TOPOLOGIES ON ALGEBRAIC SPACES

Contents

1. Introduction

In this chapter we introduce some topologies on the category of algebraic spaces. Compare with the material in [Gro71], [BLR90], [LMB00] and [Knu71]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

2. The general procedure

In this section we explain a general procedure for producing the sites we will be working with. This discussion will make little or no sense unless the reader has read Topologies, Section 2.

Let S be a base scheme. Take any category Sch_α constructed as in Sets, Lemma 9.2 starting with S and any set of schemes over S you want to be included. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained, and let $(Sch/S)_{fppf}$ denote the corresponding big fppf site of S. (The above is entirely as prescribed in Topologies, Section 7.)

Given choices as above the category of algebraic spaces over S has a set of isomorphism classes. One way to see this is to use the fact that any algebraic space over S is of the form U/R for some étale equivalence relation $j : R \to U \times_S U$ with $U, R \in \text{Ob}((Sch/S)_{fppf})$, see Spaces, Lemma 9.1. Hence we can find a full subcategory $Spaces/S$ of the category of algebraic spaces over S which has a set of objects such that each algebraic space is isomorphic to an object of $Spaces/S$. We fix a choice of such a category.

In the sections below, given a topology τ, the big site $(Spaces/S)_\tau$ (resp. the big site $(Spaces/X)_\tau$ of an algebraic space X over S) has as underlying category the category $Spaces/S$ (resp. the subcategory $Spaces/X$ of $Spaces/S$, see Categories, Example 2.13). The procedure for turning this into a site is as usual by defining a class of τ-coverings and using Sets, Lemma 11.1 to choose a sufficiently large set of coverings which defines the topology.

We point out that the *small étale site $X_{étale}$ of an algebraic space X* has already been defined in Properties of Spaces, Definition 18.1. Its objects are schemes étale over X, of which there are plenty by definition of an algebraic spaces. However, a more natural site, from the perspective of this chapter (compare Topologies, Definition 4.8) is the site $X_{spaces,étale}$ of Properties of Spaces, Definition 18.2. These two sites define the same topos, see Properties of Spaces, Lemma 18.3. We will not redefine these in this chapter; instead we will simply use them.

3. Zariski topology

In Spaces, Section 12 we introduced the notion of a Zariski covering of an algebraic space by open subspaces. Here is the corresponding notion with open subspaces replaced by open immersions.

Definition 3.1. Let S be a scheme, and let X be an algebraic space over S. A *Zariski covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces over S such that each f_i is an open immersion and such that

$$|X| = \bigcup\nolimits_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

Although Zariski coverings are occasionally useful the corresponding topology on the category of algebraic spaces is really too coarse, and not particularly useful. Still, it does define a site.

Lemma 3.2. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a Zariski covering of X.*
(2) *If $\{X_i \to X\}_{i \in I}$ is a Zariski covering and for each i we have a Zariski covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a Zariski covering.*
(3) *If $\{X_i \to X\}_{i \in I}$ is a Zariski covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a Zariski covering.*

Proof. Omitted. □

4. Étale topology

In this section we discuss the notion of a étale covering of algebraic spaces, and we define the big étale site of an algebraic space. Please compare with Topologies, Section 4.

Definition 4.1. Let S be a scheme, and let X be an algebraic space over S. An *étale covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces over S such that each f_i is étale and such that

$$|X| = \bigcup\nolimits_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 4.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a étale covering of schemes.

Lemma 4.2. *Any Zariski covering is an étale covering.*

Proof. This is clear from the definitions and the fact that an open immersion is an étale morphism (this follows from Morphisms, Lemma 36.9 via Spaces, Lemma 5.8 as immersions are representable). $\square$

Lemma 4.3. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a étale covering of X.*
(2) *If $\{X_i \to X\}_{i\in I}$ is a étale covering and for each i we have a étale covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is a étale covering.*
(3) *If $\{X_i \to X\}_{i\in I}$ is a étale covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is a étale covering.*

Proof. Omitted. $\square$

The following lemma tells us that the sites $(Spaces/X)_{étale}$ and $(Spaces/X)_{smooth}$ have the same categories of sheaves.

Lemma 4.4. *Let S be a scheme. Let X be an algebraic space over S. Let $\{X_i \to X\}_{i\in I}$ be a smooth covering of X. Then there exists an étale covering $\{U_j \to X\}_{j\in J}$ of X which refines $\{X_i \to X\}_{i\in I}$.*

Proof. First choose a scheme U and a surjective étale morphism $U \to X$. For each i choose a scheme W_i and a surjective étale morphism $W_i \to X_i$. Then $\{W_i \to X\}_{i\in I}$ is a smooth covering which refines $\{X_i \to X\}_{i\in I}$. Hence $\{W_i \times_X U \to U\}_{i\in I}$ is a smooth covering of schemes. By More on Morphisms, Lemma 38.7 we can choose an étale covering $\{U_j \to U\}$ which refines $\{W_i \times_X U \to U\}$. Then $\{U_j \to X\}_{j\in J}$ is an étale covering refining $\{X_i \to X\}_{i\in I}$. $\square$

Definition 4.5. Let S be a scheme. A big étale site $(Spaces/S)_{étale}$ is any site constructed as follows:

(1) Choose a big étale site $(Sch/S)_{étale}$ as in Topologies, Section 4.
(2) As underlying category take the category $Spaces/S$ of algebraic spaces over S (see discussion in Section 2 why this is a set).
(3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category $Spaces/S$ and the class of étale coverings of Definition 4.1.

Having defined this, we can localize to get the étale site of an algebraic space.

Definition 4.6. Let S be a scheme. Let $(Spaces/S)_{étale}$ be as in Definition 4.5. Let X be an algebraic space over S, i.e., an object of $(Spaces/S)_{étale}$. Then the big étale site $(Spaces/X)_{étale}$ of X is the localization of the site $(Spaces/S)_{étale}$ at X introduced in Sites, Section 25.

Recall that given an algebraic space X over S as in the definition, we already have defined the small étale sites $X_{spaces,étale}$ and $X_{étale}$, see Properties of Spaces, Section 18. We will silently identify the corresponding topoi using the inclusion functor $X_{étale} \subset X_{spaces,étale}$ (Properties of Spaces, Lemma 18.3) and we will call it the small étale topos of X. Next, we establish some relationships between the topoi associated to these sites.

Lemma 4.7. *Let S be a scheme. Let $f : Y \to X$ be a morphism of $(Spaces/S)_{étale}$. The inclusion functor $Y_{spaces,étale} \to (Spaces/X)_{étale}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(Y_{étale}) \longrightarrow Sh((Spaces/X)_{étale})$$

For a sheaf $\mathcal{G}$ on $(Spaces/X)_{étale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : Y_{spaces,étale} \to (Spaces/X)_{étale}$. In other words, given an étale morphism $j : U \to Y$ corresponding to an object of $Y_{spaces,étale}$ we set $u(U \to T) = (f \circ j : U \to S)$. The category $Y_{spaces,étale}$ has fibre products and equalizers and u commutes with them. It is immediate that u cocontinuous. The functor u is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 21.5 and 21.6. □

Lemma 4.8. *Let S be a scheme. Let X be an object of $(Spaces/S)_{étale}$. The inclusion functor $X_{spaces,étale} \to (Spaces/X)_{étale}$ satisfies the hypotheses of Sites, Lemma 21.8 and hence induces a morphism of sites*

$$\pi_X : (Spaces/X)_{étale} \longrightarrow X_{spaces,étale}$$

and a morphism of topoi

$$i_X : Sh(X_{étale}) \longrightarrow Sh((Spaces/X)_{étale})$$

such that $\pi_X \circ i_X = id$. Moreover, $i_X = i_{id_X}$ with i_{id_X} as in Lemma 4.7. In particular the functor $i_X^{-1} = \pi_{X,*}$ is described by the rule $i_X^{-1}(\mathcal{G})(U/X) = \mathcal{G}(U/X)$.

Proof. In this case the functor $u : X_{spaces,étale} \to (Spaces/X)_{étale}$, in addition to the properties seen in the proof of Lemma 4.7 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 21.8. □

Definition 4.9. In the situation of Lemma 4.8 the functor $i_X^{-1} = \pi_{X,*}$ is often called the *restriction to the small étale site*, and for a sheaf $\mathcal{F}$ on the big étale site we often denote $\mathcal{F}|_{X_{étale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the small site that

$$\begin{aligned}
\mathrm{Mor}_{Sh(X_{étale})}(\mathcal{F}|_{X_{étale}}, \mathcal{G}) &= \mathrm{Mor}_{Sh((Spaces/X)_{étale})}(\mathcal{F}, i_{X,*}\mathcal{G}) \\
\mathrm{Mor}_{Sh(X_{étale})}(\mathcal{G}, \mathcal{F}|_{X_{étale}}) &= \mathrm{Mor}_{Sh((Spaces/X)_{étale})}(\pi_X^{-1}\mathcal{G}, \mathcal{F})
\end{aligned}$$

Moreover, we have $(i_{X,*}\mathcal{G})|_{X_{étale}} = \mathcal{G}$ and we have $(\pi_X^{-1}\mathcal{G})|_{X_{étale}} = \mathcal{G}$.

Lemma 4.10. *Let S be a scheme. Let $f : Y \to X$ be a morphism in $(Spaces/S)_{étale}$. The functor*

$$u : (Spaces/Y)_{étale} \longrightarrow (Spaces/X)_{étale}, \quad V/Y \longmapsto V/X$$

is cocontinuous, and has a continuous right adjoint

$$v : (Spaces/X)_{étale} \longrightarrow (Spaces/Y)_{étale}, \quad (U \to X) \longmapsto (U \times_X Y \to Y).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Spaces/Y)_{étale}) \longrightarrow Sh((Spaces/X)_{étale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. We have $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 4.7). Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/Y and V/X we have $\mathrm{Mor}_X(u(U), V) = \mathrm{Mor}_Y(U, V \times_X Y)$ as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for $f_{big,*}$. ☐

Lemma 4.11. *Let S be a scheme. Let $f : Y \to X$ be a morphism in $(Spaces/S)_{étale}$.*

1. *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 4.7 and i_T as in Lemma 4.8.*
2. *The functor $X_{spaces,étale} \to T_{spaces,étale}$, $(U \to X) \mapsto (U \times_X Y \to Y)$ is continuous and induces a morphism of sites*

   $$f_{spaces,étale} : Y_{spaces,étale} \longrightarrow X_{spaces,étale}$$

   The corresponding morphism of small étale topoi is denoted

   $$f_{small} : Sh(Y_{étale}) \to Sh(X_{étale})$$

   We have $f_{small,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$.
3. *We have a commutative diagram of morphisms of sites*

   $$\begin{tikzcd} Y_{spaces,étale} \arrow[d, "f_{spaces,étale}"'] & (Spaces/Y)_{étale} \arrow[l, "\pi_Y"'] \arrow[d, "f_{big}"] \\ X_{spaces,étale} & (Spaces/X)_{étale} \arrow[l, "\pi_X"] \end{tikzcd}$$

   so that $f_{small} \circ \pi_Y = \pi_X \circ f_{big}$ as morphisms of topoi.
4. *We have $f_{small} = \pi_X \circ f_{big} \circ i_Y = \pi_X \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_Y$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : X_{spaces,étale} \to Y_{spaces,étale}$, $u(U \to X) = (U \times_X Y \to Y)$ was shown to give rise to a morphism of sites and correspong morphism of small étale topoi in Properties of Spaces, Lemma 18.8. The description of the pushforward is clear.

Part (3) follows because π_X and π_Y are given by the inclusion functors and $f_{spaces,étale}$ and f_{big} by the base change functors $U \mapsto U \times_X Y$.

Statement (4) follows from (3) by precomposing with i_Y. ☐

In the situation of the lemma, using the terminology of Definition 4.9 we have: for $\mathcal{F}$ a sheaf on the big étale site of Y

$$(f_{big,*}\mathcal{F})|_{X_{étale}} = f_{small,*}(\mathcal{F}|_{Y_{étale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of Y, resp. X is given by $\pi_{Y,*}$, resp. $\pi_{X,*}$. A similar formula involving pullbacks and restrictions is false.

Lemma 4.12. *Let S be a scheme. Given morphisms $f : X \to Y$, $g : Y \to Z$ in $(Spaces/S)_{étale}$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 4.10. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 4.11. □

Lemma 4.13. *Let S be a scheme. Consider a cartesian diagram*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

in $(Spaces/S)_{étale}$. Then $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.

Proof. Since the diagram is cartesian, we have for U'/X' that $U' \times_{X'} Y' = U' \times_X Y$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(Spaces/Y)_{étale}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{X'} Y')$ on $X'_{étale}$ (use Lemmas 4.7 and 4.10). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 28.1. □

Remark 4.14. The sites $(Spaces/X)_{étale}$ and $X_{spaces,étale}$ come with structure sheaves. For the small étale site we have seen this in Properties of Spaces, Section 21. The structure sheaf $\mathcal{O}$ on the big étale site $(Spaces/X)_{étale}$ is defined by assigning to an object U the global sections of the structure sheaf of U. This makes sense because after all U is an algebraic space itself hence has a structure sheaf. Since $\mathcal{O}_U$ is a sheaf on the étale site of U, the presheaf $\mathcal{O}$ so defined satisfies the sheaf condition for coverings of U, i.e., $\mathcal{O}$ is a sheaf. We can upgrade the morphisms i_f, π_X, i_X, f_{small}, and f_{big} defined above to morphisms of ringed sites, respectively topoi. Let us deal with these one by one.

1. (1) In Lemma 4.7 denote $\mathcal{O}$ the structure sheaf on $(Spaces/X)_{étale}$. We have $(i_f^{-1}\mathcal{O})(U/Y) = \mathcal{O}_U(U) = \mathcal{O}_Y(U)$ by construction. Hence an isomorphism $i_f^\sharp : i_f^{-1}\mathcal{O} \to \mathcal{O}_Y$.
2. (2) In Lemma 4.8 it was noted that i_X is a special case of i_f with $f = \text{id}_X$ hence we are back in case (1).
3. (3) In Lemma 4.8 the morphism π_X satisfies $(\pi_{X,*}\mathcal{O})(U) = \mathcal{O}(U) = \mathcal{O}_X(U)$. Hence we can use this to define $\pi_X^\sharp : \mathcal{O}_X \to \pi_{X,*}\mathcal{O}$.
4. (4) In Lemma 4.11 the extension of f_{small} to a morphism of ringed topoi was discussed in Properties of Spaces, Lemma 21.3.
5. (5) In Lemma 4.11 the functor f_{big}^{-1} is simply the restriction via the inclusion functor $(Spaces/Y)_{étale} \to (Spaces/X)_{étale}$. Let $\mathcal{O}_1$ be the structure sheaf on $(Spaces/X)_{étale}$ and let $\mathcal{O}_2$ be the structure sheaf on $(Spaces/Y)_{étale}$. We obtain a canonical isomorphism $f_{big}^\sharp : f_{big}^{-1}\mathcal{O}_1 \to \mathcal{O}_2$.

Moreover, with these definitions compositions work out correctly too. We omit giving a detailed statement and proof.

5. Smooth topology

In this section we discuss the notion of a smooth covering of algebraic spaces, and we define the big smooth site of an algebraic space. Please compare with Topologies, Section 5.

Definition 5.1. Let S be a scheme, and let X be an algebraic space over S. A *smooth covering of* X is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces over S such that each f_i is smooth and such that

$$|X| = \bigcup\nolimits_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 5.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a smooth covering of schemes.

Lemma 5.2. *Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.*

Proof. This is clear from the definitions, the fact that an étale morphism is smooth (Morphisms of Spaces, Lemma 39.6), and Lemma 4.2. □

Lemma 5.3. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a smooth covering of X.*
(2) *If $\{X_i \to X\}_{i \in I}$ is a smooth covering and for each i we have a smooth covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a smooth covering.*
(3) *If $\{X_i \to X\}_{i \in I}$ is a smooth covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a smooth covering.*

Proof. Omitted. □

To be continued...

6. Syntomic topology

In this section we discuss the notion of a syntomic covering of algebraic spaces, and we define the big syntomic site of an algebraic space. Please compare with Topologies, Section 6.

Definition 6.1. Let S be a scheme, and let X be an algebraic space over S. A *syntomic covering of* X is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces over S such that each f_i is syntomic and such that

$$|X| = \bigcup\nolimits_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 6.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a syntomic covering of schemes.

Lemma 6.2. *Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.*

Proof. This is clear from the definitions and the fact that a smooth morphism is syntomic (Morphisms of Spaces, Lemma 37.8), and Lemma 5.2. □

Lemma 6.3. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a syntomic covering of X.*
(2) *If $\{X_i \to X\}_{i \in I}$ is a syntomic covering and for each i we have a syntomic covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a syntomic covering.*
(3) *If $\{X_i \to X\}_{i \in I}$ is a syntomic covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a syntomic covering.*

Proof. Omitted. $\square$

To be continued...

7. Fppf topology

In this section we discuss the notion of an fppf covering of algebraic spaces, and we define the big fppf site of an algebraic space. Please compare with Topologies, Section 7.

Definition 7.1. Let S be a scheme, and let X be an algebraic space over S. An *fppf covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces over S such that each f_i is flat and locally of finite presentation and such that

$$|X| = \bigcup\nolimits_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 7.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of an fppf covering of schemes.

Lemma 7.2. *Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.*

Proof. This is clear from the definitions, the fact that a syntomic morphism is flat and locally of finite presentation (Morphisms of Spaces, Lemmas 36.5 and 36.6) and Lemma 6.2. $\square$

Lemma 7.3. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is an fppf covering of X.*
(2) *If $\{X_i \to X\}_{i \in I}$ is an fppf covering and for each i we have an fppf covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is an fppf covering.*
(3) *If $\{X_i \to X\}_{i \in I}$ is an fppf covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is an fppf covering.*

Proof. Omitted. $\square$

Lemma 7.4. *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\mathcal{U} = \{f_i : X_i \to X\}_{i \in I}$ is an fppf covering of X. Then there exists a refinement $\mathcal{V} = \{g_i : T_i \to X\}$ of $\mathcal{U}$ which is an fppf covering such that each T_i is a scheme.*

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \to X_i$. Then check that $\{T_i \to X\}$ is an fppf covering. $\square$

Lemma 7.5. *Let S be a scheme. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fppf covering of algebraic spaces over S. Then the map of sheaves*

$$\coprod X_i \longrightarrow X$$

is surjective.

Proof. This follows from Spaces, Lemma 5.9. See also Spaces, Remark 5.2 in case you are confused about the meaning of this lemma. □

Definition 7.6. Let S be a scheme. A big fppf site $(Spaces/S)_{fppf}$ is any site constructed as follows:

(1) Choose a big fppf site $(Sch/S)_{fppf}$ as in Topologies, Section 7.
(2) As underlying category take the category $Spaces/S$ of algebraic spaces over S (see discussion in Section 2 why this is a set).
(3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category $Spaces/S$ and the class of fppf coverings of Definition 7.1.

Having defined this, we can localize to get the fppf site of an algebraic space.

Definition 7.7. Let S be a scheme. Let $(Spaces/S)_{fppf}$ be as in Definition 7.6. Let X be an algebraic space over S, i.e., an object of $(Spaces/S)_{fppf}$. Then the big fppf site $(Spaces/X)_{fppf}$ of X is the localization of the site $(Spaces/S)_{fppf}$ at X introduced in Sites, Section 25.

Next, we establish some relationships between the topoi associated to these sites.

Lemma 7.8. *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The functor*

$$u : (Spaces/Y)_{fppf} \longrightarrow (Spaces/X)_{fppf}, \quad V/Y \longmapsto V/X$$

is cocontinuous, and has a continuous right adjoint

$$v : (Spaces/X)_{fppf} \longrightarrow (Spaces/Y)_{fppf}, \quad (U \to Y) \longmapsto (U \times_X Y \to Y).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Spaces/Y)_{fppf}) \longrightarrow Sh((Spaces/X)_{fppf})$$

We have $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. We have $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/X we have $\mathrm{Mor}_X(u(U), V) = \mathrm{Mor}_Y(U, V \times_X Y)$ as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for $f_{big,*}$. □

Lemma 7.9. *Let S be a scheme. Given morphisms $f : X \to Y$, $g : Y \to Z$ of algebraic spaces over S we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 7.8. □

8. The ph topology

In this section we define the ph topology. This is the topology generated by étale coverings and proper surjective morphisms, see Lemma 8.7.

Definition 8.1. Let S be a scheme and let X be an algebraic space over S. A *ph covering of* X is a family of morphisms $\{X_i \to X\}_{i \in I}$ of algebraic spaces over S such that f_i is locally of finite type and such that for every $U \to X$ with U affine there exists a standard ph covering $\{U_j \to U\}_{j=1,\ldots,m}$ refining the family $\{X_i \times_X U \to U\}_{i \in I}$.

In other words, there exists indices $i_1, \ldots, i_m \in I$ and morphisms $h_j : U_j \to X_{i_j}$ such that $f_{i_j} \circ h_j = h \circ g_j$. Note that if X and all X_i are representable, this is the same as a ph covering of schemes by Topologies, Definition 8.4.

Lemma 8.2. *Any fppf covering is a ph covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is a ph covering.*

Proof. We will show that an fppf covering is a ph covering, and then the rest follows from Lemma 7.2. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering of algebraic spaces over a base scheme S. Let U be an affine scheme and let $U \to X$ be a morphism. We can refine the fppf covering $\{X_i \times_U U \to U\}_{i \in I}$ by an fppf covering $\{T_i \to U\}_{i \in I}$ where T_i is a scheme (Lemma 7.4). Then we can find a standard ph covering $\{U_j \to U\}_{j=1,\ldots,m}$ refining $\{T_i \to U\}_{i \in I}$ by More on Morphisms, Lemma 48.7 (and the definition of ph coverings for schemes). Thus $\{X_i \to X\}_{i \in I}$ is a ph covering by definition. □

Lemma 8.3. *Let S be a scheme. Let $f : Y \to X$ be a surjective proper morphism of algebraic spaces over S. Then $\{Y \to X\}$ is a ph covering.*

Proof. Let $U \to X$ be a morphism with U affine. By Chow's lemma (in the weak form given as Cohomology of Spaces, Lemma 18.1) we see that there is a surjective proper morphism of schemes $V \to U$ which factors through $Y \times_X U \to U$. Taking any finite affine open cover of V we obtain a standard ph covering of U refining $\{X \times_Y U \to U\}$ as desired. □

Lemma 8.4. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a ph covering of X.*
(2) *If $\{X_i \to X\}_{i \in I}$ is a ph covering and for each i we have a ph covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a ph covering.*
(3) *If $\{X_i \to X\}_{i \in I}$ is a ph covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a ph covering.*

Proof. Part (1) is clear. Consider $g : X' \to X$ and $\{X_i \to X\}_{i \in I}$ a ph covering as in (3). By Morphisms of Spaces, Lemma 23.3 the morphisms $X' \times_X X_i \to X'$ are locally of finite type. If $h' : Z \to X'$ is a morphism from an affine scheme towards X', then set $h = g \circ h' : Z \to X$. The assumption on $\{X_i \to X\}_{i \in I}$ means there exists a standard ph covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to X_{i(j)}$ covering h for certain $i(j) \in I$. By the universal property of the fibre product we obtain morphisms $Z_j \to X' \times_X X_{i(j)}$ over h' also. Hence $\{X' \times_X X_i \to X'\}_{i \in I}$ is a ph covering. This proves (3).

Let $\{X_i \to X\}_{i \in I}$ and $\{X_{ij} \to X_i\}_{j \in J_i}$ be as in (2). Let $h : Z \to X$ be a morphism from an affine scheme towards X. By assumption there exists a standard ph covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $h_j : Z_j \to X_{i(j)}$ covering h for some indices $i(j) \in I$. By assumption there exist standard ph coverings $\{Z_{j,l} \to Z_j\}_{l=1,\ldots,n(j)}$ and morphisms $Z_{j,l} \to X_{i(j)j(l)}$ covering h_j for some indices $j(l) \in$

$J_{i(j)}$. By Topologies, Lemma 8.3 the family $\{Z_{j,l} \to Z\}$ can be refined by a standard ph covering. Hence we conclude that $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a ph covering. □

Definition 8.5. Let S be a scheme. A big ph site $(Spaces/S)_{ph}$ is any site constructed as follows:

(1) Choose a big ph site $(Sch/S)_{ph}$ as in Topologies, Section 8.
(2) As underlying category take the category $Spaces/S$ of algebraic spaces over S (see discussion in Section 2 why this is a set).
(3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category $Spaces/S$ and the class of ph coverings of Definition 8.1.

Having defined this, we can localize to get the ph site of an algebraic space.

Definition 8.6. Let S be a scheme. Let $(Spaces/S)_{ph}$ be as in Definition 8.5. Let X be an algebraic space over S, i.e., an object of $(Spaces/S)_{ph}$. Then the big ph site $(Spaces/X)_{ph}$ of X is the localization of the site $(Spaces/S)_{ph}$ at X introduced in Sites, Section 25.

Here is the promised characterization of ph sheaves.

Lemma 8.7. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a presheaf on $(Spaces/X)_{ph}$. Then $\mathcal{F}$ is a sheaf if and only if*

(1) *$\mathcal{F}$ satisfies the sheaf condition for étale coverings, and*
(2) *if $f : V \to U$ is a proper surjective morphism of $(Spaces/X)_{ph}$, then $\mathcal{F}(U)$ maps bijectively to the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$.*

Proof. We will show that if (1) and (2) hold, then $\mathcal{F}$ is sheaf. Let $\{T_i \to T\}$ be a ph covering, i.e., a covering in $(Spaces/X)_{ph}$. We will verify the sheaf condition for this covering. Let $s_i \in \mathcal{F}(T_i)$ be sections which restrict to the same section over $T_i \times_T T_{i'}$. We will show that there exists a unique section $s \in \mathcal{F}$ restricting to s_i over T_i. Let $\{U_j \to T\}$ be an étale covering with U_j affine. By property (1) it suffices to produce sections $s_j \in \mathcal{F}(U_j)$ which agree on $U_j \cap U_{j'}$ in order to produce s. Consider the ph coverings $\{T_i \times_T U_j \to U_j\}$. Then $s_{ji} = s_i|_{T_i \times_T U_j}$ are sections agreeing over $(T_i \times_T U_j) \times_{U_j} (T_{i'} \times_T U_j)$. Choose a proper surjective morphism $V_j \to U_j$ and a finite affine open covering $V_j = \bigcup V_{jk}$ such that the standard ph covering $\{V_{jk} \to U_j\}$ refines $\{T_i \times_T U_j \to U_j\}$. If $s_{jk} \in \mathcal{F}(V_{jk})$ denotes the pullback of s_{ji} to V_{jk} by the implied morphisms, then we find that s_{jk} glue to a section $s'_j \in \mathcal{F}(V_j)$. Using the agreement on overlaps once more, we find that s'_j is in the equalizer of the two maps $\mathcal{F}(V_j) \to \mathcal{F}(V_j \times_{U_j} V_j)$. Hence by (2) we find that s'_j comes from a unique section $s_j \in \mathcal{F}(U_j)$. We omit the verification that these sections s_j have all the desired properties. □

Next, we establish some relationships between the topoi associated to these sites.

Lemma 8.8. *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The functor*

$$u : (Spaces/Y)_{ph} \longrightarrow (Spaces/X)_{ph}, \quad V/Y \longmapsto V/X$$

is cocontinuous, and has a continuous right adjoint

$$v : (Spaces/X)_{ph} \longrightarrow (Spaces/Y)_{ph}, \quad (U \to Y) \longmapsto (U \times_X Y \to Y).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Spaces/Y)_{ph}) \longrightarrow Sh((Spaces/X)_{ph})$$

We have $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$. We have $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/X we have $\mathrm{Mor}_X(u(U), V) = \mathrm{Mor}_Y(U, V \times_X Y)$ as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for $f_{big,*}$. $\square$

Lemma 8.9. *Let S be a scheme. Given morphisms $f : X \to Y$, $g : Y \to Z$ of algebraic spaces over S we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 8.8. $\square$

Lemma 8.10. *Let S be a scheme. Let X be an algebraic space over S. Let P be a property of objects in $(Spaces/X)_{fppf}$ such that whenever $\{U_i \to U\}$ is a covering in $(Spaces/X)_{fppf}$, then*

$$P(U_{i_0} \times_U \ldots \times_U U_{i_p}) \text{ for all } p \geq 0,\ i_0, \ldots, i_p \in I \Rightarrow P(U)$$

If $P(U)$ for all U affine and flat, locally of finite presentation over X, then $P(X)$.

Proof. Let U be a separated algebraic space locally of finite presentation over X. Then we can choose an étale covering $\{U_i \to U\}_{i \in I}$ with V_i affine. Since U is separated, we conclude that $U_{i_0} \times_U \ldots \times_U U_{i_p}$ is always affine. Whence $P(U_{i_0} \times_U \ldots \times_U U_{i_p})$ always. Hence $P(U)$ holds. Choose a scheme U which is a disjoint union of affines and a surjective étale morphism $U \to X$. Then $U \times_X \ldots \times_X U$ (with $p+1$ factors) is a separated algebraic space étale over X. Hence $P(U \times_X \ldots \times_X U)$ by the above. We conclude that $P(X)$ is true. $\square$

9. Fpqc topology

We briefly discuss the notion of an fpqc covering of algebraic spaces. Please compare with Topologies, Section 9. We will show in Descent on Spaces, Proposition 4.1 that quasi-coherent sheaves descent along these.

Definition 9.1. Let S be a scheme, and let X be an algebraic space over S. An *fpqc covering of* X is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces such that each f_i is flat and such that for every affine scheme Z and morphism $h : Z \to X$ there exists a standard fpqc covering $\{g_j : Z_j \to Z\}_{j=1,\ldots,m}$ which refines the family $\{X_i \times_X Z \to Z\}_{i \in I}$.

In other words, there exists indices $i_1, \ldots, i_m \in I$ and morphisms $h_j : U_j \to X_{i_j}$ such that $f_{i_j} \circ h_j = h \circ g_j$. Note that if X and all X_i are representable, this is the same as a fpqc covering of schemes by Topologies, Lemma 9.11.

Lemma 9.2. *Any fppf covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.*

Proof. We will show that an fppf covering is an fpqc covering, and then the rest follows from Lemma 7.2. Let $\{f_i : U_i \to U\}_{i \in I}$ be an fppf covering of algebraic spaces over S. By definition this means that the f_i are flat which checks the first condition of Definition 9.1. To check the second, let $V \to U$ be a morphism with

V affine. We may choose an étale covering $\{V_{ij} \to V \times_U U_i\}$ with V_{ij} affine. Then the compositions $f_{ij} : V_{ij} \to V \times_U U_i \to V$ are flat and locally of finite presentation as compositions of such (Morphisms of Spaces, Lemmas 28.2, 30.3, 39.7, and 39.8). Hence these morphisms are open (Morphisms of Spaces, Lemma 30.6) and we see that $|V| = \bigcup_{i\in I} \bigcup_{j \in J_i} f_{ij}(|V_{ij}|)$ is an open covering of $|V|$. Since $|V|$ is quasi-compact, this covering has a finite refinement. Say $V_{i_1j_1}, \ldots, V_{i_Nj_N}$ do the job. Then $\{V_{i_kj_k} \to V\}_{k=1,\ldots,N}$ is a standard fpqc covering of V refinining the family $\{U_i \times_U V \to V\}$. This finishes the proof. □

Lemma 9.3. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is an fpqc covering of X.*
(2) *If $\{X_i \to X\}_{i\in I}$ is an fpqc covering and for each i we have an fpqc covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fpqc covering.*
(3) *If $\{X_i \to X\}_{i\in I}$ is an fpqc covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fpqc covering.*

Proof. Part (1) is clear. Consider $g : X' \to X$ and $\{X_i \to X\}_{i\in I}$ an fpqc covering as in (3). By Morphisms of Spaces, Lemma 30.4 the morphisms $X' \times_X X_i \to X'$ are flat. If $h' : Z \to X'$ is a morphism from an affine scheme towards X', then set $h = g \circ h' : Z \to X$. The assumption on $\{X_i \to X\}_{i\in I}$ means there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to X_{i(j)}$ covering h for certain $i(j) \in I$. By the universal property of the fibre product we obtain morphisms $Z_j \to X' \times_X X_{i(j)}$ over h' also. Hence $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fpqc covering. This proves (3).

Let $\{X_i \to X\}_{i\in I}$ and $\{X_{ij} \to X_i\}_{j\in J_i}$ be as in (2). Let $h : Z \to X$ be a morphism from an affine scheme towards X. By assumption there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $h_j : Z_j \to X_{i(j)}$ covering h for some indices $i(j) \in I$. By assumption there exist standard fpqc coverings $\{Z_{j,l} \to Z_j\}_{l=1,\ldots,n(j)}$ and morphisms $Z_{j,l} \to X_{i(j)j(l)}$ covering h_j for some indices $j(l) \in J_{i(j)}$. By Topologies, Lemma 9.10 the family $\{Z_{j,l} \to Z\}$ is a standard fpqc covering. Hence we conclude that $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fpqc covering. □

Lemma 9.4. *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\{f_i : X_i \to X\}_{i\in I}$ is a family of morphisms of algebraic spaces with target X. Let $U \to X$ be a surjective étale morphism from a scheme towards X. Then $\{f_i : X_i \to X\}_{i\in I}$ is an fpqc covering of X if and only if $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering of U.*

Proof. If $\{X_i \to X\}_{i\in I}$ is an fpqc covering, then so is $\{U \times_X X_i \to U\}_{i\in I}$ by Lemma 9.3. Assume that $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering. Let $h : Z \to X$ be a morphism from an affine scheme towards X. Then we see that $U \times_X Z \to Z$ is a surjective étale morphism of schemes, in particular open. Hence we can find finitely many affine opens $W_1, \ldots, W_t$ of $U \times_X Z$ whose images cover Z. For each j we may apply the condition that $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering to the morphism $W_j \to U$, and obtain a standard fpqc covering $\{W_{jl} \to W_j\}$ which refines $\{W_j \times_X X_i \to W_j\}_{i\in I}$. Hence $\{W_{jl} \to Z\}$ is a standard fpqc covering of Z (see Topologies, Lemma 9.10) which refines $\{Z \times_X X_i \to X\}$ and we win. □

Lemma 9.5. *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\mathcal{U} = \{f_i : X_i \to X\}_{i\in I}$ is an fpqc covering of X. Then there exists a refinement $\mathcal{V} = \{g_i : T_i \to X\}$ of $\mathcal{U}$ which is an fpqc covering such that each T_i is a scheme.*

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \to X_i$. Then check that $\{T_i \to X\}$ is an fpqc covering. $\square$

To be continued...

10. Other chapters

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