MODULI STACKS

Contents

1. Introduction

In this chapter we verify basic properties of moduli spaces and moduli stacks such as *Hom*, *Isom*, $\mathcal{C}oh_{X/B}$, $\mathrm{Quot}_{\mathcal{F}/X/B}$, $\mathrm{Hilb}_{X/B}$, $\mathcal{P}ic_{X/B}$, $\mathrm{Pic}_{X/B}$, $Mor_B(Z, X)$, $\mathcal{S}paces'_{fp,flat,proper}$, $\mathcal{P}olarized$, and $\mathcal{C}omplexes_{X/B}$. We have already shown these algebraic spaces or algebraic stacks under suitable hypotheses, see Quot, Sections 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, and 16. The stack of curves, denoted *Curves* and introduced in Quot, Section 15, is discussed in the chapter on moduli of curves, see Moduli of Curves, Section 3.

In some sense this chapter is following the footsteps of Grothendieck's lectures [Gro95a], [Gro95b], [Gro95e], [Gro95f], [Gro95c], and [Gro95d].

2. Conventions and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2. Unless otherwise mentioned our base scheme will be $\mathrm{Spec}(\mathbf{Z})$.

3. Properties of Hom and Isom

Let $f : X \to B$ be a morphism of algebraic spaces which is of finite presentation. Assume $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{G}$ is of finite presentation, flat over B with support proper over B, then the functor $Hom(\mathcal{F}, \mathcal{G})$ defined by

$$T/B \longmapsto \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T)$$

is an algebraic space affine over B. If $\mathcal{F}$ is of finite presentation, then $\mathit{Hom}(\mathcal{F}, \mathcal{G}) \to B$ is of finite presentation. See Quot, Proposition 3.10.

If both $\mathcal{F}$ and $\mathcal{G}$ are of finite presentation, flat over B with support proper over B, then the subfunctor

$$\mathit{Isom}(\mathcal{F}, \mathcal{G}) \subset \mathit{Hom}(\mathcal{F}, \mathcal{G})$$

is an algebraic space affine of finite presentation over B. See Quot, Proposition 4.3.

4. Properties of the stack of coherent sheaves

Let $f : X \to B$ be a morphism of algebraic spaces which is separated and of finite presentation. Then the stack $\mathcal{C}oh_{X/B}$ parametrizing flat families of coherent modules with proper support is algebraic. See Quot, Theorem 6.1.

Lemma 4.1. *The diagonal of $\mathcal{C}oh_{X/B}$ over B is affine and of finite presentation.*

Proof. The representability of the diagonal by algebraic spaces was shown in Quot, Lemma 5.3. From the proof we find that we have to show $\mathit{Isom}(\mathcal{F}, \mathcal{G}) \to T$ is affine and of finite presentation for a pair of finitely presented $\mathcal{O}_{X_T}$-modules $\mathcal{F}$, $\mathcal{G}$ flat over T with support proper over T. This was discussed in Section 3. $\square$

Lemma 4.2. *The morphism $\mathcal{C}oh_{X/B} \to B$ is quasi-separated and locally of finite presentation.*

Proof. To check $\mathcal{C}oh_{X/B} \to B$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 4.1. To prove that $\mathcal{C}oh_{X/B} \to B$ is locally of finite presentation, we have to show that $\mathcal{C}oh_{X/B} \to B$ is limit preserving, see Limits of Stacks, Proposition 3.8. This follows from Quot, Lemma 5.6 (small detail omitted). $\square$

Lemma 4.3. *Assume $X \to B$ is proper as well as of finite presentation. Then $\mathcal{C}oh_{X/B} \to B$ satisfies the existence part of the valuative criterion (Morphisms of Stacks, Definition 39.10).*

Proof. Taking base change, this immediately reduces to the following problem: given a valuation ring R with fraction field K and an algebraic space X proper over R and a coherent $\mathcal{O}_{X_K}$-module $\mathcal{F}_K$, show there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ flat over R whose generic fibre is $\mathcal{F}_K$. Observe that by Flatness on Spaces, Theorem 4.5 any finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ flat over R is of finite presentation. Denote $j : X_K \to X$ the embedding of the generic fibre. As a base change of the affine morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$ the morphism j is affine. Thus $j_*\mathcal{F}_K$ is quasi-coherent. Write

$$j_*\mathcal{F}_K = \operatorname{colim} \mathcal{F}_i$$

as a filtered colimit of its finite type quasi-coherent $\mathcal{O}_X$-submodules, see Limits of Spaces, Lemma 9.2. Since $j_*\mathcal{F}_K$ is a sheaf of K-vector spaces over X, it is flat over $\mathrm{Spec}(R)$. Thus each $\mathcal{F}_i$ is flat over R as flatness over a valuation ring is the same as being torsion free (More on Algebra, Lemma 22.10) and torsion freeness is inherited by submodules. Finally, we have to show that the map $j^*\mathcal{F}_i \to \mathcal{F}_K$ is an isomorphism for some i. Since $j^*j_*\mathcal{F}_K = \mathcal{F}_K$ (small detail omitted) and since j^* is exact, we see that $j^*\mathcal{F}_i \to \mathcal{F}_K$ is injective for all i. Since j^* commutes with colimits, we have $\mathcal{F}_K = j^*j_*\mathcal{F}_K = \operatorname{colim} j^*\mathcal{F}_i$. Since $\mathcal{F}_K$ is coherent (i.e., finitely presented), there is an i such that $j^*\mathcal{F}_i$ contains all the (finitely many) generators over an affine étale cover of X. Thus we get surjectivity of $j^*\mathcal{F}_i \to \mathcal{F}_K$ for i large enough. $\square$

Lemma 4.4. *Let B be an algebraic space. Let $\pi : X \to Y$ be a quasi-finite morphism of algebraic spaces which are separated and of finite presentation over B. Then π_* induces a morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$.*

Proof. Let $(T \to B, \mathcal{F})$ be an object of $\mathcal{C}oh_{X/B}$. We claim

(a) $(T \to B, \pi_{T,*}\mathcal{F})$ is an object of $\mathcal{C}oh_{Y/B}$ and
(b) for $T' \to T$ we have $\pi_{T',*}(X_{T'} \to X_T)^*\mathcal{F} = (Y_{T'} \to Y_T)^*\pi_{T,*}\mathcal{F}$.

Part (b) guarantees that this construction defines a functor $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ as desired.

Let $i : Z \to X_T$ be the closed subspace cut out by the zeroth fitting ideal of $\mathcal{F}$ (Divisors on Spaces, Section 5). Then $Z \to B$ is proper by assumption (see Derived Categories of Spaces, Section 7). On the other hand i is of finite presentation (Divisors on Spaces, Lemma 5.2 and Morphisms of Spaces, Lemma 28.12). There exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ of finite type with $i_*\mathcal{G} = \mathcal{F}$ (Divisors on Spaces, Lemma 5.3). In fact $\mathcal{G}$ is of finite presentation as an $\mathcal{O}_Z$-module by Descent on Spaces, Lemma 6.7. Observe that $\mathcal{G}$ is flat over B, for example because the stalks of $\mathcal{G}$ and $\mathcal{F}$ agree (Morphisms of Spaces, Lemma 13.6). Observe that $\pi_T \circ i : Z \to Y_T$ is quasi-finite as a composition of quasi-finite morphisms and that $\pi_{T,*}\mathcal{F} = (\pi_T \circ i)_*\mathcal{G})$. Since i is affine, formation of i_* commutes with base change (Cohomology of Spaces, Lemma 11.1). Therefore we may replace B by T, X by Z, $\mathcal{F}$ by $\mathcal{G}$, and Y by Y_T to reduce to the case discussed in the next paragraph.

Assume that $X \to B$ is proper. Then π is proper by Morphisms of Spaces, Lemma 40.6 and hence finite by More on Morphisms of Spaces, Lemma 35.1. Since a finite morphism is affine we see that (b) holds by Cohomology of Spaces, Lemma 11.1. On the other hand, π is of finite presentation by Morphisms of Spaces, Lemma 28.9. Thus $\pi_{T,*}\mathcal{F}$ is of finite presentation by Descent on Spaces, Lemma 6.7. Finally, $\pi_{T,*}\mathcal{F}$ is flat over B for example by looking at stalks using Cohomology of Spaces, Lemma 4.2. $\square$

Lemma 4.5. *Let B be an algebraic space. Let $\pi : X \to Y$ be an open immersion of algebraic spaces which are separated and of finite presentation over B. Then the morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ of Lemma 4.4 is an open immersion.*

Proof. Omitted. Hint: If $\mathcal{F}$ is an object of $\mathcal{C}oh_{Y/B}$ over T and for $t \in T$ we have $\text{Supp}(\mathcal{F}_t) \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. $\square$

Lemma 4.6. *Let B be an algebraic space. Let $\pi : X \to Y$ be a closed immersion of algebraic spaces which are separated and of finite presentation over B. Then the morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ of Lemma 4.4 is a closed immersion.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the sheaf of ideals cutting out X as a closed subspace of Y. Recall that π_* induces an equivalence between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of quasi-coherent $\mathcal{O}_Y$-modules annihilated by $\mathcal{I}$, see Morphisms of Spaces, Lemma 14.1. The same, mutatis mutandis, is true after base by $T \to B$ with $\mathcal{I}$ replaced by the ideal sheaf $\mathcal{I}_T = \text{Im}((Y_T \to Y)^*\mathcal{I} \to \mathcal{O}_{Y_T})$. Analyzing the proof of Lemma 4.4 we find that the essential image of $\mathcal{C}oh_{X/B} \to$

$\mathcal{C}oh_{Y/B}$ is exactly the objects $\xi = (T \to B, \mathcal{F})$ where $\mathcal{F}$ is annihilated by $\mathcal{I}_T$. In other words, ξ is in the essential image if and only if the multiplication map

$$\mathcal{F} \otimes_{\mathcal{O}_{Y_T}} (Y_T \to Y)^*\mathcal{I} \longrightarrow \mathcal{F}$$

is zero and similarly after any further base change $T' \to T$. Note that

$$(Y_{T'} \to Y_T)^*(\mathcal{F} \otimes_{\mathcal{O}_{Y_T}} (Y_T \to Y)^*\mathcal{I}) = (Y_{T'} \to Y_T)^*\mathcal{F} \otimes_{\mathcal{O}_{Y_{T'}}} (Y_{T'} \to Y)^*\mathcal{I})$$

Hence the vanishing of the multiplication map on T' is representable by a closed subspace of T by Flatness on Spaces, Lemma 8.6. □

Situation 4.7 (Numerical invariants)**.** Let $f : X \to B$ be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_X)$ be perfect. Given an object $(T \to B, \mathcal{F})$ of $\mathcal{C}oh_{X/B}$ denote $E_{i,T}$ the derived pullback of E_i to X_T. The object

$$K_i = Rf_{T,*}(E_{i,T} \otimes^{\mathbf{L}}_{\mathcal{O}_{X_T}} \mathcal{F})$$

of $D(\mathcal{O}_T)$ is perfect and its formation commutes with base change, see Derived Categories of Spaces, Lemma 25.1. Thus the function

$$\chi_i : |T| \longrightarrow \mathbf{Z}, \quad \chi_i(t) = \chi(X_t, E_{i,t} \otimes^{\mathbf{L}}_{\mathcal{O}_{X_t}} \mathcal{F}_t) = \chi(K_i \otimes^{\mathbf{L}}_{\mathcal{O}_T} \kappa(t))$$

is locally constant by Derived Categories of Spaces, Lemma 26.3. Let $P : I \to \mathbf{Z}$ be a map. Consider the substack

$$\mathcal{C}oh^P_{X/B} \subset \mathcal{C}oh_{X/B}$$

consisting of flat families of coherent sheaves with proper support whose numerical invariants agree with P. More precisely, an object $(T \to B, \mathcal{F})$ of $\mathcal{C}oh_{X/B}$ is in $\mathcal{C}oh^P_{X/B}$ if and only if $\chi_i(t) = P(i)$ for all $i \in I$ and $t \in T$.

Lemma 4.8. *In Situation 4.7 the stack $\mathcal{C}oh^P_{X/B}$ is algebraic and*

$$\mathcal{C}oh^P_{X/B} \longrightarrow \mathcal{C}oh_{X/B}$$

is a flat closed immersion. If I is finite or B is locally Noetherian, then $\mathcal{C}oh^P_{X/B}$ is an open and closed substack of $\mathcal{C}oh_{X/B}$.

Proof. This is immediately clear if I is finite, because the functions $t \mapsto \chi_i(t)$ are locally constant. If I is infinite, then we write

$$I = \bigcup\nolimits_{I' \subset I \text{ finite}} I'$$

and we denote $P' = P|_{I'}$. Then we have

$$\mathcal{C}oh^P_{X/B} = \bigcap\nolimits_{I' \subset I \text{ finite}} \mathcal{C}oh^{P'}_{X/B}$$

Therefore, $\mathcal{C}oh^P_{X/B}$ is always an algebraic stack and the morphism $\mathcal{C}oh^P_{X/B} \subset \mathcal{C}oh_{X/B}$ is always a flat closed immersion, but it may no longer be an open substack. (We leave it to the reader to make examples). However, if B is locally Noetherian, then so is $\mathcal{C}oh_{X/B}$ by Lemma 4.2 and Morphisms of Stacks, Lemma 17.5. Hence if $U \to \mathcal{C}oh_{X/B}$ is a smooth surjective morphism where U is a locally Noetherian scheme, then the inverse images of the open and closed substacks $\mathcal{C}oh^{P'}_{X/B}$ have an open intersection in U (because connected components of locally Noetherian topological spaces are open). Thus the result in this case. □

Lemma 4.9. *Let $f : X \to B$ be as in the introduction to this section. Let $E_1, \ldots, E_r \in D(\mathcal{O}_X)$ be perfect. Let $I = \mathbf{Z}^{\oplus r}$ and consider the map*

$$I \longrightarrow D(\mathcal{O}_X), \quad (n_1, \ldots, n_r) \longmapsto E_1^{\otimes n_1} \otimes \ldots \otimes E_r^{\otimes n_r}$$

Let $P : I \to \mathbf{Z}$ be a map. Then $\mathcal{C}oh^P_{X/B} \subset \mathcal{C}oh_{X/B}$ as defined in Situation 4.7 is an open and closed substack.

Proof. We may work étale locally on B, hence we may assume that B is affine. In this case we may perform absolute Noetherian reduction; we suggest the reader skip the proof. Namely, say $B = \mathrm{Spec}(\Lambda)$. Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \to \mathrm{Spec}(\Lambda_i)$ which is separated and of finite presentation and whose base change to Λ is X. See Limits of Spaces, Lemmas 7.1 and 6.9. Then after increasing i we may assume there exist perfect objects $E_{1,i}, \ldots, E_{r,i}$ in $D(\mathcal{O}_{X_i})$ whose derived pullback to X are isomorphic to $E_1, \ldots, E_r$, see Derived Categories of Spaces, Lemma 24.3. Clearly we have a cartesian square

$$\begin{tikzcd}
\mathcal{C}oh^P_{X/B} \arrow[r] \arrow[d] & \mathcal{C}oh_{X/B} \arrow[d] \\
\mathcal{C}oh^P_{X_i/\mathrm{Spec}(\Lambda_i)} \arrow[r] & \mathcal{C}oh_{X_i/\mathrm{Spec}(\Lambda_i)}
\end{tikzcd}$$

and hence we may appeal to Lemma 4.8 to finish the proof. □

Example 4.10 (Coherent sheaves with fixed Hilbert polynomial)**.** Let $f : X \to B$ be as in the introduction to this section. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $P : \mathbf{Z} \to \mathbf{Z}$ be a numerical polynomial. Then we can consider the open and closed algebraic substack

$$\mathcal{C}oh^P_{X/B} = \mathcal{C}oh^{P,\mathcal{L}}_{X/B} \subset \mathcal{C}oh_{X/B}$$

consisting of flat families of coherent sheaves with proper support whose numerical invariants agree with P: an object $(T \to B, \mathcal{F})$ of $\mathcal{C}oh_{X/B}$ lies in $\mathcal{C}oh^P_{X/B}$ if and only if

$$P(n) = \chi(X_t, \mathcal{F}_t \otimes_{\mathcal{O}_{X_t}} \mathcal{L}_t^{\otimes n})$$

for all $n \in \mathbf{Z}$ and $t \in T$. Of course this is a special case of Situation 4.7 where $I = \mathbf{Z} \to D(\mathcal{O}_X)$ is given by $n \mapsto \mathcal{L}^{\otimes n}$. It follows from Lemma 4.9 that this is an open and closed substack. Since the functions $n \mapsto \chi(X_t, \mathcal{F}_t \otimes_{\mathcal{O}_{X_t}} \mathcal{L}_t^{\otimes n})$ are always numerical polynomials (Spaces over Fields, Lemma 18.1) we conclude that

$$\mathcal{C}oh_{X/B} = \coprod\nolimits_{P \text{ numerical polynomial}} \mathcal{C}oh^P_{X/B}$$

is a disjoint union decomposition.

5. Properties of Quot

Let $f : X \to B$ be a morphism of algebraic spaces which is separated and of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathrm{Quot}_{\mathcal{F}/X/B}$ is an algebraic space. If $\mathcal{F}$ is of finite presentation, then $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is locally of finite presentation. See Quot, Proposition 8.4.

Lemma 5.1. *The diagonal of $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is a closed immersion. If $\mathcal{F}$ is of finite type, then the diagonal is a closed immersion of finite presentation.*

Proof. Suppose we have a scheme T/B and two quotients $\mathcal{F}_T \to \mathcal{Q}_i$, $i = 1, 2$ corresponding to T-valued points of $\mathrm{Quot}_{\mathcal{F}/X/B}$ over B. Denote $\mathcal{K}_1$ the kernel of the first one and set $u : \mathcal{K}_1 \to \mathcal{Q}_2$ the composition. By Flatness on Spaces, Lemma 8.6 there is a closed subspace of T such that $T' \to T$ factors through it if and only if the pullback $u_{T'}$ is zero. This proves the diagonal is a closed immersion. Moreover, if $\mathcal{F}$ is of finite type, then $\mathcal{K}_1$ is of finite type (Modules on Sites, Lemma 24.1) and we see that the diagonal is of finite presentation by the same lemma. $\square$

Lemma 5.2. *The morphism* $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ *is separated. If* $\mathcal{F}$ *is of finite presentation, then it is also locally of finite presentation.*

Proof. To check $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is separated we have to show that its diagonal is a closed immersion. This is true by Lemma 5.1. The second statement is part of Quot, Proposition 8.4. $\square$

Lemma 5.3. *Assume* $X \to B$ *is proper as well as of finite presentation and* $\mathcal{F}$ *quasi-coherent of finite type. Then* $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ *satisfies the existence part of the valuative criterion (Morphisms of Spaces, Definition 41.1).*

Proof. Taking base change, this immediately reduces to the following problem: given a valuation ring R with fraction field K, an algebraic space X proper over R, a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, and a coherent quotient $\mathcal{F}_K \to \mathcal{Q}_K$, show there exists a quotient $\mathcal{F} \to \mathcal{Q}$ where $\mathcal{Q}$ is a finitely presented $\mathcal{O}_X$-module flat over R whose generic fibre is $\mathcal{Q}_K$. Observe that by Flatness on Spaces, Theorem 4.5 any finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ flat over R is of finite presentation. We first solve the existence of $\mathcal{Q}$ affine locally.

Affine locally we arrive at the following problem: let $R \to A$ be a finitely presented ring map, let M be a finite A-module, let $\varphi : M_K \to N_K$ be an A_K-quotient module. Then we may consider

$$L = \{x \in M \mid \varphi(x \otimes 1) = 0\}$$

The $M \to M/L$ is an A-module quotient which is torsion free as an R-module. Hence it is flat as an R-module (More on Algebra, Lemma 22.10). Since M is finite as an A-module so is L and we conclude that L is of finite presentation as an A-module (by the reference above). Clearly M/L is the unqiue such quotient with $(M/L)_K = N_K$.

The uniqueness in the construction of the previous paragraph guarantees these quotients glue and give the desired $\mathcal{Q}$. Here is a bit more detail. Choose a surjective étale morphism $U \to X$ where U is an affine scheme. Use the above construction to construct a quotient $\mathcal{F}|_U \to \mathcal{Q}_U$ which is quasi-coherent, is flat over R, and recovers $\mathcal{Q}_K|U$ on the generic fibre. Since X is separated, we see that $U \times_X U$ is an affine scheme étale over X as well. Then $\mathcal{F}|_{U \times_X U} \to \mathrm{pr}_1^*\mathcal{Q}_U$ and $\mathcal{F}|_{U \times_X U} \to \mathrm{pr}_2^*\mathcal{Q}_U$ agree as quotients by the uniquess in the construction. Hence we may descend $\mathcal{F}|_U \to \mathcal{Q}_U$ to a surjection $\mathcal{F} \to \mathcal{Q}$ as desired (Properties of Spaces, Proposition 32.1). $\square$

Lemma 5.4. *Let* B *be an algebraic space. Let* $\pi : X \to Y$ *be an affine quasi-finite morphism of algebraic spaces which are separated and of finite presentation over* B. *Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_X$*-module. Then* π_* *induces a morphism* $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}$.

Proof. Set $\mathcal{G} = \pi_*\mathcal{F}$. Since π is affine we see that for any scheme T over B we have $\mathcal{G}_T = \pi_{T,*}\mathcal{F}_T$ by Cohomology of Spaces, Lemma 11.1. Moreover π_T is affine, hence $\pi_{T,*}$ is exact and transforms quotients into quotients. Observe that a quasi-coherent quotient $\mathcal{F}_T \to \mathcal{Q}$ defines a point of $\mathrm{Quot}_{X/B}$ if and only if $\mathcal{Q}$ defines an object of $\mathcal{C}oh_{X/B}$ over T (similarly for $\mathcal{G}$ and Y). Since we've seen in Lemma 4.4 that π_* induces a morphism $\mathcal{C}oh_{X/B} \to \mathcal{C}oh_{Y/B}$ we see that if $\mathcal{F}_T \to \mathcal{Q}$ is in $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$, then $\mathcal{G}_T \to \pi_{T,*}\mathcal{Q}$ is in $\mathrm{Quot}_{\mathcal{G}/Y/B}(T)$. $\square$

Lemma 5.5. *Let B be an algebraic space. Let $\pi : X \to Y$ be an affine open immersion of algebraic spaces which are separated and of finite presentation over B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the morphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}$ of Lemma 5.4 is an open immersion.*

Proof. Omitted. Hint: If $(\pi_*\mathcal{F})_T \to \mathcal{Q}$ is an element of $\mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}(T)$ and for $t \in T$ we have $\mathrm{Supp}(\mathcal{Q}_t) \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. $\square$

Lemma 5.6. *Let B be an algebraic space. Let $j : X \to Y$ be an open immersion of algebraic spaces which are separated and of finite presentation over B. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module and set $\mathcal{F} = j^*\mathcal{G}$. Then there is an open immersion*

$$\mathrm{Quot}_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}_{\mathcal{G}/Y/B}$$

of algebraic spaces over B.

Proof. If $\mathcal{F}_T \to \mathcal{Q}$ is an element of $\mathrm{Quot}_{\mathcal{F}/X/B}(T)$ then we can consider $\mathcal{G}_T \to j_{T,*}\mathcal{F}_T \to j_{T,*}\mathcal{Q}$. Looking at stalks one finds that this is surjective. By Lemma 4.4 we see that $j_{T,*}\mathcal{Q}$ is finitely presented, flat over B with support proper over B. Thus we obtain a T-valued point of $\mathrm{Quot}_{\mathcal{G}/Y/B}$. This defines the morphism of the lemma. We omit the proof that this is an open immersion. Hint: If $\mathcal{G}_T \to \mathcal{Q}$ is an element of $\mathrm{Quot}_{\mathcal{G}/Y/B}(T)$ and for $t \in T$ we have $\mathrm{Supp}(\mathcal{Q}_t) \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. $\square$

Lemma 5.7. *Let B be an algebraic space. Let $\pi : X \to Y$ be a closed immersion of algebraic spaces which are separated and of finite presentation over B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the morphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}$ of Lemma 5.4 is an isomorphism.*

Proof. For every scheme T over B the morphism $\pi_T : X_T \to Y_T$ is a closed immersion. Then $\pi_{T,*}$ is an equivalence of categories between $QCoh(\mathcal{O}_{X_T})$ and the full subcategory of $QCoh(\mathcal{O}_{Y_T})$ whose objects are those quasi-coherent modules annihilated by the ideal sheaf of X_T, see Morphisms of Spaces, Lemma 14.1. Since a qotient of $(\pi_*\mathcal{F})_T$ is annihilated by this ideal we obtain the bijectivity of the map $\mathrm{Quot}_{\mathcal{F}/X/B}(T) \to \mathrm{Quot}_{\pi_*\mathcal{F}/Y/B}(T)$ for all T as desired. $\square$

Lemma 5.8. *Let $X \to B$ be as in the introduction to this section. Let $\mathcal{F} \to \mathcal{G}$ be a surjection of quasi-coherent $\mathcal{O}_X$-modules. Then there is a canonical closed immersion $\mathrm{Quot}_{\mathcal{G}/X/B} \to \mathrm{Quot}_{\mathcal{F}/X/B}$.*

Proof. Let $\mathcal{K} = \mathrm{Ker}(\mathcal{F} \to \mathcal{G})$. By right exactness of pullbacks we find that $\mathcal{K}_T \to \mathcal{F}_T \to \mathcal{G}_T \to 0$ is an exact sequecnce for all schemes T over B. In particular, a quotient of $\mathcal{G}_T$ determines a quotient of $\mathcal{F}_T$ and we obtain our transformation of functors $\mathrm{Quot}_{\mathcal{G}/X/B} \to \mathrm{Quot}_{\mathcal{F}/X/B}$. This transformation is a closed immersion by Flatness on Spaces, Lemma 8.6. Namely, given an element $\mathcal{F}_T \to \mathcal{Q}$ of $\text{Quot}_{\mathcal{F}/X/B}(T)$, then we see that the pull back to T'/T is in the image of the transformation if and only if $\mathcal{K}_{T'} \to \mathcal{Q}_{T'}$ is zero. □

Remark 5.9 (Numerical invariants)**.** Let $f : X \to B$ and $\mathcal{F}$ be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_X)$ be perfect. Let $P : I \to \mathbf{Z}$ be a function. Recall that we have a morphism

$$\text{Quot}_{\mathcal{F}/X/B} \longrightarrow \mathcal{C}oh_{X/B}$$

which sends the element $\mathcal{F}_T \to \mathcal{Q}$ of $\text{Quot}_{\mathcal{F}/X/B}(T)$ to the object $\mathcal{Q}$ of $\mathcal{C}oh_{X/B}$ over T, see proof of Quot, Proposition 8.4. Hence we can form the fibre product diagram

$$\begin{tikzcd} \text{Quot}^P_{\mathcal{F}/X/B} \arrow[r] \arrow[d] & \mathcal{C}oh^P_{X/B} \arrow[d] \\ \text{Quot}_{\mathcal{F}/X/B} \arrow[r] & \mathcal{C}oh_{X/B} \end{tikzcd}$$

This is the defining diagram for the algebraic space in the upper left corner. The left vertical arrow is a flat closed immersion which is an open and closed immersion for example if I is finite, or B is locally Noetherian, or $I = \mathbf{Z}$ and $E_i = \mathcal{L}^{\otimes i}$ for some invertible $\mathcal{O}_X$-module $\mathcal{L}$ (in the last case we sometimes use the notation $\text{Quot}^{P,\mathcal{L}}_{\mathcal{F}/X/B}$). See Situation 4.7 and Lemmas 4.8 and 4.9 and Example 4.10.

Lemma 5.10. *Let $f : X \to B$ and $\mathcal{F}$ be as in the introduction to this section. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then tensoring with $\mathcal{L}$ defines an isomorphism*

$$\text{Quot}_{\mathcal{F}/X/B} \longrightarrow \text{Quot}_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}/X/B}$$

Given a numerical polynomial $P(t)$, then setting $P'(t) = P(t+1)$ this map induces an isomorphism $\text{Quot}^P_{\mathcal{F}/X/B} \longrightarrow \text{Quot}^{P'}_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}/X/B}$ of open and closed substacks.

Proof. Set $\mathcal{G} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$. Observe that $\mathcal{G}_T = \mathcal{F}_T \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T$. If $\mathcal{F}_T \to \mathcal{Q}$ is an element of $\text{Quot}_{\mathcal{F}/X/B}(T)$, then we send it to the element $\mathcal{G}_T \to \mathcal{Q} \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T$ of $\text{Quot}_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}/X/B}(T)$. This is compatible with pullbacks and hence defines a transformation of functors as desired. Since there is an obvious inverse transformation, it is an isomorphism. We omit the proof of the final statement. □

Lemma 5.11. *Let $f : X \to B$ and $\mathcal{F}$ be as in the introduction to this section. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then*

$$\text{Quot}^{P,\mathcal{L}}_{\mathcal{F}/X/B} = \text{Quot}^{P',\mathcal{L}^{\otimes n}}_{\mathcal{F}/X/B}$$

where $P'(t) = P(nt)$.

Proof. Follows immediately after unwinding all the definitions. □

6. Boundedness for Quot

Contrary to what happens classically, we already know the Quot functor is an algebraic space, but we don't know that it is ever represented by a finite type algebraic space.

Lemma 6.1. *Let $n \geq 0$, $r \geq 1$, $P \in \mathbf{Q}[t]$. The algebraic space*

$$X = \mathrm{Quot}^P_{\mathcal{O}^{\oplus r}_{\mathbf{P}^n_{\mathbf{Z}}}/\mathbf{P}^n_{\mathbf{Z}}/\mathbf{Z}}$$

parametrizing quotients of $\mathcal{O}^{\oplus r}_{\mathbf{P}^n_{\mathbf{Z}}}$ with Hilbert polynomial P is proper over $\mathrm{Spec}(\mathbf{Z})$.

Proof. We already know that $X \to \mathrm{Spec}(\mathbf{Z})$ is separated and locally of finite presentation (Lemma 5.2). We also know that $X \to \mathrm{Spec}(\mathbf{Z})$ satisfies the existence part of the valuative criterion, see Lemma 5.3. By the valuative criterion for properness, it suffices to prove our Quot space is quasi-compact, see Morphisms of Spaces, Lemma 44.1. Thus it suffices to find a quasi-compact scheme T and a surjective morphism $T \to X$. Let m be the integer found in Varieties, Lemma 35.18. Let

$$N = r\binom{m+n}{n} - P(m)$$

We will write $\mathbf{P}^n$ for $\mathbf{P}^n_{\mathbf{Z}} = \mathrm{Proj}(\mathbf{Z}[T_0, \ldots, T_n])$ and unadorned products will mean products over $\mathrm{Spec}(\mathbf{Z})$. The idea of the proof is to construct a "universal" map

$$\Psi : \mathcal{O}_{T \times \mathbf{P}^n}(-m)^{\oplus N} \longrightarrow \mathcal{O}^{\oplus r}_{T \times \mathbf{P}^n}$$

over an affine scheme T and show that every point of X corresponds to a cokernel of this in some point of T.

Definition of T and Ψ. We take $T = \mathrm{Spec}(A)$ where

$$A = \mathbf{Z}[a_{i,j,E}]$$

where $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, N\}$ and $E = (e_0, \ldots, e_n)$ runs through the multi-indices of total degree $|E| = \sum_{k=0,\ldots n} e_k = m$. Then we define Ψ to be the map whose (i,j) matrix entry is the map

$$\sum\nolimits_{E=(e_0,\ldots,e_n)} a_{i,j,E} T_0^{e_0} \ldots T_n^{e_n} : \mathcal{O}_{T \times \mathbf{P}^n}(-m) \longrightarrow \mathcal{O}_{T \times \mathbf{P}^n}$$

where the sum is over E as above (but i and j are fixed of course).

Consider the quotient $\mathcal{Q} = \mathrm{Coker}(\Psi)$ on $T \times \mathbf{P}^n$. By More on Morphisms, Lemma 54.1 there exists a $t \geq 0$ and closed subschemes

$$T = T_0 \supset T_1 \supset \ldots \supset T_t = \emptyset$$

such that the pullback $\mathcal{Q}_p$ of $\mathcal{Q}$ to $(T_p \setminus T_{p+1}) \times \mathbf{P}^n$ is flat over $T_p \setminus T_{p+1}$. Observe that we have an exact sequence

$$\mathcal{O}_{(T_p \setminus T_{p+1}) \times \mathbf{P}^n}(-m)^{\oplus N} \to \mathcal{O}^{\oplus r}_{(T_p \setminus T_{p+1}) \times \mathbf{P}^n} \to \mathcal{Q}_p \to 0$$

by pulling back the exact sequence defining $\mathcal{Q} = \mathrm{Coker}(\Psi)$. Therefore we obtain a morphism

$$\coprod (T_p \setminus T_{p+1}) \longrightarrow \mathrm{Quot}_{\mathcal{O}^{\oplus r}/\mathbf{P}/\mathbf{Z}} \supset \mathrm{Quot}^P_{\mathcal{O}^{\oplus r}/\mathbf{P}/\mathbf{Z}} = X$$

Since the left hand side is a Noetherian scheme and the inclusion on the right hand side is open, it suffices to show that any point of X is in the image of this morphism.

Let k be a field and let $x \in X(k)$. Then x corresponds to a surjection $\mathcal{O}^{\oplus r}_{\mathbf{P}^n_k} \to \mathcal{F}$ of coherent $\mathcal{O}_{\mathbf{P}^n_k}$-modules such that the Hilbert polynomial of $\mathcal{F}$ is P. Consider the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus r}_{\mathbf{P}^n_k} \to \mathcal{F} \to 0$$

By Varieties, Lemma 35.18 and our choice of m we see that $\mathcal{K}$ is m-regular. By Varieties, Lemma 35.12 we see that $\mathcal{K}(m)$ is globally generated. By Varieties, Lemma 35.10 and the definition of m-regularity we see that $H^i(\mathbf{P}^n_k, \mathcal{K}(m)) = 0$ for $i > 0$. Hence we see that

$$\dim_k H^0(\mathbf{P}^n_k, \mathcal{K}(m)) = \chi(\mathcal{K}(m)) = \chi(\mathcal{O}_{\mathbf{P}^n_k}(m)^{\oplus r}) - \chi(\mathcal{F}(m)) = N$$

by our choice of N. This gives a surjection

$$\mathcal{O}_{\mathbf{P}^n_k}^{\oplus N} \longrightarrow \mathcal{K}(m)$$

Twisting back down and using the short exact sequence above we see that $\mathcal{F}$ is the cokernel of a map

$$\Psi_x : \mathcal{O}_{\mathbf{P}^n_k}(-m)^{\oplus N} \to \mathcal{O}_{\mathbf{P}^n_k}^{\oplus r}$$

There is a unique ring map $\tau : A \to k$ such that the base change of Ψ by the corresponding morphism $t = \mathrm{Spec}(\tau) : \mathrm{Spec}(k) \to T$ is Ψ_x. This is true because the entries of the $N \times r$ matrix defining Ψ_x are homogeneous polynomials $\sum \lambda_{i,j,E} T_0^{e_0} \ldots T_n^{e_n}$ of degree m in $T_0, \ldots, T_n$ with coefficients $\lambda_{i,j,E} \in k$ and we can set $\tau(a_{i,j,E}) = \lambda_{i,j,E}$. Then $t \in T_p \setminus T_{p+1}$ for some p and the image of t under the morphism above is x as desired. $\square$

Lemma 6.2. *Let B be an algebraic space. Let $X = B \times \mathbf{P}^n_{\mathbf{Z}}$. Let $\mathcal{L}$ be the pullback of $\mathcal{O}_{\mathbf{P}^n}(1)$ to X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. The algebraic space $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ parametrizing quotients of $\mathcal{F}$ having Hilbert polynomial P with respect to $\mathcal{L}$ is proper over B.*

Proof. The question is étale local over B, see Morphisms of Spaces, Lemma 40.2. Thus we may assume B is an affine scheme. In this case $\mathcal{L}$ is an ample invertible module on X (by Constructions, Lemma 10.6 and the definition of ample invertible modules in Properties, Definition 26.1). Thus we can find $r' \geq 0$ and $r \geq 0$ and a surjection

$$\mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r'}$$

by Properties, Proposition 26.13. By Lemma 5.10 we may replace $\mathcal{F}$ by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r'}$ and $P(t)$ by $P(t + r')$. By Lemma 5.8 we obtain a closed immersion

$$\mathrm{Quot}^P_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}^P_{\mathcal{O}_X^{\oplus r}/X/B}$$

Since we've shown that $\mathrm{Quot}^P_{\mathcal{O}_X^{\oplus r}/X/B} \to B$ is proper in Lemma 6.1 we conclude. $\square$

Lemma 6.3. *Let $f : X \to B$ be a proper morphism of finite presentation of algebraic spaces. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 14.1. The algebraic space $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ parametrizing quotients of $\mathcal{F}$ having Hilbert polynomial P with respect to $\mathcal{L}$ is proper over B.*

Proof. The question is étale local over B, see Morphisms of Spaces, Lemma 40.2. Thus we may assume B is an affine scheme. Then we can find a closed immersion $i : X \to \mathbf{P}^n_B$ such that $i^*\mathcal{O}_{\mathbf{P}^n_B}(1) \cong \mathcal{L}^{\otimes d}$ for some $d \geq 1$. See Morphisms, Lemma 39.3. Changing $\mathcal{L}$ into $\mathcal{L}^{\otimes d}$ and the numerical polynomial $P(t)$ into $P(dt)$ leaves $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ unaffected; some details omitted. Hence we may assume $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_B}(1)$. Then the isomorphism $\mathrm{Quot}_{\mathcal{F}/X/B} \to \mathrm{Quot}_{i_*\mathcal{F}/\mathbf{P}^n_B/B}$ of Lemma 5.7 induces an isomorphism $\mathrm{Quot}^P_{\mathcal{F}/X/B} \cong \mathrm{Quot}^P_{i_*\mathcal{F}/\mathbf{P}^n_B/B}$. Since $\mathrm{Quot}^P_{i_*\mathcal{F}/\mathbf{P}^n_B/B}$ is proper over B by Lemma 6.2 we conclude. $\square$

Lemma 6.4. *Let $f : X \to B$ be a separated morphism of finite presentation of algebraic spaces. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 14.1. The algebraic space $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ parametrizing quotients of $\mathcal{F}$ having Hilbert polynomial P with respect to $\mathcal{L}$ is separated of finite presentation over B.*

Proof. We have already seen that $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is separated and locally of finite presentation, see Lemma 5.2. Thus it suffices to show that the open subspace $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ of Remark 5.9 is quasi-compact over B.

The question is étale local on B (Morphisms of Spaces, Lemma 8.8). Thus we may assume B is affine.

Assume $B = \mathrm{Spec}(\Lambda)$. Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as the colimit of its finite type $\mathbf{Z}$-subalgebras. Then we can find an i and a system $X_i, \mathcal{F}_i, \mathcal{L}_i$ as in the lemma over $B_i = \mathrm{Spec}(\Lambda_i)$ whose base change to B gives $X, \mathcal{F}, \mathcal{L}$. This follows from Limits of Spaces, Lemmas 7.1 (to find X_i), 7.2 (to find $\mathcal{F}_i$), 7.3 (to find $\mathcal{L}_i$), and 5.9 (to make X_i separated). Because

$$\mathrm{Quot}_{\mathcal{F}/X/B} = B \times_{B_i} \mathrm{Quot}_{\mathcal{F}_i/X_i/B_i}$$

and similarly for $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ we reduce to the case discussed in the next paragraph.

Assume B is affine and Noetherian. We may replace $\mathcal{L}$ by a positive power, see Lemma 5.11. Thus we may assume there exists an immersion $i : X \to \mathbf{P}^n_B$ such that $i^*\mathcal{O}_{\mathbf{P}^n}(1) = \mathcal{L}$. By Morphisms, Lemma 7.7 there exists a closed subscheme $X' \subset \mathbf{P}^n_B$ such that i factors through an open immersion $j : X \to X'$. By Properties, Lemma 22.5 there exists a finitely presented $\mathcal{O}_{X'}$-module $\mathcal{G}$ such that $j^*\mathcal{G} = \mathcal{F}$. Thus we obtain an open immersion

$$\mathrm{Quot}_{\mathcal{F}/X/B} \longrightarrow \mathrm{Quot}_{\mathcal{G}/X'/B}$$

by Lemma 5.6. Clearly this open immersion sends $\mathrm{Quot}^P_{\mathcal{F}/X/B}$ into $\mathrm{Quot}^P_{\mathcal{G}/X'/B}$. Now $\mathrm{Quot}^P_{\mathcal{G}/X'/B}$ is proper over B by Lemma 6.3. Therefore it is Noetherian and since any open of a Noetherian algebraic space is quasi-compact we win. $\square$

7. Properties of the Hilbert functor

Let $f : X \to B$ be a morphism of algebraic spaces which is separated and of finite presentation. Then $\mathrm{Hilb}_{X/B}$ is an algebraic space locally of finite presentation over B. See Quot, Proposition 9.4.

Lemma 7.1. *The diagonal of $\mathrm{Hilb}_{X/B} \to B$ is a closed immersion of finite presentation.*

Proof. In Quot, Lemma 9.2 we have seen that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$. Hence this follows from Lemma 5.1. $\square$

Lemma 7.2. *The morphism $\mathrm{Hilb}_{X/B} \to B$ is separated and locally of finite presentation.*

Proof. To check $\mathrm{Hilb}_{X/B} \to B$ is separated we have to show that its diagonal is a closed immersion. This is true by Lemma 7.1. The second statement is part of Quot, Proposition 9.4. $\square$

Lemma 7.3. *Assume $X \to B$ is proper as well as of finite presentation. Then $\mathrm{Hilb}_{X/B} \to B$ satisfies the existence part of the valuative criterion (Morphisms of Spaces, Definition 41.1).*

Proof. In Quot, Lemma 9.2 we have seen that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$. Hence this follows from Lemma 5.3. $\square$

Lemma 7.4. *Let B be an algebraic space. Let $\pi : X \to Y$ be an open immersion of algebraic spaces which are separated and of finite presentation over B. Then π induces an open immersion $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$.*

Proof. Omitted. Hint: If $Z \subset X_T$ is a closed subscheme which is proper over T, then Z is also closed in Y_T. Thus we obtain the transformation $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$. If $Z \subset Y_T$ is an element of $\mathrm{Hilb}_{Y/B}(T)$ and for $t \in T$ we have $|Z_t| \subset |X_t|$, then the same is true for $t' \in T$ in a neighbourhood of t. $\square$

Lemma 7.5. *Let B be an algebraic space. Let $\pi : X \to Y$ be a closed immersion of algebraic spaces which are separated and of finite presentation over B. Then π induces a closed immersion $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$.*

Proof. Since π is a closed immersion, it is immediate that given a closed subscheme $Z \subset X_T$, we can view Z as a closed subscheme of X_T. Thus we obtain the transformation $\mathrm{Hilb}_{X/B} \to \mathrm{Hilb}_{Y/B}$. This transformation is immediately seen to be a monomorphism. To prove that it is a closed immersion, you can use Lemma 5.8 for the map $\mathcal{O}_Y \to \mathcal{O}_X$ and the identifications $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, $\mathrm{Hilb}_{Y/B} = \mathrm{Quot}_{\mathcal{O}_Y/Y/B}$ of Quot, Lemma 9.2. $\square$

Remark 7.6 (Numerical invariants)**.** Let $f : X \to B$ be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_X)$ be perfect. Let $P : I \to \mathbf{Z}$ be a function. Recall that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, see Quot, Lemma 9.2. Thus we can define

$$\mathrm{Hilb}^P_{X/B} = \mathrm{Quot}^P_{\mathcal{O}_X/X/B}$$

where $\mathrm{Quot}^P_{\mathcal{O}_X/X/B}$ is as in Remark 5.9. The morphism

$$\mathrm{Hilb}^P_{X/B} \longrightarrow \mathrm{Hilb}_{X/B}$$

is a flat closed immersion which is an open and closed immersion for example if I is finite, or B is locally Noetherian, or $I = \mathbf{Z}$ and $E_i = \mathcal{L}^{\otimes i}$ for some invertible $\mathcal{O}_X$-module $\mathcal{L}$. In the last case we sometimes use the notation $\mathrm{Hilb}^{P,\mathcal{L}}_{X/B}$.

Lemma 7.7. *Let $f : X \to B$ be a proper morphism of finite presentation of algebraic spaces. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 14.1. The algebraic space $\mathrm{Hilb}^P_{X/B}$ parametrizing closed subschemes having Hilbert polynomial P with respect to $\mathcal{L}$ is proper over B.*

Proof. Recall that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, see Quot, Lemma 9.2. Thus this lemma is an immediate consequence of Lemma 6.3. $\square$

Lemma 7.8. *Let $f : X \to B$ be a separated morphism of finite presentation of algebraic spaces. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module ample on X/B, see Divisors on Spaces, Definition 14.1. The algebraic space $\mathrm{Hilb}^P_{X/B}$ parametrizing closed subschemes having Hilbert polynomial P with respect to $\mathcal{L}$ is separated of finite presentation over B.*

Proof. Recall that $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$, see Quot, Lemma 9.2. Thus this lemma is an immediate consequence of Lemma 6.4. $\square$

8. Properties of the Picard stack

Let $f : X \to B$ be a morphism of algebraic spaces which is flat, proper, and of finite presentation. Then the stack $\mathcal{P}ic_{X/B}$ parametrizing invertible sheaves on X/B is algebraic, see Quot, Proposition 10.2.

Lemma 8.1. *The diagonal of $\mathcal{P}ic_{X/B}$ over B is affine and of finite presentation.*

Proof. In Quot, Lemma 10.1 we have seen that $\mathcal{P}ic_{X/B}$ is an open substack of $\mathcal{C}oh_{X/B}$. Hence this follows from Lemma 4.1. $\square$

Lemma 8.2. *The morphism $\mathcal{P}ic_{X/B} \to B$ is quasi-separated and locally of finite presentation.*

Proof. In Quot, Lemma 10.1 we have seen that $\mathcal{P}ic_{X/B}$ is an open substack of $\mathcal{C}oh_{X/B}$. Hence this follows from Lemma 4.2. $\square$

Lemma 8.3. *Assume $X \to B$ is smooth in addition to being proper. Then $\mathcal{P}ic_{X/B} \to B$ satisfies the existence part of the valuative criterion (Morphisms of Stacks, Definition 39.10).*

Proof. Taking base change, this immediately reduces to the following problem: given a valuation ring R with fraction field K and an algebraic space X proper and smooth over R and an invertible $\mathcal{O}_{X_K}$-module $\mathcal{L}_K$, show there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ whose generic fibre is $\mathcal{L}_K$. Observe that X_K is Noetherian, separated, and regular (use Morphisms of Spaces, Lemma 28.6 and Spaces over Fields, Lemma 16.1). Thus we can write $\mathcal{L}_K$ as the difference in the Picard group of $\mathcal{O}_{X_K}(D_K)$ and $\mathcal{O}_{X_K}(D'_K)$ for two effective Cartier divisors D_K, D'_K in X_K, see Divisors on Spaces, Lemma 8.4. Finally, we know that D_K and D'_K are restrictions of effective Cartier divisors $D, D' \subset X$, see Divisors on Spaces, Lemma 8.5. $\square$

Lemma 8.4. *Assume $f_{T,*}\mathcal{O}_{X_T} \cong \mathcal{O}_T$ for all schemes T over B. Then the inertia stack of $\mathcal{P}ic_{X/B}$ is equal to $\mathbf{G}_m \times \mathcal{P}ic_{X/B}$.*

Proof. This is explained in Examples of Stacks, Example 17.2. $\square$

Lemma 8.5. *Assume $f : X \to B$ has relative dimension ≤ 1 in addition to the other assumptions in this section. Then $\mathcal{P}ic_{X/B} \to B$ is smooth.*

Proof. We already know that $\mathcal{P}ic_{X/B} \to B$ is locally of finite presentation, see Lemma 8.2. Thus it suffices to show that $\mathcal{P}ic_{X/B} \to B$ is formally smooth, see More on Morphisms of Stacks, Lemma 8.7. Taking base change, this immediately reduces to the following problem: given a first order thickening $T \subset T'$ of affine schemes, given $X' \to T'$ proper, flat, of finite presentation and of relative dimension ≤ 1, and for $X = T \times_{T'} X'$ given an invertible $\mathcal{O}_X$-module $\mathcal{L}$, prove that there exists an invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ whose restriction to X is $\mathcal{L}$. Since $T \subset T'$ is a first order thickening, the same is true for $X \subset X'$, see More on Morphisms of Spaces, Lemma 9.8. By More on Morphisms of Spaces, Lemma 11.1 we see that it suffices to show $H^2(X, \mathcal{I}) = 0$ where $\mathcal{I}$ is the quasi-coherent ideal cutting out X in X'. Denote $f : X \to T$ the structure morphism. By Cohomology of Spaces, Lemma 22.9 we see that $R^pf_*\mathcal{I} = 0$ for $p > 1$. Hence we get the desired vanishing by Cohomology of Spaces, Lemma 3.2 (here we finally use that T is affine). $\square$

9. Properties of the Picard functor

Let $f : X \to B$ be a morphism of algebraic spaces which is flat, proper, and of finite presentation such that moreover for every T/B the canonical map

$$\mathcal{O}_T \longrightarrow f_{T,*}\mathcal{O}_{X_T}$$

is an isomorphism. Then the Picard functor $\mathrm{Pic}_{X/B}$ is an algebraic space, see Quot, Proposition 11.8. There is a closed relationship with the Picard stack.

Lemma 9.1. *The morphism $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ turns the Picard stack into a gerbe over the Picard functor.*

Proof. The definition of $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ being a gerbe is given in Morphisms of Stacks, Definition 28.1, which in turn refers to Stacks, Definition 11.4. To prove it, we will check conditions (2)(a) and (2)(b) of Stacks, Lemma 11.3. This follows immediately from Quot, Lemma 11.2; here is a detailed explanation.

Condition (2)(a). Suppose that $\xi \in \mathrm{Pic}_{X/B}(U)$ for some scheme U over B. Since $\mathrm{Pic}_{X/B}$ is the fppf sheafification of the rule $T \mapsto \mathrm{Pic}(X_T)$ on schemes over B (Quot, Situation 11.1), we see that there exists an fppf covering $\{U_i \to U\}$ such that $\xi|_{U_i}$ corresponds to some invertible module $\mathcal{L}_i$ on X_{U_i}. Then $(U_i \to B, \mathcal{L}_i)$ is an object of $\mathcal{P}ic_{X/B}$ over U_i mapping to $\xi|_{U_i}$.

Condition (2)(b). Suppose that U is a scheme over B and $\mathcal{L}, \mathcal{N}$ are invertible modules on X_U which map to the same element of $\mathrm{Pic}_{X/B}(U)$. Then there exists an fppf covering $\{U_i \to U\}$ such that $\mathcal{L}|_{X_{U_i}}$ is isomorphic to $\mathcal{N}|_{X_{U_i}}$. Thus we find isomorphisms between $(U \to B, \mathcal{L})|_{U_i} \to (U \to B, \mathcal{N})|_{U_i}$ as desired. $\square$

Lemma 9.2. *The diagonal of $\mathrm{Pic}_{X/B}$ over B is a quasi-compact immersion.*

Proof. The diagonal is an immersion by Quot, Lemma 11.9. To finish we show that the diagonal is quasi-compact. The diagonal of $\mathcal{P}ic_{X/B}$ is quasi-compact by Lemma 8.1 and $\mathcal{P}ic_{X/B}$ is a gerbe over $\mathrm{Pic}_{X/B}$ by Lemma 9.1. We conclude by Morphisms of Stacks, Lemma 28.14. $\square$

Lemma 9.3. *The morphism $\mathrm{Pic}_{X/B} \to B$ is quasi-separated and locally of finite presentation.*

Proof. To check $\mathrm{Pic}_{X/B} \to B$ is quasi-separated we have to show that its diagonal is quasi-compact. This is immediate from Lemma 9.2. Since the morphism $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ is surjective, flat, and locally of finite presentation (by Lemma 9.1 and Morphisms of Stacks, Lemma 28.8) it suffices to prove that $\mathcal{P}ic_{X/B} \to B$ is locally of finite presentation, see Morphisms of Stacks, Lemma 27.12. This follows from Lemma 8.2. $\square$

Lemma 9.4. *Assume the geometric fibres of $X \to B$ are integral in addition to the other assumptions in this section. Then $\mathrm{Pic}_{X/B} \to B$ is separated.*

Proof. Since $\mathrm{Pic}_{X/B} \to B$ is quasi-separated, it suffices to check the uniqueness part of the valuative criterion, see Morphisms of Spaces, Lemma 43.2. This immediately reduces to the following problem: given

(1) a valuation ring R with fraction field K,
(2) an algebraic space X proper and flat over R with integral geometric fibre,
(3) an element $a \in \mathrm{Pic}_{X/R}(R)$ with $a|_{\mathrm{Spec}(K)} = 0$,

then we have to prove $a = 0$. Applying Morphisms of Stacks, Lemma 25.6 to the surjective flat morphism $\mathcal{P}ic_{X/R} \to \mathrm{Pic}_{X/R}$ (surjective and flat by Lemma 9.1 and Morphisms of Stacks, Lemma 28.8) after replacing R by an extension we may assume a is given by an invertible $\mathcal{O}_X$-module $\mathcal{L}$. Since $a|_{\mathrm{Spec}(K)} = 0$ we find $\mathcal{L}_K \cong \mathcal{O}_{X_K}$ by Quot, Lemma 11.3.

Denote $f : X \to \mathrm{Spec}(R)$ the structure morphism. Let $\eta, 0 \in \mathrm{Spec}(R)$ be the generic and closed point. Consider the perfect complexes $K = Rf_*\mathcal{L}$ and $M = Rf_*(\mathcal{L}^{\otimes -1})$ on $\mathrm{Spec}(R)$, see Derived Categories of Spaces, Lemma 25.4. Consider the functions $\beta_{K,i}, \beta_{M,i} : \mathrm{Spec}(R) \to \mathbf{Z}$ of Derived Categories of Spaces, Lemma 26.1 associated to K and M. Since the formation of K amd M commutes with base change (see lemma cited above) we find $\beta_{K,0}(\eta) = \beta_{M,0}(\beta) = 1$ by Spaces over Fields, Lemma 14.3 and our assumption on the fibres of f. By upper semi-continuity we find $\beta_{K,0}(0) \geq 1$ and $\beta_{M,0} \geq 1$. By Spaces over Fields, Lemma 14.4 we conclude that the restriction of $\mathcal{L}$ to the special fibre X_0 is trivial. In turn this gives $\beta_{K,0}(0) = \beta_{M,0} = 1$ as above. Then by More on Algebra, Lemma 75.5 we can represent K by a complex of the form

$$\ldots \to 0 \to R \to R^{\oplus \beta_{K,1}(0)} \to R^{\oplus \beta_{K,2}(0)} \to \ldots$$

Now $R \to R^{\oplus \beta_{K,1}(0)}$ is zero because $\beta_{K,0}(\eta) = 1$. In other words $K = R \oplus \tau_{\geq 1}(K)$ in $D(R)$ where $\tau_{\geq 1}(K)$ has tor amplitude in $[1, b]$ for some $b \in \mathbf{Z}$. Hence there is a global section $s \in H^0(X, \mathcal{L})$ whose restriction s_0 to X_0 is nonvanishing (again because formation of K commutes with base change). Then $s : \mathcal{O}_X \to \mathcal{L}$ is a map of invertible sheaves whose restriction to X_0 is an isomorphism and hence is an isomorphism as desired. □

Lemma 9.5. *Assume $f : X \to B$ has relative dimension ≤ 1 in addition to the other assumptions in this section. Then $\mathrm{Pic}_{X/B} \to B$ is smooth.*

Proof. By Lemma 8.5 we know that $\mathcal{P}ic_{X/B} \to B$ is smooth. The morphism $\mathcal{P}ic_{X/B} \to \mathrm{Pic}_{X/B}$ is surjective and smooth by combining Lemma 9.1 with Morphisms of Stacks, Lemma 33.8. Thus if U is a scheme and $U \to \mathcal{P}ic_{X/B}$ is surjective and smooth, then $U \to \mathrm{Pic}_{X/B}$ is surjective and smooth and $U \to B$ is surjective and smooth (because these properties are preserved by composition). Thus $\mathrm{Pic}_{X/B} \to B$ is smooth for example by Descent on Spaces, Lemma 8.3. □

10. Properties of relative morphisms

Let B be an algebraic space. Let X and Y be algebraic spaces over B such that $Y \to B$ is flat, proper, and of finite presentation and $X \to B$ is separated and of finite presentation. Then the functor $Mor_B(Y, X)$ of relative morphisms is an algebraic space locally of finite presentation over B. See Quot, Proposition 12.3.

Lemma 10.1. *The diagonal of $Mor_B(Y, X) \to B$ is a closed immersion of finite presentation.*

Proof. There is an open immersion $Mor_B(Y, X) \to \mathrm{Hilb}_{Y\times_B X/B}$, see Quot, Lemma 12.2. Thus the lemma follows from Lemma 7.1. $\square$

Lemma 10.2. *The morphism $Mor_B(Y, X) \to B$ is separated and locally of finite presentation.*

Proof. To check $Mor_B(Y, X) \to B$ is separated we have to show that its diagonal is a closed immersion. This is true by Lemma 10.1. The second statement is part of Quot, Proposition 12.3. $\square$

Lemma 10.3. *With B, X, Y as in the introduction of this section, in addition assume $X \to B$ is proper. Then the subfunctor $Isom_B(Y, X) \subset Mor_B(Y, X)$ of isomorphisms is an open subspace.*

Proof. Follows immediately from More on Morphisms of Spaces, Lemma 49.6. $\square$

Remark 10.4 (Numerical invariants)**.** Let B, X, Y be as in the introduction to this section. Let I be a set and for $i \in I$ let $E_i \in D(\mathcal{O}_{Y\times_B X})$ be perfect. Let $P : I \to \mathbf{Z}$ be a function. Recall that

$$Mor_B(Y, X) \subset \mathrm{Hilb}_{Y\times_B X/B}$$

is an open subspace, see Quot, Lemma 12.2. Thus we can define

$$Mor^P_B(Y, X) = Mor_B(Y, X) \cap \mathrm{Hilb}^P_{Y\times_B X/B}$$

where $\mathrm{Hilb}^P_{Y\times_B X/B}$ is as in Remark 7.6. The morphism

$$Mor^P_B(Y, X) \longrightarrow Mor_B(Y, X)$$

is a flat closed immersion which is an open and closed immersion for example if I is finite, or B is locally Noetherian, or $I = \mathbf{Z}$, $E_i = \mathcal{L}^{\otimes i}$ for some invertible $\mathcal{O}_{Y\times_B X}$-module $\mathcal{L}$. In the last case we sometimes use the notation $Mor^{P,\mathcal{L}}_B(Y, X)$.

Lemma 10.5. *With B, X, Y as in the introduction of this section, let $\mathcal{L}$ be ample on X/B and let $\mathcal{N}$ be ample on Y/B. See Divisors on Spaces, Definition 14.1. Let P be a numerical polynomial. Then*

$$Mor^{P,\mathcal{M}}_B(Y, X) \longrightarrow B$$

is separated and of finite presentation where $\mathcal{M} = pr_1^*\mathcal{N} \otimes_{\mathcal{O}_{Y\times_B X}} pr_2^*\mathcal{L}$.

Proof. By Lemma 10.2 the morphism $Mor_B(Y, X) \to B$ is separated and locally of finite presentation. Thus it suffices to show that the open and closed subspace $Mor^{P,\mathcal{M}}_B(Y, X)$ of Remark 10.4 is quasi-compact over B.

The question is étale local on B (Morphisms of Spaces, Lemma 8.8). Thus we may assume B is affine.

Assume $B = \mathrm{Spec}(\Lambda)$. Note that X and Y are schemes and that $\mathcal{L}$ and $\mathcal{N}$ are ample invertible sheaves on X and Y (this follows immediately from the definitions). Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as the colimit of its finite type $\mathbf{Z}$-subalgebras. Then we can find an i and a system $X_i, Y_i, \mathcal{L}_i, \mathcal{N}_i$ as in the lemma over $B_i = \mathrm{Spec}(\Lambda_i)$ whose base change to B gives $X, Y, \mathcal{L}, \mathcal{N}$. This follows from Limits, Lemmas 10.1 (to find X_i, Y_i), 10.3 (to find $\mathcal{L}_i$, $\mathcal{N}_i$), 8.6 (to make $X_i \to B_i$ separated), 13.1 (to make $Y_i \to B_i$ proper), and 4.15 (to make $\mathcal{L}_i$, $\mathcal{N}_i$ ample). Because

$$Mor_B(Y, X) = B \times_{B_i} Mor_{B_i}(Y_i, X_i)$$

and similarly for $Mor^P_B(Y, X)$ we reduce to the case discussed in the next paragraph.

Assume B is a Noetherian affine scheme. By Properties, Lemma 26.15 we see that $\mathcal{M}$ is ample. By Lemma 7.8 we see that $\mathrm{Hilb}^{P,\mathcal{M}}_{Y\times_B X/B}$ is of finite presentation over B and hence Noetherian. By construction

$$Mor^{P,\mathcal{M}}_B(Y, X) = Mor_B(Y, X) \cap \mathrm{Hilb}^{P,\mathcal{M}}_{Y\times_B X/B}$$

is an open subspace of $\mathrm{Hilb}^{P,\mathcal{M}}_{Y\times_B X/B}$ and hence quasi-compact (as an open of a Noetherian algebraic space is quasi-compact). $\square$

11. Properties of the stack of polarized proper schemes

In this section we discuss properties of the moduli stack

$$\mathcal{P}olarized \longrightarrow \mathrm{Spec}(\mathbf{Z})$$

whose category of sections over a scheme S is the category of proper, flat, finitely presented scheme over S endowed with a relatively ample invertible sheaf. This is an algebraic stack by Quot, Theorem 14.15.

Lemma 11.1. *The diagonal of $\mathcal{P}olarized$ is separated and of finite presentation.*

Proof. Recall that $\mathcal{P}olarized$ is a limit preserving algebraic stack, see Quot, Lemma 14.8. By Limits of Stacks, Lemma 3.6 this implies that $\Delta : \mathcal{P}olarized \to \mathcal{P}olarized \times \mathcal{P}olarized$ is limit preserving. Hence Δ is locally of finite presentation by Limits of Stacks, Proposition 3.8.

Let us prove that Δ is separated. To see this, it suffices to show that given an affine scheme U and two objects $\upsilon = (Y, \mathcal{N})$ and $\chi = (X, \mathcal{L})$ of $\mathcal{P}olarized$ over U, the algebraic space

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi)$$

is separated. The rule which to an isomorphism $\upsilon_T \to \chi_T$ assigns the underlying isomorphism $Y_T \to X_T$ defines a morphism

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi) \longrightarrow Isom_U(Y, X)$$

Since we have seen in Lemmas 10.2 and 10.3 that the target is a separated algebraic space, it suffices to prove that this morphism is separated. Given an isomorphism $f : Y_T \to X_T$ over some scheme T/U, then clearly

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi) \times_{Isom_U(Y,X),[f]} T = Isom(\mathcal{N}_T, f^*\mathcal{L}_T)$$

Here $[f] : T \to Isom_U(Y, X)$ indicates the T-valued point corresponding to f and $Isom(\mathcal{N}_T, f^*\mathcal{L}_T)$ is the algebraic space discussed in Section 3. Since this algebraic space is affine over U, the claim implies Δ is separated.

To finish the proof we show that Δ is quasi-compact. Since Δ is representable by algebraic spaces, it suffice to check the base change of Δ by a surjective smooth morphism $U \to \mathcal{P}olarized \times \mathcal{P}olarized$ is quasi-compact (see for example Properties of Stacks, Lemma 3.3). We can assume $U = \coprod U_i$ is a disjoint union of affine opens. Since $\mathcal{P}olarized$ is limit preserving (see above), we see that $\mathcal{P}olarized \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, hence $U_i \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation (Limits of Stacks, Proposition 3.8 and Morphisms of Stacks, Lemmas 27.2 and 33.5). In particular, U_i is Noetherian affine. This reduces us to the case discussed in the next paragraph.

In this paragraph, given a Noetherian affine scheme U and two objects $\upsilon = (Y, \mathcal{N})$ and $\chi = (X, \mathcal{L})$ of *Polarized* over U, we show the algebraic space

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi)$$

is quasi-compact. Since the connected components of U are open and closed we may replace U by these. Thus we may and do assume U is connected. Let $u \in U$ be a point. Let P be the Hilbert polynomial $n \mapsto \chi(Y_u, \mathcal{N}_u^{\otimes n})$, see Varieties, Lemma 45.1. Since U is connected and since the functions $u \mapsto \chi(Y_u, \mathcal{N}_u^{\otimes n})$ are locally constant (see Derived Categories of Schemes, Lemma 32.2) we see that we get the same Hilbert polynomial in every point of U. Set $\mathcal{M} = \text{pr}_1^*\mathcal{N} \otimes_{\mathcal{O}_{Y \times_U X}} \text{pr}_2^*\mathcal{L}$ on $Y \times_U X$. Given $(f, \varphi) \in Isom_{\mathcal{P}olarized}(\upsilon, \chi)(T)$ for some scheme T over U then for every $t \in T$ we have

$$\chi(Y_t, (\text{id} \times f)^*\mathcal{M}^{\otimes n}) = \chi(Y_t, \mathcal{N}_t^{\otimes n} \otimes_{\mathcal{O}_{Y_t}} f_t^*\mathcal{L}_t^{\otimes n}) = \chi(Y_t, \mathcal{N}_t^{\otimes 2n}) = P(2n)$$

where in the middle equality we use the isomorphism $\varphi : f^*\mathcal{L}_T \to \mathcal{N}_T$. Setting $P'(t) = P(2t)$ we find that the morphism

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi) \longrightarrow Isom_U(Y, X)$$

(see earlier) has image contained in the intersection

$$Isom_U(Y, X) \cap Mor_U^{P', \mathcal{M}}(Y, X)$$

The intersection is an intersection of open subspaces of $Mor_U(Y, X)$ (see Lemma 10.3 and Remark 10.4). Now $Mor_U^{P', \mathcal{M}}(Y, X)$ is a Noetherian algebraic space as it is of finite presentation over U by Lemma 10.5. Thus the intersection is a Noetherian algebraic space too. Since the morphism

$$Isom_{\mathcal{P}olarized}(\upsilon, \chi) \longrightarrow Isom_U(Y, X) \cap Mor_U^{P', \mathcal{M}}(Y, X)$$

is affine (see above) we conclude. $\square$

Lemma 11.2. *The morphism $\mathcal{P}olarized \to \text{Spec}(\mathbf{Z})$ is quasi-separated and locally of finite presentation.*

Proof. To check $\mathcal{P}olarized \to \text{Spec}(\mathbf{Z})$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 11.1. To prove that $\mathcal{P}olarized \to \text{Spec}(\mathbf{Z})$ is locally of finite presentation, it suffices to show that *Polarized* is limit preserving, see Limits of Stacks, Proposition 3.8. This is Quot, Lemma 14.8. $\square$

Lemma 11.3. *Let $n \geq 1$ be an integer and let P be a numerical polynomial. Let*

$$T \subset |\mathcal{P}olarized|$$

be a subset with the following property: for every $\xi \in T$ there exists a field k and an object $(X, \mathcal{L})$ of Polarized over k representing ξ such that

(1) *the Hilbert polynomial of $\mathcal{L}$ on X is P, and*
(2) *there exists a closed immersion $i : X \to \mathbf{P}^n_k$ such that $i^*\mathcal{O}_{\mathbf{P}^n}(1) \cong \mathcal{L}$.*

Then T is a Noetherian topological space, in particular quasi-compact.

Proof. Observe that $|\mathcal{P}olarized|$ is a locally Noetherian topological space, see Morphisms of Stacks, Lemma 8.3 (this also uses that $\text{Spec}(\mathbf{Z})$ is Noetherian and hence *Polarized* is a locally Noetherian algebraic stack by Lemma 11.2 and Morphisms of Stacks, Lemma 17.5). Thus any quasi-compact subset of $|\mathcal{P}olarized|$ is a Noetherian topological space and any subset of such is also Noetherian, see Topology, Lemmas 9.4 and 9.2. Thus all we have to do is a find a quasi-compact subset containing T.

By Lemma 7.7 the algebraic space

$$H = \mathrm{Hilb}^{P, \mathcal{O}(1)}_{\mathbf{P}^n_{\mathbf{Z}}/\mathrm{Spec}(\mathbf{Z})}$$

is proper over $\mathrm{Spec}(\mathbf{Z})$. By Quot, Lemma 9.3[1] the identity morphism of H corresponds to a closed subspace

$$Z \subset \mathbf{P}^n_H$$

which is proper, flat, and of finite presentation over H and such that the restriction $\mathcal{N} = \mathcal{O}(1)|_Z$ is relatively ample on Z/H and has Hilbert polynomial P on the fibres of $Z \to H$. In particular, the pair $(Z \to H, \mathcal{N})$ defines a morphism

$$H \longrightarrow \mathcal{P}\mathit{olarized}$$

which sends a morphism of schemes $U \to H$ to the classifying morphism of the family $(Z_U \to U, \mathcal{N}_U)$, see Quot, Lemma 14.4. Since H is a Noetherian algebraic space (as it is proper over $\mathbf{Z}$)) we see that $|H|$ is Noetherian and hence quasi-compact. The map

$$|H| \longrightarrow |\mathcal{P}\mathit{olarized}|$$

is continuous, hence the image is quasi-compact. Thus it suffices to prove T is contained in the image of $|H| \to |\mathcal{P}\mathit{olarized}|$. However, assumptions (1) and (2) exactly express the fact that this is the case: any choice of a closed immersion $i : X \to \mathbf{P}^n_k$ with $i^*\mathcal{O}_{\mathbf{P}^n}(1) \cong \mathcal{L}$ we get a k-valued point of H by the moduli interpretation of H. This finishes the proof of the lemma. $\square$

12. Properties of moduli of complexes on a proper morphism

Let $f : X \to B$ be a morphism of algebraic spaces which is proper, flat, and of finite presentation. Then the stack $\mathcal{C}\mathit{omplexes}_{X/B}$ parametrizing relatively perfect complexes with vanishing negative self-exts is algebraic. See Quot, Theorem 16.12.

Lemma 12.1. *The diagonal of $\mathcal{C}\mathit{omplexes}_{X/B}$ over B is affine and of finite presentation.*

Proof. The representability of the diagonal by algebraic spaces was shown in Quot, Lemma 16.5. From the proof we find that we have to show: given a scheme T over B and objects $E, E' \in D(\mathcal{O}_{X_T})$ such that (T, E) and (T, E') are objects of the fibre category of $\mathcal{C}\mathit{omplexes}_{X/B}$ over T, then $\mathit{Isom}(E, E') \to T$ is affine and of finite presentation. Here $\mathit{Isom}(E, E')$ is the functor

$$(\mathit{Sch}/T)^{opp} \to \mathit{Sets}, \quad T' \mapsto \{\varphi : E_{T'} \to E'_{T'} \text{ isomorphism in } D(\mathcal{O}_{X_{T'}})\}$$

where $E_{T'}$ and $E'_{T'}$ are the derived pullbacks of E and E' to $X_{T'}$. Consider the functor $H = \mathcal{H}\mathit{om}(E, E')$ defined by the rule

$$(\mathit{Sch}/T)^{opp} \to \mathit{Sets}, \quad T' \mapsto \mathrm{Hom}_{\mathcal{O}_{X_{T'}}}(E_T, E'_T)$$

By Quot, Lemma 16.1 this is an algebraic space affine and of finite presentation over T. The same is true for $H' = \mathcal{H}\mathit{om}(E', E)$, $I = \mathcal{H}\mathit{om}(E, E)$, and $I' = \mathcal{H}\mathit{om}(E', E')$. Therefore we see that

$$\mathit{Isom}(E, E') = (H' \times_T H) \times_{c, I \times_T I', \sigma} T$$

[1]We will see later (insert future reference here) that H is a scheme and hence the use of this lemma and Quot, Lemma 14.4 isn't necessary.

where $c(\varphi', \varphi) = (\varphi \circ \varphi', \varphi' \circ \varphi)$ and $\sigma = (\mathrm{id}, \mathrm{id})$ (compare with the proof of Quot, Proposition 4.3). Thus $\mathit{Isom}(E, E')$ is affine over T as a fibre product of schemes affine over T. Similarly, $\mathit{Isom}(E, E')$ is of finite presentation over T. $\square$

Lemma 12.2. *The morphism $\mathcal{C}omplexes_{X/B} \to B$ is quasi-separated and locally of finite presentation.*

Proof. To check $\mathcal{C}omplexes_{X/B} \to B$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 12.1. To prove that $\mathcal{C}omplexes_{X/B} \to B$ is locally of finite presentation, we have to show that $\mathcal{C}omplexes_{X/B} \to B$ is limit preserving, see Limits of Stacks, Proposition 3.8. This follows from Quot, Lemma 16.8 (small detail omitted). $\square$

13. Other chapters

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