RELATIVE CYCLES

Contents

1. Introduction

A foundational reference is [SV00].

In this chapter we only define what are called the universally integral relative cycles in [SV00]. This choice makes the theory somewhat simpler to develop than in the original, but of course we also lose something.

Fix a morphism $X \to S$ of finite type between Noetherian schemes. A family α of r-cycles on fibres of X/S is simply a collection $\alpha = (\alpha_s)_{s \in S}$ where $\alpha_s \in Z_r(X_s)$. It is immediately clear how to base change $g^*\alpha$ of α along any morphism $g : S' \to S$. Then we say α is a *relative r-cycle on X/S* if α is compatible with specializations, i.e., for any morphism $g : S' \to S$ where S' is the spectrum of a discrete valuation ring, we require the generic fibre of $g^*\alpha$ to specialize to the closed fibre of $g^*\alpha$. See Section 6.

2. Conventions and notation

Please consult the chapter on Chow Homology and Chern Classes for our conventions and notation regarding cycles on schemes locally of finite type over a fixed Noetherian base, see Chow Homology, Section 7 ff.

In particular, if X is locally of finite type over a field k, then $Z_r(X)$ denotes the group of cycles of dimension r, see Chow Homology, Example 7.2 and Section 8. Given an integral closed subscheme $Z \subset X$ with $\dim(Z) = r$ we have $[Z] \in Z_r(X)$ and if X is quasi-compact, then $Z_r(X)$ is free abelian on these classes.

3. Cycles relative to fields

Let k be a field. Let X be a locally algebraic scheme over k. Let $r \geq 0$ be an integer. In this setting we have the group $Z_r(X)$ of r-cycles on X, see Section 2.

Base change. For any field extension k'/k there is a base change map $Z_r(X) \to Z_r(X_{k'})$, see Chow Homology, Section 67. Namely, given an integral closed subscheme $Z \subset X$ of dimension r we send $[Z] \in Z_r(X)$ to the r-cycle $[Z_{k'}]_r \in Z_r(X_{k'})$ associated to the closed subscheme $Z_{k'} \subset X_{k'}$ (of course in general $Z_{k'}$ is neither irreducible nor reduced). The base change map $Z_r(X) \to Z_r(X_{k'})$ is always injective.

Lemma 3.1. *Let K/k be a field extension. Let Z be an integral locally algebraic scheme over k. The multiplicity m_{Z',Z_K} of an irreducible component $Z' \subset Z_K$ is 1 or a power of the characteristic of k.*

Proof. If the characteristic of k is zero, then k is perfect and the multiplicity is always 1 since X_K is reduced by Varieties, Lemma 6.4. Assume the characteristic of k is $p > 0$. Let L be the function field of Z. Since Z is locally algebraic over k, the field extension L/k is finitely generated. The ring $K \otimes_k L$ is Noetherian (Algebra, Lemma 31.8). Translated into algebra, we have to show that the length of the artinian local ring $(K \otimes_k L)_\mathfrak{q}$ is a power of p for every minimal prime ideal $\mathfrak{q}$.

Let L'/L be a finite purely inseparable extension, say of degree p^n. Then $K \otimes_k L \subset K \otimes_k L'$ is a finite free ring map of degree p^n which induces a homeomorphism on spectra and purely inseparable residue field extensions. Hence for every minimal prime $\mathfrak{q}$ as above there is a unique minimal prime $\mathfrak{q}' \subset K \otimes_k L'$ lying over it and

$$p^n \text{length}((K \otimes_k L)_\mathfrak{q}) = [\kappa(\mathfrak{q}') : \kappa(\mathfrak{q})] \text{length}((K \otimes_k L')_{\mathfrak{q}'})$$

by Algebra, Lemma 52.12 applied to $M = (K \otimes_k L')_{\mathfrak{q}'} \cong (K \otimes_k L)_\mathfrak{q}^{\oplus p^n}$. Since $[\kappa(\mathfrak{q}') : \kappa(\mathfrak{q})]$ is a power of p we conclude that it suffices to prove the statement for L' and $\mathfrak{q}'$.

By the previous paragraph and Algebra, Lemma 45.3 we may assume that we have a subfield $L/k'/k$ such that L/k' is separable and k'/k is finite purely inseparable. Then $K \otimes_k k'$ is an Artinian local ring. The argument of the preceding paragraph (applied to $L = k$ and $L' = k'$) shows that $\text{length}(K \otimes_k k')$ is a power of p. Since L/k' is the localization of a smooth k'-algebra (Algebra, Lemma 158.10). Hence $S = (K \otimes_k L)_\mathfrak{q}$ is the localization of a smooth $R = K \otimes_k k'$-algebra at a minimal prime. Thus $R \to S$ is a flat local homomorphism of Artinian local rings and $\mathfrak{m}_R S = \mathfrak{m}_S$. It follows from Algebra, Lemma 52.13 that $\text{length}(K \otimes_k k') = \text{length}(R) = \text{length}(S) = \text{length}((K \otimes_k L)_\mathfrak{q})$ and the proof is finished. $\square$

Lemma 3.2. *Let k be a field of characteristic $p > 0$ with perfect closure k^{perf}. Let X be an algebraic scheme over k. Let $r \geq 0$ be an integer. The cokernel of the injective map $Z_r(X) \to Z_r(X_{k^{perf}})$ is a p-power torsion module (More on Algebra, Definition 88.1).*

Proof. Since X is quasi-compact, the abelian group $Z_r(X)$ is free with basis given by the integral closed subschemes of dimension r. Similarly for $Z_r(X_{k^{perf}})$. Since $X_{k^{perf}} \to X$ is a homeomorphism, it follows that $Z_r(X) \to Z_r(X_{k^{perf}})$ is injective with torsion cokernel. Every element in the cokernel is p-power torsion by Lemma 3.1. $\square$

4. Specialization of cycles

Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $r \geq 0$. There is a specialization map

$$sp_{X/R} : Z_r(X_K) \longrightarrow Z_r(X_\kappa)$$

defined as follows. For an integral closed subscheme $Z \subset X_K$ of dimension r we denote $\overline{Z}$ the scheme theoretic image of $Z \to X$. Then we let $sp_{X/R}$ be the unique $\mathbf{Z}$-linear map such that

$$sp_{X/R}([Z]) = [\overline{Z}_\kappa]_r$$

We briefly discuss why this is well defined. First, observe that the morphism $X_K \to X$ is quasi-compact and hence the morphism $Z \to X$ is quasi-compact. Thus taking the scheme theoretic image of $Z \to X$ commutes with flat base change by Morphisms, Lemma 25.16. In particular, base changing back to X_K we see that $Z = \overline{Z}_K$. Since Z is integral, of course $\overline{Z}$ is integral too and in fact is equal to the unique integral closed subscheme whose generic point is the (image of the) generic point of Z. It follows from Varieties, Lemma 19.2 that Z_κ is equidimensional of dimension r.

Lemma 4.1. *Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $r \geq 0$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module flat over R. Assume $\dim(Supp(\mathcal{F}_K)) \leq r$. Then $\dim(Supp(\mathcal{F}_\kappa)) \leq r$ and*

$$sp_{X/R}([\mathcal{F}_K]_r) = [\mathcal{F}_\kappa]_r$$

Proof. The statement on dimension follows from More on Morphisms, Lemma 18.4. Let x be a generic point of an integral closed subscheme $Z \subset X_\kappa$ of dimension r. To finish the proof we wil show that the coefficient of $[Z]$ in the left (L) and right hand side (R) of equality are the same.

Let $A = \mathcal{O}_{X,x}$ and $M = \mathcal{F}_x$. Observe that M is a finite A-module flat over R. Let $\pi \in R$ be a uniformizer so that $A/\pi A = \mathcal{O}_{X_\kappa, x}$. By Chow Homology, Lemma 3.2 we have

$$\sum\nolimits_i \text{length}_A(A/(\pi, \mathfrak{q}_i))\text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) = \text{length}_A(M/\pi M)$$

where the sum is over the minimal primes $\mathfrak{q}_i$ in the support of M. Since π is a nonzerodivisor on M we see that $\pi \not\in \mathfrak{q}_i$ and hence these primes correspond to those generic points $y_i \in X_K$ of the support of $\mathcal{F}_K$ which specialize to our chosen $x \in X_\kappa$. Thus the left hand side is the coefficient of $[Z]$ in (L). Of course $\text{length}_A(M/\pi M)$ is the coefficient of $[Z]$ in (R). This finishes the proof. $\square$

Lemma 4.2. *Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $r \geq 0$. Let $W \subset X$ be a closed subscheme flat over R. Assume $\dim(W_K) \leq r$. Then $\dim(W_\kappa) \leq r$ and*

$$sp_{X/R}([W_K]_r) = [W_\kappa]_r$$

Proof. Taking $\mathcal{F} = \mathcal{O}_W$ this is a special case of Lemma 4.1. See Chow Homology, Lemma 10.3. $\square$

Lemma 4.3. *Let R'/R be an extension of discrete valuation rings inducing fraction field extension K'/K and residue field extension κ'/κ (More on Algebra, Definition 111.1). Let X be locally of finite type over R. Denote $X' = X_{R'}$. Then the diagram*

$$\begin{tikzcd} Z_r(X'_{K'}) \arrow[r, "sp_{X'/R'}"'] & Z_r(X'_{\kappa'}) \\ Z_r(X_K) \arrow[u] \arrow[r, "sp_{X/R}"] & Z_r(X_\kappa) \arrow[u] \end{tikzcd}$$

commutes where $r \geq 0$ and the vertical arrows are base change maps.

Proof. Observe that $X'_{K'} = X_{K'} = X_K \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K')$ and similarly for closed fibres, so that the vertical arrows indeed make sense (see Section 3). Now if $Z \subset X_K$ is an integral closed subscheme with scheme theoretic image $\overline{Z} \subset X$, then we see that $Z_{K'} \subset X_{K'}$ is a closed subscheme with scheme theoretic image $\overline{Z}_{R'} \subset X_{R'}$. The base change of $[Z]$ is $[Z_{K'}]_r = [\overline{Z}_{K'}]_r$ by definition. We have

$$sp_{X/R}([Z]) = [\overline{Z}_\kappa]_r \quad \text{and} \quad sp_{X'/R'}([\overline{Z}_{K'}]_r) = [(\overline{Z}_{R'})_{\kappa'}]_r$$

by Lemma 4.1. Since $(\overline{Z}_{R'})_{\kappa'} = (\overline{Z}_\kappa)_{\kappa'}$ we conclude. $\square$

Lemma 4.4. *Let R be a discrete valuation ring with fraction field K and residue field κ. Let X be a scheme locally of finite type over R. Let $f : X' \to X$ be a morphism which is locally of finite type, flat, and of relative dimension e. Then the diagram*

$$\begin{tikzcd} Z_{r+e}(X'_K) \arrow[r, "sp_{X'/R}"'] & Z_{r+e}(X'_\kappa) \\ Z_r(X_K) \arrow[u] \arrow[r, "sp_{X/R}"] & Z_r(X_\kappa) \arrow[u] \end{tikzcd}$$

commutes where $r \geq 0$ and the vertical arrows are given by flat pullback.

Proof. Let $Z \subset X$ be an integral closed subscheme dominating R. By the construction of $sp_{X/R}$ we have $sp_{X/R}([Z_K]) = [Z_\kappa]_r$ and this characterizes the specialization map. Set $Z' = f^{-1}(Z) = X' \times_X Z$. Since R is a valuation ring, Z is flat over R. Hence Z' is flat over R and $sp_{X'/R}([Z'_K]_{r+e}) = [Z'_\kappa]_{r+e}$ by Lemma 4.2. Since by Chow Homology, Lemma 14.4 we have $f_K^*[Z_K] = [Z'_K]_{r+e}$ and $f_\kappa^*[Z_\kappa]_r = [Z'_\kappa]_{r+e}$ we win. $\square$

Lemma 4.5. *Let R be a discrete valuation ring with fraction field K and residue field κ. Let $f : X \to Y$ be a proper morphism of schemes locally of finite type over R. Then the diagram*

$$\begin{tikzcd} Z_r(X_K) \arrow[r, "sp_{X/R}"'] \arrow[d] & Z_r(X_\kappa) \arrow[d] \\ Z_r(Y_K) \arrow[r, "sp_{Y/R}"] & Z_r(Y_\kappa) \end{tikzcd}$$

commutes where $r \geq 0$ and the vertical arrows are given by proper pushforward.

Proof. Let $Z \subset X$ be an integral closed subscheme dominating R. By the construction of $sp_{X/R}$ we have $sp_{X/R}([Z_K]) = [Z_\kappa]_r$ and this characterizes the specialization map. Set $Z' = f(Z) \subset Y$. Then Z' is an integral closed subscheme of Y dominating R. Thus $sp_{Y/R}([Z'_K]) = [Z'_\kappa]_r$.

We can think of $[Z]$ as an element of $Z_{r+1}(X)$. By definition we have $f_*[Z] = 0$ if $\dim(Z') < r+1$ and $f_*[Z] = d[Z']$ if $Z \to Z'$ is generically finite of degree d. Since proper pushforward commutes with flat pullback by $Y_K \to Y$ (Chow Homology, Lemma 15.1) we see that correspondingly $f_{K,*}[Z_K] = 0$ or $f_{K,*}[Z_K] = d[Z'_K]$. Let us apply Chow Homology, Lemma 29.8 to the commutative diagram

$$\begin{tikzcd} X_\kappa \arrow[r, "i"'] \arrow[d] & X \arrow[d] \\ Y_\kappa \arrow[r, "j"] & Y \end{tikzcd}$$

We obtain that $f_{\kappa,*}[Z_\kappa]_r = 0$ or $f_{\kappa,*}[Z_\kappa] = d[Z'_\kappa]_r$ because clearly $i^*[Z] = [Z_k]_r$ and $j^*[Z'] = [Z'_\kappa]_r$. Putting everything together we conclude. $\square$

5. Families of cycles on fibres

Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. A *family α of r-cycles on fibres of X/S* is a family

$$\alpha = (\alpha_s)_{s \in S}$$

indexed by the points s of the scheme S where $\alpha_s \in Z_r(X_s)$ is an r cycle on the scheme theoretic fibre X_s of f at s. There are various constructions we can perform on families of r-cycles on fibres.

Base change. Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a catesian square of morphisms of schemes with f locally of finite type. Let $r \geq 0$ be an integer. Given a family α of r-cycles on fibres of X/S we define the *base change* $g^*\alpha$ of α to be the family

$$g^*\alpha = (\alpha'_{s'})_{s' \in S'}$$

where $\alpha'_{s'} \in Z_r(X'_{s'})$ is the base change of the cycle α_s with $s' = g(s)$ as in Section 3 via the identitification $X'_{s'} = X_s \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(s'))$ of scheme theoretic fibres.

Restriction. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $U \subset X$ and $V \subset S$ be open subschemes with $f(U) \subset V$. Given a family α of r-cycles on fibres of X/S we can define the *restriction* $\alpha|_U$ of α to be the family of r-cycles on fibres of U/V

$$\alpha|_U = (\alpha_s|_{U_s})_{s \in V}$$

of restrictions to scheme theoretic fibres.

Flat pullback. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $r, e \geq 0$ be integers. Let $f : X' \to X$ be a flat morphism, locally of finite type, and of relative dimension e. Given a family α of r-cycles on fibres of X/S we define the *flat pullback* $f^*\alpha$ of α to be the family of $(r+e)$-cycles on fibres

$$f^*\alpha = (f_s^*\alpha_s)_{s \in S}$$

where $f_s^*\alpha_s \in Z_{r+e}(X'_s)$ is the flat pullback of the cycle α_s in $Z_r(X_s)$ by the flat morphism $f_s : X'_s \to X_s$ of relative dimension e of scheme theoretic fibres.

Proper pushforward. Let

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes with X and Y locally of finite type over S and f proper. Let $r \geq 0$ be an integer. Given a family α of r-cycles on fibres of X/S we define the *proper pushforward* $f_*\alpha$ of α to be the family of r-cycles on fibres of Y/S by

$$f_*\alpha = (f_{s,*}\alpha_s)_{s \in S}$$

where $f_{s,*}\alpha_s \in Z_r(Y_s)$ is the proper pushforward of the cycle α_s in $Z_r(X_s)$ by the proper morphism $f_s : X_s \to Y_s$ of scheme theoretic fibres.

Lemma 5.1. *We have the following compatibilities between the operations above: (1) base change is functorial, (2) restriction is a combination of base change and (a special case of) flat pullback, (3) flat pullback commutes with base change, (4) flat pullback is functorial, (5) proper pushforward commutes with base change, (6) proper pushforward is functorial, and (7) proper pushforward commutes with flat pullback.*

Proof. Each of these compatibilities follows directly from the corresponding results proved in the chapter on Chow homology applied to the fibres over S of the schemes in question. We omit the precise statements and the detailed proofs. Here are some references. Part (1): Chow Homology, Lemma 67.9. Part (2): Obvious. Part (3): Chow Homology, Lemma 67.5. Part (4): Chow Homology, Lemma 14.3. Part (5): Chow Homology, Lemma 67.6. Part (6): Chow Homology, Lemma 12.2. Part (7): Chow Homology, Lemma 15.1. $\square$

Example 5.2. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For $s \in S$ denote $\mathcal{F}_s$ the pullback of $\mathcal{F}$ to X_s. Assume $\dim(\text{Supp}(\mathcal{F}_s)) \leq r$ for all $s \in S$. Then we can associate to $\mathcal{F}$ the family $[\mathcal{F}/X/S]_r$ of r-cycles on fibres of X/S defined by the formula

$$[\mathcal{F}/X/S]_r = ([\mathcal{F}_s]_r)_{s \in S}$$

where $[\mathcal{F}_s]_r$ is given by Chow Homology, Definition 10.2.

Lemma 5.3. *The construction in Example 5.2 is compatible with base change, restriction, and flat pullback.*

Proof. See Chow Homology, Lemmas 67.3 and 14.4. $\square$

Example 5.4. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $Z \subset X$ be a closed subscheme. For $s \in S$ denote Z_s the inverse image of Z in X_s or equivalently the scheme theoretic fibre of Z at

s viewed as a closed subscheme of X_s. Assume $\dim(Z_s) \leq r$ for all $s \in S$. Then we can associate to Z the family $[Z/X/S]_r$ of r-cycles on fibres of X/S defined by the formula

$$[Z/X/S]_r = ([Z_s]_r)_{s \in S}$$

where $[Z_s]_r$ is given by Chow Homology, Definition 9.2.

Lemma 5.5. *The construction in Example 5.4 is compatible with base change, restriction, and flat pullback.*

Proof. Taking $\mathcal{F} = (Z \to X)_* \mathcal{O}_Z$ this is a special case of Lemma 5.3. See Chow Homology, Lemma 10.3. □

Remark 5.6 (Support)**.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let α be a family of r-cycles on fibres of X/S. We define the *support* of α to be

$$\text{Supp}(\alpha) = \bigcup\nolimits_{s \in S} \text{Supp}(\alpha_s) \subset X$$

Here $\text{Supp}(\alpha_s) \subset X_s$ is the support of the cycle α_s, see Chow Homology, Definition 8.3. The support $\text{Supp}(\alpha)$ is rarely a closed subset of X.

Lemma 5.7. *Taking the support as in Remark 5.6 is compatible with base change, restriction, and flat pullback.*

Proof. Omitted. □

Lemma 5.8. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $g : S' \to S$ be a surjective morphism of schemes. Set $S'' = S' \times_S S'$ and let $f' : X' \to S'$ and $f'' : X'' \to S''$ be the base changes of f. Let $x \in X$ with $trdeg_{\kappa(f(x))}(\kappa(x)) = r$.*

1. *There exists an $x' \in X'$ mapping to x with $trdeg_{\kappa(f'(x'))}(\kappa(x')) = r$.*
2. *If $x'_1, x'_2 \in X'$ are both as in (1), then there exists an $x'' \in X''$ with $trdeg_{\kappa(f''(x''))}(\kappa(x'')) = r$ and $pr_i(x'') = x'_i$.*

Proof. Part (1) is Morphisms, Lemma 28.3. Let x'_1, x'_2 be as in (2). Then since $X'' = X' \times_X X'$ we see that there exists a $x'' \in X''$ mapping to both x'_1 and x'_2 (see for example Descent, Lemma 13.1). Denote $s'' \in S''$, $s'_i \in S'$, and $s \in S$ the images of x'', x'_i, and x. Denote $k = \kappa(s)$ and let $Z \subset X_k$ be the integral closed subscheme whose generic point is x. Then x'_i is a generic point of an irreducible component of $Z_{\kappa(s'_i)}$. Let $Z'' \subset Z_{\kappa(s'')}$ be an irreducible component containing x''. Denote $\xi'' \in Z''$ the generic point. Since $\xi'' \rightsquigarrow x''$ we see that ξ'' must also map to x'_i under the two projections. On the other hand, we see that $\text{trdeg}_{\kappa(s'')}(\kappa(\xi'')) = r$ because it is a generic point of an irreducible component of the base change of Z. □

Lemma 5.9. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. Let $g : S' \to S$ be a morphism of schemes and $X' = S' \times_S X$. Assume that for every $s \in S$ there exists a point $s' \in S'$ with $g(s') = s$ and such that $\kappa(s')/\kappa(s)$ is a separable extension of fields. Then*

1. *For families α_1 and α_2 of r-cycles on fibres of X/S if $g^*\alpha_1 = g^*\alpha_2$, then $\alpha_1 = \alpha_2$.*
2. *Given a family α' of r-cycles on fibres of X'/S' if $pr_1^*\alpha' = pr_2^*\alpha'$ as families of r-cycles on fibres of $(S' \times_S S') \times_S X/(S' \times_S S')$, then there is a unique family α of r-cycles on fibres of X/S such that $g^*\alpha = \alpha'$.*

Proof. Part (1) follows from the injectivity of the base change map discussed in Section 3. (This argument works as long as $S' \to S$ is surjective.)

Let α' be as in (2). Denote $\alpha'' = \mathrm{pr}_1^*\alpha' = \mathrm{pr}_2^*\alpha'$ the common value.

Let $(X/S)^{(r)}$ be the set of $x \in X$ with $\mathrm{trdeg}_{\kappa(f(x))}(\kappa(x)) = r$ and similarly define $(X'/S')^{(r)}$ and $(X''/S'')^{(r)}$ Taking coefficients, we may think of α' and α'' as functions $\alpha' : (X'/S')^{(r)} \to \mathbf{Z}$ and $\alpha'' : (X''/S'')^{(r)} \to \mathbf{Z}$. Given a function

$$\varphi : (X/S)^{(r)} \to \mathbf{Z}$$

we define $g^*\varphi : (X'/S')^{(r)} \to \mathbf{Z}$ by analogy with our base change operation. Namely, say $x' \in (X'/S')^{(r)}$ maps to $x \in X$, $s' \in S'$, and $s \in Z$. Denote $Z' \subset X'_{s'}$ and $Z \subset X_s$ the integral closed subschemes with generic points x' and x. Note that $\dim(Z') = r$. If $\dim(Z) < r$, then we set $(g^*\varphi)(x') = 0$. If $\dim(Z) = r$, then Z' is an irreducible component of $Z_{s'}$ and hence has a multiplicity $m_{Z', Z_{s'}}$. Call this $m(x', g)$. Then we define

$$(g^*\varphi)(x') = m(x', g)\varphi(x)$$

Note that the coefficients $m(x', g)$ are always positive integers (see for example Lemma 3.1). We similarly have base change maps

$$\mathrm{pr}_1^*, \mathrm{pr}_2^* : \mathrm{Map}((X'/S')^{(r)}, \mathbf{Z}) \longrightarrow \mathrm{Map}((X''/S'')^{(r)}, \mathbf{Z})$$

It follows from the associativity of base change that we have $\mathrm{pr}_1^* \circ g^* = \mathrm{pr}_2^* \circ g^*$ (small detail omitted). To be explicity, in terms of the maps of sets this equality just means that for $x'' \in (X''/S'')^{(r)}$ we have

$$m(x'', \mathrm{pr}_1)m(\mathrm{pr}_1(x''), g) = m(x'', \mathrm{pr}_2)m(\mathrm{pr}_2(x''), g)$$

provided that $\mathrm{pr}_1(x'')$ and $\mathrm{pr}_2(x'')$ are in $(X''/S'')^{(r)}$. By Lemma 5.8 and an elementary argument[1] using the previous displayed equation, it follows that there exists a unique map

$$\alpha : (X/S)^{(r)} \to \mathbf{Q}$$

such that $g^*\alpha = \alpha'$. To finish the proof it suffices to show that α has integer values (small detail omitted: one needs to see that α determines a locally finite sum on each fibre which follows from the corresponding fact for α'). Given any $x \in (X/S)^{(r)}$ with image $s \in S$ we can pick a point $s' \in S'$ such that $\kappa(s')/\kappa(s)$ is separable. Then we may choose $x' \in (X'/S')^{(r)}$ mapping to s and x and we see that $m(x', g) = 1$ because $Z_{s'}$ is reduced in this case. Whence $\alpha(x) = \alpha'(x')$ is an integer. □

Lemma 5.10. *Let $g : S' \to S$ be a bijective morphism of schemes which induces isomorphisms of residue fields. Let $f : X \to S$ be locally of finite type. Set $X' = S' \times_S X$. Let $r \geq 0$. Then base change by g determines a bijection between the group of families of r-cycles on fibres of X/S and the group of families of r-cycles on fibres of X'/S'.*

Proof. Omitted. □

[1]Given $x \in (X/S)^{(r)}$ pick $x' \in (X'/S')^{(r)}$ mapping to x and set $\alpha(x) = \alpha'(x')/m(x', g)$. This is well defined by the formula and the lemma.

6. Relative cycles

Here is the definition we will work with; see Section 15 for a comparison with the definitions in [SV00].

Definition 6.1. Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $r \geq 0$ be an integer. A *relative r-cycle on X/S* is a family α of r-cycles on fibres of X/S such that for every morphism $g : S' \to S$ where S' is the spectrum of a discrete valuation ring we have

$$sp_{X'/S'}(\alpha_\eta) = \alpha_0$$

where $sp_{X'/S'}$ is as in Section 4 and α_η (resp. α_0) is the value of the base change $g^*\alpha$ of α at the generic (resp. closed) point of S'. The group of all relative r-cycles on X/S is denoted $z(X/S, r)$.

Lemma 6.2. *Let α be a relative r-cycle on X/S as in Definition 6.1. Then any restriction, base change, flat pullback, or proper pushforward of α is a relative r-cycle.*

Proof. For flat pullback use Lemma 4.4. Restriction is a special case of flat pullback. To see it holds for base change use that base change is transitive. For proper pushforward use Lemma 4.5. $\square$

Lemma 6.3. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a family of r-cycles on fibres of X/S. Let $\{g_i : S_i \to S\}$ be a h covering (More on Flatness, Definition 34.2). Then α is a relative r-cycle if and only if each base change $g_i^*\alpha$ is a relative r-cycle.*

Proof. If α is a relative r-cycle, then each base change $g_i^*\alpha$ is a relative r-cycle by Lemma 6.2. Assume each $g_i^*\alpha$ is a relative r-cycle. Let $g : S' \to S$ be a morphism where S' is the spectrum of a discrete valuation ring. After replacing S by S', X by $X' = X \times_S S'$, and α by $\alpha' = g^*\alpha$ and using that the base change of a h covering is a h covering (More on Flatness, Lemma 34.9) we reduce to the problem studied in the next paragraph.

Assume S is the spectrum of a discrete valuation ring with closed point 0 and generic point η. We have to show that $sp_{X/S}(\alpha_\eta) = \alpha_0$. Since a h covering is a V covering (by definition), there is an i and a specialization $s' \rightsquigarrow s$ of points of S_i with $g_i(s') = \eta$ and $g_i(s) = 0$, see Topologies, Lemma 10.13. By Properties, Lemma 5.10 we can find a morphism $h : S' \to S_i$ from the spectrum S' of a discrete valuation ring which maps the generic point η' to s' and maps the closed point $0'$ to s. Denote $\alpha' = h^*g_i^*\alpha$. By assumption we have $sp_{X'/S'}(\alpha'_{\eta'}) = \alpha'_{0'}$. Since $g = g_i \circ h : S' \to S$ is the morphism of schemes induced by an extension of discrete valuation rings we conclude that $sp_{X/S}$ and $sp_{X'/S'}$ are compatible with base change maps on the fibres, see Lemma 4.3. We conclude that $sp_{X/S}(\alpha_\eta) = \alpha_0$ because the base change map $Z_r(X_0) \to Z_r(X'_{0'})$ is injective as discussed in Section 3. $\square$

Lemma 6.4. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a family of r-cycles on fibres of X/S. Let $\{f_i : X_i \to X\}$ be a jointly surjective family of flat morphisms, locally of finite type, and of relative dimension e. Then α is a relative r-cycle if and only if each flat pullback $f_i^*\alpha$ is a relative r-cycle.*

Proof. If α is a relative r-cycle, then each pull back $f_i^*\alpha$ is a relative r-cycle by Lemma 6.2. Assume each $f_i^*\alpha$ is a relative r-cycle. Let $g : S' \to S$ be a morphism where S' is the spectrum of a discrete valuation ring. After replacing S by S', X by $X' = X \times_S S'$, and α by $\alpha' = g^*\alpha$ we reduce to the problem studied in the next paragraph.

Assume S is the spectrum of a discrete valuation ring with closed point 0 and generic point η. We have to show that $sp_{X/S}(\alpha_\eta) = \alpha_0$. Denote $f_{i,0} : X_{i,0} \to X_0$ the base change of f_i to the closed point of S. Similarly for $f_{i,\eta}$. Observe that

$$f_{i,0}^* sp_{X/S}(\alpha_\eta) = sp_{X_i/S}(f_{i,\eta}^*\alpha_\eta) = f_{i,0}^*\alpha_0$$

Namely, the first equality holds by Lemma 4.4 and the second by assumption. Since the family of maps $f_{i,0}^* : Z_r(X_0) \to Z_r(X_{i,0})$ is jointly injective (due to the fact that $f_{i,0}$ is jointly surjective), we conclude what we want. □

Lemma 6.5. *Let S be a locally Noetherian scheme. Let $i : X \to Y$ be a closed immersion of schemes locally of finite type over S. Let $r \geq 0$. Let α be a family of r-cycles on fibres of X/S. Then α is a relative r-cycle on X/S if and only if $i_*\alpha$ is a relative r-cycle on Y/S.*

Proof. Since base change commutes with i_* (Lemma 5.1) it suffices to prove the following: if S is the spectrum of a discrete valuation ring with generic point η and closed point 0, then $sp_{X/S}(\alpha_\eta) = \alpha_0$ if and only if $sp_{Y/S}(i_{\eta,*}\alpha_\eta) = i_{0,*}\alpha_0$. This is true because $i_{0,*} : Z_r(X_0) \to Z_r(Y_0)$ is injective and because $i_{0,*}sp_{X/S}(\alpha_\eta) = sp_{Y/S}(i_{\eta,*}\alpha_\eta)$ by Lemma 4.5. □

The following lemma will be strengthened in Lemma 6.12.

Lemma 6.6. *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f locally of finite type. Let $r \geq 0$. Let α and β be relative r-cycles on X/S. The following are equivalent*

1. *$\alpha = \beta$, and*
2. *$\alpha_\eta = \beta_\eta$ for any generic point $\eta \in S$ of an irreducible component of S.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2). For every $s \in S$ we can find an η as in (2) which specializes to s. By Properties, Lemma 5.10 we can find a morphism $g : S' \to S$ from the spectrum S' of a discrete valuation ring which maps the generic point η' to η and maps the closed point 0 to s. Then α_s and β_s are elements of $Z_r(X_s)$ which base change to the same element of $Z_r(X_{0'})$, namely $sp_{X_{S'}/S'}(\alpha_{\eta'})$ where $\alpha_{\eta'}$ is the base change of α_η. Since the base change map $Z_r(X_s) \to Z_r(X_{0'})$ is injective as discussed in Section 3 we conclude $\alpha_s = \beta_s$. □

Lemma 6.7. *In the situation of Example 5.2 assume S is locally Noetherian and $\mathcal{F}$ is flat over S in dimensions $\geq r$ (More on Flatness, Definition 20.10). Then $[\mathcal{F}/X/S]_r$ is a relative r-cycle on X/S.*

Proof. By More on Flatness, Lemma 20.9 the hypothesis on $\mathcal{F}$ is preserved by any base change. Also, formation of $[\mathcal{F}/X/S]_r$ is compatible with any base change by Lemma 5.3. Since the condition of being compatible with specializations is checked after base change to the spectrum of a discrete valuation ring, this reduces us to the case where S is the spectrum of a valuation ring. In this case the set $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X by More on Flatness, Lemma 13.11. Since the complement of U in X has fibres of dimension $< r$ over S by assumption, we see that restriction along the inclusion $U \subset X$ induces an isomorphism on the groups of r-cycles on fibres after any base change, compatible with specialization maps and with formation of the relative cycle associated to $\mathcal{F}$. Thus it suffices to show compability with specializations for $[\mathcal{F}|_U/U/S]_r$. Since $\mathcal{F}|_U$ is flat over S, this follows from Lemma 4.1 and the definitions. $\square$

Lemma 6.8. *In the situation of Example 5.4 assume S is locally Noetherian and Z is flat over S in dimensions $\geq r$. Then $[Z/X/S]_r$ is a relative r-cycle on X/S.*

Proof. The assumption means that $\mathcal{O}_Z$ is flat over S in dimensions $\geq r$. Thus applying Lemma 6.7 with $\mathcal{F} = (Z \to X)_*\mathcal{O}_Z$ we conclude. $\square$

Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism which is of finite type. Let $r \geq 0$. Denote $Hilb(X/S, r)$ the set of closed subschemes $Z \subset X$ such that $Z \to S$ is flat and of relative dimension $\leq r$. By Lemma 6.8 for each $Z \in Hilb(X/S, r)$ we have an element $[Z/X/S]_r \in z(X/S, r)$. Thus we obtain a group homomorphism

$$\text{free abelian group on } Hilb(X/S, r) \longrightarrow z(X/S, r) \tag{6.8.1}$$

sending $\sum n_i[Z_i]$ to $\sum n_i[Z_i/X/S]_r$. A key feature of relative r-cycles is that they are locally (on X and S in suitable topologies) in the image of this map.

Lemma 6.9. *Let $f : X \to S$ be a finite type morphism of schemes with S Noetherian. Let $r \geq 0$. Let α be a relative r-cycle on X/S. Then there is a proper, completely decomposed (More on Morphisms, Definition 78.1) morphism $g : S' \to S$ such that $g^*\alpha$ is in the image of (6.8.1).*

Proof. By Noetherian induction, we may assume the result holds for the pullback of α by any closed immersion $g : S' \to S$ which is not an isomorphism.

Let $S_1 \subset S$ be an irreducible component (viewed as an integral closed subscheme). Let $S_2 \subset S$ be the closure of the complement of S' (viewed as a reduced closed subscheme). If $S_2 \neq \emptyset$, then the result holds for the pullback of α by $S_1 \to S$ and $S_2 \to S$. If $g_1 : S'_1 \to S_1$ and $g_2 : S'_2 \to S_2$ are the corresponding completely decomposed proper morphisms, then $S' = S'_1 \amalg S'_2 \to S$ is a completely decomposed proper morphism and we see the result holds for S[2]. Thus we may assume $S' \to S$ is bijective and we reduce to the case described in the next paragraph.

Assume S is integral. Let $\eta \in S$ be the generic point and let $K = \kappa(\eta)$ be the function field of S. Then α_η is an r-cycle on X_K. Write $\alpha_\eta = \sum n_i[Y_i]$. Taking the closure of Y_i we obtain integral closed subschemes $Z_i \subset X$ whose base change to η is Y_i. By generic flatness (for example Morphisms, Proposition 27.1), we see that Z_i is flat over a nonempty open U of S for each i. Applying More on Flatness, Lemma 31.1 we can find a U-admissible blowing up $g : S' \to S$ such that the strict transform $Z'_i \subset X_{S'}$ of Z_i is flat over S'. Then $\beta = \sum n_i[Z'_i/X_{S'}/S']_r$ is in the image of (6.8.1) and $\beta = g^*\alpha$ by Lemma 6.6.

However, this does not finish the proof as $S' \to S$ may not be completely decomposed. This is easily fixed: denoting $T \subset S$ the complement of U (viewed as a closed subscheme), by Noetherian induction we can find a completely decomposed

[2]Namely, any closed subscheme of $S'_1 \times_S X$ flat and of relative dimension $\leq r$ over S'_1 may be viewed as a closed subscheme of $S' \times_S X$ flat and of relative dimension $\leq r$ over S'.

proper morphism $T' \to T$ such that $(T' \to S)^*\alpha$ is in the image of (6.8.1). Then $S' \amalg T' \to S$ does the job. $\square$

Lemma 6.10. *Let $f : X \to S$ be a finite type morphism of schemes with S the spectrum of a discrete valuation ring. Let $r \geq 0$. Then (6.8.1) is surjective.*

Proof. This of course follows from Lemma 6.9 but we can also see it directly as follows. Say α is a relative r-cycle on X/S. Write $\alpha_\eta = \sum n_i[Z_i]$ (the sum is finite). Denote $\overline{Z}_i \subset X$ the closure of Z_i as in Section 4. Then $\alpha = \sum n_i[\overline{Z}_i/X/S]$. $\square$

Lemma 6.11. *Let $f : X \to S$ be a morphism of schemes. Let $r \geq 0$. Assume S locally Noetherian and f smooth of relative dimension r. Let $\alpha \in z(X/S, r)$. Then the support of α is open and closed in X (see proof for a more precise result).*

Proof. Let $x \in X$ with image $s \in S$. Since f is smooth, there is a unique irreducible component $Z(x)$ of X_s which contains x. Then $\dim(Z(x)) = r$. Let n_x be the coefficient of $Z(x)$ in the cycle α_s. We will show the function $x \mapsto n_x$ is locally constant on X.

Let $g : S' \to S$ be a morphism of locally Noetherian schemes. Let X' be the base change of X and let $\alpha' = g^*\alpha$ be the base change of α. Let $x' \in X'$ map to $s' \in S'$, $x \in X$, and $s \in S$. We claim $n_{x'} = n_x$. Namely, since $Z(x)$ is smooth over $\kappa(s)$ we see that $Z(x) \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(s'))$ is reduced. Since $Z(x')$ is an irreducible component of this scheme, we see that the coefficient $n_{x'}$ of $Z(x')$ in $\alpha'_{s'}$ is the same as the coefficient n_x of $Z(x)$ in α_s by the definition of base change in Section 3 thereby proving the claim.

Since X is locally Noetherian, to show that $x \mapsto n_x$ is locally constant, it suffices to show: if $x' \rightsquigarrow x$ is a specialization in X, then $n_{x'} = n_x$. Choose a morphism $S' \to X$ where S' is the spectrum of a discrete valuation ring mapping the generic point η to x' and the closed point 0 to x. See Properties, Lemma 5.10. Then the base change $X' \to S'$ of f by $S' \to S$ has a section $\sigma : S' \to X'$ such that $\sigma(\eta) \rightsquigarrow \sigma(0)$ is a specialization of points of X' mapping to $x' \rightsquigarrow x$ in X. Thus we reduce to the claim in the next paragraph.

Let S be the spectrum of a discrete valuation ring with generic point η and closed point 0 and we have a section $\sigma : S \to X$. Claim: $n_{\sigma(\eta)} = n_{\sigma(0)}$. By the discussion in More on Morphisms, Section 29 and especially More on Morphisms, Lemma 29.6 after replacing X by an open subscheme, we may assume the fibres of $X \to S$ are connected. Since these fibres are smooth, they are irreducible. Then we see that $\alpha_\eta = n[X_\eta]$ with $n = n_{\sigma(\eta)}$ and the relation $sp_{X/S}(\alpha_\eta) = \alpha_0$ implies $\alpha_0 = n[X_0]$, i.e., $n_{\sigma(0)} = n$ as desired. $\square$

Lemma 6.12. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ and $\alpha, \beta \in z(X/S, r)$. The set $E = \{s \in S : \alpha_s = \beta_s\}$ is closed in S.*

Proof. The question is local on S, thus we may assume S is affine. Let $X = \bigcup U_i$ be an affine open covering. Let $E_i = \{s \in S : \alpha_s|_{U_{i,s}} = \beta_s|_{U_{i,s}}\}$. Then $E = \bigcap E_i$. Hence it suffices to prove the lemma for $U_i \to S$ and the restriction of α and β to U_i. This reduces us to the case discussed in the next paragraph.

Assume X and S are quasi-compact. Set $\gamma = \alpha - \beta$. Then $E = \{s \in S : \gamma_s = 0\}$. By Lemma 6.8 there exists a jointly surjective finite family of proper morphisms

$\{g_i : S_i \to S\}$ such that $g_i^*\gamma$ is in the image of (6.8.1). Observe that $E_i = g_i^{-1}(E)$ is the set of point $t \in S_i$ such that $(g_i^*\gamma)_t = 0$. If E_i is closed for all i, then $E = \bigcup g_i(E_i)$ is closed as well. This reduces us to the case discussed in the next paragraph.

Assume X and S are quasi-compact and $\gamma = \sum n_i[Z_i/X/S]_r$ for a finite number of closed subschemes $Z_i \subset X$ flat and of relative dimension $\leq r$ over S. Set $X' = \bigcup Z_i$ (scheme theoretic union). Then $i : X' \to X$ is a closed immersion and X' has relative dimension $\leq r$ over S. Also $\gamma = i_*\gamma'$ where $\gamma' = \sum n_i[Z_i/X'/S]_r$. Since clearly $E = E' = \{s \in S : \gamma'_s = 0\}$ we reduce to the case discussed in the next paragraph.

Assume X has relative dimension $\leq r$ over S. Let $s \in S$, $s \notin E$. We will show that there exists an open neighbourhood $V \subset S$ of s such that $E \cap V$ is empty. The assumption $s \notin E$ means there exists an integral closed subscheme $Z \subset X_s$ of dimension r such that the coefficient n of $[Z]$ in γ_s is nonzero. Let $x \in Z$ be the generic point. Since $\dim(Z) = r$ we see that x is a generic point of an irreducible component (namely Z) of X_s. Thus after replacing X by an open neighbourhood of x, we may assume that Z is the only irreducible component of X_s. In particular, we have $\gamma_s = n[Z]$.

At this point we apply More on Morphisms, Lemma 47.1 and we obtain a diagram

$$\begin{tikzcd}
X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] & & x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] \\
& Y \arrow[d, "h"] & & & y \arrow[d, mapsto] \\
S \arrow[r, equal] & S & & s \arrow[r, equal] & s
\end{tikzcd}$$

with all the properties listed there. Let $\gamma' = g^*\gamma$ be the flat pullback. Note that $E \subset E' = \{s \in S : \gamma'_s = 0\}$ and that $s \notin E'$ because the coefficient of Z' in γ'_s is nonzero, where $Z' \subset X'_s$ is the closure of x'. Similarly, set $\gamma'' = \pi_*\gamma'$. Then we have $E' \subset E'' = \{s \in S : \gamma''_s = 0\}$ and $s \notin E''$ because the coefficient of Z'' in γ''_s is nonzero, where $Z'' \subset Y_s$ is the closure of y. By Lemma 6.11 and openess of $Y \to S$ we see that an open neighbourhood of s is disjoint from E'' and the proof is complete. □

Lemma 6.13. *Let $S = \lim_{i \in I} S_i$ be the limit of a directed inverse system of Noetherian schemes with affine transition morphisms. Let $0 \in I$ and let $X_0 \to S_0$ be a finite type morphism of schemes. For $i \geq 0$ set $X_i = S_i \times_{S_0} X_0$ and set $X = S \times_{S_0} X_0$. If S is Noetherian too, then*

$$z(X/S, r) = \operatorname{colim}_{i \geq 0} z(X_i/S_i, r)$$

where the transition maps are given by base change of relative r-cycles.

Proof. Suppose that $i \geq 0$ and $\alpha_i, \beta_i \in z(X_i/S_i, r)$ map to the same element of $z(X/S, r)$. Then $S \to S_i$ maps into the closed subset $E \subset S_i$ of Lemma 6.12. Hence for some $j \geq i$ the morphism $S_j \to S_i$ maps into E, see Limits, Lemma 4.10. It follows that the base change of α_i and β_i to S_j agree. Thus the map is injective.

Let $\alpha \in z(X/S, r)$. Applying Lemma 6.9 a completely decomposed proper morphism $g : S' \to S$ such that $g^*\alpha$ is in the image of (6.8.1). Set $X' = S' \times_S X$. We write $g^*\alpha = \sum n_a[Z_a/X'/S']_r$ for some $Z_a \subset X'$ closed subscheme flat and of relative dimension $\leq r$ over S'.

Now we bring the machinery of Limits, Section 10 ff to bear. We can find an $i \geq 0$ such that there exist

(1) a completely decomposed proper morphism $g_i : S'_i \to S_i$ whose base change to S is $g : S' \to S$,
(2) setting $X'_i = S'_i \times_{S_i} X_i$ closed subschemes $Z_{ai} \subset X'_i$ flat and of relative dimension $\leq r$ over S'_i whose base change to S' is Z_a.

To do this one uses Limits, Lemmas 10.1, 8.5, 8.7, 13.1, and 18.1 and More on Morphisms, Lemma 78.5. Consider $\alpha'_i = \sum n_a[Z_{ai}/X'_i/S'_i]_r \in z(X'_i/S'_i, r)$. The image of α'_i in $z(X'/S', r)$ agrees with the base change $g^*\alpha$ by construction.

Set $S''_i = S'_i \times_{S_i} S'_i$ and $X''_i = S''_i \times_{S_i} X_i$ and set $S'' = S' \times_S S'$ and $X'' = S'' \times_S X$. We denote $\text{pr}_1, \text{pr}_2 : S'' \to S'$ and $\text{pr}_1, \text{pr}_2 : S''_i \to S'_i$ the projections. The two base changes $\text{pr}_1^*\alpha'_i$ and $\text{pr}_1^*\alpha'_i$ map to the same element of $z(X''/S'', r)$ because $\text{pr}_1^*g^*\alpha = \text{pr}_1^*g^*\alpha$. Hence after increasing i we may assume that $\text{pr}_1^*\alpha'_i = \text{pr}_1^*\alpha'_i$ by the first paragraph of the proof. By Lemma 5.9 we obtain a unique family α_i of r-cycles on fibres of X_i/S_i with $g_i^*\alpha_i = \alpha'_i$ (this uses that $S'_i \to S_i$ is completely decomposed). By Lemma 6.3 we see that $\alpha_i \in z(X_i/S_i, r)$. The uniqueness in Lemma 5.9 implies that the image of α_i in $z(X/S, r)$ is α and the proof is complete. □

Lemma 6.14. *Let S be a locally Noetherian scheme. Let $i : X \to X'$ be a thickening of schemes locally of finite type over S. Let $r \geq 0$. Then $i_* : z(X/S, r) \to z(X'/S, r)$ is a bijection.*

Proof. Since $i_s : X_s \to X'_s$ is a thickening it is clear that i_* induces a bijection between families of r-cycles on the fibres of X/S and families of r-cycles on the fibres of X'/S. Also, given a family α of r-cycles on the fibres of X/S $\alpha \in z(X/S, r) \Leftrightarrow i_*\alpha \in z(X'/S, r)$ by Lemma 6.5. The lemma follows. □

Lemma 6.15. *Let S be a locally Noetherian scheme. Let X be a scheme locally of finite type over S. Let $r \geq 0$. Let $U \subset X$ be an open such that $X \setminus U$ has relative dimension $< r$ over S, i.e., $\dim(X_s \setminus U_s) < r$ for all $s \in S$. Then restriction defines a bijection $z(X/S, r) \to z(U/S, r)$.*

Proof. Since $Z_r(X_s) \to Z_r(U_s)$ is a bijection by the dimension assumption, we see that restriction induces a bijection between families of r-cycles on the fibres of X/S and families of r-cycles on the fibres of U/S. These restriction maps $Z_r(X_s) \to Z_r(U_s)$ are compatible with base change and with specializations, see Lemma 5.1 and 4.4. The lemma follows easily from this; details omitted. □

Lemma 6.16. *Let $g : S' \to S$ be a universal homeomorphism of locally Noetherian schemes which induces isomorphisms of residue fields. Let $f : X \to S$ be locally of finite type. Set $X' = S' \times_S X$. Let $r \geq 0$. Then base change by g determines a bijection $z(X/S, r) \to z(X'/S', r)$.*

Proof. By Lemma 5.10 we have a bijection between the group of families of r-cycles on fibres of X/S and the group of families of r-cycles on fibres of X'/S'. Say α is a families of r-cycles on fibres of X/S and $\alpha' = g^*\alpha$ is the base change.

If R is a discrete valuation ring, then any morphism $h : \mathrm{Spec}(R) \to S$ factors as $g \circ h'$ for some unique morphism $h' : \mathrm{Spec}(R) \to S'$. Namely, the morphism $S' \times_S \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is a univeral homomorphism inducing bijections on residue fields, and hence has a section (for example because R is a seminormal ring, see Morphisms, Section 47). Thus the condition that α is compatible with specializations (i.e., is a relative r-cycle) is equivalent to the condition that α' is compatible with specializations. $\square$

7. Equidimensional relative cycles

Here is the definition.

Definition 7.1. Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S *equidimensional* if the support of α (Remark 5.6) is contained in a closed subset $W \subset X$ whose relative dimension over S is $\leq r$. The group of all equidimensional relative r-cycles on X/S is denoted $z_{equi}(X/S, r)$.

Example 7.2. There exist relative r-cycles which are not equidimensional. Namely, let k be a field and let $X = \mathrm{Spec}(k[x, y, t])$ over $S = \mathrm{Spec}(k[x, y])$. Let s be a point of S and denote $a, b \in \kappa(s)$ the images of x and y. Consider the family α of 0-cycles on X/S defined by

(1) $\alpha_s = 0$ if $b = 0$ and otherwise
(2) $\alpha_s = [p] - [q]$ where p, resp. q is the $\kappa(s)$-rational point of $\mathrm{Spec}(\kappa(s)[t])$ with $t = a/b$, resp. $t = (a + b^2)/b$.

We leave it to the reader to show that this is compatible with specializations; the idea is that a/b and $(a + b^2)/b = a/b + b$ limit to the same point in $\mathbf{P}^1$ over the residue field of any valuation v on $\kappa(s)$ with $v(b) > 0$. On the other hand, the closure of the support of α containes the whole fibre over $(0, 0)$.

Lemma 7.3. *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. If α is equidimensional, then any restriction, base change, or flat pullback of α is equidimensional.*

Proof. Omitted. $\square$

Lemma 7.4. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Then to check that α is equidimensional we may work Zariski locally on X and S.*

Proof. Namely, the condition that α is equidimensional just means that the closure of the support of α has relative dimension $\leq r$ over S. Since taking closures commutes with restriction to opens, the lemma follows (small detail omitted). $\square$

Lemma 7.5. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Let $\{g_i : S_i \to S\}$ be an fppf covering. Then α is equidimensional if and only if each base change $g_i^*\alpha$ is equidimensional.*

Proof. If α is equidimensional, then each $g_i^*\alpha$ is too by Lemma 7.3. Assume each $g_i^*\alpha$ is equidimensional. Denote W the closure of $\mathrm{Supp}(\alpha)$ in X. Since $g_i : S_i \to S$

is universally open (being flat and locally of finite presentation), so is the morphism $f_i : X_i = S_i \times_S X \to X$. Denote $\alpha_i = g_i^*\alpha$. We have $\text{Supp}(\alpha_i) = f_i^{-1}(\text{Supp}(\alpha))$ by Lemma 5.7. Since f_i is open, we see that $W_i = f_i^{-1}(W)$ is the closure of $\text{Supp}(\alpha_i)$. Hence by assumption the morphism $W_i \to S_i$ has relative dimension $\leq r$. By Morphisms, Lemma 28.3 (and the fact that the morphisms $S_i \to S$ are jointly surjective) we conclude that $W \to S$ has relative dimension $\leq r$. $\square$

Lemma 7.6. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. Let $\{f_i : X_i \to X\}$ be a jointly surjective family of flat morphisms, locally of finite type, and of relative dimension e. Then α is equidimensional if and only if each flat pullback $f_i^*\alpha$ is equidimensional.*

Proof. Omitted. Hint: As in the proof of Lemma 7.5 one shows that the inverse image by f_i of the closure W of the support of α is the closure W_i of the support of $f_i^*\alpha$. Then $W \to S$ has relative dimension $\leq r$ holds if $W_i \to S$ has relative dimension $\leq r + e$ for all i. $\square$

Let S be a locally Noetherian scheme. Let $f : X \to S$ be a locally quasi-finite morphism of schemes. Then we have $z(X/S, 0) = z_{equi}(X/S, 0)$ and $z(X/S, r) = 0$ for $r > 0$. Given $\alpha \in z(X/S, 0)$ let us define a map

$$w_\alpha : X \longrightarrow \mathbf{Z}, \quad x \mapsto \alpha(x)[\kappa(x) : \kappa(s)]_i \quad \text{where } s = f(x)$$

Here $\alpha(x)$ denotes the coefficient of x in the 0-cycle α_s on the fibre X_s and $[K : k]_i$ denotes the inseparable degree of a finite field extension. The following lemma shows that this map is a weighting of f (More on Morphisms, Definition 75.2) and that every weighting is of this form up to taking a multiple.

Lemma 7.7. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a locally quasi-finite morphism of schemes. Let $\alpha \in z(X/S, 0)$. The map $w_\alpha : X \to \mathbf{Z}$ constructed above is a weighting. Conversely, if X is quasi-compact, then given a weighting $w : X \to \mathbf{Z}$ there exists an integer $n > 0$ such that $nw = w_\alpha$ for some $\alpha \in z(X/S, 0)$. Finally, the integer n may be chosen to be a power of the prime p if S is a scheme over $\mathbf{F}_p$.*

Proof. First, let us show that the construction is compatible with base change: if $g : S' \to S$ is a morphism of locally Noetherian schemes, then $w_{g^*\alpha} = w_\alpha \circ g'$ where $g' : X' \to X$ is the projection $X' = S' \times_S X \to X$. Namely, let $x' \in X'$ with images s', s, x in S', S, X. Then the coefficient of $[x']$ in the base change of $[x]$ by $\kappa(s')/\kappa(s)$ is the length of the local ring $(\kappa(s') \otimes_{\kappa(s)} \kappa(x))_\mathfrak{q}$. Here $\mathfrak{q}$ is the prime ideal corresponding to x'. Thus compatibility with base change follows if

$$[\kappa(x) : \kappa(s)]_i = \text{length}((\kappa(s') \otimes_{\kappa(s)} \kappa(x))_\mathfrak{q})[\kappa(x') : \kappa(s')]_i$$

Let $k/\kappa(s')$ be an algebraically closure. Choose a prime $\mathfrak{p} \subset k \otimes_{\kappa(s)} \kappa(x)$ lying over $\mathfrak{q}$. Suppose we can show that

$$[\kappa(x) : \kappa(s)]_i = \text{length}((k \otimes_{\kappa(s)} \kappa(x))_\mathfrak{p}) \quad \text{and} \quad [\kappa(x') : \kappa(s')]_i = \text{length}((k \otimes_{\kappa(s')} \kappa(x'))_\mathfrak{p})$$

Then we win because

$$\text{length}((\kappa(s') \otimes_{\kappa(s)} \kappa(x))_\mathfrak{q})\text{length}((k \otimes_{\kappa(s')} \kappa(x'))_\mathfrak{p}) = \text{length}((k \otimes_{\kappa(s)} \kappa(x))_\mathfrak{p})$$

by Algebra, Lemma 52.13 and flatness of $\kappa(s') \otimes_{\kappa(s)} \kappa(x) \to k \otimes_{\kappa(s)} \kappa(x)$. To show the two equalities, it suffices to prove the first. Let $\kappa(x)/\kappa/\kappa(s)$ be the subfield constructed in Fields, Lemma 14.6. Then we see that

$$k \otimes_{\kappa(s)} \kappa(x) = \prod\nolimits_{\sigma : \kappa \to k} k \otimes_{\sigma, \kappa} \kappa(x)$$

and each of the factors is local of degree $[\kappa(x) : \kappa] = [\kappa(x) : \kappa(s)]_i$ as desired.

Let $\alpha \in z(X/S, 0)$ and choose a diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & U \arrow[l, "h"'] \arrow[d, "\pi"] \\ Y & V \arrow[l, "g"] \end{tikzcd}$$

as in More on Morphisms, Definition 75.2. Denote $\beta \in z(U/V, 0)$ the restriction of the base change $g^*\alpha$. By the compatibility with base change above we have $w_\beta = w_\alpha \circ h$ and it suffices to show that $\int_\pi w_\beta$ is locally constant on V. Next, note that

$$\begin{aligned} \left(\int_\pi w_\beta\right)(v) &= \sum\nolimits_{u \in U, \pi(u) = v} \beta(u)[\kappa(u) : \kappa(v)]_i[\kappa(u) : \kappa(v)]_s \\ &= \sum\nolimits_{u \in U, \pi(u) = v} \beta(u)[\kappa(u) : \kappa(v)] \end{aligned}$$

This last expression is the coefficient of v in $\pi_*\beta \in z(V/V, 0)$. By Lemma 6.11 this function is locally constand on V.

Conversely, let $w : X \to S$ be a weighting and X quasi-compact. Choose a sufficiently divisible integer n. Let α be the family of 0-cycles on fibres of X/S such that for $s \in S$ we have

$$\alpha_s = \sum\nolimits_{f(x) = s} \frac{nw(x)}{[\kappa(x) : \kappa(s)]_i}[x]$$

as a zero cycle on X_s. This makes sense since the fibres of f are universally bounded (Morphisms, Lemma 57.9) hence we can find n such that the right hand side is an integer for all $s \in S$. The final statement of the lemma also follows, provided we show α is a relative 0-cycle. To do this we have to show that α is compatible with specializations along discrete valuation rings. By the first paragraph of the proof our construction is compatible with base change (small detail omitted; it is the “inverse” construction we are discussing here). Also, the base change of a weighting is a weighting, see More on Morphisms, Lemma 75.3. Thus we reduce to the problem studied in the next paragraph.

Assume S is the spectrum of a discrete valuation ring with generic point η and closed point 0. Let $w : X \to S$ be a weighting with X quasi-finite over S. Let α be the family of 0-cycles on fibres of X/S constructed in the previous paragraph (for a suitable n). We have to show that $sp_{X/S}(\alpha_\eta) = \alpha_0$. Let $\beta \in z(X/S, 0)$ be the relative 0-cycle on X/S with $\beta_\eta = \alpha_\eta$ and $\beta_0 = sp_{X/S}(\alpha_\eta)$. Then $w' = w_\beta - nw : X \to \mathbf{Z}$ is a weighting (using the result above) and zero in the points of X which map to η. Now it is easy to see that a weighting which is zero on all points of X mapping to η has to be zero; details omitted. Hence $w' = 0$, i.e., $w_\beta = nw$, hence $\alpha = \beta$ as desired. $\square$

8. Effective relative cycles

Here is the definition.

Definition 8.1. Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S *effective* if α_s is an effective cycle (Chow Homology, Definition 8.4) for all $s \in S$. The monoid of all effective relative r-cycles on X/S is denoted $z^{eff}(X/S, r)$.

Below we will show that an effective relative cycle is equidimensional, see Lemma 8.7.

Lemma 8.2. *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. If α is effective, then any restriction, base change, flat pullback, or proper pushforward of α is effective.*

Proof. Omitted. □

Lemma 8.3. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Then to check that α is effective we may work Zariski locally on X and S.*

Proof. Omitted. □

Lemma 8.4. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Let $g : S' \to S$ be a surjective morphism. Then α is effective if and only if the base change $g^*\alpha$ is effective.*

Proof. Omitted. □

Lemma 8.5. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. Let $\{f_i : X_i \to X\}$ be a jointly surjective family of flat morphisms, locally of finite type, and of relative dimension e. Then α is effective if and only if each flat pullback $f_i^*\alpha$ is effective.*

Proof. Omitted. □

Lemma 8.6. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. If α is effective, then Supp(α) is closed in X.*

Proof. Let $g : S' \to S$ be the inclusion of an irreducible component viewed as an integral closed subscheme. By Lemmas 8.2 and 5.7 it suffices to show that the support of the base change $g^*\alpha$ is closed in $S' \times_S S$. Thus we may assume S is an integral scheme with generic point η. We will show that $\text{Supp}(\alpha)$ is the closure of $\text{Supp}(\alpha_\eta)$. To do this, pick any $s \in S$. We can find a morphism $g : S' \to S$ where S' is the spectrum of a discrete valuation ring mapping the generic point $\eta' \in S'$ to η and the closed point $0 \in S'$ to s, see Properties, Lemma 5.10. Then it suffices to prove that the support of $g^*\alpha$ is equal to the closure of $\text{Supp}((g^\alpha)_{\eta'})$. This reduces us to the case discussed in the next paragraph.

Here S is the spectrum of a discrete valuation ring with generic point η and closed point 0. We have to show that $\text{Supp}(\alpha)$ is the closure of $\text{Supp}(\alpha_\eta)$. Since α is effective we may write $\alpha_\eta = \sum n_i[Z_i]$ with $n_i > 0$ and $Z_i \subset X_\eta$ integral closed of dimension r. Since $\alpha_0 = sp_{X/S}(\alpha_\eta)$ we know that $\alpha_0 = \sum n_i[\overline{Z}_{i,0}]_r$ where $\overline{Z}_i$ is the closure of Z_i. By Varieties, Lemma 19.2 we see that $\overline{Z}_{i,0}$ is equidimensional of dimension r. Since $n_i > 0$ we conclude that $\text{Supp}(\alpha_0)$ is equal to the union of the $\overline{Z}_{i,0}$ which is the fibre over 0 of $\bigcup \overline{Z}_i$ which in turn is the closure of $\bigcup Z_i$ as desired. ☐

Lemma 8.7. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r, e \geq 0$ be integers. Let α be a relative r-cycle on X/S. If α is effective, then α is equidimensional.*

Proof. Assume α is effective. By Lemma 8.6 the support $\text{Supp}(\alpha)$ is closed in X. Thus α is equidimensional as the fibres of $\text{Supp}(\alpha) \to S$ are the supports of the cycles α_s and hence have dimension r. ☐

Remark 8.8. Let $f : X \to S$ be a morphism of schemes with S locally Noetherian and f locally of finite type. We can ask if the contravariant functor

$$\begin{matrix}\text{schemes } S' \text{ locally} \\ \text{of finite type over } S\end{matrix} \longrightarrow z^{eff}(X'/S', r) \text{ where } X' = S' \times_S X$$

is representable. Since $z(X'/S', r) = z(X'_{red}/S'_{red}, r)$ this cannot be true (we leave it to the reader to make an actual counter example). A better question would be if we can find a subcategory of the left hand side on which the functor is representable. Lemma 6.16 suggests we should restrict at least to the category of seminormal schemes over S.

If $S/\text{Spec}(\mathbf{Q})$ is Nagata and f is a projective morphism, then it turns out that $S' \mapsto z^{eff}(X'/S', r)$ is representable on the category of seminormal S'. Roughly speaking this is the content of [Kol96, Theorem 3.21].

If S has points of positive characteristic, then this no longer works even if we replace seminormality with weak normality; a locally Noetherian scheme T is weakly normal if any birational universal homeomorphism $T' \to T$ has a section. An example is to consider 0-cycles of degree 2 on $X = \mathbf{A}^2_k$ over $S = \text{Spec}(k)$ where k is a field of characteristic 2. Namely, over $W = X \times_S X$ we have a canonical relative 0-cycle $\alpha \in z^{eff}(X_W/W, 0)$: for $w = (x_1, x_2) \in W = X^2$ we have the cycle $\alpha_w = [x_1] + [x_2]$. This cycle is invariant under the involution $\sigma : W \to W$ switching the factors. Since W is smooth (hence normal, hence weakly normal), if $z(-/-, r)$ was representable by M on the category of weakly normal schemes of finite type over k we would get a σ-invariant morphism from W to M. This in turn would define a morphism from the quotient scheme $\text{Sym}^2_S(X) = W/\langle\sigma\rangle$ to M. Since $\text{Sym}^2_S(X)$ is normal, we would by the moduli property of M obtain a relative 0-cycle β on $X \times_S \text{Sym}^2_S(X)/\text{Sym}^2_S(X)$ whose pullback to W is α. However, there is no such cycle β. Namely, writing $X = \text{Spec}(k[u, v])$ the scheme $\text{Sym}^2_S(X)$ is the spectrum of

$$k[u_1 + u_2, u_1u_2, v_1 + v_2, v_1v_2, u_1v_1 + u_2v_2] \subset k[u_1, u_2, v_1, v_2]$$

The image of the diagonal $u_1 = u_2, v_1 = v_2$ in $\text{Sym}^2_S(X)$ is the closed subscheme $V = \text{Spec}(k[u_1^2, v_1^2])$; here we use that the characteristic of k is 2. Looking at the generic point η of V, the cycle β_η would be a zero cycle of degree 2 on $\mathbf{A}^2_{k(u_1^2, v_1^2)}$

whose pullback to $\mathbf{A}^2_{k(u_1,u_2)}$ whould be 2[the point with coordinates(u_1, v_2)]. This is clearly impossible.

The discussion above does not contradict [Kol96, Theorem 4.13] as the Chow variety in that theorem only coarsely represents a functor (in fact 2 distinct functors, only one of which agrees with ours for projective X as one can see with some work). Similarly, in [SV00, Section 4.4] it is shown that for projective X/S the h-sheafification of the presheaf $S' \mapsto z^{eff}(S' \times_S X/S', r)$ is equal to the h-sheafification of a representable functor.

Remark 8.9. Let $f : X \to S$ be a morphism of schemes. Let $r \geq 0$. Let $Z \subset X$ be a closed subscheme. Assume

(1) S is Noetherian and geometrically unibranch,
(2) f is of finite type, and
(3) $Z \to S$ has relative dimension $\leq r$.

Then for all sufficiently divisible integers $n \geq 1$ there exists a unique effective relative r-cycle α on X/S such that $\alpha_\eta = n[Z_\eta]_r$ for every generic point η of S. This is a reformulation of [SV00, Theorem 3.4.2]. If we ever need this result, we will precisely state and prove it here.

9. Proper relative cycles

In our setting, the following is probably the correct definition.

Definition 9.1. Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S is a *proper relative cycle* if the support of α (Remark 5.6) is contained in a closed subset $W \subset X$ proper over S (Cohomology of Schemes, Definition 26.2). The group of all proper relative r-cycles on X/S is denoted $c(X/S, r)$.

By Cohomology of Schemes, Lemma 26.3 this just means that the closure of the support is proper over the base. To see that these form a group, use Cohomology of Schemes, Lemma 26.6.

Lemma 9.2. *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. If α is proper, then any base change α is proper.*

Proof. Omitted. $\square$

Lemma 9.3. *Let $f : X \to S$ be a morphism of schemes. Assume S locally Noetherian and f locally of finite type. Let $r \geq 0$ be an integer. Let α be a relative r-cycle on X/S. Let $\{g_i : S_i \to S\}$ be a h covering. Then α is proper if and only if each base change $g_i^*\alpha$ is proper.*

Proof. If α is proper, then each $g_i^*\alpha$ is too by Lemma 9.2. Assume each $g_i^*\alpha$ is proper. To prove that α is proper, it clearly suffices to work affine locally on S. Thus we may and do assume that S is affine. Then we can refine our covering $\{S_i \to S\}$ by a family $\{T_j \to S\}$ where $g : T \to S$ is a proper surjective morphism and $T = \bigcup T_j$ is an open covering. It follows that $\beta = g^*\alpha$ is proper on $Y = T \times_S X$ over T. By Lemma 5.7 we find that the support of β is the inverse image of the support of α by the morphism $f : Y \to X$. Hence the closure $W \subset Y$ of $f^{-1}\text{Supp}(\alpha)$ is proper over T. Since the morphism $T \to S$ is proper, it follows that W is proper over S. Then by Cohomology of Schemes, Lemma 26.5 the image $f(W) \subset X$ is a closed subset proper over S. Since $f(W)$ contains $\text{Supp}(\alpha)$ we conclude α is proper. $\square$

10. Proper and equidimensional relative cycles

Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian and f is locally of finite type. Let $r \geq 0$ be an integer. We say a relative r-cycle α on X/S is a *proper and equidimensional relative cycle* if α is both equidimensional (Definition 7.1) and proper (Definition 9.1). The group of all proper, equidimensional relative r-cycles on X/S is denoted $c_{equi}(X/S, r)$.

Similarly we say a relative r-cycle α on X/S is a *proper and effective relative cycle* if α is both effective (Definition 8.1) and proper (Definition 9.1). The monoid of all proper, effective relative r-cycles on X/S is denoted $c^{eff}(X/S, r)$. Observe that these are equidimensional by Lemma 8.7.

Thus we have the following diagram of inclusion maps

$$\begin{array}{ccccc} c^{eff}(X/S, r) & \longrightarrow & c_{equi}(X/S, r) & \longrightarrow & c(X/S, r) \\ \downarrow & & \downarrow & & \downarrow \\ z^{eff}(X/S, r) & \longrightarrow & z_{equi}(X/S, r) & \longrightarrow & z(X/S, r) \end{array}$$

11. Action on cycles

Let S be a locally Noetherian, universally catenary scheme endowed with a dimension function δ, see Chow Homology, Section 7. Let $X \to Y$ be a morphism of schemes over S, both locally of finite type over S. Let $r \geq 0$. Finally, let α be a family of r-cycles on fibres of X/Y. For $e \in \mathbf{Z}$ we will construct an operation

$$\alpha \cap - : Z_e(Y) \longrightarrow Z_{r+e}(X)$$

Namely, given $\beta \in Z_e(Y)$ write $\beta = \sum n_i[Z_i]$ where $Z_i \subset Y$ is an integral closed subscheme of δ-dimension e and the family Z_i is locally finite in the scheme Y. Let $y_i \in Z_i$ be the generic point. Write $\alpha_{y_i} = \sum m_{ij}[V_{ij}]$. Thus $V_{ij} \subset X_{y_i}$ is an integral closed subscheme of dimension r and the family V_{ij} is locally finite in the scheme X_{y_i}. Then we set

$$\alpha \cap \beta = \sum n_i m_{ij} [\overline{V}_{ij}] \quad \in \quad Z_{r+e}(X)$$

Here $\overline{V}_{ij} \subset X$ is the scheme theoretic image of the morphism $V_{ij} \to X_{y_i} \to X$ or equivalently, $\overline{V}_{ij} \subset X$ is an integral closed subscheme mapping dominantly to $Z_i \subset Y$ whose generic fibre is V_{ij}. It follows readily that $\dim_\delta(\overline{V}_{ij}) = r + e$ and that the family of closed subschemes $\overline{V}_{ij} \subset X$ is locally finite (we omit the verifications). Hence $\alpha \cap \beta$ is indeed an element of $Z_{r+e}(X)$.

Lemma 11.1. *The construction above is bilinear, i.e., we have $(\alpha_1 + \alpha_2) \cap \beta = \alpha_1 \cap \beta + \alpha_2 \cap \beta$ and $\alpha \cap (\beta_1 + \beta_2) = \alpha \cap \beta_1 + \alpha \cap \beta_2$.*

Proof. Omitted. $\square$

Lemma 11.2. *If $U \subset X$ and $V \subset Y$ are open and $f(U) \subset V$, then $(\alpha \cap \beta)|_U$ is equal to $\alpha|_U \cap \beta|_V$.*

Proof. Immediate from the explict description of $\alpha \cap \beta$ given above. $\square$

Lemma 11.3. *Forming $\alpha \cap \beta$ is compatible with flat base change and flat pullback (see proof for elucidation).*

Proof. Let (S, δ), (S', δ'), $g : S' \to S$, and $c \in \mathbf{Z}$ be as in Chow Homology, Situation 67.1. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Denote $X' \to Y'$ the base change of $X \to Y$ by g. Let α be a family of r-cycles on the fibres of X/Y. Let $\beta \in Z_e(Y)$. Denote α' the base change of α by $Y' \to Y$. Denote $\beta' = g^*\beta \in Z_{e+c}(Y')$ the pullback of β by g, see Chow Homology, Section 67. Compatibility with base change means $\alpha' \cap \beta'$ is the base change of $\alpha \cap \beta$.

Proof of compatibility with base change. Since we are proving an equality of cycles on X', we may work locally on Y, see Lemma 11.2. Thus we may assume Y is affine. In particular β is a finite linear combination of prime cycles. Since $- \cap -$ is linear in the second variable (Lemma 11.1), it suffices to prove the equality when $\beta = [Z]$ for some integral closed subscheme $Z \subset Y$ of δ-dimension e.

Let $y \in Z$ be the generic point. Write $\alpha_y = \sum m_j[V_j]$. Let $\overline{V}_j$ be the closure of V_j in X. Then we have

$$\alpha \cap \beta = \sum m_j[\overline{V}_j]$$

The base change of β is $\beta' = \sum [Z \times_S S']_{e+c}$ as a cycle on $Y' = Y \times_S S'$. Let $Z'_a \subset Z \times_S S'$ be the irreducible components, denote $y'_a \in Z'_a$ their generic points, and denote n_a the multiplicity of Z'_a in $Z \times_S S'$. We have

$$\beta' = \sum [Z \times_S S']_{e+c} = \sum n_a[Z'_a]$$

We have $\alpha'_{y'_a} = \sum m_j[V_{j,\kappa(y'_a)}]_r$ because α' is the base change of α by $Y' \to Y$. Let $V'_{jab} \subset V_{j,\kappa(y'_a)}$ be the irreducible components and denote m_{jab} the multiplicity of V'_{jab} in $V_{j,\kappa(y'_a)}$. We have

$$\alpha'_{y'_a} = \sum m_j[V_{j,\kappa(y'_a)}]_r = \sum m_j m_{jab}[V'_{jab}]$$

Thus we we have

$$\alpha' \cap \beta' = \sum n_a m_j m_{jab}[\overline{V}'_{jab}]$$

where $\overline{V}'_{jab}$ is the closure of V'_{jab} in X'. Thus to prove the desired equality it suffices to prove

(1) the irreducible components of $\overline{V}_j \times_S S'$ are the schemes $\overline{V}'_{jab}$ and
(2) the multiplicity of $\overline{V}'_{jab}$ in $\overline{V}_j \times_S S'$ is equal to $n_a m_{jab}$.

Note that $V_j \to \overline{V}_j$ is a birational morphism of integral schemes. The morphisms $V_j \times_S S' \to V_j$ and $\overline{V}_j \times_S S' \to \overline{V}_j$ are flat and hence map generic points of irreducible components to the (unique) generic points of V_j and $\overline{V}_j$. It follows that $V_j \times_S S' \to \overline{V}_j \times_S S'$ is a birational morphisms hence induces a bijection on irreducible components and identifies their multiplicities. This means that it suffices to prove that the irreducible components of $V_j \times_S S'$ are the schemes V'_{jab} and the multiplicity of V'_{jab} in $V_j \times_S S'$ is equal to $n_a m_{jab}$. However, then we are just saying that the diagram

$$\begin{tikzcd} Z_r(V_j) \arrow[r] & Z_{r+c}(V_j \times_S S') \\ Z_0(\mathrm{Spec}(\kappa(y))) \arrow[r] \arrow[u] & Z_c(\mathrm{Spec}(\kappa(y)) \times_S S') \arrow[u] \end{tikzcd}$$

is commutative where the horizontal arrows are base change by $\mathrm{Spec}(\kappa(y)) \times_S S' \to \mathrm{Spec}(\kappa(y))$ and the vertical arrows are flat pullback. This was shown in Chow Homology, Lemma 67.5.

The statement in the lemma on flat pullback means the following. Let (S, δ), $X \to Y$, α, and β be as in the constuction of $\alpha \cap \beta$ above. Let $Y' \to Y$ be a flat morphism, locally of finite type, and of relative dimension c. Then we can let α' be the base change of α by $Y' \to Y$ and β' the flat pullback of β. Compatibility with flat pullback means $\alpha' \cap \beta'$ is the flat pullback of $\alpha \cap \beta$ by $X \times_Y Y' \to Y$. This is actually a special case of the discussion above if we set $S = Y$ and $S' = Y'$. □

Lemma 11.4. *Let (S, δ) and $f : X \to Y$ be as above. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with $\dim(Supp(\mathcal{F}_y)) \leq r$ for all $y \in Y$. Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module with $\dim_\delta(Supp(\mathcal{G})) \leq e$. Set $\alpha = [\mathcal{F}/X/Y]_r$ (Example 5.2) and $\beta = [\mathcal{G}]_e$ (Chow Homology, Definition 10.2). If $\mathcal{F}$ is flat over Y, then $\alpha \cap \beta = [\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}]_{r+e}$.*

Proof. Observe that

$$\mathrm{Supp}(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \mathrm{Supp}(\mathcal{F}) \cap f^{-1}\mathrm{Supp}(\mathcal{G}) = \bigcup\nolimits_{y \in \mathrm{Supp}(\mathcal{G})} \mathrm{Supp}(\mathcal{F}_y)$$

It follows that this is a closed subset of δ-dimension $\leq r+e$. Whence the expression $[\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}]_{r+e}$ makes sense.

We will use the notation $\beta = \sum n_i[Z_i]$, $y_i \in Z_i$, $\alpha_{y_i} = \sum m_{ij}[V_{ij}]$, and $\overline{V}_{ij}$ introduced in the construction of $\alpha \cap \beta$. Since $\beta = [\mathcal{G}]_e$ we see that the Z_i are the irreducible components of $\mathrm{Supp}(\mathcal{G})$ which have δ-dimension e. Similarly, the V_{ij} are the irreducible components of $\mathrm{Supp}(\mathcal{F}_{y_i})$ having dimension r. It follows from this and the equation in the first paragraph that $\overline{V}_{ij}$ are the irreducible components of $\mathrm{Supp}(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ having δ-dimension $r + e$. Thus to prove the lemma it now suffices to show that

$$\mathrm{length}_{\mathcal{O}_{X,\xi_{ij}}}((\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})_{\xi_{ij}}) = \mathrm{length}_{\mathcal{O}_{X_{y_i},\xi_{ij}}}((\mathcal{F}_{y_i})_{\xi_{ij}}) \cdot \mathrm{length}_{\mathcal{O}_{Y,y_i}}(\mathcal{G}_{y_i})$$

By the first paragraph of the proof the left hand side is equal to the lenth of the $B = \mathcal{O}_{X,\xi_{ij}}$-module

$$\mathcal{G}_{y_i} \otimes_{\mathcal{O}_{Y,y_i}} \mathcal{F}_{\xi_{ij}} = M \otimes_A N$$

Here $M = \mathcal{G}_{y_i}$ is a finite length $A = \mathcal{O}_{Y,y_i}$-module and $N = \mathcal{F}_{\xi_{ij}}$ is a finite B-module such that $N/\mathfrak{m}_A N$ has finite length. Since $\mathcal{F}$ is flat over Y the module N is A-flat. The right hand side of the formula is equal to

$$\mathrm{length}_B(N/\mathfrak{m}_A N) \cdot \mathrm{length}_A(M)$$

Thus the right and left hand side of the formula are additive in M (use flatness of N over A). Thus it suffices to prove the formula with $M = \kappa_A$ is the residue field in which case it is immediate. □

Lemma 11.5. *Let (S, δ) and $f : X \to Y$ be as above. Let $Z \subset X$ be a closed subscheme of relative dimension $\leq r$ over Y. Set $\alpha = [Z/X/Y]_r$ (Example 5.4). Let $W \subset Y$ be a closed subscheme of δ-dimension $\leq e$. Set $\beta = [W]_e$ (Chow Homology, Definition 9.2). If Z is flat over Y, then $\alpha \cap \beta = [Z \times_Y W]_{r+e}$.*

Proof. This is a special case of Lemma 11.4 if we take $\mathcal{F} = \mathcal{O}_Z$ and $\mathcal{F} = \mathcal{O}_W$. $\square$

Lemma 11.6. *Let (S, δ) be as above. Let*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of schemes locally of finite type over S with g proper. Let $r, e \geq 0$. Let α be a family of r-cycles on the fibres of X/Y. Let $\beta' \in Z_e(Y')$. Then we have $f_*(g^*\alpha \cap \beta') = \alpha \cap g_*\beta'$.

Proof. Since we are proving an equality of cycles on X, we may work locally on Y, see Lemma 11.2. Thus we may assume Y is affine. Thus Y' is quasi-compact. In particular β' is a finite linear combination of prime cycles. Since $- \cap -$ is linear in the second variable (Lemma 11.1), it suffices to prove the equality when $\beta' = [Z']$ for some integral closed subscheme $Z' \subset Y'$ of δ-dimension e. Set $Z = g(Z')$. This is an integral closed subscheme of Y of δ-dimension $\leq e$. For simplicity we are going to assume Z has δ-dimension equal to e and leave the other case (which is easier) to the reader. Let $y \in Z$ and $y' \in Z'$ be the generic points. Write $\alpha_y = \sum m_j[V_j]$ with $V_j \subset X_y$ integral closed subschemes of dimension r.

Assume first g is a closed immersion. Then $g_*\beta' = [Z]$ and $(g^*\alpha)_{y'} = \sum n_j[V_j]$; this makes sense because V_j is contained in the closed subscheme $X'_{y'}$ of X_y. Thus in this case the equality is obvious: in both cases we obtain $\sum m_j[\overline{V}_j]$ where $\overline{V}_j$ is the closure of V_j in the closed subscheme $X' \subset X$.

Back to the general case with $\beta' = [Z']$ as above. Set $W = Z \times_X Y$ and $W' = Z' \times_{X'} Y'$. Consider the cartesian squares

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd} \quad \begin{tikzcd} W' \arrow[r] \arrow[d] & X' \arrow[d] \\ Z' \arrow[r] & Y' \end{tikzcd} \quad \begin{tikzcd} W' \arrow[r] \arrow[d] & W \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

Since we know the result for the first two squares with by the previous paragraph, a formal argument shows that it suffices to prove the result for the last square and the element $\beta' = [Z'] \in Z_e(Z')$. This reduces us to the case discussed in the next paragraph.

Assume $Y' \to Y$ is a generically finite morphism of integral schemes of δ-dimension e and $\beta' = [Y']$. In this case both $f_*(g^*\alpha \cap \beta')$ and $\alpha \cap g_*\beta'$ are cycles which can be written as a sum of prime cycles dominant over Y. Thus we may replace Y by a nonempty open subscheme in order to check the equality. After such a replacement we may assume g is finite and flat, say of degree $d \geq 1$. Of course, this means that $g_*\beta' = g_*[Y'] = d[Y]$. Also $\beta' = [Y'] = g^*[Y]$. Hence

$$f_*(g^*\alpha \cap \beta') = f_*(g^*\alpha \cap g^*[Y]) = f_*f^*(\alpha \cap [Y]) = d(\alpha \cap [Y]) = \alpha \cap g_*\beta')$$

as desired. The second equality is Lemma 11.3 and the third equality is Chow Homology, Lemma 15.2. □

12. Action on chow groups

When α is a relative r-cycle, the operation $\alpha \cap -$ of Section 11 factors through rational equivalence and defines a bivariant class.

Lemma 12.1. *Let (S, δ) be as in Section 11. Let $f : X' \to X$ be a proper morphism of schemes locally of finite type over S. Let $(\mathcal{L}, s, i : D \to X)$ be as in Chow Homology, Definition 29.1. Form the diagram*

$$\begin{tikzcd} D' \arrow[r, "i'"'] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

as in Chow Homology, Remark 29.7. If $\mathcal{L}|_D \cong \mathcal{O}_D$, then $i^* f_* \alpha' = g_*(i')^* \alpha'$ in $Z_k(D)$ for any $\alpha' \in Z_{k+1}(X')$.

Proof. The statement makes sense as all operations are defined on the level of cycles, see Chow Homology, Remark 29.6 for the gysin maps. Suppose $\alpha = [W']$ for some integral closed subscheme $W' \subset X'$. Let $W = f(W') \subset X$. In case $W' \not\subset D'$, then $W \not\subset D$ and we see that

$$[W' \cap D']_k = \text{div}_{\mathcal{L}'|_{W'}}(s'|_{W'}) \quad \text{and} \quad [W \cap D]_k = \text{div}_{\mathcal{L}|_W}(s|_W)$$

and hence f_* of the first cycle equals the second cycle by Chow Homology, Lemma 26.3. Hence the equality holds as cycles. In case $W' \subset D'$, then $W \subset D$ and both sides are zero by construction. □

Lemma 12.2. *Let (S, δ) be as in Section 11. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Let $r \geq 0$ and let $\alpha \in z(X/Y, r)$ be a relative r-cycle on X/Y. Let $(\mathcal{L}, s, i : D \to Y)$ be as in Chow Homology, Definition 29.1. Form the cartesian diagram*

$$\begin{tikzcd} E \arrow[r, "j"'] \arrow[d] & X \arrow[d] \\ D \arrow[r, "i"] & Y \end{tikzcd}$$

See Chow Homology, Remark 29.7. If $\mathcal{L}|_D \cong \mathcal{O}_D$, then for $e \in \mathbf{Z}$ the diagram

$$\begin{tikzcd} Z_e(D) \arrow[r, "i^*\alpha\cap -"'] & Z_{e+r}(E) \\ Z_{e+1}(Y) \arrow[u, "i^*"] \arrow[r, "\alpha\cap -"] & Z_{r+e+1}(X) \arrow[u, "j^*"'] \end{tikzcd}$$

commutes where the vertical arrows i^* and j^* are the gysin maps on cycles as in Chow Homology, Remark 29.6.

Proof. Preliminary remark. Suppose that $g : Y' \to Y$ is an envelope (Chow Homology, Definition 22.1). Denote $D', i', E', j', X', \alpha'$ the base changes of D, i, E, j, X, α

by g and denote $f : X' \to X$ the projection. Assume the lemma holds for $D', i', E', j', X', Y', \alpha'$. Then, if $\beta' \in Z_{e+1}(Y')$, we have

$$\begin{aligned} i^*\alpha \cap i^*g_*\beta' &= i^*\alpha \cap f_*(i')^*\beta' \\ &= f_*(f^*i^*\alpha \cap (i')^*\beta') \\ &= f_*((i')^*\alpha' \cap (i')^*\beta') \\ &= f_*((j')^*(\alpha' \cap \beta')) \\ &= j^*(f_*(f^*\alpha \cap \beta')) \\ &= j^*(\alpha \cap g_*\beta') \end{aligned}$$

Here the first equality is Lemma 12.1, the second equality is Lemma 11.6, the third equality is the definition of α', the fourth equality is the assumption that our lemma holds for $D', i', E', j', X', \alpha'$, the fifth equality is Lemma 12.1, and the sixth equality is Lemma 11.6. Thus we see that our lemma holds for the image of $g_* : Z_{e+1}(Y') \to Z_e(Y)$. However, since g is completely decomposed this map is surjective and we conclude the lemma holds for D, i, E, j, X, Y, α.

Let $\beta \in Z_{e+1}(Y)$. We have to show that $(D \to Y)^*\alpha \cap i^*\beta = j^*(\alpha \cap \beta)$ as cycles on E. This question is local on E hence we can replace X and Y by open subschemes. (This uses that formation of the operators i^*, j^*, $\alpha \cap -$ and $(D \to Y)^*\alpha \cap -$ commute with localization. This is obvious for the gysin maps and follows from Lemma 11.2 for the others.) Thus we may assume that X and Y are affine and we reduce to the case discussed in the next paragraph.

Assume X and Y are quasi-compact. By the first paragraph of the proof and Lemma 6.9 we may in addition assume that α is in the image of (6.8.1). By linearity of the operations in question, we may assume that $\alpha = [Z/X/Y]_r$ for some closed subscheme $Z \subset X$ which is flat and of relative dimension $\leq r$ over Y. Also, as Y is quasi-compact, the cycle β is a finite linear combination of prime cycles. Since the operations in question are linear, it suffices to prove the equality when $\beta = [W]$ for some integral closed subscheme $W \subset Y$ of δ-dimension $e + 1$.

If $W \subset D$, then on the one hand $i^*[W] = 0$ and on the other hand $\alpha \cap [W]$ is supported on E so also $j^*(\alpha \cap [W]) = 0$. Thus the equality holds in this case.

Say $W \not\subset D$. Then $i^*[W] = [D \cap W]_e$. Note that the pullback $i^*\alpha$ of $\alpha = [Z/X/Y]_r$ by i is $[(E \cap Z)/E/D]_r$ and that $(E \cap Z) = E \times_Y Z = D \times_Y Z$ is flat over D. Hence by Lemma 11.5 used twice we have

$$i^*\alpha \cap i^*[W] = [(E \cap Z) \times_D (D \cap W)]_{r+e} = [E \cap (Z \times_Y W)]_{r+e} = j^*(\alpha \cap [W])$$

as desired. □

Proposition 12.3. *Let (S, δ) be as in Section 11. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Let $r \geq 0$ and let $\alpha \in z(X/Y, r)$ be a relative r-cycle on X/Y. The rule that to every morphism $g : Y' \to Y$ locally of finite type and every $e \in \mathbf{Z}$ associates the operation*

$$g^*\alpha \cap - : Z_e(Y') \to Z_{r+e}(X')$$

where $X' = Y' \times_Y X$ factors through rational equivalence to define a bivariant class $c(\alpha) \in A^{-r}(X \to Y)$.

Proof. The operation factors through rational equivalence by Lemma 12.2 and Chow Homology, Lemma 35.1. The resulting operation on chow groups is a bivariant class by Chow Homology, Lemma 35.2 and Lemmas 11.6, 11.3, and 12.2. ☐

Remark 12.4. Let (S, δ) be as in Section 11. Let $X \to Y$ be a morphism of schemes locally of finite type over S. Let $r \geq 0$. Let c be a rule that to every morphism $g : Y' \to Y$ locally of finite type and every $e \in \mathbf{Z}$ associates an operation

$$c \cap - : Z_e(Y') \to Z_{r+e}(X')$$

compatible with proper pushforward, flat pullback, and gysin maps as in Lemma 12.2. Then we claim there is a relative r-cycle α on X/Y such that $c\cap = g^*\alpha \cap -$ for every g as above. If we ever need this, we will carefully state and prove this here.

13. Composition of families of cycles on fibres

Let $X \to Y \to S$ be morphisms of schemes, both locally of finite type. Let $r, e \geq 0$. Let α be a family of r-cycles on fibres of X/Y and let β be a family of e-cycles on fibres of Y/S. Then we obtain a family of of $(r+e)$-cycles $\alpha \circ \beta$ on the fibres of X/S by setting

$$(\alpha \circ \beta)_s = (Y_s \to Y)^*\alpha \cap \beta_s$$

More precisely, the expression $(Y_s \to Y)^*\alpha$ denotes the base change of α by $Y_s \to Y$ to a family of r-cycles on the fibres of X_s/Y_s and the operation $- \cap -$ was defined and studied in Section 11[3].

Lemma 13.1. *The construction above is bilinear, i.e., we have* $(\alpha_1 + \alpha_2) \circ \beta\alpha_1 \circ \beta + \alpha_1 \circ \beta$ *and* $\alpha \circ (\beta_1 + \beta_2) = \alpha \circ \beta_1 + \alpha \circ \beta_2$.

Proof. Omitted. Hint: on fibres the construction is bilinear by Lemma 11.1. ☐

Lemma 13.2. *If* $U \subset X$ *and* $V \subset Y$ *are open and* $f(U) \subset V$*, then* $(\alpha \circ \beta)|_U$ *is equal to* $\alpha|_U \circ \beta|_V$.

Proof. Omitted. Hint: on fibres use Lemma 11.2. ☐

Lemma 13.3. *The formation of* $\alpha \circ \beta$ *is compatible with base change.*

Proof. Let $g : S' \to S$ be a morphism of schemes. Denote $X' \to Y'$ the base change of $X \to Y$ by g. Denote α' the base change of α with respect to $Y' \to Y$. Denote β' the base change of β with respect to $S' \to S$. The assertion means that $\alpha' \circ \beta'$ is the base change of $\alpha \circ \beta$ by $g : S' \to S$.

Let $s' \in S'$ be a point with image $s \in S$. Then

$$(\alpha' \circ \beta')_{s'} = (Y'_{s'} \to Y')^*\alpha' \cap \beta'_{s'}$$

We observe that

$$(Y'_{s'} \to Y')^*\alpha' = (Y'_{s'} \to Y')^*(Y' \to Y)^*\alpha = (Y'_{s'} \to Y_s)^*(Y_s \to Y)^*\alpha$$

and that $\beta'_{s'}$ is the base change of β_s by $s' = \mathrm{Spec}(\kappa(s')) \to \mathrm{Spec}(\kappa(s)) = s$. Hence the result follows from Lemma 11.3 applied to $(Y_s \to Y)^*\alpha$, β_s, $X_s \to Y_s \to s$, and base change by $s' \to s$. ☐

[3]To be sure, we use $s = \mathrm{Spec}(\kappa(s))$ as the base scheme with $\delta(s) = 0$.

Lemma 13.4. *Let $f : X \to Y$ and $Y \to S$ be morphisms of schemes, both locally of finite type. Let $r, e \geq 0$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type, with $\dim(Supp(\mathcal{F}_y)) \leq r$ for all $y \in Y$. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module of finite type, with $\dim(Supp(\mathcal{G}_s)) \leq e$ for all $s \in S$. If $\alpha = [\mathcal{F}/X/Y]_r$ and $\beta = [\mathcal{G}/Y/S]_e$ (Example 5.2) and $\mathcal{F}$ is flat over Y, then $\alpha \circ \beta = [\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}/X/S]_{r+e}$.*

Proof. First we observe that $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Observe that

$$(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})_s = \mathcal{F}_s \otimes_{\mathcal{O}_{X_s}} f_s^*\mathcal{G}_s$$

by right exactness of tensor products. Moreover $\mathcal{F}_s$ is flat over Y_s as a base change of a flat module. Thus the equality $(\alpha \circ \beta)_s = [(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})_s]_{r+e}$ follows from Lemma 11.4. □

Lemma 13.5. *Let $f : X \to Y$ and $Y \to S$ be morphisms of schemes, both locally of finite type. Let $r, e \geq 0$. Let $Z \subset X$ be a closed subscheme of relative dimension $\leq r$ over Y. Let $W \subset Y$ be a closed subscheme of relative dimension $\leq e$ over S. If $\alpha = [Z/X/Y]_r$ and $\beta = [W/Y/S]_e$ (Example 5.4) and Z is flat over Y, then $\alpha \circ \beta = [Z \times_Y W/X/S]_{r+e}$.*

Proof. This is a special case of Lemma 13.4 if we take $\mathcal{F} = \mathcal{O}_Z$ and $\mathcal{F} = \mathcal{O}_W$. □

Lemma 13.6. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of schemes locally of finite type over S with g proper. Let $r, e \geq 0$. Let α be a family of r-cycles on the fibres of X/Y. Let β' be a family of e-cycles on the fibres of Y'/S. Then we have $f_*(g^*(\alpha) \circ \beta') = \alpha \circ g_*\beta'$.

Proof. Unwinding the definitions, this follows from Lemma 11.6. □

Lemma 13.7. *Let (S, δ) be as in Chow Homology, Situation 7.1. Let $X \to Y \to Z$ be morphisms of schemes locally of finite type over S. Let $r, s, e \geq 0$. Then*

$$(\alpha \circ \beta) \cap \gamma = \alpha \cap (\beta \cap \gamma) \quad \textit{in} \quad Z_{r+s+e}(X)$$

where α is a family of r-cycles on fibres of X/Y, β is a family of s-cycles on fibres of Y/Z, and $\gamma \in Z_e(Z)$.

Proof. Since we are proving an equality of cycles on X, we may work locally on Z, see Lemma 11.2. Thus we may assume Z is affine. In particular γ is a finite linear combination of prime cycles. Since $- \cap -$ is linear in the second variable (Lemma 11.1), it suffices to prove the equality when $\gamma = [W]$ for some integral closed subscheme $W \subset Z$ of δ-dimension e.

Let $z \in W$ be the generic point. Write $\beta_z = \sum m_j[V_j]$ in $Z_s(Y_z)$. Then $\beta \cap \gamma$ is equal to $\sum m_j[\overline{V}_j]$ where $\overline{V}_j \subset Y$ is an integral closed subscheme mapped by $Y \to Z$ into W with generic fibre V_j. Let $y_j \in V_j$ be the generic point. We may and do view also as the generic point of $\overline{V}_j$ (mapping to z in W). Write $\alpha_{y_j} = \sum n_{jk}[W_{jk}]$ in $Z_r(X_{y_j})$. Then $\alpha \cap (\beta \cap \gamma)$ is equal to

$$\sum m_j n_{jk}[\overline{W}_{jk}]$$

where $\overline{W}_{jk} \subset X$ is an integral closed subscheme mapped by $X \to Y$ into $\overline{V}_j$ with generic fibre W_{jk}.

On the other hand, let us consider

$$(\alpha \circ \beta)_z = (Y_z \to Y)^*\alpha \cap \beta_z = (Y_z \to Y)^*\alpha \cap (\sum m_j[V_j])$$

By the construction of $- \cap -$ this is equal to the cycle

$$\sum m_j n_{jk}[(\overline{W}_{jk})_z]$$

on X_z. Thus by definition we obtain

$$(\alpha \circ \beta) \cap [W] = \sum m_j n_{jk}[\widetilde{W}_{jk}]$$

where $\widetilde{W}_{jk} \subset X$ is an integral closed subscheme which is mapped by $X \to Z$ into W with generic fibre $(\overline{W}_{jk})_z$. Clearly, we must have $\widetilde{W}_{jk} = \overline{W}_{jk}$ and the proof is complete. $\square$

14. Composition of relative cycles

Let S be a locally Noetherian scheme. Let $X \to Y$ be a morphism of schemes locally of finite type over S. We are going to define a map

$$z(X/Y, r) \otimes_{\mathbf{Z}} z(Y/S, e) \longrightarrow z(X/S, r + e), \quad \alpha \otimes \beta \longmapsto \alpha \circ \beta$$

using the construction in Section 13. We already know the construction is bilinear (Lemma 13.1) hence we obtain the displayed arrow once we show the following.

Lemma 14.1. *If α and β are relative cycles, then so is $\alpha \circ \beta$.*

Proof. The formation of $\alpha \circ \beta$ is compatible with base change by Lemma 13.3. Thus we may assume S is the spectrum of a discrete valuation ring with generic point η and closed point 0 and we have to show that $sp_{X/S}((\alpha \circ \beta)_\eta) = (\alpha \circ \beta)_0$. Since we are trying to prove an equality of cycles, we may work locally on Y and X (this uses Lemmas 13.2 and 4.4 to see that the constructions commute with restriction). Thus we may assume X and Y are affine. By Lemma 6.9 we can find a completely decomposed proper morphism $g : Y' \to Y$ such that $g^*\alpha$ is in the image of (6.8.1).

Since the family of morphisms $g_\eta : Y'_\eta \to Y_\eta$ is completely decomposed, we can find $\beta'_\eta \in Z_e(Y'_\eta)$ such that $\beta_\eta = \sum g_{\eta,*}\beta'_\eta$, see Chow Homology, Lemma 22.4. Set $\beta'_0 = sp_{Y'/S}(\beta'_\eta)$ so that $\beta' = (\beta'_\eta, \beta'_0)$ is a relative e-cycle on Y'/S. Then $g_*\beta'$ and β are relative e-cycles on Y/S (Lemma 6.2) which have the same value at η and hence are equal (Lemma 6.6). By linearity (Lemma 13.1) it suffices to show that $\alpha \circ g_*\beta'$ is a relative $(r + e)$-cycle.

Set $X' = X \times_Y Y'$ and denote $f : X' \to X$ the projection. By Lemma 13.6 we see that $\alpha \circ g_*\beta' = f_*(g^*\alpha \circ \beta')$. By Lemma 6.2 it suffices to show that $g^*\alpha \circ \beta'$ is a relative $(r+e)$-cycle. Using Lemma 6.10 and bilinearity this reduces us to the case discussed in the next paragraph.

Assume $\alpha = [Z/X/Y]_r$ and $\beta = [W/Y/S]$ where $Z \subset X$ is a closed subscheme flat and of relative dimension $\leq r$ over Y and $W \subset Y$ is a closed subscheme flat and of relative dimension $\leq e$ over S. By Lemma 13.5 we see that

$$\alpha \circ \beta = [Z \times_X W/X/S]_{r+e}$$

and $Z \times_X W \subset X$ is a closed subscheme flat over S of relative dimension $\leq r + e$. This is a relative $(r+e)$-cycle by Lemma 6.8. $\square$

Lemma 14.2. *Let $f : X \to Y$ and $g : Y \to S$ be a morphisms of schemes. Assume S locally Noetherian, g locally of finite type and flat of relative dimension $e \geq 0$, and f locally of finite type and flat of relative dimension $r \geq 0$. Then $[X/X/Y]_r \circ [Y/Y/S]_e = [X/X/S]_{r+e}$ in $z(X/S, r+e)$.*

Proof. Special case of Lemma 13.5. $\square$

15. Comparison with Suslin and Voevodsky

We have tried to use the same notation as in [SV00], except that our notation for cycles is taken from Chow Homology, Section 8 ff. Here is a comparison:

(1) In [SV00, Section 3.1] there is a notion of a "relative cycle", of a "relative cycle of dimension r", and of a "equidimensional relative cycle of dimension r". There is no corresponding notion in this chapter. Consequently, the groups $Cycl(X/S, r)$, $Cycl_{equi}(X/S, r)$, $PropCycl(X/S, r)$, and $PropCycl_{equi}(X/S, r)$, have no counter parts in this chapter.
(2) On the bottom of [SV00, page 36] the groups $z(X/S, r)$, $c(X/S, r)$, $z_{equi}(X/S, r)$, $c_{equi}(X/S, r)$ are defined. These agree with our notions when S is separated Noetherian and $X \to S$ is separated and of finite type.
(3) In [SV00] the symbol $z(X/S, r)$ is sometimes used for the presheaf $S' \mapsto z(S' \times_S X/S', r)$ on the category of schemes of finite type over S. Similarly for $c(X/S, r)$, $z_{equi}(X/S, r)$, and $c_{equi}(X/S, r)$.
(4) Base change, flat pullback, and proper pushforward as defined in [SV00] agrees with ours when both apply.
(5) For $\alpha \in z(X/S, r)$ the operation $\alpha \cap - : Z_e(S) \to Z_{e+r}(X)$ defined in Section 11 agrees with the operation $Cor(\alpha, -)$ in [SV00, Section 3.7] when both are defined.
(6) For $X \to Y \to S$ the composition law $z(X/Y, r) \otimes_{\mathbf{Z}} z(Y/S, e) \longrightarrow z(X/S, r+e)$ defined in Section 14 agrees with the opration $Cor_{X/Y}(-, -)$ in [SV00, Corollary 3.7.5].

16. Relative cycles in the non-Noetherian case

We urge the reader to skip this section.

Let $f : X \to S$ be a morphism of schemes of finite presentation. Let $r \geq 0$. Denote $Hilb(X/S, r)$ the set of closed subschemes $Z \subset X$ such that $Z \to S$ is flat, of finite presentation, and of relative dimension $\leq r$. We consider the group homomorphism

$$\begin{matrix} \text{free abelian group} \\ \text{on } Hilb(X/S, r) \end{matrix} \longrightarrow \begin{matrix} \text{families of } r\text{-cycles} \\ \text{on fibres of } X/S \end{matrix} \tag{16.0.1}$$

sending $\sum n_i [Z_i]$ to $\sum n_i [Z_i/X/S]_r$.

Lemma 16.1. *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of finite presentation. Let $r \geq 0$ and let α be a family of r-cycles on fibres of X/S. The following are equivalent*

(1) *there exists a cartesian diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_0 \arrow[d] \\ S \arrow[r] & S_0 \end{tikzcd}$$

where $X_0 \to S_0$ is a finite type morphism of Noetherian schemes and $\alpha_0 \in z(X_0/S_0, r)$ such that α is the base change of α_0 by $S \to S_0$

(2) *there exists a completely decomposed proper morphism $g : S' \to S$ of finite presentation such that $g^*\alpha$ is in the image of (16.0.1).*

Proof. Let a diagram and $\alpha_0 \in z(X_0/S_0, r)$ as in (1) be given. By Lemma 6.9 there exists a proper surjective morphism $g_0 : S'_0 \to S_0$ such that $g_0^*\alpha_0$ is in the image of (16.0.1). Namely, since S'_0 is Noetherian, every closed subscheme of $S'_0 \times_{S_0} X_0$ is of finite presentation over S'_0. Setting $S' = S \times_{S_0} S'_0$ and using base change by $S' \to S'_0$ we see that (2) holds.

Conversely, assume that (2) holds. Choose a surjective proper morphism $g : S' \to S$ of finite presentation such that $g^*\alpha$ is in the image of (16.0.1). Set $X' = S' \times_S X$. Write $g^*\alpha = \sum n_a[Z_a/X'/S']_r$ for some $Z_a \subset X'$ closed subscheme flat, of finite presentation, and of relative dimension $\leq r$ over S'.

Write $S = \lim S_i$ as a directed limit with affine transition morphisms with S_i of finite type over $\mathbf{Z}$, see Limits, Proposition 5.4. We can find an i large enough such that there exist

(1) a completely decomposed proper morphism $g_i : S'_i \to S_i$ whose base change to S is $g : S' \to S$,
(2) setting $X'_i = S'_i \times_{S_i} X_i$ closed subschemes $Z_{ai} \subset X'_i$ flat and of relative dimension $\leq r$ over S'_i whose base change to S' is Z_a.

To do this one uses Limits, Lemmas 10.1, 8.5, 8.7, 8.15, 13.1, and 18.1 and and More on Morphisms, Lemma 78.5. Consider $\alpha'_i = \sum n_a[Z_{ai}/X'_i/S_i]_r \in z(X'_i/S'_i, r)$. The base change of α'_i to a family of r-cycles on fibres of X'/S' agrees with the base change $g^*\alpha$ by construction.

Set $S''_i = S'_i \times_{S_i} S'_i$ and $X''_i = S''_i \times_{S_i} X_i$ and set $S'' = S' \times_S S'$ and $X'' = S'' \times_S X$. We denote $\text{pr}_1, \text{pr}_2 : S'' \to S'$ and $\text{pr}_1, \text{pr}_2 : S''_i \to S'_i$ the projections. The relative r-cycles $\text{pr}_1^*\alpha'_i$ and $\text{pr}_1^*\alpha'_i$ on X''_i/S''_i base change to the same family of r-cycles on fibres of X''/S'' because $\text{pr}_1^*g^*\alpha = \text{pr}_1^*g^*\alpha$. Hence the morphism $S'' \to S''_i$ maps into $E = \{s \in S''_i : (\text{pr}_1^*\alpha'_i)_s = (\text{pr}_1^*\alpha'_i)_s\}$. By Lemma 6.12 this is a closed subset. Since $S'' = \lim_{i' \geq i} S''_{i'}$ we see from Limits, Lemma 4.10 that for some $i' \geq i$ the morphism $S''_{i'} \to S''_i$ maps into E. Therefore, after replacing i by i', we may assume that $\text{pr}_1^*\alpha'_i = \text{pr}_1^*\alpha'_i$. By Lemma 5.9 we obtain a unique family α_i of r-cycles on fibres of X_i/S_i with $g_i^*\alpha_i = \alpha'_i$ (this uses that $S'_i \to S_i$ is completely decomposed). By Lemma 6.3 we see that $\alpha_i \in z(X_i/S_i, r)$. The uniqueness in Lemma 5.9 implies that the base change of α_i is α and we see (1) holds. $\square$

Discussion. If $f : X \to S$, r, and α are as in Lemma 16.1, then it makes sense to say that α is a *relative r-cycle on X/S* if the equivalent conditions (1) and (2) of Lemma 16.1 hold. This definition has many good properties; for example it doesn't conflict with the earlier definition in case S is Noetherian and most of the results of Section 6 generalize to this setting.

We may still generalize further as follows. Assume S is arbitrary and $f : X \to S$ is locally of finite presentation. Let $r \geq 0$ and let α be a family of r-cycles α on fibres of X/S. Then α is an *relative r-cycle on X/S* if for $U \subset X$ and $V \subset S$ affine open with $f(U) \subset V$ the restriction $\alpha|_U$ is a relative r-cycle on U/V as defined in the previous paragraph. Again many of the earlier results generalize to this setting.

If we ever need these generalizations we will carefully state and prove them here.

17. Other chapters

References

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