MORPHISMS OF SCHEMES

Contents

1. Introduction

In this chapter we introduce some types of morphisms of schemes. A basic reference is [DG67].

2. Closed immersions

In this section we elucidate some of the results obtained previously on closed immersions of schemes. Recall that a morphism of schemes $i : Z \to X$ is defined to be a closed immersion if (a) i induces a homeomorphism onto a closed subset of X, (b) $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and (c) the kernel of $i^\sharp$ is locally generated by sections, see Schemes, Definitions 10.2 and 4.1. It turns out that, given that Z and X are schemes, there are many different ways of characterizing a closed immersion.

Lemma 2.1. *Let $i : Z \to X$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism i is a closed immersion.*
(2) *For every affine open $\mathrm{Spec}(R) = U \subset X$, there exists an ideal $I \subset R$ such that $i^{-1}(U) = \mathrm{Spec}(R/I)$ as schemes over $U = \mathrm{Spec}(R)$.*
(3) *There exists an affine open covering $X = \bigcup_{j \in J} U_j$, $U_j = \mathrm{Spec}(R_j)$ and for every $j \in J$ there exists an ideal $I_j \subset R_j$ such that $i^{-1}(U_j) = \mathrm{Spec}(R_j/I_j)$ as schemes over $U_j = \mathrm{Spec}(R_j)$.*
(4) *The morphism i induces a homeomorphism of Z with a closed subset of X and $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.*
(5) *The morphism i induces a homeomorphism of Z with a closed subset of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $\mathrm{Ker}(i^\sharp) \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals.*

(6) *The morphism i induces a homeomorphism of Z with a closed subset of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $\mathrm{Ker}(i^\sharp) \subset \mathcal{O}_X$ is a sheaf of ideals which is locally generated by sections.*

Proof. Condition (6) is our definition of a closed immersion, see Schemes, Definitions 4.1 and 10.2. So (6) $\Leftrightarrow$ (1). We have (1) $\Rightarrow$ (2) by Schemes, Lemma 10.1. Trivially (2) $\Rightarrow$ (3).

Assume (3). Each of the morphisms $\mathrm{Spec}(R_j/I_j) \to \mathrm{Spec}(R_j)$ is a closed immersion, see Schemes, Example 8.1. Hence $i^{-1}(U_j) \to U_j$ is a homeomorphism onto its image and $i^\sharp|_{U_j}$ is surjective. Hence i is a homeomorphism onto its image and $i^\sharp$ is surjective since this may be checked locally. We conclude that (3) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (1) is Schemes, Lemma 24.2. The implication (5) $\Rightarrow$ (6) is trivial. And the implication (6) $\Rightarrow$ (5) follows from Schemes, Lemma 10.1. $\square$

Lemma 2.2. *Let X be a scheme. Let $i : Z \to X$ and $i' : Z' \to X$ be closed immersions and consider the ideal sheaves $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ and $\mathcal{I}' = \mathrm{Ker}((i')^\sharp)$ of $\mathcal{O}_X$.*

(1) *The morphism $i : Z \to X$ factors as $Z \to Z' \to X$ for some $a : Z \to Z'$ if and only if $\mathcal{I}' \subset \mathcal{I}$. If this happens, then a is a closed immersion.*
(2) *We have $Z \cong Z'$ over X if and only if $\mathcal{I} = \mathcal{I}'$.*

Proof. This follows from our discussion of closed subspaces in Schemes, Section 4 especially Schemes, Lemmas 4.5 and 4.6. It also follows in a straightforward way from characterization (3) in Lemma 2.1 above. $\square$

Lemma 2.3. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals. The following are equivalent:*

(1) *$\mathcal{I}$ is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules,*
(2) *$\mathcal{I}$ is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules, and*
(3) *there exists a closed immersion $i : Z \to X$ of schemes whose corresponding sheaf of ideals $\mathrm{Ker}(i^\sharp)$ is equal to $\mathcal{I}$.*

Proof. The equivalence of (1) and (2) is immediate from Schemes, Lemma 10.1. If (1) holds, then there is a closed subspace $i : Z \to X$ with $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ by Schemes, Definition 4.4 and Example 4.3. By Schemes, Lemma 10.1 this is a closed immersion of schemes and (3) holds. Conversely, if (3) holds, then (2) holds by Schemes, Lemma 10.1 (which applies because a closed immersion of schemes is a fortiori a closed immersion of locally ringed spaces). $\square$

Lemma 2.4. *The base change of a closed immersion is a closed immersion.*

Proof. See Schemes, Lemma 18.2. $\square$

Lemma 2.5. *A composition of closed immersions is a closed immersion.*

Proof. We have seen this in Schemes, Lemma 24.3, but here is another proof. Namely, it follows from the characterization (3) of closed immersions in Lemma 2.1. Since if $I \subset R$ is an ideal, and $\overline{J} \subset R/I$ is an ideal, then $\overline{J} = J/I$ for some ideal $J \subset R$ which contains I and $(R/I)/\overline{J} = R/J$. $\square$

Lemma 2.6. *A closed immersion is quasi-compact.*

Proof. This lemma is a duplicate of Schemes, Lemma 19.5. $\square$

Lemma 2.7. *A closed immersion is separated.*

Proof. This lemma is a special case of Schemes, Lemma 23.8. $\square$

3. Immersions

In this section we collect some facts on immersions.

Lemma 3.1. *Let $Z \to Y \to X$ be morphisms of schemes.*

(1) *If $Z \to X$ is an immersion, then $Z \to Y$ is an immersion.*
(2) *If $Z \to X$ is a quasi-compact immersion and $Y \to X$ is quasi-separated, then $Z \to Y$ is a quasi-compact immersion.*
(3) *If $Z \to X$ is a closed immersion and $Y \to X$ is separated, then $Z \to Y$ is a closed immersion.*

Proof. In each case the proof is to contemplate the commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[rd] & Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d] \\ & Y \arrow[r] & X \end{tikzcd}$$

where the composition of the top horizontal arrows is the identity. Let us prove (1). The first horizontal arrow is a section of $Y \times_X Z \to Z$, whence an immersion by Schemes, Lemma 21.11. The arrow $Y \times_X Z \to Y$ is a base change of $Z \to X$ hence an immersion (Schemes, Lemma 18.2). Finally, a composition of immersions is an immersion (Schemes, Lemma 24.3). This proves (1). The other two results are proved in exactly the same manner. $\square$

Lemma 3.2. *Let $h : Z \to X$ be an immersion. If h is quasi-compact, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : \overline{Z} \to X$ a closed immersion.*

Proof. Note that h is quasi-compact and quasi-separated (see Schemes, Lemma 23.8). Hence $h_*\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules by Schemes, Lemma 24.1. This implies that $\mathcal{I} = \text{Ker}(\mathcal{O}_X \to h_*\mathcal{O}_Z)$ is a quasi-coherent sheaf of ideals, see Schemes, Section 24. Let $\overline{Z} \subset X$ be the closed subscheme corresponding to $\mathcal{I}$, see Lemma 2.3. By Schemes, Lemma 4.6 the morphism h factors as $h = i \circ j$ where $i : \overline{Z} \to X$ is the inclusion morphism. To see that j is an open immersion, choose an open subscheme $U \subset X$ such that h induces a closed immersion of Z into U. Then it is clear that $\mathcal{I}|_U$ is the sheaf of ideals corresponding to the closed immersion $Z \to U$. Hence we see that $Z = \overline{Z} \cap U$. $\square$

Lemma 3.3. *Let $h : Z \to X$ be an immersion. If Z is reduced, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : \overline{Z} \to X$ a closed immersion.*

Proof. Let $\overline{Z} \subset X$ be the closure of $h(Z)$ with the reduced induced closed subscheme structure, see Schemes, Definition 12.5. By Schemes, Lemma 12.7 the morphism h factors as $h = i \circ j$ with $i : \overline{Z} \to X$ the inclusion morphism and $j : Z \to \overline{Z}$. From the definition of an immersion we see there exists an open subscheme $U \subset X$ such that h factors through a closed immersion into U. Hence $\overline{Z} \cap U$ and $h(Z)$ are reduced closed subschemes of U with the same underlying closed set. Hence by the uniqueness in Schemes, Lemma 12.4 we see that $h(Z) \cong \overline{Z} \cap U$. So j induces an isomorphism of Z with $\overline{Z} \cap U$. In other words j is an open immersion. $\square$

Example 3.4. Here is an example of an immersion which is not a composition of an open immersion followed by a closed immersion. Let k be a field. Let $X = \text{Spec}(k[x_1, x_2, x_3, \ldots])$. Let $U = \bigcup_{n=1}^{\infty} D(x_n)$. Then $U \to X$ is an open immersion. Consider the ideals

$$I_n = (x_1^n, x_2^n, \ldots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \ldots) \subset k[x_1, x_2, x_3, \ldots][1/x_n].$$

Note that $I_n k[x_1, x_2, x_3, \ldots][1/x_n x_m] = (1)$ for any $m \neq n$. Hence the quasi-coherent ideals $\widetilde{I_n}$ on $D(x_n)$ agree on $D(x_n x_m)$, namely $\widetilde{I_n}|_{D(x_n x_m)} = \mathcal{O}_{D(x_n x_m)}$ if $n \neq m$. Hence these ideals glue to a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_U$. Let $Z \subset U$ be the closed subscheme corresponding to $\mathcal{I}$. Thus $Z \to X$ is an immersion.

We claim that we cannot factor $Z \to X$ as $Z \to \overline{Z} \to X$, where $\overline{Z} \to X$ is closed and $Z \to \overline{Z}$ is open. Namely, $\overline{Z}$ would have to be defined by an ideal $I \subset k[x_1, x_2, x_3, \ldots]$ such that $I_n = Ik[x_1, x_2, x_3, \ldots][1/x_n]$. But the only element $f \in k[x_1, x_2, x_3, \ldots]$ which ends up in all I_n is 0! Hence I does not exist.

Lemma 3.5. *Let $f : Y \to X$ be a morphism of schemes. If for all $y \in Y$ there is an open subscheme $f(y) \in U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is an immersion, then f is an immersion.*

Proof. This statement follows readily from the discussion of closed subschemes at the end of Schemes, Section 10 but we will also give a detailed proof. Let $Z \subset X$ be the closure of $f(Y)$. Since taking closures commutes with restricting to opens, we see from the assumption that $f(Y) \subset Z$ is open. Hence $Z' = Z \setminus f(Y)$ is closed. Hence $X' = X \setminus Z'$ is an open subscheme of X and f factors as $f : Y \to X'$ followed by the inclusion. If $y \in Y$ and $U \subset X$ is as in the statement of the lemma, then $U' = X' \cap U$ is an open neighbourhood of $f'(y)$ such that $(f')^{-1}(U') \to U'$ is an immersion (Lemma 3.1) with closed image. Hence it is a closed immersion, see Schemes, Lemma 10.4. Since being a closed immersion is local on the target (for example by Lemma 2.1) we conclude that f' is a closed immersion as desired. $\square$

4. Closed immersions and quasi-coherent sheaves

The following lemma finally does for quasi-coherent sheaves on schemes what Modules, Lemma 6.1 does for abelian sheaves. See also the discussion in Modules, Section 13.

Lemma 4.1. *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor*

$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. A closed immersion is quasi-compact and separated, see Lemmas 2.6 and 2.7. Hence Schemes, Lemma 24.1 applies and the pushforward of a quasi-coherent sheaf on Z is indeed a quasi-coherent sheaf on X.

By Modules, Lemma 13.4 the functor i_* is fully faithful.

Now we turn to the description of the essential image of the functor i_*. We have $\mathcal{I}(i_*\mathcal{F}) = 0$ for any quasi-coherent $\mathcal{O}_Z$-module, for example by Modules, Lemma

13.4. Next, suppose that $\mathcal{G}$ is any quasi-coherent $\mathcal{O}_X$-module such that $\mathcal{I}\mathcal{G} = 0$. It suffices to show that the canonical map

$$\mathcal{G} \longrightarrow i_*i^*\mathcal{G}$$

is an isomorphism[1]. In the case of schemes and quasi-coherent modules, working affine locally on X and using Lemma 2.1 and Schemes, Lemma 7.3 it suffices to prove the following algebraic statement: Given a ring R, an ideal I and an R-module N such that $IN = 0$ the canonical map

$$N \longrightarrow N \otimes_R R/I, \quad n \longmapsto n \otimes 1$$

is an isomorphism of R-modules. Proof of this easy algebra fact is omitted. □

Let $i : Z \to X$ be a closed immersion. Because of the lemma above we often, by abuse of notation, denote $\mathcal{F}$ the sheaf $i_*\mathcal{F}$ on X.

Lemma 4.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}$ be a $\mathcal{O}_X$-submodule. There exists a unique quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}' \subset \mathcal{G}$ with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ the map*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular $\mathcal{G}'$ is the largest quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ contained in $\mathcal{G}$.

Proof. Let $\mathcal{G}_a$, $a \in A$ be the set of quasi-coherent $\mathcal{O}_X$-submodules contained in $\mathcal{G}$. Then the image $\mathcal{G}'$ of

$$\bigoplus\nolimits_{a \in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 24. The module $\mathcal{G}'$ is contained in $\mathcal{G}$. Hence this is the largest quasi-coherent $\mathcal{O}_X$-module contained in $\mathcal{G}$.

To prove the formula, let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\alpha : \mathcal{H} \to \mathcal{G}$ be an $\mathcal{O}_X$-module map. The image of the composition $\mathcal{H} \to \mathcal{G} \to \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in $\mathcal{G}'$. Hence α factors through $\mathcal{G}'$ as desired. □

Lemma 4.3. *Let $i : Z \to X$ be a closed immersion of schemes. There is a functor*[2] *$i^! : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ which is a right adjoint to i_*. (Compare Modules, Lemma 6.3.)*

Proof. Given quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ we consider the subsheaf $\mathcal{H}_Z(\mathcal{G})$ of $\mathcal{G}$ of local sections annihilated by $\mathcal{I}$. By Lemma 4.2 there is a canonical largest quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}_Z(\mathcal{G})'$. By construction we have

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}_Z(\mathcal{G})') = \mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G})$$

for any quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$. Hence we can set $i^!\mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$. Details omitted. □

[1] This was proved in a more general situation in the proof of Modules, Lemma 13.4.
[2] This is likely nonstandard notation.

Using the 1-to-1 corresponding between quasi-coherent sheaves of ideals and closed subschemes (see Lemma 2.3) we can define scheme theoretic intersections and unions of closed subschemes.

Definition 4.4. Let X be a scheme. Let $Z, Y \subset X$ be closed subschemes corresponding to quasi-coherent ideal sheaves $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$. The *scheme theoretic intersection* of Z and Y is the closed subscheme of X cut out by $\mathcal{I} + \mathcal{J}$. The *scheme theoretic union* of Z and Y is the closed subscheme of X cut out by $\mathcal{I} \cap \mathcal{J}$.

Lemma 4.5. *Let X be a scheme. Let $Z, Y \subset X$ be closed subschemes. Let $Z \cap Y$ be the scheme theoretic intersection of Z and Y. Then $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions and*

$$\begin{tikzcd} Z \cap Y \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

is a cartesian diagram of schemes, i.e., $Z \cap Y = Z \times_X Y$.

Proof. The morphisms $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions by Lemma 2.2. Let $U = \text{Spec}(A)$ be an affine open of X and let $Z \cap U$ and $Y \cap U$ correspond to the ideals $I \subset A$ and $J \subset A$. Then $Z \cap Y \cap U$ corresponds to $I + J \subset A$. Since $A/I \otimes_A A/J = A/(I + J)$ we see that the diagram is cartesian by our description of fibre products of schemes in Schemes, Section 17. $\square$

Lemma 4.6. *Let S be a scheme. Let $X, Y \subset S$ be closed subschemes. Let $X \cup Y$ be the scheme theoretic union of X and Y. Let $X \cap Y$ be the scheme theoretic intersection of X and Y. Then $X \to X \cup Y$ and $Y \to X \cup Y$ are closed immersions, there is a short exact sequence*

$$0 \to \mathcal{O}_{X \cup Y} \to \mathcal{O}_X \times \mathcal{O}_Y \to \mathcal{O}_{X \cap Y} \to 0$$

of $\mathcal{O}_S$-modules, and the diagram

$$\begin{tikzcd} X \cap Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \cup Y \end{tikzcd}$$

is cocartesian in the category of schemes, i.e., $X \cup Y = X \amalg_{X \cap Y} Y$.

Proof. The morphisms $X \to X \cup Y$ and $Y \to X \cup Y$ are closed immersions by Lemma 2.2. In the short exact sequence we use the equivalence of Lemma 4.1 to think of quasi-coherent modules on closed subschemes of S as quasi-coherent modules on S. For the first map in the sequence we use the canonical maps $\mathcal{O}_{X \cup Y} \to \mathcal{O}_X$ and $\mathcal{O}_{X \cup Y} \to \mathcal{O}_Y$ and for the second map we use the canonical map $\mathcal{O}_X \to \mathcal{O}_{X \cap Y}$ and the negative of the canonical map $\mathcal{O}_Y \to \mathcal{O}_{X \cap Y}$. Then to check exactness we may work affine locally. Let $U = \text{Spec}(A)$ be an affine open of S and let $X \cap U$ and $Y \cap U$ correspond to the ideals $I \subset A$ and $J \subset A$. Then $(X \cup Y) \cap U$ corresponds to $I \cap J \subset A$ and $X \cap Y \cap U$ corresponds to $I + J \subset A$. Thus exactness follows from the exactness of

$$0 \to A/I \cap J \to A/I \times A/J \to A/(I + J) \to 0$$

To show the diagram is cocartesian, suppose we are given a scheme T and morphisms of schemes $f : X \to T$, $g : Y \to T$ agreeing as morphisms $X \cap Y \to T$.

Goal: Show there exists a unique morphism $h : X \cup Y \to T$ agreeing with f and g. To construct h we may work affine locally on $X \cup Y$, see Schemes, Section 14. If $s \in X$, $s \notin Y$, then $X \to X \cup Y$ is an isomorphism in a neighbourhood of s and it is clear how to construct h. Similarly for $s \in Y$, $s \notin X$. For $s \in X \cap Y$ we can pick an affine open $V = \mathrm{Spec}(B) \subset T$ containing $f(s) = g(s)$. Then we can choose an affine open $U = \mathrm{Spec}(A) \subset S$ containing s such that $f(X \cap U)$ and $g(Y \cap U)$ are contained in V. The morphisms $f|_{X\cap U}$ and $g|_{Y\cap V}$ into V correspond to ring maps

$$B \to A/I \quad \text{and} \quad B \to A/J$$

which agree as maps into $A/(I + J)$. By the short exact sequence displayed above there is a unique lift of these ring homomorphism to a ring map $B \to A/I \cap J$ as desired. ☐

5. Supports of modules

In this section we collect some elementary results on supports of quasi-coherent modules on schemes. Recall that the support of a sheaf of modules has been defined in Modules, Section 5. On the other hand, the support of a module was defined in Algebra, Section 62. These match.

Lemma 5.1. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent*

1. *$\mathfrak{p}$ is in the support of M, and*
2. *x is in the support of $\mathcal{F}$.*

Proof. This follows from the equality $\mathcal{F}_x = M_{\mathfrak{p}}$, see Schemes, Lemma 5.4 and the definitions. ☐

Lemma 5.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. The support of $\mathcal{F}$ is closed under specialization.*

Proof. If $x' \rightsquigarrow x$ is a specialization and $\mathcal{F}_x = 0$ then $\mathcal{F}_{x'}$ is zero, as $\mathcal{F}_{x'}$ is a localization of the module $\mathcal{F}_x$. Hence the complement of $\mathrm{Supp}(\mathcal{F})$ is closed under generalization. ☐

For finite type quasi-coherent modules the support is closed, can be checked on fibres, and commutes with base change.

Lemma 5.3. *Let $\mathcal{F}$ be a finite type quasi-coherent module on a scheme X. Then*

1. *The support of $\mathcal{F}$ is closed.*
2. *For $x \in X$ we have*

$$x \in \mathit{Supp}(\mathcal{F}) \Leftrightarrow \mathcal{F}_x \neq 0 \Leftrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0.$$

3. *For any morphism of schemes $f : Y \to X$ the pullback $f^*\mathcal{F}$ is of finite type as well and we have $\mathit{Supp}(f^*\mathcal{F}) = f^{-1}(\mathit{Supp}(\mathcal{F}))$.*

Proof. Part (1) is a reformulation of Modules, Lemma 9.6. You can also combine Lemma 5.1, Properties, Lemma 16.1, and Algebra, Lemma 40.5 to see this. The first equivalence in (2) is the definition of support, and the second equivalence follows from Nakayama's lemma, see Algebra, Lemma 20.1. Let $f : Y \to X$ be a morphism of schemes. Note that $f^*\mathcal{F}$ is of finite type by Modules, Lemma 9.2. For the final assertion, let $y \in Y$ with image $x \in X$. Recall that

$$(f^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y},$$

see Sheaves, Lemma 26.4. Hence $(f^*\mathcal{F})_y \otimes \kappa(y)$ is nonzero if and only if $\mathcal{F}_x \otimes \kappa(x)$ is nonzero. By (2) this implies $x \in \text{Supp}(\mathcal{F})$ if and only if $y \in \text{Supp}(f^*\mathcal{F})$, which is the content of assertion (3). $\square$

Lemma 5.4. *Let $\mathcal{F}$ be a finite type quasi-coherent module on a scheme X. There exists a smallest closed subscheme $i : Z \to X$ such that there exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with $i_*\mathcal{G} \cong \mathcal{F}$. Moreover:*

1. *If $\text{Spec}(A) \subset X$ is any affine open, and $\mathcal{F}|_{\text{Spec}(A)} = \widetilde{M}$ then $Z \cap \text{Spec}(A) = \text{Spec}(A/I)$ where $I = Ann_A(M)$.*
2. *The quasi-coherent sheaf $\mathcal{G}$ is unique up to unique isomorphism.*
3. *The quasi-coherent sheaf $\mathcal{G}$ is of finite type.*
4. *The support of $\mathcal{G}$ and of $\mathcal{F}$ is Z.*

Proof. Suppose that $i' : Z' \to X$ is a closed subscheme which satisfies the description on open affines from the lemma. Then by Lemma 4.1 we see that $\mathcal{F} \cong i'_*\mathcal{G}'$ for some unique quasi-coherent sheaf $\mathcal{G}'$ on Z'. Furthermore, it is clear that Z' is the smallest closed subscheme with this property (by the same lemma). Finally, using Properties, Lemma 16.1 and Algebra, Lemma 5.5 it follows that $\mathcal{G}'$ is of finite type. We have $\text{Supp}(\mathcal{G}') = Z$ by Algebra, Lemma 40.5. Hence, in order to prove the lemma it suffices to show that the characterization in (1) actually does define a closed subscheme. And, in order to do this it suffices to prove that the given rule produces a quasi-coherent sheaf of ideals, see Lemma 2.3. This comes down to the following algebra fact: If A is a ring, $f \in A$, and M is a finite A-module, then $\text{Ann}_A(M)_f = \text{Ann}_{A_f}(M_f)$. We omit the proof. $\square$

Definition 5.5. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. The *scheme theoretic support of $\mathcal{F}$* is the closed subscheme $Z \subset X$ constructed in Lemma 5.4.

In this situation we often think of $\mathcal{F}$ as a quasi-coherent sheaf of finite type on Z (via the equivalence of categories of Lemma 4.1).

6. Scheme theoretic image

Caution: Some of the material in this section is ultra-general and behaves differently from what you might expect.

Lemma 6.1. *Let $f : X \to Y$ be a morphism of schemes. There exists a closed subscheme $Z \subset Y$ such that f factors through Z and such that for any other closed subscheme $Z' \subset Y$ such that f factors through Z' we have $Z \subset Z'$.*

Proof. Let $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. If $\mathcal{I}$ is quasi-coherent then we just take Z to be the closed subscheme determined by $\mathcal{I}$, see Lemma 2.3. This works by Schemes, Lemma 4.6. In general the same lemma requires us to show that there exists a largest quasi-coherent sheaf of ideals $\mathcal{I}'$ contained in $\mathcal{I}$. This follows from Lemma 4.2. $\square$

Definition 6.2. Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme $Z \subset Y$ through which f factors, see Lemma 6.1 above.

For a morphism $f : X \to Y$ of schemes with scheme theoretic image Z we often denote $f : X \to Z$ the factorization of f through its scheme theoretic image. If the morphism f is not quasi-compact, then (in general)

(1) the set theoretic inclusion $\overline{f(X)} \subset Z$ is not an equality, i.e., $f(X) \subset Z$ is not a dense subset, and
(2) the construction of the scheme theoretic image does not commute with restriction to open subschemes to Y.

In Examples, Section 23 the reader finds an example for both phenomena. These phenomena can arise even for immersions, see Examples, Section 25. However, the next lemma shows that both disasters are avoided when the morphism is quasi-compact.

Lemma 6.3. *Let $f : X \to Y$ be a morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of f. If f is quasi-compact then*

(1) *the sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent,*
(2) *the scheme theoretic image Z is the closed subscheme determined by $\mathcal{I}$,*
(3) *for any open $U \subset Y$ the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is equal to $Z \cap U$, and*
(4) *the image $f(X) \subset Z$ is a dense subset of Z, in other words the morphism $X \to Z$ is dominant (see Definition 8.1).*

Proof. Part (4) follows from part (3). To show (3) it suffices to prove (1) since the formation of $\mathcal{I}$ commutes with restriction to open subschemes of Y. And if (1) holds then in the proof of Lemma 6.1 we showed (2). Thus it suffices to prove that $\mathcal{I}$ is quasi-coherent. Since the property of being quasi-coherent is local we may assume Y is affine. As f is quasi-compact, we can find a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. Denote f' the composition

$$X' = \coprod U_i \longrightarrow X \longrightarrow Y.$$

Then $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_{X'}$, and hence $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f'_*\mathcal{O}_{X'})$. By Schemes, Lemma 24.1 the sheaf $f'_*\mathcal{O}_{X'}$ is quasi-coherent on Y. Hence we win. $\square$

Example 6.4. If $A \to B$ is a ring map with kernel I, then the scheme theoretic image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is the closed subscheme $\mathrm{Spec}(A/I)$ of $\mathrm{Spec}(A)$. This follows from Lemma 6.3.

If the morphism is quasi-compact, then the scheme theoretic image only adds points which are specializations of points in the image.

Lemma 6.5. *Let $f : X \to Y$ be a quasi-compact morphism. Let Z be the scheme theoretic image of f. Let $z \in Z$[3]. There exists a valuation ring A with fraction field*

[3]By Lemma 6.3 set-theoretically Z agrees with the closure of $f(X)$ in Y.

K and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[rr] \arrow[d] & & X \arrow[dl] \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Z \arrow[r] & Y \end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A)$ *maps to z. In particular any point of Z is the specialization of a point of $f(X)$.*

Proof. Let $z \in \mathrm{Spec}(R) = V \subset Y$ be an affine open neighbourhood of z. By Lemma 6.3 the intersection $Z \cap V$ is the scheme theoretic image of $f^{-1}(V) \to V$. Hence we may replace Y by V and assume $Y = \mathrm{Spec}(R)$ is affine. In this case X is quasi-compact as f is quasi-compact. Say $X = U_1 \cup \ldots \cup U_n$ is a finite affine open covering. Write $U_i = \mathrm{Spec}(A_i)$. Let $I = \mathrm{Ker}(R \to A_1 \times \ldots \times A_n)$. By Lemma 6.3 again we see that Z corresponds to the closed subscheme $\mathrm{Spec}(R/I)$ of Y. If $\mathfrak p \subset R$ is the prime corresponding to z, then we see that $I_{\mathfrak p} \subset R_{\mathfrak p}$ is not an equality. Hence (as localization is exact, see Algebra, Proposition 9.12) we see that $R_{\mathfrak p} \to (A_1)_{\mathfrak p} \times \ldots \times (A_n)_{\mathfrak p}$ is not zero. Hence one of the rings $(A_i)_{\mathfrak p}$ is not zero. Hence there exists an i and a prime $\mathfrak q_i \subset A_i$ lying over a prime $\mathfrak p_i \subset \mathfrak p$. By Algebra, Lemma 50.2 we can choose a valuation ring $A \subset K = \kappa(\mathfrak q_i)$ dominating the local ring $R_{\mathfrak p}/\mathfrak p_i R_{\mathfrak p} \subset \kappa(\mathfrak q_i)$. This gives the desired diagram. Some details omitted. $\square$

Lemma 6.6. *Let*

$$\begin{tikzcd} X_1 \arrow[r, "f_1"'] \arrow[d] & Y_1 \arrow[d] \\ X_2 \arrow[r, "f_2"] & Y_2 \end{tikzcd}$$

be a commutative diagram of schemes. Let $Z_i \subset Y_i$, $i = 1, 2$ be the scheme theoretic image of f_i. Then the morphism $Y_1 \to Y_2$ induces a morphism $Z_1 \to Z_2$ and a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ X_2 \arrow[r] & Z_2 \arrow[r] & Y_2 \end{tikzcd}$$

Proof. The scheme theoretic inverse image of Z_2 in Y_1 is a closed subscheme of Y_1 through which f_1 factors. Hence Z_1 is contained in this. This proves the lemma. $\square$

Lemma 6.7. *Let $f : X \to Y$ be a morphism of schemes. If X is reduced, then the scheme theoretic image of f is the reduced induced scheme structure on $\overline{f(X)}$.*

Proof. This is true because the reduced induced scheme structure on $\overline{f(X)}$ is clearly the smallest closed subscheme of Y through which f factors, see Schemes, Lemma 12.7. $\square$

Lemma 6.8. *Let $f : X \to Y$ be a separated morphism of schemes. Let $V \subset Y$ be a retrocompact open. Let $s : V \to X$ be a morphism such that $f \circ s = id_V$. Let Y' be the scheme theoretic image of s. Then $Y' \to Y$ is an isomorphism over V.*

Proof. The assumption that V is retrocompact in Y (Topology, Definition 12.1) means that $V \to Y$ is a quasi-compact morphism. By Schemes, Lemma 21.14 the morphism $s : V \to X$ is quasi-compact. Hence the construction of the scheme theoretic image Y' of s commutes with restriction to opens by Lemma 6.3. In particular, we see that $Y' \cap f^{-1}(V)$ is the scheme theoretic image of a section of the separated morphism $f^{-1}(V) \to V$. Since a section of a separated morphism is a closed immersion (Schemes, Lemma 21.11), we conclude that $Y' \cap f^{-1}(V) \to V$ is an isomorphism as desired. □

7. Scheme theoretic closure and density

We take the following definition from [DG67, IV, Definition 11.10.2].

Definition 7.1. Let X be a scheme. Let $U \subset X$ be an open subscheme.

(1) The scheme theoretic image of the morphism $U \to X$ is called the *scheme theoretic closure of U in X*.
(2) We say U is *scheme theoretically dense in X* if for every open $V \subset X$ the scheme theoretic closure of $U \cap V$ in V is equal to V.

With this definition it is **not** the case that U is scheme theoretically dense in X if and only if the scheme theoretic closure of U is X, see Example 7.2. This is somewhat inelegant; but see Lemmas 7.3 and 7.8 below. On the other hand, with this definition U is scheme theoretically dense in X if and only if for every $V \subset X$ open the ring map $\mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is injective, see Lemma 7.5 below. In particular we see that scheme theoretically dense implies dense which is pleasing.

Example 7.2. Here is an example where scheme theoretic closure being X does not imply dense for the underlying topological spaces. Let k be a field. Set $A = k[x, z_1, z_2, \ldots]/(x^n z_n)$ Set $I = (z_1, z_2, \ldots) \subset A$. Consider the affine scheme $X = \text{Spec}(A)$ and the open subscheme $U = X \setminus V(I)$. Since $A \to \prod_n A_{z_n}$ is injective we see that the scheme theoretic closure of U is X. Consider the morphism $X \to \text{Spec}(k[x])$. This morphism is surjective (set all $z_n = 0$ to see this). But the restriction of this morphism to U is not surjective because it maps to the point $x = 0$. Hence U cannot be topologically dense in X.

Lemma 7.3. *Let X be a scheme. Let $U \subset X$ be an open subscheme. If the inclusion morphism $U \to X$ is quasi-compact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X.*

Proof. Follows from Lemma 6.3 part (3). □

Example 7.4. Let A be a ring and $X = \text{Spec}(A)$. Let $f_1, \ldots, f_n \in A$ and let $U = D(f_1) \cup \ldots \cup D(f_n)$. Let $I = \text{Ker}(A \to \prod A_{f_i})$. Then the scheme theoretic closure of U in X is the closed subscheme $\text{Spec}(A/I)$ of X. Note that $U \to X$ is quasi-compact. Hence by Lemma 7.3 we see U is scheme theoretically dense in X if and only if $I = 0$.

Lemma 7.5. *Let $j : U \to X$ be an open immersion of schemes. Then U is scheme theoretically dense in X if and only if $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective.*

Proof. If $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective, then the same is true when restricted to any open V of X. Hence the scheme theoretic closure of $U \cap V$ in V is equal to V, see proof of Lemma 6.1. Conversely, suppose that the scheme theoretic closure of

$U \cap V$ is equal to V for all opens V. Suppose that $\mathcal{O}_X \to j_*\mathcal{O}_U$ is not injective. Then we can find an affine open, say $\mathrm{Spec}(A) = V \subset X$ and a nonzero element $f \in A$ such that f maps to zero in $\Gamma(V \cap U, \mathcal{O}_X)$. In this case the scheme theoretic closure of $V \cap U$ in V is clearly contained in $\mathrm{Spec}(A/(f))$ a contradiction. □

Lemma 7.6. *Let X be a scheme. If U, V are scheme theoretically dense open subschemes of X, then so is $U \cap V$.*

Proof. Let $W \subset X$ be any open. Consider the map $\mathcal{O}_X(W) \to \mathcal{O}_X(W \cap V) \to \mathcal{O}_X(W \cap V \cap U)$. By Lemma 7.5 both maps are injective. Hence the composite is injective. Hence by Lemma 7.5 $U \cap V$ is scheme theoretically dense in X. □

Lemma 7.7. *Let $h : Z \to X$ be an immersion. Assume either h is quasi-compact or Z is reduced. Let $\overline{Z} \subset X$ be the scheme theoretic image of h. Then the morphism $Z \to \overline{Z}$ is an open immersion which identifies Z with a scheme theoretically dense open subscheme of $\overline{Z}$. Moreover, Z is topologically dense in $\overline{Z}$.*

Proof. By Lemma 3.2 or Lemma 3.3 we can factor $Z \to X$ as $Z \to \overline{Z}_1 \to X$ with $Z \to \overline{Z}_1$ open and $\overline{Z}_1 \to X$ closed. On the other hand, let $Z \to \overline{Z} \subset X$ be the scheme theoretic closure of $Z \to X$. We conclude that $\overline{Z} \subset \overline{Z}_1$. Since Z is an open subscheme of $\overline{Z}_1$ it follows that Z is an open subscheme of $\overline{Z}$ as well. In the case that Z is reduced we know that $Z \subset \overline{Z}_1$ is topologically dense by the construction of $\overline{Z}_1$ in the proof of Lemma 3.3. Hence $\overline{Z}_1$ and $\overline{Z}$ have the same underlying topological spaces. Thus $\overline{Z} \subset \overline{Z}_1$ is a closed immersion into a reduced scheme which induces a bijection on underlying topological spaces, and hence it is an isomorphism. In the case that $Z \to X$ is quasi-compact we argue as follows: The assertion that Z is scheme theoretically dense in $\overline{Z}$ follows from Lemma 6.3 part (3). The last assertion follows from Lemma 6.3 part (4). □

Lemma 7.8. *Let X be a reduced scheme and let $U \subset X$ be an open subscheme. Then the following are equivalent*

(1) *U is topologically dense in X,*
(2) *the scheme theoretic closure of U in X is X, and*
(3) *U is scheme theoretically dense in X.*

Proof. This follows from Lemma 7.7 and the fact that a closed subscheme Z of X whose underlying topological space equals X must be equal to X as a scheme. □

Lemma 7.9. *Let X be a scheme and let $U \subset X$ be a reduced open subscheme. Then the following are equivalent*

(1) *the scheme theoretic closure of U in X is X, and*
(2) *U is scheme theoretically dense in X.*

If this holds then X is a reduced scheme.

Proof. This follows from Lemma 7.7 and the fact that the scheme theoretic closure of U in X is reduced by Lemma 6.7. □

Lemma 7.10. *Let S be a scheme. Let X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms of schemes over S. Let $U \subset X$ be an open subscheme such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is X and $Y \to S$ is separated, then $f = g$.*

Proof. Follows from the definitions and Schemes, Lemma 21.5. □

8. Dominant morphisms

The definition of a morphism of schemes being dominant is a little different from what you might expect if you are used to the notion of a dominant morphism of varieties.

Definition 8.1. A morphism $f : X \to S$ of schemes is called *dominant* if the image of f is a dense subset of S.

So for example, if k is an infinite field and $\lambda_1, \lambda_2, \ldots$ is a countable collection of distinct elements of k, then the morphism

$$\coprod\nolimits_{i=1,2,\ldots} \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k[x])$$

with ith factor mapping to the point $x = \lambda_i$ is dominant.

Lemma 8.2. *Let $f : X \to S$ be a morphism of schemes. If every generic point of every irreducible component of S is in the image of f, then f is dominant.*

Proof. This is a topological fact which follows directly from the fact that the topological space underlying a scheme is sober, see Schemes, Lemma 11.1, and that every point of S is contained in an irreducible component of S, see Topology, Lemma 8.3. $\square$

The expectation that morphisms are dominant only if generic points of the target are in the image does hold if the morphism is quasi-compact.

Lemma 8.3. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is dominant if and only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f.*

Proof. Let $V \subset S$ be an affine open. Because f is quasi-compact we may choose finitely many affine opens $U_i \subset f^{-1}(V)$, $i = 1, \ldots, n$ covering $f^{-1}(V)$. Consider the morphism of affines

$$f' : \coprod\nolimits_{i=1,\ldots,n} U_i \longrightarrow V.$$

A disjoint union of affines is affine, see Schemes, Lemma 6.8. Generic points of irreducible components of V are exactly the generic points of the irreducible components of S that meet V. Also, f is dominant if and only if f' is dominant no matter what choices of V, n, U_i we make above. Thus we have reduced the lemma to the case of a morphism of affine schemes. The affine case is Algebra, Lemma 30.6. $\square$

Lemma 8.4. *Let $f : X \to S$ be a quasi-compact dominant morphism of schemes. Let $g : S' \to S$ be a morphism of schemes and denote $f' : X' \to S'$ the base change of f by g. If generalizations lift along g, then f' is dominant.*

Proof. Observe that f' is quasi-compact by Schemes, Lemma 19.3. Let $\eta' \in S'$ be the generic point of an irreducible component of S'. If generalizations lift along g, then $\eta = g(\eta')$ is the generic point of an irreducible component of S. By Lemma 8.3 we see that η is in the image of f. Hence η' is in the image of f' by Schemes, Lemma 17.5. It follows that f' is dominant by Lemma 8.3. $\square$

Lemma 8.5. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $\eta \in S$ be a generic point of an irreducible component of S. If $\eta \notin f(X)$ then there exists an open neighbourhood $V \subset S$ of η such that $f^{-1}(V) = \emptyset$.*

Proof. Let $Z \subset S$ be the scheme theoretic image of f. We have to show that $\eta \notin Z$. This follows from Lemma 6.5 but can also be seen as follows. By Lemma 6.3 the morphism $X \to Z$ is dominant, which by Lemma 8.3 means all the generic points of all irreducible components of Z are in the image of $X \to Z$. By assumption we see that $\eta \notin Z$ since η would be the generic point of some irreducible component of Z if it were in Z. $\square$

There is another case where dominant is the same as having all generic points of irreducible components in the image.

Lemma 8.6. *Let $f : X \to S$ be a morphism of schemes. Suppose that X has finitely many irreducible components. Then f is dominant (if and) only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f. If so, then S has finitely many irreducible components as well.*

Proof. Assume f is dominant. Say $X = Z_1 \cup Z_2 \cup \ldots \cup Z_n$ is the decomposition of X into irreducible components. Let $\xi_i \in Z_i$ be its generic point, so $Z_i = \overline{\{\xi_i\}}$. Note that $f(Z_i)$ is an irreducible subset of S. Hence

$$S = \overline{f(X)} = \bigcup \overline{f(Z_i)} = \bigcup \overline{\{f(\xi_i)\}}$$

is a finite union of irreducible subsets whose generic points are in the image of f. The lemma follows. $\square$

Lemma 8.7. *Let $f : X \to Y$ be a morphism of integral schemes. The following are equivalent*

1. *f is dominant,*
2. *f maps the generic point of X to the generic point of Y,*
3. *for some nonempty affine opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective,*
4. *for all nonempty affine opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective,*
5. *for some $x \in X$ with image $y = f(x) \in Y$ the local ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective, and*
6. *for all $x \in X$ with image $y = f(x) \in Y$ the local ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective.*

Proof. The equivalence of (1) and (2) follows from Lemma 8.6. Let $U \subset X$ and $V \subset Y$ be nonempty affine opens with $f(U) \subset V$. Recall that the rings $A = \mathcal{O}_X(U)$ and $B = \mathcal{O}_Y(V)$ are integral domains. The morphism $f|_U : U \to V$ corresponds to a ring map $\varphi : B \to A$. The generic points of X and Y correspond to the prime ideals $(0) \subset A$ and $(0) \subset B$. Thus (2) is equivalent to the condition $(0) = \varphi^{-1}((0))$, i.e., to the condition that φ is injective. In this way we see that (2), (3), and (4) are equivalent. Similarly, given x and y as in (5) the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are domains and the prime ideals $(0) \subset \mathcal{O}_{X,x}$ and $(0) \subset \mathcal{O}_{Y,y}$ correspond to the generic points of X and Y (via the identification of the spectrum of the local ring at x with the set of points specializing to x, see Schemes, Lemma 13.2). Thus we can argue in the exact same manner as above to see that (2), (5), and (6) are equivalent. $\square$

9. Surjective morphisms

Definition 9.1. A morphism of schemes is said to be *surjective* if it is surjective on underlying topological spaces.

Lemma 9.2. *The composition of surjective morphisms is surjective.*

Proof. Omitted. $\square$

Lemma 9.3. *Let X and Y be schemes over a base scheme S. Given points $x \in X$ and $y \in Y$, there is a point of $X \times_S Y$ mapping to x and y under the projections if and only if x and y lie above the same point of S.*

Proof. The condition is obviously necessary, and the converse follows from the proof of Schemes, Lemma 17.5. $\square$

Lemma 9.4. *The base change of a surjective morphism is surjective.*

Proof. Let $f : X \to Y$ be a morphism of schemes over a base scheme S. If $S' \to S$ is a morphism of schemes, let $p : X_{S'} \to X$ and $q : Y_{S'} \to Y$ be the canonical projections. The commutative square

$$\begin{tikzcd} X_{S'} \arrow[r, "p"'] \arrow[d, "f_{S'}"'] & X \arrow[d, "f"] \\ Y_{S'} \arrow[r, "q"] & Y. \end{tikzcd}$$

identifies $X_{S'}$ as a fibre product of $X \to Y$ and $Y_{S'} \to Y$. Let Z be a subset of the underlying topological space of X. Then $q^{-1}(f(Z)) = f_{S'}(p^{-1}(Z))$, because $y' \in q^{-1}(f(Z))$ if and only if $q(y') = f(x)$ for some $x \in Z$, if and only if, by Lemma 9.3, there exists $x' \in X_{S'}$ such that $f_{S'}(x') = y'$ and $p(x') = x$. In particular taking $Z = X$ we see that if f is surjective so is the base change $f_{S'} : X_{S'} \to Y_{S'}$. $\square$

Example 9.5. Bijectivity is not stable under base change, and so neither is injectivity. For example consider the bijection $\mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{R})$. The base change $\mathrm{Spec}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}) \to \mathrm{Spec}(\mathbf{C})$ is not injective, since there is an isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \times \mathbf{C}$ (the decomposition comes from the idempotent $\frac{1\otimes 1 + i \otimes i}{2}$) and hence $\mathrm{Spec}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C})$ has two points.

Lemma 9.6. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $W \subset Z$ be a quasi-compact open. By assumption $p^{-1}(W)$ is quasi-compact. Hence by Topology, Lemma 12.7 the inverse image $q^{-1}(W) = f(p^{-1}(W))$ is quasi-compact too. This proves the lemma. $\square$

10. Radicial and universally injective morphisms

In this section we define what it means for a morphism of schemes to be *radicial* and what it means for a morphism of schemes to be *universally injective*. We then show that these notions agree. The reason for introducing both is that in the case of algebraic spaces there are corresponding notions which may not always agree.

Definition 10.1. Let $f : X \to S$ be a morphism.

(1) We say that f is *universally injective* if and only if for any morphism of schemes $S' \to S$ the base change $f' : X_{S'} \to S'$ is injective (on underlying topological spaces).
(2) We say f is *radicial* if f is injective as a map of topological spaces, and for every $x \in X$ the field extension $\kappa(x)/\kappa(f(x))$ is purely inseparable.

Lemma 10.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *For every field K the induced map* $\mathrm{Mor}(\mathrm{Spec}(K), X) \to \mathrm{Mor}(\mathrm{Spec}(K), S)$ *is injective.*
(2) *The morphism f is universally injective.*
(3) *The morphism f is radicial.*
(4) *The diagonal morphism $\Delta_{X/S} : X \longrightarrow X \times_S X$ is surjective.*

Proof. Let K be a field, and let $s : \mathrm{Spec}(K) \to S$ be a morphism. Giving a morphism $x : \mathrm{Spec}(K) \to X$ such that $f \circ x = s$ is the same as giving a section of the projection $X_K = \mathrm{Spec}(K) \times_S X \to \mathrm{Spec}(K)$, which in turn is the same as giving a point $x \in X_K$ whose residue field is K. Hence we see that (2) implies (1).

Conversely, suppose that (1) holds. Assume that $x, x' \in X_{S'}$ map to the same point $s' \in S'$. Choose a commutative diagram

$$\begin{tikzcd} K & \kappa(x) \arrow[l] \\ \kappa(x') \arrow[u] & \kappa(s') \arrow[l] \arrow[u] \end{tikzcd}$$

of fields. By Schemes, Lemma 13.3 we get two morphisms $a, a' : \mathrm{Spec}(K) \to X_{S'}$. One corresponding to the point x and the embedding $\kappa(x) \subset K$ and the other corresponding to the point x' and the embedding $\kappa(x') \subset K$. Also we have $f' \circ a = f' \circ a'$. Condition (1) now implies that the compositions of a and a' with $X_{S'} \to X$ are equal. Since $X_{S'}$ is the fibre product of S' and X over S we see that $a = a'$. Hence $x = x'$. Thus (1) implies (2).

If there are two different points $x, x' \in X$ mapping to the same point of s then (2) is violated. If for some $s = f(x)$, $x \in X$ the field extension $\kappa(x)/\kappa(s)$ is not purely inseparable, then we may find a field extension $K/\kappa(s)$ such that $\kappa(x)$ has two $\kappa(s)$-homomorphisms into K. By Schemes, Lemma 13.3 this implies that the map $\mathrm{Mor}(\mathrm{Spec}(K), X) \to \mathrm{Mor}(\mathrm{Spec}(K), S)$ is not injective, and hence (1) is violated. Thus we see that the equivalent conditions (1) and (2) imply f is radicial, i.e., they imply (3).

Assume (3). By Schemes, Lemma 13.3 a morphism $\mathrm{Spec}(K) \to X$ is given by a pair $(x, \kappa(x) \to K)$. Property (3) says exactly that associating to the pair $(x, \kappa(x) \to K)$ the pair $(s, \kappa(s) \to \kappa(x) \to K)$ is injective. In other words (1) holds. At this point we know that (1), (2) and (3) are all equivalent.

Finally, we prove the equivalence of (4) with (1), (2) and (3). A point of $X \times_S X$ is given by a quadruple $(x_1, x_2, s, \mathfrak{p})$, where $x_1, x_2 \in X$, $f(x_1) = f(x_2) = s$ and $\mathfrak{p} \subset \kappa(x_1) \otimes_{\kappa(s)} \kappa(x_2)$ is a prime ideal, see Schemes, Lemma 17.5. If f is universally injective, then by taking $S' = X$ in the definition of universally injective, $\Delta_{X/S}$ must be surjective since it is a section of the injective morphism $X \times_S X \longrightarrow X$. Conversely, if $\Delta_{X/S}$ is surjective, then always $x_1 = x_2 = x$ and there is exactly one such prime ideal $\mathfrak{p}$, which means that $\kappa(s) \subset \kappa(x)$ is purely inseparable. Hence f is radicial. Alternatively, if $\Delta_{X/S}$ is surjective, then for any $S' \to S$ the base change $\Delta_{X_{S'}/S'}$ is surjective which implies that f is universally injective. This finishes the proof of the lemma. $\square$

Lemma 10.3. *A universally injective morphism is separated.*

Proof. Combine Lemma 10.2 with the remark that $X \to S$ is separated if and only if the image of $\Delta_{X/S}$ is closed in $X \times_S X$, see Schemes, Definition 21.3 and the discussion following it. $\square$

Lemma 10.4. *A base change of a universally injective morphism is universally injective.*

Proof. This is formal. $\square$

Lemma 10.5. *A composition of radicial morphisms is radicial, and so the same holds for the equivalent condition of being universally injective.*

Proof. Omitted. $\square$

11. Affine morphisms

Definition 11.1. A morphism of schemes $f : X \to S$ is called *affine* if the inverse image of every affine open of S is an affine open of X.

Lemma 11.2. *An affine morphism is separated and quasi-compact.*

Proof. Let $f : X \to S$ be affine. Quasi-compactness is immediate from Schemes, Lemma 19.2. We will show f is separated using Schemes, Lemma 21.7. Let $x_1, x_2 \in X$ be points of X which map to the same point $s \in S$. Choose any affine open $W \subset S$ containing s. By assumption $f^{-1}(W)$ is affine. Apply the lemma cited with $U = V = f^{-1}(W)$. $\square$

Lemma 11.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *The morphism f is affine.*
2. *There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is affine.*
3. *There exists a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and an isomorphism $X \cong \underline{\mathrm{Spec}}_S(\mathcal{A})$ of schemes over S. See Constructions, Section 4 for notation.*

Moreover, in this case $X = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$.

Proof. It is obvious that (1) implies (2).

Assume $S = \bigcup_{j \in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. By Schemes, Lemma 19.2 we see that f is quasi-compact. By Schemes, Lemma 21.6 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Thus we have the scheme $g : Y = \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ over S. The identity map $\mathrm{id} : \mathcal{A} = f_*\mathcal{O}_X \to f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $can : X \to Y$ over S, see Constructions, Lemma 4.7. By assumption and the lemma just cited the restriction $can|_{f^{-1}(W_j)} : f^{-1}(W_j) \to g^{-1}(W_j)$ is an isomorphism. Thus can is an isomorphism. We have shown that (2) implies (3).

Assume (3). By Constructions, Lemma 4.6 we see that the inverse image of every affine open is affine, and hence the morphism is affine by definition. $\square$

Remark 11.4. We can also argue directly that (2) implies (1) in Lemma 11.3 above as follows. Assume $S = \bigcup W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. First argue that $\mathcal{A} = f_*\mathcal{O}_X$ is quasi-coherent as in the proof above. Let $\mathrm{Spec}(R) = V \subset S$ be affine open. We have to show that $f^{-1}(V)$ is affine. Set $A = \mathcal{A}(V) = f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$. By Schemes, Lemma 6.4 there is a canonical morphism $\psi : f^{-1}(V) \to \mathrm{Spec}(A)$ over $\mathrm{Spec}(R) = V$. By Schemes, Lemma 11.6 there exists an integer $n \geq 0$, a standard open covering $V = \bigcup_{i=1,\ldots,n} D(h_i)$, $h_i \in R$, and a map $a : \{1, \ldots, n\} \to J$ such that each $D(h_i)$ is also a standard open of the affine scheme $W_{a(i)}$. The inverse image of a standard open under a morphism of affine schemes is standard open, see Algebra, Lemma 17.4. Hence we see that $f^{-1}(D(h_i))$ is a standard open of $f^{-1}(W_{a(i)})$, in particular that $f^{-1}(D(h_i))$ is affine. Because $\mathcal{A}$ is quasi-coherent we have $A_{h_i} = \mathcal{A}(D(h_i)) = \mathcal{O}_X(f^{-1}(D(h_i)))$, so $f^{-1}(D(h_i))$ is the spectrum of A_{h_i}. It follows that the morphism ψ induces an isomorphism of the open $f^{-1}(D(h_i))$ with the open $\mathrm{Spec}(A_{h_i})$ of $\mathrm{Spec}(A)$. Since $f^{-1}(V) = \bigcup f^{-1}(D(h_i))$ and $\mathrm{Spec}(A) = \bigcup \mathrm{Spec}(A_{h_i})$ we win.

Lemma 11.5. *Let S be a scheme. There is an anti-equivalence of categories*

$$\begin{matrix} \textit{Schemes affine} \\ \textit{over } S \end{matrix} \longleftrightarrow \begin{matrix} \textit{quasi-coherent sheaves} \\ \textit{of } \mathcal{O}_S\textit{-algebras} \end{matrix}$$

which associates to $f : X \to S$ the sheaf $f_*\mathcal{O}_X$. Moreover, this equivalence is compatible with arbitrary base change.

Proof. The functor from right to left is given by $\underline{\mathrm{Spec}}_S$. The two functors are mutually inverse by Lemma 11.3 and Constructions, Lemma 4.6 part (3). The final statement is Constructions, Lemma 4.6 part (2). $\square$

Lemma 11.6. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{A} = f_*\mathcal{O}_X$. The functor $\mathcal{F} \mapsto f_*\mathcal{F}$ induces an equivalence of categories*

$$\left\{\begin{matrix} \textit{category of quasi-coherent} \\ \mathcal{O}_X\textit{-modules} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \textit{category of quasi-coherent} \\ \mathcal{A}\textit{-modules} \end{matrix}\right\}$$

Moreover, an $\mathcal{A}$-module is quasi-coherent as an $\mathcal{O}_S$-module if and only if it is quasi-coherent as an $\mathcal{A}$-module.

Proof. Omitted. $\square$

Lemma 11.7. *The composition of affine morphisms is affine.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is affine by assumption on g. Whereupon $f^{-1}(g^{-1}(U))$ is affine by assumption on f. Hence $(g \circ f)^{-1}(U)$ is affine. $\square$

Lemma 11.8. *The base change of an affine morphism is affine.*

Proof. Let $f : X \to S$ be an affine morphism. Let $S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V \subset S'$ which maps into some open affine $U \subset S$. By assumption $f^{-1}(U)$ is affine. By the material in Schemes, Section 17 we see that $f^{-1}(U)_V = V \times_U f^{-1}(U)$ is affine and equal to $(f')^{-1}(V)$. This proves that S' has an open covering by affines whose inverse image under f' is affine. We conclude by Lemma 11.3 above. □

Lemma 11.9. *A closed immersion is affine.*

Proof. The first indication of this is Schemes, Lemma 8.2. See Schemes, Lemma 10.1 for a complete statement. □

Lemma 11.10. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. The inclusion morphism $j : X_s \to X$ is affine.*

Proof. This follows from Properties, Lemma 26.4 and the definition. □

Lemma 11.11. *Suppose $g : X \to Y$ is a morphism of schemes over S.*

(1) *If X is affine over S and $\Delta : Y \to Y \times_S Y$ is affine, then g is affine.*
(2) *If X is affine over S and Y is separated over S, then g is affine.*
(3) *A morphism from an affine scheme to a scheme with affine diagonal is affine.*
(4) *A morphism from an affine scheme to a separated scheme is affine.*

Proof. Proof of (1). The base change $X \times_S Y \to Y$ is affine by Lemma 11.8. The morphism $(1, g) : X \to X \times_S Y$ is the base change of $Y \to Y \times_S Y$ by the morphism $X \times_S Y \to Y \times_S Y$. Hence it is affine by Lemma 11.8. The composition of affine morphisms is affine (see Lemma 11.7) and (1) follows. Part (2) follows from (1) as a closed immersion is affine (see Lemma 11.9) and Y/S separated means Δ is a closed immersion. Parts (3) and (4) are special cases of (1) and (2). □

Lemma 11.12. *A morphism between affine schemes is affine.*

Proof. Immediate from Lemma 11.11 with $S = \mathrm{Spec}(\mathbf{Z})$. It also follows directly from the equivalence of (1) and (2) in Lemma 11.3. □

Lemma 11.13. *Let S be a scheme. Let A be an Artinian ring. Any morphism $\mathrm{Spec}(A) \to S$ is affine.*

Proof. Omitted. □

Lemma 11.14. *Let $j : Y \to X$ be an immersion of schemes. Assume there exists an open $U \subset X$ with complement $Z = X \setminus U$ such that*

(1) *$U \to X$ is affine,*
(2) *$j^{-1}(U) \to U$ is affine, and*
(3) *$j(Y) \cap Z$ is closed.*

Then j is affine. In particular, if X is affine, so is Y.

Proof. By Schemes, Definition 10.2 there exists an open subscheme $W \subset X$ such that j factors as a closed immersion $i : Y \to W$ followed by the inclusion morphism $W \to X$. Since a closed immersion is affine (Lemma 11.9), we see that for every $x \in W$ there is an affine open neighbourhood of x in X whose inverse image under j is affine. If $x \in U$, then the same thing is true by assumption (2). Finally, assume $x \in Z$ and $x \notin W$. Then $x \notin j(Y) \cap Z$. By assumption (3) we can find an affine open neighbourhood $V \subset X$ of x which does not meet $j(Y) \cap Z$. Then $j^{-1}(V) = j^{-1}(V \cap U)$ which is affine by assumptions (1) and (2). It follows that j is affine by Lemma 11.3. □

12. Families of ample invertible modules

A short section on the notion of a family of ample invertible modules.

Definition 12.1. Let X be a scheme. Let $\{\mathcal{L}_i\}_{i \in I}$ be a family of invertible $\mathcal{O}_X$-modules. We say $\{\mathcal{L}_i\}_{i \in I}$ is an *ample family of invertible modules on* X if

(1) X is quasi-compact, and
(2) for every $x \in X$ there exists an $i \in I$, an $n \geq 1$, and $s \in \Gamma(X, \mathcal{L}_i^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

If $\{\mathcal{L}_i\}_{i \in I}$ is an ample family of invertible modules on a scheme X, then there exists a finite subset $I' \subset I$ such that $\{\mathcal{L}_i\}_{i \in I'}$ is an ample family of invertible modules on X (follows immediately from quasi-compactness). A scheme having an ample family of invertible modules has an affine diagonal by the next lemma and hence is a fortiori quasi-separated.

Lemma 12.2. *Let X be a scheme such that for every point $x \in X$ there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a global section $s \in \Gamma(X, \mathcal{L})$ such that $x \in X_s$ and X_s is affine. Then the diagonal of X is an affine morphism.*

Proof. Given invertible $\mathcal{O}_X$-modules $\mathcal{L}$, $\mathcal{M}$ and global sections $s \in \Gamma(X, \mathcal{L})$, $t \in \Gamma(X, \mathcal{M})$ such that X_s and X_t are affine we have to prove $X_s \cap X_t$ is affine. Namely, then Lemma 11.3 applied to $\Delta : X \to X \times X$ and the fact that $\Delta^{-1}(X_s \times X_t) = X_s \cap X_t$ shows that Δ is affine. The fact that $X_s \cap X_t$ is affine follows from Properties, Lemma 26.4. $\square$

Remark 12.3. In Properties, Lemma 26.7 we see that a scheme which has an ample invertible module is separated. This is wrong for schemes having an ample family of invertible modules. Namely, let X be as in Schemes, Example 14.3 with $n = 1$, i.e., the affine line with zero doubled. We use the notation of that example except that we write x for x_1 and y for y_1. There is, for every integer n, an invertible sheaf $\mathcal{L}_n$ on X which is trivial on X_1 and X_2 and whose transition function $U_{12} \to U_{21}$ is $f(x) \mapsto y^n f(y)$. The global sections of $\mathcal{L}_n$ are pairs $(f(x), g(y)) \in k[x] \oplus k[y]$ such that $y^n f(y) = g(y)$. The sections $s = (1, y)$ of $\mathcal{L}_1$ and $t = (x, 1)$ of $\mathcal{L}_{-1}$ determine an open affine cover because $X_s = X_1$ and $X_t = X_2$. Therefore X has an ample family of invertible modules but it is not separated.

13. Quasi-affine morphisms

Recall that a scheme X is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme, see Properties, Definition 18.1.

Definition 13.1. A morphism of schemes $f : X \to S$ is called *quasi-affine* if the inverse image of every affine open of S is a quasi-affine scheme.

Lemma 13.2. *A quasi-affine morphism is separated and quasi-compact.*

Proof. Let $f : X \to S$ be quasi-affine. Quasi-compactness is immediate from Schemes, Lemma 19.2. Let $U \subset S$ be an affine open. If we can show that $f^{-1}(U)$ is a separated scheme, then f is separated (Schemes, Lemma 21.7 shows that being separated is local on the base). By assumption $f^{-1}(U)$ is isomorphic to an open subscheme of an affine scheme. An affine scheme is separated and hence every open subscheme of an affine scheme is separated as desired. $\square$

Lemma 13.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *The morphism f is quasi-affine.*
2. *There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is quasi-affine.*
3. *There exists a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and a quasi-compact open immersion*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \underline{\mathrm{Spec}}_S(\mathcal{A}) \arrow[dl] \\ & S & \end{tikzcd}$$

   over S.

4. *Same as in (3) but with $\mathcal{A} = f_*\mathcal{O}_X$ and the horizontal arrow the canonical morphism of Constructions, Lemma 4.7.*

Proof. It is obvious that (1) implies (2) and that (4) implies (3).

Assume $S = \bigcup_{j \in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is quasi-affine. By Schemes, Lemma 19.2 we see that f is quasi-compact. By Schemes, Lemma 21.6 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Thus we have the scheme $g : Y = \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ over S. The identity map $\mathrm{id} : \mathcal{A} = f_*\mathcal{O}_X \to f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $can : X \to Y$ over S, see Constructions, Lemma 4.7. By assumption, the lemma just cited, and Properties, Lemma 18.4 the restriction $can|_{f^{-1}(W_j)} : f^{-1}(W_j) \to g^{-1}(W_j)$ is a quasi-compact open immersion. Thus can is a quasi-compact open immersion. We have shown that (2) implies (4).

Assume (3). Choose any affine open $U \subset S$. By Constructions, Lemma 4.6 we see that the inverse image of U in the relative spectrum is affine. Hence we conclude that $f^{-1}(U)$ is quasi-affine (note that quasi-compactness is encoded in (3) as well). Thus (3) implies (1). $\square$

Lemma 13.4. *The composition of quasi-affine morphisms is quasi-affine.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be quasi-affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is quasi-affine by assumption on g. Let $j : g^{-1}(U) \to V$ be a quasi-compact open immersion into an affine scheme V. By Lemma 13.3 above we see that $f^{-1}(g^{-1}(U))$ is a quasi-compact open subscheme of the relative spectrum $\underline{\mathrm{Spec}}_{g^{-1}(U)}(\mathcal{A})$ for some quasi-coherent sheaf of $\mathcal{O}_{g^{-1}(U)}$-algebras $\mathcal{A}$. By Schemes, Lemma 24.1 the sheaf $\mathcal{A}' = j_*\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_V$-algebras with the property that $j^*\mathcal{A}' = \mathcal{A}$. Hence we get a commutative diagram

$$\begin{tikzcd} f^{-1}(g^{-1}(U)) \arrow[r] & \underline{\mathrm{Spec}}_{g^{-1}(U)}(\mathcal{A}) \arrow[r] \arrow[d] & \underline{\mathrm{Spec}}_V(\mathcal{A}') \arrow[d] \\ & g^{-1}(U) \arrow[r, "j"] & V \end{tikzcd}$$

with the square being a fibre square, see Constructions, Lemma 4.6. Note that the upper right corner is an affine scheme. Hence $(g \circ f)^{-1}(U)$ is quasi-affine. $\square$

Lemma 13.5. *The base change of a quasi-affine morphism is quasi-affine.*

Proof. Let $f : X \to S$ be a quasi-affine morphism. By Lemma 13.3 above we can find a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and a quasi-compact open immersion $X \to \underline{\mathrm{Spec}}_S(\mathcal{A})$ over S. Let $g : S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. Since the base change of a quasi-compact open immersion is a quasi-compact open immersion we see that $X_{S'} \to \underline{\mathrm{Spec}}_{S'}(g^*\mathcal{A})$ is a quasi-compact open immersion (we have used Schemes, Lemmas 19.3 and 18.2 and Constructions, Lemma 4.6). By Lemma 13.3 again we conclude that $X_{S'} \to S'$ is quasi-affine. $\square$

Lemma 13.6. *A quasi-compact immersion is quasi-affine.*

Proof. Let $X \to S$ be a quasi-compact immersion. We have to show the inverse image of every affine open is quasi-affine. Hence, assuming S is an affine scheme, we have to show X is quasi-affine. By Lemma 7.7 the morphism $X \to S$ factors as $X \to Z \to S$ where Z is a closed subscheme of S and $X \subset Z$ is a quasi-compact open. Since S is affine Lemma 2.1 implies Z is affine. Hence we win. $\square$

Lemma 13.7. *Let S be a scheme. Let X be an affine scheme. A morphism $f : X \to S$ is quasi-affine if and only if it is quasi-compact. In particular any morphism from an affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. Let $V \subset S$ be an affine open. Then $f^{-1}(V)$ is an open subscheme of the affine scheme X, hence quasi-affine if and only if it is quasi-compact. This proves the first assertion. The quasi-compactness of any $f : X \to S$ where X is affine and S quasi-separated follows from Schemes, Lemma 21.14 applied to $X \to S \to \mathrm{Spec}(\mathbf{Z})$. $\square$

Lemma 13.8. *Suppose $g : X \to Y$ is a morphism of schemes over S. If X is quasi-affine over S and Y is quasi-separated over S, then g is quasi-affine. In particular, any morphism from a quasi-affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. The base change $X \times_S Y \to Y$ is quasi-affine by Lemma 13.5. The morphism $X \to X \times_S Y$ is a quasi-compact immersion as $Y \to S$ is quasi-separated, see Schemes, Lemma 21.11. A quasi-compact immersion is quasi-affine by Lemma 13.6 and the composition of quasi-affine morphisms is quasi-affine (see Lemma 13.4). Thus we win. $\square$

14. Types of morphisms defined by properties of ring maps

In this section we study what properties of ring maps allow one to define local properties of morphisms of schemes.

Definition 14.1. Let P be a property of ring maps.

1. We say that P is *local* if the following hold:
   (a) For any ring map $R \to A$, and any $f \in R$ we have $P(R \to A) \Rightarrow P(R_f \to A_f)$.
   (b) For any rings R, A, any $f \in R$, $a \in A$, and any ring map $R_f \to A$ we have $P(R_f \to A) \Rightarrow P(R \to A_a)$.
   (c) For any ring map $R \to A$, and $a_i \in A$ such that $(a_1, \ldots, a_n) = A$ then $\forall i, P(R \to A_{a_i}) \Rightarrow P(R \to A)$.

(2) We say that P is *stable under base change* if for any ring maps $R \to A$, $R \to R'$ we have $P(R \to A) \Rightarrow P(R' \to R' \otimes_R A)$.
(3) We say that P is *stable under composition* if for any ring maps $A \to B$, $B \to C$ we have $P(A \to B) \wedge P(B \to C) \Rightarrow P(A \to C)$.

Definition 14.2. Let P be a property of ring maps. Let $f : X \to S$ be a morphism of schemes. We say f is *locally of type P* if for any $x \in X$ there exists an affine open neighbourhood U of x in X which maps into an affine open $V \subset S$ such that the induced ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ has property P.

This is not a "good" definition unless the property P is a local property. Even if P is a local property we will not automatically use this definition to say that a morphism is "locally of type P" unless we also explicitly state the definition elsewhere.

Lemma 14.3. *Let $f : X \to S$ be a morphism of schemes. Let P be a property of ring maps. Let U be an affine open of X, and V an affine open of S such that $f(U) \subset V$. If f is locally of type P and P is local, then $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$ holds.*

Proof. As f is locally of type P for every $u \in U$ there exists an affine open $U_u \subset X$ mapping into an affine open $V_u \subset S$ such that $P(\mathcal{O}_S(V_u) \to \mathcal{O}_X(U_u))$ holds. Choose an open neighbourhood $U'_u \subset U \cap U_u$ of u which is standard affine open in both U and U_u, see Schemes, Lemma 11.5. By Definition 14.1 (1)(b) we see that $P(\mathcal{O}_S(V_u) \to \mathcal{O}_X(U'_u))$ holds. Hence we may assume that $U_u \subset U$ is a standard affine open. Choose an open neighbourhood $V'_u \subset V \cap V_u$ of $f(u)$ which is standard affine open in both V and V_u, see Schemes, Lemma 11.5. Then $U'_u = f^{-1}(V'_u) \cap U_u$ is a standard affine open of U_u (hence of U) and we have $P(\mathcal{O}_S(V'_u) \to \mathcal{O}_X(U'_u))$ by Definition 14.1 (1)(a). Hence we may assume both $U_u \subset U$ and $V_u \subset V$ are standard affine open. Applying Definition 14.1 (1)(b) one more time we conclude that $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U_u))$ holds. Because U is quasi-compact we may choose a finite number of points $u_1, \ldots, u_n \in U$ such that

$$U = U_{u_1} \cup \ldots \cup U_{u_n}.$$

By Definition 14.1 (1)(c) we conclude that $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$ holds. □

Lemma 14.4. *Let P be a local property of ring maps. Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally of type P.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ we have $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of type P.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $P(\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i))$ holds, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of type P then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of type P.

Proof. This follows from Lemma 14.3 above. □

Lemma 14.5. *Let P be a property of ring maps. Assume P is local and stable under composition. The composition of morphisms locally of type P is locally of type P.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms locally of type P. Let $x \in X$. Choose an affine open neighbourhood $W \subset Z$ of $g(f(x))$. Choose an affine open neighbourhood $V \subset g^{-1}(W)$ of $f(x)$. Choose an affine open neighbourhood $U \subset f^{-1}(V)$ of x. By Lemma 14.4 the ring maps $\mathcal{O}_Z(W) \to \mathcal{O}_Y(V)$ and $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ satisfy P. Hence $\mathcal{O}_Z(W) \to \mathcal{O}_X(U)$ satisfies P as P is assumed stable under composition. $\square$

Lemma 14.6. *Let P be a property of ring maps. Assume P is local and stable under base change. The base change of a morphism locally of type P is locally of type P.*

Proof. Let $f : X \to S$ be a morphism locally of type P. Let $S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V' \subset S'$ which maps into some open affine $V \subset S$. By Lemma 14.4 the open $f^{-1}(V)$ is a union of affines U_i such that the ring maps $\mathcal{O}_S(V) \to \mathcal{O}_X(U_i)$ all satisfy P. By the material in Schemes, Section 17 we see that $f^{-1}(U)_{V'} = V' \times_V f^{-1}(V)$ is the union of the affine opens $V' \times_V U_i$. Since $\mathcal{O}_{X_{S'}}(V' \times_V U_i) = \mathcal{O}_{S'}(V') \otimes_{\mathcal{O}_S(V)} \mathcal{O}_X(U_i)$ we see that the ring maps $\mathcal{O}_{S'}(V') \to \mathcal{O}_{X_{S'}}(V' \times_V U_i)$ satisfy P as P is assumed stable under base change. $\square$

Lemma 14.7. *The following properties of a ring map $R \to A$ are local.*

1. *(Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_\mathfrak{p} \to A_\mathfrak{q}$.*
2. *(Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.*
3. *(Reduced fibres.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ is reduced.*
4. *(Fibres of dimension at most n.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ has Krull dimension at most n.*
5. *(Locally Noetherian on the target.) The ring map $R \to A$ has the property that A is Noetherian.*
6. *Add more here as needed*[4].

Proof. Omitted. $\square$

Lemma 14.8. *The following properties of ring maps are stable under base change.*

1. *(Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_\mathfrak{p} \to A_\mathfrak{q}$.*
2. *(Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.*
3. *Add more here as needed*[5].

Proof. Omitted. $\square$

Lemma 14.9. *The following properties of ring maps are stable under composition.*

[4] But only those properties that are not already dealt with separately elsewhere.
[5] But only those properties that are not already dealt with separately elsewhere.

(1) *(Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_{\mathfrak{p}} \to A_{\mathfrak{q}}$.*
(2) *(Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.*
(3) *(Locally Noetherian on the target.) The ring map $R \to A$ has the property that A is Noetherian.*
(4) *Add more here as needed*[6].

Proof. Omitted. $\square$

15. Morphisms of finite type

Recall that a ring map $R \to A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \ldots, x_n]$ as an R-algebra, see Algebra, Definition 6.1.

Definition 15.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is of *finite type at $x \in X$* if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.
(2) We say that f is *locally of finite type* if it is of finite type at every point of X.
(3) We say that f is of *finite type* if it is locally of finite type and quasi-compact.

Lemma 15.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally of finite type.*
(2) *For all affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite type.*
(3) *There exist an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of finite type.*
(4) *There exist an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of finite type, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of finite type then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of finite type.

Proof. This follows from Lemma 14.3 if we show that the property "$R \to A$ is of finite type" is local. We check conditions (a), (b) and (c) of Definition 14.1. By Algebra, Lemma 14.2 being of finite type is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 6.2 being of finite type is stable under composition and trivially for any ring R the ring map $R \to R_f$ is of finite type. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 23.3. $\square$

Lemma 15.3. *The composition of two morphisms which are locally of finite type is locally of finite type. The same is true for morphisms of finite type.*

[6]But only those properties that are not already dealt with separately elsewhere.

Proof. In the proof of Lemma 15.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 14.5 combined with the fact that being of finite type is a property of ring maps that is stable under composition, see Algebra, Lemma 6.2. By the above and the fact that compositions of quasi-compact morphisms are quasi-compact, see Schemes, Lemma 19.4 we see that the composition of morphisms of finite type is of finite type. □

Lemma 15.4. *The base change of a morphism which is locally of finite type is locally of finite type. The same is true for morphisms of finite type.*

Proof. In the proof of Lemma 15.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 14.6 combined with the fact that being of finite type is a property of ring maps that is stable under base change, see Algebra, Lemma 14.2. By the above and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 19.3 we see that the base change of a morphism of finite type is a morphism of finite type. □

Lemma 15.5. *A closed immersion is of finite type. An immersion is locally of finite type.*

Proof. This is true because an open immersion is a local isomorphism, and a closed immersion is obviously of finite type. □

Lemma 15.6. *Let $f : X \to S$ be a morphism. If S is (locally) Noetherian and f (locally) of finite type then X is (locally) Noetherian.*

Proof. This follows immediately from the fact that a ring of finite type over a Noetherian ring is Noetherian, see Algebra, Lemma 31.1. (Also: use the fact that the source of a quasi-compact morphism with quasi-compact target is quasi-compact.) □

Lemma 15.7. *Let $f : X \to S$ be locally of finite type with S locally Noetherian. Then f is quasi-separated.*

Proof. In fact, it is true that X is quasi-separated, see Properties, Lemma 5.4 and Lemma 15.6 above. Then apply Schemes, Lemma 21.13 to conclude that f is quasi-separated. □

Lemma 15.8. *Let $X \to Y$ be a morphism of schemes over a base scheme S. If X is locally of finite type over S, then $X \to Y$ is locally of finite type.*

Proof. Via Lemma 15.2 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is of finite type, then $B \to C$ is of finite type. (See Algebra, Lemma 6.2). □

16. Points of finite type and Jacobson schemes

Let S be a scheme. A finite type point s of S is a point such that the morphism $\mathrm{Spec}(\kappa(s)) \to S$ is of finite type. The reason for studying this is that finite type points can replace closed points in a certain sense and in certain situations. There are always enough of them for example. Moreover, a scheme is Jacobson if and only if all finite type points are closed points.

Lemma 16.1. *Let S be a scheme. Let k be a field. Let $f : \mathrm{Spec}(k) \to S$ be a morphism. The following are equivalent:*

1. *The morphism f is of finite type.*
2. *The morphism f is locally of finite type.*
3. *There exists an affine open $U = \mathrm{Spec}(R)$ of S such that f corresponds to a finite ring map $R \to k$.*
4. *There exists an affine open $U = \mathrm{Spec}(R)$ of S such that the image of f consists of a closed point u in U and the field extension $k/\kappa(u)$ is finite.*

Proof. The equivalence of (1) and (2) is obvious as $\mathrm{Spec}(k)$ is a singleton and hence any morphism from it is quasi-compact.

Suppose f is locally of finite type. Choose any affine open $\mathrm{Spec}(R) = U \subset S$ such that the image of f is contained in U, and the ring map $R \to k$ is of finite type. Let $\mathfrak{p} \subset R$ be the kernel. Then $R/\mathfrak{p} \subset k$ is of finite type. By Algebra, Lemma 34.2 there exist a $\overline{f} \in R/\mathfrak{p}$ such that $(R/\mathfrak{p})_{\overline{f}}$ is a field and $(R/\mathfrak{p})_{\overline{f}} \to k$ is a finite field extension. If $f \in R$ is a lift of $\overline{f}$, then we see that k is a finite R_f-module. Thus (2) $\Rightarrow$ (3).

Suppose that $\mathrm{Spec}(R) = U \subset S$ is an affine open such that f corresponds to a finite ring map $R \to k$. Then f is locally of finite type by Lemma 15.2. Thus (3) $\Rightarrow$ (2).

Suppose $R \to k$ is finite. The image of $R \to k$ is a field over which k is finite by Algebra, Lemma 36.18. Hence the kernel of $R \to k$ is a maximal ideal. Thus (3) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (3) is immediate. $\square$

Lemma 16.2. *Let S be a scheme. Let A be an Artinian local ring with residue field κ. Let $f : \mathrm{Spec}(A) \to S$ be a morphism of schemes. Then f is of finite type if and only if the composition $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(A) \to S$ is of finite type.*

Proof. Since the morphism $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(A)$ is of finite type it is clear that if f is of finite type so is the composition $\mathrm{Spec}(\kappa) \to S$ (see Lemma 15.3). For the converse, note that $\mathrm{Spec}(A) \to S$ maps into some affine open $U = \mathrm{Spec}(B)$ of S as $\mathrm{Spec}(A)$ has only one point. To finish apply Algebra, Lemma 54.4 to $B \to A$. $\square$

Recall that given a point s of a scheme S there is a canonical morphism $\mathrm{Spec}(\kappa(s)) \to S$, see Schemes, Section 13.

Definition 16.3. Let S be a scheme. Let us say that a point s of S is a *finite type point* if the canonical morphism $\mathrm{Spec}(\kappa(s)) \to S$ is of finite type. We denote $S_{\text{ft-pts}}$ the set of finite type points of S.

We can describe the set of finite type points as follows.

Lemma 16.4. *Let S be a scheme. We have*

$$S_{\textit{ft-pts}} = \bigcup\nolimits_{U \subset S \ open} U_0$$

where U_0 is the set of closed points of U. Here we may let U range over all opens or over all affine opens of S.

Proof. Immediate from Lemma 16.1. $\square$

Lemma 16.5. *Let $f : T \to S$ be a morphism of schemes. If f is locally of finite type, then $f(T_{\textit{ft-pts}}) \subset S_{\textit{ft-pts}}$.*

Proof. If T is the spectrum of a field this is Lemma 16.1. In general it follows since the composition of morphisms locally of finite type is locally of finite type (Lemma 15.3). $\square$

Lemma 16.6. *Let $f : T \to S$ be a morphism of schemes. If f is locally of finite type and surjective, then $f(T_{ft\text{-}pts}) = S_{ft\text{-}pts}$.*

Proof. We have $f(T_{\text{ft-pts}}) \subset S_{\text{ft-pts}}$ by Lemma 16.5. Let $s \in S$ be a finite type point. As f is surjective the scheme $T_s = \text{Spec}(\kappa(s)) \times_S T$ is nonempty, therefore has a finite type point $t \in T_s$ by Lemma 16.4. Now $T_s \to T$ is a morphism of finite type as a base change of $s \to S$ (Lemma 15.4). Hence the image of t in T is a finite type point by Lemma 16.5 which maps to s by construction. $\square$

Lemma 16.7. *Let S be a scheme. For any locally closed subset $T \subset S$ we have*

$$T \neq \emptyset \Rightarrow T \cap S_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset S$ we see that $T \cap S_{ft\text{-}pts}$ is dense in T.

Proof. Note that T carries a scheme structure (see Schemes, Lemma 12.4) such that $T \to S$ is a locally closed immersion. Any locally closed immersion is locally of finite type, see Lemma 15.5. Hence by Lemma 16.5 we see $T_{\text{ft-pts}} \subset S_{\text{ft-pts}}$. Finally, any nonempty affine open of T has at least one closed point which is a finite type point of T by Lemma 16.4. $\square$

It follows that most of the material from Topology, Section 18 goes through with the set of closed points replaced by the set of points of finite type. In fact, if S is Jacobson then we recover the closed points as the finite type points.

Lemma 16.8. *Let S be a scheme. The following are equivalent:*

(1) *the scheme S is Jacobson,*
(2) *$S_{ft\text{-}pts}$ is the set of closed points of S,*
(3) *for all $T \to S$ locally of finite type closed points map to closed points, and*
(4) *for all $T \to S$ locally of finite type closed points $t \in T$ map to closed points $s \in S$ with $\kappa(s) \subset \kappa(t)$ finite.*

Proof. We have trivially (4) $\Rightarrow$ (3) $\Rightarrow$ (2). Lemma 16.7 shows that (2) implies (1). Hence it suffices to show that (1) implies (4). Suppose that $T \to S$ is locally of finite type. Choose $t \in T$ closed and let $s \in S$ be the image. Choose affine open neighbourhoods $\text{Spec}(R) = U \subset S$ of s and $\text{Spec}(A) = V \subset T$ of t with V mapping into U. The induced ring map $R \to A$ is of finite type (see Lemma 15.2) and R is Jacobson by Properties, Lemma 6.3. Thus the result follows from Algebra, Proposition 35.19. $\square$

Lemma 16.9. *Let S be a Jacobson scheme. Any scheme locally of finite type over S is Jacobson.*

Proof. This is clear from Algebra, Proposition 35.19 (and Properties, Lemma 6.3 and Lemma 15.2). $\square$

Lemma 16.10. *The following types of schemes are Jacobson.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over $\mathbf{Z}$.*
(3) *Any scheme locally of finite type over a 1-dimensional Noetherian domain with infinitely many primes.*

(4) *A scheme of the form* $\mathrm{Spec}(R) \setminus \{\mathfrak m\}$ *where* $(R, \mathfrak m)$ *is a Noetherian local ring. Also any scheme locally of finite type over it.*

Proof. We will use Lemma 16.9 without mention. The spectrum of a field is clearly Jacobson. The spectrum of $\mathbf{Z}$ is Jacobson, see Algebra, Lemma 35.6. For (3) see Algebra, Lemma 61.4. For (4) see Properties, Lemma 6.4. $\square$

17. Universally catenary schemes

Recall that a topological space X is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 11.4. Recall that a scheme is catenary if its underlying topological space is catenary. See Properties, Definition 11.1.

Definition 17.1. Let S be a scheme. Assume S is locally Noetherian. We say S is *universally catenary* if for every morphism $X \to S$ locally of finite type the scheme X is catenary.

This is a "better" notion than catenary as there exist Noetherian schemes which are catenary but not universally catenary. See Examples, Section 18. Many schemes are universally catenary, see Lemma 17.5 below.

Recall that a ring A is called *catenary* if for any pair of prime ideals $\mathfrak p \subset \mathfrak q$ there exists a maximal chain of primes

$$\mathfrak p = \mathfrak p_0 \subset \ldots \subset \mathfrak p_e = \mathfrak q$$

and all of these have the same length. See Algebra, Definition 105.1. We have seen the relationship between catenary schemes and catenary rings in Properties, Section 11. Recall that a ring A is called *universally catenary* if A is Noetherian and for every finite type ring map $A \to B$ the ring B is catenary. See Algebra, Definition 105.3. Many interesting rings which come up in algebraic geometry satisfy this property.

Lemma 17.2. *Let S be a locally Noetherian scheme. The following are equivalent*

(1) *S is universally catenary,*
(2) *there exists an open covering of S all of whose members are universally catenary schemes,*
(3) *for every affine open* $\mathrm{Spec}(R) = U \subset S$ *the ring R is universally catenary, and*
(4) *there exists an affine open covering $S = \bigcup U_i$ such that each U_i is the spectrum of a universally catenary ring.*

Moreover, in this case any scheme locally of finite type over S is universally catenary as well.

Proof. By Lemma 15.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 15.3). Thus it is clear that if S is universally catenary then any open and any scheme locally of finite type over S is universally catenary as well. This proves the final statement of the lemma and that (1) implies (2).

If $\mathrm{Spec}(R)$ is a universally catenary scheme, then every scheme $\mathrm{Spec}(A)$ with A a finite type R-algebra is catenary. Hence all these rings A are catenary by Algebra, Lemma 105.2. Thus R is universally catenary. Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (3) implies (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $X \to S$ be a morphism locally of finite type. We can find an affine open covering $X = \bigcup V_j$ such that each $V_j \to S$ maps into one of the U_i. By Lemma 15.2 the induced ring map $\mathcal{O}(U_i) \to \mathcal{O}(V_j)$ is of finite type. Hence $\mathcal{O}(V_j)$ is catenary. Hence X is catenary by Properties, Lemma 11.2. □

Lemma 17.3. *Let S be a locally Noetherian scheme. The following are equivalent:*

1. *S is universally catenary, and*
2. *all local rings $\mathcal{O}_{S,s}$ of S are universally catenary.*

Proof. Assume that all local rings of S are universally catenary. Let $f : X \to S$ be locally of finite type. We know that X is catenary if and only if $\mathcal{O}_{X,x}$ is catenary for all $x \in X$. If $f(x) = s$, then $\mathcal{O}_{X,x}$ is essentially of finite type over $\mathcal{O}_{S,s}$. Hence $\mathcal{O}_{X,x}$ is catenary by the assumption that $\mathcal{O}_{S,s}$ is universally catenary.

Conversely, assume that S is universally catenary. Let $s \in S$. We may replace S by an affine open neighbourhood of s by Lemma 17.2. Say $S = \mathrm{Spec}(R)$ and s corresponds to the prime ideal $\mathfrak{p}$. Any finite type $R_\mathfrak{p}$-algebra A' is of the form $A_\mathfrak{p}$ for some finite type R-algebra A. By assumption (and Lemma 17.2 if you like) the ring A is catenary, and hence A' (a localization of A) is catenary. Thus $R_\mathfrak{p}$ is universally catenary. □

Lemma 17.4. *Let S be a locally Noetherian scheme. Then S is universally catenary if and only if the irreducible components of S are universally catenary.*

Proof. Omitted. For the affine case, please see Algebra, Lemma 105.8. □

Lemma 17.5. *The following types of schemes are universally catenary.*

1. *Any scheme locally of finite type over a field.*
2. *Any scheme locally of finite type over a Cohen-Macaulay scheme.*
3. *Any scheme locally of finite type over $\mathbf{Z}$.*
4. *Any scheme locally of finite type over a 1-dimensional Noetherian domain.*
5. *And so on.*

Proof. All of these follow from the fact that a Cohen-Macaulay ring is universally catenary, see Algebra, Lemma 105.9. Also, use the last assertion of Lemma 17.2. Some details omitted. □

18. Nagata schemes, reprise

See Properties, Section 13 for the definitions and basic properties of Nagata and universally Japanese schemes.

Lemma 18.1. *Let $f : X \to S$ be a morphism. If S is Nagata and f locally of finite type then X is Nagata. If S is universally Japanese and f locally of finite type then X is universally Japanese.*

Proof. For "universally Japanese" this follows from Algebra, Lemma 162.4. For "Nagata" this follows from Algebra, Proposition 162.15. □

Lemma 18.2. *The following types of schemes are Nagata.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over a Noetherian complete local ring.*
(3) *Any scheme locally of finite type over* $\mathbf{Z}$*.*
(4) *Any scheme locally of finite type over a Dedekind ring of characteristic zero.*
(5) *And so on.*

Proof. By Lemma 18.1 we only need to show that the rings mentioned above are Nagata rings. For this see Algebra, Proposition 162.16. □

19. The singular locus, reprise

We look for a criterion that implies openness of the regular locus for any scheme locally of finite type over the base. Here is the definition.

Definition 19.1. Let X be a locally Noetherian scheme. We say X is *J-2* if for every morphism $Y \to X$ which is locally of finite type the regular locus $\text{Reg}(Y)$ is open in Y.

This is the analogue of the corresponding notion for Noetherian rings, see More on Algebra, Definition 47.1.

Lemma 19.2. *Let X be a locally Noetherian scheme. The following are equivalent*

(1) *X is J-2,*
(2) *there exists an open covering of X all of whose members are J-2 schemes,*
(3) *for every affine open* $\text{Spec}(R) = U \subset X$ *the ring R is J-2, and*
(4) *there exists an affine open covering $S = \bigcup U_i$ such that each $\mathcal{O}(U_i)$ is J-2 for all i.*

Moreover, in this case any scheme locally of finite type over X is J-2 as well.

Proof. By Lemma 15.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 15.3). Thus it is clear that if X is J-2 then any open and any scheme locally of finite type over X is J-2 as well. This proves the final statement of the lemma.

If $\text{Spec}(R)$ is J-2, then for every finite type R-algebra A the regular locus of the scheme $\text{Spec}(A)$ is open. Hence R is J-2, by definition (see More on Algebra, Definition 47.1). Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $Y \to X$ be a morphism locally of finite type. We can find an affine open covering $Y = \bigcup V_j$ such that each $V_j \to X$ maps into one of the U_i. By Lemma 15.2 the induced ring map $\mathcal{O}(U_i) \to \mathcal{O}(V_j)$ is of finite type. Hence the regular locus of $V_j = \text{Spec}(\mathcal{O}(V_j))$ is open. Since $\text{Reg}(Y) \cap V_j = \text{Reg}(V_j)$ we conclude that $\text{Reg}(Y)$ is open as desired. □

Lemma 19.3. *The following types of schemes are J-2.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over a Noetherian complete local ring.*
(3) *Any scheme locally of finite type over* $\mathbf{Z}$*.*
(4) *Any scheme locally of finite type over a Noetherian local ring of dimension* 1*.*
(5) *Any scheme locally of finite type over a Nagata ring of dimension* 1*.*

(6) *Any scheme locally of finite type over a Dedekind ring of characteristic zero.*
(7) *And so on.*

Proof. By Lemma 19.2 we only need to show that the rings mentioned above are J-2. For this see More on Algebra, Proposition 48.7. □

20. Quasi-finite morphisms

A solid treatment of quasi-finite morphisms is the basis of many developments further down the road. It will lead to various versions of Zariski's Main Theorem, behaviour of dimensions of fibres, descent for étale morphisms, etc, etc. Before reading this section it may be a good idea to take a look at the algebra results in Algebra, Section 122.

Recall that a finite type ring map $R \to A$ is quasi-finite at a prime $\mathfrak{q}$ if $\mathfrak{q}$ defines an isolated point of its fibre, see Algebra, Definition 122.3.

Definition 20.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *quasi-finite at a point* $x \in X$ if there exist an affine neighbourhood $\text{Spec}(A) = U \subset X$ of x and an affine open $\text{Spec}(R) = V \subset S$ such that $f(U) \subset V$, the ring map $R \to A$ is of finite type, and $R \to A$ is quasi-finite at the prime of A corresponding to x (see above).
(2) We say f is *locally quasi-finite* if f is quasi-finite at every point x of X.
(3) We say that f is *quasi-finite* if f is of finite type and every point x is an isolated point of its fibre.

Trivially, a locally quasi-finite morphism is locally of finite type. We will see below that a morphism f which is locally of finite type is quasi-finite at x if and only if x is isolated in its fibre. Moreover, the set of points at which a morphism is quasi-finite is open; we will see this in Section 56 on Zariski's Main Theorem.

Lemma 20.2. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If $\kappa(x)/\kappa(s)$ is an algebraic field extension, then*

(1) *x is a closed point of its fibre, and*
(2) *if in addition s is a closed point of S, then x is a closed point of X.*

Proof. The second statement follows from the first by elementary topology. According to Schemes, Lemma 18.5 to prove the first statement we may replace X by X_s and S by $\text{Spec}(\kappa(s))$. Thus we may assume that $S = \text{Spec}(k)$ is the spectrum of a field. In this case, let $\text{Spec}(A) = U \subset X$ be any affine open containing x. The point x corresponds to a prime ideal $\mathfrak{q} \subset A$ such that $\kappa(\mathfrak{q})/k$ is an algebraic field extension. By Algebra, Lemma 35.9 we see that $\mathfrak{q}$ is a maximal ideal, i.e., $x \in U$ is a closed point. Since the affine opens form a basis of the topology of X we conclude that $\{x\}$ is closed. □

The following lemma is a version of the Hilbert Nullstellensatz.

Lemma 20.3. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Assume f is locally of finite type. Then x is a closed point of its fibre if and only if $\kappa(x)/\kappa(s)$ is a finite field extension.*

Proof. If the extension is finite, then x is a closed point of the fibre by Lemma 20.2 above. For the converse, assume that x is a closed point of its fibre. Choose affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ such that $f(U) \subset V$. By

Lemma 15.2 the ring map $R \to A$ is of finite type. Let $\mathfrak{q} \subset A$, resp. $\mathfrak{p} \subset R$ be the prime ideal corresponding to x, resp. s. Consider the fibre ring $\overline{A} = A \otimes_R \kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{A}$ corresponding to $\mathfrak{q}$. The assumption that x is a closed point of its fibre implies that $\overline{\mathfrak{q}}$ is a maximal ideal of $\overline{A}$. Since $\overline{A}$ is an algebra of finite type over the field $\kappa(\mathfrak{p})$ we see by the Hilbert Nullstellensatz, see Algebra, Theorem 34.1, that $\kappa(\overline{\mathfrak{q}})$ is a finite extension of $\kappa(\mathfrak{p})$. Since $\kappa(s) = \kappa(\mathfrak{p})$ and $\kappa(x) = \kappa(\mathfrak{q}) = \kappa(\overline{\mathfrak{q}})$ we win. □

Lemma 20.4. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $g : S' \to S$ be any morphism. Denote $f' : X' \to S'$ the base change. If $x' \in X'$ maps to a point $x \in X$ which is closed in $X_{f(x)}$ then x' is closed in $X'_{f'(x')}$.*

Proof. The residue field $\kappa(x')$ is a quotient of $\kappa(f'(x')) \otimes_{\kappa(f(x))} \kappa(x)$, see Schemes, Lemma 17.5. Hence it is a finite extension of $\kappa(f'(x'))$ as $\kappa(x)$ is a finite extension of $\kappa(f(x))$ by Lemma 20.3. Thus we see that x' is closed in its fibre by applying that lemma one more time. □

Lemma 20.5. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If f is quasi-finite at x, then the residue field extension $\kappa(x)/\kappa(s)$ is finite.*

Proof. This is clear from Algebra, Definition 122.3. □

Lemma 20.6. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Let X_s be the fibre of f at s. Assume f is locally of finite type. The following are equivalent:*

(1) *The morphism f is quasi-finite at x.*
(2) *The point x is isolated in X_s.*
(3) *The point x is closed in X_s and there is no point $x' \in X_s$, $x' \neq x$ which specializes to x.*
(4) *For any pair of affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ and $x \in U$ corresponding to $\mathfrak{q} \subset A$ the ring map $R \to A$ is quasi-finite at $\mathfrak{q}$.*

Proof. Assume f is quasi-finite at x. By assumption there exist opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$, $x \in U$ and x an isolated point of U_s. Hence $\{x\} \subset U_s$ is an open subset. Since $U_s = U \cap X_s \subset X_s$ is also open we conclude that $\{x\} \subset X_s$ is an open subset also. Thus we conclude that x is an isolated point of X_s.

Note that X_s is a Jacobson scheme by Lemma 16.10 (and Lemma 15.4). If x is isolated in X_s, i.e., $\{x\} \subset X_s$ is open, then $\{x\}$ contains a closed point (by the Jacobson property), hence x is closed in X_s. It is clear that there is no point $x' \in X_s$, distinct from x, specializing to x.

Assume that x is closed in X_s and that there is no point $x' \in X_s$, distinct from x, specializing to x. Consider a pair of affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ and $x \in U$. Let $\mathfrak{q} \subset A$ correspond to x and $\mathfrak{p} \subset R$ correspond to s. By Lemma 15.2 the ring map $R \to A$ is of finite type. Consider the fibre ring $\overline{A} = A \otimes_R \kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{A}$ corresponding to $\mathfrak{q}$. Since $\mathrm{Spec}(\overline{A})$ is an open subscheme of the fibre X_s we see that $\overline{q}$ is a maximal ideal of $\overline{A}$ and that there is no point of $\mathrm{Spec}(\overline{A})$ specializing to $\overline{\mathfrak{q}}$. This implies that $\dim(\overline{A}_{\overline{q}}) = 0$. Hence by Algebra, Definition 122.3 we see that $R \to A$ is quasi-finite at $\mathfrak{q}$, i.e., $X \to S$ is quasi-finite at x by definition.

At this point we have shown conditions (1) – (3) are all equivalent. It is clear that (4) implies (1). And it is also clear that (2) implies (4) since if x is an isolated point of X_s then it is also an isolated point of U_s for any open U which contains it. $\square$

Lemma 20.7. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

1. *f is locally of finite type, and*
2. *$f^{-1}(\{s\})$ is a finite set.*

Then X_s is a finite discrete topological space, and f is quasi-finite at each point of X lying over s.

Proof. Suppose T is a scheme which (a) is locally of finite type over a field k, and (b) has finitely many points. Then Lemma 16.10 shows T is a Jacobson scheme. A finite Jacobson space is discrete, see Topology, Lemma 18.6. Apply this remark to the fibre X_s which is locally of finite type over $\text{Spec}(\kappa(s))$ to see the first statement. Finally, apply Lemma 20.6 to see the second. $\square$

Lemma 20.8. *Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. Then the following are equivalent*

1. *f is locally quasi-finite,*
2. *for every $s \in S$ the fibre X_s is a discrete topological space, and*
3. *for every morphism $\text{Spec}(k) \to S$ where k is a field the base change X_k has an underlying discrete topological space.*

Proof. It is immediate that (3) implies (2). Lemma 20.6 shows that (2) is equivalent to (1). Assume (2) and let $\text{Spec}(k) \to S$ be as in (3). Denote $s \in S$ the image of $\text{Spec}(k) \to S$. Then X_k is the base change of X_s via $\text{Spec}(k) \to \text{Spec}(\kappa(s))$. Hence every point of X_k is closed by Lemma 20.4. As $X_k \to \text{Spec}(k)$ is locally of finite type (by Lemma 15.4), we may apply Lemma 20.6 to conclude that every point of X_k is isolated, i.e., X_k has a discrete underlying topological space. $\square$

Lemma 20.9. *Let $f : X \to S$ be a morphism of schemes. Then f is quasi-finite if and only if f is locally quasi-finite and quasi-compact.*

Proof. Assume f is quasi-finite. It is quasi-compact by Definition 15.1. Let $x \in X$. We see that f is quasi-finite at x by Lemma 20.6. Hence f is quasi-compact and locally quasi-finite.

Assume f is quasi-compact and locally quasi-finite. Then f is of finite type. Let $x \in X$ be a point. By Lemma 20.6 we see that x is an isolated point of its fibre. The lemma is proved. $\square$

Lemma 20.10. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *f is quasi-finite, and*
2. *f is locally of finite type, quasi-compact, and has finite fibres.*

Proof. Assume f is quasi-finite. In particular f is locally of finite type and quasi-compact (since it is of finite type). Let $s \in S$. Since every $x \in X_s$ is isolated in X_s we see that $X_s = \bigcup_{x \in X_s} \{x\}$ is an open covering. As f is quasi-compact, the fibre X_s is quasi-compact. Hence we see that X_s is finite.

Conversely, assume f is locally of finite type, quasi-compact and has finite fibres. Then it is locally quasi-finite by Lemma 20.7. Hence it is quasi-finite by Lemma 20.9. $\square$

Recall that a ring map $R \to A$ is quasi-finite if it is of finite type and quasi-finite at *all* primes of A, see Algebra, Definition 122.3.

Lemma 20.11. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally quasi-finite.*
(2) *For every pair of affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is quasi-finite.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally quasi-finite.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is quasi-finite, for all $j \in J, i \in I_j$.*

Moreover, if f is locally quasi-finite then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally quasi-finite.

Proof. For a ring map $R \to A$ let us define $P(R \to A)$ to mean "$R \to A$ is quasi-finite" (see remark above lemma). We claim that P is a local property of ring maps. We check conditions (a), (b) and (c) of Definition 14.1. In the proof of Lemma 15.2 we have seen that (a), (b) and (c) hold for the property of being "of finite type". Note that, for a finite type ring map $R \to A$, the property $R \to A$ is quasi-finite at $\mathfrak{q}$ depends only on the local ring $A_\mathfrak{q}$ as an algebra over $R_\mathfrak{p}$ where $\mathfrak{p} = R \cap \mathfrak{q}$ (usual abuse of notation). Using these remarks (a), (b) and (c) of Definition 14.1 follow immediately. For example, suppose $R \to A$ is a ring map such that all of the ring maps $R \to A_{a_i}$ are quasi-finite for $a_1, \ldots, a_n \in A$ generating the unit ideal. We conclude that $R \to A$ is of finite type. Also, for any prime $\mathfrak{q} \subset A$ the local ring $A_\mathfrak{q}$ is isomorphic as an R-algebra to the local ring $(A_{a_i})_{\mathfrak{q}_i}$ for some i and some $\mathfrak{q}_i \subset A_{a_i}$. Hence we conclude that $R \to A$ is quasi-finite at $\mathfrak{q}$.

We conclude that Lemma 14.3 applies with P as in the previous paragraph. Hence it suffices to prove that f is locally quasi-finite is equivalent to f is locally of type P. Since $P(R \to A)$ is "$R \to A$ is quasi-finite" which means $R \to A$ is quasi-finite at every prime of A, this follows from Lemma 20.6. □

Lemma 20.12. *The composition of two morphisms which are locally quasi-finite is locally quasi-finite. The same is true for quasi-finite morphisms.*

Proof. In the proof of Lemma 20.11 we saw that P ="quasi-finite" is a local property of ring maps, and that a morphism of schemes is locally quasi-finite if and only if it is locally of type P as in Definition 14.2. Hence the first statement of the lemma follows from Lemma 14.5 combined with the fact that being quasi-finite is a property of ring maps that is stable under composition, see Algebra, Lemma 122.7. By the above, Lemma 20.9 and the fact that compositions of quasi-compact morphisms are quasi-compact, see Schemes, Lemma 19.4 we see that the composition of quasi-finite morphisms is quasi-finite. □

We will see later (Lemma 56.2) that the set U of the following lemma is open.

Lemma 20.13. *Let $f : X \to S$ be a morphism of schemes. Let $g : S' \to S$ be a morphism of schemes. Denote $f' : X' \to S'$ the base change of f by g and denote $g' : X' \to X$ the projection. Assume X is locally of finite type over S.*

(1) *Let $U \subset X$ (resp. $U' \subset X'$) be the set of points where f (resp. f') is quasi-finite. Then $U' = U \times_S S' = (g')^{-1}(U)$.*
(2) *The base change of a locally quasi-finite morphism is locally quasi-finite.*
(3) *The base change of a quasi-finite morphism is quasi-finite.*

Proof. The first and second assertion follow from the corresponding algebra result, see Algebra, Lemma 122.8 (combined with the fact that f' is also locally of finite type by Lemma 15.4). By the above, Lemma 20.9 and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 19.3 we see that the base change of a quasi-finite morphism is quasi-finite. $\square$

Lemma 20.14. *Let $f : X \to S$ be a morphism of schemes of finite type. Let $s \in S$. There are at most finitely many points of X lying over s at which f is quasi-finite.*

Proof. The fibre X_s is a scheme of finite type over a field, hence Noetherian (Lemma 15.6). Hence the topology on X_s is Noetherian (Properties, Lemma 5.5) and can have at most a finite number of isolated points (by elementary topology). Thus our lemma follows from Lemma 20.6. $\square$

Lemma 20.15. *Let $f : X \to Y$ be a morphism of schemes. If f is locally of finite type and a monomorphism, then f is separated and locally quasi-finite.*

Proof. A monomorphism is separated by Schemes, Lemma 23.3. A monomorphism is injective, hence we get f is quasi-finite at every $x \in X$ for example by Lemma 20.6. $\square$

Lemma 20.16. *Any immersion is locally quasi-finite.*

Proof. This is true because an open immersion is a local isomorphism and a closed immersion is clearly quasi-finite. $\square$

Lemma 20.17. *Let $X \to Y$ be a morphism of schemes over a base scheme S. Let $x \in X$. If $X \to S$ is quasi-finite at x, then $X \to Y$ is quasi-finite at x. If X is locally quasi-finite over S, then $X \to Y$ is locally quasi-finite.*

Proof. Via Lemma 20.11 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is quasi-finite, then $B \to C$ is quasi-finite. This follows from Algebra, Lemma 122.6 with $R = A$, $S = S' = C$ and $R' = B$. $\square$

Lemma 20.18. *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. If f is surjective, $g \circ f$ locally quasi-finite, and g locally of finite type, then $g : Y \to S$ is locally quasi-finite.*

Proof. Let $x \in X$ with images $y \in Y$ and $s \in S$. Since $g \circ f$ is locally quasi-finite by Lemma 20.5 the extension $\kappa(x)/\kappa(s)$ is finite. Hence $\kappa(y)/\kappa(s)$ is finite. Hence y is a closed point of Y_s by Lemma 20.2. Since f is surjective, we see that every point of Y is closed in its fibre over S. Thus by Lemma 20.6 we conclude that g is quasi-finite at every point. $\square$

21. Morphisms of finite presentation

Recall that a ring map $R \to A$ is of finite presentation if A is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ as an R-algebra for some n, m and some polynomials f_j, see Algebra, Definition 6.1.

Definition 21.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is of *finite presentation at $x \in X$* if there exists an affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite presentation.
(2) We say that f is *locally of finite presentation* if it is of finite presentation at every point of X.
(3) We say that f is of *finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation. Later we will characterize morphisms which are locally of finite presentation as those morphisms such that

$$\text{colim} \, \text{Mor}_S(T_i, X) = \text{Mor}_S(\lim T_i, X)$$

for any directed system of affine schemes T_i over S. See Limits, Proposition 6.1. In Limits, Section 10 we show that, if $S = \lim_i S_i$ is a limit of affine schemes, any scheme X of finite presentation over S descends to a scheme X_i over S_i for some i.

Lemma 21.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally of finite presentation.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation.*
(3) *There exist an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of finite presentation.*
(4) *There exist an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of finite presentation, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of finite presentation then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of finite presentation.

Proof. This follows from Lemma 14.4 if we show that the property "$R \to A$ is of finite presentation" is local. We check conditions (a), (b) and (c) of Definition 14.1. By Algebra, Lemma 14.2 being of finite presentation is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 6.2 being of finite presentation is stable under composition and trivially for any ring R the ring map $R \to R_f$ is of finite presentation. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 23.3. $\square$

Lemma 21.3. *The composition of two morphisms which are locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.*

Proof. In the proof of Lemma 21.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 14.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under composition, see Algebra, Lemma 6.2. By the above and the fact that compositions of quasi-compact, quasi-separated morphisms are quasi-compact and quasi-separated, see Schemes, Lemmas 19.4 and 21.12 we see that the composition of morphisms of finite presentation is of finite presentation. □

Lemma 21.4. *The base change of a morphism which is locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.*

Proof. In the proof of Lemma 21.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 14.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under base change, see Algebra, Lemma 14.2. By the above and the fact that a base change of a quasi-compact, quasi-separated morphism is quasi-compact and quasi-separated, see Schemes, Lemmas 19.3 and 21.12 we see that the base change of a morphism of finite presentation is a morphism of finite presentation. □

Lemma 21.5. *Any open immersion is locally of finite presentation.*

Proof. This is true because an open immersion is a local isomorphism. □

Lemma 21.6. *Any open immersion is of finite presentation if and only if it is quasi-compact.*

Proof. We have seen (Lemma 21.5) that an open immersion is locally of finite presentation. We have seen (Schemes, Lemma 23.8) that an immersion is separated and hence quasi-separated. From this and Definition 21.1 the lemma follows. □

Lemma 21.7. *A closed immersion $i : Z \to X$ is of finite presentation if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).*

Proof. On any affine open $\mathrm{Spec}(R) \subset X$ we have $i^{-1}(\mathrm{Spec}(R)) = \mathrm{Spec}(R/I)$ and $\mathcal{I} = \widetilde{I}$. Moreover, $\mathcal{I}$ is of finite type if and only if I is a finite R-module for every such affine open (see Properties, Lemma 16.1). And R/I is of finite presentation over R if and only if I is a finite R-module. Hence we win. □

Lemma 21.8. *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Omitted. □

Lemma 21.9. *Let $f : X \to S$ be a morphism.*

(1) *If S is locally Noetherian and f locally of finite type then f is locally of finite presentation.*
(2) *If S is locally Noetherian and f of finite type then f is of finite presentation.*

Proof. The first statement follows from the fact that a ring of finite type over a Noetherian ring is of finite presentation, see Algebra, Lemma 31.4. Suppose that f is of finite type and S is locally Noetherian. Then f is quasi-compact and locally of finite presentation by (1). Hence it suffices to prove that f is quasi-separated. This follows from Lemma 15.7 (and Lemma 21.8). □

Lemma 21.10. *Let S be a scheme which is quasi-compact and quasi-separated. If X is of finite presentation over S, then X is quasi-compact and quasi-separated.*

Proof. Omitted. $\square$

Lemma 21.11. *Let $f : X \to Y$ be a morphism of schemes over S.*

(1) *If X is locally of finite presentation over S and Y is locally of finite type over S, then f is locally of finite presentation.*
(2) *If X is of finite presentation over S and Y is quasi-separated and locally of finite type over S, then f is of finite presentation.*

Proof. Proof of (1). Via Lemma 21.2 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is of finite presentation and $A \to B$ is of finite type, then $B \to C$ is of finite presentation. See Algebra, Lemma 6.2.

Part (2) follows from (1) and Schemes, Lemmas 21.13 and 21.14. $\square$

Lemma 21.12. *Let $f : X \to Y$ be a morphism of schemes with diagonal $\Delta : X \to X \times_Y X$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.*

Proof. Note that Δ is a morphism of schemes over X (via the second projection $X \times_Y X \to X$). Assume f is locally of finite type. Note that X is of finite presentation over X and $X \times_Y X$ is locally of finite type over X (by Lemma 15.4). Thus the first statement holds by Lemma 21.11. The second statement follows from the first, the definitions, and the fact that a diagonal morphism is a monomorphism, hence separated (Schemes, Lemma 23.3). $\square$

22. Constructible sets

Constructible and locally constructible sets of schemes have been discussed in Properties, Section 2. In this section we prove some results concerning images and inverse images of (locally) constructible sets. The main result is Chevalley's theorem which states that the image of a locally constructible set under a morphism of finite presentation is locally constructible.

Lemma 22.1. *Let $f : X \to Y$ be a morphism of schemes. Let $E \subset Y$ be a subset. If E is (locally) constructible in Y, then $f^{-1}(E)$ is (locally) constructible in X.*

Proof. To show that the inverse image of every constructible subset is constructible it suffices to show that the inverse image of every retrocompact open V of Y is retrocompact in X, see Topology, Lemma 15.3. The significance of V being retrocompact in Y is just that the open immersion $V \to Y$ is quasi-compact. Hence the base change $f^{-1}(V) = X \times_Y V \to X$ is quasi-compact too, see Schemes, Lemma 19.3. Hence we see $f^{-1}(V)$ is retrocompact in X. Suppose E is locally constructible in Y. Choose $x \in X$. Choose an affine neighbourhood V of $f(x)$ and an affine neighbourhood $U \subset X$ of x such that $f(U) \subset V$. Thus we think of $f|_U : U \to V$ as a morphism into V. By Properties, Lemma 2.1 we see that $E \cap V$ is constructible in V. By the constructible case we see that $(f|_U)^{-1}(E \cap V)$ is constructible in U. Since $(f|_U)^{-1}(E \cap V) = f^{-1}(E) \cap U$ we win. $\square$

Lemma 22.2. *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *f is quasi-compact and locally of finite presentation, and*
(2) *Y is quasi-compact and quasi-separated.*

Then the image of every constructible subset of X is constructible in Y.

Proof. By Properties, Lemma 2.5 it suffices to prove this lemma in case Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \ldots \cup U_n$ with each U_i affine open in X. If $E \subset X$ is constructible, then each $E \cap U_i$ is constructible too, see Topology, Lemma 15.4. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 29.10. $\square$

Theorem 22.3 (Chevalley's Theorem)**.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is quasi-compact and locally of finite presentation. Then the image of every locally constructible subset is locally constructible.*

Proof. Let $E \subset X$ be locally constructible. We have to show that $f(E)$ is locally constructible too. We will show that $f(E) \cap V$ is constructible for any affine open $V \subset Y$. Thus we reduce to the case where Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \ldots \cup U_n$ with each U_i affine open in X. If $E \subset X$ is locally constructible, then each $E \cap U_i$ is constructible, see Properties, Lemma 2.1. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 29.10. $\square$

Lemma 22.4. *Let X be a scheme. Let $x \in X$. Let $E \subset X$ be a locally constructible subset. If $\{x' \mid x' \leadsto x\} \subset E$, then E contains an open neighbourhood of x.*

Proof. Assume $\{x' \mid x' \leadsto x\} \subset E$. We may assume X is affine. In this case E is constructible, see Properties, Lemma 2.1. In particular, also the complement E^c is constructible. By Algebra, Lemma 29.4 we can find a morphism of affine schemes $f : Y \to X$ such that $E^c = f(Y)$. Let $Z \subset X$ be the scheme theoretic image of f. By Lemma 6.5 and the assumption $\{x' \mid x' \leadsto x\} \subset E$ we see that $x \notin Z$. Hence $X \setminus Z \subset E$ is an open neighbourhood of x contained in E. $\square$

23. Open morphisms

Definition 23.1. Let $f : X \to S$ be a morphism.

(1) We say f is *open* if the map on underlying topological spaces is open.
(2) We say f is *universally open* if for any morphism of schemes $S' \to S$ the base change $f' : X_{S'} \to S'$ is open.

According to Topology, Lemma 19.7 generalizations lift along certain types of open maps of topological spaces. In fact generalizations lift along any open morphism of schemes (see Lemma 23.5). Also, we will see that generalizations lift along flat morphisms of schemes (Lemma 25.9). This sometimes in turn implies that the morphism is open.

Lemma 23.2. *Let $f : X \to S$ be a morphism.*

(1) *If f is locally of finite presentation and generalizations lift along f, then f is open.*
(2) *If f is locally of finite presentation and generalizations lift along every base change of f, then f is universally open.*

Proof. It suffices to prove the first assertion. This reduces to the case where both X and S are affine. In this case the result follows from Algebra, Lemma 41.3 and Proposition 41.8. $\square$

See also Lemma 25.10 for the case of a morphism flat of finite presentation.

Lemma 23.3. *A composition of (universally) open morphisms is (universally) open.*

Proof. Omitted. □

Lemma 23.4. *Let k be a field. Let X be a scheme over k. The structure morphism $X \to \mathrm{Spec}(k)$ is universally open.*

Proof. Let $S \to \mathrm{Spec}(k)$ be a morphism. We have to show that the base change $X_S \to S$ is open. The question is local on S and X, hence we may assume that S and X are affine. In this case the result is Algebra, Lemma 41.10. □

Lemma 23.5. *Let $\varphi : X \to Y$ be a morphism of schemes. If φ is open, then φ is generizing (i.e., generalizations lift along φ). If φ is universally open, then φ is universally generizing.*

Proof. Assume φ is open. Let $y' \rightsquigarrow y$ be a specialization of points of Y. Let $x \in X$ with $\varphi(x) = y$. Choose affine opens $U \subset X$ and $V \subset Y$ such that $\varphi(U) \subset V$ and $x \in U$. Then also $y' \in V$. Hence we may replace X by U and Y by V and assume X, Y affine. The affine case is Algebra, Lemma 41.2 (combined with Algebra, Lemma 41.3). □

Lemma 23.6. *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be open and surjective such that the base change $f' : X' \to Y'$ is quasi-compact. Then f is quasi-compact.*

Proof. Let $V \subset Y$ be a quasi-compact open. As g is open and surjective we can find a quasi-compact open $W' \subset W$ such that $g(W') = V$. By assumption $(f')^{-1}(W')$ is quasi-compact. The image of $(f')^{-1}(W')$ in X is equal to $f^{-1}(V)$, see Lemma 9.3. Hence $f^{-1}(V)$ is quasi-compact as the image of a quasi-compact space, see Topology, Lemma 12.7. Thus f is quasi-compact. □

24. Submersive morphisms

Definition 24.1. Let $f : X \to Y$ be a morphism of schemes.

(1) We say f is *submersive*[7] if the continuous map of underlying topological spaces is submersive, see Topology, Definition 6.3.
(2) We say f is *universally submersive* if for every morphism of schemes $Y' \to Y$ the base change $Y' \times_Y X \to Y'$ is submersive.

We note that a submersive morphism is in particular surjective.

Lemma 24.2. *The base change of a universally submersive morphism of schemes by any morphism of schemes is universally submersive.*

Proof. This is immediate from the definition. □

Lemma 24.3. *The composition of a pair of (universally) submersive morphisms of schemes is (universally) submersive.*

Proof. Omitted. □

[7] This is very different from the notion of a submersion of differential manifolds.

25. Flat morphisms

Flatness is one of the most important technical tools in algebraic geometry. In this section we introduce this notion. We intentionally limit the discussion to straightforward observations, apart from Lemma 25.10. A very important class of results, namely criteria for flatness, are discussed in Algebra, Sections 99, 101, 128, and More on Morphisms, Section 16. There is a chapter dedicated to advanced material on flat morphisms of schemes, namely More on Flatness, Section 1.

Recall that a module M over a ring R is *flat* if the functor $- \otimes_R M : \text{Mod}_R \to \text{Mod}_R$ is exact. A ring map $R \to A$ is said to be *flat* if A is flat as an R-module. See Algebra, Definition 39.1.

Definition 25.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

(1) We say f is *flat at a point* $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is flat over the local ring $\mathcal{O}_{S,f(x)}$.
(2) We say that $\mathcal{F}$ is *flat over* S *at a point* $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module.
(3) We say f is *flat* if f is flat at every point of X.
(4) We say that $\mathcal{F}$ is *flat over* S if $\mathcal{F}$ is flat over S at every point x of X.

Thus we see that f is flat if and only if the structure sheaf $\mathcal{O}_X$ is flat over S.

Lemma 25.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. The following are equivalent*

(1) *The sheaf $\mathcal{F}$ is flat over S.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the $\mathcal{O}_S(V)$-module $\mathcal{F}(U)$ is flat.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the modules $\mathcal{F}|_{U_i}$ is flat over V_j, for all $j \in J, i \in I_j$.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $\mathcal{F}(U_i)$ is a flat $\mathcal{O}_S(V_j)$-module, for all $j \in J, i \in I_j$.*

Moreover, if $\mathcal{F}$ is flat over S then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $\mathcal{F}|_U$ is flat over V.

Proof. Let $R \to A$ be a ring map. Let M be an A-module. If M is R-flat, then for all primes $\mathfrak{q}$ the module $M_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ with $\mathfrak{p}$ the prime of R lying under $\mathfrak{q}$. Conversely, if $M_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ for all primes $\mathfrak{q}$ of A, then M is flat over R. See Algebra, Lemma 39.18. This equivalence easily implies the statements of the lemma. $\square$

Lemma 25.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is flat.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is flat.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is flat.*

(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is flat, for all $j \in J, i \in I_j$.*

Moreover, if f is flat then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is flat.

Proof. This is a special case of Lemma 25.2 above. $\square$

Lemma 25.4. *Let $f : X \to Y$ be an affine morphism of schemes over a base scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is flat over S if and only if $f_*\mathcal{F}$ is flat over S.*

Proof. By Lemma 25.2 and the fact that f is an affine morphism, this reduces us to the affine case. Say $X \to Y \to S$ corresponds to the ring maps $C \leftarrow B \leftarrow A$. Let N be the C-module corresponding to $\mathcal{F}$. Recall that $f_*\mathcal{F}$ corresponds to N viewed as a B-module, see Schemes, Lemma 7.3. Thus the result is clear. $\square$

Lemma 25.5. *Let $X \to Y \to Z$ be morphisms of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image y in Y. If $\mathcal{F}$ is flat over Y at x, and Y is flat over Z at y, then $\mathcal{F}$ is flat over Z at x.*

Proof. See Algebra, Lemma 39.4. $\square$

Lemma 25.6. *The composition of flat morphisms is flat.*

Proof. This is a special case of Lemma 25.5. $\square$

Lemma 25.7. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Let $g : S' \to S$ be a morphism of schemes. Denote $g' : X' = X_{S'} \to X$ the projection. Let $x' \in X'$ be a point with image $x = g'(x') \in X$. If $\mathcal{F}$ is flat over S at x, then $(g')^*\mathcal{F}$ is flat over S' at x'. In particular, if $\mathcal{F}$ is flat over S, then $(g')^*\mathcal{F}$ is flat over S'.*

Proof. See Algebra, Lemma 39.7. $\square$

Lemma 25.8. *The base change of a flat morphism is flat.*

Proof. This is a special case of Lemma 25.7. $\square$

Lemma 25.9. *Let $f : X \to S$ be a flat morphism of schemes. Then generalizations lift along f, see Topology, Definition 19.4.*

Proof. See Algebra, Section 41. $\square$

Lemma 25.10. *A flat morphism locally of finite presentation is universally open.*

Proof. This follows from Lemmas 25.9 and Lemma 23.2 above. We can also argue directly as follows.

Let $f : X \to S$ be flat and locally of finite presentation. By Lemmas 25.8 and 21.4 any base change of f is flat and locally of finite presentation. Hence it suffices to show f is open. To show f is open it suffices to show that we may cover X by open affines $X = \bigcup U_i$ such that $U_i \to S$ is open. We may cover X by affine opens $U_i \subset X$ such that each U_i maps into an affine open $V_i \subset S$ and such that the induced ring map $\mathcal{O}_S(V_i) \to \mathcal{O}_X(U_i)$ is flat and of finite presentation (Lemmas 25.3 and 21.2). Then $U_i \to V_i$ is open by Algebra, Proposition 41.8 and the proof is complete. $\square$

Lemma 25.11. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f locally finite presentation, $\mathcal{F}$ of finite type, $X = Supp(\mathcal{F})$, and $\mathcal{F}$ flat over Y. Then f is universally open.*

Proof. By Lemmas 25.7, 21.4, and 5.3 the assumptions are preserved under base change. By Lemma 23.2 it suffices to show that generalizations lift along f. This follows from Algebra, Lemma 41.12. $\square$

Lemma 25.12. *Let $f : X \to Y$ be a quasi-compact, surjective, flat morphism. A subset $T \subset Y$ is open (resp. closed) if and only $f^{-1}(T)$ is open (resp. closed). In other words, f is a submersive morphism.*

Proof. The question is local on Y, hence we may assume that Y is affine. In this case X is quasi-compact as f is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ as a finite union of affine opens. Then $f' : X' = X_1 \amalg \ldots \amalg X_n \to Y$ is a surjective flat morphism of affine schemes. Note that for $T \subset Y$ we have $(f')^{-1}(T) = f^{-1}(T) \cap X_1 \amalg \ldots \amalg f^{-1}(T) \cap X_n$. Hence, $f^{-1}(T)$ is open if and only if $(f')^{-1}(T)$ is open. Thus we may assume both X and Y are affine.

Let $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a surjective morphism of affine schemes corresponding to a flat ring map $A \to B$. Suppose that $f^{-1}(T)$ is closed, say $f^{-1}(T) = V(J)$ for $J \subset B$ an ideal. Then $T = f(f^{-1}(T)) = f(V(J))$ is the image of $\mathrm{Spec}(B/J) \to \mathrm{Spec}(A)$ (here we use that f is surjective). On the other hand, generalizations lift along f (Lemma 25.9). Hence by Topology, Lemma 19.6 we see that $Y \setminus T = f(X \setminus f^{-1}(T))$ is stable under generalization. Hence T is stable under specialization (Topology, Lemma 19.2). Thus T is closed by Algebra, Lemma 41.5. $\square$

Lemma 25.13. *Let $h : X \to Y$ be a morphism of schemes over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Let $x \in X$ with $y = h(x) \in Y$. If h is flat at x, then*

$$\mathcal{G} \text{ flat over } S \text{ at } y \Leftrightarrow h^*\mathcal{G} \text{ flat over } S \text{ at } x.$$

In particular: If h is surjective and flat, then $\mathcal{G}$ is flat over S, if and only if $h^*\mathcal{G}$ is flat over S. If h is surjective and flat, and X is flat over S, then Y is flat over S.

Proof. You can prove this by applying Algebra, Lemma 39.9. Here is a direct proof. Let $s \in S$ be the image of y. Consider the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. By assumption the ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is faithfully flat, see Algebra, Lemma 39.17. Let $N = \mathcal{G}_y$. Note that $h^*\mathcal{G}_x = N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$, see Sheaves, Lemma 26.4. Let $M' \to M$ be an injection of $\mathcal{O}_{S,s}$-modules. By the faithful flatness mentioned above we have

$$\begin{aligned} &\mathrm{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \to M \otimes_{\mathcal{O}_{S,s}} N) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \\ = &\mathrm{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to M \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}) \end{aligned}$$

Hence the equivalence of the lemma follows from the second characterization of flatness in Algebra, Lemma 39.5. $\square$

Lemma 25.14. *Let $f : Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module with scheme theoretic support $Z \subset X$. If f is flat, then $f^{-1}(Z)$ is the scheme theoretic support of $f^*\mathcal{F}$.*

Proof. Using the characterization of scheme theoretic support on affines as given in Lemma 5.4 we reduce to Algebra, Lemma 40.4. $\square$

Lemma 25.15. *Let $f : X \to Y$ be a flat morphism of schemes. Let $V \subset Y$ be a retrocompact open which is scheme theoretically dense. Then $f^{-1}V$ is scheme theoretically dense in X.*

Proof. We will use the characterization of Lemma 7.5. We have to show that for any open $U \subset X$ the map $\mathcal{O}_X(U) \to \mathcal{O}_X(U \cap f^{-1}V)$ is injective. It suffices to prove this when U is an affine open which maps into an affine open $W \subset Y$. Say $W = \text{Spec}(A)$ and $U = \text{Spec}(B)$. Then $V \cap W = D(f_1) \cup \ldots \cup D(f_n)$ for some $f_i \in A$, see Algebra, Lemma 29.1. Thus we have to show that $B \to B_{f_1} \times \ldots \times B_{f_n}$ is injective. We are given that $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective and that $A \to B$ is flat. Since $B_{f_i} = A_{f_i} \otimes_A B$ we win. $\square$

Lemma 25.16. *Let $f : X \to Y$ be a flat morphism of schemes. Let $g : V \to Y$ be a quasi-compact morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of g and let $Z' \subset X$ be the scheme theoretic image of the base change $V \times_Y X \to X$. Then $Z' = f^{-1}Z$.*

Proof. Recall that Z is cut out by $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \to g_*\mathcal{O}_V)$ and Z' is cut out by $\mathcal{I}' = \text{Ker}(\mathcal{O}_X \to (V \times_Y X \to X)_*\mathcal{O}_{V \times_Y X})$, see Lemma 6.3. Hence the question is local on X and Y and we may assume X and Y affine. Note that we may replace V by $\coprod V_i$ where $V = V_1 \cup \ldots \cup V_n$ is a finite affine open covering. Hence we may assume g is affine. In this case $(V \times_Y X \to X)_*\mathcal{O}_{V \times_Y X}$ is the pullback of $g_*\mathcal{O}_V$ by f. Since f is flat we conclude that $f^*\mathcal{I} = \mathcal{I}'$ and the lemma holds. $\square$

26. Flat closed immersions

Connected components of schemes are not always open. But they do always have a canonical scheme structure. We explain this in this section.

Lemma 26.1. *Let X be a scheme. The rule which associates to a closed subscheme of X its underlying closed subset defines a bijection*

$$\left\{\begin{matrix} \textit{closed subschemes } Z \subset X \\ \textit{such that } Z \to X \textit{ is flat} \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \textit{closed subsets } Z \subset X \\ \textit{closed under generalizations} \end{matrix}\right\}$$

If $Z \subset X$ is such a closed subscheme, every morphism of schemes $g : Y \to X$ with $g(Y) \subset Z$ set theoretically factors (scheme theoretically) through Z.

Proof. The affine case of the bijection is Algebra, Lemma 108.4. For general schemes X the bijection follows by covering X by affines and glueing. Details omitted. For the final assertion, observe that the projection $Z \times_{X,g} Y \to Y$ is a flat (Lemma 25.8) closed immersion which is bijective on underlying topological spaces and hence must be an isomorphism by the bijection esthablished in the first part of the proof. $\square$

Lemma 26.2. *A flat closed immersion of finite presentation is the open immersion of an open and closed subscheme.*

Proof. The affine case is Algebra, Lemma 108.5. In general the lemma follows by covering X by affines. Details omitted. $\square$

Note that a connected component T of a scheme X is a closed subset stable under generalization. Hence the following definition makes sense.

Definition 26.3. Let X be a scheme. Let $T \subset X$ be a connected component. The *canonical scheme structure on T* is the unique scheme structure on T such that the closed immersion $T \to X$ is flat, see Lemma 26.1.

It turns out that we can determine when every finite flat $\mathcal{O}_X$-module is finite locally free using the previous lemma.

Lemma 26.4. *Let X be a scheme. The following are equivalent*

(1) *every finite flat quasi-coherent $\mathcal{O}_X$-module is finite locally free, and*
(2) *every closed subset $Z \subset X$ which is closed under generalizations is open.*

Proof. In the affine case this is Algebra, Lemma 108.6. The scheme case does not follow directly from the affine case, so we simply repeat the arguments.

Assume (1). Consider a closed immersion $i : Z \to X$ such that i is flat. Then $i_*\mathcal{O}_Z$ is quasi-coherent and flat, hence finite locally free by (1). Thus $Z = \text{Supp}(i_*\mathcal{O}_Z)$ is also open and we see that (2) holds. Hence the implication (1) $\Rightarrow$ (2) follows from the characterization of flat closed immersions in Lemma 26.1.

For the converse assume that X satisfies (2). Let $\mathcal{F}$ be a finite flat quasi-coherent $\mathcal{O}_X$-module. The support $Z = \text{Supp}(\mathcal{F})$ of $\mathcal{F}$ is closed, see Modules, Lemma 9.6. On the other hand, if $x \rightsquigarrow x'$ is a specialization, then by Algebra, Lemma 78.5 the module $\mathcal{F}_{x'}$ is free over $\mathcal{O}_{X,x'}$, and

$$\mathcal{F}_x = \mathcal{F}_{x'} \otimes_{\mathcal{O}_{X,x'}} \mathcal{O}_{X,x}.$$

Hence $x' \in \text{Supp}(\mathcal{F}) \Rightarrow x \in \text{Supp}(\mathcal{F})$, in other words, the support is closed under generalization. As X satisfies (2) we see that the support of $\mathcal{F}$ is open and closed. The modules $\wedge^i(\mathcal{F})$, $i = 1, 2, 3, \ldots$ are finite flat quasi-coherent $\mathcal{O}_X$-modules also, see Modules, Section 21. Note that $\text{Supp}(\wedge^{i+1}(\mathcal{F})) \subset \text{Supp}(\wedge^i(\mathcal{F}))$. Thus we see that there exists a decomposition

$$X = U_0 \amalg U_1 \amalg U_2 \amalg \ldots$$

by open and closed subsets such that the support of $\wedge^i(\mathcal{F})$ is $U_i \cup U_{i+1} \cup \ldots$ for all i. Let x be a point of X, and say $x \in U_r$. Note that $\wedge^i(\mathcal{F})_x \otimes \kappa(x) = \wedge^i(\mathcal{F}_x \otimes \kappa(x))$. Hence, $x \in U_r$ implies that $\mathcal{F}_x \otimes \kappa(x)$ is a vector space of dimension r. By Nakayama's lemma, see Algebra, Lemma 20.1 we can choose an affine open neighbourhood $U \subset U_r \subset X$ of x and sections $s_1, \ldots, s_r \in \mathcal{F}(U)$ such that the induced map

$$\mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F}|_U, \quad (f_1, \ldots, f_r) \longmapsto \sum f_i s_i$$

is surjective. This means that $\wedge^r(\mathcal{F}|_U)$ is a finite flat quasi-coherent $\mathcal{O}_U$-module whose support is all of U. By the above it is generated by a single element, namely $s_1 \wedge \ldots \wedge s_r$. Hence $\wedge^r(\mathcal{F}|_U) \cong \mathcal{O}_U/\mathcal{I}$ for some quasi-coherent sheaf of ideals $\mathcal{I}$ such that $\mathcal{O}_U/\mathcal{I}$ is flat over $\mathcal{O}_U$ and such that $V(\mathcal{I}) = U$. It follows that $\mathcal{I} = 0$ by applying Lemma 26.1. Thus $s_1 \wedge \ldots \wedge s_r$ is a basis for $\wedge^r(\mathcal{F}|_U)$ and it follows that the displayed map is injective as well as surjective. This proves that $\mathcal{F}$ is finite locally free as desired. $\square$

27. Generic flatness

A scheme of finite type over an integral base is flat over a dense open of the base. In Algebra, Section 118 we proved a Noetherian version, a version for morphisms of finite presentation, and a general version. We only state and prove the general version here. However, it turns out that this will be superseded by Proposition 27.2 which shows the result holds if we only assume the base is reduced.

Proposition 27.1 (Generic flatness)**.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

1. *S is integral,*
2. *f is of finite type, and*
3. *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over $\mathcal{O}_{X_U}$.

Proof. As S is integral it is irreducible (see Properties, Lemma 3.4) and any nonempty open is dense. Hence we may replace S by an affine open of S and assume that $S = \text{Spec}(A)$ is affine. As S is integral we see that A is a domain. As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1,\ldots,n} X_i$. Write $X_i = \text{Spec}(B_i)$. Then B_i is a finite type A-algebra, see Lemma 15.2. Moreover there are finite type B_i-modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i-module M_i, see Properties, Lemma 16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = \text{Spec}(B_i)$. Set $M_{ij} = B_i/I_{ij}$ and think of it as a B_i-module. Then $V(I_{ij}) = \text{Supp}(M_{ij})$ and M_{ij} is a finite B_i-module.

At this point we apply Algebra, Lemma 118.3 the pairs $(A \to B_i, M_{ij})$ and to the pairs $(A \to B_i, M_i)$. Thus we obtain nonzero $f_{ij}, f_i \in A$ such that (a) $A_{f_{ij}} \to B_{i,f_{ij}}$ is flat and of finite presentation and $M_{ij,f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation over $B_{i,f_{ij}}$, and (b) B_{i,f_i} is flat and of finite presentation over A_f and M_{i,f_i} is flat and of finite presentation over B_{i,f_i}. Set $f = (\prod f_i)(\prod f_{ij})$. We claim that taking $U = D(f)$ works.

To prove our claim we may replace A by A_f, i.e., perform the base change by $U = \text{Spec}(A_f) \to S$. After this base change we see that each of $A \to B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i. This already proves that $X \to S$ is quasi-compact, locally of finite presentation, flat, and that $\mathcal{F}$ is flat over S and of finite presentation over $\mathcal{O}_X$, see Lemma 21.2 and Properties, Lemma 16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \text{Supp}(M_{ij})$ is a quasi-compact open of X_i, see Algebra, Lemma 40.8. Hence we see that $X \to S$ is quasi-separated by Schemes, Lemma 21.6. This proves the proposition. □

It actually turns out that there is also a version of generic flatness over an arbitrary reduced base. Here it is.

Proposition 27.2 (Generic flatness, reduced case)**.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

1. *S is reduced,*
2. *f is of finite type, and*
3. *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over $\mathcal{O}_{X_U}$.

Proof. For the impatient reader: This proof is a repeat of the proof of Proposition 27.1 using Algebra, Lemma 118.7 instead of Algebra, Lemma 118.3.

Since being flat and being of finite presentation is local on the base, see Lemmas 25.2 and 21.2, we may work affine locally on S. Thus we may assume that $S = \mathrm{Spec}(A)$, where A is a reduced ring (see Properties, Lemma 3.2). As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1,\ldots,n} X_i$. Write $X_i = \mathrm{Spec}(B_i)$. Then B_i is a finite type A-algebra, see Lemma 15.2. Moreover there are finite type B_i-modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i-module M_i, see Properties, Lemma 16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = \mathrm{Spec}(B_i)$. Set $M_{ij} = B_i/I_{ij}$ and think of it as a B_i-module. Then $V(I_{ij}) = \mathrm{Supp}(M_{ij})$ and M_{ij} is a finite B_i-module.

At this point we apply Algebra, Lemma 118.7 the pairs $(A \to B_i, M_{ij})$ and to the pairs $(A \to B_i, M_i)$. Thus we obtain dense opens $U(A \to B_i, M_{ij}) \subset S$ and dense opens $U(A \to B_i, M_i) \subset S$ with notation as in Algebra, Equation (118.3.2). Since a finite intersection of dense opens is dense open, we see that

$$U = \bigcap\nolimits_{i,j} U(A \to B_i, M_{ij}) \quad \cap \quad \bigcap\nolimits_i U(A \to B_i, M_i)$$

is open and dense in S. We claim that U is the desired open.

Pick $u \in U$. By definition of the loci $U(A \to B_i, M_{ij})$ and $U(A \to B, M_i)$ there exist $f_{ij}, f_i \in A$ such that (a) $u \in D(f_i)$ and $u \in D(f_{ij})$, (b) $A_{f_{ij}} \to B_{i,f_{ij}}$ is flat and of finite presentation and $M_{ij,f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation over $B_{i,f_{ij}}$, and (c) B_{i,f_i} is flat and of finite presentation over A_f and M_{i,f_i} is flat and of finite presentation over B_{i,f_i}. Set $f = (\prod f_i)(\prod f_{ij})$. Now it suffices to prove that $X \to S$ is flat and of finite presentation over $D(f)$ and that $\mathcal{F}$ restricted to $X_{D(f)}$ is flat over $D(f)$ and of finite presentation over the structure sheaf of $X_{D(f)}$.

Hence we may replace A by A_f, i.e., perform the base change by $\mathrm{Spec}(A_f) \to S$. After this base change we see that each of $A \to B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i. This already proves that $X \to S$ is quasi-compact, locally of finite presentation, flat, and that $\mathcal{F}$ is flat over S and of finite presentation over $\mathcal{O}_X$, see Lemma 21.2 and Properties, Lemma 16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \mathrm{Supp}(M_{ij})$ is a quasi-compact open of X_i, see Algebra, Lemma 40.8. Hence we see that $X \to S$ is quasi-separated by Schemes, Lemma 21.6. This proves the proposition. $\square$

Remark 27.3. The results above are a first step towards more refined flattening techniques for morphisms of schemes. The article [GR71] by Raynaud and Gruson contains many wonderful results in this direction.

28. Morphisms and dimensions of fibres

Let X be a topological space, and $x \in X$. Recall that we have defined $\dim_x(X)$ as the minimum of the dimensions of the open neighbourhoods of x in X. See Topology, Definition 10.1.

Lemma 28.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ and set $s = f(x)$. Assume f is locally of finite type. Then*

$$\dim_x(X_s) = \dim(\mathcal{O}_{X_s,x}) + trdeg_{\kappa(s)}(\kappa(x)).$$

Proof. This immediately reduces to the case $S = s$, and X affine. In this case the result follows from Algebra, Lemma 116.3. $\square$

Lemma 28.2. *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. Let $x \in X$ and set $y = f(x)$, $s = g(y)$. Assume f and g locally of finite type. Then*

$$\dim_x(X_s) \leq \dim_x(X_y) + \dim_y(Y_s).$$

Moreover, equality holds if $\mathcal{O}_{X_s,x}$ is flat over $\mathcal{O}_{Y_s,y}$, which holds for example if $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$.

Proof. Note that $\text{trdeg}_{\kappa(s)}(\kappa(x)) = \text{trdeg}_{\kappa(y)}(\kappa(x)) + \text{trdeg}_{\kappa(s)}(\kappa(y))$. Thus by Lemma 28.1 the statement is equivalent to

$$\dim(\mathcal{O}_{X_s,x}) \leq \dim(\mathcal{O}_{X_y,x}) + \dim(\mathcal{O}_{Y_s,y}).$$

For this see Algebra, Lemma 112.6. For the flat case see Algebra, Lemma 112.7. $\square$

Lemma 28.3. *Let*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a fibre product diagram of schemes. Assume f locally of finite type. Suppose that $x' \in X'$, $x = g'(x')$, $s' = f'(x')$ and $s = g(s') = f(x)$. Then

(1) $\dim_x(X_s) = \dim_{x'}(X'_{s'})$,
(2) *if F is the fibre of the morphism $X'_{s'} \to X_s$ over x, then*

$$\dim(\mathcal{O}_{F,x'}) = \dim(\mathcal{O}_{X'_{s'},x'}) - \dim(\mathcal{O}_{X_s,x}) = trdeg_{\kappa(s)}(\kappa(x)) - trdeg_{\kappa(s')}(\kappa(x'))$$

In particular $\dim(\mathcal{O}_{X'_{s'},x'}) \geq \dim(\mathcal{O}_{X_s,x})$ and $trdeg_{\kappa(s)}(\kappa(x)) \geq trdeg_{\kappa(s')}(\kappa(x'))$.
(3) *given s', s, x there exists a choice of x' such that $\dim(\mathcal{O}_{X'_{s'},x'}) = \dim(\mathcal{O}_{X_s,x})$ and $trdeg_{\kappa(s)}(\kappa(x)) = trdeg_{\kappa(s')}(\kappa(x'))$.*

Proof. Part (1) follows immediately from Algebra, Lemma 116.6. Parts (2) and (3) from Algebra, Lemma 116.7. $\square$

The following lemma follows from a nontrivial algebraic result. Namely, the algebraic version of Zariski's main theorem.

Lemma 28.4. *Let $f : X \to S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite type. The set*

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

is open in X.

Proof. This is immediate from Algebra, Lemma 125.6 $\square$

Lemma 28.5. *Let $f : X \to Y$ be a morphism of finite type with Y quasi-compact. Then the dimension of the fibres of f is bounded.*

Proof. By Lemma 28.4 the set $U_n \subset X$ of points where the dimension of the fibre is $\leq n$ is open. Since f is of finite type, every point is contained in some U_n (because the dimension of a finite type algebra over a field is finite). Since Y is quasi-compact and f is of finite type, we see that X is quasi-compact. Hence $X = U_n$ for some n. $\square$

Lemma 28.6. *Let $f : X \to S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite presentation. The open*

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

of Lemma 28.4 is retrocompact in X. (See Topology, Definition 12.1.)

Proof. The topological space X has a basis for its topology consisting of affine opens $U \subset X$ such that the induced morphism $f|_U : U \to S$ factors through an affine open $V \subset S$. Hence it is enough to show that $U \cap U_n$ is quasi-compact for such a U. Note that $U_n \cap U$ is the same as the open $\{x \in U \mid \dim_x U_{f(x)} \leq n\}$. This reduces us to the case where X and S are affine. In this case the lemma follows from Algebra, Lemma 125.8 (and Lemma 21.2). $\square$

Lemma 28.7. *Let $f : X \to S$ be a morphism of schemes. Let $x \rightsquigarrow x'$ be a nontrivial specialization of points in X lying over the same point $s \in S$. Assume f is locally of finite type. Then*

1. $\dim_x(X_s) \leq \dim_{x'}(X_s)$,
2. $\dim(\mathcal{O}_{X_s,x}) < \dim(\mathcal{O}_{X_s,x'})$, *and*
3. $trdeg_{\kappa(s)}(\kappa(x)) > trdeg_{\kappa(s)}(\kappa(x'))$.

Proof. Part (1) follows from the fact that any open of X_s containing x' also contains x. Part (2) follows since $\mathcal{O}_{X_s,x}$ is a localization of $\mathcal{O}_{X_s,x'}$ at a prime ideal, hence any chain of prime ideals in $\mathcal{O}_{X_s,x}$ is part of a strictly longer chain of primes in $\mathcal{O}_{X_s,x'}$. The last inequality follows from Algebra, Lemma 116.2. $\square$

29. Morphisms of given relative dimension

In order to be able to speak comfortably about morphisms of a given relative dimension we introduce the following notion.

Definition 29.1. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type.

1. We say f is of *relative dimension $\leq d$ at x* if $\dim_x(X_{f(x)}) \leq d$.
2. We say f is of *relative dimension $\leq d$* if $\dim_x(X_{f(x)}) \leq d$ for all $x \in X$.
3. We say f is of *relative dimension d* if all nonempty fibres X_s are equidimensional of dimension d.

This is not a particularly well behaved notion, but it works well in a number of situations.

Lemma 29.2. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. If f has relative dimension d, then so does any base change of f. Same for relative dimension $\leq d$.*

Proof. This is immediate from Lemma 28.3. $\square$

Lemma 29.3. *Let $f : X \to Y$, $g : Y \to Z$ be locally of finite type. If f has relative dimension $\leq d$ and g has relative dimension $\leq e$ then $g \circ f$ has relative dimension $\leq d + e$. If*

1. *f has relative dimension d,*
2. *g has relative dimension e, and*
3. *f is flat,*

then $g \circ f$ has relative dimension $d + e$.

Proof. This is immediate from Lemma 28.2. $\square$

In general it is not possible to decompose a morphism into its pieces where the relative dimension is a given one. However, it is possible if the morphism has Cohen-Macaulay fibres and is flat of finite presentation.

Lemma 29.4. *Let $f : X \to S$ be a morphism of schemes. Assume that*

1. *f is flat,*
2. *f is locally of finite presentation, and*
3. *for all $s \in S$ the fibre X_s is Cohen-Macaulay (Properties, Definition 8.1)*

Then there exist open and closed subschemes $X_d \subset X$ such that $X = \coprod_{d \geq 0} X_d$ and $f|_{X_d} : X_d \to S$ has relative dimension d.

Proof. This is immediate from Algebra, Lemma 130.8. $\square$

Lemma 29.5. *Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. Let $x \in X$ with $s = f(x)$. Then f is quasi-finite at x if and only if $\dim_x(X_s) = 0$. In particular, f is locally quasi-finite if and only if f has relative dimension 0.*

Proof. If f is quasi-finite at x then $\kappa(x)$ is a finite extension of $\kappa(s)$ (by Lemma 20.5) and x is isolated in X_s (by Lemma 20.6), hence $\dim_x(X_s) = 0$ by Lemma 28.1. Conversely, if $\dim_x(X_s) = 0$ then by Lemma 28.1 we see $\kappa(s) \subset \kappa(x)$ is algebraic and there are no other points of X_s specializing to x. Hence x is closed in its fibre by Lemma 20.2 and by Lemma 20.6 (3) we conclude that f is quasi-finite at x. $\square$

Lemma 29.6. *Let $f : X \to Y$ be a morphism of locally Noetherian schemes which is flat, locally of finite type and of relative dimension d. For every point x in X with image y in Y we have $\dim_x(X) = \dim_y(Y) + d$.*

Proof. After shrinking X and Y to open neighborhoods of x and y, we can assume that $\dim(X) = \dim_x(X)$ and $\dim(Y) = \dim_y(Y)$, by definition of the dimension of a scheme at a point (Properties, Definition 10.1). The morphism f is open by Lemmas 21.9 and 25.10. Hence we can shrink Y to arrange that f is surjective. It remains to show that $\dim(X) = \dim(Y) + d$.

Let a be a point in X with image b in Y. By Algebra, Lemma 112.7,

$$\dim(\mathcal{O}_{X,a}) = \dim(\mathcal{O}_{Y,b}) + \dim(\mathcal{O}_{X_b,a}).$$

Taking the supremum over all points a in X, it follows that $\dim(X) = \dim(Y) + d$, as we want, see Properties, Lemma 10.2. $\square$

30. Syntomic morphisms

An algebra A over a field k is called a *global complete intersection over k* if $A \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $\dim(A) = n - c$. An algebra A over a field k is called a *local complete intersection* if $\text{Spec}(A)$ can be covered by standard opens each of which are global complete intersections over k. See Algebra, Section 135. Recall that a ring map $R \to A$ is *syntomic* if it is of finite presentation, flat with local complete intersection rings as fibres, see Algebra, Definition 136.1.

Definition 30.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *syntomic at $x \in X$* if there exists an affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is syntomic.
(2) We say that f is *syntomic* if it is syntomic at every point of X.
(3) If $S = \text{Spec}(k)$ and f is syntomic, then we say that X is a *local complete intersection over k*.
(4) A morphism of affine schemes $f : X \to S$ is called *standard syntomic* if there exists a global relative complete intersection $R \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ (see Algebra, Definition 136.5) such that $X \to S$ is isomorphic to
$$\text{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \text{Spec}(R).$$

In the literature a syntomic morphism is sometimes referred to as a *flat local complete intersection morphism*. It turns out this is a convenient class of morphisms. For example one can define a syntomic topology using these, which is finer than the smooth and étale topologies, but has many of the same formal properties.

A global relative complete intersection (which we used to define standard syntomic ring maps) is in particular flat. In More on Morphisms, Section 62 we will consider morphisms $X \to S$ which locally are of the form
$$\text{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \text{Spec}(R).$$
for some Koszul-regular sequence $f_1, \ldots, f_r$ in $R[x_1, \ldots, x_n]$. Such a morphism will be called a *local complete intersection morphism*. Once we have this definition in place it will be the case that a morphism is syntomic if and only if it is a flat, local complete intersection morphism.

Note that there is no separation or quasi-compactness hypotheses in the definition of a syntomic morphism. Hence the question of being syntomic is local in nature on the source. Here is the precise result.

Lemma 30.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is syntomic.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is syntomic.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is syntomic.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is syntomic, for all $j \in J, i \in I_j$.*

Moreover, if f is syntomic then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is syntomic.

Proof. This follows from Lemma 14.3 if we show that the property "$R \to A$ is syntomic" is local. We check conditions (a), (b) and (c) of Definition 14.1. By Algebra, Lemma 136.3 being syntomic is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 136.17 being syntomic is stable under composition and trivially for any ring R the ring map $R \to R_f$ is syntomic. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 136.4. ☐

Lemma 30.3. *The composition of two morphisms which are syntomic is syntomic.*

Proof. In the proof of Lemma 30.2 we saw that being syntomic is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 14.5 combined with the fact that being syntomic is a property of ring maps that is stable under composition, see Algebra, Lemma 136.17. ☐

Lemma 30.4. *The base change of a morphism which is syntomic is syntomic.*

Proof. In the proof of Lemma 30.2 we saw that being syntomic is a local property of ring maps. Hence the lemma follows from Lemma 14.5 combined with the fact that being syntomic is a property of ring maps that is stable under base change, see Algebra, Lemma 136.3. ☐

Lemma 30.5. *Any open immersion is syntomic.*

Proof. This is true because an open immersion is a local isomorphism. ☐

Lemma 30.6. *A syntomic morphism is locally of finite presentation.*

Proof. True because a syntomic ring map is of finite presentation by definition. ☐

Lemma 30.7. *A syntomic morphism is flat.*

Proof. True because a syntomic ring map is flat by definition. ☐

Lemma 30.8. *A syntomic morphism is universally open.*

Proof. Combine Lemmas 30.6, 30.7, and 25.10. ☐

Let k be a field. Let A be a local k-algebra essentially of finite type over k. Recall that A is called a *complete intersection over k* if we can write $A \cong R/(f_1, \ldots, f_c)$ where R is a regular local ring essentially of finite type over k, and $f_1, \ldots, f_c$ is a regular sequence in R, see Algebra, Definition 135.5.

Lemma 30.9. *Let k be a field. Let X be a scheme locally of finite type over k. The following are equivalent:*

(1) *X is a local complete intersection over k,*
(2) *for every $x \in X$ there exists an affine open $U = \text{Spec}(R) \subset X$ neighbourhood of x such that $R \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a global complete intersection over k, and*
(3) *for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a complete intersection over k.*

Proof. The corresponding algebra results can be found in Algebra, Lemmas 135.8 and 135.9. ☐

The following lemma says locally any syntomic morphism is standard syntomic. Hence we can use standard syntomic morphisms as a *local model* for a syntomic morphism. Moreover, it says that a flat morphism of finite presentation is syntomic if and only if the fibres are local complete intersection schemes.

Lemma 30.10. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s = f(x)$. Let $V \subset S$ be an affine open neighbourhood of s. The following are equivalent*

(1) *The morphism f is syntomic at x.*
(2) *There exist an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ such that $f|_U : U \to V$ is standard syntomic.*
(3) *The morphism f is of finite presentation at x, the local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is a complete intersection over $\kappa(s)$ (see Algebra, Definition 135.5).*

Proof. Follows from the definitions and Algebra, Lemma 136.15. □

Lemma 30.11. *Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and all fibres X_s are local complete intersections, then f is syntomic.*

Proof. Clear from Lemmas 30.9 and 30.10 and the isomorphisms of local rings $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x} \cong \mathcal{O}_{X_s,x}$. □

Lemma 30.12. *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the set*

$$T = \{x \in X \mid \mathcal{O}_{X_{f(x)},x} \text{ is a complete intersection over } \kappa(f(x))\}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. In particular, if f is assumed flat, and locally of finite presentation then the same holds for the open set of points where f is syntomic.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \mathrm{Spec}(\kappa(s')) \times_{\mathrm{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. Hence the first part is equivalent to Algebra, Lemma 135.10. The second part follows from the first because in that case T is the set of points where f is syntomic according to Lemma 30.10. □

Lemma 30.13. *Let R be a ring. Let $R \to A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. Set $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$. Consider the morphism $f : X \to S$ associated to the ring map $R \to A$. The function $x \mapsto \dim_x(X_{f(x)})$ is constant with value $n - c$.*

Proof. By Algebra, Definition 136.5 $R \to A$ being a relative global complete intersection means all nonzero fibre rings have dimension $n - c$. Thus for a prime $\mathfrak{p}$ of R the fibre ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ is either zero or a global complete intersection ring of dimension $n - c$. By the discussion following Algebra, Definition 135.1 this implies it is equidimensional of dimension $n - c$. Whence the lemma. □

Lemma 30.14. *Let $f : X \to S$ be a syntomic morphism. The function $x \mapsto \dim_x(X_{f(x)})$ is locally constant on X.*

Proof. By Lemma 30.10 the morphism f locally looks like a standard syntomic morphism of affines. Hence the result follows from Lemma 30.13. $\square$

Lemma 30.14 says that the following definition makes sense.

Definition 30.15. Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \to S$ is *syntomic of relative dimension d* if f is syntomic and the function $\dim_x(X_{f(x)}) = d$ for all $x \in X$.

In other words, f is syntomic and the nonempty fibres are equidimensional of dimension d.

Lemma 30.16. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective and syntomic,*
(2) *p is syntomic, and*
(3) *q is locally of finite presentation*[8].

Then q is syntomic.

Proof. By Lemma 25.13 we see that q is flat. Hence it suffices to show that the fibres of $Y \to S$ are local complete intersections, see Lemma 30.11. Let $s \in S$. Consider the morphism $X_s \to Y_s$. This is a base change of the morphism $X \to Y$ and hence surjective, and syntomic (Lemma 30.4). For the same reason X_s is syntomic over $\kappa(s)$. Moreover, Y_s is locally of finite type over $\kappa(s)$ (Lemma 15.4). In this way we reduce to the case where S is the spectrum of a field.

Assume $S = \text{Spec}(k)$. Let $y \in Y$. Choose an affine open $\text{Spec}(A) \subset Y$ neighbourhood of y. Let $\text{Spec}(B) \subset X$ be an affine open such that $f(\text{Spec}(B)) \subset \text{Spec}(A)$, containing a point $x \in X$ such that $f(x) = y$. Choose a surjection $k[x_1, \ldots, x_n] \to A$ with kernel I. Choose a surjection $A[y_1, \ldots, y_m] \to B$, which gives rise in turn to a surjection $k[x_i, y_j] \to B$ with kernel J. Let $\mathfrak{q} \subset k[x_i, y_j]$ be the prime corresponding to $y \in \text{Spec}(B)$ and let $\mathfrak{p} \subset k[x_i]$ the prime corresponding to $x \in \text{Spec}(A)$. Since x maps to y we have $\mathfrak{p} = \mathfrak{q} \cap k[x_i]$. Consider the following commutative diagram of local rings:

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, equal] & B_{\mathfrak{q}} & k[x_1, \ldots, x_n, y_1, \ldots, y_m]_{\mathfrak{q}} \arrow[l] \\ \mathcal{O}_{Y,y} \arrow[r, equal] \arrow[u] & A_{\mathfrak{p}} \arrow[u] & k[x_1, \ldots, x_n]_{\mathfrak{p}} \arrow[l] \arrow[u] \end{tikzcd}$$

We claim that the hypotheses of Algebra, Lemma 135.12 are satisfied. Conditions (1) and (2) are trivial. Condition (4) follows as $X \to Y$ is flat. Condition (3) follows as the rings $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ are complete intersection rings by our assumptions that f and p are syntomic, see Lemma 30.10. The output of Algebra,

[8]In fact, if f is surjective, flat, and locally of finite presentation and p is syntomic, then both q and f are syntomic, see Descent, Lemma 14.7.

Lemma 135.12 is exactly that $\mathcal{O}_{Y,y}$ is a complete intersection ring! Hence by Lemma 30.10 again we see that Y is syntomic over k at y as desired. $\square$

31. Conormal sheaf of an immersion

Let $i : Z \to X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the short exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

of quasi-coherent sheaves on X. Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by $\mathcal{I}$ it corresponds to a sheaf on Z by Lemma 4.1. This quasi-coherent $\mathcal{O}_Z$-module is called the *conormal sheaf of Z in X* and is often simply denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Section 4.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, where $\partial Z = \overline{Z} \setminus Z$. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 31.1. Let $i : Z \to X$ be an immersion. The *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X* or the *conormal sheaf of i* is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above.

In [DG67, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the *normal sheaf of the immersion*. It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

Lemma 31.2. *Let $i : Z \to X$ be an immersion. The conormal sheaf of i has the following properties:*

(1) *Let $U \subset X$ be any open subscheme such that i factors as $Z \xrightarrow{i'} U \to X$ where i' is a closed immersion. Let $\mathcal{I} = \mathrm{Ker}((i')^\sharp) \subset \mathcal{O}_U$. Then*
$$\mathcal{C}_{Z/X} = (i')^*\mathcal{I} \quad \text{and} \quad i'_*\mathcal{C}_{Z/X} = \mathcal{I}/\mathcal{I}^2$$
(2) *For any affine open $\mathrm{Spec}(R) = U \subset X$ such that $Z \cap U = \mathrm{Spec}(R/I)$ there is a canonical isomorphism $\Gamma(Z \cap U, \mathcal{C}_{Z/X}) = I/I^2$.*

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I/I^2 = I \otimes_R R/I$. Details omitted. $\square$

Lemma 31.3. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram in the category of schemes. Assume i, i' immersions. There is a canonical map of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

characterized by the following property: For every pair of affine opens $(\mathrm{Spec}(R) = U \subset X, \mathrm{Spec}(R') = U' \subset X')$ *with* $f(U) \subset U'$ *such that* $Z \cap U = \mathrm{Spec}(R/I)$ *and* $Z' \cap U' = \mathrm{Spec}(R'/I')$ *the induced map*

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X'}) = I'/I'^2 \longrightarrow I/I^2 = \Gamma(Z \cap U, \mathcal{C}_{Z/X})$$

is the one induced by the ring map $f^\sharp : R' \to R$ *which has the property* $f^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X. Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus \big(g^{-1}(\partial Z') \cup \partial Z\big)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into $\mathcal{I}$ under the canonical map $g^*\mathcal{I}' \to \mathcal{O}_X$, see Schemes, Lemmas 4.6 and 4.7. Hence we get an induced map of quasi-coherent sheaves $g^*(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{I}/\mathcal{I}^2$. Pulling back by i gives $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) \to i^*(\mathcal{I}/\mathcal{I}^2)$. Note that $i^*(\mathcal{I}/\mathcal{I}^2) = \mathcal{C}_{Z/X}$. On the other hand, $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*(i')^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*\mathcal{C}_{Z'/X'}$. This gives the desired map.

Checking that the map is locally described as the given map $I'/(I')^2 \to I/I^2$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). □

Lemma 31.4. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a fibre product diagram in the category of schemes with i, i' *immersions. Then the canonical map* $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ *of Lemma 31.3 is surjective. If* g *is flat, then it is an isomorphism.*

Proof. Let $R' \to R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $I'/(I')^2 \otimes_{R'} R \to I/I^2$ is surjective. If $R' \to R$ is flat, then $I = I' \otimes_{R'} R$ and $I^2 = (I')^2 \otimes_{R'} R$ and we see the map is an isomorphism. □

Lemma 31.5. *Let* $Z \to Y \to X$ *be immersions of schemes. Then there is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 31.3 and $i : Z \to Y$ *is the first morphism.*

Proof. Via Lemma 31.3 this translates into the following algebra fact. Suppose that $C \to B \to A$ are surjective ring maps. Let $I = \mathrm{Ker}(B \to A)$, $J = \mathrm{Ker}(C \to A)$ and $K = \mathrm{Ker}(C \to B)$. Then there is an exact sequence

$$K/K^2 \otimes_B A \to J/J^2 \to I/I^2 \to 0.$$

This follows immediately from the observation that $I = J/K$. □

32. Sheaf of differentials of a morphism

We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 131) and the corresponding section in the chapter on sheaves of modules (Modules, Section 28).

Definition 32.1. Let $f : X \to S$ be a morphism of schemes. The *sheaf of differentials $\Omega_{X/S}$ of X over S* is the sheaf of differentials of f viewed as a morphism of ringed spaces (Modules, Definition 28.10) equipped with its *universal S-derivation*

$$\mathrm{d}_{X/S} : \mathcal{O}_X \longrightarrow \Omega_{X/S}.$$

It turns out that $\Omega_{X/S}$ is a quasi-coherent $\mathcal{O}_X$-module for example as it is isomorphic to the conormal sheaf of the diagonal morphism $\Delta : X \to X \times_S X$ (Lemma 32.7). We have defined the module of differentials of X over S using a universal property, namely as the receptacle of the universal derivation. If you have any other construction of the sheaf of relative differentials which satisfies this universal property then, by the Yoneda lemma, it will be canonically isomorphic to the one defined above. For convenience we restate the universal property here.

Lemma 32.2. *Let $f : X \to S$ be a morphism of schemes. The map*

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{F}) \longrightarrow \mathit{Der}_S(\mathcal{O}_X, \mathcal{F}), \quad \alpha \longmapsto \alpha \circ d_{X/S}$$

is an isomorphism of functors $\mathit{Mod}(\mathcal{O}_X) \to \mathit{Sets}$.

Proof. This is just a restatement of the definition. □

Lemma 32.3. *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$, $V \subset S$ be open subschemes such that $f(U) \subset V$. Then there is a unique isomorphism $\Omega_{X/S}|_U = \Omega_{U/V}$ of $\mathcal{O}_U$-modules such that $d_{X/S}|_U = d_{U/V}$.*

Proof. This is a special case of Modules, Lemma 28.5 if we use the canonical identification $f^{-1}\mathcal{O}_S|_U = (f|_U)^{-1}\mathcal{O}_V$. □

From now on we will use these canonical identifications and simply write $\Omega_{U/S}$ or $\Omega_{U/V}$ for the restriction of $\Omega_{X/S}$ to U.

Lemma 32.4. *Let $R \to A$ be a ring map. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X = \mathrm{Spec}(A)$. Set $S = \mathrm{Spec}(R)$. The rule which associates to an S-derivation on $\mathcal{F}$ its action on global sections defines a bijection between the set of S-derivations of $\mathcal{F}$ and the set of R-derivations on $M = \Gamma(X, \mathcal{F})$.*

Proof. Let $D : A \to M$ be an R-derivation. We have to show there exists a unique S-derivation on $\mathcal{F}$ which gives rise to D on global sections. Let $U = D(f) \subset X$ be a standard affine open. Any element of $\Gamma(U, \mathcal{O}_X)$ is of the form a/f^n for some $a \in A$ and $n \geq 0$. By the Leibniz rule we have

$$D(a)|_U = a/f^n D(f^n)|_U + f^n D(a/f^n)$$

in $\Gamma(U, \mathcal{F})$. Since f acts invertibly on $\Gamma(U, \mathcal{F})$ this completely determines the value of $D(a/f^n) \in \Gamma(U, \mathcal{F})$. This proves uniqueness. Existence follows by simply defining

$$D(a/f^n) := (1/f^n)D(a)|_U - a/f^{2n} D(f^n)|_U$$

and proving this has all the desired properties (on the basis of standard opens of X). Details omitted. □

Lemma 32.5. *Let $f : X \to S$ be a morphism of schemes. For any pair of affine opens* $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ *with $f(U) \subset V$ there is a unique isomorphism*

$$\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}.$$

compatible with $\mathrm{d}_{X/S}$ and $d : A \to \Omega_{A/R}$.

Proof. By Lemma 32.3 we may replace X and S by U and V. Thus we may assume $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ and we have to show the lemma with $U = X$ and $V = S$. Consider the A-module $M = \Gamma(X, \Omega_{X/S})$ together with the R-derivation $\mathrm{d}_{X/S} : A \to M$. Let N be another A-module and denote $\widetilde{N}$ the quasi-coherent $\mathcal{O}_X$-module associated to N, see Schemes, Section 7. Precomposing by $\mathrm{d}_{X/S} : A \to M$ we get an arrow

$$\alpha : \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Der}_R(A, N)$$

Using Lemmas 32.2 and 32.4 we get identifications

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \widetilde{N}) = \mathrm{Der}_S(\mathcal{O}_X, \widetilde{N}) = \mathrm{Der}_R(A, N)$$

Taking global sections determines an arrow $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \widetilde{N}) \to \mathrm{Hom}_R(M, N)$. Combining this arrow and the identifications above we get an arrow

$$\beta : \mathrm{Der}_R(A, N) \longrightarrow \mathrm{Hom}_R(M, N)$$

Checking what happens on global sections, we find that α and β are each others inverse. Hence we see that $\mathrm{d}_{X/S} : A \to M$ satisfies the same universal property as $\mathrm{d} : A \to \Omega_{A/R}$, see Algebra, Lemma 131.3. Thus the Yoneda lemma (Categories, Lemma 3.5) implies there is a unique isomorphism of A-modules $M \cong \Omega_{A/R}$ compatible with derivations. $\square$

Remark 32.6. The lemma above gives a second way of constructing the module of differentials. Namely, let $f : X \to S$ be a morphism of schemes. Consider the collection of all affine opens $U \subset X$ which map into an affine open of S. These form a basis for the topology on X. Thus it suffices to define $\Gamma(U, \Omega_{X/S})$ for such U. We simply set $\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}$ if A, R are as in Lemma 32.5 above. This works, but it takes somewhat more algebraic preliminaries to construct the restriction mappings and to verify the sheaf condition with this ansatz.

The following lemma gives yet another way to define the sheaf of differentials and it in particular shows that $\Omega_{X/S}$ is quasi-coherent if X and S are schemes.

Lemma 32.7. *Let $f : X \to S$ be a morphism of schemes. There is a canonical isomorphism between $\Omega_{X/S}$ and the conormal sheaf of the diagonal morphism $\Delta_{X/S} : X \longrightarrow X \times_S X$.*

Proof. We first establish the existence of a couple of "global" sheaves and global maps of sheaves, and further down we describe the constructions over some affine opens.

Recall that $\Delta = \Delta_{X/S} : X \to X \times_S X$ is an immersion, see Schemes, Lemma 21.2. Let $\mathcal{J}$ be the ideal sheaf of the immersion which lives over some open subscheme W of $X \times_S X$ such that $\Delta(X) \subset W$ is closed. Let us take the one that was found in the proof of Schemes, Lemma 21.2. Note that the sheaf of rings $\mathcal{O}_W/\mathcal{J}^2$ is supported on $\Delta(X)$. Moreover it sits in a short exact sequence of sheaves

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_W/\mathcal{J}^2 \to \Delta_*\mathcal{O}_X \to 0.$$

Using Δ^{-1} we can think of this as a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras with kernel the conormal sheaf of Δ (see Definition 31.1 and Lemma 31.2).

$$0 \to \mathcal{C}_{X/X\times_S X} \to \Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2) \to \mathcal{O}_X \to 0$$

This places us in the situation of Modules, Lemma 28.11. The projection morphisms $p_i : X \times_S X \to X$, $i = 1, 2$ induce maps of sheaves of rings $(p_i)^\sharp : (p_i)^{-1}\mathcal{O}_X \to \mathcal{O}_{X\times_S X}$. We may restrict to W and quotient by $\mathcal{J}^2$ to get $(p_i)^{-1}\mathcal{O}_X \to \mathcal{O}_W/\mathcal{J}^2$. Since $\Delta^{-1}p_i^{-1}\mathcal{O}_X = \mathcal{O}_X$ we get maps

$$s_i : \mathcal{O}_X \to \Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2).$$

Both s_1 and s_2 are sections to the map $\Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2) \to \mathcal{O}_X$, as in Modules, Lemma 28.11. Thus we get an S-derivation $\mathrm{d} = s_2 - s_1 : \mathcal{O}_X \to \mathcal{C}_{X/X\times_S X}$. By the universal property of the module of differentials we find a unique $\mathcal{O}_X$-linear map

$$\Omega_{X/S} \longrightarrow \mathcal{C}_{X/X\times_S X}, \quad f\mathrm{d}g \longmapsto fs_2(g) - fs_1(g)$$

To see the map is an isomorphism, let us work this out over suitable affine opens. We can cover X by affine opens $\mathrm{Spec}(A) = U \subset X$ whose image is contained in an affine open $\mathrm{Spec}(R) = V \subset S$. According to the proof of Schemes, Lemma 21.2 $U \times_V U \subset X \times_S X$ is an affine open contained in the open W mentioned above. Also $U \times_V U = \mathrm{Spec}(A \otimes_R A)$. The sheaf $\mathcal{J}$ corresponds to the ideal $J = \mathrm{Ker}(A \otimes_R A \to A)$. The short exact sequence to the short exact sequence of $A \otimes_R A$-modules

$$0 \to J/J^2 \to (A \otimes_R A)/J^2 \to A \to 0$$

The sections s_i correspond to the ring maps

$$A \longrightarrow (A \otimes_R A)/J^2, \quad s_1 : a \mapsto a \otimes 1, \quad s_2 : a \mapsto 1 \otimes a.$$

By Lemma 31.2 we have $\Gamma(U, \mathcal{C}_{X/X\times_S X}) = J/J^2$ and by Lemma 32.5 we have $\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}$. The map above is the map $a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$ which is shown to be an isomorphism in Algebra, Lemma 131.13. $\square$

Lemma 32.8. *Let*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

be a commutative diagram of schemes. The canonical map $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*d_{X'/S'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/S'}$ is a S-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/S} \to f_*\Omega_{X'/S'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/S} \longrightarrow \Omega_{X'/S'}.$$

It is uniquely characterized by the property that $f^*d_{X/S}(h)$ maps to $d_{X'/S'}(f^*h)$ for any local section h of $\mathcal{O}_X$.

Proof. This is a special case of Modules, Lemma 28.12. In the case of schemes we can also use the functoriality of the conormal sheaves (see Lemma 31.3) and Lemma 32.7 to define c_f. Or we can use the characterization in the last line of the lemma to glue maps defined on affine patches (see Algebra, Equation (131.4.1)). $\square$

Lemma 32.9. *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Then there is a canonical exact sequence*

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 32.8.

Proof. This is the sheafified version of Algebra, Lemma 131.7. □

Lemma 32.10. *Let $X \to S$ be a morphism of schemes. Let $g : S' \to S$ be a morphism of schemes. Let $X' = X_{S'}$ be the base change of X. Denote $g' : X' \to X$ the projection. Then the map*

$$(g')^*\Omega_{X/S} \to \Omega_{X'/S'}$$

of Lemma 32.8 is an isomorphism.

Proof. This is the sheafified version of Algebra, Lemma 131.12. □

Lemma 32.11. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projection morphisms. The maps from Lemma 32.8*

$$p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \longrightarrow \Omega_{X \times_S Y/S}$$

give an isomorphism.

Proof. By Lemma 32.10 the composition $p^*\Omega_{X/S} \to \Omega_{X \times_S Y/S} \to \Omega_{X \times_S Y/Y}$ is an isomorphism, and similarly for q. Moreover, the cokernel of $p^*\Omega_{X/S} \to \Omega_{X \times_S Y/S}$ is $\Omega_{X \times_S Y/X}$ by Lemma 32.9. The result follows. □

Lemma 32.12. *Let $f : X \to S$ be a morphism of schemes. If f is locally of finite type, then $\Omega_{X/S}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Immediate from Algebra, Lemma 131.16, Lemma 32.5, Lemma 15.2, and Properties, Lemma 16.1. □

Lemma 32.13. *Let $f : X \to S$ be a morphism of schemes. If f is locally of finite presentation, then $\Omega_{X/S}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Immediate from Algebra, Lemma 131.15, Lemma 32.5, Lemma 21.2, and Properties, Lemma 16.2. □

Lemma 32.14. *If $X \to S$ is an immersion, or more generally a monomorphism, then $\Omega_{X/S}$ is zero.*

Proof. This is true because $\Delta_{X/S}$ is an isomorphism in this case and hence has trivial conormal sheaf. Hence $\Omega_{X/S} = 0$ by Lemma 32.7. The algebraic version is Algebra, Lemma 131.4. □

Lemma 32.15. *Let $i : Z \to X$ be an immersion of schemes over S. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

where the first arrow is induced by $\mathrm{d}_{X/S}$ and the second arrow comes from Lemma 32.8.

Proof. This is the sheafified version of Algebra, Lemma 131.9. However we should make sure we can define the first arrow globally. Hence we explain the meaning of "induced by $\mathrm{d}_{X/S}$" here. Namely, we may assume that i is a closed immersion by shrinking X. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals corresponding to $Z \subset X$. Then $\mathrm{d}_{X/S} : \mathcal{I} \to \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$ which is $\mathcal{O}_X/\mathcal{I}$-linear. By Lemma 4.1 this corresponds to a map $\mathcal{C}_{Z/X} \to i^*\Omega_{X/S}$ as desired. □

Lemma 32.16. *Let $i : Z \to X$ be an immersion of schemes over S, and assume i (locally) has a left inverse. Then the canonical sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 32.15 is (locally) split exact. In particular, if $s : S \to X$ is a section of the structure morphism $X \to S$ then the map $\mathcal{C}_{S/X} \to s^*\Omega_{X/S}$ induced by $\mathrm{d}_{X/S}$ is an isomorphism.

Proof. Follows from Algebra, Lemma 131.10. Clarification: if $g : X \to Z$ is a left inverse of i, then i^*c_g is a right inverse of the map $i^*\Omega_{X/S} \to \Omega_{Z/S}$. Also, if s is a section, then it is an immersion $s : Z = S \to X$ over S (see Schemes, Lemma 21.11) and in that case $\Omega_{Z/S} = 0$. □

Remark 32.17. Let $X \to S$ be a morphism of schemes. According to Lemma 32.11 we have

$$\Omega_{X \times_S X/S} = \mathrm{pr}_1^*\Omega_{X/S} \oplus \mathrm{pr}_2^*\Omega_{X/S}$$

On the other hand, the diagonal morphism $\Delta : X \to X \times_S X$ is an immersion, which locally has a left inverse. Hence by Lemma 32.16 we obtain a canonical short exact sequence

$$0 \to \mathcal{C}_{X/X \times_S X} \to \Omega_{X/S} \oplus \Omega_{X/S} \to \Omega_{X/S} \to 0$$

Note that the right arrow is $(1, 1)$ which is indeed a split surjection. On the other hand, by Lemma 32.7 we have an identification $\Omega_{X/S} = \mathcal{C}_{X/X \times_S X}$. Because we chose $\mathrm{d}_{X/S}(f) = s_2(f) - s_1(f)$ in this identification it turns out that the left arrow is the map $(-1, 1)$[9].

Lemma 32.18. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are immersions. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 31.3 and the second from Lemma 32.15.

Proof. The algebraic version of this is Algebra, Lemma 134.7. □

[9]Namely, the local section $\mathrm{d}_{X/S}(f) = 1 \otimes f - f \otimes 1$ of the ideal sheaf of Δ maps via $\mathrm{d}_{X \times_S X/X}$ to the local section $1\otimes1\otimes1\otimes f - 1\otimes f\otimes1\otimes1 - 1\otimes1\otimes f\otimes1 + f\otimes1\otimes1\otimes1 = \mathrm{pr}_2^*\mathrm{d}_{X/S}(f) - \mathrm{pr}_1^*\mathrm{d}_{X/S}(f)$.

33. Finite order differential operators

We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 133) and the corresponding section in the chapter on sheaves of modules (Modules, Section 29).

Lemma 33.1. *Let $R \to A$ be a ring map. Denote $f : X \to S$ the corresponding morphism of affine schemes. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is quasi-coherent then the map*

$$Diff^k_{X/S}(\mathcal{F}, \mathcal{G}) \to Diff^k_{A/R}(\Gamma(X, \mathcal{F}), \Gamma(X, \mathcal{G}))$$

sending a differential operator to its action on global sections is bijective.

Proof. Write $\mathcal{F} = \widetilde{M}$ for some A-module M. Set $N = \Gamma(X, \mathcal{G})$. Let $D : M \to N$ be a differential operator of order k. We have to show there exists a unique differential operator $\mathcal{F} \to \mathcal{G}$ of order k which gives rise to D on global sections. Let $U = D(f) \subset X$ be a standard affine open. Then $\mathcal{F}(U) = M_f$ is the localization. By Algebra, Lemma 133.10 the differential operator D extends to a unique differential operator

$$D_f : \mathcal{F}(U) = \widetilde{M}(U) = M_f \to N_f = \widetilde{N}(U)$$

The uniqueness shows that these maps D_f glue to give a map of sheaves $\widetilde{M} \to \widetilde{N}$ on the basis of all standard opens of X. Hence we get a unique map of sheaves $\widetilde{D} : \widetilde{M} \to \widetilde{N}$ agreeing with these maps by the material in Sheaves, Section 30. Since $\widetilde{D}$ is given by differential operators of order k on the standard opens, we find that $\widetilde{D}$ is a differential operator of order k (small detail omitted). Finally, we can post-compose with the canonical $\mathcal{O}_X$-module map $c : \widetilde{N} \to \mathcal{G}$ (Schemes, Lemma 7.1) to get $c \circ \widetilde{D} : \mathcal{F} \to \mathcal{G}$ which is a differential operator of order k by Modules, Lemma 29.2. This proves existence. We omit the proof of uniqueness. $\square$

Lemma 33.2. *Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes. Let $\mathcal{F}$ and $\mathcal{F}'$ be quasi-coherent $\mathcal{O}_X$-modules. Let $D : \mathcal{F} \to \mathcal{F}'$ be a differential operator of order k on X/S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Then there is a unique differential operator*

$$D' : pr_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} pr_2^*\mathcal{G} \longrightarrow pr_1^*\mathcal{F}' \otimes_{\mathcal{O}_{X \times_S Y}} pr_2^*\mathcal{G}$$

of order k on $X \times_S Y/Y$ such that $D'(s \otimes t) = D(s) \otimes t$ for local sections s of $\mathcal{F}$ and t of $\mathcal{G}$.

Proof. In case X, Y, and S are affine, this follows, via Lemma 33.1, from the corresponding algebra result, see Algebra, Lemma 133.11. In general, one uses coverings by affines (for example as in Schemes, Lemma 17.4) to construct D' globally. Details omitted. $\square$

Remark 33.3. Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes. Denote $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ the projections. In this remark, given an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$ let us set

$$\mathcal{F} \boxtimes \mathcal{G} = p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}$$

Denote $\mathcal{A}_{X/S}$ the additive category whose objects are quasi-coherent $\mathcal{O}_X$-modules and whose morphisms are differential operators of finite order on X/S. Similarly for $\mathcal{A}_{Y/S}$ and $\mathcal{A}_{X \times_S Y/S}$. The construction of Lemma 33.2 determines a functor

$$\boxtimes : \mathcal{A}_{X/S} \times \mathcal{A}_{Y/S} \longrightarrow \mathcal{A}_{X \times_S Y/S}, \quad (\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \boxtimes \mathcal{G}$$

which is bilinear on morphisms. If $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $S = \mathrm{Spec}(R)$, then via the identification of quasi-coherent sheaves with modules this functor is given by $(M, N) \mapsto M \otimes_R N$ on objects and sends the morphism $(D, D') : (M, N) \to (M', N')$ to $D \otimes D' : M \otimes_R N \to M' \otimes_R N'$.

34. Smooth morphisms

Let $f : X \to Y$ be a continuous map of topological spaces. Consider the following condition: For every $x \in X$ there exist open neighbourhoods $x \in U \subset X$ and $f(x) \in V \subset Y$, and an integer d such that $f(U) \subset V$ and such that we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "\pi"'] \arrow[d] & V \times \mathbf{R}^d \arrow[dl] \\ Y & V \arrow[l] & \end{tikzcd}$$

where π is a homeomorphism onto an open subset. Smooth morphisms of schemes are the analogue of these maps in the category of schemes. See Lemma 34.11 and Lemma 36.20.

Contrary to expectations (perhaps) the notion of a smooth ring map is not defined solely in terms of the module of differentials. Namely, recall that $R \to A$ is a *smooth ring map* if A is of finite presentation over R and if the naive cotangent complex of A over R is quasi-isomorphic to a projective module placed in degree 0, see Algebra, Definition 137.1.

Definition 34.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *smooth at $x \in X$* if there exist an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is smooth.
(2) We say that f is *smooth* if it is smooth at every point of X.
(3) A morphism of affine schemes $f : X \to S$ is called *standard smooth* if there exists a standard smooth ring map $R \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ (see Algebra, Definition 137.6) such that $X \to S$ is isomorphic to

$$\mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \mathrm{Spec}(R).$$

A pleasing feature of this definition is that the set of points where a morphism is smooth is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being smooth is local in nature on the source. Here is the precise result.

Lemma 34.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is smooth.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is smooth.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is smooth.*

(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is smooth, for all $j \in J, i \in I_j$.*

Moreover, if f is smooth then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is smooth.

Proof. This follows from Lemma 14.3 if we show that the property "$R \to A$ is smooth" is local. We check conditions (a), (b) and (c) of Definition 14.1. By Algebra, Lemma 137.4 being smooth is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 137.14 being smooth is stable under composition and for any ring R the ring map $R \to R_f$ is (standard) smooth. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 137.13. □

The following lemma characterizes a smooth morphism as a flat, finitely presented morphism with smooth fibres. Note that schemes smooth over a field are discussed in more detail in Varieties, Section 25.

Lemma 34.3. *Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and all fibres X_s are smooth, then f is smooth.*

Proof. Follows from Algebra, Lemma 137.17. □

Lemma 34.4. *The composition of two morphisms which are smooth is smooth.*

Proof. In the proof of Lemma 34.2 we saw that being smooth is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 14.5 combined with the fact that being smooth is a property of ring maps that is stable under composition, see Algebra, Lemma 137.14. □

Lemma 34.5. *The base change of a morphism which is smooth is smooth.*

Proof. In the proof of Lemma 34.2 we saw that being smooth is a local property of ring maps. Hence the lemma follows from Lemma 14.5 combined with the fact that being smooth is a property of ring maps that is stable under base change, see Algebra, Lemma 137.4. □

Lemma 34.6. *Any open immersion is smooth.*

Proof. This is true because an open immersion is a local isomorphism. □

Lemma 34.7. *A smooth morphism is syntomic.*

Proof. See Algebra, Lemma 137.10. □

Lemma 34.8. *A smooth morphism is locally of finite presentation.*

Proof. True because a smooth ring map is of finite presentation by definition. □

Lemma 34.9. *A smooth morphism is flat.*

Proof. Combine Lemmas 30.7 and 34.7. □

Lemma 34.10. *A smooth morphism is universally open.*

Proof. Combine Lemmas 34.9, 34.8, and 25.10. Or alternatively, combine Lemmas 34.7, 30.8. □

The following lemma says locally any smooth morphism is standard smooth. Hence we can use standard smooth morphisms as a *local model* for a smooth morphism.

Lemma 34.11. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Let $V \subset S$ be an affine open neighbourhood of $f(x)$. The following are equivalent*

1. *The morphism f is smooth at x.*
2. *There exists an affine open $U \subset X$, with $x \in U$ and $f(U) \subset V$ such that the induced morphism $f|_U : U \to V$ is standard smooth.*

Proof. Follows from the definitions and Algebra, Lemmas 137.7 and 137.10. □

Lemma 34.12. *Let $f : X \to S$ be a morphism of schemes. Assume f is smooth. Then the module of differentials $\Omega_{X/S}$ of X over S is finite locally free and*

$$\mathit{rank}_x(\Omega_{X/S}) = \dim_x(X_{f(x)})$$

for every $x \in X$.

Proof. The statement is local on X and S. By Lemma 34.11 above we may assume that f is a standard smooth morphism of affines. In this case the result follows from Algebra, Lemma 137.7 (and the definition of a relative global complete intersection, see Algebra, Definition 136.5). □

Lemma 34.12 says that the following definition makes sense.

Definition 34.13. Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \to S$ is *smooth of relative dimension d* if f is smooth and $\Omega_{X/S}$ is finite locally free of constant rank d.

In other words, f is smooth and the nonempty fibres are equidimensional of dimension d. By Lemma 34.14 below this is also the same as requiring: (a) f is locally of finite presentation, (b) f is flat, (c) all nonempty fibres equidimensional of dimension d, and (d) $\Omega_{X/S}$ finite locally free of rank d. It is not enough to simply assume that f is flat, of finite presentation, and $\Omega_{X/S}$ is finite locally free of rank d. A counter example is given by $\mathrm{Spec}(\mathbf{F}_p[t]) \to \mathrm{Spec}(\mathbf{F}_p[t^p])$.

Here is a differential criterion of smoothness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

Lemma 34.14. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:*

1. *The morphism f is smooth at x.*
2. *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $X_s \to \mathrm{Spec}(\kappa(s))$ is smooth at x.*
3. *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ can be generated by at most $\dim_x(X_{f(x)})$ elements.*
4. *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
can be generated by at most $\dim_x(X_{f(x)})$ elements.
5. *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth.*

(6) *There exist affine opens* $\mathrm{Spec}(A) = U \subset X$ *and* $\mathrm{Spec}(R) = V \subset S$ *with* $x \in U$ *corresponding to* $\mathfrak{q} \subset A$*, and* $f(U) \subset V$ *such that there exists a presentation*
$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$
with
$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$
mapping to an element of A *not in* $\mathfrak{q}$.

Proof. Note that if f is smooth at x, then we see from Lemma 34.11 that (5) holds, and (6) is a slightly weakened version of (5). Moreover, f smooth implies that the ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat (see Lemma 34.9) and that $\Omega_{X/S}$ is finite locally free of rank equal to $\dim_x(X_s)$ (see Lemma 34.12). Thus (1) implies (3) and (4). By Lemma 34.5 we also see that (1) implies (2).

By Lemma 32.10 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S. Hence the displayed equality in part (4) of the lemma. By Lemma 32.12 these modules are of finite type. Hence the minimal number of generators of the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ is the same and equal to the dimension of this $\kappa(x)$-vector space by Nakayama's Lemma (Algebra, Lemma 20.1). This in particular shows that (3) and (4) are equivalent.

Algebra, Lemma 137.17 shows that (2) implies (1). Algebra, Lemma 140.3 shows that (3) and (4) imply (2). Finally, (6) implies (5) see for example Algebra, Example 137.8 and (5) implies (1) by Algebra, Lemma 137.7. □

Lemma 34.15. *Let*
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$
be a cartesian diagram of schemes. Let $W \subset X$*, resp.* $W' \subset X'$ *be the open subscheme of points where* f*, resp.* f' *is smooth. Then* $W' = (g')^{-1}(W)$ *if*

(1) f *is flat and locally of finite presentation, or*
(2) f *is locally of finite presentation and* g *is flat.*

Proof. Assume first that f locally of finite type. Consider the set
$$T = \{x \in X \mid X_{f(x)} \text{ is smooth over } \kappa(f(x)) \text{ at } x\}$$
and the corresponding set $T' \subset X'$ for f'. Then we claim $T' = (g')^{-1}(T)$. Namely, let $s' \in S'$ be a point, and let $s = g(s')$. Then we have
$$X'_{s'} = \mathrm{Spec}(\kappa(s')) \times_{\mathrm{Spec}(\kappa(s))} X_s$$
In other words the fibres of the base change are the base changes of the fibres. Hence the claim is equivalent to Algebra, Lemma 137.19.

Thus case (1) follows because in case (1) T is the (open) set of points where f is smooth by Lemma 34.14.

In case (2) let $x' \in W'$. Then g' is flat at x' (Lemma 25.7) and $g \circ f$ is flat at x' (Lemma 25.5). It follows that f is flat at $x = g'(x')$ by Lemma 25.13. On the other hand, since $x' \in T'$ (Lemma 34.5) we see that $x \in T$. Hence f is smooth at x by Lemma 34.14. $\square$

Here is a lemma that actually uses the vanishing of H^{-1} of the naive cotangent complex for a smooth ring map.

Lemma 34.16. *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume f is smooth. Then*

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Lemma 32.9) is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 131.7 is exact. This is Algebra, Lemma 139.1. $\square$

Lemma 34.17. *Let $i : Z \to X$ be an immersion of schemes over S. Assume that Z is smooth over S. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 32.15 is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $A \to C$ smooth and $B \to C$ surjective with kernel J, then the sequence

$$0 \to J/J^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0$$

of Algebra, Lemma 131.9 is exact. This is Algebra, Lemma 139.2. $\square$

Lemma 34.18. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are immersions and $X \to Y$ is smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 32.18 is exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $A \to C$ surjective and $A \to B$ smooth, then the sequence

$$0 \to I/I^2 \to J/J^2 \to C \otimes_B \Omega_{B/A} \to 0$$

of Algebra, Lemma 134.7 is exact. This is Algebra, Lemma 139.3. $\square$

Lemma 34.19. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, and smooth,*

(2) *p is smooth, and*
(3) *q is locally of finite presentation*[10].

Then q is smooth.

Proof. By Lemma 25.13 we see that q is flat. Pick a point $y \in Y$. Pick a point $x \in X$ mapping to y. Suppose f has relative dimension a at x and p has relative dimension b at x. By Lemma 34.12 this means that $\Omega_{X/S,x}$ is free of rank b and $\Omega_{X/Y,x}$ is free of rank a. By the short exact sequence of Lemma 34.16 this means that $(f^*\Omega_{Y/S})_x$ is free of rank $b - a$. By Nakayama's Lemma this implies that $\Omega_{Y/S,y}$ can be generated by $b - a$ elements. Also, by Lemma 28.2 we see that $\dim_y(Y_s) = b - a$. Hence we conclude that $Y \to S$ is smooth at y by Lemma 34.14 part (2). □

In the situation of the following lemma the image of σ is locally on X cut out by a regular sequence, see Divisors, Lemma 22.8.

Lemma 34.20. *Let $f : X \to S$ be a morphism of schemes. Let $\sigma : S \to X$ be a section of f. Let $s \in S$ be a point such that f is smooth at $x = \sigma(s)$. Then there exist affine open neighbourhoods $\mathrm{Spec}(A) = U \subset S$ of s and $\mathrm{Spec}(B) = V \subset X$ of x such that*

(1) *$f(V) \subset U$ and $\sigma(U) \subset V$,*
(2) *with $I = \mathrm{Ker}(\sigma^\# : B \to A)$ the module I/I^2 is a free A-module, and*
(3) *$B^\wedge \cong A[[x_1, \ldots, x_d]]$ as A-algebras where $B^\wedge$ denotes the completion of B with respect to I.*

Proof. Pick an affine open $U \subset S$ containing s Pick an affine open $V \subset f^{-1}(U)$ containing x. Pick an affine open $U' \subset \sigma^{-1}(V)$ containing s. Note that $V' = f^{-1}(U') \cap V$ is affine as it is equal to the fibre product $V' = U' \times_U V$. Then U' and V' satisfy (1). Write $U' = \mathrm{Spec}(A')$ and $V' = \mathrm{Spec}(B')$. By Algebra, Lemma 139.4 the module $I'/(I')^2$ is finite locally free as a A'-module. Hence after replacing U' by a smaller affine open $U'' \subset U'$ and V' by $V'' = V' \cap f^{-1}(U'')$ we obtain the situation where $I''/(I'')^2$ is free, i.e., (2) holds. In this case (3) holds also by Algebra, Lemma 139.4. □

The dimension of a scheme X at a point x (Properties, Definition 10.1) is just the dimension of X at x as a topological space, see Topology, Definition 10.1. This is not the dimension of the local ring $\mathcal{O}_{X,x}$, in general.

Lemma 34.21. *Let $f : X \to Y$ be a smooth morphism of locally Noetherian schemes. For every point x in X with image y in Y,*

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y),$$

where X_y denotes the fiber over y.

Proof. After replacing X by an open neighborhood of x, there is a natural number d such that all fibers of $X \to Y$ have dimension d at every point, see Lemma 34.12. Then f is flat (Lemma 34.9), locally of finite type (Lemma 34.8), and of relative dimension d. Hence the result follows from Lemma 29.6. □

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[10] In fact this is implied by (1) and (2), see Descent, Lemma 14.3. Moreover, it suffices to assume f is surjective, flat and locally of finite presentation, see Descent, Lemma 14.5.

35. Unramified morphisms

We briefly discuss unramified morphisms before the (perhaps) more interesting class of étale morphisms. Recall that a ring map $R \to A$ is *unramified* if it is of finite type and $\Omega_{A/R} = 0$ (this is the definition of [Ray70]). A ring map $R \to A$ is called *G-unramified* if it is of finite presentation and $\Omega_{A/R} = 0$ (this is the definition of [DG67]). See Algebra, Definition 151.1.

Definition 35.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *unramified at $x \in X$* if there exists an affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is unramified.
(2) We say that f is *G-unramified at $x \in X$* if there exists an affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is G-unramified.
(3) We say that f is *unramified* if it is unramified at every point of X.
(4) We say that f is *G-unramified* if it is G-unramified at every point of X.

Note that a G-unramified morphism is unramified. Hence any result for unramified morphisms implies the corresponding result for G-unramified morphisms. Moreover, if S is locally Noetherian then there is no difference between G-unramified and unramified morphisms, see Lemma 35.6. A pleasing feature of this definition is that the set of points where a morphism is unramified (resp. G-unramified) is automatically open.

Lemma 35.2. *Let $f : X \to S$ be a morphism of schemes. Then*

(1) *f is unramified if and only if f is locally of finite type and $\Omega_{X/S} = 0$, and*
(2) *f is G-unramified if and only if f is locally of finite presentation and $\Omega_{X/S} = 0$.*

Proof. By definition a ring map $R \to A$ is unramified (resp. G-unramified) if and only if it is of finite type (resp. finite presentation) and $\Omega_{A/R} = 0$. Hence the lemma follows directly from the definitions and Lemma 32.5. $\square$

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being unramified is local in nature on the source. Here is the precise result.

Lemma 35.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is unramified (resp. G-unramified).*
(2) *For every affine open $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is unramified (resp. G-unramified).*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is unramified (resp. G-unramified).*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is unramified (resp. G-unramified), for all $j \in J, i \in I_j$.*

Moreover, if f is unramified (resp. G-unramified) then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is unramified (resp. G-unramified).

Proof. This follows from Lemma 14.3 if we show that the property "$R \to A$ is unramified" is local. We check conditions (a), (b) and (c) of Definition 14.1. These properties are proved in Algebra, Lemma 151.3. $\square$

Lemma 35.4. *The composition of two morphisms which are unramified is unramified. The same holds for G-unramified morphisms.*

Proof. The proof of Lemma 35.3 shows that being unramified (resp. G-unramified) is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 14.5 combined with the fact that being unramified (resp. G-unramified) is a property of ring maps that is stable under composition, see Algebra, Lemma 151.3. $\square$

Lemma 35.5. *The base change of a morphism which is unramified is unramified. The same holds for G-unramified morphisms.*

Proof. The proof of Lemma 35.3 shows that being unramified (resp. G-unramified) is a local property of ring maps. Hence the lemma follows from Lemma 14.6 combined with the fact that being unramified (resp. G-unramified) is a property of ring maps that is stable under base change, see Algebra, Lemma 151.3. $\square$

Lemma 35.6. *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian. Then f is unramified if and only if f is G-unramified.*

Proof. Follows from the definitions and Lemma 21.9. $\square$

Lemma 35.7. *Any open immersion is G-unramified.*

Proof. This is true because an open immersion is a local isomorphism. $\square$

Lemma 35.8. *A closed immersion $i : Z \to X$ is unramified. It is G-unramified if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).*

Proof. Follows from Lemma 21.7 and Algebra, Lemma 151.3. $\square$

Lemma 35.9. *An unramified morphism is locally of finite type. A G-unramified morphism is locally of finite presentation.*

Proof. An unramified ring map is of finite type by definition. A G-unramified ring map is of finite presentation by definition. $\square$

Lemma 35.10. *Let $f : X \to S$ be a morphism of schemes. If f is unramified at x then f is quasi-finite at x. In particular, an unramified morphism is locally quasi-finite.*

Proof. See Algebra, Lemma 151.6. $\square$

Lemma 35.11. *Fibres of unramified morphisms.*

(1) *Let X be a scheme over a field k. The structure morphism $X \to \mathrm{Spec}(k)$ is unramified if and only if X is a disjoint union of spectra of finite separable field extensions of k.*
(2) *If $f : X \to S$ is an unramified morphism then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.*

Proof. Part (2) follows from part (1) and Lemma 35.5. Let us prove part (1). We first use Algebra, Lemma 151.7. This lemma implies that if X is a disjoint union of spectra of finite separable field extensions of k then $X \to \mathrm{Spec}(k)$ is unramified. Conversely, suppose that $X \to \mathrm{Spec}(k)$ is unramified. By Algebra, Lemma 151.5 for every $x \in X$ the residue field extension $\kappa(x)/k$ is finite separable. Since $X \to \mathrm{Spec}(k)$ is locally quasi-finite (Lemma 35.10) we see that all points of X are isolated closed points, see Lemma 20.6. Thus X is a discrete space, in particular the disjoint union of the spectra of its local rings. By Algebra, Lemma 151.5 again these local rings are fields, and we win. □

The following lemma characterizes an unramified morphisms as morphisms locally of finite type with unramified fibres.

Lemma 35.12. *Let $f : X \to S$ be a morphism of schemes.*

1. *If f is unramified then for any $x \in X$ the field extension $\kappa(x)/\kappa(f(x))$ is finite separable.*
2. *If f is locally of finite type, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is unramified.*
3. *If f is locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is G-unramified.*

Proof. Follows from Algebra, Lemmas 151.5 and 151.7. □

Here is a characterization of unramified morphisms in terms of the diagonal morphism.

Lemma 35.13. *Let $f : X \to S$ be a morphism.*

1. *If f is unramified, then the diagonal morphism $\Delta : X \to X \times_S X$ is an open immersion.*
2. *If f is locally of finite type and Δ is an open immersion, then f is unramified.*
3. *If f is locally of finite presentation and Δ is an open immersion, then f is G-unramified.*

Proof. The first statement follows from Algebra, Lemma 151.4. The second statement from the fact that $\Omega_{X/S}$ is the conormal sheaf of the diagonal morphism (Lemma 32.7) and hence clearly zero if Δ is an open immersion. □

Lemma 35.14. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite type (resp. locally of finite presentation). The following are equivalent:*

1. *The morphism f is unramified (resp. G-unramified) at x.*
2. *The fibre X_s is unramified over $\kappa(s)$ at x.*
3. *The $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ is zero.*
4. *The $\mathcal{O}_{X_s,x}$-module $\Omega_{X_s/s,x}$ is zero.*
5. *The $\kappa(x)$-vector space*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is zero.
6. *We have $\mathfrak{m}_s \mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(x)/\kappa(s)$ is finite separable.*

Proof. Note that if f is unramified at x, then we see that $\Omega_{X/S} = 0$ in a neighbourhood of x by the definitions and the results on modules of differentials in Section 32. Hence (1) implies (3) and the vanishing of the right hand vector space in (5). It also implies (2) because by Lemma 32.10 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S. This fact on modules of differentials also implies the displayed equality of vector spaces in part (4). By Lemma 32.12 the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are of finite type. Hence the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are zero if and only if the corresponding $\kappa(x)$-vector space in (4) is zero by Nakayama's Lemma (Algebra, Lemma 20.1). This in particular shows that (3), (4) and (5) are equivalent. The support of $\Omega_{X/S}$ is closed in X, see Modules, Lemma 9.6. Assumption (3) implies that x is not in the support. Hence $\Omega_{X/S}$ is zero in a neighbourhood of x, which implies (1). The equivalence of (1) and (3) applied to $X_s \to s$ implies the equivalence of (2) and (4). At this point we have seen that (1) – (5) are equivalent.

Alternatively you can use Algebra, Lemma 151.3 to see the equivalence of (1) – (5) more directly.

The equivalence of (1) and (6) follows from Lemma 35.12. It also follows more directly from Algebra, Lemmas 151.5 and 151.7. □

Lemma 35.15. *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the open set*

$$\begin{aligned} T &= \{x \in X \mid X_{f(x)} \text{ is unramified over } \kappa(f(x)) \text{ at } x\} \\ &= \{x \in X \mid X \text{ is unramified over } S \text{ at } x\} \end{aligned}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. If f is assumed locally of finite presentation then the same holds for the open set of points where f is G-unramified.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \mathrm{Spec}(\kappa(s')) \times_{\mathrm{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. In particular

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x') = \Omega_{X'_{s'}/s',x'} \otimes_{\mathcal{O}_{X'_{s'},x'}} \kappa(x')$$

see Lemma 32.10. Whence $x' \in T'$ if and only if $x \in T$ by Lemma 35.14. The second part follows from the first because in that case T is the (open) set of points where f is G-unramified according to Lemma 35.14. □

Lemma 35.16. *Let $f : X \to Y$ be a morphism of schemes over S.*

(1) *If X is unramified over S, then f is unramified.*
(2) *If X is G-unramified over S and Y is locally of finite type over S, then f is G-unramified.*

Proof. Assume that X is unramified over S. By Lemma 15.8 we see that f is locally of finite type. By assumption we have $\Omega_{X/S} = 0$. Hence $\Omega_{X/Y} = 0$ by Lemma 32.9. Thus f is unramified. If X is G-unramified over S and Y is locally of finite type over S, then by Lemma 21.11 we see that f is locally of finite presentation and we conclude that f is G-unramified. $\square$

Lemma 35.17. *Let S be a scheme. Let X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms over S. Let $x \in X$. Assume that*

1. *the structure morphism $Y \to S$ is unramified,*
2. *$f(x) = g(x)$ in Y, say $y = f(x) = g(x)$, and*
3. *the induced maps $f^\sharp, g^\sharp : \kappa(y) \to \kappa(x)$ are equal.*

Then there exists an open neighbourhood of x in X on which f and g are equal.

Proof. Consider the morphism $(f, g) : X \to Y \times_S Y$. By assumption (1) and Lemma 35.13 the inverse image of $\Delta_{Y/S}(Y)$ is open in X. And assumptions (2) and (3) imply that x is in this open subset. $\square$

36. Étale morphisms

The Zariski topology of a scheme is a very coarse topology. This is particularly clear when looking at varieties over $\mathbf{C}$. It turns out that declaring an étale morphism to be the analogue of a local isomorphism in topology introduces a much finer topology. On varieties over $\mathbf{C}$ this topology gives rise to the "correct" Betti numbers when computing cohomology with finite coefficients. Another observable is that if $f : X \to Y$ is an étale morphism of varieties over $\mathbf{C}$, and if x is a closed point of X, then f induces an isomorphism $\mathcal{O}^\wedge_{Y,f(x)} \to \mathcal{O}^\wedge_{X,x}$ of complete local rings.

In this section we start our study of these matters. In fact we deliberately restrict our discussion to a minimum since we will discuss more interesting results elsewhere. Recall that a ring map $R \to A$ is said to be *étale* if it is smooth and $\Omega_{A/R} = 0$, see Algebra, Definition 143.1.

Definition 36.1. Let $f : X \to S$ be a morphism of schemes.

1. We say that f is *étale at $x \in X$* if there exists an affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is étale.
2. We say that f is *étale* if it is étale at every point of X.
3. A morphism of affine schemes $f : X \to S$ is called *standard étale* if $X \to S$ is isomorphic to
$$\text{Spec}(R[x]_h/(g)) \to \text{Spec}(R)$$
where $R \to R[x]_h/(g)$ is a standard étale ring map, see Algebra, Definition 144.1, i.e., g is monic and g' invertible in $R[x]_h/(g)$.

A morphism is étale if and only if it is smooth of relative dimension 0 (see Definition 34.13). A pleasing feature of the definition is that the set of points where a morphism is étale is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being étale is local in nature on the source. Here is the precise result.

Lemma 36.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *The morphism f is étale.*

(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is étale.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is étale.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is étale, for all $j \in J, i \in I_j$.*

Moreover, if f is étale then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is étale.

Proof. This follows from Lemma 14.3 if we show that the property "$R \to A$ is étale" is local. We check conditions (a), (b) and (c) of Definition 14.1. These all follow from Algebra, Lemma 143.3. □

Lemma 36.3. *The composition of two morphisms which are étale is étale.*

Proof. In the proof of Lemma 36.2 we saw that being étale is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 14.5 combined with the fact that being étale is a property of ring maps that is stable under composition, see Algebra, Lemma 143.3. □

Lemma 36.4. *The base change of a morphism which is étale is étale.*

Proof. In the proof of Lemma 36.2 we saw that being étale is a local property of ring maps. Hence the lemma follows from Lemma 14.5 combined with the fact that being étale is a property of ring maps that is stable under base change, see Algebra, Lemma 143.3. □

Lemma 36.5. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Then f is étale at x if and only if f is smooth and unramified at x.*

Proof. This follows immediately from the definitions. □

Lemma 36.6. *An étale morphism is locally quasi-finite.*

Proof. By Lemma 36.5 an étale morphism is unramified. By Lemma 35.10 an unramified morphism is locally quasi-finite. □

Lemma 36.7. *Fibres of étale morphisms.*

(1) *Let X be a scheme over a field k. The structure morphism $X \to \mathrm{Spec}(k)$ is étale if and only if X is a disjoint union of spectra of finite separable field extensions of k.*
(2) *If $f : X \to S$ is an étale morphism, then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.*

Proof. You can deduce this from Lemma 35.11 via Lemma 36.5 above. Here is a direct proof.

We will use Algebra, Lemma 143.4. Hence it is clear that if X is a disjoint union of spectra of finite separable field extensions of k then $X \to \mathrm{Spec}(k)$ is étale. Conversely, suppose that $X \to \mathrm{Spec}(k)$ is étale. Then for any affine open $U \subset X$ we see that U is a finite disjoint union of spectra of finite separable field extensions of k. Hence all points of X are closed points (see Lemma 20.2 for example). Thus X is a discrete space and we win. □

The following lemma characterizes an étale morphism as a flat, finitely presented morphism with "étale fibres".

Lemma 36.8. *Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$, then f is étale.*

Proof. You can deduce this from Algebra, Lemma 143.7. Here is another proof.

By Lemma 36.7 a fibre X_s is étale and hence smooth over s. By Lemma 34.3 we see that $X \to S$ is smooth. By Lemma 35.12 we see that f is unramified. We conclude by Lemma 36.5. □

Lemma 36.9. *Any open immersion is étale.*

Proof. This is true because an open immersion is a local isomorphism. □

Lemma 36.10. *An étale morphism is syntomic.*

Proof. See Algebra, Lemma 137.10 and use that an étale morphism is the same as a smooth morphism of relative dimension 0. □

Lemma 36.11. *An étale morphism is locally of finite presentation.*

Proof. True because an étale ring map is of finite presentation by definition. □

Lemma 36.12. *An étale morphism is flat.*

Proof. Combine Lemmas 30.7 and 36.10. □

Lemma 36.13. *An étale morphism is open.*

Proof. Combine Lemmas 36.12, 36.11, and 25.10. □

The following lemma says locally any étale morphism is standard étale. This is actually kind of a tricky result to prove in complete generality. The tricky parts are hidden in the chapter on commutative algebra. Hence a standard étale morphism is a *local model* for a general étale morphism.

Lemma 36.14. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Let $V \subset S$ be an affine open neighbourhood of $f(x)$. The following are equivalent*

(1) *The morphism f is étale at x.*
(2) *There exist an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ such that the induced morphism $f|_U : U \to V$ is standard étale (see Definition 36.1).*

Proof. Follows from the definitions and Algebra, Proposition 144.4. □

Here is a differential criterion of étaleness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

Lemma 36.15. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:*

(1) *The morphism f is étale at x.*
(2) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $X_s \to \mathrm{Spec}(\kappa(s))$ is étale at x.*
(3) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $X_s \to \mathrm{Spec}(\kappa(s))$ is unramified at x.*

(4) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ is zero.*
(5) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is zero.
(6) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat, we have $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(x)/\kappa(s)$ is finite separable.*
(7) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth of relative dimension 0.*
(8) *There exist affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*
$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$
with
$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \ldots & \partial f_n/\partial x_n \end{pmatrix}$$
mapping to an element of A not in $\mathfrak{q}$.
(9) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard étale.*
(10) *There exist affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*
$$A = R[x]_Q/(P) = R[x, 1/Q]/(P)$$
with $P, Q \in R[x]$, P monic and $P' = dP/dx$ mapping to an element of A not in $\mathfrak{q}$.

Proof. Use Lemma 36.14 and the definitions to see that (1) implies all of the other conditions. For each of the conditions (2) – (10) combine Lemmas 34.14 and 35.14 to see that (1) holds by showing f is both smooth and unramified at x and applying Lemma 36.5. Some details omitted. □

Lemma 36.16. *A morphism is étale at a point if and only if it is flat and G-unramified at that point. A morphism is étale if and only if it is flat and G-unramified.*

Proof. This is clear from Lemmas 36.15 and 35.14. □

Lemma 36.17. *Let*
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$
be a cartesian diagram of schemes. Let $W \subset X$, resp. $W' \subset X'$ be the open subscheme of points where f, resp. f' is étale. Then $W' = (g')^{-1}(W)$ if
(1) *f is flat and locally of finite presentation, or*
(2) *f is locally of finite presentation and g is flat.*

Proof. Assume first that f locally of finite type. Consider the set

$$T = \{x \in X \mid f \text{ is unramified at } x\}$$

and the corresponding set $T' \subset X'$ for f'. Then $T' = (g')^{-1}(T)$ by Lemma 35.15.

Thus case (1) follows because in case (1) T is the (open) set of points where f is étale by Lemma 36.16.

In case (2) let $x' \in W'$. Then g' is flat at x' (Lemma 25.7) and $g \circ f'$ is flat at x' (Lemma 25.5). It follows that f is flat at $x = g'(x')$ by Lemma 25.13. On the other hand, since $x' \in T'$ (Lemma 34.5) we see that $x \in T$. Hence f is étale at x by Lemma 36.15. $\square$

Our proof of the following lemma is somewhat complicated. It uses the "Critère de platitude par fibres" to see that a morphism $X \to Y$ over S between schemes étale over S is automatically flat. The details are in the chapter on commutative algebra.

Lemma 36.18. *Let $f : X \to Y$ be a morphism of schemes over S. If X and Y are étale over S, then f is étale.*

Proof. See Algebra, Lemma 143.8. $\square$

Lemma 36.19. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, and étale,*
(2) *p is étale, and*
(3) *q is locally of finite presentation*[11].

Then q is étale.

Proof. By Lemma 34.19 we see that q is smooth. Thus we only need to see that q has relative dimension 0. This follows from Lemma 28.2 and the fact that f and p have relative dimension 0. $\square$

A final characterization of smooth morphisms is that a smooth morphism $f : X \to S$ is locally the composition of an étale morphism by a projection $\mathbf{A}^d_S \to S$.

Lemma 36.20. *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. Let $V \subset Y$ be an affine open neighbourhood of $\varphi(x)$. If φ is smooth at x, then there exists an integer $d \geq 0$ and an affine open $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "\pi"] \arrow[d] & \mathbf{A}^d_V \arrow[dl] \\ Y & V \arrow[l] & \end{tikzcd}$$

where π is étale.

[11]In fact this is implied by (1) and (2), see Descent, Lemma 14.3. Moreover, it suffices to assume that f is surjective, flat and locally of finite presentation, see Descent, Lemma 14.5.

Proof. By Lemma 34.11 we can find an affine open U as in the lemma such that $\varphi|_U : U \to V$ is standard smooth. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(R)$ so that we can write

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

with

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

mapping to an invertible element of A. Then it is clear that $R[x_{c+1}, \ldots, x_n] \to A$ is standard smooth of relative dimension 0. Hence it is smooth of relative dimension 0. In other words the ring map $R[x_{c+1}, \ldots, x_n] \to A$ is étale. As $\mathbf{A}^{n-c}_V = \mathrm{Spec}(R[x_{c+1}, \ldots, x_n])$ the lemma with $d = n - c$. $\square$

37. Relatively ample sheaves

Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L}$ is ample on X if X is quasi-compact and every point of X is contained in an affine open of the form X_s, where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, see Properties, Definition 26.1. We turn this into a relative notion as follows.

Definition 37.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively ample*, or *f-relatively ample*, or *ample on X/S*, or *f-ample* if $f : X \to S$ is quasi-compact, and if for every affine open $V \subset S$ the restriction of $\mathcal{L}$ to the open subscheme $f^{-1}(V)$ of X is ample.

We note that the existence of a relatively ample sheaf on X does not force the morphism $X \to S$ to be of finite type.

Lemma 37.2. *Let $X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $n \geq 1$. Then $\mathcal{L}$ is f-ample if and only if $\mathcal{L}^{\otimes n}$ is f-ample.*

Proof. This follows from Properties, Lemma 26.2. $\square$

Lemma 37.3. *Let $f : X \to S$ be a morphism of schemes. If there exists an f-ample invertible sheaf, then f is separated.*

Proof. Being separated is local on the base (see Schemes, Lemma 21.7 for example; it also follows easily from the definition). Hence we may assume S is affine and X has an ample invertible sheaf. In this case the result follows from Properties, Lemma 26.8. $\square$

There are many ways to characterize relatively ample invertible sheaves, analogous to the equivalent conditions in Properties, Proposition 26.13. We will add these here as needed.

Lemma 37.4. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. The following are equivalent:*

1. *The invertible sheaf $\mathcal{L}$ is f-ample.*
2. *There exists an open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample relative to $f^{-1}(V_i) \to V_i$.*
3. *There exists an affine open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample.*

(4) *There exists a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ and a map of graded $\mathcal{O}_X$-algebras $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ such that $U(\psi) = X$ and*
$$r_{\mathcal{L},\psi} : X \longrightarrow \underline{Proj}_S(\mathcal{A})$$
is an open immersion (see Constructions, Lemma 19.1 for notation).
(5) *The morphism f is quasi-separated and part (4) above holds with $\mathcal{A} = f_*(\bigoplus_{d \geq 0} \mathcal{L}^{\otimes d})$ and ψ the adjunction mapping.*
(6) *Same as (4) but just requiring $r_{\mathcal{L},\psi}$ to be an immersion.*

Proof. It is immediate from the definition that (1) implies (2) and (2) implies (3). It is clear that (5) implies (4).

Assume (3) holds for the affine open covering $S = \bigcup V_i$. We are going to show (5) holds. Since each $f^{-1}(V_i)$ has an ample invertible sheaf we see that $f^{-1}(V_i)$ is separated (Properties, Lemma 26.8). Hence f is separated. By Schemes, Lemma 24.1 we see that $\mathcal{A} = f_*(\bigoplus_{d \geq 0} \mathcal{L}^{\otimes d})$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. Denote $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ the adjunction mapping. The description of the open $U(\psi)$ in Constructions, Section 19 and the definition of ampleness of $\mathcal{L}|_{f^{-1}(V_i)}$ show that $U(\psi) = X$. Moreover, Constructions, Lemma 19.1 part (3) shows that the restriction of $r_{\mathcal{L},\psi}$ to $f^{-1}(V_i)$ is the same as the morphism from Properties, Lemma 26.9 which is an open immersion according to Properties, Lemma 26.11. Hence (5) holds.

Let us show that (4) implies (1). Assume (4). Denote $\pi : \underline{\mathrm{Proj}}_S(\mathcal{A}) \to S$ the structure morphism. Choose $V \subset S$ affine open. By Constructions, Definition 16.7 we see that $\pi^{-1}(V) \subset \underline{\mathrm{Proj}}_S(\mathcal{A})$ is equal to $\mathrm{Proj}(A)$ where $A = \mathcal{A}(V)$ as a graded ring. Hence $r_{\mathcal{L},\psi}$ maps $f^{-1}(V)$ isomorphically onto a quasi-compact open of $\mathrm{Proj}(A)$. Moreover, $\mathcal{L}^{\otimes d}$ is isomorphic to the pullback of $\mathcal{O}_{\mathrm{Proj}(A)}(d)$ for some $d \geq 1$. (See part (3) of Constructions, Lemma 19.1 and the final statement of Constructions, Lemma 14.1.) This implies that $\mathcal{L}|_{f^{-1}(V)}$ is ample by Properties, Lemmas 26.12 and 26.2.

Assume (6). By the equivalence of (1) - (5) above we see that the property of being relatively ample on X/S is local on S. Hence we may assume that S is affine, and we have to show that $\mathcal{L}$ is ample on X. In this case the morphism $r_{\mathcal{L},\psi}$ is identified with the morphism, also denoted $r_{\mathcal{L},\psi} : X \to \mathrm{Proj}(A)$ associated to the map $\psi : A = \mathcal{A}(V) \to \Gamma_*(X, \mathcal{L})$. (See references above.) As above we also see that $\mathcal{L}^{\otimes d}$ is the pullback of the sheaf $\mathcal{O}_{\mathrm{Proj}(A)}(d)$ for some $d \geq 1$. Moreover, since X is quasi-compact we see that X gets identified with a closed subscheme of a quasi-compact open subscheme $Y \subset \mathrm{Proj}(A)$. By Constructions, Lemma 10.6 (see also Properties, Lemma 26.12) we see that $\mathcal{O}_Y(d')$ is an ample invertible sheaf on Y for some $d' \geq 1$. Since the restriction of an ample sheaf to a closed subscheme is ample, see Properties, Lemma 26.3 we conclude that the pullback of $\mathcal{O}_Y(d')$ is ample. Combining these results with Properties, Lemma 26.2 we conclude that $\mathcal{L}$ is ample as desired. $\square$

Lemma 37.5. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S affine. Then $\mathcal{L}$ is f-relatively ample if and only if $\mathcal{L}$ is ample on X.*

Proof. Immediate from Lemma 37.4 and the definitions. $\square$

Lemma 37.6. *Let $f : X \to S$ be a morphism of schemes. Then f is quasi-affine if and only if $\mathcal{O}_X$ is f-relatively ample.*

Proof. Follows from Properties, Lemma 27.1 and the definitions. $\square$

Lemma 37.7. *Let $f : X \to Y$ be a morphism of schemes, $\mathcal{M}$ an invertible $\mathcal{O}_Y$-module, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.*

1. *If $\mathcal{L}$ is f-ample and $\mathcal{M}$ is ample, then $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample for $a \gg 0$.*
2. *If $\mathcal{M}$ is ample and f quasi-affine, then $f^*\mathcal{M}$ is ample.*

Proof. Assume $\mathcal{L}$ is f-ample and $\mathcal{M}$ ample. By assumption Y and f are quasi-compact (see Definition 37.1 and Properties, Definition 26.1). Hence X is quasi-compact. By Properties, Lemma 26.8 the scheme Y is separated and by Lemma 37.3 the morphism f is separated. Hence X is separated by Schemes, Lemma 21.12. Pick $x \in X$. We can choose $m \geq 1$ and $t \in \Gamma(Y, \mathcal{M}^{\otimes m})$ such that Y_t is affine and $f(x) \in Y_t$. Since $\mathcal{L}$ restricts to an ample invertible sheaf on $f^{-1}(Y_t) = X_{f^*t}$ we can choose $n \geq 1$ and $s \in \Gamma(X_{f^*t}, \mathcal{L}^{\otimes n})$ with $x \in (X_{f^*t})_s$ with $(X_{f^*t})_s$ affine. By Properties, Lemma 17.2 part (2) whose assumptions are satisfied by the above, there exists an integer $e \geq 1$ and a section $s' \in \Gamma(X, \mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes em})$ which restricts to $s(f^*t)^e$ on X_{f^*t}. For any $b > 0$ consider the section $s'' = s'(f^*t)^b$ of $\mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes (e+b)m}$. Then $X_{s''} = (X_{f^*t})_s$ is an affine open of X containing x. Picking b such that n divides $e + b$ we see $\mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes (e+b)m}$ is the nth power of $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ for some a and we can get any a divisible by m and big enough. Since X is quasi-compact a finite number of these affine opens cover X. We conclude that for some a sufficiently divisible and large enough the invertible sheaf $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample on X. On the other hand, we know that $\mathcal{M}^{\otimes c}$ (and hence its pullback to X) is globally generated for all $c \gg 0$ by Properties, Proposition 26.13. Thus $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a+c}$ is ample (Properties, Lemma 26.5) for $c \gg 0$ and (1) is proved.

Part (2) follows from Lemma 37.6, Properties, Lemma 26.2, and part (1). $\square$

Lemma 37.8. *Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. Let $\mathcal{M}$ be an invertible $\mathcal{O}_Y$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If S is quasi-compact, $\mathcal{M}$ is g-ample, and $\mathcal{L}$ is f-ample, then $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is $g \circ f$-ample for $a \gg 0$.*

Proof. Let $S = \bigcup_{i=1,\ldots,n} V_i$ be a finite affine open covering. By Lemma 37.4 it suffices to prove that $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample on $(g \circ f)^{-1}(V_i)$ for $i = 1, \ldots, n$. Thus the lemma follows from Lemma 37.7. $\square$

Lemma 37.9. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes. Let $f' : X' \to S'$ be the base change of f and denote $\mathcal{L}'$ the pullback of $\mathcal{L}$ to X'. If $\mathcal{L}$ is f-ample, then $\mathcal{L}'$ is f'-ample.*

Proof. By Lemma 37.4 it suffices to find an affine open covering $S' = \bigcup U'_i$ such that $\mathcal{L}'$ restricts to an ample invertible sheaf on $(f')^{-1}(U'_i)$ for all i. We may choose U'_i mapping into an affine open $U_i \subset S$. In this case the morphism $(f')^{-1}(U'_i) \to f^{-1}(U_i)$ is affine as a base change of the affine morphism $U'_i \to U_i$ (Lemma 11.8). Thus $\mathcal{L}'|_{(f')^{-1}(U'_i)}$ is ample by Lemma 37.7. $\square$

Lemma 37.10. *Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\mathcal{L}$ is $g \circ f$-ample and f is quasi-compact*[12] *then $\mathcal{L}$ is f-ample.*

Proof. Assume f is quasi-compact and $\mathcal{L}$ is $g \circ f$-ample. Let $U \subset S$ be an affine open and let $V \subset Y$ be an affine open with $g(V) \subset U$. Then $\mathcal{L}|_{(g\circ f)^{-1}(U)}$ is ample on $(g \circ f)^{-1}(U)$ by assumption. Since $f^{-1}(V) \subset (g \circ f)^{-1}(U)$ we see that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)$ by Properties, Lemma 26.14. Namely, $f^{-1}(V) \to (g \circ f)^{-1}(U)$ is a quasi-compact open immersion by Schemes, Lemma 21.14 as $(g \circ f)^{-1}(U)$ is separated (Properties, Lemma 26.8) and $f^{-1}(V)$ is quasi-compact (as f is quasi-compact). Thus we conclude that $\mathcal{L}$ is f-ample by Lemma 37.4. $\square$

38. Very ample sheaves

Recall that given a quasi-coherent sheaf $\mathcal{E}$ on a scheme S the *projective bundle* associated to $\mathcal{E}$ is the morphism $\mathbf{P}(\mathcal{E}) \to S$, where $\mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\text{Sym}(\mathcal{E}))$, see Constructions, Definition 21.1.

Definition 38.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively very ample* or more precisely *f-relatively very ample*, or *very ample on X/S*, or *f-very ample* if there exist a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Since there is no assumption of quasi-compactness in this definition it is not true in general that a relatively very ample invertible sheaf is a relatively ample invertible sheaf.

Lemma 38.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If f is quasi-compact and $\mathcal{L}$ is a relatively very ample invertible sheaf, then $\mathcal{L}$ is a relatively ample invertible sheaf.*

Proof. By definition there exists quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Set $\mathcal{A} = \text{Sym}(\mathcal{E})$, so $\mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\mathcal{A})$ by definition. The graded $\mathcal{O}_S$-algebra $\mathcal{A}$ comes equipped with a map

$$\psi : \mathcal{A} \to \bigoplus\nolimits_{n\geq 0} \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n) \to \bigoplus\nolimits_{n\geq 0} f_*\mathcal{L}^{\otimes n}$$

where the second arrow uses the identification $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. By adjointness of f_* and f^* we get a morphism $\psi : f^*\mathcal{A} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$. We omit the verification that the morphism $r_{\mathcal{L},\psi}$ associated to this map is exactly the immersion i. Hence the result follows from part (6) of Lemma 37.4. $\square$

To arrive at the correct converse of this lemma we ask whether given a relatively ample invertible sheaf $\mathcal{L}$ there exists an integer $n \geq 1$ such that $\mathcal{L}^{\otimes n}$ is relatively very ample? In general this is false. There are several things that prevent this from being true:

1. Even if S is affine, it can happen that no finite integer n works because $X \to S$ is not of finite type, see Example 38.4.
2. The base not being quasi-compact means the result can be prevented from being true even with f finite type. Namely, given a field k there exists a scheme X_d of finite type over k with an ample invertible sheaf $\mathcal{O}_{X_d}(1)$ so

[12] This follows if g is quasi-separated by Schemes, Lemma 21.14.

that the smallest tensor power of $\mathcal{O}_{X_d}(1)$ which is very ample is the dth power. See Example 38.5. Taking f to be the disjoint union of the schemes X_d mapping to the disjoint union of copies of $\mathrm{Spec}(k)$ gives an example.

To see our version of the converse take a look at Lemma 39.5 below. We will do some preliminary work before proving it.

Example 38.3. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Set $X = \underline{\mathrm{Proj}}_S(\mathcal{A})$. In this case $\mathcal{O}_X(1)$ is a very ample invertible sheaf on X. Namely, the morphism associated to the graded $\mathcal{O}_S$-algebra map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion $X \to \mathbf{P}(\mathcal{A}_1)$ which pulls back $\mathcal{O}_{\mathbf{P}(\mathcal{A}_1)}(1)$ to $\mathcal{O}_X(1)$, see Constructions, Lemma 18.5.

Example 38.4. Let k be a field. Consider the graded k-algebra

$$A = k[U, V, Z_1, Z_2, Z_3, \ldots]/I \quad \text{with} \quad I = (U^2 - Z_1^2, U^4 - Z_2^2, U^6 - Z_3^2, \ldots)$$

with grading given by $\deg(U) = \deg(V) = \deg(Z_1) = 1$ and $\deg(Z_d) = d$. Note that $X = \mathrm{Proj}(A)$ is covered by $D_+(U)$ and $D_+(V)$. Hence the sheaves $\mathcal{O}_X(n)$ are all invertible and isomorphic to $\mathcal{O}_X(1)^{\otimes n}$. In particular $\mathcal{O}_X(1)$ is ample and f-ample for the morphism $f : X \to \mathrm{Spec}(k)$. We claim that no power of $\mathcal{O}_X(1)$ is f-relatively very ample. Namely, it is easy to see that $\Gamma(X, \mathcal{O}_X(n))$ is the degree n summand of the algebra A. Hence if $\mathcal{O}_X(n)$ were very ample, then X would be a closed subscheme of a projective space over k and hence of finite type over k. On the other hand $D_+(V)$ is the spectrum of $k[t, t_1, t_2, \ldots]/(t^2 - t_1^2, t^4 - t_2^2, t^6 - t_3^2, \ldots)$ which is not of finite type over k.

Example 38.5. Let k be an infinite field. Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be pairwise distinct elements of k^*. (This is not strictly necessary, and in fact the example works perfectly well even if all λ_i are equal to 1.) Consider the graded k-algebra

$$A_d = k[U, V, Z]/I_d \quad \text{with} \quad I_d = (Z^2 - \prod\nolimits_{i=1}^{2d}(U - \lambda_i V)).$$

with grading given by $\deg(U) = \deg(V) = 1$ and $\deg(Z) = d$. Then $X_d = \mathrm{Proj}(A_d)$ has ample invertible sheaf $\mathcal{O}_{X_d}(1)$. We claim that if $\mathcal{O}_{X_d}(n)$ is very ample, then $n \geq d$. The reason for this is that Z has degree d, and hence $\Gamma(X_d, \mathcal{O}_{X_d}(n)) = k[U, V]_n$ for $n < d$. Details omitted.

Lemma 38.6. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. If $\mathcal{L}$ is relatively very ample on X/S then f is separated.*

Proof. Being separated is local on the base (see Schemes, Section 21). An immersion is separated (see Schemes, Lemma 23.8). Hence the lemma follows since locally X has an immersion into the homogeneous spectrum of a graded ring which is separated, see Constructions, Lemma 8.8. $\square$

Lemma 38.7. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is quasi-compact. The following are equivalent*

1. *$\mathcal{L}$ is relatively very ample on X/S,*
2. *there exists an open covering $S = \bigcup V_j$ such that $\mathcal{L}|_{f^{-1}(V_j)}$ is relatively very ample on $f^{-1}(V_j)/V_j$ for all j,*

(3) *there exists a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras $\mathcal{A}$ generated in degree 1 over $\mathcal{O}_S$ and a map of graded $\mathcal{O}_X$-algebras $\psi : f^*\mathcal{A} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ such that $f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective and the associated morphism $r_{\mathcal{L},\psi} : X \to \underline{Proj}_S(\mathcal{A})$ is an immersion, and*

(4) *f is quasi-separated, the canonical map $\psi : f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective, and the associated map $r_{\mathcal{L},\psi} : X \to \mathbf{P}(f_*\mathcal{L})$ is an immersion.*

Proof. It is clear that (1) implies (2). It is also clear that (4) implies (1); the hypothesis of quasi-separation in (4) is used to guarantee that $f_*\mathcal{L}$ is quasi-coherent via Schemes, Lemma 24.1.

Assume (2). We will prove (4). Let $S = \bigcup V_j$ be an open covering as in (2). Set $X_j = f^{-1}(V_j)$ and $f_j : X_j \to V_j$ the restriction of f. We see that f is separated by Lemma 38.6 (as being separated is local on the base). By assumption there exists a quasi-coherent $\mathcal{O}_{V_j}$-module $\mathcal{E}_j$ and an immersion $i_j : X_j \to \mathbf{P}(\mathcal{E}_j)$ with $\mathcal{L}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}(\mathcal{E}_j)}(1)$. The morphism i_j corresponds to a surjection $f_j^*\mathcal{E}_j \to \mathcal{L}|_{X_j}$, see Constructions, Section 21. This map is adjoint to a map $\mathcal{E}_j \to f_*\mathcal{L}|_{V_j}$ such that the composition

$$f_j^*\mathcal{E}_j \to (f^*f_*\mathcal{L})|_{X_j} \to \mathcal{L}|_{X_j}$$

is surjective. We conclude that $\psi : f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective. Let $r_{\mathcal{L},\psi} : X \to \mathbf{P}(f_*\mathcal{L})$ be the associated morphism. We still have to show that $r_{\mathcal{L},\psi}$ is an immersion; we urge the reader to prove this for themselves. The $\mathcal{O}_{V_j}$-module map $\mathcal{E}_j \to f_*\mathcal{L}|_{V_j}$ determines a homomorphism on symmetric algebras, which in turn defines a morphism

$$\mathbf{P}(f_*\mathcal{L}|_{V_j}) \supset U_j \longrightarrow \mathbf{P}(\mathcal{E}_j)$$

where U_j is the open subscheme of Constructions, Lemma 18.1. The compatibility of ψ with $\mathcal{E}_j \to f_*\mathcal{L}|_{V_j}$ shows that $r_{\mathcal{L},\psi}(X_j) \subset U_j$ and that there is a factorization

$$X_j \xrightarrow{r_{\mathcal{L},\psi}} U_j \longrightarrow \mathbf{P}(\mathcal{E}_j)$$

We omit the verification. This shows that $r_{\mathcal{L},\psi}$ is an immersion.

At this point we see that (1), (2) and (4) are equivalent. Clearly (4) implies (3). Assume (3). We will prove (1). Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras generated in degree 1 over $\mathcal{O}_S$. Consider the map of graded $\mathcal{O}_S$-algebras $\text{Sym}(\mathcal{A}_1) \to \mathcal{A}$. This is surjective by hypothesis and hence induces a closed immersion

$$\underline{\text{Proj}}_S(\mathcal{A}) \longrightarrow \mathbf{P}(\mathcal{A}_1)$$

which pulls back $\mathcal{O}(1)$ to $\mathcal{O}(1)$, see Constructions, Lemma 18.5. Hence it is clear that (3) implies (1). $\square$

Lemma 38.8. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes. Let $f' : X' \to S'$ be the base change of f and denote $\mathcal{L}'$ the pullback of $\mathcal{L}$ to X'. If $\mathcal{L}$ is f-very ample, then $\mathcal{L}'$ is f'-very ample.*

Proof. By Definition 38.1 there exists there exist a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. The base change of $\mathbf{P}(\mathcal{E})$ to S' is the projective bundle associated to the pullback $\mathcal{E}'$ of $\mathcal{E}$ and the pullback of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is $\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1)$, see Constructions, Lemma 16.10. Finally, the base change of an immersion is an immersion (Schemes, Lemma 18.2). $\square$

39. Ample and very ample sheaves relative to finite type morphisms

In fact most of the material in this section is about the notion of a (quasi-)projective morphism which we have not defined yet.

Lemma 39.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

(1) *the invertible sheaf $\mathcal{L}$ is very ample on X/S,*
(2) *the morphism $X \to S$ is of finite type, and*
(3) *S is affine.*

Then there exist an $n \geq 0$ and an immersion $i : X \to \mathbf{P}^n_S$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. Assume (1), (2) and (3). Condition (3) means $S = \mathrm{Spec}(R)$ for some ring R. Condition (1) means by definition there exists a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $\alpha : X \to \mathbf{P}(\mathcal{E})$ such that $\mathcal{L} = \alpha^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Write $\mathcal{E} = \widetilde{M}$ for some R-module M. Thus we have

$$\mathbf{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}_R(M)).$$

Since α is an immersion, and since the topology of $\mathrm{Proj}(\mathrm{Sym}_R(M))$ is generated by the standard opens $D_+(f)$, $f \in \mathrm{Sym}^d_R(M)$, $d \geq 1$, we can find for each $x \in X$ an $f \in \mathrm{Sym}^d_R(M)$, $d \geq 1$, with $\alpha(x) \in D_+(f)$ such that

$$\alpha|_{\alpha^{-1}(D_+(f))} : \alpha^{-1}(D_+(f)) \to D_+(f)$$

is a closed immersion. Condition (2) implies X is quasi-compact. Hence we can find a finite collection of elements $f_j \in \mathrm{Sym}^{d_j}_R(M)$, $d_j \geq 1$ such that for each $f = f_j$ the displayed map above is a closed immersion and such that $\alpha(X) \subset \bigcup D_+(f_j)$. Write $U_j = \alpha^{-1}(D_+(f_j))$. Note that U_j is affine as a closed subscheme of the affine scheme $D_+(f_j)$. Write $U_j = \mathrm{Spec}(A_j)$. Condition (2) also implies that A_j is of finite type over R, see Lemma 15.2. Choose finitely many $x_{j,k} \in A_j$ which generate A_j as a R-algebra. Since $\alpha|_{U_j}$ is a closed immersion we see that $x_{j,k}$ is the image of an element

$$f_{j,k}/f_j^{e_{j,k}} \in \mathrm{Sym}_R(M)_{(f_j)} = \Gamma(D_+(f_j), \mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}).$$

Finally, choose $n \geq 1$ and elements $y_0, \ldots, y_n \in M$ such that each of the polynomials $f_j, f_{j,k} \in \mathrm{Sym}_R(M)$ is a polynomial in the elements y_t with coefficients in R. Consider the graded ring map

$$\psi : R[Y_0, \ldots, Y_n] \longrightarrow \mathrm{Sym}_R(M), \quad Y_i \longmapsto y_i.$$

Denote F_j, $F_{j,k}$ the elements of $R[Y_0, \ldots, Y_n]$ such that $\psi(F_j) = f_j$ and $\psi(F_{j,k}) = f_{j,k}$. By Constructions, Lemma 11.1 we obtain an open subscheme

$$U(\psi) \subset \mathrm{Proj}(\mathrm{Sym}_R(M))$$

and a morphism $r_\psi : U(\psi) \to \mathbf{P}^n_R$. This morphism satisfies $r_\psi^{-1}(D_+(F_j)) = D_+(f_j)$, and hence we see that $\alpha(X) \subset U(\psi)$. Moreover, it is clear that

$$i = r_\psi \circ \alpha : X \longrightarrow \mathbf{P}^n_R$$

is still an immersion since $i^\sharp(F_{j,k}/F_j^{e_{j,k}}) = x_{j,k} \in A_j = \Gamma(U_j, \mathcal{O}_X)$ by construction. Moreover, the morphism r_ψ comes equipped with a map $\theta : r_\psi^*\mathcal{O}_{\mathbf{P}^n_R}(1) \to \mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}(1)|_{U(\psi)}$ which is an isomorphism in this case (for construction θ see lemma cited above; some details omitted). Since the original map α was assumed to have the property that $\mathcal{L} = \alpha^*\mathcal{O}_{\text{Proj}(\text{Sym}_R(M))}(1)$ we win. $\square$

Lemma 39.2. *Let $\pi : X \to S$ be a morphism of schemes. Assume that X is quasi-affine and that π is locally of finite type. Then there exist $n \geq 0$ and an immersion $i : X \to \mathbf{A}^n_S$ over S.*

Proof. Let $A = \Gamma(X, \mathcal{O}_X)$. By assumption X is quasi-compact and is identified with an open subscheme of $\text{Spec}(A)$, see Properties, Lemma 18.4. Moreover, the set of opens X_f, for those $f \in A$ such that X_f is affine, forms a basis for the topology of X, see the proof of Properties, Lemma 18.4. Hence we can find a finite number of $f_j \in A$, $j = 1, \ldots, m$ such that $X = \bigcup X_{f_j}$, and such that $\pi(X_{f_j}) \subset V_j$ for some affine open $V_j \subset S$. By Lemma 15.2 the ring maps $\mathcal{O}(V_j) \to \mathcal{O}(X_{f_j}) = A_{f_j}$ are of finite type. Thus we may choose $a_1, \ldots, a_N \in A$ such that the elements $a_1, \ldots, a_N, 1/f_j$ generate A_{f_j} over $\mathcal{O}(V_j)$ for each j. Take $n = m + N$ and let

$$i : X \longrightarrow \mathbf{A}^n_S$$

be the morphism given by the global sections $f_1, \ldots, f_m, a_1, \ldots, a_N$ of the structure sheaf of X. Let $D(x_j) \subset \mathbf{A}^n_S$ be the open subscheme where the jth coordinate function is nonzero. Then for $1 \leq j \leq m$ we have $i^{-1}(D(x_j)) = X_{f_j}$ and the induced morphism $X_{f_j} \to D(x_j)$ factors through the affine open $\text{Spec}(\mathcal{O}(V_j)[x_1, \ldots, x_n, 1/x_j])$ of $D(x_j)$. Since the ring map $\mathcal{O}(V_j)[x_1, \ldots, x_n, 1/x_j] \to A_{f_j}$ is surjective by construction we conclude that $i^{-1}(D(x_j)) \to D(x_j)$ is an immersion as desired. $\square$

Lemma 39.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

1. *the invertible sheaf $\mathcal{L}$ is ample on X, and*
2. *the morphism $X \to S$ is locally of finite type.*

Then there exists a $d_0 \geq 1$ such that for every $d \geq d_0$ there exist an $n \geq 0$ and an immersion $i : X \to \mathbf{P}^n_S$ over S such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. Let $A = \Gamma_*(X, \mathcal{L}) = \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^{\otimes d})$. By Properties, Proposition 26.13 the set of affine opens X_a with $a \in A_+$ homogeneous forms a basis for the topology of X. Hence we can find finitely many such elements $a_0, \ldots, a_n \in A_+$ such that

1. we have $X = \bigcup_{i=0,\ldots,n} X_{a_i}$,
2. each X_{a_i} is affine, and
3. each X_{a_i} maps into an affine open $V_i \subset S$.

By Lemma 15.2 we see that the ring maps $\mathcal{O}_S(V_i) \to \mathcal{O}_X(X_{a_i})$ are of finite type. Hence we can find finitely many elements $f_{ij} \in \mathcal{O}_X(X_{a_i})$, $j = 1, \ldots, n_i$ which generate $\mathcal{O}_X(X_{a_i})$ as an $\mathcal{O}_S(V_i)$-algebra. By Properties, Lemma 17.2 we may write each f_{ij} as $a_{ij}/a_i^{e_{ij}}$ for some $a_{ij} \in A_+$ homogeneous. Let N be a positive integer which is a common multiple of all the degrees of the elements a_i, a_{ij}. Consider the elements

$$a_i^{N/\deg(a_i)},\ a_{ij}a_i^{(N/\deg(a_i))-e_{ij}} \in A_N.$$

By construction these generate the invertible sheaf $\mathcal{L}^{\otimes N}$ over X. Hence they give rise to a morphism

$$j : X \longrightarrow \mathbf{P}^m_S \quad \text{with } m = n + \sum n_i$$

over S, see Constructions, Lemma 13.1 and Definition 13.2. Moreover, $j^*\mathcal{O}_{\mathbf{P}_S}(1) = \mathcal{L}^{\otimes N}$. We name the homogeneous coordinates $T_0, \ldots, T_n, T_{ij}$ instead of $T_0, \ldots, T_m$.

For $i = 0, \ldots, n$ we have $i^{-1}(D_+(T_i)) = X_{a_i}$. Moreover, pulling back the element T_{ij}/T_i via $j^\sharp$ we get the element $f_{ij} \in \mathcal{O}_X(X_{a_i})$. Hence the morphism j restricted to X_{a_i} gives a closed immersion of X_{a_i} into the affine open $D_+(T_i) \cap \mathbf{P}^m_{V_i}$ of $\mathbf{P}^N_S$. Hence we conclude that the morphism j is an immersion. This implies the lemma holds for some d and n which is enough in virtually all applications.

This proves that for one $d_2 \geq 1$ (namely $d_2 = N$ above), some $m \geq 0$ there exists some immersion $j : X \to \mathbf{P}^m_S$ given by global sections $s'_0, \ldots, s'_m \in \Gamma(X, \mathcal{L}^{\otimes d_2})$. By Properties, Proposition 26.13 we know there exists an integer d_1 such that $\mathcal{L}^{\otimes d}$ is globally generated for all $d \geq d_1$. Set $d_0 = d_1 + d_2$. We claim that the lemma holds with this value of d_0. Namely, given an integer $d \geq d_0$ we may choose $s''_1, \ldots, s''_t \in \Gamma(X, \mathcal{L}^{\otimes d - d_2})$ which generate $\mathcal{L}^{\otimes d - d_2}$ over X. Set $k = (m+1)t$ and denote $s_0, \ldots, s_k$ the collection of sections $s'_\alpha s''_\beta$, $\alpha = 0, \ldots, m$, $\beta = 1, \ldots, t$. These generate $\mathcal{L}^{\otimes d}$ over X and therefore define a morphism

$$i : X \longrightarrow \mathbf{P}^{k-1}_S$$

such that $i^*\mathcal{O}_{\mathbf{P}^n_S}(1) \cong \mathcal{L}^{\otimes d}$. To see that i is an immersion, observe that i is the composition

$$X \longrightarrow \mathbf{P}^m_S \times_S \mathbf{P}^{t-1}_S \longrightarrow \mathbf{P}^{k-1}_S$$

where the first morphism is (j, j') with j' given by $s''_1, \ldots, s''_t$ and the second morphism is the Segre embedding (Constructions, Lemma 13.6). Since j is an immersion, so is (j, j') (apply Lemma 3.1 to $X \to \mathbf{P}^m_S \times_S \mathbf{P}^{t-1}_S \to \mathbf{P}^m_S$). Thus i is a composition of immersions and hence an immersion (Schemes, Lemma 24.3). $\square$

Lemma 39.4. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S affine and f of finite type. The following are equivalent*

1. *$\mathcal{L}$ is ample on X,*
2. *$\mathcal{L}$ is f-ample,*
3. *$\mathcal{L}^{\otimes d}$ is f-very ample for some $d \geq 1$,*
4. *$\mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$,*
5. *for some $d \geq 1$ there exist $n \geq 1$ and an immersion $i : X \to \mathbf{P}^n_S$ such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$, and*
6. *for all $d \gg 1$ there exist $n \geq 1$ and an immersion $i : X \to \mathbf{P}^n_S$ such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.*

Proof. The equivalence of (1) and (2) is Lemma 37.5. The implication (2) $\Rightarrow$ (6) is Lemma 39.3. Trivially (6) implies (5). As $\mathbf{P}^n_S$ is a projective bundle over S (see Constructions, Lemma 21.5) we see that (5) implies (3) and (6) implies (4) from the definition of a relatively very ample sheaf. Trivially (4) implies (3). To finish we have to show that (3) implies (2) which follows from Lemma 38.2 and Lemma 37.2. $\square$

Lemma 39.5. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S quasi-compact and f of finite type. The following are equivalent*

1. *$\mathcal{L}$ is f-ample,*
2. *$\mathcal{L}^{\otimes d}$ is f-very ample for some $d \geq 1$,*
3. *$\mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$.*

Proof. Trivially (3) implies (2). Lemma 38.2 guarantees that (2) implies (1) since a morphism of finite type is quasi-compact by definition. Assume that $\mathcal{L}$ is f-ample. Choose a finite affine open covering $S = V_1 \cup \ldots \cup V_m$. Write $X_i = f^{-1}(V_i)$. By Lemma 39.4 above we see there exists a d_0 such that $\mathcal{L}^{\otimes d}$ is relatively very ample on X_i/V_i for all $d \geq d_0$. Hence we conclude (1) implies (3) by Lemma 38.7. ☐

The following two lemmas provide the most used and most useful characterizations of relatively very ample and relatively ample invertible sheaves when the morphism is of finite type.

Lemma 39.6. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is of finite type. The following are equivalent:*

(1) *$\mathcal{L}$ is f-relatively very ample, and*
(2) *there exist an open covering $S = \bigcup V_j$, for each j an integer n_j, and immersions*
$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}^{n_j}_{V_j}$$
over V_j such that $\mathcal{L}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}^{n_j}_{V_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 39.1 to each of the restrictions of f and $\mathcal{L}$. We see that (2) implies (1) by Lemma 38.7. ☐

Lemma 39.7. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is of finite type. The following are equivalent:*

(1) *$\mathcal{L}$ is f-relatively ample, and*
(2) *there exist an open covering $S = \bigcup V_j$, for each j an integers $d_j \geq 1$, $n_j \geq 0$, and immersions*
$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}^{n_j}_{V_j}$$
over V_j such that $\mathcal{L}^{\otimes d_j}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}^{n_j}_{V_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 39.4 to each of the restrictions of f and $\mathcal{L}$. We see that (2) implies (1) by Lemma 37.4. ☐

Lemma 39.8. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{N}$, $\mathcal{L}$ be invertible $\mathcal{O}_X$-modules. Assume S is quasi-compact, f is of finite type, and $\mathcal{L}$ is f-ample. Then $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$.*

Proof. By Lemma 39.6 we reduce to the case S is affine. Combining Lemma 39.4 and Properties, Proposition 26.13 we can find an integer d_0 such that $\mathcal{N} \otimes \mathcal{L}^{\otimes d_0}$ is globally generated. Choose global sections $s_0, \ldots, s_n$ of $\mathcal{N} \otimes \mathcal{L}^{\otimes d_0}$ which generate it. This determines a morphism $j : X \to \mathbf{P}^n_S$ over S. By Lemma 39.4 we can also pick an integer d_1 such that for all $d \geq d_1$ there exist sections $t_{d,0}, \ldots, t_{d,n(d)}$ of $\mathcal{L}^{\otimes d}$ which generate it and define an immersion
$$j_d = \varphi_{\mathcal{L}^{\otimes d}, t_{d,0}, \ldots, t_{d,n(d)}} : X \longrightarrow \mathbf{P}^{n(d)}_S$$
over S. Then for $d \geq d_0 + d_1$ we can consider the morphism
$$\varphi_{\mathcal{N} \otimes \mathcal{L}^{\otimes d}, s_j \otimes t_{d-d_0,i}} : X \longrightarrow \mathbf{P}^{(n+1)(n(d-d_0)+1)-1}_S$$

This morphism is an immersion as it is the composition

$$X \to \mathbf{P}^n_S \times_S \mathbf{P}^{n(d-d_0)}_S \to \mathbf{P}^{(n+1)(n(d-d_0)+1)-1}_S$$

where the first morphism is (j, j_{d-d_0}) and the second is the Segre embedding (Constructions, Lemma 13.6). Since j is an immersion, so is (j, j_{d-d_0}) (apply Lemma 3.1). We have a composition of immersions and hence an immersion (Schemes, Lemma 24.3). □

40. Quasi-projective morphisms

The discussion in the previous section suggests the following definitions. We take our definition of quasi-projective from [DG67]. The version with the letter "H" is the definition in [Har77].

Definition 40.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say f is *quasi-projective* if f is of finite type and there exists an f-relatively ample invertible $\mathcal{O}_X$-module.
(2) We say f is *H-quasi-projective* if there exists a quasi-compact immersion $X \to \mathbf{P}^n_S$ over S for some n.[13]
(3) We say f is *locally quasi-projective* if there exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \to V_j$ is quasi-projective.

As this definition suggests the property of being quasi-projective is not local on S. At a later stage we will be able to say more about the category of quasi-projective schemes, see More on Morphisms, Section 49.

Lemma 40.2. *A base change of a quasi-projective morphism is quasi-projective.*

Proof. This follows from Lemmas 15.4 and 37.9. □

Lemma 40.3. *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. If S is quasi-compact and f and g are quasi-projective, then $g \circ f$ is quasi-projective.*

Proof. This follows from Lemmas 15.3 and 37.8. □

Lemma 40.4. *Let $f : X \to S$ be a morphism of schemes. If f is quasi-projective, or H-quasi-projective or locally quasi-projective, then f is separated of finite type.*

Proof. Omitted. □

Lemma 40.5. *A H-quasi-projective morphism is quasi-projective.*

Proof. Omitted. □

Lemma 40.6. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is locally quasi-projective.*
(2) *There exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \to V_j$ is H-quasi-projective.*

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[13] This is not exactly the same as the definition in Hartshorne. Namely, the definition in Hartshorne (8th corrected printing, 1997) is that f should be the composition of an open immersion followed by a H-projective morphism (see Definition 43.1), which does not imply f is quasi-compact. See Lemma 43.11 for the implication in the other direction.

Proof. By Lemma 40.5 we see that (2) implies (1). Assume (1). The question is local on S and hence we may assume S is affine, X of finite type over S and $\mathcal{L}$ is a relatively ample invertible sheaf on X/S. By Lemma 39.4 we may assume $\mathcal{L}$ is ample on X. By Lemma 39.3 we see that there exists an immersion of X into a projective space over S, i.e., X is H-quasi-projective over S as desired. $\square$

Lemma 40.7. *A quasi-affine morphism of finite type is quasi-projective.*

Proof. This follows from Lemma 37.6. $\square$

Lemma 40.8. *Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. If $g \circ f$ is quasi-projective and f is quasi-compact*[14]*, then f is quasi-projective.*

Proof. Observe that f is of finite type by Lemma 15.8. Thus the lemma follows from Lemma 37.10 and the definitions. $\square$

41. Proper morphisms

The notion of a proper morphism plays an important role in algebraic geometry. An important example of a proper morphism will be the structure morphism $\mathbf{P}^n_S \to S$ of projective n-space, and this is in fact the motivating example leading to the definition.

Definition 41.1. Let $f : X \to S$ be a morphism of schemes. We say f is *proper* if f is separated, finite type, and universally closed.

The morphism from the affine line with zero doubled to the affine line is of finite type and universally closed, so the separation condition is necessary in the definition above. In the rest of this section we prove some of the basic properties of proper morphisms and of universally closed morphisms.

Lemma 41.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is universally closed.*
2. *There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \to V_j$ is universally closed for all indices j.*

Proof. This is clear from the definition. $\square$

Lemma 41.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is proper.*
2. *There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \to V_j$ is proper for all indices j.*

Proof. Omitted. $\square$

Lemma 41.4. *The composition of proper morphisms is proper. The same is true for universally closed morphisms.*

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[14] This follows if g is quasi-separated by Schemes, Lemma 21.14.

Proof. A composition of closed morphisms is closed. If $X \to Y \to Z$ are universally closed morphisms and $Z' \to Z$ is any morphism, then we see that $Z' \times_Z X = (Z' \times_Z Y) \times_Y X \to Z' \times_Z Y$ is closed and $Z' \times_Z Y \to Z'$ is closed. Hence the result for universally closed morphisms. We have seen that "separated" and "finite type" are preserved under compositions (Schemes, Lemma 21.12 and Lemma 15.3). Hence the result for proper morphisms. ☐

Lemma 41.5. *The base change of a proper morphism is proper. The same is true for universally closed morphisms.*

Proof. This is true by definition for universally closed morphisms. It is true for separated morphisms (Schemes, Lemma 21.12). It is true for morphisms of finite type (Lemma 15.4). Hence it is true for proper morphisms. ☐

Lemma 41.6. *A closed immersion is proper, hence a fortiori universally closed.*

Proof. The base change of a closed immersion is a closed immersion (Schemes, Lemma 18.2). Hence it is universally closed. A closed immersion is separated (Schemes, Lemma 23.8). A closed immersion is of finite type (Lemma 15.5). Hence a closed immersion is proper. ☐

Lemma 41.7. *Suppose given a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

with Y separated over S.
1. *If $X \to S$ is universally closed, then the morphism $X \to Y$ is universally closed.*
2. *If X is proper over S, then the morphism $X \to Y$ is proper.*

In particular, in both cases the image of X in Y is closed.

Proof. Assume that $X \to S$ is universally closed (resp. proper). We factor the morphism as $X \to X \times_S Y \to Y$. The first morphism is a closed immersion, see Schemes, Lemma 21.10. Hence the first morphism is proper (Lemma 41.6). The projection $X \times_S Y \to Y$ is the base change of a universally closed (resp. proper) morphism and hence universally closed (resp. proper), see Lemma 41.5. Thus $X \to Y$ is universally closed (resp. proper) as the composition of universally closed (resp. proper) morphisms (Lemma 41.4). ☐

The proof of the following lemma is due to Bjorn Poonen, see this location.

Lemma 41.8. *A universally closed morphism of schemes is quasi-compact.*

Proof. Let $f : X \to S$ be a morphism. Assume that f is not quasi-compact. Our goal is to show that f is not universally closed. By Schemes, Lemma 19.2 there exists an affine open $V \subset S$ such that $f^{-1}(V)$ is not quasi-compact. To achieve our goal it suffices to show that $f^{-1}(V) \to V$ is not universally closed, hence we may assume that $S = \operatorname{Spec}(A)$ for some ring A.

Write $X = \bigcup_{i \in I} X_i$ where the X_i are affine open subschemes of X. Let $T = \operatorname{Spec}(A[y_i; i \in I])$. Let $T_i = D(y_i) \subset T$. Let Z be the closed set $(X \times_S T) -$

$\bigcup_{i\in I}(X_i \times_S T_i)$. It suffices to prove that the image $f_T(Z)$ of Z under $f_T : X \times_S T \to T$ is not closed.

There exists a point $s \in S$ such that there is no neighborhood U of s in S such that X_U is quasi-compact. Otherwise we could cover S with finitely many such U and Schemes, Lemma 19.2 would imply f quasi-compact. Fix such an $s \in S$.

First we check that $f_T(Z_s) \neq T_s$. Let $t \in T$ be the point lying over s with $\kappa(t) = \kappa(s)$ such that $y_i = 1$ in $\kappa(t)$ for all i. Then $t \in T_i$ for all i, and the fiber of $Z_s \to T_s$ above t is isomorphic to $(X - \bigcup_{i\in I} X_i)_s$, which is empty. Thus $t \in T_s - f_T(Z_s)$.

Assume $f_T(Z)$ is closed in T. Then there exists an element $g \in A[y_i; i \in I]$ with $f_T(Z) \subset V(g)$ but $t \notin V(g)$. Hence the image of g in $\kappa(t)$ is nonzero. In particular some coefficient of g has nonzero image in $\kappa(s)$. Hence this coefficient is invertible on some neighborhood U of s. Let J be the finite set of $j \in I$ such that y_j appears in g. Since X_U is not quasi-compact, we may choose a point $x \in X - \bigcup_{j\in J} X_j$ lying above some $u \in U$. Since g has a coefficient that is invertible on U, we can find a point $t' \in T$ lying above u such that $t' \notin V(g)$ and $t' \in V(y_i)$ for all $i \notin J$. This is true because $V(y_i; i \in I, i \notin J) = \mathrm{Spec}(A[t_j; j \in J])$ and the set of points of this scheme lying over u is bijective with $\mathrm{Spec}(\kappa(u)[t_j; j \in J])$. In other words $t' \notin T_i$ for each $i \notin J$. By Schemes, Lemma 17.5 we can find a point z of $X \times_S T$ mapping to $x \in X$ and to $t' \in T$. Since $x \notin X_j$ for $j \in J$ and $t' \notin T_i$ for $i \in I \setminus J$ we see that $z \in Z$. On the other hand $f_T(z) = t' \notin V(g)$ which contradicts $f_T(Z) \subset V(g)$. Thus the assumption "$f_T(Z)$ closed" is wrong and we conclude indeed that f_T is not closed, as desired. □

The following lemma says that the image of a proper scheme (in a separated scheme of finite type over the base) is proper.

Lemma 41.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. If X is universally closed over S and f is surjective then Y is universally closed over S. In particular, if also Y is separated and locally of finite type over S, then Y is proper over S.*

Proof. Assume X is universally closed and f surjective. Denote $p : X \to S$, $q : Y \to S$ the structure morphisms. Let $S' \to S$ be a morphism of schemes. The base change $f' : X_{S'} \to Y_{S'}$ is surjective (Lemma 9.4), and the base change $p' : X_{S'} \to S'$ is closed. If $T \subset Y_{S'}$ is closed, then $(f')^{-1}(T) \subset X_{S'}$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. This proves the first statement. Thus $Y \to S$ is quasi-compact by Lemma 41.8 and hence $Y \to S$ is proper by definition if in addition $Y \to S$ is locally of finite type and separated. □

Lemma 41.10. *Suppose given a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Assume

(1) *$X \to S$ is a universally closed (for example proper) morphism, and*
(2) *$Y \to S$ is separated and locally of finite type.*

Then the scheme theoretic image $Z \subset Y$ of h is proper over S and $X \to Z$ is surjective.

Proof. The scheme theoretic image of h is constructed in Section 6. Since f is quasi-compact (Lemma 41.8) we find that h is quasi-compact (Schemes, Lemma 21.14). Hence $h(X) \subset Z$ is dense (Lemma 6.3). On the other hand $h(X)$ is closed in Y (Lemma 41.7) hence $X \to Z$ is surjective. Thus $Z \to S$ is a proper (Lemma 41.9). $\square$

The target of a separated scheme under a surjective universally closed morphism is separated.

Lemma 41.11. *Let S be a scheme. Let $f : X \to Y$ be a surjective universally closed morphism of schemes over S.*

1. *If X is quasi-separated, then Y is quasi-separated.*
2. *If X is separated, then Y is separated.*
3. *If X is quasi-separated over S, then Y is quasi-separated over S.*
4. *If X is separated over S, then Y is separated over S.*

Proof. Parts (1) and (2) are a consequence of (3) and (4) for $S = \text{Spec}(\mathbf{Z})$ (see Schemes, Definition 21.3). Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/S}"'] \arrow[d] & X \times_S X \arrow[d] \\ Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y \end{tikzcd}$$

The left vertical arrow is surjective (i.e., universally surjective). The right vertical arrow is universally closed as a composition of the universally closed morphisms $X \times_S X \to X \times_S Y \to Y \times_S Y$. Hence it is also quasi-compact, see Lemma 41.8.

Assume X is quasi-separated over S, i.e., $\Delta_{X/S}$ is quasi-compact. If $V \subset Y \times_S Y$ is a quasi-compact open, then $V \times_{Y \times_S Y} X \to \Delta_{Y/S}^{-1}(V)$ is surjective and $V \times_{Y \times_S Y} X$ is quasi-compact by our remarks above. We conclude that $\Delta_{Y/S}$ is quasi-compact, i.e., Y is quasi-separated over S.

Assume X is separated over S, i.e., $\Delta_{X/S}$ is a closed immersion. Then $X \to Y \times_S Y$ is closed as a composition of closed morphisms. Since $X \to Y$ is surjective, it follows that $\Delta_{Y/S}(Y)$ is closed in $Y \times_S Y$. Hence Y is separated over S by the discussion following Schemes, Definition 21.3. $\square$

42. Valuative criteria

We have already discussed the valuative criterion for universal closedness and for separatedness in Schemes, Sections 20 and 22. In this section we will discuss some consequences and variants. In Limits, Section 15 we will show that it suffices to consider discrete valuation rings when working with locally Noetherian schemes and morphisms of finite type.

Lemma 42.1 (Valuative criterion for properness)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume f is of finite type and quasi-separated. Then the following are equivalent*

1. *f is proper,*

(2) *f satisfies the valuative criterion (Schemes, Definition 20.3),*
(3) *given any commutative solid diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute.

Proof. Part (3) is a reformulation of (2). Thus the lemma is a formal consequence of Schemes, Proposition 20.6 and Lemma 22.2 and the definitions. ☐

One usually does not have to consider all possible diagrams when testing the valuative criterion. We will call a valuative criterion as in the next lemma a "refined valuative criterion".

Lemma 42.2. *Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that f and h are quasi-compact and that $h(U)$ is dense in X. If given any commutative solid diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, dashed] & & S \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute, then f is universally closed. If moreover f is quasi-separated, then f is separated.

Proof. To prove f is universally closed we will verify the existence part of the valuative criterion for f which suffices by Schemes, Proposition 20.6. To do this, consider a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & S \end{tikzcd}$$

where A is a valuation ring and K is the fraction field of A. Note that since valuation rings and fields are reduced, we may replace U, X, and S by their respective reductions by Schemes, Lemma 12.7. In this case the assumption that $h(U)$ is dense means that the scheme theoretic image of $h : U \to X$ is X, see Lemma 6.7. We may also replace S by an affine open through which the morphism $\mathrm{Spec}(A) \to S$ factors. Thus we may assume that $S = \mathrm{Spec}(R)$.

Let $\mathrm{Spec}(B) \subset X$ be an affine open through which the morphism $\mathrm{Spec}(K) \to X$ factors. Choose a polynomial algebra P over B and a B-algebra surjection $P \to K$. Then $\mathrm{Spec}(P) \to X$ is flat. Hence the scheme theoretic image of the morphism $U \times_X \mathrm{Spec}(P) \to \mathrm{Spec}(P)$ is $\mathrm{Spec}(P)$ by Lemma 25.16. By Lemma 6.5 we can find a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & U \times_X \mathrm{Spec}(P) \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] & \mathrm{Spec}(P) \end{tikzcd}$$

where A' is a valuation ring and K' is the fraction field of A' such that the closed point of $\mathrm{Spec}(A')$ maps to $\mathrm{Spec}(K) \subset \mathrm{Spec}(P)$. In other words, there is a B-algebra map $\varphi : K \to A'/\mathfrak{m}_{A'}$. Choose a valuation ring $A'' \subset A'/\mathfrak{m}_{A'}$ dominating $\varphi(A)$ with field of fractions $K'' = A'/\mathfrak{m}_{A'}$ (Algebra, Lemma 50.2). We set

$$C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A''\}.$$

which is a valuation ring by Algebra, Lemma 50.10. As C is an R-algebra with fraction field K', we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & U \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(C) \arrow[rr] \arrow[urr, dashed] & & S \end{tikzcd}$$

as in the statement of the lemma. Thus a dotted arrow fitting into the diagram as indicated. By the uniqueness assumption of the lemma the composition $\mathrm{Spec}(A') \to \mathrm{Spec}(C) \to X$ agrees with the given morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(P) \to \mathrm{Spec}(B) \subset X$. Hence the restriction of the morphism to the spectrum of $C/\mathfrak{m}_{A'} = A''$ induces the given morphism $\mathrm{Spec}(K'') = \mathrm{Spec}(A'/\mathfrak{m}_{A'}) \to \mathrm{Spec}(K) \to X$. Let $x \in X$ be the image of the closed point of $\mathrm{Spec}(A'') \to X$. The image of the induced ring map $\mathcal{O}_{X,x} \to A''$ is a local subring which is contained in $K \subset K''$. Since A is maximal for the relation of domination in K and since $A \subset A''$, we have $A = K \cap A''$. We conclude that $\mathcal{O}_{X,x} \to A''$ factors through $A \subset A''$. In this way we obtain our desired arrow $\mathrm{Spec}(A) \to X$.

Finally, assume f is quasi-separated. Then $\Delta : X \to X \times_S X$ is quasi-compact. Given a solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[urr, dashed] & & X \times_S X \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute. Namely, the lower horizontal arrow is the same thing as a pair of morphisms $\mathrm{Spec}(A) \to X$ which can serve as the dotted arrow in the diagram of the lemma. Thus the required uniqueness shows that the lower horizontal arrow factors through Δ. Hence we can apply the result we just proved to $\Delta : X \to X \times_S X$ and $h : U \to X$ and conclude that Δ is universally closed. Clearly this means that f is separated. $\square$

Remark 42.3. The assumption on uniqueness of the dotted arrows in Lemma 42.2 is necessary (details omitted). Of course, uniqueness is guaranteed if f is separated (Schemes, Lemma 22.1).

Lemma 42.4. *Let S be a scheme. Let X, Y be schemes over S. Let $s \in S$ and $x \in X$, $y \in Y$ points over s.*

(1) *Let $f, g : X \to Y$ be morphisms over S such that $f(x) = g(x) = y$ and $f_x^\sharp = g_x^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Then there is an open neighbourhood $U \subset X$ with $f|_U = g|_U$ in the following cases*
   (a) *Y is locally of finite type over S,*
   (b) *X is integral,*
   (c) *X is locally Noetherian, or*
   (d) *X is reduced with finitely many irreducible components.*

(2) *Let $\varphi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ be a local $\mathcal{O}_{S,s}$-algebra map. Then there exists an open neighbourhood $U \subset X$ of x and a morphism $f : U \to Y$ mapping x to y with $f_x^\sharp = \varphi$ in the following cases*
   (a) *Y is locally of finite presentation over S,*
   (b) *Y is locally of finite type and X is integral,*
   (c) *Y is locally of finite type and X is locally Noetherian, or*
   (d) *Y is locally of finite type and X is reduced with finitely many irreducible components.*

Proof. Proof of (1). We may replace X, Y, S by suitable affine open neighbourhoods of x, y, s and reduce to the following algebra problem: given a ring R, two R-algebra maps $\varphi, \psi : B \to A$ such that

(1) $R \to B$ is of finite type, or A is a domain, or A is Noetherian, or A is reduced and has finitely many minimal primes,
(2) the two maps $B \to A_\mathfrak{p}$ are the same for some prime $\mathfrak{p} \subset A$,

show that φ, ψ define the same map $B \to A_g$ for a suitable $g \in A$, $g \notin \mathfrak{p}$. If $R \to B$ is of finite type, let $t_1, \ldots, t_m \in B$ be generators of B as an R-algebra. For each j we can find $g_j \in A$, $g_j \notin \mathfrak{p}$ such that $\varphi(t_j)$ and $\psi(t_j)$ have the same image in A_{g_j}. Then we set $g = \prod g_j$. In the other cases (if A is a domain, Noetherian, or reduced with finitely many minimal primes), we can find a $g \in A$, $g \notin \mathfrak{p}$ such that $A_g \subset A_\mathfrak{p}$. See Algebra, Lemma 31.9. Thus the maps $B \to A_g$ are equal as desired.

Proof of (2). To do this we may replace X, Y, and S by suitable affine opens. Say $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$, and $S = \text{Spec}(R)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. Let $\mathfrak{q} \subset B$ be the prime corresponding to y. Then φ is a local R-algebra map $\varphi : B_\mathfrak{q} \to A_\mathfrak{p}$. If $R \to B$ is a ring map of finite presentation, then there exists a $g \in A \setminus \mathfrak{p}$ and an R-algebra map $B \to A_g$ such that

$$\begin{tikzcd} B_\mathfrak{q} \arrow[r, "\varphi"'] & A_\mathfrak{p} \\ B \arrow[u] \arrow[r] & A_g \arrow[u] \end{tikzcd}$$

commutes, see Algebra, Lemmas 127.3 and 9.9. The induced morphism $\text{Spec}(A_g) \to \text{Spec}(B)$ works. If B is of finite type over R, let $t_1, \ldots, t_m \in B$ be generators of B as an R-algebra. Then we can choose $g_j \in A$, $g_j \notin \mathfrak{p}$ such that $\varphi(t_j) \in \text{Im}(A_{g_j} \to A_\mathfrak{p})$. Thus after replacing A by $A[1/\prod g_j]$ we may assume that B maps into the image of $A \to A_\mathfrak{p}$. If we can find a $g \in A$, $g \notin \mathfrak{p}$ such that $A_g \to A_\mathfrak{p}$ is injective, then we'll get the desired R-algebra map $B \to A_g$. Thus the proof is finished by another application of See Algebra, Lemma 31.9. $\square$

Lemma 42.5. *Let S be a scheme. Let X, Y be schemes over S. Let $x \in X$. Let $U \subset X$ be an open and let $f : U \to Y$ be a morphism over S. Assume*

1. *x is in the closure of U,*
2. *X is reduced with finitely many irreducible components or X is Noetherian,*
3. *$\mathcal{O}_{X,x}$ is a valuation ring,*
4. *$Y \to S$ is proper*

Then there exists an open $U \subset U' \subset X$ containing x and an S-morphism $f' : U' \to Y$ extending f.

Proof. It is harmless to replace X by an open neighbourhood of x in X (small detail omitted). By Properties, Lemma 29.8 we may assume X is affine with $\Gamma(X, \mathcal{O}_X) \subset \mathcal{O}_{X,x}$. In particular X is integral with a unique generic point ξ whose residue field is the fraction field K of the valuation ring $\mathcal{O}_{X,x}$. Since x is in the closure of U we see that U is not empty, hence U contains ξ. Thus by the valuative criterion of properness (Lemma 42.1) there is a morphism $t : \mathrm{Spec}(\mathcal{O}_{X,x}) \to Y$ fitting into a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d, "\xi"'] & \mathrm{Spec}(\mathcal{O}_{X,x}) \arrow[d, "t"] \\ U \arrow[r, "f"] & Y \end{tikzcd}$$

of morphisms of schemes over S. Applying Lemma 42.4 with $y = t(x)$ and $\varphi = t_x^\sharp$ we obtain an open neighbourhood $V \subset X$ of x and a morphism $g : V \to Y$ over S which sends x to y and such that $g_x^\sharp = t_x^\sharp$. As $Y \to S$ is separated, the equalizer E of $f|_{U \cap V}$ and $g|_{U \cap V}$ is a closed subscheme of $U \cap V$, see Schemes, Lemma 21.5. Since f and g determine the same morphism $\mathrm{Spec}(K) \to Y$ by construction we see that E contains the generic point of the integral scheme $U \cap V$. Hence $E = U \cap V$ and we conclude that f and g glue to a morphism $U' = U \cup V \to Y$ as desired. $\square$

43. Projective morphisms

We will use the definition of a projective morphism from [DG67]. The version of the definition with the "H" is the one from [Har77]. The resulting definitions are different. Both are useful.

Definition 43.1. Let $f : X \to S$ be a morphism of schemes.

1. We say f is *projective* if X is isomorphic as an S-scheme to a closed subscheme of a projective bundle $\mathbf{P}(\mathcal{E})$ for some quasi-coherent, finite type $\mathcal{O}_S$-module $\mathcal{E}$.
2. We say f is *H-projective* if there exists an integer n and a closed immersion $X \to \mathbf{P}^n_S$ over S.
3. We say f is *locally projective* if there exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is projective.

As expected, a projective morphism is quasi-projective, see Lemma 43.10. Conversely, quasi-projective morphisms are often compositions of open immersions and projective morphisms, see Lemma 43.12. For an overview of properties of projective morphisms over a quasi-projective base, see More on Morphisms, Section 50.

Example 43.2. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Assume furthermore that $\mathcal{A}_1$ is of finite type over $\mathcal{O}_S$.

Set $X = \underline{\text{Proj}}_S(\mathcal{A})$. In this case $X \to S$ is projective. Namely, the morphism associated to the graded $\mathcal{O}_S$-algebra map

$$\text{Sym}^*_{\mathcal{O}_X}(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion, see Constructions, Lemma 18.5.

Lemma 43.3. *An H-projective morphism is H-quasi-projective. An H-projective morphism is projective.*

Proof. The first statement is immediate from the definitions. The second holds as $\mathbf{P}^n_S$ is a projective bundle over S, see Constructions, Lemma 21.5. $\square$

Lemma 43.4. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is locally projective.*
(2) *There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is H-projective.*

Proof. By Lemma 43.3 we see that (2) implies (1). Assume (1). For every point $s \in S$ we can find $\text{Spec}(R) = U \subset S$ an affine open neighbourhood of s such that X_U is isomorphic to a closed subscheme of $\mathbf{P}(\mathcal{E})$ for some finite type, quasi-coherent sheaf of $\mathcal{O}_U$-modules $\mathcal{E}$. Write $\mathcal{E} = \widetilde{M}$ for some finite type R-module M (see Properties, Lemma 16.1). Choose generators $x_0, \ldots, x_n \in M$ of M as an R-module. Consider the surjective graded R-algebra map

$$R[X_0, \ldots, X_n] \longrightarrow \text{Sym}_R(M).$$

According to Constructions, Lemma 11.3 the corresponding morphism

$$\mathbf{P}(\mathcal{E}) \to \mathbf{P}^n_R$$

is a closed immersion. Hence we conclude that $f^{-1}(U)$ is isomorphic to a closed subscheme of $\mathbf{P}^n_U$ (as a scheme over U). In other words: (2) holds. $\square$

Lemma 43.5. *A locally projective morphism is proper.*

Proof. Let $f : X \to S$ be locally projective. In order to show that f is proper we may work locally on the base, see Lemma 41.3. Hence, by Lemma 43.4 above we may assume there exists a closed immersion $X \to \mathbf{P}^n_S$. By Lemmas 41.4 and 41.6 it suffices to prove that $\mathbf{P}^n_S \to S$ is proper. Since $\mathbf{P}^n_S \to S$ is the base change of $\mathbf{P}^n_{\mathbf{Z}} \to \text{Spec}(\mathbf{Z})$ it suffices to show that $\mathbf{P}^n_{\mathbf{Z}} \to \text{Spec}(\mathbf{Z})$ is proper, see Lemma 41.5. By Constructions, Lemma 8.8 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ is separated. By Constructions, Lemma 8.9 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ is quasi-compact. It is clear that $\mathbf{P}^n_{\mathbf{Z}} \to \text{Spec}(\mathbf{Z})$ is locally of finite type since $\mathbf{P}^n_{\mathbf{Z}}$ is covered by the affine opens $D_+(X_i)$ each of which is the spectrum of the finite type $\mathbf{Z}$-algebra

$$\mathbf{Z}[X_0/X_i, \ldots, X_n/X_i].$$

Finally, we have to show that $\mathbf{P}^n_{\mathbf{Z}} \to \text{Spec}(\mathbf{Z})$ is universally closed. This follows from Constructions, Lemma 8.11 and the valuative criterion (see Schemes, Proposition 20.6). $\square$

Lemma 43.6. *Let $f : X \to S$ be a proper morphism of schemes. If there exists an f-ample invertible sheaf on X, then f is locally projective.*

Proof. If there exists an f-ample invertible sheaf, then we can locally on S find an immersion $i : X \to \mathbf{P}^n_S$, see Lemma 39.4. Since $X \to S$ is proper the morphism i is a closed immersion, see Lemma 41.7. $\square$

Lemma 43.7. *A composition of H-projective morphisms is H-projective.*

Proof. Suppose $X \to Y$ and $Y \to Z$ are H-projective. Then there exist closed immersions $X \to \mathbf{P}^n_Y$ over Y, and $Y \to \mathbf{P}^m_Z$ over Z. Consider the following diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[r] \arrow[dl] & \mathbf{P}^n_{\mathbf{P}^m_Z} \arrow[r, equal] \arrow[dl] & \mathbf{P}^n_Z \times_Z \mathbf{P}^m_Z \arrow[r] & \mathbf{P}^{nm+n+m}_Z \arrow[ddllll] \\
Y \arrow[r] \arrow[d] & \mathbf{P}^m_Z \arrow[dl] & & & \\
Z & & & &
\end{tikzcd}$$

Here the rightmost top horizontal arrow is the Segre embedding, see Constructions, Lemma 13.6. The diagram identifies X as a closed subscheme of $\mathbf{P}^{nm+n+m}_Z$ as desired. $\square$

Lemma 43.8. *A base change of a H-projective morphism is H-projective.*

Proof. This is true because the base change of projective space over a scheme is projective space, and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 18.2. $\square$

Lemma 43.9. *A base change of a (locally) projective morphism is (locally) projective.*

Proof. This is true because the base change of a projective bundle over a scheme is a projective bundle, the pullback of a finite type $\mathcal{O}$-module is of finite type (Modules, Lemma 9.2) and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 18.2. Some details omitted. $\square$

Lemma 43.10. *A projective morphism is quasi-projective.*

Proof. Let $f : X \to S$ be a projective morphism. Choose a closed immersion $i : X \to \mathbf{P}(\mathcal{E})$ where $\mathcal{E}$ is a quasi-coherent, finite type $\mathcal{O}_S$-module. Then $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is f-very ample. Since f is proper (Lemma 43.5) it is quasi-compact. Hence Lemma 38.2 implies that $\mathcal{L}$ is f-ample. Since f is proper it is of finite type. Thus we've checked all the defining properties of quasi-projective holds and we win. $\square$

Lemma 43.11. *Let $f : X \to S$ be a H-quasi-projective morphism. Then f factors as $X \to X' \to S$ where $X \to X'$ is an open immersion and $X' \to S$ is H-projective.*

Proof. By definition we can factor f as a quasi-compact immersion $i : X \to \mathbf{P}^n_S$ followed by the projection $\mathbf{P}^n_S \to S$. By Lemma 7.7 there exists a closed subscheme $X' \subset \mathbf{P}^n_S$ such that i factors through an open immersion $X \to X'$. The lemma follows. $\square$

Lemma 43.12. *Let $f : X \to S$ be a quasi-projective morphism with S quasi-compact and quasi-separated. Then f factors as $X \to X' \to S$ where $X \to X'$ is an open immersion and $X' \to S$ is projective.*

Proof. Let $\mathcal{L}$ be f-ample. Since f is of finite type and S is quasi-compact $\mathcal{L}^{\otimes n}$ is f-very ample for some $n > 0$, see Lemma 39.5. Replace $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$. Write $\mathcal{F} = f_*\mathcal{L}$. This is a quasi-coherent $\mathcal{O}_S$-module by Schemes, Lemma 24.1 (quasi-projective morphisms are quasi-compact and separated, see Lemma 40.4). By Properties, Lemma 22.7 we can find a directed set I and a system of finite type quasi-coherent $\mathcal{O}_S$-modules $\mathcal{E}_i$ over I such that $\mathcal{F} = \operatorname{colim} \mathcal{E}_i$. Consider the compositions $\psi_i : f^*\mathcal{E}_i \to f^*\mathcal{F} \to \mathcal{L}$. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,m} V_j$. For each j we can choose sections

$$s_{j,0}, \ldots, s_{j,n_j} \in \Gamma(f^{-1}(V_j), \mathcal{L}) = f_*\mathcal{L}(V_j) = \mathcal{F}(V_j)$$

which generate $\mathcal{L}$ over $f^{-1}V_j$ and define an immersion

$$f^{-1}V_j \longrightarrow \mathbf{P}^{n_j}_{V_j},$$

see Lemma 39.1. Choose i such that there exist sections $e_{j,t} \in \mathcal{E}_i(V_j)$ mapping to $s_{j,t}$ in $\mathcal{F}$ for all $j = 1, \ldots, m$ and $t = 1, \ldots, n_j$. Then the map ψ_i is surjective as the sections $f^*e_{j,t}$ have the same image as the sections $s_{j,t}$ which generate $\mathcal{L}|_{f^{-1}V_j}$. Whence we obtain a morphism

$$r_{\mathcal{L},\psi_i} : X \longrightarrow \mathbf{P}(\mathcal{E}_i)$$

over S such that over V_j we have a factorization

$$f^{-1}V_j \to \mathbf{P}(\mathcal{E}_i)|_{V_j} \to \mathbf{P}^{n_j}_{V_j}$$

of the immersion given above. It follows that $r_{\mathcal{L},\psi_i}|_{V_j}$ is an immersion, see Lemma 3.1. Since $S = \bigcup V_j$ we conclude that $r_{\mathcal{L},\psi_i}$ is an immersion. Note that $r_{\mathcal{L},\psi_i}$ is quasi-compact as $X \to S$ is quasi-compact and $\mathbf{P}(\mathcal{E}_i) \to S$ is separated (see Schemes, Lemma 21.14). By Lemma 7.7 there exists a closed subscheme $X' \subset \mathbf{P}(\mathcal{E}_i)$ such that i factors through an open immersion $X \to X'$. Then $X' \to S$ is projective by definition and we win. $\square$

Lemma 43.13. *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of schemes. Then*

1. *f is projective if and only if f is quasi-projective and proper, and*
2. *f is H-projective if and only if f is H-quasi-projective and proper.*

Proof. If f is projective, then f is quasi-projective by Lemma 43.10 and proper by Lemma 43.5. Conversely, if $X \to S$ is quasi-projective and proper, then we can choose an open immersion $X \to X'$ with $X' \to S$ projective by Lemma 43.12. Since $X \to S$ is proper, we see that X is closed in X' (Lemma 41.7), i.e., $X \to X'$ is a (open and) closed immersion. Since X' is isomorphic to a closed subscheme of a projective bundle over S (Definition 43.1) we see that the same thing is true for X, i.e., $X \to S$ is a projective morphism. This proves (1). The proof of (2) is the same, except it uses Lemmas 43.3 and 43.11. $\square$

Lemma 43.14. *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. If S is quasi-compact and quasi-separated and f and g are projective, then $g \circ f$ is projective.*

Proof. By Lemmas 43.10 and 43.5 we see that f and g are quasi-projective and proper. By Lemmas 41.4 and 40.3 we see that $g \circ f$ is proper and quasi-projective. Thus $g \circ f$ is projective by Lemma 43.13. $\square$

Lemma 43.15. *Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. If $g \circ f$ is projective and g is separated, then f is projective.*

Proof. Choose a closed immersion $X \to \mathbf{P}(\mathcal{E})$ where $\mathcal{E}$ is a quasi-coherent, finite type $\mathcal{O}_S$-module. Then we get a morphism $X \to \mathbf{P}(\mathcal{E}) \times_S Y$. This morphism is a closed immersion because it is the composition

$$X \to X \times_S Y \to \mathbf{P}(\mathcal{E}) \times_S Y$$

where the first morphism is a closed immersion by Schemes, Lemma 21.10 (and the fact that g is separated) and the second as the base change of a closed immersion. Finally, the fibre product $\mathbf{P}(\mathcal{E}) \times_S Y$ is isomorphic to $\mathbf{P}(g^*\mathcal{E})$ and pullback preserves quasi-coherent, finite type modules. □

Lemma 43.16. *Let S be a scheme which admits an ample invertible sheaf. Then*

1. *any projective morphism $X \to S$ is H-projective, and*
2. *any quasi-projective morphism $X \to S$ is H-quasi-projective.*

Proof. The assumptions on S imply that S is quasi-compact and separated, see Properties, Definition 26.1 and Lemma 26.11 and Constructions, Lemma 8.8. Hence Lemma 43.12 applies and we see that (1) implies (2). Let $\mathcal{E}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. By our definition of projective morphisms it suffices to show that $\mathbf{P}(\mathcal{E}) \to S$ is H-projective. If $\mathcal{E}$ is generated by finitely many global sections, then the corresponding surjection $\mathcal{O}_S^{\oplus n} \to \mathcal{E}$ induces a closed immersion

$$\mathbf{P}(\mathcal{E}) \longrightarrow \mathbf{P}(\mathcal{O}_S^{\oplus n}) = \mathbf{P}^n_S$$

as desired. In general, let $\mathcal{L}$ be an invertible sheaf on S. By Properties, Proposition 26.13 there exists an integer n such that $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes n}$ is globally generated by finitely many sections. Since $\mathbf{P}(\mathcal{E}) = \mathbf{P}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes n})$ by Constructions, Lemma 20.1 this finishes the proof. □

Lemma 43.17. *Let $f : X \to S$ be a universally closed morphism. Let $\mathcal{L}$ be an f-ample invertible $\mathcal{O}_X$-module. Then the canonical morphism*

$$r : X \longrightarrow \underline{Proj}_S \left(\bigoplus\nolimits_{d \geq 0} f_*\mathcal{L}^{\otimes d}\right)$$

of Lemma 37.4 is an isomorphism.

Proof. Observe that f is quasi-compact because the existence of an f-ample invertible module forces f to be quasi-compact. By the lemma cited the morphism r is an open immersion. On the other hand, the image of r is closed by Lemma 41.7 (the target of r is separated over S by Constructions, Lemma 16.9). Finally, the image of r is dense by Properties, Lemma 26.11 (here we also use that it was shown in the proof of Lemma 37.4 that the morphism r over affine opens of S is given by the canonical morphism of Properties, Lemma 26.9). Thus we conclude that r is a surjective open immersion, i.e., an isomorphism. □

Lemma 43.18. *Let $f : X \to S$ be a universally closed morphism. Let $\mathcal{L}$ be an f-ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Then $X_s \to S$ is an affine morphism.*

Proof. The question is local on S (Lemma 11.3) hence we may assume S is affine. By Lemma 43.17 we can write $X = \text{Proj}(A)$ where A is a graded ring and s corresponds to $f \in A_1$ and $X_s = D_+(f)$ (Properties, Lemma 26.9) which proves the lemma by construction of $\text{Proj}(A)$, see Constructions, Section 8. □

44. Integral and finite morphisms

Recall that a ring map $R \to A$ is said to be *integral* if every element of A satisfies a monic equation with coefficients in R. Recall that a ring map $R \to A$ is said to be *finite* if A is finite as an R-module. See Algebra, Definition 36.1.

Definition 44.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *integral* if f is affine and if for every affine open $\text{Spec}(R) = V \subset S$ with inverse image $\text{Spec}(A) = f^{-1}(V) \subset X$ the associated ring map $R \to A$ is integral.
(2) We say that f is *finite* if f is affine and if for every affine open $\text{Spec}(R) = V \subset S$ with inverse image $\text{Spec}(A) = f^{-1}(V) \subset X$ the associated ring map $R \to A$ is finite.

It is clear that integral/finite morphisms are separated and quasi-compact. It is also clear that a finite morphism is a morphism of finite type. Most of the lemmas in this section are completely standard. But note the fun Lemma 44.7 at the end of the section.

Lemma 44.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is integral.*
(2) *There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ is integral.*
(3) *There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is integral.*

Moreover, if f is integral then for every open subscheme $U \subset S$ the morphism $f : f^{-1}(U) \to U$ is integral.

Proof. See Algebra, Lemma 36.14. Some details omitted. □

Lemma 44.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is finite.*
(2) *There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ is finite.*
(3) *There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is finite.*

Moreover, if f is finite then for every open subscheme $U \subset S$ the morphism $f : f^{-1}(U) \to U$ is finite.

Proof. See Algebra, Lemma 36.14. Some details omitted. □

Lemma 44.4. *A finite morphism is integral. An integral morphism which is locally of finite type is finite.*

Proof. See Algebra, Lemma 36.3 and Lemma 36.5. □

Lemma 44.5. *A composition of finite morphisms is finite. Same is true for integral morphisms.*

Proof. See Algebra, Lemmas 7.3 and 36.6. □

Lemma 44.6. *A base change of a finite morphism is finite. Same is true for integral morphisms.*

Proof. See Algebra, Lemma 36.13. $\square$

Lemma 44.7. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *f is integral, and*
2. *f is affine and universally closed.*

Proof. Assume (1). An integral morphism is affine by definition. A base change of an integral morphism is integral so in order to prove (2) it suffices to show that an integral morphism is closed. This follows from Algebra, Lemmas 36.22 and 41.6.

Assume (2). We may assume f is the morphism $f : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ coming from a ring map $R \to A$. Let a be an element of A. We have to show that a is integral over R, i.e. that in the kernel I of the map $R[x] \to A$ sending x to a there is a monic polynomial. Consider the ring $B = A[x]/(ax-1)$ and let J be the kernel of the composition $R[x] \to A[x] \to B$. If $f \in J$ there exists $q \in A[x]$ such that $f = (ax-1)q$ in $A[x]$ so if $f = \sum_i f_i x^i$ and $q = \sum_i q_i x^i$, for all $i \geq 0$ we have $f_i = aq_{i-1} - q_i$. For $n \geq \deg q + 1$ the polynomial

$$\sum\nolimits_{i\geq 0} f_i x^{n-i} = \sum\nolimits_{i\geq 0}(aq_{i-1} - q_i)x^{n-i} = (a-x)\sum\nolimits_{i\geq 0} q_i x^{n-i-1}$$

is clearly in I; if $f_0 = 1$ this polynomial is also monic, so we are reduced to prove that J contains a polynomial with constant term 1. We do it by proving $\mathrm{Spec}(R[x]/(J + (x))$ is empty.

Since f is universally closed the base change $\mathrm{Spec}(A[x]) \to \mathrm{Spec}(R[x])$ is closed. Hence the image of the closed subset $\mathrm{Spec}(B) \subset \mathrm{Spec}(A[x])$ is the closed subset $\mathrm{Spec}(R[x]/J) \subset \mathrm{Spec}(R[x])$, see Example 6.4 and Lemma 6.3. In particular $\mathrm{Spec}(B) \to \mathrm{Spec}(R[x]/J)$ is surjective. Consider the following diagram where every square is a pullback:

$$\begin{tikzcd}
\mathrm{Spec}(B) \arrow[r, two heads, "g"] & \mathrm{Spec}(R[x]/J) \arrow[r] & \mathrm{Spec}(R[x]) \\
\emptyset \arrow[u] \arrow[r] & \mathrm{Spec}(R[x]/(J+(x))) \arrow[u] \arrow[r] & \mathrm{Spec}(R) \arrow[u, "0"']
\end{tikzcd}$$

The bottom left corner is empty because it is the spectrum of $R \otimes_{R[x]} B$ where the map $R[x] \to B$ sends x to an invertible element and $R[x] \to R$ sends x to 0. Since g is surjective this implies $\mathrm{Spec}(R[x]/(J + (x)))$ is empty, as we wanted to show. $\square$

Lemma 44.8. *Let $f : X \to S$ be an integral morphism. Then every point of X is closed in its fibre.*

Proof. See Algebra, Lemma 36.20. $\square$

Lemma 44.9. *Let $f : X \to Y$ be an integral morphism. Then* $\dim(X) \leq \dim(Y)$*. If f is surjective then* $\dim(X) = \dim(Y)$*.*

Proof. Since the dimension of X and Y is the supremum of the dimensions of the members of an affine open covering, we may assume Y and X are affine. The inequality follows from Algebra, Lemma 112.3. The equality then follows from Algebra, Lemmas 112.1 and 36.22. $\square$

Lemma 44.10. *A finite morphism is quasi-finite.*

Proof. This is implied by Algebra, Lemma 122.4 and Lemma 20.9. Alternatively, all points in fibres are closed points by Lemma 44.8 (and the fact that a finite morphism is integral) and use Lemma 20.6 (3) to see that f is quasi-finite at x for all $x \in X$. $\square$

Lemma 44.11. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is finite, and*
(2) *f is affine and proper.*

Proof. This follows formally from Lemma 44.7, the fact that a finite morphism is integral and separated, the fact that a proper morphism is the same thing as a finite type, separated, universally closed morphism, and the fact that an integral morphism of finite type is finite (Lemma 44.4). $\square$

Lemma 44.12. *A closed immersion is finite (and a fortiori integral).*

Proof. True because a closed immersion is affine (Lemma 11.9) and a surjective ring map is finite and integral. $\square$

Lemma 44.13. *Let $X_i \to Y$, $i = 1, \ldots, n$ be finite morphisms of schemes. Then $X_1 \amalg \ldots \amalg X_n \to Y$ is finite too.*

Proof. Follows from the algebra fact that if $R \to A_i$, $i = 1, \ldots, n$ are finite ring maps, then $R \to A_1 \times \ldots \times A_n$ is finite too. $\square$

Lemma 44.14. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms.*

(1) *If $g \circ f$ is finite and g separated then f is finite.*
(2) *If $g \circ f$ is integral and g separated then f is integral.*

Proof. Assume $g \circ f$ is finite (resp. integral) and g separated. The base change $X \times_Z Y \to Y$ is finite (resp. integral) by Lemma 44.6. The morphism $X \to X \times_Z Y$ is a closed immersion as $Y \to Z$ is separated, see Schemes, Lemma 21.11. A closed immersion is finite (resp. integral), see Lemma 44.12. The composition of finite (resp. integral) morphisms is finite (resp. integral), see Lemma 44.5. Thus we win. $\square$

Lemma 44.15. *Let $f : X \to Y$ be a morphism of schemes. If f is finite and a monomorphism, then f is a closed immersion.*

Proof. This reduces to Algebra, Lemma 107.6. $\square$

Lemma 44.16. *A finite morphism is projective.*

Proof. Let $f : X \to S$ be a finite morphism. Then $f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_S$-module (Lemma 11.5) of finite type (by our definition of finite morphisms and Properties, Lemma 16.1). We claim there is a closed immersion

$$\sigma : X \longrightarrow \mathbf{P}(f_*\mathcal{O}_X) = \underline{\mathrm{Proj}}_S(\mathrm{Sym}^*_{\mathcal{O}_S}(f_*\mathcal{O}_X))$$

over S, which finishes the proof. Namely, we let σ be the morphism which corresponds (via Constructions, Lemma 16.11) to the surjection

$$f^*f_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$$

coming from the adjunction map $f^*f_* \to \text{id}$. Then σ is a closed immersion by Schemes, Lemma 21.11 and Constructions, Lemma 21.4. $\square$

45. Universal homeomorphisms

The following definition is really superfluous since a universal homeomorphism is really just an integral, universally injective and surjective morphism, see Lemma 45.5.

Definition 45.1. A morphism $f : X \to Y$ of schemes is called a *universal homeomorphism* if the base change $f' : Y' \times_Y X \to Y'$ is a homeomorphism for every morphism $Y' \to Y$.

First we state the obligatory lemmas.

Lemma 45.2. *The base change of a universal homeomorphism of schemes by any morphism of schemes is a universal homeomorphism.*

Proof. This is immediate from the definition. $\square$

Lemma 45.3. *The composition of a pair of universal homeomorphisms of schemes is a universal homeomorphism.*

Proof. Omitted. $\square$

The following simple lemma is the key to characterizing universal homeomorphisms.

Lemma 45.4. *Let $f : X \to Y$ be a morphism of schemes. If f is a homeomorphism onto a closed subset of Y then f is affine.*

Proof. Let $y \in Y$ be a point. If $y \notin f(X)$, then there exists an affine neighbourhood of y which is disjoint from $f(X)$. If $y \in f(X)$, let $x \in X$ be the unique point of X mapping to y. Let $y \in V$ be an affine open neighbourhood. Let $U \subset X$ be an affine open neighbourhood of x which maps into V. Since $f(U) \subset V \cap f(X)$ is open in the induced topology by our assumption on f we may choose a $h \in \Gamma(V, \mathcal{O}_Y)$ such that $y \in D(h)$ and $D(h) \cap f(X) \subset f(U)$. Denote $h' \in \Gamma(U, \mathcal{O}_X)$ the restriction of $f^\sharp(h)$ to U. Then we see that $D(h') \subset U$ is equal to $f^{-1}(D(h))$. In other words, every point of Y has an open neighbourhood whose inverse image is affine. Thus f is affine, see Lemma 11.3. $\square$

Lemma 45.5. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(1) *f is a universal homeomorphism, and*
(2) *f is integral, universally injective and surjective.*

Proof. Assume f is a universal homeomorphism. By Lemma 45.4 we see that f is affine. Since f is clearly universally closed we see that f is integral by Lemma 44.7. It is also clear that f is universally injective and surjective.

Assume f is integral, universally injective and surjective. By Lemma 44.7 f is universally closed. Since it is also universally bijective (see Lemma 9.4) we see that it is a universal homeomorphism. $\square$

Lemma 45.6. *Let X be a scheme. The canonical closed immersion $X_{red} \to X$ (see Schemes, Definition 12.5) is a universal homeomorphism.*

Proof. Omitted. $\square$

Lemma 45.7. *Let $f : X \to S$ and $S' \to S$ be morphisms of schemes. Assume*

1. *$S' \to S$ is a closed immersion,*
2. *$S' \to S$ is bijective on points,*
3. *$X \times_S S' \to S'$ is a closed immersion, and*
4. *$X \to S$ is of finite type or $S' \to S$ is of finite presentation.*

Then $f : X \to S$ is a closed immersion.

Proof. Assumptions (1) and (2) imply that $S' \to S$ is a universal homeomorphism (for example because $S_{red} = S'_{red}$ and using Lemma 45.6). Hence (3) implies that $X \to S$ is homeomorphism onto a closed subset of S. Then $X \to S$ is affine by Lemma 45.4. Let $U \subset S$ be an affine open, say $U = \mathrm{Spec}(A)$. Then $S' = \mathrm{Spec}(A/I)$ by (1) for a locally nilpotent ideal I by (2). As f is affine we see that $f^{-1}(U) = \mathrm{Spec}(B)$. Assumption (4) tells us B is a finite type A-algebra (Lemma 15.2) or that I is finitely generated (Lemma 21.7). Assumption (3) is that $A/I \to B/IB$ is surjective. From Algebra, Lemma 126.9 if $A \to B$ is of finite type or Algebra, Lemma 20.1 if I is finitely generated and hence nilpotent we deduce that $A \to B$ is surjective. This means that f is a closed immersion, see Lemma 2.1. □

Lemma 45.8. *Let $f : X \to Z$ be the composition of two morphisms $g : X \to Y$ and $h : Y \to Z$. If two of the morphisms $\{f, g, h\}$ are universal homeomorphisms, so is the third morphism.*

Proof. If both of g and h are universal homeomorphisms, so is f by Lemma 45.3.

Suppose both of f and g are universal homeomorphisms. We want to show that h is also. Now base change the diagram along an arbitrary morphism $\alpha : Z' \to Z$ of schemes, we get the following diagram with all squares Cartesian:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & Y' \arrow[r, "h'"] \arrow[d] & Z' \arrow[d] \\ X \arrow[r, "g"] & Y \arrow[r, "h"] & Z. \end{tikzcd}$$

Our assumption implies that the composition $f' = h' \circ g' : X' \to Z'$ and $g' : X' \to Y'$ are homeomorphisms, therefore so is h'. This finishes the proof of h being a universal homeomorphism.

Finally, assume f and h are universal homeomorphisms. We want to show that g is a universal homeomorphism. Let $\beta : Y' \to Y$ be an arbitrary morphism of schemes. We get the following diagram with all squares Cartesian:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & Y' \arrow[d, "\gamma"] & \\ X'' \arrow[r, "g''"] \arrow[d] & Y'' \arrow[r, "h''"] \arrow[d] & Y' \arrow[d, "h\circ\beta"] \\ X \arrow[r, "g"] & Y \arrow[r, "h"] & Z. \end{tikzcd}$$

Here the morphism $\gamma : Y' \to Y''$ is defined by the universal property of fiber products and the two morphisms $id_{Y'} : Y' \to Y'$ and $\beta : Y' \to Y$. We shall prove that g' is a homeomorphism. Since the property of being a homeomorphism has 2-out-of-3 property, we see that g'' is a homeomorphism. Staring at the top square, it suffices to prove that γ is a universal homeomorphism. Since h'' is a homeomorphism, we see that it is an affine morphism by Lemma 45.4 and a fortiori separated (Lemma 11.2). Since $h'' \circ \gamma$ is the identity, we see that γ is a closed immersion by Schemes, Lemma 21.11. Since h'' is bijective, it follows that γ is a bijective closed immersion and hence a universal homeomorphism (for example by the characterization in Lemma 45.5) as desired. $\square$

46. Universal homeomorphisms of affine schemes

In this section we characterize universal homeomorphisms of affine schemes.

Lemma 46.1. *Let $A \to B$ be a ring map such that the induced morphism of schemes $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism, resp. a universal homeomorphism inducing isomorphisms on residue fields, resp. universally closed, resp. universally closed and universally injective. Then for any A-subalgebra $B' \subset B$ the same thing is true for $f' : \mathrm{Spec}(B') \to \mathrm{Spec}(A)$.*

Proof. If f is universally closed, then B is integral over A by Lemma 44.7. Hence B' is integral over A and f' is universally closed (by the same lemma). This proves the case where f is universally closed.

Continuing, we see that B is integral over B' (Algebra, Lemma 36.15) which implies $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is surjective (Algebra, Lemma 36.17). Thus if $A \to B$ induces purely inseparable extensions of residue fields, then the same is true for $A \to B'$. This proves the case where f is universally closed and universally injective, see Lemma 10.2.

The case where f is a universal homeomorphism follows from the remarks above, Lemma 45.5, and the obvious observation that if f is surjective, then so is f'.

If $A \to B$ induces isomorphisms on residue fields, then so does $A \to B'$ (see argument in second paragraph). In this way we see that the lemma holds in the remaining case. $\square$

Lemma 46.2. *Let A be a ring. Let $B = \mathrm{colim}\, B_\lambda$ be a filtered colimit of A-algebras. If each $f_\lambda : \mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A)$ is a universal homeomorphism, resp. a universal homeomorphism inducing isomorphisms on residue fields, resp. universally closed, resp. universally closed and universally injective, then the same thing is true for $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.*

Proof. If f_λ is universally closed, then B_λ is integral over A by Lemma 44.7. Hence B is integral over A and f is universally closed (by the same lemma). This proves the case where each f_λ is universally closed.

For a prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ denote $\mathfrak{q}_\lambda \subset B_\lambda$ the inverse image. Then $\kappa(\mathfrak{q}) = \mathrm{colim}\, \kappa(\mathfrak{q}_\lambda)$. Thus if $A \to B_\lambda$ induces purely inseparable extensions of residue fields, then the same is true for $A \to B$. This proves the case where f_λ is universally closed and universally injective, see Lemma 10.2.

The case where f is a universal homeomorphism follows from the remarks above and Lemma 45.5 combined with the fact that prime ideals in B are the same thing as compatible sequences of prime ideals in all of the B_λ.

If $A \to B_\lambda$ induces isomorphisms on residue fields, then so does $A \to B$ (see argument in second paragraph). In this way we see that the lemma holds in the remaining case. $\square$

Lemma 46.3. *Let $A \subset B$ be a ring extension. Let $S \subset A$ be a multiplicative subset. Let $n \geq 1$ and $b_i \in B$ for $1 \leq i \leq n$. Any $x \in S^{-1}B$ such that*

$$x \notin S^{-1}A \text{ and } b_i x^i \in S^{-1}A \text{ for } i = 1, \ldots, n$$

is equal to $s^{-1}y$ with $s \in S$ and $y \in B$ such that

$$y \notin A \text{ and } b_i y^i \in A \text{ for } i = 1, \ldots, n$$

Proof. Omitted. Hint: clear denominators. $\square$

Lemma 46.4. *Let $A \subset B$ be a ring extension. If there exists $b \in B$, $b \notin A$ and an integer $n \geq 2$ with $b^n \in A$ and $b^{n+1} \in A$, then there exists a $b' \in B$, $b' \notin A$ with $(b')^2 \in A$ and $(b')^3 \in A$.*

Proof. Let b and n be as in the lemma. Then all sufficiently large powers of b are in A. Namely, $(b^n)^k(b^{n+1})^i = b^{(k+i)n+i}$ which implies any power b^m with $m \geq n^2$ is in A. Hence if $i \geq 1$ is the largest integer such that $b^i \notin A$, then $(b^i)^2 \in A$ and $(b^i)^3 \in A$. $\square$

Lemma 46.5. *Let $A \subset B$ be a ring extension such that $\text{Spec}(B) \to \text{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields. If $A \neq B$, then there exists a $b \in B$, $b \notin A$ with $b^2 \in A$ and $b^3 \in A$.*

Proof. Recall that $A \subset B$ is integral (Lemma 44.7). By Lemma 46.1 we may assume that B is generated by a single element over A. Hence B is finite over A (Algebra, Lemma 36.5). Hence the support of B/A as an A-module is closed and not empty (Algebra, Lemmas 40.5 and 40.2). Let $\mathfrak{p} \subset A$ be a minimal prime of the support. After replacing $A \subset B$ by $A_\mathfrak{p} \subset B_\mathfrak{p}$ (permissible by Lemma 46.3) we may assume that $(A, \mathfrak{m})$ is a local ring, that B is finite over A, and that B/A has support $\{\mathfrak{m}\}$ as an A-module. Since B/A is a finite module, we see that $I = \text{Ann}_A(B/A)$ satisfies $\mathfrak{m} = \sqrt{I}$ (Algebra, Lemma 40.5). Let $\mathfrak{m}' \subset B$ be the unique prime ideal lying over $\mathfrak{m}$. Because $\text{Spec}(B) \to \text{Spec}(A)$ is a homeomorphism, we find that $\mathfrak{m}' = \sqrt{IB}$. For $f \in \mathfrak{m}'$ pick $n \geq 1$ such that $f^n \in IB$. Then also $f^{n+1} \in IB$. Since $IB \subset A$ by our choice of I we conclude that $f^n, f^{n+1} \in A$. Using Lemma 46.4 we conclude our lemma is true if $\mathfrak{m}' \not\subset A$. However, if $\mathfrak{m}' \subset A$, then $\mathfrak{m}' = \mathfrak{m}$ and we conclude that $A = B$ as the residue fields are isomorphic as well by assumption. This contradiction finishes the proof. $\square$

Lemma 46.6. *Let $A \subset B$ be a ring extension such that $\text{Spec}(B) \to \text{Spec}(A)$ is a universal homeomorphism. If $A \neq B$, then either there exists a $b \in B$, $b \notin A$ with $b^2 \in A$ and $b^3 \in A$ or there exists a prime number p and a $b \in B$, $b \notin A$ with $pb \in A$ and $b^p \in A$.*

Proof. The argument is almost exactly the same as in the proof of Lemma 46.5 but we write everything out to make sure it works.

Recall that $A \subset B$ is integral (Lemma 44.7). By Lemma 46.1 we may assume that B is generated by a single element over A. Hence B is finite over A (Algebra, Lemma 36.5). Hence the support of B/A as an A-module is closed and not empty (Algebra, Lemmas 40.5 and 40.2). Let $\mathfrak{p} \subset A$ be a minimal prime of the support.

After replacing $A \subset B$ by $A_\mathfrak{p} \subset B_\mathfrak{p}$ (permissible by Lemma 46.3) we may assume that $(A, \mathfrak{m})$ is a local ring, that B is finite over A, and that B/A has support $\{\mathfrak{m}\}$ as an A-module. Since B/A is a finite module, we see that $I = \text{Ann}_A(B/A)$ satisfies $\mathfrak{m} = \sqrt{I}$ (Algebra, Lemma 40.5). Let $\mathfrak{m}' \subset B$ be the unique prime ideal lying over $\mathfrak{m}$. Because $\text{Spec}(B) \to \text{Spec}(A)$ is a homeomorphism, we find that $\mathfrak{m}' = \sqrt{IB}$. For $f \in \mathfrak{m}'$ pick $n \geq 1$ such that $f^n \in IB$. Then also $f^{n+1} \in IB$. Since $IB \subset A$ by our choice of I we conclude that $f^n, f^{n+1} \in A$. Using Lemma 46.4 we conclude our lemma is true if $\mathfrak{m}' \not\subset A$. If $\mathfrak{m}' \subset A$, then $\mathfrak{m}' = \mathfrak{m}$. Since $A \neq B$ we conclude the map $\kappa = A/\mathfrak{m} \to B/\mathfrak{m}' = \kappa'$ of residue fields cannot be an isomorphism. By Lemma 10.2 we conclude that the characteristic of κ is a prime number p and that the extension κ'/κ is purely inseparable. Pick $b \in B$ whose image in κ' is an element not contained in κ but whose pth power is in κ. Then $b \notin A$, $b^p \in A$, and $pb \in A$ (because $pb \in \mathfrak{m}' = \mathfrak{m} \subset A$) as desired. $\square$

Proposition 46.7. *Let $A \subset B$ be a ring extension. The following are equivalent*

(1) $\text{Spec}(B) \to \text{Spec}(A)$ *is a universal homeomorphism inducing isomorphisms on residue fields, and*
(2) *every finite subset $E \subset B$ is contained in an extension*
$$A[b_1, \ldots, b_n] \subset B$$
such that $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$ for $i = 1, \ldots, n$.

Proof. Assume (1). Using transfinite recursion we construct for each ordinal α an A-subalgebra $B_\alpha \subset B$ as follows. Set $B_0 = A$. If α is a limit ordinal, then we set $B_\alpha = \text{colim}_{\beta < \alpha} B_\beta$. If $\alpha = \beta + 1$, then either $B_\beta = B$ in which case we set $B_\alpha = B_\beta$ or $B_\beta \neq B$, in which case we apply Lemma 46.5 to choose a $b_\alpha \in B$, $b_\alpha \notin B_\beta$ with $b_\alpha^2, b_\alpha^3 \in B_\beta$ and we set $B_\alpha = B_\beta[b_\alpha] \subset B$. Clearly, $B = \text{colim}\, B_\alpha$ (in fact $B = B_\alpha$ for some ordinal α as one sees by looking at cardinalities). We will prove, by transfinite induction, that (2) holds for $A \to B_\alpha$ for every ordinal α. It is clear for $\alpha = 0$. Assume the statement holds for every $\beta < \alpha$ and let $E \subset B_\alpha$ be a finite subset. If α is a limit ordinal, then $B_\alpha = \bigcup_{\beta < \alpha} B_\beta$ and we see that $E \subset B_\beta$ for some $\beta < \alpha$ which proves the result in this case. If $\alpha = \beta + 1$, then $B_\alpha = B_\beta[b_\alpha]$. Thus any $e \in E$ can be written as a polynomial $e = \sum d_{e,i} b_\alpha^i$ with $d_{e,i} \in B_\beta$. Let $D \subset B_\beta$ be the set $D = \{d_{e,i}\} \cup \{b_\alpha^2, b_\alpha^3\}$. By induction assumption there exists an A-subalgebra $A[b_1, \ldots, b_n] \subset B_\beta$ as in the statement of the lemma containing D. Then $A[b_1, \ldots, b_n, b_\alpha] \subset B_\alpha$ is an A-subalgebra of B_α as in the statement of the lemma containing E.

Assume (2). Write $B = \text{colim}\, B_\lambda$ as the colimit of its finite A-subalgebras. By Lemma 46.2 it suffices to show that $\text{Spec}(B_\lambda) \to \text{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields. Compositions of universally closed morphisms are universally closed and the same thing for morphisms which induce isomorphisms on residue fields. Thus it suffices to show that if $A \subset B$ and B is generated by a single element b with $b^2, b^3 \in A$, then (1) holds. Such an extension is integral and hence $\text{Spec}(B) \to \text{Spec}(A)$ is universally closed and surjective (Lemma 44.7 and Algebra, Lemma 36.17). Note that $(b^2)^3 = (b^3)^2$ in A. For any ring map $\varphi : A \to K$ to a field K we see that there exists a $\lambda \in K$ with $\varphi(b^2) = \lambda^2$ and $\varphi(b^3) = \lambda^3$. Namely, $\lambda = 0$ if $\varphi(b^2) = 0$ and $\lambda = \varphi(b^3)/\varphi(b^2)$ if not. Thus $B \otimes_A K$ is a quotient of $K[x]/(x^2 - \lambda^2, x^3 - \lambda^3)$. This ring has exactly one prime with residue field K. This implies that $\text{Spec}(B) \to \text{Spec}(A)$ is bijective and induces isomorphisms on residue fields. Combined with universal closedness this shows (1) is true, see Lemmas 45.5 and 10.2. □

Proposition 46.8. *Let $A \subset B$ be a ring extension. The following are equivalent*

1. $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is a universal homeomorphism, and*
2. *every finite subset $E \subset B$ is contained in an extension*
$$A[b_1, \ldots, b_n] \subset B$$
such that for $i = 1, \ldots, n$ we have
   (a) $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$, *or*
   (b) *there exists a prime number p with $pb_i, b_i^p \in A[b_1, \ldots, b_{i-1}]$.*

Proof. The proof is exactly the same as the proof of Proposition 46.7 except for the following changes:

1. Use Lemma 46.6 instead of Lemma 46.5 which means that for each successor ordinal $\alpha = \beta + 1$ we either have $b_\alpha^2, b_\alpha^3 \in B_\beta$ or we have a prime p and $pb_\alpha, b_\alpha^p \in B_\beta$.
2. If α is a successor ordinal, then take $D = \{d_{e,i}\} \cup \{b_\alpha^2, b_\alpha^3\}$ or take $D = \{d_{e,i}\} \cup \{pb_\alpha, b_\alpha^p\}$ depending on which case α falls into.
3. In the proof of (2) $\Rightarrow$ (1) we also need to consider the case where B is generated over A by a single element b with $pb, b^p \in B$ for some prime number p. Here $A \subset B$ induces a universal homeomorphism on spectra for example by Algebra, Lemma 46.7.

This finishes the proof. □

Lemma 46.9. *Let p be a prime number. Let $A \to B$ be a ring map which induces an isomorphism $A[1/p] \to B[1/p]$ (for example if p is nilpotent in A). The following are equivalent*

1. $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is a universal homeomorphism, and*
2. *the kernel of $A \to B$ is a locally nilpotent ideal and for every $b \in B$ there exists a p-power q with qb and b^q in the image of $A \to B$.*

Proof. If (2) holds, then (1) holds by Algebra, Lemma 46.7. Assume (1). Then the kernel of $A \to B$ consists of nilpotent elements by Algebra, Lemma 30.6. Thus we may replace A by the image of $A \to B$ and assume that $A \subset B$. By Algebra, Lemma 46.5 the set

$$B' = \{b \in B \mid p^n b, b^{p^n} \in A \text{ for some } n \geq 0\}$$

is an A-subalgebra of B (being closed under products is trivial). We have to show $B' = B$. If not, then according to Lemma 46.6 there exists a $b \in B$, $b \notin B'$ with either $b^2, b^3 \in B'$ or there exists a prime number ℓ with $\ell b, b^\ell \in B'$. We will show both cases lead to a contradiction, thereby proving the lemma.

Since $A[1/p] = B[1/p]$ we can choose a p-power q such that $qb \in A$.

If $b^2, b^3 \in B'$ then also $b^q \in B'$. By definition of B' we find that $(b^q)^{q'} \in A$ for some p-power q'. Then $qq'b, b^{qq'} \in A$ whence $b \in B'$ which is a contradiction.

Assume now there exists a prime number ℓ with $\ell b, b^\ell \in B'$. If $\ell \neq p$ then $\ell b \in B'$ and $qb \in A \subset B'$ imply $b \in B'$ a contradiction. Thus $\ell = p$ and $b^p \in B'$ and we get a contradiction exactly as before. □

Lemma 46.10. *Let A be a ring. Let $x, y \in A$.*

(1) *If $x^3 = y^2$ in A, then $A \to B = A[t]/(t^2 - x, t^3 - y)$ induces bijections on residue fields and a universal homeomorphism on spectra.*
(2) *If there is a prime number p such that $p^p x = y^p$ in A, then $A \to B = A[t]/(t^p - x, pt - y)$ induces a universal homeomorphism on spectra.*

Proof. We will use the criterion of Lemma 45.5 to check this. In both cases the ring map is integral. Thus it suffices to show that given a field k and a ring map $\varphi : A \to k$ the k-algebra $B \otimes_A k$ has a unique prime ideal whose residue field is equal to k in case (1) and purely inseparable over k in case (2). See Lemma 10.2.

In case (1) set $\lambda = 0$ if $\varphi(x) = 0$ and set $\lambda = \varphi(y)/\varphi(x)$ if not. Then $B = k[t]/(t^2 - \lambda^2, t^3 - \lambda^2)$. Thus the result is clear.

In case (2) if the characteristic of k is p, then we obtain $\varphi(y) = 0$ and $B = k[t]/(t^p - \varphi(x))$ which is a local Artinian k-algebra whose residue field is either k or a degree p purely inseparable extension of k. If the characteristic of k is not p, then setting $\lambda = \varphi(y)/p$ we see $B = k[t]/(t - \lambda) = k$ and we conclude as well. $\square$

Lemma 46.11. *Let $A \to B$ be a ring map.*
(1) *If $A \to B$ induces a universal homeomorphism on spectra, then $B = \operatorname{colim} B_i$ is a filtered colimit of finitely presented A-algebras B_i such that $A \to B_i$ induces a universal homeomorphism on spectra.*
(2) *If $A \to B$ induces isomorphisms on residue fields and a universal homeomorphism on spectra, then $B = \operatorname{colim} B_i$ is a filtered colimit of finitely presented A-algebras B_i such that $A \to B_i$ induces isomorphisms on residue fields and a universal homeomorphism on spectra.*

Proof. Proof of (1). We will use the criterion of Algebra, Lemma 127.4. Let $A \to C$ be of finite presentation and let $\varphi : C \to B$ be an A-algebra map. Let $B' = \varphi(C) \subset B$ be the image. Then $A \to B'$ induces a universal homeomorphism on spectra by Lemma 46.1. By Algebra, Lemma 127.2 we can write $B' = \operatorname{colim}_{i \in I} B_i$ with $A \to B_i$ of finite presentation and surjective transition maps. By Algebra, Lemma 127.3 we can choose an index $0 \in I$ and a factorization $C \to B_0 \to B'$ of the map $C \to B'$. We claim that $\operatorname{Spec}(B_i) \to \operatorname{Spec}(A)$ is a universal homeomorphism for i sufficiently large. The claim finishes the proof of (1).

Proof of the claim. By Lemma 45.6 the ring map $A_{red} \to B'_{red}$ induces a universal homeomorphism on spectra. Thus $A_{red} \subset B'_{red}$ by Algebra, Lemma 30.6. Setting $A' = \operatorname{Im}(A \to B')$ we have surjections $A \to A' \to A_{red}$ inducing bijections $\operatorname{Spec}(A_{red}) = \operatorname{Spec}(A') = \operatorname{Spec}(A)$. Thus $A' \subset B'$ induces a universal homeomorphism on spectra. By Proposition 46.8 and the fact that B' is finite type over A' we can find n and $b'_1, \ldots, b'_n \in B'$ such that $B' = A'[b'_1, \ldots, b'_n]$ and such that for $j = 1, \ldots, n$ we have
(1) $(b'_j)^2, (b'_j)^3 \in A'[b'_1, \ldots, b'_{j-1}]$, or
(2) there exists a prime number p with $pb'_j, (b'_j)^p \in A'[b'_1, \ldots, b'_{j-1}]$.

Choose $b_1, \ldots, b_n \in B_0$ lifting $b'_1, \ldots, b'_n$. For $i \geq 0$ denote $b_{j,i}$ the image of b_j in B_i. For large enough i we will have for $j = 1, \ldots, n$
(1) $b_{j,i}^2, b_{j,i}^3 \in A_i[b_{1,i}, \ldots, b_{j-1,i}]$, or
(2) there exists a prime number p with $pb_{j,i}, b_{j,i}^p \in A_i[b_{1,i}, \ldots, b_{j-1,i}]$.

Here $A_i \subset B_i$ is the image of $A \to B_i$. Observe that $A \to A_i$ is a surjective ring map whose kernel is a locally nilpotent ideal. After increasing i more if necessary, we may assume B_i is generated by $b_1, \ldots, b_n$ over A_i, in other words $B_i = A_i[b_1, \ldots, b_n]$. By Algebra, Lemmas 46.7 and 46.4 we conclude that $A \to A_i \to A_i[b_1] \to \ldots \to A_i[b_1, \ldots, b_n] = B_i$ induce universal homeomorphisms on spectra. This finishes the proof of the claim.

The proof of (2) is exactly the same. □

47. Absolute weak normalization and seminormalization

Motivated by the results proved in the previous section we give the following definition.

Definition 47.1. Let A be a ring.

(1) We say A is *seminormal* if for all $x, y \in A$ with $x^3 = y^2$ there is a unique $a \in A$ with $x = a^2$ and $y = a^3$.
(2) We say A is *absolutely weakly normal* if (a) A is seminormal and (b) for any prime number p and $x, y \in A$ with $p^p x = y^p$ there is a unique $a \in A$ with $x = a^p$ and $y = pa$.

An amusing observation, see [Cos82], is that in the definition of seminormal rings it suffices[15] to assume the existence of a. Absolutely weakly normal schemes were defined in [Ryd07, Appendix B].

Lemma 47.2. *Being seminormal or being absolutely weakly normal is a local property of rings, see Properties, Definition 4.1.*

Proof. Suppose that A is seminormal and $f \in A$. Let $x', y' \in A_f$ with $(x')^3 = (y')^2$. Write $x' = x/f^{2n}$ and $y' = y/f^{3n}$ for some $n \geq 0$ and $x, y \in A$. After replacing x, y by $f^{2m}x, f^{3m}y$ and n by $n + m$, we see that $x^3 = y^2$ in A. Then we find a unique $a \in A$ with $x = a^2$ and $y = a^3$. Setting $a' = a/f^n$ we get $x' = (a')^2$ and $y' = (a')^3$ as desired. Uniqueness of a' follows from uniqueness of a. In exactly the same manner the reader shows that if A is absolutely weakly normal, then A_f is absolutely weakly normal.

Assume A is a ring and $f_1, \ldots, f_n \in A$ generate the unit ideal. Assume A_{f_i} is seminormal for each i. Let $x, y \in A$ with $x^3 = y^2$. For each i we find a unique $a_i \in A_{f_i}$ with $x = a_i^2$ and $y = a_i^3$ in A_{f_i}. By the uniqueness and the result of the first paragraph (which tells us that $A_{f_i f_j}$ is seminormal) we see that a_i and a_j map to the same element of $A_{f_i f_j}$. By Algebra, Lemma 24.2 we find a unique $a \in A$ mapping to a_i in A_{f_i} for all i. Then $x = a^2$ and $y = a^3$ by the same token. Clearly this a is unique. Thus A is seminormal. If we assume A_{f_i} is absolutely weakly normal, then the exact same argument shows that A is absolutely weakly normal. □

Next we define seminormal schemes and absolutely weakly normal schemes.

Definition 47.3. Let X be a scheme.

(1) We say X is *seminormal* if every $x \in X$ has an affine open neighbourhood $\mathrm{Spec}(R) = U \subset X$ such that the ring R is seminormal.

[15] Let A be a ring such that for all $x, y \in A$ with $x^3 = y^2$ there is an $a \in A$ with $x = a^2$ and $y = a^3$. Then A is reduced: if $x^2 = 0$, then $x^2 = x^3$ and hence there exists an a such that $x = a^3$ and $x = a^2$. Then $x = a^3 = ax = a^4 = x^2 = 0$. Finally, if $a_1^2 = a_2^2$ and $a_1^3 = a_2^3$ for a_1, a_2 in a reduced ring, then $(a_1 - a_2)^3 = a_1^3 - 3a_1^2 a_2 + 3a_1 a_2^2 - a_2^3 = (1 - 3 + 3 - 1)a_1^3 = 0$ and hence $a_1 = a_2$.

(2) We say X is *absolutely weakly normal* if every $x \in X$ has an affine open neighbourhood $\text{Spec}(R) = U \subset X$ such that the ring R is absolutely weakly normal.

Here is the obligatory lemma.

Lemma 47.4. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is seminormal.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is seminormal.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is seminormal.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is seminormal.*

Moreover, if X is seminormal then every open subscheme is seminormal. The same statements are true with "seminormal" replaced by "absolutely weakly normal".

Proof. Combine Properties, Lemma 4.3 and Lemma 47.2. $\square$

Lemma 47.5. *A seminormal scheme or ring is reduced. A fortiori the same is true for absolutely weakly normal schemes or rings.*

Proof. Let A be a ring. If $a \in A$ is nonzero but $a^2 = 0$, then $a^2 = 0^2$ and $a^3 = 0^3$ and hence A is not seminormal. $\square$

Lemma 47.6. *Let A be a ring.*

(1) *The category of ring maps $A \to B$ inducing a universal homeomorphism on spectra has a final object $A \to A^{awn}$.*
(2) *Given $A \to B$ in the category of (1) the resulting map $B \to A^{awn}$ is an isomorphism if and only if B is absolutely weakly normal.*
(3) *The category of ring maps $A \to B$ inducing isomorphisms on residue fields and a universal homeomorphism on spectra has a final object $A \to A^{sn}$.*
(4) *Given $A \to B$ in the category of (3) the resulting map $B \to A^{sn}$ is an isomorphism if and only if B is seminormal.*

For any ring map $\varphi : A \to A'$ there are unique maps $\varphi^{awn} : A^{awn} \to (A')^{awn}$ and $\varphi^{sn} : A^{sn} \to (A')^{sn}$ compatible with φ.

Proof. We prove (1) and (2) and we omit the proof of (3) and (4) and the final statement. Consider the category of A-algebras of the form

$$B = A[x_1, \ldots, x_n]/J$$

where J is a finitely generated ideal such that $A \to B$ defines a universal homeomorphism on spectra. We claim this category is directed (Categories, Definition 19.1). Namely, given

$$B = A[x_1, \ldots, x_n]/J \quad \text{and} \quad B' = A[x_1, \ldots, x_{n'}]/J'$$

then we can consider

$$B'' = A[x_1, \ldots, x_{n+n'}]/J''$$

where J'' is generated by the elements of J and the elements $f(x_{n+1}, \ldots, x_{n+n'})$ where $f \in J'$. Then we have A-algebra homomorphisms $B \to B''$ and $B' \to B''$ which induce an isomorphism $B \otimes_A B' \to B''$. It follows from Lemmas 45.2 and 45.3 that $\text{Spec}(B'') \to \text{Spec}(A)$ is a universal homeomorphism and hence $A \to B''$ is in our category. Finally, given $\varphi, \varphi' : B \to B'$ in our category with B as displayed above, then we consider the quotient B'' of B' by the ideal generated by $\varphi(x_i) - \varphi'(x_i)$, $i = 1, \ldots, n$. Since $\text{Spec}(B') = \text{Spec}(B)$ we see that $\text{Spec}(B'') \to \text{Spec}(B')$ is a bijective closed immersion hence a universal homeomorphism. Thus B'' is in our category and φ, φ' are equalized by $B' \to B''$. This completes the proof of our claim. We set

$$A^{awn} = \operatorname{colim} B$$

where the colimit is over the category just described. Observe that $A \to A^{awn}$ induces a universal homeomorphism on spectra by Lemma 46.2 (this is where we use the category is directed).

Given a ring map $A \to B$ of finite presentation inducing a universal homeomorphism on spectra, we get a canonical map $B \to A^{awn}$ by the very construction of A^{awn}. Since every $A \to B$ as in (1) is a filtered colimit of $A \to B$ as in (1) of finite presentation (Lemma 46.11), we see that $A \to A^{awn}$ is final in the category (1).

Let $x, y \in A^{awn}$ be elements such that $x^3 = y^2$. Then $A^{awn} \to A^{awn}[t]/(t^2 - x, t^3 - y)$ induces a universal homeomorphism on spectra by Lemma 46.10. Thus $A \to A^{awn}[t]/(t^2 - x, t^3 - y)$ is in the category (1) and we obtain a unique A-algebra map $A^{awn}[t]/(t^2 - x, t^3 - y) \to A^{awn}$. The image $a \in A^{awn}$ of t is therefore the unique element such that $a^2 = x$ and $a^3 = y$ in A^{awn}. In exactly the same manner, given a prime p and $x, y \in A^{awn}$ with $p^p x = y^p$ we find a unique $a \in A^{awn}$ with $a^p = x$ and $pq = y$. Thus A^{awn} is absolutely weakly normal by definition.

Finally, let $A \to B$ be in the category (1) with B absolutely weakly normal. Since $A^{awn} \to B^{awn}$ induces a universal homeomorphism on spectra and since A^{awn} is reduced (Lemma 47.5) we find $A^{awn} \subset B^{awn}$ (see Algebra, Lemma 30.6). If this inclusion is not an equality, then Lemma 46.6 implies there is an element $b \in B^{awn}$, $b \notin A^{awn}$ such that either $b^2, b^3 \in A^{awn}$ or $pb, b^p \in A^{awn}$ for some prime number p. However, by the existence and uniqueness in Definition 47.1 this forces $b \in A^{awn}$ and hence we obtain the contradiction that finishes the proof. $\square$

Lemma 47.7. *Let X be a scheme.*

1. *The category of universal homeomorphisms $Y \to X$ has an initial object $X^{awn} \to X$.*
2. *Given $Y \to X$ in the category of (1) the resulting morphism $X^{awn} \to Y$ is an isomorphism if and only if Y is absolutely weakly normal.*
3. *The category of universal homeomorphisms $Y \to X$ which induce ismomorphisms on residue fields has an initial object $X^{sn} \to X$.*
4. *Given $Y \to X$ in the category of (3) the resulting morphism $X^{sn} \to Y$ is an isomorphism if and only if Y is seminormal.*

For any morphism $h : X' \to X$ of schemes there are unique morphisms $h^{awn} : (X')^{awn} \to X^{awn}$ and $h^{sn} : (X')^{sn} \to X^{sn}$ compatible with h.

Proof. We will prove (1) and (2) and omit the proof of (3) and (4). Let $h : X' \to X$ be a morphism of schemes. If (1) holds for X and X', then $X' \times_X X^{awn} \to X'$ is a universal homeomorphism and hence we get a unique morphism $(X')^{awn} \to X' \times_X X^{awn}$ over X' by the universal property of $(X')^{awn} \to X'$. Composed with the projection $X' \times_X X^{awn} \to X^{awn}$ we obtain h^{awn}. If in addition (2) holds for X and X' and h is an open immersion, then $X' \times_X X^{awn}$ is absolutely weakly normal (Lemma 47.4) and we deduce that $(X')^{awn} \to X' \times_X X^{awn}$ is an isomorphism.

Recall that any universal homeomorphism is affine, see Lemma 45.4. Thus if X is affine then (1) and (2) follow immediately from Lemma 47.6. Let X be a scheme and let $\mathcal{B}$ be the set of affine opens of X. For each $U \in \mathcal{B}$ we obtain $U^{awn} \to U$ and for $V \subset U$, $V, U \in \mathcal{B}$ we obtain a canonical isomorphism $\rho_{V,U} : V^{awn} \to V \times_U U^{awn}$ by the discussion in the previous paragraph. Thus by relative glueing (Constructions, Lemma 2.1) we obtain a morphism $X^{awn} \to X$ which restricts to U^{awn} over U compatibly with the $\rho_{V,U}$. Next, let $Y \to X$ be a universal homeomorphism. Then $U \times_X Y \to U$ is a universal homeomorphism for $U \in \mathcal{B}$ and we obtain a unique morphism $g_U : U^{awn} \to U \times_X Y$ over U. These g_U are compatible with the morphisms $\rho_{V,U}$; details omitted. Hence there is a unique morphism $g : X^{awn} \to Y$ over X agreeing with g_U over U, see Constructions, Remark 2.3. This proves (1) for X. Part (2) follows because it holds affine locally. $\square$

Definition 47.8. Let X be a scheme.

(1) The morphism $X^{sn} \to X$ constructed in Lemma 47.7 is the *seminormalization* of X.
(2) The morphism $X^{awn} \to X$ constructed in Lemma 47.7 is the *absolute weak normalization* of X.

To be sure, the seminormalization X^{sn} of X is a seminormal scheme and the absolute weak normalization X^{awn} is an absolutely weakly normal scheme. Moreover, for any morphism $h : Y \to X$ of schemes we obtain a canonical commutative diagram

$$\begin{tikzcd} Y^{awn} \arrow[r] \arrow[d, "h^{awn}"] & Y^{sn} \arrow[r] \arrow[d, "h^{sn}"] & Y \arrow[d, "h"] \\ X^{awn} \arrow[r] & X^{sn} \arrow[r] & X \end{tikzcd}$$

of schemes; the arrows h^{sn} and h^{awn} are the unique ones compatible with h.

Lemma 47.9. *Let X be a scheme. The following are equivalent*

(1) *X is seminormal,*
(2) *X is equal to its own seminormalization, i.e., the morphism $X^{sn} \to X$ is an isomorphism,*
(3) *if $\pi : Y \to X$ is a universal homomorphism inducing isomorphisms on residue fields with Y reduced, then π is an isomorphism.*

Proof. The equivalence of (1) and (2) is clear from Lemma 47.7. If (3) holds, then $X^{sn} \to X$ is an isomorphism and we see that (2) holds.

Assume (2) holds and let $\pi : Y \to X$ be a universal homomorphism inducing isomorphisms on residue fields with Y reduced. Then there exists a factorization $X \to Y \to X$ of id_X by Lemma 47.7. Then $X \to Y$ is a closed immersion (by Schemes, Lemma 21.11 and the fact that π is separated for example by Lemma 10.3). Since $X \to Y$ is also a bijection on points, the reducedness of Y shows that it has to be an isomorphism. This finishes the proof. $\square$

Lemma 47.10. *Let X be a scheme. The following are equivalent*

(1) *X is absolutely weakly normal,*
(2) *X is equal to its own absolute weak normalization, i.e., the morphism $X^{awn} \to X$ is an isomorphism,*

(3) *if $\pi : Y \to X$ is a universal homomorphism with Y reduced, then π is an isomorphism.*

Proof. This is proved in exactly the same manner as Lemma 47.9. $\square$

48. Finite locally free morphisms

In many papers the authors use finite flat morphisms when they really mean finite locally free morphisms. The reason is that if the base is locally Noetherian then this is the same thing. But in general it is not, see Exercises, Exercise 5.3.

Definition 48.1. Let $f : X \to S$ be a morphism of schemes. We say f is *finite locally free* if f is affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module. In this case we say f is has *rank* or *degree* d if the sheaf $f_*\mathcal{O}_X$ is finite locally free of degree d.

Note that if $f : X \to S$ is finite locally free then S is the disjoint union of open and closed subschemes S_d such that $f^{-1}(S_d) \to S_d$ is finite locally free of degree d.

Lemma 48.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is finite locally free,*
(2) *f is finite, flat, and locally of finite presentation.*

If S is locally Noetherian these are also equivalent to

(3) *f is finite and flat.*

Proof. Let $V \subset S$ be affine open. In all three cases the morphism is affine hence $f^{-1}(V)$ is affine. Thus we may write $V = \mathrm{Spec}(R)$ and $f^{-1}(V) = \mathrm{Spec}(A)$ for some R-algebra A. Assume (1). This means we can cover S by affine opens $V = \mathrm{Spec}(R)$ such that A is finite free as an R-module. Then $R \to A$ is of finite presentation by Algebra, Lemma 7.4. Thus (2) holds. Conversely, assume (2). For every affine open $V = \mathrm{Spec}(R)$ of S the ring map $R \to A$ is finite and of finite presentation and A is flat as an R-module. By Algebra, Lemma 36.23 we see that A is finitely presented as an R-module. Thus Algebra, Lemma 78.2 implies A is finite locally free. Thus (1) holds. The Noetherian case follows as a finite module over a Noetherian ring is a finitely presented module, see Algebra, Lemma 31.4. $\square$

Lemma 48.3. *A composition of finite locally free morphisms is finite locally free.*

Proof. Omitted. $\square$

Lemma 48.4. *A base change of a finite locally free morphism is finite locally free.*

Proof. Omitted. $\square$

Lemma 48.5. *Let $f : X \to S$ be a finite locally free morphism of schemes. There exists a disjoint union decomposition $S = \coprod_{d \geq 0} S_d$ by open and closed subschemes such that setting $X_d = f^{-1}(S_d)$ the restrictions $f|_{X_d}$ are finite locally free morphisms $X_d \to S_d$ of degree d.*

Proof. This is true because a finite locally free sheaf locally has a well defined rank. Details omitted. $\square$

Lemma 48.6. *Let $f : Y \to X$ be a finite morphism with X affine. There exists a diagram*

$$\begin{tikzcd} Z' \arrow[rd] & Y' \arrow[l, "i"'] \arrow[r] \arrow[d] & Y \arrow[d] \\ & X' \arrow[r] & X \end{tikzcd}$$

where

1. *$Y' \to Y$ and $X' \to X$ are surjective finite locally free,*
2. *$Y' = X' \times_X Y$,*
3. *$i : Y' \to Z'$ is a closed immersion,*
4. *$Z' \to X'$ is finite locally free, and*
5. *$Z' = \bigcup_{j=1,\ldots,m} Z'_j$ is a (set theoretic) finite union of closed subschemes, each of which maps isomorphically to X'.*

Proof. Write $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. See also More on Algebra, Section 21. Let $x_1, \ldots, x_n \in B$ be generators of B over A. For each i we can choose a monic polynomial $P_i(T) \in A[T]$ such that $P(x_i) = 0$ in B. By Algebra, Lemma 136.14 (applied n times) there exists a finite locally free ring extension $A \subset A'$ such that each P_i splits completely:

$$P_i(T) = \prod\nolimits_{k=1,\ldots,d_i} (T - \alpha_{ik})$$

for certain $\alpha_{ik} \in A'$. Set

$$C = A'[T_1, \ldots, T_n]/(P_1(T_1), \ldots, P_n(T_n))$$

and $B' = A' \otimes_A B$. The map $C \to B'$, $T_i \mapsto 1 \otimes x_i$ is an A'-algebra surjection. Setting $X' = \mathrm{Spec}(A')$, $Y' = \mathrm{Spec}(B')$ and $Z' = \mathrm{Spec}(C)$ we see that (1) – (4) hold. Part (5) holds because set theoretically $\mathrm{Spec}(C)$ is the union of the closed subschemes cut out by the ideals

$$(T_1 - \alpha_{1k_1}, T_2 - \alpha_{2k_2}, \ldots, T_n - \alpha_{nk_n})$$

for any $1 \leq k_i \leq d_i$. $\square$

The following lemma is stated in the correct generality in Lemma 56.4 below.

Lemma 48.7. *Let $f : Y \to X$ be a finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y. Then $f(T) \subset X$ is a closed nowhere dense subset of X.*

Proof. By Lemma 44.11 we know that $f(T) \subset X$ is closed. Let $X = \bigcup X_i$ be an affine covering. Since T is nowhere dense in Y, we see that also $T \cap f^{-1}(X_i)$ is nowhere dense in $f^{-1}(X_i)$. Hence if we can prove the theorem in the affine case, then we see that $f(T) \cap X_i$ is nowhere dense. This then implies that T is nowhere dense in X by Topology, Lemma 21.4.

Assume X is affine. Choose a diagram

$$\begin{tikzcd} Z' \arrow[rd] & Y' \arrow[l, "i"'] \arrow[r, "a"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ & X' \arrow[r, "b"] & X \end{tikzcd}$$

as in Lemma 48.6. The morphisms a, b are open since they are finite locally free (Lemmas 48.2 and 25.10). Hence $T' = a^{-1}(T)$ is nowhere dense, see Topology, Lemma 21.6. The morphism b is surjective and open. Hence, if we can prove $f'(T') = b^{-1}(f(T))$ is nowhere dense, then $f(T)$ is nowhere dense, see Topology, Lemma 21.6. As i is a closed immersion, by Topology, Lemma 21.5 we see that $i(T') \subset Z'$ is closed and nowhere dense. Thus we have reduced the problem to the case discussed in the following paragraph.

Assume that $Y = \bigcup_{i=1,\ldots,n} Y_i$ is a finite union of closed subsets, each mapping isomorphically to X. Consider $T_i = Y_i \cap T$. If each of the T_i is nowhere dense in Y_i, then each $f(T_i)$ is nowhere dense in X as $Y_i \to X$ is an isomorphism. Hence $f(T) = f(T_i)$ is a finite union of nowhere dense closed subsets of X and we win, see Topology, Lemma 21.2. Suppose not, say T_1 contains a nonempty open $V \subset Y_1$. We are going to show this leads to a contradiction. Consider $Y_2 \cap V \subset V$. This is either a proper closed subset, or equal to V. In the first case we replace V by $V \setminus V \cap Y_2$, so $V \subset T_1$ is open in Y_1 and does not meet Y_2. In the second case we have $V \subset Y_1 \cap Y_2$ is open in both Y_1 and Y_2. Repeat sequentially with $i = 3, \ldots, n$. The result is a disjoint union decomposition

$$\{1, \ldots, n\} = I_1 \amalg I_2, \quad 1 \in I_1$$

and an open V of Y_1 contained in T_1 such that $V \subset Y_i$ for $i \in I_1$ and $V \cap Y_i = \emptyset$ for $i \in I_2$. Set $U = f(V)$. This is an open of X since $f|_{Y_1} : Y_1 \to X$ is an isomorphism. Then

$$f^{-1}(U) = V \amalg \bigcup\nolimits_{i \in I_2} (Y_i \cap f^{-1}(U))$$

As $\bigcup_{i \in I_2} Y_i$ is closed, this implies that $V \subset f^{-1}(U)$ is open, hence $V \subset Y$ is open. This contradicts the assumption that T is nowhere dense in Y, as desired. □

49. Rational maps

Let X be a scheme. Note that if U, V are dense open in X, then so is $U \cap V$.

Definition 49.1. Let X, Y be schemes.

(1) Let $f : U \to Y$, $g : V \to Y$ be morphisms of schemes defined on dense open subsets U, V of X. We say that f is *equivalent* to g if $f|_W = g|_W$ for some $W \subset U \cap V$ dense open in X.
(2) A *rational map from X to Y* is an equivalence class for the equivalence relation defined in (1).
(3) If X, Y are schemes over a base scheme S we say that a rational map from X to Y is an *S-rational map from X to Y* if there exists a representative $f : U \to Y$ of the equivalence class which is an S-morphism.

We say that two morphisms f, g as in (1) of the definition define the same rational map instead of saying that they are equivalent. In some cases rational maps are determined by maps on local rings at generic points.

Lemma 49.2. *Let S be a scheme. Let X and Y be schemes over S. Assume X has finitely many irreducible components with generic points $x_1, \ldots, x_n$. Let $s_i \in S$ be the image of x_i. Consider the map*

$$\left\{\begin{matrix} S\text{-rational maps} \\ \text{from } X \text{ to } Y \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} (y_1, \varphi_1, \ldots, y_n, \varphi_n) \text{ where } y_i \in Y \text{ lies over } s_i \text{ and} \\ \varphi_i : \mathcal{O}_{Y,y_i} \to \mathcal{O}_{X,x_i} \text{ is a local } \mathcal{O}_{S,s_i}\text{-algebra map} \end{matrix}\right\}$$

which sends $f : U \to Y$ to the $2n$-tuple with $y_i = f(x_i)$ and $\varphi_i = f^\sharp_{x_i}$. Then

(1) *If $Y \to S$ is locally of finite type, then the map is injective.*
(2) *If $Y \to S$ is locally of finite presentation, then the map is bijective.*
(3) *If $Y \to S$ is locally of finite type and X reduced, then the map is bijective.*

Proof. Observe that any dense open of X contains the points x_i so the construction makes sense. To prove (1) or (2) we may replace X by any dense open. Thus if $Z_1, \ldots, Z_n$ are the irreducible components of X, then we may replace X by $X \setminus \bigcup_{i \neq j} Z_i \cap Z_j$. After doing this X is the disjoint union of its irreducible components (viewed as open and closed subschemes). Then both the right hand side and the left hand side of the arrow are products over the irreducible components and we reduce to the case where X is irreducible.

Assume X is irreducible with generic point x lying over $s \in S$. Part (1) follows from part (1) of Lemma 42.4. Parts (2) and (3) follow from part (2) of the same lemma. $\square$

Definition 49.3. Let X be a scheme. A *rational function on* X is a rational map from X to $\mathbf{A}^1_{\mathbf{Z}}$.

See Constructions, Definition 5.1 for the definition of the affine line $\mathbf{A}^1$. Let X be a scheme over S. For any open $U \subset X$ a morphism $U \to \mathbf{A}^1_{\mathbf{Z}}$ is the same as a morphism $U \to \mathbf{A}^1_S$ over S. Hence a rational function is also the same as a S-rational map from X into $\mathbf{A}^1_S$.

Recall that we have the canonical identification $\text{Mor}(T, \mathbf{A}^1_{\mathbf{Z}}) = \Gamma(T, \mathcal{O}_T)$ for any scheme T, see Schemes, Example 15.2. Hence $\mathbf{A}^1_{\mathbf{Z}}$ is a ring-object in the category of schemes. More precisely, the morphisms

$$\begin{array}{rcl} + : \mathbf{A}^1_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}} & \longrightarrow & \mathbf{A}^1_{\mathbf{Z}} \\ (f, g) & \longmapsto & f + g \\ * : \mathbf{A}^1_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}} & \longrightarrow & \mathbf{A}^1_{\mathbf{Z}} \\ (f, g) & \longmapsto & fg \end{array}$$

satisfy all the axioms of the addition and multiplication in a ring (commutative with 1 as always). Hence also the set of rational maps into $\mathbf{A}^1_{\mathbf{Z}}$ has a natural ring structure.

Definition 49.4. Let X be a scheme. The *ring of rational functions on* X is the ring $R(X)$ whose elements are rational functions with addition and multiplication as just described.

For schemes with finitely many irreducible components we can compute this.

Lemma 49.5. *Let X be a scheme with finitely many irreducible components $X_1, \ldots, X_n$. If $\eta_i \in X_i$ is the generic point, then*

$$R(X) = \mathcal{O}_{X,\eta_1} \times \ldots \times \mathcal{O}_{X,\eta_n}$$

If X is reduced this is equal to $\prod \kappa(\eta_i)$. If X is integral then $R(X) = \mathcal{O}_{X,\eta} = \kappa(\eta)$ is a field.

Proof. Let $U \subset X$ be an open dense subset. Then $U_i = (U \cap X_i) \setminus (\bigcup_{j \neq i} X_j)$ is nonempty open as it contained η_i, contained in X_i, and $\bigcup U_i \subset U \subset X$ is dense.

Thus the identification in the lemma comes from the string of equalities

$$
\begin{aligned}
R(X) &= \mathrm{colim}_{U \subset X \text{ open dense}} \mathrm{Mor}(U, \mathbf{A}^1_{\mathbf{Z}}) \\
&= \mathrm{colim}_{U \subset X \text{ open dense}} \mathcal{O}_X(U) \\
&= \mathrm{colim}_{\eta_i \in U_i \subset X \text{ open}} \prod \mathcal{O}_X(U_i) \\
&= \prod \mathrm{colim}_{\eta_i \in U_i \subset X \text{ open}} \mathcal{O}_X(U_i) \\
&= \prod \mathcal{O}_{X,\eta_i}
\end{aligned}
$$

where the second equality is Schemes, Example 15.2. The final statement follows from Algebra, Lemma 25.1. □

Definition 49.6. Let X be an integral scheme. The *function field*, or the *field of rational functions* of X is the field $R(X)$.

We may occasionally indicate this field $k(X)$ instead of $R(X)$. We can use the notion of the function field to elucidate the separation condition on an integral scheme. Note that by Lemma 49.5 on an integral scheme every local ring $\mathcal{O}_{X,x}$ may be viewed as a local subring of $R(X)$.

Lemma 49.7. *Let X be an integral separated scheme. Let Z_1, Z_2 be distinct irreducible closed subsets of X. Let η_i be the generic point of Z_i. If $Z_1 \not\subset Z_2$, then $\mathcal{O}_{X,\eta_1} \not\subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. In particular, if $Z_1 = \{x\}$ consists of one closed point x, there exists a function regular in a neighborhood of x which is not in $\mathcal{O}_{X,\eta_2}$.*

Proof. First observe that under the assumption of X being separated, there is a unique map of schemes $\mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \to X$ over X such that the composition

$$\mathrm{Spec}(R(X)) \longrightarrow \mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \longrightarrow X$$

is the canonical map $\mathrm{Spec}(R(X)) \to X$. Namely, there is the canonical map $can : \mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \to X$, see Schemes, Equation (13.1.1). Given a second morphism a to X, we have that a agrees with can on the generic point of $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$ by assumption. Now X being separated guarantees that the subset in $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$ where these two maps agree is closed, see Schemes, Lemma 21.5. Hence $a = can$ on all of $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$.

Assume $Z_1 \not\subset Z_2$ and assume on the contrary that $\mathcal{O}_{X,\eta_1} \subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. Then we would obtain a second morphism

$$\mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \longrightarrow \mathrm{Spec}(\mathcal{O}_{X,\eta_1}) \longrightarrow X.$$

By the above this composition would have to be equal to can. This implies that η_2 specializes to η_1 (see Schemes, Lemma 13.2). But this contradicts our assumption $Z_1 \not\subset Z_2$. □

Definition 49.8. Let φ be a rational map between two schemes X and Y. We say φ is *defined in a point $x \in X$* if there exists a representative (U, f) of φ with $x \in U$. The *domain of definition* of φ is the set of all points where φ is defined.

With this definition it isn't true in general that φ has a representative which is defined on all of the domain of definition.

Lemma 49.9. *Let X and Y be schemes. Assume X reduced and Y separated. Let φ be a rational map from X to Y with domain of definition $U \subset X$. Then there exists a unique morphism $f : U \to Y$ representing φ. If X and Y are schemes over a separated scheme S and if φ is an S-rational map, then f is a morphism over S.*

Proof. Let (V, g) and (V', g') be representatives of φ. Then g, g' agree on a dense open subscheme $W \subset V \cap V'$. On the other hand, the equalizer E of $g|_{V \cap V'}$ and $g'|_{V \cap V'}$ is a closed subscheme of $V \cap V'$ (Schemes, Lemma 21.5). Now $W \subset E$ implies that $E = V \cap V'$ set theoretically. As $V \cap V'$ is reduced we conclude $E = V \cap V'$ scheme theoretically, i.e., $g|_{V \cap V'} = g'|_{V \cap V'}$. It follows that we can glue the representatives $g : V \to Y$ of φ to a morphism $f : U \to Y$, see Schemes, Lemma 14.1. We omit the proof of the final statement. $\square$

In general it does not make sense to compose rational maps. The reason is that the image of a representative of the first rational map may have empty intersection with the domain of definition of the second. However, if we assume that our schemes are irreducible and we look at dominant rational maps, then we can compose rational maps.

Definition 49.10. Let X and Y be irreducible schemes. A rational map from X to Y is called *dominant* if any representative $f : U \to Y$ is a dominant morphism of schemes.

By Lemma 8.6 it is equivalent to require that the generic point $\eta \in X$ maps to the generic point ξ of Y, i.e., $f(\eta) = \xi$ for any representative $f : U \to Y$. We can compose a dominant rational map φ between irreducible schemes X and Y with an arbitrary rational map ψ from Y to Z. Namely, choose representatives $f : U \to Y$ with $U \subset X$ open dense and $g : V \to Z$ with $V \subset Y$ open dense. Then $W = f^{-1}(V) \subset X$ is open nonempty (because it contains the generic point of X) and we let $\psi \circ \varphi$ be the equivalence class of $g \circ f|_W : W \to Z$. We omit the verification that this is well defined.

In this way we obtain a category whose objects are irreducible schemes and whose morphisms are dominant rational maps. Given a base scheme S we can similarly define a category whose objects are irreducible schemes over S and whose morphisms are dominant S-rational maps.

Definition 49.11. Let X and Y be irreducible schemes.

(1) We say X and Y are *birational* if X and Y are isomorphic in the category of irreducible schemes and dominant rational maps.
(2) Assume X and Y are schemes over a base scheme S. We say X and Y are *S-birational* if X and Y are isomorphic in the category of irreducible schemes over S and dominant S-rational maps.

If X and Y are birational irreducible schemes, then the set of rational maps from X to Z is bijective with the set of rational map from Y to Z for all schemes Z (functorially in Z). For "general" irreducible schemes this is just one possible definition. Another would be to require X and Y have isomorphic rings of rational functions. For varieties these conditions are equivalent, see Lemma 50.6.

Lemma 49.12. *Let X and Y be irreducible schemes.*

(1) *The schemes X and Y are birational if and only if they have isomorphic nonempty opens.*

(2) *Assume X and Y are schemes over a base scheme S. Then X and Y are S-birational if and only if there are nonempty opens $U \subset X$ and $V \subset Y$ which are S-isomorphic.*

Proof. Assume X and Y are birational. Let $f : U \to Y$ and $g : V \to X$ define inverse dominant rational maps from X to Y and from Y to X. We may assume V affine. We may replace U by an affine open of $f^{-1}(V)$. As $g \circ f$ is the identity as a dominant rational map, we see that the composition $U \to V \to X$ is the identity on a dense open of U. Thus after replacing U by a smaller affine open we may assume that $U \to V \to X$ is the inclusion of U into X. It follows that $U \to V$ is an immersion (apply Schemes, Lemma 21.11 to $U \to g^{-1}(U) \to U$). However, switching the roles of U and V and redoing the argument above, we see that there exists a nonempty affine open $V' \subset V$ such that the inclusion factors as $V' \to U \to V$. Then $V' \to U$ is necessarily an open immersion. Namely, $V' \to f^{-1}(V') \to V'$ are monomorphisms (Schemes, Lemma 23.8) composing to the identity, hence isomorphisms. Thus V' is isomorphic to an open of both X and Y. In the S-rational maps case, the exact same argument works. $\square$

Remark 49.13. Here is a generalization of the category of irreducible schemes and dominant rational maps. For a scheme X denote X^0 the set of points $x \in X$ with $\dim(\mathcal{O}_{X,x}) = 0$, in other words, X^0 is the set of generic points of irreducible components of X. Then we can consider the category with

(1) objects are schemes X such that every quasi-compact open has finitely many irreducible components, and
(2) morphisms from X to Y are rational maps $f : U \to Y$ from X to Y such that $f(U^0) = Y^0$.

If $U \subset X$ is a dense open of a scheme, then $U^0 \subset X^0$ need not be an equality, but if X is an object of our category, then this is the case. Thus given two morphisms in our category, the composition is well defined and a morphism in our category.

Remark 49.14. There is a variant of Definition 49.1 where we consider only those morphism $U \to Y$ defined on scheme theoretically dense open subschemes $U \subset X$. We use Lemma 7.6 to see that we obtain an equivalence relation. An equivalence class of these is called a *pseudo-morphism from X to Y*. If X is reduced the two notions coincide.

50. Birational morphisms

You may be used to the notion of a birational map of varieties having the property that it is an isomorphism over an open subset of the target. However, in general a birational morphism may not be an isomorphism over any nonempty open, see Example 50.4. Here is the formal definition.

Definition 50.1. Let X, Y be schemes. Assume X and Y have finitely many irreducible components. We say a morphism $f : X \to Y$ is *birational* if

(1) f induces a bijection between the set of generic points of irreducible components of X and the set of generic points of the irreducible components of Y, and
(2) for every generic point $\eta \in X$ of an irreducible component of X the local ring map $\mathcal{O}_{Y,f(\eta)} \to \mathcal{O}_{X,\eta}$ is an isomorphism.

We will see below that the fibres of a birational morphism over generic points are singletons. Moreover, we will see that in most cases one encounters in practice the existence a birational morphism between irreducible schemes X and Y implies X and Y are birational schemes.

Lemma 50.2. *Let $f : X \to Y$ be a morphism of schemes having finitely many irreducible components. If f is birational then f is dominant.*

Proof. Follows from Lemma 8.2 and the definition. $\square$

Lemma 50.3. *Let $f : X \to Y$ be a birational morphism of schemes having finitely many irreducible components. If $y \in Y$ is the generic point of an irreducible component, then the base change $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ is an isomorphism.*

Proof. We may assume $Y = \mathrm{Spec}(B)$ is affine and irreducible. Then X is irreducible too. If we prove the result for any nonempty affine open $U \subset X$, then the result holds for X (small argument omitted). Hence we may assume X is affine too, say $X = \mathrm{Spec}(A)$. Let $y \in Y$ correspond to the minimal prime $\mathfrak{q} \subset B$. By assumption A has a unique minimal prime $\mathfrak{p}$ lying over $\mathfrak{q}$ and $B_\mathfrak{q} \to A_\mathfrak{p}$ is an isomorphism. It follows that $A_\mathfrak{q} \to \kappa(\mathfrak{p})$ is surjective, hence $\mathfrak{p}A_\mathfrak{q}$ is a maximal ideal. On the other hand $\mathfrak{p}A_\mathfrak{q}$ is the unique minimal prime of $A_\mathfrak{q}$. We conclude that $\mathfrak{p}A_\mathfrak{q}$ is the unique prime of $A_\mathfrak{q}$ and that $A_\mathfrak{q} = A_\mathfrak{p}$. Since $A_\mathfrak{q} = A \otimes_B B_\mathfrak{q}$ the lemma follows. $\square$

Example 50.4. Here are two examples of birational morphisms which are not isomorphisms over any open of the target.

First example. Let k be an infinite field. Let $A = k[x]$. Let $B = k[x, \{y_\alpha\}_{\alpha \in k}]/((x-\alpha)y_\alpha, y_\alpha y_\beta)$. There is an inclusion $A \subset B$ and a retraction $B \to A$ setting all y_α equal to zero. Both the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ and the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are birational but not an isomorphism over any open.

Second example. Let A be a domain. Let $S \subset A$ be a multiplicative subset not containing 0. With $B = S^{-1}A$ the morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is birational. If there exists an open U of $\mathrm{Spec}(A)$ such that $f^{-1}(U) \to U$ is an isomorphism, then there exists an $a \in A$ such that each every element of S becomes invertible in the principal localization A_a. Taking $A = \mathbf{Z}$ and S the set of odd integers give a counter example.

Lemma 50.5. *Let $f : X \to Y$ be a birational morphism of schemes having finitely many irreducible components over a base scheme S. Assume one of the following conditions is satisfied*

1. *f is locally of finite type and Y reduced,*
2. *f is locally of finite presentation.*

Then there exist dense opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is an isomorphism. In particular if X and Y are irreducible, then X and Y are S-birational.

Proof. There is an immediate reduction to the case where X and Y are irreducible which we omit. Moreover, after shrinking further and we may assume X and Y are affine, say $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. By assumption A, resp. B has a unique minimal prime $\mathfrak{p}$, resp. $\mathfrak{q}$, the prime $\mathfrak{p}$ lies over $\mathfrak{q}$, and $B_\mathfrak{q} = A_\mathfrak{p}$. By Lemma 50.3 we have $B_\mathfrak{q} = A_\mathfrak{q} = A_\mathfrak{p}$.

Suppose $B \to A$ is of finite type, say $A = B[x_1, \ldots, x_n]$. There exist a $b_i \in B$ and $g_i \in B \setminus \mathfrak{q}$ such that b_i/g_i maps to the image of x_i in $A_\mathfrak{q}$. Hence $b_i - g_i x_i$ maps to zero in $A_{g_i'}$ for some $g_i' \in B \setminus \mathfrak{q}$. Setting $g = \prod g_i g_i'$ we see that $B_g \to A_g$ is surjective. If moreover Y is reduced, then the map $B_g \to B_\mathfrak{q}$ is injective and hence $B_g \to A_g$ is injective as well. This proves case (1).

Proof of (2). By the argument given in the previous paragraph we may assume that $B \to A$ is surjective. As f is locally of finite presentation the kernel $J \subset B$ is a finitely generated ideal. Say $J = (b_1, \ldots, b_r)$. Since $B_\mathfrak{q} = A_\mathfrak{q}$ there exist $g_i \in B \setminus \mathfrak{q}$ such that $g_i b_i = 0$. Setting $g = \prod g_i$ we see that $B_g \to A_g$ is an isomorphism. $\square$

Lemma 50.6. *Let S be a scheme. Let X and Y be irreducible schemes locally of finite presentation over S. Let $x \in X$ and $y \in Y$ be the generic points. The following are equivalent*

1. *X and Y are S-birational,*
2. *there exist nonempty opens of X and Y which are S-isomorphic, and*
3. *x and y map to the same point s of S and $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic as $\mathcal{O}_{S,s}$-algebras.*

Proof. We have seen the equivalence of (1) and (2) in Lemma 49.12. It is immediate that (2) implies (3). To finish we assume (3) holds and we prove (1). By Lemma 49.2 there is a rational map $f : U \to Y$ which sends $x \in U$ to y and induces the given isomorphism $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$. Thus f is a birational morphism and hence induces an isomorphism on nonempty opens by Lemma 50.5. This finishes the proof. $\square$

Lemma 50.7. *Let S be a scheme. Let X and Y be integral schemes locally of finite type over S. Let $x \in X$ and $y \in Y$ be the generic points. The following are equivalent*

1. *X and Y are S-birational,*
2. *there exist nonempty opens of X and Y which are S-isomorphic, and*
3. *x and y map to the same point $s \in S$ and $\kappa(x) \cong \kappa(y)$ as $\kappa(s)$-extensions.*

Proof. We have seen the equivalence of (1) and (2) in Lemma 49.12. It is immediate that (2) implies (3). To finish we assume (3) holds and we prove (1). Observe that $\mathcal{O}_{X,x} = \kappa(x)$ and $\mathcal{O}_{Y,y} = \kappa(y)$ by Algebra, Lemma 25.1. By Lemma 49.2 there is a rational map $f : U \to Y$ which sends $x \in U$ to y and induces the given isomorphism $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$. Thus f is a birational morphism and hence induces an isomorphism on nonempty opens by Lemma 50.5. This finishes the proof. $\square$

51. Generically finite morphisms

In this section we characterize maps between schemes which are locally of finite type and which are "generically finite" in some sense.

Lemma 51.1. *Let X, Y be schemes. Let $f : X \to Y$ be locally of finite type. Let $\eta \in Y$ be a generic point of an irreducible component of Y. The following are equivalent:*

1. *the set $f^{-1}(\{\eta\})$ is finite,*
2. *there exist affine opens $U_i \subset X$, $i = 1, \ldots, n$ and $V \subset Y$ with $f(U_i) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset \bigcup U_i$ such that each $f|_{U_i} : U_i \to V$ is finite.*

If f is quasi-separated, then these are also equivalent to

(3) *there exist affine opens $V \subset Y$, and $U \subset X$ with $f(U) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset U$ such that $f|_U : U \to V$ is finite.*

If f is quasi-compact and quasi-separated, then these are also equivalent to

(4) *there exists an affine open $V \subset Y$, $\eta \in V$ such that $f^{-1}(V) \to V$ is finite.*

Proof. The question is local on the base. Hence we may replace Y by an affine neighbourhood of η, and we may and do assume throughout the proof below that Y is affine, say $Y = \mathrm{Spec}(R)$.

It is clear that (2) implies (1). Assume that $f^{-1}(\{\eta\}) = \{\xi_1, \ldots, \xi_n\}$ is finite. Choose affine opens $U_i \subset X$ with $\xi_i \in U_i$. By Algebra, Lemma 122.10 we see that after replacing Y by a standard open in Y each of the morphisms $U_i \to Y$ is finite. In other words (2) holds.

It is clear that (3) implies (1). Assume f is quasi-separated and (1). Write $f^{-1}(\{\eta\}) = \{\xi_1, \ldots, \xi_n\}$. There are no specializations among the ξ_i by Lemma 20.7. Since each ξ_i maps to the generic point η of an irreducible component of Y, there cannot be a nontrivial specialization $\xi \rightsquigarrow \xi_i$ in X (since ξ would map to η as well). We conclude each ξ_i is a generic point of an irreducible component of X. Since Y is affine and f quasi-separated we see X is quasi-separated. By Properties, Lemma 29.1 we can find an affine open $U \subset X$ containing each ξ_i. By Algebra, Lemma 122.10 we see that after replacing Y by a standard open in Y the morphisms $U \to Y$ is finite. In other words (3) holds.

It is clear that (4) implies all of (1) – (3) with no further assumptions on f. Suppose that f is quasi-compact and quasi-separated. We have to show that the equivalent conditions (1) – (3) imply (4). Let U, V be as in (3). Replace Y by V. Since f is quasi-compact and Y is quasi-compact (being affine) we see that X is quasi-compact. Hence $Z = X \setminus U$ is quasi-compact, hence the morphism $f|_Z : Z \to Y$ is quasi-compact. By construction of Z we see that $\eta \notin f(Z)$. Hence by Lemma 8.5 we see that there exists an affine open neighbourhood V' of η in Y such that $f^{-1}(V') \cap Z = \emptyset$. Then we have $f^{-1}(V') \subset U$ and this means that $f^{-1}(V') \to V'$ is finite. $\square$

Example 51.2. Let $A = \prod_{n \in \mathbf{N}} \mathbf{F}_2$. Every element of A is an idempotent. Hence every prime ideal is maximal with residue field $\mathbf{F}_2$. Thus the topology on $X = \mathrm{Spec}(A)$ is totally disconnected and quasi-compact. The projection maps $A \to \mathbf{F}_2$ define open points of $\mathrm{Spec}(A)$. It cannot be the case that all the points of X are open since X is quasi-compact. Let $x \in X$ be a closed point which is not open. Then we can form a scheme Y which is two copies of X glued along $X \setminus \{x\}$. In other words, this is X with x doubled, compare Schemes, Example 14.3. The morphism $f : Y \to X$ is quasi-compact, finite type and has finite fibres but is not quasi-separated. The point $x \in X$ is a generic point of an irreducible component of X (since X is totally disconnected). But properties (3) and (4) of Lemma 51.1 do not hold. The reason is that for any open neighbourhood $x \in U \subset X$ the inverse image $f^{-1}(U)$ is not affine because functions on $f^{-1}(U)$ cannot separate the two points lying over x (proof omitted; this is a nice exercise). Hence the condition that f is quasi-separated is necessary in parts (3) and (4) of the lemma.

Remark 51.3. An alternative to Lemma 51.1 is the statement that a quasi-finite morphism is finite over a dense open of the target. This will be shown in More on Morphisms, Lemma 45.2.

Lemma 51.4. *Let X, Y be schemes. Let $f : X \to Y$ be locally of finite type. Let X^0, resp. Y^0 denote the set of generic points of irreducible components of X, resp. Y. Let $\eta \in Y^0$. The following are equivalent*

(1) $f^{-1}(\{\eta\}) \subset X^0$,
(2) *f is quasi-finite at all points lying over η,*
(3) *f is quasi-finite at all $\xi \in X^0$ lying over η.*

Proof. Condition (1) implies there are no specializations among the points of the fibre X_η. Hence (2) holds by Lemma 20.6. The implication (2) $\Rightarrow$ (3) is immediate. Since η is a generic point of Y, the generic points of X_η are generic points of X. Hence (3) and Lemma 20.6 imply the generic points of X_η are also closed. Thus all points of X_η are generic and we see that (1) holds. □

Lemma 51.5. *Let X, Y be schemes. Let $f : X \to Y$ be locally of finite type. Let X^0, resp. Y^0 denote the set of generic points of irreducible components of X, resp. Y. Assume*

(1) *X^0 and Y^0 are finite and $f^{-1}(Y^0) = X^0$,*
(2) *either f is quasi-compact or f is separated.*

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.

Proof. Since Y has finitely many irreducible components, we can find a dense open which is a disjoint union of its irreducible components. Thus we may assume Y is irreducible affine with generic point η. Then the fibre over η is finite as X^0 is finite.

Assume f is separated and Y irreducible affine. Choose $V \subset Y$ and $U \subset X$ as in Lemma 51.1 part (3). Since $f|_U : U \to V$ is finite, we see that $U \subset f^{-1}(V)$ is closed as well as open (Lemmas 41.7 and 44.11). Thus $f^{-1}(V) = U \amalg W$ for some open subscheme W of X. However, since U contains all the generic points of X we conclude that $W = \emptyset$ as desired.

Assume f is quasi-compact and Y irreducible affine. Then X is quasi-compact, hence there exists a dense open subscheme $U \subset X$ which is separated (Properties, Lemma 29.3). Since the set of generic points X^0 is finite, we see that $X^0 \subset U$. Thus $\eta \notin f(X \setminus U)$. Since $X \setminus U \to Y$ is quasi-compact, we conclude that there is a nonempty open $V \subset Y$ such that $f^{-1}(V) \subset U$, see Lemma 8.3. After replacing X by $f^{-1}(V)$ and Y by V we reduce to the separated case which we dealt with in the preceding paragraph. □

Lemma 51.6. *Let X, Y be schemes. Let $f : X \to Y$ be a birational morphism between schemes which have finitely many irreducible components. Assume*

(1) *either f is quasi-compact or f is separated, and*
(2) *either f is locally of finite type and Y is reduced or f is locally of finite presentation.*

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is an isomorphism.

Proof. By Lemma 51.5 we may assume that f is finite. Since Y has finitely many irreducible components, we can find a dense open which is a disjoint union of its irreducible components. Thus we may assume Y is irreducible. By Lemma 50.5 we find a nonempty open $U \subset X$ such that $f|_U : U \to Y$ is an open immersion. After removing the closed (as f finite) subset $f(X \setminus U)$ from Y we see that f is an isomorphism. □

Lemma 51.7. *Let X, Y be integral schemes. Let $f : X \to Y$ be locally of finite type. Assume f is dominant. The following are equivalent:*

1. *the extension $R(Y) \subset R(X)$ has transcendence degree 0,*
2. *the extension $R(Y) \subset R(X)$ is finite,*
3. *there exist nonempty affine opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is finite, and*
4. *the generic point of X is the only point of X mapping to the generic point of Y.*

If f is separated or if f is quasi-compact, then these are also equivalent to

5. *there exists a nonempty affine open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.*

Proof. Choose any affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset Y$ such that $f(U) \subset V$. Then R and A are domains by definition. The ring map $R \to A$ is of finite type (Lemma 15.2). By Lemma 8.6 the generic point of X maps to the generic point of Y hence $R \to A$ is injective. Let $K = R(Y)$ be the fraction field of R and $L = R(X)$ the fraction field of A. Then L/K is a finitely generated field extension. Hence we see that (1) is equivalent to (2).

Suppose (2) holds. Let $x_1, \ldots, x_n \in A$ be generators of A over R. By assumption there exist nonzero polynomials $P_i(X) \in R[X]$ such that $P_i(x_i) = 0$. Let $f_i \in R$ be the leading coefficient of P_i. Then we conclude that $R_{f_1 \ldots f_n} \to A_{f_1 \ldots f_n}$ is finite, i.e., (3) holds. Note that (3) implies (2). So now we see that (1), (2) and (3) are all equivalent.

Let η be the generic point of X, and let $\eta' \in Y$ be the generic point of Y. Assume (4). Then $\dim_\eta(X_{\eta'}) = 0$ and we see that $R(X) = \kappa(\eta)$ has transcendence degree 0 over $R(Y) = \kappa(\eta')$ by Lemma 28.1. In other words (1) holds. Assume the equivalent conditions (1), (2) and (3). Suppose that $x \in X$ is a point mapping to η'. As x is a specialization of η, this gives inclusions $R(Y) \subset \mathcal{O}_{X,x} \subset R(X)$, which implies $\mathcal{O}_{X,x}$ is a field, see Algebra, Lemma 36.19. Hence $x = \eta$. Thus we see that (1) – (4) are all equivalent.

It is clear that (5) implies (3) with no additional assumptions on f. What remains is to prove that if f is either separated or quasi-compact, then the equivalent conditions (1) – (4) imply (5). This follows from Lemma 51.5. $\square$

Definition 51.8. Let X and Y be integral schemes. Let $f : X \to Y$ be locally of finite type and dominant. Assume $[R(X) : R(Y)] < \infty$, or any other of the equivalent conditions (1) – (4) of Lemma 51.7. Then the positive integer

$$\deg(X/Y) = [R(X) : R(Y)]$$

is called the *degree of X over Y*.

It is possible to extend this notion to a morphism $f : X \to Y$ if (a) Y is integral with generic point η, (b) f is locally of finite type, and (c) $f^{-1}(\{\eta\})$ is finite. In this case we can define

$$\deg(X/Y) = \sum\nolimits_{\xi \in X,\ f(\xi) = \eta} \dim_{R(Y)}(\mathcal{O}_{X,\xi}).$$

Namely, given that $R(Y) = \kappa(\eta) = \mathcal{O}_{Y,\eta}$ (Lemma 49.5) the dimensions above are finite by Lemma 51.1 above. However, for most applications the definition given above is the right one.

Lemma 51.9. *Let X, Y, Z be integral schemes. Let $f : X \to Y$ and $g : Y \to Z$ be dominant morphisms locally of finite type. Assume that $[R(X) : R(Y)] < \infty$ and $[R(Y) : R(Z)] < \infty$. Then*

$$\deg(X/Z) = \deg(X/Y)\deg(Y/Z).$$

Proof. This comes from the multiplicativity of degrees in towers of finite extensions of fields, see Fields, Lemma 7.7. □

Remark 51.10. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. There are (at least) two properties that we could use to define *generically finite* morphisms. These correspond to whether you want the property to be local on the source or local on the target:

(1) (Local on the target; suggested by Ravi Vakil.) Assume every quasi-compact open of Y has finitely many irreducible components (for example if Y is locally Noetherian). The requirement is that the inverse image of each generic point is finite, see Lemma 51.1.
(2) (Local on the source.) The requirement is that there exists a dense open $U \subset X$ such that $U \to Y$ is locally quasi-finite.

In case (1) the requirement can be formulated without the auxiliary condition on Y, but probably doesn't give the right notion for general schemes. Property (2) as formulated doesn't imply that the fibres over generic points are finite; however, if f is quasi-compact and Y is as in (1) then it does.

Definition 51.11. Let X be an integral scheme. A *modification of* X is a birational proper morphism $f : X' \to X$ with X' integral.

Let $f : X' \to X$ be a modification as in the definition. By Lemma 51.7 there exists a nonempty $U \subset X$ such that $f^{-1}(U) \to U$ is finite. By generic flatness (Proposition 27.1) we may assume $f^{-1}(U) \to U$ is flat and of finite presentation. So $f^{-1}(U) \to U$ is finite locally free (Lemma 48.2). Since f is birational, the degree of X' over X is 1. Hence $f^{-1}(U) \to U$ is finite locally free of degree 1, in other words it is an isomorphism. Thus we can *redefine* a modification to be a proper morphism $f : X' \to X$ of integral schemes such that $f^{-1}(U) \to U$ is an isomorphism for some nonempty open $U \subset X$.

Definition 51.12. Let X be an integral scheme. An *alteration of* X is a proper dominant morphism $f : Y \to X$ with Y integral such that $f^{-1}(U) \to U$ is finite for some nonempty open $U \subset X$.

This is the definition as given in [dJ96], except that here we do not require X and Y to be Noetherian. Arguing as above we see that an alteration is a proper dominant morphism $f : Y \to X$ of integral schemes which induces a finite extension of function fields, i.e., such that the equivalent conditions of Lemma 51.7 hold.

52. The dimension formula

For morphisms between Noetherian schemes we can say a little more about dimensions of local rings. Here is an important (and not so hard to prove) result. Recall that $R(X)$ denotes the function field of an integral scheme X.

Lemma 52.1. *Let S be a scheme. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$, and set $s = f(x)$. Assume*

(1) *S is locally Noetherian,*
(2) *f is locally of finite type,*
(3) *X and S integral, and*
(4) *f dominant.*

We have

$$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{S,s}) + \mathit{trdeg}_{R(S)} R(X) - \mathit{trdeg}_{\kappa(s)}\kappa(x). \tag{52.1.1}$$

Moreover, equality holds if S is universally catenary.

Proof. The corresponding algebra statement is Algebra, Lemma 113.1. □

Lemma 52.2. *Let S be a scheme. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$, and set $s = f(x)$. Assume S is locally Noetherian and f is locally of finite type, We have*

$$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{S,s}) + E - \mathit{trdeg}_{\kappa(s)}\kappa(x). \tag{52.2.1}$$

where E is the maximum of $\mathit{trdeg}_{\kappa(f(\xi))}(\kappa(\xi))$ where ξ runs over the generic points of irreducible components of X containing x.

Proof. Let $X_1, \ldots, X_n$ be the irreducible components of X containing x endowed with their reduced induced scheme structure. These correspond to the minimal primes $\mathfrak{q}_i$ of $\mathcal{O}_{X,x}$ and hence there are finitely many of them (Schemes, Lemma 13.2 and Algebra, Lemma 31.6). Then $\dim(\mathcal{O}_{X,x}) = \max \dim(\mathcal{O}_{X,x}/\mathfrak{q}_i) = \max \dim(\mathcal{O}_{X_i,x})$. The ξ's occurring in the definition of E are exactly the generic points $\xi_i \in X_i$. Let $Z_i = \overline{\{f(\xi_i)\}} \subset S$ endowed with the reduced induced scheme structure. The composition $X_i \to X \to S$ factors through Z_i (Schemes, Lemma 12.7). Thus we may apply the dimension formula (Lemma 52.1) to see that $\dim(\mathcal{O}_{X_i,x}) \leq \dim(\mathcal{O}_{Z_i,x}) + \mathrm{trdeg}_{\kappa(f(\xi))}(\kappa(\xi)) - \mathrm{trdeg}_{\kappa(s)}\kappa(x)$. Putting everything together we obtain the lemma. □

An application is the construction of a dimension function on any scheme of finite type over a universally catenary scheme endowed with a dimension function. For the definition of dimension functions, see Topology, Definition 20.1.

Lemma 52.3. *Let S be a locally Noetherian and universally catenary scheme. Let $\delta : S \to \mathbf{Z}$ be a dimension function. Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Then the map*

$$\begin{aligned} \delta = \delta_{X/S} : X &\longrightarrow \mathbf{Z} \\ x &\longmapsto \delta(f(x)) + \mathit{trdeg}_{\kappa(f(x))}\kappa(x) \end{aligned}$$

is a dimension function on X.

Proof. Let $f : X \to S$ be locally of finite type. Let $x \rightsquigarrow y$, $x \neq y$ be a specialization in X. We have to show that $\delta_{X/S}(x) > \delta_{X/S}(y)$ and that $\delta_{X/S}(x) = \delta_{X/S}(y) + 1$ if y is an immediate specialization of x.

Choose an affine open $V \subset S$ containing the image of y and choose an affine open $U \subset X$ mapping into V and containing y. We may clearly replace X by U and S by V. Thus we may assume that $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ and that f is given by a ring map $R \to A$. The ring R is universally catenary (Lemma 17.2) and the map $R \to A$ is of finite type (Lemma 15.2).

Let $\mathfrak{q} \subset A$ be the prime ideal corresponding to the point x and let $\mathfrak{p} \subset R$ be the prime ideal corresponding to $f(x)$. The restriction δ' of δ to $S' = \mathrm{Spec}(R/\mathfrak{p}) \subset S$ is a dimension function. The ring $R/\mathfrak{p}$ is universally catenary. The restriction of $\delta_{X/S}$ to $X' = \mathrm{Spec}(A/\mathfrak{q})$ is clearly equal to the function $\delta_{X'/S'}$ constructed using the dimension function δ'. Hence we may assume in addition to the above that $R \subset A$ are domains, in other words that X and S are integral schemes, and that x is the generic point of X and $f(x)$ is the generic point of S.

Note that $\mathcal{O}_{X,x} = R(X)$ and that since $x \rightsquigarrow y$, $x \neq y$, the spectrum of $\mathcal{O}_{X,y}$ has at least two points (Schemes, Lemma 13.2) hence $\dim(\mathcal{O}_{X,y}) > 0$. If y is an immediate specialization of x, then $\mathrm{Spec}(\mathcal{O}_{X,y}) = \{x, y\}$ and $\dim(\mathcal{O}_{X,y}) = 1$.

Write $s = f(x)$ and $t = f(y)$. We compute

$$\begin{aligned}
\delta_{X/S}(x) - \delta_{X/S}(y) &= \delta(s) + \mathrm{trdeg}_{\kappa(s)}\kappa(x) - \delta(t) - \mathrm{trdeg}_{\kappa(t)}\kappa(y) \\
&= \delta(s) - \delta(t) + \mathrm{trdeg}_{R(S)}R(X) - \mathrm{trdeg}_{\kappa(t)}\kappa(y) \\
&= \delta(s) - \delta(t) + \dim(\mathcal{O}_{X,y}) - \dim(\mathcal{O}_{S,t})
\end{aligned}$$

where we use equality in (52.1.1) in the last step. Since δ is a dimension function on the scheme S and $s \in S$ is the generic point, the difference $\delta(s) - \delta(t)$ is equal to $\mathrm{codim}(\overline{\{t\}}, S)$ by Topology, Lemma 20.2. This is equal to $\dim(\mathcal{O}_{S,t})$ by Properties, Lemma 10.3. Hence we conclude that

$$\delta_{X/S}(x) - \delta_{X/S}(y) = \dim(\mathcal{O}_{X,y})$$

and the lemma follows from what we said above about $\dim(\mathcal{O}_{X,y})$. □

Another application of the dimension formula is that the dimension does not change under "alterations" (to be defined later).

Lemma 52.4. *Let $f : X \to Y$ be a morphism of schemes. Assume that*

1. *Y is locally Noetherian,*
2. *X and Y are integral schemes,*
3. *f is dominant, and*
4. *f is locally of finite type.*

Then we have

$$\dim(X) \leq \dim(Y) + \mathit{trdeg}_{R(Y)}R(X).$$

If f is closed[16] *then equality holds.*

Proof. Let $f : X \to Y$ be as in the lemma. Let $\xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \ldots \rightsquigarrow \xi_e$ be a sequence of specializations in X. Set $x = \xi_e$ and $y = f(x)$. Observe that $e \leq \dim(\mathcal{O}_{X,x})$ as the given specializations occur in the spectrum of $\mathcal{O}_{X,x}$, see Schemes, Lemma 13.2. By the dimension formula, Lemma 52.1, we see that

$$\begin{aligned}
e &\leq \dim(\mathcal{O}_{X,x}) \\
&\leq \dim(\mathcal{O}_{Y,y}) + \mathrm{trdeg}_{R(Y)}R(X) - \mathrm{trdeg}_{\kappa(y)}\kappa(x) \\
&\leq \dim(\mathcal{O}_{Y,y}) + \mathrm{trdeg}_{R(Y)}R(X)
\end{aligned}$$

Hence we conclude that $e \leq \dim(Y) + \mathrm{trdeg}_{R(Y)}R(X)$ as desired.

Next, assume f is also closed. Say $\overline{\xi}_0 \rightsquigarrow \overline{\xi}_1 \rightsquigarrow \ldots \rightsquigarrow \overline{\xi}_d$ is a sequence of specializations in Y. We want to show that $\dim(X) \geq d + r$. We may assume that $\overline{\xi}_0 = \eta$

[16] For example if f is proper, see Definition 41.1.

is the generic point of Y. The generic fibre X_η is a scheme locally of finite type over $\kappa(\eta) = R(Y)$. It is nonempty as f is dominant. Hence by Lemma 16.10 it is a Jacobson scheme. Thus by Lemma 16.8 we can find a closed point $\xi_0 \in X_\eta$ and the extension $\kappa(\eta) \subset \kappa(\xi_0)$ is a finite extension. Note that $\mathcal{O}_{X,\xi_0} = \mathcal{O}_{X_\eta,\xi_0}$ because η is the generic point of Y. Hence we see that $\dim(\mathcal{O}_{X,\xi_0}) = r$ by Lemma 52.1 applied to the scheme X_η over the universally catenary scheme $\mathrm{Spec}(\kappa(\eta))$ (see Lemma 17.5) and the point ξ_0. This means that we can find $\xi_{-r} \leadsto \ldots \leadsto \xi_{-1} \leadsto \xi_0$ in X. On the other hand, as f is closed specializations lift along f, see Topology, Lemma 19.7. Thus, as ξ_0 lies over $\eta = \overline{\xi}_0$ we can find specializations $\xi_0 \leadsto \xi_1 \leadsto \ldots \leadsto \xi_d$ lying over $\overline{\xi}_0 \leadsto \overline{\xi}_1 \leadsto \ldots \leadsto \overline{\xi}_d$. In other words we have

$$\xi_{-r} \leadsto \ldots \leadsto \xi_{-1} \leadsto \xi_0 \leadsto \xi_1 \leadsto \ldots \leadsto \xi_d$$

which means that $\dim(X) \geq d + r$ as desired. □

Lemma 52.5. *Let $f : X \to Y$ be a morphism of schemes. Assume that Y is locally Noetherian and f is locally of finite type. Then*

$$\dim(X) \leq \dim(Y) + E$$

where E is the supremum of $trdeg_{\kappa(f(\xi))}(\kappa(\xi))$ where ξ runs through the generic points of the irreducible components of X.

Proof. Immediate consequence of Lemma 52.2 and Properties, Lemma 10.2. □

53. Relative normalization

In this section we construct the *normalization of one scheme in another.*

Lemma 53.1. *Let X be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The subsheaf $\mathcal{A}' \subset \mathcal{A}$ defined by the rule*

$$U \longmapsto \{f \in \mathcal{A}(U) \mid f_x \in \mathcal{A}_x \textit{ integral over } \mathcal{O}_{X,x} \textit{ for all } x \in U\}$$

is a quasi-coherent $\mathcal{O}_X$-algebra, the stalk $\mathcal{A}'_x$ is the integral closure of $\mathcal{O}_{X,x}$ in $\mathcal{A}_x$, and for any affine open $U \subset X$ the ring $\mathcal{A}'(U) \subset \mathcal{A}(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{A}(U)$.

Proof. This is a subsheaf by the local nature of the conditions. It is an $\mathcal{O}_X$-algebra by Algebra, Lemma 36.7. Let $U \subset X$ be an affine open. Say $U = \mathrm{Spec}(R)$ and say $\mathcal{A}$ is the quasi-coherent sheaf associated to the R-algebra A. Then according to Algebra, Lemma 36.12 the value of $\mathcal{A}'$ over U is given by the integral closure A' of R in A. This proves the last assertion of the lemma. To prove that $\mathcal{A}'$ is quasi-coherent, it suffices to show that $\mathcal{A}'(D(f)) = A'_f$. This follows from the fact that integral closure and localization commute, see Algebra, Lemma 36.11. The same fact shows that the stalks are as advertised. □

Definition 53.2. Let X be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *integral closure of $\mathcal{O}_X$ in $\mathcal{A}$* is the quasi-coherent $\mathcal{O}_X$-subalgebra $\mathcal{A}' \subset \mathcal{A}$ constructed in Lemma 53.1 above.

In the setting of the definition above we can consider the morphism of relative spectra

$$\begin{tikzcd} Y = \underline{\mathrm{Spec}}_X(\mathcal{A}) \arrow[rr] \arrow[dr] & & X' = \underline{\mathrm{Spec}}_X(\mathcal{A}') \arrow[dl] \\ & X & \end{tikzcd}$$

see Lemma 11.5. The scheme $X' \to X$ will be the normalization of X in the scheme Y. Here is a slightly more general setting. Suppose we have a quasi-compact and quasi-separated morphism $f : Y \to X$ of schemes. In this case the sheaf of $\mathcal{O}_X$-algebras $f_*\mathcal{O}_Y$ is quasi-coherent, see Schemes, Lemma 24.1. Taking the integral closure $\mathcal{O}' \subset f_*\mathcal{O}_Y$ we obtain a quasi-coherent sheaf of $\mathcal{O}_X$-algebras whose relative spectrum is the normalization of X in Y. Here is the formal definition.

Definition 53.3. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The *normalization of X in Y* is the scheme[17]

$$\nu : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}') \to X$$

over X. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

The factorization is the composition of the canonical morphism $Y \to \underline{\mathrm{Spec}}(f_*\mathcal{O}_Y)$ (see Constructions, Lemma 4.7) and the morphism of relative spectra coming from the inclusion map $\mathcal{O}' \to f_*\mathcal{O}_Y$. We can characterize the normalization as follows.

Lemma 53.4. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:*

(1) *the morphism ν is integral, and*
(2) *for any factorization $f = \pi \circ g$, with $\pi : Z \to X$ integral, there exists a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f'"'] & Z \arrow[d, "\pi"] \\ X' \arrow[r, "\nu"] \arrow[ur, "h"] & X \end{tikzcd}$$

for some unique morphism $h : X' \to Z$.

Moreover, the morphism $f' : Y \to X'$ is dominant and in (2) the morphism $h : X' \to Z$ is the normalization of Z in Y.

Proof. Let $\mathcal{O}' \subset f_*\mathcal{O}_Y$ be the integral closure of $\mathcal{O}_X$ as in Definition 53.3. The morphism ν is integral by construction, which proves (1). Assume given a factorization $f = \pi \circ g$ with $\pi : Z \to X$ integral as in (2). By Definition 44.1 π is affine, and hence Z is the relative spectrum of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{B}$. The morphism $g : Y \to Z$ corresponds to a map of $\mathcal{O}_X$-algebras $\chi : \mathcal{B} \to f_*\mathcal{O}_Y$. Since $\mathcal{B}(U)$ is integral over $\mathcal{O}_X(U)$ for every affine open $U \subset X$ (by Definition 44.1) we see from Lemma 53.1 that $\chi(\mathcal{B}) \subset \mathcal{O}'$. By the functoriality of the relative spectrum

[17]The scheme X' need not be normal, for example if $Y = X$ and $f = \mathrm{id}_X$, then $X' = X$.

Lemma 11.5 this provides us with a unique morphism $h : X' \to Z$. We omit the verification that the diagram commutes.

It is clear that (1) and (2) characterize the factorization $f = \nu \circ f'$ since it characterizes it as an initial object in a category.

From the universal property in (2) we see that f' does not factor through a proper closed subscheme of X'. Hence the scheme theoretic image of f' is X'. Since f' is quasi-compact (by Schemes, Lemma 21.14 and the fact that ν is separated as an affine morphism) we see that $f'(Y)$ is dense in X'. Hence f' is dominant.

The morphism h in (2) is integral by Lemma 44.14. Given a factorization $g = \pi' \circ g'$ with $\pi' : Z' \to Z$ integral, we get a factorization $f = (\pi \circ \pi') \circ g'$ and we get a morphism $h' : X' \to Z'$. Uniqueness implies that $\pi' \circ h' = h$. Hence the characterization (1), (2) applies to the morphism $h : X' \to Z$ which gives the last statement of the lemma. □

Lemma 53.5. *Let*

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d, "f_2"'] & Y_1 \arrow[d, "f_1"] \\ X_2 \arrow[r] & X_1 \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume f_1, f_2 quasi-compact and quasi-separated. Let $f_i = \nu_i \circ f_i'$, $i = 1, 2$ be the canonical factorizations, where $\nu_i : X_i' \to X_i$ is the normalization of X_i in Y_i. Then there exists a unique arrow $X_2' \to X_1'$ fitting into a commutative diagram

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d, "f_2'"'] & Y_1 \arrow[d, "f_1'"] \\ X_2' \arrow[r] \arrow[d, "\nu_2"'] & X_1' \arrow[d, "\nu_1"] \\ X_2 \arrow[r] & X_1 \end{tikzcd}$$

Proof. By Lemmas 53.4 (1) and 44.6 the base change $X_2 \times_{X_1} X_1' \to X_2$ is integral. Note that f_2 factors through this morphism. Hence we get a unique morphism $X_2' \to X_2 \times_{X_1} X_1'$ from Lemma 53.4 (2). This gives the arrow $X_2' \to X_1'$ fitting into the commutative diagram and uniqueness follows as well. □

Lemma 53.6. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $U \subset X$ be an open subscheme and set $V = f^{-1}(U)$. Then the normalization of U in V is the inverse image of U in the normalization of X in Y.*

Proof. Clear from the construction. □

Lemma 53.7. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let X' be the normalization of X in Y. Then the normalization of X' in Y is X'.*

Proof. If $Y \to X'' \to X'$ is the normalization of X' in Y, then we can apply Lemma 53.4 to the composition $X'' \to X$ to get a canonical morphism $h : X' \to X''$ over

X. We omit the verification that the morphisms h and $X'' \to X'$ are mutually inverse (using uniqueness of the factorization in the lemma). $\square$

Lemma 53.8. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $X' \to X$ be the normalization of X in Y. If Y is reduced, so is X'.*

Proof. This follows from the fact that a subring of a reduced ring is reduced. Some details omitted. $\square$

Lemma 53.9. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $X' \to X$ be the normalization of X in Y. Every generic point of an irreducible component of X' is the image of a generic point of an irreducible component of Y.*

Proof. By Lemma 53.6 we may assume $X = \mathrm{Spec}(A)$ is affine. Choose a finite affine open covering $Y = \bigcup \mathrm{Spec}(B_i)$. Then $X' = \mathrm{Spec}(A')$ and the morphisms $\mathrm{Spec}(B_i) \to Y \to X'$ jointly define an injective A-algebra map $A' \to \prod B_i$. Thus the lemma follows from Algebra, Lemma 30.5. $\square$

Lemma 53.10. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Suppose that $Y = Y_1 \amalg Y_2$ is a disjoint union of two schemes. Write $f_i = f|_{Y_i}$. Let X'_i be the normalization of X in Y_i. Then $X'_1 \amalg X'_2$ is the normalization of X in Y.*

Proof. In terms of integral closures this corresponds to the following fact: Let $A \to B$ be a ring map. Suppose that $B = B_1 \times B_2$. Let A'_i be the integral closure of A in B_i. Then $A'_1 \times A'_2$ is the integral closure of A in B. The reason this works is that the elements $(1, 0)$ and $(0, 1)$ of B are idempotents and hence integral over A. Thus the integral closure A' of A in B is a product and it is not hard to see that the factors are the integral closures A'_i as described above (some details omitted). $\square$

Lemma 53.11. *Let $f : X \to S$ be a quasi-compact, quasi-separated and universally closed morphisms of schemes. Then $f_*\mathcal{O}_X$ is integral over $\mathcal{O}_S$. In other words, the normalization of S in X is equal to the factorization*

$$X \longrightarrow \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X) \longrightarrow S$$

of Constructions, Lemma 4.7.

Proof. The question is local on S, hence we may assume $S = \mathrm{Spec}(R)$ is affine. Let $h \in \Gamma(X, \mathcal{O}_X)$. We have to show that h satisfies a monic equation over R. Think of h as a morphism as in the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "h"'] \arrow[dr, "f"'] & & \mathbf{A}^1_S \arrow[dl] \\ & S & \end{tikzcd}$$

Let $Z \subset \mathbf{A}^1_S$ be the scheme theoretic image of h, see Definition 6.2. The morphism h is quasi-compact as f is quasi-compact and $\mathbf{A}^1_S \to S$ is separated, see Schemes, Lemma 21.14. By Lemma 6.3 the morphism $X \to Z$ is dominant. By Lemma 41.7 the morphism $X \to Z$ is closed. Hence $h(X) = Z$ (set theoretically). Thus we can use Lemma 41.9 to conclude that $Z \to S$ is universally closed (and even proper). Since $Z \subset \mathbf{A}^1_S$, we see that $Z \to S$ is affine and proper, hence integral by Lemma

44.7. Writing $\mathbf{A}^1_S = \mathrm{Spec}(R[T])$ we conclude that the ideal $I \subset R[T]$ of Z contains a monic polynomial $P(T) \in R[T]$. Hence $P(h) = 0$ and we win. $\square$

Lemma 53.12. *Let $f : Y \to X$ be an integral morphism. Then the normalization of X in Y is equal to Y.*

Proof. By Lemma 44.7 this is a special case of Lemma 53.11. $\square$

Lemma 53.13. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let X' be the normalization of X in Y. Assume*

1. *Y is a normal scheme,*
2. *quasi-compact opens of Y have finitely many irreducible components.*

Then X' is a disjoint union of integral normal schemes. Moreover, the morphism $Y \to X'$ is dominant and induces a bijection of irreducible components.

Proof. Let $U \subset X$ be an affine open. Consider the inverse image U' of U in X'. Set $V = f^{-1}(U)$. By Lemma 53.6 we $V \to U' \to U$ is the normalization of U in V. Say $U = \mathrm{Spec}(A)$. Then V is quasi-compact, and hence has a finite number of irreducible components by assumption. Hence $V = \coprod_{i=1,\ldots n} V_i$ is a finite disjoint union of normal integral schemes by Properties, Lemma 7.5. By Lemma 53.10 we see that $U' = \coprod_{i=1,\ldots,n} U'_i$, where U'_i is the normalization of U in V_i. By Properties, Lemma 7.9 we see that $B_i = \Gamma(V_i, \mathcal{O}_{V_i})$ is a normal domain. Note that $U'_i = \mathrm{Spec}(A'_i)$, where $A'_i \subset B_i$ is the integral closure of A in B_i, see Lemma 53.1. By Algebra, Lemma 37.2 we see that $A'_i \subset B_i$ is a normal domain. Hence $U' = \coprod U'_i$ is a finite union of normal integral schemes and hence is normal.

As X' has an open covering by the schemes U' we conclude from Properties, Lemma 7.2 that X' is normal. On the other hand, each U' is a finite disjoint union of irreducible schemes, hence every quasi-compact open of X' has finitely many irreducible components (by a topological argument which we omit). Thus X' is a disjoint union of normal integral schemes by Properties, Lemma 7.5. It is clear from the description of X' above that $Y \to X'$ is dominant and induces a bijection on irreducible components $V \to U'$ for every affine open $U \subset X$. The bijection of irreducible components for the morphism $Y \to X'$ follows from this by a topological argument (omitted). $\square$

Lemma 53.14. *Let $f : X \to S$ be a morphism. Assume that*

1. *S is a Nagata scheme,*
2. *f is quasi-compact and quasi-separated,*
3. *quasi-compact opens of X have finitely many irreducible components,*
4. *if $x \in X$ is a generic point of an irreducible component, then the field extension $\kappa(x)/\kappa(f(x))$ is finitely generated, and*
5. *X is reduced.*

Then the normalization $\nu : S' \to S$ of S in X is finite.

Proof. There is an immediate reduction to the case $S = \mathrm{Spec}(R)$ where R is a Nagata ring by assumption (1). We have to show that the integral closure A of R in $\Gamma(X, \mathcal{O}_X)$ is finite over R. Since f is quasi-compact by assumption (2) we can write $X = \bigcup_{i=1,\ldots,n} U_i$ with each U_i affine. Say $U_i = \mathrm{Spec}(B_i)$. Each B_i is reduced by assumption (5) and has finitely many minimal primes $\mathfrak{q}_{i1}, \ldots, \mathfrak{q}_{im_i}$ by assumption (3) and Algebra, Lemma 26.1. We have

$$\Gamma(X, \mathcal{O}_X) \subset B_1 \times \ldots \times B_n \subset \prod\nolimits_{i=1,\ldots,n} \prod\nolimits_{j=1,\ldots,m_i} (B_i)_{\mathfrak{q}_{ij}}$$

the second inclusion by Algebra, Lemma 25.2. We have $\kappa(\mathfrak{q}_{ij}) = (B_i)_{\mathfrak{q}_{ij}}$ by Algebra, Lemma 25.1. Hence the integral closure A of R in $\Gamma(X, \mathcal{O}_X)$ is contained in the product of the integral closures A_{ij} of R in $\kappa(\mathfrak{q}_{ij})$. Since R is Noetherian it suffices to show that A_{ij} is a finite R-module for each i, j. Let $\mathfrak{p}_{ij} \subset R$ be the image of $\mathfrak{q}_{ij}$. As $\kappa(\mathfrak{q}_{ij})/\kappa(\mathfrak{p}_{ij})$ is a finitely generated field extension by assumption (4), we see that $R \to \kappa(\mathfrak{q}_{ij})$ is essentially of finite type. Thus $R \to A_{ij}$ is finite by Algebra, Lemma 162.2. ☐

Lemma 53.15. *Let $f : X \to S$ be a morphism. Assume that*

(1) *S is a Nagata scheme,*
(2) *f is of finite type,*
(3) *X is reduced.*

Then the normalization $\nu : S' \to S$ of S in X is finite.

Proof. This is a special case of Lemma 53.14. Namely, (2) holds as the finite type morphism f is quasi-compact by definition and quasi-separated by Lemma 15.7. Condition (3) holds because X is locally Noetherian by Lemma 15.6. Finally, condition (4) holds because a finite type morphism induces finitely generated residue field extensions. ☐

Lemma 53.16. *Let $f : Y \to X$ be a finite type morphism of schemes with Y reduced and X Nagata. Let X' be the normalization of X in Y. Let $x' \in X'$ be a point such that*

(1) *$\dim(\mathcal{O}_{X',x'}) = 1$, and*
(2) *the fibre of $Y \to X'$ over x' is empty.*

Then $\mathcal{O}_{X',x'}$ is a discrete valuation ring.

Proof. We can replace X by an affine neighbourhood of the image of x'. Hence we may assume $X = \text{Spec}(A)$ with A Nagata. By Lemma 53.15 the morphism $X' \to X$ is finite. Hence we can write $X' = \text{Spec}(A')$ for a finite A-algebra A'. By Lemma 53.7 after replacing X by X' we reduce to the case described in the next paragraph.

The case $X = X' = \text{Spec}(A)$ with A Noetherian. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to our point x'. Choose $g \in \mathfrak{p}$ not contained in any minimal prime of A (use prime avoidance and the fact that A has finitely many minimal primes, see Algebra, Lemmas 15.2 and 31.6). Set $Z = f^{-1}V(g) \subset Y$; it is a closed subscheme of Y. Then $f(Z)$ does not contain any generic point by choice of g and does not contain x' because x' is not in the image of f. The closure of $f(Z)$ is the set of specializations of points of $f(Z)$ by Lemma 6.5. Thus the closure of $f(Z)$ does not contain x' because the condition $\dim(\mathcal{O}_{X',x'}) = 1$ implies only the generic points of $X = X'$ specialize to x'. In other words, after replacing X by an affine open neighbourhood of x' we may assume that $f^{-1}V(g) = \emptyset$. Thus g maps to an invertible global function on Y and we obtain a factorization

$$A \to A_g \to \Gamma(Y, \mathcal{O}_Y)$$

Since $X = X'$ this implies that A is equal to the integral closure of A in A_g. By Algebra, Lemma 36.11 we conclude that $A_\mathfrak{p}$ is the integral closure of $A_\mathfrak{p}$ in $A_\mathfrak{p}[1/g]$. By our choice of g, since $\dim(A_\mathfrak{p}) = 1$ and since A is reduced we see that $A_\mathfrak{p}[1/g]$ is a finite product of fields (the product of the residue fields of the minimal primes contained in $\mathfrak{p}$). Hence $A_\mathfrak{p}$ is normal (Algebra, Lemma 37.16) and the proof is complete. Some details omitted. $\square$

54. Normalization

Next, we come to the normalization of a scheme X. We only define/construct it when X has locally finitely many irreducible components. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $X^{(0)} \subset X$ be the set of generic points of irreducible components of X. Let

$$f : Y = \coprod\nolimits_{\eta \in X^{(0)}} \mathrm{Spec}(\kappa(\eta)) \longrightarrow X \tag{54.0.1}$$

be the inclusion of the generic points into X using the canonical maps of Schemes, Section 13. Note that this morphism is quasi-compact by assumption and quasi-separated as Y is separated (see Schemes, Section 21).

Definition 54.1. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We define the *normalization* of X as the morphism

$$\nu : X^\nu \longrightarrow X$$

which is the normalization of X in the morphism $f : Y \to X$ (54.0.1) constructed above.

Any locally Noetherian scheme has a locally finite set of irreducible components and the definition applies to it. Usually the normalization is defined only for reduced schemes. With the definition above the normalization of X is the same as the normalization of the reduction X_{red} of X.

Lemma 54.2. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. The normalization morphism ν factors through the reduction X_{red} and $X^\nu \to X_{red}$ is the normalization of X_{red}.*

Proof. Let $f : Y \to X$ be the morphism (54.0.1). We get a factorization $Y \to X_{red} \to X$ of f from Schemes, Lemma 12.7. By Lemma 53.4 we obtain a canonical morphism $X^\nu \to X_{red}$ and that X^ν is the normalization of X_{red} in Y. The lemma follows as $Y \to X_{red}$ is identical to the morphism (54.0.1) constructed for X_{red}. $\square$

If X is reduced, then the normalization of X is the same as the relative spectrum of the integral closure of $\mathcal{O}_X$ in the sheaf of meromorphic functions $\mathcal{K}_X$ (see Divisors, Section 23). Namely, $\mathcal{K}_X = f_*\mathcal{O}_Y$ in this case, see Divisors, Lemma 25.1 and its proof. We describe this here explicitly.

Lemma 54.3. *Let X be a reduced scheme such that every quasi-compact open has finitely many irreducible components. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open. Then*

1. *A has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$,*
2. *the total ring of fractions $Q(A)$ of A is $Q(A/\mathfrak{q}_1) \times \ldots \times Q(A/\mathfrak{q}_t)$,*
3. *the integral closure A' of A in $Q(A)$ is the product of the integral closures of the domains $A/\mathfrak{q}_i$ in the fields $Q(A/\mathfrak{q}_i)$, and*

(4) *$\nu^{-1}(U)$ is identified with the spectrum of A' where $\nu : X^\nu \to X$ is the normalization morphism.*

Proof. Minimal primes correspond to irreducible components (Algebra, Lemma 26.1), hence we have (1) by assumption. Then $(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_t$ because A is reduced (Algebra, Lemma 17.2). Then we have $Q(A) = \prod A_{\mathfrak{q}_i} = \prod \kappa(\mathfrak{q}_i)$ by Algebra, Lemmas 25.4 and 25.1. This proves (2). Part (3) follows from Algebra, Lemma 37.16, or Lemma 53.10. Part (4) holds because it is clear that $f^{-1}(U) \to U$ is the morphism

$$\mathrm{Spec}\left(\prod \kappa(\mathfrak{q}_i)\right) \longrightarrow \mathrm{Spec}(A)$$

where $f : Y \to X$ is the morphism (54.0.1). $\square$

Lemma 54.4. *Let X be a scheme such that every quasi-compact open has a finite number of irreducible components. Let $\nu : X^\nu \to X$ be the normalization of X. Let $x \in X$. Then the following are canonically isomorphic as $\mathcal{O}_{X,x}$-algebras*

(1) *the stalk $(\nu_*\mathcal{O}_{X^\nu})_x$,*
(2) *the integral closure of $\mathcal{O}_{X,x}$ in the total ring of fractions of $(\mathcal{O}_{X,x})_{red}$,*
(3) *the integral closure of $\mathcal{O}_{X,x}$ in the product of the residue fields of the minimal primes of $\mathcal{O}_{X,x}$ (and there are finitely many of these).*

Proof. After replacing X by an affine open neighbourhood of x we may assume that X has finitely many irreducible components and that x is contained in each of them. Then the stalk $(\nu_*\mathcal{O}_{X^\nu})_x$ is the integral closure of $A = \mathcal{O}_{X,x}$ in the product L of the residue fields of the minimal primes of A. This follows from the construction of the normalization and Lemma 53.1. Alternatively, you can use Lemma 54.3 and the fact that normalization commutes with localization (Algebra, Lemma 36.11). Since A_{red} has finitely many minimal primes (because these correspond exactly to the generic points of the irreducible components of X passing through x) we see that L is the total ring of fractions of A_{red} (Algebra, Lemma 25.4). Thus our ring is also the integral closure of A in the total ring of fractions of A_{red}. $\square$

Lemma 54.5. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components.*

(1) *The normalization X^ν is a disjoint union of integral normal schemes.*
(2) *The morphism $\nu : X^\nu \to X$ is integral, surjective, and induces a bijection on irreducible components.*
(3) *For any integral morphism $\alpha : X' \to X$ such that for $U \subset X$ quasi-compact open the inverse image $\alpha^{-1}(U)$ has finitely many irreducible components and $\alpha|_{\alpha^{-1}(U)} : \alpha^{-1}(U) \to U$ is birational*[18] *there exists a factorization $X^\nu \to X' \to X$ and $X^\nu \to X'$ is the normalization of X'.*
(4) *For any morphism $Z \to X$ with Z a normal scheme such that each irreducible component of Z dominates an irreducible component of X there exists a unique factorization $Z \to X^\nu \to X$.*

Proof. Let $f : Y \to X$ be as in (54.0.1). The scheme X^ν is a disjoint union of normal integral schemes because Y is normal and every affine open of Y has finitely

[18]This awkward formulation is necessary as we've only defined what it means for a morphism to be birational if the source and target have finitely many irreducible components. It suffices if $X'_{red} \to X_{red}$ satisfies the condition.

many irreducible components, see Lemma 53.13. This proves (1). Alternatively one can deduce (1) from Lemmas 54.2 and 54.3.

The morphism ν is integral by Lemma 53.4. By Lemma 53.13 the morphism $Y \to X^\nu$ induces a bijection on irreducible components, and by construction of Y this implies that $X^\nu \to X$ induces a bijection on irreducible components. By construction $f : Y \to X$ is dominant, hence also ν is dominant. Since an integral morphism is closed (Lemma 44.7) this implies that ν is surjective. This proves (2).

Suppose that $\alpha : X' \to X$ is as in (3). It is clear that X' satisfies the assumptions under which the normalization is defined. Let $f' : Y' \to X'$ be the morphism (54.0.1) constructed starting with X'. As α is locally birational it is clear that $Y' = Y$ and $f = \alpha \circ f'$. Hence the factorization $X^\nu \to X' \to X$ exists and $X^\nu \to X'$ is the normalization of X' by Lemma 53.4. This proves (3).

Let $g : Z \to X$ be a morphism whose domain is a normal scheme and such that every irreducible component dominates an irreducible component of X. By Lemma 54.2 we have $X^\nu = X^\nu_{red}$ and by Schemes, Lemma 12.7 $Z \to X$ factors through X_{red}. Hence we may replace X by X_{red} and assume X is reduced. Moreover, as the factorization is unique it suffices to construct it locally on Z. Let $W \subset Z$ and $U \subset X$ be affine opens such that $g(W) \subset U$. Write $U = \text{Spec}(A)$ and $W = \text{Spec}(B)$, with $g|_W$ given by $\varphi : A \to B$. We will use the results of Lemma 54.3 freely. Let $\mathfrak p_1, \ldots, \mathfrak p_t$ be the minimal primes of A. As Z is normal, we see that B is a normal ring, in particular reduced. Moreover, by assumption any minimal prime $\mathfrak q \subset B$ we have that $\varphi^{-1}(\mathfrak q)$ is a minimal prime of A. Hence if $x \in A$ is a nonzerodivisor, i.e., $x \notin \bigcup \mathfrak p_i$, then $\varphi(x)$ is a nonzerodivisor in B. Thus we obtain a canonical ring map $Q(A) \to Q(B)$. As B is normal it is equal to its integral closure in $Q(B)$ (see Algebra, Lemma 37.12). Hence we see that the integral closure $A' \subset Q(A)$ of A maps into B via the canonical map $Q(A) \to Q(B)$. Since $\nu^{-1}(U) = \text{Spec}(A')$ this gives the canonical factorization $W \to \nu^{-1}(U) \to U$ of $\nu|_W$. We omit the verification that it is unique. $\square$

Lemma 54.6. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $Z_i \subset X$, $i \in I$ be the irreducible components of X endowed with the reduced induced structure. Let $Z_i^\nu \to Z_i$ be the normalization. Then $\coprod_{i \in I} Z_i^\nu \to X$ is the normalization of X.*

Proof. We may assume X is reduced, see Lemma 54.2. Then the lemma follows either from the local description in Lemma 54.3 or from Lemma 54.5 part (3) because $\coprod Z_i \to X$ is integral and locally birational (as X is reduced and has locally finitely many irreducible components). $\square$

Lemma 54.7. *Let X be a reduced scheme with finitely many irreducible components. Then the normalization morphism $X^\nu \to X$ is birational.*

Proof. The normalization induces a bijection of irreducible components by Lemma 54.5. Let $\eta \in X$ be a generic point of an irreducible component of X and let $\eta^\nu \in X^\nu$ be the generic point of the corresponding irreducible component of X^ν. Then $\eta^\nu \mapsto \eta$ and to finish the proof we have to show that $\mathcal{O}_{X,\eta} \to \mathcal{O}_{X^\nu,\eta^\nu}$ is an isomorphism, see Definition 50.1. Because X and X^ν are reduced, we see that both local rings are equal to their residue fields (Algebra, Lemma 25.1). On the other hand, by the construction of the normalization as the normalization of X

in $Y = \coprod \text{Spec}(\kappa(\eta))$ we see that we have $\kappa(\eta) \subset \kappa(\eta^\nu) \subset \kappa(\eta)$ and the proof is complete. $\square$

Lemma 54.8. *A finite (or even integral) birational morphism $f : X \to Y$ of integral schemes with Y normal is an isomorphism.*

Proof. Let $V \subset Y$ be an affine open with inverse image $U \subset X$ which is an affine open too. Since f is a birational morphism of integral schemes, the homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is an injective map of domains which induces an isomorphism of fraction fields. As Y is normal, the ring $\mathcal{O}_Y(V)$ is integrally closed in the fraction field. Since f is finite (or integral) every element of $\mathcal{O}_X(U)$ is integral over $\mathcal{O}_Y(V)$. We conclude that $\mathcal{O}_Y(V) = \mathcal{O}_X(U)$. This proves that f is an isomorphism as desired. $\square$

Lemma 54.9. *Let X be an integral, Japanese scheme. The normalization $\nu : X^\nu \to X$ is a finite morphism.*

Proof. Follows from the definition (Properties, Definition 13.1) and Lemma 54.3. Namely, in this case the lemma says that $\nu^{-1}(\text{Spec}(A))$ is the spectrum of the integral closure of A in its field of fractions. $\square$

Lemma 54.10. *Let X be a Nagata scheme. The normalization $\nu : X^\nu \to X$ is a finite morphism.*

Proof. Note that a Nagata scheme is locally Noetherian, thus Definition 54.1 does apply. The lemma is now a special case of Lemma 53.14 but we can also prove it directly as follows. Write $X^\nu \to X$ as the composition $X^\nu \to X_{red} \to X$. As $X_{red} \to X$ is a closed immersion it is finite. Hence it suffices to prove the lemma for a reduced Nagata scheme (by Lemma 44.5). Let $\text{Spec}(A) = U \subset X$ be an affine open. By Lemma 54.3 we have $\nu^{-1}(U) = \text{Spec}(\prod A_i')$ where A_i' is the integral closure of $A/\mathfrak{q}_i$ in its fraction field. As A is a Nagata ring (see Properties, Lemma 13.6) each of the ring extensions $A/\mathfrak{q}_i \subset A_i'$ are finite. Hence $A \to \prod A_i'$ is a finite ring map and we win. $\square$

Lemma 54.11. *Let X be an irreducible, geometrically unibranch scheme. The normalization morphism $\nu : X^\nu \to X$ is a universal homeomorphism.*

Proof. We have to show that ν is integral, universally injective, and surjective, see Lemma 45.5. By Lemma 54.5 the morphism ν is integral. Let $x \in X$ and set $A = \mathcal{O}_{X,x}$. Since X is irreducible we see that A has a single minimal prime $\mathfrak{p}$ and $A_{red} = A/\mathfrak{p}$. By Lemma 54.4 the stalk $A' = (\nu_*\mathcal{O}_{X^\nu})_x$ is the integral closure of A in the fraction field of A_{red}. By More on Algebra, Definition 106.1 we see that A' has a single prime $\mathfrak{m}'$ lying over $\mathfrak{m}_x \subset A$ and $\kappa(\mathfrak{m}')/\kappa(x)$ is purely inseparable. Hence ν is bijective (hence surjective) and universally injective by Lemma 10.2. $\square$

55. Weak normalization

We will only define the weak normalization of a scheme when it locally has finitely many irreducible components; similar to the case of normalization.

Lemma 55.1. *Let $A \to B$ be a ring map inducing a dominant morphism $\text{Spec}(B) \to \text{Spec}(A)$ of spectra. There exists an A-subalgebra $B' \subset B$ such that*

1. $\text{Spec}(B') \to \text{Spec}(A)$ *is a universal homeomorphism,*

(2) *given a factorization $A \to C \to B$ such that* $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ *is a universal homeomorphism, the image of $C \to B$ is contained in B'.*

Proof. We will use Lemma 45.6 without further mention. Consider the commutative diagram

$$\begin{tikzcd} B \arrow[r] & B_{red} \\ A \arrow[u] \arrow[r] & A_{red} \arrow[u] \end{tikzcd}$$

For any factorization $A \to C \to B$ of $A \to B$ as in (2), we see that $A_{red} \to C_{red} \to B_{red}$ is a factorization of $A_{red} \to B_{red}$ as in (2). It follows that if the lemma holds for $A_{red} \to B_{red}$ and produces the A_{red}-subalgebra $B'_{red} \subset B_{red}$, then setting $B' \subset B$ equal to the inverse image of B'_{red} solves the lemma for $A \to B$. This reduces us to the case discussed in the next paragraph.

Assume A and B are reduced. In this case $A \subset B$ by Algebra, Lemma 30.6. Let $A \to C \to B$ be a factorization as in (2). Then we may apply Proposition 46.8 to $A \subset C$ to see that every element of C is contained in an extension $A[c_1, \ldots, c_n] \subset C$ such that for $i = 1, \ldots, n$ we have

(1) $c_i^2, c_i^3 \in A[c_1, \ldots, c_{i-1}]$, or
(2) there exists a prime number p with $pc_i, c_i^p \in A[c_1, \ldots, c_{i-1}]$.

Thus property (2) holds if we define $B' \subset B$ to be the subset of elements $b \in B$ which are contained in an extension $A[b_1, \ldots, b_n] \subset B$ such that (*) holds: for $i = 1, \ldots, n$ we have

(1) $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$, or
(2) there exists a prime number p with $pb_i, b_i^p \in A[b_1, \ldots, b_{i-1}]$.

There are only two things to check: (a) B' is an A-subalgebra, and (b) $\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is a universal homeomorphism. Part (a) follows because given $n \geq 0$ and $b_1, \ldots, b_n \in B$ satisfying (*) and $m \geq 0$ and $b'_1, \ldots, b'_m \in B$ satisfying (*), the integer $n + m$ and $b_1, \ldots, b_n, b'_1, \ldots, b'_m \in B$ also satisfies (*). Finally, part (b) holds by Proposition 46.8 and our construction of B'. $\square$

Lemma 55.2. *Let $A \to B$ be a ring map inducing a dominant morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of spectra. Formation of the A-subalgebra $B' \subset B$ in Lemma 55.1 commutes with localization (see proof for explanation).*

Proof. Let $S \subset A$ be a multiplicative subset. Then $S^{-1}A \to S^{-1}B$ is a ring map which induces a dominant morphism $\mathrm{Spec}(S^{-1}B) \to \mathrm{Spec}(S^{-1}A)$ as well (see Lemmas 8.4 and 25.9). Hence Lemma 55.1 produces an $S^{-1}A$-subalgebra $(S^{-1}B)' \subset S^{-1}B$. The statement means that $S^{-1}B' = (S^{-1}B)'$ as $S^{-1}A$-subalgebras of $S^{-1}B$.

To see this is true, we will use the construction of B' and $(S^{-1}B)'$ in the proof of Lemma 55.1. In the first step, we see that B' is the inverse image of the A_{red}-subalgebra $B'_{red} \subset B_{red}$ constructed for the ring map $A_{red} \to B_{red}$ and similarly for $(S^{-1}B)'$. Noting that $S^{-1}B_{red} = (S^{-1}B)_{red}$ this reduces us to the case discussed in the next paragraph.

If A and B are reduced, we have constructed B' as the union of the subalgebras $A[b_1, \ldots, b_n]$ such that for $i = 1, \ldots, n$ we have

(1) $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$, or
(2) there exists a prime number p with $pb_i, b_i^p \in A[b_1, \ldots, b_{i-1}]$.

Similarly for $(S^{-1}B)' \subset S^{-1}B$. Thus it is clear that the image of $B' \to B \to S^{-1}B$ is contained in $(S^{-1}B)'$. To show that the corresponding map $S^{-1}B' \to (S^{-1}B)'$ is surjective, one uses Lemma 46.3 to clear denominators successively; we omit the details. ☐

Lemma 55.3. *Let $A \to B$ be a ring map inducing a dominant morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of spectra. There exists an A-subalgebra $B' \subset B$ such that*

(1) $\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ *is a universal homeomorphism inducing isomorphisms on residue fields,*
(2) *given a factorization $A \to C \to B$ such that* $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ *is a universal homeomorphism inducing isomorphisms on residue fields, the image of $C \to B$ is contained in B'.*

Proof. This proof is exactly the same as the proof of Lemma 55.1 except we use Proposition 46.7 in stead of Proposition 46.8 ☐

Lemma 55.4. *Let $A \to B$ be a ring map inducing a dominant morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of spectra. Formation of the A-subalgebra $B' \subset B$ in Lemma 55.3 commutes with localization (see proof for explanation).*

Proof. The proof is the same as the proof of Lemma 55.2. ☐

Lemma 55.5. *Let $f : Y \to X$ be a quasi-compact, quasi-separated, and dominant morphism of schemes.*

(1) *The category of factorizations $Y \to X' \to X$ where $X' \to X$ is a universal homeomorphism has an initial object $Y \to X^{Y/wn} \to X$.*
(2) *The category of factorizations $Y \to X' \to X$ where $X' \to X$ is a universal homeomorphism inducing isomorphisms on residue fields has an initial object $Y \to X^{Y/sn} \to X$.*

Moreover, formation of the factorization $Y \to X^{Y/wn} \to X$ and $Y \to X^{Y/sn} \to X$ commutes with base change to open subschemes of X.

Proof. We will prove (1) and omit the proof of (2); also the final assertion will follow from the construction of the factorization. We will use Lemma 45.5 without further mention. First, let $Y \to X^{Y/n} \to X$ be the normalization of X in Y, see Definition 53.3. For $Y \to X' \to X$ as in (1), we obtain a unique morphism $X^{Y/n} \to X'$ compatible with the given morphisms, see Lemma 53.4. Thus it suffices to prove the lemma with f replaced by $X^{Y/n} \to X$. This reduces us to the case studied in the next paragraph.

Assume f is integral (the rest of the proof works more generally if f is affine). Let $U = \mathrm{Spec}(A)$ be an affine open of X and let $V = f^{-1}(U) = \mathrm{Spec}(B)$ be the inverse image in Y. Then $A \to B$ is a ring map which induces a dominant morphism on spectra. By Lemma 55.1 we obtain an A-subalgebra $B' \subset B$ such that setting $U^{V/wn} = \mathrm{Spec}(B')$ the factorization $V \to U^{V/wn} \to U$ is initial in the category of factorizations $V \to U' \to U$ where $U' \to U$ is a universal homeomorphism.

If $U_1 \subset U_2 \subset X$ are affine opens, then setting $V_i = f^{-1}(U_i)$ we obtain a canonical morphism

$$\rho^{U_2}_{U_1} : U_1^{V_1/wn} \to U_1 \times_{U_2} U_2^{V_2/wn}$$

over U_1 by the universal property of $U_1^{V_1/wn}$. These morphisms satisfy a natural functoriality which we leave to the reader to formulate and prove. Furthermore, the morphism $\rho_{U_1}^{U_2}$ is an isomorphism; this follows from Lemma 55.2 provided that $U_1 \subset U_2$ is a standard open and in the general case can be reduced to this case by the functorial nature of these maps and Schemes, Lemma 11.5 (details omitted). Thus by relative glueing (Constructions, Lemma 2.1) we obtain a morphism $X^{Y/wn} \to X$ which restricts to $U^{V/wn} \to U$ over U compatibly with the $\rho_{U_1}^{U_2}$. Of course, the morphisms $V \to U^{V/wn}$ glue to a morphism $Y \to X^{Y/wn}$ (see Constructions, Remark 2.3) and we get our factorization $Y \to X^{Y/wn} \to X$ where the second morphism is a universal homeomorphism.

Finally, let $Y \to X' \to X$ be a factorization as in (1). With $V \to U^{V/wn} \to U \subset X$ as above, we obtain a factorization $V \to U \times_X X' \to U$ where the second arrow is a universal homeomorphism and we obtain a unique morphism $g_U : U^{V/wn} \to U \times_X X'$ over U by the universal property of $U^{V/wn}$. These g_U are compatible with the morphisms $\rho_{U_1}^{U_2}$; details omitted. Hence there is a unique morphism $g : X^{Y/wn} \to X'$ over X agreeing with g_U over U, see Constructions, Remark 2.3. This proves that $Y \to X^{Y/wn} \to X$ is initial in our category and the proof is complete. $\square$

Definition 55.6. Let $f : Y \to X$ be a quasi-compact, quasi-separated, and dominant morphism of schemes.

(1) The factorization $Y \to X^{Y/sn} \to X$ constructed in Lemma 55.5 part (2) is the *seminormalization of X in Y*.
(2) The factorization $Y \to X^{Y/wn} \to X$ constructed in Lemma 55.5 part (1) is the *weak normalization of X in Y*.

Here is a way to reinterpret the seminormalization of a scheme which locally has finitely many irreducible components.

Lemma 55.7. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $\nu : X^\nu \to X$ be the normalization of X. Then the seminormalization of X in X^ν is is the seminormalization of X. In a formula: $X^{sn} = X^{X^\nu/sn}$.*

Proof. Let $f : Y \to X$ be as in (54.0.1) so that X^ν is the normalization of X in Y. The seminormalization $X^{sn} \to X$ of X is the initial object in the category of universal homeomorphisms $X' \to X$ inducing isomorphisms on residue fields. Since Y is the disjoint union of the spectra of the residue fields at the generic points of irreducible components of X, we see that for any $X' \to X$ in this category we obtain a canonical lift $f' : Y \to X'$ of f. Then by Lemma 53.4 we obtain a canonical morphism $X^\nu \to X'$. Whence in turn a canonical morphism $X^{X^\nu/sn} \to X'$ by the universal property of $X^{X^\nu/sn}$. In this way we see that $X^{X^\nu/sn}$ satisfies the same universal property that X^{sn} has and we conclude. $\square$

Lemma 55.7 motivates the following definition. Since we have only constructed the normalization in case X locally has finitely many irreducible components, we will also restrict ourselves to that case for the weak normalization.

Definition 55.8. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We define the *weak normalization* of X as the weak normalization

$$X^\nu \longrightarrow X^{wn} \longrightarrow X$$

of X in the normalization X^ν of X (Definition 54.1). In a formula: $X^{wn} = X^{X^\nu/wn}$.

Combined with Lemma 55.7 we see that for a scheme X which locally has finitely many irreducible components there are canonical morphisms

$$X^\nu \to X^{wn} \to X^{sn} \to X$$

Having made this definition, we can say what it means for a scheme to be weakly normal (provided it has locally finitely many irreducible components).

Definition 55.9. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We say X is *weakly normal* if the weak normalization $X^{wn} \to X$ is an isomorphism (Definition 55.8).

It follows immediately from the definitions that for a scheme X such that every quasi-compact open has finitely many irreducible components we have

$$X \text{ normal} \Rightarrow X \text{ weakly normal} \Rightarrow X \text{ seminormal}$$

We can work out the meaning of weak normality in the affine case as follows.

Lemma 55.10. *Let $X = \mathrm{Spec}(A)$ be an affine scheme which has finitely many irreducible components. Then X is weakly normal if and only if*

1. *A is seminormal (Definition 47.1),*
2. *for a prime number p and $z, w \in A$ such that (a) z is a nonzerodivisor, (b) w^p is divisible by z^p, and (c) pw is divisible by z, then w is divisible by z.*

Proof. Assume X is weakly normal. Since a weakly normal scheme is seminormal, we see that (1) holds (by our definition of weakly normal schemes). In particular A is reduced. Let p, z, w be as in (2). Choose $x, y \in A$ such that $z^p x = w^p$ and $zy = pw$. Then $p^p x = y^p$. The ring map $A \to C = A[t]/(t^p - x, pt - y)$ induces a universal homeomorphism on spectra. The normalization X^ν of X is the spectrum of the integral closure A' of A in the total ring of fractions of A, see Lemma 54.3. Note that $a = w/z \in A'$ because $a^p = x$. Hence we have an A-algebra homomorphism $A \to C \to A'$ sending t to a. At this point the defining property $X = X^{wn} = X^{X^\nu/wn}$ of being weakly normal tells us that $C \to A'$ maps into A. Thus we find $a \in A$ as desired.

Conversely, assume (1) and (2). Let A' be as in the previous paragraph. We have to show that $X^{X^\nu/wn} = X$. By construction in the proof of Lemma 55.1, the scheme $X^{X^\nu/wn}$ is the spectrum of the subring of A' which is the union of the subrings $A[a_1, \ldots, a_n] \subset A'$ such that for $i = 1, \ldots, n$ we have

(a) $a_i^2, a_i^3 \in A[a_1, \ldots, a_{i-1}]$, or
(b) there exists a prime number p with $pa_i, a_i^p \in A[a_1, \ldots, a_{i-1}]$.

Then we can use (1) and (2) to inductively see that $a_1, \ldots, a_n \in A$; we omit the details. Consequently, we have $X = X^{X^\nu/wn}$ and hence X is weakly normal. $\square$

Here is the obligatory lemma.

Lemma 55.11. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. The following are equivalent:*

1. *The scheme X is weakly normal.*
2. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ satisfies conditions (1) and (2) of Lemma 55.10.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each ring $\mathcal{O}_X(U_i)$ satisfies conditions (1) and (2) of Lemma 55.10.*

(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is weakly normal.*

Moreover, if X is weakly normal then every open subscheme is weakly normal.

Proof. The condition to X be weakly normal is that the morphism $X^{wn} = X^{X^\nu/wn} \to X$ is an isomorphism. Since the construction of $X^\nu \to X$ commutes with base change to open subschemes and since the construction of $X^{X^\nu/wn}$ commutes with base change to open subschemes of X (Lemma 55.5) the lemma is clear. $\square$

56. Zariski's Main Theorem (algebraic version)

This is the version you can prove using purely algebraic methods. Before we can prove more powerful versions (for non-affine morphisms) we need to develop more tools. See Cohomology of Schemes, Section 21 and More on Morphisms, Section 43.

Theorem 56.1 (Algebraic version of Zariski's Main Theorem)**.** *Let $f : Y \to X$ be an affine morphism of schemes. Assume f is of finite type. Let X' be the normalization of X in Y. Picture:*

$$\begin{tikzcd} Y \arrow[rr, "f'"] \arrow[dr, "f"'] & & X' \arrow[dl, "\nu"] \\ & X & \end{tikzcd}$$

Then there exists an open subscheme $U' \subset X'$ such that

(1) *$(f')^{-1}(U') \to U'$ is an isomorphism, and*
(2) *$(f')^{-1}(U') \subset Y$ is the set of points at which f is quasi-finite.*

Proof. There is an immediate reduction to the case where X and hence Y are affine. Say $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(A)$. Then $X' = \mathrm{Spec}(A')$, where A' is the integral closure of R in A, see Definitions 53.2 and 53.3. By Algebra, Theorem 123.12 for every $y \in Y$ at which f is quasi-finite, there exists an open $U'_y \subset X'$ such that $(f')^{-1}(U'_y) \to U'_y$ is an isomorphism. Set $U' = \bigcup U'_y$ where $y \in Y$ ranges over all points where f is quasi-finite. It remains to show that f is quasi-finite at all points of $(f')^{-1}(U')$. If $y \in (f')^{-1}(U')$ with image $x \in X$, then we see that $Y_x \to X'_x$ is an isomorphism in a neighbourhood of y. Hence there is no point of Y_x which specializes to y, since this is true for $f'(y)$ in X'_x, see Lemma 44.8. By Lemma 20.6 part (3) this implies f is quasi-finite at y. $\square$

We can use the algebraic version of Zariski's Main Theorem to show that the set of points where a morphism is quasi-finite is open.

Lemma 56.2. *Let $f : X \to S$ be a morphism of schemes. The set of points of X where f is quasi-finite is an open $U \subset X$. The induced morphism $U \to S$ is locally quasi-finite.*

Proof. Suppose f is quasi-finite at x. Let $x \in U = \mathrm{Spec}(A) \subset X$, $V = \mathrm{Spec}(R) \subset S$ be affine opens as in Definition 20.1. By either Theorem 56.1 above or Algebra, Lemma 123.13, the set of primes $\mathfrak{q}$ at which $R \to A$ is quasi-finite is open in $\mathrm{Spec}(A)$. Since these all correspond to points of X where f is quasi-finite we get the first statement. The second statement is obvious. $\square$

We will improve the following lemma to general quasi-finite separated morphisms later, see More on Morphisms, Lemma 43.3.

Lemma 56.3. *Let $f : Y \to X$ be a morphism of schemes. Assume*

1. *X and Y are affine, and*
2. *f is quasi-finite.*

Then there exists a diagram

$$\begin{tikzcd} Y \arrow[rr, "j"] \arrow[dr, "f"'] & & Z \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

with Z affine, π finite and j an open immersion.

Proof. This is Algebra, Lemma 123.14 reformulated in the language of schemes. □

Lemma 56.4. *Let $f : Y \to X$ be a quasi-finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y. Then $f(T) \subset X$ is a nowhere dense subset of X.*

Proof. As in the proof of Lemma 48.7 this reduces immediately to the case where the base X is affine. In this case $Y = \bigcup_{i=1,\ldots,n} Y_i$ is a finite union of affine opens (as f is quasi-compact). Since each $T \cap Y_i$ is nowhere dense, and since a finite union of nowhere dense sets is nowhere dense (see Topology, Lemma 21.2), it suffices to prove that the image $f(T \cap Y_i)$ is nowhere dense in X. This reduces us to the case where both X and Y are affine. At this point we apply Lemma 56.3 above to get a diagram

$$\begin{tikzcd} Y \arrow[rr, "j"] \arrow[dr, "f"'] & & Z \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

with Z affine, π finite and j an open immersion. Set $\overline{T} = \overline{j(T)} \subset Z$. By Topology, Lemma 21.3 we see $\overline{T}$ is nowhere dense in Z. Since $f(T) \subset \pi(\overline{T})$ the lemma follows from the corresponding result in the finite case, see Lemma 48.7. □

57. Universally bounded fibres

Let X be a scheme over a field k. If X is finite over k, then $X = \mathrm{Spec}(A)$ where A is a finite k-algebra. Another way to say this is that X is finite locally free over $\mathrm{Spec}(k)$, see Definition 48.1. Hence $X \to \mathrm{Spec}(k)$ has a *degree* which is an integer $d \geq 0$, namely $d = \dim_k(A)$. We sometime call this the *degree* of the (finite) scheme X over k.

Definition 57.1. Let $f : X \to Y$ be a morphism of schemes.

1. We say the integer n *bounds the degrees of the fibres of f* if for all $y \in Y$ the fibre X_y is a finite scheme over $\kappa(y)$ whose degree over $\kappa(y)$ is $\leq n$.
2. We say the *fibres of f are universally bounded*[19] if there exists an integer n which bounds the degrees of the fibres of f.

[19]This is probably nonstandard notation.

Note that in particular the number of points in a fibre is bounded by n as well. (The converse does not hold, even if all fibres are finite reduced schemes.)

Lemma 57.2. *Let $f : X \to Y$ be a morphism of schemes. Let $n \geq 0$. The following are equivalent:*

1. *the integer n bounds the degrees of the fibres of f, and*
2. *for every morphism $\mathrm{Spec}(k) \to Y$, where k is a field, the fibre product $X_k = \mathrm{Spec}(k) \times_Y X$ is finite over k of degree $\leq n$.*

In this case the fibres of f are universally bounded and the schemes X_k have at most n points. More precisely, if $X_k = \{x_1, \ldots, x_t\}$, then we have

$$n \geq \sum\nolimits_{i=1,\ldots,t} [\kappa(x_i) : k]$$

Proof. The implication (2) $\Rightarrow$ (1) is trivial. The other implication holds because if the image of $\mathrm{Spec}(k) \to Y$ is y, then $X_k = \mathrm{Spec}(k) \times_{\mathrm{Spec}(\kappa(y))} X_y$. By definition the fibres of f being universally bounded means that some n exists. Finally, suppose that $X_k = \mathrm{Spec}(A)$. Then $\dim_k A = n$. Hence A is Artinian, all prime ideals are maximal ideals $\mathfrak m_i$, and A is the product of the localizations at these maximal ideals. See Algebra, Lemmas 53.2 and 53.6. Then $\mathfrak m_i$ corresponds to x_i, we have $A_{\mathfrak m_i} = \mathcal{O}_{X_k, x_i}$ and hence there is a surjection $A \to \bigoplus \kappa(\mathfrak m_i) = \bigoplus \kappa(x_i)$ which implies the inequality in the statement of the lemma by linear algebra. $\square$

Lemma 57.3. *If f is a finite locally free morphism of degree d, then d bounds the degree of the fibres of f.*

Proof. This is true because any base change of f is finite locally free of degree d (Lemma 48.4) and hence the fibres of f all have degree d. $\square$

Lemma 57.4. *A composition of morphisms with universally bounded fibres is a morphism with universally bounded fibres. More precisely, assume that n bounds the degrees of the fibres of $f : X \to Y$ and m bounds the degrees of $g : Y \to Z$. Then nm bounds the degrees of the fibres of $g \circ f : X \to Z$.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ have universally bounded fibres. Say that $\deg(X_y/\kappa(y)) \leq n$ for all $y \in Y$, and that $\deg(Y_z/\kappa(z)) \leq m$ for all $z \in Z$. Let $z \in Z$ be a point. By assumption the scheme Y_z is finite over $\mathrm{Spec}(\kappa(z))$. In particular, the underlying topological space of Y_z is a finite discrete set. The fibres of the morphism $f_z : X_z \to Y_z$ are the fibres of f at the corresponding points of Y, which are finite discrete sets by the reasoning above. Hence we conclude that the underlying topological space of X_z is a finite discrete set as well. Thus X_z is an affine scheme (this is a nice exercise; it also follows for example from Properties, Lemma 29.1 applied to the set of all points of X_z). Write $X_z = \mathrm{Spec}(A)$, $Y_z = \mathrm{Spec}(B)$, and $k = \kappa(z)$. Then $k \to B \to A$ and we know that (a) $\dim_k(B) \leq m$, and (b) for every maximal ideal $\mathfrak m \subset B$ we have $\dim_{\kappa(\mathfrak m)}(A/\mathfrak m A) \leq n$. We claim this implies that $\dim_k(A) \leq nm$. Note that B is the product of its localizations $B_{\mathfrak m}$, for example because Y_z is a disjoint union of 1-point schemes, or by Algebra, Lemmas 53.2 and 53.6. So we see that $\dim_k(B) = \sum_{\mathfrak m} \dim_k(B_{\mathfrak m})$ and $\dim_k(A) = \sum_{\mathfrak m} \dim_k(A_{\mathfrak m})$ where in both cases $\mathfrak m$ runs over the maximal ideals of B (not of A). By the above, and Nakayama's Lemma (Algebra, Lemma 20.1) we see that each $A_{\mathfrak m}$ is a quotient of $B_{\mathfrak m}^{\oplus n}$ as a $B_{\mathfrak m}$-module. Hence $\dim_k(A_{\mathfrak m}) \leq n \dim_k(B_{\mathfrak m})$. Putting everything together we see that

$$\dim_k(A) = \sum\nolimits_{\mathfrak m} \dim_t a(A_{\mathfrak m}) \leq \sum\nolimits_{\mathfrak m} n \dim_k(B_{\mathfrak m}) = n \dim_k(B) \leq nm$$

as desired. □

Lemma 57.5. *A base change of a morphism with universally bounded fibres is a morphism with universally bounded fibres. More precisely, if n bounds the degrees of the fibres of $f : X \to Y$ and $Y' \to Y$ is any morphism, then the degrees of the fibres of the base change $f' : Y' \times_Y X \to Y'$ is also bounded by n.*

Proof. This is clear from the result of Lemma 57.2. □

Lemma 57.6. *Let $f : X \to Y$ be a morphism of schemes. Let $Y' \to Y$ be a morphism of schemes, and let $f' : X' = X_{Y'} \to Y'$ be the base change of f. If $Y' \to Y$ is surjective and f' has universally bounded fibres, then f has universally bounded fibres. More precisely, if n bounds the degree of the fibres of f', then also n bounds the degrees of the fibres of f.*

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibres of f'. We claim that n works for f also. Namely, if $y \in Y$ is a point, then choose a point $y' \in Y'$ lying over y and observe that

$$X'_{y'} = \mathrm{Spec}(\kappa(y')) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Since $X'_{y'}$ is assumed finite of degree $\leq n$ over $\kappa(y')$ it follows that also X_y is finite of degree $\leq n$ over $\kappa(y)$. (Some details omitted.) □

Lemma 57.7. *An immersion has universally bounded fibres.*

Proof. The integer $n = 1$ works in the definition. □

Lemma 57.8. *Let $f : X \to Y$ be an étale morphism of schemes. Let $n \geq 0$. The following are equivalent*

1. *the integer n bounds the degrees of the fibres,*
2. *for every field k and morphism $\mathrm{Spec}(k) \to Y$ the base change $X_k = \mathrm{Spec}(k) \times_Y X$ has at most n points, and*
3. *for every $y \in Y$ and every separable algebraic closure $\kappa(y) \subset \kappa(y)^{sep}$ the scheme $X_{\kappa(y)^{sep}}$ has at most n points.*

Proof. This follows from Lemma 57.2 and the fact that the fibres X_y are disjoint unions of spectra of finite separable field extensions of $\kappa(y)$, see Lemma 36.7. □

Having universally bounded fibres is an absolute notion and not a relative notion. This is why the condition in the following lemma is that X is quasi-compact, and not that f is quasi-compact.

Lemma 57.9. *Let $f : X \to Y$ be a morphism of schemes. Assume that*

1. *f is locally quasi-finite, and*
2. *X is quasi-compact.*

Then f has universally bounded fibres.

Proof. Since X is quasi-compact, there exists a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$ and affine opens $V_i \subset Y$, $i = 1, \ldots, n$ such that $f(U_i) \subset V_i$. Because of the local nature of "local quasi-finiteness" (see Lemma 20.6 part (4)) we see that the morphisms $f|_{U_i} : U_i \to V_i$ are locally quasi-finite morphisms of affines, hence quasi-finite, see Lemma 20.9. For $y \in Y$ it is clear that $X_y = \bigcup_{y \in V_i} (U_i)_y$ is an open covering. Hence it suffices to prove the lemma for a quasi-finite morphism of affines (namely, if n_i works for the morphism $f|_{U_i} : U_i \to V_i$, then $\sum n_i$ works for f).

Assume $f : X \to Y$ is a quasi-finite morphism of affines. By Lemma 56.3 we can find a diagram

$$\begin{tikzcd} X \arrow[rr, "j"'] \arrow[dr, "f"'] & & Z \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

with Z affine, π finite and j an open immersion. Since j has universally bounded fibres (Lemma 57.7) this reduces us to showing that π has universally bounded fibres (Lemma 57.4).

This reduces us to a morphism of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ where $A \to B$ is finite. Say B is generated by $x_1, \ldots, x_n$ over A and say $P_i(T) \in A[T]$ is a monic polynomial of degree d_i such that $P_i(x_i) = 0$ in B (a finite ring extension is integral, see Algebra, Lemma 36.3). With these notations it is clear that

$$\bigoplus\nolimits_{0 \leq e_i < d_i, i = 1, \ldots n} A \longrightarrow B, \quad (a_{(e_1, \ldots, e_n)}) \longmapsto \sum a_{(e_1, \ldots, e_n)} x_1^{e_1} \ldots x_n^{e_n}$$

is a surjective A-module map. Thus for any prime $\mathfrak p \subset A$ this induces a surjective map $\kappa(\mathfrak p)$-vector spaces

$$\kappa(\mathfrak p)^{\oplus d_1 \ldots d_n} \longrightarrow B \otimes_A \kappa(\mathfrak p)$$

In other words, the integer $d_1 \ldots d_n$ works in the definition of a morphism with universally bounded fibres. $\square$

Lemma 57.10. *Consider a commutative diagram of morphisms of schemes*

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres. More precisely, if n bounds the degree of the fibres of g, then also n bounds the degree of the fibres of h.

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say the degree of the fibres of g is bounded by $n \in \mathbf{N}$. We claim n also works for h. Let $z \in Z$. Consider the morphism of schemes $X_z \to Y_z$. It is flat and surjective. By assumption X_z is a finite scheme over $\kappa(z)$, in particular it is the spectrum of an Artinian ring (by Algebra, Lemma 53.2). By Lemma 11.13 the morphism $X_z \to Y_z$ is affine in particular quasi-compact. It follows from Lemma 25.12 that Y_z is a finite discrete as this holds for X_z. Hence Y_z is an affine scheme (this is a nice exercise; it also follows for example from Properties, Lemma 29.1 applied to the set of all points of Y_z). Write $Y_z = \mathrm{Spec}(B)$ and $X_z = \mathrm{Spec}(A)$. Then A is faithfully flat over B, so $B \subset A$. Hence $\dim_k(B) \leq \dim_k(A) \leq n$ as desired. $\square$

58. Miscellany

Results which do not fit elsewhere.

Lemma 58.1. *Let $f : Y \to X$ be a morphism of schemes. Let $x \in X$ be a point. Assume that Y is reduced and $f(Y)$ is set-theoretically contained in $\{x\}$. Then f factors through the canonical morphism $x = \mathrm{Spec}(\kappa(x)) \to X$.*

Proof. Omitted. Hints: working affine locally one reduces to a commutative algebra lemma. Given a ring map $A \to B$ with B reduced such that there exists a unique prime ideal $\mathfrak{p} \subset A$ in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, then $A \to B$ factors through $\kappa(\mathfrak{p})$. This is a nice exercise. $\square$

Lemma 58.2. *Let $f : Y \to X$ be a morphism of schemes. Let $E \subset X$. Assume X is locally Noetherian, there are no nontrivial specializations among the elements of E, Y is reduced, and $f(Y) \subset E$. Then f factors through $\coprod_{x \in E} x \to X$.*

Proof. When E is a singleton this follows from Lemma 58.1. If E is finite, then E (with the induced topology of X) is a finite discrete space by our assumption on specializations. Hence this case reduces to the singleton case. In general, there is a reduction to the case where X and Y are affine schemes. Say $f : Y \to X$ corresponds to the ring map $\varphi : A \to B$. Denote $A' \subset B$ the image of φ. Let $E' \subset \mathrm{Spec}(A') \subset \mathrm{Spec}(A)$ be the set of minimal primes of A'. By Algebra, Lemma 30.5 the set E' is contained in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A') \subset \mathrm{Spec}(A)$. We conclude that $E' \subset E$. Since A' is Noetherian we have E' is finite by Algebra, Lemma 31.6. Since any other point in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a specialization of an element of E' and in E, we conclude that the image is contained in E' (by our assumption on specializations between points of E). Thus we reduce to the case where E is finite which we dealt with above. $\square$

59. Other chapters

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