DERIVED CATEGORIES OF SCHEMES

Contents

1. Introduction

In this chapter we discuss derived categories of modules on schemes. Most of the material discussed here can be found in [TT90], [BN93], [BV03], and [LN07]. Of course there are many other references.

2. Conventions

If $\mathcal{A}$ is an abelian category and M is an object of $\mathcal{A}$ then we also denote M the object of $K(\mathcal{A})$ and/or $D(\mathcal{A})$ corresponding to the complex which has M in degree 0 and is zero in all other degrees.

If we have a ring A, then $K(A)$ denotes the homotopy category of complexes of A-modules and $D(A)$ the associated derived category. Similarly, if we have a ringed space $(X, \mathcal{O}_X)$ the symbol $K(\mathcal{O}_X)$ denotes the homotopy category of complexes of $\mathcal{O}_X$-modules and $D(\mathcal{O}_X)$ the associated derived category.

3. Derived category of quasi-coherent modules

In this section we discuss the relationship between quasi-coherent modules and all modules on a scheme X. A reference is [TT90, Appendix B]. By the discussion in Schemes, Section 24 the embedding $\mathit{QCoh}(\mathcal{O}_X) \subset \mathit{Mod}(\mathcal{O}_X)$ exhibits $\mathit{QCoh}(\mathcal{O}_X)$ as a weak Serre subcategory of the category of $\mathcal{O}_X$-modules. Denote

$$D_{QCoh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are quasi-coherent, see Derived Categories, Section 17. Thus we obtain a canonical functor

$$D(\mathit{QCoh}(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X) \tag{3.0.1}$$

see Derived Categories, Equation (17.1.1).

Lemma 3.1. *Let X be a scheme. Then $D_{QCoh}(\mathcal{O}_X)$ has direct sums.*

Proof. By Injectives, Lemma 13.4 the derived category $D(\mathcal{O}_X)$ has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any Grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 24. □

We will need some information on derived limits. We warn the reader that in the lemma below the derived limit will typically not be an object of D_{QCoh}.

Lemma 3.2. *Let X be a scheme. Let (K_n) be an inverse system of $D_{QCoh}(\mathcal{O}_X)$ with derived limit $K = R\lim K_n$ in $D(\mathcal{O}_X)$. Assume $H^q(K_{n+1}) \to H^q(K_n)$ is surjective for all $q \in \mathbf{Z}$ and $n \geq 1$. Then*

1. *$H^q(K) = \lim H^q(K_n)$,*
2. *$R\lim H^q(K_n) = \lim H^q(K_n)$, and*

(3) *for every affine open $U \subset X$ we have $H^p(U, \lim H^q(K_n)) = 0$ for $p > 0$.*

Proof. Let $\mathcal{B}$ be the set of affine opens of X. Since $H^q(K_n)$ is quasi-coherent we have $H^p(U, H^q(K_n)) = 0$ for $U \in \mathcal{B}$ by Cohomology of Schemes, Lemma 2.2. Moreover, the maps $H^0(U, H^q(K_{n+1})) \to H^0(U, H^q(K_n))$ are surjective for $U \in \mathcal{B}$ by Schemes, Lemma 7.5. Part (1) follows from Cohomology, Lemma 37.11 whose conditions we have just verified. Parts (2) and (3) follow from Cohomology, Lemma 37.4. $\square$

The following lemma will help us to "compute" a right derived functor on an object of $D_{QCoh}(\mathcal{O}_X)$.

Lemma 3.3. *Let X be a scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then the canonical map $E \to R\lim \tau_{\geq -n}E$ is an isomorphism*[1].

Proof. Denote $\mathcal{H}^i = H^i(E)$ the ith cohomology sheaf of E. Let $\mathcal{B}$ be the set of affine open subsets of X. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Cohomology of Schemes, Lemma 2.2. Thus the lemma follows from Cohomology, Lemma 37.9. $\square$

Lemma 3.4. *Let X be a scheme. Let $F : Mod(\mathcal{O}_X) \to Ab$ be an additive functor and $N \geq 0$ an integer. Assume that*

(1) *F commutes with countable direct products,*
(2) *$R^pF(\mathcal{F}) = 0$ for all $p \geq N$ and $\mathcal{F}$ quasi-coherent.*

Then for $E \in D_{QCoh}(\mathcal{O}_X)$

(1) *$H^i(RF(\tau_{\leq a}E)) \to H^i(RF(E))$ is an isomorphism for $i \leq a$,*
(2) *$H^i(RF(E)) \to H^i(RF(\tau_{\geq b-N+1}E))$ is an isomorphism for $i \geq b$,*
(3) *if $H^i(E) = 0$ for $i \notin [a, b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(RF(E)) = 0$ for $i \notin [a, b + N - 1]$.*

Proof. Statement (1) is Derived Categories, Lemma 16.1.

Proof of statement (2). Write $E_n = \tau_{\geq -n}E$. We have $E = R\lim E_n$, see Lemma 3.3. Thus $RF(E) = R\lim RF(E_n)$ in $D(Ab)$ by Injectives, Lemma 13.6. Thus for every $i \in \mathbf{Z}$ we have a short exact sequence

$$0 \to R^1 \lim H^{i-1}(RF(E_n)) \to H^i(RF(E)) \to \lim H^i(RF(E_n)) \to 0$$

see More on Algebra, Remark 86.10. To prove (2) we will show that the term on the left is zero and that the term on the right equals $H^i(RF(E_{-b+N-1})$ for any b with $i \geq b$.

For every n we have a distinguished triangle

$$H^{-n}(E)[n] \to E_n \to E_{n-1} \to H^{-n}(E)[n+1]$$

(Derived Categories, Remark 12.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have

$$H^i(RF(H^{-n}(E)[n])) = R^{i+n}F(H^{-n}(E)) = 0$$

for $i + n \geq N$ and

$$H^i(RF(H^{-n}(E)[n+1])) = R^{i+n+1}F(H^{-n}(E)) = 0$$

[1]In particular, E has a K-injective representative as in Cohomology, Lemma 38.1.

for $i + n + 1 \geq N$. We conclude that

$$H^i(RF(E_n)) \to H^i(RF(E_{n-1}))$$

is an isomorphism for $n \geq N - i$. Thus the systems $H^i(RF(E_n))$ all satisfy the ML condition and the $R^1 \lim$ term in our short exact sequence is zero (see discussion in More on Algebra, Section 86). Moreover, the system $H^i(RF(E_n))$ is constant starting with $n = N - i - 1$ as desired.

Proof of (3). Under the assumption on E we have $\tau_{\leq a-1}E = 0$ and we get the vanishing of $H^i(RF(E))$ for $i \leq a - 1$ from (1). Similarly, we have $\tau_{\geq b+1}E = 0$ and hence we get the vanishing of $H^i(RF(E))$ for $i \geq b + n$ from part (2). $\square$

The following lemma is the key ingredient to many of the results in this chapter.

Lemma 3.5. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. All the functors in the diagram*

$$\begin{tikzcd}[column sep=small] D(QCoh(\mathcal{O}_X)) \arrow[rr, "(3.0.1)"'] & & D_{QCoh}(\mathcal{O}_X) \arrow[dl, "R\Gamma(X{,}-)"] \\ & D(A) \arrow[ul, "\sim"] & \end{tikzcd}$$

are equivalences of triangulated categories. Moreover, for E in $D_{QCoh}(\mathcal{O}_X)$ we have $H^0(X, E) = H^0(X, H^0(E))$.

Proof. The functor $R\Gamma(X, -)$ gives a functor $D(\mathcal{O}_X) \to D(A)$ and hence by restriction a functor

$$R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \longrightarrow D(A). \tag{3.5.1}$$

We will show this functor is quasi-inverse to (3.0.1) via the equivalence between quasi-coherent modules on X and the category of A-modules.

Elucidation. Denote $(Y, \mathcal{O}_Y)$ the one point space with sheaf of rings given by A. Denote $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ the obvious morphism of ringed spaces. Then $R\Gamma(X, -)$ can be identified with $R\pi_*$ and the functor (3.0.1) via the equivalence $\mathit{Mod}(\mathcal{O}_Y) = \mathrm{Mod}_A = QCoh(\mathcal{O}_X)$ can be identified with $L\pi^* = \pi^* = \widetilde{\ }$ (see Modules, Lemma 10.5 and Schemes, Lemmas 7.1 and 7.5). Thus the functors

$$\begin{tikzcd} D(A) \arrow[r, shift left] & D(\mathcal{O}_X) \arrow[l, shift left] \end{tikzcd}$$

are adjoint (by Cohomology, Lemma 28.1). In particular we obtain canonical adjunction mappings

$$a : \widetilde{R\Gamma(X, E)} \longrightarrow E$$

for E in $D(\mathcal{O}_X)$ and

$$b : M^\bullet \longrightarrow R\Gamma(X, \widetilde{M^\bullet})$$

for $M^\bullet$ a complex of A-modules.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. We may apply Lemma 3.4 to the functor $F(-) = \Gamma(X, -)$ with $N = 1$ by Cohomology of Schemes, Lemma 2.2. Hence

$$H^0(R\Gamma(X, E)) = H^0(R\Gamma(X, \tau_{\geq 0}E)) = \Gamma(X, H^0(E))$$

(the last equality by definition of the canonical truncation). Using this we will show that the adjunction mappings a and b induce isomorphisms $H^0(a)$ and $H^0(b)$. Thus a and b are quasi-isomorphisms (as the statement is invariant under shifts) and the lemma is proved.

In both cases we use that $\widetilde{\ }$ is an exact functor (Schemes, Lemma 5.4). Namely, this implies that

$$H^0\left(\widetilde{R\Gamma(X, E)}\right) = H^0(\widetilde{R\Gamma(X, E)}) = \Gamma(X, \widetilde{H^0(E)})$$

which is equal to $H^0(E)$ because $H^0(E)$ is quasi-coherent. Thus $H^0(a)$ is an isomorphism. For the other direction we have

$$H^0(R\Gamma(X, \widetilde{M^\bullet})) = \Gamma(X, H^0(\widetilde{M^\bullet})) = \Gamma(X, \widetilde{H^0(M^\bullet)}) = H^0(M^\bullet)$$

which proves that $H^0(b)$ is an isomorphism. □

Lemma 3.6. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. If $K^\bullet$ is a K-flat complex of A-modules, then $\widetilde{K^\bullet}$ is a K-flat complex of $\mathcal{O}_X$-modules.*

Proof. By More on Algebra, Lemma 59.3 we see that $K^\bullet \otimes_A A_\mathfrak{p}$ is a K-flat complex of $A_\mathfrak{p}$-modules for every $\mathfrak{p} \in \mathrm{Spec}(A)$. Hence we conclude from Cohomology, Lemma 26.4 (and Schemes, Lemma 5.4) that $\widetilde{K^\bullet}$ is K-flat. □

Lemma 3.7. *If $f : X \to Y$ is a morphism of affine schemes given by the ring map $A \to B$, then the diagram*

$$\begin{tikzcd} D(B) \arrow[r] \arrow[d] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(A) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

commutes.

Proof. Follows from Lemma 3.5 using that $R\Gamma(Y, Rf_*K) = R\Gamma(X, K)$ by Cohomology, Lemma 32.5. □

Lemma 3.8. *Let $f : Y \to X$ be a morphism of schemes.*

(1) *The functor Lf^* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.*
(2) *If X and Y are affine and f is given by the ring map $A \to B$, then the diagram*

$$\begin{tikzcd} D(B) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \\ D(A) \arrow[r] \arrow[u, "- \otimes_A^{\mathbf{L}} B"] & D_{QCoh}(\mathcal{O}_X) \arrow[u, "Lf^*"'] \end{tikzcd}$$

commutes.

Proof. We first prove the diagram

$$\begin{tikzcd} D(B) \arrow[r] & D(\mathcal{O}_Y) \\ D(A) \arrow[r] \arrow[u, "- \otimes_A^{\mathbf{L}} B"] & D(\mathcal{O}_X) \arrow[u, "Lf^*"'] \end{tikzcd}$$

commutes. This is clear from Lemma 3.6 and the constructions of the functors in question. To see (1) let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To see that Lf^*E has quasi-coherent cohomology sheaves we may work locally on X. Note that Lf^* is compatible with restricting to open subschemes. Hence we can assume that f is a morphism of affine schemes as in (2). Then we can apply Lemma 3.5 to see that E

comes from a complex of A-modules. By the commutativity of the first diagram of the proof the same holds for Lf^*E and we conclude (1) is true. $\square$

Lemma 3.9. *Let X be a scheme.*

(1) *For objects K, L of $D_{QCoh}(\mathcal{O}_X)$ the derived tensor product $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(\mathcal{O}_X)$.*
(2) *If $X = \mathrm{Spec}(A)$ is affine then*

$$\widetilde{M^\bullet} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{K^\bullet} = \widetilde{M^\bullet \otimes^{\mathbf{L}}_A K^\bullet}$$

for any pair of complexes of A-modules $K^\bullet$, $M^\bullet$.

Proof. The equality of (2) follows immediately from Lemma 3.6 and the construction of the derived tensor product. To see (1) let K, L be objects of $D_{QCoh}(\mathcal{O}_X)$. To check that $K \otimes^{\mathbf{L}} L$ is in $D_{QCoh}(\mathcal{O}_X)$ we may work locally on X, hence we may assume $X = \mathrm{Spec}(A)$ is affine. By Lemma 3.5 we may represent K and L by complexes of A-modules. Then part (2) implies the result. $\square$

4. Total direct image

The following lemma is the analogue of Cohomology of Schemes, Lemma 4.5.

Lemma 4.1. *Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact.*

(1) *The functor Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_S)$.*
(2) *If S is quasi-compact, there exists an integer $N = N(X, S, f)$ such that for an object E of $D_{QCoh}(\mathcal{O}_X)$ with $H^m(E) = 0$ for $m > 0$ we have $H^m(Rf_*E) = 0$ for $m \geq N$.*
(3) *In fact, if S is quasi-compact we can find $N = N(X, S, f)$ such that for every morphism of schemes $S' \to S$ the same conclusion holds for the functor $R(f')_*$ where $f' : X' \to S'$ is the base change of f.*

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To prove (1) we have to show that Rf_*E has quasi-coherent cohomology sheaves. The question is local on S, hence we may assume S is quasi-compact. Pick $N = N(X, S, f)$ as in Cohomology of Schemes, Lemma 4.5. Thus $R^pf_*\mathcal{F} = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $p \geq N$ and the same remains true after base change.

First, assume E is bounded below. We will show (1) and (2) and (3) hold for such E with our choice of N. In this case we can for example use the spectral sequence

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E$$

(Derived Categories, Lemma 21.3), the quasi-coherence of $R^pf_*H^q(E)$, and the vanishing of $R^pf_*H^q(E)$ for $p \geq N$ to see that (1), (2), and (3) hold in this case.

Next we prove (2) and (3). Say $H^m(E) = 0$ for $m > 0$. Let $U \subset S$ be affine open. By Cohomology of Schemes, Lemma 4.6 and our choice of N we have $H^p(f^{-1}(U), \mathcal{F}) = 0$ for $p \geq N$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Hence we may apply Lemma 3.4 to the functor $\Gamma(f^{-1}(U), -)$ to see that

$$R\Gamma(U, Rf_*E) = R\Gamma(f^{-1}(U), E)$$

has vanishing cohomology in degrees $\geq N$. Since this holds for all $U \subset S$ affine open we conclude that $H^m(Rf_*E) = 0$ for $m \geq N$.

Next, we prove (1) in the general case. Recall that there is a distinguished triangle

$$\tau_{\leq -n-1}E \to E \to \tau_{\geq -n}E \to (\tau_{\leq -n-1}E)[1]$$

in $D(\mathcal{O}_X)$, see Derived Categories, Remark 12.4. By (2) we see that $Rf_*\tau_{\leq -n-1}E$ has vanishing cohomology sheaves in degrees $\geq -n+N$. Thus, given an integer q we see that R^qf_*E is equal to $R^qf_*\tau_{\geq -n}E$ for some n and the result above applies. □

Lemma 4.2. *Let $f : X \to S$ be a quasi-separated and quasi-compact morphism of schemes. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules each of which is right acyclic for f_*. Then $f_*\mathcal{F}^\bullet$ represents $Rf_*\mathcal{F}^\bullet$ in $D(\mathcal{O}_S)$.*

Proof. There is always a canonical map $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$. Our task is to show that this is an isomorphism on cohomology sheaves. As the statement is invariant under shifts it suffices to show that $H^0(f_*(\mathcal{F}^\bullet)) \to R^0f_*\mathcal{F}^\bullet$ is an isomorphism. The statement is local on S hence we may assume S affine. By Lemma 4.1 we have $R^0f_*\mathcal{F}^\bullet = R^0f_*\tau_{\geq -n}\mathcal{F}^\bullet$ for all sufficiently large n. Thus we may assume $\mathcal{F}^\bullet$ bounded below. As each $\mathcal{F}^n$ is right f_*-acyclic by assumption we see that $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$ is a quasi-isomorphism by Leray's acyclicity lemma (Derived Categories, Lemma 16.7). □

Lemma 4.3. *Let X be a quasi-separated and quasi-compact scheme. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules each of which is right acyclic for $\Gamma(X, -)$. Then $\Gamma(X, \mathcal{F}^\bullet)$ represents $R\Gamma(X, \mathcal{F}^\bullet)$ in $D(\Gamma(X, \mathcal{O}_X)$.*

Proof. Apply Lemma 4.2 to the canonical morphism $X \to \text{Spec}(\Gamma(X, \mathcal{O}_X))$. Some details omitted. □

Lemma 4.4. *Let X be a quasi-separated and quasi-compact scheme. For any object K of $D_{QCoh}(\mathcal{O}_X)$ the spectral sequence*

$$E_2^{i,j} = H^i(X, H^j(K)) \Rightarrow H^{i+j}(X, K)$$

of Cohomology, Example 29.3 is bounded and converges.

Proof. By the construction of the spectral sequence via Cohomology, Lemma 29.1 using the filtration given by $\tau_{\leq -p}K$, we see that suffices to show that given $n \in \mathbf{Z}$ we have

$$H^n(X, \tau_{\leq -p}K) = 0 \text{ for } p \gg 0$$

and

$$H^n(X, K) = H^n(X, \tau_{\leq -p}K) \text{ for } p \ll 0$$

The first follows from Lemma 3.4 applied with $F = \Gamma(X, -)$ and the bound in Cohomology of Schemes, Lemma 4.5. The second holds whenever $-p \leq n$ for any ringed space $(X, \mathcal{O}_X)$ and any $K \in D(\mathcal{O}_X)$. □

Lemma 4.5. *Let $f : X \to S$ be a quasi-separated and quasi-compact morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ commutes with direct sums.*

Proof. Let E_i be a family of objects of $D_{QCoh}(\mathcal{O}_X)$ and set $E = \bigoplus E_i$. We want to show that the map

$$\bigoplus Rf_*E_i \longrightarrow Rf_*E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. Choose an integer N as in Lemma 4.1. Then $R^0f_*E = R^0f_*\tau_{\geq -N}E$ and $R^0f_*E_i = R^0f_*\tau_{\geq -N}E_i$ by the lemma cited.

Observe that $\tau_{\geq -N}E = \bigoplus \tau_{\geq -N}E_i$. Thus we may assume all of the E_i have vanishing cohomology sheaves in degrees $< -N$. Next we use the spectral sequences

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E \quad \text{and} \quad R^pf_*H^q(E_i) \Rightarrow R^{p+q}f_*E_i$$

(Derived Categories, Lemma 21.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Schemes, Lemma 6.1. □

5. Affine morphisms

In this section we collect some information about pushforward along an affine morphism of schemes.

Lemma 5.1. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Then $f_*\mathcal{F}^\bullet = Rf_*\mathcal{F}^\bullet$.*

Proof. Combine Lemma 4.2 with Cohomology of Schemes, Lemma 2.3. An alternative proof is to work affine locally on S and use Lemma 3.7. □

Lemma 5.2. *Let $f : X \to S$ be an affine morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ reflects isomorphisms.*

Proof. The statement means that a morphism $\alpha : E \to F$ of $D_{QCoh}(\mathcal{O}_X)$ is an isomorphism if $Rf_*\alpha$ is an isomorphism. We may check this on cohomology sheaves. In particular, the question is local on S. Hence we may assume S and therefore X is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ given in Lemma 3.5. Some details omitted. □

Lemma 5.3. *Let $f : X \to S$ be an affine morphism of schemes. For E in $D_{QCoh}(\mathcal{O}_S)$ we have $Rf_*Lf^*E = E \otimes^{\mathbf{L}}_{\mathcal{O}_S} f_*\mathcal{O}_X$.*

Proof. Since f is affine the map $f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$ is an isomorphism (Cohomology of Schemes, Lemma 2.3). There is a canonical map $E \otimes^{\mathbf{L}} f_*\mathcal{O}_X = E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X \to Rf_*Lf^*E$ adjoint to the map

$$Lf^*(E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X) = Lf^*E \otimes^{\mathbf{L}} Lf^*Rf_*\mathcal{O}_X \longrightarrow Lf^*E \otimes^{\mathbf{L}} \mathcal{O}_X = Lf^*E$$

coming from $1 : Lf^*E \to Lf^*E$ and the canonical map $Lf^*Rf_*\mathcal{O}_X \to \mathcal{O}_X$. To check the map so constructed is an isomorphism we may work locally on S. Hence we may assume S and therefore X is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ and the functor Lf^* given in Lemmas 3.5 and 3.8. Some details omitted. □

Let Y be a scheme. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_Y$-algebras. We will denote $D_{QCoh}(\mathcal{A})$ the inverse image of $D_{QCoh}(\mathcal{O}_X)$ under the restriction functor $D(\mathcal{A}) \to D(\mathcal{O}_X)$. In other words, $K \in D(\mathcal{A})$ is in $D_{QCoh}(\mathcal{A})$ if and only if its cohomology sheaves are quasi-coherent as $\mathcal{O}_X$-modules. If $\mathcal{A}$ is quasi-coherent itself this is the same as asking the cohomology sheaves to be quasi-coherent as $\mathcal{A}$-modules, see Morphisms, Lemma 11.6.

Lemma 5.4. *Let $f : X \to Y$ be an affine morphism of schemes. Then f_* induces an equivalence*

$$\Phi : D_{QCoh}(\mathcal{O}_X) \longrightarrow D_{QCoh}(f_*\mathcal{O}_X)$$

whose composition with $D_{QCoh}(f_*\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$.

Proof. Recall that Rf_* is computed on an object $K \in D_{QCoh}(\mathcal{O}_X)$ by choosing a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules representing K and taking $f_*\mathcal{I}^\bullet$. Thus we let $\Phi(K)$ be the complex $f_*\mathcal{I}^\bullet$ viewed as a complex of $f_*\mathcal{O}_X$-modules. Denote $g : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$ the obvious morphism of ringed spaces. Then g is a flat morphism of ringed spaces (see below for a description of the stalks) and Φ is the restriction of Rg_* to $D_{QCoh}(\mathcal{O}_X)$. We claim that Lg^* is a quasi-inverse. First, observe that Lg^* sends $D_{QCoh}(f_*\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_X)$ because g^* transforms quasi-coherent modules into quasi-coherent modules (Modules, Lemma 10.4). To finish the proof it suffices to show that the adjunction mappings

$$Lg^*\Phi(K) = Lg^*Rg_*K \to K \quad \text{and} \quad M \to Rg_*Lg^*M = \Phi(Lg^*M)$$

are isomorphisms for $K \in D_{QCoh}(\mathcal{O}_X)$ and $M \in D_{QCoh}(f_*\mathcal{O}_X)$. This is a local question, hence we may assume Y and therefore X are affine.

Assume $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$. Let $\mathfrak{p} = x \in \mathrm{Spec}(A) = X$ be a point mapping to $\mathfrak{q} = y \in \mathrm{Spec}(B) = Y$. Then $(f_*\mathcal{O}_X)_y = A_\mathfrak{q}$ and $\mathcal{O}_{X,x} = A_\mathfrak{p}$ hence g is flat. Hence g^* is exact and $H^i(Lg^*M) = g^*H^i(M)$ for any M in $D(f_*\mathcal{O}_X)$. For $K \in D_{QCoh}(\mathcal{O}_X)$ we see that

$$H^i(\Phi(K)) = H^i(Rf_*K) = f_*H^i(K)$$

by the vanishing of higher direct images (Cohomology of Schemes, Lemma 2.3) and Lemma 3.4 (small detail omitted). Thus it suffice to show that

$$g^*g_*\mathcal{F} \to \mathcal{F} \quad \text{and} \quad \mathcal{G} \to g_*g^*\mathcal{F}$$

are isomorphisms where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and $\mathcal{G}$ is a quasi-coherent $f_*\mathcal{O}_X$-module. This follows from Morphisms, Lemma 11.6. □

6. Cohomology with support in a closed subset

We elaborate on the material in Cohomology, Sections 21 and 34 for schemes and quasi-coherent modules.

Definition 6.1. Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Let $T \subset X$ be a closed subset. We say E is *supported on* T if the cohomology sheaves $H^i(E)$ are supported on T.

We repeat some of the discussion from Cohomology, Section 34 in the situation of the definition. Let X be a scheme. Let $T \subset X$ be a closed subset. The category of $\mathcal{O}_X$-modules whose support is contained in T is a Serre subcategory of the category of all $\mathcal{O}_X$-modules, see Homology, Definition 10.1 and Modules, Lemma 5.2. In the following we will denote $D_T(\mathcal{O}_X)$ the strictly full, saturated triangulated subcategory of $D(\mathcal{O}_X)$ consisting of objects supported on T, see Derived Categories, Section 17.

In the situation of Definition 6.1 denote $i : T \to X$ the inclusion map. Recall from Cohomology, Section 34 that in this situation we have a functor $R\mathcal{H}_T : D(\mathcal{O}_X) \to D(i^{-1}\mathcal{O}_X)$ which is right adjoint to $i_* : D(i^{-1}\mathcal{O}_X) \to D(\mathcal{O}_X)$.

Lemma 6.2. *Let X be a scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is a retrocompact open of X. Let $i : T \to X$ be the inclusion.*

1. *For E in $D_{QCoh}(\mathcal{O}_X)$ we have $i_*R\mathcal{H}_T(E)$ in $D_{QCoh,T}(\mathcal{O}_X)$.*
2. *The functor $i_* \circ R\mathcal{H}_T : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh,T}(\mathcal{O}_X)$ is right adjoint to the inclusion functor $D_{QCoh,T}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_X)$.*

Proof. Set $U = X \setminus T$ and denote $j : U \to X$ the inclusion. By Cohomology, Lemma 34.6 there is a distinguished triangle

$$i_*R\mathcal{H}_T(E) \to E \to Rj_*(E|_U) \to i_*R\mathcal{H}_Z(E)[1]$$

in $D(\mathcal{O}_X)$. By Lemma 4.1 the complex $Rj_*(E|_U)$ has quasi-coherent cohomology sheaves (this is where we use that U is retrocompact in X). Thus we see that (1) is true. Part (2) follows from this and the adjointness of functors in Cohomology, Lemma 34.2. $\square$

Lemma 6.3. *Let X be a scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is a retrocompact open of X. Then for a family of objects E_i, $i \in I$ of $D_{QCoh}(\mathcal{O}_X)$ we have $R\mathcal{H}_T(\bigoplus E_i) = \bigoplus R\mathcal{H}_T(E_i)$.*

Proof. Set $U = X \setminus T$ and denote $j : U \to X$ the inclusion. By Cohomology, Lemma 34.6 there is a distinguished triangle

$$i_*R\mathcal{H}_T(E) \to E \to Rj_*(E|_U) \to i_*R\mathcal{H}_Z(E)[1]$$

in $D(\mathcal{O}_X)$ for any E in $D(\mathcal{O}_X)$. The functor $E \mapsto Rj_*(E|_U)$ commutes with direct sums on $D_{QCoh}(\mathcal{O}_X)$ by Lemma 4.5. It follows that the same is true for the functor $i_* \circ R\mathcal{H}_T$ (details omitted). Since $i_* : D(i^{-1}\mathcal{O}_X) \to D_T(\mathcal{O}_X)$ is an equivalence (Cohomology, Lemma 34.2) we conclude. $\square$

Remark 6.4. Let X be a scheme. Let $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$. Denote $Z \subset X$ the closed subscheme cut out by $f_1, \ldots, f_c$. For $0 \leq p < c$ and $1 \leq i_0 < \ldots < i_p \leq c$ we denote $U_{i_0 \ldots i_p} \subset X$ the open subscheme where $f_{i_0} \ldots f_{i_p}$ is invertible. For any $\mathcal{O}_X$-module $\mathcal{F}$ we set

$$\mathcal{F}_{i_0 \ldots i_p} = (U_{i_0 \ldots i_p} \to X)_*(\mathcal{F}|_{U_{i_0 \ldots i_p}})$$

In this situation the *extended alternating Čech complex* is the complex of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F} \to \bigoplus\nolimits_{i_0} \mathcal{F}_{i_0} \to \ldots \to \bigoplus\nolimits_{i_0 < \ldots < i_p} \mathcal{F}_{i_0 \ldots i_p} \to \ldots \to \mathcal{F}_{1 \ldots c} \to 0 \tag{6.4.1}$$

where $\mathcal{F}$ is put in degree 0. The maps are constructed as follows. Given $1 \leq i_0 < \ldots < i_{p+1} \leq c$ and $0 \leq j \leq p+1$ we have the canonical map

$$\mathcal{F}_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \to \mathcal{F}_{i_0 \ldots i_p}$$

coming from the inclusion $U_{i_0 \ldots i_p} \subset U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$. The differentials in the extended alternating complex use these canonical maps with sign $(-1)^j$.

Lemma 6.5. *With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 6.4 the complex (6.4.1) restricts to an acyclic complex over $X \setminus Z$.*

We remark that this lemma holds more generally for any extended alternating Čech complex defined as in Remark 6.4 starting with a finite open covering $X \setminus Z = U_1 \cup \ldots \cup U_c$.

Proof. Let $W \subset X \setminus Z$ be an open subset. Evaluating the complex of sheaves (6.4.1) on W we obtain the complex

$$\mathcal{F}(W) \to \bigoplus\nolimits_{i_0} \mathcal{F}(U_{i_0} \cap W) \to \bigoplus\nolimits_{i_0 < i_1} \mathcal{F}(U_{i_0 i_1} \cap W) \to \ldots$$

In other words, we obtain the extended ordered Čech complex for the covering $W = \bigcup U_i \cap W$ and the standard ordering on $\{1, \ldots, c\}$, see Cohomology, Section

23. By Cohomology, Lemma 23.7 this complex is homotopic to zero as soon as W is contained in $V(f_i)$ for some $1 \leq i \leq c$. This finishes the proof. $\square$

Remark 6.6. Let X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ be as in Remark 6.4. Denote $\mathcal{F}^\bullet$ the complex (6.4.1). By Lemma 6.5 the cohomology sheaves of $\mathcal{F}^\bullet$ are supported on Z hence $\mathcal{F}^\bullet$ is an object of $D_Z(\mathcal{O}_X)$. On the other hand, the equality $\mathcal{F}^0 = \mathcal{F}$ determines a canonical map $\mathcal{F}^\bullet \to \mathcal{F}$ in $D(\mathcal{O}_X)$. As $i_* \circ R\mathcal{H}_Z$ is a right adjoint to the inclusion functor $D_Z(\mathcal{O}_X) \to D(\mathcal{O}_X)$, see Cohomology, Lemma 34.2, we obtain a canonical commutative diagram

$$\begin{tikzcd} \mathcal{F}^\bullet \arrow[rr] \arrow[dr] & & \mathcal{F} \\ & i_*R\mathcal{H}_Z(\mathcal{F}) \arrow[ur] & \end{tikzcd}$$

in $D(\mathcal{O}_X)$ functorial in the $\mathcal{O}_X$-module $\mathcal{F}$.

Lemma 6.7. *With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 6.4. If $\mathcal{F}$ is quasi-coherent, then the complex (6.4.1) represents $i_*R\mathcal{H}_Z(\mathcal{F})$ in $D_Z(\mathcal{O}_X)$.*

Proof. Let us denote $\mathcal{F}^\bullet$ the complex (6.4.1). The statement of the lemma means that the map $\mathcal{F}^\bullet \to i_*R\mathcal{H}_Z(\mathcal{F})$ of Remark 6.6 is an isomorphism. Since $\mathcal{F}^\bullet$ is in $D_Z(\mathcal{O}_X)$ (see remark cited), we see that $i_*R\mathcal{H}_Z(\mathcal{F}^\bullet) = \mathcal{F}^\bullet$ by Cohomology, Lemma 34.2. The morphism $U_{i_0 \ldots i_p} \to X$ is affine as it is given over affine opens of X by inverting the function $f_{i_0} \ldots f_{i_p}$. Thus we see that

$$\mathcal{F}_{i_0 \ldots i_p} = (U_{i_0 \ldots i_p} \to X)_* \mathcal{F}|_{U_{i_0 \ldots i_p}} = R(U_{i_0 \ldots i_p} \to X)_* \mathcal{F}|_{U_{i_0 \ldots i_p}}$$

by Cohomology of Schemes, Lemma 2.3 and the assumption that $\mathcal{F}$ is quasi-coherent. We conclude that $R\mathcal{H}_Z(\mathcal{F}_{i_0 \ldots i_p}) = 0$ by Cohomology, Lemma 34.7. Thus $i_*R\mathcal{H}_Z(\mathcal{F}^p) = 0$ for $p > 0$. Putting everything together we obtain

$$\mathcal{F}^\bullet = i_*R\mathcal{H}_Z(\mathcal{F}^\bullet) = i_*R\mathcal{H}_Z(\mathcal{F})$$

as desired. $\square$

Lemma 6.8. *Let X be a scheme. Let $T \subset X$ be a closed subset which can locally be cut out by at most c elements of the structure sheaf. Then $\mathcal{H}^i_Z(\mathcal{F}) = 0$ for $i > c$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. This follows immediately from the local description of $R\mathcal{H}_T(\mathcal{F})$ given in Lemma 6.7. $\square$

Lemma 6.9. *Let X be a scheme. Let $T \subset X$ be a closed subset which can locally be cut out by a Koszul regular sequence having c elements. Then $\mathcal{H}^i_Z(\mathcal{F}) = 0$ for $i \neq c$ for every flat, quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. By the description of $R\mathcal{H}_Z(\mathcal{F})$ given in Lemma 6.7 this boils down to the following algebra statement: given a ring R, a Koszul regular sequence $f_1, \ldots, f_c \in R$, and a flat R-module M, the extended alternating Čech complex $M \to \bigoplus_{i_0} M_{f_{i_0}} \to \bigoplus_{i_0 < i_1} M_{f_{i_0} f_{i_1}} \to \ldots \to M_{f_1 \ldots f_c}$ from More on Algebra, Section 29 only has cohomology in degree c. By More on Algebra, Lemma 31.1 we obtain the desired vanishing for the extended alternating Čech complex of R. Since the complex for M is obtained by tensoring this with the flat R-module M (More on Algebra, Lemma 29.2) we conclude. $\square$

Remark 6.10. With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 6.4. There is a canonical $\mathcal{O}_X|_Z$-linear map

$$c_{f_1,\ldots,f_c} : i^*\mathcal{F} \longrightarrow \mathcal{H}^c_Z(\mathcal{F})$$

functorial in $\mathcal{F}$. Namely, denoting $\mathcal{F}^\bullet$ the extended alternating Čech complex (6.4.1) we have the canonical map $\mathcal{F}^\bullet \to i_*R\mathcal{H}_Z(\mathcal{F})$ of Remark 6.6. This determines a canonical map

$$\text{Coker}\left(\bigoplus \mathcal{F}_{1\ldots\hat{i}\ldots c} \to \mathcal{F}_{1\ldots c}\right) \longrightarrow i_*\mathcal{H}^c_Z(\mathcal{F})$$

on cohomology sheaves in degree c. Given a local section s of $\mathcal{F}$ we can consider the local section

$$\frac{s}{f_1 \ldots f_c}$$

of $\mathcal{F}_{1\ldots c}$. The class of this section in the cokernel displayed above depends only on s modulo the image of $(f_1, \ldots, f_c) : \mathcal{F}^{\oplus c} \to \mathcal{F}$. Since $i_*i^*\mathcal{F}$ is equal to the cokernel of $(f_1, \ldots, f_c) : \mathcal{F}^{\oplus c} \to \mathcal{F}$ we see that we get an $\mathcal{O}_X$-module map $i_*i^*\mathcal{F} \to i_*\mathcal{H}^c_Z(\mathcal{F})$. As i_* is fully faithful we get the map $c_{f_1,\ldots,f_c}$.

Example 6.11. Let $X = \text{Spec}(A)$ be affine, $f_1, \ldots, f_c \in A$, and let $\mathcal{F} = \widetilde{M}$ for some A-module M. The map $c_{f_1,\ldots,f_c}$ of Remark 6.10 can be described as the map

$$M/(f_1, \ldots, f_c)M \longrightarrow \text{Coker}\left(\bigoplus M_{f_1\ldots\hat{f}_i\ldots f_c} \to M_{f_1\ldots f_c}\right)$$

sending the class of $s \in M$ to the class of $s/f_1 \ldots f_c$ in the cokernel.

Lemma 6.12. *With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 6.4. Let $a_{ji} \in \Gamma(X, \mathcal{O}_X)$ for $1 \leq i, j \leq c$ and set $g_j = \sum_{i=1,\ldots,c} a_{ji}f_i$. Assume $g_1, \ldots, g_c$ scheme theoretically cut out Z. If $\mathcal{F}$ is quasi-coherent, then*

$$c_{f_1,\ldots,f_c} = \det(a_{ji})c_{g_1,\ldots,g_c}$$

where $c_{f_1,\ldots,f_c}$ and $c_{g_1,\ldots,g_c}$ are as in Remark 6.10.

Proof. We will prove that $c_{f_1,\ldots,f_c}(s) = \det(a_{ij})c_{g_1,\ldots,g_c}(s)$ as global sections of $\mathcal{H}_Z(\mathcal{F})$ for any $s \in \mathcal{F}(X)$. This is sufficient since we then obtain the same result for section over any open subscheme of X. To do this, for $1 \leq i_0 < \ldots < i_p \leq c$ and $1 \leq j_0 < \ldots < j_q \leq c$ we denote $U_{i_0\ldots i_p} \subset X$, $V_{j_0\ldots j_q} \subset X$, and $W_{i_0\ldots i_p,j_0\ldots j_q} \subset X$ the open subscheme where $f_{i_0} \ldots f_{i_p}$ is invertible, $g_{j_0} \ldots g_{j_q}$ is invertible, and where $f_{i_0} \ldots f_{i_p}g_{j_0} \ldots g_{j_q}$ is invertible. We denote $\mathcal{F}_{i_0\ldots i_p}$, resp. $\mathcal{F}'_{j_0\ldots j_q}$ $\mathcal{F}''_{i_0\ldots i_p,j_0\ldots j_q}$ the pushforward to X of the restriction of $\mathcal{F}$ to $U_{i_0\ldots i_p}$, resp. $V_{j_0\ldots j_q}$, resp. $W_{i_0\ldots i_p,j_0\ldots j_q}$. Then we obtain three extended alternating Čech complexes

$$\mathcal{F}^\bullet : \mathcal{F} \to \bigoplus\nolimits_{i_0} \mathcal{F}_{i_0} \to \bigoplus\nolimits_{i_0<i_1} \mathcal{F}_{i_0i_1} \to \ldots$$

and

$$(\mathcal{F}')^\bullet : \mathcal{F} \to \bigoplus\nolimits_{j_0} \mathcal{F}'_{j_0} \to \bigoplus\nolimits_{j_0<j_1} \mathcal{F}'_{j_0j_1} \to \ldots$$

and

$$(\mathcal{F}'')^\bullet : \mathcal{F} \to \bigoplus\nolimits_{i_0} \mathcal{F}_{i_0} \oplus \bigoplus\nolimits_{j_0} \mathcal{F}'_{j_0} \to \bigoplus\nolimits_{i_0<i_1} \mathcal{F}_{i_0i_1} \oplus \bigoplus\nolimits_{i_0,j_0} \mathcal{F}''_{i_0,j_0} \oplus \bigoplus\nolimits_{j_0<j_1} \mathcal{F}'_{j_0j_1} \to \ldots$$

whose differentials are those used in defining (6.4.1). There are maps of complexes

$$(\mathcal{F}'')^\bullet \to \mathcal{F}^\bullet \quad \text{and} \quad (\mathcal{F}'')^\bullet \to (\mathcal{F}')^\bullet$$

given by the projection maps on the terms (and hence inducing the identity map in degree 0). Observe that by Lemma 6.7 each of these complexes represents $i_*R\mathcal{H}_Z(\mathcal{F})$ and these maps represent the identity on this object. Thus it suffices to find an element

$$\sigma \in H^c((\mathcal{F}'')^\bullet(X))$$

mapping to $c_{f_1,\ldots,f_c}(s)$ and $\det(a_{ji})c_{g_1,\ldots,g_c}(s)$ by these two maps. It turns out we can explicitly give a cocycle for σ. Namely, we take

$$\sigma_{1\ldots c} = \frac{s}{f_1 \ldots f_c} \in \mathcal{F}_{1\ldots c}(X) \quad \text{and} \quad \sigma'_{1\ldots c} = \frac{\det(a_{ji})s}{g_1 \ldots g_c} \in \mathcal{F}'_{1\ldots c}(X)$$

and we take

$$\sigma_{i_0\ldots i_p, j_0\ldots j_{c-p-2}} = \frac{\lambda(i_0\ldots i_p, j_0 \ldots j_{c-p-2})s}{f_{i_0}\ldots f_{i_p} g_{j_0} \ldots g_{j_{c-p-2}}} \in \mathcal{F}''_{i_0\ldots i_p, j_0\ldots j_{c-p-2}}(X)$$

where $\lambda(i_0 \ldots i_p, j_0 \ldots j_{c-p-2})$ is the coefficient of $e_1 \wedge \ldots \wedge e_c$ in the formal expression

$$e_{i_0} \wedge \ldots \wedge e_{i_p} \wedge (a_{j_0 1}e_1 + \ldots + a_{j_0 c}e_c) \wedge \ldots \wedge (a_{j_{c-p-2}1}e_1 + \ldots + a_{j_{c-p-2}c}e_c)$$

To verify that σ is a cocycle, we have to show for $1 \leq i_0 < \ldots < i_p \leq c$ and $1 \leq j_0 < \ldots < j_{c-p-1} \leq c$ that we have

$$\begin{aligned} 0 = & \sum\nolimits_{a=0,\ldots,p} (-1)^a f_{i_a} \lambda(i_0 \ldots \hat{i}_a \ldots i_p, j_0 \ldots j_{c-p-1}) \\ & + \sum\nolimits_{b=0,\ldots,c-p-1} (-1)^{p+b+1} g_{j_b} \lambda(i_0 \ldots i_p, j_0 \ldots \hat{j}_b \ldots j_{c-p-1}) \end{aligned}$$

The easiest way to see this is perhaps to argue that the formal expression

$$\xi = e_{i_0} \wedge \ldots \wedge e_{i_p} \wedge (a_{j_0 1}e_1 + \ldots + a_{j_0 c}e_c) \wedge \ldots \wedge (a_{j_{c-p-1}1}e_1 + \ldots + a_{j_{c-p-1}c}e_c)$$

is 0 as it is an element of the $(c+1)$st wedge power of the free module on $e_1, \ldots, e_c$ and that the expression above is the image of ξ under the Koszul differential sending $e_i \to f_i$. Some details omitted. □

Lemma 6.13. *Let X be a scheme. Let $Z \to X$ be a closed immersion of finite presentation whose conormal sheaf $\mathcal{C}_{Z/X}$ is locally free of rank c. Then there is a canonical map*

$$c : \wedge^c(\mathcal{C}_{Z/X})^\vee \otimes_{\mathcal{O}_Z} i^*\mathcal{F} \longrightarrow \mathcal{H}^c_Z(\mathcal{F})$$

functorial in the quasi-coherent module $\mathcal{F}$.

Proof. Follows from the construction in Remark 6.10 and the independence of the choice of generators of the ideal sheaf shown in Lemma 6.12. Some details omitted. □

Remark 6.14. Let $g : X' \to X$ be a morphism of schemes. Let $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$. Set $f'_i = g^\sharp(f_i) \in \Gamma(X', \mathcal{O}_{X'})$. Denote $Z \subset X$, resp. $Z' \subset X'$ the closed subscheme cut out by $f_1, \ldots, f_c$, resp. $f'_1, \ldots, f'_c$. Then $Z' = Z \times_X X'$. Denote $h : Z' \to Z$ the induced morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Set $\mathcal{F}' = g^*\mathcal{F}$. In this setting, if $\mathcal{F}$ is quasi-coherent, then the diagram

$$\begin{tikzcd} (i')^{-1}\mathcal{O}_{X'} \otimes_{h^{-1}i^{-1}\mathcal{O}_X} h^{-1}\mathcal{H}^c_Z(\mathcal{F}) \arrow[r] & \mathcal{H}^c_{Z'}(\mathcal{F}') \\ h^*i^*\mathcal{F} \arrow[u, "c_{f_1,\ldots,f_c}"] \arrow[r] & (i')^*\mathcal{F}' \arrow[u, "c_{f'_1,\ldots,f'_c}"'] \end{tikzcd}$$

is commutative where the top horizonal arrow is the map of Cohomology, Remark 34.12 on cohomology sheaves in degree c. Namely, denote $\mathcal{F}^\bullet$, resp. $(\mathcal{F}')^\bullet$ the extended alternating Čech complex constructed in Remark 6.4 using $\mathcal{F}, f_1, \ldots, f_c$, resp. $\mathcal{F}', f_1', \ldots, f_c'$. Note that $(\mathcal{F}')^\bullet = g^*\mathcal{F}^\bullet$. Then, without assuming $\mathcal{F}$ is quasi-coherent, the diagram

$$\begin{tikzcd}
i'_*L(g|_{Z'})^*R\mathcal{H}_Z(\mathcal{F}) \arrow[r] \arrow[d, equal] & i'_*R\mathcal{H}_{Z'}(Lg^*\mathcal{F}) \arrow[d] \\
Lg^*i_*R\mathcal{H}_Z(\mathcal{F}) & i'_*R\mathcal{H}_{Z'}(\mathcal{F}') \\
Lg^*(\mathcal{F}^\bullet) \arrow[r] \arrow[u] & (\mathcal{F}')^\bullet \arrow[u]
\end{tikzcd}$$

is commutative where $g|_{Z'} : (Z', (i')^{-1}\mathcal{O}_{X'}) \to (Z, i^{-1}\mathcal{O}_X)$ is the induced morphism of ringed spaces. Here the top horizontal arrow is given in Cohomology, Remark 34.12 as is the explanation for the equal sign. The arrows pointing up are from Remark 6.6. The lower horizonal arrow is the map $Lg^*\mathcal{F}^\bullet \to g^*\mathcal{F}^\bullet = (\mathcal{F}')^\bullet$ and the arrow pointing down is induced by $Lg^*\mathcal{F} \to g^*\mathcal{F} = \mathcal{F}'$. The diagram commutes because going around the diagram both ways we obtain two arrows $Lg^*\mathcal{F}^\bullet \to i'_*R\mathcal{H}_{Z'}(\mathcal{F}')$ whose composition with $i'_*R\mathcal{H}_{Z'}(\mathcal{F}') \to \mathcal{F}'$ is the canonical map $Lg^*\mathcal{F}^\bullet \to \mathcal{F}'$. Some details omitted. Now the commutativity of the first diagram follows by looking at this diagram on cohomology sheaves in degree c and using that the construction of the map $i^*\mathcal{F} \to \text{Coker}(\bigoplus \mathcal{F}_{1\ldots\hat{i}\ldots c} \to \mathcal{F}_{1\ldots c})$ used in Remark 6.10 is compatible with pullbacks.

7. The coherator

Let X be a scheme. The *coherator* is a functor

$$Q_X : Mod(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

which is right adjoint to the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$. It exists for any scheme X and moreover the adjunction mapping $Q_X(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every quasi-coherent module $\mathcal{F}$, see Properties, Proposition 23.4. Since Q_X is left exact (as a right adjoint) we can consider its right derived extension

$$RQ_X : D(\mathcal{O}_X) \longrightarrow D(QCoh(\mathcal{O}_X)).$$

Since Q_X is right adjoint to the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ we see that RQ_X is right adjoint to the canonical functor $D(QCoh(\mathcal{O}_X)) \to D(\mathcal{O}_X)$ by Derived Categories, Lemma 30.3.

In this section we will study the functor RQ_X. In Section 21 we will study the (closely related) right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X)$ (when it exists).

Lemma 7.1. *Let $f : X \to Y$ be an affine morphism of schemes. Then f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$. This functor has the* *property that*

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "f_*"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

commutes.

Proof. The functor $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ is exact, see Cohomology of Schemes, Lemma 2.3. Hence f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by simply applying f_* to any representative complex, see Derived Categories, Lemma 16.9. The diagram commutes by Lemma 5.1. $\square$

Lemma 7.2. *Let $f : X \to Y$ be a morphism of schemes. Assume f is quasi-compact, quasi-separated, and flat. Then, denoting*

$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ we have $RQ_Y \circ Rf_* = \Phi \circ RQ_X$.

Proof. We will prove this by showing that $RQ_Y \circ Rf_*$ and $\Phi \circ RQ_X$ are right adjoint to the same functor $D(QCoh(\mathcal{O}_Y)) \to D(\mathcal{O}_X)$.

Since f is quasi-compact and quasi-separated, we see that f_* preserves quasi-coherence, see Schemes, Lemma 24.1. Recall that $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 23.4). Hence any K in $D(QCoh(\mathcal{O}_X))$ can be represented by a K-injective complex $\mathcal{I}^\bullet$ of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 12.6. Then we can define $\Phi(K) = f_*\mathcal{I}^\bullet$.

Since f is flat, the functor f^* is exact. Hence f^* defines $f^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ and also $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$. The functor $f^* = Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ is left adjoint to $Rf_* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$, see Cohomology, Lemma 28.1. Similarly, the functor $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$ is left adjoint to $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by Derived Categories, Lemma 30.3.

Let A be an object of $D(QCoh(\mathcal{O}_Y))$ and E an object of $D(\mathcal{O}_X)$. Then

$$\begin{aligned} \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, RQ_Y(Rf_*E)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(A, Rf_*E) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*A, E) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(f^*A, RQ_X(E)) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, \Phi(RQ_X(E))) \end{aligned}$$

This implies what we want. $\square$

Lemma 7.3. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Then*

1. *$Q_X : Mod(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$ is the functor which sends $\mathcal{F}$ to the quasi-coherent $\mathcal{O}_X$-module associated to the A-module $\Gamma(X, \mathcal{F})$,*
2. *$RQ_X : D(\mathcal{O}_X) \to D(QCoh(\mathcal{O}_X))$ is the functor which sends E to the complex of quasi-coherent $\mathcal{O}_X$-modules associated to the object $R\Gamma(X, E)$ of $D(A)$,*
3. *restricted to $D_{QCoh}(\mathcal{O}_X)$ the functor RQ_X defines a quasi-inverse to (3.0.1).*

Proof. The functor Q_X is the functor

$$\mathcal{F} \mapsto \widetilde{\Gamma(X, \mathcal{F})}$$

by Schemes, Lemma 7.1. This immediately implies (1) and (2). The third assertion follows from (the proof of) Lemma 3.5. $\square$

At this point we are ready to prove a criterion for when the functor $D(\mathit{QCoh}(\mathcal{O}_X)) \to D_{\mathit{QCoh}}(\mathcal{O}_X)$ is an equivalence.

Lemma 7.4. *Let X be a quasi-compact and quasi-separated scheme. Suppose that for every affine open $U \subset X$ the right derived functor*

$$\Phi : D(\mathit{QCoh}(\mathcal{O}_U)) \to D(\mathit{QCoh}(\mathcal{O}_X))$$

of the left exact functor $j_* : \mathit{QCoh}(\mathcal{O}_U) \to \mathit{QCoh}(\mathcal{O}_X)$ fits into a commutative diagram

$$\begin{tikzcd} D(\mathit{QCoh}(\mathcal{O}_U)) \arrow[r, "i_U"'] \arrow[d, "\Phi"'] & D_{\mathit{QCoh}}(\mathcal{O}_U) \arrow[d, "Rj_*"] \\ D(\mathit{QCoh}(\mathcal{O}_X)) \arrow[r, "i_X"] & D_{\mathit{QCoh}}(\mathcal{O}_X) \end{tikzcd}$$

Then the functor (3.0.1)

$$D(\mathit{QCoh}(\mathcal{O}_X)) \longrightarrow D_{\mathit{QCoh}}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. Let E be an object of $D_{\mathit{QCoh}}(\mathcal{O}_X)$ and let A be an object of $D(\mathit{QCoh}(\mathcal{O}_X))$. We have to show that the adjunction maps

$$RQ_X(i_X(A)) \to A \quad \text{and} \quad E \to i_X(RQ_X(E))$$

are isomorphisms. Consider the hypothesis H_n: the adjunction maps above are isomorphisms whenever E and $i_X(A)$ are supported (Definition 6.1) on a closed subset of X which is contained in the union of n affine opens of X. We will prove H_n by induction on n.

Base case: $n = 0$. In this case $E = 0$, hence the map $E \to i_X(RQ_X(E))$ is an isomorphism. Similarly $i_X(A) = 0$. Thus the cohomology sheaves of $i_X(A)$ are zero. Since the inclusion functor $\mathit{QCoh}(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$ is fully faithful and exact, we conclude that the cohomology objects of A are zero, i.e., $A = 0$ and $RQ_X(i_X(A)) \to A$ is an isomorphism as well.

Induction step. Suppose that E and $i_X(A)$ are supported on a closed subset T of X contained in $U_1 \cup \ldots \cup U_n$ with $U_i \subset X$ affine open. Set $U = U_n$. Consider the distinguished triangles

$$A \to \Phi(A|_U) \to A' \to A[1] \quad \text{and} \quad E \to Rj_*(E|_U) \to E' \to E[1]$$

where Φ is as in the statement of the lemma. Note that $E \to Rj_*(E|_U)$ is a quasi-isomorphism over $U = U_n$. Since $i_X \circ \Phi = Rj_* \circ i_U$ by assumption and since $i_X(A)|_U = i_U(A|_U)$ we see that $i_X(A) \to i_X(\Phi(A|_U))$ is a quasi-isomorphism over U. Hence $i_X(A')$ and E' are supported on the closed subset $T \setminus U$ of X which is contained in $U_1 \cup \ldots \cup U_{n-1}$. By induction hypothesis the statement is true for A' and E'. By Derived Categories, Lemma 4.3 it suffices to prove the maps

$$RQ_X(i_X(\Phi(A|_U))) \to \Phi(A|_U) \quad \text{and} \quad Rj_*(E|_U) \to i_X(RQ_X(Rj_*E|_U))$$

are isomorphisms. By assumption and by Lemma 7.2 (the inclusion morphism $j : U \to X$ is flat, quasi-compact, and quasi-separated) we have

$$RQ_X(i_X(\Phi(A|_U))) = RQ_X(Rj_*(i_U(A|_U))) = \Phi(RQ_U(i_U(A|_U)))$$

and

$$i_X(RQ_X(Rj_*(E|_U))) = i_X(\Phi(RQ_U(E|_U))) = Rj_*(i_U(RQ_U(E|_U)))$$

Finally, the maps

$$RQ_U(i_U(A|_U)) \to A|_U \quad \text{and} \quad E|_U \to i_U(RQ_U(E|_U))$$

are isomorphisms by Lemma 7.3. The result follows. □

Proposition 7.5. *Let X be a quasi-compact scheme with affine diagonal. Then the functor (3.0.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. Let $U \subset X$ be an affine open. Then the morphism $U \to X$ is affine by Morphisms, Lemma 11.11. Thus the assumption of Lemma 7.4 holds by Lemma 7.1 and we win. □

Lemma 7.6. *Let $f : X \to Y$ be a morphism of schemes. Assume X and Y are quasi-compact and have affine diagonal. Then, denoting*

$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ the diagram

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "\Phi"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

is commutative.

Proof. Observe that the horizontal arrows in the diagram are equivalences of categories by Proposition 7.5. Hence we can identify these categories (and similarly for other quasi-compact schemes with affine diagonal). The statement of the lemma is that the canonical map $\Phi(K) \to Rf_*(K)$ is an isomorphism for all K in $D(QCoh(\mathcal{O}_X))$. Note that if $K_1 \to K_2 \to K_3 \to K_1[1]$ is a distinguished triangle in $D(QCoh(\mathcal{O}_X))$ and the statement is true for two-out-of-three, then it is true for the third.

Let $U \subset X$ be an affine open. Since the diagonal of X is affine, the inclusion morphism $j : U \to X$ is affine (Morphisms, Lemma 11.11). Similarly, the composition $g = f \circ j : U \to Y$ is affine. Let $\mathcal{I}^\bullet$ be a K-injective complex in $QCoh(\mathcal{O}_U)$. Since $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$ has an exact left adjoint $j^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ we see that $j_*\mathcal{I}^\bullet$ is a K-injective complex in $QCoh(\mathcal{O}_X)$, see Derived Categories, Lemma 31.9. It follows that

$$\Phi(j_*\mathcal{I}^\bullet) = f_*j_*\mathcal{I}^\bullet = g_*\mathcal{I}^\bullet$$

By Lemma 7.1 we see that $j_*\mathcal{I}^\bullet$ represents $Rj_*\mathcal{I}^\bullet$ and $g_*\mathcal{I}^\bullet$ represents $Rg_*\mathcal{I}^\bullet$. On the other hand, we have $Rf_* \circ Rj_* = Rg_*$. Hence $f_*j_*\mathcal{I}^\bullet$ represents $Rf_*(j_*\mathcal{I}^\bullet)$. We conclude that the lemma is true for any complex of the form $j_*\mathcal{G}^\bullet$ with $\mathcal{G}^\bullet$ a complex of quasi-coherent modules on U. (Note that if $\mathcal{G}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism, then $j_*\mathcal{G}^\bullet \to j_*\mathcal{I}^\bullet$ is a quasi-isomorphism as well since j_* is an exact functor on quasi-coherent modules.)

Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Let $T \subset X$ be a closed subset such that the support of $\mathcal{F}^p$ is contained in T for all p. We will use induction on the minimal number n of affine opens $U_1, \ldots, U_n$ such that $T \subset U_1 \cup \ldots \cup U_n$. The base case $n = 0$ is trivial. If $n \geq 1$, then set $U = U_1$ and denote $j : U \to X$ the open immersion as above. We consider the map of complexes $c : \mathcal{F}^\bullet \to j_*j^*\mathcal{F}^\bullet$. We obtain two short exact sequences of complexes:

$$0 \to \mathrm{Ker}(c) \to \mathcal{F}^\bullet \to \mathrm{Im}(c) \to 0$$

and

$$0 \to \mathrm{Im}(c) \to j_*j^*\mathcal{F}^\bullet \to \mathrm{Coker}(c) \to 0$$

The complexes $\mathrm{Ker}(c)$ and $\mathrm{Coker}(c)$ are supported on $T \setminus U \subset U_2 \cup \ldots \cup U_n$ and the result holds for them by induction. The result holds for $j_*j^*\mathcal{F}^\bullet$ by the discussion in the preceding paragraph. We conclude by looking at the distinguished triangles associated to the short exact sequences and using the initial remark of the proof. $\square$

Remark 7.7 (Warning)**.** Let X be a quasi-compact scheme with affine diagonal. Even though we know that $D(QCoh(\mathcal{O}_X)) = D_{QCoh}(\mathcal{O}_X)$ by Proposition 7.5 strange things can happen and it is easy to make mistakes with this material. One pitfall is to carelessly assume that this equality means derived functors are the same. For example, suppose we have a quasi-compact open $U \subset X$. Then we can consider the higher right derived functors

$$R^i(QCoh)\Gamma(U, -) : QCoh(\mathcal{O}_X) \to Ab$$

of the left exact functor $\Gamma(U, -)$. Since this is a universal δ-functor, and since the functors $H^i(U, -)$ (defined for all abelian sheaves on X) restricted to $QCoh(\mathcal{O}_X)$ form a δ-functor, we obtain canonical tranformations

$$t^i : R^i(QCoh)\Gamma(U, -) \to H^i(U, -).$$

These transformations aren't in general isomorphisms even if $X = \mathrm{Spec}(A)$ is affine! Namely, we have $R^1(QCoh)\Gamma(U, \widetilde{I}) = 0$ if I an injective A-module by construction of right derived functors and the equivalence of $QCoh(\mathcal{O}_X)$ and Mod_A. But Examples, Lemma 46.2 shows there exists A, I, and U such that $H^1(U, \widetilde{I}) \neq 0$.

8. The coherator for Noetherian schemes

In the case of Noetherian schemes we can use the following lemma.

Lemma 8.1. *Let X be a Noetherian scheme. Let $\mathcal{J}$ be an injective object of $QCoh(\mathcal{O}_X)$. Then $\mathcal{J}$ is a flasque sheaf of $\mathcal{O}_X$-modules.*

Proof. Let $U \subset X$ be an open subset and let $s \in \mathcal{J}(U)$ be a section. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals defining the reduced induced scheme structure on $X \setminus U$ (see Schemes, Definition 12.5). By Cohomology of Schemes, Lemma 10.5 the section s corresponds to a map $\sigma : \mathcal{I}^n \to \mathcal{J}$ for some n. As $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$ we can extend σ to a map $\tilde{s} : \mathcal{O}_X \to \mathcal{J}$. Then $\tilde{s}$ corresponds to a global section of $\mathcal{J}$ restricting to s. $\square$

Lemma 8.2. *Let $f : X \to Y$ be a morphism of Noetherian schemes. Then f_* on quasi-coherent sheaves has a right derived extension $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ such that the diagram*

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "\Phi"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

commutes.

Proof. Since X and Y are Noetherian schemes the morphism is quasi-compact and quasi-separated (see Properties, Lemma 5.4 and Schemes, Remark 21.18). Thus f_* preserve quasi-coherence, see Schemes, Lemma 24.1. Next, let K be an object of $D(QCoh(\mathcal{O}_X))$. Since $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 23.4), we can represent K by a K-injective complex $\mathcal{I}^\bullet$ such that each $\mathcal{I}^n$ is an injective object of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 12.6. Thus we see that the functor Φ is defined by setting

$$\Phi(K) = f_*\mathcal{I}^\bullet$$

where the right hand side is viewed as an object of $D(QCoh(\mathcal{O}_Y))$. To finish the proof of the lemma it suffices to show that the canonical map

$$f_*\mathcal{I}^\bullet \longrightarrow Rf_*\mathcal{I}^\bullet$$

is an isomorphism in $D(\mathcal{O}_Y)$. To see this by Lemma 4.2 it suffices to show that $\mathcal{I}^n$ is right f_*-acyclic for all $n \in \mathbf{Z}$. This is true because $\mathcal{I}^n$ is flasque by Lemma 8.1 and flasque modules are right f_*-acyclic by Cohomology, Lemma 12.5. □

Proposition 8.3. *Let X be a Noetherian scheme. Then the functor (3.0.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. This follows from Lemma 7.4 and Lemma 8.2. □

9. Koszul complexes

Let A be a ring and let $f_1, \ldots, f_r$ be a sequence of elements of A. We have defined the Koszul complex $K_\bullet(f_1, \ldots, f_r)$ in More on Algebra, Definition 28.2. It is a chain complex sitting in degrees $r, \ldots, 0$. We turn this into a cochain complex $K^\bullet(f_1, \ldots, f_r)$ by setting $K^{-n}(f_1, \ldots, f_r) = K_n(f_1, \ldots, f_r)$ and using the same differentials. In the rest of this section all the complexes will be cochain complexes.

We define a complex $I^\bullet(f_1, \ldots, f_r)$ such that we have a distinguished triangle

$$I^\bullet(f_1, \ldots, f_r) \to A \to K^\bullet(f_1, \ldots, f_r) \to I^\bullet(f_1, \ldots, f_r)[1]$$

in $K(A)$. In other words, we set

$$I^i(f_1, \ldots, f_r) = \begin{cases} K^{i-1}(f_1, \ldots, f_r) & \text{if } i \leq 0 \\ 0 & \text{else} \end{cases}$$

and we use the negative of the differential on $K^\bullet(f_1, \ldots, f_r)$. The maps in the distinguished triangle are the obvious ones. Note that $I^0(f_1, \ldots, f_r) = A^{\oplus r} \to A$ is given by multiplication by f_i on the ith factor. Hence $I^\bullet(f_1, \ldots, f_r) \to A$ factors as

$$I^\bullet(f_1, \ldots, f_r) \to I \to A$$

where $I = (f_1, \ldots, f_r)$. In fact, there is a short exact sequence

$$0 \to H^{-1}(K^\bullet(f_1, \ldots, f_s)) \to H^0(I^\bullet(f_1, \ldots, f_s)) \to I \to 0$$

and for every $i < 0$ we have $H^i(I^\bullet(f_1, \ldots, f_r)) = H^{i-1}(K^\bullet(f_1, \ldots, f_r)$. Observe that given a second sequence $g_1, \ldots, g_r$ of elements of A there are canonical maps

$$I^\bullet(f_1g_1, \ldots, f_rg_r) \to I^\bullet(f_1, \ldots, f_r) \quad \text{and} \quad K^\bullet(f_1g_1, \ldots, f_rg_r) \to K^\bullet(f_1, \ldots, f_r)$$

compatible with the maps described above. The first of these maps is given by multiplication by g_i on the ith summand of $I^0(f_1g_1, \ldots, f_rg_r) = A^{\oplus r}$. In particular, given $f_1, \ldots, f_r$ we obtain an inverse system of complexes

$$(9.0.1) \qquad I^\bullet(f_1, \ldots, f_r) \leftarrow I^\bullet(f_1^2, \ldots, f_r^2) \leftarrow I^\bullet(f_1^3, \ldots, f_r^3) \leftarrow \ldots$$

which will play an important role in that which is to follow. To easily formulate the following lemmas we fix some notation.

Situation 9.1. Here A is a ring and $f_1, \ldots, f_r$ is a sequence of elements of A. We set $X = \text{Spec}(A)$ and $U = D(f_1) \cup \ldots \cup D(f_r) \subset X$. We denote $\mathcal{U} : U = \bigcup_{i=1,\ldots,r} D(f_i)$ the given open covering of U.

Our first lemma is that the complexes above can be used to compute the cohomology of quasi-coherent sheaves on U. Suppose given a complex $I^\bullet$ of A-modules and an A-module M. Then we define $\text{Hom}_A(I^\bullet, M)$ to be the complex with nth term $\text{Hom}_A(I^{-n}, M)$ and differentials given as the contragredients of the differentials on $I^\bullet$.

Lemma 9.2. *In Situation 9.1. Let M be an A-module and denote $\mathcal{F}$ the associated $\mathcal{O}_X$-module. Then there is a canonical isomorphism of complexes*

$$\text{colim}_e \text{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M) \longrightarrow \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

functorial in M.

Proof. Recall that the alternating Čech complex is the subcomplex of the usual Čech complex given by alternating cochains, see Cohomology, Section 23. As usual we view a p-cochain in $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ as an alternating function s on $\{1, \ldots, r\}^{p+1}$ whose value $s_{i_0\ldots i_p}$ at $(i_0, \ldots, i_p)$ lies in $M_{f_{i_0}\cdots f_{i_p}} = \mathcal{F}(U_{i_0\ldots i_p})$. On the other hand, a p-cochain t in $\text{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M)$ is given by a map $t : \wedge^{p+1}(A^{\oplus r}) \to M$. Write $[i] \in A^{\oplus r}$ for the ith basis element and write

$$[i_0, \ldots, i_p] = [i_0] \wedge \ldots \wedge [i_p] \in \wedge^{p+1}(A^{\oplus r})$$

Then we send t as above to s with

$$s_{i_0\ldots i_p} = \frac{t([i_0, \ldots, i_p])}{f_{i_0}^e \ldots f_{i_p}^e}$$

It is clear that s so defined is an alternating cochain. The construction of this map is compatible with the transition maps of the system as the transition map

$$I^\bullet(f_1^e, \ldots, f_r^e) \leftarrow I^\bullet(f_1^{e+1}, \ldots, f_r^{e+1}),$$

of the (9.0.1) sends $[i_0, \ldots, i_p]$ to $f_{i_0} \ldots f_{i_p}[i_0, \ldots, i_p]$. It is clear from the description of the localizations $M_{f_{i_0}\cdots f_{i_p}}$ in Algebra, Lemma 9.9 that these maps define an isomorphism of cochain modules in degree p in the limit. To finish the proof we have to show that the map is compatible with differentials. To see this recall that

$$\begin{aligned} d(s)_{i_0\ldots i_{p+1}} &= \sum\nolimits_{j=0}^{p+1}(-1)^j s_{i_0\ldots\hat{i}_j\ldots i_p} \\ &= \sum\nolimits_{j=0}^{p+1}(-1)^j \frac{t([i_0,\ldots,\hat{i}_j,\ldots i_{p+1}])}{f_{i_0}^e\ldots\hat{f}_{i_j}^e\ldots f_{i_{p+1}}^e} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{d(t)([i_0,\ldots,i_{p+1}])}{f_{i_0}^e\ldots f_{i_{p+1}}^e} &= \frac{t(d[i_0,\ldots,i_{p+1}])}{f_{i_0}^e\ldots f_{i_{p+1}}^e} \\ &= \frac{\sum_j(-1)^j f_{i_j}^e t([i_0,\ldots,\hat{i}_j,\ldots i_{p+1}])}{f_{i_0}^e\ldots f_{i_{p+1}}^e} \end{aligned}$$

The two formulas agree by inspection. □

Suppose given a finite complex $I^\bullet$ of A-modules and a complex of A-modules $M^\bullet$. We obtain a double complex $H^{\bullet,\bullet} = \mathrm{Hom}_A(I^\bullet, M^\bullet)$ where $H^{p,q} = \mathrm{Hom}_A(I^p, M^q)$. The first differential comes from the differential on $\mathrm{Hom}_A(I^\bullet, M^q)$ and the second from the differential on $M^\bullet$. Associated to this double complex is the total complex with degree n term given by

$$\bigoplus\nolimits_{p+q=n}\mathrm{Hom}_A(I^p, M^q)$$

and differential as in Homology, Definition 18.3. As our complex $I^\bullet$ has only finitely many nonzero terms, the direct sum displayed above is finite. The conventions for taking the total complex associated to a Čech complex of a complex are as in Cohomology, Section 25.

Lemma 9.3. *In Situation 9.1. Let $M^\bullet$ be a complex of A-modules and denote $\mathcal{F}^\bullet$ the associated complex of $\mathcal{O}_X$-modules. Then there is a canonical isomorphism of complexes*

$$\mathrm{colim}_e\ \mathit{Tot}(\mathrm{Hom}_A(I^\bullet(f_1^e,\ldots,f_r^e), M^\bullet)) \longrightarrow \mathit{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$$

functorial in $M^\bullet$.

Proof. Immediate from Lemma 9.2 and our conventions for taking associated total complexes. □

Lemma 9.4. *In Situation 9.1. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Then there is a canonical isomorphism*

$$\mathit{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(U, \mathcal{F}^\bullet)$$

in $D(A)$ functorial in $\mathcal{F}^\bullet$.

Proof. Let $\mathcal{B}$ be the set of affine opens of U. Since the higher cohomology groups of a quasi-coherent module on an affine scheme are zero (Cohomology of Schemes, Lemma 2.2) this is a special case of Cohomology, Lemma 40.2. □

In Situation 9.1 denote I_e the object of $D(\mathcal{O}_X)$ corresponding to the complex of A-modules $I^\bullet(f_1^e,\ldots,f_r^e)$ via the equivalence of Lemma 3.5. The maps (9.0.1) give a system

$$I_1 \leftarrow I_2 \leftarrow I_3 \leftarrow \ldots$$

Moreover, there is a compatible system of maps $I_e \to \mathcal{O}_X$ which become isomorphisms when restricted to U. Thus we see that for every object E of $D(\mathcal{O}_X)$ there is a canonical map

$$\operatorname{colim}_e \operatorname{Hom}_{D(\mathcal{O}_X)}(I_e, E) \longrightarrow H^0(U, E) \tag{9.4.1}$$

constructed by sending a map $I_e \to E$ to its restriction to U and using that $\operatorname{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_U, E|_U) = H^0(U, E)$.

Proposition 9.5. *In Situation 9.1. For every object E of $D_{QCoh}(\mathcal{O}_X)$ the map (9.4.1) is an isomorphism.*

Proof. By Lemma 3.5 we may assume that E is given by a complex of quasi-coherent sheaves $\mathcal{F}^\bullet$. Let $M^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ be the corresponding complex of A-modules. By Lemmas 9.3 and 9.4 we have quasi-isomorphisms

$$\operatorname{colim}_e \operatorname{Tot}(\operatorname{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet)) \longrightarrow \operatorname{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(U, \mathcal{F}^\bullet)$$

Taking H^0 on both sides we obtain

$$\operatorname{colim}_e \operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) = H^0(U, E)$$

Since $\operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) = \operatorname{Hom}_{D(\mathcal{O}_X)}(I_e, E)$ by Lemma 3.5 the lemma follows. $\square$

In Situation 9.1 denote K_e the object of $D(\mathcal{O}_X)$ corresponding to the complex of A-modules $K^\bullet(f_1^e, \ldots, f_r^e)$ via the equivalence of Lemma 3.5. Thus we have distinguished triangles

$$I_e \to \mathcal{O}_X \to K_e \to I_e[1]$$

and a system

$$K_1 \leftarrow K_2 \leftarrow K_3 \leftarrow \ldots$$

compatible with the system (I_e). Moreover, there is a compatible system of maps

$$K_e \to H^0(K_e) = \mathcal{O}_X/(f_1^e, \ldots, f_r^e)$$

Lemma 9.6. *In Situation 9.1. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Assume that $H^i(E)|_U = 0$ for $i = -r+1, \ldots, 0$. Then given $s \in H^0(X, E)$ there exists an $e \geq 0$ and a morphism $K_e \to E$ such that s is in the image of $H^0(X, K_e) \to H^0(X, E)$.*

Proof. Since U is covered by r affine opens we have $H^j(U, \mathcal{F}) = 0$ for $j \geq r$ and any quasi-coherent module (Cohomology of Schemes, Lemma 4.2). By Lemma 3.4 we see that $H^0(U, E)$ is equal to $H^0(U, \tau_{\geq -r+1}E)$. There is a spectral sequence

$$H^j(U, H^i(\tau_{\geq -r+1}E)) \Rightarrow H^{i+j}(U, \tau_{\geq -N}E)$$

see Derived Categories, Lemma 21.3. Hence $H^0(U, E) = 0$ by our assumed vanishing of cohomology sheaves of E. We conclude that $s|_U = 0$. Think of s as a morphism $\mathcal{O}_X \to E$ in $D(\mathcal{O}_X)$. By Proposition 9.5 the composition $I_e \to \mathcal{O}_X \to E$ is zero for some e. By the distinguished triangle $I_e \to \mathcal{O}_X \to K_e \to I_e[1]$ we obtain a morphism $K_e \to E$ such that s is the composition $\mathcal{O}_X \to K_e \to E$. $\square$

10. Pseudo-coherent and perfect complexes

In this section we make the connection between the general notions defined in Cohomology, Sections 46, 47, 48, and 49 and the corresponding notions for complexes of modules in More on Algebra, Sections 64, 66, and 74.

Lemma 10.1. *Let X be a scheme. If E is an m-pseudo-coherent object of $D(\mathcal{O}_X)$, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$ and $H^m(E)$ is a quotient of a quasi-coherent $\mathcal{O}_X$-module. If E is pseudo-coherent, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Locally on X there exists a strictly perfect complex $\mathcal{E}^\bullet$ such that $H^i(E)$ is isomorphic to $H^i(\mathcal{E}^\bullet)$ for $i > m$ and $H^m(E)$ is a quotient of $H^m(\mathcal{E}^\bullet)$. The sheaves $\mathcal{E}^i$ are direct summands of finite free modules, hence quasi-coherent. The lemma follows. $\square$

Lemma 10.2. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then E is an m-pseudo-coherent (resp. pseudo-coherent) as an object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of A-modules.*

Proof. It is immediate from the definitions that if $M^\bullet$ is m-pseudo-coherent, so is E. To prove the converse, assume E is m-pseudo-coherent. As $X = \operatorname{Spec}(A)$ is quasi-compact with a basis for the topology given by standard opens, we can find a standard open covering $X = D(f_1) \cup \ldots \cup D(f_n)$ and strictly perfect complexes $\mathcal{E}_i^\bullet$ on $D(f_i)$ and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ inducing isomorphisms on H^j for $j > m$ and surjections on H^m. By Cohomology, Lemma 46.8 after refining the open covering we may assume α_i is given by a map of complexes $\mathcal{E}_i^\bullet \to \widetilde{M^\bullet}|_{U_i}$ for each i. By Modules, Lemma 14.6 the terms $\mathcal{E}_i^n$ are finite locally free modules. Hence after refining the open covering we may assume each $\mathcal{E}_i^n$ is a finite free $\mathcal{O}_{U_i}$-module. From the definition it follows that $M^\bullet_{f_i}$ is an m-pseudo-coherent complex of A_{f_i}-modules. We conclude by applying More on Algebra, Lemma 64.14.

The case "pseudo-coherent" follows from the fact that E is pseudo-coherent if and only if E is m-pseudo-coherent for all m (by definition) and the same is true for $M^\bullet$ by More on Algebra, Lemma 64.5. $\square$

Lemma 10.3. *Let X be a Noetherian scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. For $m \in \mathbf{Z}$ the following are equivalent*

1. *$H^i(E)$ is coherent for $i \geq m$ and zero for $i \gg 0$, and*
2. *E is m-pseudo-coherent.*

In particular, E is pseudo-coherent if and only if E is an object of $D^-_{Coh}(\mathcal{O}_X)$.

Proof. As X is quasi-compact we see that in both (1) and (2) the object E is bounded above. Thus the question is local on X and we may assume X is affine. Say $X = \operatorname{Spec}(A)$ for some Noetherian ring A. In this case E corresponds to a complex of A-modules $M^\bullet$ by Lemma 3.5. By Lemma 10.2 we see that E is m-pseudo-coherent if and only if $M^\bullet$ is m-pseudo-coherent. On the other hand, $H^i(E)$ is coherent if and only if $H^i(M^\bullet)$ is a finite A-module (Properties, Lemma 16.1). Thus the result follows from More on Algebra, Lemma 64.17. $\square$

Lemma 10.4. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then*

(1) *E has tor amplitude in $[a, b]$ if and only if $M^\bullet$ has tor amplitude in $[a, b]$.*
(2) *E has finite tor dimension if and only if $M^\bullet$ has finite tor dimension.*

Proof. Part (2) follows trivially from part (1). In the proof of (1) we will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 3.5 without further mention. Assume $M^\bullet$ has tor amplitude in $[a, b]$. Then $K^\bullet$ is isomorphic in $D(A)$ to a complex $K^\bullet$ of flat A-modules with $K^i = 0$ for $i \not\in [a, b]$, see More on Algebra, Lemma 66.3. Then E is isomorphic to $\widetilde{K^\bullet}$. Since each $\widetilde{K^i}$ is a flat $\mathcal{O}_X$-module, we see that E has tor amplitude in $[a, b]$ by Cohomology, Lemma 48.3.

Assume that E has tor amplitude in $[a, b]$. Then E is bounded whence $M^\bullet$ is in $K^-(A)$. Thus we may replace $M^\bullet$ by a bounded above complex of A-modules. We may even choose a projective resolution and assume that $M^\bullet$ is a bounded above complex of free A-modules. Then for any A-module N we have

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M^\bullet} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M^\bullet \otimes_A N}$$

in $D(\mathcal{O}_X)$. Thus the vanishing of cohomology sheaves of the left hand side implies $M^\bullet$ has tor amplitude in $[a, b]$. □

Lemma 10.5. *Let $f : X \to S$ be a morphism of affine schemes corresponding to the ring map $R \to A$. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then*

(1) *E as an object of $D(f^{-1}\mathcal{O}_S)$ has tor amplitude in $[a, b]$ if and only if $M^\bullet$ has tor amplitude in $[a, b]$ as an object of $D(R)$.*
(2) *E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ if and only if $M^\bullet$ has finite tor dimension as an object of $D(R)$.*

Proof. Consider a prime $\mathfrak{q} \subset A$ lying over $\mathfrak{p} \subset R$. Let $x \in X$ and $s = f(x) \in S$ be the corresponding points. Then $(f^{-1}\mathcal{O}_S)_x = \mathcal{O}_{S,s} = R_\mathfrak{p}$ and $E_x = M^\bullet_\mathfrak{q}$. Keeping this in mind we can see the equivalence as follows.

If $M^\bullet$ has tor amplitude in $[a, b]$ as a complex of R-modules, then the same is true for the localization of $M^\bullet$ at any prime of A. Then we conclude by Cohomology, Lemma 48.5 that E has tor amplitude in $[a, b]$ as a complex of sheaves of $f^{-1}\mathcal{O}_S$-modules. Conversely, assume that E has tor amplitude in $[a, b]$ as an object of $D(f^{-1}\mathcal{O}_S)$. We conclude (using the last cited lemma) that $M^\bullet_\mathfrak{q}$ has tor amplitude in $[a, b]$ as a complex of $R_\mathfrak{p}$-modules for every prime $\mathfrak{q} \subset A$ lying over $\mathfrak{p} \subset R$. By More on Algebra, Lemma 66.15 we find that $M^\bullet$ has tor amplitude in $[a, b]$ as a complex of R-modules. This finishes the proof of (1).

Since X is quasi-compact, if E locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_S$-modules, then actually E has tor amplitude in $[a, b]$ for some a, b as a complex of $f^{-1}\mathcal{O}_S$-modules. Thus (2) follows from (1). □

Lemma 10.6. *Let X be a quasi-separated scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $a \leq b$. The following are equivalent*

(1) *E has tor amplitude in $[a, b]$, and*
(2) *for all $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$ we have $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for $i \not\in [a, b]$.*

Proof. It is clear that (1) implies (2). Assume (2). Let $U \subset X$ be an affine open. As X is quasi-separated the morphism $j : U \to X$ is quasi-compact and separated, hence j_* transforms quasi-coherent modules into quasi-coherent modules (Schemes, Lemma 24.1). Thus the functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ is essentially surjective. It follows that condition (2) implies the vanishing of $H^i(E|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{G})$ for $i \notin [a, b]$ for all quasi-coherent $\mathcal{O}_U$-modules $\mathcal{G}$. Write $U = \text{Spec}(A)$ and let $M^\bullet$ be the complex of A-modules corresponding to $E|_U$ by Lemma 3.5. We have just shown that $M^\bullet \otimes^{\mathbf{L}}_A N$ has vanishing cohomology groups outside the range $[a, b]$, in other words $M^\bullet$ has tor amplitude in $[a, b]$. By Lemma 10.4 we conclude that $E|_U$ has tor amplitude in $[a, b]$. This proves the lemma. □

Lemma 10.7. *Let $X = \text{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then E is a perfect object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is perfect as an object of $D(A)$.*

Proof. This is a logical consequence of Lemmas 10.2 and 10.4, Cohomology, Lemma 49.5, and More on Algebra, Lemma 74.2. □

As a consequence of our description of pseudo-coherent complexes on schemes we can prove certain internal homs are quasi-coherent.

Lemma 10.8. *Let X be a scheme.*

1. *If L is in $D^+_{QCoh}(\mathcal{O}_X)$ and K in $D(\mathcal{O}_X)$ is pseudo-coherent, then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$ and locally bounded below.*
2. *If L is in $D_{QCoh}(\mathcal{O}_X)$ and K in $D(\mathcal{O}_X)$ is perfect, then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$.*
3. *If $X = \text{Spec}(A)$ is affine and $K, L \in D(A)$ then*
$$R\mathcal{H}om(\widetilde{K}, \widetilde{L}) = \widetilde{R\text{Hom}_A(K, L)}$$
in the following two cases
   (a) *K is pseudo-coherent and L is bounded below,*
   (b) *K is perfect and L arbitrary.*
4. *If $X = \text{Spec}(A)$ and K, L are in $D(A)$, then the nth cohomology sheaf of $R\mathcal{H}om(\widetilde{K}, \widetilde{L})$ is the sheaf associated to the presheaf*
$$X \supset D(f) \longmapsto \text{Ext}^n_{A_f}(K \otimes_A A_f, L \otimes_A A_f)$$
for $f \in A$.

Proof. The construction of the internal hom in the derived category of $\mathcal{O}_X$ commutes with localization (see Cohomology, Section 42). Hence to prove (1) and (2) we may replace X by an affine open. By Lemmas 3.5, 10.2, and 10.7 in order to prove (1) and (2) it suffices to prove (3).

Part (3) follows from the computation of the internal hom of Cohomology, Lemma 46.11 by representing K by a bounded above (resp. finite) complex of finite projective A-modules and L by a bounded below (resp. arbitrary) complex of A-modules.

To prove (4) recall that on any ringed space the nth cohomology sheaf of $R\mathcal{H}om(A, B)$ is the sheaf associated to the presheaf
$$U \mapsto \text{Hom}_{D(U)}(A|_U, B|_U[n]) = \text{Ext}^n_{D(\mathcal{O}_U)}(A|_U, B|_U)$$
See Cohomology, Section 42. On the other hand, the restriction of $\widetilde{K}$ to a principal open $D(f)$ is the image of $K \otimes_A A_f$ and similarly for L. Hence (4) follows from the equivalence of categories of Lemma 3.5. □

Lemma 10.9. *Let X be a scheme. Let K, L, M be objects of $D_{QCoh}(\mathcal{O}_X)$. The map*
$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$
of Cohomology, Lemma 42.6 is an isomorphism in the following cases

1. *M perfect, or*
2. *K is perfect, or*
3. *M is pseudo-coherent, $L \in D^+(\mathcal{O}_X)$, and K has finite tor dimension.*

Proof. Lemma 10.8 reduces cases (1) and (3) to the affine case which is treated in More on Algebra, Lemma 98.3. (You also have to use Lemmas 10.2, 10.7, and 10.4 to do the translation into algebra.) If K is perfect but no other assumptions are made, then we do not know that either side of the arrow is in $D_{QCoh}(\mathcal{O}_X)$ but the result is still true because we can work locally and reduce to the case that K is a finite complex of finite free modules in which case it is clear. $\square$

11. Derived category of coherent modules

Let X be a locally Noetherian scheme. In this case the category $Coh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules is a weak Serre subcategory, see Homology, Section 10 and Cohomology of Schemes, Lemma 9.2. Denote
$$D_{Coh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$
the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 17. Thus we obtain a canonical functor
$$D(Coh(\mathcal{O}_X)) \longrightarrow D_{Coh}(\mathcal{O}_X) \tag{11.0.1}$$
see Derived Categories, Equation (17.1.1).

Lemma 11.1. *Let X be a Noetherian scheme. Then the functor*
$$D^-(Coh(\mathcal{O}_X)) \longrightarrow D^-_{Coh(\mathcal{O}_X)}(QCoh(\mathcal{O}_X))$$
is an equivalence.

Proof. Observe that $Coh(\mathcal{O}_X) \subset QCoh(\mathcal{O}_X)$ is a Serre subcategory, see Homology, Definition 10.1 and Lemma 10.2 and Cohomology of Schemes, Lemmas 9.2 and 9.3. On the other hand, if $\mathcal{G} \to \mathcal{F}$ is a surjection from a quasi-coherent $\mathcal{O}_X$-module to a coherent $\mathcal{O}_X$-module, then there exists a coherent submodule $\mathcal{G}' \subset \mathcal{G}$ which surjects onto $\mathcal{F}$. Namely, we can write $\mathcal{G}$ as the filtered union of its coherent submodules by Properties, Lemma 22.3 and then one of these will do the job. Thus the lemma follows from Derived Categories, Lemma 17.4. $\square$

Proposition 11.2. *Let X be a Noetherian scheme. Then the functors*
$$D^-(Coh(\mathcal{O}_X)) \longrightarrow D^-_{Coh}(\mathcal{O}_X) \quad \textit{and} \quad D^b(Coh(\mathcal{O}_X)) \longrightarrow D^b_{Coh}(\mathcal{O}_X)$$
are equivalences.

Proof. Consider the commutative diagram
$$\begin{tikzcd} D^-(Coh(\mathcal{O}_X)) \arrow[r] \arrow[d] & D^-_{Coh}(\mathcal{O}_X) \arrow[d] \\ D^-(QCoh(\mathcal{O}_X)) \arrow[r] & D^-_{QCoh}(\mathcal{O}_X) \end{tikzcd}$$

By Lemma 11.1 the left vertical arrow is fully faithful. By Proposition 8.3 the bottom arrow is an equivalence. By construction the right vertical arrow is fully faithful. We conclude that the top horizontal arrow is fully faithful. If K is an object of $D^-_{Coh}(\mathcal{O}_X)$ then the object K' of $D^-(QCoh(\mathcal{O}_X))$ which corresponds to it by Proposition 8.3 will have coherent cohomology sheaves. Hence K' is in the essential image of the left vertical arrow by Lemma 11.1 and we find that the top horizontal arrow is essentially surjective. This finishes the proof for the bounded above case. The bounded case follows immediately from the bounded above case. □

Lemma 11.3. *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D^b_{Coh}(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over S for all i. Then Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_S)$.*

Proof. Consider the spectral sequence

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E$$

see Derived Categories, Lemma 21.3. By assumption and Cohomology of Schemes, Lemma 26.10 the sheaves $R^pf_*H^q(E)$ are coherent. Hence $R^{p+q}f_*E$ is coherent, i.e., $Rf_*E \in D_{Coh}(\mathcal{O}_S)$. Boundedness from below is trivial. Boundedness from above follows from Cohomology of Schemes, Lemma 4.5 or from Lemma 4.1. □

Lemma 11.4. *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D^+_{Coh}(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over S for all i. Then Rf_*E is an object of $D^+_{Coh}(\mathcal{O}_S)$.*

Proof. The proof is the same as the proof of Lemma 11.3. You can also deduce it from Lemma 11.3 by considering what the exact functor Rf_* does to the distinguished triangles $\tau_{\leq a}E \to E \to \tau_{\geq a+1}E \to \tau_{\leq a}E[1]$. □

Lemma 11.5. *Let X be a locally Noetherian scheme. If L is in $D^+_{Coh}(\mathcal{O}_X)$ and K in $D^-_{Coh}(\mathcal{O}_X)$, then $R\mathcal{H}om(K, L)$ is in $D^+_{Coh}(\mathcal{O}_X)$.*

Proof. It suffices to prove this when X is the spectrum of a Noetherian ring A. By Lemma 10.3 we see that K is pseudo-coherent. Then we can use Lemma 10.8 to translate the problem into the following algebra problem: for $L \in D^+_{Coh}(A)$ and K in $D^-_{Coh}(A)$, then $R\operatorname{Hom}_A(K, L)$ is in $D^+_{Coh}(A)$. Since L is bounded below and K is bounded below there is a convergent spectral sequence

$$\operatorname{Ext}^p_A(K, H^q(L)) \Rightarrow \operatorname{Ext}^{p+q}_A(K, L)$$

and there are convergent spectral sequences

$$\operatorname{Ext}^i_A(H^{-j}(K), H^q(L)) \Rightarrow \operatorname{Ext}^{i+j}_A(K, H^q(L))$$

See Injectives, Remarks 13.9 and 13.11. This finishes the proof as the modules $\operatorname{Ext}^p_A(M, N)$ are finite for finite A-modules M, N by Algebra, Lemma 71.9. □

Lemma 11.6. *Let X be a Noetherian scheme. Let E in $D(\mathcal{O}_X)$ be perfect. Then*

1. *E is in $D^b_{Coh}(\mathcal{O}_X)$,*
2. *if L is in $D_{Coh}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D_{Coh}(\mathcal{O}_X)$,*
3. *if L is in $D^b_{Coh}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D^b_{Coh}(\mathcal{O}_X)$,*

(4) *if L is in $D^+_{Coh}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D^+_{Coh}(\mathcal{O}_X)$,*
(5) *if L is in $D^-_{Coh}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D^-_{Coh}(\mathcal{O}_X)$.*

Proof. Since X is quasi-compact, each of these statements can be checked over the members of any open covering of X. Thus we may assume E is represented by a bounded complex $\mathcal{E}^\bullet$ of finite free modules, see Cohomology, Lemma 49.3. In this case each of the statements is clear as both $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ and $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ can be computed on the level of complexes using $\mathcal{E}^\bullet$, see Cohomology, Lemmas 46.9 and 26.9. Some details omitted. $\square$

Lemma 11.7. *Let A be a Noetherian ring. Let X be a proper scheme over A. For L in $D^+_{Coh}(\mathcal{O}_X)$ and K in $D^-_{Coh}(\mathcal{O}_X)$, the A-modules $\text{Ext}^n_{\mathcal{O}_X}(K, L)$ are finite.*

Proof. Recall that

$$\text{Ext}^n_{\mathcal{O}_X}(K, L) = H^n(X, R\mathcal{H}om_{\mathcal{O}_X}(K, L)) = H^n(\text{Spec}(A), Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, L))$$

see Cohomology, Lemma 42.1 and Cohomology, Section 13. Thus the result follows from Lemmas 11.5 and 11.4. $\square$

Lemma 11.8. *Let X be a locally Noetherian regular scheme. Then every object of $D^b_{Coh}(\mathcal{O}_X)$ is perfect. If X is quasi-compact, i.e., Noetherian regular, then conversely every perfect object of $D(\mathcal{O}_X)$ is in $D^b_{Coh}(\mathcal{O}_X)$.*

Proof. Let K be an object of $D^b_{Coh}(\mathcal{O}_X)$. To check that K is perfect, we may work affine locally on X (see Cohomology, Section 49). Then K is perfect by Lemma 10.7 and More on Algebra, Lemma 74.14. The converse is Lemma 11.6. $\square$

12. Descent finiteness properties of complexes

This section is the analogue of Descent, Section 7 for objects of the derived category of a scheme. The easiest such result is probably the following.

Lemma 12.1. *Let $f : X \to Y$ be a surjective flat morphism of schemes (or more generally locally ringed spaces). Let $E \in D(\mathcal{O}_Y)$. Let $a, b \in \mathbf{Z}$. Then E has tor-amplitude in $[a, b]$ if and only if Lf^*E has tor-amplitude in $[a, b]$.*

Proof. Pullback always preserves tor-amplitude, see Cohomology, Lemma 48.4. We may check tor-amplitude in $[a, b]$ on stalks, see Cohomology, Lemma 48.5. A flat local ring homomorphism is faithfully flat by Algebra, Lemma 39.17. Thus the result follows from More on Algebra, Lemma 66.17. $\square$

Lemma 12.2. *Let $\{f_i : X_i \to X\}$ be an fpqc covering of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent if and only if each Lf_i^*E is m-pseudo-coherent.*

Proof. Pullback always preserves m-pseudo-coherence, see Cohomology, Lemma 47.3. Conversely, assume that Lf_i^*E is m-pseudo-coherent for all i. Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is m-pseudo-coherent. Since $\{f_i : X_i \to X\}$ is an fpqc covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_i$. Thus we may replace X by U and $\{f_i : X_i \to X\}$ by $\{V \to U\}$ and assume that X is affine and our covering is given by a single surjective flat morphism $\{f : Y \to X\}$ of affine schemes. In this case the result follows from More on Algebra, Lemma 64.15 via Lemmas 3.5 and 10.2. $\square$

Lemma 12.3. *Let $\{f_i : X_i \to X\}$ be an fppf covering of schemes. Let $E \in D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent if and only if each Lf_i^*E is m-pseudo-coherent.*

Proof. Pullback always preserves m-pseudo-coherence, see Cohomology, Lemma 47.3. Conversely, assume that Lf_i^*E is m-pseudo-coherent for all i. Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is m-pseudo-coherent. Since $\{f_i : X_i \to X\}$ is an fppf covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_i$. Thus we may replace X by U and $\{f_i : X_i \to X\}$ by $\{V \to U\}$ and assume that X is affine and our covering is given by a single surjective flat morphism $\{f : Y \to X\}$ of finite presentation.

Since f is flat the derived functor Lf^* is just given by f^* and f^* is exact. Hence $H^i(Lf^*E) = f^*H^i(E)$. Since Lf^*E is m-pseudo-coherent, we see that $Lf^*E \in D^-(\mathcal{O}_Y)$. Since f is surjective and flat, we see that $E \in D^-(\mathcal{O}_X)$. Let $i \in \mathbf{Z}$ be the largest integer such that $H^i(E)$ is nonzero. If $i < m$, then we are done. Otherwise, $f^*H^i(E)$ is a finite type $\mathcal{O}_Y$-module by Cohomology, Lemma 47.9. Then by Descent, Lemma 7.2 the $\mathcal{O}_X$-module $H^i(E)$ is of finite type. Thus, after replacing X by the members of a finite affine open covering, we may assume there exists a map

$$\alpha : \mathcal{O}_X^{\oplus n}[-i] \longrightarrow E$$

such that $H^i(\alpha)$ is a surjection. Let C be the cone of α in $D(\mathcal{O}_X)$. Pulling back to Y and using Cohomology, Lemma 47.4 we find that Lf^*C is m-pseudo-coherent. Moreover $H^j(C) = 0$ for $j \geq i$. Thus by induction on i we see that C is m-pseudo-coherent. Using Cohomology, Lemma 47.4 again we conclude. □

Lemma 12.4. *Let $\{f_i : X_i \to X\}$ be an fpqc covering of schemes. Let $E \in D(\mathcal{O}_X)$. Then E is perfect if and only if each Lf_i^*E is perfect.*

Proof. Pullback always preserves perfect complexes, see Cohomology, Lemma 49.6. Conversely, assume that Lf_i^*E is perfect for all i. Then the cohomology sheaves of each Lf_i^*E are quasi-coherent, see Lemma 10.1 and Cohomology, Lemma 49.5. Since the morphisms f_i is flat we see that $H^p(Lf_i^*E) = f_i^*H^p(E)$. Thus the cohomology sheaves of E are quasi-coherent by Descent, Proposition 5.2. Having said this the lemma follows formally from Cohomology, Lemma 49.5 and Lemmas 12.1 and 12.2. □

Lemma 12.5. *Let $i : Z \to X$ be a morphism of ringed spaces such that i is a closed immersion of underlying topological spaces and such that $i_*\mathcal{O}_Z$ is pseudo-coherent as an $\mathcal{O}_X$-module. Let $E \in D(\mathcal{O}_Z)$. Then E is m-pseudo-coherent if and only if Ri_*E is m-pseudo-coherent.*

Proof. Throughout this proof we will use that i_* is an exact functor, and hence that $Ri_* = i_*$, see Modules, Lemma 6.1.

Assume E is m-pseudo-coherent. Let $x \in X$. We will find a neighbourhood of x such that i_*E is m-pseudo-coherent on it. If $x \notin Z$ then this is clear. Thus we may assume $x \in Z$. We will use that $U \cap Z$ for $x \in U \subset X$ open form a fundamental system of neighbourhoods of x in Z. After shrinking X we may assume E is bounded above. We will argue by induction on the largest integer p such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(E)$ is an $\mathcal{O}_Z$-module of finite type, see Cohomology, Lemma 47.9. Thus we may choose, after shrinking X, a map $\mathcal{O}_Z^{\oplus n}[-p] \to E$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \to H^p(E)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \to E \to C \to \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$ and that C is m-pseudo-coherent by Cohomology, Lemma 47.4. By induction we see that i_*C is m-pseudo-coherent on X. Since $i_*\mathcal{O}_Z$ is m-pseudo-coherent on X as well, we conclude from the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

and Cohomology, Lemma 47.4 that i_*E is m-pseudo-coherent.

Assume that i_*E is m-pseudo-coherent. Let $z \in Z$. We will find a neighbourhood of z such that E is m-pseudo-coherent on it. We will use that $U \cap Z$ for $z \in U \subset X$ open form a fundamental system of neighbourhoods of z in Z. After shrinking X we may assume i_*E and hence E is bounded above. We will argue by induction on the largest integer p such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(i_*E) = i_*H^p(E)$ is an $\mathcal{O}_X$-module of finite type, see Cohomology, Lemma 47.9. Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Z$-modules representing E. We may choose, after shrinking X, a map $\alpha : \mathcal{O}_X^{\oplus n}[-p] \to i_*\mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_X^{\oplus n} \to i_*H^p(\mathcal{E}^\bullet)$. By adjunction we find a map $\alpha : \mathcal{O}_Z^{\oplus n}[-p] \to \mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \to H^p(\mathcal{E}^\bullet)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \to E \to C \to \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$. From the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

the fact that $i_*\mathcal{O}_Z$ is pseudo-coherent and Cohomology, Lemma 47.4 we conclude that i_*C is m-pseudo-coherent. By induction we conclude that C is m-pseudo-coherent. By Cohomology, Lemma 47.4 again we conclude that E is m-pseudo-coherent. $\square$

Lemma 12.6. *Let $f : X \to Y$ be a finite morphism of schemes such that $f_*\mathcal{O}_X$ is pseudo-coherent as an $\mathcal{O}_Y$-module*[2]*. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Then E is m-pseudo-coherent if and only if Rf_*E is m-pseudo-coherent.*

Proof. This is a translation of More on Algebra, Lemma 64.11 into the language of schemes. To do the translation, use Lemmas 3.5 and 10.2. $\square$

13. Lifting complexes

Let $U \subset X$ be an open subspace of a ringed space and denote $j : U \to X$ the inclusion morphism. The functor $D(\mathcal{O}_X) \to D(\mathcal{O}_U)$ is essentially surjective as Rj_* is a right inverse to restriction. In this section we extend this to complexes with quasi-coherent cohomology sheaves, etc.

Lemma 13.1. *Let X be a scheme and let $j : U \to X$ be a quasi-compact open immersion. The functors*

$$D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_U) \quad \textit{and} \quad D^+_{QCoh}(\mathcal{O}_X) \to D^+_{QCoh}(\mathcal{O}_U)$$

are essentially surjective. If X is quasi-compact, then the functors

$$D^-_{QCoh}(\mathcal{O}_X) \to D^-_{QCoh}(\mathcal{O}_U) \quad \textit{and} \quad D^b_{QCoh}(\mathcal{O}_X) \to D^b_{QCoh}(\mathcal{O}_U)$$

[2]This means that f is pseudo-coherent, see More on Morphisms, Lemma 60.8.

are essentially surjective.

Proof. The argument preceding the lemma applies for the first case because Rj_* maps $D_{QCoh}(\mathcal{O}_U)$ into $D_{QCoh}(\mathcal{O}_X)$ by Lemma 4.1. It is clear that Rj_* maps $D^+_{QCoh}(\mathcal{O}_U)$ into $D^+_{QCoh}(\mathcal{O}_X)$ which implies the statement on bounded below complexes. Finally, Lemma 4.1 guarantees that Rj_* maps $D^-_{QCoh}(\mathcal{O}_U)$ into $D^-_{QCoh}(\mathcal{O}_X)$ if X is quasi-compact. Combining these two we obtain the last statement. $\square$

Lemma 13.2. *Let X be a Noetherian scheme and let $j : U \to X$ be an open immersion. The functor $D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_U)$ is essentially surjective.*

Proof. Let K be an object of $D^b_{Coh}(\mathcal{O}_U)$. By Proposition 11.2 we can represent K by a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_U$-modules. Say $\mathcal{F}^i = 0$ for $i \not\in [a, b]$ for some $a \leq b$. Since j is quasi-compact and separated, the terms of the bounded complex $j_*\mathcal{F}^\bullet$ are quasi-coherent modules on X, see Schemes, Lemma 24.1. We inductively pick a coherent submodule $\mathcal{G}^i \subset j_*\mathcal{F}^i$ as follows. For $i = a$ we pick any coherent submodule $\mathcal{G}^a \subset j_*\mathcal{F}^a$ whose restriction to U is $\mathcal{F}^a$. This is possible by Properties, Lemma 22.2. For $i > a$ we first pick any coherent submodule $\mathcal{H}^i \subset j_*\mathcal{F}^i$ whose restriction to U is $\mathcal{F}^i$ and then we set $\mathcal{G}^i = \mathrm{Im}(\mathcal{H}^i \oplus \mathcal{G}^{i-1} \to j_*\mathcal{F}^i)$. It is clear that $\mathcal{G}^\bullet \subset j_*\mathcal{F}^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$-modules whose restriction to U is $\mathcal{F}^\bullet$ as desired. $\square$

Lemma 13.3. *Let X be an affine scheme and let $U \subset X$ be a quasi-compact open subscheme. For any pseudo-coherent object E of $D(\mathcal{O}_U)$ there exists a bounded above complex of finite free $\mathcal{O}_X$-modules whose restriction to U is isomorphic to E.*

Proof. By Lemma 10.1 we see that E is an object of $D_{QCoh}(\mathcal{O}_U)$. By Lemma 13.1 we may assume $E = E'|U$ for some object E' of $D_{QCoh}(\mathcal{O}_X)$. Write $X = \mathrm{Spec}(A)$. By Lemma 3.5 we can find a complex $M^\bullet$ of A-modules whose associated complex of $\mathcal{O}_X$-modules is a representative of E'.

Choose $f_1, \ldots, f_r \in A$ such that $U = D(f_1) \cup \ldots \cup D(f_r)$. By Lemma 10.2 the complexes $M^\bullet_{f_j}$ are pseudo-coherent complexes of A_{f_j}-modules. Let n be an integer. Assume we have a map of complexes $\alpha : F^\bullet \to M^\bullet$ where $F^\bullet$ is bounded above, $F^i = 0$ for $i < n$, each F^i is a finite free R-module, such that

$$H^i(\alpha_{f_j}) : H^i(F^\bullet_{f_j}) \to H^i(M^\bullet_{f_j})$$

is an isomorphism for $i > n$ and surjective for $i = n$. Picture

$$\begin{tikzcd} & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\ M^{n-1} \arrow[r] & M^n \arrow[r] & M^{n+1} \arrow[r] & \ldots \end{tikzcd}$$

Since each $M^\bullet_{f_j}$ has vanishing cohomology in large degrees we can find such a map for $n \gg 0$. By induction on n we are going to extend this to a map of complexes $F^\bullet \to M^\bullet$ such that $H^i(\alpha_{f_j})$ is an isomorphism for all i. The lemma will follow by taking $\widetilde{F^\bullet}$.

The induction step will be to extend the diagram above by adding F^{n-1}. Let $C^\bullet$ be the cone on α (Derived Categories, Definition 9.1). The long exact sequence of cohomology shows that $H^i(C^\bullet_{f_j}) = 0$ for $i \geq n$. By More on Algebra, Lemma 64.2 we see that $C^\bullet_{f_j}$ is $(n-1)$-pseudo-coherent. By More on Algebra, Lemma 64.3 we see that $H^{-1}(C^\bullet_{f_j})$ is a finite A_{f_j}-module. Choose a finite free A-module F^{n-1} and an A-module $\beta : F^{n-1} \to C^{-1}$ such that the composition $F^{n-1} \to C^{n-1} \to C^n$ is zero and such that $F^{n-1}_{f_j}$ surjects onto $H^{n-1}(C^\bullet_{f_j})$. (Some details omitted; hint: clear denominators.) Since $C^{n-1} = M^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$ implies these maps fit into a morphism of complexes

$$\begin{tikzcd} & F^{n-1} \arrow[r, "d^{n-1}"'] \arrow[d, "\alpha^{n-1}"] & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\ \ldots \arrow[r] & M^{n-1} \arrow[r] & M^n \arrow[r] & M^{n+1} \arrow[r] & \ldots \end{tikzcd}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{tikzcd} (F^n \to \ldots) \arrow[r] \arrow[d] & (F^{n-1} \to \ldots) \arrow[r] \arrow[d] & F^{n-1} \arrow[r] \arrow[d, "\beta"'] & (F^n \to \ldots)[1] \arrow[d] \\ (F^n \to \ldots) \arrow[r] & M^\bullet \arrow[r] & C^\bullet \arrow[r] & (F^n \to \ldots)[1] \end{tikzcd}$$

Hence our choice of β implies that the map of complexes $(F^{-1} \to \ldots) \to M^\bullet$ induces an isomorphism on cohomology localized at f_j in degrees $\geq n$ and a surjection in degree -1. This finishes the proof of the lemma. $\square$

Lemma 13.4. *Let X be a quasi-compact and quasi-separated scheme. Let $E \in D^b_{QCoh}(\mathcal{O}_X)$. There exists an integer $n_0 > 0$ such that $\text{Ext}^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = 0$ for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and every $n \geq n_0$.*

Proof. Recall that $\text{Ext}^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n])$. We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 33.3. Thus if $X = U \cup V$ and the result of the lemma holds for $E|_U$, $E|_V$, and $E|_{U \cap V}$ for some bound n_0, then the result holds for E with bound $n_0 + 1$. Thus it suffices to prove the lemma when X is affine, see Cohomology of Schemes, Lemma 4.1.

Assume $X = \text{Spec}(A)$ is affine. Choose a complex of A-modules $M^\bullet$ whose associated complex of quasi-coherent modules represents E, see Lemma 3.5. Write $\mathcal{E} = \widetilde{P}$ for some A-module P. Since $\mathcal{E}$ is finite locally free, we see that P is a finite projective A-module. We have

$$\begin{aligned} \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n]) &= \text{Hom}_{D(A)}(P, M^\bullet[n]) \\ &= \text{Hom}_{K(A)}(P, M^\bullet[n]) \\ &= \text{Hom}_A(P, H^n(M^\bullet)) \end{aligned}$$

The first equality by Lemma 3.5, the second equality by Derived Categories, Lemma 19.8, and the final equality because $\text{Hom}_A(P, -)$ is an exact functor. As E and hence $M^\bullet$ is bounded we get zero for all sufficiently large n. $\square$

Lemma 13.5. *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object E of $D(\mathcal{O}_U)$ there exists an integer r and a finite locally free sheaf $\mathcal{F}$ on U such that $\mathcal{F}[-r] \oplus E$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. Say $X = \text{Spec}(A)$. Recall that a perfect complex is pseudo-coherent, see Cohomology, Lemma 49.5. By Lemma 13.3 we can find a bounded above complex $\mathcal{F}^\bullet$ of finite free A-modules such that E is isomorphic to $\mathcal{F}^\bullet|_U$ in $D(\mathcal{O}_U)$. By

Cohomology, Lemma 49.5 and since U is quasi-compact, we see that E has finite tor dimension, say E has tor amplitude in $[a, b]$. Pick $r < a$ and set

$$\mathcal{F} = \mathrm{Ker}(\mathcal{F}^r \to \mathcal{F}^{r+1}) = \mathrm{Im}(\mathcal{F}^{r-1} \to \mathcal{F}^r).$$

Since E has tor amplitude in $[a, b]$ we see that $\mathcal{F}|_U$ is flat (Cohomology, Lemma 48.2). Hence $\mathcal{F}|_U$ is flat and of finite presentation, thus finite locally free (Properties, Lemma 20.2). It follows that

$$(\mathcal{F} \to \mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)|_U$$

is a strictly perfect complex on U representing E. We obtain a distinguished triangle

$$\mathcal{F}|_U[-r-1] \to E \to (\mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)|_U \to \mathcal{F}|_U[-r]$$

Note that $(\mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)$ is a perfect complex on X. To finish the proof it suffices to pick r such that the map $\mathcal{F}|_U[-r-1] \to E$ is zero in $D(\mathcal{O}_U)$, see Derived Categories, Lemma 4.11. By Lemma 13.4 this holds if $r \ll 0$. ☐

Lemma 13.6. *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. Let E, E' be objects of $D_{QCoh}(\mathcal{O}_X)$ with E perfect. For every map $\alpha : E|_U \to E'|_U$ there exist maps*

$$E \xleftarrow{\beta} E_1 \xrightarrow{\gamma} E'$$

of perfect complexes on X such that $\beta : E_1 \to E$ restricts to an isomorphism on U and such that $\alpha = \gamma|_U \circ \beta|_U^{-1}$. Moreover we can assume $E_1 = E \otimes^{\mathbf{L}}_{\mathcal{O}_X} I$ for some perfect complex I on X.

Proof. Write $X = \mathrm{Spec}(A)$. Write $U = D(f_1) \cup \ldots \cup D(f_r)$. Choose finite complex of finite projective A-modules $M^\bullet$ representing E (Lemma 10.7). Choose a complex of A-modules $(M')^\bullet$ representing E' (Lemma 3.5). In this case the complex $H^\bullet = \mathrm{Hom}_A(M^\bullet, (M')^\bullet)$ is a complex of A-modules whose associated complex of quasi-coherent $\mathcal{O}_X$-modules represents $R\mathcal{H}om(E, E')$, see Cohomology, Lemma 46.9. Then α determines an element s of $H^0(U, R\mathcal{H}om(E, E'))$, see Cohomology, Lemma 42.1. There exists an e and a map

$$\xi : I^\bullet(f_1^e, \ldots, f_r^e) \to \mathrm{Hom}_A(M^\bullet, (M')^\bullet)$$

corresponding to s, see Proposition 9.5. Letting E_1 be the object corresponding to complex of quasi-coherent $\mathcal{O}_X$-modules associated to

$$\mathrm{Tot}(I^\bullet(f_1^e, \ldots, f_r^e) \otimes_A M^\bullet)$$

we obtain $E_1 \to E$ using the canonical map $I^\bullet(f_1^e, \ldots, f_r^e) \to A$ and $E_1 \to E'$ using ξ and Cohomology, Lemma 42.1. ☐

Lemma 13.7. *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object F of $D(\mathcal{O}_U)$ the object $F \oplus F[1]$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. By Lemma 13.5 we can find a perfect object E of $D(\mathcal{O}_X)$ such that $E|_U = \mathcal{F}[r] \oplus F$ for some finite locally free $\mathcal{O}_U$-module $\mathcal{F}$. By Lemma 13.6 we can find a morphism of perfect complexes $\alpha : E_1 \to E$ such that $(E_1)|_U \cong E|_U$ and such that $\alpha|_U$ is the map

$$\begin{pmatrix} \mathrm{id}_{\mathcal{F}[r]} & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{F}[r] \oplus F \to \mathcal{F}[r] \oplus F$$

Then the cone on α is a solution. ☐

Lemma 13.8. *Let X be a quasi-compact and quasi-separated scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. For any morphism $\alpha : E \to E'$ in $D_{QCoh}(\mathcal{O}_X)$ such that*

1. *E is perfect, and*
2. *E' is supported on $T = V(f)$*

there exists an $n \geq 0$ such that $f^n\alpha = 0$.

Proof. We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 33.3. Thus if $X = U \cup V$ and the result of the lemma holds for $f|_U$, $f|_V$, and $f|_{U\cap V}$, then the result holds for f. Thus it suffices to prove the lemma when X is affine, see Cohomology of Schemes, Lemma 4.1.

Let $X = \text{Spec}(A)$. Then $f \in A$. We will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 3.5 without further mention. Represent E by a finite complex of finite projective A-modules $P^\bullet$. This is possible by Lemma 10.7. Let t be the largest integer such that P^t is nonzero. The distinguished triangle

$$P^t[-t] \to P^\bullet \to \sigma_{\leq t-1}P^\bullet \to P^t[-t+1]$$

shows that by induction on the length of the complex $P^\bullet$ we can reduce to the case where $P^\bullet$ has a single nonzero term. This and the shift functor reduces us to the case where $P^\bullet$ consists of a single finite projective A-module P in degree 0. Represent E' by a complex $M^\bullet$ of A-modules. Then α corresponds to a map $P \to H^0(M^\bullet)$. Since the module $H^0(M^\bullet)$ is supported on $V(f)$ by assumption (2) we see that every element of $H^0(M^\bullet)$ is annihilated by a power of f. Since P is a finite A-module the map $f^n\alpha : P \to H^0(M^\bullet)$ is zero for some n as desired. $\square$

Lemma 13.9. *Let X be an affine scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $U \subset X$ be a quasi-compact open. For every perfect object F of $D(\mathcal{O}_U)$ supported on $T \cap U$ the object $F \oplus F[1]$ is the restriction of a perfect object E of $D(\mathcal{O}_X)$ supported in T.*

Proof. Say $T = V(g_1, \ldots, g_s)$. After replacing g_j by a power we may assume multiplication by g_j is zero on F, see Lemma 13.8. Choose E as in Lemma 13.7. Note that $g_j : E \to E$ restricts to zero on U. Choose a distinguished triangle

$$E \xrightarrow{g_1} E \to C_1 \to E[1]$$

By Derived Categories, Lemma 4.11 the object C_1 restricts to $F \oplus F[1] \oplus F[1] \oplus F[2]$ on U. Moreover, $g_1 : C_1 \to C_1$ has square zero by Derived Categories, Lemma 4.5. Namely, the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "0"] & C_1 \arrow[r] \arrow[d, "g_1"] & E[1] \arrow[d, "0"] \\ E \arrow[r] & C_1 \arrow[r] & E[1] \end{tikzcd}$$

is commutative since the compositions $E \xrightarrow{g_1} E \to C_1$ and $C_1 \to E[1] \xrightarrow{g_1} E[1]$ are zero. Continuing, setting C_{i+1} equal to the cone of the map $g_i : C_i \to C_i$ we obtain a perfect complex C_s on X supported on T whose restriction to U gives

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s]$$

Choose morphisms of perfect complexes $\beta : C' \to C_s$ and $\gamma : C' \to C_s$ as in Lemma 13.6 such that $\beta|_U$ is an isomorphism and such that $\gamma|_U \circ \beta|_U^{-1}$ is the morphism

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s] \to F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s]$$

which is the identity on all summands except for F where it is zero. By Lemma 13.6 we also have $C' = C_s \otimes^{\mathbf{L}} I$ for some perfect complex I on X. Hence the nullity of $g_j^2 \mathrm{id}_{C_s}$ implies the same thing for C'. Thus C' is supported on T as well. Then $\mathrm{Cone}(\gamma)$ is a solution. □

A special case of the following lemma can be found in [Nee96].

Lemma 13.10. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in X. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\alpha : P \to E|_U$ be a map where P is a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$. Then there exists a map $\beta : R \to E$ where R is a perfect object of $D(\mathcal{O}_X)$ supported on T such that P is a direct summand of $R|_U$ in $D(\mathcal{O}_U)$ compatible α and $\beta|_U$.*

Proof. Since X is quasi-compact there exists an integer m such that $X = U \cup V_1 \cup \ldots \cup V_m$ for some affine opens V_j of X. Arguing by induction on m we see that we may assume $m = 1$. In other words, we may assume that $X = U \cup V$ with V affine. By Lemma 13.9 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V}$. By Lemma 13.6 we can replace Q by $Q \otimes^{\mathbf{L}} I$ (still supported on $T \cap V$) and assume that the map

$$Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V} \longrightarrow P|_{U \cap V} \longrightarrow E|_{U \cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma 45.1 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T as desired. □

Remark 13.11. The proof of Lemma 13.10 shows that

$$R|_U = P \oplus P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m]$$

for some $m \geq 0$ and $n_j \geq 0$. Thus the highest degree cohomology sheaf of $R|_U$ equals that of P. By repeating the construction for the map $P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m] \to R|_U$, taking cones, and using induction we can achieve equality of cohomology sheaves of $R|_U$ and P above any given degree.

14. Approximation by perfect complexes

In this section we discuss the observation, due to Neeman and Lipman, that a pseudo-coherent complex can be "approximated" by perfect complexes.

Definition 14.1. Let X be a scheme. Consider triples (T, E, m) where

(1) $T \subset X$ is a closed subset,
(2) E is an object of $D_{QCoh}(\mathcal{O}_X)$, and
(3) $m \in \mathbf{Z}$.

We say *approximation holds for the triple* (T, E, m) if there exists a perfect object P of $D(\mathcal{O}_X)$ supported on T and a map $\alpha : P \to E$ which induces isomorphisms $H^i(P) \to H^i(E)$ for $i > m$ and a surjection $H^m(P) \to H^m(E)$.

Approximation cannot hold for every triple. Namely, it is clear that if approximation holds for the triple (T, E, m), then

(1) E is m-pseudo-coherent, see Cohomology, Definition 47.1, and
(2) the cohomology sheaves $H^i(E)$ are supported on T for $i \geq m$.

Moreover, the "support" of a perfect complex is a closed subscheme whose complement is retrocompact in X (details omitted). Hence we cannot expect approximation to hold without this assumption on T. This partly explains the conditions in the following definition.

Definition 14.2. Let X be a scheme. We say *approximation by perfect complexes holds* on X if for any closed subset $T \subset X$ with $X \setminus T$ retro-compact in X there exists an integer r such that for every triple (T, E, m) as in Definition 14.1 with

(1) E is $(m - r)$-pseudo-coherent, and
(2) $H^i(E)$ is supported on T for $i \geq m - r$

approximation holds.

We will prove that approximation by perfect complexes holds for quasi-compact and quasi-separated schemes. It seems that the second condition is necessary for our method of proof. It is possible that the first condition may be weakened to "E is m-pseudo-coherent" by carefuly analyzing the arguments below.

Lemma 14.3. *Let X be a scheme. Let $U \subset X$ be an open subscheme. Let (T, E, m) be a triple as in Definition 14.1. If*

(1) *$T \subset U$,*
(2) *approximation holds for $(T, E|_U, m)$, and*
(3) *the sheaves $H^i(E)$ for $i \geq m$ are supported on T,*

then approximation holds for (T, E, m).

Proof. Let $j : U \to X$ be the inclusion morphism. If $P \to E|_U$ is an approximation of the triple $(T, E|_U, m)$ over U, then $j_!P = Rj_*P \to j_!(E|_U) \to E$ is an approximation of (T, E, m) over X. See Cohomology, Lemmas 33.6 and 49.10. □

Lemma 14.4. *Let X be an affine scheme. Then approximation holds for every triple (T, E, m) as in Definition 14.1 such that there exists an integer $r \geq 0$ with*

(1) *E is m-pseudo-coherent,*
(2) *$H^i(E)$ is supported on T for $i \geq m - r + 1$,*
(3) *$X \setminus T$ is the union of r affine opens.*

In particular, approximation by perfect complexes holds for affine schemes.

Proof. Say $X = \mathrm{Spec}(A)$. Write $T = V(f_1, \ldots, f_r)$. (The case $r = 0$, i.e., $T = X$ follows immediately from Lemma 10.2 and the definitions.) Let (T, E, m) be a triple as in the lemma. Let t be the largest integer such that $H^t(E)$ is nonzero. We will proceed by induction on t. The base case is $t < m$; in this case the result is trivial. Now suppose that $t \geq m$. By Cohomology, Lemma 47.9 the sheaf $H^t(E)$ is of finite type. Since it is quasi-coherent it is generated by finitely many sections (Properties, Lemma 16.1). For every $s \in \Gamma(X, H^t(E)) = H^t(X, E)$ (see proof of Lemma 3.5) we can find an $e > 0$ and a morphism $K_e[-t] \to E$ such that s is in the image of $H^0(K_e) = H^t(K_e[-t]) \to H^t(E)$, see Lemma 9.6. Taking a finite direct sum of these maps we obtain a map $P \to E$ where P is a perfect complex supported on T, where $H^i(P) = 0$ for $i > t$, and where $H^t(P) \to E$ is surjective. Choose a distinguished triangle

$$P \to E \to E' \to P[1]$$

Then E' is m-pseudo-coherent (Cohomology, Lemma 47.4), $H^i(E') = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r + 1$. By induction we find an approximation $P' \to E'$ of (T, E', m). Fit the composition $P' \to E' \to P[1]$ into a distinguished triangle $P \to P'' \to P' \to P[1]$ and extend the morphisms $P' \to E'$ and $P[1] \to P[1]$ into a morphism of distinguished triangles

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & P'' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & P[1] \arrow[d] \\
P \arrow[r] & E \arrow[r] & E' \arrow[r] & P[1]
\end{tikzcd}$$

using TR3. Then P'' is a perfect complex (Cohomology, Lemma 49.7) supported on T. An easy diagram chase shows that $P'' \to E$ is the desired approximation. $\square$

Lemma 14.5. *Let X be a scheme. Let $X = U \cup V$ be an open covering with U quasi-compact, V affine, and $U \cap V$ quasi-compact. If approximation by perfect complexes holds on U, then approximation holds on X.*

Proof. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in X. Let r_U be the integer of Definition 14.2 adapted to the pair $(U, T \cap U)$. Set $T' = T \setminus U$. Note that $T' \subset V$ and that $V \setminus T' = (X \setminus T) \cap U \cap V$ is quasi-compact by our assumption on T. Let r' be the number of affines needed to cover $V \setminus T'$. We claim that $r = \max(r_U, r')$ works for the pair (X, T).

To see this choose a triple (T, E, m) such that E is $(m - r)$-pseudo-coherent and $H^i(E)$ is supported on T for $i \geq m - r$. Let t be the largest integer such that $H^t(E)|_U$ is nonzero. (Such an integer exists as U is quasi-compact and $E|_U$ is $(m-r)$-pseudo-coherent.) We will prove that E can be approximated by induction on t.

Base case: $t \leq m - r'$. This means that $H^i(E)$ is supported on T' for $i \geq m - r'$. Hence Lemma 14.4 guarantees the existence of an approximation $P \to E|_V$ of $(T', E|_V, m)$ on V. Applying Lemma 14.3 we see that (T', E, m) can be approximated. Such an approximation is also an approximation of (T, E, m).

Induction step. Choose an approximation $P \to E|_U$ of $(T \cap U, E|_U, m)$. This in particular gives a surjection $H^t(P) \to H^t(E|_U)$. By Lemma 13.9 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V}$. By Lemma 13.6 we can replace Q by $Q \otimes^{\mathbf{L}} I$ and assume that the map

$$Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V} \longrightarrow P|_{U \cap V} \longrightarrow E|_{U \cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma 45.1 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T and the map $H^t(R) \to H^t(E)$ is surjective on restriction to U. Choose a distinguished triangle

$$R \to E \to E' \to R[1]$$

Then E' is $(m - r)$-pseudo-coherent (Cohomology, Lemma 47.4), $H^i(E')|_U = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r$. By induction we find an approximation $R' \to E'$ of (T, E', m). Fit the composition $R' \to E' \to R[1]$ into a distinguished triangle $R \to R'' \to R' \to R[1]$ and extend the morphisms $R' \to E'$ and $R[1] \to R[1]$ into a morphism of distinguished triangles

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R'' \arrow[r] \arrow[d] & R' \arrow[r] \arrow[d] & R[1] \arrow[d] \\ R \arrow[r] & E \arrow[r] & E' \arrow[r] & R[1] \end{tikzcd}$$

using TR3. Then R'' is a perfect complex (Cohomology, Lemma 49.7) supported on T. An easy diagram chase shows that $R'' \to E$ is the desired approximation. $\square$

Theorem 14.6. *Let X be a quasi-compact and quasi-separated scheme. Then approximation by perfect complexes holds on X.*

Proof. This follows from the induction principle of Cohomology of Schemes, Lemma 4.1 and Lemmas 14.5 and 14.4. $\square$

15. Generating derived categories

In this section we prove that the derived category $D_{QCoh}(\mathcal{O}_X)$ of a quasi-compact and quasi-separated scheme can be generated by a single perfect object. We urge the reader to read the proof of this result in the wonderful paper by Bondal and van den Bergh, see [BV03].

Lemma 15.1. *Let X be a quasi-compact and quasi-separated scheme. Let U be a quasi-compact open subscheme. Let P be a perfect object of $D(\mathcal{O}_U)$. Then P is a direct summand of the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. Special case of Lemma 13.10. $\square$

Lemma 15.2. *In Situation 9.1 denote $j : U \to X$ the open immersion and let K be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. For $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent*

(1) *$E = Rj_*(E|_U)$, and*
(2) *$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ for all $n \in \mathbf{Z}$.*

Proof. Choose a distinguished triangle $E \to Rj_*(E|_U) \to N \to E[1]$. Observe that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], Rj_*(E|_U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U[n], E) = 0$$

for all n as $K|_U = 0$. Thus it suffices to prove the result for N. In other words, we may assume that E restricts to zero on U. Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i+e''_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e''_i}, \ldots, f_r^{e_r}) \to \ldots$$

of Koszul complexes, see More on Algebra, Lemma 28.11. Hence if $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ for all $n \in \mathbf{Z}$ then the same thing is true for the K replaced by K_e as in Lemma 9.6. Thus our lemma follows immediately from that one and the fact that E is determined by the complex of A-modules $R\Gamma(X, E)$, see Lemma 3.5. $\square$

Theorem 15.3. *Let X be a quasi-compact and quasi-separated scheme. The category $D_{QCoh}(\mathcal{O}_X)$ can be generated by a single perfect object. More precisely, there exists a perfect object P of $D(\mathcal{O}_X)$ such that for $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent*

(1) $E = 0$, *and*
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[n], E) = 0$ *for all* $n \in \mathbf{Z}$.

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 4.1.

If X is affine, then $\mathcal{O}_X$ is a perfect generator. This follows from Lemma 3.5.

Assume that $X = U \cup V$ is an open covering with U quasi-compact such that the theorem holds for U and V is an affine open. Let P be a perfect object of $D(\mathcal{O}_U)$ which is a generator for $D_{QCoh}(\mathcal{O}_U)$. Using Lemma 15.1 we may choose a perfect object Q of $D(\mathcal{O}_X)$ whose restriction to U is a direct sum one of whose summands is P. Say $V = \mathrm{Spec}(A)$. Let $Z = X \setminus U$. This is a closed subset of V with $V \setminus Z$ quasi-compact. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on $Z \subset V$ closed, the pushforward $K' = R(V \to X)_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Cohomology, Lemma 49.10). We claim that $Q \oplus K'$ is a generator for $D_{QCoh}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Cohomology, Lemma 33.6 we have $K' = R(V \to X)_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Lemma 15.2 the vanishing of these groups implies that $E|_V$ is isomorphic to $R(U \cap V \to V)_*E|_{U\cap V}$. This implies that $E = R(U \to X)_*E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. $\square$

The following result is an strengthening of Theorem 15.3 proved using exactly the same methods. Recall that for a closed subset T of a scheme X we denote $D_T(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory of $D(\mathcal{O}_X)$ consisting of objects supported on T (Definition 6.1). We similarly denote $D_{QCoh,T}(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory of $D(\mathcal{O}_X)$ consisting of those complexes whose cohomology sheaves are quasi-coherent and are suppported on T.

Lemma 15.4. *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. With notation as above, the category $D_{QCoh,T}(\mathcal{O}_X)$ is generated by a single perfect object.*

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 4.1.

Assume $X = \mathrm{Spec}(A)$ is affine. In this case there exist $f_1, \ldots, f_r \in A$ such that $T = V(f_1, \ldots, f_r)$. Let K be the Koszul complex on $f_1, \ldots, f_r$ as in Lemma 15.2. Then K is a perfect object with cohomology supported on T and hence a perfect object of $D_{QCoh,T}(\mathcal{O}_X)$. On the other hand, if $E \in D_{QCoh,T}(\mathcal{O}_X)$ and $\mathrm{Hom}(K, E[n]) = 0$ for all n, then Lemma 15.2 tells us that $E = Rj_*(E|_{X\setminus T}) = 0$. Hence K generates $D_{QCoh,T}(\mathcal{O}_X)$, (by our definition of generators of triangulated categories in Derived Categories, Definition 36.3).

Assume that $X = U \cup V$ is an open covering with V affine and U quasi-compact such that the lemma holds for U. Let P be a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$ which is a generator for $D_{QCoh, T\cap U}(\mathcal{O}_U)$. Using Lemma 13.10 we may choose a perfect object Q of $D(\mathcal{O}_X)$ supported on T whose restriction to U is a direct sum one of whose summands is P. Write $V = \mathrm{Spec}(B)$. Let $Z = X \setminus U$. Then Z is a closed subset of V such that $V \setminus Z$ is quasi-compact. As X is quasi-separated, it follows that $Z \cap T$ is a closed subset of V such that $W = V \setminus (Z \cap T)$ is quasi-compact. Thus we can choose $g_1, \ldots, g_s \in B$ such that $Z \cap T = V(g_1, \ldots, g_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $g_1, \ldots, g_s$ over B. Note that since K is supported on $(Z \cap T) \subset V$ closed, the pushforward $K' = R(V \to X)_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Cohomology, Lemma 49.10). We claim that $Q \oplus K'$ is a generator for $D_{QCoh,T}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh,T}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Cohomology, Lemma 33.6 we have $K' = R(V \to X)_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Lemma 15.2 we have $E|_V = Rj_*E|_W$ where $j : W \to V$ is the inclusion. Picture

$$\begin{tikzcd}
W \arrow[r, "j"] & V & Z \cap T \arrow[l] \arrow[d] \\
U \cap V \arrow[u, "j'"] \arrow[ur, "j''"'] & & Z \arrow[ul]
\end{tikzcd}$$

Since E is supported on T we see that $E|_W$ is supported on $T \cap W = T \cap U \cap V$ which is closed in W. We conclude that

$$E|_V = Rj_*(E|_W) = Rj_*(Rj'_*(E|_{U\cap V})) = Rj''_*(E|_{U\cap V})$$

where the second equality is part (1) of Cohomology, Lemma 33.6. This implies that $E = R(U \to X)_*E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. $\square$

16. An example generator

In this section we prove that the derived category of projective space over a ring is generated by a vector bundle, in fact a direct sum of shifts of the structure sheaf.

The following lemma says that $\bigoplus_{n\geq 0} \mathcal{L}^{\otimes -n}$ is a generator if $\mathcal{L}$ is ample.

Lemma 16.1. *Let X be a scheme and $\mathcal{L}$ an ample invertible $\mathcal{O}_X$-module. If K is a nonzero object of $D_{QCoh}(\mathcal{O}_X)$, then for some $n \geq 0$ and $p \in \mathbf{Z}$ the cohomology group $H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ is nonzero.*

Proof. Recall that as X has an ample invertible sheaf, it is quasi-compact and separated (Properties, Definition 26.1 and Lemma 26.7). Thus we may apply Proposition 7.5 and represent K by a complex $\mathcal{F}^\bullet$ of quasi-coherent modules. Pick any p such that $\mathcal{H}^p = \mathrm{Ker}(\mathcal{F}^p \to \mathcal{F}^{p+1}) / \mathrm{Im}(\mathcal{F}^{p-1} \to \mathcal{F}^p)$ is nonzero. Choose a point $x \in X$ such that the stalk $\mathcal{H}^p_x$ is nonzero. Choose an $n \geq 0$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is an affine open neighbourhood of x. Choose $\tau \in \mathcal{H}^p(X_s)$ which maps to a nonzero element of the stalk $\mathcal{H}^p_x$; this is possible as $\mathcal{H}^p$ is quasi-coherent and X_s is affine. Since taking sections over X_s is an exact functor on quasi-coherent modules, we can find a section $\tau' \in \mathcal{F}^p(X_s)$ mapping to zero in $\mathcal{F}^{p+1}(X_s)$ and mapping to τ in $\mathcal{H}^p(X_s)$. By Properties, Lemma 17.2 there exists an m such that $\tau' \otimes s^{\otimes m}$ is the image of a section $\tau'' \in \Gamma(X, \mathcal{F}^p \otimes \mathcal{L}^{\otimes mn})$. Applying the same lemma once more, we find $l \geq 0$ such that $\tau'' \otimes s^{\otimes l}$ maps to zero in $\mathcal{F}^{p+1} \otimes \mathcal{L}^{\otimes (m+l)n}$. Then τ'' gives a nonzero class in $H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{(m+l)n})$ as desired. □

Lemma 16.2. *Let A be a ring. Let $X = \mathbf{P}^n_A$. For every $a \in \mathbf{Z}$ there exists an exact complex*

$$0 \to \mathcal{O}_X(a) \to \ldots \to \mathcal{O}_X(a+i)^{\oplus \binom{n+1}{i}} \to \ldots \to \mathcal{O}_X(a+n+1) \to 0$$

of vector bundles on X.

Proof. Recall that $\mathbf{P}^n_A$ is $\text{Proj}(A[X_0, \ldots, X_n])$, see Constructions, Definition 13.2. Consider the Koszul complex

$$K_\bullet = K_\bullet(A[X_0, \ldots, X_n], X_0, \ldots, X_n)$$

over $S = A[X_0, \ldots, X_n]$ on $X_0, \ldots, X_n$. Since $X_0, \ldots, X_n$ is clearly a regular sequence in the polynomial ring S, we see that (More on Algebra, Lemma 30.2) that the Koszul complex $K_\bullet$ is exact, except in degree 0 where the cohomology is $S/(X_0, \ldots, X_n)$. Note that $K_\bullet$ becomes a complex of graded modules if we put the generators of K_i in degree $+i$. In other words an exact complex

$$0 \to S(-n-1) \to \ldots \to S(-n-1+i)^{\oplus \binom{n}{i}} \to \ldots \to S \to S/(X_0, \ldots, X_n) \to 0$$

Applying the exact functor $\tilde{}$ functor of Constructions, Lemma 8.4 and using that the last term is in the kernel of this functor, we obtain the exact complex

$$0 \to \mathcal{O}_X(-n-1) \to \ldots \to \mathcal{O}_X(-n-1+i)^{\oplus \binom{n+1}{i}} \to \ldots \to \mathcal{O}_X \to 0$$

Twisting by the invertible sheaves $\mathcal{O}_X(n+a)$ we get the exact complexes of the lemma. □

Lemma 16.3. *Let A be a ring. Let $X = \mathbf{P}^n_A$. Then*

$$E = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \ldots \oplus \mathcal{O}_X(-n)$$

is a generator (Derived Categories, Definition 36.3) of $D_{QCoh}(X)$.

Proof. Let $K \in D_{QCoh}(\mathcal{O}_X)$. Assume $\text{Hom}(E, K[p]) = 0$ for all $p \in \mathbf{Z}$. We have to show that $K = 0$. By Derived Categories, Lemma 36.4 we see that $\text{Hom}(E', K[p])$ is zero for all $E' \in \langle E \rangle$ and $p \in \mathbf{Z}$. By Lemma 16.2 applied with $a = -n-1$ we see that $\mathcal{O}_X(-n-1) \in \langle E \rangle$ because it is quasi-isomorphic to a finite complex whose terms are finite direct sums of summands of E. Repeating the argument with $a = -n-2$ we see that $\mathcal{O}_X(-n-2) \in \langle E \rangle$. Arguing by induction we find that $\mathcal{O}_X(-m) \in \langle E \rangle$ for all $m \geq 0$. Since

$$\text{Hom}(\mathcal{O}_X(-m), K[p]) = H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X(m)) = H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes m})$$

we conclude that $K = 0$ by Lemma 16.1. (This also uses that $\mathcal{O}_X(1)$ is an ample invertible sheaf on X which follows from Properties, Lemma 26.12.) □

Remark 16.4. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $E \in D_{QCoh}(\mathcal{O}_Y)$ be a generator (see Theorem 15.3). Then the following are equivalent

(1) for $K \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf_*K = 0$ if and only if $K = 0$,
(2) $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ reflects isomorphisms, and
(3) Lf^*E is a generator for $D_{QCoh}(\mathcal{O}_X)$.

The equivalence between (1) and (2) is a formal consequence of the fact that $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is an exact functor of triangulated categories. Similarly, the equivalence between (1) and (3) follows formally from the fact that Lf^* is the left adjoint to Rf_*. These conditions hold if f is affine (Lemma 5.2) or if f is an open immersion, or if f is a composition of such. We conclude that

(1) if X is a quasi-affine scheme then $\mathcal{O}_X$ is a generator for $D_{QCoh}(\mathcal{O}_X)$,
(2) if $X \subset \mathbf{P}^n_A$ is a quasi-compact locally closed subscheme, then $\mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \ldots \oplus \mathcal{O}_X(-n)$ is a generator for $D_{QCoh}(\mathcal{O}_X)$ by Lemma 16.3.

17. Compact and perfect objects

Let X be a Noetherian scheme of finite dimension. By Cohomology, Proposition 20.7 and Cohomology on Sites, Lemma 52.5 the sheaves of modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O}_X)$ for all opens $U \subset X$. These sheaves are typically not quasi-coherent, hence these do not give perfect objects of the derived category $D(\mathcal{O}_X)$. However, if we restrict ourselves to complexes with quasi-coherent cohomology sheaves, then this does not happen. Here is the precise statement.

Proposition 17.1. *Let X be a quasi-compact and quasi-separated scheme. An object of $D_{QCoh}(\mathcal{O}_X)$ is compact if and only if it is perfect.*

Proof. If K is a perfect object of $D(\mathcal{O}_X)$ with dual $K^\vee$ (Cohomology, Lemma 50.5) we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in M. Since $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums and since $H^0(X, -)$ commutes with direct sums on $D_{QCoh}(\mathcal{O}_X)$ by Lemma 4.5 we conclude that K is compact in $D_{QCoh}(\mathcal{O}_X)$.

Conversely, let K be a compact object of $D_{QCoh}(\mathcal{O}_X)$. To show that K is perfect, it suffices to show that $K|_U$ is perfect for every affine open $U \subset X$, see Cohomology, Lemma 49.2. Observe that $j : U \to X$ is a quasi-compact and separated morphism. Hence $Rj_* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_X)$ commutes with direct sums, see Lemma 4.5. Thus the adjointness of restriction to U and Rj_* implies that $K|_U$ is a compact object of $D_{QCoh}(\mathcal{O}_U)$. Hence we reduce to the case that X is affine.

Assume $X = \mathrm{Spec}(A)$ is affine. By Lemma 3.5 the problem is translated into the same problem for $D(A)$. For $D(A)$ the result is More on Algebra, Proposition 78.3. $\square$

Remark 17.2. Let X be a quasi-compact and quasi-separated scheme. Let G be a perfect object of $D(\mathcal{O}_X)$ which is a generator for $D_{QCoh}(\mathcal{O}_X)$. By Theorem 15.3 there is at least one of these. Combining Lemma 3.1 with Proposition 17.1 and with Derived Categories, Proposition 37.6 we see that G is a classical generator for $D_{perf}(\mathcal{O}_X)$.

The following result is a strengthening of Proposition 17.1. Let $T \subset X$ be a closed subset of a scheme X. As before $D_T(\mathcal{O}_X)$ denotes the strictly full, saturated, triangulated subcategory of $D(\mathcal{O}_X)$ consisting of objects supported on T (Definition

6.1). Since taking direct sums commutes with taking cohomology sheaves, it follows that $D_T(\mathcal{O}_X)$ has direct sums and that they are equal to direct sums in $D(\mathcal{O}_X)$.

Lemma 17.3. *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. An object of $D_{QCoh,T}(\mathcal{O}_X)$ is compact if and only if it is perfect as an object of $D(\mathcal{O}_X)$.*

Proof. We observe that $D_{QCoh,T}(\mathcal{O}_X)$ is a triangulated category with direct sums by the remark preceding the lemma. By Proposition 17.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$ hence a fortiori of any subcategory preserved under taking direct sums. For the converse we will use there exists a generator $E \in D_{QCoh,T}(\mathcal{O}_X)$ which is a perfect complex of $\mathcal{O}_X$-modules, see Lemma 15.4. Hence by the above, E is compact. Then it follows from Derived Categories, Proposition 37.6 that E is a classical generator of the full subcategory of compact objects of $D_{QCoh,T}(\mathcal{O}_X)$. Thus any compact object can be constructed out of E by a finite sequence of operations consisting of (a) taking shifts, (b) taking finite direct sums, (c) taking cones, and (d) taking direct summands. Each of these operations preserves the property of being perfect and the result follows. $\square$

Remark 17.4. Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let G be a perfect object of $D_{QCoh,T}(\mathcal{O}_X)$ which is a generator for $D_{QCoh,T}(\mathcal{O}_X)$. By Lemma 15.4 there is at least one of these. Combining the fact that $D_{QCoh,T}(\mathcal{O}_X)$ has direct sums with Lemma 17.3 and with Derived Categories, Proposition 37.6 we see that G is a classical generator for $D_{perf,T}(\mathcal{O}_X)$.

The following lemma is an application of the ideas that go into the proof of the preceding lemma.

Lemma 17.5. *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $U = X \setminus T$ is quasi-compact. Let $\alpha : P \to E$ be a morphism of $D_{QCoh}(\mathcal{O}_X)$ with either*

(1) *P is perfect and E supported on T, or*
(2) *P pseudo-coherent, E supported on T, and E bounded below.*

Then there exists a perfect complex of $\mathcal{O}_X$-modules I and a map $I \to \mathcal{O}_X[0]$ such that $I \otimes^{\mathbf{L}} P \to E$ is zero and such that $I|_U \to \mathcal{O}_U[0]$ is an isomorphism.

Proof. Set $\mathcal{D} = D_{QCoh,T}(\mathcal{O}_X)$. In both cases the complex $K = R\mathcal{H}om(P, E)$ is an object of $\mathcal{D}$. See Lemma 10.8 for quasi-coherence. It is clear that K is supported on T as formation of $R\mathcal{H}om$ commutes with restriction to opens. The map α defines an element of $H^0(K) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[0], K)$. Then it suffices to prove the result for the map $\alpha : \mathcal{O}_X[0] \to K$.

Let $E \in \mathcal{D}$ be a perfect generator, see Lemma 15.4. Write

$$K = \text{hocolim} K_n$$

as in Derived Categories, Lemma 37.3 using the generator E. Since the functor $\mathcal{D} \to D(\mathcal{O}_X)$ commutes with direct sums, we see that $K = \text{hocolim} K_n$ holds in $D(\mathcal{O}_X)$. Since $\mathcal{O}_X$ is a compact object of $D(\mathcal{O}_X)$ we find an n and a morphism $\alpha_n : \mathcal{O}_X \to K_n$ which gives rise to α, see Derived Categories, Lemma 33.9. By Derived Categories, Lemma 37.4 applied to the morphism $\mathcal{O}_X[0] \to K_n$ in the ambient category $D(\mathcal{O}_X)$ we see that α_n factors as $\mathcal{O}_X[0] \to Q \to K_n$ where Q is an object of $\langle E \rangle$. We conclude that Q is a perfect complex supported on T.

Choose a distinguished triangle

$$I \to \mathcal{O}_X[0] \to Q \to I[1]$$

By construction I is perfect, the map $I \to \mathcal{O}_X[0]$ restricts to an isomorphism over U, and the composition $I \to K$ is zero as α factors through Q. This proves the lemma. $\square$

18. Derived categories as module categories

In this section we draw some conclusions of what has gone before. Before we do so we need a couple more lemmas.

Lemma 18.1. *Let X be a scheme. Let $K^\bullet$ be a complex of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent. Let $(E, d) = \mathrm{Hom}_{Comp^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$ be the endomorphism differential graded algebra. Then the functor*

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, d) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 35.3 has image contained in $D_{QCoh}(\mathcal{O}_X)$.

Proof. Let P be a differential graded E-module with property (P) and let $F_\bullet$ be a filtration on P as in Differential Graded Algebra, Section 20. Then we have

$$P \otimes_E K^\bullet = \mathrm{hocolim}\ F_i P \otimes_E K^\bullet$$

Each of the $F_i P$ has a finite filtration whose graded pieces are direct sums of $E[k]$. The result follows easily. $\square$

The following lemma can be strengthened (there is a uniformity in the vanishing over all L with nonzero cohomology sheaves only in a fixed range).

Lemma 18.2. *Let X be a quasi-compact and quasi-separated scheme. Let K be a perfect object of $D(\mathcal{O}_X)$. Then*

1. *there exist integers $a \leq b$ such that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L) = 0$ for $L \in D_{QCoh}(\mathcal{O}_X)$ with $H^i(L) = 0$ for $i \in [a, b]$, and*
2. *if L is bounded, then $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(K, L)$ is zero for all but finitely many n.*

Proof. Part (2) follows from (1) as $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(K, L) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L[n])$. We prove (1). Since K is perfect we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L) = H^0(X, K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$$

where $K^\vee$ is the "dual" perfect complex to K, see Cohomology, Lemma 50.5. Note that $K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L$ is in $D_{QCoh}(X)$ by Lemmas 3.9 and 10.1 (to see that a perfect complex has quasi-coherent cohomology sheaves). Say $K^\vee$ has tor amplitude in $[a, b]$. Then the spectral sequence

$$E_1^{p,q} = H^p(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} H^q(L)) \Rightarrow H^{p+q}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$$

shows that $H^j(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$ is zero if $H^q(L) = 0$ for $q \in [j - b, j - a]$. Let N be the integer d of Cohomology of Schemes, Lemma 4.4. Then $H^0(X, K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$ vanishes if the cohomology sheaves

$$H^{-N}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L),\ H^{-N+1}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L),\ \ldots,\ H^0(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$$

are zero. Namely, by the lemma cited and Lemma 3.4, we have

$$H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = H^0(X, \tau_{\geq -N}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L))$$

and by the vanishing of cohomology sheaves, this is equal to $H^0(X, \tau_{\geq 1}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L))$ which is zero by Derived Categories, Lemma 16.1. It follows that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L)$ is zero if $H^i(L) = 0$ for $i \in [-b-N, -a]$. □

The following result is taken from [BV03].

Theorem 18.3. *Let X be a quasi-compact and quasi-separated scheme. Then there exist a differential graded algebra (E, d) with only a finite number of nonzero cohomology groups $H^i(E)$ such that $D_{QCoh}(\mathcal{O}_X)$ is equivalent to $D(E, d)$.*

Proof. Let $K^\bullet$ be a K-injective complex of $\mathcal{O}$-modules which is perfect and generates $D_{QCoh}(\mathcal{O}_X)$. Such a thing exists by Theorem 15.3 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

where $\mathrm{Comp}^{dg}(\mathcal{O}_X)$ is the differential graded category of complexes of $\mathcal{O}$-modules. Please see Differential Graded Algebra, Section 35. Since $K^\bullet$ is K-injective we have

$$H^n(E) = \mathrm{Ext}^n_{D(\mathcal{O}_X)}(K^\bullet, K^\bullet) \tag{18.3.1}$$

for all $n \in \mathbf{Z}$. Only a finite number of these Exts are nonzero by Lemma 18.2. Consider the functor

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 35.3. Since $K^\bullet$ is perfect, it defines a compact object of $D(\mathcal{O}_X)$, see Proposition 17.1. Combined with (18.3.1) the functor above is fully faithful as follows from Differential Graded Algebra, Lemmas 35.6. It has a right adjoint

$$R\,\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}_X) \longrightarrow D(E, \mathrm{d})$$

by Differential Graded Algebra, Lemmas 35.5 which is a left quasi-inverse functor by generalities on adjoint functors. On the other hand, it follows from Lemma 18.1 that we obtain

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

and by our choice of $K^\bullet$ as a generator of $D_{QCoh}(\mathcal{O}_X)$ the kernel of the adjoint restricted to $D_{QCoh}(\mathcal{O}_X)$ is zero. A formal argument shows that we obtain the desired equivalence, see Derived Categories, Lemma 7.2. □

Remark 18.4 (Variant with support)**.** Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. The analogue of Theorem 18.3 holds for $D_{QCoh,T}(\mathcal{O}_X)$. This follows from the exact same argument as in the proof of the theorem, using Lemmas 15.4 and 17.3 and a variant of Lemma 18.1 with supports. If we ever need this, we will precisely state the result here and give a detailed proof.

Remark 18.5 (Uniqueness of dga)**.** Let X be a quasi-compact and quasi-separated scheme over a ring R. By the construction of the proof of Theorem 18.3 there exists a differential graded algebra (A, d) over R such that $D_{QCoh}(X)$ is R-linearly equivalent to $D(A, \mathrm{d})$ as a triangulated category. One may ask: how unique is (A, d)? The answer is (only) slightly better than just saying that (A, d) is well defined up to derived equivalence. Namely, suppose that (B, d) is a second such pair. Then we have

$$(A, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

and

$$(B, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, L^\bullet)$$

for some K-injective complexes $K^\bullet$ and $L^\bullet$ of $\mathcal{O}_X$-modules corresponding to perfect generators of $D_{QCoh}(\mathcal{O}_X)$. Set

$$\Omega = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, L^\bullet) \quad \Omega' = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, K^\bullet)$$

Then Ω is a differential graded $B^{opp} \otimes_R A$-module and Ω' is a differential graded $A^{opp} \otimes_R B$-module. Moreover, the equivalence

$$D(A, \mathrm{d}) \to D_{QCoh}(\mathcal{O}_X) \to D(B, \mathrm{d})$$

is given by the functor $- \otimes_A^{\mathbf{L}} \Omega'$ and similarly for the quasi-inverse. Thus we are in the situation of Differential Graded Algebra, Remark 37.10. If we ever need this remark we will provide a precise statement with a detailed proof here.

19. Characterizing pseudo-coherent complexes, I

We can use the methods above to characterize pseudo-coherent objects as derived homotopy limits of approximations by perfect objects.

Lemma 19.1. *Let X be a quasi-compact and quasi-separated scheme. Let $K \in D(\mathcal{O}_X)$. The following are equivalent*

1. *K is pseudo-coherent, and*
2. *$K = hocolim K_n$ where K_n is perfect and $\tau_{\geq -n}K_n \to \tau_{\geq -n}K$ is an isomorphism for all n.*

Proof. The implication (2) $\Rightarrow$ (1) is true on any ringed space. Namely, assume (2) holds. Recall that a perfect object of the derived category is pseudo-coherent, see Cohomology, Lemma 49.5. Then it follows from the definitions that $\tau_{\geq -n}K_n$ is $(-n+1)$-pseudo-coherent and hence $\tau_{\geq -n}K$ is $(-n+1)$-pseudo-coherent, hence K is $(-n+1)$-pseudo-coherent. This is true for all n, hence K is pseudo-coherent, see Cohomology, Definition 47.1.

Assume (1). We start by choosing an approximation $K_1 \to K$ of $(X, K, -2)$ by a perfect complex K_1, see Definitions 14.1 and 14.2 and Theorem 14.6. Suppose by induction we have

$$K_1 \to K_2 \to \ldots \to K_n \to K$$

with K_i perfect such that such that $\tau_{\geq -i}K_i \to \tau_{\geq -i}K$ is an isomorphism for all $1 \leq i \leq n$. Then we pick $a \leq b$ as in Lemma 18.2 for the perfect object K_n. Choose an approximation $K_{n+1} \to K$ of $(X, K, \min(a-1, -n-1))$. Choose a distinguished triangle

$$K_{n+1} \to K \to C \to K_{n+1}[1]$$

Then we see that $C \in D_{QCoh}(\mathcal{O}_X)$ has $H^i(C) = 0$ for $i \geq a$. Thus by our choice of a, b we see that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K_n, C) = 0$. Hence the composition $K_n \to K \to C$ is zero. Hence by Derived Categories, Lemma 4.2 we can factor $K_n \to K$ through K_{n+1} proving the induction step.

We still have to prove that $K = \text{hocolim} K_n$. This follows by an application of Derived Categories, Lemma 33.8 to the functors $H^i(-) : D(\mathcal{O}_X) \to \textit{Mod}(\mathcal{O}_X)$ and our choice of K_n. □

Lemma 19.2. *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $K \in D(\mathcal{O}_X)$ supported on T. The following are equivalent*

(1) *K is pseudo-coherent, and*
(2) *$K = \text{hocolim} K_n$ where K_n is perfect, supported on T, and $\tau_{\geq -n} K_n \to \tau_{\geq -n} K$ is an isomorphism for all n.*

Proof. The proof of this lemma is exactly the same as the proof of Lemma 19.1 except that in the choice of the approximations we use the triples (T, K, m). □

20. An example equivalence

In Section 16 we proved that the derived category of projective space $\mathbf{P}^n_A$ over a ring A is generated by a vector bundle, in fact a direct sum of shifts of the structure sheaf. In this section we prove this determines an equivalence of $D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_A})$ with the derived category of an A-algebra.

Before we can state the result we need some notation. Let A be a ring. Let $X = \mathbf{P}^n_A = \text{Proj}(S)$ where $S = A[X_0, \ldots, X_n]$. By Lemma 16.3 we know that

$$P = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \ldots \oplus \mathcal{O}_X(-n) \tag{20.0.1}$$

is a perfect generator of $D_{QCoh}(\mathcal{O}_X)$. Consider the (noncommutative) A-algebra

$$R = \text{Hom}_{\mathcal{O}_X}(P, P) = \begin{pmatrix} S_0 & S_1 & S_2 & \ldots & \ldots \\ 0 & S_0 & S_1 & \ldots & \ldots \\ 0 & 0 & S_0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & S_0 \end{pmatrix} \tag{20.0.2}$$

with obvious multiplication and addition. If we view P as a complex of $\mathcal{O}_X$-modules in the usual way (i.e., with P in degree 0 and zero in every other degree), then we have

$$R = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(P, P)$$

where on the right hand side we view R as a differential graded algebra over A with zero differential (i.e., with R in degree 0 and zero in every other degree). According to the discussion in Differential Graded Algebra, Section 35 we obtain a derived functor

$$- \otimes^{\mathbf{L}}_R P : D(R) \longrightarrow D(\mathcal{O}_X),$$

see especially Differential Graded Algebra, Lemma 35.3. By Lemma 18.1 we see that the essential image of this functor is contained in $D_{QCoh}(\mathcal{O}_X)$.

Lemma 20.1. *Let A be a ring. Let $X = \mathbf{P}^n_A = \textit{Proj}(S)$ where $S = A[X_0, \ldots, X_n]$. With P as in (20.0.1) and R as in (20.0.2) the functor*

$$- \otimes^{\mathbf{L}}_R P : D(R) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an A-linear equivalence of triangulated categories sending R to P.

In words: the derived category of quasi-coherent modules on projective space is equivalent to the derived category of modules over a (noncommutative) algebra. This property of projective space appears to be quite unusual among all projective schemes over A.

Proof. To prove that our functor is fully faithful it suffices to prove that $\text{Ext}^i_X(P, P)$ is zero for $i \neq 0$ and equal to R for $i = 0$, see Differential Graded Algebra, Lemma 35.6. As in the proof of Lemma 18.2 we see that

$$\text{Ext}^i_X(P, P) = H^i(X, P^\wedge \otimes P) = \bigoplus\nolimits_{0 \leq a, b \leq n} H^i(X, \mathcal{O}_X(a - b))$$

By the computation of cohomology of projective space (Cohomology of Schemes, Lemma 8.1) we find that these Ext-groups are zero unless $i = 0$. For $i = 0$ we recover R because this is how we defined R in (20.0.2). By Differential Graded Algebra, Lemma 35.5 our functor has a right adjoint, namely $R\,\text{Hom}(P, -) : D_{QCoh}(\mathcal{O}_X) \to D(R)$. Since P is a generator for $D_{QCoh}(\mathcal{O}_X)$ by Lemma 16.3 we see that the kernel of $R\,\text{Hom}(P, -)$ is zero. Hence our functor is an equivalence of triangulated categories by Derived Categories, Lemma 7.2. $\square$

21. The coherator revisited

In Section 7 we constructed and studied the right adjoint RQ_X to the canonical functor $D(QCoh(\mathcal{O}_X)) \to D(\mathcal{O}_X)$. It was constructed as the right derived extension of the coherator $Q_X : Mod(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$. In this section, we study when the inclusion functor

$$D_{QCoh}(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

has a right adjoint. If this right adjoint exists, we will denote[3] it

$$DQ_X : D(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

It turns out that quasi-compact and quasi-separated schemes have such a right adjoint.

Lemma 21.1. *Let X be a quasi-compact and quasi-separated scheme. The inclusion functor $D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X)$ has a right adjoint DQ_X.*

First proof. We will use the induction principle as in Cohomology of Schemes, Lemma 4.1 to prove this. If $D(QCoh(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ is an equivalence, then the lemma is true because the functor RQ_X of Section 7 is a right adjoint to the functor $D(QCoh(\mathcal{O}_X)) \to D(\mathcal{O}_X)$. In particular, our lemma is true for affine schemes, see Lemma 7.3. Thus we see that it suffices to show: if $X = U \cup V$ is a union of two quasi-compact opens and the lemma holds for U, V, and $U \cap V$, then the lemma holds for X.

The adjoint exists if and only if for every object K of $D(\mathcal{O}_X)$ we can find a distinguished triangle

$$E' \to E \to K \to E'[1]$$

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\text{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. See Derived Categories, Lemma 40.7. Consider the distinguished triangle

$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U \cap V,*}E|_{U \cap V} \to E[1]$$

[3]This is probably nonstandard notation. However, we have already used Q_X for the coherator and RQ_X for its derived extension.

in $D(\mathcal{O}_X)$ of Cohomology, Lemma 33.2. By Derived Categories, Lemma 40.5 it suffices to construct the desired distinguished triangles for $Rj_{U,*}E|_U$, $Rj_{V,*}E|_V$, and $Rj_{U\cap V,*}E|_{U\cap V}$. This reduces us to the statement discussed in the next paragraph.

Let $j : U \to X$ be an open immersion corresponding with U a quasi-compact open for which the lemma is true. Let L be an object of $D(\mathcal{O}_U)$. Then there exists a distinguished triangle

$$E' \to Rj_*L \to K \to E'[1]$$

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\mathrm{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. To see this we choose a distinguished triangle

$$L' \to L \to Q \to L'[1]$$

in $D(\mathcal{O}_U)$ such that L' is in $D_{QCoh}(\mathcal{O}_U)$ and such that $\mathrm{Hom}(N, Q) = 0$ for all N in $D_{QCoh}(\mathcal{O}_U)$. This is possible because the statement in Derived Categories, Lemma 40.7 is an if and only if. We obtain a distinguished triangle

$$Rj_*L' \to Rj_*L \to Rj_*Q \to Rj_*L'[1]$$

in $D(\mathcal{O}_X)$. Observe that Rj_*L' is in $D_{QCoh}(\mathcal{O}_X)$ by Lemma 4.1. On the other hand, if M in $D_{QCoh}(\mathcal{O}_X)$, then

$$\mathrm{Hom}(M, Rj_*Q) = \mathrm{Hom}(Lj^*M, Q) = 0$$

because Lj^*M is in $D_{QCoh}(\mathcal{O}_U)$ by Lemma 3.8. This finishes the proof. $\square$

Second proof. The adjoint exists by Derived Categories, Proposition 38.2. The hypotheses are satisfied: First, note that $D_{QCoh}(\mathcal{O}_X)$ has direct sums and direct sums commute with the inclusion functor (Lemma 3.1). On the other hand, $D_{QCoh}(\mathcal{O}_X)$ is compactly generated because it has a perfect generator Theorem 15.3 and because perfect objects are compact by Proposition 17.1. $\square$

Lemma 21.2. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. If the right adjoints DQ_X and DQ_Y of the inclusion functors $D_{QCoh} \to D$ exist for X and Y, then*

$$Rf_* \circ DQ_X = DQ_Y \circ Rf_*$$

Proof. The statement makes sense because Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ by Lemma 4.1. The statement is true because Lf^* similarly maps $D_{QCoh}(\mathcal{O}_Y)$ into $D_{QCoh}(\mathcal{O}_X)$ (Lemma 3.8) and hence both $Rf_* \circ DQ_X$ and $DQ_Y \circ Rf_*$ are right adjoint to $Lf^* : D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_X)$. $\square$

Remark 21.3. Let X be a quasi-compact and quasi-separated scheme. Let $X = U \cup V$ with U and V quasi-compact open. By Lemma 21.1 the functors DQ_X, DQ_U, DQ_V, $DQ_{U\cap V}$ exist. Moreover, there is a canonical distinguished triangle

$$DQ_X(K) \to Rj_{U,*}DQ_U(K|_U) \oplus Rj_{V,*}DQ_V(K|_V) \to Rj_{U\cap V,*}DQ_{U\cap V}(K|_{U\cap V}) \to$$

for any $K \in D(\mathcal{O}_X)$. This follows by applying the exact functor DQ_X to the distinguished triangle of Cohomology, Lemma 33.2 and using Lemma 21.2 three times.

Lemma 21.4. *Let X be a quasi-compact and quasi-separated scheme. The functor DQ_X of Lemma 21.1 has the following boundedness property: there exists an integer $N = N(X)$ such that, if K in $D(\mathcal{O}_X)$ with $H^i(U, K) = 0$ for U affine open in X and $i \notin [a, b]$, then the cohomology sheaves $H^i(DQ_X(K))$ are zero for $i \notin [a, b + N]$.*

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 4.1.

If X is affine, then the lemma is true with $N = 0$ because then $RQ_X = DQ_X$ is given by taking the complex of quasi-coherent sheaves associated to $R\Gamma(X, K)$. See Lemmas 3.5 and 7.3.

Asssume U, V are quasi-compact open in X and the lemma holds for U, V, and $U \cap V$. Say with integers $N(U)$, $N(V)$, and $N(U \cap V)$. Now suppose K is in $D(\mathcal{O}_X)$ with $H^i(W, K) = 0$ for all affine open $W \subset X$ and all $i \notin [a, b]$. Then $K|_U$, $K|_V$, $K|_{U\cap V}$ have the same property. Hence we see that $RQ_U(K|_U)$ and $RQ_V(K|_V)$ and $RQ_{U\cap V}(K|_{U\cap V})$ have vanishing cohomology sheaves outside the inverval $[a, b + \max(N(U), N(V), N(U \cap V))$. Since the functors $Rj_{U,*}$, $Rj_{V,*}$, $Rj_{U\cap V,*}$ have finite cohomological dimension on D_{QCoh} by Lemma 4.1 we see that there exists an N such that $Rj_{U,*}DQ_U(K|_U)$, $Rj_{V,*}DQ_V(K|_V)$, and $Rj_{U\cap V,*}DQ_{U\cap V}(K|_{U\cap V})$ have vanishing cohomology sheaves outside the interval $[a, b + N]$. Then finally we conclude by the distinguished triangle of Remark 21.3. $\square$

Example 21.5. Let X be a quasi-compact and quasi-separated scheme. Let $(\mathcal{F}_n)$ be an inverse system of quasi-coherent sheaves. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence we see that

$$DQ_X(R\lim \mathcal{F}_n) = (R\lim \text{ in } D_{QCoh}(\mathcal{O}_X))(\mathcal{F}_n)$$

where the first $R\lim$ is taken in $D(\mathcal{O}_X)$. In fact, let's write $K = R\lim \mathcal{F}_n$ for this. For any affine open $U \subset X$ we have

$$H^i(U, K) = H^i(R\Gamma(U, R\lim \mathcal{F}_n)) = H^i(R\lim R\Gamma(U, \mathcal{F}_n)) = H^i(R\lim \Gamma(U, \mathcal{F}_n))$$

since cohomology commutes with derived limits and since the quasi-coherent sheaves $\mathcal{F}_n$ have no higher cohomology on affines. By the computation of $R\lim$ in the category of abelian groups, we see that $H^i(U, K) = 0$ unless $i \in [0, 1]$. Then finally we conclude that the $R\lim$ in $D_{QCoh}(\mathcal{O}_X)$, which is $DQ_X(K)$ by the above, is in $D^b_{QCoh}(\mathcal{O}_X)$ by Lemma 21.4.

22. Cohomology and base change, IV

This section continues the discussion of Cohomology of Schemes, Section 22. First, we have a very general version of the projection formula for quasi-compact and quasi-separated morphisms of schemes and complexes with quasi-coherent cohomology sheaves.

Lemma 22.1. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. For E in $D_{QCoh}(\mathcal{O}_X)$ and K in $D_{QCoh}(\mathcal{O}_Y)$ the map*

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

defined in Cohomology, Equation (54.2.1) is an isomorphism.

Proof. To check the map is an isomorphism we may work locally on Y. Hence we reduce to the case that Y is affine.

Suppose that $K = \bigoplus K_i$ is a direct sum of some complexes $K_i \in D_{QCoh}(\mathcal{O}_Y)$. If the statement holds for each K_i, then it holds for K. Namely, the functors Lf^* and $\otimes^{\mathbf{L}}$ preserve direct sums by construction and Rf_* commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 4.5. Moreover, suppose that $K \to L \to M \to K[1]$ is a distinguished triangle in $D_{QCoh}(Y)$. Then if the statement of the lemma holds for two of K, L, M, then it holds for the third (as the functors involved are exact functors of triangulated categories).

Assume Y affine, say $Y = \mathrm{Spec}(A)$. The functor $\widetilde{\ } : D(A) \to D_{QCoh}(\mathcal{O}_Y)$ is an equivalence (Lemma 3.5). Let T be the property for $K \in D(A)$ that the statement of the lemma holds for $\widetilde{K}$. The discussion above and More on Algebra, Remark 59.11 shows that it suffices to prove T holds for $A[k]$. This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf. ☐

Definition 22.2. Let S be a scheme. Let X, Y be schemes over S. We say X and Y are *Tor independent over S* if for every $x \in X$ and $y \in Y$ mapping to the same point $s \in S$ the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are Tor independent over $\mathcal{O}_{S,s}$ (see More on Algebra, Definition 61.1).

Lemma 22.3. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. The following are equivalent*

1. *X and Y are tor independent over S, and*
2. *for every affine opens $U \subset X$, $V \subset Y$, $W \subset S$ with $f(U) \subset W$ and $g(V) \subset W$ the rings $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(V)$ are tor independent over $\mathcal{O}_S(W)$.*
3. *there exists an affine open overing $S = \bigcup W_i$ and for each i affine open coverings $f^{-1}(W_i) = \bigcup U_{ij}$ and $g^{-1}(W_i) = \bigcup V_{ik}$ such that the rings $\mathcal{O}_X(U_{ij})$ and $\mathcal{O}_Y(V_{ik})$ are tor independent over $\mathcal{O}_S(W_i)$ for all i, j, k.*

Proof. Omitted. Hint: use More on Algebra, Lemma 61.6. ☐

Lemma 22.4. *Let $X \to S$ and $Y \to S$ be morphisms of schemes. Let $S' \to S$ be a morphism of schemes and denote $X' = X \times_S S'$ and $Y' = Y \times_S S'$. If X and Y are tor independent over S and $S' \to S$ is flat, then X' and Y' are tor independent over S'.*

Proof. Omitted. Hint: use Lemma 22.3 and on affine opens use More on Algebra, Lemma 61.4. ☐

Lemma 22.5. *Let $g : S' \to S$ be a morphism of schemes. Let $f : X \to S$ be quasi-compact and quasi-separated. Consider the base change diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

If X and S' are Tor independent over S, then for all $E \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf'_*L(g')^*E = Lg^*Rf_*E$.

Proof. For any object E of $D(\mathcal{O}_X)$ we can use Cohomology, Remark 28.3 to get a canonical base change map $Lg^*Rf_*E \to Rf'_*L(g')^*E$. To check this is an isomorphism we may work locally on S'. Hence we may assume $g : S' \to S$ is a morphism of affine schemes. In particular, g is affine and it suffices to show that

$$Rg_*Lg^*Rf_*E \to Rg_*Rf'_*L(g')^*E = Rf_*(Rg'_*L(g')^*E)$$

is an isomorphism, see Lemma 5.2 (and use Lemmas 3.8, 3.9, and 4.1 to see that the objects $Rf'_*L(g')^*E$ and Lg^*Rf_*E have quasi-coherent cohomology sheaves).

Note that g' is affine as well (Morphisms, Lemma 11.8). By Lemma 5.3 the map becomes a map

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_S} g_*\mathcal{O}_{S'} \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

Observe that $g'_*\mathcal{O}_{X'} = f^*g_*\mathcal{O}_{S'}$. Thus by Lemma 22.1 it suffices to prove that $Lf^*g_*\mathcal{O}_{S'} = f^*g_*\mathcal{O}_{S'}$. This follows from our assumption that X and S' are Tor independent over S. Namely, to check it we may work locally on X, hence we may also assume X is affine. Say $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$. Our assumption implies that A and R' are Tor independent over R (More on Algebra, Lemma 61.6), i.e., $\mathrm{Tor}_i^R(A, R') = 0$ for $i > 0$. In other words $A \otimes^{\mathbf{L}}_R R' = A \otimes_R R'$ which exactly means that $Lf^*g_*\mathcal{O}_{S'} = f^*g_*\mathcal{O}_{S'}$ (use Lemma 3.8). $\square$

The following lemma will be used in the chapter on dualizing complexes.

Lemma 22.6. *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of quasi-compact and quasi-separated schemes. Assume g and f Tor independent and $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$ affine. For $M, K \in D(\mathcal{O}_X)$ the canonical map

$$R\mathrm{Hom}_X(M, K) \otimes^{\mathbf{L}}_R R' \longrightarrow R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*K)$$

in $D(R')$ is an isomorphism in the following two cases

1. *$M \in D(\mathcal{O}_X)$ is perfect and $K \in D_{QCoh}(X)$, or*
2. *$M \in D(\mathcal{O}_X)$ is pseudo-coherent, $K \in D^+_{QCoh}(X)$, and R' has finite tor dimension over R.*

Proof. There is a canonical map $R\mathrm{Hom}_X(M, K) \to R\mathrm{Hom}_{X'}(L(g')^*M, L(g')^*K)$ in $D(\Gamma(X, \mathcal{O}_X))$ of global hom complexes, see Cohomology, Section 44. Restricting scalars we can view this as a map in $D(R)$. Then we can use the adjointness of restriction and $- \otimes^{\mathbf{L}}_R R'$ to get the displayed map of the lemma. Having defined the map it suffices to prove it is an isomorphism in the derived category of abelian groups.

The right hand side is equal to

$$R\mathrm{Hom}_X(M, R(g')_*L(g')^*K) = R\mathrm{Hom}_X(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

by Lemma 5.3. In both cases the complex $R\mathcal{H}om(M, K)$ is an object of $D_{QCoh}(\mathcal{O}_X)$ by Lemma 10.8. There is a natural map

$$R\mathcal{H}om(M, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'} \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

which is an isomorphism in both cases by Lemma 10.9. To see that this lemma applies in case (2) we note that $g'_*\mathcal{O}_{X'} = Rg'_*\mathcal{O}_{X'} = Lf^*g_*\mathcal{O}_X$ the second equality by Lemma 22.5. Using Lemma 10.4 and Cohomology, Lemma 48.4 we conclude that $g'_*\mathcal{O}_{X'}$ has finite Tor dimension. Hence, in both cases by replacing K by $R\mathcal{H}om(M, K)$ we reduce to proving

$$R\Gamma(X, K) \otimes^{\mathbf{L}}_A A' \longrightarrow R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

is an isomorphism. Note that the left hand side is equal to $R\Gamma(X', L(g')^*K)$ by Lemma 5.3. Hence the result follows from Lemma 22.5. $\square$

Remark 22.7. With notation as in Lemma 22.6. The diagram

$$\begin{tikzcd} R\operatorname{Hom}_X(M, Rg'_*L) \otimes_R^{\mathbf{L}} R' \arrow[r] \arrow[d, "\mu"'] & R\operatorname{Hom}_{X'}(L(g')^*M, L(g')^*Rg'_*L) \arrow[d, "a"] \\ R\operatorname{Hom}_X(M, R(g')_*L) \arrow[r, equal] & R\operatorname{Hom}_{X'}(L(g')^*M, L) \end{tikzcd}$$

is commutative where the top horizontal arrow is the map from the lemma, μ is the multiplication map, and a comes from the adjunction map $L(g')^*Rg'_*L \to L$. The multiplication map is the adjunction map $K' \otimes_R^{\mathbf{L}} R' \to K'$ for any $K' \in D(R')$.

Lemma 22.8. *Consider a cartesian square of schemes*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Assume g and f Tor independent.

1. *If $E \in D(\mathcal{O}_X)$ has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_S$-modules, then $L(g')^*E$ has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_{S'}$-modules.*
2. *If $\mathcal{G}$ is an $\mathcal{O}_X$-module flat over S, then $L(g')^*\mathcal{G} = (g')^*\mathcal{G}$.*

Proof. We can compute tor dimension at stalks, see Cohomology, Lemma 48.5. If $x' \in X'$ with image $x \in X$, then

$$(L(g')^*E)_{x'} = E_x \otimes_{\mathcal{O}_{X,x}}^{\mathbf{L}} \mathcal{O}_{X',x'}$$

Let $s' \in S'$ and $s \in S$ be the image of x' and x. Since X and S' are tor independent over S, we can apply More on Algebra, Lemma 61.2 to see that the right hand side of the displayed formula is equal to $E_x \otimes_{\mathcal{O}_{S,s}}^{\mathbf{L}} \mathcal{O}_{S',s'}$ in $D(\mathcal{O}_{S',s'})$. Thus (1) follows from More on Algebra, Lemma 66.13. To see (2) observe that flatness of $\mathcal{G}$ is equivalent to the condition that $\mathcal{G}[0]$ has tor amplitude in $[0, 0]$. Applying (1) we conclude. □

Lemma 22.9. *Consider a cartesian diagram of schemes*

$$\begin{tikzcd} Z' \arrow[r, "i'"'] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where i is a closed immersion. If Z and X' are tor independent over X, then $Ri'_* \circ Lg^* = Lf^* \circ Ri_*$ as functors $D(\mathcal{O}_Z) \to D(\mathcal{O}_{X'})$.

Proof. Note that the lemma is supposed to hold for all $K \in D(\mathcal{O}_Z)$. Observe that i_* and i'_* are exact functors and hence Ri_* and Ri'_* are computed by applying i_* and i'_* to any representatives. Thus the base change map

$$Lf^*(Ri_*(K)) \longrightarrow Ri'_*(Lg^*(K))$$

on stalks at a point $z' \in Z'$ with image $z \in Z$ is given by

$$K_z \otimes_{\mathcal{O}_{X,z}}^{\mathbf{L}} \mathcal{O}_{X',z'} \longrightarrow K_z \otimes_{\mathcal{O}_{Z,z}}^{\mathbf{L}} \mathcal{O}_{Z',z'}$$

This map is an isomorphism by More on Algebra, Lemma 61.2 and the assumed tor independence. □

23. Künneth formula, II

For the case where the base is a field, please see Varieties, Section 29. Consider a cartesian diagram of schemes

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "p"'] \arrow[d, "f"] \arrow[dr, "q"] & \\ X \arrow[dr, "a"'] & S & Y \arrow[dl, "b"] \\ & & \end{tikzcd}$$

Let $K \in D(\mathcal{O}_X)$ and $M \in D(\mathcal{O}_Y)$. There is a canonical map

$$Ra_*K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*M \longrightarrow Rf_*(Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M) \tag{23.0.1}$$

Namely, we can use the maps $Ra_*K \to Ra_*Rp_*Lp^*K = Rf_*Lp^*K$ and $Rb_*M \to Rb_*Rq_*Lq^*M = Rf_*Lq^*M$ and then we can use the relative cup product (Cohomology, Remark 28.7).

Set $A = \Gamma(S, \mathcal{O}_S)$. There is a global Künneth map

$$R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M) \longrightarrow R\Gamma(X \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M) \tag{23.0.2}$$

in $D(A)$. This map is constructed using the pullback maps $R\Gamma(X, K) \to R\Gamma(X \times_S Y, Lp^*K)$ and $R\Gamma(Y, M) \to R\Gamma(X \times_S Y, Lq^*M)$ and the cup product constructed in Cohomology, Section 31.

Lemma 23.1. *In the situation above, if a and b are quasi-compact and quasi-separated and X and Y are tor-independent over S, then (23.0.1) is an isomorphism for $K \in D_{QCoh}(\mathcal{O}_X)$ and $M \in D_{QCoh}(\mathcal{O}_Y)$. If in addition $S = \mathrm{Spec}(A)$ is affine, then the map (23.0.2) is an isomorphism.*

First proof. This follows from the following sequence of isomorphisms

$$\begin{aligned} Rf_*(Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M) &= Ra_*Rp_*(Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M) \\ &= Ra_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Rp_*Lq^*M) \\ &= Ra_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} La^*Rb_*M) \\ &= Ra_*K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*M \end{aligned}$$

The first equality holds because $f = a \circ p$. The second equality by Lemma 22.1. The third equality by Lemma 22.5. The fourth equality by Lemma 22.1. We omit the verification that the composition of these isomorphisms is the same as the map (23.0.1). If S is affine, then the source and target of the arrow (23.0.2) are the result of applying $R\Gamma(S, -)$ to the source and target of (23.0.1) and we obtain the final statement; details omitted. $\square$

Second proof. The construction of the arrow (23.0.1) is compatible with restricting to open subschemes of S as is immediate from the construction of the relative cup product. Thus it suffices to prove that (23.0.1) is an isomorphism when S is affine.

Assume $S = \mathrm{Spec}(A)$ is affine. By Leray we have $R\Gamma(S, Rf_*K) = R\Gamma(X, K)$ and similarly for the other cases. By Cohomology, Lemma 31.7 the map (23.0.1) induces the map (23.0.2) on taking $R\Gamma(S, -)$. On the other hand, by Lemmas 4.1 and 3.9 the source and target of the map (23.0.1) are in $D_{QCoh}(\mathcal{O}_S)$. Thus, by Lemma 3.5, it suffices to prove that (23.0.2) is an isomorphism.

Assume $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$ and $Y = \text{Spec}(C)$ are all affine. We will use Lemma 3.5 without further mention. In this case we can choose a K-flat complex $K^\bullet$ of B-modules whose terms are flat such that K is represented by $\widetilde{K}^\bullet$. Similarly, we can choose a K-flat complex $M^\bullet$ of C-modules whose terms are flat such that M is represented by $\widetilde{M}^\bullet$. See More on Algebra, Lemma 59.10. Then $\widetilde{K}^\bullet$ is a K-flat complex of $\mathcal{O}_X$-modules and similarly for $\widetilde{M}^\bullet$, see Lemma 3.6. Thus La^*K is represented by

$$a^*\widetilde{K}^\bullet = \widetilde{K^\bullet \otimes_A C}$$

and similarly for Lb^*M. This in turn is a K-flat complex of $\mathcal{O}_{X \times_S Y}$-modules by the lemma cited above and More on Algebra, Lemma 59.3. Thus we finally see that the complex of $\mathcal{O}_{X \times_S Y}$-modules associated to

$$\text{Tot}((K^\bullet \otimes_A C) \otimes_{B \otimes_A C} (B \otimes_A M^\bullet)) = \text{Tot}(K^\bullet \otimes_A M^\bullet)$$

represents $La^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lb^*M$ in the derived category of $X \times_S Y$. Taking global sections we obtain $\text{Tot}(K^\bullet \otimes_A M^\bullet)$ which of course is also the complex representing $R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M)$. The fact that the isomorphism is given by cup product follows from the relationship between the genuine cup product and the naive one in Cohomology, Section 31 (and in particular Cohomology, Lemma 31.3 and the discussion following it).

Assume $S = \text{Spec}(A)$ and Y are affine. We will use the induction principle of Cohomology of Schemes, Lemma 4.1 to prove the statement. To do this we only have to show: if $X = U \cup V$ is an open covering with U and V quasi-compact and if the map (23.0.2)

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M) \longrightarrow R\Gamma(U \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M)$$

for U and Y over S, the map (23.0.2)

$$R\Gamma(V, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M) \longrightarrow R\Gamma(V \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M)$$

for V and Y over S, and the map (23.0.2)

$$R\Gamma(U \cap V, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M) \longrightarrow R\Gamma((U \cap V) \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M)$$

for $U \cap V$ and Y over S are isomorphisms, then so is the map (23.0.2) for X and Y over S. However, by Cohomology, Lemma 33.7 these maps fit into a map of distinguished triangles with (23.0.2) the final leg and hence we conclude by Derived Categories, Lemma 4.3.

Assume $S = \text{Spec}(A)$ is affine. To finish the proof we can use the induction principle of Cohomology of Schemes, Lemma 4.1 on Y. Namely, by the above we already know that our map is an isomorphism when Y is affine. The rest of the argument is exactly the same as in the previous paragraph but with the roles of X and Y switched. $\square$

Lemma 23.2. *Let $a : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}^\bullet$ be a locally bounded complex of $a^{-1}\mathcal{O}_S$-modules. Assume for all $n \in \mathbf{Z}$ the sheaf $\mathcal{F}^n$ is a flat $a^{-1}\mathcal{O}_S$-module and $\mathcal{F}^n$ has the structure of a* *quasi-coherent $\mathcal{O}_X$-module compatible with the given $a^{-1}\mathcal{O}_S$-module structure (but the differentials in the complex $\mathcal{F}^\bullet$ need not be $\mathcal{O}_X$-linear). Then the following hold*

1. $Ra_*\mathcal{F}^\bullet$ *is locally bounded,*
2. $Ra_*\mathcal{F}^\bullet$ *is in* $D_{QCoh}(\mathcal{O}_S)$*,*
3. $Ra_*\mathcal{F}^\bullet$ *locally has finite tor dimension,*
4. $\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Ra_*\mathcal{F}^\bullet = Ra_*(a^{-1}\mathcal{G} \otimes_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$ *for* $\mathcal{G} \in QCoh(\mathcal{O}_S)$*, and*
5. $K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Ra_*\mathcal{F}^\bullet = Ra_*(a^{-1}K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$ *for* $K \in D_{QCoh}(\mathcal{O}_S)$*.*

Proof. Parts (1), (2), (3) are local on S hence we may and do assume S is affine. Since a is quasi-compact we conclude that X is quasi-compact. Since $\mathcal{F}^\bullet$ is locally bounded, we conclude that $\mathcal{F}^\bullet$ is bounded.

For (1) and (2) we can use the first spectral sequence $R^pa_*\mathcal{F}^q \Rightarrow R^{p+q}a_*\mathcal{F}^\bullet$ of Derived Categories, Lemma 21.3. Combining Cohomology of Schemes, Lemma 4.5 and Homology, Lemma 24.11 we conclude.

Let us prove (3) by the induction principle of Cohomology of Schemes, Lemma 4.1. Namely, for a quasi-compact open of U of X consider the condition that $R(a|_U)_*(\mathcal{F}^\bullet|_U)$ has finite tor dimension. If U, V are quasi-compact open in X, then we have a relative Mayer-Vietoris distinguished triangle

$$R(a|_{U\cup V})_*\mathcal{F}^\bullet|_{U\cup V} \to R(a|_U)_*\mathcal{F}^\bullet|_U \oplus R(a|_V)_*\mathcal{F}^\bullet|_V \to R(a|_{U\cap V})_*\mathcal{F}^\bullet|_{U\cap V} \to$$

by Cohomology, Lemma 33.5. By the behaviour of tor amplitude in distinguished triangles (see Cohomology, Lemma 48.6) we see that if we know the result for U, V, $U \cap V$, then the result holds for $U \cup V$. This reduces us to the case where X is affine. In this case we have

$$Ra_*\mathcal{F}^\bullet = a_*\mathcal{F}^\bullet$$

by Leray's acyclicity lemma (Derived Categories, Lemma 16.7) and the vanishing of higher direct images of quasi-coherent modules under an affine morphism (Cohomology of Schemes, Lemma 2.3). Since $\mathcal{F}^n$ is S-flat by assumption and X affine, the modules $a_*\mathcal{F}^n$ are flat for all n. Hence $a_*\mathcal{F}^\bullet$ is a bounded complex of flat $\mathcal{O}_S$-modules and hence has finite tor dimension.

Proof of part (5). Denote $a' : (X, a^{-1}\mathcal{O}_S) \to (S, \mathcal{O}_S)$ the obvious flat morphism of ringed spaces. Part (5) says that

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Ra'_*\mathcal{F}^\bullet = Ra'_*(L(a')^*K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$$

Thus Cohomology, Equation (54.2.1) gives a functorial map from the left to the right and we want to show this map is an isomorphism. This question is local on S hence we may and do assume S is affine. The rest of the proof is *exactly* the same as the proof of Lemma 22.1 except that we have to show that the functor $K \mapsto Ra'_*(L(a')^*K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$ commutes with direct sums. This is where we will use $\mathcal{F}^n$ has the structure of a quasi-coherent $\mathcal{O}_X$-module. Namely, observe that $K \mapsto L(a')^*K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet$ commutes with arbitrary direct sums. Next, if $\mathcal{F}^\bullet$ consists of a single quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}^\bullet = \mathcal{F}^n[-n]$ then we have $L(a')^*G \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet = La^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}^n[-n]$, see Cohomology, Lemma 27.4. Hence in this case the commutation with direct sums follows from Lemma 4.5. Now, in general, since S is affine (hence X quasi-compact) and $\mathcal{F}^\bullet$ is locally bounded, we see that

$$\mathcal{F}^\bullet = (\mathcal{F}^a \to \ldots \to \mathcal{F}^b)$$

is bounded. Arguing by induction on $b-a$ and considering the distinguished triangle

$$\mathcal{F}^b[-b] \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^b) \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1}) \to \mathcal{F}^b[-b+1]$$

the proof of this part is finished. Some details omitted.

Proof of part (4). Let $a' : (X, a^{-1}\mathcal{O}_S) \to (S, \mathcal{O}_S)$ be as above. Since $\mathcal{F}^\bullet$ is a locally bounded complex of flat $a^{-1}\mathcal{O}_S$-modules we see the complex $a^{-1}\mathcal{G} \otimes_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet$ represents $L(a')^*\mathcal{G} \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet$ in $D(a^{-1}\mathcal{O}_S)$. Hence (4) follows from (5). □

Lemma 23.3. *Let $f : X \to Y$ be a morphism of schemes with $Y = \mathrm{Spec}(A)$ affine. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a finite affine open covering such that all the finite intersections $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ are affine. Let $\mathcal{F}^\bullet$ be a bounded complex of $f^{-1}\mathcal{O}_Y$-modules. Assume for all $n \in \mathbf{Z}$ the sheaf $\mathcal{F}^n$ is a flat $f^{-1}\mathcal{O}_Y$-module and $\mathcal{F}^n$ has the structure of a quasi-coherent $\mathcal{O}_X$-module compatible with the given $p^{-1}\mathcal{O}_Y$-module structure (but the differentials in the complex $\mathcal{F}^\bullet$ need not be $\mathcal{O}_X$-linear). Then the complex Tot($\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$) is K-flat as a complex of A-modules.*

Proof. We may write

$$\mathcal{F}^\bullet = (\mathcal{F}^a \to \ldots \to \mathcal{F}^b)$$

Arguing by induction on $b - a$ and considering the distinguished triangle

$$\mathcal{F}^b[-b] \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^b) \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1}) \to \mathcal{F}^b[-b+1]$$

and using More on Algebra, Lemma 59.5 we reduce to the case where $\mathcal{F}^\bullet$ consists of a single quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ placed in degree 0. In this case the Čech complex for $\mathcal{F}$ and $\mathcal{U}$ is homotopy equivalent to the alternating Čech complex, see Cohomology, Lemma 23.6. Since $U_{i_0 \ldots i_p}$ is always affine, we see that $\mathcal{F}(U_{i_0 \ldots i_p})$ is A-flat. Hence $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ is a bounded complex of flat A-modules and hence K-flat by More on Algebra, Lemma 59.7. □

Let X, Y, S, a, b, p, q, f be as in the introduction to this section. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Set $A = \Gamma(S, \mathcal{O}_S)$. Consider the map

$$R\Gamma(X, \mathcal{F}) \otimes^{\mathbf{L}}_A R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \qquad (23.3.1)$$

in $D(A)$. This map is constructed using the pullback maps $R\Gamma(X, \mathcal{F}) \to R\Gamma(X \times_S Y, p^*\mathcal{F})$ and $R\Gamma(Y, \mathcal{G}) \to R\Gamma(X \times_S Y, q^*\mathcal{G})$, the cup product constructed in Cohomology, Section 31, and the canonical map $p^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G} \to p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}$.

Lemma 23.4. *In the situation above the map (23.3.1) is an isomorphism if S is affine, $\mathcal{F}$ and $\mathcal{G}$ are S-flat and quasi-coherent and X and Y are quasi-compact with affine diagonal.*

Proof. We strongly urge the reader to read the proof of Varieties, Lemma 29.1 first. Choose finite affine open coverings $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : Y = \bigcup_{j \in J} V_j$. This determines an affine open covering $\mathcal{W} : X \times_S Y = \bigcup_{(i,j) \in I \times J} U_i \times_S V_j$. Note that $\mathcal{W}$ is a refinement of $\mathrm{pr}_1^{-1}\mathcal{U}$ and of $\mathrm{pr}_2^{-1}\mathcal{V}$. Thus by the discussion in Cohomology, Section 25 we obtain maps

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}) \quad \text{and} \quad \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, q^*\mathcal{G})$$

well defined up to homotopy and compatible with pullback maps on cohomology. In Cohomology, Equation (25.3.2) we have constructed a map of complexes

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{W}, q^*\mathcal{G})) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G})$$

which is compatible with the cup product on cohomology by Cohomology, Lemma 31.4. Combining the above we obtain a map of complexes

$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \tag{23.4.1}$$

We claim this is the map in the statement of the lemma, i.e., the source and target of this arrow are the same as the source and target of (23.3.1). Namely, by Cohomology of Schemes, Lemma 2.2 and Cohomology, Lemma 25.2 the canonical maps

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to R\Gamma(X, \mathcal{F}), \quad \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \to R\Gamma(Y, \mathcal{G})$$

and

$$\check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to R\Gamma(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G})$$

are isomorphisms. On the other hand, the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is K-flat by Lemma 23.3 and we conclude that $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}))$ represents the derived tensor product $R\Gamma(X, \mathcal{F}) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G})$ as claimed.

We still have to show that (23.4.1) is a quasi-isomorphism. We will do this using dimension shifting. Set $d(\mathcal{F}) = \max\{d \mid H^d(X, \mathcal{F}) \neq 0\}$. Assume $d(\mathcal{F}) > 0$. Set $U = \coprod_{i \in I} U_i$. This is an affine scheme as I is finite. Denote $j : U \to X$ the morphism which is the inclusion $U_i \to X$ on each U_i. Since the diagonal of X is affine, the morphism j is affine, see Morphisms, Lemma 11.11. It follows that $\mathcal{F}' = j_*j^*\mathcal{F}$ is S-flat, see Morphisms, Lemma 25.4. It also follows that $d(\mathcal{F}') = 0$ by combining Cohomology of Schemes, Lemmas 2.4 and 2.2. For all $x \in X$ we have $\mathcal{F}_x \to \mathcal{F}'_x$ is the inclusion of a direct summand: if $x \in U_i$, then $\mathcal{F}' \to (U_i \to X)_*\mathcal{F}|_{U_i}$ gives a splitting. We conclude that $\mathcal{F} \to \mathcal{F}'$ is injective and $\mathcal{F}'' = \mathcal{F}'/\mathcal{F}$ is S-flat as well. The short exact sequence $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of flat quasi-coherent $\mathcal{O}_X$-modules produces a short exact sequence of complexes

$$0 \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}') \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}'') \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to 0$$

and a short exact sequence of complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}' \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}'' \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to 0$$

Moreover, the maps (23.4.1) between these are compatible with these short exact sequences. Hence it suffices to prove (23.4.1) is an isomorphism for $\mathcal{F}'$ and $\mathcal{F}''$. Finally, we have $d(\mathcal{F}'') < d(\mathcal{F})$. In this way we reduce to the case $d(\mathcal{F}) = 0$.

Arguing in the same fashion for $\mathcal{G}$ we find that we may assume that both $\mathcal{F}$ and $\mathcal{G}$ have nonzero cohomology only in degree 0. Observe that this means that $\Gamma(X, \mathcal{F})$ is quasi-isomorphic to the K-flat complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of A-modules sitting in degrees ≥ 0. It follows that $\Gamma(X, \mathcal{F})$ is a flat A-module (because we can compute higher Tor's against this module by tensoring with the Cech complex). Let $V \subset Y$ be an affine open. Consider the affine open covering $\mathcal{U}_V : X \times_S V = \bigcup_{i \in I} U_i \times_S V$. It is immediate that

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \mathcal{G}(V) = \check{\mathcal{C}}^\bullet(\mathcal{U}_V, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G})$$

(equality of complexes). By the flatness of $\mathcal{G}(V)$ over A we see that $\Gamma(X, \mathcal{F}) \otimes_A \mathcal{G}(V) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \mathcal{G}(V)$ is a quasi-isomorphism. Since the sheafification of $V \mapsto \check{\mathcal{C}}^\bullet(\mathcal{U}_V, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G})$ represents $Rq_*(p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G})$ by Cohomology of Schemes, Lemma 7.1 we conclude that

$$Rq_*(p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G}) \cong \Gamma(X, \mathcal{F}) \otimes_A \mathcal{G}$$

on Y where the notation on the right hand side indicates the module

$$b^*\widetilde{\Gamma(X, \mathcal{F})} \otimes_{\mathcal{O}_Y} \mathcal{G}$$

Using the Leray spectral sequence for q we find

$$H^n(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G}) = H^n(Y, b^*\widetilde{\Gamma(X, \mathcal{F})} \otimes_{\mathcal{O}_Y} \mathcal{G})$$

Using Lemma 22.1 for the morphism $b : Y \to S = \mathrm{Spec}(A)$ and using that $\Gamma(X, \mathcal{F})$ is A-flat we conclude that $H^n(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G})$ is zero for $n > 0$ and isomorphic to $H^0(X, \mathcal{F}) \otimes_A H^0(Y, \mathcal{G})$ for $n = 0$. Of course, here we also use that $\mathcal{G}$ only has cohomology in degree 0. This finishes the proof (except that we should check that the isomorphism is indeed given by cup product in degree 0; we omit the verification). $\square$

Remark 23.5. Let $S = \mathrm{Spec}(A)$ be an affine scheme. Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules and let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $\xi \in H^i(X, \mathcal{G})$ with pullback $p^*\xi \in H^i(X \times_S Y, p^*\mathcal{G})$. Then the following diagram is commutative

$$\begin{tikzcd}
R\Gamma(X, \mathcal{F})[-i] \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{E}) \arrow[r, "\xi \otimes \mathrm{id}"'] \arrow[d] & R\Gamma(X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{E}) \arrow[d] \\
R\Gamma(X \times_S Y, p^*\mathcal{F} \otimes q^*\mathcal{E})[-i] \arrow[r, "p^*\xi"] & R\Gamma(X \times_S Y, p^*(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes q^*\mathcal{E})
\end{tikzcd}$$

where the unadorned tensor products are over $\mathcal{O}_{X \times_S Y}$. The horizontal arrows are from Cohomology, Remark 31.2 and the vertical arrows are (23.0.2) hence given by pulling back followed by cup product on $X \times_S Y$. The diagram commutes because the global cup product (on $X \times_S Y$ with the sheaves $p^*\mathcal{G}$, $p^*\mathcal{F}$, and $q^*\mathcal{E}$) is associative, see Cohomology, Lemma 31.5.

24. Künneth formula, III

Let X, Y, S, a, b, p, q, f be as in the introduction to Section 23. In this section, given an $\mathcal{O}_X$-module $\mathcal{F}$ and a $\mathcal{O}_Y$-module $\mathcal{G}$ let us set

$$\mathcal{F} \boxtimes \mathcal{G} = p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}$$

Note that, contrary to what happens in a future section, we take the nonderived tensor product here.

On X let $\mathcal{F}^\bullet$ be a complex of sheaves of abelian groups whose terms are quasi-coherent $\mathcal{O}_X$-modules such that the differentials $d^i_{\mathcal{F}} : \mathcal{F}^i \to \mathcal{F}^{i+1}$ are differential operators on X/S of finite order, see Morphisms, Section 33. Simlarly, on Y let $\mathcal{G}^\bullet$ be a complex of sheaves of abelian groups whose terms are quasi-coherent $\mathcal{O}_Y$-modules such that the differentials $d^j_{\mathcal{G}} : \mathcal{G}^j \to \mathcal{G}^{j+1}$ are differential operators on Y/S of finite order. Applying the construction of Morphisms, Lemma 33.2 we obtain a double complex

$$\begin{tikzcd}
\ldots & \ldots & \ldots & \ldots \\
\ldots \arrow[r] & \mathcal{F}^i \boxtimes \mathcal{G}^{j+1} \arrow[r, "d_1^{i,j+1}"] \arrow[u] & \mathcal{F}^{i+1} \boxtimes \mathcal{G}^{j+1} \arrow[r] \arrow[u] & \ldots \\
\ldots \arrow[r] & \mathcal{F}^i \boxtimes \mathcal{G}^j \arrow[r, "d_1^{i,j}"] \arrow[u, "d_2^{i,j}"] & \mathcal{F}^{i+1} \boxtimes \mathcal{G}^j \arrow[r] \arrow[u, "d_2^{i+1,j}"'] & \ldots \\
\ldots & \ldots \arrow[u] & \ldots \arrow[u] & \ldots
\end{tikzcd}$$

of quasi-coherent modules whose maps are differential operators of finite order on $X \times_S Y/S$. Please see the discussion in Morphisms, Remark 33.3 and Homology, Example 18.2. To be explicit, we set

$$d_1^{i,j} = d_{\mathcal{F}}^i \boxtimes 1 \quad \text{and} \quad d_2^{i,j} = 1 \boxtimes d_{\mathcal{G}}^j$$

In the discussion below the notation

$$\mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$$

refers to the total complex associated to this double complex. This complex has terms which are quasi-coherent $\mathcal{O}_{X \times_S Y}$-modules and whose differentials are differential operators of finite order on $X \times_S Y/S$.

In the situation above there exists a "relative cup product" map

$$Ra_*(\mathcal{F}^\bullet) \otimes_{\mathcal{O}_S}^{\mathbf{L}} Rb_*(\mathcal{G}^\bullet) \longrightarrow Rf_*\left(\mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)\right) \tag{24.0.1}$$

Namely, we can construct this map by combining

(1) $Ra_*(\mathcal{F}^\bullet) \to Rf_*(p^{-1}\mathcal{F}^\bullet)$,
(2) $Rb_*(\mathcal{G}^\bullet) \to Rf_*(q^{-1}\mathcal{G}^\bullet)$,
(3) $Rf_*(p^{-1}\mathcal{F}^\bullet) \otimes_{\mathcal{O}_S}^{\mathbf{L}} Rf_*(q^{-1}\mathcal{G}^\bullet) \to Rf_*(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} q^{-1}\mathcal{G}^\bullet)$,
(4) $p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} q^{-1}\mathcal{G}^\bullet \to \mathrm{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet)$
(5) $\mathrm{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet) \to \mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$.

Maps (1) and (2) are pullback maps, map (3) is the relative cup product, see Cohomology, Remark 28.7, map (4) compares the derived and nonderived tensor products, and map (5) is given by the obvious maps $p^{-1}\mathcal{F}^i \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^j \to \mathcal{F}^i \boxtimes \mathcal{G}^j$ on the underlying double complexes.

Set $A = \Gamma(S, \mathcal{O}_S)$. There exists a "global cup product" map

$$R\Gamma(X, \mathcal{F}^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G}^\bullet) \longrightarrow R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)) \tag{24.0.2}$$

in $D(A)$. This is constructed similarly to the relative cup product above using

(1) $R\Gamma(X, \mathcal{F}^\bullet) \to R\Gamma(X \times_S Y, p^{-1}\mathcal{F}^\bullet)$
(2) $R\Gamma(Y, \mathcal{G}^\bullet) \to R\Gamma(X \times_S Y, q^{-1}\mathcal{G}^\bullet)$,
(3) $R\Gamma(X \times_S Y, p^{-1}\mathcal{F}^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(X \times_S Y, q^{-1}\mathcal{G}^\bullet) \to R\Gamma(X \times_S Y, p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} q^{-1}\mathcal{G}^\bullet)$,
(4) $p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} q^{-1}\mathcal{G}^\bullet \to \mathrm{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet)$
(5) $\mathrm{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet) \to \mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$.

Here maps (1) and (2) are the pullback maps, map (3) is the cup product constructed in Cohomology, Section 31. Maps (4) and (5) are as indicated in the previous paragraph.

Lemma 24.1. *In the situation above the cup product (24.0.2) is an isomorphism in $D(A)$ if the following assumptions hold*

1. $S = \mathrm{Spec}(A)$ *is affine,*
2. X *and* Y *are quasi-compact with affine diagonal,*
3. $\mathcal{F}^\bullet$ *is bounded,*
4. $\mathcal{G}^\bullet$ *is bounded below,*
5. $\mathcal{F}^n$ *is S-flat, and*
6. $\mathcal{G}^m$ *is S-flat.*

Proof. We will use the notation $\mathcal{A}_{X/S}$ and $\mathcal{A}_{Y/S}$ introduced in Morphisms, Remark 33.3. Suppose that we have maps of complexes

$$\mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \to \mathcal{F}_1^\bullet[1]$$

in the category $\mathcal{A}_{X/S}$. Then by the functoriality of the cup product we obtain a commutative diagram

$$\begin{tikzcd}
R\Gamma(X, \mathcal{F}_1^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G}^\bullet) \arrow[r] \arrow[d] & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_1^\bullet \boxtimes \mathcal{G}^\bullet)) \arrow[d] \\
R\Gamma(X, \mathcal{F}_2^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G}^\bullet) \arrow[r] \arrow[d] & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_2^\bullet \boxtimes \mathcal{G}^\bullet)) \arrow[d] \\
R\Gamma(X, \mathcal{F}_3^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G}^\bullet) \arrow[r] \arrow[d] & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_3^\bullet \boxtimes \mathcal{G}^\bullet)) \arrow[d] \\
R\Gamma(X, \mathcal{F}_1^\bullet[1]) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G}^\bullet) \arrow[r] & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_1^\bullet[1] \boxtimes \mathcal{G}^\bullet))
\end{tikzcd}$$

If the original maps form a distinguished triangle in the homotopy category of $\mathcal{A}_{X/S}$, then the columns of this diagram form distinguished triangles in $D(A)$.

In the situation of the lemma, suppose that $\mathcal{F}^n = 0$ for $n < i$. Then we may consider the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq i+1}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \mathcal{F}^i[-i] \to 0$$

where the truncation is as in Homology, Section 15. This produces the distinguished triangle

$$\sigma_{\geq i+1}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \mathcal{F}^i[-i] \to (\sigma_{\geq i+1}\mathcal{F}^\bullet)[1]$$

in the homotopy category of $\mathcal{A}_{X/S}$ where the final arrow is given by the boundary map $\mathcal{F}^i \to \mathcal{F}^{i+1}$. It follows from the discussion above that it suffices to prove the lemma for $\mathcal{F}^i[-i]$ and $\sigma_{\geq i+1}\mathcal{F}^\bullet$. Since $\sigma_{\geq i+1}\mathcal{F}^\bullet$ has fewer nonzero terms, by induction, if we can prove the lemma if $\mathcal{F}^\bullet$ is nonzero only in single degree, then the lemma follows. Thus we may assume $\mathcal{F}^\bullet$ is nonzero only in one degree.

Assume $\mathcal{F}^\bullet$ is the complex which has an S-flat quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ sitting in degree 0 and is zero in other degrees. Observe that $R\Gamma(X, \mathcal{F})$ has finite tor dimension by Lemma 23.2 for example. Say it has tor amplitude in $[i, j]$. Pick $N \gg 0$ and consider the distinguished triangle

$$\sigma_{\geq N+1}\mathcal{G}^\bullet \to \mathcal{G}^\bullet \to \sigma_{\leq N}\mathcal{G}^\bullet \to (\sigma_{\geq N+1}\mathcal{G}^\bullet)[1]$$

in the homotopy category of $\mathcal{A}_{Y/S}$. Now observe that both

$$R\Gamma(X, \mathcal{F}) \otimes_A^{\mathbf{L}} R\Gamma(Y, \sigma_{\geq N+1}\mathcal{G}^\bullet) \quad \text{and} \quad R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F} \boxtimes \sigma_{\geq N+1}\mathcal{G}^\bullet))$$

have vanishing cohomology in degrees $\leq N + i$. Thus, using the arguments given above, if we want to prove our statement in a given degree, then we may assume $\mathcal{G}^\bullet$ is bounded. Repeating the arguments above one more time we may also assume $\mathcal{G}^\bullet$ is nonzero only in one degree. This case is handled by Lemma 23.4. $\square$

25. Künneth formula for Ext

Consider a cartesian diagram of schemes

$$\begin{tikzcd} & X \times_S Y \arrow[ld, "p"'] \arrow[d, "f"] \arrow[rd, "q"] & \\ X \arrow[rd, "a"'] & S & Y \arrow[ld, "b"] \\ & & \end{tikzcd}$$

For $K \in D(\mathcal{O}_X)$ and $M \in D(\mathcal{O}_Y)$ in this section let us define

$$K \boxtimes M = Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M$$

We claim there is a canonical map

(25.0.1)
$$Ra_*R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*R\mathcal{H}om(M, M') \longrightarrow Rf_*(R\mathcal{H}om(K \boxtimes M, K' \boxtimes M'))$$

for $K, K' \in D(\mathcal{O}_X)$ and $M, M' \in D(\mathcal{O}_Y)$. Namely, we can take the map adjoint to the map

$$\begin{gathered} Lf^*\left(Ra_*R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*R\mathcal{H}om(M, M')\right) = \\ Lf^*Ra_*R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lf^*Rb_*R\mathcal{H}om(M, M') = \\ Lp^*La^*Ra_*R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*Lb^*Rb_*R\mathcal{H}om(M, M') \to \\ Lp^*R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*R\mathcal{H}om(M, M') \to \\ R\mathcal{H}om(Lp^*K, Lp^*K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} R\mathcal{H}om(Lq^*M, Lq^*M') \to \\ R\mathcal{H}om(K \boxtimes M, K' \boxtimes M') \end{gathered}$$

Here the first equality is compatibility of pullbacks with tensor products, Cohomology, Lemma 27.3. The second equality is $f = a \circ p = b \circ q$ and composition of pullbacks, Cohomology, Lemma 27.2. The first arrow is given by the adjunction maps $La^*Ra_* \to \mathrm{id}$ and $Lb^*Rb_* \to \mathrm{id}$ because pushforward and pullback are adjoint, Cohomology, Lemma 28.1. The second arrow is given by Cohomology, Remark 42.13. The third and final arrow is Cohomology, Remark 42.10. A simple special case of this is the following result.

Lemma 25.1. *In the situation above, assume a and b are quasi-compact and quasi-separated and X and Y are tor independent over S. If K is perfect, $K' \in D_{QCoh}(\mathcal{O}_X)$, M is perfect, and $M' \in D_{QCoh}(\mathcal{O}_Y)$, then (25.0.1) is an isomorphism.*

Proof. In this case we have $R\mathcal{H}om(K, K') = K' \otimes^{\mathbf{L}} K^\vee$, $R\mathcal{H}om(M, M') = M' \otimes^{\mathbf{L}} M^\vee$, and

$$R\mathcal{H}om(K \boxtimes M, K' \boxtimes M') = (K' \otimes^{\mathbf{L}} K^\vee) \boxtimes (M' \otimes^{\mathbf{L}} M^\vee)$$

See Cohomology, Lemma 50.5 and we also use that being perfect is preserved by pullback and by tensor products. Hence this case follows from Lemma 23.1. (We omit the verification that with these identifications we obtain the same map.) $\square$

26. Cohomology and base change, V

In Section 22 we saw a base change theorem holds when the morphisms are tor independent. Even in the affine case there cannot be a base change theorem without such a condition, see More on Algebra, Section 61. In this section we analyze when one can get a base change result "one complex at a time".

To make this work, suppose we have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of schemes (usually we will assume it is cartesian). Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. For a point $x' \in X'$ set $x = g'(x') \in X$, $s' = f'(x') \in S'$ and $s = f(x) = g(s')$. Then we can consider the maps

$$K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'} \to K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'} \to K'_{x'}$$

where the first arrow is More on Algebra, Equation (61.0.1) and the second comes from $(L(g')^*K)_{x'} = K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$ and the given map $L(g')^*K \to K'$. For each $i \in \mathbf{Z}$ we obtain a $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$-module structure on $H^i(K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'})$. Putting everything together we obtain canonical maps

$$H^i(K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}) \otimes_{(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'})} \mathcal{O}_{X',x'} \longrightarrow H^i(K'_{x'}) \tag{26.0.1}$$

of $\mathcal{O}_{X',x'}$-modules.

Lemma 26.1. *Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. The following are equivalent

1. *for any $x' \in X'$ and $i \in \mathbf{Z}$ the map (26.0.1) is an isomorphism,*
2. *for $U \subset X$, $V' \subset S'$ affine open both mapping into the affine open $V \subset S$ with $U' = V' \times_V U$ the composition*
$$R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}_S(U)} \mathcal{O}_{S'}(V') \to R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X(U)} \mathcal{O}_{X'}(U') \to R\Gamma(U', K')$$
is an isomorphism in $D(\mathcal{O}_{S'}(V'))$, and
3. *there is a set I of quadruples U_i, V'_i, V_i, U'_i, $i \in I$ as in (2) with $X' = \bigcup U'_i$.*

Proof. The second arrow in (2) comes from the equality

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X(U)} \mathcal{O}_{X'}(U') = R\Gamma(U', L(g')^*K)$$

of Lemma 3.8 and the given arrow $L(g')^*K \to K'$. The first arrow of (2) is More on Algebra, Equation (61.0.1). It is clear that (2) implies (3). Observe that (1) is local on X'. Therefore it suffices to show that if X, S, S', X' are affine, then (1) is equivalent to the condition that

$$R\Gamma(X, K) \otimes^{\mathbf{L}}_{\mathcal{O}_S(S)} \mathcal{O}_{S'}(S') \to R\Gamma(X, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X(X)} \mathcal{O}_{X'}(X') \to R\Gamma(X', K')$$

is an isomorphism in $D(\mathcal{O}_{S'}(S'))$. Say $S = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(R')$, $X' = \mathrm{Spec}(A')$, K corresponds to the complex $M^\bullet$ of A-modules, and K' corresponds to the complex $N^\bullet$ of A'-modules. Note that $A' = A \otimes_R R'$. The condition above is that the composition

$$M^\bullet \otimes^{\mathbf{L}}_R R' \to M^\bullet \otimes^{\mathbf{L}}_A A' \to N^\bullet$$

is an isomorphism in $D(R')$. Equivalently, it is that for all $i \in \mathbf{Z}$ the map

$$H^i(M^\bullet \otimes^{\mathbf{L}}_R R') \to H^i(M^\bullet \otimes^{\mathbf{L}}_A A') \to H^i(N^\bullet)$$

is an isomorphism. Observe that this is a map of $A \otimes_R R'$-modules, i.e., of A'-modules. On the other hand, (1) is the requirement that for compatible primes $\mathfrak{q}' \subset A'$, $\mathfrak{q} \subset A$, $\mathfrak{p}' \subset R'$, $\mathfrak{p} \subset R$ the composition

$$H^i(M^\bullet_{\mathfrak{q}} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} R'_{\mathfrak{p}'}) \otimes_{(A_\mathfrak{q} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'})} A'_{\mathfrak{q}'} \to H^i(M^\bullet_{\mathfrak{q}} \otimes^{\mathbf{L}}_{A_\mathfrak{q}} A'_{\mathfrak{q}'}) \to H^i(N^\bullet_{\mathfrak{q}'})$$

is an isomorphism. Since

$$H^i(M^\bullet_{\mathfrak{q}} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} R'_{\mathfrak{p}'}) \otimes_{(A_\mathfrak{q} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'})} A'_{\mathfrak{q}'} = H^i(M^\bullet \otimes^{\mathbf{L}}_R R') \otimes_{A'} A'_{\mathfrak{q}'}$$

is the localization at $\mathfrak{q}'$, we see that these two conditions are equivalent by Algebra, Lemma 23.1. □

Lemma 26.2. *Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If

1. *the equivalent conditions of Lemma 26.1 hold, and*
2. *f is quasi-compact and quasi-separated,*

then the composition $Lg^*Rf_*K \to Rf'_*L(g')^*K \to Rf'_*K'$ is an isomorphism.

Proof. We could prove this using the same method as in the proof of Lemma 22.5 but instead we will prove it using the induction principle and relative Mayer-Vietoris.

To check the map is an isomorphism we may work locally on S'. Hence we may assume $g : S' \to S$ is a morphism of affine schemes. In particular X is a quasi-compact and quasi-separated scheme. We will use the induction principle of Cohomology of Schemes, Lemma 4.1 to prove that for any quasi-compact open $U \subset X$ the similarly constructed map $Lg^*R(U \to S)_*K|_U \to R(U' \to S')_*K'|_{U'}$ is an isomorphism. Here $U' = (g')^{-1}(U)$.

If $U \subset X$ is an affine open, then we find that the result is true by assumption, see Lemma 26.1 part (2) and the translation into algebra afforded to us by Lemmas 3.5 and 3.8.

The induction step. Suppose that $X = U \cup V$ is an open covering with U, V, $U \cap V$ quasi-compact such that the result holds for U, V, and $U \cap V$. Denote $a = f|_U$, $b = f|_V$ and $c = f|_{U \cap V}$. Let $a' : U' \to S'$, $b' : V' \to S'$ and $c' : U' \cap V' \to S'$ be the base changes of a, b, and c. Using the distinguished triangles from relative Mayer-Vietoris (Cohomology, Lemma 33.5) we obtain a commutative diagram

$$\begin{tikzcd}
Lg^*Rf_*K \arrow[r] \arrow[d] & Rf'_*K' \arrow[d] \\
Lg^*Ra_*K|_U \oplus Lg^*Rb_*K|_V \arrow[r] \arrow[d] & Ra'_*K'|_{U'} \oplus Rb'_*K'|_{V'} \arrow[d] \\
Lg^*Rc_*K|_{U \cap V} \arrow[r] \arrow[d] & Rc'_*K'|_{U' \cap V'} \arrow[d] \\
Lg^*Rf_*K[1] \arrow[r] & Rf'_*K'[1]
\end{tikzcd}$$

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 4.3) and the proof of the lemma is finished. $\square$

Lemma 26.3. *Let*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If the equivalent conditions of Lemma 26.1 hold, then

1. *for $E \in D_{QCoh}(\mathcal{O}_X)$ the equivalent conditions of Lemma 26.1 hold for $L(g')^*(E \otimes^{\mathbf{L}} K) \to L(g')^*E \otimes^{\mathbf{L}} K'$,*
2. *if E in $D(\mathcal{O}_X)$ is perfect the equivalent conditions of Lemma 26.1 hold for $L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, K')$, and*
3. *if K is bounded below and E in $D(\mathcal{O}_X)$ pseudo-coherent the equivalent conditions of Lemma 26.1 hold for $L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, K')$.*

Proof. The statement makes sense as the complexes involved have quasi-coherent cohomology sheaves by Lemmas 3.8, 3.9, and 10.8 and Cohomology, Lemmas 47.3 and 49.6. Having said this, we can check the maps (26.0.1) are isomorphisms in case (1) by computing the source and target of (26.0.1) using the transitive property of tensor product, see More on Algebra, Lemma 59.15. The map in (2) and (3) is the composition

$$L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, L(g')^*K) \to R\mathcal{H}om(L(g')^*E, K')$$

where the first arrow is Cohomology, Remark 42.13 and the second arrow comes from the given map $L(g')^*K \to K'$. To prove the maps (26.0.1) are isomorphisms one represents E_x by a bounded complex of finite projective $\mathcal{O}_{X,x}$-modules in case

(2) or by a bounded above complex of finite free modules in case (3) and computes the source and target of the arrow. Some details omitted. □

Lemma 26.4. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $\mathcal{G}^\bullet$ be a bounded above complex of quasi-coherent $\mathcal{O}_X$-modules flat over S. Then formation of*

$$Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : S' \to S$ be a morphism of schemes and consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

in other words $X' = S' \times_S X$. The lemma asserts that

$$Lg^* Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet) \longrightarrow Rf'_*\left(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} (g')^*\mathcal{G}^\bullet\right)$$

is an isomorphism. Observe that on the right hand side we do **not** use the derived pullback on $\mathcal{G}^\bullet$. To prove this, we apply Lemmas 26.2 and 26.3 to see that it suffices to prove the canonical map

$$L(g')^*\mathcal{G}^\bullet \to (g')^*\mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 26.1. This follows by checking the condition on stalks, where it immediately follows from the fact that $\mathcal{G}^\bullet_x \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ computes the derived tensor product by our assumptions on the complex $\mathcal{G}^\bullet$. □

Lemma 26.5. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let E be an object of $D(\mathcal{O}_X)$. Let $\mathcal{G}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. If*

(1) *E is perfect, $\mathcal{G}^\bullet$ is a bounded above, and $\mathcal{G}^n$ is flat over S, or*
(2) *E is pseudo-coherent, $\mathcal{G}^\bullet$ is bounded, and $\mathcal{G}^n$ is flat over S,*

then formation of

$$Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : S' \to S$ be a morphism of schemes and consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

in other words $X' = S' \times_S X$. The lemma asserts that

$$Lg^* Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet) \longrightarrow R(f')_* R\mathcal{H}om(L(g')^*E, (g')^*\mathcal{G}^\bullet)$$

is an isomorphism. Observe that on the right hand side we do **not** use the derived pullback on $\mathcal{G}^\bullet$. To prove this, we apply Lemmas 26.2 and 26.3 to see that it suffices to prove the canonical map

$$L(g')^*\mathcal{G}^\bullet \to (g')^*\mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 26.1. This was shown in the proof of Lemma 26.4. $\square$

27. Producing perfect complexes

The following lemma is our main technical tool for producing perfect complexes. Later versions of this result will reduce to this by Noetherian approximation, see Section 30.

Lemma 27.1. *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ such that*

1. *$E \in D^b_{Coh}(\mathcal{O}_X)$,*
2. *the support of $H^i(E)$ is proper over S for all i, and*
3. *E has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$.*

Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$.

Proof. By Lemma 11.3 we see that Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_S)$. Hence Rf_*E is pseudo-coherent (Lemma 10.3). Hence it suffices to show that Rf_*E has finite tor dimension, see Cohomology, Lemma 49.5. By Lemma 10.6 it suffices to check that $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}$ has universally bounded cohomology for all quasi-coherent sheaves $\mathcal{F}$ on S. Bounded from above is clear as $Rf_*(E)$ is bounded from above. Let $T \subset X$ be the union of the supports of $H^i(E)$ for all i. Then T is proper over S by assumptions (1) and (2), see Cohomology of Schemes, Lemma 26.6. In particular there exists a quasi-compact open $X' \subset X$ containing T. Setting $f' = f|_{X'}$ we have $Rf_*(E) = Rf'_*(E|_{X'})$ because E restricts to zero on $X \setminus T$. Thus we may replace X by X' and assume f is quasi-compact. Moreover, f is quasi-separated by Morphisms, Lemma 15.7. Now

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F} = Rf_*\left(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}\right) = Rf_*\left(E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F}\right)$$

by Lemma 22.1 and Cohomology, Lemma 27.4. By assumption (3) the complex $E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F}$ has cohomology sheaves in a given finite range, say $[a, b]$. Then Rf_* of it has cohomology in the range $[a, \infty)$ and we win. $\square$

Lemma 27.2. *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules flat over S with support proper over S. Then $K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$ is a perfect object of $D(\mathcal{O}_S)$.*

Proof. The object K is perfect by Lemma 27.1. We check the lemma applies: Locally E is isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules. Hence locally $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$ is isomorphic to a finite complex whose terms are of the form

$$\bigoplus\nolimits_{i=a,\ldots,b} (\mathcal{G}^i)^{\oplus r_i}$$

for some integers $a, b, r_a, \ldots, r_b$. This immediately implies the cohomology sheaves $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$ are coherent. The hypothesis on the tor dimension also follows as $\mathcal{G}^i$ is flat over $f^{-1}\mathcal{O}_S$. $\square$

Lemma 27.3. *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules flat over S with support proper over S. Then $K = Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet)$ is a perfect object of $D(\mathcal{O}_S)$.*

Proof. Since E is a perfect complex there exists a dual perfect complex $E^\vee$, see Cohomology, Lemma 50.5. Observe that $R\mathcal{H}om(E, \mathcal{G}^\bullet) = E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$. Thus the perfectness of K follows from Lemma 27.2. $\square$

We will generalize the following lemma to flat and proper morphisms over general bases in Lemma 30.4 and to perfect proper morphisms in More on Morphisms, Lemma 61.13.

Lemma 27.4. *Let S be a Noetherian scheme. Let $f : X \to S$ be a flat proper morphism of schemes. Let $E \in D(\mathcal{O}_X)$ be perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$.*

Proof. We claim that Lemma 27.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on X. Thus we may assume X and S affine and E represented by a strictly perfect complex of $\mathcal{O}_X$-modules. Since $\mathcal{O}_X$ is flat as a sheaf of $f^{-1}\mathcal{O}_S$-modules we find that condition (3) is satisfied. $\square$

28. A projection formula for Ext

Lemma 28.3 (or similar results in the literature) is sometimes used to verify one of Artin's criteria for Quot functors, Hilbert schemes, and other moduli problems. Suppose that $f : X \to S$ is a proper, flat, finitely presented morphism of schemes and $E \in D(\mathcal{O}_X)$ is perfect. Here the lemma says

$$\mathrm{Ext}^i_X(E, f^*\mathcal{F}) = \mathrm{Ext}^i_S((Rf_*E^\vee)^\vee, \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S. Writing it this way makes it look like a projection formula for Ext and indeed the result follows rather easily from Lemma 22.1.

Lemma 28.1. *Assumptions and notation as in Lemma 27.2. Then there are functorial isomorphisms*

$$H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \longrightarrow H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps (see proof).

Proof. We have

$$\mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F} = \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}$$

the first equality by Cohomology, Lemma 27.4, the second as $\mathcal{G}^n$ is a flat $f^{-1}\mathcal{O}_S$-module, and the third by definition of pullbacks. Hence we obtain

$$\begin{aligned} H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})) &= H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}) \\ &= H^i(S, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})) \\ &= H^i(S, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \\ &= H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \end{aligned}$$

The first equality by the above, the second by Leray (Cohomology, Lemma 13.1), and the third equality by Lemma 22.1. The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of quasi-coherent $\mathcal{O}_S$-modules, the isomorphisms fit into commutative diagrams

$$\begin{tikzcd} H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3) \arrow[r] \arrow[d, "\delta"'] & H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3)) \arrow[d, "\delta"] \\ H^{i+1}(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1) \arrow[r] & H^{i+1}(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1)) \end{tikzcd}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

of complexes of $\mathcal{O}_X$-modules. This sequence is exact because $\mathcal{G}^n$ is flat over S. We omit the verification of the commutativity of the displayed diagram. $\square$

Lemma 28.2. *Assumptions and notation as in Lemma 27.3. Then there are functorial isomorphisms*

$$H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps (see proof).

Proof. As in the proof of Lemma 27.3 let $E^\vee$ be the dual perfect complex and recall that $K = Rf_*(E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$. Since we also have

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) = H^i(X, E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

by construction of $E^\vee$, the existence of the isomorphisms follows from Lemma 28.1 applied to $E^\vee$ and $\mathcal{G}^\bullet$. The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms fit into commutative diagrams

$$\begin{tikzcd} H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3) \arrow[r] \arrow[d, "\delta"'] & \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3) \arrow[d, "\delta"] \\ H^{i+1}(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1) \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1) \end{tikzcd}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

of complexes. This sequence is exact because $\mathcal{G}$ is flat over S. We omit the verification of the commutativity of the displayed diagram. $\square$

Lemma 28.3. *Let $f : X \to S$ be a morphism of schemes, $E \in D(\mathcal{O}_X)$ and $\mathcal{G}^\bullet$ a complex of $\mathcal{O}_X$-modules. Assume*

1. *S is Noetherian,*
2. *f is locally of finite type,*
3. *$E \in D^-_{Coh}(\mathcal{O}_X)$,*

(4) *$\mathcal{G}^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$-modules flat over S with support proper over S.*

Then the following two statements are true

(A) *for every $m \in \mathbf{Z}$ there exists a perfect object K of $D(\mathcal{O}_S)$ and functorial maps*

$$\alpha^i_\mathcal{F} : \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps (see proof) such that $\alpha^i_\mathcal{F}$ is an isomorphism for $i \leq m$

(B) *there exists a pseudo-coherent $L \in D(\mathcal{O}_S)$ and functorial isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{O}_S}(L, \mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps.

Proof. Proof of (A). Suppose $\mathcal{G}^i$ is nonzero only for $i \in [a, b]$. We may replace X by a quasi-compact open neighbourhood of the union of the supports of $\mathcal{G}^i$. Hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, -m-1+a)$ (possible by Theorem 14.6). Then the induced map

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(P, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

is an isomorphism for $i \leq m$. Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of

$$\mathrm{Ext}^i_{\mathcal{O}_X}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \quad \text{resp.} \quad \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

where C is the cone of $P \to E$. Since C has vanishing cohomology sheaves in degrees $\geq -m-1+a$ these Ext-groups are zero for $i \leq m+1$ by Derived Categories, Lemma 27.3. This reduces us to the case that E is a perfect complex which is Lemma 28.2. The statement on boundaries is explained in the proof of Lemma 28.2.

Proof of (B). As in the proof of (A) we may assume X is Noetherian. Observe that E is pseudo-coherent by Lemma 10.3. By Lemma 19.1 we can write $E = \mathrm{hocolim} E_n$ with E_n perfect and $E_n \to E$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $E_n^\vee$ be the dual perfect complex (Cohomology, Lemma 50.5). We obtain an inverse system $\ldots \to E_3^\vee \to E_2^\vee \to E_1^\vee$ of perfect objects. This in turn gives rise to an inverse system

$$\ldots \to K_3 \to K_2 \to K_1 \quad \text{with} \quad K_n = Rf_*(E_n^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

perfect on S, see Lemma 27.2. By Lemma 28.2 and its proof and by the arguments in the previous paragraph (with $P = E_n$) for any quasi-coherent $\mathcal{F}$ on S we have functorial canonical maps

$$\begin{tikzcd} & \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \arrow[dl] \arrow[dr] & \\ H^i(S, K_{n+1} \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \arrow[rr] & & H^i(S, K_n \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \end{tikzcd}$$

which are isomorphisms for $i \leq n + a$. Let $L_n = K_n^\vee$ be the dual perfect complex. Then we see that $L_1 \to L_2 \to L_3 \to \ldots$ is a system of perfect objects in $D(\mathcal{O}_S)$ such that for any quasi-coherent $\mathcal{F}$ on S the maps

$$\mathrm{Ext}^i_{\mathcal{O}_S}(L_{n+1}, \mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_S}(L_n, \mathcal{F})$$

are isomorphisms for $i \leq n + a - 1$. This implies that $L_n \to L_{n+1}$ induces an isomorphism on truncations $\tau_{\geq -n-a+2}$ (hint: take cone of $L_n \to L_{n+1}$ and look at its last nonvanishing cohomology sheaf). Thus $L = \text{hocolim} L_n$ is pseudo-coherent, see Lemma 19.1. The mapping property of homotopy colimits gives that $\text{Ext}^i_{\mathcal{O}_S}(L, \mathcal{F}) = \text{Ext}^i_{\mathcal{O}_S}(L_n, \mathcal{F})$ for $i \leq n + a - 3$ which finishes the proof. □

Remark 28.4. The pseudo-coherent complex L of part (B) of Lemma 28.3 is canonically associated to the situation. For example, formation of L as in (B) is compatible with base change. In other words, given a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of schemes we have canonical functorial isomorphisms

$$\text{Ext}^i_{\mathcal{O}_{S'}}(Lg^*L, \mathcal{F}') \longrightarrow \text{Ext}^i_{\mathcal{O}_X}(L(g')^*E, (g')^*\mathcal{G}^\bullet \otimes_{\mathcal{O}_{X'}} (f')^*\mathcal{F}')$$

for $\mathcal{F}'$ quasi-coherent on S'. Obsere that we do **not** use derived pullback on $\mathcal{G}^\bullet$ on the right hand side. If we ever need this, we will formulate a precise result here and give a detailed proof.

29. Limits and derived categories

In this section we collect some results about the derived category of a scheme which is the limit of an inverse system of schemes. More precisely, we will work in the following setting.

Situation 29.1. Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{i'i} : S_{i'} \to S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \to S_i$ the projection. We also fix an element $0 \in I$.

Lemma 29.2. *In Situation 29.1. Let E_0 and K_0 be objects of $D(\mathcal{O}_{S_0})$. Set $E_i = Lf_{i0}^*E_0$ and $K_i = Lf_{i0}^*K_0$ for $i \geq 0$ and set $E = Lf_0^*E_0$ and $K = Lf_0^*K_0$. Then the map*

$$\text{colim}_{i \geq 0} \text{Hom}_{D(\mathcal{O}_{S_i})}(E_i, K_i) \longrightarrow \text{Hom}_{D(\mathcal{O}_S)}(E, K)$$

is an isomorphism if either

1. *E_0 is perfect and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$, or*
2. *E_0 is pseudo-coherent and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$ has finite tor dimension.*

Proof. For every open $U_0 \subset S_0$ consider the condition P that the canonical map

$$\text{colim}_{i \geq 0} \text{Hom}_{D(\mathcal{O}_{U_i})}(E_i|_{U_i}, K_i|_{U_i}) \longrightarrow \text{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove P holds for all quasi-compact opens U_0 by the induction principle of Cohomology of Schemes, Lemma 4.1. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 33.3. Thus it suffices to prove the lemma when S_0 is affine.

Assume S_0 is affine. Say $S_0 = \text{Spec}(A_0)$, $S_i = \text{Spec}(A_i)$, and $S = \text{Spec}(A)$. We will use Lemma 3.5 without further mention.

In case (1) the object $E_0^\bullet$ corresponds to a finite complex of finite projective A_0-modules, see Lemma 10.7. We may represent the object K_0 by a K-flat complex $K_0^\bullet$ of A_0-modules. In this situation we are trying to prove

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(A_i)}(E_0^\bullet \otimes_{A_0} A_i, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \mathrm{Hom}_{D(A)}(E_0^\bullet \otimes_{A_0} A, K_0^\bullet \otimes_{A_0} A)$$

Because $E_0^\bullet$ is a bounded above complex of projective modules we can rewrite this as

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

Since there are only a finite number of nonzero modules E_0^n and since these are all finitely presented modules, this map is an isomorphism.

In case (2) the object E_0 corresponds to a bounded above complex $E_0^\bullet$ of finite free A_0-modules, see Lemma 10.2. We may represent K_0 by a finite complex $K_0^\bullet$ of flat A_0-modules, see Lemma 10.4 and More on Algebra, Lemma 66.3. In particular $K_0^\bullet$ is K-flat and we can argue as before to arrive at the map

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

It is clear that this map is an isomorphism (only a finite number of terms are involved since $K_0^\bullet$ is bounded). $\square$

Lemma 29.3. *In Situation 29.1 the category of perfect objects of $D(\mathcal{O}_S)$ is the colimit of the categories of perfect objects of $D(\mathcal{O}_{S_i})$.*

Proof. For every open $U_0 \subset S_0$ consider the condition P that the functor

$$\mathrm{colim}_{i \geq 0} D_{perf}(\mathcal{O}_{U_i}) \longrightarrow D_{perf}(\mathcal{O}_U)$$

is an equivalence where $_{perf}$ indicates the full subcategory of perfect objects and where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove P holds for all quasi-compact opens U_0 by the induction principle of Cohomology of Schemes, Lemma 4.1. First, we observe that we already know the functor is fully faithful by Lemma 29.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have $S_0 = U_0 \cup V_0$ and that P holds for U_0, V_0, and $U_0 \cap V_0$. Let E be a perfect object of $D(\mathcal{O}_S)$. We can find $i \geq 0$ and $E_{U,i}$ perfect on U_i and $E_{V,i}$ perfect on V_i whose pullback to U and V are isomorphic to $E|_U$ and $E|_V$. Denote

$$a : E_{U,i} \to (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \cap V_i} \to E_{V,i}|_{U_i \cap V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U \cap V} \to E|_{U \cap V} \to Lf_i^*E_{V,i}|_{U \cap V}.$$

Apply Cohomology, Lemma 45.1 to get an object E_i on S_i and a map $d : E_i \to Rf_{i,*}E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is perfect and that d is adjoint to an isomorphism $Lf_i^*E_i \to E$.

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when S_0 is affine. Say $S_0 = \mathrm{Spec}(A_0)$, $S_i = \mathrm{Spec}(A_i)$, and $S = \mathrm{Spec}(A)$. Using Lemmas 3.5 and 10.7 we see that we have to show that

$$D_{perf}(A) = \mathrm{colim}\, D_{perf}(A_i)$$

This is clear from the fact that perfect complexes over rings are given by finite complexes of finite projective (hence finitely presented) modules. See More on Algebra, Lemma 74.17 for details. □

30. Cohomology and base change, VI

A final section on cohomology and base change continuing the discussion of Sections 22, 26, and 27. An easy to grok special case is given in Remark 30.2.

Lemma 30.1. *Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over S, with support proper over S. Then*

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 26.4. Thus it suffices to show that K is a perfect object. If S is Noetherian, then this follows from Lemma 27.2. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on S, hence we may assume S is affine. Say $S = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits, Lemma 10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X. By Limits, Lemma 10.2 we may assume after increasing i, that there exists a bounded complex of finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{G}_i^\bullet$ whose pullback to X is $\mathcal{G}^\bullet$. After increasing i we may assume $\mathcal{G}_i^n$ is flat over R_i, see Limits, Lemma 10.4. After increasing i we may assume the support of $\mathcal{G}_i^n$ is proper over R_i, see Limits, Lemma 13.5 and Cohomology of Schemes, Lemma 26.7. Finally, by Lemma 29.3 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 27.2 to $X_i \to \mathrm{Spec}(R_i)$, E_i, $\mathcal{G}_i^\bullet$ and using the base change property already shown we obtain the result. □

Remark 30.2. Let R be a ring. Let X be a scheme of finite presentation over R. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_X$-module flat over R with support proper over R. By Lemma 30.1 there exists a finite complex of finite projective R-modules $M^\bullet$ such that we have

$$R\Gamma(X_{R'}, \mathcal{G}_{R'}) = M^\bullet \otimes_R R'$$

functorially in the R-algebra R'.

Lemma 30.3. *Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a pseudo-coherent object. Let $\mathcal{G}^\bullet$ be a bounded above complex of finitely presented $\mathcal{O}_X$-modules, flat over S, with support proper over S. Then*

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is a pseudo-coherent object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 26.4. Thus it suffices to show that K is a pseudo-coherent object. This will follow from Lemma 30.1 by approximation by perfect complexes. We encourage the reader to skip the rest of the proof.

The question is local on S, hence we may assume S is affine. Then X is quasi-compact and quasi-separated. Moreover, there exists an integer N such that total direct image $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ has cohomological dimension N as explained in Lemma 4.1. Choose an integer b such that $\mathcal{G}^i = 0$ for $i > b$. It suffices to show that K is m-pseudo-coherent for every m. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, m - N - 1 - b)$. This is possible by Theorem 14.6. Choose a distinguished triangle

$$P \to E \to C \to P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m - N - 1 - b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ are zero in degrees $\geq m - N - 1$. Thus the cohomology sheaves of $Rf_*(C \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$ are zero in degrees $\geq m - 1$. Hence

$$Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet) \to Rf_*(E \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. Next, suppose that $H^i(P) = 0$ for $i > a$. Then $P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a}\mathcal{G}^\bullet \longrightarrow P \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ is an isomorphism on cohomology sheaves in degrees $\geq m - N - 1$. Thus again we find that

$$Rf_*(P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a}\mathcal{G}^\bullet) \to Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. By Lemma 30.1 the source is a perfect complex. We conclude that K is m-pseudo-coherent as desired. □

Lemma 30.4. *Let S be a scheme. Let $f : X \to S$ be a proper morphism of finite presentation.*

1. *Let $E \in D(\mathcal{O}_X)$ be perfect and f flat. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*
2. *Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. Then $Rf_*\mathcal{G}$ is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*

Proof. Special cases of Lemma 30.1 applied with (1) $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0 and (2) $E = \mathcal{O}_X$ and $\mathcal{G}^\bullet$ consisting of $\mathcal{G}$ sitting in degree 0. □

Lemma 30.5. *Let S be a scheme. Let $f : X \to S$ be a flat proper morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. Then Rf_*E is a pseudo-coherent object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*

More generally, if $f : X \to S$ is proper and E on X is pseudo-coherent relative to S (More on Morphisms, Definition 59.2), then Rf_*E is pseudo-coherent (but formation does not commute with base change in this generality). See [Kie72].

Proof. Special case of Lemma 30.3 applied with $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0. □

Lemma 30.6. *Let R be a ring. Let X be a scheme and let $f : X \to \mathrm{Spec}(R)$ be proper, flat, and of finite presentation. Let (M_n) be an inverse system of R-modules with surjective transition maps. Then the canonical map*

$$\mathcal{O}_X \otimes_R (\lim M_n) \longrightarrow \lim \mathcal{O}_X \otimes_R M_n$$

induces an isomorphism from the source to DQ_X applied to the target.

Proof. The statement means that for any object E of $D_{QCoh}(\mathcal{O}_X)$ the induced map

$$\mathrm{Hom}(E, \mathcal{O}_X \otimes_R (\lim M_n)) \longrightarrow \mathrm{Hom}(E, \lim \mathcal{O}_X \otimes_R M_n)$$

is an isomorphism. Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Theorem 15.3) it suffices to check this for perfect E. By Lemma 3.2 we have $\lim \mathcal{O}_X \otimes_R M_n = R\lim \mathcal{O}_X \otimes_R M_n$. The exact functor $R\operatorname{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \to D(R)$ of Cohomology, Section 44 commutes with products and hence with derived limits, whence

$$R\operatorname{Hom}_X(E, \lim \mathcal{O}_X \otimes_R M_n) = R\lim R\operatorname{Hom}_X(E, \mathcal{O}_X \otimes_R M_n)$$

Let $E^\vee$ be the dual perfect complex, see Cohomology, Lemma 50.5. We have

$$R\operatorname{Hom}_X(E, \mathcal{O}_X \otimes_R M_n) = R\Gamma(X, E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*M_n) = R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R M_n$$

by Lemma 22.1. From Lemma 30.4 we see $R\Gamma(X, E^\vee)$ is a perfect complex of R-modules. In particular it is a pseudo-coherent complex and by More on Algebra, Lemma 102.3 we obtain

$$R\lim R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R M_n = R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R \lim M_n$$

as desired. □

Lemma 30.7. *Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over S, with support proper over S. Then*

$$K = Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 26.5. Thus it suffices to show that K is a perfect object. If S is Noetherian, then this follows from Lemma 27.3. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on S, hence we may assume S is affine. Say $S = \operatorname{Spec}(R)$. We write $R = \operatorname{colim} R_i$ as a filtered colimit of Noetherian rings R_i. By Limits, Lemma 10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X. By Limits, Lemma 10.2 we may assume after increasing i, that there exists a bounded complex of finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{G}_i^\bullet$ whose pullback to X is $\mathcal{G}^\bullet$. After increasing i we may assume $\mathcal{G}_i^n$ is flat over R_i, see Limits, Lemma 10.4. After increasing i we may assume the support of $\mathcal{G}_i^n$ is proper over R_i, see Limits, Lemma 13.5 and Cohomology of Schemes, Lemma 26.7. Finally, by Lemma 29.3 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 27.3 to $X_i \to \operatorname{Spec}(R_i)$, E_i, $\mathcal{G}_i^\bullet$ and using the base change property already shown we obtain the result. □

31. Perfect complexes

We first talk about jumping loci for betti numbers of perfect complexes. Given a complex E on a scheme X and a point x of X we often write $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x)$ instead of the more correct Li_x^*E, where $i_x : x \to X$ is the canonical morphism.

Lemma 31.1. *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent (for example perfect). For any $i \in \mathbf{Z}$ consider the function*

$$\beta_i : X \longrightarrow \{0, 1, 2, \ldots\}, \quad x \longmapsto \dim_{\kappa(x)} H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x))$$

Then we have

(1) *formation of β_i commutes with arbitrary base change,*

(2) *the functions β_i are upper semi-continuous, and*
(3) *the level sets of β_i are locally constructible in X.*

Proof. Consider a morphism of schemes $f : Y \to X$ and a point $y \in Y$. Let x be the image of y and consider the commutative diagram

$$\begin{tikzcd} y \arrow[r, "j"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ x \arrow[r, "i"] & X \end{tikzcd}$$

Then we see that $Lg^* \circ Li^* = Lj^* \circ Lf^*$. This implies that the function β'_i associated to the pseudo-coherent complex Lf^*E is the pullback of the function β_i, in a formula: $\beta'_i = \beta_i \circ f$. This is the meaning of (1).

Fix i and let $x \in X$. It is enough to prove (2) and (3) holds in an open neighbourhood of x, hence we may assume X affine. Then we can represent E by a bounded above complex $\mathcal{F}^\bullet$ of finite free modules (Lemma 13.3). Then $P = \sigma_{\geq i-1}\mathcal{F}^\bullet$ is a perfect object and $P \to E$ induces an isomorphism

$$H^i(P \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x')) \to H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x'))$$

for all $x' \in X$. Thus we may assume E is perfect. In this case by More on Algebra, Lemma 75.6 there exists an affine open neighbourhood U of x and $a \leq b$ such that $E|_U$ is represented by a complex

$$\ldots \to 0 \to \mathcal{O}_U^{\oplus \beta_a(x)} \to \mathcal{O}_U^{\oplus \beta_{a+1}(x)} \to \ldots \to \mathcal{O}_U^{\oplus \beta_{b-1}(x)} \to \mathcal{O}_U^{\oplus \beta_b(x)} \to 0 \to \ldots$$

(This also uses earlier results to turn the problem into algebra, for example Lemmas 3.5 and 10.7.) It follows immediately that $\beta_i(x') \leq \beta_i(x)$ for all $x' \in U$. This proves that β_i is upper semi-continuous.

To prove (3) we may assume that X is affine and E is given by a complex of finite free $\mathcal{O}_X$-modules (for example by arguing as in the previous paragraph, or by using Cohomology, Lemma 49.3). Thus we have to show that given a complex

$$\mathcal{O}_X^{\oplus a} \to \mathcal{O}_X^{\oplus b} \to \mathcal{O}_X^{\oplus c}$$

the function associated to a point $x \in X$ the dimension of the cohomology of $\kappa_x^{\oplus a} \to \kappa_x^{\oplus b} \to \kappa_x^{\oplus c}$ in the middle has constructible level sets. Let $A \in \text{Mat}(a \times b, \Gamma(X, \mathcal{O}_X))$ be the matrix of the first arrow. The rank of the image of A in $\text{Mat}(a \times b, \kappa(x))$ is equal to r if all $(r+1) \times (r+1)$-minors of A vanish at x and there is some $r \times r$-minor of A which does not vanish at x. Thus the set of points where the rank is r is a constructible locally closed set. Arguing similarly for the second arrow and putting everything together we obtain the desired result. $\square$

Lemma 31.2. *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. The function*

$$\chi_E : X \longrightarrow \mathbf{Z}, \quad x \longmapsto \sum (-1)^i \dim_{\kappa(x)} H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x))$$

is locally constant on X.

Proof. By Cohomology, Lemma 49.3 we see that we can, locally on X, represent E by a finite complex $\mathcal{E}^\bullet$ of finite free $\mathcal{O}_X$-modules. On such an open the function χ_E is constant with value $\sum(-1)^i \text{rank}(\mathcal{E}^i)$. $\square$

Lemma 31.3. *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. Given $i, r \in \mathbf{Z}$, there exists an open subscheme $U \subset X$ characterized by the following*

(1) *$E|_U \cong H^i(E|_U)[-i]$ and $H^i(E|_U)$ is a locally free $\mathcal{O}_U$-module of rank r,*
(2) *a morphism $f : Y \to X$ factors through U if and only if Lf^*E is isomorphic to a locally free module of rank r placed in degree i.*

Proof. Let $\beta_j : X \to \{0, 1, 2, \ldots\}$ for $j \in \mathbf{Z}$ be the functions of Lemma 31.1. Then the set

$$W = \{x \in X \mid \beta_j(x) \leq 0 \text{ for all } j \neq i\}$$

is open in X and its formation commutes with pullback to any Y over X. This follows from the lemma using that apriori in a neighbourhood of any point only a finite number of the β_j are nonzero. Thus we may replace X by W and assume that $\beta_j(x) = 0$ for all $x \in X$ and all $j \neq i$. In this case $H^i(E)$ is a finite locally free module and $E \cong H^i(E)[-i]$, see for example More on Algebra, Lemma 75.6. Thus X is the disjoint union of the open subschemes where the rank of $H^i(E)$ is fixed and we win. ☐

Lemma 31.4. *Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect of tor-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Let $r \geq 0$. Then there exists a locally closed subscheme $j : Z \to X$ characterized by the following*

(1) *$H^a(Lj^*E)$ is a locally free $\mathcal{O}_Z$-module of rank r, and*
(2) *a morphism $f : Y \to X$ factors through Z if and only if for all morphisms $g : Y' \to Y$ the $\mathcal{O}_{Y'}$-module $H^a(L(f \circ g)^*E)$ is locally free of rank r.*

Moreover, $j : Z \to X$ is of finite presentation and we have

(3) *if $f : Y \to X$ factors as $Y \xrightarrow{g} Z \to X$, then $H^a(Lf^*E) = g^*H^a(Lj^*E)$,*
(4) *if $\beta_a(x) \leq r$ for all $x \in X$, then j is a closed immersion and given $f : Y \to X$ the following are equivalent*
    (a) *$f : Y \to X$ factors through Z,*
    (b) *$H^0(Lf^*E)$ is a locally free $\mathcal{O}_Y$-module of rank r,*
    and if $r = 1$ these are also equivalent to
    (c) *$\mathcal{O}_Y \to \mathcal{H}om_{\mathcal{O}_Y}(H^0(Lf^*E), H^0(Lf^*E))$ is injective.*

Proof. First, let $U \subset X$ be the locally constructible open subscheme where the function β_a of Lemma 31.1 has values $\leq r$. Let $f : Y \to X$ be as in (2). Then for any $y \in Y$ we have $\beta_a(Lf^*E) = r$ hence y maps into U by Lemma 31.1. Hence f as in (2) factors through U. Thus we may replace X by U and assume that $\beta_a(x) \in \{0, 1, \ldots, r\}$ for all $x \in X$. We will show that in this case there is a closed subscheme $Z \subset X$ cut out by a finite type quasi-coherent ideal characterized by the equivalence of (4) (a), (b) and (4)(c) if $r = 1$ and that (3) holds. This will finish the proof because it will a fortiori show that morphisms as in (2) factor through Z.

If $x \in X$ and $\beta_a(x) < r$, then there is an open neighbourhood of x where $\beta_a < r$ (Lemma 31.1). In this way we see that set theoretically at least Z is a closed subset.

To get a scheme theoretic structure, consider a point $x \in X$ with $\beta_a(x) = r$. Set $\beta = \beta_{a+1}(x)$. By More on Algebra, Lemma 75.6 there exists an affine open neighbourhood U of x such that $K|_U$ is represented by a complex

$$\ldots \to 0 \to \mathcal{O}_U^{\oplus r} \xrightarrow{(f_{ij})} \mathcal{O}_U^{\oplus \beta} \to \ldots \to \mathcal{O}_U^{\oplus \beta_{b-1}(x)} \to \mathcal{O}_U^{\oplus \beta_b(x)} \to 0 \to \ldots$$

(This also uses earlier results to turn the problem into algebra, for example Lemmas 3.5 and 10.7.) Now, if $g : Y \to U$ is any morphism of schemes such that $g^\sharp(f_{ij})$ is nonzero for some pair i, j, then $H^0(Lg^*E)$ is not a locally free $\mathcal{O}_Y$-module of rank r.

See More on Algebra, Lemma 15.7. Trivially $H^0(Lg^*E)$ is a locally free $\mathcal{O}_Y$-module if $g^\sharp(f_{ij}) = 0$ for all i, j. Thus we see that over U the closed subscheme cut out by all f_{ij} satisfies (3) and we have the equivalence of (4)(a) and (b). The characterization of Z shows that the locally constructed patches glue (details omitted). Finally, if $r = 1$ then (4)(c) is equivalent to (4)(b) because in this case locally $H^0(Lg^*E) \subset \mathcal{O}_Y$ is the annihilator of the ideal generated by the elements $g^\sharp(f_{ij})$. $\square$

32. Applications

Mostly applications of cohomology and base change. In the future we may generalize these results to the situation discussed in Lemma 30.1.

Lemma 32.1. *Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. For fixed $i \in \mathbf{Z}$ consider the function*

$$\beta_i : S \to \{0, 1, 2, \ldots\}, \quad s \longmapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$$

Then we have

1. *formation of β_i commutes with arbitrary base change,*
2. *the functions β_i are upper semi-continuous, and*
3. *the level sets of β_i are locally constructible in S.*

Proof. By cohomology and base change (more precisely by Lemma 30.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. In particular we have

$$H^i(X_s, \mathcal{F}_s) = H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s))$$

Thus the lemma follows from Lemma 31.1. $\square$

Lemma 32.2. *Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. The function*

$$s \longmapsto \chi(X_s, \mathcal{F}_s)$$

is locally constant on S. Formation of this function commutes with base change.

Proof. By cohomology and base change (more precisely by Lemma 30.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. Thus we have to show the map

$$s \longmapsto \sum (-1)^i \dim_{\kappa(s)} H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s))$$

is locally constant on S. This is Lemma 31.2. $\square$

Lemma 32.3. *Let $f : X \to S$ be a proper morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. Fix $i, r \in \mathbf{Z}$. Then there exists an open subscheme $U \subset S$ with the following property: A morphism $T \to S$ factors through U if and only if $Rf_{T,*}\mathcal{F}_T$ is isomorphic to a finite locally free module of rank r placed in degree i.*

Proof. By cohomology and base change (more precisely by Lemma 30.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. Thus this lemma follows immediately from Lemma 31.3. $\square$

Lemma 32.4. *Let $f : X \to S$ be a morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S with support proper over S. If $R^i f_* \mathcal{F} = 0$ for $i > 0$, then $f_*\mathcal{F}$ is locally free and its formation commutes with arbitrary base change (see proof for explanation).*

Proof. By Lemma 30.1 the object $E = Rf_*\mathcal{F}$ of $D(\mathcal{O}_S)$ is perfect and its formation commutes with arbitrary base change, in the sense that $Rf'_*(g')^*\mathcal{F} = Lg^*E$ for any cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of schemes. Since there is never any cohomology in degrees < 0, we see that E (locally) has tor-amplitude in $[0, b]$ for some b. If $H^i(E) = R^i f_*\mathcal{F} = 0$ for $i > 0$, then E has tor amplitude in $[0, 0]$. Whence $E = H^0(E)[0]$. We conclude $H^0(E) = f_*\mathcal{F}$ is finite locally free by More on Algebra, Lemma 74.2 (and the characterization of finite projective modules in Algebra, Lemma 78.2). Commutation with base change means that $g^*f_*\mathcal{F} = f'_*(g')^*\mathcal{F}$ for a diagram as above and it follows from the already established commutation of base change for E. $\square$

Lemma 32.5. *Let $f : X \to S$ be a morphism of schemes. Assume*

1. *f is proper, flat, and of finite presentation, and*
2. *for all $s \in S$ we have $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$.*

Then we have

(a) *$f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change,*
(b) *locally on S we have*

$$Rf_*\mathcal{O}_X = \mathcal{O}_S \oplus P$$

in $D(\mathcal{O}_S)$ where P is perfect of tor amplitude in $[1, \infty)$.

Proof. By cohomology and base change (Lemma 30.4) the complex $E = Rf_*\mathcal{O}_X$ is perfect and its formation commutes with arbitrary base change. This first implies that E has tor aplitude in $[0, \infty)$. Second, it implies that for $s \in S$ we have $H^0(E \otimes^{\mathbf{L}} \kappa(s)) = H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$. It follows that the map $\mathcal{O}_S \to Rf_*\mathcal{O}_X = E$ induces an isomorphism $\mathcal{O}_S \otimes \kappa(s) \to H^0(E \otimes^{\mathbf{L}} \kappa(s))$. Hence $H^0(E) \otimes \kappa(s) \to H^0(E \otimes^{\mathbf{L}} \kappa(s))$ is surjective and we may apply More on Algebra, Lemma 76.2 to see that, after replacing S by an affine open neighbourhood of s, we have a decomposition $E = H^0(E) \oplus \tau_{\geq 1}E$ with $\tau_{\geq 1}E$ perfect of tor amplitude in $[1, \infty)$. Since E has tor amplitude in $[0, \infty)$ we find that $H^0(E)$ is a flat $\mathcal{O}_S$-module. It follows that $H^0(E)$ is a flat, perfect $\mathcal{O}_S$-module, hence finite locally free, see More on Algebra, Lemma 74.2 (and the fact that finite projective modules are finite locally free by Algebra, Lemma 78.2). It follows that the map $\mathcal{O}_S \to H^0(E)$ is an isomorphism as we can check this after tensoring with residue fields (Algebra, Lemma 79.4). $\square$

Lemma 32.6. *Let $f : X \to S$ be a morphism of schemes. Assume*

1. *f is proper, flat, and of finite presentation, and*
2. *the geometric fibres of f are reduced and connected.*

Then $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change.

Proof. By Lemma 32.5 it suffices to show that $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$. This follows from Varieties, Lemma 9.3 and the fact that X_s is geometrically connected and geometrically reduced. $\square$

Lemma 32.7. *Let $f : X \to S$ be a proper morphism of schemes. Let $s \in S$ and let $e \in H^0(X_s, \mathcal{O}_{X_s})$ be an idempotent. Then e is in the image of the map $(f_*\mathcal{O}_X)_s \to H^0(X_s, \mathcal{O}_{X_s})$.*

Proof. Let $X_s = T_1 \amalg T_2$ be the disjoint union decomposition with T_1 and T_2 nonempty and open and closed in X_s corresponding to e, i.e., such that e is identically 1 on T_1 and identically 0 on T_2.

Assume S is Noetherian. We will use the theorem on formal functions in the form of Cohomology of Schemes, Lemma 20.7. It tells us that

$$(f_*\mathcal{O}_X)^\wedge_s = \lim_n H^0(X_n, \mathcal{O}_{X_n})$$

where X_n is the nth infinitesimal neighbourhood of X_s. Since the underlying topological space of X_n is equal to that of X_s we obtain for all n a disjoint union decomposition of schemes $X_n = T_{1,n} \amalg T_{2,n}$ where the underlying topological space of $T_{i,n}$ is T_i for $i = 1, 2$. This means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent e_n, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that e_{n+1} restricts to e_n on X_n. Hence $e_\infty = \lim e_n$ is a nontrivial idempotent of the limit. Thus e_∞ is an element of the completion of $(f_*\mathcal{O}_X)_s$ mapping to e in $H^0(X_s, \mathcal{O}_{X_s})$. Since the map $(f_*\mathcal{O}_X)^\wedge_s \to H^0(X_s, \mathcal{O}_{X_s})$ factors through $(f_*\mathcal{O}_X)^\wedge_s/\mathfrak{m}_s(f_*\mathcal{O}_X)^\wedge_s = (f_*\mathcal{O}_X)_s/\mathfrak{m}_s(f_*\mathcal{O}_X)_s$ (Algebra, Lemma 96.3) we conclude that e is in the image of the map $(f_*\mathcal{O}_X)_s \to H^0(X_s, \mathcal{O}_{X_s})$ as desired.

General case: we reduce the general case to the Noetherian case by limit arguments. We urge the reader to skip the proof. We may replace S by an affine open neighbourhood of s. Thus we may and do assume that S is affine. By Limits, Lemma 13.3 we can write $(f : X \to S) = \lim(f_i : X_i \to S_i)$ with f_i proper and S_i Noetherian. Denote $s_i \in S_i$ the image of s. Then $s = \lim s_i$, see Limits, Lemma 4.4. Then $X_s = X \times_S s = \lim X_i \times_{S_i} s_i = \lim X_{i,s_i}$ because limits commute with limits (Categories, Lemma 14.10). Hence e is the image of some idempotent $e_i \in H^0(X_{i,s_i}, \mathcal{O}_{X_{i,s_i}})$ by Limits, Lemma 4.7. By the Noetherian case there is an element $\tilde{e}_i$ in the stalk $(f_{i,*}\mathcal{O}_{X_i})_{s_i}$ mapping to e_i. Taking the pullback of $\tilde{e}_i$ we get an element $\tilde{e}$ of $(f_*\mathcal{O}_X)_s$ mapping to e and the proof is complete. $\square$

Lemma 32.8. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume*

(1) *f is proper, flat, and of finite presentation, and*
(2) *the fibre X_s is geometrically reduced.*

Then, after replacing S by an open neighbourhood of s, there exists a direct sum decomposition $Rf_*\mathcal{O}_X = f_*\mathcal{O}_X \oplus P$ in $D(\mathcal{O}_S)$ where $f_*\mathcal{O}_X$ is a finite étale $\mathcal{O}_S$-algebra and P is a perfect of tor amplitude in $[1, \infty)$.

Proof. The proof of this lemma is similar to the proof of Lemma 32.5 which we suggest the reader read first. By cohomology and base change (Lemma 30.4) the complex $E = Rf_*\mathcal{O}_X$ is perfect and its formation commutes with arbitrary base change. This first implies that E has tor aplitude in $[0, \infty)$.

We claim that after replacing S by an open neighbourhood of s we can find a direct sum decomposition $E = H^0(E) \oplus \tau_{\geq 1}E$ in $D(\mathcal{O}_S)$ with $\tau_{\geq 1}E$ of tor amplitude in $[1, \infty)$. Assume the claim is true for now and assume we've made the replacement so we have the direct sum decomposition. Since E has tor amplitude in $[0, \infty)$ we find that $H^0(E)$ is a flat $\mathcal{O}_S$-module. Hence $H^0(E)$ is a flat, perfect $\mathcal{O}_S$-module, hence finite locally free, see More on Algebra, Lemma 74.2 (and the fact that finite projective modules are finite locally free by Algebra, Lemma 78.2). Of course $H^0(E) = f_*\mathcal{O}_X$ is an $\mathcal{O}_S$-algebra. By cohomology and base change we obtain $H^0(E) \otimes \kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$. By Varieties, Lemma 9.3 and the assumption that X_s is geometrically reduced, we see that $\kappa(s) \to H^0(E) \otimes \kappa(s)$ is finite étale. By Morphisms, Lemma 36.17 applied to the finite locally free morphism $\underline{\mathrm{Spec}}_S(H^0(E)) \to S$, we conclude that after shrinking S the $\mathcal{O}_S$-algebra $H^0(E)$ is finite étale.

It remains to prove the claim. For this it suffices to prove that the map

$$(f_*\mathcal{O}_X)_s \longrightarrow H^0(X_s, \mathcal{O}_{X_s}) = H^0(E \otimes^{\mathbf{L}} \kappa(s))$$

is surjective, see More on Algebra, Lemma 76.2. Choose a flat local ring homomorphism $\mathcal{O}_{S,s} \to A$ such that the residue field k of A is algebraically closed, see Algebra, Lemma 159.1. By flat base change (Cohomology of Schemes, Lemma 5.2) we get $H^0(X_A, \mathcal{O}_{X_A}) = (f_*\mathcal{O}_X)_s \otimes_{\mathcal{O}_{S,s}} A$ and $H^0(X_k, \mathcal{O}_{X_k}) = H^0(X_s, \mathcal{O}_{X_s}) \otimes_{\kappa(s)} k$. Hence it suffices to prove that $H^0(X_A, \mathcal{O}_{X_A}) \to H^0(X_k, \mathcal{O}_{X_k})$ is surjective. Since X_k is a reduced proper scheme over k and since k is algebraically closed, we see that $H^0(X_k, \mathcal{O}_{X_k})$ is a finite product of copies of k by the already used Varieties, Lemma 9.3. Since by Lemma 32.7 the idempotents of this k-algebra are in the image of $H^0(X_A, \mathcal{O}_{X_A}) \to H^0(X_k, \mathcal{O}_{X_k})$ we conclude. □

33. Other applications

In this section we state and prove some results that can be deduced from the theory worked out above.

Lemma 33.1. *Let R be a coherent ring. Let X be a scheme of finite presentation over R. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation, flat over R, with support proper over R. Then $H^i(X, \mathcal{G})$ is a coherent R-module.*

Proof. Combine Lemma 30.1 with More on Algebra, Lemmas 64.18 and 74.2. □

Lemma 33.2. *Let X be a quasi-compact and quasi-separated scheme. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$ such that the cohomology sheaves $H^i(K)$ have countable sets of sections over affine opens. Then for any quasi-compact open $U \subset X$ and any perfect object E in $D(\mathcal{O}_X)$ the sets*

$$H^i(U, K \otimes^{\mathbf{L}} E), \quad \mathrm{Ext}^i(E|_U, K|_U)$$

are countable.

Proof. Using Cohomology, Lemma 50.5 we see that it suffices to prove the result for the groups $H^i(U, K \otimes^{\mathbf{L}} E)$. We will use the induction principle to prove the lemma, see Cohomology of Schemes, Lemma 4.1.

First we show that it holds when $U = \mathrm{Spec}(A)$ is affine. Namely, we can represent K by a complex of A-modules $K^\bullet$ and E by a finite complex of finite projective A-modules $P^\bullet$. See Lemmas 3.5 and 10.7 and our definition of perfect complexes of A-modules (More on Algebra, Definition 74.1). Then $(E \otimes^{\mathbf{L}} K)|_U$ is represented by the total complex associated to the double complex $P^\bullet \otimes_A K^\bullet$ (Lemma 3.9). Using induction on the length of the complex $P^\bullet$ (or using a suitable spectral sequence) we see that it suffices to show that $H^i(P^a \otimes_A K^\bullet)$ is countable for each a. Since P^a is a direct summand of $A^{\oplus n}$ for some n this follows from the assumption that the cohomology group $H^i(K^\bullet)$ is countable.

To finish the proof it suffices to show: if $U = V \cup W$ and the result holds for V, W, and $V \cap W$, then the result holds for U. This is an immediate consquence of the Mayer-Vietoris sequence, see Cohomology, Lemma 33.4. □

Lemma 33.3. *Let X be a quasi-compact and quasi-separated scheme such that the sets of sections of $\mathcal{O}_X$ over affine opens are countable. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

1. *$K = hocolim E_n$ with E_n a perfect object of $D(\mathcal{O}_X)$, and*
2. *the cohomology sheaves $H^i(K)$ have countable sets of sections over affine opens.*

Proof. If (1) is true, then (2) is true because homotopy colimits commutes with taking cohomology sheaves (by Derived Categories, Lemma 33.8) and because a perfect complex is locally isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules and therefore satisfies (2) by assumption on X.

Assume (2). Choose a K-injective complex $\mathcal{K}^\bullet$ representing K. Choose a perfect generator E of $D_{QCoh}(\mathcal{O}_X)$ and represent it by a K-injective complex $\mathcal{I}^\bullet$. According to Theorem 18.3 and its proof there is an equivalence of triangulated categories $F : D_{QCoh}(\mathcal{O}_X) \to D(A, \mathrm{d})$ where (A, d) is the differential graded algebra

$$(A, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$$

which maps K to the differential graded module

$$M = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{K}^\bullet)$$

Note that $H^i(A) = \mathrm{Ext}^i(E, E)$ and $H^i(M) = \mathrm{Ext}^i(E, K)$. Moreover, since F is an equivalence it and its quasi-inverse commute with homotopy colimits. Therefore, it suffices to write M as a homotopy colimit of compact objects of $D(A, \mathrm{d})$. By Differential Graded Algebra, Lemma 38.3 it suffices show that $\mathrm{Ext}^i(E, E)$ and $\mathrm{Ext}^i(E, K)$ are countable for each i. This follows from Lemma 33.2. □

Lemma 33.4. *Let A be a ring. Let X be a scheme of finite presentation over A. Let $f : U \to X$ be a flat morphism of finite presentation. Then*

1. *there exists an inverse system of perfect objects L_n of $D(\mathcal{O}_X)$ such that*
$$R\Gamma(U, Lf^*K) = hocolim\ R\,\mathrm{Hom}_X(L_n, K)$$
in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$, and
2. *there exists a system of perfect objects E_n of $D(\mathcal{O}_X)$ such that*
$$R\Gamma(U, Lf^*K) = hocolim\ R\Gamma(X, E_n \otimes^{\mathbf{L}} K)$$
in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$.

Proof. By Lemma 22.1 we have

$$R\Gamma(U, Lf^*K) = R\Gamma(X, Rf_*\mathcal{O}_U \otimes^{\mathbf{L}} K)$$

functorially in K. Observe that $R\Gamma(X, -)$ commutes with homotopy colimits because it commutes with direct sums by Lemma 4.5. Similarly, $- \otimes^{\mathbf{L}} K$ commutes with derived colimits because $- \otimes^{\mathbf{L}} K$ commutes with direct sums (because direct sums in $D(\mathcal{O}_X)$ are given by direct sums of representing complexes). Hence to prove

(2) it suffices to write $Rf_*\mathcal{O}_U = \text{hocolim} E_n$ for a system of perfect objects E_n of $D(\mathcal{O}_X)$. Once this is done we obtain (1) by setting $L_n = E_n^\vee$, see Cohomology, Lemma 50.5.

Write $A = \operatorname{colim} A_i$ with A_i of finite type over $\mathbf{Z}$. By Limits, Lemma 10.1 we can find an i and morphisms $U_i \to X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to $\text{Spec}(A)$ recovers $U \to X \to \text{Spec}(A)$. After increasing i we may assume that $f_i : U_i \to X_i$ is flat, see Limits, Lemma 8.7. By Lemma 22.5 the derived pullback of $Rf_{i,*}\mathcal{O}_{U_i}$ by $g : X \to X_i$ is equal to $Rf_*\mathcal{O}_U$. Since Lg^* commutes with derived colimits, it suffices to prove what we want for f_i. Hence we may assume that U and X are of finite type over $\mathbf{Z}$.

Assume $f : U \to X$ is a morphism of schemes of finite type over $\mathbf{Z}$. To finish the proof we will show that $Rf_*\mathcal{O}_U$ is a homotopy colimit of perfect complexes. To see this we apply Lemma 33.3. Thus it suffices to show that $R^if_*\mathcal{O}_U$ has countable sets of sections over affine opens. This follows from Lemma 33.2 applied to the structure sheaf. $\square$

34. Characterizing pseudo-coherent complexes, II

This section is a continuation of Section 19. In this section we discuss characterizations of pseudo-coherent complexes in terms of cohomology. More results of this nature can be found in More on Morphisms, Section 69.

Lemma 34.1. *Let A be a ring. Let R be a (possibly noncommutative) A-algebra which is finite free as an A-module. Then any object M of $D(R)$ which is pseudo-coherent in $D(A)$ can be represented by a bounded above complex of finite free (right) R-modules.*

Proof. Choose a complex $M^\bullet$ of right R-modules representing M. Since M is pseudo-coherent we have $H^i(M) = 0$ for large enough i. Let m be the smallest index such that $H^m(M)$ is nonzero. Then $H^m(M)$ is a finite A-module by More on Algebra, Lemma 64.3. Thus we can choose a finite free R-module F^m and a map $F^m \to M^m$ such that $F^m \to M^m \to M^{m+1}$ is zero and such that $F^m \to H^m(M)$ is surjective. Picture:

$$\begin{tikzcd} & F^m \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & \ldots \\ M^{m-1} \arrow[r] & M^m \arrow[r] & M^{m+1} \arrow[r] & \ldots \end{tikzcd}$$

By descending induction on $n \leq m$ we are going to construct finite free R-modules F^i for $i \geq n$, differentials $d^i : F^i \to F^{i+1}$ for $i \geq n$, maps $\alpha : F^i \to K^i$ compatible with differentials, such that (1) $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$, and (2) $F^i = 0$ for $i > m$. Picture

$$\begin{tikzcd} & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \arrow[r] & F^i \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & \ldots \\ M^{n-1} \arrow[r] & M^n \arrow[r] & M^{n+1} \arrow[r] & \ldots \arrow[r] & M^i \arrow[r] & M^{i+1} \arrow[r] & \ldots \end{tikzcd}$$

The base case is $n = m$ which we've done above. Induction step. Let $C^\bullet$ be the cone on α (Derived Categories, Definition 9.1). The long exact sequence of cohomology shows that $H^i(C^\bullet) = 0$ for $i \geq n$. Observe that $F^\bullet$ is pseudo-coherent as a complex of A-modules because R is finite free as an A-module. Hence by More on Algebra, Lemma 64.2 we see that $C^\bullet$ is $(n-1)$-pseudo-coherent as a complex of A-modules. By More on Algebra, Lemma 64.3 we see that $H^{n-1}(C^\bullet)$ is a finite A-module. Choose a finite free R-module F^{n-1} and a map $\beta : F^{n-1} \to C^{n-1}$ such that the composition $F^{n-1} \to C^{n-1} \to C^n$ is zero and such that F^{n-1} surjects onto $H^{n-1}(C^\bullet)$. Since $C^{n-1} = M^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$ implies these maps fit into a morphism of complexes

$$\begin{tikzcd} & F^{n-1} \arrow[r, "d^{n-1}"'] \arrow[d, "\alpha^{n-1}"] & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\ \ldots \arrow[r] & M^{n-1} \arrow[r] & M^n \arrow[r] & M^{n+1} \arrow[r] & \ldots \end{tikzcd}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{tikzcd} (F^n \to \ldots) \arrow[r] \arrow[d] & (F^{n-1} \to \ldots) \arrow[r] \arrow[d] & F^{n-1} \arrow[r] \arrow[d, "\beta"'] & (F^n \to \ldots)[1] \arrow[d] \\ (F^n \to \ldots) \arrow[r] & M^\bullet \arrow[r] & C^\bullet \arrow[r] & (F^n \to \ldots)[1] \end{tikzcd}$$

Hence our choice of β implies that the map of complexes $(F^{n-1} \to \ldots) \to M^\bullet$ induces an isomorphism on cohomology in degrees $\geq n$ and a surjection in degree $n-1$. This finishes the proof of the lemma. $\square$

Lemma 34.2. *Let A be a ring. Let $n \geq 0$. Let $K \in D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_A})$. The following are equivalent*

1. *K is pseudo-coherent,*
2. *$R\Gamma(\mathbf{P}^n_A, E \otimes^{\mathbf{L}} K)$ is a pseudo-coherent object of $D(A)$ for each pseudo-coherent object E of $D(\mathcal{O}_{\mathbf{P}^n_A})$,*
3. *$R\Gamma(\mathbf{P}^n_A, E \otimes^{\mathbf{L}} K)$ is a pseudo-coherent object of $D(A)$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_A})$,*
4. *$R\operatorname{Hom}_{\mathbf{P}^n_A}(E, K)$ is a pseudo-coherent object of $D(A)$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_A})$,*
5. *$R\Gamma(\mathbf{P}^n_A, K \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}^n_A}(d))$ is pseudo-coherent object of $D(A)$ for $d = 0, 1, \ldots, n$.*

Proof. Recall that

$$R\operatorname{Hom}_{\mathbf{P}^n_A}(E, K) = R\Gamma(\mathbf{P}^n_A, R\mathcal{H}om_{\mathcal{O}_{\mathbf{P}^n_A}}(E, K))$$

by definition, see Cohomology, Section 44. Thus parts (4) and (3) are equivalent by Cohomology, Lemma 50.5.

Since every perfect complex is pseudo-coherent, it is clear that (2) implies (3).

Assume (1) holds. Then $E \otimes^{\mathbf{L}} K$ is pseudo-coherent for every pseudo-coherent E, see Cohomology, Lemma 47.5. By Lemma 30.5 the direct image of such a pseudo-coherent complex is pseudo-coherent and we see that (2) is true.

Part (3) implies (5) because we can take $E = \mathcal{O}_{\mathbf{P}^n_A}(d)$ for $d = 0, 1, \ldots, n$.

To finish the proof we have to show that (5) implies (1). Let P be as in (20.0.1) and R as in (20.0.2). By Lemma 20.1 we have an equivalence

$$- \otimes^{\mathbf{L}}_R P : D(R) \longrightarrow D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_A})$$

Let $M \in D(R)$ be an object such that $M \otimes^{\mathbf{L}} P = K$. By Differential Graded Algebra, Lemma 35.4 there is an isomorphism

$$R\operatorname{Hom}(R, M) = R\operatorname{Hom}_{\mathbf{P}^n_A}(P, K)$$

in $D(A)$. Arguing as above we obtain

$$R\operatorname{Hom}_{\mathbf{P}^n_A}(P, K) = R\Gamma(\mathbf{P}^n_A, R\mathcal{H}om_{\mathcal{O}_{\mathbf{P}^n_A}}(E, K)) = R\Gamma(\mathbf{P}^n_A, P^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{P}^n_A}} K).$$

Using that $P^\vee$ is the direct sum of $\mathcal{O}_{\mathbf{P}^n_A}(d)$ for $d = 0, 1, \ldots, n$ and (5) we conclude $R\operatorname{Hom}(R, M)$ is pseudo-coherent as a complex of A-modules. Of course $M = R\operatorname{Hom}(R, M)$ in $D(A)$. Thus M is pseudo-coherent as a complex of A-modules. By Lemma 34.1 we may represent M by a bounded above complex $F^\bullet$ of finite free R-modules. Then $F^\bullet = \bigcup_{p \geq 0} \sigma_{\geq p} F^\bullet$ is a filtration which shows that $F^\bullet$ is a differential graded R-module with property (P), see Differential Graded Algebra, Section 20. Hence $K = M \otimes^{\mathbf{L}}_R P$ is represented by $F^\bullet \otimes_R P$ (follows from the construction of the derived tensor functor, see for example the proof of Differential Graded Algebra, Lemma 35.3). Since $F^\bullet \otimes_R P$ is a bounded above complex whose terms are direct sums of copies of P we conclude that the lemma is true. $\square$

Lemma 34.3. *Let A be a ring. Let X be a scheme over A which is quasi-compact and quasi-separated. Let $K \in D^-_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect E in $D(\mathcal{O}_X)$, then $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$.*

Proof. There exists an integer N such that $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ has cohomological dimension N as explained in Lemma 4.1. Let $b \in \mathbf{Z}$ be such that $H^i(K) = 0$ for $i > b$. Let E be pseudo-coherent on X. It suffices to show that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m-pseudo-coherent for every m. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, m - N - 1 - b)$. This is possible by Theorem 14.6. Choose a distinguished triangle

$$P \to E \to C \to P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m - N - 1 - b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} K$ are zero in degrees $\geq m - N - 1$. Thus the cohomology of $R\Gamma(X, C \otimes^{\mathbf{L}} K)$ are zero in degrees $\geq m - 1$. Hence

$$R\Gamma(X, P \otimes^{\mathbf{L}} K) \to R\Gamma(X, E \otimes^{\mathbf{L}} K)$$

is an isomorphism on cohomology in degrees $\geq m$. By assumption the source is pseudo-coherent. We conclude that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m-pseudo-coherent as desired. $\square$

35. Relatively perfect objects

In this section we introduce a notion from [Lie06].

Definition 35.1. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. An object E of $D(\mathcal{O}_X)$ is *perfect relative to S* or *S-perfect* if E is pseudo-coherent (Cohomology, Definition 47.1) and E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ (Cohomology, Definition 48.1).

Please see Remark 35.14 for a discussion.

Example 35.2. Let k be a field. Let X be a scheme of finite presentation over k (in particular X is quasi-compact). Then an object E of $D(\mathcal{O}_X)$ is k-perfect if and only if it is bounded and pseudo-coherent (by definition), i.e., if and only if it is in $D^b_{Coh}(X)$ (by Lemma 10.3). Thus being relatively perfect does **not** mean "perfect on the fibres".

The corresponding algebra concept is studied in More on Algebra, Section 83. We can link the notion for schemes with the algebraic notion as follows.

Lemma 35.3. *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

1. *E is S-perfect,*
2. *for any affine open $U \subset X$ mapping into an affine open $V \subset S$ the complex $R\Gamma(U, E)$ is $\mathcal{O}_S(V)$-perfect.*
3. *there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the complex $R\Gamma(U_{ij}, E)$ is $\mathcal{O}_S(V_i)$-perfect.*

Proof. Being pseudo-coherent is a local property and "locally having finite tor dimension" is a local property. Hence this lemma immediately reduces to the statement: if X and S are affine, then E is S-perfect if and only if $K = R\Gamma(X, E)$ is $\mathcal{O}_S(S)$-perfect. Say $X = \text{Spec}(A)$, $S = \text{Spec}(R)$ and E corresponds to $K \in D(A)$, i.e., $K = R\Gamma(X, E)$, see Lemma 3.5.

Observe that K is R-perfect if and only if K is pseudo-coherent and has finite tor dimension as a complex of R-modules (More on Algebra, Definition 83.1). By Lemma 10.2 we see that E is pseudo-coherent if and only if K is pseudo-coherent. By Lemma 10.5 we see that E has finite tor dimension over $f^{-1}\mathcal{O}_S$ if and only if K has finite tor dimension as a complex of R-modules. $\square$

Lemma 35.4. *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. The full subcategory of $D(\mathcal{O}_X)$ consisting of S-perfect objects is a saturated[4] triangulated subcategory.*

Proof. This follows from Cohomology, Lemmas 47.4, 47.6, 48.6, and 48.8. $\square$

Lemma 35.5. *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. A perfect object of $D(\mathcal{O}_X)$ is S-perfect. If $K, M \in D(\mathcal{O}_X)$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$ is S-perfect if K is perfect and M is S-perfect.*

Proof. First proof: reduce to the affine case using Lemma 35.3 and then apply More on Algebra, Lemma 83.3. $\square$

Lemma 35.6. *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $g : S' \to S$ be a morphism of schemes. Set $X' = S' \times_S X$ and denote $g' : X' \to X$ the projection. If $K \in D(\mathcal{O}_X)$ is S-perfect, then $L(g')^*K$ is S'-perfect.*

Proof. First proof: reduce to the affine case using Lemma 35.3 and then apply More on Algebra, Lemma 83.5.

Second proof: $L(g')^*K$ is pseudo-coherent by Cohomology, Lemma 47.3 and the bounded tor dimension property follows from Lemma 22.8. $\square$

[4]Derived Categories, Definition 6.1.

Situation 35.7. Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $g_{i'i} : S_{i'} \to S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $g_i : S \to S_i$ the projection. We fix an element $0 \in I$ and a flat morphism of finite presentation $X_0 \to S_0$. We set $X_i = S_i \times_{S_0} X_0$ and $X = S \times_{S_0} X_0$ and we denote the transition morphisms $f_{i'i} : X_{i'} \to X_i$ and $f_i : X \to X_i$ the projections.

Lemma 35.8. *In Situation 35.7. Let K_0 and L_0 be objects of $D(\mathcal{O}_{X_0})$. Set $K_i = Lf_{i0}^*K_0$ and $L_i = Lf_{i0}^*L_0$ for $i \geq 0$ and set $K = Lf_0^*K_0$ and $L = Lf_0^*L_0$. Then the map*

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{X_i})}(K_i, L_i) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L)$$

is an isomorphism if K_0 is pseudo-coherent and $L_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ has (locally) finite tor dimension as an object of $D((X_0 \to S_0)^{-1}\mathcal{O}_{S_0})$

Proof. For every quasi-compact open $U_0 \subset X_0$ consider the condition P that

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{U_i})}(K_i|_{U_i}, L_i|_{U_i}) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U, L|_U)$$

is an isomorphism where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. If P holds for U_0, V_0 and $U_0 \cap V_0$, then it holds for $U_0 \cup V_0$ by Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 33.3.

Denote $\pi_0 : X_0 \to S_0$ the given morphism. Then we can first consider $U_0 = \pi_0^{-1}(W_0)$ with $W_0 \subset S_0$ quasi-compact open. By the induction principle of Cohomology of Schemes, Lemma 4.1 applied to quasi-compact opens of S_0 and the remark above, we find that it is enough to prove P for $U_0 = \pi_0^{-1}(W_0)$ with W_0 affine. In other words, we have reduced to the case where S_0 is affine. Next, we apply the induction principle again, this time to all quasi-compact and quasi-separated opens of X_0, to reduce to the case where X_0 is affine as well.

If X_0 and S_0 are affine, the result follows from More on Algebra, Lemma 83.7. Namely, by Lemmas 10.1 and 3.5 the statement is translated into computations of homs in the derived categories of modules. Then Lemma 10.2 shows that the complex of modules corresponding to K_0 is pseudo-coherent. And Lemma 10.5 shows that the complex of modules corresponding to L_0 has finite tor dimension over $\mathcal{O}_{S_0}(S_0)$. Thus the assumptions of More on Algebra, Lemma 83.7 are satisfied and we win. $\square$

Lemma 35.9. *In Situation 35.7 the category of S-perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of S_i-perfect objects of $D(\mathcal{O}_{X_i})$.*

Proof. For every quasi-compact open $U_0 \subset X_0$ consider the condition P that the functor

$$\mathrm{colim}_{i \geq 0} D_{S_i\text{-perfect}}(\mathcal{O}_{U_i}) \longrightarrow D_{S\text{-perfect}}(\mathcal{O}_U)$$

is an equivalence where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We observe that we already know this functor is fully faithful by Lemma 35.8. Thus it suffices to prove essential surjectivity.

Suppose that P holds for quasi-compact opens U_0, V_0 of X_0. We claim that P holds for $U_0 \cup V_0$. We will use the notation $U_i = f_{i0}^{-1}U_0$, $U = f_0^{-1}U_0$, $V_i = f_{i0}^{-1}V_0$, and $V = f_0^{-1}V_0$ and we will abusively use the symbol f_i for all the morphisms $U \to U_i$, $V \to V_i$, $U \cap V \to U_i \cap V_i$, and $U \cup V \to U_i \cup V_i$. Suppose E is an S-perfect object of $D(\mathcal{O}_{U \cup V})$. Goal: show E is in the essential image of the functor. By assumption, we can find $i \geq 0$, an S_i-perfect object $E_{U,i}$ on U_i, an S_i-perfect object $E_{V,i}$ on V_i, and isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. Let

$$a : E_{U,i} \to (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \cap V_i} \to E_{V,i}|_{U_i \cap V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U \cap V} \to E|_{U \cap V} \to Lf_i^*E_{V,i}|_{U \cap V}.$$

Apply Cohomology, Lemma 45.1 to get an object E_i on $U_i \cup V_i$ and a map $d : E_i \to Rf_{i,*}E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is S_i-perfect (because being relatively perfect is a local property) and that d is adjoint to an isomorphism $Lf_i^*E_i \to E$.

By exactly the same argument as used in the proof of Lemma 35.8 using the induction principle (Cohomology of Schemes, Lemma 4.1) we reduce to the case where both X_0 and S_0 are affine. (First work with opens in S_0 to reduce to S_0 affine, then work with opens in X_0 to reduce to X_0 affine.) In the affine case the result follows from More on Algebra, Lemma 83.7. The translation into algebra is done by Lemma 35.3. □

Lemma 35.10. *Let $f : X \to S$ be a morphism of schemes which is flat, proper, and of finite presentation. Let $E \in D(\mathcal{O}_X)$ be S-perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*

Proof. The statement on base change is Lemma 22.5. Thus it suffices to show that Rf_*E is a perfect object. We will reduce to the case where S is Noetherian affine by a limit argument.

The question is local on S, hence we may assume S is affine. Say $S = \text{Spec}(R)$. We write $R = \text{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits, Lemma 10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X. By Limits, Lemmas 13.1 and 8.7 we may assume X_i is proper and flat over R_i. By Lemma 35.9 we may assume there exists a R_i-perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 27.1 to $X_i \to \text{Spec}(R_i)$ and E_i and using the base change property already shown we obtain the result. □

Lemma 35.11. *Let $f : X \to S$ be a morphism of schemes. Let $E, K \in D(\mathcal{O}_X)$. Assume*

1. *S is quasi-compact and quasi-separated,*
2. *f is proper, flat, and of finite presentation,*
3. *E is S-perfect,*
4. *K is pseudo-coherent.*

Then there exists a pseudo-coherent $L \in D(\mathcal{O}_S)$ such that

$$Rf_*R\mathcal{H}om(K, E) = R\mathcal{H}om(L, \mathcal{O}_S)$$

and the same is true after arbitrary base change: given

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

cartesian, then we have
$$Rf'_*R\mathcal{H}om(L(g')^*K, L(g')^*E) = R\mathcal{H}om(Lg^*L, \mathcal{O}_{S'})$$

Proof. Since S is quasi-compact and quasi-separated, the same is true for X. By Lemma 19.1 we can write $K = \text{hocolim} K_n$ with K_n perfect and $K_n \to K$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $K_n^\vee$ be the dual perfect complex (Cohomology, Lemma 50.5). We obtain an inverse system $\ldots \to K_3^\vee \to K_2^\vee \to K_1^\vee$ of perfect objects. By Lemma 35.5 we see that $K_n^\vee \otimes_{\mathcal{O}_X} E$ is S-perfect. Thus we may apply Lemma 35.10 to $K_n^\vee \otimes_{\mathcal{O}_X} E$ and we obtain an inverse system

$$\ldots \to M_3 \to M_2 \to M_1$$

of perfect complexes on S with

$$M_n = Rf_*(K_n^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = Rf_* R\mathcal{H}om(K_n, E)$$

Moreover, the formation of these complexes commutes with any base change, namely $Lg^*M_n = Rf'_*((L(g')^*K_n)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} L(g')^*E) = Rf'_*R\mathcal{H}om(L(g')^*K_n, L(g')^*E)$.

As $K_n \to K$ induces an isomorphism on $\tau_{\geq -n}$, we see that $K_n \to K_{n+1}$ induces an isomorphism on $\tau_{\geq -n}$. It follows that $K_{n+1}^\vee \to K_n^\vee$ induces an isomorphism on $\tau_{\leq n}$ as $K_n^\vee = R\mathcal{H}om(K_n, \mathcal{O}_X)$. Suppose that E has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_Y$-modules. Then the same is true after any base change, see Lemma 22.8. We find that $K_{n+1}^\vee \otimes_{\mathcal{O}_X} E \to K_n^\vee \otimes_{\mathcal{O}_X} E$ induces an isomorphism on $\tau_{\leq n+a}$ and the same is true after any base change. Applying the right derived functor Rf_* we conclude the maps $M_{n+1} \to M_n$ induce isomorphisms on $\tau_{\leq n+a}$ and the same is true after any base change. Choose a distinguished triangle

$$M_{n+1} \to M_n \to C_n \to M_{n+1}[1]$$

Take S' equal to the spectrum of the residue field at a point $s \in S$ and pull back to see that $C_n \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s)$ has nonzero cohomology only in degrees $\geq n + a$. By More on Algebra, Lemma 75.6 we see that the perfect complex C_n has tor amplitude in $[n + a, m_n]$ for some integer m_n. In particular, the dual perfect complex $C_n^\vee$ has tor amplitude in $[-m_n, -n - a]$.

Let $L_n = M_n^\vee$ be the dual perfect complex. The conclusion from the discussion in the previous paragraph is that $L_n \to L_{n+1}$ induces isomorphisms on $\tau_{\geq -n-a}$. Thus $L = \text{hocolim} L_n$ is pseudo-coherent, see Lemma 19.1. Since we have

$$R\mathcal{H}om(K, E) = R\mathcal{H}om(\text{hocolim} K_n, E) = R\lim R\mathcal{H}om(K_n, E) = R\lim K_n^\vee \otimes_{\mathcal{O}_X} E$$

(Cohomology, Lemma 51.1) and since $R\lim$ commutes with Rf_* we find that

$$Rf_*R\mathcal{H}om(K, E) = R\lim M_n = R\lim R\mathcal{H}om(L_n, \mathcal{O}_S) = R\mathcal{H}om(L, \mathcal{O}_S)$$

This proves the formula over S. Since the construction of M_n is compatible with base chance, the formula continues to hold after any base change. $\square$

Remark 35.12. The reader may have noticed the similarity between Lemma 35.11 and Lemma 28.3. Indeed, the pseudo-coherent complex L of Lemma 35.11 may be characterized as the unique pseudo-coherent complex on S such that there are functorial isomorphisms

$$\text{Ext}^i_{\mathcal{O}_S}(L, \mathcal{F}) \longrightarrow \text{Ext}^i_{\mathcal{O}_X}(K, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})$$

compatible with boundary maps for $\mathcal{F}$ ranging over $QCoh(\mathcal{O}_S)$. If we ever need this we will formulate a precise result here and give a detailed proof.

Lemma 35.13. *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let E be a pseudo-coherent object of $D(\mathcal{O}_X)$. The following are equivalent*

(1) *E is S-perfect, and*
(2) *E is locally bounded below and for every point $s \in S$ the object $L(X_s \to X)^*E$ of $D(\mathcal{O}_{X_s})$ is locally bounded below.*

Proof. Since everything is local we immediately reduce to the case that X and S are affine, see Lemma 35.3. Say $X \to S$ corresponds to $\text{Spec}(A) \to \text{Spec}(R)$ and E corresponds to K in $D(A)$. If s corresponds to the prime $\mathfrak{p} \subset R$, then $L(X_s \to X)^*E$ corresponds to $K \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})$ as $R \to A$ is flat, see for example Lemma 22.5. Thus we see that our lemma follows from the corresponding algebra result, see More on Algebra, Lemma 83.10. □

Remark 35.14. Our Definition 35.1 of a relatively perfect complex is equivalent to the one given in [Lie06] whenever our definition applies[5]. Next, suppose that $f : X \to S$ is only assumed to be locally of finite type (not necessarily flat, nor locally of finite presentation). The definition in the paper cited above is that $E \in D(\mathcal{O}_X)$ is relatively perfect if

(A) locally on X the object E should be quasi-isomorphic to a finite complex of S-flat, finitely presented $\mathcal{O}_X$-modules.

On the other hand, the natural generalization of our Definition 35.1 is

(B) E is pseudo-coherent relative to S (More on Morphisms, Definition 59.2) and E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ (Cohomology, Definition 48.1).

The advantage of condition (B) is that it clearly defines a triangulated subcategory of $D(\mathcal{O}_X)$, whereas we suspect this is not the case for condition (A). The advantage of condition (A) is that it is easier to work with in particular in regards to limits.

36. The resolution property

This notion is discussed in the paper [Tot04]; the discussion is continued in [Gro10], [Gro12], and [Gro17]. It is currently not known if a proper scheme over a field always has the resolution property or if this is false. If you know the answer to this question, please email stacks.project@gmail.com.

We can make the following definition although it scarcely makes sense to consider it for general schemes.

Definition 36.1. Let X be a scheme. We say X has the *resolution property* if every quasi-coherent $\mathcal{O}_X$-module of finite type is the quotient of a finite locally free $\mathcal{O}_X$-module.

If X is a quasi-compact and quasi-separated scheme, then it suffices to check every $\mathcal{O}_X$-module module of finite presentation (automatically quasi-coherent) is the quotient of a finite locally free $\mathcal{O}_X$-module, see Properties, Lemma 22.8. If X is a Noetherian scheme, then finite type quasi-coherent modules are exactly the coherent $\mathcal{O}_X$-modules, see Cohomology of Schemes, Lemma 9.1.

Lemma 36.2. *Let X be a scheme. If X has an ample invertible $\mathcal{O}_X$-module, then X has the resolution property.*

Proof. Immediate consquence of Properties, Proposition 26.13. □

[5]To see this, use Lemma 35.3 and More on Algebra, Lemma 83.4.

Lemma 36.3. *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *Y is quasi-compact and quasi-separated and has the resolution property,*
(2) *there exists an f-ample invertible module on X.*

Then X has the resolution property.

Proof. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an f-ample invertible module. Choose an affine open covering $Y = V_1 \cup \ldots \cup V_m$. Set $U_j = f^{-1}(V_j)$. By Properties, Proposition 26.13 for each j we know there exists finitely many maps $s_{j,i} : \mathcal{L}^{\otimes n_{j,i}}|_{U_j} \to \mathcal{F}|_{U_j}$ which are jointly surjective. Consider the quasi-coherent $\mathcal{O}_Y$-modules

$$\mathcal{H}_n = f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

We may think of $s_{j,i}$ as a section over V_j of the sheaf $\mathcal{H}_{-n_{j,i}}$. Suppose we can find finite locally free $\mathcal{O}_Y$-modules $\mathcal{E}_{i,j}$ and maps $\mathcal{E}_{i,j} \to \mathcal{H}_{-n_{j,i}}$ such that $s_{j,i}$ is in the image. Then the corresponding maps

$$f^*\mathcal{E}_{i,j} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n_{i,j}} \longrightarrow \mathcal{F}$$

are going to be jointly surjective and the lemma is proved. By Properties, Lemma 22.3 for each i, j we can find a finite type quasi-coherent submodule $\mathcal{H}'_{i,j} \subset \mathcal{H}_{-n_{j,i}}$ which contains the section $s_{i,j}$ over V_j. Thus the resolution property of Y produces surjections $\mathcal{E}_{i,j} \to \mathcal{H}'_{j,i}$ and we conclude. $\square$

Lemma 36.4. *Let $f : X \to Y$ be an affine or quasi-affine morphism of schemes with Y quasi-compact and quasi-separated. If Y has the resolution property, so does X.*

Proof. By Morphisms, Lemma 37.6 this is a special case of Lemma 36.3. $\square$

Here is a case where one can prove the resolution property goes down.

Lemma 36.5. *Let $f : X \to Y$ be a surjective finite locally free morphism of schemes. If X has the resolution property, so does Y.*

Proof. The condition means that f is affine and that $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module of positive rank. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module of finite type. By assumption there exists a surjection $\mathcal{E} \to f^*\mathcal{G}$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. Since f_* is exact on quasi-coherent modules (Cohomology of Schemes, Lemma 2.3) we get a surjection

$$f_*\mathcal{E} \longrightarrow f_*f^*\mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X$$

Taking duals we get a surjection

$$f_*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathcal{G}$$

Since $f_*\mathcal{E}$ is finite locally free[6], we conclude. $\square$

Lemma 36.6. *Let X be a scheme. Suppose given*

(1) *a finite affine open covering $X = U_1 \cup \ldots \cup U_m$*
(2) *finite type quasi-coherent ideals $\mathcal{I}_j$ with $V(\mathcal{I}_j) = X \setminus U_j$*

[6] Namely, if $A \to B$ is a finite locally free ring map and N is a finite locally free B-module, then N is a finite locally free A-module. To see this, first note that N finite locally free over B implies N is flat and finitely presented as a B-module, see Algebra, Lemma 78.2. Then N is an A-module of finite presentation by Algebra, Lemma 36.23 and a flat A-module by Algebra, Lemma 39.4. Then conclude by using Algebra, Lemma 78.2 over A.

Then X has the resolution property if and only if $\mathcal{I}_j$ is the quotient of a finite locally free $\mathcal{O}_X$-module for $j = 1, \ldots, m$.

Proof. One direction of the lemma is trivial. For the other, say $\mathcal{E}_j \to \mathcal{I}_j$ is a surjection with $\mathcal{E}_j$ finite locally free. In the next paragraph, we reduce to the Noetherian case; we suggest the reader skip it.

The first observation is that $U_j \cap U_{j'}$ is quasi-compact as the complement of the zero scheme of the quasi-coherent finite type ideal $\mathcal{I}_{j'}|U_j$ on the affine scheme U_j, see Properties, Lemma 24.1. Hence X is quasi-compact and quasi-separated, see Schemes, Lemma 21.6. By Limits, Proposition 5.4 we can write $X = \lim X_i$ as the limit of a direct system of Noetherian schemes with affine transition morphisms. For each j we can find an i and a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_{i,j}$ pulling back to $\mathcal{E}_j$, see Limits, Lemma 10.3. After increasing i we may assume that the composition $\mathcal{E}_j \to \mathcal{I}_j \to \mathcal{O}_X$ is the pullback of a map $\mathcal{E}_{i,j} \to \mathcal{O}_{X_i}$, see Limits, Lemma 10.2. Denote $\mathcal{I}_{i,j} \subset \mathcal{O}_{X_i}$ the image of this map; this is a quasi-coherent ideal sheaf on the Noetherian scheme X_i whose pullback to X is $\mathcal{I}_j$. Denoting $U_{i,j} \subset X_i$ the complementary opens, we may assume these are affine for all i, j, see Limits, Lemma 4.13. If we can prove the lemma for the opens $U_{i,j}$ and the ideal sheaves $\mathcal{I}_{i,j}$ on X_i then X, being affine over X_i, will have the resolution property by Lemma 36.4. In this way we reduce to the case of a Noetherian scheme.

Assume X is Noetherian. For every coherent module $\mathcal{F}$ we can choose a finite list of sections $s_{jk} \in \mathcal{F}(U_j)$, $k = 1, \ldots, e_j$ which generate the restriction of $\mathcal{F}$ to U_j. By Cohomology of Schemes, Lemma 10.5 we can extend s_{jk} to a map $s'_{jk} : \mathcal{I}_i^{n_{jk}} \to \mathcal{F}$ for some $n_{jk} \geq 1$. Then we can consider the compositions

$$\mathcal{E}_j^{\otimes n_{jk}} \to \mathcal{I}_j^{n_{jk}} \to \mathcal{F}$$

to conclude. $\square$

Lemma 36.7. *Let X be a scheme. If X has an ample family of invertible modules (Morphisms, Definition 12.1), then X has the resolution property.*

Proof. Since X is quasi-compact, there exists n and pairs $(\mathcal{L}_i, s_i)$, $i = 1, \ldots, n$ where $\mathcal{L}_i$ is an invertible $\mathcal{O}_X$-module and $s_i \in \Gamma(X, \mathcal{L}_i)$ is a section such that the set of points $U_i \subset X$ where s_i is nonvanishing is affine and $X = U_1 \cup \ldots \cup U_n$. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the image of $s_i : \mathcal{L}_i^{\otimes -1} \to \mathcal{O}_X$. Applying Lemma 36.6 we find that X has the resolution property. $\square$

Lemma 36.8. *Let X be a quasi-compact, regular scheme with affine diagonal. Then X has the resolution property.*

Proof. Combine Divisors, Lemma 16.8 and the above Lemma 36.7. $\square$

Lemma 36.9. *Let $X = \lim X_i$ be a limit of a direct system of quasi-compact and quasi-separated schemes with affine transition morphisms. Then X has the resolution property if and only if X_i has the resolution properties for some i.*

Proof. If X_i has the resolution property, then X does by Lemma 36.4. Assume X has the resolution property. Choose $i \in I$. Choose a finite affine open covering $X_i = U_{i,1} \cup \ldots \cup U_{i,m}$. For each j choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}_{i,j} \subset \mathcal{O}_{X_i}$ such that $X_i \setminus V(\mathcal{I}_{i,j}) = U_{i,j}$, see Properties, Lemma 24.1. Denote $U_j \subset X$ the inverse image of $U_{i,j}$ and denote $\mathcal{I}_j \subset \mathcal{O}_X$ the pullback of $\mathcal{I}_{i,j}$. Since X

has the resolution property, we may choose finite locally free $\mathcal{O}_X$-modules $\mathcal{E}_j$ and surjections $\mathcal{E}_j \to \mathcal{I}_j$. By Limits, Lemmas 10.3 and 10.2 after increasing i we can find finite locally free $\mathcal{O}_{X_i}$-modules $\mathcal{E}_{i,j}$ and maps $\mathcal{E}_{i,j} \to \mathcal{O}_{X_i}$ whose base changes to X recover the compositions $\mathcal{E}_j \to \mathcal{I}_j \to \mathcal{O}_X$, $j = 1, \ldots, m$. The pullbacks of the finitely presented $\mathcal{O}_{X_i}$-modules $\mathrm{Coker}(\mathcal{E}_{i,j} \to \mathcal{O}_{X_i})$ and $\mathcal{O}_{X_i}/\mathcal{I}_{i,j}$ to X agree as quotients of $\mathcal{O}_X$. Hence by Limits, Lemma 10.2 we may assume that these agree, in other words that the image of $\mathcal{E}_{i,j} \to \mathcal{O}_{X_i}$ is equal to $\mathcal{I}_{i,j}$. Then we conclude that X_i has the resolution property by Lemma 36.6. $\square$

Lemma 36.10. *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Then X has affine diagonal.*

Proof. Combining Limits, Proposition 5.4 and Lemma 36.9 this reduces to the case where X is Noetherian (small detail omitted). Assume X is Noetherian. Recall that $X \times X$ is covered by the affine opens $U \times V$ for affine opens U, V of X, see Schemes, Section 17. Hence to show that the diagonal $\Delta : X \to X \times X$ is affine, it suffices to show that $U \cap V = \Delta^{-1}(U \times V)$ is affine for all affine opens U, V of X, see Morphisms, Lemma 11.3. In particular, it suffices to show that the inclusion morphism $j : U \to X$ is affine if U is an affine open of X. By Cohomology of Schemes, Lemma 3.4 it suffices to show that $R^1 j_* \mathcal{G} = 0$ for any quasi-coherent $\mathcal{O}_U$-module $\mathcal{G}$. By Proposition 8.3 (this is where we use that we've reduced to the Noetherian case) we can represent $Rj_*\mathcal{G}$ by a complex $\mathcal{H}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules. Assume

$$H^1(\mathcal{H}^\bullet) = \mathrm{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)/\mathrm{Im}(\mathcal{H}^0 \to \mathcal{H}^1)$$

is nonzero in order to get a contradiction. Then we can find a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a map

$$\mathcal{F} \longrightarrow \mathrm{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)$$

such that the composition with the projection onto $H^1(\mathcal{H}^\bullet)$ is nonzero. Namely, we can write $\mathrm{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)$ as the filtered union of its coherent submodules by Properties, Lemma 22.3 and then one of these will do the job. Next, we choose a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and a surjection $\mathcal{E} \to \mathcal{F}$ using the resolution property of X. This produces a map in the derived category

$$\mathcal{E}[-1] \longrightarrow Rj_*\mathcal{G}$$

which is nonzero on cohomology sheaves and hence nonzero in $D(\mathcal{O}_X)$. By adjunction, this is the same thing as a map

$$j^*\mathcal{E}[-1] \to \mathcal{G}$$

nonzero in $D(\mathcal{O}_U)$. Since $\mathcal{E}$ is finite locally free this is the same thing as a nonzero element of

$$H^1(U, j^*\mathcal{E}^\vee \otimes_{\mathcal{O}_U} \mathcal{G})$$

where $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual finite locally free module. However, this group is zero by Cohomology of Schemes, Lemma 2.2 which is the desired contradiction. (If in doubt about the step using duals, please see the more general Cohomology, Lemma 50.5.) $\square$

37. The resolution property and perfect complexes

In this section we discuss the relationship between perfect complexes and strictly perfect complexes on schemes which have the resolution property.

Lemma 37.1. *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Let $\mathcal{F}^\bullet$ be a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules representing a perfect object of $D(\mathcal{O}_X)$. Then there exists a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules and a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$.*

Proof. Let $a, b \in \mathbf{Z}$ be integers such that $\mathcal{F}^\bullet$ has tor amplitude in $[a, b]$ and such that $\mathcal{F}^n = 0$ for $n < a$. The existence of such a pair of integers follows from Cohomology, Lemma 49.5 and the fact that X is quasi-compact. If $b < a$, then $\mathcal{F}^\bullet$ is zero in the derived category and the lemma holds. We will prove by induction on $b - a \geq 0$ that there exists a complex $\mathcal{E}^a \to \ldots \to \mathcal{E}^b$ with $\mathcal{E}^i$ finite locally free and a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$.

The base case is the case $b - a = 0$. In this case $H^b(\mathcal{F}^\bullet) = H^a(\mathcal{F}^\bullet) = \mathrm{Ker}(\mathcal{F}^a \to \mathcal{F}^{a+1})$ is finite locally free. Namely, it is a finitely presented $\mathcal{O}_X$-module of tor dimension 0 and hence finite locally free. See Cohomology, Lemmas 49.5 and 47.9 and Properties, Lemma 20.2. Thus we can take $\mathcal{E}^\bullet$ to be $H^b(\mathcal{F}^\bullet)$ sitting in degree b. The rest of the proof is dedicated to the induction step.

Assume $b > a$. Observe that

$$H^b(\mathcal{F}^\bullet) = \mathrm{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1}) / \mathrm{Im}(\mathcal{F}^{b-1} \to \mathcal{F}^b)$$

is a finite type quasi-coherent $\mathcal{O}_X$-module, see Cohomology, Lemmas 49.5 and 47.9. Then we can find a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a map

$$\mathcal{F} \longrightarrow \mathrm{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1})$$

such that the composition with the projection onto $H^b(\mathcal{F}^\bullet)$ is surjective. Namely, we can write $\mathrm{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1})$ as the filtered union of its finite type quasi-coherent submodules by Properties, Lemma 22.3 and then one of these will do the job. Next, we choose a finite locally free $\mathcal{O}_X$-module $\mathcal{E}^b$ and a surjection $\mathcal{E}^b \to \mathcal{F}$ using the resolution property of X. Consider the map of complexes

$$\alpha : \mathcal{E}^b[-b] \to \mathcal{F}^\bullet$$

and its cone $C(\alpha)^\bullet$, see Derived Categories, Definition 9.1. We observe that $C(\alpha)^\bullet$ is nonzero only in degrees $\geq a$, has tor amplitude in $[a, b]$ by Cohomology, Lemma 48.6, and has $H^b(C(\alpha)^\bullet) = 0$ by construction. Thus we actually find that $C(\alpha)^\bullet$ has tor amplitude in $[a, b-1]$. Hence the induction hypothesis applies to $C(\alpha)^\bullet$ and we find a map of complexes

$$(\mathcal{E}^a \to \ldots \to \mathcal{E}^{b-1}) \longrightarrow C(\alpha)^\bullet$$

with properties as stated in the induction hypothesis. Unwinding the definition of the cone this gives a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \mathcal{E}^{b-2} \arrow[r] \arrow[d] & \mathcal{E}^{b-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & \mathcal{F}^{b-2} \arrow[r] & \mathcal{F}^{b-1} \oplus \mathcal{E}^b \arrow[r] & \mathcal{F}^b \arrow[r] & \ldots
\end{tikzcd}$$

It is clear that we obtain a map of complexes $(\mathcal{E}^a \to \ldots \to \mathcal{E}^b) \to \mathcal{F}^\bullet$. We omit the verification that this map is a quasi-isomorphism. □

Lemma 37.2. *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Then every perfect object of $D(\mathcal{O}_X)$ can be represented by a bounded complex of finite locally free $\mathcal{O}_X$-modules.*

Proof. Let E be a perfect object of $D(\mathcal{O}_X)$. By Lemma 36.10 we see that X has affine diagonal. Hence by Proposition 7.5 we can represent E by a complex $\mathcal{F}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules. Observe that E is in $D^b(\mathcal{O}_X)$ because X is quasi-compact. Hence $\tau_{\geq n}\mathcal{F}^\bullet$ is a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules which represents E if $n \ll 0$. Thus we may apply Lemma 37.1 to conclude. □

Lemma 37.3. *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Let $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ be finite complexes of finite locally free $\mathcal{O}_X$-modules. Then any $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ can be represented by a diagram*

$$\mathcal{E}^\bullet \leftarrow \mathcal{G}^\bullet \to \mathcal{F}^\bullet$$

where $\mathcal{G}^\bullet$ is a bounded complex of finite locally free $\mathcal{O}_X$-modules and where $\mathcal{G}^\bullet \to \mathcal{E}^\bullet$ is a quasi-isomorphism.

Proof. By Lemma 36.10 we see that X has affine diagonal. Hence by Proposition 7.5 we can represent α by a diagram

$$\mathcal{E}^\bullet \leftarrow \mathcal{H}^\bullet \to \mathcal{F}^\bullet$$

where $\mathcal{H}^\bullet$ is a complex of quasi-coherent $\mathcal{O}_X$-modules and where $\mathcal{H}^\bullet \to \mathcal{E}^\bullet$ is a quasi-isomorphism. For $n \ll 0$ the maps $\mathcal{H}^\bullet \to \mathcal{E}^\bullet$ and $\mathcal{H}^\bullet \to \mathcal{F}^\bullet$ factor through the quasi-isomorphism $\mathcal{H}^\bullet \to \tau_{\geq n}\mathcal{H}^\bullet$ simply because $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are bounded complexes. Thus we may replace $\mathcal{H}^\bullet$ by $\tau_{\geq n}\mathcal{H}^\bullet$ and assume that $\mathcal{H}^\bullet$ is bounded below. Then we may apply Lemma 37.1 to conclude. □

Lemma 37.4. *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Let $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ be finite complexes of finite locally free $\mathcal{O}_X$-modules. Let $\alpha^\bullet, \beta^\bullet : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be two maps of complexes defining the same map in $D(\mathcal{O}_X)$. Then there exists a quasi-isomorphism $\gamma^\bullet : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ where $\mathcal{G}^\bullet$ is a bounded complex of finite locally free $\mathcal{O}_X$-modules such that $\alpha^\bullet \circ \gamma^\bullet$ and $\beta^\bullet \circ \gamma^\bullet$ are homotopic maps of complexes.*

Proof. By Lemma 36.10 we see that X has affine diagonal. Hence by Proposition 7.5 (and the definition of the derived category) there exists a quasi-isomorphism $\gamma^\bullet : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ where $\mathcal{G}^\bullet$ is a complex of quasi-coherent $\mathcal{O}_X$-modules such that $\alpha^\bullet \circ \gamma^\bullet$ and $\beta^\bullet \circ \gamma^\bullet$ are homotopic maps of complexes. Choose a homotopy $h^i : \mathcal{G}^i \to \mathcal{F}^{i-1}$ witnessing this fact. Choose $n \ll 0$. Then the map $\gamma^\bullet$ factors canonically over the quotient map $\mathcal{G}^\bullet \to \tau_{\geq n}\mathcal{G}^\bullet$ as $\mathcal{E}^\bullet$ is bounded below. For the exact same reason the maps h^i will factor over the surjections $\mathcal{G}^i \to (\tau_{\geq n}\mathcal{G})^i$. Hence we see that we may replace $\mathcal{G}^\bullet$ by $\tau_{\geq n}\mathcal{G}^\bullet$. Then we may apply Lemma 37.1 to conclude. □

Proposition 37.5. *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Denote*

1. *$\mathcal{A}$ the additive category of finite locally free $\mathcal{O}_X$-modules,*
2. *$K^b(\mathcal{A})$ the homotopy category of bounded complexes in $\mathcal{A}$, see Derived Categories, Section 8, and*

(3) $D_{perf}(\mathcal{O}_X)$ *the strictly full, saturated, triangulated subcategory of* $D(\mathcal{O}_X)$ *consisting of perfect objects.*

With this notation the obvious functor

$$K^b(\mathcal{A}) \longrightarrow D_{perf}(\mathcal{O}_X)$$

is an exact functor of trianglated categories which factors through an equivalence $S^{-1}K^b(\mathcal{A}) \to D_{perf}(\mathcal{O}_X)$ *of triangulated categories where* S *is the saturated multiplicative system of quasi-isomorphisms in* $K^b(\mathcal{A})$*.*

Proof. If you can parse the statement of the proposition, then please skip this first paragraph. For some of the definitions used, please see Derived Categories, Definition 3.4 (triangulated subcategory), Derived Categories, Definition 6.1 (saturated triangulated subcategory), Derived Categories, Definition 5.1 (multiplicative system compatible with the triangulated structure), and Categories, Definition 27.20 (saturated multiplicative system). Observe that $D_{perf}(\mathcal{O}_X)$ is a saturated triangulated subcategory of $D(\mathcal{O}_X)$ by Cohomology, Lemmas 49.7 and 49.9. Also, note that $K^b(\mathcal{A})$ is a triangulated category, see Derived Categories, Lemma 10.5.

It is clear that the functor sends distinguished triangles to distinguished triangles, i.e., is exact. Then S is a saturated multiplicative system compatible with the triangulated structure on $K^b(\mathcal{A})$ by Derived Categories, Lemma 5.4. Hence the localization $S^{-1}K^b(\mathcal{A})$ exists and is a triangulated category by Derived Categories, Proposition 5.6. We get an exact factorization $S^{-1}K^b(\mathcal{A}) \to D_{perf}(\mathcal{O}_X)$ by Derived Categories, Lemma 5.7. By Lemmas 37.2, 37.3, and 37.4 this functor is an equivalence. Then finally the functor $S^{-1}K^b(\mathcal{A}) \to D_{perf}(\mathcal{O}_X)$ is an equivalence of triangulated categories (in the sense that distinguished triangles correspond) by Derived Categories, Lemma 4.18. $\square$

38. K-groups

A tiny bit about K_0 of various categories associated to schemes. Previous material can be found in Algebra, Section 55, Homology, Section 11, Derived Categories, Section 28, and More on Algebra, Lemma 119.2.

Analogous to Algebra, Section 55 we will define two K-groups $K'_0(X)$ and $K_0(X)$ for any Noetherian scheme X. The first will use coherent $\mathcal{O}_X$-modules and the second will use finite locally free $\mathcal{O}_X$-modules.

Lemma 38.1. *Let* X *be a Noetherian scheme. Then*

$$K_0(\mathit{Coh}(\mathcal{O}_X)) = K_0(D^b(\mathit{Coh}(\mathcal{O}_X)) = K_0(D^b_{\mathit{Coh}}(\mathcal{O}_X))$$

Proof. The first equality is Derived Categories, Lemma 28.2. We have $K_0(\mathit{Coh}(\mathcal{O}_X)) = K_0(D^b_{\mathit{Coh}}(\mathcal{O}_X))$ by Derived Categories, Lemma 28.5. This proves the lemma. (We can also use that $D^b(\mathit{Coh}(\mathcal{O}_X)) = D^b_{\mathit{Coh}}(\mathcal{O}_X)$ by Proposition 11.2 to see the second equality.) $\square$

Here is the definition.

Definition 38.2. Let X be a scheme.

(1) We denote $K_0(X)$ the *Grothendieck group of* X. It is the zeroth K-group of the strictly full, saturated, triangulated subcategory $D_{perf}(\mathcal{O}_X)$ of $D(\mathcal{O}_X)$ consisting of perfect objects. In a formula

$$K_0(X) = K_0(D_{perf}(\mathcal{O}_X))$$

(2) If X is locally Noetherian, then we denote $K'_0(X)$ the *Grothendieck group of coherent sheaves on* X. It is the is the zeroth K-group of the abelian category of coherent $\mathcal{O}_X$-modules. In a formula

$$K'_0(X) = K_0(\textit{Coh}(\mathcal{O}_X))$$

We will show that our definition of $K_0(X)$ agrees with the often used definition in terms of finite locally free modules if X has the resolution property (for example if X has an ample invertible module). See Lemma 38.5.

Lemma 38.3. *Let $X = \text{Spec}(R)$ be an affine scheme. Then $K_0(X) = K_0(R)$ and if R is Noetherian then $K'_0(X) = K'_0(R)$.*

Proof. Recall that $K'_0(R)$ and $K_0(R)$ have been defined in Algebra, Section 55.

By More on Algebra, Lemma 119.2 we have $K_0(R) = K_0(D_{perf}(R))$. By Lemmas 10.7 and 3.5 we have $D_{perf}(R) = D_{perf}(\mathcal{O}_X)$. This proves the equality $K_0(R) = K_0(X)$.

The equality $K'_0(R) = K'_0(X)$ holds because $\textit{Coh}(\mathcal{O}_X)$ is equivalent to the category of finite R-modules by Cohomology of Schemes, Lemma 9.1. Moreover it is clear that $K'_0(R)$ is the zeroth K-group of the category of finite R-modules from the definitions. $\square$

Let X be a Noetherian scheme. Then both $K'_0(X)$ and $K_0(X)$ are defined. In this case there is a canonical map

$$K_0(X) = K_0(D_{perf}(\mathcal{O}_X)) \longrightarrow K_0(D^b_{Coh}(\mathcal{O}_X)) = K'_0(X)$$

Namely, perfect complexes are in $D^b_{Coh}(\mathcal{O}_X)$ (by Lemma 10.3), the inclusion functor $D_{perf}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_X)$ induces a map on zeroth K-groups (Derived Categories, Lemma 28.3), and we have the equality on the right by Lemma 38.1.

Lemma 38.4. *Let X be a Noetherian regular scheme. Then the map $K_0(X) \to K'_0(X)$ is an isomorphism.*

Proof. Follows immediately from Lemma 11.8 and our construction of the map $K_0(X) \to K'_0(X)$ above. $\square$

Let X be a scheme. Let us denote $\textit{Vect}(X)$ the category of finite locally free $\mathcal{O}_X$-modules. Although $\textit{Vect}(X)$ isn't an abelian category in general, it is clear what a short exact sequence of $\textit{Vect}(X)$ is. Denote $K_0(\textit{Vect}(X))$ the unique abelian group with the following properties[7]:

(1) For every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ there is given an element $[\mathcal{E}]$ in $K_0(\textit{Vect}(X))$,
(2) for every short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of finite locally free $\mathcal{O}_X$-modules we have the relation $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ in $K_0(\textit{Vect}(X))$,
(3) the group $K_0(\textit{Vect}(X))$ is generated by the elements $[\mathcal{E}]$, and
(4) all relations in $K_0(\textit{Vect}(X))$ among the generators $[\mathcal{E}]$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

[7]The correct generality here would be to define K_0 for any exact category, see Injectives, Remark 9.6.

We omit the detailed construction of $K_0(\mathit{Vect}(X))$. There is a natural map

$$K_0(\mathit{Vect}(X)) \longrightarrow K_0(X)$$

Namely, given a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ let us denote $\mathcal{E}[0]$ the perfect complex on X which has $\mathcal{E}$ sitting in degree 0 and zero in other degrees. Given a short exact sequence $0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$ of finite locally free $\mathcal{O}_X$-modules we obtain a distinguished triangle $\mathcal{E}[0] \to \mathcal{E}'[0] \to \mathcal{E}''[0] \to \mathcal{E}[1]$, see Derived Categories, Section 12. This shows that we obtain a map $K_0(\mathit{Vect}(X)) \to K_0(D_{perf}(\mathcal{O}_X)) = K_0(X)$ by sending $[\mathcal{E}]$ to $[\mathcal{E}[0]]$ with apologies for the horrendous notation.

Lemma 38.5. *Let X be a quasi-compact and quasi-separated scheme with the resolution property. Then the map $K_0(\mathit{Vect}(X)) \to K_0(X)$ is an isomorphism.*

Proof. This lemma will follow in a straightforward manner from Lemmas 37.2, 37.3, and 37.4 whose results we will use without further mention. Let us construct an inverse map

$$c : K_0(X) = K_0(D_{perf}(\mathcal{O}_X)) \longrightarrow K_0(\mathit{Vect}(X))$$

Namely, any object of $D_{perf}(\mathcal{O}_X)$ can be represented by a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules. Then we set

$$c([\mathcal{E}^\bullet]) = \sum (-1)^i [\mathcal{E}^i]$$

Of course we have to show that this is well defined. For the moment we view c as a map defined on bounded complexes of finite locally free $\mathcal{O}_X$-modules.

Suppose that $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ is a surjective map of bounded complexes of finite locally free $\mathcal{O}_X$-modules. Let $\mathcal{K}^\bullet$ be the kernel. Then we obtain short exact sequences of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}^n \to \mathcal{E}^n \to \mathcal{F}^n \to 0$$

which are locally split because $\mathcal{F}^n$ is finite locally free. Hence $\mathcal{K}^\bullet$ is also a bounded complex of finite locally free $\mathcal{O}_X$-modules and we have $c(\mathcal{E}^\bullet) = c(\mathcal{K}^\bullet) + c(\mathcal{F}^\bullet)$ in $K_0(\mathit{Vect}(X))$.

Suppose given a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules which is acyclic. Say $\mathcal{E}^n = 0$ for $n \not\in [a, b]$. Then we can break $\mathcal{E}^\bullet$ into short exact sequences

$$\begin{matrix} 0 \to \mathcal{E}^a \to \mathcal{E}^{a+1} \to \mathcal{F}^{a+1} \to 0, \\ 0 \to \mathcal{F}^{a+1} \to \mathcal{E}^{a+2} \to \mathcal{F}^{a+3} \to 0, \\ \ldots \\ 0 \to \mathcal{F}^{b-3} \to \mathcal{E}^{b-2} \to \mathcal{F}^{b-2} \to 0, \\ 0 \to \mathcal{F}^{b-2} \to \mathcal{E}^{b-1} \to \mathcal{E}^b \to 0 \end{matrix}$$

Arguing by descending induction we see that $\mathcal{F}^{b-2}, \ldots, \mathcal{F}^{a+1}$ are finite locally free $\mathcal{O}_X$-modules, and

$$c(\mathcal{E}^\bullet) = \sum (-1)[\mathcal{E}^n] = \sum (-1)^n([\mathcal{F}^{n-1}] + [\mathcal{F}^n]) = 0$$

Thus our construction gives zero on acyclic complexes.

It follows from the results of the preceding two paragraphs that c is well defined. Namely, suppose the bounded complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ of finite locally free $\mathcal{O}_X$-modules represent the same object of $D(\mathcal{O}_X)$. Then we can find quasi-isomorphisms

$a : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ and $b : \mathcal{G}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{G}^\bullet$ bounded complex of finite locally free $\mathcal{O}_X$-modules. We obtain a short exact sequence of complexes

$$0 \to \mathcal{E}^\bullet \to C(a)^\bullet \to \mathcal{G}^\bullet[1] \to 0$$

see Derived Categories, Definition 9.1. Since a is a quasi-isomorphism, the cone $C(a)^\bullet$ is acyclic (this follows for example from the discussion in Derived Categories, Section 12). Hence

$$0 = c(C(f)^\bullet) = c(\mathcal{E}^\bullet) + c(\mathcal{G}^\bullet[1]) = c(\mathcal{E}^\bullet) - c(\mathcal{G}^\bullet)$$

as desired. The same argument using b shows that $0 = c(\mathcal{F}^\bullet) - c(\mathcal{G}^\bullet)$. Hence we find that $c(\mathcal{E}^\bullet) = c(\mathcal{F}^\bullet)$ and c is well defined.

A similar argument using the cone on a map $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ of bounded complexes of finite locally free $\mathcal{O}_X$-modules shows that $c(Y) = c(X) + c(Z)$ if $X \to Y \to Z$ is a distinguished triangle in $D_{perf}(\mathcal{O}_X)$. Details omitted. Thus we get the desired homomorphism of abelian groups $c : K_0(X) \to K_0(\textit{Vect}(X))$.

It is clear that the composition $K_0(\textit{Vect}(X)) \to K_0(X) \to K_0(\textit{Vect}(X))$ is the identity. On the other hand, let $\mathcal{E}^\bullet$ be a bounded complex of finite locally free $\mathcal{O}_X$-modules. Then the the existence of the distinguished triangles of "stupid truncations" (see Homology, Section 15)

$$\sigma_{\geq n}\mathcal{E}^\bullet \to \sigma_{\geq n-1}\mathcal{E}^\bullet \to \mathcal{E}^{n-1}[-n+1] \to (\sigma_{\geq n}\mathcal{E}^\bullet)[1]$$

and induction show that

$$[\mathcal{E}^\bullet] = \sum (-1)^i[\mathcal{E}^i[0]]$$

in $K_0(X) = K_0(D_{perf}(\mathcal{O}_X))$ with apologies for the notation. Hence the map $K_0(\textit{Vect}(X)) \to K_0(D_{perf}(\mathcal{O}_X)) = K_0(X)$ is surjective which finishes the proof. $\square$

Remark 38.6. Let X be a scheme. The K-group $K_0(X)$ is canonically a commutative ring. Namely, using the derived tensor product

$$\otimes = \otimes^{\mathbf{L}}_{\mathcal{O}_X} : D_{perf}(\mathcal{O}_X) \times D_{perf}(\mathcal{O}_X) \longrightarrow D_{perf}(\mathcal{O}_X)$$

and Derived Categories, Lemma 28.6 we obtain a bilinear multiplication. Since $K \otimes L \cong L \otimes K$ we see that this product is commutative. Since $(K \otimes L) \otimes M = K \otimes (L \otimes M)$ we see that this product is associative. Finally, the unit of $K_0(X)$ is the element $1 = [\mathcal{O}_X]$.

If $\textit{Vect}(X)$ and $K_0(\textit{Vect}(X))$ are as above, then it is clearly the case that $K_0(\textit{Vect}(X))$ also has a ring structure: if $\mathcal{E}$ and $\mathcal{F}$ are finite locally free $\mathcal{O}_X$-modules, then we set

$$[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}]$$

The reader easily verifies that this indeed defines a bilinear commutative, associative product. Details omitted. The map

$$K_0(\textit{Vect}(X)) \longrightarrow K_0(X)$$

constructed above is a ring map with these definitions.

Now assume X is Noetherian. The derived tensor product also produces a map

$$\otimes = \otimes^{\mathbf{L}}_{\mathcal{O}_X} : D_{perf}(\mathcal{O}_X) \times D^b_{Coh}(\mathcal{O}_X) \longrightarrow D^b_{Coh}(\mathcal{O}_X)$$

Again using Derived Categories, Lemma 28.6 we obtain a bilinear multiplication $K_0(X) \times K'_0(X) \to K'_0(X)$ since $K'_0(X) = K_0(D^b_{Coh}(\mathcal{O}_X))$ by Lemma 38.1. The reader easily shows that this gives $K'_0(X)$ the structure of a module over the ring $K_0(X)$.

Remark 38.7. Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. There is a map

$$f_* : K'_0(X) \longrightarrow K'_0(Y)$$

which sends $[\mathcal{F}]$ to

$$[\bigoplus\nolimits_{i \geq 0} R^{2i} f_*\mathcal{F}] - [\bigoplus\nolimits_{i \geq 0} R^{2i+1} f_*\mathcal{F}]$$

This is well defined because the sheaves $R^i f_*\mathcal{F}$ are coherent (Cohomology of Schemes, Proposition 19.1), because locally only a finite number are nonzero, and because a short exact sequence of coherent sheaves on X produces a long exact sequence of $R^i f_*$ on Y. If Y is quasi-compact (the only case most often used in practice), then we can rewrite the above as

$$f_*[\mathcal{F}] = \sum (-1)^i [R^i f_*\mathcal{F}] = [Rf_*\mathcal{F}]$$

where we have used the equality $K'_0(Y) = K_0(D^b_{Coh}(Y))$ from Lemma 38.1.

Lemma 38.8. *Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. Then we have $f_*(\alpha \cdot f^*\beta) = f_*\alpha \cdot \beta$ for $\alpha \in K'_0(X)$ and $\beta \in K_0(Y)$.*

Proof. Follows from Lemma 22.1, the discussion in Remark 38.7, and the definition of the product $K'_0(X) \times K_0(X) \to K'_0(X)$ in Remark 38.6. $\square$

Remark 38.9. Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Consider the strictly full, saturated, triangulated subcategory

$$D_{Z,perf}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

consisting of perfect complexes of $\mathcal{O}_X$-modules whose cohomology sheaves are set-theoretically supported on Z. The zeroth K-group $K_0(D_{Z,perf}(\mathcal{O}_X))$ of this triangulated category is sometimes denoted $K_Z(X)$ or $K_{0,Z}(X)$. Using derived tensor product exactly as in Remark 38.6 we see that $K_0(D_{Z,perf}(\mathcal{O}_X))$ has a multiplication which is associative and commutative, but in general $K_0(D_{Z,perf}(\mathcal{O}_X))$ doesn't have a unit.

39. Determinants of complexes

This section is the continuation of More on Algebra, Section 122. For any ringed space $(X, \mathcal{O}_X)$ there is a functor

$$\det : \left\{ \begin{matrix} \text{category of perfect complexes} \\ \text{morphisms are isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{category of invertible modules} \\ \text{morphisms are isomorphisms} \end{matrix} \right\}$$

Moreover, given an object (L, F) of the filtered derived category $DF(\mathcal{O}_X)$ whose filtration is finite and whose graded parts are perfect complexes, there is a canonical isomorphism $\det(\text{gr}L) \to \det(L)$. See [KM76] for the original exposition. We will add this material later (insert future reference).

For the moment we will present an ad hoc construction in the case where X is a scheme and where we consider perfect objects L in $D(\mathcal{O}_X)$ of tor-amplitude in $[-1, 0]$.

Lemma 39.1. *Let X be a scheme. There is a functor*

$$\det : \left\{ \begin{matrix} \textit{category of perfect complexes} \\ \textit{with tor amplitude in } [-1, 0] \\ \textit{morphisms are isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \textit{category of invertible modules} \\ \textit{morphisms are isomorphisms} \end{matrix} \right\}$$

In addition, given a rank 0 perfect object L of $D(\mathcal{O}_X)$ with tor-amplitude in $[-1, 0]$ there is a canonical element $\delta(L) \in \Gamma(X, \det(L))$ such that for any isomorphism $a : L \to K$ in $D(\mathcal{O}_X)$ we have $\det(a)(\delta(L)) = \delta(K)$. Moreover, the construction is affine locally given by the construction of More on Algebra, Section 122.

Proof. Let L be an object of the left hand side. If $\text{Spec}(A) = U \subset X$ is an affine open, then $L|_U$ corresponds to a perfect complex $L^\bullet$ of A-modules with tor-amplitude in $[-1, 0]$, see Lemmas 3.5, 10.4, and 10.7. Then we can consider the invertible A-module $\det(L^\bullet)$ constructed in More on Algebra, Lemma 122.4. If $\text{Spec}(B) = V \subset U$ is another affine open contained in U, then $\det(L^\bullet) \otimes_A B = \det(L^\bullet \otimes_A B)$ and hence this construction is compatible with restriction mappings (see Lemma 3.8 and note $A \to B$ is flat). Thus we can glue these invertible modules to obtain an invertible module $\det(L)$ on X. The functoriality and canonical sections are constructed in exactly the same manner. Details omitted. $\square$

Remark 39.2. The construction of Lemma 39.1 is compatible with pullbacks. More precisely, given a morphism $f : X \to Y$ of schemes and a perfect object K of $D(\mathcal{O}_Y)$ of tor-amplitude in $[-1, 0]$ then Lf^*K is a perfect object K of $D(\mathcal{O}_X)$ of tor-amplitude in $[-1, 0]$ and we have a canonical identification

$$f^* \det(K) \longrightarrow \det(Lf^*K)$$

Moreover, if K has rank 0, then $\delta(K)$ pulls back to $\delta(Lf^*K)$ via this map. This is clear from the affine local construction of the determinant.

40. Detecting Boundedness

In this section, we show that compact generators of D_{QCoh} of a quasi-compact, quasi-separated scheme, as constructed in Section 15, have a special property. We recommend reading that section first as it is very similar to this one.

Lemma 40.1. *In Situation 9.1 denote $j : U \to X$ the open immersion and let K be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Let $E \in D_{QCoh}(\mathcal{O}_X)$ and $a \in \mathbf{Z}$. Consider the following conditions*

1. *The canonical map $\tau_{\geq a} E \to \tau_{\geq a} Rj_*(E|_U)$ is an isomorphism.*
2. *We have $\text{Hom}_{D(\mathcal{O}_X)}(K[-n], E) = 0$ for all $n \geq a$.*

Then (2) implies (1) and (1) implies (2) with a replaced by $a + 1$.

Proof. Choose a distinguished triangle $N \to E \to Rj_*(E|_U) \to N[1]$. Then (1) implies $\tau_{\geq a+1} N = 0$ and (1) is implied by $\tau_{\geq a} N = 0$. Observe that

$$\text{Hom}_{D(\mathcal{O}_X)}(K[-n], Rj_*(E|_U)) = \text{Hom}_{D(\mathcal{O}_U)}(K|_U[-n], E) = 0$$

for all n as $K|_U = 0$. Thus (2) is equivalent to $\text{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \geq a$. Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i + e''_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e''_i}, \ldots, f_r^{e_r}) \to \ldots$$

of Koszul complexes, see More on Algebra, Lemma 28.11. Hence $\text{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \geq a$ is equivalent to $\text{Hom}_{D(\mathcal{O}_X)}(K_e[-n], N) = 0$ for all $n \geq a$ and all

$e \geq 1$ with K_e as in Lemma 9.6. Since $N|_U = 0$, that lemma implies that this in turn is equivalent to $H^n(X, N) = 0$ for $n \geq a$. We conclude that (2) is equivalent to $\tau_{\geq a} N = 0$ since N is determined by the complex of A-modules $R\Gamma(X, N)$, see Lemma 3.5. Thus we find that our lemma is true. $\square$

Lemma 40.2. *In Situation 9.1 denote $j : U \to X$ the open immersion and let K be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Let $E \in D_{QCoh}(\mathcal{O}_X)$ and $a \in \mathbf{Z}$. Consider the following conditions*

1. *The canonical map $\tau_{\leq a} E \to \tau_{\leq a} Rj_*(E|_U)$ is an isomorphism, and*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], E) = 0$ for all $n \leq a$.*

Then (2) implies (1) and (1) implies (2) with a replaced by $a - 1$.

Proof. Choose a distinguished triangle $E \to Rj_*(E|_U) \to N \to E[1]$. Then (1) implies $\tau_{\leq a-1} N = 0$ and (1) is implied by $\tau_{\leq a} N = 0$. Observe that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], Rj_*(E|_U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U[-n], E) = 0$$

for all n as $K|_U = 0$. Thus (2) is equivalent to $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \leq a$. Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i + e''_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e''_i}, \ldots, f_r^{e_r}) \to \ldots$$

of Koszul complexes, see More on Algebra, Lemma 28.11. Hence $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \leq a$ is equivalent to $\mathrm{Hom}_{D(\mathcal{O}_X)}(K_e[-n], N) = 0$ for all $n \leq a$ and all $e \geq 1$ with K_e as in Lemma 9.6. Since $N|_U = 0$, that lemma implies that this in turn is equivalent to $H^n(X, N) = 0$ for $n \leq a$. We conclude that (2) is equivalent to $\tau_{\leq a} N = 0$ since N is determined by the complex of A-modules $R\Gamma(X, N)$, see Lemma 3.5. Thus we find that our lemma is true. $\square$

Lemma 40.3. *Let X be a quasi-compact and quasi-separated scheme. Let $P \in D_{perf}(\mathcal{O}_X)$ and $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a \in \mathbf{Z}$. The following are equivalent*

1. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$ for $i \gg 0$, and*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\geq a} E) = 0$ for $i \gg 0$.*

Proof. Using the triangle $\tau_{<a} E \to E \to \tau_{\geq a} E \to$ we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{<a} E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{<a} E)[i]) = 0$$

for $i \gg 0$. As P is perfect this is true by Lemma 18.2. $\square$

Lemma 40.4. *Let X be a quasi-compact and quasi-separated scheme. Let $P \in D_{perf}(\mathcal{O}_X)$ and $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a \in \mathbf{Z}$. The following are equivalent*

1. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$ for $i \ll 0$, and*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\leq a} E) = 0$ for $i \ll 0$.*

Proof. Using the triangle $\tau_{\leq a} E \to E \to \tau_{>a} E \to$ we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{>a} E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{>a} E)[i]) = 0$$

for $i \ll 0$. As P is perfect this is true by Lemma 18.2. $\square$

Proposition 40.5. *Let X be a quasi-compact and quasi-separated scheme. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

(1) *$E \in D^-_{QCoh}(\mathcal{O}_X)$,*
(2) *$\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$ for $i \gg 0$,*
(3) *$\mathrm{Ext}^i_X(G, E) = 0$ for $i \gg 0$,*
(4) *$R\,\mathrm{Hom}_X(G, E)$ is in $D^-(\mathbf{Z})$,*
(5) *$H^i(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \gg 0$,*
(6) *$R\Gamma(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^-(\mathbf{Z})$,*
(7) *for every perfect object P of $D(\mathcal{O}_X)$*
    (a) *the assertions (2), (3), (4) hold with G replaced by P, and*
    (b) *$H^i(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \gg 0$,*
    (c) *$R\Gamma(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^-(\mathbf{Z})$.*

Proof. Assume (1). Since $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(G, E[i])$ we see that this is zero for $i \gg 0$ by Lemma 18.2. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology, Section 44. Part (5) and (6) are equivalent as $H^i(X, -) = H^i(R\Gamma(X, -))$ by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology, Lemma 50.5 and the fact that $(G[-i])^\vee = G^\vee[i]$.

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that G is a classical generator for $D_{perf}(\mathcal{O}_X)$ by Remark 17.2. For $P \in D_{perf}(\mathcal{O}_X)$ let $T(P)$ be the assertion that $R\,\mathrm{Hom}_X(P, E)$ is in $D^-(\mathbf{Z})$. Clearly, T is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is perserved by shifts. Hence by Derived Categories, Remark 36.7 we see that (4) implies T holds on all of $D_{perf}(\mathcal{O}_X)$. The same argument works for all other properties, except that for property (7)(b) and (7)(c) we also use that $P \mapsto P^\vee$ is a self equivalence of $D_{perf}(\mathcal{O}_X)$. Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Cohomology of Schemes, Lemma 4.1.

First, we prove (2) – (7) $\Rightarrow$ (1) if X is affine. Set $P = \mathcal{O}_X[0]$. From (7) we obtain $H^i(X, E) = 0$ for $i \gg 0$. Hence (1) follows since E is determined by $R\Gamma(X, E)$, see Lemma 3.5.

Now assume $X = U \cup V$ with U a quasi-compact open of X and V an affine open, and assume the implication (2) – (7) $\Rightarrow$ (1) is known for the schemes U, V, and $U \cap V$. Suppose $E \in D_{QCoh}(\mathcal{O}_X)$ satisfies (2) – (7). By Lemma 15.1 and Theorem 15.3 there exists a perfect complex Q on X such that $Q|_U$ generates $D_{QCoh}(\mathcal{O}_U)$. Let $f_1, \ldots, f_r \in \Gamma(V, \mathcal{O}_V)$ be such that $V \setminus U = V(f_1, \ldots, f_r)$ as subsets of V. Let $K \in D_{perf}(\mathcal{O}_V)$ be the object corresponding to the Koszul complex on $f_1, \ldots, f_r$. Let $K' \in D_{perf}(\mathcal{O}_X)$ be

$$K' = R(V \to X)_* K = R(V \to X)_! K, \tag{40.5.1}$$

see Cohomology, Lemmas 33.6 and 49.10. This is a perfect complex on X supported on the closed set $X \setminus U \subset V$ and isomorphic to K on V. By assumption, we know $R\,\mathrm{Hom}_{\mathcal{O}_X}(Q, E)$ and $R\,\mathrm{Hom}_{\mathcal{O}_X}(K', E)$ are bounded above.

By the second description of K' in (40.5.1) we have

$$\mathrm{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = 0$$

for $i \gg 0$. Therefore, we may apply Lemma 40.1 to $E|_V$ to obtain an integer a such that $\tau_{\geq a}(E|_V) = \tau_{\geq a} R(U \cap V \to V)_*(E|_{U\cap V})$. Then $\tau_{\geq a}E = \tau_{\geq a}R(U \to X)_*(E|_U)$ (check that the canonical map is an isomorphism after restricting to U and to V). Hence using Lemma 40.3 twice we see that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \mathrm{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \to X)_*(E|_U)) = 0$$

for $i \gg 0$. Since the Proposition holds for U and the generator $Q|_U$, we have $E|_U \in D^-_{QCoh}(\mathcal{O}_U)$. But then since the functor $R(U \to X)_*$ preserves D^-_{QCoh} (by Lemma 4.1), we get $\tau_{\geq a}E \in D^-_{QCoh}(\mathcal{O}_X)$. Thus $E \in D^-_{QCoh}(\mathcal{O}_X)$. □

Proposition 40.6. *Let X be a quasi-compact and quasi-separated scheme. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent*

1. *$E \in D^+_{QCoh}(\mathcal{O}_X)$,*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$ for $i \ll 0$,*
3. *$\mathrm{Ext}^i_X(G, E) = 0$ for $i \ll 0$,*
4. *$R\,\mathrm{Hom}_X(G, E)$ is in $D^+(\mathbf{Z})$,*
5. *$H^i(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \ll 0$,*
6. *$R\Gamma(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^+(\mathbf{Z})$,*
7. *for every perfect object P of $D(\mathcal{O}_X)$*
   (a) *the assertions (2), (3), (4) hold with G replaced by P, and*
   (b) *$H^i(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \ll 0$,*
   (c) *$R\Gamma(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^+(\mathbf{Z})$.*

Proof. Assume (1). Since $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(G, E[i])$ we see that this is zero for $i \ll 0$ by Lemma 18.2. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology, Section 44. Part (5) and (6) are equivalent as $H^i(X, -) = H^i(R\Gamma(X, -))$ by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology, Lemma 50.5 and the fact that $(G[-i])^\vee = G^\vee[i]$.

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that G is a classical generator for $D_{perf}(\mathcal{O}_X)$ by Remark 17.2. For $P \in D_{perf}(\mathcal{O}_X)$ let $T(P)$ be the assertion that $R\,\mathrm{Hom}_X(P, E)$ is in $D^+(\mathbf{Z})$. Clearly, T is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is perserved by shifts. Hence by Derived Categories, Remark 36.7 we see that (4) implies T holds on all of $D_{perf}(\mathcal{O}_X)$. The same argument works for all other properties, except that for property (7)(b) and (7)(c) we also use that $P \mapsto P^\vee$ is a self equivalence of $D_{perf}(\mathcal{O}_X)$. Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Cohomology of Schemes, Lemma 4.1.

First, we prove (2) – (7) ⇒ (1) if X is affine. Let $P = \mathcal{O}_X[0]$. From (7) we obtain $H^i(X, E) = 0$ for $i \ll 0$. Hence (1) follows since E is determined by $R\Gamma(X, E)$, see Lemma 3.5.

Now assume $X = U \cup V$ with U a quasi-compact open of X and V an affine open, and assume the implication (2) – (7) ⇒ (1) is known for the schemes U, V, and $U \cap V$. Suppose $E \in D_{QCoh}(\mathcal{O}_X)$ satisfies (2) – (7). By Lemma 15.1 and Theorem

15.3 there exists a perfect complex Q on X such that $Q|_U$ generates $D_{QCoh}(\mathcal{O}_U)$. Let $f_1, \ldots, f_r \in \Gamma(V, \mathcal{O}_V)$ be such that $V \setminus U = V(f_1, \ldots, f_r)$ as subsets of V. Let $K \in D_{perf}(\mathcal{O}_V)$ be the object corresponding to the Koszul complex on $f_1, \ldots, f_r$. Let $K' \in D_{perf}(\mathcal{O}_X)$ be

$$K' = R(V \to X)_* K = R(V \to X)_! K, \tag{40.6.1}$$

see Cohomology, Lemmas 33.6 and 49.10. This is a perfect complex on X supported on the closed set $X \setminus U \subset V$ and isomorphic to K on V. By assumption, we know $R\operatorname{Hom}_{\mathcal{O}_X}(Q, E)$ and $R\operatorname{Hom}_{\mathcal{O}_X}(K', E)$ are bounded below.

By the second description of K' in (40.6.1) we have

$$\operatorname{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = \operatorname{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = 0$$

for $i \ll 0$. Therefore, we may apply Lemma 40.2 to $E|_V$ to obtain an integer a such that $\tau_{\leq a}(E|_V) = \tau_{\leq a} R(U \cap V \to V)_*(E|_{U \cap V})$. Then $\tau_{\leq a} E = \tau_{\leq a} R(U \to X)_*(E|_U)$ (check that the canonical map is an isomorphism after restricting to U and to V). Hence using Lemma 40.4 twice we see that

$$\operatorname{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \operatorname{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \to X)_*(E|_U)) = 0$$

for $i \ll 0$. Since the Proposition holds for U and the generator $Q|_U$, we have $E|_U \in D^+_{QCoh}(\mathcal{O}_U)$. But then since the functor $R(U \to X)_*$ preserves bounded below objects (see Cohomology, Section 3) we get $\tau_{\leq a} E \in D^+_{QCoh}(\mathcal{O}_X)$. Thus $E \in D^+_{QCoh}(\mathcal{O}_X)$. $\square$

41. Quasi-coherent objects in the derived category

Let X be a scheme. Recall that $X_{affine,Zar}$ denotes the category of affine opens of X with topology given by standard Zariski coverings, see Topologies, Definition 3.7. We remind the reader that the topos of $X_{affine,Zar}$ is the small Zariski topos of X, see Topologies, Lemma 3.11. The site $X_{affine,Zar}$ comes with a structure sheaf $\mathcal{O}$ and there is an equivalence of ringed topoi

$$(Sh(X_{affine,Zar}), \mathcal{O}) \longrightarrow (Sh(X_{Zar}), \mathcal{O})$$

See Descent, Equation (11.1.1) and the discussion in Descent, Section 11 surrounding it where a slightly different notation is used.

In this section we denote X_{affine} the underlying category of $X_{affine,Zar}$ endowed with the chaotic topology, i.e., such that sheaves agree with presheaves. In particular, the structure sheaf $\mathcal{O}$ becomes a sheaf on X_{affine} as well. We obtain a morphisms of ringed sites

$$\epsilon : (X_{affine,Zar}, \mathcal{O}) \longrightarrow (X_{affine}, \mathcal{O})$$

as in Cohomology on Sites, Section 27. In this section we will identify $D_{QCoh}(\mathcal{O}_X)$ with the category $QC(X_{affine}, \mathcal{O})$ introduced in Cohomology on Sites, Section 43.

Lemma 41.1. *In the sitation above there are canonical exact equivalences between the following triangulated categories*

1. $D_{QCoh}(\mathcal{O}_X)$,
2. $D_{QCoh}(X_{Zar}, \mathcal{O})$,
3. $D_{QCoh}(X_{affine,Zar}, \mathcal{O})$,
4. $D_{QCoh}(X_{affine}, \mathcal{O}_X)$, *and*
5. $QC(X_{affine}, \mathcal{O})$.

Proof. If $U \subset V \subset X$ are affine open, then the ring map $\mathcal{O}(V) \to \mathcal{O}(U)$ is flat. Hence the equivalence between (4) and (5) is a special case of Cohomology on Sites, Lemma 43.11 (the proof also clarifies the statement).

The ringed site $(X_{Zar}, \mathcal{O})$ and the ringed space $(X, \mathcal{O}_X)$ have the same categories of modules by Descent, Remark 8.3. Via this equivalence the quasi-coherent modules correspond by Descent, Proposition 8.9. Hence we get a canonical exact equivalence between the triangulated categories in (1) and (2).

The discussion preceding the lemma shows that we have an equivalence of ringed topoi $(Sh(X_{affine,Zar}), \mathcal{O}) \to (Sh(X_{Zar}), \mathcal{O})$ and hence an equivalence between abelian categories of modules. Since the notion of quasi-coherent modules is intrinsic (Modules on Sites, Lemma 23.2) we see that this equivalence preserves the subcategories of quasi-coherent modules. Thus we get a canonical exact equivalence between the triangulated categories in (2) and (3).

To get an exact equivalence between the triangulated categories in (3) and (4) we will apply Cohomology on Sites, Lemma 29.1 to the morphism $\epsilon : (X_{affine,Zar}, \mathcal{O}) \to (X_{affine}, \mathcal{O})$ above. We take $\mathcal{B} = \text{Ob}(X_{affine})$ and we take $\mathcal{A} \subset PMod(X_{affine}, \mathcal{O})$ to be the full subcategory of those presheaves $\mathcal{F}$ such that $\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \to \mathcal{F}(U)$ is an isomorphism. Observe that by Descent, Lemma 11.2 objects of $\mathcal{A}$ are exactly those sheaves in the Zariski topology which are quasi-coherent modules on $(X_{affine,Zar}, \mathcal{O})$. On the other hand, by Modules on Sites, Lemma 24.2, the objects of $\mathcal{A}$ are exactly the quasi-coherent modules on $(X_{affine}, \mathcal{O})$, i.e., in the chaotic topology. Thus if we show that Cohomology on Sites, Lemma 29.1 applies, then we do indeed get the canonical equivalence between the categories of (3) and (4) using ϵ^* and $R\epsilon_*$.

We have to verify 4 conditions:

(1) Every object of $\mathcal{A}$ is a sheaf for the Zariski topology. This we have seen above.
(2) $\mathcal{A}$ is a weak Serre subcategory of $Mod(X_{affine,Zar}, \mathcal{O})$. Above we have seen that $\mathcal{A} = QCoh(X_{affine,Zar}, \mathcal{O})$ and we have seen above that these, via the equivalence $Mod(X_{affine,Zar}, \mathcal{O}) = Mod(X, \mathcal{O}_X)$, correspond to the quasi-coherent modules on X. Thus the result by the discussion in Schemes, Section 24.
(3) Every object of X_{affine} has a covering in the chaotic topology whose members are elements of $\mathcal{B}$. This holds because $\mathcal{B}$ contains all objects.
(4) For every object U of X_{affine} and $\mathcal{F}$ in $\mathcal{A}$ we have $H^p_{Zar}(U, \mathcal{F}) = 0$ for $p > 0$. This holds by the vanishing of cohomology of quasi-coherent modules on affines, see Cohomology of Schemes, Lemma 2.2.

This finishes the proof. □

Remark 41.2. Let S be a scheme. We will later show that also $QC((Aff/S), \mathcal{O})$ is canonically equivalent to $D_{QCoh}(\mathcal{O}_S)$. See Sheaves on Stacks, Proposition 26.4.

42. Other chapters

Preliminaries

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