FUNCTORS AND MORPHISMS

Contents

1. Introduction

Let X and Y be schemes. This chapter circles around the relationship between functors $QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_X)$ and morphisms of schemes $X \to Y$. More broadly speaking we study the relationship between $QCoh(\mathcal{O}_X)$ and X or, if X is Noetherian, the relationship between $Coh(\mathcal{O}_X)$ and X. This relationship was studied in [Gab62].

2. Functors on module categories

For a ring A let us denote Mod_A^{fp} the category of finitely presented A-modules.

Lemma 2.1. *Let A be a ring. Let $\mathcal{B}$ be a category having filtered colimits. Let $F : Mod_A^{fp} \to \mathcal{B}$ be a functor. Then F extends uniquely to a functor $F' : Mod_A \to \mathcal{B}$ which commutes with filtered colimits.*

Proof. This follows from Categories, Lemma 26.2. To see that the lemma applies observe that finitely presented A-modules are categorically compact objects of Mod_A by Algebra, Lemma 11.4. Also, every A-module is a filtered colimit of finitely presented A-modules by Algebra, Lemma 11.3. □

If a category $\mathcal{B}$ is additive and has filtered colimits, then $\mathcal{B}$ has arbitrary direct sums: any direct sum can be written as a filtered colimit of finite direct sums.

Lemma 2.2. *Let A, $\mathcal{B}$, F be as in Lemma 2.1. Assume $\mathcal{B}$ is additive and F is additive. Then F' is additive and commutes with arbitrary direct sums.*

Proof. To show that F' is additive it suffices to show that $F'(M) \oplus F'(M') \to F'(M \oplus M')$ is an isomorphism for any A-modules M, M', see Homology, Lemma 7.1. Write $M = \text{colim}_i M_i$ and $M' = \text{colim}_j M'_j$ as filtered colimits of finitely presented A-modules M_i. Then $F'(M) = \mathrm{colim}_i F(M_i)$, $F'(M') = \mathrm{colim}_j F(M'_j)$, and

$$\begin{aligned} F'(M \oplus M') &= F'(\mathrm{colim}_{i,j} M_i \oplus M'_j) \\ &= \mathrm{colim}_{i,j} F(M_i \oplus M'_j) \\ &= \mathrm{colim}_{i,j} F(M_i) \oplus F(M'_j) \\ &= F'(M) \oplus F'(M') \end{aligned}$$

as desired. To show that F' commutes with direct sums, assume we have $M = \bigoplus_{i \in I} M_i$. Then $M = \mathrm{colim}_{I' \subset I \text{ finite}} \bigoplus_{i \in I'} M_i$ is a filtered colimit. We obtain

$$\begin{aligned} F'(M) &= \mathrm{colim}_{I' \subset I \text{ finite}} F'(\bigoplus\nolimits_{i \in I'} M_i) \\ &= \mathrm{colim}_{I' \subset I \text{ finite}} \bigoplus\nolimits_{i \in I'} F'(M_i) \\ &= \bigoplus\nolimits_{i \in I} F'(M_i) \end{aligned}$$

The second equality holds by the additivity of F' already shown. $\square$

If a category $\mathcal{B}$ is additive, has filtered colimits, and has cokernels, then $\mathcal{B}$ has arbitrary colimits, see discussion above and Categories, Lemma 14.12.

Lemma 2.3. *Let A, $\mathcal{B}$, F be as in Lemma 2.1. Assume $\mathcal{B}$ is additive, has cokernels, and F is right exact. Then F' is additive, right exact, and commutes with arbitrary direct sums.*

Proof. Since F is right exact, F commutes with coproducts of pairs, which are represented by direct sums. Hence F is additive by Homology, Lemma 7.1. Hence F' is additive and commutes with direct sums by Lemma 2.2. We urge the reader to prove that F' is right exact themselves instead of reading the proof below.

To show that F' is right exact, it suffices to show that F' commutes with coequalizers, see Categories, Lemma 23.3. Now, if $a, b : K \to L$ are maps of A-modules, then the coequalizer of a and b is the cokernel of $a - b : K \to L$. Thus let $K \to L \to M \to 0$ be an exact sequence of A-modules. We have to show that in

$$F'(K) \to F'(L) \to F'(M) \to 0$$

the second arrow is a cokernel for the first arrow in $\mathcal{B}$ (if $\mathcal{B}$ were abelian we would say that the displayed sequence is exact). Write $M = \mathrm{colim}_{i \in I} M_i$ as a filtered colimit of finitely presented A-modules, see Algebra, Lemma 11.3. Let $L_i = L \times_M M_i$. We obtain a system of exact sequences $K \to L_i \to M_i \to 0$ over I. Since colimits commute with colimits by Categories, Lemma 14.10 and since cokernels are a type of coequalizer, it suffices to show that $F'(L_i) \to F(M_i)$ is a cokernel of $F'(K) \to F'(L_i)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume M is finitely presented. Write $L = \mathrm{colim}_{i \in I} L_i$ as a filtered colimit of finitely presented A-modules with the property that each L_i surjects onto M. Let $K_i = K \times_L L_i$. We obtain a system of short exact sequences $K_i \to L_i \to M \to 0$ over I. Repeating the argument already given, we reduce to showing $F(L_i) \to F(M_i)$ is a cokernel of $F'(K) \to F(L_i)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume both L and M are finitely presented A-modules. In this case the module $\mathrm{Ker}(L \to M)$ is finite (Algebra, Lemma 5.3). Thus we can write $K = \mathrm{colim}_{i \in I} K_i$ as a filtered colimit of finitely presented A-modules each surjecting onto $\mathrm{Ker}(L \to M)$. We obtain a system of short exact sequences $K_i \to L \to M \to 0$ over I. Repeating the argument already given, we reduce to showing $F(L) \to F(M)$ is a cokernel of $F(K_i) \to F(L)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume K, L, and M are finitely presented A-modules. This final case follows from the assumption that F is right exact. $\square$

If a category $\mathcal{B}$ is additive and has kernels, then $\mathcal{B}$ has finite limits. Namely, finite products are direct sums which exist and the equalizer of $a, b : L \to M$ is the kernel of $a - b : K \to L$ which exists. Thus all finite limits exist by Categories, Lemma 18.4.

Lemma 2.4. *Let A, $\mathcal{B}$, F be as in Lemma 2.1. Assume A is a coherent ring (Algebra, Definition 90.1), $\mathcal{B}$ is additive, has kernels, filtered colimits commute with taking kernels, and F is left exact. Then F' is additive, left exact, and commutes with arbitrary direct sums.*

Proof. Since A is coherent, the category Mod_A^{fp} is abelian with same kernels and cokernels as in Mod_A, see Algebra, Lemmas 90.4 and 90.3. Hence all finite limits exist in Mod_A^{fp} and Categories, Definition 23.1 applies. Since F is left exact, F commutes with products of pairs, which are represented by direct sums. Hence F is additive by Homology, Lemma 7.1. Hence F' is additive and commutes with direct sums by Lemma 2.2. We urge the reader to prove that F' is left exact themselves instead of reading the proof below.

To show that F' is left exact, it suffices to show that F' commutes with equalizers, see Categories, Lemma 23.2. Now, if $a, b : L \to M$ are maps of A-modules, then the equalizer of a and b is the kernel of $a - b : L \to M$. Thus let $0 \to K \to L \to M$ be an exact sequence of A-modules. We have to show that in

$$0 \to F'(K) \to F'(L) \to F'(M)$$

the arrow $F'(K) \to F'(L)$ is a kernel for $F'(L) \to F'(M)$ in $\mathcal{B}$ (if $\mathcal{B}$ were abelian we would say that the displayed sequence is exact). Write $M = \text{colim}_{i \in I} M_i$ as a filtered colimit of finitely presented A-modules, see Algebra, Lemma 11.3. Let $L_i = L \times_M M_i$. We obtain a system of exact sequences $0 \to K \to L_i \to M_i$ over I. Since filtered colimits commute with taking kernels in $\mathcal{B}$ by assumption, it suffices to show that $F'(K) \to F'(L_i)$ is a kernel of $F'(L_i) \to F(M_i)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume M is finitely presented. Write $L = \text{colim}_{i \in I} L_i$ as a filtered colimit of finitely presented A-modules. Let $K_i = K \times_L L_i$. We obtain a system of short exact sequences $0 \to K_i \to L_i \to M$ over I. Repeating the argument already given, we reduce to showing $F'(K_i) \to F(L_i)$ is a kernel of $F(L_i) \to F(M)$ in $\mathcal{B}$ for all $i \in I$. In other words, we may assume both L and M are finitely presented A-modules. Since A is coherent, the A-module $K = \text{Ker}(L \to M)$ is of finite presentation as the category of finitely presented A-modules is abelian (see references given above). In other words, all three modules K, L, and M are finitely presented A-modules. This final case follows from the assumption that F is left exact. $\square$

If a category $\mathcal{B}$ is additive and has cokernels, then $\mathcal{B}$ has finite colimits. Namely, finite coproducts are direct sums which exist and the coequalizer of $a, b : K \to L$ is the cokernel of $a - b : K \to L$ which exists. Thus all finite colimits exist by Categories, Lemma 18.7.

Lemma 2.5. *Let A be a ring. Let $\mathcal{B}$ be an additive category with cokernels. There is an equivalence of categories between*

1. *the category of functors $F : Mod_A^{fp} \to \mathcal{B}$ which are right exact, and*
2. *the category of pairs (K, κ) where $K \in \mathrm{Ob}(\mathcal{B})$ and $\kappa : A \to End_{\mathcal{B}}(K)$ is a ring homomorphism*

given by the rule sending F to $F(A)$ with its natural A-action.

Proof. Let (K, κ) be as in (2). We will construct a functor $F : \mathrm{Mod}_A^{fp} \to \mathcal{B}$ such that $F(A) = K$ endowed with the given A-action κ. Namely, given an integer $n \geq 0$ let us set

$$F(A^{\oplus n}) = K^{\oplus n}$$

Given an A-linear map $\varphi : A^{\oplus m} \to A^{\oplus n}$ with matrix $(a_{ij}) \in \mathrm{Mat}(n \times m, A)$ we define

$$F(\varphi) : F(A^{\oplus m}) = K^{\oplus m} \longrightarrow K^{\oplus n} = F(A^{\oplus n})$$

to be the map with matrix $(\kappa(a_{ij}))$. This defines an additive functor F from the full subcategory of Mod_A^{fp} with objects 0, A, $A^{\oplus 2}$, $\ldots$ to $\mathcal{B}$; we omit the verification.

For each object M of Mod_A^{fp} choose a presentation

$$A^{\oplus m_M} \xrightarrow{\varphi_M} A^{\oplus n_M} \to M \to 0$$

of M as an A-module. Let us use the trivial presentation $0 \to A^{\oplus n} \xrightarrow{1} A^{\oplus n} \to 0$ if $M = A^{\oplus n}$ (this isn't necessary but simplifies the exposition). For each morphism $f : M \to N$ of Mod_A^{fp} we can choose a commutative diagram

$$\tag{2.5.1}
\begin{tikzcd}
A^{\oplus m_M} \arrow[r, "\varphi_M"'] \arrow[d, "\psi_f"'] & A^{\oplus n_M} \arrow[r] \arrow[d, "\chi_f"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
A^{\oplus m_N} \arrow[r, "\varphi_N"] & A^{\oplus n_N} \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Having made these choices we can define: for an object M of Mod_A^{fp} we set

$$F(M) = \mathrm{Coker}(F(\varphi_M) : F(A^{\oplus m_M}) \to F(A^{\oplus n_M}))$$

and for a morphism $f : M \to N$ of Mod_A^{fp} we set

$$F(f) = \text{the map } F(M) \to F(N) \text{ induced by } F(\psi_f) \text{ and } F(\chi_f) \text{ on cokernels}$$

Note that this rule extends the given functor F on the full subcategory consisting of the free modules $A^{\oplus n}$. We still have to show that F is a functor, that F is additive, and that F is right exact.

Let $f : M \to N$ be a morphism Mod_A^{fp}. We claim that the map $F(f)$ defined above is independent of the choices of ψ_f and χ_f in (2.5.1). Namely, say

$$\begin{tikzcd}
A^{\oplus m_M} \arrow[r, "\varphi_M"'] \arrow[d, "\psi"'] & A^{\oplus n_M} \arrow[r] \arrow[d, "\chi"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
A^{\oplus m_N} \arrow[r, "\varphi_N"] & A^{\oplus n_N} \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

is also commutative. Denote $F(f)' : F(M) \to F(N)$ the map induced by $F(\psi)$ and $F(\chi)$. Looking at the commutative diagrams, by elementary commutative algebra there exists a map $\omega : A^{\oplus n_M} \to A^{\oplus m_N}$ such that $\chi = \chi_f + \varphi_N \circ \omega$. Applying F we find that $F(\chi) = F(\chi_f) + F(\varphi_N) \circ F(\omega)$. As $F(N)$ is the cokernel of $F(\varphi_N)$ we find that the map $F(A^{\oplus n_M}) \to F(M)$ equalizes $F(f)$ and $F(f)'$. Since a cokernel is an epimorphism, we conclude that $F(f) = F(f)'$.

Let us prove F is a functor. First, observe that $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$ because we may pick the identities for ψ_f and χ_f in the diagram above in case $f = \mathrm{id}_M$. Second, suppose we have $f : M \to N$ and $g : L \to M$. Then we see that $\psi = \psi_f \circ \psi_g$ and $\chi = \chi_f \circ \chi_g$ fit into (2.5.1) for $f \circ g$. Hence these induce the correct map which exactly says that $F(f) \circ F(g) = F(f \circ g)$.

Let us prove that F is additive. Namely, suppose we have $f, g : M \to N$. Then we see that $\psi = \psi_f + \psi_g$ and $\chi = \chi_f + \chi_g$ fit into (2.5.1) for $f + g$. Hence these induce the correct map which exactly says that $F(f) + F(g) = F(f + g)$.

Finally, let us prove that F is right exact. It suffices to show that F commutes with coequalizers, see Categories, Lemma 23.3. For this, it suffices to prove that F commutes with cokernels. Let $K \to L \to M \to 0$ be an exact sequence of A-modules with K, L, M finitely presented. Since F is an additive functor, this certainly gives a complex

$$F(K) \to F(L) \to F(M) \to 0$$

and we have to show that the second arrow is the cokernel of the first in $\mathcal{B}$. In any case, we obtain a map $\mathrm{Coker}(F(K) \to F(L)) \to F(M)$. By elementary commutative algebra there exists a commutative diagram

$$\begin{tikzcd}
A^{\oplus m_M} \arrow[r, "\varphi_M"'] \arrow[d, "\psi"'] & A^{\oplus n_M} \arrow[r] \arrow[d, "\chi"'] & M \arrow[r] \arrow[d, "1"'] & 0 \\
K \arrow[r] & L \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Applying F to this diagram and using the construction of $F(M)$ as the cokernel of $F(\varphi_M)$ we find there exists a map $F(M) \to \mathrm{Coker}(F(K) \to F(L))$ which is a right inverse to the map $\mathrm{Coker}(F(K) \to F(L)) \to F(M)$. This first implies that $F(L) \to F(M)$ is an epimorphism always. Next, the above shows we have

$$\mathrm{Coker}(F(K) \to F(L)) = F(M) \oplus E$$

where the direct sum decomposition is compatible with both $F(M) \to \mathrm{Coker}(F(K) \to F(L))$ and $\mathrm{Coker}(F(K) \to F(L)) \to F(M)$. However, then the epimorphism $p : F(L) \to E$ becomes zero both after composition with $F(K) \to F(L)$ and after composition with $F(A^{n_M}) \to F(L)$. However, since $K \oplus A^{n_M} \to L$ is surjective (algebra argument omitted), we conclude that $F(K \oplus A^{n_M}) \to F(L)$ is an epimorphism (by the above) whence $E = 0$. This finishes the proof. $\square$

Lemma 2.6. *Let A be a ring. Let $\mathcal{B}$ be an additive category with arbitrary direct sums and cokernels. There is an equivalence of categories between*

1. *the category of functors $F : Mod_A \to \mathcal{B}$ which are right exact and commute with arbitrary direct sums, and*
2. *the category of pairs (K, κ) where $K \in \mathrm{Ob}(\mathcal{B})$ and $\kappa : A \to End_{\mathcal{B}}(K)$ is a ring homomorphism*

given by the rule sending F to $F(A)$ with its natural A-action.

Proof. Combine Lemmas 2.5 and 2.3. $\square$

3. Functors between categories of modules

The following lemma is archetypical of the results in this chapter.

Lemma 3.1. *Let A and B be rings. Let $F : Mod_A \to Mod_B$ be a functor. The following are equivalent*

1. *F is isomorphic to the functor $M \mapsto M \otimes_A K$ for some $A \otimes_{\mathbf{Z}} B$-module K,*
2. *F is right exact and commutes with all direct sums,*
3. *F commutes with all colimits,*
4. *F has a right adjoint G.*

Proof. If (1), then (4) as a right adjoint for $M \mapsto M \otimes_A K$ is $N \mapsto \mathrm{Hom}_B(K, N)$, see Differential Graded Algebra, Lemma 30.3. If (4), then (3) by Categories, Lemma 24.5. The implication (3) $\Rightarrow$ (2) is immediate from the definitions.

Assume (2). We will prove (1). By the discussion in Homology, Section 7 the functor F is additive. Hence F induces a ring map $A \to \mathrm{End}_B(F(M))$, $a \mapsto F(a \cdot \mathrm{id}_M)$ for every A-module M. We conclude that $F(M)$ is an $A \otimes_{\mathbf{Z}} B$-module functorially in M. Set $K = F(A)$. Define

$$M \otimes_A K = M \otimes_A F(A) \longrightarrow F(M), \quad m \otimes k \longmapsto F(\varphi_m)(k)$$

Here $\varphi_m : A \to M$ sends $a \to am$. The rule $(m, k) \mapsto F(\varphi_m)(k)$ is A-bilinear (and B-linear on the right) as required to obtain the displayed $A \otimes_{\mathbf{Z}} B$-linear map. This construction is functorial in M, hence defines a transformation of functors $- \otimes_A K \to F(-)$ which is an isomorphism when evaluated on A. For every A-module M we can choose an exact sequence

$$\bigoplus\nolimits_{j \in J} A \to \bigoplus\nolimits_{i \in I} A \to M \to 0$$

Using the maps constructed above we find a commutative diagram

$$\begin{tikzcd}
(\bigoplus_{j \in J} A) \otimes_A K \arrow[r] \arrow[d] & (\bigoplus_{i \in I} A) \otimes_A K \arrow[r] \arrow[d] & M \otimes_A K \arrow[r] \arrow[d] & 0 \\
F(\bigoplus_{j \in J} A) \arrow[r] & F(\bigoplus_{i \in I} A) \arrow[r] & F(M) \arrow[r] & 0
\end{tikzcd}$$

The lower row is exact as F is right exact. The upper row is exact as tensor product with K is right exact. Since F commutes with direct sums the left two vertical arrows are bijections. Hence we conclude. $\square$

Example 3.2. Let R be a ring. Let A and B be R-algebras. Let K be a $A \otimes_R B$-module. Then we can consider the functor

$$F : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B, \quad M \longmapsto M \otimes_A K \tag{3.2.1}$$

This functor is R-linear, right exact, commutes with arbitrary direct sums, commutes with all colimits, has a right adjoint (Lemma 3.1).

Lemma 3.3. *Let R be a ring. Let A and B be R-algebras. There is an equivalence of categories between*

1. *the category of R-linear functors $F : Mod_A \to Mod_B$ which are right exact and commute with arbitrary direct sums, and*
2. *the category $Mod_{A \otimes_R B}$.*

given by sending K to the functor F in (3.2.1).

Proof. Let F be an object of the first category. By Lemma 3.1 we may assume $F(M) = M \otimes_A K$ functorially in M for some $A \otimes_{\mathbf{Z}} B$-module K. The R-linearity of F immediately implies that the $A \otimes_{\mathbf{Z}} B$-module structure on K comes from a (unique) $A \otimes_R B$-module structure on K. Thus we see that sending K to F as in (3.2.1) is essentially surjective.

To prove that our functor is fully faithful, we have to show that given $A \otimes_R B$-modules K and K' any transformation $t : F \to F'$ between the corresponding functors, comes from a unique $\varphi : K \to K'$. Since $K = F(A)$ and $K' = F'(A)$ we can take φ to be the value $t_A : F(A) \to F'(A)$ of t at A. This maps is $A \otimes_R B$-linear by the definition of the $A \otimes B$-module structure on $F(A)$ and $F'(A)$ given in the proof of Lemma 3.1. □

Remark 3.4. Let R be a ring. Let A, B, C be R-algebras. Let $F : \mathrm{Mod}_A \to \mathrm{Mod}_B$ and $F' : \mathrm{Mod}_B \to \mathrm{Mod}_C$ be R-linear, right exact functors which commute with arbitrary direct sums. If by the equivalence of Lemma 3.3 the object K in $\mathrm{Mod}_{A \otimes_R B}$ corresponds to F and the object K' in $\mathrm{Mod}_{B \otimes_R C}$ corresponds to F', then $K \otimes_B K'$ viewed as an object of $\mathrm{Mod}_{A \otimes_R C}$ corresponds to $F' \circ F$.

Remark 3.5. In the situation of Lemma 3.3 suppose that F corresponds to K. Then F is exact $\Leftrightarrow$ K is flat over A.

Remark 3.6. In the situation of Lemma 3.3 suppose that F corresponds to K. Then F sends finite A-modules to finite B-modules $\Leftrightarrow$ K is finite as a B-module.

Remark 3.7. In the situation of Lemma 3.3 suppose that F corresponds to K. Then F sends finitely presented A-modules to finitely presented B-modules $\Leftrightarrow$ K is finitely presented as a B-module.

Lemma 3.8. *Let A and B be rings. If*

$$F : Mod_A \longrightarrow Mod_B$$

is an equivalence of categories, then there exists an isomorphism $A \to B$ of rings and an invertible B-module L such that F is isomorphic to the functor $M \mapsto (M \otimes_A B) \otimes_B L$.

Proof. Since an equivalence commutes with all colimits, we see that Lemmas 3.1 applies. Let K be the $A \otimes_{\mathbf{Z}} B$-module such that F is isomorphic to the functor $M \mapsto M \otimes_A K$. Let K' be the $B \otimes_{\mathbf{Z}} A$-module such that a quasi-inverse of F is isomorphic to the functor $N \mapsto N \otimes_B K'$. By Remark 3.4 and Lemma 3.3 we have an isomorphism

$$\psi : K \otimes_B K' \longrightarrow A$$

of $A \otimes_{\mathbf{Z}} A$-modules. Similarly, we have an isomorphism

$$\psi' : K' \otimes_A K \longrightarrow B$$

of $B \otimes_{\mathbf{Z}} B$-modules. Choose an element $\xi = \sum_{i=1,\ldots,n} x_i \otimes y_i \in K \otimes_B K'$ such that $\psi(\xi) = 1$. Consider the isomorphisms

$$K \xrightarrow{\psi^{-1} \otimes \mathrm{id}_K} K \otimes_B K' \otimes_A K \xrightarrow{\mathrm{id}_K \otimes \psi'} K$$

The composition is an isomorphism and given by

$$k \longmapsto \sum x_i \psi'(y_i \otimes k)$$

We conclude this automorphism factors as

$$K \to B^{\oplus n} \to K$$

as a map of B-modules. It follows that K is finite projective as a B-module.

We claim that K is invertible as a B-module. This is equivalent to asking the rank of K as a B-module to have the constant value 1, see More on Algebra, Lemma 117.2 and Algebra, Lemma 78.2. If not, then there exists a maximal ideal $\mathfrak{m} \subset B$ such that either (a) $K \otimes_B B/\mathfrak{m} = 0$ or (b) there is a surjection $K \to (B/\mathfrak{m})^{\oplus 2}$ of B-modules. Case (a) is absurd as $K' \otimes_A K \otimes_B N = N$ for all B-modules N. Case (b) would imply we get a surjection

$$A = K \otimes_B K' \longrightarrow (B/\mathfrak{m} \otimes_B K')^{\oplus 2}$$

of (right) A-modules. This is impossible as the target is an A-module which needs at least two generators: $B/\mathfrak{m} \otimes_B K'$ is nonzero as the image of the nonzero module $B/\mathfrak{m}$ under the quasi-inverse of F.

Since K is invertible as a B-module we see that $\text{Hom}_B(K, K) = B$. Since $K = F(A)$ the action of A on K defines a ring isomorphism $A \to B$. The lemma follows. $\square$

Lemma 3.9. *Let R be a ring. Let A and B be R-algebras. If*

$$F : Mod_A \longrightarrow Mod_B$$

is an R-linear equivalence of categories, then there exists an isomorphism $A \to B$ of R-algebras and an invertible B-module L such that F is isomorphic to the functor $M \mapsto (M \otimes_A B) \otimes_B L$.

Proof. We get $A \to B$ and L from Lemma 3.8. To finish the proof, we need to show that the R-linearity of F forces $A \to B$ to be an R-algebra map. We omit the details. $\square$

Remark 3.10. Let A and B be rings. Let us endow Mod_A and Mod_B with the usual monoidal structure given by tensor products of modules. Let $F : \text{Mod}_A \to \text{Mod}_B$ be a functor of monoidal categories, see Categories, Definition 43.2. Here are some comments:

(1) Since $F(A)$ is a unit (by our definitions) we have $F(A) = B$.
(2) We obtain a multiplicative map $\varphi : A \to B$ by sending $a \in A$ to its action on $F(A) = B$.
(3) Take $A = B$ and $F(M) = M \otimes_A M$. In this case $\varphi(a) = a^2$.
(4) If F is additive, then φ is a ring map.
(5) Take $A = B = \mathbf{Z}$ and $F(M) = M/\text{torsion}$. Then $\varphi = \text{id}_{\mathbf{Z}}$ but F is not the identity functor.
(6) If F is right exact and commutes with direct sums, then $F(M) = M \otimes_{A,\varphi} B$ by Lemma 3.1.

In other words, ring maps $A \to B$ are in bijection with isomorphism classes of functors of monoidal categories $\text{Mod}_A \to \text{Mod}_B$ which commute with all colimits.

4. Extending functors on categories of modules

For a ring A let us denote Mod_A^{fp} the category of finitely presented A-modules.

Lemma 4.1. *Let A and B be rings. Let $F : Mod_A^{fp} \to Mod_B^{fp}$ be a functor. Then F extends uniquely to a functor $F' : Mod_A \to Mod_B$ which commutes with filtered colimits.*

Proof. Special case of Lemma 2.1. ☐

Remark 4.2. With A, B, F, and F' as in Lemma 4.1. Observe that the tensor product of two finitely presented modules is finitely presented, see Algebra, Lemma 12.14. Thus we may endow Mod_A^{fp}, Mod_B^{fp}, Mod_A, and Mod_B with the usual monoidal structure given by tensor products of modules. In this case, if F is a functor of monoidal categories, so is F'. This follows immediately from the fact that tensor products of modules commutes with filtered colimits.

Lemma 4.3. *With A, B, F, and F' as in Lemma 4.1.*

1. *If F is additive, then F' is additive and commutes with arbitrary direct sums, and*
2. *if F is right exact, then F' is right exact.*

Proof. Follows from Lemmas 2.2 and 2.3. ☐

Remark 4.4. Combining Remarks 3.10 and 4.2 and Lemma 4.3 we find the following. Given rings A and B the set of ring maps $A \to B$ is in bijection with the set of isomorphism classes of functors of monoidal categories $\text{Mod}_A^{fp} \to \text{Mod}_B^{fp}$ which are right exact.

Lemma 4.5. *With A, B, F, and F' as in Lemma 4.1. Assume A is a coherent ring (Algebra, Definition 90.1). If F is left exact, then F' is left exact.*

Proof. Special case of Lemma 2.4. ☐

For a ring A let us denote Mod_A^{fg} the category of finitely generated A-modules (AKA finite A-modules).

Lemma 4.6. *Let A and B be Noetherian rings. Let $F : Mod_A^{fg} \to Mod_B^{fg}$ be a functor. Then F extends uniquely to a functor $F' : Mod_A \to Mod_B$ which commutes with filtered colimits. If F is additive, then F' is additive and commutes with arbitrary direct sums. If F is exact, left exact, or right exact, so is F'.*

Proof. See Lemmas 4.3 and 4.5. Also, use the finite A-modules are finitely presented A-modules, see Algebra, Lemma 31.4, and use that Noetherian rings are coherent, see Algebra, Lemma 90.5. ☐

5. Functors between categories of quasi-coherent modules

In this section we briefly study functors between categories of quasi-coherent modules.

Example 5.1. Let R be a ring. Let X and Y be schemes over R with X quasi-compact and quasi-separated. Let $\mathcal{K}$ be a quasi-coherent $\mathcal{O}_{X \times_R Y}$-module. Then we can consider the functor

$$F : QCoh(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_Y), \quad \mathcal{F} \longmapsto \text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}) \tag{5.1.1}$$

The morphism pr_2 is quasi-compact and quasi-separated (Schemes, Lemmas 19.3 and 21.12). Hence pushforward along this morphism preserves quasi-coherent modules, see Schemes, Lemma 24.1. Moreover, our functor is R-linear and commutes with arbitrary direct sums, see Cohomology of Schemes, Lemma 6.1.

The following lemma is a natural generalization of Lemma 3.3.

Lemma 5.2. *Let R be a ring. Let X and Y be schemes over R with X affine. There is an equivalence of categories between*

1. *the category of R-linear functors $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which are right exact and commute with arbitrary direct sums, and*
2. *the category $QCoh(\mathcal{O}_{X\times_R Y})$*

given by sending $\mathcal{K}$ to the functor F in (5.1.1).

Proof. Let $\mathcal{K}$ be an object of $QCoh(\mathcal{O}_{X\times_R Y})$ and $F_\mathcal{K}$ the functor (5.1.1). By the discussion in Example 5.1 we already know that F is R-linear and commutes with arbitrary direct sums. Since $\mathrm{pr}_2 : X \times_R Y \to Y$ is affine (Morphisms, Lemma 11.8) the functor $\mathrm{pr}_{2,*}$ is exact, see Cohomology of Schemes, Lemma 2.3. Hence F is right exact as well, in other words F is as in (1).

Let F be as in (1). Say $X = \mathrm{Spec}(A)$. Consider the quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G} = F(\mathcal{O}_X)$. The functor F induces an R-linear map $A \to \mathrm{End}_{\mathcal{O}_Y}(\mathcal{G})$, $a \mapsto F(a\cdot\mathrm{id})$. Thus $\mathcal{G}$ is a sheaf of modules over

$$A \otimes_R \mathcal{O}_Y = \mathrm{pr}_{2,*}\mathcal{O}_{X\times_R Y}$$

By Morphisms, Lemma 11.6 we find that there is a unique quasi-coherent module $\mathcal{K}$ on $X\times_R Y$ such that $F(\mathcal{O}_X) = \mathcal{G} = \mathrm{pr}_{2,*}\mathcal{K}$ compatible with action of A and $\mathcal{O}_Y$. Denote $F_\mathcal{K}$ the functor given by (5.1.1). There is an equivalence $\mathrm{Mod}_A \to QCoh(\mathcal{O}_X)$ sending A to $\mathcal{O}_X$, see Schemes, Lemma 7.5. Hence we find an isomorphism $F \cong F_\mathcal{K}$ by Lemma 2.6 because we have an isomorphism $F(\mathcal{O}_X) \cong F_\mathcal{K}(\mathcal{O}_X)$ compatible with A-action by construction.

This shows that the functor sending $\mathcal{K}$ to $F_\mathcal{K}$ is essentially surjective. We omit the verification of fully faithfulness. □

Remark 5.3. Below we will use that for an affine morphism $h : T \to S$ we have $h_*\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{H} = h_*(\mathcal{G} \otimes_{\mathcal{O}_T} h^*\mathcal{H})$ for $\mathcal{G} \in QCoh(\mathcal{O}_T)$ and $\mathcal{H} \in QCoh(\mathcal{O}_S)$. This follows immediately on translating into algebra.

Lemma 5.4. *In Lemma 5.2 let F correspond to $\mathcal{K}$ in $QCoh(\mathcal{O}_{X\times_R Y})$. We have*

1. *If $f : X' \to X$ is an affine morphism, then $F \circ f_*$ corresponds to $(f \times id_Y)^*\mathcal{K}$.*
2. *If $g : Y' \to Y$ is a flat morphism, then $g^* \circ F$ corresponds to $(id_X \times g)^*\mathcal{K}$.*
3. *If $j : V \to Y$ is an open immersion, then $j^* \circ F$ corresponds to $\mathcal{K}|_{X\times_R V}$.*

Proof. Proof of (1). Consider the commutative diagram

$$\begin{tikzcd}
X' \times_R Y \arrow[dr, "f\times \mathrm{id}_Y"'] \arrow[drr, "\mathrm{pr}'_2"] \arrow[dd, "\mathrm{pr}'_1"'] & & \\
& X \times_R Y \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"'] & Y \\
X' \arrow[r, "f"] & X &
\end{tikzcd}$$

Let $\mathcal{F}'$ be a quasi-coherent module on X'. We have

$$\begin{aligned}
\mathrm{pr}_{2,*}(\mathrm{pr}_1^* f_*\mathcal{F}' \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K}) &= \mathrm{pr}_{2,*}((f\times \mathrm{id}_Y)_*(\mathrm{pr}_1')^*\mathcal{F}' \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K}) \\
&= \mathrm{pr}_{2,*}(f\times \mathrm{id}_Y)_*\left((\mathrm{pr}_1')^*\mathcal{F}' \otimes_{\mathcal{O}_{X'\times_R Y}} (f\times \mathrm{id}_Y)^*\mathcal{K})\right) \\
&= \mathrm{pr}_{2,*}'((\mathrm{pr}_1')^*\mathcal{F}' \otimes_{\mathcal{O}_{X'\times_R Y}} (f\times \mathrm{id}_Y)^*\mathcal{K})
\end{aligned}$$

Here the first equality is affine base change for the left hand square in the diagram, see Cohomology of Schemes, Lemma 5.1. The second equality hold by Remark 5.3. The third equality is functoriality of pushforwards for modules. This proves (1).

Proof of (2). Consider the commutative diagram

$$\begin{tikzcd}
X\times_R Y' \arrow[rr, "\mathrm{pr}_2'"] \arrow[dr, "\mathrm{id}_X\times g"] \arrow[ddr, "\mathrm{pr}_1'"'] & & Y' \arrow[d, "g"] \\
& X\times_R Y \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] & Y \\
& X &
\end{tikzcd}$$

We have

$$\begin{aligned}
g^*\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K}) &= \mathrm{pr}_{2,*}'((\mathrm{id}_X\times g)^*(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K})) \\
&= \mathrm{pr}_{2,*}'((\mathrm{pr}_1')^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_R Y'}} (\mathrm{id}_X\times g)^*\mathcal{K})
\end{aligned}$$

The first equality by flat base change for the square in the diagram, see Cohomology of Schemes, Lemma 5.2. The second equality by functoriality of pullback and the fact that a pullback of tensor products it the tensor product of the pullbacks.

Part (3) is a special case of (2). ☐

Lemma 5.5. *Let R be a ring. Let X and Y be schemes over R. Assume X is quasi-compact with affine diagonal. Let $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ be an R-linear, right exact functor which commutes with arbitrary direct sums. Then we can construct*

1. *a quasi-coherent module $\mathcal{K}$ on $X\times_R Y$, and*
2. *a natural transformation $t : F \to F_{\mathcal{K}}$ where $F_{\mathcal{K}}$ denotes the functor (5.1.1)*

such that $t : F\circ f_* \to F_{\mathcal{K}}\circ f_*$ is an isomorphism for every morphism $f : X' \to X$ whose source is an affine scheme.

Proof. Consider a morphism $f' : X' \to X$ with X' affine. Since the diagonal of X is affine, we see that f' is an affine morphism (Morphisms, Lemma 11.11). Thus $f'_* : QCoh(\mathcal{O}_{X'}) \to QCoh(\mathcal{O}_X)$ is an R-linear exact functor (Cohomology of Schemes, Lemma 2.3) which commutes with direct sums (Cohomology of Schemes, Lemma 6.1). Thus $F\circ f'_*$ is an R-linear, right exact functor which commutes with arbitrary direct sums. Whence $F\circ f'_* = F_{\mathcal{K}'}$ for some $\mathcal{K}'$ on $X'\times_R Y$ by Lemma 5.2. Moreover, given a morphism $f'' : X'' \to X'$ with X'' affine we obtain a canonical identification $(f''\times \mathrm{id}_Y)^*\mathcal{K}' = \mathcal{K}''$ by the references already given combined with Lemma 5.4. These identifications satisfy a cocycle condition given another morphism $f''' : X''' \to X''$ which we leave it to the reader to spell out.

Choose an affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. Since the diagonal of X is affine, we see that the intersections $U_{i_0\ldots i_p} = U_{i_0}\cap \ldots \cap U_{i_p}$ are affine. As above the inclusion morphisms $j_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to X$ are affine. Denote $\mathcal{K}_{i_0 \ldots i_p}$ the quasi-coherent module on $U_{i_0 \ldots i_p} \times_R Y$ corresponding to $F \circ j_{i_0 \ldots i_p *}$ as above. By the above we obtain identifications

$$\mathcal{K}_{i_0 \ldots i_p} = \mathcal{K}_{i_0 \ldots \hat{i}_j \ldots i_p}|_{U_{i_0 \ldots i_p} \times_R Y}$$

which satisfy the usual compatibilites for glueing. In other words, we obtain a unique quasi-coherent module $\mathcal{K}$ on $X \times_R Y$ whose restriction to $U_{i_0 \ldots i_p} \times_R Y$ is $\mathcal{K}_{i_0 \ldots i_p}$ compatible with the displayed identifications.

Next, we construct the transformation t. Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ denote $\mathcal{F}_{i_0 \ldots i_p}$ the restriction of $\mathcal{F}$ to $U_{i_0 \ldots i_p}$ and denote $(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 \ldots i_p}$ the restriction of $\text{pr}_1^*\mathcal{F} \otimes \mathcal{K}$ to $U_{i_0 \ldots i_p} \times_R Y$. Observe that

$$\begin{aligned} F(j_{i_0 \ldots i_p *}\mathcal{F}_{i_0 \ldots i_p}) &= \text{pr}_{i_0 \ldots i_p, 2, *}(\text{pr}_{i_0 \ldots i_p, 1}^*\mathcal{F}_{i_0 \ldots i_p} \otimes \mathcal{K}_{i_0 \ldots i_p}) \\ &= \text{pr}_{i_0 \ldots i_p, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 \ldots i_p} \end{aligned}$$

where $\text{pr}_{i_0 \ldots i_p, 2} : U_{i_0 \ldots i_p} \times_R Y \to Y$ is the projection and similarly for the other projection. Moreover, these identifications are compatible with the displayed identifications in the previous paragraph. Recall, from Cohomology of Schemes, Lemma 7.1 that the relative Čech complex

$$\bigoplus \text{pr}_{i_0, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0} \to \bigoplus \text{pr}_{i_0 i_1, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 i_1} \to \bigoplus \text{pr}_{i_0 i_1 i_2, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 i_1 i_2} \to \ldots$$

computes $R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})$. Hence the cohomology sheaf in degree 0 is $F_{\mathcal{K}}(\mathcal{F})$. Thus we obtain the desired map $t : F(\mathcal{F}) \to F_{\mathcal{K}}(\mathcal{F})$ by contemplating the following commutative diagram

$$\begin{tikzcd} & F(\mathcal{F}) \arrow[r] \arrow[d, dotted] & \bigoplus F(j_{i_0 *}\mathcal{F}_{i_0}) \arrow[r] \arrow[d] & \bigoplus F(j_{i_0 i_1 *}\mathcal{F}_{i_0 i_1}) \arrow[d] \\ 0 \arrow[r] & F_{\mathcal{K}}(\mathcal{F}) \arrow[r] & \bigoplus \text{pr}_{i_0, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0} \arrow[r] & \bigoplus \text{pr}_{i_0 i_1, 2, *}(\text{pr}_1^*\mathcal{F} \otimes \mathcal{K})_{i_0 i_1} \end{tikzcd}$$

We obtain the top row by applying F to the (exact) complex $0 \to \mathcal{F} \to \bigoplus j_{i_0 *}\mathcal{F}_{i_0} \to \bigoplus j_{i_0 i_1 *}\mathcal{F}_{i_0 i_1}$ (but since F is not exact, the top row is just a complex and not necessarily exact). The solid vertical arrows are the identifications above. This does indeed define the dotted arrow as desired. The arrow is functorial in $\mathcal{F}$; we omit the details.

We still have to prove the final assertion. Let $f : X' \to X$ be as in the statement of the lemma and let $\mathcal{K}'$ be the quasi-coherent module on $X' \times_R Y$ constructed in the first paragraph of the proof. If the morphism $f : X' \to X$ maps into one of the opens U_i, then the result follows from Lemma 5.4 because in this case we know that $\mathcal{K}_i = \mathcal{K}|_{U_i \times_R Y}$ pulls back to $\mathcal{K}$. In general, we obtain an affine open covering $X' = \bigcup U_i'$ with $U_i' = f^{-1}(U_i)$ and we obtain isomorphisms $\mathcal{K}'|_{U_i'} = f_i^*\mathcal{K}_i$ where $f_i : U_i' \to U_i$ is the induced morphism. These morphisms satisfy the compatibility conditions needed to glue to an isomorphism $\mathcal{K}' = f^*\mathcal{K}$ and we conclude. Some details omitted. $\square$

Lemma 5.6. *In Lemma 5.2 or in Lemma 5.5 if F is an exact functor, then the corresponding object $\mathcal{K}$ of $QCoh(\mathcal{O}_{X \times_R Y})$ is flat over X.*

Proof. We may assume X is affine, so we are in the case of Lemma 5.2. By Lemma 5.4 we may assume Y is affine. In the affine case the statement translates into Remark 3.5. □

Lemma 5.7. *Let R be a ring. Let X and Y be schemes over R. Assume X is quasi-compact with affine diagonal. There is an equivalence of categories between*

1. *the category of R-linear exact functors $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which commute with arbitrary direct sums, and*
2. *the full subcategory of $QCoh(\mathcal{O}_{X\times_R Y})$ consisting of $\mathcal{K}$ such that*
   (a) *$\mathcal{K}$ is flat over X,*
   (b) *for $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have $R^q pr_{2,*}(pr_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K}) = 0$ for $q > 0$.*

given by sending $\mathcal{K}$ to the functor F in (5.1.1).

Proof. Let $\mathcal{K}$ be as in (2). The functor F in (5.1.1) commutes with direct sums. Since by (1) (a) the modules $\mathcal{K}$ is X-flat, we see that given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we obtain a short exact sequence

$$0 \to \mathrm{pr}_1^*\mathcal{F}_1 \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K} \to \mathrm{pr}_1^*\mathcal{F}_2 \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K} \to \mathrm{pr}_1^*\mathcal{F}_3 \otimes_{\mathcal{O}_{X\times_R Y}} \mathcal{K} \to 0$$

Since by (2)(b) the higher direct image $R^1\mathrm{pr}_{2,*}$ on the first term is zero, we conclude that $0 \to F(\mathcal{F}_1) \to F(\mathcal{F}_2) \to F(\mathcal{F}_3) \to 0$ is exact and we see that F is as in (1).

Let F be as in (1). Let $\mathcal{K}$ and $t : F \to F_{\mathcal{K}}$ be as in Lemma 5.5. By Lemma 5.6 we see that $\mathcal{K}$ is flat over X. To finish the proof we have to show that t is an isomorphism and the statement on higher direct images. Both of these follow from the fact that the relative Čech complex

$$\bigoplus \mathrm{pr}_{i_0,2,*}(\mathrm{pr}_1^*\mathcal{F}\otimes\mathcal{K})_{i_0} \to \bigoplus \mathrm{pr}_{i_0i_1,2,*}(\mathrm{pr}_1^*\mathcal{F}\otimes\mathcal{K})_{i_0i_1} \to \bigoplus \mathrm{pr}_{i_0i_1i_2,2,*}(\mathrm{pr}_1^*\mathcal{F}\otimes\mathcal{K})_{i_0i_1i_2} \to \ldots$$

computes $R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F}\otimes\mathcal{K})$. Please see proof of Lemma 5.5 for notation and for the reason why this is so. In the proof of Lemma 5.5 we also found that this complex is equal to F applied to the complex

$$\bigoplus j_{i_0*}\mathcal{F}_{i_0} \to \bigoplus j_{i_0i_1*}\mathcal{F}_{i_0i_1} \to \bigoplus j_{i_0i_1i_2*}\mathcal{F}_{i_0i_1i_2} \to \ldots$$

This complex is exact except in degree zero with cohomology sheaf equal to $\mathcal{F}$. Hence since F is an exact functor we conclude $F = F_{\mathcal{K}}$ and that (2)(b) holds.

We omit the proof that the construction that sends F to $\mathcal{K}$ is functorial and a quasi-inverse to the functor sending $\mathcal{K}$ to the functor $F_{\mathcal{K}}$ determined by (5.1.1). □

Remark 5.8. Let R be a ring. Let X and Y be schemes over R. Assume X is quasi-compact with affine diagonal. Lemma 5.7 may be generalized as follows: the functors (5.1.1) associated to quasi-coherent modules on $X \times_R Y$ are exactly those $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which have the following properties

1. F is R-linear and commutes with arbitrary direct sums,
2. $F \circ j_*$ is right exact when $j : U \to X$ is the inclusion of an affine open, and
3. $0 \to F(\mathcal{F}) \to F(\mathcal{G}) \to F(\mathcal{H})$ is exact whenever $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence such that for all $x \in X$ the sequence on stalks $0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$ is a split short exact sequence.

Namely, these assumptions are enough to get construct a transformation $t : F \to F_{\mathcal{K}}$ as in Lemma 5.5 and to show that it is an isomorphism. Moreover, properties (1), (2), and (3) do hold for functors (5.1.1). If we ever need this we will carefully state and prove this here.

Lemma 5.9. *Let R be a ring. Let X, Y, Z be schemes over R. Assume X and Y are quasi-compact and have affine diagonal. Let*

$$F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y) \quad \text{and} \quad G : QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_Z)$$

be R-linear exact functors which commute with arbitrary direct sums. Let $\mathcal{K}$ in $QCoh(\mathcal{O}_{X \times_R Y})$ and $\mathcal{L}$ in $QCoh(\mathcal{O}_{Y \times_R Z})$ be the corresponding "kernels", see Lemma 5.7. Then $G \circ F$ corresponds to $pr_{13,*}(pr_{12}^*\mathcal{K} \otimes_{\mathcal{O}_{X \times_R Y \times_R Z}} pr_{23}^*\mathcal{L})$ in $QCoh(\mathcal{O}_{X \times_R Z})$.

Proof. Since $G \circ F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ is R-linear, exact, and commutes with arbitrary direct sums, we find by Lemma 5.7 that there exists an $\mathcal{M}$ in $QCoh(\mathcal{O}_{X \times_R Z})$ corresponding to $G \circ F$. On the other hand, denote $\mathcal{E} = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*\mathcal{K} \otimes \mathrm{pr}_{23}^*\mathcal{L})$. Here and in the rest of the proof we omit the subscript from the tensor products. Let $U \subset X$ and $W \subset Z$ be affine open subschemes. To prove the lemma, we will construct an isomorphism

$$\Gamma(U \times_R W, \mathcal{E}) \cong \Gamma(U \times_R W, \mathcal{M})$$

compatible with restriction mappings for varying U and W.

First, we observe that

$$\Gamma(U \times_R W, \mathcal{E}) = \Gamma(U \times_R Y \times_R W, \mathrm{pr}_{12}^*\mathcal{K} \otimes \mathrm{pr}_{23}^*\mathcal{L})$$

by construction. Thus we have to show that the same thing is true for $\mathcal{M}$.

Write $U = \mathrm{Spec}(A)$ and denote $j : U \to X$ the inclusion morphism. Recall from the construction of $\mathcal{M}$ in the proof of Lemma 5.2 that

$$\Gamma(U \times_R W, \mathcal{M}) = \Gamma(W, G(F(j_*\mathcal{O}_U)))$$

where the A-module action on the right hand side is given by the action of A on $\mathcal{O}_U$. The correspondence between F and $\mathcal{K}$ tells us that $F(j_*\mathcal{O}_U) = b_*(a^*j_*\mathcal{O}_U \otimes \mathcal{K})$ where $a : X \times_R Y \to X$ and $b : X \times_R Y \to Y$ are the projection morphisms. Since j is an affine morphism, we have $a^*j_*\mathcal{O}_U = (j \times \mathrm{id}_Y)_*\mathcal{O}_{U \times_R Y}$ by Cohomology of Schemes, Lemma 5.1. Next, we have $(j \times \mathrm{id}_Y)_*\mathcal{O}_{U \times_R Y} \otimes \mathcal{K} = (j \times \mathrm{id}_Y)_*\mathcal{K}|_{U \times_R Y}$ by Remark 5.3 for example. Putting what we have found together we find

$$F(j_*\mathcal{O}_U) = (U \times_R Y \to Y)_*\mathcal{K}|_{U \times_R Y}$$

with obvious A-action. (This formula is implicit in the proof of Lemma 5.2.) Applying the functor G we obtain

$$G(F(j_*\mathcal{O}_U)) = t_*(s^*((U \times_R Y \to Y)_*\mathcal{K}|_{U \times_R Y}) \otimes \mathcal{L})$$

where $s : Y \times_R Z \to Y$ and $t : Y \times_R Z \to Z$ are the projection morphisms. Again using affine base change (Cohomology of Schemes, Lemma 5.1) but this time for the square

$$\begin{tikzcd} U \times_R Y \times_R Z \arrow[r] \arrow[d] & U \times_R Y \arrow[d] \\ Y \times_R Z \arrow[r] & Y \end{tikzcd}$$

we obtain

$$s^*((U \times_R Y \to Y)_*\mathcal{K}|_{U \times_R Y}) = (U \times_R Y \times_R Z \to Y \times_R Z)_*\mathrm{pr}_{12}^*\mathcal{K}|_{U \times_R Y \times_R Z}$$

Using Remark 5.3 again we find

$$\begin{aligned}&(U \times_R Y \times_R Z \to Y \times_R Z)_* \mathrm{pr}_{12}^*\mathcal{K}|_{U \times_R Y \times_R Z} \otimes \mathcal{L} \\ = {}& (U \times_R Y \times_R Z \to Y \times_R Z)_* \left(\mathrm{pr}_{12}^*\mathcal{K} \otimes \mathrm{pr}_{23}^*\mathcal{L}\right)|_{U \times_R Y \times_R Z}\end{aligned}$$

Applying the functor $\Gamma(W, t_*(-)) = \Gamma(Y \times_R W, -)$ to this we obtain

$$\begin{aligned}\Gamma(U \times_R W, \mathcal{M}) &= \Gamma(W, G(F(j_*\mathcal{O}_U))) \\ &= \Gamma(Y \times_R W, (U \times_R Y \times_R Z \to Y \times_R Z)_*(\mathrm{pr}_{12}^*\mathcal{K} \otimes \mathrm{pr}_{23}^*\mathcal{L})|_{U \times_R Y \times_R Z}) \\ &= \Gamma(U \times_R Y \times_R W, \mathrm{pr}_{12}^*\mathcal{K} \otimes \mathrm{pr}_{23}^*\mathcal{L})\end{aligned}$$

as desired. We omit the verication that these isomorphisms are compatible with restriction mappings. □

Lemma 5.10. *Let R, X, Y, and $\mathcal{K}$ be as in Lemma 5.7 part (2). Then for any scheme T over R we have*

$$R^q pr_{13,*}(pr_{12}^*\mathcal{F} \otimes_{\mathcal{O}_{T \times_R X \times_R Y}} pr_{23}^*\mathcal{K}) = 0$$

for $\mathcal{F}$ quasi-coherent on $T \times_R X$ and $q > 0$.

Proof. The question is local on T hence we may assume T is affine. In this case we can consider the diagram

$$\begin{tikzcd} T \times_R X \arrow[d] & T \times_R X \times_R Y \arrow[l] \arrow[r] \arrow[d] & T \times_R Y \arrow[d] \\ X & X \times_R Y \arrow[l] \arrow[r] & Y \end{tikzcd}$$

whose vertical arrows are affine. In particular the pushforward along $T \times_R Y \to Y$ is faithful and exact (Cohomology of Schemes, Lemma 2.3 and Morphisms, Lemma 11.6). Chasing around in the diagram using that higher direct images along affine morphisms vanish (see reference above) we see that it suffices to prove

$$R^q\mathrm{pr}_{2,*}(\mathrm{pr}_{23,*}(\mathrm{pr}_{12}^*\mathcal{F} \otimes_{\mathcal{O}_{T \times_R X \times_R Y}} \mathrm{pr}_{23}^*\mathcal{K})) = R^q\mathrm{pr}_{2,*}(\mathrm{pr}_{23,*}(\mathrm{pr}_{12}^*\mathcal{F}) \otimes_{\mathcal{O}_{X \times_R Y}} \mathcal{K}))$$

is zero which is true by assumption on $\mathcal{K}$. The equality holds by Remark 5.3. □

Lemma 5.11. *In Lemma 5.7 let F and $\mathcal{K}$ correspond. If X is separated and flat over R, then there is a surjection $\mathcal{O}_X \boxtimes F(\mathcal{O}_X) \to \mathcal{K}$.*

Proof. Let $\Delta : X \to X \times_R X$ be the diagonal morphism and set $\mathcal{O}_\Delta = \Delta_*\mathcal{O}_X$. Since Δ is a closed immersion have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X \times_R X} \to \mathcal{O}_\Delta \to 0$$

Since $\mathcal{K}$ is flat over X, the pullback $\mathrm{pr}_{23}^*\mathcal{K}$ to $X \times_R X \times_R Y$ is flat over $X \times_R X$. We obtain a short exact sequence

$$0 \to \mathrm{pr}_{12}^*\mathcal{I} \otimes \mathrm{pr}_{23}^*\mathcal{K} \to \mathrm{pr}_{23}^*\mathcal{K} \to \mathrm{pr}_{12}^*\mathcal{O}_\Delta \otimes \mathrm{pr}_{23}^*\mathcal{K} \to 0$$

on $X \times_R X \times_R Y$, see Modules, Lemma 20.4. Thus, by Lemma 5.10 we obtain a surjection

$$\mathrm{pr}_{13,*}(\mathrm{pr}_{23}^*\mathcal{K}) \to \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*\mathcal{O}_\Delta \otimes \mathrm{pr}_{23}^*\mathcal{K})$$

By flat base change (Cohomology of Schemes, Lemma 5.2) the source of this arrow is equal to $\mathrm{pr}_2^*\mathrm{pr}_{2,*}\mathcal{K} = \mathcal{O}_X \boxtimes F(\mathcal{O}_X)$. On the other hand the target is equal to

$$\mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*\mathcal{O}_\Delta \otimes \mathrm{pr}_{23}^*\mathcal{K}) = \mathrm{pr}_{13,*}(\Delta \times \mathrm{id}_Y)_*\mathcal{K} = \mathcal{K}$$

which finishes the proof. The first equality holds for example by Cohomology, Lemma 54.4 and the fact that $\mathrm{pr}_{12}^*\mathcal{O}_\Delta = (\Delta \times \mathrm{id}_Y)_*\mathcal{O}_{X \times_R Y}$. □

6. Gabriel-Rosenberg reconstruction

The title of this section refers to results like Proposition 6.6. Besides Gabriel's original paper [Gab62], please consult [Bra18] which has a proof of the result for quasi-separated schemes and discusses the literature. In this section we will only prove Gabriel-Rosenberg reconstruction for quasi-compact and quasi-separated schemes.

Lemma 6.1. *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is a categorically compact object of $QCoh(\mathcal{O}_X)$ if and only if $\mathcal{F}$ is of finite presentation.*

Proof. See Categories, Definition 26.1 for our notion of categorically compact objects in a category. If $\mathcal{F}$ is of finite presentation then it is categorically compact by Modules, Lemma 22.8. Conversely, any quasi-coherent module $\mathcal{F}$ can be written as a filtered colimit $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ of finitely presented (hence quasi-coherent) $\mathcal{O}_X$-modules, see Properties, Lemma 22.7. If $\mathcal{F}$ is categorically compact, then we find some i and a morphism $\mathcal{F} \to \mathcal{F}_i$ which is a right inverse to the given map $\mathcal{F}_i \to \mathcal{F}$. We conclude that $\mathcal{F}$ is a direct summand of a finitely presented module, and hence finitely presented itself. □

Lemma 6.2. *Let X be an affine scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{E}$ be a nonzero quasi-coherent $\mathcal{O}_X$-module. If $Supp(\mathcal{E}) \subset Supp(\mathcal{F})$, then there exists a nonzero map $\mathcal{F} \to \mathcal{E}$.*

Proof. Let us translate the statement into algebra. Let A be a ring. Let M be a finitely presented A-module. Let N be a nonzero A-module. Assume $\mathrm{Supp}(N) \subset \mathrm{Supp}(M)$. To show: $\mathrm{Hom}_A(M, N)$ is nonzero. We may assume $N = A/I$ is cyclic (replace N by any nonzero cyclic submodule). Choose a presentation

$$A^{\oplus m} \xrightarrow{T} A^{\oplus n} \to M \to 0$$

Recall that $\mathrm{Supp}(M)$ is cut out by $\mathrm{Fit}_0(M)$ which is the ideal generated by the $n \times n$ minors of the matrix T. See More on Algebra, Lemma 8.4. The assumption $\mathrm{Supp}(N) \subset \mathrm{Supp}(M)$ now means that the elements of $\mathrm{Fit}_0(M)$ are nilpotent in A/I. Consider the exact sequence

$$0 \to \mathrm{Hom}_A(M, A/I) \to (A/I)^{\oplus n} \xrightarrow{T^t} (A/I)^{\oplus m}$$

We have to show that T^t cannot be injective; we urge the reader to find their own proof of this using the nilpotency of elements of $\mathrm{Fit}_0(M)$ in A/I. Here is our proof. Since $\mathrm{Fit}_0(M)$ is finitely generated, the nilpotency means that the annihilator $J \subset A/I$ of $\mathrm{Fit}_0(M)$ in A/I is nonzero. To show the non-injectivity of T^t we may localize at a prime. Choosing a suitable prime we may assume A is local and J is still nonzero. Then T^t has a nonzero kernel by More on Algebra, Lemma 15.6. □

Lemma 6.3. *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. The following two subcategories of $QCoh(\mathcal{O}_X)$ are equal*

1. *the full subcategory $\mathcal{A} \subset QCoh(\mathcal{O}_X)$ whose objects are the quasi-coherent modules whose support is (set theoretically) contained in $Supp(\mathcal{F})$,*

(2) *the smallest Serre subcategory $\mathcal{B} \subset QCoh(\mathcal{O}_X)$ containing $\mathcal{F}$ closed under extensions and arbitrary direct sums.*

Proof. Observe that the statement makes sense as finitely presented $\mathcal{O}_X$-modules are quasi-coherent. Since $\mathcal{A}$ is a Serre subcategory closed under extensions and direct sums and since $\mathcal{F}$ is an object of $\mathcal{A}$ we see that $\mathcal{B} \subset \mathcal{A}$. Thus it remains to show that $\mathcal{A}$ is contained in $\mathcal{B}$.

Let $\mathcal{E}$ be an object of $\mathcal{A}$. There exists a maximal submodule $\mathcal{E}' \subset \mathcal{E}$ which is in $\mathcal{B}$. Namely, suppose $\mathcal{E}_i \subset \mathcal{E}$, $i \in I$ is the set of subobjects which are objects of $\mathcal{B}$. Then $\bigoplus \mathcal{E}_i$ is in $\mathcal{B}$ and so is

$$\mathcal{E}' = \mathrm{Im}(\bigoplus \mathcal{E}_i \longrightarrow \mathcal{E})$$

This is clearly the maximal submodule we were looking for.

Now suppose that we have a nonzero map $\mathcal{G} \to \mathcal{E}/\mathcal{E}'$ with $\mathcal{G}$ in $\mathcal{B}$. Then $\mathcal{G}' = \mathcal{E} \times_{\mathcal{E}/\mathcal{E}'} \mathcal{G}$ is in $\mathcal{B}$ as an extension of $\mathcal{E}'$ and $\mathcal{G}$. Then the image $\mathcal{G}' \to \mathcal{E}$ would be strictly bigger than $\mathcal{E}'$, contradicting the maximality of $\mathcal{E}'$. Thus it suffices to show the claim in the following paragraph.

Let $\mathcal{E}$ be an nonzero object of $\mathcal{A}$. We claim that there is a nonzero map $\mathcal{G} \to \mathcal{E}$ with $\mathcal{G}$ in $\mathcal{B}$. We will prove this by induction on the minimal number n of affine opens U_i of X such that $\mathrm{Supp}(\mathcal{E}) \subset U_1 \cup \ldots \cup U_n$. Set $U = U_n$ and denote $j : U \to X$ the inclusion morphism. Denote $\mathcal{E}' = \mathrm{Im}(\mathcal{E} \to j_*\mathcal{E}|_U)$. Then the kernel $\mathcal{E}''$ of the surjection $\mathcal{E} \to \mathcal{E}'$ has support contained in $U_1 \cup \ldots \cup U_{n-1}$. Thus if $\mathcal{E}''$ is nonzero, then we win. In other words, we may assume that $\mathcal{E} \subset j_*\mathcal{E}|_U$. In particular, we see that $\mathcal{E}|_U$ is nonzero. By Lemma 6.2 there exists a nonzero map $\mathcal{F}|_U \to \mathcal{E}|_U$. This corresponds to a map

$$\varphi : \mathcal{F} \longrightarrow j_*(\mathcal{E}|_U)$$

whose restriction to U is nonzero. Setting $\mathcal{G} = \varphi^{-1}(\mathcal{E})$ we conclude. $\square$

Lemma 6.4. *Let X be a quasi-compact and quasi-separated scheme. Let $Z \subset X$ be a closed subset such that $U = X \setminus Z$ is quasi-compact. Let $\mathcal{A} \subset QCoh(\mathcal{O}_X)$ be the full subcategory whose objects are the quasi-coherent modules supported on Z. Then the restriction functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ induces an equivalence $QCoh(\mathcal{O}_X)/\mathcal{A} \cong QCoh(\mathcal{O}_U)$.*

Proof. By the universal property of the quotient construction (Homology, Lemma 10.6) we certainly obtain an induced functor $QCoh(\mathcal{O}_X)/\mathcal{A} \cong QCoh(\mathcal{O}_U)$. Denote $j : U \to X$ the inclusion morphism. Since j is quasi-compact and quasi-separated we obtain a functor $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$. The reader shows that this defines a quasi-inverse; details omitted. $\square$

Lemma 6.5. *Let X be a quasi-compact and quasi-separated scheme. If $QCoh(\mathcal{O}_X)$ is equivalent to the category of modules over a ring, then X is affine.*

Proof. Say $F : \mathrm{Mod}_R \to QCoh(\mathcal{O}_X)$ is an equivalence. Then $\mathcal{F} = F(R)$ has the following properties:

(1) it is a finitely presented $\mathcal{O}_X$-module (Lemma 6.1),
(2) $\mathrm{Hom}_X(\mathcal{F}, -)$ is exact,
(3) $\mathrm{Hom}_X(\mathcal{F}, \mathcal{F})$ is a commutative ring,
(4) every object of $QCoh(\mathcal{O}_X)$ is a quotient of a direct sum of copies of $\mathcal{F}$.

Let $x \in X$ be a closed point. Consider the surjection

$$\mathcal{O}_X \to i_*\kappa(x)$$

where the target is the pushforward of $\kappa(x)$ by the inclusion morphism $i : x \to X$. We have

$$\mathrm{Hom}_X(\mathcal{F}, i_*\kappa(x)) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \kappa(x))$$

This first by (4) implies that $\mathcal{F}_x$ is nonzero. From (2) we deduce that every map $\mathcal{F}_x \to \kappa(x)$ lifts to a map $\mathcal{F}_x \to \mathcal{O}_{X,x}$ (as it even lifts to a global map $\mathcal{F} \to \mathcal{O}_X$). Since $\mathcal{F}_x$ is a finite $\mathcal{O}_{X,x}$-module, this implies that $\mathcal{F}_x$ is a (nonzero) finite free $\mathcal{O}_{X,x}$-module. Then since $\mathcal{F}$ is of finite presentation, this implies that $\mathcal{F}$ is finite free of positive rank in an open neighbourhood of x (Modules, Lemma 11.6). Since every closed subset of X contains a closed point (Topology, Lemma 12.8) this implies that $\mathcal{F}$ is finite locally free of positive rank. Similarly, the map

$$\mathrm{Hom}_X(\mathcal{F}, \mathcal{F}) \to \mathrm{Hom}_X(\mathcal{F}, i_*i^*\mathcal{F}) = \mathrm{Hom}_{\kappa(x)}(\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x, \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x)$$

is surjective. By property (3) we conclude that the rank $\mathcal{F}_x$ must be 1. Hence $\mathcal{F}$ is an invertible $\mathcal{O}_X$-module. But then we conclude that the functor

$$\mathcal{H} \longmapsto \Gamma(X, \mathcal{H}) = \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{H}) = \mathrm{Hom}_X(\mathcal{F}, \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{F})$$

on $QCoh(\mathcal{O}_X)$ is exact too. This implies that the first Ext group

$$\mathrm{Ext}^1_{QCoh(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{H}) = 0$$

computed in the abelian category $QCoh(\mathcal{O}_X)$ vanishes for all $\mathcal{H}$ in $QCoh(\mathcal{O}_X)$. However, since $QCoh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ is closed under extensions (Schemes, Section 24) we see that Ext^1 between quasi-coherent modules computed in $QCoh(\mathcal{O}_X)$ is the same as computed in $Mod(\mathcal{O}_X)$. Hence we conclude that

$$H^1(X, \mathcal{H}) = \mathrm{Ext}^1_{Mod(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{H}) = 0$$

for all $\mathcal{H}$ in $QCoh(\mathcal{O}_X)$. This implies that X is affine for example by Cohomology of Schemes, Lemma 3.1. $\square$

Proposition 6.6. *Let X and Y be quasi-compact and quasi-separated schemes. If $F : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ is an equivalence, then there exists an isomorphism $f : Y \to X$ of schemes and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$.*

Proof. Of course F is additive, exact, commutes with all limits, commutes with all colimits, commutes with direct sums, etc. Let $U \subset X$ be an affine open subscheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type quasi-coherent sheaf of ideals such that $Z = V(\mathcal{I})$ is the complement of U in X, see Properties, Lemma 24.1. Then $\mathcal{O}_X/\mathcal{I}$ is a finitely presented $\mathcal{O}_X$-module. Hence $\mathcal{G} = F(\mathcal{O}_X/\mathcal{I})$ is a finitely presented $\mathcal{O}_Y$-module by Lemma 6.1. Denote $T \subset Y$ the support of $\mathcal{G}$ and set $V = Y \setminus T$. Since $\mathcal{G}$ is of finite presentation, the scheme V is a quasi-compact open of Y. By Lemma 6.3 we see that F induces an equivalence between

(1) the full subcategory of $QCoh(\mathcal{O}_X)$ consisting of modules supported on Z, and
(2) the full subcategory of $QCoh(\mathcal{O}_Y)$ consisting of modules supported on T.

By Lemma 6.4 we obtain a commutative diagram

$$\begin{tikzcd} QCoh(\mathcal{O}_X) \arrow[r, "F"'] \arrow[d] & QCoh(\mathcal{O}_Y) \arrow[d] \\ QCoh(\mathcal{O}_U) \arrow[r, "F_U"] & QCoh(\mathcal{O}_V) \end{tikzcd}$$

where the vertical arrows are the restruction functors and the horizontal arrows are equivalences. By Lemma 6.5 we conclude that V is affine. For the affine case we have Lemma 3.8. Thus we find that there is an isomorphism $f_U : V \to U$ and an invertible $\mathcal{O}_V$-module $\mathcal{L}_U$ such that F_U is the functor $\mathcal{F} \mapsto f_U^*\mathcal{F} \otimes \mathcal{L}_U$.

The proof can be finished by noticing that the diagrams above satisfy an obvious compatibility with regards to inclusions of affine open subschemes of X. Thus the morphisms f_U and the invertible modules $\mathcal{L}_U$ glue. We omit the details. □

7. Functors between categories of coherent modules

The following lemma guarantees that we can use the material on functors between categories of quasi-coherent modules when we are given a functor between categories of coherent modules.

Lemma 7.1. *Let X and Y be Noetherian schemes. Let $F : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_Y)$ be a functor. Then F extends uniquely to a functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ which commutes with filtered colimits. If F is additive, then its extension commutes with arbitrary direct sums. If F is exact, left exact, or right exact, so is its extension.*

Proof. The existence and uniqueness of the extension is a general fact, see Categories, Lemma 26.2. To see that the lemma applies observe that coherent modules are of finite presentation (Modules, Lemma 12.2) and hence categorically compact objects of $Mod(\mathcal{O}_X)$ by Modules, Lemma 22.8. Finally, every quasi-coherent module is a filtered colimit of coherent ones for example by Properties, Lemma 22.3.

Assume F is additive. If $\mathcal{F} = \bigoplus_{j \in J} \mathcal{H}_j$ with $\mathcal{H}_j$ quasi-coherent, then $\mathcal{F} = \mathrm{colim}_{J' \subset J \text{ finite}} \bigoplus_{j \in J'} \mathcal{H}_j$. Denoting the extension of F also by F we obtain

$$\begin{aligned} F(\mathcal{F}) &= \mathrm{colim}_{J' \subset J \text{ finite}} F(\bigoplus\nolimits_{j \in J'} \mathcal{H}_j) \\ &= \mathrm{colim}_{J' \subset J \text{ finite}} \bigoplus\nolimits_{j \in J'} F(\mathcal{H}_j) \\ &= \bigoplus\nolimits_{j \in J} F(\mathcal{H}_j) \end{aligned}$$

Thus F commutes with arbitrary direct sums.

Suppose $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. Then we write $\mathcal{F}' = \bigcup \mathcal{F}'_i$ as the union of its coherent submodules, see Properties, Lemma 22.3. Denote $\mathcal{F}''_i \subset \mathcal{F}''$ the image of $\mathcal{F}'_i$ and denote $\mathcal{F}_i = \mathcal{F} \cap \mathcal{F}'_i = \mathrm{Ker}(\mathcal{F}'_i \to \mathcal{F}''_i)$. Then it is clear that $\mathcal{F} = \bigcup \mathcal{F}_i$ and $\mathcal{F}'' = \bigcup \mathcal{F}''_i$ and that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{F}'_i \to \mathcal{F}''_i \to 0$$

Since the extension commutes with filtered colimits we have $F(\mathcal{F}) = \mathrm{colim}_{i \in I} F(\mathcal{F}_i)$, $F(\mathcal{F}') = \mathrm{colim}_{i \in I} F(\mathcal{F}'_i)$, and $F(\mathcal{F}'') = \mathrm{colim}_{i \in I} F(\mathcal{F}''_i)$. Since filtered colimits are exact (Modules, Lemma 3.2) we conclude that exactness properties of F are inherited by its extension. $\square$

Lemma 7.2. *Let X and Y be Noetherian schemes. Let $F : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_Y)$ be an equivalence of categories. Then there is an isomorphism $f : Y \to X$ and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$.*

Proof. By Lemma 7.1 we obtain a unique functor $F' : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ extending F. The same is true for the quasi-inverse of F and by the uniqueness we conclude that F' is an equivalence. By Proposition 6.6 we find an isomorphism $f : Y \to X$ and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F'(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$. Then f and $\mathcal{L}$ work for F as well. $\square$

Remark 7.3. In Lemma 7.2 if X and Y are defined over a common base ring R and F is R-linear, then the isomorphism f will be a morphism of schemes over R.

Lemma 7.4. *Let $f : V \to X$ be a quasi-finite separated morphism of Noetherian schemes. If there exists a coherent $\mathcal{O}_V$-module $\mathcal{K}$ whose support is V such that $f_*\mathcal{K}$ is coherent and $R^qf_*\mathcal{K} = 0$, then f is finite.*

Proof. By Zariski's main theorem we can find an open immersion $j : V \to Y$ over X with $\pi : Y \to X$ finite, see More on Morphisms, Lemma 43.3. Since π is affine the functor π_* is exact and faithful on the category of coherent $\mathcal{O}_X$-modules. Hence we see that $j_*\mathcal{K}$ is coherent and that $R^qj_*\mathcal{K}$ is zero for $q > 0$. In other words, we reduce to the case discussed in the next paragraph.

Assume f is an open immersion. We may replace X by the scheme theoretic closure of V. Assume $X \setminus V$ is nonempty to get a contradiction. Choose a generic point $\xi \in X \setminus V$ of an irreducible component of $X \setminus V$. Looking at the situation after base change by $\mathrm{Spec}(\mathcal{O}_{X,\xi}) \to X$ using flat base change and using Local Cohomology, Lemma 8.2 we reduce to the algebra problem discussed in the next paragraph.

Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let M be a finite A-module whose support is $\mathrm{Spec}(A)$. Then $H^i_{\mathfrak{m}}(M) \neq 0$ for some i. This is true by Dualizing Complexes, Lemma 11.1 and the fact that M is not zero hence has finite depth. $\square$

The next lemma can be generalized to the case where k is a Noetherian ring and X flat over k (all other assumptions stay the same).

Lemma 7.5. *Let k be a field. Let X, Y be finite type schemes over k with X separated. There is an equivalence of categories between*

1. *the category of k-linear exact functors $F : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_Y)$, and*
2. *the category of coherent $\mathcal{O}_{X\times Y}$-modules $\mathcal{K}$ which are flat over X and have support finite over Y*

given by sending $\mathcal{K}$ to the restriction of the functor (5.1.1) to $Coh(\mathcal{O}_X)$.

Proof. Let $\mathcal{K}$ be as in (2). By Lemma 5.7 the functor F given by (5.1.1) is exact and k-linear. Moreover, F sends $Coh(\mathcal{O}_X)$ into $Coh(\mathcal{O}_Y)$ for example by Cohomology of Schemes, Lemma 26.10.

Let us construct the quasi-inverse to the construction. Let F be as in (1). By Lemma 7.1 we can extend F to a k-linear exact functor on the categories of quasi-coherent modules which commutes with arbitrary direct sums. By Lemma 5.7 the extension corresponds to a unique quasi-coherent module $\mathcal{K}$, flat over X, such that

$R^q\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{K}) = 0$ for $q > 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$. Since $F(\mathcal{O}_X)$ is a coherent $\mathcal{O}_Y$-module, we conclude from Lemma 5.11 that $\mathcal{K}$ is coherent.

For a closed point $x \in X$ denote $\mathcal{O}_x$ the skyscraper sheaf at x with value the residue field of x. We have

$$F(\mathcal{O}_x) = \mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{O}_x \otimes \mathcal{K}) = (x \times Y \to Y)_*(\mathcal{K}|_{x\times Y})$$

Since $x \times Y \to Y$ is finite, we see that the pushforward along this morphism is faithful. Hence if $y \in Y$ is in the image of the support of $\mathcal{K}|_{x\times Y}$, then y is in the support of $F(\mathcal{O}_x)$.

Let $Z \subset X \times Y$ be the scheme theoretic support Z of $\mathcal{K}$, see Morphisms, Definition 5.5. We first prove that $Z \to Y$ is quasi-finite, by proving that its fibres over closed points are finite. Namely, if the fibre of $Z \to Y$ over a closed point $y \in Y$ has dimension > 0, then we can find infinitely many pairwise distinct closed points $x_1, x_2, \ldots$ in the image of $Z_y \to X$. Since we have a surjection $\mathcal{O}_X \to \bigoplus_{i=1,\ldots,n} \mathcal{O}_{x_i}$ we obtain a surjection

$$F(\mathcal{O}_X) \to \bigoplus\nolimits_{i=1,\ldots,n} F(\mathcal{O}_{x_i})$$

By what we said above, the point y is in the support of each of the coherent modules $F(\mathcal{O}_{x_i})$. Since $F(\mathcal{O}_X)$ is a coherent module, this will lead to a contradiction because the stalk of $F(\mathcal{O}_X)$ at y will be generated by $< n$ elements if n is large enough. Hence $Z \to Y$ is quasi-finite. Since $\mathrm{pr}_{2,*}\mathcal{K}$ is coherent and $R^q\mathrm{pr}_{2,*}\mathcal{K} = 0$ for $q > 0$ we conclude that $Z \to Y$ is finite by Lemma 7.4. □

Lemma 7.6. *Let $f : X \to Y$ be a finite type separated morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent module on X with support finite over Y and with $\mathcal{L} = f_*\mathcal{F}$ an invertible $\mathcal{O}_X$-module. Then there exists a section $s : Y \to X$ such that $\mathcal{F} \cong s_*\mathcal{L}$.*

Proof. Looking affine locally this translates into the following algebra problem. Let $A \to B$ be a ring map and let N be a B-module which is invertible as an A-module. Then the annihilator J of N in B has the property that $A \to B/J$ is an isomorphism. We omit the details. □

Lemma 7.7. *Let $f : X \to Y$ be a finite type separated morphism of schemes with a section $s : Y \to X$. Let $\mathcal{F}$ be a finite type quasi-coherent module on X, set theoretically supported on $s(Y)$ with $\mathcal{L} = f_*\mathcal{F}$ an invertible $\mathcal{O}_X$-module. If Y is reduced, then $\mathcal{F} \cong s_*\mathcal{L}$.*

Proof. By Lemma 7.6 there exists a section $s' : Y \to X$ such that $\mathcal{F} = s'_*\mathcal{L}$. Since $s'(Y)$ and $s(Y)$ have the same underlying closed subset and since both are reduced closed subschemes of X, they have to be equal. Hence $s = s'$ and the lemma holds. □

Lemma 7.8. *Let k be a field. Let X, Y be finite type schemes over k with X separated and Y reduced. If there is a k-linear equivalence $F : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_Y)$ of categories, then there is an isomorphism $f : Y \to X$ over k and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ such that $F(\mathcal{F}) = f^*\mathcal{F} \otimes \mathcal{L}$.*

Proof using Gabriel-Rosenberg reconstruction. This lemma is a weak form of the results discussed in Lemma 7.2 and Remark 7.3. □

Proof not relying on Gabriel-Rosenberg reconstruction. By Lemma 7.5 we obtain a coherent $\mathcal{O}_{X\times Y}$-module $\mathcal{K}$ which is flat over X with support finite over Y such that F is given by the restriction of the functor (5.1.1) to $Coh(\mathcal{O}_X)$. If we can show that $F(\mathcal{O}_X)$ is an invertible $\mathcal{O}_Y$-module, then by Lemma 7.6 we see that $\mathcal{K} = s_*\mathcal{L}$ for some section $s : Y \to X \times Y$ of pr_2 and some invertible $\mathcal{O}_Y$-module $\mathcal{L}$. This will show that F has the form indicated with $f = \mathrm{pr}_1 \circ s$. Some details omitted.

It remains to show that $F(\mathcal{O}_X)$ is invertible. We only sketch the proof and we omit some of the details. For a closed point $x \in X$ we denote $\mathcal{O}_x$ in $Coh(\mathcal{O}_X)$ the skyscraper sheaf at x with value $\kappa(x)$. First we observe that the only simple objects of the category $Coh(\mathcal{O}_X)$ are these skyscraper sheaves $\mathcal{O}_x$. The same is true for Y. Hence for every closed point $y \in Y$ there exists a closed point $x \in X$ such that $\mathcal{O}_y \cong F(\mathcal{O}_x)$. Moreover, looking at endomorphisms we find that $\kappa(x) \cong \kappa(y)$ as finite extensions of k. Then

$$\mathrm{Hom}_Y(F(\mathcal{O}_X), \mathcal{O}_y) \cong \mathrm{Hom}_Y(F(\mathcal{O}_X), F(\mathcal{O}_x)) \cong \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_x) \cong \kappa(x) \cong \kappa(y)$$

This implies that the stalk of the coherent $\mathcal{O}_Y$-module $F(\mathcal{O}_X)$ at $y \in Y$ can be generated by 1 generator (and no less) for each closed point $y \in Y$. It follows immediately that $F(\mathcal{O}_X)$ is locally generated by 1 element (and no less) and since Y is reduced this indeed tells us it is an invertible module. $\square$

8. Other chapters

References

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