ALGEBRAIZATION OF FORMAL SPACES

Contents

1. Introduction

The main goal of this chapter is to prove Artin's theorem on dilatations, see Theorem 29.1; the result on contractions will be discussed in Artin's Axioms, Section 27. Both results use some material on formal algebraic spaces, hence in the middle part of this chapter, we continue the discussion of formal algebraic spaces from the previous chapter, see Formal Spaces, Section 1. The first part of this chapter is dedicated to algebraic preliminaries, mostly dealing with algebraization of rig-étale algebras.

Let A be a Noetherian ring and let $I \subset A$ be an ideal. In the first part of this chapter (Sections 2 – 10) we discuss the category of I-adically complete algebras B topologically of finite type over a Noetherian ring A. It is shown that $B = A\{x_1, \ldots, x_n\}/J$ for some (closed) ideal J in the restricted power series ring (where A is endowed with the I-adic topology). We show there is a good notion of a naive cotangent complex $NL^\wedge_{B/A}$. If some power of I annihilates $NL^\wedge_{B/A}$, then we say B is a rig-étale algebra over (A, I); there is a similar notion of rig-smooth algebras. If A is a G-ring, then we can show, using Popescu's theorem, that any rig-smooth algebra B over (A, I) is the completion of a finite type A-algebra; informally we say that we can "algebraize" B. However, the main result of the first part is that any rig-étale algebra B over (A, I) can be algebraized, see Lemma 10.2. One thing to note here is that we prove this without assuming the ring A is a G-ring.

Many of the results discussed in the first part can be found in the paper [Elk73]. Other general references for this part are [DG67], [Abb10], and [FK].

In the second part of this chapter (Sections 12 – 24) we talk about types of morphisms of formal algebraic spaces in a reasonable level of generality (mostly for locally Noetherian formal algebraic spaces). The most interesting of these is the notion of a "formal modification" in the last section. We carefully check that our definition agrees with Artin's definition in [Art70].

Finally, in the third and last part of this chapter (Sections 25 – 30) we prove the main theorem and we give a few applications. In fact, we deduce Artin's theorem from a stronger result, namely, Theorem 27.4. This theorem says very roughly: if $f : \mathfrak{X} \to \mathfrak{X}'$ is a rig-étale morphism and $\mathfrak{X}'$ is the formal completion of a locally Noetherian algebraic space, then so is $\mathfrak{X}$. In Artin's work the morphism f is assumed proper and rig-surjective.

2. Two categories

Let A be a ring and let $I \subset A$ be an ideal. In this section $^\wedge$ will mean I-adic completion. Set $A_n = A/I^n$ so that the I-adic completion of A is $A^\wedge = \lim A_n$. Let $\mathcal{C}$ be the category

$$\mathcal{C} = \left\{ \begin{matrix} \text{inverse systems } \ldots \to B_3 \to B_2 \to B_1 \\ \text{where } B_n \text{ is a finite type } A_n\text{-algebra,} \\ B_{n+1} \to B_n \text{ is an } A_{n+1}\text{-algebra map} \\ \text{which induces } B_{n+1}/I^n B_{n+1} \cong B_n \end{matrix} \right\} \tag{2.0.1}$$

Morphisms in $\mathcal{C}$ are given by systems of homomorphisms. Let $\mathcal{C}'$ be the category

$$\mathcal{C}' = \left\{ \begin{matrix} A\text{-algebras } B \text{ which are } I\text{-adically complete} \\ \text{such that } B/IB \text{ is of finite type over } A/I \end{matrix} \right\} \tag{2.0.2}$$

Morphisms in $\mathcal{C}'$ are A-algebra maps. There is a functor

$$\mathcal{C}' \longrightarrow \mathcal{C}, \quad B \longmapsto (B/I^n B) \tag{2.0.3}$$

Indeed, since B/IB is of finite type over A/I the ring maps $A_n = A/I^n \to B/I^n B$ are of finite type by Algebra, Lemma 126.8.

Lemma 2.1. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functor*

$$\mathcal{C} \longrightarrow \mathcal{C}', \quad (B_n) \longmapsto B = \lim B_n$$

is a quasi-inverse to (2.0.3). The completions $A[x_1, \ldots, x_r]^\wedge$ are in $\mathcal{C}'$ and any object of $\mathcal{C}'$ is of the form

$$B = A[x_1, \ldots, x_r]^\wedge/J$$

for some ideal $J \subset A[x_1, \ldots, x_r]^\wedge$.

Proof. Let (B_n) be an object of $\mathcal{C}$. By Algebra, Lemma 98.2 we see that $B = \lim B_n$ is I-adically complete and $B/I^nB = B_n$. Hence we see that B is an object of $\mathcal{C}'$ and that we can recover the object (B_n) by taking the quotients. Conversely, if B is an object of $\mathcal{C}'$, then $B = \lim B/I^nB$ by assumption. Thus $B \mapsto (B/I^nB)$ is a quasi-inverse to the functor of the lemma.

Since $A[x_1, \ldots, x_r]^\wedge = \lim A_n[x_1, \ldots, x_r]$ it is an object of $\mathcal{C}'$ by the first statement of the lemma. Finally, let B be an object of $\mathcal{C}'$. Choose $b_1, \ldots, b_r \in B$ whose images in B/IB generate B/IB as an algebra over A/I. Since B is I-adically complete, the A-algebra map $A[x_1, \ldots, x_r] \to B$, $x_i \mapsto b_i$ extends to an A-algebra map $A[x_1, \ldots, x_r]^\wedge \to B$. To finish the proof we have to show this map is surjective which follows from Algebra, Lemma 96.1 as our map $A[x_1, \ldots, x_r] \to B$ is surjective modulo I and as $B = B^\wedge$. $\square$

We warn the reader that, in case A is not Noetherian, the quotient of an object of $\mathcal{C}'$ may not be an object of $\mathcal{C}'$. See Examples, Lemma 8.1. Next we show this does not happen when A is Noetherian.

Lemma 2.2. *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Then*

(1) *every object of the category $\mathcal{C}'$ (2.0.2) is Noetherian,*
(2) *if $B \in \mathrm{Ob}(\mathcal{C}')$ and $J \subset B$ is an ideal, then B/J is an object of $\mathcal{C}'$,*
(3) *for a finite type A-algebra C the I-adic completion $C^\wedge$ is in $\mathcal{C}'$,*
(4) *in particular the completion $A[x_1, \ldots, x_r]^\wedge$ is in $\mathcal{C}'$.*

Proof. Part (4) follows from Algebra, Lemma 97.6 as $A[x_1, \ldots, x_r]$ is Noetherian (Algebra, Lemma 31.1). To see (1) by Lemma 2.1 we reduce to the case of the completion of the polynomial ring which we just proved. Part (2) follows from Algebra, Lemma 97.1 which tells us that ever finite B-module is IB-adically complete. Part (3) follows in the same manner as part (4). $\square$

Remark 2.3 (Base change)**.** Let $\varphi : A_1 \to A_2$ be a ring map and let $I_i \subset A_i$ be ideals such that $\varphi(I_1^c) \subset I_2$ for some $c \geq 1$. This induces ring maps $A_{1,cn} = A_1/I_1^{cn} \to A_2/I_2^n = A_{2,n}$ for all $n \geq 1$. Let $\mathcal{C}_i$ be the category (2.0.1) for (A_i, I_i). There is a base change functor

$$\mathcal{C}_1 \longrightarrow \mathcal{C}_2, \quad (B_n) \longmapsto (B_{cn} \otimes_{A_{1,cn}} A_{2,n}) \tag{2.3.1}$$

Let $\mathcal{C}_i'$ be the category (2.0.2) for (A_i, I_i). If I_2 is finitely generated, then there is a base change functor

$$\mathcal{C}_1' \longrightarrow \mathcal{C}_2', \quad B \longmapsto (B \otimes_{A_1} A_2)^\wedge \tag{2.3.2}$$

because in this case the completion is complete (Algebra, Lemma 96.3). If both I_1 and I_2 are finitely generated, then the two base change functors agree via the functors (2.0.3) which are equivalences by Lemma 2.1.

Remark 2.4 (Base change by closed immersion)**.** Let A be a Noetherian ring and $I \subset A$ an ideal. Let $\mathfrak{a} \subset A$ be an ideal. Denote $\bar{A} = A/\mathfrak{a}$. Let $\bar{I} \subset \bar{A}$ be an ideal such that $I^c\bar{A} \subset \bar{I}$ and $\bar{I}^d \subset I\bar{A}$ for some $c, d \geq 1$. In this case the base change functor (2.3.2) for (A, I) to $(\bar{A}, \bar{I})$ is given by $B \mapsto \bar{B} = B/\mathfrak{a}B$. Namely, we have

$$\bar{B} = (B \otimes_A \bar{A})^\wedge = (B/\mathfrak{a}B)^\wedge = B/\mathfrak{a}B \tag{2.4.1}$$

the last equality because any finite B-module is I-adically complete by Algebra, Lemma 97.1 and if annihilated by $\mathfrak{a}$ also $\bar{I}$-adically complete by Algebra, Lemma 96.9.

3. A naive cotangent complex

Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let B be an A-algebra which is I-adically complete such that $A/I \to B/IB$ is of finite type, i.e., an object of (2.0.2). By Lemma 2.2 we can write

$$B = A[x_1, \ldots, x_r]^\wedge/J$$

for some finitely generated ideal J. For a choice of presentation as above we define the *naive cotangent complex* in this setting by the formula

$$NL^\wedge_{B/A} = (J/J^2 \longrightarrow \bigoplus B\mathrm{d}x_i) \tag{3.0.1}$$

with terms sitting in degrees -1 and 0 where the map sends the residue class of $g \in J$ to the differential $\mathrm{d}g = \sum(\partial g/\partial x_i)\mathrm{d}x_i$. Here the partial derivative is taken by thinking of g as a power series. The following lemma shows that $NL^\wedge_{B/A}$ is well defined up to homotopy.

Lemma 3.1. *Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let B be an object of (2.0.2). The naive cotangent complex $NL^\wedge_{B/A}$ is well defined in $K(B)$.*

Proof. The lemma signifies that given a second presentation $B = A[y_1, \ldots, y_s]^\wedge/K$ the complexes of B-modules

$$(J/J^2 \to B\mathrm{d}x_i) \quad \text{and} \quad (K/K^2 \to \bigoplus B\mathrm{d}y_j)$$

are homotopy equivalent. To see this, we can argue exactly as in the proof of Algebra, Lemma 134.2.

Step 1. If we choose $g_i(y_1, \ldots, y_s) \in A[y_1, \ldots, y_s]^\wedge$ mapping to the image of x_i in B, then we obtain a (unique) continuous A-algebra homomorphism

$$A[x_1, \ldots, x_r]^\wedge \to A[y_1, \ldots, y_s]^\wedge, \quad x_i \mapsto g_i(y_1, \ldots, y_s)$$

compatible with the given surjections to B. Such a map is called a morphism of presentations. It induces a map from J into K and hence induces a B-module map $J/J^2 \to K/K^2$. Sending $\mathrm{d}x_i$ to $\sum(\partial g_i/\partial y_j)\mathrm{d}y_j$ we obtain a map of complexes

$$(J/J^2 \to \bigoplus B\mathrm{d}x_i) \longrightarrow (K/K^2 \to \bigoplus B\mathrm{d}y_j)$$

Of course we can do the same thing with the roles of the two presentations exchanged to get a map of complexes in the other direction.

Step 2. The construction above is compatible with compositions of morphsms of presentations. Hence to finish the proof it suffices to show: given $g_i(x_1, \ldots, x_r) \in A[x_1, \ldots, x_n]^\wedge$ mapping to the image of x_i in B, the induced map of complexes

$$(J/J^2 \to \bigoplus B\mathrm{d}x_i) \longrightarrow (J/J^2 \to \bigoplus B\mathrm{d}x_i)$$

is homotopic to the identity map. To see this consider the map $h : \bigoplus B\mathrm{d}x_i \to J/J^2$ given by the rule $\mathrm{d}x_i \mapsto g_i(x_1, \ldots, x_n) - x_i$ and compute. $\square$

Lemma 3.2. *Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let $A \to B$ be a finite type ring map. Choose a presentation $\alpha : A[x_1, \ldots, x_n] \to B$. Then $NL^\wedge_{B^\wedge/A} = \lim NL(\alpha) \otimes_B B^\wedge$ as complexes and $NL^\wedge_{B^\wedge/A} = NL_{B/A} \otimes_B^{\mathbf{L}} B^\wedge$ in $D(B^\wedge)$.*

Proof. The statement makes sense as $B^\wedge$ is an object of (2.0.2) by Lemma 2.2. Let $J = \mathrm{Ker}(\alpha)$. The functor of taking I-adic completion is exact on finite modules over $A[x_1, \ldots, x_n]$ and agrees with the functor $M \mapsto M \otimes_{A[x_1,\ldots,x_n]} A[x_1, \ldots, x_n]^\wedge$, see Algebra, Lemmas 97.1 and 97.2. Moreover, the ring maps $A[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]^\wedge$ and $B \to B^\wedge$ are flat. Hence $B^\wedge = A[x_1, \ldots, x_n]^\wedge/J^\wedge$ and

$$(J/J^2) \otimes_B B^\wedge = (J/J^2)^\wedge = J^\wedge/(J^\wedge)^2$$

Since $NL(\alpha) = (J/J^2 \to \bigoplus B\mathrm{d}x_i)$, see Algebra, Section 134, we conclude the complex $NL^\wedge_{B^\wedge/A}$ is equal to $NL(\alpha) \otimes_B B^\wedge$. The final statement follows as $NL_{B/A}$ is homotopy equivalent to $NL(\alpha)$ and because the ring map $B \to B^\wedge$ is flat (so derived base change along $B \to B^\wedge$ is just base change). $\square$

Lemma 3.3. *Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let B be an object of (2.0.2). Then*

1. *the pro-objects $\{NL^\wedge_{B/A} \otimes_B B/I^nB\}$ and $\{NL_{B_n/A_n}\}$ of $D(B)$ are strictly isomorphic (see proof for elucidation),*
2. *$NL^\wedge_{B/A} = R\lim NL_{B_n/A_n}$ in $D(B)$.*

Here B_n and A_n are as in Section 2.

Proof. The statement means the following: for every n we have a well defined complex NL_{B_n/A_n} of B_n-modules and we have transition maps $NL_{B_{n+1}/A_{n+1}} \to NL_{B_n/A_n}$. See Algebra, Section 134. Thus we can consider

$$\ldots \to NL_{B_3/A_3} \to NL_{B_2/A_2} \to NL_{B_1/A_1}$$

as an inverse system of complexes of B-modules and a fortiori as an inverse system in $D(B)$. Furthermore $R\lim NL_{B_n/A_n}$ is a homotopy limit of this inverse system, see Derived Categories, Section 34.

Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. This defines presentations

$$B_n = B/I^nB = A_n[x_1, \ldots, x_r]/J_n$$

where

$$J_n = JA_n[x_1, \ldots, x_r] = J/(J \cap I^nA[x_1, \ldots, x_r]^\wedge)$$

The two term complex $J_n/J_n^2 \longrightarrow \bigoplus B_n\mathrm{d}x_i$ represents NL_{B_n/A_n}, see Algebra, Section 134. By Artin-Rees (Algebra, Lemma 51.2) in the Noetherian ring $A[x_1, \ldots, x_r]^\wedge$ (Lemma 2.2) we find a $c \geq 0$ such that we have canonical surjections

$$J/I^nJ \to J_n \to J/I^{n-c}J \to J_{n-c}, \quad n \geq c$$

for all $n \geq c$. A moment's thought shows that these maps are compatible with differentials and we obtain maps of complexes

$$NL^\wedge_{B/A} \otimes_B B/I^nB \to NL_{B_n/A_n} \to NL^\wedge_{B/A} \otimes_B B/I^{n-c}B \to NL_{B_{n-c}/A_{n-c}}$$

compatible with the transition maps of the inverse systems $\{NL^\wedge_{B/A} \otimes_B B/I^nB\}$ and $\{NL_{B_n/A_n}\}$. This proves part (1) of the lemma.

By part (1) and since pro-isomorphic systems have the same $R\lim$ in order to prove (2) it suffices to show that $NL^\wedge_{B/A}$ is equal to $R\lim NL^\wedge_{B/A} \otimes_B B/I^nB$. However, $NL^\wedge_{B/A}$ is a two term complex $M^\bullet$ of finite B-modules which are I-adically complete for example by Algebra, Lemma 97.1. Hence $M^\bullet = \lim M^\bullet/I^nM^\bullet = R\lim M^\bullet/I^nM^\bullet$, see More on Algebra, Lemma 87.1 and Remark 87.6. $\square$

Lemma 3.4. *Let $(A_1, I_1) \to (A_2, I_2)$ be as in Remark 2.3 with A_1 and A_2 Noetherian. Let B_1 be in (2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1. Then there is a canonical map*

$$NL_{B_1/A_1} \otimes_{B_2} B_1 \to NL_{B_2/A_2}$$

which induces and isomorphism on H^0 and a surjection on H^{-1}.

Proof. Choose a presentation $B_1 = A_1[x_1, \ldots, x_r]^\wedge/J_1$. Since $A_2/I_2^n[x_1, \ldots, x_r] = A_1/I_1^{cn}[x_1, \ldots, x_r] \otimes_{A_1/I_1^{cn}} A_2/I_2^n$ we have

$$A_2[x_1, \ldots, x_r]^\wedge = (A_1[x_1, \ldots, x_r]^\wedge \otimes_{A_1} A_2)^\wedge$$

where we use I_2-adic completion on both sides (but of course I_1-adic completion for $A_1[x_1, \ldots, x_r]^\wedge$). Set $J_2 = J_1A_2[x_1, \ldots, x_r]^\wedge$. Arguing similarly we get the presentation

$$\begin{aligned}
B_2 &= (B_1 \otimes_{A_1} A_2)^\wedge \\
&= \lim \frac{A_1/I_1^{cn}[x_1, \ldots, x_r]}{J_1(A_1/I_1^{cn}[x_1, \ldots, x_r])} \otimes_{A_1/I_1^{cn}} A_2/I_2^n \\
&= \lim \frac{A_2/I_2^n[x_1, \ldots, x_r]}{J_2(A_2/I_2^n[x_1, \ldots, x_r])} \\
&= A_2[x_1, \ldots, x_r]^\wedge/J_2
\end{aligned}$$

for B_2 over A_2. As a consequence obtain a commutative diagram

$$\begin{tikzcd}
NL^\wedge_{B_1/A_1} : \arrow[d] & J_1/J_1^2 \arrow[r, "\mathrm{d}"'] \arrow[d] & \bigoplus B_1 \mathrm{d}x_i \arrow[d] \\
NL^\wedge_{B_2/A_2} : & J_2/J_2^2 \arrow[r, "\mathrm{d}"] & \bigoplus B_2 \mathrm{d}x_i
\end{tikzcd}$$

The induced arrow $J_1/J_1^2 \otimes_{B_1} B_2 \to J_2/J_2^2$ is surjective because J_2 is generated by the image of J_1. This determines the arrow displayed in the lemma. We omit the proof that this arrow is well defined up to homotopy (i.e., indepedent of the choice of the presentations up to homotopy). The statement about the induced map on cohomology modules follows easily from the discussion (details omitted). $\square$

Lemma 3.5. *Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let $B \to C$ be morphism of (2.0.2). Then there is an exact sequence*

$$\begin{tikzcd}
C \otimes_B H^0(NL^\wedge_{B/A}) \arrow[r] & H^0(NL^\wedge_{C/A}) \arrow[r] & H^0(NL^\wedge_{C/B}) \arrow[r] & 0 \\
H^{-1}(NL^\wedge_{B/A} \otimes_B C) \arrow[r] & H^{-1}(NL^\wedge_{C/A}) \arrow[r] & H^{-1}(NL^\wedge_{C/B}) \arrow[llu]
\end{tikzcd}$$

See proof for elucidation.

Proof. Observe that taking the tensor product $NL^\wedge_{B/A} \otimes_B C$ makes sense as $NL^\wedge_{B/A}$ is well defined up to homotopy by Lemma 3.1. Also, (B, IB) is pair where B is a Noetherian ring (Lemma 2.2) and C is in the corresponding category (2.0.2). Thus all the terms in the 6-term sequence are (well) defined.

Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Choose a presentation $C = B[y_1, \ldots, y_s]^\wedge/J'$. Combinging these presentations gives a presentation

$$C = A[x_1, \ldots, x_r, y_1, \ldots, y_s]^\wedge/K$$

Then the reader verifies that we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus C\mathrm{d}x_i \arrow[r] & \bigoplus C\mathrm{d}x_i \oplus \bigoplus C\mathrm{d}y_j \arrow[r] & \bigoplus C\mathrm{d}y_j \arrow[r] & 0 \\ & J/J^2 \otimes_B C \arrow[r] \arrow[u] & K/K^2 \arrow[r] \arrow[u] & J'/(J')^2 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. Note that the vertical arrow on the left hand side is the tensor product of the arrow defining $NL^\wedge_{B/A}$ with id_C. The lemma follows by applying the snake lemma (Algebra, Lemma 4.1). □

Lemma 3.6. *With assumptions as in Lemma 3.5 assume that $B/I^nB \to C/I^nC$ is a local complete intersection homomorphism for all n. Then $H^{-1}(NL^\wedge_{B/A} \otimes_B C) \to H^{-1}(NL^\wedge_{C/A})$ is injective.*

Proof. For each $n \geq 1$ we set $A_n = A/I^n$, $B_n = B/I^nB$, and $C_n = C/I^nC$. We have

$$\begin{aligned} H^{-1}(NL^\wedge_{B/A} \otimes_B C) &= \lim H^{-1}(NL^\wedge_{B/A} \otimes_B C_n) \\ &= \lim H^{-1}(NL^\wedge_{B/A} \otimes_B B_n \otimes_{B_n} C_n) \\ &= \lim H^{-1}(NL_{B_n/A_n} \otimes_{B_n} C_n) \end{aligned}$$

The first equality follows from More on Algebra, Lemma 100.1 and the fact that $H^{-1}(NL^\wedge_{B/A} \otimes_B C)$ is a finite C-module and hence I-adically complete for example by Algebra, Lemma 97.1. The second equality is trivial. The third holds by Lemma 3.3. The maps $H^{-1}(NL_{B_n/A_n} \otimes_{B_n} C_n) \to H^{-1}(NL_{C_n/A_n})$ are injective by More on Algebra, Lemma 33.6. The proof is finished because we also have $H^{-1}(NL^\wedge_{C/A}) = \lim H^{-1}(NL_{C_n/A_n})$ similarly to the above. □

4. Rig-smooth algebras

As motivation for the following definition, please take a look at More on Algebra, Remark 84.2.

Definition 4.1. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (2.0.2). We say B is *rig-smooth over* (A, I) if there exists an integer $c \geq 0$ such that I^c annihilates $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ for every B-module N.

Let us work out what this means.

Lemma 4.2. *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (2.0.2). Write $B = A[x_1, \ldots, x_r]^\wedge/J$ (Lemma 2.2) and let $NL^\wedge_{B/A} = (J/J^2 \to \bigoplus B dx_i)$ be its naive cotangent complex (3.0.1). The following are equivalent*

(1) *B is rig-smooth over (A, I),*
(2) *the object $NL^\wedge_{B/A}$ of $D(B)$ satisfies the equivalent conditions (1) – (4) of More on Algebra, Lemma 84.10 with respect to the ideal IB,*
(3) *there exists a $c \geq 0$ such that for all $a \in I^c$ there is a map $h : \bigoplus B dx_i \to J/J^2$ such that $a : J/J^2 \to J/J^2$ is equal to $h \circ d$,*
(4) *there exist $b_1, \ldots, b_s \in B$ such that $V(b_1, \ldots, b_s) \subset V(IB)$ and such that for every $l = 1, \ldots, s$ there exist $m \geq 0$, $f_1, \ldots, f_m \in J$, and subset $T \subset \{1, \ldots, n\}$ with $|T| = m$ such that*
   (a) *$\det_{i \in T, j \leq m}(\partial f_j / \partial x_i)$ divides b_l in B, and*
   (b) *$b_l J \subset (f_1, \ldots, f_m) + J^2$.*

Proof. The equivalence of (1), (2), and (3) is immediate from More on Algebra, Lemma 84.10.

Assume $b_1, \ldots, b_s$ are as in (4). Since B is Noetherian the inclusion $V(b_1, \ldots, b_s) \subset V(IB)$ implies $I^c B \subset (b_1, \ldots, b_s)$ for some $c \geq 0$ (for example by Algebra, Lemma 62.4). Pick $1 \leq l \leq s$ and $m \geq 0$ and $f_1, \ldots, f_m \in J$ and $T \subset \{1, \ldots, n\}$ with $|T| = m$ satisfying (4)(a) and (b). Then if we invert b_l we see that

$$NL^\wedge_{B/A} \otimes_B B_{b_l} = \left(\bigoplus\nolimits_{j \leq m} B_{b_l} f_j \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} B_{b_l} \mathrm{d}x_i \right)$$

and moreover the arrow is isomorphic to the inclusion of the direct summand $\bigoplus_{i \in T} B_{b_l} \mathrm{d}x_i$. We conclude that $H^{-1}(NL^\wedge_{B/A})$ is b_l-power torsion and that $H^0(NL^\wedge_{B/A})$ becomes finite free after inverting b_l. Combined with the inclusion $I^c B \subset (b_1, \ldots, b_s)$ we see that $H^{-1}(NL^\wedge_{B/A})$ is IB-power torsion. Hence we see that condition (4) of More on Algebra, Lemma 84.10 holds. In this way we see that (4) implies (2).

Assume the equivalent conditions (1), (2), and (3) hold. We will prove that (4) holds, but we strongly urge the reader to convince themselves of this. The complex $NL^\wedge_{B/A}$ determines an object of $D^b_{Coh}(\mathrm{Spec}(B))$ whose restriction to the Zariski open $U = \mathrm{Spec}(B) \setminus V(IB)$ is a finite locally free module $\mathcal{E}$ placed in degree 0 (this follows for example from the the fourth equivalent condition in More on Algebra, Lemma 84.10). Choose generators $f_1, \ldots, f_M$ for J. This determines an exact sequence

$$\bigoplus\nolimits_{j=1,\ldots,M} \mathcal{O}_U \cdot f_j \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \cdot \mathrm{d}x_i \to \mathcal{E} \to 0$$

Let $U = \bigcup_{l=1,\ldots,s} U_l$ be a finite affine open covering such that $\mathcal{E}|_{U_l}$ is free of rank $r_l = n - m_l$ for some integer $n \geq m_l \geq 0$. After replacing each U_l by an affine open covering we may assume there exists a subset $T_l \subset \{1, \ldots, n\}$ such that the elements $\mathrm{d}x_i$, $i \in \{1, \ldots, n\} \setminus T_l$ map to a basis for $\mathcal{E}|_{U_l}$. Repeating the argument, we may assume there exists a subset $T'_l \subset \{1, \ldots, M\}$ of cardinality m_l such that f_j, $j \in T'_l$ map to a basis of the kernel of $\mathcal{O}_{U_l} \cdot \mathrm{d}x_i \to \mathcal{E}|_{U_l}$. Finally, since the open covering $U = \bigcup U_l$ may be refined by a open covering by standard opens (Algebra, Lemma 17.2) we may assume $U_l = D(g_l)$ for some $g_l \in B$. In particular we have $V(g_1, \ldots, g_s) = V(IB)$. A linear algebra argument using our choices above shows that $\det_{i \in T_l, j \in T'_l}(\partial f_j / \partial x_i)$ maps to an invertible element of B_{b_l}. Similarly, the vanishing of cohomology of $NL^\wedge_{B/A}$ in degree -1 over U_l shows that $J/J^2 + (f_j; j \in T')$ is annihilated by a power of b_l. After replacing each g_l by a suitable power we obtain conditions (4)(a) and (4)(b) of the lemma. Some details omitted. $\square$

Lemma 4.3. *Let A be a Noetherian ring and let I be an ideal. Let B be a finite type A-algebra.*

(1) *If* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is smooth over* $\mathrm{Spec}(A) \setminus V(I)$*, then* $B^\wedge$ *is rig-smooth over* (A, I)*.*
(2) *If* $B^\wedge$ *is rig-smooth over* (A, I)*, then there exists* $g \in 1 + IB$ *such that* $\mathrm{Spec}(B_g)$ *is smooth over* $\mathrm{Spec}(A) \setminus V(I)$*.*

Proof. We will use Lemma 4.2 without further mention.

Assume (1). Recall that formation of $NL_{B/A}$ commutes with localization, see Algebra, Lemma 134.13. Hence by the very definition of smooth ring maps (in terms of the naive cotangent complex being quasi-isomorphic to a finite projective module placed in degree 0), we see that $NL_{B/A}$ satisfies the fourth equivalent condition of More on Algebra, Lemma 84.10 with respect to the ideal IB (small detail omitted). Since $NL^\wedge_{B^\wedge/A} = NL_{B/A} \otimes_B B^\wedge$ by Lemma 3.2 we conclude (2) holds by More on Algebra, Lemma 84.7.

Assume (2). Choose a presentation $B = A[x_1, \ldots, x_n]/J$, set $N = J/J^2$, and consider the element $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, J/J^2)$ determined by the identity map on J/J^2. Using again that $NL^\wedge_{B^\wedge/A} = NL_{B/A} \otimes_B B^\wedge$ we find that our assumption implies the image

$$\xi \otimes 1 \in \mathrm{Ext}^1_{B^\wedge}(NL_{B/A} \otimes_B B^\wedge, N \otimes_B B^\wedge) = \mathrm{Ext}^1_{B^\wedge}(NL_{B/A}, N) \otimes_B B^\wedge$$

is annihilated by I^c for some integer $c \geq 0$. The equality holds for example by More on Algebra, Lemma 99.2 (but can also easily be deduced from the much simpler More on Algebra, Lemma 65.4). Thus $M = I^c B\xi \subset \mathrm{Ext}^1_B(NL_{B/A}, N)$ is a finite submodule which maps to zero in $\mathrm{Ext}^1_B(NL_{B/A}, N) \otimes_B B^\wedge$. Since $B \to B^\wedge$ is flat this means that $M \otimes_B B^\wedge$ is zero. By Nakayama's lemma (Algebra, Lemma 20.1) this means that $M = I^c B\xi$ is annihilated by an element of the form $g = 1 + x$ with $x \in IB$. This implies that for every $b \in I^c B$ there is a B-linear dotted arrow making the diagram commute

$$\begin{tikzcd} J/J^2 \arrow[r] \arrow[d, "b"] & \bigoplus B\mathrm{d}x_i \arrow[d, dotted, "h"] \\ J/J^2 \arrow[r] & (J/J^2)_g \end{tikzcd}$$

Thus $(NL_{B/A})_{gb}$ is quasi-isomorphic to a finite projective module; small detail omitted. Since $(NL_{B/A})_{gb} = NL_{B_{gb}/A}$ in $D(B_{gb})$ this shows that B_{gb} is smooth over $\mathrm{Spec}(A)$. As this holds for all $b \in I^c B$ we conclude that $\mathrm{Spec}(B_g) \to \mathrm{Spec}(A)$ is smooth over $\mathrm{Spec}(A) \setminus V(I)$ as desired. □

Lemma 4.4. *Let* $(A_1, I_1) \to (A_2, I_2)$ *be as in Remark 2.3 with* A_1 *and* A_2 *Noetherian. Let* B_1 *be in (2.0.2) for* (A_1, I_1)*. Let* B_2 *be the base change of* B_1*. Let* $f_1 \in B_1$ *with image* $f_2 \in B_2$*. If* $\mathrm{Ext}^1_{B_1}(NL^\wedge_{B_1/A_1}, N_1)$ *is annihilated by* f_1 *for every* B_1*-module* N_1*, then* $\mathrm{Ext}^1_{B_2}(NL^\wedge_{B_2/A_2}, N_2)$ *is annihilated by* f_2 *for every* B_2*-module* N_2*.*

Proof. By Lemma 3.4 there is a map

$$NL_{B_1/A_1} \otimes_{B_2} B_1 \to NL_{B_2/A_2}$$

which induces and isomorphism on H^0 and a surjection on H^{-1}. Thus the result by More on Algebra, Lemmas 84.6, 84.7, and 84.9 the last two applied with the principal ideals $(f_1) \subset B_1$ and $(f_2) \subset B_2$. □

Lemma 4.5. *Let $A_1 \to A_2$ be a map of Noetherian rings. Let $I_i \subset A_i$ be an ideal such that $V(I_1A_2) = V(I_2)$. Let B_1 be in (2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1 as in Remark 2.3. If B_1 is rig-smooth over (A_1, I_1), then B_2 is rig-smooth over (A_2, I_2).*

Proof. Follows from Lemma 4.4 and Definition 4.1 and the fact that I_2^c is contained in I_1A_2 for some $c \geq 0$ as A_2 is Noetherian. □

5. Deformations of ring homomorphisms

Some work on lifting ring homomorphisms from rig-smooth algebras.

Remark 5.1 (Linear approximation)**.** Let A be a ring and $I \subset A$ be a finitely generated ideal. Let C be an I-adically complete A-algebra. Let $\psi : A[x_1, \ldots, x_r]^\wedge \to C$ be a continuous A-algebra map. Suppose given $\delta_i \in C$, $i = 1, \ldots, r$. Then we can consider

$$\psi' : A[x_1, \ldots, x_r]^\wedge \to C, \quad x_i \longmapsto \psi(x_i) + \delta_i$$

see Formal Spaces, Remark 28.1. Then we have

$$\psi'(g) = \psi(g) + \sum \psi(\partial g/\partial x_i)\delta_i + \xi$$

with error term $\xi \in (\delta_i\delta_j)$. This follows by writing g as a power series and working term by term. Convergence is automatic as the coefficients of g tend to zero. Details omitted.

Remark 5.2 (Lifting maps)**.** Let A be a Noetherian ring and $I \subset A$ be an ideal. Let B be an object of (2.0.2). Let C be an I-adically complete A-algebra. Let $\psi_n : B \to C/I^nC$ be an A-algebra homomorphism. The obstruction to lifting ψ_n to an A-algebra homomorphism into $C/I^{2n}C$ is an element

$$o(\psi_n) \in \operatorname{Ext}^1_B(NL^\wedge_{B/A}, I^nC/I^{2n}C)$$

as we will explain. Namely, choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Choose a lift $\psi : A[x_1, \ldots, x_r]^\wedge \to C$ of ψ_n. Since $\psi(J) \subset I^nC$ we get $\psi(J^2) \subset I^{2n}C$ and hence we get a B-linear homomorphism

$$o(\psi) : J/J^2 \longrightarrow I^nC/I^{2n}C, \quad g \longmapsto \psi(g)$$

which of course extends to a C-linear map $J/J^2 \otimes_B C \to I^nC/I^{2n}C$. Since $NL^\wedge_{B/A} = (J/J^2 \to \bigoplus B\mathrm{d}x_i)$ we get $o(\psi_n)$ as the image of $o(\psi)$ by the identification

$$\begin{aligned}&\operatorname{Ext}^1_B(NL^\wedge_{B/A}, I^nC/I^{2n}C)\\ &= \operatorname{Coker}\left(\operatorname{Hom}_B(\bigoplus B\mathrm{d}x_i, I^nC/I^{2n}C) \to \operatorname{Hom}_B(J/J^2, I^nC/I^{2n}C)\right)\end{aligned}$$

See More on Algebra, Lemma 84.4 part (1) for the equality.

Suppose that $o(\psi_n)$ maps to zero in $\operatorname{Ext}^1_B(NL^\wedge_{B/A}, I^{n'}C/I^{2n'}C)$ for some integer n' with $n > n' > n/2$. We claim that this means we can find an A-algebra homomorphism $\psi'_{2n'} : B \to C/I^{2n'}C$ which agrees with ψ_n as maps into $C/I^{n'}C$. The extreme case $n' = n$ explains why we previously said $o(\psi_n)$ is the obstruction to lifting ψ_n to $C/I^{2n}C$. Proof of the claim: the hypothesis that $o(\psi_n)$ maps to zero tells us we can find a B-module map

$$h : \bigoplus B\mathrm{d}x_i \longrightarrow I^{n'}C/I^{2n'}C$$

such that $o(\psi)$ and $h \circ \mathrm{d}$ agree as maps into $I^{n'}C/I^{2n'}C$. Say $h(\mathrm{d}x_i) = \delta_i \bmod I^{2n'}C$ for some $\delta_i \in I^{n'}C$. Then we look at the map

$$\psi' : A[x_1, \ldots, x_r]^\wedge \to C, \quad x_i \longmapsto \psi(x_i) - \delta_i$$

A computation with power series shows that $\psi'(J) \subset I^{2n'}C$. Namely, for $g \in J$ we get

$$\psi'(g) \equiv \psi(g) - \sum \psi(\partial g/\partial x_i)\delta_i \equiv o(\psi)(g) - (h \circ \mathrm{d})(g) \equiv 0 \bmod I^{2n'}C$$

See Remark 5.1 for the first equality. Hence ψ' induces an A-algebra homomorphism $\psi'_{2n'} : B \to C/I^{2n'}C$ as desired.

Lemma 5.3. *Assume given the following data*

(1) *an integer $c \geq 0$,*
(2) *an ideal I of a Noetherian ring A,*
(3) *B in (2.0.2) for (A, I) such that I^c annihilates $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ for any B-module N,*
(4) *a Noetherian I-adically complete A-algebra C; denote $d = d(Gr_I(C))$ and $q_0 = q(Gr_I(C))$ the integers found in Local Cohomology, Section 22,*
(5) *an integer $n \geq \max(q_0 + (d+1)c, 2(d+1)c + 1)$, and*
(6) *an A-algebra homomorphism $\psi_n : B \to C/I^nC$.*

Then there exists a map $\varphi : B \to C$ of A-algebras such that $\psi_n \bmod I^{n-(d+1)c} = \varphi \bmod I^{n-(d+1)c}$.

Proof. Consider the obstruction class

$$o(\psi_n) \in \mathrm{Ext}^1_B(NL^\wedge_{B/A}, I^nC/I^{2n}C)$$

of Remark 5.2. For any C/I^nC-module N we have

$$\begin{aligned}\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N) &= \mathrm{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes^{\mathbf{L}}_B C/I^nC, N) \\ &= \mathrm{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes_B C/I^nC, N)\end{aligned}$$

The first equality by More on Algebra, Lemma 99.1 and the second one by More on Algebra, Lemma 84.6. In particular, we see that $\mathrm{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes_B C/I^nC, N)$ is annihilated by I^cC for all C/I^nC-modules N. It follows that we may apply Local Cohomology, Lemma 22.7 to see that $o(\psi_n)$ maps to zero in

$$\mathrm{Ext}^1_{C/I^nC}(NL^\wedge_{B/A} \otimes_B C/I^nC, I^{n'}C/I^{2n'}C) = \mathrm{Ext}^1_B(NL^\wedge_{B/A}, I^{n'}C/I^{2n'}C) =$$

where $n' = n - (d+1)c$. By the discussion in Remark 5.2 we obtain a map

$$\psi'_{2n'} : B \to C/I^{2n'}C$$

which agrees with ψ_n modulo $I^{n'}$. Observe that $2n' > n$ because $n \geq 2(d+1)c+1$.

We may repeat this procedure. Starting with $n_0 = n$ and $\psi^0 = \psi_n$ we end up getting a strictly increasing sequence of integers

$$n_0 < n_1 < n_2 < \ldots$$

and A-algebra homorphisms $\psi^i : B \to C/I^{n_i}C$ such that ψ^{i+1} and ψ^i agree modulo $I^{n_i - tc}$. Since C is I-adically complete we can take φ to be the limit of the maps $\psi^i \bmod I^{n_i-(d+1)c} : B \to C/I^{n_i-(d+1)c}C$ and the lemma follows. $\square$

We suggest the reader skip ahead to the next section. Namely, the following two lemmas are consequences of the result above if the algebra C in them is assumed Noetherian.

Lemma 5.4. *Let $I = (a)$ be a principal ideal of a Noetherian ring A. Let B be an object of (2.0.2). Assume given an integer $c \geq 0$ such that $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by a^c for all B-modules N. Let C be an I-adically complete A-algebra such that a is a nonzerodivisor on C. Let $n > 2c$. For any A-algebra map $\psi_n : B \to C/a^nC$ there exists an A-algebra map $\varphi : B \to C$ such that $\psi_n \bmod a^{n-c}C = \varphi \bmod a^{n-c}C$.*

Proof. Consider the obstruction class

$$o(\psi_n) \in \mathrm{Ext}^1_B(NL^\wedge_{B/A}, a^nC/a^{2n}C)$$

of Remark 5.2. Since a is a nonzerodivisor on C the map $a^c : a^nC/a^{2n}C \to a^nC/a^{2n}C$ is isomorphic to the map $a^nC/a^{2n}C \to a^{n-c}C/a^{2n-c}C$ in the category of C-modules. Hence by our assumption on $NL^\wedge_{B/A}$ we conclude that the class $o(\psi_n)$ maps to zero in

$$\mathrm{Ext}^1_B(NL^\wedge_{B/A}, a^{n-c}C/a^{2n-c}C)$$

and a fortiori in

$$\mathrm{Ext}^1_B(NL^\wedge_{B/A}, a^{n-c}C/a^{2n-2c}C)$$

By the discussion in Remark 5.2 we obtain a map

$$\psi_{2n-2c} : B \to C/a^{2n-2c}C$$

which agrees with ψ_n modulo $a^{n-c}C$. Observe that $2n - 2c > n$ because $n > 2c$.

We may repeat this procedure. Starting with $n_0 = n$ and $\psi^0 = \psi_n$ we end up getting a strictly increasing sequence of integers

$$n_0 < n_1 < n_2 < \ldots$$

and A-algebra homorphisms $\psi^i : B \to C/a^{n_i}C$ such that ψ^{i+1} and ψ^i agree modulo $a^{n_i-c}C$. Since C is I-adically complete we can take φ to be the limit of the maps $\psi^i \bmod a^{n_i-c}C : B \to C/a^{n_i-c}C$ and the lemma follows. □

Lemma 5.5. *Let $I = (a)$ be a principal ideal of a Noetherian ring A. Let B be an object of (2.0.2). Assume given an integer $c \geq 0$ such that $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by a^c for all B-modules N. Let C be an I-adically complete A-algebra. Assume given an integer $d \geq 0$ such that $C[a^\infty] \cap a^dC = 0$. Let $n > \max(2c, c+d)$. For any A-algebra map $\psi_n : B \to C/a^nC$ there exists an A-algebra map $\varphi : B \to C$ such that $\psi_n \bmod a^{n-c} = \varphi \bmod a^{n-c}$.*

If C is Noetherian we have $C[a^\infty] = C[a^e]$ for some $e \geq 0$. By Artin-Rees (Algebra, Lemma 51.2) there exists an integer f such that $a^nC \cap C[a^\infty] \subset a^{n-f}C[a^\infty]$ for all $n \geq f$. Then $d = e + f$ is an integer as in the lemma. This argument works in particular if C is an object of (2.0.2) by Lemma 2.2.

Proof. Let $C \to C'$ be the quotient of C by $C[a^\infty]$. The A-algebra C' is I-adically complete by Algebra, Lemma 96.10 and the fact that $\bigcap(C[a^\infty] + a^nC) = C[a^\infty]$

because for $n \geq d$ the sum $C[a^\infty] + a^n C$ is direct. For $m \geq d$ the diagram

$$\begin{tikzcd} 0 \arrow[r] & C[a^\infty] \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C[a^\infty] \arrow[r] & C/a^m C \arrow[r] & C'/a^m C' \arrow[r] & 0 \end{tikzcd}$$

has exact rows. Thus C is the fibre product of C' and $C/a^m C$ over $C'/a^m C'$ for all $m \geq d$. By Lemma 5.4 we can choose a homomorphism $\varphi' : B \to C'$ such that φ' and ψ_n agree as maps into $C'/a^{n-c}C'$. We obtain a homomorphism $(\varphi', \psi_n \bmod a^{n-c}C) : B \to C' \times_{C'/a^{n-c}C'} C/a^{n-c}C$. Since $n - c \geq d$ this is the same thing as a homomorphism $\varphi : B \to C$. This finishes the proof. $\square$

6. Algebraization of rig-smooth algebras over G-rings

If the base ring A is a Noetherian G-ring, then we can prove [Elk73, III Theorem 7] for arbitrary rig-smooth algebras with respect to any ideal $I \subset A$ (not necessarily principal).

Lemma 6.1. *Let I be an ideal of a Noetherian ring A. Let $r \geq 0$ and write $P = A[x_1, \ldots, x_r]$ the I-adic completion. Consider a resolution*

$$P^{\oplus t} \xrightarrow{K} P^{\oplus m} \xrightarrow{g_1, \ldots, g_m} P \to B \to 0$$

of a quotient of P. Assume B is rig-smooth over (A, I). Then there exists an integer n such that for any complex

$$P^{\oplus t} \xrightarrow{K'} P^{\oplus m} \xrightarrow{g'_1, \ldots, g'_m} P$$

with $g_i - g'_i \in I^n P$ and $K - K' \in I^n Mat(m \times t, P)$ there exists an isomorphism $B \to B'$ of A-algebras where $B' = P/(g'_1, \ldots, g'_m)$.

Proof. (A) By Definition 4.1 we can choose a $c \geq 0$ such that I^c annihilates $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ for all B-modules N.

(B) By More on Algebra, Lemmas 4.1 and 4.2 there exists a constant $c_1 = c(g_1, \ldots, g_m, K)$ such that for $n \geq c_1 + 1$ the complex

$$P^{\oplus t} \xrightarrow{K'} P^{\oplus m} \xrightarrow{g'_1, \ldots, g'_m} P \to B' \to 0$$

is exact and $\mathrm{Gr}_I(B) \cong \mathrm{Gr}_I(B')$.

(C) Let $d_0 = d(\mathrm{Gr}_I(B))$ and $q_0 = q(\mathrm{Gr}_I(B))$ be the integers found in Local Cohomology, Section 22.

We claim that $n = \max(c_1 + 1, q_0 + (d_0 + 1)c, 2(d_0 + 1)c + 1)$ works where c is as in (A), c_1 is as in (B), and q_0, d_0 are as in (C).

Let $g'_1, \ldots, g'_m$ and K' be as in the lemma. Since $g_i = g'_i \in I^n P$ we obtain a canonical A-algebra homomorphism

$$\psi_n : B \longrightarrow B'/I^n B'$$

which induces an isomorphism $B/I^n B \to B'/I^n B'$. Since $\mathrm{Gr}_I(B) \cong \mathrm{Gr}_I(B')$ we have $d_0 = d(\mathrm{Gr}_I(B'))$ and $q_0 = q(\mathrm{Gr}_I(B'))$ and since $n \geq \max(q_0 + (1 + d_0)c, 2(d_0 + 1)c + 1)$ we may apply Lemma 5.3 to find an A-algebra homomorphism

$$\varphi : B \longrightarrow B'$$

such that $\varphi \bmod I^{n-(d_0+1)c}B' = \psi_n \bmod I^{n-(d_0+1)c}B'$. Since $n - (d_0+1)c > 0$ we see that φ is an A-algebra homomorphism which modulo I induces the isomorphism $B/IB \to B'/IB'$ we found above. The rest of the proof shows that these facts force φ to be an isomorphism; we suggest the reader find their own proof of this.

Namely, it follows that φ is surjective for example by applying Algebra, Lemma 96.1 part (1) using the fact that B and B' are complete. Thus φ induces a surjection $\mathrm{Gr}_I(B) \to \mathrm{Gr}_I(B')$ which has to be an isomorphism because the source and target are isomorphic Noetherian rings, see Algebra, Lemma 31.10 (of course you can show φ induces the isomorphism we found above but that would need a tiny argument). Thus φ induces injective maps $I^eB/I^{e+1}B \to I^eB'/I^{e+1}B'$ for all $e \geq 0$. This implies φ is injective since for any $b \in B$ there exists an $e \geq 0$ such that $b \in I^eB$, $b \notin I^{e+1}B$ by Krull's intersection theorem (Algebra, Lemma 51.4). This finishes the proof. □

Lemma 6.2. *Let I be an ideal of a Noetherian ring A. Let C^h be the henselization of a finite type A-algebra C with respect to the ideal IC. Let $J \subset C^h$ be an ideal. Then there exists a finite type A-algebra B such that $B^\wedge \cong (C^h/J)^\wedge$.*

Proof. By More on Algebra, Lemma 12.4 the ring C^h is Noetherian. Say $J = (g_1, \ldots, g_m)$. The ring C^h is a filtered colimit of étale C algebras C' such that $C/IC \to C'/IC'$ is an isomorphism (see proof of More on Algebra, Lemma 12.1). Pick an C' such that $g_1, \ldots, g_m$ are the images of $g'_1, \ldots, g'_m \in C'$. Setting $B = C'/(g'_1, \ldots, g'_m)$ we get a finite type A-algebra. Of course (C, IC) and $C', IC')$ have the same henselizations and the same completions. It follows easily from this that $B^\wedge = (C^h/J)^\wedge$. □

Proposition 6.3. *Let I be an ideal of a Noetherian G-ring A. Let B be an object of (2.0.2). If B is rig-smooth over (A, I), then there exists a finite type A-algebra C and an isomorphism $B \cong C^\wedge$ of A-algebras.*

Proof. Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Write $P = A[x_1, \ldots, x_r]^\wedge$. Choose generators $g_1, \ldots, g_m \in J$. Choose generators $k_1, \ldots, k_t$ of the module of relations between $g_1, \ldots, g_m$, i.e., such that

$$P^{\oplus t} \xrightarrow{k_1, \ldots, k_t} P^{\oplus m} \xrightarrow{g_1, \ldots, g_m} P \to B \to 0$$

is a resolution. Write $k_i = (k_{i1}, \ldots, k_{im})$ so that we have

$$\sum\nolimits_j k_{ij} g_j = 0 \tag{6.3.1}$$

for $i = 1, \ldots, t$. Denote $K = (k_{ij})$ the $m \times t$-matrix with entries k_{ij}.

Let $A[x_1, \ldots, x_r]^h$ be the henselization of the pair $(A[x_1, \ldots, x_r], IA[x_1, \ldots, x_r])$, see More on Algebra, Lemma 12.1. We may and do think of $A[x_1, \ldots, x_r]^h$ as a subring of $P = A[x_1, \ldots, x_r]^\wedge$, see More on Algebra, Lemma 12.4. Since A is a Noetherian G-ring, so is $A[x_1, \ldots, x_r]$, see More on Algebra, Proposition 50.10. Hence we have approximation for the map $A[x_1, \ldots, x_r]^h \to A[x_1, \ldots, x_r]^\wedge = P$ with respect to the ideal generated by I, see Smoothing Ring Maps, Lemma 14.1. Choose a large enough integer n as in Lemma 6.1. By the approximation property we may choose $g'_1, \ldots, g'_m \in A[x_1, \ldots, x_r]^h$ and a matrix $K' = (k'_{ij}) \in \mathrm{Mat}(m \times t, A[x_1, \ldots, x_r]^h)$

such that $\sum_j k'_{ij} g'_j = 0$ in $A[x_1, \ldots, x_r]^h$ and such that $g_i - g'_i \in I^n P$ and $K - K' \in I^n \text{Mat}(m \times t, P)$. By our choice of n we conclude that there is an isomorphism

$$B \to P/(g'_1, \ldots, g'_m) = \left(A[x_1, \ldots, x_r]^h/(g'_1, \ldots, g'_m)\right)^\wedge$$

This finishes the proof by Lemma 6.2. □

The following lemma isn't true in general if A is not a G-ring but just Noetherian. Namely, if $(A, \mathfrak{m})$ is local and $I = \mathfrak{m}$, then the lemma is equivalent to Artin approximation for A^h (as in Smoothing Ring Maps, Theorem 13.1) which does not hold for every Noetherian local ring.

Lemma 6.4. *Let A be a Noetherian G-ring. Let $I \subset A$ be an ideal. Let B, C be finite type A-algebras. For any A-algebra map $\varphi : B^\wedge \to C^\wedge$ of I-adic completions and any $N \geq 1$ there exist*

(1) *an étale ring map $C \to C'$ which induces an isomorphism $C/IC \to C'/IC'$,*
(2) *an A-algebra map $\varphi : B \to C'$*

such that φ and ψ agree modulo I^N into $C^\wedge = (C')^\wedge$.

Proof. The statement of the lemma makes sense as $C \to C'$ is flat (Algebra, Lemma 143.3) hence induces an isomorphism $C/I^nC \to C'/I^nC'$ for all n (More on Algebra, Lemma 89.2) and hence an isomorphism on completions. Let C^h be the henselization of the pair (C, IC), see More on Algebra, Lemma 12.1. Then C^h is the filtered colimit of the algebras C' and the maps $C \to C' \to C^h$ induce isomorphism on completions (More on Algebra, Lemma 12.4). Thus it suffices to prove there exists an A-algebra map $B \to C^h$ which is congruent to ψ modulo I^N. Write $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. The ring map ψ corresponds to elements $\hat{c}_1, \ldots, \hat{c}_n \in C^\wedge$ with $f_j(\hat{c}_1, \ldots, \hat{c}_n) = 0$ for $j = 1, \ldots, m$. Namely, as A is a Noetherian G-ring, so is C, see More on Algebra, Proposition 50.10. Thus Smoothing Ring Maps, Lemma 14.1 applies to give elements $c_1, \ldots, c_n \in C^h$ such that $f_j(c_1, \ldots, c_n) = 0$ for $j = 1, \ldots, m$ and such that $\hat{c}_i - c_i \in I^N C^h$. This determines the map $B \to C^h$ as desired. □

7. Algebraization of rig-smooth algebras

It turns out that if the rig-smooth algebra has a specific presentation, then it is straightforward to algebraize it. Please also see Remark 7.3 for a discussion.

Lemma 7.1. *Let A be a ring. Let $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ and set $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Assume $m \leq n$ and set $g = \det_{1 \leq i,j \leq m}(\partial f_j/\partial x_i)$. Then*

(1) *g annihilates $\text{Ext}^1_B(NL_{B/A}, N)$ for every B-module N,*
(2) *if $n = m$, then multiplication by g on $NL_{B/A}$ is 0 in $D(B)$.*

Proof. Let T be the $m \times m$ matrix with entries $\partial f_j/\partial x_i$ for $1 \leq i, j \leq n$. Let $K \in D(B)$ be represented by the complex $T : B^{\oplus m} \to B^{\oplus m}$ with terms sitting in degrees -1 and 0. By More on Algebra, Lemmas 84.12 we have $g : K \to K$ is zero in $D(B)$. Set $J = (f_1, \ldots, f_m)$. Recall that $NL_{B/A}$ is homotopy equivalent to $J/J^2 \to \bigoplus_{i=1,\ldots,n} B\text{d}x_i$, see Algebra, Section 134. Denote L the complex $J/J^2 \to \bigoplus_{i=1,\ldots,m} B\text{d}x_i$ to that we have the quotient map $NL_{B/A} \to L$. We also have a surjective map of complexes $K \to L$ by sending the jth basis element in the term $B^{\oplus m}$ in degree -1 to the class of f_j in J/J^2. Picture

$$NL_{B/A} \to L \leftarrow K$$

From More on Algebra, Lemma 84.8 we conclude that multiplication by g on L is 0 in $D(B)$. On the other hand, the distinguished triangle $B^{\oplus n-m}[0] \to NL_{B/A} \to L$ shows that $\text{Ext}^1_B(L, N) \to \text{Ext}^1_B(NL_{B/A}, N)$ is surjective for every B-module N and hence annihilated by g. This proves part (1). If $n = m$ then $NL_{B/A} = L$ and we see that (2) holds. □

Lemma 7.2. *Let I be an ideal of a Noetherian ring A. Let B be an object of (2.0.2). Let $B = A[x_1, \ldots, x_r]^\wedge/J$ be a presentation. Assume there exists an element $b \in B$, $0 \leq m \leq r$, and $f_1, \ldots, f_m \in J$ such that*

1. *$V(b) \subset V(IB)$ in $\text{Spec}(B)$,*
2. *the image of $\Delta = \det_{1 \leq i,j \leq m}(\partial f_j/\partial x_i)$ in B divides b, and*
3. *$bJ \subset (f_1, \ldots, f_m) + J^2$.*

Then there exists a finite type A-algebra C and an A-algebra isomorphism $B \cong C^\wedge$.

Proof. The conditions imply that B is rig-smooth over (A, I), see Lemma 4.2. Write $b'\Delta = b$ in B for some $b' \in B$. Say $I = (a_1, \ldots, a_t)$. Since $V(b) \subset V(IB)$ there exists an integer $c \geq 0$ such that $I^c B \subset bB$. Write $bb_i = a_i^c$ in B for some $b_i \in B$.

Choose an integer $n \gg 0$ (we will see later how large). Choose polynomials $f'_1, \ldots, f'_m \in A[x_1, \ldots, x_r]$ such that $f_i - f'_i \in I^n A[x_1, \ldots, x_r]^\wedge$. We set $\Delta' = \det_{1 \leq i,j \leq m}(\partial f'_j/\partial x_i)$ and we consider the finite type A-algebra

$$C = A[x_1, \ldots, x_r, z_1, \ldots, z_t]/(f'_1, \ldots, f'_m, z_1\Delta' - a_1^c, \ldots, z_t\Delta' - a_t^c)$$

We will apply Lemma 7.1 to C. We compute

$$\det \begin{pmatrix} \text{matrix of partials of} \\ f'_1, \ldots, f'_m, z_1\Delta' - a_1^c, \ldots, z_t\Delta' - a_t^c \\ \text{with respect to the variables} \\ x_1, \ldots, x_m, z_1, \ldots, z_t \end{pmatrix} = (\Delta')^{t+1}$$

Hence we see that $\text{Ext}^1_C(NL_{C/A}, N)$ is annihilated by $(\Delta')^{t+1}$ for all C-modules N. Since a_i^c is divisible by Δ' in C we see that $a_i^{(t+1)c}$ annihilates these Ext^1's also. Thus I^{c_1} annihilates $\text{Ext}^1_C(NL_{C/A}, N)$ for all C-modules N where $c_1 = 1 + t((t+1)c - 1)$. The exact value of c_1 doesn't matter for the rest of the argument; what matters is that it is independent of n.

Since $NL^\wedge_{C^\wedge/A} = NL_{C/A} \otimes_C C^\wedge$ by Lemma 3.2 we conclude that multiplication by I^{c_1} is zero on $\text{Ext}^1_{C^\wedge}(NL^\wedge_{C^\wedge/A}, N)$ for any $C^\wedge$-module N as well, see More on Algebra, Lemmas 84.7 and 84.6. In particular $C^\wedge$ is rig-smooth over (A, I).

Observe that we have a surjective A-algebra homomorphism

$$\psi_n : C \longrightarrow B/I^n B$$

sending the class of x_i to the class of x_i and sending the class of z_i to the class of $b_i b'$. This works because of our choices of b' and b_i in the first paragraph of the proof.

Let $d = d(\text{Gr}_I(B))$ and $q_0 = q(\text{Gr}_I(B))$ be the integers found in Local Cohomology, Section 22. By Lemma 5.3 if we take $n \geq \max(q_0 + (d+1)c_1, 2(d+1)c_1 + 1)$ we can find a homomorphism $\varphi : C^\wedge \to B$ of A-algebras which is congruent to ψ_n modulo $I^{n-(d+1)c_1}B$.

Since $\varphi : C^\wedge \to B$ is surjective modulo I we see that it is surjective (for example use Algebra, Lemma 96.1). To finish the proof it suffices to show that $\mathrm{Ker}(\varphi)/\mathrm{Ker}(\varphi)^2$ is annihilated by a power of I, see More on Algebra, Lemma 108.4.

Since φ is surjective we see that $NL^\wedge_{B/C^\wedge}$ has cohomology modules $H^0(NL^\wedge_{B/C^\wedge}) = 0$ and $H^{-1}(NL^\wedge_{B/C^\wedge}) = \mathrm{Ker}(\varphi)/\mathrm{Ker}(\varphi)^2$. We have an exact sequence

$$H^{-1}(NL^\wedge_{C^\wedge/A} \otimes_{C^\wedge} B) \to H^{-1}(NL^\wedge_{B/A}) \to H^{-1}(NL^\wedge_{B/C^\wedge}) \to H^0(NL^\wedge_{C^\wedge/A} \otimes_{C^\wedge} B) \to H^0(NL^\wedge_{B/A}) \to 0$$

by Lemma 3.5. The first two modules are annihilated by a power of I as B and $C^\wedge$ are rig-smooth over (A, I). Hence it suffices to show that the kernel of the surjective map $H^0(NL^\wedge_{C^\wedge/A} \otimes_{C^\wedge} B) \to H^0(NL^\wedge_{B/A})$ is annihilated by a power of I. For this it suffices to show that it is annihilated by a power of b. In other words, it suffices to show that

$$H^0(NL^\wedge_{C^\wedge/A}) \otimes_{C^\wedge} B[1/b] \longrightarrow H^0(NL^\wedge_{B/A}) \otimes_B B[1/b]$$

is an isomorphism. However, both are free $B[1/b]$ modules of rank $r - m$ with basis $\mathrm{d}x_{m+1}, \ldots, \mathrm{d}x_r$ and we conclude the proof. ☐

Remark 7.3. Let I be an ideal of a Noetherian ring A. Let B be an object of (2.0.2) which is rig-smooth over (A, I). As far as we know, it is an open question as to whether B is isomorphic to the I-adic completion of a finite type A-algebra. Here are some things we do know:

1. If A is a G-ring, then the answer is yes by Proposition 6.3.
2. If B is rig-étale over (A, I), then the answer is yes by Lemma 10.2.
3. If I is principal, then the answer is yes by [Elk73, III Theorem 7].
4. In general there exists an ideal $J = (b_1, \ldots, b_s) \subset B$ such that $V(J) \subset V(IB)$ and such that the I-adic completion of each of the affine blowup algebras $B[\frac{J}{b_i}]$ are isomorphic to the I-adic completion of a finite type A-algebra.

To see the last statement, choose $b_1, \ldots, b_s$ as in Lemma 4.2 part (4) and use the properties mentioned there to see that Lemma 7.2 applies to each completion $(B[\frac{J}{b_i}])^\wedge$. Part (4) tells us that "rig-locally a rig-smooth formal algebraic space is the completion of a finite type scheme over A" and it tells us that "there is an admissible formal blowing up of $\mathrm{Spf}(B)$ which is affine locally algebraizable".

8. Rig-étale algebras

In view of our definition of rig-smooth algebras (Definition 4.1), the following definition should not come as a surprise.

Definition 8.1. Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (2.0.2). We say B is *rig-étale over* (A, I) if there exists an integer $c \geq 0$ such that for all $a \in I^c$ multiplication by a on $NL^\wedge_{B/A}$ is zero in $D(B)$.

Condition (7) in the next lemma is one of the conditions used in [Art70] to define formal modifications. We have added it to the list of conditions to facilitate comparison with our conditions later on.

Lemma 8.2. *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (2.0.2). Write $B = A[x_1, \ldots, x_r]^\wedge/J$ (Lemma 2.2) and let $NL^\wedge_{B/A} = (J/J^2 \to \bigoplus B\mathrm{d}x_i)$ be its naive cotangent complex (3.0.1). The following are equivalent*

(1) *B is rig-étale over (A, I),*
(2) *there exists a $c \geq 0$ such that for all $a \in I^c$ multiplication by a on $NL^\wedge_{B/A}$ is zero in $D(B)$,*
(3) *there exits a $c \geq 0$ such that $H^i(NL^\wedge_{B/A})$, $i = -1, 0$ is annihilated by I^c,*
(4) *there exists a $c \geq 0$ such that $H^i(NL_{B_n/A_n})$, $i = -1, 0$ is annihilated by I^c for all $n \geq 1$ where $A_n = A/I^n$ and $B_n = B/I^nB$,*
(5) *for every $a \in I$ there exists a $c \geq 0$ such that*
   (a) *a^c annihilates $H^0(NL^\wedge_{B/A})$, and*
   (b) *there exist $f_1, \ldots, f_r \in J$ such that $a^cJ \subset (f_1, \ldots, f_r) + J^2$.*
(6) *for every $a \in I$ there exist $f_1, \ldots, f_r \in J$ and $c \geq 0$ such that*
   (a) *$\det_{1\leq i,j\leq r}(\partial f_j/\partial x_i)$ divides a^c in B, and*
   (b) *$a^cJ \subset (f_1, \ldots, f_r) + J^2$.*
(7) *choosing generaters $f_1, \ldots, f_t$ for J we have*
   (a) *the Jacobian ideal of B over A, namely the ideal in B generated by the $r \times r$ minors of the matrx $(\partial f_j/\partial x_i)_{1\leq i\leq r, 1\leq j\leq t}$, contains the ideal I^cB for some c, and*
   (b) *the Cramer ideal of B over A, namely the ideal in B generated by the image in B of the rth Fitting ideal of J as an $A[x_1, \ldots, x_r]^\wedge$-module, contains I^cB for some c.*

Proof. The equivalence of (1) and (2) is a restatement of Definition 8.1.

The equivalence of (2) and (3) follows from More on Algebra, Lemma 84.11.

The equivalence of (3) and (4) follows from the fact that the systems $\{NL_{B_n/A_n}\}$ and $NL^\wedge_{B/A} \otimes_B B_n$ are strictly isomorphic, see Lemma 3.3. Some details omitted.

Assume (2). Let $a \in I$. Let c be such that multiplication by a^c is zero on $NL^\wedge_{B/A}$. By More on Algebra, Lemma 84.4 part (1) there exists a map $\alpha : \bigoplus B\text{d}x_i \to J/J^2$ such that $\text{d} \circ \alpha$ and $\alpha \circ \text{d}$ are both multiplication by a^c. Let $f_i \in J$ be an element whose class modulo J^2 is equal to $\alpha(\text{d}x_i)$. A simple calculation gives that (6)(a), (b) hold.

We omit the verification that (6) implies (5); it is just a statement on two term complexes over B of the form $M \to B^{\oplus r}$.

Assume (5) holds. Say $I = (a_1, \ldots, a_t)$. Let $c_i \geq 0$ be the integer such that (5)(a), (b) hold for $a_i^{c_i}$. Then we see that $I^{\sum c_i}$ annihilates $H^0(NL^\wedge_{B/A})$. Let $f_{i,1}, \ldots, f_{i,r} \in J$ be as in (5)(b) for a_i. Consider the composition

$$B^{\oplus r} \to J/J^2 \to \bigoplus B\text{d}x_i$$

where the jth basis vector is mapped to the class of $f_{i,j}$ in J/J^2. By (5)(a) and (b) the cokernel of the composition is annihilated by $a_i^{2c_i}$. Thus this map is surjective after inverting $a_i^{c_i}$, and hence an isomorphism (Algebra, Lemma 16.4). Thus the kernel of $B^{\oplus r} \to \bigoplus B\text{d}x_i$ is a_i-power torsion, and hence $H^{-1}(NL^\wedge_{B/A}) = \text{Ker}(J/J^2 \to \bigoplus B\text{d}x_i)$ is a_i-power torsion. Since B is Noetherian (Lemma 2.2), all modules including $H^{-1}(NL^\wedge_{B/A})$ are finite. Thus $a_i^{d_i}$ annihilates $H^{-1}(NL^\wedge_{B/A})$ for some $d_i \geq 0$. It follows that $I^{\sum d_i}$ annihilates $H^{-1}(NL^\wedge_{B/A})$ and we see that (3) holds.

Thus conditions (2), (3), (4), (5), and (6) are equivalent. Thus it remains to show that these conditions are equivalent with (7). Observe that the Cramer ideal

$\text{Fit}_r(J)B$ is equal to $\text{Fit}_r(J/J^2)$ as $J/J^2 = J \otimes_{A[x_1,\ldots,x_r]^\wedge} B$, see More on Algebra, Lemma 8.4 part (3). Also, observe that the Jacobian ideal is just $\text{Fit}_0(H^0(NL^\wedge_{B/A}))$. Thus we see that the equivalence of (3) and (7) is a purely algebraic question which we discuss in the next paragraph.

Let R be a Noetherian ring and let $I \subset R$ be an ideal. Let $M \xrightarrow{d} R^{\oplus r}$ be a two term complex. We have to show that the following are equivalent

(A) the cohomology of $M \to R^{\oplus r}$ is annihilated by a power of I, and
(B) the ideals $\text{Fit}_r(M)$ and $\text{Fit}_0(\text{Coker}(d))$ contain a power of I.

Since R is Noetherian, we can reformulate part (2) as an inclusion of the corresponding closed subschemes, see Algebra, Lemmas 17.2 and 32.5. On the other hand, over the complement of $V(\text{Fit}_0(\text{Coker}(d)))$ the cokernel of d vanishes and over the complement of $V(\text{Fit}_r(M))$ the module M is locally generated by r elements, see More on Algebra, Lemma 8.6. Thus (B) is equivalent to

(C) away from $V(I)$ the cokernel of d vanishes and the module M is locally generated by $\leq r$ elements.

Of course this is equivalent to the condition that $M \to R^{\oplus r}$ has vanishing cohomology over $\text{Spec}(R) \setminus V(I)$ which in turn is equivalent to (A). This finishes the proof. □

Lemma 8.3. *Let A be a Noetherian ring and let I be an ideal. Let B be an object of (2.0.2). If B is rig-étale over (A, I), then B is rig-smooth over (A, I).*

Proof. Immediate from Definitions 4.1 and 8.1. □

Lemma 8.4. *Let A be a Noetherian ring and let I be an ideal. Let B be a finite type A-algebra.*

(1) *If $\text{Spec}(B) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$, then $B^\wedge$ satisfies the equivalent conditions of Lemma 8.2.*
(2) *If $B^\wedge$ satisfies the equivalent conditions of Lemma 8.2, then there exists $g \in 1 + IB$ such that $\text{Spec}(B_g)$ is étale over $\text{Spec}(A) \setminus V(I)$.*

Proof. Assume $B^\wedge$ satisfies the equivalent conditions of Lemma 8.2. The naive cotangent complex $NL_{B/A}$ is a complex of finite type B-modules and hence H^{-1} and H^0 are finite B-modules. Completion is an exact functor on finite B-modules (Algebra, Lemma 97.2) and $NL^\wedge_{B^\wedge/A}$ is the completion of the complex $NL_{B/A}$ (this is easy to see by choosing presentations). Hence the assumption implies there exists a $c \geq 0$ such that H^{-1}/I^nH^{-1} and H^0/I^nH^0 are annihilated by I^c for all n. By Nakayama's lemma (Algebra, Lemma 20.1) this means that I^cH^{-1} and I^cH^0 are annihilated by an element of the form $g = 1 + x$ with $x \in IB$. After inverting g (which does not change the quotients B/I^nB) we see that $NL_{B/A}$ has cohomology annihilated by I^c. Thus $A \to B$ is étale at any prime of B not lying over $V(I)$ by the definition of étale ring maps, see Algebra, Definition 143.1.

Conversely, assume that $\text{Spec}(B) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$. Then for every $a \in I$ there exists a $c \geq 0$ such that multiplication by a^c is zero $NL_{B/A}$. Since $NL^\wedge_{B^\wedge/A}$ is the derived completion of $NL_{B/A}$ (see Lemma 3.3) it follows that $B^\wedge$ satisfies the equivalent conditions of Lemma 8.2. □

Lemma 8.5. *Let $(A_1, I_1) \to (A_2, I_2)$ be as in Remark 2.3 with A_1 and A_2 Noetherian. Let B_1 be in (2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1. If* *multiplication by $f_1 \in B_1$ on $NL^\wedge_{B_1/A_1}$ is zero in $D(B_1)$, then multiplication by the image $f_2 \in B_2$ on $NL^\wedge_{B_2/A_2}$ is zero in $D(B_2)$.*

Proof. By Lemma 3.4 there is a map

$$NL_{B_1/A_1} \otimes_{B_2} B_1 \to NL_{B_2/A_2}$$

which induces and isomorphism on H^0 and a surjection on H^{-1}. Thus the result by More on Algebra, Lemma 84.8. □

Lemma 8.6. *Let $A_1 \to A_2$ be a map of Noetherian rings. Let $I_i \subset A_i$ be an ideal such that $V(I_1A_2) = V(I_2)$. Let B_1 be in (2.0.2) for (A_1, I_1). Let B_2 be the base change of B_1 as in Remark 2.3. If B_1 is rig-étale over (A_1, I_1), then B_2 is rig-étale over (A_2, I_2).*

Proof. Follows from Lemma 8.5 and Definition 8.1 and the fact that $I_2^c \subset I_1A_2$ for some $c \geq 0$ as A_2 is Noetherian. □

Lemma 8.7. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be a finite type A-algebra such that $\text{Spec}(B) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$. Let C be a Noetherian A-algebra. Then any A-algebra map $B^\wedge \to C^\wedge$ of I-adic completions comes from a unique A-algebra map*

$$B \longrightarrow C^h$$

where C^h is the henselization of the pair (C, IC) as in More on Algebra, Lemma 12.1. Moreover, any A-algebra homomorphism $B \to C^h$ factors through some étale C-algebra C' such that $C/IC \to C'/IC'$ is an isomorphism.

Proof. Uniqueness follows from the fact that C^h is a subring of $C^\wedge$, see for example More on Algebra, Lemma 12.4. The final assertion follows from the fact that C^h is the filtered colimit of these C-algebras C', see proof of More on Algebra, Lemma 12.1. Having said this we now turn to the proof of existence.

Let $\varphi : B^\wedge \to C^\wedge$ be the given map. This defines a section

$$\sigma : (B \otimes_A C)^\wedge \longrightarrow C^\wedge$$

of the completion of the map $C \to B \otimes_A C$. We may replace (A, I, B, C, φ) by $(C, IC, B \otimes_A C, C, \sigma)$. In this way we see that we may assume that $A = C$.

Proof of existence in the case $A = C$. In this case the map $\varphi : B^\wedge \to A^\wedge$ is necessarily surjective. By Lemmas 8.4 and 3.5 we see that the cohomology groups of $NL^\wedge_{A^\wedge/_\varphi B^\wedge}$ are annihilated by a power of I. Since φ is surjective, this implies that $\text{Ker}(\varphi)/\text{Ker}(\varphi)^2$ is annihilated by a power of I. Hence $\varphi : B^\wedge \to A^\wedge$ is the completion of a finite type B-algebra $B \to D$, see More on Algebra, Lemma 108.4. Hence $A \to D$ is a finite type algebra map which induces an isomorphism $A^\wedge \to D^\wedge$. By Lemma 8.4 we may replace D by a localization and assume that $A \to D$ is étale away from $V(I)$. Since $A^\wedge \to D^\wedge$ is an isomorphism, we see that $\text{Spec}(D) \to \text{Spec}(A)$ is also étale in a neighbourhood of $V(ID)$ (for example by More on Morphisms, Lemma 12.3). Thus $\text{Spec}(D) \to \text{Spec}(A)$ is étale. Therefore D maps to A^h and the lemma is proved. □

9. A pushout argument

The only goal in this section is to prove the following lemma which will play a key role in algebraization of rig-étale algebras. We will use a bit of the theory of algebraic spaces to prove this lemma; an earlier version of this chapter gave a (much longer) proof using algebra and a bit of deformation theory that the interested reader can find in the history of the Stacks project.

Lemma 9.1. *Let A be a Noetherian ring and $I \subset A$ an ideal. Let $J \subset A$ be a nilpotent ideal. Consider a commutative diagram*

$$\begin{tikzcd} C \arrow[r] & C_0 \arrow[r, equal] & C/JC \\ & B_0 \arrow[u] & \\ A \arrow[uu] \arrow[r] & A_0 \arrow[u] \arrow[r, equal] & A/J \end{tikzcd}$$

whose vertical arrows are of finite type such that

(1) $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ *is étale over* $\mathrm{Spec}(A) \setminus V(I)$*,*
(2) $\mathrm{Spec}(B_0) \to \mathrm{Spec}(A_0)$ *is étale over* $\mathrm{Spec}(A_0) \setminus V(IA_0)$*, and*
(3) *$B_0 \to C_0$ is étale and induces an isomorphism $B_0/IB_0 = C_0/IC_0$.*

Then we can fill in the diagram above to a commutative diagram

$$\begin{tikzcd} C \arrow[r] & C/JC \\ B \arrow[u] \arrow[r] & B_0 \arrow[u] \\ A \arrow[u] \arrow[r] & A/J \arrow[u] \end{tikzcd}$$

with $A \to B$ of finite type, $B/JB = B_0$, $B \to C$ étale, and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *étale over* $\mathrm{Spec}(A) \setminus V(I)$*.*

Proof. Set $X = \mathrm{Spec}(A)$, $X_0 = \mathrm{Spec}(A_0)$, $Y_0 = \mathrm{Spec}(B_0)$, $Z = \mathrm{Spec}(C)$, $Z_0 = \mathrm{Spec}(C_0)$. Furthermore, denote $U \subset X$, $U_0 \subset X_0$, $V_0 \subset Y_0$, $W \subset Z$, $W_0 \subset Z_0$ the complement of the vanishing set of I. Here is a picture to help visualize the situation:

$$\begin{tikzcd} Z \arrow[dd] & Z_0 \arrow[l] \arrow[d] & & W \arrow[dd] & W_0 \arrow[l] \arrow[d] \\ & Y_0 \arrow[d] & & & V_0 \arrow[d] \\ X & X_0 \arrow[l] & & U & U_0 \arrow[l] \end{tikzcd}$$

The conditions in the lemma guarantee that

$$\begin{tikzcd} W_0 \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ V_0 \arrow[r] & Y_0 \end{tikzcd}$$

is an elementary distinguished square, see Derived Categories of Spaces, Definition 9.1. In addition we know that $W_0 \to U_0$ and $V_0 \to U_0$ are étale. The morphism $X_0 \subset X$ is a finite order thickening as J is assumed nilpotent. By the topological invariance of the étale site we can find a unique étale morphism $V \to X$ of schemes with $V_0 = V \times_X X_0$ and we can lift the given morphism $W_0 \to V_0$ to a unique morphism $W \to V$ over X. See Étale Morphisms, Theorem 15.2. Since $W_0 \to V_0$ is separated, the morphism $W \to V$ is separated too, see for example More on Morphisms, Lemma 10.3. By Pushouts of Spaces, Lemma 9.2 we can construct an elementary distinguished square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d] \\ V \arrow[r] & Y \end{tikzcd}$$

in the category of algebraic spaces over X. Since the base change of an elementary distinguished square is an elementary distinguished square (Derived Categories of Spaces, Lemma 9.2) we see that

$$\begin{tikzcd} W_0 \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ V_0 \arrow[r] & Y \times_X X_0 \end{tikzcd}$$

is an elementary distinguished square. It follows that there is a unique isomorphism $Y \times_X X_0 = Y_0$ compatible with the two squares involving these spaces because elementary distinguished squares are pushouts (Pushouts of Spaces, Lemma 9.1). It follows that Y is affine by Limits of Spaces, Proposition 15.2. Write $Y = \mathrm{Spec}(B)$. It is clear that B fits into the desired diagram and satisfies all the properties required of it. $\square$

10. Algebraization of rig-étale algebras

The main goal is to prove algebraization for rig-étale algebras when the underlying Noetherian ring A is not assumed to be a G-ring and when the ideal $I \subset A$ is arbitrary – not necessarily principal. We first prove the principal ideal case and then use the result of Section 9 to finish the proof.

Lemma 10.1. *Let A be a Noetherian ring and $I = (a)$ a principal ideal. Let B be an object of (2.0.2) which is rig-étale over (A, I). Then there exists a finite type A-algebra C and an isomorphism $B \cong C^\wedge$.*

Proof. Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. By Lemma 8.2 part (6) we can find $c \geq 0$ and $f_1, \ldots, f_r \in J$ such that $\det_{1 \leq i,j \leq r}(\partial f_j/\partial x_i)$ divides a^c in B and $a^c J \subset (f_1, \ldots, f_r) + J^2$. Hence Lemma 7.2 applies. This finishes the proof, but we'd like to point out that in this case the use of Lemma 5.3 can be replaced by the much easier Lemma 5.5. $\square$

Lemma 10.2. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be an object of (2.0.2) which is rig-étale over (A, I). Then there exists a finite type A-algebra C and an isomorphism $B \cong C^\wedge$.*

Proof. We prove this lemma by induction on the number of generators of I. Say $I = (a_1, \ldots, a_t)$. If $t = 0$, then $I = 0$ and there is nothing to prove. If $t = 1$, then the lemma follows from Lemma 10.1. Assume $t > 1$.

For any $m \geq 1$ set $\bar{A}_m = A/(a_t^m)$. Consider the ideal $\bar{I}_m = (\bar{a}_1, \ldots, \bar{a}_{t-1})$ in $\bar{A}_m$. Observe that $V(I\bar{A}_m) = V(\bar{I}_m)$. Let $B_m = B/(a_t^m)$ be the base change of B for the map $(A, I) \to (\bar{A}_m, \bar{I}_m)$, see Remark 2.4. By Lemma 8.6 we find that B_m is rig-étale over $(\bar{A}_m, \bar{I}_m)$.

By induction hypothesis (on t) we can find a finite type $\bar{A}_m$-algebra C_m and a map $C_m \to B_m$ which induces an isomorphism $C_m^\wedge \cong B_m$ where the completion is with respect to $\bar{I}_m$. By Lemma 8.4 we may assume that $\mathrm{Spec}(C_m) \to \mathrm{Spec}(\bar{A}_m)$ is étale over $\mathrm{Spec}(\bar{A}_m) \setminus V(\bar{I}_m)$.

We claim that we may choose $A_m \to C_m \to B_m$ as in the previous paragraph such that moreover there are isomorphisms $C_m/(a_t^{m-1}) \to C_{m-1}$ compatible with the given A-algebra structure and the maps to $B_{m-1} = B_m/(a_t^{m-1})$. Namely, first fix a choice of $A_1 \to C_1 \to B_1$. Suppose we have found $C_{m-1} \to C_{m-2} \to \ldots \to C_1$ with the desired properties. Note that $C_m/(a_t^{m-1})$ is étale over $\mathrm{Spec}(\bar{A}_{m-1}) \setminus V(\bar{I}_{m-1})$. Hence by Lemma 8.7 there exists an étale extension $C_{m-1} \to C'_{m-1}$ which induces an isomorphism modulo $\bar{I}_{m-1}$ and an $\bar{A}_{m-1}$-algebra map $C_m/(a_t^{m-1}) \to C'_{m-1}$ inducing the isomorphism $B_m/(a_t^{m-1}) \to B_{m-1}$ on completions. Note that $C_m/(a_t^{m-1}) \to C'_{m-1}$ is étale over the complement of $V(\bar{I}_{m-1})$ by Morphisms, Lemma 36.18 and over $V(\bar{I}_{m-1})$ induces an isomorphism on completions hence is étale there too (for example by More on Morphisms, Lemma 12.3). Thus $C_m/(a_t^{m-1}) \to C'_{m-1}$ is étale. By the topological invariance of étale morphisms (Étale Morphisms, Theorem 15.2) there exists an étale ring map $C_m \to C'_m$ such that $C_m/(a_t^{m-1}) \to C'_{m-1}$ is isomorphic to $C_m/(a_t^{m-1}) \to C'_m/(a_t^{m-1})$. Observe that the $\bar{I}_m$-adic completion of C'_m is equal to the $\bar{I}_m$-adic completion of C_m, i.e., to B_m (details omitted). We apply Lemma 9.1 to the diagram

$$\begin{tikzcd}
 & C'_m \arrow[r] & C'_m/(a_t^{m-1}) \\
C''_m \arrow[ur, dotted] \arrow[rr, dotted] & & C_{m-1} \arrow[u] \\
 & \bar{A}_m \arrow[uu] \arrow[ul, dotted] \arrow[r] & \bar{A}_{m-1} \arrow[u]
\end{tikzcd}$$

to see that there exists a "lift" of C''_m of C_{m-1} to an algebra over $\bar{A}_m$ with all the desired properties.

By construction (C_m) is an object of the category (2.0.1) for the principal ideal (a_t). Thus the inverse limit $B' = \lim C_m$ is an (a_t)-adically complete A-algebra such that $B'/a_t B'$ is of finite type over $A/(a_t)$, see Lemma 2.1. By construction the I-adic completion of B' is isomorphic to B (details omitted). Consider the complex $NL^\wedge_{B'/A}$ constructed using the (a_t)-adic topology. Choosing a presentation for B' (which induces a similar presentation for B) the reader immediately sees that $NL^\wedge_{B'/A} \otimes_{B'} B = NL^\wedge_{B/A}$. Since $a_t \in I$ and since the cohomology modules of $NL^\wedge_{B'/A}$ are finite B'-modules (hence complete for the a_t-adic topology), we conclude that a_t^c acts as zero on these cohomologies as the same thing is true by assumption for $NL^\wedge_{B/A}$. Thus B' is rig-étale over $(A, (a_t))$ by Lemma 8.2. Hence finally, we may apply Lemma 10.1 to B' over $(A, (a_t))$ to finish the proof. □

Lemma 10.3. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be an I-adically complete A-algebra with $A/I \to B/IB$ of finite type. The equivalent conditions of Lemma 8.2 are also equivalent to*

(8) *there exists a finite type A-algebra C such that $\operatorname{Spec}(C) \to \operatorname{Spec}(A)$ is étale over $\operatorname{Spec}(A) \setminus V(I)$ and such that $B \cong C^\wedge$.*

Proof. Combine Lemmas 8.2, 10.2, and 8.4. Small detail omitted. □

11. Finite type morphisms

In Formal Spaces, Section 24 we have defined finite type morphisms of formal algebraic spaces. In this section we study the corresponding types of continuous ring maps of adic topological rings which have a finitely generated ideal of definition. We strongly suggest the reader skip this section.

Lemma 11.1. *Let A and B be adic topological rings which have a finitely generated ideal of definition. Let $\varphi : A \to B$ be a continuous ring homomorphism. The following are equivalent:*

(1) *φ is adic and B is topologically of finite type over A,*
(2) *φ is taut and B is topologically of finite type over A,*
(3) *there exists an ideal of definition $I \subset A$ such that the topology on B is the I-adic topology and there exist an ideal of definition $I' \subset A$ such that $A/I' \to B/I'B$ is of finite type,*
(4) *for all ideals of definition $I \subset A$ the topology on B is the I-adic topology and $A/I \to B/IB$ is of finite type,*
(5) *there exists an ideal of definition $I \subset A$ such that the topology on B is the I-adic topology and B is in the category (2.0.2),*
(6) *for all ideals of definition $I \subset A$ the topology on B is the I-adic topology and B is in the category (2.0.2),*
(7) *B as a topological A-algebra is the quotient of $A\{x_1, \ldots, x_r\}$ by a closed ideal,*
(8) *B as a topological A-algebra is the quotient of $A[x_1, \ldots, x_r]^\wedge$ by a closed ideal where $A[x_1, \ldots, x_r]^\wedge$ is the completion of $A[x_1, \ldots, x_r]$ with respect to some ideal of definition of A, and*
(9) *add more here.*

Moreover, these equivalent conditions define a local property of morphisms of $WAdm^{adic*}$ as defined in Formal Spaces, Remark 21.4.

Proof. Taut ring homomorphisms are defined in Formal Spaces, Definition 5.1. Adic ring homomorphisms are defined in Formal Spaces, Definition 6.1. The lemma follows from a combination of Formal Spaces, Lemmas 29.6, 29.7, and 23.1. We omit the details. To be sure, there is no difference between the topological rings $A[x_1, \ldots, x_n]^\wedge$ and $A\{x_1, \ldots, x_r\}$, see Formal Spaces, Remark 28.2. □

Remark 11.2. Let $A \to B$ be an arrow of WAdm^{adic*} which is adic and topologically of finite type (see Lemma 11.1). Write $B = A\{x_1, \ldots, x_r\}/J$. Then we can set[1]

$$NL^{\wedge}_{B/A} = \left(J/J^2 \longrightarrow \bigoplus B\mathrm{d}x_i\right)$$

Exactly as in the proof of Lemma 3.1 the reader can show that this complex of B-modules is well defined up to (unique isomorphism) in the homotopy category $K(B)$. Now, if A is Noetherian and $I \subset A$ is an ideal of definition, then this construction reproduces the naive cotangent complex of B over (A, I) defined by Equation (3.0.1) in Section 3 simply because $A[x_1, \ldots, x_n]^\wedge$ agrees with $A\{x_1, \ldots, x_r\}$ by Formal Spaces, Remark 28.2. In particular, we find that, still when A is an adic Noetherian topological ring, the object $NL^{\wedge}_{B/A}$ is independent of the choice of the ideal of definition $I \subset A$.

Lemma 11.3. *Consider the property P on arrows of $WAdm^{adic*}$ defined in Lemma 11.1. Then P is stable under base change as defined in Formal Spaces, Remark 21.8.*

Proof. The statement makes sense by Lemma 11.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in $WAdm^{adic*}$ and that as a topological B-algebra we have $A = B\{x_1, \ldots, x_r\}/J$ for some closed ideal J. Then $A\widehat{\otimes}_B C$ is isomorphic to the quotient of $C\{x_1, \ldots, x_r\}/J'$ where J' is the closure of $JC\{x_1, \ldots, x_r\}$. Some details omitted. $\square$

Lemma 11.4. *Consider the property P on arrows of $WAdm^{adic*}$ defined in Lemma 11.1. Then P is stable under composition as defined in Formal Spaces, Remark 21.13.*

Proof. The statement makes sense by Lemma 11.1. The easiest way to prove it is true is to show that (a) compositions of adic ring maps between adic topological rings are adic and (b) that compositions of continuous ring maps preserves the property of being topologically of finite type. We omit the details. $\square$

The following lemma says that morphisms of adic* formal algebraic spaces are locally of finite type if and only if they are étale locally given by the types of maps of topological rings described in Lemma 11.1.

Lemma 11.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally adic* formal algebraic spaces over S. The following are equivalent*

(1) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to an arrow of $WAdm^{adic*}$ which is adic and topologically of finite type,

[1] In fact, this construction works for arrows of WAdm^{count} satisfying the equivalent conditions of Formal Spaces, Lemma 29.6.

(2) *there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 11.1 such that each $X_{ji} \to Y_j$ corresponds to an arrow of WAdm^{adic*} which is adic and topologically of finite type,*
(3) *there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to an arrow of WAdm^{adic*} which is adic and topologically of finite type, and*
(4) *f is locally of finite type.*

Proof. Immediate consequence of the equivalence of (1) and (2) in Lemma 11.1 and Formal Spaces, Lemma 29.9. □

12. Finite type on reductions

In this section we talk a little bit about morphisms $X \to Y$ of locally countably indexed formal algebraic spaces such that $X_{red} \to Y_{red}$ is locally of finite type. We will translate this into an algebraic condition. To understand this algebraic condition it pays to keep in mind the following:

- If A is a weakly admissible topological ring, then the set $\mathfrak{a} \subset A$ of topological nilpotent elements is an open, radical ideal and $\text{Spf}(A)_{red} = \text{Spec}(A/\mathfrak{a})$.

See Formal Spaces, Definition 4.8, Lemma 4.10, and Example 12.2.

Lemma 12.1. *For an arrow $\varphi : A \to B$ in WAdm^{count} consider the property $P(\varphi) =$"the induced ring homomorphism $A/\mathfrak{a} \to B/\mathfrak{b}$ is of finite type" where $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ are the ideals of topologically nilpotent elements. Then P is a local property as defined in Formal Spaces, Situation 21.2.*

Proof. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & (B')^\wedge \\ A \arrow[u, "\varphi"] \arrow[r] & (A')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd}$$

as in Formal Spaces, Situation 21.2. Taking Spf of this diagram we obtain

$$\begin{tikzcd} \text{Spf}(B) \arrow[d] & \text{Spf}((B')^\wedge) \arrow[l] \arrow[d] \\ \text{Spf}(A) & \text{Spf}((A')^\wedge) \arrow[l] \end{tikzcd}$$

of affine formal algebraic spaces whose horizontal arrows are representable by algebraic spaces and étale by Formal Spaces, Lemma 19.13. Hence we obtain a commutative diagram of affine schemes

$$\begin{tikzcd} \text{Spf}(B)_{red} \arrow[d, "f"] & \text{Spf}((B')^\wedge)_{red} \arrow[l, "g"'] \arrow[d, "f'"] \\ \text{Spf}(A)_{red} & \text{Spf}((A')^\wedge)_{red} \arrow[l] \end{tikzcd}$$

whose horizontal arrows are étale by Formal Spaces, Lemma 12.3. By Formal Spaces, Example 12.2 and Lemma 19.14 conditions (1), (2), and (3) of Formal Spaces, Situation 21.2 translate into the following statements

(1) if f is locally of finite type, then f' is locally of finite type,
(2) if f' is locally of finite type and g is surjective, then f is locally of finite type, and
(3) if $T_i \to S$, $i = 1, \ldots, n$ are locally of finite type, then $\coprod_{i=1,\ldots,n} T_i \to S$ is locally of finite type.

Properties (1) and (2) follow from the fact that being locally of finite type is local on the source and target in the étale topology, see discussion in Morphisms of Spaces, Section 23. Property (3) is a straightforward consequence of the definition. $\square$

Lemma 12.2. *Consider the property P on arrows of $WAdm^{count}$ defined in Lemma 12.1. Then P is stable under base change (Formal Spaces, Situation 21.6).*

Proof. The statement makes sense by Lemma 12.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in $WAdm^{count}$ such that $B/\mathfrak{b} \to A/\mathfrak{a}$ is of finite type where $\mathfrak{b} \subset B$ and $\mathfrak{a} \subset A$ are the ideals of topologically nilpotent elements. Since A and B are weakly admissible, the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are open. Let $\mathfrak{c} \subset C$ be the (open) ideal of topologically nilpotent elements. Then we find a surjection $A \widehat{\otimes}_B C \to A/\mathfrak{a} \otimes_{B/\mathfrak{b}} C/\mathfrak{c}$ whose kernel is a weak ideal of definition and hence consists of topologically nilpotent elements (please compare with the proof of Formal Spaces, Lemma 4.12). Since already $C/\mathfrak{c} \to A/\mathfrak{a} \otimes_{B/\mathfrak{b}} C/\mathfrak{c}$ is of finite type as a base change of $B/\mathfrak{b} \to A/\mathfrak{a}$ we conclude. $\square$

Lemma 12.3. *Consider the property P on arrows of $WAdm^{count}$ defined in Lemma 12.1. Then P is stable under composition (Formal Spaces, Situation 21.11).*

Proof. Omitted. Hint: compositions of finite type ring maps are of finite type. $\square$

Lemma 12.4. *Let $\varphi : A \to B$ be an arrow of $WAdm^{count}$. If φ is taut and topologically of finite type, then φ satisfies the condition defined in Lemma 12.1.*

Proof. This is an easy consequence of the definitions. $\square$

Lemma 12.5. *Let $\varphi : A \to B$ be an arrow of $WAdm^{Noeth}$ satisfying the condition defined in Lemma 12.1. Then $A \to B$ is topologically of finite type.*

Proof. Let $\mathfrak{b} \subset B$ be the ideal of topologically nilpotent elements. Choose $b_1, \ldots, b_r \in B$ which map to generators of $B/\mathfrak{b}$ over A. Choose generators $b_{r+1}, \ldots, b_s$ of the ideal $\mathfrak{b}$. We claim that the image of

$$\varphi : A[x_1, \ldots, x_s] \longrightarrow B, \quad x_i \longmapsto b_i$$

has dense image. Namely, if $b \in \mathfrak{b}^n$ for some $n \geq 0$, then we can write $b = \sum b_E b_{r+1}^{e_{r+1}} \ldots b_s^{e_s}$ for multiindices $E = (e_{r+1}, \ldots, e_s)$ with $|E| = \sum e_i = n$ and $b_E \in B$. Next, we can write $b_E = f_E(b_1, \ldots, b_r) + b'_E$ with $b'_E \in \mathfrak{b}$ and $f_E \in A[x_1, \ldots, x_r]$. Combined we obtain $b \in \text{Im}(\varphi) + \mathfrak{b}^{n+1}$. By induction we see that $B = \text{Im}(\varphi) + \mathfrak{b}^n$ for all $n \geq 0$ which mplies what we want as $\mathfrak{b}$ is an ideal of definition of B. $\square$

Lemma 12.6. *Let $\varphi : A \to B$ be an arrow of $WAdm^{Noeth}$. If φ is adic the following are equivalent*

(1) *φ satisfies the condition defined in Lemma 12.1 and*
(2) *φ satisfies the condition defined in Lemma 11.1.*

Proof. Omitted. Hint: For the proof of (1) $\Rightarrow$ (2) use Lemma 12.5. $\square$

Lemma 12.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally countably indexed formal algebraic spaces over S. The following are equivalent*

(1) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to an arrow of $\textit{WAdm}^{count}$ satisfying the property defined in Lemma 12.1,

(2) *there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 11.1 such that each $X_{ji} \to Y_j$ corresponds to an arrow of $\textit{WAdm}^{count}$ satisfying the property defined in Lemma 12.1,*

(3) *there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to an arrow of $\textit{WAdm}^{count}$ satisfying the property defined in Lemma 12.1, and*

(4) *the morphism $f_{red} : X_{red} \to Y_{red}$ is locally of finite type.*

Proof. The equivalence of (1), (2), and (3) follows from Lemma 12.1 and an application of Formal Spaces, Lemma 21.3. Let Y_j and X_{ji} be as in (2). Then

- The families $\{Y_{j,red} \to Y_{red}\}$ and $\{X_{ji,red} \to X_{red}\}$ are étale coverings by affine schemes. This follows from the discussion in the proof of Formal Spaces, Lemma 12.1 or directly from Formal Spaces, Lemma 12.3.
- If $X_{ji} \to Y_j$ corresponds to the morphism $B_j \to A_{ji}$ of $\textit{WAdm}^{count}$, then $X_{ji,red} \to Y_{j,red}$ corresponds to the ring map $B_j/\mathfrak{b}_j \to A_{ji}/\mathfrak{a}_{ji}$ where $\mathfrak{b}_j$ and $\mathfrak{a}_{ji}$ are the ideals of topologically nilpotent elements. This follows from Formal Spaces, Example 12.2. Hence $X_{ji,red} \to Y_{j,red}$ is locally of finite type if and only if $B_j \to A_{ji}$ satisfies the property defined in Lemma 12.1.

The equivalence of (2) and (4) follows from these remarks because being locally of finite type is a property of morphisms of algebraic spaces which is étale local on source and target, see discussion in Morphisms of Spaces, Section 23. $\square$

13. Flat morphisms

In this section we define flat morphisms of locally Noetherian formal algebraic spaces.

Lemma 13.1. *The property $P(\varphi)=$"φ is flat" on arrows of $\textit{WAdm}^{Noeth}$ is a local property as defined in Formal Spaces, Remark 21.5.*

Proof. Let us recall what the statement signifies. First, $\textit{WAdm}^{Noeth}$ is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & (B')^\wedge \\ A \arrow[u, "\varphi"] \arrow[r] & (A')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 21.1. We have to show

(1) φ is flat $\Rightarrow \varphi'$ is flat,
(2) if $B \to B'$ faithfully flat, then φ' is flat $\Rightarrow \varphi$ is flat, and
(3) if $A \to B_i$ is flat for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is flat.

We will use without further mention that completions of Noetherian rings are flat (Algebra, Lemma 97.2). Since of course $A \to A'$ and $B \to B'$ are flat, we see in particular that the horizontal arrows in the diagram are flat.

Proof of (1). If φ is flat, then the composition $A \to (A')^\wedge \to (B')^\wedge$ is flat. Hence $A' \to (B')^\wedge$ is flat by More on Flatness, Lemma 2.3. Hence we see that $(A')^\wedge \to (B')^\wedge$ is flat by applying More on Algebra, Lemma 27.5 with $R = A'$, with ideal $I(A')$, and with $M = (B')^\wedge = M^\wedge$.

Proof of (2). Assume φ' is flat and $B \to B'$ is faithfully flat. Then the composition $A \to (A')^\wedge \to (B')^\wedge$ is flat. Also we see that $B \to (B')^\wedge$ is faithfully flat by Formal Spaces, Lemma 19.14. Hence by Algebra, Lemma 39.9 we find that $\varphi : A \to B$ is flat.

Proof of (3). Omitted. $\square$

Lemma 13.2. *Denote P the property of arrows of WAdm^{Noeth} defined in Lemma 13.1. Denote Q the property defined in Lemma 12.1 viewed as a property of arrows of WAdm^{Noeth}. Denote R the property defined in Lemma 11.1 viewed as a property of arrows of WAdm^{Noeth}. Then*

(1) *P is stable under base change by Q (Formal Spaces, Remark 21.10), and*
(2) *$P + R$ is stable under base change (Formal Spaces, Remark 21.9).*

Proof. The statement makes sense as each of the properties P, Q, and R is a local property of morphisms of WAdm^{Noeth}. Let $\varphi : B \to A$ and $\psi : B \to C$ be morphisms of WAdm^{Noeth}. If either $Q(\varphi)$ or $Q(\psi)$ then we see that $A\widehat{\otimes}_B C$ is Noetherian by Formal Spaces, Lemma 4.12. Since R implies Q (Lemma 12.4), we find that this holds in both cases (1) and (2). This is the first thing we have to check. It remains to show that $C \to A\widehat{\otimes}_B C$ is flat.

Proof of (1). Fix ideals of definition $I \subset A$ and $J \subset B$. By Lemma 12.5 the ring map $B \to C$ is topologically of finite type. Hence $B \to C/J^n$ is of finite type for all $n \geq 1$. Hence $A \otimes_B C/J^n$ is Noetherian as a ring (because it is of finite type over A and A is Noetherian). Thus the I-adic completion $A\widehat{\otimes}_B C/J^n$ of $A \otimes_B C/J^n$ is flat over C/J^n because $C/J^n \to A \otimes_B C/J^n$ is flat as a base change of $B \to A$ and because $A \otimes_B C/J^n \to A\widehat{\otimes}_B C/J^n$ is flat by Algebra, Lemma 97.2 Observe that $A\widehat{\otimes}_B C/J^n = (A\widehat{\otimes}_B C)/J^n(A\widehat{\otimes}_B C)$; details omitted. We conclude that $M = A\widehat{\otimes}_B C$ is a C-module which is complete with respect to the J-adic topology such that M/J^nM is flat over C/J^n for all $n \geq 1$. This implies that M is flat over C by More on Algebra, Lemma 27.4.

Proof of (2). In this case $B \to A$ is adic and hence we have just $A\widehat{\otimes}_B C = \lim A \otimes_B C/J^n$. The rings $A \otimes_B C/J^n$ are Noetherian by an application of Formal Spaces, Lemma 4.12 with C replaced by C/J^n. We conclude in the same manner as before. □

Lemma 13.3. *Denote P the property of arrows of $WAdm^{Noeth}$ defined in Lemma 13.1. Then P is stable under composition (Formal Spaces, Remark 21.14).*

Proof. This is true because compositions of flat ring maps are flat. □

Definition 13.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is *flat* if for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a flat map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on the source and target.

Lemma 13.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent*

1. *f is flat,*
2. *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

   with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a flat map in $WAdm^{Noeth}$,
3. *there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 11.1 such that each $X_{ji} \to Y_j$ corresponds to a flat map in $WAdm^{Noeth}$, and*
4. *there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a flat map in $WAdm^{Noeth}$.*

Proof. The equivalence of (1) and (2) is Definition 13.4. The equivalence of (2), (3), and (4) follows from the fact that being flat is a local property of arrows of WAdm^{Noeth} by Lemma 13.1 and an application of the variant of Formal Spaces, Lemma 21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 21.5. ☐

Lemma 13.6. *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S.*

(1) *If f is flat and $g_{red} : Z_{red} \to Y_{red}$ is locally of finite type, then the base change $X \times_Y Z \to Z$ is flat.*
(2) *If f is flat and locally of finite type, then the base change $X \times_Y Z \to Z$ is flat and locally of finite type.*

Proof. Part (1) follows from a combination of Formal Spaces, Remark 21.10, Lemma 13.2 part (1), Lemma 13.5, and Lemma 12.7.

Part (2) follows from a combination of Formal Spaces, Remark 21.9, Lemma 13.2 part (2), Lemma 13.5, and Lemma 11.5. ☐

Lemma 13.7. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are flat, then so is $g \circ f$.*

Proof. Combine Formal Spaces, Remark 21.14 and Lemma 13.3. ☐

Lemma 13.8. *Let S be a scheme. Let $f : X \to Y$ be a morphisms of locally Noetherian formal algebraic spaces over S. If f is representable by algebraic spaces and flat in the sense of Bootstrap, Definition 4.1, then f is flat in the sense of Definition 13.4.*

Proof. This is a sanity check whose proof should be trivial but isn't quite. We urge the reader to skip the proof. Assume f is representable by algebraic spaces and flat in the sense of Bootstrap, Definition 4.1. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale. Then the morphism $U \to V$ corresponds to a taut map $B \to A$ of $\textit{WAdm}^{Noeth}$ by Formal Spaces, Lemma 22.2. Observe that this means $B \to A$ is adic (Formal Spaces, Lemma 23.1) and in particular for any ideal of definition $J \subset B$ the topology on A is the J-adic topology and the diagrams

$$\begin{tikzcd} \mathrm{Spec}(A/J^nA) \arrow[r] \arrow[d] & \mathrm{Spec}(B/J^n) \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

are cartesian.

Let $T \to V$ is a morphism where T is a scheme. Then

$$\begin{aligned} X \times_Y T \to T \text{ is flat} &\Rightarrow U \times_Y T \to T \text{ is flat} \\ &\Rightarrow U \times_V V \times_Y T \to T \text{ is flat} \\ &\Rightarrow U \times_V V \times_Y T \to V \times_Y T \text{ is flat} \\ &\Rightarrow U \times_V T \to T \text{ is flat} \end{aligned}$$

The first statement is the assumption on f. The first implication because $U \to X$ is étale and hence flat and compositions of flat morphisms of algebraic spaces are flat. The second impliciation because $U \times_Y T = U \times_V V \times_Y T$. The third implication by More on Flatness, Lemma 2.3. The fourth implication because we can pullback by the morphism $T \to V \times_Y T$. We conclude that $U \to V$ is flat in the sense of Bootstrap, Definition 4.1. In terms of the continuous ring map $B \to A$ this means the ring maps $B/J^n \to A/J^nA$ are flat (see diagram above).

Finally, we can conclude that $B \to A$ is flat for example by More on Algebra, Lemma 27.4. $\square$

14. Rig-closed points

We develop just enough theory to be able to use this for testing rig-flatness in a later section. The reader can find more theory in [BL93] who discuss (among other things) the case of locally Noetherian formal schemes.

Lemma 14.1. *Let A be a Noetherian adic topological ring. Let $\mathfrak q \subset A$ be a prime ideal. The following are equivalent*

(1) *for some ideal of definition $I \subset A$ we have $I \not\subset \mathfrak q$ and $\mathfrak q$ is maximal with respect to this property,*
(2) *for some ideal of definition $I \subset A$ the prime $\mathfrak q$ defines a closed point of* $\text{Spec}(A) \setminus V(I)$,
(3) *for any ideal of definition $I \subset A$ we have $I \not\subset \mathfrak q$ and $\mathfrak q$ is maximal with respect to this property,*
(4) *for any ideal of definition $I \subset A$ the prime $\mathfrak q$ defines a closed point of* $\text{Spec}(A) \setminus V(I)$,
(5) $\dim(A/\mathfrak q) = 1$ *and for some ideal of definition $I \subset A$ we have $I \not\subset \mathfrak q$,*
(6) $\dim(A/\mathfrak q) = 1$ *and for any ideal of definition $I \subset A$ we have $I \not\subset \mathfrak q$,*
(7) $\dim(A/\mathfrak q) = 1$ *and the induced topology on $A/\mathfrak q$ is nontrivial,*
(8) *$A/\mathfrak q$ is a 1-dimensional Noetherian complete local domain whose maximal ideal is the radical of the image of any ideal of definition of A, and*
(9) *add more here.*

Proof. It is clear that (1) and (2) are equivalent and for the same reason that (3) and (4) are equivalent. Since $V(I)$ is independent of the choice of the ideal of definition I of A, we see that (2) and (4) are equivalent.

Assume the equivalent conditions (1) – (4) hold. If $\dim(A/\mathfrak q) > 1$ we can choose a maximal ideal $\mathfrak q \subset \mathfrak m \subset A$ such that $\dim((A/\mathfrak q)_\mathfrak m) > 1$. Then $\text{Spec}((A/\mathfrak q)_\mathfrak m) - V(I(A/\mathfrak q)_\mathfrak m)$ would be infinite by Algebra, Lemma 61.1. This contradicts the fact that $\mathfrak q$ is closed in $\text{Spec}(A) \setminus V(I)$. Hence we see that (6) holds. Trivially (6) implies (5).

Conversely, assume (5) holds. Let $I \subset A$ be an ideal of definition. Since $A/\mathfrak q$ is complete with respect to $I(A/\mathfrak q)$ (for example by Algebra, Lemma 97.1) we see that all closed points of $\text{Spec}(A/\mathfrak q)$ are contained in $V(IA/\mathfrak q)$ by Algebra, Lemma 96.6. Since $\dim(A/\mathfrak q) = 1$ and since $I \not\subset \mathfrak q$ we conclude two things: (a) $V(IA/\mathfrak q)$ must contain all points distinct from the generic point of $\text{Spec}(A/\mathfrak q)$, and (b) $V(IA/\mathfrak q)$ must be a (finite) discrete set. From (a) we see that $\mathfrak q$ is a closed point of $\text{Spec}(A) \setminus V(I)$ and we conclude that (2) holds.

Continuing to assume (5) we see that the finite discrete space $V(IA/\mathfrak{q})$ must be a singleton by More on Algebra, Lemma 11.16 for example (and the fact that complete pairs are henselian pairs, see More on Algebra, Lemma 11.4). Hence we see that (8) is true. Conversely, it is clear that (8) implies (5).

At this point we know that (1) – (6) and (8) are equivalent. We omit the verification that these are also equivalent to (7). $\square$

In order to comfortably talk about such primes we introduce the following nonstandard notation.

Definition 14.2. Let A be a Noetherian adic topological ring. Let $\mathfrak{q} \subset A$ be a prime ideal. We say $\mathfrak{q}$ is *rig-closed* if the equivalent conditions of Lemma 14.1 are satisfied.

We will need a few lemmas which essentially tell us there are plenty of rig-closed primes even in a relative settting.

Lemma 14.3. *Let $\varphi : A \to B$ in $\textit{WAdm}^{Noeth}$. Denote $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ the ideals of topologically nilpotent elements. Assume $A/\mathfrak{a} \to B/\mathfrak{b}$ is of finite type. Let $\mathfrak{q} \subset B$ be rig-closed. The residue field κ of the local ring $B/\mathfrak{q}$ is a finite type $A/\mathfrak{a}$-algebra.*

Proof. Let $\mathfrak{q} \subset \mathfrak{m} \subset B$ be the unique maximal ideal containing $\mathfrak{q}$. Then $\mathfrak{b} \subset \mathfrak{m}$. Hence $A/\mathfrak{a} \to B/\mathfrak{b} \to B/\mathfrak{m} = \kappa$ is of finite type. $\square$

Lemma 14.4. *Let $\varphi : A \to B$ be an arrow of $\textit{WAdm}^{Noeth}$ which is adic and topologically of finite type. Let $\mathfrak{q} \subset B$ be rig-closed. Let $\mathfrak{p} = \varphi^{-1}(\mathfrak{q}) \subset A$. Let $\mathfrak{a} \subset A$ be the ideal of topologically nilpotent elements. The following are equivalent*

(1) *the residue field κ of $B/\mathfrak{q}$ is finite over $A/\mathfrak{a}$,*
(2) *$\mathfrak{p} \subset A$ is rig-closed,*
(3) *$A/\mathfrak{p} \subset B/\mathfrak{q}$ is a finite extension of rings.*

Proof. Assume (1). Recall that $B/\mathfrak{q}$ is a Noetherian local ring of dimension 1 whose topology is the adic topology coming from the maximal ideal. Since φ is adic, we see that $A \to B/\mathfrak{q}$ is adic. Hence $\varphi(\mathfrak{a})$ is a nonzero ideal in $B/\mathfrak{q}$. Hence $B/\mathfrak{q} + \varphi(\mathfrak{a})$ has finite length. Hence $B/\mathfrak{q} + \varphi(\mathfrak{a})$ is finite as an $A/\mathfrak{a}$-module by our assumption. Thus $B/\mathfrak{q}$ is finite over A by Algebra, Lemma 96.12. Thus (3) holds.

Assume (3). Then $\text{Spec}(B/\mathfrak{q}) \to \text{Spec}(A/\mathfrak{p})$ is surjective by Algebra, Lemma 36.17. This implies (2).

Assume (2). Denote κ' the residue field of $A/\mathfrak{p}$. By Lemma 14.3 (and Lemma 12.4) the extension κ/κ' is finitely generated as an algebra. By the Hilbert Nullstellensatz (Algebra, Lemma 34.2) we see that κ/κ' is a finite extension. Hence we see that (1) holds. $\square$

Lemma 14.5. *Let $\varphi : A \to B$ be an arrow of $\textit{WAdm}^{Noeth}$ which is adic and topologically of finite type. Let $\mathfrak{q} \subset B$ be rig-closed. If A/I is Jacobson for some ideal of definition $I \subset A$, then $\mathfrak{p} = \varphi^{-1}(\mathfrak{q}) \subset A$ is rig-closed.*

Proof. By Lemma 14.3 (combined with Lemma 12.4) the residue field κ of $B/\mathfrak{q}$ is of finite type over $A/\mathfrak{a}$. Since $A/\mathfrak{a}$ is Jacobson, we see that κ is finite over $A/\mathfrak{a}$ by Algebra, Lemma 35.18. We conclude by Lemma 14.4. $\square$

Lemma 14.6. *Let $\varphi : A \to B$ be an arrow of WAdm^{Noeth} which is adic and topologically of finite type. Let $\mathfrak p \subset A$ be rig-closed. Let $\mathfrak a \subset A$ and $\mathfrak b \subset B$ be the ideals of topologically nilpotent elements. If φ is flat, then the following are equivalent*

(1) *the maximal ideal of $A/\mathfrak p$ is in the image of* $\mathrm{Spec}(B/\mathfrak b) \to \mathrm{Spec}(A/\mathfrak a)$*,*
(2) *there exists a rig-closed prime ideal $\mathfrak q \subset B$ such that $\mathfrak p = \varphi^{-1}(\mathfrak q)$.*

and if so then φ, $\mathfrak p$, and $\mathfrak q$ satisfy the conclusions of Lemma 14.4.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1). To prove the existence of $\mathfrak q$ we may replace A by $A/\mathfrak p$ and B by $B/\mathfrak pB$ (some details omitted). Thus we may assume $(A, \mathfrak m, \kappa)$ is a local complete 1-dimensional Noetherian ring, $\mathfrak m = \mathfrak a$, and $\mathfrak p = (0)$. Condition (1) just says that $B_0 = B \otimes_A \kappa = B/\mathfrak mB = B/\mathfrak aB$ is nonzero. Note that B_0 is of finite type over κ. Hence we can use induction on $\dim(B_0)$. If $\dim(B_0) = 0$, then any minimal prime $\mathfrak q \subset B$ will do (flatness of $A \to B$ insures that $\mathfrak q$ will lie over $\mathfrak p = (0)$). If $\dim(B_0) > 0$ then we can find an element $b \in B$ which maps to an element $b_0 \in B_0$ which is a nonzerodivisor and a nonunit, see Algebra, Lemma 63.20. By Algebra, Lemma 99.2 the ring $B' = B/bB$ is flat over A. Since $B'_0 = B' \otimes_A \kappa = B_0/(b_0)$ is not zero, we may apply the induction hypothesis to B' and conclude. The final statement of the lemma is clear from Lemma 14.4. $\square$

We introduce some notation.

Definition 14.7. Let A be an adic topological ring which has a finitely generated ideal of definition. Let $f \in A$. The *completed principal localization* $A_{\{f\}}$ of A is the completion of $A_f = A[1/f]$ of the principal localization of A at f with respect to any ideal of definition of A.

To be sure, if f is topologically nilpotent, then $A_{\{f\}}$ is the zero ring.

Lemma 14.8. *Let A be an adic Noetherian topological ring. Let $\mathfrak p \subset A$ be a prime ideal. Let $f \in A$ be an element mapping to a unit in $A/\mathfrak p$. Then*

$$\mathfrak pA_{\{f\}} = \mathfrak p(A_f)^\wedge = \mathfrak p \otimes_A (A_f)^\wedge = (\mathfrak p_f)^\wedge$$

is a prime ideal with quotient

$$A/\mathfrak p = (A/\mathfrak p) \otimes_A (A_f)^\wedge = (A_f)^\wedge/\mathfrak p(A_f)^\wedge = A_{\{f\}}/\mathfrak pA_{\{f\}}$$

Proof. Since A_f is Noetherian the ring map $A \to A_f \to (A_f)^\wedge$ is flat. For any finite A-module M we see that $M \otimes_A (A_f)^\wedge$ is the completion of M_f. If f is a unit on M, then $M_f = M$ is already complete. See discussion in Algebra, Section 97. From these observations the results follow easily. $\square$

Lemma 14.9. *Let $\varphi : A \to B$ be an arrow of WAdm^{Noeth} which is adic and topologically of finite type. Let $\mathfrak q \subset B$ be rig-closed. There exists an $f \in A$ which maps to a unit in $B/\mathfrak q$ such that we obtain a diagram*

$$\begin{tikzcd} B \arrow[r] & B_{\{f\}} \\ A \arrow[u, "\varphi"] \arrow[r] & A_{\{f\}} \arrow[u, "\varphi_{\{f\}}"'] \end{tikzcd} \quad \text{with primes} \quad \begin{tikzcd} \mathfrak q \arrow[r, dash] \arrow[d, dash] & \mathfrak q' \arrow[r, equal] \arrow[d, dash] & \mathfrak qB_{\{f\}} \\ \mathfrak p \arrow[r, dash] & \mathfrak p' & \end{tikzcd}$$

such that $\mathfrak p'$ is rig-closed, i.e., the map $A_{\{f\}} \to B_{\{f\}}$ and the prime ideals $\mathfrak q'$ and $\mathfrak p'$ satisfy the equivalent conditions of Lemma 14.4.

Proof. Please see Lemma 14.8 for the description of $\mathfrak{q}'$. The only assertion the lemma makes is that for a suitable choice of f the prime ideal $\mathfrak{p}'$ has the property $\dim((A_f)^\wedge/\mathfrak{p}') = 1$. By Lemma 14.4 this in turn just means that the residue field κ of $B/\mathfrak{q} = (B_f)^\wedge/\mathfrak{q}'$ is finite over $(A_f)^\wedge/\mathfrak{a}' = (A/\mathfrak{a})_f$. By Lemma 14.3 we know that $A/\mathfrak{a} \to \kappa$ is a finite type algebra homomorphism. By the Hilbert Nullstellensatz in the form of Algebra, Lemma 34.2 we can find an $f \in A$ which maps to a unit in κ such that κ is finite over A_f. This finishes the proof. □

Lemma 14.10. *Let A be a Noetherian adic topological ring. Denote $A\{x_1, \ldots, x_n\}$ the restricted power series over A. Let $\mathfrak{q} \subset A\{x_1, \ldots, x_n\}$ be a prime ideal. Set $\mathfrak{q}' = A[x_1, \ldots, x_n] \cap \mathfrak{q}$ and $\mathfrak{p} = A \cap \mathfrak{q}$. If $\mathfrak{q}$ and $\mathfrak{p}$ are rig-closed, then the map*

$$A[x_1, \ldots, x_n]_{\mathfrak{q}'} \to A\{x_1, \ldots, x_n\}_{\mathfrak{q}}$$

defines an isomorphism on completions with respect to their maximal ideals.

Proof. By Lemma 14.4 the ring map $A/\mathfrak{p} \to A\{x_1, \ldots, x_n\}/\mathfrak{q}$ is finite. For every $m \geq 1$ the module $\mathfrak{q}^m/\mathfrak{q}^{m+1}$ is finite over A as it is a finite $A\{x_1, \ldots, x_n\}/\mathfrak{q}$-module. Hence $A\{x_1, \ldots x_n\}/\mathfrak{q}^m$ is a finite A-module. Hence $A[x_1, \ldots, x_n] \to A\{x_1, \ldots, x_n\}/\mathfrak{q}^m$ is surjective (as the image is dense and an A-submodule). It follows in a straightforward manner that $A[x_1, \ldots, x_n]/(\mathfrak{q}')^m \to A\{x_1, \ldots, x_n\}/\mathfrak{q}^m$ is an isomorphism for all m. From this the lemma easily follows. Hint: Pick a topologically nilpotent $g \in A$ which is not contained in $\mathfrak{p}$. Then the map of completions is the map

$$\lim\nolimits_m \left(A[x_1, \ldots, x_n]/(\mathfrak{q}')^m\right)_g \longrightarrow \left(A\{x_1, \ldots, x_n\}/\mathfrak{q}^m\right)_g$$

Some details omitted. □

Lemma 14.11. *Let $\varphi : A \to B$ be an arrow of $\textit{WAdm}^{Noeth}$. Assume φ is adic, topologically of finite type, flat, and $A/I \to B/IB$ is étale for some (resp. any) ideal of definition $I \subset A$. Let $\mathfrak{q} \subset B$ be rig-closed such that $\mathfrak{p} = A \cap \mathfrak{q}$ is rig-closed as well. Then $\mathfrak{p}B_{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{q}}$.*

Proof. Let κ be the residue field of the 1-dimensional complete local ring $A/\mathfrak{p}$. Since $A/I \to B/IB$ is étale, we see that $B \otimes_A \kappa$ is a finite product of finite separable extensions of κ, see Algebra, Lemma 143.4. One of these is the residue field of $B/\mathfrak{q}$. By Algebra, Lemma 96.12 we see that $B/\mathfrak{p}B$ is a finite $A/\mathfrak{p}$-algebra. It is also flat. Combining the above we see that $A/\mathfrak{p} \to B/\mathfrak{p}B$ is finite étale, see Algebra, Lemma 143.7. Hence $B/\mathfrak{p}B$ is reduced, which implies the statement of the lemma (details omitted). □

Lemma 14.12. *Let A be an adic Noetherian topological ring. Let $\mathfrak{p} \subset A$ be a rig-closed prime. For any $n \geq 1$ the ring map*

$$A/\mathfrak{p} \longrightarrow A\{x_1, \ldots, x_n\} \otimes_A A/\mathfrak{p} = A/\mathfrak{p}\{x_1, \ldots, x_n\}$$

is regular. In particular, the algebra $A\{x_1, \ldots, x_n\} \otimes_A \kappa(\mathfrak{p})$ is geometrically regular over $\kappa(\mathfrak{p})$.

Proof. We will use some fact on regular ring maps the reader can find in More on Algebra, Section 41. Since $A/\mathfrak{p}$ is a complete local Noetherian ring it is excellent (More on Algebra, Proposition 52.3). Hence $A/\mathfrak{p}[x_1, \ldots, x_n]$ is excellent (by the same reference). Hence $A/\mathfrak{p}[x_1, \ldots, x_n] \to A/\mathfrak{p}\{x_1, \ldots, x_n\}$ is a regular ring homomorphism by More on Algebra, Lemma 50.14. Of course $A/\mathfrak{p} \to A/\mathfrak{p}[x_1, \ldots, x_n]$ is smooth and hence regular. Since the composition of regular ring maps is regular the proof is complete. ☐

15. Rig-flat homomorphisms

In this section we define rig-flat homomorphisms of adic Noetherian topological rings.

Lemma 15.1. *Let $\varphi : A \to B$ be a morphism in $\textit{WAdm}^{adic*}$ (Formal Spaces, Section 21). Assume φ is adic. The following are equivalent:*

1. *B_f is flat over A for all topologically nilpotent $f \in A$,*
2. *B_g is flat over A for all topologically nilpotent $g \in B$,*
3. *$B_\mathfrak{q}$ is flat over A for all primes $\mathfrak{q} \subset B$ which do not contain an ideal of definition,*
4. *$B_\mathfrak{q}$ is flat over A for every rig-closed prime $\mathfrak{q} \subset B$, and*
5. *add more here.*

Proof. Follows from the definitions and Algebra, Lemma 39.18. ☐

Definition 15.2. Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of $\textit{WAdm}^{Noeth}$. We say φ is *naively rig-flat* if φ is adic, topologically of finite type, and satisfies the equivalent conditions of Lemma 15.1.

The example below shows that this notion does not "localize".

Example 15.3. By Examples, Lemma 17.1 there exists a local Noetherian 2-dimensional domain $(A, \mathfrak{m})$ complete with respect to a principal ideal $I = (a)$ and an element $f \in \mathfrak{m}$, $f \notin I$ with the following property: the ring $A_{\{f\}}[1/a]$ is nonreduced. Here $A_{\{f\}}$ is the I-adic completion $(A_f)^\wedge$ of the principal localization A_f. To be sure the ring $A_{\{f\}}[1/a]$ is nonzero. Let $B = A_{\{f\}}/\text{nil}(A_{\{f\}})$ be the quotient by its nilradical. Observe that $A \to B$ is adic and topologically of finite type. In fact, B is a quotient of $A\{x\} = A[x]^\wedge$ by the map sending x to the image of $1/f$ in B. Every prime $\mathfrak{q}$ of B not containing a must lie over $(0) \subset A$[2]. Hence $B_\mathfrak{q}$ is flat over A as it is a module over the fraction field of A. Thus $A \to B$ is naively rig-flat. On the other hand, the map

$$A_{\{f\}} \longrightarrow B_{\{f\}} = (B_f)^\wedge = B = A_{\{f\}}/\text{nil}(A_{\{f\}})$$

is not flat after inverting a because we get the nontrivial surjection $A_{\{f\}}[1/a] \to A_{\{f\}}[1/a]/\text{nil}(A_{\{f\}}[1/a])$. Hence $A_{\{f\}} \to B^\wedge_{\{f\}}$ is not naively rig-flat!

It turns out that it is easy to work around this problem by using the following definition.

Definition 15.4. Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of $\textit{WAdm}^{Noeth}$. We say φ is *rig-flat* if φ is adic, topologically of finite type, and for all $f \in A$ the induced map

$$A_{\{f\}} \longrightarrow B_{\{f\}}$$

is naively rig-flat (Definition 15.2).

---

[2]Namely, we can find $\mathfrak{q} \subset \mathfrak{q}' \subset B$ with $a \in \mathfrak{q}'$ because B is a-adically complete. Then $\mathfrak{p}' = A \cap \mathfrak{q}'$ contains a but not f hence is a height 1 prime. Then $\mathfrak{p} = A \cap \mathfrak{q}$ must be strictly contained in $\mathfrak{p}'$ as $a \notin \mathfrak{p}$. Since $\dim(A) = 2$ we see that $\mathfrak{p} = (0)$.

Setting $f = 1$ in the definition above we see that rig-flatness implies naive rig-flatness. The example shows the converse is false. However, in many situations we don't need to worry about the difference between rig-flatness and its naive version as the next lemma shows.

Lemma 15.5. *Let $\varphi : A \to B$ be an arrow of $\textit{WAdm}^{Noeth}$. If A/I is Jacobson for some (equivalently any) ideal of definition $I \subset A$ and φ is naively rig-flat, then φ is rig-flat.*

Proof. Assume φ is naively rig-flat. We first state some obvious consequences of the assumptions. Namely, let $f \in A$. Then $A, B, A_{\{f\}}, B_{\{f\}}$ are Noetherian adic topological rings. The maps $A \to A_{\{f\}} \to B_{\{f\}}$ and $A \to B \to B_{\{f\}}$ are adic and topologically of finite type. The ring maps $A \to A_{\{f\}}$ and $B \to B_{\{f\}}$ are flat as compositions of $A \to A_f$ and $B \to B_f$ and the completion maps which are flat by Algebra, Lemma 97.2. The quotients of each of the rings $A, B, A_{\{f\}}, B_{\{f\}}$ by I is of finite type over A/I and hence Jacobson too (Algebra, Proposition 35.19).

Let $\mathfrak{q}' \subset B_{\{f\}}$ be rig-closed. It suffices to prove that $(B_{\{f\}})_{\mathfrak{q}'}$ is flat over $A_{\{f\}}$, see Lemma 15.1. By Lemma 14.5 the primes $\mathfrak{q} \subset B$ and $\mathfrak{p}' \subset A_{\{f\}}$ and $\mathfrak{p} \subset A$ lying under $\mathfrak{q}'$ are rig-closed. We are going to apply Algebra, Lemma 100.2 to the diagram

$$\begin{tikzcd} B_{\mathfrak{q}} \arrow[r] & (B_{\{f\}})_{\mathfrak{q}'} \\ A_{\mathfrak{p}} \arrow[u] \arrow[r] & (A_{\{f\}})_{\mathfrak{p}'} \arrow[u] \end{tikzcd}$$

with $M = B_{\mathfrak{q}}$. The only assumption that hasn't been checked yet is the fact that $\mathfrak{p}$ generates the maximal ideal of $(A_{\{f\}})_{\mathfrak{p}'}$. This follows from Lemma 14.8; here we use that $\mathfrak{p}$ and $\mathfrak{p}'$ are rig-closed to see that f maps to a unit of $A/\mathfrak{p}$ (this is the only step in the proof that fails without the Jacobson assumption). Namely, this tells us that $A/\mathfrak{p} \to A_{\{f\}}/\mathfrak{p}'$ is a finite inclusion of local rings (Lemma 14.4) and f maps to a unit in the second one. $\square$

Lemma 15.6. *Let $\varphi : A \to B$ and $A \to C$ be arrows of $\textit{WAdm}^{Noeth}$. Assume φ is rig-flat and $A \to C$ adic and topologically of finite type. Then $C \to B\widehat{\otimes}_A C$ is rig-flat.*

Proof. Assume φ is rig-flat. Let $f \in C$ be an element. We have to show that $C_{\{f\}} \to B\widehat{\otimes}_A C_{\{f\}}$ is naively rig-flat. Since we can replace C by $C_{\{f\}}$ we it suffices to show that $C \to B\widehat{\otimes}_A C$ is naively rig-flat.

If $A \to C$ is surjective or more generally if C is finite as an A-module, then $B \otimes_A C = B\widehat{\otimes}_A C$ as a finite module over a complete Noetherian ring is complete, see Algebra, Lemma 97.1. By the usual base change for flatness (Algebra, Lemma 39.7) we see that naive rig-flatness of φ implies naive rig-flatness for $C \to B \times_A C$ in this case.

In the general case, we can factor $A \to C$ as $A \to A\{x_1, \ldots, x_n\} \to C$ where $A\{x_1, \ldots, x_n\}$ is the restricted power series ring and $A\{x_1, \ldots, x_n\} \to C$ is surjective. Thus it suffices to show $C \to B\widehat{\otimes}_A B$ is naively rig-flat in case $C = A\{x_1, \ldots, x_n\}$. Since $A\{x_1, \ldots, x_n\} = A\{x_1, \ldots, x_{n-1}\}\{x_n\}$ by induction on n we reduce to the case discussed in the next paragraph.

Here $C = A\{x\}$. Note that $B\widehat{\otimes}_A C = B\{x\}$. We have to show that $A\{x\} \to B\{x\}$ is naively rig-flat. Let $\mathfrak q \subset B\{x\}$ be a rig-closed prime ideal. We have to show that $B\{x\}_{\mathfrak q}$ is flat over $A\{x\}$. Set $\mathfrak p = A \cap \mathfrak q$. By Lemma 14.9 we can find an $f \in A$ such that f maps to a unit in $B\{x\}/\mathfrak q$ and such that the prime ideal $\mathfrak p'$ in $A_{\{f\}}$ induced is rig-closed. Below we will use that $A_{\{f\}}\{x\} = A\{x\}_{\{f\}}$ and similarly for B; details omitted. Consider the diagram

$$\begin{tikzcd} (B\{x\})_{\mathfrak q} \arrow[r] & (B_{\{f\}}\{x\})_{\mathfrak q'} \\ A\{x\} \arrow[r] \arrow[u] & A_{\{f\}}\{x\} \arrow[u] \end{tikzcd}$$

We want to show that the left vertical arrow is flat. The top horizontal arrow is faithfully flat as it is a local homomorphism of local rings and flat as $B_{\{f\}}\{x\}$ is the completion of a localization of the Noetherian ring $B\{x\}$. Similarly the bottom horizontal arrow is flat. Hence it suffices to prove that the right vertical arrow is flat. This reduces us to the case discussed in the next paragraph.

Here $C = A\{x\}$, we have a rig-closed prime ideal $\mathfrak q \subset B\{x\}$ such that $\mathfrak p = A \cap \mathfrak q$ is rig-closed as well. This implies, via Lemma 14.4, that the intermediate primes $B \cap \mathfrak q$ and $A\{x\} \cap \mathfrak q$ are rig-closed as well. Consider the diagram

$$\begin{tikzcd} (B[x])_{B[x]\cap\mathfrak q} \arrow[r] & (B\{x\})_{\mathfrak q} \\ (A[x])_{A[x]\cap\mathfrak q} \arrow[r] \arrow[u] & (A\{x\})_{A\{x\}\cap\mathfrak q} \arrow[u] \end{tikzcd}$$

of local homomorphisms of Noetherian local rings. By Lemma 14.10 the horizontal arrows define isomorphisms on completions. We already know that the left vertical arrow is flat (as $A \to B$ is naively rig-flat and hence $A[x] \to B[x]$ is flat away from the closed locus defined by an ideal of definition). Hence we finally conclude by More on Algebra, Lemma 43.8. $\square$

Lemma 15.7. *Consider a commutative diagram*

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[r] \arrow[u, "\varphi"] & A' \arrow[u, "\varphi'"'] \end{tikzcd}$$

in $WAdm^{Noeth}$ with all arrows adic and topologically of finite type. Assume $A \to A'$ and $B \to B'$ are flat. Let $I \subset A$ be an ideal of definition. If φ is rig-flat and $A/I \to A'/IA'$ is étale, then φ' is rig-flat.

Proof. Given $f \in A'$ the assumptions of the lemma remain true for the digram

$$\begin{tikzcd} B \arrow[r] & (B')_{\{f\}} \\ A \arrow[r] \arrow[u, "\varphi"] & (A')_{\{f\}} \arrow[u] \end{tikzcd}$$

Hence it suffices to prove that φ' is naively rig-flat.

Take a rig-closed prime ideal $\mathfrak{q}' \subset B'$. We have to show that $(B')_{\mathfrak{q}'}$ is flat over A'. We can choose an $f \in A$ which maps to a unit of $B'/\mathfrak{q}'$ such that the induced prime ideal $\mathfrak{p}''$ of $A_{\{f\}}$ is rig-closed, see Lemma 14.9. To be precise, here $\mathfrak{q}'' = \mathfrak{q}' B'_{\{f\}}$ and $\mathfrak{p}'' = A_{\{f\}} \cap \mathfrak{q}''$. Consider the diagram

$$\begin{tikzcd} B'_{\mathfrak{q}'} \arrow[r] & (B'_{\{f\}})_{\mathfrak{q}''} \\ A \arrow[u] \arrow[r] & A_{\{f\}} \arrow[u] \end{tikzcd}$$

We want to show that the left vertical arrow is flat. The top horizontal arrow is faithfully flat as it is a local homomorphism of local rings and flat as $B'_{\{f\}}$ is the completion of a localization of the Noetherian ring B'_f. Similarly the bottom horizontal arrow is flat. Hence it suffices to prove that the right vertical arrow is flat. Finally, all the assumptions of the lemma remain true for the diagram

$$\begin{tikzcd} B_{\{f\}} \arrow[r] & B'_{\{f\}} \\ A_{\{f\}} \arrow[u] \arrow[r] & A'_{\{f\}} \arrow[u] \end{tikzcd}$$

This reduces us to the case discussed in the next paragraph.

Take a rig-closed prime ideal $\mathfrak{q}' \subset B'$ and assume $\mathfrak{p} = A \cap \mathfrak{q}'$ is rig-closed as well. This implies also the primes $\mathfrak{q} = B \cap \mathfrak{q}'$ and $\mathfrak{p}' = A' \cap \mathfrak{q}'$ are rig-closed, see Lemma 14.4. We are going to apply Algebra, Lemma 100.2 to the diagram

$$\begin{tikzcd} B_{\mathfrak{q}} \arrow[r] & B'_{\mathfrak{q}'} \\ A_{\mathfrak{p}} \arrow[u] \arrow[r] & A'_{\mathfrak{p}'} \arrow[u] \end{tikzcd}$$

with $M = B_{\mathfrak{q}}$. The only assumption that hasn't been checked yet is the fact that $\mathfrak{p}$ generates the maximal ideal of $A'_{\mathfrak{p}'}$. This follows from Lemma 14.11. ☐

Lemma 15.8. *Consider a commutative diagram*

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u, "\varphi"] \arrow[r] & A' \arrow[u, "\varphi'"'] \end{tikzcd}$$

in $WAdm^{Noeth}$ with all arrows adic and topologically of finite type. Assume $A \to A'$ flat and $B \to B'$ faithfully flat. If φ' is rig-flat, then φ is rig-flat.

Proof. Given $f \in A$ the assumptions of the lemma remain true for the digram

$$\begin{tikzcd} B_{\{f\}} \arrow[r] & (B')_{\{f\}} \\ A_{\{f\}} \arrow[u, "\varphi"] \arrow[r] & (A')_{\{f\}} \arrow[u] \end{tikzcd}$$

(To check the condition on faithful flatness: faithful flatness of $B \to B'$ is equivalent to $B \to B'$ being flat and $\mathrm{Spec}(B'/IB') \to \mathrm{Spec}(B/IB)$ being surjective for some ideal of definition $I \subset A$.) Hence it suffices to prove that φ is naively rig-flat. However, we know that φ' is naively rig-flat and that $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is surjective. From this the result follows immediately. □

Finally, we can show that rig-flatness is a local property.

Lemma 15.9. *The property $P(\varphi) =$"φ is rig-flat" on arrows of $WAdm^{Noeth}$ is a local property as defined in Formal Spaces, Remark 21.4.*

Proof. Let us recall what the statement signifies. First, $WAdm^{Noeth}$ is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & (B')^\wedge \\ A \arrow[u, "\varphi"] \arrow[r] & (A')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 21.1. We have to show

(1) φ is rig-flat $\Rightarrow$ φ' is rig-flat,
(2) if $B \to B'$ faithfully flat, then φ' is rig-flat $\Rightarrow$ φ is rig-flat, and
(3) if $A \to B_i$ is rig-flat for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is rig-flat.

Being adic and topologically of finite type satisfies conditions (1), (2), and (3), see Lemma 11.1. Thus in verifying (1), (2), and (3) for the property "rig-flat" we may already assume our ring maps are all adic and topologically of finite type. Then (1) and (2) follow from Lemmas 15.7 and 15.8. We omit the trivial proof of (3). □

Lemma 15.10. *The property $P(\varphi) =$"φ is rig-flat" on arrows of $WAdm^{Noeth}$ is stable under composition as defined in Formal Spaces, Remark 21.14.*

Proof. The statement makes sense by Lemma 15.9. To see that it is true assume we have rig-flat morphisms $A \to B$ and $B \to C$ in $WAdm^{Noeth}$. Then $A \to C$ is adic and topologically of finite type by Lemma 11.4. To finish the proof we have to show that for all $f \in A$ the map $A_{\{f\}} \to C_{\{f\}}$ is naively rig-flat. Since $A_{\{f\}} \to B_{\{f\}}$ and $B_{\{f\}} \to C_{\{f\}}$ are naively rig-flat, it suffices to show that compositions of naively rig-flat maps are naively rig-flat. This is a consequence of Algebra, Lemma 39.4. □

16. Rig-flat morphisms

In this section we use the work done in Section 15 to define rig-flat morphisms of locally Noetherian algebraic spaces.

Definition 16.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is *rig-flat* if for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-flat map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on source and target.

Lemma 16.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent*

(1) *f is rig-flat,*
(2) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-flat map in $\textit{WAdm}^{Noeth}$,
(3) *there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 11.1 such that each $X_{ji} \to Y_j$ corresponds to a rig-flat map in $\textit{WAdm}^{Noeth}$, and*
(4) *there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a rig-flat map in $\textit{WAdm}^{Noeth}$.*

Proof. The equivalence of (1) and (2) is Definition 16.1. The equivalence of (2), (3), and (4) follows from the fact that being rig-flat is a local property of arrows of WAdm^{Noeth} by Lemma 15.9 and an application of the variant of Formal Spaces, Lemma 21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 21.5. $\square$

Lemma 16.3. *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S. If f is rig-flat and g is locally of finite type, then the base change $X \times_Y Z \to Z$ is rig-flat.*

Proof. By Formal Spaces, Remark 21.10 and the discussion in Formal Spaces, Section 23, this follows from Lemma 15.6. $\square$

Lemma 16.4. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are rig-flat, then so is $g \circ f$.*

Proof. By Formal Spaces, Remark 21.14 this follows from Lemma 15.10. $\square$

17. Rig-smooth homomorphisms

In this section we prove some properties of rig-smooth homomorphisms of adic Noetherian topological rings which are needed to introduce rig-smooth morpisms of locally Noetherian formal algebraic spaces.

Lemma 17.1. *Let $A \to B$ be a morphism in $\textit{WAdm}^{Noeth}$ (Formal Spaces, Section 21). The following are equivalent:*

(a) *$A \to B$ satisfies the equivalent conditions of Lemma 11.1 and there exists an ideal of definition $I \subset B$ such that B is rig-smooth over (A, I), and*
(b) *$A \to B$ satisfies the equivalent conditions of Lemma 11.1 and for all ideals of definition $I \subset A$ the algebra B is rig-smooth over (A, I).*

Proof. Let I and I' be ideals of definitions of A. Then there exists an integer $c \geq 0$ such that $I^c \subset I'$ and $(I')^c \subset I$. Hence B is rig-smooth over (A, I) if and only if B is rig-smooth over (A, I'). This follows from Definition 4.1, the inclusions $I^c \subset I'$ and $(I')^c \subset I$, and the fact that the naive cotangent complex $NL^\wedge_{B/A}$ is independent of the choice of ideal of definition of A by Remark 11.2. $\square$

Definition 17.2. Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of $\textit{WAdm}^{Noeth}$. We say φ is *rig-smooth* if the equivalent conditions of Lemma 17.1 hold.

This defines a local property.

Lemma 17.3. *The property $P(\varphi) =$ "φ is rig-smooth" on arrows of $\textit{WAdm}^{Noeth}$ is a local property as defined in Formal Spaces, Remark 21.5.*

Proof. Let us recall what the statement signifies. First, $\textit{WAdm}^{Noeth}$ is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & (B')^\wedge \\ A \arrow[u, "\varphi"] \arrow[r] & (A')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 21.1. We have to show

(1) φ is rig-smooth $\Rightarrow \varphi'$ is rig-smooth,
(2) if $B \to B'$ faithfully flat, then φ' is rig-smooth $\Rightarrow \varphi$ is rig-smooth, and
(3) if $A \to B_i$ is rig-smooth for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is rig-smooth.

The equivalent conditions of Lemma 11.1 satisfy conditions (1), (2), and (3). Thus in verifying (1), (2), and (3) for the property "rig-smooth" we may already assume our ring maps satisfy the equivalent conditions of Lemma 11.1 in each case.

Pick an ideal of definition $I \subset A$. By the remarks above the topology on each ring in the diagram is the I-adic topology and B, $(A')^\wedge$, and $(B')^\wedge$ are in the category (2.0.2) for (A, I). Since $A \to A'$ and $B \to B'$ are étale the complexes $NL_{A'/A}$

and $NL_{B'/B}$ are zero and hence $NL^\wedge_{(A')^\wedge/A}$ and $NL^\wedge_{(B')^\wedge/B}$ are zero by Lemma 3.2. Applying Lemma 3.5 to $A \to (A')^\wedge \to (B')^\wedge$ we get isomorphisms

$$H^i(NL^\wedge_{(B')^\wedge/(A')^\wedge}) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

Thus $NL^\wedge_{(B')^\wedge/A} \to NL_{(B')^\wedge/(A')^\wedge}$ is a quasi-isomorphism. The ring maps $B/I^nB \to B'/I^nB'$ are étale and hence are local complete intersections (Algebra, Lemma 143.2). Hence we may apply Lemmas 3.5 and 3.6 to $A \to B \to (B')^\wedge$ and we get isomorphisms

$$H^i(NL^\wedge_{B/A} \otimes_B (B')^\wedge) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

We conclude that $NL^\wedge_{B/A} \otimes_B (B')^\wedge \to NL^\wedge_{(B')^\wedge/A}$ is a quasi-isomorphism. Combining these two observations we obtain that

$$NL^\wedge_{(B')^\wedge/(A')^\wedge} \cong NL^\wedge_{B/A} \otimes_B (B')^\wedge$$

in $D((B')^\wedge)$. With these preparations out of the way we can start the actual proof.

Proof of (1). Assume φ is rig-smooth. Then there exists a $c \geq 0$ such that $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by I^c for every B-module N. By More on Algebra, Lemmas 84.6 and 84.7 this property is preserved under base change by $B \to (B')^\wedge$. Hence $\mathrm{Ext}^1_{(B')^\wedge}(NL^\wedge_{(B')^\wedge/(A')^\wedge}, N)$ is annihilated by $I^c(A')^\wedge$ for all $(B')^\wedge$-modules N which tells us that φ' is rig-smooth. This proves (1).

To prove (2) assume $B \to B'$ is faithfully flat and that φ' is rig-smooth. Then there exists a $c \geq 0$ such that $\mathrm{Ext}^1_{(B')^\wedge}(NL^\wedge_{(B')^\wedge/(A')^\wedge}, N')$ is annihilated by $I^c(B')^\wedge$ for every $(B')^\wedge$-module N'. The composition $B \to B' \to (B')^\wedge$ is flat (Algebra, Lemma 97.2) hence for any B-module N we have

$$\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N) \otimes_B (B')^\wedge = \mathrm{Ext}^1_{(B')^\wedge}(NL^\wedge_{B/A} \otimes_B (B')^\wedge, N \otimes_B (B')^\wedge)$$

by More on Algebra, Lemma 99.2 part (3) (minor details omitted). Thus we see that this module is annihilated by I^c. However, $B \to (B')^\wedge$ is actually faithfully flat by our assumption that $B \to B'$ is faithfully flat (Formal Spaces, Lemma 19.14). Thus we conclude that $\mathrm{Ext}^1_B(NL^\wedge_{B/A}, N)$ is annihilated by I^c. Hence φ is rig-smooth. This proves (2).

To prove (3), setting $B = \prod_{i=1,\ldots,n} B_i$ we just observe that $NL^\wedge_{B/A}$ is the direct sum of the complexes $NL^\wedge_{B_i/A}$ viewed as complexes of B-modules. $\square$

Lemma 17.4. *Consider the properties $P(\varphi)$ ="φ is rig-smooth" and $Q(\varphi)$="φ is adic" on arrows of $WAdm^{Noeth}$. Then P is stable under base change by Q as defined in Formal Spaces, Remark 21.10.*

Proof. The statement makes sense by Lemma 17.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in $WAdm^{Noeth}$ and that $B \to A$ is rig-smooth and $B \to C$ is adic (Formal Spaces, Definition 6.1). Then we can choose an ideal of definition $I \subset B$ such that the topology on A and C is the I-adic topology. In this situation it follows immediately that $A \widehat{\otimes}_B C$ is rig-smooth over (C, IC) by Lemma 4.5. $\square$

Lemma 17.5. *The property $P(\varphi)$ ="φ is rig-smooth" on arrows of $WAdm^{Noeth}$ is stable under composition as defined in Formal Spaces, Remark 21.14.*

Proof. We strongly urge the reader to find their own proof and not read the proof that follows. The statement makes sense by Lemma 17.1. To see that it is true assume we have rig-smooth morphisms $A \to B$ and $B \to C$ in $\textit{WAdm}^{Noeth}$. Then we can choose an ideal of definition $I \subset A$ such that the topology on C and B is the I-adic topology. By Lemma 3.5 we obtain an exact sequence

$$\begin{tikzcd} C \otimes_B H^0(NL^\wedge_{B/A}) \arrow[r] & H^0(NL^\wedge_{C/A}) \arrow[r] & H^0(NL^\wedge_{C/B}) \arrow[r] & 0 \\ H^{-1}(NL^\wedge_{B/A} \otimes_B C) \arrow[r] & H^{-1}(NL^\wedge_{C/A}) \arrow[r] & H^{-1}(NL^\wedge_{C/B}) \arrow[ull] \end{tikzcd}$$

Observe that $H^{-1}(NL^\wedge_{B/A} \otimes_B C)$ and $H^{-1}(NL^\wedge_{C/B})$ are annihilated by a power of I; this follows from Lemma 4.2 part (2) combined with More on Algebra, Lemmas 84.6 and 84.7 (to deal with the base change by $B \to C$). Hence $H^{-1}(NL^\wedge_{C/A})$ is annihilated by a power of I. Next, by the characterization of rig-smooth algebras in Lemma 4.2 part (2) which in turn refers to More on Algebra, Lemma 84.10 part (5) we can choose $f_1, \ldots, f_s \in IB$ and $g_1, \ldots, g_t \in IC$ such that $V(f_1, \ldots, f_s) = V(IB)$ and $V(g_1, \ldots, g_t) = V(IC)$ and such that $H^0(NL^\wedge_{B/A})_{f_i}$ is a finite projective B_{f_i}-module and $H^0(NL^\wedge_{C/B})_{g_j}$ is a finite projective C_{g_j}-module. Since the cohomologies in degree -1 vanish upon localization at $f_i g_j$ we get a short exact sequence

$$0 \to (C \otimes_B H^0(NL^\wedge_{B/A}))_{f_i g_j} \to H^0(NL^\wedge_{C/A})_{f_i g_j} \to H^0(NL^\wedge_{C/B})_{f_i g_j} \to 0$$

and we conclude that $H^0(NL^\wedge_{C/A})_{f_i g_j}$ is a finite projective $C_{f_i g_j}$-module as an extension of same. Thus by the criterion in Lemma 4.2 part (2) and via that the criterion in More on Algebra, Lemma 84.10 part (4) we conclude that C is rig-smooth over (A, I). $\square$

The following lemma can be interpreted as saying that a rig-smooth homomorphism is "rig-syntomic" or "rig-flat+rig-lci".

Lemma 17.6. *Let $\varphi : A \to B$ be an arrow of $\textit{WAdm}^{Noeth}$. If φ is rig-smooth, then φ is rig-flat, and for any presentation $B = A\{x_1, \ldots, x_n\}/J$ and prime $J \subset \mathfrak{q} \subset A\{x_1, \ldots, x_n\}$ not containing an ideal of definition the ideal $J_\mathfrak{q} \subset A\{x_1, \ldots, x_n\}_\mathfrak{q}$ is generated by a regular sequence.*

Proof. Let $f \in A$. To prove that φ is rig-flat we have to show that $\varphi_{\{f\}} : A_{\{f\}} \to B_{\{f\}}$ is naively rig-flat. Now either by viewing $\varphi_{\{f\}}$ as a base change of φ and using Lemma 17.4 or by using the fact that being rig-smooth is a local property (Lemma 17.3) we see that $\varphi_{\{f\}}$ is rig-smooth. Hence it suffices to show that φ is naively rig-flat.

Choose a presentation $B = A\{x_1, \ldots, x_n\}/J$. In order to check the second part of the lemma it suffices to check $J_\mathfrak{q} \subset A\{x_1, \ldots, x_n\}_\mathfrak{q}$ is generated by a regular sequence for $J \subset \mathfrak{q}$ for $\mathfrak{q}$ maximal with respect to not containing an ideal of definition, see Algebra, Lemma 68.6 (which shows that the set of primes in $V(J)$ where there is a regular sequence generating J is open). In other words, we may assume $\mathfrak{q}$ is rig-closed in $A\{x_1, \ldots, x_n\}$. And to check that B is naively rig-flat, it also suffices to check that the corresponding localizations $B_\mathfrak{q}$ are flat over A.

Let $\mathfrak{q} \subset A\{x_1, \ldots, x_n\}$ be rig-closed with $J \subset \mathfrak{q}$. By Lemma 14.9 we may choose an $f \in A$ mapping to a unit in $A\{x_1, \ldots, x_n\}/\mathfrak{q}$ and such that the prime ideal $\mathfrak{p}'$ in $A_{\{f\}}$

induced is rig-closed. Below we will use that $A_{\{f\}}\{x_1, \ldots, x_n\} = A\{x_1, \ldots, x_n\}_{\{f\}}$; details omitted. Consider the diagram

$$\begin{tikzcd}
A\{x_1, \ldots, x_n\}_\mathfrak{q}/J_\mathfrak{q} \arrow[r] & A_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}/JA_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'} \\
A\{x_1, \ldots, x_n\}_\mathfrak{q} \arrow[r] \arrow[u] & A_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'} \arrow[u] \\
A \arrow[r] \arrow[u] & A_{\{f\}} \arrow[u]
\end{tikzcd}$$

The middle horizontal arrow is faithfully flat as it is a local homomorphism of local rings and flat as $A_{\{f\}}\{x_1, \ldots, x_n\}$ is the completion of a localization of the Noetherian ring $A\{x_1, \ldots, x_n\}$. Similarly the bottom horizontal arrow is flat. Hence to show that $J_\mathfrak{q}$ is generated by a regular sequence and that $A \to A\{x_1, \ldots, x_n\}_\mathfrak{q}/J_\mathfrak{q}$ is flat, it suffices to prove the same things for $JA_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}$ and $A_{\{f\}} \to A_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}/JA_{\{f\}}\{x_1, \ldots, x_n\}_{\mathfrak{q}'}$. See Algebra, Lemma 68.5 or More on Algebra, Lemma 32.4 for the statement on regular sequences. Finally, we have already seen that $A_{\{f\}} \to B_{\{f\}}$ is rig-smooth. This reduces us to the case discussed in the next paragraph.

Let $\mathfrak{q} \subset A\{x_1, \ldots, x_n\}$ be rig-closed with $J \subset \mathfrak{q}$ such that moreover $\mathfrak{p} = A \cap \mathfrak{q}$ is rig-closed as well. By the characterization of rig-smooth algebras given in Lemma 4.2 after reordering the variables $x_1, \ldots, x_n$ we can find $m \geq 0$ and $f_1, \ldots, f_m \in J$ such that

(1) $J_\mathfrak{q}$ is generated by $f_1, \ldots, f_m$, and
(2) $\det_{1 \leq i,j \leq m}(\partial f_j/\partial x_i)$ maps to a unit in $A\{x_1, \ldots, x_n\}_\mathfrak{q}$.

By Lemma 14.12 the fibre ring

$$F = A\{x_1, \ldots, x_n\} \otimes_A \kappa(\mathfrak{p})$$

is regular. Observe that the A-derivations $\partial/\partial x_i$ extend (uniquely) to derivations $D_i : F \to F$. By More on Algebra, Lemma 48.3 we see that $f_1, \ldots, f_m$ map to a regular sequence in $F_\mathfrak{q}$. By flatness of $A \to A\{x_1, \ldots, x_n\}$ and Algebra, Lemma 99.3 this shows that $f_1, \ldots, f_m$ map to a regular sequence in $A\{x_1, \ldots, x_m\}_\mathfrak{q}$ and the quotient by these elements is flat over A. This finishes the proof. $\square$

Lemma 17.7. *Let $A \to B \to C$ be arrows in $WAdm^{Noeth}$ which are adic and topologically of finite type. If $B \to C$ is rig-smooth, then the kernel of the map*

$$H^{-1}(NL^\wedge_{B/A} \otimes_B C) \to H^{-1}(NL^\wedge_{C/A})$$

(see Lemma 3.5) is annihilated by an ideal of definition.

Proof. Let $\overline{\mathfrak{q}} \subset C$ be a prime ideal which does not contain an ideal of definition. Since the modules in question are finite it suffices to show that

$$H^{-1}(NL^\wedge_{B/A} \otimes_B C)_{\overline{\mathfrak{q}}} \to H^{-1}(NL^\wedge_{C/A})_{\overline{\mathfrak{q}}}$$

is injective. As in the proof of Lemma 3.5 choose presentations $B = A\{x_1, \ldots, x_r\}/J$, $C = B\{y_1, \ldots, y_s\}/J'$, and $C = A\{x_1, \ldots, x_r, y_1, \ldots, y_s\}/K$. Looking at the diagram in the proof of Lemma 3.5 we see that it suffices to show that $J/J^2 \otimes_B C \to K/K^2$ is injective after localization at the prime ideal $\mathfrak{q} \subset A\{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ corresponding to $\overline{\mathfrak{q}}$. Please compare with More on Algebra, Lemma 33.6 and its proof. This is the same as asking $J/KJ \to K/K^2$ to be injective after localization at $\mathfrak q$. Equivalently, we have to show that $J_{\mathfrak q} \cap K_{\mathfrak q}^2 = (KJ)_{\mathfrak q}$. By Lemma 17.6 we know that $(K/J)_{\mathfrak q} = J'_{\mathfrak q}$ is generated by a regular sequence. Hence the desired intersection property follows from More on Algebra, Lemma 32.5 (and the fact that an ideal generated by a regular sequence is H_1-regular, see More on Algebra, Section 32). ☐

18. Rig-smooth morphisms

In this section we use the work done in Section 17 to define rig-smooth morphisms of locally Noetherian algebraic spaces.

Definition 18.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is *rig-smooth* if for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-smooth map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on source and target.

Lemma 18.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent*

(1) *f is rig-smooth,*
(2) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-smooth map in WAdm^{Noeth},
(3) *there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 11.1 such that each $X_{ji} \to Y_j$ corresponds to a rig-smooth map in WAdm^{Noeth}, and*
(4) *there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a rig-smooth map in WAdm^{Noeth}.*

Proof. The equivalence of (1) and (2) is Definition 18.1. The equivalence of (2), (3), and (4) follows from the fact that being rig-smooth is a local property of arrows of WAdm^{Noeth} by Lemma 17.3 and an application of the variant of Formal Spaces, Lemma 21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 21.5. ☐

Lemma 18.3. *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S. If f is rig-smooth and g is adic, then the base change $X \times_Y Z \to Z$ is rig-smooth.*

Proof. By Formal Spaces, Remark 21.10 and the discussion in Formal Spaces, Section 23, this follows from Lemma 17.4. $\square$

Lemma 18.4. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are rig-smooth, then so is $g \circ f$.*

Proof. By Formal Spaces, Remark 21.14 this follows from Lemma 17.5. $\square$

Lemma 18.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. If f is rig-smooth, then f is rig-flat.*

Proof. Follows immediately from Lemma 17.6 and the definitions. $\square$

19. Rig-étale homomorphisms

In this section we prove some properties of rig-étale homomorphisms of adic Noetherian topological rings which are needed to introduce rig-étale morphisms of locally Noetherian algebraic spaces.

Lemma 19.1. *Let $A \to B$ be a morphism in $\textit{WAdm}^{Noeth}$ (Formal Spaces, Section 21). The following are equivalent:*

(a) *$A \to B$ satisfies the equivalent conditions of Lemma 11.1 and there exists an ideal of definition $I \subset B$ such that B is rig-étale over (A, I), and*
(b) *$A \to B$ satisfies the equivalent conditions of Lemma 11.1 and for all ideals of definition $I \subset A$ the algebra B is rig-étale over (A, I).*

Proof. Let I and I' be ideals of definitions of A. Then there exists an integer $c \geq 0$ such that $I^c \subset I'$ and $(I')^c \subset I$. Hence B is rig-étale over (A, I) if and only if B is rig-étale over (A, I'). This follows from Definition 8.1, the inclusions $I^c \subset I'$ and $(I')^c \subset I$, and the fact that the naive cotangent complex $NL^\wedge_{B/A}$ is independent of the choice of ideal of definition of A by Remark 11.2. $\square$

Definition 19.2. Let $\varphi : A \to B$ be a continuous ring homomorphism between adic Noetherian topological rings, i.e., φ is an arrow of $\textit{WAdm}^{Noeth}$. We say φ is *rig-etale* if the equivalent conditions of Lemma 19.1 hold.

This defines a local property.

Lemma 19.3. *The property $P(\varphi) =$ "φ is rig-étale" on arrows of $\textit{WAdm}^{Noeth}$ is a local property as defined in Formal Spaces, Remark 21.5.*

Proof. This proof is exactly the same as the proof of Lemma 17.3. Let us recall what the statement signifies. First, $\textit{WAdm}^{Noeth}$ is the category whose objects are adic Noetherian topological rings and whose morphisms are continuous ring homomorphisms. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & (B')^\wedge \\ A \arrow[u, "\varphi"] \arrow[r] & (A')^\wedge \arrow[u, "\varphi'"'] \end{tikzcd}$$

satisfying the following conditions: A and B are adic Noetherian topological rings, $A \to A'$ and $B \to B'$ are étale ring maps, $(A')^\wedge = \lim A'/I^nA'$ for some ideal of definition $I \subset A$, $(B')^\wedge = \lim B'/J^nB'$ for some ideal of definition $J \subset B$, and $\varphi : A \to B$ and $\varphi' : (A')^\wedge \to (B')^\wedge$ are continuous. Note that $(A')^\wedge$ and $(B')^\wedge$ are adic Noetherian topological rings by Formal Spaces, Lemma 21.1. We have to show

(1) φ is rig-étale $\Rightarrow \varphi'$ is rig-étale,
(2) if $B \to B'$ faithfully flat, then φ' is rig-étale $\Rightarrow \varphi$ is rig-étale, and
(3) if $A \to B_i$ is rig-étale for $i = 1, \ldots, n$, then $A \to \prod_{i=1,\ldots,n} B_i$ is rig-étale.

The equivalent conditions of Lemma 11.1 satisfy conditions (1), (2), and (3). Thus in verifying (1), (2), and (3) for the property "rig-étale" we may already assume our ring maps satisfy the equivalent conditions of Lemma 11.1 in each case.

Pick an ideal of definition $I \subset A$. By the remarks above the topology on each ring in the diagram is the I-adic topology and B, $(A')^\wedge$, and $(B')^\wedge$ are in the category (2.0.2) for (A, I). Since $A \to A'$ and $B \to B'$ are étale the complexes $NL_{A'/A}$ and $NL_{B'/B}$ are zero and hence $NL^\wedge_{(A')^\wedge/A}$ and $NL^\wedge_{(B')^\wedge/B}$ are zero by Lemma 3.2. Applying Lemma 3.5 to $A \to (A')^\wedge \to (B')^\wedge$ we get isomorphisms

$$H^i(NL^\wedge_{(B')^\wedge/(A')^\wedge}) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

Thus $NL^\wedge_{(B')^\wedge/A} \to NL_{(B')^\wedge/(A')^\wedge}$ is a quasi-isomorphism. The ring maps $B/I^nB \to B'/I^nB'$ are étale and hence are local complete intersections (Algebra, Lemma 143.2). Hence we may apply Lemmas 3.5 and 3.6 to $A \to B \to (B')^\wedge$ and we get isomorphisms

$$H^i(NL^\wedge_{B/A} \otimes_B (B')^\wedge) \to H^i(NL^\wedge_{(B')^\wedge/A})$$

We conclude that $NL^\wedge_{B/A} \otimes_B (B')^\wedge \to NL^\wedge_{(B')^\wedge/A}$ is a quasi-isomorphism. Combining these two observations we obtain that

$$NL^\wedge_{(B')^\wedge/(A')^\wedge} \cong NL^\wedge_{B/A} \otimes_B (B')^\wedge$$

in $D((B')^\wedge)$. With these preparations out of the way we can start the actual proof.

Proof of (1). Assume φ is rig-étale. Then there exists a $c \geq 0$ such that multiplication by $a \in I^c$ is zero on $NL^\wedge_{B/A}$ in $D(B)$. This property is preserved under base change by $B \to (B')^\wedge$, see More on Algebra, Lemmas 84.6. By the isomorphism above we find that φ' is rig-étale. This proves (1).

To prove (2) assume $B \to B'$ is faithfully flat and that φ' is rig-étale. Then there exists a $c \geq 0$ such that multiplication by $a \in I^c$ is zero on $NL^\wedge_{(B')^\wedge/(A')^\wedge}$ in $D((B')^\wedge)$. By the isomorphism above we see that a^c annihilates the cohomology modules of $NL^\wedge_{B/A} \otimes_B (B')^\wedge$. The composition $B \to (B')^\wedge$ is faithfully flat by our assumption that $B \to B'$ is faithfully flat, see Formal Spaces, Lemma 19.14. Hence the cohomology modules of $NL^\wedge_{B/A}$ are annihilated by I^c. It follows from Lemma 8.2 that φ is rig-étale. This proves (2).

To prove (3), setting $B = \prod_{i=1,\ldots,n} B_i$ we just observe that $NL^\wedge_{B/A}$ is the direct sum of the complexes $NL^\wedge_{B_i/A}$ viewed as complexes of B-modules. □

Lemma 19.4. *Consider the properties $P(\varphi)$ ="φ is rig-étale" and $Q(\varphi)$="φ is adic" on arrows of $WAdm^{Noeth}$. Then P is stable under base change by Q as defined in Formal Spaces, Remark 21.10.*

Proof. The statement makes sense by Lemma 19.1. To see that it is true assume we have morphisms $B \to A$ and $B \to C$ in $WAdm^{Noeth}$ and that $B \to A$ is rig-étale and $B \to C$ is adic (Formal Spaces, Definition 6.1). Then we can choose an ideal of definition $I \subset B$ such that the topology on A and C is the I-adic topology. In this situation it follows immediately that $A \widehat{\otimes}_B C$ is rig-étale over (C, IC) by Lemma 8.6. $\square$

Lemma 19.5. *The property $P(\varphi) =$ "φ is rig-étale" on arrows of $WAdm^{Noeth}$ is stable under composition as defined in Formal Spaces, Remark 21.14.*

Proof. The statement makes sense by Lemma 19.1. To see that it is true assume we have rig-étale morphisms $A \to B$ and $B \to C$ in $WAdm^{Noeth}$. Then we can choose an ideal of definition $I \subset A$ such that the topology on C and B is the I-adic topology. By Lemma 3.5 we obtain an exact sequence

$$\begin{tikzcd} C \otimes_B H^0(NL^\wedge_{B/A}) \arrow[r] & H^0(NL^\wedge_{C/A}) \arrow[r] & H^0(NL^\wedge_{C/B}) \arrow[r] & 0 \\ H^{-1}(NL^\wedge_{B/A} \otimes_B C) \arrow[r] & H^{-1}(NL^\wedge_{C/A}) \arrow[r] & H^{-1}(NL^\wedge_{C/B}) \arrow[ull] \end{tikzcd}$$

There exists a $c \geq 0$ such that for all $a \in I$ multiplication by a^c is zero on $NL^\wedge_{B/A}$ in $D(B)$ and $NL^\wedge_{C/B}$ in $D(C)$. Then of course multiplication by a^c is zero on $NL^\wedge_{B/A} \otimes_B C$ in $D(C)$ too. Hence $H^0(NL^\wedge_{B/A}) \otimes_A C$, $H^0(NL^\wedge_{C/B})$, $H^{-1}(NL^\wedge_{B/A} \otimes_B C)$, and $H^{-1}(NL^\wedge_{C/B})$ are annihilated by a^c. From the exact sequence we obtain that multiplication by a^{2c} is zero on $H^0(NL^\wedge_{C/A})$ and $H^{-1}(NL^\wedge_{C/A})$. It follows from Lemma 8.2 that C is rig-étale over (A, I) as desired. $\square$

Lemma 19.6. *The property $P(\varphi) =$ "φ is rig-étale" on arrows of $WAdm^{Noeth}$ has the cancellation property as defined in Formal Spaces, Remark 21.18.*

Proof. The statement makes sense by Lemma 19.1. To see that it is true assume we have maps $A \to B$ and $B \to C$ in $WAdm^{Noeth}$ with $A \to C$ and $A \to B$ rig-étale. We have to show that $B \to C$ is rig-étale. Then we can choose an ideal of definition $I \subset A$ such that the topology on C and B is the I-adic topology. By Lemma 3.5 we obtain an exact sequence

$$\begin{tikzcd} C \otimes_B H^0(NL^\wedge_{B/A}) \arrow[r] & H^0(NL^\wedge_{C/A}) \arrow[r] & H^0(NL^\wedge_{C/B}) \arrow[r] & 0 \\ H^{-1}(NL^\wedge_{B/A} \otimes_B C) \arrow[r] & H^{-1}(NL^\wedge_{C/A}) \arrow[r] & H^{-1}(NL^\wedge_{C/B}) \arrow[ull] \end{tikzcd}$$

There exists a $c \geq 0$ such that for all $a \in I$ multiplication by a^c is zero on $NL^\wedge_{B/A}$ in $D(B)$ and $NL^\wedge_{C/A}$ in $D(C)$. Hence $H^0(NL^\wedge_{B/A}) \otimes_A C$, $H^0(NL^\wedge_{C/A})$, and $H^{-1}(NL^\wedge_{C/A})$ are annihilated by a^c. From the exact sequence we obtain that multiplication by a^{2c} is zero on $H^0(NL^\wedge_{C/B})$ and $H^{-1}(NL^\wedge_{C/B})$. It follows from Lemma 8.2 that C is rig-étale over (B, IB) as desired. $\square$

20. Rig-étale morphisms

In this section we use the work done in Section 19 to define rig-étale morphisms of locally Noetherian algebraic spaces.

Definition 20.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is *rig-étale* if for every commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-étale map of adic Noetherian topological rings.

Let us prove that we can check this condition étale locally on source and target.

Lemma 20.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. The following are equivalent*

(1) *f is rig-étale,*
(2) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U and V affine formal algebraic spaces, $U \to X$ and $V \to Y$ representable by algebraic spaces and étale, the morphism $U \to V$ corresponds to a rig-étale map in $\textit{WAdm}^{Noeth}$,
(3) *there exists a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 11.1 and for each j a covering $\{X_{ji} \to Y_j \times_Y X\}$ as in Formal Spaces, Definition 11.1 such that each $X_{ji} \to Y_j$ corresponds to a rig-étale map in $\textit{WAdm}^{Noeth}$, and*
(4) *there exist a covering $\{X_i \to X\}$ as in Formal Spaces, Definition 11.1 and for each i a factorization $X_i \to Y_i \to Y$ where Y_i is an affine formal algebraic space, $Y_i \to Y$ is representable by algebraic spaces and étale, and $X_i \to Y_i$ corresponds to a rig-étale map in $\textit{WAdm}^{Noeth}$.*

Proof. The equivalence of (1) and (2) is Definition 20.1. The equivalence of (2), (3), and (4) follows from the fact that being rig-étale is a local property of arrows of WAdm^{Noeth} by Lemma 19.3 and an application of the variant of Formal Spaces, Lemma 21.3 for morphisms between locally Noetherian algebraic spaces mentioned in Formal Spaces, Remark 21.5. □

To be sure, a rig-étale morphism is locally of finite type.

Lemma 20.3. *A rig-étale morphism of locally Noetherian formal algebraic spaces is locally of finite type.*

Proof. The property P in Lemma 19.3 implies the equivalent conditions (a), (b), (c), and (d) in Formal Spaces, Lemma 29.6. Hence this follows from Formal Spaces, Lemma 29.9. □

Lemma 20.4. *A rig-étale morphism of locally Noetherian formal algebraic spaces is rig-smooth.*

Proof. Follows from the definitions and Lemma 8.3. □

Lemma 20.5. *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of locally Noetherian formal algebraic spaces over S. If f is rig-étale and g is adic, then the base change $X \times_Y Z \to Z$ is rig-étale.*

Proof. By Formal Spaces, Remark 21.10 and the discussion in Formal Spaces, Section 23, this follows from Lemma 19.4. □

Lemma 20.6. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If f and g are rig-étale, then so is $g \circ f$.*

Proof. By Formal Spaces, Remark 21.14 this follows from Lemma 19.5. □

Lemma 20.7. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be a morphism of locally Noetherian formal algebraic spaces over S. If $g \circ f$ and g are rig-étale, then so is f.*

Proof. By Formal Spaces, Remark 21.18 this follows from Lemma 19.6. □

Lemma 20.8. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. If $g \circ f$ is rig-étale and g is an adic monomorphism, then f is rig-étale.*

Proof. Use Lemma 20.5 and that f is the base change of $g \circ f$ by g. □

Lemma 20.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume that X and Y are locally Noetherian and f is a closed immersion. The following are equivalent*

(1) *f is rig-smooth,*
(2) *f is rig-étale,*
(3) *for every affine formal algebraic space V and every morphism $V \to Y$ which is representable by algebraic spaces and étale the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in WAdm^{Noeth} whose kernel J has the following property: $I(J/J^2) = 0$ for some ideal of definition I of B.*

Proof. Let us observe that given V and $V \to Y$ as in (2) without any further assumption on f we see that the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in WAdm^{Noeth} by Formal Spaces, Lemma 29.5.

We have (2) $\Rightarrow$ (1) by Lemma 20.4.

Proof of (3) $\Rightarrow$ (2). Assume (3). By Lemma 20.2 it suffices to show that the ring maps $B \to A$ occuring in (3) are rig-étale in the sense of Definition 19.2. Let I be as in (3). The naive cotangent complex $NL^\wedge_{A/B}$ of A over (B, I) is the complex of A-modules given by putting J/J^2 in degree -1. Hence A is rig-étale over (B, I) by Definition 8.1.

Assume (1) and let V and $B \to A$ be as in (3). By Definition 18.1 we see that $B \to A$ is rig-smooth. Choose any ideal of definition $I \subset B$. Then A is rig-smooth over (B, I). As above the complex $NL^\wedge_{A/B}$ is given by putting J/J^2 in degree -1. Hence by Lemma 4.2 we see that J/J^2 is annihilated by a power I^n for some $n \geq 1$. Since B is adic, we see that I^n is an ideal of definition of B and the proof is complete. □

21. Rig-surjective morphisms

For morphisms locally of finite type between locally Noetherian formal algebraic spaces a definition borrowed from [Art70] can be used. See Remark 21.2 for a discussion of what to do in more general cases.

Definition 21.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Assume that X and Y are locally Noetherian and that f is locally of finite type. We say f is *rig-surjective* if for every solid diagram

$$\begin{tikzcd} \mathrm{Spf}(R') \arrow[r, dotted] \arrow[d, dotted] & X \arrow[d, "f"] \\ \mathrm{Spf}(R) \arrow[r, "p"] & Y \end{tikzcd}$$

where R is a complete discrete valuation ring and where p is an adic morphism there exists an extension of complete discrete valuation rings $R \subset R'$ and a morphism $\mathrm{Spf}(R') \to X$ making the displayed diagram commute.

We will see in the lemmas below that this notion behaves reasonably well in the context of locally Noetherian formal algebraic spaces and morphisms which are locally of finite type. In the next remark we discuss options for modifying this definition to a wider class of morphisms of formal algebraic spaces.

Remark 21.2. The condition as formulated in Definition 21.1 is not right even for morphisms of finite type of locally adic* formal algebraic spaces. For example, if $A = (\bigcup_{n \geq 1} k[t^{1/n}])^\wedge$ where the completion is the t-adic completion, then there are no adic morphisms $\mathrm{Spf}(R) \to \mathrm{Spf}(A)$ where R is a complete discrete valuation ring. Thus any morphism $X \to \mathrm{Spf}(A)$ would be rig-surjective, but since A is a domain and $t \in A$ is not zero, we want to think of A as having at least one "rig-point", and we do not want to allow $X = \emptyset$. To cover this particular case, one can consider adic morphisms

$$\mathrm{Spf}(R) \longrightarrow Y$$

where R is a valuation ring complete with respect to a principal ideal J whose radical is $\mathfrak{m}_R = \sqrt{J}$. In this case the value group of R can be embedded into $(\mathbf{R}, +)$ and one obtains the point of view used by Berkovich in defining an analytic space associated to Y, see [Ber90]. Another approach is championed by Huber. In his theory, one drops the hypothesis that $\mathrm{Spec}(R/J)$ is a singleton, see [Hub93].

Lemma 21.3. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z are locally Noetherian and f and g locally of finite type. Then if f and g are rig-surjective, so is $g \circ f$.*

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemma 24.3). $\square$

Lemma 21.4. *Let S be a scheme. Let $f : X \to Y$ and $Z \to Y$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z are locally Noetherian and f and g locally of finite type. If f is rig-surjective, then the base change $Z \times_Y X \to Z$ is too.*

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemmas 24.9 and 24.4). $\square$

Lemma 21.5. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms locally of finite type of locally Noetherian formal algebraic spaces over S. If $g \circ f$ is rig-surjective and g is a monomorphism, then f is rig-surjective.*

Proof. Use Lemma 21.4 and that f is the base change of $g \circ f$ by g. $\square$

Lemma 21.6. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z locally Noetherian and f and g locally of finite type. If $g \circ f : X \to Z$ is rig-surjective, so is $g : Y \to Z$.*

Proof. Immediate from the definition. $\square$

Lemma 21.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces which is representable by algebraic spaces, étale, and surjective. Then f is rig-surjective.*

Proof. Let $p : \mathrm{Spf}(R) \to Y$ be an adic morphism where R is a complete discrete valuation ring. Let $Z = \mathrm{Spf}(R) \times_Y X$. Then $Z \to \mathrm{Spf}(R)$ is representable by algebraic spaces, étale, and surjective. Hence Z is nonempty. Pick a nonempty affine formal algebraic space V and an étale morphism $V \to Z$ (possible by our definitions). Then $V \to \mathrm{Spf}(R)$ corresponds to $R \to A^\wedge$ where $R \to A$ is an étale ring map, see Formal Spaces, Lemma 19.13. Since $A^\wedge \neq 0$ (as $V \neq \emptyset$) we can find a maximal ideal $\mathfrak m$ of A lying over $\mathfrak m_R$. Then $A_\mathfrak m$ is a discrete valuation ring (More on Algebra, Lemma 44.4). Then $R' = A_\mathfrak m^\wedge$ is a complete discrete valuation ring (More on Algebra, Lemma 43.5). Applying Formal Spaces, Lemma 9.10. we find the desired morphism $\mathrm{Spf}(R') \to V \to Z \to X$. $\square$

The upshot of the lemmas above is that we may check whether $f : X \to Y$ is rig-surjective, étale locally on Y.

Lemma 21.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces which is locally of finite type. Let $\{g_i : Y_i \to Y\}$ be a family of morphisms of formal algebraic spaces which are representable by algebraic spaces and étale such that $\coprod g_i$ is surjective. Then f is rig-surjective if and only if each $f_i : X \times_Y Y_i \to Y_i$ is rig-surjective.*

Proof. Namely, if f is rig-surjective, so is any base change (Lemma 21.4). Conversely, if all f_i are rig-surjective, so is $\coprod f_i : \coprod X \times_Y Y_i \to \coprod Y_i$. By Lemma 21.7 the morphism $\coprod g_i : \coprod Y_i \to Y$ is rig-surjective. Hence $\coprod X \times_Y Y_i \to Y$ is rig-surjective (Lemma 21.3). Since this morphism factors through $X \to Y$ we see that $X \to Y$ is rig-surjective by Lemma 21.6. $\square$

Lemma 21.9. *Let A be a Noetherian ring complete with respect to an ideal I. Let B be an I-adically complete A-algebra. If $A/I^n \to B/I^nB$ is of finite type and flat for all n and faithfully flat for $n = 1$, then Spf$(B) \to$ Spf(A) is rig-surjective.*

Proof. We will use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 9.10. Let $\varphi : A \to R$ be a continuous map into a complete discrete valuation ring A. This implies that $\varphi(I) \subset \mathfrak m_R$. On the other hand, since we only need to produce the lift $\varphi' : B' \to R'$ in the case that φ corresponds to an adic morphism, we may assume that $\varphi(I) \neq 0$. Thus we may consider the base change $C = B \widehat{\otimes}_A R$, see Remark 2.3 for example. Then C is an $\mathfrak m_R$-adically complete R-algebra such that $C/\mathfrak m_R^n C$ is of finite type and flat over $R/\mathfrak m_R^n$ and such that $C/\mathfrak{m}_R C$ is nonzero. Pick any maximal ideal $\mathfrak{m} \subset C$ lying over $\mathfrak{m}_R$. By flatness (which implies going down) we see that $\text{Spec}(C_\mathfrak{m}) \setminus V(\mathfrak{m}_R C_\mathfrak{m})$ is a nonempty open. Hence We can pick a prime $\mathfrak{q} \subset \mathfrak{m}$ such that $\mathfrak{q}$ defines a closed point of $\text{Spec}(C_\mathfrak{m}) \setminus \{\mathfrak{m}\}$ and such that $\mathfrak{q} \notin V(IC_\mathfrak{m})$, see Properties, Lemma 6.4. Then $C/\mathfrak{q}$ is a dimension 1-local domain and we can find $C/\mathfrak{q} \subset R'$ with R' a discrete valuation ring (Algebra, Lemma 119.13). By construction $\mathfrak{m}_R R' \subset \mathfrak{m}_{R'}$ and we see that $C \to R'$ extends to a continuous map $C \to (R')^\wedge$ (in fact we can pick R' such that $R' = (R')^\wedge$ in our current situation but we do not need this). Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption gives a commutative diagram of rings

$$\begin{tikzcd}
(R')^\wedge & C \arrow[l] & B \arrow[l] \\
R \arrow[u] & R \arrow[l] \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

which gives the desired lift. $\square$

Lemma 21.10. *Let A be a Noetherian ring complete with respect to an ideal I. Let B be an I-adically complete A-algebra. Assume that*

(1) *the I-torsion in A is 0,*
(2) *$A/I^n \to B/I^n B$ is flat and of finite type for all n.*

Then Spf(B) $\to$ Spf(A) is rig-surjective if and only if $A/I \to B/IB$ is faithfully flat.

Proof. Faithful flatness implies rig-surjectivity by Lemma 21.9. To prove the converse we will use without further mention that the vanishing of I-torsion is equivalent to the vanishing of I-power torsion (More on Algebra, Lemma 88.3). We will also use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 9.10.

Assume $\text{Spf}(B) \to \text{Spf}(A)$ is rig-surjective. Choose a maximal ideal $I \subset \mathfrak{m} \subset A$. The open $U = \text{Spec}(A_\mathfrak{m}) \setminus V(I_\mathfrak{m})$ of $\text{Spec}(A_\mathfrak{m})$ is nonempty as the $I_\mathfrak{m}$-torsion of $A_\mathfrak{m}$ is zero (use Algebra, Lemma 62.4). Thus we can find a prime $\mathfrak{q} \subset A_\mathfrak{m}$ which defines a point of U (i.e., $IA_\mathfrak{m} \not\subset \mathfrak{q}$) and which corresponds to a closed point of $\text{Spec}(A_\mathfrak{m}) \setminus \{\mathfrak{m}\}$, see Properties, Lemma 6.4. Then $A_\mathfrak{m}/\mathfrak{q}$ is a dimension 1 local domain. Thus we can find an injective local homomorphism of local rings $A_\mathfrak{m}/\mathfrak{q} \subset R$ where R is a discrete valuation ring (Algebra, Lemma 119.13). By construction $IR \subset \mathfrak{m}_R$ and we see that $A \to R$ extends to a continuous map $A \to R^\wedge$. Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption gives a commutative diagram of rings

$$\begin{tikzcd}
R' & B \arrow[l] \\
R^\wedge \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

Thus we find a prime ideal of B lying over $\mathfrak{m}$. It follows that $\text{Spec}(B/IB) \to \text{Spec}(A/I)$ is surjective, whence $A/I \to B/IB$ is faithfully flat (Algebra, Lemma 39.16). $\square$

Lemma 21.11. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume X and Y are locally Noetherian, f locally of finite type, and f a monomorphism. Then f is rig surjective if and only if every adic morphism Spf(R) $\to Y$ where R is a complete discrete valuation ring factors through X.*

Proof. One direction is trivial. For the other, suppose that $\text{Spf}(R) \to Y$ is an adic morphism such that there exists an extension of complete discrete valuation rings $R \subset R'$ with $\text{Spf}(R') \to \text{Spf}(R) \to X$ factoring through Y. Then $\text{Spec}(R'/\mathfrak{m}_R^n R') \to \text{Spec}(R/\mathfrak{m}_R^n)$ is surjective and flat, hence the morphisms $\text{Spec}(R/\mathfrak{m}_R^n) \to X$ factor through X as X satisfies the sheaf condition for fpqc coverings, see Formal Spaces, Lemma 32.1. In other words, $\text{Spf}(R) \to Y$ factors through X. $\square$

Lemma 21.12. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume that X and Y are locally Noetherian and f is a closed immersion. The following are equivalent*

1. *f is rig-surjective, and*
2. *for every affine formal algebraic space V and every morphism $V \to Y$ which is representable by algebraic spaces and étale the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in $\textit{WAdm}^{Noeth}$ whose kernel J has the following property: $IJ^n = 0$ for some ideal of definition I of B and some $n \geq 1$.*

Proof. Let us observe that given V and $V \to Y$ as in (2) without any further assumption on f we see that the morphism $X \times_Y V \to V$ corresponds to a surjective morphism $B \to A$ in $\textit{WAdm}^{Noeth}$ by Formal Spaces, Lemma 29.5.

Assume (1). By Lemma 21.4 we see that $\text{Spf}(A) \to \text{Spf}(B)$ is rig-surjective. Let $I \subset B$ be an ideal of definition. Since B is adic, $I^m \subset B$ is an ideal of definition for all $m \geq 1$. If $I^m J^n \neq 0$ for all $n, m \geq 1$, then IJ is not nilpotent, hence $V(IJ) \neq \text{Spec}(B)$. Thus we can find a prime ideal $\mathfrak{p} \subset B$ with $\mathfrak{p} \notin V(I) \cup V(J)$. Observe that $I(B/\mathfrak{p}) \neq B/\mathfrak{p}$ hence we can find a maximal ideal $\mathfrak{p} + I \subset \mathfrak{m} \subset B$. By Algebra, Lemma 119.13 we can find a discrete valuation ring R and an injective local ring homomorphism $(B/\mathfrak{p})_\mathfrak{m} \to R$. Clearly, the ring map $B \to R$ cannot factor through $A = B/J$. According to Lemma 21.11 this contradicts the fact that $\text{Spf}(A) \to \text{Spf}(B)$ is rig-surjective. Hence for some n, m we do have $I^n J^m = 0$ which shows that (2) holds.

Assume (2). By Lemma 21.8 it suffices to show that $\text{Spf}(A) \to \text{Spf}(B)$ is rig-surjective. Pick an ideal of definition $I \subset B$ and an integer n such that $IJ^n = 0$. Consider a ring map $B \to R$ where R is a discrete valuation ring and the image of I is nonzero. Since R is a domain, we conclude the image of J in R is zero. Hence $B \to R$ factors through the surjection $B \to A$ and we are done by definition of rig-surjective morphisms. $\square$

Lemma 21.13. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume that X and Y are locally Noetherian and f is a closed immersion. The following are equivalent*

1. *f is rig-smooth and rig-surjective,*
2. *f is rig-étale and rig-surjective, and*
3. *for every affine formal algebraic space V and every morphism $V \to Y$ which is representable by algebraic spaces and étale the morphism $X \times_Y V \to V$*

corresponds to a surjective morphism $B \to A$ in $\textit{WAdm}^{Noeth}$ whose kernel J has the following property: $IJ = 0$ for some ideal of definition I of B.

Proof. Let I and J be ideals of a ring B such that $IJ^n = 0$ and $I(J/J^2) = 0$. Then $I^nJ = 0$ (proof omitted). Hence this lemma follows from a trivial combination of Lemmas 20.9 and 21.12. $\square$

Lemma 21.14. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of locally Noetherian formal algebraic spaces over S. Assume*

(1) *g is locally of finite type,*
(2) *f is rig-smooth (resp. rig-étale) and rig-surjective,*
(3) *$g \circ f$ is rig-smooth (resp. rig-étale)*

then g is rig-smooth (resp. rig-étale).

Proof. We will prove this in the rig-smooth case and indicate the necessary changes to prove the rig-étale case at the end of the proof. Consider a commutative diagram

$$\begin{tikzcd} X \times_Y V \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

with V and W affine formal algebraic spaces, $V \to Y$ and $W \to Z$ representable by algebraic spaces and étale. We have to show that $V \to W$ corresponds to a rig-smooth map of adic Noetherian topological rings, see Definition 18.1. We may write $V = \mathrm{Spf}(B)$ and $W = \mathrm{Spf}(C)$ and that $V \to W$ corresponds to an adic ring map $C \to B$ which is topologically of finite type, see Lemma 11.5.

We will use below without further mention that $X \times_Y V \to V$ is rig-smooth and rig-surjective, see Lemmas 18.3 and 21.4. Also, the composition $X \times_Y V \to V \to W$ is rig-smooth since $g \circ f$ is rig-smooth.

Let $I \subset C$ be an ideal of definition. The module Assume $C \to B$ is not rig-smooth to get a contradiction. This means that there exists a prime ideal $\mathfrak{q} \subset B$ not containing IB such that either $H^{-1}(NL^\wedge_{B/C})_\mathfrak{p}$ is nonzero or $H^0(NL^\wedge_{B/C})_\mathfrak{p}$ is not a finite free $B_\mathfrak{q}$-module. See Lemma 4.2; some details omitted. We may choose a maximal ideal $IB + \mathfrak{q} \subset \mathfrak{m}$. By Algebra, Lemma 119.13 we can find a complete discrete valuation ring R and an injective local ring homomorphism $(B/\mathfrak{q})_\mathfrak{m} \to R$.

After replacing R by an extension, we may assume given a lift $\mathrm{Spf}(R) \to X \times_Y V$ of the adic morphism $\mathrm{Spf}(R) \to V = \mathrm{Spf}(B)$. Choose an étale covering $\{\mathrm{Spf}(A_i) \to X \times_Y V\}$ as in Formal Spaces, Definition 11.1. By Lemma 21.7 we may assume $\mathrm{Spf}(R) \to X \times_Y V$ lifts to a morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(A_i)$ for some i (this might require replacing R by another extension). Set $A = A_i$. Consider the ring maps

$$C \to B \to A \to R$$

Let $\mathfrak{p} \subset A$ be the kernel of the map $A \to R$ and note that $\mathfrak{p}$ lies over $\mathfrak{q}$. We know that $C \to A$ and $B \to A$ are rig-smooth. In particular the ring map $B_\mathfrak{q} \to A_\mathfrak{p}$ is flat by Lemma 17.6. Consider the associated exact sequence

$$\begin{tikzcd}
 & H^0(NL^\wedge_{B/C}) \otimes_B A_\mathfrak{p} \arrow[r] & H^0(NL^\wedge_{A/C})_\mathfrak{p} \arrow[r] & H^0(NL^\wedge_{A/B})_\mathfrak{p} \arrow[r] & 0 \\
0 \arrow[r] & H^{-1}(NL^\wedge_{B/C} \otimes_B A)_\mathfrak{p} \arrow[r] & H^{-1}(NL^\wedge_{A/C})_\mathfrak{p} \arrow[r] & H^{-1}(NL^\wedge_{A/B})_\mathfrak{p} \arrow[llu]
\end{tikzcd}$$

of Lemmas 3.5 and 17.7. Given the rig-smoothness of $C \to A$ and $B \to A$ we conclude that $H^{-1}(NL^\wedge_{B/C} \otimes_B A)_\mathfrak{p} = 0$ and that $H^0(NL^\wedge_{B/C}) \otimes_B A_\mathfrak{p}$ is finite free as a kernel of a surjection of finite free $A_\mathfrak{p}$-modules. Since $B_\mathfrak{q} \to A_\mathfrak{p}$ is flat and hence faithfully flat, this implies that $H^{-1}(NL^\wedge_{B/C})_\mathfrak{q} = 0$ and that $H^0(NL^\wedge_{B/C})_\mathfrak{q}$ is finite free which is the contradiction we were looking for.

In the rig-étale case one argues in exactly the same manner but the conclusion obtained is that both $H^{-1}(NL^\wedge_{B/C})_\mathfrak{q}$ and $H^0(NL^\wedge_{B/C})_\mathfrak{q}$ are zero. $\square$

22. Formal algebraic spaces over cdvrs

In this section we will use the following terminology: if A is a weakly admissible topological ring, then we say "*X is a formal algebraic space over A*" to mean that X is a formal algebraic space which comes equipped with a morphism $p : X \to \mathrm{Spf}(A)$ of formal algebraic spaces. In this situation we will call p the *structure morphism*.

Lemma 22.1. *Let X be a locally Noetherian formal algebraic space over a complete discrete valuation ring A. Then there exists a closed immersion $X' \to X$ of formal algebraic spaces such that X' is flat over A and such that any morphism $Y \to X$ of locally Noetherian formal algebraic spaces with Y flat over A factors through X'.*

Proof. Let $\pi \in A$ be the uniformizer. Recall that an A-module is flat if and only if the π-power torsion is 0.

First assume that X is an affine formal algebraic space. Then $X = \mathrm{Spf}(B)$ with B an adic Noetherian A-algebra. In this case we set $X' = \mathrm{Spf}(B')$ where $B' = B/\pi$-power torsion. It is clear that X' is flat over A and that $X' \to X$ is a closed immersion. Let $g : Y \to X$ be a morphism of locally Noetherian formal algebraic spaces with Y flat over A. Choose a covering $\{Y_j \to Y\}$ as in Formal Spaces, Definition 11.1. Then $Y_j = \mathrm{Spf}(C_j)$ with C_j flat over A. Hence the morphism $Y_j \to X$, which correspond to a continuous R-algebra map $B \to C_j$, factors through X' as clearly $B \to C_j$ kills the π-power torsion. Since $\{Y_j \to Y\}$ is a covering and since $X' \to X$ is a monomorphism, we conclude that g factors through X'.

Let X and $\{X_i \to X\}_{i \in I}$ be as in Formal Spaces, Definition 11.1. For each i let $X'_i \to X_i$ be the flat part as constructed above. For $i, j \in I$ the projection $X'_i \times_X X_j \to X'_i$ is an étale (by assumption) morphism of schemes (by Formal Spaces, Lemma 9.11). Hence $X'_i \times_X X_j$ is flat over A as morphisms representable by algebraic spaces and étale are flat (Lemma 13.8). Thus the projection $X'_i \times_X X_j \to X_j$ factors through X'_j by the universal property. We conclude that

$$R_{ij} = X'_i \times_X X_j = X'_i \times_X X'_j = X_i \times_X X'_j$$

because the morphisms $X'_i \to X_i$ are injections of sheaves. Set $U = \coprod X'_i$, set $R = \coprod R_{ij}$, and denote $s, t : R \to U$ the two projections. As a sheaf $R = U \times_X U$ and s and t are étale. Then $(t, s) : R \to U$ defines an étale equivalence relation by our observations above. Thus $X' = U/R$ is an algebraic space by Spaces, Theorem 10.5. By construction the diagram

$$\begin{tikzcd} \coprod X'_i \arrow[r] \arrow[d] & \coprod X_i \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

is cartesian. Since the right vertical arrow is étale surjective and the top horizontal arrow is representable and a closed immersion we conclude that $X' \to X$ is representable by Bootstrap, Lemma 5.2. Then we can use Spaces, Lemma 5.6 to conclude that $X' \to X$ is a closed immersion.

Finally, suppose that $Y \to X$ is a morphism with Y a locally Noetherian formal algebraic space flat over A. Then each $X_i \times_X Y$ is étale over Y and therefore flat over A (see above). Then $X_i \times_X Y \to X_i$ factors through X'_i. Hence $Y \to X$ factors through X' because $\{X_i \times_X Y \to Y\}$ is an étale covering. □

Lemma 22.2. *Let X be a locally Noetherian formal algebraic space which is locally of finite type over a complete discrete valuation ring A. Let $X' \subset X$ be as in Lemma 22.1. If $X \to X \times_{Spf(A)} X$ is rig-étale and rig-surjective, then $X' = Spf(A)$ or $X' = \emptyset$.*

Proof. (Aside: the diagonal is always locally of finite type by Formal Spaces, Lemma 15.5 and $X \times_{\mathrm{Spf}(A)} X$ is locally Noetherian by Formal Spaces, Lemmas 24.4 and 24.8. Thus imposing the conditions on the diagonal morphism makes sense.) The diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X' \times_{\mathrm{Spf}(A)} X' \arrow[d] \\ X \arrow[r] & X \times_{\mathrm{Spf}(A)} X \end{tikzcd}$$

is cartesian. Hence $X' \to X' \times_{\mathrm{Spf}(A)} X'$ is rig-étale and rig-surjective by Lemma 21.4. Choose an affine formal algebraic space U and a morphism $U \to X'$ which is representable by algebraic spaces and étale. Then $U = \mathrm{Spf}(B)$ where B is an adic Noetherian topological ring which is a flat A-algebra, whose topology is the π-adic topology where $\pi \in A$ is a uniformizer, and such that $A/\pi^n A \to B/\pi^n B$ is of finite type for each n. For later use, we remark that this in particular implies: if $B \neq 0$, then the map $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is a surjection of sheaves (please recall that we are using the fppf topology as always). Repeating the argument above, we see that

$$W = U \times_{X'} U = X' \times_{X' \times_{\mathrm{Spf}(A)} X'} (U \times_{\mathrm{Spf}(A)} U) \longrightarrow U \times_{\mathrm{Spf}(A)} U$$

is a closed immersion and rig-étale and rig-surjective. We have $U \times_{\mathrm{Spf}(A)} U = \mathrm{Spf}(B \widehat{\otimes}_A B)$ by Formal Spaces, Lemma 16.4. Then $B \widehat{\otimes}_A B$ is a flat A-algebra as the π-adic completion of the flat A-algebra $B \otimes_A B$. Hence $W = U \times_{\mathrm{Spf}(A)} U$ by Lemma 21.13. In other words, we have $U \times_{X'} U = U \times_{\mathrm{Spf}(A)} U$ which in turn means that the image of $U \to X'$ (as a map of sheaves) maps injectively to $\mathrm{Spf}(A)$. Choose a covering $\{U_i \to X'\}$ as in Formal Spaces, Definition 11.1. In particular $\coprod U_i \to X'$ is a surjection of sheaves. By applying the above to $U_i \coprod U_j \to X'$ (using the fact that $U_i \amalg U_j$ is an affine formal algebraic space as well) we see that $X' \to \mathrm{Spf}(A)$ is an injective map of fppf sheaves. Since X' is flat over A, either X'

is empty (if U_i is empty for all i) or the map is an isomorphism (if U_i is nonempty for some i when we have seen that $U_i \to \mathrm{Spf}(A)$ is a surjective map of sheaves) and the proof is complete. $\square$

Lemma 22.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume*

1. *X and Y are locally Noetherian,*
2. *f locally of finite type,*
3. *$\Delta_f : X \to X \times_Y X$ is rig-étale and rig-surjective.*

Then f is rig surjective if and only if every adic morphism Spf$(R) \to Y$ where R is a complete discrete valuation ring lifts to a morphism Spf$(R) \to X$.

Proof. One direction is trivial. For the other, suppose that $\mathrm{Spf}(R) \to Y$ is an adic morphism such that there exists an extension of complete discrete valuation rings $R \subset R'$ with $\mathrm{Spf}(R') \to \mathrm{Spf}(R) \to X$ factoring through Y. Consider the fibre product diagram

$$\begin{tikzcd} \mathrm{Spf}(R') \arrow[r] \arrow[dr] & \mathrm{Spf}(R) \times_Y X \arrow[r] \arrow[d, "p"] & X \arrow[d, "f"] \\ & \mathrm{Spf}(R) \arrow[r] & Y \end{tikzcd}$$

The morphism p is locally of finite type as a base change of f, see Formal Spaces, Lemma 24.4. The diagonal morphism Δ_p is the base change of Δ_f and hence is rig-étale and rig-surjective. By Lemma 22.2 the flat locus of $\mathrm{Spf}(R) \times_Y X$ over R is either $\emptyset$ or equal to $\mathrm{Spf}(R)$. However, since $\mathrm{Spf}(R')$ factors through it we conclude it is not empty and hence we get a morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(R) \times_Y X \to X$ as desired. $\square$

23. The completion functor

In this section we consider the following situation. First we fix a base scheme S. All rings, topological rings, schemes, algebraic spaces, and formal algebraic spaces and morphisms between these will be over S. Next, we fix an algebraic space X and a closed subset $T \subset |X|$. We denote $U \subset X$ be the open subspace with $|U| = |X| \setminus T$. Picture

$$U \to X \quad |X| = |U| \amalg T$$

In this situation, given an algebraic space X' over X, i.e., an algebraic space X' endowed with a morphism $f : X' \to X$, then we denote $T' \subset |X'|$ the inverse image of T and we let $U' \subset X'$ be the open subspace with $|U'| = |X'| \setminus T'$. Picture

$$U' = f^{-1}U \qquad \begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ U \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} {|U'|} \arrow[r] \arrow[d] & {|X'|} \arrow[d, "|f|"] & T' \arrow[l] \arrow[d] \\ {|U|} \arrow[r] & {|X|} & T \arrow[l] \end{tikzcd} \qquad T' = |f|^{-1}T$$

We will relate properties of f to properties of the induced morphism

$$f_{/T} : X'_{/T'} \longrightarrow X_{/T}$$

of formal completions. As indicated in the displayed formula, we will denote this morphism $f_{/T}$. We have already seen that $f_{/T}$ is representable by algebraic spaces in Formal Spaces, Lemma 14.4. In fact, as the proof of that lemma shows, the diagram

$$\begin{tikzcd} X'_{/T'} \arrow[r] \arrow[d, "f_{/T}"'] & X' \arrow[d, "f"] \\ X_{/T} \arrow[r] & X \end{tikzcd}$$

is cartesian. Please keep this fact in mind whilst reading the lemmas stated and proved below.

Lemma 23.1. *In the situation above. If f is locally of finite type, then $f_{/T}$ is locally of finite type.*

Proof. (Finite type morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 24.) Namely, suppose that $Z \to X$ is a morphism from a scheme into X such that $|Z|$ maps into T. From the cartesian square above we see that $Z \times_X X'$ is an algebraic space representing $Z \times_{X_{/T}} X'_{/T'}$. Since $Z \times_X X' \to Z$ is locally of finite type by Morphisms of Spaces, Lemma 23.3 we conclude. $\square$

Lemma 23.2. *In the situation above. If f is étale, then $f_{/T}$ is étale.*

Proof. By the same argument as in the proof of Lemma 23.1 this follows from Morphisms of Spaces, Lemma 39.4. $\square$

Lemma 23.3. *In the situation above. If f is a closed immersion, then $f_{/T}$ is a closed immersion.*

Proof. (Closed immersions of formal algebraic spaces are discussed in Formal Spaces, Section 27.) By the same argument as in the proof of Lemma 23.1 this follows from Spaces, Lemma 12.3. $\square$

Lemma 23.4. *In the situation above. If f is proper, then $f_{/T}$ is proper.*

Proof. (Proper morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 31.) By the same argument as in the proof of Lemma 23.1 this follows from Morphisms of Spaces, Lemma 40.3. $\square$

Lemma 23.5. *In the situation above. If f is quasi-compact, then $f_{/T}$ is quasi-compact.*

Proof. (Quasi-compact morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 17.) We have to show that $(X'_{/T'})_{red} \to (X_{/T})_{red}$ is a quasi-compact morphism of algebraic spaces. By Formal Spaces, Lemma 14.5 this is the morphism $Z' \to Z$ where $Z' \subset X'$, resp. $Z \subset X$ is the reduced induced algebraic space structure on T', resp. T. It follows that $Z' \to f^{-1}Z = Z \times_X X'$ is a thickening (a closed immersion defining an isomorphism on underlying topological spaces). Since $Z \times_X X' \to Z$ is quasi-compact as a base change of f (Morphisms of Spaces, Lemma 8.4) we conclude that $Z' \to Z$ is too by More on Morphisms of Spaces, Lemma 10.1. $\square$

Remark 23.6. In the situation above consider the diagonal morphisms $\Delta_f : X' \to X' \times_X X'$ and $\Delta_{f_{/T}} : X'_{/T'} \to X'_{/T'} \times_{X_{/T}} X'_{/T'}$. It is easy to see that

$$X'_{/T'} \times_{X_{/T}} X'_{/T'} = (X' \times_X X')_{/T''}$$

as subfunctors of $X' \times_X X'$ where $T'' \subset |X' \times_X X'|$ is the inverse image of T. Hence we see that $\Delta_{f_{/T}} = (\Delta_f)_{/T''}$. We will use this below to show that properties of Δ_f are inherited by $\Delta_{f_{/T}}$.

Lemma 23.7. *In the situation above. If f is (quasi-)separated, then $f_{/T}$ is too.*

Proof. (Separation conditions on morphisms of formal algebraic spaces are discussed in Formal Spaces, Section 30.) We have to show that if Δ_f is quasi-compact, resp. a closed immersion, then the same is true for $\Delta_{f_{/T}}$. This follows from the discussion in Remark 23.6 and Lemmas 23.5 and 23.3. $\square$

Lemma 23.8. *In the situation above. If X is locally Noetherian, f is locally of finite type, and $U' \to U$ is smooth, then $f_{/T}$ is rig-smooth.*

Proof. The strategy of the proof is this: reduce to the case where X and X' are affine, translate the affine case into algebra, and finally apply Lemma 4.3. We urge the reader to skip the details.

Choose a surjective étale morphism $W \to X$ with $W = \coprod W_i$ a disjoint union of affine schemes, see Properties of Spaces, Lemma 6.1. For each i choose a surjective étale morphism $W'_i \to W_i \times_X X'$ where $W'_i = \coprod W'_{ij}$ is a disjoint union of affines. In particular $\coprod W'_{ij} \to X'$ is surjective and étale. Denote $f_{ij} : W_{ij} \to W_i$ the given morphism. Denote $T_i \subset W_i$ and $T'_{ij} \subset W_{ij}$ the inverse images of T. Since taking the completion along the inverse image of T produces cartesian diagrams (see above) we have $(W_i)_{/T_i} = W_i \times_X X_{/T}$ and similarly $(W'_{ij})_{/T'_{ij}} = W'_{ij} \times_{X'} X'_{/T'}$. Moreover, recall that $(W_i)_{/T_i}$ and $(W'_{ij})_{/T'_{ij}}$ are affine formal algebraic spaces. Hence $\{W'_{ij})_{/T'_{ij}} \to X'_{/T'}\}$ is a covering as in Formal Spaces, Definition 11.1. By Lemma 18.2 we see that it suffices to prove that

$$(W'_{ij})_{/T'_{ij}} \longrightarrow (W_i)_{/T_i}$$

is rig-smooth. Observe that $W'_{ij} \to W_i$ is locally of finite type and induces a smooth morphism $W'_{ij} \setminus T'_{ij} \to W_i \setminus T_i$ (as this is true for f and these properties of morphisms are étale local on the source and target). Observe that W_i is locally Noetherian (as X is locally Noetherian and this property is étale local on the algebraic space). Hence it suffices to prove the lemma when X and X' are affine schemes.

Assume $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(A')$ are affine schemes. Since X is Noetherian, we see that A is Noetherian. The morphism f is given by a ring map $A \to A'$ of finite type. Let $I \subset A$ be an ideal cutting out T. Then IA' cuts out T'. Also $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is smooth over $\mathrm{Spec}(A) \setminus T$. Let $A^\wedge$ and $(A')^\wedge$ be the I-adic completions. We have $X_{/T} = \mathrm{Spf}(A^\wedge)$ and $X'_{/T'} = \mathrm{Spf}((A')^\wedge)$, see proof of Formal Spaces, Lemma 20.8. By Lemma 4.3 we see that $(A')^\wedge$ is rig-smooth over $(A.I)$ which in turn means that $A^\wedge \to (A')^\wedge$ is rig-smooth which finally implies that $X'_{/T'} \to X_{/T}$ is rig smooth by Lemma 18.2. $\square$

Lemma 23.9. *In the situation above. If X is locally Noetherian, f is locally of finite type, and $U' \to U$ is étale, then $f_{/T}$ is rig-étale.*

Proof. The proof is exactly the same as the proof of Lemma 23.8 except with Lemmas 4.3 and 18.2 replaced by Lemmas 8.4 and 20.2 $\square$

Lemma 23.10. *In the situation above. If X is locally Noetherian, f is proper, and $U' \to U$ is surjective, then $f_{/T}$ is rig-surjective.*

Proof. (The statement makes sense by Lemma 23.1 and Formal Spaces, Lemma 20.8.) Let R be a complete discrete valuation ring with fraction field K. Let $p : \mathrm{Spf}(R) \to X_{/T}$ be an adic morphism of formal algebraic spaces. By Formal Spaces, Lemma 33.4 the composition $\mathrm{Spf}(R) \to X_{/T} \to X$ corresponds to a morphism $q : \mathrm{Spec}(R) \to X$ which maps $\mathrm{Spec}(K)$ into U. Since $U' \to U$ is proper and surjective we see that $\mathrm{Spec}(K) \times_U U'$ is nonempty and proper over K. Hence we can choose a field extension K'/K and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & U' \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & U \arrow[r] & X \end{tikzcd}$$

Let $R' \subset K'$ be a discrete valuation ring dominating R with fraction field K', see Algebra, Lemma 119.13. Since $\mathrm{Spec}(K) \to X$ extends to $\mathrm{Spec}(R) \to X$ we see by the valuative criterion of properness (Morphisms of Spaces, Lemma 44.1) that we can extend our K'-point of U' to a morphism $\mathrm{Spec}(R') \to X'$ over $\mathrm{Spec}(R) \to X$. It follows that the inverse image of T' in $\mathrm{Spec}(R')$ is the closed point and we find an adic morphism $\mathrm{Spf}((R')^\wedge) \to X'_{/T'}$ lifting p as desired (note that $(R')^\wedge$ is a complete discrete valuation ring by More on Algebra, Lemma 43.5). $\square$

Lemma 23.11. *In the situation above. If X is locally Noetherian, f is separated and locally of finite type, and $U' \to U$ is a monomorphism, then $\Delta_{f/T}$ is rig-surjective.*

Proof. The diagonal $\Delta_f : X' \to X' \times_X X'$ is a closed immersion and the restriction $U' \to U' \times_U U'$ of Δ_f is surjective. Hence the lemma follows from the discussion in Remark 23.6 and Lemma 23.10. $\square$

24. Formal modifications

In this section we define and study Artin's notion of a formal modification of locally Noetherian formal algebraic spaces. First, here is the definition.

Definition 24.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is a *formal modification* if

(1) f is a proper morphism (Formal Spaces, Definition 31.1),
(2) f is rig-étale,
(3) f is rig-surjective,
(4) $\Delta_f : X \to X \times_Y X$ is rig-surjective.

A typical example is given in Lemma 24.3 and indeed we will later show that every formal modification is "formal locally" of this type, see Lemma 29.2. Let us compare these conditions with those in Artin's paper.

Remark 24.2. In [Art70, Definition 1.7] a formal modification is defined as a proper morphism $f : X \to Y$ of locally Noetherian formal algebraic spaces satisfying the following three conditions[3]

(i) the Cramer and Jacobian ideal of f each contain an ideal of definition of X,

[3] We will not completely translate these conditions into the language developed in the Stacks project. We hope nonetheless the discussion here will be useful to the reader.

(ii) the ideal defining the diagonal map $\Delta : X \to X \times_Y X$ is annihilated by an ideal of definition of $X \times_Y X$, and
(iii) any adic morphism $\text{Spf}(R) \to Y$ lifts to $\text{Spf}(R) \to X$ whenever R is a complete discrete valuation ring.

Let us compare these to our list of conditions above.

Ad (i). Property (i) agrees with our condition that f be a rig-étale morphism: this follows from Lemma 8.2 part (7).

Ad (ii). Assume f is rig-étale. Then $\Delta_f : X \to X \times_Y X$ is rig-étale as a morphism of locally Noetherian formal algebraic spaces which are rig-étale over X (via id_X for the first one and via pr_1 for the second one). See Lemmas 20.5 and 20.7. Hence property (ii) agrees with our condition that Δ_f be rig-surjective by Lemma 21.13.

Ad (iii). Property (iii) does not quite agree with our notion of a rig-surjective morphism, as Artin requires all adic morphisms $\text{Spf}(R) \to Y$ to lift to morphisms into X whereas our notion of rig-surjective only asserts the existence of a lift after replacing R by an extension. However, since we already have that Δ_f is rig-étale and rig-surjective by (i) and (ii), these conditions are equivalent by Lemma 22.3.

Lemma 24.3. *Let S, $f : X' \to X$, $T \subset |X|$, $U \subset X$, $T' \subset |X'|$, and $U' \subset X'$ be as in Section 23. If X is locally Noetherian, f is proper, and $U' \to U$ is an isomorphism, then $f_{/T} : X'_{/T'} \to X_{/T}$ is a formal modification.*

Proof. By Formal Spaces, Lemmas 20.8 the source and target of the arrow are locally Noetherian formal algebraic spaces. The other conditions follow from Lemmas 23.4, 23.9, 23.10, and 23.11. ☐

Lemma 24.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S which is a formal modification. Then for any adic morphism $Y' \to Y$ of locally Noetherian formal algebraic spaces, the base change $f' : X \times_Y Y' \to Y'$ is a formal modification.*

Proof. The morphism f' is proper by Formal Spaces, Lemma 31.3. The morphism f' is rig-etale by Lemma 20.5. Then morphism f' is rig-surjective by Lemma 21.4. Set $X' = X \times_Y'$. The morphism $\Delta_{f'}$ is the base change of Δ_f by the adic morphism $X' \times_{Y'} X' \to X \times_Y X$. Hence $\Delta_{f'}$ is rig-surjective by Lemma 21.4. ☐

25. Completions and morphisms, I

In this section we put some preliminary results on completions which we will use in the proof of Theorem 27.4. Although the lemmas stated and proved here are not trivial (some are based on our work on algebraization of rig-étale algebras), we still suggest the reader skip this section on a first reading.

Lemma 25.1. *Let $T \subset X$ be a closed subset of a Noetherian affine scheme X. Let W be a Noetherian affine formal algebraic space. Let $g : W \to X_{/T}$ be a rig-étale morphism. Then there exists an affine scheme X' and a finite type morphism $f : X' \to X$ étale over $X \setminus T$ such that there is an isomorphism $X'_{/f^{-1}T} \cong W$ compatible with $f_{/T}$ and g. Moreover, if $W \to X_{/T}$ is étale, then $X' \to X$ is étale.*

Proof. The existence of X' is a restatement of Lemma 10.3. The final statement follows from More on Morphisms, Lemma 12.3. ☐

Lemma 25.2. *Assume we have*

(1) *Noetherian affine schemes X, X', and Y,*
(2) *a closed subset $T \subset |X|$,*
(3) *a morphism $f : X' \to X$ locally of finite type and étale over $X \setminus T$,*
(4) *a morphism $h : Y \to X$,*
(5) *a morphism $\alpha : Y_{/T} \to X'_{/T}$ over $X_{/T}$ (see proof for notation).*

Then there exists an étale morphism $b : Y' \to Y$ of affine schemes which induces an isomorphism $b_{/T} : Y'_{/T} \to Y_{/T}$ and a morphism $a : Y' \to X'$ over X such that $\alpha = a_{/T} \circ b_{/T}^{-1}$.

Proof. The notation using the subscript $_{/T}$ in the statement refers to the construction which to a morphism of schemes $g : V \to X$ associates the morphism $g_{/T} : V_{/g^{-1}T} \to X_{/T}$ of formal algebraic spaces; it is a functor from the category of schemes over X to the category of formal algebraic spaces over $X_{/T}$, see Section 23. Having said this, the lemma is just a reformulation of Lemma 8.7. $\square$

Lemma 25.3. *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of algebraic spaces. Let $T \subset |X|$ be closed. Assume that*

(1) *X is locally Noetherian,*
(2) *g is a monomorphism and locally of finite type,*
(3) *$f|_{X \setminus T} : X \setminus T \to Y$ factors through g, and*
(4) *$f_{/T} : X_{/T} \to Y$ factors through g,*

then f factors through g.

Proof. Consider the fibre product $E = X \times_Y Z \to X$. By assumption the open immersion $X \setminus T \to X$ factors through E and any morphism $\varphi : X' \to X$ with $|\varphi|(|X'|) \subset T$ factors through E as well, see Formal Spaces, Section 14. By More on Morphisms of Spaces, Lemma 20.3 this implies that $E \to X$ is étale at every point of E mapping to a point of T. Hence $E \to X$ is an étale monomorphism, hence an open immersion (Morphisms of Spaces, Lemma 51.2). Then it follows that $E = X$ since our assumptions imply that $|X| = |E|$. $\square$

Lemma 25.4. *Let S be a scheme. Let X, W be algebraic spaces over S with X locally Noetherian. Let $T \subset |X|$ be a closed subset. Let $a, b : X \to W$ be morphisms of algebraic spaces over S such that $a|_{X \setminus T} = b|_{X \setminus T}$ and such that $a_{/T} = b_{/T}$ as morphisms $X_{/T} \to W$. Then $a = b$.*

Proof. Let E be the equalizer of a and b. Then E is an algebraic space and $E \to X$ is locally of finite type and a monomorphism, see Morphisms of Spaces, Lemma 4.1. Our assumptions imply we can apply Lemma 25.3 to the two morphisms $f = \text{id} : X \to X$ and $g : E \to X$ and the closed subset T of $|X|$. $\square$

Lemma 25.5. *Let S be a scheme. Let X, Y be locally Noetherian algebraic spaces over S. Let $T \subset |X|$ and $T' \subset |Y|$ be closed subsets. Let $a, b : X \to Y$ be morphisms of algebraic spaces over S such that $a|_{X \setminus T} = b|_{X \setminus T}$, such that $|a|(T) \subset T'$ and $|b|(T) \subset T'$, and such that $a_{/T} = b_{/T}$ as morphisms $X_{/T} \to Y_{/T'}$. Then $a = b$.*

Proof. Consequence of the more general Lemma 25.4. $\square$

Lemma 25.6. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $s, t : R \to U$ be two morphisms of algebraic spaces over X. Assume*

(1) *R, U are locally of finite type over X,*
(2) *the base change of s and t to $X \setminus T$ is an étale equivalence relation, and*
(3) *the formal completion $(t_{/T}, s_{/T}) : R_{/T} \to U_{/T} \times_{X_{/T}} U_{/T}$ is an equivalence relation too (see proof for notation).*

Then $(t, s) : R \to U \times_X U$ is an étale equivalence relation.

Proof. The notation using the subscript $_{/T}$ in the statement refers to the construction which to a morphism $f : X' \to X$ of algebraic spaces associates the morphism $f_{/T} : X'_{/f^{-1}T} \to X_{/T}$ of formal algebraic spaces, see Section 23. The morphisms $s, t : R \to U$ are étale over $X \setminus T$ by assumption. Since the formal completions of the maps $s, t : R \to U$ are étale, we see that s and t are étale for example by More on Morphisms, Lemma 12.3. Applying Lemma 25.3 to the morphisms $\text{id} : R \times_{U \times_X U} R \to R \times_{U \times_X U} R$ and $\Delta : R \to R \times_{U \times_X U} R$ we conclude that (t, s) is a monomorphism. Applying it again to $(t \circ \text{pr}_0, s \circ \text{pr}_1) : R \times_{s,U,t} R \to U \times_X U$ and $(t, s) : R \to U \times_X U$ we find that "transitivity" holds. We omit the proof of the other two axioms of an equivalence relation. $\square$

Lemma 25.7. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S and let $T \subset |X|$ be a closed subset. Let $f : X' \to X$ be a morphism of algebraic spaces which is locally of finite type and étale outside of T. There exists a factorization*

$$X' \longrightarrow X'' \longrightarrow X$$

of f with the following properties: $X'' \to X$ is locally of finite type, $X'' \to X$ is an isomorphism over $X \setminus T$, and $X'_{/T} \to X''_{/T}$ is an isomorphism (see proof for notation).

Proof. The notation using the subscript $_{/T}$ in the statement refers to the construction which to a morphism $f : X' \to X$ of algebraic spaces associates the morphism $f_{/T} : X'_{/f^{-1}T} \to X_{/T}$ of formal algebraic spaces, see Section 23. We will also use the notion $U \subset X$ and $U' \subset X'$ to denote the open subspaces with $|U| = |X| \setminus T$ and $U' = |X'| \setminus f^{-1}T$ introduced in Section 23.

After replacing X' by $X' \amalg U$ we may and do assume the image of $X' \to X$ contains U. Let

$$R = X' \amalg_{U'} (U' \times_U U')$$

be the pushout of $U' \to X'$ and the diagonal morphism $U' \to U' \times_U U' = U' \times_X U'$. Since $U' \to X$ is étale, this diagonal is an open immersion and we see that R is an algebraic space (this follows for example from Spaces, Lemma 8.5). The two projections $U' \times_U U' \to U'$ extend to R and we obtain two étale morphisms $s, t : R \to X'$. Checking on each piece separatedly we find that R is an étale equivalence relation on X'. Set $X'' = X'/R$ which is an algebraic space by Bootstrap, Theorem 10.1. By construction have the factorization as in the lemma and the morphism $X'' \to X$ is locally of finite type (as this can be checked étale locally, i.e., on X'). Since $U' \to U$ is a surjective étale morphism and since $s^{-1}(U') = t^{-1}(U') = U' \times_U U'$ we see that $U'' = U \times_X X'' \to U$ is an isomorphism. Finally, we have to show the morphism $X' \to X''$ induces an isomorphism $X'_{/T} \to X''_{/T}$. To see this, note that the formal completion of R along the inverse image of T is equal to the formal completion of X' along the inverse image of T by our choice of R! By our construction of the formal completion in Formal Spaces, Section 14 we have

$X''_{/T} = (X'_{/T})/(R_{/T})$ as sheaves. Since $X'_{/T} = R_{/T}$ we conclude that $X'_{/T} = X''_{/T}$ and this finishes the proof. □

26. Rig glueing of morphisms

Let X, W be algebraic spaces with X Noetherian. Let $Z \subset X$ be a closed subspace with open complement U. The proposition below says roughly speaking that

$$\{\text{morphisms } X \to W\} = \{\text{compatible morphisms } U \to W \text{ and } X_{/Z} \to W\}$$

where compatibility of $a : U \to W$ and $b : X_{/Z} \to W$ means that a and b define the same "morphism of rig-spaces". To introduce the category of "rig-spaces" requires a lot of work, but we don't need to do so in order to state precisely what the condition means in this case.

Proposition 26.1. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset with complementary open subspace $U \subset X$. Let $f : X' \to X$ be a proper morphism of algebraic spaces such that $f^{-1}(U) \to U$ is an isomorphism. For any algebraic space W over S the map*

$$\mathrm{Mor}_S(X, W) \longrightarrow \mathrm{Mor}_S(X', W) \times_{\mathrm{Mor}_S(X'_{/T}, W)} \mathrm{Mor}_S(X_{/T}, W)$$

is bijective.

Proof. Let $w' : X' \to W$ and $\hat{w} : X_{/T} \to W$ be morphisms which determine the same morphism $X'_{/T} \to W$ by composition with $X'_{/T} \to X$ and $X'_{/T} \to X_{/T}$. We have to prove there exists a unique morphism $w : X \to W$ whose composition with $X' \to X$ and $X_{/T} \to X$ recovers w' and $\hat{w}$. The uniqueness is immediate from Lemma 25.4.

The assumptions on T and f are preserved by base change by any étale morphism $X_1 \to X$ of algebraic spaces. Since formal algebraic spaces are sheaves for the étale topology and since we aready have the uniqueness, it suffices to prove existence after replacing X by the members of an étale covering. Thus we may assume X is an affine Noetherian scheme.

Assume X is an affine Noetherian scheme. We will construct the morphism $w : X \to W$ using the material in Pushouts of Spaces, Section 13. It makes sense to read a little bit of the material in that section before continuing the read the proof.

Set $X'' = X' \times_X X'$ and consider the two morphisms $a = w' \circ \mathrm{pr}_1 : X'' \to W$ and $b = w' \circ \mathrm{pr}_2 : X'' \to W$. Then we see that a and b agree over the open U and that $a_{/T}$ and $b_{a/T}$ agree (as these are both equal to the composition $X''_{/T} \to X_{/T} \to W$ where the second arrow is $\hat{w}$). Thus by Lemma 25.4 we see $a = b$.

Denote $Z \subset X$ the reduced induced closed subscheme structure on T. For $n \geq 1$ denote $Z_n \subset X$ the nth infinitesimal neighbourhood of Z. Denote $w_n = \hat{w}|_{Z_n} : Z_n \to W$ so that we have $\hat{w} = \mathrm{colim}\, w_n$ on $X_{/T} = \mathrm{colim}\, Z_n$. Set $Y_n = X' \amalg Z_n$. Consider the two projections

$$s_n, t_n : R_n = Y_n \times_X Y_n \longrightarrow Y_n$$

Let $Y_n \to X_n \to X$ be the coequalizer of s_n and t_n as in Pushouts of Spaces, Section 13 (in particular this coequalizer exists, has good properties, etc, see Pushouts of

Spaces, Lemma 13.1). By the result $a = b$ of the previous parapgraph and the agreement of w' and $\hat{w}$ over $X'_{/T}$ we see that the morphism

$$w' \amalg w_n : Y_n \longrightarrow W$$

equalizes the morphisms s_n and t_n. Hence we see that for all $n \geq 1$ there is a morphism $w^n : X_n \to W$ compatible with w' and w_n. Moreover, for $m \geq 1$ the composition

$$X_n \to X_{n+m} \xrightarrow{w^{n+m}} W$$

is equal to w^n by construction (as the corresponding statement holds for $w' \amalg w_{n+m}$ and $w' \amalg w_n$). By Pushouts of Spaces, Lemma 13.4 and Remark 13.5 the system of algebraic spaces X_n is essentially constant with value X and we conclude. □

27. Algebraization of rig-étale morphisms

In this section we prove a generalization of the result on dilatations from the paper of Artin [Art70].

The notation in this section will agree with the notation in Section 23 except our algebraic spaces and formal algebraic spaces will be locally Noetherian.

Thus, we first fix a base scheme S. All rings, topological rings, schemes, algebraic spaces, and formal algebraic spaces and morphisms between these will be over S. Next, we fix a locally Noetherian algebraic space X and a closed subset $T \subset |X|$. We denote $U \subset X$ be the open subspace with $|U| = |X| \setminus T$. Picture

$$U \to X \quad |X| = |U| \amalg T$$

Given a morphism of algebraic spaces $f : X' \to X$, we will use the notation $U' = f^{-1}U$, $T' = |f|^{-1}(T)$, and $f_{/T} : X'_{/T'} \to X_{/T}$ as in Section 23. We will sometimes write $X'_{/T}$ in stead of $X'_{/T'}$ and more generally for a morphism $a : X' \to X''$ of algebraic spaces over X we will denote $a_{/T} : X'_{/T} \to X''_{/T}$ the induced morphism of formal algebraic spaces obtained by completing the morphism a along the inverse images of T in X' and X''.

Given this setup we will consider the functor

$$\left\{\begin{matrix}\text{morphisms of algebraic spaces}\\ f : X' \to X \text{ which are locally}\\ \text{of finite type and such that}\\ U' \to U \text{ is an isomorphism}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{morphisms } g : W \to X_{/T}\\ \text{of formal algebraic spaces}\\ \text{with } W \text{ locally Noetherian}\\ \text{and } g \text{ rig-étale}\end{matrix}\right\} \tag{27.0.1}$$

sending $f : X' \to X$ to $f_{/T} : X'_{/T'} \to X_{/T}$. This makes sense because $f_{/T}$ is rig-étale by Lemma 23.9.

Lemma 27.1. *In the situation above, let $X_1 \to X$ be a morphism of algebraic spaces with X_1 locally Noetherian. Denote $T_1 \subset |X_1|$ the inverse image of T and $U_1 \subset X_1$ the inverse image of U. We denote*

(1) *$\mathcal{C}_{X,T}$ the category whose objects are morphisms of algebraic spaces $f : X' \to X$ which are locally of finite type and such that $U' = f^{-1}U \to U$ is an isomorphism,*
(2) *$\mathcal{C}_{X_1,T_1}$ the category whose objects are morphisms of algebraic spaces $f_1 : X'_1 \to X_1$ which are locally of finite type and such that $f_1^{-1}U_1 \to U_1$ is an isomorphism,*

(3) $\mathcal{C}_{X_{/T}}$ *the category whose objects are morphisms* $g : W \to X_{/T}$ *of formal algebraic spaces with* W *locally Noetherian and* g *rig-étale,*

(4) $\mathcal{C}_{X_{1,/T_1}}$ *the category whose objects are morphisms* $g_1 : W_1 \to X_{1,/T_1}$ *of formal algebraic spaces with* W_1 *locally Noetherian and* g_1 *rig-étale.*

Then the diagram

$$\begin{tikzcd} \mathcal{C}_{X,T} \arrow[r] \arrow[d] & \mathcal{C}_{X_{/T}} \arrow[d] \\ \mathcal{C}_{X_1,T_1} \arrow[r] & \mathcal{C}_{X_{1,/T_1}} \end{tikzcd}$$

is commutative where the horizonal arrows are given by (27.0.1) and the vertical arrows by base change along $X_1 \to X$ *and along* $X_{1,/T_1} \to X_{/T}$*.*

Proof. This follows immediately from the fact that the completion functor $(h : Y \to X) \mapsto Y_{/T} = Y_{/|h|^{-1}T}$ on the category of algebraic spaces over X commutes with fibre products. ☐

Lemma 27.2. *In the situation above. Let* $f : X' \to X$ *be a morphism of algebraic spaces which is locally of finite type and an isomorphism over* U*. Let* $g : Y \to X$ *be a morphism with* Y *locally Noetherian. Then completion defines a bijection*

$$\mathrm{Mor}_X(Y, X') \longrightarrow \mathrm{Mor}_{X_{/T}}(Y_{/T}, X'_{/T})$$

In particular, the functor (27.0.1) is fully faithful.

Proof. Let $a, b : Y \to X'$ be morphisms over X such that $a_{/T} = b_{/T}$. Then we see that a and b agree over the open subspace $g^{-1}U$ and after completion along $g^{-1}T$. Hence $a = b$ by Lemma 25.5. In other words, the completion map is always injective.

Let $\alpha : Y_{/T} \to X'_{/T}$ be a morphism of formal algebraic spaces over $X_{/T}$. We have to prove there exists a morphism $a : Y \to X'$ over X such that $\alpha = a_{/T}$. The proof proceeds by a standard but cumbersome reduction to the affine case and then applying Lemma 25.2.

Let $\{h_i : Y_i \to Y\}$ be an étale covering of algebraic spaces. If we can find for each i a morphism $a_i : Y_i \to X'$ over X whose completion $(a_i)_{/T} : (Y_i)_{/T} \to X'_{/T}$ is equal to $\alpha \circ (h_i)_{/T}$, then we get a morphism $a : Y \to X'$ with $\alpha = a_{/T}$. Namely, we first observe that $(a_i)_{/T} \circ \mathrm{pr}_1 = (a_j)_{/T} \circ \mathrm{pr}_2$ as morphisms $(Y_i \times_Y Y_j)_{/T} \to X'_{/T}$ by the agreement with α (this uses that completion $_{/T}$ commutes with fibre products). By the injectivity already proven this shows that $a_i \circ \mathrm{pr}_1 = a_j \circ \mathrm{pr}_2$ as morphisms $Y_i \times_Y Y_j \to X'$. Since X' is an fppf sheaf this means that the collection of morphisms a_i descends to a morphism $a : Y \to X'$. We have $\alpha = a_{/T}$ because $\{(a_i)_{/T} : (Y_i)_{/T} \to X'_{/T}\}$ is an étale covering.

By the result of the previous paragraph, to prove existence, we may assume that Y is affine and that $g : Y \to X$ factors as $g_1 : Y \to X_1$ and an étale morphism $X_1 \to X$ with X_1 affine. Then we can consider $T_1 \subset |X_1|$ the inverse image of T and we can set $X'_1 = X' \times_X X_1$ with projection $f_1 : X'_1 \to X_1$ and

$$\alpha_1 = (\alpha, (g_1)_{/T_1}) : Y_{/T_1} = Y_{/T} \longrightarrow X'_{/T} \times_{X_{/T}} (X_1)_{/T_1} = (X'_1)_{/T_1}$$

We conclude that it suffices to prove the existence for α_1 over X_1, in other words, we may replace $X, T, X', Y, f, g, \alpha$ by $X_1, T_1, X'_1, Y, g_1, \alpha_1$. This reduces us to the case described in the next paragraph.

Assume Y and X are affine. Recall that $(Y_{/T})_{red}$ is an affine scheme (isomorphic to the reduced induced scheme structure on $g^{-1}T \subset Y$, see Formal Spaces, Lemma 14.5). Hence $\alpha_{red} : (Y_{/T})_{red} \to (X'_{/T})_{red}$ has quasi-compact image E in $f^{-1}T$ (this is the underlying topological space of $(X'_{/T})_{red}$ by the same lemma as above). Thus we can find an affine scheme V and an étale morpism $h : V \to X'$ such that the image of h contains E. Choose a solid cartesian diagram

$$\begin{tikzcd}
Y'_{/T} \arrow[r, dotted] \arrow[rd, dotted] & W \arrow[r] \arrow[d] & V_{/T} \arrow[d, "h_{/T}"] \\
 & Y_{/T} \arrow[r, "\alpha"] & X'_{/T}
\end{tikzcd}$$

By construction, the morphism $W \to Y_{/T}$ is representable by algebraic spaces, étale, and surjective (surjectivity can be seen by looking at the reductions, see Formal Spaces, Lemma 12.4). By Lemma 25.1 we can write $W = Y'_{/T}$ for $Y' \to Y$ étale and Y' affine. This gives the dotted arrows in the diagram. Since $W \to Y_{/T}$ is surjective, we see that the image of $Y' \to Y$ contains $g^{-1}T$. Hence $\{Y' \to Y, Y \setminus g^{-1}T \to Y\}$ is an étale covering. As f is an isomorphism over U we have a (unique) morphism $Y \setminus g^{-1}T \to X'$ over X agreeing with α on completions (as the completion of $Y \setminus g^{-1}T$ is empty). Thus it suffices to prove the existence for Y' which reduces us to the case studied in the next paragraph.

By the result of the previous paragraph, we may assume that Y is affine and that α factors as $Y_{/T} \to V_{/T} \to X'_{/T}$ where V is an affine scheme étale over X'. We may still replace Y by the members of an affine étale covering. By Lemma 25.2 we may find an étale morphism $b : Y' \to Y$ of affine schemes which induces an isomorphism $b_{/T} : Y'_{/T} \to Y_{/T}$ and a morphism $c : Y' \to V$ such that $c_{/T} \circ b_{/T}^{-1}$ is the given morphism $Y_{/T} \to V_{/T}$. Setting $a' : Y' \to X'$ equal to the composition of c and $V \to X'$ we find that $a'_{/T} = \alpha \circ b_{/T}$, in other words, we have existence for Y' and $\alpha \circ b_{/T}$. Then we are done by replacing considering once more the étale covering $\{Y' \to Y, Y \setminus g^{-1}T \to Y\}$. $\square$

Lemma 27.3. *In the situation above. Assume X is affine. Then the functor (27.0.1) is an equivalence.*

Before we prove this lemma let us discuss an example. Suppose that $S = \text{Spec}(k)$, $X = \mathbf{A}^1_k$, and $T = \{0\}$. Then $X_{/T} = \text{Spf}(k[[x]])$. Let $W = \text{Spf}(k[[x]] \times k[[x]])$. Then the corresponding $f : X' \to X$ is the affine line with zero doubled mapping to the affine line (Schemes, Example 14.3). Moreover, this is the output of the construction in Lemma 25.7 starting with $X \amalg X$ over X.

Proof. We already know the functor is fully faithful, see Lemma 27.2. Essential surjectivity. Let $g : W \to X_{/T}$ be a morphism of formal algebraic spaces with W locally Noetherian and g rig-étale. We will prove W is in the essential image in a number of steps.

Step 1: W is an affine formal algebraic space. Then we can find $U \to X$ of finite type and étale over $X \setminus T$ such that $U_{/T}$ is isomorphic to W, see Lemma 25.1. Thus we see that W is in the essential image by Lemma 25.7.

Step 2: W is separated. Choose $\{W_i \to W\}$ as in Formal Spaces, Definition 11.1. By Step 1 the formal algebraic spaces W_i and $W_i \times_W W_j$ are in the essential image. Say $W_i = (X'_i)_{/T}$ and $W_i \times_W W_j = (X'_{ij})_{/T}$. By fully faithfulness we obtain morphisms $t_{ij} : X'_{ij} \to X'_i$ and $s_{ij} : X'_{ij} \to X'_j$ matching the projections $W_i \times_W W_j \to W_i$ and $W_i \times_W W_j \to W_j$. Consider the structure

$$R = \coprod X'_{ij}, \quad V = \coprod X'_i, \quad s = \coprod s_{ij}, \quad t = \coprod t_{ij}$$

(We can't use the letter U as it has already been used.) Applying Lemma 25.6 we find that $(t, s) : R \to V \times_X V$ defines an étale equivalence relation on V over X. Thus we can take the quotient $X' = V/R$ and it is an algebraic space, see Bootstrap, Theorem 10.1. Since completion commutes with fibre products and taking quotient sheaves, we find that $X'_{/T} \cong W$ as formal algebraic spaces over $X_{/T}$.

Step 3: W is general. Choose $\{W_i \to W\}$ as in Formal Spaces, Definition 11.1. The formal algebraic spaces W_i and $W_i \times_W W_j$ are separated. Hence by Step 2 the formal algebraic spaces W_i and $W_i \times_W W_j$ are in the essential image. Then we argue exactly as in the previous paragraph to see that W is in the essential image as well. This concludes the proof. $\square$

Theorem 27.4. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $U \subset X$ be the open subspace with $|U| = |X| \setminus T$. The completion functor (27.0.1)*

$$\left\{ \begin{matrix} \textit{morphisms of algebraic spaces} \\ f : X' \to X \textit{ which are locally} \\ \textit{of finite type and such that} \\ f^{-1}U \to U \textit{ is an isomorphism} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \textit{morphisms } g : W \to X_{/T} \\ \textit{of formal algebraic spaces} \\ \textit{with } W \textit{ locally Noetherian} \\ \textit{and } g \textit{ rig-étale} \end{matrix} \right\}$$

sending $f : X' \to X$ to $f_{/T} : X'_{/T'} \to X_{/T}$ is an equivalence.

Proof. The functor is fully faithful by Lemma 27.2. Let $g : W \to X_{/T}$ be a morphism of formal algebraic spaces with W locally Noetherian and g rig-étale. We will prove W is in the essential image to finish the proof.

Choose an étale covering $\{X_i \to X\}$ with X_i affine for all i. Denote $U_i \subset X_i$ the inverse image of U and denote $T_i \subset X_i$ the inverse image of T. Recall that $(X_i)_{/T_i} = (X_i)_{/T} = (X_i \times_X X)_{/T}$ and $W_i = X_i \times_X W = (X_i)_{/T} \times_{X_{/T}} W$, see Lemma 27.1. Observe that we obtain isomorphisms

$$\alpha_{ij} : W_i \times_{X_{/T}} (X_j)_{/T} \longrightarrow (X_i)_{/T} \times_{X_{/T}} W_j$$

satisfying a suitable cocycle condition. By Lemma 27.3 applied to $X_i, T_i, U_i, W_i \to (X_i)_{/T}$ there exists a morphism $X'_i \to X_i$ of algebraic spaces which is locally of finite type and an isomorphism over U_i and an isomorphism $\beta_i : (X'_i)_{/T} \cong W_i$ over $(X_i)_{/T}$. By fully faithfullness we find an isomorphism

$$a_{ij} : X'_i \times_X X_j \longrightarrow X_i \times_X X'_j$$

over $X_i \times_X X_j$ such that $\alpha_{ij} = \beta_j|_{X_i \times_X X_j} \circ (a_{ij})_{/T} \circ \beta_i^{-1}|_{X_i \times_X X_j}$. By fully faithfulness again (this time over $X_i \times_X X_j \times_X X_k$) we see that these morphisms a_{ij}

satisfy the same cocycle condition as satisfied by the α_{ij}. In other words, we obtain a descent datum (as in Descent on Spaces, Definition 22.3) (X'_i, a_{ij}) relative to the family $\{X_i \to X\}$. By Bootstrap, Lemma 11.3, this descent datum is effective. Thus we find a morphism $f : X' \to X$ of algebraic spaces and isomorphisms $h_i : X' \times_X X_i \to X'_i$ over X_i such that $a_{ij} = h_j|_{X_i \times_X X_j} \circ h_i^{-1}|_{X_i \times_X X_j}$. The reader can check that the ensuing isomorphisms

$$(X' \times_X X_i)_{/T} \xrightarrow{\beta_i \circ (h_i)_{/T}} W_i$$

over X_i glue to an isomorphism $X'_{/T} \to W$ over $X_{/T}$; some details omitted. $\square$

28. Completions and morphisms, II

To obtain Artin's theorem on dilatations, we need to match formal modifications with actual modifications in the correspondence given by Theorem 27.4. We urge the reader to skip this section.

Lemma 28.1. *With assumptions and notation as in Theorem 27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is quasi-compact if and only if g is quasi-compact.*

Proof. If f is quasi-compact, then g is quasi-compact by Lemma 23.5. Conversely, assume g is quasi-compact. Choose an étale covering $\{X_i \to X\}$ with X_i affine. It suffices to prove that the base change $X' \times_X X_i \to X_i$ is quasi-compact, see Morphisms of Spaces, Lemma 8.8. By Formal Spaces, Lemma 17.3 the base changes $W_i \times_{X_{/T}} (X_i)_{/T} \to (X_i)_{/T}$ are quasi-compact. By Lemma 27.1 we reduce to the case described in the next paragraph.

Assume X is affine and $g : W \to X_{/T}$ quasi-compact. We have to show that X' is quasi-compact. Let $V \to X'$ be a surjective étale morphism where $V = \coprod_{j \in J} V_j$ is a disjoint union of affines. Then $V_{/T} \to X'_{/T} = W$ is a surjective étale morphism. Since W is quasi-compact, then we can find a finite subset $J' \subset J$ such that $\coprod_{j \in J'} (V_j)_{/T} \to W$ is surjective. Then it follows that

$$U \amalg \coprod\nolimits_{j \in J'} V_j \longrightarrow X'$$

is surjective (and hence X' is quasi-compact). Namely, we have $|X'| = |U| \amalg |W_{red}|$ as $X'_{/T} = W$. $\square$

Lemma 28.2. *With assumptions and notation as in Theorem 27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is quasi-separated if and only if g is so.*

Proof. If f is quasi-separated, then g is quasi-separated by Lemma 23.7. Conversely, assume g is quasi-separated. We have to show that f is quasi-separated. Exactly as in the proof of Lemma 28.1 we may check this over the members of a étale covering of X by affine schemes using Morphisms of Spaces, Lemma 4.12 and Formal Spaces, Lemma 30.5. Thus we may and do assume X is affine.

Let $V \to X'$ be a surjective étale morphism where $V = \coprod_{j \in J} V_j$ is a disjoint union of affines. To show that X' is quasi-separated, it suffices to show that $V_j \times_{X'} V_{j'}$ is quasi-compact for all $j, j' \in J$. Since W is quasi-separated the fibre products

$(V_j \times_Y V_{j'})_{/T} = (V_j)_{/T} \times_{X'_{/T}} (V_{j'})_{/T}$ are quasi-compact for all $j, j' \in J$. Since X is Noetherian affine and $U' \to U$ is an isomorphism, we see that

$$(V_j \times_{X'} V_{j'}) \times_X U = (V_j \times_X V_{j'}) \times_X U$$

is quasi-compact. Hence we conclude by the equality

$$|V_j \times_{X'} V_{j'}| = |(V_j \times_{X'} V_{j'}) \times_X U| \amalg |(V_j \times_{X'} V_{j'})_{/T,red}|$$

and the fact that a formal algebraic space is quasi-compact if and only if its associated reduced algebraic space is so. □

Lemma 28.3. *With assumptions and notation as in Theorem 27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is separated $\Leftrightarrow$ g is separated and $\Delta_g : W \to W \times_{X_{/T}} W$ is rig-surjective.*

Proof. If f is separated, then g is separated and Δ_g is rig-surjective by Lemmas 23.7 and 23.11. Assume g is separated and Δ_g is rig-surjective. Exactly as in the proof of Lemma 28.1 we may check this over the members of a étale covering of X by affine schemes using Morphisms of Spaces, Lemma 4.4 (locality on the base of being separated for morphisms of algebraic spaces), Formal Spaces, Lemma 30.2 (being separated for morphisms of formal algebraic spaces is preserved by base change), and Lemma 21.4 (being rig-surjective is preserved by base change). Thus we may and do assume X is affine. Furthermore, we already know that $f : X' \to X$ is quasi-separated by Lemma 28.2.

By Cohomology of Spaces, Lemma 19.1 and Remark 19.3 it suffices to show that given any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "p"] \arrow[ur, dashed] & X' \times_X X' \end{tikzcd}$$

where R is a complete discrete valuation ring with fraction field K, there is a dotted arrow making the diagram commute (as this will give the uniqueness part of the valuative criterion). Let $h : \mathrm{Spec}(R) \to X$ be the composition of p with the morphism $Y \times_X Y \to X$. There are three cases: Case I: $h(\mathrm{Spec}(R)) \subset U$. This case is trivial because $U' = X' \times_X U \to U$ is an isomorphism. Case II: h maps $\mathrm{Spec}(R)$ into T. This case follows from our assumption that $g : W \to X_{/T}$ is separated. Namely, if Z denotes the reduced induced closed subspace structure on T, then h factors through Z and

$$W \times_{X_{/T}} Z = X' \times_X Z \longrightarrow Z$$

is separated by assumption (and for example Formal Spaces, Lemma 30.5) which implies we get the lifting property by Cohomology of Spaces, Lemma 19.1 applied to the displayed arrow. Case III: $h(\mathrm{Spec}(K))$ is not in T but h maps the closed point of $\mathrm{Spec}(R)$ into T. In this case the corresponding morphism

$$p_{/T} : \mathrm{Spf}(R) \longrightarrow (X' \times_X X')_{/T} = W \times_{X_{/T}} W$$

is an adic morphism (by Formal Spaces, Lemma 14.4 and Definition 23.2). Hence our assumption that $\Delta_g : W \to W \times_{X_{/T}} W$ is rig-surjective implies we can lift $p_{/T}$ to a morphism $\mathrm{Spf}(R) \to W = X'_{/T}$, see Lemma 21.11. Algebraizing the composition

$\mathrm{Spf}(R) \to X'$ using Formal Spaces, Lemma 33.3 we find a morphism $\mathrm{Spec}(R) \to X'$ lifting p as desired. $\square$

Lemma 28.4. *With assumptions and notation as in Theorem 27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is proper if and only if g is a formal modification (Definition 24.1).*

Proof. If f is proper, then g is a formal modification by Lemma 24.3. Assume g is a formal modification. By Lemmas 28.1 and 28.3 we see that f is quasi-compact and separated.

By Cohomology of Spaces, Lemma 19.2 and Remark 19.3 it suffices to show that given any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r, "p"] \arrow[ur, dashed] & X \end{tikzcd}$$

where R is a complete discrete valuation ring with fraction field K, there is a dotted arrow making the diagram commute. There are three cases: Case I: $p(\mathrm{Spec}(R)) \subset U$. This case is trivial because $U' \to U$ is an isomorphism. Case II: p maps $\mathrm{Spec}(R)$ into T. This case follows from our assumption that $g : W \to X_{/T}$ is proper. Namely, if Z denotes the reduced induced closed subspace structure on T, then p factors through Z and

$$W \times_{X_{/T}} Z = X' \times_X Z \longrightarrow Z$$

is proper by assumption which implies we get the lifting property by Cohomology of Spaces, Lemma 19.2 applied to the displayed arrow. Case III: $p(\mathrm{Spec}(K))$ is not in T but p maps the closed point of $\mathrm{Spec}(R)$ into T. In this case the corresponding morphism

$$p_{/T} : \mathrm{Spf}(R) \longrightarrow X'_{/T} = W$$

is an adic morphism (by Formal Spaces, Lemma 14.4 and Definition 23.2). Hence our assumption that $g : W \to X_{/T}$ be rig-surjective implies we can lift $g_{/T}$ to a morphism $\mathrm{Spf}(R') \to W = X'_{/T}$ for some extension of complete discrete valuation rings $R \subset R'$. Algebraizing the composition $\mathrm{Spf}(R') \to X'$ using Formal Spaces, Lemma 33.3 we find a morphism $\mathrm{Spec}(R') \to X'$ lifting p as desired. $\square$

Lemma 28.5. *With assumptions and notation as in Theorem 27.4 let $f : X' \to X$ correspond to $g : W \to X_{/T}$. Then f is étale if and only if g is étale.*

Proof. If f is étale, then g is étale by Lemma 23.2. Conversely, assume g is étale. Since f is an isomorphism over U we see that f is étale over U. Thus it suffices to prove that f is étale at any point of X' lying over T. Denote $Z \subset X$ the reduced closed subspace whose underlying topological space is $|Z| = T \subset |X|$, see Properties of Spaces, Definition 12.5. Letting $Z_n \subset X$ be the nth infinitesimal neighbourhood we have $X_{/T} = \mathrm{colim}\, Z_n$. Since $X'_{/T} = W \to X_{/T}$ we conclude that $f^{-1}(Z_n) = X' \times_X Z_n \to Z_n$ is étale by the assumed étaleness of g. By More on Morphisms of Spaces, Lemma 20.3 we conclude that f is étale at points lying over T. $\square$

29. Artin's theorem on dilatations

In this section we use a different font for formal algebraic spaces to stress the similarity of the statements with the corresponding statements in [Art70]. Here is the first main theorem of this chapter.

Theorem 29.1. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $\mathfrak{X} = X_{/T}$ be the formal completion of X along T. Let*

$$\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$$

be a formal modification (Definition 24.1). Then there exists a unique proper morphism $f : X' \to X$ which is an isomorphism over the complement of T in X whose completion $f_{/T}$ recovers $\mathfrak{f}$.

Proof. This follows from Theorem 27.4 and Lemma 28.4. □

Here is the characterization of formal modifcations as promised in Section 24.

Lemma 29.2. *Let S be a scheme. Let $\mathfrak{X}' \to \mathfrak{X}$ be a formal modification (Definition 24.1) of locally Noetherian formal algebraic spaces over S. Given*

(1) *any adic Noetherian topological ring A,*
(2) *any adic morphism $Spf(A) \longrightarrow \mathfrak{X}$*

there exists a proper morphism $X \to \mathrm{Spec}(A)$ of algebraic spaces and an isomorphism

$$Spf(A) \times_{\mathfrak{X}} \mathfrak{X}' \longrightarrow X_{/Z}$$

over $Spf(A)$ of the base change of $\mathfrak{X}$ with the formal completion of X along the "closed fibre" $Z = X \times_{\mathrm{Spec}(A)} Spf(A)_{red}$ of X over A.

Proof. The morphism $\mathrm{Spf}(A) \times_{\mathfrak{X}} \mathfrak{X}' \to \mathrm{Spf}(A)$ is a formal modification by Lemma 24.4. Hence this follows from Theorem 29.1. □

30. Application to modifications

Let A be a Noetherian ring and let $I \subset A$ be an ideal. We set $X = \mathrm{Spec}(A)$ and $U = X \setminus V(I)$. In this section we will consider the category

$$(30.0.1) \qquad \left\{ f : X' \longrightarrow X \;\middle|\; \begin{array}{c} X' \text{ is an algebraic space} \\ f \text{ is locally of finite type} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array} \right\}$$

A morphism from X'/X to X''/X will be a morphism of algebraic spaces $X' \to X''$ over X.

Let $A \to B$ be a homomorphism of Noetherian rings and let $J \subset B$ be an ideal such that $J = \sqrt{IB}$. Then base change along the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ gives a functor from the category (30.0.1) for A to the category (30.0.1) for B.

Lemma 30.1. *Let $A \to B$ be a ring homomorphism of Noetherian rings inducing an isomorphism on I-adic completions for some ideal $I \subset A$ (for example if B is the I-adic completion of A). Then base change defines an equivalence of categories between the category (30.0.1) for (A, I) with the category (30.0.1) for (B, IB).*

Proof. Set $X = \mathrm{Spec}(A)$ and $T = V(I)$. Set $X_1 = \mathrm{Spec}(B)$ and $T_1 = V(IB)$. By Theorem 27.4 (in fact we only need the affine case treated in Lemma 27.3) the category (30.0.1) for X and T is equivalent to the the category of rig-étale morphisms $W \to X_{/T}$ of locally Noetherian formal algebraic spaces. Similarly, the the category (30.0.1) for X_1 and T_1 is equivalent to the category of rig-étale morphisms $W_1 \to X_{1,/T_1}$ of locally Noetherian formal algebraic spaces. Since $X_{/T} = \mathrm{Spf}(A^\wedge)$ and $X_{1,/T_1} = \mathrm{Spf}(B^\wedge)$ (Formal Spaces, Lemma 14.6) we see that these categories are equivalent by our assumption that $A^\wedge \to B^\wedge$ is an isomorphism. We omit the verification that this equivalence is given by base change. ☐

Lemma 30.2. *Notation and assumptions as in Lemma 30.1. Let $f : X' \to \mathrm{Spec}(A)$ correspond to $g : Y' \to \mathrm{Spec}(B)$ via the equivalence. Then f is quasi-compact, quasi-separated, separated, proper, finite, and add more here if and only if g is so.*

Proof. You can deduce this for the statements quasi-compact, quasi-separated, separated, and proper by using Lemmas 28.1 28.2, 28.3, 28.2, and 28.4 to translate the corresponding property into a property of the formal completion and using the argument of the proof of Lemma 30.1. However, there is a direct argument using fpqc descent as follows. First, you can reduce to proving the lemma for $A \to A^\wedge$ and $B \to B^\wedge$ since $A^\wedge \to B^\wedge$ is an isomorphism. Then note that $\{U \to \mathrm{Spec}(A), \mathrm{Spec}(A^\wedge) \to \mathrm{Spec}(A)\}$ is an fpqc covering with $U = \mathrm{Spec}(A) \setminus V(I)$ as before. The base change of f by $U \to \mathrm{Spec}(A)$ is id_U by definition of our category (30.0.1). Let P be a property of morphisms of algebraic spaces which is fpqc local on the base (Descent on Spaces, Definition 10.1) such that P holds for identity morphisms. Then we see that P holds for f if and only if P holds for g. This applies to P equal to quasi-compact, quasi-separated, separated, proper, and finite by Descent on Spaces, Lemmas 11.1, 11.2, 11.18, 11.19, and 11.23. ☐

Lemma 30.3. *Let $A \to B$ be a local map of local Noetherian rings such that*
1. *$A \to B$ is flat,*
2. *$\mathfrak{m}_B = \mathfrak{m}_A B$, and*
3. *$\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$*

Then the base change functor from the category (30.0.1) for $(A, \mathfrak{m}_A)$ to the category (30.0.1) for $(B, \mathfrak{m}_B)$ is an equivalence.

Proof. The conditions signify that $A \to B$ induces an isomorphism on completions, see More on Algebra, Lemma 43.9. Hence this lemma is a special case of Lemma 30.1. ☐

Lemma 30.4. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $f : X \to S$ be an object of (30.0.1). Then there exists a U-admissible blowup $S' \to S$ which dominates X.*

Proof. Special case of More on Morphisms of Spaces, Lemma 39.5. ☐

31. Other chapters

Preliminaries

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