

DIVISORS ON ALGEBRAIC SPACES

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1. Introduction

In this chapter we study divisors on algebraic spaces and related topics. A basic reference for algebraic spaces is [Knu71].

2. Associated and weakly associated points

In the case of schemes we have introduced two competing notions of associated points. Namely, the usual associated points (Divisors, Section 2) and the weakly associated points (Divisors, Section 5). For a general algebraic space the notion of an associated point is basically useless and we don't even bother to introduce it. If the algebraic space is locally Noetherian, then we allow ourselves to use the phrase “associated point” instead of “weakly associated point” as the notions are the same for Noetherian schemes (Divisors, Lemma 5.8). Before we make our definition, we need a lemma.

Lemma 2.1. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $x \in |X|$. The following are equivalent*

- (1) for some étale morphism $f : U \rightarrow X$ with U a scheme and $u \in U$ mapping to x , the point u is weakly associated to $f^*\mathcal{F}$,
- (2) for every étale morphism $f : U \rightarrow X$ with U a scheme and $u \in U$ mapping to x , the point u is weakly associated to $f^*\mathcal{F}$,
- (3) the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ is a weakly associated prime of the stalk $\mathcal{F}_{\bar{x}}$.

If X is locally Noetherian, then these are also equivalent to

- (4) for some étale morphism $f : U \rightarrow X$ with U a scheme and $u \in U$ mapping to x , the point u is associated to $f^*\mathcal{F}$,
- (5) for every étale morphism $f : U \rightarrow X$ with U a scheme and $u \in U$ mapping to x , the point u is associated to $f^*\mathcal{F}$,
- (6) the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ is an associated prime of the stalk $\mathcal{F}_{\bar{x}}$.

Proof. Choose a scheme U with a point u and an étale morphism $f : U \rightarrow X$ mapping u to x . Lift \bar{x} to a geometric point of U over u . Recall that $\mathcal{O}_{X,\bar{x}} = \mathcal{O}_{U,u}^{sh}$ where the strict henselization is with respect to our chosen lift of \bar{x} , see Properties of Spaces, Lemma 22.1. Finally, we have

$$\mathcal{F}_{\bar{x}} = (f^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\bar{x}} = (f^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{U,u}^{sh}$$

by Properties of Spaces, Lemma 29.4. Hence the equivalence of (1), (2), and (3) follows from More on Flatness, Lemma 2.9. If X is locally Noetherian, then any U as above is locally Noetherian, hence we see that (1), resp. (2) are equivalent to (4), resp. (5) by Divisors, Lemma 5.8. On the other hand, in the locally Noetherian case the local ring $\mathcal{O}_{X,\bar{x}}$ is Noetherian too (Properties of Spaces, Lemma 24.4). Hence the equivalence of (3) and (6) by the same lemma (or by Algebra, Lemma 66.9). \square

Definition 2.2. Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a quasi-coherent sheaf on X . Let $x \in |X|$.

- (1) We say x is *weakly associated* to \mathcal{F} if the equivalent conditions (1), (2), and (3) of Lemma 2.1 are satisfied.
- (2) We denote $\text{WeakAss}(\mathcal{F})$ the set of weakly associated points of \mathcal{F} .
- (3) The *weakly associated points* of X are the weakly associated points of \mathcal{O}_X .

If X is locally Noetherian we will say x is *associated* to \mathcal{F} if and only if x is weakly associated to \mathcal{F} and we set $\text{Ass}(\mathcal{F}) = \text{WeakAss}(\mathcal{F})$. Finally (still assuming X is locally Noetherian), we will say x is an *associated point* of X if and only if x is a weakly associated point of X .

At this point we can prove the obligatory lemmas.

Lemma 2.3. Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then $\text{WeakAss}(\mathcal{F}) \subset \text{Supp}(\mathcal{F})$.

Proof. This is immediate from the definitions. The support of an abelian sheaf on X is defined in Properties of Spaces, Definition 20.3. \square

Lemma 2.4. Let S be a scheme. Let X be an algebraic space over S . Let $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ be a short exact sequence of quasi-coherent sheaves on X . Then $\text{WeakAss}(\mathcal{F}_2) \subset \text{WeakAss}(\mathcal{F}_1) \cup \text{WeakAss}(\mathcal{F}_3)$ and $\text{WeakAss}(\mathcal{F}_1) \subset \text{WeakAss}(\mathcal{F}_2)$.

Proof. For every geometric point $\bar{x} \in X$ the sequence of stalks $0 \rightarrow \mathcal{F}_{1,\bar{x}} \rightarrow \mathcal{F}_{2,\bar{x}} \rightarrow \mathcal{F}_{3,\bar{x}} \rightarrow 0$ is a short exact sequence of $\mathcal{O}_{X,\bar{x}}$ -modules. Hence the lemma follows from Algebra, Lemma 66.4. \square

Lemma 2.5. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then*

$$\mathcal{F} = (0) \Leftrightarrow \text{WeakAss}(\mathcal{F}) = \emptyset$$

Proof. Choose a scheme U and a surjective étale morphism $f : U \rightarrow X$. Then \mathcal{F} is zero if and only if $f^*\mathcal{F}$ is zero. Hence the lemma follows from the definition and the lemma in the case of schemes, see Divisors, Lemma 5.5. \square

Lemma 2.6. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $x \in |X|$. If*

- (1) $x \in \text{Supp}(\mathcal{F})$
- (2) x is a codimension 0 point of X (Properties of Spaces, Definition 10.2).

Then $x \in \text{WeakAss}(\mathcal{F})$. If \mathcal{F} is a finite type \mathcal{O}_X -module with scheme theoretic support Z (Morphisms of Spaces, Definition 15.4) and x is a codimension 0 point of Z , then $x \in \text{WeakAss}(\mathcal{F})$.

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the stalk $\mathcal{F}_{\bar{x}}$ is not zero. Hence $\text{WeakAss}(\mathcal{F}_{\bar{x}})$ is nonempty by Algebra, Lemma 66.5. On the other hand, the spectrum of $\mathcal{O}_{X,\bar{x}}$ is a singleton. Hence x is a weakly associated point of \mathcal{F} by definition. The final statement follows as $\mathcal{O}_{X,\bar{x}} \rightarrow \mathcal{O}_{Z,\bar{z}}$ is a surjection, the spectrum of $\mathcal{O}_{Z,\bar{z}}$ is a singleton, and $\mathcal{F}_{\bar{x}}$ is a nonzero module over $\mathcal{O}_{Z,\bar{z}}$. \square

Lemma 2.7. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $x \in |X|$. If*

- (1) X is decent (for example quasi-separated or locally separated),
- (2) $x \in \text{Supp}(\mathcal{F})$
- (3) x is not a specialization of another point in $\text{Supp}(\mathcal{F})$.

Then $x \in \text{WeakAss}(\mathcal{F})$.

Proof. (A quasi-separated algebraic space is decent, see Decent Spaces, Section 6. A locally separated algebraic space is decent, see Decent Spaces, Lemma 15.2.) Choose a scheme U , a point $u \in U$, and an étale morphism $f : U \rightarrow X$ mapping u to x . By Decent Spaces, Lemma 12.1 if $u' \rightsquigarrow u$ is a nontrivial specialization, then $f(u') \neq x$. Hence we see that $u \in \text{Supp}(f^*\mathcal{F})$ is not a specialization of another point of $\text{Supp}(f^*\mathcal{F})$. Hence $u \in \text{WeakAss}(f^*\mathcal{F})$ by Divisors, Lemma 2.6. \square

Lemma 2.8. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S . Let \mathcal{F} be a coherent \mathcal{O}_X -module. Then $\text{Ass}(\mathcal{F}) \cap W$ is finite for every quasi-compact open $W \subset |X|$.*

Proof. Choose a quasi-compact scheme U and an étale morphism $U \rightarrow X$ such that W is the image of $|U| \rightarrow |X|$. Then U is a Noetherian scheme and we may apply Divisors, Lemma 2.5 to conclude. \square

Lemma 2.9. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. If $U \rightarrow X$ is an étale morphism such that $\text{WeakAss}(\mathcal{F}) \subset \text{Im}(|U| \rightarrow |X|)$, then $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F})$ is injective.*

Proof. Let $s \in \Gamma(X, \mathcal{F})$ be a section which restricts to zero on U . Let $\mathcal{F}' \subset \mathcal{F}$ be the image of the map $\mathcal{O}_X \rightarrow \mathcal{F}$ defined by s . Then $\mathcal{F}'|_U = 0$. This implies that $\text{WeakAss}(\mathcal{F}') \cap \text{Im}(|U| \rightarrow |X|) = \emptyset$ (by the definition of weakly associated points). On the other hand, $\text{WeakAss}(\mathcal{F}') \subset \text{WeakAss}(\mathcal{F})$ by Lemma 2.4. We conclude $\text{WeakAss}(\mathcal{F}') = \emptyset$. Hence $\mathcal{F}' = 0$ by Lemma 2.5. \square

Lemma 2.10. *Let S be a scheme. Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $y \in |Y|$ be a point which is not in the image of $|f|$. Then y is not weakly associated to $f_*\mathcal{F}$.*

Proof. By Morphisms of Spaces, Lemma 11.2 the \mathcal{O}_Y -module $f_*\mathcal{F}$ is quasi-coherent hence the lemma makes sense. Choose an affine scheme V , a point $v \in V$, and an étale morphism $V \rightarrow Y$ mapping v to y . We may replace $f : X \rightarrow Y$, \mathcal{F} , y by $X \times_Y V \rightarrow V$, $\mathcal{F}|_{X \times_Y V}$, v . Thus we may assume Y is an affine scheme. In this case X is quasi-compact, hence we can choose an affine scheme U and a surjective étale morphism $U \rightarrow X$. Denote $g : U \rightarrow Y$ the composition. Then $f_*\mathcal{F} \subset g_*(\mathcal{F}|_U)$. By Lemma 2.4 we reduce to the case of schemes which is Divisors, Lemma 5.9. \square

Lemma 2.11. *Let S be a scheme. Let X be an algebraic space over S . Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a map of quasi-coherent \mathcal{O}_X -modules. Assume that for every $x \in |X|$ at least one of the following happens*

- (1) $\mathcal{F}_{\bar{x}} \rightarrow \mathcal{G}_{\bar{x}}$ is injective, or
- (2) $x \notin \text{WeakAss}(\mathcal{F})$.

Then φ is injective.

Proof. The assumptions imply that $\text{WeakAss}(\text{Ker}(\varphi)) = \emptyset$ and hence $\text{Ker}(\varphi) = 0$ by Lemma 2.5. \square

Lemma 2.12. *Let S be a scheme. Let X be a reduced algebraic space over S . Then the weakly associated point of X are exactly the codimension 0 points of X .*

Proof. Working étale locally this follows from Divisors, Lemma 5.12 and Properties of Spaces, Lemma 11.1. \square

3. Morphisms and weakly associated points

Lemma 3.1. *Let S be a scheme. Let $f : X \rightarrow Y$ be an affine morphism of algebraic spaces over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then we have*

$$\text{WeakAss}_Y(f_*\mathcal{F}) \subset f(\text{WeakAss}_X(\mathcal{F}))$$

Proof. Choose a scheme V and a surjective étale morphism $V \rightarrow Y$. Set $U = X \times_Y V$. Then $U \rightarrow V$ is an affine morphism of schemes. By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \rightarrow V$. This case is treated in Divisors, Lemma 6.1. \square

Lemma 3.2. *Let S be a scheme. Let $f : X \rightarrow Y$ be an affine morphism of algebraic spaces over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. If X is locally Noetherian, then we have*

$$\text{WeakAss}_Y(f_*\mathcal{F}) = f(\text{WeakAss}_X(\mathcal{F}))$$

Proof. Choose a scheme V and a surjective étale morphism $V \rightarrow Y$. Set $U = X \times_Y V$. Then $U \rightarrow V$ is an affine morphism of schemes and U is locally Noetherian. By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \rightarrow V$. This case is treated in Divisors, Lemma 6.2. \square

Lemma 3.3. *Let S be a scheme. Let $f : X \rightarrow Y$ be a finite morphism of algebraic spaces over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then $\text{WeakAss}(f_*\mathcal{F}) = f(\text{WeakAss}(\mathcal{F}))$.*

Proof. Choose a scheme V and a surjective étale morphism $V \rightarrow Y$. Set $U = X \times_Y V$. Then $U \rightarrow V$ is a finite morphism of schemes. By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \rightarrow V$. This case is treated in Divisors, Lemma 6.3. \square

Lemma 3.4. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let \mathcal{G} be a quasi-coherent \mathcal{O}_Y -module. Let $x \in |X|$ and $y = f(x) \in |Y|$. If*

- (1) $y \in \text{WeakAss}_S(\mathcal{G})$,
- (2) f is flat at x , and
- (3) *the dimension of the local ring of the fibre of f at x is zero (Morphisms of Spaces, Definition 33.1),*

then $x \in \text{WeakAss}(f^\mathcal{G})$.*

Proof. Choose a scheme V , a point $v \in V$, and an étale morphism $V \rightarrow Y$ mapping v to y . Choose a scheme U , a point $u \in U$, and an étale morphism $U \rightarrow V \times_Y X$ mapping u to a point lying over v and x . This is possible because there is a $t \in |V \times_Y X|$ mapping to (v, y) by Properties of Spaces, Lemma 4.3. By definition we see that the dimension of $\mathcal{O}_{U_v, u}$ is zero. Hence u is a generic point of the fiber U_v . By our definition of weakly associated points the problem is reduced to the morphism of schemes $U \rightarrow V$. This case is treated in Divisors, Lemma 6.4. \square

Lemma 3.5. *Let K/k be a field extension. Let X be an algebraic space over k . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $y \in X_K$ with image $x \in X$. If y is a weakly associated point of the pullback \mathcal{F}_K , then x is a weakly associated point of \mathcal{F} .*

Proof. This is the translation of Divisors, Lemma 6.5 into the language of algebraic spaces. We omit the details of the translation. \square

Lemma 3.6. *Let S be a scheme. Let $f : X \rightarrow Y$ be a finite flat morphism of algebraic spaces. Let \mathcal{G} be a quasi-coherent \mathcal{O}_Y -module. Let $x \in |X|$ be a point with image $y \in |Y|$. Then*

$$x \in \text{WeakAss}(f^*\mathcal{G}) \Leftrightarrow y \in \text{WeakAss}(\mathcal{G})$$

Proof. Follows immediately from the case of schemes (More on Flatness, Lemma 2.7) by étale localization. \square

Lemma 3.7. *Let S be a scheme. Let $f : X \rightarrow Y$ be an étale morphism of algebraic spaces. Let \mathcal{G} be a quasi-coherent \mathcal{O}_Y -module. Let $x \in |X|$ be a point with image $y \in |Y|$. Then*

$$x \in \text{WeakAss}(f^*\mathcal{G}) \Leftrightarrow y \in \text{WeakAss}(\mathcal{G})$$

Proof. This is immediate from the definition of weakly associated points and in fact the corresponding lemma for the case of schemes (More on Flatness, Lemma 2.8) is the basis for our definition. \square

4. Relative weak assassin

We need a couple of lemmas to define this gadget.

Lemma 4.1. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let $y \in |Y|$. The following are equivalent*

- (1) for some scheme V , point $v \in V$, and étale morphism $V \rightarrow Y$ mapping v to y , the algebraic space X_v is locally Noetherian,
- (2) for every scheme V , point $v \in V$, and étale morphism $V \rightarrow Y$ mapping v to y , the algebraic space X_v is locally Noetherian, and
- (3) there exists a field k and a morphism $\text{Spec}(k) \rightarrow Y$ representing y such that X_k is locally Noetherian.

If there exists a field k_0 and a monomorphism $\text{Spec}(k_0) \rightarrow Y$ representing y , then these are also equivalent to

- (4) the algebraic space X_{k_0} is locally Noetherian.

Proof. Observe that $X_v = v \times_Y X = \text{Spec}(\kappa(v)) \times_Y X$. Hence the implications (2) \Rightarrow (1) \Rightarrow (3) are clear. Assume that $\text{Spec}(k) \rightarrow Y$ is a morphism from the spectrum of a field such that X_k is locally Noetherian. Let $V \rightarrow Y$ be an étale morphism from a scheme V and let $v \in V$ a point mapping to y . Then the scheme $v \times_Y \text{Spec}(k)$ is nonempty. Choose a point $w \in v \times_Y \text{Spec}(k)$. Consider the morphisms

$$X_v \longleftarrow X_w \longrightarrow X_k$$

Since $V \rightarrow Y$ is étale and since w may be viewed as a point of $V \times_Y \text{Spec}(k)$, we see that $\kappa(w)/k$ is a finite separable extension of fields (Morphisms, Lemma 36.7). Thus $X_w \rightarrow X_k$ is a finite étale morphism as a base change of $w \rightarrow \text{Spec}(k)$. Hence X_w is locally Noetherian (Morphisms of Spaces, Lemma 23.5). The morphism $X_w \rightarrow X_v$ is a surjective, affine, flat morphism as a base change of the surjective, affine, flat morphism $w \rightarrow v$. Then the fact that X_w is locally Noetherian implies that X_v is locally Noetherian. This can be seen by picking a surjective étale morphism $U \rightarrow X$ and then using that $U_w \rightarrow U_v$ is surjective, affine, and flat. Working affine locally on the scheme U_v we conclude that U_w is locally Noetherian by Algebra, Lemma 164.1.

Finally, it suffices to prove that (3) implies (4) in case we have a monomorphism $\text{Spec}(k_0) \rightarrow Y$ in the class of y . Then $\text{Spec}(k) \rightarrow Y$ factors as $\text{Spec}(k) \rightarrow \text{Spec}(k_0) \rightarrow Y$. The argument given above then shows that X_k being locally Noetherian implies that X_{k_0} is locally Noetherian. \square

Definition 4.2. Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let $y \in |Y|$. We say *the fibre of f over y is locally Noetherian* if the equivalent conditions (1), (2), and (3) of Lemma 4.1 are satisfied. We say *the fibres of f are locally Noetherian* if this holds for every $y \in |Y|$.

Of course, the usual way to guarantee locally Noetherian fibres is to assume the morphism is locally of finite type.

Lemma 4.3. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . If f is locally of finite type, then the fibres of f are locally Noetherian.*

Proof. This follows from Morphisms of Spaces, Lemma 23.5 and the fact that the spectrum of a field is Noetherian. \square

Lemma 4.4. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let $x \in |X|$ and $y = f(x) \in |Y|$. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module.*

Consider commutative diagrams

$$\begin{array}{ccccc}
 X & \longleftarrow & X \times_Y V & \longleftarrow & X_v \\
 \downarrow & & \downarrow & & \downarrow \\
 Y & \longleftarrow & V & \longleftarrow & v
 \end{array}
 \quad
 \begin{array}{ccccc}
 X & \longleftarrow & U & \longleftarrow & U_v \\
 \downarrow & & \downarrow & & \downarrow \\
 Y & \longleftarrow & V & \longleftarrow & v
 \end{array}
 \quad
 \begin{array}{ccccc}
 x & \longleftarrow & x' & \longleftarrow & u \\
 \downarrow & & \downarrow & & \swarrow \\
 y & \longleftarrow & v & &
 \end{array}$$

where V and U are schemes, $V \rightarrow Y$ and $U \rightarrow X \times_Y V$ are étale, $v \in V$, $x' \in |X_v|$, $u \in U$ are points related as in the last diagram. Denote $\mathcal{F}|_{X_v}$ and $\mathcal{F}|_{U_v}$ the pullbacks of \mathcal{F} . The following are equivalent

- (1) for some V, v, x' as above x' is a weakly associated point of $\mathcal{F}|_{X_v}$,
- (2) for every $V \rightarrow Y, v, x'$ as above x' is a weakly associated point of $\mathcal{F}|_{X_v}$,
- (3) for some U, V, u, v as above u is a weakly associated point of $\mathcal{F}|_{U_v}$,
- (4) for every U, V, u, v as above u is a weakly associated point of $\mathcal{F}|_{U_v}$,
- (5) for some field k and morphism $\text{Spec}(k) \rightarrow Y$ representing y and some $t \in |X_k|$ mapping to x , the point t is a weakly associated point of $\mathcal{F}|_{X_k}$.

If there exists a field k_0 and a monomorphism $\text{Spec}(k_0) \rightarrow Y$ representing y , then these are also equivalent to

- (6) x_0 is a weakly associated point of $\mathcal{F}|_{X_{k_0}}$ where $x_0 \in |X_{k_0}|$ is the unique point mapping to x .

If the fibre of f over y is locally Noetherian, then in conditions (1), (2), (3), (4), and (6) we may replace “weakly associated” with “associated”.

Proof. Observe that given V, v, x' as in the lemma we can find $U \rightarrow X \times_Y V$ and $u \in U$ mapping to x' and then the morphism $U_v \rightarrow X_v$ is étale. Thus it is clear that (1) and (3) are equivalent as well as (2) and (4). Each of these implies (5). We will show that (5) implies (2). Suppose given V, v, x' as well as $\text{Spec}(k) \rightarrow X$ and $t \in |X_k|$ such that the point t is a weakly associated point of $\mathcal{F}|_{X_k}$. We can choose a point $w \in v \times_Y \text{Spec}(k)$. Then we obtain the morphisms

$$X_v \longleftarrow X_w \longrightarrow X_k$$

Since $V \rightarrow Y$ is étale and since w may be viewed as a point of $V \times_Y \text{Spec}(k)$, we see that $\kappa(w)/k$ is a finite separable extension of fields (Morphisms, Lemma 36.7). Thus $X_w \rightarrow X_k$ is a finite étale morphism as a base change of $w \rightarrow \text{Spec}(k)$. Thus any point x'' of X_w lying over t is a weakly associated point of $\mathcal{F}|_{X_w}$ by Lemma 3.7. We may pick x'' mapping to x' (Properties of Spaces, Lemma 4.3). Then Lemma 3.5 implies that x' is a weakly associated point of $\mathcal{F}|_{X_v}$.

To finish the proof it suffices to show that the equivalent conditions (1) – (5) imply (6) if we are given $\text{Spec}(k_0) \rightarrow Y$ as in (6). In this case the morphism $\text{Spec}(k) \rightarrow Y$ of (5) factors uniquely as $\text{Spec}(k) \rightarrow \text{Spec}(k_0) \rightarrow Y$. Then x_0 is the image of t under the morphism $X_k \rightarrow X_{k_0}$. Hence the same lemma as above shows that (6) is true. \square

Definition 4.5. Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. The *relative weak assassin* of \mathcal{F} in X over Y is the set $\text{WeakAss}_{X/Y}(\mathcal{F}) \subset |X|$ consisting of those $x \in |X|$ such that the equivalent conditions of Lemma 4.4 are satisfied. If the fibres of f are locally Noetherian (Definition 4.2) then we use the notation $\text{Ass}_{X/Y}(\mathcal{F})$.

With this notation we can formulate some of the results already proven for schemes.

Lemma 4.6. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let \mathcal{G} be a quasi-coherent \mathcal{O}_Y -module. Assume*

- (1) \mathcal{F} is flat over Y ,
- (2) X and Y are locally Noetherian, and
- (3) the fibres of f are locally Noetherian.

Then

$$\text{Ass}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \{x \in \text{Ass}_{X/Y}(\mathcal{F}) \text{ such that } f(x) \in \text{Ass}_Y(\mathcal{G})\}$$

Proof. Via étale localization, this is an immediate consequence of the result for schemes, see Divisors, Lemma 3.1. The result for schemes is more general only because we haven't defined associated points for non-Noetherian algebraic spaces (hence we need to assume X and the fibres of $X \rightarrow Y$ are locally Noetherian to even be able to formulate this result). \square

Lemma 4.7. *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of algebraic spaces over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module and set $\mathcal{F}' = (g')^\mathcal{F}$. If f is locally of finite type, then*

- (1) $x' \in \text{Ass}_{X'/Y'}(\mathcal{F}') \Rightarrow g'(x') \in \text{Ass}_{X/Y}(\mathcal{F})$
- (2) if $x \in \text{Ass}_{X/Y}(\mathcal{F})$, then given $y' \in |Y'|$ with $f(x) = g(y')$, there exists an $x' \in \text{Ass}_{X'/Y'}(\mathcal{F}')$ with $g'(x') = x$ and $f'(x') = y'$.

Proof. This follows from the case of schemes by étale localization. We write out the details completely. Choose a scheme V and a surjective étale morphism $V \rightarrow Y$. Choose a scheme U and a surjective étale morphism $U \rightarrow V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \rightarrow V \times_Y Y'$. Then $U' = V' \times_V U$ is a scheme and the morphism $U' \rightarrow X'$ is surjective and étale.

Proof of (1). Choose $u' \in U'$ mapping to x' . Denote $v' \in V'$ the image of u' . Then $x' \in \text{Ass}_{X'/Y'}(\mathcal{F}')$ is equivalent to $u' \in \text{Ass}(\mathcal{F}|_{U'_{v'}})$ by definition (writing Ass instead of WeakAss makes sense as $U'_{v'}$ is locally Noetherian). Applying Divisors, Lemma 7.3 we see that the image $u \in U$ of u' is in $\text{Ass}(\mathcal{F}|_{U_v})$ where $v \in V$ is the image of u . This in turn means $g'(x') \in \text{Ass}_{X/Y}(\mathcal{F})$.

Proof of (2). Choose $u \in U$ mapping to x . Denote $v \in V$ the image of u . Then $x \in \text{Ass}_{X/Y}(\mathcal{F})$ is equivalent to $u \in \text{Ass}(\mathcal{F}|_{U_v})$ by definition. Choose a point $v' \in V'$ mapping to $y' \in |Y'|$ and to $v \in V$ (possible by Properties of Spaces, Lemma 4.3). Let $t \in \text{Spec}(\kappa(v') \otimes_{\kappa(v)} \kappa(u))$ be a generic point of an irreducible component. Let $u' \in U'$ be the image of t . Applying Divisors, Lemma 7.3 we see that $u' \in \text{Ass}(\mathcal{F}'|_{U'_{v'}})$. This in turn means $x' \in \text{Ass}_{X'/Y'}(\mathcal{F}')$ where $x' \in |X'|$ is the image of u' . \square

Lemma 4.8. *With notation and assumptions as in Lemma 4.7. Assume g is locally quasi-finite, or more generally that for every $y' \in |Y'|$ the transcendence degree of $y'/g(y')$ is 0. Then $\text{Ass}_{X'/Y'}(\mathcal{F}')$ is the inverse image of $\text{Ass}_{X/Y}(\mathcal{F})$.*

Proof. The transcendence degree of a point over its image is defined in Morphisms of Spaces, Definition 33.1. Let $x' \in |X'|$ with image $x \in |X|$. Choose a scheme V and a surjective étale morphism $V \rightarrow Y$. Choose a scheme U and a surjective étale morphism $U \rightarrow V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \rightarrow V \times_Y Y'$. Then $U' = V' \times_V U$ is a scheme and the morphism $U' \rightarrow X'$ is surjective and étale. Choose $u \in U$ mapping to x . Denote $v \in V$ the image of u . Then $x \in \text{Ass}_{X/Y}(\mathcal{F})$ is equivalent to $u \in \text{Ass}(\mathcal{F}|_{U_v})$ by definition. Choose a point $u' \in U'$ mapping to $x' \in |X'|$ and to $u \in U$ (possible by Properties of Spaces, Lemma 4.3). Let $v' \in V'$ be the image of u' . Then $x' \in \text{Ass}_{X'/Y'}(\mathcal{F}')$ is equivalent to $u' \in \text{Ass}(\mathcal{F}'|_{U'_{v'}})$ by definition. Now the lemma follows from the discussion in Divisors, Remark 7.4 applied to $u' \in \text{Spec}(\kappa(v') \otimes_{\kappa(v)} \kappa(u))$. \square

Lemma 4.9. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let $i : Z \rightarrow X$ be a finite morphism. Let \mathcal{G} be a quasi-coherent \mathcal{O}_Z -module. Then $\text{WeakAss}_{X/Y}(i_*\mathcal{G}) = i(\text{WeakAss}_{Z/Y}(\mathcal{G}))$.*

Proof. Follows from the case of schemes (Divisors, Lemma 8.3) by étale localization. Details omitted. \square

Lemma 4.10. *Let Y be a scheme. Let X be an algebraic space of finite presentation over Y . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite presentation. Let $U \subset X$ be an open subspace such that $U \rightarrow Y$ is quasi-compact. Then the set*

$$E = \{y \in Y \mid \text{Ass}_{X_y}(\mathcal{F}_y) \subset |U_y|\}$$

is locally constructible in Y .

Proof. Note that since Y is a scheme, it makes sense to take the fibres $X_y = \text{Spec}(\kappa(y)) \times_Y X$. (Also, by our definitions, the set $\text{Ass}_{X_y}(\mathcal{F}_y)$ is exactly the fibre of $\text{Ass}_{X/Y}(\mathcal{F}) \rightarrow Y$ over y , but we won't need this.) The question is local on Y , indeed, we have to show that E is constructible if Y is affine. In this case X is quasi-compact. Choose an affine scheme W and a surjective étale morphism $\varphi : W \rightarrow X$. Then $\text{Ass}_{X_y}(\mathcal{F}_y)$ is the image of $\text{Ass}_{W_y}(\varphi^*\mathcal{F}_y)$ for all $y \in Y$. Hence the lemma follows from the case of schemes for the open $\varphi^{-1}(U) \subset W$ and the morphism $W \rightarrow Y$. The case of schemes is More on Morphisms, Lemma 25.5. \square

5. Fitting ideals

This section is the continuation of the discussion in Divisors, Section 9. Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a finite type, quasi-coherent \mathcal{O}_X -module. In this situation we can construct the Fitting ideals

$$0 = \text{Fit}_{-1}(\mathcal{F}) \subset \text{Fit}_0(\mathcal{F}) \subset \text{Fit}_1(\mathcal{F}) \subset \dots \subset \mathcal{O}_X$$

as the sequence of quasi-coherent sheaves ideals characterized by the following property: for every affine $U = \text{Spec}(A)$ étale over X if $\mathcal{F}|_U$ corresponds to the A -module M , then $\text{Fit}_i(\mathcal{F})|_U$ corresponds to the ideal $\text{Fit}_i(M) \subset A$. This is well defined and a quasi-coherent sheaf of ideals because if $A \rightarrow B$ is an étale ring map, then the i th Fitting ideal of $M \otimes_A B$ over B is equal to $\text{Fit}_i(M)B$ by More on Algebra, Lemma 8.4 part (3). More precisely (perhaps), the existence of the quasi-coherent sheaves of ideals $\text{Fit}_i(\mathcal{O}_X)$ follows (for example) from the description of quasi-coherent sheaves in Properties of Spaces, Lemma 29.3 and the pullback property given in Divisors, Lemma 9.1.

The advantage of constructing the Fitting ideals in this way is that we see immediately that formation of Fitting ideals commutes with étale localization hence many properties of the Fitting ideals immediately reduce to the corresponding properties in the case of schemes. Often we will use the discussion in Properties of Spaces, Section 30 to do the translation between properties of quasi-coherent sheaves on schemes and on algebraic spaces.

Lemma 5.1. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let \mathcal{F} be a finite type, quasi-coherent \mathcal{O}_Y -module. Then $f^{-1}\text{Fit}_i(\mathcal{F}) \cdot \mathcal{O}_X = \text{Fit}_i(f^*\mathcal{F})$.*

Proof. Reduces to Divisors, Lemma 9.1 by étale localization. \square

Lemma 5.2. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a finitely presented \mathcal{O}_X -module. Then $\text{Fit}_r(\mathcal{F})$ is a quasi-coherent ideal of finite type.*

Proof. Reduces to Divisors, Lemma 9.2 by étale localization. \square

Lemma 5.3. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a finite type, quasi-coherent \mathcal{O}_X -module. Let $Z_0 \subset X$ be the closed subspace cut out by $\text{Fit}_0(\mathcal{F})$. Let $Z \subset X$ be the scheme theoretic support of \mathcal{F} . Then*

- (1) $Z \subset Z_0 \subset X$ as closed subspaces,
- (2) $|Z| = |Z_0| = \text{Supp}(\mathcal{F})$ as closed subsets of $|X|$,
- (3) there exists a finite type, quasi-coherent \mathcal{O}_{Z_0} -module \mathcal{G}_0 with

$$(Z_0 \rightarrow X)_*\mathcal{G}_0 = \mathcal{F}.$$

Proof. Recall that formation of Z commutes with étale localization, see Morphisms of Spaces, Definition 15.4 (which uses Morphisms of Spaces, Lemma 15.3 to define Z). Hence (1) and (2) follow from the case of schemes, see Divisors, Lemma 9.3. To get \mathcal{G}_0 as in part (3) we can use that we have \mathcal{G} on Z as in Morphisms of Spaces, Lemma 15.3 and set $\mathcal{G}_0 = (Z \rightarrow Z_0)_*\mathcal{G}$. \square

Lemma 5.4. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a finite type, quasi-coherent \mathcal{O}_X -module. Let $x \in |X|$. Then \mathcal{F} can be generated by r elements in an étale neighbourhood of x if and only if $\text{Fit}_r(\mathcal{F})_{\bar{x}} = \mathcal{O}_{X,\bar{x}}$.*

Proof. Reduces to Divisors, Lemma 9.4 by étale localization (as well as the description of the local ring in Properties of Spaces, Section 22 and the fact that the strict henselization of a local ring is faithfully flat to see that the equality over the strict henselization is equivalent to the equality over the local ring). \square

Lemma 5.5. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a finite type, quasi-coherent \mathcal{O}_X -module. Let $r \geq 0$. The following are equivalent*

- (1) \mathcal{F} is finite locally free of rank r
- (2) $\text{Fit}_{r-1}(\mathcal{F}) = 0$ and $\text{Fit}_r(\mathcal{F}) = \mathcal{O}_X$, and
- (3) $\text{Fit}_k(\mathcal{F}) = 0$ for $k < r$ and $\text{Fit}_k(\mathcal{F}) = \mathcal{O}_X$ for $k \geq r$.

Proof. Reduces to Divisors, Lemma 9.5 by étale localization. \square

Lemma 5.6. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a finite type, quasi-coherent \mathcal{O}_X -module. The closed subspaces*

$$X = Z_{-1} \supset Z_0 \supset Z_1 \supset Z_2 \dots$$

defined by the Fitting ideals of \mathcal{F} have the following properties

- (1) The intersection $\bigcap Z_r$ is empty.
- (2) The functor $(Sch/X)^{opp} \rightarrow Sets$ defined by the rule

$$T \mapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is locally generated by } \leq r \text{ sections} \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the open subspace $X \setminus Z_r$.

- (3) The functor $F_r : (Sch/X)^{opp} \rightarrow Sets$ defined by the rule

$$T \mapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ locally free rank } r \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the locally closed subspace $Z_{r-1} \setminus Z_r$ of X .

If \mathcal{F} is of finite presentation, then $Z_r \rightarrow X$, $X \setminus Z_r \rightarrow X$, and $Z_{r-1} \setminus Z_r \rightarrow X$ are of finite presentation.

Proof. Reduces to Divisors, Lemma 9.6 by étale localization. \square

Lemma 5.7. Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be an \mathcal{O}_X -module of finite presentation. Let $X = Z_{-1} \subset Z_0 \subset Z_1 \subset \dots$ be as in Lemma 5.6. Set $X_r = Z_{r-1} \setminus Z_r$. Then $X' = \coprod_{r \geq 0} X_r$ represents the functor

$$F_{flat} : Sch/X \longrightarrow Sets, \quad T \mapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ flat over } T \\ \emptyset & \text{otherwise} \end{cases}$$

Moreover, $\mathcal{F}|_{X_r}$ is locally free of rank r and the morphisms $X_r \rightarrow X$ and $X' \rightarrow X$ are of finite presentation.

Proof. Reduces to Divisors, Lemma 9.7 by étale localization. \square

6. Effective Cartier divisors

For some reason it seem convenient to define the notion of an effective Cartier divisor before anything else. Note that in Morphisms of Spaces, Section 13 we discussed the correspondence between closed subspaces and quasi-coherent sheaves of ideals. Moreover, in Properties of Spaces, Section 30, we discussed properties of quasi-coherent modules, in particular “locally generated by 1 element”. These references show that the following definition is compatible with the definition for schemes.

Definition 6.1. Let S be a scheme. Let X be an algebraic space over S .

- (1) A *locally principal closed subspace* of X is a closed subspace whose sheaf of ideals is locally generated by 1 element.
- (2) An *effective Cartier divisor* on X is a closed subspace $D \subset X$ such that the ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$ is an invertible \mathcal{O}_X -module.

Thus an effective Cartier divisor is a locally principal closed subspace, but the converse is not always true. Effective Cartier divisors are closed subspaces of pure codimension 1 in the strongest possible sense. Namely they are locally cut out by a single element which is not a zerodivisor. In particular they are nowhere dense.

Lemma 6.2. Let S be a scheme. Let X be an algebraic space over S . Let $D \subset X$ be a closed subspace. The following are equivalent:

- (1) The subspace D is an effective Cartier divisor on X .
- (2) For some scheme U and surjective étale morphism $U \rightarrow X$ the inverse image $D \times_X U$ is an effective Cartier divisor on U .

- (3) For every scheme U and every étale morphism $U \rightarrow X$ the inverse image $D \times_X U$ is an effective Cartier divisor on U .
- (4) For every $x \in |D|$ there exists an étale morphism $(U, u) \rightarrow (X, x)$ of pointed algebraic spaces such that $U = \operatorname{Spec}(A)$ and $D \times_X U = \operatorname{Spec}(A/(f))$ with $f \in A$ not a zerodivisor.

Proof. The equivalence of (1) – (3) follows from Definition 6.1 and the references preceding it. Assume (1) and let $x \in |D|$. Choose a scheme W and a surjective étale morphism $W \rightarrow X$. Choose $w \in D \times_X W$ mapping to x . By (3) $D \times_X W$ is an effective Cartier divisor on W . Hence we can find affine étale neighbourhood U by choosing an affine open neighbourhood of w in W as in Divisors, Lemma 13.2.

Assume (4). Then we see that $\mathcal{I}_D|_U$ is invertible by Divisors, Lemma 13.2. Since we can find an étale covering of X by the collection of all such U and $X \setminus D$, we conclude that \mathcal{I}_D is an invertible \mathcal{O}_X -module. \square

Lemma 6.3. *Let S be a scheme. Let X be an algebraic space over S . Let $Z \subset X$ be a locally principal closed subspace. Let $U = X \setminus Z$. Then $U \rightarrow X$ is an affine morphism.*

Proof. The question is étale local on X , see Morphisms of Spaces, Lemmas 20.3 and Lemma 6.2. Thus this follows from the case of schemes which is Divisors, Lemma 13.3. \square

Lemma 6.4. *Let S be a scheme. Let X be an algebraic space over S . Let $D \subset X$ be an effective Cartier divisor. Let $U = X \setminus D$. Then $U \rightarrow X$ is an affine morphism and U is scheme theoretically dense in X .*

Proof. Affineness is Lemma 6.3. The density question is étale local on X by Morphisms of Spaces, Definition 17.3. Thus this follows from the case of schemes which is Divisors, Lemma 13.4. \square

Lemma 6.5. *Let S be a scheme. Let X be an algebraic space over S . Let $D \subset X$ be an effective Cartier divisor. Let $x \in |D|$. If $\dim_x(X) < \infty$, then $\dim_x(D) < \dim_x(X)$.*

Proof. Both the definition of an effective Cartier divisor and of the dimension of an algebraic space at a point (Properties of Spaces, Definition 9.1) are étale local. Hence this lemma follows from the case of schemes which is Divisors, Lemma 13.5. \square

Definition 6.6. Let S be a scheme. Let X be an algebraic space over S . Given effective Cartier divisors D_1, D_2 on X we set $D = D_1 + D_2$ equal to the closed subspace of X corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}_{D_1}\mathcal{I}_{D_2} \subset \mathcal{O}_S$. We call this the *sum of the effective Cartier divisors D_1 and D_2* .

It is clear that we may define the sum $\sum n_i D_i$ given finitely many effective Cartier divisors D_i on X and nonnegative integers n_i .

Lemma 6.7. *The sum of two effective Cartier divisors is an effective Cartier divisor.*

Proof. Omitted. Étale locally this reduces to the following simple algebra fact: if $f_1, f_2 \in A$ are nonzerodivisors of a ring A , then $f_1 f_2 \in A$ is a nonzerodivisor. \square

Lemma 6.8. *Let S be a scheme. Let X be an algebraic space over S . Let Z, Y be two closed subspaces of X with ideal sheaves \mathcal{I} and \mathcal{J} . If $\mathcal{I}\mathcal{J}$ defines an effective Cartier divisor $D \subset X$, then Z and Y are effective Cartier divisors and $D = Z + Y$.*

Proof. By Lemma 6.2 this reduces to the case of schemes which is Divisors, Lemma 13.9. \square

Recall that we have defined the inverse image of a closed subspace under any morphism of algebraic spaces in Morphisms of Spaces, Definition 13.2.

Lemma 6.9. *Let S be a scheme. Let $f : X' \rightarrow X$ be a morphism of algebraic spaces over S . Let $Z \subset X$ be a locally principal closed subspace. Then the inverse image $f^{-1}(Z)$ is a locally principal closed subspace of X' .*

Proof. Omitted. \square

Definition 6.10. Let S be a scheme. Let $f : X' \rightarrow X$ be a morphism of algebraic spaces over S . Let $D \subset X$ be an effective Cartier divisor. We say the *pullback of D by f* is defined if the closed subspace $f^{-1}(D) \subset X'$ is an effective Cartier divisor. In this case we denote it either f^*D or $f^{-1}(D)$ and we call it the *pullback of the effective Cartier divisor*.

The condition that $f^{-1}(D)$ is an effective Cartier divisor is often satisfied in practice.

Lemma 6.11. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let $D \subset Y$ be an effective Cartier divisor. The pullback of D by f is defined in each of the following cases:*

- (1) $f(x) \notin |D|$ for any weakly associated point x of X ,
- (2) f is flat, and
- (3) add more here as needed.

Proof. Working étale locally this lemma reduces to the case of schemes, see Divisors, Lemma 13.13. \square

Lemma 6.12. *Let S be a scheme. Let $f : X' \rightarrow X$ be a morphism of algebraic spaces over S . Let D_1, D_2 be effective Cartier divisors on X . If the pullbacks of D_1 and D_2 are defined then the pullback of $D = D_1 + D_2$ is defined and $f^*D = f^*D_1 + f^*D_2$.*

Proof. Omitted. \square

7. Effective Cartier divisors and invertible sheaves

Since an effective Cartier divisor has an invertible ideal sheaf (Definition 6.1) the following definition makes sense.

Definition 7.1. Let S be a scheme. Let X be an algebraic space over S and let $D \subset X$ be an effective Cartier divisor with ideal sheaf \mathcal{I}_D .

- (1) The *invertible sheaf $\mathcal{O}_X(D)$ associated to D* is defined by

$$\mathcal{O}_X(D) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X) = \mathcal{I}_D^{\otimes -1}.$$

- (2) The canonical section, usually denoted 1 or 1_D , is the global section of $\mathcal{O}_X(D)$ corresponding to the inclusion mapping $\mathcal{I}_D \rightarrow \mathcal{O}_X$.
- (3) We write $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^{\otimes -1} = \mathcal{I}_D$.

- (4) Given a second effective Cartier divisor $D' \subset X$ we define $\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D')$.

Some comments. We will see below that the assignment $D \mapsto \mathcal{O}_X(D)$ turns addition of effective Cartier divisors (Definition 6.6) into addition in the Picard group of X (Lemma 7.3). However, the expression $D - D'$ in the definition above does not have any geometric meaning. More precisely, we can think of the set of effective Cartier divisors on X as a commutative monoid $\text{EffCart}(X)$ whose zero element is the empty effective Cartier divisor. Then the assignment $(D, D') \mapsto \mathcal{O}_X(D - D')$ defines a group homomorphism

$$\text{EffCart}(X)^{gp} \longrightarrow \text{Pic}(X)$$

where the left hand side is the group completion of $\text{EffCart}(X)$. In other words, when we write $\mathcal{O}_X(D - D')$ we may think of $D - D'$ as an element of $\text{EffCart}(X)^{gp}$.

Lemma 7.2. *Let S be a scheme. Let X be an algebraic space over S . Let $D \subset X$ be an effective Cartier divisor. Then for the conormal sheaf we have $\mathcal{C}_{D/X} = \mathcal{I}_D|_D = \mathcal{O}_X(D)^{\otimes -1}|_D$.*

Proof. Omitted. □

Lemma 7.3. *Let S be a scheme. Let X be an algebraic space over S . Let D_1, D_2 be effective Cartier divisors on X . Let $D = D_1 + D_2$. Then there is a unique isomorphism*

$$\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_X(D)$$

which maps $1_{D_1} \otimes 1_{D_2}$ to 1_D .

Proof. Omitted. □

Definition 7.4. Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{L} be an invertible sheaf on X . A global section $s \in \Gamma(X, \mathcal{L})$ is called a *regular section* if the map $\mathcal{O}_X \rightarrow \mathcal{L}, f \mapsto fs$ is injective.

Lemma 7.5. *Let S be a scheme. Let X be an algebraic space over S . Let $f \in \Gamma(X, \mathcal{O}_X)$. The following are equivalent:*

- (1) f is a regular section, and
- (2) for any $x \in X$ the image $f \in \mathcal{O}_{X, \bar{x}}$ is not a zerodivisor.
- (3) for any affine $U = \text{Spec}(A)$ étale over X the restriction $f|_U$ is a nonzero-divisor of A , and
- (4) there exists a scheme U and a surjective étale morphism $U \rightarrow X$ such that $f|_U$ is a regular section of \mathcal{O}_U .

Proof. Omitted. □

Note that a global section s of an invertible \mathcal{O}_X -module \mathcal{L} may be seen as an \mathcal{O}_X -module map $s : \mathcal{O}_X \rightarrow \mathcal{L}$. Its dual is therefore a map $s : \mathcal{L}^{\otimes -1} \rightarrow \mathcal{O}_X$. (See Modules on Sites, Lemma 32.4 for the dual invertible sheaf.)

Definition 7.6. Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{L} be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$. The *zero scheme* of s is the closed subspace $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is the image of the map $s : \mathcal{L}^{\otimes -1} \rightarrow \mathcal{O}_X$.

Lemma 7.7. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $s \in \Gamma(X, \mathcal{L})$.*

- (1) *Consider closed immersions $i : Z \rightarrow X$ such that $i^*s \in \Gamma(Z, i^*\mathcal{L})$ is zero ordered by inclusion. The zero scheme $Z(s)$ is the maximal element of this ordered set.*
- (2) *For any morphism of algebraic spaces $f : Y \rightarrow X$ over S we have $f^*s = 0$ in $\Gamma(Y, f^*\mathcal{L})$ if and only if f factors through $Z(s)$.*
- (3) *The zero scheme $Z(s)$ is a locally principal closed subspace of X .*
- (4) *The zero scheme $Z(s)$ is an effective Cartier divisor on X if and only if s is a regular section of \mathcal{L} .*

Proof. Omitted. □

Lemma 7.8. *Let S be a scheme. Let X be an algebraic space over S .*

- (1) *If $D \subset X$ is an effective Cartier divisor, then the canonical section 1_D of $\mathcal{O}_X(D)$ is regular.*
- (2) *Conversely, if s is a regular section of the invertible sheaf \mathcal{L} , then there exists a unique effective Cartier divisor $D = Z(s) \subset X$ and a unique isomorphism $\mathcal{O}_X(D) \rightarrow \mathcal{L}$ which maps 1_D to s .*

The constructions $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ give mutually inverse maps

$$\{\text{effective Cartier divisors on } X\} \leftrightarrow \left\{ \begin{array}{l} \text{pairs } (\mathcal{L}, s) \text{ consisting of an invertible} \\ \mathcal{O}_X\text{-module and a regular global section} \end{array} \right\}$$

Proof. Omitted. □

8. Effective Cartier divisors on Noetherian spaces

In the locally Noetherian setting most of the discussion of effective Cartier divisors and regular sections simplifies somewhat.

Lemma 8.1. *Let S be a scheme and let X be a locally Noetherian algebraic space over S . Let $D \subset X$ be an effective Cartier divisor. If X is (S_k) , then D is (S_{k-1}) .*

Proof. By our definition of the property (S_k) for algebraic spaces (Properties of Spaces, Section 7) and Lemma 6.2 this follows from the case of schemes (Divisors, Lemma 15.5). □

Lemma 8.2. *Let S be a scheme and let X be a locally Noetherian normal algebraic space over S . Let $D \subset X$ be an effective Cartier divisor. Then D is (S_1) .*

Proof. By our definition of normality for algebraic spaces (Properties of Spaces, Section 7) and Lemma 6.2 this follows from the case of schemes (Divisors, Lemma 15.6). □

The following lemma can sometimes be used to produce effective Cartier divisors.

Lemma 8.3. *Let S be a scheme. Let X be a regular Noetherian separated algebraic space over S . Let $U \subset X$ be a dense affine open. Then there exists an effective Cartier divisor $D \subset X$ with $U = X \setminus D$.*

Proof. We claim that the reduced induced algebraic space structure D on $X \setminus U$ (Properties of Spaces, Definition 12.5) is the desired effective Cartier divisor. The construction of D commutes with étale localization, see proof of Properties of Spaces, Lemma 12.3. Let $X' \rightarrow X$ be a surjective étale morphism with X' affine.

Since X is separated, we see that $U' = X' \times_X U$ is affine. Since $|X'| \rightarrow |X|$ is open, we see that U' is dense in X' . Since $D' = X' \times_X D$ is the reduced induced scheme structure on $X' \setminus U'$, we conclude that D' is an effective Cartier divisor by Divisors, Lemma 16.6 and its proof. This is what we had to show. \square

Lemma 8.4. *Let S be a scheme. Let X be a regular Noetherian separated algebraic space over S . Then every invertible \mathcal{O}_X -module is isomorphic to*

$$\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D')^{\otimes -1}$$

for some effective Cartier divisors D, D' in X .

Proof. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Choose a dense affine open $U \subset X$ such that $\mathcal{L}|_U$ is trivial. This is possible because X has a dense open subspace which is a scheme, see Properties of Spaces, Proposition 13.3. Denote $s : \mathcal{O}_U \rightarrow \mathcal{L}|_U$ the trivialization. The complement of U is an effective Cartier divisor D . We claim that for some $n > 0$ the map s extends uniquely to a map

$$s : \mathcal{O}_X(-nD) \longrightarrow \mathcal{L}$$

The claim implies the lemma because it shows that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(nD)$ has a regular global section hence is isomorphic to $\mathcal{O}_X(D')$ for some effective Cartier divisor D' by Lemma 7.8. To prove the claim we may work étale locally. Thus we may assume X is an affine Noetherian scheme. Since $\mathcal{O}_X(-nD) = \mathcal{I}^n$ where $\mathcal{I} = \mathcal{O}_X(-D)$ is the ideal sheaf of D in X , this case follows from Cohomology of Schemes, Lemma 10.5. \square

The following lemma really belongs to a different section.

Lemma 8.5. *Let R be a valuation ring with fraction field K . Let X be an algebraic space over R such that $X \rightarrow \operatorname{Spec}(R)$ is smooth. For every effective Cartier divisor $D \subset X_K$ there exists an effective Cartier divisor $D' \subset X$ with $D'_K = D$.*

Proof. Let $D' \subset X$ be the scheme theoretic image of $D \rightarrow X_K \rightarrow X$. Since this morphism is quasi-compact, formation of D' commutes with flat base change, see Morphisms of Spaces, Lemma 30.12. In particular we find that $D'_K = D$. Hence, we may assume X is affine. Say $X = \operatorname{Spec}(A)$. Then $X_K = \operatorname{Spec}(A \otimes_R K)$ and D corresponds to an ideal $I \subset A \otimes_R K$. We have to show that $J = I \cap A$ cuts out an effective Cartier divisor in X . First, observe that A/J is flat over R (as a torsion free R -module, see More on Algebra, Lemma 22.10), hence J is finitely generated by More on Algebra, Lemma 25.6 and Algebra, Lemma 5.3. Thus it suffices to show that $J_{\mathfrak{q}} \subset A_{\mathfrak{q}}$ is generated by a single element for each prime $\mathfrak{q} \subset A$. Let $\mathfrak{p} = R \cap \mathfrak{q}$. Then $R_{\mathfrak{p}}$ is a valuation ring (Algebra, Lemma 50.9). Observe further that $A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}$ is a regular ring by Algebra, Lemma 140.3. Thus we may apply More on Algebra, Lemma 121.3 to see that $I(A_{\mathfrak{q}} \otimes_R K)$ is generated by a single element $f \in A_{\mathfrak{p}} \otimes_R K$. After clearing denominators we may assume $f \in A_{\mathfrak{q}}$. Let $\mathfrak{c} \subset R_{\mathfrak{p}}$ be the content ideal of f (see More on Algebra, Definition 24.1 and More on Flatness, Lemma 19.6). Since $R_{\mathfrak{p}}$ is a valuation ring and since \mathfrak{c} is finitely generated (More on Algebra, Lemma 24.2) we see $\mathfrak{c} = (\pi)$ for some $\pi \in R_{\mathfrak{p}}$ (Algebra, Lemma 50.15). After relacing f by $\pi^{-1}f$ we see that $f \in A_{\mathfrak{q}}$ and $f \notin \mathfrak{p}A_{\mathfrak{q}}$. Claim: $I_{\mathfrak{q}} = (f)$ which finishes the proof. To see the claim, observe that $f \in I_{\mathfrak{q}}$. Hence we have a surjection $A_{\mathfrak{q}}/(f) \rightarrow A_{\mathfrak{q}}/I_{\mathfrak{q}}$ which is an isomorphism after tensoring over R with K . Thus we are done if $A_{\mathfrak{q}}/(f)$ is $R_{\mathfrak{p}}$ -flat. This follows from Algebra, Lemma 128.5 and our choice of f . \square

9. Relative effective Cartier divisors

The following lemma shows that an effective Cartier divisor which is flat over the base is really a “family of effective Cartier divisors” over the base. For example the restriction to any fibre is an effective Cartier divisor.

Lemma 9.1. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let $D \subset X$ be a closed subspace. Assume*

- (1) *D is an effective Cartier divisor, and*
- (2) *$D \rightarrow Y$ is a flat morphism.*

Then for every morphism of schemes $g : Y' \rightarrow Y$ the pullback $(g')^{-1}D$ is an effective Cartier divisor on $X' = Y' \times_Y X$ where $g' : X' \rightarrow X$ is the projection.

Proof. Using Lemma 6.2 the property of being an effective Cartier divisor is étale local. Thus this lemma immediately reduces to the case of schemes which is Divisors, Lemma 18.1. \square

This lemma is the motivation for the following definition.

Definition 9.2. Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . A *relative effective Cartier divisor* on X/Y is an effective Cartier divisor $D \subset X$ such that $D \rightarrow Y$ is a flat morphism of algebraic spaces.

10. Meromorphic functions and sections

This section is the analogue of Divisors, Section 23. Beware: it is even easier to make mistakes with this material in the case of algebraic space, than it is in the case of schemes!

Let S be a scheme. Let X be an algebraic space over S . For any scheme U étale over X we have defined the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ of regular sections of \mathcal{O}_X over U , see Definition 7.4. The restriction of a regular section to V/U étale is regular. Hence $\mathcal{S} : U \mapsto \mathcal{S}(U)$ is a subsheaf (of sets) of \mathcal{O}_X . We sometimes denote $\mathcal{S} = \mathcal{S}_X$ if we want to indicate the dependence on X . Moreover, $\mathcal{S}(U)$ is a multiplicative subset of the ring $\mathcal{O}_X(U)$ for each U . Hence we may consider the presheaf of rings

$$U \mapsto \mathcal{S}(U)^{-1} \mathcal{O}_X(U),$$

on $X_{\text{étale}}$ and its sheafification, see Modules on Sites, Section 44.

Definition 10.1. Let S be a scheme. Let X be an algebraic space over S . The *sheaf of meromorphic functions* on X is the sheaf \mathcal{K}_X on $X_{\text{étale}}$ associated to the presheaf displayed above. A *meromorphic function* on X is a global section of \mathcal{K}_X .

Since each element of each $\mathcal{S}(U)$ is a nonzerodivisor on $\mathcal{O}_X(U)$ we see that the natural map of sheaves of rings $\mathcal{O}_X \rightarrow \mathcal{K}_X$ is injective. Moreover, by the compatibility of sheafification and taking stalks we see that

$$\mathcal{K}_{X,\bar{x}} = \mathcal{S}_{\bar{x}}^{-1} \mathcal{O}_{X,\bar{x}}$$

for any geometric point \bar{x} of X . The set $\mathcal{S}_{\bar{x}}$ is a subset of the set of nonzerodivisors of $\mathcal{O}_{X,\bar{x}}$, but in general not equal to this.

Lemma 10.2. *Let S be a scheme. Let X be an algebraic space over S . For U affine and étale over X the set $\mathcal{S}_X(U)$ is the set of nonzerodivisors in $\mathcal{O}_X(U)$.*

Proof. Follows from Lemma 7.5. \square

Next, let \mathcal{F} be a sheaf of \mathcal{O}_X -modules on $X_{\text{étale}}$. Consider the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Its sheafification is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$, see Modules on Sites, Lemma 44.2.

Definition 10.3. Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{F} be a sheaf of \mathcal{O}_X -modules on $X_{\text{étale}}$.

- (1) We denote $\mathcal{K}_X(\mathcal{F})$ the sheaf of \mathcal{K}_X -modules which is the sheafification of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Equivalently $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ (see above).
- (2) A *meromorphic section* of \mathcal{F} is a global section of $\mathcal{K}_X(\mathcal{F})$.

In particular we have

$$\mathcal{K}_X(\mathcal{F})_{\bar{x}} = \mathcal{F}_{\bar{x}} \otimes_{\mathcal{O}_{X,\bar{x}}} \mathcal{K}_{X,\bar{x}} = \mathcal{S}_{\bar{x}}^{-1} \mathcal{F}_{\bar{x}}$$

for any geometric point \bar{x} of X . However, one has to be careful since it may not be the case that $\mathcal{S}_{\bar{x}}$ is the set of nonzerodivisors in the étale local ring $\mathcal{O}_{X,\bar{x}}$ as we pointed out above. The sheaves of meromorphic sections aren't quasi-coherent modules in general, but they do have some properties in common with quasi-coherent modules.

Lemma 10.4. *Let S be a scheme. Let X be an algebraic space over S . Assume*

- (a) *every weakly associated point of X is a point of codimension 0, and*
- (b) *X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 49.1.*

Then

- (1) *\mathcal{K}_X is a quasi-coherent sheaf of \mathcal{O}_X -algebras,*
- (2) *for $U \in X_{\text{étale}}$ affine $\mathcal{K}_X(U)$ is the total ring of fractions of $\mathcal{O}_X(U)$,*
- (3) *for a geometric point \bar{x} the set $\mathcal{S}_{\bar{x}}$ the set of nonzerodivisors of $\mathcal{O}_{X,\bar{x}}$, and*
- (4) *for a geometric point \bar{x} the ring $\mathcal{K}_{X,\bar{x}}$ is the total ring of fractions of $\mathcal{O}_{X,\bar{x}}$.*

Proof. By Lemma 7.5 we see that $U \in X_{\text{étale}}$ affine $\mathcal{S}_X(U) \subset \mathcal{O}_X(U)$ is the set of nonzerodivisors in $\mathcal{O}_X(U)$. Thus the presheaf $\mathcal{S}^{-1}\mathcal{O}_X$ is equal to

$$U \mapsto Q(\mathcal{O}_X(U))$$

on $X_{\text{affine, étale}}$, with notation as in Algebra, Example 9.8. Observe that the codimension 0 points of X correspond to the generic points of U , see Properties of Spaces, Lemma 11.1. Hence if $U = \text{Spec}(A)$, then A is a ring with finitely many minimal primes such that any weakly associated prime of A is minimal. The same is true for any étale extension of A (because the spectrum of such is an affine scheme étale over X hence can play the role of A in the previous sentence). In order to show that our presheaf is a sheaf and quasi-coherent it suffices to show that

$$Q(A) \otimes_A B \longrightarrow Q(B)$$

is an isomorphism when $A \rightarrow B$ is an étale ring map, see Properties of Spaces, Lemma 29.3. (To define the displayed arrow, observe that since $A \rightarrow B$ is flat it maps nonzerodivisors to nonzerodivisors.) By Algebra, Lemmas 25.4 and 66.7. we have

$$Q(A) = \prod_{\mathfrak{p} \subset A \text{ minimal}} A_{\mathfrak{p}} \quad \text{and} \quad Q(B) = \prod_{\mathfrak{q} \subset B \text{ minimal}} B_{\mathfrak{q}}$$

Since $A \rightarrow B$ is étale, the minimal primes of B are exactly the primes of B lying over the minimal primes of A (for example by More on Algebra, Lemma 44.2). By Algebra, Lemmas 153.10, 153.3 (13), and 153.5 we see that $A_{\mathfrak{p}} \otimes_A B$ is a finite

product of local rings finite étale over $A_{\mathfrak{p}}$. This clearly implies that $A_{\mathfrak{p}} \otimes_A B = \prod_{\mathfrak{q} \text{ lies over } \mathfrak{p}} B_{\mathfrak{q}}$ as desired.

At this point we know that (1) and (2) hold. Proof of (3). Let $s \in \mathcal{O}_{X, \bar{x}}$ be a nonzerodivisor. Then we can find an étale neighbourhood $(U, \bar{u}) \rightarrow (X, \bar{x})$ and $f \in \mathcal{O}_X(U)$ mapping to s . Let $u \in U$ be the point determined by \bar{u} . Since $\mathcal{O}_{U, u} \rightarrow \mathcal{O}_{X, \bar{x}}$ is faithfully flat (as a strict henselization), we see that f maps to a nonzerodivisor in $\mathcal{O}_{U, u}$. By Divisors, Lemma 23.6 after shrinking U we find that f is a nonzerodivisor and hence a section of $\mathcal{S}_X(U)$. Part (4) follows from (3) by computing stalks. \square

Lemma 10.5. *Let S be a scheme. Let X be an algebraic space over S . Assume*

- (a) *every weakly associated point of X is a point of codimension 0, and*
- (b) *X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 49.1.*
- (c) *X is representable by a scheme X_0 (awkward but temporary notation).*

Then the sheaf of meromorphic functions \mathcal{K}_X is the quasi-coherent sheaf of \mathcal{O}_X -algebras associated to the quasi-coherent sheaf of meromorphic functions \mathcal{K}_{X_0} .

Proof. For the equivalence between $QCoh(\mathcal{O}_X)$ and $QCoh(\mathcal{O}_{X_0})$, please see Properties of Spaces, Section 29. The lemma is true because \mathcal{K}_X and \mathcal{K}_{X_0} are quasi-coherent and have the same value on corresponding affine opens of X and X_0 by Lemma 10.4 and Divisors, Lemma 23.6. \square

Definition 10.6. Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . We say that *pullbacks of meromorphic functions are defined for f* if for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array}$$

with $U \in X_{\text{étale}}$ and $V \in Y_{\text{étale}}$ and any section $s \in \mathcal{S}_Y(V)$ the pullback $f^{\#}(s) \in \mathcal{O}_X(U)$ is an element of $\mathcal{S}_X(U)$.

In this case there is an induced map $f^{\#} : f_{\text{small}}^{-1} \mathcal{K}_Y \rightarrow \mathcal{K}_X$, in other words we obtain a commutative diagram of morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(X_{\text{étale}}), \mathcal{K}_X) & \longrightarrow & (Sh(X_{\text{étale}}), \mathcal{O}_X) \\ \downarrow f_{\text{small}} & & \downarrow f_{\text{small}} \\ (Sh(Y_{\text{étale}}), \mathcal{K}_Y) & \longrightarrow & (Sh(Y_{\text{étale}}), \mathcal{O}_Y) \end{array}$$

We sometimes denote $f^*(s) = f^{\#}(s)$ for a section $s \in \Gamma(Y, \mathcal{K}_Y)$.

Lemma 10.7. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Pullbacks of meromorphic sections are defined in each of the following cases*

- (1) *weakly associated points of X are mapped to points of codimension 0 on Y ,*
- (2) *f is flat,*
- (3) *add more here as needed.*

Proof. Working étale locally, this translates into the case of schemes, see Divisors, Lemma 23.5. To do the translation use Lemma 7.5 (description of regular sections), Definition 2.2 (definition of weakly associated points), and Properties of Spaces, Lemma 11.1 (description of codimension 0 points). \square

Lemma 10.8. *Let S be a scheme. Let X be an algebraic space over S . Assume*

- (a) *every weakly associated point of X is a point of codimension 0, and*
- (b) *X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 49.1,*
- (c) *every codimension 0 point of X can be represented by a monomorphism $\mathrm{Spec}(k) \rightarrow X$.*

Let $X^0 \subset |X|$ be the set of codimension 0 points of X . Then we have

$$\mathcal{K}_X = \bigoplus_{\eta \in X^0} j_{\eta,*} \mathcal{O}_{X,\eta} = \prod_{\eta \in X^0} j_{\eta,*} \mathcal{O}_{X,\eta}$$

where $j_\eta : \mathrm{Spec}(\mathcal{O}_{X,\eta}) \rightarrow X$ is the canonical map of Schemes, Section 13; this makes sense because X^0 is contained in the schematic locus of X . Similarly, for every quasi-coherent \mathcal{O}_X -module \mathcal{F} we obtain the formula

$$\mathcal{K}_X(\mathcal{F}) = \bigoplus_{\eta \in X^0} j_{\eta,*} \mathcal{F}_\eta = \prod_{\eta \in X^0} j_{\eta,*} \mathcal{F}_\eta$$

for the sheaf of meromorphic sections of \mathcal{F} . Finally, the ring of rational functions of X is the ring of meromorphic functions on X , in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.

Proof. By Decent Spaces, Lemma 20.3 and Section 6 we see that X is decent¹. Thus $X^0 \subset |X|$ is the set of generic points of irreducible components (Decent Spaces, Lemma 20.1) and X^0 is locally finite in $|X|$ by (b). It follows that X^0 is contained in every dense open subset of $|X|$. In particular, X^0 is contained in the schematic locus (Decent Spaces, Theorem 10.2). Thus the local rings $\mathcal{O}_{X,\eta}$ and the morphisms j_η are defined.

Observe that a locally finite direct sum of sheaves of modules is equal to the product. This and the fact that X^0 is locally finite in $|X|$ explains the equalities between direct sums and products in the statement. Then since $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ we see that the second equality follows from the first.

Let $j : Y = \prod_{\eta \in X^0} \mathrm{Spec}(\mathcal{O}_{X,\eta}) \rightarrow X$ be the product of the morphisms j_η . We have to show that $\mathcal{K}_X = j_* \mathcal{O}_Y$. Observe that $\mathcal{K}_Y = \mathcal{O}_Y$ as Y is a disjoint union of spectra of local rings of dimension 0: in a local ring of dimension zero any nonzerodivisor is a unit. Next, note that pullbacks of meromorphic functions are defined for j by Lemma 10.7. This gives a map

$$\mathcal{K}_X \longrightarrow j_* \mathcal{O}_Y.$$

Let $U \in X_{\text{étale}}$ be affine. By Lemma 10.4 the left hand side evaluates to total ring of fractions of $\mathcal{O}_X(U)$. On the other hand, the right hand side is equal to the product of the local rings of U at the codimension 0 points, i.e., the generic points of U . These two rings are equal (as we already saw in the proof of Lemma 10.4) by Algebra, Lemmas 25.4 and 66.7. Thus our map is an isomorphism.

Finally, we have to show that $R(X) = \Gamma(X, \mathcal{K}_X)$. This follows from the case of schemes (Divisors, Lemma 23.6) applied to the schematic locus $X' \subset X$. Namely, the ring of rational functions of X is by definition the same as the ring of rational functions on X' as it is a dense open subspace of X (see above). Certainly, $R(X')$

¹Conversely, if X is decent, then condition (c) holds automatically.

agrees with the ring of rational functions when X' is viewed as a scheme. On the other hand, by our description of \mathcal{K}_X above, and the fact, seen above, that $X^0 \subset |X'|$ is contained in any dense open, we see that $\Gamma(X, \mathcal{K}_X) = \Gamma(X', \mathcal{K}_{X'})$. Finally, use the compatibility recorded in Lemma 10.5. \square

Definition 10.9. Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{L} be an invertible \mathcal{O}_X -module. A meromorphic section s of \mathcal{L} is said to be *regular* if the induced map $\mathcal{K}_X \rightarrow \mathcal{K}_X(\mathcal{L})$ is injective.

Let us spell out when (regular) meromorphic sections can be pulled back.

Lemma 10.10. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume that pullbacks of meromorphic functions are defined for f (see Definition 10.6).*

- (1) *Let \mathcal{F} be a sheaf of \mathcal{O}_Y -modules. There is a canonical pullback map $f^* : \Gamma(Y, \mathcal{K}_Y(\mathcal{F})) \rightarrow \Gamma(X, \mathcal{K}_X(f^*\mathcal{F}))$ for meromorphic sections of \mathcal{F} .*
- (2) *Let \mathcal{L} be an invertible \mathcal{O}_X -module. A regular meromorphic section s of \mathcal{L} pulls back to a regular meromorphic section f^*s of $f^*\mathcal{L}$.*

Proof. Omitted. \square

Lemma 10.11. *Let S be a scheme. Let X be an algebraic space over S satisfying (a), (b), and (c) of Lemma 10.8. Then every invertible \mathcal{O}_X -module \mathcal{L} has a regular meromorphic section.*

Proof. With notation as in Lemma 10.8 the stalk \mathcal{L}_η of \mathcal{L} at η is defined for all $\eta \in X^0$ and it is a rank 1 free $\mathcal{O}_{X,\eta}$ -module. Pick a generator $s_\eta \in \mathcal{L}_\eta$ for all $\eta \in X^0$. It follows immediately from the description of \mathcal{K}_X and $\mathcal{K}_X(\mathcal{L})$ in Lemma 10.8 that $s = \prod s_\eta$ is a regular meromorphic section of \mathcal{L} . \square

11. Relative Proj

This section revisits the construction of the relative proj in the setting of algebraic spaces. The material in this section corresponds to the material in Constructions, Section 16 and Divisors, Section 30 in the case of schemes.

Situation 11.1. Here S is a scheme, X is an algebraic space over S , and \mathcal{A} is a quasi-coherent graded \mathcal{O}_X -algebra.

In Situation 11.1 we are going to define a functor $F : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ which will turn out to be an algebraic space. We will follow (mutatis mutandis) the procedure of Constructions, Section 16. First, given a scheme T over S we define a *quadruple over T* to be a system $(d, f : T \rightarrow X, \mathcal{L}, \psi)$

- (1) $d \geq 1$ is an integer,
- (2) $f : T \rightarrow X$ is a morphism over S ,
- (3) \mathcal{L} is an invertible \mathcal{O}_T -module, and
- (4) $\psi : f^*\mathcal{A}^{(d)} \rightarrow \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a homomorphism of graded \mathcal{O}_T -algebras such that $f^*\mathcal{A}_d \rightarrow \mathcal{L}$ is surjective.

We say two quadruples $(d, f, \mathcal{L}, \psi)$ and $(d', f', \mathcal{L}', \psi')$ are *equivalent*² if and only if we have $f = f'$ and for some positive integer $m = ad = a'd'$ there exists an

²This definition is motivated by Constructions, Lemma 16.4. The advantage of choosing this one is that it clearly defines an equivalence relation.

isomorphism $\beta : \mathcal{L}^{\otimes a} \rightarrow (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \rightarrow \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes mn}$. Given a quadruple $(d, f, \mathcal{L}, \psi)$ and a morphism $h : T' \rightarrow T$ we have the pullback $(d, f \circ h, h^*\mathcal{L}, h^*\psi)$. Pullback preserves the equivalence relation. Finally, for a *quasi-compact* scheme T over S we set

$$F(T) = \text{the set of equivalence classes of quadruples over } T$$

and for an arbitrary scheme T over S we set

$$F(T) = \lim_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T . Thus we have defined our functor

$$(11.1.1) \quad F : \text{Sch}^{opp} \longrightarrow \text{Sets}$$

There is a morphism $F \rightarrow X$ of functors sending the quadruple $(d, f, \mathcal{L}, \psi)$ to f .

Lemma 11.2. *In Situation 11.1. The functor F above is an algebraic space. For any morphism $g : Z \rightarrow X$ where Z is a scheme there is a canonical isomorphism $\text{Proj}_Z(g^*\mathcal{A}) = Z \times_X F$ compatible with further base change.*

Proof. It suffices to prove the second assertion, see Spaces, Lemma 11.3. Let $g : Z \rightarrow X$ be a morphism where Z is a scheme. Let F' be the functor of quadruples associated to the graded quasi-coherent \mathcal{O}_Z -algebra $g^*\mathcal{A}$. Then there is a canonical isomorphism $F' = Z \times_X F$, sending a quadruple $(d, f : T \rightarrow Z, \mathcal{L}, \psi)$ for F' to $(d, g \circ f, \mathcal{L}, \psi)$ (details omitted, see proof of Constructions, Lemma 16.1). By Constructions, Lemmas 16.4, 16.5, and 16.6 and Definition 16.7 we see that F' is representable by $\text{Proj}_Z(g^*\mathcal{A})$. \square

The lemma above tells us the following definition makes sense.

Definition 11.3. Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{A} be a quasi-coherent sheaf of graded \mathcal{O}_X -algebras. The *relative homogeneous spectrum of \mathcal{A} over X* , or the *homogeneous spectrum of \mathcal{A} over X* , or the *relative Proj of \mathcal{A} over X* is the algebraic space F over X of Lemma 11.2. We denote it $\pi : \text{Proj}_X(\mathcal{A}) \rightarrow X$.

In particular the structure morphism of the relative Proj is representable by construction. We can also think about the relative Proj via glueing. Let $\varphi : U \rightarrow X$ be a surjective étale morphism, where U is a scheme. Set $R = U \times_X U$ with projection morphisms $s, t : R \rightarrow U$. By Lemma 11.2 there exists a canonical isomorphism

$$\gamma : \text{Proj}_U(\varphi^*\mathcal{A}) \longrightarrow \text{Proj}_X(\mathcal{A}) \times_X U$$

over U . Let $\alpha : t^*\varphi^*\mathcal{A} \rightarrow s^*\varphi^*\mathcal{A}$ be the canonical isomorphism of Properties of Spaces, Proposition 32.1. Then the diagram

$$\begin{array}{ccc}
 & \underline{\text{Proj}}_U(\varphi^*\mathcal{A}) \times_{U,s} R & \equiv \underline{\text{Proj}}_R(s^*\varphi^*\mathcal{A}) \\
 \nearrow s^*\gamma & & \downarrow \text{induced by } \alpha \\
 \underline{\text{Proj}}_X(\mathcal{A}) \times_X R & & \\
 \searrow t^*\gamma & & \downarrow \\
 & \underline{\text{Proj}}_U(\varphi^*\mathcal{A}) \times_{U,t} R & \equiv \underline{\text{Proj}}_R(t^*\varphi^*\mathcal{A})
 \end{array}$$

is commutative (the equal signs come from Constructions, Lemma 16.10). Thus, if we denote $\mathcal{A}_U, \mathcal{A}_R$ the pullback of \mathcal{A} to U, R , then $P = \underline{\text{Proj}}_X(\mathcal{A})$ has an étale covering by the scheme $P_U = \underline{\text{Proj}}_U(\mathcal{A}_U)$ and $P_U \times_P P_U$ is equal to $P_R = \underline{\text{Proj}}_R(\mathcal{A}_R)$. Using these remarks we can argue in the usual fashion using étale localization to transfer results on the relative proj from the case of schemes to the case of algebraic spaces.

Lemma 11.4. *In Situation 11.1. The relative Proj comes equipped with a quasi-coherent sheaf of \mathbf{Z} -graded algebras $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_X(\mathcal{A})}(n)$ and a canonical homomorphism of graded algebras*

$$\psi : \pi^*\mathcal{A} \longrightarrow \bigoplus_{n \geq 0} \mathcal{O}_{\underline{\text{Proj}}_X(\mathcal{A})}(n)$$

whose base change to any scheme over X agrees with Constructions, Lemma 15.5.

Proof. As in the discussion following Definition 11.3 choose a scheme U and a surjective étale morphism $U \rightarrow X$, set $R = U \times_X U$ with projections $s, t : R \rightarrow U$, $\mathcal{A}_U = \mathcal{A}|_U$, $\mathcal{A}_R = \mathcal{A}|_R$, and $\pi : P = \underline{\text{Proj}}_X(\mathcal{A}) \rightarrow X$, $\pi_U : P_U = \underline{\text{Proj}}_U(\mathcal{A}_U) \rightarrow U$ and $\pi_R : P_R = \underline{\text{Proj}}_R(\mathcal{A}_R) \rightarrow R$. By the Constructions, Lemma 15.5 we have a quasi-coherent sheaf of \mathbf{Z} -graded \mathcal{O}_{P_U} -algebras $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_{P_U}(n)$ and a canonical map $\psi_U : \pi_U^*\mathcal{A}_U \rightarrow \bigoplus_{n \geq 0} \mathcal{O}_{P_U}(n)$ and similarly for P_R . By Constructions, Lemma 16.10 the pullback of $\mathcal{O}_{P_U}(n)$ and ψ_U by either projection $P_R \rightarrow P_U$ is equal to $\mathcal{O}_{P_R}(n)$ and ψ_R . By Properties of Spaces, Proposition 32.1 we obtain $\mathcal{O}_P(n)$ and ψ . We omit the verification of compatibility with pullback to arbitrary schemes over X . \square

Having constructed the relative Proj we turn to some basic properties.

Lemma 11.5. *Let S be a scheme. Let $g : X' \rightarrow X$ be a morphism of algebraic spaces over S and let \mathcal{A} be a quasi-coherent sheaf of graded \mathcal{O}_X -algebras. Then there is a canonical isomorphism*

$$r : \underline{\text{Proj}}_{X'}(g^*\mathcal{A}) \longrightarrow X' \times_X \underline{\text{Proj}}_X(\mathcal{A})$$

as well as a corresponding isomorphism

$$\theta : r^*pr_2^* \left(\bigoplus_{d \in \mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_X(\mathcal{A})}(d) \right) \longrightarrow \bigoplus_{d \in \mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_{X'}(g^*\mathcal{A})}(d)$$

of \mathbf{Z} -graded $\mathcal{O}_{\underline{\text{Proj}}_{X'}(g^*\mathcal{A})}$ -algebras.

Proof. Let F be the functor (11.1.1) and let F' be the corresponding functor defined using $g^*\mathcal{A}$ on X' . We claim there is a canonical isomorphism $r : F' \rightarrow X' \times_X F$ of functors (and of course r is the isomorphism of the lemma). It suffices

to construct the bijection $r : F'(T) \rightarrow X'(T) \times_{X(T)} F(T)$ for quasi-compact schemes T over S . First, if $\xi = (d', f', \mathcal{L}', \psi')$ is a quadruple over T for F' , then we can set $r(\xi) = (f', (d', g \circ f', \mathcal{L}', \psi'))$. This makes sense as $(g \circ f')^* \mathcal{A}^{(d)} = (f')^* (g^* \mathcal{A})^{(d)}$. The inverse map sends the pair $(f', (d, f, \mathcal{L}, \psi))$ to the quadruple $(d, f', \mathcal{L}, \psi)$. We omit the proof of the final assertion (hint: reduce to the case of schemes by étale localization and apply Constructions, Lemma 16.10). \square

Lemma 11.6. *In Situation 11.1 the morphism $\pi : \underline{\text{Proj}}_X(\mathcal{A}) \rightarrow X$ is separated.*

Proof. By Morphisms of Spaces, Lemma 4.12 and the construction of the relative Proj this follows from the case of schemes which is Constructions, Lemma 16.9. \square

Lemma 11.7. *In Situation 11.1. If one of the following holds*

- (1) \mathcal{A} is of finite type as a sheaf of \mathcal{A}_0 -algebras,
- (2) \mathcal{A} is generated by \mathcal{A}_1 as an \mathcal{A}_0 -algebra and \mathcal{A}_1 is a finite type \mathcal{A}_0 -module,
- (3) there exists a finite type quasi-coherent \mathcal{A}_0 -submodule $\mathcal{F} \subset \mathcal{A}_+$ such that $\mathcal{A}_+/\mathcal{F}\mathcal{A}$ is a locally nilpotent sheaf of ideals of $\mathcal{A}/\mathcal{F}\mathcal{A}$,

then $\pi : \underline{\text{Proj}}_X(\mathcal{A}) \rightarrow X$ is quasi-compact.

Proof. By Morphisms of Spaces, Lemma 8.8 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 30.1. \square

Lemma 11.8. *In Situation 11.1. If \mathcal{A} is of finite type as a sheaf of \mathcal{O}_X -algebras, then $\pi : \underline{\text{Proj}}_X(\mathcal{A}) \rightarrow X$ is of finite type.*

Proof. By Morphisms of Spaces, Lemma 23.4 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 30.2. \square

Lemma 11.9. *In Situation 11.1. If $\mathcal{O}_X \rightarrow \mathcal{A}_0$ is an integral algebra map³ and \mathcal{A} is of finite type as an \mathcal{A}_0 -algebra, then $\pi : \underline{\text{Proj}}_X(\mathcal{A}) \rightarrow X$ is universally closed.*

Proof. By Morphisms of Spaces, Lemma 9.5 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 30.3. \square

Lemma 11.10. *In Situation 11.1. The following conditions are equivalent*

- (1) \mathcal{A}_0 is a finite type \mathcal{O}_X -module and \mathcal{A} is of finite type as an \mathcal{A}_0 -algebra,
- (2) \mathcal{A}_0 is a finite type \mathcal{O}_X -module and \mathcal{A} is of finite type as an \mathcal{O}_X -algebra.

If these conditions hold, then $\pi : \underline{\text{Proj}}_X(\mathcal{A}) \rightarrow X$ is proper.

Proof. By Morphisms of Spaces, Lemma 40.2 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 30.3. \square

Lemma 11.11. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{A} be a quasi-coherent sheaf of graded \mathcal{O}_X -modules generated as an \mathcal{A}_0 -algebra by \mathcal{A}_1 . With $P = \underline{\text{Proj}}_X(\mathcal{A})$ we have*

- (1) P represents the functor F_1 which associates to T over S the set of isomorphism classes of triples (f, \mathcal{L}, ψ) , where $f : T \rightarrow X$ is a morphism over S , \mathcal{L} is an invertible \mathcal{O}_T -module, and $\psi : f^* \mathcal{A} \rightarrow \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a map of graded \mathcal{O}_T -algebras inducing a surjection $f^* \mathcal{A}_1 \rightarrow \mathcal{L}$,
- (2) the canonical map $\pi^* \mathcal{A}_1 \rightarrow \mathcal{O}_P(1)$ is surjective, and

³In other words, the integral closure of \mathcal{O}_X in \mathcal{A}_0 , see Morphisms of Spaces, Definition 48.2, equals \mathcal{A}_0 .

- (3) each $\mathcal{O}_P(n)$ is invertible and the multiplication maps induce isomorphisms $\mathcal{O}_P(n) \otimes_{\mathcal{O}_P} \mathcal{O}_P(m) = \mathcal{O}_P(n+m)$.

Proof. Omitted. See Constructions, Lemma 16.11 for the case of schemes. \square

12. Functoriality of relative proj

This section is the analogue of Constructions, Section 18.

Lemma 12.1. *Let S be a scheme. Let X be an algebraic space over S . Let $\psi : \mathcal{A} \rightarrow \mathcal{B}$ be a map of quasi-coherent graded \mathcal{O}_X -algebras. Set $P = \underline{\text{Proj}}_X(\mathcal{A}) \rightarrow X$ and $Q = \underline{\text{Proj}}_X(\mathcal{B}) \rightarrow X$. There is a canonical open subspace $U(\psi) \subset Q$ and a canonical morphism of algebraic spaces*

$$r_\psi : U(\psi) \longrightarrow P$$

over X and a map of \mathbf{Z} -graded $\mathcal{O}_{U(\psi)}$ -algebras

$$\theta = \theta_\psi : r_\psi^* \left(\bigoplus_{d \in \mathbf{Z}} \mathcal{O}_P(d) \right) \longrightarrow \bigoplus_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the property that for any scheme W étale over X the triple

$$(U(\psi) \times_X W, \quad r_\psi|_{U(\psi) \times_X W} : U(\psi) \times_X W \rightarrow P \times_X W, \quad \theta|_{U(\psi) \times_X W})$$

is equal to the triple associated to $\psi : \mathcal{A}|_W \rightarrow \mathcal{B}|_W$ of Constructions, Lemma 18.1.

Proof. This lemma follows from étale localization and the case of schemes, see discussion following Definition 11.3. Details omitted. \square

Lemma 12.2. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{A}, \mathcal{B} , and \mathcal{C} be quasi-coherent graded \mathcal{O}_X -algebras. Set $P = \underline{\text{Proj}}_X(\mathcal{A})$, $Q = \underline{\text{Proj}}_X(\mathcal{B})$ and $R = \underline{\text{Proj}}_X(\mathcal{C})$. Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$, $\psi : \mathcal{B} \rightarrow \mathcal{C}$ be graded \mathcal{O}_X -algebra maps. Then we have*

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \text{and} \quad r_{\psi \circ \varphi} = r_\varphi \circ r_\psi|_{U(\psi \circ \varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi \circ \varphi}$$

with obvious notation.

Proof. Omitted. \square

Lemma 12.3. *With hypotheses and notation as in Lemma 12.1 above. Assume $\mathcal{A}_d \rightarrow \mathcal{B}_d$ is surjective for $d \gg 0$. Then*

- (1) $U(\psi) = Q$,
- (2) $r_\psi : Q \rightarrow P$ is a closed immersion, and
- (3) the maps $\theta : r_\psi^* \mathcal{O}_P(n) \rightarrow \mathcal{O}_Q(n)$ are surjective but not isomorphisms in general (even if $\mathcal{A} \rightarrow \mathcal{B}$ is surjective).

Proof. Follows from the case of schemes (Constructions, Lemma 18.3) by étale localization. \square

Lemma 12.4. *With hypotheses and notation as in Lemma 12.1 above. Assume $\mathcal{A}_d \rightarrow \mathcal{B}_d$ is an isomorphism for all $d \gg 0$. Then*

- (1) $U(\psi) = Q$,
- (2) $r_\psi : Q \rightarrow P$ is an isomorphism, and
- (3) the maps $\theta : r_\psi^* \mathcal{O}_P(n) \rightarrow \mathcal{O}_Q(n)$ are isomorphisms.

Proof. Follows from the case of schemes (Constructions, Lemma 18.4) by étale localization. \square

Lemma 12.5. *With hypotheses and notation as in Lemma 12.1 above. Assume $\mathcal{A}_d \rightarrow \mathcal{B}_d$ is surjective for $d \gg 0$ and that \mathcal{A} is generated by \mathcal{A}_1 over \mathcal{A}_0 . Then*

- (1) $U(\psi) = Q$,
- (2) $r_\psi : Q \rightarrow P$ is a closed immersion, and
- (3) the maps $\theta : r_\psi^* \mathcal{O}_P(n) \rightarrow \mathcal{O}_Q(n)$ are isomorphisms.

Proof. Follows from the case of schemes (Constructions, Lemma 18.5) by étale localization. \square

13. Invertible sheaves and morphisms into relative Proj

It seems that we may need the following lemma somewhere. The situation is the following:

- (1) Let S be a scheme and Y an algebraic space over S .
- (2) Let \mathcal{A} be a quasi-coherent graded \mathcal{O}_Y -algebra.
- (3) Denote $\pi : \underline{\text{Proj}}_Y(\mathcal{A}) \rightarrow Y$ the relative Proj of \mathcal{A} over Y .
- (4) Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S .
- (5) Let \mathcal{L} be an invertible \mathcal{O}_X -module.
- (6) Let $\psi : f^* \mathcal{A} \rightarrow \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ be a homomorphism of graded \mathcal{O}_X -algebras.

Given this data let $U(\psi) \subset X$ be the open subspace with

$$|U(\psi)| = \bigcup_{d \geq 1} \{\text{locus where } f^* \mathcal{A}_d \rightarrow \mathcal{L}^{\otimes d} \text{ is surjective}\}$$

Formation of $U(\psi) \subset X$ commutes with pullback by any morphism $X' \rightarrow X$.

Lemma 13.1. *With assumptions and notation as above. The morphism ψ induces a canonical morphism of algebraic spaces over Y*

$$r_{\mathcal{L}, \psi} : U(\psi) \longrightarrow \underline{\text{Proj}}_Y(\mathcal{A})$$

together with a map of graded $\mathcal{O}_{U(\psi)}$ -algebras

$$\theta : r_{\mathcal{L}, \psi}^* \left(\bigoplus_{d \geq 0} \mathcal{O}_{\underline{\text{Proj}}_Y(\mathcal{A})}(d) \right) \longrightarrow \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}|_{U(\psi)}$$

characterized by the following properties:

- (1) For $V \rightarrow Y$ étale and $d \geq 0$ the diagram

$$\begin{array}{ccc} \mathcal{A}_d(V) & \xrightarrow{\quad \psi \quad} & \Gamma(V \times_Y X, \mathcal{L}^{\otimes d}) \\ \psi \downarrow & & \downarrow \text{restrict} \\ \Gamma(V \times_Y \underline{\text{Proj}}_Y(\mathcal{A}), \mathcal{O}_{\underline{\text{Proj}}_Y(\mathcal{A})}(d)) & \xrightarrow{\quad \theta \quad} & \Gamma(V \times_Y U(\psi), \mathcal{L}^{\otimes d}) \end{array}$$

is commutative.

- (2) For any $d \geq 1$ and any morphism $W \rightarrow X$ where W is a scheme such that $\psi|_W : f^* \mathcal{A}_d|_W \rightarrow \mathcal{L}^{\otimes d}|_W$ is surjective we have (a) $W \rightarrow X$ factors through $U(\psi)$ and (b) composition of $W \rightarrow U(\psi)$ with $r_{\mathcal{L}, \psi}$ agrees with the morphism $W \rightarrow \underline{\text{Proj}}_Y(\mathcal{A})$ which exists by the construction of $\underline{\text{Proj}}_Y(\mathcal{A})$, see Definition 11.3.

(3) Consider a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & g' & \downarrow f \\ Y' & \xrightarrow{\quad} & Y \end{array}$$

where X' and Y' are schemes, set $\mathcal{A}' = g'^*\mathcal{A}$ and $\mathcal{L}' = (g')^*\mathcal{L}$ and denote $\psi' : (f')^*\mathcal{A} \rightarrow \bigoplus_{d \geq 0} (\mathcal{L}')^{\otimes d}$ the pullback of ψ . Let $U(\psi')$, $r_{\psi', \mathcal{L}'}$, and θ' be the open, morphism, and homomorphism constructed in Constructions, Lemma 13.1. Then $U(\psi') = (g')^{-1}(U(\psi))$ and $r_{\psi', \mathcal{L}'}$ agrees with the base change of $r_{\psi, \mathcal{L}}$ via the isomorphism $\underline{\text{Proj}}_{Y'}(\mathcal{A}') = Y' \times_Y \underline{\text{Proj}}_Y(\mathcal{A})$ of Lemma 11.5. Moreover, θ' is the pullback of θ .

Proof. Omitted. Hints: First we observe that for a quasi-compact scheme W over X the following are equivalent

- (1) $W \rightarrow X$ factors through $U(\psi)$, and
- (2) there exists a d such that $\psi|_W : f^*\mathcal{A}_d|_W \rightarrow \mathcal{L}^{\otimes d}|_W$ is surjective.

This gives a description of $U(\psi)$ as a subfunctor of X on our base category $(\text{Sch}/S)_{fppf}$. For such a W and d we consider the quadruple $(d, W \rightarrow Y, \mathcal{L}|_W, \psi^{(d)}|_W)$. By definition of $\underline{\text{Proj}}_Y(\mathcal{A})$ we obtain a morphism $W \rightarrow \underline{\text{Proj}}_Y(\mathcal{A})$. By our notion of equivalence of quadruples one sees that this morphism is independent of the choice of d . This clearly defines a transformation of functors $r_{\psi, \mathcal{L}} : U(\psi) \rightarrow \underline{\text{Proj}}_Y(\mathcal{A})$, i.e., a morphism of algebraic spaces. By construction this morphism satisfies (2). Since the morphism constructed in Constructions, Lemma 19.1 satisfies the same property, we see that (3) is true.

To construct θ and check the compatibility (1) of the lemma, work étale locally on Y and X , arguing as in the discussion following Definition 11.3. \square

14. Relatively ample sheaves

This section is the analogue of Morphisms, Section 37 for algebraic spaces. Our definition of a relatively ample invertible sheaf is as follows.

Definition 14.1. Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let \mathcal{L} be an invertible \mathcal{O}_X -module. We say \mathcal{L} is *relatively ample*, or *f-relatively ample*, or *ample on X/Y* , or *f-ample* if $f : X \rightarrow Y$ is representable and for every morphism $Z \rightarrow Y$ where Z is a scheme, the pullback \mathcal{L}_Z of \mathcal{L} to $X_Z = Z \times_Y X$ is ample on X_Z/Z as in Morphisms, Definition 37.1.

We will almost always reduce questions about relatively ample invertible sheaves to the case of schemes. Thus in this section we have mainly sanity checks.

Lemma 14.2. Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let \mathcal{L} be an invertible \mathcal{O}_X -module. Assume Y is a scheme. The following are equivalent

- (1) \mathcal{L} is ample on X/Y in the sense of Definition 14.1, and
- (2) X is a scheme and \mathcal{L} is ample on X/Y in the sense of Morphisms, Definition 37.1.

Proof. This follows from the definitions and Morphisms, Lemma 37.9 (which says that being relatively ample for schemes is preserved under base change). \square

Lemma 14.3. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $Y' \rightarrow Y$ be a morphism of algebraic spaces over S . Let $f' : X' \rightarrow Y'$ be the base change of f and denote \mathcal{L}' the pullback of \mathcal{L} to X' . If \mathcal{L} is f -ample, then \mathcal{L}' is f' -ample.*

Proof. This follows immediately from the definition! (Hint: transitivity of base change.) \square

Lemma 14.4. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . If there exists an f -ample invertible sheaf, then f is representable, quasi-compact, and separated.*

Proof. This is clear from the definitions and Morphisms, Lemma 37.3. (If in doubt, take a look at the principle of Algebraic Spaces, Lemma 5.8.) \square

Lemma 14.5. *Let $V \rightarrow U$ be a surjective étale morphism of affine schemes. Let X be an algebraic space over U . Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $Y = V \times_U X$ and let \mathcal{N} be the pullback of \mathcal{L} to Y . The following are equivalent*

- (1) \mathcal{L} is ample on X/U , and
- (2) \mathcal{N} is ample on Y/V .

Proof. The implication (1) \Rightarrow (2) follows from Lemma 14.3. Assume (2). This implies that $Y \rightarrow V$ is quasi-compact and separated (Lemma 14.4) and Y is a scheme. It follows that the morphism $f : X \rightarrow U$ is quasi-compact and separated (Morphisms of Spaces, Lemmas 8.8 and 4.12). Set $\mathcal{A} = \bigoplus_{d \geq 0} f_* \mathcal{L}^{\otimes d}$. This is a quasi-coherent sheaf of graded \mathcal{O}_U -algebras (Morphisms of Spaces, Lemma 11.2). By adjunction we have a map $\psi : f^* \mathcal{A} \rightarrow \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$. Applying Lemma 13.1 we obtain an open subspace $U(\psi) \subset X$ and a morphism

$$r_{\mathcal{L}, \psi} : U(\psi) \rightarrow \underline{\mathrm{Proj}}_U(\mathcal{A})$$

Since $h : V \rightarrow U$ is étale we have $\mathcal{A}|_V = (Y \rightarrow V)_* (\bigoplus_{d \geq 0} \mathcal{N}^{\otimes d})$, see Properties of Spaces, Lemma 26.2. It follows that the pullback ψ' of ψ to Y is the adjunction map for the situation $(Y \rightarrow V, \mathcal{N})$ as in Morphisms, Lemma 37.4 part (5). Since \mathcal{N} is ample on Y/V we conclude from the lemma just cited that $U(\psi') = Y$ and that $r_{\mathcal{N}, \psi'}$ is an open immersion. Since Lemma 13.1 tells us that the formation of $r_{\mathcal{L}, \psi}$ commutes with base change, we conclude that $U(\psi) = X$ and that we have a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{r'} & \underline{\mathrm{Proj}}_V(\mathcal{A}|_V) & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{r} & \underline{\mathrm{Proj}}_U(\mathcal{A}) & \longrightarrow & U \end{array}$$

whose squares are fibre products. We conclude that r is an open immersion by Morphisms of Spaces, Lemma 12.1. Thus X is a scheme. Then we can apply Morphisms, Lemma 37.4 part (5) to conclude that \mathcal{L} is ample on X/U . \square

Lemma 14.6. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Let \mathcal{L} be an invertible \mathcal{O}_X -module. The following are equivalent*

- (1) \mathcal{L} is ample on X/Y ,
- (2) for every scheme Z and every morphism $Z \rightarrow Y$ the algebraic space $X_Z = Z \times_Y X$ is a scheme and the pullback \mathcal{L}_Z is ample on X_Z/Z ,

- (3) for every affine scheme Z and every morphism $Z \rightarrow Y$ the algebraic space $X_Z = Z \times_Y X$ is a scheme and the pullback \mathcal{L}_Z is ample on X_Z/Z ,
- (4) there exists a scheme V and a surjective étale morphism $V \rightarrow Y$ such that the algebraic space $X_V = V \times_Y X$ is a scheme and the pullback \mathcal{L}_V is ample on X_V/V .

Proof. Parts (1) and (2) are equivalent by definition. The implication (2) \Rightarrow (3) is immediate. If (3) holds and $Z \rightarrow Y$ is as in (2), then we see that $X_Z \rightarrow Z$ is affine locally on Z representable. Hence X_Z is a scheme for example by Properties of Spaces, Lemma 13.1. Then it follows that \mathcal{L}_Z is ample on X_Z/Z because it holds locally on Z and we can use Morphisms, Lemma 37.4. Thus (1), (2), and (3) are equivalent. Clearly these conditions imply (4).

Assume (4). Let $Z \rightarrow Y$ be a morphism with Z affine. Then $U = V \times_Y Z \rightarrow Z$ is a surjective étale morphism such that the pullback of \mathcal{L}_Z by $X_U \rightarrow X_Z$ is relatively ample on X_U/U . Of course we may replace U by an affine open. It follows that \mathcal{L}_Z is ample on X_Z/Z by Lemma 14.5. Thus (4) \Rightarrow (3) and the proof is complete. \square

Lemma 14.7. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Then f is quasi-affine if and only if \mathcal{O}_X is f -relatively ample.*

Proof. Follows from the case of schemes, see Morphisms, Lemma 37.6. \square

15. Relative ampleness and cohomology

This section contains some results related to the results in Cohomology of Schemes, Sections 21 and 17.

The following lemma is just an example of what we can do.

Lemma 15.1. *Let R be a Noetherian ring. Let X be an algebraic space over R such that the structure morphism $f : X \rightarrow \text{Spec}(R)$ is proper. Let \mathcal{L} be an invertible \mathcal{O}_X -module. The following are equivalent*

- (1) \mathcal{L} is ample on X/R (Definition 14.1),
- (2) for every coherent \mathcal{O}_X -module \mathcal{F} there exists an $n_0 \geq 0$ such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$.

Proof. The implication (1) \Rightarrow (2) follows from Cohomology of Schemes, Lemma 16.1 because assumption (1) implies that X is a scheme. The implication (2) \Rightarrow (1) is Cohomology of Spaces, Lemma 16.9. \square

Lemma 15.2. *Let Y be a Noetherian scheme. Let X be an algebraic space over Y such that the structure morphism $f : X \rightarrow Y$ is proper. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let \mathcal{F} be a coherent \mathcal{O}_X -module. Let $y \in Y$ be a point such that X_y is a scheme and \mathcal{L}_y is ample on X_y . Then there exists a d_0 such that for all $d \geq d_0$ we have*

$$R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = 0 \text{ for } p > 0$$

and the map

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y \longrightarrow H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$$

is surjective.

Proof. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : \operatorname{Spec}(\mathcal{O}_{Y,y}) \rightarrow Y$, see Schemes, Equation (13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \rightarrow \operatorname{Spec}(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^* R^p f_* \mathcal{F} = R^p f'_* \mathcal{F}'$ by Cohomology of Spaces, Lemma 11.2. Moreover, the fibres X_y and X'_y are identified. Hence we may assume that $Y = \operatorname{Spec}(A)$ is the spectrum of a Noetherian local ring $(A, \mathfrak{m}, \kappa)$ and $y \in Y$ corresponds to \mathfrak{m} . In this case $R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ for all $p \geq 0$. Denote $f_y : X_y \rightarrow \operatorname{Spec}(\kappa)$ the projection.

Let $B = \operatorname{Gr}_{\mathfrak{m}}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$. Consider the sheaf $\mathcal{B} = f_y^* \tilde{B}$ of quasi-coherent graded \mathcal{O}_{X_y} -algebras. We will use notation as in Cohomology of Spaces, Section 22 with I replaced by \mathfrak{m} . Since X_y is the closed subspace of X cut out by $\mathfrak{m}\mathcal{O}_X$ we may think of $\mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F}$ as a coherent \mathcal{O}_{X_y} -module, see Cohomology of Spaces, Lemma 12.8. Then $\bigoplus_{n \geq 0} \mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F}$ is a quasi-coherent graded \mathcal{B} -module of finite type because it is generated in degree zero over \mathcal{B} and because the degree zero part is $\mathcal{F}_y = \mathcal{F} / \mathfrak{m} \mathcal{F}$ which is a coherent \mathcal{O}_{X_y} -module. Hence by Cohomology of Schemes, Lemma 19.3 part (2) there exists a d_0 such that

$$H^p(X_y, \mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d}) = 0$$

for all $p > 0$, $d \geq d_0$, and $n \geq 0$. By Cohomology of Spaces, Lemma 8.3 this is the same as the statement that $H^p(X, \mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$.

Consider the short exact sequences

$$0 \rightarrow \mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \rightarrow \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \rightarrow \mathcal{F} / \mathfrak{m}^n \mathcal{F} \rightarrow 0$$

of coherent \mathcal{O}_X -modules. Tensoring with $\mathcal{L}^{\otimes d}$ is an exact functor and we obtain short exact sequences

$$0 \rightarrow \mathfrak{m}^n \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \rightarrow \mathcal{F} / \mathfrak{m}^{n+1} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \rightarrow \mathcal{F} / \mathfrak{m}^n \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \rightarrow 0$$

Using the long exact cohomology sequence and the vanishing above we conclude (using induction) that

- (1) $H^p(X, \mathcal{F} / \mathfrak{m}^n \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$, and
- (2) $H^0(X, \mathcal{F} / \mathfrak{m}^n \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \rightarrow H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective for all $d \geq d_0$ and $n \geq 1$.

By the theorem on formal functions (Cohomology of Spaces, Theorem 22.5) we find that the \mathfrak{m} -adic completion of $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. Since $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is a finite A -module by Cohomology of Spaces, Lemma 20.3 it follows from Nakayama's lemma (Algebra, Lemma 20.1) that $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. For $p = 0$ we deduce from Cohomology of Spaces, Lemma 22.4 part (3) that $H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \rightarrow H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective, which gives the final statement of the lemma. \square

Lemma 15.3. (For a more general version see *Descent on Spaces*, Lemma 13.2). *Let Y be a Noetherian scheme. Let X be an algebraic space over Y such that the structure morphism $f : X \rightarrow Y$ is proper. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $y \in Y$ be a point such that X_y is a scheme and \mathcal{L}_y is ample on X_y . Then there is an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)/V$ (as in Definition 14.1).*

Proof. Pick d_0 as in Lemma 15.2 for $\mathcal{F} = \mathcal{O}_X$. Pick $d \geq d_0$ so that we can find $r \geq 0$ and sections $s_{y,0}, \dots, s_{y,r} \in H^0(X_y, \mathcal{L}_y^{\otimes d})$ which define a closed immersion

$$\varphi_y = \varphi_{\mathcal{L}_y^{\otimes d}, (s_{y,0}, \dots, s_{y,r})} : X_y \rightarrow \mathbf{P}_{\kappa(y)}^r.$$

This is possible by Morphisms, Lemma 39.4 but we also use Morphisms, Lemma 41.7 to see that φ_y is a closed immersion and Constructions, Section 13 for the description of morphisms into projective space in terms of invertible sheaves and sections. By our choice of d_0 , after replacing Y by an open neighbourhood of y , we can choose $s_0, \dots, s_r \in H^0(X, \mathcal{L}^{\otimes d})$ mapping to $s_{y,0}, \dots, s_{y,r}$. Let $X_{s_i} \subset X$ be the open subspace where s_i is a generator of $\mathcal{L}^{\otimes d}$. Since the $s_{y,i}$ generate $\mathcal{L}_y^{\otimes d}$ we see that $|X_y| \subset U = \bigcup |X_{s_i}|$. Since $X \rightarrow Y$ is closed, we see that there is an open neighbourhood $y \in V \subset Y$ such that $|f|^{-1}(V) \subset U$. After replacing Y by V we may assume that the s_i generate $\mathcal{L}^{\otimes d}$. Thus we obtain a morphism

$$\varphi = \varphi_{\mathcal{L}^{\otimes d}, (s_0, \dots, s_r)} : X \rightarrow \mathbf{P}_Y^r$$

with $\mathcal{L}^{\otimes d} \cong \varphi^* \mathcal{O}_{\mathbf{P}_Y^r}(1)$ whose base change to y gives φ_y (strictly speaking we need to write out a proof that the construction of morphisms into projective space given in Constructions, Section 13 also works to describe morphisms of algebraic spaces into projective space; we omit the details).

We will finish the proof by a sleight of hand; the “correct” proof proceeds by directly showing that φ is a closed immersion after base changing to an open neighbourhood of y . Namely, by Cohomology of Spaces, Lemma 23.2 we see that φ is a finite over an open neighbourhood of the fibre $\mathbf{P}_{\kappa(y)}^r$ of $\mathbf{P}_Y^r \rightarrow Y$ above y . Using that $\mathbf{P}_Y^r \rightarrow Y$ is closed, after shrinking Y we may assume that φ is finite. In particular X is a scheme. Then $\mathcal{L}^{\otimes d} \cong \varphi^* \mathcal{O}_{\mathbf{P}_Y^r}(1)$ is ample by the very general Morphisms, Lemma 37.7. \square

16. Closed subspaces of relative proj

Some auxiliary lemmas about closed subspaces of relative proj. This section is the analogue of Divisors, Section 31.

Lemma 16.1. *Let S be a scheme. Let X be an algebraic space over S . Let \mathcal{A} be a quasi-coherent graded \mathcal{O}_X -algebra. Let $\pi : P = \text{Proj}_X(\mathcal{A}) \rightarrow X$ be the relative Proj of \mathcal{A} . Let $i : Z \rightarrow P$ be a closed subspace. Denote $\mathcal{I} \subset \mathcal{A}$ the kernel of the canonical map*

$$\mathcal{A} \rightarrow \bigoplus_{d \geq 0} \pi_*((i_* \mathcal{O}_Z)(d))$$

If π is quasi-compact, then there is an isomorphism $Z = \text{Proj}_X(\mathcal{A}/\mathcal{I})$.

Proof. The morphism π is separated by Lemma 11.6. As π is quasi-compact, π_* transforms quasi-coherent modules into quasi-coherent modules, see Morphisms of Spaces, Lemma 11.2. Hence \mathcal{I} is a quasi-coherent \mathcal{O}_X -module. In particular, $\mathcal{B} = \mathcal{A}/\mathcal{I}$ is a quasi-coherent graded \mathcal{O}_X -algebra. The functoriality morphism $Z' = \text{Proj}_X(\mathcal{B}) \rightarrow \text{Proj}_X(\mathcal{A})$ is everywhere defined and a closed immersion, see Lemma 12.3. Hence it suffices to prove $Z = Z'$ as closed subspaces of P .

Having said this, the question is étale local on the base and we reduce to the case of schemes (Divisors, Lemma 31.1) by étale localization. \square

In case the closed subspace is locally cut out by finitely many equations we can define it by a finite type ideal sheaf of \mathcal{A} .

Lemma 16.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let \mathcal{A} be a quasi-coherent graded \mathcal{O}_X -algebra. Let $\pi : P = \underline{\text{Proj}}_X(\mathcal{A}) \rightarrow X$ be the relative Proj of \mathcal{A} . Let $i : Z \rightarrow P$ be a closed subscheme. If π is quasi-compact and i of finite presentation, then there exists a $d > 0$ and a quasi-coherent finite type \mathcal{O}_X -submodule $\mathcal{F} \subset \mathcal{A}_d$ such that $Z = \underline{\text{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$.*

Proof. The reader can redo the arguments used in the case of schemes. However, we will show the lemma follows from the case of schemes by a trick. Let $\mathcal{I} \subset \mathcal{A}$ be the quasi-coherent graded ideal cutting out Z of Lemma 16.1. Choose an affine scheme U and a surjective étale morphism $U \rightarrow X$, see Properties of Spaces, Lemma 6.3. By the case of schemes (Divisors, Lemma 31.4) there exists a $d > 0$ and a quasi-coherent finite type \mathcal{O}_U -submodule $\mathcal{F}' \subset \mathcal{I}_d|_U \subset \mathcal{A}_d|_U$ such that $Z \times_X U$ is equal to $\underline{\text{Proj}}_U(\mathcal{A}|_U/\mathcal{F}'\mathcal{A}|_U)$. By Limits of Spaces, Lemma 9.2 we can find a finite type quasi-coherent submodule $\mathcal{F} \subset \mathcal{I}_d$ such that $\mathcal{F}' \subset \mathcal{F}|_U$. Let $Z' = \underline{\text{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$. Then $Z' \rightarrow P$ is a closed immersion (Lemma 12.5) and $Z \subset Z'$ as $\mathcal{F}\mathcal{A} \subset \mathcal{I}$. On the other hand, $Z' \times_X U \subset Z \times_X U$ by our choice of \mathcal{F} . Thus $Z = Z'$ as desired. \square

Lemma 16.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let \mathcal{A} be a quasi-coherent graded \mathcal{O}_X -algebra. Let $\pi : P = \underline{\text{Proj}}_X(\mathcal{A}) \rightarrow X$ be the relative Proj of \mathcal{A} . Let $i : Z \rightarrow X$ be a closed subspace. Let $U \subset X$ be an open. Assume that*

- (1) π is quasi-compact,
- (2) i of finite presentation,
- (3) $|U| \cap |\pi(|i|(|Z|))| = \emptyset$,
- (4) U is quasi-compact,
- (5) \mathcal{A}_n is a finite type \mathcal{O}_X -module for all n .

Then there exists a $d > 0$ and a quasi-coherent finite type \mathcal{O}_X -submodule $\mathcal{F} \subset \mathcal{A}_d$ with (a) $Z = \underline{\text{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$ and (b) the support of $\mathcal{A}_d/\mathcal{F}$ is disjoint from U .

Proof. We use the same trick as in the proof of Lemma 16.2 to reduce to the case of schemes. Let $\mathcal{I} \subset \mathcal{A}$ be the quasi-coherent graded ideal cutting out Z of Lemma 16.1. Choose an affine scheme W and a surjective étale morphism $W \rightarrow X$, see Properties of Spaces, Lemma 6.3. By the case of schemes (Divisors, Lemma 31.5) there exists a $d > 0$ and a quasi-coherent finite type \mathcal{O}_W -submodule $\mathcal{F}' \subset \mathcal{I}_d|_W \subset \mathcal{A}_d|_W$ such that (a) $Z \times_X W$ is equal to $\underline{\text{Proj}}_W(\mathcal{A}|_W/\mathcal{F}'\mathcal{A}|_W)$ and (b) the support of $\mathcal{A}_d|_W/\mathcal{F}'$ is disjoint from $U \times_X W$. By Limits of Spaces, Lemma 9.2 we can find a finite type quasi-coherent submodule $\mathcal{F} \subset \mathcal{I}_d$ such that $\mathcal{F}' \subset \mathcal{F}|_W$. Let $Z' = \underline{\text{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$. Then $Z' \rightarrow P$ is a closed immersion (Lemma 12.5) and $Z \subset Z'$ as $\mathcal{F}\mathcal{A} \subset \mathcal{I}$. On the other hand, $Z' \times_X W \subset Z \times_X W$ by our choice of \mathcal{F} . Thus $Z = Z'$. Finally, we see that $\mathcal{A}_d/\mathcal{F}$ is supported on $X \setminus U$ as $\mathcal{A}_d|_W/\mathcal{F}|_W$ is a quotient of $\mathcal{A}_d|_W/\mathcal{F}'$ which is supported on $W \setminus U \times_X W$. Thus the lemma follows. \square

Lemma 16.4. *Let S be a scheme and let X be an algebraic space over S . Let \mathcal{E} be a quasi-coherent \mathcal{O}_X -module. There is a bijection*

$$\left\{ \begin{array}{l} \text{sections } \sigma \text{ of the} \\ \text{morphism } \mathbf{P}(\mathcal{E}) \rightarrow X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{surjections } \mathcal{E} \rightarrow \mathcal{L} \text{ where} \\ \mathcal{L} \text{ is an invertible } \mathcal{O}_X\text{-module} \end{array} \right\}$$

In this case σ is a closed immersion and there is a canonical isomorphism

$$\text{Ker}(\mathcal{E} \rightarrow \mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \longrightarrow \mathcal{C}_{\sigma(X)/\mathbf{P}(\mathcal{E})}$$

Both the bijection and isomorphism are compatible with base change.

Proof. Because the constructions are compatible with base change, it suffices to check the statement étale locally on X . Thus we may assume X is a scheme and the result is Divisors, Lemma 31.6. \square

17. Blowing up

Blowing up is an important tool in algebraic geometry.

Definition 17.1. Let S be a scheme. Let X be an algebraic space over S . Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be the closed subspace corresponding to \mathcal{I} (Morphisms of Spaces, Lemma 13.1). The *blowing up of X along Z* , or the *blowing up of X in the ideal sheaf \mathcal{I}* is the morphism

$$b : \text{Proj}_X \left(\bigoplus_{n \geq 0} \mathcal{I}^n \right) \longrightarrow X$$

The *exceptional divisor* of the blowup is the inverse image $b^{-1}(Z)$. Sometimes Z is called the *center* of the blowup.

We will see later that the exceptional divisor is an effective Cartier divisor. Moreover, the blowing up is characterized as the “smallest” algebraic space over X such that the inverse image of Z is an effective Cartier divisor.

If $b : X' \rightarrow X$ is the blowup of X in Z , then we often denote $\mathcal{O}_{X'}(n)$ the twists of the structure sheaf. Note that these are invertible $\mathcal{O}_{X'}$ -modules and that $\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(1)^{\otimes n}$ because X' is the relative Proj of a quasi-coherent graded \mathcal{O}_X -algebra which is generated in degree 1, see Lemma 11.11.

Lemma 17.2. Let S be a scheme. Let X be an algebraic space over S . Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $U = \text{Spec}(A)$ be an affine scheme étale over X and let $I \subset A$ be the ideal corresponding to $\mathcal{I}|_U$. If $X' \rightarrow X$ is the blowup of X in \mathcal{I} , then there is a canonical isomorphism

$$U \times_X X' = \text{Proj} \left(\bigoplus_{d \geq 0} I^d \right)$$

of schemes over U , where the right hand side is the homogeneous spectrum of the Rees algebra of I in A . Moreover, $U \times_X X'$ has an affine open covering by spectra of the affine blowup algebras $A[\frac{I}{a}]$.

Proof. Note that the restriction $\mathcal{I}|_U$ is equal to the pullback of \mathcal{I} via the morphism $U \rightarrow X$, see Properties of Spaces, Section 26. Thus the lemma follows on combining Lemma 11.2 with Divisors, Lemma 32.2. \square

Lemma 17.3. Let S be a scheme. Let $X_1 \rightarrow X_2$ be a flat morphism of algebraic spaces over S . Let $Z_2 \subset X_2$ be a closed subspace. Let Z_1 be the inverse image of Z_2 in X_1 . Let X'_i be the blowup of Z_i in X_i . Then there exists a cartesian diagram

$$\begin{array}{ccc} X'_1 & \longrightarrow & X'_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

of algebraic spaces over S .

Proof. Let \mathcal{I}_2 be the ideal sheaf of Z_2 in X_2 . Denote $g : X_1 \rightarrow X_2$ the given morphism. Then the ideal sheaf \mathcal{I}_1 of Z_1 is the image of $g^*\mathcal{I}_2 \rightarrow \mathcal{O}_{X_1}$ (see Morphisms of Spaces, Definition 13.2 and discussion following the definition). By Lemma 11.5 we see that $X_1 \times_{X_2} X'_2$ is the relative Proj of $\bigoplus_{n \geq 0} g^*\mathcal{I}_2^n$. Because g is flat the map $g^*\mathcal{I}_2^n \rightarrow \mathcal{O}_{X_1}$ is injective with image \mathcal{I}_1^n . Thus we see that $X_1 \times_{X_2} X'_2 = X'_1$. \square

Lemma 17.4. *Let S be a scheme. Let X be an algebraic space over S . Let $Z \subset X$ be a closed subspace. The blowing up $b : X' \rightarrow X$ of Z in X has the following properties:*

- (1) $b|_{b^{-1}(X \setminus Z)} : b^{-1}(X \setminus Z) \rightarrow X \setminus Z$ is an isomorphism,
- (2) the exceptional divisor $E = b^{-1}(Z)$ is an effective Cartier divisor on X' ,
- (3) there is a canonical isomorphism $\mathcal{O}_{X'}(-1) = \mathcal{O}_{X'}(E)$

Proof. Let U be a scheme and let $U \rightarrow X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 17.3) we can prove each of these statements after base change to U . This reduces us to the case of schemes. In this case the result is Divisors, Lemma 32.4. \square

Lemma 17.5 (Universal property blowing up). *Let S be a scheme. Let X be an algebraic space over S . Let $Z \subset X$ be a closed subspace. Let \mathcal{C} be the full subcategory of (Spaces/X) consisting of $Y \rightarrow X$ such that the inverse image of Z is an effective Cartier divisor on Y . Then the blowing up $b : X' \rightarrow X$ of Z in X is a final object of \mathcal{C} .*

Proof. We see that $b : X' \rightarrow X$ is an object of \mathcal{C} according to Lemma 17.4. Let $f : Y \rightarrow X$ be an object of \mathcal{C} . We have to show there exists a unique morphism $Y \rightarrow X'$ over X . Let $D = f^{-1}(Z)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z and let \mathcal{I}_D be the ideal sheaf of D . Then $f^*\mathcal{I} \rightarrow \mathcal{I}_D$ is a surjection to an invertible \mathcal{O}_Y -module. This extends to a map $\psi : \bigoplus f^*\mathcal{I}^d \rightarrow \bigoplus \mathcal{I}_D^d$ of graded \mathcal{O}_Y -algebras. (We observe that $\mathcal{I}_D^d = \mathcal{I}_D^{\otimes d}$ as D is an effective Cartier divisor.) By Lemma 11.11. the triple $(f : Y \rightarrow X, \mathcal{I}_D, \psi)$ defines a morphism $Y \rightarrow X'$ over X . The restriction

$$Y \setminus D \longrightarrow X' \setminus b^{-1}(Z) = X \setminus Z$$

is unique. The open $Y \setminus D$ is scheme theoretically dense in Y according to Lemma 6.4. Thus the morphism $Y \rightarrow X'$ is unique by Morphisms of Spaces, Lemma 17.8 (also b is separated by Lemma 11.6). \square

Lemma 17.6. *Let S be a scheme. Let X be an algebraic space over S . Let $Z \subset X$ be an effective Cartier divisor. The blowup of X in Z is the identity morphism of X .*

Proof. Immediate from the universal property of blowups (Lemma 17.5). \square

Lemma 17.7. *Let S be a scheme. Let X be an algebraic space over S . Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is reduced, then the blowup X' of X in \mathcal{I} is reduced.*

Proof. Let U be a scheme and let $U \rightarrow X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 17.3) we can prove each of these statements after base change to U . This reduces us to the case of schemes. In this case the result is Divisors, Lemma 32.8. \square

Lemma 17.8. *Let S be a scheme. Let X be an algebraic space over S . Let $b : X' \rightarrow X$ be the blowup of X in a closed subspace. If X satisfies the equivalent conditions of Morphisms of Spaces, Lemma 49.1 then so does X' .*

Proof. Follows immediately from the lemma cited in the statement, the étale local description of blowing ups in Lemma 17.2, and Divisors, Lemma 32.10. \square

Lemma 17.9. *Let S be a scheme. Let X be an algebraic space over S . Let $b : X' \rightarrow X$ be a blowup of X in a closed subspace. For any effective Cartier divisor D on X the pullback $b^{-1}D$ is defined (see Definition 6.10).*

Proof. By Lemmas 17.2 and 6.2 this reduces to the following algebra fact: Let A be a ring, $I \subset A$ an ideal, $a \in I$, and $x \in A$ a nonzerodivisor. Then the image of x in $A[\frac{I}{a}]$ is a nonzerodivisor. Namely, suppose that $x(y/a^n) = 0$ in $A[\frac{I}{a}]$. Then $a^m xy = 0$ in A for some m . Hence $a^m y = 0$ as x is a nonzerodivisor. Whence y/a^n is zero in $A[\frac{I}{a}]$ as desired. \square

Lemma 17.10. *Let S be a scheme. Let X be an algebraic space over S . Let $\mathcal{I} \subset \mathcal{O}_X$ and \mathcal{J} be quasi-coherent sheaves of ideals. Let $b : X' \rightarrow X$ be the blowing up of X in \mathcal{I} . Let $b' : X'' \rightarrow X'$ be the blowing up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$. Then $X'' \rightarrow X$ is canonically isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$.*

Proof. Let $E \subset X'$ be the exceptional divisor of b which is an effective Cartier divisor by Lemma 17.4. Then $(b')^{-1}E$ is an effective Cartier divisor on X'' by Lemma 17.9. Let $E' \subset X''$ be the exceptional divisor of b' (also an effective Cartier divisor). Consider the effective Cartier divisor $E'' = E' + (b')^{-1}E$. By construction the ideal of E'' is $(b \circ b')^{-1}\mathcal{I}(b \circ b')^{-1}\mathcal{J}\mathcal{O}_{X''}$. Hence according to Lemma 17.5 there is a canonical morphism from X'' to the blowup $c : Y \rightarrow X$ of X in $\mathcal{I}\mathcal{J}$. Conversely, as $\mathcal{I}\mathcal{J}$ pulls back to an invertible ideal we see that $c^{-1}\mathcal{I}\mathcal{O}_Y$ defines an effective Cartier divisor, see Lemma 6.8. Thus a morphism $c' : Y \rightarrow X'$ over X by Lemma 17.5. Then $(c')^{-1}b^{-1}\mathcal{J}\mathcal{O}_Y = c^{-1}\mathcal{J}\mathcal{O}_Y$ which also defines an effective Cartier divisor. Thus a morphism $c'' : Y \rightarrow X''$ over X' . We omit the verification that this morphism is inverse to the morphism $X'' \rightarrow Y$ constructed earlier. \square

Lemma 17.11. *Let S be a scheme. Let X be an algebraic space over S . Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $b : X' \rightarrow X$ be the blowing up of X in the ideal sheaf \mathcal{I} . If \mathcal{I} is of finite type, then $b : X' \rightarrow X$ is a proper morphism.*

Proof. Let U be a scheme and let $U \rightarrow X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 17.3) we can prove each of these statements after base change to U (see Morphisms of Spaces, Lemma 40.2). This reduces us to the case of schemes. In this case the morphism b is projective by Divisors, Lemma 32.13 hence proper by Morphisms, Lemma 43.5. \square

Lemma 17.12. *Let S be a scheme and let X be an algebraic space over S . Assume X is quasi-compact and quasi-separated. Let $Z \subset X$ be a closed subspace of finite presentation. Let $b : X' \rightarrow X$ be the blowing up with center Z . Let $Z' \subset X'$ be a closed subspace of finite presentation. Let $X'' \rightarrow X'$ be the blowing up with center Z' . There exists a closed subspace $Y \subset X$ of finite presentation, such that*

- (1) $|Y| = |Z| \cup |b(|Z'|)|$, and
- (2) the composition $X'' \rightarrow X$ is isomorphic to the blowing up of X in Y .

Proof. The condition that $Z \rightarrow X$ is of finite presentation means that Z is cut out by a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, see Morphisms of Spaces, Lemma 28.12. Write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ so that $X' = \underline{\text{Proj}}(\mathcal{A})$. Note that $X \setminus Z$ is a quasi-compact open subspace of X by Limits of Spaces, Lemma 14.1. Since $b^{-1}(X \setminus Z) \rightarrow X \setminus Z$ is an isomorphism (Lemma 17.4) the same result shows that $b^{-1}(X \setminus Z) \setminus Z'$ is quasi-compact open subspace in X' . Hence $U = X \setminus (Z \cup b(Z'))$ is quasi-compact open subspace in X . By Lemma 16.3 there exist a $d > 0$ and a finite type \mathcal{O}_X -submodule $\mathcal{F} \subset \mathcal{I}^d$ such that $Z' = \underline{\text{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ and such that the support of $\mathcal{I}^d/\mathcal{F}$ is contained in $X \setminus U$.

Since $\mathcal{F} \subset \mathcal{I}^d$ is an \mathcal{O}_X -submodule we may think of $\mathcal{F} \subset \mathcal{I}^d \subset \mathcal{O}_X$ as a finite type quasi-coherent sheaf of ideals on X . Let's denote this $\mathcal{J} \subset \mathcal{O}_X$ to prevent confusion. Since $\mathcal{I}^d/\mathcal{F}$ and $\mathcal{O}/\mathcal{I}^d$ are supported on $|X| \setminus |U|$ we see that $|V(\mathcal{J})|$ is contained in $|X| \setminus |U|$. Conversely, as $\mathcal{J} \subset \mathcal{I}^d$ we see that $|Z| \subset |V(\mathcal{J})|$. Over $X \setminus Z \cong X' \setminus b^{-1}(Z)$ the sheaf of ideals \mathcal{J} cuts out Z' (see displayed formula below). Hence $|V(\mathcal{J})|$ equals $|Z| \cup |b|(|Z'|)$. It follows that also $|V(\mathcal{I}\mathcal{J})| = |Z| \cup |b|(|Z'|)$. Moreover, $\mathcal{I}\mathcal{J}$ is an ideal of finite type as a product of two such. We claim that $X'' \rightarrow X$ is isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$ which finishes the proof of the lemma by setting $Y = V(\mathcal{I}\mathcal{J})$.

First, recall that the blowup of X in $\mathcal{I}\mathcal{J}$ is the same as the blowup of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$, see Lemma 17.10. Hence it suffices to show that the blowup of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$ agrees with the blowup of X' in Z' . We will show that

$$b^{-1}\mathcal{J}\mathcal{O}_{X'} = \mathcal{I}_E^d \mathcal{I}_{Z'}$$

as ideal sheaves on X'' . This will prove what we want as \mathcal{I}_E^d cuts out the effective Cartier divisor dE and we can use Lemmas 17.6 and 17.10.

To see the displayed equality of the ideals we may work locally. With notation A , I , $a \in I$ as in Lemma 17.2 we see that \mathcal{F} corresponds to an R -submodule $M \subset I^d$ mapping isomorphically to an ideal $J \subset R$. The condition $Z' = \underline{\text{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ means that $Z' \cap \text{Spec}(A[\frac{I}{a}])$ is cut out by the ideal generated by the elements m/a^d , $m \in M$. Say the element $m \in M$ corresponds to the function $f \in J$. Then in the affine blowup algebra $A' = A[\frac{I}{a}]$ we see that $f = (a^d m)/a^d = a^d(m/a^d)$. Thus the equality holds. \square

18. Strict transform

This section is the analogue of Divisors, Section 33. Let S be a scheme, let B be an algebraic space over S , and let $Z \subset B$ be a closed subspace. Let $b : B' \rightarrow B$ be the blowing up of B in Z and denote $E \subset B'$ the exceptional divisor $E = b^{-1}Z$. In the following we will often consider an algebraic space X over B and form the cartesian diagram

$$\begin{array}{ccccc} \text{pr}_{B'}^{-1}E & \longrightarrow & X \times_B B' & \xrightarrow{\text{pr}_X} & X \\ \downarrow & & \downarrow \text{pr}_{B'} & & \downarrow f \\ E & \longrightarrow & B' & \longrightarrow & B \end{array}$$

Since E is an effective Cartier divisor (Lemma 17.4) we see that $\text{pr}_{B'}^{-1}E \subset X \times_B B'$ is locally principal (Lemma 6.9). Thus the inclusion morphism of the complement

of $\mathrm{pr}_B^{-1}E$ in $X \times_B B'$ is affine and in particular quasi-compact (Lemma 6.3). Consequently, for a quasi-coherent $\mathcal{O}_{X \times_B B'}$ -module \mathcal{G} the subsheaf of sections supported on $|\mathrm{pr}_B^{-1}E|$ is a quasi-coherent submodule, see Limits of Spaces, Definition 14.6. If \mathcal{G} is a quasi-coherent sheaf of algebras, e.g., $\mathcal{G} = \mathcal{O}_{X \times_B B'}$, then this subsheaf is an ideal of \mathcal{G} .

Definition 18.1. With $Z \subset B$ and $f : X \rightarrow B$ as above.

- (1) Given a quasi-coherent \mathcal{O}_X -module \mathcal{F} the *strict transform* of \mathcal{F} with respect to the blowup of B in Z is the quotient \mathcal{F}' of $\mathrm{pr}_X^* \mathcal{F}$ by the submodule of sections supported on $|\mathrm{pr}_B^{-1}E|$.
- (2) The *strict transform* of X is the closed subspace $X' \subset X \times_B B'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X \times_B B'}$ supported on $|\mathrm{pr}_B^{-1}E|$.

Note that taking the strict transform along a blowup depends on the closed subspace used for the blowup (and not just on the morphism $B' \rightarrow B$).

Lemma 18.2 (Étale localization and strict transform). *In the situation of Definition 18.1. Let*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & B \end{array}$$

be a commutative diagram of morphisms with U and V schemes and étale horizontal arrows. Let $V' \rightarrow V$ be the blowup of V in $Z \times_B V$. Then

- (1) $V' = V \times_B B'$ and the maps $V' \rightarrow B'$ and $U \times_V V' \rightarrow X \times_B B'$ are étale,
- (2) the strict transform U' of U relative to $V' \rightarrow V$ is equal to $X' \times_X U$ where X' is the strict transform of X relative to $B' \rightarrow B$, and
- (3) for a quasi-coherent \mathcal{O}_X -module \mathcal{F} the restriction of the strict transform \mathcal{F}' to $U \times_V V'$ is the strict transform of $\mathcal{F}|_U$ relative to $V' \rightarrow V$.

Proof. Part (1) follows from the fact that blowup commutes with flat base change (Lemma 17.3), the fact that étale morphisms are flat, and that the base change of an étale morphism is étale. Part (3) then follows from the fact that taking the sheaf of sections supported on a closed commutes with pullback by étale morphisms, see Limits of Spaces, Lemma 14.5. Part (2) follows from (3) applied to $\mathcal{F} = \mathcal{O}_X$. \square

Lemma 18.3. *In the situation of Definition 18.1.*

- (1) The strict transform X' of X is the blowup of X in the closed subspace $f^{-1}Z$ of X .
- (2) For a quasi-coherent \mathcal{O}_X -module \mathcal{F} the strict transform \mathcal{F}' is canonically isomorphic to the pushforward along $X' \rightarrow X \times_B B'$ of the strict transform of \mathcal{F} relative to the blowing up $X' \rightarrow X$.

Proof. Let $X'' \rightarrow X$ be the blowup of X in $f^{-1}Z$. By the universal property of blowing up (Lemma 17.5) there exists a commutative diagram

$$\begin{array}{ccc} X'' & \longrightarrow & X \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

whence a morphism $i : X'' \rightarrow X \times_B B'$. The first assertion of the lemma is that i is a closed immersion with image X' . The second assertion of the lemma is that $\mathcal{F}' = i_* \mathcal{F}''$ where \mathcal{F}'' is the strict transform of \mathcal{F} with respect to the blowing up $X'' \rightarrow X$. We can check these assertions étale locally on X , hence we reduce to the case of schemes (Divisors, Lemma 33.2). Some details omitted. \square

Lemma 18.4. *In the situation of Definition 18.1.*

- (1) *If X is flat over B at all points lying over Z , then the strict transform of X is equal to the base change $X \times_B B'$.*
- (2) *Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. If \mathcal{F} is flat over B at all points lying over Z , then the strict transform \mathcal{F}' of \mathcal{F} is equal to the pullback $pr_X^* \mathcal{F}$.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 33.3) by étale localization (Lemma 18.2). \square

Lemma 18.5. *Let S be a scheme. Let B be an algebraic space over S . Let $Z \subset B$ be a closed subspace. Let $b : B' \rightarrow B$ be the blowing up of Z in B . Let $g : X \rightarrow Y$ be an affine morphism of spaces over B . Let \mathcal{F} be a quasi-coherent sheaf on X . Let $g' : X \times_B B' \rightarrow Y \times_B B'$ be the base change of g . Let \mathcal{F}' be the strict transform of \mathcal{F} relative to b . Then $g'_* \mathcal{F}'$ is the strict transform of $g_* \mathcal{F}$.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 33.4) by étale localization (Lemma 18.2). \square

Lemma 18.6. *Let S be a scheme. Let B be an algebraic space over S . Let $Z \subset B$ be a closed subspace. Let $D \subset B$ be an effective Cartier divisor. Let $Z' \subset B$ be the closed subspace cut out by the product of the ideal sheaves of Z and D . Let $B' \rightarrow B$ be the blowup of B in Z .*

- (1) *The blowup of B in Z' is isomorphic to $B' \rightarrow B$.*
- (2) *Let $f : X \rightarrow B$ be a morphism of algebraic spaces and let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. If the subsheaf of \mathcal{F} of sections supported on $|f^{-1}D|$ is zero, then the strict transform of \mathcal{F} relative to the blowing up in Z agrees with the strict transform of \mathcal{F} relative to the blowing up of B in Z' .*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 33.5) by étale localization (Lemma 18.2). \square

Lemma 18.7. *Let S be a scheme. Let B be an algebraic space over S . Let $Z \subset B$ be a closed subspace. Let $b : B' \rightarrow B$ be the blowing up with center Z . Let $Z' \subset B'$ be a closed subspace. Let $B'' \rightarrow B'$ be the blowing up with center Z' . Let $Y \subset B$ be a closed subscheme such that $|Y| = |Z| \cup |b|(|Z'|)$ and the composition $B'' \rightarrow B$ is isomorphic to the blowing up of B in Y . In this situation, given any scheme X over B and $\mathcal{F} \in \text{QCoh}(\mathcal{O}_X)$ we have*

- (1) *the strict transform of \mathcal{F} with respect to the blowing up of B in Y is equal to the strict transform with respect to the blowup $B'' \rightarrow B'$ in Z' of the strict transform of \mathcal{F} with respect to the blowup $B' \rightarrow B$ of B in Z , and*
- (2) *the strict transform of X with respect to the blowing up of B in Y is equal to the strict transform with respect to the blowup $B'' \rightarrow B'$ in Z' of the strict transform of X with respect to the blowup $B' \rightarrow B$ of B in Z .*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 33.6) by étale localization (Lemma 18.2). \square

Lemma 18.8. *In the situation of Definition 18.1. Suppose that*

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

is an exact sequence of quasi-coherent sheaves on X which remains exact after any base change $T \rightarrow B$. Then the strict transforms of \mathcal{F}_i' relative to any blowup $B' \rightarrow B$ form a short exact sequence $0 \rightarrow \mathcal{F}_1' \rightarrow \mathcal{F}_2' \rightarrow \mathcal{F}_3' \rightarrow 0$ too.

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 33.7) by étale localization (Lemma 18.2). \square

Lemma 18.9. *Let S be a scheme. Let B be an algebraic space over S . Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_B -module. Let $Z_k \subset S$ be the closed subscheme cut out by $\text{Fit}_k(\mathcal{F})$, see Section 5. Let $B' \rightarrow B$ be the blowup of B in Z_k and let \mathcal{F}' be the strict transform of \mathcal{F} . Then \mathcal{F}' can locally be generated by $\leq k$ sections.*

Proof. Omitted. Follows from the case of schemes (Divisors, Lemma 35.1) by étale localization (Lemma 18.2). \square

Lemma 18.10. *Let S be a scheme. Let B be an algebraic space over S . Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_B -module. Let $Z_k \subset S$ be the closed subscheme cut out by $\text{Fit}_k(\mathcal{F})$, see Section 5. Assume that \mathcal{F} is locally free of rank k on $B \setminus Z_k$. Let $B' \rightarrow B$ be the blowup of B in Z_k and let \mathcal{F}' be the strict transform of \mathcal{F} . Then \mathcal{F}' is locally free of rank k .*

Proof. Omitted. Follows from the case of schemes (Divisors, Lemma 35.2) by étale localization (Lemma 18.2). \square

19. Admissible blowups

To have a bit more control over our blowups we introduce the following standard terminology.

Definition 19.1. Let S be a scheme. Let X be an algebraic space over S . Let $U \subset X$ be an open subspace. A morphism $X' \rightarrow X$ is called a *U -admissible blowup* if there exists a closed immersion $Z \rightarrow X$ of finite presentation with Z disjoint from U such that X' is isomorphic to the blowup of X in Z .

We recall that $Z \rightarrow X$ is of finite presentation if and only if the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is of finite type, see Morphisms of Spaces, Lemma 28.12. In particular, a U -admissible blowup is a proper morphism, see Lemma 17.11. Note that there can be multiple centers which give rise to the same morphism. Hence the requirement is just the existence of some center disjoint from U which produces X' . Finally, as the morphism $b : X' \rightarrow X$ is an isomorphism over U (see Lemma 17.4) we will often abuse notation and think of U as an open subspace of X' as well.

Lemma 19.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let $U \subset X$ be a quasi-compact open subspace. Let $b : X' \rightarrow X$ be a U -admissible blowup. Let $X'' \rightarrow X'$ be a U -admissible blowup. Then the composition $X'' \rightarrow X$ is a U -admissible blowup.*

Proof. Immediate from the more precise Lemma 17.12. \square

Lemma 19.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $U, V \subset X$ be quasi-compact open subspaces. Let $b : V' \rightarrow V$ be a $U \cap V$ -admissible blowup. Then there exists a U -admissible blowup $X' \rightarrow X$ whose restriction to V is V' .*

Proof. Let $\mathcal{I} \subset \mathcal{O}_V$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I})$ is disjoint from $U \cap V$ and such that V' is isomorphic to the blowup of V in \mathcal{I} . Let $\mathcal{I}' \subset \mathcal{O}_{U \cup V}$ be the quasi-coherent sheaf of ideals whose restriction to U is \mathcal{O}_U and whose restriction to V is \mathcal{I} . By Limits of Spaces, Lemma 9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ whose restriction to $U \cup V$ is \mathcal{I}' . The lemma follows. \square

Lemma 19.4. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let $U \subset X$ be a quasi-compact open subspace. Let $b_i : X_i \rightarrow X$, $i = 1, \dots, n$ be U -admissible blowups. There exists a U -admissible blowup $b : X' \rightarrow X$ such that (a) b factors as $X' \rightarrow X_i \rightarrow X$ for $i = 1, \dots, n$ and (b) each of the morphisms $X' \rightarrow X_i$ is a U -admissible blowup.*

Proof. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_i)$ is disjoint from U and such that X_i is isomorphic to the blowup of X in \mathcal{I}_i . Set $\mathcal{I} = \mathcal{I}_1 \cdot \dots \cdot \mathcal{I}_n$ and let X' be the blowup of X in \mathcal{I} . Then $X' \rightarrow X$ factors through b_i by Lemma 17.10. \square

Lemma 19.5. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S . Let U, V be quasi-compact disjoint open subspaces of X . Then there exist a $U \cup V$ -admissible blowup $b : X' \rightarrow X$ such that X' is a disjoint union of open subspaces $X' = X'_1 \amalg X'_2$ with $b^{-1}(U) \subset X'_1$ and $b^{-1}(V) \subset X'_2$.*

Proof. Choose a finite type quasi-coherent sheaf of ideals \mathcal{I} , resp. \mathcal{J} such that $X \setminus U = V(\mathcal{I})$, resp. $X \setminus V = V(\mathcal{J})$, see Limits of Spaces, Lemma 14.1. Then $|V(\mathcal{I}\mathcal{J})| = |X|$. Hence $\mathcal{I}\mathcal{J}$ is a locally nilpotent sheaf of ideals. Since \mathcal{I} and \mathcal{J} are of finite type and X is quasi-compact there exists an $n > 0$ such that $\mathcal{I}^n \mathcal{J}^n = 0$. We may and do replace \mathcal{I} by \mathcal{I}^n and \mathcal{J} by \mathcal{J}^n . Whence $\mathcal{I}\mathcal{J} = 0$. Let $b : X' \rightarrow X$ be the blowing up in $\mathcal{I} + \mathcal{J}$. This is $U \cup V$ -admissible as $|V(\mathcal{I} + \mathcal{J})| = |X| \setminus (|U| \cup |V|)$. We will show that X' is a disjoint union of open subspaces $X' = X'_1 \amalg X'_2$ as in the statement of the lemma.

Since $|V(\mathcal{I} + \mathcal{J})|$ is the complement of $|U \cup V|$ we conclude that $V \cup U$ is scheme theoretically dense in X' , see Lemmas 17.4 and 6.4. Thus if such a decomposition $X' = X'_1 \amalg X'_2$ into open and closed subspaces exists, then X'_1 is the scheme theoretic closure of U in X' and similarly X'_2 is the scheme theoretic closure of V in X' . Since $U \rightarrow X'$ and $V \rightarrow X'$ are quasi-compact taking scheme theoretic closures commutes with étale localization (Morphisms of Spaces, Lemma 16.3). Hence to verify the existence of X'_1 and X'_2 we may work étale locally on X . This reduces us to the case of schemes which is treated in the proof of Divisors, Lemma 34.5. \square

20. Other chapters

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