LIMITS OF ALGEBRAIC STACKS

Contents

1. Introduction

In this chapter we put material related to limits of algebraic stacks. Many results on limits of algebraic stacks and algebraic spaces have been obtained by David Rydh in [Ryd08].

2. Conventions

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2.

3. Morphisms of finite presentation

This section is the analogue of Limits of Spaces, Section 3. There we defined what it means for a transformation of functors on *Sch* to be limit preserving (we suggest looking at the characterization in Limits of Spaces, Lemma 3.2). In Criteria for Representability, Section 5 we defined the notion "limit preserving on objects". Recall that in Artin's Axioms, Section 11 we have defined what it means for a category fibred in groupoids over *Sch* to be limit preserving. Combining these we get the following notion.

Definition 3.1. Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We say f is *limit preserving* if for every directed limit $U = \lim U_i$ of affine schemes over S the diagram

$$\begin{tikzcd} \mathrm{colim}\, \mathcal{X}_{U_i} \arrow[r] \arrow[d, "f"'] & \mathcal{X}_U \arrow[d, "f"] \\ \mathrm{colim}\, \mathcal{Y}_{U_i} \arrow[r] & \mathcal{Y}_U \end{tikzcd}$$

of fibre categories is 2-cartesian.

Lemma 3.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f is limit preserving (Definition 3.1), then f is limit preserving on objects (Criteria for Representability, Section 5).*

Proof. If for every directed limit $U = \lim U_i$ of affine schemes over U, the functor

$$\text{colim}\, \mathcal{X}_{U_i} \longrightarrow (\text{colim}\, \mathcal{Y}_{U_i}) \times_{\mathcal{Y}_U} \mathcal{X}_U$$

is essentially surjective, then f is limit preserving on objects. □

Lemma 3.3. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is limit preserving, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.*

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S. For each i we have

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{U_i} = \mathcal{X}_{U_i} \times_{\mathcal{Y}_{U_i}} \mathcal{Z}_{U_i}$$

Filtered colimits commute with 2-fibre products of categories (details omitted) hence if p is limit preserving we get

$$\begin{aligned} \text{colim}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{U_i} &= \text{colim}\, \mathcal{X}_{U_i} \times_{\text{colim}\, \mathcal{Y}_{U_i}} \text{colim}\, \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \text{colim}\, \mathcal{Y}_{U_i} \times_{\text{colim}\, \mathcal{Y}_{U_i}} \text{colim}\, \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \text{colim}\, \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{Z}_U \times_{\mathcal{Z}_U} \text{colim}\, \mathcal{Z}_{U_i} \\ &= (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_U \times_{\mathcal{Z}_U} \text{colim}\, \mathcal{Z}_{U_i} \end{aligned}$$

as desired. □

Lemma 3.4. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are limit preserving, then so is the composition $q \circ p$.*

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S. If p and q are limit preserving we get

$$\begin{aligned} \text{colim}\, \mathcal{X}_{U_i} &= \mathcal{X}_U \times_{\mathcal{Y}_U} \text{colim}\, \mathcal{Y}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{Y}_U \times_{\mathcal{Z}_U} \text{colim}\, \mathcal{Z}_{U_i} \\ &= \mathcal{X}_U \times_{\mathcal{Z}_U} \text{colim}\, \mathcal{Z}_{U_i} \end{aligned}$$

as desired. □

Lemma 3.5. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

(1) *p is limit preserving,*
(2) *p is limit preserving on objects, and*
(3) *p is locally of finite presentation (see Algebraic Stacks, Definition 10.1).*

Proof. In Criteria for Representability, Lemma 5.3 we have seen that (2) and (3) are equivalent. Thus it suffices to show that (1) and (2) are equivalent. One direction we saw in Lemma 3.2. For the other direction, let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S. We have to show that

$$\text{colim}\, \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \text{colim}\, \mathcal{Y}_{U_i}$$

is an equivalence. Since we are assuming (2) we know that it is essentially surjective. Hence we need to prove it is fully faithful. Since p is faithful on fibre categories (Algebraic Stacks, Lemma 9.2) we see that the functor is faithful. Let x_i and x'_i be objects in the fibre category of $\mathcal{X}$ over U_i. The functor above sends x_i to $(x_i|_U, p(x_i), can)$ where can is the canonical isomorphism $p(x_i|_U) \to p(x_i)|_U$. Thus we assume given a morphism

$$(\alpha, \beta_i) : (x_i|_U, p(x_i), can) \longrightarrow (x'_i|_U, p(x'_i), can)$$

in the category of the right hand side of the first displayed arrow of this proof. Our task is to produce an $i' \geq i$ and a morphism $x_i|_{U_{i'}} \to x'_i|_{U_{i'}}$ which maps to $(\alpha, \beta_i|_{U_{i'}})$.

Set $y_i = p(x_i)$ and $y'_i = p(x'_i)$. By (Algebraic Stacks, Lemma 9.2) the functor

$$X_{y_i} : (Sch/U_i)^{opp} \to Sets, \quad V/U_i \mapsto \{(x, \phi) \mid x \in \mathrm{Ob}(\mathcal{X}_V), \phi : f(x) \to y_i|V\}/\cong$$

is an algebraic space over U_i and the same is true for the analogously defined functor $X_{y'_i}$. Since (2) is equivalent to (3) we see that $X_{y'_i}$ is locally of finite presentation over U_i. Observe that (x_i, id) and (x'_i, id) define U_i-valued points of X_{y_i} and $X_{y'_i}$. There is a transformation of functors

$$\beta_i : X_{y_i} \to X_{y'_i}, \quad (x/V, \phi) \mapsto (x/V, \beta_i|_V \circ \phi)$$

in other words, this is a morphism of algebraic spaces over U_i. We claim that

$$\begin{tikzcd} U \arrow[rr] \arrow[d] & & U_i \arrow[d, "{(x'_i,\mathrm{id})}"] \\ U_i \arrow[r, "{(x_i,\mathrm{id})}"] & X_{y_i} \arrow[r, "\beta_i"] & X_{y'_i} \end{tikzcd}$$

commutes. Namely, this is equivalent to the condition that the pairs $(x_i|_U, \beta_i|_U)$ and $(x'_i|_U, \mathrm{id})$ as in the definition of the functor $X_{y'_i}$ are isomorphic. And the morphism $\alpha : x_i|_U \to x'_i|_U$ exactly produces such an isomorphism. Arguing backwards the reader sees that if we can find an $i' \geq i$ such that the diagram

$$\begin{tikzcd} U_{i'} \arrow[rr] \arrow[d] & & U_i \arrow[d, "{(x'_i,\mathrm{id})}"] \\ U_i \arrow[r, "{(x_i,\mathrm{id})}"] & X_{y_i} \arrow[r, "\beta_i"] & X_{y'_i} \end{tikzcd}$$

commutes, then we obtain an isomorphism $x_i|_{U_{i'}} \to x'_i|_{U_{i'}}$ which is a solution to the problem posed in the preceding paragraph. However, the diagonal morphism

$$\Delta : X_{y'_i} \to X_{y'_i} \times_{U_i} X_{y'_i}$$

is locally of finite presentation (Morphisms of Spaces, Lemma 28.10) hence the fact that $U \to U_i$ equalizes the two morphisms to $X_{y'_i}$, means that for some $i' \geq i$ the morphism $U_{i'} \to U_i$ equalizes the two morphisms, see Limits of Spaces, Proposition 3.10. $\square$

Lemma 3.6. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent*

(1) *the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is limit preserving, and*

(2) *for every directed limit $U = \lim U_i$ of affine schemes over S the functor*

$$\mathrm{colim}\, \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \mathrm{colim}\, \mathcal{Y}_{U_i}$$

is fully faithful.

In particular, if p is limit preserving, then Δ is too.

Proof. Let $U = \lim U_i$ be a directed limit of affine schemes over S. We claim that the functor

$$\mathrm{colim}\, \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{\mathcal{Y}_U} \mathrm{colim}\, \mathcal{Y}_{U_i}$$

is fully faithful if and only if the functor

$$\mathrm{colim}\, \mathcal{X}_{U_i} \longrightarrow \mathcal{X}_U \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_U} \mathrm{colim} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_{U_i}$$

is an equivalence. This will prove the lemma. Since $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_U = \mathcal{X}_U \times_{\mathcal{Y}_U} \mathcal{X}_U$ and $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})_{U_i} = \mathcal{X}_{U_i} \times_{\mathcal{Y}_{U_i}} \mathcal{X}_{U_i}$ this is a purely category theoretic assertion which we discuss in the next paragraph.

Let $\mathcal{I}$ be a filtered index category. Let $(\mathcal{C}_i)$ and $(\mathcal{D}_i)$ be systems of groupoids over $\mathcal{I}$. Let $p : (\mathcal{C}_i) \to (\mathcal{D}_i)$ be a map of systems of groupoids over $\mathcal{I}$. Suppose we have a functor $p : \mathcal{C} \to \mathcal{D}$ of groupoids and functors $f : \mathrm{colim}\, \mathcal{C}_i \to \mathcal{C}$ and $g : \mathrm{colim}\, \mathcal{D}_i \to \mathcal{D}$ fitting into a commutative diagram

$$\begin{tikzcd} \mathrm{colim}\, \mathcal{C}_i \arrow[r, "f"'] \arrow[d, "p"'] & \mathcal{C} \arrow[d, "p"] \\ \mathrm{colim}\, \mathcal{D}_i \arrow[r, "g"] & \mathcal{D} \end{tikzcd}$$

Then we claim that

$$A : \mathrm{colim}\, \mathcal{C}_i \longrightarrow \mathcal{C} \times_{\mathcal{D}} \mathrm{colim}\, \mathcal{D}_i$$

is fully faithful if and only if the functor

$$B : \mathrm{colim}\, \mathcal{C}_i \longrightarrow \mathcal{C} \times_{\Delta, \mathcal{C} \times_{\mathcal{D}} \mathcal{C}, f \times_g f} \mathrm{colim}(\mathcal{C}_i \times_{\mathcal{D}_i} \mathcal{C}_i)$$

is an equivalence. Set $\mathcal{C}' = \mathrm{colim}\, \mathcal{C}_i$ and $\mathcal{D}' = \mathrm{colim}\, \mathcal{D}_i$. Since 2-fibre products commute with filtered colimits we see that A and B become the functors

$$A' : \mathcal{C}' \to \mathcal{C} \times_{\mathcal{D}} \mathcal{D}' \quad \text{and} \quad B' : \mathcal{C}' \longrightarrow \mathcal{C} \times_{\Delta, \mathcal{C} \times_{\mathcal{D}} \mathcal{C}, f \times_g f} (\mathcal{C}' \times_{\mathcal{D}'} \mathcal{C}')$$

Thus it suffices to prove that if

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "f"'] \arrow[d, "p"'] & \mathcal{C} \arrow[d, "p"] \\ \mathcal{D}' \arrow[r, "g"] & \mathcal{D} \end{tikzcd}$$

is a commutative diagram of groupoids, then A' is fully faithful if and only if B' is an equivalence. This follows from Categories, Lemma 35.10 (with trivial, i.e., punctual, base category) because

$$\mathcal{C} \times_{\Delta, \mathcal{C} \times_{\mathcal{D}} \mathcal{C}, f \times_g f} (\mathcal{C}' \times_{\mathcal{D}'} \mathcal{C}') = \mathcal{C}' \times_{A', \mathcal{C} \times_{\mathcal{D}} \mathcal{D}', A'} \mathcal{C}'$$

This finishes the proof. $\square$

Lemma 3.7. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. If $\mathcal{X} \to S$ is locally of finite presentation, then $\mathcal{X}$ is limit preserving in the sense of Artin's Axioms, Definition 11.1 (equivalently: the morphism $\mathcal{X} \to S$ is limit preserving).*

Proof. Choose a surjective smooth morphism $U \to \mathcal{X}$ for some scheme U. Then $U \to S$ is locally of finite presentation, see Morphisms of Stacks, Section 27. We can write $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces (U, R, s, t, c), see Algebraic Stacks, Lemma 16.2. Since U is locally of finite presentation over S it follows that the algebraic space R is locally of finite presentation over S. Recall that $[U/R]$ is the stack in groupoids over $(Sch/S)_{fppf}$ obtained by stackyfying the category fibred in groupoids whose fibre category over T is the groupoid $(U(T), R(T), s, t, c)$. Since U and R are limit preserving as functors (Limits of Spaces, Proposition 3.10) this category fibred in groupoids is limit preserving. Thus it suffices to show that fppf stackyfication preserves the property of being limit preserving. This is true (hint: use Topologies, Lemma 13.2). However, we give a direct proof below using that in this case we know what the stackyfication amounts to.

Let $T = \lim T_\lambda$ be a directed limit of affine schemes over S. We have to show that the functor

$$\mathrm{colim}[U/R]_{T_\lambda} \longrightarrow [U/R]_T$$

is an equivalence of categories. Let us show this functor is essentially surjective. Let $x \in \mathrm{Ob}([U/R]_T)$. In Groupoids in Spaces, Lemma 24.1 the reader finds a description of the category $[U/R]_T$. In particular x corresponds to an fppf covering $\{T_i \to T\}_{i \in I}$ and a $[U/R]$-descent datum (u_i, r_{ij}) relative to this covering. After refining this covering we may assume it is a standard fppf covering of the affine scheme T. By Topologies, Lemma 13.2 we may choose a λ and a standard fppf covering $\{T_{\lambda,i} \to T_\lambda\}_{i \in I}$ whose base change to T is equal to $\{T_i \to T\}_{i \in I}$. For each i, after increasing λ, we can find a $u_{\lambda,i} : T_{\lambda,i} \to U$ whose composition with $T_i \to T_{\lambda,i}$ is the given morphism u_i (this is where we use that U is limit preserving). Similarly, for each i, j, after increasing λ, we can find a $r_{\lambda,ij} : T_{\lambda,i} \times_{T_\lambda} T_{\lambda,j} \to R$ whose composition with $T_{ij} \to T_{\lambda,ij}$ is the given morphism r_{ij} (this is where we use that R is limit preserving). After increasing λ we can further assume that

$$s \circ r_{\lambda,ij} = u_{\lambda,i} \circ \mathrm{pr}_0 \quad \text{and} \quad t \circ r_{\lambda,ij} = u_{\lambda,j} \circ \mathrm{pr}_1,$$

and

$$c \circ (r_{\lambda,jk} \circ \mathrm{pr}_{12}, r_{\lambda,ij} \circ \mathrm{pr}_{01}) = r_{\lambda,ik} \circ \mathrm{pr}_{02}.$$

In other words, we may assume that $(u_{\lambda,i}, r_{\lambda,ij})$ is a $[U/R]$-descent datum relative to the covering $\{T_{\lambda,i} \to T_\lambda\}_{i \in I}$. Then we obtain a corresponding object of $[U/R]$ over T_λ whose pullback to T is isomorphic to x as desired. The proof of fully faithfulness works in exactly the same way using the description of morphisms in the fibre categories of $[U/T]$ given in Groupoids in Spaces, Lemma 24.1. □

Proposition 3.8. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

1. *f is limit preserving,*
2. *f is limit preserving on objects, and*
3. *f is locally of finite presentation.*

Proof. Assume (3). Let $T = \lim T_i$ be a directed limit of affine schemes. Consider the functor

$$\mathrm{colim}\, \mathcal{X}_{T_i} \longrightarrow \mathcal{X}_T \times_{\mathcal{Y}_T} \mathrm{colim}\, \mathcal{Y}_{T_i}$$

Let (x, y_i, β) be an object on the right hand side, i.e., $x \in \mathrm{Ob}(\mathcal{X}_T)$, $y_i \in \mathrm{Ob}(\mathcal{Y}_{T_i})$, and $\beta : f(x) \to y_i|_T$ in $\mathcal{Y}_T$. Then we can consider (x, y_i, β) as an object of the algebraic stack $\mathcal{X}_{y_i} = \mathcal{X} \times_{\mathcal{Y}, y_i} T_i$ over T. Since $\mathcal{X}_{y_i} \to T_i$ is locally of finite presentation (as a base change of f) we see that it is limit preserving by Lemma 3.7. This means that (x, y_i, β) comes from an object over $T_{i'}$ for some $i' \geq i$ and unwinding the definitions we find that (x, y_i, β) is in the essential image of the displayed functor. In other words, the displayed functor is essentially surjective. Another formulation is that this means f is limit preserving on objects. Now we apply this to the diagonal Δ of f. Namely, by Morphisms of Stacks, Lemma 27.7 the morphism Δ is locally of finite presentation. Thus the argument above shows that Δ is limit preserving on objects. By Lemma 3.5 this implies that Δ is limit preserving. By Lemma 3.6 we conclude that the displayed functor above is fully faithful. Thus it is an equivalence (as we already proved essential surjectivity) and we conclude that (1) holds.

The implication (1) $\Rightarrow$ (2) is trivial. Assume (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. By Criteria for Representability, Lemma 5.1 the base change $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is limit preserving on objects. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Since a smooth morphism is locally of finite presentation, we see that $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is limit preserving (first part of the proof). By Criteria for Representability, Lemma 5.2 we find that the composition $U \to V$ is limit preserving on objects. We conclude that $U \to V$ is locally of finite presentation, see Criteria for Representability, Lemma 5.3. This is exactly the condition that f is locally of finite presentation, see Morphisms of Stacks, Definition 27.1. $\square$

4. Descending properties

This section is the analogue of Limits, Section 4.

Situation 4.1. Let $Y = \lim_{i \in I} Y_i$ be a limit of a directed system of algebraic spaces with affine transition morphisms. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

Lemma 4.2. *In Situation 4.1 assume that $\mathcal{X}_0 \to Y_0$ is a morphism from algebraic stack to Y_0. Assume $\mathcal{X}_0$ is quasi-compact and quasi-separated. If $Y \times_{Y_0} \mathcal{X}_0 \to Y$ is separated, then $Y_i \times_{Y_0} \mathcal{X}_0 \to Y_i$ is separated for all sufficiently large $i \in I$.*

Proof. Write $\mathcal{X} = Y \times_{Y_0} \mathcal{X}_0$ and $\mathcal{X}_i = Y_i \times_{Y_0} \mathcal{X}_0$. Choose an affine scheme U_0 and a surjective smooth morphism $U_0 \to \mathcal{X}_0$. Set $U = Y \times_{Y_0} U_0$ and $U_i = Y_i \times_{Y_0} U_0$. Then U and U_i are affine and $U \to \mathcal{X}$ and $U_i \to \mathcal{X}_i$ are smooth and surjective. Set $R_0 = U_0 \times_{\mathcal{X}_0} U_0$. Set $R = Y \times_{Y_0} R_0$ and $R_i = Y_i \times_{Y_0} R_0$. Then $R = U \times_{\mathcal{X}} U$ and $R_i = U_i \times_{\mathcal{X}_i} U_i$.

With this notation note that $\mathcal{X} \to Y$ is separated implies that $R \to U \times_Y U$ is proper as the base change of $\mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}$ by $U \times_Y U \to \mathcal{X} \times_Y \mathcal{X}$. Conversely, we see that $\mathcal{X}_i \to Y_i$ is separated if $R_i \to U_i \times_{Y_i} U_i$ is proper because $U_i \times_{Y_i} U_i \to \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$ is surjective and smooth, see Properties of Stacks, Lemma 3.3. Observe that $R_0 \to U_0 \times_{Y_0} U_0$ is locally of finite type and that R_0 is quasi-compact and quasi-separated. By Limits of Spaces, Lemma 6.13 we see that $R_i \to U_i \times_{Y_i} U_i$ is proper for large enough i which finishes the proof. $\square$

5. Descending relative objects

This section is the analogue of Limits of Spaces, Section 7.

Lemma 5.1. *Let I be a directed set. Let $(X_i, f_{ii'})$ be an inverse system of algebraic spaces over I. Assume*

(1) *the morphisms* $f_{ii'} : X_i \to X_{i'}$ *are affine,*
(2) *the spaces* X_i *are quasi-compact and quasi-separated.*

Let $X = \lim X_i$. *If* $\mathcal{X}$ *is an algebraic stack of finite presentation over* X, *then there exists an* $i \in I$ *and an algebraic stack* $\mathcal{X}_i$ *of finite presentation over* X_i *with* $\mathcal{X} \cong \mathcal{X}_i \times_{X_i} X$ *as algebraic stacks over* X.

Proof. By Morphisms of Stacks, Definition 27.1 the morphism $\mathcal{X} \to X$ is quasi-compact, locally of finite presentation, and quasi-separated. Since X is quasi-compact and $\mathcal{X} \to X$ is quasi-compact, we see that $\mathcal{X}$ is quasi-compact (Morphisms of Stacks, Definition 7.2). Hence we can find an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$ (Properties of Stacks, Lemma 6.2). Set $R = U \times_{\mathcal{X}} U$. We obtain a smooth groupoid in algebraic spaces (U, R, s, t, c) over X such that $\mathcal{X} = [U/R]$, see Algebraic Stacks, Lemma 16.2. Since $\mathcal{X} \to X$ is quasi-separated and X is quasi-separated we see that $\mathcal{X}$ is quasi-separated (Morphisms of Stacks, Lemma 4.10). Thus $R \to U \times U$ is quasi-compact and quasi-separated (Morphisms of Stacks, Lemma 4.7) and hence R is a quasi-separated and quasi-compact algebraic space. On the other hand $U \to X$ is locally of finite presentation and hence also $R \to X$ is locally of finite presentation (because $s : R \to U$ is smooth hence locally of finite presentation). Thus (U, R, s, t, c) is a groupoid object in the category of algebraic spaces which are of finite presentation over X. By Limits of Spaces, Lemma 7.1 there exists an i and a groupoid in algebraic spaces $(U_i, R_i, s_i, t_i, c_i)$ over X_i whose pullback to X is isomorphic to (U, R, s, t, c). After increasing i we may assume that s_i and t_i are smooth, see Limits of Spaces, Lemma 6.3. The quotient stack $\mathcal{X}_i = [U_i/R_i]$ is an algebraic stack (Algebraic Stacks, Theorem 17.3).

There is a morphism $[U/R] \to [U_i/R_i]$, see Groupoids in Spaces, Lemma 21.1. We claim that combined with the morphisms $[U/R] \to X$ and $[U_i/R_i] \to X_i$ (Groupoids in Spaces, Lemma 20.2) we obtain an isomorphism (i.e., equivalence)

$$[U/R] \longrightarrow [U_i/R_i] \times_{X_i} X$$

The corresponding map

$$[U/_pR] \longrightarrow [U_i/_pR_i] \times_{X_i} X$$

on the level of "presheaves of groupoids" as in Groupoids in Spaces, Equation (20.0.1) is an isomorphism. Thus the claim follows from the fact that stackification commutes with fibre products, see Stacks, Lemma 8.4. □

6. Finite type closed in finite presentation

This section is the analogue of Limits of Spaces, Section 11.

Lemma 6.1. *Let* $f : \mathcal{X} \to Y$ *be a morphism from an algebraic stack to an algebraic space. Assume:*

(1) f *is of finite type and quasi-separated,*
(2) Y *is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : \mathcal{X}' \to Y$ *and a closed immersion* $\mathcal{X} \to \mathcal{X}'$ *of algebraic stacks over* Y.

Proof. Write $Y = \lim_{i \in I} Y_i$ as a limit of algebraic spaces over a directed set I with affine transition morphisms and with Y_i Noetherian, see Limits of Spaces, Proposition 8.1. We will use the material from Limits of Spaces, Section 23.

Choose a presentation $\mathcal{X} = [U/R]$. Denote (U, R, s, t, c, e, i) the corresponding groupoid in algebraic spaces over Y. We may and do assume U is affine. Then U, R, $R \times_{s,U,t} R$ are quasi-separated algebraic spaces of finite type over Y. We have two morpisms $s, t : R \to U$, three morphisms $c : R \times_{s,U,t} R \to R$, $\text{pr}_1 : R \times_{s,U,t} R \to R$, $\text{pr}_2 : R \times_{s,U,t} R \to R$, a morphism $e : U \to R$, and finally a morphism $i : R \to R$. These morphisms satisfy a list of axioms which are detailed in Groupoids, Section 13.

According to Limits of Spaces, Remark 23.5 we can find an $i_0 \in I$ and inverse systems

(1) $(U_i)_{i \geq i_0}$,
(2) $(R_i)_{i \geq i_0}$,
(3) $(T_i)_{i \geq i_0}$

over $(Y_i)_{i \geq i_0}$ such that $U = \lim_{i \geq i_0} U_i$, $R = \lim_{i \geq i_0} R_i$, and $R \times_{s,U,t} R = \lim_{i \geq i_0} T_i$ and such that there exist morphisms of systems

(1) $(s_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \to (U_i)_{i \geq i_0}$,
(2) $(t_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \to (U_i)_{i \geq i_0}$,
(3) $(c_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(4) $(p_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(5) $(q_i)_{i \geq i_0} : (T_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(6) $(e_i)_{i \geq i_0} : (U_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$,
(7) $(i_i)_{i \geq i_0} : (R_i)_{i \geq i_0} \to (R_i)_{i \geq i_0}$

with $s = \lim_{i \geq i_0} s_i$, $t = \lim_{i \geq i_0} t_i$, $c = \lim_{i \geq i_0} c_i$, $\text{pr}_1 = \lim_{i \geq i_0} p_i$, $\text{pr}_2 = \lim_{i \geq i_0} q_i$, $e = \lim_{i \geq i_0} e_i$, and $i = \lim_{i \geq i_0} i_i$. By Limits of Spaces, Lemma 23.7 we see that we may assume that s_i and t_i are smooth (this may require increasing i_0). By Limits of Spaces, Lemma 23.6 we may assume that the maps $R \to U \times_{U_i, s_i} R_i$ given by s and $R \to R_i$ and $R \to U \times_{U_i, t_i} R_i$ given by t and $R \to R_i$ are isomorphisms for all $i \geq i_0$. By Limits of Spaces, Lemma 23.9 we see that we may assume that the diagrams

$$\begin{tikzcd} T_i \arrow[r, "q_i"'] \arrow[d, "p_i"'] & R_i \arrow[d, "t_i"] \\ R_i \arrow[r, "s_i"] & U_i \end{tikzcd}$$

are cartesian. The uniqueness of Limits of Spaces, Lemma 23.4 then guarantees that for a sufficiently large i the relations between the morphisms s, t, c, e, i mentioned above are satisfied by s_i, t_i, c_i, e_i, i_i. Fix such an i.

It follows that $(U_i, R_i, s_i, t_i, c_i, e_i, i_i)$ is a smooth groupoid in algebraic spaces over Y_i. Hence $\mathcal{X}_i = [U_i/R_i]$ is an algebraic stack (Algebraic Stacks, Theorem 17.3). The morphism of groupoids

$$(U, R, s, t, c, e, i) \to (U_i, R_i, s_i, t_i, c_i, e_i, i_i)$$

over $Y \to Y_i$ determines a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}_i \arrow[d] \\ Y \arrow[r] & Y_i \end{tikzcd}$$

(Groupoids in Spaces, Lemma 21.1). We claim that the morphism $\mathcal{X} \to Y \times_{Y_i} \mathcal{X}_i$ is a closed immersion. The claim finishes the proof because the algebraic stack $\mathcal{X}_i \to Y_i$ is of finite presentation by construction. To prove the claim, note that the left diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U_i \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}_i \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r] \arrow[d] & Y \times_{Y_i} U_i \arrow[d] \\ \mathcal{X} \arrow[r] & Y \times_{Y_i} \mathcal{X}_i \end{tikzcd}$$

is cartesian by Groupoids in Spaces, Lemma 25.3 and the results mentioned above. Hence the right commutative diagram is cartesian too. Then the desired result follows from the fact that $U \to Y \times_{Y_i} U_i$ is a closed immersion by construction of the inverse system (U_i) in Limits of Spaces, Lemma 23.3, the fact that $Y \times_{Y_i} U_i \to Y \times_{Y_i} \mathcal{X}_i$ is smooth and surjective, and Properties of Stacks, Lemma 9.4. $\square$

There is a version for separated algebraic stacks.

Lemma 6.2. *Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Assume:*

1. *f is of finite type and separated,*
2. *Y is quasi-compact and quasi-separated.*

Then there exists a separated morphism of finite presentation $f' : \mathcal{X}' \to Y$ and a closed immersion $\mathcal{X} \to \mathcal{X}'$ of algebraic stacks over Y.

Proof. First we use exactly the same procedure as in the proof of Lemma 6.1 (and we borrow its notation) to construct the embedding $\mathcal{X} \to \mathcal{X}'$ as a morphism $\mathcal{X} \to \mathcal{X}' = Y \times_{Y_i} \mathcal{X}_i$ with $\mathcal{X}_i = [U_i/R_i]$. Thus it is enough to show that $\mathcal{X}_i \to Y_i$ is separated for sufficiently large i. In other words, it is enough to show that $\mathcal{X}_i \to \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$ is proper for i sufficiently large. Since the morphism $U_i \times_{Y_i} U_i \to \mathcal{X}_i \times_{Y_i} \mathcal{X}_i$ is surjective and smooth and since $R_i = \mathcal{X}_i \times_{\mathcal{X}_i \times_{Y_i} \mathcal{X}_i} U_i \times_{Y_i} U_i$ it is enough to show that the morphism $(s_i, t_i) : R_i \to U_i \times_{Y_i} U_i$ is proper for i sufficiently large, see Properties of Stacks, Lemma 3.3. We prove this in the next paragraph.

Observe that $U \times_Y U \to Y$ is quasi-separated and of finite type. Hence we can use the construction of Limits of Spaces, Remark 23.5 to find an $i_1 \in I$ and an inverse system $(V_i)_{i \geq i_1}$ with $U \times_Y U = \lim_{i \geq i_1} V_i$. By Limits of Spaces, Lemma 23.9 for i sufficiently large the functoriality of the construction applied to the projections $U \times_Y U \to U$ gives closed immersions

$$V_i \to U_i \times_{Y_i} U_i$$

(There is a small mismatch here because in truth we should replace Y_i by the scheme theoretic image of $Y \to Y_i$, but clearly this does not change the fibre product.) On the other hand, by Limits of Spaces, Lemma 23.8 the functoriality applied to the proper morphism $(s, t) : R \to U \times_Y U$ (here we use that $\mathcal{X}$ is separated) leads to morphisms $R_i \to V_i$ which are proper for large enough i. Composing these morphisms we obtain a proper morphisms $R_i \to U_i \times_{Y_i} U_i$ for all i large enough. The functoriality of the construction of Limits of Spaces, Remark 23.5 shows that this is the morphism is the same as (s_i, t_i) for large enough i and the proof is complete. $\square$

7. Universally closed morphisms

This section is the analogue of Limits of Spaces, Section 20.

Lemma 7.1. *Let $g : Z \to Y$ be a morphism of affine schemes. Let $f : \mathcal{X} \to Y$ be a quasi-compact morphism of algebraic stacks. Let $z \in Z$ and let $T \subset |\mathcal{X} \times_Y Z|$ be a closed subset with $z \notin \mathrm{Im}(T \to |Z|)$. If $\mathcal{X}$ is quasi-compact, then there exist an open neighbourhood $V \subset Z$ of z, a commutative diagram*

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |X \times_Y Z'|$ such that

1. *Z' is an affine scheme of finite presentation over Y,*
2. *with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and*
3. *the inverse image of T in $|\mathcal{X} \times_Y V|$ maps into T' via $|\mathcal{X} \times_Y V| \to |\mathcal{X} \times_Y Z'|$.*

Proof. We will deduce this from the corresponding result for morphisms of schemes. Since $\mathcal{X}$ is quasi-compact, we may choose an affine scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Let $T_W \subset |W \times_Y Z|$ be the inverse image of T. Then z is not in the image of T_W. By the schemes case (Limits, Lemma 14.1) we can find an open neighbourhood $V \subset Z$ of z a commutative diagram of schemes

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |W \times_Y Z'|$ such that

1. Z' is an affine scheme of finite presentation over Y,
2. with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and
3. $T_1 = T_W \cap |W \times_Y V|$ maps into T' via $|W \times_Y V| \to |W \times_Y Z'|$.

The commutative diagram

$$\begin{tikzcd} W \times_Y Z \arrow[d] & W \times_Y V \arrow[l] \arrow[r, "a_1"'] \arrow[d, "c"'] & W \times_Y Z' \arrow[d, "q"] \\ \mathcal{X} \times_Y Z & \mathcal{X} \times_Y V \arrow[l] \arrow[r, "a_2"] & \mathcal{X} \times_Y Z' \end{tikzcd}$$

has cartesian squares and the vertical maps are surjective, smooth, and a fortiori open. Looking at the left hand square we see that $T_1 = T_W \cap |W \times_Y V|$ is the inverse image of $T_2 = T \cap |\mathcal{X} \times_Y V|$ by c. By Properties of Stacks, Lemma 4.3 we get $a_1(T_1) = q^{-1}(a_2(T_2))$. By Topology, Lemma 6.4 we get

$$q^{-1}\left(\overline{a_2(T_2)}\right) = \overline{q^{-1}(a_2(T_2))} = \overline{a_1(T_1)} \subset T'$$

As q is surjective the image of $\overline{a_2(T_2)} \to |Z'|$ does not contain z' since the same is true for T'. Thus we can take the diagram with Z', V, a, b above and the closed subset $\overline{a_2(T_2)} \subset |\mathcal{X} \times_Y Z'|$ as a solution to the problem posed by the lemma. $\square$

Lemma 7.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. The following are equivalent*

(1) *f is universally closed,*
(2) *for every morphism $Z \to \mathcal{Y}$ which is locally of finite presentation and where Z is an affine scheme the map $|\mathcal{X} \times_Y Z| \to |Z|$ is closed, and*
(3) *there exists a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$ such that $|\mathbf{A}^n \times (\mathcal{X} \times_{\mathcal{Y}} V)| \to |\mathbf{A}^n \times V|$ is closed for all $n \geq 0$.*

Proof. It is clear that (1) implies (2).

Assume (2). Choose a scheme V which is the disjoint union of affine schemes and a surjective smooth morphism $V \to \mathcal{Y}$. In order to show that f is universally closed, it suffices to show that the base change $\mathcal{X} \times_{\mathcal{Y}} V \to V$ of f is universally closed, see Morphisms of Stacks, Lemma 13.5. Note that property (2) holds for this base change. Hence in order to prove that (2) implies (1) we may assume $Y = \mathcal{Y}$ is an affine scheme.

Assume (2) and assume $\mathcal{Y} = Y$ is an affine scheme. If f is not universally closed, then there exists an affine scheme Z over Y such that $|\mathcal{X} \times_Y Z| \to |Z|$ is not closed, see Morphisms of Stacks, Lemma 13.5. This means that there exists some closed subset $T \subset |\mathcal{X} \times_Y Z|$ such that $\mathrm{Im}(T \to |Z|)$ is not closed. Pick $z \in |Z|$ in the closure of the image of T but not in the image. Apply Lemma 7.1. We find an open neighbourhood $V \subset Z$, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & Z' \arrow[d, "b"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

and a closed subset $T' \subset |\mathcal{X} \times_Y Z'|$ such that

(1) Z' is an affine scheme of finite presentation over Y,
(2) with $z' = a(z)$ we have $z' \notin \mathrm{Im}(T' \to |Z'|)$, and
(3) the inverse image of T in $|\mathcal{X} \times_Y V|$ maps into T' via $|\mathcal{X} \times_Y V| \to |\mathcal{X} \times_Y Z'|$.

We claim that z' is in the closure of $\mathrm{Im}(T' \to |Z'|)$. This implies that $|\mathcal{X} \times_Y Z'| \to |Z'|$ is not closed and this is absurd as we assumed (2), in other words, the claim shows that (2) implies (1). To see the claim is true we contemplate the following commutative diagram

$$\begin{tikzcd} \mathcal{X} \times_Y Z \arrow[d] & \mathcal{X} \times_Y V \arrow[l] \arrow[r] \arrow[d] & \mathcal{X} \times_Y Z' \arrow[d] \\ Z & V \arrow[l] \arrow[r, "a"] & Z' \end{tikzcd}$$

Let $T_V \subset |\mathcal{X} \times_Y V|$ be the inverse image of T. By Properties of Stacks, Lemma 4.3 the image of T_V in $|V|$ is the inverse image of the image of T in $|Z|$. Then since z is in the closure of the image of $T \to |Z|$ and since $|V| \to |Z|$ is open, we see that z is in the closure of the image of $T_V \to |V|$. Since the image of T_V in $|\mathcal{X} \times_Y Z'|$ is contained in $|T'|$ it follows immediately that $z' = a(z)$ is in the closure of the image of T'.

It is clear that (1) implies (3). Let $V \to \mathcal{Y}$ be as in (3). If we can show that $\mathcal{X} \times_Y V \to V$ is universally closed, then f is universally closed by Morphisms of Stacks, Lemma 13.5. Thus it suffices to show that $f : \mathcal{X} \to \mathcal{Y}$ satisfies (2) if f is a quasi-compact morphism of algebraic stacks, $\mathcal{Y} = Y$ is a scheme, and $|\mathbf{A}^n \times \mathcal{X}| \to |\mathbf{A}^n \times Y|$ is closed for all n. Let $Z \to Y$ be locally of finite presentation where Z is an affine scheme. We have to show the map $|\mathcal{X} \times_Y Z| \to |Z|$ is closed. Since Y is a scheme, Z is affine, and $Z \to Y$ is locally of finite presentation we can find an immersion $Z \to \mathbf{A}^n \times Y$, see Morphisms, Lemma 39.2. Consider the cartesian diagram

$$\begin{tikzcd} \mathcal{X} \times_Y Z \arrow[r] \arrow[d] & \mathbf{A}^n \times \mathcal{X} \arrow[d] \\ Z \arrow[r] & \mathbf{A}^n \times Y \end{tikzcd} \quad \text{inducing the cartesian square} \quad \begin{tikzcd} |\mathcal{X} \times_Y Z| \arrow[r] \arrow[d] & |\mathbf{A}^n \times \mathcal{X}| \arrow[d] \\ |Z| \arrow[r] & |\mathbf{A}^n \times Y| \end{tikzcd}$$

of topological spaces whose horizontal arrows are homeomorphisms onto locally closed subsets (Properties of Stacks, Lemma 9.6). Thus every closed subset T of $|X \times_Y Z|$ is the pullback of a closed subset T' of $|\mathbf{A}^n \times Y|$. Since the assumption is that the image of T' in $|\mathbf{A}^n \times X|$ is closed we conclude that the image of T in $|Z|$ is closed as desired. $\square$

8. Other chapters

References

[Ryd08] David Rydh, *Noetherian approximation of algebraic spaces and stacks*, math.AG/0904.0227 (2008).