Cartier divisor  $D \subset G$  such that  $G \setminus D$  is affine. We will use below that for any  $n \geq 1$  and  $g_1, \dots, g_n \in G(k)$  the complement  $G \setminus \bigcup Dg_i$  is affine. Namely, it is the intersection of the affine opens  $G \setminus Dg_i \cong G \setminus D$  in the separated scheme  $G$ .

We may choose the top row of the diagram

$$\begin{array}{ccccc} G & \xleftarrow{j} & U & \xrightarrow{\pi} & \mathbf{A}_k^d \\ & & \uparrow & & \uparrow \\ & & W & \xrightarrow{\pi'} & V \end{array}$$

such that  $U \neq \emptyset$ ,  $j : U \rightarrow G$  is an open immersion, and  $\pi$  is étale, see Morphisms, Lemma 36.20. There is a nonempty affine open  $V \subset \mathbf{A}_k^d$  such that with  $W = \pi^{-1}(V)$  the morphism  $\pi' = \pi|_W : W \rightarrow V$  is finite étale. In particular  $\pi'$  is finite locally free, say of degree  $n$ . Consider the effective Cartier divisor

$$\mathcal{D} = \{(g, w) \mid m(g, j(w)) \in D\} \subset G \times W$$

(This is the restriction to  $G \times W$  of the pullback of  $D \subset G$  under the flat morphism  $m : G \times G \rightarrow G$ .) Consider the closed subset<sup>1</sup>  $T = (1 \times \pi')(\mathcal{D}) \subset G \times V$ . Since  $\pi'$  is finite locally free, every irreducible component of  $T$  has codimension 1 in  $G \times V$ . Since  $G \times V$  is smooth over  $k$  we conclude these components are effective Cartier divisors (Divisors, Lemma 15.7 and lemmas cited above) and hence  $T$  is the support of an effective Cartier divisor  $E$  in  $G \times V$ . If  $v \in V(k)$ , then  $(\pi')^{-1}(v) = \{w_1, \dots, w_n\} \subset W(k)$  and we see that

$$E_v = \bigcup_{i=1, \dots, n} Dj(w_i)^{-1}$$

in  $G$  set theoretically. In particular we see that  $G \setminus E_v$  is affine open (see above). Moreover, if  $g \in G(k)$ , then there exists a  $v \in V$  such that  $g \notin E_v$ . Namely, the set  $W'$  of  $w \in W$  such that  $g \notin Dj(w)^{-1}$  is nonempty open and it suffices to pick  $v$  such that the fibre of  $W' \rightarrow V$  over  $v$  has  $n$  elements.

Consider the invertible sheaf  $\mathcal{M} = \mathcal{O}_{G \times V}(E)$  on  $G \times V$ . By Varieties, Lemma 30.5 the isomorphism class  $\mathcal{L}$  of the restriction  $\mathcal{M}_v = \mathcal{O}_G(E_v)$  is independent of  $v \in V(k)$ . On the other hand, for every  $g \in G(k)$  we can find a  $v$  such that  $g \notin E_v$  and such that  $G \setminus E_v$  is affine. Thus the canonical section (Divisors, Definition 14.1) of  $\mathcal{O}_G(E_v)$  corresponds to a section  $s_v$  of  $\mathcal{L}$  which does not vanish at  $g$  and such that  $G_{s_v}$  is affine. This means that  $\mathcal{L}$  is ample by definition (Properties, Definition 26.1).  $\square$

**Lemma 8.8.** *Let  $k$  be a field. Let  $G$  be a locally algebraic group scheme over  $k$ . Then the center of  $G$  is a closed subgroup scheme of  $G$ .*

**Proof.** Let  $\text{Aut}(G)$  denote the contravariant functor on the category of schemes over  $k$  which associates to  $S/k$  the set of automorphisms of the base change  $G_S$  as a group scheme over  $S$ . There is a natural transformation

$$G \longrightarrow \text{Aut}(G), \quad g \longmapsto \text{inn}_g$$

sending an  $S$ -valued point  $g$  of  $G$  to the inner automorphism of  $G$  determined by  $g$ . The center  $C$  of  $G$  is by definition the kernel of this transformation, i.e., the functor which to  $S$  associates those  $g \in G(S)$  whose associated inner automorphism is trivial. The statement of the lemma is that this functor is representable by a closed subgroup scheme of  $G$ .

<sup>1</sup>Using the material in Divisors, Section 17 we could take as effective Cartier divisor  $E$  the norm of the effective Cartier divisor  $\mathcal{D}$  along the finite locally free morphism  $1 \times \pi'$  bypassing some of the arguments.



Choose an integer  $n \geq 1$ . Let  $G_n \subset G$  be the  $n$ th infinitesimal neighbourhood of the identity element  $e$  of  $G$ . For every scheme  $S/k$  the base change  $G_{n,S}$  is the  $n$ th infinitesimal neighbourhood of  $e_S : S \rightarrow G_S$ . Thus we see that there is a natural transformation  $\text{Aut}(G) \rightarrow \text{Aut}(G_n)$  where the right hand side is the functor of automorphisms of  $G_n$  as a scheme ( $G_n$  isn't in general a group scheme). Observe that  $G_n$  is the spectrum of an artinian local ring  $A_n$  with residue field  $k$  which has finite dimension as a  $k$ -vector space (Varieties, Lemma 20.2). Since every automorphism of  $G_n$  induces in particular an invertible linear map  $A_n \rightarrow A_n$ , we obtain transformations of functors

$$G \rightarrow \text{Aut}(G) \rightarrow \text{Aut}(G_n) \rightarrow \text{GL}(A_n)$$

The final group valued functor is representable, see Example 5.4, and the last arrow is visibly injective. Thus for every  $n$  we obtain a closed subgroup scheme

$$H_n = \text{Ker}(G \rightarrow \text{Aut}(G_n)) = \text{Ker}(G \rightarrow \text{GL}(A_n)).$$

As a first approximation we set  $H = \bigcap_{n \geq 1} H_n$  (scheme theoretic intersection). This is a closed subgroup scheme which contains the center  $C$ .

Let  $h$  be an  $S$ -valued point of  $H$  with  $S$  locally Noetherian. Then the automorphism  $\text{inn}_h$  induces the identity on all the closed subschemes  $G_{n,S}$ . Consider the kernel  $K = \text{Ker}(\text{inn}_h : G_S \rightarrow G_S)$ . This is a closed subgroup scheme of  $G_S$  over  $S$  containing the closed subschemes  $G_{n,S}$  for  $n \geq 1$ . This implies that  $K$  contains an open neighbourhood of  $e(S) \subset G_S$ , see Algebra, Remark 51.6. Let  $G^0 \subset G$  be as in Proposition 7.11. Since  $G^0$  is geometrically irreducible, we conclude that  $K$  contains  $G_S^0$  (for any nonempty open  $U \subset G_{k'}^0$  and any field extension  $k'/k$  we have  $U \cdot U^{-1} = G_{k'}^0$ , see proof of Lemma 7.9). Applying this with  $S = H$  we find that  $G^0$  and  $H$  are subgroup schemes of  $G$  whose points commute: for any scheme  $S$  and any  $S$ -valued points  $g \in G^0(S)$ ,  $h \in H(S)$  we have  $gh = hg$  in  $G(S)$ .

Assume that  $k$  is algebraically closed. Then we can pick a  $k$ -valued point  $g_i$  in each irreducible component  $G_i$  of  $G$ . Observe that in this case the connected components of  $G$  are the irreducible components of  $G$  are the translates of  $G^0$  by our  $g_i$ . We claim that

$$C = H \cap \bigcap_i \text{Ker}(\text{inn}_{g_i} : G \rightarrow G) \quad (\text{scheme theoretic intersection})$$

Namely,  $C$  is contained in the right hand side. On the other hand, every  $S$ -valued point  $h$  of the right hand side commutes with  $G^0$  and with  $g_i$  hence with everything in  $G = \bigcup G^0 g_i$ .

The case of a general base field  $k$  follows from the result for the algebraic closure  $\bar{k}$  by descent. Namely, let  $A \subset G_{\bar{k}}$  the closed subgroup scheme representing the center of  $G_{\bar{k}}$ . Then we have

$$A \times_{\text{Spec}(k)} \text{Spec}(\bar{k}) = \text{Spec}(\bar{k}) \times_{\text{Spec}(k)} A$$

as closed subschemes of  $G_{\bar{k} \otimes_k \bar{k}}$  by the functorial nature of the center. Hence we see that  $A$  descends to a closed subgroup scheme  $Z \subset G$  by Descent, Lemma 37.2 (and Descent, Lemma 23.19). Then  $Z$  represents  $C$  (small argument omitted) and the proof is complete.  $\square$

### 9. Abelian varieties

An excellent reference for this material is Mumford's book on abelian varieties, see [Mum70]. We encourage the reader to look there. There are many equivalent definitions; here is one.

**Definition 9.1.** Let  $k$  be a field. An *abelian variety* is a group scheme over  $k$  which is also a proper, geometrically integral variety over  $k^2$ .

We prove a few lemmas about this notion and then we collect all the results together in Proposition 9.11.

**Lemma 9.2.** *Let  $k$  be a field. Let  $A$  be an abelian variety over  $k$ . Then  $A$  is projective.*

**Proof.** This follows from Lemma 8.7 and More on Morphisms, Lemma 50.1.  $\square$

**Lemma 9.3.** *Let  $k$  be a field. Let  $A$  be an abelian variety over  $k$ . For any field extension  $K/k$  the base change  $A_K$  is an abelian variety over  $K$ .*

**Proof.** Omitted. Note that this is why we insisted on  $A$  being geometrically integral; without that condition this lemma (and many others below) would be wrong.  $\square$

**Lemma 9.4.** *Let  $k$  be a field. Let  $A$  be an abelian variety over  $k$ . Then  $A$  is smooth over  $k$ .*

**Proof.** If  $k$  is perfect then this follows from Lemma 8.2 (characteristic zero) and Lemma 8.4 (positive characteristic). We can reduce the general case to this case by descent for smoothness (Descent, Lemma 23.27) and going to the perfect closure using Lemma 9.3.  $\square$

**Lemma 9.5.** *An abelian variety is an abelian group scheme, i.e., the group law is commutative.*

**Proof.** Let  $k$  be a field. Let  $A$  be an abelian variety over  $k$ . By Lemma 9.3 we may replace  $k$  by its algebraic closure. Consider the morphism

$$h : A \times_k A \longrightarrow A \times_k A, \quad (x, y) \longmapsto (x, xyx^{-1}y^{-1})$$

This is a morphism over  $A$  via the first projection on either side. Let  $e \in A(k)$  be the unit. Then we see that  $h|_{e \times A}$  is constant with value  $(e, e)$ . By More on Morphisms, Lemma 44.3 there exists an open neighbourhood  $U \subset A$  of  $e$  such that  $h|_{U \times A}$  factors through some  $Z \subset U \times A$  finite over  $U$ . This means that for  $x \in U(k)$  the morphism  $A \rightarrow A$ ,  $y \mapsto xyx^{-1}y^{-1}$  takes finitely many values. Of course this means it is constant with value  $e$ . Thus  $(x, y) \mapsto xyx^{-1}y^{-1}$  is constant with value  $e$  on  $U \times A$  which implies that the group law on  $A$  is abelian.  $\square$

**Lemma 9.6.** *Let  $k$  be a field. Let  $A$  be an abelian variety over  $k$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_A$ -module. Then there is an isomorphism*

$$m_{1,2,3}^* \mathcal{L} \otimes m_1^* \mathcal{L} \otimes m_2^* \mathcal{L} \otimes m_3^* \mathcal{L} \cong m_{1,2}^* \mathcal{L} \otimes m_{1,3}^* \mathcal{L} \otimes m_{2,3}^* \mathcal{L}$$

*of invertible modules on  $A \times_k A \times_k A$  where  $m_{i_1, \dots, i_t} : A \times_k A \times_k A \rightarrow A$  is the morphism  $(x_1, x_2, x_3) \mapsto \sum x_{i_j}$ .*

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<sup>2</sup>For equivalent definitions see Remark 9.12.

**Proof.** Apply the theorem of the cube (More on Morphisms, Theorem 33.8) to the difference

$$\mathcal{M} = m_{1,2,3}^* \mathcal{L} \otimes m_1^* \mathcal{L} \otimes m_2^* \mathcal{L} \otimes m_3^* \mathcal{L} \otimes m_{1,2}^* \mathcal{L}^{\otimes -1} \otimes m_{1,3}^* \mathcal{L}^{\otimes -1} \otimes m_{2,3}^* \mathcal{L}^{\otimes -1}$$

This works because the restriction of  $\mathcal{M}$  to  $A \times A \times e = A \times A$  is equal to

$$n_{1,2}^* \mathcal{L} \otimes n_1^* \mathcal{L} \otimes n_2^* \mathcal{L} \otimes n_{1,2}^* \mathcal{L}^{\otimes -1} \otimes n_1^* \mathcal{L}^{\otimes -1} \otimes n_2^* \mathcal{L}^{\otimes -1} \cong \mathcal{O}_{A \times_k A}$$

where  $n_{i_1, \dots, i_t} : A \times_k A \rightarrow A$  is the morphism  $(x_1, x_2) \mapsto \sum x_{i_j}$ . Similarly for  $A \times e \times A$  and  $e \times A \times A$ .  $\square$

**Lemma 9.7.** *Let  $k$  be a field. Let  $A$  be an abelian variety over  $k$ . Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_A$ -module. Then*

$$[n]^* \mathcal{L} \cong \mathcal{L}^{\otimes n(n+1)/2} \otimes ([-1]^* \mathcal{L})^{\otimes n(n-1)/2}$$

where  $[n] : A \rightarrow A$  sends  $x$  to  $x + x + \dots + x$  with  $n$  summands and where  $[-1] : A \rightarrow A$  is the inverse of  $A$ .

**Proof.** Consider the morphism  $A \rightarrow A \times_k A \times_k A$ ,  $x \mapsto (x, x, -x)$  where  $-x = [-1](x)$ . Pulling back the relation of Lemma 9.6 we obtain

$$\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \otimes [-1]^* \mathcal{L} \cong [2]^* \mathcal{L}$$

which proves the result for  $n = 2$ . By induction assume the result holds for  $1, 2, \dots, n$ . Then consider the morphism  $A \rightarrow A \times_k A \times_k A$ ,  $x \mapsto (x, x, [n-1]x)$ . Pulling back the relation of Lemma 9.6 we obtain

$$[n+1]^* \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \otimes [n-1]^* \mathcal{L} \cong [2]^* \mathcal{L} \otimes [n]^* \mathcal{L} \otimes [n]^* \mathcal{L}$$

and the result follows by elementary arithmetic.  $\square$

**Lemma 9.8.** *Let  $k$  be a field. Let  $A$  be an abelian variety over  $k$ . Let  $[d] : A \rightarrow A$  be the multiplication by  $d$ . Then  $[d]$  is finite locally free of degree  $d^{2 \dim(A)}$ .*

**Proof.** By Lemma 9.2 (and More on Morphisms, Lemma 50.1) we see that  $A$  has an ample invertible module  $\mathcal{L}$ . Since  $[-1] : A \rightarrow A$  is an automorphism, we see that  $[-1]^* \mathcal{L}$  is an ample invertible  $\mathcal{O}_X$ -module as well. Thus  $\mathcal{N} = \mathcal{L} \otimes [-1]^* \mathcal{L}$  is ample, see Properties, Lemma 26.5. Since  $\mathcal{N} \cong [-1]^* \mathcal{N}$  we see that  $[d]^* \mathcal{N} \cong \mathcal{N}^{\otimes d^2}$  by Lemma 9.7.

To get a contradiction, let  $C \subset X$  be a proper curve contained in a fibre of  $[d]$ . Then  $\mathcal{N}^{\otimes d^2}|_C \cong \mathcal{O}_C$  is an ample invertible  $\mathcal{O}_C$ -module of degree 0 which contradicts Varieties, Lemma 44.14 for example. (You can also use Varieties, Lemma 45.9.) Thus every fibre of  $[d]$  has dimension 0 and hence  $[d]$  is finite for example by Cohomology of Schemes, Lemma 21.1. Moreover, since  $A$  is smooth over  $k$  by Lemma 9.4 we see that  $[d] : A \rightarrow A$  is flat by Algebra, Lemma 128.1 (we also use that schemes smooth over fields are regular and that regular rings are Cohen-Macaulay, see Varieties, Lemma 25.3 and Algebra, Lemma 106.3). Thus  $[d]$  is finite flat hence finite locally free by Morphisms, Lemma 48.2.

Finally, we come to the formula for the degree. By Varieties, Lemma 45.11 we see that

$$\deg_{\mathcal{N}^{\otimes d^2}}(A) = \deg([d]) \deg_{\mathcal{N}}(A)$$

Since the degree of  $A$  with respect to  $\mathcal{N}^{\otimes d^2}$ , respectively  $\mathcal{N}$  is the coefficient of  $n^{\dim(A)}$  in the polynomial

$$n \mapsto \chi(A, \mathcal{N}^{\otimes nd^2}), \quad \text{respectively} \quad n \mapsto \chi(A, \mathcal{N}^{\otimes n})$$

we see that  $\deg([d]) = d^{2\dim(A)}$ .  $\square$

**Lemma 9.9.** *Let  $k$  be a field. Let  $A$  be a nonzero abelian variety over  $k$ . Then  $[d] : A \rightarrow A$  is étale if and only if  $d$  is invertible in  $k$ .*

**Proof.** Observe that  $[d](x+y) = [d](x) + [d](y)$ . Since translation by a point is an automorphism of  $A$ , we see that the set of points where  $[d] : A \rightarrow A$  is étale is either empty or equal to  $A$  (some details omitted). Thus it suffices to check whether  $[d]$  is étale at the unit  $e \in A(k)$ . Since we know that  $[d]$  is finite locally free (Lemma 9.8) to see that it is étale at  $e$  is equivalent to proving that  $d[d] : T_{A/k,e} \rightarrow T_{A/k,e}$  is injective. See Varieties, Lemma 16.8 and Morphisms, Lemma 36.16. By Lemma 6.4 we see that  $d[d]$  is given by multiplication by  $d$  on  $T_{A/k,e}$ .  $\square$

**Lemma 9.10.** *Let  $k$  be a field of characteristic  $p > 0$ . Let  $A$  be an abelian variety of dimension  $g$  over  $k$ . The fibre of  $[p] : A \rightarrow A$  over  $0$  has at most  $p^g$  distinct points.*

**Proof.** To prove this, we may and do replace  $k$  by the algebraic closure. By Lemma 6.4 the derivative of  $[p]$  is multiplication by  $p$  as a map  $T_{A/k,e} \rightarrow T_{A/k,e}$  and hence is zero (compare with proof of Lemma 9.9). Since  $[p]$  commutes with translation we conclude that the derivative of  $[p]$  is everywhere zero, i.e., that the induced map  $[p]^* \Omega_{A/k} \rightarrow \Omega_{A/k}$  is zero. Looking at generic points, we find that the corresponding map  $[p]^* : k(A) \rightarrow k(A)$  of function fields induces the zero map on  $\Omega_{k(A)/k}$ . Let  $t_1, \dots, t_g$  be a  $p$ -basis of  $k(A)$  over  $k$  (More on Algebra, Definition 46.1 and Lemma 46.2). Then  $[p]^*(t_i)$  has a  $p$ th root by Algebra, Lemma 158.2. We conclude that  $k(A)[x_1, \dots, x_g]/(x_1^p - t_1, \dots, x_g^p - t_g)$  is a subextension of  $[p]^* : k(A) \rightarrow k(A)$ . Thus we can find an affine open  $U \subset A$  such that  $t_i \in \mathcal{O}_A(U)$  and  $x_i \in \mathcal{O}_A([p]^{-1}(U))$ . We obtain a factorization

$$[p]^{-1}(U) \xrightarrow{\pi_1} \text{Spec}(\mathcal{O}(U)[x_1, \dots, x_g]/(x_1^p - t_1, \dots, x_g^p - t_g)) \xrightarrow{\pi_2} U$$

of  $[p]$  over  $U$ . After shrinking  $U$  we may assume that  $\pi_1$  is finite locally free (for example by generic flatness – actually it is already finite locally free in our case). By Lemma 9.8 we see that  $[p]$  has degree  $p^{2g}$ . Since  $\pi_2$  has degree  $p^g$  we see that  $\pi_1$  has degree  $p^g$  as well. The morphism  $\pi_2$  is a universal homeomorphism hence the fibres are singletons. We conclude that the (set theoretic) fibres of  $[p]^{-1}(U) \rightarrow U$  are the fibres of  $\pi_1$ . Hence they have at most  $p^g$  elements. Since  $[p]$  is a homomorphism of group schemes over  $k$ , the fibre of  $[p] : A(k) \rightarrow A(k)$  has the same cardinality for every  $a \in A(k)$  and the proof is complete.  $\square$

**Proposition 9.11.** *Let  $A$  be an abelian variety over a field  $k$ . Then*

- (1)  $A$  is projective over  $k$ ,
- (2)  $A$  is a commutative group scheme,
- (3) the morphism  $[n] : A \rightarrow A$  is surjective for all  $n \geq 1$ ,
- (4) if  $k$  is algebraically closed, then  $A(k)$  is a divisible abelian group,
- (5)  $A[n] = \text{Ker}([n] : A \rightarrow A)$  is a finite group scheme of degree  $n^{2\dim A}$  over  $k$ ,
- (6)  $A[n]$  is étale over  $k$  if and only if  $n \in k^*$ ,
- (7) if  $n \in k^*$  and  $k$  is algebraically closed, then  $A(k)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2\dim(A)}$ ,

- (8) if  $k$  is algebraically closed of characteristic  $p > 0$ , then there exists an integer  $0 \leq f \leq \dim(A)$  such that  $A(k)[p^m] \cong (\mathbf{Z}/p^m\mathbf{Z})^{\oplus f}$  for all  $m \geq 1$ .

**Proof.** Part (1) follows from Lemma 9.2. Part (2) follows from Lemma 9.5. Part (3) follows from Lemma 9.8. If  $k$  is algebraically closed then surjective morphisms of varieties over  $k$  induce surjective maps on  $k$ -rational points, hence (4) follows from (3). Part (5) follows from Lemma 9.8 and the fact that a base change of a finite locally free morphism of degree  $N$  is a finite locally free morphism of degree  $N$ . Part (6) follows from Lemma 9.9. Namely, if  $n$  is invertible in  $k$ , then  $[n]$  is étale and hence  $A[n]$  is étale over  $k$ . On the other hand, if  $n$  is not invertible in  $k$ , then  $[n]$  is not étale at  $e$  and it follows that  $A[n]$  is not étale over  $k$  at  $e$  (use Morphisms, Lemmas 36.16 and 35.15).

Assume  $k$  is algebraically closed. Set  $g = \dim(A)$ . Proof of (7). Let  $\ell$  be a prime number which is invertible in  $k$ . Then we see that

$$A[\ell](k) = A(k)[\ell]$$

is a finite abelian group, annihilated by  $\ell$ , of order  $\ell^{2g}$ . It follows that it is isomorphic to  $(\mathbf{Z}/\ell\mathbf{Z})^{2g}$  by the structure theory for finite abelian groups. Next, we consider the short exact sequence

$$0 \rightarrow A(k)[\ell] \rightarrow A(k)[\ell^2] \xrightarrow{\ell} A(k)[\ell] \rightarrow 0$$

Arguing similarly as above we conclude that  $A(k)[\ell^2] \cong (\mathbf{Z}/\ell^2\mathbf{Z})^{2g}$ . By induction on the exponent we find that  $A(k)[\ell^m] \cong (\mathbf{Z}/\ell^m\mathbf{Z})^{2g}$ . For composite integers  $n$  prime to the characteristic of  $k$  we take primary parts and we find the correct shape of the  $n$ -torsion in  $A(k)$ . The proof of (8) proceeds in exactly the same way, using that Lemma 9.10 gives  $A(k)[p] \cong (\mathbf{Z}/p\mathbf{Z})^{\oplus f}$  for some  $0 \leq f \leq g$ .  $\square$

**Remark 9.12.** Let  $k$  be a field. There are  $2 \times 4 \times 2 = 16$  equivalent definitions of abelian varieties. Let

- projective, proper,
- geometrically irreducible, irreducible, geometrically connected, connected,
- smooth, geometrically reduced

be three sets of properties, pick one from each of them, and let  $A$  be a group scheme over  $k$  with the chosen properties over  $k$ . Then  $A$  is an abelian variety. If we pick the options “proper, geometrically irreducible, geometrically reduced”, then we recover Definition 9.1 (use Varieties, Lemma 9.2). The weakest possible options would be “proper, connected, and geometrically reduced”, see for example Morphisms, Lemma 43.5 and Varieties, Lemma 25.4. So say  $A$  is a proper, connected, and geometrically reduced group scheme over  $k$ . Then  $A$  is geometrically irreducible by Lemmas 7.10 and 7.4 and hence an abelian variety. Finally, if  $A/k$  is an abelian variety, then it is projective and smooth over  $k$  (Proposition 9.11), whence satisfies the strongest possible options “projective, geometrically irreducible, smooth”.

## 10. Actions of group schemes

Let  $(G, m)$  be a group and let  $V$  be a set. Recall that a (left) action of  $G$  on  $V$  is given by a map  $a : G \times V \rightarrow V$  such that

- (1) (associativity)  $a(m(g, g'), v) = a(g, a(g', v))$  for all  $g, g' \in G$  and  $v \in V$ , and
- (2) (identity)  $a(e, v) = v$  for all  $v \in V$ .

We also say that  $V$  is a  $G$ -set (this usually means we drop the  $a$  from the notation – which is abuse of notation). A *map of  $G$ -sets*  $\psi : V \rightarrow V'$  is any set map such that  $\psi(a(g, v)) = a(g, \psi(v))$  for all  $v \in V$ .

**Definition 10.1.** Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ .

- (1) An *action of  $G$  on the scheme  $X/S$*  is a morphism  $a : G \times_S X \rightarrow X$  over  $S$  such that for every  $T/S$  the map  $a : G(T) \times X(T) \rightarrow X(T)$  defines the structure of a  $G(T)$ -set on  $X(T)$ .
- (2) Suppose that  $X, Y$  are schemes over  $S$  each endowed with an action of  $G$ . An *equivariant* or more precisely a  *$G$ -equivariant* morphism  $\psi : X \rightarrow Y$  is a morphism of schemes over  $S$  such that for every  $T/S$  the map  $\psi : X(T) \rightarrow Y(T)$  is a morphism of  $G(T)$ -sets.

In situation (1) this means that the diagrams

$$(10.1.1) \quad \begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{1_G \times a} & G \times_S X \\ m \times 1_X \downarrow & & \downarrow a \\ G \times_S X & \xrightarrow{a} & X \end{array} \quad \begin{array}{ccc} G \times_S X & \xrightarrow{a} & X \\ e \times 1_X \uparrow & \nearrow 1_X & \\ X & & \end{array}$$

are commutative. In situation (2) this just means that the diagram

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\text{id} \times \psi} & G \times_S Y \\ a \downarrow & & \downarrow a \\ X & \xrightarrow{\psi} & Y \end{array}$$

commutes.

**Definition 10.2.** Let  $S, G \rightarrow S$ , and  $X \rightarrow S$  as in Definition 10.1. Let  $a : G \times_S X \rightarrow X$  be an action of  $G$  on  $X/S$ . We say the action is *free* if for every scheme  $T$  over  $S$  the action  $a : G(T) \times X(T) \rightarrow X(T)$  is a free action of the group  $G(T)$  on the set  $X(T)$ .

**Lemma 10.3.** *Situation as in Definition 10.2, The action  $a$  is free if and only if*

$$G \times_S X \rightarrow X \times_S X, \quad (g, x) \mapsto (a(g, x), x)$$

*is a monomorphism.*

**Proof.** Immediate from the definitions. □

## 11. Principal homogeneous spaces

In Cohomology on Sites, Definition 4.1 we have defined a torsor for a sheaf of groups on a site. Suppose  $\tau \in \{\text{Zariski}, \text{étale}, \text{smooth}, \text{syntomic}, \text{fppf}\}$  is a topology and  $(G, m)$  is a group scheme over  $S$ . Since  $\tau$  is stronger than the canonical topology (see Descent, Lemma 13.7) we see that  $\underline{G}$  (see Sites, Definition 12.3) is a sheaf of groups on  $(\text{Sch}/S)_\tau$ . Hence we already know what it means to have a torsor for  $\underline{G}$  on  $(\text{Sch}/S)_\tau$ . A special situation arises if this sheaf is representable. In the following definitions we define directly what it means for the representing scheme to be a  $G$ -torsor.

**Definition 11.1.** Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ . Let  $X$  be a scheme over  $S$ , and let  $a : G \times_S X \rightarrow X$  be an action of  $G$  on  $X$ .

- (1) We say  $X$  is a *pseudo  $G$ -torsor* or that  $X$  is *formally principally homogeneous under  $G$*  if the induced morphism of schemes  $G \times_S X \rightarrow X \times_S X$ ,  $(g, x) \mapsto (a(g, x), x)$  is an isomorphism of schemes over  $S$ .
- (2) A pseudo  $G$ -torsor  $X$  is called *trivial* if there exists an  $G$ -equivariant isomorphism  $G \rightarrow X$  over  $S$  where  $G$  acts on  $G$  by left multiplication.

It is clear that if  $S' \rightarrow S$  is a morphism of schemes then the pullback  $X_{S'}$  of a pseudo  $G$ -torsor over  $S$  is a pseudo  $G_{S'}$ -torsor over  $S'$ .

**Lemma 11.2.** *In the situation of Definition 11.1.*

- (1) *The scheme  $X$  is a pseudo  $G$ -torsor if and only if for every scheme  $T$  over  $S$  the set  $X(T)$  is either empty or the action of the group  $G(T)$  on  $X(T)$  is simply transitive.*
- (2) *A pseudo  $G$ -torsor  $X$  is trivial if and only if the morphism  $X \rightarrow S$  has a section.*

**Proof.** Omitted. □

**Definition 11.3.** Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ . Let  $X$  be a pseudo  $G$ -torsor over  $S$ .

- (1) We say  $X$  is a *principal homogeneous space* or a  *$G$ -torsor* if there exists a fpqc covering<sup>3</sup>  $\{S_i \rightarrow S\}_{i \in I}$  such that each  $X_{S_i} \rightarrow S_i$  has a section (i.e., is a trivial pseudo  $G_{S_i}$ -torsor).
- (2) Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{smooth}, \text{syntomic}, \text{fppf}\}$ . We say  $X$  is a  *$G$ -torsor in the  $\tau$  topology*, or a  *$\tau$   $G$ -torsor*, or simply a  *$\tau$  torsor* if there exists a  $\tau$  covering  $\{S_i \rightarrow S\}_{i \in I}$  such that each  $X_{S_i} \rightarrow S_i$  has a section.
- (3) If  $X$  is a  $G$ -torsor, then we say that it is *quasi-isotrivial* if it is a torsor for the étale topology.
- (4) If  $X$  is a  $G$ -torsor, then we say that it is *locally trivial* if it is a torsor for the Zariski topology.

We sometimes say “let  $X$  be a  $G$ -torsor over  $S$ ” to indicate that  $X$  is a scheme over  $S$  equipped with an action of  $G$  which turns it into a principal homogeneous space over  $S$ . Next we show that this agrees with the notation introduced earlier when both apply.

**Lemma 11.4.** *Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ . Let  $X$  be a scheme over  $S$ , and let  $a : G \times_S X \rightarrow X$  be an action of  $G$  on  $X$ . Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{smooth}, \text{syntomic}, \text{fppf}\}$ . Then  $X$  is a  $G$ -torsor in the  $\tau$ -topology if and only if  $\underline{X}$  is a  $\underline{G}$ -torsor on  $(\text{Sch}/S)_\tau$ .*

**Proof.** Omitted. □

**Remark 11.5.** Let  $(G, m)$  be a group scheme over the scheme  $S$ . In this situation we have the following natural types of questions:

- (1) If  $X \rightarrow S$  is a pseudo  $G$ -torsor and  $X \rightarrow S$  is surjective, then is  $X$  necessarily a  $G$ -torsor?
- (2) Is every  $\underline{G}$ -torsor on  $(\text{Sch}/S)_{\text{fppf}}$  representable? In other words, does every  $\underline{G}$ -torsor come from a fppf  $G$ -torsor?

---

<sup>3</sup>This means that the default type of torsor is a pseudo torsor which is trivial on an fpqc covering. This is the definition in [ABD<sup>+</sup>66, Exposé IV, 6.5]. It is a little bit inconvenient for us as we most often work in the fppf topology.

(3) Is every  $G$ -torsor an fppf (resp. smooth, resp. étale, resp. Zariski) torsor?

In general the answers to these questions is no. To get a positive answer we need to impose additional conditions on  $G \rightarrow S$ . For example: If  $S$  is the spectrum of a field, then the answer to (1) is yes because then  $\{X \rightarrow S\}$  is a fpqc covering trivializing  $X$ . If  $G \rightarrow S$  is affine, then the answer to (2) is yes (this follows from Descent, Lemma 37.1). If  $G = \mathrm{GL}_{n,S}$  then the answer to (3) is yes and in fact any  $\mathrm{GL}_{n,S}$ -torsor is locally trivial (this follows from Descent, Lemma 7.6).

## 12. Equivariant quasi-coherent sheaves

We think of “functions” as dual to “space”. Thus for a morphism of spaces the map on functions goes the other way. Moreover, we think of the sections of a sheaf of modules as “functions”. This leads us naturally to the direction of the arrows chosen in the following definition.

**Definition 12.1.** Let  $S$  be a scheme, let  $(G, m)$  be a group scheme over  $S$ , and let  $a : G \times_S X \rightarrow X$  be an action of the group scheme  $G$  on  $X/S$ . A  $G$ -equivariant quasi-coherent  $\mathcal{O}_X$ -module, or simply an equivariant quasi-coherent  $\mathcal{O}_X$ -module, is a pair  $(\mathcal{F}, \alpha)$ , where  $\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_X$ -module, and  $\alpha$  is a  $\mathcal{O}_{G \times_S X}$ -module map

$$\alpha : a^* \mathcal{F} \longrightarrow \mathrm{pr}_1^* \mathcal{F}$$

where  $\mathrm{pr}_1 : G \times_S X \rightarrow X$  is the projection such that

(1) the diagram

$$\begin{array}{ccc} (1_G \times a)^* \mathrm{pr}_1^* \mathcal{F} & \xrightarrow{\quad \mathrm{pr}_{12}^* \alpha \quad} & \mathrm{pr}_2^* \mathcal{F} \\ (1_G \times a)^* \alpha \uparrow & & \uparrow (m \times 1_X)^* \alpha \\ (1_G \times a)^* a^* \mathcal{F} & \xlongequal{\quad} & (m \times 1_X)^* a^* \mathcal{F} \end{array}$$

is a commutative in the category of  $\mathcal{O}_{G \times_S G \times_S X}$ -modules, and

(2) the pullback

$$(e \times 1_X)^* \alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

For explanation compare with the relevant diagrams of Equation (10.1.1).

Note that the commutativity of the first diagram guarantees that  $(e \times 1_X)^* \alpha$  is an idempotent operator on  $\mathcal{F}$ , and hence condition (2) is just the condition that it is an isomorphism.

**Lemma 12.2.** Let  $S$  be a scheme. Let  $G$  be a group scheme over  $S$ . Let  $f : Y \rightarrow X$  be a  $G$ -equivariant morphism between  $S$ -schemes endowed with  $G$ -actions. Then pullback  $f^*$  given by  $(\mathcal{F}, \alpha) \mapsto (f^* \mathcal{F}, (1_G \times f)^* \alpha)$  defines a functor from the category of  $G$ -equivariant quasi-coherent  $\mathcal{O}_X$ -modules to the category of  $G$ -equivariant quasi-coherent  $\mathcal{O}_Y$ -modules.

**Proof.** Omitted. □

Let us give an example.



**Example 12.3.** Let  $A$  be a  $\mathbf{Z}$ -graded ring, i.e.,  $A$  comes with a direct sum decomposition  $A = \bigoplus_{n \in \mathbf{Z}} A_n$  and  $A_n \cdot A_m \subset A_{n+m}$ . Set  $X = \text{Spec}(A)$ . Then we obtain a  $\mathbf{G}_m$ -action

$$a : \mathbf{G}_m \times X \longrightarrow X$$

by the ring map  $\mu : A \rightarrow A \otimes \mathbf{Z}[x, x^{-1}]$ ,  $f \mapsto f \otimes x^{\deg(f)}$ . Namely, to check this we have to verify that

$$\begin{array}{ccc} A & \xrightarrow{\mu} & A \otimes \mathbf{Z}[x, x^{-1}] \\ \mu \downarrow & & \downarrow \mu \otimes 1 \\ A \otimes \mathbf{Z}[x, x^{-1}] & \xrightarrow{1 \otimes m} & A \otimes \mathbf{Z}[x, x^{-1}] \otimes \mathbf{Z}[x, x^{-1}] \end{array}$$

where  $m(x) = x \otimes x$ , see Example 5.1. This is immediately clear when evaluating on a homogeneous element. Suppose that  $M$  is a graded  $A$ -module. Then we obtain a  $\mathbf{G}_m$ -equivariant quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{F} = \widetilde{M}$  by using  $\alpha$  as in Definition 12.1 corresponding to the  $A \otimes \mathbf{Z}[x, x^{-1}]$ -module map

$$M \otimes_{A, \mu} (A \otimes \mathbf{Z}[x, x^{-1}]) \longrightarrow M \otimes_{A, \text{id}_A \otimes 1} (A \otimes \mathbf{Z}[x, x^{-1}])$$

sending  $m \otimes 1 \otimes 1$  to  $m \otimes 1 \otimes x^{\deg(m)}$  for  $m \in M$  homogeneous.

**Lemma 12.4.** *Let  $a : \mathbf{G}_m \times X \rightarrow X$  be an action on an affine scheme. Then  $X$  is the spectrum of a  $\mathbf{Z}$ -graded ring and the action is as in Example 12.3.*

**Proof.** Let  $f \in A = \Gamma(X, \mathcal{O}_X)$ . Then we can write

$$a^\#(f) = \sum_{n \in \mathbf{Z}} f_n \otimes x^n \quad \text{in} \quad A \otimes \mathbf{Z}[x, x^{-1}] = \Gamma(\mathbf{G}_m \times X, \mathcal{O}_{\mathbf{G}_m \times X})$$

as a finite sum with  $f_n$  in  $A$  uniquely determined. Thus we obtain maps  $A \rightarrow A$ ,  $f \mapsto f_n$ . Since  $a$  is an action, if we evaluate at  $x = 1$ , we see  $f = \sum f_n$ . Since  $a$  is an action we find that

$$\sum (f_n)_m \otimes x^m \otimes x^n = \sum f_n x^n \otimes x^n$$

(compare with computation in Example 12.3). Thus  $(f_n)_m = 0$  if  $n \neq m$  and  $(f_n)_n = f_n$ . Thus if we set

$$A_n = \{f \in A \mid f_n = f\}$$

then we get  $A = \sum A_n$ . On the other hand, the sum has to be direct since  $f = 0$  implies  $f_n = 0$  in the situation above.  $\square$

**Lemma 12.5.** *Let  $A$  be a graded ring. Let  $X = \text{Spec}(A)$  with action  $a : \mathbf{G}_m \times X \rightarrow X$  as in Example 12.3. Let  $\mathcal{F}$  be a  $\mathbf{G}_m$ -equivariant quasi-coherent  $\mathcal{O}_X$ -module. Then  $M = \Gamma(X, \mathcal{F})$  has a canonical grading such that it is a graded  $A$ -module and such that the isomorphism  $\widetilde{M} \rightarrow \mathcal{F}$  (Schemes, Lemma 7.4) is an isomorphism of  $\mathbf{G}_m$ -equivariant modules where the  $\mathbf{G}_m$ -equivariant structure on  $\widetilde{M}$  is the one from Example 12.3.*

**Proof.** You can either prove this by repeating the arguments of Lemma 12.4 for the module  $M$ . Alternatively, you can consider the scheme  $(X', \mathcal{O}_{X'}) = (X, \mathcal{O}_X \oplus \mathcal{F})$  where  $\mathcal{F}$  is viewed as an ideal of square zero. There is a natural action  $a' : \mathbf{G}_m \times X' \rightarrow X'$  defined using the action on  $X$  and on  $\mathcal{F}$ . Then apply Lemma 12.4 to  $X'$  and conclude. (The nice thing about this argument is that it immediately shows that the grading on  $A$  and  $M$  are compatible, i.e., that  $M$  is a graded  $A$ -module.) Details omitted.  $\square$

### 13. Groupoids

Recall that a groupoid is a category in which every morphism is an isomorphism, see Categories, Definition 2.5. Hence a groupoid has a set of objects  $\text{Ob}$ , a set of arrows  $\text{Arrows}$ , a *source* and *target* map  $s, t : \text{Arrows} \rightarrow \text{Ob}$ , and a *composition law*  $c : \text{Arrows} \times_{s, \text{Ob}, t} \text{Arrows} \rightarrow \text{Arrows}$ . These maps satisfy exactly the following axioms

- (1) (associativity)  $c \circ (1, c) = c \circ (c, 1)$  as maps  $\text{Arrows} \times_{s, \text{Ob}, t} \text{Arrows} \times_{s, \text{Ob}, t} \text{Arrows} \rightarrow \text{Arrows}$ ,
- (2) (identity) there exists a map  $e : \text{Ob} \rightarrow \text{Arrows}$  such that
  - (a)  $s \circ e = t \circ e = \text{id}$  as maps  $\text{Ob} \rightarrow \text{Ob}$ ,
  - (b)  $c \circ (1, e \circ s) = c \circ (e \circ t, 1) = 1$  as maps  $\text{Arrows} \rightarrow \text{Arrows}$ ,
- (3) (inverse) there exists a map  $i : \text{Arrows} \rightarrow \text{Arrows}$  such that
  - (a)  $s \circ i = t, t \circ i = s$  as maps  $\text{Arrows} \rightarrow \text{Ob}$ , and
  - (b)  $c \circ (1, i) = e \circ t$  and  $c \circ (i, 1) = e \circ s$  as maps  $\text{Arrows} \rightarrow \text{Arrows}$ .

If this is the case the maps  $e$  and  $i$  are uniquely determined and  $i$  is a bijection. Note that if  $(\text{Ob}', \text{Arrows}', s', t', c')$  is a second groupoid category, then a functor  $f : (\text{Ob}, \text{Arrows}, s, t, c) \rightarrow (\text{Ob}', \text{Arrows}', s', t', c')$  is given by a pair of set maps  $f : \text{Ob} \rightarrow \text{Ob}'$  and  $f : \text{Arrows} \rightarrow \text{Arrows}'$  such that  $s' \circ f = f \circ s, t' \circ f = f \circ t$ , and  $c' \circ (f, f) = f \circ c$ . The compatibility with identity and inverse is automatic. We will use this below. (Warning: The compatibility with identity has to be imposed in the case of general categories.)

**Definition 13.1.** Let  $S$  be a scheme.

- (1) A *groupoid scheme over  $S$* , or simply a *groupoid over  $S$*  is a quintuple  $(U, R, s, t, c)$  where  $U$  and  $R$  are schemes over  $S$ , and  $s, t : R \rightarrow U$  and  $c : R \times_{s, U, t} R \rightarrow R$  are morphisms of schemes over  $S$  with the following property: For any scheme  $T$  over  $S$  the quintuple

$$(U(T), R(T), s, t, c)$$

is a groupoid category in the sense described above.

- (2) A *morphism  $f : (U, R, s, t, c) \rightarrow (U', R', s', t', c')$  of groupoid schemes over  $S$*  is given by morphisms of schemes  $f : U \rightarrow U'$  and  $f : R \rightarrow R'$  with the following property: For any scheme  $T$  over  $S$  the maps  $f$  define a functor from the groupoid category  $(U(T), R(T), s, t, c)$  to the groupoid category  $(U'(T), R'(T), s', t', c')$ .

Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . Note that, by the remarks preceding the definition and the Yoneda lemma, there are unique morphisms of schemes  $e : U \rightarrow R$  and  $i : R \rightarrow R$  over  $S$  such that for every scheme  $T$  over  $S$  the induced map  $e : U(T) \rightarrow R(T)$  is the identity, and  $i : R(T) \rightarrow R(T)$  is the inverse of the groupoid category. The septuple  $(U, R, s, t, c, e, i)$  satisfies commutative diagrams corresponding to each of the axioms (1), (2)(a), (2)(b), (3)(a) and (3)(b) above, and conversely given a septuple with this property the quintuple  $(U, R, s, t, c)$  is a groupoid scheme. Note that  $i$  is an isomorphism, and  $e$  is a section of both  $s$  and  $t$ . Moreover, given a groupoid scheme over  $S$  we denote

$$j = (t, s) : R \longrightarrow U \times_S U$$

which is compatible with our conventions in Section 3 above. We sometimes say “let  $(U, R, s, t, c, e, i)$  be a groupoid over  $S$ ” to stress the existence of identity and inverse.

**Lemma 13.2.** *Given a groupoid scheme  $(U, R, s, t, c)$  over  $S$  the morphism  $j : R \rightarrow U \times_S U$  is a pre-equivalence relation.*

**Proof.** Omitted. This is a nice exercise in the definitions.  $\square$

**Lemma 13.3.** *Given an equivalence relation  $j : R \rightarrow U \times_S U$  over  $S$  there is a unique way to extend it to a groupoid  $(U, R, s, t, c)$  over  $S$ .*

**Proof.** Omitted. This is a nice exercise in the definitions.  $\square$

**Lemma 13.4.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . In the commutative diagram*

$$\begin{array}{ccccc}
 & & U & & \\
 & \swarrow t & & \nwarrow t & \\
 R & \xleftarrow{pr_0} & R \times_{s,U,t} R & \xrightarrow{c} & R \\
 \downarrow s & & \downarrow pr_1 & & \downarrow s \\
 U & \xleftarrow{t} & R & \xrightarrow{s} & U
 \end{array}$$

*the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.*

**Proof.** Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry.  $\square$

**Lemma 13.5.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c, e, i)$  be a groupoid over  $S$ . The diagram*

$$\begin{array}{ccccc}
 (13.5.1) & R \times_{t,U,t} R & \xrightarrow{pr_1} & R & \xrightarrow{t} & U \\
 & \downarrow (pr_0, c \circ (i, 1)) & & \downarrow id_R & & \downarrow id_U \\
 & R \times_{s,U,t} R & \xrightarrow{c} & R & \xrightarrow{t} & U \\
 & \downarrow pr_1 & & \downarrow s & & \\
 & R & \xrightarrow{s} & U & & \\
 & & & \downarrow t & & 
 \end{array}$$

*is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.*

**Proof.** The commutativity of the diagram follows from the axioms of a groupoid. Note that, in terms of groupoids, the top left vertical arrow assigns to a pair of morphisms  $(\alpha, \beta)$  with the same target, the pair of morphisms  $(\alpha, \alpha^{-1} \circ \beta)$ . In any groupoid this defines a bijection between  $\text{Arrows} \times_{t, \text{Ob}, t} \text{Arrows}$  and  $\text{Arrows} \times_{s, \text{Ob}, t} \text{Arrows}$ . Hence the second assertion of the lemma. The last assertion follows from Lemma 13.4.  $\square$

**Lemma 13.6.** *Let  $(U, R, s, t, c)$  be a groupoid over a scheme  $S$ . Let  $S' \rightarrow S$  be a morphism. Then the base changes  $U' = S' \times_S U$ ,  $R' = S' \times_S R$  endowed with the base changes  $s', t', c'$  of the morphisms  $s, t, c$  form a groupoid scheme  $(U', R', s', t', c')$  over  $S'$  and the projections determine a morphism  $(U', R', s', t', c') \rightarrow (U, R, s, t, c)$  of groupoid schemes over  $S$ .*

**Proof.** Omitted. Hint:  $R' \times_{s', U', t'} R' = S' \times_S (R \times_{s, U, t} R)$ .  $\square$

#### 14. Quasi-coherent sheaves on groupoids

See the introduction of Section 12 for our choices in direction of arrows.

**Definition 14.1.** Let  $S$  be a scheme, let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . A *quasi-coherent module* on  $(U, R, s, t, c)$  is a pair  $(\mathcal{F}, \alpha)$ , where  $\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_U$ -module, and  $\alpha$  is a  $\mathcal{O}_R$ -module map

$$\alpha : t^* \mathcal{F} \longrightarrow s^* \mathcal{F}$$

such that

- (1) the diagram

$$\begin{array}{ccc} & \text{pr}_1^* t^* \mathcal{F} \xrightarrow{\text{pr}_1^* \alpha} \text{pr}_1^* s^* \mathcal{F} & \\ & \parallel & \\ \text{pr}_0^* s^* \mathcal{F} & & c^* s^* \mathcal{F} \\ & \nwarrow \text{pr}_0^* \alpha \quad \nearrow c^* \alpha & \\ & \text{pr}_0^* t^* \mathcal{F} \xlongequal{\quad} c^* t^* \mathcal{F} & \end{array}$$

is a commutative in the category of  $\mathcal{O}_{R \times_{s, U, t} R}$ -modules, and

- (2) the pullback

$$e^* \alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

Compare with the commutative diagrams of Lemma 13.4.

The commutativity of the first diagram forces the operator  $e^* \alpha$  to be idempotent. Hence the second condition can be reformulated as saying that  $e^* \alpha$  is an isomorphism. In fact, the condition implies that  $\alpha$  is an isomorphism.

**Lemma 14.2.** *Let  $S$  be a scheme, let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . If  $(\mathcal{F}, \alpha)$  is a quasi-coherent module on  $(U, R, s, t, c)$  then  $\alpha$  is an isomorphism.*

**Proof.** Pull back the commutative diagram of Definition 14.1 by the morphism  $(i, 1) : R \rightarrow R \times_{s, U, t} R$ . Then we see that  $i^* \alpha \circ \alpha = s^* e^* \alpha$ . Pulling back by the morphism  $(1, i)$  we obtain the relation  $\alpha \circ i^* \alpha = t^* e^* \alpha$ . By the second assumption these morphisms are the identity. Hence  $i^* \alpha$  is an inverse of  $\alpha$ .  $\square$

**Lemma 14.3.** *Let  $S$  be a scheme. Consider a morphism  $f : (U, R, s, t, c) \rightarrow (U', R', s', t', c')$  of groupoid schemes over  $S$ . Then pullback  $f^*$  given by*

$$(\mathcal{F}, \alpha) \mapsto (f^* \mathcal{F}, f^* \alpha)$$

*defines a functor from the category of quasi-coherent sheaves on  $(U', R', s', t', c')$  to the category of quasi-coherent sheaves on  $(U, R, s, t, c)$ .*

**Proof.** Omitted.  $\square$

**Lemma 14.4.** *Let  $S$  be a scheme. Consider a morphism  $f : (U, R, s, t, c) \rightarrow (U', R', s', t', c')$  of groupoid schemes over  $S$ . Assume that*

- (1)  $f : U \rightarrow U'$  is quasi-compact and quasi-separated,
- (2) the square

$$\begin{array}{ccc} R & \xrightarrow{\quad} & R' \\ t \downarrow & & \downarrow t' \\ U & \xrightarrow{\quad} & U' \end{array}$$

*is cartesian, and*

- (3)  $s'$  and  $t'$  are flat.

Then pushforward  $f_*$  given by

$$(\mathcal{F}, \alpha) \mapsto (f_*\mathcal{F}, f_*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on  $(U, R, s, t, c)$  to the category of quasi-coherent sheaves on  $(U', R', s', t', c')$  which is right adjoint to pullback as defined in Lemma 14.3.

**Proof.** Since  $U \rightarrow U'$  is quasi-compact and quasi-separated we see that  $f_*$  transforms quasi-coherent sheaves into quasi-coherent sheaves (Schemes, Lemma 24.1). Moreover, since the squares

$$\begin{array}{ccc} R & \xrightarrow{\quad} & R' \\ t \downarrow & & \downarrow t' \\ U & \xrightarrow{\quad} & U' \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \xrightarrow{\quad} & R' \\ s \downarrow & & \downarrow s' \\ U & \xrightarrow{\quad} & U' \end{array}$$

are cartesian we find that  $(t')^*f_*\mathcal{F} = f_*t^*\mathcal{F}$  and  $(s')^*f_*\mathcal{F} = f_*s^*\mathcal{F}$ , see Cohomology of Schemes, Lemma 5.2. Thus it makes sense to think of  $f_*\alpha$  as a map  $(t')^*f_*\mathcal{F} \rightarrow (s')^*f_*\mathcal{F}$ . A similar argument shows that  $f_*\alpha$  satisfies the cocycle condition. The functor is adjoint to the pullback functor since pullback and pushforward on modules on ringed spaces are adjoint. Some details omitted.  $\square$

**Lemma 14.5.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . The category of quasi-coherent modules on  $(U, R, s, t, c)$  has colimits.*

**Proof.** Let  $i \mapsto (\mathcal{F}_i, \alpha_i)$  be a diagram over the index category  $\mathcal{I}$ . We can form the colimit  $\mathcal{F} = \text{colim } \mathcal{F}_i$  which is a quasi-coherent sheaf on  $U$ , see Schemes, Section 24. Since colimits commute with pullback we see that  $s^*\mathcal{F} = \text{colim } s^*\mathcal{F}_i$  and similarly  $t^*\mathcal{F} = \text{colim } t^*\mathcal{F}_i$ . Hence we can set  $\alpha = \text{colim } \alpha_i$ . We omit the proof that  $(\mathcal{F}, \alpha)$  is the colimit of the diagram in the category of quasi-coherent modules on  $(U, R, s, t, c)$ .  $\square$

**Lemma 14.6.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . If  $s, t$  are flat, then the category of quasi-coherent modules on  $(U, R, s, t, c)$  is abelian.*

**Proof.** Let  $\varphi : (\mathcal{F}, \alpha) \rightarrow (\mathcal{G}, \beta)$  be a homomorphism of quasi-coherent modules on  $(U, R, s, t, c)$ . Since  $s$  is flat we see that

$$0 \rightarrow s^* \text{Ker}(\varphi) \rightarrow s^*\mathcal{F} \rightarrow s^*\mathcal{G} \rightarrow s^* \text{Coker}(\varphi) \rightarrow 0$$

is exact and similarly for pullback by  $t$ . Hence  $\alpha$  and  $\beta$  induce isomorphisms  $\kappa : t^* \text{Ker}(\varphi) \rightarrow s^* \text{Ker}(\varphi)$  and  $\lambda : t^* \text{Coker}(\varphi) \rightarrow s^* \text{Coker}(\varphi)$  which satisfy the cocycle

condition. Then it is straightforward to verify that  $(\text{Ker}(\varphi), \kappa)$  and  $(\text{Coker}(\varphi), \lambda)$  are a kernel and cokernel in the category of quasi-coherent modules on  $(U, R, s, t, c)$ . Moreover, the condition  $\text{Coim}(\varphi) = \text{Im}(\varphi)$  follows because it holds over  $U$ .  $\square$

### 15. Colimits of quasi-coherent modules

In this section we prove some technical results saying that under suitable assumptions every quasi-coherent module on a groupoid is a filtered colimit of “small” quasi-coherent modules.

**Lemma 15.1.** *Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $s, t$  are flat, quasi-compact, and quasi-separated. For any quasi-coherent module  $\mathcal{G}$  on  $U$ , there exists a canonical isomorphism  $\alpha : t^* s_* t^* \mathcal{G} \rightarrow s^* s_* t^* \mathcal{G}$  which turns  $(s_* t^* \mathcal{G}, \alpha)$  into a quasi-coherent module on  $(U, R, s, t, c)$ . This construction defines a functor*

$$QCoh(\mathcal{O}_U) \longrightarrow QCoh(U, R, s, t, c)$$

which is a right adjoint to the forgetful functor  $(\mathcal{F}, \beta) \mapsto \mathcal{F}$ .

**Proof.** The pushforward of a quasi-coherent module along a quasi-compact and quasi-separated morphism is quasi-coherent, see Schemes, Lemma 24.1. Hence  $s_* t^* \mathcal{G}$  is quasi-coherent. With notation as in Lemma 13.4 we have

$$t^* s_* t^* \mathcal{G} = \text{pr}_{1,*} \text{pr}_0^* t^* \mathcal{G} = \text{pr}_{1,*} c^* t^* \mathcal{G} = s^* s_* t^* \mathcal{G}$$

The middle equality because  $t \circ c = t \circ \text{pr}_0$  as morphisms  $R \times_{s,U,t} R \rightarrow U$ , and the first and the last equality because we know that base change and pushforward commute in these steps by Cohomology of Schemes, Lemma 5.2.

To verify the cocycle condition of Definition 14.1 for  $\alpha$  and the adjointness property we describe the construction  $\mathcal{G} \mapsto (s_* t^* \mathcal{G}, \alpha)$  in another way. Consider the groupoid scheme  $(R, R \times_{t,U,t} R, \text{pr}_0, \text{pr}_1, \text{pr}_{02})$  associated to the equivalence relation  $R \times_{t,U,t} R$  on  $R$ , see Lemma 13.3. There is a morphism

$$f : (R, R \times_{t,U,t} R, \text{pr}_1, \text{pr}_0, \text{pr}_{02}) \longrightarrow (U, R, s, t, c)$$

of groupoid schemes given by  $s : R \rightarrow U$  and  $R \times_{t,U,t} R \rightarrow R$  given by  $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$ ; we omit the verification of the commutativity of the required diagrams. Since  $t, s : R \rightarrow U$  are quasi-compact, quasi-separated, and flat, and since we have a cartesian square

$$\begin{array}{ccc} R \times_{t,U,t} R & \xrightarrow{(r_0, r_1) \mapsto r_0^{-1} \circ r_1} & R \\ \text{pr}_0 \downarrow & & \downarrow t \\ R & \xrightarrow{s} & U \end{array}$$

by Lemma 13.5 it follows that Lemma 14.4 applies to  $f$ . Thus pushforward and pullback of quasi-coherent modules along  $f$  are adjoint functors. To finish the proof we will identify these functors with the functors described above. To do this, note that

$$t^* : QCoh(\mathcal{O}_U) \longrightarrow QCoh(R, R \times_{t,U,t} R, \text{pr}_1, \text{pr}_0, \text{pr}_{02})$$

is an equivalence by the theory of descent of quasi-coherent sheaves as  $\{t : R \rightarrow U\}$  is an fpqc covering, see Descent, Proposition 5.2.

Pushforward along  $f$  precomposed with the equivalence  $t^*$  sends  $\mathcal{G}$  to  $(s_* t^* \mathcal{G}, \alpha)$ ; we omit the verification that the isomorphism  $\alpha$  obtained in this fashion is the same as the one constructed above.

Pullback along  $f$  postcomposed with the inverse of the equivalence  $t^*$  sends  $(\mathcal{F}, \beta)$  to the descent relative to  $\{t : R \rightarrow U\}$  of the module  $s^*\mathcal{F}$  endowed with the descent datum  $\gamma$  on  $R \times_{t,U,t} R$  which is the pullback of  $\beta$  by  $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$ . Consider the isomorphism  $\beta : t^*\mathcal{F} \rightarrow s^*\mathcal{F}$ . The canonical descent datum (Descent, Definition 2.3) on  $t^*\mathcal{F}$  relative to  $\{t : R \rightarrow U\}$  translates via  $\beta$  into the map

$$\mathrm{pr}_0^* s^* \mathcal{F} \xrightarrow{\mathrm{pr}_0^* \beta^{-1}} \mathrm{pr}_0^* t^* \mathcal{F} \xrightarrow{\mathrm{can}} \mathrm{pr}_1^* t^* \mathcal{F} \xrightarrow{\mathrm{pr}_1^* \beta} \mathrm{pr}_1^* s^* \mathcal{F}$$

Since  $\beta$  satisfies the cocycle condition, this is equal to the pullback of  $\beta$  by  $(r_0, r_1) \mapsto r_0^{-1} \circ r_1$ . To see this take the actual cocycle relation in Definition 14.1 and pull it back by the morphism  $(\mathrm{pr}_0, c \circ (i, 1)) : R \times_{t,U,t} R \rightarrow R \times_{s,U,t} R$  which also plays a role in the commutative diagram of Lemma 13.5. It follows that  $(s^*\mathcal{F}, \gamma)$  is isomorphic to  $(t^*\mathcal{F}, \mathrm{can})$ . All in all, we conclude that pullback by  $f$  postcomposed with the inverse of the equivalence  $t^*$  is isomorphic to the forgetful functor  $(\mathcal{F}, \beta) \mapsto \mathcal{F}$ .  $\square$

**Remark 15.2.** In the situation of Lemma 15.1 denote

$$F : QCoh(U, R, s, t, c) \rightarrow QCoh(\mathcal{O}_U), \quad (\mathcal{F}, \beta) \mapsto \mathcal{F}$$

the forgetful functor and denote

$$G : QCoh(\mathcal{O}_U) \rightarrow QCoh(U, R, s, t, c), \quad \mathcal{G} \mapsto (s_* t^* \mathcal{G}, \alpha)$$

the right adjoint constructed in the lemma. Then the unit  $\eta : \mathrm{id} \rightarrow G \circ F$  of the adjunction evaluated on  $(\mathcal{F}, \beta)$  is given by the map

$$\mathcal{F} \rightarrow s_* s^* \mathcal{F} \xrightarrow{\beta^{-1}} s_* t^* \mathcal{F}$$

We omit the verification.

**Lemma 15.3.** *Let  $f : Y \rightarrow X$  be a morphism of schemes. Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module, let  $\mathcal{G}$  be a quasi-coherent  $\mathcal{O}_Y$ -module, and let  $\varphi : \mathcal{G} \rightarrow f^* \mathcal{F}$  be a module map. Assume*

- (1)  $\varphi$  is injective,
- (2)  $f$  is quasi-compact, quasi-separated, flat, and surjective,
- (3)  $X, Y$  are locally Noetherian, and
- (4)  $\mathcal{G}$  is a coherent  $\mathcal{O}_Y$ -module.

*Then  $\mathcal{F} \cap f_* \mathcal{G}$  defined as the pullback*

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & f_* f^* \mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F} \cap f_* \mathcal{G} & \longrightarrow & f_* \mathcal{G} \end{array}$$

*is a coherent  $\mathcal{O}_X$ -module.*

**Proof.** We will freely use the characterization of coherent modules of Cohomology of Schemes, Lemma 9.1 as well as the fact that coherent modules form a Serre subcategory of  $QCoh(\mathcal{O}_X)$ , see Cohomology of Schemes, Lemma 9.3. If  $f$  has a section  $\sigma$ , then we see that  $\mathcal{F} \cap f_* \mathcal{G}$  is contained in the image of  $\sigma^* \mathcal{G} \rightarrow \sigma^* f^* \mathcal{F} = \mathcal{F}$ , hence coherent. In general, to show that  $\mathcal{F} \cap f_* \mathcal{G}$  is coherent, it suffices to show that  $f^*(\mathcal{F} \cap f_* \mathcal{G})$  is coherent (see Descent, Lemma 7.1). Since  $f$  is flat this is equal to  $f^* \mathcal{F} \cap f^* f_* \mathcal{G}$ . Since  $f$  is flat, quasi-compact, and quasi-separated we see  $f^* f_* \mathcal{G} = p_* q^* \mathcal{G}$  where  $p, q : Y \times_X Y \rightarrow Y$  are the projections, see Cohomology of Schemes, Lemma 5.2. Since  $p$  has a section we win.  $\square$

Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid in schemes over  $S$ . Assume that  $U$  is locally Noetherian. In the lemma below we say that a quasi-coherent sheaf  $(\mathcal{F}, \alpha)$  on  $(U, R, s, t, c)$  is *coherent* if  $\mathcal{F}$  is a coherent  $\mathcal{O}_U$ -module.

**Lemma 15.4.** *Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume that*

- (1)  $U, R$  are Noetherian,
- (2)  $s, t$  are flat, quasi-compact, and quasi-separated.

*Then every quasi-coherent module  $(\mathcal{F}, \beta)$  on  $(U, R, s, t, c)$  is a filtered colimit of coherent modules.*

**Proof.** We will use the characterization of Cohomology of Schemes, Lemma 9.1 of coherent modules on locally Noetherian scheme without further mention. We can write  $\mathcal{F} = \text{colim } \mathcal{H}_i$  as the filtered colimit of coherent submodules  $\mathcal{H}_i \subset \mathcal{F}$ , see Cohomology of Schemes, Lemma 10.4. Given a quasi-coherent sheaf  $\mathcal{H}$  on  $U$  we denote  $(s_* t^* \mathcal{H}, \alpha)$  the quasi-coherent sheaf on  $(U, R, s, t, c)$  of Lemma 15.1. Consider the adjunction map  $(\mathcal{F}, \beta) \rightarrow (s_* t^* \mathcal{F}, \alpha)$  in  $QCoh(U, R, s, t, c)$ , see Remark 15.2. Set

$$(\mathcal{F}_i, \beta_i) = (\mathcal{F}, \beta) \times_{(s_* t^* \mathcal{F}, \alpha)} (s_* t^* \mathcal{H}_i, \alpha)$$

in  $QCoh(U, R, s, t, c)$ . Since restriction to  $U$  is an exact functor on  $QCoh(U, R, s, t, c)$  by the proof of Lemma 14.6 we obtain a pullback diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & s_* t^* \mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F}_i & \longrightarrow & s_* t^* \mathcal{H}_i \end{array}$$

in other words  $\mathcal{F}_i = \mathcal{F} \cap s_* t^* \mathcal{H}_i$ . By the description of the adjunction map in Remark 15.2 this diagram is isomorphic to the diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & s_* s^* \mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F}_i & \longrightarrow & s_* t^* \mathcal{H}_i \end{array}$$

where the right vertical arrow is the result of applying  $s_*$  to the map

$$t^* \mathcal{H}_i \rightarrow t^* \mathcal{F} \xrightarrow{\beta} s^* \mathcal{F}$$

This arrow is injective as  $t$  is a flat morphism. It follows that  $\mathcal{F}_i$  is coherent by Lemma 15.3. Finally, because  $s$  is quasi-compact and quasi-separated we see that  $s_*$  commutes with colimits (see Cohomology of Schemes, Lemma 6.1). Hence  $s_* t^* \mathcal{F} = \text{colim } s_* t^* \mathcal{H}_i$  and hence  $(\mathcal{F}, \beta) = \text{colim}(\mathcal{F}_i, \beta_i)$  as desired.  $\square$

Here is a curious lemma that is useful when working with groupoids on fields. In fact, this is the standard argument to prove that any representation of an algebraic group is a colimit of finite dimensional representations.

**Lemma 15.5.** *Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume that*

- (1)  $U, R$  are affine,
- (2) *there exist  $e_i \in \mathcal{O}_R(R)$  such that every element  $g \in \mathcal{O}_R(R)$  can be uniquely written as  $\sum s^*(f_i) e_i$  for some  $f_i \in \mathcal{O}_U(U)$ .*

*Then every quasi-coherent module  $(\mathcal{F}, \alpha)$  on  $(U, R, s, t, c)$  is a filtered colimit of finite type quasi-coherent modules.*



**Proof.** The assumption means that  $\mathcal{O}_R(R)$  is a free  $\mathcal{O}_U(U)$ -module via  $s$  with basis  $e_i$ . Hence for any quasi-coherent  $\mathcal{O}_U$ -module  $\mathcal{G}$  we see that  $s^*\mathcal{G}(R) = \bigoplus_i \mathcal{G}(U)e_i$ . We will write  $s(-)$  to indicate pullback of sections by  $s$  and similarly for other morphisms. Let  $(\mathcal{F}, \alpha)$  be a quasi-coherent module on  $(U, R, s, t, c)$ . Let  $\sigma \in \mathcal{F}(U)$ . By the above we can write

$$\alpha(t(\sigma)) = \sum s(\sigma_i)e_i$$

for some unique  $\sigma_i \in \mathcal{F}(U)$  (all but finitely many are zero of course). We can also write

$$c(e_i) = \sum \text{pr}_1(f_{ij})\text{pr}_0(e_j)$$

as functions on  $R \times_{s,U,t} R$ . Then the commutativity of the diagram in Definition 14.1 means that

$$\sum \text{pr}_1(\alpha(t(\sigma_i)))\text{pr}_0(e_i) = \sum \text{pr}_1(s(\sigma_i)f_{ij})\text{pr}_0(e_j)$$

(calculation omitted). Picking off the coefficients of  $\text{pr}_0(e_l)$  we see that  $\alpha(t(\sigma_l)) = \sum s(\sigma_i)f_{il}$ . Hence the submodule  $\mathcal{G} \subset \mathcal{F}$  generated by the elements  $\sigma_i$  defines a finite type quasi-coherent module preserved by  $\alpha$ . Hence it is a subobject of  $\mathcal{F}$  in  $QCoh(U, R, s, t, c)$ . This submodule contains  $\sigma$  (as one sees by pulling back the first relation by  $e$ ). Hence we win.  $\square$

We suggest the reader skip the rest of this section. Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid in schemes over  $S$ . Let  $\kappa$  be a cardinal. In the following we will say that a quasi-coherent sheaf  $(\mathcal{F}, \alpha)$  on  $(U, R, s, t, c)$  is  $\kappa$ -generated if  $\mathcal{F}$  is a  $\kappa$ -generated  $\mathcal{O}_U$ -module, see Properties, Definition 23.1.

**Lemma 15.6.** *Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $\kappa$  be a cardinal. There exists a set  $T$  and a family  $(\mathcal{F}_t, \alpha_t)_{t \in T}$  of  $\kappa$ -generated quasi-coherent modules on  $(U, R, s, t, c)$  such that every  $\kappa$ -generated quasi-coherent module on  $(U, R, s, t, c)$  is isomorphic to one of the  $(\mathcal{F}_t, \alpha_t)$ .*

**Proof.** For each quasi-coherent module  $\mathcal{F}$  on  $U$  there is a (possibly empty) set of maps  $\alpha : t^*\mathcal{F} \rightarrow s^*\mathcal{F}$  such that  $(\mathcal{F}, \alpha)$  is a quasi-coherent modules on  $(U, R, s, t, c)$ . By Properties, Lemma 23.2 there exists a set of isomorphism classes of  $\kappa$ -generated quasi-coherent  $\mathcal{O}_U$ -modules.  $\square$

**Lemma 15.7.** *Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume that  $s, t$  are flat. There exists a cardinal  $\kappa$  such that every quasi-coherent module  $(\mathcal{F}, \alpha)$  on  $(U, R, s, t, c)$  is the directed colimit of its  $\kappa$ -generated quasi-coherent submodules.*

**Proof.** In the statement of the lemma and in this proof a *submodule* of a quasi-coherent module  $(\mathcal{F}, \alpha)$  is a quasi-coherent submodule  $\mathcal{G} \subset \mathcal{F}$  such that  $\alpha(t^*\mathcal{G}) = s^*\mathcal{G}$  as subsheaves of  $s^*\mathcal{F}$ . This makes sense because since  $s, t$  are flat the pullbacks  $s^*$  and  $t^*$  are exact, i.e., preserve subsheaves. The proof will be a repeat of the proof of Properties, Lemma 23.3. We urge the reader to read that proof first.

Choose an affine open covering  $U = \bigcup_{i \in I} U_i$ . For each pair  $i, j$  choose affine open coverings

$$U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk} \quad \text{and} \quad s^{-1}(U_i) \cap t^{-1}(U_j) = \bigcup_{k \in J_{ij}} W_{ijk}.$$

Write  $U_i = \text{Spec}(A_i)$ ,  $U_{ijk} = \text{Spec}(A_{ijk})$ ,  $W_{ijk} = \text{Spec}(B_{ijk})$ . Let  $\kappa$  be any infinite cardinal  $\geq$  than the cardinality of any of the sets  $I, I_{ij}, J_{ij}$ .

Let  $(\mathcal{F}, \alpha)$  be a quasi-coherent module on  $(U, R, s, t, c)$ . Set  $M_i = \mathcal{F}(U_i)$ ,  $M_{ijk} = \mathcal{F}(U_{ijk})$ . Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}$$

and that  $\alpha$  gives isomorphisms

$$\alpha|_{W_{ijk}} : M_i \otimes_{A_i, t} B_{ijk} \longrightarrow M_j \otimes_{A_j, s} B_{ijk}$$

see Schemes, Lemma 7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every  $i, j \in I$ ,  $k \in I_{ij}$  or  $k \in J_{ij}$  and  $m \in M_i$  a finite subset  $S(i, j, k, m) \subset M_j$  such that we have

$$m \otimes 1 = \sum_{m' \in S(i, j, k, m)} m' \otimes a_{m'} \quad \text{or} \quad \alpha(m \otimes 1) = \sum_{m' \in S(i, j, k, m)} m' \otimes b_{m'}$$

in  $M_{ijk}$  for some  $a_{m'} \in A_{ijk}$  or  $b_{m'} \in B_{ijk}$ . Moreover, let's agree that  $S(i, i, k, m) = \{m\}$  for all  $i, j = i, k, m$  when  $k \in I_{ij}$ . Fix such a collection  $S(i, j, k, m)$

Given a family  $\mathcal{S} = (S_i)_{i \in I}$  of subsets  $S_i \subset M_i$  of cardinality at most  $\kappa$  we set  $\mathcal{S}' = (S'_i)$  where

$$S'_j = \bigcup_{(i, j, k, m) \text{ such that } m \in S_i} S(i, j, k, m)$$

Note that  $S_i \subset S'_i$ . Note that  $S'_i$  has cardinality at most  $\kappa$  because it is a union over a set of cardinality at most  $\kappa$  of finite sets. Set  $\mathcal{S}^{(0)} = \mathcal{S}$ ,  $\mathcal{S}^{(1)} = \mathcal{S}'$  and by induction  $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$ . Then set  $\mathcal{S}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{S}^{(n)}$ . Writing  $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$  we see that for any element  $m \in S_i^{(\infty)}$  the image of  $m$  in  $M_{ijk}$  can be written as a finite sum  $\sum m' \otimes a_{m'}$  with  $m' \in S_j^{(\infty)}$ . In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk} \quad \text{and} \quad \alpha(N_i \otimes_{A_i, t} B_{ijk}) = N_j \otimes_{A_j, s} B_{ijk}$$

as submodules of  $M_{ijk}$  or  $M_j \otimes_{A_j, s} B_{ijk}$ . Thus there exists a quasi-coherent submodule  $\mathcal{G} \subset \mathcal{F}$  with  $\mathcal{G}(U_i) = N_i$  such that  $\alpha(t^*\mathcal{G}) = s^*\mathcal{G}$  as submodules of  $s^*\mathcal{F}$ . In other words,  $(\mathcal{G}, \alpha|_{t^*\mathcal{G}})$  is a submodule of  $(\mathcal{F}, \alpha)$ . Moreover, by construction  $\mathcal{G}$  is  $\kappa$ -generated.

Let  $\{(\mathcal{G}_t, \alpha_t)\}_{t \in T}$  be the set of  $\kappa$ -generated quasi-coherent submodules of  $(\mathcal{F}, \alpha)$ . If  $t, t' \in T$  then  $\mathcal{G}_t + \mathcal{G}_{t'}$  is also a  $\kappa$ -generated quasi-coherent submodule as it is the image of the map  $\mathcal{G}_t \oplus \mathcal{G}_{t'} \rightarrow \mathcal{F}$ . Hence the system (ordered by inclusion) is directed. The arguments above show that every section of  $\mathcal{F}$  over  $U_i$  is in one of the  $\mathcal{G}_t$  (because we can start with  $\mathcal{S}$  such that the given section is an element of  $S_i$ ). Hence  $\text{colim}_t \mathcal{G}_t \rightarrow \mathcal{F}$  is both injective and surjective as desired.  $\square$

## 16. Groupoids and group schemes

There are many ways to construct a groupoid out of an action  $a$  of a group  $G$  on a set  $V$ . We choose the one where we think of an element  $g \in G$  as an arrow with source  $v$  and target  $a(g, v)$ . This leads to the following construction for group actions of schemes.

**Lemma 16.1.** *Let  $S$  be a scheme. Let  $Y$  be a scheme over  $S$ . Let  $(G, m)$  be a group scheme over  $Y$  with identity  $e_G$  and inverse  $i_G$ . Let  $X/Y$  be a scheme over  $Y$  and let  $a : G \times_Y X \rightarrow X$  be an action of  $G$  on  $X/Y$ . Then we get a groupoid scheme  $(U, R, s, t, c, e, i)$  over  $S$  in the following manner:*

- (1) *We set  $U = X$ , and  $R = G \times_Y X$ .*
- (2) *We set  $s : R \rightarrow U$  equal to  $(g, x) \mapsto x$ .*
- (3) *We set  $t : R \rightarrow U$  equal to  $(g, x) \mapsto a(g, x)$ .*
- (4) *We set  $c : R \times_{s, U, t} R \rightarrow R$  equal to  $((g, x), (g', x')) \mapsto (m(g, g'), x')$ .*
- (5) *We set  $e : U \rightarrow R$  equal to  $x \mapsto (e_G(x), x)$ .*
- (6) *We set  $i : R \rightarrow R$  equal to  $(g, x) \mapsto (i_G(g), a(g, x))$ .*

**Proof.** Omitted. Hint: It is enough to show that this works on the set level. For this use the description above the lemma describing  $g$  as an arrow from  $v$  to  $a(g, v)$ .  $\square$

**Lemma 16.2.** *Let  $S$  be a scheme. Let  $Y$  be a scheme over  $S$ . Let  $(G, m)$  be a group scheme over  $Y$ . Let  $X$  be a scheme over  $Y$  and let  $a : G \times_Y X \rightarrow X$  be an action of  $G$  on  $X$  over  $Y$ . Let  $(U, R, s, t, c)$  be the groupoid scheme constructed in Lemma 16.1. The rule  $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \alpha)$  defines an equivalence of categories between  $G$ -equivariant  $\mathcal{O}_X$ -modules and the category of quasi-coherent modules on  $(U, R, s, t, c)$ .*

**Proof.** The assertion makes sense because  $t = a$  and  $s = \text{pr}_1$  as morphisms  $R = G \times_Y X \rightarrow X$ , see Definitions 12.1 and 14.1. Using the translation in Lemma 16.1 the commutativity requirements of the two definitions match up exactly.  $\square$

## 17. The stabilizer group scheme

Given a groupoid scheme we get a group scheme as follows.

**Lemma 17.1.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . The scheme  $G$  defined by the cartesian square*

$$\begin{array}{ccc} G & \longrightarrow & R \\ \downarrow & & \downarrow j=(t,s) \\ U & \xrightarrow{\Delta} & U \times_S U \end{array}$$

*is a group scheme over  $U$  with composition law  $m$  induced by the composition law  $c$ .*

**Proof.** This is true because in a groupoid category the set of self maps of any object forms a group.  $\square$

Since  $\Delta$  is an immersion we see that  $G = j^{-1}(\Delta_{U/S})$  is a locally closed subscheme of  $R$ . Thinking of it in this way, the structure morphism  $j^{-1}(\Delta_{U/S}) \rightarrow U$  is induced by either  $s$  or  $t$  (it is the same), and  $m$  is induced by  $c$ .

**Definition 17.2.** Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . The group scheme  $j^{-1}(\Delta_{U/S}) \rightarrow U$  is called the *stabilizer of the groupoid scheme*  $(U, R, s, t, c)$ .

In the literature the stabilizer group scheme is often denoted  $S$  (because the word stabilizer starts with an “s” presumably); we cannot do this since we have already used  $S$  for the base scheme.

**Lemma 17.3.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ , and let  $G/U$  be its stabilizer. Denote  $R_t/U$  the scheme  $R$  seen as a scheme over  $U$  via the morphism  $t : R \rightarrow U$ . There is a canonical left action*

$$a : G \times_U R_t \longrightarrow R_t$$

*induced by the composition law  $c$ .*

**Proof.** In terms of points over  $T/S$  we define  $a(g, r) = c(g, r)$ .  $\square$

**Lemma 17.4.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $G$  be the stabilizer group scheme of  $R$ . Let*

$$G_0 = G \times_{U, \text{pr}_0} (U \times_S U) = G \times_S U$$

*as a group scheme over  $U \times_S U$ . The action of  $G$  on  $R$  of Lemma 17.3 induces an action of  $G_0$  on  $R$  over  $U \times_S U$  which turns  $R$  into a pseudo  $G_0$ -torsor over  $U \times_S U$ .*

**Proof.** This is true because in a groupoid category  $\mathcal{C}$  the set  $\text{Mor}_{\mathcal{C}}(x, y)$  is a principal homogeneous set under the group  $\text{Mor}_{\mathcal{C}}(y, y)$ .  $\square$

**Lemma 17.5.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $p \in U \times_S U$  be a point. Denote  $R_p$  the scheme theoretic fibre of  $j = (t, s) : R \rightarrow U \times_S U$ . If  $R_p \neq \emptyset$ , then the action*

$$G_{0, \kappa(p)} \times_{\kappa(p)} R_p \longrightarrow R_p$$

*(see Lemma 17.4) which turns  $R_p$  into a  $G_{\kappa(p)}$ -torsor over  $\kappa(p)$ .*

**Proof.** The action is a pseudo-torsor by the lemma cited in the statement. And if  $R_p$  is not the empty scheme, then  $\{R_p \rightarrow p\}$  is an fpqc covering which trivializes the pseudo-torsor.  $\square$

## 18. Restricting groupoids

Consider a (usual) groupoid  $\mathcal{C} = (\text{Ob}, \text{Arrows}, s, t, c)$ . Suppose we have a map of sets  $g : \text{Ob}' \rightarrow \text{Ob}$ . Then we can construct a groupoid  $\mathcal{C}' = (\text{Ob}', \text{Arrows}', s', t', c')$  by thinking of a morphism between elements  $x', y'$  of  $\text{Ob}'$  as a morphism in  $\mathcal{C}$  between  $g(x'), g(y')$ . In other words we set

$$\text{Arrows}' = \text{Ob}' \times_{g, \text{Ob}, t} \text{Arrows} \times_{s, \text{Ob}, g} \text{Ob}'.$$

with obvious choices for  $s', t'$ , and  $c'$ . There is a canonical functor  $\mathcal{C}' \rightarrow \mathcal{C}$  which is fully faithful, but not necessarily essentially surjective. This groupoid  $\mathcal{C}'$  endowed with the functor  $\mathcal{C}' \rightarrow \mathcal{C}$  is called the *restriction* of the groupoid  $\mathcal{C}$  to  $\text{Ob}'$ .

**Lemma 18.1.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $g : U' \rightarrow U$  be a morphism of schemes. Consider the following diagram*

$$\begin{array}{ccccc}
 & & s' & & \\
 & \curvearrowright & & \curvearrowright & \\
 R' & \xrightarrow{\quad} & R \times_{s, U} U' & \xrightarrow{\quad} & U' \\
 \downarrow & & \downarrow & & \downarrow g \\
 U' \times_{t, R} R & \xrightarrow{\quad} & R & \xrightarrow{s} & U \\
 \downarrow & & \downarrow t & & \\
 U' & \xrightarrow{g} & U & & 
 \end{array}$$

$t'$  (curved arrow from  $R'$  to  $U'$ )

where all the squares are fibre product squares. Then there is a canonical composition law  $c' : R' \times_{s', U', t'} R' \rightarrow R'$  such that  $(U', R', s', t', c')$  is a groupoid scheme over  $S$  and such that  $U' \rightarrow U$ ,  $R' \rightarrow R$  defines a morphism  $(U', R', s', t', c') \rightarrow (U, R, s, t, c)$  of groupoid schemes over  $S$ . Moreover, for any scheme  $T$  over  $S$  the functor of groupoids

$$(U'(T), R'(T), s', t', c') \rightarrow (U(T), R(T), s, t, c)$$

is the restriction (see above) of  $(U(T), R(T), s, t, c)$  via the map  $U'(T) \rightarrow U(T)$ .

**Proof.** Omitted.  $\square$

**Definition 18.2.** Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $g : U' \rightarrow U$  be a morphism of schemes. The morphism of groupoids  $(U', R', s', t', c') \rightarrow (U, R, s, t, c)$  constructed in Lemma 18.1 is called the *restriction of  $(U, R, s, t, c)$  to  $U'$* . We sometime use the notation  $R' = R|_{U'}$  in this case.

**Lemma 18.3.** The notions of restricting groupoids and (pre-)equivalence relations defined in Definitions 18.2 and 3.3 agree via the constructions of Lemmas 13.2 and 13.3.

**Proof.** What we are saying here is that  $R'$  of Lemma 18.1 is also equal to

$$R' = (U' \times_S U') \times_{U \times_S U} R \longrightarrow U' \times_S U'$$

In fact this might have been a clearer way to state that lemma.  $\square$

**Lemma 18.4.** Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $g : U' \rightarrow U$  be a morphism of schemes. Let  $(U', R', s', t', c')$  be the restriction of  $(U, R, s, t, c)$  via  $g$ . Let  $G$  be the stabilizer of  $(U, R, s, t, c)$  and let  $G'$  be the stabilizer of  $(U', R', s', t', c')$ . Then  $G'$  is the base change of  $G$  by  $g$ , i.e., there is a canonical identification  $G' = U' \times_{g, U} G$ .

**Proof.** Omitted.  $\square$

## 19. Invariant subschemes

In this section we discuss briefly the notion of an invariant subscheme.

**Definition 19.1.** Let  $(U, R, s, t, c)$  be a groupoid scheme over the base scheme  $S$ .

- (1) A subset  $W \subset U$  is *set-theoretically  $R$ -invariant* if  $t(s^{-1}(W)) \subset W$ .
- (2) An open  $W \subset U$  is  *$R$ -invariant* if  $t(s^{-1}(W)) \subset W$ .
- (3) A closed subscheme  $Z \subset U$  is called  *$R$ -invariant* if  $t^{-1}(Z) = s^{-1}(Z)$ . Here we use the scheme theoretic inverse image, see Schemes, Definition 17.7.
- (4) A monomorphism of schemes  $T \rightarrow U$  is  *$R$ -invariant* if  $T \times_{U, t} R = R \times_{s, U} T$  as schemes over  $R$ .

For subsets and open subschemes  $W \subset U$  the  $R$ -invariance is also equivalent to requiring that  $s^{-1}(W) = t^{-1}(W)$  as subsets of  $R$ . If  $W \subset U$  is an  $R$ -equivariant open subscheme then the restriction of  $R$  to  $W$  is just  $R_W = s^{-1}(W) = t^{-1}(W)$ . Similarly, if  $Z \subset U$  is an  $R$ -invariant closed subscheme, then the restriction of  $R$  to  $Z$  is just  $R_Z = s^{-1}(Z) = t^{-1}(Z)$ .

**Lemma 19.2.** Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ .

- (1) For any subset  $W \subset U$  the subset  $t(s^{-1}(W))$  is set-theoretically  $R$ -invariant.

- (2) If  $s$  and  $t$  are open, then for every open  $W \subset U$  the open  $t(s^{-1}(W))$  is an  $R$ -invariant open subscheme.
- (3) If  $s$  and  $t$  are open and quasi-compact, then  $U$  has an open covering consisting of  $R$ -invariant quasi-compact open subschemes.

**Proof.** Part (1) follows from Lemmas 3.4 and 13.2, namely,  $t(s^{-1}(W))$  is the set of points of  $U$  equivalent to a point of  $W$ . Next, assume  $s$  and  $t$  open and  $W \subset U$  open. Since  $s$  is open the set  $W' = t(s^{-1}(W))$  is an open subset of  $U$ . Finally, assume that  $s, t$  are both open and quasi-compact. Then, if  $W \subset U$  is a quasi-compact open, then also  $W' = t(s^{-1}(W))$  is a quasi-compact open, and invariant by the discussion above. Letting  $W$  range over all affine opens of  $U$  we see (3).  $\square$

**Lemma 19.3.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $s$  and  $t$  quasi-compact and flat and  $U$  quasi-separated. Let  $W \subset U$  be quasi-compact open. Then  $t(s^{-1}(W))$  is an intersection of a nonempty family of quasi-compact open subsets of  $U$ .*

**Proof.** Note that  $s^{-1}(W)$  is quasi-compact open in  $R$ . As a continuous map  $t$  maps the quasi-compact subset  $s^{-1}(W)$  to a quasi-compact subset  $t(s^{-1}(W))$ . As  $t$  is flat and  $s^{-1}(W)$  is closed under generalization, so is  $t(s^{-1}(W))$ , see (Morphisms, Lemma 25.9 and Topology, Lemma 19.6). Pick a quasi-compact open  $W' \subset U$  containing  $t(s^{-1}(W))$ . By Properties, Lemma 2.4 we see that  $W'$  is a spectral space (here we use that  $U$  is quasi-separated). Then the lemma follows from Topology, Lemma 24.7 applied to  $t(s^{-1}(W)) \subset W'$ .  $\square$

**Lemma 19.4.** *Assumptions and notation as in Lemma 19.3. There exists an  $R$ -invariant open  $V \subset U$  and a quasi-compact open  $W'$  such that  $W \subset V \subset W' \subset U$ .*

**Proof.** Set  $E = t(s^{-1}(W))$ . Recall that  $E$  is set-theoretically  $R$ -invariant (Lemma 19.2). By Lemma 19.3 there exists a quasi-compact open  $W'$  containing  $E$ . Let  $Z = U \setminus W'$  and consider  $T = t(s^{-1}(Z))$ . Observe that  $Z \subset T$  and that  $E \cap T = \emptyset$  because  $s^{-1}(E) = t^{-1}(E)$  is disjoint from  $s^{-1}(Z)$ . Since  $T$  is the image of the closed subset  $s^{-1}(Z) \subset R$  under the quasi-compact morphism  $t : R \rightarrow U$  we see that any point  $\xi$  in the closure  $\overline{T}$  is the specialization of a point of  $T$ , see Morphisms, Lemma 6.5 (and Morphisms, Lemma 6.3 to see that the scheme theoretic image is the closure of the image). Say  $\xi' \rightsquigarrow \xi$  with  $\xi' \in T$ . Suppose that  $r \in R$  and  $s(r) = \xi$ . Since  $s$  is flat we can find a specialization  $r' \rightsquigarrow r$  in  $R$  such that  $s(r') = \xi'$  (Morphisms, Lemma 25.9). Then  $t(r') \rightsquigarrow t(r)$ . We conclude that  $t(r') \in T$  as  $T$  is set-theoretically invariant by Lemma 19.2. Thus  $\overline{T}$  is a set-theoretically  $R$ -invariant closed subset and  $V = U \setminus \overline{T}$  is the open we are looking for. It is contained in  $W'$  which finishes the proof.  $\square$

## 20. Quotient sheaves

Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{smooth}, \text{syntomic}\}$ . Let  $S$  be a scheme. Let  $j : R \rightarrow U \times_S U$  be a pre-relation over  $S$ . Say  $U, R, S$  are objects of a  $\tau$ -site  $\text{Sch}_\tau$  (see Topologies, Section 2). Then we can consider the functors

$$h_U, h_R : (\text{Sch}/S)_\tau^{\text{opp}} \longrightarrow \text{Sets}.$$

These are sheaves, see Descent, Lemma 13.7. The morphism  $j$  induces a map  $j : h_R \rightarrow h_U \times h_U$ . For each object  $T \in \text{Ob}((\text{Sch}/S)_\tau)$  we can take the equivalence

relation  $\sim_T$  generated by  $j(T) : R(T) \rightarrow U(T) \times U(T)$  and consider the quotient. Hence we get a presheaf

$$(20.0.1) \quad (Sch/S)_\tau^{opp} \longrightarrow Sets, \quad T \longmapsto U(T)/\sim_T$$

**Definition 20.1.** Let  $\tau$ ,  $S$ , and the pre-relation  $j : R \rightarrow U \times_S U$  be as above. In this setting the *quotient sheaf*  $U/R$  associated to  $j$  is the sheafification of the presheaf (20.0.1) in the  $\tau$ -topology. If  $j : R \rightarrow U \times_S U$  comes from the action of a group scheme  $G/S$  on  $U$  as in Lemma 16.1 then we sometimes denote the quotient sheaf  $U/G$ .

This means exactly that the diagram

$$h_R \rightrightarrows h_U \longrightarrow U/R$$

is a coequalizer diagram in the category of sheaves of sets on  $(Sch/S)_\tau$ . Using the Yoneda embedding we may view  $(Sch/S)_\tau$  as a full subcategory of sheaves on  $(Sch/S)_\tau$  and hence identify schemes with representable functors. Using this abuse of notation we will often depict the diagram above simply

$$R \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} U \longrightarrow U/R$$

We will mostly work with the fppf topology when considering quotient sheaves of groupoids/equivalence relations.

**Definition 20.2.** In the situation of Definition 20.1. We say that the pre-relation  $j$  has a *representable quotient* if the sheaf  $U/R$  is representable. We will say a groupoid  $(U, R, s, t, c)$  has a *representable quotient* if the quotient  $U/R$  with  $j = (t, s)$  is representable.

The following lemma characterizes schemes  $M$  representing the quotient. It applies for example if  $\tau = fppf$ ,  $U \rightarrow M$  is flat, of finite presentation and surjective, and  $R \cong U \times_M U$ .

**Lemma 20.3.** *In the situation of Definition 20.1. Assume there is a scheme  $M$ , and a morphism  $U \rightarrow M$  such that*

- (1) *the morphism  $U \rightarrow M$  equalizes  $s, t$ ,*
- (2) *the morphism  $U \rightarrow M$  induces a surjection of sheaves  $h_U \rightarrow h_M$  in the  $\tau$ -topology, and*
- (3) *the induced map  $(t, s) : R \rightarrow U \times_M U$  induces a surjection of sheaves  $h_R \rightarrow h_{U \times_M U}$  in the  $\tau$ -topology.*

*In this case  $M$  represents the quotient sheaf  $U/R$ .*

**Proof.** Condition (1) says that  $h_U \rightarrow h_M$  factors through  $U/R$ . Condition (2) says that  $U/R \rightarrow h_M$  is surjective as a map of sheaves. Condition (3) says that  $U/R \rightarrow h_M$  is injective as a map of sheaves. Hence the lemma follows.  $\square$

The following lemma is wrong if we do not require  $j$  to be a pre-equivalence relation (but just a pre-relation say).

**Lemma 20.4.** *Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{smooth}, \text{syntomic}\}$ . Let  $S$  be a scheme. Let  $j : R \rightarrow U \times_S U$  be a pre-equivalence relation over  $S$ . Assume  $U, R, S$  are objects of a  $\tau$ -site  $Sch_\tau$ . For  $T \in \text{Ob}((Sch/S)_\tau)$  and  $a, b \in U(T)$  the following are equivalent:*

- (1)  $a$  and  $b$  map to the same element of  $(U/R)(T)$ , and
- (2) there exists a  $\tau$ -covering  $\{f_i : T_i \rightarrow T\}$  of  $T$  and morphisms  $r_i : T_i \rightarrow R$  such that  $a \circ f_i = s \circ r_i$  and  $b \circ f_i = t \circ r_i$ .

In other words, in this case the map of  $\tau$ -sheaves

$$h_R \longrightarrow h_U \times_{U/R} h_U$$

is surjective.

**Proof.** Omitted. Hint: The reason this works is that the presheaf (20.0.1) in this case is really given by  $T \mapsto U(T)/j(R(T))$  as  $j(R(T)) \subset U(T) \times U(T)$  is an equivalence relation, see Definition 3.1.  $\square$

**Lemma 20.5.** Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{smooth}, \text{syntomic}\}$ . Let  $S$  be a scheme. Let  $j : R \rightarrow U \times_S U$  be a pre-equivalence relation over  $S$  and  $g : U' \rightarrow U$  a morphism of schemes over  $S$ . Let  $j' : R' \rightarrow U' \times_S U'$  be the restriction of  $j$  to  $U'$ . Assume  $U, U', R, S$  are objects of a  $\tau$ -site  $\text{Sch}_\tau$ . The map of quotient sheaves

$$U'/R' \longrightarrow U/R$$

is injective. If  $g$  defines a surjection  $h_{U'} \rightarrow h_U$  of sheaves in the  $\tau$ -topology (for example if  $\{g : U' \rightarrow U\}$  is a  $\tau$ -covering), then  $U'/R' \rightarrow U/R$  is an isomorphism.

**Proof.** Suppose  $\xi, \xi' \in (U'/R')(T)$  are sections which map to the same section of  $U/R$ . Then we can find a  $\tau$ -covering  $\mathcal{T} = \{T_i \rightarrow T\}$  of  $T$  such that  $\xi|_{T_i}, \xi'|_{T_i}$  are given by  $a_i, a'_i \in U'(T_i)$ . By Lemma 20.4 and the axioms of a site we may after refining  $\mathcal{T}$  assume there exist morphisms  $r_i : T_i \rightarrow R$  such that  $g \circ a_i = s \circ r_i$ ,  $g \circ a'_i = t \circ r_i$ . Since by construction  $R' = R \times_{U \times_S U} (U' \times_S U')$  we see that  $(r_i, (a_i, a'_i)) \in R'(T_i)$  and this shows that  $a_i$  and  $a'_i$  define the same section of  $U'/R'$  over  $T_i$ . By the sheaf condition this implies  $\xi = \xi'$ .

If  $h_{U'} \rightarrow h_U$  is a surjection of sheaves, then of course  $U'/R' \rightarrow U/R$  is surjective also. If  $\{g : U' \rightarrow U\}$  is a  $\tau$ -covering, then the map of sheaves  $h_{U'} \rightarrow h_U$  is surjective, see Sites, Lemma 12.4. Hence  $U'/R' \rightarrow U/R$  is surjective also in this case.  $\square$

**Lemma 20.6.** Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{smooth}, \text{syntomic}\}$ . Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $g : U' \rightarrow U$  a morphism of schemes over  $S$ . Let  $(U', R', s', t', c')$  be the restriction of  $(U, R, s, t, c)$  to  $U'$ . Assume  $U, U', R, S$  are objects of a  $\tau$ -site  $\text{Sch}_\tau$ . The map of quotient sheaves

$$U'/R' \longrightarrow U/R$$

is injective. If the composition

$$U' \times_{g, U, t} R \xrightarrow{\quad h \quad} R \xrightarrow[\quad s \quad]{\quad pr_1 \quad} U$$

defines a surjection of sheaves in the  $\tau$ -topology then the map is bijective. This holds for example if  $\{h : U' \times_{g, U, t} R \rightarrow U\}$  is a  $\tau$ -covering, or if  $U' \rightarrow U$  defines a surjection of sheaves in the  $\tau$ -topology, or if  $\{g : U' \rightarrow U\}$  is a covering in the  $\tau$ -topology.



**Proof.** Injectivity follows on combining Lemmas 13.2 and 20.5. To see surjectivity (see Sites, Section 11 for a characterization of surjective maps of sheaves) we argue as follows. Suppose that  $T$  is a scheme and  $\sigma \in U/R(T)$ . There exists a covering  $\{T_i \rightarrow T\}$  such that  $\sigma|_{T_i}$  is the image of some element  $f_i \in U(T_i)$ . Hence we may assume that  $\sigma$  is the image of  $f \in U(T)$ . By the assumption that  $h$  is a surjection of sheaves, we can find a  $\tau$ -covering  $\{\varphi_i : T_i \rightarrow T\}$  and morphisms  $f_i : T_i \rightarrow U' \times_{g,U,t} R$  such that  $f \circ \varphi_i = h \circ f_i$ . Denote  $f'_i = \text{pr}_0 \circ f_i : T_i \rightarrow U'$ . Then we see that  $f'_i \in U'(T_i)$  maps to  $g \circ f'_i \in U(T_i)$  and that  $g \circ f'_i \sim_{T_i} h \circ f_i = f \circ \varphi_i$  notation as in (20.0.1). Namely, the element of  $R(T_i)$  giving the relation is  $\text{pr}_1 \circ f_i$ . This means that the restriction of  $\sigma$  to  $T_i$  is in the image of  $U'/R'(T_i) \rightarrow U/R(T_i)$  as desired.

If  $\{h\}$  is a  $\tau$ -covering, then it induces a surjection of sheaves, see Sites, Lemma 12.4. If  $U' \rightarrow U$  is surjective, then also  $h$  is surjective as  $s$  has a section (namely the neutral element  $e$  of the groupoid scheme).  $\square$

**Lemma 20.7.** *Let  $S$  be a scheme. Let  $f : (U, R, j) \rightarrow (U', R', j')$  be a morphism between equivalence relations over  $S$ . Assume that*

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ s \downarrow & & \downarrow s' \\ U & \xrightarrow{f} & U' \end{array}$$

*is cartesian. For any  $\tau \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{smooth}, \text{syntomic}\}$  the diagram*

$$\begin{array}{ccc} U & \xrightarrow{\quad} & U/R \\ \downarrow & & \downarrow f \\ U' & \xrightarrow{\quad} & U'/R' \end{array}$$

*is a fibre product square of  $\tau$ -sheaves.*

**Proof.** By Lemma 20.4 the quotient sheaves have a simple description which we will use below without further mention. We first show that

$$U \longrightarrow U' \times_{U'/R'} U/R$$

is injective. Namely, assume  $a, b \in U(T)$  map to the same element on the right hand side. Then  $f(a) = f(b)$ . After replacing  $T$  by the members of a  $\tau$ -covering we may assume that there exists an  $r \in R(T)$  such that  $a = s(r)$  and  $b = t(r)$ . Then  $r' = f(r)$  is a  $T$ -valued point of  $R'$  with  $s'(r') = t'(r')$ . Hence  $r' = e'(f(a))$  (where  $e'$  is the identity of the groupoid scheme associated to  $j'$ , see Lemma 13.3). Because the first diagram of the lemma is cartesian this implies that  $r$  has to equal  $e(a)$ . Thus  $a = b$ .

Finally, we show that the displayed arrow is surjective. Let  $T$  be a scheme over  $S$  and let  $(a', \bar{b})$  be a section of the sheaf  $U' \times_{U'/R'} U/R$  over  $T$ . After replacing  $T$  by the members of a  $\tau$ -covering we may assume that  $\bar{b}$  is the class of an element  $b \in U(T)$ . After replacing  $T$  by the members of a  $\tau$ -covering we may assume that there exists an  $r' \in R'(T)$  such that  $a' = t(r')$  and  $s'(r') = f(b)$ . Because the first diagram of the lemma is cartesian we can find  $r \in R(T)$  such that  $s(r) = b$  and  $f(r) = r'$ . Then it is clear that  $a = t(r) \in U(T)$  is a section which maps to  $(a', \bar{b})$ .  $\square$

## 21. Descent in terms of groupoids

Cartesian morphisms are defined as follows.

**Definition 21.1.** Let  $S$  be a scheme. Let  $f : (U', R', s', t', c') \rightarrow (U, R, s, t, c)$  be a morphism of groupoid schemes over  $S$ . We say  $f$  is *cartesian*, or that  $(U', R', s', t', c')$  is *cartesian over*  $(U, R, s, t, c)$ , if the diagram

$$\begin{array}{ccc} R' & \xrightarrow{f} & R \\ s' \downarrow & & \downarrow s \\ U' & \xrightarrow{f} & U \end{array}$$

is a fibre square in the category of schemes. A *morphism of groupoid schemes cartesian over*  $(U, R, s, t, c)$  is a morphism of groupoid schemes compatible with the structure morphisms towards  $(U, R, s, t, c)$ .

Cartesian morphisms are related to descent data. First we prove a general lemma describing the category of cartesian groupoid schemes over a fixed groupoid scheme.

**Lemma 21.2.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . The category of groupoid schemes cartesian over  $(U, R, s, t, c)$  is equivalent to the category of pairs  $(V, \varphi)$  where  $V$  is a scheme over  $U$  and*

$$\varphi : V \times_{U,t} R \longrightarrow R \times_{s,U} V$$

*is an isomorphism over  $R$  such that  $e^* \varphi = \text{id}_V$  and such that*

$$c^* \varphi = \text{pr}_1^* \varphi \circ \text{pr}_0^* \varphi$$

*as morphisms of schemes over  $R \times_{s,U,t} R$ .*

**Proof.** The pullback notation in the lemma signifies base change. The displayed formula makes sense because

$$(R \times_{s,U,t} R) \times_{\text{pr}_1, R, \text{pr}_1} (V \times_{U,t} R) = (R \times_{s,U,t} R) \times_{\text{pr}_0, R, \text{pr}_0} (R \times_{s,U} V)$$

as schemes over  $R \times_{s,U,t} R$ .

Given  $(V, \varphi)$  we set  $U' = V$  and  $R' = V \times_{U,t} R$ . We set  $t' : R' \rightarrow U'$  equal to the projection  $V \times_{U,t} R \rightarrow V$ . We set  $s'$  equal to  $\varphi$  followed by the projection  $R \times_{s,U} V \rightarrow V$ . We set  $c'$  equal to the composition

$$\begin{aligned} R' \times_{s', U', t'} R' &\xrightarrow{\varphi, 1} (R \times_{s,U} V) \times_V (V \times_{U,t} R) \\ &\rightarrow R \times_{s,U} V \times_{U,t} R \\ &\xrightarrow{\varphi^{-1}, 1} V \times_{U,t} (R \times_{s,U,t} R) \\ &\xrightarrow{1, c} V \times_{U,t} R = R' \end{aligned}$$

A computation, which we omit shows that we obtain a groupoid scheme over  $(U, R, s, t, c)$ . It is clear that this groupoid scheme is cartesian over  $(U, R, s, t, c)$ .

Conversely, given  $f : (U', R', s', t', c') \rightarrow (U, R, s, t, c)$  cartesian then the morphisms

$$U' \times_{U,t} R \xleftarrow{t', f} R' \xrightarrow{f, s'} R \times_{s,U} U'$$

are isomorphisms and we can set  $V = U'$  and  $\varphi$  equal to the composition  $(f, s') \circ (t', f)^{-1}$ . We omit the proof that  $\varphi$  satisfies the conditions in the lemma. We omit the proof that these constructions are mutually inverse.  $\square$

Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of schemes over  $S$ . Then we obtain a groupoid scheme  $(X, X \times_Y X, \text{pr}_1, \text{pr}_0, c)$  over  $S$ . Namely,  $j : X \times_Y X \rightarrow X \times_S X$  is an equivalence relation and we can take the associated groupoid, see Lemma 13.3.

**Lemma 21.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of schemes over  $S$ . The construction of Lemma 21.2 determines an equivalence*

$$\begin{array}{ccc} \text{category of groupoid schemes} & \longrightarrow & \text{category of descent data} \\ \text{cartesian over } (X, X \times_Y X, \dots) & & \text{relative to } X/Y \end{array}$$

**Proof.** This is clear from Lemma 21.2 and the definition of descent data on schemes in Descent, Definition 34.1.  $\square$

## 22. Separation conditions

This really means conditions on the morphism  $j : R \rightarrow U \times_S U$  when given a groupoid  $(U, R, s, t, c)$  over  $S$ . As in the previous section we first formulate the corresponding diagram.

**Lemma 22.1.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . Let  $G \rightarrow U$  be the stabilizer group scheme. The commutative diagram*

$$\begin{array}{ccccc} R & \xrightarrow{f \mapsto (f, s(f))} & R \times_{s, U} U & \longrightarrow & U \\ \downarrow \Delta_{R/U \times_S U} & & \downarrow & & \downarrow \\ R \times_{(U \times_S U)} R & \xrightarrow{(f, g) \mapsto (f, f^{-1} \circ g)} & R \times_{s, U} G & \longrightarrow & G \end{array}$$

*the two left horizontal arrows are isomorphisms and the right square is a fibre product square.*

**Proof.** Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry.  $\square$

**Lemma 22.2.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . Let  $G \rightarrow U$  be the stabilizer group scheme.*

- (1) *The following are equivalent*
  - (a)  $j : R \rightarrow U \times_S U$  is separated,
  - (b)  $G \rightarrow U$  is separated, and
  - (c)  $e : U \rightarrow G$  is a closed immersion.
- (2) *The following are equivalent*
  - (a)  $j : R \rightarrow U \times_S U$  is quasi-separated,
  - (b)  $G \rightarrow U$  is quasi-separated, and
  - (c)  $e : U \rightarrow G$  is quasi-compact.

**Proof.** The group scheme  $G \rightarrow U$  is the base change of  $R \rightarrow U \times_S U$  by the diagonal morphism  $U \rightarrow U \times_S U$ , see Lemma 17.1. Hence if  $j$  is separated (resp. quasi-separated), then  $G \rightarrow U$  is separated (resp. quasi-separated). (See Schemes, Lemma 21.12). Thus (a)  $\Rightarrow$  (b) in both (1) and (2).

If  $G \rightarrow U$  is separated (resp. quasi-separated), then the morphism  $U \rightarrow G$ , as a section of the structure morphism  $G \rightarrow U$  is a closed immersion (resp. quasi-compact), see Schemes, Lemma 21.11. Thus (b)  $\Rightarrow$  (a) in both (1) and (2).

By the result of Lemma 22.1 (and Schemes, Lemmas 18.2 and 19.3) we see that if  $e$  is a closed immersion (resp. quasi-compact)  $\Delta_{R/U \times_S U}$  is a closed immersion (resp. quasi-compact). Thus (c)  $\Rightarrow$  (a) in both (1) and (2).  $\square$

### 23. Finite flat groupoids, affine case

Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $U = \text{Spec}(A)$ , and  $R = \text{Spec}(B)$  are affine. In this case we get two ring maps  $s^\#, t^\# : A \rightarrow B$ . Let  $C$  be the equalizer of  $s^\#$  and  $t^\#$ . In a formula

$$(23.0.1) \quad C = \{a \in A \mid t^\#(a) = s^\#(a)\}.$$

We will sometimes call this the *ring of  $R$ -invariant functions on  $U$* . What properties does  $M = \text{Spec}(C)$  have? The first observation is that the diagram

$$\begin{array}{ccc} R & \xrightarrow{s} & U \\ t \downarrow & & \downarrow \\ U & \longrightarrow & M \end{array}$$

is commutative, i.e., the morphism  $U \rightarrow M$  equalizes  $s, t$ . Moreover, if  $T$  is any affine scheme, and if  $U \rightarrow T$  is a morphism which equalizes  $s, t$ , then  $U \rightarrow T$  factors through  $U \rightarrow M$ . In other words,  $U \rightarrow M$  is a coequalizer in the category of affine schemes.

We would like to find conditions that guarantee the morphism  $U \rightarrow M$  is really a “quotient” in the category of schemes. We will discuss this at length elsewhere (insert future reference here); here we just discuss some special cases. Namely, we will focus on the case where  $s, t$  are finite locally free.

**Example 23.1.** Let  $k$  be a field. Let  $U = \text{GL}_{2,k}$ . Let  $B \subset \text{GL}_2$  be the closed subgroup scheme of upper triangular matrices. Then the quotient sheaf  $\text{GL}_{2,k}/B$  (in the Zariski, étale or fppf topology, see Definition 20.1) is representable by the projective line:  $\mathbf{P}^1 = \text{GL}_{2,k}/B$ . (Details omitted.) On the other hand, the ring of invariant functions in this case is just  $k$ . Note that in this case the morphisms  $s, t : R = \text{GL}_{2,k} \times_k B \rightarrow \text{GL}_{2,k} = U$  are smooth of relative dimension 3.

Recall that in Exercises, Exercises 22.6 and 22.7 we have defined the determinant and the norm for finitely locally free modules and finite locally free ring extensions. If  $\varphi : A \rightarrow B$  is a finite locally free ring map, then we will denote  $\text{Norm}_\varphi(b) \in A$  the norm of  $b \in B$ . In the case of a finite locally free morphism of schemes, the norm was constructed in Divisors, Lemma 17.6.

**Lemma 23.2.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $U = \text{Spec}(A)$  and  $R = \text{Spec}(B)$  are affine and  $s, t : R \rightarrow U$  finite locally free. Let  $C$  be as in (23.0.1). Let  $f \in A$ . Then  $\text{Norm}_{s^\#}(t^\#(f)) \in C$ .*

**Proof.** Consider the commutative diagram

$$\begin{array}{ccccc}
 & & U & & \\
 & \swarrow t & & \nwarrow t & \\
 R & \xleftarrow{\text{pr}_0} & R \times_{s,U,t} R & \xrightarrow{c} & R \\
 \downarrow s & & \downarrow \text{pr}_1 & & \downarrow s \\
 U & \xleftarrow{t} & R & \xrightarrow{s} & U
 \end{array}$$

of Lemma 13.4. Think of  $f \in \Gamma(U, \mathcal{O}_U)$ . The commutativity of the top part of the diagram shows that  $\text{pr}_0^\#(t^\#(f)) = c^\#(t^\#(f))$  as elements of  $\Gamma(R \times_{s,U,t} R, \mathcal{O})$ . Looking at the right lower cartesian square the compatibility of the norm construction with base change shows that  $s^\#(\text{Norm}_{s^\#}(t^\#(f))) = \text{Norm}_{\text{pr}_1^\#}(c^\#(t^\#(f)))$ . Similarly we get  $t^\#(\text{Norm}_{s^\#}(t^\#(f))) = \text{Norm}_{\text{pr}_1^\#}(\text{pr}_0^\#(t^\#(f)))$ . Hence by the first equality of this proof we see that  $s^\#(\text{Norm}_{s^\#}(t^\#(f))) = t^\#(\text{Norm}_{s^\#}(t^\#(f)))$  as desired.  $\square$

**Lemma 23.3.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $s, t : R \rightarrow U$  finite locally free. Then*

$$U = \coprod_{r \geq 1} U_r$$

*is a disjoint union of  $R$ -invariant opens such that the restriction  $R_r$  of  $R$  to  $U_r$  has the property that  $s, t : R_r \rightarrow U_r$  are finite locally free of rank  $r$ .*

**Proof.** By Morphisms, Lemma 48.5 there exists a decomposition  $U = \coprod_{r \geq 0} U_r$  such that  $s : s^{-1}(U_r) \rightarrow U_r$  is finite locally free of rank  $r$ . As  $s$  is surjective we see that  $U_0 = \emptyset$ . Note that  $u \in U_r \Leftrightarrow$  if and only if the scheme theoretic fibre  $s^{-1}(u)$  has degree  $r$  over  $\kappa(u)$ . Now, if  $z \in R$  with  $s(z) = u$  and  $t(z) = u'$  then using notation as in Lemma 13.4

$$\text{pr}_1^{-1}(z) \rightarrow \text{Spec}(\kappa(z))$$

is the base change of both  $s^{-1}(u) \rightarrow \text{Spec}(\kappa(u))$  and  $s^{-1}(u') \rightarrow \text{Spec}(\kappa(u'))$  by the lemma cited. Hence  $u \in U_r \Leftrightarrow u' \in U_r$ , in other words, the open subsets  $U_r$  are  $R$ -invariant. In particular the restriction of  $R$  to  $U_r$  is just  $s^{-1}(U_r)$  and  $s : R_r \rightarrow U_r$  is finite locally free of rank  $r$ . As  $t : R_r \rightarrow U_r$  is isomorphic to  $s$  by the inverse of  $R_r$  we see that it has also rank  $r$ .  $\square$

**Lemma 23.4.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $U = \text{Spec}(A)$  and  $R = \text{Spec}(B)$  are affine and  $s, t : R \rightarrow U$  finite locally free. Let  $C \subset A$  be as in (23.0.1). Then  $A$  is integral over  $C$ .*

**Proof.** First, by Lemma 23.3 we know that  $(U, R, s, t, c)$  is a disjoint union of groupoid schemes  $(U_r, R_r, s, t, c)$  such that each  $s, t : R_r \rightarrow U_r$  has constant rank  $r$ . As  $U$  is quasi-compact, we have  $U_r = \emptyset$  for almost all  $r$ . It suffices to prove the lemma for each  $(U_r, R_r, s, t, c)$  and hence we may assume that  $s, t$  are finite locally free of rank  $r$ .

Assume that  $s, t$  are finite locally free of rank  $r$ . Let  $f \in A$ . Consider the element  $x - f \in A[x]$ , where we think of  $x$  as the coordinate on  $\mathbf{A}^1$ . Since

$$(U \times \mathbf{A}^1, R \times \mathbf{A}^1, s \times \text{id}_{\mathbf{A}^1}, t \times \text{id}_{\mathbf{A}^1}, c \times \text{id}_{\mathbf{A}^1})$$

is also a groupoid scheme with finite source and target, we may apply Lemma 23.2 to it and we see that  $P(x) = \text{Norm}_{s^\sharp}(t^\sharp(x - f))$  is an element of  $C[x]$ . Because  $s^\sharp : A \rightarrow B$  is finite locally free of rank  $r$  we see that  $P$  is monic of degree  $r$ . Moreover  $P(f) = 0$  by Cayley-Hamilton (Algebra, Lemma 16.1).  $\square$

**Lemma 23.5.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $U = \text{Spec}(A)$  and  $R = \text{Spec}(B)$  are affine and  $s, t : R \rightarrow U$  finite locally free. Let  $C \subset A$  be as in (23.0.1). Let  $C \rightarrow C'$  be a ring map, and set  $U' = \text{Spec}(A \otimes_C C')$ ,  $R' = \text{Spec}(B \otimes_C C')$ . Then*

- (1) *The maps  $s, t, c$  induce maps  $s', t', c'$  such that  $(U', R', s', t', c')$  is a groupoid scheme. Let  $C^1 \subset A'$  be the  $R'$ -invariant functions on  $U'$ .*
- (2) *The canonical map  $\varphi : C' \rightarrow C^1$  satisfies*
  - (a) *for every  $f \in C^1$  there exists an  $n > 0$  and a polynomial  $P \in C'[x]$  whose image in  $C^1[x]$  is  $(x - f)^n$ , and*
  - (b) *for every  $f \in \text{Ker}(\varphi)$  there exists an  $n > 0$  such that  $f^n = 0$ .*
- (3) *If  $C \rightarrow C'$  is flat then  $\varphi$  is an isomorphism.*

**Proof.** The proof of part (1) is omitted. Let us denote  $A' = A \otimes_C C'$  and  $B' = B \otimes_C C'$ . Then we have

$$C^1 = \{a \in A' \mid (t')^\sharp(a) = (s')^\sharp(a)\} = \{a \in A \otimes_C C' \mid t^\sharp \otimes 1(a) = s^\sharp \otimes 1(a)\}.$$

In other words,  $C^1$  is the kernel of the difference map  $(t^\sharp - s^\sharp) \otimes 1$  which is just the base change of the  $C$ -linear map  $t^\sharp - s^\sharp : A \rightarrow B$  by  $C \rightarrow C'$ . Hence (3) follows.

Proof of part (2)(b). Since  $C \rightarrow A$  is integral (Lemma 23.4) and injective we see that  $\text{Spec}(A) \rightarrow \text{Spec}(C)$  is surjective, see Algebra, Lemma 36.17. Thus also  $\text{Spec}(A') \rightarrow \text{Spec}(C')$  is surjective as a base change of a surjective morphism (Morphisms, Lemma 9.4). Hence  $\text{Spec}(C^1) \rightarrow \text{Spec}(C')$  is surjective also. This implies (2)(b) holds for example by Algebra, Lemma 30.6.

Proof of part (2)(a). By Lemma 23.3 our groupoid scheme  $(U, R, s, t, c)$  decomposes as a finite disjoint union of groupoid schemes  $(U_r, R_r, s, t, c)$  such that  $s, t : R_r \rightarrow U_r$  are finite locally free of rank  $r$ . Pulling back by  $U' = \text{Spec}(C') \rightarrow U$  we obtain a similar decomposition of  $U'$  and  $U^1 = \text{Spec}(C^1)$ . We will show in the next paragraph that (2)(a) holds for the corresponding system of rings  $A_r, B_r, C_r, C'_r, C_r^1$  with  $n = r$ . Then given  $f \in C^1$  let  $P_r \in C_r[x]$  be the polynomial whose image in  $C_r^1[x]$  is the image of  $(x - f)^r$ . Choosing a sufficiently divisible integer  $n$  we see that there is a polynomial  $P \in C'[x]$  whose image in  $C^1[x]$  is  $(x - f)^n$ ; namely, we take  $P$  to be the unique element of  $C'[x]$  whose image in  $C'_r[x]$  is  $P_r^{n/r}$ .

In this paragraph we prove (2)(a) in case the ring maps  $s^\sharp, t^\sharp : A \rightarrow B$  are finite locally free of a fixed rank  $r$ . Let  $f \in C^1 \subset A' = A \otimes_C C'$ . Choose a flat  $C$ -algebra  $D$  and a surjection  $D \rightarrow C'$ . Choose a lift  $g \in A \otimes_C D$  of  $f$ . Consider the polynomial

$$P = \text{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(x - g))$$

in  $(A \otimes_C D)[x]$ . By Lemma 23.2 and part (3) of the current lemma the coefficients of  $P$  are in  $D$  (compare with the proof of Lemma 23.4). On the other hand, the image of  $P$  in  $(A \otimes_C C')[x]$  is  $(x - f)^r$  because  $t^\sharp \otimes 1(x - f) = s^\sharp(x - f)$  and  $s^\sharp$  is finite locally free of rank  $r$ . This proves what we want with  $P$  as in the statement (2)(a) given by the image of our  $P$  under the map  $D[x] \rightarrow C'[x]$ .  $\square$

**Lemma 23.6.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $U = \text{Spec}(A)$  and  $R = \text{Spec}(B)$  are affine and  $s, t : R \rightarrow U$  finite locally free. Let  $C \subset A$  be as in (23.0.1). Then  $U \rightarrow M = \text{Spec}(C)$  has the following properties:*

- (1) *the map on points  $|U| \rightarrow |M|$  is surjective and  $u_0, u_1 \in |U|$  map to the same point if and only if there exists a  $r \in |R|$  with  $t(r) = u_0$  and  $s(r) = u_1$ , in a formula*

$$|M| = |U|/|R|$$

- (2) *for any algebraically closed field  $k$  we have*

$$M(k) = U(k)/R(k)$$

**Proof.** Since  $C \rightarrow A$  is integral (Lemma 23.4) and injective we see that  $\text{Spec}(A) \rightarrow \text{Spec}(C)$  is surjective, see Algebra, Lemma 36.17. Thus  $|U| \rightarrow |M|$  is surjective.

Let  $k$  be an algebraically closed field and let  $C \rightarrow k$  be a ring map. Since surjective morphisms are preserved under base change (Morphisms, Lemma 9.4) we see that  $A \otimes_C k$  is not zero. Now  $k \subset A \otimes_C k$  is a nonzero integral extension. Hence any residue field of  $A \otimes_C k$  is an algebraic extension of  $k$ , hence equal to  $k$ . Thus we see that  $U(k) \rightarrow M(k)$  is surjective.

Let  $a_0, a_1 : A \rightarrow k$  be two ring maps. If there exists a ring map  $b : B \rightarrow k$  such that  $a_0 = b \circ t^\sharp$  and  $a_1 = b \circ s^\sharp$  then we see that  $a_0|_C = a_1|_C$  by definition. Thus the map  $U(k) \rightarrow M(k)$  equalizes the two maps  $R(k) \rightarrow U(k)$ . Conversely, suppose that  $a_0|_C = a_1|_C$ . Let us name this algebra map  $c : C \rightarrow k$ . Consider the diagram

$$\begin{array}{ccc}
 & & B \\
 & \nearrow \text{dotted} & \uparrow \\
 k & \xleftarrow{a_1} & A \\
 & \xleftarrow{a_0} & \uparrow \\
 & & C \\
 & \searrow c & 
 \end{array}$$

If we can construct a dotted arrow making the diagram commute, then the proof of part (2) of the lemma is complete. Since  $s : A \rightarrow B$  is finite there exist finitely many ring maps  $b_1, \dots, b_n : B \rightarrow k$  such that  $b_i \circ s^\sharp = a_1$ . If the dotted arrow does not exist, then we see that none of the  $a'_i = b_i \circ t^\sharp$ ,  $i = 1, \dots, n$  is equal to  $a_0$ . Hence the maximal ideals

$$\mathfrak{m}'_i = \text{Ker}(a'_i \otimes 1 : A \otimes_C k \rightarrow k)$$

of  $A \otimes_C k$  are distinct from  $\mathfrak{m} = \text{Ker}(a_0 \otimes 1 : A \otimes_C k \rightarrow k)$ . By Algebra, Lemma 15.2 we would get an element  $f \in A \otimes_C k$  with  $f \in \mathfrak{m}$ , but  $f \notin \mathfrak{m}'_i$  for  $i = 1, \dots, n$ . Consider the norm

$$g = \text{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(f)) \in A \otimes_C k$$

By Lemma 23.2 this lies in the invariants  $C^1 \subset A \otimes_C k$  of the base change groupoid (base change via the map  $c : C \rightarrow k$ ). On the one hand,  $a_1(g) \in k^*$  since the value of  $t^\sharp(f)$  at all the points (which correspond to  $b_1, \dots, b_n$ ) lying over  $a_1$  is invertible (insert future reference on property determinant here). On the other hand, since  $f \in \mathfrak{m}$ , we see that  $f$  is not a unit, hence  $t^\sharp(f)$  is not a unit (as  $t^\sharp \otimes 1$  is faithfully flat), hence its norm is not a unit (insert future reference on property determinant here). We conclude that  $C^1$  contains an element which is not nilpotent and not a

unit. We will now show that this leads to a contradiction. Namely, apply Lemma 23.5 to the map  $c : C \rightarrow C' = k$ , then we see that the map of  $k$  into the invariants  $C^1$  is injective and moreover, that for any element  $x \in C^1$  there exists an integer  $n > 0$  such that  $x^n \in k$ . Hence every element of  $C^1$  is either a unit or nilpotent.

We still have to finish the proof of (1). We already know that  $|U| \rightarrow |M|$  is surjective. It is clear that  $|U| \rightarrow |M|$  is  $|R|$ -invariant. Finally, suppose  $u_0, u_1 \in U$  maps to the same point  $m \in M$ . Then the induced field extensions  $\kappa(u_0)/\kappa(m)$  and  $\kappa(u_1)/\kappa(m)$  are algebraic (as  $A$  is integral over  $C$  as used above). Hence if  $k$  is an algebraic closure of  $\kappa(m)$  then we can find  $\kappa(m)$ -embeddings  $\bar{u}_0 : \kappa(u_0) \rightarrow k$  and  $\bar{u}_1 : \kappa(u_1) \rightarrow k$ . These determine  $k$ -valued points  $\bar{u}_0, \bar{u}_1 \in U(k)$  mapping to the same point of  $M(k)$ . By part (2) we see that there exists a point  $\bar{r} \in R(k)$  with  $s(\bar{r}) = \bar{u}_0$  and  $t(\bar{r}) = \bar{u}_1$ . The image  $r \in R$  of  $\bar{r}$  is a point with  $s(r) = u_0$  and  $t(r) = u_1$  as desired.  $\square$

**Lemma 23.7.** *Let  $S$  be a scheme. Let  $f : (U', R', s', t') \rightarrow (U, R, s, t, c)$  be a morphism of groupoid schemes over  $S$ .*

- (1)  $U, R, U', R'$  are affine,
- (2)  $s, t, s', t'$  are finite locally free,
- (3) the diagrams

$$\begin{array}{ccc} R' & \xrightarrow{f} & R \\ s' \downarrow & & \downarrow s \\ U' & \xrightarrow{f} & U \end{array} \quad \begin{array}{ccc} R' & \xrightarrow{f} & R \\ t' \downarrow & & \downarrow t \\ U' & \xrightarrow{f} & U \end{array} \quad \begin{array}{ccc} G' & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ U' & \xrightarrow{f} & U \end{array}$$

are cartesian where  $G$  and  $G'$  are the stabilizer group schemes, and

- (4)  $f : U' \rightarrow U$  is étale.

Then the map  $C \rightarrow C'$  from the  $R$ -invariant functions on  $U$  to the  $R'$ -invariant functions on  $U'$  is étale and  $U' = \text{Spec}(C') \times_{\text{Spec}(C)} U$ .

**Proof.** Set  $M = \text{Spec}(C)$  and  $M' = \text{Spec}(C')$ . Write  $U = \text{Spec}(A)$ ,  $U' = \text{Spec}(A')$ ,  $R = \text{Spec}(B)$ , and  $R' = \text{Spec}(B')$ . We will use the results of Lemmas 23.4, 23.5, and 23.6 without further mention.

Assume  $C$  is a strictly henselian local ring. Let  $p \in M$  be the closed point and let  $p' \in M'$  map to  $p$ . Claim: in this case there is a disjoint union decomposition  $(U', R', s', t', c') = (U, R, s, t, c) \amalg (U'', R'', s'', t'', c'')$  over  $(U, R, s, t, c)$  such that for the corresponding disjoint union decomposition  $M' = M \amalg M''$  over  $M$  the point  $p'$  corresponds to  $p \in M$ .

The claim implies the lemma. Suppose that  $M_1 \rightarrow M$  is a flat morphism of affine schemes. Then we can base change everything to  $M_1$  without affecting the hypotheses (1) – (4). From Lemma 23.5 we see  $M_1$ , resp.  $M'_1$  is the spectrum of the  $R_1$ -invariant functions on  $U_1$ , resp. the  $R'_1$ -invariant functions on  $U'_1$ . Suppose that  $p' \in M'$  maps to  $p \in M$ . Let  $M_1$  be the spectrum of the strict henselization of  $\mathcal{O}_{M,p}$  with closed point  $p_1 \in M_1$ . Choose a point  $p'_1 \in M'_1$  mapping to  $p_1$  and  $p'$ . From the claim we get

$$(U'_1, R'_1, s'_1, t'_1, c'_1) = (U_1, R_1, s_1, t_1, c_1) \amalg (U''_1, R''_1, s''_1, t''_1, c''_1)$$

and correspondingly  $M'_1 = M_1 \amalg M''_1$  as a scheme over  $M_1$ . Write  $M_1 = \text{Spec}(C_1)$  and write  $C_1 = \text{colim } C_i$  as a filtered colimit of étale  $C$ -algebras. Set  $M_i = \text{Spec}(C_i)$ .



The  $M_1 = \lim M_i$  and similarly for the other schemes. By Limits, Lemmas 4.11 and 8.11 we can find an  $i$  such that

$$(U'_i, R'_i, s'_i, t'_i, c'_i) = (U_i, R_i, s_i, t_i, c_i) \amalg (U''_i, R''_i, s''_i, t''_i, c''_i)$$

We conclude that  $M'_i = M_i \amalg M''_i$ . In particular  $M' \rightarrow M$  becomes étale at a point over  $p'$  after an étale base change. This implies that  $M' \rightarrow M$  is étale at  $p'$  (for example by Morphisms, Lemma 36.17). We will prove  $U' \cong M' \times_M U$  after we prove the claim.

Proof of the claim. Observe that  $U_p$  and  $U'_{p'}$  have finitely many points. For  $u \in U_p$  we have  $\kappa(u)/\kappa(p)$  is algebraic, hence  $\kappa(u)$  is separably closed. As  $U' \rightarrow U$  is étale, we conclude the morphism  $U'_{p'} \rightarrow U_p$  induces isomorphisms on residue field extensions. Let  $u' \in U'_{p'}$  with image  $u \in U_p$ . By assumption (3) the morphism of scheme theoretic fibres  $(s')^{-1}(u') \rightarrow s^{-1}(u)$ ,  $(t')^{-1}(u') \rightarrow t^{-1}(u)$ , and  $G'_{u'} \rightarrow G_u$  are isomorphisms. Observing that  $U_p = t(s^{-1}(u))$  (set theoretically) we conclude that the points of  $U'_{p'}$  surject onto the points of  $U_p$ . Suppose that  $u'_1$  and  $u'_2$  are points of  $U'_{p'}$  mapping to the same point  $u$  of  $U_p$ . Then there exists a point  $r' \in R'_{p'}$  with  $s'(r') = u'_1$  and  $t'(r') = u'_2$ . Consider the two towers of fields

$$\kappa(r')/\kappa(u'_1)/\kappa(u)/\kappa(p) \quad \kappa(r')/\kappa(u'_2)/\kappa(u)/\kappa(p)$$

whose “ends” are the same as the two “ends” of the two towers

$$\kappa(r')/\kappa(u'_1)/\kappa(p')/\kappa(p) \quad \kappa(r')/\kappa(u'_2)/\kappa(p')/\kappa(p)$$

These two induce the same maps  $\kappa(p') \rightarrow \kappa(r')$  as  $(U'_{p'}, R'_{p'}, s', t', c')$  is a groupoid over  $p'$ . Since  $\kappa(u)/\kappa(p)$  is purely inseparable, we conclude that the two induced maps  $\kappa(u) \rightarrow \kappa(r')$  are the same. Therefore  $r'$  maps to a point of the fibre  $G_u$ . By assumption (3) we conclude that  $r' \in (G')_{u'_1}$ . Namely, we may think of  $G$  as a closed subscheme of  $R$  viewed as a scheme over  $U$  via  $s$  and use that the base change to  $U'$  gives  $G' \subset R'$ . In particular we have  $u'_1 = u'_2$ . We conclude that  $U'_{p'} \rightarrow U_p$  is a bijective map on points inducing isomorphisms on residue fields. It follows that  $U'_{p'}$  is a finite set of closed points (Algebra, Lemma 35.9) and hence  $U'_{p'}$  is closed in  $U'$ . Let  $J' \subset A'$  be the radical ideal cutting out  $U'_{p'}$  set theoretically.

Second part proof of the claim. Let  $\mathfrak{m} \subset C$  be the maximal ideal. Observe that  $(A, \mathfrak{m}A)$  is a henselian pair by More on Algebra, Lemma 11.8. Let  $J = \sqrt{\mathfrak{m}A}$ . Then  $(A, J)$  is a henselian pair (More on Algebra, Lemma 11.7) and the étale ring map  $A \rightarrow A'$  induces an isomorphism  $A/J \rightarrow A'/J'$  by our deliberations above. We conclude that  $A' = A \times A''$  by More on Algebra, Lemma 11.6. Consider the corresponding disjoint union decomposition  $U' = U \amalg U''$ . The open  $(s')^{-1}(U)$  is the set of points of  $R'$  specializing to a point of  $R'_{p'}$ . Similarly for  $(t')^{-1}(U)$ . Similarly we have  $(s')^{-1}(U'') = (t')^{-1}(U'')$  as this is the set of points which do not specialize to  $R'_{p'}$ . Hence we obtain a disjoint union decomposition

$$(U', R', s', t', c') = (U, R, s, t, c) \amalg (U'', R'', s'', t'', c'')$$

This immediately gives  $M' = M \amalg M''$  and the proof of the claim is complete.

We still have to prove that the canonical map  $U' \rightarrow M' \times_M U$  is an isomorphism. It is an étale morphism (Morphisms, Lemma 36.18). On the other hand, by base changing to strictly henselian local rings (as in the third paragraph of the proof) and using the bijectivity  $U'_{p'} \rightarrow U_p$  established in the course of the proof of the claim, we see that  $U' \rightarrow M' \times_M U$  is universally bijective (some details omitted). However, a

universally bijective étale morphism is an isomorphism (Descent, Lemma 25.2) and the proof is complete.  $\square$

**Lemma 23.8.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume*

- (1)  $U = \text{Spec}(A)$ , and  $R = \text{Spec}(B)$  are affine, and
- (2) there exist elements  $x_i \in A$ ,  $i \in I$  such that  $B = \bigoplus_{i \in I} s^\sharp(A)t^\sharp(x_i)$ .

Then  $A = \bigoplus_{i \in I} Cx_i$ , and  $B \cong A \otimes_C A$  where  $C \subset A$  is the  $R$ -invariant functions on  $U$  as in (23.0.1).

**Proof.** During this proof we will write  $s, t : A \rightarrow B$  instead of  $s^\sharp, t^\sharp$ , and similarly  $c : B \rightarrow B \otimes_{s,A,t} B$ . We write  $p_0 : B \rightarrow B \otimes_{s,A,t} B$ ,  $b \mapsto b \otimes 1$  and  $p_1 : B \rightarrow B \otimes_{s,A,t} B$ ,  $b \mapsto 1 \otimes b$ . By Lemma 13.5 and the definition of  $C$  we have the following commutative diagram

$$\begin{array}{ccccc} B \otimes_{s,A,t} B & \xleftarrow{c} & B & \xleftarrow{t} & A \\ p_1 \uparrow & & \uparrow s & & \uparrow \\ B & \xleftarrow{s} & A & \xleftarrow{t} & C \end{array}$$

Moreover the two left squares are cocartesian in the category of rings, and the top row is isomorphic to the diagram

$$B \otimes_{t,A,t} B \xleftarrow[p_0]{p_1} B \xleftarrow{t} A$$

which is an equalizer diagram according to Descent, Lemma 3.6 because condition (2) implies in particular that  $s$  (and hence also then isomorphic arrow  $t$ ) is faithfully flat. The lower row is an equalizer diagram by definition of  $C$ . We can use the  $x_i$  and get a commutative diagram

$$\begin{array}{ccccc} B \otimes_{s,A,t} B & \xleftarrow{c} & B & \xleftarrow{t} & A \\ p_1 \uparrow & & \uparrow s & & \uparrow \\ \bigoplus_{i \in I} Bx_i & \xleftarrow{s} & \bigoplus_{i \in I} Ax_i & \xleftarrow{t} & \bigoplus_{i \in I} Cx_i \end{array}$$

where in the right vertical arrow we map  $x_i$  to  $x_i$ , in the middle vertical arrow we map  $x_i$  to  $t(x_i)$  and in the left vertical arrow we map  $x_i$  to  $c(t(x_i)) = t(x_i) \otimes 1 = p_0(t(x_i))$  (equality by the commutativity of the top part of the diagram in Lemma 13.4). Then the diagram commutes. Moreover the middle vertical arrow is an isomorphism by assumption. Since the left two squares are cocartesian we conclude that also the left vertical arrow is an isomorphism. On the other hand, the horizontal rows are exact (i.e., they are equalizers). Hence we conclude that also the right vertical arrow is an isomorphism.  $\square$

**Proposition 23.9.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume*

- (1)  $U = \text{Spec}(A)$ , and  $R = \text{Spec}(B)$  are affine,
- (2)  $s, t : R \rightarrow U$  finite locally free, and
- (3)  $j = (t, s)$  is an equivalence.

In this case, let  $C \subset A$  be as in (23.0.1). Then  $U \rightarrow M = \operatorname{Spec}(C)$  is finite locally free and  $R = U \times_M U$ . Moreover,  $M$  represents the quotient sheaf  $U/R$  in the fppf topology (see Definition 20.1).

**Proof.** During this proof we use the notation  $s, t : A \rightarrow B$  instead of the notation  $s^\#, t^\#$ . By Lemma 20.3 it suffices to show that  $C \rightarrow A$  is finite locally free and that the map

$$t \otimes s : A \otimes_C A \longrightarrow B$$

is an isomorphism. First, note that  $j$  is a monomorphism, and also finite (since already  $s$  and  $t$  are finite). Hence we see that  $j$  is a closed immersion by Morphisms, Lemma 44.15. Hence  $A \otimes_C A \rightarrow B$  is surjective.

We will perform base change by flat ring maps  $C \rightarrow C'$  as in Lemma 23.5, and we will use that formation of invariants commutes with flat base change, see part (3) of the lemma cited. We will show below that for every prime  $\mathfrak{p} \subset C$ , there exists a local flat ring map  $C_{\mathfrak{p}} \rightarrow C'_{\mathfrak{p}}$  such that the result holds after a base change to  $C'_{\mathfrak{p}}$ . This implies immediately that  $A \otimes_C A \rightarrow B$  is injective (use Algebra, Lemma 23.1). It also implies that  $C \rightarrow A$  is flat, by combining Algebra, Lemmas 39.17, 39.18, and 39.8. Then since  $U \rightarrow \operatorname{Spec}(C)$  is surjective also (Lemma 23.6) we conclude that  $C \rightarrow A$  is faithfully flat. Then the isomorphism  $B \cong A \otimes_C A$  implies that  $A$  is a finitely presented  $C$ -module, see Algebra, Lemma 83.2. Hence  $A$  is finite locally free over  $C$ , see Algebra, Lemma 78.2.

By Lemma 23.3 we know that  $A$  is a finite product of rings  $A_r$  and  $B$  is a finite product of rings  $B_r$  such that the groupoid scheme decomposes accordingly (see the proof of Lemma 23.4). Then also  $C$  is a product of rings  $C_r$  and correspondingly  $C'$  decomposes as a product. Hence we may and do assume that the ring maps  $s, t : A \rightarrow B$  are finite locally free of a fixed rank  $r$ .

The local ring maps  $C_{\mathfrak{p}} \rightarrow C'_{\mathfrak{p}}$  we are going to use are any local flat ring maps such that the residue field of  $C'_{\mathfrak{p}}$  is infinite. By Algebra, Lemma 159.1 such local ring maps exist.

Assume  $C$  is a local ring with maximal ideal  $\mathfrak{m}$  and infinite residue field, and assume that  $s, t : A \rightarrow B$  is finite locally free of constant rank  $r > 0$ . Since  $C \subset A$  is integral (Lemma 23.4) all primes lying over  $\mathfrak{m}$  are maximal, and all maximal ideals of  $A$  lie over  $\mathfrak{m}$ . Similarly for  $C \subset B$ . Pick a maximal ideal  $\mathfrak{m}'$  of  $A$  lying over  $\mathfrak{m}$  (exists by Lemma 23.6). Since  $t : A \rightarrow B$  is finite locally free there exist at most finitely many maximal ideals of  $B$  lying over  $\mathfrak{m}'$ . Hence we conclude (by Lemma 23.6 again) that  $A$  has finitely many maximal ideals, i.e.,  $A$  is semi-local. This in turn implies that  $B$  is semi-local as well. OK, and now, because  $t \otimes s : A \otimes_C A \rightarrow B$  is surjective, we can apply Algebra, Lemma 78.8 to the ring map  $C \rightarrow A$ , the  $A$ -module  $M = B$  (seen as an  $A$ -module via  $t$ ) and the  $C$ -submodule  $s(A) \subset B$ . This lemma implies that there exist  $x_1, \dots, x_r \in A$  such that  $M$  is free over  $A$  on the basis  $s(x_1), \dots, s(x_r)$ . Hence we conclude that  $C \rightarrow A$  is finite free and  $B \cong A \otimes_C A$  by applying Lemma 23.8.  $\square$

## 24. Finite flat groupoids

In this section we prove a lemma that will help to show that the quotient of a scheme by a finite flat equivalence relation is a scheme, provided that each equivalence class is contained in an affine. See Properties of Spaces, Proposition 14.1.

**Lemma 24.1.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $s, t$  are finite locally free. Let  $u \in U$  be a point such that  $t(s^{-1}(\{u\}))$  is contained in an affine open of  $U$ . Then there exists an  $R$ -invariant affine open neighbourhood of  $u$  in  $U$ .*

**Proof.** Since  $s$  is finite locally free it has finite fibres. Hence  $t(s^{-1}(\{u\})) = \{u_1, \dots, u_n\}$  is a finite set. Note that  $u \in \{u_1, \dots, u_n\}$ . Let  $W \subset U$  be an affine open containing  $\{u_1, \dots, u_n\}$ , in particular  $u \in W$ . Consider  $Z = R \setminus s^{-1}(W) \cap t^{-1}(W)$ . This is a closed subset of  $R$ . The image  $t(Z)$  is a closed subset of  $U$  which can be loosely described as the set of points of  $U$  which are  $R$ -equivalent to a point of  $U \setminus W$ . Hence  $W' = U \setminus t(Z)$  is an  $R$ -invariant, open subscheme of  $U$  contained in  $W$ , and  $\{u_1, \dots, u_n\} \subset W'$ . Picture

$$\{u_1, \dots, u_n\} \subset W' \subset W \subset U.$$

Let  $f \in \Gamma(W, \mathcal{O}_W)$  be an element such that  $\{u_1, \dots, u_n\} \subset D(f) \subset W'$ . Such an  $f$  exists by Algebra, Lemma 15.2. By our choice of  $W'$  we have  $s^{-1}(W') \subset t^{-1}(W)$ , and hence we get a diagram

$$\begin{array}{ccc} s^{-1}(W') & \xrightarrow{t} & W \\ s \downarrow & & \\ W' & & \end{array}$$

The vertical arrow is finite locally free by assumption. Set

$$g = \text{Norm}_s(t^\# f) \in \Gamma(W', \mathcal{O}_{W'})$$

By construction  $g$  is a function on  $W'$  which is nonzero in  $u$ , as  $t^\#(f)$  is nonzero in each of the points of  $R$  lying over  $u$ , since  $f$  is nonzero in  $u_1, \dots, u_n$ . Similarly,  $D(g) \subset W'$  is equal to the set of points  $w$  such that  $f$  is not zero in any of the points equivalent to  $w$ . This means that  $D(g)$  is an  $R$ -invariant affine open of  $W'$ . The final picture is

$$\{u_1, \dots, u_n\} \subset D(g) \subset D(f) \subset W' \subset W \subset U$$

and hence we win.  $\square$

## 25. Descending quasi-projective schemes

We can use Lemma 24.1 to show that a certain type of descent datum is effective.

**Lemma 25.1.** *Let  $X \rightarrow Y$  be a surjective finite locally free morphism. Let  $V$  be a scheme over  $X$  such that for all  $(y, v_1, \dots, v_d)$  where  $y \in Y$  and  $v_1, \dots, v_d \in V_y$  there exists an affine open  $U \subset V$  with  $v_1, \dots, v_d \in U$ . Then any descent datum on  $V/X/Y$  is effective.*

**Proof.** Let  $\varphi$  be a descent datum as in Descent, Definition 34.1. Recall that the functor from schemes over  $Y$  to descent data relative to  $\{X \rightarrow Y\}$  is fully faithful, see Descent, Lemma 35.11. Thus using Constructions, Lemma 2.1 it suffices to prove the lemma in the case that  $Y$  is affine. Some details omitted (this argument can be avoided if  $Y$  is separated or has affine diagonal, because then every morphism from an affine scheme to  $X$  is affine).

Assume  $Y$  is affine. If  $V$  is also affine, then we have effectivity by Descent, Lemma 37.1. Hence by Descent, Lemma 35.13 it suffices to prove that every

point  $v$  of  $V$  has a  $\varphi$ -invariant affine open neighbourhood. Consider the groupoid  $(X, X \times_Y X, \text{pr}_1, \text{pr}_0, \text{pr}_0)$ . By Lemma 21.3 the descent datum  $\varphi$  determines and is determined by a cartesian morphism of groupoid schemes

$$(V, R, s, t, c) \longrightarrow (X, X \times_Y X, \text{pr}_1, \text{pr}_0, \text{pr}_0)$$

over  $\text{Spec}(\mathbf{Z})$ . Since  $X \rightarrow Y$  is finite locally free, we see that  $\text{pr}_i : X \times_Y X \rightarrow X$  and hence  $s$  and  $t$  are finite locally free. In particular the  $R$ -orbit  $t(s^{-1}(\{v\}))$  of our point  $v \in V$  is finite. Using the equivalence of categories of Lemma 21.3 once more we see that  $\varphi$ -invariant opens of  $V$  are the same thing as  $R$ -invariant opens of  $V$ . Our assumption shows there exists an affine open of  $V$  containing the orbit  $t(s^{-1}(\{v\}))$  as all the points in this orbit map to the same point of  $Y$ . Thus Lemma 24.1 provides an  $R$ -invariant affine open containing  $v$ .  $\square$

**Lemma 25.2.** *Let  $X \rightarrow Y$  be a surjective finite locally free morphism. Let  $V$  be a scheme over  $X$  such that one of the following holds*

- (1)  $V \rightarrow X$  is projective,
- (2)  $V \rightarrow X$  is quasi-projective,
- (3) there exists an ample invertible sheaf on  $V$ ,
- (4) there exists an  $X$ -ample invertible sheaf on  $V$ ,
- (5) there exists an  $X$ -very ample invertible sheaf on  $V$ .

*Then any descent datum on  $V/X/Y$  is effective.*

**Proof.** We check the condition in Lemma 25.1. Let  $y \in Y$  and  $v_1, \dots, v_d \in V$  points over  $y$ . Case (1) is a special case of (2), see Morphisms, Lemma 43.10. Case (2) is a special case of (4), see Morphisms, Definition 40.1. If there exists an ample invertible sheaf on  $V$ , then there exists an affine open containing  $v_1, \dots, v_d$  by Properties, Lemma 29.5. Thus (3) is true. In cases (4) and (5) it is harmless to replace  $Y$  by an affine open neighbourhood of  $y$ . Then  $X$  is affine too. In case (4) we see that  $V$  has an ample invertible sheaf by Morphisms, Definition 37.1 and the result follows from case (3). In case (5) we can replace  $V$  by a quasi-compact open containing  $v_1, \dots, v_d$  and we reduce to case (4) by Morphisms, Lemma 38.2.  $\square$

## 26. Other chapters

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