SCHEMES

Contents

1. Introduction

In this document we define schemes. A basic reference is [DG67].

2. Locally ringed spaces

Recall that we defined ringed spaces in Sheaves, Section 25. Briefly, a ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and an f-map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$. You can think of $f^\sharp$ as a map $\mathcal{O}_Y \to f_* \mathcal{O}_X$, see Sheaves, Definition 21.7 and Lemma 21.8.

A good geometric example of this to keep in mind is $\mathcal{C}^\infty$-manifolds and morphisms of $\mathcal{C}^\infty$-manifolds. Namely, if M is a $\mathcal{C}^\infty$-manifold, then the sheaf $\mathcal{C}^\infty_M$ of smooth functions is a sheaf of rings on M. And any map $f : M \to N$ of manifolds is smooth if and only if for every local section h of $\mathcal{C}^\infty_N$ the composition $h \circ f$ is a local section of $\mathcal{C}^\infty_M$. Thus a smooth map f gives rise in a natural way to a morphism of ringed spaces

$$f : (M, \mathcal{C}^\infty_M) \longrightarrow (N, \mathcal{C}^\infty_N)$$

see Sheaves, Example 25.2. It is instructive to consider what happens to stalks. Namely, let $m \in M$ with image $f(m) = n \in N$. Recall that the stalk $\mathcal{C}^\infty_{M,m}$ is the ring of germs of smooth functions at m, see Sheaves, Example 11.4. The algebra of germs of functions on (M, m) is a local ring with maximal ideal the functions which vanish at m. Similarly for $\mathcal{C}^\infty_{N,n}$. The map on stalks $f^\sharp : \mathcal{C}^\infty_{N,n} \to \mathcal{C}^\infty_{M,m}$ maps the maximal ideal into the maximal ideal, simply because $f(m) = n$.

In algebraic geometry we study schemes. On a scheme the sheaf of rings is not determined by an intrinsic property of the space. The spectrum of a ring R (see Algebra, Section 17) endowed with a sheaf of rings constructed out of R (see below), will be our basic building block. It will turn out that the stalks of $\mathcal{O}$ on $\mathrm{Spec}(R)$ are the local rings of R at its primes. There are two reasons to introduce locally ringed spaces in this setting: (1) There is in general no mechanism that assigns to a continuous map of spectra a map of the corresponding rings. This is why we add as an extra datum the map $f^\sharp$. (2) If we consider morphisms of these spectra in the category of ringed spaces, then the maps on stalks may not be local homomorphisms. Since our geometric intuition says it should we introduce locally ringed spaces as follows.

Definition 2.1. Locally ringed spaces.
  (1) A *locally ringed space* $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ all of whose stalks are local rings.
  (2) Given a locally ringed space $(X, \mathcal{O}_X)$ we say that $\mathcal{O}_{X,x}$ is the *local ring of X at x*. We denote $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$. Moreover, the *residue field of X at x* is the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.
  (3) A *morphism of locally ringed spaces* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.

We will usually suppress the sheaf of rings $\mathcal{O}_X$ in the notation when discussing locally ringed spaces. We will simply refer to "the locally ringed space X". We will by abuse of notation think of X also as the underlying topological space. Finally we will denote the corresponding sheaf of rings $\mathcal{O}_X$ as the *structure sheaf of X*. In addition, it is customary to denote the maximal ideal of the local ring $\mathcal{O}_{X,x}$ by $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$. We will say "let $f : X \to Y$ be a morphism of locally ringed spaces" thereby suppressing the structure sheaves even further. In this case, we will by abuse of notation think of $f : X \to Y$ also as the underlying continuous map of topological spaces. The f-map corresponding to f will customarily be denoted $f^\sharp$. The condition that f is a morphism of locally ringed spaces can then be expressed by saying that for every $x \in X$ the map on stalks

$$f^\sharp_x : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

maps the maximal ideal $\mathfrak{m}_{Y,f(x)}$ into $\mathfrak{m}_{X,x}$.

Let us use these notational conventions to show that the collection of locally ringed spaces and morphisms of locally ringed spaces forms a category. In order to see this we have to show that the composition of morphisms of locally ringed spaces is a morphism of locally ringed spaces. OK, so let $f : X \to Y$ and $g : Y \to Z$ be morphism of locally ringed spaces. The composition of f and g is defined in Sheaves, Definition 25.3. Let $x \in X$. By Sheaves, Lemma 21.10 the composition

$$\mathcal{O}_{Z,g(f(x))} \xrightarrow{g^{\sharp}} \mathcal{O}_{Y,f(x)} \xrightarrow{f^{\sharp}} \mathcal{O}_{X,x}$$

is the associated map on stalks for the morphism $g \circ f$. The result follows since a composition of local ring homomorphisms is a local ring homomorphism.

A pleasing feature of the definition is the fact that the functor

$$\textit{Locally ringed spaces} \longrightarrow \textit{Ringed spaces}$$

reflects isomorphisms (plus more). Here is a less abstract statement.

Lemma 2.2. *Let X, Y be locally ringed spaces. If $f : X \to Y$ is an isomorphism of ringed spaces, then f is an isomorphism of locally ringed spaces.*

Proof. This follows trivially from the corresponding fact in algebra: Suppose A, B are local rings. Any isomorphism of rings $A \to B$ is a local ring homomorphism. $\square$

3. Open immersions of locally ringed spaces

Definition 3.1. Let $f : X \to Y$ be a morphism of locally ringed spaces. We say that f is an *open immersion* if f is a homeomorphism of X onto an open subset of Y, and the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

The following construction is parallel to Sheaves, Definition 31.2 (3).

Example 3.2. Let X be a locally ringed space. Let $U \subset X$ be an open subset. Let $\mathcal{O}_U = \mathcal{O}_X|_U$ be the restriction of $\mathcal{O}_X$ to U. For $u \in U$ the stalk $\mathcal{O}_{U,u}$ is equal to the stalk $\mathcal{O}_{X,u}$, and hence is a local ring. Thus $(U, \mathcal{O}_U)$ is a locally ringed space and the morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ is an open immersion.

Definition 3.3. Let X be a locally ringed space. Let $U \subset X$ be an open subset. The locally ringed space $(U, \mathcal{O}_U)$ of Example 3.2 above is the *open subspace of X associated to U*.

Lemma 3.4. *Let $f : X \to Y$ be an open immersion of locally ringed spaces. Let $j : V = f(X) \to Y$ be the open subspace of Y associated to the image of f. There is a unique isomorphism $f' : X \cong V$ of locally ringed spaces such that $f = j \circ f'$.*

Proof. Let f' be the homeomorphism between X and V induced by f. Then $f = j \circ f'$ as maps of topological spaces. Since there is an isomorphism of sheaves $f^{\sharp} : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$, there is an isomorphism of rings $f^{\sharp} : \Gamma(U, f^{-1}(\mathcal{O}_Y)) \to \Gamma(U, \mathcal{O}_X)$ for each open subset $U \subset X$. Since $\mathcal{O}_V = j^{-1}\mathcal{O}_Y$ and $f^{-1} = f'^{-1}j^{-1}$ (Sheaves, Lemma 21.6) we see that $f^{-1}\mathcal{O}_Y = f'^{-1}\mathcal{O}_V$, hence $\Gamma(U, f'^{-1}(\mathcal{O}_V)) \to \Gamma(U, f^{-1}(\mathcal{O}_Y))$ is an isomorphism for every $U \subset X$ open. By composing these we get an isomorphism of rings

$$\Gamma(U, f'^{-1}(\mathcal{O}_V)) \to \Gamma(U, \mathcal{O}_X)$$

for each open subset $U \subset X$, and therefore an isomorphism of sheaves $f^{-1}(\mathcal{O}_V) \to \mathcal{O}_X$. In other words, we have an isomorphism $f'^{\sharp} : f'^{-1}(\mathcal{O}_V) \to \mathcal{O}_X$ and therefore

an isomorphism of locally ringed spaces $(f', f'^\sharp) : (X, \mathcal{O}_X) \to (V, \mathcal{O}_V)$ (use Lemma 2.2). Note that $f = j \circ f'$ as morphisms of locally ringed spaces by construction.

Suppose we have another morphism $f'' : (X, \mathcal{O}_X) \to (V, \mathcal{O}_V)$ such that $f = j \circ f''$. At any point $x \in X$, we have $j(f'(x)) = j(f''(x))$ from which it follows that $f'(x) = f''(x)$ since j is the inclusion map; therefore f' and f'' are the same as morphisms of topological spaces. On structure sheaves, for each open subset $U \subset X$ we have a commutative diagram

$$\Gamma(U, f^{-1}(\mathcal{O}_Y)) \xrightarrow{\cong} \Gamma(U, \mathcal{O}_X)$$

$$\Gamma(U, f'^{-1}(\mathcal{O}_V))$$

from which we see that $f'^\sharp$ and $f''^\sharp$ define the same morphism of sheaves. $\square$

From now on we do not distinguish between open subsets and their associated subspaces.

Lemma 3.5. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $U \subset X$, and $V \subset Y$ be open subsets. Suppose that $f(U) \subset V$. There exists a unique morphism of locally ringed spaces $f|_U : U \to V$ such that the following diagram is a commutative square of locally ringed spaces*

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle f|_U}\downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & Y \end{array}$$

Proof. Omitted. $\square$

In the following we will use without further mention the following fact which follows from the lemma above. Given any morphism $f : Y \to X$ of locally ringed spaces, and any open subset $U \subset X$ such that $f(Y) \subset U$, then there exists a unique morphism of locally ringed spaces $Y \to U$ such that the composition $Y \to U \to X$ is equal to f. In fact, we will even by abuse of notation write $f : Y \to U$ since this rarely gives rise to confusion.

4. Closed immersions of locally ringed spaces

We follow our conventions introduced in Modules, Definition 13.1.

Definition 4.1. Let $i : Z \to X$ be a morphism of locally ringed spaces. We say that i is a *closed immersion* if:

(1) The map i is a homeomorphism of Z onto a closed subset of X.
(2) The map $\mathcal{O}_X \to i_* \mathcal{O}_Z$ is surjective; let $\mathcal{I}$ denote the kernel.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

Lemma 4.2. *Let $f : Z \to X$ be a morphism of locally ringed spaces. In order for f to be a closed immersion it suffices that there exists an open covering $X = \bigcup U_i$ such that each $f : f^{-1} U_i \to U_i$ is a closed immersion.*

Proof. Omitted. $\square$

Example 4.3. Let X be a locally ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals which is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Let Z be the support of the sheaf of rings $\mathcal{O}_X/\mathcal{I}$. This is a closed subset of X, by Modules, Lemma 5.3. Denote $i : Z \to X$ the inclusion map. By Modules, Lemma 6.1 there is a unique sheaf of rings $\mathcal{O}_Z$ on Z with $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. For any $z \in Z$ the stalk $\mathcal{O}_{Z,z}$ is equal to a quotient $\mathcal{O}_{X,i(z)}/\mathcal{I}_{i(z)}$ of a local ring and nonzero, hence a local ring. Thus $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a closed immersion of locally ringed spaces.

Definition 4.4. Let X be a locally ringed space. Let $\mathcal{I}$ be a sheaf of ideals on X which is locally generated by sections. The locally ringed space $(Z, \mathcal{O}_Z)$ of Example 4.3 above is the *closed subspace of X associated to the sheaf of ideals $\mathcal{I}$.*

Lemma 4.5. *Let $f : X \to Y$ be a closed immersion of locally ringed spaces. Let $\mathcal{I}$ be the kernel of the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Let $i : Z \to Y$ be the closed subspace of Y associated to $\mathcal{I}$. There is a unique isomorphism $f' : X \cong Z$ of locally ringed spaces such that $f = i \circ f'$.*

Proof. Omitted. $\square$

Lemma 4.6. *Let X, Y be locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals locally generated by sections. Let $i : Z \to X$ be the associated closed subspace. A morphism $f : Y \to X$ factors through Z if and only if the map $f^*\mathcal{I} \to f^*\mathcal{O}_X = \mathcal{O}_Y$ is zero. If this is the case the morphism $g : Y \to Z$ such that $f = i \circ g$ is unique.*

Proof. Clearly if f factors as $Y \to Z \to X$ then the map $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Conversely suppose that $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Pick any $y \in Y$, and consider the ring map $f_y^\sharp : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$. Since the composition $\mathcal{I}_{f(y)} \to \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is zero by assumption and since $f_y^\sharp(1) = 1$ we see that $1 \notin \mathcal{I}_{f(y)}$, i.e., $\mathcal{I}_{f(y)} \neq \mathcal{O}_{X,f(y)}$. We conclude that $f(Y) \subset Z = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$. Hence $f = i \circ g$ where $g : Y \to Z$ is continuous. Consider the map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$. The assumption $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero implies that the composition $\mathcal{I} \to \mathcal{O}_X \to f_*\mathcal{O}_Y$ is zero by adjointness of f_* and f^*. In other words, we obtain a morphism of sheaves of rings $\overline{f^\sharp} : \mathcal{O}_X/\mathcal{I} \to f_*\mathcal{O}_Y$. Note that $f_*\mathcal{O}_Y = i_*g_*\mathcal{O}_Y$ and that $\mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$. By Sheaves, Lemma 32.4 we obtain a unique morphism of sheaves of rings $g^\sharp : \mathcal{O}_Z \to g_*\mathcal{O}_Y$ whose pushforward under i is $\overline{f^\sharp}$. We omit the verification that $(g, g^\sharp)$ defines a morphism of locally ringed spaces and that $f = i \circ g$ as a morphism of locally ringed spaces. The uniqueness of $(g, g^\sharp)$ was pointed out above. $\square$

Lemma 4.7. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals which is locally generated by sections. Let $i : Z \to Y$ be the closed subspace associated to the sheaf of ideals $\mathcal{I}$. Let $\mathcal{J}$ be the image of the map $f^*\mathcal{I} \to f^*\mathcal{O}_Y = \mathcal{O}_X$. Then this ideal is locally generated by sections. Moreover, let $i' : Z' \to X$ be the associated closed subspace of X. There exists a unique morphism of locally ringed spaces $f' : Z' \to Z$ such that the following diagram is a commutative square of locally ringed spaces*

$$
\begin{array}{ccc}
Z' & \xrightarrow{\ i'\ } & X \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Z & \xrightarrow{\ i\ } & Y
\end{array}
$$

Moreover, this diagram is a fibre square in the category of locally ringed spaces.

Proof. The ideal $\mathcal{J}$ is locally generated by sections by Modules, Lemma 8.2. The rest of the lemma follows from the characterization, in Lemma 4.6 above, of what it means for a morphism to factor through a closed subspace. $\square$

5. Affine schemes

Let R be a ring. Consider the topological space $\mathrm{Spec}(R)$ associated to R, see Algebra, Section 17. We will endow this space with a sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ and the resulting pair $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ will be an affine scheme.

Recall that $\mathrm{Spec}(R)$ has a basis of open sets $D(f)$, $f \in R$ which we call standard opens, see Algebra, Definition 17.3. In addition, the intersection of two standard opens is another: $D(f) \cap D(g) = D(fg)$, $f, g \in R$.

Lemma 5.1. *Let R be a ring. Let $f \in R$.*
  (1) *If $g \in R$ and $D(g) \subset D(f)$, then*
     (a) *f is invertible in R_g,*
     (b) *$g^e = af$ for some $e \geq 1$ and $a \in R$,*
     (c) *there is a canonical ring map $R_f \to R_g$, and*
     (d) *there is a canonical R_f-module map $M_f \to M_g$ for any R-module M.*
  (2) *Any open covering of $D(f)$ can be refined to a finite open covering of the form $D(f) = \bigcup_{i=1}^{n} D(g_i)$.*
  (3) *If $g_1, \ldots, g_n \in R$, then $D(f) \subset \bigcup D(g_i)$ if and only if $g_1, \ldots, g_n$ generate the unit ideal in R_f.*

Proof. Recall that $D(g) = \mathrm{Spec}(R_g)$ (see Algebra, Lemma 17.6). Thus (a) holds because f maps to an element of R_g which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma 17.2. Write the inverse of f in R_g as a/g^d. This means $g^d - af$ is annihilated by a power of g, whence (b). For (c), the map $R_f \to R_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. The equality $M_f = M \otimes_R R_f$ can be used to obtain the map on modules, or we can define $M_f \to M_g$ by mapping x/f^n to $a^n x/g^{ne}$.

Recall that $D(f)$ is quasi-compact, see Algebra, Lemma 29.1. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma 17.2. $\square$

In Sheaves, Section 30 we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas 30.6 and 30.9. Moreover, we showed in Sheaves, Lemma 30.4 that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

Definition 5.2. Let R be a ring.
  (1) A *standard open covering* of $\mathrm{Spec}(R)$ is a covering $\mathrm{Spec}(R) = \bigcup_{i=1}^{n} D(f_i)$, where $f_1, \ldots, f_n \in R$.
  (2) Suppose that $D(f) \subset \mathrm{Spec}(R)$ is a standard open. A *standard open covering* of $D(f)$ is a covering $D(f) = \bigcup_{i=1}^{n} D(g_i)$, where $g_1, \ldots, g_n \in R$.

Let R be a ring. Let M be an R-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset \mathrm{Spec}(R)$ is a standard open. If $f, g \in R$ are such that $D(f) = D(g)$, then by Lemma 5.1 above there are canonical maps $M_f \to M_g$ and $M_g \to M_f$ which are mutually inverse. Hence we may choose any f such that $U = D(f)$ and define

$$\widetilde{M}(U) = M_f.$$

Note that if $D(g) \subset D(f)$, then by Lemma 5.1 above we have a canonical map

$$\widetilde{M}(D(f)) = M_f \longrightarrow M_g = \widetilde{M}(D(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = R$, then $\widetilde{R}$ is a presheaf of rings on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in \mathrm{Spec}(R)$. Suppose that x corresponds to the prime $\mathfrak{p} \subset R$. By definition of the stalk we see that

$$\widetilde{M}_x = \mathrm{colim}_{f \in R, f \notin \mathfrak{p}} M_f$$

Here the set $\{f \in R, f \notin \mathfrak{p}\}$ is preordered by the rule $f \geq f' \Leftrightarrow D(f) \subset D(f')$. If $f_1, f_2 \in R \setminus \mathfrak{p}$, then we have $f_1 f_2 \geq f_1$ in this ordering. Hence by Algebra, Lemma 9.9 we conclude that

$$\widetilde{M}_x = M_{\mathfrak{p}}.$$

Next, we check the sheaf condition for the standard open coverings. If $D(f) = \bigcup_{i=1}^n D(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_f \to \bigoplus M_{g_i} \to \bigoplus M_{g_i g_j}.$$

Note that $D(g_i) = D(fg_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_f \to \bigoplus M_{fg_i} \to \bigoplus M_{fg_i g_j}.$$

In addition, by Lemma 5.1 above we see that $g_1, \ldots, g_n$ generate the unit ideal in R_f. Thus we may apply Algebra, Lemma 24.1 to the module M_f over R_f and the elements $g_1, \ldots, g_n$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section 30 that there exists a unique sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ which agrees with $\widetilde{R}$ on the standard opens. Note that by our computation of stalks above, the stalks of this sheaf of rings are all local rings.

Similarly, for any R-module M there exists a unique sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma 30.12.

Definition 5.3. Let R be a ring.
  (1) The *structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ of the spectrum of R* is the unique sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ which agrees with $\widetilde{R}$ on the basis of standard opens.
  (2) The locally ringed space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is called the *spectrum* of R and denoted $\mathrm{Spec}(R)$.
  (3) The sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules extending $\widetilde{M}$ to all opens of $\mathrm{Spec}(R)$ is called the sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

Lemma 5.4. *Let R be a ring. Let M be an R-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules associated to M.*

    (1) *We have $\Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) = R$.*
    (2) *We have $\Gamma(\mathrm{Spec}(R), \widetilde{M}) = M$ as an R-module.*
    (3) *For every $f \in R$ we have $\Gamma(D(f), \mathcal{O}_{\mathrm{Spec}(R)}) = R_f$.*
    (4) *For every $f \in R$ we have $\Gamma(D(f), \widetilde{M}) = M_f$ as an R_f-module.*
    (5) *Whenever $D(g) \subset D(f)$ the restriction mappings on $\mathcal{O}_{\mathrm{Spec}(R)}$ and $\widetilde{M}$ are the maps $R_f \to R_g$ and $M_f \to M_g$ from Lemma 5.1.*
    (6) *Let $\mathfrak{p}$ be a prime of R, and let $x \in \mathrm{Spec}(R)$ be the corresponding point. We have $\mathcal{O}_{\mathrm{Spec}(R),x} = R_{\mathfrak{p}}$.*
    (7) *Let $\mathfrak{p}$ be a prime of R, and let $x \in \mathrm{Spec}(R)$ be the corresponding point. We have $\widetilde{M}_x = M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module.*

Moreover, all these identifications are functorial in the R module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of R-modules to the category of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules.

Proof. Assertions (1) - (7) are clear from the discussion above. The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{\mathfrak{p}}$ is exact and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma 3.1. $\square$

Definition 5.5. An *affine scheme* is a locally ringed space isomorphic as a locally ringed space to $\mathrm{Spec}(R)$ for some ring R. A *morphism of affine schemes* is a morphism in the category of locally ringed spaces.

It turns out that affine schemes play a special role among all locally ringed spaces, which is what the next section is about.

6. The category of affine schemes

Note that if Y is an affine scheme, then its points are in canonical $1-1$ bijection with prime ideals in $\Gamma(Y, \mathcal{O}_Y)$.

Lemma 6.1. *Let X be a locally ringed space. Let Y be an affine scheme. Let $f \in \mathrm{Mor}(X, Y)$ be a morphism of locally ringed spaces. Given a point $x \in X$ consider the ring maps*

$$\Gamma(Y, \mathcal{O}_Y) \xrightarrow{f^\sharp} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$$

Let $\mathfrak{p} \subset \Gamma(Y, \mathcal{O}_Y)$ denote the inverse image of $\mathfrak{m}_x$. Let $y \in Y$ be the corresponding point. Then $f(x) = y$.

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
\Gamma(X, \mathcal{O}_X) & \longrightarrow & \mathcal{O}_{X,x} \\
\uparrow & & \uparrow \\
\Gamma(Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{O}_{Y,f(x)}
\end{array}
$$

(see the discussion of f-maps below Sheaves, Definition 21.7). Since the right vertical arrow is local we see that $\mathfrak{m}_{f(x)}$ is the inverse image of $\mathfrak{m}_x$. The result follows. $\square$

Lemma 6.2. *Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The set*

$$D(f) = \{x \in X \mid image\ f \notin \mathfrak{m}_x\}$$

is open. Moreover $f|_{D(f)}$ has an inverse.

Proof. This is a special case of Modules, Lemma 25.10, but we also give a direct proof. Suppose that $U \subset X$ and $V \subset X$ are two open subsets such that $f|_U$ has an inverse g and $f|_V$ has an inverse h. Then clearly $g|_{U \cap V} = h|_{U \cap V}$. Thus it suffices to show that f is invertible in an open neighbourhood of any $x \in D(f)$. This is clear because $f \notin \mathfrak{m}_x$ implies that $f \in \mathcal{O}_{X,x}$ has an inverse $g \in \mathcal{O}_{X,x}$ which means there is some open neighbourhood $x \in U \subset X$ so that $g \in \mathcal{O}_X(U)$ and $g \cdot f|_U = 1$. $\square$

Lemma 6.3. *In Lemma 6.2 above, if X is an affine scheme, then the open $D(f)$ agrees with the standard open $D(f)$ defined previously (in Algebra, Definition 17.1).*

Proof. Omitted. $\square$

Lemma 6.4. *Let X be a locally ringed space. Let Y be an affine scheme. The map*

$$\mathrm{Mor}(X, Y) \longrightarrow \mathrm{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

which maps f to $f^\sharp$ (on global sections) is bijective.

Proof. Since Y is affine we have $(Y, \mathcal{O}_Y) \cong (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for some ring R. During the proof we will use facts about Y and its structure sheaf which are direct consequences of things we know about the spectrum of a ring, see e.g. Lemma 5.4.

Motivated by the lemmas above we construct the inverse map. Let $\psi_Y : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ be a ring map. First, we define the corresponding map of spaces

$$\Psi : X \longrightarrow Y$$

by the rule of Lemma 6.1. In other words, given $x \in X$ we define $\Psi(x)$ to be the point of Y corresponding to the prime in $\Gamma(Y, \mathcal{O}_Y)$ which is the inverse image of $\mathfrak{m}_x$ under the composition $\Gamma(Y, \mathcal{O}_Y) \xrightarrow{\psi_Y} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$.

We claim that the map $\Psi : X \to Y$ is continuous. The standard opens $D(g)$, for $g \in \Gamma(Y, \mathcal{O}_Y)$ are a basis for the topology of Y. Thus it suffices to prove that $\Psi^{-1}(D(g))$ is open. By construction of Ψ the inverse image $\Psi^{-1}(D(g))$ is exactly the set $D(\psi_Y(g)) \subset X$ which is open by Lemma 6.2. Hence Ψ is continuous.

Next we construct a Ψ-map of sheaves from $\mathcal{O}_Y$ to $\mathcal{O}_X$. By Sheaves, Lemma 30.14 it suffices to define ring maps $\psi_{D(g)} : \Gamma(D(g), \mathcal{O}_Y) \to \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X)$ compatible with restriction maps. We have a canonical isomorphism $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, because Y is an affine scheme. Because $\psi_Y(g)$ is invertible on $D(\psi_Y(g))$ we see that there is a canonical map

$$\Gamma(Y, \mathcal{O}_Y)_g \longrightarrow \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X) = \Gamma(D(\psi_Y(g)), \mathcal{O}_X)$$

extending the map ψ_Y by the universal property of localization. Note that there is no choice but to take the canonical map here! And we take this, combined with the canonical identification $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, to be $\psi_{D(g)}$. This is compatible

with localization since the restriction mapping on the affine schemes are defined in terms of the universal properties of localization also, see Lemmas 5.4 and 5.1.

Thus we have defined a morphism of ringed spaces $(\Psi, \psi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ recovering ψ_Y on global sections. To see that it is a morphism of locally ringed spaces we have to show that the induced maps on local rings

$$\psi_x : \mathcal{O}_{Y,\Psi(x)} \longrightarrow \mathcal{O}_{X,x}$$

are local. This follows immediately from the commutative diagram of the proof of Lemma 6.1 and the definition of Ψ.

Finally, we have to show that the constructions $(\Psi, \psi) \mapsto \psi_Y$ and the construction $\psi_Y \mapsto (\Psi, \psi)$ are inverse to each other. Clearly, $\psi_Y \mapsto (\Psi, \psi) \mapsto \psi_Y$. Hence the only thing to prove is that given ψ_Y there is at most one pair (Ψ, ψ) giving rise to it. The uniqueness of Ψ was shown in Lemma 6.1 and given the uniqueness of Ψ the uniqueness of the map ψ was pointed out during the course of the proof above. $\square$

Lemma 6.5. *The category of affine schemes is equivalent to the opposite of the category of rings. The equivalence is given by the functor that associates to an affine scheme the global sections of its structure sheaf.*

Proof. This is now clear from Definition 5.5 and Lemma 6.4. $\square$

Lemma 6.6. *Let Y be an affine scheme. Let $f \in \Gamma(Y, \mathcal{O}_Y)$. The open subspace $D(f)$ is an affine scheme.*

Proof. We may assume that $Y = \mathrm{Spec}(R)$ and $f \in R$. Consider the morphism of affine schemes $\phi : U = \mathrm{Spec}(R_f) \to \mathrm{Spec}(R) = Y$ induced by the ring map $R \to R_f$. By Algebra, Lemma 17.6 we know that it is a homeomorphism onto $D(f)$. On the other hand, the map $\phi^{-1}\mathcal{O}_Y \to \mathcal{O}_U$ is an isomorphism on stalks, hence an isomorphism. Thus we see that ϕ is an open immersion. We conclude that $D(f)$ is isomorphic to U by Lemma 3.4. $\square$

Lemma 6.7. *The category of affine schemes has finite products, and fibre products. In other words, it has finite limits. Moreover, the products and fibre products in the category of affine schemes are the same as in the category of locally ringed spaces. In a formula, we have (in the category of locally ringed spaces)*

$$\mathrm{Spec}(R) \times \mathrm{Spec}(S) = \mathrm{Spec}(R \otimes_{\mathbf{Z}} S)$$

and given ring maps $R \to A$, $R \to B$ we have

$$\mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(B) = \mathrm{Spec}(A \otimes_R B).$$

Proof. This is just an application of Lemma 6.4. First of all, by that lemma, the affine scheme $\mathrm{Spec}(\mathbf{Z})$ is the final object in the category of locally ringed spaces. Thus the first displayed formula follows from the second. To prove the second note that for any locally ringed space X we have

$$
\begin{aligned}
\mathrm{Mor}(X, \mathrm{Spec}(A \otimes_R B)) &= \mathrm{Hom}(A \otimes_R B, \mathcal{O}_X(X)) \\
&= \mathrm{Hom}(A, \mathcal{O}_X(X)) \times_{\mathrm{Hom}(R, \mathcal{O}_X(X))} \mathrm{Hom}(B, \mathcal{O}_X(X)) \\
&= \mathrm{Mor}(X, \mathrm{Spec}(A)) \times_{\mathrm{Mor}(X, \mathrm{Spec}(R))} \mathrm{Mor}(X, \mathrm{Spec}(B))
\end{aligned}
$$

which proves the formula. See Categories, Section 6 for the relevant definitions. $\square$

Lemma 6.8. *Let X be a locally ringed space. Assume $X = U \amalg V$ with U and V open and such that U, V are affine schemes. Then X is an affine scheme.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. Note that $R = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ by the sheaf property. By Lemma 6.4 there is a canonical morphism of locally ringed spaces $X \to \mathrm{Spec}(R)$. By Algebra, Lemma 21.2 we see that as a topological space $\mathrm{Spec}(\mathcal{O}_X(U)) \amalg \mathrm{Spec}(\mathcal{O}_X(V)) = \mathrm{Spec}(R)$ with the maps coming from the ring homomorphisms $R \to \mathcal{O}_X(U)$ and $R \to \mathcal{O}_X(V)$. This of course means that $\mathrm{Spec}(R)$ is the coproduct in the category of locally ringed spaces as well. By assumption the morphism $X \to \mathrm{Spec}(R)$ induces an isomorphism of $\mathrm{Spec}(\mathcal{O}_X(U))$ with U and similarly for V. Hence $X \to \mathrm{Spec}(R)$ is an isomorphism. $\square$

7. Quasi-coherent sheaves on affines

Recall that we have defined the abstract notion of a quasi-coherent sheaf in Modules, Definition 10.1. In this section we show that any quasi-coherent sheaf on an affine scheme $\mathrm{Spec}(R)$ corresponds to the sheaf $\widetilde{M}$ associated to an R-module M.

Lemma 7.1. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let M be an R-module. There exists a canonical isomorphism between the sheaf $\widetilde{M}$ associated to the R-module M (Definition 5.3) and the sheaf $\mathcal{F}_M$ associated to the R-module M (Modules, Definition 10.6). This isomorphism is functorial in M. In particular, the sheaves $\widetilde{M}$ are quasi-coherent. Moreover, they are characterized by the following mapping property*

$$\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) = \mathrm{Hom}_R(M, \Gamma(X, \mathcal{F}))$$

for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Here a map $\alpha : \widetilde{M} \to \mathcal{F}$ corresponds to its effect on global sections.

Proof. By Modules, Lemma 10.5 we have a morphism $\mathcal{F}_M \to \widetilde{M}$ corresponding to the map $M \to \Gamma(X, \widetilde{M}) = M$. Let $x \in X$ correspond to the prime $\mathfrak{p} \subset R$. The induced map on stalks are the maps $\mathcal{O}_{X,x} \otimes_R M \to M_{\mathfrak{p}}$ which are isomorphisms because $R_{\mathfrak{p}} \otimes_R M = M_{\mathfrak{p}}$. Hence the map $\mathcal{F}_M \to \widetilde{M}$ is an isomorphism. The mapping property follows from the mapping property of the sheaves $\mathcal{F}_M$. $\square$

Lemma 7.2. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. There are canonical isomorphisms*

(1) *$\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$, see Modules, Section 16.*
(2) *$\widetilde{T^n(M)} \cong T^n(\widetilde{M})$, $\widetilde{\mathrm{Sym}^n(M)} \cong \mathrm{Sym}^n(\widetilde{M})$, and $\widetilde{\wedge^n(M)} \cong \wedge^n(\widetilde{M})$, see Modules, Section 21.*
(3) *if M is a finitely presented R-module, then $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\mathrm{Hom}_R(M, N)}$, see Modules, Section 22.*

First proof. Using Lemma 7.1 and Modules, Lemma 10.5 we see that the functor $M \mapsto \widetilde{M}$ can be viewed as π^* for a morphism π of ringed spaces. And pulling back modules commutes with tensor constructions by Modules, Lemmas 16.4 and 21.3. The morphism $\pi : (X, \mathcal{O}_X) \to (\{*\}, R)$ is flat for example because the stalks of $\mathcal{O}_X$ are localizations of R (Lemma 5.4) and hence flat over R. Thus pullback by π commutes with internal hom if the first module is finitely presented by Modules, Lemma 22.5. $\square$

Second proof. Proof of (1). By Lemma 7.1 to give a map $\widetilde{M \otimes_R N}$ into $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ we have to give a map on global sections $M \otimes_R N \to \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})$ which exists by definition of the tensor product of sheaves of modules. To see that this map is an isomorphism it suffices to check that it is an isomorphism on stalks. And this follows from the description of the stalks of $\widetilde{M}$ (either in Lemma 5.4 or in Modules, Lemma 10.5), the fact that tensor product commutes with localization (Algebra, Lemma 12.16) and Modules, Lemma 16.1.

Proof of (2). This is similar to the proof of (1), using Algebra, Lemma 13.6 and Modules, Lemma 21.2.

Proof of (3). Since the construction $M \mapsto \widetilde{M}$ is functorial there is an R-linear map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. The target of this map is the global sections of $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. Hence by Lemma 7.1 we obtain a map of $\mathcal{O}_X$-modules $\widetilde{\mathrm{Hom}_R(M, N)} \to \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. We check that this is an isomorphism by comparing stalks. If M is finitely presented as an R-module then $\widetilde{M}$ has a global finite presentation as an $\mathcal{O}_X$-module. Hence we conclude using Algebra, Lemma 10.2 and Modules, Lemma 22.4. $\square$

Third proof of part (1). For any $\mathcal{O}_X$-module $\mathcal{F}$ we have the following isomorphisms functorial in M, N, and $\mathcal{F}$

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}, \mathcal{F}) &= \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F})) \\
&= \mathrm{Hom}_R(M, \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F}))) \\
&= \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F})) \\
&= \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, \Gamma(X, \mathcal{F}))) \\
&= \mathrm{Hom}_R(M \otimes_R N, \Gamma(X, \mathcal{F})) \\
&= \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M \otimes_R N}, \mathcal{F})
\end{aligned}
$$

The first equality is Modules, Lemma 22.1. The second equality is the universal property of $\widetilde{M}$, see Lemma 7.1. The third equality holds by definition of $\mathcal{H}om$. The fourth equality is the universal property of $\widetilde{N}$. Then fifth equality is Algebra, Lemma 12.8. The final equality is the universal property of $\widetilde{M \otimes_R N}$. By the Yoneda lemma (Categories, Lemma 3.5) we obtain (1). $\square$

Lemma 7.3. *Let* $(X, \mathcal{O}_X) = (\mathrm{Spec}(S), \mathcal{O}_{\mathrm{Spec}(S)})$, $(Y, \mathcal{O}_Y) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ *be affine schemes. Let* $\psi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *be a morphism of affine schemes, corresponding to the ring map* $\psi^\sharp : R \to S$ *(see Lemma 6.5).*

(1) *We have* $\psi^* \widetilde{M} = \widetilde{S \otimes_R M}$ *functorially in the R-module M.*
(2) *We have* $\psi_* \widetilde{N} = \widetilde{N_R}$ *functorially in the S-module N.*

Proof. The first assertion follows from the identification in Lemma 7.1 and the result of Modules, Lemma 10.7. The second assertion follows from the fact that $\psi^{-1}(D(f)) = D(\psi^\sharp(f))$ and hence

$$
\psi_* \widetilde{N}(D(f)) = \widetilde{N}(D(\psi^\sharp(f))) = N_{\psi^\sharp(f)} = (N_R)_f = \widetilde{N_R}(D(f))
$$

as desired. $\square$

Lemma 7.3 above says in particular that if you restrict the sheaf $\widetilde{M}$ to a standard affine open subspace $D(f)$, then you get $\widetilde{M_f}$. We will use this from now on without further mention.

Lemma 7.4. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is isomorphic to the sheaf associated to the R-module $\Gamma(X, \mathcal{F})$.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Since every standard open $D(f)$ is quasi-compact we see that X is a locally quasi-compact, i.e., every point has a fundamental system of quasi-compact neighbourhoods, see Topology, Definition 13.1. Hence by Modules, Lemma 10.8 for every prime $\mathfrak{p} \subset R$ corresponding to $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the quasi-coherent sheaf associated to some $\mathcal{O}_X(U)$-module M. In other words, we get an open covering by U's with this property. By Lemma 5.1 for example we can refine this covering to a standard open covering. Thus we get a covering $\mathrm{Spec}(R) = \bigcup D(f_i)$ and R_{f_i}-modules M_i and isomorphisms $\varphi_i : \mathcal{F}|_{D(f_i)} \to \mathcal{F}_{M_i}$ for some R_{f_i}-module M_i. On the overlaps we get isomorphisms

$$\mathcal{F}_{M_i}|_{D(f_i f_j)} \xrightarrow{\ \varphi_i^{-1}|_{D(f_i f_j)}\ } \mathcal{F}|_{D(f_i f_j)} \xrightarrow{\ \varphi_j|_{D(f_i f_j)}\ } \mathcal{F}_{M_j}|_{D(f_i f_j)}.$$

Let us denote these ψ_{ij}. It is clear that we have the cocycle condition

$$\psi_{jk}|_{D(f_i f_j f_k)} \circ \psi_{ij}|_{D(f_i f_j f_k)} = \psi_{ik}|_{D(f_i f_j f_k)}$$

on triple overlaps.

Recall that each of the open subspaces $D(f_i)$, $D(f_i f_j)$, $D(f_i f_j f_k)$ is an affine scheme. Hence the sheaves $\mathcal{F}_{M_i}$ are isomorphic to the sheaves $\widetilde{M_i}$ by Lemma 7.1 above. In particular we see that $\mathcal{F}_{M_i}(D(f_i f_j)) = (M_i)_{f_j}$, etc. Also by Lemma 7.1 above we see that ψ_{ij} corresponds to a unique $R_{f_i f_j}$-module isomorphism

$$\psi_{ij} : (M_i)_{f_j} \longrightarrow (M_j)_{f_i}$$

namely, the effect of ψ_{ij} on sections over $D(f_i f_j)$. Moreover these then satisfy the cocycle condition that

commutes (for any triple i, j, k).

Now Algebra, Lemma 24.5 shows that there exist an R-module M such that $M_i = M_{f_i}$ compatible with the morphisms ψ_{ij}. Consider $\mathcal{F}_M = \widetilde{M}$. At this point it is a formality to show that $\widetilde{M}$ is isomorphic to the quasi-coherent sheaf $\mathcal{F}$ we started out with. Namely, the sheaves $\mathcal{F}$ and $\widetilde{M}$ give rise to isomorphic sets of glueing data of sheaves of $\mathcal{O}_X$-modules with respect to the covering $X = \bigcup D(f_i)$, see Sheaves, Section 33 and in particular Lemma 33.4. Explicitly, in the current situation, this boils down to the following argument: Let us construct an R-module map

$$M \longrightarrow \Gamma(X, \mathcal{F}).$$

Namely, given $m \in M$ we get $m_i = m/1 \in M_{f_i} = M_i$ by construction of M. By construction of M_i this corresponds to a section $s_i \in \mathcal{F}(U_i)$. (Namely, $\varphi_i^{-1}(m_i)$.) We claim that $s_i|_{D(f_i f_j)} = s_j|_{D(f_i f_j)}$. This is true because, by construction of M, we have $\psi_{ij}(m_i) = m_j$, and by the construction of the ψ_{ij}. By the sheaf condition of $\mathcal{F}$ this collection of sections gives rise to a unique section s of $\mathcal{F}$ over X. We leave it to the reader to show that $m \mapsto s$ is a R-module map. By Lemma 7.1 we obtain an associated $\mathcal{O}_X$-module map

$$\widetilde{M} \longrightarrow \mathcal{F}.$$

By construction this map reduces to the isomorphisms φ_i^{-1} on each $D(f_i)$ and hence is an isomorphism.   □

Lemma 7.5. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. The functors $M \mapsto \widetilde{M}$ and $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ define quasi-inverse equivalences of categories*

$$QCoh(\mathcal{O}_X) \xrightarrow{\quad\longrightarrow\quad} Mod_R$$

between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of R-modules.

Proof. See Lemmas 7.1 and 7.4 above.   □

From now on we will not distinguish between quasi-coherent sheaves on affine schemes and sheaves of the form $\widetilde{M}$.

Lemma 7.6. *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Kernels and cokernels of maps of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

Proof. This follows from the exactness of the functor $^\sim$ since by Lemma 7.1 we know that any map $\psi : \widetilde{M} \to \widetilde{N}$ comes from an R-module map $\varphi : M \to N$. (So we have $\mathrm{Ker}(\psi) = \widetilde{\mathrm{Ker}(\varphi)}$ and $\mathrm{Coker}(\psi) = \widetilde{\mathrm{Coker}(\varphi)}$.)   □

Lemma 7.7. *Let $X = \mathrm{Spec}(R)$ be an affine scheme. The direct sum of an arbitrary collection of quasi-coherent sheaves on X is quasi-coherent. The same holds for colimits.*

Proof. Suppose $\mathcal{F}_i$, $i \in I$ is a collection of quasi-coherent sheaves on X. By Lemma 7.5 above we can write $\mathcal{F}_i = \widetilde{M_i}$ for some R-module M_i. Set $M = \bigoplus M_i$. Consider the sheaf $\widetilde{M}$. For each standard open $D(f)$ we have

$$\widetilde{M}(D(f)) = M_f = \left(\bigoplus M_i\right)_f = \bigoplus M_{i,f}.$$

Hence we see that the quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$ is the direct sum of the sheaves $\mathcal{F}_i$. A similar argument works for general colimits.   □

Lemma 7.8. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If two out of three are quasi-coherent then so is the third.

Proof. This is clear in case both $\mathcal{F}_1$ and $\mathcal{F}_2$ are quasi-coherent because the functor $M \mapsto \widetilde{M}$ is exact, see Lemma 5.4. Similarly in case both $\mathcal{F}_2$ and $\mathcal{F}_3$ are quasi-coherent. Now, suppose that $\mathcal{F}_1 = \widetilde{M}_1$ and $\mathcal{F}_3 = \widetilde{M}_3$ are quasi-coherent. Set $M_2 = \Gamma(X, \mathcal{F}_2)$. We claim it suffices to show that the sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact. Namely, if this is the case, then (by using the mapping property of Lemma 7.1) we get a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{M}_1 & \longrightarrow & \widetilde{M}_2 & \longrightarrow & \widetilde{M}_3 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0
\end{array}
$$

and we win by the snake lemma.

The "correct" argument here would be to show first that $H^1(X, \mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$. This is actually not all that hard, but it is perhaps better to postpone this till later. Instead we use a small trick.

Pick $m \in M_3 = \Gamma(X, \mathcal{F}_3)$. Consider the following set

$$I = \{ f \in R \mid \text{the element } fm \text{ comes from } M_2 \}.$$

Clearly this is an ideal. It suffices to show $1 \in I$. Hence it suffices to show that for any prime $\mathfrak{p}$ there exists an $f \in I$, $f \notin \mathfrak{p}$. Let $x \in X$ be the point corresponding to $\mathfrak{p}$. Because surjectivity can be checked on stalks there exists an open neighbourhood U of x such that $m|_U$ comes from a local section $s \in \mathcal{F}_2(U)$. In fact we may assume that $U = D(f)$ is a standard open, i.e., $f \in R$, $f \notin \mathfrak{p}$. We will show that for some $N \gg 0$ we have $f^N \in I$, which will finish the proof.

Take any point $z \in V(f)$, say corresponding to the prime $\mathfrak{q} \subset R$. We can also find a $g \in R$, $g \notin \mathfrak{q}$ such that $m|_{D(g)}$ lifts to some $s' \in \mathcal{F}_2(D(g))$. Consider the difference $s|_{D(fg)} - s'|_{D(fg)}$. This is an element m' of $\mathcal{F}_1(D(fg)) = (M_1)_{fg}$. For some integer $n = n(z)$ the element $f^n m'$ comes from some $m'_1 \in (M_1)_g$. We see that $f^n s$ extends to a section σ of $\mathcal{F}_2$ on $D(f) \cup D(g)$ because it agrees with the restriction of $f^n s' + m'_1$ on $D(f) \cap D(g) = D(fg)$. Moreover, σ maps to the restriction of $f^n m$ to $D(f) \cup D(g)$.

Since $V(f)$ is quasi-compact, there exists a finite list of elements $g_1, \ldots, g_m \in R$ such that $V(f) \subset \bigcup D(g_j)$, an integer $n > 0$ and sections $\sigma_j \in \mathcal{F}_2(D(f) \cup D(g_j))$ such that $\sigma_j|_{D(f)} = f^n s$ and σ_j maps to the section $f^n m|_{D(f) \cup D(g_j)}$ of $\mathcal{F}_3$. Consider the differences

$$\sigma_j|_{D(f) \cup D(g_j g_k)} - \sigma_k|_{D(f) \cup D(g_j g_k)}.$$

These correspond to sections of $\mathcal{F}_1$ over $D(f) \cup D(g_j g_k)$ which are zero on $D(f)$. In particular their images in $\mathcal{F}_1(D(g_j g_k)) = (M_1)_{g_j g_k}$ are zero in $(M_1)_{g_j g_k f}$. Thus some high power of f kills each and every one of these. In other words, the elements $f^N \sigma_j$, for some $N \gg 0$ satisfy the glueing condition of the sheaf property and give rise to a section σ of $\mathcal{F}_2$ over $\bigcup(D(f) \cup D(g_j)) = X$ as desired. $\qquad\square$

8. Closed subspaces of affine schemes

Example 8.1. Let R be a ring. Let $I \subset R$ be an ideal. Consider the morphism of affine schemes $i : Z = \mathrm{Spec}(R/I) \to \mathrm{Spec}(R) = X$. By Algebra, Lemma 17.7 this is a homeomorphism of Z onto a closed subset of X. Moreover, if $I \subset \mathfrak{p} \subset R$ is a prime corresponding to a point $x = i(z)$, $x \in X$, $z \in Z$, then on stalks we get the map

$$\mathcal{O}_{X,x} = R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}/IR_{\mathfrak{p}} = \mathcal{O}_{Z,z}$$

Thus we see that i is a closed immersion of locally ringed spaces, see Definition 4.1. Clearly, this is (isomorphic) to the closed subspace associated to the quasi-coherent sheaf of ideals $\widetilde{I}$, as in Example 4.3.

Lemma 8.2. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let $i : Z \to X$ be any closed immersion of locally ringed spaces. Then there exists a unique ideal $I \subset R$ such that the morphism $i : Z \to X$ can be identified with the closed immersion $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ constructed in Example 8.1 above.*

Proof. This is kind of silly! Namely, by Lemma 4.5 we can identify $Z \to X$ with the closed subspace associated to a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ as in Definition 4.4 and Example 4.3. By our conventions this sheaf of ideals is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Hence the quotient sheaf $\mathcal{O}_X/\mathcal{I}$ is locally on X the cokernel of a map $\bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{O}_U$. Thus by definition, $\mathcal{O}_X/\mathcal{I}$ is quasi-coherent. By our results in Section 7 it is of the form $\widetilde{S}$ for some R-module S. Moreover, since $\mathcal{O}_X = \widetilde{R} \to \widetilde{S}$ is surjective we see by Lemma 7.8 that also $\mathcal{I}$ is quasi-coherent, say $\mathcal{I} = \widetilde{I}$. Of course $I \subset R$ and $S = R/I$ and everything is clear. $\square$

9. Schemes

Definition 9.1. A *scheme* is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme. A *morphism of schemes* is a morphism of locally ringed spaces. The category of schemes will be denoted *Sch*.

Let X be a scheme. We will use the following (very slight) abuse of language. We will say $U \subset X$ is an *affine open*, or an *open affine* if the open subspace U is an affine scheme. We will often write $U = \mathrm{Spec}(R)$ to indicate that U is isomorphic to $\mathrm{Spec}(R)$ and moreover that we will identify (temporarily) U and $\mathrm{Spec}(R)$.

Lemma 9.2. *Let X be a scheme. Let $j : U \to X$ be an open immersion of locally ringed spaces. Then U is a scheme. In particular, any open subspace of X is a scheme.*

Proof. Let $U \subset X$. Let $u \in U$. Pick an affine open neighbourhood $u \in V \subset X$. Because standard opens of V form a basis of the topology on V we see that there exists a $f \in \mathcal{O}_V(V)$ such that $u \in D(f) \subset U$. And $D(f)$ is an affine scheme by Lemma 6.6. This proves that every point of U has an open neighbourhood which is affine. $\square$

Clearly the lemma (or its proof) shows that any scheme X has a basis (see Topology, Section 5) for the topology consisting of affine opens.

Example 9.3. Let k be a field. An example of a scheme which is not affine is given by the open subspace $U = \mathrm{Spec}(k[x,y]) \setminus \{(x,y)\}$ of the affine scheme $X = \mathrm{Spec}(k[x,y])$. It is covered by two affines, namely $D(x) = \mathrm{Spec}(k[x,y,1/x])$ and $D(y) = \mathrm{Spec}(k[x,y,1/y])$ whose intersection is $D(xy) = \mathrm{Spec}(k[x,y,1/xy])$. By the sheaf property for $\mathcal{O}_U$ there is an exact sequence

$$0 \to \Gamma(U, \mathcal{O}_U) \to k[x,y,1/x] \times k[x,y,1/y] \to k[x,y,1/xy]$$

We conclude that the map $k[x,y] \to \Gamma(U, \mathcal{O}_U)$ (coming from the morphism $U \to X$) is an isomorphism. Therefore U cannot be affine since if it was then by Lemma 6.5 we would have $U \cong X$.

10. Immersions of schemes

In Lemma 9.2 we saw that any open subspace of a scheme is a scheme. Below we will prove that the same holds for a closed subspace of a scheme.

Note that the notion of a quasi-coherent sheaf of $\mathcal{O}_X$-modules is defined for any ringed space X in particular when X is a scheme. By our efforts in Section 7 we know that such a sheaf is on any affine open $U \subset X$ of the form $\widetilde{M}$ for some $\mathcal{O}_X(U)$-module M.

Lemma 10.1. *Let X be a scheme. Let $i : Z \to X$ be a closed immersion of locally ringed spaces.*

1. *The locally ringed space Z is a scheme,*
2. *the kernel $\mathcal{I}$ of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of ideals,*
3. *for any affine open $U = \mathrm{Spec}(R)$ of X the morphism $i^{-1}(U) \to U$ can be identified with $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ for some ideal $I \subset R$, and*
4. *we have $\mathcal{I}|_U = \widetilde{I}$.*

In particular, any sheaf of ideals locally generated by sections is a quasi-coherent sheaf of ideals (and vice versa), and any closed subspace of X is a scheme.

Proof. Let $i : Z \to X$ be a closed immersion. Let $z \in Z$ be a point. Choose any affine open neighbourhood $i(z) \in U \subset X$. Say $U = \mathrm{Spec}(R)$. By Lemma 8.2 we know that $i^{-1}(U) \to U$ can be identified with the morphism of affine schemes $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$. First of all this implies that $z \in i^{-1}(U) \subset Z$ is an affine neighbourhood of z. Thus Z is a scheme. Second this implies that $\mathcal{I}|_U$ is $\widetilde{I}$. In other words for every point $x \in i(Z)$ there exists an open neighbourhood such that $\mathcal{I}$ is quasi-coherent in that neighbourhood. Note that $\mathcal{I}|_{X \setminus i(Z)} \cong \mathcal{O}_{X \setminus i(Z)}$. Thus the restriction of the sheaf of ideals is quasi-coherent on $X \setminus i(Z)$ also. We conclude that $\mathcal{I}$ is quasi-coherent. $\square$

Definition 10.2. Let X be a scheme.

1. A morphism of schemes is called an *open immersion* if it is an open immersion of locally ringed spaces (see Definition 3.1).
2. An *open subscheme* of X is an open subspace of X in the sense of Definition 3.3; an open subscheme of X is a scheme by Lemma 9.2.
3. A morphism of schemes is called a *closed immersion* if it is a closed immersion of locally ringed spaces (see Definition 4.1).
4. A *closed subscheme* of X is a closed subspace of X in the sense of Definition 4.4; a closed subscheme is a scheme by Lemma 10.1.

(5) A morphism of schemes $f : X \to Y$ is called an *immersion*, or a *locally closed immersion* if it can be factored as $j \circ i$ where i is a closed immersion and j is an open immersion.

It follows from the lemmas in Sections 3 and 4 that any open (resp. closed) immersion of schemes is isomorphic to the inclusion of an open (resp. closed) subscheme of the target.

Our definition of a closed immersion is halfway between Hartshorne and EGA. Hartshorne defines a closed immersion as a morphism $f : X \to Y$ of schemes which induces a homeomorphism of X onto a closed subset of Y such that $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective, see [Har77, Page 85]. We will show this is equivalent to our notion in Lemma 24.2. In [DG67], Grothendieck and Dieudonné first define closed subschemes via the construction of Example 4.3 using quasi-coherent sheaves of ideals and then define a closed immersion as a morphism $f : X \to Y$ which induces an isomorphism with a closed subscheme. It follows from Lemma 10.1 that this agrees with our notion.

Pedagogically speaking the definition above is a disaster/nightmare. In teaching this material to students, we have found it often convenient to define a closed immersion as an affine morphism $f : X \to Y$ of schemes such that $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective. Namely, it turns out that the notion of an affine morphism (Morphisms, Section 11) is quite natural and easy to understand.

For more information on closed immersions we suggest the reader visit Morphisms, Sections 2 and 4.

We will discuss locally closed subschemes and immersions at the end of this section.

Remark 10.3. If $f : X \to Y$ is an immersion of schemes, then it is in general not possible to factor f as an open immersion followed by a closed immersion. See Morphisms, Example 3.4.

Lemma 10.4. *Let $f : Y \to X$ be an immersion of schemes. Then f is a closed immersion if and only if $f(Y) \subset X$ is a closed subset.*

Proof. If f is a closed immersion then $f(Y)$ is closed by definition. Conversely, suppose that $f(Y)$ is closed. By definition there exists an open subscheme $U \subset X$ such that f is the composition of a closed immersion $i : Y \to U$ and the open immersion $j : U \to X$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals associated to the closed immersion i. Note that $\mathcal{I}|_{U \setminus i(Y)} = \mathcal{O}_{U \setminus i(Y)} = \mathcal{O}_{X \setminus i(Y)}|_{U \setminus i(Y)}$. Thus we may glue (see Sheaves, Section 33) $\mathcal{I}$ and $\mathcal{O}_{X \setminus i(Y)}$ to a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$. Since every point of X has a neighbourhood where $\mathcal{J}$ is quasi-coherent, we see that $\mathcal{J}$ is quasi-coherent (in particular locally generated by sections). By construction $\mathcal{O}_X / \mathcal{J}$ is supported on U and equal to $\mathcal{O}_U / \mathcal{I}$. Thus we see that the closed subspaces associated to $\mathcal{I}$ and $\mathcal{J}$ are canonically isomorphic, see Example 4.3. In particular the closed subspace of U associated to $\mathcal{I}$ is isomorphic to a closed subspace of X. Since $Y \to U$ is identified with the closed subspace associated to $\mathcal{I}$, see Lemma 4.5, we conclude that $Y \to U \to X$ is a closed immersion. $\square$

Let $f : Y \to X$ be an immersion. Let $Z = \overline{f(Y)} \setminus f(Y)$ which is a closed subset of X. Let $U = X \setminus Z$. The lemma implies that U is the biggest open subspace of X such that $f : Y \to X$ factors through a closed immersion into U. We define a *locally closed subscheme of X* as a pair (Z, U) consisting of a closed subscheme Z

of an open subscheme U of X such that in addition $\overline{Z} \cup U = X$. We usually just say "let Z be a locally closed subscheme of X" since we may recover U from the morphism $Z \to X$. The above then shows that any immersion $f : Y \to X$ factors uniquely as $Y \to Z \to X$ where Z is a locally closed subspace of X and $Y \to Z$ is an isomorphism.

The interest of this is that the collection of locally closed subschemes of X forms a set. We may define a partial ordering on this set, which we call inclusion for obvious reasons. To be explicit, if $Z \to X$ and $Z' \to X$ are two locally closed subschemes of X, then we say that Z *is contained in* Z' simply if the morphism $Z \to X$ factors through Z'. If it does, then of course Z is identified with a unique locally closed subscheme of Z', and so on.

For more information on immersions, we refer the reader to Morphisms, Section 3.

11. Zariski topology of schemes

See Topology, Section 1 for some basic material in topology adapted to the Zariski topology of schemes.

Lemma 11.1. *Let X be a scheme. Any irreducible closed subset of X has a unique generic point. In other words, X is a sober topological space, see Topology, Definition 8.6.*

Proof. Let $Z \subset X$ be an irreducible closed subset. For every affine open $U \subset X$, $U = \mathrm{Spec}(R)$ we know that $Z \cap U = V(I)$ for a unique radical ideal $I \subset R$. Note that $Z \cap U$ is either empty or irreducible. In the second case (which occurs for at least one U) we see that $I = \mathfrak{p}$ is a prime ideal, which is a generic point ξ of $Z \cap U$. It follows that $Z = \overline{\{\xi\}}$, in other words ξ is a generic point of Z. If ξ' was a second generic point, then $\xi' \in Z \cap U$ and it follows immediately that $\xi' = \xi$. $\square$

Lemma 11.2. *Let X be a scheme. The collection of affine opens of X forms a basis for the topology on X.*

Proof. This follows from the discussion on open subschemes in Section 9. $\square$

Remark 11.3. In general the intersection of two affine opens in X is not affine open. See Example 14.3.

Lemma 11.4. *The underlying topological space of any scheme is locally quasi-compact, see Topology, Definition 13.1.*

Proof. This follows from Lemma 11.2 above and the fact that the spectrum of ring is quasi-compact, see Algebra, Lemma 17.10. $\square$

Lemma 11.5. *Let X be a scheme. Let U, V be affine opens of X, and let $x \in U \cap V$. There exists an affine open neighbourhood W of x such that W is a standard open of both U and V.*

Proof. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$. Say x corresponds to the prime $\mathfrak{p} \subset A$ and the prime $\mathfrak{q} \subset B$. We may choose an $f \in A$, $f \notin \mathfrak{p}$ such that $D(f) \subset U \cap V$. Note that any standard open of $D(f)$ is a standard open of $\mathrm{Spec}(A) = U$. Hence we may assume that $U \subset V$. In other words, now we may think of U as an affine open of V. Next we choose a $g \in B$, $g \notin \mathfrak{q}$ such that $D(g) \subset U$. In this case we see that $D(g) = D(g_A)$ where $g_A \in A$ denotes the image of g by the map $B \to A$. Thus the lemma is proved. $\square$

Lemma 11.6. *Let X be a scheme. Let $X = \bigcup_i U_i$ be an affine open covering. Let $V \subset X$ be an affine open. There exists a standard open covering $V = \bigcup_{j=1,\ldots,m} V_j$ (see Definition 5.2) such that each V_j is a standard open in one of the U_i.*

Proof. Pick $v \in V$. Then $v \in U_i$ for some i. By Lemma 11.5 above there exists an open $v \in W_v \subset V \cap U_i$ such that W_v is a standard open in both V and U_i. Since V is quasi-compact the lemma follows. $\square$

Lemma 11.7. *Let X be a scheme. Let $\mathcal{B}$ be the set of affine opens of X. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$, see Sheaves, Definition 30.1. The following are equivalent*

(1) *$\mathcal{F}$ is the restriction of a sheaf on X to $\mathcal{B}$,*
(2) *$\mathcal{F}$ is a sheaf on $\mathcal{B}$, and*
(3) *$\mathcal{F}(\emptyset)$ is a singleton and whenever $U = V \cup W$ with $U, V, W \in \mathcal{B}$ and $V, W \subset U$ standard open (Algebra, Definition 17.3) the map*

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \times \mathcal{F}(W)$$

is injective with image the set of pairs (s,t) such that $s|_{V \cap W} = t|_{V \cap W}$.

Proof. The equivalence of (1) and (2) is Sheaves, Lemma 30.7. It is clear that (2) implies (3). Hence it suffices to prove that (3) implies (2). By Sheaves, Lemma 30.4 and Lemma 5.1 it suffices to prove the sheaf condition holds for standard open coverings (Definition 5.2) of elements of $\mathcal{B}$. Let $U = U_1 \cup \ldots \cup U_n$ be a standard open covering with $U \subset X$ affine open. We will prove the sheaf condition for this covering by induction on n. If $n = 0$, then U is empty and we get the sheaf condition by assumption. If $n = 1$, then there is nothing to prove. If $n = 2$, then this is assumption (3). If $n > 2$, then we write $U_i = D(f_i)$ for $f_i \in A = \mathcal{O}_X(U)$. Suppose that $s_i \in \mathcal{F}(U_i)$ are sections such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $1 \leq i < j \leq n$. Since $U = U_1 \cup \ldots \cup U_n$ we have $1 = \sum_{i=1,\ldots,n} a_i f_i$ in A for some $a_i \in A$, see Algebra, Lemma 17.2. Set $g = \sum_{i=1,\ldots,n-1} a_i f_i$. Then $U = D(g) \cup D(f_n)$. Observe that $D(g) = D(gf_1) \cup \ldots \cup D(gf_{n-1})$ is a standard open covering. By induction there is a unique section $s' \in \mathcal{F}(D(g))$ which agrees with $s_i|_{D(gf_i)}$ for $i = 1, \ldots, n-1$. We claim that s' and s_n have the same restriction to $D(gf_n)$. This is true by induction and the covering $D(gf_n) = D(gf_n f_1) \cup \ldots \cup D(gf_n f_{n-1})$. Thus there is a unique section $s \in \mathcal{F}(U)$ whose restriction to $D(g)$ is s' and whose restriction to $D(f_n)$ is s_n. We omit the verification that s restricts to s_i on $D(f_i)$ for $i = 1, \ldots, n-1$ and we omit the verification that s is unique. $\square$

Lemma 11.8. *Let X be a scheme whose underlying topological space is a finite discrete set. Then X is affine.*

Proof. Say $X = \{x_1, \ldots, x_n\}$. Then $U_i = \{x_i\}$ is an open neighbourhood of x_i. By Lemma 11.2 it is affine. Hence X is a finite disjoint union of affine schemes, and hence is affine by Lemma 6.8. $\square$

Example 11.9. There exists a scheme without closed points. Namely, let R be a local domain whose spectrum looks like $(0) = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{m}$. Then the open subscheme $\operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ does not have a closed point. To see that such a ring R exists, we use that given any totally ordered group $(\Gamma, \geq)$ there exists a valuation ring A with valuation group $(\Gamma, \geq)$, see [Kru32]. See Algebra, Section 50 for notation. We take $\Gamma = \mathbf{Z}x_1 \oplus \mathbf{Z}x_2 \oplus \mathbf{Z}x_3 \oplus \ldots$ and we define $\sum_i a_i x_i \geq 0$ if and only if the first nonzero a_i is > 0, or all $a_i = 0$. So $x_1 \geq x_2 \geq x_3 \geq \ldots \geq 0$. The

subsets $x_i + \Gamma_{\geq 0}$ are prime ideals of $(\Gamma, \geq)$, see Algebra, notation above Lemma 50.17. These together with $\emptyset$ and $\Gamma_{\geq 0}$ are the only prime ideals. Hence A is an example of a ring with the given structure of its spectrum, by Algebra, Lemma 50.17.

12. Reduced schemes

Definition 12.1. Let X be a scheme. We say X is *reduced* if every local ring $\mathcal{O}_{X,x}$ is reduced.

Lemma 12.2. *A scheme X is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for all $U \subset X$ open.*

Proof. Assume that X is reduced. Let $f \in \mathcal{O}_X(U)$ be a section such that $f^n = 0$. Then the image of f in $\mathcal{O}_{U,u}$ is zero for all $u \in U$. Hence f is zero, see Sheaves, Lemma 11.1. Conversely, assume that $\mathcal{O}_X(U)$ is reduced for all opens U. Pick any nonzero element $f \in \mathcal{O}_{X,x}$. Any representative $(U, f \in \mathcal{O}(U))$ of f is nonzero and hence not nilpotent. Hence f is not nilpotent in $\mathcal{O}_{X,x}$. $\square$

Lemma 12.3. *An affine scheme $\mathrm{Spec}(R)$ is reduced if and only if R is reduced.*

Proof. The direct implication follows immediately from Lemma 12.2 above. In the other direction it follows since any localization of a reduced ring is reduced, and in particular the local rings of a reduced ring are reduced. $\square$

Lemma 12.4. *Let X be a scheme. Let $T \subset X$ be a closed subset. There exists a unique closed subscheme $Z \subset X$ with the following properties: (a) the underlying topological space of Z is equal to T, and (b) Z is reduced.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sub presheaf defined by the rule

$$\mathcal{I}(U) = \{f \in \mathcal{O}_X(U) \mid f(t) = 0 \text{ for all } t \in T \cap U\}$$

Here we use $f(t)$ to indicate the image of f in the residue field $\kappa(t)$ of X at t. Because of the local nature of the condition it is clear that $\mathcal{I}$ is a sheaf of ideals. Moreover, let $U = \mathrm{Spec}(R)$ be an affine open. We may write $T \cap U = V(I)$ for a unique radical ideal $I \subset R$. Given a prime $\mathfrak{p} \in V(I)$ corresponding to $t \in T \cap U$ and an element $f \in R$ we have $f(t) = 0 \Leftrightarrow f \in \mathfrak{p}$. Hence $\mathcal{I}(U) = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = I$ by Algebra, Lemma 17.2. Moreover, for any standard open $D(g) \subset \mathrm{Spec}(R) = U$ we have $\mathcal{I}(D(g)) = I_g$ by the same reasoning. Thus $\widetilde{I}$ and $\mathcal{I}|_U$ agree (as ideals) on a basis of opens and hence are equal. Therefore $\mathcal{I}$ is a quasi-coherent sheaf of ideals.

At this point we may define Z as the closed subspace associated to the sheaf of ideals $\mathcal{I}$. For every affine open $U = \mathrm{Spec}(R)$ of X we see that $Z \cap U = \mathrm{Spec}(R/I)$ where I is a radical ideal and hence Z is reduced (by Lemma 12.3 above). By construction the underlying closed subset of Z is T. Hence we have found a closed subscheme with properties (a) and (b).

Let $Z' \subset X$ be a second closed subscheme with properties (a) and (b). For every affine open $U = \mathrm{Spec}(R)$ of X we see that $Z' \cap U = \mathrm{Spec}(R/I')$ for some ideal $I' \subset R$. By Lemma 12.3 the ring R/I' is reduced and hence I' is radical. Since $V(I') = T \cap U = V(I)$ we deduced that $I = I'$ by Algebra, Lemma 17.2. Hence Z' and Z are defined by the same sheaf of ideals and hence are equal. $\square$

Definition 12.5. Let X be a scheme. Let $Z \subset X$ be a closed subset. A *scheme structure on Z* is given by a closed subscheme Z' of X whose underlying set is equal to Z. We often say "let $(Z, \mathcal{O}_Z)$ be a scheme structure on Z" to indicate this. The *reduced induced scheme structure* on Z is the one constructed in Lemma 12.4. The *reduction X_{red} of X* is the reduced induced scheme structure on X itself.

Often when we say "let $Z \subset X$ be an irreducible component of X" we think of Z as a reduced closed subscheme of X using the reduced induced scheme structure.

Remark 12.6. Let X be a scheme. Let $T \subset X$ be a locally closed subset. In this situation we sometimes also use the phrase "reduced induced scheme structure on T". It refers to the reduced induced scheme structure from Definition 12.5 when we view T as a closed subset of the open subscheme $X \setminus \partial T$ of X. Here $\partial T = \overline{T} \setminus T$ is the "boundary" of T in the topological space of X.

Lemma 12.7. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let Y be a reduced scheme. A morphism $f : Y \to X$ factors through Z if and only if $f(Y) \subset Z$ (set theoretically). In particular, any morphism $Y \to X$ factors as $Y \to X_{red} \to X$.*

Proof. Assume $f(Y) \subset Z$ (set theoretically). Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. For any affine opens $U \subset X$, $\mathrm{Spec}(B) = V \subset Y$ with $f(V) \subset U$ and any $g \in \mathcal{I}(U)$ the pullback $b = f^\sharp(g) \in \Gamma(V, \mathcal{O}_Y) = B$ maps to zero in the residue field of any $y \in V$. In other words $b \in \bigcap_{\mathfrak{p} \subset B} \mathfrak{p}$. This implies $b = 0$ as B is reduced (Lemma 12.2, and Algebra, Lemma 17.2). Hence f factors through Z by Lemma 4.6. $\square$

13. Points of schemes

Given a scheme X we can define a functor

$$h_X : Sch^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}(T, X).$$

See Categories, Example 3.4. This is called the *functor of points of X*. A fun part of scheme theory is to find descriptions of the internal geometry of X in terms of this functor h_X. In this section we find a simple way to describe points of X.

Let X be a scheme. Let R be a local ring with maximal ideal $\mathfrak{m} \subset R$. Suppose that $f : \mathrm{Spec}(R) \to X$ is a morphism of schemes. Let $x \in X$ be the image of the closed point $\mathfrak{m} \in \mathrm{Spec}(R)$. Then we obtain a local homomorphism of local rings

$$f^\sharp : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{\mathrm{Spec}(R), \mathfrak{m}} = R.$$

Lemma 13.1. *Let X be a scheme. Let R be a local ring. The construction above gives a bijective correspondence between morphisms $\mathrm{Spec}(R) \to X$ and pairs (x, φ) consisting of a point $x \in X$ and a local homomorphism of local rings $\varphi : \mathcal{O}_{X,x} \to R$.*

Proof. Let A be a ring. For any ring homomorphism $\psi : A \to R$ there exists a unique prime ideal $\mathfrak{p} \subset A$ and a factorization $A \to A_\mathfrak{p} \to R$ where the last map is a local homomorphism of local rings. Namely, $\mathfrak{p} = \psi^{-1}(\mathfrak{m})$. Via Lemma 6.4 this proves that the lemma holds if X is an affine scheme.

Let X be a general scheme. Any $x \in X$ is contained in an open affine $U \subset X$. By the affine case we conclude that every pair (x, φ) occurs as the end product of the construction above the lemma.

To finish the proof it suffices to show that any morphism $f : \mathrm{Spec}(R) \to X$ has image contained in any affine open containing the image x of the closed point of

$\mathrm{Spec}(R)$. In fact, let $x \in V \subset X$ be any open neighbourhood containing x. Then $f^{-1}(V) \subset \mathrm{Spec}(R)$ is an open containing the unique closed point and hence equal to $\mathrm{Spec}(R)$. $\qquad\square$

As a special case of the lemma above we obtain for every point x of a scheme X a canonical morphism

$$(13.1.1) \qquad\qquad \mathrm{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$$

corresponding to the identity map on the local ring of X at x. We may reformulate the lemma above as saying that for any morphism $f : \mathrm{Spec}(R) \to X$ there exists a unique point $x \in X$ such that f factors as $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ where the first map comes from a local homomorphism $\mathcal{O}_{X,x} \to R$.

In case we have a morphism of schemes $f : X \to S$, and a point x mapping to a point $s \in S$ we obtain a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(\mathcal{O}_{X,x}) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S
\end{array}
$$

where the left vertical map corresponds to the local ring map $f_x^\sharp : \mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$.

Lemma 13.2. *Let X be a scheme. Let $x, x' \in X$ be points of X. Then $x' \in X$ is a generalization of x if and only if x' is in the image of the canonical morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$.*

Proof. A continuous map preserves the relation of specialization/generalization. Since every point of $\mathrm{Spec}(\mathcal{O}_{X,x})$ is a generalization of the closed point we see every point in the image of $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ is a generalization of x. Conversely, suppose that x' is a generalization of x. Choose an affine open neighbourhood $U = \mathrm{Spec}(R)$ of x. Then $x' \in U$. Say $\mathfrak{p} \subset R$ and $\mathfrak{p}' \subset R$ are the primes corresponding to x and x'. Since x' is a generalization of x we see that $\mathfrak{p}' \subset \mathfrak{p}$. This means that $\mathfrak{p}'$ is in the image of the morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) = \mathrm{Spec}(R_\mathfrak{p}) \to \mathrm{Spec}(R) = U \subset X$ as desired. $\qquad\square$

Now, let us discuss morphisms from spectra of fields. Let $(R, \mathfrak{m}, \kappa)$ be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let K be a field. A local homomorphism $R \to K$ by definition factors as $R \to \kappa \to K$, i.e., is the same thing as a morphism $\kappa \to K$. Thus we see that morphisms

$$\mathrm{Spec}(K) \longrightarrow X$$

correspond to pairs $(x, \kappa(x) \to K)$. We may define a preorder on morphisms of spectra of fields to X by saying that $\mathrm{Spec}(K) \to X$ dominates $\mathrm{Spec}(L) \to X$ if $\mathrm{Spec}(K) \to X$ factors through $\mathrm{Spec}(L) \to X$. This suggests the following notion: Let us temporarily say that two morphisms $p : \mathrm{Spec}(K) \to X$ and $q : \mathrm{Spec}(L) \to X$ are *equivalent* if there exists a third field Ω and a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(\Omega) & \longrightarrow & \mathrm{Spec}(L) \\
\downarrow & & \downarrow{\scriptstyle q} \\
\mathrm{Spec}(K) & \xrightarrow{\ p\ } & X
\end{array}
$$

Of course this immediately implies that the unique points of all three of the schemes $\mathrm{Spec}(K)$, $\mathrm{Spec}(L)$, and $\mathrm{Spec}(\Omega)$ map to the same $x \in X$. Thus a diagram (by the remarks above) corresponds to a point $x \in X$ and a commutative diagram

$$
\begin{array}{ccc}
\Omega & \longleftarrow & L \\
\uparrow & & \uparrow \\
K & \longleftarrow & \kappa(x)
\end{array}
$$

of fields. This defines an equivalence relation, because given any set of field extensions K_i/κ there exists some field extension Ω/κ such that all the field extensions K_i are contained in the extension Ω.

Lemma 13.3. *Let X be a scheme. Points of X correspond bijectively to equivalence classes of morphisms from spectra of fields into X. Moreover, each equivalence class contains a (unique up to unique isomorphism) smallest element $\mathrm{Spec}(\kappa(x)) \to X$.*

Proof. Follows from the discussion above. $\square$

Of course the morphisms $\mathrm{Spec}(\kappa(x)) \to X$ factor through the canonical morphisms $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$. And the content of Lemma 13.2 is in this setting that the morphism $\mathrm{Spec}(\kappa(x')) \to X$ factors as $\mathrm{Spec}(\kappa(x')) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ whenever x' is a generalization of x. In case we have a morphism of schemes $f : X \to S$, and a point x mapping to a point $s \in S$ we obtain a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(\kappa(x)) \longrightarrow \mathrm{Spec}(\mathcal{O}_{X,x}) \longrightarrow X \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\
\mathrm{Spec}(\kappa(s)) \longrightarrow \mathrm{Spec}(\mathcal{O}_{S,s}) \longrightarrow S.
\end{array}
$$

14. Glueing schemes

Let I be a set. For each $i \in I$ let $(X_i, \mathcal{O}_i)$ be a locally ringed space. (Actually the construction that follows works equally well for ringed spaces.) For each pair $i, j \in I$ let $U_{ij} \subset X_i$ be an open subspace. For each pair $i, j \in I$, let

$$\varphi_{ij} : U_{ij} \to U_{ji}$$

be an isomorphism of locally ringed spaces. For convenience we assume that $U_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}_{X_i}$. For each triple $i, j, k \in I$ assume that

(1) we have $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$, and
(2) the diagram

$$
\begin{array}{ccc}
U_{ij} \cap U_{ik} & \xrightarrow{\quad\varphi_{ik}\quad} & U_{ki} \cap U_{kj} \\
{\scriptstyle\varphi_{ij}}\searrow & & \nearrow{\scriptstyle\varphi_{jk}} \\
& U_{ji} \cap U_{jk} &
\end{array}
$$

is commutative.

Let us call a collection $(I, (X_i)_{i \in I}, (U_{ij})_{i,j \in I}, (\varphi_{ij})_{i,j \in I})$ satisfying the conditions above a glueing data.

Lemma 14.1. *Given any glueing data of locally ringed spaces there exists a locally ringed space X and open subspaces $U_i \subset X$ together with isomorphisms $\varphi_i : X_i \to U_i$ of locally ringed spaces such that*

(1) $X = \bigcup_{i \in I} U_i$,
(2) $\varphi_i(U_{ij}) = U_i \cap U_j$, *and*
(3) $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$.

The locally ringed space X is characterized by the following mapping properties: Given a locally ringed space Y we have

$$
\begin{aligned}
\operatorname{Mor}(X, Y) &= \{(f_i)_{i \in I} \mid f_i : X_i \to Y,\ f_j \circ \varphi_{ij} = f_i|_{U_{ij}}\} \\
f &\mapsto (f|_{U_i} \circ \varphi_i)_{i \in I} \\
\operatorname{Mor}(Y, X) &= \left\{ \begin{array}{c} \text{open covering } Y = \bigcup_{i \in I} V_i \text{ and } (g_i : V_i \to X_i)_{i \in I} \text{ such that} \\ g_i^{-1}(U_{ij}) = V_i \cap V_j \text{ and } g_j|_{V_i \cap V_j} = \varphi_{ij} \circ g_i|_{V_i \cap V_j} \end{array} \right\} \\
g &\mapsto V_i = g^{-1}(U_i),\ g_i = \varphi_i^{-1} \circ g|_{V_i}
\end{aligned}
$$

Proof. We construct X in stages. As a set we take

$$
X = \left(\coprod X_i \right) / \sim .
$$

Here given $x \in X_i$ and $x' \in X_j$ we say $x \sim x'$ if and only if $x \in U_{ij}$, $x' \in U_{ji}$ and $\varphi_{ij}(x) = x'$. This is an equivalence relation since if $x \in X_i$, $x' \in X_j$, $x'' \in X_k$, and $x \sim x'$ and $x' \sim x''$, then $x' \in U_{ji} \cap U_{jk}$, hence by condition (1) of a glueing data also $x \in U_{ij} \cap U_{ik}$ and $x'' \in U_{ki} \cap U_{kj}$ and by condition (2) we see that $\varphi_{ik}(x) = x''$. (Reflexivity and symmetry follows from our assumptions that $U_{ii} = X_i$ and $\varphi_{ii} = \operatorname{id}_{X_i}$.) Denote $\varphi_i : X_i \to X$ the natural maps. Denote $U_i = \varphi_i(X_i) \subset X$. Note that $\varphi_i : X_i \to U_i$ is a bijection.

The topology on X is defined by the rule that $U \subset X$ is open if and only if $\varphi_i^{-1}(U)$ is open for all i. We leave it to the reader to verify that this does indeed define a topology. Note that in particular U_i is open since $\varphi_j^{-1}(U_i) = U_{ji}$ which is open in X_j for all j. Moreover, for any open set $W \subset X_i$ the image $\varphi_i(W) \subset U_i$ is open because $\varphi_j^{-1}(\varphi_i(W)) = \varphi_{ji}^{-1}(W \cap U_{ij})$. Therefore $\varphi_i : X_i \to U_i$ is a homeomorphism.

To obtain a locally ringed space we have to construct the sheaf of rings $\mathcal{O}_X$. We do this by glueing the sheaves of rings $\mathcal{O}_{U_i} := \varphi_{i,*} \mathcal{O}_i$. Namely, in the commutative diagram

$$
\begin{array}{ccc}
U_{ij} & \xrightarrow{\ \ \varphi_{ij}\ \ } & U_{ji} \\
& \searrow^{\varphi_i|_{U_{ij}}} \quad \swarrow_{\varphi_j|_{U_{ji}}} & \\
& U_i \cap U_j &
\end{array}
$$

the arrow on top is an isomorphism of ringed spaces, and hence we get unique isomorphisms of sheaves of rings

$$
\mathcal{O}_{U_i}|_{U_i \cap U_j} \longrightarrow \mathcal{O}_{U_j}|_{U_i \cap U_j}.
$$

These satisfy a cocycle condition as in Sheaves, Section 33. By the results of that section we obtain a sheaf of rings $\mathcal{O}_X$ on X such that $\mathcal{O}_X|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}$ compatibly with the glueing maps displayed above. In particular $(X, \mathcal{O}_X)$ is a locally ringed space since the stalks of $\mathcal{O}_X$ are equal to the stalks of $\mathcal{O}_i$ at corresponding points.

The proof of the mapping properties is omitted. $\qquad \square$

Lemma 14.2. *In Lemma 14.1 above, assume that all X_i are schemes. Then the resulting locally ringed space X is a scheme.*

Proof. This is clear since each of the U_i is a scheme and hence every $x \in X$ has an affine neighbourhood. $\qquad\square$

It is customary to think of X_i as an open subspace of X via the isomorphisms φ_i. We will do this in the next two examples.

Example 14.3 (Affine space with zero doubled). Let k be a field. Let $n \geq 1$. Let $X_1 = \mathrm{Spec}(k[x_1, \ldots, x_n])$, let $X_2 = \mathrm{Spec}(k[y_1, \ldots, y_n])$. Let $0_1 \in X_1$ be the point corresponding to the maximal ideal $(x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$. Let $0_2 \in X_2$ be the point corresponding to the maximal ideal $(y_1, \ldots, y_n) \subset k[y_1, \ldots, y_n]$. Let $U_{12} = X_1 \setminus \{0_1\}$ and let $U_{21} = X_2 \setminus \{0_2\}$. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism coming from the isomorphism of k-algebras $k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$ mapping y_i to x_i (which induces $X_1 \cong X_2$ mapping 0_1 to 0_2). Let X be the scheme obtained from the glueing data $(X_1, X_2, U_{12}, U_{21}, \varphi_{12}, \varphi_{21} = \varphi_{12}^{-1})$. Via the slight abuse of notation introduced above the example we think of $X_1, X_2 \subset X$ as open subschemes. There is a morphism $f : X \to \mathrm{Spec}(k[t_1, \ldots, t_n])$ which on X_1 (resp. X_2) corresponds to k algebra map $k[t_1, \ldots, t_n] \to k[x_1, \ldots, x_n]$ (resp. $k[t_1, \ldots, t_n] \to k[y_1, \ldots, y_n]$) mapping t_i to x_i (resp. t_i to y_i). It is easy to see that this morphism identifies $k[t_1, \ldots, t_n]$ with $\Gamma(X, \mathcal{O}_X)$. Since $f(0_1) = f(0_2)$ we see that X is not affine.

Note that X_1 and X_2 are affine opens of X. But, if $n = 2$, then $X_1 \cap X_2$ is the scheme described in Example 9.3 and hence not affine. Thus in general the intersection of affine opens of a scheme is not affine. (This fact holds more generally for any $n > 1$.)

Another curious feature of this example is the following. If $n > 1$ there are many irreducible closed subsets $T \subset X$ (take the closure of any non closed point in X_1 for example). But unless $T = \{0_1\}$, or $T = \{0_2\}$ we have $0_1 \in T \Leftrightarrow 0_2 \in T$. Proof omitted.

Example 14.4 (Projective line). Let k be a field. Let $X_1 = \mathrm{Spec}(k[x])$, let $X_2 = \mathrm{Spec}(k[y])$. Let $0 \in X_1$ be the point corresponding to the maximal ideal $(x) \subset k[x]$. Let $\infty \in X_2$ be the point corresponding to the maximal ideal $(y) \subset k[y]$. Let $U_{12} = X_1 \setminus \{0\} = D(x) = \mathrm{Spec}(k[x, 1/x])$ and let $U_{21} = X_2 \setminus \{\infty\} = D(y) = \mathrm{Spec}(k[y, 1/y])$. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism coming from the isomorphism of k-algebras $k[y, 1/y] \to k[x, 1/x]$ mapping y to $1/x$. Let $\mathbf{P}^1_k$ be the scheme obtained from the glueing data $(X_1, X_2, U_{12}, U_{21}, \varphi_{12}, \varphi_{21} = \varphi_{12}^{-1})$. Via the slight abuse of notation introduced above the example we think of $X_i \subset \mathbf{P}^1_k$ as open subschemes. In this case we see that $\Gamma(\mathbf{P}^1_k, \mathcal{O}) = k$ because the only polynomials $g(x)$ in x such that $g(1/y)$ is also a polynomial in y are constant polynomials. Since $\mathbf{P}^1_k$ is infinite we see that $\mathbf{P}^1_k$ is not affine.

We claim that there exists an affine open $U \subset \mathbf{P}^1_k$ which contains both 0 and ∞. Namely, let $U = \mathbf{P}^1_k \setminus \{1\}$, where 1 is the point of X_1 corresponding to the maximal ideal $(x - 1)$ and also the point of X_2 corresponding to the maximal ideal $(y - 1)$. Then it is easy to see that $s = 1/(x - 1) = y/(1 - y) \in \Gamma(U, \mathcal{O}_U)$. In fact you can show that $\Gamma(U, \mathcal{O}_U)$ is equal to the polynomial ring $k[s]$ and that the corresponding morphism $U \to \mathrm{Spec}(k[s])$ is an isomorphism of schemes. Details omitted.

15. A representability criterion

In this section we reformulate the glueing lemma of Section 14 in terms of functors. We recall some of the material from Categories, Section 3. Recall that given a scheme X we can define a functor

$$h_X : Sch^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}(T, X).$$

This is called the *functor of points of X*.

Let F be a contravariant functor from the category of schemes to the category of sets. In a formula

$$F : Sch^{opp} \longrightarrow Sets.$$

We will use the same terminology as in Sites, Section 2. Namely, given a scheme T, an element $\xi \in F(T)$, and a morphism $f : T' \to T$ we will denote $f^*\xi$ the element $F(f)(\xi)$, and sometimes we will even use the notation $\xi|_{T'}$

Definition 15.1. (See Categories, Definition 3.6.) Let F be a contravariant functor from the category of schemes to the category of sets (as above). We say that F is *representable by a scheme* or *representable* if there exists a scheme X such that $h_X \cong F$.

Suppose that F is representable by the scheme X and that $s : h_X \to F$ is an isomorphism. By Categories, Yoneda Lemma 3.5 the pair $(X, s : h_X \to F)$ is unique up to unique isomorphism if it exists. Moreover, the Yoneda lemma says that given any contravariant functor F as above and any scheme Y, we have a bijection

$$\mathrm{Mor}_{\mathrm{Fun}(Sch^{opp}, Sets)}(h_Y, F) \longrightarrow F(Y), \quad s \longmapsto s(\mathrm{id}_Y).$$

Here is the reverse construction. Given any $\xi \in F(Y)$ the transformation of functors $s_\xi : h_Y \to F$ associates to any morphism $f : T \to Y$ the element $f^*\xi \in F(T)$.

In particular, in the case that F is representable, there exists a scheme X and an element $\xi \in F(X)$ such that the corresponding morphism $h_X \to F$ is an isomorphism. In this case we also say *the pair (X, ξ) represents F*. The element $\xi \in F(X)$ is often called the *"universal family"* for reasons that will become more clear when we talk about algebraic stacks (insert future reference here). For the moment we simply observe that the fact that if the pair (X, ξ) represents F, then every element $\xi' \in F(T)$ for any T is of the form $\xi' = f^*\xi$ for a unique morphism $f : T \to X$.

Example 15.2. Consider the rule which associates to every scheme T the set $F(T) = \Gamma(T, \mathcal{O}_T)$. We can turn this into a contravariant functor by using for a morphism $f : T' \to T$ the pullback map $f^\sharp : \Gamma(T, \mathcal{O}_T) \to \Gamma(T', \mathcal{O}_{T'})$. Given a ring R and an element $t \in R$ there exists a unique ring homomorphism $\mathbf{Z}[x] \to R$ which maps x to t. Thus, using Lemma 6.4, we see that

$$\mathrm{Mor}(T, \mathrm{Spec}(\mathbf{Z}[x])) = \mathrm{Hom}(\mathbf{Z}[x], \Gamma(T, \mathcal{O}_T)) = \Gamma(T, \mathcal{O}_T).$$

This does indeed give an isomorphism $h_{\mathrm{Spec}(\mathbf{Z}[x])} \to F$. What is the "universal family" ξ? To get it we have to apply the identifications above to $\mathrm{id}_{\mathrm{Spec}(\mathbf{Z}[x])}$. Clearly under the identifications above this gives that $\xi = x \in \Gamma(\mathrm{Spec}(\mathbf{Z}[x]), \mathcal{O}_{\mathrm{Spec}(\mathbf{Z}[x])}) = \mathbf{Z}[x]$ as expected.

Definition 15.3. Let F be a contravariant functor on the category of schemes with values in sets.

(1) We say that F *satisfies the sheaf property for the Zariski topology* if for every scheme T and every open covering $T = \bigcup_{i \in I} U_i$, and for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.

(2) A *subfunctor* $H \subset F$ is a rule that associates to every scheme T a subset $H(T) \subset F(T)$ such that the maps $F(f) : F(T) \to F(T')$ maps $H(T)$ into $H(T')$ for all morphisms of schemes $f : T' \to T$.

(3) Let $H \subset F$ be a subfunctor. We say that $H \subset F$ is *representable by open immersions* if for all pairs (T, ξ), where T is a scheme and $\xi \in F(T)$ there exists an open subscheme $U_\xi \subset T$ with the following property:

(*) A morphism $f : T' \to T$ factors through U_ξ if and only if $f^*\xi \in H(T')$.

(4) Let I be a set. For each $i \in I$ let $H_i \subset F$ be a subfunctor. We say that the collection $(H_i)_{i \in I}$ *covers* F if and only if for every $\xi \in F(T)$ there exists an open covering $T = \bigcup U_i$ such that $\xi|_{U_i} \in H_i(U_i)$.

In condition (4), if $H_i \subset F$ is representable by open immersions for all i, then to check $(H_i)_{i \in I}$ covers F, it suffices to check $F(T) = \bigcup H_i(T)$ whenever T is the spectrum of a field.

Lemma 15.4. *Let F be a contravariant functor on the category of schemes with values in the category of sets. Suppose that*

(1) *F satisfies the sheaf property for the Zariski topology,*

(2) *there exists a set I and a collection of subfunctors $F_i \subset F$ such that*

    (a) *each F_i is representable,*

    (b) *each $F_i \subset F$ is representable by open immersions, and*

    (c) *the collection $(F_i)_{i \in I}$ covers F.*

Then F is representable.

Proof. Let X_i be a scheme representing F_i and let $\xi_i \in F_i(X_i) \subset F(X_i)$ be the "universal family". Because $F_j \subset F$ is representable by open immersions, there exists an open $U_{ij} \subset X_i$ such that $T \to X_i$ factors through U_{ij} if and only if $\xi_i|_T \in F_j(T)$. In particular $\xi_i|_{U_{ij}} \in F_j(U_{ij})$ and therefore we obtain a canonical morphism $\varphi_{ij} : U_{ij} \to X_j$ such that $\varphi_{ij}^*\xi_j = \xi_i|_{U_{ij}}$. By definition of U_{ji} this implies that φ_{ij} factors through U_{ji}. Since $(\varphi_{ij} \circ \varphi_{ji})^*\xi_j = \varphi_{ji}^*(\varphi_{ij}^*\xi_j) = \varphi_{ji}^*\xi_i = \xi_j$ we conclude that $\varphi_{ij} \circ \varphi_{ji} = \mathrm{id}_{U_{ji}}$ because the pair (X_j, ξ_j) represents F_j. In particular the maps $\varphi_{ij} : U_{ij} \to U_{ji}$ are isomorphisms of schemes. Next we have to show that $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$. This is true because (a) $U_{ji} \cap U_{jk}$ is the largest open of U_{ji} such that ξ_j restricts to an element of F_k, (b) $U_{ij} \cap U_{ik}$ is the largest open of U_{ij} such that ξ_i restricts to an element of F_k, and (c) $\varphi_{ij}^*\xi_j = \xi_i$. Moreover, the cocycle condition in Section 14 follows because both $\varphi_{jk}|_{U_{ji} \cap U_{jk}} \circ \varphi_{ij}|_{U_{ij} \cap U_{ik}}$ and $\varphi_{ik}|_{U_{ij} \cap U_{ik}}$ pullback ξ_k to the element ξ_i. Thus we may apply Lemma 14.2 to obtain a scheme X with an open covering $X = \bigcup U_i$ and isomorphisms $\varphi_i : X_i \to U_i$ with properties as in Lemma 14.1. Let $\xi_i' = (\varphi_i^{-1})^*\xi_i$. The conditions of Lemma 14.1 imply that $\xi_i'|_{U_i \cap U_j} = \xi_j'|_{U_i \cap U_j}$. Therefore, by the condition that F satisfies the sheaf condition in the Zariski topology we see that there exists an element $\xi' \in F(X)$ such that $\xi_i = \varphi_i^*\xi'|_{U_i}$ for all i. Since φ_i is an isomorphism we also get that $(U_i, \xi'|_{U_i})$ represents the functor F_i.

We claim that the pair (X, ξ') represents the functor F. To show this, let T be a scheme and let $\xi \in F(T)$. We will construct a unique morphism $g : T \to X$ such

that $g^*\xi' = \xi$. Namely, by the condition that the subfunctors F_i cover F there exists an open covering $T = \bigcup V_i$ such that for each i the restriction $\xi|_{V_i} \in F_i(V_i)$. Moreover, since each of the inclusions $F_i \subset F$ are representable by open immersions we may assume that each $V_i \subset T$ is maximal open with this property. Because, (U_i, ξ'_{U_i}) represents the functor F_i we get a unique morphism $g_i : V_i \to U_i$ such that $g_i^* \xi'|_{U_i} = \xi|_{V_i}$. On the overlaps $V_i \cap V_j$ the morphisms g_i and g_j agree, for example because they both pull back $\xi'|_{U_i \cap U_j} \in F_i(U_i \cap U_j)$ to the same element. Thus the morphisms g_i glue to a unique morphism from $T \to X$ as desired. $\square$

Remark 15.5. Suppose the functor F is defined on all locally ringed spaces, and if conditions of Lemma 15.4 are replaced by the following:

(1) F satisfies the sheaf property on the category of locally ringed spaces,
(2) there exists a set I and a collection of subfunctors $F_i \subset F$ such that
   (a) each F_i is representable by a scheme,
   (b) each $F_i \subset F$ is representable by open immersions on the category of locally ringed spaces, and
   (c) the collection $(F_i)_{i \in I}$ covers F as a functor on the category of locally ringed spaces.

We leave it to the reader to spell this out further. Then the end result is that the functor F is representable in the category of locally ringed spaces and that the representing object is a scheme.

16. Existence of fibre products of schemes

A very basic question is whether or not products and fibre products exist on the category of schemes. We first prove abstractly that products and fibre products exist, and in the next section we show how we may think in a reasonable way about fibre products of schemes.

Lemma 16.1. *The category of schemes has a final object, products and fibre products. In other words, the category of schemes has finite limits, see Categories, Lemma 18.4.*

Proof. Please skip this proof. It is more important to learn how to work with the fibre product which is explained in the next section.

By Lemma 6.4 the scheme $\mathrm{Spec}(\mathbf{Z})$ is a final object in the category of locally ringed spaces. Thus it suffices to prove that fibred products exist.

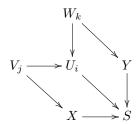Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. We have to show that the functor

$$\begin{aligned} F : Sch^{opp} &\longrightarrow Sets \\ T &\longmapsto \mathrm{Mor}(T, X) \times_{\mathrm{Mor}(T,S)} \mathrm{Mor}(T, Y) \end{aligned}$$

is representable. We claim that Lemma 15.4 applies to the functor F. If we prove this then the lemma is proved.

First we show that F satisfies the sheaf property in the Zariski topology. Namely, suppose that T is a scheme, $T = \bigcup_{i \in I} U_i$ is an open covering, and $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ for all pairs i, j. By definition ξ_i corresponds to a pair (a_i, b_i) where $a_i : U_i \to X$ and $b_i : U_i \to Y$ are morphisms of schemes such that $f \circ a_i = g \circ b_i$. The glueing condition says that $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ and

$b_i|_{U_i \cap U_j} = b_j|_{U_i \cap U_j}$. Thus by glueing the morphisms a_i we obtain a morphism of locally ringed spaces (i.e., a morphism of schemes) $a : T \to X$ and similarly $b : T \to Y$ (see for example the mapping property of Lemma 14.1). Moreover, on the members of an open covering the compositions $f \circ a$ and $g \circ b$ agree. Therefore $f \circ a = g \circ b$ and the pair (a, b) defines an element of $F(T)$ which restricts to the pairs (a_i, b_i) on each U_i. The sheaf condition is verified.

Next, we construct the family of subfunctors. Choose an open covering by open affines $S = \bigcup_{i \in I} U_i$. For every $i \in I$ choose open coverings by open affines $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ and $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$. Note that $X = \bigcup_{i \in I} \bigcup_{j \in J_i} V_j$ is an open covering and similarly for Y. For any $i \in I$ and each pair $(j, k) \in J_i \times K_i$ we have a commutative diagram

$$
\begin{array}{ccc}
& W_k & \\
& \downarrow \searrow & \\
V_j \longrightarrow U_i & & Y \\
\searrow & \searrow & \downarrow \\
& X \longrightarrow S &
\end{array}
$$

where all the skew arrows are open immersions. For such a triple we get a functor

$$
\begin{aligned}
F_{i,j,k} : Sch^{opp} & \longrightarrow Sets \\
T & \longmapsto \mathrm{Mor}(T, V_j) \times_{\mathrm{Mor}(T, U_i)} \mathrm{Mor}(T, W_k).
\end{aligned}
$$

There is an obvious transformation of functors $F_{i,j,k} \to F$ (coming from the huge commutative diagram above) which is injective, so we may think of $F_{i,j,k}$ as a subfunctor of F.

We check condition (2)(a) of Lemma 15.4. This follows directly from Lemma 6.7. (Note that we use here that the fibre products in the category of affine schemes are also fibre products in the whole category of locally ringed spaces.)

We check condition (2)(b) of Lemma 15.4. Let T be a scheme and let $\xi \in F(T)$. In other words, $\xi = (a, b)$ where $a : T \to X$ and $b : T \to Y$ are morphisms of schemes such that $f \circ a = g \circ b$. Set $V_{i,j,k} = a^{-1}(V_j) \cap b^{-1}(W_k)$. For any further morphism $h : T' \to T$ we have $h^*\xi = (a \circ h, b \circ h)$. Hence we see that $h^*\xi \in F_{i,j,k}(T')$ if and only if $a(h(T')) \subset V_j$ and $b(h(T')) \subset W_k$. In other words, if and only if $h(T') \subset V_{i,j,k}$. This proves condition (2)(b).

We check condition (2)(c) of Lemma 15.4. Let T be a scheme and let $\xi = (a, b) \in F(T)$ as above. Set $V_{i,j,k} = a^{-1}(V_j) \cap b^{-1}(W_k)$ as above. Condition (2)(c) just means that $T = \bigcup V_{i,j,k}$ which is evident. Thus the lemma is proved and fibre products exist. $\qquad\square$

Remark 16.2. Using Remark 15.5 you can show that the fibre product of morphisms of schemes exists in the category of locally ringed spaces and is a scheme.
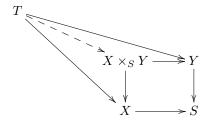
17. Fibre products of schemes

Here is a review of the general definition, even though we have already shown that fibre products of schemes exist.

Definition 17.1. Given morphisms of schemes $f : X \to S$ and $g : Y \to S$ the *fibre product* is a scheme $X \times_S Y$ together with projection morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ sitting into the following commutative diagram

$$
\begin{array}{ccc}
X \times_S Y & \xrightarrow{\ q\ } & Y \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
X & \xrightarrow{\ f\ } & S
\end{array}
$$

which is universal among all diagrams of this sort, see Categories, Definition 6.1.

In other words, given any solid commutative diagram of morphisms of schemes

there exists a unique dotted arrow making the diagram commute. We will prove some lemmas which will tell us how to think about fibre products.

Lemma 17.2. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. If X, Y, S are all affine then $X \times_S Y$ is affine.*

Proof. Suppose that $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. By Lemma 6.7 the affine scheme $\mathrm{Spec}(A \otimes_R B)$ is the fibre product $X \times_S Y$ in the category of locally ringed spaces. Hence it is a fortiori the fibre product in the category of schemes. $\square$

Lemma 17.3. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $X \times_S Y$, p, q be the fibre product. Suppose that $U \subset S$, $V \subset X$, $W \subset Y$ are open subschemes such that $f(V) \subset U$ and $g(W) \subset U$. Then the canonical morphism $V \times_U W \to X \times_S Y$ is an open immersion which identifies $V \times_U W$ with $p^{-1}(V) \cap q^{-1}(W)$.*

Proof. Let T be a scheme. Suppose $a : T \to V$ and $b : T \to W$ are morphisms such that $f \circ a = g \circ b$ as morphisms into U. Then they agree as morphisms into S. By the universal property of the fibre product we get a unique morphism $T \to X \times_S Y$. Of course this morphism has image contained in the open $p^{-1}(V) \cap q^{-1}(W)$. Thus $p^{-1}(V) \cap q^{-1}(W)$ is a fibre product of V and W over U. The result follows from the uniqueness of fibre products, see Categories, Section 6. $\square$

In particular this shows that $V \times_U W = V \times_S W$ in the situation of the lemma. Moreover, if U, V, W are all affine, then we know that $V \times_U W$ is affine. And of course we may cover $X \times_S Y$ by such affine opens $V \times_U W$. We formulate this as a lemma.

Lemma 17.4. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $S = \bigcup U_i$ be any affine open covering of S. For each $i \in I$,*

let $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ *be an affine open covering of* $f^{-1}(U_i)$ *and let* $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$ *be an affine open covering of* $g^{-1}(U_i)$. *Then*

$$X \times_S Y = \bigcup_{i \in I} \bigcup_{j \in J_i, \ k \in K_i} V_j \times_{U_i} W_k$$

is an affine open covering of $X \times_S Y$.

Proof. See discussion above the lemma. $\square$

In other words, we might have used the previous lemma to construct the fibre product directly by glueing the affine schemes. (Which is of course exactly what we did in the proof of Lemma 16.1 anyway.) Here is a way to describe the set of points of a fibre product of schemes.

Lemma 17.5. *Let* $f : X \to S$ *and* $g : Y \to S$ *be morphisms of schemes with the same target. Points* z *of* $X \times_S Y$ *are in bijective correspondence to quadruples*

$$(x, y, s, \mathfrak{p})$$

where $x \in X$, $y \in Y$, $s \in S$ *are points with* $f(x) = s$, $g(y) = s$ *and* $\mathfrak{p}$ *is a prime ideal of the ring* $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$. *The residue field of* z *corresponds to the residue field of the prime* $\mathfrak{p}$.

Proof. Let z be a point of $X \times_S Y$ and let us construct a quadruple as above. Recall that we may think of z as a morphism $\mathrm{Spec}(\kappa(z)) \to X \times_S Y$, see Lemma 13.3. This morphism corresponds to morphisms $a : \mathrm{Spec}(\kappa(z)) \to X$ and $b : \mathrm{Spec}(\kappa(z)) \to Y$ such that $f \circ a = g \circ b$. By the same lemma again we get points $x \in X$, $y \in Y$ lying over the same point $s \in S$ as well as field maps $\kappa(x) \to \kappa(z)$, $\kappa(y) \to \kappa(z)$ such that the compositions $\kappa(s) \to \kappa(x) \to \kappa(z)$ and $\kappa(s) \to \kappa(y) \to \kappa(z)$ are the same. In other words we get a ring map $\kappa(x) \otimes_{\kappa(s)} \kappa(y) \to \kappa(z)$. We let $\mathfrak{p}$ be the kernel of this map.

Conversely, given a quadruple $(x, y, s, \mathfrak{p})$ we get a commutative solid diagram

see the discussion in Section 13. Thus we get the dotted arrow. The corresponding point z of $X \times_S Y$ is the image of the generic point of $\mathrm{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)/\mathfrak{p})$. We omit the verification that the two constructions are inverse to each other. $\square$

Lemma 17.6. *Let* $f : X \to S$ *and* $g : Y \to S$ *be morphisms of schemes with the same target.*

(1) *If* $f : X \to S$ *is a closed immersion, then* $X \times_S Y \to Y$ *is a closed immersion. Moreover, if* $X \to S$ *corresponds to the quasi-coherent sheaf*

> *of ideals $\mathcal{I} \subset \mathcal{O}_S$, then $X \times_S Y \to Y$ corresponds to the sheaf of ideals $\mathrm{Im}(g^*\mathcal{I} \to \mathcal{O}_Y)$.*
>
> (2) *If $f : X \to S$ is an open immersion, then $X \times_S Y \to Y$ is an open immersion.*
>
> (3) *If $f : X \to S$ is an immersion, then $X \times_S Y \to Y$ is an immersion.*

Proof. Assume that $X \to S$ is a closed immersion corresponding to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$. By Lemma 4.7 the closed subspace $Z \subset Y$ defined by the sheaf of ideals $\mathrm{Im}(g^*\mathcal{I} \to \mathcal{O}_Y)$ is the fibre product in the category of locally ringed spaces. By Lemma 10.1 Z is a scheme. Hence $Z = X \times_S Y$ and the first statement follows. The second follows from Lemma 17.3 for example. The third is a combination of the first two. $\square$

Definition 17.7. Let $f : X \to Y$ be a morphism of schemes. Let $Z \subset Y$ be a closed subscheme of Y. The *inverse image $f^{-1}(Z)$ of the closed subscheme Z* is the closed subscheme $Z \times_Y X$ of X. See Lemma 17.6 above.

We may occasionally also use this terminology with locally closed and open sub-schemes.

18. Base change in algebraic geometry

One motivation for the introduction of the language of schemes is that it gives a very precise notion of what it means to define a variety over a particular field. For example a variety X over $\mathbf{Q}$ is synonymous (Varieties, Definition 3.1) with $X \to \mathrm{Spec}(\mathbf{Q})$ which is of finite type, separated, irreducible and reduced[1]. In any case, the idea is more generally to work with schemes over a given *base scheme*, often denoted S. We use the language: "let X be a scheme over S" to mean simply that X comes equipped with a morphism $X \to S$. In diagrams we will try to picture the *structure morphism $X \to S$* as a downward arrow from X to S. We are often more interested in the properties of X relative to S rather than the internal geometry of X. For example, we would like to know things about the fibres of $X \to S$, what happens to X after base change, and so on.

We introduce some of the language that is customarily used. Of course this language is just a special case of thinking about the category of objects over a given object in a category, see Categories, Example 2.13.

Definition 18.1. Let S be a scheme.

(1) We say X is a *scheme over S* to mean that X comes equipped with a morphism of schemes $X \to S$. The morphism $X \to S$ is sometimes called the *structure morphism*.

(2) If R is a ring we say X is a *scheme over R* instead of X is a scheme over $\mathrm{Spec}(R)$.

(3) A *morphism $f : X \to Y$ of schemes over S* is a morphism of schemes such that the composition $X \to Y \to S$ of f with the structure morphism of Y is equal to the structure morphism of X.

(4) We denote $\mathrm{Mor}_S(X, Y)$ the set of all morphisms from X to Y over S.

(5) Let X be a scheme over S. Let $S' \to S$ be a morphism of schemes. The *base change* of X is the scheme $X_{S'} = S' \times_S X$ over S'.

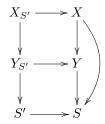[1]Of course algebraic geometers still quibble over whether one should require X to be geometrically irreducible over $\mathbf{Q}$.

(6) Let $f : X \to Y$ be a morphism of schemes over S. Let $S' \to S$ be a morphism of schemes. The *base change* of f is the induced morphism $f' : X_{S'} \to Y_{S'}$ (namely the morphism $\mathrm{id}_{S'} \times_{\mathrm{id}_S} f$).

(7) Let R be a ring. Let X be a scheme over R. Let $R \to R'$ be a ring map. The *base change* $X_{R'}$ is the scheme $\mathrm{Spec}(R') \times_{\mathrm{Spec}(R)} X$ over R'.

Here is a typical result.

Lemma 18.2. *Let S be a scheme. Let $f : X \to Y$ be an immersion (resp. closed immersion, resp. open immersion) of schemes over S. Then any base change of f is an immersion (resp. closed immersion, resp. open immersion).*

Proof. We can think of the base change of f via the morphism $S' \to S$ as the top left vertical arrow in the following commutative diagram:

$$
\begin{array}{ccc}
X_{S'} & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y_{S'} & \longrightarrow & Y \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S
\end{array}
$$

The diagram implies $X_{S'} \cong Y_{S'} \times_Y X$, and the lemma follows from Lemma 17.6. $\square$

In fact this type of result is so typical that there is a piece of language to express it. Here it is.

Definition 18.3. Properties and base change.

(1) Let $\mathcal{P}$ be a property of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that $\mathcal{P}$ is *preserved under base change* if whenever X/S has $\mathcal{P}$, any base change $X_{S'}/S'$ has $\mathcal{P}$.

(2) Let $\mathcal{P}$ be a property of morphisms of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that *preserved under base change* if whenever $f : X \to Y$ over S has $\mathcal{P}$, any base change $f' : X_{S'} \to Y_{S'}$ over S' has $\mathcal{P}$.

At this point we can say that "being a closed immersion" is preserved under arbitrary base change.

Definition 18.4. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$ be a point. The *scheme theoretic fibre X_s of f over s*, or simply the *fibre of f over s*, is the scheme fitting in the following fibre product diagram
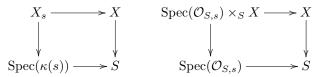
$$
\begin{array}{ccc}
X_s = \mathrm{Spec}(\kappa(s)) \times_S X & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\kappa(s)) & \longrightarrow & S
\end{array}
$$

We think of the fibre X_s always as a scheme over $\kappa(s)$.

Lemma 18.5. *Let $f : X \to S$ be a morphism of schemes. Consider the diagrams*

$$
\begin{array}{ccc}
X_s & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\kappa(s)) & \longrightarrow & S
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{Spec}(\mathcal{O}_{S,s}) \times_S X & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S
\end{array}
$$

In both cases the top horizontal arrow is a homeomorphism onto its image.

Proof. Choose an open affine $U \subset S$ that contains s. The bottom horizontal morphisms factor through U, see Lemma 13.1 for example. Thus we may assume that S is affine. If X is also affine, then the result follows from Algebra, Remark 17.8. In the general case the result follows by covering X by open affines. $\square$

19. Quasi-compact morphisms

A scheme is *quasi-compact* if its underlying topological space is quasi-compact. There is a relative notion which is defined as follows.

Definition 19.1. A morphism of schemes is called *quasi-compact* if the underlying map of topological spaces is quasi-compact, see Topology, Definition 12.1.

Lemma 19.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *$f : X \to S$ is quasi-compact,*
(2) *the inverse image of every affine open is quasi-compact, and*
(3) *there exists some affine open covering $S = \bigcup_{i \in I} U_i$ such that $f^{-1}(U_i)$ is quasi-compact for all i.*

Proof. Suppose we are given a covering $S = \bigcup_{i \in I} U_i$ as in (3). First, let $U \subset S$ be any affine open. For any $u \in U$ we can find an index $i(u) \in I$ such that $u \in U_{i(u)}$. As standard opens form a basis for the topology on $U_{i(u)}$ we can find $W_u \subset U \cap U_{i(u)}$ which is standard open in $U_{i(u)}$. By compactness we can find finitely many points $u_1, \ldots, u_n \in U$ such that $U = \bigcup_{j=1}^n W_{u_j}$. For each j write $f^{-1} U_{i(u_j)} = \bigcup_{k \in K_j} V_{jk}$ as a finite union of affine opens. Since $W_{u_j} \subset U_{i(u_j)}$ is a standard open we see that $f^{-1}(W_{u_j}) \cap V_{jk}$ is a standard open of V_{jk}, see Algebra, Lemma 17.4. Hence $f^{-1}(W_{u_j}) \cap V_{jk}$ is affine, and so $f^{-1}(W_{u_j})$ is a finite union of affines. This proves that the inverse image of any affine open is a finite union of affine opens.

Next, assume that the inverse image of every affine open is a finite union of affine opens. Let $K \subset S$ be any quasi-compact open. Since S has a basis of the topology consisting of affine opens we see that K is a finite union of affine opens. Hence the inverse image of K is a finite union of affine opens. Hence f is quasi-compact.

Finally, assume that f is quasi-compact. In this case the argument of the previous paragraph shows that the inverse image of any affine is a finite union of affine opens. $\square$

Lemma 19.3. *Being quasi-compact is a property of morphisms of schemes over a base which is preserved under arbitrary base change.*

Proof. Omitted. $\square$

Lemma 19.4. *The composition of quasi-compact morphisms is quasi-compact.*

Proof. This follows from the definitions and Topology, Lemma 12.2. $\qquad\square$

Lemma 19.5. *A closed immersion is quasi-compact.*

Proof. Follows from the definitions and Topology, Lemma 12.3. $\qquad\square$

Example 19.6. An open immersion is in general not quasi-compact. The standard example of this is the open subspace $U \subset X$, where $X = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots])$, where U is $X \setminus \{0\}$, and where 0 is the point of X corresponding to the maximal ideal $(x_1, x_2, x_3, \ldots)$.

Lemma 19.7. *Let $f : X \to S$ be a quasi-compact morphism of schemes. The following are equivalent*

  (1) $f(X) \subset S$ *is closed, and*
  (2) $f(X) \subset S$ *is stable under specialization.*

Proof. We have (1) $\Rightarrow$ (2) by Topology, Lemma 19.2. Assume (2). Let $U \subset S$ be an affine open. It suffices to prove that $f(X) \cap U$ is closed. Since $U \cap f(X)$ is stable under specializations in U, we have reduced to the case where S is affine. Because f is quasi-compact we deduce that $X = f^{-1}(S)$ is quasi-compact as S is affine. Thus we may write $X = \bigcup_{i=1}^n U_i$ with $U_i \subset X$ open affine. Say $S = \mathrm{Spec}(R)$ and $U_i = \mathrm{Spec}(A_i)$ for some R-algebra A_i. Then $f(X) = \mathrm{Im}(\mathrm{Spec}(A_1 \times \ldots \times A_n) \to \mathrm{Spec}(R))$. Thus the lemma follows from Algebra, Lemma 41.5. $\qquad\square$

Lemma 19.8. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is closed if and only if specializations lift along f, see Topology, Definition 19.4.*

Proof. According to Topology, Lemma 19.7 if f is closed then specializations lift along f. Conversely, suppose that specializations lift along f. Let $Z \subset X$ be a closed subset. We may think of Z as a scheme with the reduced induced scheme structure, see Definition 12.5. Since $Z \subset X$ is closed the restriction of f to Z is still quasi-compact. Moreover specializations lift along $Z \to S$ as well, see Topology, Lemma 19.5. Hence it suffices to prove $f(X)$ is closed if specializations lift along f. In particular $f(X)$ is stable under specializations, see Topology, Lemma 19.6. Thus $f(X)$ is closed by Lemma 19.7. $\qquad\square$

20. Valuative criterion for universal closedness

In Topology, Section 17 there is a discussion of proper maps as closed maps of topological spaces all of whose fibres are quasi-compact, or as maps such that all base changes are closed maps. Here is the corresponding notion in algebraic geometry.

Definition 20.1. A morphism of schemes $f : X \to S$ is said to be *universally closed* if every base change $f' : X_{S'} \to S'$ is closed.

In fact the adjective "universally" is often used in this way. In other words, given a property $\mathcal{P}$ of morphisms then we say that "$X \to S$ is universally $\mathcal{P}$" if and only if every base change $X_{S'} \to S'$ has $\mathcal{P}$.
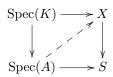
Please take a look at Morphisms, Section 41 for a more detailed discussion of the properties of universally closed morphisms. In this section we restrict the discussion to the relationship between universal closed morphisms and morphisms satisfying the existence part of the valuative criterion.

Lemma 20.2. *Let $f : X \to S$ be a morphism of schemes.*

(1) *If f is universally closed then specializations lift along any base change of f, see Topology, Definition 19.4.*
(2) *If f is quasi-compact and specializations lift along any base change of f, then f is universally closed.*

Proof. Part (1) is a direct consequence of Topology, Lemma 19.7. Part (2) follows from Lemmas 19.8 and 19.3. $\square$

Definition 20.3. Let $f : X \to S$ be a morphism of schemes. We say f *satisfies the existence part of the valuative criterion* if given any commutative solid diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(K) & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\mathrm{Spec}(A) & \longrightarrow & S
\end{array}
$$

where A is a valuation ring with field of fractions K, the dotted arrow exists. We say f *satisfies the uniqueness part of the valuative criterion* if there is at most one dotted arrow given any diagram as above (without requiring existence of course).

A *valuation ring* is a local domain maximal among the relation of domination in its fraction field, see Algebra, Definition 50.1. Hence the spectrum of a valuation ring has a unique generic point η and a unique closed point 0, and of course we have the specialization $\eta \rightsquigarrow 0$. The significance of valuation rings is that any specialization of points in any scheme is the image of $\eta \rightsquigarrow 0$ under some morphism from the spectrum of some valuation ring. Here is the precise result.

Lemma 20.4. *Let S be a scheme. Let $s' \rightsquigarrow s$ be a specialization of points of S. Then*

(1) *there exists a valuation ring A and a morphism $f : \mathrm{Spec}(A) \to S$ such that the generic point η of $\mathrm{Spec}(A)$ maps to s' and the special point maps to s, and*
(2) *given a field extension $K/\kappa(s')$ we may arrange it so that the extension $\kappa(\eta)/\kappa(s')$ induced by f is isomorphic to the given extension.*

Proof. Let $s' \rightsquigarrow s$ be a specialization in S, and let $K/\kappa(s')$ be an extension of fields. By Lemma 13.2 and the discussion following Lemma 13.3 this leads to ring maps $\mathcal{O}_{S,s} \to \kappa(s') \to K$. Let $A \subset K$ be any valuation ring whose field of fractions is K and which dominates the image of $\mathcal{O}_{S,s} \to K$, see Algebra, Lemma 50.2. The ring map $\mathcal{O}_{S,s} \to A$ induces the morphism $f : \mathrm{Spec}(A) \to S$, see Lemma 13.1. This morphism has all the desired properties by construction. $\square$

Lemma 20.5. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *Specializations lift along any base change of f*
(2) *The morphism f satisfies the existence part of the valuative criterion.*

Proof. Assume (1) holds. Let a solid diagram as in Definition 20.3 be given. In order to find the dotted arrow we may replace $X \to S$ by $X_{\mathrm{Spec}(A)} \to \mathrm{Spec}(A)$ since after all the assumption is stable under base change. Thus we may assume $S = \mathrm{Spec}(A)$. Let $x' \in X$ be the image of $\mathrm{Spec}(K) \to X$, so that we have

$\kappa(x') \subset K$, see Lemma 13.3. By assumption there exists a specialization $x' \rightsquigarrow x$ in X such that x maps to the closed point of $S = \mathrm{Spec}(A)$. We get a local ring map $A \to \mathcal{O}_{X,x}$ and a ring map $\mathcal{O}_{X,x} \to \kappa(x')$, see Lemma 13.2 and the discussion following Lemma 13.3. The composition $A \to \mathcal{O}_{X,x} \to \kappa(x') \to K$ is the given injection $A \to K$. Since $A \to \mathcal{O}_{X,x}$ is local, the image of $\mathcal{O}_{X,x} \to K$ dominates A and hence is equal to A, by Algebra, Definition 50.1. Thus we obtain a ring map $\mathcal{O}_{X,x} \to A$ and hence a morphism $\mathrm{Spec}(A) \to X$ (see Lemma 13.1 and discussion following it). This proves (2).

Conversely, assume (2) holds. It is immediate that the existence part of the valuative criterion holds for any base change $X_{S'} \to S'$ of f by considering the following commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Spec}(K) & \longrightarrow & X_{S'} & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(A) & \longrightarrow & S' & \longrightarrow & S
\end{array}
$$

Namely, the more horizontal dotted arrow will lead to the other one by definition of the fibre product. OK, so it clearly suffices to show that specializations lift along f. Let $s' \rightsquigarrow s$ be a specialization in S, and let $x' \in X$ be a point lying over s'. Apply Lemma 20.4 to $s' \rightsquigarrow s$ and the extension of fields $K = \kappa(x')/\kappa(s')$. We get a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(K) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S
\end{array}
$$

and by condition (2) we get the dotted arrow. The image x of the closed point of $\mathrm{Spec}(A)$ in X will be a solution to our problem, i.e., x is a specialization of x' and maps to s. $\square$

Proposition 20.6 (Valuative criterion of universal closedness)**.** *Let f be a quasi-compact morphism of schemes. Then f is universally closed if and only if f satisfies the existence part of the valuative criterion.*

Proof. This is a formal consequence of Lemmas 20.2 and 20.5 above. $\square$

Example 20.7. Let k be a field. Consider the structure morphism $p : \mathbf{P}^1_k \to \mathrm{Spec}(k)$ of the projective line over k, see Example 14.4. Let us use the valuative criterion above to prove that p is universally closed. By construction $\mathbf{P}^1_k$ is covered by two affine opens and hence p is quasi-compact. Let a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(K) & \xrightarrow{\ \xi\ } & \mathbf{P}^1_k \\
\downarrow & & \downarrow \\
\mathrm{Spec}(A) & \xrightarrow{\ \varphi\ } & \mathrm{Spec}(k)
\end{array}
$$

be given, where A is a valuation ring and K is its field of fractions. Recall that $\mathbf{P}^1_k$ is gotten by glueing $\mathrm{Spec}(k[x])$ to $\mathrm{Spec}(k[y])$ by glueing $D(x)$ to $D(y)$ via $x = y^{-1}$ (or more symmetrically $xy = 1$). To show there is a morphism $\mathrm{Spec}(A) \to \mathbf{P}^1_k$ fitting diagonally into the diagram above we may assume that ξ maps into the

open $\mathrm{Spec}(k[x])$ (by symmetry). This gives the following commutative diagram of rings

$$\begin{array}{ccc} K & \xleftarrow{\quad \xi^\sharp \quad} & k[x] \\ \uparrow & & \uparrow \\ A & \xleftarrow{\quad \varphi^\sharp \quad} & k \end{array}$$

By Algebra, Lemma 50.4 we see that either $\xi^\sharp(x) \in A$ or $\xi^\sharp(x)^{-1} \in A$. In the first case we get a ring map

$$k[x] \to A, \ \lambda \mapsto \varphi^\sharp(\lambda), \ x \mapsto \xi^\sharp(x)$$

fitting into the diagram of rings above, and we win. In the second case we see that we get a ring map

$$k[y] \to A, \ \lambda \mapsto \varphi^\sharp(\lambda), \ y \mapsto \xi^\sharp(x)^{-1}.$$

This gives a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(k[y]) \to \mathbf{P}^1_k$ which fits diagonally into the initial commutative diagram of this example (check omitted).

21. Separation axioms

A topological space X is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is a closed subset. The analogue in algebraic geometry is, given a scheme X over a base scheme S, to consider the diagonal morphism

$$\Delta_{X/S} : X \longrightarrow X \times_S X.$$

This is the unique morphism of schemes such that $\mathrm{pr}_1 \circ \Delta_{X/S} = \mathrm{id}_X$ and $\mathrm{pr}_2 \circ \Delta_{X/S} = \mathrm{id}_X$ (it exists in any category with fibre products).

Lemma 21.1. *The diagonal morphism of a morphism between affines is closed.*

Proof. The diagonal morphism associated to the morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is the morphism on spectra corresponding to the ring map $S \otimes_R S \to S$, $a \otimes b \mapsto ab$. This map is clearly surjective, so $S \cong S \otimes_R S/J$ for some ideal $J \subset S \otimes_R S$. Hence Δ is a closed immersion according to Example 8.1 $\qquad \square$

Lemma 21.2. *Let X be a scheme over S. The diagonal morphism $\Delta_{X/S}$ is an immersion.*

Proof. Recall that if $V \subset X$ is affine open and maps into $U \subset S$ affine open, then $V \times_U V$ is affine open in $X \times_S X$, see Lemmas 17.2 and 17.3. Consider the open subscheme W of $X \times_S X$ which is the union of these affine opens $V \times_U V$. By Lemma 4.2 it is enough to show that each morphism $\Delta_{X/S}^{-1}(V \times_U V) \to V \times_U V$ is a closed immersion. Since $V = \Delta_{X/S}^{-1}(V \times_U V)$ we are just checking that $\Delta_{V/U}$ is a closed immersion, which is Lemma 21.1. $\qquad \square$

Definition 21.3. Let $f : X \to S$ be a morphism of schemes.

(1) We say f is *separated* if the diagonal morphism $\Delta_{X/S}$ is a closed immersion.
(2) We say f is *quasi-separated* if the diagonal morphism $\Delta_{X/S}$ is a quasi-compact morphism.
(3) We say a scheme Y is *separated* if the morphism $Y \to \mathrm{Spec}(\mathbf{Z})$ is separated.
(4) We say a scheme Y is *quasi-separated* if the morphism $Y \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated.

By Lemmas 21.2 and 10.4 we see that $\Delta_{X/S}$ is a closed immersion if an only if $\Delta_{X/S}(X) \subset X \times_S X$ is a closed subset. Moreover, by Lemma 19.5 we see that a separated morphism is quasi-separated. The reason for introducing quasi-separated morphisms is that nonseparated morphisms come up naturally in studying algebraic varieties (especially when doing moduli, algebraic stacks, etc). But most often they are still quasi-separated.

Example 21.4. Here is an example of a non-quasi-separated morphism. Suppose $X = X_1 \cup X_2 \to S = \mathrm{Spec}(k)$ with $X_1 = X_2 = \mathrm{Spec}(k[t_1, t_2, t_3, \ldots])$ glued along the complement of $\{0\} = \{(t_1, t_2, t_3, \ldots)\}$ (glued as in Example 14.3). In this case the inverse image of the affine scheme $X_1 \times_S X_2$ under $\Delta_{X/S}$ is the scheme $\mathrm{Spec}(k[t_1, t_2, t_3, \ldots]) \setminus \{0\}$ which is not quasi-compact.

Lemma 21.5. *Let X, Y be schemes over S. Let $a, b : X \to Y$ be morphisms of schemes over S. There exists a largest locally closed subscheme $Z \subset X$ such that $a|_Z = b|_Z$. In fact Z is the equalizer of (a, b). Moreover, if Y is separated over S, then Z is a closed subscheme.*

Proof. The equalizer of (a, b) is for categorical reasons the fibre product Z in the following diagram

$$
\begin{array}{ccc}
Z = Y \times_{(Y \times_S Y)} X & \longrightarrow & X \\
\downarrow & & \downarrow {\scriptstyle (a,b)} \\
Y & \xrightarrow{\ \Delta_{Y/S}\ } & Y \times_S Y
\end{array}
$$

Thus the lemma follows from Lemmas 18.2, 21.2 and Definition 21.3. $\square$

Lemma 21.6. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is quasi-separated.*
(2) *For every pair of affine opens $U, V \subset X$ which map into a common affine open of S the intersection $U \cap V$ is a finite union of affine opens of X.*
(3) *There exists an affine open covering $S = \bigcup_{i \in I} U_i$ and for each i an affine open covering $f^{-1}U_i = \bigcup_{j \in I_i} V_j$ such that for each i and each pair $j, j' \in I_i$ the intersection $V_j \cap V_{j'}$ is a finite union of affine opens of X.*

Proof. Let us prove that (3) implies (1). By Lemma 17.4 the covering $X \times_S X = \bigcup_i \bigcup_{j,j'} V_j \times_{U_i} V_{j'}$ is an affine open covering of $X \times_S X$. Moreover, $\Delta_{X/S}^{-1}(V_j \times_{U_i} V_{j'}) = V_j \cap V_{j'}$. Hence the implication follows from Lemma 19.2.

The implication (1) $\Rightarrow$ (2) follows from the fact that under the hypotheses of (2) the fibre product $U \times_S V$ is an affine open of $X \times_S X$. The implication (2) $\Rightarrow$ (3) is trivial. $\square$

Lemma 21.7. *Let $f : X \to S$ be a morphism of schemes.*

(1) *If f is separated then for every pair of affine opens (U, V) of X which map into a common affine open of S we have*
   (a) *the intersection $U \cap V$ is affine.*
   (b) *the ring map $\mathcal{O}_X(U) \otimes_{\mathbf{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.*
(2) *If any pair of points $x_1, x_2 \in X$ lying over a common point $s \in S$ are contained in affine opens $x_1 \in U$, $x_2 \in V$ which map into a common affine open of S such that (a), (b) hold, then f is separated.*

Proof. Assume f separated. Suppose (U, V) is a pair as in (1). Let $W = \mathrm{Spec}(R)$ be an affine open of S containing both $f(U)$ and $f(V)$. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ for R-algebras A and B. By Lemma 17.3 we see that $U \times_S V = U \times_W V = \mathrm{Spec}(A \otimes_R B)$ is an affine open of $X \times_S X$. Hence, by Lemma 10.1 we see that $\Delta^{-1}(U \times_S V) \to U \times_S V$ can be identified with $\mathrm{Spec}((A \otimes_R B)/J)$ for some ideal $J \subset A \otimes_R B$. Thus $U \cap V = \Delta^{-1}(U \times_S V)$ is affine. Assertion (1)(b) holds because $A \otimes_{\mathbf{Z}} B \to (A \otimes_R B)/J$ is surjective.

Assume the hypothesis formulated in (2) holds. Clearly the collection of affine opens $U \times_S V$ for pairs (U, V) as in (2) form an affine open covering of $X \times_S X$ (see e.g. Lemma 17.4). Hence it suffices to show that each morphism $U \cap V = \Delta_{X/S}^{-1}(U \times_S V) \to U \times_S V$ is a closed immersion, see Lemma 4.2. By assumption (a) we have $U \cap V = \mathrm{Spec}(C)$ for some ring C. After choosing an affine open $W = \mathrm{Spec}(R)$ of S into which both U and V map and writing $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(B)$ we see that the assumption (b) means that the composition

$$A \otimes_{\mathbf{Z}} B \to A \otimes_R B \to C$$

is surjective. Hence $A \otimes_R B \to C$ is surjective and we conclude that $\mathrm{Spec}(C) \to \mathrm{Spec}(A \otimes_R B)$ is a closed immersion. $\square$

Example 21.8. Let k be a field. Consider the structure morphism $p : \mathbf{P}_k^1 \to \mathrm{Spec}(k)$ of the projective line over k, see Example 14.4. Let us use the lemma above to prove that p is separated. By construction $\mathbf{P}_k^1$ is covered by two affine opens $U = \mathrm{Spec}(k[x])$ and $V = \mathrm{Spec}(k[y])$ with intersection $U \cap V = \mathrm{Spec}(k[x, y]/(xy-1))$ (using obvious notation). Thus it suffices to check that conditions (2)(a) and (2)(b) of Lemma 21.7 hold for the pairs of affine opens (U, U), (U, V), (V, U) and (V, V). For the pairs (U, U) and (V, V) this is trivial. For the pair (U, V) this amounts to proving that $U \cap V$ is affine, which is true, and that the ring map

$$k[x] \otimes_{\mathbf{Z}} k[y] \longrightarrow k[x, y]/(xy - 1)$$

is surjective. This is clear because any element in the right hand side can be written as a sum of a polynomial in x and a polynomial in y.

Lemma 21.9. *Let $f : X \to T$ and $g : Y \to T$ be morphisms of schemes with the same target. Let $h : T \to S$ be a morphism of schemes. Then the induced morphism $i : X \times_T Y \to X \times_S Y$ is an immersion. If $T \to S$ is separated, then i is a closed immersion. If $T \to S$ is quasi-separated, then i is a quasi-compact morphism.*

Proof. By general category theory the following diagram

$$
\begin{array}{ccc}
X \times_T Y & \longrightarrow & X \times_S Y \\
\downarrow & & \downarrow \\
T & \xrightarrow{\;\Delta_{T/S}\;} & T \times_S T
\end{array}
$$

is a fibre product diagram. The lemma follows from Lemmas 21.2, 17.6 and 19.3. $\square$

Lemma 21.10. *Let $g : X \to Y$ be a morphism of schemes over S. The morphism $i : X \to X \times_S Y$ is an immersion. If Y is separated over S it is a closed immersion. If Y is quasi-separated over S it is quasi-compact.*

Proof. This is a special case of Lemma 21.9 applied to the morphism $X = X \times_Y Y \to X \times_S Y$. $\square$

Lemma 21.11. *Let $f : X \to S$ be a morphism of schemes. Let $s : S \to X$ be a section of f (in a formula $f \circ s = id_S$). Then s is an immersion. If f is separated then s is a closed immersion. If f is quasi-separated, then s is quasi-compact.*

Proof. This is a special case of Lemma 21.10 applied to $g = s$ so the morphism $i = s : S \to S \times_S X$. $\square$

Lemma 21.12. *Permanence properties.*

(1) *A composition of separated morphisms is separated.*
(2) *A composition of quasi-separated morphisms is quasi-separated.*
(3) *The base change of a separated morphism is separated.*
(4) *The base change of a quasi-separated morphism is quasi-separated.*
(5) *A (fibre) product of separated morphisms is separated.*
(6) *A (fibre) product of quasi-separated morphisms is quasi-separated.*

Proof. Let $X \to Y \to Z$ be morphisms. Assume that $X \to Y$ and $Y \to Z$ are separated. The composition

$$X \to X \times_Y X \to X \times_Z X$$

is closed because the first one is by assumption and the second one by Lemma 21.9. The same argument works for "quasi-separated" (with the same references).

Let $f : X \to Y$ be a morphism of schemes over a base S. Let $S' \to S$ be a morphism of schemes. Let $f' : X_{S'} \to Y_{S'}$ be the base change of f. Then the diagonal morphism of f' is a morphism

$$\Delta_{f'} : X_{S'} = S' \times_S X \longrightarrow X_{S'} \times_{Y_{S'}} X_{S'} = S' \times_S (X \times_Y X)$$

which is easily seen to be the base change of Δ_f. Thus (3) and (4) follow from the fact that closed immersions and quasi-compact morphisms are preserved under arbitrary base change (Lemmas 17.6 and 19.3).

If $f : X \to Y$ and $g : U \to V$ are morphisms of schemes over a base S, then $f \times g$ is the composition of $X \times_S U \to X \times_S V$ (a base change of g) and $X \times_S V \to Y \times_S V$ (a base change of f). Hence (5) and (6) follow from (1) – (4). $\square$

Lemma 21.13. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is separated then so is f. If $g \circ f$ is quasi-separated then so is f.*

Proof. Assume that $g \circ f$ is separated. Consider the factorization $X \to X \times_Y X \to X \times_Z X$ of the diagonal morphism of $g \circ f$. By Lemma 21.9 the last morphism is an immersion. By assumption the image of X in $X \times_Z X$ is closed. Hence it is also closed in $X \times_Y X$. Thus we see that $X \to X \times_Y X$ is a closed immersion by Lemma 10.4.

Assume that $g \circ f$ is quasi-separated. Let $V \subset Y$ be an affine open which maps into an affine open of Z. Let $U_1, U_2 \subset X$ be affine opens which map into V. Then $U_1 \cap U_2$ is a finite union of affine opens because U_1, U_2 map into a common affine open of Z. Since we may cover Y by affine opens like V we deduce the lemma from Lemma 21.6. $\square$

Lemma 21.14. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : X \to X \times_Z Y \to Y$. The first map is quasi-compact by Lemma 21.11 because it is a section of the quasi-separated morphism $X \times_Z Y \to X$ (a base change of g, see Lemma 21.12). The second map is quasi-compact as it is the base change of $g \circ f$, see Lemma 19.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 19.4. $\square$

Lemma 21.15. *An affine scheme is separated. A morphism from an affine scheme to another scheme is separated.*

Proof. Let $U = \operatorname{Spec}(A)$ be an affine scheme. Then $U \to \operatorname{Spec}(\mathbf{Z})$ has closed diagonal by Lemma 21.1. Thus U is separated by Definition 21.3. If $U \to X$ is a morphism of schemes, then we can apply Lemma 21.13 to the morphisms $U \to X \to \operatorname{Spec}(\mathbf{Z})$ to conclude that $U \to X$ is separated. $\square$

You may have been wondering whether the condition of only considering pairs of affine opens whose image is contained in an affine open is really necessary to be able to conclude that their intersection is affine. Often it isn't!

Lemma 21.16. *Let $f : X \to S$ be a morphism. Assume f is separated and S is a separated scheme. Suppose $U \subset X$ and $V \subset X$ are affine. Then $U \cap V$ is affine (and a closed subscheme of $U \times V$).*

Proof. In this case X is separated by Lemma 21.12. Hence $U \cap V$ is affine by applying Lemma 21.7 to the morphism $X \to \operatorname{Spec}(\mathbf{Z})$. $\square$

On the other hand, the following example shows that we cannot expect the image of an affine to be contained in an affine.

Example 21.17. Consider the nonaffine scheme $U = \operatorname{Spec}(k[x, y]) \setminus \{(x, y)\}$ of Example 9.3. On the other hand, consider the scheme

$$\mathbf{GL}_{2,k} = \operatorname{Spec}(k[a, b, c, d, 1/ad - bc]).$$

There is a morphism $\mathbf{GL}_{2,k} \to U$ corresponding to the ring map $x \mapsto a$, $y \mapsto b$. It is easy to see that this is a surjective morphism, and hence the image is not contained in any affine open of U. In fact, the affine scheme $\mathbf{GL}_{2,k}$ also surjects onto $\mathbf{P}^1_k$, and $\mathbf{P}^1_k$ does not even have an immersion into *any* affine scheme.

Remark 21.18. The category of quasi-compact and quasi-separated schemes $\mathcal{C}$ has the following properties. If $X, Y \in \operatorname{Ob}(\mathcal{C})$, then any morphism of schemes $f : X \to Y$ is quasi-compact and quasi-separated by Lemmas 21.14 and 21.13 with $Z = \operatorname{Spec}(\mathbf{Z})$. Moreover, if $X \to Y$ and $Z \to Y$ are morphisms $\mathcal{C}$, then $X \times_Y Z$ is an object of $\mathcal{C}$ too. Namely, the projection $X \times_Y Z \to Z$ is quasi-compact and quasi-separated as a base change of the morphism $Z \to Y$, see Lemmas 21.12 and 19.3. Hence the composition $X \times_Y Z \to Z \to \operatorname{Spec}(\mathbf{Z})$ is quasi-compact and quasi-separated, see Lemmas 21.12 and 19.4.

22. Valuative criterion of separatedness

Lemma 22.1. *Let $f : X \to S$ be a morphism of schemes. If f is separated, then f satisfies the uniqueness part of the valuative criterion.*
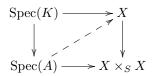
Proof. Let a diagram as in Definition 20.3 be given. Suppose there are two morphisms $a, b : \operatorname{Spec}(A) \to X$ fitting into the diagram. Let $Z \subset \operatorname{Spec}(A)$ be the

equalizer of a and b. By Lemma 21.5 this is a closed subscheme of $\mathrm{Spec}(A)$. By assumption it contains the generic point of $\mathrm{Spec}(A)$. Since A is a domain this implies $Z = \mathrm{Spec}(A)$. Hence $a = b$ as desired. $\qquad\square$

Lemma 22.2 (Valuative criterion separatedness)**.** *Let $f : X \to S$ be a morphism. Assume*

    (1) *the morphism f is quasi-separated, and*

    (2) *the morphism f satisfies the uniqueness part of the valuative criterion.*

Then f is separated.

Proof. By assumption (1), Proposition 20.6, and Lemmas 21.2 and 10.4 we see that it suffices to prove the morphism $\Delta_{X/S} : X \to X \times_S X$ satisfies the existence part of the valuative criterion. Let a solid commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(K) & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\mathrm{Spec}(A) & \longrightarrow & X \times_S X
\end{array}
$$

be given. The lower right arrow corresponds to a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ over S. By (2) we see that $a = b$. Hence using a as the dotted arrow works. $\qquad\square$

23. Monomorphisms

Definition 23.1. A morphism of schemes is called a *monomorphism* if it is a monomorphism in the category of schemes, see Categories, Definition 13.1.

Lemma 23.2. *Let $j : X \to Y$ be a morphism of schemes. Then j is a monomorphism if and only if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.*

Proof. This is true in any category with fibre products. $\qquad\square$

Lemma 23.3. *A monomorphism of schemes is separated.*

Proof. This is true because an isomorphism is a closed immersion, and Lemma 23.2 above. $\qquad\square$

Lemma 23.4. *A composition of monomorphisms is a monomorphism.*

Proof. True in any category. $\qquad\square$

Lemma 23.5. *The base change of a monomorphism is a monomorphism.*

Proof. True in any category with fibre products. $\qquad\square$

Lemma 23.6. *Let $j : X \to Y$ be a morphism of schemes. If j is injective on points, then j is separated.*

Proof. Let z be a point of $X \times_Y X$. Then $x = \mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ are the same because j maps these points to the same point y of Y. Then we can choose an affine open neighbourhood $V \subset Y$ of y and an affine open neighbourhood $U \subset X$ of x with $j(U) \subset V$. Then $z \in U \times_V U \subset X \times_Y X$. Hence $X \times_Y X$ is the union of the affine opens $U \times_V U$. Since $\Delta_{X/Y}^{-1}(U \times_V U) = U$ and since $U \to U \times_V U$ is a closed immersion, we conclude that $\Delta_{X/Y}$ is a closed immersion (see argument in the proof of Lemma 21.2). $\qquad\square$

Lemma 23.7. *Let $j : X \to Y$ be a morphism of schemes. If*

(1) *j is injective on points, and*

(2) *for any $x \in X$ the ring map $j_x^\sharp : \mathcal{O}_{Y,j(x)} \to \mathcal{O}_{X,x}$ is surjective,*

then j is a monomorphism.

Proof. Let $a, b : Z \to X$ be two morphisms of schemes such that $j \circ a = j \circ b$. Then (1) implies $a = b$ as underlying maps of topological spaces. For any $z \in Z$ we have $a_z^\sharp \circ j_{a(z)}^\sharp = b_z^\sharp \circ j_{b(z)}^\sharp$ as maps $\mathcal{O}_{Y,j(a(z))} \to \mathcal{O}_{Z,z}$. The surjectivity of the maps $j_x^\sharp$ forces $a_z^\sharp = b_z^\sharp$, $\forall z \in Z$. This implies that $a^\sharp = b^\sharp$. Hence we conclude $a = b$ as morphisms of schemes as desired. $\square$

Lemma 23.8. *An immersion of schemes is a monomorphism. In particular, any immersion is separated.*

Proof. We can see this by checking that the criterion of Lemma 23.7 applies. More elegantly perhaps, we can use that Lemmas 3.5 and 4.6 imply that open and closed immersions are monomorphisms and hence any immersion (which is a composition of such) is a monomorphism. $\square$

Lemma 23.9. *Let $f : X \to S$ be a separated morphism. Any locally closed subscheme $Z \subset X$ is separated over S.*

Proof. Follows from Lemma 23.8 and the fact that a composition of separated morphisms is separated (Lemma 21.12). $\square$

Example 23.10. The morphism $\mathrm{Spec}(\mathbf{Q}) \to \mathrm{Spec}(\mathbf{Z})$ is a monomorphism. This is true because $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}$. More generally, for any scheme S and any point $s \in S$ the canonical morphism

$$\mathrm{Spec}(\mathcal{O}_{S,s}) \longrightarrow S$$

is a monomorphism.

Lemma 23.11. *Let $k_1, \ldots, k_n$ be fields. For any monomorphism of schemes $X \to \mathrm{Spec}(k_1 \times \ldots \times k_n)$ there exists a subset $I \subset \{1, \ldots, n\}$ such that $X \cong \mathrm{Spec}(\prod_{i \in I} k_i)$ as schemes over $\mathrm{Spec}(k_1 \times \ldots \times k_n)$. More generally, if $X = \coprod_{i \in I} \mathrm{Spec}(k_i)$ is a disjoint union of spectra of fields and $Y \to X$ is a monomorphism, then there exists a subset $J \subset I$ such that $Y = \coprod_{i \in J} \mathrm{Spec}(k_i)$.*

Proof. First reduce to the case $n = 1$ (or $\#I = 1$) by taking the inverse images of the open and closed subschemes $\mathrm{Spec}(k_i)$. In this case X has only one point hence is affine. The corresponding algebra problem is this: If $k \to R$ is an algebra map with $R \otimes_k R \cong R$, then $R \cong k$ or $R = 0$. This holds for dimension reasons. See also Algebra, Lemma 107.8 $\square$

24. Functoriality for quasi-coherent modules

Let X be a scheme. We denote $QCoh(\mathcal{O}_X)$ the category of quasi-coherent $\mathcal{O}_X$-modules as defined in Modules, Definition 10.1. We have seen in Section 7 that the category $QCoh(\mathcal{O}_X)$ has a lot of good properties when X is affine. Since the property of being quasi-coherent is local on X, these properties are inherited by the category of quasi-coherent sheaves on any scheme X. We enumerate them here.

(1) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the restriction of $\mathcal{F}$ to each affine open $U = \mathrm{Spec}(R)$ is of the form $\widetilde{M}$ for some R-module M.

(2) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the restriction of $\mathcal{F}$ to each of the members of an affine open covering is quasi-coherent.
(3) Any direct sum of quasi-coherent sheaves is quasi-coherent.
(4) Any colimit of quasi-coherent sheaves is quasi-coherent.
(5) The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.
(6) Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are quasi-coherent so is the third.
(7) Given a morphism of schemes $f : Y \to X$ the pullback of a quasi-coherent $\mathcal{O}_X$-module is a quasi-coherent $\mathcal{O}_Y$-module. See Modules, Lemma 10.4.
(8) Given two quasi-coherent $\mathcal{O}_X$-modules the tensor product is quasi-coherent, see Modules, Lemma 16.6.
(9) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the tensor, symmetric and exterior algebras on $\mathcal{F}$ are quasi-coherent, see Modules, Lemma 21.6.
(10) Given two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is of finite presentation, then the internal hom $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, see Modules, Lemma 22.6 and (5) above.

On the other hand, it is in general not the case that the pushforward of a quasi-coherent module is quasi-coherent. Here is a case where this does hold.

Lemma 24.1. *Let $f : X \to S$ be a morphism of schemes. If f is quasi-compact and quasi-separated then f_* transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_S$-modules.*

Proof. The question is local on S and hence we may assume that S is affine. Because X is quasi-compact we may write $X = \bigcup_{i=1}^{n} U_i$ with each U_i open affine. Because f is quasi-separated we may write $U_i \cap U_j = \bigcup_{k=1}^{n_{ij}} U_{ijk}$ for some affine open U_{ijk}, see Lemma 21.6. Denote $f_i : U_i \to S$ and $f_{ijk} : U_{ijk} \to S$ the restrictions of f. For any open V of S and any sheaf $\mathcal{F}$ on X we have

$$
\begin{aligned}
f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}V) \\
&= \mathrm{Ker}\left(\bigoplus_i \mathcal{F}(f^{-1}V \cap U_i) \to \bigoplus_{i,j,k} \mathcal{F}(f^{-1}V \cap U_{ijk}) \right) \\
&= \mathrm{Ker}\left(\bigoplus_i f_{i,*}(\mathcal{F}|_{U_i})(V) \to \bigoplus_{i,j,k} f_{ijk,*}(\mathcal{F}|_{U_{ijk}})(V) \right) \\
&= \mathrm{Ker}\left(\bigoplus_i f_{i,*}(\mathcal{F}|_{U_i}) \to \bigoplus_{i,j,k} f_{ijk,*}(\mathcal{F}|_{U_{ijk}}) \right)(V)
\end{aligned}
$$

In other words there is an exact sequence of sheaves

$$
0 \to f_*\mathcal{F} \to \bigoplus f_{i,*}\mathcal{F}_i \to \bigoplus f_{ijk,*}\mathcal{F}_{ijk}
$$

where $\mathcal{F}_i, \mathcal{F}_{ijk}$ denotes the restriction of $\mathcal{F}$ to the corresponding open. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module then $\mathcal{F}_i$ is a quasi-coherent $\mathcal{O}_{U_i}$-module and $\mathcal{F}_{ijk}$ is a quasi-coherent $\mathcal{O}_{U_{ijk}}$-module. Hence by Lemma 7.3 we see that the second and third term of the exact sequence are quasi-coherent $\mathcal{O}_S$-modules. Thus we conclude that $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-module. $\square$

Using this we can characterize (closed) immersions of schemes as follows.

Lemma 24.2. *Let $f : X \to Y$ be a morphism of schemes. Suppose that*
(1) *f induces a homeomorphism of X with a closed subset of Y, and*
(2) *$f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.*

Then f is a closed immersion of schemes.

Proof. Assume (1) and (2). By (1) the morphism f is quasi-compact (see Topology, Lemma 12.3). Conditions (1) and (2) imply conditions (1) and (2) of Lemma 23.7. Hence $f : X \to Y$ is a monomorphism. In particular, f is separated, see Lemma 23.3. Hence Lemma 24.1 above applies and we conclude that $f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module. Therefore the kernel of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is quasi-coherent by Lemma 7.8. Since a quasi-coherent sheaf is locally generated by sections (see Modules, Definition 10.1) this implies that f is a closed immersion, see Definition 4.1. $\square$

We can use this lemma to prove the following lemma.

Lemma 24.3. *A composition of immersions of schemes is an immersion, a composition of closed immersions of schemes is a closed immersion, and a composition of open immersions of schemes is an open immersion.*

Proof. This is clear for the case of open immersions since an open subspace of an open subspace is also an open subspace.

Suppose $a : Z \to Y$ and $b : Y \to X$ are closed immersions of schemes. We will verify that $c = b \circ a$ is also a closed immersion. The assumption implies that a and b are homeomorphisms onto closed subsets, and hence also $c = b \circ a$ is a homeomorphism onto a closed subset. Moreover, the map $\mathcal{O}_X \to c_*\mathcal{O}_Z$ is surjective since it factors as the composition of the surjective maps $\mathcal{O}_X \to b_*\mathcal{O}_Y$ and $b_*\mathcal{O}_Y \to b_*a_*\mathcal{O}_Z$ (surjective as b_* is exact, see Modules, Lemma 6.1). Hence by Lemma 24.2 above c is a closed immersion.

Finally, we come to the case of immersions. Suppose $a : Z \to Y$ and $b : Y \to X$ are immersions of schemes. This means there exist open subschemes $V \subset Y$ and $U \subset X$ such that $a(Z) \subset V$, $b(Y) \subset U$ and $a : Z \to V$ and $b : Y \to U$ are closed immersions. Since the topology on Y is induced from the topology on U we can find an open $U' \subset U$ such that $V = b^{-1}(U')$. Then we see that $Z \to V = b^{-1}(U') \to U'$ is a composition of closed immersions and hence a closed immersion. This proves that $Z \to X$ is an immersion and we win. $\square$

25. Other chapters

Preliminaries

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