TOPOLOGY

Contents

1. Introduction

Basic topology will be explained in this document. A reference is [Eng77].

2. Basic notions

The following is a list of basic notions in topology. Some of these notions are discussed in more detail in the text that follows and some are defined in the list, but others are considered basic and will not be defined. If you are not familiar with most of the italicized concepts, then we suggest looking at an introductory text on topology before continuing.

(1) X is a *topological space*,
(2) $x \in X$ is a *point*,
(3) $E \subset X$ is a *locally closed* subset,
(4) $x \in X$ is a *closed point*,
(5) $E \subset X$ is a *dense* subset,
(6) $f : X_1 \to X_2$ is *continuous*,
(7) an extended real function $f : X \to \mathbf{R} \cup \{\infty, -\infty\}$ is *upper semi-continuous* if $\{x \in X \mid f(x) < a\}$ is open for all $a \in \mathbf{R}$,
(8) an extended real function $f : X \to \mathbf{R} \cup \{\infty, -\infty\}$ is *lower semi-continuous* if $\{x \in X \mid f(x) > a\}$ is open for all $a \in \mathbf{R}$,
(9) a continuous map of spaces $f : X \to Y$ is *open* if $f(U)$ is open in Y for $U \subset X$ open,
(10) a continuous map of spaces $f : X \to Y$ is *closed* if $f(Z)$ is closed in Y for $Z \subset X$ closed,
(11) a *neighbourhood of* $x \in X$ is any subset $E \subset X$ which contains an open subset that contains x,
(12) the *induced topology* on a subset $E \subset X$,
(13) $\mathcal{U} : U = \bigcup_{i \in I} U_i$ is an *open covering of* U (note: we allow any U_i to be empty and we even allow, in case U is empty, the empty set for I),
(14) a *subcover* of a covering as in (13) is an open covering $\mathcal{U}' : U = \bigcup_{i \in I'} U_i$ where $I' \subset I$,
(15) the open covering $\mathcal{V}$ is a *refinement* of the open covering $\mathcal{U}$ (if $\mathcal{V} : U = \bigcup_{j \in J} V_j$ and $\mathcal{U} : U = \bigcup_{i \in I} U_i$ this means each V_j is completely contained in one of the U_i),
(16) $\{E_i\}_{i \in I}$ *is a fundamental system of neighbourhoods of* x *in* X,
(17) a topological space X is called *Hausdorff* or *separated* if and only if for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U$, $y \in V$,
(18) the *product* of two topological spaces,
(19) the *fibre product* $X \times_Y Z$ of a pair of continuous maps $f : X \to Y$ and $g : Z \to Y$,
(20) the *discrete topology* and the *indiscrete topology* on a set,
(21) etc.

3. Hausdorff spaces

The category of topological spaces has finite products.

Lemma 3.1. *Let X be a topological space. The following are equivalent:*

(1) *X is Hausdorff,*
(2) *the diagonal $\Delta(X) \subset X \times X$ is closed.*

Proof. We suppose that X is Hausdorff. Let $(x, y) \notin \Delta(X)$, i.e., $x \neq y$. There are U and V disjoint open sets of X such that $x \in U$ and $y \in V$. This implies that $U \times V \subset (X \times X) \setminus \Delta(X)$. This shows that $(X \times X) \setminus \Delta(X)$ is an open set of $X \times X$ which is equivalent to say that the diagonal $\Delta(X) \subset X \times X$ is closed in $X \times X$. The converse is similar: The complement $(X \times X) \setminus \Delta(X)$ consist precisely of $(x, y) \in X \times X$ with $x \neq y$. Thus, if $\Delta(X) \subset X \times X$ is closed, then, by the definition of the product topology, for every such (x, y), there are opens $U, V \subset X$ with $(x, y) \in U \times V$ and $(U \times V) \cap \Delta(X) = \emptyset$. In other words, with $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$. $\square$

Lemma 3.2. *Let $f : X \to Y$ be a continuous map of topological spaces. If Y is Hausdorff, then the graph of f is closed in $X \times Y$.*

Proof. The graph is the inverse image of the diagonal under the map $X \times Y \to Y \times Y$. Thus the lemma follows from Lemma 3.1. $\square$

Lemma 3.3. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $s : Y \to X$ be a continuous map such that $f \circ s = id_Y$. If X is Hausdorff, then $s(Y)$ is closed.*

Proof. This follows from Lemma 3.1 as $s(Y) = \{x \in X \mid x = s(f(x))\}$. $\square$

Lemma 3.4. *Let $X \to Z$ and $Y \to Z$ be continuous maps of topological spaces. If Z is Hausdorff, then $X \times_Z Y$ is closed in $X \times Y$.*

Proof. This follows from Lemma 3.1 as $X \times_Z Y$ is the inverse image of $\Delta(Z)$ under $X \times Y \to Z \times Z$. $\square$

4. Separated maps

Just the definition and some simple lemmas.

Definition 4.1. A continuous map $f : X \to Y$ of topological spaces is called *separated* if and only if the diagonal $\Delta : X \to X \times_Y X$ is a closed map.

Lemma 4.2. *Let $f : X \to Y$ be continuous map of topological spaces. The following are equivalent:*

(1) *f is separated,*
(2) *$\Delta(X) \subset X \times_Y X$ is a closed subset,*
(3) *given distinct points $x, x' \in X$ mapping to the same point of Y, there exist disjoint open neighbourhoods of x and x'.*

Proof. If f is separated, by Definition 4.1, Δ is a closed map. The fact that X is closed in X gives us that $\Delta(X)$ is closed in $X \times_Y X$. Thus (1) implies (2).

Assune $\Delta(X) \subset X \times_Y X$ is a closed subset and denote U the complementary open. This means we have an open set $W \subset X \times X$ such that $W \cap (X \times_Y X) = U$. However, by definition of the product topology, if $(x, x') \in W \cap (X \times_Y X)$, we have V and V' open sets of X such that $x \in V$, $x' \in V'$ and $V \times V' \subset W$. If we had $V \cap V' \neq \emptyset$, we would have $z \in V \cap V'$. However, $(z, z) \in X \times_Y X$, so $(z, z) \in (V \times V') \cap (X \times_Y X) \subset U$, which is absurd. Therefore $V \cap V' = \emptyset$, and we have two disjoint open neighborhoods for x and x'. It proves that (2) implies (3).

Finally, we suppose that given distinct points $x, x' \in X$ mapping to the same point of Y, there exist disjoint open neighbourhoods of x and x'. Let F be a closed set of X. We will show that $\Delta(F)$ is a closed subset of $X \times_Y X$. Let $(x, x') \in X \times_Y X$ be a point not contained in $\Delta(F)$. Then either $x \neq x'$ or $x \notin F$. In the first case, we choose disjoint open neighbourhoods $V, V' \subset X$ of x, x' and we see that $(V \times V') \cap X \times_Y X$ is an open neighbourood of (x, x') not meeting $\Delta(F)$. In the second case, we see that $((X \setminus F) \times X) \cap X \times_Y X$ is an open neighbourood of (x, x') not meeting $\Delta(F)$. We have shown that (3) implies (1). $\square$

Lemma 4.3. *Let $f : X \to Y$ be continuous map of topological spaces. If X is Hausdorff, then f is separated.*

Proof. Clear from Lemmas 4.2 and 3.1 as $\Delta(X)$ closed in $X \times X$ implies $\Delta(X)$ closed in $X \times_Y X$. $\square$

Lemma 4.4. *Let $f : X \to Y$ and $Y' \to Y$ be continuous maps of topological spaces. If f is separated, then $f' : Y' \times_Y X \to Z$ is separated.*

Proof. Follows from characterization (3) of Lemma 4.2. Namely, with $X' = Y' \times_Y X$ the image $\Delta(X')$ of the diagonal in the fibre product $X' \times_{Y'} X'$ is the inverse image of $\Delta(X)$ in $X \times_Y X$. $\square$

5. Bases

Basic material on bases for topological spaces.

Definition 5.1. Let X be a topological space. A collection of subsets $\mathcal{B}$ of X is called a *base for the topology on X* or a *basis for the topology on X* if the following conditions hold:

1. Every element $B \in \mathcal{B}$ is open in X.
2. For every open $U \subset X$ and every $x \in U$, there exists an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

The following lemma is sometimes used to define a topology.

Lemma 5.2. *Let X be a set and let $\mathcal{B}$ be a collection of subsets. Assume that $X = \bigcup_{B \in \mathcal{B}} B$ and that given $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$ there is a $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$. Then there is a unique topology on X such that $\mathcal{B}$ is a basis for this topology.*

Proof. Let $\sigma(\mathcal{B})$ be the set of subsets of X which can be written as unions of elements of $\mathcal{B}$. We claim $\sigma(\mathcal{B})$ is a topology. Namely, the empty set is an element of $\sigma(\mathcal{B})$ (as an empty union) and X is an element of $\sigma(\mathcal{B})$ (as the union of all elements of $\mathcal{B}$). It is clear that $\sigma(\mathcal{B})$ is preserved under unions. Finally, if $U, V \in \sigma(\mathcal{B})$ then write $U = \bigcup_{i \in I} U_i$ and $V = \bigcup_{j \in J} V_j$ with $U_i, V_j \in \mathcal{B}$ for all $i \in I$ and $j \in J$. Then

$$U \cap V = \bigcup\nolimits_{i \in I, j \in J} U_i \cap V_j$$

The assumption in the lemma tells us that $U_i \cap V_j \in \sigma(\mathcal{B})$ hence we see that $U \cap V$ is too. Thus $\sigma(\mathcal{B}$ is a topology. Properties (1) and (2) of Definition 5.1 are immediate for this topology. To prove the uniqueness of this topology let $\mathcal{T}$ be a topology on X such that $\mathcal{B}$ is a base for it. Then of course every element of $\mathcal{B}$ is in $\mathcal{T}$ by (1) of Definition 5.1 and hence $\sigma(\mathcal{B} \subset \mathcal{T}$. Conversely, part (2) of Definition 5.1 tells us that every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$, i.e., $\mathcal{T} \subset \sigma(\mathcal{B})$. This finishes the proof. $\square$

Lemma 5.3. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{U} : U = \bigcup_i U_i$ be an open covering of $U \subset X$. There exists an open covering $U = \bigcup V_j$ which is a refinement of $\mathcal{U}$ such that each V_j is an element of the basis $\mathcal{B}$.*

Proof. If $x \in U = \bigcup_{i \in I} U_i$, there is an $i_x \in I$ such that $x \in U_{i_x}$. Thus we have a $B_{i_x} \in \mathcal{B}$ verifying $x \in B_{i_x} \subset U_{i_x}$. Set $J = \{i_x | x \in U\}$ and for $j = i_x \in J$ set $V_j = B_{i_x}$. This gives the desired open covering of U by $\{V_j\}_{j \in J}$. $\square$

Definition 5.4. Let X be a topological space. A collection of subsets $\mathcal{B}$ of X is called a *subbase for the topology on X* or a *subbasis for the topology on X* if the finite intersections of elements of $\mathcal{B}$ form a basis for the topology on X.

In particular every element of $\mathcal{B}$ is open.

Lemma 5.5. *Let X be a set. Given any collection $\mathcal{B}$ of subsets of X there is a unique topology on X such that $\mathcal{B}$ is a subbase for this topology.*

Proof. By convention $\bigcap_\emptyset B = X$. Thus we can apply Lemma 5.2 to the set of finite intersections of elements from $\mathcal{B}$. $\square$

Lemma 5.6. *Let X be a topological space. Let $\mathcal{B}$ be a collection of opens of X. Assume $X = \bigcup_{U \in \mathcal{B}} U$ and for $U, V \in \mathcal{B}$ we have $U \cap V = \bigcup_{W \in \mathcal{B}, W \subset U \cap V} W$. Then there is a continuous map $f : X \to Y$ of topological spaces such that*
1. *for $U \in \mathcal{B}$ the image $f(U)$ is open,*
2. *for $U \in \mathcal{B}$ we have $f^{-1}(f(U)) = U$, and*
3. *the opens $f(U)$, $U \in \mathcal{B}$ form a basis for the topology on Y.*

Proof. Define an equivalence relation $\sim$ on points of X by the rule

$$x \sim y \Leftrightarrow (\forall U \in \mathcal{B} : x \in U \Leftrightarrow y \in U)$$

Let Y be the set of equivalence classes and $f : X \to Y$ the natural map. Part (2) holds by construction. The assumptions on $\mathcal{B}$ exactly mirror the assumptions in Lemma 5.2 on the set of subsets $f(U)$, $U \in \mathcal{B}$. Hence there is a unique topology on Y such that (3) holds. Then (1) is clear as well. $\square$

6. Submersive maps

If X is a topological space and $E \subset X$ is a subset, then we usually endow E with the *induced topology*.

Lemma 6.1. *Let X be a topological space. Let Y be a set and let $f : Y \to X$ be an injective map of sets. The induced topology on Y is the topology characterized by each of the following statements:*
1. *it is the weakest topology on Y such that f is continuous,*
2. *the open subsets of Y are $f^{-1}(U)$ for $U \subset X$ open,*
3. *the closed subsets of Y are the sets $f^{-1}(Z)$ for $Z \subset X$ closed.*

Proof. The set $\mathcal{T} = \{f^{-1}(U) | U \subset X \text{ open}\}$ is a topology on Y. Firstly, $\emptyset = f^{-1}(\emptyset)$ and $f^{-1}(X) = Y$. So $\mathcal{T}$ contains $\emptyset$ and Y.

Now let $\{V_i\}_{i \in I}$ be a collection of open subsets where $V_i \in \mathcal{T}$ and write $V_i = f^{-1}(U_i)$ where U_i is an open subset of X, then

$$\bigcup\nolimits_{i \in I} V_i = \bigcup\nolimits_{i \in I} f^{-1}(U_i) = f^{-1}\left(\bigcup\nolimits_{i \in I} U_i\right)$$

So $\bigcup_{i\in I} V_i \in \mathcal{T}$ as $\bigcup_{i\in I} U_i$ is open in X. Now let $V_1, V_2 \in \mathcal{T}$. We have U_1, U_2 open in X such that $V_1 = f^{-1}(U_1)$ and $V_2 = f^{-1}(U_2)$. Then

$$V_1 \cap V_2 = f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2)$$

So $V_1 \cap V_2 \in \mathcal{T}$ because $U_1 \cap U_2$ is open in X.

Any topology on Y such that f is continuous contains $\mathcal{T}$ according to the definition of a continuous map. Thus $\mathcal{T}$ is indeed the weakest topology on Y such that f is continuous. This proves that (1) and (2) are equivalent.

The equivalence of (2) and (3) follows from the equality $f^{-1}(X \setminus E) = Y \setminus f^{-1}(E)$ for all subsets $E \subset X$. ☐

Dually, if X is a topological space and $X \to Y$ is a surjection of sets, then Y can be endowed with the *quotient topology.*

Lemma 6.2. *Let X be a topological space. Let Y be a set and let $f : X \to Y$ be a surjective map of sets. The quotient topology on Y is the topology characterized by each of the following statements:*

(1) *it is the strongest topology on Y such that f is continuous,*
(2) *a subset V of Y is open if and only if $f^{-1}(V)$ is open,*
(3) *a subset Z of Y is closed if and only if $f^{-1}(Z)$ is closed.*

Proof. The set $\mathcal{T} = \{V \subset Y | f^{-1}(V) \text{ is open}\}$ is a topology on Y. Firstly $\emptyset = f^{-1}(\emptyset)$ and $f^{-1}(Y) = X$. So $\mathcal{T}$ contains $\emptyset$ and Y.

Let $(V_i)_{i\in I}$ be a family of elements $V_i \in \mathcal{T}$. Then

$$\bigcup_{i\in I} f^{-1}(V_i) = f^{-1}\left(\bigcup_{i\in I} V_i\right)$$

Thus $\bigcup_{i\in I} V_i \in \mathcal{T}$ as $\bigcup_{i\in I} f^{-1}(V_i)$ is open in X. Furthermore if $V_1, V_2 \in \mathcal{T}$ then

$$f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2)$$

So $V_1 \cap V_2 \in \mathcal{T}$ because $f^{-1}(V_1) \cap f^{-1}(V_2)$ is open in X.

Finally a topology on Y such that f is continuous is included in $\mathcal{T}$ according to the definition of a continuous function, so $\mathcal{T}$ is the strongest topology on Y such that f is continuous. It proves that (1) and (2) are equivalent.

Finally, (2) and (3) equivalence follows from $f^{-1}(X \setminus E) = Y \setminus f^{-1}(E)$ for all subsets $E \subset X$. ☐

Let $f : X \to Y$ be a continuous map of topological spaces. In this case we obtain a factorization $X \to f(X) \to Y$ of maps of sets. We can endow $f(X)$ with the quotient topology coming from the surjection $X \to f(X)$ or with the induced topology coming from the injection $f(X) \to Y$. The map

$$(f(X), \text{quotient topology}) \longrightarrow (f(X), \text{induced topology})$$

is continuous.

Definition 6.3. Let $f : X \to Y$ be a continuous map of topological spaces.

(1) We say f is a *strict map of topological spaces* if the induced topology and the quotient topology on $f(X)$ agree (see discussion above).
(2) We say f is *submersive*[1] if f is surjective and strict.

[1]This is very different from the notion of a submersion between differential manifolds! It is probably a good idea to use "strict and surjective" in stead of "submersive".

Thus a continuous map $f : X \to Y$ is submersive if f is a surjection and for any $T \subset Y$ we have T is open or closed if and only if $f^{-1}(T)$ is so. In other words, Y has the quotient topology relative to the surjection $X \to Y$.

Lemma 6.4. *Let $f : X \to Y$ be surjective, open, continuous map of topological spaces. Let $T \subset Y$ be a subset. Then*

1. $f^{-1}(\overline{T}) = \overline{f^{-1}(T)}$,
2. *$T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed,*
3. *$T \subset Y$ is open if and only if $f^{-1}(T)$ is open, and*
4. *$T \subset Y$ is locally closed if and only if $f^{-1}(T)$ is locally closed.*

In particular we see that f is submersive.

Proof. It is clear that $\overline{f^{-1}(T)} \subset f^{-1}(\overline{T})$. If $x \in X$, and $x \notin \overline{f^{-1}(T)}$, then there exists an open neighbourhood $x \in U \subset X$ with $U \cap f^{-1}(T) = \emptyset$. Since f is open we see that $f(U)$ is an open neighbourhood of $f(x)$ not meeting T. Hence $x \notin f^{-1}(\overline{T})$. This proves (1). Part (2) is an easy consequence of (1). Part (3) is obvious from the fact that f is open and surjective. For (4), if $f^{-1}(T)$ is locally closed, then $f^{-1}(T) \subset \overline{f^{-1}(T)} = f^{-1}(\overline{T})$ is open, and hence by (3) applied to the map $f^{-1}(\overline{T}) \to \overline{T}$ we see that T is open in $\overline{T}$, i.e., T is locally closed. $\square$

Lemma 6.5. *Let $f : X \to Y$ be surjective, closed, continuous map of topological spaces. Let $T \subset Y$ be a subset. Then*

1. $\overline{T} = f(\overline{f^{-1}(T)})$,
2. *$T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed,*
3. *$T \subset Y$ is open if and only if $f^{-1}(T)$ is open, and*
4. *$T \subset Y$ is locally closed if and only if $f^{-1}(T)$ is locally closed.*

In particular we see that f is submersive.

Proof. It is clear that $\overline{f^{-1}(T)} \subset f^{-1}(\overline{T})$. Then $T \subset f(\overline{f^{-1}(T)}) \subset \overline{T}$ is a closed subset, hence we get (1). Part (2) is obvious from the fact that f is closed and surjective. Part (3) follows from (2) applied to the complement of T. For (4), if $f^{-1}(T)$ is locally closed, then $f^{-1}(T) \subset \overline{f^{-1}(T)}$ is open. Since the map $\overline{f^{-1}(T)} \to \overline{T}$ is surjective by (1) we can apply part (3) to the map $\overline{f^{-1}(T)} \to \overline{T}$ induced by f to conclude that T is open in $\overline{T}$, i.e., T is locally closed. $\square$

7. Connected components

Definition 7.1. Let X be a topological space.

1. We say X is *connected* if X is not empty and whenever $X = T_1 \amalg T_2$ with $T_i \subset X$ open and closed, then either $T_1 = \emptyset$ or $T_2 = \emptyset$.
2. We say $T \subset X$ is a *connected component* of X if T is a maximal connected subset of X.

The empty space is not connected.

Lemma 7.2. *Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is a connected subset, then $f(E) \subset Y$ is connected as well.*

Proof. Let $A \subset f(E)$ an open and closed subset of $f(E)$. Because f is continuous, $f^{-1}(A)$ is an open and closed subset of E. As E is connected, $f^{-1}(A) = \emptyset$ or $f^{-1}(A) = E$. However, $A \subset f(E)$ implies that $A = f(f^{-1}(A))$. Indeed, if $x \in f(f^{-1}(A))$ then there is $y \in f^{-1}(A)$ such that $f(y) = x$ and because $y \in f^{-1}(A)$, we have $f(y) \in A$ i.e. $x \in A$. Reciprocally, if $x \in A$, $A \subset f(E)$ implies that there is $y \in E$ such that $f(y) = x$. Therefore $y \in f^{-1}(A)$, and then $x \in f(f^{-1}(A))$. Thus $A = \emptyset$ or $A = f(E)$ proving that $f(E)$ is connected. $\square$

Lemma 7.3. *Let X be a topological space.*

1. *If $T \subset X$ is connected, then so is its closure.*
2. *Any connected component of X is closed (but not necessarily open).*
3. *Every connected subset of X is contained in a unique connected component of X.*
4. *Every point of X is contained in a unique connected component, in other words, X is the disjoint union of its connected components.*

Proof. Let $\overline{T}$ be the closure of the connected subset T. Suppose $\overline{T} = T_1 \amalg T_2$ with $T_i \subset \overline{T}$ open and closed. Then $T = (T \cap T_1) \amalg (T \cap T_2)$. Hence T equals one of the two, say $T = T_1 \cap T$. Thus $\overline{T} \subset T_1$. This implies (1) and (2).

Let A be a nonempty set of connected subsets of X such that $\Omega = \bigcap_{T \in A} T$ is nonempty. We claim $E = \bigcup_{T \in A} T$ is connected. Namely, E is nonempty as it contains Ω. Say $E = E_1 \amalg E_2$ with E_i closed in E. We may assume E_1 meets Ω (after renumbering). Then each $T \in A$ meets E_1 and hence must be contained in E_1 as T is connected. Hence $E \subset E_1$ which proves the claim.

Let $W \subset X$ be a nonempty connected subset. If we apply the result of the previous paragraph to the set of all connected subsets of X containing W, then we see that E is a connected component of X. This implies existence and uniqueness in (3).

Let $x \in X$. Taking $W = \{x\}$ in the previous paragraph we see that x is contained in a unique connected component of X. Any two distinct connected components must be disjoint (by the result of the second paragraph).

To get an example where connected components are not open, just take an infinite product $\prod_{n \in \mathbf{N}} \{0, 1\}$ with the product topology. Its connected components are singletons, which are not open. $\square$

Remark 7.4. Let X be a topological space and $x \in X$. Let $Z \subset X$ be the connected component of X passing through x. Consider the intersection E of all open and closed subsets of X containing x. It is clear that $Z \subset E$. In general $Z \neq E$. For example, let $X = \{x, y, z_1, z_2, \ldots\}$ with the topology with the following basis of opens, $\{z_n\}$, $\{x, z_n, z_{n+1}, \ldots\}$, and $\{y, z_n, z_{n+1}, \ldots\}$ for all n. Then $Z = \{x\}$ and $E = \{x, y\}$. We omit the details.

Lemma 7.5. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that*

1. *all fibres of f are connected, and*
2. *a set $T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed.*

Then f induces a bijection between the sets of connected components of X and Y.

Proof. Let $T \subset Y$ be a connected component. Note that T is closed, see Lemma 7.3. The lemma follows if we show that $f^{-1}(T)$ is connected because any connected subset of X maps into a connected component of Y by Lemma 7.2. Suppose that $f^{-1}(T) = Z_1 \amalg Z_2$ with Z_1, Z_2 closed. For any $t \in T$ we see that $f^{-1}(\{t\}) = Z_1 \cap f^{-1}(\{t\}) \amalg Z_2 \cap f^{-1}(\{t\})$. By (1) we see $f^{-1}(\{t\})$ is connected we conclude that either $f^{-1}(\{t\}) \subset Z_1$ or $f^{-1}(\{t\}) \subset Z_2$. In other words $T = T_1 \amalg T_2$ with

$f^{-1}(T_i) = Z_i$. By (2) we conclude that T_i is closed in Y. Hence either $T_1 = \emptyset$ or $T_2 = \emptyset$ as desired. $\square$

Lemma 7.6. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) f is open, (b) all fibres of f are connected. Then f induces a bijection between the sets of connected components of X and Y.*

Proof. This is a special case of Lemma 7.5. $\square$

Lemma 7.7. *Let $f : X \to Y$ be a continuous map of nonempty topological spaces. Assume that (a) Y is connected, (b) f is open and closed, and (c) there is a point $y \in Y$ such that the fiber $f^{-1}(y)$ is a finite set. Then X has at most $|f^{-1}(y)|$ connected components. Hence any connected component T of X is open and closed, and $f(T)$ is a nonempty open and closed subset of Y, which is therefore equal to Y.*

Proof. If the topological space X has at least N connected components for some $N \in \mathbf{N}$, we find by induction a decomposition $X = X_1 \amalg \ldots \amalg X_N$ of X as a disjoint union of N nonempty open and closed subsets $X_1, \ldots, X_N$ of X. As f is open and closed, each $f(X_i)$ is a nonempty open and closed subset of Y and is hence equal to Y. In particular the intersection $X_i \cap f^{-1}(y)$ is nonempty for each $1 \leq i \leq N$. Hence $f^{-1}(y)$ has at least N elements. $\square$

Definition 7.8. A topological space is *totally disconnected* if the connected components are all singletons.

A discrete space is totally disconnected. A totally disconnected space need not be discrete, for example $\mathbf{Q} \subset \mathbf{R}$ is totally disconnected but not discrete.

Lemma 7.9. *Let X be a topological space. Let $\pi_0(X)$ be the set of connected components of X. Let $X \to \pi_0(X)$ be the map which sends $x \in X$ to the connected component of X passing through x. Endow $\pi_0(X)$ with the quotient topology. Then $\pi_0(X)$ is a totally disconnected space and any continuous map $X \to Y$ from X to a totally disconnected space Y factors through $\pi_0(X)$.*

Proof. By Lemma 7.5 the connected components of $\pi_0(X)$ are the singletons. We omit the proof of the second statement. $\square$

Definition 7.10. A topological space X is called *locally connected* if every point $x \in X$ has a fundamental system of connected neighbourhoods.

Lemma 7.11. *Let X be a topological space. If X is locally connected, then*

(1) *any open subset of X is locally connected, and*
(2) *the connected components of X are open.*

So also the connected components of open subsets of X are open. In particular, every point has a fundamental system of open connected neighbourhoods.

Proof. For all $x \in X$ let write $\mathcal{N}(x)$ the fundamental system of connected neighbourhoods of x and let $U \subset X$ be an open subset of X. Then for all $x \in U$, U is a neighbourhood of x, so the set $\{V \in \mathcal{N}(x) | V \subset U\}$ is not empty and is a fundamental system of connected neighbourhoods of x in U. Thus U is locally connected and it proves (1).

Let $x \in \mathcal{C} \subset X$ where $\mathcal{C}$ is the connected component of x. Because X is locally connected, there exists $\mathcal{N}$ a connected neighbourhood of x. Therefore by the definition of a connected component, we have $\mathcal{N} \subset \mathcal{C}$ and then $\mathcal{C}$ is a neighbourhood of x. It implies that $\mathcal{C}$ is a neighbourhood of each of his point, in other words $\mathcal{C}$ is open and (2) is proven. □

8. Irreducible components

Definition 8.1. Let X be a topological space.

(1) We say X is *irreducible*, if X is not empty, and whenever $X = Z_1 \cup Z_2$ with Z_i closed, we have $X = Z_1$ or $X = Z_2$.
(2) We say $Z \subset X$ is an *irreducible component* of X if Z is a maximal irreducible subset of X.

An irreducible space is obviously connected.

Lemma 8.2. *Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is an irreducible subset, then $f(E) \subset Y$ is irreducible as well.*

Proof. Clearly we may assume $E = X$ (i.e., X irreducible) and $f(E) = Y$ (i.e., f surjective). First, $Y \neq \emptyset$ since $X \neq \emptyset$. Next, assume $Y = Y_1 \cup Y_2$ with Y_1, Y_2 closed. Then $X = X_1 \cup X_2$ with $X_i = f^{-1}(Y_i)$ closed in X. By assumption on X, we must have $X = X_1$ or $X = X_2$, hence $Y = Y_1$ or $Y = Y_2$ since f is surjective. □

Lemma 8.3. *Let X be a topological space.*

(1) *If $T \subset X$ is irreducible so is its closure in X.*
(2) *Any irreducible component of X is closed.*
(3) *Any irreducible subset of X is contained in an irreducible component of X.*
(4) *Every point of X is contained in some irreducible component of X, in other words, X is the union of its irreducible components.*

Proof. Let $\overline{T}$ be the closure of the irreducible subset T. If $\overline{T} = Z_1 \cup Z_2$ with $Z_i \subset \overline{T}$ closed, then $T = (T \cap Z_1) \cup (T \cap Z_2)$ and hence T equals one of the two, say $T = Z_1 \cap T$. Thus clearly $\overline{T} \subset Z_1$. This proves (1). Part (2) follows immediately from (1) and the definition of irreducible components.

Let $T \subset X$ be irreducible. Consider the set A of irreducible subsets $T \subset T_\alpha \subset X$. Note that A is nonempty since $T \in A$. There is a partial ordering on A coming from inclusion: $\alpha \leq \alpha' \Leftrightarrow T_\alpha \subset T_{\alpha'}$. Choose a maximal totally ordered subset $A' \subset A$, and let $T' = \bigcup_{\alpha \in A'} T_\alpha$. We claim that T' is irreducible. Namely, suppose that $T' = Z_1 \cup Z_2$ is a union of two closed subsets of T'. For each $\alpha \in A'$ we have either $T_\alpha \subset Z_1$ or $T_\alpha \subset Z_2$, by irreducibility of T_α. Suppose that for some $\alpha_0 \in A'$ we have $T_{\alpha_0} \not\subset Z_1$ (say, if not we're done anyway). Then, since A' is totally ordered we see immediately that $T_\alpha \subset Z_2$ for all $\alpha \in A'$. Hence $T' = Z_2$. This proves (3). Part (4) is an immediate consequence of (3) as a singleton space is irreducible. □

Lemma 8.4. *Let X be a topological space and suppose $X = \bigcup_{i=1,\ldots,n} X_i$ where each X_i is an irreducible closed subset of X and no X_i is contained in the union of the other members. Then each X_i is an irreducible component of X and each irreducible component of X is one of the X_i.*

Proof. Let Y be an irreducible component of X. Write $Y = \bigcup_{i=1,\ldots,n}(Y \cap X_i)$ and note that each $Y \cap X_i$ is closed in Y since X_i is closed in X. By irreducibility of Y we see that Y is equal to one of the $Y \cap X_i$, i.e., $Y \subset X_i$. By maximality of irreducible components we get $Y = X_i$.

Conversely, take one of the X_i and expand it to an irreducible component Y, which we have already shown is one of the X_j. So $X_i \subset X_j$ and since the original union does not have redundant members, $X_i = X_j$, which is an irreducible component. $\square$

Lemma 8.5. *Let $f : X \to Y$ be a surjective, continuous map of topological spaces. If X has a finite number, say n, of irreducible components, then Y has $\leq n$ irreducible components.*

Proof. Say $X_1, \ldots, X_n$ are the irreducible components of X. By Lemmas 8.2 and 8.3 the closure $Y_i \subset Y$ of $f(X_i)$ is irreducible. Since f is surjective, we see that Y is the union of the Y_i. We may choose a minimal subset $I \subset \{1, \ldots, n\}$ such that $Y = \bigcup_{i \in I} Y_i$. Then we may apply Lemma 8.4 to see that the Y_i for $i \in I$ are the irreducible components of Y. $\square$

A singleton is irreducible. Thus if $x \in X$ is a point then the closure $\overline{\{x\}}$ is an irreducible closed subset of X.

Definition 8.6. Let X be a topological space.

(1) Let $Z \subset X$ be an irreducible closed subset. A *generic point* of Z is a point $\xi \in Z$ such that $Z = \overline{\{\xi\}}$.
(2) The space X is called *Kolmogorov*, if for every $x, x' \in X$, $x \neq x'$ there exists a closed subset of X which contains exactly one of the two points.
(3) The space X is called *quasi-sober* if every irreducible closed subset has a generic point.
(4) The space X is called *sober* if every irreducible closed subset has a unique generic point.

A topological space X is Kolmogorov, quasi-sober, resp. sober if and only if the map $x \mapsto \overline{\{x\}}$ from X to the set of irreducible closed subsets of X is injective, surjective, resp. bijective. Hence we see that a topological space is sober if and only if it is quasi-sober and Kolmogorov.

Lemma 8.7. *Let X be a topological space and let $Y \subset X$.*

(1) *If X is Kolmogorov then so is Y.*
(2) *Suppose Y is locally closed in X. If X is quasi-sober then so is Y.*
(3) *Suppose Y is locally closed in X. If X is sober then so is Y.*

Proof. Proof of (1). Suppose X is Kolmogorov. Let $x, y \in Y$ with $x \neq y$. Then $\overline{\{x\}} \cap Y = \overline{\{x\}} \neq \overline{\{y\}} = \overline{\{y\}} \cap Y$. Hence $\overline{\{x\}} \cap Y \neq \overline{\{y\}} \cap Y$. This shows that Y is Kolmogorov.

Proof of (2). Suppose X is quasi-sober. It suffices to consider the cases Y is closed and Y is open. First, suppose Y is closed. Let Z be an irreducible closed subset of Y. Then Z is an irreducible closed subset of X. Hence there exists $x \in Z$ with $\overline{\{x\}} = Z$. It follows $\overline{\{x\}} \cap Y = Z$. This shows Y is quasi-sober. Second, suppose Y is open. Let Z be an irreducible closed subset of Y. Then $\overline{Z}$ is an irreducible closed subset of X. Hence there exists $x \in \overline{Z}$ with $\overline{\{x\}} = \overline{Z}$. If $x \notin Y$ we get the contradiction $Z = Z \cap Y \subset \overline{Z} \cap Y = \overline{\{x\}} \cap Y = \emptyset$. Therefore $x \in Y$. It follows $Z = \overline{Z} \cap Y = \overline{\{x\}} \cap Y$. This shows Y is quasi-sober.

Proof of (3). Immediately from (1) and (2). $\square$

Lemma 8.8. *Let X be a topological space and let $(X_i)_{i \in I}$ be a covering of X.*

(1) *Suppose X_i is locally closed in X for every $i \in I$. Then, X is Kolmogorov if and only if X_i is Kolmogorov for every $i \in I$.*
(2) *Suppose X_i is open in X for every $i \in I$. Then, X is quasi-sober if and only if X_i is quasi-sober for every $i \in I$.*
(3) *Suppose X_i is open in X for every $i \in I$. Then, X is sober if and only if X_i is sober for every $i \in I$.*

Proof. Proof of (1). If X is Kolmogorov then so is X_i for every $i \in I$ by Lemma 8.7. Suppose X_i is Kolmogorov for every $i \in I$. Let $x, y \in X$ with $\overline{\{x\}} = \overline{\{y\}}$. There exists $i \in I$ with $x \in X_i$. There exists an open subset $U \subset X$ such that X_i is a closed subset of U. If $y \notin U$ we get the contradiction $x \in \overline{\{x\}} \cap U = \overline{\{y\}} \cap U = \emptyset$. Hence $y \in U$. It follows $y \in \overline{\{y\}} \cap U = \overline{\{x\}} \cap U \subset X_i$. This shows $y \in X_i$. It follows $\overline{\{x\}} \cap X_i = \overline{\{y\}} \cap X_i$. Since X_i is Kolmogorov we get $x = y$. This shows X is Kolmogorov.

Proof of (2). If X is quasi-sober then so is X_i for every $i \in I$ by Lemma 8.7. Suppose X_i is quasi-sober for every $i \in I$. Let Y be an irreducible closed subset of X. As $Y \neq \emptyset$ there exists $i \in I$ with $X_i \cap Y \neq \emptyset$. As X_i is open in X it follows $X_i \cap Y$ is non-empty and open in Y, hence irreducible and dense in Y. Thus $X_i \cap Y$ is an irreducible closed subset of X_i. As X_i is quasi-sober there exists $x \in X_i \cap Y$ with $X_i \cap Y = \overline{\{x\}} \cap X_i \subset \overline{\{x\}}$. Since $X_i \cap Y$ is dense in Y and Y is closed in X it follows $Y = \overline{X_i \cap Y} \cap Y \subset \overline{X_i \cap Y} \subset \overline{\{x\}} \subset Y$. Therefore $Y = \overline{\{x\}}$. This shows X is quasi-sober.

Proof of (3). Immediately from (1) and (2). $\square$

Example 8.9. Let X be an indiscrete space of cardinality at least 2. Then X is quasi-sober but not Kolmogorov. Moreover, the family of its singletons is a covering of X by discrete and hence Kolmogorov spaces.

Example 8.10. Let Y be an infinite set, furnished with the topology whose closed sets are Y and the finite subsets of Y. Then Y is Kolmogorov but not quasi-sober. However, the family of its singletons (which are its irreducible components) is a covering by discrete and hence sober spaces.

Example 8.11. Let X and Y be as in Example 8.9 and Example 8.10. Then, $X \amalg Y$ is neither Kolmogorov nor quasi-sober.

Example 8.12. Let Z be an infinite set and let $z \in Z$. We furnish Z with the topology whose closed sets are Z and the finite subsets of $Z \setminus \{z\}$. Then Z is sober but its subspace $Z \setminus \{z\}$ is not quasi-sober.

Example 8.13. Recall that a topological space X is Hausdorff iff for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U$, $y \in V$. In this case X is irreducible if and only if X is a singleton. Similarly, any subset of X is irreducible if and only if it is a singleton. Hence a Hausdorff space is sober.

Lemma 8.14. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) Y is irreducible, (b) f is open, and (c) there exists a dense collection of points $y \in Y$ such that $f^{-1}(y)$ is irreducible. Then X is irreducible.*

Proof. Suppose $X = Z_1 \cup Z_2$ with Z_i closed. Consider the open sets $U_1 = Z_1 \setminus Z_2 = X \setminus Z_2$ and $U_2 = Z_2 \setminus Z_1 = X \setminus Z_1$. To get a contradiction assume that U_1 and U_2 are both nonempty. By (b) we see that $f(U_i)$ is open. By (a) we have Y irreducible and hence $f(U_1) \cap f(U_2) \neq \emptyset$. By (c) there is a point y which corresponds to a point of this intersection such that the fibre $X_y = f^{-1}(y)$ is irreducible. Then $X_y \cap U_1$ and $X_y \cap U_2$ are nonempty disjoint open subsets of X_y which is a contradiction. $\square$

Lemma 8.15. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) f is open, and (b) for every $y \in Y$ the fibre $f^{-1}(y)$ is irreducible. Then f induces a bijection between irreducible components.*

Proof. We point out that assumption (b) implies that f is surjective (see Definition 8.1). Let $T \subset Y$ be an irreducible component. Note that T is closed, see Lemma 8.3. The lemma follows if we show that $f^{-1}(T)$ is irreducible because any irreducible subset of X maps into an irreducible component of Y by Lemma 8.2. Note that $f^{-1}(T) \to T$ satisfies the assumptions of Lemma 8.14. Hence we win. $\square$

The construction of the following lemma is sometimes called the "soberification".

Lemma 8.16. *Let X be a topological space. There is a canonical continuous map*

$$c : X \longrightarrow X'$$

from X to a sober topological space X' which is universal among continuous maps from X to sober topological spaces. Moreover, the assignment $U' \mapsto c^{-1}(U')$ is a bijection between opens of X' and X which commutes with finite intersections and arbitrary unions. The image $c(X)$ is a Kolmogorov topological space and the map $c : X \to c(X)$ is universal for maps of X into Kolmogorov spaces.

Proof. Let X' be the set of irreducible closed subsets of X and let

$$c : X \to X', \quad x \mapsto \overline{\{x\}}.$$

For $U \subset X$ open, let $U' \subset X'$ denote the set of irreducible closed subsets of X which meet U. Then $c^{-1}(U') = U$. In particular, if $U_1 \neq U_2$ are open in X, then $U_1' \neq U_2'$. Hence c induces a bijection between the subsets of X' of the form U' and the opens of X.

Let U_1, U_2 be open in X. Suppose that $Z \in U_1'$ and $Z \in U_2'$. Then $Z \cap U_1$ and $Z \cap U_2$ are nonempty open subsets of the irreducible space Z and hence $Z \cap U_1 \cap U_2$ is nonempty. Thus $(U_1 \cap U_2)' = U_1' \cap U_2'$. The rule $U \mapsto U'$ is also compatible with arbitrary unions (details omitted). Thus it is clear that the collection of U' form a topology on X' and that we have a bijection as stated in the lemma.

Next we show that X' is sober. Let $T \subset X'$ be an irreducible closed subset. Let $U \subset X$ be the open such that $X' \setminus T = U'$. Then $Z = X \setminus U$ is irreducible because of the properties of the bijection of the lemma. We claim that $Z \in T$ is the unique generic point. Namely, any open of the form $V' \subset X'$ which does not contain Z must come from an open $V \subset X$ which misses Z, i.e., is contained in U.

Finally, we check the universal property. Let $f : X \to Y$ be a continuous map to a sober topological space. Then we let $f' : X' \to Y$ be the map which sends the irreducible closed $Z \subset X$ to the unique generic point of $\overline{f(Z)}$. It follows immediately that $f' \circ c = f$ as maps of sets, and the properties of c imply that f' is continuous. We omit the verification that the continuous map f' is unique. We also omit the proof of the statements on Kolmogorov spaces. $\square$

Lemma 8.17. *Let X be a connected topological space with a finite number of irreducible components $X_1, \ldots, X_n$. If $n > 1$ there is an $1 \leq j \leq n$ such that $X' = \bigcup_{i \neq j} X_i$ is connected.*

Proof. This is a graph theory problem. Let Γ be the graph with vertices $V = \{1, \ldots, n\}$ and an edge between i and j if and only if $X_i \cap X_j$ is nonempty. Connectedness of X means that Γ is connected. Our problem is to find $1 \leq j \leq n$ such that $\Gamma \setminus \{j\}$ is still connected. You can do this by choosing $j, j' \in E$ with maximal distance and then j works (choose a leaf!). Details omitted. $\square$

9. Noetherian topological spaces

Definition 9.1. A topological space is called *Noetherian* if the descending chain condition holds for closed subsets of X. A topological space is called *locally Noetherian* if every point has a neighbourhood which is Noetherian.

Lemma 9.2. *Let X be a Noetherian topological space.*

(1) *Any subset of X with the induced topology is Noetherian.*
(2) *The space X has finitely many irreducible components.*
(3) *Each irreducible component of X contains a nonempty open of X.*

Proof. Let $T \subset X$ be a subset of X. Let $T_1 \supset T_2 \supset \ldots$ be a descending chain of closed subsets of T. Write $T_i = T \cap Z_i$ with $Z_i \subset X$ closed. Consider the descending chain of closed subsets $Z_1 \supset Z_1 \cap Z_2 \supset Z_1 \cap Z_2 \cap Z_3 \ldots$ This stabilizes by assumption and hence the original sequence of T_i stabilizes. Thus T is Noetherian.

Let A be the set of closed subsets of X which do not have finitely many irreducible components. Assume that A is not empty to arrive at a contradiction. The set A is partially ordered by inclusion: $\alpha \leq \alpha' \Leftrightarrow Z_\alpha \subset Z_{\alpha'}$. By the descending chain condition we may find a smallest element of A, say Z. As Z is not a finite union of irreducible components, it is not irreducible. Hence we can write $Z = Z' \cup Z''$ and both are strictly smaller closed subsets. By construction $Z' = \bigcup Z'_i$ and $Z'' = \bigcup Z''_j$ are finite unions of their irreducible components. Hence $Z = \bigcup Z'_i \cup \bigcup Z''_j$ is a finite union of irreducible closed subsets. After removing redundant members of this expression, this will be the decomposition of Z into its irreducible components (Lemma 8.4), a contradiction.

Let $Z \subset X$ be an irreducible component of X. Let $Z_1, \ldots, Z_n$ be the other irreducible components of X. Consider $U = Z \setminus (Z_1 \cup \ldots \cup Z_n)$. This is not empty since otherwise the irreducible space Z would be contained in one of the other Z_i. Because $X = Z \cup Z_1 \cup \ldots Z_n$ (see Lemma 8.3), also $U = X \setminus (Z_1 \cup \ldots \cup Z_n)$ and hence open in X. Thus Z contains a nonempty open of X. $\square$

Lemma 9.3. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If X is Noetherian, then $f(X)$ is Noetherian.*
(2) *If X is locally Noetherian and f open, then $f(X)$ is locally Noetherian.*

Proof. In case (1), suppose that $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ is a descending chain of closed subsets of $f(X)$ (as usual with the induced topology as a subset of Y). Then $f^{-1}(Z_1) \supset f^{-1}(Z_2) \supset f^{-1}(Z_3) \supset \ldots$ is a descending chain of closed subsets of X. Hence this chain stabilizes. Since $f(f^{-1}(Z_i)) = Z_i$ we conclude that $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ stabilizes also. In case (2), let $y \in f(X)$. Choose $x \in X$ with $f(x) = y$. By assumption there exists a neighbourhood $E \subset X$ of x which is Noetherian. Then $f(E) \subset f(X)$ is a neighbourhood which is Noetherian by part (1). $\square$

Lemma 9.4. *Let X be a topological space. Let $X_i \subset X$, $i = 1, \ldots, n$ be a finite collection of subsets. If each X_i is Noetherian (with the induced topology), then $\bigcup_{i=1,\ldots,n} X_i$ is Noetherian (with the induced topology).*

Proof. Let $\{F_m\}_{m\in\mathbf{N}}$ a decreasing sequence of closed subsets of $X' = \bigcup_{i=1,\ldots,n} X_i$ with the induced topology. Then we can find a decreasing sequence $\{G_m\}_{m\in\mathbf{N}}$ of closed subsets of X verifying $F_m = G_m \cap X'$ for all m (small detail omitted). As X_i is noetherian and $\{G_m \cap X_i\}_{m\in\mathbf{N}}$ a decreasing sequence of closed subsets of X_i, there exists $m_i \in \mathbf{N}$ such that for all $m \geq m_i$ we have $G_m \cap X_i = G_{m_i} \cap X_i$. Let $m_0 = \max_{i=1,\ldots,n} m_i$. Then clearly

$$F_m = G_m \cap X' = G_m \cap (X_1 \cup \ldots \cup X_n) = (G_m \cap X_1) \cup \ldots (G_m \cap X_n)$$

stabilizes for $m \geq m_0$ and the proof is complete. $\square$

Example 9.5. Any nonempty, Kolmogorov Noetherian topological space has a closed point (combine Lemmas 12.8 and 12.13). Let $X = \{1, 2, 3, \ldots\}$. Define a topology on X with opens $\emptyset$, $\{1, 2, \ldots, n\}$, $n \geq 1$ and X. Thus X is a locally Noetherian topological space, without any closed points. This space cannot be the underlying topological space of a locally Noetherian scheme, see Properties, Lemma 5.9.

Lemma 9.6. *Let X be a locally Noetherian topological space. Then X is locally connected.*

Proof. Let $x \in X$. Let E be a neighbourhood of x. We have to find a connected neighbourhood of x contained in E. By assumption there exists a neighbourhood E' of x which is Noetherian. Then $E \cap E'$ is Noetherian, see Lemma 9.2. Let $E \cap E' = Y_1 \cup \ldots \cup Y_n$ be the decomposition into irreducible components, see Lemma 9.2. Let $E'' = \bigcup_{x\in Y_i} Y_i$. This is a connected subset of $E \cap E'$ containing x. It contains the open $E \cap E' \setminus (\bigcup_{x\notin Y_i} Y_i)$ of $E \cap E'$ and hence it is a neighbourhood of x in X. This proves the lemma. $\square$

10. Krull dimension

Definition 10.1. Let X be a topological space.

(1) A *chain of irreducible closed subsets* of X is a sequence $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ with Z_i closed irreducible and $Z_i \neq Z_{i+1}$ for $i = 0, \ldots, n-1$.
(2) The *length* of a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ of irreducible closed subsets of X is the integer n.
(3) The *dimension* or more precisely the *Krull dimension* $\dim(X)$ of X is the element of $\{-\infty, 0, 1, 2, 3, \ldots, \infty\}$ defined by the formula:
$$\dim(X) = \sup\{\text{lengths of chains of irreducible closed subsets}\}$$
Thus $\dim(X) = -\infty$ if and only if X is the empty space.
(4) Let $x \in X$. The *Krull dimension of X at x* is defined as
$$\dim_x(X) = \min\{\dim(U), x \in U \subset X \text{ open}\}$$
the minimum of $\dim(U)$ where U runs over the open neighbourhoods of x in X.

Note that if $U' \subset U \subset X$ are open then $\dim(U') \leq \dim(U)$. Hence if $\dim_x(X) = d$ then x has a fundamental system of open neighbourhoods U with $\dim(U) = \dim_x(X)$.

Lemma 10.2. *Let X be a topological space. Then $\dim(X) = \sup \dim_x(X)$ where the supremum runs over the points x of X.*

Proof. It is clear that $\dim(X) \geq \dim_x(X)$ for all $x \in X$ (see discussion following Definition 10.1). Thus an inequality in one direction. For the converse, let $n \geq 0$ and suppose that $\dim(X) \geq n$. Then we can find a chain of irreducible closed subsets $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$. Pick $x \in Z_0$. For every open neighbourhood U of x we get a chain of irreducible closed subsets

$$Z_0 \cap U \subset Z_1 \cap U \subset \ldots \subset Z_n \cap U$$

in U. Namely, the sets $U \cap Z_i$ are irreducible closed in U and the inclusions are strict (details omitted; hint: the closure of $U \cap Z_i$ is Z_i). In this way we see that $\dim_x(X) \geq n$ which proves the other inequality. □

Example 10.3. The Krull dimension of the usual Euclidean space $\mathbf{R}^n$ is 0.

Example 10.4. Let $X = \{s, \eta\}$ with open sets given by $\{\emptyset, \{\eta\}, \{s, \eta\}\}$. In this case a maximal chain of irreducible closed subsets is $\{s\} \subset \{s, \eta\}$. Hence $\dim(X) = 1$. It is easy to generalize this example to get a $(n + 1)$-element topological space of Krull dimension n.

Definition 10.5. Let X be a topological space. We say that X is *equidimensional* if every irreducible component of X has the same dimension.

11. Codimension and catenary spaces

We only define the codimension of irreducible closed subsets.

Definition 11.1. Let X be a topological space. Let $Y \subset X$ be an irreducible closed subset. The *codimension* of Y in X is the supremum of the lengths e of chains

$$Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e \subset X$$

of irreducible closed subsets in X starting with Y. We will denote this $\text{codim}(Y, X)$.

The codimension is an element of $\{0, 1, 2, \ldots\} \cup \{\infty\}$. If $\text{codim}(Y, X) < \infty$, then every chain can be extended to a maximal chain (but these do not all have to have the same length).

Lemma 11.2. *Let X be a topological space. Let $Y \subset X$ be an irreducible closed subset. Let $U \subset X$ be an open subset such that $Y \cap U$ is nonempty. Then*

$$\mathit{codim}(Y, X) = \mathit{codim}(Y \cap U, U)$$

Proof. The rule $T \mapsto \overline{T}$ defines a bijective inclusion preserving map between the closed irreducible subsets of U and the closed irreducible subsets of X which meet U. Using this the lemma easily follows. Details omitted. □

Example 11.3. Let $X = [0, 1]$ be the unit interval with the following topology: The sets $[0, 1]$, $(1 - 1/n, 1]$ for $n \in \mathbf{N}$, and $\emptyset$ are open. So the closed sets are $\emptyset$, $\{0\}$, $[0, 1 - 1/n]$ for $n > 1$ and $[0, 1]$. This is clearly a Noetherian topological space. But the irreducible closed subset $Y = \{0\}$ has infinite codimension $\text{codim}(Y, X) = \infty$. To see this we just remark that all the closed sets $[0, 1 - 1/n]$ are irreducible.

Definition 11.4. Let X be a topological space. We say X is *catenary* if for every pair of irreducible closed subsets $T \subset T'$ we have $\operatorname{codim}(T, T') < \infty$ and every maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

has the same length (equal to the codimension).

Lemma 11.5. *Let X be a topological space. The following are equivalent:*

(1) *X is catenary,*
(2) *X has an open covering by catenary spaces.*

Moreover, in this case any locally closed subspace of X is catenary.

Proof. Suppose that X is catenary and that $U \subset X$ is an open subset. The rule $T \mapsto \overline{T}$ defines a bijective inclusion preserving map between the closed irreducible subsets of U and the closed irreducible subsets of X which meet U. Using this the lemma easily follows. Details omitted. $\square$

Lemma 11.6. *Let X be a topological space. The following are equivalent:*

(1) *X is catenary, and*
(2) *for every pair of irreducible closed subsets $Y \subset Y'$ we have $codim(Y, Y') < \infty$ and for every triple $Y \subset Y' \subset Y''$ of irreducible closed subsets we have*

$$codim(Y, Y'') = codim(Y, Y') + codim(Y', Y'').$$

Proof. Let suppose that X is catenary. According to Definition 11.4, for every pair of irreducible closed subsets $Y \subset Y'$ we have $\operatorname{codim}(Y, Y') < \infty$. Let $Y \subset Y' \subset Y''$ be a triple of irreducible closed subsets of X. Let

$$Y = Y_0 \subset Y_1 \subset ... \subset Y_{e_1} = Y'$$

be a maximal chain of irreducible closed subsets between Y and Y' where $e_1 = \operatorname{codim}(Y, Y')$. Let also

$$Y' = Y_{e_1} \subset Y_{e_1+1} \subset ... \subset Y_{e_1+e_2} = Y''$$

be a maximal chain of irreducible closed subsets between Y' and Y'' where $e_2 = \operatorname{codim}(Y', Y'')$. As the two chains are maximal, the concatenation

$$Y = Y_0 \subset Y_1 \subset ... \subset Y_{e_1} = Y' = Y_{e_1} \subset Y_{e_1+1} \subset ... \subset Y_{e_1+e_2} = Y''$$

is maximal too (between Y and Y'') and its length equals to $e_1 + e_2$. As X is catenary, each maximal chain has the same length equals to the codimension. Thus the point (2) that $\operatorname{codim}(Y, Y'') = e_1 + e_2 = \operatorname{codim}(Y, Y') + \operatorname{codim}(Y', Y'')$ is verified.

For the reciprocal, we show by induction that : if $Y = Y_1 \subset ... \subset Y_n = Y'$, then $\operatorname{codim}(Y, Y') = \operatorname{codim}(Y_1, Y_2) + ... + \operatorname{codim}(Y_{n-1}, Y_n)$. Therefore, it forces maximal chains to have the same length. $\square$

12. Quasi-compact spaces and maps

The phrase "compact" will be reserved for Hausdorff topological spaces. And many spaces occurring in algebraic geometry are not Hausdorff.

Definition 12.1. Quasi-compactness.

(1) We say that a topological space X is *quasi-compact* if every open covering of X has a finite subcover.

(2) We say that a continuous map $f : X \to Y$ is *quasi-compact* if the inverse image $f^{-1}(V)$ of every quasi-compact open $V \subset Y$ is quasi-compact.
(3) We say a subset $Z \subset X$ is *retrocompact* if the inclusion map $Z \to X$ is quasi-compact.

In many texts on topology a space is called *compact* if it is quasi-compact and Hausdorff; and in other texts the Hausdorff condition is omitted. To avoid confusion in algebraic geometry we use the term quasi-compact. The notion of quasi-compactness of a map is very different from the notion of a “proper map”, since there we require (besides closedness and separatedness) the inverse image of any quasi-compact subset of the target to be quasi-compact, whereas in the definition above we only consider quasi-compact *open* sets.

Lemma 12.2. *A composition of quasi-compact maps is quasi-compact.*

Proof. This is immediate from the definition. □

Lemma 12.3. *A closed subset of a quasi-compact topological space is quasi-compact.*

Proof. Let $E \subset X$ be a closed subset of the quasi-compact space X. Let $E = \bigcup V_j$ be an open covering. Choose $U_j \subset X$ open such that $V_j = E \cap U_j$. Then $X = (X \setminus E) \cup \bigcup U_j$ is an open covering of X. Hence $X = (X \setminus E) \cup U_{j_1} \cup \ldots \cup U_{j_n}$ for some n and indices j_i. Thus $E = V_{j_1} \cup \ldots \cup V_{j_n}$ as desired. □

Lemma 12.4. *Let X be a Hausdorff topological space.*
(1) *If $E \subset X$ is quasi-compact, then it is closed.*
(2) *If $E_1, E_2 \subset X$ are disjoint quasi-compact subsets then there exists opens $E_i \subset U_i$ with $U_1 \cap U_2 = \emptyset$.*

Proof. Proof of (1). Let $x \in X$, $x \notin E$. For every $e \in E$ we can find disjoint opens V_e and U_e with $e \in V_e$ and $x \in U_e$. Since $E \subset \bigcup V_e$ we can find finitely many $e_1, \ldots, e_n$ such that $E \subset V_{e_1} \cup \ldots \cup V_{e_n}$. Then $U = U_{e_1} \cap \ldots \cap U_{e_n}$ is an open neighbourhood of x which avoids $V_{e_1} \cup \ldots \cup V_{e_n}$. In particular it avoids E. Thus E is closed.

Proof of (2). In the proof of (1) we have seen that given $x \in E_1$ we can find an open neighbourhood $x \in U_x$ and an open $E_2 \subset V_x$ such that $U_x \cap V_x = \emptyset$. Because E_1 is quasi-compact we can find a finite number $x_i \in E_1$ such that $E_1 \subset U = U_{x_1} \cup \ldots \cup U_{x_n}$. We take $V = V_{x_1} \cap \ldots \cap V_{x_n}$ to finish the proof. □

Lemma 12.5. *Let X be a quasi-compact Hausdorff space. Let $E \subset X$. The following are equivalent: (a) E is closed in X, (b) E is quasi-compact.*

Proof. The implication (a) $\Rightarrow$ (b) is Lemma 12.3. The implication (b) $\Rightarrow$ (a) is Lemma 12.4. □

The following is really a reformulation of the quasi-compact property.

Lemma 12.6. *Let X be a quasi-compact topological space. If $\{Z_\alpha\}_{\alpha \in A}$ is a collection of closed subsets such that the intersection of each finite subcollection is nonempty, then $\bigcap_{\alpha \in A} Z_\alpha$ is nonempty.*

Proof. We suppose that $\bigcap_{\alpha \in A} Z_\alpha = \emptyset$. So we have $\bigcup_{\alpha \in A}(X \setminus Z_\alpha) = X$ by complementation. As the subsets Z_α are closed, $\bigcup_{\alpha \in A}(X \setminus Z_\alpha)$ is an open covering of the quasi-compact space X. Thus there exists a finite subset $J \subset A$ such that $X =$

$\bigcup_{\alpha \in J}(X \setminus Z_\alpha)$. The complementary is then empty, which means that $\bigcap_{\alpha \in J} Z_\alpha = \emptyset$. It proves there exists a finite subcollection of $\{Z_\alpha\}_{\alpha \in J}$ verifying $\bigcap_{\alpha \in J} Z_\alpha = \emptyset$, which concludes by contraposition. $\square$

Lemma 12.7. *Let $f : X \to Y$ be a continuous map of topological spaces.*

1. *If X is quasi-compact, then $f(X)$ is quasi-compact.*
2. *If f is quasi-compact, then $f(X)$ is retrocompact.*

Proof. If $f(X) = \bigcup V_i$ is an open covering, then $X = \bigcup f^{-1}(V_i)$ is an open covering. Hence if X is quasi-compact then $X = f^{-1}(V_{i_1}) \cup \ldots \cup f^{-1}(V_{i_n})$ for some $i_1, \ldots, i_n \in I$ and hence $f(X) = V_{i_1} \cup \ldots \cup V_{i_n}$. This proves (1). Assume f is quasi-compact, and let $V \subset Y$ be quasi-compact open. Then $f^{-1}(V)$ is quasi-compact, hence by (1) we see that $f(f^{-1}(V)) = f(X) \cap V$ is quasi-compact. Hence $f(X)$ is retrocompact. $\square$

Lemma 12.8. *Let X be a topological space. Assume that*

1. *X is nonempty,*
2. *X is quasi-compact, and*
3. *X is Kolmogorov.*

Then X has a closed point.

Proof. Consider the set

$$\mathcal{T} = \{Z \subset X \mid Z = \overline{\{x\}} \text{ for some } x \in X\}$$

of all closures of singletons in X. It is nonempty since X is nonempty. Make $\mathcal{T}$ into a partially ordered set using the relation of inclusion. Suppose Z_α, $\alpha \in A$ is a totally ordered subset of $\mathcal{T}$. By Lemma 12.6 we see that $\bigcap_{\alpha \in A} Z_\alpha \neq \emptyset$. Hence there exists some $x \in \bigcap_{\alpha \in A} Z_\alpha$ and we see that $Z = \overline{\{x\}} \in \mathcal{T}$ is a lower bound for the family. By Zorn's lemma there exists a minimal element $Z \in \mathcal{T}$. As X is Kolmogorov we conclude that $Z = \{x\}$ for some x and $x \in X$ is a closed point. $\square$

Lemma 12.9. *Let X be a quasi-compact Kolmogorov space. Then the set X_0 of closed points of X is quasi-compact.*

Proof. Let $X_0 = \bigcup U_{i,0}$ be an open covering. Write $U_{i,0} = X_0 \cap U_i$ for some open $U_i \subset X$. Consider the complement Z of $\bigcup U_i$. This is a closed subset of X, hence quasi-compact (Lemma 12.3) and Kolmogorov. By Lemma 12.8 if Z is nonempty it would have a closed point which contradicts the fact that $X_0 \subset \bigcup U_i$. Hence $Z = \emptyset$ and $X = \bigcup U_i$. Since X is quasi-compact this covering has a finite subcover and we conclude. $\square$

Lemma 12.10. *Let X be a topological space. Assume*

1. *X is quasi-compact,*
2. *X has a basis for the topology consisting of quasi-compact opens, and*
3. *the intersection of two quasi-compact opens is quasi-compact.*

For any $x \in X$ the connected component of X containing x is the intersection of all open and closed subsets of X containing x.

Proof. Let T be the connected component containing x. Let $S = \bigcap_{\alpha \in A} Z_\alpha$ be the intersection of all open and closed subsets Z_α of X containing x. Note that S is closed in X. Note that any finite intersection of Z_α's is a Z_α. Because T is connected and $x \in T$ we have $T \subset S$. It suffices to show that S is connected. If not, then there exists a disjoint union decomposition $S = B \amalg C$ with B and C open and closed in S. In particular, B and C are closed in X, and so quasi-compact by Lemma 12.3 and assumption (1). By assumption (2) there exist quasi-compact opens $U, V \subset X$ with $B = S \cap U$ and $C = S \cap V$ (details omitted). Then $U \cap V \cap S = \emptyset$. Hence $\bigcap_\alpha U \cap V \cap Z_\alpha = \emptyset$. By assumption (3) the intersection $U \cap V$ is quasi-compact. By Lemma 12.6 for some $\alpha' \in A$ we have $U \cap V \cap Z_{\alpha'} = \emptyset$. Since $X \setminus (U \cup V)$ is disjoint from S and closed in X hence quasi-compact, we can use the same lemma to see that $Z_{\alpha''} \subset U \cup V$ for some $\alpha'' \in A$. Then $Z_\alpha = Z_{\alpha'} \cap Z_{\alpha''}$ is contained in $U \cup V$ and disjoint from $U \cap V$. Hence $Z_\alpha = U \cap Z_\alpha \amalg V \cap Z_\alpha$ is a decomposition into two open pieces, hence $U \cap Z_\alpha$ and $V \cap Z_\alpha$ are open and closed in X. Thus, if $x \in B$ say, then we see that $S \subset U \cap Z_\alpha$ and we conclude that $C = \emptyset$. □

Lemma 12.11. *Let X be a topological space. Assume X is quasi-compact and Hausdorff. For any $x \in X$ the connected component of X containing x is the intersection of all open and closed subsets of X containing x.*

Proof. Let T be the connected component containing x. Let $S = \bigcap_{\alpha \in A} Z_\alpha$ be the intersection of all open and closed subsets Z_α of X containing x. Note that S is closed in X. Note that any finite intersection of Z_α's is a Z_α. Because T is connected and $x \in T$ we have $T \subset S$. It suffices to show that S is connected. If not, then there exists a disjoint union decomposition $S = B \amalg C$ with B and C open and closed in S. In particular, B and C are closed in X, and so quasi-compact by Lemma 12.3. By Lemma 12.4 there exist disjoint opens $U, V \subset X$ with $B \subset U$ and $C \subset V$. Then $X \setminus U \cup V$ is closed in X hence quasi-compact (Lemma 12.3). It follows that $(X \setminus U \cup V) \cap Z_\alpha = \emptyset$ for some α by Lemma 12.6. In other words, $Z_\alpha \subset U \cup V$. Thus $Z_\alpha = Z_\alpha \cap V \amalg Z_\alpha \cap U$ is a decomposition into two open pieces, hence $U \cap Z_\alpha$ and $V \cap Z_\alpha$ are open and closed in X. Thus, if $x \in B$ say, then we see that $S \subset U \cap Z_\alpha$ and we conclude that $C = \emptyset$. □

Lemma 12.12. *Let X be a topological space. Assume*

1. *X is quasi-compact,*
2. *X has a basis for the topology consisting of quasi-compact opens, and*
3. *the intersection of two quasi-compact opens is quasi-compact.*

For a subset $T \subset X$ the following are equivalent:

- (a) *T is an intersection of open and closed subsets of X, and*
- (b) *T is closed in X and is a union of connected components of X.*

Proof. It is clear that (a) implies (b). Assume (b). Let $x \in X$, $x \notin T$. Let $x \in C \subset X$ be the connected component of X containing x. By Lemma 12.10 we see that $C = \bigcap V_\alpha$ is the intersection of all open and closed subsets V_α of X which contain C. In particular, any pairwise intersection $V_\alpha \cap V_\beta$ occurs as a V_α. As T is a union of connected components of X we see that $C \cap T = \emptyset$. Hence $T \cap \bigcap V_\alpha = \emptyset$. Since T is quasi-compact as a closed subset of a quasi-compact space (see Lemma 12.3) we deduce that $T \cap V_\alpha = \emptyset$ for some α, see Lemma 12.6. For this α we see that $U_\alpha = X \setminus V_\alpha$ is an open and closed subset of X which contains T and not x. The lemma follows. □

Lemma 12.13. *Let X be a Noetherian topological space.*

1. *The space X is quasi-compact.*
2. *Any subset of X is retrocompact.*

Proof. Suppose $X = \bigcup U_i$ is an open covering of X indexed by the set I which does not have a refinement by a finite open covering. Choose $i_1, i_2, \ldots$ elements of I inductively in the following way: Choose i_{n+1} such that $U_{i_{n+1}}$ is not contained in $U_{i_1} \cup \ldots \cup U_{i_n}$. Thus we see that $X \supset (X \setminus U_{i_1}) \supset (X \setminus U_{i_1} \cup U_{i_2}) \supset \ldots$ is a strictly decreasing infinite sequence of closed subsets. This contradicts the fact that X is Noetherian. This proves the first assertion. The second assertion is now clear since every subset of X is Noetherian by Lemma 9.2. ☐

Lemma 12.14. *A quasi-compact locally Noetherian space is Noetherian.*

Proof. The conditions imply immediately that X has a finite covering by Noetherian subsets, and hence is Noetherian by Lemma 9.4. ☐

Lemma 12.15 (Alexander subbase theorem)**.** *Let X be a topological space. Let $\mathcal{B}$ be a subbase for X. If every covering of X by elements of $\mathcal{B}$ has a finite refinement, then X is quasi-compact.*

Proof. Assume there is an open covering of X which does not have a finite refinement. Using Zorn's lemma we can choose a maximal open covering $X = \bigcup_{i \in I} U_i$ which does not have a finite refinement (details omitted). In other words, if $U \subset X$ is any open which does not occur as one of the U_i, then the covering $X = U \cup \bigcup_{i \in I} U_i$ does have a finite refinement. Let $I' \subset I$ be the set of indices such that $U_i \in \mathcal{B}$. Then $\bigcup_{i \in I'} U_i \neq X$, since otherwise we would get a finite refinement covering X by our assumption on $\mathcal{B}$. Pick $x \in X$, $x \notin \bigcup_{i \in I'} U_i$. Pick $i \in I$ with $x \in U_i$. Pick $V_1, \ldots, V_n \in \mathcal{B}$ such that $x \in V_1 \cap \ldots \cap V_n \subset U_i$. This is possible as $\mathcal{B}$ is a subbasis for X. Note that V_j does not occur as a U_i. By maximality of the chosen covering we see that for each j there exist $i_{j,1}, \ldots, i_{j,n_j} \in I$ such that $X = V_j \cup U_{i_{j,1}} \cup \ldots \cup U_{i_{j,n_j}}$. Since $V_1 \cap \ldots \cap V_n \subset U_i$ we conclude that $X = U_i \cup \bigcup U_{i_{j,l}}$ a contradiction. ☐

13. Locally quasi-compact spaces

Recall that a neighbourhood of a point need not be open.

Definition 13.1. A topological space X is called *locally quasi-compact*[2] if every point has a fundamental system of quasi-compact neighbourhoods.

The term *locally compact space* in the literature often refers to a space as in the following lemma.

Lemma 13.2. *A Hausdorff space is locally quasi-compact if and only if every point has a quasi-compact neighbourhood.*

Proof. Let X be a Hausdorff space. Let $x \in X$ and let $x \in E \subset X$ be a quasi-compact neighbourhood. Then E is closed by Lemma 12.4. Suppose that $x \in U \subset X$ is an open neighbourhood of x. Then $Z = E \setminus U$ is a closed subset of E not containing x. Hence we can find a pair of disjoint open subsets $W, V \subset E$ of E such that $x \in V$ and $Z \subset W$, see Lemma 12.4. It follows that $\overline{V} \subset E$ is a closed neighbourhood of x contained in $E \cap U$. Also $\overline{V}$ is quasi-compact as a closed subset of E (Lemma 12.3). In this way we obtain a fundamental system of quasi-compact neighbourhoods of x. ☐

[2]This may not be standard notation. Alternative notions used in the literature are: (1) Every point has some quasi-compact neighbourhood, and (2) Every point has a closed quasi-compact neighbourhood. A scheme has the property that every point has a fundamental system of open quasi-compact neighbourhoods.

Lemma 13.3 (Baire category theorem)**.** *Let X be a locally quasi-compact Hausdorff space. Let $U_n \subset X$, $n \geq 1$ be dense open subsets. Then $\bigcap_{n \geq 1} U_n$ is dense in X.*

Proof. After replacing U_n by $\bigcap_{i=1,\ldots,n} U_i$ we may assume that $U_1 \supset U_2 \supset \ldots$. Let $x \in X$. We will show that x is in the closure of $\bigcap_{n \geq 1} U_n$. Thus let E be a neighbourhood of x. To show that $E \cap \bigcap_{n \geq 1} U_n$ is nonempty we may replace E by a smaller neighbourhood. After replacing E by a smaller neighbourhood, we may assume that E is quasi-compact.

Set $x_0 = x$ and $E_0 = E$. Below, we will inductively choose a point $x_i \in E_{i-1} \cap U_i$ and a quasi-compact neighbourhood E_i of x_i with $E_i \subset E_{i-1} \cap U_i$. Because X is Hausdorff, the subsets $E_i \subset X$ are closed (Lemma 12.4). Since the E_i are also nonempty we conclude that $\bigcap_{i \geq 1} E_i$ is nonempty (Lemma 12.6). Since $\bigcap_{i \geq 1} E_i \subset E \cap \bigcap_{n \geq 1} U_n$ this proves the lemma.

The base case $i = 0$ we have done above. Induction step. Since E_{i-1} is a neighbourhood of x_{i-1} we can find an open $x_{i-1} \in W \subset E_{i-1}$. Since U_i is dense in X we see that $W \cap U_i$ is nonempty. Pick any $x_i \in W \cap U_i$. By definition of locally quasi-compact spaces we can find a quasi-compact neighbourhood E_i of x_i contained in $W \cap U_i$. Then $E_i \subset E_{i-1} \cap U_i$ as desired. □

Lemma 13.4. *Let X be a Hausdorff and quasi-compact space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Then there exists an open covering $X = \bigcup_{i \in I} V_i$ such that $\overline{V_i} \subset U_i$ for all i.*

Proof. Let $x \in X$. Choose an $i(x) \in I$ such that $x \in U_{i(x)}$. Since $X \setminus U_{i(x)}$ and $\{x\}$ are disjoint closed subsets of X, by Lemmas 12.3 and 12.4 there exists an open neighbourhood U_x of x whose closure is disjoint from $X \setminus U_{i(x)}$. Thus $\overline{U_x} \subset U_{i(x)}$. Since X is quasi-compact, there is a finite list of points $x_1, \ldots, x_m$ such that $X = U_{x_1} \cup \ldots \cup U_{x_m}$. Setting $V_i = \bigcup_{i = i(x_j)} U_{x_j}$ the proof is finished. □

Lemma 13.5. *Let X be a Hausdorff and quasi-compact space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Suppose given an integer $p \geq 0$ and for every $(p+1)$-tuple $i_0, \ldots, i_p$ of I an open covering $U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$. Then there exists an open covering $X = \bigcup_{j \in J} V_j$ and a map $\alpha : J \to I$ such that $\overline{V_j} \subset U_{\alpha(j)}$ and such that each $V_{j_0} \cap \ldots \cap V_{j_p}$ is contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for some k.*

Proof. Since X is quasi-compact, there is a reduction to the case where I is finite (details omitted). We prove the result for I finite by induction on p. The base case $p = 0$ is immediate by taking a covering as in Lemma 13.4 refining the open covering $X = \bigcup W_{i_0, k}$.

Induction step. Assume the lemma proven for $p - 1$. For all p-tuples $i'_0, \ldots, i'_{p-1}$ of I let $U_{i'_0} \cap \ldots \cap U_{i'_{p-1}} = \bigcup W_{i'_0 \ldots i'_{p-1}, k}$ be a common refinement of the coverings $U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$ for those $(p+1)$-tuples such that $\{i'_0, \ldots, i'_{p-1}\} = \{i_0, \ldots, i_p\}$ (equality of sets). (There are finitely many of these as I is finite.) By induction there exists a solution for these opens, say $X = \bigcup V_j$ and $\alpha : J \to I$. At this point the covering $X = \bigcup_{j \in J} V_j$ and α satisfy $\overline{V_j} \subset U_{\alpha(j)}$ and each $V_{j_0} \cap \ldots \cap V_{j_p}$ is contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for some k if there is a repetition in $\alpha(j_0), \ldots, \alpha(j_p)$. Of course, we may and do assume that J is finite.

Fix $i_0, \ldots, i_p \in I$ pairwise distinct. Consider $(p+1)$-tuples $j_0, \ldots, j_p \in J$ with $i_0 = \alpha(j_0), \ldots, i_p = \alpha(j_p)$ such that $V_{j_0} \cap \ldots \cap V_{j_p}$ is **not** contained in $W_{\alpha(j_0)\ldots\alpha(j_p),k}$ for any k. Let N be the number of such $(p+1)$-tuples. We will show how to decrease N. Since

$$\overline{V_{j_0}} \cap \ldots \cap \overline{V_{j_p}} \subset U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0\ldots i_p,k}$$

we find a finite set $K = \{k_1, \ldots, k_t\}$ such that the LHS is contained in $\bigcup_{k \in K} W_{i_0\ldots i_p,k}$. Then we consider the open covering

$$V_{j_0} = (V_{j_0} \setminus (\overline{V_{j_1}} \cap \ldots \cap \overline{V_{j_p}})) \cup (\bigcup\nolimits_{k \in K} V_{j_0} \cap W_{i_0\ldots i_p,k})$$

The first open on the RHS intersects $V_{j_1} \cap \ldots \cap V_{j_p}$ in the empty set and the other opens $V_{j_0,k}$ of the RHS satisfy $V_{j_0,k} \cap V_{j_1} \ldots \cap V_{j_p} \subset W_{\alpha(j_0)\ldots\alpha(j_p),k}$. Set $J' = J \amalg K$. For $j \in J$ set $V'_j = V_j$ if $j \neq j_0$ and set $V'_{j_0} = V_{j_0} \setminus (\overline{V_{j_1}} \cap \ldots \cap \overline{V_{j_p}})$. For $k \in K$ set $V'_k = V_{j_0,k}$. Finally, the map $\alpha' : J' \to I$ is given by α on J and maps every element of K to i_0. A simple check shows that N has decreased by one under this replacement. Repeating this procedure N times we arrive at the situation where $N = 0$.

To finish the proof we argue by induction on the number M of $(p+1)$-tuples $i_0, \ldots, i_p \in I$ with pairwise distinct entries for which there exists a $(p+1)$-tuple $j_0, \ldots, j_p \in J$ with $i_0 = \alpha(j_0), \ldots, i_p = \alpha(j_p)$ such that $V_{j_0} \cap \ldots \cap V_{j_p}$ is **not** contained in $W_{\alpha(j_0)\ldots\alpha(j_p),k}$ for any k. To do this, we claim that the operation performed in the previous paragraph does not increase M. This follows formally from the fact that the map $\alpha' : J' \to I$ factors through a map $\beta : J' \to J$ such that $V'_{j'} \subset V_{\beta(j')}$. □

Lemma 13.6. *Let X be a Hausdorff and locally quasi-compact space. Let $Z \subset X$ be a quasi-compact (hence closed) subset. Suppose given an integer $p \geq 0$, a set I, for every $i \in I$ an open $U_i \subset X$, and for every $(p+1)$-tuple $i_0, \ldots, i_p$ of I an open $W_{i_0\ldots i_p} \subset U_{i_0} \cap \ldots \cap U_{i_p}$ such that*

1. *$Z \subset \bigcup U_i$, and*
2. *for every $i_0, \ldots, i_p$ we have $W_{i_0\ldots i_p} \cap Z = U_{i_0} \cap \ldots \cap U_{i_p} \cap Z$.*

Then there exist opens V_i of X such that we have $Z \subset \bigcup V_i$, for all i we have $\overline{V_i} \subset U_i$, and we have $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0\ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$.

Proof. Since Z is quasi-compact, there is a reduction to the case where I is finite (details omitted). Because X is locally quasi-compact and Z is quasi-compact, we can find a neighbourhood $Z \subset E$ which is quasi-compact, i.e., E is quasi-compact and contains an open neighbourhood of Z in X. If we prove the result after replacing X by E, then the result follows. Hence we may assume X is quasi-compact.

We prove the result in case I is finite and X is quasi-compact by induction on p. The base case is $p = 0$. In this case we have $X = (X \setminus Z) \cup \bigcup W_i$. By Lemma 13.4 we can find a covering $X = V \cup \bigcup V_i$ by opens $V_i \subset W_i$ and $V \subset X \setminus Z$ with $\overline{V_i} \subset W_i$ for all i. Then we see that we obtain a solution of the problem posed by the lemma.

Induction step. Assume the lemma proven for $p-1$. Set $W_{j_0\ldots j_{p-1}}$ equal to the intersection of all $W_{i_0\ldots i_p}$ with $\{j_0, \ldots, j_{p-1}\} = \{i_0, \ldots, i_p\}$ (equality of sets). By induction there exists a solution for these opens, say $V_i \subset U_i$. It follows from our choice of $W_{j_0\ldots j_{p-1}}$ that we have $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0\ldots i_p}$ for all $(p+1)$-tuples

$i_0, \ldots, i_p$ where $i_a = i_b$ for some $0 \leq a < b \leq p$. Thus we only need to modify our choice of V_i if $V_{i_0} \cap \ldots \cap V_{i_p} \not\subset W_{i_0 \ldots i_p}$ for some $(p+1)$-tuple $i_0, \ldots, i_p$ with pairwise distinct elements. In this case we have

$$T = \overline{V_{i_0} \cap \ldots \cap V_{i_p} \setminus W_{i_0 \ldots i_p}} \subset \overline{V_{i_0}} \cap \ldots \cap \overline{V_{i_p}} \setminus W_{i_0 \ldots i_p}$$

is a closed subset of X contained in $U_{i_0} \cap \ldots \cap U_{i_p}$ not meeting Z. Hence we can replace V_{i_0} by $V_{i_0} \setminus T$ to "fix" the problem. After repeating this finitely many times for each of the problem tuples, the lemma is proven. □

Lemma 13.7. *Let X be a topological space. Let $Z \subset X$ be a quasi-compact subset such that any two points of Z have disjoint open neighbourhoods in X. Suppose given an integer $p \geq 0$, a set I, for every $i \in I$ an open $U_i \subset X$, and for every $(p+1)$-tuple $i_0, \ldots, i_p$ of I an open $W_{i_0 \ldots i_p} \subset U_{i_0} \cap \ldots \cap U_{i_p}$ such that*

1. *$Z \subset \bigcup U_i$, and*
2. *for every $i_0, \ldots, i_p$ we have $W_{i_0 \ldots i_p} \cap Z = U_{i_0} \cap \ldots \cap U_{i_p} \cap Z$.*

Then there exist opens V_i of X such that

1. *$Z \subset \bigcup V_i$,*
2. *$V_i \subset U_i$ for all i,*
3. *$\overline{V_i} \cap Z \subset U_i$ for all i, and*
4. *$V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0 \ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$.*

Proof. Since Z is quasi-compact, there is a reduction to the case where I is finite (details omitted). We prove the result in case I is finite by induction on p.

The base case is $p = 0$. For $z \in Z \cap U_i$ and $z' \in Z \setminus U_i$ there exist disjoint opens $z \in V_{z,z'}$ and $z' \in W_{z,z'}$ of X. Since $Z \setminus U_i$ is quasi-compact (Lemma 12.3), we can choose a finite nunber $z'_1, \ldots, z'_r$ such that $Z \setminus U_i \subset W_{z,z'_1} \cup \ldots \cup W_{z,z'_r}$. Then we see that $V_z = V_{z,z'_1} \cap \ldots \cap V_{z,z'_r} \cap U_i$ is an open neighbourhood of z contained in U_i with the property that $\overline{V_z} \cap Z \subset U_i$. Since z and i where arbitrary and since Z is quasi-compact we can find a finite list $z_1, i_1, \ldots, z_t, i_t$ and opens $V_{z_j} \subset U_{i_j}$ with $\overline{V_{z_j}} \cap Z \subset U_{i_j}$ and $Z \subset \bigcup V_{z_j}$. Then we can set $V_i = W_i \cap (\bigcup_{j : i = i_j} V_{z_j})$ to solve the problem for $p = 0$.

Induction step. Assume the lemma proven for $p - 1$. Set $W_{j_0 \ldots j_{p-1}}$ equal to the intersection of all $W_{i_0 \ldots i_p}$ with $\{j_0, \ldots, j_{p-1}\} = \{i_0, \ldots, i_p\}$ (equality of sets). By induction there exists a solution for these opens, say $V_i \subset U_i$. It follows from our choice of $W_{j_0 \ldots j_{p-1}}$ that we have $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0 \ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$ where $i_a = i_b$ for some $0 \leq a < b \leq p$. Thus we only need to modify our choice of V_i if $V_{i_0} \cap \ldots \cap V_{i_p} \not\subset W_{i_0 \ldots i_p}$ for some $(p+1)$-tuple $i_0, \ldots, i_p$ with pairwise distinct elements. In this case we have

$$T = \overline{V_{i_0} \cap \ldots \cap V_{i_p} \setminus W_{i_0 \ldots i_p}} \subset \overline{V_{i_0}} \cap \ldots \cap \overline{V_{i_p}} \setminus W_{i_0 \ldots i_p}$$

is a closed subset of X not meeting Z by our property (3) of the opens V_i. Hence we can replace V_{i_0} by $V_{i_0} \setminus T$ to "fix" the problem. After repeating this finitely many times for each of the problem tuples, the lemma is proven. □

14. Limits of spaces

The category of topological spaces has products. Namely, if I is a set and for $i \in I$ we are given a topological space X_i then we endow $\prod_{i \in I} X_i$ with the *product* *topology*. As a basis for the topology we use sets of the form $\prod U_i$ where $U_i \subset X_i$ is open and $U_i = X_i$ for almost all i.

The category of topological spaces has equalizers. Namely, if $a, b : X \to Y$ are morphisms of topological spaces, then the equalizer of a and b is the subset $\{x \in X \mid a(x) = b(x)\} \subset X$ endowed with the induced topology.

Lemma 14.1. *The category of topological spaces has limits and the forgetful functor to sets commutes with them.*

Proof. This follows from the discussion above and Categories, Lemma 14.11. It follows from the description above that the forgetful functor commutes with limits. Another way to see this is to use Categories, Lemma 24.5 and use that the forgetful functor has a left adjoint, namely the functor which assigns to a set the corresponding discrete topological space. □

Lemma 14.2. *Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of topological spaces over $\mathcal{I}$. Let $X = \lim X_i$ be the limit with projection maps $f_i : X \to X_i$.*

(1) *Any open of X is of the form $\bigcup_{j \in J} f_j^{-1}(U_j)$ for some subset $J \subset I$ and opens $U_j \subset X_j$.*
(2) *Any quasi-compact open of X is of the form $f_i^{-1}(U_i)$ for some i and some $U_i \subset X_i$ open.*

Proof. The construction of the limit given above shows that $X \subset \prod X_i$ with the induced topology. A basis for the topology of $\prod X_i$ are the opens $\prod U_i$ where $U_i \subset X_i$ is open and $U_i = X_i$ for almost all i. Say $i_1, \ldots, i_n \in \mathrm{Ob}(\mathcal{I})$ are the objects such that $U_{i_j} \neq X_{i_j}$. Then

$$X \cap \prod U_i = f_{i_1}^{-1}(U_{i_1}) \cap \ldots \cap f_{i_n}^{-1}(U_{i_n})$$

For a general limit of topological spaces these form a basis for the topology on X. However, if $\mathcal{I}$ is cofiltered as in the statement of the lemma, then we can pick a $j \in \mathrm{Ob}(\mathcal{I})$ and morphisms $j \to i_l$, $l = 1, \ldots, n$. Let

$$U_j = (X_j \to X_{i_1})^{-1}(U_{i_1}) \cap \ldots \cap (X_j \to X_{i_n})^{-1}(U_{i_n})$$

Then it is clear that $X \cap \prod U_i = f_j^{-1}(U_j)$. Thus for any open W of X there is a set A and a map $\alpha : A \to \mathrm{Ob}(\mathcal{I})$ and opens $U_a \subset X_{\alpha(a)}$ such that $W = \bigcup f_{\alpha(a)}^{-1}(U_a)$. Set $J = \mathrm{Im}(\alpha)$ and for $j \in J$ set $U_j = \bigcup_{\alpha(a) = j} U_a$ to see that $W = \bigcup_{j \in J} f_j^{-1}(U_j)$. This proves (1).

To see (2) suppose that $\bigcup_{j \in J} f_j^{-1}(U_j)$ is quasi-compact. Then it is equal to $f_{j_1}^{-1}(U_{j_1}) \cup \ldots \cup f_{j_m}^{-1}(U_{j_m})$ for some $j_1, \ldots, j_m \in J$. Since $\mathcal{I}$ is cofiltered, we can pick a $i \in \mathrm{Ob}(\mathcal{I})$ and morphisms $i \to j_l$, $l = 1, \ldots, m$. Let

$$U_i = (X_i \to X_{j_1})^{-1}(U_{j_1}) \cup \ldots \cup (X_i \to X_{j_m})^{-1}(U_{j_m})$$

Then our open equals $f_i^{-1}(U_i)$ as desired. □

Lemma 14.3. *Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of topological spaces over $\mathcal{I}$. Let X be a topological space such that*

(1) *$X = \lim X_i$ as a set (denote f_i the projection maps),*
(2) *the sets $f_i^{-1}(U_i)$ for $i \in \mathrm{Ob}(\mathcal{I})$ and $U_i \subset X_i$ open form a basis for the topology of X.*

Then X is the limit of the X_i as a topological space.

Proof. Follows from the description of the limit topology in Lemma 14.2. $\square$

Theorem 14.4 (Tychonov)**.** *A product of quasi-compact spaces is quasi-compact.*

Proof. Let I be a set and for $i \in I$ let X_i be a quasi-compact topological space. Set $X = \prod X_i$. Let $\mathcal{B}$ be the set of subsets of X of the form $U_i \times \prod_{i' \in I, i' \neq i} X_{i'}$ where $U_i \subset X_i$ is open. By construction this family is a subbasis for the topology on X. By Lemma 12.15 it suffices to show that any covering $X = \bigcup_{j \in J} B_j$ by elements B_j of $\mathcal{B}$ has a finite refinement. We can decompose $J = \coprod J_i$ so that if $j \in J_i$, then $B_j = U_j \times \prod_{i' \neq i} X_{i'}$ with $U_j \subset X_i$ open. If $X_i = \bigcup_{j \in J_i} U_j$, then there is a finite refinement and we conclude that $X = \bigcup_{j \in J} B_j$ has a finite refinement. If this is not the case, then for every i we can choose an point $x_i \in X_i$ which is not in $\bigcup_{j \in J_i} U_j$. But then the point $x = (x_i)_{i \in I}$ is an element of X not contained in $\bigcup_{j \in J} B_j$, a contradiction. $\square$

The following lemma does not hold if one drops the assumption that the spaces X_i are Hausdorff, see Examples, Section 4.

Lemma 14.5. *Let $\mathcal{I}$ be a category and let $i \mapsto X_i$ be a diagram over $\mathcal{I}$ in the category of topological spaces. If each X_i is quasi-compact and Hausdorff, then $\lim X_i$ is quasi-compact.*

Proof. Recall that $\lim X_i$ is a subspace of $\prod X_i$. By Theorem 14.4 this product is quasi-compact. Hence it suffices to show that $\lim X_i$ is a closed subspace of $\prod X_i$ (Lemma 12.3). If $\varphi : j \to k$ is a morphism of $\mathcal{I}$, then let $\Gamma_\varphi \subset X_j \times X_k$ denote the graph of the corresponding continuous map $X_j \to X_k$. By Lemma 3.2 this graph is closed. It is clear that $\lim X_i$ is the intersection of the closed subsets

$$\Gamma_\varphi \times \prod\nolimits_{l \neq j,k} X_l \subset \prod X_i$$

Thus the result follows. $\square$

The following lemma generalizes Categories, Lemma 21.7 and partially generalizes Lemma 12.6.

Lemma 14.6. *Let $\mathcal{I}$ be a cofiltered category and let $i \mapsto X_i$ be a diagram over $\mathcal{I}$ in the category of topological spaces. If each X_i is quasi-compact, Hausdorff, and nonempty, then $\lim X_i$ is nonempty.*

Proof. In the proof of Lemma 14.5 we have seen that $X = \lim X_i$ is the intersection of the closed subsets

$$Z_\varphi = \Gamma_\varphi \times \prod\nolimits_{l \neq j,k} X_l$$

inside the quasi-compact space $\prod X_i$ where $\varphi : j \to k$ is a morphism of $\mathcal{I}$ and $\Gamma_\varphi \subset X_j \times X_k$ is the graph of the corresponding morphism $X_j \to X_k$. Hence by Lemma 12.6 it suffices to show any finite intersection of these subsets is nonempty. Assume $\varphi_t : j_t \to k_t$, $t = 1, \ldots, n$ is a finite collection of morphisms of $\mathcal{I}$. Since $\mathcal{I}$ is cofiltered, we can pick an object j and a morphism $\psi_t : j \to j_t$ for each t. For each pair t, t' such that either (a) $j_t = j_{t'}$, or (b) $j_t = k_{t'}$, or (c) $k_t = k_{t'}$ we obtain two morphisms $j \to l$ with $l = j_t$ in case (a), (b) or $l = k_t$ in case (c). Because $\mathcal{I}$ is cofiltered and since there are finitely many pairs (t, t') we may choose a map $j' \to j$ which equalizes these two morphisms for all such pairs (t, t'). Pick an element

$x \in X_{j'}$ and for each t let x_{j_t}, resp. x_{k_t} be the image of x under the morphism $X_{j'} \to X_j \to X_{j_t}$, resp. $X_{j'} \to X_j \to X_{j_t} \to X_{k_t}$. For any index $l \in \mathrm{Ob}(\mathcal{I})$ which is not equal to j_t or k_t for some t we pick an arbitrary element $x_l \in X_l$ (using the axiom of choice). Then $(x_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ is in the intersection

$$Z_{\varphi_1} \cap \ldots \cap Z_{\varphi_n}$$

by construction and the proof is complete. $\square$

15. Constructible sets

Definition 15.1. Let X be a topological space. Let $E \subset X$ be a subset of X.

(1) We say E is *constructible*[3] in X if E is a finite union of subsets of the form $U \cap V^c$ where $U, V \subset X$ are open and retrocompact in X.
(2) We say E is *locally constructible* in X if there exists an open covering $X = \bigcup V_i$ such that each $E \cap V_i$ is constructible in V_i.

Lemma 15.2. *The collection of constructible sets is closed under finite intersections, finite unions and complements.*

Proof. Note that if U_1, U_2 are open and retrocompact in X then so is $U_1 \cup U_2$ because the union of two quasi-compact subsets of X is quasi-compact. It is also true that $U_1 \cap U_2$ is retrocompact. Namely, suppose $U \subset X$ is quasi-compact open, then $U_2 \cap U$ is quasi-compact because U_2 is retrocompact in X, and then we conclude $U_1 \cap (U_2 \cap U)$ is quasi-compact because U_1 is retrocompact in X. From this it is formal to show that the complement of a constructible set is constructible, that finite unions of constructibles are constructible, and that finite intersections of constructibles are constructible. $\square$

Lemma 15.3. *Let $f : X \to Y$ be a continuous map of topological spaces. If the inverse image of every retrocompact open subset of Y is retrocompact in X, then inverse images of constructible sets are constructible.*

Proof. This is true because $f^{-1}(U \cap V^c) = f^{-1}(U) \cap f^{-1}(V)^c$, combined with the definition of constructible sets. $\square$

Lemma 15.4. *Let $U \subset X$ be open. For a constructible set $E \subset X$ the intersection $E \cap U$ is constructible in U.*

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap U$ is retrocompact in U by Lemma 15.3. To show this let $W \subset U$ be open and quasi-compact. Then W is open and quasi-compact in X. Hence $V \cap W = V \cap U \cap W$ is quasi-compact as V is retrocompact in X. $\square$

Lemma 15.5. *Let $U \subset X$ be a retrocompact open. Let $E \subset U$. If E is constructible in U, then E is constructible in X.*

Proof. Suppose that $V, W \subset U$ are retrocompact open in U. Then V, W are retrocompact open in X (Lemma 12.2). Hence $V \cap (U \setminus W) = V \cap (X \setminus W)$ is constructible in X. We conclude since every constructible subset of U is a finite union of subsets of the form $V \cap (U \setminus W)$. $\square$

[3]In the second edition of EGA I [GD71] this was called a "globally constructible" set and a the terminology "constructible" was used for what we call a locally constructible set.

Lemma 15.6. *Let X be a topological space. Let $E \subset X$ be a subset. Let $X = V_1 \cup \ldots \cup V_m$ be a finite covering by retrocompact opens. Then E is constructible in X if and only if $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$.*

Proof. If E is constructible in X, then by Lemma 15.4 we see that $E \cap V_j$ is constructible in V_j for all j. Conversely, suppose that $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$. Then $E = \bigcup E \cap V_j$ is a finite union of constructible sets by Lemma 15.5 and hence constructible. $\square$

Lemma 15.7. *Let X be a topological space. Let $Z \subset X$ be a closed subset such that $X \setminus Z$ is quasi-compact. Then for a constructible set $E \subset X$ the intersection $E \cap Z$ is constructible in Z.*

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap Z$ is retrocompact in Z by Lemma 15.3. To show this let $W \subset Z$ be open and quasi-compact. The subset $W' = W \cup (X \setminus Z)$ is quasi-compact, open, and $W = Z \cap W'$. Hence $V \cap Z \cap W = V \cap Z \cap W'$ is a closed subset of the quasi-compact open $V \cap W'$ as V is retrocompact in X. Thus $V \cap Z \cap W$ is quasi-compact by Lemma 12.3. $\square$

Lemma 15.8. *Let X be a topological space. Let $T \subset X$ be a subset. Suppose*

1. *T is retrocompact in X,*
2. *quasi-compact opens form a basis for the topology on X.*

Then for a constructible set $E \subset X$ the intersection $E \cap T$ is constructible in T.

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap T$ is retrocompact in T by Lemma 15.3. To show this let $W \subset T$ be open and quasi-compact. By assumption (2) we can find a quasi-compact open $W' \subset X$ such that $W = T \cap W'$ (details omitted). Hence $V \cap T \cap W = V \cap T \cap W'$ is the intersection of T with the quasi-compact open $V \cap W'$ as V is retrocompact in X. Thus $V \cap T \cap W$ is quasi-compact. $\square$

Lemma 15.9. *Let $Z \subset X$ be a closed subset whose complement is retrocompact open. Let $E \subset Z$. If E is constructible in Z, then E is constructible in X.*

Proof. Suppose that $V \subset Z$ is retrocompact open in Z. Consider the open subset $\tilde{V} = V \cup (X \setminus Z)$ of X. Let $W \subset X$ be quasi-compact open. Then

$$W \cap \tilde{V} = (V \cap W) \cup ((X \setminus Z) \cap W).$$

The first part is quasi-compact as $V \cap W = V \cap (Z \cap W)$ and $(Z \cap W)$ is quasi-compact open in Z (Lemma 12.3) and V is retrocompact in Z. The second part is quasi-compact as $(X \setminus Z)$ is retrocompact in X. In this way we see that $\tilde{V}$ is retrocompact in X. Thus if $V_1, V_2 \subset Z$ are retrocompact open, then

$$V_1 \cap (Z \setminus V_2) = \tilde{V}_1 \cap (X \setminus \tilde{V}_2)$$

is constructible in X. We conclude since every constructible subset of Z is a finite union of subsets of the form $V_1 \cap (Z \setminus V_2)$. $\square$

Lemma 15.10. *Let X be a topological space. Every constructible subset of X is retrocompact.*

Proof. Let $E = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c$ with U_i, V_i retrocompact open in X. Let $W \subset X$ be quasi-compact open. Then $E \cap W = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c \cap W$. Thus it suffices to show that $U \cap V^c \cap W$ is quasi-compact if U, V are retrocompact open and W is quasi-compact open. This is true because $U \cap V^c \cap W$ is a closed subset of the quasi-compact $U \cap W$ so Lemma 12.3 applies. □

Question: Does the following lemma also hold if we assume X is a quasi-compact topological space? Compare with Lemma 15.7.

Lemma 15.11. *Let X be a topological space. Assume X has a basis consisting of quasi-compact opens. For E, E' constructible in X, the intersection $E \cap E'$ is constructible in E.*

Proof. Combine Lemmas 15.8 and 15.10. □

Lemma 15.12. *Let X be a topological space. Assume X has a basis consisting of quasi-compact opens. Let E be constructible in X and $F \subset E$ constructible in E. Then F is constructible in X.*

Proof. Observe that any retrocompact subset T of X has a basis for the induced topology consisting of quasi-compact opens. In particular this holds for any constructible subset (Lemma 15.10). Write $E = E_1 \cup \ldots \cup E_n$ with $E_i = U_i \cap V_i^c$ where $U_i, V_i \subset X$ are retrocompact open. Note that $E_i = E \cap E_i$ is constructible in E by Lemma 15.11. Hence $F \cap E_i$ is constructible in E_i by Lemma 15.11. Thus it suffices to prove the lemma in case $E = U \cap V^c$ where $U, V \subset X$ are retrocompact open. In this case the inclusion $E \subset X$ is a composition

$$E = U \cap V^c \to U \to X$$

Then we can apply Lemma 15.9 to the first inclusion and Lemma 15.5 to the second. □

Lemma 15.13. *Let X be a quasi-compact topological space having a basis consisting of quasi-compact opens such that the intersection of any two quasi-compact opens is quasi-compact. Let $T \subset X$ be a locally closed subset such that T is quasi-compact and T^c is retrocompact in X. Then T is constructible in X.*

Proof. Note that T is quasi-compact and open in $\overline{T}$. Using our basis of quasi-compact opens we can write $T = U \cap \overline{T}$ where U is quasi-compact open in X. Then $V = U \setminus T = U \cap T^c$ is retrocompact in U as T^c is retrocompact in X. Hence V is quasi-compact. Since the intersection of any two quasi-compact opens is quasi-compact any quasi-compact open of X is retrocompact. Thus $T = U \cap V^c$ with U and $V = U \setminus T$ retrocompact opens of X. A fortiori, T is constructible in X. □

Lemma 15.14. *Let X be a topological space which has a basis for the topology consisting of quasi-compact opens. Let $E \subset X$ be a subset. Let $X = E_1 \cup \ldots \cup E_m$ be a finite covering by constructible subsets. Then E is constructible in X if and only if $E \cap E_j$ is constructible in E_j for each $j = 1, \ldots, m$.*

Proof. Combine Lemmas 15.11 and 15.12. □

Lemma 15.15. *Let X be a topological space. Suppose that $Z \subset X$ is irreducible. Let $E \subset X$ be a finite union of locally closed subsets (e.g. E is constructible). The following are equivalent*

(1) *The intersection $E \cap Z$ contains an open dense subset of Z.*

(2) *The intersection $E \cap Z$ is dense in Z.*

If Z has a generic point ξ, then this is also equivalent to

(3) *We have $\xi \in E$.*

Proof. The implication (1) $\Rightarrow$ (2) is clear. Assume (2). Note that $E \cap Z$ is a finite union of locally closed subsets Z_i of Z. Since Z is irreducible, one of the Z_i must be dense in Z. Then this Z_i is dense open in Z as it is open in its closure. Hence (1) holds.

Suppose that $\xi \in Z$ is a generic point. If the equivalent conditions (1) and (2) hold, then $\xi \in E$. Conversely, if $\xi \in E$ then $\xi \in E \cap Z$ and hence $E \cap Z$ is dense in Z. $\square$

16. Constructible sets and Noetherian spaces

Lemma 16.1. *Let X be a Noetherian topological space. The constructible sets in X are precisely the finite unions of locally closed subsets of X.*

Proof. This follows immediately from Lemma 12.13. $\square$

Lemma 16.2. *Let $f : X \to Y$ be a continuous map of Noetherian topological spaces. If $E \subset Y$ is constructible in Y, then $f^{-1}(E)$ is constructible in X.*

Proof. Follows immediately from Lemma 16.1 and the definition of a continuous map. $\square$

Lemma 16.3. *Let X be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent:*

(1) *E is constructible in X, and*
(2) *for every irreducible closed $Z \subset X$ the intersection $E \cap Z$ either contains a nonempty open of Z or is not dense in Z.*

Proof. Assume E is constructible and $Z \subset X$ irreducible closed. Then $E \cap Z$ is constructible in Z by Lemma 16.2. Hence $E \cap Z$ is a finite union of nonempty locally closed subsets T_i of Z. Clearly if none of the T_i is open in Z, then $E \cap Z$ is not dense in Z. In this way we see that (1) implies (2).

Conversely, assume (2) holds. Consider the set $\mathcal{S}$ of closed subsets Y of X such that $E \cap Y$ is not constructible in Y. If $\mathcal{S} \neq \emptyset$, then it has a smallest element Y as X is Noetherian. Let $Y = Y_1 \cup \ldots \cup Y_r$ be the decomposition of Y into its irreducible components, see Lemma 9.2. If $r > 1$, then each $Y_i \cap E$ is constructible in Y_i and hence a finite union of locally closed subsets of Y_i. Thus $E \cap Y$ is a finite union of locally closed subsets of Y too and we conclude that $E \cap Y$ is constructible in Y by Lemma 16.1. This is a contradiction and so $r = 1$. If $r = 1$, then Y is irreducible, and by assumption (2) we see that $E \cap Y$ either (a) contains an open V of Y or (b) is not dense in Y. In case (a) we see, by minimality of Y, that $E \cap (Y \setminus V)$ is a finite union of locally closed subsets of $Y \setminus V$. Thus $E \cap Y$ is a finite union of locally closed subsets of Y and is constructible by Lemma 16.1. This is a contradiction and so we must be in case (b). In case (b) we see that $E \cap Y = E \cap Y'$ for some proper closed subset $Y' \subset Y$. By minimality of Y we see that $E \cap Y'$ is a finite union of locally closed subsets of Y' and we see that $E \cap Y' = E \cap Y$ is a finite union of locally closed subsets of Y and is constructible by Lemma 16.1. This contradiction finishes the proof of the lemma. $\square$

Lemma 16.4. *Let X be a Noetherian topological space. Let $x \in X$. Let $E \subset X$ be constructible in X. The following are equivalent:*

(1) *E is a neighbourhood of x, and*
(2) *for every irreducible closed subset Y of X which contains x the intersection $E \cap Y$ is dense in Y.*

Proof. It is clear that (1) implies (2). Assume (2). Consider the set $\mathcal{S}$ of closed subsets Y of X containing x such that $E \cap Y$ is not a neighbourhood of x in Y. If $\mathcal{S} \neq \emptyset$, then it has a minimal element Y as X is Noetherian. Suppose $Y = Y_1 \cup Y_2$ with two smaller nonempty closed subsets Y_1, Y_2. If $x \in Y_i$ for $i = 1, 2$, then $Y_i \cap E$ is a neighbourhood of x in Y_i and we conclude $Y \cap E$ is a neighbourhood of x in Y which is a contradiction. If $x \in Y_1$ but $x \notin Y_2$ (say), then $Y_1 \cap E$ is a neighbourhood of x in Y_1 and hence also in Y, which is a contradiction as well. We conclude that Y is irreducible closed. By assumption (2) we see that $E \cap Y$ is dense in Y. Thus $E \cap Y$ contains an open V of Y, see Lemma 16.3. If $x \in V$ then $E \cap Y$ is a neighbourhood of x in Y which is a contradiction. If $x \notin V$, then $Y' = Y \setminus V$ is a proper closed subset of Y containing x. By minimality of Y we see that $E \cap Y'$ contains an open neighbourhood $V' \subset Y'$ of x in Y'. But then $V' \cup V$ is an open neighbourhood of x in Y contained in E, a contradiction. This contradiction finishes the proof of the lemma. □

Lemma 16.5. *Let X be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent:*

(1) *E is open in X, and*
(2) *for every irreducible closed subset Y of X the intersection $E \cap Y$ is either empty or contains a nonempty open of Y.*

Proof. This follows formally from Lemmas 16.3 and 16.4. □

17. Characterizing proper maps

We include a section discussing the notion of a proper map in usual topology. We define a continuous map of topological spaces to be *proper* if it is universally closed and separated. Although this matches well with the definition of a proper morphism in algebraic geometry, this is different from the definition in Bourbaki. With our definition of a proper map of topological spaces, the proper base change theorem (Cohomology, Theorem 18.2) holds without any further assumptions. Furthermore, given a morphism $f : X \to Y$ of finite type schemes over $\mathbf{C}$ one has: f is proper as a morphism of schemes if and only if the continuous map $f : X(\mathbf{C}) \to Y(\mathbf{C})$ on $\mathbf{C}$-points with the classical topology is proper. This is explained in [Gro71, Exp. XII, Prop. 3.2(v)] which also has a footnote pointing out that they take properness in topology to be Bourbaki's notion with separatedness added on.

We find it useful to have names for three distinct concepts: separated, universally closed, and both of those together (i.e., properness). For a continuous map $f : X \to Y$ of locally compact Hausdorff spaces the word "proper" has long been used for the notion "f^{-1}(compact) = compact" and this is equivalent to universal closedness for such nice spaces. In fact, we will see the preimage condition formulated for clarity using the word "quasi-compact" is equivalent to universal closedness in general, if one includes the assumption of the map being closed. See also [Lan93, Exercises 22-26 in Chapter II] but beware that Lang uses "proper" as a synonym for "universally closed", like Bourbaki does.

Lemma 17.1 (Tube lemma). *Let X and Y be topological spaces. Let $A \subset X$ and $B \subset Y$ be quasi-compact subsets. Let $A \times B \subset W \subset X \times Y$ with W open in $X \times Y$. Then there exists opens $A \subset U \subset X$ and $B \subset V \subset Y$ such that $U \times V \subset W$.*

Proof. For every $a \in A$ and $b \in B$ there exist opens $U_{(a,b)}$ of X and $V_{(a,b)}$ of Y such that $(a,b) \in U_{(a,b)} \times V_{(a,b)} \subset W$. Fix b and we see there exist a finite number $a_1, \ldots, a_n$ such that $A \subset U_{(a_1,b)} \cup \ldots \cup U_{(a_n,b)}$. Hence

$$A \times \{b\} \subset (U_{(a_1,b)} \cup \ldots \cup U_{(a_n,b)}) \times (V_{(a_1,b)} \cap \ldots \cap V_{(a_n,b)}) \subset W.$$

Thus for every $b \in B$ there exists opens $U_b \subset X$ and $V_b \subset Y$ such that $A \times \{b\} \subset U_b \times V_b \subset W$. As above there exist a finite number $b_1, \ldots, b_m$ such that $B \subset V_{b_1} \cup \ldots \cup V_{b_m}$. Then we win because $A \times B \subset (U_{b_1} \cap \ldots \cap U_{b_m}) \times (V_{b_1} \cup \ldots \cup V_{b_m})$. $\square$

The notation in the following definition may be slightly different from what you are used to.

Definition 17.2. Let $f : X \to Y$ be a continuous map between topological spaces.

(1) We say that the map f is *closed* if the image of every closed subset is closed.
(2) We say that the map f is *Bourbaki-proper*[4] if the map $Z \times X \to Z \times Y$ is closed for any topological space Z.
(3) We say that the map f is *quasi-proper* if the inverse image $f^{-1}(V)$ of every quasi-compact subset $V \subset Y$ is quasi-compact.
(4) We say that f is *universally closed* if the map $f' : Z \times_Y X \to Z$ is closed for any continuous map $g : Z \to Y$.
(5) We say that f is *proper* if f is separated and universally closed.

The following lemma is useful later.

Lemma 17.3. *A topological space X is quasi-compact if and only if the projection map $Z \times X \to Z$ is closed for any topological space Z.*

Proof. (See also remark below.) If X is not quasi-compact, there exists an open covering $X = \bigcup_{i \in I} U_i$ such that no finite number of U_i cover X. Let Z be the subset of the power set $\mathcal{P}(I)$ of I consisting of I and all nonempty finite subsets of I. Define a topology on Z with as a basis for the topology the following sets:

(1) All subsets of $Z \setminus \{I\}$.
(2) For every finite subset K of I the set $U_K := \{J \subset I \mid J \in Z,\ K \subset J\})$.

It is left to the reader to verify this is the basis for a topology. Consider the subset of $Z \times X$ defined by the formula

$$M = \{(J, x) \mid J \in Z,\ x \in \bigcap\nolimits_{i \in J} U_i^c)\}$$

If $(J, x) \notin M$, then $x \in U_i$ for some $i \in J$. Hence $U_{\{i\}} \times U_i \subset Z \times X$ is an open subset containing (J, x) and not intersecting M. Hence M is closed. The projection of M to Z is $Z - \{I\}$ which is not closed. Hence $Z \times X \to Z$ is not closed.

[4]This is the terminology used in [Bou71]. Sometimes this property may be called "universally closed" in the literature.

Assume X is quasi-compact. Let Z be a topological space. Let $M \subset Z \times X$ be closed. Let $z \in Z$ be a point which is not in $\mathrm{pr}_1(M)$. By the Tube Lemma 17.1 there exists an open $U \subset Z$ such that $U \times X$ is contained in the complement of M. Hence $\mathrm{pr}_1(M)$ is closed. □

Remark 17.4. Lemma 17.3 is a combination of [Bou71, I, p. 75, Lemme 1] and [Bou71, I, p. 76, Corollaire 1].

Theorem 17.5. *Let $f : X \to Y$ be a continuous map between topological spaces. The following conditions are equivalent:*

1. *The map f is quasi-proper and closed.*
2. *The map f is Bourbaki-proper.*
3. *The map f is universally closed.*
4. *The map f is closed and $f^{-1}(y)$ is quasi-compact for any $y \in Y$.*

Proof. (See also the remark below.) If the map f satisfies (1), it automatically satisfies (4) because any single point is quasi-compact.

Assume map f satisfies (4). We will prove it is universally closed, i.e., (3) holds. Let $g : Z \to Y$ be a continuous map of topological spaces and consider the diagram

$$\begin{tikzcd} Z \times_Y X \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

During the proof we will use that $Z \times_Y X \to Z \times X$ is a homeomorphism onto its image, i.e., that we may identify $Z \times_Y X$ with the corresponding subset of $Z \times X$ with the induced topology. The image of $f' : Z \times_Y X \to Z$ is $\mathrm{Im}(f') = \{z : g(z) \in f(X)\}$. Because $f(X)$ is closed, we see that $\mathrm{Im}(f')$ is a closed subspace of Z. Consider a closed subset $P \subset Z \times_Y X$. Let $z \in Z$, $z \notin f'(P)$. If $z \notin \mathrm{Im}(f')$, then $Z \setminus \mathrm{Im}(f')$ is an open neighbourhood which avoids $f'(P)$. If z is in $\mathrm{Im}(f')$ then $(f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\}$ and $f^{-1}\{g(z)\}$ is quasi-compact by assumption. Because P is a closed subset of $Z \times_Y X$, we have a closed P' of $Z \times X$ such that $P = P' \cap Z \times_Y X$. Since $(f')^{-1}\{z\}$ is a subset of $P^c = P'^c \cup (Z \times_Y X)^c$, and since $(f')^{-1}\{z\}$ is disjoint from $(Z \times_Y X)^c$ we see that $(f')^{-1}\{z\}$ is contained in P'^c. We may apply the Tube Lemma 17.1 to $(f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\} \subset (P')^c \subset Z \times X$. This gives $V \times U$ containing $(f')^{-1}\{z\}$ where U and V are open sets in X and Z respectively and $V \times U$ has empty intersection with P'. Then the set $V \cap g^{-1}(Y - f(U^c))$ is open in Z since f is closed, contains z, and has empty intersection with the image of P. Thus $f'(P)$ is closed. In other words, the map f is universally closed.

The implication (3) ⇒ (2) is trivial. Namely, given any topological space Z consider the projection morphism $g : Z \times Y \to Y$. Then it is easy to see that f' is the map $Z \times X \to Z \times Y$, in other words that $(Z \times Y) \times_Y X = Z \times X$. (This identification is a purely categorical property having nothing to do with topological spaces per se.)

Assume f satisfies (2). We will prove it satisfies (1). Note that f is closed as f can be identified with the map $\{pt\} \times X \to \{pt\} \times Y$ which is assumed closed. Choose any quasi-compact subset $K \subset Y$. Let Z be any topological space. Because $Z \times X \to Z \times Y$ is closed we see the map $Z \times f^{-1}(K) \to Z \times K$ is closed (if T is closed in $Z \times f^{-1}(K)$, write $T = Z \times f^{-1}(K) \cap T'$ for some closed $T' \subset Z \times X$).

Because K is quasi-compact, $K \times Z \to Z$ is closed by Lemma 17.3. Hence the composition $Z \times f^{-1}(K) \to Z \times K \to Z$ is closed and therefore $f^{-1}(K)$ must be quasi-compact by Lemma 17.3 again. $\square$

Remark 17.6. Here are some references to the literature. In [Bou71, I, p. 75, Theorem 1] you can find: (2) $\Leftrightarrow$ (4). In [Bou71, I, p. 77, Proposition 6] you can find: (2) $\Rightarrow$ (1). Of course, trivially we have (1) $\Rightarrow$ (4). Thus (1), (2) and (4) are equivalent. The equivalence of (3) and (4) is [Lan93, Chapter II, Exercise 25].

Lemma 17.7. *Let $f : X \to Y$ be a continuous map of topological spaces. If X is quasi-compact and Y is Hausdorff, then f is universally closed.*

Proof. Since every point of Y is closed, we see from Lemma 12.3 that the closed subset $f^{-1}(y)$ of X is quasi-compact for all $y \in Y$. Thus, by Theorem 17.5 it suffices to show that f is closed. If $E \subset X$ is closed, then it is quasi-compact (Lemma 12.3), hence $f(E) \subset Y$ is quasi-compact (Lemma 12.7), hence $f(E)$ is closed in Y (Lemma 12.4). $\square$

Lemma 17.8. *Let $f : X \to Y$ be a continuous map of topological spaces. If f is bijective, X is quasi-compact, and Y is Hausdorff, then f is a homeomorphism.*

Proof. It suffices to prove f is closed, because this implies that f^{-1} is continuous. If $T \subset X$ is closed, then T is quasi-compact by Lemma 12.3, hence $f(T)$ is quasi-compact by Lemma 12.7, hence $f(T)$ is closed by Lemma 12.4. $\square$

18. Jacobson spaces

Definition 18.1. Let X be a topological space. Let X_0 be the set of closed points of X. We say that X is *Jacobson* if every closed subset $Z \subset X$ is the closure of $Z \cap X_0$.

Note that a topological space X is Jacobson if and only if every nonempty locally closed subset of X has a point closed in X.

Let X be a Jacobson space and let X_0 be the set of closed points of X with the induced topology. Clearly, the definition implies that the morphism $X_0 \to X$ induces a bijection between the closed subsets of X_0 and the closed subsets of X. Thus many properties of X are inherited by X_0. For example, the Krull dimensions of X and X_0 are the same.

Lemma 18.2. *Let X be a topological space. Let X_0 be the set of closed points of X. Suppose that for every point $x \in X$ the intersection $X_0 \cap \overline{\{x\}}$ is dense in $\overline{\{x\}}$. Then X is Jacobson.*

Proof. Let Z be closed subset of X and U be and open subset of X such that $U \cap Z$ is nonempty. Then for $x \in U \cap Z$ we have that $\overline{\{x\}} \cap U$ is a nonempty subset of $Z \cap U$, and by hypothesis it contains a point closed in X as required. $\square$

Lemma 18.3. *Let X be a Kolmogorov topological space with a basis of quasi-compact open sets. If X is not Jacobson, then there exists a non-closed point $x \in X$ such that $\{x\}$ is locally closed.*

Proof. As X is not Jacobson there exists a closed set Z and an open set U in X such that $Z \cap U$ is nonempty and does not contain points closed in X. As X has a basis of quasi-compact open sets we may replace U by an open quasi-compact neighborhood of a point in $Z \cap U$ and so we may assume that U is quasi-compact open. By Lemma 12.8, there exists a point $x \in Z \cap U$ closed in $Z \cap U$, and so $\{x\}$ is locally closed but not closed in X. □

Lemma 18.4. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Then X is Jacobson if and only if each U_i is Jacobson. Moreover, in this case $X_0 = \bigcup U_{i,0}$.*

Proof. Let X be a topological space. Let X_0 be the set of closed points of X. Let $U_{i,0}$ be the set of closed points of U_i. Then $X_0 \cap U_i \subset U_{i,0}$ but equality may not hold in general.

First, assume that each U_i is Jacobson. We claim that in this case $X_0 \cap U_i = U_{i,0}$. Namely, suppose that $x \in U_{i,0}$, i.e., x is closed in U_i. Let $\overline{\{x\}}$ be the closure in X. Consider $\overline{\{x\}} \cap U_j$. If $x \notin U_j$, then $\overline{\{x\}} \cap U_j = \emptyset$. If $x \in U_j$, then $U_i \cap U_j \subset U_j$ is an open subset of U_j containing x. Let $T' = U_j \setminus U_i \cap U_j$ and $T = \{x\} \amalg T'$. Then T, T' are closed subsets of U_j and T contains x. As U_j is Jacobson we see that the closed points of U_j are dense in T. Because $T = \{x\} \amalg T'$ this can only be the case if x is closed in U_j. Hence $\overline{\{x\}} \cap U_j = \{x\}$. We conclude that $\overline{\{x\}} = \{x\}$ as desired.

Let $Z \subset X$ be a closed subset (still assuming each U_i is Jacobson). Since now we know that $X_0 \cap Z \cap U_i = U_{i,0} \cap Z$ are dense in $Z \cap U_i$ it follows immediately that $X_0 \cap Z$ is dense in Z.

Conversely, assume that X is Jacobson. Let $Z \subset U_i$ be closed. Then $X_0 \cap \overline{Z}$ is dense in $\overline{Z}$. Hence also $X_0 \cap Z$ is dense in Z, because $\overline{Z} \setminus Z$ is closed. As $X_0 \cap U_i \subset U_{i,0}$ we see that $U_{i,0} \cap Z$ is dense in Z. Thus U_i is Jacobson as desired. □

Lemma 18.5. *Let X be Jacobson. The following types of subsets $T \subset X$ are Jacobson:*

(1) *Open subspaces.*
(2) *Closed subspaces.*
(3) *Locally closed subspaces.*
(4) *Unions of locally closed subspaces.*
(5) *Constructible sets.*
(6) *Any subset $T \subset X$ which locally on X is a union of locally closed subsets.*

In each of these cases closed points of T are closed in X.

Proof. Let X_0 be the set of closed points of X. For any subset $T \subset X$ we let $(*)$ denote the property:

(*) Every nonempty locally closed subset of T has a point closed in X.

Note that always $X_0 \cap T \subset T_0$. Hence property $(*)$ implies that T is Jacobson. In addition it clearly implies that every closed point of T is closed in X.

Suppose that $T = \bigcup_i T_i$ with T_i locally closed in X. Take $A \subset T$ a locally closed nonempty subset in T, then there exists a T_i such that $A \cap T_i$ is nonempty, it is locally closed in T_i and so in X. As X is Jacobson A has a point closed in X. □

Lemma 18.6. *A finite Jacobson space is discrete.*

Proof. If X is finite Jacobson, $X_0 \subset X$ the subset of closed points, then, on the one hand, $\overline{X_0} = X$. On the other hand, X, and hence X_0 is finite, so $X_0 =$

$\{x_1, \ldots, x_n\} = \bigcup_{i=1,\ldots,n} \{x_i\}$ is a finite union of closed sets, hence closed, so $X = \overline{X_0} = X_0$. Every point is closed, and by finiteness, every point is open. $\square$

Lemma 18.7. *Suppose X is a Jacobson topological space. Let X_0 be the set of closed points of X. There is a bijective, inclusion preserving correspondence*

$$\{\textit{finite unions loc. closed subsets of } X\} \leftrightarrow \{\textit{finite unions loc. closed subsets of } X_0\}$$

given by $E \mapsto E \cap X_0$. This correspondence preserves the subsets of locally closed, of open and of closed subsets.

Proof. We just prove that the correspondence $E \mapsto E \cap X_0$ is injective. Indeed if $E \neq E'$ then without loss of generality $E \setminus E'$ is nonempty, and it is a finite union of locally closed sets (details omitted). As X is Jacobson, we see that $(E \setminus E') \cap X_0 = E \cap X_0 \setminus E' \cap X_0$ is not empty. $\square$

Lemma 18.8. *Suppose X is a Jacobson topological space. Let X_0 be the set of closed points of X. There is a bijective, inclusion preserving correspondence*

$$\{\textit{constructible subsets of } X\} \leftrightarrow \{\textit{constructible subsets of } X_0\}$$

given by $E \mapsto E \cap X_0$. This correspondence preserves the subset of retrocompact open subsets, as well as complements of these.

Proof. From Lemma 18.7 above, we just have to see that if U is open in X then $U \cap X_0$ is retrocompact in X_0 if and only if U is retrocompact in X. This follows if we prove that for U open in X then $U \cap X_0$ is quasi-compact if and only if U is quasi-compact. From Lemma 18.5 it follows that we may replace X by U and assume that $U = X$. Finally notice that any collection of opens $\mathcal{U}$ of X cover X if and only if they cover X_0, using the Jacobson property of X in the closed $X \setminus \bigcup \mathcal{U}$ to find a point in X_0 if it were nonempty. $\square$

19. Specialization

Definition 19.1. Let X be a topological space.

(1) If $x, x' \in X$ then we say x is a *specialization* of x', or x' is a *generalization* of x if $x \in \overline{\{x'\}}$. Notation: $x' \rightsquigarrow x$.
(2) A subset $T \subset X$ is *stable under specialization* if for all $x' \in T$ and every specialization $x' \rightsquigarrow x$ we have $x \in T$.
(3) A subset $T \subset X$ is *stable under generalization* if for all $x \in T$ and every generalization $x' \rightsquigarrow x$ we have $x' \in T$.

Lemma 19.2. *Let X be a topological space.*

(1) *Any closed subset of X is stable under specialization.*
(2) *Any open subset of X is stable under generalization.*
(3) *A subset $T \subset X$ is stable under specialization if and only if the complement T^c is stable under generalization.*

Proof. Let F be a closed subset of X, if $y \in F$ then $\{y\} \subset F$, so $\overline{\{y\}} \subset \overline{F} = F$ as F is closed. Thus for all specialization x of y, we have $x \in F$.

Let $x, y \in X$ such that $x \in \overline{\{y\}}$ and let T be a subset of X. Saying that T is stable under specialization means that $y \in T$ implies $x \in T$ and reciprocally saying that T is stable under generalization means that $x \in T$ implies $y \in T$. Therefore (3) is proven using contraposition.

The second property follows from (1) and (3) by considering the complement. $\square$

Lemma 19.3. *Let $T \subset X$ be a subset of a topological space X. The following are equivalent*

(1) *T is stable under specialization, and*
(2) *T is a (directed) union of closed subsets of X.*

Proof. Suppose that T is stable under specialization, then for all $y \in T$ we have $\overline{\{y\}} \subset T$. Thus $T = \bigcup_{y \in T} \overline{\{y\}}$ which is an union of closed subsets of X. Reciprocally, suppose that $T = \bigcup_{i \in I} F_i$ where F_i are closed subsets of X. If $y \in T$ then there exists $i \in I$ such that $y \in F_i$. As F_i is closed, we have $\overline{\{y\}} \subset F_i \subset T$, which proves that T is stable under specialization. $\square$

Definition 19.4. Let $f : X \to Y$ be a continuous map of topological spaces.

(1) We say that *specializations lift along f* or that f is *specializing* if given $y' \leadsto y$ in Y and any $x' \in X$ with $f(x') = y'$ there exists a specialization $x' \leadsto x$ of x' in X such that $f(x) = y$.
(2) We say that *generalizations lift along f* or that f is *generalizing* if given $y' \leadsto y$ in Y and any $x \in X$ with $f(x) = y$ there exists a generalization $x' \leadsto x$ of x in X such that $f(x') = y'$.

Lemma 19.5. *Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. If specializations lift along both f and g then specializations lift along $g \circ f$. Similarly for "generalizations lift along".*

Proof. Let $z' \leadsto z$ be a specialization in Z and let $x' \in X$ such as $g \circ f(x') = z'$. Then because specializations lift along g, there exists a specialization $f(x') \leadsto y$ of $f(x')$ in Y such that $g(y) = z$. Likewise, because specializations lift along f, there exists a specialization $x' \leadsto x$ of x' in X such that $f(x) = y$. It provides a specialization $x' \leadsto x$ of x' in X such that $g \circ f(x) = z$. In other words, specialization lift along $g \circ f$. $\square$

Lemma 19.6. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If specializations lift along f, and if $T \subset X$ is stable under specialization, then $f(T) \subset Y$ is stable under specialization.*
(2) *If generalizations lift along f, and if $T \subset X$ is stable under generalization, then $f(T) \subset Y$ is stable under generalization.*

Proof. Let $y' \leadsto y$ be a specialization in Y where $y' \in f(T)$ and let $x' \in T$ such that $f(x') = y'$. Because specialization lift along f, there exists a specialization $x' \leadsto x$ of x' in X such that $f(x) = y$. But T is stable under specialization so $x \in T$ and then $y \in f(T)$. Therefore $f(T)$ is stable under specialization.

The proof of (2) is identical, using that generalizations lift along f. $\square$

Lemma 19.7. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If f is closed then specializations lift along f.*
(2) *If f is open, X is a Noetherian topological space, each irreducible closed subset of X has a generic point, and Y is Kolmogorov then generalizations lift along f.*

Proof. Assume f is closed. Let $y' \rightsquigarrow y$ in Y and any $x' \in X$ with $f(x') = y'$ be given. Consider the closed subset $T = \overline{\{x'\}}$ of X. Then $f(T) \subset Y$ is a closed subset, and $y' \in f(T)$. Hence also $y \in f(T)$. Hence $y = f(x)$ with $x \in T$, i.e., $x' \rightsquigarrow x$.

Assume f is open, X Noetherian, every irreducible closed subset of X has a generic point, and Y is Kolmogorov. Let $y' \rightsquigarrow y$ in Y and any $x \in X$ with $f(x) = y$ be given. Consider $T = f^{-1}(\{y'\}) \subset X$. Take an open neighbourhood $x \in U \subset X$ of x. Then $f(U) \subset Y$ is open and $y \in f(U)$. Hence also $y' \in f(U)$. In other words, $T \cap U \neq \emptyset$. This proves that $x \in \overline{T}$. Since X is Noetherian, T is Noetherian (Lemma 9.2). Hence it has a decomposition $T = T_1 \cup \ldots \cup T_n$ into irreducible components. Then correspondingly $\overline{T} = \overline{T_1} \cup \ldots \cup \overline{T_n}$. By the above $x \in \overline{T_i}$ for some i. By assumption there exists a generic point $x' \in \overline{T_i}$, and we see that $x' \rightsquigarrow x$. As $x' \in \overline{T}$ we see that $f(x') \in \overline{\{y'\}}$. Note that $f(\overline{T_i}) = f(\overline{\{x'\}}) \subset \overline{\{f(x')\}}$. If $f(x') \neq y'$, then since Y is Kolmogorov $f(x')$ is not a generic point of the irreducible closed subset $\overline{\{y'\}}$ and the inclusion $\overline{\{f(x')\}} \subset \overline{\{y'\}}$ is strict, i.e., $y' \notin f(\overline{T_i})$. This contradicts the fact that $f(T_i) = \{y'\}$. Hence $f(x') = y'$ and we win. $\square$

Lemma 19.8. *Suppose that $s, t : R \to U$ and $\pi : U \to X$ are continuous maps of topological spaces such that*

1. *π is open,*
2. *U is sober,*
3. *s, t have finite fibres,*
4. *generalizations lift along s, t,*
5. *$(t, s)(R) \subset U \times U$ is an equivalence relation on U and X is the quotient of U by this equivalence relation (as a set).*

Then X is Kolmogorov.

Proof. Properties (3) and (5) imply that a point x corresponds to an finite equivalence class $\{u_1, \ldots, u_n\} \subset U$ of the equivalence relation. Suppose that $x' \in X$ is a second point corresponding to the equivalence class $\{u'_1, \ldots, u'_m\} \subset U$. Suppose that $u_i \rightsquigarrow u'_j$ for some i, j. Then for any $r' \in R$ with $s(r') = u'_j$ by (4) we can find $r \rightsquigarrow r'$ with $s(r) = u_i$. Hence $t(r) \rightsquigarrow t(r')$. Since $\{u'_1, \ldots, u'_m\} = t(s^{-1}(\{u'_j\}))$ we conclude that every element of $\{u'_1, \ldots, u'_m\}$ is the specialization of an element of $\{u_1, \ldots, u_n\}$. Thus $\overline{\{u_1\}} \cup \ldots \cup \overline{\{u_n\}}$ is a union of equivalence classes, hence of the form $\pi^{-1}(Z)$ for some subset $Z \subset X$. By (1) we see that Z is closed in X and in fact $Z = \overline{\{x\}}$ because $\pi(\overline{\{u_i\}}) \subset \overline{\{x\}}$ for each i. In other words, $x \rightsquigarrow x'$ if and only if some lift of x in U specializes to some lift of x' in U, if and only if every lift of x' in U is a specialization of some lift of x in U.

Suppose that both $x \rightsquigarrow x'$ and $x' \rightsquigarrow x$. Say x corresponds to $\{u_1, \ldots, u_n\}$ and x' corresponds to $\{u'_1, \ldots, u'_m\}$ as above. Then, by the results of the preceding paragraph, we can find a sequence

$$\ldots \rightsquigarrow u'_{j_3} \rightsquigarrow u_{i_3} \rightsquigarrow u'_{j_2} \rightsquigarrow u_{i_2} \rightsquigarrow u'_{j_1} \rightsquigarrow u_{i_1}$$

which must repeat, hence by (2) we conclude that $\{u_1, \ldots, u_n\} = \{u'_1, \ldots, u'_m\}$, i.e., $x = x'$. Thus X is Kolmogorov. $\square$

Lemma 19.9. *Let $f : X \to Y$ be a morphism of topological spaces. Suppose that Y is a sober topological space, and f is surjective. If either specializations or generalizations lift along f, then* $\dim(X) \geq \dim(Y)$.

Proof. Assume specializations lift along f. Let $Z_0 \subset Z_1 \subset \ldots Z_e \subset Y$ be a chain of irreducible closed subsets of X. Let $\xi_e \in X$ be a point mapping to the generic point of Z_e. By assumption there exists a specialization $\xi_e \leadsto \xi_{e-1}$ in X such that ξ_{e-1} maps to the generic point of Z_{e-1}. Continuing in this manner we find a sequence of specializations

$$\xi_e \leadsto \xi_{e-1} \leadsto \ldots \leadsto \xi_0$$

with ξ_i mapping to the generic point of Z_i. This clearly implies the sequence of irreducible closed subsets

$$\overline{\{\xi_0\}} \subset \overline{\{\xi_1\}} \subset \ldots \overline{\{\xi_e\}}$$

is a chain of length e in X. The case when generalizations lift along f is similar. □

Lemma 19.10. *Let X be a Noetherian sober topological space. Let $E \subset X$ be a subset of X.*

1. *If E is constructible and stable under specialization, then E is closed.*
2. *If E is constructible and stable under generalization, then E is open.*

Proof. Let E be constructible and stable under generalization. Let $Y \subset X$ be an irreducible closed subset with generic point $\xi \in Y$. If $E \cap Y$ is nonempty, then it contains ξ (by stability under generalization) and hence is dense in Y, hence it contains a nonempty open of Y, see Lemma 16.3. Thus E is open by Lemma 16.5. This proves (2). To prove (1) apply (2) to the complement of E in X. □

20. Dimension functions

It scarcely makes sense to consider dimension functions unless the space considered is sober (Definition 8.6). Thus the definition below can be improved by considering the sober topological space associated to X. Since the underlying topological space of a scheme is sober we do not bother with this improvement.

Definition 20.1. Let X be a topological space.

1. Let $x, y \in X$, $x \neq y$. Suppose $x \leadsto y$, that is y is a specialization of x. We say y is an *immediate specialization* of x if there is no $z \in X \setminus \{x, y\}$ with $x \leadsto z$ and $z \leadsto y$.
2. A map $\delta : X \to \mathbf{Z}$ is called a *dimension function*[5] if
   (a) whenever $x \leadsto y$ and $x \neq y$ we have $\delta(x) > \delta(y)$, and
   (b) for every immediate specialization $x \leadsto y$ in X we have $\delta(x) = \delta(y)+1$.

It is clear that if δ is a dimension function, then so is $\delta + t$ for any $t \in \mathbf{Z}$. Here is a fun lemma.

Lemma 20.2. *Let X be a topological space. If X is sober and has a dimension function, then X is catenary. Moreover, for any $x \leadsto y$ we have*

$$\delta(x) - \delta(y) = \mathit{codim}\left(\overline{\{y\}},\ \overline{\{x\}}\right).$$

Proof. Suppose $Y \subset Y' \subset X$ are irreducible closed subsets. Let $\xi \in Y$, $\xi' \in Y'$ be their generic points. Then we see immediately from the definitions that $\text{codim}(Y, Y') \leq \delta(\xi) - \delta(\xi') < \infty$. In fact the first inequality is an equality. Namely, suppose

$$Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e = Y'$$

[5]This is likely nonstandard notation. This notion is usually introduced only for (locally) Noetherian schemes, in which case condition (a) is implied by (b).

is any maximal chain of irreducible closed subsets. Let $\xi_i \in Y_i$ denote the generic point. Then we see that $\xi_i \rightsquigarrow \xi_{i+1}$ is an immediate specialization. Hence we see that $e = \delta(\xi) - \delta(\xi')$ as desired. This also proves the last statement of the lemma. $\square$

Lemma 20.3. *Let X be a topological space. Let δ, δ' be two dimension functions on X. If X is locally Noetherian and sober then $\delta - \delta'$ is locally constant on X.*

Proof. Let $x \in X$ be a point. We will show that $\delta - \delta'$ is constant in a neighbourhood of x. We may replace X by an open neighbourhood of x in X which is Noetherian. Hence we may assume X is Noetherian and sober. Let $Z_1, \ldots, Z_r$ be the irreducible components of X passing through x. (There are finitely many as X is Noetherian, see Lemma 9.2.) Let $\xi_i \in Z_i$ be the generic point. Note $Z_1 \cup \ldots \cup Z_r$ is a neighbourhood of x in X (not necessarily closed). We claim that $\delta - \delta'$ is constant on $Z_1 \cup \ldots \cup Z_r$. Namely, if $y \in Z_i$, then

$$\delta(x) - \delta(y) = \delta(x) - \delta(\xi_i) + \delta(\xi_i) - \delta(y) = -\text{codim}(\overline{\{x\}}, Z_i) + \text{codim}(\overline{\{y\}}, Z_i)$$

by Lemma 20.2. Similarly for δ'. Whence the result. $\square$

Lemma 20.4. *Let X be locally Noetherian, sober and catenary. Then any point has an open neighbourhood $U \subset X$ which has a dimension function.*

Proof. We will use repeatedly that an open subspace of a catenary space is catenary, see Lemma 11.5 and that a Noetherian topological space has finitely many irreducible components, see Lemma 9.2. In the proof of Lemma 20.3 we saw how to construct such a function. Namely, we first replace X by a Noetherian open neighbourhood of x. Next, we let $Z_1, \ldots, Z_r \subset X$ be the irreducible components of X. Let

$$Z_i \cap Z_j = \bigcup Z_{ijk}$$

be the decomposition into irreducible components. We replace X by

$$X \setminus \left(\bigcup\nolimits_{x \notin Z_i} Z_i \cup \bigcup\nolimits_{x \notin Z_{ijk}} Z_{ijk}\right)$$

so that we may assume $x \in Z_i$ for all i and $x \in Z_{ijk}$ for all i, j, k. For $y \in X$ choose any i such that $y \in Z_i$ and set

$$\delta(y) = -\text{codim}(\overline{\{x\}}, Z_i) + \text{codim}(\overline{\{y\}}, Z_i).$$

We claim this is a dimension function. First we show that it is well defined, i.e., independent of the choice of i. Namely, suppose that $y \in Z_{ijk}$ for some i, j, k. Then we have (using Lemma 11.6)

$$\begin{aligned}\delta(y) &= -\text{codim}(\overline{\{x\}}, Z_i) + \text{codim}(\overline{\{y\}}, Z_i) \\ &= -\text{codim}(\overline{\{x\}}, Z_{ijk}) - \text{codim}(Z_{ijk}, Z_i) + \text{codim}(\overline{\{y\}}, Z_{ijk}) + \text{codim}(Z_{ijk}, Z_i) \\ &= -\text{codim}(\overline{\{x\}}, Z_{ijk}) + \text{codim}(\overline{\{y\}}, Z_{ijk})\end{aligned}$$

which is symmetric in i and j. We omit the proof that it is a dimension function. $\square$

Remark 20.5. Combining Lemmas 20.3 and 20.4 we see that on a catenary, locally Noetherian, sober topological space the obstruction to having a dimension function is an element of $H^1(X, \mathbf{Z})$.

21. Nowhere dense sets

Definition 21.1. Let X be a topological space.

(1) Given a subset $T \subset X$ the *interior* of T is the largest open subset of X contained in T.
(2) A subset $T \subset X$ is called *nowhere dense* if the closure of T has empty interior.

Lemma 21.2. *Let X be a topological space. The union of a finite number of nowhere dense sets is a nowhere dense set.*

Proof. Omitted. ☐

Lemma 21.3. *Let X be a topological space. Let $U \subset X$ be an open. Let $T \subset U$ be a subset. If T is nowhere dense in U, then T is nowhere dense in X.*

Proof. Assume T is nowhere dense in U. Suppose that $x \in X$ is an interior point of the closure $\overline{T}$ of T in X. Say $x \in V \subset \overline{T}$ with $V \subset X$ open in X. Note that $\overline{T} \cap U$ is the closure of T in U. Hence the interior of $\overline{T} \cap U$ being empty implies $V \cap U = \emptyset$. Thus x cannot be in the closure of U, a fortiori cannot be in the closure of T, a contradiction. ☐

Lemma 21.4. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $T \subset X$ be a subset. If $T \cap U_i$ is nowhere dense in U_i for all i, then T is nowhere dense in X.*

Proof. Denote $\overline{T}_i$ the closure of $T \cap U_i$ in U_i. We have $\overline{T} \cap U_i = \overline{T}_i$. Taking the interior commutes with intersection with opens, thus

$$(\text{interior of } \overline{T}) \cap U_i = \text{interior of } (\overline{T} \cap U_i) = \text{interior in } U_i \text{ of } \overline{T}_i$$

By assumption the last of these is empty. Hence T is nowhere dense in X. ☐

Lemma 21.5. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset X$ be a subset. If f is a homeomorphism of X onto a closed subset of Y and T is nowhere dense in X, then also $f(T)$ is nowhere dense in Y.*

Proof. Omitted. ☐

Lemma 21.6. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset Y$ be a subset. If f is open and T is a closed nowhere dense subset of Y, then also $f^{-1}(T)$ is a closed nowhere dense subset of X. If f is surjective and open, then T is closed nowhere dense if and only if $f^{-1}(T)$ is closed nowhere dense.*

Proof. Omitted. (Hint: In the first case the interior of $f^{-1}(T)$ maps into the interior of T, and in the second case the interior of $f^{-1}(T)$ maps onto the interior of T.) ☐

22. Profinite spaces

Here is the definition.

Definition 22.1. A topological space is *profinite* if it is homeomorphic to a limit of a diagram of finite discrete spaces.

This is not the most convenient characterization of a profinite space.

Lemma 22.2. *Let X be a topological space. The following are equivalent*

(1) *X is a profinite space, and*
(2) *X is Hausdorff, quasi-compact, and totally disconnected.*

If this is true, then X is a cofiltered limit of finite discrete spaces.

Proof. Assume (1). Choose a diagram $i \mapsto X_i$ of finite discrete spaces such that $X = \lim X_i$. As each X_i is Hausdorff and quasi-compact we find that X is quasi-compact by Lemma 14.5. If $x, x' \in X$ are distinct points, then x and x' map to distinct points in some X_i. Hence x and x' have disjoint open neighbourhoods, i.e., X is Hausdorff. In exactly the same way we see that X is totally disconnected.

Assume (2). Let $\mathcal{I}$ be the set of finite disjoint union decompositions $X = \coprod_{i \in I} U_i$ with U_i nonempty open (and closed) for all $i \in I$. For each $I \in \mathcal{I}$ there is a continuous map $X \to I$ sending a point of U_i to i. We define a partial ordering: $I \leq I'$ for $I, I' \in \mathcal{I}$ if and only if the covering corresponding to I' refines the covering corresponding to I. In this case we obtain a canonical map $I' \to I$. In other words we obtain an inverse system of finite discrete spaces over $\mathcal{I}$. The maps $X \to I$ fit together and we obtain a continuous map

$$X \longrightarrow \lim_{I \in \mathcal{I}} I$$

We claim this map is a homeomorphism, which finishes the proof. (The final assertion follows too as the partially ordered set $\mathcal{I}$ is directed: given two disjoint union decompositions of X we can find a third refining both.) Namely, the map is injective as X is totally disconnected and hence $\{x\}$ is the intersection of all open and closed subsets of X containing x (Lemma 12.11) and the map is surjective by Lemma 12.6. By Lemma 17.8 the map is a homeomorphism. $\square$

Lemma 22.3. *A limit of profinite spaces is profinite.*

Proof. Let $i \mapsto X_i$ be a diagram of profinite spaces over the index category $\mathcal{I}$. Let us use the characterization of profinite spaces in Lemma 22.2. In particular each X_i is Hausdorff, quasi-compact, and totally disconnected. By Lemma 14.1 the limit $X = \lim X_i$ exists. By Lemma 14.5 the limit X is quasi-compact. Let $x, x' \in X$ be distinct points. Then there exists an i such that x and x' have distinct images x_i and x'_i in X_i under the projection $X \to X_i$. Then x_i and x'_i have disjoint open neighbourhoods in X_i. Taking the inverse images of these opens we conclude that X is Hausdorff. Similarly, x_i and x'_i are in distinct connected components of X_i whence necessarily x and x' must be in distinct connected components of X. Hence X is totally disconnected. This finishes the proof. $\square$

Lemma 22.4. *Let X be a profinite space. Every open covering of X has a refinement by a finite covering $X = \coprod U_i$ with U_i open and closed.*

Proof. Write $X = \lim X_i$ as a limit of an inverse system of finite discrete spaces over a directed set I (Lemma 22.2). Denote $f_i : X \to X_i$ the projection. For every point $x = (x_i) \in X$ a fundamental system of open neighbourhoods is the collection $f_i^{-1}(\{x_i\})$. Thus, as X is quasi-compact, we may assume we have an open covering

$$X = f_{i_1}^{-1}(\{x_{i_1}\}) \cup \ldots \cup f_{i_n}^{-1}(\{x_{i_n}\})$$

Choose $i \in I$ with $i \geq i_j$ for $j = 1, \ldots, n$ (this is possible as I is a directed set). Then we see that the covering

$$X = \coprod_{t \in X_i} f_i^{-1}(\{t\})$$

refines the given covering and is of the desired form. $\square$

Lemma 22.5. *Let X be a topological space. If X is quasi-compact and every connected component of X is the intersection of the open and closed subsets containing it, then $\pi_0(X)$ is a profinite space.*

Proof. We will use Lemma 22.2 to prove this. Since $\pi_0(X)$ is the image of a quasi-compact space it is quasi-compact (Lemma 12.7). It is totally disconnected by construction (Lemma 7.9). Let $C, D \subset X$ be distinct connected components of X. Write $C = \bigcap U_\alpha$ as the intersection of the open and closed subsets of X containing C. Any finite intersection of U_α's is another. Since $\bigcap U_\alpha \cap D = \emptyset$ we conclude that $U_\alpha \cap D = \emptyset$ for some α (use Lemmas 7.3, 12.3 and 12.6) Since U_α is open and closed, it is the union of the connected components it contains, i.e., U_α is the inverse image of some open and closed subset $V_\alpha \subset \pi_0(X)$. This proves that the points corresponding to C and D are contained in disjoint open subsets, i.e., $\pi_0(X)$ is Hausdorff. $\square$

23. Spectral spaces

The material in this section is taken from [Hoc69] and [Hoc67]. In his thesis Hochster proves (among other things) that the spectral spaces are exactly the topological spaces that occur as the spectrum of a ring.

Definition 23.1. A topological space X is called *spectral* if it is sober, quasi-compact, the intersection of two quasi-compact opens is quasi-compact, and the collection of quasi-compact opens forms a basis for the topology. A continuous map $f : X \to Y$ of spectral spaces is called *spectral* if the inverse image of a quasi-compact open is quasi-compact.

In other words a continuous map of spectral spaces is spectral if and only if it is quasi-compact (Definition 12.1).

Let X be a spectral space. The *constructible topology* on X is the topology which has as a subbase of opens the sets U and U^c where U is a quasi-compact open of X. Note that since X is spectral an open $U \subset X$ is retrocompact if and only if U is quasi-compact. Hence the constructible topology can also be characterized as the coarsest topology such that every constructible subset of X is both open and closed (see Section 15 for definitions and properties of constructible sets). It follows that a subset of X is open, resp. closed in the constructible topology if and only if it is a union, resp. intersection of constructible subsets. Since the collection of quasi-compact opens is a basis for the topology on X we see that the constructible topology is stronger than the given topology on X.

Lemma 23.2. *Let X be a spectral space. The constructible topology is Hausdorff, totally disconnected, and quasi-compact.*

Proof. Let $x, y \in X$ with $x \neq y$. Since X is sober, there is an open subset U containing exactly one of the two points x, y. Say $x \in U$. We may replace U by a quasi-compact open neighbourhood of x contained in U. Then U and U^c are open and closed in the constructible topology. Hence X is Hausdorff in the constructible topology because $x \in U$ and $y \in U^c$ are disjoint opens in the constructible topology. The existence of U also implies x and y are in distinct connected components in the constructible topology, whence X is totally disconnected in the constructible topology.

Let $\mathcal{B}$ be the collection of subsets $B \subset X$ with B either quasi-compact open or closed with quasi-compact complement. If $B \in \mathcal{B}$ then $B^c \in \mathcal{B}$. It suffices to show every covering $X = \bigcup_{i \in I} B_i$ with $B_i \in \mathcal{B}$ has a finite refinement, see Lemma 12.15. Taking complements we see that we have to show that any family $\{B_i\}_{i \in I}$ of elements of $\mathcal{B}$ such that $B_{i_1} \cap \ldots \cap B_{i_n} \neq \emptyset$ for all n and all $i_1, \ldots, i_n \in I$ has a common point of intersection. We may and do assume $B_i \neq B_{i'}$ for $i \neq i'$.

To get a contradiction assume $\{B_i\}_{i \in I}$ is a family of elements of $\mathcal{B}$ having the finite intersection property but empty intersection. An application of Zorn's lemma shows that we may assume our family is maximal (details omitted). Let $I' \subset I$ be those indices such that B_i is closed and set $Z = \bigcap_{i \in I'} B_i$. This is a closed subset of X which is nonempty by Lemma 12.6. If Z is reducible, then we can write $Z = Z' \cup Z''$ as a union of two closed subsets, neither equal to Z. This means in particular that we can find a quasi-compact open $U' \subset X$ meeting Z' but not Z''. Similarly, we can find a quasi-compact open $U'' \subset X$ meeting Z'' but not Z'. Set $B' = X \setminus U'$ and $B'' = X \setminus U''$. Note that $Z'' \subset B'$ and $Z' \subset B''$. If there exist a finite number of indices $i_1, \ldots, i_n \in I$ such that $B' \cap B_{i_1} \cap \ldots \cap B_{i_n} = \emptyset$ as well as a finite number of indices $j_1, \ldots, j_m \in I$ such that $B'' \cap B_{j_1} \cap \ldots \cap B_{j_m} = \emptyset$ then we find that $Z \cap B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m} = \emptyset$. However, the set $B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m}$ is quasi-compact hence we would find a finite number of indices $i'_1, \ldots, i'_l \in I'$ with $B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m} \cap B_{i'_1} \cap \ldots \cap B_{i'_l} = \emptyset$, a contradiction. Thus we see that we may add either B' or B'' to the given family contradicting maximality. We conclude that Z is irreducible. However, this leads to a contradiction as well, as now every nonempty (by the same argument as above) open $Z \cap B_i$ for $i \in I \setminus I'$ contains the unique generic point of Z. This contradiction proves the lemma. □

Lemma 23.3. *Let $f : X \to Y$ be a spectral map of spectral spaces. Then*
(1) *f is continuous in the constructible topology,*
(2) *the fibres of f are quasi-compact, and*
(3) *the image is closed in the constructible topology.*

Proof. Let X' and Y' denote X and Y endowed with the constructible topology which are quasi-compact Hausdorff spaces by Lemma 23.2. Part (1) says $X' \to Y'$ is continuous and follows immediately from the definitions. Part (3) follows as $f(X')$ is a quasi-compact subset of the Hausdorff space Y', see Lemma 12.4. We have a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

of continuous maps of topological spaces. Since Y' is Hausdorff we see that the fibres X'_y are closed in X'. As X' is quasi-compact we see that X'_y is quasi-compact (Lemma 12.3). As $X'_y \to X_y$ is a surjective continuous map we conclude that X_y is quasi-compact (Lemma 12.7). □

Lemma 23.4. *Let X and Y be spectral spaces. Let $f : X \to Y$ be a continuous map. Then f is spectral if and only if f is continuous in the constructible topology.*

Proof. The only if part of this is Lemma 23.3. Assume f is continuous in the constructible topology. Let $V \subset Y$ be quasi-compact open. Then V is open and closed in the constructible topology. Hence $f^{-1}(V)$ is open and closed in the constructible topology. Hence $f^{-1}(V)$ is quasi-compact in the constructible topology as X is quasi-compact in the constructible topology by Lemma 23.2. Since the identity $f^{-1}(V) \to f^{-1}(V)$ is surjective and continuous from the constructible topology to the usual topology, we conclude that $f^{-1}(V)$ is quasi-compact in the topology of X by Lemma 12.7. This finishes the proof. $\square$

Lemma 23.5. *Let X be a spectral space. Let $E \subset X$ be closed in the constructible topology (for example constructible or closed). Then E with the induced topology is a spectral space.*

Proof. Let $Z \subset E$ be a closed irreducible subset. Let η be the generic point of the closure $\overline{Z}$ of Z in X. To prove that E is sober, we show that $\eta \in E$. If not, then since E is closed in the constructible topology, there exists a constructible subset $F \subset X$ such that $\eta \in F$ and $F \cap E = \emptyset$. By Lemma 15.15 this implies $F \cap \overline{Z}$ contains a nonempty open subset of $\overline{Z}$. But this is impossible as $\overline{Z}$ is the closure of Z and $Z \cap F = \emptyset$.

Since E is closed in the constructible topology, it is quasi-compact in the constructible topology (Lemmas 12.3 and 23.2). Hence a fortiori it is quasi-compact in the topology coming from X. If $U \subset X$ is a quasi-compact open, then $E \cap U$ is closed in the constructible topology, hence quasi-compact (as seen above). It follows that the quasi-compact open subsets of E are the intersections $E \cap U$ with U quasi-compact open in X. These form a basis for the topology. Finally, given two $U, U' \subset X$ quasi-compact opens, the intersection $(E \cap U) \cap (E \cap U') = E \cap (U \cap U')$ and $U \cap U'$ is quasi-compact as X is spectral. This finishes the proof. $\square$

Lemma 23.6. *Let X be a spectral space. Let $E \subset X$ be a subset closed in the constructible topology (for example constructible).*

1. *If $x \in \overline{E}$, then x is the specialization of a point of E.*
2. *If E is stable under specialization, then E is closed.*
3. *If $E' \subset X$ is open in the constructible topology (for example constructible) and stable under generalization, then E' is open.*

Proof. Proof of (1). Let $x \in \overline{E}$. Let $\{U_i\}$ be the set of quasi-compact open neighbourhoods of x. A finite intersection of the U_i is another one. The intersection $U_i \cap E$ is nonempty for all i. Since the subsets $U_i \cap E$ are closed in the constructible topology we see that $\bigcap(U_i \cap E)$ is nonempty by Lemma 23.2 and Lemma 12.6. Since $\{U_i\}$ is a fundamental system of open neighbourhoods of x, we see that $\bigcap U_i$ is the set of generalizations of x. Thus x is a specialization of a point of E.

Part (2) is immediate from (1).

Proof of (3). Assume E' is as in (3). The complement of E' is closed in the constructible topology (Lemma 15.2) and closed under specialization (Lemma 19.2). Hence the complement is closed by (2), i.e., E' is open. $\square$

Lemma 23.7. *Let X be a spectral space. Let $x, y \in X$. Then either there exists a third point specializing to both x and y, or there exist disjoint open neighbourhoods containing x and y.*

Proof. Let $\{U_i\}$ be the set of quasi-compact open neighbourhoods of x. A finite intersection of the U_i is another one. Let $\{V_j\}$ be the set of quasi-compact open neighbourhoods of y. A finite intersection of the V_j is another one. If $U_i \cap V_j$ is empty for some i, j we are done. If not, then the intersection $U_i \cap V_j$ is nonempty for all i and j. The sets $U_i \cap V_j$ are closed in the constructible topology on X. By Lemma 23.2 we see that $\bigcap(U_i \cap V_j)$ is nonempty (Lemma 12.6). Since X is a sober space and $\{U_i\}$ is a fundamental system of open neighbourhoods of x, we see that $\bigcap U_i$ is the set of generalizations of x. Similarly, $\bigcap V_j$ is the set of generalizations of y. Thus any element of $\bigcap(U_i \cap V_j)$ specializes to both x and y. $\square$

Lemma 23.8. *Let X be a spectral space. The following are equivalent:*

1. *X is profinite,*
2. *X is Hausdorff,*
3. *X is totally disconnected,*
4. *every quasi-compact open is closed,*
5. *there are no nontrivial specializations between points,*
6. *every point of X is closed,*
7. *every point of X is the generic point of an irreducible component of X,*
8. *the constructible topology equals the given topology on X, and*
9. *add more here.*

Proof. Lemma 22.2 shows the implication (1) $\Rightarrow$ (3). Irreducible components are closed, so if X is totally disconnected, then every point is closed. So (3) implies (6). The equivalence of (6) and (5) is immediate, and (6) $\Leftrightarrow$ (7) holds because X is sober. Assume (5). Then all constructible subsets of X are closed (Lemma 23.6), in particular all quasi-compact opens are closed. So (5) implies (4). Since X is sober, for any two points there is a quasi-compact open containing exactly one of them, hence (4) implies (2). Parts (4) and (8) are equivalent by the definition of the constructible topology. It remains to prove (2) implies (1). Suppose X is Hausdorff. Every quasi-compact open is also closed (Lemma 12.4). This implies X is totally disconnected. Hence it is profinite, by Lemma 22.2. $\square$

Lemma 23.9. *If X is a spectral space, then $\pi_0(X)$ is a profinite space.*

Proof. Combine Lemmas 12.10 and 22.5. $\square$

Lemma 23.10. *The product of two spectral spaces is spectral.*

Proof. Let X, Y be spectral spaces. Denote $p : X \times Y \to X$ and $q : X \times Y \to Y$ the projections. Let $Z \subset X \times Y$ be a closed irreducible subset. Then $p(Z) \subset X$ is irreducible and $q(Z) \subset Y$ is irreducible. Let $x \in X$ be the generic point of the closure of $p(X)$ and let $y \in Y$ be the generic point of the closure of $q(Y)$. If $(x, y) \notin Z$, then there exist opens $x \in U \subset X$, $y \in V \subset Y$ such that $Z \cap U \times V = \emptyset$. Hence Z is contained in $(X \setminus U) \times Y \cup X \times (Y \setminus V)$. Since Z is irreducible, we see that either $Z \subset (X \setminus U) \times Y$ or $Z \subset X \times (Y \setminus V)$. In the first case $p(Z) \subset (X \setminus U)$ and in the second case $q(Z) \subset (Y \setminus V)$. Both cases are absurd as x is in the closure of $p(Z)$ and y is in the closure of $q(Z)$. Thus we conclude that $(x, y) \in Z$, which means that (x, y) is the generic point for Z.

A basis of the topology of $X \times Y$ are the opens of the form $U \times V$ with $U \subset X$ and $V \subset Y$ quasi-compact open (here we use that X and Y are spectral). Then $U \times V$ is quasi-compact as the product of quasi-compact spaces is quasi-compact.

Moreover, any quasi-compact open of $X \times Y$ is a finite union of such quasi-compact rectangles $U \times V$. It follows that the intersection of two such is again quasi-compact (since X and Y are spectral). This concludes the proof. $\square$

Lemma 23.11. *Let $f : X \to Y$ be a continuous map of topological spaces. If*

(1) *X and Y are spectral,*
(2) *f is spectral and bijective, and*
(3) *generalizations (resp. specializations) lift along f.*

Then f is a homeomorphism.

Proof. Since f is spectral it defines a continuous map between X and Y in the constructible topology. By Lemmas 23.2 and 17.8 it follows that $X \to Y$ is a homeomorphism in the constructible topology. Let $U \subset X$ be quasi-compact open. Then $f(U)$ is constructible in Y. Let $y \in Y$ specialize to a point in $f(U)$. By the last assumption we see that $f^{-1}(y)$ specializes to a point of U. Hence $f^{-1}(y) \in U$. Thus $y \in f(U)$. It follows that $f(U)$ is open, see Lemma 23.6. Whence f is a homeomorphism. To prove the lemma in case specializations lift along f one shows instead that $f(Z)$ is closed if $X \setminus Z$ is a quasi-compact open of X. $\square$

Lemma 23.12. *The inverse limit of a directed inverse system of finite sober topological spaces is a spectral topological space.*

Proof. Let I be a directed set. Let X_i be an inverse system of finite sober spaces over I. Let $X = \lim X_i$ which exists by Lemma 14.1. As a set $X = \lim X_i$. Denote $p_i : X \to X_i$ the projection. Because I is directed we may apply Lemma 14.2. A basis for the topology is given by the opens $p_i^{-1}(U_i)$ for $U_i \subset X_i$ open. Since an open covering of $p_i^{-1}(U_i)$ is in particular an open covering in the profinite topology, we conclude that $p_i^{-1}(U_i)$ is quasi-compact. Given $U_i \subset X_i$ and $U_j \subset X_j$, then $p_i^{-1}(U_i) \cap p_j^{-1}(U_j) = p_k^{-1}(U_k)$ for some $k \geq i, j$ and open $U_k \subset X_k$. Finally, if $Z \subset X$ is irreducible and closed, then $p_i(Z) \subset X_i$ is irreducible and therefore has a unique generic point ξ_i (because X_i is a finite sober topological space). Then $\xi = \lim \xi_i$ is a generic point of Z (it is a point of Z as Z is closed). This finishes the proof. $\square$

Lemma 23.13. *Let W be the topological space with two points, one closed, the other not. A topological space is spectral if and only if it is homeomorphic to a subspace of a product of copies of W which is closed in the constructible topology.*

Proof. Write $W = \{0, 1\}$ where 0 is a specialization of 1 but not vice versa. Let I be a set. The space $\prod_{i \in I} W$ is spectral by Lemma 23.12. Thus we see that a subspace of $\prod_{i \in I} W$ closed in the constructible topology is a spectral space by Lemma 23.5.

For the converse, let X be a spectral space. Let $U \subset X$ be a quasi-compact open. Consider the continuous map

$$f_U : X \longrightarrow W$$

which maps every point in U to 1 and every point in $X \setminus U$ to 0. Taking the product of these maps we obtain a continuous map

$$f = \prod f_U : X \longrightarrow \prod\nolimits_U W$$

By construction the map $f : X \to Y$ is spectral. By Lemma 23.3 the image of f is closed in the constructible topology. If $x', x \in X$ are distinct, then since X is sober either x' is not a specialization of x or conversely. In either case (as the quasi-compact opens form a basis for the topology of X) there exists a quasi-compact open $U \subset X$ such that $f_U(x') \neq f_U(x)$. Thus f is injective. Let $Y = f(X)$ endowed with the induced topology. Let $y' \leadsto y$ be a specialization in Y and say $f(x') = y'$ and $f(x) = y$. Arguing as above we see that $x' \leadsto x$, since otherwise there is a U such that $x \in U$ and $x' \notin U$, which would imply $f_U(x') \not\leadsto f_U(x)$. We conclude that $f : X \to Y$ is a homeomorphism by Lemma 23.11. $\square$

Lemma 23.14. *A topological space is spectral if and only if it is a directed inverse limit of finite sober topological spaces.*

Proof. One direction is given by Lemma 23.12. For the converse, assume X is spectral. Then we may assume $X \subset \prod_{i \in I} W$ is a subset closed in the constructible topology where $W = \{0, 1\}$ as in Lemma 23.13. We can write

$$\prod\nolimits_{i \in I} W = \lim\nolimits_{J \subset I \text{ finite}} \prod\nolimits_{j \in J} W$$

as a cofiltered limit. For each J, let $X_J \subset \prod_{j \in J} W$ be the image of X. Then we see that $X = \lim X_J$ as sets because X is closed in the product with the constructible topology (detail omitted). A formal argument (omitted) on limits shows that $X = \lim X_J$ as topological spaces. $\square$

Lemma 23.15. *Let X be a topological space and let $c : X \to X'$ be the universal map from X to a sober topological space, see Lemma 8.16.*
(1) *If X is quasi-compact, so is X'.*
(2) *If X is quasi-compact, has a basis of quasi-compact opens, and the intersection of two quasi-compact opens is quasi-compact, then X' is spectral.*
(3) *If X is Noetherian, then X' is a Noetherian spectral space.*

Proof. Let $U \subset X$ be open and let $U' \subset X'$ be the corresponding open, i.e., the open such that $c^{-1}(U') = U$. Then U is quasi-compact if and only if U' is quasi-compact, as pulling back by c is a bijection between the opens of X and X' which commutes with unions. This in particular proves (1).

Proof of (2). It follows from the above that X' has a basis of quasi-compact opens. Since c^{-1} also commutes with intersections of pairs of opens, we see that the intersection of two quasi-compact opens X' is quasi-compact. Finally, X' is quasi-compact by (1) and sober by construction. Hence X' is spectral.

Proof of (3). It is immediate that X' is Noetherian as this is defined in terms of the acc for open subsets which holds for X. We have already seen in (2) that X' is spectral. $\square$

24. Limits of spectral spaces

Lemma 23.14 tells us that every spectral space is a cofiltered limit of finite sober spaces. Every finite sober space is a spectral space and every continuous map of finite sober spaces is a spectral map of spectral spaces. In this section we prove some lemmas concerning limits of systems of spectral topological spaces along spectral maps.

Lemma 24.1. *Let $\mathcal{I}$ be a category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral.*

(1) *Given subsets $Z_i \subset X_i$ closed in the constructible topology with $f_a(Z_j) \subset Z_i$ for all $a : j \to i$ in $\mathcal{I}$, then $\lim Z_i$ is quasi-compact.*
(2) *The space $X = \lim X_i$ is quasi-compact.*

Proof. The limit $Z = \lim Z_i$ exists by Lemma 14.1. Denote X'_i the space X_i endowed with the constructible topology and Z'_i the corresponding subspace of X'_i. Let $a : j \to i$ in $\mathcal{I}$ be a morphism. As f_a is spectral it defines a continuous map $f_a : X'_j \to X'_i$. Thus $f_a|_{Z_j} : Z'_j \to Z'_i$ is a continuous map of quasi-compact Hausdorff spaces (by Lemmas 23.2 and 12.3). Thus $Z' = \lim Z_i$ is quasi-compact by Lemma 14.5. The maps $Z'_i \to Z_i$ are continuous, hence $Z' \to Z$ is continuous and a bijection on underlying sets. Hence Z is quasi-compact as the image of the surjective continuous map $Z' \to Z$ (Lemma 12.7). □

Lemma 24.2. *Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral.*

(1) *Given nonempty subsets $Z_i \subset X_i$ closed in the constructible topology with $f_a(Z_j) \subset Z_i$ for all $a : j \to i$ in $\mathcal{I}$, then $\lim Z_i$ is nonempty.*
(2) *If each X_i is nonempty, then $X = \lim X_i$ is nonempty.*

Proof. Denote X'_i the space X_i endowed with the constructible topology and Z'_i the corresponding subspace of X'_i. Let $a : j \to i$ in $\mathcal{I}$ be a morphism. As f_a is spectral it defines a continuous map $f_a : X'_j \to X'_i$. Thus $f_a|_{Z_j} : Z'_j \to Z'_i$ is a continuous map of quasi-compact Hausdorff spaces (by Lemmas 23.2 and 12.3). By Lemma 14.6 the space $\lim Z'_i$ is nonempty. Since $\lim Z'_i = \lim Z_i$ as sets we conclude. □

Lemma 24.3. *Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Let $X = \lim X_i$ with projections $p_i : X \to X_i$. Let $i \in \mathrm{Ob}(\mathcal{I})$ and let $E, F \subset X_i$ be subsets with E closed in the constructible topology and F open in the constructible topology. Then $p_i^{-1}(E) \subset p_i^{-1}(F)$ if and only if there is a morphism $a : j \to i$ in $\mathcal{I}$ such that $f_a^{-1}(E) \subset f_a^{-1}(F)$.*

Proof. Observe that

$$p_i^{-1}(E) \setminus p_i^{-1}(F) = \lim_{a:j\to i} f_a^{-1}(E) \setminus f_a^{-1}(F)$$

Since f_a is a spectral map, it is continuous in the constructible topology hence the set $f_a^{-1}(E) \setminus f_a^{-1}(F)$ is closed in the constructible topology. Hence Lemma 24.2 applies to show that the LHS is nonempty if and only if each of the spaces of the RHS is nonempty. □

Lemma 24.4. *Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Let $X = \lim X_i$ with projections $p_i : X \to X_i$. Let $E \subset X$ be a constructible subset. Then there exists an $i \in \mathrm{Ob}(\mathcal{I})$ and a constructible subset $E_i \subset X_i$ such that $p_i^{-1}(E_i) = E$. If E is open, resp. closed, we may choose E_i open, resp. closed.*

Proof. Assume E is a quasi-compact open of X. By Lemma 14.2 we can write $E = p_i^{-1}(U_i)$ for some i and some open $U_i \subset X_i$. Write $U_i = \bigcup U_{i,\alpha}$ as a union of quasi-compact opens. As E is quasi-compact we can find $\alpha_1, \ldots, \alpha_n$ such that $E = p_i^{-1}(U_{i,\alpha_1} \cup \ldots \cup U_{i,\alpha_n})$. Hence $E_i = U_{i,\alpha_1} \cup \ldots \cup U_{i,\alpha_n}$ works.

Assume E is a constructible closed subset. Then E^c is quasi-compact open. So $E^c = p_i^{-1}(F_i)$ for some i and quasi-compact open $F_i \subset X_i$ by the result of the previous paragraph. Then $E = p_i^{-1}(F_i^c)$ as desired.

If E is general we can write $E = \bigcup_{l=1,\ldots,n} U_l \cap Z_l$ with U_l constructible open and Z_l constructible closed. By the result of the previous paragraphs we may write $U_l = p_{i_l}^{-1}(U_{l,i_l})$ and $Z_l = p_{j_l}^{-1}(Z_{l,j_l})$ with $U_{l,i_l} \subset X_{i_l}$ constructible open and $Z_{l,j_l} \subset X_{j_l}$ constructible closed. As $\mathcal{I}$ is cofiltered we may choose an object k of $\mathcal{I}$ and morphism $a_l : k \to i_l$ and $b_l : k \to j_l$. Then taking $E_k = \bigcup_{l=1,\ldots,n} f_{a_l}^{-1}(U_{l,i_l}) \cap f_{b_l}^{-1}(Z_{l,j_l})$ we obtain a constructible subset of X_k whose inverse image in X is E. $\square$

Lemma 24.5. *Let $\mathcal{I}$ be a cofiltered index category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Then the inverse limit $X = \lim X_i$ is a spectral topological space and the projection maps $p_i : X \to X_i$ are spectral.*

Proof. The limit $X = \lim X_i$ exists (Lemma 14.1) and is quasi-compact by Lemma 24.1.

Denote $p_i : X \to X_i$ the projection. Because $\mathcal{I}$ is cofiltered we can apply Lemma 14.2. Hence a basis for the topology on X is given by the opens $p_i^{-1}(U_i)$ for $U_i \subset X_i$ open. Since a basis for the topology of X_i is given by the quasi-compact open, we conclude that a basis for the topology on X is given by $p_i^{-1}(U_i)$ with $U_i \subset X_i$ quasi-compact open. A formal argument shows that

$$p_i^{-1}(U_i) = \lim_{a:j \to i} f_a^{-1}(U_i)$$

as topological spaces. Since each f_a is spectral the sets $f_a^{-1}(U_i)$ are closed in the constructible topology of X_j and hence $p_i^{-1}(U_i)$ is quasi-compact by Lemma 24.1. Thus X has a basis for the topology consisting of quasi-compact opens.

Any quasi-compact open U of X is of the form $U = p_i^{-1}(U_i)$ for some i and some quasi-compact open $U_i \subset X_i$ (see Lemma 24.4). Given $U_i \subset X_i$ and $U_j \subset X_j$ quasi-compact open, then $p_i^{-1}(U_i) \cap p_j^{-1}(U_j) = p_k^{-1}(U_k)$ for some k and quasi-compact open $U_k \subset X_k$. Namely, choose k and morphisms $k \to i$ and $k \to j$ and let U_k be the intersection of the pullbacks of U_i and U_j to X_k. Thus we see that the intersection of two quasi-compact opens of X is quasi-compact open.

Finally, let $Z \subset X$ be irreducible and closed. Then $p_i(Z) \subset X_i$ is irreducible and therefore $Z_i = \overline{p_i(Z)}$ has a unique generic point ξ_i (because X_i is a spectral space). Then $f_a(\xi_j) = \xi_i$ for $a : j \to i$ in $\mathcal{I}$ because $\overline{f_a(Z_j)} = Z_i$. Hence $\xi = \lim \xi_i$ is a point of X. Claim: $\xi \in Z$. Namely, if not we can find a quasi-compact open containing ξ disjoint from Z. This would be of the form $p_i^{-1}(U_i)$ for some i and quasi-compact open $U_i \subset X_i$. Then $\xi_i \in U_i$ but $p_i(Z) \cap U_i = \emptyset$ which contradicts $\xi_i \in \overline{p_i(Z)}$. So $\xi \in Z$ and hence $\overline{\{\xi\}} \subset Z$. Conversely, every $z \in Z$ is in the closure of ξ. Namely, given a quasi-compact open neighbourhood U of z we write $U = p_i^{-1}(U_i)$ for some i and quasi-compact open $U_i \subset X_i$. We see that $p_i(z) \in U_i$ hence $\xi_i \in U_i$ hence $\xi \in U$. Thus ξ is a generic point of Z. We omit the proof that ξ is the unique generic point of Z (hint: show that a second generic point has to be equal to ξ by showing that it has to map to ξ_i in X_i since by spectrality of X_i the irreducible Z_i has a unique generic point). This finishes the proof. $\square$

Lemma 24.6. *Let $\mathcal{I}$ be a cofiltered index category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Set $X = \lim X_i$ and denote $p_i : X \to X_i$ the projection.*

(1) *Given any quasi-compact open $U \subset X$ there exists an $i \in \mathrm{Ob}(\mathcal{I})$ and a quasi-compact open $U_i \subset X_i$ such that $p_i^{-1}(U_i) = U$.*
(2) *Given $U_i \subset X_i$ and $U_j \subset X_j$ quasi-compact opens such that $p_i^{-1}(U_i) \subset p_j^{-1}(U_j)$ there exist $k \in \mathrm{Ob}(\mathcal{I})$ and morphisms $a : k \to i$ and $b : k \to j$ such that $f_a^{-1}(U_i) \subset f_b^{-1}(U_j)$.*
(3) *If $U_i, U_{1,i}, \ldots, U_{n,i} \subset X_i$ are quasi-compact opens and $p_i^{-1}(U_i) = p_i^{-1}(U_{1,i}) \cup \ldots \cup p_i^{-1}(U_{n,i})$ then $f_a^{-1}(U_i) = f_a^{-1}(U_{1,i}) \cup \ldots \cup f_a^{-1}(U_{n,i})$ for some morphism $a : j \to i$ in $\mathcal{I}$.*
(4) *Same statement as in (3) but for intersections.*

Proof. Part (1) is a special case of Lemma 24.4. Part (2) is a special case of Lemma 24.3 as quasi-compact opens are both open and closed in the constructible topology. Parts (3) and (4) follow formally from (1) and (2) and the fact that taking inverse images of subsets commutes with taking unions and intersections. $\square$

Lemma 24.7. *Let W be a subset of a spectral space X. The following are equivalent:*

(1) *W is an intersection of constructible sets and closed under generalizations,*
(2) *W is quasi-compact and closed under generalizations,*
(3) *there exists a quasi-compact subset $E \subset X$ such that W is the set of points specializing to E,*
(4) *W is an intersection of quasi-compact open subsets,*
(5) *there exists a nonempty set I and quasi-compact opens $U_i \subset X$, $i \in I$ such that $W = \bigcap U_i$ and for all $i, j \in I$ there exists a $k \in I$ with $U_k \subset U_i \cap U_j$.*

In this case we have (a) W is a spectral space, (b) $W = \lim U_i$ as topological spaces, and (c) for any open U containing W there exists an i with $U_i \subset U$.

Proof. Let $W \subset X$ satisfy (1). Then W is closed in the constructible topology, hence quasi-compact in the constructible topology (by Lemmas 23.2 and 12.3), hence quasi-compact in the topology of X (because opens in X are open in the constructible topology). Thus (2) holds.

It is clear that (2) implies (3) by taking $E = W$.

Let X be a spectral space and let $E \subset W$ be as in (3). Since every point of W specializes to a point of E we see that an open of W which contains E is equal to W. Hence since E is quasi-compact, so is W. If $x \in X$, $x \notin W$, then $Z = \overline{\{x\}}$ is disjoint from W. Since W is quasi-compact we can find a quasi-compact open U with $W \subset U$ and $U \cap Z = \emptyset$. We conclude that (4) holds.

If $W = \bigcap_{j \in J} U_j$ then setting I equal to the set of finite subsets of J and $U_i = U_{j_1} \cap \ldots \cap U_{j_r}$ for $i = \{j_1, \ldots, j_r\}$ shows that (4) implies (5). It is immediate that (5) implies (1).

Let I and U_i be as in (5). Since $W = \bigcap U_i$ we have $W = \lim U_i$ by the universal property of limits. Then W is a spectral space by Lemma 24.5. Let $U \subset X$ be an open neighbourhood of W. Then $E_i = U_i \cap (X \setminus U)$ is a family of constructible subsets of the spectral space $Z = X \setminus U$ with empty intersection. Using that the spectral topology on Z is quasi-compact (Lemma 23.2) we conclude from Lemma 12.6 that $E_i = \emptyset$ for some i. $\square$

Lemma 24.8. *Let X be a spectral space. Let $E \subset X$ be a constructible subset. Let $W \subset X$ be the set of points of X which specialize to a point of E. Then $W \setminus E$ is a spectral space. If $W = \bigcap U_i$ with U_i as in Lemma 24.7 (5) then $W \setminus E = \lim (U_i \setminus E)$.*

Proof. Since E is constructible, it is quasi-compact and hence Lemma 24.7 applies to W. If E is constructible, then E is constructible in U_i for all $i \in I$. Hence $U_i \setminus E$ is spectral by Lemma 23.5. Since $W \setminus E = \bigcap (U_i \setminus E)$ we have $W \setminus E = \lim U_i \setminus E$ by the universal property of limits. Then $W \setminus E$ is a spectral space by Lemma 24.5. $\square$

25. Stone-Čech compactification

The Stone-Čech compactification of a topological space X is a map $X \to \beta(X)$ from X to a Hausdorff quasi-compact space $\beta(X)$ which is universal for such maps. We prove this exists by a standard argument using the following simple lemma.

Lemma 25.1. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that $f(X)$ is dense in Y and that Y is Hausdorff. Then the cardinality of Y is at most the cardinality of $P(P(X))$ where P is the power set operation.*

Proof. Let $S = f(X) \subset Y$. Let $\mathcal{D}$ be the set of all closed domains of Y, i.e., subsets $D \subset Y$ which equal the closure of its interior. Note that the closure of an open subset of Y is a closed domain. For $y \in Y$ consider the set

$$I_y = \{T \subset S \mid \text{ there exists } D \in \mathcal{D} \text{ with } T = S \cap D \text{ and } y \in D\}.$$

Since S is dense in Y for every closed domain D we see that $S \cap D$ is dense in D. Hence, if $D \cap S = D' \cap S$ for $D, D' \in \mathcal{D}$, then $D = D'$. Thus $I_y = I_{y'}$ implies that $y = y'$ because the Hausdorff condition assures us that we can find a closed domain containing y but not y'. The result follows. $\square$

Let X be a topological space. By Lemma 25.1, there is a set I of isomorphism classes of continuous maps $f : X \to Y$ which have dense image and where Y is Hausdorff and quasi-compact. For $i \in I$ choose a representative $f_i : X \to Y_i$. Consider the map

$$\prod f_i : X \longrightarrow \prod\nolimits_{i \in I} Y_i$$

and denote $\beta(X)$ the closure of the image. Since each Y_i is Hausdorff, so is $\beta(X)$. Since each Y_i is quasi-compact, so is $\beta(X)$ (use Theorem 14.4 and Lemma 12.3).

Let us show the canonical map $X \to \beta(X)$ satisfies the universal property with respect to maps to Hausdorff, quasi-compact spaces. Namely, let $f : X \to Y$ be such a morphism. Let $Z \subset Y$ be the closure of $f(X)$. Then $X \to Z$ is isomorphic to one of the maps $f_i : X \to Y_i$, say $f_{i_0} : X \to Y_{i_0}$. Thus f factors as $X \to \beta(X) \to \prod Y_i \to Y_{i_0} \cong Z \to Y$ as desired.

Lemma 25.2. *Let X be a Hausdorff, locally quasi-compact space. There exists a map $X \to X^*$ which identifies X as an open subspace of a quasi-compact Hausdorff space X^* such that $X^* \setminus X$ is a singleton (one point compactification). In particular, the map $X \to \beta(X)$ identifies X with an open subspace of $\beta(X)$.*

Proof. Set $X^* = X \amalg \{\infty\}$. We declare a subset V of X^* to be open if either $V \subset X$ is open in X, or $\infty \in V$ and $U = V \cap X$ is an open of X such that $X \setminus U$ is quasi-compact. We omit the verification that this defines a topology. It is clear that $X \to X^*$ identifies X with an open subspace of X.

Since X is locally quasi-compact, every point $x \in X$ has a quasi-compact neighbourhood $x \in E \subset X$. Then E is closed (Lemma 12.4 part (1)) and $V = (X \setminus E) \amalg \{\infty\}$ is an open neighbourhood of ∞ disjoint from the interior of E. Thus X^* is Hausdorff.

Let $X^* = \bigcup V_i$ be an open covering. Then for some i, say i_0, we have $\infty \in V_{i_0}$. By construction $Z = X^* \setminus V_{i_0}$ is quasi-compact. Hence the covering $Z \subset \bigcup_{i \neq i_0} Z \cap V_i$ has a finite refinement which implies that the given covering of X^* has a finite refinement. Thus X^* is quasi-compact.

The map $X \to X^*$ factors as $X \to \beta(X) \to X^*$ by the universal property of the Stone-Čech compactification. Let $\varphi : \beta(X) \to X^*$ be this factorization. Then $X \to \varphi^{-1}(X)$ is a section to $\varphi^{-1}(X) \to X$ hence has closed image (Lemma 3.3). Since the image of $X \to \beta(X)$ is dense we conclude that $X = \varphi^{-1}(X)$. □

26. Extremally disconnected spaces

The material in this section is taken from [Gle58] (with a slight modification as in [Rai59]). In Gleason's paper it is shown that in the category of quasi-compact Hausdorff spaces, the "projective objects" are exactly the extremally disconnected spaces.

Definition 26.1. A topological space X is called *extremally disconnected* if the closure of every open subset of X is open.

If X is Hausdorff and extremally disconnected, then X is totally disconnected (this isn't true in general). If X is quasi-compact, Hausdorff, and extremally disconnected, then X is profinite by Lemma 22.2, but the converse does not hold in general. For example the p-adic integers $\mathbf{Z}_p = \lim \mathbf{Z}/p^n\mathbf{Z}$ is a profinite space which is not extremally disconnected. Namely, if $U \subset \mathbf{Z}_p$ is the set of nonzero elements whose valuation is even, then U is open but its closure is $U \cup \{0\}$ which is not open.

Lemma 26.2. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume f is surjective and $f(E) \neq Y$ for all proper closed subsets $E \subset X$. Then for $U \subset X$ open the subset $f(U)$ is contained in the closure of $Y \setminus f(X \setminus U)$.*

Proof. Pick $y \in f(U)$ and let $V \subset Y$ be any open neighbourhood of y. We will show that V intersects $Y \setminus f(X \setminus U)$. Note that $W = U \cap f^{-1}(V)$ is a nonempty open subset of X, hence $f(X \setminus W) \neq Y$. Take $y' \in Y$, $y' \notin f(X \setminus W)$. It is elementary to show that $y' \in V$ and $y' \in Y \setminus f(X \setminus U)$. □

Lemma 26.3. *Let X be an extremally disconnected space. If $U, V \subset X$ are disjoint open subsets, then $\overline{U}$ and $\overline{V}$ are disjoint too.*

Proof. By assumption $\overline{U}$ is open, hence $V \cap \overline{U}$ is open and disjoint from U, hence empty because $\overline{U}$ is the intersection of all the closed subsets of X containing U. This means the open $\overline{V} \cap \overline{U}$ avoids V hence is empty by the same argument. □

Lemma 26.4. *Let $f : X \to Y$ be a continuous map of Hausdorff quasi-compact topological spaces. If Y is extremally disconnected, f is surjective, and $f(Z) \neq Y$ for every proper closed subset Z of X, then f is a homeomorphism.*

Proof. By Lemma 17.8 it suffices to show that f is injective. Suppose that $x, x' \in X$ are distinct points with $y = f(x) = f(x')$. Choose disjoint open neighbourhoods $U, U' \subset X$ of x, x'. Observe that f is closed (Lemma 17.7) hence $T = f(X \setminus U)$ and $T' = f(X \setminus U')$ are closed in Y. Since X is the union of $X \setminus U$ and $X \setminus U'$ we see that $Y = T \cup T'$. By Lemma 26.2 we see that y is contained in the closure of $Y \setminus T$ and the closure of $Y \setminus T'$. On the other hand, by Lemma 26.3, this intersection is empty. In this way we obtain the desired contradiction. $\square$

Lemma 26.5. *Let $f : X \to Y$ be a continuous surjective map of Hausdorff quasi-compact topological spaces. There exists a quasi-compact subset $E \subset X$ such that $f(E) = Y$ but $f(E') \neq Y$ for all proper closed subsets $E' \subset E$.*

Proof. We will use without further mention that the quasi-compact subsets of X are exactly the closed subsets (Lemma 12.5). Consider the collection $\mathcal{E}$ of all quasi-compact subsets $E \subset X$ with $f(E) = Y$ ordered by inclusion. We will use Zorn's lemma to show that $\mathcal{E}$ has a minimal element. To do this it suffices to show that given a totally ordered family E_λ of elements of $\mathcal{E}$ the intersection $\bigcap E_\lambda$ is an element of $\mathcal{E}$. It is quasi-compact as it is closed. For every $y \in Y$ the sets $E_\lambda \cap f^{-1}(\{y\})$ are nonempty and closed, hence the intersection $\bigcap E_\lambda \cap f^{-1}(\{y\}) = \bigcap (E_\lambda \cap f^{-1}(\{y\}))$ is nonempty by Lemma 12.6. This finishes the proof. $\square$

Proposition 26.6. *Let X be a Hausdorff, quasi-compact topological space. The following are equivalent*

(1) *X is extremally disconnected,*
(2) *for any surjective continuous map $f : Y \to X$ with Y Hausdorff quasi-compact there exists a continuous section, and*
(3) *for any solid commutative diagram*

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dotted] & Z \end{tikzcd}$$

of continuous maps of quasi-compact Hausdorff spaces with $Y \to Z$ surjective, there is a dotted arrow in the category of topological spaces making the diagram commute.

Proof. It is clear that (3) implies (2). On the other hand, if (2) holds and $X \to Z$ and $Y \to Z$ are as in (3), then (2) assures there is a section to the projection $X \times_Z Y \to X$ which implies a suitable dotted arrow exists (details omitted). Thus (3) is equivalent to (2).

Assume X is extremally disconnected and let $f : Y \to X$ be as in (2). By Lemma 26.5 there exists a quasi-compact subset $E \subset Y$ such that $f(E) = X$ but $f(E') \neq X$ for all proper closed subsets $E' \subset E$. By Lemma 26.4 we find that $f|_E : E \to X$ is a homeomorphism, the inverse of which gives the desired section.

Assume (2). Let $U \subset X$ be open with complement Z. Consider the continuous surjection $f : \overline{U} \amalg Z \to X$. Let σ be a section. Then $\overline{U} = \sigma^{-1}(\overline{U})$ is open. Thus X is extremally disconnected. $\square$

Lemma 26.7. *Let $f : X \to X$ be a surjective continuous selfmap of a Hausdorff topological space. If f is not id_X, then there exists a proper closed subset $E \subset X$ such that $X = E \cup f(E)$.*

Proof. Pick $p \in X$ with $f(p) \neq p$. Choose disjoint open neighbourhoods $p \in U$, $f(p) \in V$ and set $E = X \setminus U \cap f^{-1}(V)$. Then $p \notin E$ hence E is a proper closed subset. If $x \in X$, then either $x \in E$, or if not, then $x \in U \cap f^{-1}(V)$ and writing $x = f(y)$ (possible as f is surjective) we find $y \in V \subset E$ and $x \in f(E)$. $\square$

Example 26.8. We can use Proposition 26.6 to see that the Stone-Čech compactification $\beta(X)$ of a discrete space X is extremally disconnected. Namely, let $f : Y \to \beta(X)$ be a continuous surjection where Y is quasi-compact and Hausdorff. Then we can lift the map $X \to \beta(X)$ to a continuous (!) map $X \to Y$ as X is discrete. By the universal property of the Stone-Čech compactification we see that we obtain a factorization $X \to \beta(X) \to Y$. Since $\beta(X) \to Y \to \beta(X)$ equals the identity on the dense subset X we conclude that we get a section. In particular, we conclude that the Stone-Čech compactification of a discrete space is totally disconnected, whence profinite (see discussion following Definition 26.1 and Lemma 22.2).

Using the supply of extremally disconnected spaces given by Example 26.8 we can prove that every quasi-compact Hausdorff space has a "projective cover" in the category of quasi-compact Hausdorff spaces.

Lemma 26.9. *Let X be a quasi-compact Hausdorff space. There exists a continuous surjection $X' \to X$ with X' quasi-compact, Hausdorff, and extremally disconnected. If we require that every proper closed subset of X' does not map onto X, then X' is unique up to isomorphism.*

Proof. Let $Y = X$ but endowed with the discrete topology. Let $X' = \beta(Y)$. The continuous map $Y \to X$ factors as $Y \to X' \to X$. This proves the first statement of the lemma by Example 26.8.

By Lemma 26.5 we can find a quasi-compact subset $E \subset X'$ surjecting onto X such that no proper closed subset of E surjects onto X. Because X' is extremally disconnected there exists a continuous map $f : X' \to E$ over X (Proposition 26.6). Composing f with the map $E \to X'$ gives a continuous selfmap $f|_E : E \to E$. Observe that $f|_E$ has to be surjective as otherwise the image would be a proper closed subset surjecting onto X. Hence $f|_E$ has to be id_E as otherwise Lemma 26.7 shows that E isn't minimal. Thus the id_E factors through the extremally disconnected space X'. A formal, categorical argument (using the characterization of Proposition 26.6) shows that E is extremally disconnected.

To prove uniqueness, suppose we have a second $X'' \to X$ minimal cover. By the lifting property proven in Proposition 26.6 we can find a continuous map $g : X' \to X''$ over X. Observe that g is a closed map (Lemma 17.7). Hence $g(X') \subset X''$ is a closed subset surjecting onto X and we conclude $g(X') = X''$ by minimality of X''. On the other hand, if $E \subset X'$ is a proper closed subset, then $g(E) \neq X''$ as E does not map onto X by minimality of X'. By Lemma 26.4 we see that g is an isomorphism. $\square$

Remark 26.10. Let X be a quasi-compact Hausdorff space. Let κ be an infinite cardinal bigger or equal than the cardinality of X. Then the cardinality of the minimal quasi-compact, Hausdorff, extremally disconnected cover $X' \to X$ (Lemma 26.9) is at most 2^{2^κ}. Namely, choose a subset $S \subset X'$ mapping bijectively to X. By minimality of X' the set S is dense in X'. Thus $|X'| \leq 2^{2^\kappa}$ by Lemma 25.1.

27. Miscellany

The following lemma applies to the underlying topological space associated to a quasi-separated scheme.

Lemma 27.1. *Let X be a topological space which*

1. *has a basis of the topology consisting of quasi-compact opens, and*
2. *has the property that the intersection of any two quasi-compact opens is quasi-compact.*

Then

1. *X is locally quasi-compact,*
2. *a quasi-compact open $U \subset X$ is retrocompact,*
3. *any quasi-compact open $U \subset X$ has a cofinal system of open coverings $\mathcal{U} : U = \bigcup_{j \in J} U_j$ with J finite and all U_j and $U_j \cap U_{j'}$ quasi-compact,*
4. *add more here.*

Proof. Omitted. $\square$

Definition 27.2. Let X be a topological space. We say $x \in X$ is an *isolated point* of X if $\{x\}$ is open in X.

28. Partitions and stratifications

Stratifications can be defined in many different ways. We welcome comments on the choice of definitions in this section.

Definition 28.1. Let X be a topological space. A *partition* of X is a decomposition $X = \coprod X_i$ into locally closed subsets X_i. The X_i are called the *parts* of the partition. Given two partitions of X we say one *refines* the other if the parts of one are unions of parts of the other.

Any topological space X has a partition into connected components. If X has finitely many irreducible components $Z_1, \ldots, Z_r$, then there is a partition with parts $X_I = \bigcap_{i \in I} Z_i \setminus (\bigcup_{i \not\in I} Z_i)$ whose indices are subsets $I \subset \{1, \ldots, r\}$ which refines the partition into connected components.

Definition 28.2. Let X be a topological space. A *good stratification* of X is a partition $X = \coprod X_i$ such that for all $i, j \in I$ we have

$$X_i \cap \overline{X_j} \neq \emptyset \Rightarrow X_i \subset \overline{X_j}.$$

Given a good stratification $X = \coprod_{i \in I} X_i$ we obtain a partial ordering on I by setting $i \leq j$ if and only if $X_i \subset \overline{X_j}$. Then we see that

$$\overline{X_j} = \bigcup\nolimits_{i \leq j} X_i$$

However, what often happens in algebraic geometry is that one just has that the left hand side is a subset of the right hand side in the last displayed formula. This leads to the following definition.

Definition 28.3. Let X be a topological space. A *stratification* of X is given by a partition $X = \coprod_{i \in I} X_i$ and a partial ordering on I such that for each $j \in I$ we have

$$\overline{X_j} \subset \bigcup\nolimits_{i \leq j} X_i$$

The parts X_i are called the *strata* of the stratification.

We often impose additional conditions on the stratification. For example, stratifications are particularly nice if they are *locally finite*, which means that every point has a neighbourhood which meets only finitely many strata. More generally we introduce the following definition.

Definition 28.4. Let X be a topological space. Let I be a set and for $i \in I$ let $E_i \subset X$ be a subset. We say the collection $\{E_i\}_{i\in I}$ is *locally finite* if for all $x \in X$ there exists an open neighbourhood U of x such that $\{i \in I | E_i \cap U \neq \emptyset\}$ is finite.

Remark 28.5. Given a locally finite stratification $X = \coprod X_i$ of a topological space X, we obtain a family of closed subsets $Z_i = \bigcup_{j\leq i} X_j$ of X indexed by I such that

$$Z_i \cap Z_j = \bigcup\nolimits_{k\leq i,j} Z_k$$

Conversely, given closed subsets $Z_i \subset X$ indexed by a partially ordered set I such that $X = \bigcup Z_i$, such that every point has a neighbourhood meeting only finitely many Z_i, and such that the displayed formula holds, then we obtain a locally finite stratification of X by setting $X_i = Z_i \setminus \bigcup_{j<i} Z_j$.

Lemma 28.6. *Let X be a topological space. Let $X = \coprod X_i$ be a finite partition of X. Then there exists a finite stratification of X refining it.*

Proof. Let $T_i = \overline{X_i}$ and $\Delta_i = T_i \setminus X_i$. Let S be the set of all intersections of T_i and Δ_i. (For example $T_1 \cap T_2 \cap \Delta_4$ is an element of S.) Then $S = \{Z_s\}$ is a finite collection of closed subsets of X such that $Z_s \cap Z_{s'} \in S$ for all $s, s' \in S$. Define a partial ordering on S by inclusion. Then set $Y_s = Z_s \setminus \bigcup_{s'<s} Z_{s'}$ to get the desired stratification. $\square$

Lemma 28.7. *Let X be a topological space. Suppose $X = T_1 \cup \ldots \cup T_n$ is written as a union of constructible subsets. There exists a finite stratification $X = \coprod X_i$ with each X_i constructible such that each T_k is a union of strata.*

Proof. By definition of constructible subsets, we can write each T_i as a finite union of $U \cap V^c$ with $U, V \subset X$ retrocompact open. Hence we may assume that $T_i = U_i \cap V_i^c$ with $U_i, V_i \subset X$ retrocompact open. Let S be the finite set of closed subsets of X consisting of $\emptyset, X, U_i^c, V_i^c$ and finite intersections of these. If $Z \in S$, then Z is constructible in X (Lemma 15.2). Moreover, $Z \cap Z' \in S$ for all $Z, Z' \in S$. Define a partial ordering on S by inclusion. For $Z \in S$ set $X_Z = Z \setminus \bigcup_{Z'<Z,\ Z'\in S} Z'$ to get a stratification $X = \coprod_{Z\in S} X_Z$ satisfying the properties stated in the lemma. $\square$

Lemma 28.8. *Let X be a Noetherian topological space. Any finite partition of X can be refined by a finite good stratification.*

Proof. Let $X = \coprod X_i$ be a finite partition of X. Let Z be an irreducible component of X. Since $X = \bigcup \overline{X_i}$ with finite index set, there is an i such that $Z \subset \overline{X_i}$. Since X_i is locally closed this implies that $Z \cap X_i$ contains an open of Z. Thus $Z \cap X_i$ contains an open U of X (Lemma 9.2). Write $X_i = U \amalg X_i^1 \amalg X_i^2$ with $X_i^1 = (X_i \setminus U) \cap \overline{U}$ and $X_i^2 = (X_i \setminus U) \cap \overline{U}^c$. For $i' \neq i$ we set $X_{i'}^1 = X_{i'} \cap \overline{U}$ and $X_{i'}^2 = X_{i'} \cap \overline{U}^c$. Then

$$X \setminus U = \coprod X_l^k$$

is a partition such that $\overline{U} \setminus U = \bigcup X_l^1$. Note that $X \setminus U$ is closed and strictly smaller than X. By Noetherian induction we can refine this partition by a finite good stratification $X \setminus U = \coprod_{\alpha\in A} T_\alpha$. Then $X = U \amalg \coprod_{\alpha\in A} T_\alpha$ is a finite good stratification of X refining the partition we started with. $\square$

29. Colimits of spaces

The category of topological spaces has coproducts. Namely, if I is a set and for $i \in I$ we are given a topological space X_i then we endow the set $\coprod_{i \in I} X_i$ with the *coproduct topology*. As a basis for this topology we use sets of the form U_i where $U_i \subset X_i$ is open.

The category of topological spaces has coequalizers. Namely, if $a, b : X \to Y$ are morphisms of topological spaces, then the coequalizer of a and b is the coequalizer $Y/\sim$ in the category of sets endowed with the quotient topology (Section 6).

Lemma 29.1. *The category of topological spaces has colimits and the forgetful functor to sets commutes with them.*

Proof. This follows from the discussion above and Categories, Lemma 14.12. Another proof of existence of colimits is sketched in Categories, Remark 25.2. It follows from the above that the forgetful functor commutes with colimits. Another way to see this is to use Categories, Lemma 24.5 and use that the forgetful functor has a right adjoint, namely the functor which assigns to a set the corresponding chaotic (or indiscrete) topological space. □

30. Topological groups, rings, modules

This is just a short section with definitions and elementary properties.

Definition 30.1. A *topological group* is a group G endowed with a topology such that multiplication $G \times G \to G$, $(x, y) \mapsto xy$ and inverse $G \to G$, $x \mapsto x^{-1}$ are continuous. A *homomorphism of topological groups* is a homomorphism of groups which is continuous.

If G is a topological group and $H \subset G$ is a subgroup, then H with the induced topology is a topological group. If G is a topological group and $G \to H$ is a surjection of groups, then H endowed with the quotient topology is a topological group.

Example 30.2. Let E be a set. We can endow the set of self maps $\text{Map}(E, E)$ with the compact open topology, i.e., the topology such that given $f : E \to E$ a fundamental system of neighbourhoods of f is given by the sets $U_S(f) = \{f' : E \to E \mid f'|_S = f|_S\}$ where $S \subset E$ is finite. With this topology the action

$$\text{Map}(E, E) \times E \longrightarrow E, \quad (f, e) \longmapsto f(e)$$

is continuous when E is given the discrete topology. If X is a topological space and $X \times E \to E$ is a continuous map, then the map $X \to \text{Map}(E, E)$ is continuous. In other words, the compact open topology is the coarsest topology such that the "action" map displayed above is continuous. The composition

$$\text{Map}(E, E) \times \text{Map}(E, E) \to \text{Map}(E, E)$$

is continuous as well (as is easily verified using the description of neighbourhoods above). Finally, if $\text{Aut}(E) \subset \text{Map}(E, E)$ is the subset of invertible maps, then the inverse $i : \text{Aut}(E) \to \text{Aut}(E)$, $f \mapsto f^{-1}$ is continuous too. Namely, say $S \subset E$ is finite, then $i^{-1}(U_S(f^{-1})) = U_{f^{-1}(S)}(f)$. Hence $\text{Aut}(E)$ is a topological group as in Definition 30.1.

Lemma 30.3. *The category of topological groups has limits and limits commute with the forgetful functors to (a) the category of topological spaces and (b) the category of groups.*

Proof. It is enough to prove the existence and commutation for products and equalizers, see Categories, Lemma 14.11. Let G_i, $i \in I$ be a collection of topological groups. Take the usual product $G = \prod G_i$ with the product topology. Since $G \times G = \prod(G_i \times G_i)$ as a topological space (because products commutes with products in any category), we see that multiplication on G is continuous. Similarly for the inverse map. Let $a, b : G \to H$ be two homomorphisms of topological groups. Then as the equalizer we can simply take the equalizer of a and b as maps of topological spaces, which is the same thing as the equalizer as maps of groups endowed with the induced topology. $\square$

Lemma 30.4. *Let G be a topological group. The following are equivalent*

(1) *G as a topological space is profinite,*
(2) *G is a limit of a diagram of finite discrete topological groups,*
(3) *G is a cofiltered limit of finite discrete topological groups.*

Proof. We have the corresponding result for topological spaces, see Lemma 22.2. Combined with Lemma 30.3 we see that it suffices to prove that (1) implies (3).

We first prove that every neighbourhood E of the neutral element e contains an open subgroup. Namely, since G is the cofiltered limit of finite discrete topological spaces (Lemma 22.2), we can choose a continuous map $f : G \to T$ to a finite discrete space T such that $f^{-1}(f(\{e\})) \subset E$. Consider

$$H = \{g \in G \mid f(gg') = f(g') \text{ for all } g' \in G\}$$

This is a subgroup of G and contained in E. Thus it suffices to show that H is open. Pick $t \in T$ and set $W = f^{-1}(\{t\})$. Observe that $W \subset G$ is open and closed, in particular quasi-compact. For each $w \in W$ there exist open neighbourhoods $e \in U_w \subset G$ and $w \in U'_w \subset W$ such that $U_w U'_w \subset W$. By quasi-compactness we can find $w_1, \ldots, w_n$ such that $W = \bigcup U'_{w_i}$. Then $U_t = U_{w_1} \cap \ldots \cap U_{w_n}$ is an open neighbourhood of e such that $f(gw) = t$ for all $w \in W$. Since T is finite we see that $\bigcap_{t \in T} U_t \subset H$ is an open neighbourhood of e. Since $H \subset G$ is a subgroup it follows that H is open.

Suppose that $H \subset G$ is an open subgroup. Since G is quasi-compact we see that the index of H in G is finite. Say $G = Hg_1 \cup \ldots \cup Hg_n$. Then $N = \bigcap_{i=1,\ldots,n} g_i H g_i^{-1}$ is an open normal subgroup contained in H. Since N also has finite index we see that $G \to G/N$ is a surjection to a finite discrete topological group.

Consider the map

$$G \longrightarrow \lim_{N \subset G \text{ open and normal}} G/N$$

We claim that this map is an isomorphism of topological groups. This finishes the proof of the lemma as the limit on the right is cofiltered (the intersection of two open normal subgroups is open and normal). The map is continuous as each $G \to G/N$ is continuous. The map is injective as G is Hausdorff and every neighbourhood of e contains an N by the arguments above. The map is surjective by Lemma 12.6. By Lemma 17.8 the map is a homeomorphism. $\square$

Definition 30.5. A topological group is called a *profinite group* if it satisfies the equivalent conditions of Lemma 30.4.

If $G_1 \to G_2 \to G_3 \to \ldots$ is a system of topological groups then the colimit $G = \mathrm{colim}\, G_n$ as a topological group (Lemma 30.6) is in general different from the colimit as a topological space (Lemma 29.1) even though these have the same underlying set. See Examples, Section 77.

Lemma 30.6. *The category of topological groups has colimits and colimits commute with the forgetful functor to the category of groups.*

Proof. We will use the argument of Categories, Remark 25.2 to prove existence of colimits. Namely, suppose that $\mathcal{I} \to \mathit{Top}$, $i \mapsto G_i$ is a functor into the category *TopGroup* of topological groups. Then we can consider

$$F : \mathit{TopGroup} \longrightarrow \mathit{Sets}, \quad H \longmapsto \lim_{\mathcal{I}} \mathrm{Mor}_{\mathit{TopGroup}}(G_i, H)$$

This functor commutes with limits. Moreover, given any topological group H and an element $(\varphi_i : G_i \to H)$ of $F(H)$, there is a subgroup $H' \subset H$ of cardinality at most $|\coprod G_i|$ (coproduct in the category of groups, i.e., the free product on the G_i) such that the morphisms φ_i map into H'. Namely, we can take the induced topology on the subgroup generated by the images of the φ_i. Thus it is clear that the hypotheses of Categories, Lemma 25.1 are satisfied and we find a topological group G representing the functor F, which precisely means that G is the colimit of the diagram $i \mapsto G_i$.

To see the statement on commutation with the forgetful functor to groups we will use Categories, Lemma 24.5. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a group the corresponding chaotic (or indiscrete) topological group. □

Definition 30.7. A *topological ring* is a ring R endowed with a topology such that addition $R \times R \to R$, $(x, y) \mapsto x + y$ and multiplication $R \times R \to R$, $(x, y) \mapsto xy$ are continuous. A *homomorphism of topological rings* is a homomorphism of rings which is continuous.

In the Stacks project rings are commutative with 1. If R is a topological ring, then $(R, +)$ is a topological group since $x \mapsto -x$ is continuous. If R is a topological ring and $R' \subset R$ is a subring, then R' with the induced topology is a topological ring. If R is a topological ring and $R \to R'$ is a surjection of rings, then R' endowed with the quotient topology is a topological ring.

Lemma 30.8. *The category of topological rings has limits and limits commute with the forgetful functors to (a) the category of topological spaces and (b) the category of rings.*

Proof. It is enough to prove the existence and commutation for products and equalizers, see Categories, Lemma 14.11. Let R_i, $i \in I$ be a collection of topological rings. Take the usual product $R = \prod R_i$ with the product topology. Since $R \times R = \prod (R_i \times R_i)$ as a topological space (because products commutes with products in any category), we see that addition and multiplication on R are continuous. Let $a, b : R \to R'$ be two homomorphisms of topological rings. Then as the equalizer we can simply take the equalizer of a and b as maps of topological spaces, which is the same thing as the equalizer as maps of rings endowed with the induced topology. □

Lemma 30.9. *The category of topological rings has colimits and colimits commute with the forgetful functor to the category of rings.*

Proof. The exact same argument as used in the proof of Lemma 30.6 shows existence of colimits. To see the statement on commutation with the forgetful functor to rings we will use Categories, Lemma 24.5. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a ring the corresponding chaotic (or indiscrete) topological ring. □

Definition 30.10. Let R be a topological ring. A *topological module* is an R-module M endowed with a topology such that addition $M \times M \to M$ and scalar multiplication $R \times M \to M$ are continuous. A *homomorphism of topological modules* is a homomorphism of modules which is continuous.

If R is a topological ring and M is a topological module, then $(M, +)$ is a topological group since $x \mapsto -x$ is continuous. If R is a topological ring, M is a topological module and $M' \subset M$ is a submodule, then M' with the induced topology is a topological module. If R is a topological ring, M is a topological module, and $M \to M'$ is a surjection of modules, then M' endowed with the quotient topology is a topological module.

Lemma 30.11. *Let R be a topological ring. The category of topological modules over R has limits and limits commute with the forgetful functors to (a) the category of topological spaces and (b) the category of R-modules.*

Proof. It is enough to prove the existence and commutation for products and equalizers, see Categories, Lemma 14.11. Let M_i, $i \in I$ be a collection of topological modules over R. Take the usual product $M = \prod M_i$ with the product topology. Since $M \times M = \prod(M_i \times M_i)$ as a topological space (because products commutes with products in any category), we see that addition on M is continuous. Similarly for multiplication $R \times M \to M$. Let $a, b : M \to M'$ be two homomorphisms of topological modules over R. Then as the equalizer we can simply take the equalizer of a and b as maps of topological spaces, which is the same thing as the equalizer as maps of modules endowed with the induced topology. □

Lemma 30.12. *Let R be a topological ring. The category of topological modules over R has colimits and colimits commute with the forgetful functor to the category of modules over R.*

Proof. The exact same argument as used in the proof of Lemma 30.6 shows existence of colimits. To see the statement on commutation with the forgetful functor to R-modules we will use Categories, Lemma 24.5. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a module the corresponding chaotic (or indiscrete) topological module. □

31. Other chapters

Preliminaries

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