INJECTIVES

Contents

1. Introduction

In future chapters we will use the existence of injectives and K-injective complexes to do cohomology of sheaves of modules on ringed sites. In this chapter we explain how to produce injectives and K-injective complexes first for modules on sites and later more generally for Grothendieck abelian categories.

We observe that we already know that the category of abelian groups and the category of modules over a ring have enough injectives, see More on Algebra, Sections 54 and 55

2. Baer's argument for modules

There is another, more set-theoretic approach to showing that any R-module M can be imbedded in an injective module. This approach constructs the injective module by a transfinite colimit of push-outs. While this method is somewhat abstract and more complicated than the one of More on Algebra, Section 55, it is also more general. Apparently this method originates with Baer, and was revisited by Cartan and Eilenberg in [CE56] and by Grothendieck in [Gro57]. There Grothendieck uses it to show that many other abelian categories have enough injectives. We will get back to the general case later (Section 11).

We begin with a few set theoretic remarks. Let $\{B_\beta\}_{\beta \in \alpha}$ be an inductive system of objects in some category $\mathcal{C}$, indexed by an ordinal α. Assume that $\operatorname{colim}_{\beta \in \alpha} B_\beta$ exists in $\mathcal{C}$. If A is an object of $\mathcal{C}$, then there is a natural map

$$(2.0.1) \qquad \operatorname{colim}_{\beta \in \alpha} \operatorname{Mor}_{\mathcal{C}}(A, B_\beta) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(A, \operatorname{colim}_{\beta \in \alpha} B_\beta).$$

because if one is given a map $A \to B_\beta$ for some β, one naturally gets a map from A into the colimit by composing with $B_\beta \to \operatorname{colim}_{\beta \in \alpha} B_\alpha$. Note that the left colimit is one of sets! In general, (2.0.1) is neither injective or surjective.

Example 2.1. Consider the category of sets. Let $A = \mathbf{N}$ and $B_n = \{1, \ldots, n\}$ be the inductive system indexed by the natural numbers where $B_n \to B_m$ for $n \leq m$ is the obvious map. Then $\operatorname{colim} B_n = \mathbf{N}$, so there is a map $A \to \operatorname{colim} B_n$, which does not factor as $A \to B_m$ for any m. Consequently, $\operatorname{colim} \operatorname{Mor}(A, B_n) \to \operatorname{Mor}(A, \operatorname{colim} B_n)$ is not surjective.

Example 2.2. Next we give an example where the map fails to be injective. Let $B_n = \mathbf{N}/\{1, 2, \ldots, n\}$, that is, the quotient set of $\mathbf{N}$ with the first n elements collapsed to one element. There are natural maps $B_n \to B_m$ for $n \leq m$, so the $\{B_n\}$ form a system of sets over $\mathbf{N}$. It is easy to see that $\operatorname{colim} B_n = \{*\}$: it is the one-point set. So it follows that $\operatorname{Mor}(A, \operatorname{colim} B_n)$ is a one-element set for every set A. However, $\operatorname{colim} \operatorname{Mor}(A, B_n)$ is **not** a one-element set. Consider the family of maps $A \to B_n$ which are just the natural projections $\mathbf{N} \to \mathbf{N}/\{1, 2, \ldots, n\}$ and the family of maps $A \to B_n$ which map the whole of A to the class of 1. These two families of maps are distinct at each step and thus are distinct in $\operatorname{colim} \operatorname{Mor}(A, B_n)$, but they induce the same map $A \to \operatorname{colim} B_n$.

Nonetheless, if we map out of a finite set then (2.0.1) is an isomorphism always.

Lemma 2.3. *Suppose that, in (2.0.1), $\mathcal{C}$ is the category of sets and A is a finite set, then the map is a bijection.*

Proof. Let $f : A \to \operatorname{colim} B_\beta$. The range of f is finite, containing say elements $c_1, \ldots, c_r \in \operatorname{colim} B_\beta$. These all come from some elements in B_β for $\beta \in \alpha$ large by definition of the colimit. Thus we can define $\widetilde{f} : A \to B_\beta$ lifting f at a finite stage. This proves that (2.0.1) is surjective. Next, suppose two maps $f : A \to B_\gamma, f' : A \to B_{\gamma'}$ define the same map $A \to \operatorname{colim} B_\beta$. Then each of the finitely many elements of A gets sent to the same point in the colimit. By definition of the colimit for sets, there is $\beta \geq \gamma, \gamma'$ such that the finitely many elements of A get sent to the same points in B_β under f and f'. This proves that (2.0.1) is injective. $\square$

The most interesting case of the lemma is when $\alpha = \omega$, i.e., when the system $\{B_\beta\}$ is a system $\{B_n\}_{n \in \mathbf{N}}$ over the natural numbers as in Examples 2.1 and 2.2. The essential idea is that A is "small" relative to the long chain of compositions $B_1 \to B_2 \to \ldots$, so that it has to factor through a finite step. A more general version of this lemma can be found in Sets, Lemma 7.1. Next, we generalize this to the category of modules.

Definition 2.4. Let $\mathcal{C}$ be a category, let $I \subset \operatorname{Arrows}(\mathcal{C})$, and let α be an ordinal. An object A of $\mathcal{C}$ is said to be α-*small with respect to I* if whenever $\{B_\beta\}$ is a system over α with transition maps in I, then the map (2.0.1) is an isomorphism.

In the rest of this section we shall restrict ourselves to the category of R-modules for a fixed commutative ring R. We shall also take I to be the collection of injective

maps, i.e., the *monomorphisms* in the category of modules over R. In this case, for any system $\{B_\beta\}$ as in the definition each of the maps

$$B_\beta \to \operatorname{colim}_{\beta \in \alpha} B_\beta$$

is an injection. It follows that the map (2.0.1) is an *injection*. We can in fact interpret the B_β's as submodules of the module $B = \operatorname{colim}_{\beta \in \alpha} B_\beta$, and then we have $B = \bigcup_{\beta \in \alpha} B_\beta$. This is not an abuse of notation if we identify B_α with the image in the colimit. We now want to show that modules are always small for "large" ordinals α.

Proposition 2.5. *Let R be a ring. Let M be an R-module. Let κ the cardinality of the set of submodules of M. If α is an ordinal whose cofinality is bigger than κ, then M is α-small with respect to injections.*

Proof. The proof is straightforward, but let us first think about a special case. If M is finite, then the claim is that for any inductive system $\{B_\beta\}$ with injections between them, parametrized by a limit ordinal, any map $M \to \operatorname{colim} B_\beta$ factors through one of the B_β. And this we proved in Lemma 2.3.

Now we start the proof in the general case. We need only show that the map (2.0.1) is a surjection. Let $f : M \to \operatorname{colim} B_\beta$ be a map. Consider the subobjects $\{f^{-1}(B_\beta)\}$ of M, where B_β is considered as a subobject of the colimit $B = \bigcup_\beta B_\beta$. If one of these, say $f^{-1}(B_\beta)$, fills M, then the map factors through B_β.

So suppose to the contrary that all of the $f^{-1}(B_\beta)$ were proper subobjects of M. However, we know that

$$\bigcup f^{-1}(B_\beta) = f^{-1}\left(\bigcup B_\beta\right) = M.$$

Now there are at most κ different subobjects of M that occur among the $f^{-1}(B_\alpha)$, by hypothesis. Thus we can find a subset $S \subset \alpha$ of cardinality at most κ such that as β' ranges over S, the $f^{-1}(B_{\beta'})$ range over *all* the $f^{-1}(B_\alpha)$.

However, S has an upper bound $\widetilde{\alpha} < \alpha$ as α has cofinality bigger than κ. In particular, all the $f^{-1}(B_{\beta'})$, $\beta' \in S$ are contained in $f^{-1}(B_{\widetilde{\alpha}})$. It follows that $f^{-1}(B_{\widetilde{\alpha}}) = M$. In particular, the map f factors through $B_{\widetilde{\alpha}}$. $\qquad\square$

From this lemma we will be able to deduce the existence of lots of injectives. Let us recall Baer's criterion.

Lemma 2.6 (Baer's criterion)**.** *Let R be a ring. An R-module Q is injective if and only if in every commutative diagram*

for $\mathfrak{a} \subset R$ an ideal, the dotted arrow exists.

Proof. This is the equivalence of (1) and (3) in More on Algebra, Lemma 55.4; please observe that the proof given there is elementary (and does not use Ext groups or the existence of injectives or projectives in the category of R-modules). $\qquad\square$

If M is an R-module, then in general we may have a semi-complete diagram as in Lemma 2.6. In it, we can form the *push-out*

$$
\begin{array}{ccc}
\mathfrak{a} & \longrightarrow & M \\
\downarrow & & \downarrow \\
R & \longrightarrow & R \oplus_{\mathfrak{a}} M.
\end{array}
$$

Here the vertical map is injective, and the diagram commutes. The point is that we can extend $\mathfrak{a} \to M$ to R *if* we extend M to the larger module $R \oplus_{\mathfrak{a}} M$.

The key point of Baer's argument is to repeat this procedure transfinitely many times. To do this we first define, given an R-module M the following (huge) pushout

(2.6.1)
$$
\begin{array}{ccc}
\bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \operatorname{Hom}_R(\mathfrak{a}, M)} \mathfrak{a} & \longrightarrow & M \\
\downarrow & & \downarrow \\
\bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \operatorname{Hom}_R(\mathfrak{a}, M)} R & \longrightarrow & \mathbf{M}(M).
\end{array}
$$

Here the top horizontal arrow maps the element $a \in \mathfrak{a}$ in the summand corresponding to φ to the element $\varphi(a) \in M$. The left vertical arrow maps $a \in \mathfrak{a}$ in the summand corresponding to φ simply to the element $a \in R$ in the summand corresponding to φ. The fundamental properties of this construction are formulated in the following lemma.
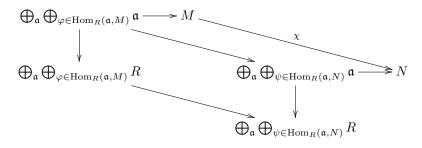
Lemma 2.7. *Let R be a ring.*

(1) *The construction $M \mapsto (M \to \mathbf{M}(M))$ is functorial in M.*
(2) *The map $M \to \mathbf{M}(M)$ is injective.*
(3) *For any ideal $\mathfrak{a}$ and any R-module map $\varphi : \mathfrak{a} \to M$ there is an R-module map $\varphi' : R \to \mathbf{M}(M)$ such that*

$$
\begin{array}{ccc}
\mathfrak{a} & \xrightarrow{\ \varphi\ } & M \\
\downarrow & & \downarrow \\
R & \xrightarrow{\ \varphi'\ } & \mathbf{M}(M)
\end{array}
$$

commutes.

Proof. Parts (2) and (3) are immediate from the construction. To see (1), let $\chi : M \to N$ be an R-module map. We claim there exists a canonical commutative diagram

which induces the desired map $\mathbf{M}(M) \to \mathbf{M}(N)$. The middle east-south-east arrow maps the summand $\mathfrak{a}$ corresponding to φ via $\mathrm{id}_{\mathfrak{a}}$ to the summand $\mathfrak{a}$ corresponding to $\psi = \chi \circ \varphi$. Similarly for the lower east-south-east arrow. Details omitted. $\square$

The idea will now be to apply the functor $\mathbf{M}$ a transfinite number of times. We define for each ordinal α a functor $\mathbf{M}_\alpha$ on the category of R-modules, together with a natural injection $N \to \mathbf{M}_\alpha(N)$. We do this by transfinite recursion. First, $\mathbf{M}_1 = \mathbf{M}$ is the functor defined above. Now, suppose given an ordinal α, and suppose $\mathbf{M}_{\alpha'}$ is defined for $\alpha' < \alpha$. If α has an immediate predecessor $\widetilde{\alpha}$, we let

$$\mathbf{M}_\alpha = \mathbf{M} \circ \mathbf{M}_{\widetilde{\alpha}}.$$

If not, i.e., if α is a limit ordinal, we let

$$\mathbf{M}_\alpha(N) = \mathrm{colim}_{\alpha' < \alpha} \mathbf{M}_{\alpha'}(N).$$

It is clear (e.g., inductively) that the $\mathbf{M}_\alpha(N)$ form an inductive system over ordinals, so this is reasonable.

Theorem 2.8. *Let κ be the cardinality of the set of ideals in R, and let α be an ordinal whose cofinality is greater than κ. Then $\mathbf{M}_\alpha(N)$ is an injective R-module, and $N \to \mathbf{M}_\alpha(N)$ is a functorial injective embedding.*

Proof. By Baer's criterion Lemma 2.6, it suffices to show that if $\mathfrak{a} \subset R$ is an ideal, then any map $f : \mathfrak{a} \to \mathbf{M}_\alpha(N)$ extends to $R \to \mathbf{M}_\alpha(N)$. However, we know since α is a limit ordinal that

$$\mathbf{M}_\alpha(N) = \mathrm{colim}_{\beta < \alpha} \mathbf{M}_\beta(N),$$

so by Proposition 2.5, we find that

$$\mathrm{Hom}_R(\mathfrak{a}, \mathbf{M}_\alpha(N)) = \mathrm{colim}_{\beta < \alpha} \mathrm{Hom}_R(\mathfrak{a}, \mathbf{M}_\beta(N)).$$

This means in particular that there is some $\beta' < \alpha$ such that f factors through the submodule $\mathbf{M}_{\beta'}(N)$, as

$$f : \mathfrak{a} \to \mathbf{M}_{\beta'}(N) \to \mathbf{M}_\alpha(N).$$

However, by the fundamental property of the functor $\mathbf{M}$, see Lemma 2.7 part (3), we know that the map $\mathfrak{a} \to \mathbf{M}_{\beta'}(N)$ can be extended to

$$R \to \mathbf{M}(\mathbf{M}_{\beta'}(N)) = \mathbf{M}_{\beta'+1}(N),$$

and the last object imbeds in $\mathbf{M}_\alpha(N)$ (as $\beta' + 1 < \alpha$ since α is a limit ordinal). In particular, f can be extended to $\mathbf{M}_\alpha(N)$. $\square$

3. G-modules

We will see later (Differential Graded Algebra, Section 17) that the category of modules over an algebra has functorial injective embeddings. The construction is exactly the same as the construction in More on Algebra, Section 55.

Lemma 3.1. *Let G be a topological group. Let R be a ring. The category $Mod_{R,G}$ of R-G-modules, see Étale Cohomology, Definition 57.1, has functorial injective hulls. In particular this holds for the category of discrete G-modules.*

Proof. By the remark above the lemma the category $\text{Mod}_{R[G]}$ has functorial injective embeddings. Consider the forgetful functor $v : \text{Mod}_{R,G} \to \text{Mod}_{R[G]}$. This functor is fully faithful, transforms injective maps into injective maps and has a right adjoint, namely

$$u : M \mapsto u(M) = \{x \in M \mid \text{stabilizer of } x \text{ is open}\}$$

Since $v(M) = 0 \Rightarrow M = 0$ we conclude by Homology, Lemma 29.5. $\qquad\square$

4. Abelian sheaves on a space

Lemma 4.1. *Let X be a topological space. The category of abelian sheaves on X has enough injectives. In fact it has functorial injective embeddings.*

Proof. For an abelian group A we denote $j : A \to J(A)$ the functorial injective embedding constructed in More on Algebra, Section 55. Let $\mathcal{F}$ be an abelian sheaf on X. By Sheaves, Example 7.5 the assignment

$$\mathcal{I} : U \mapsto \mathcal{I}(U) = \prod_{x \in U} J(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map $\mathcal{F} \to \mathcal{I}$ given by mapping $s \in \mathcal{F}(U)$ to $\prod_{x \in U} j(s_x)$ where $s_x \in \mathcal{F}_x$ denotes the germ of s at x. This map is injective, see Sheaves, Lemma 11.1 for example.

It remains to prove the following: Given a rule $x \mapsto I_x$ which assigns to each point $x \in X$ an injective abelian group the sheaf $\mathcal{I} : U \mapsto \prod_{x \in U} I_x$ is injective. Note that

$$\mathcal{I} = \prod_{x \in X} i_{x,*} I_x$$

is the product of the skyscraper sheaves $i_{x,*} I_x$ (see Sheaves, Section 27 for notation.) We have

$$\text{Mor}_{Ab}(\mathcal{F}_x, I_x) = \text{Mor}_{Ab(X)}(\mathcal{F}, i_{x,*} I_x).$$

see Sheaves, Lemma 27.3. Hence it is clear that each $i_{x,*} I_x$ is injective. Hence the injectivity of $\mathcal{I}$ follows from Homology, Lemma 27.3. $\qquad\square$

5. Sheaves of modules on a ringed space

Lemma 5.1. *Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Section 25. The category of sheaves of $\mathcal{O}_X$-modules on X has enough injectives. In fact it has functorial injective embeddings.*

Proof. For any ring R and any R-module M we denote $j : M \to J_R(M)$ the functorial injective embedding constructed in More on Algebra, Section 55. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X. By Sheaves, Examples 7.5 and 15.6 the assignment

$$\mathcal{I} : U \mapsto \mathcal{I}(U) = \prod_{x \in U} J_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map $\mathcal{F} \to \mathcal{I}$ given by mapping $s \in \mathcal{F}(U)$ to $\prod_{x \in U} j(s_x)$ where $s_x \in \mathcal{F}_x$ denotes the germ of s at x. This map is injective, see Sheaves, Lemma 11.1 for example.

It remains to prove the following: Given a rule $x \mapsto I_x$ which assigns to each point $x \in X$ an injective $\mathcal{O}_{X,x}$-module the sheaf $\mathcal{I} : U \mapsto \prod_{x \in U} I_x$ is injective. Note that

$$\mathcal{I} = \prod_{x \in X} i_{x,*} I_x$$

is the product of the skyscraper sheaves $i_{x,*}I_x$ (see Sheaves, Section 27 for notation.) We have

$$\operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I_x) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_{x,*}I_x).$$

see Sheaves, Lemma 27.3. Hence it is clear that each $i_{x,*}I_x$ is an injective $\mathcal{O}_X$-module (see Homology, Lemma 29.1 or argue directly). Hence the injectivity of $\mathcal{I}$ follows from Homology, Lemma 27.3. □

6. Abelian presheaves on a category

Let $\mathcal{C}$ be a category. Recall that this means that $\operatorname{Ob}(\mathcal{C})$ is a set. On the one hand, consider abelian presheaves on $\mathcal{C}$, see Sites, Section 2. On the other hand, consider families of abelian groups indexed by elements of $\operatorname{Ob}(\mathcal{C})$; in other words presheaves on the discrete category with underlying set of objects $\operatorname{Ob}(\mathcal{C})$. Let us denote this discrete category simply $\operatorname{Ob}(\mathcal{C})$. There is a natural functor

$$i : \operatorname{Ob}(\mathcal{C}) \longrightarrow \mathcal{C}$$

and hence there is a natural restriction or forgetful functor

$$v = i^p : PAb(\mathcal{C}) \longrightarrow PAb(\operatorname{Ob}(\mathcal{C}))$$

compare Sites, Section 5. We will denote presheaves on $\mathcal{C}$ by B and presheaves on $\operatorname{Ob}(\mathcal{C})$ by A.

There are also two functors, namely i_p and $_pi$ which assign an abelian presheaf on $\mathcal{C}$ to an abelian presheaf on $\operatorname{Ob}(\mathcal{C})$, see Sites, Sections 5 and 19. Here we will use $u = {_pi}$ which is defined (in the case at hand) as follows:

$$uA(U) = \prod_{U' \to U} A(U').$$

So an element is a family $(a_\phi)_\phi$ with ϕ ranging through all morphisms in $\mathcal{C}$ with target U. The restriction map on uA corresponding to $g : V \to U$ maps our element $(a_\phi)_\phi$ to the element $(a_{g \circ \psi})_\psi$.

There is a canonical surjective map $vuA \to A$ and a canonical injective map $B \to uvB$. We leave it to the reader to show that

$$\operatorname{Mor}_{PAb(\mathcal{C})}(B, uA) = \operatorname{Mor}_{PAb(\operatorname{Ob}(\mathcal{C}))}(vB, A).$$

in this simple case; the general case is in Sites, Section 5. Thus the pair (u, v) is an example of a pair of adjoint functors, see Categories, Section 24.

At this point we can list the following facts about the situation above.

(1) The functors u and v are exact. This follows from the explicit description of these functors given above.
(2) In particular the functor v transforms injective maps into injective maps.
(3) The category $PAb(\operatorname{Ob}(\mathcal{C}))$ has enough injectives.
(4) In fact there is a functorial injective embedding $A \mapsto \big(A \to J(A)\big)$ as in Homology, Definition 27.5. Namely, we can take $J(A)$ to be the presheaf $U \mapsto J(A(U))$, where $J(-)$ is the functor constructed in More on Algebra, Section 55 for the ring $\mathbf{Z}$.

Putting all of this together gives us the following procedure for embedding objects B of $PAb(\mathcal{C}))$ into an injective object: $B \to uJ(vB)$. See Homology, Lemma 29.5.

Proposition 6.1. *For abelian presheaves on a category there is a functorial injective embedding.*

Proof. See discussion above. $\square$

7. Abelian Sheaves on a site

Let $\mathcal{C}$ be a site. In this section we prove that there are enough injectives for abelian sheaves on $\mathcal{C}$.

Denote $i : Ab(\mathcal{C}) \longrightarrow PAb(\mathcal{C})$ the forgetful functor from abelian sheaves to abelian presheaves. Let $^\# : PAb(\mathcal{C}) \longrightarrow Ab(\mathcal{C})$ denote the sheafification functor. Recall that $^\#$ is a left adjoint to i, that $^\#$ is exact, and that $i\mathcal{F}^\# = \mathcal{F}$ for any abelian sheaf $\mathcal{F}$. Finally, let $\mathcal{G} \to J(\mathcal{G})$ denote the canonical embedding into an injective presheaf we found in Section 6.

For any sheaf $\mathcal{F}$ in $Ab(\mathcal{C})$ and any ordinal β we define a sheaf $J_\beta(\mathcal{F})$ by transfinite recursion. We set $J_0(\mathcal{F}) = \mathcal{F}$. We define $J_1(\mathcal{F}) = J(i\mathcal{F})^\#$. Sheafification of the canonical map $i\mathcal{F} \to J(i\mathcal{F})$ gives a functorial map

$$\mathcal{F} \longrightarrow J_1(\mathcal{F})$$

which is injective as $\#$ is exact. We set $J_{\alpha+1}(\mathcal{F}) = J_1(J_\alpha(\mathcal{F}))$. So that there are canonical injective maps $J_\alpha(\mathcal{F}) \to J_{\alpha+1}(\mathcal{F})$. For a limit ordinal β, we define

$$J_\beta(\mathcal{F}) = \operatorname{colim}_{\alpha < \beta} J_\alpha(\mathcal{F}).$$

Note that this is a directed colimit. Hence for any ordinals $\alpha < \beta$ we have an injective map $J_\alpha(\mathcal{F}) \to J_\beta(\mathcal{F})$.

Lemma 7.1. *With notation as above. Suppose that $\mathcal{G}_1 \to \mathcal{G}_2$ is an injective map of abelian sheaves on $\mathcal{C}$. Let α be an ordinal and let $\mathcal{G}_1 \to J_\alpha(\mathcal{F})$ be a morphism of sheaves. There exists a morphism $\mathcal{G}_2 \to J_{\alpha+1}(\mathcal{F})$ such that the following diagram commutes*

$$
\begin{array}{ccc}
\mathcal{G}_1 & \longrightarrow & \mathcal{G}_2 \\
\downarrow & & \downarrow \\
J_\alpha(\mathcal{F}) & \longrightarrow & J_{\alpha+1}(\mathcal{F})
\end{array}
$$

Proof. This is because the map $i\mathcal{G}_1 \to i\mathcal{G}_2$ is injective and hence $i\mathcal{G}_1 \to iJ_\alpha(\mathcal{F})$ extends to $i\mathcal{G}_2 \to J(iJ_\alpha(\mathcal{F}))$ which gives the desired map after applying the sheafification functor. $\square$

This lemma says that somehow the system $\{J_\alpha(\mathcal{F})\}$ is an injective embedding of $\mathcal{F}$. Of course we cannot take the limit over all α because they form a class and not a set. However, the idea is now that you don't have to check injectivity on all injections $\mathcal{G}_1 \to \mathcal{G}_2$, plus the following lemma.

Lemma 7.2. *Suppose that $\mathcal{G}_i$, $i \in I$ is set of abelian sheaves on $\mathcal{C}$. There exists an ordinal β such that for any sheaf $\mathcal{F}$, any $i \in I$, and any map $\varphi : \mathcal{G}_i \to J_\beta(\mathcal{F})$ there exists an $\alpha < \beta$ such that φ factors through $J_\alpha(\mathcal{F})$.*

Proof. This reduces to the case of a single sheaf $\mathcal{G}$ by taking the direct sum of all the $\mathcal{G}_i$.

Consider the sets

$$S = \coprod_{U \in \mathrm{Ob}(\mathcal{C})} \mathcal{G}(U).$$

and
$$T_\beta = \coprod_{U \in \mathrm{Ob}(\mathcal{C})} J_\beta(\mathcal{F})(U)$$

The transition maps between the sets T_β are injective. If the cofinality of β is large enough, then $T_\beta = \mathrm{colim}_{\alpha < \beta} T_\alpha$, see Sites, Lemma 17.10. A morphism $\mathcal{G} \to J_\beta(\mathcal{F})$ factors through $J_\alpha(\mathcal{F})$ if and only if the associated map $S \to T_\beta$ factors through T_α. By Sets, Lemma 7.1 if the cofinality of β is bigger than the cardinality of S, then the result of the lemma is true. Hence the lemma follows from the fact that there are ordinals with arbitrarily large cofinality, see Sets, Proposition 7.2. $\square$

Recall that for an object X of $\mathcal{C}$ we denote $\mathbf{Z}_X$ the presheaf of abelian groups $\Gamma(U, \mathbf{Z}_X) = \oplus_{U \to X} \mathbf{Z}$, see Modules on Sites, Section 4. The sheaf associated to this presheaf is denoted $\mathbf{Z}_X^\#$, see Modules on Sites, Section 5. It can be characterized by the property

$$(7.2.1) \qquad \mathrm{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}_X^\#, \mathcal{G}) = \mathcal{G}(X)$$

where the element φ of the left hand side is mapped to $\varphi(1 \cdot \mathrm{id}_X)$ in the right hand side. We can use these sheaves to characterize injective abelian sheaves.

Lemma 7.3. *Suppose $\mathcal{J}$ is a sheaf of abelian groups with the following property: For all $X \in \mathrm{Ob}(\mathcal{C})$, for any abelian subsheaf $\mathcal{S} \subset \mathbf{Z}_X^\#$ and any morphism $\varphi : \mathcal{S} \to \mathcal{J}$, there exists a morphism $\mathbf{Z}_X^\# \to \mathcal{J}$ extending φ. Then $\mathcal{J}$ is an injective sheaf of abelian groups.*

Proof. Let $\mathcal{F} \to \mathcal{G}$ be an injective map of abelian sheaves. Suppose $\varphi : \mathcal{F} \to \mathcal{J}$ is a morphism. Arguing as in the proof of More on Algebra, Lemma 54.1 we see that it suffices to prove that if $\mathcal{F} \neq \mathcal{G}$, then we can find an abelian sheaf $\mathcal{F}'$, $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{G}$ such that (a) the inclusion $\mathcal{F} \subset \mathcal{F}'$ is strict, and (b) φ can be extended to $\mathcal{F}'$. To find $\mathcal{F}'$, let X be an object of $\mathcal{C}$ such that the inclusion $\mathcal{F}(X) \subset \mathcal{G}(X)$ is strict. Pick $s \in \mathcal{G}(X)$, $s \notin \mathcal{F}(X)$. Let $\psi : \mathbf{Z}_X^\# \to \mathcal{G}$ be the morphism corresponding to the section s via (7.2.1). Set $\mathcal{S} = \psi^{-1}(\mathcal{F})$. By assumption the morphism

$$\mathcal{S} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{J}$$

can be extended to a morphism $\varphi' : \mathbf{Z}_X^\# \to \mathcal{J}$. Note that φ' annihilates the kernel of ψ (as this is true for φ). Thus φ' gives rise to a morphism $\varphi'' : \mathrm{Im}(\psi) \to \mathcal{J}$ which agrees with φ on the intersection $\mathcal{F} \cap \mathrm{Im}(\psi)$ by construction. Thus φ and φ'' glue to give an extension of φ to the strictly bigger subsheaf $\mathcal{F}' = \mathcal{F} + \mathrm{Im}(\psi)$. $\square$

Theorem 7.4. *The category of sheaves of abelian groups on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 27.5.*

Proof. Let $\mathcal{G}_i$, $i \in I$ be a set of abelian sheaves such that every subsheaf of every $\mathbf{Z}_X^\#$ occurs as one of the $\mathcal{G}_i$. Apply Lemma 7.2 to this collection to get an ordinal β. We claim that for any sheaf of abelian groups $\mathcal{F}$ the map $\mathcal{F} \to J_\beta(\mathcal{F})$ is an injection of $\mathcal{F}$ into an injective. Note that by construction the assignment $\mathcal{F} \mapsto (\mathcal{F} \to J_\beta(\mathcal{F}))$ is indeed functorial.

The proof of the claim comes from the fact that by Lemma 7.3 it suffices to extend any morphism $\gamma : \mathcal{G} \to J_\beta(\mathcal{F})$ from a subsheaf $\mathcal{G}$ of some $\mathbf{Z}_X^\#$ to all of $\mathbf{Z}_X^\#$. Then by Lemma 7.2 the map γ lifts into $J_\alpha(\mathcal{F})$ for some $\alpha < \beta$. Finally, we apply Lemma 7.1 to get the desired extension of γ to a morphism into $J_{\alpha+1}(\mathcal{F}) \to J_\beta(\mathcal{F})$. $\square$

8. Modules on a ringed site

Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. By analogy with More on Algebra, Section 55 let us try to prove that there are enough injective $\mathcal{O}$-modules. First of all, we pick an injective embedding

$$\bigoplus\nolimits_{U,\mathcal{I}} j_{U!}\mathcal{O}_U/\mathcal{I} \longrightarrow \mathcal{J}$$

where $\mathcal{J}$ is an injective abelian sheaf (which exists by the previous section). Here the direct sum is over all objects U of $\mathcal{C}$ and over all $\mathcal{O}$-submodules $\mathcal{I} \subset j_{U!}\mathcal{O}_U$. Please see Modules on Sites, Section 19 to read about the functors restriction and extension by 0 for the localization functor $j_U : \mathcal{C}/U \to \mathcal{C}$.

For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ denote

$$\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{J})$$

with its natural $\mathcal{O}$-module structure. Insert here future reference to internal hom. We will also need a canonical flat resolution of a sheaf of $\mathcal{O}$-modules. This we can do as follows: For any $\mathcal{O}$-module $\mathcal{F}$ we denote

$$F(\mathcal{F}) = \bigoplus\nolimits_{U \in \mathrm{Ob}(\mathcal{C}), s \in \mathcal{F}(U)} j_{U!}\mathcal{O}_U.$$

This is a flat sheaf of $\mathcal{O}$-modules which comes equipped with a canonical surjection $F(\mathcal{F}) \to \mathcal{F}$, see Modules on Sites, Lemma 28.8. Moreover the construction $\mathcal{F} \mapsto F(\mathcal{F})$ is functorial in $\mathcal{F}$.

Lemma 8.1. *The functor $\mathcal{F} \mapsto \mathcal{F}^\vee$ is exact.*

Proof. This because $\mathcal{J}$ is an injective abelian sheaf. $\qquad\square$

There is a canonical map $ev : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ given by evaluation: given $x \in \mathcal{F}(U)$ we let $ev(x) \in (\mathcal{F}^\vee)^\vee = \mathcal{H}om(\mathcal{F}^\vee, \mathcal{J})$ be the map $\varphi \mapsto \varphi(x)$.

Lemma 8.2. *For any $\mathcal{O}$-module $\mathcal{F}$ the evaluation map $ev : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ is injective.*

Proof. You can check this using the definition of $\mathcal{J}$. Namely, if $s \in \mathcal{F}(U)$ is not zero, then let $j_{U!}\mathcal{O}_U \to \mathcal{F}$ be the map of $\mathcal{O}$-modules it corresponds to via adjunction. Let $\mathcal{I}$ be the kernel of this map. There exists a nonzero map $\mathcal{F} \supset j_{U!}\mathcal{O}_U/\mathcal{I} \to \mathcal{J}$ which does not annihilate s. As $\mathcal{J}$ is an injective $\mathcal{O}$-module, this extends to a map $\varphi : \mathcal{F} \to \mathcal{J}$. Then $ev(s)(\varphi) = \varphi(s) \neq 0$ which is what we had to prove. $\qquad\square$

The canonical surjection $F(\mathcal{F}) \to \mathcal{F}$ of $\mathcal{O}$-modules turns into a canonical injection, see above, of $\mathcal{O}$-modules

$$(\mathcal{F}^\vee)^\vee \longrightarrow (F(\mathcal{F}^\vee))^\vee.$$

Set $J(\mathcal{F}) = (F(\mathcal{F}^\vee))^\vee$. The composition of ev with this the displayed map gives $\mathcal{F} \to J(\mathcal{F})$ functorially in $\mathcal{F}$.

Lemma 8.3. *Let $\mathcal{O}$ be a sheaf of rings. For every $\mathcal{O}$-module $\mathcal{F}$ the $\mathcal{O}$-module $J(\mathcal{F})$ is injective.*

Proof. We have to show that the functor $\mathrm{Hom}_\mathcal{O}(\mathcal{G}, J(\mathcal{F}))$ is exact. Note that

$$
\begin{aligned}
\mathrm{Hom}_\mathcal{O}(\mathcal{G}, J(\mathcal{F})) &= \mathrm{Hom}_\mathcal{O}(\mathcal{G}, (F(\mathcal{F}^\vee))^\vee) \\
&= \mathrm{Hom}_\mathcal{O}(\mathcal{G}, \mathcal{H}om(F(\mathcal{F}^\vee), \mathcal{J})) \\
&= \mathrm{Hom}(\mathcal{G} \otimes_\mathcal{O} F(\mathcal{F}^\vee), \mathcal{J})
\end{aligned}
$$

Thus what we want follows from the fact that $F(\mathcal{F}^\vee)$ is flat and $\mathcal{J}$ is injective. $\quad\square$

Theorem 8.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. The category of sheaves of $\mathcal{O}$-modules on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 27.5.*

Proof. From the discussion in this section. $\square$

Proposition 8.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. The category $PMod(\mathcal{O})$ of presheaves of $\mathcal{O}$-modules has functorial injective embeddings.*

Proof. We could prove this along the lines of the discussion in Section 6. But instead we argue using the theorem above. Endow $\mathcal{C}$ with the structure of a site by letting the set of coverings of an object U consist of all singletons $\{f : V \to U\}$ where f is an isomorphism. We omit the verification that this defines a site. A sheaf for this topology is the same as a presheaf (proof omitted). Hence the theorem applies. $\square$

9. Embedding abelian categories

In this section we show that an abelian category embeds in the category of abelian sheaves on a site having enough points. The site will be the one described in the following lemma.

Lemma 9.1. *Let $\mathcal{A}$ be an abelian category. Let*
$$Cov = \{\{f : V \to U\} \mid f \text{ is surjective}\}.$$
Then $(\mathcal{A}, Cov)$ is a site, see Sites, Definition 6.2.

Proof. Note that $\mathrm{Ob}(\mathcal{A})$ is a set by our conventions about categories. An isomorphism is a surjective morphism. The composition of surjective morphisms is surjective. And the base change of a surjective morphism in $\mathcal{A}$ is surjective, see Homology, Lemma 5.14. $\square$

Let $\mathcal{A}$ be a pre-additive category. In this case the Yoneda embedding $\mathcal{A} \to PSh(\mathcal{A})$, $X \mapsto h_X$ factors through a functor $\mathcal{A} \to PAb(\mathcal{A})$.

Lemma 9.2. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} = (\mathcal{A}, Cov)$ be the site defined in Lemma 9.1. Then $X \mapsto h_X$ defines a fully faithful, exact functor*
$$\mathcal{A} \longrightarrow Ab(\mathcal{C}).$$
Moreover, the site $\mathcal{C}$ has enough points.

Proof. Suppose that $f : V \to U$ is a surjective morphism of $\mathcal{A}$. Let $K = \mathrm{Ker}(f)$. Recall that $V \times_U V = \mathrm{Ker}((f, -f) : V \oplus V \to U)$, see Homology, Example 5.6. In particular there exists an injection $K \oplus K \to V \times_U V$. Let $p, q : V \times_U V \to V$ be the two projection morphisms. Note that $p - q : V \times_U V \to V$ is a morphism such that $f \circ (p - q) = 0$. Hence $p - q$ factors through $K \to V$. Let us denote this morphism by $c : V \times_U V \to K$. And since the composition $K \oplus K \to V \times_U V \to K$ is surjective, we conclude that c is surjective. It follows that
$$V \times_U V \xrightarrow{p-q} V \to U \to 0$$
is an exact sequence of $\mathcal{A}$. Hence for an object X of $\mathcal{A}$ the sequence
$$0 \to \mathrm{Hom}_{\mathcal{A}}(U, X) \to \mathrm{Hom}_{\mathcal{A}}(V, X) \to \mathrm{Hom}_{\mathcal{A}}(V \times_U V, X)$$
is an exact sequence of abelian groups, see Homology, Lemma 5.8. This means that h_X satisfies the sheaf condition on $\mathcal{C}$.

The functor is fully faithful by Categories, Lemma 3.5. The functor is a left exact functor between abelian categories by Homology, Lemma 5.8. To show that it is right exact, let $X \to Y$ be a surjective morphism of $\mathcal{A}$. Let U be an object of $\mathcal{A}$, and let $s \in h_Y(U) = \mathrm{Mor}_{\mathcal{A}}(U, Y)$ be a section of h_Y over U. By Homology, Lemma 5.14 the projection $U \times_Y X \to U$ is surjective. Hence $\{V = U \times_Y X \to U\}$ is a covering of U such that $s|_V$ lifts to a section of h_X. This proves that $h_X \to h_Y$ is a surjection of abelian sheaves, see Sites, Lemma 11.2.

The site $\mathcal{C}$ has enough points by Sites, Proposition 39.3. $\square$

Remark 9.3. The Freyd-Mitchell embedding theorem says there exists a fully faithful exact functor from any abelian category $\mathcal{A}$ to the category of modules over a ring. Lemma 9.2 is not quite as strong. But the result is suitable for the Stacks project as we have to understand sheaves of abelian groups on sites in detail anyway. Moreover, "diagram chasing" works in the category of abelian sheaves on $\mathcal{C}$, for example by working with sections over objects, or by working on the level of stalks using that $\mathcal{C}$ has enough points. To see how to deduce the Freyd-Mitchell embedding theorem from Lemma 9.2 see Remark 9.5.

Remark 9.4. If $\mathcal{A}$ is a "big" abelian category, i.e., if $\mathcal{A}$ has a class of objects, then Lemma 9.2 does not work. In this case, given any set of objects $E \subset \mathrm{Ob}(\mathcal{A})$ there exists an abelian full subcategory $\mathcal{A}' \subset \mathcal{A}$ such that $\mathrm{Ob}(\mathcal{A}')$ is a set and $E \subset \mathrm{Ob}(\mathcal{A}')$. Then one can apply Lemma 9.2 to $\mathcal{A}'$. One can use this to prove that results depending on a diagram chase hold in $\mathcal{A}$.

Remark 9.5. Let $\mathcal{C}$ be a site. Note that $Ab(\mathcal{C})$ has enough injectives, see Theorem 7.4. (In the case that $\mathcal{C}$ has enough points this is straightforward because $p_* I$ is an injective sheaf if I is an injective $\mathbf{Z}$-module and p is a point.) Also, $Ab(\mathcal{C})$ has a cogenerator (details omitted). Hence Lemma 9.2 proves that we have a fully faithful, exact embedding $\mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ has a cogenerator and enough injectives. We can apply this to $\mathcal{A}^{opp}$ and we get a fully faithful exact functor $i : \mathcal{A} \to \mathcal{D} = \mathcal{B}^{opp}$ where $\mathcal{D}$ has enough projectives and a generator. Hence $\mathcal{D}$ has a projective generator P. Set $R = \mathrm{Mor}_{\mathcal{D}}(P, P)$. Then

$$\mathcal{A} \longrightarrow \mathrm{Mod}_R, \quad X \longmapsto \mathrm{Hom}_{\mathcal{D}}(P, X).$$

One can check this is a fully faithful, exact functor. In other words, one retrieves the Freyd-Mitchell theorem mentioned in Remark 9.3 above.

Remark 9.6. The arguments proving Lemmas 9.1 and 9.2 work also for *exact categories*, see [Büh10, Appendix A] and [BBD82, 1.1.4]. We quickly review this here and we add more details if we ever need it in the Stacks project.

Let $\mathcal{A}$ be an additive category. A *kernel-cokernel* pair is a pair (i, p) of morphisms of $\mathcal{A}$ with $i : A \to B$, $p : B \to C$ such that i is the kernel of p and p is the cokernel of i. Given a set $\mathcal{E}$ of kernel-cokernel pairs we say $i : A \to B$ is an *admissible monomorphism* if $(i, p) \in \mathcal{E}$ for some morphism p. Similarly we say a morphism $p : B \to C$ is an *admissible epimorphism* if $(i, p) \in \mathcal{E}$ for some morphism i. The pair $(\mathcal{A}, \mathcal{E})$ is said to be an *exact category* if the following axioms hold

(1) $\mathcal{E}$ is closed under isomorphisms of kernel-cokernel pairs,
(2) for any object A the morphism 1_A is both an admissible epimorphism and an admissible monomorphism,
(3) admissible monomorphisms are stable under composition,

(4) admissible epimorphisms are stable under composition,
(5) the push-out of an admissible monomorphism $i : A \to B$ via any morphism $A \to A'$ exist and the induced morphism $i' : A' \to B'$ is an admissible monomorphism, and
(6) the base change of an admissible epimorphism $p : B \to C$ via any morphism $C' \to C$ exist and the induced morphism $p' : B' \to C'$ is an admissible epimorphism.

Given such a structure let $\mathcal{C} = (\mathcal{A}, \mathrm{Cov})$ where coverings (i.e., elements of Cov) are given by admissible epimorphisms. The axioms listed above immediately imply that this is a site. Consider the functor

$$F : \mathcal{A} \longrightarrow Ab(\mathcal{C}), \quad X \longmapsto h_X$$

exactly as in Lemma 9.2. It turns out that this functor is fully faithful, exact, and reflects exactness. Moreover, any extension of objects in the essential image of F is in the essential image of F.

10. Grothendieck's AB conditions

This and the next few sections are mostly interesting for "big" abelian categories, i.e., those categories listed in Categories, Remark 2.2. A good case to keep in mind is the category of sheaves of modules on a ringed site.

Grothendieck proved the existence of injectives in great generality in the paper [Gro57]. He used the following conditions to single out abelian categories with special properties.

Definition 10.1. Let $\mathcal{A}$ be an abelian category. We name some conditions

AB3 $\mathcal{A}$ has direct sums,
AB4 $\mathcal{A}$ has AB3 and direct sums are exact,
AB5 $\mathcal{A}$ has AB3 and filtered colimits are exact.

Here are the dual notions

AB3* $\mathcal{A}$ has products,
AB4* $\mathcal{A}$ has AB3* and products are exact,
AB5* $\mathcal{A}$ has AB3* and cofiltered limits are exact.

We say an object U of $\mathcal{A}$ is a *generator* if for every $N \subset M$, $N \neq M$ in $\mathcal{A}$ there exists a morphism $U \to M$ which does not factor through N. We say $\mathcal{A}$ is a *Grothendieck abelian category* if it has AB5 and a generator.

Discussion: A direct sum in an abelian category is a coproduct. If an abelian category has direct sums (i.e., AB3), then it has colimits, see Categories, Lemma 14.12. Similarly if $\mathcal{A}$ has AB3* then it has limits, see Categories, Lemma 14.11. Exactness of direct sums means the following: given an index set I and short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0, \quad i \in I$$

in $\mathcal{A}$ then the sequence

$$0 \to \bigoplus_{i \in I} A_i \to \bigoplus_{i \in I} B_i \to \bigoplus_{i \in I} C_i \to 0$$

is exact as well. Without assuming AB4 it is only true in general that the sequence is exact on the right (i.e., taking direct sums is a right exact functor if direct sums

exist). Similarly, exactness of filtered colimits means the following: given a directed set I and a system of short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

over I in $\mathcal{A}$ then the sequence

$$0 \to \mathrm{colim}_{i \in I} A_i \to \mathrm{colim}_{i \in I} B_i \to \mathrm{colim}_{i \in I} C_i \to 0$$

is exact as well. Without assuming AB5 it is only true in general that the sequence is exact on the right (i.e., taking colimits is a right exact functor if colimits exist). A similar explanation holds for AB4* and AB5*.

11. Injectives in Grothendieck categories

The existence of a generator implies that given an object M of a Grothendieck abelian category $\mathcal{A}$ there is a set of subobjects. (This may not be true for a general "big" abelian category.)

Lemma 11.1. *Let $\mathcal{A}$ be an abelian category with a generator U and X and object of $\mathcal{A}$. If κ is the cardinality of $\mathrm{Mor}(U, X)$ then*

    (1) *There does not exist a strictly increasing (or strictly decreasing) chain of subobjects of X indexed by a cardinal bigger than κ.*
    (2) *If α is an ordinal of cofinality $> \kappa$ then any increasing (or decreasing) sequence of subobjects of X indexed by α is eventually constant.*
    (3) *The cardinality of the set of subobjects of X is $\leq 2^{\kappa}$.*

Proof. For (1) assume $\kappa' > \kappa$ is a cardinal and assume X_i, $i \in \kappa'$ is strictly increasing. Then take for each i a $\phi_i \in \mathrm{Mor}(U, X)$ such that ϕ_i factors through X_{i+1} but not through X_i. Then the morphisms ϕ_i are distinct, which contradicts the definition of κ.

Part (2) follows from the definition of cofinality and (1).

Proof of (3). For any subobject $Y \subset X$ define $S_Y \in \mathcal{P}(\mathrm{Mor}(U, X))$ (power set) as $S_Y = \{\phi \in \mathrm{Mor}(U, X) : \phi)$ factors through $Y\}$. Then $Y = Y'$ if and only if $S_Y = S_{Y'}$. Hence the cardinality of the set of subobjects is at most the cardinality of this power set. $\square$

By Lemma 11.1 the following definition makes sense.

Definition 11.2. Let $\mathcal{A}$ be a Grothendieck abelian category. Let M be an object of $\mathcal{A}$. The *size* $|M|$ of M is the cardinality of the set of subobjects of M.

Lemma 11.3. *Let $\mathcal{A}$ be a Grothendieck abelian category. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $\mathcal{A}$, then $|M'|, |M''| \leq |M|$.*

Proof. Immediate from the definitions. $\square$

Lemma 11.4. *Let $\mathcal{A}$ be a Grothendieck abelian category with generator U.*

    (1) *If $|M| \leq \kappa$, then M is the quotient of a direct sum of at most κ copies of U.*
    (2) *For every cardinal κ there exists a set of isomorphism classes of objects M with $|M| \leq \kappa$.*

Proof. For (1) choose for every proper subobject $M' \subset M$ a morphism $\varphi_{M'} : U \to M$ whose image is not contained in M'. Then $\bigoplus_{M' \subset M} \varphi_{M'} : \bigoplus_{M' \subset M} U \to M$ is surjective. It is clear that (1) implies (2). $\square$

Proposition 11.5. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let M be an object of $\mathcal{A}$. Let $\kappa = |M|$. If α is an ordinal whose cofinality is bigger than κ, then M is α-small with respect to injections.*

Proof. Please compare with Proposition 2.5. We need only show that the map (2.0.1) is a surjection. Let $f : M \to \operatorname{colim} B_\beta$ be a map. Consider the subobjects $\{f^{-1}(B_\beta)\}$ of M, where B_β is considered as a subobject of the colimit $B = \bigcup_\beta B_\beta$. If one of these, say $f^{-1}(B_\beta)$, fills M, then the map factors through B_β.

So suppose to the contrary that all of the $f^{-1}(B_\beta)$ were proper subobjects of M. However, because $\mathcal{A}$ has AB5 we have

$$\operatorname{colim} f^{-1}(B_\beta) = f^{-1}\left(\operatorname{colim} B_\beta\right) = M.$$

Now there are at most κ different subobjects of M that occur among the $f^{-1}(B_\alpha)$, by hypothesis. Thus we can find a subset $S \subset \alpha$ of cardinality at most κ such that as β' ranges over S, the $f^{-1}(B_{\beta'})$ range over *all* the $f^{-1}(B_\alpha)$.

However, S has an upper bound $\widetilde{\alpha} < \alpha$ as α has cofinality bigger than κ. In particular, all the $f^{-1}(B_{\beta'})$, $\beta' \in S$ are contained in $f^{-1}(B_{\widetilde{\alpha}})$. It follows that $f^{-1}(B_{\widetilde{\alpha}}) = M$. In particular, the map f factors through $B_{\widetilde{\alpha}}$. $\square$

Lemma 11.6. *Let $\mathcal{A}$ be a Grothendieck abelian category with generator U. An object I of $\mathcal{A}$ is injective if and only if in every commutative diagram*

$$
\begin{array}{ccc}
M & \longrightarrow & I \\
\downarrow & \nearrow & \\
U & &
\end{array}
$$

for $M \subset U$ a subobject, the dotted arrow exists.

Proof. Please see Lemma 2.6 for the case of modules. Choose an injection $A \subset B$ and a morphism $\varphi : A \to I$. Consider the set S of pairs (A', φ') consisting of subobjects $A \subset A' \subset B$ and a morphism $\varphi' : A' \to I$ extending φ. Define a partial ordering on this set in the obvious manner. Choose a totally ordered subset $T \subset S$. Then

$$A' = \operatorname{colim}_{t \in T} A_t \xrightarrow{\operatorname{colim}_{t \in T} \varphi_t} I$$

is an upper bound. Hence by Zorn's lemma the set S has a maximal element (A', φ'). We claim that $A' = B$. If not, then choose a morphism $\psi : U \to B$ which does not factor through A'. Set $N = A' \cap \psi(U)$. Set $M = \psi^{-1}(N)$. Then the map

$$M \to N \to A' \xrightarrow{\varphi'} I$$

can be extended to a morphism $\chi : U \to I$. Since $\chi|_{\operatorname{Ker}(\psi)} = 0$ we see that χ factors as

$$U \to \operatorname{Im}(\psi) \xrightarrow{\varphi''} I$$

Since φ' and φ'' agree on $N = A' \cap \operatorname{Im}(\psi)$ we see that combined the define a morphism $A' + \operatorname{Im}(\psi) \to I$ contradicting the assumed maximality of A'. $\square$

Theorem 11.7. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\mathcal{A}$ has functorial injective embeddings.*

Proof. Please compare with the proof of Theorem 2.8. Choose a generator U of $\mathcal{A}$. For an object M we define $\mathbf{M}(M)$ by the following pushout diagram

$$
\begin{array}{ccc}
\bigoplus_{N \subset U} \bigoplus_{\varphi \in \mathrm{Hom}(N,M)} N & \longrightarrow & M \\
\downarrow & & \downarrow \\
\bigoplus_{N \subset U} \bigoplus_{\varphi \in \mathrm{Hom}(N,M)} U & \longrightarrow & \mathbf{M}(M).
\end{array}
$$

Note that $M \to \mathbf{M}(N)$ is a functor and that there exist functorial injective maps $M \to \mathbf{M}(M)$. By transfinite induction we define functors $\mathbf{M}_\alpha(M)$ for every ordinal α. Namely, set $\mathbf{M}_0(M) = M$. Given $\mathbf{M}_\alpha(M)$ set $\mathbf{M}_{\alpha+1}(M) = \mathbf{M}(\mathbf{M}_\alpha(M))$. For a limit ordinal β set

$$
\mathbf{M}_\beta(M) = \mathrm{colim}_{\alpha < \beta} \mathbf{M}_\alpha(M).
$$

Finally, pick any ordinal α whose cofinality is greater than $|U|$. Such an ordinal exists by Sets, Proposition 7.2. We claim that $M \to \mathbf{M}_\alpha(M)$ is the desired functorial injective embedding. Namely, if $N \subset U$ is a subobject and $\varphi : N \to \mathbf{M}_\alpha(M)$ is a morphism, then we see that φ factors through $\mathbf{M}_{\alpha'}(M)$ for some $\alpha' < \alpha$ by Proposition 11.5. By construction of $\mathbf{M}(-)$ we see that φ extends to a morphism from U into $\mathbf{M}_{\alpha'+1}(M)$ and hence into $\mathbf{M}_\alpha(M)$. By Lemma 11.6 we conclude that $\mathbf{M}_\alpha(M)$ is injective. $\qquad\square$

12. K-injectives in Grothendieck categories

The material in this section is taken from the paper [Ser03] authored by Serpé. This paper generalizes some of the results of [Spa88] by Spaltenstein to general Grothendieck abelian categories. Our Lemma 12.3 is only implicit in the paper by Serpé. Our approach is to mimic Grothendieck's proof of Theorem 11.7.

Lemma 12.1. *Let $\mathcal{A}$ be a Grothendieck abelian category with generator U. Let c be the function on cardinals defined by $c(\kappa) = |\bigoplus_{\alpha \in \kappa} U|$. If $\pi : M \to N$ is a surjection then there exists a subobject $M' \subset M$ which surjects onto N with $|M'| \leq c(|N|)$.*

Proof. For every proper subobject $N' \subset N$ choose a morphism $\varphi_{N'} : U \to M$ such that $U \to M \to N$ does not factor through N'. Set

$$
M' = \mathrm{Im}\left(\bigoplus_{N' \subset N} \varphi_{N'} : \bigoplus_{N' \subset N} U \longrightarrow M \right)
$$

Then M' works. $\qquad\square$

Lemma 12.2. *Let $\mathcal{A}$ be a Grothendieck abelian category. There exists a cardinal κ such that given any acyclic complex $M^\bullet$ we have*

(1) *if $M^\bullet$ is nonzero, there is a nonzero subcomplex $N^\bullet$ which is bounded above, acyclic, and $|N^n| \leq \kappa$,*

(2) *there exists a surjection of complexes*

$$
\bigoplus_{i \in I} M_i^\bullet \longrightarrow M^\bullet
$$

where $M_i^\bullet$ is bounded above, acyclic, and $|M_i^n| \leq \kappa$.

Proof. Choose a generator U of $\mathcal{A}$. Denote c the function of Lemma 12.1. Set $\kappa = \sup\{c^n(|U|), n = 1, 2, 3, \ldots\}$. Let $n \in \mathbf{Z}$ and let $\psi : U \to M^n$ be a morphism. In order to prove (1) and (2) it suffices to prove there exists a subcomplex $N^\bullet \subset M^\bullet$ which is bounded above, acyclic, and $|N^m| \leq \kappa$, such that ψ factors through N^n. To do this set $N^n = \mathrm{Im}(\psi)$, $N^{n+1} = \mathrm{Im}(U \to M^n \to M^{n+1})$, and $N^m = 0$ for $m \geq n + 2$. Suppose we have constructed $N^m \subset M^m$ for all $m \geq k$ such that

(1) $\mathrm{d}(N^m) \subset N^{m+1}$, $m \geq k$,
(2) $\mathrm{Im}(N^{m-1} \to N^m) = \mathrm{Ker}(N^m \to N^{m+1})$ for all $m \geq k + 1$, and
(3) $|N^m| \leq c^{\max\{n-m,0\}}(|U|)$.

for some $k \leq n$. Because $M^\bullet$ is acyclic, we see that the subobject $\mathrm{d}^{-1}(\mathrm{Ker}(N^k \to N^{k+1})) \subset M^{k-1}$ surjects onto $\mathrm{Ker}(N^k \to N^{k+1})$. Thus we can choose $N^{k-1} \subset M^{k-1}$ surjecting onto $\mathrm{Ker}(N^k \to N^{k+1})$ with $|N^{k-1}| \leq c^{n-k+1}(|U|)$ by Lemma 12.1. The proof is finished by induction on k. $\qquad\square$

Lemma 12.3. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let κ be a cardinal as in Lemma 12.2. Suppose that $I^\bullet$ is a complex such that*

(1) *each I^j is injective, and*
(2) *for every bounded above acyclic complex $M^\bullet$ such that $|M^n| \leq \kappa$ we have* $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$.

Then $I^\bullet$ is an K-injective complex.

Proof. Let $M^\bullet$ be an acyclic complex. We are going to construct by induction on the ordinal α an acyclic subcomplex $K_\alpha^\bullet \subset M^\bullet$ as follows. For $\alpha = 0$ we set $K_0^\bullet = 0$. For $\alpha > 0$ we proceed as follows:

(1) If $\alpha = \beta + 1$ and $K_\beta^\bullet = M^\bullet$ then we choose $K_\alpha^\bullet = K_\beta^\bullet$.
(2) If $\alpha = \beta + 1$ and $K_\beta^\bullet \neq M^\bullet$ then $M^\bullet / K_\beta^\bullet$ is a nonzero acyclic complex. We choose a subcomplex $N_\alpha^\bullet \subset M^\bullet / K_\beta^\bullet$ as in Lemma 12.2. Finally, we let $K_\alpha^\bullet \subset M^\bullet$ be the inverse image of $N_\alpha^\bullet$.
(3) If α is a limit ordinal we set $K_\beta^\bullet = \mathrm{colim}\, K_\alpha^\bullet$.

It is clear that $M^\bullet = K_\alpha^\bullet$ for a suitably large ordinal α. We will prove that

$$\mathrm{Hom}_{K(\mathcal{A})}(K_\alpha^\bullet, I^\bullet)$$

is zero by transfinite induction on α. It holds for $\alpha = 0$ since $K_0^\bullet$ is zero. Suppose it holds for β and $\alpha = \beta + 1$. In case (1) of the list above the result is clear. In case (2) there is a short exact sequence of complexes

$$0 \to K_\beta^\bullet \to K_\alpha^\bullet \to N_\alpha^\bullet \to 0$$

Since each component of $I^\bullet$ is injective we see that we obtain an exact sequence

$$\mathrm{Hom}_{K(\mathcal{A})}(K_\beta^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(K_\alpha^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(N_\alpha^\bullet, I^\bullet)$$

By induction the term on the left is zero and by assumption on $I^\bullet$ the term on the right is zero. Thus the middle group is zero too. Finally, suppose that α is a limit ordinal. Then we see that

$$\mathrm{Hom}^\bullet(K_\alpha^\bullet, I^\bullet) = \lim_{\beta < \alpha} \mathrm{Hom}^\bullet(K_\beta^\bullet, I^\bullet)$$

with notation as in More on Algebra, Section 71. These complexes compute morphisms in $K(\mathcal{A})$ by More on Algebra, Equation (71.0.1). Note that the transition maps in the system are surjective because I^j is surjective for each j. Moreover, for

a limit ordinal α we have equality of limit and value (see displayed formula above). Thus we may apply Homology, Lemma 31.8 to conclude. $\square$

Lemma 12.4. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $(K_i^\bullet)_{i \in I}$ be a set of acyclic complexes. There exists a functor $M^\bullet \mapsto \mathbf{M}^\bullet(M^\bullet)$ and a natural transformation $j_{M^\bullet} : M^\bullet \to \mathbf{M}^\bullet(M^\bullet)$ such*

(1) *$j_{M^\bullet}$ is a (termwise) injective quasi-isomorphism, and*
(2) *for every $i \in I$ and $w : K_i^\bullet \to M^\bullet$ the morphism $j_{M^\bullet} \circ w$ is homotopic to zero.*

Proof. For every $i \in I$ choose a (termwise) injective map of complexes $K_i^\bullet \to L_i^\bullet$ which is homotopic to zero with $L_i^\bullet$ quasi-isomorphic to zero. For example, take $L_i^\bullet$ to be the cone on the identity of $K_i^\bullet$. We define $\mathbf{M}^\bullet(M^\bullet)$ by the following pushout diagram

$$
\begin{array}{ccc}
\bigoplus_{i \in I} \bigoplus_{w:K_i^\bullet \to M^\bullet} K_i^\bullet & \longrightarrow & M^\bullet \\
\downarrow & & \downarrow \\
\bigoplus_{i \in I} \bigoplus_{w:K_i^\bullet \to M^\bullet} L_i^\bullet & \longrightarrow & \mathbf{M}^\bullet(M^\bullet).
\end{array}
$$

Then $M^\bullet \to \mathbf{M}^\bullet(M^\bullet)$ is a functor. The right vertical arrow defines the functorial injective map $j_{M^\bullet}$. The cokernel of $j_{M^\bullet}$ is isomorphic to the direct sum of the cokernels of the maps $K_i^\bullet \to L_i^\bullet$ hence acyclic. Thus $j_{M^\bullet}$ is a quasi-isomorphism. Part (2) holds by construction. $\square$

Lemma 12.5. *Let $\mathcal{A}$ be a Grothendieck abelian category. There exists a functor $M^\bullet \mapsto \mathbf{N}^\bullet(M^\bullet)$ and a natural transformation $j_{M^\bullet} : M^\bullet \to \mathbf{N}^\bullet(M^\bullet)$ such*

(1) *$j_{M^\bullet}$ is a (termwise) injective quasi-isomorphism, and*
(2) *for every $n \in \mathbf{Z}$ the map $M^n \to \mathbf{N}^n(M^\bullet)$ factors through a subobject $I^n \subset \mathbf{N}^n(M^\bullet)$ where I^n is an injective object of $\mathcal{A}$.*

Proof. Choose a functorial injective embeddings $i_M : M \to I(M)$, see Theorem 11.7. For every complex $M^\bullet$ denote $J^\bullet(M^\bullet)$ the complex with terms $J^n(M^\bullet) = I(M^n) \oplus I(M^{n+1})$ and differential

$$
d_{J^\bullet(M^\bullet)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
$$

There exists a canonical injective map of complexes $u_{M^\bullet} : M^\bullet \to J^\bullet(M^\bullet)$ by mapping M^n to $I(M^n) \oplus I(M^{n+1})$ via the maps $i_{M^n} : M^n \to I(M^n)$ and $i_{M^{n+1}} \circ d : M^n \to M^{n+1} \to I(M^{n+1})$. Hence a short exact sequence of complexes

$$
0 \to M^\bullet \xrightarrow{u_{M^\bullet}} J^\bullet(M^\bullet) \xrightarrow{v_{M^\bullet}} Q^\bullet(M^\bullet) \to 0
$$

functorial in $M^\bullet$. Set

$$
\mathbf{N}^\bullet(M^\bullet) = C(v_{M^\bullet})^\bullet[-1].
$$

Note that

$$
\mathbf{N}^n(M^\bullet) = Q^{n-1}(M^\bullet) \oplus J^n(M^\bullet)
$$

with differential

$$
\begin{pmatrix} -d_{Q^\bullet(M^\bullet)}^{n-1} & -v_{M^\bullet}^n \\ 0 & d_{J^\bullet(M)}^n \end{pmatrix}
$$

Hence we see that there is a map of complexes $j_{M^\bullet} : M^\bullet \to \mathbf{N}^\bullet(M^\bullet)$ induced by u. It is injective and factors through an injective subobject by construction. The map $j_{M^\bullet}$ is a quasi-isomorphism as one can prove by looking at the long exact sequence of cohomology associated to the short exact sequences of complexes above. $\square$

Theorem 12.6. *Let $\mathcal{A}$ be a Grothendieck abelian category. For every complex $M^\bullet$ there exists a quasi-isomorphism $M^\bullet \to I^\bullet$ such that $M^n \to I^n$ is injective and I^n is an injective object of $\mathcal{A}$ for all n and $I^\bullet$ is a K-injective complex. Moreover, the construction is functorial in $M^\bullet$.*

Proof. Please compare with the proof of Theorem 2.8 and Theorem 11.7. Choose a cardinal κ as in Lemmas 12.2 and 12.3. Choose a set $(K_i^\bullet)_{i \in I}$ of bounded above, acyclic complexes such that every bounded above acyclic complex $K^\bullet$ such that $|K^n| \leq \kappa$ is isomorphic to $K_i^\bullet$ for some $i \in I$. This is possible by Lemma 11.4. Denote $\mathbf{M}^\bullet(-)$ the functor constructed in Lemma 12.4. Denote $\mathbf{N}^\bullet(-)$ the functor constructed in Lemma 12.5. Both of these functors come with injective transformations $\mathrm{id} \to \mathbf{M}$ and $\mathrm{id} \to \mathbf{N}$.

Using transfinite recursion we define a sequence of functors $\mathbf{T}_\alpha(-)$ and corresponding transformations $\mathrm{id} \to \mathbf{T}_\alpha$. Namely we set $\mathbf{T}_0(M^\bullet) = M^\bullet$. If $\mathbf{T}_\alpha$ is given then we set

$$\mathbf{T}_{\alpha+1}(M^\bullet) = \mathbf{N}^\bullet(\mathbf{M}^\bullet(\mathbf{T}_\alpha(M^\bullet)))$$

If β is a limit ordinal we set

$$\mathbf{T}_\beta(M^\bullet) = \mathrm{colim}_{\alpha < \beta} \mathbf{T}_\alpha(M^\bullet)$$

The transition maps of the system are injective quasi-isomorphisms. By AB5 we see that the colimit is still quasi-isomorphic to $M^\bullet$. We claim that $M^\bullet \to \mathbf{T}_\alpha(M^\bullet)$ does the job if the cofinality of α is larger than $\max(\kappa, |U|)$ where U is a generator of $\mathcal{A}$. Namely, it suffices to check conditions (1) and (2) of Lemma 12.3.

For (1) we use the criterion of Lemma 11.6. Suppose that $M \subset U$ and $\varphi : M \to \mathbf{T}_\alpha^n(M^\bullet)$ is a morphism for some $n \in \mathbf{Z}$. By Proposition 11.5 we see that φ factor through $\mathbf{T}_{\alpha'}^n(M^\bullet)$ for some $\alpha' < \alpha$. In particular, by the construction of the functor $\mathbf{N}^\bullet(-)$ we see that φ factors through an injective object of $\mathcal{A}$ which shows that φ lifts to a morphism on U.

For (2) let $w : K^\bullet \to \mathbf{T}_\alpha(M^\bullet)$ be a morphism of complexes where $K^\bullet$ is a bounded above acyclic complex such that $|K^n| \leq \kappa$. Then $K^\bullet \cong K_i^\bullet$ for some $i \in I$. Moreover, by Proposition 11.5 once again we see that w factor through $\mathbf{T}_{\alpha'}^n(M^\bullet)$ for some $\alpha' < \alpha$. In particular, by the construction of the functor $\mathbf{M}^\bullet(-)$ we see that w is homotopic to zero. This finishes the proof. $\square$

13. Additional remarks on Grothendieck abelian categories

In this section we put some results on Grothendieck abelian categories which are folklore.

Lemma 13.1. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $F : \mathcal{A}^{opp} \to Sets$ be a functor. Then F is representable if and only if F commutes with colimits, i.e.,*

$$F(\mathrm{colim}_i N_i) = \lim F(N_i)$$

for any diagram $\mathcal{I} \to \mathcal{A}$, $i \in \mathcal{I}$.

Proof. If F is representable, then it commutes with colimits by definition of colimits.

Assume that F commutes with colimits. Then $F(M \oplus N) = F(M) \times F(N)$ and we can use this to define a group structure on $F(M)$. Hence we get $F : \mathcal{A} \to Ab$ which is additive and right exact, i.e., transforms a short exact sequence $0 \to K \to L \to M \to 0$ into an exact sequence $F(K) \leftarrow F(L) \leftarrow F(M) \leftarrow 0$ (compare with Homology, Section 7).

Let U be a generator for $\mathcal{A}$. Set $A = \bigoplus_{s \in F(U)} U$. Let $s_{univ} = (s)_{s \in F(U)} \in F(A) = \prod_{s \in F(U)} F(U)$. Let $A' \subset A$ be the largest subobject such that s_{univ} restricts to zero on A'. This exists because $\mathcal{A}$ is a Grothendieck category and because F commutes with colimits. Because F commutes with colimits there exists a unique element $\bar{s}_{univ} \in F(A/A')$ which maps to s_{univ} in $F(A)$. We claim that A/A' represents F, in other words, the Yoneda map

$$\bar{s}_{univ} : h_{A/A'} \longrightarrow F$$

is an isomorphism. Let $M \in \mathrm{Ob}(\mathcal{A})$ and $s \in F(M)$. Consider the surjection

$$c_M : A_M = \bigoplus_{\varphi \in \mathrm{Hom}_{\mathcal{A}}(U,M)} U \longrightarrow M.$$

This gives $F(c_M)(s) = (s_\varphi) \in \prod_\varphi F(U)$. Consider the map

$$\psi : A_M = \bigoplus_{\varphi \in \mathrm{Hom}_{\mathcal{A}}(U,M)} U \longrightarrow \bigoplus_{s \in F(U)} U = A$$

which maps the summand corresponding to φ to the summand corresponding to s_φ by the identity map on U. Then s_{univ} maps to $(s_\varphi)_\varphi$ by construction. in other words the right square in the diagram

$$
\begin{array}{ccccc}
A' & \longrightarrow & A & \xdashrightarrow{\ s_{univ}\ } & F \\
\Big\uparrow{\scriptstyle ?} & & \Big\uparrow{\scriptstyle \psi} & & \Big\uparrow{\scriptstyle s} \\
K & \longrightarrow & A_M & \longrightarrow & M
\end{array}
$$

commutes. Let $K = \mathrm{Ker}(A_M \to M)$. Since s restricts to zero on K we see that $\psi(K) \subset A'$ by definition of A'. Hence there is an induced morphism $M \to A/A'$. This construction gives an inverse to the map $h_{A/A'}(M) \to F(M)$ (details omitted). $\square$

Lemma 13.2. *A Grothendieck abelian category has Ab3*.*

Proof. Let M_i, $i \in I$ be a family of objects of $\mathcal{A}$ indexed by a set I. The functor $F = \prod_{i \in I} h_{M_i}$ commutes with colimits. Hence Lemma 13.1 applies. $\square$

Remark 13.3. In the chapter on derived categories we consistently work with "small" abelian categories (as is the convention in the Stacks project). For a "big" abelian category $\mathcal{A}$ it isn't clear that the derived category $D(\mathcal{A})$ exists because it isn't clear that morphisms in the derived category are sets. In general this isn't true, see Examples, Lemma 61.1. However, if $\mathcal{A}$ is a Grothendieck abelian category, and given $K^\bullet, L^\bullet$ in $K(\mathcal{A})$, then by Theorem 12.6 there exists a quasi-isomorphism $L^\bullet \to I^\bullet$ to a K-injective complex $I^\bullet$ and Derived Categories, Lemma 31.2 shows that

$$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$$

which is a set. Some examples of Grothendieck abelian categories are the category of modules over a ring, or more generally the category of sheaves of modules on a ringed site.

Lemma 13.4. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then*

(1) *$D(\mathcal{A})$ has both direct sums and products,*
(2) *direct sums are obtained by taking termwise direct sums of any complexes,*
(3) *products are obtained by taking termwise products of K-injective complexes.*

Proof. Let $K_i^\bullet$, $i \in I$ be a family of objects of $D(\mathcal{A})$ indexed by a set I. We claim that the termwise direct sum $\bigoplus_{i \in I} K_i^\bullet$ is a direct sum in $D(\mathcal{A})$. Namely, let $I^\bullet$ be a K-injective complex. Then we have

$$\mathrm{Hom}_{D(\mathcal{A})}(\bigoplus_{i \in I} K_i^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(\bigoplus_{i \in I} K_i^\bullet, I^\bullet)$$
$$= \prod_{i \in I} \mathrm{Hom}_{K(\mathcal{A})}(K_i^\bullet, I^\bullet)$$
$$= \prod_{i \in I} \mathrm{Hom}_{D(\mathcal{A})}(K_i^\bullet, I^\bullet)$$

as desired. This is sufficient since any complex can be represented by a K-injective complex by Theorem 12.6. To construct the product, choose a K-injective resolution $K_i^\bullet \to I_i^\bullet$ for each i. Then we claim that $\prod_{i \in I} I_i^\bullet$ is a product in $D(\mathcal{A})$. This follows from Derived Categories, Lemma 31.5. $\square$

Remark 13.5. Let R be a ring. Suppose that M_n, $n \in \mathbf{Z}$ are R-modules. Denote $E_n = M_n[-n] \in D(R)$. We claim that $E = \bigoplus M_n[-n]$ is *both* the direct sum and the product of the objects E_n in $D(R)$. To see that it is the direct sum, take a look at the proof of Lemma 13.4. To see that it is the direct product, take injective resolutions $M_n \to I_n^\bullet$. By the proof of Lemma 13.4 we have

$$\prod E_n = \prod I_n^\bullet[-n]$$

in $D(R)$. Since products in Mod_R are exact, we see that $\prod I_n^\bullet[-n]$ is quasi-isomorphic to E. This works more generally in $D(\mathcal{A})$ where $\mathcal{A}$ is a Grothendieck abelian category with Ab4*.

Lemma 13.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Assume*

(1) *$\mathcal{A}$ is a Grothendieck abelian category,*
(2) *$\mathcal{B}$ has exact countable products, and*
(3) *F commutes with countable products.*

Then $RF : D(\mathcal{A}) \to D(\mathcal{B})$ commutes with derived limits.

Proof. Observe that RF exists as $\mathcal{A}$ has enough K-injectives (Theorem 12.6 and Derived Categories, Lemma 31.6). The statement means that if $K = R\lim K_n$, then $RF(K) = R\lim RF(K_n)$. See Derived Categories, Definition 34.1 for notation. Since RF is an exact functor of triangulated categories it suffices to see that RF commutes with countable products of objects of $D(\mathcal{A})$. In the proof of Lemma 13.4 we have seen that products in $D(\mathcal{A})$ are computed by taking products of K-injective complexes and moreover that a product of K-injective complexes is K-injective. Moreover, in Derived Categories, Lemma 34.2 we have seen that products in $D(\mathcal{B})$ are computed by taking termwise products. Since RF is computed by applying F to a K-injective representative and since we've assumed F commutes with countable products, the lemma follows. $\square$
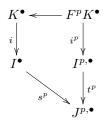
The following lemma is some kind of generalization of the existence of Cartan-Eilenberg resolutions (Derived Categories, Section 21).

Lemma 13.7. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $K^\bullet$ be a filtered complex of $\mathcal{A}$, see Homology, Definition 24.1. Then there exists a morphism $j : K^\bullet \to J^\bullet$ of filtered complexes of $\mathcal{A}$ such that*

  (1) *J^n, $F^p J^n$, $J^n/F^p J^n$ and $F^p J^n / F^{p'} J^n$ are injective objects of $\mathcal{A}$,*
  (2) *$J^\bullet$, $F^p J^\bullet$, $J^\bullet/F^p J^\bullet$, and $F^p J^\bullet / F^{p'} J^\bullet$ are K-injective complexes,*
  (3) *j induces quasi-isomorphisms $K^\bullet \to J^\bullet$, $F^p K^\bullet \to F^p J^\bullet$, $K^\bullet/F^p K^\bullet \to J^\bullet/F^p J^\bullet$, and $F^p K^\bullet/F^{p'} K^\bullet \to F^p J^\bullet/F^{p'} J^\bullet$.*

Proof. By Theorem 12.6 we obtain quasi-isomorphisms $i : K^\bullet \to I^\bullet$ and $i^p : F^p K^\bullet \to I^{p,\bullet}$ as well as commutative diagrams

$$
\begin{array}{ccc}
K^\bullet & \longleftarrow & F^p K^\bullet \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i^p} \\
I^\bullet & \xleftarrow{\ \alpha^p\ } & I^{p,\bullet}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
F^{p'} K^\bullet & \longleftarrow & F^p K^\bullet \\
{\scriptstyle i^{p'}}\downarrow & & \downarrow{\scriptstyle i^p} \\
I^{p',\bullet} & \xleftarrow{\ \alpha^{pp'}\ } & I^{p,\bullet}
\end{array}
\qquad \text{for } p' \le p
$$

such that $\alpha^p \circ \alpha^{p'p} = \alpha^{p'}$ and $\alpha^{p'p''} \circ \alpha^{pp'} = \alpha^{pp''}$. The problem is that the maps $\alpha^p : I^{p,\bullet} \to I^\bullet$ need not be injective. For each p we choose an injection $t^p : I^{p,\bullet} \to J^{p,\bullet}$ into an acyclic K-injective complex $J^{p,\bullet}$ whose terms are injective objects of $\mathcal{A}$ (first map to the cone on the identity and then use the theorem). Choose a map of complexes $s^p : I^\bullet \to J^{p,\bullet}$ such that the following diagram commutes

$$
\begin{array}{ccc}
K^\bullet & \longleftarrow & F^p K^\bullet \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i^p} \\
I^\bullet & & I^{p,\bullet} \\
& {\scriptstyle s^p}\searrow & \downarrow{\scriptstyle t^p} \\
& & J^{p,\bullet}
\end{array}
$$

This is possible: the composition $F^p K^\bullet \to J^{p,\bullet}$ is homotopic to zero because $J^{p,\bullet}$ is acyclic and K-injective (Derived Categories, Lemma 31.2). Since the objects $J^{p,n-1}$ are injective and since $F^p K^n \to K^n \to I^n$ are injective morphisms, we can lift the maps $F^p K^n \to J^{p,n-1}$ giving the homotopy to a map $h^n : I^n \to J^{p,n-1}$. Then we set s^p equal to $h \circ \mathrm{d} + \mathrm{d} \circ h$. (Warning: It will not be the case that $t^p = s^p \circ \alpha^p$, so we have to be careful not to use this below.)

Consider

$$
J^\bullet = I^\bullet \times \prod_p J^{p,\bullet}
$$

Because products in $D(\mathcal{A})$ are given by taking products of K-injective complexes (Lemma 13.4) and since $J^{p,\bullet}$ is isomorphic to 0 in $D(\mathcal{A})$ we see that $J^\bullet \to I^\bullet$ is an isomorphism in $D(\mathcal{A})$. Consider the map

$$
j = i \times (s^p \circ i)_{p \in \mathbf{Z}} : K^\bullet \longrightarrow I^\bullet \times \prod_p J^{p,\bullet} = J^\bullet
$$

By our remarks above this is a quasi-isomorphism. It is also injective. For $p \in \mathbf{Z}$ we let $F^p J^\bullet \subset J^\bullet$ be

$$
\mathrm{Im}\left(\alpha^p \times (t^{p'} \circ \alpha^{pp'})_{p' \le p} : I^{p,\bullet} \to I^\bullet \times \prod_{p' \le p} J^{p',\bullet} \right) \times \prod_{p' > p} J^{p',\bullet}
$$

This complex is isomorphic to the complex $I^{p,\bullet} \times \prod_{p'>p} J^{p,\bullet}$ as $\alpha^{pp} = \mathrm{id}$ and t^p is injective. Hence $F^p J^\bullet$ is quasi-isomorphic to $I^{p,\bullet}$ (argue as above). We have $j(F^p K^\bullet) \subset F^p J^\bullet$ because of the commutativity of the diagram above. The corresponding map of complexes $F^p K^\bullet \to F^p J^\bullet$ is a quasi-isomorphism by what we just said. Finally, to see that $F^{p+1} J^\bullet \subset F^p J^\bullet$ use that $\alpha^{p+1 p} \circ \alpha^{pp'} = \alpha^{p+1 p'}$ and the commutativity of the first displayed diagram in the first paragraph of the proof.

We claim that $j : K^\bullet \to J^\bullet$ is a solution to the problem posed by the lemma. Namely, $F^p J^n$ is an injective object of $\mathcal{A}$ because it is isomorphic to $I^{p,n} \times \prod_{p'>p} J^{p',n}$ and products of injectives are injective. Then the injective map $F^p J^n \to J^n$ splits and hence the quotient $J^n / F^p J^n$ is injective as well as a direct summand of the injective object J^n. Similarly for $F^p J^n / F^{p'} J^n$. This in particular means that $0 \to F^p J^\bullet \to J^\bullet \to J^\bullet / F^p J^\bullet \to 0$ is a termwise split short exact sequence of complexes, hence defines a distinguished triangle in $K(\mathcal{A})$ by fiat. Since $J^\bullet$ and $F^p J^\bullet$ are K-injective complexes we see that the same is true for $J^\bullet / F^p J^\bullet$ by Derived Categories, Lemma 31.3. A similar argument shows that $F^p J^\bullet / F^{p'} J^\bullet$ is K-injective. By construction $j : K^\bullet \to J^\bullet$ and the induced maps $F^p K^\bullet \to F^p J^\bullet$ are quasi-isomorphisms. Using the long exact cohomology sequences of the complexes in play we find that the same holds for $K^\bullet / F^p K^\bullet \to J^\bullet / F^p J^\bullet$ and $F^p K^\bullet / F^{p'} K^\bullet \to F^p J^\bullet / F^{p'} J^\bullet$. $\qquad\square$

Remark 13.8. Let $\mathcal{A}$ be a Grothendieck abelian category. Let $K^\bullet$ be a filtered complex of $\mathcal{A}$, see Homology, Definition 24.1. For ease of notation denote K, $F^p K$, $\mathrm{gr}^p K$ the object of $D(\mathcal{A})$ represented by $K^\bullet$, $F^p K^\bullet$, $\mathrm{gr}^p K^\bullet$. Let $M \in D(\mathcal{A})$. Using Lemma 13.7 we can construct a spectral sequence $(E_r, d_r)_{r \geq 1}$ of bigraded objects of $\mathcal{A}$ with d_r of bidgree $(r, -r+1)$ and with

$$E_1^{p,q} = \mathrm{Ext}^{p+q}(M, \mathrm{gr}^p K)$$

If for every n we have

$$\mathrm{Ext}^n(M, F^p K) = 0 \text{ for } p \gg 0 \quad \text{and} \quad \mathrm{Ext}^n(M, F^p K) = \mathrm{Ext}^n(M, K) \text{ for } p \ll 0$$

then the spectral sequence is bounded and converges to $\mathrm{Ext}^{p+q}(M, K)$. Namely, choose any complex $M^\bullet$ representing M, choose $j : K^\bullet \to J^\bullet$ as in the lemma, and consider the complex

$$\mathrm{Hom}^\bullet(M^\bullet, I^\bullet)$$

defined exactly as in More on Algebra, Section 71. Setting $F^p \mathrm{Hom}^\bullet(M^\bullet, I^\bullet) = \mathrm{Hom}^\bullet(M^\bullet, F^p I^\bullet)$ we obtain a filtered complex. The spectral sequence of Homology, Section 24 has differentials and terms as described above; details omitted. The boundedness and convergence follows from Homology, Lemma 24.13.

Remark 13.9. Let $\mathcal{A}$ be a Grothendieck abelian category. Let M, K be objects of $D(\mathcal{A})$. For any choice of complex $K^\bullet$ representing K we can use the filtration $F^p K^\bullet = \tau_{\leq -p} K^\bullet$ and the discussion in Remark 13.8 to get a spectral sequence with

$$E_1^{p,q} = \mathrm{Ext}^{2p+q}(M, H^{-p}(K))$$

This spectral sequence is independent of the choice of complex $K^\bullet$ representing K. After renumbering $p = -j$ and $q = i + 2j$ we find a spectral sequence $(E_r', d_r')_{r \geq 2}$ with d_r' of bidegree $(r, -r+1)$, with

$$(E_2')^{i,j} = \mathrm{Ext}^i(M, H^j(K))$$

If $M \in D^-(\mathcal{A})$ and $K \in D^+(\mathcal{A})$ then both E_r and E'_r are bounded and converge to $\mathrm{Ext}^{p+q}(M, K)$. If we use the filtration $F^p K^\bullet = \sigma_{\geq p} K^\bullet$ then we get

$$E_1^{p,q} = \mathrm{Ext}^q(M, K^p)$$

If $M \in D^-(\mathcal{A})$ and $K^\bullet$ is bounded below, then this spectral sequence is bounded and converges to $\mathrm{Ext}^{p+q}(M, K)$.

Remark 13.10. Let $\mathcal{A}$ be a Grothendieck abelian category. Let $K \in D(\mathcal{A})$. Let $M^\bullet$ be a filtered complex of $\mathcal{A}$, see Homology, Definition 24.1. For ease of notation denote M, $M/F^p M$, $\mathrm{gr}^p M$ the object of $D(\mathcal{A})$ represented by $M^\bullet$, $M^\bullet/F^p M^\bullet$, $\mathrm{gr}^p M^\bullet$. Dually to Remark 13.8 we can construct a spectral sequence $(E_r, d_r)_{r \geq 1}$ of bigraded objects of $\mathcal{A}$ with d_r of bidegree $(r, -r + 1)$ and with

$$E_1^{p,q} = \mathrm{Ext}^{p+q}(\mathrm{gr}^{-p} M, K)$$

If for every n we have

$$\mathrm{Ext}^n(M/F^p M, K) = 0 \text{ for } p \ll 0 \quad \text{and} \quad \mathrm{Ext}^n(M/F^p M, K) = \mathrm{Ext}^n(M, K) \text{ for } p \gg 0$$

then the spectral sequence is bounded and converges to $\mathrm{Ext}^{p+q}(M, K)$. Namely, choose a K-injective complex $I^\bullet$ with injective terms representing K, see Theorem 12.6. Consider the complex

$$\mathrm{Hom}^\bullet(M^\bullet, I^\bullet)$$

defined exactly as in More on Algebra, Section 71. Setting

$$F^p \mathrm{Hom}^\bullet(M^\bullet, I^\bullet) = \mathrm{Hom}^\bullet(M^\bullet/F^{-p+1} M^\bullet, I^\bullet)$$

we obtain a filtered complex (note sign and shift in filtration). The spectral sequence of Homology, Section 24 has differentials and terms as described above; details omitted. The boundedness and convergence follows from Homology, Lemma 24.13.

Remark 13.11. Let $\mathcal{A}$ be a Grothendieck abelian category. Let M, K be objects of $D(\mathcal{A})$. For any choice of complex $M^\bullet$ representing M we can use the filtration $F^p M^\bullet = \tau_{\leq -p} M^\bullet$ and the discussion in Remark 13.8 to get a spectral sequence with

$$E_1^{p,q} = \mathrm{Ext}^{2p+q}(H^p(M), K)$$

This spectral sequence is independent of the choice of complex $M^\bullet$ representing M. After renumbering $p = -j$ and $q = i + 2j$ we find a spectral sequence $(E'_r, d'_r)_{r \geq 2}$ with d'_r of bidegree $(r, -r + 1)$, with

$$(E'_2)^{i,j} = \mathrm{Ext}^i(H^{-j}(M), K)$$

If $M \in D^-(\mathcal{A})$ and $K \in D^+(\mathcal{A})$ then E_r and E'_r are bounded and converge to $\mathrm{Ext}^{p+q}(M, K)$. If we use the filtration $F^p M^\bullet = \sigma_{\geq p} M^\bullet$ then we get
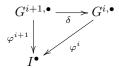
$$E_1^{p,q} = \mathrm{Ext}^q(M^{-p}, K)$$

If $K \in D^+(\mathcal{A})$ and $M^\bullet$ is bounded above, then this spectral sequence is bounded and converges to $\mathrm{Ext}^{p+q}(M, K)$.

Lemma 13.12. *Let $\mathcal{A}$ be a Grothendieck abelian category. Suppose given an object $E \in D(\mathcal{A})$ and an inverse system $\{E^i\}_{i \in \mathbf{Z}}$ of objects of $D(\mathcal{A})$ over $\mathbf{Z}$ together with a compatible system of maps $E^i \to E$. Picture:*

$$\ldots \to E^{i+1} \to E^i \to E^{i-1} \to \ldots \to E$$

Then there exists a filtered complex $K^\bullet$ of $\mathcal{A}$ (Homology, Definition 24.1) such that $K^\bullet$ represents E and $F^i K^\bullet$ represents E^i compatibly with the given maps.

Proof. By Theorem 12.6 we can choose a K-injective complex $I^\bullet$ representing E all of whose terms I^n are injective objects of $\mathcal{A}$. Choose a complex $G^{0,\bullet}$ representing E^0. Choose a map of complexes $\varphi^0 : G^{0,\bullet} \to I^\bullet$ representing $E^0 \to E$. For $i > 0$ we inductively represent $E^i \to E^{i-1}$ by a map of complexes $\delta : G^{i,\bullet} \to G^{i-1,\bullet}$ and we set $\varphi^i = \delta \circ \varphi^{i-1}$. For $i < 0$ we inductively represent $E^{i+1} \to E^i$ by a termwise injective map of complexes $\delta : G^{i+1,\bullet} \to G^{i,\bullet}$ (for example you can use Derived Categories, Lemma 9.6). Claim: we can find a map of complexes $\varphi^i : G^{i,\bullet} \to I^\bullet$ representing the map $E^i \to E$ and fitting into the commutative diagram

$$
\begin{array}{ccc}
G^{i+1,\bullet} & \xrightarrow{\ \delta\ } & G^{i,\bullet} \\
{\scriptstyle \varphi^{i+1}}\big\downarrow & & \swarrow{\scriptstyle \varphi^i} \\
I^\bullet & &
\end{array}
$$

Namely, we first choose any map of complexes $\varphi : G^{i,\bullet} \to I^\bullet$ representing the map $E^i \to E$. Then we see that $\varphi \circ \delta$ and φ^{i+1} are homotopic by some homotopy $h^p : G^{i+1,p} \to I^{p-1}$. Since the terms of $I^\bullet$ are injective and since δ is termwise injective, we can lift h^p to $(h')^p : G^{i,p} \to I^{p-1}$. Then we set $\varphi^i = \varphi + h' \circ d + d \circ h'$ and we get what we claimed.

Next, we choose for every i a termwise injective map of complexes $a^i : G^{i,\bullet} \to J^{i,\bullet}$ with $J^{i,\bullet}$ acyclic, K-injective, with $J^{i,p}$ injective objects of $\mathcal{A}$. To do this first map $G^{i,\bullet}$ to the cone on the identity and then apply the theorem cited above. Arguing as above we can find maps of complexes $\delta' : J^{i,\bullet} \to J^{i-1,\bullet}$ such that the diagrams

$$
\begin{array}{ccc}
G^{i,\bullet} & \xrightarrow{\ \delta\ } & G^{i-1,\bullet} \\
{\scriptstyle a^i}\big\downarrow & & \big\downarrow{\scriptstyle a^{i-1}} \\
J^{i,\bullet} & \xrightarrow{\ \delta'\ } & J^{i-1,\bullet}
\end{array}
$$

commute. (You could also use the functoriality of cones plus the functoriality in the theorem to get this.) Then we consider the maps

$$
G^{i+1,\bullet} \times \prod_{p>i+1} J^{p,\bullet} \longrightarrow G^{i,\bullet} \times \prod_{p>i} J^{p,\bullet} \longrightarrow G^{i-1,\bullet} \times \prod_{p>i-1} J^{p,\bullet}
$$

$$
I^\bullet \times \prod_p J^{p,\bullet}
$$

Here the arrows on $J^{p,\bullet}$ are the obvious ones (identity or zero). On the factor $G^{i,\bullet}$ we use $\delta : G^{i,\bullet} \to G^{i-1,\bullet}$, the map $\varphi^i : G^{i,\bullet} \to I^\bullet$, the zero map $0 : G^{i,\bullet} \to J^{p,\bullet}$ for $p > i$, the map $a^i : G^{i,\bullet} \to J^{p,\bullet}$ for $p = i$, and $(\delta')^{i-p} \circ a^i = a^p \circ \delta^{i-p} : G^{i,\bullet} \to J^{p,\bullet}$ for $p < i$. We omit the verification that all the arrows in the diagram are termwise injective. Thus we obtain a filtered complex. Because products in $D(\mathcal{A})$ are given by taking products of K-injective complexes (Lemma 13.4) and because $J^{p,\bullet}$ is zero in $D(\mathcal{A})$ we conclude this diagram represents the given diagram in the derived category. This finishes the proof. $\qquad\square$

Lemma 13.13. *In the situation of Lemma 13.12 assume we have a second inverse system $\{(E')^i\}_{i\in\mathbf{Z}}$ and a compatible system of maps $(E')^i \to E$. Then there exists a bi-filtered complex $K^\bullet$ of $\mathcal{A}$ such that $K^\bullet$ represents E, $F^i K^\bullet$ represents E^i, and $(F')^i K^\bullet$ represents $(E')^i$ compatibly with the given maps.*

Proof. Using the lemma we can first choose $K^\bullet$ and F. Then we can choose $(K')^\bullet$ and F' which work for $\{(E')^i\}_{i \in \mathbf{Z}}$ and the maps $(E')^i \to E$. Using Lemma 13.7 we can assume $K^\bullet$ is a K-injective complex. Then we can choose a map of complexes $(K')^\bullet \to K^\bullet$ corresponding to the given identifications $(K')^\bullet \cong E \cong K^\bullet$. We can additionally choose a termwise injective map $(K')^\bullet \to J^\bullet$ with $J^\bullet$ acyclic and K-injective. (To do this first map $(K')^\bullet$ to the cone on the identity and then apply Theorem 12.6.) Then $(K')^\bullet \to K^\bullet \times J^\bullet$ and $K^\bullet \to K^\bullet \times J^\bullet$ are both termwise injective and quasi-isomorphisms (as the product represents E by Lemma 13.4). Then we can simply take the images of the filtrations on $K^\bullet$ and $(K')^\bullet$ under these maps to conclude. $\square$

14. The Gabriel-Popescu theorem

In this section we discuss the main theorem of [PG64]. The method of proof follows a write-up by Jacob Lurie and another by Akhil Mathew who in turn follow the presentation by Kuhn in [Kuh94]. See also [Tak71].

Let $\mathcal{A}$ be a Grothendieck abelian category and let U be a generator for $\mathcal{A}$, see Definition 10.1. Let $R = \mathrm{Hom}_{\mathcal{A}}(U, U)$. Consider the functor $G : \mathcal{A} \to \mathrm{Mod}_R$ given by

$$G(A) = \mathrm{Hom}_{\mathcal{A}}(U, A)$$

endowed with its canonical right R-module structure.

Lemma 14.1. *The functor G above has a left adjoint $F : Mod_R \to \mathcal{A}$.*

Proof. We will give two proofs of this lemma.

The first proof will use the adjoint functor theorem, see Categories, Theorem 25.3. Observe that that $G : \mathcal{A} \to \mathrm{Mod}_R$ is left exact and sends products to products. Hence G commutes with limits. To check the set theoretical condition in the theorem, suppose that M is an object of Mod_R. Choose a suitably large cardinal κ and denote E a set of objects of $\mathcal{A}$ such that every object A with $|A| \leq \kappa$ is isomorphic to an element of E. This is possible by Lemma 11.4. Set $I = \coprod_{A \in E} \mathrm{Hom}_R(M, G(A))$. We think of an element $i \in I$ as a pair (A_i, f_i). Finally, let A be an arbitrary object of $\mathcal{A}$ and $f : M \to G(A)$ arbitrary. We are going to think of elements of $\mathrm{Im}(f) \subset G(A) = \mathrm{Hom}_{\mathcal{A}}(U, A)$ as maps $u : U \to A$. Set

$$A' = \mathrm{Im}(\bigoplus\nolimits_{u \in \mathrm{Im}(f)} U \xrightarrow{u} A)$$

Since G is left exact, we see that $G(A') \subset G(A)$ contains $\mathrm{Im}(f)$ and we get $f' : M \to G(A')$ factoring f. On the other hand, the object A' is the quotient of a direct sum of at most $|M|$ copies of U. Hence if $\kappa = |\bigoplus_{|M|} U|$, then we see that (A', f') is isomorphic to an element (A_i, f_i) of E and we conclude that f factors as $M \xrightarrow{f_i} G(A_i) \to G(A)$ as desired.

The second proof will give a construction of F which will show that "$F(M) = M \otimes_R U$" in some sense. Namely, for any R-module M we can choose a resolution

$$\bigoplus\nolimits_{j \in J} R \to \bigoplus\nolimits_{i \in I} R \to M \to 0$$

Then we define $F(M)$ by the corresponding exact sequence

$$\bigoplus\nolimits_{j \in J} U \to \bigoplus\nolimits_{i \in I} U \to F(M) \to 0$$

This construction is independent of the choice of the resolution and is functorial; we omit the details. For any A in $\mathcal{A}$ we obtain an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(F(M), A) \to \prod_{i \in I} G(A) \to \prod_{j \in J} G(A)$$

which is isomorphic to the sequence

$$0 \to \operatorname{Hom}_R(M, G(A)) \to \operatorname{Hom}_R(\bigoplus_{i \in I} R, G(A)) \to \operatorname{Hom}_R(\bigoplus_{j \in J} R, G(A))$$

which shows that F is the left adjoint to G. $\qquad\square$

Lemma 14.2. *Let $f : M \to G(A)$ be an injective map in Mod_R. Then the adjoint map $f' : F(M) \to A$ is injective too.*

Proof. Choose a map $R^{\oplus n} \to M$ and consider the corresponding map $U^{\oplus n} \to F(M)$. Consider a map $v : U \to U^{\oplus n}$ such that the composition $U \to U^{\oplus n} \to F(M) \to A$ is 0. Then this arrow $v : U \to U^{\oplus n}$ is an element v of $R^{\oplus n}$ mapping to zero in $G(A)$. Since f is injective, we conclude that v maps to zero in M which means that $U \to U^{\oplus n} \to F(M)$ is zero by construction of $F(M)$ in the proof of Lemma 14.1. Since U is a generator we conclude that

$$\operatorname{Ker}(U^{\oplus n} \to F(M) \to A) = \operatorname{Ker}(U^{\oplus n} \to F(M))$$

To finish the proof we choose a surjection $\bigoplus_{i \in I} R \to M$ and we consider the corresponding surjection

$$\pi : \bigoplus_{i \in I} U \longrightarrow F(M)$$

To prove f' is injective it suffices to show that $\operatorname{Ker}(\pi) = \operatorname{Ker}(f' \circ \pi)$ as subobjects of $\bigoplus_{i \in I} U$. However, now we can write $\bigoplus_{i \in I} U$ as the filtered colimit of its subobjects $\bigoplus_{i \in I'} U$ where $I' \subset I$ ranges over the finite subsets. Since filtered colimits are exact by AB5 for $\mathcal{A}$, we see that

$$\operatorname{Ker}(\pi) = \operatorname{colim}_{I' \subset I \text{ finite}} \left(\bigoplus_{i \in I'} U\right) \bigcap \operatorname{Ker}(\pi)$$

and

$$\operatorname{Ker}(f' \circ \pi) = \operatorname{colim}_{I' \subset I \text{ finite}} \left(\bigoplus_{i \in I'} U\right) \bigcap \operatorname{Ker}(f' \circ \pi)$$

and we get equality because the same is true for each I' by the first displayed equality above. $\qquad\square$

Theorem 14.3. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then there exists a (noncommutative) ring R and functors $G : \mathcal{A} \to \mathrm{Mod}_R$ and $F : \mathrm{Mod}_R \to \mathcal{A}$ such that*

(1) *F is the left adjoint to G,*
(2) *G is fully faithful, and*
(3) *F is exact.*

Moreover, the functors are the ones constructed above.

Proof. We first prove G is fully faithful, or equivalently that $F \circ G \to \mathrm{id}$ is an isomorphism, see Categories, Lemma 24.4. First, given an object A the map $F(G(A)) \to A$ is surjective, because every map of $U \to A$ factors through $F(G(A))$ by construction. On the other hand, the map $F(G(A)) \to A$ is the adjoint of the map $\mathrm{id} : G(A) \to G(A)$ and hence injective by Lemma 14.2.

The functor F is right exact as it is a left adjoint. Since Mod_R has enough projectives, to show that F is exact, it is enough to show that the first left derived

functor L_1F is zero. To prove $L_1F(M) = 0$ for some R-module M choose an exact sequence $0 \to K \to P \to M \to 0$ of R-modules with P free. It suffices to show $F(K) \to F(P)$ is injective. Now we can write this sequence as a filtered colimit of sequences $0 \to K_i \to P_i \to M_i \to 0$ with P_i a finite free R-module: just write P in this manner and set $K_i = K \cap P_i$ and $M_i = \text{Im}(P_i \to M)$. Because F is a left adjoint it commutes with colimits and because $\mathcal{A}$ is a Grothendieck abelian category, we find that $F(K) \to F(P)$ is injective if each $F(K_i) \to F(P_i)$ is injective. Thus it suffices to check $F(K) \to F(P)$ is injective when $K \subset P = R^{\oplus n}$. Thus $F(K) \to U^{\oplus n}$ is injective by an application of Lemma 14.2. $\qquad\square$

Lemma 14.4. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let R, F, G be as in the Gabriel-Popescu theorem (Theorem 14.3). Then we obtain derived functors*

$$RG : D(\mathcal{A}) \to D(Mod_R) \quad and \quad F : D(Mod_R) \to D(\mathcal{A})$$

such that F is left adjoint to RG, RG is fully faithful, and $F \circ RG = \text{id}$.

Proof. The existence and adjointness of the functors follows from Theorems 14.3 and 12.6 and Derived Categories, Lemmas 31.6, 16.9, and 30.3. The statement $F \circ RG = \text{id}$ follows because we can compute RG on an object of $D(\mathcal{A})$ by applying G to a suitable representative complex $I^\bullet$ (for example a K-injective one) and then $F(G(I^\bullet)) = I^\bullet$ because $F \circ G = \text{id}$. Fully faithfulness of RG follows from this by Categories, Lemma 24.4. $\qquad\square$

15. Brown representability and Grothendieck abelian categories

In this section we quickly prove a representability theorem for derived categories of Grothendieck abelian categories. The reader should first read the case of compactly generated triangulated categories in Derived Categories, Section 38. After that, instead of reading this section, it makes sense to consult the literature for more general results of this nature, for example see [Fra01], [Nee01], [Kra02], or take a look at Derived Categories, Section 39.

Lemma 15.1. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $H : D(\mathcal{A}) \to Ab$ be a contravariant cohomological functor which transforms direct sums into products. Then H is representable.*

Proof. Let R, F, G, RG be as in Lemma 14.4 and consider the functor $H \circ F : D(\text{Mod}_R) \to Ab$. Observe that since F is a left adjoint it sends direct sums to direct sums and hence $H \circ F$ transforms direct sums into products. On the other hand, the derived category $D(\text{Mod}_R)$ is generated by a single compact object, namely R. By Derived Categories, Lemma 38.1 we see that $H \circ F$ is representable, say by $L \in D(\text{Mod}_R)$. Choose a distinguished triangle

$$M \to L \to RG(F(L)) \to M[1]$$

in $D(\text{Mod}_R)$. Then $F(M) = 0$ because $F \circ RG = \text{id}$. Hence $H(F(M)) = 0$ hence $\text{Hom}(M, L) = 0$. It follows that $L \to RG(F(L))$ is the inclusion of a direct

summand, see Derived Categories, Lemma 4.11. For A in $D(\mathcal{A})$ we obtain

$$\begin{aligned}
H(A) &= H(F(RG(A)) \\
&= \mathrm{Hom}(RG(A), L) \\
&\to \mathrm{Hom}(RG(A), RG(F(L))) \\
&= \mathrm{Hom}(F(RG(A)), F(L)) \\
&= \mathrm{Hom}(A, F(L))
\end{aligned}$$

where the arrow has a left inverse functorial in A. In other words, we find that H is the direct summand of a representable functor. Since $D(\mathcal{A})$ is Karoubian (Derived Categories, Lemma 4.14) we conclude. $\square$

Proposition 15.2. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $\mathcal{D}$ be a triangulated category. Let $F : D(\mathcal{A}) \to \mathcal{D}$ be an exact functor of triangulated categories which transforms direct sums into direct sums. Then F has an exact right adjoint.*

Proof. For an object Y of $\mathcal{D}$ consider the contravariant functor

$$D(\mathcal{A}) \to Ab, \quad W \mapsto \mathrm{Hom}_{\mathcal{D}}(F(W), Y)$$

This is a cohomological functor as F is exact and transforms direct sums into products as F transforms direct sums into direct sums. Thus by Lemma 15.1 we find an object X of $D(\mathcal{A})$ such that $\mathrm{Hom}_{D(\mathcal{A})}(W, X) = \mathrm{Hom}_{\mathcal{D}}(F(W), Y)$. The existence of the adjoint follows from Categories, Lemma 24.2. Exactness follows from Derived Categories, Lemma 7.1. $\square$

16. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
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(23) Divided Power Algebra
(24) Differential Graded Sheaves
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Schemes

(26) Schemes
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(40) More on Groupoid Schemes
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Topics in Scheme Theory

(42) Chow Homology

Algebraic Spaces

Topics in Geometry

Deformation Theory

Algebraic Stacks

Topics in Moduli Theory

Miscellany

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