DERIVED CATEGORIES

Contents

1. Introduction

We first discuss triangulated categories and localization in triangulated categories. Next, we prove that the homotopy category of complexes in an additive category is a triangulated category. Once this is done we define the derived category of an abelian category as the localization of the homotopy category with respect to quasi-isomorphisms. A good reference is Verdier's thesis [Ver96].

2. Triangulated categories

Triangulated categories are a convenient tool to describe the type of structure inherent in the derived category of an abelian category. Some references are [Ver96], [KS06], and [Nee01].

3. The definition of a triangulated category

In this section we collect most of the definitions concerning triangulated and pre-triangulated categories.

Definition 3.1. Let $\mathcal{D}$ be an additive category. Let $[1] : \mathcal{D} \to \mathcal{D}$, $E \mapsto E[1]$ be an additive functor which is an auto-equivalence of $\mathcal{D}$.

(1) A *triangle* is a sextuple (X, Y, Z, f, g, h) where $X, Y, Z \in \mathrm{Ob}(\mathcal{D})$ and $f : X \to Y$, $g : Y \to Z$ and $h : Z \to X[1]$ are morphisms of $\mathcal{D}$.
(2) A *morphism of triangles* $(X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$ is given by morphisms $a : X \to X'$, $b : Y \to Y'$ and $c : Z \to Z'$ of $\mathcal{D}$ such that $b \circ f = f' \circ a$, $c \circ g = g' \circ b$ and $a[1] \circ h = h' \circ c$.

A morphism of triangles is visualized by the following commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "a"] & Y \arrow[r] \arrow[d, "b"] & Z \arrow[r] \arrow[d, "c"] & X[1] \arrow[d, "a[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

In the setting of Definition 3.1, we write $[0] = \mathrm{id}$, for $n > 0$ we denote $[n]$ the n-fold composition of $[1]$, we choose a quasi-inverse $[-1]$ of $[1]$, and we set $[-n]$ equal to the n-fold composition of $[-1]$. Then $\{[n]\}_{n \in \mathbf{Z}}$ is a collection of additive auto-equivalences of $\mathcal{D}$ indexed by $n \in \mathbf{Z}$ such that we are given isomorphisms of functors $[n] \circ [m] \cong [n + m]$.

Here is the definition of a triangulated category as given in Verdier's thesis.

Definition 3.2. A *triangulated category* consists of a triple $(\mathcal{D}, \{[n]\}_{n \in \mathbf{Z}}, \mathcal{T})$ where

(1) $\mathcal{D}$ is an additive category,
(2) $[1] : \mathcal{D} \to \mathcal{D}$, $E \mapsto E[1]$ is an additive auto-equivalence and $[n]$ for $n \in \mathbf{Z}$ is as discussed above, and

(3) $\mathcal{T}$ is a set of triangles (Definition 3.1) called the *distinguished triangles*

subject to the following conditions

TR1 Any triangle isomorphic to a distinguished triangle is a distinguished triangle. Any triangle of the form $(X, X, 0, \mathrm{id}, 0, 0)$ is distinguished. For any morphism $f : X \to Y$ of $\mathcal{D}$ there exists a distinguished triangle of the form (X, Y, Z, f, g, h).

TR2 The triangle (X, Y, Z, f, g, h) is distinguished if and only if the triangle $(Y, Z, X[1], g, h, -f[1])$ is.

TR3 Given a solid diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d, dashed] & X[1] \arrow[d, "a[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

whose rows are distinguished triangles and which satisfies $b \circ f = f' \circ a$, there exists a morphism $c : Z \to Z'$ such that (a, b, c) is a morphism of triangles.

TR4 Given objects X, Y, Z of $\mathcal{D}$, and morphisms $f : X \to Y$, $g : Y \to Z$, and distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3), there exist morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$ such that
   (a) $(Q_1, Q_2, Q_3, a, b, p_1[1] \circ d_3)$ is a distinguished triangle,
   (b) the triple (id_X, g, a) is a morphism of triangles $(X, Y, Q_1, f, p_1, d_1) \to (X, Z, Q_2, g \circ f, p_2, d_2)$, and
   (c) the triple (f, id_Z, b) is a morphism of triangles $(X, Z, Q_2, g \circ f, p_2, d_2) \to (Y, Z, Q_3, g, p_3, d_3)$.

We will call $(\mathcal{D}, [\ ], \mathcal{T})$ a *pre-triangulated category* if TR1, TR2 and TR3 hold.[1]

The explanation of TR4 is that if you think of Q_1 as Y/X, Q_2 as Z/X and Q_3 as Z/Y, then TR4(a) expresses the isomorphism $(Z/X)/(Y/X) \cong Z/Y$ and TR4(b) and TR4(c) express that we can compare the triangles $X \to Y \to Q_1 \to X[1]$ etc with morphisms of triangles. For a more precise reformulation of this idea see the proof of Lemma 10.2.

The sign in TR2 means that if (X, Y, Z, f, g, h) is a distinguished triangle then in the long sequence

$$(3.2.1) \quad \ldots \to Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \to \ldots$$

each four term sequence gives a distinguished triangle.

As usual we abuse notation and we simply speak of a (pre-)triangulated category $\mathcal{D}$ without explicitly introducing notation for the additional data. The notion of a pre-triangulated category is useful in finding statements equivalent to TR4.

We have the following definition of a triangulated functor.

[1] We use $[\ ]$ as an abbreviation for the family $\{[n]\}_{n \in \mathbf{Z}}$.

Definition 3.3. Let $\mathcal{D}$, $\mathcal{D}'$ be pre-triangulated categories. An *exact functor*, or a *triangulated functor* from $\mathcal{D}$ to $\mathcal{D}'$ is a functor $F : \mathcal{D} \to \mathcal{D}'$ together with given functorial isomorphisms $\xi_X : F(X[1]) \to F(X)[1]$ such that for every distinguished triangle (X, Y, Z, f, g, h) of $\mathcal{D}$ the triangle $(F(X), F(Y), F(Z), F(f), F(g), \xi_X \circ F(h))$ is a distinguished triangle of $\mathcal{D}'$.

An exact functor is additive, see Lemma 4.17. When we say two triangulated categories are equivalent we mean that they are equivalent in the 2-category of triangulated categories. A 2-morphism $a : (F, \xi) \to (F', \xi')$ in this 2-category is simply a transformation of functors $a : F \to F'$ which is compatible with ξ and ξ', i.e.,

$$\begin{tikzcd} F \circ [1] \arrow[r, "\xi"'] \arrow[d, "a \star 1"'] & {[1] \circ F} \arrow[d, "1 \star a"] \\ F' \circ [1] \arrow[r, "\xi'"] & {[1] \circ F'} \end{tikzcd}$$

commutes.

Definition 3.4. Let $(\mathcal{D}, [\ ], \mathcal{T})$ be a pre-triangulated category. A *pre-triangulated subcategory*[2] is a pair $(\mathcal{D}', \mathcal{T}')$ such that

(1) $\mathcal{D}'$ is an additive subcategory of $\mathcal{D}$ which is preserved under $[1]$ and such that $[1] : \mathcal{D}' \to \mathcal{D}'$ is an auto-equivalence,
(2) $\mathcal{T}' \subset \mathcal{T}$ is a subset such that for every $(X, Y, Z, f, g, h) \in \mathcal{T}'$ we have $X, Y, Z \in \mathrm{Ob}(\mathcal{D}')$ and $f, g, h \in \mathrm{Arrows}(\mathcal{D}')$, and
(3) $(\mathcal{D}', [\ ], \mathcal{T}')$ is a pre-triangulated category.

If $\mathcal{D}$ is a triangulated category, then we say $(\mathcal{D}', \mathcal{T}')$ is a *triangulated subcategory* if it is a pre-triangulated subcategory and $(\mathcal{D}', [\ ], \mathcal{T}')$ is a triangulated category.

In this situation the inclusion functor $\mathcal{D}' \to \mathcal{D}$ is an exact functor with $\xi_X : X[1] \to X[1]$ given by the identity on $X[1]$.

We will see in Lemma 4.1 that for a distinguished triangle (X, Y, Z, f, g, h) in a pre-triangulated category the composition $g \circ f : X \to Z$ is zero. Thus the sequence (3.2.1) is a complex. A homological functor is one that turns this complex into a long exact sequence.

Definition 3.5. Let $\mathcal{D}$ be a pre-triangulated category. Let $\mathcal{A}$ be an abelian category. An additive functor $H : \mathcal{D} \to \mathcal{A}$ is called *homological* if for every distinguished triangle (X, Y, Z, f, g, h) the sequence

$$H(X) \to H(Y) \to H(Z)$$

is exact in the abelian category $\mathcal{A}$. An additive functor $H : \mathcal{D}^{opp} \to \mathcal{A}$ is called *cohomological* if the corresponding functor $\mathcal{D} \to \mathcal{A}^{opp}$ is homological.

If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor we often write $H^n(X) = H(X[n])$ so that $H(X) = H^0(X)$. Our discussion of TR2 above implies that a distinguished triangle (X, Y, Z, f, g, h) determines a long exact sequence

(3.5.1)
$$H^{-1}(Z) \xrightarrow{H(h[-1])} H^0(X) \xrightarrow{H(f)} H^0(Y) \xrightarrow{H(g)} H^0(Z) \xrightarrow{H(h)} H^1(X)$$

[2] This definition may be nonstandard. If $\mathcal{D}'$ is a full subcategory then $\mathcal{T}'$ is the intersection of the set of triangles in $\mathcal{D}'$ with $\mathcal{T}$, see Lemma 4.16. In this case we drop $\mathcal{T}'$ from the notation.

This will be called the *long exact sequence* associated to the distinguished triangle and the homological functor. As indicated we will not use any signs for the morphisms in the long exact sequence. This has the side effect that maps in the long exact sequence associated to the rotation (TR2) of a distinguished triangle differ from the maps in the sequence above by some signs.

Definition 3.6. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D}$ be a triangulated category. A *δ-functor from $\mathcal{A}$ to $\mathcal{D}$* is given by a functor $G : \mathcal{A} \to \mathcal{D}$ and a rule which assigns to every short exact sequence

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

a morphism $\delta = \delta_{A \to B \to C} : G(C) \to G(A)[1]$ such that

(1) the triangle $(G(A), G(B), G(C), G(a), G(b), \delta_{A \to B \to C})$ is a distinguished triangle of $\mathcal{D}$ for any short exact sequence as above, and
(2) for every morphism $(A \to B \to C) \to (A' \to B' \to C')$ of short exact sequences the diagram

$$\begin{tikzcd} G(C) \arrow[r, "\delta_{A \to B \to C}"'] \arrow[d] & G(A)[1] \arrow[d] \\ G(C') \arrow[r, "\delta_{A' \to B' \to C'}"] & G(A')[1] \end{tikzcd}$$

is commutative.

In this situation we call $(G(A), G(B), G(C), G(a), G(b), \delta_{A \to B \to C})$ the *image of the short exact sequence under the given δ-functor*.

Note how a δ-functor comes equipped with additional structure. Strictly speaking it does not make sense to say that a given functor $\mathcal{A} \to \mathcal{D}$ is a δ-functor, but we will often do so anyway.

4. Elementary results on triangulated categories

Most of the results in this section are proved for pre-triangulated categories and a fortiori hold in any triangulated category.

Lemma 4.1. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. Then $g \circ f = 0$, $h \circ g = 0$ and $f[1] \circ h = 0$.*

Proof. By TR1 we know $(X, X, 0, 1, 0, 0)$ is a distinguished triangle. Apply TR3 to

$$\begin{tikzcd} X \arrow[r] \arrow[d, "1"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "1[1]"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

Of course the dotted arrow is the zero map. Hence the commutativity of the diagram implies that $g \circ f = 0$. For the other cases rotate the triangle, i.e., apply TR2. □

Lemma 4.2. *Let $\mathcal{D}$ be a pre-triangulated category. For any object W of $\mathcal{D}$ the functor* $\mathrm{Hom}_{\mathcal{D}}(W, -)$ *is homological, and the functor* $\mathrm{Hom}_{\mathcal{D}}(-, W)$ *is cohomological.*

Proof. Consider a distinguished triangle (X, Y, Z, f, g, h). We have already seen that $g \circ f = 0$, see Lemma 4.1. Suppose $a : W \to Y$ is a morphism such that $g \circ a = 0$. Then we get a commutative diagram

$$\begin{tikzcd} W \arrow[r, "1"'] \arrow[d, dotted, "b"] & W \arrow[r] \arrow[d, "a"] & 0 \arrow[r] \arrow[d, "0"] & W[1] \arrow[d, dotted, "b[1]"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

Both rows are distinguished triangles (use TR1 for the top row). Hence we can fill the dotted arrow b (first rotate using TR2, then apply TR3, and then rotate back). This proves the lemma. $\square$

Lemma 4.3. *Let $\mathcal{D}$ be a pre-triangulated category. Let*

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

be a morphism of distinguished triangles. If two among a, b, c are isomorphisms so is the third.

Proof. Assume that a and c are isomorphisms. For any object W of $\mathcal{D}$ write $H_W(-) = \mathrm{Hom}_\mathcal{D}(W, -)$. Then we get a commutative diagram of abelian groups

$$\begin{tikzcd} H_W(Z[-1]) \arrow[r] \arrow[d] & H_W(X) \arrow[r] \arrow[d] & H_W(Y) \arrow[r] \arrow[d] & H_W(Z) \arrow[r] \arrow[d] & H_W(X[1]) \arrow[d] \\ H_W(Z'[-1]) \arrow[r] & H_W(X') \arrow[r] & H_W(Y') \arrow[r] & H_W(Z') \arrow[r] & H_W(X'[1]) \end{tikzcd}$$

By assumption the right two and left two vertical arrows are bijective. As H_W is homological by Lemma 4.2 and the five lemma (Homology, Lemma 5.20) it follows that the middle vertical arrow is an isomorphism. Hence by Yoneda's lemma, see Categories, Lemma 3.5 we see that b is an isomorphism. This implies the other cases by rotating (using TR2). $\square$

Remark 4.4. Let $\mathcal{D}$ be an additive category with translation functors $[n]$ as in Definition 3.1. Let us call a triangle (X, Y, Z, f, g, h) *special*[3] if for every object W of $\mathcal{D}$ the long sequence of abelian groups

$$\ldots \to \mathrm{Hom}_\mathcal{D}(W, X) \to \mathrm{Hom}_\mathcal{D}(W, Y) \to \mathrm{Hom}_\mathcal{D}(W, Z) \to \mathrm{Hom}_\mathcal{D}(W, X[1]) \to \ldots$$

is exact. The proof of Lemma 4.3 shows that if

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

is a morphism of special triangles and if two among a, b, c are isomorphisms so is the third. There is a dual statement for *co-special* triangles, i.e., triangles which turn into long exact sequences on applying the functor $\mathrm{Hom}_\mathcal{D}(-, W)$. Thus distinguished triangles are special and co-special, but in general there are many more (co-)special triangles, than there are distinguished triangles.

Lemma 4.5. *Let $\mathcal{D}$ be a pre-triangulated category. Let*

$$(0, b, 0), (0, b', 0) : (X, Y, Z, f, g, h) \to (X, Y, Z, f, g, h)$$

be endomorphisms of a distinguished triangle. Then $bb' = 0$.

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[3]This is nonstandard notation.

Proof. Picture

$$\begin{tikzcd} X \arrow[r] \arrow[d, "0"] & Y \arrow[r] \arrow[d, "{b,b'}"] \arrow[dl, dotted, "\alpha"] & Z \arrow[r] \arrow[d, "0"] \arrow[dl, dotted, "\beta"] & X[1] \arrow[d, "0"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

Applying Lemma 4.2 we find dotted arrows α and β such that $b' = f \circ \alpha$ and $b = \beta \circ g$. Then $bb' = \beta \circ g \circ f \circ \alpha = 0$ as $g \circ f = 0$ by Lemma 4.1. $\square$

Lemma 4.6. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. If*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "c"] & X[1] \arrow[d, "{a[1]}"] \\ Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

is commutative and $a^2 = a$, $c^2 = c$, then there exists a morphism $b : Y \to Y$ with $b^2 = b$ such that (a, b, c) is an endomorphism of the triangle (X, Y, Z, f, g, h).

Proof. By TR3 there exists a morphism b' such that (a, b', c) is an endomorphism of (X, Y, Z, f, g, h). Then $(0, (b')^2 - b', 0)$ is also an endomorphism. By Lemma 4.5 we see that $(b')^2 - b'$ has square zero. Set $b = b' - (2b' - 1)((b')^2 - b') = 3(b')^2 - 2(b')^3$. A computation shows that (a, b, c) is an endomorphism and that $b^2 - b = (4(b')^2 - 4b' - 3)((b')^2 - b')^2 = 0$. $\square$

Lemma 4.7. *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. There exists a distinguished triangle (X, Y, Z, f, g, h) which is unique up to (nonunique) isomorphism of triangles. More precisely, given a second such distinguished triangle (X, Y, Z', f, g', h') there exists an isomorphism*

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Z', f, g', h')$$

Proof. Existence by TR1. Uniqueness up to isomorphism by TR3 and Lemma 4.3. $\square$

Lemma 4.8. *Let $\mathcal{D}$ be a pre-triangulated category. Let*

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

be a morphism of distinguished triangles. If one of the following conditions holds

(1) $\mathrm{Hom}(Y, X') = 0$,
(2) $\mathrm{Hom}(Z, Y') = 0$,
(3) $\mathrm{Hom}(X, X') = \mathrm{Hom}(Z, X') = 0$,
(4) $\mathrm{Hom}(Z, X') = \mathrm{Hom}(Z, Z') = 0$, *or*
(5) $\mathrm{Hom}(X[1], Z') = \mathrm{Hom}(Z, X') = 0$

then b is the unique morphism from $Y \to Y'$ such that (a, b, c) is a morphism of triangles.

Proof. If we have a second morphism of triangles (a, b', c) then $(0, b - b', 0)$ is a morphism of triangles. Hence we have to show: the only morphism $b : Y \to Y'$ such that $X \to Y \to Y'$ and $Y \to Y' \to Z'$ are zero is 0. We will use Lemma 4.2 without further mention. In particular, condition (3) implies (1). Given condition (1) if the composition $g' \circ b : Y \to Y' \to Z'$ is zero, then b lifts to a morphism $Y \to X'$ which has to be zero. This proves (1).

The proof of (2) and (4) are dual to this argument.

Assume (5). Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[d, "0"] & Y \arrow[r, "g"'] \arrow[d, "b"] & Z \arrow[r, "h"'] \arrow[d, "0"] \arrow[dl, dotted, "\epsilon"] & X[1] \arrow[d, "0"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

We may choose ϵ such that $b = \epsilon \circ g$. Then $g' \circ \epsilon \circ g = 0$ which implies that $g' \circ \epsilon = \delta \circ h$ for some $\delta \in \mathrm{Hom}(X[1], Z')$. Since $\mathrm{Hom}(X[1], Z') = 0$ we conclude that $g' \circ \epsilon = 0$. Hence $\epsilon = f' \circ \gamma$ for some $\gamma \in \mathrm{Hom}(Z, X')$. Since $\mathrm{Hom}(Z, X') = 0$ we conclude that $\epsilon = 0$ and hence $b = 0$ as desired. □

Lemma 4.9. *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. The following are equivalent*

1. *f is an isomorphism,*
2. *$(X, Y, 0, f, 0, 0)$ is a distinguished triangle, and*
3. *for any distinguished triangle (X, Y, Z, f, g, h) we have $Z = 0$.*

Proof. By TR1 the triangle $(X, X, 0, 1, 0, 0)$ is distinguished. Let (X, Y, Z, f, g, h) be a distinguished triangle. By TR3 there is a map of distinguished triangles $(1, f, 0) : (X, X, 0) \to (X, Y, Z)$. If f is an isomorphism, then $(1, f, 0)$ is an isomorphism of triangles by Lemma 4.3 and $Z = 0$. Conversely, if $Z = 0$, then $(1, f, 0)$ is an isomorphism of triangles as well, hence f is an isomorphism. □

Lemma 4.10. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be triangles. The following are equivalent*

1. *$(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$ is a distinguished triangle,*
2. *both (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') are distinguished triangles.*

Proof. Assume (2). By TR1 we may choose a distinguished triangle $(X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. By TR3 we can find morphisms of distinguished triangles $(X, Y, Z, f, g, h) \to (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$ and $(X', Y', Z', f', g', h') \to (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. Taking the direct sum of these morphisms we obtain a morphism of triangles

$$\begin{tikzcd}
(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h') \arrow[d, "{(1,1,c)}"] \\
(X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'').
\end{tikzcd}$$

In the terminology of Remark 4.4 this is a map of special triangles (because a direct sum of special triangles is special) and we conclude that c is an isomorphism. Thus (1) holds.

Assume (1). We will show that (X, Y, Z, f, g, h) is a distinguished triangle. First observe that (X, Y, Z, f, g, h) is a special triangle (terminology from Remark 4.4) as a direct summand of the distinguished hence special triangle $(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$. Using TR1 let (X, Y, Q, f, g'', h'') be a distinguished triangle. By TR3 there exists a morphism of distinguished triangles $(X \oplus X', Y \oplus$

$Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h') \to (X, Y, Q, f, g'', h'')$. Composing this with the inclusion map we get a morphism of triangles

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Q, f, g'', h'')$$

By Remark 4.4 we find that c is an isomorphism and we conclude that (2) holds. $\square$

Lemma 4.11. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle.*

(1) *If $h = 0$, then there exists a right inverse $s : Z \to Y$ to g.*
(2) *For any right inverse $s : Z \to Y$ of g the map $f \oplus s : X \oplus Z \to Y$ is an isomorphism.*
(3) *For any objects X', Z' of $\mathcal{D}$ the triangle $(X', X' \oplus Z', Z', (1, 0), (0, 1), 0)$ is distinguished.*

Proof. To see (1) use that $\mathrm{Hom}_\mathcal{D}(Z, Y) \to \mathrm{Hom}_\mathcal{D}(Z, Z) \to \mathrm{Hom}_\mathcal{D}(Z, X[1])$ is exact by Lemma 4.2. By the same token, if s is as in (2), then $h = 0$ and the sequence

$$0 \to \mathrm{Hom}_\mathcal{D}(W, X) \to \mathrm{Hom}_\mathcal{D}(W, Y) \to \mathrm{Hom}_\mathcal{D}(W, Z) \to 0$$

is split exact (split by $s : Z \to Y$). Hence by Yoneda's lemma we see that $X \oplus Z \to Y$ is an isomorphism. The last assertion follows from TR1 and Lemma 4.10. $\square$

Lemma 4.12. *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. The following are equivalent*

(1) *f has a kernel,*
(2) *f has a cokernel,*
(3) *f is the isomorphic to a composition $K \oplus Z \to Z \to Z \oplus Q$ of a projection and coprojection for some objects K, Z, Q of $\mathcal{D}$.*

Proof. Any morphism isomorphic to a map of the form $X' \oplus Z \to Z \oplus Y'$ has both a kernel and a cokernel. Hence (3) $\Rightarrow$ (1), (2). Next we prove (1) $\Rightarrow$ (3). Suppose first that $f : X \to Y$ is a monomorphism, i.e., its kernel is zero. By TR1 there exists a distinguished triangle (X, Y, Z, f, g, h). By Lemma 4.1 the composition $f \circ h[-1] = 0$. As f is a monomorphism we see that $h[-1] = 0$ and hence $h = 0$. Then Lemma 4.11 implies that $Y = X \oplus Z$, i.e., we see that (3) holds. Next, assume f has a kernel K. As $K \to X$ is a monomorphism we conclude $X = K \oplus X'$ and $f|_{X'} : X' \to Y$ is a monomorphism. Hence $Y = X' \oplus Y'$ and we win. The implication (2) $\Rightarrow$ (3) is dual to this. $\square$

Lemma 4.13. *Let $\mathcal{D}$ be a pre-triangulated category. Let I be a set.*

(1) *Let X_i, $i \in I$ be a family of objects of $\mathcal{D}$.*
    (a) *If $\prod X_i$ exists, then $(\prod X_i)[1] = \prod X_i[1]$.*
    (b) *If $\bigoplus X_i$ exists, then $(\bigoplus X_i)[1] = \bigoplus X_i[1]$.*
(2) *Let $X_i \to Y_i \to Z_i \to X_i[1]$ be a family of distinguished triangles of $\mathcal{D}$.*
    (a) *If $\prod X_i$, $\prod Y_i$, $\prod Z_i$ exist, then $\prod X_i \to \prod Y_i \to \prod Z_i \to \prod X_i[1]$ is a distinguished triangle.*
    (b) *If $\bigoplus X_i$, $\bigoplus Y_i$, $\bigoplus Z_i$ exist, then $\bigoplus X_i \to \bigoplus Y_i \to \bigoplus Z_i \to \bigoplus X_i[1]$ is a distinguished triangle.*

Proof. Part (1) is true because [1] is an autoequivalence of $\mathcal{D}$ and because direct sums and products are defined in terms of the category structure. Let us prove (2)(a). Choose a distinguished triangle $\prod X_i \to \prod Y_i \to Z \to \prod X_i[1]$. For each j we can use TR3 to choose a morphism $p_j : Z \to Z_j$ fitting into a morphism of distinguished triangles with the projection maps $\prod X_i \to X_j$ and $\prod Y_i \to Y_j$. Using the definition of products we obtain a map $\prod p_i : Z \to \prod Z_i$ fitting into a morphism of triangles from the distinguished triangle to the triangle made out of the products. Observe that the "product" triangle $\prod X_i \to \prod Y_i \to \prod Z_i \to \prod X_i[1]$ is special in the terminology of Remark 4.4 because products of exact sequences of abelian groups are exact. Hence Remark 4.4 shows that the morphism of triangles is an isomorphism and we conclude by TR1. The proof of (2)(b) is dual. □

Lemma 4.14. *Let $\mathcal{D}$ be a pre-triangulated category. If $\mathcal{D}$ has countable products, then $\mathcal{D}$ is Karoubian. If $\mathcal{D}$ has countable coproducts, then $\mathcal{D}$ is Karoubian.*

Proof. Assume $\mathcal{D}$ has countable products. By Homology, Lemma 4.3 it suffices to check that morphisms which have a right inverse have kernels. Any morphism which has a right inverse is an epimorphism, hence has a kernel by Lemma 4.12. The second statement is dual to the first. □

The following lemma makes it slightly easier to prove that a pre-triangulated category is triangulated.

Lemma 4.15. *Let $\mathcal{D}$ be a pre-triangulated category. In order to prove TR4 it suffices to show that given any pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ there exist*

(1) *isomorphisms $i : X' \to X$, $j : Y' \to Y$ and $k : Z' \to Z$, and then setting $f' = j^{-1}fi : X' \to Y'$ and $g' = k^{-1}gj : Y' \to Z'$ there exist*
(2) *distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$, such that the assertion of TR4 holds.*

Proof. The replacement of X, Y, Z by X', Y', Z' is harmless by our definition of distinguished triangles and their isomorphisms. The lemma follows from the fact that the distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$ are unique up to isomorphism by Lemma 4.7. □

Lemma 4.16. *Let $\mathcal{D}$ be a pre-triangulated category. Assume that $\mathcal{D}'$ is an additive full subcategory of $\mathcal{D}$. The following are equivalent*

(1) *there exists a set of triangles $\mathcal{T}'$ such that $(\mathcal{D}', \mathcal{T}')$ is a pre-triangulated subcategory of $\mathcal{D}$,*
(2) *$\mathcal{D}'$ is preserved under $[1]$ and $[1] : \mathcal{D}' \to \mathcal{D}'$ is an auto-equivalence and given any morphism $f : X \to Y$ in $\mathcal{D}'$ there exists a distinguished triangle (X, Y, Z, f, g, h) in $\mathcal{D}$ such that Z is isomorphic to an object of $\mathcal{D}'$.*

In this case $\mathcal{T}'$ as in (1) is the set of distinguished triangles (X, Y, Z, f, g, h) of $\mathcal{D}$ such that $X, Y, Z \in \mathrm{Ob}(\mathcal{D}')$. Finally, if $\mathcal{D}$ is a triangulated category, then (1) and (2) are also equivalent to

(3) *$\mathcal{D}'$ is a triangulated subcategory.*

Proof. Omitted. □

Lemma 4.17. *An exact functor of pre-triangulated categories is additive.*

Proof. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Since $(0, 0, 0, 1_0, 1_0, 0)$ is a distinguished triangle of $\mathcal{D}$ the triangle

$$(F(0), F(0), F(0), 1_{F(0)}, 1_{F(0)}, F(0))$$

is distinguished in $\mathcal{D}'$. This implies that $1_{F(0)} \circ 1_{F(0)}$ is zero, see Lemma 4.1. Hence $F(0)$ is the zero object of $\mathcal{D}'$. This also implies that F applied to any zero morphism is zero (since a morphism in an additive category is zero if and only if it factors through the zero object). Next, using that $(X, X \oplus Y, Y, (1,0), (0,1), 0)$ is a distinguished triangle by Lemma 4.11 part (3), we see that $(F(X), F(X \oplus Y), F(Y), F(1,0), F(0,1), 0)$ is one too. This implies that the map $F(X) \oplus F(Y) \to F(X \oplus Y)$ is an isomorphism by Lemma 4.11 part (2). To finish we apply Homology, Lemma 7.1. □

Lemma 4.18. *Let $F : \mathcal{D} \to \mathcal{D}'$ be a fully faithful exact functor of pre-triangulated categories. Then a triangle (X, Y, Z, f, g, h) of $\mathcal{D}$ is distinguished if and only if $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished in $\mathcal{D}'$.*

Proof. The "only if" part is clear. Assume $(F(X), F(Y), F(Z))$ is distinguished in $\mathcal{D}'$. Pick a distinguished triangle (X, Y, Z', f, g', h') in $\mathcal{D}$. By Lemma 4.7 there exists an isomorphism of triangles

$$(1, 1, c') : (F(X), F(Y), F(Z)) \longrightarrow (F(X), F(Y), F(Z')).$$

Since F is fully faithful, there exists a morphism $c : Z \to Z'$ such that $F(c) = c'$. Then $(1, 1, c)$ is an isomorphism between (X, Y, Z) and (X, Y, Z'). Hence (X, Y, Z) is distinguished by TR1. □

Lemma 4.19. *Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be pre-triangulated categories. Let $F : \mathcal{D} \to \mathcal{D}'$ and $F' : \mathcal{D}' \to \mathcal{D}''$ be exact functors. Then $F' \circ F$ is an exact functor.*

Proof. Omitted. □

Lemma 4.20. *Let $\mathcal{D}$ be a pre-triangulated category. Let $\mathcal{A}$ be an abelian category. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor.*

1. *Let $\mathcal{D}'$ be a pre-triangulated category. Let $F : \mathcal{D}' \to \mathcal{D}$ be an exact functor. Then the composition $H \circ F$ is a homological functor as well.*
2. *Let $\mathcal{A}'$ be an abelian category. Let $G : \mathcal{A} \to \mathcal{A}'$ be an exact functor. Then $G \circ H$ is a homological functor as well.*

Proof. Omitted. □

Lemma 4.21. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ be an abelian category. Let $G : \mathcal{A} \to \mathcal{D}$ be a δ-functor.*

1. *Let $\mathcal{D}'$ be a triangulated category. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor. Then the composition $F \circ G$ is a δ-functor as well.*
2. *Let $\mathcal{A}'$ be an abelian category. Let $H : \mathcal{A}' \to \mathcal{A}$ be an exact functor. Then $G \circ H$ is a δ-functor as well.*

Proof. Omitted. □

Lemma 4.22. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $G : \mathcal{A} \to \mathcal{D}$ be a δ-functor. Let $H : \mathcal{D} \to \mathcal{B}$ be a homological functor. Assume that $H^{-1}(G(A)) = 0$ for all A in $\mathcal{A}$. Then the collection*

$$\{H^n \circ G, H^n(\delta_{A \to B \to C})\}_{n \geq 0}$$

is a δ-functor from $\mathcal{A} \to \mathcal{B}$, see Homology, Definition 12.1.

Proof. The notation signifies the following. If $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ is a short exact sequence in $\mathcal{A}$, then

$$\delta = \delta_{A \to B \to C} : G(C) \to G(A)[1]$$

is a morphism in $\mathcal{D}$ such that $(G(A), G(B), G(C), a, b, \delta)$ is a distinguished triangle, see Definition 3.6. Then $H^n(\delta) : H^n(G(C)) \to H^n(G(A)[1]) = H^{n+1}(G(A))$ is clearly functorial in the short exact sequence. Finally, the long exact cohomology sequence (3.5.1) combined with the vanishing of $H^{-1}(G(C))$ gives a long exact sequence

$$0 \to H^0(G(A)) \to H^0(G(B)) \to H^0(G(C)) \xrightarrow{H^0(\delta)} H^1(G(A)) \to \ldots$$

in $\mathcal{B}$ as desired. $\square$

The proof of the following result uses TR4.

Proposition 4.23. *Let $\mathcal{D}$ be a triangulated category. Any commutative diagram*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

can be extended to a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[1] \arrow[d] \\
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & X''[1] \arrow[d] \\
X[1] \arrow[r] & Y[1] \arrow[r] & Z[1] \arrow[r] & X[2]
\end{tikzcd}$$

where all the squares are commutative, except for the lower right square which is anticommutative. Moreover, each of the rows and columns are distinguished triangles. Finally, the morphisms on the bottom row (resp. right column) are obtained from the morphisms of the top row (resp. left column) by applying [1].

Proof. During this proof we avoid writing the arrows in order to make the proof legible. Choose distinguished triangles (X, Y, Z), (X', Y', Z'), (X, X', X''), (Y, Y', Y''), and (X, Y', A). Note that the morphism $X \to Y'$ is both equal to the composition $X \to Y \to Y'$ and equal to the composition $X \to X' \to Y'$. Hence, we can find morphisms

(1) $a : Z \to A$ and $b : A \to Y''$, and
(2) $a' : X'' \to A$ and $b' : A \to Z'$

as in TR4. Denote $c : Y'' \to Z[1]$ the composition $Y'' \to Y[1] \to Z[1]$ and denote $c' : Z' \to X''[1]$ the composition $Z' \to X'[1] \to X''[1]$. The conclusion of our application TR4 are that

(1) (Z, A, Y'', a, b, c), (X'', A, Z', a', b', c') are distinguished triangles,

(2) $(X, Y, Z) \to (X, Y', A)$, $(X, Y', A) \to (Y, Y', Y'')$, $(X, X', X'') \to (X, Y', A)$, $(X, Y', A) \to (X', Y', Z')$ are morphisms of triangles.

First using that $(X, X', X'') \to (X, Y', A)$ and $(X, Y', A) \to (Y, Y', Y'')$. are morphisms of triangles we see the first of the diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X'' \arrow[r, "b \circ a'"] \arrow[d] & Y'' \arrow[d] \\ X[1] \arrow[r] & Y[1] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "b' \circ a"] & X[1] \arrow[d] \\ Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

is commutative. The second is commutative too using that $(X, Y, Z) \to (X, Y', A)$ and $(X, Y', A) \to (X', Y', Z')$ are morphisms of triangles. At this point we choose a distinguished triangle (X'', Y'', Z'') starting with the map $b \circ a' : X'' \to Y''$.

Next we apply TR4 one more time to the morphisms $X'' \to A \to Y''$ and the triangles (X'', A, Z', a', b', c'), (X'', Y'', Z''), and $(A, Y'', Z[1], b, c, -a[1])$ to get morphisms $a'' : Z' \to Z''$ and $b'' : Z'' \to Z[1]$. Then $(Z', Z'', Z[1], a'', b'', -b'[1] \circ a[1])$ is a distinguished triangle, hence also $(Z, Z', Z'', -b' \circ a, a'', -b'')$ and hence also $(Z, Z', Z'', b' \circ a, a'', b'')$. Moreover, $(X'', A, Z') \to (X'', Y'', Z'')$ and $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ are morphisms of triangles. At this point we have defined all the distinguished triangles and all the morphisms, and all that's left is to verify some commutativity relations.

To see that the middle square in the diagram commutes, note that the arrow $Y' \to Z'$ factors as $Y' \to A \to Z'$ because $(X, Y', A) \to (X', Y', Z')$ is a morphism of triangles. Similarly, the morphism $Y' \to Y''$ factors as $Y' \to A \to Y''$ because $(X, Y', A) \to (Y, Y', Y'')$ is a morphism of triangles. Hence the middle square commutes because the square with sides (A, Z', Z'', Y'') commutes as $(X'', A, Z') \to (X'', Y'', Z'')$ is a morphism of triangles (by TR4). The square with sides $(Y'', Z'', Y[1], Z[1])$ commutes because $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and $c : Y'' \to Z[1]$ is the composition $Y'' \to Y[1] \to Z[1]$. The square with sides $(Z', X'[1], X''[1], Z'')$ is commutative because $(X'', A, Z') \to (X'', Y'', Z'')$ is a morphism of triangles and $c' : Z' \to X''[1]$ is the composition $Z' \to X'[1] \to X''[1]$. Finally, we have to show that the square with sides $(Z'', X''[1], Z[1], X[2])$ anticommutes. This holds because $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and we're done. $\square$

5. Localization of triangulated categories

In order to construct the derived category starting from the homotopy category of complexes, we will use a localization process.

Definition 5.1. Let $\mathcal{D}$ be a pre-triangulated category. We say a multiplicative system S is *compatible with the triangulated structure* if the following two conditions hold:

MS5 For a morphism f of $\mathcal{D}$ we have $f \in S \Leftrightarrow f[1] \in S$[4].

[4]See Remark 5.3.

MS6 Given a solid commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"] & Y \arrow[r] \arrow[d, "s'"] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "s[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

whose rows are distinguished triangles with $s, s' \in S$ there exists a morphism $s'' : Z \to Z'$ in S such that (s, s', s'') is a morphism of triangles.

It turns out that these axioms are not independent of the axioms defining multiplicative systems.

Lemma 5.2. *Let $\mathcal{D}$ be a pre-triangulated category. Let $S \subset Arrows(\mathcal{D})$.*

(1) *If S contains all identities and MS6 holds (Definition 5.1), then every isomorphism of $\mathcal{D}$ is in S.*
(2) *If MS1, MS5 (Categories, Definition 27.1) and MS6 hold, then MS2 holds.*

Proof. Assume S contains all identities and MS6 holds. Let $f : X \to Y$ be an isomorphism of $\mathcal{D}$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "1"] & X \arrow[r, "1"'] \arrow[d, "1"] & X \arrow[r] \arrow[d, dashed] & 0[1] \arrow[d, "1[1]"] \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r] & 0[1] \end{tikzcd}$$

The rows are distinguished triangles by Lemma 4.9. By MS6 we see that the dotted arrow exists and is in S, so f is in S.

Assume MS1, MS5, MS6. Suppose that $f : X \to Y$ is a morphism of $\mathcal{D}$ and $t : X \to X'$ an element of S. Choose a distinguished triangle (X, Y, Z, f, g, h). Next, choose a distinguished triangle $(X', Y', Z, f', g', t[1] \circ h)$ (here we use TR1 and TR2). By MS5, MS6 (and TR2 to rotate) we can find the dotted arrow in the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "t"] & Y \arrow[r] \arrow[d, dotted, "s'"] & Z \arrow[r] \arrow[d, "1"] & X[1] \arrow[d, "t[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z \arrow[r] & X'[1] \end{tikzcd}$$

with moreover $s' \in S$. This proves LMS2. The proof of RMS2 is dual. $\square$

Remark 5.3. In the presence of MS1 and MS6, condition MS5 is equivalent to asking $s[n] \in S$ for all $s \in S$ and $n \in \mathbf{Z}$. For example, suppose MS5 holds, we have $s \in S$, and we want to show $s[-1] \in S$. This isn't immediate because $s[-1][1]$ is not equal to s, only isomorphic to s as an arrow of $\mathcal{D}$. Still, this does imply that $s[-1][1] = f \circ s \circ g$ for isomorphisms f, g. By Lemma 5.2 (1) we find $f, g \in S$, hence $s[-1][1] \in S$ by MS1, hence $s[-1] \in S$ by MS5. We leave a complete proof to the reader as an exercise.

Lemma 5.4. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let*

$$S = \{f \in Arrows(\mathcal{D}) \mid F(f) \text{ is an isomorphism}\}$$

Then S is a saturated (see Categories, Definition 27.20) multiplicative system compatible with the triangulated structure on $\mathcal{D}$.

Proof. We have to prove axioms MS1 – MS6, see Categories, Definitions 27.1 and 27.20 and Definition 5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and Lemma 4.3. By Lemma 5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds. To do this let $f, g : X \to Y$ be morphisms of $\mathcal{D}$, and let $t : Z \to X$ be an element of S such that $f \circ t = g \circ t$. As $\mathcal{D}$ is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, d, h) using TR1. Since $a \circ t = 0$ we see by Lemma 4.2 there exists a morphism $i : Q \to Y$ such that $i \circ d = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k). Here is a picture

$$\begin{tikzcd}
Z \arrow[r, "t"'] & X \arrow[r, "d"'] \arrow[d, "1"] & Q \arrow[r] \arrow[d, "i"] & Z[1] \\
 & X \arrow[r, "a"'] & Y \arrow[d, "j"] & \\
 & & W &
\end{tikzcd}$$

OK, and now we apply the functor F to this diagram. Since $t \in S$ we see that $F(Q) = 0$, see Lemma 4.9. Hence $F(j)$ is an isomorphism by the same lemma, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ d = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. $\square$

Lemma 5.5. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor between a pre-triangulated category and an abelian category. Let*

$$S = \{f \in Arrows(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z}\}$$

Then S is a saturated (see Categories, Definition 27.20) multiplicative system compatible with the triangulated structure on $\mathcal{D}$.

Proof. We have to prove axioms MS1 – MS6, see Categories, Definitions 27.1 and 27.20 and Definition 5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and the long exact cohomology sequence (3.5.1). By Lemma 5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds. To do this let $f, g : X \to Y$ be morphisms of $\mathcal{D}$, and let $t : Z \to X$ be an element of S such that $f \circ t = g \circ t$. As $\mathcal{D}$ is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 4.2 there exists a morphism $i : Q \to Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k). Here is a picture

$$\begin{tikzcd}
Z \arrow[r, "t"'] & X \arrow[r, "g"'] \arrow[d, "1"] & Q \arrow[r] \arrow[d, "i"] & Z[1] \\
 & X \arrow[r, "a"'] & Y \arrow[d, "j"] & \\
 & & W &
\end{tikzcd}$$

OK, and now we apply the functors H^i to this diagram. Since $t \in S$ we see that $H^i(Q) = 0$ by the long exact cohomology sequence (3.5.1). Hence $H^i(j)$ is an isomorphism for all i by the same argument, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ g = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. $\square$

Proposition 5.6. *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Then there exists a unique structure of a pre-triangulated category on $S^{-1}\mathcal{D}$ such that $[1] \circ Q = Q \circ [1]$ and the localization functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ is exact. Moreover, if $\mathcal{D}$ is a triangulated category, so is $S^{-1}\mathcal{D}$.*

Proof. We have seen that $S^{-1}\mathcal{D}$ is an additive category and that the localization functor Q is additive in Homology, Lemma 8.2. It follows from MS5 that there is a unique additive auto-equivalence $[1] : S^{-1}\mathcal{D} \to S^{-1}\mathcal{D}$ such that $Q \circ [1] = [1] \circ Q$ (equality of functors); we omit the details. We say a triangle of $S^{-1}\mathcal{D}$ is distinguished if it is isomorphic to the image of a distinguished triangle under the localization functor Q.

Proof of TR1. The only thing to prove here is that if $a : Q(X) \to Q(Y)$ is a morphism of $S^{-1}\mathcal{D}$, then a fits into a distinguished triangle. Write $a = Q(s)^{-1} \circ Q(f)$ for some $s : Y \to Y'$ in S and $f : X \to Y'$. Choose a distinguished triangle (X, Y', Z, f, g, h) in $\mathcal{D}$. Then we see that $(Q(X), Q(Y), Q(Z), a, Q(g) \circ Q(s), Q(h))$ is a distinguished triangle of $S^{-1}\mathcal{D}$.

Proof of TR2. This is immediate from the definitions.

Proof of TR3. Note that the existence of the dotted arrow which is required to exist may be proven after replacing the two triangles by isomorphic triangles. Hence we may assume given distinguished triangles (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') of $\mathcal{D}$ and a commutative diagram

$$\begin{tikzcd} Q(X) \arrow[r, "Q(f)"'] \arrow[d, "a"'] & Q(Y) \arrow[d, "b"] \\ Q(X') \arrow[r, "Q(f')"] & Q(Y') \end{tikzcd}$$

in $S^{-1}\mathcal{D}$. Now we apply Categories, Lemma 27.10 to find a morphism $f'' : X'' \to Y''$ in $\mathcal{D}$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "k"'] \arrow[d, "f"'] & X'' \arrow[d, "f''"] & X' \arrow[l, "s"'] \arrow[d, "f'"] \\ Y \arrow[r, "l"] & Y'' & Y' \arrow[l, "t"'] \end{tikzcd}$$

in $\mathcal{D}$ with $s, t \in S$ and $a = s^{-1}k$, $b = t^{-1}l$. At this point we can use TR3 for $\mathcal{D}$ and MS6 to find a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "k"] & Y \arrow[r] \arrow[d, "l"] & Z \arrow[r] \arrow[d, "m"] & X[1] \arrow[d, "g[1]"] \\ X'' \arrow[r] & Y'' \arrow[r] & Z'' \arrow[r] & X''[1] \\ X' \arrow[r] \arrow[u, "s"] & Y' \arrow[r] \arrow[u, "t"] & Z' \arrow[r] \arrow[u, "r"] & X'[1] \arrow[u, "s[1]"] \end{tikzcd}$$

with $r \in S$. It follows that setting $c = Q(r)^{-1}Q(m)$ we obtain the desired morphism of triangles

$$\begin{tikzcd} (Q(X), Q(Y), Q(Z), Q(f), Q(g), Q(h)) \arrow[d, "(a,b,c)"] \\ (Q(X'), Q(Y'), Q(Z'), Q(f'), Q(g'), Q(h')) \end{tikzcd}$$

This proves the first statement of the lemma. If $\mathcal{D}$ is also a triangulated category, then we still have to prove TR4 in order to show that $S^{-1}\mathcal{D}$ is triangulated as well. To do this we reduce by Lemma 4.15 to the following statement: Given composable morphisms $a : Q(X) \to Q(Y)$ and $b : Q(Y) \to Q(Z)$ we have to produce an octahedron after possibly replacing $Q(X), Q(Y), Q(Z)$ by isomorphic objects. To do this we may first replace Y by an object such that $a = Q(f)$ for some morphism $f : X \to Y$ in $\mathcal{D}$. (More precisely, write $a = s^{-1}f$ with $s : Y \to Y'$ in S and $f : X \to Y'$. Then replace Y by Y'.) After this we similarly replace Z by an object such that $b = Q(g)$ for some morphism $g : Y \to Z$. Now we can find distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3) in $\mathcal{D}$ (by TR1), and morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$ as in TR4. Then it is immediately verified that applying the functor Q to all these data gives a corresponding structure in $S^{-1}\mathcal{D}$ □

The universal property of the localization of a triangulated category is as follows (we formulate this for pre-triangulated categories, hence it holds a fortiori for triangulated categories).

Lemma 5.7. *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Let $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ be the localization functor, see Proposition 5.6.*

(1) *If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor into an abelian category $\mathcal{A}$ such that $H(s)$ is an isomorphism for all $s \in S$, then the unique factorization $H' : S^{-1}\mathcal{D} \to \mathcal{A}$ such that $H = H' \circ Q$ (see Categories, Lemma 27.8) is a homological functor too.*
(2) *If $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor into a pre-triangulated category $\mathcal{D}'$ such that $F(s)$ is an isomorphism for all $s \in S$, then the unique factorization $F' : S^{-1}\mathcal{D} \to \mathcal{D}'$ such that $F = F' \circ Q$ (see Categories, Lemma 27.8) is an exact functor too.*

Proof. This lemma proves itself. Details omitted. □

Lemma 5.8. *Let $\mathcal{D}$ be a pre-triangulated category and let $\mathcal{D}' \subset \mathcal{D}$ be a full, pre-triangulated subcategory. Let S be a saturated multiplicative system of $\mathcal{D}$ compatible with the triangulated structure. Assume that for each X in $\mathcal{D}$ there exists an $s : X' \to X$ in S such that X' is an object of $\mathcal{D}'$. Then $S' = S \cap Arrows(\mathcal{D}')$ is a saturated multiplicative system compatible with the triangulated structure and the functor*

$$(S')^{-1}\mathcal{D}' \longrightarrow S^{-1}\mathcal{D}$$

is an equivalence of pre-triangulated categories.

Proof. Consider the quotient functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ of Proposition 5.6. Since S is saturated we have that a morphism $f : X \to Y$ is in S if and only if $Q(f)$ is invertible, see Categories, Lemma 27.21. Thus S' is the collection of arrows which are turned into isomorphisms by the composition $\mathcal{D}' \to \mathcal{D} \to S^{-1}\mathcal{D}$. Hence S' is is a saturated multiplicative system compatible with the triangulated structure by Lemma 5.4. By Lemma 5.7 we obtain the exact functor $(S')^{-1}\mathcal{D}' \to S^{-1}\mathcal{D}$ of pre-triangulated categories. By assumption this functor is essentially surjective. Let X', Y' be objects of $\mathcal{D}'$. By Categories, Remark 27.15 we have

$$\mathrm{Mor}_{S^{-1}\mathcal{D}}(X', Y') = \mathrm{colim}_{s : X \to X' \text{ in } S} \mathrm{Mor}_{\mathcal{D}}(X, Y')$$

Our assumption implies that for any $s : X \to X'$ in S we can find a morphism $s' : X'' \to X$ in S with X'' in $\mathcal{D}'$. Then $s \circ s' : X'' \to X'$ is in S'. Hence the colimit above is equal to

$$\mathrm{colim}_{s'' : X'' \to X' \text{ in } S'} \mathrm{Mor}_{\mathcal{D}'}(X'', Y') = \mathrm{Mor}_{(S')^{-1}\mathcal{D}'}(X', Y')$$

This proves our functor is also fully faithful and the proof is complete. □

The following lemma describes the kernel (see Definition 6.5) of the localization functor.

Lemma 5.9. *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Let Z be an object of $\mathcal{D}$. The following are equivalent*

(1) $Q(Z) = 0$ *in* $S^{-1}\mathcal{D}$,
(2) *there exists* $Z' \in \mathrm{Ob}(\mathcal{D})$ *such that* $0 : Z \to Z'$ *is an element of* S,
(3) *there exists* $Z' \in \mathrm{Ob}(\mathcal{D})$ *such that* $0 : Z' \to Z$ *is an element of* S, *and*
(4) *there exists an object* Z' *and a distinguished triangle* $(X, Y, Z \oplus Z', f, g, h)$ *such that* $f \in S$.

If S is saturated, then these are also equivalent to

(5) *the morphism* $0 \to Z$ *is an element of* S,
(6) *the morphism* $Z \to 0$ *is an element of* S,
(7) *there exists a distinguished triangle* (X, Y, Z, f, g, h) *such that* $f \in S$.

Proof. The equivalence of (1), (2), and (3) is Homology, Lemma 8.3. If (2) holds, then $(Z'[-1], Z'[-1] \oplus Z, Z, (1, 0), (0, 1), 0)$ is a distinguished triangle (see Lemma 4.11) with "$0 \in S$". By rotating we conclude that (4) holds. If $(X, Y, Z \oplus Z', f, g, h)$ is a distinguished triangle with $f \in S$ then $Q(f)$ is an isomorphism hence $Q(Z \oplus Z') = 0$ hence $Q(Z) = 0$. Thus (1) – (4) are all equivalent.

Next, assume that S is saturated. Note that each of (5), (6), (7) implies one of the equivalent conditions (1) – (4). Suppose that $Q(Z) = 0$. Then $0 \to Z$ is a morphism of $\mathcal{D}$ which becomes an isomorphism in $S^{-1}\mathcal{D}$. According to Categories, Lemma 27.21 the fact that S is saturated implies that $0 \to Z$ is in S. Hence (1) $\Rightarrow$ (5). Dually (1) $\Rightarrow$ (6). Finally, if $0 \to Z$ is in S, then the triangle $(0, Z, Z, 0, \mathrm{id}_Z, 0)$ is distinguished by TR1 and TR2 and is a triangle as in (4). □

Lemma 5.10. *Let $\mathcal{D}$ be a triangulated category. Let S be a saturated multiplicative system in $\mathcal{D}$ that is compatible with the triangulated structure. Let (X, Y, Z, f, g, h) be a distinguished triangle in $\mathcal{D}$. Consider the category of morphisms of triangles*

$$\mathcal{I} = \{(s, s', s'') : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h') \mid s, s', s'' \in S\}$$

Then $\mathcal{I}$ is a filtered category and the functors $\mathcal{I} \to X/S$, $\mathcal{I} \to Y/S$, and $\mathcal{I} \to Z/S$ are cofinal.

Proof. We strongly suggest the reader skip the proof of this lemma and instead work it out on a napkin.

The first remark is that using rotation of distinguished triangles (TR2) gives an equivalence of categories between $\mathcal{I}$ and the corresponding category for the distinguished triangle $(Y, Z, X[1], g, h, -f[1])$. Using this we see for example that if we prove the functor $\mathcal{I} \to X/S$ is cofinal, then the same thing is true for the functors $\mathcal{I} \to Y/S$ and $\mathcal{I} \to Z/S$.

Note that if $s : X \to X'$ is a morphism of S, then using MS2 we can find $s' : Y \to Y'$ and $f' : X' \to Y'$ such that $f' \circ s = s' \circ f$, whereupon we can use MS6 to complete this into an object of $\mathcal{I}$. Hence the functor $\mathcal{I} \to X/S$ is surjective on objects. Using rotation as above this implies the same thing is true for the functors $\mathcal{I} \to Y/S$ and $\mathcal{I} \to Z/S$.

Suppose given objects $s_1 : X \to X_1$ and $s_2 : X \to X_2$ in X/S and a morphism $a : X_1 \to X_2$ in X/S. Since S is saturated, we see that $a \in S$, see Categories, Lemma 27.21. By the argument of the previous paragraph we can complete $s_1 : X \to X_1$ to an object $(s_1, s_1', s_1'') : (X, Y, Z, f, g, h) \to (X_1, Y_1, Z_1, f_1, g_1, h_1)$ in $\mathcal{I}$. Then we can repeat and find $(a, b, c) : (X_1, Y_1, Z_1, f_1, g_1, h_1) \to (X_2, Y_2, Z_2, f_2, g_2, h_2)$ with $a, b, c \in S$ completing the given $a : X_1 \to X_2$. But then (a, b, c) is a morphism in $\mathcal{I}$. In this way we conclude that the functor $\mathcal{I} \to X/S$ is also surjective on arrows. Using rotation as above, this implies the same thing is true for the functors $\mathcal{I} \to Y/S$ and $\mathcal{I} \to Z/S$.

The category $\mathcal{I}$ is nonempty as the identity provides an object. This proves the condition (1) of the definition of a filtered category, see Categories, Definition 19.1.

We check condition (2) of Categories, Definition 19.1 for the category $\mathcal{I}$. Suppose given objects $(s_1, s_1', s_1'') : (X, Y, Z, f, g, h) \to (X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(s_2, s_2', s_2'') : (X, Y, Z, f, g, h) \to (X_2, Y_2, Z_2, f_2, g_2, h_2)$ in $\mathcal{I}$. We want to find an object of $\mathcal{I}$ which is the target of an arrow from both $(X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(X_2, Y_2, Z_2, f_2, g_2, h_2)$. By Categories, Remark 27.7 the categories X/S, Y/S, Z/S are filtered. Thus we can find $X \to X_3$ in X/S and morphisms $s : X_2 \to X_3$ and $a : X_1 \to X_3$. By the above we can find a morphism $(s, s', s'') : (X_2, Y_2, Z_2, f_2, g_2, h_2) \to (X_3, Y_3, Z_3, f_3, g_3, h_3)$ with $s', s'' \in S$. After replacing (X_2, Y_2, Z_2) by (X_3, Y_3, Z_3) we may assume that there exists a morphism $a : X_1 \to X_2$ in X/S. Repeating the argument for Y and Z (by rotating as above) we may assume there is a morphism $a : X_1 \to X_2$ in X/S, $b : Y_1 \to Y_2$ in Y/S, and $c : Z_1 \to Z_2$ in Z/S. However, these morphisms do not necessarily give rise to a morphism of distinguished triangles. On the other hand, the necessary diagrams do commute in $S^{-1}\mathcal{D}$. Hence we see (for example) that there exists a morphism $s_2' : Y_2 \to Y_3$ in S such that $s_2' \circ f_2 \circ a = s_2' \circ b \circ f_1$. Another replacement of (X_2, Y_2, Z_2) as above then gets us to the situation where $f_2 \circ a = b \circ f_1$. Rotating and applying the same argument two more times we see that we may assume (a, b, c) is a morphism of triangles. This proves condition (2).

Next we check condition (3) of Categories, Definition 19.1. Suppose $(s_1, s_1', s_1'') : (X, Y, Z) \to (X_1, Y_1, Z_1)$ and $(s_2, s_2', s_2'') : (X, Y, Z) \to (X_2, Y_2, Z_2)$ are objects of $\mathcal{I}$, and suppose $(a, b, c), (a', b', c')$ are two morphisms between them. Since $a \circ s_1 = a' \circ s_1$ there exists a morphism $s_3 : X_2 \to X_3$ such that $s_3 \circ a = s_3 \circ a'$. Using the surjectivity statement we can complete this to a morphism of triangles $(s_3, s_3', s_3'') : (X_2, Y_2, Z_2) \to (X_3, Y_3, Z_3)$ with $s_3, s_3', s_3'' \in S$. Thus $(s_3 \circ s_2, s_3' \circ s_2', s_3'' \circ s_2'') : (X, Y, Z) \to (X_3, Y_3, Z_3)$ is also an object of $\mathcal{I}$ and after composing the maps $(a, b, c), (a', b', c')$ with (s_3, s_3', s_3'') we obtain $a = a'$. By rotating we may do the same to get $b = b'$ and $c = c'$.

Finally, we check that $\mathcal{I} \to X/S$ is cofinal, see Categories, Definition 17.1. The first condition is true as the functor is surjective. Suppose that we have an object $s : X \to X'$ in X/S and two objects $(s_1, s_1', s_1'') : (X, Y, Z, f, g, h) \to (X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(s_2, s_2', s_2'') : (X, Y, Z, f, g, h) \to (X_2, Y_2, Z_2, f_2, g_2, h_2)$ in $\mathcal{I}$ as well as morphisms $t_1 : X' \to X_1$ and $t_2 : X' \to X_2$ in X/S. By property (2) of $\mathcal{I}$ proved above we can find morphisms $(s_3, s_3', s_3'') : (X_1, Y_1, Z_1, f_1, g_1, h_1) \to (X_3, Y_3, Z_3, f_3, g_3, h_3)$ and $(s_4, s_4', s_4'') : (X_2, Y_2, Z_2, f_2, g_2, h_2) \to (X_3, Y_3, Z_3, f_3, g_3, h_3)$ in $\mathcal{I}$. We would be done if the compositions $X' \to X_1 \to X_3$ and $X' \to X_2 \to X_3$ where equal (see displayed equation in Categories, Definition 17.1). If not, then, because X/S is filtered, we can choose a morphism $X_3 \to X_4$ in S such that the compositions $X' \to X_1 \to X_3 \to X_4$ and $X' \to X_2 \to X_3 \to X_4$ are equal. Then we finally complete $X_3 \to X_4$ to a morphism $(X_3, Y_3, Z_3) \to (X_4, Y_4, Z_4)$ in $\mathcal{I}$ and compose with that morphism to see that the result is true. $\square$

6. Quotients of triangulated categories

Given a triangulated category and a triangulated subcategory we can construct another triangulated category by taking the "quotient". The construction uses a localization. This is similar to the quotient of an abelian category by a Serre subcategory, see Homology, Section 10. Before we do the actual construction we briefly discuss kernels of exact functors.

Definition 6.1. Let $\mathcal{D}$ be a pre-triangulated category. We say a full pre-triangulated subcategory $\mathcal{D}'$ of $\mathcal{D}$ is *saturated* if whenever $X \oplus Y$ is isomorphic to an object of $\mathcal{D}'$ then both X and Y are isomorphic to objects of $\mathcal{D}'$.

A saturated triangulated subcategory is sometimes called a *thick triangulated subcategory*. In some references, this is only used for strictly full triangulated subcategories (and sometimes the definition is written such that it implies strictness). There is another notion, that of an *épaisse triangulated subcategory*. The definition is that given a commutative diagram

$$\begin{tikzcd} & S \arrow[dr] & & & \\ X \arrow[ur] \arrow[rr] & & Y \arrow[r] & T \arrow[r] & X[1] \end{tikzcd}$$

where the second line is a distinguished triangle and S and T are isomorphic to objects of $\mathcal{D}'$, then also X and Y are isomorphic to objects of $\mathcal{D}'$. It turns out that this is equivalent to being saturated (this is elementary and can be found in [Ric89]) and the notion of a saturated category is easier to work with.

Lemma 6.2. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let $\mathcal{D}''$ be the full subcategory of $\mathcal{D}$ with objects*

$$\mathrm{Ob}(\mathcal{D}'') = \{X \in \mathrm{Ob}(\mathcal{D}) \mid F(X) = 0\}$$

Then $\mathcal{D}''$ is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then $\mathcal{D}''$ is a triangulated subcategory.

Proof. It is clear that $\mathcal{D}''$ is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $F(X) = F(Y) = 0$, then also $F(Z) = 0$ as $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished. Hence we may apply Lemma 4.16 to see that $\mathcal{D}''$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The final assertion of being saturated follows from $F(X) \oplus F(Y) = 0 \Rightarrow F(X) = F(Y) = 0$. □

Lemma 6.3. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ with objects*

$$\mathrm{Ob}(\mathcal{D}') = \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \textit{ for all } n \in \mathbf{Z}\}$$

Then $\mathcal{D}'$ is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then $\mathcal{D}'$ is a triangulated subcategory.

Proof. It is clear that $\mathcal{D}'$ is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $H(X[n]) = H(Y[n]) = 0$ for all n, then also $H(Z[n]) = 0$ for all n by the long exact sequence (3.5.1). Hence we may apply Lemma 4.16 to see that $\mathcal{D}'$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The assertion of being saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

Lemma 6.4. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let $\mathcal{D}^+_H, \mathcal{D}^-_H, \mathcal{D}^b_H$ be the full subcategory of $\mathcal{D}$ with objects*

$$\begin{aligned} \mathrm{Ob}(\mathcal{D}^+_H) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \textit{ for all } n \ll 0\} \\ \mathrm{Ob}(\mathcal{D}^-_H) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \textit{ for all } n \gg 0\} \\ \mathrm{Ob}(\mathcal{D}^b_H) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \textit{ for all } |n| \gg 0\} \end{aligned}$$

Each of these is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then each is a triangulated subcategory.

Proof. Let us prove this for $\mathcal{D}^+_H$. It is clear that it is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $H(X[n]) = H(Y[n]) = 0$ for all $n \ll 0$, then also $H(Z[n]) = 0$ for all $n \ll 0$ by the long exact sequence (3.5.1). Hence we may apply Lemma 4.16 to see that $\mathcal{D}^+_H$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The assertion of being saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

Definition 6.5. Let $\mathcal{D}$ be a (pre-)triangulated category.

(1) Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor. The *kernel of F* is the strictly full saturated (pre-)triangulated subcategory described in Lemma 6.2.
(2) Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor. The *kernel of H* is the strictly full saturated (pre-)triangulated subcategory described in Lemma 6.3.

These are sometimes denoted $\text{Ker}(F)$ or $\text{Ker}(H)$.

The proof of the following lemma uses TR4.

Lemma 6.6. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{D}' \subset \mathcal{D}$ be a full triangulated subcategory. Set*

$$(6.6.1) \qquad S = \left\{ \begin{matrix} f \in \mathit{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{D}' \end{matrix} \right\}$$

Then S is a multiplicative system compatible with the triangulated structure on $\mathcal{D}$. In this situation the following are equivalent

(1) *S is a saturated multiplicative system,*
(2) *$\mathcal{D}'$ is a saturated triangulated subcategory.*

Proof. To prove the first assertion we have to prove that MS1, MS2, MS3 and MS5, MS6 hold.

Proof of MS1. It is clear that identities are in S because $(X, X, 0, 1, 0, 0)$ is distinguished for every object X of $\mathcal{D}$ and because 0 is an object of $\mathcal{D}'$. Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms contained in S. Choose distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3). By assumption we know that Q_1 and Q_3 are isomorphic to objects of $\mathcal{D}'$. By TR4 we know there exists a distinguished triangle (Q_1, Q_2, Q_3, a, b, c). Since $\mathcal{D}'$ is a triangulated subcategory we conclude that Q_2 is isomorphic to an object of $\mathcal{D}'$. Hence $g \circ f \in S$.

Proof of MS3. Let $a : X \to Y$ be a morphism and let $t : Z \to X$ be an element of S such that $a \circ t = 0$. To prove LMS3 it suffices to find an $s \in S$ such that $s \circ a = 0$, compare with the proof of Lemma 5.4. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 4.2 there exists a morphism $i : Q \to Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, s, k). Here is a picture

$$\begin{tikzcd}
Z \arrow[r, "t"'] & X \arrow[r, "g"'] \arrow[d, "1"] & Q \arrow[r] \arrow[d, "i"] & Z[1] \\
 & X \arrow[r, "a"'] & Y \arrow[d, "s"] & \\
 & & W &
\end{tikzcd}$$

Since $t \in S$ we see that Q is isomorphic to an object of $\mathcal{D}'$. Hence $s \in S$. Finally, $s \circ a = s \circ i \circ g = 0$ as $s \circ i = 0$ by Lemma 4.1. We conclude that LMS3 holds. The proof of RMS3 is dual.

Proof of MS5. Follows as distinguished triangles and $\mathcal{D}'$ are stable under translations

Proof of MS6. Suppose given a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "s"] & Y \arrow[d, "s'"] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

with $s, s' \in S$. By Proposition 4.23 we can extend this to a nine square diagram. As s, s' are elements of S we see that X'', Y'' are isomorphic to objects of $\mathcal{D}'$. Since $\mathcal{D}'$ is a full triangulated subcategory we see that Z'' is also isomorphic to an object of $\mathcal{D}'$. Whence the morphism $Z \to Z'$ is an element of S. This proves MS6.

MS2 is a formal consequence of MS1, MS5, and MS6, see Lemma 5.2. This finishes the proof of the first assertion of the lemma.

Let's assume that S is saturated. (In the following we will use rotation of distinguished triangles without further mention.) Let $X \oplus Y$ be an object isomorphic to an object of $\mathcal{D}'$. Consider the morphism $f : 0 \to X$. The composition $0 \to X \to X \oplus Y$ is an element of S as $(0, X \oplus Y, X \oplus Y, 0, 1, 0)$ is a distinguished triangle. The composition $Y[-1] \to 0 \to X$ is an element of S as $(X, X \oplus Y, Y, (1, 0), (0, 1), 0)$ is a distinguished triangle, see Lemma 4.11. Hence $0 \to X$ is an element of S (as S is saturated). Thus X is isomorphic to an object of $\mathcal{D}'$ as desired.

Finally, assume $\mathcal{D}'$ is a saturated triangulated subcategory. Let

$$W \xrightarrow{h} X \xrightarrow{g} Y \xrightarrow{f} Z$$

be composable morphisms of $\mathcal{D}$ such that $fg, gh \in S$. We will build up a picture of objects as in the diagram below.

$$\begin{tikzcd}
 & & Q_{12} \arrow[dr] & & Q_{23} \arrow[dr] & & \\
 & Q_1 \arrow[ur] \arrow[dl, "+1"'] & & Q_2 \arrow[ll, "+1"'] \arrow[ur] \arrow[dl, "+1"'] & & Q_3 \arrow[ll, "+1"'] \arrow[dl, "+1"'] & \\
W \arrow[rr] & & X \arrow[ul] \arrow[rr] & & Y \arrow[ul] \arrow[rr] & & Z \arrow[ul]
\end{tikzcd}$$

First choose distinguished triangles (W, X, Q_1), (X, Y, Q_2), (Y, Z, Q_3) (W, Y, Q_{12}), and (X, Z, Q_{23}). Denote $s : Q_2 \to Q_1[1]$ the composition $Q_2 \to X[1] \to Q_1[1]$. Denote $t : Q_3 \to Q_2[1]$ the composition $Q_3 \to Y[1] \to Q_2[1]$. By TR4 applied to the composition $W \to X \to Y$ and the composition $X \to Y \to Z$ there exist distinguished triangles (Q_1, Q_{12}, Q_2) and (Q_2, Q_{23}, Q_3) which use the morphisms s and t. The objects Q_{12} and Q_{23} are isomorphic to objects of $\mathcal{D}'$ as $W \to Y$ and $X \to Z$ are assumed in S. Hence also $s[1]t$ is an element of S as S is closed under compositions and shifts. Note that $s[1]t = 0$ as $Y[1] \to Q_2[1] \to X[2]$ is zero, see Lemma 4.1. Hence $Q_3[1] \oplus Q_1[2]$ is isomorphic to an object of $\mathcal{D}'$, see Lemma 4.11. By assumption on $\mathcal{D}'$ we conclude that Q_3 and Q_1 are isomorphic to objects of $\mathcal{D}'$. Looking at the distinguished triangle (Q_1, Q_{12}, Q_2) we conclude that Q_2 is also isomorphic to an object of $\mathcal{D}'$. Looking at the distinguished triangle (X, Y, Q_2) we finally conclude that $g \in S$. (It is also follows that $h, f \in S$, but we don't need this.) $\square$

Definition 6.7. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory. We define the *quotient category* $\mathcal{D}/\mathcal{B}$ by the formula $\mathcal{D}/\mathcal{B} = S^{-1}\mathcal{D}$, where S is the multiplicative system of $\mathcal{D}$ associated to $\mathcal{B}$ via Lemma 6.6. The localization functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is called the *quotient functor* in this case.

Note that the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is an exact functor of triangulated categories, see Proposition 5.6. The universal property of this construction is the following.

Lemma 6.8. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory of $\mathcal{D}$. Let $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ be the quotient functor.*

1. *If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor into an abelian category $\mathcal{A}$ such that $\mathcal{B} \subset \mathrm{Ker}(H)$ then there exists a unique factorization $H' : \mathcal{D}/\mathcal{B} \to \mathcal{A}$ such that $H = H' \circ Q$ and H' is a homological functor too.*
2. *If $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor into a pre-triangulated category $\mathcal{D}'$ such that $\mathcal{B} \subset \mathrm{Ker}(F)$ then there exists a unique factorization $F' : \mathcal{D}/\mathcal{B} \to \mathcal{D}'$ such that $F = F' \circ Q$ and F' is an exact functor too.*

Proof. This lemma follows from Lemma 5.7. Namely, if $f : X \to Y$ is a morphism of $\mathcal{D}$ such that for some distinguished triangle (X, Y, Z, f, g, h) the object Z is isomorphic to an object of $\mathcal{B}$, then $H(f)$, resp. $F(f)$ is an isomorphism under the assumptions of (1), resp. (2). Details omitted. $\square$

The kernel of the quotient functor can be described as follows.

Lemma 6.9. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory. The kernel of the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is the strictly full subcategory of $\mathcal{D}$ whose objects are*

$$\mathrm{Ob}(\mathrm{Ker}(Q)) = \left\{ \begin{matrix} Z \in \mathrm{Ob}(\mathcal{D}) \text{ such that there exists a } Z' \in \mathrm{Ob}(\mathcal{D}) \\ \text{such that } Z \oplus Z' \text{ is isomorphic to an object of } \mathcal{B} \end{matrix} \right\}$$

In other words it is the smallest strictly full saturated triangulated subcategory of $\mathcal{D}$ containing $\mathcal{B}$.

Proof. First note that the kernel is automatically a strictly full triangulated subcategory containing summands of any of its objects, see Lemma 6.2. The description of its objects follows from the definitions and Lemma 5.9 part (4). $\square$

Let $\mathcal{D}$ be a triangulated category. At this point we have constructions which induce order preserving maps between

(1) the partially ordered set of multiplicative systems S in $\mathcal{D}$ compatible with the triangulated structure, and
(2) the partially ordered set of full triangulated subcategories $\mathcal{B} \subset \mathcal{D}$.

Namely, the constructions are given by $S \mapsto \mathcal{B}(S) = \mathrm{Ker}(Q : \mathcal{D} \to S^{-1}\mathcal{D})$ and $\mathcal{B} \mapsto S(\mathcal{B})$ where $S(\mathcal{B})$ is the multiplicative set of (6.6.1), i.e.,

$$S(\mathcal{B}) = \left\{ \begin{matrix} f \in \mathrm{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{B} \end{matrix} \right\}$$

Note that it is not the case that these operations are mutually inverse.

Lemma 6.10. *Let $\mathcal{D}$ be a triangulated category. The operations described above have the following properties*

(1) *$S(\mathcal{B}(S))$ is the "saturation" of S, i.e., it is the smallest saturated multiplicative system in $\mathcal{D}$ containing S, and*
(2) *$\mathcal{B}(S(\mathcal{B}))$ is the "saturation" of $\mathcal{B}$, i.e., it is the smallest strictly full saturated triangulated subcategory of $\mathcal{D}$ containing $\mathcal{B}$.*

In particular, the constructions define mutually inverse maps between the (partially ordered) set of saturated multiplicative systems in $\mathcal{D}$ compatible with the triangulated structure on $\mathcal{D}$ and the (partially ordered) set of strictly full saturated triangulated subcategories of $\mathcal{D}$.

Proof. First, let's start with a full triangulated subcategory $\mathcal{B}$. Then $\mathcal{B}(S(\mathcal{B})) = \mathrm{Ker}(Q : \mathcal{D} \to \mathcal{D}/\mathcal{B})$ and hence (2) is the content of Lemma 6.9.

Next, suppose that S is multiplicative system in $\mathcal{D}$ compatible with the triangulation on $\mathcal{D}$. Then $\mathcal{B}(S) = \mathrm{Ker}(Q : \mathcal{D} \to S^{-1}\mathcal{D})$. Hence (using Lemma 4.9 in the localized category)

$$\begin{aligned} S(\mathcal{B}(S)) & = \left\{ \begin{matrix} f \in \mathrm{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished} \\ \text{triangle } (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Q(Z) = 0 \end{matrix} \right\} \\ & = \{ f \in \mathrm{Arrows}(\mathcal{D}) \mid Q(f) \text{ is an isomorphism} \} \\ & = \hat{S} = S' \end{aligned}$$

in the notation of Categories, Lemma 27.21. The final statement of that lemma finishes the proof. □

Lemma 6.11. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{A}$, see Definition 3.5. The subcategory $\mathrm{Ker}(H)$ of $\mathcal{D}$ is a strictly full saturated triangulated subcategory of $\mathcal{D}$ whose corresponding saturated multiplicative system (see Lemma 6.10) is the set*

$$S = \{ f \in \mathit{Arrows}(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z} \}.$$

The functor H factors through the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathrm{Ker}(H)$.

Proof. The category $\mathrm{Ker}(H)$ is a strictly full saturated triangulated subcategory of $\mathcal{D}$ by Lemma 6.3. The set S is a saturated multiplicative system compatible with the triangulated structure by Lemma 5.5. Recall that the multiplicative system corresponding to $\mathrm{Ker}(H)$ is the set

$$\left\{ \begin{matrix} f \in \mathrm{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ with } H^i(Z) = 0 \text{ for all } i \end{matrix} \right\}$$

By the long exact cohomology sequence, see (3.5.1), it is clear that f is an element of this set if and only if f is an element of S. Finally, the factorization of H through Q is a consequence of Lemma 6.8. $\square$

7. Adjoints for exact functors

Results on adjoint functors between triangulated categories.

Lemma 7.1. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories. If F admits a right adjoint $G : \mathcal{D}' \to \mathcal{D}$, then G is also an exact functor.*

Proof. Let X be an object of $\mathcal{D}$ and A an object of $\mathcal{D}'$. Since F is an exact functor we see that

$$\begin{aligned} \mathrm{Mor}_{\mathcal{D}}(X, G(A[1])) &= \mathrm{Mor}_{\mathcal{D}'}(F(X), A[1]) \\ &= \mathrm{Mor}_{\mathcal{D}'}(F(X)[-1], A) \\ &= \mathrm{Mor}_{\mathcal{D}'}(F(X[-1]), A) \\ &= \mathrm{Mor}_{\mathcal{D}}(X[-1], G(A)) \\ &= \mathrm{Mor}_{\mathcal{D}}(X, G(A)[1]) \end{aligned}$$

By Yoneda's lemma (Categories, Lemma 3.5) we obtain a canonical isomorphism $G(A)[1] = G(A[1])$. Let $A \to B \to C \to A[1]$ be a distinguished triangle in $\mathcal{D}'$. Choose a distinguished triangle

$$G(A) \to G(B) \to X \to G(A)[1]$$

in $\mathcal{D}$. Then $F(G(A)) \to F(G(B)) \to F(X) \to F(G(A))[1]$ is a distinguished triangle in $\mathcal{D}'$. By TR3 we can choose a morphism of distinguished triangles

$$\begin{tikzcd} F(G(A)) \arrow[r] \arrow[d] & F(G(B)) \arrow[r] \arrow[d] & F(X) \arrow[r] \arrow[d] & F(G(A))[1] \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1] \end{tikzcd}$$

Since G is the adjoint the new morphism determines a morphism $X \to G(C)$ such that the diagram

$$\begin{tikzcd} G(A) \arrow[r] \arrow[d] & G(B) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & G(A)[1] \arrow[d] \\ G(A) \arrow[r] & G(B) \arrow[r] & G(C) \arrow[r] & G(A)[1] \end{tikzcd}$$

commutes. Applying the homological functor $\mathrm{Hom}_{\mathcal{D}'}(W, -)$ for an object W of $\mathcal{D}'$ we deduce from the 5 lemma that

$$\mathrm{Hom}_{\mathcal{D}'}(W, X) \to \mathrm{Hom}_{\mathcal{D}'}(W, G(C))$$

is a bijection and using the Yoneda lemma once more we conclude that $X \to G(C)$ is an isomorphism. Hence we conclude that $G(A) \to G(B) \to G(C) \to G(A)[1]$ is a distinguished triangle which is what we wanted to show. $\square$

Lemma 7.2. *Let $\mathcal{D}$, $\mathcal{D}'$ be triangulated categories. Let $F : \mathcal{D} \to \mathcal{D}'$ and $G : \mathcal{D}' \to \mathcal{D}$ be functors. Assume that*

(1) *F and G are exact functors,*
(2) *F is fully faithful,*

(3) *G is a right adjoint to F, and*
(4) *the kernel of G is zero.*

Then F is an equivalence of categories.

Proof. Since F is fully faithful the adjunction map $\mathrm{id} \to G \circ F$ is an isomorphism (Categories, Lemma 24.4). Let X be an object of $\mathcal{D}'$. Choose a distinguished triangle

$$F(G(X)) \to X \to Y \to F(G(X))[1]$$

in $\mathcal{D}'$. Applying G and using that $G(F(G(X))) = G(X)$ we find a distinguished triangle

$$G(X) \to G(X) \to G(Y) \to G(X)[1]$$

Hence $G(Y) = 0$. Thus $Y = 0$. Thus $F(G(X)) \to X$ is an isomorphism. $\square$

8. The homotopy category

Let $\mathcal{A}$ be an additive category. The homotopy category $K(\mathcal{A})$ of $\mathcal{A}$ is the category of complexes of $\mathcal{A}$ with morphisms given by morphisms of complexes up to homotopy. Here is the formal definition.

Definition 8.1. Let $\mathcal{A}$ be an additive category.

(1) We set $\mathrm{Comp}(\mathcal{A}) = \mathrm{CoCh}(\mathcal{A})$ be the *category of (cochain) complexes*.
(2) A complex $K^\bullet$ is said to be *bounded below* if $K^n = 0$ for all $n \ll 0$.
(3) A complex $K^\bullet$ is said to be *bounded above* if $K^n = 0$ for all $n \gg 0$.
(4) A complex $K^\bullet$ is said to be *bounded* if $K^n = 0$ for all $|n| \gg 0$.
(5) We let $\mathrm{Comp}^+(\mathcal{A})$, $\mathrm{Comp}^-(\mathcal{A})$, resp. $\mathrm{Comp}^b(\mathcal{A})$ be the full subcategory of $\mathrm{Comp}(\mathcal{A})$ whose objects are the complexes which are bounded below, bounded above, resp. bounded.
(6) We let $K(\mathcal{A})$ be the category with the same objects as $\mathrm{Comp}(\mathcal{A})$ but as morphisms homotopy classes of maps of complexes (see Homology, Lemma 13.7).
(7) We let $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ be the full subcategory of $K(\mathcal{A})$ whose objects are bounded below, bounded above, resp. bounded complexes of $\mathcal{A}$.

It will turn out that the categories $K(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. To prove this we first develop some machinery related to cones and split exact sequences.

9. Cones and termwise split sequences

Let $\mathcal{A}$ be an additive category, and let $K(\mathcal{A})$ denote the category of complexes of $\mathcal{A}$ with morphisms given by morphisms of complexes up to homotopy. Note that the shift functors $[n]$ on complexes, see Homology, Definition 14.7, give rise to functors $[n] : K(\mathcal{A}) \to K(\mathcal{A})$ such that $[n] \circ [m] = [n+m]$ and $[0] = \mathrm{id}$.

Definition 9.1. Let $\mathcal{A}$ be an additive category. Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. The *cone* of f is the complex $C(f)^\bullet$ given by $C(f)^n = L^n \oplus K^{n+1}$ and differential

$$d^n_{C(f)} = \begin{pmatrix} d^n_L & f^{n+1} \\ 0 & -d^{n+1}_K \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : L^\bullet \to C(f)^\bullet$ and $p : C(f)^\bullet \to K^\bullet[1]$ induced by the obvious maps $L^n \to C(f)^n \to K^{n+1}$.

In other words $(K, L, C(f), f, i, p)$ forms a triangle:

$$K^\bullet \to L^\bullet \to C(f)^\bullet \to K^\bullet[1]$$

The formation of this triangle is functorial in the following sense.

Lemma 9.2. *Suppose that*

$$\begin{tikzcd} K_1^\bullet \arrow[r, "f_1"'] \arrow[d, "a"'] & L_1^\bullet \arrow[d, "b"] \\ K_2^\bullet \arrow[r, "f_2"] & L_2^\bullet \end{tikzcd}$$

is a diagram of morphisms of complexes which is commutative up to homotopy. Then there exists a morphism $c : C(f_1)^\bullet \to C(f_2)^\bullet$ which gives rise to a morphism of triangles $(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \to (K_2^\bullet, L_2^\bullet, C(f_2)^\bullet, f_2, i_2, p_2)$ of $K(\mathcal{A})$.

Proof. Let $h^n : K_1^n \to L_2^{n-1}$ be a family of morphisms such that $b \circ f_1 - f_2 \circ a = d \circ h + h \circ d$. Define c^n by the matrix

$$c^n = \begin{pmatrix} b^n & h^{n+1} \\ 0 & a^{n+1} \end{pmatrix} : L_1^n \oplus K_1^{n+1} \to L_2^n \oplus K_2^{n+1}$$

A matrix computation show that c is a morphism of complexes. It is trivial that $c \circ i_1 = i_2 \circ b$, and it is trivial also to check that $p_2 \circ c = a \circ p_1$. $\square$

Note that the morphism $c : C(f_1)^\bullet \to C(f_2)^\bullet$ constructed in the proof of Lemma 9.2 in general depends on the chosen homotopy h between $f_2 \circ a$ and $b \circ f_1$.

Lemma 9.3. *Suppose that $f : K^\bullet \to L^\bullet$ and $g : L^\bullet \to M^\bullet$ are morphisms of complexes such that $g \circ f$ is homotopic to zero. Then*

(1) *g factors through a morphism $C(f)^\bullet \to M^\bullet$, and*
(2) *f factors through a morphism $K^\bullet \to C(g)^\bullet[-1]$.*

Proof. The assumptions say that the diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "f"'] \arrow[d] & L^\bullet \arrow[d, "g"] \\ 0 \arrow[r] & M^\bullet \end{tikzcd}$$

commutes up to homotopy. Since the cone on $0 \to M^\bullet$ is $M^\bullet$ the map $C(f)^\bullet \to C(0 \to M^\bullet) = M^\bullet$ of Lemma 9.2 is the map in (1). The cone on $K^\bullet \to 0$ is $K^\bullet[1]$ and applying Lemma 9.2 gives a map $K^\bullet[1] \to C(g)^\bullet$. Applying $[-1]$ we obtain the map in (2). $\square$

Note that the morphisms $C(f)^\bullet \to M^\bullet$ and $K^\bullet \to C(g)^\bullet[-1]$ constructed in the proof of Lemma 9.3 in general depend on the chosen homotopy.

Definition 9.4. Let $\mathcal{A}$ be an additive category. A *termwise split injection* $\alpha : A^\bullet \to B^\bullet$ is a morphism of complexes such that each $A^n \to B^n$ is isomorphic to the inclusion of a direct summand. A *termwise split surjection* $\beta : B^\bullet \to C^\bullet$ is a morphism of complexes such that each $B^n \to C^n$ is isomorphic to the projection onto a direct summand.

Lemma 9.5. *Let $\mathcal{A}$ be an additive category. Let*

$$\begin{tikzcd} A^\bullet \arrow[r, "f"'] \arrow[d, "a"'] & B^\bullet \arrow[d, "b"] \\ C^\bullet \arrow[r, "g"] & D^\bullet \end{tikzcd}$$

be a diagram of morphisms of complexes commuting up to homotopy. If f is a termwise split injection, then b is homotopic to a morphism which makes the diagram commute. If g is a termwise split surjection, then a is homotopic to a morphism which makes the diagram commute.

Proof. Let $h^n : A^n \to D^{n-1}$ be a collection of morphisms such that $bf - ga = dh + hd$. Suppose that $\pi^n : B^n \to A^n$ are morphisms splitting the morphisms f^n. Take $b' = b - dh\pi - h\pi d$. Suppose $s^n : D^n \to C^n$ are morphisms splitting the morphisms $g^n : C^n \to D^n$. Take $a' = a + dsh + shd$. Computations omitted. □

The following lemma can be used to replace a morphism of complexes by a morphism where in each degree the map is the injection of a direct summand.

Lemma 9.6. *Let $\mathcal{A}$ be an additive category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. There exists a factorization*

$$\begin{tikzcd} K^\bullet \arrow[r, "\tilde{\alpha}"] \arrow[rr, bend right, "\alpha"'] & \tilde{L}^\bullet \arrow[r, "\pi"] & L^\bullet \end{tikzcd}$$

such that

1. (1) *$\tilde{\alpha}$ is a termwise split injection (see Definition 9.4),*
2. (2) *there is a map of complexes $s : L^\bullet \to \tilde{L}^\bullet$ such that $\pi \circ s = id_{L^\bullet}$ and such that $s \circ \pi$ is homotopic to $id_{\tilde{L}^\bullet}$.*

Moreover, if both $K^\bullet$ and $L^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so is $\tilde{L}^\bullet$.

Proof. We set

$$\tilde{L}^n = L^n \oplus K^n \oplus K^{n+1}$$

and we define

$$d^n_{\tilde{L}} = \begin{pmatrix} d^n_L & 0 & 0 \\ 0 & d^n_K & \mathrm{id}_{K^{n+1}} \\ 0 & 0 & -d^{n+1}_K \end{pmatrix}$$

In other words, $\tilde{L}^\bullet = L^\bullet \oplus C(1_{K^\bullet})$. Moreover, we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha \\ \mathrm{id}_{K^n} \\ 0 \end{pmatrix}$$

which is clearly a split injection. It is also clear that it defines a morphism of complexes. We define

$$\pi = \begin{pmatrix} \mathrm{id}_{L^n} & 0 & 0 \end{pmatrix}$$

so that clearly $\pi \circ \tilde{\alpha} = \alpha$. We set

$$s = \begin{pmatrix} \mathrm{id}_{L^n} \\ 0 \\ 0 \end{pmatrix}$$

so that $\pi \circ s = \mathrm{id}_{L^\bullet}$. Finally, let $h^n : \tilde{L}^n \to \tilde{L}^{n-1}$ be the map which maps the summand K^n of $\tilde{L}^n$ via the identity morphism to the summand K^n of $\tilde{L}^{n-1}$. Then it is a trivial matter (see computations in remark below) to prove that

$$\mathrm{id}_{\tilde{L}^\bullet} - s \circ \pi = d \circ h + h \circ d$$

which finishes the proof of the lemma. □

Remark 9.7. To see the last displayed equality in the proof above we can argue with elements as follows. We have $s\pi(l, k, k^+) = (l, 0, 0)$. Hence the morphism of the left hand side maps (l, k, k^+) to $(0, k, k^+)$. On the other hand $h(l, k, k^+) = (0, 0, k)$ and $d(l, k, k^+) = (dl, dk + k^+, -dk^+)$. Hence $(dh + hd)(l, k, k^+) = d(0, 0, k) + h(dl, dk + k^+, -dk^+) = (0, k, -dk) + (0, 0, dk + k^+) = (0, k, k^+)$ as desired.

Lemma 9.8. *Let $\mathcal{A}$ be an additive category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. There exists a factorization*

$$\begin{tikzcd} K^\bullet \arrow[r, "i"] \arrow[rr, bend right, "\alpha"'] & \tilde{K}^\bullet \arrow[r, "\tilde{\alpha}"] & L^\bullet \end{tikzcd}$$

such that

(1) *$\tilde{\alpha}$ is a termwise split surjection (see Definition 9.4),*
(2) *there is a map of complexes $s : \tilde{K}^\bullet \to K^\bullet$ such that $s \circ i = id_{K^\bullet}$ and such that $i \circ s$ is homotopic to $id_{\tilde{K}^\bullet}$.*

Moreover, if both $K^\bullet$ and $L^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so is $\tilde{K}^\bullet$.

Proof. Dual to Lemma 9.6. Take

$$\tilde{K}^n = K^n \oplus L^{n-1} \oplus L^n$$

and we define

$$d^n_{\tilde{K}} = \begin{pmatrix} d^n_K & 0 & 0 \\ 0 & -d^{n-1}_L & \mathrm{id}_{L^n} \\ 0 & 0 & d^n_L \end{pmatrix}$$

in other words $\tilde{K}^\bullet = K^\bullet \oplus C(1_{L^\bullet[-1]})$. Moreover, we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha & 0 & \mathrm{id}_{L^n} \end{pmatrix}$$

which is clearly a split surjection. It is also clear that it defines a morphism of complexes. We define

$$i = \begin{pmatrix} \mathrm{id}_{K^n} \\ 0 \\ 0 \end{pmatrix}$$

so that clearly $\tilde{\alpha} \circ i = \alpha$. We set

$$s = \begin{pmatrix} \mathrm{id}_{K^n} & 0 & 0 \end{pmatrix}$$

so that $s \circ i = \mathrm{id}_{K^\bullet}$. Finally, let $h^n : \tilde{K}^n \to \tilde{K}^{n-1}$ be the map which maps the summand L^{n-1} of $\tilde{K}^n$ via the identity morphism to the summand L^{n-1} of $\tilde{K}^{n-1}$. Then it is a trivial matter to prove that

$$\mathrm{id}_{\tilde{K}^\bullet} - i \circ s = d \circ h + h \circ d$$

which finishes the proof of the lemma. □

Definition 9.9. Let $\mathcal{A}$ be an additive category. A *termwise split exact sequence of complexes of* $\mathcal{A}$ is a complex of complexes

$$0 \to A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \to 0$$

together with given direct sum decompositions $B^n = A^n \oplus C^n$ compatible with α^n and β^n. We often write $s^n : C^n \to B^n$ and $\pi^n : B^n \to A^n$ for the maps induced by the direct sum decompositions. According to Homology, Lemma 14.10 we get an associated morphism of complexes

$$\delta : C^\bullet \longrightarrow A^\bullet[1]$$

which in degree n is the map $\pi^{n+1} \circ d_B^n \circ s^n$. In other words $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ forms a triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

This will be the *triangle associated to the termwise split sequence of complexes.*

Lemma 9.10. *Let $\mathcal{A}$ be an additive category. Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be termwise split exact sequences as in Definition 9.9. Let $(\pi')^n$, $(s')^n$ be a second collection of splittings. Denote $\delta' : C^\bullet \longrightarrow A^\bullet[1]$ the morphism associated to this second set of splittings. Then*

$$(1, 1, 1) : (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta) \longrightarrow (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta')$$

is an isomorphism of triangles in $K(\mathcal{A})$.

Proof. The statement simply means that δ and δ' are homotopic maps of complexes. This is Homology, Lemma 14.12. $\square$

Remark 9.11. Let $\mathcal{A}$ be an additive category. Let $0 \to A_i^\bullet \to B_i^\bullet \to C_i^\bullet \to 0$, $i = 1, 2$ be termwise split exact sequences. Suppose that $a : A_1^\bullet \to A_2^\bullet$, $b : B_1^\bullet \to B_2^\bullet$, and $c : C_1^\bullet \to C_2^\bullet$ are morphisms of complexes such that

$$\begin{tikzcd} A_1^\bullet \arrow[r] \arrow[d, "a"'] & B_1^\bullet \arrow[r] \arrow[d, "b"'] & C_1^\bullet \arrow[d, "c"'] \\ A_2^\bullet \arrow[r] & B_2^\bullet \arrow[r] & C_2^\bullet \end{tikzcd}$$

commutes in $K(\mathcal{A})$. In general, there does **not** exist a morphism $b' : B_1^\bullet \to B_2^\bullet$ which is homotopic to b such that the diagram above commutes in the category of complexes. Namely, consider Examples, Equation (63.0.1). If we could replace the middle map there by a homotopic one such that the diagram commutes, then we would have additivity of traces which we do not.

Lemma 9.12. *Let $\mathcal{A}$ be an additive category. Let $0 \to A_i^\bullet \to B_i^\bullet \to C_i^\bullet \to 0$, $i = 1, 2, 3$ be termwise split exact sequences of complexes. Let $b : B_1^\bullet \to B_2^\bullet$ and $b' : B_2^\bullet \to B_3^\bullet$ be morphisms of complexes such that*

$$\begin{tikzcd} A_1^\bullet \arrow[r] \arrow[d, "0"'] & B_1^\bullet \arrow[r] \arrow[d, "b"'] & C_1^\bullet \arrow[d, "0"'] \\ A_2^\bullet \arrow[r] & B_2^\bullet \arrow[r] & C_2^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_2^\bullet \arrow[r] \arrow[d, "0"] & B_2^\bullet \arrow[r] \arrow[d, "b'"] & C_2^\bullet \arrow[d, "0"] \\ A_3^\bullet \arrow[r] & B_3^\bullet \arrow[r] & C_3^\bullet \end{tikzcd}$$

commute in $K(\mathcal{A})$. Then $b' \circ b = 0$ in $K(\mathcal{A})$.

Proof. By Lemma 9.5 we can replace b and b' by homotopic maps such that the right square of the left diagram commutes and the left square of the right diagram commutes. In other words, we have $\mathrm{Im}(b^n) \subset \mathrm{Im}(A_2^n \to B_2^n)$ and $\mathrm{Ker}((b')^n) \supset \mathrm{Im}(A_2^n \to B_2^n)$. Then $b' \circ b = 0$ as a map of complexes. $\square$

Lemma 9.13. *Let $\mathcal{A}$ be an additive category. Let $f_1 : K_1^\bullet \to L_1^\bullet$ and $f_2 : K_2^\bullet \to L_2^\bullet$ be morphisms of complexes. Let*

$$(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \longrightarrow (K_2^\bullet, L_2^\bullet, C(f_2)^\bullet, f_2, i_2, p_2)$$

be any morphism of triangles of $K(\mathcal{A})$. If a and b are homotopy equivalences then so is c.

Proof. Let $a^{-1} : K_2^\bullet \to K_1^\bullet$ be a morphism of complexes which is inverse to a in $K(\mathcal{A})$. Let $b^{-1} : L_2^\bullet \to L_1^\bullet$ be a morphism of complexes which is inverse to b in $K(\mathcal{A})$. Let $c' : C(f_2)^\bullet \to C(f_1)^\bullet$ be the morphism from Lemma 9.2 applied to $f_1 \circ a^{-1} = b^{-1} \circ f_2$. If we can show that $c \circ c'$ and $c' \circ c$ are isomorphisms in $K(\mathcal{A})$ then we win. Hence it suffices to prove the following: Given a morphism of triangles $(1, 1, c) : (K^\bullet, L^\bullet, C(f)^\bullet, f, i, p)$ in $K(\mathcal{A})$ the morphism c is an isomorphism in $K(\mathcal{A})$. By assumption the two squares in the diagram

$$\begin{tikzcd}
L^\bullet \arrow[r] \arrow[d, "1"'] & C(f)^\bullet \arrow[r] \arrow[d, "c"'] & K^\bullet[1] \arrow[d, "1"'] \\
L^\bullet \arrow[r] & C(f)^\bullet \arrow[r] & K^\bullet[1]
\end{tikzcd}$$

commute up to homotopy. By construction of $C(f)^\bullet$ the rows form termwise split sequences of complexes. Thus we see that $(c-1)^2 = 0$ in $K(\mathcal{A})$ by Lemma 9.12. Hence c is an isomorphism in $K(\mathcal{A})$ with inverse $2 - c$. $\square$

Hence if a and b are homotopy equivalences then the resulting morphism of triangles is an isomorphism of triangles in $K(\mathcal{A})$. It turns out that the collection of triangles of $K(\mathcal{A})$ given by cones and the collection of triangles of $K(\mathcal{A})$ given by termwise split sequences of complexes are the same up to isomorphisms, at least up to sign!

Lemma 9.14. *Let $\mathcal{A}$ be an additive category.*

(1) *Given a termwise split sequence of complexes $(\alpha : A^\bullet \to B^\bullet, \beta : B^\bullet \to C^\bullet, s^n, \pi^n)$ there exists a homotopy equivalence $C(\alpha)^\bullet \to C^\bullet$ such that the diagram*

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C(\alpha)^\bullet \arrow[r, "-p"'] \arrow[d] & A^\bullet[1] \arrow[d] \\
A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r, "\delta"] & A^\bullet[1]
\end{tikzcd}$$

defines an isomorphism of triangles in $K(\mathcal{A})$.

(2) *Given a morphism of complexes $f : K^\bullet \to L^\bullet$ there exists an isomorphism of triangles*

$$\begin{tikzcd}
K^\bullet \arrow[r] \arrow[d] & \tilde{L}^\bullet \arrow[r] \arrow[d] & M^\bullet \arrow[r, "\delta"'] \arrow[d] & K^\bullet[1] \arrow[d] \\
K^\bullet \arrow[r] & L^\bullet \arrow[r] & C(f)^\bullet \arrow[r, "-p"] & K^\bullet[1]
\end{tikzcd}$$

where the upper triangle is the triangle associated to a termwise split exact sequence $K^\bullet \to \tilde{L}^\bullet \to M^\bullet$.

Proof. Proof of (1). We have $C(\alpha)^n = B^n \oplus A^{n+1}$ and we simply define $C(\alpha)^n \to C^n$ via the projection onto B^n followed by β^n. This defines a morphism of complexes because the compositions $A^{n+1} \to B^{n+1} \to C^{n+1}$ are zero. To get a homotopy inverse we take $C^\bullet \to C(\alpha)^\bullet$ given by $(s^n, -\delta^n)$ in degree n. This is a morphism of complexes because the morphism δ^n can be characterized as the unique morphism $C^n \to A^{n+1}$ such that $d \circ s^n - s^{n+1} \circ d = \alpha \circ \delta^n$, see proof of Homology, Lemma 14.10. The composition $C^\bullet \to C(\alpha)^\bullet \to C^\bullet$ is the identity. The composition $C(\alpha)^\bullet \to C^\bullet \to C(\alpha)^\bullet$ is equal to the morphism

$$\begin{pmatrix} s^n \circ \beta^n & 0 \\ -\delta^n \circ \beta^n & 0 \end{pmatrix}$$

To see that this is homotopic to the identity map use the homotopy $h^n : C(\alpha)^n \to C(\alpha)^{n-1}$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} : C(\alpha)^n = B^n \oplus A^{n+1} \to B^{n-1} \oplus A^n = C(\alpha)^{n-1}$$

It is trivial to verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s^n \\ -\delta^n \end{pmatrix} \begin{pmatrix} \beta^n & 0 \end{pmatrix} = \begin{pmatrix} d & \alpha^n \\ 0 & -d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi^{n+1} & 0 \end{pmatrix} \begin{pmatrix} d & \alpha^{n+1} \\ 0 & -d \end{pmatrix}$$

To finish the proof of (1) we have to show that the morphisms $-p : C(\alpha)^\bullet \to A^\bullet[1]$ (see Definition 9.1) and $C(\alpha)^\bullet \to C^\bullet \to A^\bullet[1]$ agree up to homotopy. This is clear from the above. Namely, we can use the homotopy inverse $(s, -\delta) : C^\bullet \to C(\alpha)^\bullet$ and check instead that the two maps $C^\bullet \to A^\bullet[1]$ agree. And note that $p \circ (s, -\delta) = -\delta$ as desired.

Proof of (2). We let $\tilde{f} : K^\bullet \to \tilde{L}^\bullet$, $s : L^\bullet \to \tilde{L}^\bullet$ and $\pi : \tilde{L}^\bullet \to L^\bullet$ be as in Lemma 9.6. By Lemmas 9.2 and 9.13 the triangles $(K^\bullet, L^\bullet, C(f), i, p)$ and $(K^\bullet, \tilde{L}^\bullet, C(\tilde{f}), \tilde{i}, \tilde{p})$ are isomorphic. Note that we can compose isomorphisms of triangles. Thus we may replace $L^\bullet$ by $\tilde{L}^\bullet$ and f by $\tilde{f}$. In other words we may assume that f is a termwise split injection. In this case the result follows from part (1). $\square$

Lemma 9.15. *Let $\mathcal{A}$ be an additive category. Let $A_1^\bullet \to A_2^\bullet \to \ldots \to A_n^\bullet$ be a sequence of composable morphisms of complexes. There exists a commutative diagram*

$$\begin{tikzcd} A_1^\bullet \arrow[r] & A_2^\bullet \arrow[r] & \ldots \arrow[r] & A_n^\bullet \\ B_1^\bullet \arrow[r] \arrow[u] & B_2^\bullet \arrow[r] \arrow[u] & \ldots \arrow[r] & B_n^\bullet \arrow[u] \end{tikzcd}$$

such that each morphism $B_i^\bullet \to B_{i+1}^\bullet$ is a split injection and each $B_i^\bullet \to A_i^\bullet$ is a homotopy equivalence. Moreover, if all $A_i^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so are the $B_i^\bullet$.

Proof. The case $n = 1$ is without content. Lemma 9.6 is the case $n = 2$. Suppose we have constructed the diagram except for $B_n^\bullet$. Apply Lemma 9.6 to the composition $B_{n-1}^\bullet \to A_{n-1}^\bullet \to A_n^\bullet$. The result is a factorization $B_{n-1}^\bullet \to B_n^\bullet \to A_n^\bullet$ as desired. $\square$

Lemma 9.16. *Let $\mathcal{A}$ be an additive category. Let $(\alpha : A^\bullet \to B^\bullet, \beta : B^\bullet \to C^\bullet, s^n, \pi^n)$ be a termwise split sequence of complexes. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the associated triangle. Then the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to the triangle $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$.*

Proof. We write $B^n = A^n \oplus C^n$ and we identify α^n and β^n with the natural inclusion and projection maps. By construction of δ we have

$$d_B^n = \begin{pmatrix} d_A^n & \delta^n \\ 0 & d_C^n \end{pmatrix}$$

On the other hand the cone of $\delta[-1] : C^\bullet[-1] \to A^\bullet$ is given as $C(\delta[-1])^n = A^n \oplus C^n$ with differential identical with the matrix above! Whence the lemma. $\square$

Lemma 9.17. *Let $\mathcal{A}$ be an additive category. Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes. The triangle $(L^\bullet, C(f)^\bullet, K^\bullet[1], i, p, f[1])$ is the triangle associated to the termwise split sequence*

$$0 \to L^\bullet \to C(f)^\bullet \to K^\bullet[1] \to 0$$

coming from the definition of the cone of f.

Proof. Immediate from the definitions. $\square$

10. Distinguished triangles in the homotopy category

Since we want our boundary maps in long exact sequences of cohomology to be given by the maps in the snake lemma without signs we define distinguished triangles in the homotopy category as follows.

Definition 10.1. Let $\mathcal{A}$ be an additive category. A triangle (X, Y, Z, f, g, h) of $K(\mathcal{A})$ is called a *distinguished triangle of* $K(\mathcal{A})$ if it is isomorphic to the triangle associated to a termwise split exact sequence of complexes, see Definition 9.9. Same definition for $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$.

Note that according to Lemma 9.14 a triangle of the form $(K^\bullet, L^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle. This does indeed lead to a triangulated category, see Proposition 10.3. Before we can prove the proposition we need one more lemma in order to be able to prove TR4.

Lemma 10.2. *Let $\mathcal{A}$ be an additive category. Suppose that $\alpha : A^\bullet \to B^\bullet$ and $\beta : B^\bullet \to C^\bullet$ are split injections of complexes. Then there exist distinguished triangles $(A^\bullet, B^\bullet, Q_1^\bullet, \alpha, p_1, d_1)$, $(A^\bullet, C^\bullet, Q_2^\bullet, \beta \circ \alpha, p_2, d_2)$ and $(B^\bullet, C^\bullet, Q_3^\bullet, \beta, p_3, d_3)$ for which TR4 holds.*

Proof. Say $\pi_1^n : B^n \to A^n$, and $\pi_3^n : C^n \to B^n$ are the splittings. Then also $A^\bullet \to C^\bullet$ is a split injection with splittings $\pi_2^n = \pi_1^n \circ \pi_3^n$. Let us write $Q_1^\bullet$, $Q_2^\bullet$ and $Q_3^\bullet$ for the "quotient" complexes. In other words, $Q_1^n = \mathrm{Ker}(\pi_1^n)$, $Q_3^n = \mathrm{Ker}(\pi_3^n)$ and $Q_2^n = \mathrm{Ker}(\pi_2^n)$. Note that the kernels exist. Then $B^n = A^n \oplus Q_1^n$ and $C_n = B^n \oplus Q_3^n$, where we think of A^n as a subobject of B^n and so on. This implies $C^n = A^n \oplus Q_1^n \oplus Q_3^n$. Note that $\pi_2^n = \pi_1^n \circ \pi_3^n$ is zero on both Q_1^n and Q_3^n. Hence

$Q_2^n = Q_1^n \oplus Q_3^n$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A^\bullet & \to & B^\bullet & \to & Q_1^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^\bullet & \to & C^\bullet & \to & Q_2^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B^\bullet & \to & C^\bullet & \to & Q_3^\bullet & \to & 0 \end{array}$$

The rows of this diagram are termwise split exact sequences, and hence determine distinguished triangles by definition. Moreover downward arrows in the diagram above are compatible with the chosen splittings and hence define morphisms of triangles

$$(A^\bullet \to B^\bullet \to Q_1^\bullet \to A^\bullet[1]) \longrightarrow (A^\bullet \to C^\bullet \to Q_2^\bullet \to A^\bullet[1])$$

and

$$(A^\bullet \to C^\bullet \to Q_2^\bullet \to A^\bullet[1]) \longrightarrow (B^\bullet \to C^\bullet \to Q_3^\bullet \to B^\bullet[1]).$$

Note that the splittings $Q_3^n \to C^n$ of the bottom split sequence in the diagram provides a splitting for the split sequence $0 \to Q_1^\bullet \to Q_2^\bullet \to Q_3^\bullet \to 0$ upon composing with $C^n \to Q_2^n$. It follows easily from this that the morphism $\delta : Q_3^\bullet \to Q_1^\bullet[1]$ in the corresponding distinguished triangle

$$(Q_1^\bullet \to Q_2^\bullet \to Q_3^\bullet \to Q_1^\bullet[1])$$

is equal to the composition $Q_3^\bullet \to B^\bullet[1] \to Q_1^\bullet[1]$. Hence we get a structure as in the conclusion of axiom TR4. $\square$

Proposition 10.3. *Let $\mathcal{A}$ be an additive category. The category $K(\mathcal{A})$ of complexes up to homotopy with its natural translation functors and distinguished triangles as defined above is a triangulated category.*

Proof. Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Also, any triangle $(A^\bullet, A^\bullet, 0, 1, 0, 0)$ is distinguished since $0 \to A^\bullet \to A^\bullet \to 0 \to 0$ is a termwise split sequence of complexes. Finally, given any morphism of complexes $f : K^\bullet \to L^\bullet$ the triangle $(K, L, C(f), f, i, -p)$ is distinguished by Lemma 9.14.

Proof of TR2. Let (X, Y, Z, f, g, h) be a triangle. Assume $(Y, Z, X[1], g, h, -f[1])$ is distinguished. Then there exists a termwise split sequence of complexes $A^\bullet \to B^\bullet \to C^\bullet$ such that the associated triangle $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is isomorphic to $(Y, Z, X[1], g, h, -f[1])$. Rotating back we see that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, B^\bullet, -\delta[-1], \alpha, \beta)$. It follows from Lemma 9.16 that the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$. Precomposing the previous isomorphism of triangles with -1 on Y it follows that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, -p)$. Hence it is distinguished by Lemma 9.14. On the other hand, suppose that (X, Y, Z, f, g, h) is distinguished. By Lemma 9.14 this means that it is isomorphic to a triangle of the form $(K^\bullet, L^\bullet, C(f), f, i, -p)$ for some morphism of complexes f. Then the rotated triangle $(Y, Z, X[1], g, h, -f[1])$ is isomorphic to $(L^\bullet, C(f), K^\bullet[1], i, -p, -f[1])$ which is isomorphic to the triangle $(L^\bullet, C(f), K^\bullet[1], i, p, f[1])$. By Lemma 9.17 this triangle is distinguished. Hence $(Y, Z, X[1], g, h, -f[1])$ is distinguished as desired.

Proof of TR3. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be distinguished triangles of $K(\mathcal{A})$ and let $a : X \to X'$ and $b : Y \to Y'$ be morphisms such that $f' \circ a =$

$b \circ f$. By Lemma 9.14 we may assume that $(X, Y, Z, f, g, h) = (X, Y, C(f), f, i, -p)$ and $(X', Y', Z', f', g', h') = (X', Y', C(f'), f', i', -p')$. At this point we simply apply Lemma 9.2 to the commutative diagram given by f, f', a, b.

Proof of TR4. At this point we know that $K(\mathcal{A})$ is a pre-triangulated category. Hence we can use Lemma 4.15. Let $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ be composable morphisms of $K(\mathcal{A})$. By Lemma 9.15 we may assume that $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ are split injective morphisms. In this case the result follows from Lemma 10.2. $\square$

Remark 10.4. Let $\mathcal{A}$ be an additive category. Exactly the same proof as the proof of Proposition 10.3 shows that the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. Namely, the cone of a morphism between bounded (above, below) is bounded (above, below). But we prove below that these are triangulated subcategories of $K(\mathcal{A})$ which gives another proof.

Lemma 10.5. *Let $\mathcal{A}$ be an additive category. The categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are full triangulated subcategories of $K(\mathcal{A})$.*

Proof. Each of the categories mentioned is a full additive subcategory. We use the criterion of Lemma 4.16 to show that they are triangulated subcategories. It is clear that each of the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ is preserved under the shift functors $[1], [-1]$. Finally, suppose that $f : A^\bullet \to B^\bullet$ is a morphism in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$. Then $(A^\bullet, B^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle of $K(\mathcal{A})$ with $C(f)^\bullet \in K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$ as is clear from the construction of the cone. Thus the lemma is proved. (Alternatively, $K^\bullet \to L^\bullet$ is isomorphic to an termwise split injection of complexes in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, see Lemma 9.6 and then one can directly take the associated distinguished triangle.) $\square$

Lemma 10.6. *Let $\mathcal{A}$, $\mathcal{B}$ be additive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. The induced functors*

$$\begin{aligned} F : K(\mathcal{A}) &\longrightarrow K(\mathcal{B}) \\ F : K^+(\mathcal{A}) &\longrightarrow K^+(\mathcal{B}) \\ F : K^-(\mathcal{A}) &\longrightarrow K^-(\mathcal{B}) \\ F : K^b(\mathcal{A}) &\longrightarrow K^b(\mathcal{B}) \end{aligned}$$

are exact functors of triangulated categories.

Proof. Suppose $A^\bullet \to B^\bullet \to C^\bullet$ is a termwise split sequence of complexes of $\mathcal{A}$ with splittings (s^n, π^n) and associated morphism $\delta : C^\bullet \to A^\bullet[1]$, see Definition 9.9. Then $F(A^\bullet) \to F(B^\bullet) \to F(C^\bullet)$ is a termwise split sequence of complexes with splittings $(F(s^n), F(\pi^n))$ and associated morphism $F(\delta) : F(C^\bullet) \to F(A^\bullet)[1]$. Thus F transforms distinguished triangles into distinguished triangles. $\square$

Lemma 10.7. *Let $\mathcal{A}$ be an additive category. Let $(A^\bullet, B^\bullet, C^\bullet, a, b, c)$ be a distinguished triangle in $K(\mathcal{A})$. Then there exists an isomorphic distinguished triangle $(A^\bullet, (B')^\bullet, C^\bullet, a', b', c)$ such that $0 \to A^n \to (B')^n \to C^n \to 0$ is a split short exact sequence for all n.*

Proof. We will use that $K(\mathcal{A})$ is a triangulated category by Proposition 10.3. Let $W^\bullet$ be the cone on $c : C^\bullet \to A^\bullet[1]$ with its maps $i : A^\bullet[1] \to W^\bullet$ and $p : W^\bullet \to C^\bullet[1]$. Then $(C^\bullet, A^\bullet[1], W^\bullet, c, i, -p)$ is a distinguished triangle by Lemma 9.14. Rotating backwards twice we see that $(A^\bullet, W^\bullet[-1], C^\bullet, -i[-1], p[-1], c)$ is a distinguished triangle. By TR3 there is a morphism of distinguished triangles

$(\mathrm{id}, \beta, \mathrm{id}) : (A^\bullet, B^\bullet, C^\bullet, a, b, c) \to (A^\bullet, W^\bullet[-1], C^\bullet, -i[-1], p[-1], c)$ which must be an isomorphism by Lemma 4.3. This finishes the proof because $0 \to A^\bullet \to W^\bullet[-1] \to C^\bullet \to 0$ is a termwise split short exact sequence of complexes by the very construction of cones in Section 9. □

Remark 10.8. Let $\mathcal{A}$ be an additive category with countable direct sums. Let $\text{DoubleComp}(\mathcal{A})$ denote the category of double complexes in $\mathcal{A}$, see Homology, Section 18. We can use this category to construct two triangulated categories.

(1) We can consider an object $A^{\bullet,\bullet}$ of $\text{DoubleComp}(\mathcal{A})$ as a complex of complexes as follows

$$\ldots \to A^{\bullet,-1} \to A^{\bullet,0} \to A^{\bullet,1} \to \ldots$$

and take the homotopy category $K_{first}(\text{DoubleComp}(\mathcal{A}))$ with the corresponding triangulated structure given by Proposition 10.3. By Homology, Remark 18.6 the functor

$$\text{Tot} : K_{first}(\text{DoubleComp}(\mathcal{A})) \longrightarrow K(\mathcal{A})$$

is an exact functor of triangulated categories.

(2) We can consider an object $A^{\bullet,\bullet}$ of $\text{DoubleComp}(\mathcal{A})$ as a complex of complexes as follows

$$\ldots \to A^{-1,\bullet} \to A^{0,\bullet} \to A^{1,\bullet} \to \ldots$$

and take the homotopy category $K_{second}(\text{DoubleComp}(\mathcal{A}))$ with the corresponding triangulated structure given by Proposition 10.3. By Homology, Remark 18.7 the functor

$$\text{Tot} : K_{second}(\text{DoubleComp}(\mathcal{A})) \longrightarrow K(\mathcal{A})$$

is an exact functor of triangulated categories.

Remark 10.9. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be additive categories and assume $\mathcal{C}$ has countable direct sums. Suppose that

$$\otimes : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}, \quad (X, Y) \longmapsto X \otimes Y$$

is a functor which is bilinear on morphisms. This determines a functor

$$\text{Comp}(\mathcal{A}) \times \text{Comp}(\mathcal{B}) \longrightarrow \text{DoubleComp}(\mathcal{C}), \quad (X^\bullet, Y^\bullet) \longmapsto X^\bullet \otimes Y^\bullet$$

See Homology, Example 18.2.

(1) For a fixed object $X^\bullet$ of $\text{Comp}(\mathcal{A})$ the functor

$$K(\mathcal{B}) \longrightarrow K(\mathcal{C}), \quad Y^\bullet \longmapsto \text{Tot}(X^\bullet \otimes Y^\bullet)$$

is an exact functor of triangulated categories.

(2) For a fixed object $Y^\bullet$ of $\text{Comp}(\mathcal{B})$ the functor

$$K(\mathcal{A}) \longrightarrow K(\mathcal{C}), \quad X^\bullet \longmapsto \text{Tot}(X^\bullet \otimes Y^\bullet)$$

is an exact functor of triangulated categories.

This follows from Remark 10.8 since the functors $\text{Comp}(\mathcal{A}) \to \text{DoubleComp}(\mathcal{C})$, $Y^\bullet \mapsto X^\bullet \otimes Y^\bullet$ and $\text{Comp}(\mathcal{B}) \to \text{DoubleComp}(\mathcal{C})$, $X^\bullet \mapsto X^\bullet \otimes Y^\bullet$ are immediately seen to be compatible with homotopies and termwise split short exact sequences and hence induce exact functors of triangulated categories

$$K(\mathcal{B}) \to K_{first}(\text{DoubleComp}(\mathcal{C})) \quad \text{and} \quad K(\mathcal{A}) \to K_{second}(\text{DoubleComp}(\mathcal{C}))$$

Observe that for the first of the two the isomorphism

$$\mathrm{Tot}(X^\bullet \otimes Y^\bullet[1]) \cong \mathrm{Tot}(X^\bullet \otimes Y^\bullet)[1]$$

involves signs (this goes back to the signs chosen in Homology, Remark 18.5).

11. Derived categories

In this section we construct the derived category of an abelian category $\mathcal{A}$ by inverting the quasi-isomorphisms in $K(\mathcal{A})$. Before we do this recall that the functors $H^i : \mathrm{Comp}(\mathcal{A}) \to \mathcal{A}$ factor through $K(\mathcal{A})$, see Homology, Lemma 13.11. Moreover, in Homology, Definition 14.8 we have defined identifications $H^i(K^\bullet[n]) = H^{i+n}(K^\bullet)$. At this point it makes sense to redefine

$$H^i(K^\bullet) = H^0(K^\bullet[i])$$

in order to avoid confusion and possible sign errors.

Lemma 11.1. *Let $\mathcal{A}$ be an abelian category. The functor*

$$H^0 : K(\mathcal{A}) \longrightarrow \mathcal{A}$$

is homological.

Proof. Because H^0 is a functor, and by our definition of distinguished triangles it suffices to prove that given a termwise split short exact sequence of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ the sequence $H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet)$ is exact. This follows from Homology, Lemma 13.12. □

In particular, this lemma implies that a distinguished triangle (X, Y, Z, f, g, h) in $K(\mathcal{A})$ gives rise to a long exact cohomology sequence

$$(11.1.1) \qquad \ldots \longrightarrow H^i(X) \xrightarrow{H^i(f)} H^i(Y) \xrightarrow{H^i(g)} H^i(Z) \xrightarrow{H^i(h)} H^{i+1}(X) \longrightarrow \ldots$$

see (3.5.1). Moreover, there is a compatibility with the long exact sequence of cohomology associated to a short exact sequence of complexes (insert future reference here). For example, if $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is the distinguished triangle associated to a termwise split exact sequence of complexes (see Definition 9.9), then the cohomology sequence above agrees with the one defined using the snake lemma, see Homology, Lemma 13.12 and for agreement of sequences, see Homology, Lemma 14.11.

Recall that a complex $K^\bullet$ is *acyclic* if $H^i(K^\bullet) = 0$ for all $i \in \mathbf{Z}$. Moreover, recall that a morphism of complexes $f : K^\bullet \to L^\bullet$ is a *quasi-isomorphism* if and only if $H^i(f)$ is an isomorphism for all i. See Homology, Definition 13.10.

Lemma 11.2. *Let $\mathcal{A}$ be an abelian category. The full subcategory $Ac(\mathcal{A})$ of $K(\mathcal{A})$ consisting of acyclic complexes is a strictly full saturated triangulated subcategory of $K(\mathcal{A})$. The corresponding saturated multiplicative system (see Lemma 6.10) of $K(\mathcal{A})$ is the set $Qis(\mathcal{A})$ of quasi-isomorphisms. In particular, the kernel of the localization functor $Q : K(\mathcal{A}) \to Qis(\mathcal{A})^{-1}K(\mathcal{A})$ is $Ac(\mathcal{A})$ and the functor H^0 factors through Q.*

Proof. We know that H^0 is a homological functor by Lemma 11.1. Thus this lemma is a special case of Lemma 6.11. □

Definition 11.3. Let $\mathcal{A}$ be an abelian category. Let $\text{Ac}(\mathcal{A})$ and $\text{Qis}(\mathcal{A})$ be as in Lemma 11.2. The *derived category of* $\mathcal{A}$ is the triangulated category

$$D(\mathcal{A}) = K(\mathcal{A})/\text{Ac}(\mathcal{A}) = \text{Qis}(\mathcal{A})^{-1}K(\mathcal{A}).$$

We denote $H^0 : D(\mathcal{A}) \to \mathcal{A}$ the unique functor whose composition with the quotient functor gives back the functor H^0 defined above. Using Lemma 6.4 we introduce the strictly full saturated triangulated subcategories $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ whose sets of objects are

$$\begin{matrix} \text{Ob}(D^+(\mathcal{A})) = \{X \in \text{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \ll 0\} \\ \text{Ob}(D^-(\mathcal{A})) = \{X \in \text{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \gg 0\} \\ \text{Ob}(D^b(\mathcal{A})) = \{X \in \text{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } |n| \gg 0\} \end{matrix}$$

The category $D^b(\mathcal{A})$ is called the *bounded derived category* of $\mathcal{A}$.

If $K^\bullet$ and $L^\bullet$ are complexes of $\mathcal{A}$ then we sometimes say "$K^\bullet$ is *quasi-isomorphic* to $L^\bullet$" to indicate that $K^\bullet$ and $L^\bullet$ are isomorphic objects of $D(\mathcal{A})$.

Remark 11.4. In this chapter, we consistently work with "small" abelian categories (as is the convention in the Stacks project). For a "big" abelian category $\mathcal{A}$, it isn't clear that the derived category $D(\mathcal{A})$ exists, because it isn't clear that morphisms in the derived category are sets. In fact, in general they aren't, see Examples, Lemma 61.1. However, if $\mathcal{A}$ is a Grothendieck abelian category, and given $K^\bullet, L^\bullet$ in $K(\mathcal{A})$, then by Injectives, Theorem 12.6 there exists a quasi-isomorphism $L^\bullet \to I^\bullet$ to a K-injective complex $I^\bullet$ and Lemma 31.2 shows that

$$\text{Hom}_{D(\mathcal{A})}(K^\bullet, L^\bullet) = \text{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$$

which is a set. Some examples of Grothendieck abelian categories are the category of modules over a ring, or more generally the category of sheaves of modules on a ringed site.

Each of the variants $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ can be constructed as a localization of the corresponding homotopy category. This relies on the following simple lemma.

Lemma 11.5. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex.*

(1) *If $H^n(K^\bullet) = 0$ for all $n \ll 0$, then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^\bullet$ bounded below.*
(2) *If $H^n(K^\bullet) = 0$ for all $n \gg 0$, then there exists a quasi-isomorphism $M^\bullet \to K^\bullet$ with $M^\bullet$ bounded above.*
(3) *If $H^n(K^\bullet) = 0$ for all $|n| \gg 0$, then there exists a commutative diagram of morphisms of complexes*

$$\begin{tikzcd} K^\bullet \arrow[r] & L^\bullet \\ M^\bullet \arrow[u] \arrow[r] & N^\bullet \arrow[u] \end{tikzcd}$$

where all the arrows are quasi-isomorphisms, $L^\bullet$ bounded below, $M^\bullet$ bounded above, and $N^\bullet$ a bounded complex.

Proof. Pick $a \ll 0 \ll b$ and set $M^\bullet = \tau_{\leq b}K^\bullet$, $L^\bullet = \tau_{\geq a}K^\bullet$, and $N^\bullet = \tau_{\leq b}L^\bullet = \tau_{\geq a}M^\bullet$. See Homology, Section 15 for the truncation functors. $\square$

To state the following lemma denote $\text{Ac}^+(\mathcal{A})$, $\text{Ac}^-(\mathcal{A})$, resp. $\text{Ac}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\text{Ac}(\mathcal{A})$. Denote $\text{Qis}^+(\mathcal{A})$, $\text{Qis}^-(\mathcal{A})$, resp. $\text{Qis}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\text{Qis}(\mathcal{A})$.

Lemma 11.6. *Let $\mathcal{A}$ be an abelian category. The subcategories $Ac^+(\mathcal{A})$, $Ac^-(\mathcal{A})$, resp. $Ac^b(\mathcal{A})$ are strictly full saturated triangulated subcategories of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$. The corresponding saturated multiplicative systems (see Lemma 6.10) are the sets $Qis^+(\mathcal{A})$, $Qis^-(\mathcal{A})$, resp. $Qis^b(\mathcal{A})$.*

(1) *The kernel of the functor $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ is $Ac^+(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^+(\mathcal{A})/Ac^+(\mathcal{A}) = Qis^+(\mathcal{A})^{-1}K^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$$
(2) *The kernel of the functor $K^-(\mathcal{A}) \to D^-(\mathcal{A})$ is $Ac^-(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^-(\mathcal{A})/Ac^-(\mathcal{A}) = Qis^-(\mathcal{A})^{-1}K^-(\mathcal{A}) \longrightarrow D^-(\mathcal{A})$$
(3) *The kernel of the functor $K^b(\mathcal{A}) \to D^b(\mathcal{A})$ is $Ac^b(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^b(\mathcal{A})/Ac^b(\mathcal{A}) = Qis^b(\mathcal{A})^{-1}K^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})$$

Proof. The initial statements follow from Lemma 6.11 by considering the restriction of the homological functor H^0. The statement on kernels in (1), (2), (3) is a consequence of the definitions in each case. Each of the functors is essentially surjective by Lemma 11.5. To finish the proof we have to show the functors are fully faithful. We first do this for the bounded below version.

Suppose that $K^\bullet, L^\bullet$ are bounded above complexes. A morphism between these in $D(\mathcal{A})$ is of the form $s^{-1}f$ for a pair $f : K^\bullet \to (L')^\bullet$, $s : L^\bullet \to (L')^\bullet$ where s is a quasi-isomorphism. This implies that $(L')^\bullet$ has cohomology bounded below. Hence by Lemma 11.5 we can choose a quasi-isomorphism $s' : (L')^\bullet \to (L'')^\bullet$ with $(L'')^\bullet$ bounded below. Then the pair $(s' \circ f, s' \circ s)$ defines a morphism in $\text{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$. Hence the functor is "full". Finally, suppose that the pair $f : K^\bullet \to (L')^\bullet$, $s : L^\bullet \to (L')^\bullet$ defines a morphism in $\text{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$ which is zero in $D(\mathcal{A})$. This means that there exists a quasi-isomorphism $s' : (L')^\bullet \to (L'')^\bullet$ such that $s' \circ f = 0$. Using Lemma 11.5 once more we obtain a quasi-isomorphism $s'' : (L'')^\bullet \to (L''')^\bullet$ with $(L''')^\bullet$ bounded below. Thus we see that $s'' \circ s' \circ f = 0$ which implies that $s^{-1}f$ is zero in $\text{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$. This finishes the proof that the functor in (1) is an equivalence.

The proof of (2) is dual to the proof of (1). To prove (3) we may use the result of (2). Hence it suffices to prove that the functor $\text{Qis}^b(\mathcal{A})^{-1}K^b(\mathcal{A}) \to \text{Qis}^-(\mathcal{A})^{-1}K^-(\mathcal{A})$ is fully faithful. The argument given in the previous paragraph applies directly to show this where we consistently work with complexes which are already bounded above. □

12. The canonical delta-functor

The derived category should be the receptacle for the universal cohomology functor. In order to state the result we use the notion of a δ-functor from an abelian category into a triangulated category, see Definition 3.6.

Consider the functor $\text{Comp}(\mathcal{A}) \to K(\mathcal{A})$. This functor is **not** a δ-functor in general. The easiest way to see this is to consider a nonsplit short exact sequence $0 \to A \to B \to C \to 0$ of objects of $\mathcal{A}$. Since $\text{Hom}_{K(\mathcal{A})}(C[0], A[1]) = 0$ we see that any distinguished triangle arising from this short exact sequence would look like $(A[0], B[0], C[0], a, b, 0)$. But the existence of such a distinguished triangle in $K(\mathcal{A})$ implies that the extension is split. A contradiction.

It turns out that the functor $\text{Comp}(\mathcal{A}) \to D(\mathcal{A})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to A^\bullet \xrightarrow{a} B^\bullet \xrightarrow{b} C^\bullet \to 0$$

of complexes in the abelian category $\mathcal{A}$. Consider the cone $C(a)^\bullet$ of the morphism a. We have $C(a)^n = B^n \oplus A^{n+1}$ and we define $q^n : C(a)^n \to C^n$ via the projection to B^n followed by b^n. Hence a morphism of complexes

$$q : C(a)^\bullet \longrightarrow C^\bullet.$$

It is clear that $q \circ i = b$ where i is as in Definition 9.1. Note that, as $a^\bullet$ is injective in each degree, the kernel of q is identified with the cone of $\text{id}_{A^\bullet}$ which is acyclic. Hence we see that q is a quasi-isomorphism. According to Lemma 9.14 the triangle

$$(A, B, C(a), a, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$. As the localization functor $K(\mathcal{A}) \to D(\mathcal{A})$ is exact we see that $(A, B, C(a), a, i, -p)$ is a distinguished triangle in $D(\mathcal{A})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(\mathcal{A})$. Hence we deduce that

$$(A, B, C, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(\mathcal{A})$. This suggests the following lemma.

Lemma 12.1. *Let $\mathcal{A}$ be an abelian category. The functor $\text{Comp}(\mathcal{A}) \to D(\mathcal{A})$ defined has the natural structure of a δ-functor, with*

$$\delta_{A^\bullet \to B^\bullet \to C^\bullet} = -p \circ q^{-1}$$

with p and q as explained above. The same construction turns the functors $\text{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$, $\text{Comp}^-(\mathcal{A}) \to D^-(\mathcal{A})$, and $\text{Comp}^b(\mathcal{A}) \to D^b(\mathcal{A})$ into δ-functors.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show that given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r, "a"'] \arrow[d, "f"'] & B^\bullet \arrow[r, "b"'] \arrow[d, "g"'] & C^\bullet \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & (A')^\bullet \arrow[r, "a'"] & (B')^\bullet \arrow[r, "b'"] & (C')^\bullet \arrow[r] & 0
\end{tikzcd}$$

we get the desired commutative diagram of Definition 3.6 (2). By Lemma 9.2 the pair (f, g) induces a canonical morphism $c : C(a)^\bullet \to C(a')^\bullet$. It is a simple computation to show that $q' \circ c = h \circ q$ and $f[1] \circ p = p' \circ c$. From this the result follows directly. $\square$

Lemma 12.2. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} 0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D^\bullet \arrow[r] & E^\bullet \arrow[r] & F^\bullet \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of morphisms of complexes such that the rows are short exact sequences of complexes, and the vertical arrows are quasi-isomorphisms. The δ-functor of Lemma 12.1 above maps the short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ and $0 \to D^\bullet \to E^\bullet \to F^\bullet \to 0$ to isomorphic distinguished triangles.

Proof. Trivial from the fact that $K(\mathcal{A}) \to D(\mathcal{A})$ transforms quasi-isomorphisms into isomorphisms and that the associated distinguished triangles are functorial. $\square$

Lemma 12.3. *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

be a short exact sequences of complexes. Assume this short exact sequence is termwise split. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the distinguished triangle of $K(\mathcal{A})$ associated to the sequence. The δ-functor of Lemma 12.1 above maps the short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ to a triangle isomorphic to the distinguished triangle

$$(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta).$$

Proof. Follows from Lemma 9.14. $\square$

Remark 12.4. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$. Let $a \in \mathbf{Z}$. We claim there is a canonical distinguished triangle

$$\tau_{\leq a}K^\bullet \to K^\bullet \to \tau_{\geq a+1}K^\bullet \to (\tau_{\leq a}K^\bullet)[1]$$

in $D(\mathcal{A})$. Here we have used the canonical truncation functors τ from Homology, Section 15. Namely, we first take the distinguished triangle associated by our δ-functor (Lemma 12.1) to the short exact sequence of complexes

$$0 \to \tau_{\leq a}K^\bullet \to K^\bullet \to K^\bullet/\tau_{\leq a}K^\bullet \to 0$$

Next, we use that the map $K^\bullet \to \tau_{\geq a+1}K^\bullet$ factors through a quasi-isomorphism $K^\bullet/\tau_{\leq a}K^\bullet \to \tau_{\geq a+1}K^\bullet$ by the description of cohomology groups in Homology, Section 15. In a similar way we obtain canonical distinguished triangles

$$\tau_{\leq a}K^\bullet \to \tau_{\leq a+1}K^\bullet \to H^{a+1}(K^\bullet)[-a-1] \to (\tau_{\leq a}K^\bullet)[1]$$

and

$$H^a(K^\bullet)[-a] \to \tau_{\geq a}K^\bullet \to \tau_{\geq a+1}K^\bullet \to H^a(K^\bullet)[-a+1]$$

Lemma 12.5. *Let $\mathcal{A}$ be an abelian category. Let*

$$K_0^\bullet \to K_1^\bullet \to \ldots \to K_n^\bullet$$

be maps of complexes such that

(1) $H^i(K_0^\bullet) = 0$ *for* $i > 0$,
(2) $H^{-j}(K_j^\bullet) \to H^{-j}(K_{j+1}^\bullet)$ *is zero.*

Then the composition $K_0^\bullet \to K_n^\bullet$ factors through $\tau_{\leq -n}K_n^\bullet \to K_n^\bullet$ in $D(\mathcal{A})$. Dually, given maps of complexes

$$K_n^\bullet \to K_{n-1}^\bullet \to \ldots \to K_0^\bullet$$

such that

(1) $H^i(K_0^\bullet) = 0$ *for* $i < 0$,
(2) $H^j(K_{j+1}^\bullet) \to H^j(K_j^\bullet)$ *is zero,*

then the composition $K_n^\bullet \to K_0^\bullet$ factors through $K_n^\bullet \to \tau_{\geq n}K_n^\bullet$ in $D(\mathcal{A})$.

Proof. The case $n = 1$. Since $\tau_{\leq 0}K_0^\bullet = K_0^\bullet$ in $D(\mathcal{A})$ we can replace $K_0^\bullet$ by $\tau_{\leq 0}K_0^\bullet$ and $K_1^\bullet$ by $\tau_{\leq 0}K_1^\bullet$. Consider the distinguished triangle

$$\tau_{\leq -1}K_1^\bullet \to K_1^\bullet \to H^0(K_1^\bullet)[0] \to (\tau_{\leq -1}K_1^\bullet)[1]$$

(Remark 12.4). The composition $K_0^\bullet \to K_1^\bullet \to H^0(K_1^\bullet)[0]$ is zero as it is equal to $K_0^\bullet \to H^0(K_0^\bullet)[0] \to H^0(K_1^\bullet)[0]$ which is zero by assumption. The fact that $\mathrm{Hom}_{D(\mathcal{A})}(K_0^\bullet, -)$ is a homological functor (Lemma 4.2), allows us to find the desired factorization. For $n = 2$ we get a factorization $K_0^\bullet \to \tau_{\leq -1}K_1^\bullet$ by the case $n = 1$ and we can apply the case $n = 1$ to the map of complexes $\tau_{\leq -1}K_1^\bullet \to \tau_{\leq -1}K_2^\bullet$ to get a factorization $\tau_{\leq -1}K_1^\bullet \to \tau_{\leq -2}K_2^\bullet$. The general case is proved in exactly the same manner. $\square$

13. Filtered derived categories

A reference for this section is [Ill72, I, Chapter V]. Let $\mathcal{A}$ be an abelian category. In this section we will define the filtered derived category $DF(\mathcal{A})$ of $\mathcal{A}$. In short, we will define it as the derived category of the exact category of objects of $\mathcal{A}$ endowed with a finite filtration. (Thus our construction is a special case of a more general construction of the derived category of an exact category, see for example [Büh10], [Kel90].) Illusie's filtered derived category is the full subcategory of ours consisting of those objects whose filtration is finite. (In our category the filtration is still finite in each degree, but may not be uniformly bounded.) The rationale for our choice is that it is not harder and it allows us to apply the discussion to the spectral sequences of Lemma 21.3, see also Remark 21.4.

We will use the notation regarding filtered objects introduced in Homology, Section 19. The category of filtered objects of $\mathcal{A}$ is denoted $\mathrm{Fil}(\mathcal{A})$. All filtrations will be decreasing by fiat.

Definition 13.1. Let $\mathcal{A}$ be an abelian category. The *category of finite filtered objects of* $\mathcal{A}$ is the category of filtered objects (A, F) of $\mathcal{A}$ whose filtration F is finite. We denote it $\mathrm{Fil}^f(\mathcal{A})$.

Thus $\mathrm{Fil}^f(\mathcal{A})$ is a full subcategory of $\mathrm{Fil}(\mathcal{A})$. For each $p \in \mathbf{Z}$ there is a functor $\mathrm{gr}^p : \mathrm{Fil}^f(\mathcal{A}) \to \mathcal{A}$. There is a functor

$$\mathrm{gr} = \bigoplus\nolimits_{p\in\mathbf{Z}} \mathrm{gr}^p : \mathrm{Fil}^f(\mathcal{A}) \to \mathrm{Gr}(\mathcal{A})$$

where $\mathrm{Gr}(\mathcal{A})$ is the category of graded objects of $\mathcal{A}$, see Homology, Definition 16.1. Finally, there is a functor

$$(\text{forget } F) : \mathrm{Fil}^f(\mathcal{A}) \longrightarrow \mathcal{A}$$

which associates to the filtered object (A, F) the underlying object of $\mathcal{A}$. The category $\text{Fil}^f(\mathcal{A})$ is an additive category, but not abelian in general, see Homology, Example 3.13.

Because the functors gr^p, gr, (forget F) are additive they induce exact functors of triangulated categories

$$\text{gr}^p, (\text{forget } F) : K(\text{Fil}^f(\mathcal{A})) \to K(\mathcal{A}) \quad \text{and} \quad \text{gr} : K(\text{Fil}^f(\mathcal{A})) \to K(\text{Gr}(\mathcal{A}))$$

by Lemma 10.6. By analogy with the case of the homotopy category of an abelian category we make the following definitions.

Definition 13.2. Let $\mathcal{A}$ be an abelian category.

(1) Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of $K(\text{Fil}^f(\mathcal{A}))$. We say that α is a *filtered quasi-isomorphism* if the morphism $\text{gr}(\alpha)$ is a quasi-isomorphism.
(2) Let $K^\bullet$ be an object of $K(\text{Fil}^f(\mathcal{A}))$. We say that $K^\bullet$ is *filtered acyclic* if the complex $\text{gr}(K^\bullet)$ is acyclic.

Note that $\alpha : K^\bullet \to L^\bullet$ is a filtered quasi-isomorphism if and only if each $\text{gr}^p(\alpha)$ is a quasi-isomorphism. Similarly a complex $K^\bullet$ is filtered acyclic if and only if each $\text{gr}^p(K^\bullet)$ is acyclic.

Lemma 13.3. *Let $\mathcal{A}$ be an abelian category.*

(1) *The functor $K(Fil^f(\mathcal{A})) \longrightarrow Gr(\mathcal{A})$, $K^\bullet \longmapsto H^0(gr(K^\bullet))$ is homological.*
(2) *The functor $K(Fil^f(\mathcal{A})) \to \mathcal{A}$, $K^\bullet \longmapsto H^0(gr^p(K^\bullet))$ is homological.*
(3) *The functor $K(Fil^f(\mathcal{A})) \longrightarrow \mathcal{A}$, $K^\bullet \longmapsto H^0((forget\ F)K^\bullet)$ is homological.*

Proof. This follows from the fact that $H^0 : K(\mathcal{A}) \to \mathcal{A}$ is homological, see Lemma 11.1 and the fact that the functors gr, gr^p, (forget F) are exact functors of triangulated categories. See Lemma 4.20. $\square$

Lemma 13.4. *Let $\mathcal{A}$ be an abelian category. The full subcategory $FAc(\mathcal{A})$ of $K(Fil^f(\mathcal{A}))$ consisting of filtered acyclic complexes is a strictly full saturated triangulated subcategory of $K(Fil^f(\mathcal{A}))$. The corresponding saturated multiplicative system (see Lemma 6.10) of $K(Fil^f(\mathcal{A}))$ is the set $FQis(\mathcal{A})$ of filtered quasi-isomorphisms. In particular, the kernel of the localization functor*

$$Q : K(Fil^f(\mathcal{A})) \longrightarrow FQis(\mathcal{A})^{-1}K(Fil^f(\mathcal{A}))$$

is $FAc(\mathcal{A})$ and the functor $H^0 \circ gr$ factors through Q.

Proof. We know that $H^0 \circ \text{gr}$ is a homological functor by Lemma 13.3. Thus this lemma is a special case of Lemma 6.11. $\square$

Definition 13.5. Let $\mathcal{A}$ be an abelian category. Let $\text{FAc}(\mathcal{A})$ and $\text{FQis}(\mathcal{A})$ be as in Lemma 13.4. The *filtered derived category of $\mathcal{A}$* is the triangulated category

$$DF(\mathcal{A}) = K(\text{Fil}^f(\mathcal{A}))/\text{FAc}(\mathcal{A}) = \text{FQis}(\mathcal{A})^{-1}K(\text{Fil}^f(\mathcal{A})).$$

Lemma 13.6. *The functors $gr^p, gr, (forget\ F)$ induce canonical exact functors*

$$gr^p, gr, (forget\ F) : DF(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

which commute with the localization functors.

Proof. This follows from the universal property of localization, see Lemma 5.7, provided we can show that a filtered quasi-isomorphism is turned into a quasi-isomorphism by each of the functors $\text{gr}^p, \text{gr}, (\text{forget } F)$. This is true by definition for the first two. For the last one the statement we have to do a little bit of work. Let $f : K^\bullet \to L^\bullet$ be a filtered quasi-isomorphism in $K(\text{Fil}^f(\mathcal{A}))$. Choose a distinguished triangle $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ which contains f. Then $M^\bullet$ is filtered acyclic, see Lemma 13.4. Hence by the corresponding lemma for $K(\mathcal{A})$ it suffices to show that a filtered acyclic complex is an acyclic complex if we forget the filtration. This follows from Homology, Lemma 19.15. $\square$

Definition 13.7. Let $\mathcal{A}$ be an abelian category. The *bounded filtered derived category* $DF^b(\mathcal{A})$ is the full subcategory of $DF(\mathcal{A})$ with objects those X such that $\text{gr}(X) \in D^b(\mathcal{A})$. Similarly for the bounded below filtered derived category $DF^+(\mathcal{A})$ and the bounded above filtered derived category $DF^-(\mathcal{A})$.

Lemma 13.8. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet \in K(\text{Fil}^f(\mathcal{A}))$.*

(1) *If $H^n(\text{gr}(K^\bullet)) = 0$ for all $n < a$, then there exists a filtered quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^n = 0$ for all $n < a$.*
(2) *If $H^n(\text{gr}(K^\bullet)) = 0$ for all $n > b$, then there exists a filtered quasi-isomorphism $M^\bullet \to K^\bullet$ with $M^n = 0$ for all $n > b$.*
(3) *If $H^n(\text{gr}(K^\bullet)) = 0$ for all $|n| \gg 0$, then there exists a commutative diagram of morphisms of complexes*

$$\begin{tikzcd} K^\bullet \arrow[r] & L^\bullet \\ M^\bullet \arrow[u] \arrow[r] & N^\bullet \arrow[u] \end{tikzcd}$$

where all the arrows are filtered quasi-isomorphisms, $L^\bullet$ bounded below, $M^\bullet$ bounded above, and $N^\bullet$ a bounded complex.

Proof. Suppose that $H^n(\text{gr}(K^\bullet)) = 0$ for all $n < a$. By Homology, Lemma 19.15 the sequence

$$K^{a-1} \xrightarrow{d^{a-2}} K^{a-1} \xrightarrow{d^{a-1}} K^a$$

is an exact sequence of objects of $\mathcal{A}$ and the morphisms d^{a-2} and d^{a-1} are strict. Hence $\text{Coim}(d^{a-1}) = \text{Im}(d^{a-1})$ in $\text{Fil}^f(\mathcal{A})$ and the map $\text{gr}(\text{Im}(d^{a-1})) \to \text{gr}(K^a)$ is injective with image equal to the image of $\text{gr}(K^{a-1}) \to \text{gr}(K^a)$, see Homology, Lemma 19.13. This means that the map $K^\bullet \to \tau_{\geq a}K^\bullet$ into the truncation

$$\tau_{\geq a}K^\bullet = (\ldots \to 0 \to K^a/\text{Im}(d^{a-1}) \to K^{a+1} \to \ldots)$$

is a filtered quasi-isomorphism. This proves (1). The proof of (2) is dual to the proof of (1). Part (3) follows formally from (1) and (2). $\square$

To state the following lemma denote $\text{FAc}^+(\mathcal{A})$, $\text{FAc}^-(\mathcal{A})$, resp. $\text{FAc}^b(\mathcal{A})$ the intersection of $K^+(\text{Fil}^f\mathcal{A})$, $K^-(\text{Fil}^f\mathcal{A})$, resp. $K^b(\text{Fil}^f\mathcal{A})$ with $\text{FAc}(\mathcal{A})$. Denote $\text{FQis}^+(\mathcal{A})$, $\text{FQis}^-(\mathcal{A})$, resp. $\text{FQis}^b(\mathcal{A})$ the intersection of $K^+(\text{Fil}^f\mathcal{A})$, $K^-(\text{Fil}^f\mathcal{A})$, resp. $K^b(\text{Fil}^f\mathcal{A})$ with $\text{FQis}(\mathcal{A})$.

Lemma 13.9. *Let $\mathcal{A}$ be an abelian category. The subcategories $FAc^+(\mathcal{A})$, $FAc^-(\mathcal{A})$, resp. $FAc^b(\mathcal{A})$ are strictly full saturated triangulated subcategories of $K^+(Fil^f\mathcal{A})$, $K^-(Fil^f\mathcal{A})$, resp. $K^b(Fil^f\mathcal{A})$. The corresponding saturated multiplicative systems (see Lemma 6.10) are the sets $FQis^+(\mathcal{A})$, $FQis^-(\mathcal{A})$, resp. $FQis^b(\mathcal{A})$.*

(1) *The kernel of the functor* $K^+(Fil^f\mathcal{A}) \to DF^+(\mathcal{A})$ *is* $FAc^+(\mathcal{A})$ *and this induces an equivalence of triangulated categories*
$$K^+(Fil^f\mathcal{A})/FAc^+(\mathcal{A}) = FQis^+(\mathcal{A})^{-1}K^+(Fil^f\mathcal{A}) \longrightarrow DF^+(\mathcal{A})$$
(2) *The kernel of the functor* $K^-(Fil^f\mathcal{A}) \to DF^-(\mathcal{A})$ *is* $FAc^-(\mathcal{A})$ *and this induces an equivalence of triangulated categories*
$$K^-(Fil^f\mathcal{A})/FAc^-(\mathcal{A}) = FQis^-(\mathcal{A})^{-1}K^-(Fil^f\mathcal{A}) \longrightarrow DF^-(\mathcal{A})$$
(3) *The kernel of the functor* $K^b(Fil^f\mathcal{A}) \to DF^b(\mathcal{A})$ *is* $FAc^b(\mathcal{A})$ *and this induces an equivalence of triangulated categories*
$$K^b(Fil^f\mathcal{A})/FAc^b(\mathcal{A}) = FQis^b(\mathcal{A})^{-1}K^b(Fil^f\mathcal{A}) \longrightarrow DF^b(\mathcal{A})$$

Proof. This follows from the results above, in particular Lemma 13.8, by exactly the same arguments as used in the proof of Lemma 11.6. $\square$

14. Derived functors in general

A reference for this section is Deligne's exposé XVII in [AGV71]. A very general notion of right and left derived functors exists where we have an exact functor between triangulated categories, a multiplicative system in the source category and we want to find the "correct" extension of the exact functor to the localized category.

Situation 14.1. Here $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor of triangulated categories and S is a saturated multiplicative system in $\mathcal{D}$ compatible with the structure of triangulated category on $\mathcal{D}$.

Let $X \in \text{Ob}(\mathcal{D})$. Recall from Categories, Remark 27.7 the filtered category X/S of arrows $s : X \to X'$ in S with source X. Dually, in Categories, Remark 27.15 we defined the cofiltered category S/X of arrows $s : X' \to X$ in S with target X.

Definition 14.2. Assumptions and notation as in Situation 14.1. Let $X \in \text{Ob}(\mathcal{D})$.

(1) we say the *right derived functor RF is defined at X* if the ind-object
$$(X/S) \longrightarrow \mathcal{D}', \quad (s : X \to X') \longmapsto F(X')$$
is essentially constant[5]; in this case the value Y in $\mathcal{D}'$ is called the *value of RF at X*.
(2) we say the *left derived functor LF is defined at X* if the pro-object
$$(S/X) \longrightarrow \mathcal{D}', \quad (s : X' \to X) \longmapsto F(X')$$
is essentially constant; in this case the value Y in $\mathcal{D}'$ is called the *value of LF at X*.

By abuse of notation we often denote the values simply $RF(X)$ or $LF(X)$.

It will turn out that the full subcategory of $\mathcal{D}$ consisting of objects where RF is defined is a triangulated subcategory, and RF will define a functor on this subcategory which transforms morphisms of S into isomorphisms.

Lemma 14.3. *Assumptions and notation as in Situation 14.1. Let $f : X \to Y$ be a morphism of $\mathcal{D}$.*

[5] For a discussion of when an ind-object or pro-object of a category is essentially constant we refer to Categories, Section 22.

(1) *If RF is defined at X and Y then there exists a unique morphism $RF(f) : RF(X) \to RF(Y)$ between the values such that for any commutative diagram*

$$\begin{tikzcd} X \arrow[r, "s"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "s'"] & Y' \end{tikzcd}$$

with $s, s' \in S$ the diagram

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(X') \arrow[r] \arrow[d] & RF(X) \arrow[d] \\ F(Y) \arrow[r] & F(Y') \arrow[r] & RF(Y) \end{tikzcd}$$

commutes.

(2) *If LF is defined at X and Y then there exists a unique morphism $LF(f) : LF(X) \to LF(Y)$ between the values such that for any commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "s"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "s'"] & Y \end{tikzcd}$$

with s, s' in S the diagram

$$\begin{tikzcd} LF(X) \arrow[r] \arrow[d] & F(X') \arrow[r] \arrow[d] & F(X) \arrow[d] \\ LF(Y) \arrow[r] & F(Y') \arrow[r] & F(Y) \end{tikzcd}$$

commutes.

Proof. Part (1) holds if we only assume that the colimits

$$RF(X) = \mathrm{colim}_{s:X\to X'} F(X') \quad \text{and} \quad RF(Y) = \mathrm{colim}_{s':Y\to Y'} F(Y')$$

exist. Namely, to give a morphism $RF(X) \to RF(Y)$ between the colimits is the same thing as giving for each $s : X \to X'$ in $\mathrm{Ob}(X/S)$ a morphism $F(X') \to RF(Y)$ compatible with morphisms in the category X/S. To get the morphism we choose a commutative diagram

$$\begin{tikzcd} X \arrow[r, "s"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "s'"] & Y' \end{tikzcd}$$

with s, s' in S as is possible by MS2 and we set $F(X') \to RF(Y)$ equal to the composition $F(X') \to F(Y') \to RF(Y)$. To see that this is independent of the choice of the diagram above use MS3. Details omitted. The proof of (2) is dual. $\square$

Lemma 14.4. *Assumptions and notation as in Situation 14.1. Let $s : X \to Y$ be an element of S.*

(1) *RF is defined at X if and only if it is defined at Y. In this case the map $RF(s) : RF(X) \to RF(Y)$ between values is an isomorphism.*

(2) *LF is defined at X if and only if it is defined at Y. In this case the map $LF(s) : LF(X) \to LF(Y)$ between values is an isomorphism.*

Proof. Omitted. $\square$

Lemma 14.5. *Assumptions and notation as in Situation 14.1. Let X be an object of $\mathcal{D}$ and $n \in \mathbf{Z}$.*

(1) *RF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $RF(X)[n] = RF(X[n])$ between values.*
(2) *LF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $LF(X)[n] \to LF(X[n])$ between values.*

Proof. Omitted. $\square$

Lemma 14.6. *Assumptions and notation as in Situation 14.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$. If RF is defined at two out of three of X, Y, Z, then it is defined at the third. Moreover, in this case*

$$(RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h))$$

is a distinguished triangle in $\mathcal{D}'$. Similarly for LF.

Proof. Say RF is defined at X, Y with values A, B. Let $RF(f) : A \to B$ be the induced morphism, see Lemma 14.3. We may choose a distinguished triangle $(A, B, C, RF(f), b, c)$ in $\mathcal{D}'$. We claim that C is a value of RF at Z.

To see this pick $s : X \to X'$ in S such that there exists a morphism $\alpha : A \to F(X')$ as in Categories, Definition 22.1. We may choose a commutative diagram

$$\begin{tikzcd} X \arrow[r, "s"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "s'"] & Y' \end{tikzcd}$$

with $s' \in S$ by MS2. Using that Y/S is filtered we can (after replacing s' by some $s'' : Y \to Y''$ in S) assume that there exists a morphism $\beta : B \to F(Y')$ as in Categories, Definition 22.1. Picture

$$\begin{tikzcd} A \arrow[r, "\alpha"'] \arrow[d, "RF(f)"'] & F(X') \arrow[r] \arrow[d, "F(f')"] & A \arrow[d, "RF(f)"] \\ B \arrow[r, "\beta"] & F(Y') \arrow[r] & B \end{tikzcd}$$

It may not be true that the left square commutes, but the outer and right squares commute. The assumption that the ind-object $\{F(Y')\}_{s' : Y' \to Y}$ is essentially constant means that there exists a $s'' : Y \to Y''$ in S and a morphism $h : Y' \to Y''$ such that $s'' = h \circ s'$ and such that $F(h)$ equal to $F(Y') \to B \to F(Y') \to F(Y'')$. Hence after replacing Y' by Y'' and β by $F(h) \circ \beta$ the diagram will commute (by direct computation with arrows).

Using MS6 choose a morphism of triangles

$$(s, s', s'') : (X, Y, Z, f, g, h) \longrightarrow (X', Y', Z', f', g', h')$$

with $s'' \in S$. By TR3 choose a morphism of triangles

$$(\alpha, \beta, \gamma) : (A, B, C, RF(f), b, c) \longrightarrow (F(X'), F(Y'), F(Z'), F(f'), F(g'), F(h'))$$

By Lemma 14.4 it suffices to prove that $RF(Z')$ is defined and has value C. Consider the category $\mathcal{I}$ of Lemma 5.10 of triangles

$$\mathcal{I} = \{(t, t', t'') : (X', Y', Z', f', g', h') \to (X'', Y'', Z'', f'', g'', h'') \mid (t, t', t'') \in S\}$$

To show that the system $F(Z'')$ is essentially constant over the category Z'/S is equivalent to showing that the system of $F(Z'')$ is essentially constant over $\mathcal{I}$ because $\mathcal{I} \to Z'/S$ is cofinal, see Categories, Lemma 22.11 (cofinality is proven in Lemma 5.10). For any object W in $\mathcal{D}'$ we consider the diagram

$$\begin{tikzcd}
\mathrm{colim}_\mathcal{I} \mathrm{Mor}_{\mathcal{D}'}(W, F(X'')) & \mathrm{Mor}_{\mathcal{D}'}(W, A) \arrow[l] \\
\mathrm{colim}_\mathcal{I} \mathrm{Mor}_{\mathcal{D}'}(W, F(Y'')) \arrow[u] & \mathrm{Mor}_{\mathcal{D}'}(W, B) \arrow[l] \arrow[u] \\
\mathrm{colim}_\mathcal{I} \mathrm{Mor}_{\mathcal{D}'}(W, F(Z'')) \arrow[u] & \mathrm{Mor}_{\mathcal{D}'}(W, C) \arrow[l] \arrow[u] \\
\mathrm{colim}_\mathcal{I} \mathrm{Mor}_{\mathcal{D}'}(W, F(X''[1])) \arrow[u] & \mathrm{Mor}_{\mathcal{D}'}(W, A[1]) \arrow[l] \arrow[u] \\
\mathrm{colim}_\mathcal{I} \mathrm{Mor}_{\mathcal{D}'}(W, F(Y''[1])) \arrow[u] & \mathrm{Mor}_{\mathcal{D}'}(W, B[1]) \arrow[l] \arrow[u]
\end{tikzcd}$$

where the horizontal arrows are given by composing with (α, β, γ). Since filtered colimits are exact (Algebra, Lemma 8.8) the left column is an exact sequence. Thus the 5 lemma (Homology, Lemma 5.20) tells us the

$$\mathrm{colim}_\mathcal{I} \mathrm{Mor}_{\mathcal{D}'}(W, F(Z'')) \longrightarrow \mathrm{Mor}_{\mathcal{D}'}(W, C)$$

is bijective. Choose an object $(t, t', t'') : (X', Y', Z') \to (X'', Y'', Z'')$ of $\mathcal{I}$. Applying what we just showed to $W = F(Z'')$ and the element $\mathrm{id}_{F(X'')}$ of the colimit we find a unique morphism $c_{(X'', Y'', Z'')} : F(Z'') \to C$ such that for some $(X'', Y'', Z'') \to (X''', Y''', Z'')$ in $\mathcal{I}$

$$F(Z'') \xrightarrow{c_{(X'', Y'', Z'')}} C \xrightarrow{\gamma} F(Z') \to F(Z'') \to F(Z''') \quad \text{equals} \quad F(Z'') \to F(Z''')$$

The family of morphisms $c_{(X'', Y'', Z'')}$ form an element c of $\lim_\mathcal{I} \mathrm{Mor}_{\mathcal{D}'}(F(Z''), C)$ by uniqueness (computation omitted). Finally, we show that $\mathrm{colim}_\mathcal{I} F(Z'') = C$ via the morphisms $c_{(X'', Y'', Z'')}$ which will finish the proof by Categories, Lemma 22.9. Namely, let W be an object of $\mathcal{D}'$ and let $d_{(X'', Y'', Z'')} : F(Z'') \to W$ be a family of maps corresponding to an element of $\lim_\mathcal{I} \mathrm{Mor}_{\mathcal{D}'}(F(Z''), W)$. If $d_{(X', Y', Z')} \circ \gamma = 0$, then for every object (X'', Y'', Z'') of $\mathcal{I}$ the morphism $d_{(X'', Y'', Z'')}$ is zero by the existence of $c_{(X'', Y'', Z'')}$ and the morphism $(X'', Y'', Z'') \to (X''', Y''', Z'')$ in $\mathcal{I}$ satisfying the displayed equality above. Hence the map

$$\lim_\mathcal{I} \mathrm{Mor}_{\mathcal{D}'}(F(Z''), W) \longrightarrow \mathrm{Mor}_{\mathcal{D}'}(C, W)$$

(coming from precomposing by γ) is injective. However, it is also surjective because the element c gives a left inverse. We conclude that C is the colimit by Categories, Remark 14.4. $\square$

Lemma 14.7. *Assumptions and notation as in Situation 14.1. Let X, Y be objects of $\mathcal{D}$.*

1. *If RF is defined at X and Y, then RF is defined at $X \oplus Y$.*
2. *If $\mathcal{D}'$ is Karoubian and RF is defined at $X \oplus Y$, then RF is defined at both X and Y.*

In either case we have $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. Similarly for LF.

Proof. If RF is defined at X and Y, then the distinguished triangle $X \to X \oplus Y \to Y \to X[1]$ (Lemma 4.11) and Lemma 14.6 shows that RF is defined at $X \oplus Y$ and that we have a distinguished triangle $RF(X) \to RF(X \oplus Y) \to RF(Y) \to RF(X)[1]$. Applying Lemma 4.11 to this once more we find that $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. This proves (1) and the final assertion.

Conversely, assume that RF is defined at $X \oplus Y$ and that $\mathcal{D}'$ is Karoubian. Since S is a saturated system S is the set of arrows which become invertible under the additive localization functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$, see Categories, Lemma 27.21. Thus for any $s : X \to X'$ and $s' : Y \to Y'$ in S the morphism $s \oplus s' : X \oplus Y \to X' \oplus Y'$ is an element of S. In this way we obtain a functor

$$X/S \times Y/S \longrightarrow (X \oplus Y)/S$$

Recall that the categories $X/S, Y/S, (X \oplus Y)/S$ are filtered (Categories, Remark 27.7). By Categories, Lemma 22.12 $X/S \times Y/S$ is filtered and $F|_{X/S} : X/S \to \mathcal{D}'$ (resp. $G|_{Y/S} : Y/S \to \mathcal{D}'$) is essentially constant if and only if $F|_{X/S} \circ \text{pr}_1 : X/S \times Y/S \to \mathcal{D}'$ (resp. $G|_{Y/S} \circ \text{pr}_2 : X/S \times Y/S \to \mathcal{D}'$) is essentially constant. Below we will show that the displayed functor is cofinal, hence by Categories, Lemma 22.11. we see that $F|_{(X \oplus Y)/S}$ is essentially constant implies that $F|_{X/S} \circ \text{pr}_1 \oplus F|_{Y/S} \circ \text{pr}_2 : X/S \times Y/S \to \mathcal{D}'$ is essentially constant. By Homology, Lemma 30.3 (and this is where we use that $\mathcal{D}'$ is Karoubian) we see that $F|_{X/S} \circ \text{pr}_1 \oplus F|_{Y/S} \circ \text{pr}_2$ being essentially constant implies $F|_{X/S} \circ \text{pr}_1$ and $F|_{Y/S} \circ \text{pr}_2$ are essentially constant proving that RF is defined at X and Y.

Proof that the displayed functor is cofinal. To do this pick any $t : X \oplus Y \to Z$ in S. Using MS2 we can find morphisms $Z \to X'$, $Z \to Y'$ and $s : X \to X'$, $s' : Y \to Y'$ in S such that

$$\begin{tikzcd} X \arrow[d, "s"] & X \oplus Y \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d, "s'"'] \\ X' & Z \arrow[l] \arrow[r] & Y' \end{tikzcd}$$

commutes. This proves there is a map $Z \to X' \oplus Y'$ in $(X \oplus Y)/S$, i.e., we get part (1) of Categories, Definition 17.1. To prove part (2) it suffices to prove that given $t : X \oplus Y \to Z$ and morphisms $s_i \oplus s'_i : Z \to X'_i \oplus Y'_i$, $i = 1, 2$ in $(X \oplus Y)/S$ we can find morphisms $a : X'_1 \to X'$, $b : X'_2 \to X'$, $c : Y'_1 \to Y'$, $d : Y'_2 \to Y'$ in S such that $a \circ s_1 = b \circ s_2$ and $c \circ s'_1 = d \circ s'_2$. To do this we first choose any X' and Y' and maps a, b, c, d in S; this is possible as X/S and Y/S are filtered. Then the two maps $a \circ s_1, b \circ s_2 : Z \to X'$ become equal in $S^{-1}\mathcal{D}$. Hence we can find a morphism $X' \to X''$ in S equalizing them. Similarly we find $Y' \to Y''$ in S equalizing $c \circ s'_1$ and $d \circ s'_2$. Replacing X' by X'' and Y' by Y'' we get $a \circ s_1 = b \circ s_2$ and $c \circ s'_1 = d \circ s'_2$.

The proof of the corresponding statements for LF are dual. $\square$

Proposition 14.8. *Assumptions and notation as in Situation 14.1.*

(1) *The full subcategory $\mathcal{E}$ of $\mathcal{D}$ consisting of objects at which RF is defined is a strictly full triangulated subcategory of $\mathcal{D}$.*
(2) *We obtain an exact functor $RF : \mathcal{E} \longrightarrow \mathcal{D}'$ of triangulated categories.*
(3) *Elements of S with either source or target in $\mathcal{E}$ are morphisms of $\mathcal{E}$.*
(4) *The functor $S_{\mathcal{E}}^{-1}\mathcal{E} \to S^{-1}\mathcal{D}$ is a fully faithful exact functor of triangulated categories.*
(5) *Any element of $S_{\mathcal{E}} = Arrows(\mathcal{E}) \cap S$ is mapped to an isomorphism by RF.*
(6) *We obtain an exact functor*
$$RF : S_{\mathcal{E}}^{-1}\mathcal{E} \longrightarrow \mathcal{D}'.$$
(7) *If $\mathcal{D}'$ is Karoubian, then $\mathcal{E}$ is a saturated triangulated subcategory of $\mathcal{D}$.*

A similar result holds for LF.

Proof. Since S is saturated it contains all isomorphisms (see remark following Categories, Definition 27.20). Hence (1) follows from Lemmas 14.4, 14.6, and 14.5. We get (2) from Lemmas 14.3, 14.5, and 14.6. We get (3) from Lemma 14.4. The fully faithfulness in (4) follows from (3) and the definitions. The fact that $S_{\mathcal{E}}^{-1}\mathcal{E} \to S^{-1}\mathcal{D}$ is exact follows from the fact that a triangle in $S_{\mathcal{E}}^{-1}\mathcal{E}$ is distinguished if and only if it is isomorphic to the image of a distinguished triangle in $\mathcal{E}$, see proof of Proposition 5.6. Part (5) follows from Lemma 14.4. The factorization of $RF : \mathcal{E} \to \mathcal{D}'$ through an exact functor $S_{\mathcal{E}}^{-1}\mathcal{E} \to \mathcal{D}'$ follows from Lemma 5.7. Part (7) follows from Lemma 14.7. $\square$

Proposition 14.8 tells us that RF lives on a maximal strictly full triangulated subcategory of $S^{-1}\mathcal{D}$ and is an exact functor on this triangulated category. Picture:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "F"'] \arrow[d, "Q"'] & & \mathcal{D}' \\
S^{-1}\mathcal{D} & & S_{\mathcal{E}}^{-1}\mathcal{E} \arrow[ll, "\text{fully faithful}"', "\text{exact}"] \arrow[u, "RF"']
\end{tikzcd}$$

Definition 14.9. In Situation 14.1. We say F is *right derivable*, or that RF *everywhere defined* if RF is defined at every object of $\mathcal{D}$. We say F is *left derivable*, or that LF *everywhere defined* if LF is defined at every object of $\mathcal{D}$.

In this case we obtain a right (resp. left) derived functor

$$RF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}', \quad (\text{resp. } LF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}'), \tag{14.9.1}$$

see Proposition 14.8. In most interesting situations it is not the case that $RF \circ Q$ is equal to F. In fact, it might happen that the canonical map $F(X) \to RF(X)$ is never an isomorphism. In practice this does not happen, because in practice we only know how to prove F is right derivable by showing that RF can be computed by evaluating F at judiciously chosen objects of the triangulated category $\mathcal{D}$. This warrants a definition.

Definition 14.10. In Situation 14.1.

(1) An object X of $\mathcal{D}$ *computes* RF if RF is defined at X and the canonical map $F(X) \to RF(X)$ is an isomorphism.
(2) An object X of $\mathcal{D}$ *computes* LF if LF is defined at X and the canonical map $LF(X) \to F(X)$ is an isomorphism.

Lemma 14.11. *Assumptions and notation as in Situation 14.1. Let X be an object of $\mathcal{D}$ and $n \in \mathbf{Z}$.*

1. *X computes RF if and only if $X[n]$ computes RF.*
2. *X computes LF if and only if $X[n]$ computes LF.*

Proof. Omitted. □

Lemma 14.12. *Assumptions and notation as in Situation 14.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$. If X, Y compute RF then so does Z. Similar for LF.*

Proof. By Lemma 14.6 we know that RF is defined at Z and that RF applied to the triangle produces a distinguished triangle. Consider the morphism of distinguished triangles

$$\begin{tikzcd} (F(X), F(Y), F(Z), F(f), F(g), F(h)) \arrow[d] \\ (RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h)) \end{tikzcd}$$

Two out of three maps are isomorphisms, hence so is the third. □

Lemma 14.13. *Assumptions and notation as in Situation 14.1. Let X, Y be objects of $\mathcal{D}$. If $X \oplus Y$ computes RF, then X and Y compute RF. Similarly for LF.*

Proof. If $X \oplus Y$ computes RF, then $RF(X \oplus Y) = F(X) \oplus F(Y)$. In the proof of Lemma 14.7 we have seen that the functor $X/S \times Y/S \to (X \oplus Y)/S$, $(s, s') \mapsto s \oplus s'$ is cofinal. We will use this without further mention. Let $s : X \to X'$ be an element of S. Then $F(X) \to F(X')$ has a section, namely,

$$F(X') \to F(X' \oplus Y) \to RF(X' \oplus Y) = RF(X \oplus Y) = F(X) \oplus F(Y) \to F(X).$$

where we have used Lemma 14.4. Hence $F(X') = F(X) \oplus E$ for some object E of $\mathcal{D}'$ such that $E \to F(X' \oplus Y) \to RF(X' \oplus Y) = RF(X \oplus Y)$ is zero (Lemma 4.12). Because RF is defined at $X' \oplus Y$ with value $F(X) \oplus F(Y)$ we can find a morphism $t : X' \oplus Y \to Z$ of S such that $F(t)$ annihilates E. We may assume $Z = X'' \oplus Y''$ and $t = t' \oplus t''$ with $t', t'' \in S$. Then $F(t')$ annihilates E. It follows that F is essentially constant on X/S with value $F(X)$ as desired. □

Lemma 14.14. *Assumptions and notation as in Situation 14.1.*

1. *If for every object $X \in \mathrm{Ob}(\mathcal{D})$ there exists an arrow $s : X \to X'$ in S such that X' computes RF, then RF is everywhere defined.*
2. *If for every object $X \in \mathrm{Ob}(\mathcal{D})$ there exists an arrow $s : X' \to X$ in S such that X' computes LF, then LF is everywhere defined.*

Proof. This is clear from the definitions. □

Lemma 14.15. *Assumptions and notation as in Situation 14.1. If there exists a subset $\mathcal{I} \subset \mathrm{Ob}(\mathcal{D})$ such that*

1. *for all $X \in \mathrm{Ob}(\mathcal{D})$ there exists $s : X \to X'$ in S with $X' \in \mathcal{I}$, and*
2. *for every arrow $s : X \to X'$ in S with $X, X' \in \mathcal{I}$ the map $F(s) : F(X) \to F(X')$ is an isomorphism,*

then RF is everywhere defined and every $X \in \mathcal{I}$ computes RF. Dually, if there exists a subset $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ such that

(1) *for all* $X \in \mathrm{Ob}(\mathcal{D})$ *there exists* $s : X' \to X$ *in* S *with* $X' \in \mathcal{P}$*, and*
(2) *for every arrow* $s : X \to X'$ *in* S *with* $X, X' \in \mathcal{P}$ *the map* $F(s) : F(X) \to F(X')$ *is an isomorphism,*

then LF *is everywhere defined and every* $X \in \mathcal{P}$ *computes* LF*.*

Proof. Let X be an object of $\mathcal{D}$. Assumption (1) implies that the arrows $s : X \to X'$ in S with $X' \in \mathcal{I}$ are cofinal in the category X/S. Assumption (2) implies that F is constant on this cofinal subcategory. Clearly this implies that $F : (X/S) \to \mathcal{D}'$ is essentially constant with value $F(X')$ for any $s : X \to X'$ in S with $X' \in \mathcal{I}$. □

Lemma 14.16. *Let* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *be triangulated categories. Let* S*, resp.* S' *be a saturated multiplicative system in* $\mathcal{A}$*, resp.* $\mathcal{B}$ *compatible with the triangulated structure. Let* $F : \mathcal{A} \to \mathcal{B}$ *and* $G : \mathcal{B} \to \mathcal{C}$ *be exact functors. Denote* $F' : \mathcal{A} \to (S')^{-1}\mathcal{B}$ *the composition of* F *with the localization functor.*

(1) *If* RF'*,* RG*,* $R(G \circ F)$ *are everywhere defined, then there is a canonical transformation of functors* $t : R(G \circ F) \longrightarrow RG \circ RF'$*.*
(2) *If* LF'*,* LG*,* $L(G \circ F)$ *are everywhere defined, then there is a canonical transformation of functors* $t : LG \circ LF' \to L(G \circ F)$*.*

Proof. In this proof we try to be careful. Hence let us think of the derived functors as the functors

$$RF' : S^{-1}\mathcal{A} \to (S')^{-1}\mathcal{B}, \quad R(G \circ F) : S^{-1}\mathcal{A} \to \mathcal{C}, \quad RG : (S')^{-1}\mathcal{B} \to \mathcal{C}.$$

Let us denote $Q_A : \mathcal{A} \to S^{-1}\mathcal{A}$ and $Q_B : \mathcal{B} \to (S')^{-1}\mathcal{B}$ the localization functors. Then $F' = Q_B \circ F$. Note that for every object Y of $\mathcal{B}$ there is a canonical map

$$G(Y) \longrightarrow RG(Q_B(Y))$$

in other words, there is a transformation of functors $t' : G \to RG \circ Q_B$. Let X be an object of $\mathcal{A}$. We have

$$\begin{aligned} R(G \circ F)(Q_A(X)) &= \mathrm{colim}_{s:X \to X' \in S}\, G(F(X')) \\ &\xrightarrow{t'} \mathrm{colim}_{s:X \to X' \in S}\, RG(Q_B(F(X'))) \\ &= \mathrm{colim}_{s:X \to X' \in S}\, RG(F'(X')) \\ &= RG(\mathrm{colim}_{s:X \to X' \in S}\, F'(X')) \\ &= RG(RF'(X)). \end{aligned}$$

The system $F'(X')$ is essentially constant in the category $(S')^{-1}\mathcal{B}$. Hence we may pull the colimit inside the functor RG in the third equality of the diagram above, see Categories, Lemma 22.8 and its proof. We omit the proof this defines a transformation of functors. The case of left derived functors is similar. □

15. Derived functors on derived categories

In practice derived functors come about most often when given an additive functor between abelian categories.

Situation 15.1. Here $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories. This induces exact functors

$$F : K(\mathcal{A}) \to K(\mathcal{B}), \quad K^+(\mathcal{A}) \to K^+(\mathcal{B}), \quad K^-(\mathcal{A}) \to K^-(\mathcal{B}).$$

See Lemma 10.6. We also denote F the composition $K(\mathcal{A}) \to D(\mathcal{B})$, $K^+(\mathcal{A}) \to D^+(\mathcal{B})$, and $K^-(\mathcal{A}) \to D^-(\mathcal{B})$ of F with the localization functor $K(\mathcal{B}) \to D(\mathcal{B})$, etc. This situation leads to four derived functors we will consider in the following.

(1) The right derived functor of $F : K(\mathcal{A}) \to D(\mathcal{B})$ relative to the multiplicative system $\text{Qis}(\mathcal{A})$.
(2) The right derived functor of $F : K^+(\mathcal{A}) \to D^+(\mathcal{B})$ relative to the multiplicative system $\text{Qis}^+(\mathcal{A})$.
(3) The left derived functor of $F : K(\mathcal{A}) \to D(\mathcal{B})$ relative to the multiplicative system $\text{Qis}(\mathcal{A})$.
(4) The left derived functor of $F : K^-(\mathcal{A}) \to D^-(\mathcal{B})$ relative to the multiplicative system $\text{Qis}^-(\mathcal{A})$.

Each of these cases is an example of Situation 14.1.

Some of the ambiguity that may arise is alleviated by the following.

Lemma 15.2. *In Situation 15.1.*

(1) *Let X be an object of $K^+(\mathcal{A})$. The right derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ is defined at X if and only if the right derived functor of $K^+(\mathcal{A}) \to D^+(\mathcal{B})$ is defined at X. Moreover, the values are canonically isomorphic.*
(2) *Let X be an object of $K^+(\mathcal{A})$. Then X computes the right derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ if and only if X computes the right derived functor of $K^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(3) *Let X be an object of $K^-(\mathcal{A})$. The left derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ is defined at X if and only if the left derived functor of $K^-(\mathcal{A}) \to D^-(\mathcal{B})$ is defined at X. Moreover, the values are canonically isomorphic.*
(4) *Let X be an object of $K^-(\mathcal{A})$. Then X computes the left derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ if and only if X computes the left derived functor of $K^-(\mathcal{A}) \to D^-(\mathcal{B})$.*

Proof. Let X be an object of $K^+(\mathcal{A})$. Consider a quasi-isomorphism $s : X \to X'$ in $K(\mathcal{A})$. By Lemma 11.5 there exists quasi-isomorphism $X' \to X''$ with X'' bounded below. Hence we see that $X/\text{Qis}^+(\mathcal{A})$ is cofinal in $X/\text{Qis}(\mathcal{A})$. Thus it is clear that (1) holds. Part (2) follows directly from part (1). Parts (3) and (4) are dual to parts (1) and (2). $\square$

Given an object A of an abelian category $\mathcal{A}$ we get a complex

$$A[0] = (\ldots \to 0 \to A \to 0 \to \ldots)$$

where A is placed in degree zero. Hence a functor $\mathcal{A} \to K(\mathcal{A})$, $A \mapsto A[0]$. Let us temporarily say that a partial functor is one that is defined on a subcategory.

Definition 15.3. In Situation 15.1.

(1) The *right derived functors of F* are the partial functors RF associated to cases (1) and (2) of Situation 15.1.
(2) The *left derived functors of F* are the partial functors LF associated to cases (3) and (4) of Situation 15.1.
(3) An object A of $\mathcal{A}$ is said to be *right acyclic for F*, or *acyclic for RF* if $A[0]$ computes RF.
(4) An object A of $\mathcal{A}$ is said to be *left acyclic for F*, or *acyclic for LF* if $A[0]$ computes LF.

The following few lemmas give some criteria for the existence of enough acyclics.

Lemma 15.4. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset containing 0 such that every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$. Let $a \in \mathbf{Z}$.*

1. *Given $K^\bullet$ with $K^n = 0$ for $n > a$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^n \in \mathcal{P}$ and $P^n \to K^n$ surjective for all n and $P^n = 0$ for $n > a$.*
2. *Given $K^\bullet$ with $H^n(K^\bullet) = 0$ for $n > a$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^n \in \mathcal{P}$ for all n and $P^n = 0$ for $n > a$.*

Proof. Proof of part (1). Consider the following induction hypothesis IH_n: There are $P^j \in \mathcal{P}$, $j \geq n$, with $P^j = 0$ for $j > a$, maps $d^j : P^j \to P^{j+1}$ for $j \geq n$, and surjective maps $\alpha^j : P^j \to K^j$ for $j \geq n$ such that the diagram

$$\begin{tikzcd} & P^n \arrow[r] \arrow[d, "\alpha"] & P^{n+1} \arrow[r] \arrow[d, "\alpha"] & P^{n+2} \arrow[r] \arrow[d, "\alpha"] & \ldots \\ \ldots \arrow[r] K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & K^{n+2} \arrow[r] & \ldots \end{tikzcd}$$

is commutative, such that $d^{j+1} \circ d^j = 0$ for $j \geq n$, such that α induces isomorphisms $H^j(K^\bullet) \to \mathrm{Ker}(d^j)/\mathrm{Im}(d^{j-1})$ for $j > n$, and such that $\alpha : \mathrm{Ker}(d^n) \to \mathrm{Ker}(d^n_K)$ is surjective. Then we choose a surjection

$$P^{n-1} \longrightarrow K^{n-1} \times_{K^n} \mathrm{Ker}(d^n) = K^{n-1} \times_{\mathrm{Ker}(d^n_K)} \mathrm{Ker}(d^n)$$

with P^{n-1} in $\mathcal{P}$. This allows us to extend the diagram above to

$$\begin{tikzcd} & P^{n-1} \arrow[r] \arrow[d, "\alpha"] & P^n \arrow[r] \arrow[d, "\alpha"] & P^{n+1} \arrow[r] \arrow[d, "\alpha"] & P^{n+2} \arrow[r] \arrow[d, "\alpha"] & \ldots \\ \ldots \arrow[r] & K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & K^{n+2} \arrow[r] & \ldots \end{tikzcd}$$

The reader easily checks that IH_{n-1} holds with this choice.

We finish the proof of (1) as follows. First we note that IH_n is true for $n = a + 1$ since we can just take $P^j = 0$ for $j > a$. Hence we see that proceeding by descending induction we produce a complex $P^\bullet$ with $P^n = 0$ for $n > a$ consisting of objects from $\mathcal{P}$, and a termwise surjective quasi-isomorphism $\alpha : P^\bullet \to K^\bullet$ as desired.

Proof of part (2). The assumption implies that the morphism $\tau_{\leq a}K^\bullet \to K^\bullet$ (Homology, Section 15) is a quasi-isomorphism. Apply part (1) to find $P^\bullet \to \tau_{\leq a}K^\bullet$. The composition $P^\bullet \to K^\bullet$ is the desired quasi-isomorphism. □

Lemma 15.5. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subset \mathrm{Ob}(\mathcal{A})$ be a subset containing 0 such that every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$. Let $a \in \mathbf{Z}$.*

1. *Given $K^\bullet$ with $K^n = 0$ for $n < a$ there exists a quasi-isomorphism $K^\bullet \to I^\bullet$ with $K^n \to I^n$ injective and $I^n \in \mathcal{I}$ for all n and $I^n = 0$ for $n < a$,*
2. *Given $K^\bullet$ with $H^n(K^\bullet) = 0$ for $n < a$ there exists a quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^n \in \mathcal{I}$ and $I^n = 0$ for $n < a$.*

Proof. This lemma is dual to Lemma 15.4. □

Lemma 15.6. *In Situation 15.1. Let $\mathcal{I} \subset \mathrm{Ob}(\mathcal{A})$ be a subset with the following properties:*

1. *every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$,*

(2) *for any short exact sequence* $0 \to P \to Q \to R \to 0$ *of* $\mathcal{A}$ *with* $P, Q \in \mathcal{I}$, *then* $R \in \mathcal{I}$, *and* $0 \to F(P) \to F(Q) \to F(R) \to 0$ *is exact.*

Then every object of $\mathcal{I}$ *is acyclic for* RF.

Proof. We may add 0 to $\mathcal{I}$ if necessary. Pick $A \in \mathcal{I}$. Let $A[0] \to K^\bullet$ be a quasi-isomorphism with $K^\bullet$ bounded below. Then we can find a quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below and each $I^n \in \mathcal{I}$, see Lemma 15.5. Hence we see that these resolutions are cofinal in the category $A[0]/\text{Qis}^+(\mathcal{A})$. To finish the proof it therefore suffices to show that for any quasi-isomorphism $A[0] \to I^\bullet$ with $I^\bullet$ bounded below and $I^n \in \mathcal{I}$ we have $F(A)[0] \to F(I^\bullet)$ is a quasi-isomorphism. To see this suppose that $I^n = 0$ for $n < n_0$. Of course we may assume that $n_0 < 0$. Starting with $n = n_0$ we prove inductively that $\text{Im}(d^{n-1}) = \text{Ker}(d^n)$ and $\text{Im}(d^{-1})$ are elements of $\mathcal{I}$ using property (2) and the exact sequences

$$0 \to \text{Ker}(d^n) \to I^n \to \text{Im}(d^n) \to 0.$$

Moreover, property (2) also guarantees that the complex

$$0 \to F(I^{n_0}) \to F(I^{n_0+1}) \to \ldots \to F(I^{-1}) \to F(\text{Im}(d^{-1})) \to 0$$

is exact. The exact sequence $0 \to \text{Im}(d^{-1}) \to I^0 \to I^0/\text{Im}(d^{-1}) \to 0$ implies that $I^0/\text{Im}(d^{-1})$ is an element of $\mathcal{I}$. The exact sequence $0 \to A \to I^0/\text{Im}(d^{-1}) \to \text{Im}(d^0) \to 0$ then implies that $\text{Im}(d^0) = \text{Ker}(d^1)$ is an elements of $\mathcal{I}$ and from then on one continues as before to show that $\text{Im}(d^{n-1}) = \text{Ker}(d^n)$ is an element of $\mathcal{I}$ for all $n > 0$. Applying F to each of the short exact sequences mentioned above and using (2) we observe that $F(A)[0] \to F(I^\bullet)$ is an isomorphism as desired. □

Lemma 15.7. *In Situation 15.1. Let* $\mathcal{P} \subset \text{Ob}(\mathcal{A})$ *be a subset with the following properties:*

(1) *every object of* $\mathcal{A}$ *is a quotient of an element of* $\mathcal{P}$,
(2) *for any short exact sequence* $0 \to P \to Q \to R \to 0$ *of* $\mathcal{A}$ *with* $Q, R \in \mathcal{P}$, *then* $P \in \mathcal{P}$, *and* $0 \to F(P) \to F(Q) \to F(R) \to 0$ *is exact.*

Then every object of $\mathcal{P}$ *is acyclic for* LF.

Proof. Dual to the proof of Lemma 15.6. □

16. Higher derived functors

The following simple lemma shows that right derived functors "move to the right".

Lemma 16.1. *Let* $F : \mathcal{A} \to \mathcal{B}$ *be an additive functor between abelian categories. Let* $K^\bullet$ *be a complex of* $\mathcal{A}$ *and* $a \in \mathbf{Z}$.

(1) *If* $H^i(K^\bullet) = 0$ *for all* $i < a$ *and* RF *is defined at* $K^\bullet$, *then* $H^i(RF(K^\bullet)) = 0$ *for all* $i < a$.
(2) *If* RF *is defined at* $K^\bullet$ *and* $\tau_{\leq a}K^\bullet$, *then* $H^i(RF(\tau_{\leq a}K^\bullet)) = H^i(RF(K^\bullet))$ *for all* $i \leq a$.

Proof. Assume $K^\bullet$ satisfies the assumptions of (1). Let $K^\bullet \to L^\bullet$ be any quasi-isomorphism. Then it is also true that $K^\bullet \to \tau_{\geq a}L^\bullet$ is a quasi-isomorphism by our assumption on $K^\bullet$. Hence in the category $K^\bullet/\text{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : K^\bullet \to L^\bullet$ with $L^n = 0$ for $n < a$ are cofinal. Thus RF is the value of the essentially constant ind-object $F(L^\bullet)$ for these s it follows that $H^i(RF(K^\bullet)) = 0$ for $i < a$.

To prove (2) we use the distinguished triangle

$$\tau_{\leq a}K^\bullet \to K^\bullet \to \tau_{\geq a+1}K^\bullet \to (\tau_{\leq a}K^\bullet)[1]$$

of Remark 12.4 to conclude via Lemma 14.6 that RF is defined at $\tau_{\geq a+1}K^\bullet$ as well and that we have a distinguished triangle

$$RF(\tau_{\leq a}K^\bullet) \to RF(K^\bullet) \to RF(\tau_{\geq a+1}K^\bullet) \to RF(\tau_{\leq a}K^\bullet)[1]$$

in $D(\mathcal{B})$. By part (1) we see that $RF(\tau_{\geq a+1}K^\bullet)$ has vanishing cohomology in degrees $< a+1$. The long exact cohomology sequence of this distinguished triangle then shows what we want. $\square$

Definition 16.2. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $i \in \mathbf{Z}$. The *ith right derived functor R^iF of F* is the functor

$$R^iF = H^i \circ RF : \mathcal{A} \longrightarrow \mathcal{B}$$

The following lemma shows that it really does not make a lot of sense to take the right derived functor unless the functor is left exact.

Lemma 16.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.*

(1) *We have $R^iF = 0$ for $i < 0$,*
(2) *R^0F is left exact,*
(3) *the map $F \to R^0F$ is an isomorphism if and only if F is left exact.*

Proof. Let A be an object of $\mathcal{A}$. Let $A[0] \to K^\bullet$ be any quasi-isomorphism. Then it is also true that $A[0] \to \tau_{\geq 0}K^\bullet$ is a quasi-isomorphism. Hence in the category $A[0]/\text{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : A[0] \to K^\bullet$ with $K^n = 0$ for $n < 0$ are cofinal. Thus it is clear that $H^i(RF(A[0])) = 0$ for $i < 0$. Moreover, for such an s the sequence

$$0 \to A \to K^0 \to K^1$$

is exact. Hence if F is left exact, then $0 \to F(A) \to F(K^0) \to F(K^1)$ is exact as well, and we see that $F(A) \to H^0(F(K^\bullet))$ is an isomorphism for every $s : A[0] \to K^\bullet$ as above which implies that $H^0(RF(A[0])) = F(A)$.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathcal{A}$. By Lemma 12.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $D^+(\mathcal{A})$. From the long exact cohomology sequence (and the vanishing for $i < 0$ proved above) we deduce that $0 \to R^0F(A) \to R^0F(B) \to R^0F(C)$ is exact. Hence R^0F is left exact. Of course this also proves that if $F \to R^0F$ is an isomorphism, then F is left exact. $\square$

Lemma 16.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let A be an object of $\mathcal{A}$.*

(1) *A is right acyclic for F if and only if $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for all $i > 0$,*
(2) *if F is left exact, then A is right acyclic for F if and only if $R^iF(A) = 0$ for all $i > 0$.*

Proof. If A is right acyclic for F, then $RF(A[0]) = F(A)[0]$ and in particular $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for $i \neq 0$. Conversely, if $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for all $i > 0$ then $F(A[0]) \to$

$RF(A[0])$ is a quasi-isomorphism by Lemma 16.3 part (1) and hence A is acyclic. If F is left exact then $F = R^0F$, see Lemma 16.3. $\square$

Lemma 16.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathcal{A}$.*

1. *If A and C are right acyclic for F then so is B.*
2. *If A and B are right acyclic for F then so is C.*
3. *If B and C are right acyclic for F and $F(B) \to F(C)$ is surjective then A is right acyclic for F.*

In each of the three cases

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is a short exact sequence of $\mathcal{B}$.

Proof. By Lemma 12.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $K^+(\mathcal{A})$. As RF is an exact functor and since $R^iF = 0$ for $i < 0$ and $R^0F = F$ (Lemma 16.3) we obtain an exact cohomology sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to \ldots$$

in the abelian category $\mathcal{B}$. Thus the lemma follows from the characterization of acyclic objects in Lemma 16.4. $\square$

Lemma 16.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.*

1. *The functors R^iF, $i \geq 0$ come equipped with a canonical structure of a δ-functor from $\mathcal{A} \to \mathcal{B}$, see Homology, Definition 12.1.*
2. *If every object of $\mathcal{A}$ is a subobject of a right acyclic object for F, then $\{R^iF, \delta\}_{i \geq 0}$ is a universal δ-functor, see Homology, Definition 12.3.*

Proof. The functor $\mathcal{A} \to \text{Comp}^+(\mathcal{A})$, $A \mapsto A[0]$ is exact. The functor $\text{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$ is a δ-functor, see Lemma 12.1. The functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is exact. Finally, the functor $H^0 : D^+(\mathcal{B}) \to \mathcal{B}$ is a homological functor, see Definition 11.3. Hence we get the structure of a δ-functor from Lemma 4.22 and Lemma 4.21. Part (2) follows from Homology, Lemma 12.4 and the description of acyclics in Lemma 16.4. $\square$

Lemma 16.7 (Leray's acyclicity lemma)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Let $A^\bullet$ be a bounded below complex of right F-acyclic objects such that RF is defined at $A^\bullet$[6]. The canonical map*

$$F(A^\bullet) \longrightarrow RF(A^\bullet)$$

is an isomorphism in $D^+(\mathcal{B})$, i.e., $A^\bullet$ computes RF.

Proof. Let $A^\bullet$ be a bounded complex of right F-acyclic objects. We claim that RF is defined at $A^\bullet$ and that $F(A^\bullet) \to RF(A^\bullet)$ is an isomorphism in $D^+(\mathcal{B})$. Namely, it holds for complexes with at most one nonzero right F-acyclic object for example by Lemma 16.4. Next, suppose that $A^n = 0$ for $n \notin [a, b]$. Using the "stupid" truncations we obtain a termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1}A^\bullet \to A^\bullet \to \sigma_{\leq a}A^\bullet \to 0$$

[6]For example this holds if $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.

see Homology, Section 15. Thus a distinguished triangle $(\sigma_{\geq a+1}A^\bullet, A^\bullet, \sigma_{\leq a}A^\bullet)$. By induction hypothesis RF is defined for the two outer complexes and these complexes compute RF. Then the same is true for the middle one by Lemma 14.12.

Suppose that $A^\bullet$ is a bounded below complex of acyclic objects such that RF is defined at $A^\bullet$. To show that $F(A^\bullet) \to RF(A^\bullet)$ is an isomorphism in $D^+(\mathcal{B})$ it suffices to show that $H^i(F(A^\bullet)) \to H^i(RF(A^\bullet))$ is an isomorphism for all i. Pick i. Consider the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq i+2}A^\bullet \to A^\bullet \to \sigma_{\leq i+1}A^\bullet \to 0.$$

Note that this induces a termwise split short exact sequence

$$0 \to \sigma_{\geq i+2}F(A^\bullet) \to F(A^\bullet) \to \sigma_{\leq i+1}F(A^\bullet) \to 0.$$

Hence we get distinguished triangles

$$(\sigma_{\geq i+2}A^\bullet, A^\bullet, \sigma_{\leq i+1}A^\bullet) \quad \text{and} \quad (\sigma_{\geq i+2}F(A^\bullet), F(A^\bullet), \sigma_{\leq i+1}F(A^\bullet))$$

Since RF is defined at $A^\bullet$ (by assumption) and at $\sigma_{\leq i+1}A^\bullet$ (by the first paragraph) we see that RF is defined at $\sigma_{\geq i+1}A^\bullet$ and we get a distinghuished triangle

$$(RF(\sigma_{\geq i+2}A^\bullet), RF(A^\bullet), RF(\sigma_{\leq i+1}A^\bullet))$$

See Lemma 14.6. Using these distinguished triangles we obtain a map of exact sequences

$$\begin{tikzcd}
H^i(\sigma_{\geq i+2}F(A^\bullet)) \arrow[r] \arrow[d] & H^i(F(A^\bullet)) \arrow[r] \arrow[d, "\alpha"] & H^i(\sigma_{\leq i+1}F(A^\bullet)) \arrow[r] \arrow[d, "\beta"] & H^{i+1}(\sigma_{\geq i+2}F(A^\bullet)) \arrow[d] \\
H^i(RF(\sigma_{\geq i+2}A^\bullet)) \arrow[r] & H^i(RF(A^\bullet)) \arrow[r] & H^i(RF(\sigma_{\leq i+1}A^\bullet)) \arrow[r] & H^{i+1}(RF(\sigma_{\geq i+2}A^\bullet))
\end{tikzcd}$$

By the results of the first paragraph the map β is an isomorphism. By inspection the objects on the upper left and the upper right are zero. Hence to finish the proof it suffices to show that $H^i(RF(\sigma_{\geq i+2}A^\bullet)) = 0$ and $H^{i+1}(RF(\sigma_{\geq i+2}A^\bullet)) = 0$. This follows immediately from Lemma 16.1. $\square$

Proposition 16.8. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories.*

(1) *If every object of $\mathcal{A}$ injects into an object acyclic for RF, then RF is defined on all of $K^+(\mathcal{A})$ and we obtain an exact functor*

$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$

see (14.9.1). Moreover, any bounded below complex $A^\bullet$ whose terms are acyclic for RF computes RF.

(2) *If every object of $\mathcal{A}$ is quotient of an object acyclic for LF, then LF is defined on all of $K^-(\mathcal{A})$ and we obtain an exact functor*

$$LF : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B})$$

see (14.9.1). Moreover, any bounded above complex $A^\bullet$ whose terms are acyclic for LF computes LF.

Proof. Assume every object of $\mathcal{A}$ injects into an object acyclic for RF. Let $\mathcal{I}$ be the set of objects acyclic for RF. Let $K^\bullet$ be a bounded below complex in $\mathcal{A}$. By Lemma 15.5 there exists a quasi-isomorphism $\alpha : K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below and $I^n \in \mathcal{I}$. Hence in order to prove (1) it suffices to show that $F(I^\bullet) \to F((I')^\bullet)$ is a quasi-isomorphism when $s : I^\bullet \to (I')^\bullet$ is a quasi-isomorphism of bounded below complexes of objects from $\mathcal{I}$, see Lemma 14.15. Note that the cone $C(s)^\bullet$ is an acyclic bounded below complex all of whose terms are in $\mathcal{I}$. Hence it suffices to show: given an acyclic bounded below complex $I^\bullet$ all of whose terms are in $\mathcal{I}$ the complex $F(I^\bullet)$ is acyclic.

Say $I^n = 0$ for $n < n_0$. Setting $J^n = \mathrm{Im}(d^n)$ we break $I^\bullet$ into short exact sequences $0 \to J^n \to I^{n+1} \to J^{n+1} \to 0$ for $n \geq n_0$. These sequences induce distinguished triangles (J^n, I^{n+1}, J^{n+1}) in $D^+(\mathcal{A})$ by Lemma 12.1. For each $k \in \mathbf{Z}$ denote H_k the assertion: For all $n \leq k$ the right derived functor RF is defined at J^n and $R^iF(J^n) = 0$ for $i \neq 0$. Then H_k holds trivially for $k \leq n_0$. If H_n holds, then, using Proposition 14.8, we see that RF is defined at J^{n+1} and $(RF(J^n), RF(I^{n+1}), RF(J^{n+1}))$ is a distinguished triangle of $D^+(\mathcal{B})$. Thus the long exact cohomology sequence (11.1.1) associated to this triangle gives an exact sequence

$$0 \to R^{-1}F(J^{n+1}) \to R^0F(J^n) \to F(I^{n+1}) \to R^0F(J^{n+1}) \to 0$$

and gives that $R^iF(J^{n+1}) = 0$ for $i \notin \{-1, 0\}$. By Lemma 16.1 we see that $R^{-1}F(J^{n+1}) = 0$. This proves that H_{n+1} is true hence H_k holds for all k. We also conclude that

$$0 \to R^0F(J^n) \to F(I^{n+1}) \to R^0F(J^{n+1}) \to 0$$

is short exact for all n. This in turn proves that $F(I^\bullet)$ is exact.

The proof in the case of LF is dual. □

Lemma 16.9. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of abelian categories. Then*

1. *every object of $\mathcal{A}$ is right acyclic for F,*
2. *$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined,*
3. *$RF : D(\mathcal{A}) \to D(\mathcal{B})$ is everywhere defined,*
4. *every complex computes RF, in other words, the canonical map $F(K^\bullet) \to RF(K^\bullet)$ is an isomorphism for all complexes, and*
5. *$R^iF = 0$ for $i \neq 0$.*

Proof. This is true because F transforms acyclic complexes into acyclic complexes and quasi-isomorphisms into quasi-isomorphisms. Details omitted. □

17. Triangulated subcategories of the derived category

Let $\mathcal{A}$ be an abelian category. In this section we look at certain strictly full saturated triangulated subcategories $\mathcal{D}' \subset D(\mathcal{A})$.

Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory, see Homology, Definition 10.1 and Lemma 10.3. We let $D_{\mathcal{B}}(\mathcal{A})$ the full subcategory of $D(\mathcal{A})$ whose objects are

$$\mathrm{Ob}(D_{\mathcal{B}}(\mathcal{A})) = \{X \in \mathrm{Ob}(D(\mathcal{A})) \mid H^n(X) \text{ is an object of } \mathcal{B} \text{ for all } n\}$$

We also define $D^+_{\mathcal{B}}(\mathcal{A}) = D^+(\mathcal{A}) \cap D_{\mathcal{B}}(\mathcal{A})$ and similarly for the other bounded versions.

Lemma 17.1. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory. The category $D_{\mathcal{B}}(\mathcal{A})$ is a strictly full saturated triangulated subcategory of $D(\mathcal{A})$. Similarly for the bounded versions.*

Proof. It is clear that $D_\mathcal{B}(\mathcal{A})$ is an additive subcategory preserved under the translation functors. If $X \oplus Y$ is in $D_\mathcal{B}(\mathcal{A})$, then both $H^n(X)$ and $H^n(Y)$ are kernels of maps between maps of objects of $\mathcal{B}$ as $H^n(X \oplus Y) = H^n(X) \oplus H^n(Y)$. Hence both X and Y are in $D_\mathcal{B}(\mathcal{A})$. By Lemma 4.16 it therefore suffices to show that given a distinguished triangle (X, Y, Z, f, g, h) such that X and Y are in $D_\mathcal{B}(\mathcal{A})$ then Z is an object of $D_\mathcal{B}(\mathcal{A})$. The long exact cohomology sequence (11.1.1) and the definition of a weak Serre subcategory (see Homology, Definition 10.1) show that $H^n(Z)$ is an object of $\mathcal{B}$ for all n. Thus Z is an object of $D_\mathcal{B}(\mathcal{A})$. $\square$

We continue to assume that $\mathcal{B}$ is a weak Serre subcategory of the abelian category $\mathcal{A}$. Then $\mathcal{B}$ is an abelian category and the inclusion functor $\mathcal{B} \to \mathcal{A}$ is exact. Hence we obtain a derived functor $D(\mathcal{B}) \to D(\mathcal{A})$, see Lemma 16.9. Clearly the functor $D(\mathcal{B}) \to D(\mathcal{A})$ factors through a canonical exact functor

$$D(\mathcal{B}) \longrightarrow D_\mathcal{B}(\mathcal{A}) \tag{17.1.1}$$

After all a complex made from objects of $\mathcal{B}$ certainly gives rise to an object of $D_\mathcal{B}(\mathcal{A})$ and as distinguished triangles in $D_\mathcal{B}(\mathcal{A})$ are exactly the distinguished triangles of $D(\mathcal{A})$ whose vertices are in $D_\mathcal{B}(\mathcal{A})$ we see that the functor is exact since $D(\mathcal{B}) \to D(\mathcal{A})$ is exact. Similarly we obtain functors $D^+(\mathcal{B}) \to D^+_\mathcal{B}(\mathcal{A})$, $D^-(\mathcal{B}) \to D^-_\mathcal{B}(\mathcal{A})$, and $D^b(\mathcal{B}) \to D^b_\mathcal{B}(\mathcal{A})$ for the bounded versions. A key question in many cases is whether the displayed functor is an equivalence.

Now, suppose that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$. In this case we have the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$, see Homology, Lemma 10.6. In this case $D_\mathcal{B}(\mathcal{A})$ is the kernel of the functor $D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$. Thus we obtain a canonical functor

$$D(\mathcal{A})/D_\mathcal{B}(\mathcal{A}) \longrightarrow D(\mathcal{A}/\mathcal{B})$$

by Lemma 6.8. Similarly for the bounded versions.

Lemma 17.2. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Then $D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$ is essentially surjective.*

Proof. We will use the description of the category $\mathcal{A}/\mathcal{B}$ in the proof of Homology, Lemma 10.6. Let $(X^\bullet, d^\bullet)$ be a complex of $\mathcal{A}/\mathcal{B}$. This means that X^i is an object of $\mathcal{A}$ and $d^i : X^i \to X^{i+1}$ is a morphism in $\mathcal{A}/\mathcal{B}$ such that $d^i \circ d^{i-1} = 0$ in $\mathcal{A}/\mathcal{B}$.

For $i \geq 0$ we may write $d^i = (s^i, f^i)$ where $s^i : Y^i \to X^i$ is a morphism of $\mathcal{A}$ whose kernel and cokernel are in $\mathcal{B}$ (equivalently s^i becomes an isomorphism in the quotient category) and $f^i : Y^i \to X^{i+1}$ is a morphism of $\mathcal{A}$. By induction we will construct a commutative diagram

$$\begin{tikzcd}
 & (X')^1 \arrow[rr, dotted] & & (X')^2 \arrow[rr, dotted] & & \ldots \\
X^0 \arrow[ur, dotted] & & X^1 \arrow[ul, dotted] & & X^2 \arrow[ul, dotted] & & \ldots \\
Y^0 \arrow[u, "s^0"] \arrow[urr, "f^0"'] & & Y^1 \arrow[u, "s^1"] \arrow[urr, "f^1"'] & & Y^2 \arrow[u, "s^2"] \arrow[urr, "f^2"'] & & \ldots
\end{tikzcd}$$

where the vertical arrows $X^i \to (X')^i$ become isomorphisms in the quotient category. Namely, we first let $(X')^1 = \text{Coker}(Y^0 \to X^0 \oplus X^1)$ (or rather the pushout of the diagram with arrows s^0 and f^0) which gives the first commutative diagram.

Next, we take $(X')^2 = \mathrm{Coker}(Y^1 \to (X')^1 \oplus X^2)$. And so on. Setting additionally $(X')^n = X^n$ for $n \leq 0$ we see that the map $(X^\bullet, d^\bullet) \to ((X')^\bullet, (d')^\bullet)$ is an isomorphism of complexes in $\mathcal{A}/\mathcal{B}$. Hence we may assume $d^n : X^n \to X^{n+1}$ is given by a map $X^n \to X^{n+1}$ in $\mathcal{A}$ for $n \geq 0$.

Dually, for $i < 0$ we may write $d^i = (g^i, t^{i+1})$ where $t^{i+1} : X^{i+1} \to Z^{i+1}$ is an isomorphism in the quotient category and $g^i : X^i \to Z^{i+1}$ is a morphism. By induction we will construct a commutative diagram

$$\begin{tikzcd}
\ldots & Z^{-2} & Z^{-1} & Z^0 \\
\ldots & X^{-2} \arrow[u, "t_{-2}"] \arrow[ur, "g_{-2}"'] & X^{-1} \arrow[u, "t_{-1}"] \arrow[ur, "g_{-1}"'] & X^0 \arrow[u, "t^0"'] \\
\ldots & (X')^{-2} \arrow[u, dotted] \arrow[r, dotted] & (X')^{-1} \arrow[u, dotted] \arrow[ur, dotted] &
\end{tikzcd}$$

where the vertical arrows $(X')^i \to X^i$ become isomorphisms in the quotient category. Namely, we take $(X')^{-1} = X^{-1} \times_{Z^0} X^0$. Then we take $(X')^{-2} = X^{-2} \times_{Z^{-1}} (X')^{-1}$. And so on. Setting additionally $(X')^n = X^n$ for $n \geq 0$ we see that the map $((X')^\bullet, (d')^\bullet) \to (X^\bullet, d^\bullet)$ is an isomorphism of complexes in $\mathcal{A}/\mathcal{B}$. Hence we may assume $d^n : X^n \to X^{n+1}$ is given by a map $d^n : X^n \to X^{n+1}$ in $\mathcal{A}$ for all $n \in \mathbf{Z}$.

In this case we know the compositions $d^n \circ d^{n-1}$ are zero in $\mathcal{A}/\mathcal{B}$. If for $n > 0$ we replace X^n by

$$(X')^n = X^n / \sum\nolimits_{0 < k \leq n} \mathrm{Im}(\mathrm{Im}(X^{k-2} \to X^k) \to X^n)$$

then the compositions $d^n \circ d^{n-1}$ are zero for $n \geq 0$. (Similarly to the second paragraph above we obtain an isomorphism of complexes $(X^\bullet, d^\bullet) \to ((X')^\bullet, (d')^\bullet)$.) Finally, for $n < 0$ we replace X^n by

$$(X')^n = \bigcap\nolimits_{n \leq k < 0} (X^n \to X^k)^{-1} \mathrm{Ker}(X^k \to X^{k+2})$$

and we argue in the same manner to get a complex in $\mathcal{A}$ whose image in $\mathcal{A}/\mathcal{B}$ is isomorphic to the given one. $\square$

Lemma 17.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Suppose that the functor $v : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ has a left adjoint $u : \mathcal{A}/\mathcal{B} \to \mathcal{A}$ such that $vu \cong id$. Then*

$$D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A}) = D(\mathcal{A}/\mathcal{B})$$

and similarly for the bounded versions.

Proof. The functor $D(v) : D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$ is essentially surjective by Lemma 17.2. For an object X of $D(\mathcal{A})$ the adjunction mapping $c_X : uvX \to X$ maps to an isomorphism in $D(\mathcal{A}/\mathcal{B})$ because $vuv \cong v$ by the assumption that $vu \cong \mathrm{id}$. Thus in a distinguished triangle (uvX, X, Z, c_X, g, h) the object Z is an object of $D_{\mathcal{B}}(\mathcal{A})$ as we see by looking at the long exact cohomology sequence. Hence c_X is an element of the multiplicative system used to define the quotient category $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. Thus $uvX \cong X$ in $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. For $X, Y \in \mathrm{Ob}(\mathcal{A}))$ the map

$$\mathrm{Hom}_{D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})}(X, Y) \longrightarrow \mathrm{Hom}_{D(\mathcal{A}/\mathcal{B})}(vX, vY)$$

is bijective because u gives an inverse (by the remarks above). $\square$

For certain Serre subcategories $\mathcal{B} \subset \mathcal{A}$ we can prove that the functor $D(\mathcal{B}) \to D_{\mathcal{B}}(\mathcal{A})$ is fully faithful.

Lemma 17.4. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Assume that for every surjection $X \to Y$ with $X \in \mathrm{Ob}(\mathcal{A})$ and $Y \in \mathrm{Ob}(\mathcal{B})$ there exists $X' \subset X$, $X' \in \mathrm{Ob}(\mathcal{B})$ which surjects onto Y. Then the functor $D^-(\mathcal{B}) \to D^-_{\mathcal{B}}(\mathcal{A})$ of (17.1.1) is an equivalence.*

Proof. Let $X^\bullet$ be a bounded above complex of $\mathcal{A}$ such that $H^i(X^\bullet) \in \mathrm{Ob}(\mathcal{B})$ for all $i \in \mathbf{Z}$. Moreover, suppose we are given $B^i \subset X^i$, $B^i \in \mathrm{Ob}(\mathcal{B})$ for all $i \in \mathbf{Z}$. Claim: there exists a subcomplex $Y^\bullet \subset X^\bullet$ such that

1. $Y^\bullet \to X^\bullet$ is a quasi-isomorphism,
2. $Y^i \in \mathrm{Ob}(\mathcal{B})$ for all $i \in \mathbf{Z}$, and
3. $B^i \subset Y^i$ for all $i \in \mathbf{Z}$.

To prove the claim, using the assumption of the lemma we first choose $C^i \subset \mathrm{Ker}(d^i : X^i \to X^{i+1})$, $C^i \in \mathrm{Ob}(\mathcal{B})$ surjecting onto $H^i(X^\bullet)$. Setting $D^i = C^i + d^{i-1}(B^{i-1}) + B^i$ we find a subcomplex $D^\bullet$ satisfying (2) and (3) such that $H^i(D^\bullet) \to H^i(X^\bullet)$ is surjective for all $i \in \mathbf{Z}$. For any choice of $E^i \subset X^i$ with $E^i \in \mathrm{Ob}(\mathcal{B})$ and $d^i(E^i) \subset D^{i+1} + E^{i+1}$ we see that setting $Y^i = D^i + E^i$ gives a subcomplex whose terms are in $\mathcal{B}$ and whose cohomology surjects onto the cohomology of $X^\bullet$. Clearly, if $d^i(E^i) = (D^{i+1} + E^{i+1}) \cap \mathrm{Im}(d^i)$ then we see that the map on cohomology is also injective. For $n \gg 0$ we can take E^n equal to 0. By descending induction we can choose E^i for all i with the desired property. Namely, given $E^{i+1}, E^{i+2}, \ldots$ we choose $E^i \subset X^i$ such that $d^i(E^i) = (D^{i+1} + E^{i+1}) \cap \mathrm{Im}(d^i)$. This is possible by our assumption in the lemma combined with the fact that $(D^{i+1} + E^{i+1}) \cap \mathrm{Im}(d^i)$ is in $\mathcal{B}$ as $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$.

The claim above implies the lemma. Essential surjectivity is immediate from the claim. Let us prove faithfulness. Namely, suppose we have a morphism $f : U^\bullet \to V^\bullet$ of bounded above complexes of $\mathcal{B}$ whose image in $D(\mathcal{A})$ is zero. Then there exists a quasi-isomorphism $s : V^\bullet \to X^\bullet$ into a bounded above complex of $\mathcal{A}$ such that $s \circ f$ is homotopic to zero. Choose a homotopy $h^i : U^i \to X^{i-1}$ between 0 and $s \circ f$. Apply the claim with $B^i = h^{i+1}(U^{i+1}) + s^i(V^i)$. The resulting map $s' : V^\bullet \to Y^\bullet$ is a quasi-isomorphism as well and $s' \circ f$ is homotopic to zero as is clear from the fact that h^i factors through Y^{i-1}. This proves faithfulness. Fully faithfulness is proved in the exact same manner. $\square$

18. Injective resolutions

In this section we prove some lemmas regarding the existence of injective resolutions in abelian categories having enough injectives.

Definition 18.1. Let $\mathcal{A}$ be an abelian category. Let $A \in \mathrm{Ob}(\mathcal{A})$. An *injective resolution of A* is a complex $I^\bullet$ together with a map $A \to I^0$ such that:

1. We have $I^n = 0$ for $n < 0$.
2. Each I^n is an injective object of $\mathcal{A}$.
3. The map $A \to I^0$ is an isomorphism onto $\mathrm{Ker}(d^0)$.
4. We have $H^i(I^\bullet) = 0$ for $i > 0$.

Hence $A[0] \to I^\bullet$ is a quasi-isomorphism. In other words the complex

$$\ldots \to 0 \to A \to I^0 \to I^1 \to \ldots$$

is acyclic. Let $K^\bullet$ be a complex in $\mathcal{A}$. An *injective resolution of* $K^\bullet$ is a complex $I^\bullet$ together with a map $\alpha : K^\bullet \to I^\bullet$ of complexes such that

(1) We have $I^n = 0$ for $n \ll 0$, i.e., $I^\bullet$ is bounded below.
(2) Each I^n is an injective object of $\mathcal{A}$.
(3) The map $\alpha : K^\bullet \to I^\bullet$ is a quasi-isomorphism.

In other words an injective resolution $K^\bullet \to I^\bullet$ gives rise to a diagram

$$\begin{tikzcd}
\ldots \arrow[r] & K^{n-1} \arrow[r] \arrow[d] & K^n \arrow[r] \arrow[d] & K^{n+1} \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & I^{n-1} \arrow[r] & I^n \arrow[r] & I^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

which induces an isomorphism on cohomology objects in each degree. An injective resolution of an object A of $\mathcal{A}$ is almost the same thing as an injective resolution of the complex $A[0]$.

Lemma 18.2. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$.*

(1) *If $K^\bullet$ has an injective resolution then $H^n(K^\bullet) = 0$ for $n \ll 0$.*
(2) *If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^\bullet$ bounded below.*

Proof. Omitted. For the second statement use $L^\bullet = \tau_{\geq n} K^\bullet$ for some $n \ll 0$. See Homology, Section 15 for the definition of the truncation $\tau_{\geq n}$. $\square$

Lemma 18.3. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives.*

(1) *Any object of $\mathcal{A}$ has an injective resolution.*
(2) *If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then $K^\bullet$ has an injective resolution.*
(3) *If $K^\bullet$ is a complex with $K^n = 0$ for $n < a$, then there exists an injective resolution $\alpha : K^\bullet \to I^\bullet$ with $I^n = 0$ for $n < a$ such that each $\alpha^n : K^n \to I^n$ is injective.*

Proof. Proof of (1). First choose an injection $A \to I^0$ of A into an injective object of $\mathcal{A}$. Next, choose an injection $I_0/A \to I^1$ into an injective object of $\mathcal{A}$. Denote d^0 the induced map $I^0 \to I^1$. Next, choose an injection $I^1/\mathrm{Im}(d^0) \to I^2$ into an injective object of $\mathcal{A}$. Denote d^1 the induced map $I^1 \to I^2$. And so on. By Lemma 18.2 part (2) follows from part (3). Part (3) is a special case of Lemma 15.5. $\square$

Lemma 18.4. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be an acyclic complex. Let $I^\bullet$ be bounded below and consisting of injective objects. Any morphism $K^\bullet \to I^\bullet$ is homotopic to zero.*

Proof. Let $\alpha : K^\bullet \to I^\bullet$ be a morphism of complexes. Assume that $\alpha^j = 0$ for $j < n$. We will show that there exists a morphism $h : K^{n+1} \to I^n$ such that $\alpha^n = h \circ d$. Thus α will be homotopic to the morphism of complexes β defined by

$$\beta^j = \begin{cases} 0 & \text{if} \quad j \leq n \\ \alpha^{n+1} - d \circ h & \text{if} \quad j = n+1 \\ \alpha^j & \text{if} \quad j > n+1 \end{cases}$$

This will clearly prove the lemma (by induction). To prove the existence of h note that $\alpha^n|_{d^{n-1}(K^{n-1})} = 0$ since $\alpha^{n-1} = 0$. Since $K^\bullet$ is acyclic we have $d^{n-1}(K^{n-1}) = \mathrm{Ker}(K^n \to K^{n+1})$. Hence we can think of α^n as a map into I^n defined on the subobject $\mathrm{Im}(K^n \to K^{n+1})$ of K^{n+1}. By injectivity of the object I^n we can extend this to a map $h : K^{n+1} \to I^n$ as desired. □

Remark 18.5. Let $\mathcal{A}$ be an abelian category. Using the fact that $K(\mathcal{A})$ is a triangulated category we may use Lemma 18.4 to obtain proofs of some of the lemmas below which are usually proved by chasing through diagrams. Namely, suppose that $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism of complexes. Then

$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$ (Lemma 9.14) and $C(\alpha)^\bullet$ is an acyclic complex (Lemma 11.2). Next, let $I^\bullet$ be a bounded below complex of injective objects. Then

$$\begin{tikzcd} \mathrm{Hom}_{K(\mathcal{A})}(C(\alpha)^\bullet, I^\bullet) \arrow[r] & \mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) \arrow[r] & \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet) \arrow[dll] \\ \mathrm{Hom}_{K(\mathcal{A})}(C(\alpha)^\bullet[-1], I^\bullet) & & \end{tikzcd}$$

is an exact sequence of abelian groups, see Lemma 4.2. At this point Lemma 18.4 guarantees that the outer two groups are zero and hence $\mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

Lemma 18.6. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"'] \arrow[d, "\gamma"'] & L^\bullet \arrow[dl, dashed, "\beta"] \\ I^\bullet & \end{tikzcd}$$

where $I^\bullet$ is bounded below and consists of injective objects, and α is a quasi-isomorphism.

(1) *There exists a map of complexes β making the diagram commute up to homotopy.*
(2) *If α is injective in every degree then we can find a β which makes the diagram commute.*

Proof. The "correct" proof of part (1) is explained in Remark 18.5. We also give a direct proof here.

We first show that (2) implies (1). Namely, let $\tilde{\alpha} : K \to \tilde{L}^\bullet$, π, s be as in Lemma 9.6. Since $\tilde{\alpha}$ is injective by (2) there exists a morphism $\tilde{\beta} : \tilde{L}^\bullet \to I^\bullet$ such that $\gamma = \tilde{\beta} \circ \tilde{\alpha}$. Set $\beta = \tilde{\beta} \circ s$. Then we have

$$\beta \circ \alpha = \tilde{\beta} \circ s \circ \pi \circ \tilde{\alpha} \sim \tilde{\beta} \circ \tilde{\alpha} = \gamma$$

as desired.

Assume that $\alpha : K^\bullet \to L^\bullet$ is injective. Suppose we have already defined β in all degrees $\leq n - 1$ compatible with differentials and such that $\gamma^j = \beta^j \circ \alpha^j$ for all

$j \leq n-1$. Consider the commutative solid diagram

$$\begin{tikzcd} K^{n-1} \arrow[r] \arrow[d, "\alpha"] \arrow[dd, bend right, "\gamma"'] & K^n \arrow[d, "\alpha"] \arrow[dd, bend left, "\gamma"] \\ L^{n-1} \arrow[r] \arrow[d, "\beta"] & L^n \arrow[d, dashed] \\ I^{n-1} \arrow[r] & I^n \end{tikzcd}$$

Thus we see that the dotted arrow is prescribed on the subobjects $\alpha(K^n)$ and $d^{n-1}(L^{n-1})$. Moreover, these two arrows agree on $\alpha(d^{n-1}(K^{n-1}))$. Hence if

$$\alpha(d^{n-1}(K^{n-1})) = \alpha(K^n) \cap d^{n-1}(L^{n-1}) \tag{18.6.1}$$

then these morphisms glue to a morphism $\alpha(K^n) + d^{n-1}(L^{n-1}) \to I^n$ and, using the injectivity of I^n, we can extend this to a morphism from all of L^n into I^n. After this by induction we get the morphism β for all n simultaneously (note that we can set $\beta^n = 0$ for all $n \ll 0$ since $I^\bullet$ is bounded below – in this way starting the induction).

It remains to prove the equality (18.6.1). The reader is encouraged to argue this for themselves with a suitable diagram chase. Nonetheless here is our argument. Note that the inclusion $\alpha(d^{n-1}(K^{n-1})) \subset \alpha(K^n) \cap d^{n-1}(L^{n-1})$ is obvious. Take an object T of $\mathcal{A}$ and a morphism $x : T \to L^n$ whose image is contained in the subobject $\alpha(K^n) \cap d^{n-1}(L^{n-1})$. Since α is injective we see that $x = \alpha \circ x'$ for some $x' : T \to K^n$. Moreover, since x lies in $d^{n-1}(L^{n-1})$ we see that $d^n \circ x = 0$. Hence using injectivity of α again we see that $d^n \circ x' = 0$. Thus x' gives a morphism $[x'] : T \to H^n(K^\bullet)$. On the other hand the corresponding map $[x] : T \to H^n(L^\bullet)$ induced by x is zero by assumption. Since α is a quasi-isomorphism we conclude that $[x'] = 0$. This of course means exactly that the image of x' is contained in $d^{n-1}(K^{n-1})$ and we win. □

Lemma 18.7. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & L^\bullet \arrow[dl, dashed, "\beta_i"] \\ I^\bullet & \end{tikzcd}$$

where $I^\bullet$ is bounded below and consists of injective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. This follows from Remark 18.5. We also give a direct argument here.

Let $\tilde{\alpha} : K \to \tilde{L}^\bullet$, π, s be as in Lemma 9.6. If we can show that $\beta_1 \circ \pi$ is homotopic to $\beta_2 \circ \pi$, then we deduce that $\beta_1 \sim \beta_2$ because $\pi \circ s$ is the identity. Hence we may assume $\alpha^n : K^n \to L^n$ is the inclusion of a direct summand for all n. Thus we get a short exact sequence of complexes

$$0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$$

which is termwise split and such that $M^\bullet$ is acyclic. We choose splittings $L^n = K^n \oplus M^n$, so we have $\beta_i^n : K^n \oplus M^n \to I^n$ and $\gamma^n : K^n \to I^n$. In this case the condition on β_i is that there are morphisms $h_i^n : K^n \to I^{n-1}$ such that

$$\gamma^n - \beta_i^n|_{K^n} = d \circ h_i^n + h_i^{n+1} \circ d$$

Thus we see that

$$\beta_1^n|_{K^n} - \beta_2^n|_{K^n} = d \circ (h_1^n - h_2^n) + (h_1^{n+1} - h_2^{n+1}) \circ d$$

Consider the map $h^n : K^n \oplus M^n \to I^{n-1}$ which equals $h_1^n - h_2^n$ on the first summand and zero on the second. Then we see that

$$\beta_1^n - \beta_2^n - (d \circ h^n + h^{n+1}) \circ d$$

is a morphism of complexes $L^\bullet \to I^\bullet$ which is identically zero on the subcomplex $K^\bullet$. Hence it factors as $L^\bullet \to M^\bullet \to I^\bullet$. Thus the result of the lemma follows from Lemma 18.4. $\square$

Lemma 18.8. *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet$ be bounded below complex consisting of injective objects. Let $L^\bullet \in K(\mathcal{A})$. Then*

$$\mathrm{Mor}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Mor}_{D(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism and $\gamma : K^\bullet \to I^\bullet$ is a map of complexes. By Lemma 18.6 we can find a morphism $\beta : L^\bullet \to I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ. This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \to I^\bullet$ such that there exists a quasi-isomorphism $\alpha : K^\bullet \to L^\bullet$ with $\beta \circ \alpha$ homotopic to zero. Then Lemma 18.7 shows that β is homotopic to zero also, i.e., $b = 0$. $\square$

Lemma 18.9. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of Comp$^+(\mathcal{A})$ there exists a commutative diagram in Comp$^+(\mathcal{A})$*

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_1^\bullet \arrow[r] & I_2^\bullet \arrow[r] & I_3^\bullet \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are injective resolutions and the rows are short exact sequences of complexes. In fact, given any injective resolution $A^\bullet \to I^\bullet$ we may assume $I_1^\bullet = I^\bullet$.

Proof. Step 1. Choose an injective resolution $A^\bullet \to I^\bullet$ (see Lemma 18.3) or use the given one. Recall that Comp$^+(\mathcal{A})$ is an abelian category, see Homology, Lemma 13.9. Hence we may form the pushout along the map $A^\bullet \to I^\bullet$ to get

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^\bullet \arrow[r] & E^\bullet \arrow[r] & C^\bullet \arrow[r] & 0
\end{tikzcd}$$

Because of the 5-lemma and the last assertion of Homology, Lemma 13.12 the map $B^\bullet \to A^\bullet$ is a quasi-isomorphism. Note that the lower short exact sequence is termwise split, see Homology, Lemma 27.2. Hence it suffices to prove the lemma when $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is termwise split.

Step 2. Choose splittings. In other words, write $B^n = A^n \oplus C^n$. Denote $\delta : C^\bullet \to A^\bullet[1]$ the morphism as in Homology, Lemma 14.10. Choose injective resolutions $f_1 : A^\bullet \to I_1^\bullet$ and $f_3 : C^\bullet \to I_3^\bullet$. (If $A^\bullet$ is a complex of injectives, then use $I_1^\bullet = A^\bullet$.) We may assume f_3 is injective in every degree. By Lemma 18.6 we may find a morphism $\delta' : I_3^\bullet \to I_1^\bullet[1]$ such that $\delta' \circ f_3 = f_1[1] \circ \delta$ (equality of morphisms of complexes). Set $I_2^n = I_1^n \oplus I_3^n$. Define

$$d_{I_2}^n = \begin{pmatrix} d_{I_1}^n & (\delta')^n \\ 0 & d_{I_3}^n \end{pmatrix}$$

and define the maps $B^n \to I_2^n$ to be given as the sum of the maps $A^n \to I_1^n$ and $C^n \to I_3^n$. Everything is clear. □

19. Projective resolutions

This section is dual to Section 18. We give definitions and state results, but we do not reprove the lemmas.

Definition 19.1. Let $\mathcal{A}$ be an abelian category. Let $A \in \mathrm{Ob}(\mathcal{A})$. An *projective resolution of A* is a complex $P^\bullet$ together with a map $P^0 \to A$ such that:

1. We have $P^n = 0$ for $n > 0$.
2. Each P^n is an projective object of $\mathcal{A}$.
3. The map $P^0 \to A$ induces an isomorphism $\mathrm{Coker}(d^{-1}) \to A$.
4. We have $H^i(P^\bullet) = 0$ for $i < 0$.

Hence $P^\bullet \to A[0]$ is a quasi-isomorphism. In other words the complex

$$\ldots \to P^{-1} \to P^0 \to A \to 0 \to \ldots$$

is acyclic. Let $K^\bullet$ be a complex in $\mathcal{A}$. An *projective resolution of $K^\bullet$* is a complex $P^\bullet$ together with a map $\alpha : P^\bullet \to K^\bullet$ of complexes such that

1. We have $P^n = 0$ for $n \gg 0$, i.e., $P^\bullet$ is bounded above.
2. Each P^n is an projective object of $\mathcal{A}$.
3. The map $\alpha : P^\bullet \to K^\bullet$ is a quasi-isomorphism.

Lemma 19.2. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$.*

1. *If $K^\bullet$ has a projective resolution then $H^n(K^\bullet) = 0$ for $n \gg 0$.*
2. *If $H^n(K^\bullet) = 0$ for $n \gg 0$ then there exists a quasi-isomorphism $L^\bullet \to K^\bullet$ with $L^\bullet$ bounded above.*

Proof. Dual to Lemma 18.2. □

Lemma 19.3. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives.*

1. *Any object of $\mathcal{A}$ has a projective resolution.*
2. *If $H^n(K^\bullet) = 0$ for all $n \gg 0$ then $K^\bullet$ has a projective resolution.*
3. *If $K^\bullet$ is a complex with $K^n = 0$ for $n > a$, then there exists a projective resolution $\alpha : P^\bullet \to K^\bullet$ with $P^n = 0$ for $n > a$ such that each $\alpha^n : P^n \to K^n$ is surjective.*

Proof. Dual to Lemma 18.3. □

Lemma 19.4. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be an acyclic complex. Let $P^\bullet$ be bounded above and consisting of projective objects. Any morphism $P^\bullet \to K^\bullet$ is homotopic to zero.*

Proof. Dual to Lemma 18.4. □

Remark 19.5. Let $\mathcal{A}$ be an abelian category. Suppose that $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism of complexes. Let $P^\bullet$ be a bounded above complex of projectives. Then

$$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, L^\bullet)$$

is an isomorphism. This is dual to Remark 18.5.

Lemma 19.6. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{tikzcd} K^\bullet & L^\bullet \arrow[l, "\alpha"'] \\ P^\bullet \arrow[u] \arrow[ur, dashed, "\beta"'] & \end{tikzcd}$$

where $P^\bullet$ is bounded above and consists of projective objects, and α is a quasi-isomorphism.

1. *There exists a map of complexes β making the diagram commute up to homotopy.*
2. *If α is surjective in every degree then we can find a β which makes the diagram commute.*

Proof. Dual to Lemma 18.6. □

Lemma 19.7. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{tikzcd} K^\bullet & L^\bullet \arrow[l, "\alpha"'] \\ P^\bullet \arrow[u] \arrow[ur, dashed, "\beta_i"'] & \end{tikzcd}$$

where $P^\bullet$ is bounded above and consists of projective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. Dual to Lemma 18.7. □

Lemma 19.8. *Let $\mathcal{A}$ be an abelian category. Let $P^\bullet$ be bounded above complex consisting of projective objects. Let $L^\bullet \in K(\mathcal{A})$. Then*

$$\mathrm{Mor}_{K(\mathcal{A})}(P^\bullet, L^\bullet) = \mathrm{Mor}_{D(\mathcal{A})}(P^\bullet, L^\bullet).$$

Proof. Dual to Lemma 18.8. □

Lemma 19.9. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives. For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of Comp$^+(\mathcal{A})$ there exists a commutative diagram in Comp$^+(\mathcal{A})$*

$$\begin{tikzcd} 0 \arrow[r] & P_1^\bullet \arrow[r] \arrow[d] & P_2^\bullet \arrow[r] \arrow[d] & P_3^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r] & 0 \end{tikzcd}$$

where the vertical arrows are projective resolutions and the rows are short exact sequences of complexes. In fact, given any projective resolution $P^\bullet \to C^\bullet$ we may assume $P_3^\bullet = P^\bullet$.

Proof. Dual to Lemma 18.9. □

Lemma 19.10. *Let $\mathcal{A}$ be an abelian category. Let $P^\bullet$, $K^\bullet$ be complexes. Let $n \in \mathbf{Z}$. Assume that*

(1) *$P^\bullet$ is a bounded complex consisting of projective objects,*
(2) *$P^i = 0$ for $i < n$, and*
(3) *$H^i(K^\bullet) = 0$ for $i \geq n$.*

Then $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet) = \mathrm{Hom}_{D(\mathcal{A})}(P^\bullet, K^\bullet) = 0$.

Proof. The first equality follows from Lemma 19.8. Note that there is a distinguished triangle

$$(\tau_{\leq n-1}K^\bullet, K^\bullet, \tau_{\geq n}K^\bullet, f, g, h)$$

by Remark 12.4. Hence, by Lemma 4.2 it suffices to prove $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, \tau_{\leq n-1}K^\bullet) = 0$ and $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, \tau_{\geq n}K^\bullet) = 0$. The first vanishing is trivial and the second is Lemma 19.4. □

Lemma 19.11. *Let $\mathcal{A}$ be an abelian category. Let $\beta : P^\bullet \to L^\bullet$ and $\alpha : E^\bullet \to L^\bullet$ be maps of complexes. Let $n \in \mathbf{Z}$. Assume*

(1) *$P^\bullet$ is a bounded complex of projectives and $P^i = 0$ for $i < n$,*
(2) *$H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$.*

Then there exists a map of complexes $\gamma : P^\bullet \to E^\bullet$ such that $\alpha \circ \gamma$ and β are homotopic.

Proof. Consider the cone $C^\bullet = C(\alpha)^\bullet$ with map $i : L^\bullet \to C^\bullet$. Note that $i \circ \beta$ is zero by Lemma 19.10. Hence we can lift β to $E^\bullet$ by Lemma 4.2. □

20. Right derived functors and injective resolutions

At this point we can use the material above to define the right derived functors of an additive functor between an abelian category having enough injectives and a general abelian category.

Lemma 20.1. *Let $\mathcal{A}$ be an abelian category. Let $I \in \mathrm{Ob}(\mathcal{A})$ be an injective object. Let $I^\bullet$ be a bounded below complex of injectives in $\mathcal{A}$.*

(1) *$I^\bullet$ computes RF relative to $Qis^+(\mathcal{A})$ for any exact functor $F : K^+(\mathcal{A}) \to \mathcal{D}$ into any triangulated category $\mathcal{D}$.*
(2) *I is right acyclic for any additive functor $F : \mathcal{A} \to \mathcal{B}$ into any abelian category $\mathcal{B}$.*

Proof. Part (2) is a direct consequences of part (1) and Definition 15.3. To prove (1) let $\alpha : I^\bullet \to K^\bullet$ be a quasi-isomorphism into a complex. By Lemma 18.6 we see that α has a left inverse. Hence the category $I^\bullet/\mathrm{Qis}^+(\mathcal{A})$ is essentially constant with value $\mathrm{id} : I^\bullet \to I^\bullet$. Thus also the ind-object

$$I^\bullet/\mathrm{Qis}^+(\mathcal{A}) \longrightarrow \mathcal{D}, \quad (I^\bullet \to K^\bullet) \longmapsto F(K^\bullet)$$

is essentially constant with value $F(I^\bullet)$. This proves (1), see Definitions 14.2 and 14.10. □

Lemma 20.2. *Let $\mathcal{A}$ be an abelian category with enough injectives.*

(1) *For any exact functor $F : K^+(\mathcal{A}) \to \mathcal{D}$ into a triangulated category $\mathcal{D}$ the right derived functor*

$$RF : D^+(\mathcal{A}) \longrightarrow \mathcal{D}$$

is everywhere defined.

(2) *For any additive functor $F : \mathcal{A} \to \mathcal{B}$ into an abelian category $\mathcal{B}$ the right derived functor*
$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$
is everywhere defined.

Proof. Combine Lemma 20.1 and Proposition 16.8 for the second assertion. To see the first assertion combine Lemma 18.3, Lemma 20.1, Lemma 14.14, and Equation (14.9.1). $\square$

Lemma 20.3. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor.*

(1) *The functor RF is an exact functor $D^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(2) *The functor RF induces an exact functor $K^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(3) *The functor RF induces a δ-functor $Comp^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(4) *The functor RF induces a δ-functor $\mathcal{A} \to D^+(\mathcal{B})$.*

Proof. This lemma simply reviews some of the results obtained so far. Note that by Lemma 20.2 RF is everywhere defined. Here are some references:

(1) The derived functor is exact: This boils down to Lemma 14.6.
(2) This is true because $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ is exact and compositions of exact functors are exact.
(3) This is true because $\text{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$ is a δ-functor, see Lemma 12.1.
(4) This is true because $\mathcal{A} \to \text{Comp}^+(\mathcal{A})$ is exact and precomposing a δ-functor by an exact functor gives a δ-functor.

$\square$

Lemma 20.4. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor.*

(1) *For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of complexes in $Comp^+(\mathcal{A})$ there is an associated long exact sequence*
$$\ldots \to H^i(RF(A^\bullet)) \to H^i(RF(B^\bullet)) \to H^i(RF(C^\bullet)) \to H^{i+1}(RF(A^\bullet)) \to \ldots$$
(2) *The functors $R^iF : \mathcal{A} \to \mathcal{B}$ are zero for $i < 0$. Also $R^0F = F : \mathcal{A} \to \mathcal{B}$.*
(3) *We have $R^iF(I) = 0$ for $i > 0$ and I injective.*
(4) *The sequence (R^iF, δ) forms a universal δ-functor (see Homology, Definition 12.3) from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. This lemma simply reviews some of the results obtained so far. Note that by Lemma 20.2 RF is everywhere defined. Here are some references:

(1) This follows from Lemma 20.3 part (3) combined with the long exact cohomology sequence (11.1.1) for $D^+(\mathcal{B})$.
(2) This is Lemma 16.3.
(3) This is the fact that injective objects are acyclic.
(4) This is Lemma 16.6.

$\square$

21. Cartan-Eilenberg resolutions

This section can be expanded. The material can be generalized and applied in more cases. Resolutions need not use injectives and the method also works in the unbounded case in some situations.

Definition 21.1. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a bounded below complex. A *Cartan-Eilenberg resolution* of $K^\bullet$ is given by a double complex $I^{\bullet,\bullet}$ and a morphism of complexes $\epsilon : K^\bullet \to I^{\bullet,0}$ with the following properties:

1. There exists a $i \ll 0$ such that $I^{p,q} = 0$ for all $p < i$ and all q.
2. We have $I^{p,q} = 0$ if $q < 0$.
3. The complex $I^{p,\bullet}$ is an injective resolution of K^p.
4. The complex $\text{Ker}(d_1^{p,\bullet})$ is an injective resolution of $\text{Ker}(d_K^p)$.
5. The complex $\text{Im}(d_1^{p,\bullet})$ is an injective resolution of $\text{Im}(d_K^p)$.
6. The complex $H_I^p(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$.

Lemma 21.2. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $K^\bullet$ be a bounded below complex. There exists a Cartan-Eilenberg resolution of $K^\bullet$.*

Proof. Suppose that $K^p = 0$ for $p < n$. Decompose $K^\bullet$ into short exact sequences as follows: Set $Z^p = \text{Ker}(d^p)$, $B^p = \text{Im}(d^{p-1})$, $H^p = Z^p/B^p$, and consider

$$\begin{matrix} 0 \to Z^n \to K^n \to B^{n+1} \to 0 \\ 0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0 \\ 0 \to Z^{n+1} \to K^{n+1} \to B^{n+2} \to 0 \\ 0 \to B^{n+2} \to Z^{n+2} \to H^{n+2} \to 0 \\ \ldots \end{matrix}$$

Set $I^{p,q} = 0$ for $p < n$. Inductively we choose injective resolutions as follows:

1. Choose an injective resolution $Z^n \to J_Z^{n,\bullet}$.
2. Using Lemma 18.9 choose injective resolutions $K^n \to I^{n,\bullet}$, $B^{n+1} \to J_B^{n+1,\bullet}$, and an exact sequence of complexes $0 \to J_Z^{n,\bullet} \to I^{n,\bullet} \to J_B^{n+1,\bullet} \to 0$ compatible with the short exact sequence $0 \to Z^n \to K^n \to B^{n+1} \to 0$.
3. Using Lemma 18.9 choose injective resolutions $Z^{n+1} \to J_Z^{n+1,\bullet}$, $H^{n+1} \to J_H^{n+1,\bullet}$, and an exact sequence of complexes $0 \to J_B^{n+1,\bullet} \to J_Z^{n+1,\bullet} \to J_H^{n+1,\bullet} \to 0$ compatible with the short exact sequence $0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0$.
4. Etc.

Taking as maps $d_1^\bullet : I^{p,\bullet} \to I^{p+1,\bullet}$ the compositions $I^{p,\bullet} \to J_B^{p+1,\bullet} \to J_Z^{p+1,\bullet} \to I^{p+1,\bullet}$ everything is clear. □

Lemma 21.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution for $K^\bullet$. The spectral sequences $({}'E_r, {}'d_r)_{r\geq 0}$ and $({}''E_r, {}''d_r)_{r\geq 0}$ associated to the double complex $F(I^{\bullet,\bullet})$ satisfy the relations*

$${}'E_1^{p,q} = R^qF(K^p) \quad \textit{and} \quad {}''E_2^{p,q} = R^pF(H^q(K^\bullet))$$

Moreover, these spectral sequences are bounded, converge to $H^*(RF(K^\bullet))$, and the associated induced filtrations on $H^n(RF(K^\bullet))$ are finite.

Proof. We will use the following remarks without further mention:

(1) As $I^{p,\bullet}$ is an injective resolution of K^p we see that RF is defined at $K^p[0]$ with value $F(I^{p,\bullet})$.
(2) As $H^p_I(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$ the derived functor RF is defined at $H^p(K^\bullet)[0]$ with value $F(H^p_I(I^{\bullet,\bullet}))$.
(3) By Homology, Lemma 25.4 the total complex $\text{Tot}(I^{\bullet,\bullet})$ is an injective resolution of $K^\bullet$. Hence RF is defined at $K^\bullet$ with value $F(\text{Tot}(I^{\bullet,\bullet}))$.

Consider the two spectral sequences associated to the double complex $L^{\bullet,\bullet} = F(I^{\bullet,\bullet})$, see Homology, Lemma 25.1. These are both bounded, converge to $H^*(\text{Tot}(L^{\bullet,\bullet}))$, and induce finite filtrations on $H^n(\text{Tot}(L^{\bullet,\bullet}))$, see Homology, Lemma 25.3. Since $\text{Tot}(L^{\bullet,\bullet}) = \text{Tot}(F(I^{\bullet,\bullet})) = F(\text{Tot}(I^{\bullet,\bullet}))$ computes $H^n(RF(K^\bullet))$ we find the final assertion of the lemma holds true.

Computation of the first spectral sequence. We have $'E_1^{p,q} = H^q(L^{p,\bullet})$ in other words

$$'E_1^{p,q} = H^q(F(I^{p,\bullet})) = R^qF(K^p)$$

as desired. Observe for later use that the maps $'d_1^{p,q} : {'E_1^{p,q}} \to {'E_1^{p+1,q}}$ are the maps $R^qF(K^p) \to R^qF(K^{p+1})$ induced by $K^p \to K^{p+1}$ and the fact that R^qF is a functor.

Computation of the second spectral sequence. We have $''E_1^{p,q} = H^q(L^{\bullet,p}) = H^q(F(I^{\bullet,p}))$. Note that the complex $I^{\bullet,p}$ is bounded below, consists of injectives, and moreover each kernel, image, and cohomology group of the differentials is an injective object of $\mathcal{A}$. Hence we can split the differentials, i.e., each differential is a split surjection onto a direct summand. It follows that the same is true after applying F. Hence $''E_1^{p,q} = F(H^q(I^{\bullet,p})) = F(H^q_I(I^{\bullet,p}))$. The differentials on this are $(-1)^q$ times F applied to the differential of the complex $H^p_I(I^{\bullet,\bullet})$ which is an injective resolution of $H^p(K^\bullet)$. Hence the description of the E_2 terms. □

Remark 21.4. The spectral sequences of Lemma 21.3 are functorial in the complex $K^\bullet$. This follows from functoriality properties of Cartan-Eilenberg resolutions. On the other hand, they are both examples of a more general spectral sequence which may be associated to a filtered complex of $\mathcal{A}$. The functoriality will follow from its construction. We will return to this in the section on the filtered derived category, see Remark 26.15.

22. Composition of right derived functors

Sometimes we can compute the right derived functor of a composition. Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume that the right derived functors $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, $RG : D^+(\mathcal{B}) \to D^+(\mathcal{C})$, and $R(G \circ F) : D^+(\mathcal{A}) \to D^+(\mathcal{C})$ are everywhere defined. Then there exists a canonical transformation

$$t : R(G \circ F) \longrightarrow RG \circ RF$$

of functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$, see Lemma 14.16. This transformation need not always be an isomorphism.

Lemma 22.1. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume $\mathcal{A}$, $\mathcal{B}$ have enough injectives. The following are equivalent*

(1) *$F(I)$ is right acyclic for G for each injective object I of $\mathcal{A}$, and*

(2) *the canonical map*
$$t : R(G \circ F) \longrightarrow RG \circ RF.$$
is isomorphism of functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$.

Proof. If (2) holds, then (1) follows by evaluating the isomorphism t on $RF(I) = F(I)$. Conversely, assume (1) holds. Let $A^\bullet$ be a bounded below complex of $\mathcal{A}$. Choose an injective resolution $A^\bullet \to I^\bullet$. The map t is given (see proof of Lemma 14.16) by the maps
$$R(G \circ F)(A^\bullet) = (G \circ F)(I^\bullet) = G(F(I^\bullet))) \to RG(F(I^\bullet)) = RG(RF(A^\bullet))$$
where the arrow is an isomorphism by Lemma 16.7. □

Lemma 22.2 (Grothendieck spectral sequence)**.** *With assumptions as in Lemma 22.1 and assuming the equivalent conditions (1) and (2) hold. Let X be an object of $D^+(\mathcal{A})$. There exists a spectral sequence $(E_r, d_r)_{r\geq 0}$ consisting of bigraded objects E_r of $\mathcal{C}$ with d_r of bidegree $(r, -r+1)$ and with*
$$E_2^{p,q} = R^pG(H^q(RF(X)))$$
Moreover, this spectral sequence is bounded, converges to $H^*(R(G \circ F)(X))$, and induces a finite filtration on each $H^n(R(G \circ F)(X))$.

For an object A of $\mathcal{A}$ we get $E_2^{p,q} = R^pG(R^qF(A))$ converging to $R^{p+q}(G \circ F)(A)$.

Proof. We may represent X by a bounded below complex $A^\bullet$. Choose an injective resolution $A^\bullet \to I^\bullet$. Choose a Cartan-Eilenberg resolution $F(I^\bullet) \to I^{\bullet,\bullet}$ using Lemma 21.2. Apply the second spectral sequence of Lemma 21.3. □

23. Resolution functors

Let $\mathcal{A}$ be an abelian category with enough injectives. Denote $\mathcal{I}$ the full additive subcategory of $\mathcal{A}$ whose objects are the injective objects of $\mathcal{A}$. It turns out that $K^+(\mathcal{I})$ and $D^+(\mathcal{A})$ are equivalent in this case (see Proposition 23.1). For many purposes it therefore makes sense to think of $D^+(\mathcal{A})$ as the (easier to grok) category $K^+(\mathcal{I})$ in this case.

Proposition 23.1. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Denote $\mathcal{I} \subset \mathcal{A}$ the strictly full additive subcategory whose objects are the injective objects of $\mathcal{A}$. The functor*
$$K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$$
is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories.

Proof. It is clear that the functor is exact. It is essentially surjective by Lemma 18.3. Fully faithfulness is a consequence of Lemma 18.8. □

Proposition 23.1 implies that we can find resolution functors. It turns out that we can prove resolution functors exist even in some cases where the abelian category $\mathcal{A}$ is a "big" category, i.e., has a class of objects.

Definition 23.2. Let $\mathcal{A}$ be an abelian category with enough injectives. A *resolution functor*[7] for $\mathcal{A}$ is given by the following data:

[7]This is likely nonstandard terminology.

(1) for all $K^\bullet \in \mathrm{Ob}(K^+(\mathcal{A}))$ a bounded below complex of injectives $j(K^\bullet)$, and
(2) for all $K^\bullet \in \mathrm{Ob}(K^+(\mathcal{A}))$ a quasi-isomorphism $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$.

Lemma 23.3. *Let $\mathcal{A}$ be an abelian category with enough injectives. Given a resolution functor (j, i) there is a unique way to turn j into a functor and i into a 2-isomorphism producing a 2-commutative diagram*

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[rr, "j"'] \arrow[dr] & & K^+(\mathcal{I}) \arrow[dl] \\ & D^+(\mathcal{A}) & \end{tikzcd}$$

where $\mathcal{I}$ is the full additive subcategory of $\mathcal{A}$ consisting of injective objects.

Proof. For every morphism $\alpha : K^\bullet \to L^\bullet$ of $K^+(\mathcal{A})$ there is a unique morphism $j(\alpha) : j(K^\bullet) \to j(L^\bullet)$ in $K^+(\mathcal{I})$ such that

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"'] \arrow[d, "i_{K^\bullet}"'] & L^\bullet \arrow[d, "i_{L^\bullet}"] \\ j(K^\bullet) \arrow[r, "j(\alpha)"] & j(L^\bullet) \end{tikzcd}$$

is commutative in $K^+(\mathcal{A})$. To see this either use Lemmas 18.6 and 18.7 or the equivalent Lemma 18.8. The uniqueness implies that j is a functor, and the commutativity of the diagram implies that i gives a 2-morphism which witnesses the 2-commutativity of the diagram of categories in the statement of the lemma. $\square$

Lemma 23.4. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Then a resolution functor j exists and is unique up to unique isomorphism of functors.*

Proof. Consider the set of all objects $K^\bullet$ of $K^+(\mathcal{A})$. (Recall that by our conventions any category has a set of objects unless mentioned otherwise.) By Lemma 18.3 every object has an injective resolution. By the axiom of choice we can choose for each $K^\bullet$ an injective resolution $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$. $\square$

Lemma 23.5. *Let $\mathcal{A}$ be an abelian category with enough injectives. Any resolution functor $j : K^+(\mathcal{A}) \to K^+(\mathcal{I})$ is exact.*

Proof. Denote $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$ the canonical maps of Definition 23.2. First we discuss the existence of the functorial isomorphism $j(K^\bullet[1]) \to j(K^\bullet)[1]$. Consider the diagram

$$\begin{tikzcd} K^\bullet[1] \arrow[r, equal] \arrow[d, "i_{K^\bullet[1]}"] & K^\bullet[1] \arrow[d, "i_{K^\bullet}[1]"] \\ j(K^\bullet[1]) \arrow[r, dotted, "\xi_{K^\bullet}"] & j(K^\bullet)[1] \end{tikzcd}$$

By Lemmas 18.6 and 18.7 there exists a unique dotted arrow $\xi_{K^\bullet}$ in $K^+(\mathcal{I})$ making the diagram commute in $K^+(\mathcal{A})$. We omit the verification that this gives a functorial isomorphism. (Hint: use Lemma 18.7 again.)

Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle of $K^+(\mathcal{A})$. We have to show that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle of $K^+(\mathcal{I})$.

Note that we have a commutative diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "f"'] \arrow[d] & L^\bullet \arrow[r, "g"'] \arrow[d] & M^\bullet \arrow[r, "h"'] \arrow[d] & K^\bullet[1] \arrow[d] \\ j(K^\bullet) \arrow[r, "j(f)"] & j(L^\bullet) \arrow[r, "j(g)"] & j(M^\bullet) \arrow[r, "\xi_{K^\bullet} \circ j(h)"] & j(K^\bullet)[1] \end{tikzcd}$$

in $K^+(\mathcal{A})$ whose vertical arrows are the quasi-isomorphisms i_K, i_L, i_M. Hence we see that the image of $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ in $D^+(\mathcal{A})$ is isomorphic to a distinguished triangle and hence a distinguished triangle by TR1. Thus we see from Lemma 4.18 that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle in $K^+(\mathcal{I})$. $\square$

Lemma 23.6. *Let $\mathcal{A}$ be an abelian category which has enough injectives. Let j be a resolution functor. Write $Q : K^+(\mathcal{A}) \to D^+(\mathcal{A})$ for the natural functor. Then $j = j' \circ Q$ for a unique functor $j' : D^+(\mathcal{A}) \to K^+(\mathcal{I})$ which is quasi-inverse to the canonical functor $K^+(\mathcal{I}) \to D^+(\mathcal{A})$.*

Proof. By Lemma 11.6 Q is a localization functor. To prove the existence of j' it suffices to show that any element of $\text{Qis}^+(\mathcal{A})$ is mapped to an isomorphism under the functor j, see Lemma 5.7. This is true by the remarks following Definition 23.2. $\square$

Remark 23.7. Suppose that $\mathcal{A}$ is a "big" abelian category with enough injectives such as the category of abelian groups. In this case we have to be slightly more careful in constructing our resolution functor since we cannot use the axiom of choice with a quantifier ranging over a class. But note that the proof of the lemma does show that any two localization functors are canonically isomorphic. Namely, given quasi-isomorphisms $i : K^\bullet \to I^\bullet$ and $i' : K^\bullet \to J^\bullet$ of a bounded below complex $K^\bullet$ into bounded below complexes of injectives there exists a unique(!) morphism $a : I^\bullet \to J^\bullet$ in $K^+(\mathcal{I})$ such that $i' = i \circ a$ as morphisms in $K^+(\mathcal{I})$. Hence the only issue is existence, and we will see how to deal with this in the next section.

24. Functorial injective embeddings and resolution functors

In this section we redo the construction of a resolution functor $K^+(\mathcal{A}) \to K^+(\mathcal{I})$ in case the category $\mathcal{A}$ has functorial injective embeddings. There are two reasons for this: (1) the proof is easier and (2) the construction also works if $\mathcal{A}$ is a "big" abelian category. See Remark 24.3 below.

Let $\mathcal{A}$ be an abelian category. As before denote $\mathcal{I}$ the additive full subcategory of $\mathcal{A}$ consisting of injective objects. Consider the category $\text{InjRes}(\mathcal{A})$ of arrows $\alpha : K^\bullet \to I^\bullet$ where $K^\bullet$ is a bounded below complex of $\mathcal{A}$, $I^\bullet$ is a bounded below complex of injectives of $\mathcal{A}$ and α is a quasi-isomorphism. In other words, α is an injective resolution and $K^\bullet$ is bounded below. There is an obvious functor

$$s : \text{InjRes}(\mathcal{A}) \longrightarrow \text{Comp}^+(\mathcal{A})$$

defined by $(\alpha : K^\bullet \to I^\bullet) \mapsto K^\bullet$. There is also a functor

$$t : \text{InjRes}(\mathcal{A}) \longrightarrow K^+(\mathcal{I})$$

defined by $(\alpha : K^\bullet \to I^\bullet) \mapsto I^\bullet$.

Lemma 24.1. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has functorial injective embeddings, see Homology, Definition 27.5.*

1. *There exists a functor $inj : Comp^+(\mathcal{A}) \to InjRes(\mathcal{A})$ such that $s \circ inj = id$.*
2. *For any functor $inj : Comp^+(\mathcal{A}) \to InjRes(\mathcal{A})$ such that $s \circ inj = id$ we obtain a resolution functor, see Definition 23.2.*

Proof. Let $A \mapsto (A \to J(A))$ be a functorial injective embedding, see Homology, Definition 27.5. We first note that we may assume $J(0) = 0$. Namely, if not then for any object A we have $0 \to A \to 0$ which gives a direct sum decomposition $J(A) = J(0) \oplus \mathrm{Ker}(J(A) \to J(0))$. Note that the functorial morphism $A \to J(A)$ has to map into the second summand. Hence we can replace our functor by $J'(A) = \mathrm{Ker}(J(A) \to J(0))$ if needed.

Let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. Say $K^p = 0$ if $p < B$. We are going to construct a double complex $I^{\bullet,\bullet}$ of injectives, together with a map $\alpha : K^\bullet \to I^{\bullet,0}$ such that α induces a quasi-isomorphism of $K^\bullet$ with the associated total complex of $I^{\bullet,\bullet}$. First we set $I^{p,q} = 0$ whenever $q < 0$. Next, we set $I^{p,0} = J(K^p)$ and $\alpha^p : K^p \to I^{p,0}$ the functorial embedding. Since J is a functor we see that $I^{\bullet,0}$ is a complex and that α is a morphism of complexes. Each α^p is injective. And $I^{p,0} = 0$ for $p < B$ because $J(0) = 0$. Next, we set $I^{p,1} = J(\mathrm{Coker}(K^p \to I^{p,0}))$. Again by functoriality we see that $I^{\bullet,1}$ is a complex. And again we get that $I^{p,1} = 0$ for $p < B$. It is also clear that K^p maps isomorphically onto $\mathrm{Ker}(I^{p,0} \to I^{p,1})$. As our third step we take $I^{p,2} = J(\mathrm{Coker}(I^{p,0} \to I^{p,1}))$. And so on and so forth.

At this point we can apply Homology, Lemma 25.4 to get that the map

$$\alpha : K^\bullet \longrightarrow \mathrm{Tot}(I^{\bullet,\bullet})$$

is a quasi-isomorphism. To prove we get a functor inj it rests to show that the construction above is functorial. This verification is omitted.

Suppose we have a functor inj such that $s \circ inj = \mathrm{id}$. For every object $K^\bullet$ of $\mathrm{Comp}^+(\mathcal{A})$ we can write

$$inj(K^\bullet) = (i_{K^\bullet} : K^\bullet \to j(K^\bullet))$$

This provides us with a resolution functor as in Definition 23.2. $\square$

Remark 24.2. Suppose inj is a functor such that $s \circ inj = \mathrm{id}$ as in part (2) of Lemma 24.1. Write $inj(K^\bullet) = (i_{K^\bullet} : K^\bullet \to j(K^\bullet))$ as in the proof of that lemma. Suppose $\alpha : K^\bullet \to L^\bullet$ is a map of bounded below complexes. Consider the map $inj(\alpha)$ in the category $\mathrm{InjRes}(\mathcal{A})$. It induces a commutative diagram

$$\begin{tikzcd}
K^\bullet \arrow[r, "\alpha"] \arrow[d, "i_K"'] & L^\bullet \arrow[d, "i_L"] \\
j(K)^\bullet \arrow[r, "inj(\alpha)"] & j(L)^\bullet
\end{tikzcd}$$

of morphisms of complexes. Hence, looking at the proof of Lemma 23.3 we see that the functor $j : K^+(\mathcal{A}) \to K^+(\mathcal{I})$ is given by the rule

$$j(\alpha \text{ up to homotopy}) = inj(\alpha) \text{ up to homotopy} \in \mathrm{Hom}_{K^+(\mathcal{I})}(j(K^\bullet), j(L^\bullet))$$

Hence we see that j matches $t \circ inj$ in this case, i.e., the diagram

$$\begin{tikzcd} \mathrm{Comp}^+(\mathcal{A}) \arrow[rr, "t \circ inj"'] \arrow[dr] & & K^+(\mathcal{I}) \\ & K^+(\mathcal{A}) \arrow[ur, "j"'] & \end{tikzcd}$$

is commutative.

Remark 24.3. Let $Mod(\mathcal{O}_X)$ be the category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ (or more generally on a ringed site). We will see later that $Mod(\mathcal{O}_X)$ has enough injectives and in fact functorial injective embeddings, see Injectives, Theorem 8.4. Note that the proof of Lemma 23.4 does not apply to $Mod(\mathcal{O}_X)$. But the proof of Lemma 24.1 does apply to $Mod(\mathcal{O}_X)$. Thus we obtain

$$j : K^+(Mod(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

which is a resolution functor where $\mathcal{I}$ is the additive category of injective $\mathcal{O}_X$-modules. This argument also works in the following cases:

(1) The category Mod_R of R-modules over a ring R.
(2) The category $PMod(\mathcal{O})$ of presheaves of $\mathcal{O}$-modules on a site endowed with a presheaf of rings.
(3) The category $Mod(\mathcal{O})$ of sheaves of $\mathcal{O}$-modules on a ringed site.
(4) Add more here as needed.

25. Right derived functors via resolution functors

The content of the following lemma is that we can simply define $RF(K^\bullet) = F(j(K^\bullet))$ if we are given a resolution functor j.

Lemma 25.1. *Let $\mathcal{A}$ be an abelian category with enough injectives Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor into an abelian category. Let (i, j) be a resolution functor, see Definition 23.2. The right derived functor RF of F fits into the following 2-commutative diagram*

$$\begin{tikzcd} D^+(\mathcal{A}) \arrow[rr, "j'"] \arrow[dr, "RF"'] & & K^+(\mathcal{I}) \arrow[dl, "F"] \\ & D^+(\mathcal{B}) & \end{tikzcd}$$

where j' is the functor from Lemma 23.6.

Proof. By Lemma 20.1 we have $RF(K^\bullet) = F(j(K^\bullet))$. $\square$

Remark 25.2. In the situation of Lemma 25.1 we see that we have actually lifted the right derived functor to an exact functor $F \circ j' : D^+(\mathcal{A}) \to K^+(\mathcal{B})$. It is occasionally useful to use such a factorization.

26. Filtered derived category and injective resolutions

Let $\mathcal{A}$ be an abelian category. In this section we will show that if $\mathcal{A}$ has enough injectives, then so does the category $\mathrm{Fil}^f(\mathcal{A})$ in some sense. One can use this observation to compute in the filtered derived category of $\mathcal{A}$.

The category $\text{Fil}^f(\mathcal{A})$ is an example of an exact category, see Injectives, Remark 9.6. A special role is played by the strict morphisms, see Homology, Definition 19.3, i.e., the morphisms f such that $\text{Coim}(f) = \text{Im}(f)$. We will say that a complex $A \to B \to C$ in $\text{Fil}^f(\mathcal{A})$ is *exact* if the sequence $\text{gr}(A) \to \text{gr}(B) \to \text{gr}(C)$ is exact in $\mathcal{A}$. This implies that $A \to B$ and $B \to C$ are strict morphisms, see Homology, Lemma 19.15.

Definition 26.1. Let $\mathcal{A}$ be an abelian category. We say an object I of $\text{Fil}^f(\mathcal{A})$ is *filtered injective* if each $\text{gr}^p(I)$ is an injective object of $\mathcal{A}$.

Lemma 26.2. *Let $\mathcal{A}$ be an abelian category. An object I of $\text{Fil}^f(\mathcal{A})$ is filtered injective if and only if there exist $a \leq b$, injective objects I_n, $a \leq n \leq b$ of $\mathcal{A}$ and an isomorphism $I \cong \bigoplus_{a \leq n \leq b} I_n$ such that $F^pI = \bigoplus_{n \geq p} I_n$.*

Proof. Follows from the fact that any injection $J \to M$ of $\mathcal{A}$ is split if J is an injective object. Details omitted. □

Lemma 26.3. *Let $\mathcal{A}$ be an abelian category. Any strict monomorphism $u : I \to A$ of $\text{Fil}^f(\mathcal{A})$ where I is a filtered injective object is a split injection.*

Proof. Let p be the largest integer such that $F^pI \neq 0$. In particular $\text{gr}^p(I) = F^pI$. Let I' be the object of $\text{Fil}^f(\mathcal{A})$ whose underlying object of $\mathcal{A}$ is F^pI and with filtration given by $F^nI' = 0$ for $n > p$ and $F^nI' = I' = F^pI$ for $n \leq p$. Note that $I' \to I$ is a strict monomorphism too. The fact that u is a strict monomorphism implies that $F^pI \to A/F^{p+1}(A)$ is injective, see Homology, Lemma 19.13. Choose a splitting $s : A/F^{p+1}A \to F^pI$ in $\mathcal{A}$. The induced morphism $s' : A \to I'$ is a strict morphism of filtered objects splitting the composition $I' \to I \to A$. Hence we can write $A = I' \oplus \text{Ker}(s')$ and $I = I' \oplus \text{Ker}(s'|_I)$. Note that $\text{Ker}(s'|_I) \to \text{Ker}(s')$ is a strict monomorphism and that $\text{Ker}(s'|_I)$ is a filtered injective object. By induction on the length of the filtration on I the map $\text{Ker}(s'|_I) \to \text{Ker}(s')$ is a split injection. Thus we win. □

Lemma 26.4. *Let $\mathcal{A}$ be an abelian category. Let $u : A \to B$ be a strict monomorphism of $\text{Fil}^f(\mathcal{A})$ and $f : A \to I$ a morphism from A into a filtered injective object in $\text{Fil}^f(\mathcal{A})$. Then there exists a morphism $g : B \to I$ such that $f = g \circ u$.*

Proof. The pushout $f' : I \to I \amalg_A B$ of f by u is a strict monomorphism, see Homology, Lemma 19.10. Hence the result follows formally from Lemma 26.3. □

Lemma 26.5. *Let $\mathcal{A}$ be an abelian category with enough injectives. For any object A of $\text{Fil}^f(\mathcal{A})$ there exists a strict monomorphism $A \to I$ where I is a filtered injective object.*

Proof. Pick $a \leq b$ such that $\text{gr}^p(A) = 0$ unless $p \in \{a, a+1, \ldots, b\}$. For each $n \in \{a, a+1, \ldots, b\}$ choose an injection $u_n : A/F^{n+1}A \to I_n$ with I_n an injective object. Set $I = \bigoplus_{a \leq n \leq b} I_n$ with filtration $F^pI = \bigoplus_{n \geq p} I_n$ and set $u : A \to I$ equal to the direct sum of the maps u_n. □

Lemma 26.6. *Let $\mathcal{A}$ be an abelian category with enough injectives. For any object A of $\text{Fil}^f(\mathcal{A})$ there exists a filtered quasi-isomorphism $A[0] \to I^\bullet$ where $I^\bullet$ is a complex of filtered injective objects with $I^n = 0$ for $n < 0$.*

Proof. First choose a strict monomorphism $u_0 : A \to I^0$ of A into a filtered injective object, see Lemma 26.5. Next, choose a strict monomorphism $u_1 : \text{Coker}(u_0) \to I^1$ into a filtered injective object of $\mathcal{A}$. Denote d^0 the induced map $I^0 \to I^1$. Next, choose a strict monomorphism $u_2 : \text{Coker}(u_1) \to I^2$ into a filtered injective object of $\mathcal{A}$. Denote d^1 the induced map $I^1 \to I^2$. And so on. This works because each of the sequences

$$0 \to \text{Coker}(u_n) \to I^{n+1} \to \text{Coker}(u_{n+1}) \to 0$$

is short exact, i.e., induces a short exact sequence on applying gr. To see this use Homology, Lemma 19.13. □

Lemma 26.7. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $f : A \to B$ be a morphism of $\text{Fil}^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \to I^\bullet$ and $B[0] \to J^\bullet$ where $I^\bullet, J^\bullet$ are complexes of filtered injective objects with $I^n = J^n = 0$ for $n < 0$, then there exists a commutative diagram*

$$\begin{tikzcd} A[0] \arrow[r] \arrow[d] & B[0] \arrow[d] \\ I^\bullet \arrow[r] & J^\bullet \end{tikzcd}$$

Proof. As $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \to I^0$, $b : B \to J^0$ and all the morphisms d_I^n, d_J^n are strict, see Homology, Lemma 19.15. We will inductively construct the maps f^n in the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"'] \arrow[d, "f"'] & I^0 \arrow[r] \arrow[d, "f^0"] & I^1 \arrow[r] \arrow[d, "f^1"] & I^2 \arrow[r] \arrow[d, "f^2"] & \ldots \\ B \arrow[r, "b"] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \ldots \end{tikzcd}$$

Because $A \to I^0$ is a strict monomorphism and because J^0 is filtered injective, we can find a morphism $f^0 : I^0 \to J^0$ such that $f^0 \circ a = b \circ f$, see Lemma 26.4. The composition $d_J^0 \circ b \circ f$ is zero, hence $d_J^0 \circ f^0 \circ a = 0$, hence $d_J^0 \circ f^0$ factors through a unique morphism

$$\text{Coker}(a) = \text{Coim}(d_I^0) = \text{Im}(d_I^0) \longrightarrow J^1.$$

As $\text{Im}(d_I^0) \to I^1$ is a strict monomorphism we can extend the displayed arrow to a morphism $f^1 : I^1 \to J^1$ by Lemma 26.4 again. And so on. □

Lemma 26.8. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\text{Fil}^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ where $I^\bullet, J^\bullet$ are complexes of filtered injective objects with $I^n = J^n = 0$ for $n < 0$, then there exists a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & A[0] \arrow[r] \arrow[d] & B[0] \arrow[r] \arrow[d] & C[0] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^\bullet \arrow[r] & M^\bullet \arrow[r] & J^\bullet \arrow[r] & 0 \end{tikzcd}$$

where the lower row is a termwise split sequence of complexes.

Proof. As $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \to I^0$, $c : C \to J^0$ and all the morphisms d_I^n, d_J^n are strict, see Homology, Lemma 13.4. We are going to step by step construct the south-east and the south arrows in the following commutative diagram

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[dr, "b"] & C \arrow[r, "c"] \arrow[dr, "\overline{b}"] & J^0 \arrow[r] \arrow[d, "\delta^0"] & J^1 \arrow[r] \arrow[d, "\delta^1"] & \ldots \\ A \arrow[u, "\alpha"] \arrow[r, "a"] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \ldots \end{tikzcd}$$

As $A \to B$ is a strict monomorphism, we can find a morphism $b : B \to I^0$ such that $b \circ \alpha = a$, see Lemma 26.4. As A is the kernel of the strict morphism $I^0 \to I^1$ and $\beta = \text{Coker}(\alpha)$ we obtain a unique morphism $\overline{b} : C \to I^1$ fitting into the diagram. As c is a strict monomorphism and I^1 is filtered injective we can find $\delta^0 : J^0 \to I^1$, see Lemma 26.4. Because $B \to C$ is a strict epimorphism and because $B \to I^0 \to I^1 \to I^2$ is zero, we see that $C \to I^1 \to I^2$ is zero. Hence $d_I^1 \circ \delta^0$ is zero on $C \cong \text{Im}(c)$. Hence $d_I^1 \circ \delta^0$ factors through a unique morphism

$$\text{Coker}(c) = \text{Coim}(d_J^0) = \text{Im}(d_J^0) \longrightarrow I^2.$$

As I^2 is filtered injective and $\text{Im}(d_J^0) \to J^1$ is a strict monomorphism we can extend the displayed morphism to a morphism $\delta^1 : J^1 \to I^2$, see Lemma 26.4. And so on. We set $M^\bullet = I^\bullet \oplus J^\bullet$ with differential

$$d_M^n = \begin{pmatrix} d_I^n & (-1)^{n+1}\delta^n \\ 0 & d_J^n \end{pmatrix}$$

Finally, the map $B[0] \to M^\bullet$ is given by $b \oplus c \circ \beta : M \to I^0 \oplus J^0$. $\square$

Lemma 26.9. *Let $\mathcal{A}$ be an abelian category with enough injectives. For every $K^\bullet \in K^+(Fil^f(\mathcal{A}))$ there exists a filtered quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below, each I^n a filtered injective object, and each $K^n \to I^n$ a strict monomorphism.*

Proof. After replacing $K^\bullet$ by a shift (which is harmless for the proof) we may assume that $K^n = 0$ for $n < 0$. Consider the short exact sequences

$$\begin{matrix} 0 \to \text{Ker}(d_K^0) \to K^0 \to \text{Coim}(d_K^0) \to 0 \\ 0 \to \text{Ker}(d_K^1) \to K^1 \to \text{Coim}(d_K^1) \to 0 \\ 0 \to \text{Ker}(d_K^2) \to K^2 \to \text{Coim}(d_K^2) \to 0 \\ \ldots \end{matrix}$$

of the exact category $\text{Fil}^f(\mathcal{A})$ and the maps $u_i : \text{Coim}(d_K^i) \to \text{Ker}(d_K^{i+1})$. For each $i \geq 0$ we may choose filtered quasi-isomorphisms

$$\begin{matrix} \text{Ker}(d_K^i)[0] \to I_{ker,i}^\bullet \\ \text{Coim}(d_K^i)[0] \to I_{coim,i}^\bullet \end{matrix}$$

with $I_{ker,i}^n, I_{coim,i}^n$ filtered injective and zero for $n < 0$, see Lemma 26.6. By Lemma 26.7 we may lift u_i to a morphism of complexes $u_i^\bullet : I_{coim,i}^\bullet \to I_{ker,i+1}^\bullet$. Finally, for each $i \geq 0$ we may complete the diagrams

$$\begin{tikzcd} 0 \arrow[r] & \text{Ker}(d_K^i)[0] \arrow[r] \arrow[d] & K^i[0] \arrow[r] \arrow[d] & \text{Coim}(d_K^i)[0] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I_{ker,i}^\bullet \arrow[r, "\alpha_i"] & I_i^\bullet \arrow[r, "\beta_i"] & I_{coim,i}^\bullet \arrow[r] & 0 \end{tikzcd}$$

with the lower sequence a termwise split exact sequence, see Lemma 26.8. For $i \geq 0$ set $d_i : I_i^\bullet \to I_{i+1}^\bullet$ equal to $d_i = \alpha_{i+1} \circ u_i^\bullet \circ \beta_i$. Note that $d_i \circ d_{i-1} = 0$ because $\beta_i \circ \alpha_i = 0$. Hence we have constructed a commutative diagram

$$\begin{tikzcd} I_0^\bullet \arrow[r] & I_1^\bullet \arrow[r] & I_2^\bullet \arrow[r] & \ldots \\ K^0[0] \arrow[r] \arrow[u] & K^1[0] \arrow[r] \arrow[u] & K^2[0] \arrow[r] \arrow[u] & \ldots \end{tikzcd}$$

Here the vertical arrows are filtered quasi-isomorphisms. The upper row is a complex of complexes and each complex consists of filtered injective objects with no nonzero objects in degree < 0. Thus we obtain a double complex by setting $I^{a,b} = I_a^b$ and using

$$d_1^{a,b} : I^{a,b} = I_a^b \to I_{a+1}^b = I^{a+1,b}$$

the map d_a^b and using for

$$d_2^{a,b} : I^{a,b} = I_a^b \to I_a^{b+1} = I^{a,b+1}$$

the map $d_{I_a}^b$. Denote $\text{Tot}(I^{\bullet,\bullet})$ the total complex associated to this double complex, see Homology, Definition 18.3. Observe that the maps $K^n[0] \to I_n^\bullet$ come from maps $K^n \to I^{n,0}$ which give rise to a map of complexes

$$K^\bullet \longrightarrow \text{Tot}(I^{\bullet,\bullet})$$

We claim this is a filtered quasi-isomorphism. As $\text{gr}(-)$ is an additive functor, we see that $\text{gr}(\text{Tot}(I^{\bullet,\bullet})) = \text{Tot}(\text{gr}(I^{\bullet,\bullet}))$. Thus we can use Homology, Lemma 25.4 to conclude that $\text{gr}(K^\bullet) \to \text{gr}(\text{Tot}(I^{\bullet,\bullet}))$ is a quasi-isomorphism as desired. $\square$

Lemma 26.10. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet, I^\bullet \in K(Fil^f(\mathcal{A}))$. Assume $K^\bullet$ is filtered acyclic and $I^\bullet$ bounded below and consisting of filtered injective objects. Any morphism $K^\bullet \to I^\bullet$ is homotopic to zero:* $\text{Hom}_{K(Fil^f(\mathcal{A}))}(K^\bullet, I^\bullet) = 0$.

Proof. Let $\alpha : K^\bullet \to I^\bullet$ be a morphism of complexes. Assume that $\alpha^j = 0$ for $j < n$. We will show that there exists a morphism $h : K^{n+1} \to I^n$ such that $\alpha^n = h \circ d$. Thus α will be homotopic to the morphism of complexes β defined by

$$\beta^j = \begin{cases} 0 & \text{if} \quad j \leq n \\ \alpha^{n+1} - d \circ h & \text{if} \quad j = n+1 \\ \alpha^j & \text{if} \quad j > n+1 \end{cases}$$

This will clearly prove the lemma (by induction). To prove the existence of h note that $\alpha^n \circ d_K^{n-1} = 0$ since $\alpha^{n-1} = 0$. Since $K^\bullet$ is filtered acyclic we see that d_K^{n-1} and d_K^n are strict and that

$$0 \to \text{Im}(d_K^{n-1}) \to K^n \to \text{Im}(d_K^n) \to 0$$

is an exact sequence of the exact category $\text{Fil}^f(\mathcal{A})$, see Homology, Lemma 19.15. Hence we can think of α^n as a map into I^n defined on $\text{Im}(d_K^n)$. Using that $\text{Im}(d_K^n) \to K^{n+1}$ is a strict monomorphism and that I^n is filtered injective we may lift this map to a map $h : K^{n+1} \to I^n$ as desired, see Lemma 26.4. $\square$

Lemma 26.11. *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet \in K(Fil^f(\mathcal{A}))$ be a bounded below complex consisting of filtered injective objects.*

(1) *Let* $\alpha : K^\bullet \to L^\bullet$ *in* $K(Fil^f(\mathcal{A}))$ *be a filtered quasi-isomorphism. Then the map*
$$\mathrm{Hom}_{K(Fil^f(\mathcal{A}))}(L^\bullet, I^\bullet) \to \mathrm{Hom}_{K(Fil^f(\mathcal{A}))}(K^\bullet, I^\bullet)$$
is bijective.

(2) *Let* $L^\bullet \in K(Fil^f(\mathcal{A}))$. *Then*
$$\mathrm{Hom}_{K(Fil^f(\mathcal{A}))}(L^\bullet, I^\bullet) = \mathrm{Hom}_{DF(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Proof of (1). Note that
$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$
is a distinguished triangle in $K(\mathrm{Fil}^f(\mathcal{A}))$ (Lemma 9.14) and $C(\alpha)^\bullet$ is a filtered acyclic complex (Lemma 13.4). Then

$$\begin{tikzcd} \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet, I^\bullet) \arrow[r] & \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(L^\bullet, I^\bullet) \arrow[r] & \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(K^\bullet, I^\bullet) \arrow[dll] \\ \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet[-1], I^\bullet) & & \end{tikzcd}$$

is an exact sequence of abelian groups, see Lemma 4.2. At this point Lemma 26.10 guarantees that the outer two groups are zero and hence $\mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

Proof of (2). Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \to L^\bullet$ is a filtered quasi-isomorphism and $\gamma : K^\bullet \to I^\bullet$ is a map of complexes. By part (1) we can find a morphism $\beta : L^\bullet \to I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ. This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \to I^\bullet$ such that there exists a filtered quasi-isomorphism $\alpha : K^\bullet \to L^\bullet$ with $\beta \circ \alpha$ homotopic to zero. Then part (1) shows that β is homotopic to zero also, i.e., $b = 0$. □

Lemma 26.12. *Let* $\mathcal{A}$ *be an abelian category with enough injectives. Let* $\mathcal{I}^f \subset Fil^f(\mathcal{A})$ *denote the strictly full additive subcategory whose objects are the filtered injective objects. The canonical functor*
$$K^+(\mathcal{I}^f) \longrightarrow DF^+(\mathcal{A})$$
is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories. Furthermore the diagrams
$$\begin{tikzcd} K^+(\mathcal{I}^f) \arrow[r] \arrow[d, "gr^p"'] & DF^+(\mathcal{A}) \arrow[d, "gr^p"'] \\ K^+(\mathcal{I}) \arrow[r] & D^+(\mathcal{A}) \end{tikzcd} \qquad \begin{tikzcd} K^+(\mathcal{I}^f) \arrow[r] \arrow[d, "forget\ F"] & DF^+(\mathcal{A}) \arrow[d, "forget\ F"] \\ K^+(\mathcal{I}) \arrow[r] & D^+(\mathcal{A}) \end{tikzcd}$$
are commutative, where $\mathcal{I} \subset \mathcal{A}$ *is the strictly full additive subcategory whose objects are the injective objects.*

Proof. The functor $K^+(\mathcal{I}^f) \to DF^+(\mathcal{A})$ is essentially surjective by Lemma 26.9. It is fully faithful by Lemma 26.11. It is an exact functor by our definitions regarding distinguished triangles. The commutativity of the squares is immediate. □

Remark 26.13. We can invert the arrow of the lemma only if $\mathcal{A}$ is a category in our sense, namely if it has a set of objects. However, suppose given a big abelian category $\mathcal{A}$ with enough injectives, such as $\mathit{Mod}(\mathcal{O}_X)$ for example. Then for any given set of objects $\{A_i\}_{i \in I}$ there is an abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ containing all of them and having enough injectives, see Sets, Lemma 12.1. Thus we may use the lemma above for $\mathcal{A}'$. This essentially means that if we use a set worth of diagrams, etc then we will never run into trouble using the lemma.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. (We cannot use the letter F for the functor since this would conflict too much with our use of the letter F to indicate filtrations.) Note that T induces an additive functor

$$T : \text{Fil}^f(\mathcal{A}) \to \text{Fil}^f(\mathcal{B})$$

by the rule $T(A, F) = (T(A), F)$ where $F^pT(A) = T(F^pA)$ which makes sense as T is left exact. (Warning: It may not be the case that $\text{gr}(T(A)) = T(\text{gr}(A))$.) This induces functors of triangulated categories

$$T : K^+(\text{Fil}^f(\mathcal{A})) \longrightarrow K^+(\text{Fil}^f(\mathcal{B})) \tag{26.13.1}$$

The filtered right derived functor of T is the right derived functor of Definition 14.2 for this exact functor composed with the exact functor $K^+(\text{Fil}^f(\mathcal{B})) \to DF^+(\mathcal{B})$ and the multiplicative set $\text{FQis}^+(\mathcal{A})$. Assume $\mathcal{A}$ has enough injectives. At this point we can redo the discussion of Section 20 to define the *filtered right derived functors*

$$RT : DF^+(\mathcal{A}) \longrightarrow DF^+(\mathcal{B}) \tag{26.13.2}$$

of our functor T.

However, instead we will proceed as in Section 25, and it will turn out that we can define RT even if T is just additive. Namely, we first choose a quasi-inverse $j' : DF^+(\mathcal{A}) \to K^+(\mathcal{I}^f)$ of the equivalence of Lemma 26.12. By Lemma 4.18 we see that j' is an exact functor of triangulated categories. Next, we note that for a filtered injective object I we have a (noncanonical) decomposition

$$I \cong \bigoplus\nolimits_{p \in \mathbf{Z}} I_p, \quad \text{with} \quad F^pI = \bigoplus\nolimits_{q \geq p} I_q \tag{26.13.3}$$

by Lemma 26.2. Hence if T is any additive functor $T : \mathcal{A} \to \mathcal{B}$ then we get an additive functor

$$T_{ext} : \mathcal{I}^f \to \text{Fil}^f(\mathcal{B}) \tag{26.13.4}$$

by setting $T_{ext}(I) = \bigoplus T(I_p)$ with $F^pT_{ext}(I) = \bigoplus_{q \geq p} T(I_q)$. Note that we have the property $\text{gr}(T_{ext}(I)) = T(\text{gr}(I))$ by construction. Hence we obtain a functor

$$T_{ext} : K^+(\mathcal{I}^f) \to K^+(\text{Fil}^f(\mathcal{B})) \tag{26.13.5}$$

which commutes with gr. Then we define (26.13.2) by the composition

$$RT = T_{ext} \circ j'. \tag{26.13.6}$$

Since $RT : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is computed by injective resolutions as well, see Lemmas 20.1, the commutation of T with gr, and the commutative diagrams of Lemma 26.12 imply that

$$\text{gr}^p \circ RT \cong RT \circ \text{gr}^p \tag{26.13.7}$$

and

$$(\text{forget } F) \circ RT \cong RT \circ (\text{forget } F) \tag{26.13.8}$$

as functors $DF^+(\mathcal{A}) \to D^+(\mathcal{B})$.

The filtered derived functor RT (26.13.2) induces functors

$$\begin{gathered} RT : \text{Fil}^f(\mathcal{A}) \to DF^+(\mathcal{B}), \\ RT : \text{Comp}^+(\text{Fil}^f(\mathcal{A})) \to DF^+(\mathcal{B}), \\ RT : KF^+(\mathcal{A}) \to DF^+(\mathcal{B}). \end{gathered}$$

Note that since $\text{Fil}^f(\mathcal{A})$, and $\text{Comp}^+(\text{Fil}^f(\mathcal{A}))$ are no longer abelian it does not make sense to say that RT restricts to a δ-functor on them. (This can be repaired by thinking of these categories as exact categories and formulating the notion of a δ-functor from an exact category into a triangulated category.) But it does make sense, and it is true by construction, that RT is an exact functor on the triangulated category $KF^+(\mathcal{A})$.

Lemma 26.14. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume $\mathcal{A}$ has enough injectives. Let $(K^\bullet, F)$ be an object of $Comp^+(Fil^f(\mathcal{A}))$. There exists a spectral sequence $(E_r, d_r)_{r \geq 0}$ consisting of bigraded objects E_r of $\mathcal{B}$ and d_r of bidegree $(r, -r+1)$ and with*

$$E_1^{p,q} = R^{p+q}T(gr^p(K^\bullet))$$

Moreover, this spectral sequence is bounded, converges to $R^*T(K^\bullet)$, and induces a finite filtration on each $R^nT(K^\bullet)$. The construction of this spectral sequence is functorial in the object $K^\bullet$ of $Comp^+(Fil^f(\mathcal{A}))$ and the terms (E_r, d_r) for $r \geq 1$ do not depend on any choices.

Proof. Choose a filtered quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ a bounded below complex of filtered injective objects, see Lemma 26.9. Consider the complex $RT(K^\bullet) = T_{ext}(I^\bullet)$, see (26.13.6). Thus we can consider the spectral sequence $(E_r, d_r)_{r \geq 0}$ associated to this as a filtered complex in $\mathcal{B}$, see Homology, Section 24. By Homology, Lemma 24.2 we have $E_1^{p,q} = H^{p+q}(\text{gr}^p(T(I^\bullet)))$. By Equation (26.13.3) we have $E_1^{p,q} = H^{p+q}(T(\text{gr}^p(I^\bullet)))$, and by definition of a filtered injective resolution the map $\text{gr}^p(K^\bullet) \to \text{gr}^p(I^\bullet)$ is an injective resolution. Hence $E_1^{p,q} = R^{p+q}T(\text{gr}^p(K^\bullet))$.

On the other hand, each I^n has a finite filtration and hence each $T(I^n)$ has a finite filtration. Thus we may apply Homology, Lemma 24.11 to conclude that the spectral sequence is bounded, converges to $H^n(T(I^\bullet)) = R^nT(K^\bullet)$ moreover inducing finite filtrations on each of the terms.

Suppose that $K^\bullet \to L^\bullet$ is a morphism of $\text{Comp}^+(\text{Fil}^f(\mathcal{A}))$. Choose a filtered quasi-isomorphism $L^\bullet \to J^\bullet$ with $J^\bullet$ a bounded below complex of filtered injective objects, see Lemma 26.9. By our results above, for example Lemma 26.11, there exists a diagram

$$\begin{tikzcd} K^\bullet \arrow[r] \arrow[d] & L^\bullet \arrow[d] \\ I^\bullet \arrow[r] & J^\bullet \end{tikzcd}$$

which commutes up to homotopy. Hence we get a morphism of filtered complexes $T(I^\bullet) \to T(J^\bullet)$ which gives rise to the morphism of spectral sequences, see Homology, Lemma 24.4. The last statement follows from this. $\square$

Remark 26.15. As promised in Remark 21.4 we discuss the connection of the lemma above with the constructions using Cartan-Eilenberg resolutions. Namely, let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, assume $\mathcal{A}$ has enough injectives, and let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. We give an alternative construction of the spectral sequences $'E$ and $''E$ of Lemma 21.3.

First spectral sequence. Consider the "stupid" filtration on $K^\bullet$ obtained by setting $F^p(K^\bullet) = \sigma_{\geq p}(K^\bullet)$, see Homology, Section 15. Note that this stupid in the sense that $d(F^p(K^\bullet)) \subset F^{p+1}(K^\bullet)$, compare Homology, Lemma 24.3. Note that $\text{gr}^p(K^\bullet) = K^p[-p]$ with this filtration. According to Lemma 26.14 there is a spectral sequence with E_1 term

$$E_1^{p,q} = R^{p+q}T(K^p[-p]) = R^qT(K^p)$$

as in the spectral sequence $'E_r$. Observe moreover that the differentials $E_1^{p,q} \to E_1^{p+1,q}$ agree with the differentials in $'E_1$, see Homology, Lemma 24.3 part (2) and the description of $'d_1$ in the proof of Lemma 21.3.

Second spectral sequence. Consider the filtration on the complex $K^\bullet$ obtained by setting $F^p(K^\bullet) = \tau_{\leq -p}(K^\bullet)$, see Homology, Section 15. The minus sign is necessary to get a decreasing filtration. Note that $\text{gr}^p(K^\bullet)$ is quasi-isomorphic to $H^{-p}(K^\bullet)[p]$ with this filtration. According to Lemma 26.14 there is a spectral sequence with E_1 term

$$E_1^{p,q} = R^{p+q}T(H^{-p}(K^\bullet)[p]) = R^{2p+q}T(H^{-p}(K^\bullet)) = {}''E_2^{i,j}$$

with $i = 2p+q$ and $j = -p$. (This looks unnatural, but note that we could just have well developed the whole theory of filtered complexes using increasing filtrations, with the end result that this then looks natural, but the other one doesn't.) We leave it to the reader to see that the differentials match up.

Actually, given a Cartan-Eilenberg resolution $K^\bullet \to I^{\bullet,\bullet}$ the induced morphism $K^\bullet \to \text{Tot}(I^{\bullet,\bullet})$ into the associated total complex will be a filtered injective resolution for either filtration using suitable filtrations on $\text{Tot}(I^{\bullet,\bullet})$. This can be used to match up the spectral sequences exactly.

27. Ext groups

In this section we start describing the Ext groups of objects of an abelian category. First we have the following very general definition.

Definition 27.1. Let $\mathcal{A}$ be an abelian category. Let $i \in \mathbf{Z}$. Let X, Y be objects of $D(\mathcal{A})$. The *ith extension group* of X by Y is the group

$$\text{Ext}^i_{\mathcal{A}}(X, Y) = \text{Hom}_{D(\mathcal{A})}(X, Y[i]) = \text{Hom}_{D(\mathcal{A})}(X[-i], Y).$$

If $A, B \in \text{Ob}(\mathcal{A})$ we set $\text{Ext}^i_{\mathcal{A}}(A, B) = \text{Ext}^i_{\mathcal{A}}(A[0], B[0])$.

Since $\text{Hom}_{D(\mathcal{A})}(X, -)$, resp. $\text{Hom}_{D(\mathcal{A})}(-, Y)$ is a homological, resp. cohomological functor, see Lemma 4.2, we see that a distinguished triangle (Y, Y', Y''), resp. (X, X', X'') leads to a long exact sequence

$$\ldots \to \text{Ext}^i_{\mathcal{A}}(X, Y) \to \text{Ext}^i_{\mathcal{A}}(X, Y') \to \text{Ext}^i_{\mathcal{A}}(X, Y'') \to \text{Ext}^{i+1}_{\mathcal{A}}(X, Y) \to \ldots$$

respectively

$$\ldots \to \mathrm{Ext}^i_{\mathcal{A}}(X'', Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X', Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X, Y) \to \mathrm{Ext}^{i+1}_{\mathcal{A}}(X'', Y) \to \ldots$$

Note that since $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$ are full subcategories we may compute the Ext groups by Hom groups in these categories provided X, Y are contained in them.

In case the category $\mathcal{A}$ has enough injectives or enough projectives we can compute the Ext groups using injective or projective resolutions. To avoid confusion, recall that having an injective (resp. projective) resolution implies vanishing of homology in all low (resp. high) degrees, see Lemmas 18.2 and 19.2.

Lemma 27.2. *Let $\mathcal{A}$ be an abelian category. Let $X^\bullet, Y^\bullet \in \mathrm{Ob}(K(\mathcal{A}))$.*

(1) *Let $Y^\bullet \to I^\bullet$ be an injective resolution (Definition 18.1). Then*
$$\mathrm{Ext}^i_{\mathcal{A}}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, I^\bullet[i]).$$
(2) *Let $P^\bullet \to X^\bullet$ be a projective resolution (Definition 19.1). Then*
$$\mathrm{Ext}^i_{\mathcal{A}}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet[-i], Y^\bullet).$$

Proof. Follows immediately from Lemma 18.8 and Lemma 19.8. $\square$

In the rest of this section we discuss extensions of objects of the abelian category itself. First we observe the following.

Lemma 27.3. *Let $\mathcal{A}$ be an abelian category.*

(1) *Let X, Y be objects of $D(\mathcal{A})$. Given $a, b \in \mathbf{Z}$ such that $H^i(X) = 0$ for $i > a$ and $H^j(Y) = 0$ for $j < b$, we have $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) = 0$ for $n < b - a$ and*
$$\mathrm{Ext}^{b-a}_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(H^a(X), H^b(Y))$$
(2) *Let $A, B \in \mathrm{Ob}(\mathcal{A})$. For $i < 0$ we have $\mathrm{Ext}^i_{\mathcal{A}}(B, A) = 0$. We have $\mathrm{Ext}^0_{\mathcal{A}}(B, A) = \mathrm{Hom}_{\mathcal{A}}(B, A)$.*

Proof. Choose complexes $X^\bullet$ and $Y^\bullet$ representing X and Y. Since $Y^\bullet \to \tau_{\geq b} Y^\bullet$ is a quasi-isomorphism, we may assume that $Y^j = 0$ for $j < b$. Let $L^\bullet \to X^\bullet$ be any quasi-isomorphism. Then $\tau_{\leq a} L^\bullet \to X^\bullet$ is a quasi-isomorphism. Hence a morphism $X \to Y[n]$ in $D(\mathcal{A})$ can be represented as fs^{-1} where $s : L^\bullet \to X^\bullet$ is a quasi-isomorphism, $f : L^\bullet \to Y^\bullet[n]$ a morphism, and $L^i = 0$ for $i < a$. Note that f maps L^i to Y^{i+n}. Thus $f = 0$ if $n < b - a$ because always either L^i or Y^{i+n} is zero. If $n = b - a$, then f corresponds exactly to a morphism $H^a(X) \to H^b(Y)$. Part (2) is a special case of (1). $\square$

Let $\mathcal{A}$ be an abelian category. Suppose that $0 \to A \to A' \to A'' \to 0$ is a short exact sequence of objects of $\mathcal{A}$. Then $0 \to A[0] \to A'[0] \to A''[0] \to 0$ leads to a distinguished triangle in $D(\mathcal{A})$ (see Lemma 12.1) hence a long exact sequence of Ext groups

$$0 \to \mathrm{Ext}^0_{\mathcal{A}}(B, A) \to \mathrm{Ext}^0_{\mathcal{A}}(B, A') \to \mathrm{Ext}^0_{\mathcal{A}}(B, A'') \to \mathrm{Ext}^1_{\mathcal{A}}(B, A) \to \ldots$$

Similarly, given a short exact sequence $0 \to B \to B' \to B'' \to 0$ we obtain a long exact sequence of Ext groups

$$0 \to \mathrm{Ext}^0_{\mathcal{A}}(B'', A) \to \mathrm{Ext}^0_{\mathcal{A}}(B', A) \to \mathrm{Ext}^0_{\mathcal{A}}(B, A) \to \mathrm{Ext}^1_{\mathcal{A}}(B'', A) \to \ldots$$

We may view these Ext groups as an application of the construction of the derived category. It shows one can define Ext groups and construct the long exact sequence of Ext groups without needing the existence of enough injectives or projectives. There is an alternative construction of the Ext groups due to Yoneda which avoids the use of the derived category, see [Yon60].

Definition 27.4. Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathrm{Ob}(\mathcal{A})$. A degree i *Yoneda extension* of B by A is an exact sequence

$$E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$$

in $\mathcal{A}$. We say two Yoneda extensions E and E' of the same degree are *equivalent* if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & Z_{i-1} \arrow[r] & \ldots \arrow[r] & Z_0 \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] & Z''_{i-1} \arrow[r] \arrow[u] \arrow[d] & \ldots \arrow[r] & Z''_0 \arrow[r] \arrow[u] \arrow[d] & B \arrow[r] \arrow[u, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & A \arrow[r] & Z'_{i-1} \arrow[r] & \ldots \arrow[r] & Z'_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

where the middle row is a Yoneda extension as well.

It is not immediately clear that the equivalence of the definition is an equivalence relation. Although it is instructive to prove this directly this will also follow from Lemma 27.5 below.

Let $\mathcal{A}$ be an abelian category with objects A, B. Given a Yoneda extension $E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$ we define an associated element $\delta(E) \in \mathrm{Ext}^i(B, A)$ as the morphism $\delta(E) = fs^{-1} : B[0] \to A[i]$ where s is the quasi-isomorphism

$$(\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots) \longrightarrow B[0]$$

and f is the morphism of complexes

$$(\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots) \longrightarrow A[i]$$

We call $\delta(E) = fs^{-1}$ the *class* of the Yoneda extension. It turns out that this class characterizes the equivalence class of the Yoneda extension.

Lemma 27.5. *Let $\mathcal{A}$ be an abelian category with objects A, B. Any element in $\mathrm{Ext}^i_{\mathcal{A}}(B, A)$ is $\delta(E)$ for some degree i Yoneda extension of B by A. Given two Yoneda extensions E, E' of the same degree then E is equivalent to E' if and only if $\delta(E) = \delta(E')$.*

Proof. Let $\xi : B[0] \to A[i]$ be an element of $\mathrm{Ext}^i_{\mathcal{A}}(B, A)$. We may write $\xi = fs^{-1}$ for some quasi-isomorphism $s : L^\bullet \to B[0]$ and map $f : L^\bullet \to A[i]$. After replacing $L^\bullet$ by $\tau_{\leq 0} L^\bullet$ we may assume that $L^j = 0$ for $j > 0$. Picture

$$\begin{tikzcd}
L^{-i-1} \arrow[r] & L^{-i} \arrow[r] \arrow[d] & \ldots \arrow[r] & L^0 \arrow[r] & B \arrow[r] & 0 \\
& A & & & &
\end{tikzcd}$$

Then setting $Z_{i-1} = (L^{-i+1} \oplus A)/L^{-i}$ and $Z_j = L^{-j}$ for $j = i-2, \ldots, 0$ we see that we obtain a degree i extension E of B by A whose class $\delta(E)$ equals ξ.

It is immediate from the definitions that equivalent Yoneda extensions have the same class. Suppose that $E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$ and $E' : 0 \to A \to Z'_{i-1} \to Z'_{i-2} \to \ldots \to Z'_0 \to B \to 0$ are Yoneda extensions with the same class. By construction of $D(\mathcal{A})$ as the localization of $K(\mathcal{A})$ at the set of quasi-isomorphisms, this means there exists a complex $L^\bullet$ and quasi-isomorphisms

$$t : L^\bullet \to (\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots)$$

and

$$t' : L^\bullet \to (\ldots \to 0 \to A \to Z'_{i-1} \to \ldots \to Z'_0 \to 0 \to \ldots)$$

such that $s \circ t = s' \circ t'$ and $f \circ t = f' \circ t'$, see Categories, Section 27. Let E'' be the degree i extension of B by A constructed from the pair $L^\bullet \to B[0]$ and $L^\bullet \to A[i]$ in the first paragraph of the proof. Then the reader sees readily that there exists "morphisms" of degree i Yoneda extensions $E'' \to E$ and $E'' \to E'$ as in the definition of equivalent Yoneda extensions (details omitted). This finishes the proof. □

Lemma 27.6. *Let $\mathcal{A}$ be an abelian category. Let A, B be objects of $\mathcal{A}$. Then $\mathrm{Ext}^1_{\mathcal{A}}(B, A)$ is the group $\mathrm{Ext}_{\mathcal{A}}(B, A)$ constructed in Homology, Definition 6.2.*

Proof. This is the case $i = 1$ of Lemma 27.5. □

Lemma 27.7. *Let $\mathcal{A}$ be an abelian category. Let $0 \to A \to Z \to B \to 0$ and $0 \to B \to Z' \to C \to 0$ be short exact sequences in $\mathcal{A}$. Denote $[Z] \in \mathrm{Ext}^1(B, A)$ and $[Z'] \in \mathrm{Ext}^1(C, B)$ their classes. Then $[Z] \circ [Z'] \in \mathrm{Ext}^2_{\mathcal{A}}(C, A)$ is 0 if and only if there exists a commutative diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "1"] & Z \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & W \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & 0 \\
 & & C \arrow[r, "1"] \arrow[d] & C \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns in $\mathcal{A}$.

Proof. Omitted. Hints: You can argue this using the result of Lemma 27.5 and working out what it means for a 2-extension class to be zero. Or you can use that if $[Z] \circ [Z'] \in \mathrm{Ext}^2_{\mathcal{A}}(C, A)$ is zero, then by the long exact cohomology sequence of Ext the element $[Z] \in \mathrm{Ext}^1(B, A)$ is the image of some element in $\mathrm{Ext}^1(W', A)$. □

Lemma 27.8. *Let $\mathcal{A}$ be an abelian category and let $p \geq 0$. If $\mathrm{Ext}^p_{\mathcal{A}}(B, A) = 0$ for any pair of objects A, B of $\mathcal{A}$, then $\mathrm{Ext}^i_{\mathcal{A}}(B, A) = 0$ for $i \geq p$ and any pair of objects A, B of $\mathcal{A}$.*

Proof. For $i > p$ write any class ξ as $\delta(E)$ where E is a Yoneda extension

$$E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$$

This is possible by Lemma 27.5. Set $C = \mathrm{Ker}(Z_{p-1} \to Z_p) = \mathrm{Im}(Z_p \to Z_{p-1})$. Then $\delta(E)$ is the composition of $\delta(E')$ and $\delta(E'')$ where

$$E' : 0 \to C \to Z_{p-1} \to \ldots \to Z_0 \to B \to 0$$

and

$$E'' : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_p \to C \to 0$$

Since $\delta(E') \in \mathrm{Ext}^p_{\mathcal{A}}(B, C) = 0$ we conclude. □

Lemma 27.9. *Let $\mathcal{A}$ be an abelian category. Let K be an object of $D^b(\mathcal{A})$ such that $\mathrm{Ext}^p_{\mathcal{A}}(H^i(K), H^j(K)) = 0$ for all $p \geq 2$ and $i > j$. Then K is isomorphic to the direct sum of its cohomologies: $K \cong \bigoplus H^i(K)[-i]$.*

Proof. Choose a, b such that $H^i(K) = 0$ for $i \not\in [a, b]$. We will prove the lemma by induction on $b - a$. If $b - a \leq 0$, then the result is clear. If $b - a > 0$, then we look at the distinguished triangle of truncations

$$\tau_{\leq b-1}K \to K \to H^b(K)[-b] \to (\tau_{\leq b-1}K)[1]$$

see Remark 12.4. By Lemma 4.11 if the last arrow is zero, then $K \cong \tau_{\leq b-1}K \oplus H^b(K)[-b]$ and we win by induction. Again using induction we see that

$$\mathrm{Hom}_{D(\mathcal{A})}(H^b(K)[-b], (\tau_{\leq b-1}K)[1]) = \bigoplus\nolimits_{i<b} \mathrm{Ext}^{b-i+1}_{\mathcal{A}}(H^b(K), H^i(K))$$

By assumption the direct sum is zero and the proof is complete. □

Lemma 27.10. *Let $\mathcal{A}$ be an abelian category. Assume $\mathrm{Ext}^2_{\mathcal{A}}(B, A) = 0$ for any pair of objects A, B of $\mathcal{A}$. Then any object K of $D^b(\mathcal{A})$ is isomorphic to the direct sum of its cohomologies: $K \cong \bigoplus H^i(K)[-i]$.*

Proof. The assumption implies that $\mathrm{Ext}^i_{\mathcal{A}}(B, A) = 0$ for $i \geq 2$ and any pair of objects A, B of $\mathcal{A}$ by Lemma 27.8. Hence this lemma is a special case of Lemma 27.9. □

28. K-groups

A tiny bit about K_0 of a triangulated category.

Definition 28.1. Let $\mathcal{D}$ be a triangulated category. We denote $K_0(\mathcal{D})$ the *zeroth K-group of $\mathcal{D}$*. It is the abelian group constructed as follows. Take the free abelian group on the objects on $\mathcal{D}$ and for every distinguished triangle $X \to Y \to Z$ impose the relation $[Y] - [X] - [Z] = 0$.

Observe that this implies that $[X[n]] = (-1)^n[X]$ because we have the distinguished triangle $(X, 0, X[1], 0, 0, -\mathrm{id}[1])$.

Lemma 28.2. *Let $\mathcal{A}$ be an abelian category. Then there is a canonical identification $K_0(D^b(\mathcal{A})) = K_0(\mathcal{A})$ of zeroth K-groups.*

Proof. Given an object A of $\mathcal{A}$ denote $A[0]$ the object A viewed as a complex sitting in degree 0. If $0 \to A \to A' \to A'' \to 0$ is a short exact sequence, then we get a distinguished triangle $A[0] \to A'[0] \to A''[0] \to A[1]$, see Section 12. This shows that we obtain a map $K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A}))$ by sending $[A]$ to $[A[0]]$ with apologies for the horrendous notation.

On the other hand, given an object X of $D^b(\mathcal{A})$ we can consider the element

$$c(X) = \sum (-1)^i [H^i(X)] \in K_0(\mathcal{A})$$

Given a distinguished triangle $X \to Y \to Z$ the long exact sequence of cohomology (11.1.1) and the relations in $K_0(\mathcal{A})$ show that $c(Y) = c(X) + c(Z)$. Thus c factors through a map $c : K_0(D^b(\mathcal{A})) \to K_0(\mathcal{A})$.

We want to show that the two maps above are mutually inverse. It is clear that the composition $K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A})) \to K_0(\mathcal{A})$ is the identity. Suppose that $X^\bullet$ is a bounded complex of $\mathcal{A}$. The existence of the distinguished triangles of "stupid truncations" (see Homology, Section 15)

$$\sigma_{\geq n} X^\bullet \to \sigma_{\geq n-1} X^\bullet \to X^{n-1}[-n+1] \to (\sigma_{\geq n} X^\bullet)[1]$$

and induction show that

$$[X^\bullet] = \sum (-1)^i [X^i[0]]$$

in $K_0(D^b(\mathcal{A}))$ (with again apologies for the notation). It follows that the composition $K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A}))$ is surjective which finishes the proof. □

Lemma 28.3. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories. Then F induces a group homomorphism $K_0(\mathcal{D}) \to K_0(\mathcal{D}')$.*

Proof. Omitted. □

Lemma 28.4. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor from a triangulated category to an abelian category. Assume that for any X in $\mathcal{D}$ only a finite number of the objects $H(X[i])$ are nonzero in $\mathcal{A}$. Then H induces a group homomorphism $K_0(\mathcal{D}) \to K_0(\mathcal{A})$ sending $[X]$ to $\sum (-1)^i [H(X[i])]$.*

Proof. Omitted. □

Lemma 28.5. *Let $\mathcal{B}$ be a weak Serre subcategory of the abelian category $\mathcal{A}$. There is a canonical isomorphism*

$$K_0(\mathcal{B}) \longrightarrow K_0(D^b_{\mathcal{B}}(\mathcal{A})), \quad [B] \longmapsto [B[0]]$$

The inverse sends the class $[X]$ of X to the element $\sum (-1)^i [H^i(X)]$.

Proof. We omit the verification that the rule for the inverse gives a well defined map $K_0(D^b_{\mathcal{B}}(\mathcal{A})) \to K_0(\mathcal{B})$. It is immediate that the composition $K_0(\mathcal{B}) \to K_0(D^b_{\mathcal{B}}(\mathcal{A})) \to K_0(\mathcal{B})$ is the identity. On the other hand, using the distinguished triangles of Remark 12.4 and an induction argument the reader may show that the displayed arrow in the statement of the lemma is surjective (details omitted). The lemma follows. □

Lemma 28.6. *Let $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$ be triangulated categories. Let*

$$\otimes : \mathcal{D} \times \mathcal{D}' \longrightarrow \mathcal{D}''$$

be a functor such that for fixed X in $\mathcal{D}$ the functor $X \otimes - : \mathcal{D}' \to \mathcal{D}''$ is an exact functor and for fixed X' in $\mathcal{D}'$ the functor $- \otimes X' : \mathcal{D} \to \mathcal{D}''$ is an exact functor. Then $\otimes$ induces a bilinear map $K_0(\mathcal{D}) \times K_0(\mathcal{D}') \to K_0(\mathcal{D}'')$ which sends $([X], [X'])$ to $[X \otimes X']$.

Proof. Omitted. □

29. Unbounded complexes

A reference for the material in this section is [Spa88]. The following lemma is useful to find "good" left resolutions of unbounded complexes.

Lemma 29.1. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset. Assume $\mathcal{P}$ contains 0, is closed under (finite) direct sums, and every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$. Let $K^\bullet$ be a complex. There exists a commutative diagram*

$$\begin{tikzcd} P_1^\bullet \arrow[r] \arrow[d] & P_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}K^\bullet \arrow[r] & \tau_{\leq 2}K^\bullet \arrow[r] & \ldots \end{tikzcd}$$

in the category of complexes such that

1. *the vertical arrows are quasi-isomorphisms and termwise surjective,*
2. *$P_n^\bullet$ is a bounded above complex with terms in $\mathcal{P}$,*
3. *the arrows $P_n^\bullet \to P_{n+1}^\bullet$ are termwise split injections and each cokernel P^i_{n+1}/P^i_n is an element of $\mathcal{P}$.*

Proof. We are going to use that the homotopy category $K(\mathcal{A})$ is a triangulated category, see Proposition 10.3. By Lemma 15.4 we can find a termwise surjective map of complexes $P_1^\bullet \to \tau_{\leq 1}K^\bullet$ which is a quasi-isomorphism such that the terms of $P_1^\bullet$ are in $\mathcal{P}$. By induction it suffices, given $P_1^\bullet, \ldots, P_n^\bullet$ to construct $P_{n+1}^\bullet$ and the maps $P_n^\bullet \to P_{n+1}^\bullet$ and $P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$.

Choose a distinguished triangle $P_n^\bullet \to \tau_{\leq n+1}K^\bullet \to C^\bullet \to P_n^\bullet[1]$ in $K(\mathcal{A})$. Applying Lemma 15.4 we choose a map of complexes $Q^\bullet \to C^\bullet$ which is a quasi-isomorphism such that the terms of $Q^\bullet$ are in $\mathcal{P}$. By the axioms of triangulated categories we may fit the composition $Q^\bullet \to C^\bullet \to P_n^\bullet[1]$ into a distinguished triangle $P_n^\bullet \to P_{n+1}^\bullet \to Q^\bullet \to P_n^\bullet[1]$ in $K(\mathcal{A})$. By Lemma 10.7 we may and do assume $0 \to P_n^\bullet \to P_{n+1}^\bullet \to Q^\bullet \to 0$ is a termwise split short exact sequence. This implies that the terms of $P_{n+1}^\bullet$ are in $\mathcal{P}$ and that $P_n^\bullet \to P_{n+1}^\bullet$ is a termwise split injection whose cokernels are in $\mathcal{P}$. By the axioms of triangulated categories we obtain a map of distinguished triangles

$$\begin{tikzcd} P_n^\bullet \arrow[r] \arrow[d] & P_{n+1}^\bullet \arrow[r] \arrow[d] & Q^\bullet \arrow[r] \arrow[d] & P_n^\bullet[1] \arrow[d] \\ P_n^\bullet \arrow[r] & \tau_{\leq n+1}K^\bullet \arrow[r] & C^\bullet \arrow[r] & P_n^\bullet[1] \end{tikzcd}$$

in the triangulated category $K(\mathcal{A})$. Choose an actual morphism of complexes $f : P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$. The left square of the diagram above commutes up to homotopy, but as $P_n^\bullet \to P_{n+1}^\bullet$ is a termwise split injection we can lift the homotopy and modify our choice of f to make it commute. Finally, f is a quasi-isomorphism, because both $P_n^\bullet \to P_n^\bullet$ and $Q^\bullet \to C^\bullet$ are.

At this point we have all the properties we want, except we don't know that the map $f : P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$ is termwise surjective. Since we have the commutative diagram

$$\begin{tikzcd} P_n^\bullet \arrow[r] \arrow[d] & P_{n+1}^\bullet \arrow[d] \\ \tau_{\leq n}K^\bullet \arrow[r] & \tau_{\leq n+1}K^\bullet \end{tikzcd}$$

of complexes, by induction hypothesis we see that f is surjective on terms in all degrees except possibly n and $n+1$. Choose an object $P \in \mathcal{P}$ and a surjection $q : P \to K^n$. Consider the map

$$g : P^\bullet = (\ldots \to 0 \to P \xrightarrow{1} P \to 0 \to \ldots) \longrightarrow \tau_{\leq n+1}K^\bullet$$

with first copy of P in degree n and maps given by q in degree n and $d_K \circ q$ in degree $n+1$. This is a surjection in degree n and the cokernel in degree $n+1$ is $H^{n+1}(\tau_{\leq n+1}K^\bullet)$; to see this recall that $\tau_{\leq n+1}K^\bullet$ has $\mathrm{Ker}(d_K^{n+1})$ in degree $n+1$. However, since f is a quasi-isomorphism we know that $H^{n+1}(f)$ is surjective. Hence after replacing $f : P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$ by $f \oplus g : P_{n+1}^\bullet \oplus P^\bullet \to \tau_{\leq n+1}K^\bullet$ we win. $\square$

In some cases we can use the lemma above to show that a left derived functor is everywhere defined.

Proposition 29.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset. Assume*

1. *$\mathcal{P}$ contains 0, is closed under (finite) direct sums, and every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$,*
2. *for any bounded above acyclic complex $P^\bullet$ of $\mathcal{A}$ with $P^n \in \mathcal{P}$ for all n the complex $F(P^\bullet)$ is exact,*
3. *$\mathcal{A}$ and $\mathcal{B}$ have colimits of systems over $\mathbf{N}$,*
4. *colimits over $\mathbf{N}$ are exact in both $\mathcal{A}$ and $\mathcal{B}$, and*
5. *F commutes with colimits over $\mathbf{N}$.*

Then LF is defined on all of $D(\mathcal{A})$.

Proof. By (1) and Lemma 15.4 for any bounded above complex $K^\bullet$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^\bullet$ bounded above and $P^n \in \mathcal{P}$ for all n. Suppose that $s : P^\bullet \to (P')^\bullet$ is a quasi-isomorphism of bounded above complexes consisting of objects of $\mathcal{P}$. Then $F(P^\bullet) \to F((P')^\bullet)$ is a quasi-isomorphism because $F(C(s)^\bullet)$ is acyclic by assumption (2). This already shows that LF is defined on $D^-(\mathcal{A})$ and that a bounded above complex consisting of objects of $\mathcal{P}$ computes LF, see Lemma 14.15.

Next, let $K^\bullet$ be an arbitrary complex of $\mathcal{A}$. Choose a diagram

$$\begin{tikzcd} P_1^\bullet \arrow[r] \arrow[d] & P_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}K^\bullet \arrow[r] & \tau_{\leq 2}K^\bullet \arrow[r] & \ldots \end{tikzcd}$$

as in Lemma 29.1. Note that the map $\mathrm{colim}\, P_n^\bullet \to K^\bullet$ is a quasi-isomorphism because colimits over $\mathbf{N}$ in $\mathcal{A}$ are exact and $H^i(P_n^\bullet) = H^i(K^\bullet)$ for $n > i$. We claim that

$$F(\mathrm{colim}\, P_n^\bullet) = \mathrm{colim}\, F(P_n^\bullet)$$

(termwise colimits) is $LF(K^\bullet)$, i.e., that $\operatorname{colim} P_n^\bullet$ computes LF. To see this, by Lemma 14.15, it suffices to prove the following claim. Suppose that

$$\operatorname{colim} Q_n^\bullet = Q^\bullet \xrightarrow{\alpha} P^\bullet = \operatorname{colim} P_n^\bullet$$

is a quasi-isomorphism of complexes, such that each $P_n^\bullet$, $Q_n^\bullet$ is a bounded above complex whose terms are in $\mathcal{P}$ and the maps $P_n^\bullet \to \tau_{\leq n} P^\bullet$ and $Q_n^\bullet \to \tau_{\leq n} Q^\bullet$ are quasi-isomorphisms. Claim: $F(\alpha)$ is a quasi-isomorphism.

The problem is that we do not assume that α is given as a colimit of maps between the complexes $P_n^\bullet$ and $Q_n^\bullet$. However, for each n we know that the solid arrows in the diagram

$$\begin{tikzcd} & R^\bullet \arrow[d, dotted] & \\ P_n^\bullet \arrow[d] & L^\bullet \arrow[l, dotted] \arrow[r, dotted] & Q_n^\bullet \arrow[d] \\ \tau_{\leq n} P^\bullet \arrow[rr, "\tau_{\leq n}\alpha"] & & \tau_{\leq n} Q^\bullet \end{tikzcd}$$

are quasi-isomorphisms. Because quasi-isomorphisms form a multiplicative system in $K(\mathcal{A})$ (see Lemma 11.2) we can find a quasi-isomorphism $L^\bullet \to P_n^\bullet$ and map of complexes $L^\bullet \to Q_n^\bullet$ such that the diagram above commutes up to homotopy. Then $\tau_{\leq n} L^\bullet \to L^\bullet$ is a quasi-isomorphism. Hence (by the first part of the proof) we can find a bounded above complex $R^\bullet$ whose terms are in $\mathcal{P}$ and a quasi-isomorphism $R^\bullet \to L^\bullet$ (as indicated in the diagram). Using the result of the first paragraph of the proof we see that $F(R^\bullet) \to F(P_n^\bullet)$ and $F(R^\bullet) \to F(Q_n^\bullet)$ are quasi-isomorphisms. Thus we obtain a isomorphisms $H^i(F(P_n^\bullet)) \to H^i(F(Q_n^\bullet))$ fitting into the commutative diagram

$$\begin{tikzcd} H^i(F(P_n^\bullet)) \arrow[r] \arrow[d] & H^i(F(Q_n^\bullet)) \arrow[d] \\ H^i(F(P^\bullet)) \arrow[r] & H^i(F(Q^\bullet)) \end{tikzcd}$$

The exact same argument shows that these maps are also compatible as n varies. Since by (4) and (5) we have

$$H^i(F(P^\bullet)) = H^i(F(\operatorname{colim} P_n^\bullet)) = H^i(\operatorname{colim} F(P_n^\bullet)) = \operatorname{colim} H^i(F(P_n^\bullet))$$

and similarly for $Q^\bullet$ we conclude that $H^i(\alpha) : H^i(F(P^\bullet) \to H^i(F(Q^\bullet)$ is an isomorphism and the claim follows. $\square$

Lemma 29.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subset \operatorname{Ob}(\mathcal{A})$ be a subset. Assume $\mathcal{I}$ contains 0, is closed under (finite) products, and every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$. Let $K^\bullet$ be a complex. There exists a commutative diagram*

$$\begin{tikzcd} \ldots \arrow[r] & \tau_{\geq -2} K^\bullet \arrow[r] \arrow[d] & \tau_{\geq -1} K^\bullet \arrow[d] \\ \ldots \arrow[r] & I_2^\bullet \arrow[r] & I_1^\bullet \end{tikzcd}$$

in the category of complexes such that

(1) *the vertical arrows are quasi-isomorphisms and termwise injective,*
(2) $I_n^\bullet$ *is a bounded below complex with terms in* $\mathcal{I}$*,*
(3) *the arrows* $I_{n+1}^\bullet \to I_n^\bullet$ *are termwise split surjections and* $\mathrm{Ker}(I_{n+1}^i \to I_n^i)$ *is an element of* $\mathcal{I}$*.*

Proof. This lemma is dual to Lemma 29.1. $\square$

30. Deriving adjoints

Let $F : \mathcal{D} \to \mathcal{D}'$ and $G : \mathcal{D}' \to \mathcal{D}$ be exact functors of triangulated categories. Let S, resp. S' be a multiplicative system for $\mathcal{D}$, resp. $\mathcal{D}'$ compatible with the triangulated structure. Denote $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ and $Q' : \mathcal{D}' \to (S')^{-1}\mathcal{D}'$ the localization functors. In this situation, by abuse of notation, one often denotes RF the partially defined right derived functor corresponding to $Q' \circ F : \mathcal{D} \to (S')^{-1}\mathcal{D}'$ and the multiplicative system S. Similarly one denotes LG the partially defined left derived functor corresponding to $Q \circ G : \mathcal{D}' \to S^{-1}\mathcal{D}$ and the multiplicative system S'. Picture

$$\begin{tikzcd} \mathcal{D} \arrow[r, "F"'] \arrow[d, "Q"'] & \mathcal{D}' \arrow[d, "Q'"] \\ S^{-1}\mathcal{D} \arrow[r, dotted, "RF"] & (S')^{-1}\mathcal{D}' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{D}' \arrow[r, "G"'] \arrow[d, "Q'"'] & \mathcal{D} \arrow[d, "Q"] \\ (S')^{-1}\mathcal{D}' \arrow[r, dotted, "LG"] & S^{-1}\mathcal{D} \end{tikzcd}$$

Lemma 30.1. *In the situation above assume F is right adjoint to G. Let $K \in \mathrm{Ob}(\mathcal{D})$ and $M \in \mathrm{Ob}(\mathcal{D}')$. If RF is defined at K and LG is defined at M, then there is a canonical isomorphism*

$$\mathrm{Hom}_{(S')^{-1}\mathcal{D}'}(M, RF(K)) = \mathrm{Hom}_{S^{-1}\mathcal{D}}(LG(M), K)$$

This isomorphism is functorial in both variables on the triangulated subcategories of $S^{-1}\mathcal{D}$ and $(S')^{-1}\mathcal{D}'$ where RF and LG are defined.

Proof. Since RF is defined at K, we see that the rule which assigns to an $s : K \to I$ in S the object $F(I)$ is essentially constant as an ind-object of $(S')^{-1}\mathcal{D}'$ with value $RF(K)$. Similarly, the rule which assigns to a $t : P \to M$ in S' the object $G(P)$ is essentially constant as a pro-object of $S^{-1}\mathcal{D}$ with value $LG(M)$. Thus we have

$$\begin{aligned} \mathrm{Hom}_{(S')^{-1}\mathcal{D}'}(M, RF(K)) &= \mathrm{colim}_{s:K\to I} \mathrm{Hom}_{(S')^{-1}\mathcal{D}'}(M, F(I)) \\ &= \mathrm{colim}_{s:K\to I} \mathrm{colim}_{t:P\to M} \mathrm{Hom}_{\mathcal{D}'}(P, F(I)) \\ &= \mathrm{colim}_{t:P\to M} \mathrm{colim}_{s:K\to I} \mathrm{Hom}_{\mathcal{D}'}(P, F(I)) \\ &= \mathrm{colim}_{t:P\to M} \mathrm{colim}_{s:K\to I} \mathrm{Hom}_{\mathcal{D}}(G(P), I) \\ &= \mathrm{colim}_{t:P\to M} \mathrm{Hom}_{S^{-1}\mathcal{D}}(G(P), K) \\ &= \mathrm{Hom}_{S^{-1}\mathcal{D}}(LG(M), K) \end{aligned}$$

The first equality holds by Categories, Lemma 22.9. The second equality holds by the definition of morphisms in $D(\mathcal{B})$, see Categories, Remark 27.15. The third equality holds by Categories, Lemma 14.10. The fourth equality holds because F and G are adjoint. The fifth equality holds by definition of morphism in $D(\mathcal{A})$, see Categories, Remark 27.7. The sixth equality holds by Categories, Lemma 22.10. We omit the proof of functoriality. $\square$

Lemma 30.2. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors of abelian categories such that F is a right adjoint to G. Let $K^\bullet$ be a complex of $\mathcal{A}$ and let $M^\bullet$ be a complex of $\mathcal{B}$. If RF is defined at $K^\bullet$ and LG is defined at $M^\bullet$, then there is a canonical isomorphism*

$$\mathrm{Hom}_{D(\mathcal{B})}(M^\bullet, RF(K^\bullet)) = \mathrm{Hom}_{D(\mathcal{A})}(LG(M^\bullet), K^\bullet)$$

This isomorphism is functorial in both variables on the triangulated subcategories of $D(\mathcal{A})$ and $D(\mathcal{B})$ where RF and LG are defined.

Proof. This is a special case of the very general Lemma 30.1. □

The following lemma is an example of why it is easier to work with unbounded derived categories. Namely, without having the unbounded derived functors, the lemma could not even be stated.

Lemma 30.3. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors of abelian categories such that F is a right adjoint to G. If the derived functors $RF : D(\mathcal{A}) \to D(\mathcal{B})$ and $LG : D(\mathcal{B}) \to D(\mathcal{A})$ exist, then RF is a right adjoint to LG.*

Proof. Immediate from Lemma 30.2. □

31. K-injective complexes

The following types of complexes can be used to compute right derived functors on the unbounded derived category.

Definition 31.1. Let $\mathcal{A}$ be an abelian category. A complex $I^\bullet$ is *K-injective* if for every acyclic complex $M^\bullet$ we have $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$.

In the situation of the definition we have in fact $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet[i], I^\bullet) = 0$ for all i as the translate of an acyclic complex is acyclic.

Lemma 31.2. *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet$ be a complex. The following are equivalent*

1. *$I^\bullet$ is K-injective,*
2. *for every quasi-isomorphism $M^\bullet \to N^\bullet$ the map*
$$\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$$
is bijective, and
3. *for every complex $N^\bullet$ the map*
$$\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(N^\bullet, I^\bullet)$$
is an isomorphism.

Proof. Assume (1). Then (2) holds because the functor $\mathrm{Hom}_{K(\mathcal{A})}(-, I^\bullet)$ is cohomological and the cone on a quasi-isomorphism is acyclic.

Assume (2). A morphism $N^\bullet \to I^\bullet$ in $D(\mathcal{A})$ is of the form $fs^{-1} : N^\bullet \to I^\bullet$ where $s : M^\bullet \to N^\bullet$ is a quasi-isomorphism and $f : M^\bullet \to I^\bullet$ is a map. By (2) this corresponds to a unique morphism $N^\bullet \to I^\bullet$ in $K(\mathcal{A})$, i.e., (3) holds.

Assume (3). If $M^\bullet$ is acyclic then $M^\bullet$ is isomorphic to the zero complex in $D(\mathcal{A})$ hence $\mathrm{Hom}_{D(\mathcal{A})}(M^\bullet, I^\bullet) = 0$, whence $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$ by (3), i.e., (1) holds. □

Lemma 31.3. *Let $\mathcal{A}$ be an abelian category. Let (K, L, M, f, g, h) be a distinguished triangle of $K(\mathcal{A})$. If two out of K, L, M are K-injective complexes, then the third is too.*

Proof. Follows from the definition, Lemma 4.2, and the fact that $K(\mathcal{A})$ is a triangulated category (Proposition 10.3). $\square$

Lemma 31.4. *Let $\mathcal{A}$ be an abelian category. A bounded below complex of injectives is K-injective.*

Proof. Follows from Lemmas 31.2 and 18.8. $\square$

Lemma 31.5. *Let $\mathcal{A}$ be an abelian category. Let T be a set and for each $t \in T$ let $I_t^\bullet$ be a K-injective complex. If $I^n = \prod_t I_t^n$ exists for all n, then $I^\bullet$ is a K-injective complex. Moreover, $I^\bullet$ represents the product of the objects $I_t^\bullet$ in $D(\mathcal{A})$.*

Proof. Let $K^\bullet$ be an complex. Observe that the complex

$$C : \prod\nolimits_b \mathrm{Hom}(K^{-b}, I^{b-1}) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I^b) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I^{b+1})$$

has cohomology $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$ in the middle. Similarly, the complex

$$C_t : \prod\nolimits_b \mathrm{Hom}(K^{-b}, I_t^{b-1}) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I_t^b) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I_t^{b+1})$$

computes $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I_t^\bullet)$. Next, observe that we have

$$C = \prod\nolimits_{t \in T} C_t$$

as complexes of abelian groups by our choice of I. Taking products is an exact functor on the category of abelian groups. Hence if $K^\bullet$ is acyclic, then $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I_t^\bullet) = 0$, hence C_t is acyclic, hence C is acyclic, hence we get $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet) = 0$. Thus we find that $I^\bullet$ is K-injective. Having said this, we can use Lemma 31.2 to conclude that

$$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, I^\bullet) = \prod\nolimits_{t \in T} \mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, I_t^\bullet)$$

and indeed $I^\bullet$ represents the product in the derived category. $\square$

Lemma 31.6. *Let $\mathcal{A}$ be an abelian category. Let $F : K(\mathcal{A}) \to \mathcal{D}'$ be an exact functor of triangulated categories. Then RF is defined at every complex in $K(\mathcal{A})$ which is quasi-isomorphic to a K-injective complex. In fact, every K-injective complex computes RF.*

Proof. By Lemma 14.4 it suffices to show that RF is defined at a K-injective complex, i.e., it suffices to show a K-injective complex $I^\bullet$ computes RF. Any quasi-isomorphism $I^\bullet \to N^\bullet$ is a homotopy equivalence as it has an inverse by Lemma 31.2. Thus $I^\bullet \to I^\bullet$ is a final object of $I^\bullet/\mathrm{Qis}(\mathcal{A})$ and we win. $\square$

Lemma 31.7. *Let $\mathcal{A}$ be an abelian category. Assume every complex has a quasi-isomorphism towards a K-injective complex. Then any exact functor $F : K(\mathcal{A}) \to \mathcal{D}'$ of triangulated categories has a right derived functor*

$$RF : D(\mathcal{A}) \longrightarrow \mathcal{D}'$$

and $RF(I^\bullet) = F(I^\bullet)$ for K-injective complexes $I^\bullet$.

Proof. To see this we apply Lemma 14.15 with $\mathcal{I}$ the collection of K-injective complexes. Since (1) holds by assumption, it suffices to prove that if $I^\bullet \to J^\bullet$ is a quasi-isomorphism of K-injective complexes, then $F(I^\bullet) \to F(J^\bullet)$ is an isomorphism. This is clear because $I^\bullet \to J^\bullet$ is a homotopy equivalence, i.e., an isomorphism in $K(\mathcal{A})$, by Lemma 31.2. □

The following lemma can be generalized to limits over bigger ordinals.

Lemma 31.8. *Let $\mathcal{A}$ be an abelian category. Let*

$$\ldots \to I_3^\bullet \to I_2^\bullet \to I_1^\bullet$$

be an inverse system of complexes. Assume

1. *each $I_n^\bullet$ is K-injective,*
2. *each map $I_{n+1}^m \to I_n^m$ is a split surjection,*
3. *the limits $I^m = \lim I_n^m$ exist.*

Then the complex $I^\bullet$ is K-injective.

Proof. We urge the reader to skip the proof of this lemma. Let $M^\bullet$ be an acyclic complex. Let us abbreviate $H_n(a,b) = \mathrm{Hom}_\mathcal{A}(M^a, I_n^b)$. With this notation $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$ is the cohomology of the complex

$$\prod_m \lim_n H_n(m, m{-}2) \to \prod_m \lim_n H_n(m, m{-}1) \to \prod_m \lim_n H_n(m, m) \to \prod_m \lim_n H_n(m, m{+}1)$$

in the third spot from the left. We may exchange the order of $\prod$ and lim and each of the complexes

$$\prod_m H_n(m, m-2) \to \prod_m H_n(m, m-1) \to \prod_m H_n(m, m) \to \prod_m H_n(m, m+1)$$

is exact by assumption (1). By assumption (2) the maps in the systems

$$\ldots \to \prod_m H_3(m, m-2) \to \prod_m H_2(m, m-2) \to \prod_m H_1(m, m-2)$$

are surjective. Thus the lemma follows from Homology, Lemma 31.4. □

It appears that a combination of Lemmas 29.3, 31.4, and 31.8 produces "enough K-injectives" for any abelian category with enough injectives and countable products. Actually, this may not work! See Lemma 34.4 for an explanation.

Lemma 31.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

1. *u is right adjoint to v, and*
2. *v is exact.*

Then u transforms K-injective complexes into K-injective complexes.

Proof. Let $I^\bullet$ be a K-injective complex of $\mathcal{A}$. Let $M^\bullet$ be a acyclic complex of $\mathcal{B}$. As v is exact we see that $v(M^\bullet)$ is an acyclic complex. By adjointness we get

$$0 = \mathrm{Hom}_{K(\mathcal{A})}(v(M^\bullet), I^\bullet) = \mathrm{Hom}_{K(\mathcal{B})}(M^\bullet, u(I^\bullet))$$

hence the lemma follows. □

32. Bounded cohomological dimension

There is another case where the unbounded derived functor exists. Namely, when the functor has bounded cohomological dimension.

Lemma 32.1. *Let $\mathcal{A}$ be an abelian category. Let $d : \text{Ob}(\mathcal{A}) \to \{0, 1, 2, \ldots, \infty\}$ be a function. Assume that*

1. *every object of $\mathcal{A}$ is a subobject of an object A with $d(A) = 0$,*
2. *$d(A \oplus B) \leq \max\{d(A), d(B)\}$ for $A, B \in \mathcal{A}$, and*
3. *if $0 \to A \to B \to C \to 0$ is short exact, then $d(C) \leq \max\{d(A) - 1, d(B)\}$.*

Let $K^\bullet$ be a complex such that $n + d(K^n)$ tends to $-\infty$ as $n \to -\infty$. Then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $d(L^n) = 0$ for all $n \in \mathbf{Z}$.

Proof. By Lemma 15.5 we can find a quasi-isomorphism $\sigma_{\geq 0}K^\bullet \to M^\bullet$ with $M^n = 0$ for $n < 0$ and $d(M^n) = 0$ for $n \geq 0$. Then $K^\bullet$ is quasi-isomorphic to the complex

$$\ldots \to K^{-2} \to K^{-1} \to M^0 \to M^1 \to \ldots$$

Hence we may assume that $d(K^n) = 0$ for $n \gg 0$. Note that the condition $n + d(K^n) \to -\infty$ as $n \to -\infty$ is not violated by this replacement.

We are going to improve $K^\bullet$ by an (infinite) sequence of elementary replacements. An *elementary replacement* is the following. Choose an index n such that $d(K^n) > 0$. Choose an injection $K^n \to M$ where $d(M) = 0$. Set $M' = \text{Coker}(K^n \to M \oplus K^{n+1})$. Consider the map of complexes

$$\begin{tikzcd} K^\bullet : \arrow[d] & K^{n-1} \arrow[r] \arrow[d] & K^n \arrow[r] \arrow[d] & K^{n+1} \arrow[r] \arrow[d] & K^{n+2} \arrow[d] \\ (K')^\bullet : & K^{n-1} \arrow[r] & M \arrow[r] & M' \arrow[r] & K^{n+2} \end{tikzcd}$$

It is clear that $K^\bullet \to (K')^\bullet$ is a quasi-isomorphism. Moreover, it is clear that $d((K')^n) = 0$ and

$$d((K')^{n+1}) \leq \max\{d(K^n) - 1, d(M \oplus K^{n+1})\} \leq \max\{d(K^n) - 1, d(K^{n+1})\}$$

and the other values are unchanged.

To finish the proof we carefuly choose the order in which to do the elementary replacements so that for every integer m the complex $\sigma_{\geq m}K^\bullet$ is changed only a finite number of times. To do this set

$$\xi(K^\bullet) = \max\{n + d(K^n) \mid d(K^n) > 0\}$$

and

$$I = \{n \in \mathbf{Z} \mid \xi(K^\bullet) = n + d(K^n) \text{ and } d(K^n) > 0\}$$

Our assumption that $n + d(K^n)$ tends to $-\infty$ as $n \to -\infty$ and the fact that $d(K^n) = 0$ for $n >> 0$ implies $\xi(K^\bullet) < +\infty$ and that I is a finite set. It is clear that $\xi((K')^\bullet) \leq \xi(K^\bullet)$ for an elementary transformation as above. An elementary transformation changes the complex in degrees $\leq \xi(K^\bullet) + 1$. Hence if we can find finite sequence of elementary transformations which decrease $\xi(K^\bullet)$, then we win. However, note that if we do an elementary transformation starting with the smallest element $n \in I$, then we either decrease the size of I, or we increase $\min I$. Since every element of I is $\leq \xi(K^\bullet)$ we see that we win after a finite number of steps. $\square$

Lemma 32.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Assume*

(1) *every object of $\mathcal{A}$ is a subobject of an object which is right acyclic for F,*
(2) *there exists an integer $n \geq 0$ such that $R^nF = 0$,*

Then

(1) $RF : D(\mathcal{A}) \to D(\mathcal{B})$ *exists,*
(2) *any complex consisting of right acyclic objects for F computes RF,*
(3) *any complex is the source of a quasi-isomorphism into a complex consisting of right acyclic objects for F,*
(4) *for $E \in D(\mathcal{A})$*
    (a) $H^i(RF(\tau_{\leq a}E) \to H^i(RF(E))$ *is an isomorphism for $i \leq a$,*
    (b) $H^i(RF(E)) \to H^i(RF(\tau_{\geq b-n+1}E))$ *is an isomorphism for $i \geq b$,*
    (c) *if $H^i(E) = 0$ for $i \notin [a,b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(RF(E)) = 0$ for $i \notin [a, b+n-1]$.*

Proof. Note that the first assumption implies that $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists, see Proposition 16.8. Let A be an object of $\mathcal{A}$. Choose an injection $A \to A'$ with A' acyclic. Then we see that $R^{n+1}F(A) = R^nF(A'/A) = 0$ by the long exact cohomology sequence. Hence we conclude that $R^{n+1}F = 0$. Continuing like this using induction we find that $R^mF = 0$ for all $m \geq n$.

We are going to use Lemma 32.1 with the function $d : \mathrm{Ob}(\mathcal{A}) \to \{0, 1, 2, \ldots\}$ given by $d(A) = \max\{0\} \cup \{i \mid R^iF(A) \neq 0\}$. The first assumption of Lemma 32.1 is our assumption (1). The second assumption of Lemma 32.1 follows from the fact that $RF(A \oplus B) = RF(A) \oplus RF(B)$. The third assumption of Lemma 32.1 follows from the long exact cohomology sequence. Hence for every complex $K^\bullet$ there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ into a complex of objects right acyclic for F. This proves statement (3).

We claim that if $L^\bullet \to M^\bullet$ is a quasi-isomorphism of complexes of right acyclic objects for F, then $F(L^\bullet) \to F(M^\bullet)$ is a quasi-isomorphism. If we prove this claim then we get statements (1) and (2) of the lemma by Lemma 14.15. To prove the claim pick an integer $i \in \mathbf{Z}$. Consider the distinguished triangle

$$\sigma_{\geq i-n-1}L^\bullet \to \sigma_{\geq i-n-1}M^\bullet \to Q^\bullet,$$

i.e., let $Q^\bullet$ be the cone of the first map. Note that $Q^\bullet$ is bounded below and that $H^j(Q^\bullet)$ is zero except possibly for $j = i-n-1$ or $j = i-n-2$. We may apply RF to $Q^\bullet$. Using the second spectral sequence of Lemma 21.3 and the assumed vanishing of cohomology (2) we conclude that $H^j(RF(Q^\bullet))$ is zero except possibly for $j \in \{i-n-2, \ldots, i-1\}$. Hence we see that $RF(\sigma_{\geq i-n-1}L^\bullet) \to RF(\sigma_{\geq i-n-1}M^\bullet)$ induces an isomorphism of cohomology objects in degrees $\geq i$. By Proposition 16.8 we know that $RF(\sigma_{\geq i-n-1}L^\bullet) = \sigma_{\geq i-n-1}F(L^\bullet)$ and $RF(\sigma_{\geq i-n-1}M^\bullet) = \sigma_{\geq i-n-1}F(M^\bullet)$. We conclude that $F(L^\bullet) \to F(M^\bullet)$ is an isomorphism in degree i as desired.

Part (4)(a) follows from Lemma 16.1.

For part (4)(b) let E be represented by the complex $L^\bullet$ of objects right acyclic for F. By part (2) $RF(E)$ is represented by the complex $F(L^\bullet)$ and $RF(\sigma_{\geq c}L^\bullet)$ is represented by $\sigma_{\geq c}F(L^\bullet)$. Consider the distinguished triangle

$$H^{b-n}(L^\bullet)[n-b] \to \tau_{\geq b-n}L^\bullet \to \tau_{\geq b-n+1}L^\bullet$$

of Remark 12.4. The vanishing established above gives that $H^i(RF(\tau_{\geq b-n}L^\bullet))$ agrees with $H^i(RF(\tau_{\geq b-n+1}L^\bullet))$ for $i \geq b$. Consider the short exact sequence of complexes

$$0 \to \mathrm{Im}(L^{b-n-1} \to L^{b-n})[n-b] \to \sigma_{\geq b-n}L^\bullet \to \tau_{\geq b-n}L^\bullet \to 0$$

Using the distinguished triangle associated to this (see Section 12) and the vanishing as before we conclude that $H^i(RF(\tau_{\geq b-n}L^\bullet))$ agrees with $H^i(RF(\sigma_{\geq b-n}L^\bullet))$ for $i \geq b$. Since the map $RF(\sigma_{\geq b-n}L^\bullet) \to RF(L^\bullet)$ is represented by $\sigma_{\geq b-n}F(L^\bullet) \to F(L^\bullet)$ we conclude that this in turn agrees with $H^i(RF(L^\bullet))$ for $i \geq b$ as desired.

Proof of (4)(c). Under the assumption on E we have $\tau_{\leq a-1}E = 0$ and we get the vanishing of $H^i(RF(E))$ for $i \leq a-1$ from part (4)(a). Similarly, we have $\tau_{\geq b+1}E = 0$ and hence we get the vanishing of $H^i(RF(E))$ for $i \geq b+n$ from part (4)(b). ☐

Lemma 32.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories. If*

1. *every object of $\mathcal{A}$ is a quotient of an object which is left acyclic for F,*
2. *there exists an integer $n \geq 0$ such that $L^nF = 0$,*

Then

1. *$LF : D(\mathcal{A}) \to D(\mathcal{B})$ exists,*
2. *any complex consisting of left acyclic objects for F computes LF,*
3. *any complex is the target of a quasi-isomorphism from a complex consisting of left acyclic objects for F,*
4. *for $E \in D(\mathcal{A})$*
   (a) *$H^i(LF(\tau_{\leq a+n-1}E) \to H^i(LF(E))$ is an isomorphism for $i \leq a$,*
   (b) *$H^i(LF(E)) \to H^i(LF(\tau_{\geq b}E))$ is an isomorphism for $i \geq b$,*
   (c) *if $H^i(E) = 0$ for $i \notin [a,b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(LF(E)) = 0$ for $i \notin [a-n+1, b]$.*

Proof. This is dual to Lemma 32.2. ☐

33. Derived colimits

In a triangulated category there is a notion of derived colimit.

Definition 33.1. Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be a system of objects of $\mathcal{D}$. We say an object K is a *derived colimit*, or a *homotopy colimit* of the system (K_n) if the direct sum $\bigoplus K_n$ exists and there is a distinguished triangle

$$\bigoplus K_n \to \bigoplus K_n \to K \to \bigoplus K_n[1]$$

where the map $\bigoplus K_n \to \bigoplus K_n$ is given by $1 - f_n$ in degree n. If this is the case, then we sometimes indicate this by the notation $K = \mathrm{hocolim} K_n$.

By TR3 a derived colimit, if it exists, is unique up to (non-unique) isomorphism. Moreover, by TR1 a derived colimit of K_n exists as soon as $\bigoplus K_n$ exists. The derived category $D(Ab)$ of the category of abelian groups is an example of a triangulated category where all homotopy colimits exist.

The nonuniqueness makes it hard to pin down the derived colimit. In More on Algebra, Lemma 86.5 the reader finds an exact sequence

$$0 \to R^1 \lim \mathrm{Hom}(K_n, L[-1]) \to \mathrm{Hom}(\mathrm{hocolim} K_n, L) \to \lim \mathrm{Hom}(K_n, L) \to 0$$

describing the Homs out of a homotopy colimit in terms of the usual Homs.

Remark 33.2. Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be a system of objects of $\mathcal{D}$. We may think of a derived colimit as an object K of $\mathcal{D}$ endowed with morphisms $i_n : K_n \to K$ such that $i_{n+1} \circ f_n = i_n$ and such that there exists a morphism $c : K \to \bigoplus K_n$ with the property that

$$\bigoplus K_n \xrightarrow{1-f_n} \bigoplus K_n \xrightarrow{i_n} K \xrightarrow{c} \bigoplus K_n[1]$$

is a distinguished triangle. If (K', i'_n, c') is a second derived colimit, then there exists an isomorphism $\varphi : K \to K'$ such that $\varphi \circ i_n = i'_n$ and $c' \circ \varphi = c$. The existence of φ is TR3 and the fact that φ is an isomorphism is Lemma 4.3.

Remark 33.3. Let $\mathcal{D}$ be a triangulated category. Let $(a_n) : (K_n, f_n) \to (L_n, g_n)$ be a morphism of systems of objects of $\mathcal{D}$. Let (K, i_n, c) be a derived colimit of the first system and let (L, j_n, d) be a derived colimit of the second system with notation as in Remark 33.2. Then there exists a morphism $a : K \to L$ such that $a \circ i_n = j_n$ and $d \circ a = (a_n[1]) \circ c$. This follows from TR3 applied to the defining distinguished triangles.

Lemma 33.4. *Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be a system of objects of $\mathcal{D}$. Let $n_1 < n_2 < n_3 < \ldots$ be a sequence of integers. Assume $\bigoplus K_n$ and $\bigoplus K_{n_i}$ exist. Then there exists an isomorphism $hocolim K_{n_i} \to hocolim K_n$ such that*

$$\begin{tikzcd} K_{n_i} \arrow[r] \arrow[d, "id"'] & hocolim K_{n_i} \arrow[d] \\ K_{n_i} \arrow[r] & hocolim K_n \end{tikzcd}$$

commutes for all i.

Proof. Let $g_i : K_{n_i} \to K_{n_{i+1}}$ be the composition $f_{n_{i+1}-1} \circ \ldots \circ f_{n_i}$. We construct commutative diagrams

$$\begin{tikzcd} \bigoplus_i K_{n_i} \arrow[r, "1-g_i"'] \arrow[d, "b"'] & \bigoplus_i K_{n_i} \arrow[d, "a"] \\ \bigoplus_n K_n \arrow[r, "1-f_n"] & \bigoplus_n K_n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \bigoplus_n K_n \arrow[r, "1-f_n"'] \arrow[d, "d"'] & \bigoplus_n K_n \arrow[d, "c"] \\ \bigoplus_i K_{n_i} \arrow[r, "1-g_i"] & \bigoplus_i K_{n_i} \end{tikzcd}$$

as follows. Let $a_i = a|_{K_{n_i}}$ be the inclusion of K_{n_i} into the direct sum. In other words, a is the natural inclusion. Let $b_i = b|_{K_{n_i}}$ be the map

$$K_{n_i} \xrightarrow{1,\ f_{n_i},\ f_{n_i+1} \circ f_{n_i},\ \ldots,\ f_{n_{i+1}-2} \circ \ldots \circ f_{n_i}} K_{n_i} \oplus K_{n_i+1} \oplus \ldots \oplus K_{n_{i+1}-1}$$

If $n_{i-1} < j \leq n_i$, then we let $c_j = c|_{K_j}$ be the map

$$K_j \xrightarrow{f_{n_i-1} \circ \ldots \circ f_j} K_{n_i}$$

We let $d_j = d|_{K_j}$ be zero if $j \neq n_i$ for any i and we let d_{n_i} be the natural inclusion of K_{n_i} into the direct sum. In other words, d is the natural projection. By TR3 these diagrams define morphisms

$$\varphi : \text{hocolim} K_{n_i} \to \text{hocolim} K_n \quad \text{and} \quad \psi : \text{hocolim} K_n \to \text{hocolim} K_{n_i}$$

Since $c \circ a$ and $d \circ b$ are the identity maps we see that $\varphi \circ \psi$ is an isomorphism by Lemma 4.3. The other way around we get the morphisms $a \circ c$ and $b \circ d$. Consider the morphism $h = (h_j) : \bigoplus K_n \to \bigoplus K_n$ given by the rule: for $n_{i-1} < j < n_i$ we set

$$h_j : K_j \xrightarrow{1,\ f_j,\ f_{j+1}\circ f_j,\ \ldots,\ f_{n_i-1}\circ\ldots\circ f_j} K_j \oplus \ldots \oplus K_{n_i}$$

Then the reader verifies that $(1-f)\circ h = \mathrm{id} - a\circ c$ and $h\circ(1-f) = \mathrm{id} - b\circ d$. This means that $\mathrm{id} - \psi\circ\varphi$ has square zero by Lemma 4.5 (small argument omitted). In other words, $\psi\circ\varphi$ differs from the identity by a nilpotent endomorphism, hence is an isomorphism. Thus φ and ψ are isomorphisms as desired. □

Lemma 33.5. *Let $\mathcal{A}$ be an abelian category. If $\mathcal{A}$ has exact countable direct sums, then $D(\mathcal{A})$ has countable direct sums. In fact given a collection of complexes $K_i^\bullet$ indexed by a countable index set I the termwise direct sum $\bigoplus K_i^\bullet$ is the direct sum of $K_i^\bullet$ in $D(\mathcal{A})$.*

Proof. Let $L^\bullet$ be a complex. Suppose given maps $\alpha_i : K_i^\bullet \to L^\bullet$ in $D(\mathcal{A})$. This means there exist quasi-isomorphisms $s_i : M_i^\bullet \to K_i^\bullet$ of complexes and maps of complexes $f_i : M_i^\bullet \to L^\bullet$ such that $\alpha_i = f_i s_i^{-1}$. By assumption the map of complexes

$$s : \bigoplus M_i^\bullet \longrightarrow \bigoplus K_i^\bullet$$

is a quasi-isomorphism. Hence setting $f = \bigoplus f_i$ we see that $\alpha = fs^{-1}$ is a map in $D(\mathcal{A})$ whose composition with the coprojection $K_i^\bullet \to \bigoplus K_i^\bullet$ is α_i. We omit the verification that α is unique. □

Lemma 33.6. *Let $\mathcal{A}$ be an abelian category. Assume colimits over $\mathbf{N}$ exist and are exact. Then countable direct sums exists and are exact. Moreover, if (A_n, f_n) is a system over $\mathbf{N}$, then there is a short exact sequence*

$$0 \to \bigoplus A_n \to \bigoplus A_n \to \mathrm{colim}\, A_n \to 0$$

where the first map in degree n is given by $1 - f_n$.

Proof. The first statement follows from $\bigoplus A_n = \mathrm{colim}(A_1 \oplus \ldots \oplus A_n)$. For the second, note that for each n we have the short exact sequence

$$0 \to A_1 \oplus \ldots \oplus A_{n-1} \to A_1 \oplus \ldots \oplus A_n \to A_n \to 0$$

where the first map is given by the maps $1 - f_i$ and the second map is the sum of the transition maps. Take the colimit to get the sequence of the lemma. □

Lemma 33.7. *Let $\mathcal{A}$ be an abelian category. Let $L_n^\bullet$ be a system of complexes of $\mathcal{A}$. Assume colimits over $\mathbf{N}$ exist and are exact in $\mathcal{A}$. Then the termwise colimit $L^\bullet = \mathrm{colim}\, L_n^\bullet$ is a homotopy colimit of the system in $D(\mathcal{A})$.*

Proof. We have an exact sequence of complexes

$$0 \to \bigoplus L_n^\bullet \to \bigoplus L_n^\bullet \to L^\bullet \to 0$$

by Lemma 33.6. The direct sums are direct sums in $D(\mathcal{A})$ by Lemma 33.5. Thus the result follows from the definition of derived colimits in Definition 33.1 and the fact that a short exact sequence of complexes gives a distinguished triangle (Lemma 12.1). □

Lemma 33.8. *Let $\mathcal{D}$ be a triangulated category having countable direct sums. Let $\mathcal{A}$ be an abelian category with exact colimits over $\mathbf{N}$. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor commuting with countable direct sums. Then $H(hocolim K_n) =$ $\mathrm{colim}\, H(K_n)$ for any system of objects of $\mathcal{D}$.*

Proof. Write $K = \text{hocolim} K_n$. Apply H to the defining distinguished triangle to get

$$\bigoplus H(K_n) \to \bigoplus H(K_n) \to H(K) \to \bigoplus H(K_n[1]) \to \bigoplus H(K_n[1])$$

where the first map is given by $1 - H(f_n)$ and the last map is given by $1 - H(f_n[1])$. Apply Lemma 33.6 to see that this proves the lemma. $\square$

The following lemma tells us that taking maps out of a compact object (to be defined later) commutes with derived colimits.

Lemma 33.9. *Let $\mathcal{D}$ be a triangulated category with countable direct sums. Let $K \in \mathcal{D}$ be an object such that for every countable set of objects $E_n \in \mathcal{D}$ the canonical map*

$$\bigoplus \text{Hom}_{\mathcal{D}}(K, E_n) \longrightarrow \text{Hom}_{\mathcal{D}}(K, \bigoplus E_n)$$

is a bijection. Then, given any system L_n of $\mathcal{D}$ over $\mathbf{N}$ whose derived colimit $L = hocolim L_n$ exists we have that

$$\text{colim}\, \text{Hom}_{\mathcal{D}}(K, L_n) \longrightarrow \text{Hom}_{\mathcal{D}}(K, L)$$

is a bijection.

Proof. Consider the defining distinguished triangle

$$\bigoplus L_n \to \bigoplus L_n \to L \to \bigoplus L_n[1]$$

Apply the cohomological functor $\text{Hom}_{\mathcal{D}}(K, -)$ (see Lemma 4.2). By elementary considerations concerning colimits of abelian groups we get the result. $\square$

34. Derived limits

In a triangulated category there is a notion of derived limit.

Definition 34.1. Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be an inverse system of objects of $\mathcal{D}$. We say an object K is a *derived limit*, or a *homotopy limit* of the system (K_n) if the product $\prod K_n$ exists and there is a distinguished triangle

$$K \to \prod K_n \to \prod K_n \to K[1]$$

where the map $\prod K_n \to \prod K_n$ is given by $(k_n) \mapsto (k_n - f_{n+1}(k_{n+1}))$. If this is the case, then we sometimes indicate this by the notation $K = R\lim K_n$.

By TR3 a derived limit, if it exists, is unique up to (non-unique) isomorphism. Moreover, by TR1 a derived limit $R\lim K_n$ exists as soon as $\prod K_n$ exists. The derived category $D(Ab)$ of the category of abelian groups is an example of a triangulated category where all derived limits exist.

The nonuniqueness makes it hard to pin down the derived limit. In More on Algebra, Lemma 86.4 the reader finds an exact sequence

$$0 \to R^1 \lim \text{Hom}(L, K_n[-1]) \to \text{Hom}(L, R\lim K_n) \to \lim \text{Hom}(L, K_n) \to 0$$

describing the Homs into a derived limit in terms of the usual Homs.

Lemma 34.2. *Let $\mathcal{A}$ be an abelian category with exact countable products. Then*

1. *$D(\mathcal{A})$ has countable products,*
2. *countable products $\prod K_i$ in $D(\mathcal{A})$ are obtained by taking termwise products of any complexes representing the K_i, and*

(3) $H^p(\prod K_i) = \prod H^p(K_i)$.

Proof. Let $K_i^\bullet$ be a complex representing K_i in $D(\mathcal{A})$. Let $L^\bullet$ be a complex. Suppose given maps $\alpha_i : L^\bullet \to K_i^\bullet$ in $D(\mathcal{A})$. This means there exist quasi-isomorphisms $s_i : K_i^\bullet \to M_i^\bullet$ of complexes and maps of complexes $f_i : L^\bullet \to M_i^\bullet$ such that $\alpha_i = s_i^{-1} f_i$. By assumption the map of complexes

$$s : \prod K_i^\bullet \longrightarrow \prod M_i^\bullet$$

is a quasi-isomorphism. Hence setting $f = \prod f_i$ we see that $\alpha = s^{-1} f$ is a map in $D(\mathcal{A})$ whose composition with the projection $\prod K_i^\bullet \to K_i^\bullet$ is α_i. We omit the verification that α is unique. □

The duals of Lemmas 33.6, 33.7, and 33.9 should be stated here and proved. However, we do not know any applications of these lemmas for now.

Lemma 34.3. *Let $\mathcal{A}$ be an abelian category with countable products and enough injectives. Let (K_n) be an inverse system of $D^+(\mathcal{A})$. Then $R\lim K_n$ exists.*

Proof. It suffices to show that $\prod K_n$ exists in $D(\mathcal{A})$. For every n we can represent K_n by a bounded below complex $I_n^\bullet$ of injectives (Lemma 18.3). Then $\prod K_n$ is represented by $\prod I_n^\bullet$, see Lemma 31.5. □

Lemma 34.4. *Let $\mathcal{A}$ be an abelian category with countable products and enough injectives. Let $K^\bullet$ be a complex. Let $I_n^\bullet$ be the inverse system of bounded below complexes of injectives produced by Lemma 29.3. Then $I^\bullet = \lim I_n^\bullet$ exists, is K-injective, and the following are equivalent*

(1) *the map $K^\bullet \to I^\bullet$ is a quasi-isomorphism,*
(2) *the canonical map $K^\bullet \to R\lim \tau_{\geq -n} K^\bullet$ is an isomorphism in $D(\mathcal{A})$.*

Proof. The statement of the lemma makes sense as $R\lim \tau_{\geq -n} K^\bullet$ exists by Lemma 34.3. Each complex $I_n^\bullet$ is K-injective by Lemma 31.4. Choose direct sum decompositions $I_{n+1}^p = C_{n+1}^p \oplus I_n^p$ for all $n \geq 1$. Set $C_1^p = I_1^p$. The complex $I^\bullet = \lim I_n^\bullet$ exists because we can take $I^p = \prod_{n \geq 1} C_n^p$. Fix $p \in \mathbf{Z}$. We claim there is a split short exact sequence

$$0 \to I^p \to \prod I_n^p \to \prod I_n^p \to 0$$

of objects of $\mathcal{A}$. Here the first map is given by the projection maps $I^p \to I_n^p$ and the second map by $(x_n) \mapsto (x_n - f_{n+1}^p(x_{n+1}))$ where $f_n^p : I_n^p \to I_{n-1}^p$ are the transition maps. The splitting comes from the map $\prod I_n^p \to \prod C_n^p = I^p$. We obtain a termwise split short exact sequence of complexes

$$0 \to I^\bullet \to \prod I_n^\bullet \to \prod I_n^\bullet \to 0$$

Hence a corresponding distinguished triangle in $K(\mathcal{A})$ and $D(\mathcal{A})$. By Lemma 31.5 the products are K-injective and represent the corresponding products in $D(\mathcal{A})$. It follows that $I^\bullet$ represents $R\lim I_n^\bullet$ (Definition 34.1). Moreover, it follows that $I^\bullet$ is K-injective by Lemma 31.3. By the commutative diagram of Lemma 29.3 we obtain a corresponding commutative diagram

$$\begin{tikzcd} K^\bullet \arrow[r] \arrow[d] & R\lim \tau_{\geq -n} K^\bullet \arrow[d] \\ I^\bullet \arrow[r] & R\lim I_n^\bullet \end{tikzcd}$$

in $D(\mathcal{A})$. Since the right vertical arrow is an isomorphism (as derived limits are defined on the level of the derived category and since $\tau_{\geq -n}K^\bullet \to I_n^\bullet$ is a quasi-isomorphism), the lemma follows. $\square$

Lemma 34.5. *Let $\mathcal{A}$ be an abelian category having enough injectives and exact countable products. Then for every complex there is a quasi-isomorphism to a K-injective complex.*

Proof. By Lemma 34.4 it suffices to show that $K \to R\lim \tau_{\geq -n}K$ is an isomorphism for all K in $D(\mathcal{A})$. Consider the defining distinguished triangle

$$R\lim \tau_{\geq -n}K \to \prod \tau_{\geq -n}K \to \prod \tau_{\geq -n}K \to (R\lim \tau_{\geq -n}K)[1]$$

By Lemma 34.2 we have

$$H^p(\prod \tau_{\geq -n}K) = \prod\nolimits_{p \geq -n} H^p(K)$$

It follows in a straightforward manner from the long exact cohomology sequence of the displayed distinguished triangle that $H^p(R\lim \tau_{\geq -n}K) = H^p(K)$. $\square$

35. Operations on full subcategories

Let $\mathcal{T}$ be a triangulated category. We will identify full subcategories of $\mathcal{T}$ with subsets of $\text{Ob}(\mathcal{T})$. Given full subcategories $\mathcal{A}, \mathcal{B}, \ldots$ we let

1. $\mathcal{A}[a,b]$ for $-\infty \leq a \leq b \leq \infty$ be the full subcategory of $\mathcal{T}$ consisting of all objects $A[-i]$ with $i \in [a,b] \cap \mathbf{Z}$ with $A \in \text{Ob}(\mathcal{A})$ (note the minus sign!),
2. $smd(\mathcal{A})$ be the full subcategory of $\mathcal{T}$ consisting of all objects which are isomorphic to direct summands of objects of $\mathcal{A}$,
3. $add(\mathcal{A})$ be the full subcategory of $\mathcal{T}$ consisting of all objects which are isomorphic to finite direct sums of objects of $\mathcal{A}$,
4. $\mathcal{A} \star \mathcal{B}$ be the full subcategory of $\mathcal{T}$ consisting of all objects X of $\mathcal{T}$ which fit into a distinguished triangle $A \to X \to B$ with $A \in \text{Ob}(\mathcal{A})$ and $B \in \text{Ob}(\mathcal{B})$,
5. $\mathcal{A}^{\star n} = \mathcal{A} \star \ldots \star \mathcal{A}$ with $n \geq 1$ factors (we will see $\star$ is associative below),
6. $smd(add(\mathcal{A})^{\star n}) = smd(add(\mathcal{A}) \star \ldots \star add(\mathcal{A}))$ with $n \geq 1$ factors.

If E is an object of $\mathcal{T}$, then we think of E sometimes also as the full subcategory of $\mathcal{T}$ whose single object is E. Then we can consider things like $add(E[-1,2])$ and so on and so forth. We warn the reader that this notation is not universally accepted.

Lemma 35.1. *Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ we have $(\mathcal{A} \star \mathcal{B}) \star \mathcal{C} = \mathcal{A} \star (\mathcal{B} \star \mathcal{C})$.*

Proof. If we have distinguished triangles $A \to X \to B$ and $X \to Y \to C$ then by Axiom TR4 we have distinguished triangles $A \to Y \to Z$ and $B \to Z \to C$. $\square$

Lemma 35.2. *Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}$, $\mathcal{B}$ we have $smd(\mathcal{A}) \star smd(\mathcal{B}) \subset smd(\mathcal{A} \star \mathcal{B})$ and $smd(smd(\mathcal{A}) \star smd(\mathcal{B})) = smd(\mathcal{A} \star \mathcal{B})$.*

Proof. Suppose we have a distinguished triangle $A_1 \to X \to B_1$ where $A_1 \oplus A_2 \in \text{Ob}(\mathcal{A})$ and $B_1 \oplus B_2 \in \text{Ob}(\mathcal{B})$. Then we obtain a distinguished triangle $A_1 \oplus A_2 \to A_2 \oplus X \oplus B_2 \to B_1 \oplus B_2$ which proves that X is in $smd(\mathcal{A} \star \mathcal{B})$. This proves the inclusion. The equality follows trivially from this. $\square$

Lemma 35.3. *Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}$, $\mathcal{B}$ the full subcategories $add(\mathcal{A}) \star add(\mathcal{B})$ and $smd(add(\mathcal{A}))$ are closed under direct sums.*

Proof. Namely, if $A \to X \to B$ and $A' \to X' \to B'$ are distinguished triangles and $A, A' \in add(\mathcal{A})$ and $B, B' \in add(\mathcal{B})$ then $A \oplus A' \to X \oplus X' \to B \oplus B'$ is a distinguished triangle with $A \oplus A' \in add(\mathcal{A})$ and $B \oplus B' \in add(\mathcal{B})$. The result for $smd(add(\mathcal{A}))$ is trivial. □

Lemma 35.4. *Let $\mathcal{T}$ be a triangulated category. Given a full subcategory $\mathcal{A}$ for $n \geq 1$ the subcategory*

$$\mathcal{C}_n = smd(add(\mathcal{A})^{\star n}) = smd(add(\mathcal{A}) \star \ldots \star add(\mathcal{A}))$$

defined above is a strictly full subcategory of $\mathcal{T}$ closed under direct sums and direct summands and $\mathcal{C}_{n+m} = smd(\mathcal{C}_n \star \mathcal{C}_m)$ for all $n, m \geq 1$.

Proof. Immediate from Lemmas 35.1, 35.2, and 35.3. □

Remark 35.5. Let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor of triangulated categories. Given a full subcategory $\mathcal{A}$ of $\mathcal{T}$ we denote $F(\mathcal{A})$ the full subcategory of $\mathcal{T}'$ whose objects consists of all objects $F(A)$ with $A \in \mathrm{Ob}(\mathcal{A})$. We have

$$\begin{gathered} F(\mathcal{A}[a,b]) = F(\mathcal{A})[a,b] \\ F(smd(\mathcal{A})) \subset smd(F(\mathcal{A})), \\ F(add(\mathcal{A})) \subset add(F(\mathcal{A})), \\ F(\mathcal{A} \star \mathcal{B}) \subset F(\mathcal{A}) \star F(\mathcal{B}), \\ F(\mathcal{A}^{\star n}) \subset F(\mathcal{A})^{\star n}. \end{gathered}$$

We omit the trivial verifications.

Remark 35.6. Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \ldots$ and $\mathcal{B}$ of $\mathcal{T}$ we have

$$\begin{gathered} \left(\bigcup \mathcal{A}_i\right)[a,b] = \bigcup \mathcal{A}_i[a,b] \\ smd\left(\bigcup \mathcal{A}_i\right) = \bigcup smd(\mathcal{A}_i), \\ add\left(\bigcup \mathcal{A}_i\right) = \bigcup add(\mathcal{A}_i), \\ \left(\bigcup \mathcal{A}_i\right) \star \mathcal{B} = \bigcup \mathcal{A}_i \star \mathcal{B}, \\ \mathcal{B} \star \left(\bigcup \mathcal{A}_i\right) = \bigcup \mathcal{B} \star \mathcal{A}_i, \\ \left(\bigcup \mathcal{A}_i\right)^{\star n} = \bigcup \mathcal{A}_i^{\star n}. \end{gathered}$$

We omit the trivial verifications.

Lemma 35.7. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D} = D(\mathcal{A})$. Let $\mathcal{E} \subset \mathrm{Ob}(\mathcal{A})$ be a subset which we view as a subset of $\mathrm{Ob}(\mathcal{D})$ also. Let K be an object of $\mathcal{D}$.*

1. *Let $b \geq a$ and assume $H^i(K)$ is zero for $i \notin [a,b]$ and $H^i(K) \in \mathcal{E}$ if $i \in [a,b]$. Then K is in $smd(add(\mathcal{E}[a,b])^{\star(b-a+1)})$.*
2. *Let $b \geq a$ and assume $H^i(K)$ is zero for $i \notin [a,b]$ and $H^i(K) \in smd(add(\mathcal{E}))$ if $i \in [a,b]$. Then K is in $smd(add(\mathcal{E}[a,b])^{\star(b-a+1)})$.*
3. *Let $b \geq a$ and assume K can be represented by a complex $K^\bullet$ with $K^i = 0$ for $i \notin [a,b]$ and $K^i \in \mathcal{E}$ for $i \in [a,b]$. Then K is in $smd(add(\mathcal{E}[a,b])^{\star(b-a+1)})$.*
4. *Let $b \geq a$ and assume K can be represented by a complex $K^\bullet$ with $K^i = 0$ for $i \notin [a,b]$ and $K^i \in smd(add(\mathcal{E}))$ for $i \in [a,b]$. Then K is in $smd(add(\mathcal{E}[a,b])^{\star(b-a+1)})$.*

Proof. We will use Lemma 35.4 without further mention. We will prove (2) which trivially implies (1). We use induction on $b - a$. If $b - a = 0$, then K is isomorphic to $H^i(K)[-a]$ in $\mathcal{D}$ and the result is immediate. If $b - a > 0$, then we consider the distinguished triangle

$$\tau_{\leq b-1}K^\bullet \to K^\bullet \to K^b[-b]$$

and we conclude by induction on $b - a$. We omit the proof of (3) and (4). $\square$

Lemma 35.8. *Let $\mathcal{T}$ be a triangulated category. Let $H : \mathcal{T} \to \mathcal{A}$ be a homological functor to an abelian category $\mathcal{A}$. Let $a \leq b$ and $\mathcal{E} \subset \mathrm{Ob}(\mathcal{T})$ be a subset such that $H^i(E) = 0$ for $E \in \mathcal{E}$ and $i \notin [a, b]$. Then for $X \in smd(add(\mathcal{E}[-m, m])^{\star n})$ we have $H^i(X) = 0$ for $i \notin [-m + na, m + nb]$.*

Proof. Omitted. Pleasant exercise in the definitions. $\square$

36. Generators of triangulated categories

In this section we briefly introduce a few of the different notions of a generator for a triangulated category. Our terminology is taken from [BV03] (except that we use "saturated" for what they call "épaisse", see Definition 6.1, and our definition of $add(\mathcal{A})$ is different).

Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. Denote $\langle E \rangle_1$ the strictly full subcategory of $\mathcal{D}$ consisting of objects in $\mathcal{D}$ isomorphic to direct summands of finite direct sums

$$\bigoplus\nolimits_{i=1,\ldots,r} E[n_i]$$

of shifts of E. It is clear that in the notation of Section 35 we have

$$\langle E \rangle_1 = smd(add(E[-\infty, \infty]))$$

For $n > 1$ let $\langle E \rangle_n$ denote the full subcategory of $\mathcal{D}$ consisting of objects of $\mathcal{D}$ isomorphic to direct summands of objects X which fit into a distinguished triangle

$$A \to X \to B \to A[1]$$

where A is an object of $\langle E \rangle_1$ and B an object of $\langle E \rangle_{n-1}$. In the notation of Section 35 we have

$$\langle E \rangle_n = smd(\langle E \rangle_1 \star \langle E \rangle_{n-1})$$

Each of the categories $\langle E \rangle_n$ is a strictly full additive (by Lemma 35.3) subcategory of $\mathcal{D}$ preserved under shifts and under taking summands. But, $\langle E \rangle_n$ is not necessarily closed under "taking cones" or "extensions", hence not necessarily a triangulated subcategory. This will be true for the subcategory

$$\langle E \rangle = \bigcup\nolimits_n \langle E \rangle_n$$

as will be shown in the lemmas below.

Lemma 36.1. *Let $\mathcal{T}$ be a triangulated category. Let E be an object of $\mathcal{T}$. For $n \geq 1$ we have*

$$\langle E \rangle_n = smd(\langle E \rangle_1 \star \ldots \star \langle E \rangle_1) = smd({\langle E \rangle_1}^{\star n}) = \bigcup\nolimits_{m \geq 1} smd(add(E[-m, m])^{\star n})$$

For $n, n' \geq 1$ we have $\langle E \rangle_{n+n'} = smd(\langle E \rangle_n \star \langle E \rangle_{n'})$.

Proof. The left equality in the displayed formula follows from Lemmas 35.1 and 35.2 and induction. The middle equality is a matter of notation. Since $\langle E \rangle_1 = smd(add(E[-\infty, \infty]))$ and since $E[-\infty, \infty] = \bigcup_{m \geq 1} E[-m, m]$ we see from Remark 35.6 and Lemma 35.2 that we get the equality on the right. Then the final statement follows from the remark and the corresponding statement of Lemma 35.4. $\square$

Lemma 36.2. *Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. The subcategory*

$$\langle E \rangle = \bigcup\nolimits_n \langle E \rangle_n = \bigcup\nolimits_{n,m \geq 1} smd(add(E[-m, m])^{\star n})$$

is a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ and it is the smallest such subcategory of $\mathcal{D}$ containing the object E.

Proof. The equality on the right follows from Lemma 36.1. It is clear that $\langle E \rangle = \bigcup \langle E \rangle_n$ contains E, is preserved under shifts, direct sums, direct summands. If $A \in \langle E \rangle_a$ and $B \in \langle E \rangle_b$ and if $A \to X \to B \to A[1]$ is a distinguished triangle, then $X \in \langle E \rangle_{a+b}$ by Lemma 36.1. Hence $\bigcup \langle E \rangle_n$ is also preserved under extensions and it follows that it is a triangulated subcategory.

Finally, let $\mathcal{D}' \subset \mathcal{D}$ be a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ containing E. Then $\mathcal{D}'[-\infty, \infty] \subset \mathcal{D}'$, $add(\mathcal{D}) \subset \mathcal{D}'$, $smd(\mathcal{D}') \subset \mathcal{D}'$, and $\mathcal{D}' \star \mathcal{D}' \subset \mathcal{D}'$. In other words, all the operations we used to construct $\langle E \rangle$ out of E preserve $\mathcal{D}'$. Hence $\langle E \rangle \subset \mathcal{D}'$ and this finishes the proof. $\square$

Definition 36.3. Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$.

(1) We say E is a *classical generator* of $\mathcal{D}$ if the smallest strictly full, saturated, triangulated subcategory of $\mathcal{D}$ containing E is equal to $\mathcal{D}$, in other words, if $\langle E \rangle = \mathcal{D}$.
(2) We say E is a *strong generator* of $\mathcal{D}$ if $\langle E \rangle_n = \mathcal{D}$ for some $n \geq 1$.
(3) We say E is a *weak generator* or a *generator* of $\mathcal{D}$ if for any nonzero object K of $\mathcal{D}$ there exists an integer n and a nonzero map $E \to K[n]$.

This definition can be generalized to the case of a family of objects.

Lemma 36.4. *Let $\mathcal{D}$ be a triangulated category. Let E, K be objects of $\mathcal{D}$. The following are equivalent*

(1) $\text{Hom}(E, K[i]) = 0$ *for all* $i \in \mathbf{Z}$,
(2) $\text{Hom}(E', K) = 0$ *for all* $E' \in \langle E \rangle$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Conversely, assume (1). Then $\text{Hom}(X, K) = 0$ for all X in $\langle E \rangle_1$. Arguing by induction on n and using Lemma 4.2 we see that $\text{Hom}(X, K) = 0$ for all X in $\langle E \rangle_n$. $\square$

Lemma 36.5. *Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. If E is a classical generator of $\mathcal{D}$, then E is a generator.*

Proof. Assume E is a classical generator. Let K be an object of $\mathcal{D}$ such that $\text{Hom}(E, K[i]) = 0$ for all $i \in \mathbf{Z}$. By Lemma 36.4 $\text{Hom}(E', K) = 0$ for all E' in $\langle E \rangle$. However, since $\mathcal{D} = \langle E \rangle$ we conclude that $\text{id}_K = 0$, i.e., $K = 0$. $\square$

Lemma 36.6. *Let $\mathcal{D}$ be a triangulated category which has a strong generator. Let E be an object of $\mathcal{D}$. If E is a classical generator of $\mathcal{D}$, then E is a strong generator.*

Proof. Let E' be an object of $\mathcal{D}$ such that $\mathcal{D} = \langle E' \rangle_n$. Since $\mathcal{D} = \langle E \rangle$ we see that $E' \in \langle E \rangle_m$ for some $m \geq 1$ by Lemma 36.2. Then $\langle E' \rangle_1 \subset \langle E \rangle_m$ hence

$$\mathcal{D} = \langle E' \rangle_n = smd(\langle E' \rangle_1 \star \ldots \star \langle E' \rangle_1) \subset smd(\langle E \rangle_m \star \ldots \star \langle E \rangle_m) = \langle E \rangle_{nm}$$

as desired. Here we used Lemma 36.1. □

Remark 36.7. Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. Let T be a property of objects of $\mathcal{D}$. Suppose that

1. if $K_i \in D(A)$, $i = 1, \ldots, r$ with $T(K_i)$ for $i = 1, \ldots, r$, then $T(\bigoplus K_i)$,
2. if $K \to L \to M \to K[1]$ is a distinguished triangle and T holds for two, then T holds for the third object,
3. if $T(K \oplus L)$ then $T(K)$ and $T(L)$, and
4. $T(E[n])$ holds for all n.

Then T holds for all objects of $\langle E \rangle$.

37. Compact objects

Here is the definition.

Definition 37.1. Let $\mathcal{D}$ be an additive category with arbitrary direct sums. A *compact object* of $\mathcal{D}$ is an object K such that the map

$$\bigoplus\nolimits_{i \in I} \mathrm{Hom}_\mathcal{D}(K, E_i) \longrightarrow \mathrm{Hom}_\mathcal{D}(K, \bigoplus\nolimits_{i \in I} E_i)$$

is bijective for any set I and objects $E_i \in \mathrm{Ob}(\mathcal{D})$ parametrized by $i \in I$.

This notion turns out to be very useful in algebraic geometry. It is an intrinsic condition on objects that forces the objects to be, well, compact.

Lemma 37.2. *Let $\mathcal{D}$ be a (pre-)triangulated category with direct sums. Then the compact objects of $\mathcal{D}$ form the objects of a Karoubian, saturated, strictly full, (pre-)triangulated subcategory $\mathcal{D}_c$ of $\mathcal{D}$.*

Proof. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$ with X and Y compact. Then it follows from Lemma 4.2 and the five lemma (Homology, Lemma 5.20) that Z is a compact object too. It is clear that if $X \oplus Y$ is compact, then X, Y are compact objects too. Hence $\mathcal{D}_c$ is a saturated triangulated subcategory. Since $\mathcal{D}$ is Karoubian by Lemma 4.14 we conclude that the same is true for $\mathcal{D}_c$. □

Lemma 37.3. *Let $\mathcal{D}$ be a triangulated category with direct sums. Let E_i, $i \in I$ be a family of compact objects of $\mathcal{D}$ such that $\bigoplus E_i$ generates $\mathcal{D}$. Then every object X of $\mathcal{D}$ can be written as*

$$X = hocolim X_n$$

where X_1 is a direct sum of shifts of the E_i and each transition morphism fits into a distinguished triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$ where Y_n is a direct sum of shifts of the E_i.

Proof. Set $X_1 = \bigoplus_{(i,m,\varphi)} E_i[m]$ where the direct sum is over all triples (i, m, φ) such that $i \in I$, $m \in \mathbf{Z}$ and $\varphi : E_i[m] \to X$. Then X_1 comes equipped with a canonical morphism $X_1 \to X$. Given $X_n \to X$ we set $Y_n = \bigoplus_{(i,m,\varphi)} E_i[m]$ where the direct sum is over all triples (i, m, φ) such that $i \in I$, $m \in \mathbf{Z}$, and $\varphi : E_i[m] \to X_n$ is a morphism such that $E_i[m] \to X_n \to X$ is zero. Choose a distinguished triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$ and let $X_{n+1} \to X$ be any morphism such that $X_n \to X_{n+1} \to X$ is the given one; such a morphism exists by our choice of Y_n. We obtain a morphism $\text{hocolim} X_n \to X$ by the construction of our maps $X_n \to X$. Choose a distinguished triangle

$$C \to \text{hocolim} X_n \to X \to C[1]$$

Let $E_i[m] \to C$ be a morphism. Since E_i is compact, the composition $E_i[m] \to C \to \text{hocolim} X_n$ factors through X_n for some n, say by $E_i[m] \to X_n$. Then the construction of Y_n shows that the composition $E_i[m] \to X_n \to X_{n+1}$ is zero. In other words, the composition $E_i[m] \to C \to \text{hocolim} X_n$ is zero. This means that our morphism $E_i[m] \to C$ comes from a morphism $E_i[m] \to X[-1]$. The construction of X_1 then shows that such morphism lifts to $\text{hocolim} X_n$ and we conclude that our morphism $E_i[m] \to C$ is zero. The assumption that $\bigoplus E_i$ generates $\mathcal{D}$ implies that C is zero and the proof is done. $\square$

Lemma 37.4. *With assumptions and notation as in Lemma 37.3. If C is a compact object and $C \to X_n$ is a morphism, then there is a factorization $C \to E \to X_n$ where E is an object of $\langle E_{i_1} \oplus \ldots \oplus E_{i_t} \rangle$ for some $i_1, \ldots, i_t \in I$.*

Proof. We prove this by induction on n. The base case $n = 1$ is clear. If $n > 1$ consider the composition $C \to X_n \to Y_{n-1}[1]$. This can be factored through some $E'[1] \to Y_{n-1}[1]$ where E' is a finite direct sum of shifts of the E_i. Let $I' \subset I$ be the finite set of indices that occur in this direct sum. Thus we obtain

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & E'[1] \arrow[d] \\ Y_{n-1} \arrow[r] & X_{n-1} \arrow[r] & X_n \arrow[r] & Y_{n-1}[1] \end{tikzcd}$$

By induction the morphism $C' \to X_{n-1}$ factors through $E'' \to X_{n-1}$ with E'' an object of $\langle \bigoplus_{i \in I''} E_i \rangle$ for some finite subset $I'' \subset I$. Choose a distinguished triangle

$$E' \to E'' \to E \to E'[1]$$

then E is an object of $\langle \bigoplus_{i \in I' \cup I''} E_i \rangle$. By construction and the axioms of a triangulated category we can choose morphisms $C \to E$ and a morphism $E \to X_n$ fitting into morphisms of triangles $(E', C', C) \to (E', E'', E)$ and $(E', E'', E) \to (Y_{n-1}, X_{n-1}, X_n)$. The composition $C \to E \to X_n$ may not equal the given morphism $C \to X_n$, but the compositions into Y_{n-1} are equal. Let $C \to X_{n-1}$ be a morphism that lifts the difference. By induction assumption we can factor this through a morphism $E''' \to X_{n-1}$ with E'' an object of $\langle \bigoplus_{i \in I'''} E_i \rangle$ for some finite subset $I' \subset I$. Thus we see that we get a solution on considering $E \oplus E''' \to X_n$ because $E \oplus E'''$ is an object of $\langle \bigoplus_{i \in I' \cup I'' \cup I'''} E_i \rangle$. $\square$

Definition 37.5. Let $\mathcal{D}$ be a triangulated category with arbitrary direct sums. We say $\mathcal{D}$ is *compactly generated* if there exists a set E_i, $i \in I$ of compact objects such that $\bigoplus E_i$ generates $\mathcal{D}$.

The following proposition clarifies the relationship between classical generators and weak generators.

Proposition 37.6. *Let $\mathcal{D}$ be a triangulated category with direct sums. Let E be a compact object of $\mathcal{D}$. The following are equivalent*

1. *E is a classical generator for $\mathcal{D}_c$ and $\mathcal{D}$ is compactly generated, and*

(2) *E is a generator for $\mathcal{D}$.*

Proof. If E is a classical generator for $\mathcal{D}_c$, then $\mathcal{D}_c = \langle E \rangle$. It follows formally from the assumption that $\mathcal{D}$ is compactly generated and Lemma 36.4 that E is a generator for $\mathcal{D}$.

The converse is more interesting. Assume that E is a generator for $\mathcal{D}$. Let X be a compact object of $\mathcal{D}$. Apply Lemma 37.3 with $I = \{1\}$ and $E_1 = E$ to write

$$X = \text{hocolim} X_n$$

as in the lemma. Since X is compact we find that $X \to \text{hocolim} X_n$ factors through X_n for some n (Lemma 33.9). Thus X is a direct summand of X_n. By Lemma 37.4 we see that X is an object of $\langle E \rangle$ and the lemma is proven. □

38. Brown representability

A reference for the material in this section is [Nee96].

Lemma 38.1. *Let $\mathcal{D}$ be a triangulated category with direct sums which is compactly generated. Let $H : \mathcal{D} \to Ab$ be a contravariant cohomological functor which transforms direct sums into products. Then H is representable.*

Proof. Let E_i, $i \in I$ be a set of compact objects such that $\bigoplus_{i \in I} E_i$ generates $\mathcal{D}$. We may and do assume that the set of objects $\{E_i\}$ is preserved under shifts. Consider pairs (i, a) where $i \in I$ and $a \in H(E_i)$ and set

$$X_1 = \bigoplus\nolimits_{(i,a)} E_i$$

Since $H(X_1) = \prod_{(i,a)} H(E_i)$ we see that $(a)_{(i,a)}$ defines an element $a_1 \in H(X_1)$. Set $H_1 = \text{Hom}_{\mathcal{D}}(-, X_1)$. By Yoneda's lemma (Categories, Lemma 3.5) the element a_1 defines a natural transformation $H_1 \to H$.

We are going to inductively construct X_n and transformations $a_n : H_n \to H$ where $H_n = \text{Hom}_{\mathcal{D}}(-, X_n)$. Namely, we apply the procedure above to the functor $\text{Ker}(H_n \to H)$ to get an object

$$K_{n+1} = \bigoplus\nolimits_{(i,k),\ k \in \text{Ker}(H_n(E_i) \to H(E_i))} E_i$$

and a transformation $\text{Hom}_{\mathcal{D}}(-, K_{n+1}) \to \text{Ker}(H_n \to H)$. By Yoneda's lemma the composition $\text{Hom}_{\mathcal{D}}(-, K_{n+1}) \to H_n$ gives a morphism $K_{n+1} \to X_n$. We choose a distinguished triangle

$$K_{n+1} \to X_n \to X_{n+1} \to K_{n+1}[1]$$

in $\mathcal{D}$. The element $a_n \in H(X_n)$ maps to zero in $H(K_{n+1})$ by construction. Since H is cohomological we can lift it to an element $a_{n+1} \in H(X_{n+1})$.

We claim that $X = \text{hocolim} X_n$ represents H. Applying H to the defining distinguished triangle

$$\bigoplus X_n \to \bigoplus X_n \to X \to \bigoplus X_n[1]$$

we obtain an exact sequence

$$\prod H(X_n) \leftarrow \prod H(X_n) \leftarrow H(X)$$

Thus there exists an element $a \in H(X)$ mapping to (a_n) in $\prod H(X_n)$. Hence a natural transformation $\mathrm{Hom}_{\mathcal{D}}(-, X) \to H$ such that

$$\mathrm{Hom}_{\mathcal{D}}(-, X_1) \to \mathrm{Hom}_{\mathcal{D}}(-, X_2) \to \mathrm{Hom}_{\mathcal{D}}(-, X_3) \to \ldots \to \mathrm{Hom}_{\mathcal{D}}(-, X) \to H$$

commutes. For each i the map $\mathrm{Hom}_{\mathcal{D}}(E_i, X) \to H(E_i)$ is surjective, by construction of X_1. On the other hand, by construction of $X_n \to X_{n+1}$ the kernel of $\mathrm{Hom}_{\mathcal{D}}(E_i, X_n) \to H(E_i)$ is killed by the map $\mathrm{Hom}_{\mathcal{D}}(E_i, X_n) \to \mathrm{Hom}_{\mathcal{D}}(E_i, X_{n+1})$. Since

$$\mathrm{Hom}_{\mathcal{D}}(E_i, X) = \mathrm{colim}\, \mathrm{Hom}_{\mathcal{D}}(E_i, X_n)$$

by Lemma 33.9 we see that $\mathrm{Hom}_{\mathcal{D}}(E_i, X) \to H(E_i)$ is injective.

To finish the proof, consider the subcategory

$$\mathcal{D}' = \{Y \in \mathrm{Ob}(\mathcal{D}) \mid \mathrm{Hom}_{\mathcal{D}}(Y[n], X) \to H(Y[n]) \text{ is an isomorphism for all } n\}$$

As $\mathrm{Hom}_{\mathcal{D}}(-, X) \to H$ is a transformation between cohomological functors, the subcategory $\mathcal{D}'$ is a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ (details omitted; see proof of Lemma 6.3). Moreover, as both H and $\mathrm{Hom}_{\mathcal{D}}(-, X)$ transform direct sums into products, we see that direct sums of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Thus derived colimits of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Since $\{E_i\}$ is preserved under shifts, we see that E_i is an object of $\mathcal{D}'$ for all i. It follows from Lemma 37.3 that $\mathcal{D}' = \mathcal{D}$ and the proof is complete. $\square$

Proposition 38.2. *Let $\mathcal{D}$ be a triangulated category with direct sums which is compactly generated. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories which transforms direct sums into direct sums. Then F has an exact right adjoint.*

Proof. For an object Y of $\mathcal{D}'$ consider the contravariant functor

$$\mathcal{D} \to Ab, \quad W \mapsto \mathrm{Hom}_{\mathcal{D}'}(F(W), Y)$$

This is a cohomological functor as F is exact and transforms direct sums into products as F transforms direct sums into direct sums. Thus by Lemma 38.1 we find an object X of $\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(W, X) = \mathrm{Hom}_{\mathcal{D}'}(F(W), Y)$. The existence of the adjoint follows from Categories, Lemma 24.2. Exactness follows from Lemma 7.1. $\square$

39. Brown representability, bis

In this section we explain a version of Brown representability for triangulated categories which have a suitable set of generators; for other versions, please see [Fra01], [Nee01], and [Kra02].

Lemma 39.1. *Let $\mathcal{D}$ be a triangulated category with direct sums. Suppose given a set $\mathcal{E}$ of objects of $\mathcal{D}$ such that*

(1) *if X is a nonzero object of $\mathcal{D}$, then there exists an $E \in \mathcal{E}$ and a nonzero map $E \to X$, and*
(2) *given objects X_n, $n \in \mathbf{N}$ of $\mathcal{D}$, $E \in \mathcal{E}$, and $\alpha : E \to \bigoplus X_n$, there exist $E_n \in \mathcal{E}$ and $\beta_n : E_n \to X_n$ and a morphism $\gamma : E \to \bigoplus E_n$ such that $\alpha = (\bigoplus \beta_n) \circ \gamma$.*

Let $H : \mathcal{D} \to Ab$ be a contravariant cohomological functor which transforms direct sums into products. Then H is representable.

Proof. This proof is very similar to the proof of Lemma 38.1. We may replace $\mathcal{E}$ by $\bigcup_{i \in \mathbf{Z}} \mathcal{E}[i]$ and assume that $\mathcal{E}$ is preserved by shifts. Consider pairs (E, a) where $E \in \mathcal{E}$ and $a \in H(E)$ and set

$$X_1 = \bigoplus\nolimits_{(E,a)} E$$

Since $H(X_1) = \prod_{(E,a)} H(E)$ we see that $(a)_{(E,a)}$ defines an element $a_1 \in H(X_1)$. Set $H_1 = \mathrm{Hom}_\mathcal{D}(-, X_1)$. By Yoneda's lemma (Categories, Lemma 3.5) the element a_1 defines a natural transformation $H_1 \to H$.

We are going to inductively construct X_n and transformations $a_n : H_n \to H$ where $H_n = \mathrm{Hom}_\mathcal{D}(-, X_n)$. Namely, we apply the procedure above to the functor $\mathrm{Ker}(H_n \to H)$ to get an object

$$K_{n+1} = \bigoplus\nolimits_{(E,k),\ k \in \mathrm{Ker}(H_n(E) \to H(E))} E$$

and a transformation $\mathrm{Hom}_\mathcal{D}(-, K_{n+1}) \to \mathrm{Ker}(H_n \to H)$. By Yoneda's lemma the composition $\mathrm{Hom}_\mathcal{D}(-, K_{n+1}) \to H_n$ gives a morphism $K_{n+1} \to X_n$. We choose a distinguished triangle

$$K_{n+1} \to X_n \to X_{n+1} \to K_{n+1}[1]$$

in $\mathcal{D}$. The element $a_n \in H(X_n)$ maps to zero in $H(K_{n+1})$ by construction. Since H is cohomological we can lift it to an element $a_{n+1} \in H(X_{n+1})$.

Set $X = \mathrm{hocolim} X_n$. Applying H to the defining distinguished triangle

$$\bigoplus X_n \to \bigoplus X_n \to X \to \bigoplus X_n[1]$$

we obtain an exact sequence

$$\prod H(X_n) \leftarrow \prod H(X_n) \leftarrow H(X)$$

Thus there exists an element $a \in H(X)$ mapping to (a_n) in $\prod H(X_n)$. Hence a natural transformation $\mathrm{Hom}_\mathcal{D}(-, X) \to H$ such that

$$\mathrm{Hom}_\mathcal{D}(-, X_1) \to \mathrm{Hom}_\mathcal{D}(-, X_2) \to \mathrm{Hom}_\mathcal{D}(-, X_3) \to \ldots \to \mathrm{Hom}_\mathcal{D}(-, X) \to H$$

commutes. We claim that $\mathrm{Hom}_\mathcal{D}(-, X) \to H(-)$ is an isomorphism.

Let $E \in \mathcal{E}$. Let us show that

$$\mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n) \to \mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n)$$

is injective. Namely, let $\alpha : E \to \bigoplus X_n$. Then by assumption (2) we obtain a factorization $\alpha = (\bigoplus \beta_n) \circ \gamma$. Since $E_n \to X_n \to X_{n+1}$ is zero by construction, we see that the composition $\bigoplus E_n \to \bigoplus X_n \to \bigoplus X_n$ is equal to $\bigoplus \beta_n$. Hence also the composition $E \to \bigoplus X_n \to \bigoplus X_n$ is equal to α. This proves the stated injectivity and hence also

$$\mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n[1]) \to \mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n[1])$$

is injective. It follows that we have an exact sequence

$$\mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n) \to \mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n) \to \mathrm{Hom}_\mathcal{D}(E, X) \to 0$$

for all $E \in \mathcal{E}$.

Let $E \in \mathcal{E}$ and let $f : E \to X$ be a morphism. By the previous paragraph, we may choose $\alpha : E \to \bigoplus X_n$ lifting f. Then by assumption (2) we obtain a factorization

$\alpha = (\bigoplus \beta_n) \circ \gamma$. For each n there is a morphism $\delta_n : E_n \to X_1$ such that δ_n and β_n map to the same element of $H(E_n)$. Then the compositions

$$E_n \to X_n \to X_{n+1} \quad \text{and} \quad E_n \to X_1 \to X_{n+1}$$

are equal by construction of $X_n \to X_{n+1}$. It follows that

$$\bigoplus E_n \to \bigoplus X_n \to X \quad \text{and} \quad \bigoplus E_n \to \bigoplus X_1 \to X$$

are the same too. Observing that $\bigoplus X_1 \to X$ factors as $\bigoplus X_1 \to X_1 \to X$, we conclude that

$$\mathrm{Hom}_{\mathcal{D}}(E, X_1) \to \mathrm{Hom}_{\mathcal{D}}(E, X)$$

is surjective. Since by construction the map $\mathrm{Hom}_{\mathcal{D}}(E, X_1) \to H(E)$ is surjective and by construction the kernel of this map is annihilated by $\mathrm{Hom}_{\mathcal{D}}(E, X_1) \to \mathrm{Hom}_{\mathcal{D}}(E, X)$ we conclude that $\mathrm{Hom}_{\mathcal{D}}(E, X) \to H(E)$ is a bijection for all $E \in \mathcal{E}$.

To finish the proof, consider the subcategory

$$\mathcal{D}' = \{Y \in \mathrm{Ob}(\mathcal{D}) \mid \mathrm{Hom}_{\mathcal{D}}(Y[n], X) \to H(Y[n]) \text{ is an isomorphism for all } n\}$$

As $\mathrm{Hom}_{\mathcal{D}}(-, X) \to H$ is a transformation between cohomological functors, the subcategory $\mathcal{D}'$ is a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ (details omitted; see proof of Lemma 6.3). Moreover, as both H and $\mathrm{Hom}_{\mathcal{D}}(-, X)$ transform direct sums into products, we see that direct sums of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Thus derived colimits of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Since $\mathcal{E}$ is preserved by shifts, we conclude that $\mathcal{E} \subset \mathrm{Ob}(\mathcal{D}')$ by the result of the previous paragraph. To finish the proof we have to show that $\mathcal{D}' = \mathcal{D}$.

Let Y be an object of $\mathcal{D}$ and set $H(-) = \mathrm{Hom}_{\mathcal{D}}(-, Y)$. Then H is a cohomological functor which transforms direct sums into products. By the construction in the first part of the proof we obtain a morphism $\mathrm{colim}\, X_n = X \to Y$ such that $\mathrm{Hom}_{\mathcal{D}}(E, X) \to \mathrm{Hom}_{\mathcal{D}}(E, Y)$ is bijective for all $E \in \mathcal{E}$. Then assumption (1) tells us that $X \to Y$ is an isomorphism! On the other hand, by construction $X_1, X_2, \ldots$ are in $\mathcal{D}'$ and so is X. Thus $Y \in \mathcal{D}'$ and the proof is complete. □

Proposition 39.2. *Let $\mathcal{D}$ be a triangulated category with direct sums. Assume there exists a set $\mathcal{E}$ of objects of $\mathcal{D}$ satisfying conditions (1) and (2) of Lemma 39.1. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories which transforms direct sums into direct sums. Then F has an exact right adjoint.*

Proof. For an object Y of $\mathcal{D}'$ consider the contravariant functor

$$\mathcal{D} \to Ab, \quad W \mapsto \mathrm{Hom}_{\mathcal{D}'}(F(W), Y)$$

This is a cohomological functor as F is exact and transforms direct sums into products as F transforms direct sums into direct sums. Thus by Lemma 39.1 we find an object X of $\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(W, X) = \mathrm{Hom}_{\mathcal{D}'}(F(W), Y)$. The existence of the adjoint follows from Categories, Lemma 24.2. Exactness follows from Lemma 7.1. □

40. Admissible subcategories

A reference for this section is [BK89, Section 1].

Definition 40.1. Let $\mathcal{D}$ be an additive category. Let $\mathcal{A} \subset \mathcal{D}$ be a full subcategory. The *right orthogonal* $\mathcal{A}^\perp$ of $\mathcal{A}$ is the full subcategory consisting of the objects X of $\mathcal{D}$ such that $\mathrm{Hom}(A, X) = 0$ for all $A \in \mathrm{Ob}(\mathcal{A})$. The *left orthogonal* ${}^\perp\mathcal{A}$ of $\mathcal{A}$ is the full subcategory consisting of the objects X of $\mathcal{D}$ such that $\mathrm{Hom}(X, A) = 0$ for all $A \in \mathrm{Ob}(\mathcal{A})$.

Lemma 40.2. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ be a full subcategory invariant under all shifts. Consider a distinguished triangle*

$$X \to Y \to Z \to X[1]$$

of $\mathcal{D}$. The following are equivalent

(1) *Z is in $\mathcal{A}^\perp$, and*
(2) $\mathrm{Hom}(A, X) = \mathrm{Hom}(A, Y)$ *for all* $A \in \mathrm{Ob}(\mathcal{A})$.

Proof. By Lemma 4.2 the functor $\mathrm{Hom}(A, -)$ is homological and hence we get a long exact sequence as in (3.5.1). Assume (1) and let $A \in \mathrm{Ob}(\mathcal{A})$. Then we consider the exact sequence

$$\mathrm{Hom}(A[1], Z) \to \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y) \to \mathrm{Hom}(A, Z)$$

Since $A[1] \in \mathrm{Ob}(\mathcal{A})$ we see that the first and last groups are zero. Thus we get (2). Assume (2) and let $A \in \mathrm{Ob}(\mathcal{A})$. Then we consider the exact sequence

$$\mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y) \to \mathrm{Hom}(A, Z) \to \mathrm{Hom}(A[-1], X) \to \mathrm{Hom}(A[-1], Y)$$

and we conclude that $\mathrm{Hom}(A, Z) = 0$ as desired. □

Lemma 40.3. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B} \subset \mathcal{D}$ be a full subcategory invariant under all shifts. Consider a distinguished triangle*

$$X \to Y \to Z \to X[1]$$

of $\mathcal{D}$. The following are equivalent

(1) *X is in ${}^\perp\mathcal{B}$, and*
(2) $\mathrm{Hom}(Y, B) = \mathrm{Hom}(Z, B)$ *for all* $B \in \mathrm{Ob}(\mathcal{B})$.

Proof. Dual to Lemma 40.2. □

Lemma 40.4. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ be a full subcategory invariant under all shifts. Then both the right orthogonal $\mathcal{A}^\perp$ and the left orthogonal ${}^\perp\mathcal{A}$ of $\mathcal{A}$ are strictly full, saturated*[8]*, triangulated subcagories of $\mathcal{D}$.*

Proof. It is immediate from the definitions that the orthogonals are preserved under taking shifts, direct sums, and direct summands. Consider a distinguished triangle

$$X \to Y \to Z \to X[1]$$

of $\mathcal{D}$. By Lemma 4.16 it suffices to show that if X and Y are in $\mathcal{A}^\perp$, then Z is in $\mathcal{A}^\perp$. This is immediate from Lemma 40.2. □

Lemma 40.5. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ be a full triangulated subcategory of $\mathcal{D}$. For an object X of $\mathcal{D}$ consider the property $P(X)$: there exists a distinguished triangle $A \to X \to B \to A[1]$ in $\mathcal{D}$ with A in $\mathcal{A}$ and B in $\mathcal{A}^\perp$.*

[8] Definition 6.1.

(1) *If $X_1 \to X_2 \to X_3 \to X_1[1]$ is a distinguished triangle and P holds for two out of three, then it holds for the third.*
(2) *If P holds for X_1 and X_2, then it holds for $X_1 \oplus X_2$.*

Proof. Let $X_1 \to X_2 \to X_3 \to X_1[1]$ be a distinguished triangle and assume P holds for X_1 and X_2. Choose distinguished triangles

$$A_1 \to X_1 \to B_1 \to A_1[1] \quad \text{and} \quad A_2 \to X_2 \to B_2 \to A_2[1]$$

as in condition P. Since $\mathrm{Hom}(A_1, A_2) = \mathrm{Hom}(A_1, X_2)$ by Lemma 40.2 there is a unique morphism $A_1 \to A_2$ such that the diagram

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ A_2 \arrow[r] & X_2 \end{tikzcd}$$

commutes. Choose an extension of this to a diagram

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & A_1[1] \arrow[d] \\ A_2 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & Q_2 \arrow[r] \arrow[d] & A_2[1] \arrow[d] \\ A_3 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & Q_3 \arrow[r] \arrow[d] & A_3[1] \arrow[d] \\ A_1[1] \arrow[r] & X_1[1] \arrow[r] & Q_1[1] \arrow[r] & A_1[2] \end{tikzcd}$$

as in Proposition 4.23. By TR3 we see that $Q_1 \cong B_1$ and $Q_2 \cong B_2$ and hence $Q_1, Q_2 \in \mathrm{Ob}(\mathcal{A}^\perp)$. As $Q_1 \to Q_2 \to Q_3 \to Q_1[1]$ is a distinguished triangle we see that $Q_3 \in \mathrm{Ob}(\mathcal{A}^\perp)$ by Lemma 40.4. Since $\mathcal{A}$ is a full triangulated subcategory, we see that A_3 is isomorphic to an object of $\mathcal{A}$. Thus X_3 satisfies P. The other cases of (1) follow from this case by translation. Part (2) is a special case of (1) via Lemma 4.11. □

Lemma 40.6. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory of $\mathcal{D}$. For an object X of $\mathcal{D}$ consider the property $P(X)$: there exists a distinguished triangle $A \to X \to B \to A[1]$ in $\mathcal{D}$ with B in $\mathcal{B}$ and A in ${}^\perp\mathcal{B}$.*

(1) *If $X_1 \to X_2 \to X_3 \to X_1[1]$ is a distinguished triangle and P holds for two out of three, then it holds for the third.*
(2) *If P holds for X_1 and X_2, then it holds for $X_1 \oplus X_2$.*

Proof. Dual to Lemma 40.5. □

Lemma 40.7. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ be a full triangulated subcategory. The following are equivalent*

(1) *the inclusion functor $\mathcal{A} \to \mathcal{D}$ has a right adjoint, and*
(2) *for every X in $\mathcal{D}$ there exists a distinguished triangle*

$$A \to X \to B \to A[1]$$

in $\mathcal{D}$ with $A \in \mathrm{Ob}(\mathcal{A})$ and $B \in \mathrm{Ob}(\mathcal{A}^\perp)$.

If this holds, then $\mathcal{A}$ is saturated (Definition 6.1) and if $\mathcal{A}$ is strictly full in $\mathcal{D}$, then $\mathcal{A} = {}^\perp(\mathcal{A}^\perp)$.

Proof. Assume (1) and denote $v : \mathcal{D} \to \mathcal{A}$ the right adjoint. Let $X \in \mathrm{Ob}(\mathcal{D})$. Set $A = v(X)$. We may extend the adjunction mapping $A \to X$ to a distinguished triangle $A \to X \to B \to A[1]$. Since

$$\mathrm{Hom}_{\mathcal{A}}(A', A) = \mathrm{Hom}_{\mathcal{A}}(A', v(X)) = \mathrm{Hom}_{\mathcal{D}}(A', X)$$

for $A' \in \mathrm{Ob}(\mathcal{A})$, we conclude that $B \in \mathrm{Ob}(\mathcal{A}^\perp)$ by Lemma 40.2.

Assume (2). We will contruct the adjoint v explictly. Let $X \in \mathrm{Ob}(\mathcal{D})$. Choose $A \to X \to B \to A[1]$ as in (2). Set $v(X) = A$. Let $f : X \to Y$ be a morphism in $\mathcal{D}$. Choose $A' \to Y \to B' \to A'[1]$ as in (2). Since $\mathrm{Hom}(A, A') = \mathrm{Hom}(A, Y)$ by Lemma 40.2 there is a unique morphism $f' : A \to A'$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ A' \arrow[r] & Y \end{tikzcd}$$

commutes. Hence we can set $v(f) = f'$ to get a functor. To see that v is adjoint to the inclusion morphism use Lemma 40.2 again.

Proof of the final statement. In order to prove that $\mathcal{A}$ is saturated we may replace $\mathcal{A}$ by the strictly full subcategory having the same isomorphism classes as $\mathcal{A}$; details omitted. Assume $\mathcal{A}$ is strictly full. If we show that $\mathcal{A} = {}^\perp(\mathcal{A}^\perp)$, then $\mathcal{A}$ will be saturated by Lemma 40.4. Since the incusion $\mathcal{A} \subset {}^\perp(\mathcal{A}^\perp)$ is clear it suffices to prove the other inclusion. Let X be an object of ${}^\perp(\mathcal{A}^\perp)$. Choose a distinguished triangle $A \to X \to B \to A[1]$ as in (2). As $\mathrm{Hom}(X, B) = 0$ by assumption we see that $A \cong X \oplus B[-1]$ by Lemma 4.11. Since $\mathrm{Hom}(A, B[-1]) = 0$ as $B \in \mathcal{A}^\perp$ this implies $B[-1] = 0$ and $A \cong X$ as desired. □

Lemma 40.8. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B} \subset \mathcal{D}$ be a full triangulated subcategory. The following are equivalent*

1. *the inclusion functor $\mathcal{B} \to \mathcal{D}$ has a left adjoint, and*
2. *for every X in $\mathcal{D}$ there exists a distinguished triangle*
$$A \to X \to B \to A[1]$$
in $\mathcal{D}$ with $B \in \mathrm{Ob}(\mathcal{B})$ and $A \in \mathrm{Ob}({}^\perp\mathcal{B})$.

If this holds, then $\mathcal{B}$ is saturated (Definition 6.1) and if $\mathcal{B}$ is strictly full in $\mathcal{D}$, then $\mathcal{B} = ({}^\perp\mathcal{B})^\perp$.

Proof. Dual to Lemma 40.7. □

Definition 40.9. Let $\mathcal{D}$ be a triangulated category. A *right admissible* subcategory of $\mathcal{D}$ is a strictly full triangulated subcategory satisfying the equivalent conditions of Lemma 40.7. A *left admissible* subcategory of $\mathcal{D}$ is a strictly full triangulated subcategory satisfying the equivalent conditions of Lemma 40.8. A *two-sided admissible* subcategory is one which is both right and left admissible.

Let $\mathcal{A}$ be a right admissible subcategory of the triangulated category $\mathcal{D}$. Then we observe that for $X \in \mathcal{D}$ the distinguished triangle

$$A \to X \to B \to A[1]$$

with $A \in \mathcal{A}$ and $B \in \mathcal{A}^\perp$ is canonical in the following sense: for any other distinguished triangle $A' \to X \to B' \to A'[1]$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{A}^\perp$ there is an isomorphism $(\alpha, \mathrm{id}_X, \beta) : (A, X, B) \to (A', X, B')$ of triangles. The following proposition summarizes what was said above.

Proposition 40.10. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ and $\mathcal{B} \subset \mathcal{D}$ be subcategories. The following are equivalent*

(1) *$\mathcal{A}$ is right admissible and $\mathcal{B} = \mathcal{A}^\perp$,*
(2) *$\mathcal{B}$ is left admissible and $\mathcal{A} = {}^\perp\mathcal{B}$,*
(3) $\mathrm{Hom}(A, B) = 0$ *for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and for every X in $\mathcal{D}$ there exists a distinguished triangle $A \to X \to B \to A[1]$ in $\mathcal{D}$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

If this is true, then $\mathcal{A} \to \mathcal{D}/\mathcal{B}$ and $\mathcal{B} \to \mathcal{D}/\mathcal{A}$ are equivalences of triangulated categories, the right adjoint to the inclusion functor $\mathcal{A} \to \mathcal{D}$ is $\mathcal{D} \to \mathcal{D}/\mathcal{B} \to \mathcal{A}$, and the left adjoint to the inclusion functor $\mathcal{B} \to \mathcal{D}$ is $\mathcal{D} \to \mathcal{D}/\mathcal{A} \to \mathcal{B}$.

Proof. The equivalence between (1), (2), and (3) follows in a straighforward manner from Lemmas 40.7 and 40.8 (small detail omitted). Denote $v : \mathcal{D} \to \mathcal{A}$ the right adjoint of the inclusion functor $i : \mathcal{A} \to \mathcal{D}$. It is immediate that $\mathrm{Ker}(v) = \mathcal{A}^\perp = \mathcal{B}$. Thus v factors over a functor $\overline{v} : \mathcal{D}/\mathcal{B} \to \mathcal{A}$ by the universal property of the quotient. Since $v \circ i = \mathrm{id}_\mathcal{A}$ by Categories, Lemma 24.4 we see that $\overline{v}$ is a left quasi-inverse to $\overline{i} : \mathcal{A} \to \mathcal{D}/\mathcal{B}$. We claim also the composition $\overline{i} \circ \overline{v}$ is isomorphic to $\mathrm{id}_{\mathcal{D}/\mathcal{B}}$. Namely, suppose we have X fitting into a distinguished triangle $A \to X \to B \to A[1]$ as in (3). Then $v(X) = A$ as was seen in the proof of Lemma 40.7. Viewing X as an object of $\mathcal{D}/\mathcal{B}$ we have $\overline{i}(\overline{v}(X)) = A$ and there is a functorial isomorphism $\overline{i}(\overline{v}(X)) = A \to X$ in $\mathcal{D}/\mathcal{B}$. Thus we find that indeed $\overline{v} : \mathcal{D}/\mathcal{B} \to \mathcal{A}$ is an equivalence. To show that $\mathcal{B} \to \mathcal{D}/\mathcal{A}$ is an equivalence and the left adjoint to the inclusion functor $\mathcal{B} \to \mathcal{D}$ is $\mathcal{D} \to \mathcal{D}/\mathcal{A} \to \mathcal{B}$ is dual to what we just said. ☐

41. Postnikov systems

A reference for this section is [Orl97]. Let $\mathcal{D}$ be a triangulated category. Let

$$X_n \to X_{n-1} \to \ldots \to X_0$$

be a complex in $\mathcal{D}$. In this section we consider the problem of constructing a "totalization" of this complex.

Definition 41.1. Let $\mathcal{D}$ be a triangulated category. Let

$$X_n \to X_{n-1} \to \ldots \to X_0$$

be a complex in $\mathcal{D}$. A *Postnikov system* is defined inductively as follows.

(1) If $n = 0$, then it is an isomorphism $Y_0 \to X_0$.
(2) If $n = 1$, then it is a choice of an isomorphism $Y_0 \to X_0$ and a choice of a distinguished triangle

$$Y_1 \to X_1 \to Y_0 \to Y_1[1]$$

where $X_1 \to Y_0$ composed with $Y_0 \to X_0$ is the given morphism $X_1 \to X_0$.
(3) If $n > 1$, then it is a choice of a Postnikov system for $X_{n-1} \to \ldots \to X_0$ and a choice of a distinguished triangle

$$Y_n \to X_n \to Y_{n-1} \to Y_n[1]$$

where the morphism $X_n \to Y_{n-1}$ composed with $Y_{n-1} \to X_{n-1}$ is the given morphism $X_n \to X_{n-1}$.

Given a morphism

$$
\begin{tikzcd}
X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & \ldots \arrow[r] & X_0 \arrow[d] \\
X'_n \arrow[r] & X'_{n-1} \arrow[r] & \ldots \arrow[r] & X'_0
\end{tikzcd}
\tag{41.1.1}
$$

between complexes of the same length in $\mathcal{D}$ there is an obvious notion of a *morphism of Postnikov systems*.

Here is a key example.

Example 41.2. Let $\mathcal{A}$ be an abelian category. Let $\ldots \to A_2 \to A_1 \to A_0$ be a chain complex in $\mathcal{A}$. Then we can consider the objects

$$X_n = A_n \quad \text{and} \quad Y_n = (A_n \to A_{n-1} \to \ldots \to A_0)[-n]$$

of $D(\mathcal{A})$. With the evident canonical maps $Y_n \to X_n$ and $Y_0 \to Y_1[1] \to Y_2[2] \to \ldots$ the distinguished triangles $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ define a Postnikov system as in Definition 41.1 for $\ldots \to X_2 \to X_1 \to X_0$. Here we are using the obvious extension of Postnikov systems for an infinite complex of $D(\mathcal{A})$. Finally, if colimits over $\mathbf{N}$ exist and are exact in $\mathcal{A}$ then

$$\text{hocolim} Y_n[n] = (\ldots \to A_2 \to A_1 \to A_0 \to 0 \to \ldots)$$

in $D(\mathcal{A})$. This follows immediately from Lemma 33.7.

Given a complex $X_n \to X_{n-1} \to \ldots \to X_0$ and a Postnikov system as in Definition 41.1 we can consider the maps

$$Y_0 \to Y_1[1] \to \ldots \to Y_n[n]$$

These maps fit together in certain distinguished triangles and fit with the given maps between the X_i. Here is a picture for $n = 3$:

$$
\begin{tikzcd}
Y_0 \arrow[rr] & & Y_1[1] \arrow[rr] \arrow[dl] & & Y_2[2] \arrow[rr] \arrow[dl] & & Y_3[3] \arrow[dl] \\
& X_1[1] \arrow[ul, "+1"'] & & X_2[2] \arrow[ll, "+1"'] \arrow[ul, "+1"'] & & X_3[3] \arrow[ll, "+1"'] \arrow[ul, "+1"'] &
\end{tikzcd}
$$

We encourage the reader to think of $Y_n[n]$ as obtained from $X_0, X_1[1], \ldots, X_n[n]$; for example if the maps $X_i \to X_{i-1}$ are zero, then we can take $Y_n[n] = \bigoplus_{i=0,\ldots,n} X_i[i]$. Postnikov systems do not always exist. Here is a simple lemma for low n.

Lemma 41.3. *Let $\mathcal{D}$ be a triangulated category. Consider Postnikov systems for complexes of length n.*

(1) *For $n = 0$ Postnikov systems always exist and any morphism (41.1.1) of complexes extends to a unique morphism of Postnikov systems.*
(2) *For $n = 1$ Postnikov systems always exist and any morphism (41.1.1) of complexes extends to a (nonunique) morphism of Postnikov systems.*
(3) *For $n = 2$ Postnikov systems always exist but morphisms (41.1.1) of complexes in general do not extend to morphisms of Postnikov systems.*
(4) *For $n > 2$ Postnikov systems do not always exist.*

Proof. The case $n = 0$ is immediate as isomorphisms are invertible. The case $n = 1$ follows immediately from TR1 (existence of triangles) and TR3 (extending morphisms to triangles). For the case $n = 2$ we argue as follows. Set $Y_0 = X_0$. By the case $n = 1$ we can choose a Postnikov system

$$Y_1 \to X_1 \to Y_0 \to Y_1[1]$$

Since the composition $X_2 \to X_1 \to X_0$ is zero, we can factor $X_2 \to X_1$ (nonuniquely) as $X_2 \to Y_1 \to X_1$ by Lemma 4.2. Then we simply fit the morphism $X_2 \to Y_1$ into a distinguished triangle

$$Y_2 \to X_2 \to Y_1 \to Y_2[1]$$

to get the Postnikov system for $n = 2$. For $n > 2$ we cannot argue similarly, as we do not know whether the composition $X_n \to X_{n-1} \to Y_{n-1}$ is zero in $\mathcal{D}$. $\square$

Lemma 41.4. *Let $\mathcal{D}$ be a triangulated category. Given a map (41.1.1) consider the condition*

$$\text{Hom}(X_i[i-j-1], X'_j) = 0 \textit{ for } i > j+1 \quad (41.4.1)$$

Then

(1) *If we have a Postnikov system for $X'_n \to X'_{n-1} \to \ldots \to X'_0$ then property (41.4.1) implies that*
$$\text{Hom}(X_i[i-j-1], Y'_j) = 0 \textit{ for } i > j+1$$
(2) *If we are given Postnikov systems for both complexes and we have (41.4.1), then the map extends to a (nonunique) map of Postnikov systems.*

Proof. We first prove (1) by induction on j. For the base case $j = 0$ there is nothing to prove as $Y'_0 \to X'_0$ is an isomorphism. Say the result holds for $j-1$. We consider the distinguished triangle

$$Y'_j \to X'_j \to Y'_{j-1} \to Y'_j[1]$$

The long exact sequence of Lemma 4.2 gives an exact sequence

$$\text{Hom}(X_i[i-j-1], Y'_{j-1}[-1]) \to \text{Hom}(X_i[i-j-1], Y'_j) \to \text{Hom}(X_i[i-j-1], X'_j)$$

From the induction hypothesis and (41.4.1) we conclude the outer groups are zero and we win.

Proof of (2). For $n = 1$ the existence of morphisms has been established in Lemma 41.3. For $n > 1$ by induction, we may assume given the map of Postnikov systems of length $n-1$. The problem is that we do not know whether the diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\ X'_n \arrow[r] & Y'_{n-1} \end{tikzcd}$$

is commutative. Denote $\alpha : X_n \to Y'_{n-1}$ the difference. Then we do know that the composition of α with $Y'_{n-1} \to X'_{n-1}$ is zero (because of what it means to be a map of Postnikov systems of length $n-1$). By the distinguished triangle $Y'_{n-1} \to X'_{n-1} \to Y'_{n-2} \to Y'_{n-1}[1]$, this means that α is the composition of $Y'_{n-2}[-1] \to Y'_{n-1}$ with a map $\alpha' : X_n \to Y'_{n-2}[-1]$. Then (41.4.1) guarantees α' is zero by part (1) of the lemma. Thus α is zero. To finish the proof of existence, the commutativity guarantees we can choose the dotted arrow fitting into the diagram

$$\begin{tikzcd} Y_{n-1}[-1] \arrow[r] \arrow[d] & Y_n \arrow[r] \arrow[d, dotted] & X_n \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\ Y'_{n-1}[-1] \arrow[r] & Y'_n \arrow[r] & X'_n \arrow[r] & Y'_{n-1} \end{tikzcd}$$

by TR3. $\square$

Lemma 41.5. *Let $\mathcal{D}$ be a triangulated category. Given a map (41.1.1) assume we are given Postnikov systems for both complexes. If*

(1) $\mathrm{Hom}(X_i[i], Y'_n[n]) = 0$ *for* $i = 1, \ldots, n$, *or*
(2) $\mathrm{Hom}(Y_n[n], X'_{n-i}[n-i]) = 0$ *for* $i = 1, \ldots, n$, *or*
(3) $\mathrm{Hom}(X_{j-i}[-i+1], X'_j) = 0$ *and* $\mathrm{Hom}(X_j, X'_{j-i}[-i]) = 0$ *for* $j \geq i > 0$,

then there exists at most one morphism between these Postnikov systems.

Proof. Proof of (1). Look at the following diagram

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & Y_1[1] \arrow[r] \arrow[dl] & Y_2[2] \arrow[r] \arrow[dll] & \ldots \arrow[r] & Y_n[n] \arrow[dllll] \\ Y'_n[n] \end{tikzcd}$$

The arrows are the composition of the morphism $Y_n[n] \to Y'_n[n]$ and the morphism $Y_i[i] \to Y_n[n]$. The arrow $Y_0 \to Y'_n[n]$ is determined as it is the composition $Y_0 = X_0 \to X'_0 = Y'_0 \to Y'_n[n]$. Since we have the distinguished triangle $Y_0 \to Y_1[1] \to X_1[1]$ we see that $\mathrm{Hom}(X_1[1], Y'_n[n]) = 0$ guarantees that the second vertical arrow is unique. Since we have the distinguished triangle $Y_1[1] \to Y_2[2] \to X_2[2]$ we see that $\mathrm{Hom}(X_2[2], Y'_n[n]) = 0$ guarantees that the third vertical arrow is unique. And so on.

Proof of (2). The composition $Y_n[n] \to Y'_n[n] \to X_n[n]$ is the same as the composition $Y_n[n] \to X_n[n] \to X'_n[n]$ and hence is unique. Then using the distinguished triangle $Y'_{n-1}[n-1] \to Y'_n[n] \to X'_n[n]$ we see that it suffices to show $\mathrm{Hom}(Y_n[n], Y'_{n-1}[n-1]) = 0$. Using the distinguished triangles

$$Y'_{n-i-1}[n-i-1] \to Y'_{n-i}[n-i] \to X'_{n-i}[n-i]$$

we get this vanishing from our assumption. Small details omitted.

Proof of (3). Looking at the proof of Lemma 41.4 and arguing by induction on n it suffices to show that the dotted arrow in the morphism of triangles

$$\begin{tikzcd} Y_{n-1}[-1] \arrow[r] \arrow[d] & Y_n \arrow[r] \arrow[d, dotted] & X_n \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\ Y'_{n-1}[-1] \arrow[r] & Y'_n \arrow[r] & X'_n \arrow[r] & Y'_{n-1} \end{tikzcd}$$

is unique. By Lemma 4.8 part (5) it suffices to show that $\mathrm{Hom}(Y_{n-1}, X'_n) = 0$ and $\mathrm{Hom}(X_n, Y'_{n-1}[-1]) = 0$. To prove the first vanishing we use the distinguished triangles $Y_{n-i-1}[-i] \to Y_{n-i}[-(i-1)] \to X_{n-i}[-(i-1)]$ for $i > 0$ and induction on i to see that the assumed vanishing of $\mathrm{Hom}(X_{n-i}[-i+1], X'_n)$ is enough. For the second we similarly use the distinguished triangles $Y'_{n-i-1}[-i-1] \to Y'_{n-i}[-i] \to X'_{n-i}[-i]$ to see that the assumed vanishing of $\text{Hom}(X_n, X'_{n-i}[-i])$ is enough as well. $\square$

Lemma 41.6. *Let $\mathcal{D}$ be a triangulated category. Let $X_n \to X_{n-1} \to \ldots \to X_0$ be a complex in $\mathcal{D}$. If*

$$\text{Hom}(X_i[i-j-2], X_j) = 0 \textit{ for } i > j+2$$

then there exists a Postnikov system. If we have

$$\text{Hom}(X_i[i-j-1], X_j) = 0 \textit{ for } i > j+1$$

then any two Postnikov systems are isomorphic.

Proof. We argue by induction on n. The cases $n = 0, 1, 2$ follow from Lemma 41.3. Assume $n > 2$. Suppose given a Postnikov system for the complex $X_{n-1} \to X_{n-2} \to \ldots \to X_0$. The only obstruction to extending this to a Postnikov system of length n is that we have to find a morphism $X_n \to Y_{n-1}$ such that the composition $X_n \to Y_{n-1} \to X_{n-1}$ is equal to the given map $X_n \to X_{n-1}$. Considering the distinguished triangle

$$Y_{n-1} \to X_{n-1} \to Y_{n-2} \to Y_{n-1}[1]$$

and the associated long exact sequence coming from this and the functor $\text{Hom}(X_n, -)$ (see Lemma 4.2) we find that it suffices to show that the composition $X_n \to X_{n-1} \to Y_{n-2}$ is zero. Since we know that $X_n \to X_{n-1} \to X_{n-2}$ is zero we can apply the distinguished triangle

$$Y_{n-2} \to X_{n-2} \to Y_{n-3} \to Y_{n-2}[1]$$

to see that it suffices if $\text{Hom}(X_n, Y_{n-3}[-1]) = 0$. Arguing exactly as in the proof of Lemma 41.4 part (1) the reader easily sees this follows from the condition stated in the lemma.

The statement on isomorphisms follows from the existence of a map between the Postnikov systems extending the identity on the complex proven in Lemma 41.4 part (2) and Lemma 4.3 to show all the maps are isomorphisms. $\square$

42. Essentially constant systems

Some preliminary lemmas on essentially constant systems in triangulated categories.

Lemma 42.1. *Let $\mathcal{D}$ be a triangulated category. Let (A_i) be an inverse system in $\mathcal{D}$. Then (A_i) is essentially constant (see Categories, Definition 22.1) if and only if there exists an i and for all $j \geq i$ a direct sum decomposition $A_j = A \oplus Z_j$ such that (a) the maps $A_{j'} \to A_j$ are compatible with the direct sum decompositions and identity on A, (b) for all $j \geq i$ there exists some $j' \geq j$ such that $Z_{j'} \to Z_j$ is zero.*

Proof. Assume (A_i) is essentially constant with value A. Then $A = \lim A_i$ and there exists an i and a morphism $A_i \to A$ such that (1) the composition $A \to A_i \to A$ is the identity on A and (2) for all $j \geq i$ there exists a $j' \geq j$ such that $A_{j'} \to A_j$ factors as $A_{j'} \to A_i \to A \to A_j$. From (1) we conclude that for $j \geq i$ the maps $A \to A_j$ and $A_j \to A_i \to A$ compose to the identity on A. It follows that $A_j \to A$ has a kernel Z_j and that the map $A \oplus Z_j \to A_j$ is an isomorphism, see Lemmas 4.12 and 4.11. These direct sum decompositions clearly satisfy (a). From (2) we conclude that for all j there is a $j' \geq j$ such that $Z_{j'} \to Z_j$ is zero, so (b) holds. Proof of the converse is omitted. $\square$

Lemma 42.2. *Let $\mathcal{D}$ be a triangulated category. Let*

$$A_n \to B_n \to C_n \to A_n[1]$$

be an inverse system of distinguished triangles in $\mathcal{D}$. If (A_n) and (C_n) are essentially constant, then (B_n) is essentially constant and their values fit into a distinguished triangle $A \to B \to C \to A[1]$ such that for some $n \geq 1$ there is a map

$$\begin{tikzcd} A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & A_n[1] \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1] \end{tikzcd}$$

of distinguished triangles which induces an isomorphism $\lim_{n' \geq n} A_{n'} \to A$ and similarly for B and C.

Proof. After renumbering we may assume that $A_n = A \oplus A'_n$ and $C_n = C \oplus C'_n$ for inverse systems (A'_n) and (C'_n) which are essentially zero, see Lemma 42.1. In particular, the morphism

$$C \oplus C'_n \to (A \oplus A'_n)[1]$$

maps the summand C into the summand $A[1]$ for all n by a map $\delta : C \to A[1]$ which is independent of n. Choose a distinguished triangle

$$A \to B \to C \xrightarrow{\delta} A[1]$$

Next, choose a morphism of distingished triangles

$$(A_1 \to B_1 \to C_1 \to A_1[1]) \to (A \to B \to C \to A[1])$$

which is possible by TR3. For any object D of $\mathcal{D}$ this induces a commutative diagram

$$\begin{tikzcd} \ldots \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(C, D) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(B, D) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(A, D) \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & \mathrm{colim}\, \mathrm{Hom}_{\mathcal{D}}(C_n, D) \arrow[r] & \mathrm{colim}\, \mathrm{Hom}_{\mathcal{D}}(B_n, D) \arrow[r] & \mathrm{colim}\, \mathrm{Hom}_{\mathcal{D}}(A_n, D) \arrow[r] & \ldots \end{tikzcd}$$

The left and right vertical arrows are isomorphisms and so are the ones to the left and right of those. Thus by the 5-lemma we conclude that the middle arrow is an isomorphism. It follows that (B_n) is isomorphic to the constant inverse system with value B by the discussion in Categories, Remark 22.7. Since this is equivalent to (B_n) being essentially constant with value B by Categories, Remark 22.5 the proof is complete. $\square$

Lemma 42.3. *Let $\mathcal{A}$ be an abelian category. Let A_n be an inverse system of objects of $D(\mathcal{A})$. Assume*

1. *there exist integers $a \leq b$ such that $H^i(A_n) = 0$ for $i \notin [a, b]$, and*
2. *the inverse systems $H^i(A_n)$ of $\mathcal{A}$ are essentially constant for all $i \in \mathbf{Z}$.*

Then A_n is an essentially constant system of $D(\mathcal{A})$ whose value A satisfies that $H^i(A)$ is the value of the constant system $H^i(A_n)$ for each $i \in \mathbf{Z}$.

Proof. By Remark 12.4 we obtain an inverse system of distinguished triangles

$$\tau_{\leq a}A_n \to A_n \to \tau_{\geq a+1}A_n \to (\tau_{\leq a}A_n)[1]$$

Of course we have $\tau_{\leq a}A_n = H^a(A_n)[-a]$ in $D(\mathcal{A})$. Thus by assumption these form an essentially constant system. By induction on $b - a$ we find that the inverse system $\tau_{\geq a+1}A_n$ is essentially constant, say with value A'. By Lemma 42.2 we find that A_n is an essentially constant system. We omit the proof of the statement on cohomologies (hint: use the final part of Lemma 42.2). $\square$

Lemma 42.4. *Let $\mathcal{D}$ be a triangulated category. Let*

$$A_n \to B_n \to C_n \to A_n[1]$$

be an inverse system of distinguished triangles. If the system C_n is pro-zero (essentially constant with value 0), then the maps $A_n \to B_n$ determine a pro-isomorphism between the pro-object (A_n) and the pro-object (B_n).

Proof. For any object X of $\mathcal{D}$ consider the exact sequence

$$\operatorname{colim} \operatorname{Hom}(C_n, X) \to \operatorname{colim} \operatorname{Hom}(B_n, X) \to \operatorname{colim} \operatorname{Hom}(A_n, X) \to \operatorname{colim} \operatorname{Hom}(C_n[-1], X) \to$$

Exactness follows from Lemma 4.2 combined with Algebra, Lemma 8.8. By assumption the first and last term are zero. Hence the map $\operatorname{colim} \operatorname{Hom}(B_n, X) \to \operatorname{colim} \operatorname{Hom}(A_n, X)$ is an isomorphism for all X. The lemma follows from this and Categories, Remark 22.7. $\square$

Lemma 42.5. *Let $\mathcal{A}$ be an abelian category.*

$$A_n \to B_n$$

be an inverse system of maps of $D(\mathcal{A})$. Assume

1. *there exist integers $a \leq b$ such that $H^i(A_n) = 0$ and $H^i(B_n) = 0$ for $i \notin [a, b]$, and*
2. *the inverse system of maps $H^i(A_n) \to H^i(B_n)$ of $\mathcal{A}$ define an isomorphism of pro-objects of $\mathcal{A}$ for all $i \in \mathbf{Z}$.*

Then the maps $A_n \to B_n$ determine a pro-isomorphism between the pro-object (A_n) and the pro-object (B_n).

Proof. We can inductively extend the maps $A_n \to B_n$ to an inverse system of distinguished triangles $A_n \to B_n \to C_n \to A_n[1]$ by axiom TR3. By Lemma 42.4 it suffices to prove that C_n is pro-zero. By Lemma 42.3 it suffices to show that $H^p(C_n)$ is pro-zero for each p. This follows from assumption (2) and the long exact sequences

$$H^p(A_n) \xrightarrow{\alpha_n} H^p(B_n) \xrightarrow{\beta_n} H^p(C_n) \xrightarrow{\delta_n} H^{p+1}(A_n) \xrightarrow{\epsilon_n} H^{p+1}(B_n)$$

Namely, for every n we can find an $m > n$ such that $\operatorname{Im}(\beta_m)$ maps to zero in $H^p(C_n)$ because we may choose m such that $H^p(B_m) \to H^p(B_n)$ factors through $\alpha_n : H^p(A_n) \to H^p(B_n)$. For a similar reason we may then choose $k > m$ such that $\operatorname{Im}(\delta_k)$ maps to zero in $H^{p+1}(A_m)$. Then $H^p(C_k) \to H^p(C_n)$ is zero because $H^p(C_k) \to H^p(C_m)$ maps into $\operatorname{Ker}(\delta_m)$ and $H^p(C_m) \to H^p(C_n)$ annihilates $\operatorname{Ker}(\delta_m) = \operatorname{Im}(\beta_m)$. $\square$

43. Other chapters

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