

# LIMITS OF SCHEMES

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## 1. Introduction

In this chapter we put material related to limits of schemes. We mostly study limits of inverse systems over directed sets (Categories, Definition 21.1) with affine transition maps. We discuss absolute Noetherian approximation. We characterize schemes locally of finite presentation over a base as those whose associated functor of points is limit preserving. As an application of absolute Noetherian approximation we prove that the image of an affine under an integral morphism is affine. Moreover, we prove some very general variants of Chow's lemma. A basic reference is [DG67].

## 2. Directed limits of schemes with affine transition maps

In this section we construct the limit.

**Lemma 2.1.** *Let  $I$  be a directed set. Let  $(S_i, f_{ii'})$  be an inverse system of schemes over  $I$ . If all the schemes  $S_i$  are affine, then the limit  $S = \lim_i S_i$  exists in the category of schemes. In fact  $S$  is affine and  $S = \text{Spec}(\text{colim}_i R_i)$  with  $R_i = \Gamma(S_i, \mathcal{O})$ .*

**Proof.** Just define  $S = \text{Spec}(\text{colim}_i R_i)$ . It follows from Schemes, Lemma 6.4 that  $S$  is the limit even in the category of locally ringed spaces.  $\square$

**Lemma 2.2.** *Let  $I$  be a directed set. Let  $(S_i, f_{ii'})$  be an inverse system of schemes over  $I$ . If all the morphisms  $f_{ii'} : S_i \rightarrow S_{i'}$  are affine, then the limit  $S = \lim_i S_i$  exists in the category of schemes. Moreover,*

- (1) *each of the morphisms  $f_i : S \rightarrow S_i$  is affine,*
- (2) *for an element  $0 \in I$  and any open subscheme  $U_0 \subset S_0$  we have*

$$f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$$

*in the category of schemes.*

**Proof.** Choose an element  $0 \in I$ . Note that  $I$  is nonempty as the limit is directed. For every  $i \geq 0$  consider the quasi-coherent sheaf of  $\mathcal{O}_{S_0}$ -algebras  $\mathcal{A}_i = f_{i0,*} \mathcal{O}_{S_i}$ . Recall that  $S_i = \underline{\text{Spec}}_{S_0}(\mathcal{A}_i)$ , see Morphisms, Lemma 11.3. Set  $\mathcal{A} = \text{colim}_{i \geq 0} \mathcal{A}_i$ . This is a quasi-coherent sheaf of  $\mathcal{O}_{S_0}$ -algebras, see Schemes, Section 24. Set  $S = \underline{\text{Spec}}_{S_0}(\mathcal{A})$ . By Morphisms, Lemma 11.5 we get for  $i \geq 0$  morphisms  $f_i : S \rightarrow S_i$  compatible with the transition morphisms. Note that the morphisms  $f_i$  are affine by Morphisms, Lemma 11.11 for example. By Lemma 2.1 above we see that for any affine open  $U_0 \subset S_0$  the inverse image  $U = f_0^{-1}(U_0) \subset S$  is the limit of the system of opens  $U_i = f_{i0}^{-1}(U_0)$ ,  $i \geq 0$  in the category of schemes.

Let  $T$  be a scheme. Let  $g_i : T \rightarrow S_i$  be a compatible system of morphisms. To show that  $S = \lim_i S_i$  we have to prove there is a unique morphism  $g : T \rightarrow S$  with  $g_i = f_i \circ g$  for all  $i \in I$ . For every  $t \in T$  there exists an affine open  $U_0 \subset S_0$  containing  $g_0(t)$ . Let  $V \subset g_0^{-1}(U_0)$  be an affine open neighbourhood containing  $t$ . By the remarks above we obtain a unique morphism  $g_V : V \rightarrow U = f_0^{-1}(U_0)$  such that  $f_i \circ g_V = g_i|_{U_i}$  for all  $i$ . The open sets  $V \subset T$  so constructed form a basis for the topology of  $T$ . The morphisms  $g_V$  glue to a morphism  $g : T \rightarrow S$  because of the uniqueness property. This gives the desired morphism  $g : T \rightarrow S$ .

The final statement is clear from the construction of the limit above.  $\square$

**Lemma 2.3.** *Let  $I$  be a directed set. Let  $(S_i, f_{ii'})$  be an inverse system of schemes over  $I$ . Assume all the morphisms  $f_{ii'} : S_i \rightarrow S_{i'}$  are affine. Let  $S = \lim_i S_i$ . Let  $0 \in I$ . Suppose that  $T$  is a scheme over  $S_0$ . Then*

$$T \times_{S_0} S = \lim_{i \geq 0} T \times_{S_0} S_i$$

**Proof.** The right hand side is a scheme by Lemma 2.2. The equality is formal, see Categories, Lemma 14.10.  $\square$

### 3. Infinite products

Infinite products of schemes usually do not exist. For example in Examples, Section 55 it is shown that an infinite product of copies of  $\mathbf{P}^1$  is not even an algebraic space.

On the other hand, infinite products of affine schemes do exist and are affine. Using Schemes, Lemma 6.4 this corresponds to the fact that in the category of rings we

have infinite coproducts: if  $I$  is a set and  $R_i$  is a ring for each  $i$ , then we can consider the ring

$$R = \otimes R_i = \operatorname{colim}_{\{i_1, \dots, i_n\} \subset I} R_{i_1} \otimes_{\mathbf{Z}} \dots \otimes_{\mathbf{Z}} R_{i_n}$$

Given another ring  $A$  a map  $R \rightarrow A$  is the same thing as a collection of ring maps  $R_i \rightarrow A$  for all  $i \in I$  as follows from the corresponding property of finite tensor products.

**Lemma 3.1.** *Let  $S$  be a scheme. Let  $I$  be a set and for each  $i \in I$  let  $f_i : T_i \rightarrow S$  be an affine morphism. Then the product  $T = \prod T_i$  exists in the category of schemes over  $S$ . In fact, we have*

$$T = \lim_{\{i_1, \dots, i_n\} \subset I} T_{i_1} \times_S \dots \times_S T_{i_n}$$

and the projection morphisms  $T \rightarrow T_{i_1} \times_S \dots \times_S T_{i_n}$  are affine.

**Proof.** Omitted. Hint: Argue as in the discussion preceding the lemma and use Lemma 2.2 for existence of the limit.  $\square$

**Lemma 3.2.** *Let  $S$  be a scheme. Let  $I$  be a set and for each  $i \in I$  let  $f_i : T_i \rightarrow S$  be a surjective affine morphism. Then the product  $T = \prod T_i$  in the category of schemes over  $S$  (Lemma 3.1) maps surjectively to  $S$ .*

**Proof.** Let  $s \in S$ . Choose  $t_i \in T_i$  mapping to  $s$ . Choose a huge field extension  $K/\kappa(s)$  such that  $\kappa(s_i)$  embeds into  $K$  for each  $i$ . Then we get morphisms  $\operatorname{Spec}(K) \rightarrow T_i$  with image  $s_i$  agreeing as morphisms to  $S$ . Whence a morphism  $\operatorname{Spec}(K) \rightarrow T$  which proves there is a point of  $T$  mapping to  $s$ .  $\square$

**Lemma 3.3.** *Let  $S$  be a scheme. Let  $I$  be a set and for each  $i \in I$  let  $f_i : T_i \rightarrow S$  be an integral morphism. Then the product  $T = \prod T_i$  in the category of schemes over  $S$  (Lemma 3.1) is integral over  $S$ .*

**Proof.** Omitted. Hint: On affine pieces this reduces to the following algebra fact: if  $A \rightarrow B_i$  is integral for all  $i$ , then  $A \rightarrow \otimes_A B_i$  is integral.  $\square$

#### 4. Descending properties

First some basic lemmas describing the topology of a limit.

**Lemma 4.1.** *Let  $S = \lim S_i$  be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). Then  $S_{\text{set}} = \lim_i S_{i, \text{set}}$  where  $S_{\text{set}}$  indicates the underlying set of the scheme  $S$ .*

**Proof.** Pick  $i \in I$ . Take  $U_i \subset S_i$  an affine open. Denote  $U_{i'} = f_{i'i}^{-1}(U_i)$  and  $U = f_i^{-1}(U_i)$ . Here  $f_{i'i} : S_{i'} \rightarrow S_i$  is the transition morphism and  $f_i : S \rightarrow S_i$  is the projection. By Lemma 2.2 we have  $U = \lim_{i' \geq i} U_{i'}$ . Suppose we can show that  $U_{\text{set}} = \lim_{i' \geq i} U_{i', \text{set}}$ . Then the lemma follows by a simple argument using an affine covering of  $S_i$ . Hence we may assume all  $S_i$  and  $S$  affine. This reduces us to the algebra question considered in the next paragraph.

Suppose given a system of rings  $(A_i, \varphi_{ii'})$  over  $I$ . Set  $A = \operatorname{colim}_i A_i$  with canonical maps  $\varphi_i : A_i \rightarrow A$ . Then

$$\operatorname{Spec}(A) = \lim_i \operatorname{Spec}(A_i)$$

Namely, suppose that we are given primes  $\mathfrak{p}_i \subset A_i$  such that  $\mathfrak{p}_i = \varphi_{ii'}^{-1}(\mathfrak{p}_{i'})$  for all  $i' \geq i$ . Then we simply set

$$\mathfrak{p} = \{x \in A \mid \exists i, x_i \in \mathfrak{p}_i \text{ with } \varphi_i(x_i) = x\}$$

It is clear that this is an ideal and has the property that  $\varphi_i^{-1}(\mathfrak{p}) = \mathfrak{p}_i$ . Then it follows easily that it is a prime ideal as well.  $\square$

**Lemma 4.2.** *Let  $S = \lim S_i$  be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). Then  $S_{\text{top}} = \lim_i S_{i,\text{top}}$  where  $S_{\text{top}}$  indicates the underlying topological space of the scheme  $S$ .*

**Proof.** We will use the criterion of Topology, Lemma 14.3. We have seen that  $S_{\text{set}} = \lim_i S_{i,\text{set}}$  in Lemma 4.1. The maps  $f_i : S \rightarrow S_i$  are morphisms of schemes hence continuous. Thus  $f_i^{-1}(U_i)$  is open for each open  $U_i \subset S_i$ . Finally, let  $s \in S$  and let  $s \in V \subset S$  be an open neighbourhood. Choose  $0 \in I$  and choose an affine open neighbourhood  $U_0 \subset S_0$  of the image of  $s$ . Then  $f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$ , see Lemma 2.2. Then  $f_0^{-1}(U_0)$  and  $f_{i0}^{-1}(U_0)$  are affine and

$$\mathcal{O}_S(f_0^{-1}(U_0)) = \text{colim}_{i \geq 0} \mathcal{O}_{S_i}(f_{i0}^{-1}(U_0))$$

either by the proof of Lemma 2.2 or by Lemma 2.1. Choose  $a \in \mathcal{O}_S(f_0^{-1}(U_0))$  such that  $s \in D(a) \subset V$ . This is possible because the principal opens form a basis for the topology on the affine scheme  $f_0^{-1}(U_0)$ . Then we can pick an  $i \geq 0$  and  $a_i \in \mathcal{O}_{S_i}(f_{i0}^{-1}(U_0))$  mapping to  $a$ . It follows that  $D(a_i) \subset f_{i0}^{-1}(U_0) \subset S_i$  is an open subset whose inverse image in  $S$  is  $D(a)$ . This finishes the proof.  $\square$

**Lemma 4.3.** *Let  $S = \lim S_i$  be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). If all the schemes  $S_i$  are nonempty and quasi-compact, then the limit  $S = \lim_i S_i$  is nonempty.*

**Proof.** Choose  $0 \in I$ . Note that  $I$  is nonempty as the limit is directed. Choose an affine open covering  $S_0 = \bigcup_{j=1,\dots,m} U_j$ . Since  $I$  is directed there exists a  $j \in \{1, \dots, m\}$  such that  $f_{i0}^{-1}(U_j) \neq \emptyset$  for all  $i \geq 0$ . Hence  $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$  is not empty since a directed colimit of nonzero rings is nonzero (because  $1 \neq 0$ ). As  $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$  is an open subscheme of the limit we win.  $\square$

**Lemma 4.4.** *Let  $S = \lim S_i$  be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 2.2). Let  $s \in S$  with images  $s_i \in S_i$ . Then*

- (1)  $s = \lim s_i$  as schemes, i.e.,  $\kappa(s) = \text{colim } \kappa(s_i)$ ,
- (2)  $\overline{\{s\}} = \lim \overline{\{s_i\}}$  as sets, and
- (3)  $\overline{\{s\}} = \lim \overline{\{s_i\}}$  as schemes where  $\overline{\{s\}}$  and  $\overline{\{s_i\}}$  are endowed with the reduced induced scheme structure.

**Proof.** Choose  $0 \in I$  and an affine open covering  $S_0 = \bigcup_{j \in J} U_{0,j}$ . For  $i \geq 0$  let  $U_{i,j} = f_{i0}^{-1}(U_{0,j})$  and set  $U_j = f_{00}^{-1}(U_{0,j})$ . Here  $f_{i'i} : S_{i'} \rightarrow S_i$  is the transition morphism and  $f_i : S \rightarrow S_i$  is the projection. For  $j \in J$  the following are equivalent: (a)  $s \in U_j$ , (b)  $s_0 \in U_{0,j}$ , (c)  $s_i \in U_{i,j}$  for all  $i \geq 0$ . Let  $J' \subset J$  be the set of indices for which (a), (b), (c) are true. Then  $\overline{\{s\}} = \bigcup_{j \in J'} (\overline{\{s\}} \cap U_j)$  and similarly for  $\overline{\{s_i\}}$  for  $i \geq 0$ . Note that  $\overline{\{s\}} \cap U_j$  is the closure of the set  $\{s\}$  in the topological space  $U_j$ . Similarly for  $\overline{\{s_i\}} \cap U_{i,j}$  for  $i \geq 0$ . Hence it suffices to prove the lemma in the

case  $S$  and  $S_i$  affine for all  $i$ . This reduces us to the algebra question considered in the next paragraph.

Suppose given a system of rings  $(A_i, \varphi_{ii'})$  over  $I$ . Set  $A = \operatorname{colim}_i A_i$  with canonical maps  $\varphi_i : A_i \rightarrow A$ . Let  $\mathfrak{p} \subset A$  be a prime and set  $\mathfrak{p}_i = \varphi_i^{-1}(\mathfrak{p})$ . Then

$$V(\mathfrak{p}) = \lim_i V(\mathfrak{p}_i)$$

This follows from Lemma 4.1 because  $A/\mathfrak{p} = \operatorname{colim} A_i/\mathfrak{p}_i$ . This equality of rings also shows the final statement about reduced induced scheme structures holds true. The equality  $\kappa(\mathfrak{p}) = \operatorname{colim} \kappa(\mathfrak{p}_i)$  follows from the statement as well.  $\square$

In the rest of this section we work in the following situation.

**Situation 4.5.** Let  $S = \lim_{i \in I} S_i$  be the limit of a directed system of schemes with affine transition morphisms  $f_{i'i} : S_{i'} \rightarrow S_i$  (Lemma 2.2). We assume that  $S_i$  is quasi-compact and quasi-separated for all  $i \in I$ . We denote  $f_i : S \rightarrow S_i$  the projection. We also choose an element  $0 \in I$ .

In this situation the morphism  $S \rightarrow S_0$  is affine. It follows that  $S$  is quasi-compact and quasi-separated<sup>1</sup>. The type of result we are looking for is the following: If we have an object over  $S$ , then for some  $i$  there is a similar object over  $S_i$ .

**Lemma 4.6.** *In Situation 4.5.*

- (1) *We have  $S_{\text{set}} = \lim_i S_{i,\text{set}}$  where  $S_{\text{set}}$  indicates the underlying set of the scheme  $S$ .*
- (2) *We have  $S_{\text{top}} = \lim_i S_{i,\text{top}}$  where  $S_{\text{top}}$  indicates the underlying topological space of the scheme  $S$ .*
- (3) *If  $s, s' \in S$  and  $s'$  is not a specialization of  $s$  then for some  $i \in I$  the image  $s'_i \in S_i$  of  $s'$  is not a specialization of the image  $s_i \in S_i$  of  $s$ .*
- (4) *Add more easy facts on topology of  $S$  here. (Requirement: whatever is added should be easy in the affine case.)*

**Proof.** Part (1) is a special case of Lemma 4.1.

Part (2) is a special case of Lemma 4.2.

Part (3) is a special case of Lemma 4.4.  $\square$

**Lemma 4.7.** *In Situation 4.5. Suppose that  $\mathcal{F}_0$  is a quasi-coherent sheaf on  $S_0$ . Set  $\mathcal{F}_i = f_{i0}^* \mathcal{F}_0$  for  $i \geq 0$  and set  $\mathcal{F} = f_0^* \mathcal{F}_0$ . Then*

$$\Gamma(S, \mathcal{F}) = \operatorname{colim}_{i \geq 0} \Gamma(S_i, \mathcal{F}_i)$$

**Proof.** Write  $\mathcal{A}_j = f_{j0,*} \mathcal{O}_{S_j}$ . This is a quasi-coherent sheaf of  $\mathcal{O}_{S_0}$ -algebras (see Morphisms, Lemma 11.5) and  $S_i$  is the relative spectrum of  $\mathcal{A}_i$  over  $S_0$ . In the proof of Lemma 2.2 we constructed  $S$  as the relative spectrum of  $\mathcal{A} = \operatorname{colim}_{i \geq 0} \mathcal{A}_i$  over  $S_0$ . Set

$$\mathcal{M}_i = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}_i$$

and

$$\mathcal{M} = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}.$$

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<sup>1</sup>Follows from Morphisms, Lemma 11.2, Topology, Definition 12.1, and Schemes, Lemma 21.12.

Then we have  $f_{i0,*}\mathcal{F}_i = \mathcal{M}_i$  and  $f_{0,*}\mathcal{F} = \mathcal{M}$ . Since  $\mathcal{A}$  is the colimit of the sheaves  $\mathcal{A}_i$  and since tensor product commutes with directed colimits, we conclude that  $\mathcal{M} = \operatorname{colim}_{i \geq 0} \mathcal{M}_i$ . Since  $S_0$  is quasi-compact and quasi-separated we see that

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= \Gamma(S_0, \mathcal{M}) \\ &= \Gamma(S_0, \operatorname{colim}_{i \geq 0} \mathcal{M}_i) \\ &= \operatorname{colim}_{i \geq 0} \Gamma(S_0, \mathcal{M}_i) \\ &= \operatorname{colim}_{i \geq 0} \Gamma(S_i, \mathcal{F}_i) \end{aligned}$$

see Sheaves, Lemma 29.1 and Topology, Lemma 27.1 for the middle equality.  $\square$

**Lemma 4.8.** *In Situation 4.5. Suppose for each  $i$  we are given a nonempty closed subset  $Z_i \subset S_i$  with  $f_{i'i}(Z_{i'}) \subset Z_i$  for all  $i' \geq i$ . Then there exists a point  $s \in S$  with  $f_i(s) \in Z_i$  for all  $i$ .*

**Proof.** Let  $Z_i \subset S_i$  also denote the reduced closed subscheme associated to  $Z_i$ , see Schemes, Definition 12.5. A closed immersion is affine, and a composition of affine morphisms is affine (see Morphisms, Lemmas 11.9 and 11.7), and hence  $Z_{i'} \rightarrow S_i$  is affine when  $i' \geq i$ . We conclude that the morphism  $f_{i'i} : Z_{i'} \rightarrow Z_i$  is affine by Morphisms, Lemma 11.11. Each of the schemes  $Z_i$  is quasi-compact as a closed subscheme of a quasi-compact scheme. Hence we may apply Lemma 4.3 to see that  $Z = \lim_i Z_i$  is nonempty. Since there is a canonical morphism  $Z \rightarrow S$  we win.  $\square$

**Lemma 4.9.** *In Situation 4.5. Suppose we are given an  $i$  and a morphism  $T \rightarrow S_i$  such that*

- (1)  $T \times_{S_i} S = \emptyset$ , and
- (2)  $T$  is quasi-compact.

*Then  $T \times_{S_i} S_{i'} = \emptyset$  for all sufficiently large  $i'$ .*

**Proof.** By Lemma 2.3 we see that  $T \times_{S_i} S = \lim_{i' \geq i} T \times_{S_i} S_{i'}$ . Hence the result follows from Lemma 4.3.  $\square$

**Lemma 4.10.** *In Situation 4.5. Suppose we are given an  $i$  and a locally constructible subset  $E \subset S_i$  such that  $f_i(S) \subset E$ . Then  $f_{i'i}(S_{i'}) \subset E$  for all sufficiently large  $i'$ .*

**Proof.** Writing  $S_i$  as a finite union of open affine subschemes reduces the question to the case that  $S_i$  is affine and  $E$  is constructible, see Lemma 2.2 and Properties, Lemma 2.1. In this case the complement  $S_i \setminus E$  is constructible too. Hence there exists an affine scheme  $T$  and a morphism  $T \rightarrow S_i$  whose image is  $S_i \setminus E$ , see Algebra, Lemma 29.4. By Lemma 4.9 we see that  $T \times_{S_i} S_{i'}$  is empty for all sufficiently large  $i'$ , and hence  $f_{i'i}(S_{i'}) \subset E$  for all sufficiently large  $i'$ .  $\square$

**Lemma 4.11.** *In Situation 4.5 we have the following:*

- (1) *Given any quasi-compact open  $V \subset S = \lim_i S_i$  there exists an  $i \in I$  and a quasi-compact open  $V_i \subset S_i$  such that  $f_i^{-1}(V_i) = V$ .*
- (2) *Given  $V_i \subset S_i$  and  $V_{i'} \subset S_{i'}$  quasi-compact opens such that  $f_i^{-1}(V_i) = f_{i'i'}^{-1}(V_{i'})$  there exists an index  $i'' \geq i, i'$  such that  $f_{i''i}^{-1}(V_i) = f_{i''i'}^{-1}(V_{i'})$ .*
- (3) *If  $V_{1,i}, \dots, V_{n,i} \subset S_i$  are quasi-compact opens and  $S = f_i^{-1}(V_{1,i}) \cup \dots \cup f_i^{-1}(V_{n,i})$  then  $S_{i'} = f_{i'i}^{-1}(V_{1,i}) \cup \dots \cup f_{i'i}^{-1}(V_{n,i})$  for some  $i' \geq i$ .*

**Proof.** Choose  $i_0 \in I$ . Note that  $I$  is nonempty as the limit is directed. For convenience we write  $S_0 = S_{i_0}$  and  $i_0 = 0$ . Choose an affine open covering  $S_0 = U_{1,0} \cup \dots \cup U_{m,0}$ . Denote  $U_{j,i} \subset S_i$  the inverse image of  $U_{j,0}$  under the transition morphism for  $i \geq 0$ . Denote  $U_j$  the inverse image of  $U_{j,0}$  in  $S$ . Note that  $U_j = \lim_i U_{j,i}$  is a limit of affine schemes.

We first prove the uniqueness statement: Let  $V_i \subset S_i$  and  $V_{i'} \subset S_{i'}$  quasi-compact opens such that  $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$ . It suffices to show that  $f_{i''}^{-1}(V_i \cap U_{j,i''})$  and  $f_{i''}^{-1}(V_{i'} \cap U_{j,i''})$  become equal for  $i''$  large enough. Hence we reduce to the case of a limit of affine schemes. In this case write  $S = \text{Spec}(R)$  and  $S_i = \text{Spec}(R_i)$  for all  $i \in I$ . We may write  $V_i = S_i \setminus V(h_1, \dots, h_m)$  and  $V_{i'} = S_{i'} \setminus V(g_1, \dots, g_n)$ . The assumption means that the ideals  $\sum g_j R$  and  $\sum h_j R$  have the same radical in  $R$ . This means that  $g_j^N = \sum a_{jj'} h_{j'}$  and  $h_j^N = \sum b_{jj'} g_{j'}$  for some  $N \gg 0$  and  $a_{jj'}$  and  $b_{jj'}$  in  $R$ . Since  $R = \text{colim}_i R_i$  we can choose an index  $i'' \geq i$  such that the equations  $g_j^N = \sum a_{jj'} h_{j'}$  and  $h_j^N = \sum b_{jj'} g_{j'}$  hold in  $R_{i''}$  for some  $a_{jj'}$  and  $b_{jj'}$  in  $R_{i''}$ . This implies that the ideals  $\sum g_j R_{i''}$  and  $\sum h_j R_{i''}$  have the same radical in  $R_{i''}$  as desired.

We prove existence: If  $S_0$  is affine, then  $S_i = \text{Spec}(R_i)$  for all  $i \geq 0$  and  $S = \text{Spec}(R)$  with  $R = \text{colim}_i R_i$ . Then  $V = S \setminus V(g_1, \dots, g_n)$  for some  $g_1, \dots, g_n \in R$ . Choose any  $i$  large enough so that each of the  $g_j$  comes from an element  $g_{j,i} \in R_i$  and take  $V_i = S_i \setminus V(g_{1,i}, \dots, g_{n,i})$ . If  $S_0$  is general, then the opens  $V \cap U_j$  are quasi-compact because  $S$  is quasi-separated. Hence by the affine case we see that for each  $j = 1, \dots, m$  there exists an  $i_j \in I$  and a quasi-compact open  $V_{i_j} \subset U_{j,i_j}$  whose inverse image in  $U_j$  is  $V \cap U_j$ . Set  $i = \max(i_1, \dots, i_m)$  and let  $V_i = \bigcup f_{ii_j}^{-1}(V_{i_j})$ .

The statement on coverings follows from the uniqueness statement for the opens  $V_{1,i} \cup \dots \cup V_{n,i}$  and  $S_i$  of  $S_i$ .  $\square$

**Lemma 4.12.** *In Situation 4.5 if  $S$  is quasi-affine, then for some  $i_0 \in I$  the schemes  $S_i$  for  $i \geq i_0$  are quasi-affine.*

**Proof.** Choose  $i_0 \in I$ . Note that  $I$  is nonempty as the limit is directed. For convenience we write  $S_0 = S_{i_0}$  and  $i_0 = 0$ . Let  $s \in S$ . We may choose an affine open  $U_0 \subset S_0$  containing  $f_0(s)$ . Since  $S$  is quasi-affine we may choose an element  $a \in \Gamma(S, \mathcal{O}_S)$  such that  $s \in D(a) \subset f_0^{-1}(U_0)$ , and such that  $D(a)$  is affine. By Lemma 4.7 there exists an  $i \geq 0$  such that  $a$  comes from an element  $a_i \in \Gamma(S_i, \mathcal{O}_{S_i})$ . For any index  $j \geq i$  we denote  $a_j$  the image of  $a_i$  in the global sections of the structure sheaf of  $S_j$ . Consider the opens  $D(a_j) \subset S_j$  and  $U_j = f_{j0}^{-1}(U_0)$ . Note that  $U_j$  is affine and  $D(a_j)$  is a quasi-compact open of  $S_j$ , see Properties, Lemma 26.4 for example. Hence we may apply Lemma 4.11 to the opens  $U_j$  and  $U_j \cup D(a_j)$  to conclude that  $D(a_j) \subset U_j$  for some  $j \geq i$ . For such an index  $j$  we see that  $D(a_j) \subset S_j$  is an affine open (because  $D(a_j)$  is a standard affine open of the affine open  $U_j$ ) containing the image  $f_j(s)$ .

We conclude that for every  $s \in S$  there exist an index  $i \in I$ , and a global section  $a \in \Gamma(S_i, \mathcal{O}_{S_i})$  such that  $D(a) \subset S_i$  is an affine open containing  $f_i(s)$ . Because  $S$  is quasi-compact we may choose a single index  $i \in I$  and global sections  $a_1, \dots, a_m \in \Gamma(S_i, \mathcal{O}_{S_i})$  such that each  $D(a_j) \subset S_i$  is affine open and such that  $f_i : S \rightarrow S_i$  has image contained in the union  $W_i = \bigcup_{j=1, \dots, m} D(a_j)$ . For  $i' \geq i$  set  $W_{i'} = f_{i'i}^{-1}(W_i)$ . Since  $f_i^{-1}(W_i)$  is all of  $S$  we see (by Lemma 4.11 again) that for a suitable  $i' \geq i$  we

have  $S_{i'} = W_{i'}$ . Thus we may replace  $i$  by  $i'$  and assume that  $S_i = \bigcup_{j=1, \dots, m} D(a_j)$ . This implies that  $\mathcal{O}_{S_i}$  is an ample invertible sheaf on  $S_i$  (see Properties, Definition 26.1) and hence that  $S_i$  is quasi-affine, see Properties, Lemma 27.1. Hence we win.  $\square$

**Lemma 4.13.** *In Situation 4.5 if  $S$  is affine, then for some  $i_0 \in I$  the schemes  $S_i$  for  $i \geq i_0$  are affine.*

**Proof.** By Lemma 4.12 we may assume that  $S_0$  is quasi-affine for some  $0 \in I$ . Set  $R_0 = \Gamma(S_0, \mathcal{O}_{S_0})$ . Then  $S_0$  is a quasi-compact open of  $T_0 = \text{Spec}(R_0)$ . Denote  $j_0 : S_0 \rightarrow T_0$  the corresponding quasi-compact open immersion. For  $i \geq 0$  set  $\mathcal{A}_i = f_{i0,*} \mathcal{O}_{S_i}$ . Since  $f_{i0}$  is affine we see that  $S_i = \text{Spec}_{S_0}(\mathcal{A}_i)$ . Set  $T_i = \text{Spec}_{T_0}(j_{0,*} \mathcal{A}_i)$ . Then  $T_i \rightarrow T_0$  is affine, hence  $T_i$  is affine. Thus  $T_i$  is the spectrum of

$$R_i = \Gamma(T_0, j_{0,*} \mathcal{A}_i) = \Gamma(S_0, \mathcal{A}_i) = \Gamma(S_i, \mathcal{O}_{S_i}).$$

Write  $S = \text{Spec}(R)$ . We have  $R = \text{colim}_i R_i$  by Lemma 4.7. Hence also  $S = \lim_i T_i$ . As formation of the relative spectrum commutes with base change, the inverse image of the open  $S_0 \subset T_0$  in  $T_i$  is  $S_i$ . Let  $Z_0 = T_0 \setminus S_0$  and let  $Z_i \subset T_i$  be the inverse image of  $Z_0$ . As  $S_i = T_i \setminus Z_i$ , it suffices to show that  $Z_i$  is empty for some  $i$ . Assume  $Z_i$  is nonempty for all  $i$  to get a contradiction. By Lemma 4.8 there exists a point  $s$  of  $S = \lim_i T_i$  which maps to a point of  $Z_i$  for every  $i$ . But  $S = \lim_i S_i$ , and hence we arrive at a contradiction by Lemma 4.6.  $\square$

**Lemma 4.14.** *In Situation 4.5 if  $S$  is separated, then for some  $i_0 \in I$  the schemes  $S_i$  for  $i \geq i_0$  are separated.*

**Proof.** Choose a finite affine open covering  $S_0 = U_{0,1} \cup \dots \cup U_{0,m}$ . Set  $U_{i,j} \subset S_i$  and  $U_j \subset S$  equal to the inverse image of  $U_{0,j}$ . Note that  $U_{i,j}$  and  $U_j$  are affine. As  $S$  is separated the intersections  $U_{j_1} \cap U_{j_2}$  are affine. Since  $U_{j_1} \cap U_{j_2} = \lim_{i \geq 0} U_{i,j_1} \cap U_{i,j_2}$  we see that  $U_{i,j_1} \cap U_{i,j_2}$  is affine for large  $i$  by Lemma 4.13. To show that  $S_i$  is separated for large  $i$  it now suffices to show that

$$\mathcal{O}_{S_i}(U_{i,j_1}) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_{S_i}(U_{i,j_2}) \longrightarrow \mathcal{O}_{S_i}(U_{i,j_1} \cap U_{i,j_2})$$

is surjective for large  $i$  (Schemes, Lemma 21.7).

To get rid of the annoying indices, assume we have affine opens  $U, V \subset S_0$  such that  $U \cap V$  is affine too. Let  $U_i, V_i \subset S_i$ , resp.  $U, V \subset S$  be the inverse images. We have to show that  $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i) \rightarrow \mathcal{O}(U_i \cap V_i)$  is surjective for  $i$  large enough and we know that  $\mathcal{O}(U) \otimes \mathcal{O}(V) \rightarrow \mathcal{O}(U \cap V)$  is surjective. Note that  $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0) \rightarrow \mathcal{O}(U_0 \cap V_0)$  is of finite type, as the diagonal morphism  $S_i \rightarrow S_i \times S_i$  is an immersion (Schemes, Lemma 21.2) hence locally of finite type (Morphisms, Lemmas 15.2 and 15.5). Thus we can choose elements  $f_{0,1}, \dots, f_{0,n} \in \mathcal{O}(U_0 \cap V_0)$  which generate  $\mathcal{O}(U_0 \cap V_0)$  over  $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0)$ . Observe that for  $i \geq 0$  the diagram of schemes

$$\begin{array}{ccc} U_i \cap V_i & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ U_0 \cap V_0 & \longrightarrow & U_0 \end{array}$$

is cartesian. Thus we see that the images  $f_{i,1}, \dots, f_{i,n} \in \mathcal{O}(U_i \cap V_i)$  generate  $\mathcal{O}(U_i \cap V_i)$  over  $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$  and a fortiori over  $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$ . By assumption the images  $f_1, \dots, f_n \in \mathcal{O}(U \cap V)$  are in the image of the map  $\mathcal{O}(U) \otimes \mathcal{O}(V) \rightarrow \mathcal{O}(U \cap V)$ .



Since  $\mathcal{O}(U) \otimes \mathcal{O}(V) = \text{colim } \mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$  we see that they are in the image of the map at some finite level and the lemma is proved.  $\square$

**Lemma 4.15.** *In Situation 4.5 let  $\mathcal{L}_0$  be an invertible sheaf of modules on  $S_0$ . If the pullback  $\mathcal{L}$  to  $S$  is ample, then for some  $i \in I$  the pullback  $\mathcal{L}_i$  to  $S_i$  is ample.*

**Proof.** The assumption means there are finitely many sections  $s_1, \dots, s_m \in \Gamma(S, \mathcal{L})$  such that  $S_{s_j}$  is affine and such that  $S = \bigcup S_{s_j}$ , see Properties, Definition 26.1. By Lemma 4.7 we can find an  $i \in I$  and sections  $s_{i,j} \in \Gamma(S_i, \mathcal{L}_i)$  mapping to  $s_j$ . By Lemma 4.13 we may, after increasing  $i$ , assume that  $(S_i)_{s_{i,j}}$  is affine for  $j = 1, \dots, m$ . By Lemma 4.11 we may, after increasing  $i$  a last time, assume that  $S_i = \bigcup (S_i)_{s_{i,j}}$ . Then  $\mathcal{L}_i$  is ample by definition.  $\square$

**Lemma 4.16.** *Let  $S$  be a scheme. Let  $X = \lim X_i$  be a directed limit of schemes over  $S$  with affine transition morphisms. Let  $Y \rightarrow X$  be a morphism of schemes over  $S$ .*

- (1) *If  $Y \rightarrow X$  is a closed immersion,  $X_i$  quasi-compact, and  $Y$  locally of finite type over  $S$ , then  $Y \rightarrow X_i$  is a closed immersion for  $i$  large enough.*
- (2) *If  $Y \rightarrow X$  is an immersion,  $X_i$  quasi-separated,  $Y \rightarrow S$  locally of finite type, and  $Y$  quasi-compact, then  $Y \rightarrow X_i$  is an immersion for  $i$  large enough.*
- (3) *If  $Y \rightarrow X$  is an isomorphism,  $X_i$  quasi-compact,  $X_i \rightarrow S$  locally of finite type, the transition morphisms  $X_{i'} \rightarrow X_i$  are closed immersions, and  $Y \rightarrow S$  is locally of finite presentation, then  $Y \rightarrow X_i$  is an isomorphism for  $i$  large enough.*

**Proof.** Proof of (1). Choose  $0 \in I$  and a finite affine open covering  $X_0 = U_{0,1} \cup \dots \cup U_{0,m}$  with the property that  $U_{0,j}$  maps into an affine open  $W_j \subset S$ . Let  $V_j \subset Y$ , resp.  $U_{i,j} \subset X_i$ ,  $i \geq 0$ , resp.  $U_j \subset X$  be the inverse image of  $U_{0,j}$ . It suffices to prove that  $V_j \rightarrow U_{i,j}$  is a closed immersion for  $i$  sufficiently large and we know that  $V_j \rightarrow U_j$  is a closed immersion. Thus we reduce to the following algebra fact: If  $A = \text{colim } A_i$  is a directed colimit of  $R$ -algebras,  $A \rightarrow B$  is a surjection of  $R$ -algebras, and  $B$  is a finitely generated  $R$ -algebra, then  $A_i \rightarrow B$  is surjective for  $i$  sufficiently large.

Proof of (2). Choose  $0 \in I$ . Choose a quasi-compact open  $X'_0 \subset X_0$  such that  $Y \rightarrow X_0$  factors through  $X'_0$ . After replacing  $X_i$  by the inverse image of  $X'_0$  for  $i \geq 0$  we may assume all  $X'_i$  are quasi-compact and quasi-separated. Let  $U \subset X$  be a quasi-compact open such that  $Y \rightarrow X$  factors through a closed immersion  $Y \rightarrow U$  ( $U$  exists as  $Y$  is quasi-compact). By Lemma 4.11 we may assume that  $U = \lim U_i$  with  $U_i \subset X_i$  quasi-compact open. By part (1) we see that  $Y \rightarrow U_i$  is a closed immersion for some  $i$ . Thus (2) holds.

Proof of (3). Working affine locally on  $X_0$  for some  $0 \in I$  as in the proof of (1) we reduce to the following algebra fact: If  $A = \lim A_i$  is a directed colimit of  $R$ -algebras with surjective transition maps and  $A$  of finite presentation over  $A_0$ , then  $A = A_i$  for some  $i$ . Namely, write  $A = A_0/(f_1, \dots, f_n)$ . Pick  $i$  such that  $f_1, \dots, f_n$  map to zero under the surjective map  $A_0 \rightarrow A_i$ .  $\square$

**Lemma 4.17.** *Let  $S$  be a scheme. Let  $X = \lim X_i$  be a directed limit of schemes over  $S$  with affine transition morphisms. Assume*

- (1)  *$S$  quasi-separated,*
- (2)  *$X_i$  quasi-compact and quasi-separated,*

(3)  $X \rightarrow S$  separated.

Then  $X_i \rightarrow S$  is separated for all  $i$  large enough.

**Proof.** Let  $0 \in I$ . Note that  $I$  is nonempty as the limit is directed. As  $X_0$  is quasi-compact we can find finitely many affine opens  $U_1, \dots, U_n \subset S$  such that  $X_0 \rightarrow S$  maps into  $U_1 \cup \dots \cup U_n$ . Denote  $h_i : X_i \rightarrow S$  the structure morphism. It suffices to check that for some  $i \geq 0$  the morphisms  $h_i^{-1}(U_j) \rightarrow U_j$  are separated for  $j = 1, \dots, n$ . Since  $S$  is quasi-separated the morphisms  $U_j \rightarrow S$  are quasi-compact. Hence  $h_i^{-1}(U_j)$  is quasi-compact and quasi-separated. In this way we reduce to the case  $S$  affine. In this case we have to show that  $X_i$  is separated and we know that  $X$  is separated. Thus the lemma follows from Lemma 4.14.  $\square$

**Lemma 4.18.** *Let  $S$  be a scheme. Let  $X = \lim X_i$  be a directed limit of schemes over  $S$  with affine transition morphisms. Assume*

- (1)  $S$  quasi-compact and quasi-separated,
- (2)  $X_i$  quasi-compact and quasi-separated,
- (3)  $X \rightarrow S$  affine.

Then  $X_i \rightarrow S$  is affine for  $i$  large enough.

**Proof.** Choose a finite affine open covering  $S = \bigcup_{j=1, \dots, n} V_j$ . Denote  $f : X \rightarrow S$  and  $f_i : X_i \rightarrow S$  the structure morphisms. For each  $j$  the scheme  $f^{-1}(V_j) = \lim_i f_i^{-1}(V_j)$  is affine (as a finite morphism is affine by definition). Hence by Lemma 4.13 there exists an  $i \in I$  such that each  $f_i^{-1}(V_j)$  is affine. In other words,  $f_i : X_i \rightarrow S$  is affine for  $i$  large enough, see Morphisms, Lemma 11.3.  $\square$

**Lemma 4.19.** *Let  $S$  be a scheme. Let  $X = \lim X_i$  be a directed limit of schemes over  $S$  with affine transition morphisms. Assume*

- (1)  $S$  quasi-compact and quasi-separated,
- (2)  $X_i$  quasi-compact and quasi-separated,
- (3) the transition morphisms  $X_{i'} \rightarrow X_i$  are finite,
- (4)  $X_i \rightarrow S$  locally of finite type
- (5)  $X \rightarrow S$  integral.

Then  $X_i \rightarrow S$  is finite for  $i$  large enough.

**Proof.** By Lemma 4.18 we may assume  $X_i \rightarrow S$  is affine for all  $i$ . Choose a finite affine open covering  $S = \bigcup_{j=1, \dots, n} V_j$ . Denote  $f : X \rightarrow S$  and  $f_i : X_i \rightarrow S$  the structure morphisms. It suffices to show that there exists an  $i$  such that  $f_i^{-1}(V_j)$  is finite over  $V_j$  for  $j = 1, \dots, n$  (Morphisms, Lemma 44.3). Namely, for  $i' \geq i$  the composition  $X_{i'} \rightarrow X_i \rightarrow S$  will be finite as a composition of finite morphisms (Morphisms, Lemma 44.5). This reduces us to the affine case: Let  $R$  be a ring and  $A = \text{colim } A_i$  with  $R \rightarrow A$  integral and  $A_i \rightarrow A_{i'}$  finite for all  $i \leq i'$ . Moreover  $R \rightarrow A_i$  is of finite type for all  $i$ . Goal: Show that  $A_i$  is finite over  $R$  for some  $i$ . To prove this choose an  $i \in I$  and pick generators  $x_1, \dots, x_m \in A_i$  of  $A_i$  as an  $R$ -algebra. Since  $A$  is integral over  $R$  we can find monic polynomials  $P_j \in R[T]$  such that  $P_j(x_j) = 0$  in  $A$ . Thus there exists an  $i' \geq i$  such that  $P_j(x_j) = 0$  in  $A_{i'}$  for  $j = 1, \dots, m$ . Then the image  $A'_i$  of  $A_i$  in  $A_{i'}$  is finite over  $R$  by Algebra, Lemma 36.5. Since  $A'_i \subset A_{i'}$  is finite too we conclude that  $A_{i'}$  is finite over  $R$  by Algebra, Lemma 7.3.  $\square$

**Lemma 4.20.** *Let  $S$  be a scheme. Let  $X = \lim X_i$  be a directed limit of schemes over  $S$  with affine transition morphisms. Assume*

- (1)  $S$  quasi-compact and quasi-separated,
- (2)  $X_i$  quasi-compact and quasi-separated,
- (3) the transition morphisms  $X_{i'} \rightarrow X_i$  are closed immersions,
- (4)  $X_i \rightarrow S$  locally of finite type
- (5)  $X \rightarrow S$  a closed immersion.

*Then  $X_i \rightarrow S$  is a closed immersion for  $i$  large enough.*

**Proof.** By Lemma 4.18 we may assume  $X_i \rightarrow S$  is affine for all  $i$ . Choose a finite affine open covering  $S = \bigcup_{j=1, \dots, n} V_j$ . Denote  $f : X \rightarrow S$  and  $f_i : X_i \rightarrow S$  the structure morphisms. It suffices to show that there exists an  $i$  such that  $f_i^{-1}(V_j)$  is a closed subscheme of  $V_j$  for  $j = 1, \dots, n$  (Morphisms, Lemma 2.1). This reduces us to the affine case: Let  $R$  be a ring and  $A = \text{colim } A_i$  with  $R \rightarrow A$  surjective and  $A_i \rightarrow A_{i'}$  surjective for all  $i \leq i'$ . Moreover  $R \rightarrow A_i$  is of finite type for all  $i$ . Goal: Show that  $R \rightarrow A_i$  is surjective for some  $i$ . To prove this choose an  $i \in I$  and pick generators  $x_1, \dots, x_m \in A_i$  of  $A_i$  as an  $R$ -algebra. Since  $R \rightarrow A$  is surjective we can find  $r_j \in R$  such that  $r_j$  maps to  $x_j$  in  $A$ . Thus there exists an  $i' \geq i$  such that  $r_j$  maps to the image of  $x_j$  in  $A_{i'}$  for  $j = 1, \dots, m$ . Since  $A_i \rightarrow A_{i'}$  is surjective this implies that  $R \rightarrow A_{i'}$  is surjective.  $\square$

**Lemma 4.21.** *Let  $S$  be a scheme. Let  $X = \lim X_i$  be a directed limit of schemes over  $S$  with affine transition morphisms. Assume*

- (1)  $S$  quasi-separated,
- (2)  $X_i$  quasi-compact and quasi-separated,
- (3) the transition morphisms  $X_{i'} \rightarrow X_i$  are closed immersions,
- (4)  $X_i \rightarrow S$  locally of finite type, and
- (5)  $X \rightarrow S$  an immersion.

*Then  $X_i \rightarrow S$  is an immersion for  $i$  large enough.*

**Proof.** Choose an open subscheme  $U \subset S$  such that  $X \rightarrow S$  factors as a closed immersion  $X \rightarrow U$  composed with the inclusion morphism  $U \rightarrow S$ . Since  $X$  is quasi-compact, we may shrink  $U$  and assume  $U$  is quasi-compact. Denote  $V_i \subset X_i$  the inverse image of  $U$ . Since  $V_i$  pulls back to  $X$  we see that  $V_i = X_i$  for all  $i$  large enough by Lemma 4.11. Thus we may assume  $X = \lim X_i$  in the category of schemes over  $U$ . Then we see that  $X_i \rightarrow U$  is a closed immersion for  $i$  large enough by Lemma 4.20. This proves the lemma.  $\square$

## 5. Absolute Noetherian Approximation

A nice reference for this section is Appendix C of the article by Thomason and Trobaugh [TT90]. See Categories, Section 21 for our conventions regarding directed systems. We will use the existence result and properties of the limit from Section 2 without further mention.

**Lemma 5.1.** *Let  $W$  be a quasi-affine scheme of finite type over  $\mathbf{Z}$ . Suppose  $W \rightarrow \text{Spec}(R)$  is an open immersion into an affine scheme. There exists a finite type  $\mathbf{Z}$ -algebra  $A \subset R$  which induces an open immersion  $W \rightarrow \text{Spec}(A)$ . Moreover,  $R$  is the directed colimit of such subalgebras.*

**Proof.** Choose an affine open covering  $W = \bigcup_{i=1,\dots,n} W_i$  such that each  $W_i$  is a standard affine open in  $\text{Spec}(R)$ . In other words, if we write  $W_i = \text{Spec}(R_i)$  then  $R_i = R_{f_i}$  for some  $f_i \in R$ . Choose finitely many  $x_{ij} \in R_i$  which generate  $R_i$  over  $\mathbf{Z}$ . Pick an  $N \gg 0$  such that each  $f_i^N x_{ij}$  comes from an element of  $R$ , say  $y_{ij} \in R$ . Set  $A$  equal to the  $\mathbf{Z}$ -algebra generated by the  $f_i$  and the  $y_{ij}$  and (optionally) finitely many additional elements of  $R$ . Then  $A$  works. Details omitted.  $\square$

**Lemma 5.2.** *Suppose given a cartesian diagram of rings*

$$\begin{array}{ccc} B & \xrightarrow{s} & R \\ \uparrow & & \uparrow t \\ B' & \longrightarrow & R' \end{array}$$

*Let  $W' \subset \text{Spec}(R')$  be an open of the form  $W' = D(f_1) \cup \dots \cup D(f_n)$  such that  $t(f_i) = s(g_i)$  for some  $g_i \in B$  and  $B_{g_i} \cong R_{s(g_i)}$ . Then  $B' \rightarrow R'$  induces an open immersion of  $W'$  into  $\text{Spec}(B')$ .*

**Proof.** Set  $h_i = (g_i, f_i) \in B'$ . More on Algebra, Lemma 5.3 shows that  $(B')_{h_i} \cong (R')_{f_i}$  as desired.  $\square$

The following lemma is a precise statement of Noetherian approximation.

**Lemma 5.3.** *Let  $S$  be a quasi-compact and quasi-separated scheme. Let  $V \subset S$  be a quasi-compact open. Let  $I$  be a directed set and let  $(V_i, f_{ii'})$  be an inverse system of schemes over  $I$  with affine transition maps, with each  $V_i$  of finite type over  $\mathbf{Z}$ , and with  $V = \lim V_i$ . Then there exist*

- (1) *a directed set  $J$ ,*
- (2) *an inverse system of schemes  $(S_j, g_{jj'})$  over  $J$ ,*
- (3) *an order preserving map  $\alpha : J \rightarrow I$ ,*
- (4) *open subschemes  $V'_j \subset S_j$ , and*
- (5) *isomorphisms  $V'_j \rightarrow V_{\alpha(j)}$*

*such that*

- (1) *the transition morphisms  $g_{jj'} : S_j \rightarrow S_{j'}$  are affine,*
- (2) *each  $S_j$  is of finite type over  $\mathbf{Z}$ ,*
- (3)  *$g_{jj'}^{-1}(V'_{j'}) = V'_j$ ,*
- (4)  *$S = \lim S_j$  and  $V = \lim V'_j$ , and*
- (5) *the diagrams*

$$\begin{array}{ccc} V & & \\ \downarrow & \searrow & \\ V'_j & \longrightarrow & V_{\alpha(j)} \end{array} \quad \text{and} \quad \begin{array}{ccc} V'_j & \longrightarrow & V_{\alpha(j)} \\ \downarrow & & \downarrow \\ V'_{j'} & \longrightarrow & V_{\alpha(j')} \end{array}$$

*are commutative.*

**Proof.** Set  $Z = S \setminus V$ . Choose affine opens  $U_1, \dots, U_m \subset S$  such that  $Z \subset \bigcup_{l=1,\dots,m} U_l$ . Consider the opens

$$V \subset V \cup U_1 \subset V \cup U_1 \cup U_2 \subset \dots \subset V \cup \bigcup_{l=1,\dots,m} U_l = S$$

If we can prove the lemma successively for each of the cases

$$V \cup U_1 \cup \dots \cup U_l \subset V \cup U_1 \cup \dots \cup U_{l+1}$$

then the lemma will follow for  $V \subset S$ . In each case we are adding one affine open. Thus we may assume

- (1)  $S = U \cup V$ ,
- (2)  $U$  affine open in  $S$ ,
- (3)  $V$  quasi-compact open in  $S$ , and
- (4)  $V = \lim_i V_i$  with  $(V_i, f_{ii'})$  an inverse system over a directed set  $I$ , each  $f_{ii'}$  affine and each  $V_i$  of finite type over  $\mathbf{Z}$ .

Denote  $f_i : V \rightarrow V_i$  the projections. Set  $W = U \cap V$ . As  $S$  is quasi-separated, this is a quasi-compact open of  $V$ . By Lemma 4.11 (and after shrinking  $I$ ) we may assume that there exist opens  $W_i \subset V_i$  such that  $f_{ii'}^{-1}(W_{i'}) = W_i$  and such that  $f_i^{-1}(W_i) = W$ . Since  $W$  is a quasi-compact open of  $U$  it is quasi-affine. Hence we may assume (after shrinking  $I$  again) that  $W_i$  is quasi-affine for all  $i$ , see Lemma 4.12.

Write  $U = \text{Spec}(B)$ . Set  $R = \Gamma(W, \mathcal{O}_W)$ , and  $R_i = \Gamma(W_i, \mathcal{O}_{W_i})$ . By Lemma 4.7 we have  $R = \text{colim}_i R_i$ . Now we have the maps of rings

$$\begin{array}{ccc} B & \xrightarrow{s} & R \\ & & \uparrow t_i \\ & & R_i \end{array}$$

We set  $B_i = \{(b, r) \in B \times R_i \mid s(b) = t_i(r)\}$  so that we have a cartesian diagram

$$\begin{array}{ccc} B & \xrightarrow{s} & R \\ \uparrow & & \uparrow t_i \\ B_i & \longrightarrow & R_i \end{array}$$

for each  $i$ . The transition maps  $R_i \rightarrow R_{i'}$  induce maps  $B_i \rightarrow B_{i'}$ . It is clear that  $B = \text{colim}_i B_i$ . In the next paragraph we show that for all sufficiently large  $i$  the composition  $W_i \rightarrow \text{Spec}(R_i) \rightarrow \text{Spec}(B_i)$  is an open immersion.

As  $W$  is a quasi-compact open of  $U = \text{Spec}(B)$  we can find a finitely many elements  $g_l \in B$ ,  $l = 1, \dots, m$  such that  $D(g_l) \subset W$  and such that  $W = \bigcup_{l=1, \dots, m} D(g_l)$ . Note that this implies  $D(g_l) = W_{s(g_l)}$  as open subsets of  $U$ , where  $W_{s(g_l)}$  denotes the largest open subset of  $W$  on which  $s(g_l)$  is invertible. Hence

$$B_{g_l} = \Gamma(D(g_l), \mathcal{O}_U) = \Gamma(W_{s(g_l)}, \mathcal{O}_W) = R_{s(g_l)},$$

where the last equality is Properties, Lemma 17.1. Since  $W_{s(g_l)}$  is affine this also implies that  $D(s(g_l)) = W_{s(g_l)}$  as open subsets of  $\text{Spec}(R)$ . Since  $R = \text{colim}_i R_i$  we can (after shrinking  $I$ ) assume there exist  $g_{l,i} \in R_i$  for all  $i \in I$  such that  $s(g_l) = t_i(g_{l,i})$ . Of course we choose the  $g_{l,i}$  such that  $g_{l,i}$  maps to  $g_{l,i'}$  under the transition maps  $R_i \rightarrow R_{i'}$ . Then, by Lemma 4.11 we can (after shrinking  $I$  again) assume the corresponding opens  $D(g_{l,i}) \subset \text{Spec}(R_i)$  are contained in  $W_i$  for  $l = 1, \dots, m$  and cover  $W_i$ . We conclude that the morphism  $W_i \rightarrow \text{Spec}(R_i) \rightarrow \text{Spec}(B_i)$  is an open immersion, see Lemma 5.2.

By Lemma 5.1 we can write  $B_i$  as a directed colimit of subalgebras  $A_{i,p} \subset B_i$ ,  $p \in P_i$  each of finite type over  $\mathbf{Z}$  and such that  $W_i$  is identified with an open subscheme of  $\text{Spec}(A_{i,p})$ . Let  $S_{i,p}$  be the scheme obtained by glueing  $V_i$  and  $\text{Spec}(A_{i,p})$  along the open  $W_i$ , see Schemes, Section 14. Here is the resulting commutative diagram of schemes:

$$\begin{array}{ccccc}
 & & V & \longleftarrow & W \\
 & \swarrow & \downarrow & \searrow & \downarrow \\
 V_i & \longleftarrow & W_i & \longleftarrow & S \\
 \downarrow & & \downarrow & \swarrow & \downarrow \\
 S_{i,p} & \longleftarrow & \text{Spec}(A_{i,p}) & \longleftarrow & U
 \end{array}$$

The morphism  $S \rightarrow S_{i,p}$  arises because the upper right square is a pushout in the category of schemes. Note that  $S_{i,p}$  is of finite type over  $\mathbf{Z}$  since it has a finite affine open covering whose members are spectra of finite type  $\mathbf{Z}$ -algebras. We define a preorder on  $J = \coprod_{i \in I} P_i$  by the rule  $(i', p') \geq (i, p)$  if and only if  $i' \geq i$  and the map  $B_i \rightarrow B_{i'}$  maps  $A_{i,p}$  into  $A_{i',p'}$ . This is exactly the condition needed to define a morphism  $S_{i',p'} \rightarrow S_{i,p}$ : namely make a commutative diagram as above using the transition morphisms  $V_{i'} \rightarrow V_i$  and  $W_{i'} \rightarrow W_i$  and the morphism  $\text{Spec}(A_{i',p'}) \rightarrow \text{Spec}(A_{i,p})$  induced by the ring map  $A_{i,p} \rightarrow A_{i',p'}$ . The relevant commutativities have been built into the constructions. We claim that  $S$  is the directed limit of the schemes  $S_{i,p}$ . Since by construction the schemes  $V_i$  have limit  $V$  this boils down to the fact that  $B$  is the limit of the rings  $A_{i,p}$  which is true by construction. The map  $\alpha : J \rightarrow I$  is given by the rule  $j = (i, p) \mapsto i$ . The open subscheme  $V_j'$  is just the image of  $V_i \rightarrow S_{i,p}$  above. The commutativity of the diagrams in (5) is clear from the construction. This finishes the proof of the lemma.  $\square$

**Proposition 5.4.** *Let  $S$  be a quasi-compact and quasi-separated scheme. There exist a directed set  $I$  and an inverse system of schemes  $(S_i, f_{ii'})$  over  $I$  such that*

- (1) *the transition morphisms  $f_{ii'}$  are affine*
- (2) *each  $S_i$  is of finite type over  $\mathbf{Z}$ , and*
- (3)  *$S = \lim_i S_i$ .*

**Proof.** This is a special case of Lemma 5.3 with  $V = \emptyset$ .  $\square$

## 6. Limits and morphisms of finite presentation

The following is a generalization of Algebra, Lemma 127.3.

**Proposition 6.1.** *Let  $f : X \rightarrow S$  be a morphism of schemes. The following are equivalent:*

- (1) *The morphism  $f$  is locally of finite presentation.*
- (2) *For any directed set  $I$ , and any inverse system  $(T_i, f_{ii'})$  of  $S$ -schemes over  $I$  with each  $T_i$  affine, we have*

$$\text{Mor}_S(\lim_i T_i, X) = \text{colim}_i \text{Mor}_S(T_i, X)$$

- (3) *For any directed set  $I$ , and any inverse system  $(T_i, f_{ii'})$  of  $S$ -schemes over  $I$  with each  $f_{ii'}$  affine and every  $T_i$  quasi-compact and quasi-separated as a*

scheme, we have

$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X)$$

**Proof.** It is clear that (3) implies (2).

Let us prove that (2) implies (1). Assume (2). Choose any affine opens  $U \subset X$  and  $V \subset S$  such that  $f(U) \subset V$ . We have to show that  $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$  is of finite presentation. Let  $(A_i, \varphi_{ii'})$  be a directed system of  $\mathcal{O}_S(V)$ -algebras. Set  $A = \mathrm{colim}_i A_i$ . According to Algebra, Lemma 127.3 we have to show that

$$\mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A) = \mathrm{colim}_i \mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i)$$

Consider the schemes  $T_i = \mathrm{Spec}(A_i)$ . They form an inverse system of  $V$ -schemes over  $I$  with transition morphisms  $f_{ii'} : T_i \rightarrow T_{i'}$  induced by the  $\mathcal{O}_S(V)$ -algebra maps  $\varphi_{ii'}$ . Set  $T := \mathrm{Spec}(A) = \lim_i T_i$ . The formula above becomes in terms of morphism sets of schemes

$$\mathrm{Mor}_V(\lim_i T_i, U) = \mathrm{colim}_i \mathrm{Mor}_V(T_i, U).$$

We first observe that  $\mathrm{Mor}_V(T_i, U) = \mathrm{Mor}_S(T_i, U)$  and  $\mathrm{Mor}_V(T, U) = \mathrm{Mor}_S(T, U)$ . Hence we have to show that

$$\mathrm{Mor}_S(\lim_i T_i, U) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, U)$$

and we are given that

$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X).$$

Hence it suffices to prove that given a morphism  $g_i : T_i \rightarrow X$  over  $S$  such that the composition  $T \rightarrow T_i \rightarrow X$  ends up in  $U$  there exists some  $i' \geq i$  such that the composition  $g_{i'} : T_{i'} \rightarrow T_i \rightarrow X$  ends up in  $U$ . Denote  $Z_{i'} = g_{i'}^{-1}(X \setminus U)$ . Assume each  $Z_{i'}$  is nonempty to get a contradiction. By Lemma 4.8 there exists a point  $t$  of  $T$  which is mapped into  $Z_{i'}$  for all  $i' \geq i$ . Such a point is not mapped into  $U$ . A contradiction.

Finally, let us prove that (1) implies (3). Assume (1). Let an inverse directed system  $(T_i, f_{ii'})$  of  $S$ -schemes be given. Assume the morphisms  $f_{ii'}$  are affine and each  $T_i$  is quasi-compact and quasi-separated as a scheme. Let  $T = \lim_i T_i$ . Denote  $f_i : T \rightarrow T_i$  the projection morphisms. We have to show:

- (a) Given morphisms  $g_i, g'_i : T_i \rightarrow X$  over  $S$  such that  $g_i \circ f_i = g'_i \circ f_i$ , then there exists an  $i' \geq i$  such that  $g_i \circ f_{i'i} = g'_i \circ f_{i'i}$ .
- (b) Given any morphism  $g : T \rightarrow X$  over  $S$  there exists an  $i \in I$  and a morphism  $g_i : T_i \rightarrow X$  such that  $g = f_i \circ g_i$ .

First let us prove the uniqueness part (a). Let  $g_i, g'_i : T_i \rightarrow X$  be morphisms such that  $g_i \circ f_i = g'_i \circ f_i$ . For any  $i' \geq i$  we set  $g_{i'} = g_i \circ f_{i'i}$  and  $g'_{i'} = g'_i \circ f_{i'i}$ . We also set  $g = g_i \circ f_i = g'_i \circ f_i$ . Consider the morphism  $(g_i, g'_i) : T_i \rightarrow X \times_S X$ . Set

$$W = \bigcup_{U \subset X \text{ affine open}, V \subset S \text{ affine open}, f(U) \subset V} U \times_V U.$$

This is an open in  $X \times_S X$ , with the property that the morphism  $\Delta_{X/S}$  factors through a closed immersion into  $W$ , see the proof of Schemes, Lemma 21.2. Note that the composition  $(g_i, g'_i) \circ f_i : T \rightarrow X \times_S X$  is a morphism into  $W$  because it factors through the diagonal by assumption. Set  $Z_{i'} = (g_{i'}, g'_{i'})^{-1}(X \times_S X \setminus W)$ . If each  $Z_{i'}$  is nonempty, then by Lemma 4.8 there exists a point  $t \in T$  which maps to  $Z_{i'}$  for all  $i' \geq i$ . This is a contradiction with the fact that  $T$  maps into  $W$ .

Hence we may increase  $i$  and assume that  $(g_i, g'_i) : T_i \rightarrow X \times_S X$  is a morphism into  $W$ . By construction of  $W$ , and since  $T_i$  is quasi-compact we can find a finite affine open covering  $T_i = T_{1,i} \cup \dots \cup T_{n,i}$  such that  $(g_i, g'_i)|_{T_{j,i}}$  is a morphism into  $U \times_V U$  for some pair  $(U, V)$  as in the definition of  $W$  above. Since it suffices to prove that  $g_{i'}$  and  $g'_{i'}$  agree on each of the  $f_{i'i}^{-1}(T_{j,i})$  this reduces us to the affine case. The affine case follows from Algebra, Lemma 127.3 and the fact that the ring map  $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$  is of finite presentation (see Morphisms, Lemma 21.2).

Finally, we prove the existence part (b). Let  $g : T \rightarrow X$  be a morphism of schemes over  $S$ . We can find a finite affine open covering  $T = W_1 \cup \dots \cup W_n$  such that for each  $j \in \{1, \dots, n\}$  there exist affine opens  $U_j \subset X$  and  $V_j \subset S$  with  $f(U_j) \subset V_j$  and  $g(W_j) \subset U_j$ . By Lemmas 4.11 and 4.13 (after possibly shrinking  $I$ ) we may assume that there exist affine open coverings  $T_i = W_{1,i} \cup \dots \cup W_{n,i}$  compatible with transition maps such that  $W_j = \lim_i W_{j,i}$ . We apply Algebra, Lemma 127.3 to the rings corresponding to the affine schemes  $U_j$ ,  $V_j$ ,  $W_{j,i}$  and  $W_j$  using that  $\mathcal{O}_S(V_j) \rightarrow \mathcal{O}_X(U_j)$  is of finite presentation (see Morphisms, Lemma 21.2). Thus we can find for each  $j$  an index  $i_j \in I$  and a morphism  $g_{j,i_j} : W_{j,i_j} \rightarrow X$  such that  $g_{j,i_j} \circ f_i|_{W_j} : W_j \rightarrow W_{j,i_j} \rightarrow X$  equals  $g|_{W_j}$ . By part (a) proved above, using the quasi-compactness of  $W_{j_1,i} \cap W_{j_2,i}$  which follows as  $T_i$  is quasi-separated, we can find an index  $i' \in I$  larger than all  $i_j$  such that

$$g_{j_1,i_{j_1}} \circ f_{i'i_{j_1}}|_{W_{j_1,i'} \cap W_{j_2,i'}} = g_{j_2,i_{j_2}} \circ f_{i'i_{j_2}}|_{W_{j_1,i'} \cap W_{j_2,i'}}$$

for all  $j_1, j_2 \in \{1, \dots, n\}$ . Hence the morphisms  $g_{j,i_j} \circ f_{i'i_j}|_{W_{j,i'}}$  glue to given the desired morphism  $T_{i'} \rightarrow X$ .  $\square$

**Remark 6.2.** Let  $S$  be a scheme. Let us say that a functor  $F : (\text{Sch}/S)^{\text{opp}} \rightarrow \text{Sets}$  is *limit preserving* if for every directed inverse system  $\{T_i\}_{i \in I}$  of affine schemes with limit  $T$  we have  $F(T) = \text{colim}_i F(T_i)$ . Let  $X$  be a scheme over  $S$ , and let  $h_X : (\text{Sch}/S)^{\text{opp}} \rightarrow \text{Sets}$  be its functor of points, see Schemes, Section 15. In this terminology Proposition 6.1 says that a scheme  $X$  is locally of finite presentation over  $S$  if and only if  $h_X$  is limit preserving.

**Lemma 6.3.** *Let  $f : X \rightarrow S$  be a morphism of schemes. If for every directed limit  $T = \lim_{i \in I} T_i$  of affine schemes over  $S$  the map*

$$\text{colim}_i \text{Mor}_S(T_i, X) \longrightarrow \text{Mor}_S(T, X)$$

*is surjective, then  $f$  is locally of finite presentation. In other words, in Proposition 6.1 parts (2) and (3) it suffices to check surjectivity of the map.*

**Proof.** The proof is exactly the same as the proof of the implication “(2) implies (1)” in Proposition 6.1. Choose any affine opens  $U \subset X$  and  $V \subset S$  such that  $f(U) \subset V$ . We have to show that  $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$  is of finite presentation. Let  $(A_i, \varphi_{ii'})$  be a directed system of  $\mathcal{O}_S(V)$ -algebras. Set  $A = \text{colim}_i A_i$ . According to Algebra, Lemma 127.3 it suffices to show that

$$\text{colim}_i \text{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i) \rightarrow \text{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A)$$

is surjective. Consider the schemes  $T_i = \text{Spec}(A_i)$ . They form an inverse system of  $V$ -schemes over  $I$  with transition morphisms  $f_{ii'} : T_i \rightarrow T_{i'}$  induced by the  $\mathcal{O}_S(V)$ -algebra maps  $\varphi_{ii'}$ . Set  $T := \text{Spec}(A) = \lim_i T_i$ . The formula above becomes in terms of morphism sets of schemes

$$\text{colim}_i \text{Mor}_V(T_i, U) \rightarrow \text{Mor}_V(\lim_i T_i, U)$$



We first observe that  $\text{Mor}_V(T_i, U) = \text{Mor}_S(T_i, U)$  and  $\text{Mor}_V(T, U) = \text{Mor}_S(T, U)$ . Hence we have to show that

$$\text{colim}_i \text{Mor}_S(T_i, U) \rightarrow \text{Mor}_S(\lim_i T_i, U)$$

is surjective and we are given that

$$\text{colim}_i \text{Mor}_S(T_i, X) \rightarrow \text{Mor}_S(\lim_i T_i, X)$$

is surjective. Hence it suffices to prove that given a morphism  $g_i : T_i \rightarrow X$  over  $S$  such that the composition  $T \rightarrow T_i \rightarrow X$  ends up in  $U$  there exists some  $i' \geq i$  such that the composition  $g_{i'} : T_{i'} \rightarrow T_i \rightarrow X$  ends up in  $U$ . Denote  $Z_{i'} = g_{i'}^{-1}(X \setminus U)$ . Assume each  $Z_{i'}$  is nonempty to get a contradiction. By Lemma 4.8 there exists a point  $t$  of  $T$  which is mapped into  $Z_{i'}$  for all  $i' \geq i$ . Such a point is not mapped into  $U$ . A contradiction.  $\square$

The following is an example application of Proposition 6.1.

**Lemma 6.4.** *Let  $S$  be a scheme. Let  $X$  and  $Y$  be schemes over  $S$ . Assume  $Y$  is locally of finite presentation over  $S$ . Let  $x \in X$  be a closed point such that  $U = X \setminus \{x\} \rightarrow X$  is quasi-compact. With  $V = \text{Spec}(\mathcal{O}_{X,x}) \setminus \{x\}$  there is a bijection*

$$\{\text{morphisms } X \rightarrow Y \text{ over } S\} \longrightarrow \left\{ (a, b) \text{ where } a : U \rightarrow Y \text{ and } b : \text{Spec}(\mathcal{O}_{X,x}) \rightarrow Y \right. \\ \left. \text{are morphisms over } S \text{ which agree over } V \right\}$$

**Proof.** Let  $W \subset X$  be an open neighbourhood of  $x$ . By glueing of schemes, see Schemes, Section 14 the result holds if we consider pairs of morphisms  $a : U \rightarrow Y$  and  $c : W \rightarrow Y$  which agree over  $U \cap W$ . We have  $\mathcal{O}_{X,x} = \text{colim } \mathcal{O}_W(W)$  where  $W$  runs over the affine open neighbourhoods of  $x$  in  $X$ . Hence  $\text{Spec}(\mathcal{O}_{X,x}) = \lim W$  where  $W$  runs over the affine open neighbourhoods of  $s$ . Thus by Proposition 6.1 any morphism  $b : \text{Spec}(\mathcal{O}_{X,x}) \rightarrow Y$  over  $S$  comes from a morphism  $c : W \rightarrow Y$  for some  $W$  as above (and  $c$  is unique up to further shrinking  $W$ ). For every affine open  $x \in W$  we see that  $U \cap W$  is quasi-compact as  $U \rightarrow X$  is quasi-compact. Hence  $V = \lim W \cap U = \lim W \setminus \{x\}$  is a limit of quasi-compact and quasi-separated schemes (see Lemma 2.2). Thus if  $a$  and  $b$  agree over  $V$ , then after shrinking  $W$  we see that  $a$  and  $c$  agree over  $U \cap W$  (by the same proposition). The lemma follows.  $\square$

## 7. Relative approximation

We discuss variants of Proposition 5.4 over a base.

**Lemma 7.1.** *Let  $f : X \rightarrow S$  be a morphism of quasi-compact and quasi-separated schemes. Then there exists a direct set  $I$  and an inverse system  $(f_i : X_i \rightarrow S_i)$  of morphisms schemes over  $I$ , such that the transition morphisms  $X_i \rightarrow X_{i'}$  and  $S_i \rightarrow S_{i'}$  are affine, such that  $X_i$  and  $S_i$  are of finite type over  $\mathbf{Z}$ , and such that  $(X \rightarrow S) = \lim(X_i \rightarrow S_i)$ .*

**Proof.** Write  $X = \lim_{a \in A} X_a$  and  $S = \lim_{b \in B} S_b$  as in Proposition 5.4, i.e., with  $X_a$  and  $S_b$  of finite type over  $\mathbf{Z}$  and with affine transition morphisms.

Fix  $b \in B$ . By Proposition 6.1 applied to  $S_b$  and  $X = \lim X_a$  over  $\mathbf{Z}$  we find there exists an  $a \in A$  and a morphism  $f_{a,b} : X_a \rightarrow S_b$  making the diagram

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow & & \downarrow \\ X_a & \longrightarrow & S_b \end{array}$$

commute. Let  $I$  be the set of triples  $(a, b, f_{a,b})$  we obtain in this manner.

Let  $(a, b, f_{a,b})$  and  $(a', b', f_{a',b'})$  be in  $I$ . Let  $b'' \leq \min(b, b')$ . By Proposition 6.1 again, there exists an  $a'' \geq \max(a, a')$  such that the compositions  $X_{a''} \rightarrow X_a \rightarrow S_b \rightarrow S_{b''}$  and  $X_{a''} \rightarrow X_{a'} \rightarrow S_{b'} \rightarrow S_{b''}$  are equal. We endow  $I$  with the preorder

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a', \quad b \geq b', \quad \text{and} \quad g_{b,b'} \circ f_{a,b} = f_{a',b'} \circ h_{a,a'}$$

where  $h_{a,a'} : X_a \rightarrow X_{a'}$  and  $g_{b,b'} : S_b \rightarrow S_{b'}$  are the transition morphisms. The remarks above show that  $I$  is directed and that the maps  $I \rightarrow A$ ,  $(a, b, f_{a,b}) \mapsto a$  and  $I \rightarrow B$ ,  $(a, b, f_{a,b}) \mapsto b$  are cofinal. If for  $i = (a, b, f_{a,b})$  we set  $X_i = X_a$ ,  $S_i = S_b$ , and  $f_i = f_{a,b}$ , then we get an inverse system of morphisms over  $I$  and we have

$$\lim_{i \in I} X_i = \lim_{a \in A} X_a = X \quad \text{and} \quad \lim_{i \in I} S_i = \lim_{b \in B} S_b = S$$

by Categories, Lemma 17.4 (recall that limits over  $I$  are really limits over the opposite category associated to  $I$  and hence cofinal turns into initial). This finishes the proof.  $\square$

**Lemma 7.2.** *Let  $f : X \rightarrow S$  be a morphism of schemes. Assume that*

- (1)  *$X$  is quasi-compact and quasi-separated, and*
- (2)  *$S$  is quasi-separated.*

*Then  $X = \lim X_i$  is a limit of a directed system of schemes  $X_i$  of finite presentation over  $S$  with affine transition morphisms over  $S$ .*

**Proof.** Since  $f(X)$  is quasi-compact we may replace  $S$  by a quasi-compact open containing  $f(X)$ . Hence we may assume  $S$  is quasi-compact. By Lemma 7.1 we can write  $(X \rightarrow S) = \lim (X_i \rightarrow S_i)$  for some directed inverse system of morphisms of finite type schemes over  $\mathbf{Z}$  with affine transition morphisms. Since limits commute with limits (Categories, Lemma 14.10) we have  $X = \lim X_i \times_{S_i} S$ . Let  $i \geq i'$  in  $I$ . The morphism  $X_i \times_{S_i} S \rightarrow X_{i'} \times_{S_{i'}} S$  is affine as the composition

$$X_i \times_{S_i} S \rightarrow X_i \times_{S_{i'}} S \rightarrow X_{i'} \times_{S_{i'}} S$$

where the first morphism is a closed immersion (by Schemes, Lemma 21.9) and the second is a base change of an affine morphism (Morphisms, Lemma 11.8) and the composition of affine morphisms is affine (Morphisms, Lemma 11.7). The morphisms  $f_i$  are of finite presentation (Morphisms, Lemmas 21.9 and 21.11) and hence the base changes  $X_i \times_{f_i, S_i} S \rightarrow S$  are of finite presentation (Morphisms, Lemma 21.4).  $\square$

**Lemma 7.3.** *Let  $X \rightarrow S$  be an integral morphism with  $S$  quasi-compact and quasi-separated. Then  $X = \lim X_i$  with  $X_i \rightarrow S$  finite and of finite presentation.*

**Proof.** Consider the sheaf  $\mathcal{A} = f_* \mathcal{O}_X$ . This is a quasi-coherent sheaf of  $\mathcal{O}_S$ -algebras, see Schemes, Lemma 24.1. Combining Properties, Lemma 22.13 we can

write  $\mathcal{A} = \operatorname{colim}_i \mathcal{A}_i$  as a filtered colimit of finite and finitely presented  $\mathcal{O}_S$ -algebras. Then

$$X_i = \operatorname{Spec}_S(\mathcal{A}_i) \longrightarrow S$$

is a finite and finitely presented morphism of schemes. By construction  $X = \lim_i X_i$  which proves the lemma.  $\square$

## 8. Descending properties of morphisms

This section is the analogue of Section 4 for properties of morphisms over  $S$ . We will work in the following situation.

**Situation 8.1.** Let  $S = \lim S_i$  be a limit of a directed system of schemes with affine transition morphisms (Lemma 2.2). Let  $0 \in I$  and let  $f_0 : X_0 \rightarrow Y_0$  be a morphism of schemes over  $S_0$ . Assume  $S_0, X_0, Y_0$  are quasi-compact and quasi-separated. Let  $f_i : X_i \rightarrow Y_i$  be the base change of  $f_0$  to  $S_i$  and let  $f : X \rightarrow Y$  be the base change of  $f_0$  to  $S$ .

**Lemma 8.2.** *Notation and assumptions as in Situation 8.1. If  $f$  is affine, then there exists an index  $i \geq 0$  such that  $f_i$  is affine.*

**Proof.** Let  $Y_0 = \bigcup_{j=1, \dots, m} V_{j,0}$  be a finite affine open covering. Set  $U_{j,0} = f_0^{-1}(V_{j,0})$ . For  $i \geq 0$  we denote  $V_{j,i}$  the inverse image of  $V_{j,0}$  in  $Y_i$  and  $U_{j,i} = f_i^{-1}(V_{j,i})$ . Similarly we have  $U_j = f^{-1}(V_j)$ . Then  $U_j = \lim_{i \geq 0} U_{j,i}$  (see Lemma 2.2). Since  $U_j$  is affine by assumption we see that each  $U_{j,i}$  is affine for  $i$  large enough, see Lemma 4.13. As there are finitely many  $j$  we can pick an  $i$  which works for all  $j$ . Thus  $f_i$  is affine for  $i$  large enough, see Morphisms, Lemma 11.3.  $\square$

**Lemma 8.3.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is a finite morphism, and*
- (2)  *$f_0$  is locally of finite type,*

*then there exists an  $i \geq 0$  such that  $f_i$  is finite.*

**Proof.** A finite morphism is affine, see Morphisms, Definition 44.1. Hence by Lemma 8.2 above after increasing 0 we may assume that  $f_0$  is affine. By writing  $Y_0$  as a finite union of affines we reduce to proving the result when  $X_0$  and  $Y_0$  are affine and map into a common affine  $W \subset S_0$ . The corresponding algebra statement follows from Algebra, Lemma 168.3.  $\square$

**Lemma 8.4.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is unramified, and*
- (2)  *$f_0$  is locally of finite type,*

*then there exists an  $i \geq 0$  such that  $f_i$  is unramified.*

**Proof.** Choose a finite affine open covering  $Y_0 = \bigcup_{j=1, \dots, m} Y_{j,0}$  such that each  $Y_{j,0}$  maps into an affine open  $S_{j,0} \subset S_0$ . For each  $j$  let  $f_0^{-1}Y_{j,0} = \bigcup_{k=1, \dots, n_j} X_{k,0}$  be a finite affine open covering. Since the property of being unramified is local we see that it suffices to prove the lemma for the morphisms of affines  $X_{k,i} \rightarrow Y_{j,i} \rightarrow S_{j,i}$  which are the base changes of  $X_{k,0} \rightarrow Y_{j,0} \rightarrow S_{j,0}$  to  $S_i$ . Thus we reduce to the case that  $X_0, Y_0, S_0$  are affine

In the affine case we reduce to the following algebra result. Suppose that  $R = \operatorname{colim}_{i \in I} R_i$ . For some  $0 \in I$  suppose given an  $R_0$ -algebra map  $A_i \rightarrow B_i$  of finite

type. If  $R \otimes_{R_0} A_0 \rightarrow R \otimes_{R_0} B_0$  is unramified, then for some  $i \geq 0$  the map  $R_i \otimes_{R_0} A_0 \rightarrow R_i \otimes_{R_0} B_0$  is unramified. This follows from Algebra, Lemma 168.5.  $\square$

**Lemma 8.5.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is a closed immersion, and*
- (2)  *$f_0$  is locally of finite type,*

*then there exists an  $i \geq 0$  such that  $f_i$  is a closed immersion.*

**Proof.** A closed immersion is affine, see Morphisms, Lemma 11.9. Hence by Lemma 8.2 above after increasing 0 we may assume that  $f_0$  is affine. By writing  $Y_0$  as a finite union of affines we reduce to proving the result when  $X_0$  and  $Y_0$  are affine and map into a common affine  $W \subset S_0$ . The corresponding algebra statement is a consequence of Algebra, Lemma 168.4.  $\square$

**Lemma 8.6.** *Notation and assumptions as in Situation 8.1. If  $f$  is separated, then  $f_i$  is separated for some  $i \geq 0$ .*

**Proof.** Apply Lemma 8.5 to the diagonal morphism  $\Delta_{X_0/S_0} : X_0 \rightarrow X_0 \times_{S_0} X_0$ . (This is permissible as diagonal morphisms are locally of finite type and the fibre product  $X_0 \times_{S_0} X_0$  is quasi-compact and quasi-separated, see Schemes, Lemma 21.2, Morphisms, Lemma 15.5, and Schemes, Remark 21.18.  $\square$

**Lemma 8.7.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is flat,*
- (2)  *$f_0$  is locally of finite presentation,*

*then  $f_i$  is flat for some  $i \geq 0$ .*

**Proof.** Choose a finite affine open covering  $Y_0 = \bigcup_{j=1, \dots, m} Y_{j,0}$  such that each  $Y_{j,0}$  maps into an affine open  $S_{j,0} \subset S_0$ . For each  $j$  let  $f_0^{-1}Y_{j,0} = \bigcup_{k=1, \dots, n_j} X_{k,0}$  be a finite affine open covering. Since the property of being flat is local we see that it suffices to prove the lemma for the morphisms of affines  $X_{k,i} \rightarrow Y_{j,i} \rightarrow S_{j,i}$  which are the base changes of  $X_{k,0} \rightarrow Y_{j,0} \rightarrow S_{j,0}$  to  $S_i$ . Thus we reduce to the case that  $X_0, Y_0, S_0$  are affine

In the affine case we reduce to the following algebra result. Suppose that  $R = \text{colim}_{i \in I} R_i$ . For some  $0 \in I$  suppose given an  $R_0$ -algebra map  $A_i \rightarrow B_i$  of finite presentation. If  $R \otimes_{R_0} A_0 \rightarrow R \otimes_{R_0} B_0$  is flat, then for some  $i \geq 0$  the map  $R_i \otimes_{R_0} A_0 \rightarrow R_i \otimes_{R_0} B_0$  is flat. This follows from Algebra, Lemma 168.1 part (3).  $\square$

**Lemma 8.8.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is finite locally free (of degree  $d$ ),*
- (2)  *$f_0$  is locally of finite presentation,*

*then  $f_i$  is finite locally free (of degree  $d$ ) for some  $i \geq 0$ .*

**Proof.** By Lemmas 8.7 and 8.3 we find an  $i$  such that  $f_i$  is flat and finite. On the other hand,  $f_i$  is locally of finite presentation. Hence  $f_i$  is finite locally free by Morphisms, Lemma 48.2. If moreover  $f$  is finite locally free of degree  $d$ , then the image of  $Y \rightarrow Y_i$  is contained in the open and closed locus  $W_d \subset Y_i$  over which  $f_i$  has degree  $d$ . By Lemma 4.10 we see that for some  $i' \geq i$  the image of  $Y_{i'} \rightarrow Y_i$  is contained in  $W_d$ . Then  $f_{i'}$  will be finite locally free of degree  $d$ .  $\square$

**Lemma 8.9.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is smooth,*
- (2)  *$f_0$  is locally of finite presentation,*

*then  $f_i$  is smooth for some  $i \geq 0$ .*

**Proof.** Being smooth is local on the source and the target (Morphisms, Lemma 34.2) hence we may assume  $S_0, X_0, Y_0$  affine (details omitted). The corresponding algebra fact is Algebra, Lemma 168.8.  $\square$

**Lemma 8.10.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is étale,*
- (2)  *$f_0$  is locally of finite presentation,*

*then  $f_i$  is étale for some  $i \geq 0$ .*

**Proof.** Being étale is local on the source and the target (Morphisms, Lemma 36.2) hence we may assume  $S_0, X_0, Y_0$  affine (details omitted). The corresponding algebra fact is Algebra, Lemma 168.7.  $\square$

**Lemma 8.11.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is an isomorphism, and*
- (2)  *$f_0$  is locally of finite presentation,*

*then  $f_i$  is an isomorphism for some  $i \geq 0$ .*

**Proof.** By Lemmas 8.10 and 8.5 we can find an  $i$  such that  $f_i$  is flat and a closed immersion. Then  $f_i$  identifies  $X_i$  with an open and closed subscheme of  $Y_i$ , see Morphisms, Lemma 26.2. By assumption the image of  $Y \rightarrow Y_i$  maps into  $f_i(X_i)$ . Thus by Lemma 4.10 we find that  $Y_{i'}$  maps into  $f_i(X_i)$  for some  $i' \geq i$ . It follows that  $X_{i'} \rightarrow Y_{i'}$  is surjective and we win.  $\square$

**Lemma 8.12.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is an open immersion, and*
- (2)  *$f_0$  is locally of finite presentation,*

*then  $f_i$  is an open immersion for some  $i \geq 0$ .*

**Proof.** By Lemma 8.10 we can find an  $i$  such that  $f_i$  is étale. Then  $V_i = f_i(X_i)$  is a quasi-compact open subscheme of  $Y_i$  (Morphisms, Lemma 36.13). let  $V$  and  $V_{i'}$  for  $i' \geq i$  be the inverse image of  $V_i$  in  $Y$  and  $Y_{i'}$ . Then  $f : X \rightarrow V$  is an isomorphism (namely it is a surjective open immersion). Hence by Lemma 8.11 we see that  $X_{i'} \rightarrow V_{i'}$  is an isomorphism for some  $i' \geq i$  as desired.  $\square$

**Lemma 8.13.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is an immersion, and*
- (2)  *$f_0$  is locally of finite type,*

*then  $f_i$  is an immersion for some  $i \geq 0$ .*

**Proof.** There exists an open  $V \subset Y$  such that the morphism  $f$  factors as  $X \rightarrow V \rightarrow Y$  and such that  $X \rightarrow V$  is a closed immersion, see discussion in Schemes, Section 10. Since  $X$  is quasi-compact, we may and do assume  $V$  is a quasi-compact open of  $Y$ . By Lemma 4.11 after increasing 0 we can find a quasi-compact open  $V_0 \subset Y_0$  such that  $V$  is the inverse image of  $V_0$ . Then the inverse image of  $V_0$  in  $X_0$  is a quasi-compact open whose inverse image in  $X$  is  $X$ . Hence by the same

lemma applied to  $X = \lim X_i$  we may assume after increasing 0 that we have the factorization  $X_0 \rightarrow V_0 \rightarrow Y_0$ . Then for large enough  $i \geq 0$  the morphism  $X_i \rightarrow V_i$  where  $V_i = Y_i \times_{Y_0} V_0$  is a closed immersion by Lemma 8.5 and the proof is complete.  $\square$

**Lemma 8.14.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is a monomorphism, and*
- (2)  *$f_0$  is locally of finite type,*

*then  $f_i$  is a monomorphism for some  $i \geq 0$ .*

**Proof.** Recall that a morphism of schemes  $V \rightarrow W$  is a monomorphism if and only if the diagonal  $V \rightarrow V \times_W V$  is an isomorphism (Schemes, Lemma 23.2). The morphism  $X_0 \rightarrow X_0 \times_{Y_0} X_0$  is locally of finite presentation by Morphisms, Lemma 21.12. Since  $X_0 \times_{Y_0} X_0$  is quasi-compact and quasi-separated (Schemes, Remark 21.18) we conclude from Lemma 8.11 that  $\Delta_i : X_i \rightarrow X_i \times_{Y_i} X_i$  is an isomorphism for some  $i \geq 0$ . For this  $i$  the morphism  $f_i$  is a monomorphism.  $\square$

**Lemma 8.15.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is surjective, and*
- (2)  *$f_0$  is locally of finite presentation,*

*then there exists an  $i \geq 0$  such that  $f_i$  is surjective.*

**Proof.** The morphism  $f_0$  is of finite presentation. Hence  $E = f_0(X_0)$  is a constructible subset of  $Y_0$ , see Morphisms, Lemma 22.2. Since  $f_i$  is the base change of  $f_0$  by  $Y_i \rightarrow Y_0$  we see that the image of  $f_i$  is the inverse image of  $E$  in  $Y_i$ . Moreover, we know that  $Y \rightarrow Y_0$  maps into  $E$ . Hence we win by Lemma 4.10.  $\square$

**Lemma 8.16.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is syntomic, and*
- (2)  *$f_0$  is locally of finite presentation,*

*then there exists an  $i \geq 0$  such that  $f_i$  is syntomic.*

**Proof.** Choose a finite affine open covering  $Y_0 = \bigcup_{j=1,\dots,m} Y_{j,0}$  such that each  $Y_{j,0}$  maps into an affine open  $S_{j,0} \subset S_0$ . For each  $j$  let  $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\dots,n_j} X_{k,0}$  be a finite affine open covering. Since the property of being syntomic is local we see that it suffices to prove the lemma for the morphisms of affines  $X_{k,i} \rightarrow Y_{j,i} \rightarrow S_{j,i}$  which are the base changes of  $X_{k,0} \rightarrow Y_{j,0} \rightarrow S_{j,0}$  to  $S_i$ . Thus we reduce to the case that  $X_0, Y_0, S_0$  are affine

In the affine case we reduce to the following algebra result. Suppose that  $R = \operatorname{colim}_{i \in I} R_i$ . For some  $0 \in I$  suppose given an  $R_0$ -algebra map  $A_i \rightarrow B_i$  of finite presentation. If  $R \otimes_{R_0} A_0 \rightarrow R \otimes_{R_0} B_0$  is syntomic, then for some  $i \geq 0$  the map  $R_i \otimes_{R_0} A_0 \rightarrow R_i \otimes_{R_0} B_0$  is syntomic. This follows from Algebra, Lemma 168.9.  $\square$

## 9. Finite type closed in finite presentation

A result of this type is [Kie72, Satz 2.10]. Another reference is [Con07].

**Lemma 9.1.** *Let  $f : X \rightarrow S$  be a morphism of schemes. Assume:*

- (1) *The morphism  $f$  is locally of finite type.*
- (2) *The scheme  $X$  is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation  $f' : X' \rightarrow S$  and an immersion  $X \rightarrow X'$  of schemes over  $S$ .

**Proof.** By Proposition 5.4 we can write  $X = \lim_i X_i$  with each  $X_i$  of finite type over  $\mathbf{Z}$  and with transition morphisms  $f_{ii'} : X_i \rightarrow X_{i'}$  affine. Consider the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_{i,S} & \longrightarrow & X_i \\ & \searrow & \downarrow & & \downarrow \\ & & S & \longrightarrow & \mathrm{Spec}(\mathbf{Z}) \end{array}$$

Note that  $X_i$  is of finite presentation over  $\mathrm{Spec}(\mathbf{Z})$ , see Morphisms, Lemma 21.9. Hence the base change  $X_{i,S} \rightarrow S$  is of finite presentation by Morphisms, Lemma 21.4. Thus it suffices to show that the arrow  $X \rightarrow X_{i,S}$  is an immersion for  $i$  sufficiently large.

To do this we choose a finite affine open covering  $X = V_1 \cup \dots \cup V_n$  such that  $f$  maps each  $V_j$  into an affine open  $U_j \subset S$ . Let  $h_{j,a} \in \mathcal{O}_X(V_j)$  be a finite set of elements which generate  $\mathcal{O}_X(V_j)$  as an  $\mathcal{O}_S(U_j)$ -algebra, see Morphisms, Lemma 15.2. By Lemmas 4.11 and 4.13 (after possibly shrinking  $I$ ) we may assume that there exist affine open coverings  $X_i = V_{1,i} \cup \dots \cup V_{n,i}$  compatible with transition maps such that  $V_j = \lim_i V_{j,i}$ . By Lemma 4.7 we can choose  $i$  so large that each  $h_{j,a}$  comes from an element  $h_{j,a,i} \in \mathcal{O}_{X_i}(V_{j,i})$ . Thus the arrow in

$$V_j \longrightarrow U_j \times_{\mathrm{Spec}(\mathbf{Z})} V_{j,i} = (V_{j,i})_{U_j} \subset (V_{j,i})_S \subset X_{i,S}$$

is a closed immersion. Since  $\bigcup (V_{j,i})_{U_j}$  forms an open of  $X_{i,S}$  and since the inverse image of  $(V_{j,i})_{U_j}$  in  $X$  is  $V_j$  it follows that  $X \rightarrow X_{i,S}$  is an immersion.  $\square$

**Remark 9.2.** We cannot do better than this if we do not assume more on  $S$  and the morphism  $f : X \rightarrow S$ . For example, in general it will not be possible to find a *closed* immersion  $X \rightarrow X'$  as in the lemma. The reason is that this would imply that  $f$  is quasi-compact which may not be the case. An example is to take  $S$  to be infinite dimensional affine space with 0 doubled and  $X$  to be one of the two infinite dimensional affine spaces.

**Lemma 9.3.** *Let  $f : X \rightarrow S$  be a morphism of schemes. Assume:*

- (1) *The morphism  $f$  is of locally of finite type.*
- (2) *The scheme  $X$  is quasi-compact and quasi-separated, and*
- (3) *The scheme  $S$  is quasi-separated.*

*Then there exists a morphism of finite presentation  $f' : X' \rightarrow S$  and a closed immersion  $X \rightarrow X'$  of schemes over  $S$ .*

**Proof.** By Lemma 9.1 above there exists a morphism  $Y \rightarrow S$  of finite presentation and an immersion  $i : X \rightarrow Y$  of schemes over  $S$ . For every point  $x \in X$ , there exists an affine open  $V_x \subset Y$  such that  $i^{-1}(V_x) \rightarrow V_x$  is a closed immersion. Since  $X$  is quasi-compact we can find finitely many affine opens  $V_1, \dots, V_n \subset Y$  such that  $i(X) \subset V_1 \cup \dots \cup V_n$  and  $i^{-1}(V_j) \rightarrow V_j$  is a closed immersion. In other words such that  $i : X \rightarrow X' = V_1 \cup \dots \cup V_n$  is a closed immersion of schemes over  $S$ . Since  $S$  is quasi-separated and  $Y$  is quasi-separated over  $S$  we deduce that  $Y$  is quasi-separated, see Schemes, Lemma 21.12. Hence the open immersion  $X' = V_1 \cup \dots \cup V_n \rightarrow Y$  is quasi-compact. This implies that  $X' \rightarrow Y$  is of finite

presentation, see Morphisms, Lemma 21.6. We conclude since then  $X' \rightarrow Y \rightarrow S$  is a composition of morphisms of finite presentation, and hence of finite presentation (see Morphisms, Lemma 21.3).  $\square$

**Lemma 9.4.** *Let  $X \rightarrow Y$  be a closed immersion of schemes. Assume  $Y$  quasi-compact and quasi-separated. Then  $X$  can be written as a directed limit  $X = \lim X_i$  of schemes over  $Y$  where  $X_i \rightarrow Y$  is a closed immersion of finite presentation.*

**Proof.** Let  $\mathcal{I} \subset \mathcal{O}_Y$  be the quasi-coherent sheaf of ideals defining  $X$  as a closed subscheme of  $Y$ . By Properties, Lemma 22.3 we can write  $\mathcal{I}$  as a directed colimit  $\mathcal{I} = \operatorname{colim}_{i \in I} \mathcal{I}_i$  of its quasi-coherent sheaves of ideals of finite type. Let  $X_i \subset Y$  be the closed subscheme defined by  $\mathcal{I}_i$ . These form an inverse system of schemes indexed by  $I$ . The transition morphisms  $X_i \rightarrow X_{i'}$  are affine because they are closed immersions. Each  $X_i$  is quasi-compact and quasi-separated since it is a closed subscheme of  $Y$  and  $Y$  is quasi-compact and quasi-separated by our assumptions. We have  $X = \lim_i X_i$  as follows directly from the fact that  $\mathcal{I} = \operatorname{colim}_{i \in I} \mathcal{I}_i$ . Each of the morphisms  $X_i \rightarrow Y$  is of finite presentation, see Morphisms, Lemma 21.7.  $\square$

**Lemma 9.5.** *Let  $f : X \rightarrow S$  be a morphism of schemes. Assume*

- (1) *The morphism  $f$  is of locally of finite type.*
- (2) *The scheme  $X$  is quasi-compact and quasi-separated, and*
- (3) *The scheme  $S$  is quasi-separated.*

*Then  $X = \lim X_i$  where the  $X_i \rightarrow S$  are of finite presentation, the  $X_i$  are quasi-compact and quasi-separated, and the transition morphisms  $X_{i'} \rightarrow X_i$  are closed immersions (which implies that  $X \rightarrow X_i$  are closed immersions for all  $i$ ).*

**Proof.** By Lemma 9.3 there is a closed immersion  $X \rightarrow Y$  with  $Y \rightarrow S$  of finite presentation. Then  $Y$  is quasi-separated by Schemes, Lemma 21.12. Since  $X$  is quasi-compact, we may assume  $Y$  is quasi-compact by replacing  $Y$  with a quasi-compact open containing  $X$ . We see that  $X = \lim X_i$  with  $X_i \rightarrow Y$  a closed immersion of finite presentation by Lemma 9.4. The morphisms  $X_i \rightarrow S$  are of finite presentation by Morphisms, Lemma 21.3.  $\square$

**Proposition 9.6.** *Let  $f : X \rightarrow S$  be a morphism of schemes. Assume*

- (1)  *$f$  is of finite type and separated, and*
- (2)  *$S$  is quasi-compact and quasi-separated.*

*Then there exists a separated morphism of finite presentation  $f' : X' \rightarrow S$  and a closed immersion  $X \rightarrow X'$  of schemes over  $S$ .*

**Proof.** Apply Lemma 9.5 and note that  $X_i \rightarrow S$  is separated for large  $i$  by Lemma 4.17 as we have assumed that  $X \rightarrow S$  is separated.  $\square$

**Lemma 9.7.** *Let  $f : X \rightarrow S$  be a morphism of schemes. Assume*

- (1)  *$f$  is finite, and*
- (2)  *$S$  is quasi-compact and quasi-separated.*

*Then there exists a morphism which is finite and of finite presentation  $f' : X' \rightarrow S$  and a closed immersion  $X \rightarrow X'$  of schemes over  $S$ .*

**Proof.** We may write  $X = \lim X_i$  as in Lemma 9.5. Applying Lemma 4.19 we see that  $X_i \rightarrow S$  is finite for large enough  $i$ .  $\square$

**Lemma 9.8.** *Let  $f : X \rightarrow S$  be a morphism of schemes. Assume*



- (1)  $f$  is finite, and
- (2)  $S$  quasi-compact and quasi-separated.

Then  $X$  is a directed limit  $X = \lim X_i$  where the transition maps are closed immersions and the objects  $X_i$  are finite and of finite presentation over  $S$ .

**Proof.** We may write  $X = \lim X_i$  as in Lemma 9.5. Applying Lemma 4.19 we see that  $X_i \rightarrow S$  is finite for large enough  $i$ .  $\square$

## 10. Descending relative objects

The following lemma is typical of the type of results in this section. We write out the “standard” proof completely. It may be faster to convince yourself that the result is true than to read this proof.

**Lemma 10.1.** *Let  $I$  be a directed set. Let  $(S_i, f_{ii'})$  be an inverse system of schemes over  $I$ . Assume*

- (1) *the morphisms  $f_{ii'} : S_i \rightarrow S_{i'}$  are affine,*
- (2) *the schemes  $S_i$  are quasi-compact and quasi-separated.*

*Let  $S = \lim_i S_i$ . Then we have the following:*

- (1) *For any morphism of finite presentation  $X \rightarrow S$  there exists an index  $i \in I$  and a morphism of finite presentation  $X_i \rightarrow S_i$  such that  $X \cong X_{i,S}$  as schemes over  $S$ .*
- (2) *Given an index  $i \in I$ , schemes  $X_i, Y_i$  of finite presentation over  $S_i$ , and a morphism  $\varphi : X_{i,S} \rightarrow Y_{i,S}$  over  $S$ , there exists an index  $i' \geq i$  and a morphism  $\varphi_{i'} : X_{i,S_{i'}} \rightarrow Y_{i,S_{i'}}$  whose base change to  $S$  is  $\varphi$ .*
- (3) *Given an index  $i \in I$ , schemes  $X_i, Y_i$  of finite presentation over  $S_i$  and a pair of morphisms  $\varphi_i, \psi_i : X_i \rightarrow Y_i$  whose base changes  $\varphi_{i,S} = \psi_{i,S}$  are equal, there exists an index  $i' \geq i$  such that  $\varphi_{i,S_{i'}} = \psi_{i,S_{i'}}$ .*

*In other words, the category of schemes of finite presentation over  $S$  is the colimit over  $I$  of the categories of schemes of finite presentation over  $S_i$ .*

**Proof.** In case each of the schemes  $S_i$  is affine, and we consider only affine schemes of finite presentation over  $S_i$ , resp.  $S$  this lemma is equivalent to Algebra, Lemma 127.8. We claim that the affine case implies the lemma in general.

Let us prove (3). Suppose given an index  $i \in I$ , schemes  $X_i, Y_i$  of finite presentation over  $S_i$  and a pair of morphisms  $\varphi_i, \psi_i : X_i \rightarrow Y_i$ . Assume that the base changes are equal:  $\varphi_{i,S} = \psi_{i,S}$ . We will use the notation  $X_{i'} = X_{i,S_{i'}}$  and  $Y_{i'} = Y_{i,S_{i'}}$  for  $i' \geq i$ . We also set  $X = X_{i,S}$  and  $Y = Y_{i,S}$ . Note that according to Lemma 2.3 we have  $X = \lim_{i' \geq i} X_{i'}$  and similarly for  $Y$ . Additionally we denote  $\varphi_{i'}$  and  $\psi_{i'}$  (resp.  $\varphi$  and  $\psi$ ) the base change of  $\varphi_i$  and  $\psi_i$  to  $S_{i'}$  (resp.  $S$ ). So our assumption means that  $\varphi = \psi$ . Since  $Y_i$  and  $X_i$  are of finite presentation over  $S_i$ , and since  $S_i$  is quasi-compact and quasi-separated, also  $X_i$  and  $Y_i$  are quasi-compact and quasi-separated (see Morphisms, Lemma 21.10). Hence we may choose a finite affine open covering  $Y_i = \bigcup V_{j,i}$  such that each  $V_{j,i}$  maps into an affine open of  $S$ . As above, denote  $V_{j,i'}$  the inverse image of  $V_{j,i}$  in  $Y_{i'}$  and  $V_j$  the inverse image in  $Y$ . The immersions  $V_{j,i'} \rightarrow Y_{i'}$  are quasi-compact, and the inverse images  $U_{j,i'} = \varphi_i^{-1}(V_{j,i'})$  and  $U'_{j,i'} = \psi_i^{-1}(V_{j,i'})$  are quasi-compact opens of  $X_{i'}$ . By assumption the inverse images of  $V_j$  under  $\varphi$  and  $\psi$  in  $X$  are equal. Hence by Lemma 4.11 there exists an index  $i' \geq i$  such that  $U_{j,i'} = U'_{j,i'}$  in  $X_{i'}$ . Choose an finite affine open covering

$U_{j,i'} = U'_{j,i'} = \bigcup W_{j,k,i'}$  which induce coverings  $U_{j,i''} = U'_{j,i''} = \bigcup W_{j,k,i''}$  for all  $i'' \geq i'$ . By the affine case there exists an index  $i''$  such that  $\varphi_{i''}|_{W_{j,k,i''}} = \psi_{i''}|_{W_{j,k,i''}}$  for all  $j, k$ . Then  $i''$  is an index such that  $\varphi_{i''} = \psi_{i''}$  and (3) is proved.

Let us prove (2). Suppose given an index  $i \in I$ , schemes  $X_i, Y_i$  of finite presentation over  $S_i$  and a morphism  $\varphi : X_{i,S} \rightarrow Y_{i,S}$ . We will use the notation  $X_{i'} = X_{i,S_{i'}}$  and  $Y_{i'} = Y_{i,S_{i'}}$  for  $i' \geq i$ . We also set  $X = X_{i,S}$  and  $Y = Y_{i,S}$ . Note that according to Lemma 2.3 we have  $X = \lim_{i' \geq i} X_{i'}$  and similarly for  $Y$ . Since  $Y_i$  and  $X_i$  are of finite presentation over  $S_i$ , and since  $S_i$  is quasi-compact and quasi-separated, also  $X_i$  and  $Y_i$  are quasi-compact and quasi-separated (see Morphisms, Lemma 21.10). Hence we may choose a finite affine open covering  $Y_i = \bigcup V_{j,i}$  such that each  $V_{j,i}$  maps into an affine open of  $S$ . As above, denote  $V_{j,i'}$  the inverse image of  $V_{j,i}$  in  $Y_{i'}$  and  $V_j$  the inverse image in  $Y$ . The immersions  $V_j \rightarrow Y$  are quasi-compact, and the inverse images  $U_j = \varphi^{-1}(V_j)$  are quasi-compact opens of  $X$ . Hence by Lemma 4.11 there exists an index  $i' \geq i$  and quasi-compact opens  $U_{j,i'}$  of  $X_{i'}$  whose inverse image in  $X$  is  $U_j$ . Choose an finite affine open covering  $U_{j,i'} = \bigcup W_{j,k,i'}$  which induce affine open coverings  $U_{j,i''} = \bigcup W_{j,k,i''}$  for all  $i'' \geq i'$  and an affine open covering  $U_j = \bigcup W_{j,k}$ . By the affine case there exists an index  $i''$  and morphisms  $\varphi_{j,k,i''} : W_{j,k,i''} \rightarrow V_{j,i''}$  such that  $\varphi|_{W_{j,k}} = \varphi_{j,k,i''}|_{W_{j,k}}$  for all  $j, k$ . By part (3) proved above, there is a further index  $i''' \geq i''$  such that

$$\varphi_{j_1,k_1,i''}|_{W_{j_1,k_1,i''} \cap W_{j_2,k_2,i''}} = \varphi_{j_2,k_2,i''}|_{W_{j_1,k_1,i''} \cap W_{j_2,k_2,i''}}$$

for all  $j_1, j_2, k_1, k_2$ . Then  $i'''$  is an index such that there exists a morphism  $\varphi_{i'''} : X_{i'''} \rightarrow Y_{i'''}$  whose base change to  $S$  gives  $\varphi$ . Hence (2) holds.

Let us prove (1). Suppose given a scheme  $X$  of finite presentation over  $S$ . Since  $X$  is of finite presentation over  $S$ , and since  $S$  is quasi-compact and quasi-separated, also  $X$  is quasi-compact and quasi-separated (see Morphisms, Lemma 21.10). Choose a finite affine open covering  $X = \bigcup U_j$  such that each  $U_j$  maps into an affine open  $V_j \subset S$ . Denote  $U_{j_1 j_2} = U_{j_1} \cap U_{j_2}$  and  $U_{j_1 j_2 j_3} = U_{j_1} \cap U_{j_2} \cap U_{j_3}$ . By Lemmas 4.11 and 4.13 we can find an index  $i_1$  and affine opens  $V_{j,i_1} \subset S_{i_1}$  such that each  $V_j$  is the inverse of this in  $S$ . Let  $V_{j,i}$  be the inverse image of  $V_{j,i_1}$  in  $S_i$  for  $i \geq i_1$ . By the affine case we may find an index  $i_2 \geq i_1$  and affine schemes  $U_{j,i_2} \rightarrow V_{j,i_2}$  such that  $U_j = S \times_{S_{i_2}} U_{j,i_2}$  is the base change. Denote  $U_{j,i} = S_i \times_{S_{i_2}} U_{j,i_2}$  for  $i \geq i_2$ . By Lemma 4.11 there exists an index  $i_3 \geq i_2$  and open subschemes  $W_{j_1,j_2,i_3} \subset U_{j_1,i_3}$  whose base change to  $S$  is equal to  $U_{j_1 j_2}$ . Denote  $W_{j_1,j_2,i} = S_i \times_{S_{i_3}} W_{j_1,j_2,i_3}$  for  $i \geq i_3$ . By part (2) shown above there exists an index  $i_4 \geq i_3$  and morphisms  $\varphi_{j_1,j_2,i_4} : W_{j_1,j_2,i_4} \rightarrow W_{j_2,j_1,i_4}$  whose base change to  $S$  gives the identity morphism  $U_{j_1 j_2} = U_{j_2 j_1}$  for all  $j_1, j_2$ . For all  $i \geq i_4$  denote  $\varphi_{j_1,j_2,i} = \text{id}_S \times \varphi_{j_1,j_2,i_4}$  the base change. We claim that for some  $i_5 \geq i_4$  the system  $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$  forms a glueing datum as in Schemes, Section 14. In order to see this we have to verify that for  $i$  large enough we have

$$\varphi_{j_1,j_2,i}^{-1}(W_{j_1,j_2,i} \cap W_{j_1,j_3,i}) = W_{j_1,j_2,i} \cap W_{j_1,j_3,i}$$

and that for large enough  $i$  the cocycle condition holds. The first condition follows from Lemma 4.11 and the fact that  $U_{j_2 j_1 j_3} = U_{j_1 j_2 j_3}$ . The second from part (1) of the lemma proved above and the fact that the cocycle condition holds for the maps  $\text{id} : U_{j_1 j_2} \rightarrow U_{j_2 j_1}$ . Ok, so now we can use Schemes, Lemma 14.2 to glue the system  $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$  to get a scheme  $X_{i_5} \rightarrow S_{i_5}$ . By

construction the base change of  $X_{i_5}$  to  $S$  is formed by glueing the open affines  $U_j$  along the opens  $U_{j_1} \leftarrow U_{j_1 j_2} \rightarrow U_{j_2}$ . Hence  $S \times_{S_{i_5}} X_{i_5} \cong X$  as desired.  $\square$

**Lemma 10.2.** *Let  $I$  be a directed set. Let  $(S_i, f_{ii'})$  be an inverse system of schemes over  $I$ . Assume*

- (1) *all the morphisms  $f_{ii'} : S_i \rightarrow S_{i'}$  are affine,*
- (2) *all the schemes  $S_i$  are quasi-compact and quasi-separated.*

*Let  $S = \lim_i S_i$ . Then we have the following:*

- (1) *For any sheaf of  $\mathcal{O}_S$ -modules  $\mathcal{F}$  of finite presentation there exists an index  $i \in I$  and a sheaf of  $\mathcal{O}_{S_i}$ -modules of finite presentation  $\mathcal{F}_i$  such that  $\mathcal{F} \cong f_i^* \mathcal{F}_i$ .*
- (2) *Suppose given an index  $i \in I$ , sheaves of  $\mathcal{O}_{S_i}$ -modules  $\mathcal{F}_i, \mathcal{G}_i$  of finite presentation and a morphism  $\varphi : f_i^* \mathcal{F}_i \rightarrow f_i^* \mathcal{G}_i$  over  $S$ . Then there exists an index  $i' \geq i$  and a morphism  $\varphi_{i'} : f_{i'}^* \mathcal{F}_i \rightarrow f_{i'}^* \mathcal{G}_i$  whose base change to  $S$  is  $\varphi$ .*
- (3) *Suppose given an index  $i \in I$ , sheaves of  $\mathcal{O}_{S_i}$ -modules  $\mathcal{F}_i, \mathcal{G}_i$  of finite presentation and a pair of morphisms  $\varphi_i, \psi_i : \mathcal{F}_i \rightarrow \mathcal{G}_i$ . Assume that the base changes are equal:  $f_i^* \varphi_i = f_i^* \psi_i$ . Then there exists an index  $i' \geq i$  such that  $f_{i'}^* \varphi_i = f_{i'}^* \psi_i$ .*

*In other words, the category of modules of finite presentation over  $S$  is the colimit over  $I$  of the categories modules of finite presentation over  $S_i$ .*

**Proof.** We sketch two proofs, but we omit the details.

First proof. If  $S$  and  $S_i$  are affine schemes, then this lemma is equivalent to Algebra, Lemma 127.6. In the general case, use Zariski glueing to deduce it from the affine case.

Second proof. We use

- (1) there is an equivalence of categories between quasi-coherent  $\mathcal{O}_S$ -modules and vector bundles over  $S$ , see Constructions, Section 6, and
- (2) a vector bundle  $\mathbf{V}(\mathcal{F}) \rightarrow S$  is of finite presentation over  $S$  if and only if  $\mathcal{F}$  is an  $\mathcal{O}_S$ -module of finite presentation.

Having said this, we can use Lemma 10.1 to show that the category of vector bundles of finite presentation over  $S$  is the colimit over  $I$  of the categories of vector bundles over  $S_i$ .  $\square$

**Lemma 10.3.** *Let  $S = \lim S_i$  be the limit of a directed system of quasi-compact and quasi-separated schemes  $S_i$  with affine transition morphisms. Then*

- (1) *any finite locally free  $\mathcal{O}_S$ -module is the pullback of a finite locally free  $\mathcal{O}_{S_i}$ -module for some  $i$ ,*
- (2) *any invertible  $\mathcal{O}_S$ -module is the pullback of an invertible  $\mathcal{O}_{S_i}$ -module for some  $i$ , and*
- (3) *any finite type quasi-coherent ideal  $\mathcal{I} \subset \mathcal{O}_S$  is of the form  $\mathcal{I}_i \cdot \mathcal{O}_S$  for some  $i$  and some finite type quasi-coherent ideal  $\mathcal{I}_i \subset \mathcal{O}_{S_i}$ .*

**Proof.** Let  $\mathcal{E}$  be a finite locally free  $\mathcal{O}_S$ -module. Since finite locally free modules are of finite presentation we can find an  $i$  and an  $\mathcal{O}_{S_i}$ -module  $\mathcal{E}_i$  of finite presentation such that  $f_i^* \mathcal{E}_i \cong \mathcal{E}$ , see Lemma 10.2. After increasing  $i$  we may assume  $\mathcal{E}_i$  is a flat  $\mathcal{O}_{S_i}$ -module, see Algebra, Lemma 168.1. (Using this lemma is not necessary, but it is convenient.) Then  $\mathcal{E}_i$  is finite locally free by Algebra, Lemma 78.2.

If  $\mathcal{L}$  is an invertible  $\mathcal{O}_S$ -module, then by the above we can find an  $i$  and finite locally free  $\mathcal{O}_{S_i}$ -modules  $\mathcal{L}_i$  and  $\mathcal{N}_i$  pulling back to  $\mathcal{L}$  and  $\mathcal{L}^{\otimes -1}$ . After possibly increasing  $i$  we see that the map  $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \rightarrow \mathcal{O}_X$  descends to a map  $\mathcal{L}_i \otimes_{\mathcal{O}_{S_i}} \mathcal{N}_i \rightarrow \mathcal{O}_{S_i}$ . And after increasing  $i$  further, we may assume it is an isomorphism. It follows that  $\mathcal{L}_i$  is an invertible module (Modules, Lemma 25.2) and the proof of (2) is complete.

Given  $\mathcal{I}$  as in (3) we see that  $\mathcal{O}_S \rightarrow \mathcal{O}_S/\mathcal{I}$  is a map of finitely presented  $\mathcal{O}_S$ -modules. Hence by Lemma 10.2 this is the pullback of some map  $\mathcal{O}_{S_i} \rightarrow \mathcal{F}_i$  of finitely presented  $\mathcal{O}_{S_i}$ -modules. After increasing  $i$  we may assume this map is surjective (details omitted; hint: use Algebra, Lemma 127.5 on affine open cover). Then the kernel of  $\mathcal{O}_{S_i} \rightarrow \mathcal{F}_i$  is a finite type quasi-coherent ideal in  $\mathcal{O}_{S_i}$  whose pullback gives  $\mathcal{I}$ .  $\square$

**Lemma 10.4.** *With notation and assumptions as in Lemma 10.1. Let  $i \in I$ . Suppose that  $\varphi_i : X_i \rightarrow Y_i$  is a morphism of schemes of finite presentation over  $S_i$  and that  $\mathcal{F}_i$  is a quasi-coherent  $\mathcal{O}_{X_i}$ -module of finite presentation. If the pullback of  $\mathcal{F}_i$  to  $X_i \times_{S_i} S$  is flat over  $Y_i \times_{S_i} S$ , then there exists an index  $i' \geq i$  such that the pullback of  $\mathcal{F}_i$  to  $X_i \times_{S_i} S_{i'}$  is flat over  $Y_i \times_{S_i} S_{i'}$ .*

**Proof.** (This lemma is the analogue of Lemma 8.7 for modules.) For  $i' \geq i$  denote  $X_{i'} = S_{i'} \times_{S_i} X_i$ ,  $\mathcal{F}_{i'} = (X_{i'} \rightarrow X_i)^* \mathcal{F}_i$  and similarly for  $Y_{i'}$ . Denote  $\varphi_{i'}$  the base change of  $\varphi_i$  to  $S_{i'}$ . Also set  $X = S \times_{S_i} X_i$ ,  $Y = S \times_{S_i} Y_i$ ,  $\mathcal{F} = (X \rightarrow X_i)^* \mathcal{F}_i$  and  $\varphi$  the base change of  $\varphi_i$  to  $S$ . Let  $Y_i = \bigcup_{j=1, \dots, m} V_{j,i}$  be a finite affine open covering such that each  $V_{j,i}$  maps into some affine open of  $S_i$ . For each  $j = 1, \dots, m$  let  $\varphi_i^{-1}(V_{j,i}) = \bigcup_{k=1, \dots, m(j)} U_{k,j,i}$  be a finite affine open covering. For  $i' \geq i$  we denote  $V_{j,i'}$  the inverse image of  $V_{j,i}$  in  $Y_{i'}$  and  $U_{k,j,i'}$  the inverse image of  $U_{k,j,i}$  in  $X_{i'}$ . Similarly we have  $U_{k,j} \subset X$  and  $V_j \subset Y$ . Then  $U_{k,j} = \lim_{i' \geq i} U_{k,j,i'}$  and  $V_j = \lim_{i' \geq i} V_{j,i}$  (see Lemma 2.2). Since  $X_{i'} = \bigcup_{k,j} U_{k,j,i'}$  is a finite open covering it suffices to prove the lemma for each of the morphisms  $U_{k,j,i} \rightarrow V_{j,i}$  and the sheaf  $\mathcal{F}_i|_{U_{k,j,i}}$ . Hence we see that the lemma reduces to the case that  $X_i$  and  $Y_i$  are affine and map into an affine open of  $S_i$ , i.e., we may also assume that  $S$  is affine.

In the affine case we reduce to the following algebra result. Suppose that  $R = \text{colim}_{i \in I} R_i$ . For some  $i \in I$  suppose given a map  $A_i \rightarrow B_i$  of finitely presented  $R_i$ -algebras. Let  $N_i$  be a finitely presented  $B_i$ -module. Then, if  $R \otimes_{R_i} N_i$  is flat over  $R \otimes_{R_i} A_i$ , then for some  $i' \geq i$  the module  $R_{i'} \otimes_{R_i} N_i$  is flat over  $R_{i'} \otimes_{R_i} A_i$ . This is exactly the result proved in Algebra, Lemma 168.1 part (3).  $\square$

**Lemma 10.5.** *For a scheme  $T$  denote  $\mathcal{C}_T$  the full subcategory of schemes  $W$  over  $T$  such that  $W$  is quasi-compact and quasi-separated and such that the structure morphism  $W \rightarrow T$  is locally of finite presentation. Let  $S = \lim S_i$  be a directed limit of schemes with affine transition morphisms. Then there is an equivalence of categories*

$$\text{colim } \mathcal{C}_{S_i} \longrightarrow \mathcal{C}_S$$

*given by the base change functors.*

Warning: do not use this lemma if you do not understand the difference between this lemma and Lemma 10.1.

**Proof.** Fully faithfulness. Suppose we have  $i \in I$  and objects  $X_i, Y_i$  of  $\mathcal{C}_{S_i}$ . Denote  $X = X_i \times_{S_i} S$  and  $Y = Y_i \times_{S_i} S$ . Suppose given a morphism  $f : X \rightarrow Y$  over  $S$ . We can choose a finite affine open covering  $Y_i = V_{i,1} \cup \dots \cup V_{i,m}$  such that

$V_{i,j} \rightarrow Y_i \rightarrow S_i$  maps into an affine open  $W_{i,j}$  of  $S_i$ . Denote  $Y = V_1 \cup \dots \cup V_m$  the induced affine open covering of  $Y$ . Since  $f : X \rightarrow Y$  is quasi-compact (Schemes, Lemma 21.14) after increasing  $i$  we may assume that there is a finite open covering  $X_i = U_{i,1} \cup \dots \cup U_{i,m}$  by quasi-compact opens such that the inverse image of  $U_{i,j}$  in  $Y$  is  $f^{-1}(V_j)$ , see Lemma 4.11. By Lemma 10.1 applied to  $f|_{f^{-1}(V_j)}$  over  $W_j$  we may assume, after increasing  $i$ , that there is a morphism  $f_{i,j} : V_{i,j} \rightarrow U_{i,j}$  over  $S$  whose base change to  $S$  is  $f|_{f^{-1}(V_j)}$ . Increasing  $i$  more we may assume  $f_{i,j}$  and  $f_{i,j'}$  agree on the quasi-compact open  $U_{i,j} \cap U_{i,j'}$ . Then we can glue these morphisms to get the desired morphism  $f_i : X_i \rightarrow Y_i$ . This morphism is unique (up to increasing  $i$ ) because this is true for the morphisms  $f_{i,j}$ .

To show that the functor is essentially surjective we argue in exactly the same way. Namely, suppose that  $X$  is an object of  $\mathcal{C}_S$ . Pick  $i \in I$ . We can choose a finite affine open covering  $X = U_1 \cup \dots \cup U_m$  such that  $U_j \rightarrow X \rightarrow S \rightarrow S_i$  factors through an affine open  $W_{i,j} \subset S_i$ . Set  $W_j = W_{i,j} \times_{S_i} S$ . This is an affine open of  $S$ . By Lemma 10.1, after increasing  $i$ , we may assume there exist  $U_{i,j} \rightarrow W_{i,j}$  of finite presentation whose base change to  $W_j$  is  $U_j$ . After increasing  $i$  we may assume there exist quasi-compact opens  $U_{i,j,j'} \subset U_{i,j}$  whose base changes to  $S$  are equal to  $U_j \cap U_{j'}$ . Claim: after increasing  $i$  we may assume the image of the morphism  $U_{i,j,j'} \rightarrow U_{i,j} \rightarrow W_{i,j}$  ends up in  $W_{i,j} \cap W_{i,j'}$ . Namely, because the complement of  $W_{i,j} \cap W_{i,j'}$  is closed in the affine scheme  $W_{i,j}$  it is affine. Since  $U_j \cap U_{j'} = \lim U_{i,j,j'}$  does map into  $W_{i,j} \cap W_{i,j'}$  we can apply Lemma 4.9 to get the claim. Thus we can view both

$$U_{i,j,j'} \quad \text{and} \quad U_{i,j',j}$$

as schemes over  $W_{i,j'}$  whose base changes to  $W_{j'}$  recover  $U_j \cap U_{j'}$ . Hence after increasing  $i$ , using Lemma 10.1, we may assume there are isomorphisms  $U_{i,j,j'} \rightarrow U_{i,j',j}$  over  $W_{i,j'}$  and hence over  $S_i$ . Increasing  $i$  further (details omitted) we may assume these isomorphisms satisfy the cocycle condition mentioned in Schemes, Section 14. Applying Schemes, Lemma 14.1 we obtain an object  $X_i$  of  $\mathcal{C}_{S_i}$  whose base change to  $S$  is isomorphic to  $X$ ; we omit some of the verifications.  $\square$

## 11. Characterizing affine schemes

If  $f : X \rightarrow S$  is a surjective integral morphism of schemes such that  $X$  is an affine scheme then  $S$  is affine too. See [Con07, A.2]. Our proof relies on the Noetherian case which we stated and proved in Cohomology of Schemes, Lemma 13.3. See also [DG67, II 6.7.1].

**Lemma 11.1.** *Let  $f : X \rightarrow S$  be a morphism of schemes. Assume that  $f$  is surjective and finite, and assume that  $X$  is affine. Then  $S$  is affine.*

**Proof.** Since  $f$  is surjective and  $X$  is quasi-compact we see that  $S$  is quasi-compact. Since  $X$  is separated and  $f$  is surjective and universally closed (Morphisms, Lemma 44.7), we see that  $S$  is separated (Morphisms, Lemma 41.11).

By Lemma 9.8 we can write  $X = \lim_a X_a$  with  $X_a \rightarrow S$  finite and of finite presentation. By Lemma 4.13 we see that  $X_a$  is affine for some  $a \in A$ . Replacing  $X$  by  $X_a$  we may assume that  $X \rightarrow S$  is surjective, finite, of finite presentation and that  $X$  is affine.

By Proposition 5.4 we may write  $S = \lim_{i \in I} S_i$  as a directed limits of schemes of finite type over  $\mathbf{Z}$ . By Lemma 10.1 we can after shrinking  $I$  assume there exist

schemes  $X_i \rightarrow S_i$  of finite presentation such that  $X_{i'} = X_i \times_S S_{i'}$  for  $i' \geq i$  and such that  $X = \lim_i X_i$ . By Lemma 8.3 we may assume that  $X_i \rightarrow S_i$  is finite for all  $i \in I$  as well. By Lemma 4.13 once again we may assume that  $X_i$  is affine for all  $i \in I$ . Hence the result follows from the Noetherian case, see Cohomology of Schemes, Lemma 13.3.  $\square$

**Proposition 11.2.** *Let  $f : X \rightarrow S$  be a morphism of schemes. Assume  $X$  is affine and that  $f$  is surjective and universally closed<sup>2</sup>. Then  $S$  is affine.*

**Proof.** By Morphisms, Lemma 41.11 the scheme  $S$  is separated. Then by Morphisms, Lemma 11.11 we find that  $f$  is affine. Whereupon by Morphisms, Lemma 44.7 we see that  $f$  is integral.

By the preceding paragraph, we may assume  $f : X \rightarrow S$  is surjective and integral,  $X$  is affine, and  $S$  is separated. Since  $f$  is surjective and  $X$  is quasi-compact we also deduce that  $S$  is quasi-compact.

By Lemma 7.3 we can write  $X = \lim_i X_i$  with  $X_i \rightarrow S$  finite. By Lemma 4.13 we see that for  $i$  sufficiently large the scheme  $X_i$  is affine. Moreover, since  $X \rightarrow S$  factors through each  $X_i$  we see that  $X_i \rightarrow S$  is surjective. Hence we conclude that  $S$  is affine by Lemma 11.1.  $\square$

**Lemma 11.3.** *Let  $X$  be a scheme which is set theoretically the union of finitely many affine closed subschemes. Then  $X$  is affine.*

**Proof.** Let  $Z_i \subset X$ ,  $i = 1, \dots, n$  be affine closed subschemes such that  $X = \bigcup Z_i$  set theoretically. Then  $\coprod Z_i \rightarrow X$  is surjective and integral with affine source. Hence  $X$  is affine by Proposition 11.2.  $\square$

**Lemma 11.4.** *Let  $i : Z \rightarrow X$  be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Let  $\mathcal{L}$  be an invertible sheaf on  $X$ . Then  $i^*\mathcal{L}$  is ample on  $Z$ , if and only if  $\mathcal{L}$  is ample on  $X$ .*

**Proof.** If  $\mathcal{L}$  is ample, then  $i^*\mathcal{L}$  is ample for example by Morphisms, Lemma 37.7. Assume  $i^*\mathcal{L}$  is ample. Then  $Z$  is quasi-compact (Properties, Definition 26.1) and separated (Properties, Lemma 26.8). Since  $i$  is surjective, we see that  $X$  is quasi-compact. Since  $i$  is universally closed and surjective, we see that  $X$  is separated (Morphisms, Lemma 41.11).

By Proposition 5.4 we can write  $X = \lim X_i$  as a directed limit of finite type schemes over  $\mathbf{Z}$  with affine transition morphisms. We can find an  $i$  and an invertible sheaf  $\mathcal{L}_i$  on  $X_i$  whose pullback to  $X$  is isomorphic to  $\mathcal{L}$ , see Lemma 10.2.

For each  $i$  let  $Z_i \subset X_i$  be the scheme theoretic image of the morphism  $Z \rightarrow X_i$ . If  $\text{Spec}(A_i) \subset X_i$  is an affine open subscheme with inverse image of  $\text{Spec}(A)$  in  $X$  and if  $Z \cap \text{Spec}(A)$  is defined by the ideal  $I \subset A$ , then  $Z_i \cap \text{Spec}(A_i)$  is defined by the ideal  $I_i \subset A_i$  which is the inverse image of  $I$  in  $A_i$  under the ring map  $A_i \rightarrow A$ , see Morphisms, Example 6.4. Since  $\text{colim } A_i/I_i = A/I$  it follows that  $\lim Z_i = Z$ . By Lemma 4.15 we see that  $\mathcal{L}_i|_{Z_i}$  is ample for some  $i$ . Since  $Z$  and hence  $X$  maps into  $Z_i$  set theoretically, we see that  $X_{i'} \rightarrow X_i$  maps into  $Z_i$  set theoretically for some  $i' \geq i$ , see Lemma 4.10. (Observe that since  $X_i$  is Noetherian, every closed subset of  $X_i$  is constructible.) Let  $T \subset X_{i'}$  be the scheme theoretic inverse image of  $Z_i$  in  $X_{i'}$ . Observe that  $\mathcal{L}_{i'}|_T$  is the pullback of  $\mathcal{L}_i|_{Z_i}$  and hence ample by Morphisms,

<sup>2</sup>An integral morphism is universally closed, see Morphisms, Lemma 44.7.

Lemma 37.7 and the fact that  $T \rightarrow Z_i$  is an affine morphism. Thus we see that  $\mathcal{L}_{i'}$  is ample on  $X_{i'}$  by Cohomology of Schemes, Lemma 17.5. Pulling back to  $X$  (using the same lemma as above) we find that  $\mathcal{L}$  is ample.  $\square$

**Lemma 11.5.** *Let  $i : Z \rightarrow X$  be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Then  $X$  is quasi-affine if and only if  $Z$  is quasi-affine.*

**Proof.** Recall that a scheme is quasi-affine if and only if the structure sheaf is ample, see Properties, Lemma 27.1. Hence if  $Z$  is quasi-affine, then  $\mathcal{O}_Z$  is ample, hence  $\mathcal{O}_X$  is ample by Lemma 11.4, hence  $X$  is quasi-affine. A proof of the converse, which can also be seen in an elementary way, is gotten by reading the argument just given backwards.  $\square$

The following lemma does not really belong in this section.

**Lemma 11.6.** *Let  $X$  be a scheme. Let  $\mathcal{L}$  be an ample invertible sheaf on  $X$ . Assume we have morphisms of schemes*

$$\mathrm{Spec}(k) \leftarrow \mathrm{Spec}(A) \rightarrow W \subset X$$

*where  $k$  is a field,  $A$  is an integral  $k$ -algebra,  $W$  is open in  $X$ . Then there exists an  $n > 0$  and a section  $s \in \Gamma(X, \mathcal{L}^{\otimes n})$  such that  $X_s$  is affine,  $X_s \subset W$ , and  $\mathrm{Spec}(A) \rightarrow W$  factors through  $X_s$*

**Proof.** Since  $\mathrm{Spec}(A)$  is quasi-compact, we may replace  $W$  by a quasi-compact open still containing the image of  $\mathrm{Spec}(A) \rightarrow X$ . Recall that  $X$  is quasi-separated and quasi-compact by dint of having an ample invertible sheaf, see Properties, Definition 26.1 and Lemma 26.7. By Proposition 5.4 we can write  $X = \lim X_i$  as a limit of a directed system of schemes of finite type over  $\mathbf{Z}$  with affine transition morphisms. For some  $i$  the ample invertible sheaf  $\mathcal{L}$  on  $X$  descends to an ample invertible sheaf  $\mathcal{L}_i$  on  $X_i$  and the open  $W$  is the inverse image of a quasi-compact open  $W_i \subset X_i$ , see Lemmas 4.15, 10.3, and 4.11. We may replace  $X, W, \mathcal{L}$  by  $X_i, W_i, \mathcal{L}_i$  and assume  $X$  is of finite presentation over  $\mathbf{Z}$ . Write  $A = \mathrm{colim} A_j$  as the colimit of its finite  $k$ -subalgebras. Then for some  $j$  the morphism  $\mathrm{Spec}(A) \rightarrow X$  factors through a morphism  $\mathrm{Spec}(A_j) \rightarrow X$ , see Proposition 6.1. Since  $\mathrm{Spec}(A_j)$  is finite this reduces the lemma to Properties, Lemma 29.6.  $\square$

## 12. Variants of Chow's Lemma

In this section we prove a number of variants of Chow's lemma. The most interesting version is probably just the Noetherian case, which we stated and proved in Cohomology of Schemes, Section 18.

**Lemma 12.1.** *Let  $S$  be a quasi-compact and quasi-separated scheme. Let  $f : X \rightarrow S$  be a separated morphism of finite type. Then there exists an  $n \geq 0$  and a diagram*

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \longrightarrow & \mathbf{P}_S^n \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

*where  $X' \rightarrow \mathbf{P}_S^n$  is an immersion, and  $\pi : X' \rightarrow X$  is proper and surjective.*

**Proof.** By Proposition 9.6 we can find a closed immersion  $X \rightarrow Y$  where  $Y$  is separated and of finite presentation over  $S$ . Clearly, if we prove the assertion for  $Y$ , then the result follows for  $X$ . Hence we may assume that  $X$  is of finite presentation over  $S$ .

Write  $S = \lim_i S_i$  as a directed limit of Noetherian schemes, see Proposition 5.4. By Lemma 10.1 we can find an index  $i \in I$  and a scheme  $X_i \rightarrow S_i$  of finite presentation so that  $X = S \times_{S_i} X_i$ . By Lemma 8.6 we may assume that  $X_i \rightarrow S_i$  is separated. Clearly, if we prove the assertion for  $X_i$  over  $S_i$ , then the assertion holds for  $X$ . The case  $X_i \rightarrow S_i$  is treated by Cohomology of Schemes, Lemma 18.1.  $\square$

**Remark 12.2.** In the situation of Chow's Lemma 12.1:

- (1) The morphism  $\pi$  is actually H-projective (hence projective, see Morphisms, Lemma 43.3) since the morphism  $X' \rightarrow \mathbf{P}_S^n \times_S X = \mathbf{P}_X^n$  is a closed immersion (use the fact that  $\pi$  is proper, see Morphisms, Lemma 41.7).
- (2) We may assume that  $X'$  is reduced as we can replace  $X'$  by its reduction without changing the other assertions of the lemma.
- (3) We may assume that  $X' \rightarrow X$  is of finite presentation without changing the other assertions of the lemma. This can be deduced from the proof of Lemma 12.1 but we can also prove this directly as follows. By (1) we have a closed immersion  $X' \rightarrow \mathbf{P}_X^n$ . By Lemma 9.4 we can write  $X' = \lim X'_i$  where  $X'_i \rightarrow \mathbf{P}_X^n$  is a closed immersion of finite presentation. In particular  $X'_i \rightarrow X$  is of finite presentation, proper, and surjective. For large enough  $i$  the morphism  $X'_i \rightarrow \mathbf{P}_S^n$  is an immersion by Lemma 4.16. Replacing  $X'$  by  $X'_i$  we get what we want.

Of course in general we can't simultaneously achieve both (2) and (3).

Here is a variant of Chow's lemma where we assume the scheme on top has finitely many irreducible components.

**Lemma 12.3.** *Let  $S$  be a quasi-compact and quasi-separated scheme. Let  $f : X \rightarrow S$  be a separated morphism of finite type. Assume that  $X$  has finitely many irreducible components. Then there exists an  $n \geq 0$  and a diagram*

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \longrightarrow & \mathbf{P}_S^n \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

where  $X' \rightarrow \mathbf{P}_S^n$  is an immersion, and  $\pi : X' \rightarrow X$  is proper and surjective. Moreover, there exists an open dense subscheme  $U \subset X$  such that  $\pi^{-1}(U) \rightarrow U$  is an isomorphism of schemes.

**Proof.** Let  $X = Z_1 \cup \dots \cup Z_n$  be the decomposition of  $X$  into irreducible components. Let  $\eta_j \in Z_j$  be the generic point.

There are (at least) two ways to proceed with the proof. The first is to redo the proof of Cohomology of Schemes, Lemma 18.1 using the general Properties, Lemma 29.4 to find suitable affine opens in  $X$ . (This is the "standard" proof.) The second is to use absolute Noetherian approximation as in the proof of Lemma 12.1 above. This is what we will do here.



By Proposition 9.6 we can find a closed immersion  $X \rightarrow Y$  where  $Y$  is separated and of finite presentation over  $S$ . Write  $S = \lim_i S_i$  as a directed limit of Noetherian schemes, see Proposition 5.4. By Lemma 10.1 we can find an index  $i \in I$  and a scheme  $Y_i \rightarrow S_i$  of finite presentation so that  $Y = S \times_{S_i} Y_i$ . By Lemma 8.6 we may assume that  $Y_i \rightarrow S_i$  is separated. We have the following diagram

$$\begin{array}{ccccccc} \eta_j \in Z_j & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Y_i \\ & & & \searrow & \downarrow & & \downarrow \\ & & & & S & \longrightarrow & S_i \end{array}$$

Denote  $h : X \rightarrow Y_i$  the composition.

For  $i' \geq i$  write  $Y_{i'} = S_{i'} \times_{S_i} Y_i$ . Then  $Y = \lim_{i' \geq i} Y_{i'}$ , see Lemma 2.3. Choose  $j, j' \in \{1, \dots, n\}$ ,  $j \neq j'$ . Note that  $\eta_j$  is not a specialization of  $\eta_{j'}$ . By Lemma 4.6 we can replace  $i$  by a bigger index and assume that  $h(\eta_j)$  is not a specialization of  $h(\eta_{j'})$  for all pairs  $(j, j')$  as above. For such an index, let  $Y' \subset Y_i$  be the scheme theoretic image of  $h : X \rightarrow Y_i$ , see Morphisms, Definition 6.2. The morphism  $h$  is quasi-compact as the composition of the quasi-compact morphisms  $X \rightarrow Y$  and  $Y \rightarrow Y_i$  (which is affine). Hence by Morphisms, Lemma 6.3 the morphism  $X \rightarrow Y'$  is dominant. Thus the generic points of  $Y'$  are all contained in the set  $\{h(\eta_1), \dots, h(\eta_n)\}$ , see Morphisms, Lemma 8.3. Since none of the  $h(\eta_j)$  is the specialization of another we see that the points  $h(\eta_1), \dots, h(\eta_n)$  are pairwise distinct and are each a generic point of  $Y'$ .

We apply Cohomology of Schemes, Lemma 18.1 above to the morphism  $Y' \rightarrow S_i$ . This gives a diagram

$$\begin{array}{ccccc} Y' & \xleftarrow{\pi} & Y^* & \longrightarrow & \mathbf{P}_{S_i}^n \\ & \searrow & \downarrow & \swarrow & \\ & & S_i & & \end{array}$$

such that  $\pi$  is proper and surjective and an isomorphism over a dense open subscheme  $V \subset Y'$ . By our choice of  $i$  above we know that  $h(\eta_1), \dots, h(\eta_n) \in V$ . Consider the commutative diagram

$$\begin{array}{ccccccc} X' & \xlongequal{\quad} & X \times_{Y'} Y^* & \longrightarrow & Y^* & \longrightarrow & \mathbf{P}_{S_i}^n \\ & & \downarrow & & \downarrow & & \swarrow \\ & & X & \longrightarrow & Y' & & \\ & & \downarrow & & \downarrow & & \\ & & S & \longrightarrow & S_i & & \end{array}$$

Note that  $X' \rightarrow X$  is an isomorphism over the open subscheme  $U = h^{-1}(V)$  which contains each of the  $\eta_j$  and hence is dense in  $X$ . We conclude  $X \leftarrow X' \rightarrow \mathbf{P}_S^n$  is a solution to the problem posed in the lemma.  $\square$

### 13. Applications of Chow's lemma

Here is a first application of Chow's lemma.

**Lemma 13.1.** *Assumptions and notation as in Situation 8.1. If*

- (1)  $f$  is proper, and
- (2)  $f_0$  is locally of finite type,

then there exists an  $i$  such that  $f_i$  is proper.

**Proof.** By Lemma 8.6 we see that  $f_i$  is separated for some  $i \geq 0$ . Replacing 0 by  $i$  we may assume that  $f_0$  is separated. Observe that  $f_0$  is quasi-compact, see Schemes, Lemma 21.14. By Lemma 12.1 we can choose a diagram

$$\begin{array}{ccccc} X_0 & \xleftarrow{\pi} & X'_0 & \longrightarrow & \mathbf{P}_{Y_0}^n \\ & \searrow & \downarrow & \swarrow & \\ & & Y_0 & & \end{array}$$

where  $X'_0 \rightarrow \mathbf{P}_{Y_0}^n$  is an immersion, and  $\pi : X'_0 \rightarrow X_0$  is proper and surjective. Introduce  $X' = X'_0 \times_{Y_0} Y$  and  $X'_i = X'_0 \times_{Y_0} Y_i$ . By Morphisms, Lemmas 41.4 and 41.5 we see that  $X' \rightarrow Y$  is proper. Hence  $X' \rightarrow \mathbf{P}_Y^n$  is a closed immersion (Morphisms, Lemma 41.7). By Morphisms, Lemma 41.9 it suffices to prove that  $X'_i \rightarrow Y_i$  is proper for some  $i$ . By Lemma 8.5 we find that  $X'_i \rightarrow \mathbf{P}_{Y_i}^n$  is a closed immersion for  $i$  large enough. Then  $X'_i \rightarrow Y_i$  is proper and we win.  $\square$

**Lemma 13.2.** *Let  $f : X \rightarrow S$  be a proper morphism with  $S$  quasi-compact and quasi-separated. Then  $X = \lim X_i$  is a directed limit of schemes  $X_i$  proper and of finite presentation over  $S$  such that all transition morphisms and the morphisms  $X \rightarrow X_i$  are closed immersions.*

**Proof.** By Proposition 9.6 we can find a closed immersion  $X \rightarrow Y$  with  $Y$  separated and of finite presentation over  $S$ . By Lemma 12.1 we can find a diagram

$$\begin{array}{ccccc} Y & \xleftarrow{\pi} & Y' & \longrightarrow & \mathbf{P}_S^n \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

where  $Y' \rightarrow \mathbf{P}_S^n$  is an immersion, and  $\pi : Y' \rightarrow Y$  is proper and surjective. By Lemma 9.4 we can write  $X = \lim X_i$  with  $X_i \rightarrow Y$  a closed immersion of finite presentation. Denote  $X'_i \subset Y'$ , resp.  $X' \subset Y'$  the scheme theoretic inverse image of  $X_i \subset Y$ , resp.  $X \subset Y$ . Then  $\lim X'_i = X'$ . Since  $X' \rightarrow S$  is proper (Morphisms, Lemmas 41.4), we see that  $X' \rightarrow \mathbf{P}_S^n$  is a closed immersion (Morphisms, Lemma 41.7). Hence for  $i$  large enough we find that  $X'_i \rightarrow \mathbf{P}_S^n$  is a closed immersion by Lemma 4.20. Thus  $X'_i$  is proper over  $S$ . For such  $i$  the morphism  $X_i \rightarrow S$  is proper by Morphisms, Lemma 41.9.  $\square$

**Lemma 13.3.** *Let  $f : X \rightarrow S$  be a proper morphism with  $S$  quasi-compact and quasi-separated. Then there exists a directed set  $I$ , an inverse system  $(f_i : X_i \rightarrow S_i)$  of morphisms of schemes over  $I$ , such that the transition morphisms  $X_i \rightarrow X_{i'}$  and  $S_i \rightarrow S_{i'}$  are affine, such that  $f_i$  is proper, such that  $S_i$  is of finite type over  $\mathbf{Z}$ , and such that  $(X \rightarrow S) = \lim(X_i \rightarrow S_i)$ .*

**Proof.** By Lemma 13.2 we can write  $X = \lim_{k \in K} X_k$  with  $X_k \rightarrow S$  proper and of finite presentation. Next, by absolute Noetherian approximation (Proposition 5.4) we can write  $S = \lim_{j \in J} S_j$  with  $S_j$  of finite type over  $\mathbf{Z}$ . For each  $k$  there exists a  $j$  and a morphism  $X_{k,j} \rightarrow S_j$  of finite presentation with  $X_k \cong S \times_{S_j} X_{k,j}$  as schemes over  $S$ , see Lemma 10.1. After increasing  $j$  we may assume  $X_{k,j} \rightarrow S_j$

is proper, see Lemma 13.1. The set  $I$  will consist of these pairs  $(k, j)$  and the corresponding morphism is  $X_{k,j} \rightarrow S_j$ . For every  $k' \geq k$  we can find a  $j' \geq j$  and a morphism  $X_{j',k'} \rightarrow X_{j,k}$  over  $S_{j'} \rightarrow S_j$  whose base change to  $S$  gives the morphism  $X_{k'} \rightarrow X_k$  (follows again from Lemma 10.1). These morphisms form the transition morphisms of the system. Some details omitted.  $\square$

**Lemma 13.4.** *Let  $S$  be a scheme. Let  $X = \lim X_i$  be a directed limit of schemes over  $S$  with affine transition morphisms. Let  $Y \rightarrow X$  be a morphism of schemes over  $S$ . If  $Y \rightarrow X$  is proper,  $X_i$  quasi-compact and quasi-separated, and  $Y$  locally of finite type over  $S$ , then  $Y \rightarrow X_i$  is proper for  $i$  large enough.*

**Proof.** Choose a closed immersion  $Y \rightarrow Y'$  with  $Y'$  proper and of finite presentation over  $X$ , see Lemma 13.2. Then choose an  $i$  and a proper morphism  $Y'_i \rightarrow X_i$  such that  $Y' = X \times_{X_i} Y'_i$ . This is possible by Lemmas 10.1 and 13.1. Then after replacing  $i$  by a larger index we have that  $Y \rightarrow Y'_i$  is a closed immersion, see Lemma 4.16.  $\square$

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms, Definition 5.5.

**Lemma 13.5.** *Assumptions and notation as in Situation 8.1. Let  $\mathcal{F}_0$  be a quasi-coherent  $\mathcal{O}_{X_0}$ -module. Denote  $\mathcal{F}$  and  $\mathcal{F}_i$  the pullbacks of  $\mathcal{F}_0$  to  $X$  and  $X_i$ . Assume*

- (1)  $f_0$  is locally of finite type,
- (2)  $\mathcal{F}_0$  is of finite type,
- (3) the scheme theoretic support of  $\mathcal{F}$  is proper over  $Y$ .

*Then the scheme theoretic support of  $\mathcal{F}_i$  is proper over  $Y_i$  for some  $i$ .*

**Proof.** We may replace  $X_0$  by the scheme theoretic support of  $\mathcal{F}_0$ . By Morphisms, Lemma 5.3 this guarantees that  $X_i$  is the support of  $\mathcal{F}_i$  and  $X$  is the support of  $\mathcal{F}$ . Then, if  $Z \subset X$  denotes the scheme theoretic support of  $\mathcal{F}$ , we see that  $Z \rightarrow X$  is a universal homeomorphism. We conclude that  $X \rightarrow Y$  is proper as this is true for  $Z \rightarrow Y$  by assumption, see Morphisms, Lemma 41.9. By Lemma 13.1 we see that  $X_i \rightarrow Y$  is proper for some  $i$ . Then it follows that the scheme theoretic support  $Z_i$  of  $\mathcal{F}_i$  is proper over  $Y$  by Morphisms, Lemmas 41.6 and 41.4.  $\square$

## 14. Universally closed morphisms

In this section we discuss when a quasi-compact (but not necessarily separated) morphism is universally closed. We first prove a lemma which will allow us to check universal closedness after a base change which is locally of finite presentation.

**Lemma 14.1.** *Let  $f : X \rightarrow S$  be a quasi-compact morphism of schemes. Let  $g : T \rightarrow S$  be a morphism of schemes. Let  $t \in T$  be a point and  $Z \subset X_t$  be a closed subscheme such that  $Z \cap X_t = \emptyset$ . Then there exists an open neighbourhood  $V \subset T$  of  $t$ , a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

*and a closed subscheme  $Z' \subset X_{T'}$  such that*

- (1) the morphism  $b : T' \rightarrow S$  is locally of finite presentation,

- (2) with  $t' = a(t)$  we have  $Z' \cap X_{t'} = \emptyset$ , and
- (3)  $Z \cap X_V$  maps into  $Z'$  via the morphism  $X_V \rightarrow X_{T'}$ .

Moreover, we may assume  $V$  and  $T'$  are affine.

**Proof.** Let  $s = g(t)$ . During the proof we may always replace  $T$  by an open neighbourhood of  $t$ . Hence we may also replace  $S$  by an open neighbourhood of  $s$ . Thus we may and do assume that  $T$  and  $S$  are affine. Say  $S = \text{Spec}(A)$ ,  $T = \text{Spec}(B)$ ,  $g$  is given by the ring map  $A \rightarrow B$ , and  $t$  correspond to the prime ideal  $\mathfrak{q} \subset B$ .

As  $X \rightarrow S$  is quasi-compact and  $S$  is affine we may write  $X = \bigcup_{i=1, \dots, n} U_i$  as a finite union of affine opens. Write  $U_i = \text{Spec}(C_i)$ . In particular we have  $X_T = \bigcup_{i=1, \dots, n} U_{i,T} = \bigcup_{i=1, \dots, n} \text{Spec}(C_i \otimes_A B)$ . Let  $I_i \subset C_i \otimes_A B$  be the ideal corresponding to the closed subscheme  $Z \cap U_{i,T}$ . The condition that  $Z \cap X_t = \emptyset$  signifies that  $I_i$  generates the unit ideal in the ring

$$C_i \otimes_A \kappa(\mathfrak{q}) = (B \setminus \mathfrak{q})^{-1} (C_i \otimes_A B / \mathfrak{q} C_i \otimes_A B)$$

Since  $I_i(B \setminus \mathfrak{q})^{-1} (C_i \otimes_A B) = (B \setminus \mathfrak{q})^{-1} I_i$  this means that  $1 = x_i/g_i$  for some  $x_i \in I_i$  and  $g_i \in B$ ,  $g_i \notin \mathfrak{q}$ . Thus, clearing denominators we can find a relation of the form

$$x_i + \sum_j f_{i,j} c_{i,j} = g_i$$

with  $x_i \in I_i$ ,  $f_{i,j} \in \mathfrak{q}$ ,  $c_{i,j} \in C_i \otimes_A B$ , and  $g_i \in B$ ,  $g_i \notin \mathfrak{q}$ . After replacing  $B$  by  $B_{g_1 \dots g_n}$ , i.e., after replacing  $T$  by a smaller affine neighbourhood of  $t$ , we may assume the equations read

$$x_i + \sum_j f_{i,j} c_{i,j} = 1$$

with  $x_i \in I_i$ ,  $f_{i,j} \in \mathfrak{q}$ ,  $c_{i,j} \in C_i \otimes_A B$ .

To finish the argument write  $B$  as a colimit of finitely presented  $A$ -algebras  $B_\lambda$  over a directed set  $\Lambda$ . For each  $\lambda$  set  $\mathfrak{q}_\lambda = (B_\lambda \rightarrow B)^{-1}(\mathfrak{q})$ . For sufficiently large  $\lambda \in \Lambda$  we can find

- (1) an element  $x_{i,\lambda} \in C_i \otimes_A B_\lambda$  which maps to  $x_i$ ,
- (2) elements  $f_{i,j,\lambda} \in \mathfrak{q}_{i,\lambda}$  mapping to  $f_{i,j}$ , and
- (3) elements  $c_{i,j,\lambda} \in C_i \otimes_A B_\lambda$  mapping to  $c_{i,j}$ .

After increasing  $\lambda$  a bit more the equation

$$x_{i,\lambda} + \sum_j f_{i,j,\lambda} c_{i,j,\lambda} = 1$$

will hold. Fix such a  $\lambda$  and set  $T' = \text{Spec}(B_\lambda)$ . Then  $t' \in T'$  is the point corresponding to the prime  $\mathfrak{q}_\lambda$ . Finally, let  $Z' \subset X_{T'}$  be the scheme theoretic image of  $Z \rightarrow X_T \rightarrow X_{T'}$ . As  $X_T \rightarrow X_{T'}$  is affine, we can compute  $Z'$  on the affine open pieces  $U_{i,T'}$  as the closed subscheme associated to  $\text{Ker}(C_i \otimes_A B_\lambda \rightarrow C_i \otimes_A B / I_i)$ , see Morphisms, Example 6.4. Hence  $x_{i,\lambda}$  is in the ideal defining  $Z'$ . Thus the last displayed equation shows that  $Z' \cap X_{t'}$  is empty.  $\square$

**Lemma 14.2.** *Let  $f : X \rightarrow S$  be a quasi-compact morphism of schemes. The following are equivalent*

- (1)  $f$  is universally closed,
- (2) for every morphism  $S' \rightarrow S$  which is locally of finite presentation the base change  $X_{S'} \rightarrow S'$  is closed, and
- (3) for every  $n$  the morphism  $\mathbf{A}^n \times X \rightarrow \mathbf{A}^n \times S$  is closed.

**Proof.** It is clear that (1) implies (2). Let us prove that (2) implies (1). Suppose that the base change  $X_T \rightarrow T$  is not closed for some scheme  $T$  over  $S$ . By Schemes, Lemma 19.8 this means that there exists some specialization  $t_1 \rightsquigarrow t$  in  $T$  and a point  $\xi \in X_T$  mapping to  $t_1$  such that  $\xi$  does not specialize to a point in the fibre over  $t$ . Set  $Z = \overline{\{\xi\}} \subset X_T$ . Then  $Z \cap X_t = \emptyset$ . Apply Lemma 14.1. We find an open neighbourhood  $V \subset T$  of  $t$ , a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme  $Z' \subset X_{T'}$  such that

- (1) the morphism  $b : T' \rightarrow S$  is locally of finite presentation,
- (2) with  $t' = a(t)$  we have  $Z' \cap X_{t'} = \emptyset$ , and
- (3)  $Z \cap X_V$  maps into  $Z'$  via the morphism  $X_V \rightarrow X_{T'}$ .

Clearly this means that  $X_{T'} \rightarrow T'$  maps the closed subset  $Z'$  to a subset of  $T'$  which contains  $a(t_1)$  but not  $t' = a(t)$ . Since  $a(t_1) \rightsquigarrow a(t) = t'$  we conclude that  $X_{T'} \rightarrow T'$  is not closed. Hence we have shown that  $X \rightarrow S$  not universally closed implies that  $X_{T'} \rightarrow T'$  is not closed for some  $T' \rightarrow S$  which is locally of finite presentation. In order words (2) implies (1).

Assume that  $\mathbf{A}^n \times X \rightarrow \mathbf{A}^n \times S$  is closed for every integer  $n$ . We want to prove that  $X_T \rightarrow T$  is closed for every scheme  $T$  which is locally of finite presentation over  $S$ . We may of course assume that  $T$  is affine and maps into an affine open  $V$  of  $S$  (since  $X_T \rightarrow T$  being a closed is local on  $T$ ). In this case there exists a closed immersion  $T \rightarrow \mathbf{A}^n \times V$  because  $\mathcal{O}_T(T)$  is a finitely presented  $\mathcal{O}_S(V)$ -algebra, see Morphisms, Lemma 21.2. Then  $T \rightarrow \mathbf{A}^n \times S$  is a locally closed immersion. Hence we get a cartesian diagram

$$\begin{array}{ccc} X_T & \longrightarrow & \mathbf{A}^n \times X \\ f_T \downarrow & & \downarrow f_n \\ T & \longrightarrow & \mathbf{A}^n \times S \end{array}$$

of schemes where the horizontal arrows are locally closed immersions. Hence any closed subset  $Z \subset X_T$  can be written as  $X_T \cap Z'$  for some closed subset  $Z' \subset \mathbf{A}^n \times X$ . Then  $f_T(Z) = T \cap f_n(Z')$  and we see that if  $f_n$  is closed, then also  $f_T$  is closed.  $\square$

**Lemma 14.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow S$  be a separated morphism of finite type. The following are equivalent:*

- (1) *The morphism  $f$  is proper.*
- (2) *For any morphism  $S' \rightarrow S$  which is locally of finite type the base change  $X_{S'} \rightarrow S'$  is closed.*
- (3) *For every  $n \geq 0$  the morphism  $\mathbf{A}^n \times X \rightarrow \mathbf{A}^n \times S$  is closed.*

**First proof.** In view of the fact that a proper morphism is the same thing as a separated, finite type, and universally closed morphism, this lemma is a special case of Lemma 14.2.  $\square$

**Second proof.** Clearly (1) implies (2), and (2) implies (3), so we just need to show (3) implies (1). First we reduce to the case when  $S$  is affine. Assume that (3)

implies (1) when the base is affine. Now let  $f : X \rightarrow S$  be a separated morphism of finite type. Being proper is local on the base (see Morphisms, Lemma 41.3), so if  $S = \bigcup_{\alpha} S_{\alpha}$  is an open affine cover, and if we denote  $X_{\alpha} := f^{-1}(S_{\alpha})$ , then it is enough to show that  $f|_{X_{\alpha}} : X_{\alpha} \rightarrow S_{\alpha}$  is proper for all  $\alpha$ . Since  $S_{\alpha}$  is affine, if the map  $f|_{X_{\alpha}}$  satisfies (3), then it will satisfy (1) by assumption, and will be proper. To finish the reduction to the case  $S$  is affine, we must show that if  $f : X \rightarrow S$  is separated of finite type satisfying (3), then  $f|_{X_{\alpha}} : X_{\alpha} \rightarrow S_{\alpha}$  is separated of finite type satisfying (3). Separatedness and finite type are clear. To see (3), notice that  $\mathbf{A}^n \times X_{\alpha}$  is the open preimage of  $\mathbf{A}^n \times S_{\alpha}$  under the map  $1 \times f$ . Fix a closed set  $Z \subset \mathbf{A}^n \times X_{\alpha}$ . Let  $\bar{Z}$  denote the closure of  $Z$  in  $\mathbf{A}^n \times X$ . Then for topological reasons,

$$1 \times f(\bar{Z}) \cap \mathbf{A}^n \times S_{\alpha} = 1 \times f(Z).$$

Hence  $1 \times f(Z)$  is closed, and we have reduced the proof of (3)  $\Rightarrow$  (1) to the affine case.

Assume  $S$  affine, and  $f : X \rightarrow S$  separated of finite type. We can apply Chow's Lemma 12.1 to get  $\pi : X' \rightarrow X$  proper surjective and  $X' \rightarrow \mathbf{P}_S^n$  an immersion. If  $X$  is proper over  $S$ , then  $X' \rightarrow S$  is proper (Morphisms, Lemma 41.4). Since  $\mathbf{P}_S^n \rightarrow S$  is separated, we conclude that  $X' \rightarrow \mathbf{P}_S^n$  is proper (Morphisms, Lemma 41.7) and hence a closed immersion (Schemes, Lemma 10.4). Conversely, assume  $X' \rightarrow \mathbf{P}_S^n$  is a closed immersion. Consider the diagram:

$$(14.3.1) \quad \begin{array}{ccc} X' & \longrightarrow & \mathbf{P}_S^n \\ \pi \downarrow & & \downarrow \\ X & \xrightarrow{f} & S \end{array}$$

All maps are a priori proper except for  $X \rightarrow S$ . Hence we conclude that  $X \rightarrow S$  is proper by Morphisms, Lemma 41.9. Therefore, we have shown that  $X \rightarrow S$  is proper if and only if  $X' \rightarrow \mathbf{P}_S^n$  is a closed immersion.

Assume  $S$  is affine and (3) holds, and let  $n, X', \pi$  be as above. Since being a closed morphism is local on the base, the map  $X \times \mathbf{P}^n \rightarrow S \times \mathbf{P}^n$  is closed since by (3)  $X \times \mathbf{A}^n \rightarrow S \times \mathbf{A}^n$  is closed and since projective space is covered by copies of affine  $n$ -space, see Constructions, Lemma 13.3. By Morphisms, Lemma 41.5 the morphism

$$X' \times_S \mathbf{P}_S^n \rightarrow X \times_S \mathbf{P}_S^n = X \times \mathbf{P}^n$$

is proper. Since  $\mathbf{P}^n$  is separated, the projection

$$X' \times_S \mathbf{P}_S^n = \mathbf{P}_{X'}^n \rightarrow X'$$

will be separated as it is just a base change of a separated morphism. Therefore, the map  $X' \rightarrow X' \times_S \mathbf{P}_S^n$  is proper, since it is a section to a separated map (see Schemes, Lemma 21.11). Composing these morphisms

$$X' \rightarrow X' \times_S \mathbf{P}_S^n \rightarrow X \times_S \mathbf{P}_S^n = X \times \mathbf{P}^n \rightarrow S \times \mathbf{P}^n = \mathbf{P}_S^n$$

we find that the immersion  $X' \rightarrow \mathbf{P}_S^n$  is closed, and hence a closed immersion.  $\square$

### 15. Noetherian valuative criterion

If the base is Noetherian we can show that the valuative criterion holds using only discrete valuation rings.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

**Lemma 15.1.** *Let  $f : X \rightarrow Y$  be a morphism of schemes. Assume  $f$  finite type and  $Y$  locally Noetherian. Let  $y \in Y$  be a point in the closure of the image of  $f$ . Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where  $A$  is a discrete valuation ring and  $K$  is its field of fractions mapping the closed point of  $\mathrm{Spec}(A)$  to  $y$ . Moreover, we can assume that the image point of  $\mathrm{Spec}(K) \rightarrow X$  is a generic point  $\eta$  of an irreducible component of  $X$  and that  $K = \kappa(\eta)$ .

**Proof.** By the non-Noetherian version of this lemma (Morphisms, Lemma 6.5) there exists a point  $x \in X$  such that  $f(x)$  specializes to  $y$ . We may replace  $x$  by any point specializing to  $x$ , hence we may assume that  $x$  is a generic point of an irreducible component of  $X$ . This produces a ring map  $\mathcal{O}_{Y,y} \rightarrow \kappa(x)$  (see Schemes, Section 13). Let  $R \subset \kappa(x)$  be the image. Then  $R$  is Noetherian as a quotient of the Noetherian local ring  $\mathcal{O}_{Y,y}$ . On the other hand, the extension  $\kappa(x)$  is a finitely generated extension of the fraction field of  $R$  as  $f$  is of finite type. Thus there exists a discrete valuation ring  $A \subset \kappa(x)$  with fraction field  $\kappa(x)$  dominating  $R$  by Algebra, Lemma 119.13. Then

$$\begin{array}{ccccccc} \mathrm{Spec}(\kappa(x)) & \xrightarrow{\hspace{4cm}} & & & X & & \\ \downarrow & & & & \downarrow & & \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(R) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{Y,y}) & \longrightarrow & Y \end{array}$$

gives the desired diagram.  $\square$

First we state the result concerning separation. We will often use solid commutative diagrams of morphisms of schemes having the following shape

$$(15.1.1) \quad \begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

with  $A$  a valuation ring and  $K$  its field of fractions.

**Lemma 15.2.** *Let  $S$  be a locally Noetherian scheme. Let  $f : X \rightarrow S$  be a morphism of schemes. Assume  $f$  is locally of finite type. The following are equivalent:*

- (1) *The morphism  $f$  is separated.*
- (2) *For any diagram (15.1.1) there is at most one dotted arrow.*

- (3) For all diagrams (15.1.1) with  $A$  a discrete valuation ring there is at most one dotted arrow.
- (4) For any irreducible component  $X_0$  of  $X$  with generic point  $\eta \in X_0$ , for any discrete valuation ring  $A \subset K = \kappa(\eta)$  with fraction field  $K$  and any diagram (15.1.1) such that the morphism  $\text{Spec}(K) \rightarrow X$  is the canonical one (see Schemes, Section 13) there is at most one dotted arrow.

**Proof.** Clearly (1) implies (2), (2) implies (3), and (3) implies (4). It remains to show (4) implies (1). Assume (4). We begin by reducing to  $S$  affine. Being separated is a local on the base (see Schemes, Lemma 21.7). Hence, if we can show that whenever  $X \rightarrow S$  has (4) that the restriction  $X_\alpha \rightarrow S_\alpha$  has (4) where  $S_\alpha \subset S$  is an (affine) open subset and  $X_\alpha := f^{-1}(S_\alpha)$ , then we will be done. The generic points of the irreducible components of  $X_\alpha$  will be the generic points of irreducible components of  $X$ , since  $X_\alpha$  is open in  $X$ . Therefore, any two distinct dotted arrows in the diagram

$$(15.2.1) \quad \begin{array}{ccc} \text{Spec}(K) & \longrightarrow & X_\alpha \\ \downarrow & \nearrow \text{---} & \downarrow \\ \text{Spec}(A) & \longrightarrow & S_\alpha \end{array}$$

would then give two distinct arrows in diagram (15.1.1) via the maps  $X_\alpha \rightarrow X$  and  $S_\alpha \rightarrow S$ , which is a contradiction. Thus we have reduced to the case  $S$  is affine. We remark that in the course of this reduction, we prove that if  $X \rightarrow S$  has (4) then the restriction  $U \rightarrow V$  has (4) for opens  $U \subset X$  and  $V \subset S$  with  $f(U) \subset V$ .

We next wish to reduce to the case  $X \rightarrow S$  is finite type. Assume that we know (4) implies (1) when  $X$  is finite type. Since  $S$  is Noetherian and  $X$  is locally of finite type over  $S$  we see  $X$  is locally Noetherian as well (see Morphisms, Lemma 15.6). Thus,  $X \rightarrow S$  is quasi-separated (see Properties, Lemma 5.4), and therefore we may apply the valuative criterion to check whether  $X$  is separated (see Schemes, Lemma 22.2). Let  $X = \bigcup_\alpha X_\alpha$  be an affine open cover of  $X$ . Given any two dotted arrows, in a diagram (15.1.1), the image of the closed points of  $\text{Spec } A$  will fall in two sets  $X_\alpha$  and  $X_\beta$ . Since  $X_\alpha \cup X_\beta$  is open, for topological reasons it must contain the image of  $\text{Spec}(A)$  under both maps. Therefore, the two dotted arrows factor through  $X_\alpha \cup X_\beta \rightarrow X$ , which is a scheme of finite type over  $S$ . Since  $X_\alpha \cup X_\beta$  is an open subset of  $X$ , by our previous remark,  $X_\alpha \cup X_\beta$  satisfies (4), so by assumption, is separated. This implies the two given dotted arrows are the same. Therefore, we have reduced to  $X \rightarrow S$  is finite type.

Assume  $X \rightarrow S$  of finite type and assume (4). Since  $X \rightarrow S$  is finite type, and  $S$  is an affine Noetherian scheme,  $X$  is also Noetherian (see Morphisms, Lemma 15.6). Therefore,  $X \rightarrow X \times_S X$  will be a quasi-compact immersion of Noetherian schemes. We proceed by contradiction. Assume that  $X \rightarrow X \times_S X$  is not closed. Then, there is some  $y \in X \times_S X$  in the closure of the image that is not in the image. As  $X$  is Noetherian it has finitely many irreducible components. Therefore,  $y$  is in the closure of the image of one of the irreducible components  $X_0 \subset X$ . Give  $X_0$  the reduced induced structure. The composition  $X_0 \rightarrow X \rightarrow X \times_S X$  factors through the closed subscheme  $X_0 \times_S X_0 \subset X \times_S X$ . Denote the closure of  $\Delta(X_0)$  in  $X_0 \times_S X_0$  by  $\bar{X}_0$  (again as a reduced closed subscheme). Thus  $y \in \bar{X}_0$ . Since  $X_0 \rightarrow X_0 \times_S X_0$  is an immersion, the image of  $X_0$  will be open in  $\bar{X}_0$ . Hence  $X_0$



and  $\bar{X}_0$  are birational. Since  $\bar{X}_0$  is a closed subscheme of a Noetherian scheme, it is Noetherian. Thus, the local ring  $\mathcal{O}_{\bar{X}_0, y}$  is a local Noetherian domain with fraction field  $K$  equal to the function field of  $X_0$ . By the Krull-Akizuki theorem (see Algebra, Lemma 119.13), there exists a discrete valuation ring  $A$  dominating  $\mathcal{O}_{\bar{X}_0, y}$  with fraction field  $K$ . This allows to construct a diagram:

$$(15.2.2) \quad \begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X_0 \\ \downarrow & \nearrow \text{dotted} & \downarrow \Delta \\ \mathrm{Spec}(A) & \longrightarrow & X_0 \times_S X_0 \end{array}$$

which sends  $\mathrm{Spec} K$  to the generic point of  $\Delta(X_0)$  and the closed point of  $A$  to  $y \in X_0 \times_S X_0$  (use the material in Schemes, Section 13 to construct the arrows). There cannot even exist a set theoretic dotted arrow, since  $y$  is not in the image of  $\Delta$  by our choice of  $y$ . By categorical means, the existence of the dotted arrow in the above diagram is equivalent to the uniqueness of the dotted arrow in the following diagram:

$$(15.2.3) \quad \begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X_0 \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

Therefore, we have non-uniqueness in this latter diagram by the nonexistence in the first. Therefore,  $X_0$  does not satisfy uniqueness for discrete valuation rings, and since  $X_0$  is an irreducible component of  $X$ , we have that  $X \rightarrow S$  does not satisfy (4). Therefore, we have shown (4) implies (1).  $\square$

**Lemma 15.3.** *Let  $S$  be a locally Noetherian scheme. Let  $f : X \rightarrow S$  be a morphism of finite type. The following are equivalent:*

- (1) *The morphism  $f$  is proper.*
- (2) *For any diagram (15.1.1) there exists exactly one dotted arrow.*
- (3) *For all diagrams (15.1.1) with  $A$  a discrete valuation ring there exists exactly one dotted arrow.*
- (4) *For any irreducible component  $X_0$  of  $X$  with generic point  $\eta \in X_0$ , for any discrete valuation ring  $A \subset K = \kappa(\eta)$  with fraction field  $K$  and any diagram (15.1.1) such that the morphism  $\mathrm{Spec}(K) \rightarrow X$  is the canonical one (see Schemes, Section 13) there exists exactly one dotted arrow.*

**Proof.** (1) implies (2) implies (3) implies (4). We will now show (4) implies (1). As in the proof of Lemma 15.2, we can reduce to the case  $S$  is affine, since properness is local on the base, and if  $X \rightarrow S$  satisfies (4), then  $X_\alpha \rightarrow S_\alpha$  does as well for open  $S_\alpha \subset S$  and  $X_\alpha = f^{-1}(S_\alpha)$ .

Now  $S$  is a Noetherian scheme, and so  $X$  is as well, since  $X \rightarrow S$  is of finite type. Now we may use Chow's lemma (Cohomology of Schemes, Lemma 18.1) to get a surjective, proper, birational  $X' \rightarrow X$  and an immersion  $X' \rightarrow \mathbf{P}_S^n$ . We wish to show  $X \rightarrow S$  is universally closed. As in the proof of Lemma 14.3, it is enough to check that  $X' \rightarrow \mathbf{P}_S^n$  is a closed immersion. For the sake of contradiction, assume that  $X' \rightarrow \mathbf{P}_S^n$  is not a closed immersion. Then there is some  $y \in \mathbf{P}_S^n$  that is in the closure of the image of  $X'$ , but is not in the image. So  $y$  is in the closure of the

image of an irreducible component  $X'_0$  of  $X'$ , but not in the image. Let  $\bar{X}'_0 \subset \mathbf{P}_S^n$  be the closure of the image of  $X'_0$ . As  $X' \rightarrow \mathbf{P}_S^n$  is an immersion of Noetherian schemes, the morphism  $X'_0 \rightarrow \bar{X}'_0$  is open and dense. By Algebra, Lemma 119.13 or Properties, Lemma 5.10 we can find a discrete valuation ring  $A$  dominating  $\mathcal{O}_{\bar{X}'_0, y}$  and with identical field of fractions  $K$ . It is clear that  $K$  is the residue field at the generic point of  $X'_0$ . Thus the solid commutative diagram

$$(15.3.1) \quad \begin{array}{ccccc} \mathrm{Spec} K & \longrightarrow & X' & \longrightarrow & \mathbf{P}_S^n \\ \downarrow & \nearrow \text{dotted} & \downarrow & \nearrow & \downarrow \\ \mathrm{Spec} A & \dashrightarrow & X & \longrightarrow & S \end{array}$$

Note that the closed point of  $A$  maps to  $y \in \mathbf{P}_S^n$ . By construction, there does not exist a set theoretic lift to  $X'$ . As  $X' \rightarrow X$  is birational, the image of  $X'_0$  in  $X$  is an irreducible component  $X_0$  of  $X$  and  $K$  is also identified with the function field of  $X_0$ . Hence, as  $X \rightarrow S$  is assumed to satisfy (4), the dotted arrow  $\mathrm{Spec}(A) \rightarrow X$  exists. Since  $X' \rightarrow X$  is proper, the dotted arrow lifts to the dotted arrow  $\mathrm{Spec}(A) \rightarrow X'$  (use Schemes, Proposition 20.6). We can compose this with the immersion  $X' \rightarrow \mathbf{P}_S^n$  to obtain another morphism (not depicted in the diagram) from  $\mathrm{Spec}(A) \rightarrow \mathbf{P}_S^n$ . Since  $\mathbf{P}_S^n$  is proper over  $S$ , it satisfies (2), and so these two morphisms agree. This is a contradiction, for we have constructed the forbidden lift of our original map  $\mathrm{Spec}(A) \rightarrow \mathbf{P}_S^n$  to  $X'$ .  $\square$

**Lemma 15.4.** *Let  $f : X \rightarrow S$  be a finite type morphism of schemes. Assume  $S$  is locally Noetherian. Then the following are equivalent*

- (1)  *$f$  is universally closed,*
- (2) *for every  $n$  the morphism  $\mathbf{A}^n \times X \rightarrow \mathbf{A}^n \times S$  is closed,*
- (3) *for any diagram (15.1.1) there exists some dotted arrow,*
- (4) *for all diagrams (15.1.1) with  $A$  a discrete valuation ring there exists some dotted arrow.*

**Proof.** The equivalence of (1) and (2) is a special case of Lemma 14.2. The equivalence of (1) and (3) is a special case of Schemes, Proposition 20.6. Trivially (3) implies (4). Thus all we have to do is prove that (4) implies (2). We will prove that  $\mathbf{A}^n \times X \rightarrow \mathbf{A}^n \times S$  is closed by the criterion of Schemes, Lemma 19.8. Pick  $n$  and a specialization  $z \rightsquigarrow z'$  of points in  $\mathbf{A}^n \times S$  and a point  $y \in \mathbf{A}^n \times X$  lying over  $z$ . Note that  $\kappa(y)$  is a finitely generated field extension of  $\kappa(z)$  as  $\mathbf{A}^n \times X \rightarrow \mathbf{A}^n \times S$  is of finite type. Hence by Properties, Lemma 5.10 or Algebra, Lemma 119.13 implies that there exists a discrete valuation ring  $A \subset \kappa(y)$  with fraction field  $\kappa(z)$  dominating the image of  $\mathcal{O}_{\mathbf{A}^n \times S, z'}$  in  $\kappa(z)$ . This gives a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\kappa(y)) & \longrightarrow & \mathbf{A}^n \times X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathbf{A}^n \times S & \longrightarrow & S \end{array}$$

Now property (4) implies that there exists a morphism  $\mathrm{Spec}(A) \rightarrow X$  which fits into this diagram. Since we already have the morphism  $\mathrm{Spec}(A) \rightarrow \mathbf{A}^n$  from the left lower horizontal arrow we also get a morphism  $\mathrm{Spec}(A) \rightarrow \mathbf{A}^n \times X$  fitting into the left square. Thus the image  $y' \in \mathbf{A}^n \times X$  of the closed point is a specialization

of  $y$  lying over  $z'$ . This proves that specializations lift along  $\mathbf{A}^n \times X \rightarrow \mathbf{A}^n \times S$  and we win.  $\square$

## 16. Refined Noetherian valuative criteria

One usually does not have to consider all possible diagrams with valuation rings when checking valuative criteria. An example is given by Morphisms, Lemma 42.2. In the Noetherian setting, we have also seen this in Lemmas 15.2 and 15.3. Here is another variant.

**Lemma 16.1.** *Let  $f : X \rightarrow S$  and  $h : U \rightarrow X$  be morphisms of schemes. Assume that  $S$  is locally Noetherian, that  $f$  and  $h$  are of finite type, that  $f$  is separated, and that  $h(U)$  is dense in  $X$ . If given any commutative solid diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & \nearrow \text{dotted} & & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & & & S \end{array}$$

where  $A$  is a discrete valuation ring with field of fractions  $K$ , there exists a dotted arrow making the diagram commute, then  $f$  is proper.

**Proof.** There is an immediate reduction to the case where  $S$  is affine. Then  $U$  is quasi-compact. Let  $U = U_1 \cup \dots \cup U_n$  be an affine open covering. We may replace  $U$  by  $U_1 \amalg \dots \amalg U_n$  without changing the assumptions, hence we may assume  $U$  is affine. Thus we can find an open immersion  $U \rightarrow Y$  over  $X$  with  $Y$  proper over  $X$ . (First put  $U$  inside  $\mathbf{A}_X^n$  using Morphisms, Lemma 39.2 and then take the closure inside  $\mathbf{P}_X^n$ , or you can directly use Morphisms, Lemma 43.12.) We can assume  $U$  is dense in  $Y$  (replace  $Y$  by the scheme theoretic closure of  $U$  if necessary, see Morphisms, Section 7). Note that  $g : Y \rightarrow X$  is surjective as the image is closed and contains the dense subset  $h(U)$ . We will show that  $Y \rightarrow S$  is proper. This will imply that  $X \rightarrow S$  is proper by Morphisms, Lemma 41.9 thereby finishing the proof. To show that  $Y \rightarrow S$  is proper we will use part (4) of Lemma 15.3. To do this consider a diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{y} & Y \\ \downarrow & \nearrow \text{dotted} & \downarrow f \circ g \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

where  $A$  is a discrete valuation ring with fraction field  $K$  and where  $y : \mathrm{Spec}(K) \rightarrow Y$  is the inclusion of a generic point. We have to show there exists a unique dotted arrow. Uniqueness holds by the converse to the valuative criterion for separatedness (Schemes, Lemma 22.1) since  $Y \rightarrow S$  is separated as the composition of the separated morphisms  $Y \rightarrow X$  and  $X \rightarrow S$  (Schemes, Lemma 21.12). Existence can be seen as follows. As  $y$  is a generic point of  $Y$ , it is contained in  $U$ . By assumption of the lemma there exists a morphism  $a : \mathrm{Spec}(A) \rightarrow X$  such that

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \xrightarrow{y} & U & \longrightarrow & X \\ \downarrow & & \nearrow a & & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & & & S \end{array}$$

is commutative. Then since  $Y \rightarrow X$  is proper, we can apply the valuative criterion for properness (Morphisms, Lemma 42.1) to find a morphism  $b : \text{Spec}(A) \rightarrow Y$  such that

$$\begin{array}{ccc} \text{Spec}(K) & \xrightarrow{y} & Y \\ \downarrow & \nearrow b & \downarrow g \\ \text{Spec}(A) & \xrightarrow{a} & X \end{array}$$

is commutative. This finishes the proof since  $b$  can serve as the dotted arrow above.  $\square$

**Lemma 16.2.** *Let  $f : X \rightarrow S$  and  $h : U \rightarrow X$  be morphisms of schemes. Assume that  $S$  is locally Noetherian, that  $f$  is locally of finite type, that  $h$  is of finite type, and that  $h(U)$  is dense in  $X$ . If given any commutative solid diagram*

$$\begin{array}{ccccc} \text{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow f \\ \text{Spec}(A) & \longrightarrow & & & S \end{array}$$

where  $A$  is a discrete valuation ring with field of fractions  $K$ , there exists at most one dotted arrow making the diagram commute, then  $f$  is separated.

**Proof.** We will apply Lemma 16.1 to the morphisms  $U \rightarrow X$  and  $\Delta : X \rightarrow X \times_S X$ . We check the conditions. Observe that  $\Delta$  is quasi-compact by Properties, Lemma 5.4 (and Schemes, Lemma 21.13). Of course  $\Delta$  is locally of finite type and separated (true for any diagonal morphism). Finally, suppose given a commutative solid diagram

$$\begin{array}{ccccc} \text{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow \Delta \\ \text{Spec}(A) & \xrightarrow{(a,b)} & & & X \times_S X \end{array}$$

where  $A$  is a discrete valuation ring with field of fractions  $K$ . Then  $a$  and  $b$  give two dotted arrows in the diagram of the lemma and have to be equal. Hence as dotted arrow we can use  $a = b$  which gives existence. This finishes the proof.  $\square$

**Lemma 16.3.** *Let  $f : X \rightarrow S$  and  $h : U \rightarrow X$  be morphisms of schemes. Assume that  $S$  is locally Noetherian, that  $f$  and  $h$  are of finite type, and that  $h(U)$  is dense in  $X$ . If given any commutative solid diagram*

$$\begin{array}{ccccc} \text{Spec}(K) & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & & \nearrow & \downarrow f \\ \text{Spec}(A) & \longrightarrow & & & S \end{array}$$

where  $A$  is a discrete valuation ring with field of fractions  $K$ , there exists a unique dotted arrow making the diagram commute, then  $f$  is proper.

**Proof.** Combine Lemmas 16.2 and 16.1.  $\square$

### 17. Valuative criteria over a Nagata base

When working with schemes locally of finite type over a Nagata base we can reduce to discrete valuation rings which are essentially of finite type over the base. The following are just some example results one can get.

**Lemma 17.1.** *Let  $S$  be a Nagata scheme (and in particular locally Noetherian). Let  $f : X \rightarrow Y$  be a quasi-compact morphism of schemes locally of finite type over  $S$ . The following are equivalent*

- (1)  $f$  is universally closed,
- (2) for every  $n$  the morphism  $\mathbf{A}^n \times X \rightarrow \mathbf{A}^n \times Y$  is closed,
- (3) for any commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ C & \longrightarrow & Y \end{array}$$

of schemes over  $S$  such that

- (a)  $C$  is a normal integral scheme of finite type over  $S$ ,
- (b)  $U = C \setminus \{c\}$  for some closed point  $c \in C$ ,
- (c)  $A = \mathcal{O}_{C,c}$  has dimension 1<sup>3</sup>

then in the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where  $K = \mathrm{Frac}(A)$  some dotted arrow exists<sup>4</sup> making the diagram commute.

**Proof.** We have seen the equivalence of (1) and (2) and the fact that these imply (3) in Lemma 15.4. Thus it suffices to prove that (3) implies (2). Observe that if condition (3) holds for  $f : X \rightarrow Y$ , then condition (3) holds for  $1 \times f : \mathbf{A}^n \times X \rightarrow \mathbf{A}^n \times Y$  (see argument in the proof of Lemma 15.4). Hence it suffices to show that (3) implies that  $f$  is closed.

Reduction to the case where  $Y$  and  $S$  are affine; we suggest skipping this paragraph. Let  $S' \subset S$  be an affine open and let  $Y' \subset Y$  be an affine open mapping into  $S'$ . Set  $X' = f^{-1}(Y')$ . Then we claim that the restriction  $f' : X' \rightarrow Y'$  of  $f$  viewed as a morphism of schemes over  $S'$  has property (3) also. We omit the details. Now if we can prove that  $f'$  is closed for all choices of  $S'$  and  $Y'$ , then it follows that  $f$  is closed. This reduces us to the case discussed in the next paragraph.

Assume  $S$  and  $Y$  affine. Let  $Z \subset X$  be a closed subset. We may and do view  $Z$  as a reduced closed subscheme of  $X$ . We have to show that  $E = f(Z)$  is closed. Pick  $y \in Y$  a closed point contained in the closure of  $f(Z)$ . It suffices to show  $y \in E$ . We assume  $y \notin E$  to get a contradiction. The image  $s \in S$  of  $y$  is a finite type point of

<sup>3</sup>It follows that  $A$  is a discrete valuation ring, see Algebra, Lemma 119.7. Moreover,  $c$  maps to a finite type point  $s \in S$  and  $A$  is essentially of finite type over  $\mathcal{O}_{S,s}$ .

<sup>4</sup>By Lemma 6.4 this is equivalent to asking for the existence of dotted arrow making the first commutative diagram commute.

$S$ , see Morphisms, Lemma 16.5. Recall that  $E$  is constructible (Morphisms, Lemma 22.2). Consider the intersection  $\mathrm{Spec}(\mathcal{O}_{Y,y}) \cap E$ . This is a constructible subset of the spectrum (Morphisms, Lemma 22.1) which doesn't contain the closed point. Since the punctured spectrum  $\mathrm{Spec}(\mathcal{O}_{Y,y}) \setminus \{y\}$  is Jacobson (Morphisms, Lemma 16.10), we find a closed point  $t \in \mathrm{Spec}(\mathcal{O}_{Y,y}) \setminus \{y\}$  with  $t \in E$  (see Topology, Lemma 18.5). In other words,  $t \in E$  is a point of  $Y$  which has an immediate specialization  $t \rightsquigarrow y$ . As  $t \in E$  the scheme theoretic fibre  $Z_t$  is nonempty. Choose a closed point  $x \in Z_t$ . In particular we have  $[\kappa(x) : \kappa(t)] < \infty$  by the Hilbert Nullstellensatz (Morphisms, Lemma 20.3).

Denote  $T = \overline{\{t\}} \subset Y$  the integral closed subscheme whose underlying topological space is as indicated (Schemes, Definition 12.5). Then  $t \in T$  is the generic point. Denote  $C \rightarrow T$  the normalization of  $T$  in  $\kappa(x)$ , see Morphisms, Section 53 (more precisely,  $C \rightarrow T$  is the normalization of  $T$  in  $x$  where we view  $x = \mathrm{Spec}(\kappa(x)) \rightarrow T$  as a scheme over  $T$ ). Since  $S$  is a Nagata scheme, so is  $T$  (Morphisms, Lemma 18.1). Hence we see that  $C \rightarrow T$  is finite (Morphisms, Lemma 53.14). As  $t$  is in the image we see that  $C \rightarrow T$  is surjective (because the image is closed and  $T$  is the closure of  $t$  in  $Y$ ). Choose a point  $c \in C$  mapping to  $y \in T$ . Since  $y$  is a closed point of  $T$  we see that  $c$  is a closed point of  $C$ . Since  $\dim(\mathcal{O}_{T,y}) = 1$  we see that  $\dim(\mathcal{O}_{C,c}) = 1$  (the dimension is at least 1 as  $c$  is not the generic point of  $C$  and at most 1 as  $C \rightarrow T$  is finite). As the function field of  $C$  is  $\kappa(x)$  and as  $x$  is a point of  $X$ , we have a  $Y$ -rational map from  $C$  to  $X$  (see for example Morphisms, Lemma 49.2). Let  $C \supset U \rightarrow X$  be a representative (in particular  $U$  is nonempty). We may assume  $c \notin U$  (replace  $U$  by  $U \setminus \{c\}$ ). Since  $c$  is a closed point of codimension 1 in the integral scheme  $C$  we have  $C = U \amalg \{c\} \amalg \Sigma$  for some proper closed subset  $\Sigma \subset C$ . After replacing  $C$  by  $C \setminus \Sigma$  we have constructed a commutative diagram as in part (3). By the 2nd footnote in the statement of the lemma, the existence of the dotted arrow produces an extension of the rational map to all of  $C$  and we get the contradiction because the image of  $c$  will be a point of  $Z$  mapping to  $y$ .  $\square$

**Lemma 17.2.** *Let  $S$  be a Nagata scheme (and in particular locally Noetherian). Let  $f : X \rightarrow Y$  be a morphism of schemes locally of finite type over  $S$ . The following are equivalent*

- (1)  $f$  separated,
- (2) for any commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ C & \longrightarrow & Y \end{array}$$

of schemes over  $S$  such that

- (a)  $C$  is a normal integral scheme of finite type over  $S$ ,
- (b)  $U = C \setminus \{c\}$  for some closed point  $c \in C$ ,
- (c)  $A = \mathcal{O}_{C,c}$  has dimension 1<sup>5</sup>

<sup>5</sup>It follows that  $A$  is a discrete valuation ring, see Algebra, Lemma 119.7. Moreover,  $c$  maps to a finite type point  $s \in S$  and  $A$  is essentially of finite type over  $\mathcal{O}_{S,s}$ .

then in the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where  $K = \mathrm{Frac}(A)$  there exists at most one dotted arrow<sup>6</sup> making the diagram commute.

**Proof.** By Lemma 15.2 we see that (1) implies (2). Assume (2). In order to show that  $f$  is separated, we have to show that  $\Delta : X \rightarrow X \times_Y X$  is closed. By Morphisms, Lemma 15.7 the morphism  $\Delta$  is quasi-compact. By Lemma 17.1 it suffices to show: for any commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \Delta \\ C & \xrightarrow{(a_1, a_2)} & X \times_Y X \end{array}$$

of schemes over  $S$  such that

- (1)  $C$  is a normal integral scheme of finite type over  $S$ ,
- (2)  $U = C \setminus \{c\}$  for some closed point  $c \in C$ ,
- (3)  $A = \mathcal{O}_{C,c}$  has dimension 1.

then in the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \Delta \\ \mathrm{Spec}(A) & \longrightarrow & X \times_Y X \end{array}$$

where  $K = \mathrm{Frac}(A)$  there exists some dotted arrow making the diagram commute. By Lemma 6.4 the existence of the dotted arrow in the second diagram is equivalent to the existence of the dotted arrow in the first diagram. Moreover, the existence there is the same as asking  $a_1 = a_2$ . However  $a_1|_U = a_2|_U$ , so by the uniqueness assumption (2) we see that this is true and the proof is complete.  $\square$

**Lemma 17.3.** *Let  $S$  be a Nagata scheme (and in particular locally Noetherian). Let  $f : X \rightarrow Y$  be a quasi-compact morphism of schemes locally of finite type over  $S$ . The following are equivalent*

- (1)  $f$  proper,
- (2) for any commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ C & \longrightarrow & Y \end{array}$$

of schemes over  $S$  such that

- (a)  $C$  is a normal integral scheme of finite type over  $S$ ,

<sup>6</sup>By Lemma 6.4 this is equivalent to asking there to be at most one dotted arrow making the first commutative diagram commute.

- (b)  $U = C \setminus \{c\}$  for some closed point  $c \in C$ ,  
(c)  $A = \mathcal{O}_{C,c}$  has dimension 1<sup>7</sup>  
then in the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where  $K = \mathrm{Frac}(A)$  there exists exactly one dotted arrow<sup>8</sup> making the diagram commute.

**Proof.** This is formal from Lemmas 17.1 and 17.2 and the definition of proper morphisms as being finite type, separated, and universally closed.  $\square$

### 18. Limits and dimensions of fibres

The following lemma is most often used in the situation of Lemma 10.1 to assure that if the fibres of the limit have dimension  $\leq d$ , then the fibres at some finite stage have dimension  $\leq d$ .

**Lemma 18.1.** *Let  $I$  be a directed set. Let  $(f_i : X_i \rightarrow S_i)$  be an inverse system of morphisms of schemes over  $I$ . Assume*

- (1) *all the morphisms  $S_{i'} \rightarrow S_i$  are affine,*
- (2) *all the schemes  $S_i$  are quasi-compact and quasi-separated,*
- (3) *the morphisms  $f_i$  are of finite type, and*
- (4) *the morphisms  $X_{i'} \rightarrow X_i \times_{S_i} S_{i'}$  are closed immersions.*

*Let  $f : X = \lim_i X_i \rightarrow S = \lim_i S_i$  be the limit. Let  $d \geq 0$ . If every fibre of  $f$  has dimension  $\leq d$ , then for some  $i$  every fibre of  $f_i$  has dimension  $\leq d$ .*

**Proof.** For each  $i$  let  $U_i = \{x \in X_i \mid \dim_x((X_i)_{f_i(x)}) \leq d\}$ . This is an open subset of  $X_i$ , see Morphisms, Lemma 28.4. Set  $Z_i = X_i \setminus U_i$  (with reduced induced scheme structure). We have to show that  $Z_i = \emptyset$  for some  $i$ . If not, then  $Z = \lim Z_i \neq \emptyset$ , see Lemma 4.3. Say  $z \in Z$  is a point. Note that  $Z \subset X$  is a closed subscheme. Set  $s = f(z)$ . For each  $i$  let  $s_i \in S_i$  be the image of  $s$ . We remark that  $Z_s$  is the limit of the schemes  $(Z_i)_{s_i}$  and  $Z_s$  is also the limit of the schemes  $(Z_i)_{s_i}$  base changed to  $\kappa(s)$ . Moreover, all the morphisms

$$Z_s \longrightarrow (Z_{i'})_{s_{i'}} \times_{\mathrm{Spec}(\kappa(s_{i'}))} \mathrm{Spec}(\kappa(s)) \longrightarrow (Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s)) \longrightarrow X_s$$

are closed immersions by assumption (4). Hence  $Z_s$  is the scheme theoretic intersection of the closed subschemes  $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$  in  $X_s$ . Since all the irreducible components of the schemes  $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$  have dimension  $> d$  and contain  $z$  we conclude that  $Z_s$  contains an irreducible component of dimension  $> d$  passing through  $z$  which contradicts the fact that  $Z_s \subset X_s$  and  $\dim(X_s) \leq d$ .  $\square$

**Lemma 18.2.** *Notation and assumptions as in Situation 8.1. If*

- (1)  *$f$  is a quasi-finite morphism, and*

<sup>7</sup>It follows that  $A$  is a discrete valuation ring, see Algebra, Lemma 119.7. Moreover,  $c$  maps to a finite type point  $s \in S$  and  $A$  is essentially of finite type over  $\mathcal{O}_{S,s}$ .

<sup>8</sup>By Lemma 6.4 this is equivalent to asking for the existence and uniqueness of the dotted arrow making the first commutative diagram commute.



(2)  $f_0$  is locally of finite type,  
 then there exists an  $i \geq 0$  such that  $f_i$  is quasi-finite.

**Proof.** Follows immediately from Lemma 18.1.  $\square$

**Lemma 18.3.** *Assumptions and notation as in Situation 8.1. Let  $d \geq 0$ . If*

- (1)  $f$  has relative dimension  $\leq d$  (Morphisms, Definition 29.1), and
- (2)  $f_0$  is locally of finite type,

*then there exists an  $i$  such that  $f_i$  has relative dimension  $\leq d$ .*

**Proof.** Follows immediately from Lemma 18.1.  $\square$

**Lemma 18.4.** *Notation and assumptions as in Situation 8.1. If*

- (1)  $f$  has relative dimension  $d$ , and
- (2)  $f_0$  is locally of finite presentation,

*then there exists an  $i \geq 0$  such that  $f_i$  has relative dimension  $d$ .*

**Proof.** By Lemma 18.1 we may assume all fibres of  $f_0$  have dimension  $\leq d$ . By Morphisms, Lemma 28.6 the set  $U_0 \subset X_0$  of points  $x \in X_0$  such that the dimension of the fibre of  $X_0 \rightarrow Y_0$  at  $x$  is  $\leq d - 1$  is open and retrocompact in  $X_0$ . Hence the complement  $E = X_0 \setminus U_0$  is constructible. Moreover the image of  $X \rightarrow X_0$  is contained in  $E$  by Morphisms, Lemma 28.3. Thus for  $i \gg 0$  we have that the image of  $X_i \rightarrow X_0$  is contained in  $E$  (Lemma 4.10). Then all fibres of  $X_i \rightarrow Y_i$  have dimension  $d$  by the aforementioned Morphisms, Lemma 28.3.  $\square$

**Lemma 18.5.** *Let  $S$  be a quasi-compact and quasi-separated scheme. Let  $f : X \rightarrow S$  be a morphism of finite presentation. Let  $d \geq 0$  be an integer. If  $Z \subset X$  be a closed subscheme such that  $\dim(Z_s) \leq d$  for all  $s \in S$ , then there exists a closed subscheme  $Z' \subset X$  such that*

- (1)  $Z \subset Z'$ ,
- (2)  $Z' \rightarrow X$  is of finite presentation, and
- (3)  $\dim(Z'_s) \leq d$  for all  $s \in S$ .

**Proof.** By Proposition 5.4 we can write  $S = \lim S_i$  as the limit of a directed inverse system of Noetherian schemes with affine transition maps. By Lemma 10.1 we may assume that there exist a system of morphisms  $f_i : X_i \rightarrow S_i$  of finite presentation such that  $X_{i'} = X_i \times_{S_i} S_{i'}$  for all  $i' \geq i$  and such that  $X = X_i \times_{S_i} S$ . Let  $Z_i \subset X_i$  be the scheme theoretic image of  $Z \rightarrow X \rightarrow X_i$ . Then for  $i' \geq i$  the morphism  $X_{i'} \rightarrow X_i$  maps  $Z_{i'}$  into  $Z_i$  and the induced morphism  $Z_{i'} \rightarrow Z_i \times_{S_i} S_{i'}$  is a closed immersion. By Lemma 18.1 we see that the dimension of the fibres of  $Z_i \rightarrow S_i$  all have dimension  $\leq d$  for a suitable  $i \in I$ . Fix such an  $i$  and set  $Z' = Z_i \times_{S_i} S \subset X$ . Since  $S_i$  is Noetherian, we see that  $X_i$  is Noetherian, and hence the morphism  $Z_i \rightarrow X_i$  is of finite presentation. Therefore also the base change  $Z' \rightarrow X$  is of finite presentation. Moreover, the fibres of  $Z' \rightarrow S$  are base changes of the fibres of  $Z_i \rightarrow S_i$  and hence have dimension  $\leq d$ .  $\square$

## 19. Base change in top degree

For a proper morphism and a finite type quasi-coherent module the base change map is an isomorphism in top degree.

**Lemma 19.1.** *Let  $f : X \rightarrow Y$  be a morphism of schemes. Let  $d \geq 0$ . Assume*

- (1)  $X$  and  $Y$  are quasi-compact and quasi-separated, and
- (2)  $R^i f_* \mathcal{F} = 0$  for  $i > d$  and every quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$ .

Then we have

- (a) for any base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

we have  $R^i f'_* \mathcal{F}' = 0$  for  $i > d$  and any quasi-coherent  $\mathcal{O}_{X'}$ -module  $\mathcal{F}'$ ,

- (b)  $R^d f'_* (\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \mathcal{G}') = R^d f'_* \mathcal{F}' \otimes_{\mathcal{O}_{Y'}} \mathcal{G}'$  for any quasi-coherent  $\mathcal{O}_{Y'}$ -module  $\mathcal{G}'$ ,
- (c) formation of  $R^d f'_* \mathcal{F}'$  commutes with arbitrary further base change (see proof for explanation).

**Proof.** Before giving the proofs, we explain the meaning of (c). Suppose we have an additional cartesian square

$$\begin{array}{ccccc} X'' & \xrightarrow{\quad} & X' & \xrightarrow{g'} & X \\ f'' \downarrow & & f' \downarrow & & \downarrow f \\ Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y \end{array}$$

tacked onto our given diagram. If (a) holds, then there is a canonical map  $\gamma : h^* R^d f'_* \mathcal{F}' \rightarrow R^d f''_* (h')^* \mathcal{F}'$ . Namely,  $\gamma$  is the map on degree  $d$  cohomology sheaves induced by the composition

$$Lh^* Rf'_* \mathcal{F}' \longrightarrow Rf''_* L(h')^* \mathcal{F}' \longrightarrow Rf''_* (h')^* \mathcal{F}'$$

Here the first arrow is the base change map (Cohomology, Remark 28.3) and the second arrow complex from the canonical map  $L(g')^* \mathcal{F} \rightarrow (g')^* \mathcal{F}$ . Similarly, since  $Rf'_* \mathcal{F}$  has no nonzero cohomology sheaves in degrees  $> d$  by (a) we have  $H^d(Lh^* Rf'_* \mathcal{F}') = h^* R^d f_* \mathcal{F}$ . The content of (c) is that  $\gamma$  is an isomorphism.

Having said this, we can check (a), (b), and (c) locally on  $Y'$  and  $Y''$ . Suppose that  $V \subset Y$  is a quasi-compact open subscheme. Then we claim (1) and (2) hold for  $f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V$ . Namely, (1) is immediate and (2) follows because any quasi-coherent module on  $f^{-1}(V)$  is the restriction of a quasi-coherent module on  $X$  (Properties, Lemma 22.1) and formation of higher direct images commutes with restriction to opens. Thus we may also work locally on  $Y$ . In other words, we may assume  $Y''$ ,  $Y'$ , and  $Y$  are affine schemes.

Proof of (a) when  $Y'$  and  $Y$  are affine. In this case the morphisms  $g$  and  $g'$  are affine. Thus  $g_* = Rg_*$  and  $g'_* = Rg'_*$  (Cohomology of Schemes, Lemma 2.3) and  $g_*$  is identified with the restriction functor on modules (Schemes, Lemma 7.3). Then

$$g_*(R^i f'_* \mathcal{F}') = H^i(Rg_* Rf'_* \mathcal{F}') = H^i(Rf_* Rg'_* \mathcal{F}') = H^i(Rf_* g'_* \mathcal{F}') = Rf_*^i g'_* \mathcal{F}'$$

which is zero by assumption (2). Hence (a) by our description of  $g_*$ .

Proof of (b) when  $Y'$  is affine, say  $Y' = \text{Spec}(R')$ . By part (a) we have  $H^{d+1}(X', \mathcal{F}') = 0$  for any quasi-coherent  $\mathcal{O}_{X'}$ -module  $\mathcal{F}'$ , see Cohomology of Schemes, Lemma 4.6. Consider the functor  $F$  on  $R'$ -modules defined by the rule

$$F(M) = H^d(X', \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{M})$$

By Cohomology, Lemma 19.1 this functor commutes with direct sums (this is where we use that  $X$  and hence  $X'$  is quasi-compact and quasi-separated). On the other hand, if  $M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  is an exact sequence, then

$$\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{M}_1 \rightarrow \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{M}_2 \rightarrow \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{M}_3 \rightarrow 0$$

is an exact sequence of quasi-coherent modules on  $X'$  and by the vanishing of higher cohomology given above we get an exact sequence

$$F(M_1) \rightarrow F(M_2) \rightarrow F(M_3) \rightarrow 0$$

In other words,  $F$  is right exact. Any right exact  $R'$ -linear functor  $F : \text{Mod}_{R'} \rightarrow \text{Mod}_{R'}$  which commutes with direct sums is given by tensoring with an  $R'$ -module (omitted; left as exercise for the reader). Thus we obtain  $F(M) = H^d(X', \mathcal{F}') \otimes_{R'} M$ . Since  $R^d(f')_* \mathcal{F}'$  and  $R^d(f')_*(\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{M})$  are quasi-coherent (Cohomology of Schemes, Lemma 4.5), the fact that  $F(M) = H^d(X', \mathcal{F}') \otimes_{R'} M$  translates into the statement given in (b).

Proof of (c) when  $Y'' \rightarrow Y' \rightarrow Y$  are morphisms of affine schemes. Say  $Y'' = \text{Spec}(R'')$  and  $Y' = \text{Spec}(R')$ . Then we see that  $R^d f''_*(h')^* \mathcal{F}'$  is the quasi-coherent module on  $Y'$  associated to the  $R''$ -module  $H^d(X'', (h')^* \mathcal{F}')$ . Now  $h' : X'' \rightarrow X'$  is affine hence  $H^d(X'', (h')^* \mathcal{F}') = H^d(X, h'_*(h')^* \mathcal{F}')$  by the already used Cohomology of Schemes, Lemma 2.4. We have

$$h'_*(h')^* \mathcal{F}' = \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \widetilde{R''}$$

as the reader sees by checking on an affine open covering. Thus  $H^d(X'', (h')^* \mathcal{F}') = H^d(X', \mathcal{F}') \otimes_{R'} R''$  by part (b) applied to  $f'$  and the proof is complete.  $\square$

**Lemma 19.2.** *Let  $f : X \rightarrow Y$  be a morphism of schemes. Let  $y \in Y$ . Assume  $f$  is proper and  $\dim(X_y) = d$ . Then*

- (1) *for  $\mathcal{F} \in QCoh(\mathcal{O}_X)$  we have  $(R^i f_* \mathcal{F})_y = 0$  for all  $i > d$ ,*
- (2) *there is an affine open neighbourhood  $V \subset Y$  of  $y$  such that  $f^{-1}(V) \rightarrow V$  and  $d$  satisfy the assumptions and conclusions of Lemma 19.1.*

**Proof.** By Morphisms, Lemma 28.4 and the fact that  $f$  is closed, we can find an affine open neighbourhood  $V$  of  $y$  such that the fibres over points of  $V$  all have dimension  $\leq d$ . Thus we may assume  $X \rightarrow Y$  is a proper morphism all of whose fibres have dimension  $\leq d$  with  $Y$  affine. We will show that (2) holds, which will immediately imply (1) for all  $y \in Y$ .

By Lemma 13.2 we can write  $X = \lim X_i$  as a cofiltered limit with  $X_i \rightarrow Y$  proper and of finite presentation and such that both  $X \rightarrow X_i$  and transition morphisms are closed immersions. For some  $i$  we have that  $X_i \rightarrow Y$  has fibres of dimension  $\leq d$ , see Lemma 18.1. For a quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  we have  $R^p f_* \mathcal{F} = R^p f_{i,*} (X \rightarrow X_i)_* \mathcal{F}$  by Cohomology of Schemes, Lemma 2.3 and Leray (Cohomology, Lemma 13.8). Thus we may replace  $X$  by  $X_i$  and reduce to the case discussed in the next paragraph.

Assume  $Y$  is affine and  $f : X \rightarrow Y$  is proper and of finite presentation and all fibres have dimension  $\leq d$ . It suffices to show that  $H^p(X, \mathcal{F}) = 0$  for  $p > d$ . Namely, by Cohomology of Schemes, Lemma 4.6 we have  $H^p(X, \mathcal{F}) = H^0(Y, R^p f_* \mathcal{F})$ . On the other hand,  $R^p f_* \mathcal{F}$  is quasi-coherent on  $Y$  by Cohomology of Schemes, Lemma 4.5, hence vanishing of global sections implies vanishing. Write  $Y = \lim_{i \in I} Y_i$  as a cofiltered limit of affine schemes with  $Y_i$  the spectrum of a Noetherian ring

(for example a finite type  $\mathbf{Z}$ -algebra). We can choose an element  $0 \in I$  and a finite type morphism  $X_0 \rightarrow Y_0$  such that  $X \cong Y \times_{Y_0} X_0$ , see Lemma 10.1. After increasing  $0$  we may assume  $X_0 \rightarrow Y_0$  is proper (Lemma 13.1) and that the fibres of  $X_0 \rightarrow Y_0$  have dimension  $\leq d$  (Lemma 18.1). Since  $X \rightarrow X_0$  is affine, we find that  $H^p(X, \mathcal{F}) = H^p(X_0, (X \rightarrow X_0)_* \mathcal{F})$  by Cohomology of Schemes, Lemma 2.4. This reduces us to the case discussed in the next paragraph.

Assume  $Y$  is affine Noetherian and  $f : X \rightarrow Y$  is proper and all fibres have dimension  $\leq d$ . In this case we can write  $\mathcal{F} = \text{colim } \mathcal{F}_i$  as a filtered colimit of coherent  $\mathcal{O}_X$ -modules, see Properties, Lemma 22.7. Then  $H^p(X, \mathcal{F}) = \text{colim } H^p(X, \mathcal{F}_i)$  by Cohomology, Lemma 19.1. Thus we may assume  $\mathcal{F}$  is coherent. In this case we see that  $(R^p f_* \mathcal{F})_y = 0$  for all  $y \in Y$  by Cohomology of Schemes, Lemma 20.9. Thus  $R^p f_* \mathcal{F} = 0$  and therefore  $H^p(X, \mathcal{F}) = 0$  (see above) and we win.  $\square$

**Lemma 19.3.** *Let  $f : X \rightarrow Y$  be a morphism of schemes. Let  $d \geq 0$ . Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module. Assume*

- (1)  *$f$  is a proper morphism all of whose fibres have dimension  $\leq d$ ,*
- (2)  *$\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_X$ -module of finite type.*

*Then  $R^d f_* \mathcal{F}$  is a quasi-coherent  $\mathcal{O}_Y$ -module of finite type.*

**Proof.** The module  $R^d f_* \mathcal{F}$  is quasi-coherent by Cohomology of Schemes, Lemma 4.5. The question is local on  $Y$  hence we may assume  $Y$  is affine. Say  $Y = \text{Spec}(R)$ . Then it suffices to prove that  $H^d(X, \mathcal{F})$  is a finite  $R$ -module.

By Lemma 13.2 we can write  $X = \lim X_i$  as a cofiltered limit with  $X_i \rightarrow Y$  proper and of finite presentation and such that both  $X \rightarrow X_i$  and transition morphisms are closed immersions. For some  $i$  we have that  $X_i \rightarrow Y$  has fibres of dimension  $\leq d$ , see Lemma 18.1. We have  $R^p f_* \mathcal{F} = R^p f_{i,*} (X \rightarrow X_i)_* \mathcal{F}$  by Cohomology of Schemes, Lemma 2.3 and Leray (Cohomology, Lemma 13.8). Thus we may replace  $X$  by  $X_i$  and reduce to the case discussed in the next paragraph.

Assume  $Y$  is affine and  $f : X \rightarrow Y$  is proper and of finite presentation and all fibres have dimension  $\leq d$ . We can write  $\mathcal{F}$  as a quotient of a finitely presented  $\mathcal{O}_X$ -module  $\mathcal{F}'$ , see Properties, Lemma 22.8. The map  $H^d(X, \mathcal{F}') \rightarrow H^d(X, \mathcal{F})$  is surjective, as we have  $H^{d+1}(X, \text{Ker}(\mathcal{F}' \rightarrow \mathcal{F})) = 0$  by the vanishing of higher cohomology seen in Lemma 19.2 (or its proof). Thus we reduce to the case discussed in the next paragraph.

Assume  $Y = \text{Spec}(R)$  is affine and  $f : X \rightarrow Y$  is proper and of finite presentation and all fibres have dimension  $\leq d$  and  $\mathcal{F}$  is an  $\mathcal{O}_X$ -module of finite presentation. Write  $Y = \lim_{i \in I} Y_i$  as a cofiltered limit of affine schemes with  $Y_i = \text{Spec}(R_i)$  the spectrum of a Noetherian ring (for example a finite type  $\mathbf{Z}$ -algebra). We can choose an element  $0 \in I$  and a finite type morphism  $X_0 \rightarrow Y_0$  such that  $X \cong Y \times_{Y_0} X_0$ , see Lemma 10.1. After increasing  $0$  we may assume  $X_0 \rightarrow Y_0$  is proper (Lemma 13.1) and that the fibres of  $X_0 \rightarrow Y_0$  have dimension  $\leq d$  (Lemma 18.1). After increasing  $0$  we can assume there is a coherent  $\mathcal{O}_{X_0}$ -module  $\mathcal{F}_0$  which pulls back to  $\mathcal{F}$ , see Lemma 10.2. By Lemma 19.1 we have

$$H^d(X, \mathcal{F}) = H^d(X_0, \mathcal{F}_0) \otimes_{R_0} R$$

This finishes the proof because the cohomology module  $H^d(X_0, \mathcal{F}_0)$  is finite by Cohomology of Schemes, Lemma 19.2.  $\square$

**Lemma 19.4.** *Let  $f : X \rightarrow Y$  be a morphism of schemes. Let  $d \geq 0$ . Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module. Assume*

- (1)  *$f$  is a proper morphism of finite presentation all of whose fibres have dimension  $\leq d$ ,*
- (2)  *$\mathcal{F}$  is an  $\mathcal{O}_X$ -module of finite presentation.*

*Then  $R^d f_* \mathcal{F}$  is an  $\mathcal{O}_Y$ -module of finite presentation.*

**Proof.** The proof is exactly the same as the proof of Lemma 19.3 except that the third paragraph can be skipped. We omit the details.  $\square$

## 20. Glueing in closed fibres

Applying our theory above to the spectrum of a local ring we obtain the following pleasing glueing result for relative schemes.

**Lemma 20.1.** *Let  $S$  be a scheme. Let  $s \in S$  be a closed point such that  $U = S \setminus \{s\} \rightarrow S$  is quasi-compact. With  $V = \text{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$  there is an equivalence of categories*

$$\{X \rightarrow S \text{ of finite presentation}\} \longrightarrow \left\{ \begin{array}{ccccc} X' & \longleftarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & V & \longrightarrow & \text{Spec}(\mathcal{O}_{S,s}) \end{array} \right\}$$

*where on the right hand side we consider commutative diagrams whose squares are cartesian and whose vertical arrows are of finite presentation.*

**Proof.** Let  $W \subset S$  be an open neighbourhood of  $s$ . By glueing of relative schemes, see Constructions, Section 2, the functor

$$\{X \rightarrow S \text{ of finite presentation}\} \longrightarrow \left\{ \begin{array}{ccccc} X' & \longleftarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & W \setminus \{s\} & \longrightarrow & W \end{array} \right\}$$

is an equivalence of categories. We have  $\mathcal{O}_{S,s} = \text{colim } \mathcal{O}_W(W)$  where  $W$  runs over the affine open neighbourhoods of  $s$ . Hence  $\text{Spec}(\mathcal{O}_{S,s}) = \lim W$  where  $W$  runs over the affine open neighbourhoods of  $s$ . Thus the category of schemes of finite presentation over  $\text{Spec}(\mathcal{O}_{S,s})$  is the limit of the category of schemes of finite presentation over  $W$  where  $W$  runs over the affine open neighbourhoods of  $s$ , see Lemma 10.1. For every affine open  $s \in W$  we see that  $U \cap W$  is quasi-compact as  $U \rightarrow S$  is quasi-compact. Hence  $V = \lim W \cap U = \lim W \setminus \{s\}$  is a limit of quasi-compact and quasi-separated schemes (see Lemma 2.2). Thus also the category of schemes of finite presentation over  $V$  is the limit of the categories of schemes of finite presentation over  $W \cap U$  where  $W$  runs over the affine open neighbourhoods of  $s$ . The lemma follows formally from a combination of these results.  $\square$

**Lemma 20.2.** *Let  $S$  be a scheme. Let  $s \in S$  be a closed point such that  $U = S \setminus \{s\} \rightarrow S$  is quasi-compact. With  $V = \text{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$  there is an equivalence of categories*

$$\{\mathcal{O}_S\text{-modules } \mathcal{F} \text{ of finite presentation}\} \longrightarrow \{(\mathcal{G}, \mathcal{H}, \alpha)\}$$

where on the right hand side we consider triples consisting of a  $\mathcal{O}_U$ -module  $\mathcal{G}$  of finite presentation, a  $\mathcal{O}_{\mathrm{Spec}(\mathcal{O}_{S,s})}$ -module  $\mathcal{H}$  of finite presentation, and an isomorphism  $\alpha : \mathcal{G}|_V \rightarrow \mathcal{H}|_V$  of  $\mathcal{O}_V$ -modules.

**Proof.** You can either prove this by redoing the proof of Lemma 20.1 using Lemma 10.2 or you can deduce it from Lemma 20.1 using the equivalence between quasi-coherent modules and “vector bundles” from Constructions, Section 6. We omit the details.  $\square$

**Lemma 20.3.** *Let  $S$  be a scheme. Let  $U \subset S$  be a retrocompact open. Let  $s \in S$  be a point in the complement of  $U$ . With  $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \cap U$  there is an equivalence of categories*

$$\mathrm{colim}_{s \in U' \supset U \text{ open}} \left\{ \begin{array}{c} X \\ \downarrow \\ U' \end{array} \right\} \rightarrow \left\{ \begin{array}{ccccc} X' & \longleftarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & V & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,s}) \end{array} \right\}$$

where on the left hand side the vertical arrow is of finite presentation and on the right hand side we consider commutative diagrams whose squares are cartesian and whose vertical arrows are of finite presentation.

**Proof.** Let  $W \subset S$  be an open neighbourhood of  $s$ . By glueing of relative schemes, see Constructions, Section 2, the functor

$$\{X \rightarrow U' = U \cup W \text{ of finite presentation}\} \rightarrow \left\{ \begin{array}{ccccc} X' & \longleftarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & W \cap U & \longrightarrow & W \end{array} \right\}$$

is an equivalence of categories. We have  $\mathcal{O}_{S,s} = \mathrm{colim} \mathcal{O}_W(W)$  where  $W$  runs over the affine open neighbourhoods of  $s$ . Hence  $\mathrm{Spec}(\mathcal{O}_{S,s}) = \lim W$  where  $W$  runs over the affine open neighbourhoods of  $s$ . Thus the category of schemes of finite presentation over  $\mathrm{Spec}(\mathcal{O}_{S,s})$  is the limit of the category of schemes of finite presentation over  $W$  where  $W$  runs over the affine open neighbourhoods of  $s$ , see Lemma 10.1. For every affine open  $s \in W$  we see that  $U \cap W$  is quasi-compact as  $U \rightarrow S$  is quasi-compact. Hence  $V = \lim W \cap U$  is a limit of quasi-compact and quasi-separated schemes (see Lemma 2.2). Thus also the category of schemes of finite presentation over  $V$  is the limit of the categories of schemes of finite presentation over  $W \cap U$  where  $W$  runs over the affine open neighbourhoods of  $s$ . The lemma follows formally from a combination of these results.  $\square$

**Lemma 20.4.** *Notation and assumptions as in Lemma 20.3. Let  $U \subset U' \subset X$  be an open containing  $s$ .*

- (1) *Let  $f' : X \rightarrow U'$  correspond to  $f : X' \rightarrow U$  and  $g : Y \rightarrow \mathrm{Spec}(\mathcal{O}_{S,s})$  via the equivalence. If  $f$  and  $g$  are separated, proper, finite, étale, then after possibly shrinking  $U'$  the morphism  $f'$  has the same property.*
- (2) *Let  $a : X_1 \rightarrow X_2$  be a morphism of schemes of finite presentation over  $U'$  with base change  $a' : X'_1 \rightarrow X'_2$  over  $U$  and  $b : Y_1 \rightarrow Y_2$  over  $\mathrm{Spec}(\mathcal{O}_{S,s})$ . If  $a'$  and  $b$  are separated, proper, finite, étale, then after possibly shrinking  $U'$  the morphism  $a$  has the same property.*

**Proof.** Proof of (1). Recall that  $\text{Spec}(\mathcal{O}_{S,s})$  is the limit of the affine open neighbourhoods of  $s$  in  $S$ . Since  $g$  has the property in question, then the restriction of  $f'$  to one of these affine open neighbourhoods does too, see Lemmas 8.6, 13.1, 8.3, and 8.10. Since  $f'$  has the given property over  $U$  as  $f$  does, we conclude as one can check the property locally on the base.

Proof of (2). If we write  $\text{Spec}(\mathcal{O}_{S,s}) = \lim W$  where  $W$  runs over the affine open neighbourhoods of  $s$  in  $S$ , then we have  $Y_i = \lim W \times_S X_i$ . Thus we can use exactly the same arguments as in the proof of (1).  $\square$

**Lemma 20.5.** *Let  $S$  be a scheme. Let  $s_1, \dots, s_n \in S$  be pairwise distinct closed points such that  $U = S \setminus \{s_1, \dots, s_n\} \rightarrow S$  is quasi-compact. With  $S_i = \text{Spec}(\mathcal{O}_{S,s_i})$  and  $U_i = S_i \setminus \{s_i\}$  there is an equivalence of categories*

$$FP_S \longrightarrow FP_U \times_{(FP_{U_1} \times \dots \times FP_{U_n})} (FP_{S_1} \times \dots \times FP_{S_n})$$

where  $FP_T$  is the category of schemes of finite presentation over the scheme  $T$ .

**Proof.** For  $n = 1$  this is Lemma 20.1. For  $n > 1$  the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that  $f_i : X_i \rightarrow S_i$  are objects of  $FP_{S_i}$  and  $f : X \rightarrow U$  is an object of  $FP_U$  and we're given isomorphisms  $X_i \times_{S_i} U_i = X \times_U U_i$ . By Lemma 20.1 we can find a morphism  $f' : X' \rightarrow U' = S \setminus \{s_1, \dots, s_{n-1}\}$  which is of finite presentation, which is isomorphic to  $X_i$  over  $S_i$ , which is isomorphic to  $X$  over  $U$ , and these isomorphisms are compatible with the given isomorphism  $X_i \times_{S_i} U_n = X \times_U U_n$ . Then we can apply induction to  $f_i : X_i \rightarrow S_i$ ,  $i \leq n-1$ ,  $f' : X' \rightarrow U'$ , and the induced isomorphisms  $X_i \times_{S_i} U_i = X' \times_{U'} U_i$ ,  $i \leq n-1$ . This shows essential surjectivity. We omit the proof of fully faithfulness.  $\square$

## 21. Application to modifications

Using the results from Section 20 we can describe the category of modifications of a scheme over a closed point in terms of the local ring.

**Lemma 21.1.** *Let  $S$  be a scheme. Let  $s \in S$  be a closed point such that  $U = S \setminus \{s\} \rightarrow S$  is quasi-compact. With  $V = \text{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$  the base change functor*

$$\left\{ \begin{array}{l} f : X \rightarrow S \text{ of finite presentation} \\ f^{-1}(U) \rightarrow U \text{ is an isomorphism} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} g : Y \rightarrow \text{Spec}(\mathcal{O}_{S,s}) \text{ of finite presentation} \\ g^{-1}(V) \rightarrow V \text{ is an isomorphism} \end{array} \right\}$$

is an equivalence of categories.

**Proof.** This is a special case of Lemma 20.1.  $\square$

**Lemma 21.2.** *Notation and assumptions as in Lemma 21.1. Let  $f : X \rightarrow S$  correspond to  $g : Y \rightarrow \text{Spec}(\mathcal{O}_{S,s})$  via the equivalence. Then  $f$  is separated, proper, finite, étale and add more here if and only if  $g$  is so.*

**Proof.** The property of being separated, proper, integral, finite, etc is stable under base change. See Schemes, Lemma 21.12 and Morphisms, Lemmas 41.5 and 44.6. Hence if  $f$  has the property, then so does  $g$ . The converse follows from Lemma 20.4 but we also give a direct proof here. Namely, if  $g$  has to property, then  $f$  does in a neighbourhood of  $s$  by Lemmas 8.6, 13.1, 8.3, and 8.10. Since  $f$  clearly has the given property over  $S \setminus \{s\}$  we conclude as one can check the property locally on the base.  $\square$

**Remark 21.3.** The lemma above can be generalized as follows. Let  $S$  be a scheme and let  $T \subset S$  be a closed subset. Assume there exists a cofinal system of open neighbourhoods  $T \subset W_i$  such that (1)  $W_i \setminus T$  is quasi-compact and (2)  $W_i \subset W_j$  is an affine morphism. Then  $W = \lim W_i$  is a scheme which contains  $T$  as a closed subscheme. Set  $U = X \setminus T$  and  $V = W \setminus T$ . Then the base change functor

$$\left\{ \begin{array}{l} f : X \rightarrow S \text{ of finite presentation} \\ f^{-1}(U) \rightarrow U \text{ is an isomorphism} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} g : Y \rightarrow W \text{ of finite presentation} \\ g^{-1}(V) \rightarrow V \text{ is an isomorphism} \end{array} \right\}$$

is an equivalence of categories. If we ever need this we will change this remark into a lemma and provide a detailed proof.

## 22. Descending finite type schemes

This section continues the theme of Section 9 in the spirit of the results discussed in Section 10.

**Situation 22.1.** Let  $S = \lim_{i \in I} S_i$  be the limit of a directed system of Noetherian schemes with affine transition morphisms  $S_{i'} \rightarrow S_i$  for  $i' \geq i$ .

**Lemma 22.2.** *In Situation 22.1. Let  $X \rightarrow S$  be quasi-separated and of finite type. Then there exists an  $i \in I$  and a diagram*

$$(22.2.1) \quad \begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_i \end{array}$$

*such that  $W \rightarrow S_i$  is of finite type and such that the induced morphism  $X \rightarrow S \times_{S_i} W$  is a closed immersion.*

**Proof.** By Lemma 9.3 we can find a closed immersion  $X \rightarrow X'$  over  $S$  where  $X'$  is a scheme of finite presentation over  $S$ . By Lemma 10.1 we can find an  $i$  and a morphism of finite presentation  $X'_i \rightarrow S_i$  whose pull back is  $X'$ . Set  $W = X'_i$ .  $\square$

**Lemma 22.3.** *In Situation 22.1. Let  $X \rightarrow S$  be quasi-separated and of finite type. Given  $i \in I$  and a diagram*

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_i \end{array}$$

*as in (22.2.1) for  $i' \geq i$  let  $X_{i'}$  be the scheme theoretic image of  $X \rightarrow S_{i'} \times_{S_i} W$ . Then  $X = \lim_{i' \geq i} X_{i'}$ .*

**Proof.** Since  $X$  is quasi-compact and quasi-separated formation of the scheme theoretic image of  $X \rightarrow S_{i'} \times_{S_i} W$  commutes with restriction to open subschemes (Morphisms, Lemma 6.3). Hence we may and do assume  $W$  is affine and maps into an affine open  $U_i$  of  $S_i$ . Let  $U \subset S$ ,  $U_{i'} \subset S_{i'}$  be the inverse image of  $U_i$ . Then  $U$ ,  $U_{i'}$ ,  $S_{i'} \times_{S_i} W = U_{i'} \times_{U_i} W$ , and  $S \times_{S_i} W = U \times_{U_i} W$  are all affine. This implies  $X$  is affine because  $X \rightarrow S \times_{S_i} W$  is a closed immersion. This also shows the ring map

$$\mathcal{O}(U) \otimes_{\mathcal{O}(U_i)} \mathcal{O}(W) \rightarrow \mathcal{O}(X)$$



is surjective. Let  $I$  be the kernel. Then we see that  $X_{i'}$  is the spectrum of the ring

$$\mathcal{O}(X_{i'}) = \mathcal{O}(U_{i'}) \otimes_{\mathcal{O}(U_i)} \mathcal{O}(W) / I_{i'}$$

where  $I_{i'}$  is the inverse image of the ideal  $I$  (see Morphisms, Example 6.4). Since  $\mathcal{O}(U) = \text{colim } \mathcal{O}(U_{i'})$  we see that  $I = \text{colim } I_{i'}$  and we conclude that  $\text{colim } \mathcal{O}(X_{i'}) = \mathcal{O}(X)$ .  $\square$

**Lemma 22.4.** *In Situation 22.1. Let  $f : X \rightarrow Y$  be a morphism of schemes quasi-separated and of finite type over  $S$ . Let*

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_{i_1} \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \longrightarrow & V \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_{i_2} \end{array}$$

be diagrams as in (22.2.1). Let  $X = \lim_{i \geq i_1} X_i$  and  $Y = \lim_{i \geq i_2} Y_i$  be the corresponding limit descriptions as in Lemma 22.3. Then there exists an  $i_0 \geq \max(i_1, i_2)$  and a morphism

$$(f_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \rightarrow (Y_i)_{i \geq i_0}$$

of inverse systems over  $(S_i)_{i \geq i_0}$  such that  $f = \lim_{i \geq i_0} f_i$ . If  $(g_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \rightarrow (Y_i)_{i \geq i_0}$  is a second morphism of inverse systems over  $(S_i)_{i \geq i_0}$  such that  $f = \lim_{i \geq i_0} g_i$  then  $f_i = g_i$  for all  $i \gg i_0$ .

**Proof.** Since  $V \rightarrow S_{i_2}$  is of finite presentation and  $X = \lim_{i \geq i_1} X_i$  we can appeal to Proposition 6.1 to find an  $i_0 \geq \max(i_1, i_2)$  and a morphism  $h : X_{i_0} \rightarrow V$  over  $S_{i_2}$  such that  $X \rightarrow X_{i_0} \rightarrow V$  is equal to  $X \rightarrow Y \rightarrow V$ . For  $i \geq i_0$  we get a commutative solid diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_i & \longrightarrow & X_{i_0} \\ \downarrow & \searrow & \downarrow & & \downarrow h \\ Y & \longrightarrow & Y_i & \longrightarrow & V \\ \downarrow & \searrow & \downarrow & & \downarrow \\ S & \longrightarrow & S_i & \longrightarrow & S_{i_0} \end{array}$$

Since  $X \rightarrow X_i$  has scheme theoretically dense image and since  $Y_i$  is the scheme theoretic image of  $Y \rightarrow S_i \times_{S_{i_2}} V$  we find that the morphism  $X_i \rightarrow S_i \times_{S_{i_2}} V$  induced by the diagram factors through  $Y_i$  (Morphisms, Lemma 6.6). This proves existence.

Uniqueness. Let  $E_i \subset X_i$  be the equalizer of  $f_i$  and  $g_i$  for  $i \geq i_0$ . By Schemes, Lemma 21.5  $E_i$  is a locally closed subscheme of  $X_i$ . Since  $X_i$  is a closed subscheme of  $S_i \times_{S_{i_0}} X_{i_0}$  and similarly for  $Y_i$  we see that

$$E_i = X_i \times_{(S_i \times_{S_{i_0}} X_{i_0})} (S_i \times_{S_{i_0}} E_{i_0})$$

Thus to finish the proof it suffices to show that  $X_i \rightarrow X_{i_0}$  factors through  $E_{i_0}$  for some  $i \geq i_0$ . To do this we will use that  $X \rightarrow X_{i_0}$  factors through  $E_{i_0}$  as both  $f_{i_0}$  and  $g_{i_0}$  are compatible with  $f$ . Since  $X_i$  is Noetherian, we see that the underlying topological space  $|E_{i_0}|$  is a constructible subset of  $|X_{i_0}|$  (Topology, Lemma 16.1). Hence  $X_i \rightarrow X_{i_0}$  factors through  $E_{i_0}$  set theoretically for large enough  $i$  by Lemma 4.10. For such an  $i$  the scheme theoretic inverse image  $(X_i \rightarrow X_{i_0})^{-1}(E_{i_0})$  is

a closed subscheme of  $X_i$  through which  $X$  factors and hence equal to  $X_i$  since  $X \rightarrow X_i$  has scheme theoretically dense image by construction. This concludes the proof.  $\square$

**Remark 22.5.** In Situation 22.1 Lemmas 22.2, 22.3, and 22.4 tell us that the category of schemes quasi-separated and of finite type over  $S$  is equivalent to certain types of inverse systems of schemes over  $(S_i)_{i \in I}$ , namely the ones produced by applying Lemma 22.3 to a diagram of the form (22.2.1). For example, given  $X \rightarrow S$  finite type and quasi-separated if we choose two different diagrams  $X \rightarrow V_1 \rightarrow S_{i_1}$  and  $X \rightarrow V_2 \rightarrow S_{i_2}$  as in (22.2.1), then applying Lemma 22.4 to  $\text{id}_X$  (in two directions) we see that the corresponding limit descriptions of  $X$  are canonically isomorphic (up to shrinking the directed set  $I$ ). And so on and so forth.

**Lemma 22.6.** *Notation and assumptions as in Lemma 22.4. If  $f$  is flat and of finite presentation, then there exists an  $i_3 \geq i_0$  such that for  $i \geq i_3$  we have  $f_i$  is flat,  $X_i = Y_i \times_{Y_{i_3}} X_{i_3}$ , and  $X = Y \times_{Y_{i_3}} X_{i_3}$ .*

**Proof.** By Lemma 10.1 we can choose an  $i \geq i_2$  and a morphism  $U \rightarrow Y_i$  of finite presentation such that  $X = Y \times_{Y_i} U$  (this is where we use that  $f$  is of finite presentation). After increasing  $i$  we may assume that  $U \rightarrow Y_i$  is flat, see Lemma 8.7. As discussed in Remark 22.5 we may and do replace the initial diagram used to define the system  $(X_i)_{i \geq i_1}$  by the system corresponding to  $X \rightarrow U \rightarrow S_i$ . Thus  $X_{i'}$  for  $i' \geq i$  is defined as the scheme theoretic image of  $X \rightarrow S_{i'} \times_{S_i} U$ .

Because  $U \rightarrow Y_i$  is flat (this is where we use that  $f$  is flat), because  $X = Y \times_{Y_i} U$ , and because the scheme theoretic image of  $Y \rightarrow Y_i$  is  $Y_i$ , we see that the scheme theoretic image of  $X \rightarrow U$  is  $U$  (Morphisms, Lemma 25.16). Observe that  $Y_{i'} \rightarrow S_{i'} \times_{S_i} Y_i$  is a closed immersion for  $i' \geq i$  by construction of the system of  $Y_j$ . Then the same argument as above shows that the scheme theoretic image of  $X \rightarrow S_{i'} \times_{S_i} U$  is equal to the closed subscheme  $Y_{i'} \times_{Y_i} U$ . Thus we see that  $X_{i'} = Y_{i'} \times_{Y_i} U$  for all  $i' \geq i$  and hence the lemma holds with  $i_3 = i$ .  $\square$

**Lemma 22.7.** *Notation and assumptions as in Lemma 22.4. If  $f$  is smooth, then there exists an  $i_3 \geq i_0$  such that for  $i \geq i_3$  we have  $f_i$  is smooth.*

**Proof.** Combine Lemmas 22.6 and 8.9.  $\square$

**Lemma 22.8.** *Notation and assumptions as in Lemma 22.4. If  $f$  is proper, then there exists an  $i_3 \geq i_0$  such that for  $i \geq i_3$  we have  $f_i$  is proper.*

**Proof.** By the discussion in Remark 22.5 the choice of  $i_1$  and  $W$  fitting into a diagram as in (22.2.1) is immaterial for the truth of the lemma. Thus we choose  $W$  as follows. First we choose a closed immersion  $X \rightarrow X'$  with  $X' \rightarrow S$  proper and of finite presentation, see Lemma 13.2. Then we choose an  $i_3 \geq i_2$  and a proper morphism  $W \rightarrow Y_{i_3}$  such that  $X' = Y \times_{Y_{i_3}} W$ . This is possible because  $Y = \lim_{i \geq i_2} Y_i$  and Lemmas 10.1 and 13.1. With this choice of  $W$  it is immediate from the construction that for  $i \geq i_3$  the scheme  $X_i$  is a closed subscheme of  $Y_i \times_{Y_{i_3}} W \subset S_i \times_{S_{i_3}} W$  and hence proper over  $Y_i$ .  $\square$

**Lemma 22.9.** *In Situation 22.1 suppose that we have a cartesian diagram*

$$\begin{array}{ccc} X^1 & \xrightarrow{p} & X^3 \\ q \downarrow & & \downarrow a \\ X^2 & \xrightarrow{b} & X^4 \end{array}$$

*of schemes quasi-separated and of finite type over  $S$ . For each  $j = 1, 2, 3, 4$  choose  $i_j \in I$  and a diagram*

$$\begin{array}{ccc} X^j & \longrightarrow & W^j \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_{i_j} \end{array}$$

*as in (22.2.1). Let  $X^j = \lim_{i \geq i_j} X_i^j$  be the corresponding limit descriptions as in Lemma 22.4. Let  $(a_i)_{i \geq i_5}$ ,  $(b_i)_{i \geq i_6}$ ,  $(p_i)_{i \geq i_7}$ , and  $(q_i)_{i \geq i_8}$  be the corresponding morphisms of systems constructed in Lemma 22.4. Then there exists an  $i_9 \geq \max(i_5, i_6, i_7, i_8)$  such that for  $i \geq i_9$  we have  $a_i \circ p_i = b_i \circ q_i$  and such that*

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

*is a closed immersion. If  $a$  and  $b$  are flat and of finite presentation, then there exists an  $i_{10} \geq \max(i_5, i_6, i_7, i_8, i_9)$  such that for  $i \geq i_{10}$  the last displayed morphism is an isomorphism.*

**Proof.** According to the discussion in Remark 22.5 the choice of  $W^1$  fitting into a diagram as in (22.2.1) is immaterial for the truth of the lemma. Thus we may choose  $W^1 = W^2 \times_{W^4} W^3$ . Then it is immediate from the construction of  $X_i^1$  that  $a_i \circ p_i = b_i \circ q_i$  and that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion.

If  $a$  and  $b$  are flat and of finite presentation, then so are  $p$  and  $q$  as base changes of  $a$  and  $b$ . Thus we can apply Lemma 22.6 to each of  $a$ ,  $b$ ,  $p$ ,  $q$ , and  $a \circ p = b \circ q$ . It follows that there exists an  $i_9 \in I$  such that

$$(q_i, p_i) : X_i^1 \rightarrow X_i^2 \times_{X_i^4} X_i^3$$

is the base change of  $(q_{i_9}, p_{i_9})$  by the morphism by the morphism  $X_i^4 \rightarrow X_{i_9}^4$  for all  $i \geq i_9$ . We conclude that  $(q_i, p_i)$  is an isomorphism for all sufficiently large  $i$  by Lemma 8.11.  $\square$

## 23. Other chapters

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