VARIETIES

Contents

1. Introduction

In this chapter we start studying varieties and more generally schemes over a field. A fundamental reference is [DG67].

2. Notation

Throughout this chapter we use the letter k to denote the ground field.

3. Varieties

In the Stacks project we will use the following as our definition of a variety.

Definition 3.1. Let k be a field. A *variety* is a scheme X over k such that X is integral and the structure morphism $X \to \mathrm{Spec}(k)$ is separated and of finite type.

This definition has the following drawback. Suppose that k'/k is an extension of fields. Suppose that X is a variety over k. Then the base change $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is not necessarily a variety over k'. This phenomenon (in greater generality) will be discussed in detail in the following sections. The product of two varieties need not be a variety (this is really the same phenomenon). Here is an example.

Example 3.2. Let $k = \mathbf{Q}$. Let $X = \mathrm{Spec}(\mathbf{Q}(i))$ and $Y = \mathrm{Spec}(\mathbf{Q}(i))$. Then the product $X \times_{\mathrm{Spec}(k)} Y$ of the varieties X and Y is not a variety, since it is reducible. (It is isomorphic to the disjoint union of two copies of X.)

If the ground field is algebraically closed however, then the product of varieties is a variety. This follows from the results in the algebra chapter, but there we treat much more general situations. There is also a simple direct proof of it which we present here.

Lemma 3.3. *Let k be an algebraically closed field. Let X, Y be varieties over k. Then $X \times_{\mathrm{Spec}(k)} Y$ is a variety over k.*

Proof. The morphism $X \times_{\mathrm{Spec}(k)} Y \to \mathrm{Spec}(k)$ is of finite type and separated because it is the composition of the morphisms $X \times_{\mathrm{Spec}(k)} Y \to Y \to \mathrm{Spec}(k)$ which are separated and of finite type, see Morphisms, Lemmas 15.4 and 15.3 and Schemes, Lemma 21.12. To finish the proof it suffices to show that $X \times_{\mathrm{Spec}(k)} Y$ is integral. Let $X = \bigcup_{i=1,\ldots,n} U_i$, $Y = \bigcup_{j=1,\ldots,m} V_j$ be finite affine open coverings. If we can show that each $U_i \times_{\mathrm{Spec}(k)} V_j$ is integral, then we are done by Properties, Lemmas 3.2, 3.3, and 3.4. This reduces us to the affine case.

The affine case translates into the following algebra statement: Suppose that A, B are integral domains and finitely generated k-algebras. Then $A \otimes_k B$ is an integral domain. To get a contradiction suppose that

$$(\sum\nolimits_{i=1,\ldots,n} a_i \otimes b_i)(\sum\nolimits_{j=1,\ldots,m} c_j \otimes d_j) = 0$$

in $A \otimes_k B$ with both factors nonzero in $A \otimes_k B$. We may assume that $b_1, \ldots, b_n$ are k-linearly independent in B, and that $d_1, \ldots, d_m$ are k-linearly independent in B. Of course we may also assume that a_1 and c_1 are nonzero in A. Hence $D(a_1 c_1) \subset \mathrm{Spec}(A)$ is nonempty. By the Hilbert Nullstellensatz (Algebra, Theorem 34.1) we can find a maximal ideal $\mathfrak{m} \subset A$ contained in $D(a_1 c_1)$ and $A/\mathfrak{m} = k$ as k is algebraically closed. Denote $\bar{a}_i, \bar{c}_j$ the residue classes of a_i, c_j in $A/\mathfrak{m} = k$. The equation above becomes

$$(\sum\nolimits_{i=1,\ldots,n} \bar{a}_i b_i)(\sum\nolimits_{j=1,\ldots,m} \bar{c}_j d_j) = 0$$

which is a contradiction with $\mathfrak{m} \in D(a_1 c_1)$, the linear independence of $b_1, \ldots, b_n$ and $d_1, \ldots, d_m$, and the fact that B is a domain. $\square$

4. Varieties and rational maps

Let k be a field. Let X and Y be varieties over k. We will use the phrase *rational map of varieties from X to Y* to mean a $\mathrm{Spec}(k)$-rational map from the scheme X to the scheme Y as defined in Morphisms, Definition 49.1. As is customary, the phrase "rational map of varieties" does not refer to the (common) base field of the varieties, even though for general schemes we make the distinction between rational maps and rational maps over a given base.

The title of this section refers to the following fundamental theorem.

Theorem 4.1. *Let k be a field. The category of varieties and dominant rational maps is equivalent to the category of finitely generated field extensions K/k.*

Proof. Let X and Y be varieties with generic points $x \in X$ and $y \in Y$. Recall that dominant rational maps from X to Y are exactly those rational maps which map x to y (Morphisms, Definition 49.10 and discussion following). Thus given a dominant rational map $X \supset U \to Y$ we obtain a map of function fields

$$k(Y) = \kappa(y) = \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x} = \kappa(x) = k(X)$$

Conversely, such a k-algebra map (which is automatically local as the source and target are fields) determines (uniquely) a dominant rational map by Morphisms, Lemma 49.2. In this way we obtain a fully faithful functor. To finish the proof it suffices to show that every finitely generated field extension K/k is in the essential image. Since K/k is finitely generated, there exists a finite type k-algebra $A \subset K$ such that K is the fraction field of A. Then $X = \mathrm{Spec}(A)$ is a variety whose function field is K. $\square$

Let k be a field. Let X and Y be varieties over k. We will use the phrase X
and Y are birational varieties to mean X and Y are $\mathrm{Spec}(k)$-birational as defined
in Morphisms, Definition 50.1. As is customary, the phrase "birational varieties"
does not refer to the (common) base field of the varieties, even though for general
irreducible schemes we make the distinction between being birational and being
birational over a given base.

Lemma 4.2. *Let X and Y be varieties over a field k. The following are equivalent*

(1) *X and Y are birational varieties,*
(2) *the function fields $k(X)$ and $k(Y)$ are isomorphic,*
(3) *there exist nonempty opens of X and Y which are isomorphic as varieties,*
(4) *there exists an open $U \subset X$ and a birational morphism $U \to Y$ of varieties.*

Proof. This is a special case of Morphisms, Lemma 50.6. $\square$

5. Change of fields and local rings

Some preliminary results on what happens to local rings under an extension of
ground fields.

Lemma 5.1. *Let K/k be an extension of fields. Let X be scheme over k and set
$Y = X_K$. If $y \in Y$ with image $x \in X$, then*

(1) *$\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a faithfully flat local ring homomorphism,*
(2) *with $\mathfrak{p}_0 = \mathrm{Ker}(\kappa(x) \otimes_k K \to \kappa(y))$ we have $\kappa(y) = \kappa(\mathfrak{p}_0)$,*
(3) *$\mathcal{O}_{Y,y} = (\mathcal{O}_{X,x} \otimes_k K)_\mathfrak{p}$ where $\mathfrak{p} \subset \mathcal{O}_{X,x} \otimes_k K$ is the inverse image of $\mathfrak{p}_0$.*
(4) *we have $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y} = (\kappa(x) \otimes_k K)_{\mathfrak{p}_0}$*

Proof. We may assume $X = \mathrm{Spec}(A)$ is affine. Then $Y = \mathrm{Spec}(A \otimes_k K)$. Since K
is flat over k, we see that $A \to A \otimes_k K$ is flat. Hence $Y \to X$ is flat and we get the
first statement if we also use Algebra, Lemma 39.17. The second statement follows
from Schemes, Lemma 17.5. Now y corresponds to a prime ideal $\mathfrak{q} \subset A \otimes_k K$ and
x to $\mathfrak{r} = A \cap \mathfrak{q}$. Then $\mathfrak{p}_0$ is the kernel of the induced map $\kappa(\mathfrak{r}) \otimes_k K \to \kappa(\mathfrak{q})$. The
map on local rings is
$$A_\mathfrak{r} \longrightarrow (A \otimes_k K)_\mathfrak{q}$$
We can factor this map through $A_\mathfrak{r} \otimes_k K = (A \otimes_k K)_\mathfrak{r}$ to get
$$A_\mathfrak{r} \longrightarrow A_\mathfrak{r} \otimes_k K \longrightarrow (A \otimes_k K)_\mathfrak{q}$$
and then the second arrow is a localization at some prime. This prime ideal is the
inverse image of $\mathfrak{p}_0$ (details omitted) and this proves (3). To see (4) use (3) and
that localization and $- \otimes_k K$ are exact functors. $\square$

Lemma 5.2. *Notation as in Lemma 5.1. Assume X is locally of finite type over
k. Then*
$$\dim(\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}) = trdeg_k(\kappa(x)) - trdeg_K(\kappa(y)) = \dim(\mathcal{O}_{Y,y}) - \dim(\mathcal{O}_{X,x})$$

Proof. This is a restatement of Algebra, Lemma 116.7. $\square$

Lemma 5.3. *Notation as in Lemma 5.1. Assume X is locally of finite type over k,
that $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$ and that $\kappa(x) \otimes_k K$ is reduced (for example if $\kappa(x)/k$
is separable or K/k is separable). Then $\mathfrak{m}_x\mathcal{O}_{Y,y} = \mathfrak{m}_y$.*

Proof. (The parenthetical statement follows from Algebra, Lemma 43.6.) Combin-
ing Lemmas 5.1 and 5.2 we see that $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ has dimension 0 and is reduced.
Hence it is a field. $\square$

6. Geometrically reduced schemes

If X is a reduced scheme over a field, then it can happen that X becomes nonreduced after extending the ground field. This does not happen for geometrically reduced schemes.

Definition 6.1. Let k be a field. Let X be a scheme over k.
  (1) Let $x \in X$ be a point. We say X is *geometrically reduced at x* if for any field extension k'/k and any point $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is reduced.
  (2) We say X is *geometrically reduced* over k if X is geometrically reduced at every point of X.

This may seem a little mysterious at first, but it is really the same thing as the notion discussed in the algebra chapter. Here are some basic results explaining the connection.

Lemma 6.2. *Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent*
  (1) *X is geometrically reduced at x, and*
  (2) *the ring $\mathcal{O}_{X,x}$ is geometrically reduced over k (see Algebra, Definition 43.1).*

Proof. Assume (1). This in particular implies that $\mathcal{O}_{X,x}$ is reduced. Let k'/k be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 46.7 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 18.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a reduced ring. Hence we deduce (2) by Algebra, Lemma 44.3.

Assume (2). Let k'/k be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is reduced by assumption and (1) is proved. $\square$

The notion isn't interesting in characteristic zero.

Lemma 6.3. *Let X be a scheme over a perfect field k (e.g. k has characteristic zero). Let $x \in X$. If $\mathcal{O}_{X,x}$ is reduced, then X is geometrically reduced at x. If X is reduced, then X is geometrically reduced over k.*

Proof. The first statement follows from Lemma 6.2 and Algebra, Lemma 43.6 and the definition of a perfect field (Algebra, Definition 45.1). The second statement follows from the first. $\square$

Lemma 6.4. *Let k be a field of characteristic $p > 0$. Let X be a scheme over k. The following are equivalent*
  (1) *X is geometrically reduced,*
  (2) *$X_{k'}$ is reduced for every field extension k'/k,*
  (3) *$X_{k'}$ is reduced for every finite purely inseparable field extension k'/k,*
  (4) *$X_{k^{1/p}}$ is reduced,*
  (5) *$X_{k^{perf}}$ is reduced,*
  (6) *$X_{\bar{k}}$ is reduced,*
  (7) *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically reduced (see Algebra, Definition 43.1).*

Proof. Assume (1). Then for every field extension k'/k and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is reduced. In other words $X_{k'}$ is reduced. Hence (2).

Assume (2). Let $U \subset X$ be an affine open. Then for every field extension k'/k the scheme $X_{k'}$ is reduced, hence $U_{k'} = \mathrm{Spec}(\mathcal{O}(U) \otimes_k k')$ is reduced, hence $\mathcal{O}(U) \otimes_k k'$ is reduced (see Properties, Section 3). In other words $\mathcal{O}(U)$ is geometrically reduced, so (7) holds.

Assume (7). For any field extension k'/k the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 17). Hence $X_{k'}$ is reduced. So (1) holds.

This proves that (1), (2), and (7) are equivalent. These are equivalent to (3), (4), (5), and (6) because we can apply Algebra, Lemma 44.3 to $\mathcal{O}_X(U)$ for $U \subset X$ affine open. $\square$

Lemma 6.5. *Let k be a field of characteristic $p > 0$. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

    (1) *X is geometrically reduced at x,*
    (2) *$\mathcal{O}_{X_{k'},x'}$ is reduced for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ the unique point lying over x,*
    (3) *$\mathcal{O}_{X_{k^{1/p}},x'}$ is reduced for $x' \in X_{k^{1/p}}$ the unique point lying over x, and*
    (4) *$\mathcal{O}_{X_{k^{perf}},x'}$ is reduced for $x' \in X_{k^{perf}}$ the unique point lying over x.*

Proof. Note that if k'/k is purely inseparable, then $X_{k'} \to X$ induces a homeomorphism on underlying topological spaces, see Algebra, Lemma 46.7. Whence the uniqueness of x' lying over x mentioned in the statement. Moreover, in this case $\mathcal{O}_{X_{k'},x'} = \mathcal{O}_{X,x} \otimes_k k'$. Hence the lemma follows from Lemma 6.2 above and Algebra, Lemma 44.3. $\square$

Lemma 6.6. *Let k be a field. Let X be a scheme over k. Let k'/k be a field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

    (1) *X is geometrically reduced at x,*
    (2) *$X_{k'}$ is geometrically reduced at x'.*

In particular, X is geometrically reduced over k if and only if $X_{k'}$ is geometrically reduced over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let k''/k be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common field extension k'''/k (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x'''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is reduced. Thus by Algebra, Lemma 164.2 we conclude that $\mathcal{O}_{X_{k''},x''}$ is reduced. Thus by Lemma 6.5 we conclude that X is geometrically reduced at x. $\square$

Lemma 6.7. *Let k be a field. Let X, Y be schemes over k.*

    (1) *If X is geometrically reduced at x, and Y reduced, then $X \times_k Y$ is reduced at every point lying over x.*

(2) *If X geometrically reduced over k and Y reduced. Then $X \times_k Y$ is reduced.*

Proof. Combine, Lemmas 6.2 and 6.4 and Algebra, Lemma 43.5. $\qquad\square$

Lemma 6.8. *Let k be a field. Let X be a scheme over k.*

(1) *If $x' \rightsquigarrow x$ is a specialization and X is geometrically reduced at x, then X is geometrically reduced at x'.*

(2) *If $x \in X$ such that (a) $\mathcal{O}_{X,x}$ is reduced, and (b) for each specialization $x' \rightsquigarrow x$ where x' is a generic point of an irreducible component of X the scheme X is geometrically reduced at x', then X is geometrically reduced at x.*

(3) *If X is reduced and geometrically reduced at all generic points of irreducible components of X, then X is geometrically reduced.*

Proof. Part (1) follows from Lemma 6.2 and the fact that if A is a geometrically reduced k-algebra, then $S^{-1}A$ is a geometrically reduced k-algebra for any multiplicative subset S of A, see Algebra, Lemma 43.3.

Let $A = \mathcal{O}_{X,x}$. The assumptions (a) and (b) of (2) imply that A is reduced, and that $A_\mathfrak{q}$ is geometrically reduced over k for every minimal prime $\mathfrak{q}$ of A. Hence A is geometrically reduced over k, see Algebra, Lemma 43.7. Thus X is geometrically reduced at x, see Lemma 6.2.

Part (3) follows trivially from part (2). $\qquad\square$

Lemma 6.9. *Let k be a field. Let X be a scheme over k. Let $x \in X$. Assume X locally Noetherian and geometrically reduced at x. Then there exists an open neighbourhood $U \subset X$ of x which is geometrically reduced over k.*

Proof. Assume X locally Noetherian and geometrically reduced at x. By Properties, Lemma 29.8 we can find an affine open neighbourhood $U \subset X$ of x such that $R = \mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is injective. By Lemma 6.2 the assumption means that $\mathcal{O}_{X,x}$ is geometrically reduced over k. By Algebra, Lemma 43.2 this implies that R is geometrically reduced over k, which in turn implies that U is geometrically reduced. $\qquad\square$

Example 6.10. Let $k = \mathbf{F}_p(s,t)$, i.e., a purely transcendental extension of the prime field. Consider the variety $X = \mathrm{Spec}(k[x,y]/(1 + sx^p + ty^p))$. Let k'/k be any extension such that both s and t have a pth root in k'. Then the base change $X_{k'}$ is not reduced. Namely, the ring $k'[x,y]/(1 + sx^p + ty^p)$ contains the element $1 + s^{1/p}x + t^{1/p}y$ whose pth power is zero but which is not zero (since the ideal $(1 + sx^p + ty^p)$ certainly does not contain any nonzero element of degree $< p$).

Lemma 6.11. *Let k be a field. Let $X \to \mathrm{Spec}(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite purely inseparable extension k'/k such that $(X_{k'})_{red}$ is geometrically reduced over k'.*

Proof. To prove this lemma we may replace X by its reduction X_{red}. Hence we may assume that X is reduced and locally of finite type over k. Let $x_1, \ldots, x_n \in X$ be the generic points of the irreducible components of X. Note that for every purely inseparable algebraic extension k'/k the morphism $(X_{k'})_{red} \to X$ is a homeomorphism, see Algebra, Lemma 46.7. Hence the points $x'_1, \ldots, x'_n$ lying over $x_1, \ldots, x_n$ are the generic points of the irreducible components of $(X_{k'})_{red}$. As X is reduced the local rings $K_i = \mathcal{O}_{X,x_i}$ are fields, see Algebra, Lemma 25.1. As X is locally of

finite type over k the field extensions K_i/k are finitely generated field extensions. Finally, the local rings $\mathcal{O}_{(X_{k'})_{red},x_i'}$ are the fields $(K_i \otimes_k k')_{red}$. By Algebra, Lemma 45.3 we can find a finite purely inseparable extension k'/k such that $(K_i \otimes_k k')_{red}$ are separable field extensions of k'. In particular each $(K_i \otimes_k k')_{red}$ is geometrically reduced over k' by Algebra, Lemma 44.1. At this point Lemma 6.8 part (3) implies that $(X_{k'})_{red}$ is geometrically reduced. $\qquad\square$

7. Geometrically connected schemes

If X is a connected scheme over a field, then it can happen that X becomes disconnected after extending the ground field. This does not happen for geometrically connected schemes.

Definition 7.1. Let X be a scheme over the field k. We say X is *geometrically connected* over k if the scheme $X_{k'}$ is connected for every field extension k' of k.

By convention a connected topological space is nonempty; hence a fortiori geometrically connected schemes are nonempty. Here is an example of a variety which is not geometrically connected.

Example 7.2. Let $k = \mathbf{Q}$. The scheme $X = \mathrm{Spec}(\mathbf{Q}(i))$ is a variety over $\mathrm{Spec}(\mathbf{Q})$. But the base change $X_{\mathbf{C}}$ is the spectrum of $\mathbf{C} \otimes_{\mathbf{Q}} \mathbf{Q}(i) \cong \mathbf{C} \times \mathbf{C}$ which is the disjoint union of two copies of $\mathrm{Spec}(\mathbf{C})$. So in fact, this is an example of a non-geometrically connected variety.

Lemma 7.3. *Let X be a scheme over the field k. Let k'/k be a field extension. Then X is geometrically connected over k if and only if $X_{k'}$ is geometrically connected over k'.*

Proof. If X is geometrically connected over k, then it is clear that $X_{k'}$ is geometrically connected over k'. For the converse, note that for any field extension k''/k there exists a common field extension k'''/k' and k'''/k''. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the connectedness of $X_{k'''}$ implies the connectedness of $X_{k''}$. Thus if $X_{k'}$ is geometrically connected over k' then X is geometrically connected over k. $\qquad\square$

Lemma 7.4. *Let k be a field. Let X, Y be schemes over k. Assume X is geometrically connected over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between connected components.

Proof. The scheme theoretic fibres of p are connected, since they are base changes of the geometrically connected scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 18.5. By Morphisms, Lemma 23.4 the map p is open. Thus we may apply Topology, Lemma 7.6 to conclude. $\qquad\square$

Lemma 7.5. *Let k be a field. Let A be a k-algebra. Then $X = \mathrm{Spec}(A)$ is geometrically connected over k if and only if A is geometrically connected over k (see Algebra, Definition 48.3).*

Proof. Immediate from the definitions. $\qquad\square$

Lemma 7.6. *Let k'/k be an extension of fields. Let X be a scheme over k. Assume k separably algebraically closed. Then the morphism $X_{k'} \to X$ induces a bijection of connected components. In particular, X is geometrically connected over k if and only if X is connected.*

Proof. Since k is separably algebraically closed we see that k' is geometrically connected over k, see Algebra, Lemma 48.4. Hence $Z = \operatorname{Spec}(k')$ is geometrically connected over k by Lemma 7.5 above. Since $X_{k'} = Z \times_k X$ the result is a special case of Lemma 7.4. $\square$

Lemma 7.7. *Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a separable algebraic closure of k. Then X is geometrically connected if and only if the base change $X_{\overline{k}}$ is connected.*

Proof. Assume $X_{\overline{k}}$ is connected. Let k'/k be a field extension. There exists a field extension $\overline{k}'/\overline{k}$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 7.6 we see that $X_{\overline{k}'}$ is connected. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is connected as desired. $\square$

Lemma 7.8. *Let k be a field. Let X be a scheme over k. Let A be a k-algebra. Let $V \subset X_A$ be a quasi-compact open. Then there exists a finitely generated k-subalgebra $A' \subset A$ and a quasi-compact open $V' \subset X_{A'}$ such that $V = V'_A$.*

Proof. We remark that if X is also quasi-separated this follows from Limits, Lemma 4.11. Let $U_1, \ldots, U_n$ be finitely many affine opens of X such that $V \subset \bigcup U_{i,A}$. Say $U_i = \operatorname{Spec}(R_i)$. Since V is quasi-compact we can find finitely many $f_{ij} \in R_i \otimes_k A$, $j = 1, \ldots, n_i$ such that $V = \bigcup_i \bigcup_{j=1,\ldots,n_i} D(f_{ij})$ where $D(f_{ij}) \subset U_{i,A}$ is the corresponding standard open. (We do not claim that $V \cap U_{i,A}$ is the union of the $D(f_{ij})$, $j = 1, \ldots, n_i$.) It is clear that we can find a finitely generated k-subalgebra $A' \subset A$ such that f_{ij} is the image of some $f'_{ij} \in R_i \otimes_k A'$. Set $V' = \bigcup D(f'_{ij})$ which is a quasi-compact open of $X_{A'}$. Denote $\pi : X_A \to X_{A'}$ the canonical morphism. We have $\pi(V) \subset V'$ as $\pi(D(f_{ij})) \subset D(f'_{ij})$. If $x \in X_A$ with $\pi(x) \in V'$, then $\pi(x) \in D(f'_{ij})$ for some i, j and we see that $x \in D(f_{ij})$ as f'_{ij} maps to f_{ij}. Thus we see that $V = \pi^{-1}(V')$ as desired. $\square$

Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. For example $\overline{k}$ could be the separable algebraic closure of k. For any $\sigma \in \operatorname{Gal}(\overline{k}/k)$ we get a corresponding automorphism $\operatorname{Spec}(\sigma) : \operatorname{Spec}(\overline{k}) \longrightarrow \operatorname{Spec}(\overline{k})$. Note that $\operatorname{Spec}(\sigma) \circ \operatorname{Spec}(\tau) = \operatorname{Spec}(\tau \circ \sigma)$. Hence we get an action

$$\operatorname{Gal}(\overline{k}/k)^{opp} \times \operatorname{Spec}(\overline{k}) \longrightarrow \operatorname{Spec}(\overline{k})$$

of the opposite group on the scheme $\operatorname{Spec}(\overline{k})$. Let X be a scheme over k. Since $X_{\overline{k}} = \operatorname{Spec}(\overline{k}) \times_{\operatorname{Spec}(k)} X$ by definition we see that the action above induces a canonical action

(7.8.1) $$\operatorname{Gal}(\overline{k}/k)^{opp} \times X_{\overline{k}} \longrightarrow X_{\overline{k}}.$$

Lemma 7.9. *Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a (possibly infinite) Galois extension of k. Let $V \subset X_{\overline{k}}$ be a quasi-compact open. Then*

    (1) *there exists a finite subextension $\overline{k}/k'/k$ and a quasi-compact open $V' \subset X_{k'}$ such that $V = (V')_{\overline{k}}$,*

(2) *there exists an open subgroup $H \subset \mathrm{Gal}(\bar{k}/k)$ such that $\sigma(V) = V$ for all $\sigma \in H$.*

Proof. By Lemma 7.8 there exists a finite subextension $k'/k \subset \bar{k}$ and an open $V' \subset X_{k'}$ which pulls back to V. This proves (1). Since $\mathrm{Gal}(\bar{k}/k')$ is open in $\mathrm{Gal}(\bar{k}/k)$ part (2) is clear as well. $\square$

Lemma 7.10. *Let k be a field. Let $\bar{k}/k$ be a (possibly infinite) Galois extension. Let X be a scheme over k. Let $\overline{T} \subset X_{\bar{k}}$ have the following properties*

(1) *$\overline{T}$ is a closed subset of $X_{\bar{k}}$,*
(2) *for every $\sigma \in \mathrm{Gal}(\bar{k}/k)$ we have $\sigma(\overline{T}) = \overline{T}$.*

Then there exists a closed subset $T \subset X$ whose inverse image in $X_{\bar{k}}$ is $\overline{T}$.

Proof. This lemma immediately reduces to the case where $X = \mathrm{Spec}(A)$ is affine. In this case, let $\overline{I} \subset A \otimes_k \bar{k}$ be the radical ideal corresponding to $\overline{T}$. Assumption (2) implies that $\sigma(\overline{I}) = \overline{I}$ for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Pick $x \in \overline{I}$. There exists a finite Galois extension k'/k contained in $\bar{k}$ such that $x \in A \otimes_k k'$. Set $G = \mathrm{Gal}(k'/k)$. Set

$$P(T) = \prod_{\sigma \in G} (T - \sigma(x)) \in (A \otimes_k k')[T]$$

It is clear that $P(T)$ is monic and is actually an element of $(A \otimes_k k')^G[T] = A[T]$ (by basic Galois theory). Moreover, if we write $P(T) = T^d + a_1 T^{d-1} + \ldots + a_d$ the we see that $a_i \in I := A \cap \overline{I}$. Combining $P(x) = 0$ and $a_i \in I$ we find $x^d = -a_1 x^{d-1} - \ldots - a_d \in I(A \otimes_k \bar{k})$. Thus x is contained in the radical of $I(A \otimes_k \bar{k})$. Hence $\overline{I}$ is the radical of $I(A \otimes_k \bar{k})$ and setting $T = V(I)$ is a solution. $\square$

Lemma 7.11. *Let k be a field. Let X be a scheme over k. The following are equivalent*

(1) *X is geometrically connected,*
(2) *for every finite separable field extension k'/k the scheme $X_{k'}$ is connected.*

Proof. It follows immediately from the definition that (1) implies (2). Assume that X is not geometrically connected. Let $k \subset \bar{k}$ be a separable algebraic closure of k. By Lemma 7.7 it follows that $X_{\bar{k}}$ is disconnected. Say $X_{\bar{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open, closed, and nonempty.

Suppose that $W \subset X$ is any quasi-compact open. Then $W_{\bar{k}} \cap \overline{U}$ and $W_{\bar{k}} \cap \overline{V}$ are open and closed in $W_{\bar{k}}$. In particular $W_{\bar{k}} \cap \overline{U}$ and $W_{\bar{k}} \cap \overline{V}$ are quasi-compact, and by Lemma 7.9 both $W_{\bar{k}} \cap \overline{U}$ and $W_{\bar{k}} \cap \overline{V}$ are defined over a finite subextension and invariant under an open subgroup of $\mathrm{Gal}(\bar{k}/k)$. We will use this without further mention in the following.

Pick $W_0 \subset X$ quasi-compact open such that both $W_{0,\bar{k}} \cap \overline{U}$ and $W_{0,\bar{k}} \cap \overline{V}$ are nonempty. Choose a finite subextension $\bar{k}/k'/k$ and a decomposition $W_{0,k'} = U_0' \amalg V_0'$ into open and closed subsets such that $W_{0,\bar{k}} \cap \overline{U} = (U_0')_{\bar{k}}$ and $W_{0,\bar{k}} \cap \overline{V} = (V_0')_{\bar{k}}$. Let $H = \mathrm{Gal}(\bar{k}/k') \subset \mathrm{Gal}(\bar{k}/k)$. In particular $\sigma(W_{0,\bar{k}} \cap \overline{U}) = W_{0,\bar{k}} \cap \overline{U}$ and similarly for $\overline{V}$.

Having chosen W_0, k' as above, for every quasi-compact open $W \subset X$ we set

$$U_W = \bigcap_{\sigma \in H} \sigma(W_{\bar{k}} \cap \overline{U}), \quad V_W = \bigcup_{\sigma \in H} \sigma(W_{\bar{k}} \cap \overline{V}).$$

Now, since $W_{\bar{k}} \cap \overline{U}$ and $W_{\bar{k}} \cap \overline{V}$ are fixed by an open subgroup of $\mathrm{Gal}(\bar{k}/k)$ we see that the union and intersection above are finite. Hence U_W and V_W are both open and closed. Also, by construction $W_{\bar{k}} = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W_{\bar{k}} \cap U_{W'} = U_W$ and $W_{\bar{k}} \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X_{\bar{k}} = U \amalg V$ is a disjoint union of open and closed subsets. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $W_0 \cap \overline{U}$ by construction. Finally, $U, V \subset X_{\bar{k}}$ are closed and H-invariant by construction. Hence by Lemma 7.10 we have $U = (U')_{\bar{k}}$, and $V = (V')_{\bar{k}}$ for some closed $U', V' \subset X_{k'}$. Clearly $X_{k'} = U' \amalg V'$ and we see that $X_{k'}$ is disconnected as desired. $\square$

Lemma 7.12. *Let k be a field. Let $\bar{k}/k$ be a (possibly infinite) Galois extension. Let $f : T \to X$ be a morphism of schemes over k. Assume $T_{\bar{k}}$ connected and $X_{\bar{k}}$ disconnected. Then X is disconnected.*

Proof. Write $X_{\bar{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open and closed. Denote $\overline{f} : T_{\bar{k}} \to X_{\bar{k}}$ the base change of f. Since $T_{\bar{k}}$ is connected we see that $T_{\bar{k}}$ is contained in either $\overline{f}^{-1}(\overline{U})$ or $\overline{f}^{-1}(\overline{V})$. Say $T_{\bar{k}} \subset \overline{f}^{-1}(\overline{U})$.

Fix a quasi-compact open $W \subset X$. There exists a finite Galois subextension $\bar{k}/k'/k$ such that $\overline{U} \cap W_{\bar{k}}$ and $\overline{V} \cap W_{\bar{k}}$ come from quasi-compact opens $U', V' \subset W_{k'}$. Then also $W_{k'} = U' \amalg V'$. Consider

$$U'' = \bigcap_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(U'), \quad V'' = \bigcup_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(V').$$

These are Galois invariant, open and closed, and $W_{k'} = U'' \amalg V''$. By Lemma 7.10 we get open and closed subsets $U_W, V_W \subset W$ such that $U'' = (U_W)_{k'}$, $V'' = (V_W)_{k'}$ and $W = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W \cap U_{W'} = U_W$ and $W \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X = U \amalg V$. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $f(T)$ by construction. $\square$

Lemma 7.13. *Let k be a field. Let $T \to X$ be a morphism of schemes over k. Assume T is geometrically connected and X connected. Then X is geometrically connected.*

Proof. This is a reformulation of Lemma 7.12. $\square$

Lemma 7.14. *Let k be a field. Let X be a scheme over k. Assume X is connected and has a point x such that k is algebraically closed in $\kappa(x)$. Then X is geometrically connected. In particular, if X has a k-rational point and X is connected, then X is geometrically connected.*

Proof. Set $T = \mathrm{Spec}(\kappa(x))$. Let $\bar{k}$ be a separable algebraic closure of k. The assumption on $\kappa(x)/k$ implies that $T_{\bar{k}}$ is irreducible, see Algebra, Lemma 47.8. Hence by Lemma 7.13 we see that $X_{\bar{k}}$ is connected. By Lemma 7.7 we conclude that X is geometrically connected. $\square$

Lemma 7.15. *Let K/k be an extension of fields. Let X be a scheme over k. For every connected component T of X the inverse image $T_K \subset X_K$ is a union of connected components of X_K.*

Proof. This is a purely topological statement. Denote $p : X_K \to X$ the projection morphism. Let $T \subset X$ be a connected component of X. Let $t \in T_K = p^{-1}(T)$. Let $C \subset X_K$ be a connected component containing t. Then $p(C)$ is a connected subset of X which meets T, hence $p(C) \subset T$. Hence $C \subset T_K$. $\square$

The following lemma will be superseded by the stronger Lemma 7.17 below.

Lemma 7.16. *Let K/k be a finite extension of fields and let X be a scheme over k. Denote by $p : X_K \to X$ the projection morphism. For every connected component T of X_K the image $p(T)$ is a connected component of X.*

Proof. The image $p(T)$ is contained in some connected component X' of X. Consider X' as a closed subscheme of X in any way. Then T is also a connected component of $X'_K = p^{-1}(X')$ and we may therefore assume that X is connected. The morphism p is open (Morphisms, Lemma 23.4), closed (Morphisms, Lemma 44.7) and the fibers of p are finite sets (Morphisms, Lemma 44.10). Thus we may apply Topology, Lemma 7.7 to conclude. $\square$

Lemma 7.17 (Gabber). *Let K/k be an extension of fields. Let X be a scheme over k. Denote $p : X_K \to X$ the projection morphism. Let $\overline{T} \subset X_K$ be a connected component. Then $p(\overline{T})$ is a connected component of X.*

Proof. When K/k is finite this is Lemma 7.16. In general the proof is more difficult.

Let $T \subset X$ be the connected component of X containing the image of $\overline{T}$. We may replace X by T (with the induced reduced subscheme structure). Thus we may assume X is connected. Let $A = H^0(X, \mathcal{O}_X)$. Let $L \subset A$ be the maximal weakly étale k-subalgebra, see More on Algebra, Lemma 105.2. Since A does not have any nontrivial idempotents we see that L is a field and a separable algebraic extension of k by More on Algebra, Lemma 105.1. Observe that L is also the maximal weakly étale L-subalgebra of A (because any weakly étale L-algebra is weakly étale over k by More on Algebra, Lemma 104.9). By Schemes, Lemma 6.4 we obtain a factorization $X \to \operatorname{Spec}(L) \to \operatorname{Spec}(k)$ of the structure morphism.

Let L'/L be a finite separable extension. By Cohomology of Schemes, Lemma 5.3 we have
$$A \otimes_L L' = H^0(X \times_{\operatorname{Spec}(L)} \operatorname{Spec}(L'), \mathcal{O}_{X \times_{\operatorname{Spec}(L)} \operatorname{Spec}(L')})$$
The maximal weakly étale L'-subalgebra of $A \otimes_L L'$ is $L \otimes_L L' = L'$ by More on Algebra, Lemma 105.4. In particular $A \otimes_L L'$ does not have nontrivial idempotents (such an idempotent would generate a weakly étale subalgebra) and we conclude that $X \times_{\operatorname{Spec}(L)} \operatorname{Spec}(L')$ is connected. By Lemma 7.11 we conclude that X is geometrically connected over L.

Let's give $\overline{T}$ the reduced induced scheme structure and consider the composition
$$\overline{T} \xrightarrow{i} X_K = X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(K) \xrightarrow{\pi} \operatorname{Spec}(L \otimes_k K)$$
The image is contained in a connected component of $\operatorname{Spec}(L \otimes_k K)$. Since $K \to L \otimes_k K$ is integral we see that the connected components of $\operatorname{Spec}(L \otimes_k K)$ are

points and all points are closed, see Algebra, Lemma 36.19. Thus we get a quotient field $L \otimes_k K \to E$ such that $\overline{T}$ maps into $\mathrm{Spec}(E) \subset \mathrm{Spec}(L \otimes_k K)$. Hence $i(\overline{T}) \subset \pi^{-1}(\mathrm{Spec}(E))$. But

$$\pi^{-1}(\mathrm{Spec}(E)) = (X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)) \times_{\mathrm{Spec}(L \otimes_k K)} \mathrm{Spec}(E) = X \times_{\mathrm{Spec}(L)} \mathrm{Spec}(E)$$

which is connected because X is geometrically connected over L. Then we get the equality $\overline{T} = X \times_{\mathrm{Spec}(L)} \mathrm{Spec}(E)$ (set theoretically) and we conclude that $\overline{T} \to X$ is surjective as desired. $\square$

Let X be a scheme. We denote $\pi_0(X)$ the set of connected components of X.

Lemma 7.18. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action*

$$Gal(\overline{k}/k)^{opp} \times \pi_0(X_{\overline{k}}) \longrightarrow \pi_0(X_{\overline{k}})$$

with the following properties:

(1) *An element $\overline{T} \in \pi_0(X_{\overline{k}})$ is fixed by the action if and only if there exists a connected component $T \subset X$, which is geometrically connected over k, such that $T_{\overline{k}} = \overline{T}$.*

(2) *For any field extension k'/k with separable algebraic closure $\overline{k}'$ the diagram*

$$\begin{array}{ccc} Gal(\overline{k}'/k') \times \pi_0(X_{\overline{k}'}) & \longrightarrow & \pi_0(X_{\overline{k}'}) \\ \downarrow & & \downarrow \\ Gal(\overline{k}/k) \times \pi_0(X_{\overline{k}}) & \longrightarrow & \pi_0(X_{\overline{k}}) \end{array}$$

is commutative (where the right vertical arrow is a bijection according to Lemma 7.6).

Proof. The action (7.8.1) of $Gal(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its connected components. Connected components are always closed (Topology, Lemma 7.3). Hence if $\overline{T}$ is as in (1), then by Lemma 7.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically connected over k, see Lemma 7.7. To see that T is a connected component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is a connected component of X. In this case $T'_{\overline{k}'}$ strictly contains $\overline{T}$ and hence is disconnected. By Lemma 7.12 this means that T' is disconnected! Contradiction.

We omit the proof of the functoriality in (2). $\square$

Lemma 7.19. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. Assume*

(1) *X is quasi-compact, and*

(2) *the connected components of $X_{\overline{k}}$ are open.*

Then

(a) *$\pi_0(X_{\overline{k}})$ is finite, and*

(b) *the action of $Gal(\overline{k}/k)$ on $\pi_0(X_{\overline{k}})$ is continuous.*

Moreover, assumptions (1) and (2) are satisfied when X is of finite type over k.

Proof. Since the connected components are open, cover $X_{\overline{k}}$ (Topology, Lemma 7.3) and $X_{\overline{k}}$ is quasi-compact, we conclude that there are only finitely many of them. Thus (a) holds. By Lemma 7.8 these connected components are each defined over a finite subextension of $\overline{k}/k$ and we get (b). If X is of finite type over k, then $X_{\overline{k}}$ is of finite type over $\overline{k}$ (Morphisms, Lemma 15.4). Hence $X_{\overline{k}}$ is a Noetherian scheme (Morphisms, Lemma 15.6). Thus $X_{\overline{k}}$ has finitely many irreducible components (Properties, Lemma 5.7) and a fortiori finitely many connected components (which are therefore open). $\square$

8. Geometrically irreducible schemes

If X is an irreducible scheme over a field, then it can happen that X becomes reducible after extending the ground field. This does not happen for geometrically irreducible schemes.

Definition 8.1. Let X be a scheme over the field k. We say X is *geometrically irreducible* over k if the scheme $X_{k'}$ is irreducible[1] for any field extension k' of k.

Lemma 8.2. *Let X be a scheme over the field k. Let k'/k be a field extension. Then X is geometrically irreducible over k if and only if $X_{k'}$ is geometrically irreducible over k'.*

Proof. If X is geometrically irreducible over k, then it is clear that $X_{k'}$ is geometrically irreducible over k'. For the converse, note that for any field extension k''/k there exists a common field extension k'''/k' and k'''/k''. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the irreducibility of $X_{k'''}$ implies the irreducibility of $X_{k''}$. Thus if $X_{k'}$ is geometrically irreducible over k' then X is geometrically irreducible over k. $\square$

Lemma 8.3. *Let X be a scheme over a separably closed field k. If X is irreducible, then X_K is irreducible for any field extension K/k. I.e., X is geometrically irreducible over k.*

Proof. Use Properties, Lemma 3.3 and Algebra, Lemma 47.2. $\square$

Lemma 8.4. *Let k be a field. Let X, Y be schemes over k. Assume X is geometrically irreducible over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between irreducible components.

Proof. First, note that the scheme theoretic fibres of p are irreducible, since they are base changes of the geometrically irreducible scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 18.5. By Morphisms, Lemma 23.4 the map p is open. Thus we may apply Topology, Lemma 8.15 to conclude. $\square$

Lemma 8.5. *Let k be a field. Let X be a scheme over k. The following are equivalent*

    (1) *X is geometrically irreducible over k,*

---

[1]An irreducible space is nonempty.

(2) *for every nonempty affine open U the k-algebra $\mathcal{O}_X(U)$ is geometrically irreducible over k (see Algebra, Definition 47.4),*

(3) *X is irreducible and there exists an affine open covering $X = \bigcup U_i$ such that each k-algebra $\mathcal{O}_X(U_i)$ is geometrically irreducible, and*

(4) *there exists an open covering $X = \bigcup_{i \in I} X_i$ with $I \neq \emptyset$ such that X_i is geometrically irreducible for each i and such that $X_i \cap X_j \neq \emptyset$ for all $i, j \in I$.*

Moreover, if X is geometrically irreducible so is every nonempty open subscheme of X.

Proof. An affine scheme $\mathrm{Spec}(A)$ over k is geometrically irreducible if and only if A is geometrically irreducible over k; this is immediate from the definitions. Recall that if a scheme is irreducible so is every nonempty open subscheme of X, any two nonempty open subsets have a nonempty intersection. Also, if every affine open is irreducible then the scheme is irreducible, see Properties, Lemma 3.3. Hence the final statement of the lemma is clear, as well as the implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (4). If (4) holds, then for any field extension k'/k the scheme $X_{k'}$ has a covering by irreducible opens which pairwise intersect. Hence $X_{k'}$ is irreducible. Hence (4) implies (1). $\square$

Lemma 8.6. *Let X be an irreducible scheme over the field k. Let $\xi \in X$ be its generic point. The following are equivalent*

(1) *X is geometrically irreducible over k, and*

(2) *$\kappa(\xi)$ is geometrically irreducible over k.*

Proof. Assume (1). Recall that $\mathcal{O}_{X,\xi}$ is the filtered colimit of $\mathcal{O}_X(U)$ where U runs over the nonempty open affine subschemes of X. Combining Lemma 8.5 and Algebra, Lemma 47.6 we see that $\mathcal{O}_{X,\xi}$ is geometrically irreducible over k. Since $\mathcal{O}_{X,\xi} \to \kappa(\xi)$ is a surjection with locally nilpotent kernel (see Algebra, Lemma 25.1) it follows that $\kappa(\xi)$ is geometrically irreducible, see Algebra, Lemma 46.7.

Assume (2). We may assume that X is reduced. Let $U \subset X$ be a nonempty affine open. Then $U = \mathrm{Spec}(A)$ where A is a domain with fraction field $\kappa(\xi)$. Thus A is a k-subalgebra of a geometrically irreducible k-algebra. Hence by Algebra, Lemma 47.6 we see that A is geometrically irreducible over k. By Lemma 8.5 we conclude that X is geometrically irreducible over k. $\square$

Lemma 8.7. *Let k'/k be an extension of fields. Let X be a scheme over k. Set $X' = X_{k'}$. Assume k separably algebraically closed. Then the morphism $X' \to X$ induces a bijection of irreducible components.*

Proof. Since k is separably algebraically closed we see that k' is geometrically irreducible over k, see Algebra, Lemma 47.5. Hence $Z = \mathrm{Spec}(k')$ is geometrically irreducible over k. by Lemma 8.5 above. Since $X' = Z \times_k X$ the result is a special case of Lemma 8.4. $\square$

Lemma 8.8. *Let k be a field. Let X be a scheme over k. The following are equivalent:*

(1) *X is geometrically irreducible over k,*

(2) *for every finite separable field extension k'/k the scheme $X_{k'}$ is irreducible, and*

(3) *$X_{\overline{k}}$ is irreducible, where $k \subset \overline{k}$ is a separable algebraic closure of k.*

Proof. Assume $X_{\overline{k}}$ is irreducible, i.e., assume (3). Let k'/k be a field extension. There exists a field extension $\overline{k}'/\overline{k}$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 8.7 we see that $X_{\overline{k}'}$ is irreducible. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is irreducible. Hence (1) holds.

Let $k \subset \overline{k}$ be a separable algebraic closure of k. Assume not (3), i.e., assume $X_{\overline{k}}$ is reducible. Our goal is to show that also $X_{k'}$ is reducible for some finite subextension $\overline{k}/k'/k$. Let $X = \bigcup_{i \in I} U_i$ be an affine open covering with U_i not empty. If for some i the scheme U_i is reducible, or if for some pair $i \neq j$ the intersection $U_i \cap U_j$ is empty, then X is reducible (Properties, Lemma 3.3) and we are done. In particular we may assume that $U_{i,\overline{k}} \cap U_{j,\overline{k}}$ for all $i, j \in I$ is nonempty and we conclude that $U_{i,\overline{k}}$ has to be reducible for some i. According to Algebra, Lemma 47.3 this means that $U_{i,k'}$ is reducible for some finite separable field extension k'/k. Hence also $X_{k'}$ is reducible. Thus we see that (2) implies (3).

The implication (1) $\Rightarrow$ (2) is immediate. This proves the lemma. $\qquad\square$

Lemma 8.9. *Let K/k be an extension of fields. Let X be a scheme over k. For every irreducible component T of X the inverse image $T_K \subset X_K$ is a union of irreducible components of X_K.*

Proof. Let $T \subset X$ be an irreducible component of X. The morphism $T_K \to T$ is flat, so generalizations lift along $T_K \to T$. Hence every $\xi \in T_K$ which is a generic point of an irreducible component of T_K maps to the generic point η of T. If $\xi' \rightsquigarrow \xi$ is a specialization in X_K then ξ' maps to η since there are no points specializing to η in X. Hence $\xi' \in T_K$ and we conclude that $\xi = \xi'$. In other words ξ is the generic point of an irreducible component of X_K. This means that the irreducible components of T_K are all irreducible components of X_K. $\qquad\square$

For a scheme X we denote $\mathrm{IrredComp}(X)$ the set of irreducible components of X.

Lemma 8.10. *Let K/k be an extension of fields. Let X be a scheme over k. For every irreducible component $\overline{T} \subset X_K$ the image of $\overline{T}$ in X is an irreducible component in X. This defines a canonical map*

$$IrredComp(X_K) \longrightarrow IrredComp(X)$$

which is surjective.

Proof. Consider the diagram

$$
\begin{array}{ccc}
X_K & \longleftarrow & X_{\overline{K}} \\
\downarrow & & \downarrow \\
X & \longleftarrow & X_{\overline{k}}
\end{array}
$$

where $\overline{K}$ is the separable algebraic closure of K, and where $\overline{k}$ is the separable algebraic closure of k. By Lemma 8.7 the morphism $X_{\overline{K}} \to X_{\overline{k}}$ induces a bijection between irreducible components. Hence it suffices to show the lemma for the morphisms $X_{\overline{k}} \to X$ and $X_{\overline{K}} \to X_K$. In other words we may assume that $K = \overline{k}$.

The morphism $p : X_{\overline{k}} \to X$ is integral, flat and surjective. Flatness implies that generalizations lift along p, see Morphisms, Lemma 25.9. Hence generic points of irreducible components of $X_{\overline{k}}$ map to generic points of irreducible components of

X. Integrality implies that p is universally closed, see Morphisms, Lemma 44.7. Hence we conclude that the image $p(\overline{T})$ of an irreducible component is a closed irreducible subset which contains a generic point of an irreducible component of X, hence $p(\overline{T})$ is an irreducible component of X. This proves the first assertion. If $T \subset X$ is an irreducible component, then $p^{-1}(T) = T_K$ is a nonempty union of irreducible components, see Lemma 8.9. Each of these necessarily maps onto T by the first part. Hence the map is surjective. $\square$

Lemma 8.11. *Let k be a field. Let X be a scheme over k. If X is irreducible and has a dense set of k-rational points, then X is geometrically irreducible.*

Proof. Let k'/k be a finite extension of fields and let $Z, Z' \subset X_{k'}$ be irreducible components. It suffices to show $Z = Z'$, see Lemma 8.8. By Lemma 8.10 we have $p(Z) = p(Z') = X$ where $p : X_{k'} \to X$ is the projection. If $Z \neq Z'$ then $Z \cap Z'$ is nowhere dense in $X_{k'}$ and hence $p(Z \cap Z')$ is not dense by Morphisms, Lemma 48.7; here we also use that p is a finite morphism as the base change of the finite morphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, see Morphisms, Lemma 44.6. Thus we can pick a k-rational point $x \in X$ with $x \notin p(Z \cap Z')$. Since the residue field of x is k we see that $p^{-1}(\{x\}) = \{x'\}$ where $x' \in X_{k'}$ is a point whose residue field is k'. Since $x \in p(Z) = p(Z')$ we conclude that $x' \in Z \cap Z'$ which is the contradiction we were looking for. $\square$

Lemma 8.12. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action*

$$Gal(\overline{k}/k)^{opp} \times IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X_{\overline{k}})$$

with the following properties:

(1) *An element $\overline{T} \in IrredComp(X_{\overline{k}})$ is fixed by the action if and only if there exists an irreducible component $T \subset X$, which is geometrically irreducible over k, such that $T_{\overline{k}} = \overline{T}$.*

(2) *For any field extension k'/k with separable algebraic closure $\overline{k}'$ the diagram*

$$
\begin{array}{ccc}
Gal(\overline{k}'/k') \times IrredComp(X_{\overline{k}'}) & \longrightarrow & IrredComp(X_{\overline{k}'}) \\
\downarrow & & \downarrow \\
Gal(\overline{k}/k) \times IrredComp(X_{\overline{k}}) & \longrightarrow & IrredComp(X_{\overline{k}})
\end{array}
$$

is commutative (where the right vertical arrow is a bijection according to Lemma 8.7).

Proof. The action (7.8.1) of $Gal(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its irreducible components. Irreducible components are always closed (Topology, Lemma 7.3). Hence if $\overline{T}$ is as in (1), then by Lemma 7.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically irreducible over k, see Lemma 8.8. To see that T is an irreducible component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is an irreducible component of X. Let $\overline{\eta}$ be the generic point of $\overline{T}$. It maps to the generic point η of T. Then the generic point $\xi \in T'$ specializes to η. As $X_{\overline{k}} \to X$ is flat there exists a point $\overline{\xi} \in X_{\overline{k}}$ which maps to ξ and specializes to $\overline{\eta}$. It follows that the closure of the singleton $\{\overline{\xi}\}$ is an irreducible closed subset of $X_{\overline{\xi}}$ which strictly contains $\overline{T}$. This is the desired contradiction.

We omit the proof of the functoriality in (2). □

Lemma 8.13. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. The fibres of the map*

$$IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X)$$

of Lemma 8.10 are exactly the orbits of $Gal(\overline{k}/k)$ under the action of Lemma 8.12.

Proof. Let $T \subset X$ be an irreducible component of X. Let $\eta \in T$ be its generic point. By Lemmas 8.9 and 8.10 the generic points of irreducible components of $\overline{T}$ which map into T map to η. By Algebra, Lemma 47.14 the Galois group acts transitively on all of the points of $X_{\overline{k}}$ mapping to η. Hence the lemma follows. □

Lemma 8.14. *Let k be a field. Assume $X \to \mathrm{Spec}(k)$ locally of finite type. In this case*

(1) *the action*

$$Gal(\overline{k}/k)^{opp} \times IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X_{\overline{k}})$$

is continuous if we give $IrredComp(X_{\overline{k}})$ the discrete topology,
(2) *every irreducible component of $X_{\overline{k}}$ can be defined over a finite extension of k, and*
(3) *given any irreducible component $T \subset X$ the scheme $T_{\overline{k}}$ is a finite union of irreducible components of $X_{\overline{k}}$ which are all in the same $Gal(\overline{k}/k)$-orbit.*

Proof. Let $\overline{T}$ be an irreducible component of $X_{\overline{k}}$. We may choose an affine open $U \subset X$ such that $\overline{T} \cap U_{\overline{k}}$ is not empty. Write $U = \mathrm{Spec}(A)$, so A is a finite type k-algebra, see Morphisms, Lemma 15.2. Hence $A_{\overline{k}}$ is a finite type $\overline{k}$-algebra, and in particular Noetherian. Let $\mathfrak{p} = (f_1, \ldots, f_n)$ be the prime ideal corresponding to $\overline{T} \cap U_{\overline{k}}$. Since $A_{\overline{k}} = A \otimes_k \overline{k}$ we see that there exists a finite subextension $\overline{k}/k'/k$ such that each $f_i \in A_{k'}$. It is clear that $Gal(\overline{k}/k')$ fixes $\overline{T}$, which proves (1).

Part (2) follows by applying Lemma 8.12 (1) to the situation over k' which implies the irreducible component $\overline{T}$ is of the form $T'_{\overline{k}}$ for some irreducible $T' \subset X_{k'}$.

To prove (3), let $T \subset X$ be an irreducible component. Choose an irreducible component $\overline{T} \subset X_{\overline{k}}$ which maps to T, see Lemma 8.10. By the above the orbit of $\overline{T}$ is finite, say it is $\overline{T}_1, \ldots, \overline{T}_n$. Then $\overline{T}_1 \cup \ldots \cup \overline{T}_n$ is a $Gal(\overline{k}/k)$-invariant closed subset of $X_{\overline{k}}$ hence of the form $W_{\overline{k}}$ for some $W \subset X$ closed by Lemma 7.10. Clearly $W = T$ and we win. □

Lemma 8.15. *Let k be a field. Let $X \to \mathrm{Spec}(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite separable extension k'/k such that every irreducible component of $X_{k'}$ is geometrically irreducible over k'.*

Proof. Let $\overline{k}$ be a separable algebraic closure of k. The assumption that X has finitely many irreducible components combined with Lemma 8.14 (3) shows that $X_{\overline{k}}$ has finitely many irreducible components $\overline{T}_1, \ldots, \overline{T}_n$. By Lemma 8.14 (2) there exists a finite extension $\overline{k}/k'/k$ and irreducible components $T_i \subset X_{k'}$ such that $\overline{T}_i = T_{i,\overline{k}}$ and we win. □

Lemma 8.16. *Let X be a scheme over the field k. Assume X has finitely many irreducible components which are all geometrically irreducible. Then X has finitely many connected components each of which is geometrically connected.*

Proof. This is clear because a connected component is a union of irreducible components. Details omitted. $\square$

9. Geometrically integral schemes

If X is an integral scheme over a field, then it can happen that X becomes either nonreduced or reducible after extending the ground field. This does not happen for geometrically integral schemes.

Definition 9.1. Let X be a scheme over the field k.
   (1) Let $x \in X$. We say X is *geometrically pointwise integral at x* if for every field extension k'/k and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'}, x'}$ is integral.
   (2) We say X is *geometrically pointwise integral* if X is geometrically pointwise integral at every point.
   (3) We say X is *geometrically integral* over k if the scheme $X_{k'}$ is integral for every field extension k' of k.

The distinction between notions (2) and (3) is necessary. For example if $k = \mathbf{R}$ and $X = \mathrm{Spec}(\mathbf{C}[x])$, then X is geometrically pointwise integral over $\mathbf{R}$ but of course not geometrically integral.

Lemma 9.2. *Let k be a field. Let X be a scheme over k. Then X is geometrically integral over k if and only if X is both geometrically reduced and geometrically irreducible over k.*

Proof. See Properties, Lemma 3.4. $\square$

Lemma 9.3. *Let k be a field. Let X be a proper scheme over k.*
   (1) *$A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra,*
   (2) *$A = \prod_{i=1,\ldots,n} A_i$ is a product of Artinian local k-algebras, one factor for each connected component of X,*
   (3) *if X is reduced, then $A = \prod_{i=1,\ldots,n} k_i$ is a product of fields, each a finite extension of k,*
   (4) *if X is geometrically reduced, then k_i is finite separable over k,*
   (5) *if X is geometrically connected, then A is geometrically irreducible over k,*
   (6) *if X is geometrically irreducible, then A is geometrically irreducible over k,*
   (7) *if X is geometrically reduced and connected, then $A = k$, and*
   (8) *if X is geometrically integral, then $A = k$.*

Proof. By Cohomology of Schemes, Lemma 19.2 we see that $A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra. This proves (1).

Then A is a product of local Artinian k-algebras by Algebra, Lemma 53.2 and Proposition 60.7. If $X = Y \amalg Z$ with Y and Z open in X, then we obtain an idempotent $e \in A$ by taking the section of $\mathcal{O}_X$ which is 1 on Y and 0 on Z. Conversely, if $e \in A$ is an idempotent, then we get a corresponding decomposition of X. Finally, as X has a Noetherian underlying topological space its connected

components are open. Hence the connected components of X correspond 1-to-1 with primitive idempotents of A. This proves (2).

If X is reduced, then A is reduced. Hence the local rings $A_i = k_i$ are reduced and therefore fields (for example by Algebra, Lemma 25.1). This proves (3).

If X is geometrically reduced, then $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ (equality by Cohomology of Schemes, Lemma 5.2) is reduced. This implies that $k_i \otimes_k \overline{k}$ is a product of fields and hence k_i/k is separable for example by Algebra, Lemmas 44.1 and 44.3. This proves (4).

If X is geometrically connected, then $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ is a zero dimensional local ring by part (2) and hence its spectrum has one point, in particular it is irreducible. Thus A is geometrically irreducible. This proves (5). Of course (5) implies (6).

If X is geometrically reduced and connected, then $A = k_1$ is a field and the extension k_1/k is finite separable and geometrically irreducible. However, then $k_1 \otimes_k \overline{k}$ is a product of $[k_1 : k]$ copies of $\overline{k}$ and we conclude that $k_1 = k$. This proves (7). Of course (7) implies (8). $\square$

Here is a baby version of Stein factorization; actual Stein factorization will be discussed in More on Morphisms, Section 53.

Lemma 9.4. *Let X be a proper scheme over a field k. Set $A = H^0(X, \mathcal{O}_X)$. The fibres of the canonical morphism $X \to \mathrm{Spec}(A)$ are geometrically connected.*

Proof. Set $S = \mathrm{Spec}(A)$. The canonical morphism $X \to S$ is the morphism corresponding to $\Gamma(S, \mathcal{O}_S) = A = \Gamma(X, \mathcal{O}_X)$ via Schemes, Lemma 6.4. The k-algebra A is a finite product $A = \prod A_i$ of local Artinian k-algebras finite over k, see Lemma 9.3. Denote $s_i \in S$ the point corresponding to the maximal ideal of A_i. Choose an algebraic closure $\overline{k}$ of k and set $\overline{A} = A \otimes_k \overline{k}$. Choose an embedding $\kappa(s_i) \to \overline{k}$ over k; this determines a $\overline{k}$-algebra map

$$\sigma_i : \overline{A} = A \otimes_k \overline{k} \to \kappa(s_i) \otimes_k \overline{k} \to \overline{k}$$

Consider the base change

$$
\begin{array}{ccc}
\overline{X} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\overline{S} & \longrightarrow & S
\end{array}
$$

of X to $\overline{S} = \mathrm{Spec}(\overline{A})$. By Cohomology of Schemes, Lemma 5.2 we have $\Gamma(\overline{X}, \mathcal{O}_{\overline{X}}) = \overline{A}$. If $\overline{s}_i \in \mathrm{Spec}(\overline{A})$ denotes the $\overline{k}$-rational point corresponding to σ_i, then we see that $\overline{s}_i$ maps to $s_i \in S$ and $\overline{X}_{\overline{s}_i}$ is the base change of X_{s_i} by $\mathrm{Spec}(\sigma_i)$. Thus we see that it suffices to prove the lemma in case k is algebraically closed.

Assume k is algebraically closed. In this case $\kappa(s_i)$ is algebraically closed and we have to show that X_{s_i} is connected. The product decomposition $A = \prod A_i$ corresponds to a disjoint union decomposition $\mathrm{Spec}(A) = \coprod \mathrm{Spec}(A_i)$, see Algebra, Lemma 21.2. Denote X_i the inverse image of $\mathrm{Spec}(A_i)$. It follows from Lemma 9.3 part (2) that $A_i = \Gamma(X_i, \mathcal{O}_{X_i})$. Observe that $X_{s_i} \to X_i$ is a closed immersion inducing an isomorphism on underlying topological spaces (because $\mathrm{Spec}(A_i)$ is a singleton). Hence if X_{s_i} isn't connected, then neither is X_i. So either X_i is empty

and $A_i = 0$ or X_i can be written as $U \amalg V$ with U and V open and nonempty which would imply that A_i has a nontrivial idempotent. Since A_i is local this is a contradiction and the proof is complete. $\square$

Lemma 9.5. *Let k be a field. Let X be a proper geometrically reduced scheme over k. The following are equivalent*

(1) $H^0(X, \mathcal{O}_X) = k$, *and*
(2) X *is geometrically connected.*

Proof. By Lemma 9.4 we have (1) $\Rightarrow$ (2). By Lemma 9.3 we have (2) $\Rightarrow$ (1). $\square$

10. Geometrically normal schemes

In Properties, Definition 7.1 we have defined the notion of a normal scheme. This notion is defined even for non-Noetherian schemes. Hence, contrary to our discussion of "geometrically regular" schemes we consider all field extensions of the ground field.

Definition 10.1. Let X be a scheme over the field k.

(1) Let $x \in X$. We say X is *geometrically normal at x* if for every field extension k'/k and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is normal.
(2) We say X is *geometrically normal* over k if X is geometrically normal at every $x \in X$.

Lemma 10.2. *Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

(1) X *is geometrically normal at x,*
(2) *for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is normal, and*
(3) *the ring $\mathcal{O}_{X,x}$ is geometrically normal over k (see Algebra, Definition 165.2).*

Proof. It is clear that (1) implies (2). Assume (2). Let k'/k be a finite purely inseparable field extension (for example $k = k'$). Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 46.7 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 18.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a normal ring. Hence we deduce (3) by Algebra, Lemma 165.1.

Assume (3). Let k'/k be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is normal by assumption and (1) is proved. $\square$

Lemma 10.3. *Let k be a field. Let X be a scheme over k. The following are equivalent*

(1) X *is geometrically normal,*
(2) $X_{k'}$ *is a normal scheme for every field extension k'/k,*
(3) $X_{k'}$ *is a normal scheme for every finitely generated field extension k'/k,*
(4) $X_{k'}$ *is a normal scheme for every finite purely inseparable field extension k'/k,*
(5) *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically normal (see Algebra, Definition 165.2), and*

(6) $X_{k^{perf}}$ *is a normal scheme.*

Proof. Assume (1). Then for every field extension k'/k and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is normal. By definition this means that $X_{k'}$ is normal. Hence (2).

It is clear that (2) implies (3) implies (4).

Assume (4) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a normal scheme for any finite purely inseparable extension k'/k (including $k = k'$). This means that $k' \otimes_k \mathcal{O}(U)$ is a normal ring for all finite purely inseparable extensions k'/k. Hence $\mathcal{O}(U)$ is a geometrically normal k-algebra by definition. Hence (4) implies (5).

Assume (5). For any field extension k'/k the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 17). Hence $X_{k'}$ is normal. So (1) holds.

The equivalence of (5) and (6) follows from the definition of geometrically normal algebras and the equivalence (just proved) of (3) and (4). $\qquad\square$

Lemma 10.4. *Let k be a field. Let X be a scheme over k. Let k'/k be a field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

(1) *X is geometrically normal at x,*
(2) *$X_{k'}$ is geometrically normal at x'.*

In particular, X is geometrically normal over k if and only if $X_{k'}$ is geometrically normal over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let k''/k be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common field extension k'''/k (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''}, x''} \longrightarrow \mathcal{O}_{X_{k'''}, x'''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is normal. Thus by Algebra, Lemma 164.3 we conclude that $\mathcal{O}_{X_{k''}, x''}$ is normal. By Lemma 10.2 we see that X is geometrically normal at x. $\qquad\square$

Lemma 10.5. *Let k be a field. Let X be a geometrically normal scheme over k and let Y be a normal scheme over k. Then $X \times_k Y$ is a normal scheme.*

Proof. This reduces to Algebra, Lemma 165.5 by Lemma 10.3. $\qquad\square$

Lemma 10.6. *Let k be a field. Let X be a normal scheme over k. Let K/k be a separable field extension. Then X_K is a normal scheme.*

Proof. Follows from Lemma 10.5 and Algebra, Lemma 165.4. $\qquad\square$

Lemma 10.7. *Let k be a field. Let X be a proper geometrically normal scheme over k. The following are equivalent*

(1) $H^0(X, \mathcal{O}_X) = k$,
(2) *X is geometrically connected,*

(3) *X is geometrically irreducible, and*
(4) *X is geometrically integral.*

Proof. By Lemma 9.5 we have the equivalence of (1) and (2). A locally Noetherian normal scheme (such as $X_{\overline{k}}$) is a disjoint union of its irreducible components (Properties, Lemma 7.6). Thus we see that (2) and (3) are equivalent. Since $X_{\overline{k}}$ is assumed reduced, we see that (3) and (4) are equivalent too. $\square$

11. Change of fields and locally Noetherian schemes

Let X a locally Noetherian scheme over a field k. It is not always that case that $X_{k'}$ is locally Noetherian too. For example if $X = \mathrm{Spec}(\overline{\mathbf{Q}})$ and $k = \mathbf{Q}$, then $X_{\overline{\mathbf{Q}}}$ is the spectrum of $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ which is not Noetherian. (Hint: It has too many idempotents). But if we only base change using finitely generated field extensions then the Noetherian property is preserved. (Or if X is locally of finite type over k, since this property is preserved under base change.)

Lemma 11.1. *Let k be a field. Let X be a scheme over k. Let k'/k be a finitely generated field extension. Then X is locally Noetherian if and only if $X_{k'}$ is locally Noetherian.*

Proof. Using Properties, Lemma 5.2 we reduce to the case where X is affine, say $X = \mathrm{Spec}(A)$. In this case we have to prove that A is Noetherian if and only if $A_{k'}$ is Noetherian. Since $A \to A_{k'} = k' \otimes_k A$ is faithfully flat, we see that if $A_{k'}$ is Noetherian, then so is A, by Algebra, Lemma 164.1. Conversely, if A is Noetherian then $A_{k'}$ is Noetherian by Algebra, Lemma 31.8. $\square$

12. Geometrically regular schemes

A geometrically regular scheme over a field k is a locally Noetherian scheme over k which remains regular upon suitable changes of base field. A finite type scheme over k is geometrically regular if and only if it is smooth over k (see Lemma 12.6). The notion of geometric regularity is most interesting in situations where smoothness cannot be used such as formal fibres (insert future reference here).

In the following definition we restrict ourselves to locally Noetherian schemes, since the property of being a regular local ring is only defined for Noetherian local rings. By Lemma 11.1 above, if we restrict ourselves to finitely generated field extensions then this property is preserved under change of base field. This comment will be used without further reference in this section. In particular the following definition makes sense.

Definition 12.1. Let k be a field. Let X be a locally Noetherian scheme over k.
  (1) Let $x \in X$. We say X is *geometrically regular at x* over k if for every finitely generated field extension k'/k and any $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular.
  (2) We say X is *geometrically regular over k* if X is geometrically regular at all of its points.

A similar definition works to define geometrically Cohen-Macaulay, (R_k), and (S_k) schemes over a field. We will add a section for these separately as needed.

Lemma 12.2. *Let k be a field. Let X be a locally Noetherian scheme over k. Let $x \in X$. The following are equivalent*

(1) X is geometrically regular at x,
(2) for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular, and
(3) the ring $\mathcal{O}_{X,x}$ is geometrically regular over k (see Algebra, Definition 166.2).

Proof. It is clear that (1) implies (2). Assume (2). This in particular implies that $\mathcal{O}_{X,x}$ is a regular local ring. Let k'/k be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 46.7 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 18.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a regular ring. Hence we deduce (3) from the definition of a geometrically regular ring.

Assume (3). Let k'/k be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is regular by assumption and (1) is proved. $\square$

Lemma 12.3. *Let k be a field. Let X be a locally Noetherian scheme over k. The following are equivalent*

(1) *X is geometrically regular,*
(2) *$X_{k'}$ is a regular scheme for every finitely generated field extension k'/k,*
(3) *$X_{k'}$ is a regular scheme for every finite purely inseparable field extension k'/k,*
(4) *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically regular (see Algebra, Definition 166.2), and*
(5) *there exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is geometrically regular over k.*

Proof. Assume (1). Then for every finitely generated field extension k'/k and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is regular. By Properties, Lemma 9.2 this means that $X_{k'}$ is regular. Hence (2).

It is clear that (2) implies (3).

Assume (3) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a regular scheme for any finite purely inseparable extension k'/k (including $k = k'$). This means that $k' \otimes_k \mathcal{O}(U)$ is a regular ring for all finite purely inseparable extensions k'/k. Hence $\mathcal{O}(U)$ is a geometrically regular k-algebra and we see that (4) holds.

It is clear that (4) implies (5). Let $X = \bigcup U_i$ be an affine open covering as in (5). For any field extension k'/k the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U_i) \otimes_k k'$ (see Schemes, Section 17). Hence $X_{k'}$ is regular. So (1) holds. $\square$

Lemma 12.4. *Let k be a field. Let X be a scheme over k. Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

(1) *X is geometrically regular at x,*
(2) *$X_{k'}$ is geometrically regular at x'.*

In particular, X is geometrically regular over k if and only if $X_{k'}$ is geometrically regular over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let k''/k be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common, finitely generated, field extension k'''/k (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x''''}.$$

This is a flat local ring homomorphism of Noetherian local rings and hence faithfully flat. By (2) we see that the local ring on the right is regular. Thus by Algebra, Lemma 110.9 we conclude that $\mathcal{O}_{X_{k''},x''}$ is regular. By Lemma 12.2 we see that X is geometrically regular at x. $\square$

The following lemma is a geometric variant of Algebra, Lemma 166.3.

Lemma 12.5. *Let k be a field. Let $f : X \to Y$ be a morphism of locally Noetherian schemes over k. Let $x \in X$ be a point and set $y = f(x)$. If X is geometrically regular at x and f is flat at x then Y is geometrically regular at y. In particular, if X is geometrically regular over k and f is flat and surjective, then Y is geometrically regular over k.*

Proof. Let k' be finite purely inseparable extension of k. Let $f' : X_{k'} \to Y_{k'}$ be the base change of f. Let $x' \in X_{k'}$ be the unique point lying over x. If we show that $Y_{k'}$ is regular at $y' = f'(x')$, then Y is geometrically regular over k at y', see Lemma 12.3. By Morphisms, Lemma 25.7 the morphism $X_{k'} \to Y_{k'}$ is flat at x'. Hence the ring map

$$\mathcal{O}_{Y_{k'},y'} \longrightarrow \mathcal{O}_{X_{k'},x'}$$

is a flat local homomorphism of local Noetherian rings with right hand side regular by assumption. Hence the left hand side is a regular local ring by Algebra, Lemma 110.9. $\square$

Lemma 12.6. *Let k be a field. Let X be a scheme locally of finite type over k. Let $x \in X$. Then X is geometrically regular at x if and only if $X \to \mathrm{Spec}(k)$ is smooth at x (Morphisms, Definition 34.1).*

Proof. The question is local around x, hence we may assume that $X = \mathrm{Spec}(A)$ for some finite type k-algebra. Let x correspond to the prime $\mathfrak{p}$.

If A is smooth over k at $\mathfrak{p}$, then we may localize A and assume that A is smooth over k. In this case $k' \otimes_k A$ is smooth over k' for all extension fields k'/k, and each of these Noetherian rings is regular by Algebra, Lemma 140.3.

Assume X is geometrically regular at x. Consider the residue field $K := \kappa(x) = \kappa(\mathfrak{p})$ of x. It is a finitely generated extension of k. By Algebra, Lemma 45.3 there exists a finite purely inseparable extension k'/k such that the compositum $k'K$ is a separable field extension of k'. Let $\mathfrak{p}' \subset A' = k' \otimes_k A$ be a prime ideal lying over $\mathfrak{p}$. It is the unique prime lying over $\mathfrak{p}$, see Algebra, Lemma 46.7. Hence the residue field $K' := \kappa(\mathfrak{p}')$ is the compositum $k'K$. By assumption the local ring $(A')_{\mathfrak{p}'}$ is regular. Hence by Algebra, Lemma 140.5 we see that $k' \to A'$ is smooth at $\mathfrak{p}'$. This in turn implies that $k \to A$ is smooth at $\mathfrak{p}$ by Algebra, Lemma 137.19. The lemma is proved. $\square$

Example 12.7. Let $k = \mathbf{F}_p(t)$. It is quite easy to give an example of a regular variety V over k which is not geometrically reduced. For example we can take $\mathrm{Spec}(k[x]/(x^p - t))$. In fact, there exists an example of a regular variety V which is geometrically reduced, but not even geometrically normal. Namely, take for $p > 2$ the scheme $V = \mathrm{Spec}(k[x,y]/(y^2 - x^p + t))$. This is a variety as the polynomial $y^2 - x^p + t \in k[x, y]$ is irreducible. The morphism $V \to \mathrm{Spec}(k)$ is smooth at all points except at the point $v_0 \in V$ corresponding to the maximal ideal $(y, x^p - t)$ (because $2y$ is invertible). In particular we see that V is (geometrically) regular at all points, except possibly v_0. The local ring

$$\mathcal{O}_{V,v_0} = \big(k[x,y]/(y^2 - x^p + t) \big)_{(y, x^p - t)}$$

is a domain of dimension 1. Its maximal ideal is generated by 1 element, namely y. Hence it is a discrete valuation ring and regular. Let $k' = k[t^{1/p}]$. Denote $t' = t^{1/p} \in k'$, $V' = V_{k'}$, $v_0' \in V'$ the unique point lying over v_0. Over k' we can write $x^p - t = (x - t')^p$, but the polynomial $y^2 - (x - t')^p$ is still irreducible and V' is still a variety. But the element

$$\frac{y}{x - t'} \in (\text{fraction field of } \mathcal{O}_{V', v_0'})$$

is integral over $\mathcal{O}_{V', v_0'}$ (just compute its square) and not contained in it, so V' is not normal at v_0'. This concludes the example.

13. Change of fields and the Cohen-Macaulay property

The following lemma says that it does not make sense to define geometrically Cohen-Macaulay schemes, since these would be the same as Cohen-Macaulay schemes.

Lemma 13.1. *Let X be a locally Noetherian scheme over the field k. Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. Then we have*

$$\mathcal{O}_{X,x} \text{ is Cohen-Macaulay} \Leftrightarrow \mathcal{O}_{X_{k'}, x'} \text{ is Cohen-Macaulay}$$

If X is locally of finite type over k, the same holds for any field extension k'/k.

Proof. The first case of the lemma follows from Algebra, Lemma 167.2. The second case of the lemma is equivalent to Algebra, Lemma 130.6. $\square$

14. Change of fields and the Jacobson property

A scheme locally of finite type over a field has plenty of closed points, namely it is Jacobson. Moreover, the residue fields are finite extensions of the ground field.

Lemma 14.1. *Let X be a scheme which is locally of finite type over k. Then*

(1) *for any closed point $x \in X$ the extension $\kappa(x)/k$ is algebraic, and*
(2) *X is a Jacobson scheme (Properties, Definition 6.1).*

Proof. A scheme is Jacobson if and only if it has an affine open covering by Jacobson schemes, see Properties, Lemma 6.3. The property on residue fields at closed points is also local on X. Hence we may assume that X is affine. In this case the result is a consequence of the Hilbert Nullstellensatz, see Algebra, Theorem 34.1. It also follows from a combination of Morphisms, Lemmas 16.8, 16.9, and 16.10. $\square$

It turns out that if X is not locally of finite type, then we can achieve the same result after making a suitably large base field extension.

Lemma 14.2. *Let X be a scheme over a field k. For any field extension K/k whose cardinality is large enough we have*

(1) *for any closed point $x \in X_K$ the extension $\kappa(x)/K$ is algebraic, and*
(2) *X_K is a Jacobson scheme (Properties, Definition 6.1).*

Proof. Choose an affine open covering $X = \bigcup U_i$. By Algebra, Lemma 35.12 and Properties, Lemma 6.2 there exist cardinals κ_i such that $U_{i,K}$ has the desired properties over K if $\#(K) \geq \kappa_i$. Set $\kappa = \max\{\kappa_i\}$. Then if the cardinality of K is larger than κ we see that each $U_{i,K}$ satisfies the conclusions of the lemma. Hence X_K is Jacobson by Properties, Lemma 6.3. The statement on residue fields at closed points of X_K follows from the corresponding statements for residue fields of closed points of the $U_{i,K}$. $\square$

15. Change of fields and ample invertible sheaves

The following result is typical for the results in this section.

Lemma 15.1. *Let k be a field. Let X be a scheme over k. If there exists an ample invertible sheaf on X_K for some field extension K/k, then X has an ample invertible sheaf.*

Proof. Let K/k be a field extension such that X_K has an ample invertible sheaf $\mathcal{L}$. The morphism $X_K \to X$ is surjective. Hence X is quasi-compact as the image of a quasi-compact scheme (Properties, Definition 26.1). Since X_K is quasi-separated (by Properties, Lemma 26.7) we see that X is quasi-separated: If $U, V \subset X$ are affine open, then $(U \cap V)_K = U_K \cap V_K$ is quasi-compact and $(U \cap V)_K \to U \cap V$ is surjective. Thus Schemes, Lemma 21.6 applies.

Write $K = \operatorname{colim} A_i$ as the colimit of the subalgebras of K which are of finite type over k. Denote $X_i = X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(A_i)$. Since $X_K = \lim X_i$ we find an i and an invertible sheaf $\mathcal{L}_i$ on X_i whose pullback to X_K is $\mathcal{L}$ (Limits, Lemma 10.3; here and below we use that X is quasi-compact and quasi-separated as just shown). By Limits, Lemma 4.15 we may assume $\mathcal{L}_i$ is ample after possibly increasing i. Fix such an i and let $\mathfrak{m} \subset A_i$ be a maximal ideal. By the Hilbert Nullstellensatz (Algebra, Theorem 34.1) the residue field $k' = A_i/\mathfrak{m}$ is a finite extension of k. Hence $X_{k'} \subset X_i$ is a closed subscheme hence has an ample invertible sheaf (Properties, Lemma 26.3). Since $X_{k'} \to X$ is finite locally free we conclude that X has an ample invertible sheaf by Divisors, Proposition 17.9. $\square$

Lemma 15.2. *Let k be a field. Let X be a scheme over k. If X_K is quasi-affine for some field extension K/k, then X is quasi-affine.*

Proof. Let K/k be a field extension such that X_K is quasi-affine. The morphism $X_K \to X$ is surjective. Hence X is quasi-compact as the image of a quasi-compact scheme (Properties, Definition 18.1). Since X_K is quasi-separated (as an open subscheme of an affine scheme) we see that X is quasi-separated: If $U, V \subset X$ are affine open, then $(U \cap V)_K = U_K \cap V_K$ is quasi-compact and $(U \cap V)_K \to U \cap V$ is surjective. Thus Schemes, Lemma 21.6 applies.

Write $K = \operatorname{colim} A_i$ as the colimit of the subalgebras of K which are of finite type over k. Denote $X_i = X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(A_i)$. Since $X_K = \lim X_i$ we find an i such that X_i is quasi-affine (Limits, Lemma 4.12; here we use that X is quasi-compact and quasi-separated as just shown). By the Hilbert Nullstellensatz (Algebra, Theorem

34.1) the residue field $k' = A_i/\mathfrak{m}$ is a finite extension of k. Hence $X_{k'} \subset X_i$ is a closed subscheme hence is quasi-affine (Properties, Lemma 27.2). Since $X_{k'} \to X$ is finite locally free we conclude by Divisors, Lemma 17.10. $\square$

Lemma 15.3. *Let k be a field. Let X be a scheme over k. If X_K is quasi-projective over K for some field extension K/k, then X is quasi-projective over k.*

Proof. By definition a morphism of schemes $g : Y \to T$ is quasi-projective if it is locally of finite type, quasi-compact, and there exists a g-ample invertible sheaf on Y. Let K/k be a field extension such that X_K is quasi-projective over K. Let $\mathrm{Spec}(A) \subset X$ be an affine open. Then U_K is an affine open subscheme of X_K, hence A_K is a K-algebra of finite type. Then A is a k-algebra of finite type by Algebra, Lemma 126.1. Hence $X \to \mathrm{Spec}(k)$ is locally of finite type. Since $X_K \to \mathrm{Spec}(K)$ is quasi-compact, we see that X_K is quasi-compact, hence X is quasi-compact, hence $X \to \mathrm{Spec}(k)$ is of finite type. By Morphisms, Lemma 39.4 we see that X_K has an ample invertible sheaf. Then X has an ample invertible sheaf by Lemma 15.1. Hence $X \to \mathrm{Spec}(k)$ is quasi-projective by Morphisms, Lemma 39.4. $\square$

The following lemma is a special case of Descent, Lemma 23.14.

Lemma 15.4. *Let k be a field. Let X be a scheme over k. If X_K is proper over K for some field extension K/k, then X is proper over k.*

Proof. Let K/k be a field extension such that X_K is proper over K. Recall that this implies X_K is separated and quasi-compact (Morphisms, Definition 41.1). The morphism $X_K \to X$ is surjective. Hence X is quasi-compact as the image of a quasi-compact scheme (Properties, Definition 26.1). Since X_K is separated we see that X is quasi-separated: If $U, V \subset X$ are affine open, then $(U \cap V)_K = U_K \cap V_K$ is quasi-compact and $(U \cap V)_K \to U \cap V$ is surjective. Thus Schemes, Lemma 21.6 applies.

Write $K = \mathrm{colim}\, A_i$ as the colimit of the subalgebras of K which are of finite type over k. Denote $X_i = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A_i)$. By Limits, Lemma 13.1 there exists an i such that $X_i \to \mathrm{Spec}(A_i)$ is proper. Here we use that X is quasi-compact and quasi-separated as just shown. Choose a maximal ideal $\mathfrak{m} \subset A_i$. By the Hilbert Nullstellensatz (Algebra, Theorem 34.1) the residue field $k' = A_i/\mathfrak{m}$ is a finite extension of k. The base change $X_{k'} \to \mathrm{Spec}(k')$ is proper (Morphisms, Lemma 41.5). Since k'/k is finite both $X_{k'} \to X$ and the composition $X_{k'} \to \mathrm{Spec}(k)$ are proper as well (Morphisms, Lemmas 44.11, 41.5, and 41.4). The first implies that X is separated over k as $X_{k'}$ is separated (Morphisms, Lemma 41.11). The second implies that $X \to \mathrm{Spec}(k)$ is proper by Morphisms, Lemma 41.9. $\square$

Lemma 15.5. *Let k be a field. Let X be a scheme over k. If X_K is projective over K for some field extension K/k, then X is projective over k.*

Proof. A scheme over k is projective over k if and only if it is quasi-projective and proper over k. See Morphisms, Lemma 43.13. Thus the lemma follows from Lemmas 15.3 and 15.4. $\square$

16. Tangent spaces

In this section we define the tangent space of a morphism of schemes at a point of the source using points with values in dual numbers.

Definition 16.1. For any ring R the *dual numbers* over R is the R-algebra denoted $R[\epsilon]$. As an R-module it is free with basis 1, ϵ and the R-algebra structure comes from setting $\epsilon^2 = 0$.

Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s = f(x)$ in S. Consider the solid commutative diagram

(16.1.1)

$$
\begin{array}{ccc}
\mathrm{Spec}(\kappa(x)) \longrightarrow \mathrm{Spec}(\kappa(x)[\epsilon]) \dashrightarrow X \\
\searrow \qquad \downarrow \qquad \downarrow \\
\mathrm{Spec}(\kappa(s)) \longrightarrow S
\end{array}
$$

with the curved arrow being the canonical morphism of $\mathrm{Spec}(\kappa(x))$ into X.

Lemma 16.2. *The set of dotted arrows making (16.1.1) commute has a canonical $\kappa(x)$-vector space structure.*

Proof. Set $\kappa = \kappa(x)$. Observe that we have a pushout in the category of schemes

$$\mathrm{Spec}(\kappa[\epsilon]) \amalg_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa[\epsilon]) = \mathrm{Spec}(\kappa[\epsilon_1, \epsilon_2])$$

where $\kappa[\epsilon_1, \epsilon_2]$ is the κ-algebra with basis $1, \epsilon_1, \epsilon_2$ and $\epsilon_1^2 = \epsilon_1\epsilon_2 = \epsilon_2^2 = 0$. This follows immediately from the corresponding result for rings and the description of morphisms from spectra of local rings to schemes in Schemes, Lemma 13.1. Given two arrows $\theta_1, \theta_2 : \mathrm{Spec}(\kappa[\epsilon]) \to X$ we can consider the morphism

$$\theta_1 + \theta_2 : \mathrm{Spec}(\kappa[\epsilon]) \to \mathrm{Spec}(\kappa[\epsilon_1, \epsilon_2]) \xrightarrow{\theta_1, \theta_2} X$$

where the first arrow is given by $\epsilon_i \mapsto \epsilon$. On the other hand, given $\lambda \in \kappa$ there is a self map of $\mathrm{Spec}(\kappa[\epsilon])$ corresponding to the κ-algebra endomorphism of $\kappa[\epsilon]$ which sends ϵ to $\lambda\epsilon$. Precomposing $\theta : \mathrm{Spec}(\kappa[\epsilon]) \to X$ by this selfmap gives $\lambda\theta$. The reader can verify the axioms of a vector space by verifying the existence of suitable commutative diagrams of schemes. We omit the details. (An alternative proof would be to express everything in terms of local rings and then verify the vector space axioms on the level of ring maps.) $\square$

Definition 16.3. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. The set of dotted arrows making (16.1.1) commute with its canonical $\kappa(x)$-vector space structure is called the *tangent space of X over S at x* and we denote it $T_{X/S,x}$. An element of this space is called a *tangent vector* of X/S at x.

Since tangent vectors at $x \in X$ live in the scheme theoretic fibre X_s of $f : X \to S$ over $s = f(x)$, we get a canonical identification

(16.3.1) $T_{X/S,x} = T_{X_s/s,x}$

This pleasing definition involving the functor of points has the following algebraic description, which suggests defining the *cotangent space of X over S at x* as the $\kappa(x)$-vector space

$$T^*_{X/S,x} = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

simply because it is canonically $\kappa(x)$-dual to the tangent space of X over S at x.

Lemma 16.4. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. There is a canonical isomorphism*

$$T_{X/S,x} = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\Omega_{X/S,x}, \kappa(x))$$

of vector spaces over $\kappa(x)$.

Proof. Set $\kappa = \kappa(x)$. Given $\theta \in T_{X/S,x}$ we obtain a map

$$\theta^* \Omega_{X/S} \to \Omega_{\mathrm{Spec}(\kappa[\epsilon])/\mathrm{Spec}(\kappa(s))} \to \Omega_{\mathrm{Spec}(\kappa[\epsilon])/\mathrm{Spec}(\kappa)}$$

Taking sections we obtain an $\mathcal{O}_{X,x}$-linear map $\xi_\theta : \Omega_{X/S,x} \to \kappa \mathrm{d}\epsilon$, i.e., an element of the right hand side of the formula of the lemma. To show that $\theta \mapsto \xi_\theta$ is an isomorphism we can replace S by s and X by the scheme theoretic fibre X_s. Indeed, both sides of the formula only depend on the scheme theoretic fibre; this is clear for $T_{X/S,x}$ and for the RHS see Morphisms, Lemma 32.10. We may also replace X by the spectrum of $\mathcal{O}_{X,x}$ as this does not change $T_{X/S,x}$ (Schemes, Lemma 13.1) nor $\Omega_{X/S,x}$ (Modules, Lemma 28.7).

Let $(A, \mathfrak{m}, \kappa)$ be a local ring over a field k. To finish the proof we have to show that any A-linear map $\xi : \Omega_{A/k} \to \kappa$ comes from a unique k-algebra map $\varphi : A \to \kappa[\epsilon]$ agreeing with the canonical map $c : A \to \kappa$ modulo ϵ. Write $\varphi(a) = c(a) + D(a)\epsilon$ the reader sees that $a \mapsto D(a)$ is a k-derivation. Using the universal property of $\Omega_{A/k}$ we see that each D corresponds to a unique ξ and vice versa. This finishes the proof. $\square$

Lemma 16.5. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point and let $s = f(x) \in S$. Assume that $\kappa(x) = \kappa(s)$. Then there are canonical isomorphisms*

$$\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

and

$$T_{X/S,x} = \mathrm{Hom}_{\kappa(x)}(\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}), \kappa(x))$$

This works more generally if $\kappa(x)/\kappa(s)$ is a separable algebraic extension.

Proof. The second isomorphism follows from the first by Lemma 16.4. For the first, we can replace S by s and X by X_s, see Morphisms, Lemma 32.10. We may also replace X by the spectrum of $\mathcal{O}_{X,x}$, see Modules, Lemma 28.7. Thus we have to show the following algebra fact: let $(A, \mathfrak{m}, \kappa)$ be a local ring over a field k such that κ/k is separable algebraic. Then the canonical map

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{A/k} \otimes \kappa$$

is an isomorphism. Observe that $\mathfrak{m}/\mathfrak{m}^2 = H_1(NL_{\kappa/A})$. By Algebra, Lemma 134.4 it suffices to show that $\Omega_{\kappa/k} = 0$ and $H_1(NL_{\kappa/k}) = 0$. Since κ is the union of its finite separable extensions in k it suffices to prove this when κ is a finite separable extension of k (Algebra, Lemma 134.9). In this case the ring map $k \to \kappa$ is étale and hence $NL_{\kappa/k} = 0$ (more or less by definition, see Algebra, Section 143). $\square$

Lemma 16.6. *Let $f : X \to Y$ be a morphism of schemes over a base scheme S. Let $x \in X$ be a point. Set $y = f(x)$. If $\kappa(y) = \kappa(x)$, then f induces a natural linear map*

$$df : T_{X/S,x} \longrightarrow T_{Y/S,y}$$

which is dual to the linear map $\Omega_{Y/S,y} \otimes \kappa(y) \to \Omega_{X/S,x}$ via the identifications of Lemma 16.4.

Proof. Omitted. $\square$

Lemma 16.7. *Let X, Y be schemes over a base S. Let $x \in X$ and $y \in Y$ with the same image point $s \in S$ such that $\kappa(s) = \kappa(x)$ and $\kappa(s) = \kappa(y)$. There is a canonical isomorphism*

$$T_{X \times_S Y/S,(x,y)} = T_{X/S,x} \oplus T_{Y/S,y}$$

The map from left to right is induced by the maps on tangent spaces coming from the projections $X \times_S Y \to X$ and $X \times_S Y \to Y$. The map from right to left is induced by the maps $1 \times y : X_s \to X_s \times_s Y_s$ and $x \times 1 : Y_s \to X_s \times_s Y_s$ via the identification (16.3.1) of tangent spaces with tangent spaces of fibres.

Proof. The direct sum decomposition follows from Morphisms, Lemma 32.11 via Lemma 16.5. Compatibility with the maps comes from Lemma 16.6. $\qquad\square$

Lemma 16.8. *Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base scheme S. Let $x \in X$ be a point. Set $y = f(x)$ and assume that $\kappa(y) = \kappa(x)$. Then the following are equivalent*

(1) *$df : T_{X/S,x} \longrightarrow T_{Y/S,y}$ is injective, and*
(2) *f is unramified at x.*

Proof. The morphism f is locally of finite type by Morphisms, Lemma 15.8. The map df is injective, if and only if $\Omega_{Y/S,y} \otimes \kappa(y) \to \Omega_{X/S,x} \otimes \kappa(x)$ is surjective (Lemma 16.6). The exact sequence $f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$ (Morphisms, Lemma 32.9) then shows that this happens if and only if $\Omega_{X/Y,x} \otimes \kappa(x) = 0$. Hence the result follows from Morphisms, Lemma 35.14. $\qquad\square$

17. Generically finite morphisms

In this section we revisit the notion of a generically finite morphism of schemes as studied in Morphisms, Section 51.

Lemma 17.1. *Let $f : X \to Y$ be locally of finite type. Let $y \in Y$ be a point such that $\mathcal{O}_{Y,y}$ is Noetherian of dimension ≤ 1. Assume in addition one of the following conditions is satisfied*

(1) *for every generic point η of an irreducible component of X the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),*
(2) *for every generic point η of an irreducible component of X such that $f(\eta) \rightsquigarrow y$ the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),*
(3) *f is quasi-finite at every generic point of an irreducible component of X,*
(4) *Y is locally Noetherian and f is quasi-finite at a dense set of points of X,*
(5) *add more here.*

Then f is quasi-finite at every point of X lying over y.

Proof. Condition (4) implies X is locally Noetherian (Morphisms, Lemma 15.6). The set of points at which morphism is quasi-finite is open (Morphisms, Lemma 56.2). A dense open of a locally Noetherian scheme contains all generic point of irreducible components, hence (4) implies (3). Condition (3) implies condition (1) by Morphisms, Lemma 20.5. Condition (1) implies condition (2). Thus it suffices to prove the lemma in case (2) holds.

Assume (2) holds. Recall that $\mathrm{Spec}(\mathcal{O}_{Y,y})$ is the set of points of Y specializing to y, see Schemes, Lemma 13.2. Combined with Morphisms, Lemma 20.13 this shows

we may replace Y by $\mathrm{Spec}(\mathcal{O}_{Y,y})$. Thus we may assume $Y = \mathrm{Spec}(B)$ where B is a Noetherian local ring of dimension ≤ 1 and y is the closed point.

Let $X = \bigcup X_i$ be the irreducible components of X viewed as reduced closed subschemes. If we can show each fibre $X_{i,y}$ is a discrete space, then $X_y = \bigcup X_{i,y}$ is discrete as well and we conclude that $X \to Y$ is quasi-finite at all points of X_y by Morphisms, Lemma 20.6. Thus we may assume X is an integral scheme.

If $X \to Y$ maps the generic point η of X to y, then X is the spectrum of a finite extension of $\kappa(y)$ and the result is true. Assume that X maps η to a point corresponding to a minimal prime $\mathfrak{q}$ of B different from $\mathfrak{m}_B$. We obtain a factorization $X \to \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(B)$. Let $x \in X$ be a point lying over y. By the dimension formula (Morphisms, Lemma 52.1) we have

$$\dim(\mathcal{O}_{X,x}) \leq \dim(B/\mathfrak{q}) + \mathrm{trdeg}_{\kappa(\mathfrak{q})}(R(X)) - \mathrm{trdeg}_{\kappa(y)}\kappa(x)$$

We know that $\dim(B/\mathfrak{q}) = 1$, that the generic point of X is not equal to x and specializes to x and that $R(X)$ is algebraic over $\kappa(\mathfrak{q})$. Thus we get

$$1 \leq 1 - \mathrm{trdeg}_{\kappa(y)}\kappa(x)$$

Hence every point x of X_y is closed in X_y by Morphisms, Lemma 20.2 and hence $X \to Y$ is quasi-finite at every point x of X_y by Morphisms, Lemma 20.6 (which also implies that X_y is a discrete topological space). $\square$

Lemma 17.2. *Let $f : X \to Y$ be a proper morphism. Let $y \in Y$ be a point such that $\mathcal{O}_{Y,y}$ is Noetherian of dimension ≤ 1. Assume in addition one of the following conditions is satisfied*

(1) *for every generic point η of an irreducible component of X the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),*
(2) *for every generic point η of an irreducible component of X such that $f(\eta) \rightsquigarrow y$ the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),*
(3) *f is quasi-finite at every generic point of X,*
(4) *Y is locally Noetherian and f is quasi-finite at a dense set of points of X,*
(5) *add more here.*

Then there exists an open neighbourhood $V \subset Y$ of y such that $f^{-1}(V) \to V$ is finite.

Proof. By Lemma 17.1 the morphism f is quasi-finite at every point of the fibre X_y. Hence X_y is a discrete topological space (Morphisms, Lemma 20.6). As f is proper the fibre X_y is quasi-compact, i.e., finite. Thus we can apply Cohomology of Schemes, Lemma 21.2 to conclude. $\square$

Lemma 17.3. *Let X be a Noetherian scheme. Let $f : Y \to X$ be a birational proper morphism of schemes with Y reduced. Let $U \subset X$ be the maximal open over which f is an isomorphism. Then U contains*

(1) *every point of codimension 0 in X,*
(2) *every $x \in X$ of codimension 1 on X such that $\mathcal{O}_{X,x}$ is a discrete valuation ring,*
(3) *every $x \in X$ such that the fibre of $Y \to X$ over x is finite and such that $\mathcal{O}_{X,x}$ is normal, and*
(4) *every $x \in X$ such that f is quasi-finite at some $y \in Y$ lying over x and $\mathcal{O}_{X,x}$ is normal.*

Proof. Part (1) follows from Morphisms, Lemma 51.6. Part (2) follows from part (3) and Lemma 17.2 (and the fact that finite morphisms have finite fibres).

Part (3) follows from part (4) and Morphisms, Lemma 20.7 but we will also give a direct proof. Let $x \in X$ be as in (3). By Cohomology of Schemes, Lemma 21.2 we may assume f is finite. We may assume X affine. This reduces us to the case of a finite birational morphism of Noetherian affine schemes $Y \to X$ and $x \in X$ such that $\mathcal{O}_{X,x}$ is a normal domain. Since $\mathcal{O}_{X,x}$ is a domain and X is Noetherian, we may replace X by an affine open of x which is integral. Then, since $Y \to X$ is birational and Y is reduced we see that Y is integral. Writing $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ we see that $A \subset B$ is a finite inclusion of domains having the same field of fractions. If $\mathfrak{p} \subset A$ is the prime corresponding to x, then $A_\mathfrak{p}$ being normal implies that $A_\mathfrak{p} \subset B_\mathfrak{p}$ is an equality. Since B is a finite A-module, we see there exists an $a \in A$, $a \notin \mathfrak{p}$ such that $A_a \to B_a$ is an isomorphism.

Let $x \in X$ and $y \in Y$ be as in (4). After replacing X by an affine open neighbourhood we may assume $X = \mathrm{Spec}(A)$ and $A \subset \mathcal{O}_{X,x}$, see Properties, Lemma 29.8. Then A is a domain and hence X is integral. Since f is birational and Y is reduced it follows that Y is integral too. Consider the ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$. This is a ring map which is essentially of finite type, the residue field extension is finite, and $\dim(\mathcal{O}_{Y,y}/\mathfrak{m}_x \mathcal{O}_{Y,y}) = 0$ (to see this trace through the definitions of quasi-finite maps in Morphisms, Definition 20.1 and Algebra, Definition 122.3). By Algebra, Lemma 124.2 $\mathcal{O}_{Y,y}$ is the localization of a finite $\mathcal{O}_{X,x}$-algebra B. Of course we may replace B by the image of B in $\mathcal{O}_{Y,y}$ and assume that B is a domain with the same fraction field as $\mathcal{O}_{Y,y}$. Then $\mathcal{O}_{X,x} \subset B$ have the same fraction field as f is birational. Since $\mathcal{O}_{X,x}$ is normal, we conclude that $\mathcal{O}_{X,x} = B$ (because finite implies integral), in particular, we see that $\mathcal{O}_{X,x} = \mathcal{O}_{Y,y}$. By Morphisms, Lemma 42.4 after shrinking X we may assume there is a section $X \to Y$ of f mapping x to y and inducing the given isomorphism on local rings. Since $X \to Y$ is closed (by Schemes, Lemma 21.11) necessarily maps the generic point of X to the generic point of Y it follows that the image of $X \to Y$ is Y. Then $Y = X$ and we've proved what we wanted to show. $\square$

18. Variants of Noether normalization

Noether normalization is the statement that if k is a field and A is a finite type k algebra of dimension d, then there exists a finite injective k-algebra homomorphism $k[x_1, \ldots, x_d] \to A$. See Algebra, Lemma 115.4. Geometrically this means there is a finite surjective morphism $\mathrm{Spec}(A) \to \mathbf{A}_k^d$ over $\mathrm{Spec}(k)$.

Lemma 18.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $V \subset S$ be an affine open neighbourhood of s. If f is locally of finite type and $\dim_x(X_s) = d$, then there exists an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ and a factorization*

$$U \xrightarrow{\pi} \mathbf{A}_V^d \to V$$

of $f|_U : U \to V$ such that π is quasi-finite.

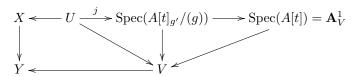**Proof.** This follows from Algebra, Lemma 125.2. $\square$

Lemma 18.2. *Let $f : X \to S$ be a finite type morphism of affine schemes. Let $s \in S$. If $\dim(X_s) = d$, then there exists a factorization*

$$X \xrightarrow{\pi} \mathbf{A}_S^d \to S$$

of f such that the morphism $\pi_s : X_s \to \mathbf{A}_{\kappa(s)}^d$ of fibres over s is finite.

Proof. Write $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ and let $A \to B$ be the ring map corresponding to f. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. We can choose a surjection $A[x_1, \ldots, x_r] \to B$. By Algebra, Lemma 115.4 there exist elements $y_1, \ldots, y_d \in A$ in the $\mathbf{Z}$-subalgebra of A generated by $x_1, \ldots, x_r$ such that the A-algebra homomorphism $A[t_1, \ldots, t_d] \to B$ sending t_i to y_i induces a finite $\kappa(\mathfrak{p})$-algebra homomorphism $\kappa(\mathfrak{p})[t_1, \ldots, t_d] \to B \otimes_A \kappa(\mathfrak{p})$. This proves the lemma. $\square$

Lemma 18.3. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Let $V = \operatorname{Spec}(A)$ be an affine open neighbourhood of $f(x)$ in S. If f is unramified at x, then there exist exists an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ such that we have a commutative diagram*

$$
\begin{array}{ccccccc}
X & \longleftarrow & U & \xrightarrow{\ j\ } & \operatorname{Spec}(A[t]_{g'}/(g)) & \longrightarrow & \operatorname{Spec}(A[t]) = \mathbf{A}_V^1 \\
\downarrow & & & \searrow & \downarrow & \swarrow & \\
Y & \longleftarrow & & & V & &
\end{array}
$$

where j is an immersion, $g \in A[t]$ is a monic polynomial, and g' is the derivative of g with respect to t. If f is étale at x, then we may choose the diagram such that j is an open immersion.

Proof. The unramified case is a translation of Algebra, Proposition 152.1. In the étale case this is a translation of Algebra, Proposition 144.4 or equivalently it follows from Morphisms, Lemma 36.14 although the statements differ slightly. $\square$

Lemma 18.4. *Let $f : X \to S$ be a finite type morphism of affine schemes. Let $x \in X$ with image $s \in S$. Let*

$$r = \dim_{\kappa(x)} \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) = \dim_{\kappa(x)} \Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \dim_{\kappa(x)} T_{X/S,x}$$

Then there exists a factorization

$$X \xrightarrow{\pi} \mathbf{A}_S^r \to S$$

of f such that π is unramified at x.

Proof. By Morphisms, Lemma 32.12 the first dimension is finite. The first equality follows as the restriction of $\Omega_{X/S}$ to the fibre is the module of differentials from Morphisms, Lemma 32.10. The last equality follows from Lemma 16.4. Thus we see that the statement makes sense.

To prove the lemma write $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ and let $A \to B$ be the ring map corresponding to f. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Choose a surjection of A-algebras $A[x_1, \ldots, x_t] \to B$. Since $\Omega_{B/A}$ is generated by $dx_1, \ldots, dx_t$ we see that their images in $\Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ generate this as a $\kappa(x)$-vector space. After renumbering we may assume that $dx_1, \ldots, dx_r$ map to a basis of $\Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$. We claim that $P = A[x_1, \ldots, x_r] \to B$ is unramified at $\mathfrak{q}$. To see this it suffices to show that $\Omega_{B/P,\mathfrak{q}} = 0$ (Algebra, Lemma 151.3). Note

that $\Omega_{B/P}$ is the quotient of $\Omega_{B/A}$ by the submodule generated by $dx_1, \ldots, dx_r$. Hence $\Omega_{B/P,\mathfrak{q}} \otimes_{B_\mathfrak{q}} \kappa(\mathfrak{q}) = 0$ by our choice of $x_1, \ldots, x_r$. By Nakayama's lemma, more precisely Algebra, Lemma 20.1 part (2) which applies as $\Omega_{B/P}$ is finite (see reference above), we conclude that $\Omega_{B/P,\mathfrak{q}} = 0$. $\qquad\square$

Lemma 18.5. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $V \subset S$ be an affine open neighbourhood of s. If f is locally of finite type and*

$$r = \dim_{\kappa(x)} \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) = \dim_{\kappa(x)} \Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \dim_{\kappa(x)} T_{X/S,x}$$

then there exist

(1) *an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ and a factorization*

$$U \xrightarrow{j} \mathbf{A}_V^{r+1} \to V$$

of $f|_U$ such that j is an immersion, or

(2) *an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ and a factorization*

$$U \xrightarrow{j} D \to V$$

of $f|_U$ such that j is a closed immersion and $D \to V$ is smooth of relative dimension r.

Proof. Pick any affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$. Apply Lemma 18.4 to $U \to V$ to get $U \to \mathbf{A}_V^r \to V$ as in the statement of that lemma. By Lemma 18.3 we get a factorization

$$U \xrightarrow{j} D \xrightarrow{j'} \mathbf{A}_V^{r+1} \xrightarrow{p} \mathbf{A}_V^r \to V$$

where j and j' are immersions, p is the projection, and $p \circ j'$ is standard étale. Thus we see in particular that (1) and (2) hold. $\qquad\square$

19. Dimension of fibres

We have already seen that dimension of fibres of finite type morphisms typically jump up. In this section we discuss the phenomenon that in codimension 1 this does not happen. More generally, we discuss how much the dimension of a fibre can jump. Here is a list of related results:

(1) For a finite type morphism $X \to S$ the set of $x \in X$ with $\dim_x(X_{f(x)}) \leq d$ is open, see Algebra, Lemma 125.6 and Morphisms, Lemma 28.4.
(2) We have the dimension formula, see Algebra, Lemma 113.1 and Morphisms, Lemma 52.1.
(3) Constant fibre dimension for an integral finite type scheme dominating a valuation ring, see Algebra, Lemma 125.9.
(4) If $X \to S$ is of finite type and is quasi-finite at every generic point of X, then $X \to S$ is quasi-finite in codimension 1, see Algebra, Lemma 113.2 and Lemma 17.1.

The last result mentioned above generalizes as follows.

Lemma 19.1. *Let $f : X \to Y$ be locally of finite type. Let $x \in X$ be a point with image $y \in Y$ such that $\mathcal{O}_{Y,y}$ is Noetherian of dimension ≤ 1. Let $d \geq 0$ be an integer such that for every generic point η of an irreducible component of X which contains x, we have $\dim_\eta(X_{f(\eta)}) = d$. Then $\dim_x(X_y) = d$.*

Proof. Recall that $\mathrm{Spec}(\mathcal{O}_{Y,y})$ is the set of points of Y specializing to y, see Schemes, Lemma 13.2. Thus we may replace Y by $\mathrm{Spec}(\mathcal{O}_{Y,y})$ and assume $Y = \mathrm{Spec}(B)$ where B is a Noetherian local ring of dimension ≤ 1 and y is the closed point. We may also replace X by an affine neighbourhood of x.

Let $X = \bigcup X_i$ be the irreducible components of X viewed as reduced closed subschemes. If we can show each fibre $X_{i,y}$ has dimension d, then $X_y = \bigcup X_{i,y}$ has dimension d as well. Thus we may assume X is an integral scheme.

If $X \to Y$ maps the generic point η of X to y, then X is a scheme over $\kappa(y)$ and the result is true by assumption. Assume that X maps η to a point $\xi \in Y$ corresponding to a minimal prime $\mathfrak{q}$ of B different from $\mathfrak{m}_B$. We obtain a factorization $X \to \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(B)$. By the dimension formula (Morphisms, Lemma 52.1) we have

$$\dim(\mathcal{O}_{X,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) \leq \dim(B/\mathfrak{q}) + \mathrm{trdeg}_{\kappa(\mathfrak{q})}(R(X))$$

We have $\dim(B/\mathfrak{q}) = 1$. We have $\mathrm{trdeg}_{\kappa(\mathfrak{q})}(R(X)) = d$ by our assumption that $\dim_\eta(X_\xi) = d$, see Morphisms, Lemma 28.1. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_s,x}$ has a kernel (as $\eta \mapsto \xi \neq y$) and since $\mathcal{O}_{X,x}$ is a Noetherian domain we see that $\dim(\mathcal{O}_{X,x}) > \dim(\mathcal{O}_{X_y,x})$. We conclude that

$$\dim_x(X_s) = \dim(\mathcal{O}_{X_s,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) \leq d$$

(Morphisms, Lemma 28.1). On the other hand, we have $\dim_x(X_s) \geq \dim_\eta(X_{f(\eta)}) = d$ by Morphisms, Lemma 28.4. $\square$

Lemma 19.2. *Let $f : X \to \mathrm{Spec}(R)$ be a morphism from an irreducible scheme to the spectrum of a valuation ring. If f is locally of finite type and surjective, then the special fibre is equidimensional of dimension equal to the dimension of the generic fibre.*

Proof. We may replace X by its reduction because this does not change the dimension of X or of the special fibre. Then X is integral and the lemma follows from Algebra, Lemma 125.9. $\square$

The following lemma generalizes Lemma 19.1.

Lemma 19.3. *Let $f : X \to Y$ be locally of finite type. Let $x \in X$ be a point with image $y \in Y$ such that $\mathcal{O}_{Y,y}$ is Noetherian. Let $d \geq 0$ be an integer such that for every generic point η of an irreducible component of X which contains x, we have $f(\eta) \neq y$ and $\dim_\eta(X_{f(\eta)}) = d$. Then $\dim_x(X_y) \leq d + \dim(\mathcal{O}_{Y,y}) - 1$.*

Proof. Exactly as in the proof of Lemma 19.1 we reduce to the case $X = \mathrm{Spec}(A)$ with A a domain and $Y = \mathrm{Spec}(B)$ where B is a Noetherian local ring whose maximal ideal corresponds to y. After replacing B by $B/\mathrm{Ker}(B \to A)$ we may assume that B is a domain and that $B \subset A$. Then we use the dimension formula (Morphisms, Lemma 52.1) to get

$$\dim(\mathcal{O}_{X,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) \leq \dim(B) + \mathrm{trdeg}_B(A)$$

We have $\mathrm{trdeg}_B(A) = d$ by our assumption that $\dim_\eta(X_\xi) = d$, see Morphisms, Lemma 28.1. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_y,x}$ has a kernel (as $f(\eta) \neq y$) and since $\mathcal{O}_{X,x}$ is a Noetherian domain we see that $\dim(\mathcal{O}_{X,x}) > \dim(\mathcal{O}_{X_y,x})$. We conclude that

$$\dim_x(X_y) = \dim(\mathcal{O}_{X_y,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) < \dim(B) + d$$

(equality by Morphisms, Lemma 28.1) which proves what we want. $\square$

20. Algebraic schemes

The following definition is taken from [DG67, I Definition 6.4.1].

Definition 20.1. Let k be a field. An *algebraic k-scheme* is a scheme X over k such that the structure morphism $X \to \mathrm{Spec}(k)$ is of finite type. A *locally algebraic k-scheme* is a scheme X over k such that the structure morphism $X \to \mathrm{Spec}(k)$ is locally of finite type.

Note that every (locally) algebraic k-scheme is (locally) Noetherian, see Morphisms, Lemma 15.6. The category of algebraic k-schemes has all products and fibre products (unlike the category of varieties over k). Similarly for the category of locally algebraic k-schemes.

Lemma 20.2. *Let k be a field. Let X be a locally algebraic k-scheme of dimension 0. Then X is a disjoint union of spectra of local Artinian k-algebras A with $\dim_k(A) < \infty$. If X is an algebraic k-scheme of dimension 0, then in addition X is affine and the morphism $X \to \mathrm{Spec}(k)$ is finite.*

Proof. Let X be a locally algebraic k-scheme of dimension 0. Let $U = \mathrm{Spec}(A) \subset X$ be an affine open subscheme. Since $\dim(X) = 0$ we see that $\dim(A) = 0$. By Noether normalization, see Algebra, Lemma 115.4 we see that there exists a finite injection $k \to A$, i.e., $\dim_k(A) < \infty$. Hence A is Artinian, see Algebra, Lemma 53.2. This implies that $A = A_1 \times \ldots \times A_r$ is a product of finitely many Artinian local rings, see Algebra, Lemma 53.6. Of course $\dim_k(A_i) < \infty$ for each i as the sum of these dimensions equals $\dim_k(A)$.

The arguments above show that X has an open covering whose members are finite discrete topological spaces. Hence X is a discrete topological space. It follows that X is isomorphic to the disjoint union of its connected components each of which is a singleton. Since a singleton scheme is affine we conclude (by the results of the paragraph above) that each of these singletons is the spectrum of a local Artinian k-algebra A with $\dim_k(A) < \infty$.

Finally, if X is an algebraic k-scheme of dimension 0, then X is quasi-compact hence is a finite disjoint union $X = \mathrm{Spec}(A_1) \amalg \ldots \amalg \mathrm{Spec}(A_r)$ hence affine (see Schemes, Lemma 6.8) and we have seen the finiteness of $X \to \mathrm{Spec}(k)$ in the first paragraph of the proof. $\square$

The following lemma collects some statements on dimension theory for locally algebraic schemes.

Lemma 20.3. *Let k be a field. Let X be a locally algebraic k-scheme.*
  (1) *The topological space of X is catenary (Topology, Definition 11.4).*
  (2) *For $x \in X$ closed, we have $\dim_x(X) = \dim(\mathcal{O}_{X,x})$.*
  (3) *For X irreducible we have $\dim(X) = \dim(U)$ for any nonempty open $U \subset X$ and $\dim(X) = \dim_x(X)$ for any $x \in X$.*
  (4) *For X irreducible any chain of irreducible closed subsets can be extended to a maximal chain and all maximal chains of irreducible closed subsets have length equal to $\dim(X)$.*
  (5) *For $x \in X$ we have $\dim_x(X) = \max \dim(Z) = \min \dim(\mathcal{O}_{X,x'})$ where the maximum is over irreducible components $Z \subset X$ containing x and the minimum is over specializations $x \rightsquigarrow x'$ with x' closed in X.*

(6) *If X is irreducible with generic point x, then $\dim(X) = trdeg_k(\kappa(x))$.*

(7) *If $x \rightsquigarrow x'$ is an immediate specialization of points of X, then we have $trdeg_k(\kappa(x)) = trdeg_k(\kappa(x')) + 1$.*

(8) *The dimension of X is the supremum of the numbers $trdeg_k(\kappa(x))$ where x runs over the generic points of the irreducible components of X.*

(9) *If $x \rightsquigarrow x'$ is a nontrivial specialization of points of X, then*
   (a) *$\dim_x(X) \le \dim_{x'}(X)$,*
   (b) *$\dim(\mathcal{O}_{X,x}) < \dim(\mathcal{O}_{X,x'})$,*
   (c) *$trdeg_k(\kappa(x)) > trdeg_k(\kappa(x'))$, and*
   (d) *any maximal chain of nontrivial specializations $x = x_0 \rightsquigarrow x_1 \rightsquigarrow \ldots \rightsquigarrow x_n = x$ has length $n = trdeg_k(\kappa(x)) - trdeg_k(\kappa(x'))$.*

(10) *For $x \in X$ we have $\dim_x(X) = trdeg_k(\kappa(x)) + \dim(\mathcal{O}_{X,x})$.*

(11) *If $x \rightsquigarrow x'$ is an immediate specialization of points of X and X is irreducible or equidimensional, then $\dim(\mathcal{O}_{X,x'}) = \dim(\mathcal{O}_{X,x}) + 1$.*

Proof. Instead on relying on the more general results proved earlier we will reduce the statements to the corresponding statements for finite type k-algebras and cite results from the chapter on commutative algebra.

Proof of (1). This is local on X by Topology, Lemma 11.5. Thus we may assume $X = \mathrm{Spec}(A)$ where A is a finite type k-algebra. We have to show that A is catenary (Algebra, Lemma 105.2). We can reduce to $k[x_1, \ldots, x_n]$ using Algebra, Lemma 105.7 and then apply Algebra, Lemma 114.3. Alternatively, this holds because k is Cohen-Macaulay (trivially) and Cohen-Macaulay rings are universally catenary (Algebra, Lemma 105.9).

Proof of (2). Choose an affine neighbourhood $U = \mathrm{Spec}(A)$ of x. Then $\dim_x(X) = \dim_x(U)$. Hence we reduce to the affine case, which is Algebra, Lemma 114.6.

Proof of (3). It suffices to show that any two nonempty affine opens $U, U' \subset X$ have the same dimension (any finite chain of irreducible subsets meets an affine open). Pick a closed point x of X with $x \in U \cap U'$. This is possible because X is irreducible, hence $U \cap U'$ is nonempty, hence there is such a closed point because X is Jacobson by Lemma 14.1. Then $\dim(U) = \dim(\mathcal{O}_{X,x}) = \dim(U')$ by Algebra, Lemma 114.4 (strictly speaking you have to replace X by its reduction before applying the lemma).

Proof of (4). Given a chain of irreducible closed subsets we can find an affine open $U \subset X$ which meets the smallest one. Thus the statement follows from Algebra, Lemma 114.4 and $\dim(U) = \dim(X)$ which we have seen in (3).

Proof of (5). Choose an affine neighbourhood $U = \mathrm{Spec}(A)$ of x. Then $\dim_x(X) = \dim_x(U)$. The rule $Z \mapsto Z \cap U$ is a bijection between irreducible components of X passing through x and irreducible components of U passing through x. Also, $\dim(Z \cap U) = \dim(Z)$ for such Z by (3). Hence the statement follows from Algebra, Lemma 114.5.

Proof of (6). By (3) this reduces to the case where $X = \mathrm{Spec}(A)$ is affine. In this case it follows from Algebra, Lemma 116.1 applied to A_{red}.

Proof of (7). Let $Z = \overline{\{x\}} \supset Z' = \overline{\{x'\}}$. Then it follows from (4) that $Z \supset Z'$ is the start of a maximal chain of irreducible closed subschemes in Z and consequently $\dim(Z) = \dim(Z') + 1$. We conclude by (6).

Proof of (8). A simple topological argument shows that $\dim(X) = \sup \dim(Z)$ where the supremum is over the irreducible components of X (hint: use Topology, Lemma 8.3). Thus this follows from (6).

Proof of (9). Part (a) follows from the fact that any open $U \subset X$ containing x' also contains x. Part (b) follows because $\mathcal{O}_{X,x}$ is a localization of $\mathcal{O}_{X,x'}$ hence any chain of primes in $\mathcal{O}_{X,x}$ corresponds to a chain of primes in $\mathcal{O}_{X,x'}$ which can be extended by adding $\mathfrak{m}_{x'}$ at the end. Both (c) and (d) follow formally from (7).

Proof of (10). Choose an affine neighbourhood $U = \operatorname{Spec}(A)$ of x. Then $\dim_x(X) = \dim_x(U)$. Hence we reduce to the affine case, which is Algebra, Lemma 116.3.

Proof of (11). If X is equidimensional (Topology, Definition 10.5) then $\dim(X)$ is equal to the dimension of every irreducible component of X, whence $\dim_x(X) = \dim(X) = \dim_{x'}(X)$ by (5). Thus this follows from (7). $\square$

Lemma 20.4. *Let k be a field. Let $f : X \to Y$ be a morphism of locally algebraic k-schemes.*
  1. *For $y \in Y$, the fibre X_y is a locally algebraic scheme over $\kappa(y)$ hence all the results of Lemma 20.3 apply.*
  2. *Assume X is irreducible. Set $Z = \overline{f(X)}$ and $d = \dim(X) - \dim(Z)$. Then*
     (a) *$\dim_x(X_{f(x)}) \geq d$ for all $x \in X$,*
     (b) *the set of $x \in X$ with $\dim_x(X_{f(x)}) = d$ is dense open,*
     (c) *if $\dim(\mathcal{O}_{Z,f(x)}) \geq 1$, then $\dim_x(X_{f(x)}) \leq d + \dim(\mathcal{O}_{Z,f(x)}) - 1$,*
     (d) *if $\dim(\mathcal{O}_{Z,f(x)}) = 1$, then $\dim_x(X_{f(x)}) = d$,*
  3. *For $x \in X$ with $y = f(x)$ we have $\dim_x(X_y) \geq \dim_x(X) - \dim_y(Y)$.*

Proof. The morphism f is locally of finite type by Morphisms, Lemma 15.8. Hence the base change $X_y \to \operatorname{Spec}(\kappa(y))$ is locally of finite type. This proves (1). In the rest of the proof we will freely use the results of Lemma 20.3 for X, Y, and the fibres of f.

Proof of (2). Let $\eta \in X$ be the generic point and set $\xi = f(\eta)$. Then $Z = \overline{\{\xi\}}$. Hence

$$d = \dim(X) - \dim(Z) = \operatorname{trdeg}_k \kappa(\eta) - \operatorname{trdeg}_k \kappa(\xi) = \operatorname{trdeg}_{\kappa(\xi)} \kappa(\eta) = \dim_\eta(X_\xi)$$

Thus parts (2)(a) and (2)(b) follow from Morphisms, Lemma 28.4. Parts (2)(c) and (2)(d) follow from Lemmas 19.3 and 19.1.

Proof of (3). Let $x \in X$. Let $X' \subset X$ be a irreducible component of X passing through x of dimension $\dim_x(X)$. Then (2) implies that $\dim_x(X_y) \geq \dim(X') - \dim(Z')$ where $Z' \subset Y$ is the closure of the image of X'. This proves (3). $\square$

Lemma 20.5. *Let k be a field. Let X, Y be locally algebraic k-schemes.*
  1. *For $z \in X \times Y$ lying over (x, y) we have $\dim_z(X \times Y) = \dim_x(X) + \dim_y(Y)$.*
  2. *We have $\dim(X \times Y) = \dim(X) + \dim(Y)$.*

Proof. Proof of (1). Consider the factorization

$$X \times Y \longrightarrow Y \longrightarrow \operatorname{Spec}(k)$$

of the structure morphism. The first morphism $p : X \times Y \to Y$ is flat as a base change of the flat morphism $X \to \operatorname{Spec}(k)$ by Morphisms, Lemma 25.8. Moreover, we have $\dim_z(p^{-1}(y)) = \dim_x(X)$ by Morphisms, Lemma 28.3. Hence

$\dim_z(X \times Y) = \dim_x(X) + \dim_y(Y)$ by Morphisms, Lemma 28.2. Part (2) is a direct consequence of (1). $\square$

21. Complete local rings

Some results on complete local rings of schemes over fields.

Lemma 21.1. *Let k be a field. Let X be a locally Noetherian scheme over k. Let $x \in X$ be a point with residue field κ. There is an isomorphism*

(21.1.1) $$\kappa[[x_1, \ldots, x_n]]/I \longrightarrow \mathcal{O}_{X,x}^\wedge$$

inducing the identity on residue fields. In general we cannot choose (21.1.1) to be a k-algebra isomorphism. However, if the extension κ/k is separable, then we can choose (21.1.1) to be an isomorphism of k-algebras.

Proof. The existence of the isomorphism is an immediate consequence of the Cohen structure theorem[2] (Algebra, Theorem 160.8).

Let p be an odd prime number, let $k = \mathbf{F}_p(t)$, and $A = k[x,y]/(y^2 + x^p - t)$. Then the completion $A^\wedge$ of A in the maximal ideal $\mathfrak{m} = (y)$ is isomorphic to $k(t^{1/p})[[z]]$ as a ring but not as a k-algebra. The reason is that $A^\wedge$ does not contain an element whose pth power is t (as the reader can see by computing modulo y^2). This also shows that any isomorphism (21.1.1) cannot be a k-algebra isomorphism.

If κ/k is separable, then there is a k-algebra homomorphism $\kappa \to \mathcal{O}_{X,x}^\wedge$ inducing the identity on residue fields by More on Algebra, Lemma 38.3. Let $f_1, \ldots, f_n \in \mathfrak{m}_x$ be generators. Consider the map

$$\kappa[[x_1, \ldots, x_n]] \longrightarrow \mathcal{O}_{X,x}^\wedge, \quad x_i \longmapsto f_i$$

Since both sides are $(x_1, \ldots, x_n)$-adically complete (the right hand side by Algebra, Lemmas 96.3) this map is surjective by Algebra, Lemma 96.1 as it is surjective modulo $(x_1, \ldots, x_n)$ by construction. $\square$

Lemma 21.2. *Let K/k be an extension of fields. Let X be a locally algebraic k-scheme. Set $Y = X_K$. Let $y \in Y$ be a point with image $x \in X$. Assume that $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$ and that $\kappa(x)/k$ is separable. Choose an isomorphism*

$$\kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \longrightarrow \mathcal{O}_{X,x}^\wedge$$

of k-algebras as in (21.1.1). Then we have an isomorphism

$$\kappa(y)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \longrightarrow \mathcal{O}_{Y,y}^\wedge$$

of K-algebras as in (21.1.1). Here we use $\kappa(x) \to \kappa(y)$ to view g_j as a power series over $\kappa(y)$.

Proof. The local ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ induces a local ring map $\mathcal{O}_{X,x}^\wedge \to \mathcal{O}_{Y,y}^\wedge$. The induced map

$$\kappa(x) \to \kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \to \mathcal{O}_{X,x}^\wedge \to \mathcal{O}_{Y,y}^\wedge$$

composed with the projection to $\kappa(y)$ is the canonical homomorphism $\kappa(x) \to \kappa(y)$. By Lemma 5.1 the residue field $\kappa(y)$ is a localization of $\kappa(x) \otimes_k K$ at the kernel $\mathfrak{p}_0$ of

---

[2]Note that if κ has characteristic p, then the theorem just says we get a surjection $\Lambda[[x_1, \ldots, x_n]] \to \mathcal{O}_{X,x}^\wedge$ where Λ is a Cohen ring for κ. But of course in this case the map factors through $\Lambda/p\Lambda[[x_1, \ldots, x_n]]$ and $\Lambda/p\Lambda = \kappa$.

$\kappa(x) \otimes_k K \to \kappa(y)$. On the other hand, by Lemma 5.3 the local ring $(\kappa(x) \otimes_k K)_{\mathfrak{p}_0}$ is equal to $\kappa(y)$. Hence the map

$$\kappa(x) \otimes_k K \to \mathcal{O}^\wedge_{Y,y}$$

factors canonically through $\kappa(y)$. We obtain a commutative diagram

$$
\begin{array}{ccc}
\kappa(y) & \longrightarrow & \mathcal{O}^\wedge_{Y,y} \\
\uparrow & & \uparrow \\
\kappa(x) \longrightarrow \kappa(x)[[x_1,\ldots,x_n]]/(g_1,\ldots,g_m) & \longrightarrow & \mathcal{O}^\wedge_{X,x}
\end{array}
$$

Let $f_i \in \mathfrak{m}^\wedge_x \subset \mathcal{O}^\wedge_{X,x}$ be the image of x_i. Observe that $\mathfrak{m}^\wedge_x = (f_1,\ldots,f_n)$ as the map is surjective. Consider the map

$$\kappa(y)[[x_1,\ldots,x_n]] \longrightarrow \mathcal{O}^\wedge_{Y,y}, \quad x_i \longmapsto f_i$$

where here f_i really means the image of f_i in $\mathfrak{m}^\wedge_y$. Since $\mathfrak{m}_x \mathcal{O}_{Y,y} = \mathfrak{m}_y$ by Lemma 5.3 we see that the right hand side is complete with respect to $(x_1,\ldots,x_n)$ (use Algebra, Lemma 96.3 to see that it is a complete local ring). Since both sides are $(x_1,\ldots,x_n)$-adically complete our map is surjective by Algebra, Lemma 96.1 as it is surjective modulo $(x_1,\ldots,x_n)$. Of course the power series $g_1,\ldots,g_m$ are mapped to zero under this map, as they already map to zero in $\mathcal{O}^\wedge_{X,x}$. Thus we have the commutative diagram

$$
\begin{array}{ccc}
\kappa(y)[[x_1,\ldots,x_n]]/(g_1,\ldots,g_m) & \longrightarrow & \mathcal{O}^\wedge_{Y,y} \\
\uparrow & & \uparrow \\
\kappa(x)[[x_1,\ldots,x_n]]/(g_1,\ldots,g_m) & \longrightarrow & \mathcal{O}^\wedge_{X,x}
\end{array}
$$

We still need to show that the top horizontal arrow is an isomorphism. We already know that it is surjective. We know that $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is flat (Lemma 5.1), which implies that $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{Y,y}$ is flat (More on Algebra, Lemma 43.8). Thus we may apply Algebra, Lemma 99.1 with $R = \kappa(x)[[x_1,\ldots,x_n]]/(g_1,\ldots,g_m)$, with $S = \kappa(y)[[x_1,\ldots,x_n]]/(g_1,\ldots,g_m)$, with $M = \mathcal{O}^\wedge_{Y,y}$, and with $N = S$ to conclude that the map is injective. $\square$

22. Global generation

Some lemmas related to global generation of quasi-coherent modules.

Lemma 22.1. *Let $X \to \mathrm{Spec}(A)$ be a morphism of schemes. Let $A \subset A'$ be a faithfully flat ring map. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is globally generated if and only if the base change $\mathcal{F}_{A'}$ is globally generated.*

Proof. More precisely, set $X_{A'} = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$. Let $\mathcal{F}_{A'} = p^* \mathcal{F}$ where $p : X_{A'} \to X$ is the projection. By Cohomology of Schemes, Lemma 5.2 we have $H^0(X_{k'}, \mathcal{F}_{A'}) = H^0(X, \mathcal{F}) \otimes_A A'$. Thus if s_i, $i \in I$ are generators for $H^0(X, \mathcal{F})$ as an A-module, then their images in $H^0(X_{A'}, \mathcal{F}_{A'})$ are generators for $H^0(X_{A'}, \mathcal{F}_{A'})$ as an A'-module. Thus we have to show that the map $\alpha : \bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$, $(f_i) \mapsto \sum f_i s_i$ is surjective if and only if $p^* \alpha$ is surjective. This we may check over an affine open $U = \mathrm{Spec}(B)$ of X. Then $\mathcal{F}|_U$ corresponds to a B-module M and $s_i|_U$ to elements $x_i \in M$. Thus we have to show that $\bigoplus_{i \in I} B \to M$ is surjective if and only if the

base change $\bigoplus_{i \in I} B \otimes_A A' \to M \otimes_A A'$ is surjective. This is true because $A \to A'$ is faithfully flat. $\qquad\square$

Lemma 22.2. *Let k be an infinite field. Let X be a scheme of finite type over k. Let $\mathcal{L}$ be a very ample invertible sheaf on X. Let $n \geq 0$ and $x, x_1, \ldots, x_n \in X$ be points with x a k-rational point, i.e., $\kappa(x) = k$, and $x \neq x_i$ for $i = 1, \ldots, n$. Then there exists an $s \in H^0(X, \mathcal{L})$ which vanishes at x but not at x_i.*

Proof. If $n = 0$ the result is trivial, hence we assume $n > 0$. By definition of a very ample invertible sheaf, the lemma immediately reduces to the case where $X = \mathbf{P}_k^r$ for some $r > 0$ and $\mathcal{L} = \mathcal{O}_X(1)$. Write $\mathbf{P}_k^r = \operatorname{Proj}(k[T_0, \ldots, T_r])$. Set $V = H^0(X, \mathcal{L}) = kT_0 \oplus \ldots \oplus kT_r$. Since x is a k-rational point, we see that the set $s \in V$ which vanish at x is a codimension 1 subspace $W \subset V$ and that W generates the homogeneous prime ideal corresponding to x. Since $x_i \neq x$ the corresponding homogeneous prime $\mathfrak{p}_i \subset k[T_0, \ldots, T_r]$ does not contain W. Since k is infinite, we then see that $W \neq \bigcup W \cap \mathfrak{q}_i$ and the proof is complete. $\qquad\square$

Lemma 22.3. *Let k be an infinite field. Let X be an algebraic k-scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $V \to \Gamma(X, \mathcal{L})$ be a linear map of k-vector spaces whose image generates $\mathcal{L}$. Then there exists a subspace $W \subset V$ with $\dim_k(W) \leq \dim(X) + 1$ which generates $\mathcal{L}$.*

Proof. Throughout the proof we will use that for every $x \in X$ the linear map

$$\psi_x : V \to \Gamma(X, \mathcal{L}) \to \mathcal{L}_x \to \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is nonzero. The proof is by induction on $\dim(X)$.

The base case is $\dim(X) = 0$. In this case X has finitely many points $X = \{x_1, \ldots, x_n\}$ (see for example Lemma 20.2). Since k is infinite there exists a vector $v \in V$ such that $\psi_{x_i}(v) \neq 0$ for all i. Then $W = k \cdot v$ does the job.

Assume $\dim(X) > 0$. Let $X_i \subset X$ be the irreducible components of dimension equal to $\dim(X)$. Since X is Noetherian there are only finitely many of these. For each i pick a point $x_i \in X_i$. As above choose $v \in V$ such that $\psi_{x_i}(v) \neq 0$ for all i. Let $Z \subset X$ be the zero scheme of the image of v in $\Gamma(X, \mathcal{L})$, see Divisors, Definition 14.8. By construction $\dim(Z) < \dim(X)$. By induction we can find $W \subset V$ with $\dim(W) \leq \dim(X)$ such that W generates $\mathcal{L}|_Z$. Then $W + k \cdot v$ generates $\mathcal{L}$. $\qquad\square$

23. Separating points and tangent vectors

This is just the following result.

Lemma 23.1. *Let k be an algebraically closed field. Let X be a proper k-scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $V \subset H^0(X, \mathcal{L})$ be a k-subvector space. If*

(1) *for every pair of distinct closed points $x, y \in X$ there is a section $s \in V$ which vanishes at x but not at y, and*

(2) *for every closed point $x \in X$ and nonzero tangent vector $\theta \in T_{X/k,x}$ there exists a section $s \in V$ which vanishes at x but whose pullback by θ is nonzero,*

then $\mathcal{L}$ is very ample and the canonical morphism $\varphi_{\mathcal{L},V} : X \to \mathbf{P}(V)$ is a closed immersion.

Proof. Condition (1) implies in particular that the elements of V generate $\mathcal{L}$ over X. Hence we get a canonical morphism

$$\varphi = \varphi_{\mathcal{L},V} : X \longrightarrow \mathbf{P}(V)$$

by Constructions, Example 21.2. The morphism φ is proper by Morphisms, Lemma 41.7. By (1) the map φ is injective on closed points (computation omitted). In particular, the fibre over any closed point of $\mathbf{P}(V)$ is a singleton (small detail omitted). Thus we see that φ is finite, for example use Cohomology of Schemes, Lemma 21.2. To finish the proof it suffices to show that the map

$$\varphi^\sharp : \mathcal{O}_{\mathbf{P}(V)} \longrightarrow \varphi_* \mathcal{O}_X$$

is surjective. This we may check on stalks at closed points. Let $x \in X$ be a closed point with image the closed point $p = \varphi(x) \in \mathbf{P}(V)$. Since $\varphi^{-1}(\{p\}) = \{x\}$ by (1) and since φ is proper (hence closed), we see that $\varphi^{-1}(U)$ runs through a fundamental system of open neighbourhoods of x as U runs through a fundamental system of open neighbourhoods of p. We conclude that on stalks at p we obtain the map

$$\varphi_x^\sharp : \mathcal{O}_{\mathbf{P}(V),p} \longrightarrow \mathcal{O}_{X,x}$$

In particular, $\mathcal{O}_{X,x}$ is a finite $\mathcal{O}_{\mathbf{P}(V),p}$-module. Moreover, the residue fields of x and p are equal to k (as k is algebraically closed – use the Hilbert Nullstellensatz). Finally, condition (2) implies that the map

$$T_{X/k,x} \longrightarrow T_{\mathbf{P}(V)/k,p}$$

is injective since any nonzero θ in the kernel of this map couldn't possibly satisfy the conclusion of (2). In terms of the map of local rings above this means that

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \longrightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$$

is surjective, see Lemma 16.5. Now the proof is finished by applying Algebra, Lemma 20.3. $\square$

Lemma 23.2. *Let k be an algebraically closed field. Let X be a proper k-scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Suppose that for every closed subscheme $Z \subset X$ of dimension 0 and degree 2 over k the map*

$$H^0(X, \mathcal{L}) \longrightarrow H^0(Z, \mathcal{L}|_Z)$$

is surjective. Then $\mathcal{L}$ is very ample on X over k.

Proof. This is a reformulation of Lemma 23.1. Namely, given distinct closed points $x, y \in X$ taking $Z = x \cup y$ (viewed as closed subscheme) we get condition (1) of the lemma. And given a nonzero tangent vector $\theta \in T_{X/k,x}$ the morphism $\theta : \mathrm{Spec}(k[\epsilon]) \to X$ is a closed immersion. Setting $Z = \mathrm{Im}(\theta)$ we obtain condition (2) of the lemma. $\square$

24. Closures of products

Some results on the relation between closure and products.

Lemma 24.1. *Let k be a field. Let X, Y be schemes over k, and let $A \subset X$, $B \subset Y$ be subsets. Set*

$$AB = \{z \in X \times_k Y \mid pr_X(z) \in A, \ pr_Y(z) \in B\} \subset X \times_k Y$$

Then set theoretically we have

$$\overline{A} \times_k \overline{B} = \overline{AB}$$

Proof. The inclusion $\overline{AB} \subset \overline{A} \times_k \overline{B}$ is immediate. We may replace X and Y by the reduced closed subschemes $\overline{A}$ and $\overline{B}$. Let $W \subset X \times_k Y$ be a nonempty open subset. By Morphisms, Lemma 23.4 the subset $U = \mathrm{pr}_X(W)$ is nonempty open in X. Hence $A \cap U$ is nonempty. Pick $a \in A \cap U$. Denote $Y_{\kappa(a)} = \{a\} \times_k Y$ the fibre of $\mathrm{pr}_X : X \times_k Y \to X$ over a. By Morphisms, Lemma 23.4 again the morphism $Y_a \to Y$ is open as $\mathrm{Spec}(\kappa(a)) \to \mathrm{Spec}(k)$ is universally open. Hence the nonempty open subset $W_a = W \times_{X \times_k Y} Y_a$ maps to a nonempty open subset of Y. We conclude there exists a $b \in B$ in the image. Hence $AB \cap W \neq \emptyset$ as desired. $\square$

Lemma 24.2. *Let k be a field. Let $f : A \to X$, $g : B \to Y$ be morphisms of schemes over k. Then set theoretically we have*

$$\overline{f(A)} \times_k \overline{g(B)} = \overline{(f \times g)(A \times_k B)}$$

Proof. This follows from Lemma 24.1 as the image of $f \times g$ is $f(A)g(B)$ in the notation of that lemma. $\square$

Lemma 24.3. *Let k be a field. Let $f : A \to X$, $g : B \to Y$ be quasi-compact morphisms of schemes over k. Let $Z \subset X$ be the scheme theoretic image of f, see Morphisms, Definition 6.2. Similarly, let $Z' \subset Y$ be the scheme theoretic image of g. Then $Z \times_k Z'$ is the scheme theoretic image of $f \times g$.*

Proof. Recall that Z is the smallest closed subscheme of X through which f factors. Similarly for Z'. Let $W \subset X \times_k Y$ be the scheme theoretic image of $f \times g$. As $f \times g$ factors through $Z \times_k Z'$ we see that $W \subset Z \times_k Z'$.

To prove the other inclusion let $U \subset X$ and $V \subset Y$ be affine opens. By Morphisms, Lemma 6.3 the scheme $Z \cap U$ is the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$, and similarly for $Z' \cap V$ and $W \cap U \times_k V$. Hence we may assume X and Y affine. As f and g are quasi-compact this implies that $A = \bigcup U_i$ is a finite union of affines and $B = \bigcup V_j$ is a finite union of affines. Then we may replace A by $\coprod U_i$ and B by $\coprod V_j$, i.e., we may assume that A and B are affine as well. In this case Z is cut out by $\mathrm{Ker}(\Gamma(X, \mathcal{O}_X) \to \Gamma(A, \mathcal{O}_A))$ and similarly for Z' and W. Hence the result follows from the equality

$$\Gamma(A \times_k B, \mathcal{O}_{A \times_k B}) = \Gamma(A, \mathcal{O}_A) \otimes_k \Gamma(B, \mathcal{O}_B)$$

which holds as A and B are affine. Details omitted. $\square$

25. Schemes smooth over fields

Here are two lemmas characterizing smooth schemes over fields.

Lemma 25.1. *Let k be a field. Let X be a scheme over k. Assume*

1. *X is locally of finite type over k,*
2. *$\Omega_{X/k}$ is locally free, and*
3. *k has characteristic zero.*

Then the structure morphism $X \to \mathrm{Spec}(k)$ is smooth.

Proof. This follows from Algebra, Lemma 140.7. $\square$

In positive characteristic there exist nonreduced schemes of finite type whose sheaf of differentials is free, for example $\mathrm{Spec}(\mathbf{F}_p[t]/(t^p))$ over $\mathrm{Spec}(\mathbf{F}_p)$. If the ground field k is nonperfect of characteristic p, there exist reduced schemes X/k with free $\Omega_{X/k}$ which are nonsmooth, for example $\mathrm{Spec}(k[t]/(t^p - a))$ where $a \in k$ is not a pth power.

Lemma 25.2. *Let k be a field. Let X be a scheme over k. Assume*

(1) *X is locally of finite type over k,*
(2) *$\Omega_{X/k}$ is locally free,*
(3) *X is reduced, and*
(4) *k is perfect.*

Then the structure morphism $X \to \mathrm{Spec}(k)$ is smooth.

Proof. Let $x \in X$ be a point. As X is locally Noetherian (see Morphisms, Lemma 15.6) there are finitely many irreducible components $X_1, \ldots, X_n$ passing through x (see Properties, Lemma 5.5 and Topology, Lemma 9.2). Let $\eta_i \in X_i$ be the generic point. As X is reduced we have $\mathcal{O}_{X,\eta_i} = \kappa(\eta_i)$, see Algebra, Lemma 25.1. Moreover, $\kappa(\eta_i)$ is a finitely generated field extension of the perfect field k hence separably generated over k (see Algebra, Section 42). It follows that $\Omega_{X/k,\eta_i} = \Omega_{\kappa(\eta_i)/k}$ is free of rank the transcendence degree of $\kappa(\eta_i)$ over k. By Morphisms, Lemma 28.1 we conclude that $\dim_{\eta_i}(X_i) = \mathrm{rank}_{\eta_i}(\Omega_{X/k})$. Since $x \in X_1 \cap \ldots \cap X_n$ we see that

$$\mathrm{rank}_x(\Omega_{X/k}) = \mathrm{rank}_{\eta_i}(\Omega_{X/k}) = \dim(X_i).$$

Therefore $\dim_x(X) = \mathrm{rank}_x(\Omega_{X/k})$, see Algebra, Lemma 114.5. It follows that $X \to \mathrm{Spec}(k)$ is smooth at x for example by Algebra, Lemma 140.3. $\square$

Lemma 25.3. *Let $X \to \mathrm{Spec}(k)$ be a smooth morphism where k is a field. Then X is a regular scheme.*

Proof. (See also Lemma 12.6.) By Algebra, Lemma 140.3 every local ring $\mathcal{O}_{X,x}$ is regular. And because X is locally of finite type over k it is locally Noetherian. Hence X is regular by Properties, Lemma 9.2. $\square$

Lemma 25.4. *Let $X \to \mathrm{Spec}(k)$ be a smooth morphism where k is a field. Then X is geometrically regular, geometrically normal, and geometrically reduced over k.*

Proof. (See also Lemma 12.6.) Let k' be a finite purely inseparable extension of k. It suffices to prove that $X_{k'}$ is regular, normal, reduced, see Lemmas 12.3, 10.3, and 6.5. By Morphisms, Lemma 34.5 the morphism $X_{k'} \to \mathrm{Spec}(k')$ is smooth too. Hence it suffices to show that a scheme X smooth over a field is regular, normal, and reduced. We see that X is regular by Lemma 25.3. Hence Properties, Lemma 9.4 guarantees that X is normal. $\square$

Lemma 25.5. *Let k be a field. Let $d \geq 0$. Let $W \subset \mathbf{A}_k^d$ be nonempty open. Then there exists a closed point $w \in W$ such that $k \subset \kappa(w)$ is finite separable.*

Proof. After possible shrinking W we may assume that $W = \mathbf{A}_k^d \setminus V(f)$ for some $f \in k[x_1, \ldots, x_d]$. If the lemma is wrong then $f(a_1, \ldots, a_d) = 0$ for all $(a_1, \ldots, a_d) \in (k^{sep})^d$. This is absurd as k^{sep} is an infinite field. $\square$

Lemma 25.6. *Let k be a field. If X is smooth over $\mathrm{Spec}(k)$ then the set*

$$\{x \in X \ closed \ such \ that \ k \subset \kappa(x) \ is \ finite \ separable\}$$

is dense in X.

Proof. It suffices to show that given a nonempty smooth X over k there exists at least one closed point whose residue field is finite separable over k. To see this, choose a diagram

$$X \longleftarrow U \xrightarrow{\ \pi\ } \mathbf{A}_k^d$$

with π étale, see Morphisms, Lemma 36.20. The morphism $\pi : U \to \mathbf{A}_k^d$ is open, see Morphisms, Lemma 36.13. By Lemma 25.5 we may choose a closed point $w \in \pi(U)$ whose residue field is finite separable over k. Pick any $x \in U$ with $\pi(x) = w$. By Morphisms, Lemma 36.7 the field extension $\kappa(x)/\kappa(w)$ is finite separable. Hence $\kappa(x)/k$ is finite separable. The point x is a closed point of X by Morphisms, Lemma 20.2. $\qquad\square$

Lemma 25.7. *Let X be a scheme over a field k. If X is locally of finite type and geometrically reduced over k then X contains a dense open which is smooth over k.*

Proof. The problem is local on X, hence we may assume X is quasi-compact. Let $X = X_1 \cup \ldots \cup X_n$ be the irreducible components of X. Then $Z = \bigcup_{i \neq j} X_i \cap X_j$ is nowhere dense in X. Hence we may replace X by $X \setminus Z$. As $X \setminus Z$ is a disjoint union of irreducible schemes, this reduces us to the case where X is irreducible. As X is irreducible and reduced, it is integral, see Properties, Lemma 3.4. Let $\eta \in X$ be its generic point. Then the function field $K = k(X) = \kappa(\eta)$ is geometrically reduced over k, hence separable over k, see Algebra, Lemma 44.1. Let $U = \mathrm{Spec}(A) \subset X$ be any nonempty affine open so that $K = A_{(0)}$ is the fraction field of A. Apply Algebra, Lemma 140.5 to conclude that A is smooth at (0) over k. By definition this means that some principal localization of A is smooth over k and we win. $\quad\square$

Lemma 25.8. *Let k be a perfect field. Let X be a locally algebraic reduced k-scheme, for example a variety over k. Then we have*

$$\{x \in X \mid X \to \mathrm{Spec}(k) \ is \ smooth \ at \ x\} = \{x \in X \mid \mathcal{O}_{X,x} \ is \ regular\}$$

and this is a dense open subscheme of X.

Proof. The equality of the two sets follows immediately from Algebra, Lemma 140.5 and the definitions (see Algebra, Definition 45.1 for the definition of a perfect field). The set is open because the set of points where a morphism of schemes is smooth is open, see Morphisms, Definition 34.1. Finally, we give two arguments to see that it is dense: (1) The generic points of X are in the set as the local rings at generic points are fields (Algebra, Lemma 25.1) hence regular. (2) We use that X is geometrically reduced by Lemma 6.3 and hence Lemma 25.7 applies. $\quad\square$

Lemma 25.9. *Let k be a field. Let $f : X \to Y$ be a morphism of schemes locally of finite type over k. Let $x \in X$ be a point and set $y = f(x)$. If $X \to \mathrm{Spec}(k)$ is smooth at x and f is flat at x then $Y \to \mathrm{Spec}(k)$ is smooth at y. In particular, if X is smooth over k and f is flat and surjective, then Y is smooth over k.*

Proof. It suffices to show that Y is geometrically regular at y, see Lemma 12.6. This follows from Lemma 12.5 (and Lemma 12.6 applied to (X, x)). $\qquad\square$

Lemma 25.10. *Let k be a field. Let X be a variety over k which has a k-rational point x such that X is smooth at x. Then X is geometrically integral over k.*

Proof. Let $U \subset X$ be the smooth locus of X. By assumption U is nonempty and hence dense and scheme theoretically dense. Then $U_{\overline{k}} \subset X_{\overline{k}}$ is dense and scheme theoretically dense as well (some details omitted). Thus it suffices to show that U is geometrically integral. Because U has a k-rational point it is geometrically connected by Lemma 7.14. On the other hand, $U_{\overline{k}}$ is reduced and normal (Lemma 25.4. Since a connected normal Noetherian scheme is integral (Properties, Lemma 7.6) the proof is complete. $\square$

Lemma 25.11. *Let X be a scheme of finite type over a field k. There exists a finite purely inseparable extension k'/k, an integer $t \geq 0$, and closed subschemes*

$$X_{k'} \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_0 = (X_{k'})_{red}$ and $Z_i \setminus Z_{i+1}$ is smooth over k' for all i.

Proof. We may use induction on $\dim(X)$. By Lemma 6.11 we can find a finite purely inseparable extension k'/k such that $(X_{k'})_{red}$ is geometrically reduced over k'. By Lemma 25.7 there is a nowhere dense closed subscheme $X' \subset (X_{k'})_{red}$ such that $(X_{k'})_{red} \setminus X'$ is smooth over k'. Then $\dim(X') < \dim(X)$. By induction hypothesis there exists a finite purely inseparable extension k''/k', an integer $t' \geq 0$, and closed subschemes

$$X'_{k''} \supset Y_0 \supset Y_1 \supset \ldots \supset Y_{t'} = \emptyset$$

such that $Y_0 = (X'_{k''})_{red}$ and $Y_i \setminus Y_{i+1}$ is smooth over k'' for all i. Then we let $t = t' + 1$ and we consider

$$X_{k''} \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

given by $Z_0 = (X_{k''})_{red}$ and $Z_i = Y_{i-1}$ for $i > 0$; this makes sense as $X'_{k''}$ is a closed subscheme of $X_{k''}$. We omit the verification that all the stated properties hold. $\square$

26. Types of varieties

Short section discussion some elementary global properties of varieties.

Definition 26.1. Let k be a field. Let X be a variety over k.
  (1) We say X is an *affine variety* if X is an affine scheme. This is equivalent to requiring X to be isomorphic to a closed subscheme of $\mathbf{A}_k^n$ for some n.
  (2) We say X is a *projective variety* if the structure morphism $X \to \mathrm{Spec}(k)$ is projective. By Morphisms, Lemma 43.4 this is true if and only if X is isomorphic to a closed subscheme of $\mathbf{P}_k^n$ for some n.
  (3) We say X is a *quasi-projective variety* if the structure morphism $X \to \mathrm{Spec}(k)$ is quasi-projective. By Morphisms, Lemma 40.6 this is true if and only if X is isomorphic to a locally closed subscheme of $\mathbf{P}_k^n$ for some n.
  (4) A *proper variety* is a variety such that the morphism $X \to \mathrm{Spec}(k)$ is proper.
  (5) A *smooth variety* is a variety such that the morphism $X \to \mathrm{Spec}(k)$ is smooth.

Note that a projective variety is a proper variety, see Morphisms, Lemma 43.5. Also, an affine variety is quasi-projective as $\mathbf{A}_k^n$ is isomorphic to an open subscheme of $\mathbf{P}_k^n$, see Constructions, Lemma 13.3.

Lemma 26.2. *Let X be a proper variety over k. Then*

(1) $K = H^0(X, \mathcal{O}_X)$ is a field which is a finite extension of the field k,
(2) if X is geometrically reduced, then K/k is separable,
(3) if X is geometrically irreducible, then K/k is purely inseparable,
(4) if X is geometrically integral, then $K = k$.

Proof. This is a special case of Lemma 9.3. $\qquad\square$

27. Normalization

Some issues associated to normalization.

Lemma 27.1. *Let k be a field. Let X be a locally algebraic scheme over k. Let $\nu : X^\nu \to X$ be the normalization morphism, see Morphisms, Definition 54.1. Then*

(1) *ν is finite, dominant, and X^ν is a disjoint union of normal irreducible locally algebraic schemes over k,*
(2) *ν factors as $X^\nu \to X_{red} \to X$ and the first morphism is the normalization morphism of X_{red},*
(3) *if X is a reduced algebraic scheme, then ν is birational,*
(4) *if X is a variety, then X^ν is a variety and ν is a finite birational morphism of varieties.*

Proof. Since X is locally of finite type over a field, we see that X is locally Noetherian (Morphisms, Lemma 15.6) hence every quasi-compact open has finitely many irreducible components (Properties, Lemma 5.7). Thus Morphisms, Definition 54.1 applies. The normalization X^ν is always a disjoint union of normal integral schemes and the normalization morphism ν is always dominant, see Morphisms, Lemma 54.5. Since X is universally Nagata (Morphisms, Lemma 18.2) we see that ν is finite (Morphisms, Lemma 54.10). Hence X^ν is locally algebraic too. At this point we have proved (1).

Part (2) is Morphisms, Lemma 54.2.

Part (3) is Morphisms, Lemma 54.7.

Part (4) follows from (1), (2), (3), and the fact that X^ν is separated as a scheme finite over a separated scheme. $\qquad\square$

Lemma 27.2. *Let k be a field. Let X be a proper scheme over k. Let $\nu : X^\nu \to X$ be the normalization morphism, see Morphisms, Definition 54.1. Then X^ν is proper over k. If X is projective over k, then X^ν is projective over k.*

Proof. By Lemma 27.1 the morphism ν is finite. Hence X^ν is proper over k by Morphisms, Lemmas 44.11 and 41.4. The morphism ν is projective by Morphisms, Lemma 44.16 and hence if X is projective over k, then X^ν is projective over k by Morphisms, Lemma 43.14. $\qquad\square$

Lemma 27.3. *Let k be a field. Let $f : Y \to X$ be a quasi-compact morphism of locally algebraic schemes over k. Let X' be the normalization of X in Y. If Y is reduced, then $X' \to X$ is finite.*

Proof. Since Y is quasi-separated (by Properties, Lemma 5.4 and Morphisms, Lemma 15.6) the morphism f is quasi-separated (Schemes, Lemma 21.13). Hence Morphisms, Definition 53.3 applies. The result follows from Morphisms, Lemma 53.14. This uses that locally algebraic schemes are locally Noetherian (hence have

locally finitely many irreducible components) and that locally algebraic schemes are Nagata (Morphisms, Lemma 18.2). Some small details omitted. $\square$

Lemma 27.4. *Let k be a field. Let X be an algebraic k-scheme. Then there exists a finite purely inseparable extension k'/k such that the normalization Y of $X_{k'}$ is geometrically normal over k'.*

Proof. Let $K = k^{perf}$ be the perfect closure. Let Y_K be the normalization of X_K, see Lemma 27.1. By Limits, Lemma 10.1 there exists a finite sub extension $K/k'/k$ and a morphism $\nu : Y \to X_{k'}$ of finite presentation whose base change to K is the normalization morphism $\nu_K : Y_K \to X_K$. Observe that Y is geometrically normal over k' (Lemma 10.3). After increasing k' we may assume $Y \to X_{k'}$ is finite (Limits, Lemma 8.3). Since $\nu_K : Y_K \to X_K$ is the normalization morphism, it induces a birational morphism $Y_K \to (X_K)_{red}$. Hence there is a dense open $V_K \subset X_K$ such that $\nu_K^{-1}(V_K) \to V_K$ is a closed immersion (inducing an isomorphism of $\nu_K^{-1}(V_K)$ with $V_{K,red}$, see for example Morphisms, Lemma 51.6). After increasing k' we find V_K is the base change of a dense open $V \subset Y$ and the morphism $\nu^{-1}(V) \to V$ is a closed immersion (Limits, Lemmas 4.11 and 8.5). It follows readily from this that ν is the normalization morphism and the proof is complete. $\square$

Lemma 27.5. *Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be an extension of fields. Let $\nu : X^\nu \to X$ be the normalization of X and let $Y^\nu \to X_K$ be the normalization of the base change. Then the canonical morphism*

$$Y^\nu \longrightarrow X^\nu \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$$

is an isomorphism if K/k is separable and a universal homeomorphism in general.

Proof. Set $Y = X_K$. Let $X^{(0)}$, resp. $Y^{(0)}$ be the set of generic points of irreducible components of X, resp. Y. Then the projection morphism $\pi : Y \to X$ satisfies $\pi(Y^{(0)}) = X^{(0)}$. This is true because π is surjective, open, and generizing, see Morphisms, Lemmas 23.4 and 23.5. If we view $X^{(0)}$, resp. $Y^{(0)}$ as (reduced) schemes, then X^ν, resp. Y^ν is the normalization of X, resp. Y in $X^{(0)}$, resp. $Y^{(0)}$. Thus Morphisms, Lemma 53.5 gives a canonical morphism $Y^\nu \to X^\nu$ over $Y \to X$ which in turn gives the canonical morphism of the lemma by the universal property of the fibre product.

To prove this morphism has the properties stated in the lemma we may assume $X = \mathrm{Spec}(A)$ is affine. Let $Q(A_{red})$ be the total ring of fractions of A_{red}. Then X^ν is the spectrum of the integral closure A' of A in $Q(A_{red})$, see Morphisms, Lemmas 54.2 and 54.3. Similarly, Y^ν is the spectrum of the integral closure B' of $A \otimes_k K$ in $Q((A \otimes_k K)_{red})$. There is a canonical map $Q(A_{red}) \to Q((A \otimes_k K)_{red})$, a canonical map $A' \to B'$, and the morphism of the lemma corresponds to the induced map

$$A' \otimes_k K \longrightarrow B'$$

of K-algebras. The kernel consists of nilpotent elements as the kernel of $Q(A_{red}) \otimes_k K \to Q((A \otimes_k K)_{red})$ is the set of nilpotent elements.

If K/k is separable, then $A' \otimes_k K$ is normal by Lemma 10.6. In particular it is reduced, whence $Q((A \otimes_k K)_{red}) = Q(A' \otimes_k K)$ and $B' = A' \otimes_k K$ by Algebra, Lemma 37.16.

Assume K/k is not separable. Then the characteristic of k is $p > 0$. We will show that for every $b \in B'$ there is a power q of p such that b^q is in the image of

$A' \otimes_k K$. This will prove that the displayed map is a universal homeomorphism by Algebra, Lemma 46.7. For a given b there is a subfield $F \subset K$ with F/k finitely generated such that b is contained in $Q((A \otimes_k F)_{red})$ and is integral over $A \otimes_k F$. Choose a monic polynomial $P = T^d + \alpha_1 T^{d-1} + \ldots + \alpha_d$ with $P(b) = 0$ and $\alpha_i \in A \otimes_k F$. Choose a transcendence basis $t_1, \ldots, t_r$ for F over k. Let $F/F'/k(t_1, \ldots, t_r)$ be the maximal separable subextension (Fields, Lemma 14.6). Since F/F' is finite purely inseparable, there is a q such that $\lambda^q \in F'$ for all $\lambda \in F$. Then b^q is in $Q((A \otimes_k F')_{red})$ and satisfies the polynomial $T^d + \alpha_1^q T^{d-1} + \ldots + \alpha_d^q$ with $\alpha_i^q \in A \otimes_k F'$. By the separable case we see that $b^q \in A' \otimes_k F'$ and the proof is complete. $\square$

Lemma 27.6. *Let k be a field. Let X be a locally algebraic k-scheme. Let $\nu : X^\nu \to X$ be the normalization of X. Let $x \in X$ be a point such that (a) $\mathcal{O}_{X,x}$ is reduced, (b) $\dim(\mathcal{O}_{X,x}) = 1$, and (c) for every $x' \in X^\nu$ with $\nu(x') = x$ the extension $\kappa(x')/k$ is separable. Then X is geometrically reduced at x and X^ν is geometrically regular at x' with $\nu(x') = x$.*

Proof. We will use the results of Lemma 27.1 without further mention. Let $x' \in X^\nu$ be a point over x. By dimension theory (Section 20) we have $\dim(\mathcal{O}_{X^\nu,x'}) = 1$. Since X^ν is normal, we see that $\mathcal{O}_{X^\nu,x'}$ is a discrete valuation ring (Properties, Lemma 12.5). Thus $\mathcal{O}_{X^\nu,x'}$ is a regular local k-algebra whose residue field is separable over k. Hence $k \to \mathcal{O}_{X^\nu,x'}$ is formally smooth in the $\mathfrak{m}_{x'}$-adic topology, see More on Algebra, Lemma 38.5. Then $\mathcal{O}_{X^\nu,x'}$ is geometrically regular over k by More on Algebra, Theorem 40.1. Thus X^ν is geometrically regular at x' by Lemma 12.2.

Since $\mathcal{O}_{X,x}$ is reduced, the family of maps $\mathcal{O}_{X,x} \to \mathcal{O}_{X^\nu,x'}$ is injective. Since $\mathcal{O}_{X^\nu,x'}$ is a geometrically reduced k-algebra, it follows immediately that $\mathcal{O}_{X,x}$ is a geometrically reduced k-algebra. Hence X is geometrically reduced at x by Lemma 6.2. $\square$

28. Groups of invertible functions

It is often (but not always) the case that $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group if X is a variety over k. We show this by a series of lemmas. Everything rests on the following special case.

Lemma 28.1. *Let k be an algebraically closed field. Let $\overline{X}$ be a proper variety over k. Let $X \subset \overline{X}$ be an open subscheme. Assume X is normal. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. Since the statement only concerns X, we may replace $\overline{X}$ by a different proper variety over k. Let $\nu : \overline{X}^\nu \to \overline{X}$ be the normalization morphism. By Lemma 27.1 we have that ν is finite and $\overline{X}^\nu$ is a variety. Since X is normal, we see that $\nu^{-1}(X) \to X$ is an isomorphism (tiny detail omitted). Finally, we see that $\overline{X}^\nu$ is proper over k as a finite morphism is proper (Morphisms, Lemma 44.11) and compositions of proper morphisms are proper (Morphisms, Lemma 41.4). Thus we may and do assume $\overline{X}$ is normal.

We will use without further mention that for any affine open U of $\overline{X}$ the ring $\mathcal{O}(U)$ is a finitely generated k-algebra, which is Noetherian, a domain and normal, see Algebra, Lemma 31.1, Properties, Definition 3.1, Properties, Lemmas 5.2 and 7.2, Morphisms, Lemma 15.2.

Let $\xi_1, \ldots, \xi_r$ be the generic points of the complement of X in $\overline{X}$. There are finitely many since $\overline{X}$ has a Noetherian underlying topological space (see Morphisms, Lemma 15.6, Properties, Lemma 5.5, and Topology, Lemma 9.2). For each i the local ring $\mathcal{O}_i = \mathcal{O}_{X,\xi_i}$ is a normal Noetherian local domain (as a localization of a Noetherian normal domain). Let $J \subset \{1, \ldots, r\}$ be the set of indices i such that $\dim(\mathcal{O}_i) = 1$. For $j \in J$ the local ring $\mathcal{O}_j$ is a discrete valuation ring, see Algebra, Lemma 119.7. Hence we obtain a valuation

$$v_j : k(\overline{X})^* \longrightarrow \mathbf{Z}$$

with the property that $v_j(f) \geq 0 \Leftrightarrow f \in \mathcal{O}_j$.

Think of $\mathcal{O}(X)$ as a sub k-algebra of $k(X) = k(\overline{X})$. We claim that the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod_{j \in J} \mathbf{Z}, \quad f \longmapsto \prod v_j(f)$$

is k^*. It is clear that this claim proves the lemma. Namely, suppose that $f \in \mathcal{O}(X)$ is an element of the kernel. Let $U = \mathrm{Spec}(B) \subset \overline{X}$ be any affine open. Then B is a Noetherian normal domain. For every height one prime $\mathfrak{q} \subset B$ with corresponding point $\xi \in X$ we see that either $\xi = \xi_j$ for some $j \in J$ or that $\xi \in X$. The reason is that $\mathrm{codim}(\overline{\{\xi\}}, \overline{X}) = 1$ by Properties, Lemma 10.3 and hence if $\xi \in \overline{X} \setminus X$ it must be a generic point of $\overline{X} \setminus X$, hence equal to some ξ_j, $j \in J$. We conclude that $f \in \mathcal{O}_{X,\xi} = B_{\mathfrak{q}}$ in either case as f is in the kernel of the map. Thus $f \in \bigcap_{\mathrm{ht}(\mathfrak{q})=1} B_{\mathfrak{q}} = B$, see Algebra, Lemma 157.6. In other words, we see that $f \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$. But since k is algebraically closed we conclude that $f \in k$ by Lemma 26.2. $\square$

Next, we generalize the case above by some elementary arguments, still keeping the field algebraically closed.

Lemma 28.2. *Let k be an algebraically closed field. Let X be an integral scheme locally of finite type over k. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. As X is integral the restriction mapping $\mathcal{O}(X) \to \mathcal{O}(U)$ is injective for any nonempty open subscheme $U \subset X$. Hence we may assume that X is affine. Choose a closed immersion $X \to \mathbf{A}_k^n$ and denote $\overline{X}$ the closure of X in $\mathbf{P}_k^n$ via the usual immersion $\mathbf{A}_k^n \to \mathbf{P}_k^n$. Thus we may assume that X is an affine open of a projective variety $\overline{X}$.

Let $\nu : \overline{X}^\nu \to \overline{X}$ be the normalization morphism, see Morphisms, Definition 54.1. We know that ν is finite, dominant, and that $\overline{X}^\nu$ is a normal irreducible scheme, see Morphisms, Lemmas 54.5, 54.9, and 18.2. It follows that $\overline{X}^\nu$ is a proper variety, because $\overline{X} \to \mathrm{Spec}(k)$ is proper as a composition of a finite and a proper morphism (see results in Morphisms, Sections 41 and 44). It also follows that ν is a surjective morphism, because the image of ν is closed and contains the generic point of $\overline{X}$. Hence setting $X^\nu = \nu^{-1}(X)$ we see that it suffices to prove the result for X^ν. In other words, we may assume that X is a nonempty open of a normal proper variety $\overline{X}$. This case is handled by Lemma 28.1. $\square$

The preceding lemma implies the following slight generalization.

Lemma 28.3. *Let k be an algebraically closed field. Let X be a connected reduced scheme which is locally of finite type over k with finitely many irreducible components. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. Let $X = \bigcup X_i$ be the irreducible components. By Lemma 28.2 we see that $\mathcal{O}(X_i)^*/k^*$ is a finitely generated abelian group. Let $f \in \mathcal{O}(X)^*$ be in the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod \mathcal{O}(X_i)^*/k^*.$$

Then for each i there exists an element $\lambda_i \in k$ such that $f|_{X_i} = \lambda_i$. By restricting to $X_i \cap X_j$ we conclude that $\lambda_i = \lambda_j$ if $X_i \cap X_j \neq \emptyset$. Since X is connected we conclude that all λ_i agree and hence that $f \in k^*$. This proves that

$$\mathcal{O}(X)^*/k^* \subset \prod \mathcal{O}(X_i)^*/k^*$$

and the lemma follows as on the right we have a product of finitely many finitely generated abelian groups. $\square$

Lemma 28.4. *Let k be a field. Let X be a scheme over k which is connected and reduced. Then the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is a field.*

Proof. Let $k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k. Then $X \to \operatorname{Spec}(k)$ factors through $\operatorname{Spec}(k')$, see Schemes, Lemma 6.4. As X is reduced we see that k' has no nonzero nilpotent elements. As $k \to k'$ is integral we see that every prime ideal of k' is both a maximal ideal and a minimal prime, and $\operatorname{Spec}(k')$ is totally disconnected, see Algebra, Lemmas 36.20 and 26.5. As X is connected the morphism $X \to \operatorname{Spec}(k')$ is constant, say with image the point corresponding to $\mathfrak{p} \subset k'$. Then any $f \in k'$, $f \notin \mathfrak{p}$ maps to an invertible element of $\mathcal{O}_X$. By definition of k' this then forces f to be a unit of k'. Hence we see that k' is local with maximal ideal $\mathfrak{p}$, see Algebra, Lemma 18.2. Since we've already seen that k' is reduced this implies that k' is a field, see Algebra, Lemma 25.1. $\square$

Proposition 28.5. *Let k be a field. Let X be a scheme over k. Assume that X is locally of finite type over k, connected, reduced, and has finitely many irreducible components. Then $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group if in addition to the conditions above at least one of the following conditions is satisfied:*

    (1) *the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is k,*
    (2) *X has a k-rational point, or*
    (3) *X is geometrically integral.*

Proof. Let $\overline{k}$ be an algebraic closure of k. Let Y be a connected component of $(X_{\overline{k}})_{red}$. Note that the canonical morphism $p : Y \to X$ is open (by Morphisms, Lemma 23.4) and closed (by Morphisms, Lemma 44.7). Hence $p(Y) = X$ as X was assumed connected. In particular, as X is reduced this implies $\mathcal{O}(X) \subset \mathcal{O}(Y)$. By Lemma 8.14 we see that Y has finitely many irreducible components. Thus Lemma 28.3 applies to Y. This implies that if $\mathcal{O}(X)^*/k^*$ is not a finitely generated abelian group, then there exist elements $f \in \mathcal{O}(X)$, $f \notin k$ which map to an element of $\overline{k}$ via the map $\mathcal{O}(X) \to \mathcal{O}(Y)$. In this case f is algebraic over k, hence integral over k. Thus, if condition (1) holds, then this cannot happen. To finish the proof we show that conditions (2) and (3) imply (1).

Let $k \subset k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k in $\Gamma(X, \mathcal{O}_X)$. By Lemma 28.4 we see that k' is a field. If $e : \operatorname{Spec}(k) \to X$ is a k-rational point, then $e^\sharp : \Gamma(X, \mathcal{O}_X) \to k$ is a section to the inclusion map $k \to \Gamma(X, \mathcal{O}_X)$. In particular the restriction of $e^\sharp$ to k' is a field map $k' \to k$ over k, which clearly shows that (2) implies (1).

If the integral closure k' of k in $\Gamma(X, \mathcal{O}_X)$ is not trivial, then we see that X is either not geometrically connected (if k'/k is not purely inseparable) or that X is not geometrically reduced (if k'/k is nontrivial purely inseparable). Details omitted. Hence (3) implies (1). $\square$

Lemma 28.6. *Let k be a field. Let X be a variety over k. The group $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group provided at least one of the following conditions holds:*

(1) *k is integrally closed in $\Gamma(X, \mathcal{O}_X)$,*
(2) *k is algebraically closed in $k(X)$,*
(3) *X is geometrically integral over k, or*
(4) *k is the "intersection" of the field extensions $\kappa(x)/k$ where x runs over the closed points of x.*

Proof. We see that (1) is enough by Proposition 28.5. We omit the verification that each of (2), (3), (4) implies (1). $\square$

29. Künneth formula, I

In this section we prove the Künneth formula when the base is a field and we are considering cohomology of quasi-coherent modules. For a more general version, please see Derived Categories of Schemes, Section 23.

Lemma 29.1. *Let k be a field. Let X and Y be schemes over k and let $\mathcal{F}$, resp. $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module, resp. $\mathcal{O}_Y$-module. Then we have a canonical isomorphism*

$$H^n(X \times_{\mathrm{Spec}(k)} Y, pr_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_{\mathrm{Spec}(k)} Y}} pr_2^*\mathcal{G}) = \bigoplus\nolimits_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G})$$

provided X and Y are quasi-compact and have affine diagonal[3] (for example if X and Y are separated).

Proof. In this proof unadorned products and tensor products are over k. As maps

$$H^p(X, \mathcal{F}) \otimes H^q(Y, \mathcal{G}) \longrightarrow H^n(X \times Y, \mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \mathrm{pr}_2^*\mathcal{G})$$

we use functoriality of cohomology to get maps $H^p(X, \mathcal{F}) \to H^p(X \times Y, \mathrm{pr}_1^*\mathcal{F})$ and $H^p(Y, \mathcal{G}) \to H^p(X \times Y, \mathrm{pr}_2^*\mathcal{G})$ and then we use the cup product

$$\cup : H^p(X \times Y, \mathrm{pr}_1^*\mathcal{F}) \otimes H^q(X \times Y, \mathrm{pr}_2^*\mathcal{G}) \longrightarrow H^n(X \times Y, \mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \mathrm{pr}_2^*\mathcal{G})$$

The result is true when X and Y are affine by the vanishing of higher cohomology groups on affines (Cohomology of Schemes, Lemma 2.2) and the definitions (of pullbacks of quasi-coherent modules and tensor products of quasi-coherent modules).

Choose finite affine open coverings $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : Y = \bigcup_{j \in J} V_j$. This determines an affine open covering $\mathcal{W} : X \times Y = \bigcup_{(i,j) \in I \times J} U_i \times V_j$. Note that $\mathcal{W}$ is a refinement of $\mathrm{pr}_1^{-1}\mathcal{U}$ and of $\mathrm{pr}_2^{-1}\mathcal{V}$. Thus by Cohomology, Lemma 15.1 we obtain maps

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, \mathrm{pr}_1^*\mathcal{F}) \quad \text{and} \quad \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, \mathrm{pr}_2^*\mathcal{G})$$

---

[3]The case where X and Y are quasi-separated will be discussed in Lemma 29.2 below.

compatible with pullback maps on cohomology. In Cohomology, Equation (25.3.2) we have constructed a map of complexes

$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F}) \otimes \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_2^*\mathcal{G})) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$$

defining the cup product on cohomology. Combining the above we obtain a map of complexes

$$(29.1.1) \qquad \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$$

We warn the reader that this map is not an isomorphism of complexes. Recall that we may compute the cohomologies of our quasi-coherent sheaves using our coverings (Cohomology of Schemes, Lemmas 2.5 and 2.6). Thus on cohomology (29.1.1) reproduces the map of the lemma.

Consider a short exact sequence $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of quasi-coherent modules. Since the construction of (29.1.1) is functorial in $\mathcal{F}$ and since the formation of the relevant Čech complexes is exact in the variable $\mathcal{F}$ (because we are taking sections over affine opens) we find a map between short exact sequence of complexes

$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}') \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}'') \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}))$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$\check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F}' \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F}'' \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$$

(we have dropped the outer zeros). Looking at long exact cohomology sequences we find that if the result of the lemma holds for 2-out-of-3 of $\mathcal{F}, \mathcal{F}', \mathcal{F}''$, then it holds for the third.

Observe that X has finite cohomological dimension for quasi-coherent modules, see Cohomology of Schemes, Lemma 4.2. Using induction on $d(\mathcal{F}) = \max\{d \mid H^d(X, \mathcal{F}) \neq 0\}$ we will reduce to the case $d(\mathcal{F}) = 0$. Assume $d(\mathcal{F}) > 0$. By Cohomology of Schemes, Lemma 4.3 we have seen that there exists an embedding $\mathcal{F} \to \mathcal{F}'$ such that $H^p(X, \mathcal{F}') = 0$ for all $p \geq 1$. Setting $\mathcal{F}'' = \text{Coker}(\mathcal{F} \to \mathcal{F}')$ we see that $d(\mathcal{F}'') < d(\mathcal{F})$. Then we can apply the result from the previous paragraph to see that it suffices to prove the lemma for $\mathcal{F}'$ and $\mathcal{F}''$ thereby proving the induction step.

Arguing in the same fashion for $\mathcal{G}$ we find that we may assume that both $\mathcal{F}$ and $\mathcal{G}$ have nonzero cohomology only in degree 0. Let $V \subset Y$ be an affine open. Consider the affine open covering $\mathcal{U}_V : X \times V = \bigcup_{i \in I} U_i \times V$. It is immediate that

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes \mathcal{G}(V) = \check{\mathcal{C}}^\bullet(\mathcal{U}_V, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$$

(equality of complexes). We conclude that

$$R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G}) \cong \Gamma(X, \mathcal{F}) \otimes_k \mathcal{G} \cong \bigoplus_{\alpha \in A} \mathcal{G}$$

on Y. Here A is a basis for the k-vector space $\Gamma(X, \mathcal{F})$. Cohomology on Y commutes with direct sums (Cohomology, Lemma 19.1). Using the Leray spectral sequence for pr_2 (via Cohomology, Lemma 13.6) we conclude that $H^n(X \times Y, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$ is zero for $n > 0$ and isomorphic to $H^0(X, \mathcal{F}) \otimes H^0(Y, \mathcal{G})$ for $n = 0$. This finishes the proof (except that we should check that the isomorphism is indeed given by cup product in degree 0; we omit the verification). $\qquad \square$

Lemma 29.2. *Let k be a field. Let X and Y be schemes over k and let $\mathcal{F}$, resp. $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module, resp. $\mathcal{O}_Y$-module. Then we have a canonical isomorphism*

$$H^n(X \times_{\mathrm{Spec}(k)} Y, pr_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_{\mathrm{Spec}(k)} Y}} pr_2^*\mathcal{G}) = \bigoplus\nolimits_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G})$$

provided X and Y are quasi-compact and quasi-separated.

Proof. If X and Y are separated or more generally have affine diagonal, then please see Lemma 29.1 for "better" proof (the feature it has over this proof is that it identifies the maps as pullbacks followed by cup products). Let X', resp. Y' be the infinitesimal thickening of X, resp. Y whose structure sheaf is $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{F}$, resp. $\mathcal{O}_{Y'} = \mathcal{O}_Y \oplus \mathcal{G}$ where $\mathcal{F}$, resp. $\mathcal{G}$ is an ideal of square zero. Then

$$\mathcal{O}_{X' \times Y'} = \mathcal{O}_{X \times Y} \oplus \mathrm{pr}_1^*\mathcal{F} \oplus \mathrm{pr}_2^*\mathcal{G} \oplus \mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \mathrm{pr}_2^*\mathcal{G}$$

as sheaves on $X \times Y$. In this way we see that it suffices to prove that

$$H^n(X \times Y, \mathcal{O}_{X \times Y}) = \bigoplus\nolimits_{p+q=n} H^p(X, \mathcal{O}_X) \otimes_k H^q(Y, \mathcal{O}_Y)$$

for any pair of quasi-compact and quasi-separated schemes over k. Some details omitted.

To prove this statement we use cohomology and base change in the form of Cohomology of Schemes, Lemma 7.3. This lemma tells us there exists a bounded below complex of k-vector spaces, i.e., a complex $\mathcal{K}^\bullet$ of quasi-coherent modules on $\mathrm{Spec}(k)$, which universally computes the cohomology of Y over $\mathrm{Spec}(k)$. In particular, we see that

$$R\mathrm{pr}_{1,*}(\mathcal{O}_{X \times Y}) \cong (X \to \mathrm{Spec}(k))^*\mathcal{K}^\bullet$$

in $D(\mathcal{O}_X)$. Up to homotopy the complex $\mathcal{K}^\bullet$ is isomorphic to $\bigoplus_{q \geq 0} H^q(Y, \mathcal{O}_Y)[-q]$ because this is true for every complex of vector spaces over a field. We conclude that

$$R\mathrm{pr}_{1,*}(\mathcal{O}_{X \times Y}) \cong \bigoplus\nolimits_{q \geq 0} H^q(Y, \mathcal{O}_Y)[-q] \otimes_k \mathcal{O}_X$$

in $D(\mathcal{O}_X)$. Then we have

$$
\begin{aligned}
R\Gamma(X \times Y, \mathcal{O}_{X \times Y}) &= R\Gamma(X, R\mathrm{pr}_{1,*}(\mathcal{O}_{X \times Y})) \\
&= R\Gamma(X, \bigoplus\nolimits_{q \geq 0} H^q(Y, \mathcal{O}_Y)[-q] \otimes_k \mathcal{O}_X) \\
&= \bigoplus\nolimits_{q \geq 0} R\Gamma(X, H^q(Y, \mathcal{O}_Y) \otimes \mathcal{O}_X)[-q] \\
&= \bigoplus\nolimits_{q \geq 0} R\Gamma(X, \mathcal{O}_X) \otimes_k H^q(Y, \mathcal{O}_Y)[-q] \\
&= \bigoplus\nolimits_{p,q \geq 0} H^p(X, \mathcal{O}_X)[-p] \otimes_k H^q(Y, \mathcal{O}_Y)[-q]
\end{aligned}
$$

as desired. The first equality by Leray for pr_1 (Cohomology, Lemma 13.1). The second by our decomposition of the total direct image given above. The third because cohomology always commutes with finite direct sums (and cohomology of Y vanishes in sufficiently large degree by Cohomology of Schemes, Lemma 4.4). The fourth because cohomology on X commutes with infinite direct sums by Cohomology, Lemma 19.1. The final equality by our remark on the derived category of a field above. $\square$

30. Picard groups of varieties

In this section we collect some elementary results on Picard groups of algebraic varieties.

Lemma 30.1. *Let $A \to B$ be a faithfully flat ring map. Let X be a quasi-compact and quasi-separated scheme over A. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose pullback to X_B is trivial. Then $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{\otimes -1})$ are invertible $H^0(X, \mathcal{O}_X)$-modules and the multiplication map induces an isomorphism*

$$H^0(X, \mathcal{L}) \otimes_{H^0(X, \mathcal{O}_X)} H^0(X, \mathcal{L}^{\otimes -1}) \longrightarrow H^0(X, \mathcal{O}_X)$$

Proof. Denote $\mathcal{L}_B$ the pullback of $\mathcal{L}$ to X_B. Choose an isomorphism $\mathcal{L}_B \to \mathcal{O}_{X_B}$. Set $R = H^0(X, \mathcal{O}_X)$, $M = H^0(X, \mathcal{L})$ and think of M as an R-module. For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ with pullback $\mathcal{F}_B$ on X_B there is a canonical isomorphism $H^0(X_B, \mathcal{F}_B) = H^0(X, \mathcal{F}) \otimes_A B$, see Cohomology of Schemes, Lemma 5.2. Thus we have

$$M \otimes_R (R \otimes_A B) = M \otimes_A B = H^0(X_B, \mathcal{L}_B) \cong H^0(X_B, \mathcal{O}_{X_B}) = R \otimes_A B$$

Since $R \to R \otimes_A B$ is faithfully flat (as the base change of the faithfully flat map $A \to B$), we conclude that M is an invertible R-module by Algebra, Proposition 83.3. Similarly $N = H^0(X, \mathcal{L}^{\otimes -1})$ is an invertible R-module. To see that the statement on tensor products is true, use that it is true after pulling back to X_B and faithful flatness of $R \to R \otimes_A B$. Some details omitted. $\square$

Lemma 30.2. *Let $A \to B$ be a faithfully flat ring map. Let X be a scheme over A such that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *$R = H^0(X, \mathcal{O}_X)$ is a semi-local ring.*

Then the pullback map $\text{Pic}(X) \to \text{Pic}(X_B)$ is injective.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose pullback $\mathcal{L}'$ to X_B is trivial. Set $M = H^0(X, \mathcal{L})$ and $N = H^0(X, \mathcal{L}^{\otimes -1})$. By Lemma 30.1 the R-modules M and N are invertible. Since R is semi-local $M \cong R$ and $N \cong R$, see Algebra, Lemma 78.7. Choose generators $s \in M$ and $t \in N$. Then $st \in R = H^0(X, \mathcal{O}_X)$ is a unit by the last part of Lemma 30.1. We conclude that s and t define trivializations of $\mathcal{L}$ and $\mathcal{L}^{\otimes -1}$ over X. $\square$

Lemma 30.3. *Let k'/k be a field extension. Let X be a scheme over k such that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *$R = H^0(X, \mathcal{O}_X)$ is semi-local, e.g., if $\dim_k R < \infty$.*

Then the pullback map $\text{Pic}(X) \to \text{Pic}(X_{k'})$ is injective.

Proof. Special case of Lemma 30.2. If $\dim_k R < \infty$, then R is Artinian and hence semi-local (Algebra, Lemmas 53.2 and 53.3). $\square$

Example 30.4. Lemma 30.3 is not true without some condition on the scheme X over the field k. Here is an example. Let k be a field. Let $t \in \mathbf{P}_k^1$ be a closed point. Set $X = \mathbf{P}^1 \setminus \{t\}$. Then we have a surjection

$$\mathbf{Z} = \text{Pic}(\mathbf{P}_k^1) \longrightarrow \text{Pic}(X)$$

The first equality by Divisors, Lemma 28.5 and surjective by Divisors, Lemma 28.3 (as $\mathbf{P}_k^1$ is smooth of dimension 1 over k and hence all its local rings are discrete

valuation rings). If $\mathcal{L}$ is in the kernel of the displayed map, then $\mathcal{L} \cong \mathcal{O}_{\mathbf{P}^1_k}(nt)$ for some $n \in \mathbf{Z}$. We leave it to the reader to show that $\mathcal{O}_{\mathbf{P}^1_k}(t) \cong \mathcal{O}_{\mathbf{P}^1_k}(d)$ where $d = [\kappa(t) : k]$. Hence

$$\operatorname{Pic}(X) = \mathbf{Z}/d\mathbf{Z}$$

Thus if t is not a k-rational point, then $d > 1$ and this Picard group is nonzero. On the other hand, if we extend the ground field k to any field extension k' such that there exists a k-embedding $\kappa(t) \to k'$, then $\mathbf{P}^1_{k'} \setminus X_{k'}$ has a k'-rational point t'. Hence $\mathcal{O}_{\mathbf{P}^1_{k'}}(1) = \mathcal{O}_{\mathbf{P}^1_{k'}}(t')$ will be in the kernel of the map $\mathbf{Z} \to \operatorname{Pic}(X_{k'})$ and it will follow in the same manner as above that $\operatorname{Pic}(X_{k'}) = 0$.

The following lemma tells us that "rationally equivalence invertible modules" are isomorphic on normal varieties.

Lemma 30.5. *Let k be a field. Let X be a normal variety over k. Let $U \subset \mathbf{A}^n_k$ be an open subscheme with k-rational points $p, q \in U(k)$. For every invertible module $\mathcal{L}$ on $X \times_{\operatorname{Spec}(k)} U$ the restrictions $\mathcal{L}|_{X \times p}$ and $\mathcal{L}|_{X \times q}$ are isomorphic.*

Proof. The fibres of $X \times_{\operatorname{Spec}(k)} U \to X$ are open subschemes of affine n-space over fields. Hence these fibres have trivial Picard groups by Divisors, Lemma 28.4. Applying Divisors, Lemma 28.1 we see that $\mathcal{L}$ is the pullback of an invertible module $\mathcal{N}$ on X. $\square$

31. Uniqueness of base field

The phrase "let X be a scheme over k" means that X is a scheme which comes equipped with a morphism $X \to \operatorname{Spec}(k)$. Now we can ask whether the field k is uniquely determined by the scheme X. Of course this is not the case, since for example $\mathbf{A}^1_{\mathbf{C}}$ which we ordinarily consider as a scheme over the field $\mathbf{C}$ of complex numbers, could also be considered as a scheme over $\mathbf{Q}$. But what if we ask that the morphism $X \to \operatorname{Spec}(k)$ does not factor as $X \to \operatorname{Spec}(k') \to \operatorname{Spec}(k)$ for any nontrivial field extension k'/k? In other words we ask that k is somehow maximal such that X lives over k.

An example to show that this still does not guarantee uniqueness of k is the scheme

$$X = \operatorname{Spec}\left(\mathbf{Q}(x)[y]\left[\frac{1}{P(y)}, P \in \mathbf{Q}[y], P \neq 0\right]\right)$$

At first sight this seems to be a scheme over $\mathbf{Q}(x)$, but on a second look it is clear that it is also a scheme over $\mathbf{Q}(y)$. Moreover, the fields $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are subfields of $R = \Gamma(X, \mathcal{O}_X)$ which are maximal among the subfields of R (details omitted). In particular, both $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are maximal in the sense above. Note that both morphisms $X \to \operatorname{Spec}(\mathbf{Q}(x))$ and $X \to \operatorname{Spec}(\mathbf{Q}(y))$ are "essentially of finite type" (i.e., the corresponding ring map is essentially of finite type). Hence X is a Noetherian scheme of finite dimension, i.e., it is not completely pathological.

Another issue that can prevent uniqueness is that the scheme X may be nonreduced. In that case there can be many different morphisms from X to the spectrum of a given field. As an explicit example consider the dual numbers $D = \mathbf{C}[y]/(y^2) = \mathbf{C} \oplus \epsilon\mathbf{C}$. Given any derivation $\theta : \mathbf{C} \to \mathbf{C}$ over $\mathbf{Q}$ we get a ring map

$$\mathbf{C} \longrightarrow D, \quad c \longmapsto c + \epsilon\theta(c).$$

The subfield of $\mathbf{C}$ on which all of these maps are the same is the algebraic closure of $\mathbf{Q}$. This means that taking the intersection of all the fields that X can live over may end up being a very small field if X is nonreduced.

One observation in this regard is the following: given a field k and two subfields k_1, k_2 of k such that k is finite over k_1 and over k_2, then in general it is *not* the case that k is finite over $k_1 \cap k_2$. An example is the field $k = \mathbf{Q}(t)$ and its subfields $k_1 = \mathbf{Q}(t^2)$ and $\mathbf{Q}((t+1)^2)$. Namely we have $k_1 \cap k_2 = \mathbf{Q}$ in this case. So in the following we have to be careful when taking intersections of fields.

Having said all of this we now show that if X is locally of finite type over a field, then some uniqueness holds. Here is the precise result.

Proposition 31.1. *Let X be a scheme. Let $a : X \to \mathrm{Spec}(k_1)$ and $b : X \to \mathrm{Spec}(k_2)$ be morphisms from X to spectra of fields. Assume a, b are locally of finite type, and X is reduced, and connected. Then we have $k_1' = k_2'$, where $k_i' \subset \Gamma(X, \mathcal{O}_X)$ is the integral closure of k_i in $\Gamma(X, \mathcal{O}_X)$.*

Proof. First, assume the lemma holds in case X is quasi-compact (we will do the quasi-compact case below). As X is locally of finite type over a field, it is locally Noetherian, see Morphisms, Lemma 15.6. In particular this means that it is locally connected, connected components of open subsets are open, and intersections of quasi-compact opens are quasi-compact, see Properties, Lemma 5.5, Topology, Lemma 7.11, Topology, Section 9, and Topology, Lemma 16.1. Pick an open covering $X = \bigcup_{i \in I} U_i$ such that each U_i is quasi-compact and connected. For each i let $K_i \subset \mathcal{O}_X(U_i)$ be the integral closure of k_1 and of k_2. For each pair $i, j \in I$ we decompose

$$U_i \cap U_j = \coprod U_{i,j,l}$$

into its finitely many connected components. Write $K_{i,j,l} \subset \mathcal{O}(U_{i,j,l})$ for the integral closure of k_1 and of k_2. By Lemma 28.4 the rings K_i and $K_{i,j,l}$ are fields. Now we claim that k_1' and k_2' both equal the kernel of the map

$$\prod K_i \longrightarrow \prod K_{i,j,l}, \quad (x_i)_i \longmapsto x_i|_{U_{i,j,l}} - x_j|_{U_{i,j,l}}$$

which proves what we want. Namely, it is clear that k_1' is contained in this kernel. On the other hand, suppose that $(x_i)_i$ is in the kernel. By the sheaf condition $(x_i)_i$ corresponds to $f \in \mathcal{O}(X)$. Pick some $i_0 \in I$ and let $P(T) \in k_1[T]$ be a monic polynomial with $P(x_{i_0}) = 0$. Then we claim that $P(f) = 0$ which proves that $f \in k_1$. To prove this we have to show that $P(x_i) = 0$ for all i. Pick $i \in I$. As X is connected there exists a sequence $i_0, i_1, \ldots, i_n = i \in I$ such that $U_{i_t} \cap U_{i_{t+1}} \neq \emptyset$. Now this means that for each t there exists an l_t such that x_{i_t} and $x_{i_{t+1}}$ map to the same element of the field $K_{i,j,l}$. Hence if $P(x_{i_t}) = 0$, then $P(x_{i_{t+1}}) = 0$. By induction, starting with $P(x_{i_0}) = 0$ we deduce that $P(x_i) = 0$ as desired.

To finish the proof of the lemma we prove the lemma under the additional hypothesis that X is quasi-compact. By Lemma 28.4 after replacing k_i by k_i' we may assume that k_i is integrally closed in $\Gamma(X, \mathcal{O}_X)$. This implies that $\mathcal{O}(X)^*/k_i^*$ is a finitely generated abelian group, see Proposition 28.5. Let $k_{12} = k_1 \cap k_2$ as a subring of $\mathcal{O}(X)$. Note that k_{12} is a field. Since

$$k_1^*/k_{12}^* \longrightarrow \mathcal{O}(X)^*/k_2^*$$

we see that k_1^*/k_{12}^* is a finitely generated abelian group as well. Hence there exist $\alpha_1, \ldots, \alpha_n \in k_1^*$ such that every element $\lambda \in k_1$ has the form

$$\lambda = c\alpha_1^{e_1} \ldots \alpha_n^{e_n}$$

for some $e_i \in \mathbf{Z}$ and $c \in k_{12}$. In particular, the ring map

$$k_{12}[x_1, \ldots, x_n, \frac{1}{x_1 \ldots x_n}] \longrightarrow k_1, \quad x_i \longmapsto \alpha_i$$

is surjective. By the Hilbert Nullstellensatz, Algebra, Theorem 34.1 we conclude that k_1 is a finite extension of k_{12}. In the same way we conclude that k_2 is a finite extension of k_{12}. In particular both k_1 and k_2 are contained in the integral closure k'_{12} of k_{12} in $\Gamma(X, \mathcal{O}_X)$. But since k'_{12} is a field by Lemma 28.4 and since we chose k_i to be integrally closed in $\Gamma(X, \mathcal{O}_X)$ we conclude that $k_1 = k_{12} = k_2$ as desired. $\square$

32. Automorphisms

A section on automorphisms of schemes over fields. For some information on (infinitesimal) automorphisms of curves, see Algebraic Curves, Section 25 and Moduli of Curves, Section 7.

Lemma 32.1. *Let X be a reduced scheme of finite type over a field k. Let $f : X \to X$ be an automorphism over k which induces the identity map on the underlying topological space of X. Then*

    (1) *$f^*\mathcal{F} \cong \mathcal{F}$ for every coherent $\mathcal{O}_X$-module, and*

    (2) *if $\dim(Z) > 0$ for every irreducible component $Z \subset X$, then f is the identity.*

Proof. Part (1) follows from part (2) and the fact that the connected components of X of dimension 0 are spectra of fields.

Let $Z \subset X$ be an irreducible component viewed as an integral closed subscheme. Clearly $f(Z) \subset Z$ and $f|_Z : Z \to Z$ is an automorphism over k which induces the identity map on the underlying topological space of Z. Since X is reduced, it suffices to show that the arrows $f|_Z : Z \to Z$ are the identity. This reduces us to the case discussed in the next paragraph.

Assume X is irreducible of dimension > 0. Choose a nonempty affine open $U \subset X$. Since $f(U) \subset U$ and since $U \subset X$ is scheme theoretically dense it suffices to prove that $f|_U : U \to U$ is the identity.

Assume $X = \operatorname{Spec}(A)$ is affine, irreducible, of dimension > 0 and k is an infinite field. Let $g \in A$ be nonconstant. The set

$$S = \bigcup_{\lambda \in k} V(g - \lambda)$$

is dense in X because it is the inverse image of the dense subset $\mathbf{A}_k^1(k)$ by the nonconstant morphism $g : X \to \mathbf{A}_k^1$. If $x \in S$, then the image $g(x)$ of g in $\kappa(x)$ is in the image of $k \to \kappa(x)$. Hence $f^\sharp : \kappa(x) \to \kappa(x)$ fixes $g(x)$. Thus the image of $f^\sharp(g)$ in $\kappa(x)$ is equal to $g(x)$. We conclude that

$$S \subset V(g - f^\sharp(g))$$

and since X is reduced and S is dense we conclude $g = f^\sharp(g)$. This proves $f^\sharp = \operatorname{id}_A$ as A is generated as a k-algebra by elements g as above (details omitted; hint: the set of constant functions is a finite dimensional k-subvector space of A). We conclude that $f = \operatorname{id}_X$.

Assume $X = \operatorname{Spec}(A)$ is affine, irreducible, of dimension > 0 and k is a finite field. If for every 1-dimensional integral closed subscheme $C \subset X$ the restriction $f|_C : C \to C$ is the identity, then f is the identity. This reduces us to the case where X is a curve. A curve over a finite field has a finite automorphism group (details omitted). Hence f has finite order, say n. Then we pick $g : X \to \mathbf{A}^1_k$ nonconstant as above and we consider

$$S = \{x \in X \text{ closed such that } [\kappa(g(x)) : k] \text{ is prime to } n\}$$

Arguing as before we find that S is dense in X. Since for $x \in X$ closed the map $f^\sharp : \kappa(x) \to \kappa(x)$ is an automorphism of order dividing n we see that for $x \in S$ this automorphism acts trivially on the subfield generated by the image of g in $\kappa(x)$. Thus we conclude that $S \subset V(g - f^\sharp(g))$ and we win as before. $\qquad\square$

33. Euler characteristics

In this section we prove some elementary properties of Euler characteristics of coherent sheaves on schemes proper over fields.

Definition 33.1. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. In this situation the *Euler characteristic of $\mathcal{F}$* is the integer

$$\chi(X, \mathcal{F}) = \sum_i (-1)^i \dim_k H^i(X, \mathcal{F}).$$

For justification of the formula see below.

In the situation of the definition only a finite number of the vector spaces $H^i(X, \mathcal{F})$ are nonzero (Cohomology of Schemes, Lemma 4.5) and each of these spaces is finite dimensional (Cohomology of Schemes, Lemma 19.2). Thus $\chi(X, \mathcal{F}) \in \mathbf{Z}$ is well defined. Observe that this definition depends on the field k and not just on the pair $(X, \mathcal{F})$.

Lemma 33.2. *Let k be a field. Let X be a proper scheme over k. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent modules on X. Then*

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$$

Proof. Consider the long exact sequence of cohomology

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1) \to \dots$$

associated to the short exact sequence of the lemma. The rank-nullity theorem in linear algebra shows that

$$0 = \dim H^0(X, \mathcal{F}_1) - \dim H^0(X, \mathcal{F}_2) + \dim H^0(X, \mathcal{F}_3) - \dim H^1(X, \mathcal{F}_1) + \dots$$

This immediately implies the lemma. $\qquad\square$

Lemma 33.3. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent sheaf with $\dim(Supp(\mathcal{F})) \leq 0$. Then*

(1) *$\mathcal{F}$ is generated by global sections,*
(2) *$H^0(X, \mathcal{F}) = \bigoplus_{x \in Supp(\mathcal{F})} \mathcal{F}_x$,*
(3) *$H^i(X, \mathcal{F}) = 0$ for $i > 0$,*
(4) *$\chi(X, \mathcal{F}) = \dim_k H^0(X, \mathcal{F})$, and*
(5) *$\chi(X, \mathcal{F} \otimes \mathcal{E}) = n\chi(X, \mathcal{F})$ for every locally free module $\mathcal{E}$ of rank n.*

Proof. By Cohomology of Schemes, Lemma 9.7 we see that $\mathcal{F} = i_* \mathcal{G}$ where $i : Z \to X$ is the inclusion of the scheme theoretic support of $\mathcal{F}$ and where $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module. By definition of the scheme theoretic support the underlying topological space of Z is $\mathrm{Supp}(\mathcal{F})$. Since the dimension of Z is 0, we see Z is affine (Properties, Lemma 10.5). Hence $\mathcal{G}$ is globally generated and the higher cohomology groups of $\mathcal{G}$ are zero (Cohomology of Schemes, Lemma 2.2). In fact, by Lemma 20.2 the scheme Z is a finite disjoint union of spectra of local Artinian rings. Thus correspondingly $H^0(Z, \mathcal{G}) = \bigoplus_{z \in Z} \mathcal{G}_z$. The cohomologies of $\mathcal{F}$ and $\mathcal{G}$ agree by Cohomology of Schemes, Lemma 2.4. Thus $H^i(X, \mathcal{F}) = 0$ for $i > 0$ and $H^0(X, \mathcal{F}) = H^0(Z, \mathcal{G})$. In particular we have (3) is true. For $z \in Z$ corresponding to $x \in \mathrm{Supp}(\mathcal{F})$ we have $\mathcal{G}_z = (i_* \mathcal{G})_x = \mathcal{F}_x$. We conclude that (2) holds. Of course (2) implies (1). We have (4) by definition of the Euler characteristic $\chi(X, \mathcal{F})$ and (3). By the projection formula (Cohomology, Lemma 54.2) we have

$$i_*(\mathcal{G} \otimes i^* \mathcal{E}) = \mathcal{F} \otimes \mathcal{E}.$$

Since Z has dimension 0 the locally free sheaf $i^* \mathcal{E}$ is isomorphic to $\mathcal{O}_Z^{\oplus n}$ and arguing as above we see that (5) holds. $\square$

Lemma 33.4. *Let k'/k be an extension of fields. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $X_{k'}$. Then $\chi(X, \mathcal{F}) = \chi(X', \mathcal{F}')$.*

Proof. This is true because

$$H^i(X_{k'}, \mathcal{F}') = H^i(X, \mathcal{F}) \otimes_k k'$$

by flat base change, see Cohomology of Schemes, Lemma 5.2. $\square$

Lemma 33.5. *Let k be a field. Let $f : Y \to X$ be a morphism of proper schemes over k. Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module. Then*

$$\chi(Y, \mathcal{G}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{G})$$

Proof. The formula makes sense: the sheaves $R^i f_* \mathcal{G}$ are coherent and only a finite number of them are nonzero, see Cohomology of Schemes, Proposition 19.1 and Lemma 4.5. By Cohomology, Lemma 13.4 there is a spectral sequence with

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{G})$$

converging to $H^{p+q}(Y, \mathcal{G})$. By finiteness of cohomology on X we see that only a finite number of $E_2^{p,q}$ are nonzero and each $E_2^{p,q}$ is a finite dimensional vector space. It follows that the same is true for $E_r^{p,q}$ for $r \geq 2$ and that

$$\sum (-1)^{p+q} \dim_k E_r^{p,q}$$

is independent of r. Since for r large enough we have $E_r^{p,q} = E_\infty^{p,q}$ and since convergence means there is a filtration on $H^n(Y, \mathcal{G})$ whose graded pieces are $E_\infty^{p,q}$ with $p + q = n$ (this is the meaning of convergence of the spectral sequence), we conclude. Compare also with the more general Homology, Lemma 24.12. $\square$

34. Projective space

Some results on projective space over a field.

Lemma 34.1. *Let k be a field and $n \geq 0$. Then $\mathbf{P}_k^n$ is a smooth projective variety of dimension n over k.*

Proof. Omitted. $\square$

Lemma 34.2. *Let k be a field and $n \geq 0$. Let $X, Y \subset \mathbf{A}_k^n$ be closed subsets. Assume that X and Y are equidimensional, $\dim(X) = r$ and $\dim(Y) = s$. Then every irreducible component of $X \cap Y$ has dimension $\geq r + s - n$.*

Proof. Consider the closed subscheme $X \times Y \subset \mathbf{A}_k^{2n}$ where we use coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. Then $X \cap Y = X \times Y \cap V(x_1 - y_1, \ldots, x_n - y_n)$. Let $t \in X \cap Y \subset X \times Y$ be a closed point. By Lemma 20.5 we have $\dim_t(X \times Y) = \dim(X) + \dim(Y)$. Thus $\dim(\mathcal{O}_{X \times Y, t}) = r + s$ by Lemma 20.3. By Algebra, Lemma 60.13 we conclude that

$$\dim(\mathcal{O}_{X \cap Y, t}) = \dim(\mathcal{O}_{X \times Y, t}/(x_1 - y_1, \ldots, x_n - y_n)) \geq r + s - n$$

This implies the result by Lemma 20.3. $\square$

Lemma 34.3. *Let k be a field and $n \geq 0$. Let $X, Y \subset \mathbf{P}_k^n$ be nonempty closed subsets. If $\dim(X) = r$ and $\dim(Y) = s$ and $r + s \geq n$, then $X \cap Y$ is nonempty and $\dim(X \cap Y) \geq r + s - n$.*

Proof. Write $\mathbf{A}^n = \operatorname{Spec}(k[x_0, \ldots, x_n])$ and $\mathbf{P}^n = \operatorname{Proj}(k[T_0, \ldots, T_n])$. Consider the morphism $\pi : \mathbf{A}^{n+1} \setminus \{0\} \to \mathbf{P}^n$ which sends $(x_0, \ldots, x_n)$ to the point $[x_0 : \ldots : x_n]$. More precisely, it is the morphism associated to the pair $(\mathcal{O}_{\mathbf{A}^{n+1} \setminus \{0\}}, (x_0, \ldots, x_n))$, see Constructions, Lemma 13.1. Over the standard affine open $D_+(T_i)$ we get the morphism associated to the ring map

$$k\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right] \longrightarrow k\left[T_0, \ldots, T_n, \frac{1}{T_i}\right] \cong k\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]\left[T_i, \frac{1}{T_i}\right]$$

which is surjective and smooth of relative dimension 1 with irreducible fibres (details omitted). Hence $\pi^{-1}(X)$ and $\pi^{-1}(Y)$ are nonempty closed subsets of dimension $r + 1$ and $s + 1$. Choose an irreducible component $V \subset \pi^{-1}(X)$ of dimension $r + 1$ and an irreducible component $W \subset \pi^{-1}(Y)$ of dimension $s + 1$. Observe that this implies V and W contain every fibre of π they meet (since π has irreducible fibres of dimension 1 and since Lemma 20.4 says the fibres of $V \to \pi(V)$ and $W \to \pi(W)$ have dimension ≥ 1). Let $\overline{V}$ and $\overline{W}$ be the closure of V and W in $\mathbf{A}^{n+1}$. Since $0 \in \mathbf{A}^{n+1}$ is in the closure of every fibre of π we see that $0 \in \overline{V} \cap \overline{W}$. By Lemma 34.2 we have $\dim(\overline{V} \cap \overline{W}) \geq r + s - n + 1$. Arguing as above using Lemma 20.4 again, we conclude that $\pi(V \cap W) \subset X \cap Y$ has dimension at least $r + s - n$ as desired. $\square$

Lemma 34.4. *Let k be a field. Let $Z \subset \mathbf{P}_k^n$ be a closed subscheme which has no embedded points such that every irreducible component of Z has dimension $n - 1$. Then the ideal $I(Z) \subset k[T_0, \ldots, T_n]$ corresponding to Z is principal.*

Proof. This is a special case of Divisors, Lemma 31.3. $\square$

35. Coherent sheaves on projective space

In this section we prove some results on the cohomology of coherent sheaves on $\mathbf{P}^n$ over a field which can be found in [Mum66]. These will be useful later when discussing Quot and Hilbert schemes.

35.1. Preliminaries. Let k be a field, $n \geq 1$, $d \geq 1$, and let $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(d))$ be a nonzero section. In this section we will write $\mathcal{O}(d)$ for the dth twist of the structure sheaf on projective space (Constructions, Definitions 10.1 and 13.2). Since $\mathbf{P}^n_k$ is a variety this section is regular, hence s is a regular section of $\mathcal{O}(d)$ and defines an effective Cartier divisor $H = Z(s) \subset \mathbf{P}^n_k$, see Divisors, Section 13. Such a divisor H is called a *hypersurface* and if $d = 1$ it is called a *hyperplane*.

Lemma 35.2. *Let k be a field. Let $n \geq 1$. Let $i : H \to \mathbf{P}^n_k$ be a hyperplane. Then there exists an isomorphism*
$$\varphi : \mathbf{P}^{n-1}_k \longrightarrow H$$
such that $i^* \mathcal{O}(1)$ pulls back to $\mathcal{O}(1)$.

Proof. We have $\mathbf{P}^n_k = \mathrm{Proj}(k[T_0, \ldots, T_n])$. The section s corresponds to a homogeneous form in $T_0, \ldots, T_n$ of degree 1, see Cohomology of Schemes, Section 8. Say $s = \sum a_i T_i$. Constructions, Lemma 13.7 gives that $H = \mathrm{Proj}(k[T_0, \ldots, T_n]/I)$ for the graded ideal I defined by setting I_d equal to the kernel of the map $\Gamma(\mathbf{P}^n_k, \mathcal{O}(d)) \to \Gamma(H, i^* \mathcal{O}(d))$. By our construction of $Z(s)$ in Divisors, Definition 14.8 we see that on $D_+(T_j)$ the ideal of H is generated by $\sum a_i T_i / T_j$ in the polynomial ring $k[T_0/T_j, \ldots, T_n/T_j]$. Thus it is clear that I is the ideal generated by $\sum a_i T_i$. Note that
$$k[T_0, \ldots, T_n]/I = k[T_0, \ldots, T_n]/(\sum a_i T_i) \cong k[S_0, \ldots, S_{n-1}]$$
as graded rings. For example, if $a_n \neq 0$, then mapping S_i equal to the class of T_i works. We obtain the desired isomorphism by functoriality of Proj. Equality of twists of structure sheaves follows for example from Constructions, Lemma 11.5. $\square$

Lemma 35.3. *Let k be an infinite field. Let $n \geq 1$. Let $\mathcal{F}$ be a coherent module on $\mathbf{P}^n_k$. Then there exist a nonzero section $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ and a short exact sequence*
$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to i_* \mathcal{G} \to 0$$
where $i : H \to \mathbf{P}^n_k$ is the hyperplane H associated to s and $\mathcal{G} = i^* \mathcal{F}$.

Proof. The map $\mathcal{F}(-1) \to \mathcal{F}$ comes from Constructions, Equation (10.1.2) with $n = 1$, $m = -1$ and the section s of $\mathcal{O}(1)$. Let's work out what this map looks like if we restrict it to $D_+(T_0)$. Write $D_+(T_0) = \mathrm{Spec}(k[x_1, \ldots, x_n])$ with $x_i = T_i/T_0$. Identify $\mathcal{O}(1)|_{D_+(T_0)}$ with $\mathcal{O}$ using the section T_0. Hence if $s = \sum a_i T_i$ then $s|_{D_+(T_0)} = a_0 + \sum a_i x_i$ with the identification chosen above. Furthermore, suppose $\mathcal{F}|_{D_+(T_0)}$ corresponds to the finite $k[x_1, \ldots, x_n]$-module M. Via the identification $\mathcal{F}(-1) = \mathcal{F} \otimes \mathcal{O}(-1)$ and our chosen trivialization of $\mathcal{O}(1)$ we see that $\mathcal{F}(-1)$ corresponds to M as well. Thus restricting $\mathcal{F}(-1) \to \mathcal{F}$ to $D_+(T_0)$ gives the map
$$M \xrightarrow{a_0 + \sum a_i x_i} M$$
To see that the arrow is injective, it suffices to pick $a_0 + \sum a_i x_i$ outside any of the associated primes of M, see Algebra, Lemma 63.9. By Algebra, Lemma 63.5 the set $\mathrm{Ass}(M)$ of associated primes of M is finite. Note that for $\mathfrak{p} \in \mathrm{Ass}(M)$ the intersection $\mathfrak{p} \cap \{a_0 + \sum a_i x_i\}$ is a proper k-subvector space. We conclude that

there is a finite family of proper sub vector spaces $V_1, \ldots, V_m \subset \Gamma(\mathbf{P}_k^n, \mathcal{O}(1))$ such that if we take s outside of $\bigcup V_i$, then multiplication by s is injective over $D_+(T_0)$. Similarly for the restriction to $D_+(T_j)$ for $j = 1, \ldots, n$. Since k is infinite, a finite union of proper sub vector spaces is never equal to the whole space, hence we may choose s such that the map is injective. The cokernel of $\mathcal{F}(-1) \to \mathcal{F}$ is annihilated by $\operatorname{Im}(s : \mathcal{O}(-1) \to \mathcal{O})$ which is the ideal sheaf of H by Divisors, Definition 14.8. Hence we obtain $\mathcal{G}$ on H using Cohomology of Schemes, Lemma 9.8. $\qquad\square$

Remark 35.4. Let k be an infinite field. Let $n \geq 1$. Given a finite number of coherent modules $\mathcal{F}_i$ on $\mathbf{P}_k^n$ we can choose a single $s \in \Gamma(\mathbf{P}_k^n, \mathcal{O}(1))$ such that the statement of Lemma 35.3 works for each of them. To prove this, just apply the lemma to $\bigoplus \mathcal{F}_i$.

Remark 35.5. In the situation of Lemmas 35.2 and 35.3 we have $H \cong \mathbf{P}_k^{n-1}$ with Serre twists $\mathcal{O}_H(d) = i^* \mathcal{O}_{\mathbf{P}_k^n}(d)$. For every $d \in \mathbf{Z}$ we have a short exact sequence

$$0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to i_*(\mathcal{G}(d)) \to 0$$

Namely, tensoring by $\mathcal{O}_{\mathbf{P}_k^n}(d)$ is an exact functor and by the projection formula (Cohomology, Lemma 54.2) we have $i_*(\mathcal{G}(d)) = i_*\mathcal{G} \otimes \mathcal{O}_{\mathbf{P}_k^n}(d)$. We obtain corresponding long exact sequences

$$H^i(\mathbf{P}_k^n, \mathcal{F}(d-1)) \to H^i(\mathbf{P}_k^n, \mathcal{F}(d)) \to H^i(H, \mathcal{G}(d)) \to H^{i+1}(\mathbf{P}_k^n, \mathcal{F}(d-1))$$

This follows from the above and the fact that we have $H^i(\mathbf{P}_k^n, i_*\mathcal{G}(d)) = H^i(H, \mathcal{G}(d))$ by Cohomology of Schemes, Lemma 2.4 (closed immersions are affine).

35.6. Regularity. Here is the definition.

Definition 35.7. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}_k^n$. We say $\mathcal{F}$ is *m-regular* if

$$H^i(\mathbf{P}_k^n, \mathcal{F}(m-i)) = 0$$

for $i = 1, \ldots, n$.

Note that $\mathcal{F} = \mathcal{O}(d)$ is m-regular if and only if $d \geq -m$. This follows from the computation of cohomology groups in Cohomology of Schemes, Equation (8.1.1). Namely, we see that $H^n(\mathbf{P}_k^n, \mathcal{O}(d)) = 0$ if and only if $d \geq -n$.

Lemma 35.8. *Let k'/k be an extension of fields. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}_k^n$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $\mathbf{P}_{k'}^n$. Then $\mathcal{F}$ is m-regular if and only if $\mathcal{F}'$ is m-regular.*

Proof. This is true because

$$H^i(\mathbf{P}_{k'}^n, \mathcal{F}') = H^i(\mathbf{P}_k^n, \mathcal{F}) \otimes_k k'$$

by flat base change, see Cohomology of Schemes, Lemma 5.2. $\qquad\square$

Lemma 35.9. *In the situation of Lemma 35.3, if $\mathcal{F}$ is m-regular, then $\mathcal{G}$ is m-regular on $H \cong \mathbf{P}_k^{n-1}$.*

Proof. Recall that $H^i(\mathbf{P}_k^n, i_*\mathcal{G}) = H^i(H, \mathcal{G})$ by Cohomology of Schemes, Lemma 2.4. Hence we see that for $i \geq 1$ we get

$$H^i(\mathbf{P}_k^n, \mathcal{F}(m-i)) \to H^i(H, \mathcal{G}(m-i)) \to H^{i+1}(\mathbf{P}_k^n, \mathcal{F}(m-1-i))$$

by Remark 35.5. The lemma follows. $\qquad\square$

Lemma 35.10. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}$ is $(m+1)$-regular.*

Proof. We prove this by induction on n. If $n = 0$ every sheaf is m-regular for all m and there is nothing to prove. By Lemma 35.8 we may replace k by an infinite overfield and assume k is infinite. Thus we may apply Lemma 35.3. By Lemma 35.9 we know that $\mathcal{G}$ is m-regular. By induction on n we see that $\mathcal{G}$ is $(m+1)$-regular. Considering the long exact cohomology sequence associated to the sequence

$$0 \to \mathcal{F}(m-i) \to \mathcal{F}(m+1-i) \to i_*\mathcal{G}(m+1-i) \to 0$$

as in Remark 35.5 the reader easily deduces for $i \geq 1$ the vanishing of $H^i(\mathbf{P}^n_k, \mathcal{F}(m+1-i))$ from the (known) vanishing of $H^i(\mathbf{P}^n_k, \mathcal{F}(m-i))$ and $H^i(\mathbf{P}^n_k, \mathcal{G}(m+1-i))$. $\square$

Lemma 35.11. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then the multiplication map*

$$H^0(\mathbf{P}^n_k, \mathcal{F}(m)) \otimes_k H^0(\mathbf{P}^n_k, \mathcal{O}(1)) \longrightarrow H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$$

is surjective.

Proof. Let k'/k be an extension of fields. Let $\mathcal{F}'$ be as in Lemma 35.8. By Cohomology of Schemes, Lemma 5.2 the base change of the linear map of the lemma to k' is the same linear map for the sheaf $\mathcal{F}'$. Since $k \to k'$ is faithfully flat it suffices to prove the lemma over k', i.e., we may assume k is infinite.

Assume k is infinite. We prove the lemma by induction on n. The case $n = 0$ is trivial as $\mathcal{O}(1) \cong \mathcal{O}$ is generated by T_0. For $n > 0$ apply Lemma 35.3 and tensor the sequence by $\mathcal{O}(m+1)$ to get

$$0 \to \mathcal{F}(m) \xrightarrow{s} \mathcal{F}(m+1) \to i_*\mathcal{G}(m+1) \to 0$$

see Remark 35.5. Let $t \in H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$. By induction the image $\bar{t} \in H^0(H, \mathcal{G}(m+1))$ is the image of $\sum g_i \otimes \bar{s}_i$ with $\bar{s}_i \in \Gamma(H, \mathcal{O}(1))$ and $g_i \in H^0(H, \mathcal{G}(m))$. Since $\mathcal{F}$ is m-regular we have $H^1(\mathbf{P}^n_k, \mathcal{F}(m-1)) = 0$, hence long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{F}(m-1) \xrightarrow{s} \mathcal{F}(m) \to i_*\mathcal{G}(m) \to 0$$

shows we can lift g_i to $f_i \in H^0(\mathbf{P}^n_k, \mathcal{F}(m))$. We can also lift $\bar{s}_i$ to $s_i \in H^0(\mathbf{P}^n_k, \mathcal{O}(1))$ (see proof of Lemma 35.2 for example). After substracting the image of $\sum f_i \otimes s_i$ from t we see that we may assume $\bar{t} = 0$. But this exactly means that t is the image of $f \otimes s$ for some $f \in H^0(\mathbf{P}^n_k, \mathcal{F}(m))$ as desired. $\square$

Lemma 35.12. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}(m)$ is globally generated.*

Proof. For all $d \gg 0$ the sheaf $\mathcal{F}(d)$ is globally generated. This follows for example from the first part of Cohomology of Schemes, Lemma 14.1. Pick $d \geq m$ such that $\mathcal{F}(d)$ is globally generated. Choose a basis $f_1, \ldots, f_r \in H^0(\mathbf{P}^n_k, \mathcal{F})$. By Lemma 35.11 every element $f \in H^0(\mathbf{P}^n_k, \mathcal{F}(d))$ can be written as $f = \sum P_i f_i$ for some $P_i \in k[T_0, \ldots, T_n]$ homogeneous of degree $d - m$. Since the sections f generate $\mathcal{F}(d)$ it follows that the sections f_i generate $\mathcal{F}(m)$. $\square$

35.13. Hilbert polynomials. The following lemma will be made obsolete by the more general Lemma 45.1.

Lemma 35.14. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}_k^n$. The function*

$$d \longmapsto \chi(\mathbf{P}_k^n, \mathcal{F}(d))$$

is a polynomial.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}_k^n = \mathrm{Spec}(k)$ and $\mathcal{F}(d) = \mathcal{F}$. Hence in this case the function is constant, i.e., a polynomial of degree 0. Assume $n > 0$. By Lemma 33.4 we may assume k is infinite. Apply Lemma 35.3. Applying Lemma 33.2 to the twisted sequences $0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to i_* \mathcal{G}(d) \to 0$ we obtain

$$\chi(\mathbf{P}_k^n, \mathcal{F}(d)) - \chi(\mathbf{P}_k^n, \mathcal{F}(d-1)) = \chi(H, \mathcal{G}(d))$$

See Remark 35.5. Since $H \cong \mathbf{P}_k^{n-1}$ by induction the right hand side is a polynomial. The lemma is finished by noting that any function $f : \mathbf{Z} \to \mathbf{Z}$ with the property that the map $d \mapsto f(d) - f(d-1)$ is a polynomial, is itself a polynomial. We omit the proof of this fact (hint: compare with Algebra, Lemma 58.5). $\square$

Definition 35.15. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}_k^n$. The function $d \mapsto \chi(\mathbf{P}_k^n, \mathcal{F}(d))$ is called the *Hilbert polynomial* of $\mathcal{F}$.

The Hilbert polynomial has coefficients in $\mathbf{Q}$ and not in general in $\mathbf{Z}$. For example the Hilbert polynomial of $\mathcal{O}_{\mathbf{P}_k^n}$ is

$$d \longmapsto \binom{d+n}{n} = \frac{d^n}{n!} + \dots$$

This follows from the following lemma and the fact that

$$H^0(\mathbf{P}_k^n, \mathcal{O}_{\mathbf{P}_k^n}(d)) = k[T_0, \dots, T_n]_d$$

(degree d part) whose dimension over k is $\binom{d+n}{n}$.

Lemma 35.16. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}_k^n$ with Hilbert polynomial $P \in \mathbf{Q}[t]$. Then*

$$P(d) = \dim_k H^0(\mathbf{P}_k^n, \mathcal{F}(d))$$

for all $d \gg 0$.

Proof. This follows from the vanishing of cohomology of high enough twists of $\mathcal{F}$. See Cohomology of Schemes, Lemma 14.1. $\square$
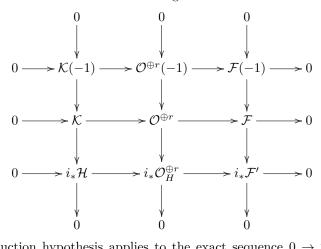
35.17. Boundedness of quotients. In this subsection we bound the regularity of quotients of a given coherent sheaf on $\mathbf{P}^n$ in terms of the Hilbert polynomial.

Lemma 35.18. *Let k be a field. Let $n \geq 0$. Let $r \geq 1$. Let $P \in \mathbf{Q}[t]$. There exists an integer m depending on n, r, and P with the following property: if*

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus r} \to \mathcal{F} \to 0$$

is a short exact sequence of coherent sheaves on $\mathbf{P}_k^n$ and $\mathcal{F}$ has Hilbert polynomial P, then $\mathcal{K}$ is m-regular.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}_k^n = \mathrm{Spec}(k)$ and any coherent module is 0-regular and any surjective map is surjective on global sections. Assume $n > 0$. Consider an exact sequence as in the lemma. Let $P' \in \mathbf{Q}[t]$ be the polynomial $P'(t) = P(t) - P(t-1)$. Let m' be the integer which works for $n-1$, r, and P'. By Lemmas 35.8 and 33.4 we may replace k by a field extension, hence we may assume k is infinite. Apply Lemma 35.3 to the coherent sheaf $\mathcal{F}$. The Hilbert polynomial of $\mathcal{F}' = i^*\mathcal{F}$ is P' (see proof of Lemma 35.14). Since i^* is right exact we see that $\mathcal{F}'$ is a quotient of $\mathcal{O}_H^{\oplus r} = i^*\mathcal{O}^{\oplus r}$. Thus the induction hypothesis applies to $\mathcal{F}'$ on $H \cong \mathbf{P}_k^{n-1}$ (Lemma 35.2). Note that the map $\mathcal{K}(-1) \to \mathcal{K}$ is injective as $\mathcal{K} \subset \mathcal{O}^{\oplus r}$ and has cokernel $i_*\mathcal{H}$ where $\mathcal{H} = i^*\mathcal{K}$. By the snake lemma (Homology, Lemma 5.17) we obtain a commutative diagram with exact columns and rows

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}(-1) & \longrightarrow & \mathcal{O}^{\oplus r}(-1) & \longrightarrow & \mathcal{F}(-1) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{O}^{\oplus r} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & i_*\mathcal{H} & \longrightarrow & i_*\mathcal{O}_H^{\oplus r} & \longrightarrow & i_*\mathcal{F}' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Thus the induction hypothesis applies to the exact sequence $0 \to \mathcal{H} \to \mathcal{O}_H^{\oplus r} \to \mathcal{F}' \to 0$ on $H \cong \mathbf{P}_k^{n-1}$ (Lemma 35.2) and $\mathcal{H}$ is m'-regular. Recall that this implies that $\mathcal{H}$ is d-regular for all $d \geq m'$ (Lemma 35.10).

Let $i \geq 2$ and $d \geq m'$. It follows from the long exact cohomology sequence associated to the left column of the diagram above and the vanishing of $H^{i-1}(H, \mathcal{H}(d))$ that the map

$$H^i(\mathbf{P}_k^n, \mathcal{K}(d-1)) \longrightarrow H^i(\mathbf{P}_k^n, \mathcal{K}(d))$$

is injective. As these groups are zero for $d \gg 0$ (Cohomology of Schemes, Lemma 14.1) we conclude $H^i(\mathbf{P}_k^n, \mathcal{K}(d))$ are zero for all $d \geq m'$ and $i \geq 2$.

We still have to control H^1. First we observe that all the maps

$$H^1(\mathbf{P}_k^n, \mathcal{K}(m'-1)) \to H^1(\mathbf{P}_k^n, \mathcal{K}(m')) \to H^1(\mathbf{P}_k^n, \mathcal{K}(m'+1)) \to \ldots$$

are surjective by the vanishing of $H^1(H, \mathcal{H}(d))$ for $d \geq m'$. Suppose $d > m'$ is such that

$$H^1(\mathbf{P}_k^n, \mathcal{K}(d-1)) \longrightarrow H^1(\mathbf{P}_k^n, \mathcal{K}(d))$$

is injective. Then $H^0(\mathbf{P}_k^n, \mathcal{K}(d)) \to H^0(H, \mathcal{H}(d))$ is surjective. Consider the commutative diagram

$$
\begin{array}{ccc}
H^0(\mathbf{P}_k^n, \mathcal{K}(d)) \otimes_k H^0(\mathbf{P}_k^n, \mathcal{O}(1)) & \longrightarrow & H^0(\mathbf{P}_k^n, \mathcal{K}(d+1)) \\
\downarrow & & \downarrow \\
H^0(H, \mathcal{H}(d)) \otimes_k H^0(H, \mathcal{O}_H(1)) & \longrightarrow & H^0(H, \mathcal{H}(d+1))
\end{array}
$$

By Lemma 35.11 we see that the bottom horizontal arrow is surjective. Hence the right vertical arrow is surjective. We conclude that

$$H^1(\mathbf{P}_k^n, \mathcal{K}(d)) \longrightarrow H^1(\mathbf{P}_k^n, \mathcal{K}(d+1))$$

is injective. By induction we see that

$$H^1(\mathbf{P}_k^n, \mathcal{K}(d-1)) \to H^1(\mathbf{P}_k^n, \mathcal{K}(d)) \to H^1(\mathbf{P}_k^n, \mathcal{K}(d+1)) \to \dots$$

are all injective and we conclude that $H^1(\mathbf{P}_k^n, \mathcal{K}(d-1)) = 0$ because of the eventual vanishing of these groups. Thus the dimensions of the groups $H^1(\mathbf{P}_k^n, \mathcal{K}(d))$ for $d \geq m'$ are strictly decreasing until they become zero. It follows that the regularity of $\mathcal{K}$ is bounded by $m' + \dim_k H^1(\mathbf{P}_k^n, \mathcal{K}(m'))$. On the other hand, by the vanishing of the higher cohomology groups we have

$$\dim_k H^1(\mathbf{P}_k^n, \mathcal{K}(m')) = -\chi(\mathbf{P}_k^n, \mathcal{K}(m')) + \dim_k H^0(\mathbf{P}_k^n, \mathcal{K}(m'))$$

Note that the H^0 has dimension bounded by the dimension of $H^0(\mathbf{P}_k^n, \mathcal{O}^{\oplus r}(m'))$ which is at most $r\binom{n+m'}{n}$ if $m' > 0$ and zero if not. Finally, the term $\chi(\mathbf{P}_k^n, \mathcal{K}(m'))$ is equal to $r\binom{n+m'}{n} - P(m')$. This gives a bound of the desired type finishing the proof of the lemma. $\square$

36. Frobenii

Let p be a prime number. If X is a scheme, then we say "X has characteristic p", or "X is of characteristic p", or "X is in characteristic p" if p is zero in $\mathcal{O}_X$.

Definition 36.1. Let p be a prime number. Let X be a scheme in characteristic p. The *absolute frobenius of X* is the morphism $F_X : X \to X$ given by the identity on the underlying topological space and with $F_X^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ given by $g \mapsto g^p$.

This makes sense because for any ring A of characteristic p the map $F_A : A \to A$, $a \mapsto a^p$ is a ring endomorphism which induces the identity on $\mathrm{Spec}(A)$. Moreover, if A is local, then F_A is a local homomorphism. In this way we see that the absolute frobenius of X is an endomorphism of X in the category of schemes. It turns out that the absolute frobenius defines a self map of the identity functor on the category of schemes in characteristic p.

Lemma 36.2. *Let $p > 0$ be a prime number. Let $f : X \to Y$ be a morphism of schemes in characteristic p. Then the diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{F_X} & X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{F_Y} & Y
\end{array}
$$

commutes.

Proof. This follows from the following trivial algebraic fact: if $\varphi : A \to B$ is a homomorphism of rings of characteristic p, then $\varphi(a^p) = \varphi(a)^p$. $\square$

Lemma 36.3. *Let $p > 0$ be a prime number. Let X be a scheme in characteristic p. Then the absolute frobenius $F_X : X \to X$ is a universal homeomorphism, is integral, and induces purely inseparable residue field extensions.*
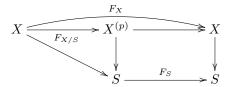
Proof. This follows from the corresponding results for the frobenius endomorphism $F_A : A \to A$ of a ring A of characteristic $p > 0$. See the discussion in Algebra, Section 46, for example Lemma 46.7. $\qquad\square$

If we are working with schemes over a fixed base, then there is a relative version of the frobenius morphism.

Definition 36.4. Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. We define

$$X^{(p)} = X^{(p/S)} = X \times_{S, F_S} S$$

viewed as a scheme over S. Applying Lemma 36.2 we see there is a unique morphism $F_{X/S} : X \longrightarrow X^{(p)}$ over S fitting into the commutative diagram

where the right square is cartesian. The morphism $F_{X/S}$ is called the *relative Frobenius morphism of X/S*.

Observe that $X \mapsto X^{(p)}$ is a functor; it is the base change functor for the absolute frobenius morphism $F_S : S \to S$. We have the same lemmas as before regarding the relative Frobenius morphism.

Lemma 36.5. *Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let $f : X \to Y$ be a morphism of schemes over S . Then the diagram*

commutes.

Proof. This follows from Lemma 36.2 and the definitions. $\qquad\square$

Lemma 36.6. *Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. Then the relative frobenius $F_{X/S} : X \to X^{(p)}$ is a universal homeomorphism, is integral, and induces purely inseparable residue field extensions.*

Proof. By Lemma 36.3 the morphisms $F_X : X \to X$ and the base change $h : X^{(p)} \to X$ of F_S are universal homeomorphisms. Since $h \circ F_{X/S} = F_X$ we conclude that $F_{X/S}$ is a universal homeomorphism (Morphisms, Lemma 45.8). By Morphisms, Lemmas 45.5 and 10.2 we conclude that $F_{X/S}$ has the other properties as well. $\qquad\square$

Lemma 36.7. *Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. Then $\Omega_{X/S} = \Omega_{X/X^{(p)}}$.*

Proof. This translates into the following algebra fact. Let $A \to B$ be a homomorphism of rings of characteristic p. Set $B' = B \otimes_{A,F_A} A$ and consider the ring map $F_{B/A} : B' \to B$, $b \otimes a \mapsto b^p a$. Then our assertion is that $\Omega_{B/A} = \Omega_{B/B'}$. This is true because $d(b^p a) = 0$ if $d : B \to \Omega_{B/A}$ is the universal derivation and hence d is a B'-derivation. $\qquad\square$

Lemma 36.8. *Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. If $X \to S$ is locally of finite type, then $F_{X/S}$ is finite.*

Proof. This translates into the following algebra fact. Let $A \to B$ be a finite type homomorphism of rings of characteristic p. Set $B' = B \otimes_{A,F_A} A$ and consider the ring map $F_{B/A} : B' \to B$, $b \otimes a \mapsto b^p a$. Then our assertion is that $F_{B/A}$ is finite. Namely, if $x_1, \ldots, x_n \in B$ are generators over A, then x_i is integral over B' because $x_i^p = F_{B/A}(x_i \otimes 1)$. Hence $F_{B/A} : B' \to B$ is finite by Algebra, Lemma 36.5. $\qquad\square$

Lemma 36.9. *Let k be a field of characteristic $p > 0$. Let X be a scheme over k. Then X is geometrically reduced if and only if $X^{(p)}$ is reduced.*

Proof. Consider the absolute frobenius $F_k : k \to k$. Then $F_k(k) = k^p$ in other words, $F_k : k \to k$ is isomorphic to the embedding of k into $k^{1/p}$. Thus the lemma follows from Lemma 6.4. $\qquad\square$

Lemma 36.10. *Let k be a field of characteristic $p > 0$. Let X be a variety over k. The following are equivalent*

(1) *$X^{(p)}$ is reduced,*
(2) *X is geometrically reduced,*
(3) *there is a nonempty open $U \subset X$ smooth over k.*

In this case $X^{(p)}$ is a variety over k and $F_{X/k} : X \to X^{(p)}$ is a finite dominant morphism of degree $p^{\dim(X)}$.

Proof. We have seen the equivalence of (1) and (2) in Lemma 36.9. We have seen that (2) implies (3) in Lemma 25.7. If (3) holds, then U is geometrically reduced (see for example Lemma 12.6) and hence X is geometrically reduced by Lemma 6.8. In this way we see that (1), (2), and (3) are equivalent.

Assume (1), (2), and (3) hold. Since $F_{X/k}$ is a homeomorphism (Lemma 36.6) we see that $X^{(p)}$ is a variety. Then $F_{X/k}$ is finite by Lemma 36.8. It is dominant as it is surjective. To compute the degree (Morphisms, Definition 51.8) it suffices to compute the degree of $F_{U/k} : U \to U^{(p)}$ (as $F_{U/k} = F_{X/k}|_U$ by Lemma 36.5). After shrinking U a bit we may assume there exists an étale morphism $h : U \to \mathbf{A}_k^n$, see Morphisms, Lemma 36.20. Of course $n = \dim(U)$ because $\mathbf{A}_k^n \to \mathrm{Spec}(k)$ is smooth of relative dimension n, the étale morphism h is smooth of relative dimension 0, and $U \to \mathrm{Spec}(k)$ is smooth of relative dimension $\dim(U)$ and relative dimensions add up correctly (Morphisms, Lemma 29.3). Observe that h is a generically finite dominant morphism of varieties, and hence $\deg(h)$ is defined. By Lemma 36.5 we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;F_{X/k}\;\;} & X^{(p)} \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h^{(p)}} \\
\mathbf{A}_k^n & \xrightarrow{\;F_{\mathbf{A}_k^n/k}\;} & (\mathbf{A}_k^n)^{(p)}
\end{array}
$$

Since $h^{(p)}$ is a base change of h it is étale as well and it follows that $h^{(p)}$ is a generically finite dominant morphism of varieties as well. The degree of $h^{(p)}$ is the degree of the extension $k(X^{(p)})/k((\mathbf{A}_k^n)^{(p)})$ which is the same as the degree of the extension $k(X)/k(\mathbf{A}_k^n)$ because $h^{(p)}$ is the base change of h (small detail omitted). By multiplicativity of degrees (Morphisms, Lemma 51.9) it suffices to show that the degree of $F_{\mathbf{A}_k^n/k}$ is p^n. To see this observe that $(\mathbf{A}_k^n)^{(p)} = \mathbf{A}_k^n$ and that $F_{\mathbf{A}_k^n/k}$ is given by the map sending the coordinates to their pth powers. $\qquad\square$

Remark 36.11. Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. For $n \geq 1$

$$X^{(p^n)} = X^{(p^n/S)} = X \times_{S, F_S^n} S$$

viewed as a scheme over S. Observe that $X \mapsto X^{(p^n)}$ is a functor. Applying Lemma 36.2 we see $F_{X/S,n} = (F_X^n, \mathrm{id}_S) : X \longrightarrow X^{(p^n)}$ is a morphism over S fitting into the commutative diagram

where the right square is cartesian. The morphism $F_{X/S,n}$ is sometimes called the *n-fold relative Frobenius morphism of X/S*. This makes sense because we have the formula

$$F_{X/S,n} = F_{X^{(p^{n-1})}/S} \circ \ldots \circ F_{X^{(p)}/S} \circ F_{X/S}$$

which shows that $F_{X/S,n}$ is the composition of n relative Frobenii. Since we have

$$F_{X^{(p^m)}/S} = F_{X^{(p^{m-1})}/S}^{(p)} = \ldots = F_{X/S}^{(p^m)}$$

(details omitted) we get also that

$$F_{X/S,n} = F_{X/S}^{(p^{n-1})} \circ \ldots \circ F_{X/S}^{(p)} \circ F_{X/S}$$

37. Glueing dimension one rings

This section contains some algebraic preliminaries to proving that a finite set of codimension 1 points of a separated scheme is contained in an affine open.

Situation 37.1. Here we are given a commutative diagram of rings

where K is a field and A, B are subrings of K with fraction field K. Finally, $R = A \times_K B = A \cap B$.

Lemma 37.2. *In Situation 37.1 assume that B is a valuation ring. Then for every unit u of A either $u \in R$ or $u^{-1} \in R$.*

Proof. Namely, if the image c of u in K is in B, then $u \in R$. Otherwise, $c^{-1} \in B$ (Algebra, Lemma 50.4) and $u^{-1} \in R$. $\qquad\square$

The following lemma explains the meaning of the condition "$A \otimes B \to K$ is surjective" which comes up quite a bit in the following.

Lemma 37.3. *In Situation 37.1 assume A is a Noetherian ring of dimension 1. The following are equivalent*

    (1) *$A \otimes B \to K$ is not surjective,*
    (2) *there exists a discrete valuation ring $\mathcal{O} \subset K$ containing both A and B.*

Proof. It is clear that (2) implies (1). On the other hand, if $A \otimes B \to K$ is not surjective, then the image $C \subset K$ is not a field hence C has a nonzero maximal ideal $\mathfrak{m}$. Choose a valuation ring $\mathcal{O} \subset K$ dominating $C_{\mathfrak{m}}$. By Algebra, Lemma 119.12 applied to $A \subset \mathcal{O}$ the ring $\mathcal{O}$ is Noetherian. Hence $\mathcal{O}$ is a discrete valuation ring by Algebra, Lemma 50.18. $\square$

Lemma 37.4. *In Situation 37.1 assume*

    (1) *A is a Noetherian semi-local domain of dimension 1,*
    (2) *B is a discrete valuation ring,*

Then we have the following two possibilities

    (a) *If A^* is not contained in R, then $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ and $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ are open immersions covering $\mathrm{Spec}(R)$ and $K = A \otimes_R B$.*
    (b) *If A^* is contained in R, then B dominates one of the local rings of A at a maximal ideal and $A \otimes B \to K$ is not surjective.*

Proof. Assumption (a) implies there is a unit u of A whose image in K lies in the maximal ideal of B. Then u is a nonzerodivisor of R and for every $a \in A$ there exists an n such that $u^n a \in R$. It follows that $A = R_u$.

Let $\mathfrak{m}_A$ be the Jacobson radical of A. Let $x \in \mathfrak{m}_A$ be a nonzero element. Since $\dim(A) = 1$ we see that $K = A_x$. After replacing x by $x^n u^m$ for some $n \geq 1$ and $m \in \mathbf{Z}$ we may assume x maps to a unit of B. We see that for every $b \in B$ we have that $x^n b$ in the image of R for some n. Thus $B = R_x$.

Let $z \in R$. If $z \notin \mathfrak{m}_A$ and z does not map to an element of $\mathfrak{m}_B$, then z is invertible. Thus $x + u$ is invertible in R. Hence $\mathrm{Spec}(R) = D(x) \cup D(u)$. We have seen above that $D(u) = \mathrm{Spec}(A)$ and $D(x) = \mathrm{Spec}(B)$.

Case (b). If $x \in \mathfrak{m}_A$, then $1 + x$ is a unit and hence $1 + x \in R$, i.e, $x \in R$. Thus we see that $\mathfrak{m}_A \subset R \subset A$. In fact, in this case A is integral over R. Namely, write $A/\mathfrak{m}_A = \kappa_1 \times \ldots \times \kappa_n$ as a product of fields. Say $x = (c_1, \ldots, c_r, 0, \ldots, 0)$ is an element with $c_i \neq 0$. Then

$$x^2 - x(c_1, \ldots, c_r, 1, \ldots, 1) = 0$$

Since R contains all units we see that $A/\mathfrak{m}_A$ is integral over the image of R in it, and hence A is integral over R. It follows that $R \subset A \subset B$ as B is integrally closed. Moreover, if $x \in \mathfrak{m}_A$ is nonzero, then $K = A_x = \bigcup x^{-n} A = \bigcup x^{-n} R$. Hence $x^{-1} \notin B$, i.e., $x \in \mathfrak{m}_B$. We conclude $\mathfrak{m}_A \subset \mathfrak{m}_B$. Thus $A \cap \mathfrak{m}_B$ is a maximal ideal of A thereby finishing the proof. $\square$

Lemma 37.5. *Let B be a semi-local Noetherian domain of dimension 1. Let B' be the integral closure of B in its fraction field. Then B' is a semi-local Dedekind domain. Let x be a nonzero element of the Jacobson radical of B'. Then for every $y \in B'$ there exists an n such that $x^n y \in B$.*

Proof. Let $\mathfrak{m}_B$ be the Jacobson radical of B. The structure of B' results from Algebra, Lemma 120.18. Given $x, y \in B'$ as in the statement of the lemma consider the subring $B \subset A \subset B'$ generated by x and y. Then A is finite over B (Algebra, Lemma 36.5). Since the fraction fields of B and A are the same we see that the finite module A/B is supported on the set of closed points of B. Thus $\mathfrak{m}_B^n A \subset B$ for a suitable n. Moreover, $\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is surjective (Algebra, Lemma 36.17), hence A is semi-local as well. It also follows that x is in the Jacobson radical $\mathfrak{m}_A$ of A. Note that $\mathfrak{m}_A = \sqrt{\mathfrak{m}_B A}$. Thus $x^m y \in \mathfrak{m}_B A$ for some m. Then $x^{nm} y \in B$. $\qquad\square$

Lemma 37.6. *In Situation 37.1 assume*

    (1) *A is a Noetherian semi-local domain of dimension 1,*
    (2) *B is a Noetherian semi-local domain of dimension 1,*
    (3) *$A \otimes B \to K$ is surjective.*

Then $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ and $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ are open immersions covering $\mathrm{Spec}(R)$ and $K = A \otimes_R B$.

Proof. Special case: B is integrally closed in K. This means that B is a Dedekind domain (Algebra, Lemma 120.17) whence all of its localizations at maximal ideals are discrete valuation rings. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals of B. We set

$$R_1 = A \times_K B_{\mathfrak{m}_1}$$

Observing that $A \otimes_{R_1} B_{\mathfrak{m}_1} \to K$ is surjective we conclude from Lemma 37.4 that A and $B_{\mathfrak{m}_1}$ define open subschemes covering $\mathrm{Spec}(R_1)$ and that $K = A \otimes_{R_1} B_{\mathfrak{m}_1}$. In particular R_1 is a semi-local Noetherian ring of dimension 1. By induction we define

$$R_{i+1} = R_i \times_K B_{\mathfrak{m}_{i+1}}$$

for $i = 1, \ldots, r-1$. Observe that $R = R_n$ because $B = B_{\mathfrak{m}_1} \cap \ldots \cap B_{\mathfrak{m}_r}$ (see Algebra, Lemma 157.6). It follows from the inductive procedure that $R \to A$ defines an open immersion $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$. On the other hand, the maximal ideals $\mathfrak{n}_i$ of R not in this open correspond to the maximal ideals $\mathfrak{m}_i$ of B and in fact the ring map $R \to B$ defines an isomorphisms $R_{\mathfrak{n}_i} \to B_{\mathfrak{m}_i}$ (details omitted; hint: in each step we added exactly one maximal ideal to $\mathrm{Spec}(R_i)$). It follows that $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ is an open immersion as desired.

General case. Let $B' \subset K$ be the integral closure of B. See Lemma 37.5. Then the special case applies to $R' = A \times_K B'$. Pick $x \in R'$ which is not contained in the maximal ideals of A and is contained in the maximal ideals of B' (see Algebra, Lemma 15.4). By Lemma 37.5 there exists an integer n such that $x^n \in R = A \times_K B$. Replace x by x^n so $x \in R$. For every $y \in R'$ there exists an integer n such that $x^n y \in R$. On the other hand, it is clear that $R'_x = A$. Thus $R_x = A$. Exchanging the roles of A and B we also find an $y \in R$ such that $B = R_y$. Note that inverting both x and y leaves no primes except (0). Thus $K = R_{xy} = R_x \otimes_R R_y$. This finishes the proof. $\qquad\square$

Lemma 37.7. *Let K be a field. Let $A_1, \ldots, A_r \subset K$ be Noetherian semi-local rings of dimension 1 with fraction field K. If $A_i \otimes A_j \to K$ is surjective for all $i \neq j$, then there exists a Noetherian semi-local domain $A \subset K$ of dimension 1 contained in $A_1, \ldots, A_r$ such that*

    (1) *$A \to A_i$ induces an open immersion $j_i : \mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$,*
    (2) *$\mathrm{Spec}(A)$ is the union of the opens $j_i(\mathrm{Spec}(A_i))$,*

(3) *each closed point of* $\mathrm{Spec}(A)$ *lies in exactly one of these opens.*

Proof. Namely, we can take $A = A_1 \cap \ldots \cap A_r$. First we note that (3), once (1) and (2) have been proven, follows from the assumption that $A_i \otimes A_j \to K$ is surjective since if $\mathfrak{m} \in j_i(\mathrm{Spec}(A_i)) \cap j_j(\mathrm{Spec}(A_j))$, then $A_i \otimes A_j \to K$ ends up in $A_\mathfrak{m}$. To prove (1) and (2) we argue by induction on r. If $r > 1$ by induction we have the results (1) and (2) for $B = A_2 \cap \ldots \cap A_r$. Then we apply Lemma 37.6 to see they hold for $A = A_1 \cap B$. $\square$

Lemma 37.8. *Let A be a domain with fraction field K. Let $B_1, \ldots, B_r \subset K$ be Noetherian 1-dimensional semi-local domains whose fraction fields are K. If $A \otimes B_i \to K$ are surjective for $i = 1, \ldots, r$, then there exists an $x \in A$ such that x^{-1} is in the Jacobson radical of B_i for $i = 1, \ldots, r$.*

Proof. Let B_i' be the integral closure of B_i in K. Suppose we find a nonzero $x \in A$ such that x^{-1} is in the Jacobson radical of B_i' for $i = 1, \ldots, r$. Then by Lemma 37.5, after replacing x by a power we get $x^{-1} \in B_i$. Since $\mathrm{Spec}(B_i') \to \mathrm{Spec}(B_i)$ is surjective we see that x^{-1} is then also in the Jacobson radical of B_i. Thus we may assume that each B_i is a semi-local Dedekind domain.

If B_i is not local, then remove B_i from the list and add back the finite collection of local rings $(B_i)_\mathfrak{m}$. Thus we may assume that B_i is a discrete valuation ring for $i = 1, \ldots, r$.

Let $v_i : K \to \mathbf{Z}$, $i = 1, \ldots, r$ be the corresponding discrete valuations (see Algebra, Lemma 120.17). We are looking for a nonzero $x \in A$ with $v_i(x) < 0$ for $i = 1, \ldots, r$. We will prove this by induction on r.

If $r = 1$ and the result is wrong, then $A \subset B$ and the map $A \otimes B \to K$ is not surjective, contradiction.

If $r > 1$, then by induction we can find a nonzero $x \in A$ such that $v_i(x) < 0$ for $i = 1, \ldots, r - 1$. If $v_r(x) < 0$ then we are done, so we may assume $v_r(x) \geq 0$. By the base case we can find $y \in A$ nonzero such that $v_r(y) < 0$. After replacing x by a power we may assume that $v_i(x) < v_i(y)$ for $i = 1, \ldots, r - 1$. Then $x + y$ is the element we are looking for. $\square$

Lemma 37.9. *Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_\mathfrak{p}$ where the product is over the minimal primes of A. Let $a_1, a_2 \in \mathfrak{m}_A$ map to the same element of L. Then $a_1^n = a_2^n$ for some $n > 0$.*

Proof. Write $a_1 = a_2 + x$. Then x maps to zero in L. Hence x is a nilpotent element of A because $\bigcap \mathfrak{p}$ is the radical of (0) and the annihilator I of x contains a power of the maximal ideal because $\mathfrak{p} \notin V(I)$ for all minimal primes. Say $x^k = 0$ and $\mathfrak{m}^n \subset I$. Then

$$a_1^{k+n} = a_2^{k+n} + \binom{n+k}{1} a_2^{n+k-1} x + \binom{n+k}{2} a_2^{n+k-2} x^2 + \ldots + \binom{n+k}{k-1} a_2^{n+1} x^{k-1} = a_2^{n+k}$$

because $a_2 \in \mathfrak{m}_A$. $\square$

Lemma 37.10. *Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_\mathfrak{p}$ and $I = \bigcap \mathfrak{p}$ where the product and intersection are over the minimal primes of A. Let $f \in L$ be an element of the form $f = i + a$ where $a \in \mathfrak{m}_A$ and $i \in IL$. Then some power of f is in the image of $A \to L$.*

Proof. Since A is Noetherian we have $I^t = 0$ for some $t > 0$. Suppose that we know that $f = a + i$ with $i \in I^k L$. Then $f^n = a^n + na^{n-1}i \mod I^{k+1}L$. Hence it suffices to show that $na^{n-1}i$ is in the image of $I^k \to I^k L$ for some $n \gg 0$. To see this, pick a $g \in A$ such that $\mathfrak{m}_A = \sqrt{(g)}$ (Algebra, Lemma 60.8). Then $L = A_g$ for example by Algebra, Proposition 60.7. On the other hand, there is an n such that $a^n \in (g)$. Hence we can clear denominators for elements of L by multiplying by a high power of a. $\qquad\square$

Lemma 37.11. *Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_{\mathfrak{p}}$ where the product is over the minimal primes of A. Let $K \to L$ be an integral ring map. Then there exist $a \in \mathfrak{m}_A$ and $x \in K$ which map to the same element of L such that $\mathfrak{m}_A = \sqrt{(a)}$.*

Proof. By Lemma 37.10 we may replace A by $A/(\bigcap \mathfrak{p})$ and assume that A is a reduced ring (some details omitted). We may also replace K by the image of $K \to L$. Then K is a reduced ring. The map $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is surjective and closed (details omitted). Hence $\mathrm{Spec}(K)$ is a finite discrete space. It follows that K is a finite product of fields.

Let $\mathfrak{p}_j$, $j = 1, \ldots, m$ be the minimal primes of A. Set L_j be the fraction field of A_j so that $L = \prod_{j=1,\ldots,m} L_j$. Let A_j be the normalization of $A/\mathfrak{p}_j$. Then A_j is a semi-local Dedekind domain with at least one maximal ideal, see Algebra, Lemma 120.18. Let n be the sum of the numbers of maximal ideals in $A_1, \ldots, A_m$. For such a maximal ideal $\mathfrak{m} \subset A_j$ we consider the function

$$v_{\mathfrak{m}} : L \to L_j \to \mathbf{Z} \cup \{\infty\}$$

where the second arrow is the discrete valuation corresponding to the discrete valuation ring $(A_j)_{\mathfrak{m}}$ extended by mapping 0 to ∞. In this way we obtain n functions $v_1, \ldots, v_n : L \to \mathbf{Z} \cup \{\infty\}$. We will find an element $x \in K$ such that $v_i(x) < 0$ for all $i = 1, \ldots, n$.

First we claim that for each i there exists an element $x \in K$ with $v_i(x) < 0$. Namely, suppose that v_i corresponds to $\mathfrak{m} \subset A_j$. If $v_i(x) \geq 0$ for all $x \in K$, then K maps into $(A_j)_{\mathfrak{m}}$ inside the fraction field L_j of A_j. The image of K in L_j is a field over L_j is algebraic by Algebra, Lemma 36.18. Combined we get a contradiction with Algebra, Lemma 50.8.

Suppose we have found an element $x \in K$ such that $v_1(x) < 0, \ldots, v_r(x) < 0$ for some $r < n$. If $v_{r+1}(x) < 0$, then x works for $r + 1$. If not, then choose some $y \in K$ with $v_{r+1}(y) < 0$ as is possible by the result of the previous paragraph. After replacing x by x^n for some $n > 0$, we may assume $v_i(x) < v_i(y)$ for $i = 1, \ldots, r$. Then $v_j(x+y) = v_j(x) < 0$ for $j = 1, \ldots, r$ by properties of valuations and similarly $v_{r+1}(x + y) = v_{r+1}(y) < 0$. Arguing by induction, we find $x \in K$ with $v_i(x) < 0$ for $i = 1, \ldots, n$.

In particular, the element $x \in K$ has nonzero projection in each factor of K (recall that K is a finite product of fields and if some component of x was zero, then one of the values $v_i(x)$ would be ∞). Hence x is invertible and $x^{-1} \in K$ is an element with $\infty > v_i(x^{-1}) > 0$ for all i. It follows from Lemma 37.5 that for some $e < 0$ the element $x^e \in K$ maps to an element of $\mathfrak{m}_A/\mathfrak{p}_j \subset A/\mathfrak{p}_j$ for all $j = 1, \ldots, m$. Observe that the cokernel of the map $\mathfrak{m}_A \to \prod \mathfrak{m}_A/\mathfrak{p}_j$ is annihilated by a power of $\mathfrak{m}_A$. Hence after replacing e by a more negative e, we find an element $a \in \mathfrak{m}_A$

whose image in $\mathfrak{m}_A/\mathfrak{p}_j$ is equal to the image of x^e. The pair (a, x^e) satisfies the conclusions of the lemma. $\square$

Lemma 37.12. *Let A be a ring. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be a finite set of a primes of A. Let $S = A \setminus \bigcup \mathfrak{p}_i$. Then S is a multiplicative system and $S^{-1}A$ is a semi-local ring whose maximal ideals correspond to the maximal elements of the set $\{\mathfrak{p}_i\}$.*

Proof. If $a, b \in A$ and $a, b \in S$, then $a, b \notin \mathfrak{p}_i$ hence $ab \notin \mathfrak{p}_i$, hence $ab \in S$. Also $1 \in S$. Thus S is a multiplicative subset of A. By the description of $\operatorname{Spec}(S^{-1}A)$ in Algebra, Lemma 17.5 and by Algebra, Lemma 15.2 we see that the primes of $S^{-1}A$ correspond to the primes of A contained in one of the $\mathfrak{p}_i$. Hence the maximal ideals of $S^{-1}A$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. $\square$

38. One dimensional Noetherian schemes

The main result of this section is that a Noetherian separated scheme of dimension 1 has an ample invertible sheaf. See Proposition 38.12.

Lemma 38.1. *Let X be a scheme all of whose local rings are Noetherian of dimension ≤ 1. Let $U \subset X$ be a retrocompact open. Denote $j : U \to X$ the inclusion morphism. Then $R^p j_* \mathcal{F} = 0$, $p > 0$ for every quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$.*

Proof. We may check the vanishing of $R^p j_* \mathcal{F}$ at stalks. Formation of $R^q j_*$ commutes with flat base change, see Cohomology of Schemes, Lemma 5.2. Thus we may assume that X is the spectrum of a Noetherian local ring of dimension ≤ 1. In this case X has a closed point x and finitely many other points $x_1, \ldots, x_n$ which specialize to x but not each other (see Algebra, Lemma 31.6). If $x \in U$, then $U = X$ and the result is clear. If not, then $U = \{x_1, \ldots, x_r\}$ for some r after possibly renumbering the points. Then U is affine (Schemes, Lemma 11.8). Thus the result follows from Cohomology of Schemes, Lemma 2.3. $\square$

Lemma 38.2. *Let X be an affine scheme all of whose local rings are Noetherian of dimension ≤ 1. Then any quasi-compact open $U \subset X$ is affine.*

Proof. Denote $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. By Lemma 38.1 the higher direct images $R^p j_* \mathcal{F}$ are zero. The $\mathcal{O}_X$-module $j_* \mathcal{F}$ is quasi-coherent (Schemes, Lemma 24.1). Hence it has vanishing higher cohomology groups by Cohomology of Schemes, Lemma 2.2. By the Leray spectral sequence Cohomology, Lemma 13.6 we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$. Thus U is affine, for example by Cohomology of Schemes, Lemma 3.1. $\square$

Lemma 38.3. *Let X be a scheme. Let $U \subset X$ be an open. Assume*

(1) *U is a retrocompact open of X,*
(2) *$X \setminus U$ is discrete, and*
(3) *for $x \in X \setminus U$ the local ring $\mathcal{O}_{X,x}$ is Noetherian of dimension ≤ 1.*

Then (1) there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a section s such that $U = X_s$ and (2) the map $\operatorname{Pic}(X) \to \operatorname{Pic}(U)$ is surjective.

Proof. Let $X \setminus U = \{x_i; i \in I\}$. Choose affine opens $U_i \subset X$ with $x_i \in U_i$ and $x_j \notin U_i$ for $j \neq i$. This is possible by condition (2). Say $U_i = \operatorname{Spec}(A_i)$. Let $\mathfrak{m}_i \subset A_i$ be the maximal ideal corresponding to x_i. By our assumption on the local rings there are only a finite number of prime ideals $\mathfrak{q} \subset \mathfrak{m}_i$, $\mathfrak{q} \neq \mathfrak{m}_i$ (see

Algebra, Lemma 31.6). Thus by prime avoidance (Algebra, Lemma 15.2) we can find $f_i \in \mathfrak{m}_i$ not contained in any of those primes. Then $V(f_i) = \{\mathfrak{m}_i\} \amalg Z_i$ for some closed subset $Z_i \subset U_i$ because Z_i is a retrocompact open subset of $V(f_i)$ closed under specialization, see Algebra, Lemma 41.7. After shrinking U_i we may assume $V(f_i) = \{x_i\}$. Then

$$\mathcal{U} : X = U \cup \bigcup U_i$$

is an open covering of X. Consider the 2-cocycle with values in $\mathcal{O}_X^*$ given by f_i on $U \cap U_i$ and by f_i/f_j on $U_i \cap U_j$. This defines a line bundle $\mathcal{L}$ such that the section s defined by 1 on U and f_i on U_i is as in the statement of the lemma.

Let $\mathcal{N}$ be an invertible $\mathcal{O}_U$-module. Let N_i be the invertible $(A_i)_{f_i}$ module such that $\mathcal{N}|_{U \cap U_i}$ is equal to $\tilde{N}_i$. Observe that $(A_{\mathfrak{m}_i})_{f_i}$ is an Artinian ring (as a dimension zero Noetherian ring, see Algebra, Lemma 60.5). Thus it is a product of local rings (Algebra, Lemma 53.6) and hence has trivial Picard group. Thus, after shrinking U_i (i.e., after replacing A_i by $(A_i)_g$ for some $g \in A_i$, $g \notin \mathfrak{m}_i$) we can assume that $N_i = (A_i)_{f_i}$, i.e., that $\mathcal{N}|_{U \cap U_i}$ is trivial. In this case it is clear how to extend $\mathcal{N}$ to an invertible sheaf over X (by extending it by a trivial invertible module over each U_i). $\square$

Lemma 38.4. *Let X be an integral separated scheme. Let $U \subset X$ be a nonempty affine open such that $X \setminus U$ is a finite set of points $x_1, \ldots, x_r$ with $\mathcal{O}_{X,x_i}$ Noetherian of dimension 1. Then there exists a globally generated invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a section s such that $U = X_s$.*

Proof. Say $U = \operatorname{Spec}(A)$ and let K be the function field of X. Write $B_i = \mathcal{O}_{X,x_i}$ and $\mathfrak{m}_i = \mathfrak{m}_{x_i}$. Since $x_i \notin U$ we see that the open $U \times_X \operatorname{Spec}(B_i)$ of $\operatorname{Spec}(B_i)$ has only one point, i.e., $U \times_X \operatorname{Spec}(B_i) = \operatorname{Spec}(K)$. Since X is separated, we find that $\operatorname{Spec}(K)$ is a closed subscheme of $U \times \operatorname{Spec}(B_i)$, i.e., the map $A \otimes B_i \to K$ is a surjection. By Lemma 37.8 we can find a nonzero $f \in A$ such that $f^{-1} \in \mathfrak{m}_i$ for $i = 1, \ldots, r$. Pick opens $x_i \in U_i \subset X$ such that $f^{-1} \in \mathcal{O}(U_i)$. Then

$$\mathcal{U} : X = U \cup \bigcup U_i$$

is an open covering of X. Consider the 2-cocycle with values in $\mathcal{O}_X^*$ given by f on $U \cap U_i$ and by 1 on $U_i \cap U_j$. This defines a line bundle $\mathcal{L}$ with two sections:

(1) a section s defined by 1 on U and f^{-1} on U_i is as in the statement of the lemma, and
(2) a section t defined by f on U and 1 on U_i.

Note that $X_t \supset U_1 \cup \ldots \cup U_r$. Hence s, t generate $\mathcal{L}$ and the lemma is proved. $\square$

Lemma 38.5. *Let X be a quasi-compact scheme. If for every $x \in X$ there exists a pair $(\mathcal{L}, s)$ consisting of a globally generated invertible sheaf $\mathcal{L}$ and a global section s such that $x \in X_s$ and X_s is affine, then X has an ample invertible sheaf.*

Proof. Since X is quasi-compact we can find a finite collection $(\mathcal{L}_i, s_i)$, $i = 1, \ldots, n$ of pairs such that $\mathcal{L}_i$ is globally generated, X_{s_i} is affine and $X = \bigcup X_{s_i}$. Again because X is quasi-compact we can find, for each i, a finite collection of sections $t_{i,j}$ of $\mathcal{L}_i$, $j = 1, \ldots, m_i$ such that $X = \bigcup X_{t_{i,j}}$. Set $t_{i,0} = s_i$. Consider the invertible sheaf

$$\mathcal{L} = \mathcal{L}_1 \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L}_n$$

and the global sections

$$\tau_J = t_{1,j_1} \otimes \ldots \otimes t_{n,j_n}$$

By Properties, Lemma 26.4 the open X_{τ_J} is affine as soon as $j_i = 0$ for some i. It is a simple matter to see that these opens cover X. Hence $\mathcal{L}$ is ample by definition. $\quad\square$

Lemma 38.6. *Let X be a Noetherian integral separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. Choose an affine open covering $X = U_1 \cup \ldots \cup U_n$. Since X is Noetherian, each of the sets $X \setminus U_i$ is finite. Thus by Lemma 38.4 we can find a pair $(\mathcal{L}_i, s_i)$ consisting of a globally generated invertible sheaf $\mathcal{L}_i$ and a global section s_i such that $U_i = X_{s_i}$. We conclude that X has an ample invertible sheaf by Lemma 38.5. $\quad\square$

Lemma 38.7. *Let $f : X \to Y$ be a finite morphism of schemes. Assume there exists an open $V \subset Y$ such that $f^{-1}(V) \to V$ is an isomorphism and $Y \setminus V$ is a discrete space. Then every invertible $\mathcal{O}_X$-module is the pullback of an invertible $\mathcal{O}_Y$-module.*

Proof. We will use that $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$, see Cohomology, Lemma 6.1. Consider the Leray spectral sequence for the abelian sheaf $\mathcal{O}_X^*$ and f, see Cohomology, Lemma 13.4. Consider the induced map

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^0(Y, R^1 f_* \mathcal{O}_X^*)$$

Divisors, Lemma 17.1 says exactly that this map is zero. Hence Leray gives $H^1(X, \mathcal{O}_X^*) = H^1(Y, f_* \mathcal{O}_X^*)$. Next we consider the map

$$f^\sharp : \mathcal{O}_Y^* \longrightarrow f_* \mathcal{O}_X^*$$

By assumption the kernel and cokernel of this map are supported on the closed subset $T = Y \setminus V$ of Y. Since T is a discrete topological space by assumption the higher cohomology groups of any abelian sheaf on Y supported on T is zero (follows from Cohomology, Lemma 20.1, Modules, Lemma 6.1, and the fact that $H^i(T, \mathcal{F}) = 0$ for any $i > 0$ and any abelian sheaf $\mathcal{F}$ on T). Breaking the displayed map into short exact sequences

$$0 \to \mathrm{Ker}(f^\sharp) \to \mathcal{O}_Y^* \to \mathrm{Im}(f^\sharp) \to 0, \quad 0 \to \mathrm{Im}(f^\sharp) \to f_* \mathcal{O}_X^* \to \mathrm{Coker}(f^\sharp) \to 0$$

we first conclude that $H^1(Y, \mathcal{O}_Y^*) \to H^1(Y, \mathrm{Im}(f^\sharp))$ is surjective and then that $H^1(Y, \mathrm{Im}(f^\sharp)) \to H^1(Y, f_* \mathcal{O}_X^*)$ is surjective. Combining all the above we find that $H^1(Y, \mathcal{O}_Y^*) \to H^1(X, \mathcal{O}_X^*)$ is surjective as desired. $\quad\square$

Lemma 38.8. *Let X be a scheme. Let $Z_1, \ldots, Z_n \subset X$ be closed subschemes. Let $\mathcal{L}_i$ be an invertible sheaf on Z_i. Assume that*

    (1) *X is reduced,*
    (2) *$X = \bigcup Z_i$ set theoretically, and*
    (3) *$Z_i \cap Z_j$ is a discrete topological space for $i \neq j$.*

Then there exists an invertible sheaf $\mathcal{L}$ on X whose restriction to Z_i is $\mathcal{L}_i$. Moreover, if we are given sections $s_i \in \Gamma(Z_i, \mathcal{L}_i)$ which are nonvanishing at the points of $Z_i \cap Z_j$, then we can choose $\mathcal{L}$ such that there exists a $s \in \Gamma(X, \mathcal{L})$ with $s|_{Z_i} = s_i$ for all i.

Proof. The existence of $\mathcal{L}$ can be deduced from Lemma 38.7 but we will also give a direct proof and we will use the direct proof to see the statement about sections is true. Set $T = \bigcup_{i \neq j} Z_i \cap Z_j$. As X is reduced we have

$$X \setminus T = \bigcup (Z_i \setminus T)$$

as schemes. Assumption (3) implies T is a discrete subset of X. Thus for each $t \in T$ we can find an open $U_t \subset X$ with $t \in U_t$ but $t' \notin U_t$ for $t' \in T$, $t' \neq t$. By shrinking U_t if necessary, we may assume that there exist isomorphisms $\varphi_{t,i} : \mathcal{L}_i|_{U_t \cap Z_i} \to \mathcal{O}_{U_t \cap Z_i}$. Furthermore, for each i choose an open covering

$$Z_i \setminus T = \bigcup_j U_{ij}$$

such that there exist isomorphisms $\varphi_{i,j} : \mathcal{L}_i|_{U_{ij}} \cong \mathcal{O}_{U_{ij}}$. Observe that

$$\mathcal{U} : X = \bigcup U_t \cup \bigcup U_{ij}$$

is an open covering of X. We claim that we can use the isomorphisms $\varphi_{t,i}$ and $\varphi_{i,j}$ to define a 2-cocycle with values in $\mathcal{O}_X^*$ for this covering that defines $\mathcal{L}$ as in the statement of the lemma.

Namely, if $i \neq i'$, then $U_{i,j} \cap U_{i',j'} = \emptyset$ and there is nothing to do. For $U_{i,j} \cap U_{i,j'}$ we have $\mathcal{O}_X(U_{i,j} \cap U_{i,j'}) = \mathcal{O}_{Z_i}(U_{i,j} \cap U_{i,j'})$ by the first remark of the proof. Thus the transition function for $\mathcal{L}_i$ (more precisely $\varphi_{i,j} \circ \varphi_{i,j'}^{-1}$) defines the value of our cocycle on this intersection. For $U_t \cap U_{i,j}$ we can do the same thing. Finally, for $t \neq t'$ we have

$$U_t \cap U_{t'} = \coprod (U_t \cap U_{t'}) \cap Z_i$$

and moreover the intersection $U_t \cap U_{t'} \cap Z_i$ is contained in $Z_i \setminus T$. Hence by the same reasoning as before we see that

$$\mathcal{O}_X(U_t \cap U_{t'}) = \prod \mathcal{O}_{Z_i}(U_t \cap U_{t'} \cap Z_i)$$

and we can use the transition functions for $\mathcal{L}_i$ (more precisely $\varphi_{t,i} \circ \varphi_{t',i}^{-1}$) to define the value of our cocycle on $U_t \cap U_{t'}$. This finishes the proof of existence of $\mathcal{L}$.

Given sections s_i as in the last assertion of the lemma, in the argument above, we choose U_t such that $s_i|_{U_t \cap Z_i}$ is nonvanishing and we choose $\varphi_{t,i}$ such that $\varphi_{t,i}(s_i|_{U_t \cap Z_i}) = 1$. Then using 1 over U_t and $\varphi_{i,j}(s_i|_{U_{i,j}})$ over $U_{i,j}$ will define a section of $\mathcal{L}$ which restricts to s_i over Z_i. $\square$

Remark 38.9. Let A be a reduced ring. Let I, J be ideals of A such that $V(I) \cup V(J) = \mathrm{Spec}(A)$. Set $B = A/J$. Then $I \to IB$ is an isomorphism of A-modules. Namely, we have $IB = I + J/J = I/(I \cap J)$ and $I \cap J$ is zero because A is reduced and $\mathrm{Spec}(A) = V(I) \cup V(J) = V(I \cap J)$. Thus for any projective A-module P we also have $IP = I(P/JP)$.

Lemma 38.10. *Let X be a Noetherian reduced separated scheme of dimension* 1. *Then X has an ample invertible sheaf.*

Proof. Let Z_i, $i = 1, \ldots, n$ be the irreducible components of X. We view these as reduced closed subschemes of X. By Lemma 38.6 there exist ample invertible sheaves $\mathcal{L}_i$ on Z_i. Set $T = \bigcup_{i \neq j} Z_i \cap Z_j$. As X is Noetherian of dimension 1, the set T is finite and consists of closed points of X. For each i we may, possibly after

replacing $\mathcal{L}_i$ by a power, choose $s_i \in \Gamma(Z_i, \mathcal{L}_i)$ such that $(Z_i)_{s_i}$ is affine and contains $T \cap Z_i$, see Properties, Lemma 29.6.

By Lemma 38.8 we can find an invertible sheaf $\mathcal{L}$ on X and $s \in \Gamma(X, \mathcal{L})$ such that $(\mathcal{L}, s)|_{Z_i} = (\mathcal{L}_i, s_i)$. Observe that X_s contains T and is set theoretically equal to the affine closed subschemes $(Z_i)_{s_i}$. Thus it is affine by Limits, Lemma 11.3. To finish the proof, it suffices to find for every $x \in X$, $x \notin T$ an integer $m > 0$ and a section $t \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that X_t is affine and $x \in X_t$. Since $x \notin T$ we see that $x \in Z_i$ for some unique i, say $i = 1$. Let $Z \subset X$ be the reduced closed subscheme whose underlying topological space is $Z_2 \cup \ldots \cup Z_n$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Denote that $\mathcal{I}_1 \subset \mathcal{O}_{Z_1}$ the inverse image of this ideal sheaf under the inclusion morphism $Z_1 \to X$. Observe that

$$\Gamma(X, \mathcal{I}\mathcal{L}^{\otimes m}) = \Gamma(Z_1, \mathcal{I}_1 \mathcal{L}_1^{\otimes m})$$

see Remark 38.9. Thus it suffices to find $m > 0$ and $t \in \Gamma(Z_1, \mathcal{I}_1 \mathcal{L}_1^{\otimes m})$ with $x \in (Z_1)_t$ affine. Since $\mathcal{L}_1$ is ample and since x is not in $Z_1 \cap T = V(\mathcal{I}_1)$ we can find a section $t_1 \in \Gamma(Z_1, \mathcal{I}_1 \mathcal{L}_1^{\otimes m_1})$ with $x \in (Z_1)_{t_1}$, see Properties, Proposition 26.13. Since $\mathcal{L}_1$ is ample we can find a section $t_2 \in \Gamma(Z_1, \mathcal{L}_1^{\otimes m_2})$ with $x \in (Z_1)_{t_2}$ and $(Z_1)_{t_2}$ affine, see Properties, Definition 26.1. Set $m = m_1 + m_2$ and $t = t_1 t_2$. Then $t \in \Gamma(Z_1, \mathcal{I}_1 \mathcal{L}_1^{\otimes m})$ with $x \in (Z_1)_t$ by construction and $(Z_1)_t$ is affine by Properties, Lemma 26.4. $\square$

Lemma 38.11. *Let $i : Z \to X$ be a closed immersion of schemes. If the underlying topological space of X is Noetherian and $\dim(X) \leq 1$, then $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is surjective.*

Proof. Consider the short exact sequence

$$0 \to (1 + \mathcal{I}) \cap \mathcal{O}_X^* \to \mathcal{O}_X^* \to i_* \mathcal{O}_Z^* \to 0$$

of sheaves of abelian groups on X where $\mathcal{I}$ is the quasi-coherent sheaf of ideals corresponding to Z. Since $\dim(X) \leq 1$ we see that $H^2(X, \mathcal{F}) = 0$ for any abelian sheaf $\mathcal{F}$, see Cohomology, Proposition 20.7. Hence the map $H^1(X, \mathcal{O}_X^*) \to H^1(X, i_* \mathcal{O}_Z^*)$ is surjective. By Cohomology, Lemma 20.1 we have $H^1(X, i_* \mathcal{O}_Z^*) = H^1(Z, \mathcal{O}_Z^*)$. This proves the lemma by Cohomology, Lemma 6.1. $\square$

Proposition 38.12. *Let X be a Noetherian separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. Let $Z \subset X$ be the reduction of X. By Lemma 38.10 the scheme Z has an ample invertible sheaf. Thus by Lemma 38.11 there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X whose restriction to Z is ample. Then $\mathcal{L}$ is ample by an application of Cohomology of Schemes, Lemma 17.5. $\square$

Remark 38.13. In fact, if X is a scheme whose reduction is a Noetherian separated scheme of dimension 1, then X has an ample invertible sheaf. The argument to prove this is the same as the proof of Proposition 38.12 except one uses Limits, Lemma 11.4 instead of Cohomology of Schemes, Lemma 17.5.

The following lemma actually holds for quasi-finite separated morphisms as the reader can see by using Zariski's main theorem (More on Morphisms, Lemma 43.3) and Lemma 38.3.

Lemma 38.14. *Let $f : X \to Y$ be a morphism of schemes. Assume Y is Noetherian of dimension ≤ 1, f is finite, and there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is a closed immersion. Then every invertible $\mathcal{O}_X$-module is the pullback of an invertible $\mathcal{O}_Y$-module.*

Proof. We factor f as $X \to Z \to Y$ where Z is the scheme theoretic image of f. Then $X \to Z$ is an isomorphism over $V \cap Z$ and Lemma 38.7 applies. On the other hand, Lemma 38.11 applies to $Z \to Y$. Some details omitted. $\qquad\square$

39. The delta invariant

In this section we define the δ-invariant of a singular point on a reduced 1-dimensional Nagata scheme.

Lemma 39.1. *Let $(A, \mathfrak{m})$ be a Noetherian 1-dimensional local ring. Let $f \in \mathfrak{m}$. The following are equivalent*

(1) $\mathfrak{m} = \sqrt{(f)}$,
(2) f *is not contained in any minimal prime of A, and*
(3) $A_f = \prod_{\mathfrak{p} \ minimal} A_\mathfrak{p}$ *as A-algebras.*

Such an $f \in \mathfrak{m}$ exists. If $depth(A) = 1$ (for example A is reduced), then (1) – (3) are also equivalent to

(4) f *is a nonzerodivisor,*
(5) A_f *is the total ring of fractions of A.*

If A is reduced, then (1) – (5) are also equivalent to

(6) A_f *is the product of the residue fields at the minimal primes of A.*

Proof. The spectrum of A has finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ besides $\mathfrak{m}$ and these are all minimal, see Algebra, Lemma 31.6. Then the equivalence of (1) and (2) follows from Algebra, Lemma 17.2. Clearly, (3) implies (2). Conversely, if (2) is true, then the spectrum of A_f is the subset $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ of $\operatorname{Spec}(A)$ with induced topology, see Algebra, Lemma 17.5. This is a finite discrete topological space. Hence $A_f = \prod_{\mathfrak{p} \ minimal} A_\mathfrak{p}$ by Algebra, Proposition 60.7. The existence of an f is asserted in Algebra, Lemma 60.8.

Assume A has depth 1. (This is the maximum by Algebra, Lemma 72.3 and holds if A is reduced by Algebra, Lemma 157.3.) Then $\mathfrak{m}$ is not an associated prime of A. Every minimal prime of A is an associated prime (Algebra, Proposition 63.6). Hence the set of nonzerodivisors of A is exactly the set of elements not contained in any of the minimal primes by Algebra, Lemma 63.9. Thus (4) is equivalent to (2). Part (5) is equivalent to (3) by Algebra, Lemma 25.4.

Then $A_\mathfrak{p}$ is a field for $\mathfrak{p} \subset A$ minimal, see Algebra, Lemma 25.1. Hence (3) is equivalent ot (6). $\qquad\square$

Lemma 39.2. *Let $(A, \mathfrak{m})$ be a reduced Nagata 1-dimensional local ring. Let A' be the integral closure of A in the total ring of fractions of A. Then A' is a normal Nagata ring, $A \to A'$ is finite, and A'/A has finite length as an A-module.*

Proof. The total ring of fractions is essentially of finite type over A hence $A \to A'$ is finite because A is Nagata, see Algebra, Lemma 162.2. The ring A' is normal for example by Algebra, Lemma 37.16 and 31.6. The ring A' is Nagata for example by Algebra, Lemma 162.5. Choose $f \in \mathfrak{m}$ as in Lemma 39.1. As $A' \subset A_f$ it is clear

that $A_f = A'_f$. Hence the support of the finite A-module A'/A is contained in $\{\mathfrak{m}\}$. It follows that it has finite length by Algebra, Lemma 62.3. $\qquad\square$

Definition 39.3. Let A be a reduced Nagata local ring of dimension 1. The *δ-invariant of A* is $\text{length}_A(A'/A)$ where A' is as in Lemma 39.2.

We prove some lemmas about the behaviour of this invariant.

Lemma 39.4. *Let A be a reduced Nagata local ring of dimension* 1*. The δ-invariant of A is* 0 *if and only if A is a discrete valuation ring.*

Proof. If A is a discrete valuation ring, then A is normal and the ring A' is equal to A. Conversely, if the δ-invariant of A is 0, then A is integrally closed in its total ring of fractions which implies that A is normal (Algebra, Lemma 37.16) and this forces A to be a discrete valuation ring by Algebra, Lemma 119.7. $\qquad\square$

Lemma 39.5. *Let A be a reduced Nagata local ring of dimension* 1*. Let $A \to A'$ be as in Lemma 39.2. Let A^h, A^{sh}, resp. $A^\wedge$ be the henselization, strict henselization, resp. completion of A. Then A^h, A^{sh}, resp. $A^\wedge$ is a reduced Nagata local ring of dimension* 1 *and $A' \otimes_A A^h$, $A' \otimes_A A^{sh}$, resp. $A' \otimes_A A^\wedge$ is the integral closure of A^h, A^{sh}, resp. $A^\wedge$ in its total ring of fractions.*

Proof. Observe that $A^\wedge$ is reduced, see More on Algebra, Lemma 43.6. The rings A^h and A^{sh} are reduced by More on Algebra, Lemma 45.4. The dimensions of A, A^h, A^{sh}, and $A^\wedge$ are the same by More on Algebra, Lemmas 43.1 and 45.7.

Recall that a Noetherian local ring is Nagata if and only if the formal fibres of A are geometrically reduced, see More on Algebra, Lemma 52.4. This property is inherited by A^h and A^{sh}, see the material in More on Algebra, Section 51 and especially Lemma 51.8. The completion is Nagata by Algebra, Lemma 162.8.

Now we come to the statement on integral closures. Before continuing let us pick $f \in \mathfrak{m}$ as in Lemma 39.1. Then the image of f in A^h, A^{sh}, and $A^\wedge$ clearly is an element satisfying properties (1) – (6) in that ring.

Since $A \to A'$ is finite we see that $A' \otimes_A A^h$ and $A' \otimes_A A^{sh}$ is the product of henselian local rings finite over A^h and A^{sh}, see Algebra, Lemma 153.4. Each of these local rings is the henselization of A' at a maximal ideal $\mathfrak{m}' \subset A'$ lying over $\mathfrak{m}$, see Algebra, Lemma 156.1 or 156.3. Hence these local rings are normal domains by More on Algebra, Lemma 45.6. It follows that $A' \otimes_A A^h$ and $A' \otimes_A A^{sh}$ are normal rings. Since $A^h \to A' \otimes_A A^h$ and $A^{sh} \to A' \otimes_A A^{sh}$ are finite (hence integral) and since $A' \otimes_A A^h \subset (A^h)_f = Q(A^h)$ and $A' \otimes_A A^{sh} \subset (A^{sh})_f = Q(A^{sh})$ we conclude that $A' \otimes_A A^h$ and $A' \otimes_A A^{sh}$ are the desired integral closures.

For the completion we argue in entirely the same manner. First, by Algebra, Lemma 97.8 we have

$$A' \otimes_A A^\wedge = (A')^\wedge = \prod (A'_{\mathfrak{m}'})^\wedge$$

The local rings $A'_{\mathfrak{m}'}$ are normal and have dimension 1 (by Algebra, Lemma 113.2 for example or the discussion in Algebra, Section 112). Thus $A'_{\mathfrak{m}'}$ is a discrete valuation ring, see Algebra, Lemma 119.7. Hence $(A'_{\mathfrak{m}'})^\wedge$ is a discrete valuation ring by More on Algebra, Lemma 43.5. It follows that $A' \otimes_A A^\wedge$ is a normal ring and we can conclude in exactly the same manner as before. $\qquad\square$

Lemma 39.6. *Let A be a reduced Nagata local ring of dimension 1. The δ-invariant of A is the same as the δ-invariant of the henselization, strict henselization, or the completion of A.*

Proof. Let us do this in case of the completion $B = A^\wedge$; the other cases are proved in exactly the same manner. Let A', resp. B' be the integral closure of A, resp. B in its total ring of fractions. Then $B' = A' \otimes_A B$ by Lemma 39.5. Hence $B'/B = A'/A \otimes_A B$. The equality now follows from Algebra, Lemma 52.13 and the fact that $B \otimes_A \kappa_A = \kappa_B$. $\qquad\qquad\square$

Definition 39.7. Let k be a field. Let X be a locally algebraic k-scheme. Let $x \in X$ be a point such that $\mathcal{O}_{X,x}$ is reduced and $\dim(\mathcal{O}_{X,x}) = 1$. The δ-*invariant of X at x* is the δ-invariant of $\mathcal{O}_{X,x}$ as defined in Definition 39.3.

This makes sense because the local ring of a locally algebraic scheme is Nagata by Algebra, Proposition 162.16. Of course, more generally we can make this definition whenever $x \in X$ is a point of a scheme such that the local ring $\mathcal{O}_{X,x}$ is reduced, Nagata of dimension 1. It follows from Lemma 39.6 that the δ-invariant of X at x is

$$\delta\text{-invariant of } X \text{ at } x = \delta\text{-invariant of } \mathcal{O}^h_{X,x} = \delta\text{-invariant of } \mathcal{O}^\wedge_{X,x}$$

We conclude that the δ-invariant is an invariant of the complete local ring of the point.

Lemma 39.8. *Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be a field extension and set $Y = X_K$. Let $y \in Y$ with image $x \in X$. Assume X is geometrically reduced at x and $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) = 1$. Then*

$$\delta\text{-invariant of } X \text{ at } x \leq \delta\text{-invariant of } Y \text{ at } y$$

Proof. Set $A = \mathcal{O}_{X,x}$ and $B = \mathcal{O}_{Y,y}$. By Lemma 6.2 we see that A is geometrically reduced. Hence B is a localization of $A \otimes_k K$. Let $A \to A'$ be as in Lemma 39.2. Then

$$B' = B \otimes_{(A \otimes_k K)} (A' \otimes_k K)$$

is finite over B and $B \to B'$ induces an isomorphism on total rings of fractions. Namely, pick $f \in \mathfrak{m}_A$ satisfying $(1) - (6)$ of Lemma 39.1; since $\dim(B) = 1$ we see that $f \in \mathfrak{m}_B$ plays the same role for B and we see that $B_f = B'_f$ because $A_f = A'_f$. Let B'' be the integral closure of B in its total ring of fractions as in Lemma 39.2. Then $B' \subset B''$. Thus the δ-invariant of Y at y is $\text{length}_B(B''/B)$ and

$$\begin{aligned}
\text{length}_B(B''/B) &\geq \text{length}_B(B'/B) \\
&= \text{length}_B((A'/A) \otimes_A B) \\
&= \text{length}_B(B/\mathfrak{m}_A B)\text{length}_A(A'/A)
\end{aligned}$$

by Algebra, Lemma 52.13 since $A \to B$ is flat (as a localization of $A \to A \otimes_k K$). Since $\text{length}_A(A'/A)$ is the δ-invariant of X at x and since $\text{length}_B(B/\mathfrak{m}_A B) \geq 1$ the lemma is proved. $\qquad\qquad\square$

Lemma 39.9. *Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be a field extension and set $Y = X_K$. Let $y \in Y$ with image $x \in X$. Assume assumptions (a), (b), (c) of Lemma 27.6 hold for $x \in X$ and that $\dim(\mathcal{O}_{Y,y}) = 1$. Then the δ-invariant of X at x is δ-invariant of Y at y.*

Proof. Set $A = \mathcal{O}_{X,x}$ and $B = \mathcal{O}_{Y,y}$. By Lemma 27.6 we see that A is geometrically reduced. Hence B is a localization of $A \otimes_k K$. Let $A \to A'$ be as in Lemma 39.2. By Lemma 27.6 we see that $A' \otimes_k K$ is normal. Hence

$$B' = B \otimes_{(A \otimes_k K)} (A' \otimes_k K)$$

is normal, finite over B, and $B \to B'$ induces an isomorphism on total rings of fractions. Namely, pick $f \in \mathfrak{m}_A$ satisfying $(1) - (6)$ of Lemma 39.1; since $\dim(B) = 1$ we see that $f \in \mathfrak{m}_B$ playes the same role for B and we see that $B_f = B'_f$ because $A_f = A'_f$. It follows that $B \to B'$ is as in Lemma 39.2 for B. Thus we have to show that $\mathrm{length}_A(A'/A) = \mathrm{length}_B(B'/B) = \mathrm{length}_B((A'/A) \otimes_A B)$. Since $A \to B$ is flat (as a localization of $A \to A \otimes_k K$) and since $\mathfrak{m}_B = \mathfrak{m}_A B$ (because $B/\mathfrak{m}_A B$ is zero dimensional by the remarks above and a localization of $K \otimes_k \kappa(x)$ which is reduced as $\kappa(x)$ is separable over k) we conclude by Algebra, Lemma 52.13. $\square$

40. The number of branches

We have defined the number of branches of a scheme at a point in Properties, Section 15.

Lemma 40.1. *Let X be a scheme. Assume every quasi-compact open of X has finitely many irreducible components. Let $\nu : X^\nu \to X$ be the normalization of X. Let $x \in X$.*

(1) *The number of branches of X at x is the number of inverse images of x in X^ν.*

(2) *The number of geometric branches of X at x is $\sum_{\nu(x^\nu)=x}[\kappa(x^\nu) : \kappa(x)]_s$.*

Proof. First note that the assumption on X exactly means that the normalization is defined, see Morphisms, Definition 54.1. Then the stalk $A' = (\nu_* \mathcal{O}_{X^\nu})_x$ is the integral closure of $A = \mathcal{O}_{X,x}$ in the total ring of fractions of A_{red}, see Morphisms, Lemma 54.4. Since ν is an integral morphism, we see that the points of X^ν lying over x correspond to the primes of A' lying over the maximal ideal $\mathfrak{m}$ of A. As $A \to A'$ is integral, this is the same thing as the maximal ideals of A' (Algebra, Lemmas 36.20 and 36.22). Thus the lemma now follows from its algebraic counterpart: More on Algebra, Lemma 106.7. $\square$

Lemma 40.2. *Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be an extension of fields. Let $y \in X_K$ be a point with image x in X. Then the number of geometric branches of X at x is the number of geometric branches of X_K at y.*

Proof. Write $Y = X_K$ and let X^ν, resp. Y^ν be the normalization of X, resp. Y. Consider the commutative diagram

$$
\begin{array}{ccccc}
Y^\nu & \longrightarrow & X_K^\nu & \longrightarrow & X^\nu \\
\downarrow & & \downarrow{\scriptstyle \nu_K} & & \downarrow{\scriptstyle \nu} \\
Y & = & Y & \longrightarrow & X
\end{array}
$$

By Lemma 27.5 we see that the left top horizontal arrow is a universal homeomorphism. Hence it induces purely inseparable residue field extensions, see Morphisms, Lemmas 45.5 and 10.2. Thus the number of geometric branches of Y at y is $\sum_{\nu_K(y')=y}[\kappa(y') : \kappa(y)]_s$ by Lemma 40.1. Similarly $\sum_{\nu(x')=x}[\kappa(x') : \kappa(x)]_s$ is the number of geometric branches of X at x. Using Schemes, Lemma 17.5 our

statement follows from the following algebra fact: given a field extension l/κ and an algebraic field extension m/κ, then

$$\sum\nolimits_{m\otimes_\kappa l\to m'}[m':l']_s = [m:\kappa]_s$$

where the sum is over the quotient fields of $m \otimes_\kappa l$. One can prove this in an elementary way, or one can use Lemma 7.6 applied to

$$\text{Spec}(m\otimes_\kappa l)\times_{\text{Spec}(l)}\text{Spec}(\bar{l}) = \text{Spec}(m)\otimes_{\text{Spec}(\kappa)}\text{Spec}(\bar{l}) \longrightarrow \text{Spec}(m)\times_{\text{Spec}(\kappa)}\text{Spec}(\bar{\kappa})$$

because one can interpret $[m : \kappa]_s$ as the number of connected components of the right hand side and the sum $\sum_{m\otimes_\kappa l\to m'}[m':l']_s$ as the number of connected components of the left hand side. $\square$

Lemma 40.3. *Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be an extension of fields. Let $y \in X_K$ be a point with image x in X. Then X is geometrically unibranch at x if and only if X_K is geometrically unibranch at y.*

Proof. Immediate from Lemma 40.2 and More on Algebra, Lemma 106.7. $\square$

Definition 40.4. Let A and A_i, $1 \le i \le n$ be local rings. We say *A is a wedge of* $A_1, \ldots, A_n$ if there exist isomorphisms

$$\kappa_{A_1} \to \kappa_{A_2} \to \ldots \to \kappa_{A_n}$$

and A is isomorphic to the ring consisting of n-tuples $(a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ which map to the same element of κ_{A_n}.

If we are given a base ring Λ and A and A_i are Λ-algebras, then we require $\kappa_{A_i} \to \kappa_{A_{i+1}}$ to be a Λ-algebra isomorphisms and A to be isomorphic as a Λ-algebra to the Λ-algebra consisting of n-tuples $(a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ which map to the same element of κ_{A_n}. In particular, if $\Lambda = k$ is a field and the maps $k \to \kappa_{A_i}$ are isomorphisms, then there is a unique choice for the isomorphisms $\kappa_{A_i} \to \kappa_{A_{i+1}}$ and we often speak of *the wedge of $A_1, \ldots, A_n$*.

Lemma 40.5. *Let $(A, \mathfrak{m})$ be a strictly henselian 1-dimensional reduced Nagata local ring. Then*

$$\delta\text{-invariant of } A \ge \text{number of geometric branches of } A - 1$$

If equality holds, then A is a wedge of $n \ge 1$ strictly henselian discrete valuation rings.

Proof. The number of geometric branches is equal to the number of branches of A (immediate from More on Algebra, Definition 106.6). Let $A \to A'$ be as in Lemma 39.2. Observe that the number of branches of A is the number of maximal ideals of A', see More on Algebra, Lemma 106.7. There is a surjection

$$A'/A \longrightarrow \left(\prod\nolimits_{\mathfrak{m}'} \kappa(\mathfrak{m}')\right)/\kappa(\mathfrak{m})$$

Since $\dim_{\kappa(\mathfrak{m})}\prod \kappa(\mathfrak{m}')$ is $\ge$ the number of branches, the inequality is obvious.

If equality holds, then $\kappa(\mathfrak{m}') = \kappa(\mathfrak{m})$ for all $\mathfrak{m}' \subset A'$ and the displayed arrow above is an isomorphism. Since A is henselian and $A \to A'$ is finite, we see that A' is a product of local henselian rings, see Algebra, Lemma 153.4. The factors are the local rings $A'_{\mathfrak{m}'}$ and as A' is normal, these factors are discrete valuation rings (Algebra, Lemma 119.7). Since the displayed arrow is an isomorphism we see that A is indeed the wedge of these local rings. $\square$

Lemma 40.6. *Let $(A, \mathfrak{m})$ be a 1-dimensional reduced Nagata local ring. Then*

$$\delta\text{-invariant of } A \geq \text{number of geometric branches of } A - 1$$

Proof. We may replace A by the strict henselization of A without changing the δ-invariant (Lemma 39.6) and without changing the number of geometric branches of A (this is immediate from the definition, see More on Algebra, Definition 106.6). Thus we may assume A is strictly henselian and we may apply Lemma 40.5. $\square$

41. Normalization of one dimensional schemes

The normalization morphism of a Noetherian scheme of dimension 1 has unexpectedly good properties by the Krull-Akizuki result.

Lemma 41.1. *Let X be a locally Noetherian scheme of dimension 1. Let $\nu : X^\nu \to X$ be the normalization. Then*

    (1) *ν is integral, surjective, and induces a bijection on irreducible components,*
    (2) *there is a factorization $X^\nu \to X_{red} \to X$ and the morphism $X^\nu \to X_{red}$ is the normalization of X_{red},*
    (3) *$X^\nu \to X_{red}$ is birational,*
    (4) *for every closed point $x \in X$ the stalk $(\nu_* \mathcal{O}_{X^\nu})_x$ is the integral closure of $\mathcal{O}_{X,x}$ in the total ring of fractions of $(\mathcal{O}_{X,x})_{red} = \mathcal{O}_{X_{red},x}$,*
    (5) *the fibres of ν are finite and the residue field extensions are finite,*
    (6) *X^ν is a disjoint union of integral normal Noetherian schemes and each affine open is the spectrum of a finite product of Dedekind domains.*

Proof. Many of the results are in fact general properties of the normalization morphism, see Morphisms, Lemmas 54.2, 54.4, 54.5, and 54.7. What is not clear is that the fibres are finite, that the induced residue field extensions are finite, and that X^ν locally looks like the spectrum of a Dedekind domain (and hence is Noetherian). To see this we may assume that $X = \text{Spec}(A)$ is affine, Noetherian, dimension 1, and that A is reduced. Then we may use the description in Morphisms, Lemma 54.3 to reduce to the case where A is a Noetherian domain of dimension 1. In this case the desired properties follow from Krull-Akizuki in the form stated in Algebra, Lemma 120.18. $\square$

Of course there is a variant of the following lemma in case X is not reduced.

Lemma 41.2. *Let X be a reduced Nagata scheme of dimension 1. Let $\nu : X^\nu \to X$ be the normalization. Let $x \in X$ denote a closed point. Then*

    (1) *$\nu : X^\nu \to X$ is finite, surjective, and birational,*
    (2) *$\mathcal{O}_X \subset \nu_* \mathcal{O}_{X^\nu}$ and $\nu_* \mathcal{O}_{X^\nu}/\mathcal{O}_X$ is a direct sum of skyscraper sheaves $\mathcal{Q}_x$ in the singular points x of X,*
    (3) *$A' = (\nu_* \mathcal{O}_{X^\nu})_x$ is the integral closure of $A = \mathcal{O}_{X,x}$ in its total ring of fractions,*
    (4) *$\mathcal{Q}_x = A'/A$ has finite length equal to the δ-invariant of X at x,*
    (5) *A' is a semi-local ring which is a finite product of Dedekind domains,*
    (6) *$A^\wedge$ is a reduced Noetherian complete local ring of dimension 1,*
    (7) *$(A')^\wedge$ is the integral closure of $A^\wedge$ in its total ring of fractions,*
    (8) *$(A')^\wedge$ is a finite product of complete discrete valuation rings, and*
    (9) *$A'/A \cong (A')^\wedge/A^\wedge$.*

Proof. We may and will use all the results of Lemma 41.1. Finiteness of ν follows from Morphisms, Lemma 54.10. Since X is reduced, Nagata, of dimension 1, we see that the regular locus is a dense open $U \subset X$ by More on Algebra, Proposition 48.7. Since a regular scheme is normal, this shows that ν is an isomorphism over U. Since $\dim(X) \leq 1$ this implies that ν is not an isomorphism over a discrete set of closed points $x \in X$. In particular we see that we have a short exact sequence

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{X^\nu} \to \bigoplus_{x \in X \setminus U} \mathcal{Q}_x \to 0$$

As we have the description of the stalks of $\nu_* \mathcal{O}_{X^\nu}$ by Lemma 41.1, we conclude that $Q_x = A'/A$ indeed has length equal to the δ-invariant of X at x. Note that $Q_x \neq 0$ exactly when x is a singular point for example by Lemma 39.4. The description of A' as a product of semi-local Dedekind domains follows from Lemma 41.1 as well. The relationship between A, A', and $(A')^\wedge$ we have see in Lemma 39.5 (and its proof). $\square$

42. Finding affine opens

We continue the discussion started in Properties, Section 29. It turns out that we can find affines containing a finite given set of codimension 1 points on a separated scheme. See Proposition 42.7.

We will improve on the following lemma in Descent, Lemma 25.4.

Lemma 42.1. *Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If*

(1) *f is separated,*
(2) *there is an open covering $X = \bigcup U_i$ such that $f|_{U_i} : U_i \to Y$ is an open immersion, and*
(3) *if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,*

then f is an open immersion.

Proof. Suppose that $y = f(x) = f(x')$. Pick a specialization $y_0 \rightsquigarrow y$ where y_0 is a generic point of an irreducible component of Y. Since f is locally on the source an isomorphism we can pick specializations $x_0 \rightsquigarrow x$ and $x_0' \rightsquigarrow x'$ mapping to $y_0 \rightsquigarrow y$. Note that $x_0, x_0' \in X^0$. Hence $x_0 = x_0'$ by assumption (3). As f is separated we conclude that $x = x'$. Thus f is an open immersion. $\square$

Lemma 42.2. *Let $X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. If*

(1) *$\mathcal{O}_{X,x} = \mathcal{O}_{S,s}$,*
(2) *X is reduced,*
(3) *$X \to S$ is of finite type, and*
(4) *S has finitely many irreducible components,*

then there exists an open neighbourhood U of x such that $f|_U$ is an open immersion.

Proof. We may remove the (finitely many) irreducible components of S which do not contain s. We may replace S by an affine open neighbourhood of s. We may replace X by an affine open neighbourhood of x. Say $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$. Let $\mathfrak{q} \subset B$, resp. $\mathfrak{p} \subset A$ be the prime ideal corresponding to x, resp. s. As A is a reduced and all of the minimal primes of A are contained in $\mathfrak{p}$ we see that $A \subset A_{\mathfrak{p}}$. As $X \to S$ is of finite type, B is of finite type over A.

Let $b_1, \ldots, b_n \in B$ be elements which generate B over A Since $A_{\mathfrak{p}} = B_{\mathfrak{q}}$ we can find $f \in A$, $f \notin \mathfrak{p}$ and $a_i \in A$ such that b_i and a_i/f have the same image in $B_{\mathfrak{q}}$. Thus we can find $g \in B$, $g \notin \mathfrak{q}$ such that $g(fb_i - a_i) = 0$ in B. It follows that the image of $A_f \to B_{fg}$ contains the images of $b_1, \ldots, b_n$, in particular also the image of g. Choose $n \geq 0$ and $f' \in A$ such that f'/f^n maps to the image of g in B_{fg}. Since $A_{\mathfrak{p}} = B_{\mathfrak{q}}$ we see that $f' \notin \mathfrak{p}$. We conclude that $A_{ff'} \to B_{fg}$ is surjective. Finally, as $A_{ff'} \subset A_{\mathfrak{p}} = B_{\mathfrak{q}}$ (see above) the map $A_{ff'} \to B_{fg}$ is injective, hence an isomorphism. $\square$

Lemma 42.3. *Let $f : T \to X$ be a morphism of schemes. Let X^0, resp. T^0 denote the sets of generic points of irreducible components. Let $t_1, \ldots, t_m \in T$ be a finite set of points with images $x_j = f(t_j)$. If*

(1) *T is affine,*
(2) *X is quasi-separated,*
(3) *X^0 is finite*
(4) *$f(T^0) \subset X^0$ and $f : T^0 \to X^0$ is injective, and*
(5) *$\mathcal{O}_{X,x_j} = \mathcal{O}_{T,t_j}$,*

then there exists an affine open of X containing $x_1, \ldots, x_r$.

Proof. Using Limits, Proposition 11.2 there is an immediate reduction to the case where X and T are reduced. Details omitted.

Assume X and T are reduced. We may write $T = \lim_{i \in I} T_i$ as a directed limit of schemes of finite presentation over X with affine transition morphisms, see Limits, Lemma 7.2. Pick $i \in I$ such that T_i is affine, see Limits, Lemma 4.13. Say $T_i = \mathrm{Spec}(R_i)$ and $T = \mathrm{Spec}(R)$. Let $R' \subset R$ be the image of $R_i \to R$. Then $T' = \mathrm{Spec}(R')$ is affine, reduced, of finite type over X, and $T \to T'$ dominant. For $j = 1, \ldots, r$ let $t'_j \in T'$ be the image of t_j. Consider the local ring maps

$$\mathcal{O}_{X,x_j} \to \mathcal{O}_{T',t'_j} \to \mathcal{O}_{T,t_j}$$

Denote $(T')^0$ the set of generic points of irreducible components of T'. Let $\xi \rightsquigarrow t'_j$ be a specialization with $\xi \in (T')^0$. As $T \to T'$ is dominant we can choose $\eta \in T^0$ mapping to ξ (warning: a priori we do not know that η specializes to t_j). Assumption (3) applied to η tells us that the image θ of ξ in X corresponds to a minimal prime of $\mathcal{O}_{X,x_j}$. Lifting ξ via the isomorphism of (5) we obtain a specialization $\eta' \rightsquigarrow t_j$ with $\eta' \in T^0$ mapping to $\theta \rightsquigarrow x_j$. The injectivity of (4) shows that $\eta = \eta'$. Thus every minimal prime of $\mathcal{O}_{T',t'_j}$ lies below a minimal prime of $\mathcal{O}_{T,t_j}$. We conclude that $\mathcal{O}_{T',t'_j} \to \mathcal{O}_{T,t_j}$ is injective, hence both maps above are isomorphisms.

By Lemma 42.2 there exists an open $U \subset T'$ containing all the points t'_j such that $U \to X$ is a local isomorphism as in Lemma 42.1. By that lemma we see that $U \to X$ is an open immersion. Finally, by Properties, Lemma 29.5 we can find an open $W \subset U \subset T'$ containing all the t'_j. The image of W in X is the desired affine open. $\square$

Lemma 42.4. *Let X be an integral separated scheme. Let $x_1, \ldots, x_r \in X$ be a finite set of points such that $\mathcal{O}_{X,x_i}$ is Noetherian of dimension ≤ 1. Then there exists an affine open subscheme of X containing all of $x_1, \ldots, x_r$.*

Proof. Let K be the field of rational functions of X. Set $A_i = \mathcal{O}_{X,x_i}$. Then $A_i \subset K$ and K is the fraction field of A_i. Since X is separated, and $x_i \neq x_j$ there cannot be a valuation ring $\mathcal{O} \subset K$ dominating both A_i and A_j. Namely, considering the diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(\mathcal{O}) & \longrightarrow & \mathrm{Spec}(A_1) \\
\downarrow & & \downarrow \\
\mathrm{Spec}(A_2) & \longrightarrow & X
\end{array}
$$

and applying the valuative criterion of separatedness (Schemes, Lemma 22.1) we would get $x_i = x_j$. Thus we see by Lemma 37.3 that $A_i \otimes A_j \to K$ is surjective for all $i \neq j$. By Lemma 37.7 we see that $A = A_1 \cap \ldots \cap A_r$ is a Noetherian semi-local ring with exactly r maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ such that $A_i = A_{\mathfrak{m}_i}$. Moreover,

$$\mathrm{Spec}(A) = \mathrm{Spec}(A_1) \cup \ldots \cup \mathrm{Spec}(A_r)$$

is an open covering and the intersection of any two pieces of this covering is $\mathrm{Spec}(K)$. Thus the given morphisms $\mathrm{Spec}(A_i) \to X$ glue to a morphism of schemes

$$\mathrm{Spec}(A) \longrightarrow X$$

mapping $\mathfrak{m}_i$ to x_i and inducing isomorphisms of local rings. Thus the result follows from Lemma 42.3. $\qquad\square$

Lemma 42.5. *Let A be a ring, $I \subset A$ an ideal, $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ primes of A, and $\overline{f} \in A/I$ an element. If $I \not\subset \mathfrak{p}_i$ for all i, then there exists an $f \in A$, $f \notin \mathfrak{p}_i$ which maps to $\overline{f}$ in A/I.*

Proof. We may assume there are no inclusion relations among the $\mathfrak{p}_i$ (by removing the smaller primes). First pick any $f \in A$ lifting $\overline{f}$. Let S be the set $s \in \{1, \ldots, r\}$ such that $f \in \mathfrak{p}_s$. If S is empty we are done. If not, consider the ideal $J = I \prod_{i \notin S} \mathfrak{p}_i$. Note that J is not contained in $\mathfrak{p}_s$ for $s \in S$ because there are no inclusions among the $\mathfrak{p}_i$ and because I is not contained in any $\mathfrak{p}_i$. Hence we can choose $g \in J$, $g \notin \mathfrak{p}_s$ for $s \in S$ by Algebra, Lemma 15.2. Then $f + g$ is a solution to the problem posed by the lemma. $\qquad\square$

Lemma 42.6. *Let X be a scheme. Let $T \subset X$ be finite set of points. Assume*

 (1) *X has finitely many irreducible components $Z_1, \ldots, Z_t$, and*
 (2) *$Z_i \cap T$ is contained in an affine open of the reduced induced subscheme corresponding to Z_i.*

Then there exists an affine open subscheme of X containing T.

Proof. Using Limits, Proposition 11.2 there is an immediate reduction to the case where X is reduced. Details omitted. In the rest of the proof we endow every closed subset of X with the induced reduced closed subscheme structure.

We argue by induction that we can find an affine open $U \subset Z_1 \cup \ldots \cup Z_r$ containing $T \cap (Z_1 \cup \ldots \cup Z_r)$. For $r = 1$ this holds by assumption. Say $r > 1$ and let $U \subset Z_1 \cup \ldots \cup Z_{r-1}$ be an affine open containing $T \cap (Z_1 \cup \ldots \cup Z_{r-1})$. Let $V \subset X_r$ be an affine open containing $T \cap Z_r$ (exists by assumption). Then $U \cap V$ contains $T \cap (Z_1 \cup \ldots \cup Z_{r-1}) \cap Z_r$. Hence

$$\Delta = (U \cap Z_r) \setminus (U \cap V)$$

does not contain any element of T. Note that Δ is a closed subset of U. By prime avoidance (Algebra, Lemma 15.2), we can find a standard open U' of U containing $T \cap U$ and avoiding Δ, i.e., $U' \cap Z_r \subset U \cap V$. After replacing U by U' we may assume that $U \cap V$ is closed in U.

Using that by the same arguments as above also the set $\Delta' = (U \cap (Z_1 \cup \ldots \cup Z_{r-1})) \setminus (U \cap V)$ does not contain any element of T we find a $h \in \mathcal{O}(V)$ such that $D(h) \subset V$ contains $T \cap V$ and such that $U \cap D(h) \subset U \cap V$. Using that $U \cap V$ is closed in U we can use Lemma 42.5 to find an element $g \in \mathcal{O}(U)$ whose restriction to $U \cap V$ equals the restriction of h to $U \cap V$ and such that $T \cap U \subset D(g)$. Then we can replace U by $D(g)$ and V by $D(h)$ to reach the situation where $U \cap V$ is closed in both U and V. In this case the scheme $U \cup V$ is affine by Limits, Lemma 11.3. This proves the induction step and thereby the lemma. $\square$

Here is a conclusion we can draw from the material above.

Proposition 42.7. *Let X be a separated scheme such that every quasi-compact open has a finite number of irreducible components. Let $x_1, \ldots, x_r \in X$ be points such that $\mathcal{O}_{X, x_i}$ is Noetherian of dimension ≤ 1. Then there exists an affine open subscheme of X containing all of $x_1, \ldots, x_r$.*

Proof. We can replace X by a quasi-compact open containing $x_1, \ldots, x_r$ hence we may assume that X has finitely many irreducible components. By Lemma 42.6 we reduce to the case where X is integral. This case is Lemma 42.4. $\square$

43. Curves

In the Stacks project we will use the following as our definition of a curve.

Definition 43.1. Let k be a field. A *curve* is a variety of dimension 1 over k.

Two standard examples of curves over k are the affine line $\mathbf{A}^1_k$ and the projective line $\mathbf{P}^1_k$. The scheme $X = \mathrm{Spec}(k[x, y]/(f))$ is a curve if and only if $f \in k[x, y]$ is irreducible.

Our definition of a curve has the same problems as our definition of a variety, see the discussion following Definition 3.1. Moreover, it means that every curve comes with a specified field of definition. For example $X = \mathrm{Spec}(\mathbf{C}[x])$ is a curve over $\mathbf{C}$ but we can also view it as a curve over $\mathbf{R}$. The scheme $\mathrm{Spec}(\mathbf{Z})$ isn't a curve, even though the schemes $\mathrm{Spec}(\mathbf{Z})$ and $\mathbf{A}^1_{\mathbf{F}_p}$ behave similarly in many respects.

Lemma 43.2. *Let X be a separated, irreducible scheme of dimension > 0 over a field k. Let $x \in X$ be a closed point. The open subscheme $X \setminus \{x\}$ is not proper over k.*

Proof. Since X is irreducible, $U = X \setminus \{x\}$ is not closed in X. In particular, the immersion $U \to X$ is not proper. By Morphisms, Lemma 41.7 (here we use X is separated), $U \to \mathrm{Spec}(k)$ is not proper either. $\square$

Lemma 43.3. *Let X be a separated finite type scheme over a field k. If $\dim(X) \leq 1$ then X is H-quasi-projective over k.*

Proof. By Proposition 38.12 the scheme X has an ample invertible sheaf $\mathcal{L}$. By Morphisms, Lemma 39.3 we see that X is isomorphic to a locally closed subscheme of $\mathbf{P}^n_k$ over $\mathrm{Spec}(k)$. This is the definition of being H-quasi-projective over k, see Morphisms, Definition 40.1. $\square$

Lemma 43.4. *Let X be a proper scheme over a field k. If $\dim(X) \leq 1$ then X is H-projective over k.*

Proof. By Lemma 43.3 we see that X is a locally closed subscheme of $\mathbf{P}_k^n$ for some field k. Since X is proper over k it follows that X is a closed subscheme of $\mathbf{P}_k^n$ (Morphisms, Lemma 41.7). $\qquad\square$

Lemma 43.5. *Let X be a separated scheme of finite type over k. If $\dim(X) \leq 1$, then there exists an open immersion $j : X \to \overline{X}$ with the following properties*

    (1) *$\overline{X}$ is H-projective over k, i.e., $\overline{X}$ is a closed subscheme of $\mathbf{P}_k^d$ for some d,*
    (2) *$j(X) \subset \overline{X}$ is dense and scheme theoretically dense,*
    (3) *$\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ for some closed points $x_i \in \overline{X}$.*

Proof. By Lemma 43.3 we may assume X is a locally closed subscheme of $\mathbf{P}_k^d$ for some d. Let $\overline{X} \subset \mathbf{P}_k^d$ be the scheme theoretic image of $X \to \mathbf{P}_k^d$, see Morphisms, Definition 6.2. The description in Morphisms, Lemma 7.7 gives properties (1) and (2). Then $\dim(X) = 1 \Rightarrow \dim(\overline{X}) = 1$ for example by looking at generic points, see Lemma 20.3. As $\overline{X}$ is Noetherian, it then follows that $\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ is a finite set of closed points. $\qquad\square$

Lemma 43.6. *Let X be a separated scheme of finite type over k. If X is reduced and $\dim(X) \leq 1$, then there exists an open immersion $j : X \to \overline{X}$ such that*

    (1) *$\overline{X}$ is H-projective over k, i.e., $\overline{X}$ is a closed subscheme of $\mathbf{P}_k^d$ for some d,*
    (2) *$j(X) \subset \overline{X}$ is dense and scheme theoretically dense,*
    (3) *$\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ for some closed points $x_i \in \overline{X}$,*
    (4) *the local rings $\mathcal{O}_{\overline{X}, x_i}$ are discrete valuation rings for $i = 1, \ldots, n$.*

Proof. Let $j : X \to \overline{X}$ be as in Lemma 43.5. Consider the normalization X' of $\overline{X}$ in X. By Lemma 27.3 the morphism $X' \to \overline{X}$ is finite. By Morphisms, Lemma 44.16 $X' \to \overline{X}$ is projective. By Morphisms, Lemma 43.16 we see that $X' \to \overline{X}$ is H-projective. By Morphisms, Lemma 43.7 we see that $X' \to \mathrm{Spec}(k)$ is H-projective. Let $\{x'_1, \ldots, x'_m\} \subset X'$ be the inverse image of $\{x_1, \ldots, x_n\} = \overline{X} \setminus X$. Then $\dim(\mathcal{O}_{X', x'_i}) = 1$ for all $1 \leq i \leq m$. Hence the local rings $\mathcal{O}_{X', x'}$ are discrete valuation rings by Morphisms, Lemma 53.16. Then $X \to X'$ and $\{x'_1, \ldots, x'_m\}$ is as desired. $\qquad\square$

Lemma 43.7. *Let X be a separated scheme of finite type over k with $\dim(X) \leq 1$. Then there exists a commutative diagram*

$$
\begin{array}{ccccccc}
\overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n & \xleftarrow{\ j\ } & Y_1 \amalg \ldots \amalg Y_n & \xrightarrow{\ \nu\ } & X_{k'} & \longrightarrow & X \\
& \searrow & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\
& & \mathrm{Spec}(k'_1) \amalg \ldots \amalg \mathrm{Spec}(k'_n) & \longrightarrow & \mathrm{Spec}(k') & \longrightarrow & \mathrm{Spec}(k)
\end{array}
$$

of schemes with the following properties:

    (1) *k'/k is a finite purely inseparable extension of fields,*
    (2) *ν is the normalization of $X_{k'}$,*
    (3) *j is an open immersion with dense image,*
    (4) *k'_i/k' is a finite separable extension for $i = 1, \ldots, n$,*
    (5) *$\overline{Y}_i$ is smooth, projective, geometrically irreducible dimension ≤ 1 over k'_i.*

Proof. As we may replace X by its reduction, we may and do assume X is reduced. Choose $X \to \overline{X}$ as in Lemma 43.6. If we can show the lemma for $\overline{X}$, then the lemma follows for X (details omitted). Thus we may and do assume X is projective.

Choose k'/k finite purely inseparable such that the normalization of $X_{k'}$ is geometrically normal over k', see Lemma 27.4. Denote $Y = (X_{k'})^\nu$ the normalization; for properties of the normalization, see Section 27. Then Y is geometrically regular as normal and regular are the same in dimension ≤ 1, see Properties, Lemma 12.6. Hence Y is smooth over k' by Lemma 12.6. Let $Y = Y_1 \amalg \ldots \amalg Y_n$ be the decomposition of Y into irreducible components. Set $k'_i = \Gamma(Y_i, \mathcal{O}_{Y_i})$. These are finite separable extensions of k' by Lemma 9.3. The proof is finished by Lemma 9.4. $\square$

Lemma 43.8. *Let k be a field. Let X be a curve over k. Let $x \in X$ be a closed point. We think of x as a (reduced) closed subscheme of X with sheaf of ideals $\mathcal{I}$. The following are equivalent*

    (1) *$\mathcal{O}_{X,x}$ is regular,*
    (2) *$\mathcal{O}_{X,x}$ is normal,*
    (3) *$\mathcal{O}_{X,x}$ is a discrete valuation ring,*
    (4) *$\mathcal{I}$ is an invertible $\mathcal{O}_X$-module,*
    (5) *x is an effective Cartier divisor on X.*

If k is perfect or if $\kappa(x)$ is separable over k, these are also equivalent to

    (6) *$X \to \mathrm{Spec}(k)$ is smooth at x.*

Proof. Since X is a curve, the local ring $\mathcal{O}_{X,x}$ is a Noetherian local domain of dimension 1 (Lemma 20.3). Parts (4) and (5) are equivalent by definition and are equivalent to $\mathcal{I}_x = \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ having one generator (Divisors, Lemma 15.2). The equivalence of (1), (2), (3), (4), and (5) therefore follows from Algebra, Lemma 119.7. The final statement follows from Lemma 25.8 in case k is perfect. If $\kappa(x)/k$ is separable, then the equivalence follows from Algebra, Lemma 140.5. $\square$

Remark 43.9. Let k be a field. Let X be a regular curve over k. By Lemmas 43.8 and 43.6 there exists a nonsingular projective curve $\overline{X}$ which is a compactification of X, i.e., there exists an open immersion $j : X \to \overline{X}$ such that the complement consists of a finite number of closed points. If k is perfect, then X and $\overline{X}$ are smooth over k and $\overline{X}$ is a smooth projective compactification of X.

Observe that if an affine scheme X over k is proper over k then X is finite over k (Morphisms, Lemma 44.11) and hence has dimension 0 (Algebra, Lemma 53.2 and Proposition 60.7). Hence a scheme of dimension > 0 over k cannot be both affine and proper over k. Thus the possibilities in the following lemma are mutually exclusive.

Lemma 43.10. *Let X be a curve over k. Then either X is an affine scheme or X is H-projective over k.*

Proof. Choose $X \to \overline{X}$ with $\overline{X} \setminus X = \{x_1, \ldots, x_r\}$ as in Lemma 43.6. Then $\overline{X}$ is a curve as well. If $r = 0$, then $X = \overline{X}$ is H-projective over k. Thus we may assume $r \geq 1$ and our goal is to show that X is affine. By Lemma 38.2 it suffices to show that $\overline{X} \setminus \{x_1\}$ is affine. This reduces us to the claim stated in the next paragraph.

Let X be an H-projective curve over k. Let $x \in X$ be a closed point such that $\mathcal{O}_{X,x}$ is a discrete valuation ring. Claim: $U = X \setminus \{x\}$ is affine. By Lemma 43.8 the

point x defines an effective Cartier divisor of X. For $n \geq 1$ denote $nx = x + \ldots + x$ the n-fold sum, see Divisors, Definition 13.6. Denote $\mathcal{O}_{nx}$ the structure sheaf of nx viewed as a coherent module on X. Since every invertible module on the local scheme nx is trivial the first short exact sequence of Divisors, Remark 14.11 reads

$$0 \to \mathcal{O}_X \xrightarrow{1} \mathcal{O}_X(nx) \to \mathcal{O}_{nx} \to 0$$

in our case. Note that $\dim_k H^0(X, \mathcal{O}_{nx}) \geq n$. Namely, by Lemma 33.3 we have $H^0(X, \mathcal{O}_{nx}) = \mathcal{O}_{X,x}/(\pi^n)$ where π in $\mathcal{O}_{X,x}$ is a uniformizer and the powers π^i map to k-linearly independent elements in $\mathcal{O}_{X,x}/(\pi^n)$ for $i = 0, 1, \ldots, n - 1$. We have $\dim_k H^1(X, \mathcal{O}_X) < \infty$ by Cohomology of Schemes, Lemma 19.2. If $n > \dim_k H^1(X, \mathcal{O}_X)$ we conclude from the long exact cohomology sequence that there exists an $s \in \Gamma(X, \mathcal{O}_X(nx))$ which is not a section of $\mathcal{O}_X$. If we take n minimal with this property, then s will map to a generator of the stalk $(\mathcal{O}_X(nx))_x$ since otherwise it would define a section of $\mathcal{O}_X((n-1)x) \subset \mathcal{O}_X(nx)$. For this n we conclude that $s_0 = 1$ and $s_1 = s$ generate the invertible module $\mathcal{L} = \mathcal{O}_X(nx)$.

Consider the corresponding morphism $f = \varphi_{\mathcal{L},(s_0,s_1)} : X \to \mathbf{P}^1_k$ of Constructions, Section 13. Observe that the inverse image of $D_+(T_0)$ is $U = X \setminus \{x\}$ as the section s_0 of $\mathcal{L}$ only vanishes at x. In particular, f is non-constant, i.e., $\mathrm{Im}(f)$ has more than one point. Hence f must map the generic point η of X to the generic point of $\mathbf{P}^1_k$. Hence if $y \in \mathbf{P}^1_k$ is a closed point, then $f^{-1}(\{y\})$ is a closed set of X not containing η, hence finite. Finally, f is proper[4]. By Cohomology of Schemes, Lemma 21.2[5] we conclude that f is finite. Hence $U = f^{-1}(D_+(T_0))$ is affine. $\square$

The following lemma combined with Lemma 43.2 tells us that given a separated scheme X of dimension 1 and of finite type over k, then $X \setminus Z$ is affine, whenever the closed subset Z meets every irreducible component of X.

Lemma 43.11. *Let X be a separated scheme of finite type over k. If $\dim(X) \leq 1$ and no irreducible component of X is proper of dimension 1, then X is affine.*

Proof. Let $X = \bigcup X_i$ be the decomposition of X into irreducible components. We think of X_i as an integral scheme (using the reduced induced scheme structure, see Schemes, Definition 12.5). In particular X_i is a singleton (hence affine) or a curve hence affine by Lemma 43.10. Then $\coprod X_i \to X$ is finite surjective and $\coprod X_i$ is affine. Thus we see that X is affine by Cohomology of Schemes, Lemma 13.3. $\square$

44. Degrees on curves

We start defining the degree of an invertible sheaf and more generally a locally free sheaf on a proper scheme of dimension 1 over a field. In Section 33 we defined the Euler characteristic of a coherent sheaf $\mathcal{F}$ on a proper scheme X over a field k by the formula

$$\chi(X, \mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

---

[4]Namely, a H-projective variety is a proper variety by Morphisms, Lemma 43.13. A morphism of varieties whose source is a proper variety is a proper morphism by Morphisms, Lemma 41.7.

[5]One can avoid using this lemma which relies on the theorem of formal functions. Namely, X is projective hence it suffices to show a proper morphism $f : X \to Y$ with finite fibres between quasi-projective schemes over k is finite. To do this, one chooses an affine open of X containing the fibre of f over a point y using that any finite set of points of a quasi-projective scheme over k is contained in an affine. Shrinking Y to a small affine neighbourhood of y one reduces to the case of a proper morphism between affines. Such a morphism is finite by Morphisms, Lemma 44.7.

Definition 44.1. Let k be a field, let X be a proper scheme of dimension ≤ 1 over k, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The *degree* of $\mathcal{L}$ is defined by

$$\deg(\mathcal{L}) = \chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X)$$

More generally, if $\mathcal{E}$ is a locally free sheaf of rank n we define the *degree* of $\mathcal{E}$ by

$$\deg(\mathcal{E}) = \chi(X, \mathcal{E}) - n\chi(X, \mathcal{O}_X)$$

Observe that this depends on the triple $\mathcal{E}/X/k$. If X is disconnected and $\mathcal{E}$ is finite locally free (but not of constant rank), then one can modify the definition by summing the degrees of the restriction of $\mathcal{E}$ to the connected components of X. If $\mathcal{E}$ is just a coherent sheaf, there are several different ways of extending the definition[6]. In a series of lemmas we show that this definition has all the properties one expects of the degree.

Lemma 44.2. *Let k'/k be an extension of fields. Let X be a proper scheme of dimension ≤ 1 over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of constant rank n. Then the degree of $\mathcal{E}/X/k$ is equal to the degree of $\mathcal{E}_{k'}/X_{k'}/k'$.*

Proof. More precisely, set $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$. Let $\mathcal{E}_{k'} = p^*\mathcal{E}$ where $p : X_{k'} \to X$ is the projection. By Cohomology of Schemes, Lemma 5.2 we have $H^i(X_{k'}, \mathcal{E}_{k'}) = H^i(X, \mathcal{E}) \otimes_k k'$ and $H^i(X_{k'}, \mathcal{O}_{X_{k'}}) = H^i(X, \mathcal{O}_X) \otimes_k k'$. Hence we see that the Euler characteristics are unchanged, hence the degree is unchanged. $\square$

Lemma 44.3. *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ be a short exact sequence of locally free $\mathcal{O}_X$-modules each of finite constant rank. Then*

$$\deg(\mathcal{E}_2) = \deg(\mathcal{E}_1) + \deg(\mathcal{E}_3)$$

Proof. Follows immediately from additivity of Euler characteristics (Lemma 33.2) and additivity of ranks. $\square$

Lemma 44.4. *Let k be a field. Let $f : X' \to X$ be a birational morphism of proper schemes of dimension ≤ 1 over k. Then*

$$\deg(f^*\mathcal{E}) = \deg(\mathcal{E})$$

for every finite locally free sheaf of constant rank. More generally it suffices if f induces a bijection between irreducible components of dimension 1 and isomorphisms of local rings at the corresponding generic points.

Proof. The morphism f is proper (Morphisms, Lemma 41.7) and has fibres of dimension ≤ 0. Hence f is finite (Cohomology of Schemes, Lemma 21.2). Thus

$$Rf_*f^*\mathcal{E} = f_*f^*\mathcal{E} = \mathcal{E} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_{X'}$$

Since f induces an isomorphism on local rings at generic points of all irreducible components of dimension 1 we see that the kernel and cokernel

$$0 \to \mathcal{K} \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \to \mathcal{Q} \to 0$$

---

[6]If X is a proper curve and $\mathcal{F}$ is a coherent sheaf on X, then one often defines the degree as $\chi(X, \mathcal{F}) - r\chi(X, \mathcal{O}_X)$ where $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is the rank of $\mathcal{F}$ at the generic point ξ of X.

have supports of dimension ≤ 0. Note that tensoring this with $\mathcal{E}$ is still an exact sequence as $\mathcal{E}$ is locally free. We obtain

$$
\begin{aligned}
\chi(X, \mathcal{E}) - \chi(X', f^*\mathcal{E}) &= \chi(X, \mathcal{E}) - \chi(X, f_*f^*\mathcal{E}) \\
&= \chi(X, \mathcal{E}) - \chi(X, \mathcal{E} \otimes f_*\mathcal{O}_{X'}) \\
&= \chi(X, \mathcal{K} \otimes \mathcal{E}) - \chi(X, \mathcal{Q} \otimes \mathcal{E}) \\
&= n\chi(X, \mathcal{K}) - n\chi(X, \mathcal{Q}) \\
&= n\chi(X, \mathcal{O}_X) - n\chi(X, f_*\mathcal{O}_{X'}) \\
&= n\chi(X, \mathcal{O}_X) - n\chi(X', \mathcal{O}_{X'})
\end{aligned}
$$

which proves what we want. The first equality as f is finite, see Cohomology of Schemes, Lemma 2.4. The second equality by projection formula, see Cohomology, Lemma 54.2. The third by additivity of Euler characteristics, see Lemma 33.2. The fourth by Lemma 33.3. $\qquad\square$

Lemma 44.5. *Let k be a field. Let X be a proper curve over k with generic point ξ. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then*

$$
\chi(X, \mathcal{E} \otimes \mathcal{F}) = r \deg(\mathcal{E}) + n\chi(X, \mathcal{F})
$$

where $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is the rank of $\mathcal{F}$.

Proof. Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X expressing that the formula of the lemma holds. We claim that the assumptions (1) and (2) of Cohomology of Schemes, Lemma 12.6 hold for $\mathcal{P}$. Namely, (1) holds because the Euler characteristic and the rank r are additive in short exact sequences of coherent sheaves. And (2) holds too: If $Z = X$ then we may take $\mathcal{G} = \mathcal{O}_X$ and $\mathcal{P}(\mathcal{O}_X)$ is true by the definition of degree. If $i : Z \to X$ is the inclusion of a closed point we may take $\mathcal{G} = i_*\mathcal{O}_Z$ and $\mathcal{P}$ holds by Lemma 33.3 and the fact that $r = 0$ in this case. $\qquad\square$

Let k be a field. Let X be a finite type scheme over k of dimension ≤ 1. Let $C_i \subset X$, $i = 1, \ldots, t$ be the irreducible components of dimension 1. We view C_i as a scheme by using the induced reduced scheme structure. Let $\xi_i \in C_i$ be the generic point. The *multiplicity of C_i in X* is defined as the length

$$
m_i = \mathrm{length}_{\mathcal{O}_{X, \xi_i}} \mathcal{O}_{X, \xi_i}
$$

This makes sense because $\mathcal{O}_{X, \xi_i}$ is a zero dimensional Noetherian local ring and hence has finite length over itself (Algebra, Proposition 60.7). See Chow Homology, Section 9 for additional information. It turns out the degree of a locally free sheaf only depends on the restriction of the irreducible components.

Lemma 44.6. *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n. Then*

$$
\deg(\mathcal{E}) = \sum m_i \deg(\mathcal{E}|_{C_i})
$$

where $C_i \subset X$, $i = 1, \ldots, t$ are the irreducible components of dimension 1 with reduced induced scheme structure and m_i is the multiplicity of C_i in X.

Proof. Observe that the statement makes sense because $C_i \to \mathrm{Spec}(k)$ is proper of dimension 1 (Morphisms, Lemmas 41.6 and 41.4). Consider the open subscheme $U_i = X \setminus (\bigcup_{j \neq i} C_j)$ and let $X_i \subset X$ be the scheme theoretic closure of U_i. Note that $X_i \cap U_i = U_i$ (scheme theoretically) and that $X_i \cap U_j = \emptyset$ (set theoretically) for

$i \neq j$; this follows from the description of scheme theoretic closure in Morphisms, Lemma 7.7. Thus we may apply Lemma 44.4 to the morphism $X' = \bigcup X_i \to X$. Since it is clear that $C_i \subset X_i$ (scheme theoretically) and that the multiplicity of C_i in X_i is equal to the multiplicity of C_i in X, we see that we reduce to the case discussed in the following paragraph.

Assume X is irreducible with generic point ξ. Let $C = X_{red}$ have multiplicity m. We have to show that $\deg(\mathcal{E}) = m \deg(\mathcal{E}|_C)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal defining the closed subscheme C. Let $e \geq 0$ be minimal such that $\mathcal{I}^{e+1} = 0$ (Cohomology of Schemes, Lemma 10.2). We argue by induction on e. If $e = 0$, then $X = C$ and the result is immediate. Otherwise we set $\mathcal{F} = \mathcal{I}^e$ viewed as a coherent $\mathcal{O}_C$-module (Cohomology of Schemes, Lemma 9.8). Let $X' \subset X$ be the closed subscheme cut out by the coherent ideal $\mathcal{I}^e$ and let m' be the multiplicity of C in X'. Taking stalks at ξ of the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

we find (use Algebra, Lemmas 52.3, 52.6, and 52.5) that

$$m = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{X,\xi} = \dim_{\kappa(\xi)} \mathcal{F}_\xi + \text{length}_{\mathcal{O}_{X',\xi}} \mathcal{O}_{X',\xi} = r + m'$$

where r is the rank of $\mathcal{F}$ as a coherent sheaf on C. Tensoring with $\mathcal{E}$ we obtain a short exact sequence

$$0 \to \mathcal{E}|_C \otimes \mathcal{F} \to \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_{X'} \to 0$$

By induction we have $\chi(\mathcal{E} \otimes \mathcal{O}_{X'}) = m' \deg(\mathcal{E}|_C)$. By Lemma 44.5 we have $\chi(\mathcal{E}|_C \otimes \mathcal{F}) = r \deg(\mathcal{E}|_C) + n\chi(\mathcal{F})$. Putting everything together we obtain the result. $\square$

Lemma 44.7. *Let k be a field, let X be a proper scheme of dimension ≤ 1 over k, and let $\mathcal{E}$, $\mathcal{V}$ be locally free $\mathcal{O}_X$-modules of constant finite rank. Then*

$$\deg(\mathcal{E} \otimes \mathcal{V}) = rank(\mathcal{E}) \deg(\mathcal{V}) + rank(\mathcal{V}) \deg(\mathcal{E})$$

Proof. By Lemma 44.6 and elementary arithmetic, we reduce to the case of a proper curve. This case follows from Lemma 44.5. $\square$

Lemma 44.8. *Let k be a field, let X be a proper scheme of dimension ≤ 1 over k, and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n. Then*

$$\deg(\mathcal{E}) = \deg(\wedge^n(\mathcal{E})) = \deg(\det(\mathcal{E}))$$

Proof. By Lemma 44.6 and elementary arithmetic, we reduce to the case of a proper curve. Then there exists a modification $f : X' \to X$ such that $f^*\mathcal{E}$ has a filtration whose successive quotients are invertible modules, see Divisors, Lemma 36.1. By Lemma 44.4 we may work on X'. Thus we may assume we have a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_n = \mathcal{E}$$

by locally free $\mathcal{O}_X$-modules with $\mathcal{L}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ is invertible. By Modules, Lemma 26.1 and induction we find $\det(\mathcal{E}) = \mathcal{L}_1 \otimes \ldots \otimes \mathcal{L}_n$. Thus the equality follows from Lemma 44.7 and additivity (Lemma 44.3). $\square$

Lemma 44.9. *Let k be a field, let X be a proper scheme of dimension ≤ 1 over k. Let D be an effective Cartier divisor on X. Then D is finite over $\text{Spec}(k)$ of degree $\deg(D) = \dim_k \Gamma(D, \mathcal{O}_D)$. For a locally free sheaf $\mathcal{E}$ of rank n we have*

$$\deg(\mathcal{E}(D)) = n \deg(D) + \deg(\mathcal{E})$$

where $\mathcal{E}(D) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$.

Proof. Since D is nowhere dense in X (Divisors, Lemma 13.4) we see that $\dim(D) \leq 0$. Hence D is finite over k by Lemma 20.2. Since k is a field, the morphism $D \to \mathrm{Spec}(k)$ is finite locally free and hence has a degree (Morphisms, Definition 48.1), which is clearly equal to $\dim_k \Gamma(D, \mathcal{O}_D)$ as stated in the lemma. By Divisors, Definition 14.1 there is a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_* i^* \mathcal{O}_X(D) \to 0$$

where $i : D \to X$ is the closed immersion. Tensoring with $\mathcal{E}$ we obtain a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}(D) \to i_* i^* \mathcal{E}(D) \to 0$$

The equation of the lemma follows from additivity of the Euler characteristic (Lemma 33.2) and Lemma 33.3. $\square$

Lemma 44.10. *Let k be a field. Let X be a proper scheme over k which is reduced and connected. Let $\kappa = H^0(X, \mathcal{O}_X)$. Then κ/k is a finite extension of fields and $w = [\kappa : k]$ divides*

(1) $\deg(\mathcal{E})$ *for all locally free $\mathcal{O}_X$-modules $\mathcal{E}$,*
(2) $[\kappa(x) : k]$ *for all closed points $x \in X$, and*
(3) $\deg(D)$ *for all closed subschemes $D \subset X$ of dimension zero.*

Proof. See Lemma 9.3 for the assertions about κ. For every quasi-coherent $\mathcal{O}_X$-module, the k-vector spaces $H^i(X, \mathcal{F})$ are κ-vector spaces. The divisibilities easily follow from this statement and the definitions. $\square$

Lemma 44.11. *Let k be a field. Let $f : X \to Y$ be a nonconstant morphism of proper curves over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module. Then*

$$\deg(f^* \mathcal{E}) = \deg(X/Y) \deg(\mathcal{E})$$

Proof. The degree of X over Y is defined in Morphisms, Definition 51.8. Thus $f_* \mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module of rank $\deg(X/Y)$, i.e., $\deg(X/Y) = \dim_{\kappa(\xi)}(f_* \mathcal{O}_X)_\xi$ where ξ is the generic point of Y. Thus we obtain

$$\begin{aligned}
\chi(X, f^* \mathcal{E}) &= \chi(Y, f_* f^* \mathcal{E}) \\
&= \chi(Y, \mathcal{E} \otimes f_* \mathcal{O}_X) \\
&= \deg(X/Y) \deg(\mathcal{E}) + n\chi(Y, f_* \mathcal{O}_X) \\
&= \deg(X/Y) \deg(\mathcal{E}) + n\chi(X, \mathcal{O}_X)
\end{aligned}$$

as desired. The first equality as f is finite, see Cohomology of Schemes, Lemma 2.4. The second equality by projection formula, see Cohomology, Lemma 54.2. The third equality by Lemma 44.5. $\square$

The following is a trivial but important consequence of the results on degrees above.

Lemma 44.12. *Let k be a field. Let X be a proper curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.*

(1) *If $\mathcal{L}$ has a nonzero section, then $\deg(\mathcal{L}) \geq 0$.*
(2) *If $\mathcal{L}$ has a nonzero section s which vanishes at a point, then $\deg(\mathcal{L}) > 0$.*
(3) *If $\mathcal{L}$ and $\mathcal{L}^{-1}$ have nonzero sections, then $\mathcal{L} \cong \mathcal{O}_X$.*
(4) *If $\deg(\mathcal{L}) \leq 0$ and $\mathcal{L}$ has a nonzero section, then $\mathcal{L} \cong \mathcal{O}_X$.*
(5) *If $\mathcal{N} \to \mathcal{L}$ is a nonzero map of invertible $\mathcal{O}_X$-modules, then $\deg(\mathcal{L}) \geq \deg(\mathcal{N})$ and if equality holds then it is an isomorphism.*

Proof. Let s be a nonzero section of $\mathcal{L}$. Since X is a curve, we see that s is a regular section. Hence there is an effective Cartier divisor $D \subset X$ and an isomorphism $\mathcal{L} \to \mathcal{O}_X(D)$ mapping s the canonical section 1 of $\mathcal{O}_X(D)$, see Divisors, Lemma 14.10. Then $\deg(\mathcal{L}) = \deg(D)$ by Lemma 44.9. As $\deg(D) \geq 0$ and $= 0$ if and only if $D = \emptyset$, this proves (1) and (2). In case (3) we see that $\deg(\mathcal{L}) = 0$ and $D = \emptyset$. Similarly for (4). To see (5) apply (1) and (4) to the invertible sheaf

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{L})$$

which has degree $\deg(\mathcal{L}) - \deg(\mathcal{N})$ by Lemma 44.7. $\qquad\qquad\square$

Lemma 44.13. *Let k be a field. Let X be a proper scheme over k which is reduced, connected, and equidimensional of dimension 1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\deg(\mathcal{L}|_C) \leq 0$ for all irreducible components C of X, then either $H^0(X, \mathcal{L}) = 0$ or $\mathcal{L} \cong \mathcal{O}_X$.*

Proof. Let $s \in H^0(X, \mathcal{L})$ be nonzero. Since X is reduced there exists an irreducible component C of X with $s|_C \neq 0$. But if $s|_C$ is nonzero, then s is nowhere vanishing on C by Lemma 44.12. This in turn implies s is nowhere vanishing on every irreducible component of X meeting C. Since X is connected, we conclude that s vanishes nowhere and the lemma follows. $\qquad\qquad\square$

Lemma 44.14. *Let k be a field. Let X be a proper curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is ample if and only if $\deg(\mathcal{L}) > 0$.*

Proof. If $\mathcal{L}$ is ample, then there exists an $n > 0$ and a section $s \in H^0(X, \mathcal{L}^{\otimes n})$ with X_s affine. Since X isn't affine (otherwise by Morphisms, Lemma 44.11 X would be finite), we see that s vanishes at some point. Hence $\deg(\mathcal{L}^{\otimes n}) > 0$ by Lemma 44.12. By Lemma 44.7 we conclude that $\deg(\mathcal{L}) = 1/n \deg(\mathcal{L}^{\otimes n}) > 0$.

Assume $\deg(\mathcal{L}) > 0$. Then

$$\dim_k H^0(X, \mathcal{L}^{\otimes n}) \geq \chi(X, \mathcal{L}^n) = n\deg(\mathcal{L}) + \chi(X, \mathcal{O}_X)$$

grows linearly with n. Hence for any finite collection of closed points $x_1, \ldots, x_t$ of X, we can find an n such that $\dim_k H^0(X, \mathcal{L}^{\otimes n}) > \sum \dim_k \kappa(x_i)$. (Recall that by Hilbert Nullstellensatz, the extension fields $\kappa(x_i)/k$ are finite, see for example Morphisms, Lemma 20.3). Hence we can find a nonzero $s \in H^0(X, \mathcal{L}^{\otimes n})$ vanishing in $x_1, \ldots, x_t$. In particular, if we choose $x_1, \ldots, x_t$ such that $X \setminus \{x_1, \ldots, x_t\}$ is affine, then X_s is affine too (for example by Properties, Lemma 26.4 although if we choose our finite set such that $\mathcal{L}|_{X \setminus \{x_1, \ldots, x_t\}}$ is trivial, then it is immediate). The conclusion is that we can find an $n > 0$ and a nonzero section $s \in H^0(X, \mathcal{L}^{\otimes n})$ such that X_s is affine.

We will show that for every quasi-coherent sheaf of ideals $\mathcal{I}$ there exists an $m > 0$ such that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes m})$ is zero. This will finish the proof by Cohomology of Schemes, Lemma 17.1. To see this we consider the maps

$$\mathcal{I} \xrightarrow{s} \mathcal{I} \otimes \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{I} \otimes \mathcal{L}^{\otimes 2n} \xrightarrow{s} \ldots$$

Since $\mathcal{I}$ is torsion free, these maps are injective and isomorphisms over X_s, hence the cokernels have vanishing H^1 (by Cohomology of Schemes, Lemma 9.10 for example). We conclude that the maps of vector spaces

$$H^1(X, \mathcal{I}) \to H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) \to H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes 2n}) \to \ldots$$

are surjective. On the other hand, the dimension of $H^1(X,\mathcal{I})$ is finite, and every element maps to zero eventually by Cohomology of Schemes, Lemma 17.4. Thus for some $e > 0$ we see that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes en})$ is zero. This finishes the proof. $\square$

Lemma 44.15. *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $C_i \subset X$, $i = 1, \ldots, t$ be the irreducible components of dimension 1. The following are equivalent:*

    (1) *$\mathcal{L}$ is ample, and*
    (2) *$\deg(\mathcal{L}|_{C_i}) > 0$ for $i = 1, \ldots, t$.*

Proof. Let $x_1, \ldots, x_r \in X$ be the isolated closed points. Think of $x_i = \mathrm{Spec}(\kappa(x_i))$ as a scheme. Consider the morphism of schemes

$$f : C_1 \amalg \ldots \amalg C_t \amalg x_1 \amalg \ldots \amalg x_r \longrightarrow X$$

This is a finite surjective morphism of schemes proper over k (details omitted). Thus $\mathcal{L}$ is ample if and only if $f^*\mathcal{L}$ is ample (Cohomology of Schemes, Lemma 17.2). Thus we conclude by Lemma 44.14. $\square$

Lemma 44.16. *Let k be an algebraically closed field. Let X be a proper curve over k. Then there exist*

    (1) *an invertible $\mathcal{O}_X$-module $\mathcal{L}$ with $\dim_k H^0(X, \mathcal{L}) = 1$ and $H^1(X, \mathcal{L}) = 0$, and*
    (2) *an invertible $\mathcal{O}_X$-module $\mathcal{N}$ with $\dim_k H^0(X, \mathcal{N}) = 0$ and $H^1(X, \mathcal{N}) = 0$.*

Proof. Choose a closed immersion $i : X \to \mathbf{P}^n_k$ (Lemma 43.4). Setting $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n}(d)$ for $d \gg 0$ we see that there exists an invertible sheaf $\mathcal{L}$ with $H^0(X, \mathcal{L}) \neq 0$ and $H^1(X, \mathcal{L}) = 0$ (see Cohomology of Schemes, Lemma 17.1 for vanishing and the references therein for nonvanishing). We will finish the proof of (1) by descending induction on $t = \dim_k H^0(X, \mathcal{L})$. The base case $t = 1$ is trivial. Assume $t > 1$.

Let $U \subset X$ be the nonempty open subset of nonsingular points studied in Lemma 25.8. Let $s \in H^0(X, \mathcal{L})$ be nonzero. There exists a closed point $x \in U$ such that s does not vanish in x. Let $\mathcal{I}$ be the ideal sheaf of $i : x \to X$ as in Lemma 43.8. Look at the short exact sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{L} \to i_*i^*\mathcal{L} \to 0$$

Observe that $H^0(X, i_*i^*\mathcal{L}) = H^0(x, i^*\mathcal{L})$ has dimension 1 as x is a k-rational point (k is algebraically closed). Since s does not vanish at x we conclude that

$$H^0(X, \mathcal{L}) \longrightarrow H^0(X, i_*i^*\mathcal{L})$$

is surjective. Hence $\dim_k H^0(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}) = t - 1$. Finally, the long exact sequence of cohomology also shows that $H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}) = 0$ thereby finishing the proof of the induction step.

To get an invertible sheaf as in (2) take an invertible sheaf $\mathcal{L}$ as in (1) and do the argument in the previous paragraph one more time. $\square$

Lemma 44.17. *Let k be an algebraically closed field. Let X be a proper curve over k. Set $g = \dim_k H^1(X, \mathcal{O}_X)$. For every invertible $\mathcal{O}_X$-module $\mathcal{L}$ with $\deg(\mathcal{L}) \geq 2g - 1$ we have $H^1(X, \mathcal{L}) = 0$.*

Proof. Let $\mathcal{N}$ be the invertible module we found in Lemma 44.16 part (2). The degree of $\mathcal{N}$ is $\chi(X, \mathcal{N}) - \chi(X, \mathcal{O}_X) = 0 - (1 - g) = g - 1$. Hence the degree of $\mathcal{L} \otimes \mathcal{N}^{\otimes -1}$ is $\deg(\mathcal{L}) - (g - 1) \geq g$. Hence $\chi(X, \mathcal{L} \otimes \mathcal{N}^{\otimes -1}) \geq g + 1 - g = 1$. Thus there is a nonzero global section s whose zero scheme is an effective Cartier divisor D of degree $\deg(\mathcal{L}) - (g - 1)$. This gives a short exact sequence

$$0 \to \mathcal{N} \xrightarrow{s} \mathcal{L} \to i_*(\mathcal{L}|_D) \to 0$$

where $i : D \to X$ is the inclusion morphism. We conclude that $H^0(X, \mathcal{L})$ maps isomorphically to $H^0(D, \mathcal{L}|_D)$ which has dimension $\deg(\mathcal{L}) - (g - 1)$. The result follows from the definition of degree. $\square$

45. Numerical intersections

In this section we play around with the Euler characteristic of coherent sheaves on proper schemes to obtain numerical intersection numbers for invertible modules. Our main tool will be the following lemma.

Lemma 45.1. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. The map*

$$(n_1, \ldots, n_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most the dimension of the support of $\mathcal{F}$.

Proof. We prove this by induction on $\dim(\mathrm{Supp}(\mathcal{F}))$. If this number is zero, then the function is constant with value $\dim_k \Gamma(X, \mathcal{F})$ by Lemma 33.3. Assume $\dim(\mathrm{Supp}(\mathcal{F})) > 0$.

If $\mathcal{F}$ has embedded associated points, then we can consider the short exact sequence $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}' \to 0$ constructed in Divisors, Lemma 4.6. Since the dimension of the support of $\mathcal{K}$ is strictly less, the result holds for $\mathcal{K}$ by induction hypothesis and with strictly smaller total degree. By additivity of the Euler characteristic (Lemma 33.2) it suffices to prove the result for $\mathcal{F}'$. Thus we may assume $\mathcal{F}$ does not have embedded associated points.

If $i : Z \to X$ is a closed immersion and $\mathcal{F} = i_*\mathcal{G}$, then we see that the result for X, $\mathcal{F}$, $\mathcal{L}_1, \ldots, \mathcal{L}_r$ is equivalent to the result for Z, $\mathcal{G}$, $i^*\mathcal{L}_1, \ldots, i^*\mathcal{L}_r$ (since the cohomologies agree, see Cohomology of Schemes, Lemma 2.4). Applying Divisors, Lemma 4.7 we may assume that X has no embedded components and $X = \mathrm{Supp}(\mathcal{F})$.

Pick a regular meromorphic section s of $\mathcal{L}_1$, see Divisors, Lemma 25.4. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s and consider the maps

$$\mathcal{I}\mathcal{F} \to \mathcal{F}, \quad \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes \mathcal{L}_1$$

of Divisors, Lemma 24.5. These are injective and have cokernels $\mathcal{Q}$, $\mathcal{Q}'$ supported on nowhere dense closed subschemes of $X = \mathrm{Supp}(\mathcal{F})$. Tensoring with the invertible module $\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}$ is exact, hence using additivity again we see that

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1 + 1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$
$$= \chi(\mathcal{Q} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \chi(\mathcal{Q}' \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$

Thus we see that the function $P(n_1, \ldots, n_r)$ of the lemma has the property that

$$P(n_1 + 1, n_2, \ldots, n_r) - P(n_1, \ldots, n_r)$$

is a numerical polynomial of total degree $<$ the dimension of the support of $\mathcal{F}$. Of course by symmetry the same thing is true for

$$P(n_1, \ldots, n_{i-1}, n_i + 1, n_{i+1}, \ldots, n_r) - P(n_1, \ldots, n_r)$$

for any $i \in \{1, \ldots, r\}$. A simple arithmetic argument shows that P is a numerical polynomial of total degree at most $\dim(\mathrm{Supp}(\mathcal{F}))$. $\square$

The following lemma roughly shows that the leading coefficient only depends on the length of the coherent module in the generic points of its support.

Lemma 45.2. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. Let $d = \dim(\mathrm{Supp}(\mathcal{F}))$. Let $Z_i \subset X$ be the irreducible components of $\mathrm{Supp}(\mathcal{F})$ of dimension d. Let $\xi_i \in Z_i$ be the generic point and set $m_i = \mathrm{length}_{\mathcal{O}_{X,\xi_i}}(\mathcal{F}_{\xi_i})$. Then*

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum_i m_i \ \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree $< d$.

Proof. Consider pairs (ξ, Z) where $Z \subset X$ is an integral closed subscheme of dimension d and ξ is its generic point. Then the finite $\mathcal{O}_{X,\xi}$-module $\mathcal{F}_\xi$ has support contained in $\{\xi\}$ hence the length $m_Z = \mathrm{length}_{\mathcal{O}_{X,\xi}}(\mathcal{F}_\xi)$ is finite (Algebra, Lemma 62.3) and zero unless $Z = Z_i$ for some i. Thus the expression of the lemma can be written as

$$E(\mathcal{F}) = \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum m_Z \ \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_Z)$$

where the sum is over integral closed subschemes $Z \subset X$ of dimension d. The assignment $\mathcal{F} \mapsto E(\mathcal{F})$ is additive in short exact sequences $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of coherent $\mathcal{O}_X$-modules whose support has dimension $\leq d$. This follows from additivity of Euler characteristics (Lemma 33.2) and additivity of lengths (Algebra, Lemma 52.3). Let us apply Cohomology of Schemes, Lemma 12.3 to find a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exists an integral closed subscheme $V_j \subset X$ and a nonzero sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{V_j}$ such that

$$\mathcal{F}_j / \mathcal{F}_{j-1} \cong (V_j \to X)_* \mathcal{I}_j$$

It follows that $V_j \subset \mathrm{Supp}(\mathcal{F})$ and hence $\dim(V_j) \leq d$. By the additivity we remarked upon above it suffices to prove the result for each of the subquotients $\mathcal{F}_j / \mathcal{F}_{j-1}$. Thus it suffices to prove the result when $\mathcal{F} = (V \to X)_* \mathcal{I}$ where $V \subset X$ is an integral closed subscheme of dimension $\leq d$ and $\mathcal{I} \subset \mathcal{O}_V$ is a nonzero coherent sheaf of ideals. If $\dim(V) < d$ and more generally for $\mathcal{F}$ whose support has dimension $< d$, then the first term in $E(\mathcal{F})$ has total degree $< d$ by Lemma 45.1 and the second term is zero. If $\dim(V) = d$, then we can use the short exact sequence

$$0 \to (V \to X)_* \mathcal{I} \to (V \to X)_* \mathcal{O}_V \to (V \to X)_* (\mathcal{O}_V / \mathcal{I}) \to 0$$

The result holds for the middle sheaf because the only Z occurring in the sum is $Z = V$ with $m_Z = 1$ and because

$$H^i(X, ((V \to X)_* \mathcal{O}_V) \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) = H^i(V, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_V)$$

by the projection formula (Cohomology, Section 54) and Cohomology of Schemes, Lemma 2.4; so in this case we actually have $E(\mathcal{F}) = 0$. The result holds for the

sheaf on the right because its support has dimension $< d$. Thus the result holds for the sheaf on the left and the lemma is proved. $\qquad\square$

Definition 45.3. Let k be a field. Let X be a proper scheme over k. Let $i : Z \to X$ be a closed subscheme of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. We define the *intersection number* $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ as the coefficient of $n_1 \ldots n_d$ in the numerical polynomial

$$\chi(X, i_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

In the special case that $\mathcal{L}_1 = \ldots = \mathcal{L}_d = \mathcal{L}$ we write $(\mathcal{L}^d \cdot Z)$.

The displayed equality in the definition follows from the projection formula (Cohomology, Section 54) and Cohomology of Schemes, Lemma 2.4. We prove a few lemmas for these intersection numbers.

Lemma 45.4. *In the situation of Definition 45.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is an integer.*

Proof. Any numerical polynomial of degree e in $n_1, \ldots, n_d$ can be written uniquely as a **Z**-linear combination of the functions $\binom{n_1}{k_1}\binom{n_2}{k_2} \ldots \binom{n_d}{k_d}$ with $k_1 + \ldots + k_d \leq e$. Apply this with $e = d$. Left as an exercise. $\qquad\square$

Lemma 45.5. *In the situation of Definition 45.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is additive: if $\mathcal{L}_i = \mathcal{L}_i' \otimes \mathcal{L}_i''$, then we have*

$$(\mathcal{L}_1 \cdots \mathcal{L}_i \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_1 \cdots \mathcal{L}_i' \cdots \mathcal{L}_d \cdot Z) + (\mathcal{L}_1 \cdots \mathcal{L}_i'' \cdots \mathcal{L}_d \cdot Z)$$

Proof. This is true because by Lemma 45.1 the function

$$(n_1, \ldots, n_{i-1}, n_i', n_i'', n_{i+1}, \ldots, n_d) \mapsto \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes (\mathcal{L}_i')^{\otimes n_i'} \otimes (\mathcal{L}_i'')^{\otimes n_i''} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

is a numerical polynomial of total degree at most d in $d+1$ variables. $\qquad\square$

Lemma 45.6. *In the situation of Definition 45.3 let $Z_i \subset Z$ be the irreducible components of dimension d. Let $m_i = \text{length}_{\mathcal{O}_{X,\xi_i}}(\mathcal{O}_{Z,\xi_i})$ where $\xi_i \in Z_i$ is the generic point. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \sum m_i (\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z_i)$$

Proof. Immediate from Lemma 45.2 and the definitions. $\qquad\square$

Lemma 45.7. *Let k be a field. Let $f : Y \to X$ be a morphism of proper schemes over k. Let $Z \subset Y$ be an integral closed subscheme of dimension d and let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then*

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot Z) = \deg(f|_Z : Z \to f(Z))(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot f(Z))$$

where $\deg(Z \to f(Z))$ is as in Morphisms, Definition 51.8 or 0 if $\dim(f(Z)) < d$.

Proof. The left hand side is computed using the coefficient of $n_1 \ldots n_d$ in the function

$$\chi(Y, \mathcal{O}_Z \otimes f^*\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes f^*\mathcal{L}_d^{\otimes n_d}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$$

The equality follows from Lemma 33.5 and the projection formula (Cohomology, Lemma 54.2). If $f(Z)$ has dimension $< d$, then the right hand side is a polynomial of total degree $< d$ by Lemma 45.1 and the result is true. Assume $\dim(f(Z)) = d$. Let $\xi \in f(Z)$ be the generic point. By dimension theory (see Lemmas 20.3 and 20.4)

the generic point of Z is the unique point of Z mapping to ξ. Then $f : Z \to f(Z)$ is finite over a nonempty open of $f(Z)$, see Morphisms, Lemma 51.1. Thus $\deg(f : Z \to f(Z))$ is defined and in fact it is equal to the length of the stalk of $f_* \mathcal{O}_Z$ at ξ over $\mathcal{O}_{X,\xi}$. Moreover, the stalk of $R^i f_* \mathcal{O}_X$ at ξ is zero for $i > 0$ because we just saw that $f|_Z$ is finite in a neighbourhood of ξ (so that Cohomology of Schemes, Lemma 9.9 gives the vanishing). Thus the terms $\chi(X, R^i f_* \mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ with $i > 0$ have total degree $< d$ and

$$\chi(X, f_* \mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \deg(f : Z \to f(Z)) \chi(f(Z), \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_{f(Z)})$$

modulo a polynomial of total degree $< d$ by Lemma 45.2. The desired result follows. $\square$

Lemma 45.8. *Let k be a field. Let X be proper over k. Let $Z \subset X$ be a closed subscheme of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Assume there exists an effective Cartier divisor $D \subset Z$ such that $\mathcal{L}_1|_Z \cong \mathcal{O}_Z(D)$. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot D)$$

Proof. We may replace X by Z and $\mathcal{L}_i$ by $\mathcal{L}_i|_Z$. Thus we may assume $X = Z$ and $\mathcal{L}_1 = \mathcal{O}_X(D)$. Then $\mathcal{L}_1^{-1}$ is the ideal sheaf of D and we can consider the short exact sequence

$$0 \to \mathcal{L}_1^{\otimes -1} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

Set $P(n_1, \ldots, n_d) = \chi(X, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ and $Q(n_1, \ldots, n_d) = \chi(D, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_D)$. We conclude from additivity that

$$P(n_1, \ldots, n_d) - P(n_1 - 1, n_2, \ldots, n_d) = Q(n_1, \ldots, n_d)$$

Because the total degree of P is at most d, we see that the coefficient of $n_1 \ldots n_d$ in P is equal to the coefficient of $n_2 \ldots n_d$ in Q. $\square$

Lemma 45.9. *Let k be a field. Let X be proper over k. Let $Z \subset X$ be a closed subscheme of dimension d. If $\mathcal{L}_1, \ldots, \mathcal{L}_d$ are ample, then $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is positive.*

Proof. We will prove this by induction on d. The case $d = 0$ follows from Lemma 33.3. Assume $d > 0$. By Lemma 45.6 we may assume that Z is an integral closed subscheme. In fact, we may replace X by Z and $\mathcal{L}_i$ by $\mathcal{L}_i|_Z$ to reduce to the case $Z = X$ is a proper variety of dimension d. By Lemma 45.5 we may replace $\mathcal{L}_1$ by a positive tensor power. Thus we may assume there exists a nonzero section $s \in \Gamma(X, \mathcal{L}_1)$ such that X_s is affine (here we use the definition of ample invertible sheaf, see Properties, Definition 26.1). Observe that X is not affine because proper and affine implies finite (Morphisms, Lemma 44.11) which contradicts $d > 0$. It follows that s has a nonempty vanishing scheme $Z(s) \subset X$. Since X is a variety, s is a regular section of $\mathcal{L}_1$, so $Z(s)$ is an effective Cartier divisor, thus $Z(s)$ has codimension 1 in X, and hence $Z(s)$ has dimension $d - 1$ (here we use material from Divisors, Sections 13, 14, and 15 and from dimension theory as in Lemma 20.3). By Lemma 45.8 we have

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot Z(s))$$

By induction the right hand side is positive and the proof is complete. $\square$

Definition 45.10. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For any closed subscheme the *degree of Z with respect to $\mathcal{L}$*, denoted $\deg_{\mathcal{L}}(Z)$, is the intersection number $(\mathcal{L}^d \cdot Z)$ where $d = \dim(Z)$.

By Lemma 45.9 the degree of a subscheme is always a positive integer. We note that $\deg_{\mathcal{L}}(Z) = d$ if and only if

$$\chi(Z, \mathcal{L}^{\otimes n}|_Z) = \frac{d}{\dim(Z)!} n^{\dim(Z)} + l.o.t$$

as can be seen using that

$$(n_1 + \ldots + n_{\dim(Z)})^{\dim(Z)} = \dim(Z)! \, n_1 \ldots n_{\dim(Z)} + \text{other terms}$$

Lemma 45.11. *Let k be a field. Let $f : Y \to X$ be a finite dominant morphism of proper varieties over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Then*

$$\deg_{f^*\mathcal{L}}(Y) = \deg(f) \deg_{\mathcal{L}}(X)$$

where $\deg(f)$ is as in Morphisms, Definition 51.8.

Proof. The statement makes sense because $f^*\mathcal{L}$ is ample by Morphisms, Lemma 37.7. Having said this the result is a special case of Lemma 45.7. $\square$

Finally we relate the intersection number with a curve to the notion of degrees of invertible modules on curves introduced in Section 44.

Lemma 45.12. *Let k be a field. Let X be a proper scheme over k. Let $Z \subset X$ be a closed subscheme of dimension ≤ 1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then*

$$(\mathcal{L} \cdot Z) = \deg(\mathcal{L}|_Z)$$

where $\deg(\mathcal{L}|_Z)$ is as in Definition 44.1. If $\mathcal{L}$ is ample, then $\deg_{\mathcal{L}}(Z) = \deg(\mathcal{L}|_Z)$.

Proof. This follows from the fact that the function $n \mapsto \chi(Z, \mathcal{L}|_Z^{\otimes n})$ has degree 1 and hence the leading coefficient is the difference of consecutive values. $\square$

Proposition 45.13 (Asymptotic Riemann-Roch)**.** *Let k be a field. Let X be a proper scheme over k of dimension d. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Then*

$$\dim_k \Gamma(X, \mathcal{L}^{\otimes n}) \sim cn^d + l.o.t.$$

where $c = \deg_{\mathcal{L}}(X)/d!$ is a positive constant.

Proof. This follows from the definitions, Lemma 45.9, and the vanishing of higher cohomology in Cohomology of Schemes, Lemma 17.1. $\square$

46. Embedding dimension

There are several ways to define the embedding dimension, but for closed points on algebraic schemes over algebraically closed fields all definitions are equivalent to the following.

Definition 46.1. Let k be an algebraically closed field. Let X be a locally algebraic k-scheme and let $x \in X$ be a closed point. The *embedding dimension of X at x* is $\dim_k \mathfrak{m}_x/\mathfrak{m}_x^2$.

Facts about embedding dimension. Let k, X, x be as in Definition 46.1.

   (1) The embedding dimension of X at x is the dimension of the tangent space $T_{X/k,x}$ (Definition 16.3) as a k-vector space.

(2) The embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists a surjection

$$k[[x_1, \ldots, x_d]] \longrightarrow \mathcal{O}_{X,x}^{\wedge}$$

of k-algebras.

(3) The embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists an open neighbourhood $U \subset X$ of x and a closed immersion $U \to Y$ where Y is a smooth variety of dimension d over k.

(4) The embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists an open neighbourhood $U \subset X$ of x and an unramified morphism $U \to \mathbf{A}_k^d$.

(5) If we are given a closed embedding $X \to Y$ with Y smooth over k, then the embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists a closed subscheme $Z \subset Y$ with $X \subset Z$, with $Z \to \mathrm{Spec}(k)$ smooth at x, and with $\dim_x(Z) = d$.

If we ever need these, we will formulate a precise result and provide a proof.

Non-algebraically closed ground fields or non-closed points. Let k be a field and let X be a locally algebraic k-scheme. If $x \in X$ is a point, then we have several options for the embedding dimension of X at x. Namely, we could use

(1) $\dim_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2)$,
(2) $\dim_{\kappa(x)}(T_{X/k,x}) = \dim_{\kappa(x)}(\Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x))$ (Lemma 16.4),
(3) the smallest integer $d \geq 0$ such that there exists an open neighbourhood $U \subset X$ of x and a closed immersion $U \to Y$ where Y is a smooth variety of dimension d over k.

In characteristic zero $(1) = (2)$ if x is a closed point; more generally this holds if $\kappa(x)$ is separable algebraic over k, see Lemma 16.5. It seems that the geometric definition (3) corresponds most closely to the geometric intuition the phrase "embedding dimension" invokes. Since one can show that (3) and (2) define the same number (this follows from Lemma 18.5) this is what we will use. In our terminology we will make clear that we are taking the embedding dimension relative to the ground field.

Definition 46.2. Let k be a field. Let X be a locally algebraic k-scheme. Let $x \in X$ be a point. The *embedding dimension of X/k at x* is $\dim_{\kappa(x)}(T_{X/k,x})$.

If $(A, \mathfrak{m}, \kappa)$ is a Noetherian local ring the *embedding dimension of A* is sometimes defined as the dimension of $\mathfrak{m}/\mathfrak{m}^2$ over κ. Above we have seen that if A is given as an algebra over a field k, it may be preferable to use $\dim_\kappa(\Omega_{A/k} \otimes_A \kappa)$. Let us call this quantity the *embedding dimension of A/k*. With this terminology in place we have

embed dim of X/k at x = embed dim of $\mathcal{O}_{X,x}/k$ = embed dim of $\mathcal{O}_{X,x}^{\wedge}/k$

if k, X, x are as in Definition 46.2.

47. Bertini theorems

In this section we prove results of the form: given a smooth projective variety X over a field k there exists an ample divisor $H \subset X$ which is smooth.

Lemma 47.1. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a closed subscheme. Then there exists an integer n_0 such that for all $n \geq n_0$ the kernel V_n of $\Gamma(X, \mathcal{L}^{\otimes n}) \to \Gamma(Z, \mathcal{L}^{\otimes n}|_Z)$ generates $\mathcal{L}^{\otimes n}|_{X \setminus Z}$ and the canonical morphism*

$$X \setminus Z \longrightarrow \mathbf{P}(V_n)$$

is an immersion of schemes over k.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf of Z. Observe that via the inclusion $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \subset \mathcal{L}^{\otimes n}$ we have $V_n = \Gamma(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$. Choose n_1 such that for $n \geq n_1$ the sheaf $\mathcal{I} \otimes \mathcal{L}^{\otimes n}$ is globally generated, see Properties, Proposition 26.13. It follows that V_n gererates $\mathcal{L}^{\otimes n}|_{X \setminus Z}$ for $n \geq n_1$.

For $n \geq n_1$ denote $\psi_n : V_n \to \Gamma(X \setminus Z, \mathcal{L}^{\otimes n}|_{X \setminus Z})$ the restriction map. We get a canonical morphism

$$\varphi = \varphi_{\mathcal{L}^{\otimes n}|_{X \setminus Z}, \psi_n} : X \setminus Z \longrightarrow \mathbf{P}(V_n)$$

by Constructions, Example 21.2. Choose n_2 such that for all $n \geq n_2$ the invertible sheaf $\mathcal{L}^{\otimes n}$ is very ample on X. We claim that $n_0 = n_1 + n_2$ works.

Proof of the claim. Say $n \geq n_0$ and write $n = n_1 + n'$. For $x \in X \setminus Z$ we can choose $s_1 \in V_1$ not vanishing at x. Set $V' = \Gamma(X, \mathcal{L}^{\otimes n'})$. By our choice of n and n' we see that the corresponding morphism $\varphi' : X \to \mathbf{P}(V')$ is a closed immersion. Thus if we choose $s' \in \Gamma(X, \mathcal{L}^{\otimes n'})$ not vanishing at x, then $X_{s'} = (\varphi')^{-1}(D_+(s'))$ (see Constructions, Lemma 14.1) is affine and $X_{s'} \to D_+(s')$ is a closed immersion. Then $s = s_1 \otimes s' \in V_n$ does not vanish at x. If $D_+(s) \subset \mathbf{P}(V_n)$ denotes the corresponding open affine space of our projective space, then $\varphi^{-1}(D_+(s)) = X_s \subset X \setminus Z$ (see reference above). The open $X_s = X_{s'} \cap X_{s_1}$ is affine, see Properties, Lemma 26.4. Consider the ring map

$$\mathrm{Sym}(V)_{(s)} \longrightarrow \mathcal{O}_X(X_s)$$

defining the morphism $X_s \to D_+(s)$. Because $X_{s'} \to D_+(s')$ is a closed immersion, the images of the elements

$$\frac{s_1 \otimes t'}{s_1 \otimes s'}$$

where $t' \in V'$ generate the image of $\mathcal{O}_X(X_{s'}) \to \mathcal{O}_X(X_s)$. Since $X_s \to X_{s'}$ is an open immersion, this implies that $X_s \to D_+(s)$ is an immersion of affine schemes (see below). Thus φ_n is an immersion by Morphisms, Lemma 3.5.

Let $a : A' \to A$ and $c : B \to A$ be ring maps such that $\mathrm{Spec}(a)$ is an immersion and $\mathrm{Im}(a) \subset \mathrm{Im}(c)$. Set $B' = A' \times_A B$ with projections $b : B' \to B$ and $c' : B' \to A'$. By assumption c' is surjective and hence $\mathrm{Spec}(c')$ is a closed immersion. Whence $\mathrm{Spec}(c') \circ \mathrm{Spec}(a)$ is an immersion (Schemes, Lemma 24.3). Then $\mathrm{Spec}(c)$ has to be an immersion because it factors the immersion $\mathrm{Spec}(c') \circ \mathrm{Spec}(a) = \mathrm{Spec}(b) \circ \mathrm{Spec}(c)$, see Morphisms, Lemma 3.1. $\square$

Situation 47.2. Let k be a field, let X be a scheme over k, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, let V be a finite dimensional k-vector space, and let $\psi : V \to \Gamma(X, \mathcal{L})$ be a k-linear map. Say $\dim(V) = r$ and we have a basis $v_1, \ldots, v_r$ of V. Then we obtain a "universal divisor"

$$H_{univ} = Z(s_{univ}) \subset \mathbf{A}^r \times_k X$$

as the zero scheme (Divisors, Definition 14.8) of the section

$$s_{univ} = \sum\nolimits_{i=1,\dots,r} x_i \psi(v_i) \in \Gamma(\mathbf{A}^r \times_k X, \mathrm{pr}_2^* \mathcal{L})$$

For a field extension k'/k the k'-points $v \in \mathbf{A}_k^r(k')$ correspond to vectors $(a_1, \dots, a_r)$ of elements of k'. Thus we may on the one hand think of v as the element $v = \sum_{i=1,\dots,r} a_i v_i \in V \otimes_k k'$ and on the other hand we may assign to v the section

$$\psi(v) = \sum\nolimits_{i=1,\dots,r} a_i \psi(v_i) \in \Gamma(X_{k'}, \mathcal{L}|_{X_{k'}})$$

With this notation it is clear that the fibre of H_{univ} over $v \in V \otimes k'$ is the zero scheme of $\psi(v)$. In a formula:

$$H_v = H_{univ,v} = Z(\psi(v))$$

We will denote this common value by H_v as indicated. Finally, in this situation let P be a property of vectors $v \in V \otimes_k k'$ for k'/k an arbitrary field extension[7]. We say P *holds for general* $v \in V \otimes_k k'$ if there exists a nonempty Zariski open $U \subset \mathbf{A}_k^r$ such that if v corresponds to a k'-point of U for any k'/k then $P(v)$ holds.

Lemma 47.3. *In Situation 47.2 assume*

    (1) *X is smooth over k,*
    (2) *the image of $\psi : V \to \Gamma(X, \mathcal{L})$ generates $\mathcal{L}$,*
    (3) *the corresponding morphism $\varphi_{\mathcal{L},\psi} : X \to \mathbf{P}(V)$ is an immersion.*

Then for general $v \in V \otimes_k k'$ the scheme H_v is smooth over k'.

Proof. (We observe that X is separated and finite type as a locally closed subscheme of a projective space.) Let us use the notation introduced above the statement of the lemma. We consider the projections

$$\mathbf{A}_k^r \times_k X \xleftarrow{\hspace{1cm}} H_{univ} \xrightarrow{\hspace{1cm}} \mathbf{A}_k^r \times_k X$$
$$\downarrow \qquad \qquad \swarrow_p \qquad \qquad \searrow_q \qquad \qquad \downarrow$$
$$X \qquad \qquad \qquad \qquad \qquad \qquad \mathbf{A}_k^r$$

Let $\Sigma \subset H_{univ}$ be the singular locus of the morphsm $q : H_{univ} \to \mathbf{A}_k^r$, i.e., the set of points where q is not smooth. Then Σ is closed because the smooth locus of a morphism is open by definition. Since the fibre of a smooth morphism is smooth, it suffices to prove $q(\Sigma)$ is contained in a proper closed subset of $\mathbf{A}_k^r$. Since Σ (with reduced induced scheme structure) is a finite type scheme over k it suffices to prove $\dim(\Sigma) < r$ This follows from Lemma 20.4. Since dimensions aren't changed by replacing k by a bigger field (Morphisms, Lemma 28.3), we may and do assume k is algebraically closed. By dimension theory (Lemma 20.4), it suffices to prove that for $x \in X \setminus Z$ closed we have $p^{-1}(\{x\}) \cap \Sigma$ has dimension $< r - \dim(X')$ where X' is the unique irreducible component of X containing x. As X is smooth over k and x is a closed point we have $\dim(X') = \dim \mathfrak{m}_x/\mathfrak{m}_x^2$ (Morphisms, Lemma 34.12 and Algebra, Lemma 140.1). Thus we win if

$$\dim p^{-1}(x) \cap \Sigma < r - \dim \mathfrak{m}_x/\mathfrak{m}_x^2$$

for all $x \in X$ closed.

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[7]For example we could consider the condition that H_v is smooth over k', or geometrically irreducible over k'.

Since V globally generated $\mathcal{L}$, for every irreducible component X' of X there is a nonempty Zariski open of $\mathbf{A}^r$ such that the fibres of q over this open do not contain X'. (For example, if $x' \in X'$ is a closed point, then we can take the open corresponding to those vectors $v \in V$ such that $\psi(v)$ does not vanish at x'. This open will be the complement of a hyperplane in $\mathbf{A}^r_k$.) Let $U \subset \mathbf{A}^r$ be the (nonempty) intersection of these opens. Then the fibres of $q^{-1}(U) \to U$ are effective Cartier divisors on the fibres of $U \times_k X \to U$ (because a nonvanishing section of an invertible module on an integral scheme is a regular section). Hence the morphism $q^{-1}(U) \to U$ is flat by Divisors, Lemma 18.9. Thus for $x \in X$ closed and $v \in V = \mathbf{A}^r_k(k)$, if $(x,v) \in H_{univ}$, i.e., if $x \in H_v$ then q is smooth at (x,v) if and only if the fibre H_v is smooth at x, see Morphisms, Lemma 34.14.

Consider the image $\psi(v)_x$ in the stalk $\mathcal{L}_x$ of the section corresponding to $v \in V$. We have
$$x \in H_v \Leftrightarrow \psi(v)_x \in \mathfrak{m}_x \mathcal{L}_x$$
If this is true, then we have
$$H_v \text{ singular at } x \Leftrightarrow \psi(v)_x \in \mathfrak{m}_x^2 \mathcal{L}_x$$
Namely, $\psi(v)_x$ is not contained in $\mathfrak{m}_x^2 \mathcal{L}_x \Leftrightarrow$ the local equation for $H_v \subset X$ at x is not contained in $\mathfrak{m}_x^2 \Leftrightarrow \mathcal{O}_{H_v,x}$ is regular (Algebra, Lemma 106.3) $\Leftrightarrow H_v$ is smooth at x over k (Algebra, Lemma 140.5). We conclude that the closed points of $p^{-1}(x) \cap \Sigma$ correspond to those $v \in V$ such that $\psi(v)_x \in \mathfrak{m}_x^2 \mathcal{L}_x$. However, as $\varphi_{\mathcal{L},\psi}$ is an immersion the map
$$V \longrightarrow \mathcal{L}_x / \mathfrak{m}_x^2 \mathcal{L}_x$$
is surjective (small detail omitted). By the above, the closed points of the locus $p^{-1}(x) \cap \Sigma$ viewed as a subspace of V is the kernel of this map and hence has dimension $r - \dim \mathfrak{m}_x / \mathfrak{m}_x^2 - 1$ as desired. $\qquad\square$

48. Enriques-Severi-Zariski

In this section we prove some results of the form: twisting by a "very negative" invertible module kills low degree cohomology. We also deduce the connectedness of a hypersurface section of a normal proper scheme of dimension ≥ 2.

Lemma 48.1. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. If $\mathrm{Ass}(\mathcal{F})$ does not contain any closed points, then $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$ for $n \ll 0$.*

Proof. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$ let $\mathcal{P}(\mathcal{F})$ be the property: there exists an $n_0 \in \mathbf{Z}$ such that for $n \leq n_0$ every section s of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ has support consisting only of closed points. Since $\mathrm{Ass}(\mathcal{F}) = \mathrm{Ass}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ we see that it suffices to prove $\mathcal{P}$ holds for all coherent modules on X. To do this we will prove that conditions (1), (2), and (3) of Cohomology of Schemes, Lemma 12.8 are satisfied.

To see condition (1) suppose that
$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$
is a short exact sequence of coherent $\mathcal{O}_X$-modules such that we have $\mathcal{P}$ for $\mathcal{F}_i$, $i = 1, 2$. Let n_1, n_2 be the cutoffs we find. Let $\mathcal{F}_2' \subset \mathcal{F}_2$ be the maximal coherent submodule whose support is a finite set of closed points. Let $\mathcal{I} \subset \mathcal{O}_X$ be the annihilator of $\mathcal{F}_2'$. Since $\mathcal{L}$ is ample, we can find an $e > 0$ such that $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes e}$ is globally generated. Set $n_0 = \min(n_2, n_1 - e)$. Let $n \leq n_0$ and let t be a global

section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$. The image of t in $\mathcal{F}_2 \otimes \mathcal{L}^{\otimes n}$ falls into $\mathcal{F}_2' \otimes \mathcal{L}^{\otimes n}$ because $n \leq n_2$. Hence for any $s \in \Gamma(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes e})$ the product $t \otimes s$ lies in $\mathcal{F}_1 \otimes \mathcal{L}^{\otimes n+e}$. Thus $t \otimes s$ has support contained in the finite set of closed points in $\mathrm{Ass}(\mathcal{F}_1)$ because $n + e \leq n_1$. Since by our choice of e we may choose s invertible in any point not in the support of $\mathcal{F}_2'$ we conclude that the support of t is contained in the union of the finite set of closed points in $\mathrm{Ass}(\mathcal{F}_1)$ and the finite set of closed points in $\mathrm{Ass}(\mathcal{F}_2)$. This finishes the proof of condition (1).

Condition (2) is immediate.

For condition (3) we choose $\mathcal{G} = \mathcal{O}_Z$. In this case, if Z is a closed point of X, then there is nothing the show. If $\dim(Z) > 0$, then we will show that $\Gamma(Z, \mathcal{L}^{\otimes n}|_Z) = 0$ for $n < 0$. Namely, let s be a nonzero section of a negative power of $\mathcal{L}|_Z$. Choose a nonzero section t of a positive power of $\mathcal{L}|_Z$ (this is possible as $\mathcal{L}$ is ample, see Properties, Proposition 26.13). Then $s^{\deg(t)} \otimes t^{\deg(s)}$ is a nonzero global section of $\mathcal{O}_Z$ (because Z is integral) and hence a unit (Lemma 9.3). This implies that t is a trivializing section of a positive power of $\mathcal{L}$. Thus the function $n \mapsto \dim_k \Gamma(X, \mathcal{L}^{\otimes n})$ is bounded on an infinite set of positive integers which contradicts asymptotic Riemann-Roch (Proposition 45.13) since $\dim(Z) > 0$. $\qquad\square$

Lemma 48.2 (Enriques-Severi-Zariski). *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that for $x \in X$ closed we have $\mathrm{depth}(\mathcal{F}_x) \geq 2$. Then $H^1(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}) = 0$ for $m \ll 0$.*

Proof. Choose a closed immersion $i : X \to \mathbf{P}_k^n$ such that $i^* \mathcal{O}(1) \cong \mathcal{L}^{\otimes e}$ for some $e > 0$ (see Morphisms, Lemma 39.4). Then it suffices to prove the lemma for

$$\mathcal{G} = i_*(\mathcal{F} \oplus \mathcal{F} \otimes \mathcal{L} \oplus \ldots \oplus \mathcal{F} \otimes \mathcal{L}^{\otimes e-1}) \quad \text{and} \quad \mathcal{O}(1)$$

on $\mathbf{P}_k^n$. Namely, we have

$$H^1(\mathbf{P}_k^n, \mathcal{G}(m)) = \bigoplus\nolimits_{j=0,\ldots,e-1} H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes j+me})$$

by Cohomology of Schemes, Lemma 2.4. Also, if $y \in \mathbf{P}_k^n$ is a closed point then $\mathrm{depth}(\mathcal{G}_y) = \infty$ if $y \notin i(X)$ and $\mathrm{depth}(\mathcal{G}_y) = \mathrm{depth}(\mathcal{F}_x)$ if $y = i(x)$ because in this case $\mathcal{G}_y \cong \mathcal{F}_x^{\oplus e}$ as a module over $\mathcal{O}_{\mathbf{P}_k^n,x}$ and we can use for example Algebra, Lemma 72.11 to get the equality.

Assume $X = \mathbf{P}_k^n$ and $\mathcal{L} = \mathcal{O}(1)$ and k is infinite. Choose $s \in H^0(\mathbf{P}_k^1, \mathcal{O}(1))$ which determines an exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{s} \mathcal{F} \to \mathcal{G} \to 0$$

as in Lemma 35.3. Since the map $\mathcal{F}(-1) \to \mathcal{F}$ is affine locally given by multiplying by a nonzerodivisor on $\mathcal{F}$ we see that for $x \in \mathbf{P}_k^n$ closed we have $\mathrm{depth}(\mathcal{G}_x) \geq 1$, see Algebra, Lemma 72.7. Hence by Lemma 48.1 we have $H^0(\mathcal{G}(m)) = 0$ for $m \ll 0$. Looking at the long exact sequence of cohomology after twisting (see Remark 35.5) we find that the sequence of numbers

$$\dim H^1(\mathbf{P}_k^n, \mathcal{F}(m))$$

stabilizes for $m \leq m_0$ for some integer m_0. Let N be the common dimension of these spaces for $m \leq m_0$. We have to show $N = 0$.

For $d > 0$ and $m \leq m_0$ consider the bilinear map

$$H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \times H^1(\mathbf{P}^n_k, \mathcal{F}(m-d)) \longrightarrow H^1(\mathbf{P}^n_k, \mathcal{F}(m))$$

By linear algebra, there is a codimension $\leq N^2$ subspace $V_m \subset H^0(\mathbf{P}^n_k, \mathcal{O}(d))$ such that multiplication by $s' \in V_m$ annihilates $H^1(\mathbf{P}^n_k, \mathcal{F}(m-d))$. Observe that for $m' < m \leq m_0$ the diagram

$$
\begin{array}{ccc}
H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \times H^1(\mathbf{P}^n_k, \mathcal{F}(m'-d)) & \longrightarrow & H^1(\mathbf{P}^n_k, \mathcal{F}(m')) \\
\Big\downarrow{\scriptstyle 1 \times s^{m'-m}} & & \Big\downarrow{\scriptstyle s^{m'-m}} \\
H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \times H^1(\mathbf{P}^n_k, \mathcal{F}(m-d)) & \longrightarrow & H^1(\mathbf{P}^n_k, \mathcal{F}(m))
\end{array}
$$

commutes with isomorphisms going vertically. Thus $V_m = V$ is independent of $m \leq m_0$. For $x \in \mathrm{Ass}(\mathcal{F})$ set $Z = \overline{\{x\}}$. For d large enough the linear map

$$H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \to H^0(Z, \mathcal{O}(d)|_Z)$$

has rank $> N^2$ because $\dim(Z) \geq 1$ (for example this follows from asymptotic Riemann-Roch and ampleness $\mathcal{O}(1)$; details omitted). Hence we can find $s' \in V$ such that s' does not vanish in any associated point of $\mathcal{F}$ (use that the set of associated points is finite). Then we obtain

$$0 \to \mathcal{F}(-d) \xrightarrow{s'} \mathcal{F} \to \mathcal{G}' \to 0$$

and as before we conclude as before that multiplication by s' on $H^1(\mathbf{P}^n_k, \mathcal{F}(m-d))$ is injective for $m \ll 0$. This contradicts the choice of s' unless $N = 0$ as desired.

We still have to treat the case where k is finite. In this case let K/k be any infinite algebraic field extension. Denote $\mathcal{F}_K$ and $\mathcal{L}_K$ the pullbacks of $\mathcal{F}$ and $\mathcal{L}$ to $X_K = \mathrm{Spec}(K) \times_{\mathrm{Spec}(k)} X$. We have

$$H^1(X_K, \mathcal{F}_K \otimes \mathcal{L}_K^{\otimes m}) = H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \otimes_k K$$

by Cohomology of Schemes, Lemma 5.2. On the other hand, a closed point x_K of X_K maps to a closed point x of X because K/k is an algebraic extension. The ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_K,x_K}$ is flat (Lemma 5.1). Hence we have

$$\mathrm{depth}(\mathcal{F}_{x_K}) = \mathrm{depth}(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_K,x_K}) \geq \mathrm{depth}(\mathcal{F}_x)$$

by Algebra, Lemma 163.1 (in fact equality holds here but we don't need it). Therefore the result over k follows from the result over the infinite field K and the proof is complete. $\square$

Lemma 48.3. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Assume*

    (1) *s is a regular section (Divisors, Definition 14.6),*
    (2) *for every closed point $x \in X$ we have $\mathrm{depth}(\mathcal{O}_{X,x}) \geq 2$, and*
    (3) *X is connected.*

Then the zero scheme $Z(s)$ of s is connected.

Proof. Since s is a regular section, so is $s^n \in \Gamma(X, \mathcal{L}^{\otimes n})$ for all $n > 1$. Moreover, the inclusion morphism $Z(s) \to Z(s^n)$ is a bijection on underlying topological spaces. Hence if $Z(s)$ is disconnected, so is $Z(s^n)$. Now consider the canonical short exact sequence

$$0 \to \mathcal{L}^{\otimes -n} \xrightarrow{s^n} \mathcal{O}_X \to \mathcal{O}_{Z(s^n)} \to 0$$

Consider the k-algebra $R_n = \Gamma(X, \mathcal{O}_{Z(s^n)})$. If $Z(s)$ is disconnected, i.e., $Z(s^n)$ is disconnected, then either R_n is zero in case $Z(s^n) = \emptyset$ or R_n contains a nontrivial idempotent in case $Z(s^n) = U \amalg V$ with $U, V \subset Z(s^n)$ open and nonempty (the reader may wish to consult Lemma 9.3). Thus the map $\Gamma(X, \mathcal{O}_X) \to R_n$ cannot be an isomorphism. It follows that either $H^0(X, \mathcal{L}^{\otimes -n})$ or $H^1(X, \mathcal{L}^{\otimes -n})$ is nonzero for infinitely many positive n. This contradicts Lemma 48.1 or 48.2 and the proof is complete. $\square$

49. Other chapters

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