DIVISORS

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1. Introduction

In this chapter we study some very basic questions related to defining divisors, etc. A basic reference is [DG67].

2. Associated points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is *associated* to M if there exists an element of M whose annihilator is $\mathfrak{p}$. See Algebra, Definition 63.1. Here is the definition of associated points for quasi-coherent sheaves on schemes as given in [DG67, IV Definition 3.1.1].

Definition 2.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) We say $x \in X$ is *associated* to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
(2) We denote $\text{Ass}(\mathcal{F})$ or $\text{Ass}_X(\mathcal{F})$ the set of associated points of $\mathcal{F}$.
(3) The *associated points of* X are the associated points of $\mathcal{O}_X$.

These definitions are most useful when X is locally Noetherian and $\mathcal{F}$ of finite type. For example it may happen that a generic point of an irreducible component of X is not associated to X, see Example 2.7. In the non-Noetherian case it may be more convenient to use weakly associated points, see Section 5. Let us link the scheme theoretic notion with the algebraic notion on affine opens; note that this correspondence works perfectly only for locally Noetherian schemes.

Lemma 2.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\text{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime.*

(1) *If $\mathfrak{p}$ is associated to M, then x is associated to $\mathcal{F}$.*
(2) *If $\mathfrak{p}$ is finitely generated, then the converse holds as well.*

In particular, if X is locally Noetherian, then the equivalence

$$\mathfrak{p} \in Ass(M) \Leftrightarrow x \in Ass(\mathcal{F})$$

holds for all pairs $(\mathfrak{p}, x)$ as above.

Proof. This follows from Algebra, Lemma 63.15. But we can also argue directly as follows. Suppose $\mathfrak{p}$ is associated to M. Then there exists an $m \in M$ whose annihilator is $\mathfrak{p}$. Since localization is exact we see that $\mathfrak{p}A_{\mathfrak{p}}$ is the annihilator of $m/1 \in M_{\mathfrak{p}}$. Since $M_{\mathfrak{p}} = \mathcal{F}_x$ (Schemes, Lemma 5.4) we conclude that x is associated to $\mathcal{F}$.

Conversely, assume that x is associated to $\mathcal{F}$, and $\mathfrak{p}$ is finitely generated. As x is associated to $\mathcal{F}$ there exists an element $m' \in M_{\mathfrak{p}}$ whose annihilator is $\mathfrak{p}A_{\mathfrak{p}}$. Write $m' = m/f$ for some $f \in A$, $f \notin \mathfrak{p}$. The annihilator I of m is an ideal of A such that $IA_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$. Hence $I \subset \mathfrak{p}$, and $(\mathfrak{p}/I)_{\mathfrak{p}} = 0$. Since $\mathfrak{p}$ is finitely generated, there exists a $g \in A$, $g \notin \mathfrak{p}$ such that $g(\mathfrak{p}/I) = 0$. Hence the annihilator of gm is $\mathfrak{p}$ and we win.

If X is locally Noetherian, then A is Noetherian (Properties, Lemma 5.2) and $\mathfrak{p}$ is always finitely generated. $\square$

Lemma 2.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $Ass(\mathcal{F}) \subset Supp(\mathcal{F})$.*

Proof. This is immediate from the definitions. $\square$

Lemma 2.4. *Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $Ass(\mathcal{F}_2) \subset Ass(\mathcal{F}_1) \cup Ass(\mathcal{F}_3)$ and $Ass(\mathcal{F}_1) \subset Ass(\mathcal{F}_2)$.*

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 63.3. □

Lemma 2.5. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $Ass(\mathcal{F}) \cap U$ is finite for every quasi-compact open $U \subset X$.*

Proof. This is true because the set of associated primes of a finite module over a Noetherian ring is finite, see Algebra, Lemma 63.5. To translate from schemes to algebra use that U is a finite union of affine opens, each of these opens is the spectrum of a Noetherian ring (Properties, Lemma 5.2), $\mathcal{F}$ corresponds to a finite module over this ring (Cohomology of Schemes, Lemma 9.1), and finally use Lemma 2.2. □

Lemma 2.6. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathcal{F} = 0 \Leftrightarrow Ass(\mathcal{F}) = \emptyset.$$

Proof. If $\mathcal{F} = 0$, then $\text{Ass}(\mathcal{F}) = \emptyset$ by definition. Conversely, if $\text{Ass}(\mathcal{F}) = \emptyset$, then $\mathcal{F} = 0$ by Algebra, Lemma 63.7. To translate from schemes to algebra, restrict to any affine and use Lemma 2.2. □

Example 2.7. Let k be a field. The ring $R = k[x_1, x_2, x_3, \ldots]/(x_i^2)$ is local with locally nilpotent maximal ideal $\mathfrak{m}$. There exists no element of R which has annihilator $\mathfrak{m}$. Hence $\text{Ass}(R) = \emptyset$, and $X = \text{Spec}(R)$ is an example of a scheme which has no associated points.

Lemma 2.8. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $U \subset X$ is open and $Ass(\mathcal{F}) \subset U$, then $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is injective.*

Proof. Let $s \in \Gamma(X, \mathcal{F})$ be a section which restricts to zero on U. Let $\mathcal{F}' \subset \mathcal{F}$ be the image of the map $\mathcal{O}_X \to \mathcal{F}$ defined by s. Then $\text{Supp}(\mathcal{F}') \cap U = \emptyset$. On the other hand, $\text{Ass}(\mathcal{F}') \subset \text{Ass}(\mathcal{F})$ by Lemma 2.4. Since also $\text{Ass}(\mathcal{F}') \subset \text{Supp}(\mathcal{F}')$ (Lemma 2.3) we conclude $\text{Ass}(\mathcal{F}') = \emptyset$. Hence $\mathcal{F}' = 0$ by Lemma 2.6. □

Lemma 2.9. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in Supp(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $Supp(\mathcal{F})$. Then $x \in Ass(\mathcal{F})$. In particular, any generic point of an irreducible component of X is an associated point of X.*

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\text{Ass}(\mathcal{F}_x) \subset \text{Spec}(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 63.7. On the other hand, by assumption $\text{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\text{Ass}(\mathcal{F}_x) \subset \text{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 63.2) we see that $\mathfrak{m}_x$ is associated to $\mathcal{F}_x$ and we win. □

The following lemma is the analogue of More on Algebra, Lemma 23.12.

Lemma 2.10. *Let X be a locally Noetherian scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume that for every $x \in X$ at least one of the following happens*

(1) $\mathcal{F}_x \to \mathcal{G}_x$ *is injective, or*
(2) $x \notin Ass(\mathcal{F})$.

Then φ *is injective.*

Proof. The assumptions imply that $\text{Ass}(\text{Ker}(\varphi)) = \emptyset$ and hence $\text{Ker}(\varphi) = 0$ by Lemma 2.6. $\square$

Lemma 2.11. *Let X be a locally Noetherian scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is coherent and that for every $x \in X$ one of the following happens*

(1) $\mathcal{F}_x \to \mathcal{G}_x$ *is an isomorphism, or*
(2) $depth(\mathcal{F}_x) \geq 2$ *and* $x \notin Ass(\mathcal{G})$.

Then φ *is an isomorphism.*

Proof. This is a translation of More on Algebra, Lemma 23.13 into the language of schemes. $\square$

3. Morphisms and associated points

Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $s \in S$ is a point, then it is often convenient to denote $\mathcal{F}_s$ the $\mathcal{O}_{X_s}$-module one gets by pulling back $\mathcal{F}$ by the morphism $i_s : X_s \to X$. Here X_s is the scheme theoretic fibre of f over s. In a formula

$$\mathcal{F}_s = i_s^*\mathcal{F}$$

Of course, this notation clashes with the already existing notation for the stalk of $\mathcal{F}$ at a point $x \in X$ if $f = \text{id}_X$. However, the notation is often convenient, as in the formulation of the following lemma.

Lemma 3.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. Then we have*

$$Ass_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \supset \bigcup\nolimits_{s \in Ass_S(\mathcal{G})} Ass_{X_s}(\mathcal{F}_s)$$

and equality holds if S is locally Noetherian (for the notation $\mathcal{F}_s$ see above).

Proof. Let $x \in X$ and let $s = f(x) \in S$. Set $B = \mathcal{O}_{X,x}$, $A = \mathcal{O}_{S,s}$, $N = \mathcal{F}_x$, and $M = \mathcal{G}_s$. Note that the stalk of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ at x is equal to the B-module $M \otimes_A N$. Hence $x \in \text{Ass}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if $\mathfrak{m}_B$ is in $\text{Ass}_B(M \otimes_A N)$. Similarly $s \in \text{Ass}_S(\mathcal{G})$ and $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $\mathfrak{m}_A \in \text{Ass}_A(M)$ and $\mathfrak{m}_B/\mathfrak{m}_A B \in \text{Ass}_{B \otimes \kappa(\mathfrak{m}_A)}(N \otimes \kappa(\mathfrak{m}_A))$. Thus the lemma follows from Algebra, Lemma 65.5. $\square$

4. Embedded points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is an *embedded associated prime* of M if it is an associated prime of M which is not minimal among the associated primes of M. See Algebra, Definition 67.1. Here is the definition of embedded associated points for quasi-coherent sheaves on schemes as given in [DG67, IV Definition 3.1.1].

Definition 4.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) An *embedded associated point* of $\mathcal{F}$ is an associated point which is not maximal among the associated points of $\mathcal{F}$, i.e., it is the specialization of another associated point of $\mathcal{F}$.

(2) A point x of X is called an *embedded point* if x is an embedded associated point of $\mathcal{O}_X$.
(3) An *embedded component* of X is an irreducible closed subset $Z = \overline{\{x\}}$ where x is an embedded point of X.

In the Noetherian case when $\mathcal{F}$ is coherent we have the following.

Lemma 4.2. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then*

(1) *the generic points of irreducible components of Supp($\mathcal{F}$) are associated points of $\mathcal{F}$, and*
(2) *an associated point of $\mathcal{F}$ is embedded if and only if it is not a generic point of an irreducible component of Supp($\mathcal{F}$).*

In particular an embedded point of X is an associated point of X which is not a generic point of an irreducible component of X.

Proof. Recall that in this case $Z = \text{Supp}(\mathcal{F})$ is closed, see Morphisms, Lemma 5.3 and that the generic points of irreducible components of Z are associated points of $\mathcal{F}$, see Lemma 2.9. Finally, we have $\text{Ass}(\mathcal{F}) \subset Z$, by Lemma 2.3. These results, combined with the fact that Z is a sober topological space and hence every point of Z is a specialization of a generic point of Z, imply (1) and (2). $\square$

Lemma 4.3. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Then the following are equivalent:*

(1) *$\mathcal{F}$ has no embedded associated points, and*
(2) *$\mathcal{F}$ has property (S_1).*

Proof. This is Algebra, Lemma 157.2, combined with Lemma 2.2 above. $\square$

Lemma 4.4. *Let X be a locally Noetherian scheme of dimension ≤ 1. The following are equivalent*

(1) *X is Cohen-Macaulay, and*
(2) *X has no embedded points.*

Proof. Follows from Lemma 4.3 and the definitions. $\square$

Lemma 4.5. *Let X be a locally Noetherian scheme. Let $U \subset X$ be an open subscheme. The following are equivalent*

(1) *U is scheme theoretically dense in X (Morphisms, Definition 7.1),*
(2) *U is dense in X and U contains all embedded points of X.*

Proof. The question is local on X, hence we may assume that $X = \text{Spec}(A)$ where A is a Noetherian ring. Then U is quasi-compact (Properties, Lemma 5.3) hence $U = D(f_1) \cup \ldots \cup D(f_n)$ (Algebra, Lemma 29.1). In this situation U is scheme theoretically dense in X if and only if $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective, see Morphisms, Example 7.4. Condition (2) translated into algebra means that for every associated prime $\mathfrak{p}$ of A there exists an i with $f_i \not\in \mathfrak{p}$.

Assume (1), i.e., $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective. If $x \in A$ has annihilator a prime $\mathfrak{p}$, then x maps to a nonzero element of A_{f_i} for some i and hence $f_i \not\in \mathfrak{p}$. Thus (2) holds. Assume (2), i.e., every associated prime $\mathfrak{p}$ of A corresponds to a prime of A_{f_i} for some i. Then $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective because $A \to \prod_{\mathfrak{p} \in \text{Ass}(A)} A_{\mathfrak{p}}$ is injective by Algebra, Lemma 63.19. $\square$

Lemma 4.6. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. The set of coherent subsheaves*

$$\{\mathcal{K} \subset \mathcal{F} \mid Supp(\mathcal{K}) \text{ is nowhere dense in } Supp(\mathcal{F})\}$$

has a maximal element $\mathcal{K}$. Setting $\mathcal{F}' = \mathcal{F}/\mathcal{K}$ we have the following

1. *$Supp(\mathcal{F}') = Supp(\mathcal{F})$,*
2. *$\mathcal{F}'$ has no embedded associated points, and*
3. *there exists a dense open $U \subset X$ such that $U \cap Supp(\mathcal{F})$ is dense in $Supp(\mathcal{F})$ and $\mathcal{F}'|_U \cong \mathcal{F}|_U$.*

Proof. This follows from Algebra, Lemmas 67.2 and 67.3. Note that U can be taken as the complement of the closure of the set of embedded associated points of $\mathcal{F}$. □

Lemma 4.7. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module without embedded associated points. Set*

$$\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})).$$

This is a coherent sheaf of ideals which defines a closed subscheme $Z \subset X$ without embedded points. Moreover there exists a coherent sheaf $\mathcal{G}$ on Z such that (a) $\mathcal{F} = (Z \to X)_*\mathcal{G}$, (b) $\mathcal{G}$ has no associated embedded points, and (c) $Supp(\mathcal{G}) = Z$ (as sets).

Proof. Some of the statements we have seen in the proof of Cohomology of Schemes, Lemma 9.7. The others follow from Algebra, Lemma 67.4. □

5. Weakly associated points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is *weakly associated* to M if there exists an element m of M such that $\mathfrak{p}$ is minimal among the primes containing the annihilator of m. See Algebra, Definition 66.1. If R is a local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is weakly associated to M if and only if there exists an element $m \in M$ whose annihilator has radical $\mathfrak{m}$, see Algebra, Lemma 66.2.

Definition 5.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

1. We say $x \in X$ is *weakly associated* to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is weakly associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
2. We denote $\mathrm{WeakAss}(\mathcal{F})$ the set of weakly associated points of $\mathcal{F}$.
3. The *weakly associated points of* X are the weakly associated points of $\mathcal{O}_X$.

In this case, on any affine open, this corresponds exactly to the weakly associated primes as defined above. Here is the precise statement.

Lemma 5.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent*

1. *$\mathfrak{p}$ is weakly associated to M, and*
2. *x is weakly associated to $\mathcal{F}$.*

Proof. This follows from Algebra, Lemma 66.2. □

Lemma 5.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$Ass(\mathcal{F}) \subset WeakAss(\mathcal{F}) \subset Supp(\mathcal{F}).$$

Proof. This is immediate from the definitions. □

Lemma 5.4. *Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $WeakAss(\mathcal{F}_2) \subset WeakAss(\mathcal{F}_1) \cup WeakAss(\mathcal{F}_3)$ and $WeakAss(\mathcal{F}_1) \subset WeakAss(\mathcal{F}_2)$.*

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 66.4. □

Lemma 5.5. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathcal{F} = (0) \Leftrightarrow WeakAss(\mathcal{F}) = \emptyset$$

Proof. Follows from Lemma 5.2 and Algebra, Lemma 66.5 □

Lemma 5.6. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $U \subset X$ is open and $WeakAss(\mathcal{F}) \subset U$, then $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is injective.*

Proof. Let $s \in \Gamma(X, \mathcal{F})$ be a section which restricts to zero on U. Let $\mathcal{F}' \subset \mathcal{F}$ be the image of the map $\mathcal{O}_X \to \mathcal{F}$ defined by s. Then $\text{Supp}(\mathcal{F}') \cap U = \emptyset$. On the other hand, $\text{WeakAss}(\mathcal{F}') \subset \text{WeakAss}(\mathcal{F})$ by Lemma 5.4. Since also $\text{WeakAss}(\mathcal{F}') \subset \text{Supp}(\mathcal{F}')$ (Lemma 5.3) we conclude $\text{WeakAss}(\mathcal{F}') = \emptyset$. Hence $\mathcal{F}' = 0$ by Lemma 5.5. □

Lemma 5.7. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in Supp(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $Supp(\mathcal{F})$. Then $x \in WeakAss(\mathcal{F})$. In particular, any generic point of an irreducible component of X is weakly associated to $\mathcal{O}_X$.*

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\text{WeakAss}(\mathcal{F}_x) \subset \text{Spec}(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 66.5. On the other hand, by assumption $\text{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\text{WeakAss}(\mathcal{F}_x) \subset \text{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 66.6) we see that $\mathfrak{m}_x$ is weakly associated to $\mathcal{F}_x$ and we win. □

Lemma 5.8. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathfrak{m}_x$ is a finitely generated ideal of $\mathcal{O}_{X,x}$, then*

$$x \in Ass(\mathcal{F}) \Leftrightarrow x \in WeakAss(\mathcal{F}).$$

In particular, if X is locally Noetherian, then $Ass(\mathcal{F}) = WeakAss(\mathcal{F})$.

Proof. See Algebra, Lemma 66.9. □

Lemma 5.9. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$ be a point which is not in the image of f. Then s is not weakly associated to $f_*\mathcal{F}$.*

Proof. Consider the base change $f' : X' \to \text{Spec}(\mathcal{O}_{S,s})$ of f by the morphism $g : \text{Spec}(\mathcal{O}_{S,s}) \to S$ and denote $g' : X' \to X$ the other projection. Then

$$(f_*\mathcal{F})_s = (g^*f_*\mathcal{F})_s = (f'_*(g')^*\mathcal{F})_s$$

The first equality because g induces an isomorphism on local rings at s and the second by flat base change (Cohomology of Schemes, Lemma 5.2). Of course $s \in$

$\mathrm{Spec}(\mathcal{O}_{S,s})$ is not in the image of f'. Thus we may assume S is the spectrum of a local ring $(A, \mathfrak{m})$ and s corresponds to $\mathfrak{m}$. By Schemes, Lemma 24.1 the sheaf $f_*\mathcal{F}$ is quasi-coherent, say corresponding to the A-module M. As s is not in the image of f we see that $X = \bigcup_{a \in \mathfrak{m}} f^{-1}D(a)$ is an open covering. Since X is quasi-compact we can find $a_1, \ldots, a_n \in \mathfrak{m}$ such that $X = f^{-1}D(a_1) \cup \ldots \cup f^{-1}D(a_n)$. It follows that

$$M \to M_{a_1} \oplus \ldots \oplus M_{a_r}$$

is injective. Hence for any nonzero element m of the stalk $M_\mathfrak{p}$ there exists an i such that $a_i^n m$ is nonzero for all $n \geq 0$. Thus $\mathfrak{m}$ is not weakly associated to M. □

Lemma 5.10. *Let X be a scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume that for every $x \in X$ at least one of the following happens*

1. *$\mathcal{F}_x \to \mathcal{G}_x$ is injective, or*
2. *$x \notin \mathit{WeakAss}(\mathcal{F})$.*

Then φ is injective.

Proof. The assumptions imply that $\mathrm{WeakAss}(\mathrm{Ker}(\varphi)) = \emptyset$ and hence $\mathrm{Ker}(\varphi) = 0$ by Lemma 5.5. □

Lemma 5.11. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $j : U \to X$ be an open subscheme such that for $x \in X \setminus U$ we have $\mathit{depth}(\mathcal{F}_x) \geq 2$. Then*

$$\mathcal{F} \longrightarrow j_*(\mathcal{F}|_U)$$

is an isomorphism and consequently $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is an isomorphism too.

Proof. We claim Lemma 2.11 applies to the map displayed in the lemma. Let $x \in X$. If $x \in U$, then the map is an isomorphism on stalks as $j_*(\mathcal{F}|_U)|_U = \mathcal{F}|_U$. If $x \in X \setminus U$, then $x \notin \mathrm{Ass}(j_*(\mathcal{F}|_U))$ (Lemmas 5.9 and 5.3). Since we've assumed $\mathrm{depth}(\mathcal{F}_x) \geq 2$ this finishes the proof. □

Lemma 5.12. *Let X be a reduced scheme. Then the weakly associated points of X are exactly the generic points of the irreducible components of X.*

Proof. Follows from Algebra, Lemma 66.3. □

6. Morphisms and weakly associated points

Lemma 6.1. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then we have*

$$\mathit{WeakAss}_S(f_*\mathcal{F}) \subset f(\mathit{WeakAss}_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. Then $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 66.11. □

Lemma 6.2. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is locally Noetherian, then we have*

$$f(\mathit{Ass}_X(\mathcal{F})) = \mathit{Ass}_S(f_*\mathcal{F}) = \mathit{WeakAss}_S(f_*\mathcal{F}) = f(\mathit{WeakAss}_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. As X is locally Noetherian the ring B is Noetherian, see Properties, Lemma 5.2. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 2.2 the associated points of $\mathcal{F}$ correspond exactly to the associated primes of M, and any associated prime of M as an A-module is an associated points of $f_*\mathcal{F}$. Hence the inclusion

$$f(\text{Ass}_X(\mathcal{F})) \subset \text{Ass}_S(f_*\mathcal{F})$$

follows from Algebra, Lemma 63.13. We have the inclusion

$$\text{Ass}_S(f_*\mathcal{F}) \subset \text{WeakAss}_S(f_*\mathcal{F})$$

by Lemma 5.3. We have the inclusion

$$\text{WeakAss}_S(f_*\mathcal{F}) \subset f(\text{WeakAss}_X(\mathcal{F}))$$

by Lemma 6.1. The outer sets are equal by Lemma 5.8 hence we have equality everywhere. $\square$

Lemma 6.3. *Let $f : X \to S$ be a finite morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\text{WeakAss}(f_*\mathcal{F}) = f(\text{WeakAss}(\mathcal{F}))$.*

Proof. We may assume X and S affine, so $X \to S$ comes from a finite ring map $A \to B$. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 66.13. $\square$

Lemma 6.4. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $x \in X$ with $s = f(x)$. If f is flat at x, the point x is a generic point of the fibre X_s, and $s \in \text{WeakAss}_S(\mathcal{G})$, then $x \in \text{WeakAss}(f^*\mathcal{G})$.*

Proof. Let $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$, and $M = \mathcal{G}_s$. Let $m \in M$ be an element whose annihilator $I = \{a \in A \mid am = 0\}$ has radical $\mathfrak{m}_A$. Then $m \otimes 1$ has annihilator IB as $A \to B$ is faithfully flat. Thus it suffices to see that $\sqrt{IB} = \mathfrak{m}_B$. This follows from the fact that the maximal ideal of $B/\mathfrak{m}_A B$ is locally nilpotent (see Algebra, Lemma 25.1) and the assumption that $\sqrt{I} = \mathfrak{m}_A$. Some details omitted. $\square$

Lemma 6.5. *Let K/k be a field extension. Let X be a scheme over k. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $y \in X_K$ with image $x \in X$. If y is a weakly associated point of the pullback $\mathcal{F}_K$, then x is a weakly associated point of $\mathcal{F}$.*

Proof. This is the translation of Algebra, Lemma 66.19 into the language of schemes. $\square$

Here is a simple lemma where we find that pushforwards often have depth at least 2.

Lemma 6.6. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$.*

1. *If $s \notin f(X)$, then s is not weakly associated to $f_*\mathcal{F}$.*
2. *If $s \notin f(X)$ and $\mathcal{O}_{S,s}$ is Noetherian, then s is not associated to $f_*\mathcal{F}$.*
3. *If $s \notin f(X)$, $(f_*\mathcal{F})_s$ is a finite $\mathcal{O}_{S,s}$-module, and $\mathcal{O}_{S,s}$ is Noetherian, then $\text{depth}((f_*\mathcal{F})_s) \geq 2$.*
4. *If $\mathcal{F}$ is flat over S and $a \in \mathfrak{m}_s$ is a nonzerodivisor, then a is a nonzerodivisor on $(f_*\mathcal{F})_s$.*

(5) *If $\mathcal{F}$ is flat over S and $a, b \in \mathfrak{m}_s$ is a regular sequence, then a is a nonzerodivisor on $(f_*\mathcal{F})_s$ and b is a nonzerodivisor on $(f_*\mathcal{F})_s/a(f_*\mathcal{F})_s$.*
(6) *If $\mathcal{F}$ is flat over S and $(f_*\mathcal{F})_s$ is a finite $\mathcal{O}_{S,s}$-module, then depth$((f_*\mathcal{F})_s) \geq$ $\min(2, \text{depth}(\mathcal{O}_{S,s}))$.*

Proof. Part (1) is Lemma 5.9. Part (2) follows from (1) and Lemma 5.8.

Proof of part (3). To show the depth is ≥ 2 it suffices to show that $\text{Hom}_{\mathcal{O}_{S,s}}(\kappa(s), (f_*\mathcal{F})_s) = 0$ and $\text{Ext}^1_{\mathcal{O}_{S,s}}(\kappa(s), (f_*\mathcal{F})_s) = 0$, see Algebra, Lemma 72.5. Using the exact sequence $0 \to \mathfrak{m}_s \to \mathcal{O}_{S,s} \to \kappa(s) \to 0$ it suffices to prove that the map

$$\text{Hom}_{\mathcal{O}_{S,s}}(\mathcal{O}_{S,s}, (f_*\mathcal{F})_s) \to \text{Hom}_{\mathcal{O}_{S,s}}(\mathfrak{m}_s, (f_*\mathcal{F})_s)$$

is an isomorphism. By flat base change (Cohomology of Schemes, Lemma 5.2) we may replace S by $\text{Spec}(\mathcal{O}_{S,s})$ and X by $\text{Spec}(\mathcal{O}_{S,s}) \times_S X$. Denote $\mathfrak{m} \subset \mathcal{O}_S$ the ideal sheaf of s. Then we see that

$$\text{Hom}_{\mathcal{O}_{S,s}}(\mathfrak{m}_s, (f_*\mathcal{F})_s) = \text{Hom}_{\mathcal{O}_S}(\mathfrak{m}, f_*\mathcal{F}) = \text{Hom}_{\mathcal{O}_X}(f^*\mathfrak{m}, \mathcal{F})$$

the first equality because S is local with closed point s and the second equality by adjunction for f^*, f_* on quasi-coherent modules. However, since $s \notin f(X)$ we see that $f^*\mathfrak{m} = \mathcal{O}_X$. Working backwards through the arguments we get the desired equality.

For the proof of (4), (5), and (6) we use flat base change (Cohomology of Schemes, Lemma 5.2) to reduce to the case where S is the spectrum of $\mathcal{O}_{S,s}$. Then a nonzerodivisor $a \in \mathcal{O}_{S,s}$ deterimines a short exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{a} \mathcal{O}_S \to \mathcal{O}_S/a\mathcal{O}_S \to 0$$

Since $\mathcal{F}$ is flat over S, we obtain an exact sequence

$$0 \to \mathcal{F} \xrightarrow{a} \mathcal{F} \to \mathcal{F}/a\mathcal{F} \to 0$$

Pushing forward we obtain an exact sequence

$$0 \to f_*\mathcal{F} \xrightarrow{a} f_*\mathcal{F} \to f_*(\mathcal{F}/a\mathcal{F})$$

This proves (4) and it shows that $f_*\mathcal{F}/af_*\mathcal{F} \subset f_*(\mathcal{F}/a\mathcal{F})$. If b is a nonzerodivisor on $\mathcal{O}_{S,s}/a\mathcal{O}_{S,s}$, then the exact same argument shows $b : \mathcal{F}/a\mathcal{F} \to \mathcal{F}/a\mathcal{F}$ is injective. Pushing forward we conclude

$$b : f_*(\mathcal{F}/a\mathcal{F}) \to f_*(\mathcal{F}/a\mathcal{F})$$

is injective and hence also $b : f_*\mathcal{F}/af_*\mathcal{F} \to f_*\mathcal{F}/af_*\mathcal{F}$ is injective. This proves (5). Part (6) follows from (4) and (5) and the definitions. □

7. Relative assassin

Let $A \to B$ be a ring map. Let N be a B-module. Recall that a prime $\mathfrak{q} \subset B$ is said to be in the relative assassin of N over B/A if $\mathfrak{q}$ is an associated prime of $N \otimes_A \kappa(\mathfrak{p})$. Here $\mathfrak{p} = A \cap \mathfrak{q}$. See Algebra, Definition 65.2. Here is the definition of the relative assassin for quasi-coherent sheaves over a morphism of schemes.

Definition 7.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The *relative assassin of $\mathcal{F}$ in X over S* is the set

$$\text{Ass}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \text{Ass}_{X_s}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

Again there is a caveat that this is best used when the fibres of f are locally Noetherian and $\mathcal{F}$ is of finite type. In the general case we should probably use the relative weak assassin (defined in the next section). Let us link the scheme theoretic notion with the algebraic notion on affine opens; note that this correspondence works perfectly only for morphisms of schemes whose fibres are locally Noetherian.

Lemma 7.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $U \subset X$ and $V \subset S$ be affine opens with $f(U) \subset V$. Write $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(R)$, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. Then*

$$\mathfrak{p} \in Ass_{A/R}(M) \Rightarrow x \in Ass_{X/S}(\mathcal{F})$$

If all fibres X_s of f are locally Noetherian, then $\mathfrak{p} \in Ass_{A/R}(M) \Leftrightarrow x \in Ass_{X/S}(\mathcal{F})$ for all pairs $(\mathfrak{p}, x)$ as above.

Proof. The set $\mathrm{Ass}_{A/R}(M)$ is defined in Algebra, Definition 65.2. Choose a pair $(\mathfrak{p}, x)$. Let $s = f(x)$. Let $\mathfrak{r} \subset R$ be the prime lying under $\mathfrak{p}$, i.e., the prime corresponding to s. Let $\mathfrak{p}' \subset A \otimes_R \kappa(\mathfrak{r})$ be the prime whose inverse image is $\mathfrak{p}$, i.e., the prime corresponding to x viewed as a point of its fibre X_s. Then $\mathfrak{p} \in \mathrm{Ass}_{A/R}(M)$ if and only if $\mathfrak{p}'$ is an associated prime of $M \otimes_R \kappa(\mathfrak{r})$, see Algebra, Lemma 65.1. Note that the ring $A \otimes_R \kappa(\mathfrak{r})$ corresponds to U_s and the module $M \otimes_R \kappa(\mathfrak{r})$ corresponds to the quasi-coherent sheaf $\mathcal{F}_s|_{U_s}$. Hence x is an associated point of $\mathcal{F}_s$ by Lemma 2.2. The reverse implication holds if $\mathfrak{p}'$ is finitely generated which is how the last sentence is seen to be true. $\square$

Lemma 7.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $g : S' \to S$ be a morphism of schemes. Consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

and set $\mathcal{F}' = (g')^*\mathcal{F}$. Let $x' \in X'$ be a point with images $x \in X$, $s' \in S'$ and $s \in S$. Assume f locally of finite type. Then $x' \in Ass_{X'/S'}(\mathcal{F}')$ if and only if $x \in Ass_{X/S}(\mathcal{F})$ and x' corresponds to a generic point of an irreducible component of $\mathrm{Spec}(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$.

Proof. Consider the morphism $X'_{s'} \to X_s$ of fibres. As $X_{s'} = X_s \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(s'))$ this is a flat morphism. Moreover $\mathcal{F}'_{s'}$ is the pullback of $\mathcal{F}_s$ via this morphism. As X_s is locally of finite type over the Noetherian scheme $\mathrm{Spec}(\kappa(s))$ we have that X_s is locally Noetherian, see Morphisms, Lemma 15.6. Thus we may apply Lemma 3.1 and we see that

$$\mathrm{Ass}_{X'_{s'}}(\mathcal{F}'_{s'}) = \bigcup\nolimits_{x \in \mathrm{Ass}(\mathcal{F}_s)} \mathrm{Ass}((X'_{s'})_x).$$

Thus to prove the lemma it suffices to show that the associated points of the fibre $(X'_{s'})_x$ of the morphism $X'_{s'} \to X_s$ over x are its generic points. Note that $(X'_{s'})_x = \mathrm{Spec}(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$ as schemes. By Algebra, Lemma 167.1 the ring $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ is a Noetherian Cohen-Macaulay ring. Hence its associated primes are its minimal primes, see Algebra, Proposition 63.6 (minimal primes are associated) and Algebra, Lemma 157.2 (no embedded primes). $\square$

Remark 7.4. With notation and assumptions as in Lemma 7.3 we see that it is always the case that $(g')^{-1}(\text{Ass}_{X/S}(\mathcal{F})) \supset \text{Ass}_{X'/S'}(\mathcal{F}')$. If the morphism $S' \to S$ is locally quasi-finite, then we actually have

$$(g')^{-1}(\text{Ass}_{X/S}(\mathcal{F})) = \text{Ass}_{X'/S'}(\mathcal{F}')$$

because in this case the field extensions $\kappa(s')/\kappa(s)$ are always finite. In fact, this holds more generally for any morphism $g : S' \to S$ such that all the field extensions $\kappa(s')/\kappa(s)$ are algebraic, because in this case all prime ideals of $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ are maximal (and minimal) primes, see Algebra, Lemma 36.19.

8. Relative weak assassin

Definition 8.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The *relative weak assassin of $\mathcal{F}$ in X over S* is the set

$$\text{WeakAss}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \text{WeakAss}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

Lemma 8.2. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\text{WeakAss}_{X/S}(\mathcal{F}) = \text{Ass}_{X/S}(\mathcal{F})$.*

Proof. This is true because the fibres of f are locally Noetherian schemes, and associated and weakly associated points agree on locally Noetherian schemes, see Lemma 5.8. □

Lemma 8.3. *Let $f : X \to S$ be a morphism of schemes. Let $i : Z \to X$ be a finite morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Z$-module. Then $\text{WeakAss}_{X/S}(i_*\mathcal{F}) = i(\text{WeakAss}_{Z/S}(\mathcal{F}))$.*

Proof. Let $i_s : Z_s \to X_s$ be the induced morphism between fibres. Then $(i_*\mathcal{F})_s = i_{s,*}(\mathcal{F}_s)$ by Cohomology of Schemes, Lemma 5.1 and the fact that i is affine. Hence we may apply Lemma 6.3 to conclude. □

9. Fitting ideals

This section is the continuation of the discussion in More on Algebra, Section 8. Let S be a scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. In this situation we can construct the Fitting ideals

$$0 = \text{Fit}_{-1}(\mathcal{F}) \subset \text{Fit}_0(\mathcal{F}) \subset \text{Fit}_1(\mathcal{F}) \subset \ldots \subset \mathcal{O}_S$$

as the sequence of quasi-coherent ideals characterized by the following property: for every affine open $U = \text{Spec}(A)$ of S if $\mathcal{F}|_U$ corresponds to the A-module M, then $\text{Fit}_i(\mathcal{F})|_U$ corresponds to the ideal $\text{Fit}_i(M) \subset A$. This is well defined and a quasi-coherent sheaf of ideals because if $f \in A$, then the ith Fitting ideal of M_f over A_f is equal to $\text{Fit}_i(M)A_f$ by More on Algebra, Lemma 8.4.

Alternatively, we can construct the Fitting ideals in terms of local presentations of $\mathcal{F}$. Namely, if $U \subset X$ is open, and

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

is a presentation of $\mathcal{F}$ over U, then $\text{Fit}_r(\mathcal{F})|_U$ is generated by the $(n-r) \times (n-r)$-minors of the matrix defining the first arrow of the presentation. This is compatible with the construction above because this is how the Fitting ideal of a module over a ring is actually defined. Some details omitted.

Lemma 9.1. *Let $f : T \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. Then $f^{-1}Fit_i(\mathcal{F}) \cdot \mathcal{O}_T = Fit_i(f^*\mathcal{F})$.*

Proof. Follows immediately from More on Algebra, Lemma 8.4 part (3). □

Lemma 9.2. *Let S be a scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_S$-module. Then $Fit_r(\mathcal{F})$ is a quasi-coherent ideal of finite type.*

Proof. Follows immediately from More on Algebra, Lemma 8.4 part (4). □

Lemma 9.3. *Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. Let $Z_0 \subset S$ be the closed subscheme cut out by $Fit_0(\mathcal{F})$. Let $Z \subset S$ be the scheme theoretic support of $\mathcal{F}$. Then*

1. *$Z \subset Z_0 \subset S$ as closed subschemes,*
2. *$Z = Z_0 = Supp(\mathcal{F})$ as closed subsets,*
3. *there exists a finite type, quasi-coherent $\mathcal{O}_{Z_0}$-module $\mathcal{G}_0$ with*
$$(Z_0 \to X)_*\mathcal{G}_0 = \mathcal{F}.$$

Proof. Recall that Z is locally cut out by the annihilator of $\mathcal{F}$, see Morphisms, Definition 5.5 (which uses Morphisms, Lemma 5.4 to define Z). Hence we see that $Z \subset Z_0$ scheme theoretically by More on Algebra, Lemma 8.4 part (6). On the other hand we have $Z = \text{Supp}(\mathcal{F})$ set theoretically by Morphisms, Lemma 5.4 and we have $Z_0 = Z$ set theoretically by More on Algebra, Lemma 8.4 part (7). Finally, to get $\mathcal{G}_0$ as in part (3) we can either use that we have $\mathcal{G}$ on Z as in Morphisms, Lemma 5.4 and set $\mathcal{G}_0 = (Z \to Z_0)_*\mathcal{G}$ or we can use Morphisms, Lemma 4.1 and the fact that $\text{Fit}_0(\mathcal{F})$ annihilates $\mathcal{F}$ by More on Algebra, Lemma 8.4 part (6). □

Lemma 9.4. *Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. Let $s \in S$. Then $\mathcal{F}$ can be generated by r elements in a neighbourhood of s if and only if $Fit_r(\mathcal{F})_s = \mathcal{O}_{S,s}$.*

Proof. Follows immediately from More on Algebra, Lemma 8.6. □

Lemma 9.5. *Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. Let $r \geq 0$. The following are equivalent*

1. *$\mathcal{F}$ is finite locally free of rank r*
2. *$Fit_{r-1}(\mathcal{F}) = 0$ and $Fit_r(\mathcal{F}) = \mathcal{O}_S$, and*
3. *$Fit_k(\mathcal{F}) = 0$ for $k < r$ and $Fit_k(\mathcal{F}) = \mathcal{O}_S$ for $k \geq r$.*

Proof. Follows immediately from More on Algebra, Lemma 8.7. □

Lemma 9.6. *Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. The closed subschemes*
$$S = Z_{-1} \supset Z_0 \supset Z_1 \supset Z_2 \ldots$$
defined by the Fitting ideals of $\mathcal{F}$ have the following properties

1. *The intersection $\bigcap Z_r$ is empty.*
2. *The functor $(Sch/S)^{opp} \to Sets$ defined by the rule*
$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is locally generated by } \leq r \text{ sections} \\ \emptyset & \text{otherwise} \end{cases}$$
is representable by the open subscheme $S \setminus Z_r$.

(3) *The functor* $F_r : (Sch/S)^{opp} \to Sets$ *defined by the rule*

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ locally free rank } r \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the locally closed subscheme $Z_{r-1} \setminus Z_r$ *of* S.

If $\mathcal{F}$ *is of finite presentation, then* $Z_r \to S$, $S \setminus Z_r \to S$, *and* $Z_{r-1} \setminus Z_r \to S$ *are of finite presentation.*

Proof. Part (1) is true because over every affine open U there is an integer n such that $\text{Fit}_n(\mathcal{F})|_U = \mathcal{O}_U$. Namely, we can take n to be the number of generators of $\mathcal{F}$ over U, see More on Algebra, Section 8.

For any morphism $g : T \to S$ we see from Lemmas 9.1 and 9.4 that $\mathcal{F}_T$ is locally generated by $\leq r$ sections if and only if $\text{Fit}_r(\mathcal{F}) \cdot \mathcal{O}_T = \mathcal{O}_T$. This proves (2).

For any morphism $g : T \to S$ we see from Lemmas 9.1 and 9.5 that $\mathcal{F}_T$ is free of rank r if and only if $\text{Fit}_r(\mathcal{F}) \cdot \mathcal{O}_T = \mathcal{O}_T$ and $\text{Fit}_{r-1}(\mathcal{F}) \cdot \mathcal{O}_T = 0$. This proves (3).

Assume $\mathcal{F}$ is of finite presentation. Then each of the morphisms $Z_r \to S$ is of finite presentation as $\text{Fit}_r(\mathcal{F})$ is of finite type (Lemma 9.2 and Morphisms, Lemma 21.7). This implies that $Z_{r-1} \setminus Z_r$ is a retrocompact open in Z_r (Properties, Lemma 24.1) and hence the morphism $Z_{r-1} \setminus Z_r \to Z_r$ is of finite presentation as well. □

Lemma 9.6 notwithstanding the following lemma does not hold if $\mathcal{F}$ is a finite type quasi-coherent module. Namely, the stratification still exists but it isn't true that it represents the functor F_{flat} in general.

Lemma 9.7. *Let* S *be a scheme. Let* $\mathcal{F}$ *be an* $\mathcal{O}_S$*-module of finite presentation. Let* $S = Z_{-1} \supset Z_0 \supset Z_1 \supset \ldots$ *be as in Lemma 9.6. Set* $S_r = Z_{r-1} \setminus Z_r$. *Then* $S' = \coprod_{r \geq 0} S_r$ *represents the functor*

$$F_{flat} : Sch/S \longrightarrow Sets, \qquad T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ flat over } T \\ \emptyset & \text{otherwise} \end{cases}$$

Moreover, $\mathcal{F}|_{S_r}$ *is locally free of rank* r *and the morphisms* $S_r \to S$ *and* $S' \to S$ *are of finite presentation.*

Proof. Suppose that $g : T \to S$ is a morphism of schemes such that the pullback $\mathcal{F}_T = g^*\mathcal{F}$ is flat. Then $\mathcal{F}_T$ is a flat $\mathcal{O}_T$-module of finite presentation. Hence $\mathcal{F}_T$ is finite locally free, see Properties, Lemma 20.2. Thus $T = \coprod_{r \geq 0} T_r$, where $\mathcal{F}_T|_{T_r}$ is locally free of rank r. This implies that

$$F_{flat} = \coprod\nolimits_{r \geq 0} F_r$$

in the category of Zariski sheaves on Sch/S where F_r is as in Lemma 9.6. It follows that F_{flat} is represented by $\coprod_{r \geq 0}(Z_{r-1} \setminus Z_r)$ where Z_r is as in Lemma 9.6. The other statements also follow from the lemma. □

Example 9.8. Let $R = \prod_{n \in \mathbf{N}} \mathbf{F}_2$. Let $I \subset R$ be the ideal of elements $a = (a_n)_{n \in \mathbf{N}}$ almost all of whose components are zero. Let $\mathfrak{m}$ be a maximal ideal containing I. Then $M = R/\mathfrak{m}$ is a finite flat R-module, because R is absolutely flat (More on Algebra, Lemma 104.6). Set $S = \text{Spec}(R)$ and $\mathcal{F} = \widetilde{M}$. The closed subschemes of Lemma 9.6 are $S = Z_{-1}$, $Z_0 = \text{Spec}(R/\mathfrak{m})$, and $Z_i = \emptyset$ for $i > 0$. But $\text{id} : S \to S$ does not factor through $(S \setminus Z_0) \amalg Z_0$ because $\mathfrak{m}$ is a nonisolated point of S. Thus Lemma 9.7 does not hold for finite type modules.

10. The singular locus of a morphism

Let $f : X \to S$ be a finite type morphism of schemes. The set U of points where f is smooth is an open of X (by Morphisms, Definition 34.1). In many situations it is useful to have a canonical closed subscheme $\text{Sing}(f) \subset X$ whose complement is U and whose formation commutes with arbitrary change of base.

If f is of finite presentation, then one choice would be to consider the closed subscheme Z cut out by functions which are affine locally "strictly standard" in the sense of Smoothing Ring Maps, Definition 2.3. It follows from Smoothing Ring Maps, Lemma 2.7 that if $f' : X' \to S'$ is the base change of f by a morphism $S' \to S$, then $Z' \subset S' \times_S Z$ where Z' is the closed subscheme of X' cut out by functions which are affine locally strictly standard. However, equality isn't clear. The notion of a strictly standard element was useful in the chapter on Popescu's theorem. The closed subscheme defined by these elements is (as far as we know) not used in the literature[1].

If f is flat, of finite presentation, and the fibres of f all are equidimensional of dimension d, then the dth fitting ideal of $\Omega_{X/S}$ is used to get a good closed subscheme. For any morphism of finite type the closed subschemes of X defined by the fitting ideals of $\Omega_{X/S}$ define a stratification of X in terms of the rank of $\Omega_{X/S}$ whose formation commutes with base change. This can be helpful; it is related to embedding dimensions of fibres, see Varieties, Section 46.

Lemma 10.1. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $X = Z_{-1} \supset Z_0 \supset Z_1 \supset \ldots$ be the closed subschemes defined by the fitting ideals of $\Omega_{X/S}$. Then the formation of Z_i commutes with arbitrary base change.*

Proof. Observe that $\Omega_{X/S}$ is a finite type quasi-coherent $\mathcal{O}_X$-module (Morphisms, Lemma 32.12) hence the fitting ideals are defined. If $f' : X' \to S'$ is the base change of f by $g : S' \to S$, then $\Omega_{X'/S'} = (g')^*\Omega_{X/S}$ where $g' : X' \to X$ is the projection (Morphisms, Lemma 32.10). Hence $(g')^{-1}\text{Fit}_i(\Omega_{X/S}) \cdot \mathcal{O}_{X'} = \text{Fit}_i(\Omega_{X'/S'})$. This means that

$$Z_i' = (g')^{-1}(Z_i) = Z_i \times_X X'$$

scheme theoretically and this is the meaning of the statement of the lemma. □

The 0th fitting ideal of Ω cuts out the "ramified locus" of the morphism.

Lemma 10.2. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. The closed subscheme $Z \subset X$ cut out by the 0th fitting ideal of $\Omega_{X/S}$ is exactly the set of points where f is not unramified.*

Proof. By Lemma 9.3 the complement of Z is exactly the locus where $\Omega_{X/S}$ is zero. This is exactly the set of points where f is unramified by Morphisms, Lemma 35.2. □

Lemma 10.3. *Let $f : X \to S$ be a morphism of schemes. Let $d \geq 0$ be an integer. Assume*

1. *f is flat,*
2. *f is locally of finite presentation, and*
3. *every nonempty fibre of f is equidimensional of dimension d.*

[1]If f is a local complete intersection morphism (More on Morphisms, Definition 62.2) then the closed subscheme cut out by the locally strictly standard elements is the correct thing to look at.

Let $Z \subset X$ be the closed subscheme cut out by the dth fitting ideal of $\Omega_{X/S}$. Then Z is exactly the set of points where f is not smooth.

Proof. By Lemma 9.6 the complement of Z is exactly the locus where $\Omega_{X/S}$ can be generated by at most d elements. Hence the lemma follows from Morphisms, Lemma 34.14. $\square$

11. Torsion free modules

This section is the analogue of More on Algebra, Section 22 for quasi-coherent modules.

Lemma 11.1. *Let X be an integral scheme with generic point η. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be nonempty open and $s \in \mathcal{F}(U)$. The following are equivalent*

1. *for some $x \in U$ the image of s in $\mathcal{F}_x$ is torsion,*
2. *for all $x \in U$ the image of s in $\mathcal{F}_x$ is torsion,*
3. *the image of s in $\mathcal{F}_\eta$ is zero,*
4. *the image of s in $j_*\mathcal{F}_\eta$ is zero, where $j : \eta \to X$ is the inclusion morphism.*

Proof. Omitted. $\square$

Definition 11.2. Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

1. We say a local section of $\mathcal{F}$ is *torsion* if it satisfies the equivalent conditions of Lemma 11.1.
2. We say $\mathcal{F}$ is *torsion free* if every torsion section of $\mathcal{F}$ is 0.

Here is the obligatory lemma comparing this to the usual algebraic notion.

Lemma 11.3. *Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is torsion free,*
2. *for $U \subset X$ affine open $\mathcal{F}(U)$ is a torsion free $\mathcal{O}(U)$-module.*

Proof. Omitted. $\square$

Lemma 11.4. *Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The torsion sections of $\mathcal{F}$ form a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{F}_{tors} \subset \mathcal{F}$. The quotient module $\mathcal{F}/\mathcal{F}_{tors}$ is torsion free.*

Proof. Omitted. See More on Algebra, Lemma 22.2 for the algebraic analogue. $\square$

Lemma 11.5. *Let X be an integral scheme. Any flat quasi-coherent $\mathcal{O}_X$-module is torsion free.*

Proof. Omitted. See More on Algebra, Lemma 22.9. $\square$

Lemma 11.6. *Let $f : X \to Y$ be a flat morphism of integral schemes. Let $\mathcal{G}$ be a torsion free quasi-coherent $\mathcal{O}_Y$-module. Then $f^*\mathcal{G}$ is a torsion free $\mathcal{O}_X$-module.*

Proof. Omitted. See More on Algebra, Lemma 23.7 for the algebraic analogue. $\square$

Lemma 11.7. *Let $f : X \to Y$ be a flat morphism of schemes. If Y is integral and the generic fibre of f is integral, then X is integral.*

Proof. The algebraic analogue is this: let A be a domain with fraction field K and let B be a flat A-algebra such that $B \otimes_A K$ is a domain. Then B is a domain. This is true because B is torsion free by More on Algebra, Lemma 22.9 and hence $B \subset B \otimes_A K$. □

Lemma 11.8. *Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is torsion free if and only if $\mathcal{F}_x$ is a torsion free $\mathcal{O}_{X,x}$-module for all $x \in X$.*

Proof. Omitted. See More on Algebra, Lemma 22.6. □

Lemma 11.9. *Let X be an integral scheme. Let $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{F}''$ are torsion free, then $\mathcal{F}'$ is torsion free.*

Proof. Omitted. See More on Algebra, Lemma 22.5 for the algebraic analogue. □

Lemma 11.10. *Let X be a locally Noetherian integral scheme with generic point η. Let $\mathcal{F}$ be a nonzero coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is torsion free,*
(2) *η is the only associated prime of $\mathcal{F}$,*
(3) *η is in the support of $\mathcal{F}$ and $\mathcal{F}$ has property (S_1), and*
(4) *η is in the support of $\mathcal{F}$ and $\mathcal{F}$ has no embedded associated prime.*

Proof. This is a translation of More on Algebra, Lemma 22.8 into the language of schemes. We omit the translation. □

Lemma 11.11. *Let X be an integral regular scheme of dimension ≤ 1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is torsion free,*
(2) *$\mathcal{F}$ is finite locally free.*

Proof. It is clear that a finite locally free module is torsion free. For the converse, we will show that if $\mathcal{F}$ is torsion free, then $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all $x \in X$. This is enough by Algebra, Lemma 78.2 and the fact that $\mathcal{F}$ is coherent. If $\dim(\mathcal{O}_{X,x}) = 0$, then $\mathcal{O}_{X,x}$ is a field and the statement is clear. If $\dim(\mathcal{O}_{X,x}) = 1$, then $\mathcal{O}_{X,x}$ is a discrete valuation ring (Algebra, Lemma 119.7) and $\mathcal{F}_x$ is torsion free. Hence $\mathcal{F}_x$ is free by More on Algebra, Lemma 22.11. □

Lemma 11.12. *Let X be an integral scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{G}$ is torsion free and $\mathcal{F}$ is of finite presentation, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is torsion free.*

Proof. The statement makes sense because $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent by Schemes, Section 24. To see the statement is true, see More on Algebra, Lemma 22.12. Some details omitted. □

Lemma 11.13. *Let X be an integral locally Noetherian scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is coherent, $\mathcal{G}$ is torsion free, and that for every $x \in X$ one of the following happens*

(1) *$\mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism, or*
(2) *$depth(\mathcal{F}_x) \geq 2$.*

Then φ is an isomorphism.

Proof. This is a translation of More on Algebra, Lemma 23.14 into the language of schemes. □

12. Reflexive modules

This section is the analogue of More on Algebra, Section 23 for coherent modules on locally Noetherian schemes. The reason for working with coherent modules is that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent for every pair of coherent $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, see Modules, Lemma 22.6.

Definition 12.1. Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The *reflexive hull* of $\mathcal{F}$ is the $\mathcal{O}_X$-module

$$\mathcal{F}^{**} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$

We say $\mathcal{F}$ is *reflexive* if the natural map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is an isomorphism.

It follows from Lemma 12.8 that the reflexive hull is a reflexive $\mathcal{O}_X$-module. You can use the same definition to define reflexive modules in more general situations, but this does not seem to be very useful. Here is the obligatory lemma comparing this to the usual algebraic notion.

Lemma 12.2. *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is reflexive,*
2. *for $U \subset X$ affine open $\mathcal{F}(U)$ is a reflexive $\mathcal{O}(U)$-module.*

Proof. Omitted. □

Remark 12.3. If X is a scheme of finite type over a field, then sometimes a different notion of reflexive modules is used (see for example [HL97, bottom of page 5 and Definition 1.1.9]). This other notion uses $R\mathcal{H}om$ into a dualizing complex $\omega_X^\bullet$ instead of into $\mathcal{O}_X$ and should probably have a different name because it can be different when X is not Gorenstein. For example, if $X = \text{Spec}(k[t^3, t^4, t^5])$, then a computation shows the dualizing sheaf ω_X is not reflexive in our sense, but it is reflexive in the other sense as $\omega_X \to \mathcal{H}om(\mathcal{H}om(\omega_X, \omega_X), \omega_X)$ is an isomorphism.

Lemma 12.4. *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

1. *If $\mathcal{F}$ is reflexive, then $\mathcal{F}$ is torsion free.*
2. *The map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is injective if and only if $\mathcal{F}$ is torsion free.*

Proof. Omitted. See More on Algebra, Lemma 23.2. □

Lemma 12.5. *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is reflexive,*
2. *$\mathcal{F}_x$ is a reflexive $\mathcal{O}_{X,x}$-module for all $x \in X$,*
3. *$\mathcal{F}_x$ is a reflexive $\mathcal{O}_{X,x}$-module for all closed points $x \in X$.*

Proof. By Modules, Lemma 22.4 we see that (1) and (2) are equivalent. Since every point of X specializes to a closed point (Properties, Lemma 5.9) we see that (2) and (3) are equivalent. □

Lemma 12.6. *Let $f : X \to Y$ be a flat morphism of integral locally Noetherian schemes. Let $\mathcal{G}$ be a coherent reflexive $\mathcal{O}_Y$-module. Then $f^*\mathcal{G}$ is a coherent reflexive $\mathcal{O}_X$-module.*

Proof. Omitted. See More on Algebra, Lemma 22.4 for the algebraic analogue. □

Lemma 12.7. *Let X be an integral locally Noetherian scheme. Let $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ be an exact sequence of coherent $\mathcal{O}_X$-modules. If $\mathcal{F}'$ is reflexive and $\mathcal{F}''$ is torsion free, then $\mathcal{F}$ is reflexive.*

Proof. Omitted. See More on Algebra, Lemma 23.5. □

Lemma 12.8. *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. If $\mathcal{G}$ is reflexive, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is reflexive.*

Proof. The statement makes sense because $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent by Cohomology of Schemes, Lemma 9.4. To see the statement is true, see More on Algebra, Lemma 23.8. Some details omitted. □

Remark 12.9. Let X be an integral locally Noetherian scheme. Thanks to Lemma 12.8 we know that the reflexive hull $\mathcal{F}^{**}$ of a coherent $\mathcal{O}_X$-module is coherent reflexive. Consider the category $\mathcal{C}$ of coherent reflexive $\mathcal{O}_X$-modules. Taking reflexive hulls gives a left adjoint to the inclusion functor $\mathcal{C} \to Coh(\mathcal{O}_X)$. Observe that $\mathcal{C}$ is an additive category with kernels and cokernels. Namely, given $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathcal{C}$, the usual kernel $\text{Ker}(\varphi)$ is reflexive (Lemma 12.7) and the reflexive hull $\text{Coker}(\varphi)^{**}$ of the usual cokernel is the cokernel in $\mathcal{C}$. Moreover $\mathcal{C}$ inherits a tensor product

$$\mathcal{F} \otimes_{\mathcal{C}} \mathcal{G} = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{**}$$

which is associative and symmetric. There is an internal Hom in the sense that for any three objects $\mathcal{F}, \mathcal{G}, \mathcal{H}$ of $\mathcal{C}$ we have the identity

$$\text{Hom}_{\mathcal{C}}(\mathcal{F} \otimes_{\mathcal{C}} \mathcal{G}, \mathcal{H}) = \text{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

see Modules, Lemma 22.1. In $\mathcal{C}$ every object $\mathcal{F}$ has a *dual object* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. Without further conditions on X it can happen that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \not\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{C}} \mathcal{G} \quad \text{and} \quad \mathcal{F} \otimes_{\mathcal{C}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \not\cong \mathcal{O}_X$$

for $\mathcal{F}, \mathcal{G}$ of rank 1 in $\mathcal{C}$. To make an example let $X = \text{Spec}(R)$ where R is as in More on Algebra, Example 23.17 and let $\mathcal{F}, \mathcal{G}$ be the modules corresponding to M. Computation omitted.

Lemma 12.10. *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is reflexive,*
2. *for each $x \in X$ one of the following happens*
   (a) *$\mathcal{F}_x$ is a reflexive $\mathcal{O}_{X,x}$-module, or*
   (b) *depth$(\mathcal{F}_x) \geq 2$.*

Proof. Omitted. See More on Algebra, Lemma 23.15. □

Lemma 12.11. *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent reflexive $\mathcal{O}_X$-module. Let $x \in X$.*

1. *If depth$(\mathcal{O}_{X,x}) \geq 2$, then depth$(\mathcal{F}_x) \geq 2$.*
2. *If X is (S_2), then $\mathcal{F}$ is (S_2).*

Proof. Omitted. See More on Algebra, Lemma 23.16. □

Lemma 12.12. *Let X be an integral locally Noetherian scheme. Let $j : U \to X$ be an open subscheme with complement Z. Assume $\mathcal{O}_{X,z}$ has depth ≥ 2 for all $z \in Z$. Then j^* and j_* define an equivalence of categories between the category of coherent reflexive $\mathcal{O}_X$-modules and the category of coherent reflexive $\mathcal{O}_U$-modules.*

Proof. Let $\mathcal{F}$ be a coherent reflexive $\mathcal{O}_X$-module. For $z \in Z$ the stalk $\mathcal{F}_z$ has depth ≥ 2 by Lemma 12.11. Thus $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism by Lemma 5.11. Conversely, let $\mathcal{G}$ be a coherent reflexive $\mathcal{O}_U$-module. It suffices to show that $j_*\mathcal{G}$ is a coherent reflexive $\mathcal{O}_X$-module. To prove this we may assume X is affine. By Properties, Lemma 22.5 there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\mathcal{G} = j^*\mathcal{F}$. After replacing $\mathcal{F}$ by its reflexive hull, we may assume $\mathcal{F}$ is reflexive (see discussion above and in particular Lemma 12.8). By the above $j_*\mathcal{G} = j_*j^*\mathcal{F} = \mathcal{F}$ as desired. □

If the scheme is normal, then reflexive is the same thing as torsion free and (S_2).

Lemma 12.13. *Let X be an integral locally Noetherian normal scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is reflexive,*
(2) *$\mathcal{F}$ is torsion free and has property (S_2), and*
(3) *there exists an open subscheme $j : U \to X$ such that*
    (a) *every irreducible component of $X \setminus U$ has codimension ≥ 2 in X,*
    (b) *$j^*\mathcal{F}$ is finite locally free, and*
    (c) *$\mathcal{F} = j_*j^*\mathcal{F}$.*

Proof. Using Lemma 12.2 the equivalence of (1) and (2) follows from More on Algebra, Lemma 23.18. Let $U \subset X$ be as in (3). By Properties, Lemma 12.5 we see that $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ for $x \notin U$. Since a finite locally free module is reflexive, we conclude (3) implies (1) by Lemma 12.12.

Assume (1). Let $U \subset X$ be the maximal open subscheme such that $j^*\mathcal{F} = \mathcal{F}|_U$ is finite locally free. So (3)(b) holds. Let $x \in X$ be a point. If $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, then $x \in U$, see Modules, Lemma 11.6. If $\dim(\mathcal{O}_{X,x}) \leq 1$, then $\mathcal{O}_{X,x}$ is either a field or a discrete valuation ring (Properties, Lemma 12.5) and hence $\mathcal{F}_x$ is free (More on Algebra, Lemma 22.11). Thus $x \notin U \Rightarrow \dim(\mathcal{O}_{X,x}) \geq 2$. Then Properties, Lemma 10.3 shows (3)(a) holds. By the already used Properties, Lemma 12.5 we also see that $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ for $x \notin U$ and hence (3)(c) follows from Lemma 12.12. □

Lemma 12.14. *Let X be an integral locally Noetherian normal scheme with generic point η. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $T : \mathcal{G}_\eta \to \mathcal{F}_\eta$ be a linear map. Then T extends to a map $\mathcal{G} \to \mathcal{F}^{**}$ of $\mathcal{O}_X$-modules if and only if*

(*) *for every $x \in X$ with $\dim(\mathcal{O}_{X,x}) = 1$ we have*

$$T\left(\text{Im}(\mathcal{G}_x \to \mathcal{G}_\eta)\right) \subset \text{Im}(\mathcal{F}_x \to \mathcal{F}_\eta).$$

Proof. Because $\mathcal{F}^{**}$ is torsion free and $\mathcal{F}_\eta = \mathcal{F}^{**}_\eta$ an extension, if it exists, is unique. Thus it suffices to prove the lemma over the members of an open covering of X, i.e., we may assume X is affine. In this case we are asking the following algebra question: Let R be a Noetherian normal domain with fraction field K, let M, N be finite R-modules, let $T : M \otimes_R K \to N \otimes_R K$ be a K-linear map. When does T extend to a map $N \to M^{**}$? By More on Algebra, Lemma 23.19 this happens if and only if $N_\mathfrak{p}$ maps into $(M/M_{tors})_\mathfrak{p}$ for every height 1 prime $\mathfrak{p}$ of R. This is exactly condition $(*)$ of the lemma. □

Lemma 12.15. *Let X be a regular scheme of dimension ≤ 2. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is reflexive,*

(2) $\mathcal{F}$ *is finite locally free.*

Proof. It is clear that a finite locally free module is reflexive. For the converse, we will show that if $\mathcal{F}$ is reflexive, then $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all $x \in X$. This is enough by Algebra, Lemma 78.2 and the fact that $\mathcal{F}$ is coherent. If $\dim(\mathcal{O}_{X,x}) = 0$, then $\mathcal{O}_{X,x}$ is a field and the statement is clear. If $\dim(\mathcal{O}_{X,x}) = 1$, then $\mathcal{O}_{X,x}$ is a discrete valuation ring (Algebra, Lemma 119.7) and $\mathcal{F}_x$ is torsion free. Hence $\mathcal{F}_x$ is free by More on Algebra, Lemma 22.11. If $\dim(\mathcal{O}_{X,x}) = 2$, then $\mathcal{O}_{X,x}$ is a regular local ring of dimension 2. By More on Algebra, Lemma 23.18 we see that $\mathcal{F}_x$ has depth ≥ 2. Hence $\mathcal{F}$ is free by Algebra, Lemma 106.6. $\square$

13. Effective Cartier divisors

We define the notion of an effective Cartier divisor before any other type of divisor.

Definition 13.1. Let S be a scheme.

(1) A *locally principal closed subscheme* of S is a closed subscheme whose sheaf of ideals is locally generated by a single element.
(2) An *effective Cartier divisor* on S is a closed subscheme $D \subset S$ whose ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_S$ is an invertible $\mathcal{O}_S$-module.

Thus an effective Cartier divisor is a locally principal closed subscheme, but the converse is not always true. Effective Cartier divisors are closed subschemes of pure codimension 1 in the strongest possible sense. Namely they are locally cut out by a single element which is a nonzerodivisor. In particular they are nowhere dense.

Lemma 13.2. *Let S be a scheme. Let $D \subset S$ be a closed subscheme. The following are equivalent:*

(1) *The subscheme D is an effective Cartier divisor on S.*
(2) *For every $x \in D$ there exists an affine open neighbourhood $\text{Spec}(A) = U \subset S$ of x such that $U \cap D = \text{Spec}(A/(f))$ with $f \in A$ a nonzerodivisor.*

Proof. Assume (1). For every $x \in D$ there exists an affine open neighbourhood $\text{Spec}(A) = U \subset S$ of x such that $\mathcal{I}_D|_U \cong \mathcal{O}_U$. In other words, there exists a section $f \in \Gamma(U, \mathcal{I}_D)$ which freely generates the restriction $\mathcal{I}_D|_U$. Hence $f \in A$, and the multiplication map $f : A \to A$ is injective. Also, since $\mathcal{I}_D$ is quasi-coherent we see that $D \cap U = \text{Spec}(A/(f))$.

Assume (2). Let $x \in D$. By assumption there exists an affine open neighbourhood $\text{Spec}(A) = U \subset S$ of x such that $U \cap D = \text{Spec}(A/(f))$ with $f \in A$ a nonzerodivisor. Then $\mathcal{I}_D|_U \cong \mathcal{O}_U$ since it is equal to $\widetilde{(f)} \cong \widetilde{A} \cong \mathcal{O}_U$. Of course $\mathcal{I}_D$ restricted to the open subscheme $S \setminus D$ is isomorphic to $\mathcal{O}_{S \setminus D}$. Hence $\mathcal{I}_D$ is an invertible $\mathcal{O}_S$-module. $\square$

Lemma 13.3. *Let S be a scheme. Let $Z \subset S$ be a locally principal closed subscheme. Let $U = S \setminus Z$. Then $U \to S$ is an affine morphism.*

Proof. The question is local on S, see Morphisms, Lemmas 11.3. Thus we may assume $S = \text{Spec}(A)$ and $Z = V(f)$ for some $f \in A$. In this case $U = D(f) = \text{Spec}(A_f)$ is affine hence $U \to S$ is affine. $\square$

Lemma 13.4. *Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor. Let $U = S \setminus D$. Then $U \to S$ is an affine morphism and U is scheme theoretically dense in S.*

Proof. Affineness is Lemma 13.3. The density question is local on S, see Morphisms, Lemma 7.5. Thus we may assume $S = \mathrm{Spec}(A)$ and D corresponding to the nonzerodivisor $f \in A$, see Lemma 13.2. Thus $A \subset A_f$ which implies that $U \subset S$ is scheme theoretically dense, see Morphisms, Example 7.4. □

Lemma 13.5. *Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor. Let $s \in D$. If $\dim_s(S) < \infty$, then $\dim_s(D) < \dim_s(S)$.*

Proof. Assume $\dim_s(S) < \infty$. Let $U = \mathrm{Spec}(A) \subset S$ be an affine open neighbourhood of s such that $\dim(U) = \dim_s(S)$ and such that $D = V(f)$ for some nonzerodivisor $f \in A$ (see Lemma 13.2). Recall that $\dim(U)$ is the Krull dimension of the ring A and that $\dim(U \cap D)$ is the Krull dimension of the ring $A/(f)$. Then f is not contained in any minimal prime of A. Hence any maximal chain of primes in $A/(f)$, viewed as a chain of primes in A, can be extended by adding a minimal prime. □

Definition 13.6. Let S be a scheme. Given effective Cartier divisors D_1, D_2 on S we set $D = D_1 + D_2$ equal to the closed subscheme of S corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}_{D_1}\mathcal{I}_{D_2} \subset \mathcal{O}_S$. We call this the *sum of the effective Cartier divisors D_1 and D_2*.

It is clear that we may define the sum $\sum n_i D_i$ given finitely many effective Cartier divisors D_i on X and nonnegative integers n_i.

Lemma 13.7. *The sum of two effective Cartier divisors is an effective Cartier divisor.*

Proof. Omitted. Locally $f_1, f_2 \in A$ are nonzerodivisors, then also $f_1 f_2 \in A$ is a nonzerodivisor. □

Lemma 13.8. *Let X be a scheme. Let D, D' be two effective Cartier divisors on X. If $D \subset D'$ (as closed subschemes of X), then there exists an effective Cartier divisor D'' such that $D' = D + D''$.*

Proof. Omitted. □

Lemma 13.9. *Let X be a scheme. Let Z, Y be two closed subschemes of X with ideal sheaves $\mathcal{I}$ and $\mathcal{J}$. If $\mathcal{I}\mathcal{J}$ defines an effective Cartier divisor $D \subset X$, then Z and Y are effective Cartier divisors and $D = Z + Y$.*

Proof. Applying Lemma 13.2 we obtain the following algebra situation: A is a ring, $I, J \subset A$ ideals and $f \in A$ a nonzerodivisor such that $IJ = (f)$. Thus the result follows from Algebra, Lemma 120.16. □

Lemma 13.10. *Let X be a scheme. Let $D, D' \subset X$ be effective Cartier divisors such that the scheme theoretic intersection $D \cap D'$ is an effective Cartier divisor on D'. Then $D + D'$ is the scheme theoretic union of D and D'.*

Proof. See Morphisms, Definition 4.4 for the definition of scheme theoretic intersection and union. To prove the lemma working locally (using Lemma 13.2) we obtain the following algebra problem: Given a ring A and nonzerodivisors $f_1, f_2 \in A$ such that f_1 maps to a nonzerodivisor in A/f_2A, show that $f_1A \cap f_2A = f_1f_2A$. We omit the straightforward argument. □

Recall that we have defined the inverse image of a closed subscheme under any morphism of schemes in Schemes, Definition 17.7.

Lemma 13.11. *Let $f : S' \to S$ be a morphism of schemes. Let $Z \subset S$ be a locally principal closed subscheme. Then the inverse image $f^{-1}(Z)$ is a locally principal closed subscheme of S'.*

Proof. Omitted. ☐

Definition 13.12. Let $f : S' \to S$ be a morphism of schemes. Let $D \subset S$ be an effective Cartier divisor. We say the *pullback of D by f is defined* if the closed subscheme $f^{-1}(D) \subset S'$ is an effective Cartier divisor. In this case we denote it either f^*D or $f^{-1}(D)$ and we call it the *pullback of the effective Cartier divisor*.

The condition that $f^{-1}(D)$ is an effective Cartier divisor is often satisfied in practice. Here is an example lemma.

Lemma 13.13. *Let $f : X \to Y$ be a morphism of schemes. Let $D \subset Y$ be an effective Cartier divisor. The pullback of D by f is defined in each of the following cases:*

1. *$f(x) \notin D$ for any weakly associated point x of X,*
2. *X, Y integral and f dominant,*
3. *X reduced and $f(\xi) \notin D$ for any generic point ξ of any irreducible component of X,*
4. *X is locally Noetherian and $f(x) \notin D$ for any associated point x of X,*
5. *X is locally Noetherian, has no embedded points, and $f(\xi) \notin D$ for any generic point ξ of an irreducible component of X,*
6. *f is flat, and*
7. *add more here as needed.*

Proof. The question is local on X, and hence we reduce to the case where $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(R)$, f is given by $\varphi : R \to A$ and $D = \mathrm{Spec}(R/(t))$ where $t \in R$ is a nonzerodivisor. The goal in each case is to show that $\varphi(t) \in A$ is a nonzerodivisor.

In case (1) this follows from Algebra, Lemma 66.7. Case (4) is a special case of (1) by Lemma 5.8. Case (5) follows from (4) and the definitions. Case (3) is a special case of (1) by Lemma 5.12. Case (2) is a special case of (3). If $R \to A$ is flat, then $t : R \to R$ being injective shows that $t : A \to A$ is injective. This proves (6). ☐

Lemma 13.14. *Let $f : S' \to S$ be a morphism of schemes. Let D_1, D_2 be effective Cartier divisors on S. If the pullbacks of D_1 and D_2 are defined then the pullback of $D = D_1 + D_2$ is defined and $f^*D = f^*D_1 + f^*D_2$.*

Proof. Omitted. ☐

14. Effective Cartier divisors and invertible sheaves

Since an effective Cartier divisor has an invertible ideal sheaf (Definition 13.1) the following definition makes sense.

Definition 14.1. Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor with ideal sheaf $\mathcal{I}_D$.

(1) The *invertible sheaf* $\mathcal{O}_S(D)$ *associated to* D is defined by
$$\mathcal{O}_S(D) = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{I}_D, \mathcal{O}_S) = \mathcal{I}_D^{\otimes -1}.$$
(2) The *canonical section*, usually denoted 1 or 1_D, is the global section of $\mathcal{O}_S(D)$ corresponding to the inclusion mapping $\mathcal{I}_D \to \mathcal{O}_S$.
(3) We write $\mathcal{O}_S(-D) = \mathcal{O}_S(D)^{\otimes -1} = \mathcal{I}_D$.
(4) Given a second effective Cartier divisor $D' \subset S$ we define $\mathcal{O}_S(D - D') = \mathcal{O}_S(D) \otimes_{\mathcal{O}_S} \mathcal{O}_S(-D')$.

Some comments. We will see below that the assignment $D \mapsto \mathcal{O}_S(D)$ turns addition of effective Cartier divisors (Definition 13.6) into addition in the Picard group of S (Lemma 14.4). However, the expression $D - D'$ in the definition above does not have any geometric meaning. More precisely, we can think of the set of effective Cartier divisors on S as a commutative monoid $\text{EffCart}(S)$ whose zero element is the empty effective Cartier divisor. Then the assignment $(D, D') \mapsto \mathcal{O}_S(D - D')$ defines a group homomorphism
$$\text{EffCart}(S)^{gp} \longrightarrow \text{Pic}(S)$$
where the left hand side is the group completion of $\text{EffCart}(S)$. In other words, when we write $\mathcal{O}_S(D - D')$ we may think of $D - D'$ as an element of $\text{EffCart}(S)^{gp}$.

Lemma 14.2. *Let S be a scheme and let $D \subset S$ be an effective Cartier divisor. Then the conormal sheaf is $\mathcal{C}_{D/S} = \mathcal{I}_D|_D = \mathcal{O}_S(-D)|_D$ and the normal sheaf is $\mathcal{N}_{D/S} = \mathcal{O}_S(D)|_D$.*

Proof. This follows from Morphisms, Lemma 31.2. $\square$

Lemma 14.3. *Let X be a scheme. Let $D, C \subset X$ be effective Cartier divisors with $C \subset D$ and let $D' = D + C$. Then there is a short exact sequence*
$$0 \to \mathcal{O}_X(-D)|_C \to \mathcal{O}_{D'} \to \mathcal{O}_D \to 0$$
of $\mathcal{O}_X$-modules.

Proof. In the statement of the lemma and in the proof we use the equivalence of Morphisms, Lemma 4.1 to think of quasi-coherent modules on closed subschemes of X as quasi-coherent modules on X. Let $\mathcal{I}$ be the ideal sheaf of D in D'. Then there is a short exact sequence
$$0 \to \mathcal{I} \to \mathcal{O}_{D'} \to \mathcal{O}_D \to 0$$
because $D \to D'$ is a closed immersion. There is a canonical surjection $\mathcal{I} \to \mathcal{I}/\mathcal{I}^2 = \mathcal{C}_{D/D'}$. We have $\mathcal{C}_{D/X} = \mathcal{O}_X(-D)|_D$ by Lemma 14.2 and there is a canonical surjective map
$$\mathcal{C}_{D/X} \longrightarrow \mathcal{C}_{D/D'}$$
see Morphisms, Lemmas 31.3 and 31.4. Thus it suffices to show: (a) $\mathcal{I}^2 = 0$ and (b) $\mathcal{I}$ is an invertible $\mathcal{O}_C$-module. Both (a) and (b) can be checked locally, hence we may assume $X = \text{Spec}(A)$, $D = \text{Spec}(A/fA)$ and $C = \text{Spec}(A/gA)$ where $f, g \in A$ are nonzerodivisors (Lemma 13.2). Since $C \subset D$ we see that $f \in gA$. Then $I = fA/fgA$ has square zero and is invertible as an A/gA-module as desired. $\square$

Lemma 14.4. *Let S be a scheme. Let D_1, D_2 be effective Cartier divisors on S. Let $D = D_1 + D_2$. Then there is a unique isomorphism*
$$\mathcal{O}_S(D_1) \otimes_{\mathcal{O}_S} \mathcal{O}_S(D_2) \longrightarrow \mathcal{O}_S(D)$$

which maps $1_{D_1} \otimes 1_{D_2}$ to 1_D.

Proof. Omitted. □

Lemma 14.5. *Let $f : S' \to S$ be a morphism of schemes. Let D be a effective Cartier divisors on S. If the pullback of D is defined then $f^*\mathcal{O}_S(D) = \mathcal{O}_{S'}(f^*D)$ and the canonical section 1_D pulls back to the canonical section 1_{f^*D}.*

Proof. Omitted. □

Definition 14.6. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{L}$ be an invertible sheaf on X. A global section $s \in \Gamma(X, \mathcal{L})$ is called a *regular section* if the map $\mathcal{O}_X \to \mathcal{L}$, $f \mapsto fs$ is injective.

Lemma 14.7. *Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The following are equivalent:*

(1) *f is a regular section, and*
(2) *for any $x \in X$ the image $f \in \mathcal{O}_{X,x}$ is a nonzerodivisor.*

If X is a scheme these are also equivalent to

(3) *for any affine open $\mathrm{Spec}(A) = U \subset X$ the image $f \in A$ is a nonzerodivisor,*
(4) *there exists an affine open covering $X = \bigcup \mathrm{Spec}(A_i)$ such that the image of f in A_i is a nonzerodivisor for all i.*

Proof. Omitted. □

Note that a global section s of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ may be seen as an $\mathcal{O}_X$-module map $s : \mathcal{O}_X \to \mathcal{L}$. Its dual is therefore a map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$. (See Modules, Definition 25.6 for the definition of the dual invertible sheaf.)

Definition 14.8. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$ be a global section. The *zero scheme* of s is the closed subscheme $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is the image of the map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

Lemma 14.9. *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$.*

(1) *Consider closed immersions $i : Z \to X$ such that $i^*s \in \Gamma(Z, i^*\mathcal{L})$ is zero ordered by inclusion. The zero scheme $Z(s)$ is the maximal element of this ordered set.*
(2) *For any morphism of schemes $f : Y \to X$ we have $f^*s = 0$ in $\Gamma(Y, f^*\mathcal{L})$ if and only if f factors through $Z(s)$.*
(3) *The zero scheme $Z(s)$ is a locally principal closed subscheme.*
(4) *The zero scheme $Z(s)$ is an effective Cartier divisor if and only if s is a regular section of $\mathcal{L}$.*

Proof. Omitted. □

Lemma 14.10. *Let X be a scheme.*

(1) *If $D \subset X$ is an effective Cartier divisor, then the canonical section 1_D of $\mathcal{O}_X(D)$ is regular.*
(2) *Conversely, if s is a regular section of the invertible sheaf $\mathcal{L}$, then there exists a unique effective Cartier divisor $D = Z(s) \subset X$ and a unique isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ which maps 1_D to s.*

The constructions $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ give mutually inverse maps

$$\left\{\text{effective Cartier divisors on } X\right\} \leftrightarrow \left\{\begin{matrix}\text{isomorphism classes of pairs } (\mathcal{L}, s) \\ \text{consisting of an invertible } \mathcal{O}_X\text{-module} \\ \mathcal{L} \text{ and a regular global section } s\end{matrix}\right\}$$

Proof. Omitted. □

Remark 14.11. Let X be a scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and s a regular section of $\mathcal{L}$. Then the zero scheme $D = Z(s)$ is an effective Cartier divisor on X and there are short exact sequences

$$0 \to \mathcal{O}_X \to \mathcal{L} \to i_*(\mathcal{L}|_D) \to 0 \quad \text{and} \quad 0 \to \mathcal{L}^{\otimes -1} \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0.$$

Given an effective Cartier divisor $D \subset X$ using Lemmas 14.10 and 14.2 we get

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*(\mathcal{N}_{D/X}) \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*(\mathcal{O}_D) \to 0$$

15. Effective Cartier divisors on Noetherian schemes

In the locally Noetherian setting most of the discussion of effective Cartier divisors and regular sections simplifies somewhat.

Lemma 15.1. *Let X be a locally Noetherian scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Then s is a regular section if and only if s does not vanish in the associated points of X.*

Proof. Omitted. Hint: reduce to the affine case and $\mathcal{L}$ trivial and then use Lemma 14.7 and Algebra, Lemma 63.9. □

Lemma 15.2. *Let X be a locally Noetherian scheme. Let $D \subset X$ be a closed subscheme corresponding to the quasi-coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$.*

1. *If for every $x \in D$ the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ can be generated by one element, then D is locally principal.*
2. *If for every $x \in D$ the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ can be generated by a single nonzerodivisor, then D is an effective Cartier divisor.*

Proof. Let $\mathrm{Spec}(A)$ be an affine neighbourhood of a point $x \in D$. Let $\mathfrak{p} \subset A$ be the prime corresponding to x. Let $I \subset A$ be the ideal defining the trace of D on $\mathrm{Spec}(A)$. Since A is Noetherian (as X is locally Noetherian) the ideal I is generated by finitely many elements, say $I = (f_1, \ldots, f_r)$. Under the assumption of (1) we have $I_\mathfrak{p} = (f)$ for some $f \in A_\mathfrak{p}$. Then $f_i = g_i f$ for some $g_i \in A_\mathfrak{p}$. Write $g_i = a_i/h_i$ and $f = f'/h$ for some $a_i, h_i, f', h \in A$, $h_i, h \notin \mathfrak{p}$. Then $I_{h_1 \ldots h_r h} \subset A_{h_1 \ldots h_r h}$ is principal, because it is generated by f'. This proves (1). For (2) we may assume $I = (f)$. The assumption implies that the image of f in $A_\mathfrak{p}$ is a nonzerodivisor. Then f is a nonzerodivisor on a neighbourhood of x by Algebra, Lemma 68.6. This proves (2). □

Lemma 15.3. *Let X be a locally Noetherian scheme.*

1. *Let $D \subset X$ be a locally principal closed subscheme. Let $\xi \in D$ be a generic point of an irreducible component of D. Then $\dim(\mathcal{O}_{X,\xi}) \leq 1$.*
2. *Let $D \subset X$ be an effective Cartier divisor. Let $\xi \in D$ be a generic point of an irreducible component of D. Then $\dim(\mathcal{O}_{X,\xi}) = 1$.*

Proof. Proof of (1). By assumption we may assume $X = \text{Spec}(A)$ and $D = \text{Spec}(A/(f))$ where A is a Noetherian ring and $f \in A$. Let ξ correspond to the prime ideal $\mathfrak{p} \subset A$. The assumption that ξ is a generic point of an irreducible component of D signifies $\mathfrak{p}$ is minimal over (f). Thus $\dim(A_\mathfrak{p}) \leq 1$ by Algebra, Lemma 60.11.

Proof of (2). By part (1) we see that $\dim(\mathcal{O}_{X,\xi}) \leq 1$. On the other hand, the local equation f is a nonzerodivisor in $A_\mathfrak{p}$ by Lemma 13.2 which implies the dimension is at least 1 (because there must be a prime in $A_\mathfrak{p}$ not containing f by the elementary Algebra, Lemma 17.2). $\square$

Lemma 15.4. *Let X be a Noetherian scheme. Let $D \subset X$ be an integral closed subscheme which is also an effective Cartier divisor. Then the local ring of X at the generic point of D is a discrete valuation ring.*

Proof. By Lemma 13.2 we may assume $X = \text{Spec}(A)$ and $D = \text{Spec}(A/(f))$ where A is a Noetherian ring and $f \in A$ is a nonzerodivisor. The assumption that D is integral signifies that (f) is prime. Hence the local ring of X at the generic point is $A_{(f)}$ which is a Noetherian local ring whose maximal ideal is generated by a nonzerodivisor. Thus it is a discrete valuation ring by Algebra, Lemma 119.7. $\square$

Lemma 15.5. *Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. If X is (S_k), then D is (S_{k-1}).*

Proof. Let $x \in D$. Then $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/(f)$ where $f \in \mathcal{O}_{X,x}$ is a nonzerodivisor. By assumption we have $\text{depth}(\mathcal{O}_{X,x}) \geq \min(\dim(\mathcal{O}_{X,x}), k)$. By Algebra, Lemma 72.7 we have $\text{depth}(\mathcal{O}_{D,x}) = \text{depth}(\mathcal{O}_{X,x}) - 1$ and by Algebra, Lemma 60.13 $\dim(\mathcal{O}_{D,x}) = \dim(\mathcal{O}_{X,x}) - 1$. It follows that $\text{depth}(\mathcal{O}_{D,x}) \geq \min(\dim(\mathcal{O}_{D,x}), k-1)$ as desired. $\square$

Lemma 15.6. *Let X be a locally Noetherian normal scheme. Let $D \subset X$ be an effective Cartier divisor. Then D is (S_1).*

Proof. By Properties, Lemma 12.5 we see that X is (S_2). Thus we conclude by Lemma 15.5. $\square$

Lemma 15.7. *Let X be a Noetherian scheme. Let $D \subset X$ be an integral closed subscheme. Assume that*

(1) *D has codimension 1 in X, and*
(2) *$\mathcal{O}_{X,x}$ is a UFD for all $x \in D$.*

Then D is an effective Cartier divisor.

Proof. Let $x \in D$ and set $A = \mathcal{O}_{X,x}$. Let $\mathfrak{p} \subset A$ correspond to the generic point of D. Then $A_\mathfrak{p}$ has dimension 1 by assumption (1). Thus $\mathfrak{p}$ is a prime ideal of height 1. Since A is a UFD this implies that $\mathfrak{p} = (f)$ for some $f \in A$. Of course f is a nonzerodivisor and we conclude by Lemma 15.2. $\square$

Lemma 15.8. *Let X be a Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Assume there exist integral effective Cartier divisors $D_i \subset X$ and a closed subset $Z' \subset X$ of codimension ≥ 2 such that $Z \subset Z' \cup \bigcup D_i$ set-theoretically. Then there exists an effective Cartier divisor of the form*

$$D = \sum a_i D_i \subset Z$$

such that $D \to Z$ is an isomorphism away from codimension 2 in X. The existence of the D_i is guaranteed if $\mathcal{O}_{X,x}$ is a UFD for all $x \in Z$ or if X is regular.

Proof. Let $\xi_i \in D_i$ be the generic point and let $\mathcal{O}_i = \mathcal{O}_{X,\xi_i}$ be the local ring which is a discrete valuation ring by Lemma 15.4. Let $a_i \geq 0$ be the minimal valuation of an element of $\mathcal{I}_{Z,\xi_i} \subset \mathcal{O}_i$. We claim that the effective Cartier divisor $D = \sum a_i D_i$ works.

Namely, suppose that $x \in X$. Let $A = \mathcal{O}_{X,x}$. Let $D_1, \ldots, D_n$ be the pairwise distinct divisors D_i such that $x \in D_i$. For $1 \leq i \leq n$ let $f_i \in A$ be a local equation for D_i. Then f_i is a prime element of A and $\mathcal{O}_i = A_{(f_i)}$. Let $I = \mathcal{I}_{Z,x} \subset A$ be the stalk of the ideal sheaf of Z. By our choice of a_i we have $IA_{(f_i)} = f_i^{a_i} A_{(f_i)}$. We claim that $I \subset (\prod_{i=1,\ldots,n} f_i^{a_i})$.

Proof of the claim. The localization map $\varphi : A/(f_i) \to A_{(f_i)}/f_i A_{(f_i)}$ is injective as the prime ideal (f_i) is the inverse image of the maximal ideal $f_i A_{(f_i)}$. By induction on n we deduce that $\varphi_n : A/(f_i^n) \to A_{(f_i)}/f_i^n A_{(f_i)}$ is also injective. Since $\varphi_{a_i}(I) = 0$, we have $I \subset (f_i^{a_i})$. Thus, for any $x \in I$, we may write $x = f_1^{a_1} x_1$ for some $x_1 \in A$. Since $D_1, \ldots, D_n$ are pairwise distinct, f_i is a unit in $A_{(f_j)}$ for $i \neq j$. Comparing x and x_1 at $A_{(f_i)}$ for $n \geq i > 1$, we still have $x_1 \in (f_i^{a_i})$. Repeating the previous process, we inductively write $x_i = f_{i+1}^{a_{i+1}} x_{i+1}$ for any $n > i \geq 1$. In conclusion, $x \in (\prod_{i=1,\ldots n} f_i^{a_i})$ for any $x \in I$ as desired.

The claim shows that $\mathcal{I}_Z \subset \mathcal{I}_D$, i.e., that $D \subset Z$. Moreover, we also see that D and Z agree at the ξ_i, which proves that $D \to Z$ is an isomorphism away from codimension 2 on X.

To see the final statements we argue as follows. A regular local ring is a UFD (More on Algebra, Lemma 121.2) hence it suffices to argue in the UFD case. In that case, let D_i be the irreducible components of Z which have codimension 1 in X. By Lemma 15.7 each D_i is an effective Cartier divisor. □

Lemma 15.9. *Let $Z \subset X$ be a closed subscheme of a Noetherian scheme. Assume*

1. *Z has no embedded points,*
2. *every irreducible component of Z has codimension 1 in X,*
3. *every local ring $\mathcal{O}_{X,x}$, $x \in Z$ is a UFD or X is regular.*

Then Z is an effective Cartier divisor.

Proof. Let $D = \sum a_i D_i$ be as in Lemma 15.8 where $D_i \subset Z$ are the irreducible components of Z. If $D \to Z$ is not an isomorphism, then $\mathcal{O}_Z \to \mathcal{O}_D$ has a nonzero kernel sitting in codimension ≥ 2. This would mean that Z has embedded points, which is forbidden by assumption (1). Hence $D \cong Z$ as desired. □

Lemma 15.10. *Let R be a Noetherian UFD. Let $I \subset R$ be an ideal such that R/I has no embedded primes and such that every minimal prime over I has height 1. Then $I = (f)$ for some $f \in R$.*

Proof. By Lemma 15.9 the ideal sheaf $\tilde{I}$ is invertible on $\text{Spec}(R)$. By More on Algebra, Lemma 117.3 it is generated by a single element. □

Lemma 15.11. *Let X be a Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. Assume that there exist integral effective Cartier divisors $D_i \subset X$ such that $D \subset \bigcup D_i$ set theoretically. Then $D = \sum a_i D_i$ for some $a_i \geq 0$. The existence of the D_i is guaranteed if $\mathcal{O}_{X,x}$ is a UFD for all $x \in D$ or if X is regular.*

Proof. Choose a_i as in Lemma 15.8 and set $D' = \sum a_i D_i$. Then $D' \to D$ is an inclusion of effective Cartier divisors which is an isomorphism away from codimension 2 on X. Pick $x \in X$. Set $A = \mathcal{O}_{X,x}$ and let $f, f' \in A$ be the nonzerodivisor generating the ideal of D, D' in A. Then $f = gf'$ for some $g \in A$. Moreover, for every prime $\mathfrak{p}$ of height ≤ 1 of A we see that g maps to a unit of $A_{\mathfrak{p}}$. This implies that g is a unit because the minimal primes over (g) have height 1 (Algebra, Lemma 60.11). □

Lemma 15.12. *Let X be a Noetherian scheme which has an ample invertible sheaf. Then every invertible $\mathcal{O}_X$-module is isomorphic to*

$$\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D')^{\otimes -1}$$

for some effective Cartier divisors D, D' in X. Moreover, given a finite subset $E \subset X$ we may choose D, D' such that $E \cap D = \emptyset$ and $E \cap D' = \emptyset$. If X is quasi-affine, then we may choose $D' = \emptyset$.

Proof. Let $x_1, \ldots, x_n$ be the associated points of X (Lemma 2.5).

If X is quasi-affine and $\mathcal{N}$ is any invertible $\mathcal{O}_X$-module, then we can pick a section t of $\mathcal{N}$ which does not vanish at any of the points of $E \cup \{x_1, \ldots, x_n\}$, see Properties, Lemma 29.7. Then t is a regular section of $\mathcal{N}$ by Lemma 15.1. Hence $\mathcal{N} \cong \mathcal{O}_X(D)$ where $D = Z(t)$ is the effective Cartier divisor corresponding to t, see Lemma 14.10. Since $E \cap D = \emptyset$ by construction we are done in this case.

Returning to the general case, let $\mathcal{L}$ be an ample invertible sheaf on X. There exists an $n > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine and such that $E \cup \{x_1, \ldots, x_n\} \subset X_s$ (Properties, Lemma 29.6).

Let $\mathcal{N}$ be an arbitrary invertible $\mathcal{O}_X$-module. By the quasi-affine case, we can find a section $t \in \mathcal{N}(X_s)$ which does not vanish at any point of $E \cup \{x_1, \ldots, x_n\}$. By Properties, Lemma 17.2 we see that for some $e \geq 0$ the section $s^e|_{X_s} t$ extends to a global section τ of $\mathcal{L}^{\otimes e} \otimes \mathcal{N}$. Thus both $\mathcal{L}^{\otimes e} \otimes \mathcal{N}$ and $\mathcal{L}^{\otimes e}$ are invertible sheaves which have global sections which do not vanish at any point of $E \cup \{x_1, \ldots, x_n\}$. Thus these are regular sections by Lemma 15.1. Hence $\mathcal{L}^{\otimes e} \otimes \mathcal{N} \cong \mathcal{O}_X(D)$ and $\mathcal{L}^{\otimes e} \cong \mathcal{O}_X(D')$ for some effective Cartier divisors D and D', see Lemma 14.10. By construction $E \cap D = \emptyset$ and $E \cap D' = \emptyset$ and the proof is complete. □

Lemma 15.13. *Let X be an integral regular scheme of dimension 2. Let $i : D \to X$ be the immersion of an effective Cartier divisor. Let $\mathcal{F} \to \mathcal{F}' \to i_*\mathcal{G} \to 0$ be an exact sequence of coherent $\mathcal{O}_X$-modules. Assume*

1. *$\mathcal{F}, \mathcal{F}'$ are locally free of rank r on a nonempty open of X,*
2. *D is an integral scheme,*
3. *$\mathcal{G}$ is a finite locally free $\mathcal{O}_D$-module of rank s.*

Then $\mathcal{L} = (\wedge^r \mathcal{F})^{**}$ and $\mathcal{L}' = (\wedge^r \mathcal{F}')^{**}$ are invertible $\mathcal{O}_X$-modules and $\mathcal{L}' \cong \mathcal{L}(kD)$ for some $k \in \{0, \ldots, \min(s, r)\}$.

Proof. The first statement follows from Lemma 12.15 as assumption (1) implies that $\mathcal{L}$ and $\mathcal{L}'$ have rank 1. Taking $\wedge^r$ and double duals are functors, hence we obtain a canonical map $\sigma : \mathcal{L} \to \mathcal{L}'$ which is an isomorphism over the nonempty open of (1), hence nonzero. To finish the proof, it suffices to see that σ viewed as a global section of $\mathcal{L}' \otimes \mathcal{L}^{\otimes -1}$ does not vanish at any codimension point of X, except at the generic point of D and there with vanishing order at most $\min(s, r)$.

Translated into algebra, we arrive at the following problem: Let $(A, \mathfrak{m}, \kappa)$ be a discrete valuation ring with fraction field K. Let $M \to M' \to N \to 0$ be an exact sequence of finite A-modules with $\dim_K(M \otimes K) = \dim_K(M' \otimes K) = r$ and with $N \cong \kappa^{\oplus s}$. Show that the induced map $L = \wedge^r(M)^{**} \to L' = \wedge^r(M')^{**}$ vanishes to order at most $\min(s, r)$. We will use the structure theorem for modules over A, see More on Algebra, Lemma 124.3 or 124.9. Dividing out a finite A-module by a torsion submodule does not change the double dual. Thus we may replace M by M/M_{tors} and M' by $M'/\text{Im}(M_{tors} \to M')$ and assume that M is torsion free. Then $M \to M'$ is injective and $M'_{tors} \to N$ is injective. Hence we may replace M' by M'/M'_{tors} and N by N/M'_{tors}. Thus we reduce to the case where M and M' are free of rank r and $N \cong \kappa^{\oplus s}$. In this case σ is the determinant of $M \to M'$ and vanishes to order s for example by Algebra, Lemma 121.7. □

16. Complements of affine opens

In this section we discuss the result that the complement of an affine open in a variety has pure codimension 1.

Lemma 16.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. The punctured spectrum $U = \text{Spec}(A) \setminus \{\mathfrak{m}\}$ of A is affine if and only if $\dim(A) \leq 1$.*

Proof. If $\dim(A) = 0$, then U is empty hence affine (equal to the spectrum of the 0 ring). If $\dim(A) = 1$, then we can choose an element $f \in \mathfrak{m}$ not contained in any of the finite number of minimal primes of A (Algebra, Lemmas 31.6 and 15.2). Then $U = \text{Spec}(A_f)$ is affine.

The converse is more interesting. We will give a somewhat nonstandard proof and discuss the standard argument in a remark below. Assume $U = \text{Spec}(B)$ is affine. Since affineness and dimension are not affecting by going to the reduction we may replace A by the quotient by its ideal of nilpotent elements and assume A is reduced. Set $Q = B/A$ viewed as an A-module. The support of Q is $\{\mathfrak{m}\}$ as $A_\mathfrak{p} = B_\mathfrak{p}$ for all nonmaximal primes $\mathfrak{p}$ of A. We may assume $\dim(A) \geq 1$, hence as above we can pick $f \in \mathfrak{m}$ not contained in any of the minimal ideals of A. Since A is reduced this implies that f is a nonzerodivisor. In particular $\dim(A/fA) = \dim(A) - 1$, see Algebra, Lemma 60.13. Applying the snake lemma to multiplication by f on the short exact sequence $0 \to A \to B \to Q \to 0$ we obtain

$$0 \to Q[f] \to A/fA \to B/fB \to Q/fQ \to 0$$

where $Q[f] = \text{Ker}(f : Q \to Q)$. This implies that $Q[f]$ is a finite A-module. Since the support of $Q[f]$ is $\{\mathfrak{m}\}$ we see $l = \text{length}_A(Q[f]) < \infty$ (Algebra, Lemma 62.3). Set $l_n = \text{length}_A(Q[f^n])$. The exact sequence

$$0 \to Q[f^n] \to Q[f^{n+1}] \xrightarrow{f^n} Q[f]$$

shows inductively that $l_n < \infty$ and that $l_n \leq l_{n+1}$. Considering the exact sequence

$$0 \to Q[f] \to Q[f^{n+1}] \xrightarrow{f} Q[f^n] \to Q/fQ$$

and we see that the image of $Q[f^n]$ in Q/fQ has length $l_n - l_{n+1} + l \leq l$. Since $Q = \bigcup Q[f^n]$ we find that the length of Q/fQ is at most l, i.e., bounded. Thus Q/fQ is a finite A-module. Hence $A/fA \to B/fB$ is a finite ring map, in particular induces a closed map on spectra (Algebra, Lemmas 36.22 and 41.6). On the other hand

Spec(B/fB) is the punctured spectrum of Spec(A/fA). This is a contradiction unless Spec$(B/fB) = \emptyset$ which means that $\dim(A/fA) = 0$ as desired. $\square$

Remark 16.2. If $(A, \mathfrak{m})$ is a Noetherian local normal domain of dimension ≥ 2 and U is the punctured spectrum of A, then $\Gamma(U, \mathcal{O}_U) = A$. This algebraic version of Hartogs's theorem follows from the fact that $A = \bigcap_{\text{height}(\mathfrak{p})=1} A_\mathfrak{p}$ we've seen in Algebra, Lemma 157.6. Thus in this case U cannot be affine (since it would force $\mathfrak{m}$ to be a point of U). This is often used as the starting point of the proof of Lemma 16.1. To reduce the case of a general Noetherian local ring to this case, we first complete (to get a Nagata local ring), then replace A by $A/\mathfrak{q}$ for a suitable minimal prime, and then normalize. Each of these steps does not change the dimension and we obtain a contradiction. You can skip the completion step, but then the normalization in general is not a Noetherian domain. However, it is still a Krull domain of the same dimension (this is proved using Krull-Akizuki) and one can apply the same argument.

Remark 16.3. It is not clear how to characterize the non-Noetherian local rings $(A, \mathfrak{m})$ whose punctured spectrum is affine. Such a ring has a finitely generated ideal I with $\mathfrak{m} = \sqrt{I}$. Of course if we can take I generated by 1 element, then A has an affine puncture spectrum; this gives lots of non-Noetherian examples. Conversely, it follows from the argument in the proof of Lemma 16.1 that such a ring cannot possess a nonzerodivisor $f \in \mathfrak{m}$ with $H^0_I(A/fA) = 0$ (so A cannot have a regular sequence of length 2). Moreover, the same holds for any ring A' which is the target of a local homomorphism of local rings $A \to A'$ such that $\mathfrak{m}_{A'} = \sqrt{\mathfrak{m}A'}$.

Lemma 16.4. *Let X be a locally Noetherian scheme. Let $U \subset X$ be an open subscheme such that the inclusion morphism $U \to X$ is affine. For every generic point ξ of an irreducible component of $X \setminus U$ the local ring $\mathcal{O}_{X,\xi}$ has dimension ≤ 1. If U is dense or if ξ is in the closure of U, then* $\dim(\mathcal{O}_{X,\xi}) = 1$.

Proof. Since ξ is a generic point of $X \setminus U$, we see that

$$U_\xi = U \times_X \text{Spec}(\mathcal{O}_{X,\xi}) \subset \text{Spec}(\mathcal{O}_{X,\xi})$$

is the punctured spectrum of $\mathcal{O}_{X,\xi}$ (hint: use Schemes, Lemma 13.2). As $U \to X$ is affine, we see that $U_\xi \to \text{Spec}(\mathcal{O}_{X,\xi})$ is affine (Morphisms, Lemma 11.8) and we conclude that U_ξ is affine. Hence $\dim(\mathcal{O}_{X,\xi}) \leq 1$ by Lemma 16.1. If $\xi \in \overline{U}$, then there is a specialization $\eta \to \xi$ where $\eta \in U$ (just take η a generic point of an irreducible component of $\overline{U}$ which contains ξ; since $\overline{U}$ is locally Noetherian, hence locally has finitely many irreducible components, we see that $\eta \in U$). Then $\eta \in \text{Spec}(\mathcal{O}_{X,\xi})$ and we see that the dimension cannot be 0. $\square$

Lemma 16.5. *Let X be a separated locally Noetherian scheme. Let $U \subset X$ be an affine open. For every generic point ξ of an irreducible component of $X \setminus U$ the local ring $\mathcal{O}_{X,\xi}$ has dimension ≤ 1. If U is dense or if ξ is in the closure of U, then* $\dim(\mathcal{O}_{X,\xi}) = 1$.

Proof. This follows from Lemma 16.4 because the morphism $U \to X$ is affine by Morphisms, Lemma 11.11. $\square$

The following lemma can sometimes be used to produce effective Cartier divisors.

Lemma 16.6. *Let X be a Noetherian separated scheme. Let $U \subset X$ be a dense affine open. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an effective Cartier divisor $D \subset X$ with $U = X \setminus D$.*

Proof. Since X is Noetherian, the complement $X \setminus U$ has finitely many irreducible components $D_1, \ldots, D_r$ (Properties, Lemma 5.7 applied to the reduced induced subscheme structure on $X \setminus U$). Each $D_i \subset X$ has codimension 1 by Lemma 16.5 (and Properties, Lemma 10.3). Thus D_i is an effective Cartier divisor by Lemma 15.7. Hence we can take $D = D_1 + \ldots + D_r$. □

Lemma 16.7. *Let X be a Noetherian scheme with affine diagonal. Let $U \subset X$ be a dense affine open. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an effective Cartier divisor $D \subset X$ with $U = X \setminus D$.*

Proof. Since X is Noetherian, the complement $X \setminus U$ has finitely many irreducible components $D_1, \ldots, D_r$ (Properties, Lemma 5.7 applied to the reduced induced subscheme structure on $X \setminus U$). We view D_i as a reduced closed subscheme of X. Let $X = \bigcup_{j \in J} X_j$ be an affine open covering of X. For all j in J, set $U_j = U \cap X_j$. Since X has affine diagonal, the scheme

$$U_j = X \times_{(X \times X)} (U \times X_j)$$

is affine. Therefore, as X_j is separated, it follows from Lemma 16.6 and its proof that for all $j \in J$ and $1 \leq i \leq r$ the intersection $D_i \cap X_j$ is either empty or an effective Cartier divisor in X_j. Thus $D_i \subset X$ is an effective Cartier divisor (as this is a local property). Hence we can take $D = D_1 + \ldots + D_r$. □

Lemma 16.8. *Let X be a quasi-compact, regular scheme with affine diagonal. Then X has an ample family of invertible modules (Morphisms, Definition 12.1.*

Proof. Observe that X is a finite disjoint union of integral schemes (Properties, Lemmas 9.4 and 7.6). Thus we may assume that X is integral as well as Noetherian, regular, and having affine diagonal. Let $x \in X$. Choose an affine open neighbourhood $U \subset X$ of x. Since X is integral, U is dense in X. By More on Algebra, Lemma 121.2 the local rings of X are UFDs. Hence by Lemma 16.7 we can find an effective Cartier divisor $D \subset X$ whose complement is U. Then the canonical section $s = 1_D$ of $\mathcal{L} = \mathcal{O}_X(D)$, see Definition 14.1, vanishes exactly along D hence $U = X_s$. Thus both conditions in Morphisms, Definition 12.1 hold and we are done. □

17. Norms

Let $\pi : X \to Y$ be a finite morphism of schemes and let $d \geq 1$ be an integer. Let us say there exists a *norm of degree d for π*[2] if there exists a multiplicative map

$$\text{Norm}_\pi : \pi_*\mathcal{O}_X \to \mathcal{O}_Y$$

of sheaves such that

(1) the composition $\mathcal{O}_Y \xrightarrow{\pi^\sharp} \pi_*\mathcal{O}_X \xrightarrow{\text{Norm}_\pi} \mathcal{O}_Y$ equals $g \mapsto g^d$, and
(2) for $V \subset Y$ open if $f \in \mathcal{O}_X(\pi^{-1}V)$ is zero at $x \in \pi^{-1}(V)$, then $\text{Norm}_\pi(f)$ is zero at $\pi(x)$.

[2]This is nonstandard notation.

We observe that condition (1) forces π to be surjective. Since Norm_π is multiplicative it sends units to units hence, given $y \in Y$, if f is a regular function on X defined at but nonvanishing at any $x \in X$ with $\pi(x) = y$, then $\text{Norm}_\pi(f)$ is defined and does not vanish at y. This holds without requiring (2); in fact, the constructions in this section will only require condition (1) and only certain vanishing properties (which are used in particular in the proof of Lemma 17.4) will require property (2).

Lemma 17.1. *Let $\pi : X \to Y$ be a finite morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $y \in Y$. There exists an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{\pi^{-1}(V)}$ is trivial.*

Proof. Clearly we may assume Y and hence X affine. Since π is finite the fibre $\pi^{-1}(\{y\})$ over y is finite. Since X is affine, we can pick $s \in \Gamma(X, \mathcal{L})$ not vanishing in any point of $\pi^{-1}(\{y\})$. This follows from Properties, Lemma 29.7 but we also give a direct argument. Namely, we can pick a finite set $E \subset X$ of closed points such that every $x \in \pi^{-1}(\{y\})$ specializes to some point of E. For $x \in E$ denote $i_x : x \to X$ the closed immersion. Then $\mathcal{L} \to \bigoplus_{x \in E} i_{x,*}i_x^*\mathcal{L}$ is a surjective map of quasi-coherent $\mathcal{O}_X$-modules, and hence the map

$$\Gamma(X, \mathcal{L}) \to \bigoplus\nolimits_{x \in E} \mathcal{L}_x/\mathfrak{m}_x\mathcal{L}_x$$

is surjective (as taking global sections is an exact functor on the category of quasi-coherent $\mathcal{O}_X$-modules, see Schemes, Lemma 7.5). Thus we can find an $s \in \Gamma(X, \mathcal{L})$ not vanishing at any point specializing to a point of E. Then $X_s \subset X$ is an open neighbourhood of $\pi^{-1}(\{y\})$. Since π is finite, hence closed, we conclude that there is an open neighbourhood $V \subset Y$ of y whose inverse image is contained in X_s as desired. $\square$

Lemma 17.2. *Let $\pi : X \to Y$ be a finite morphism of schemes. If there exists a norm of degree d for π, then there exists a homomorphism of abelian groups*

$$Norm_\pi : \text{Pic}(X) \to \text{Pic}(Y)$$

such that $Norm_\pi(\pi^*\mathcal{N}) \cong \mathcal{N}^{\otimes d}$ for all invertible $\mathcal{O}_Y$-modules $\mathcal{N}$.

Proof. We will use the correspondence between isomorphism classes of invertible $\mathcal{O}_X$-modules and elements of $H^1(X, \mathcal{O}_X^*)$ given in Cohomology, Lemma 6.1 without further mention. We explain how to take the norm of an invertible $\mathcal{O}_X$-module $\mathcal{L}$. Namely, by Lemma 17.1 there exists an open covering $Y = \bigcup V_j$ such that $\mathcal{L}|_{\pi^{-1}V_j}$ is trivial. Choose a generating section $s_j \in \mathcal{L}(\pi^{-1}V_j)$ for each j. On the overlaps $\pi^{-1}V_j \cap \pi^{-1}V_{j'}$ we can write

$$s_j = u_{jj'}s_{j'}$$

for a unique $u_{jj'} \in \mathcal{O}_X^*(\pi^{-1}V_j \cap \pi^{-1}V_{j'})$. Thus we can consider the elements

$$v_{jj'} = \text{Norm}_\pi(u_{jj'}) \in \mathcal{O}_Y^*(V_j \cap V_{j'})$$

These elements satisfy the cocycle condition (because the $u_{jj'}$ do and Norm_π is multiplicative) and therefore define an invertible $\mathcal{O}_Y$-module. We omit the verification that: this is well defined, additive on Picard groups, and satisfies the property $\text{Norm}_\pi(\pi^*\mathcal{N}) \cong \mathcal{N}^{\otimes d}$ for all invertible $\mathcal{O}_Y$-modules $\mathcal{N}$. $\square$

Lemma 17.3. *Let $\pi : X \to Y$ be a finite morphism of schemes. Assume there exists a norm of degree d for π. For any $\mathcal{O}_X$-linear map $\varphi : \mathcal{L} \to \mathcal{L}'$ of invertible $\mathcal{O}_X$-modules there is an $\mathcal{O}_Y$-linear map*

$$Norm_\pi(\varphi) : Norm_\pi(\mathcal{L}) \longrightarrow Norm_\pi(\mathcal{L}')$$

with $Norm_\pi(\mathcal{L})$, $Norm_\pi(\mathcal{L}')$ as in Lemma 17.2. Moreover, for $y \in Y$ the following are equivalent

(1) *φ is zero at a point of $x \in X$ with $\pi(x) = y$, and*
(2) *$Norm_\pi(\varphi)$ is zero at y.*

Proof. We choose an open covering $Y = \bigcup V_j$ such that $\mathcal{L}$ and $\mathcal{L}'$ are trivial over the opens $\pi^{-1}V_j$. This is possible by Lemma 17.1. Choose generating sections s_j and s'_j of $\mathcal{L}$ and $\mathcal{L}'$ over the opens $\pi^{-1}V_j$. Then $\varphi(s_j) = f_j s'_j$ for some $f_j \in \mathcal{O}_X(\pi^{-1}V_j)$. Define $\text{Norm}_\pi(\varphi)$ to be multiplication by $\text{Norm}_\pi(f_j)$ on V_j. An simple calculation involving the cocycles used to construct $\text{Norm}_\pi(\mathcal{L})$, $\text{Norm}_\pi(\mathcal{L}')$ in the proof of Lemma 17.2 shows that this defines a map as stated in the lemma. The final statement follows from condition (2) in the definition of a norm map of degree d. Some details omitted. □

Lemma 17.4. *Let $\pi : X \to Y$ be a finite morphism of schemes. Assume X has an ample invertible sheaf and there exists a norm of degree d for π. Then Y has an ample invertible sheaf.*

Proof. Let $\mathcal{L}$ be the ample invertible sheaf on X given to us by assumption. We will prove that $\mathcal{N} = \text{Norm}_\pi(\mathcal{L})$ is ample on Y.

Since X is quasi-compact (Properties, Definition 26.1) and $X \to Y$ surjective (by the existence of Norm_π) we see that Y is quasi-compact. Let $y \in Y$ be a point. To finish the proof we will show that there exists a section t of some positive tensor power of $\mathcal{N}$ which does not vanish at y such that Y_t is affine. To do this, choose an affine open neighbourhood $V \subset Y$ of y. Choose $n \gg 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that

$$\pi^{-1}(\{y\}) \subset X_s \subset \pi^{-1}V$$

by Properties, Lemma 29.6. Then $t = \text{Norm}_\pi(s)$ is a section of $\mathcal{N}^{\otimes n}$ which does not vanish at x and with $Y_t \subset V$, see Lemma 17.3. Then Y_t is affine by Properties, Lemma 26.4. □

Lemma 17.5. *Let $\pi : X \to Y$ be a finite morphism of schemes. Assume X is quasi-affine and there exists a norm of degree d for π. Then Y is quasi-affine.*

Proof. By Properties, Lemma 27.1 we see that $\mathcal{O}_X$ is an ample invertible sheaf on X. The proof of Lemma 17.4 shows that $\text{Norm}_\pi(\mathcal{O}_X) = \mathcal{O}_Y$ is an ample invertible $\mathcal{O}_Y$-module. Hence Properties, Lemma 27.1 shows that Y is quasi-affine. □

Lemma 17.6. *Let $\pi : X \to Y$ be a finite locally free morphism of degree $d \geq 1$. Then there exists a canonical norm of degree d whose formation commutes with arbitrary base change.*

Proof. Let $V \subset Y$ be an affine open such that $(\pi_*\mathcal{O}_X)|_V$ is finite free of rank d. Choosing a basis we obtain an isomorphism

$$\mathcal{O}_V^{\oplus d} \cong (\pi_*\mathcal{O}_X)|_V$$

For every $f \in \pi_*\mathcal{O}_X(V) = \mathcal{O}_X(\pi^{-1}(V))$ multiplication by f defines a $\mathcal{O}_V$-linear endomorphism m_f of the displayed free vector bundle. Thus we get a $d \times d$ matrix $M_f \in \text{Mat}(d \times d, \mathcal{O}_Y(V))$ and we can set

$$\text{Norm}_\pi(f) = \det(M_f)$$

Since the determinant of a matrix is independent of the choice of the basis chosen we see that this is well defined which also means that this construction will glue to a global map as desired. Compatibility with base change is straightforward from the construction.

Property (1) follows from the fact that the determinant of a $d \times d$ diagonal matrix with entries $g, g, \ldots, g$ is g^d. To see property (2) we may base change and assume that Y is the spectrum of a field k. Then $X = \text{Spec}(A)$ with A a k-algebra with $\dim_k(A) = d$. If there exists an $x \in X$ such that $f \in A$ vanishes at x, then there exists a map $A \to \kappa$ into a field such that f maps to zero in κ. Then $f : A \to A$ cannot be surjective, hence $\det(f : A \to A) = 0$ as desired. □

Lemma 17.7. *Let $\pi : X \to Y$ be a finite surjective morphism with X and Y integral and Y normal. Then there exists a norm of degree $[R(X) : R(Y)]$ for π.*

Proof. Let $\text{Spec}(B) \subset Y$ be an affine open subset and let $\text{Spec}(A) \subset X$ be its inverse image. Then A and B are domains. Let K be the fraction field of A and L the fraction field of B. Picture:

$$\begin{tikzcd} L \arrow[r] & K \\ B \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

Since K/L is a finite extension, there is a norm map $\text{Norm}_{K/L} : K^* \to L^*$ of degree $d = [K : L]$; this is given by mapping $f \in K$ to $\det_L(f : K \to K)$ as in the proof of Lemma 17.6. Observe that the characteristic polynomial of $f : K \to K$ is a power of the minimal polynomial of f over L; in particular $\text{Norm}_{K/L}(f)$ is a power of the constant coefficient of the minimal polynomial of f over L. Hence by Algebra, Lemma 38.6 $\text{Norm}_{K/L}$ maps A into B. This determines a compatible system of maps on sections over affines and hence a global norm map Norm_π of degree d.

Property (1) is immediate from the construction. To see property (2) let $f \in A$ be contained in the prime ideal $\mathfrak{p} \subset A$. Let $f^m + b_1 f^{m-1} + \ldots + b_m$ be the minimal polynomial of f over L. By Algebra, Lemma 38.6 we have $b_i \in B$. Hence $b_0 \in B \cap \mathfrak{p}$. Since $\text{Norm}_{K/L}(f) = b_0^{d/m}$ (see above) we conclude that the norm vanishes in the image point of $\mathfrak{p}$. □

Lemma 17.8. *Let X be a Noetherian scheme. Let p be a prime number such that $p\mathcal{O}_X = 0$. Then for some $e > 0$ there exists a norm of degree p^e for $X_{red} \to X$ where X_{red} is the reduction of X.*

Proof. Let A be a Noetherian ring with $pA = 0$. Let $I \subset A$ be the ideal of nilpotent elements. Then $I^n = 0$ for some n (Algebra, Lemma 32.5). Pick e such that $p^e \geq n$. Then

$$A/I \longrightarrow A, \quad f \bmod I \longmapsto f^{p^e}$$

is well defined. This produces a norm of degree p^e for $\text{Spec}(A/I) \to \text{Spec}(A)$. Now if X is obtained by glueing some affine schemes $\text{Spec}(A_i)$ then for some $e \gg 0$ these maps glue to a norm map for $X_{red} \to X$. Details omitted. □

Proposition 17.9. *Let $\pi : X \to Y$ be a finite surjective morphism of schemes. Assume that X has an ample invertible $\mathcal{O}_X$-module. If*

(1) *π is finite locally free, or*
(2) *Y is an integral normal scheme, or*
(3) *Y is Noetherian, $p\mathcal{O}_Y = 0$, and $X = Y_{red}$,*

then Y has an ample invertible $\mathcal{O}_Y$-module.

Proof. Case (1) follows from a combination of Lemmas 17.6 and 17.4. Case (3) follows from a combination of Lemmas 17.8 and 17.4. In case (2) we first replace X by an irreducible component of X which dominates Y (viewed as a reduced closed subscheme of X). Then we can apply Lemma 17.7. □

Lemma 17.10. *Let $\pi : X \to Y$ be a finite surjective morphism of schemes. Assume that X is quasi-affine. If either*

(1) *π is finite locally free, or*
(2) *Y is an integral normal scheme*

then Y is quasi-affine.

Proof. Case (1) follows from a combination of Lemmas 17.6 and 17.5. In case (2) we first replace X by an irreducible component of X which dominates Y (viewed as a reduced closed subscheme of X). Then we can apply Lemma 17.7. □

18. Relative effective Cartier divisors

The following lemma shows that an effective Cartier divisor which is flat over the base is really a "family of effective Cartier divisors" over the base. For example the restriction to any fibre is an effective Cartier divisor.

Lemma 18.1. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a closed subscheme. Assume*

(1) *D is an effective Cartier divisor, and*
(2) *$D \to S$ is a flat morphism.*

Then for every morphism of schemes $g : S' \to S$ the pullback $(g')^{-1}D$ is an effective Cartier divisor on $X' = S' \times_S X$ where $g' : X' \to X$ is the projection.

Proof. Using Lemma 13.2 we translate this as follows into algebra. Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzerodivisor and that B/hB is flat over A. Then

$$0 \to B \xrightarrow{h} B \to B/hB \to 0$$

is a short exact sequence of A-modules with B/hB flat over A. By Algebra, Lemma 39.12 this sequence remains exact on tensoring over A with any module, in particular with any A-algebra A'. □

This lemma is the motivation for the following definition.

Definition 18.2. Let $f : X \to S$ be a morphism of schemes. A *relative effective Cartier divisor* on X/S is an effective Cartier divisor $D \subset X$ such that $D \to S$ is a flat morphism of schemes.

We warn the reader that this may be nonstandard notation. In particular, in [DG67, IV, Section 21.15] the notion of a relative divisor is discussed only when $X \to S$ is flat and locally of finite presentation. Our definition is a bit more general. However, it turns out that if $x \in D$ then $X \to S$ is flat at x in many cases (but not always).

Lemma 18.3. *Let $f : X \to S$ be a morphism of schemes. If $D_1, D_2 \subset X$ are relative effective Cartier divisor on X/S then so is $D_1 + D_2$ (Definition 13.6).*

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h_1, h_2 \in B$. Assume the h_i are nonzerodivisors and that B/h_iB is flat over A. Then h_1h_2 is a nonzerodivisor and B/h_1h_2B is flat over A. The reason is that we have a short exact sequence

$$0 \to B/h_1B \to B/h_1h_2B \to B/h_2B \to 0$$

where the first arrow is given by multiplication by h_2. Since the outer two are flat modules over A, so is the middle one, see Algebra, Lemma 39.13. □

Lemma 18.4. *Let $f : X \to S$ be a morphism of schemes. If $D_1, D_2 \subset X$ are relative effective Cartier divisor on X/S and $D_1 \subset D_2$ as closed subschemes, then the effective Cartier divisor D such that $D_2 = D_1 + D$ (Lemma 13.8) is a relative effective Cartier divisor on X/S.*

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h_1, h_2 \in B$. Assume the h_i are nonzerodivisors, that B/h_iB is flat over A, and that $(h_2) \subset (h_1)$. Then we can write $h_2 = hh_1$ where $h \in B$ is a nonzerodivisor. We get a short exact sequence

$$0 \to B/hB \to B/h_2B \to B/h_1B \to 0$$

where the first arrow is given by multiplication by h_1. Since the right two are flat modules over A, so is the middle one, see Algebra, Lemma 39.13. □

Lemma 18.5. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If $x \in D$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.*

Proof. Set $A = \mathcal{O}_{S,f(x)}$ and $B = \mathcal{O}_{X,x}$. Let $h \in B$ be an element which generates the ideal of D. Then h is a nonzerodivisor in B such that B/hB is a flat local A-algebra. Let $I \subset A$ be a finitely generated ideal. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "h"'] & B \arrow[r] & B/hB \arrow[r] & 0 \\
0 \arrow[r] & B \otimes_A I \arrow[r, "h"] \arrow[u] & B \otimes_A I \arrow[r] \arrow[u] & B/hB \otimes_A I \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The lower sequence is short exact as B/hB is flat over A, see Algebra, Lemma 39.12. The right vertical arrow is injective as B/hB is flat over A, see Algebra, Lemma 39.5. Hence multiplication by h is surjective on the kernel K of the middle vertical arrow. By Nakayama's lemma, see Algebra, Lemma 20.1 we conclude that $K = 0$. Hence B is flat over A, see Algebra, Lemma 39.5. □

The following lemma relies on the algebraic version of openness of the flat locus. The scheme theoretic version can be found in More on Morphisms, Section 15.

Lemma 18.6. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor. If f is locally of finite presentation, then there exists an open subscheme $U \subset X$ such that $D \subset U$ and such that $f|_U : U \to S$ is flat.*

Proof. Pick $x \in D$. It suffices to find an open neighbourhood $U \subset X$ of x such that $f|_U$ is flat. Hence the lemma reduces to the case that $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine and that D is given by a nonzerodivisor $h \in B$. By assumption B is a finitely presented A-algebra and B/hB is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume h is the image of $h' \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $h', g_1, \ldots, g_m$ are in A_0. Then we can set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and h_0 the image of h' in B_0. Then $B = B_0 \otimes_{A_0} A$ and $B/hB = B_0/h_0B_0 \otimes_{A_0} A$. By Algebra, Lemma 168.1 we may, after enlarging A_0, assume that B_0/h_0B_0 is flat over A_0. Let $K_0 = \mathrm{Ker}(h_0 : B_0 \to B_0)$. As B_0 is of finite type over $\mathbf{Z}$ we see that K_0 is a finitely generated ideal. Let $A_1 \subset A$ be a finite type $\mathbf{Z}$-subalgebra containing A_0 and denote B_1, h_1, K_1 the corresponding objects over A_1. By More on Algebra, Lemma 31.3 the map $K_0 \otimes_{A_0} A_1 \to K_1$ is surjective. On the other hand, the kernel of $h : B \to B$ is zero by assumption. Hence every element of K_0 maps to zero in K_1 for sufficiently large subrings $A_1 \subset A$. Since K_0 is finitely generated, we conclude that $K_1 = 0$ for a suitable choice of A_1.

Set $f_1 : X_1 \to S_1$ equal to Spec of the ring map $A_1 \to B_1$. Set $D_1 = \mathrm{Spec}(B_1/h_1B_1)$. Since $B = B_1 \otimes_{A_1} A$, i.e., $X = X_1 \times_{S_1} S$, it now suffices to prove the lemma for $X_1 \to S_1$ and the relative effective Cartier divisor D_1, see Morphisms, Lemma 25.7. Hence we have reduced to the case where A is a Noetherian ring. In this case we know that the ring map $A \to B$ is flat at every prime $\mathfrak{q}$ of $V(h)$ by Lemma 18.5. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 129.4 we win. $\square$

There is also the following lemma (whose idea is apparently due to Michael Artin, see [Nob77]) which needs no finiteness assumptions at all.

Lemma 18.7. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If f is flat at all points of $X \setminus D$, then f is flat.*

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzerodivisor, that B/hB is flat over A, and that the localization B_h is flat over A. Then B is flat over A. The reason is that we have a short exact sequence

$$0 \to B \to B_h \to \mathrm{colim}_n(1/h^n)B/B \to 0$$

and that the second and third terms are flat over A, which implies that B is flat over A (see Algebra, Lemma 39.13). Note that a filtered colimit of flat modules is flat (see Algebra, Lemma 39.3) and that by induction on n each $(1/h^n)B/B \cong B/h^nB$ is flat over A since it fits into the short exact sequence

$$0 \to B/h^{n-1}B \xrightarrow{h} B/h^nB \to B/hB \to 0$$

Some details omitted. $\square$

Example 18.8. Here is an example of a relative effective Cartier divisor D where the ambient scheme is not flat in a neighbourhood of D. Namely, let $A = k[t]$ and

$$B = k[t, x, y, x^{-1}y, x^{-2}y, \ldots]/(ty, tx^{-1}y, tx^{-2}y, \ldots)$$

Then B is not flat over A but $B/xB \cong A$ is flat over A. Moreover x is a nonzerodivisor and hence defines a relative effective Cartier divisor in $\mathrm{Spec}(B)$ over $\mathrm{Spec}(A)$.

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative effective Cartier divisor in terms of its fibres. See also More on Morphisms, Lemma 23.1 for a slightly different take on this lemma.

Lemma 18.9. *Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $Z \subset X$ be a closed subscheme. Let $x \in Z$ with image $s \in S$.*

(1) *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, then there exists an open $U \subset X$ and a relative effective Cartier divisor $D \subset U$ such that $Z \cap U \subset D$ and $Z_s \cap U = D_s$.*
(2) *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, the morphism $Z \to X$ is of finite presentation, and $Z \to S$ is flat at x, then we can choose U and D such that $Z \cap U = D$.*
(3) *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x and Z is a locally principal closed subscheme of X in a neighbourhood of x, then we can choose U and D such that $Z \cap U = D$.*

In particular, if $Z \to S$ is locally of finite presentation and flat and all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor. Similarly, if Z is a locally principal closed subscheme of X such that all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor.

Proof. Choose affine open neighbourhoods $\mathrm{Spec}(A)$ of s and $\mathrm{Spec}(B)$ of x such that $\varphi(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to Z. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $\mathrm{Spec}(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a single element which is a nonzerodivisor in $B \otimes_A \kappa(\mathfrak{p})$. Say $f \in I$ maps to this generator. We claim that after inverting an element $g \in B$, $g \notin \mathfrak{q}$ the closed subscheme $D = V(f) \subset \mathrm{Spec}(B_g)$ is a relative effective Cartier divisor.

By Algebra, Lemma 168.1 we can find a flat finite type ring map $A_0 \to B_0$ of Noetherian rings, an element $f_0 \in B_0$, a ring map $A_0 \to A$ and an isomorphism $A \otimes_{A_0} B_0 \cong B$. If $\mathfrak{p}_0 = A_0 \cap \mathfrak{p}$ then we see that

$$B \otimes_A \kappa(\mathfrak{p}) = (B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)) \otimes_{\kappa(\mathfrak{p}_0))} \kappa(\mathfrak{p})$$

hence f_0 is a nonzerodivisor in $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)$. By Algebra, Lemma 99.2 we see that f_0 is a nonzerodivisor in $(B_0)_{\mathfrak{q}_0}$ where $\mathfrak{q}_0 = B_0 \cap \mathfrak{q}$ and that $(B_0/f_0B_0)_{\mathfrak{q}_0}$ is flat over A_0. Hence by Algebra, Lemma 68.6 and Algebra, Theorem 129.4 there exists a $g_0 \in B_0$, $g_0 \notin \mathfrak{q}_0$ such that f_0 is a nonzerodivisor in $(B_0)_{g_0}$ and such that $(B_0/f_0B_0)_{g_0}$ is flat over A_0. Hence we see that $D_0 = V(f_0) \subset \mathrm{Spec}((B_0)_{g_0})$ is a relative effective Cartier divisor. Since we know that this property is preserved under base change, see Lemma 18.1, we obtain the claim mentioned above with g equal to the image of g_0 in B.

At this point we have proved (1). To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 128.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 126.6. To see (3), after possibly shrinking U we may assume that the ideal of D is generated by a single nonzerodivisor f and the ideal of Z is generated by an element g. Then $f = gh$. But $g|_{U_s}$ and $f|_{U_s}$ cut out the same effective Cartier divisor in a neighbourhood of x. Hence $h|_{X_s}$ is a unit in $\mathcal{O}_{X_s,x}$, hence h is a unit in $\mathcal{O}_{X,x}$ hence h is a unit in an open neighbourhood of x. I.e., $Z \cap U = D$ after shrinking U.

The final statements of the lemma follow immediately from parts (2) and (3), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 21.3 and 21.11. □

19. The normal cone of an immersion

Let $i : Z \to X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. Since the sheaves $\mathcal{I}^n/\mathcal{I}^{n+1}$ are each annihilated by $\mathcal{I}$ this graded algebra corresponds to a quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras by Morphisms, Lemma 4.1. This quasi-coherent graded $\mathcal{O}_Z$-algebra is called the *conormal algebra of Z in X* and is often simply denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ by the abuse of notation mentioned in Morphisms, Section 4.

Let $f : Z \to X$ be an immersion. We define the conormal algebra of f as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, where $\partial Z = \overline{Z} \setminus Z$. It is often denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 19.1. Let $f : Z \to X$ be an immersion. The *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X* or the *conormal algebra of f* is the quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ described above.

Thus $\mathcal{C}_{Z/X,1} = \mathcal{C}_{Z/X}$ is the conormal sheaf of the immersion. Also $\mathcal{C}_{Z/X,0} = \mathcal{O}_Z$ and $\mathcal{C}_{Z/X,n}$ is a quasi-coherent $\mathcal{O}_Z$-module characterized by the property

$$i_*\mathcal{C}_{Z/X,n} = \mathcal{I}^n/\mathcal{I}^{n+1} \tag{19.1.1}$$

where $i : Z \to X \setminus \partial Z$ and $\mathcal{I}$ is the ideal sheaf of i as above. Finally, note that there is a canonical surjective map

$$\text{Sym}^*(\mathcal{C}_{Z/X}) \longrightarrow \mathcal{C}_{Z/X,*} \tag{19.1.2}$$

of quasi-coherent graded $\mathcal{O}_Z$-algebras which is an isomorphism in degrees 0 and 1.

Lemma 19.2. *Let $i : Z \to X$ be an immersion. The conormal algebra of i has the following properties:*

(1) *Let $U \subset X$ be any open such that $i(Z)$ is a closed subset of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals corresponding to the closed subscheme $i(Z) \subset U$. Then*

$$\mathcal{C}_{Z/X,*} = i^*\left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n\right) = i^{-1}\left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$

(2) *For any affine open* $\mathrm{Spec}(R) = U \subset X$ *such that* $Z \cap U = \mathrm{Spec}(R/I)$ *there is a canonical isomorphism* $\Gamma(Z \cap U, \mathcal{C}_{Z/X,*}) = \bigoplus_{n \geq 0} I^n/I^{n+1}$.

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I^n/I^{n+1} = I^n \otimes_R R/I$. Details omitted. $\square$

Lemma 19.3. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram in the category of schemes. Assume i, i' immersions. There is a canonical map of graded $\mathcal{O}_Z$-algebras

$$f^*\mathcal{C}_{Z'/X',*} \longrightarrow \mathcal{C}_{Z/X,*}$$

characterized by the following property: For every pair of affine opens $(\mathrm{Spec}(R) = U \subset X, \mathrm{Spec}(R') = U' \subset X')$ *with* $f(U) \subset U'$ *such that* $Z \cap U = \mathrm{Spec}(R/I)$ *and* $Z' \cap U' = \mathrm{Spec}(R'/I')$ *the induced map*

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X',*}) = \bigoplus (I')^n/(I')^{n+1} \longrightarrow \bigoplus\nolimits_{n \geq 0} I^n/I^{n+1} = \Gamma(Z \cap U, \mathcal{C}_{Z/X,*})$$

is the one induced by the ring map $f^\sharp : R' \to R$ *which has the property* $f^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X. Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus \big(g^{-1}(\partial Z') \cup \partial Z\big)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into $\mathcal{I}$ under the canonical map $g^*\mathcal{I}' \to \mathcal{O}_X$, see Schemes, Lemmas 4.6 and 4.7. Hence we get an induced map of quasi-coherent sheaves $g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$. Pulling back by i gives $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to i^*(\mathcal{I}^n/\mathcal{I}^{n+1})$. Note that $i^*(\mathcal{I}^n/\mathcal{I}^{n+1}) = \mathcal{C}_{Z/X,n}$. On the other hand, $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*(i')^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*\mathcal{C}_{Z'/X',n}$. This gives the desired map.

Checking that the map is locally described as the given map $(I')^n/(I')^{n+1} \to I^n/I^{n+1}$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). $\square$

Lemma 19.4. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a fibre product diagram in the category of schemes with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X',*} \to \mathcal{C}_{Z/X,*}$ *of Lemma 19.3 is surjective. If g is flat, then it is an isomorphism.*

Proof. Let $R' \to R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $(I')^n/(I')^{n+1} \otimes_{R'} R \to I^n/I^{n+1}$ is surjective. If $R' \to R$ is flat, then $I^n = (I')^n \otimes_{R'} R$ and we see the map is an isomorphism. $\square$

Definition 19.5. Let $i : Z \to X$ be an immersion of schemes. The *normal cone* $C_Z X$ of Z in X is

$$C_Z X = \underline{\mathrm{Spec}}_Z(\mathcal{C}_{Z/X,*})$$

see Constructions, Definitions 7.1 and 7.2. The *normal bundle* of Z in X is the vector bundle

$$N_Z X = \underline{\mathrm{Spec}}_Z(\mathrm{Sym}(\mathcal{C}_{Z/X}))$$

see Constructions, Definitions 6.1 and 6.2.

Thus $C_Z X \to Z$ is a cone over Z and $N_Z X \to Z$ is a vector bundle over Z (recall that in our terminology this does not imply that the conormal sheaf is a finite locally free sheaf). Moreover, the canonical surjection (19.1.2) of graded algebras defines a canonical closed immersion

$$C_Z X \longrightarrow N_Z X \tag{19.5.1}$$

of cones over Z.

20. Regular ideal sheaves

In this section we generalize the notion of an effective Cartier divisor to higher codimension. Recall that a sequence of elements $f_1, \ldots, f_r$ of a ring R is a *regular sequence* if for each $i = 1, \ldots, r$ the element f_i is a nonzerodivisor on $R/(f_1, \ldots, f_{i-1})$ and $R/(f_1, \ldots, f_r) \neq 0$, see Algebra, Definition 68.1. There are three closely related weaker conditions that we can impose. The first is to assume that $f_1, \ldots, f_r$ is a *Koszul-regular sequence*, i.e., that $H_i(K_\bullet(f_1, \ldots, f_r)) = 0$ for $i > 0$, see More on Algebra, Definition 30.1. The sequence is called an *H_1-regular sequence* if $H_1(K_\bullet(f_1, \ldots, f_r)) = 0$. Another condition we can impose is that with $J = (f_1, \ldots, f_r)$, the map

$$R/J[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{n \geq 0} J^n/J^{n+1}$$

which maps T_i to $f_i \bmod J^2$ is an isomorphism. In this case we say that $f_1, \ldots, f_r$ is a *quasi-regular sequence*, see Algebra, Definition 69.1. Given an R-module M there is also a notion of M-regular and M-quasi-regular sequence.

We can generalize this to the case of ringed spaces as follows. Let X be a ringed space and let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We say that $f_1, \ldots, f_r$ is a *regular sequence* if for each $i = 1, \ldots, r$ the map

$$f_i : \mathcal{O}_X/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_X/(f_1, \ldots, f_{i-1}) \tag{20.0.1}$$

is an injective map of sheaves. We say that $f_1, \ldots, f_r$ is a *Koszul-regular sequence* if the Koszul complex

$$K_\bullet(\mathcal{O}_X, f_\bullet), \tag{20.0.2}$$

see Modules, Definition 24.2, is acyclic in degrees > 0. We say that $f_1, \ldots, f_r$ is a *H_1-regular sequence* if the Koszul complex $K_\bullet(\mathcal{O}_X, f_\bullet)$ is exact in degree 1. Finally, we say that $f_1, \ldots, f_r$ is a *quasi-regular* sequence if the map

$$\mathcal{O}_X/\mathcal{J}[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{J}^d/\mathcal{J}^{d+1} \tag{20.0.3}$$

is an isomorphism of sheaves where $\mathcal{J} \subset \mathcal{O}_X$ is the sheaf of ideals generated by $f_1, \ldots, f_r$. (There is also a notion of $\mathcal{F}$-regular and $\mathcal{F}$-quasi-regular sequence for a given $\mathcal{O}_X$-module $\mathcal{F}$ which we will introduce here if we ever need it.)

Lemma 20.1. *Let X be a ringed space. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We have the following implications $f_1, \ldots, f_r$ is a regular sequence $\Rightarrow f_1, \ldots, f_r$ is a Koszul-regular sequence $\Rightarrow f_1, \ldots, f_r$ is an H_1-regular sequence $\Rightarrow f_1, \ldots, f_r$ is a quasi-regular sequence.*

Proof. Since we may check exactness at stalks, a sequence $f_1, \ldots, f_r$ is a regular sequence if and only if the maps

$$f_i : \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1})$$

are injective for all $x \in X$. In other words, the image of the sequence $f_1, \ldots, f_r$ in the ring $\mathcal{O}_{X,x}$ is a regular sequence for all $x \in X$. The other types of regularity can be checked stalkwise as well (details omitted). Hence the implications follow from More on Algebra, Lemmas 30.2, 30.3, and 30.6. □

Definition 20.2. Let X be a ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals.

(1) We say $\mathcal{J}$ is *regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(2) We say $\mathcal{J}$ is *Koszul-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a Koszul-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(3) We say $\mathcal{J}$ is *H_1-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a H_1-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(4) We say $\mathcal{J}$ is *quasi-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.

Many properties of this notion immediately follow from the corresponding notions for regular and quasi-regular sequences in rings.

Lemma 20.3. *Let X be a ringed space. Let $\mathcal{J}$ be a sheaf of ideals. We have the following implications: $\mathcal{J}$ is regular $\Rightarrow \mathcal{J}$ is Koszul-regular $\Rightarrow \mathcal{J}$ is H_1-regular $\Rightarrow \mathcal{J}$ is quasi-regular.*

Proof. The lemma immediately reduces to Lemma 20.1. □

Lemma 20.4. *Let X be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Then $\mathcal{J}$ is quasi-regular if and only if the following conditions are satisfied:*

(1) *$\mathcal{J}$ is an $\mathcal{O}_X$-module of finite type,*
(2) *$\mathcal{J}/\mathcal{J}^2$ is a finite locally free $\mathcal{O}_X/\mathcal{J}$-module, and*
(3) *the canonical maps*

$$Sym^n_{\mathcal{O}_X/\mathcal{J}}(\mathcal{J}/\mathcal{J}^2) \longrightarrow \mathcal{J}^n/\mathcal{J}^{n+1}$$

are isomorphisms for all $n \geq 0$.

Proof. It is clear that if $U \subset X$ is an open such that $\mathcal{J}|_U$ is generated by a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ then $\mathcal{J}|_U$ is of finite type, $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free with basis $f_1, \ldots, f_r$, and the maps in (3) are isomorphisms because they are coordinate free formulation of the degree n part of (20.0.3). Hence it is clear that being quasi-regular implies conditions (1), (2), and (3).

Conversely, suppose that (1), (2), and (3) hold. Pick a point $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$. Then there exists a neighbourhood $U \subset X$ of x such that $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free of rank r over $\mathcal{O}_U/\mathcal{J}|_U$. After possibly shrinking U we may assume there exist $f_1, \ldots, f_r \in \mathcal{J}(U)$ which map to a basis of $\mathcal{J}|_U/\mathcal{J}^2|_U$ as an $\mathcal{O}_U/\mathcal{J}|_U$-module. In particular we see that the images of $f_1, \ldots, f_r$ in $\mathcal{J}_x/\mathcal{J}_x^2$ generate. Hence by Nakayama's lemma (Algebra, Lemma 20.1) we see that $f_1, \ldots, f_r$ generate the stalk $\mathcal{J}_x$. Hence, since $\mathcal{J}$ is of finite type, by Modules, Lemma 9.4 after shrinking U we may assume that $f_1, \ldots, f_r$ generate $\mathcal{J}$. Finally, from (3) and the isomorphism $\mathcal{J}|_U/\mathcal{J}^2|_U = \bigoplus \mathcal{O}_U/\mathcal{J}|_U f_i$ it is clear that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ is a quasi-regular sequence. □

Lemma 20.5. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Let $x \in X$ and $f_1, \ldots, f_r \in \mathcal{J}_x$ whose images give a basis for the $\kappa(x)$-vector space $\mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$.*

1. *If $\mathcal{J}$ is quasi-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form a quasi-regular sequence generating $\mathcal{J}|_U$.*
2. *If $\mathcal{J}$ is H_1-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an H_1-regular sequence generating $\mathcal{J}|_U$.*
3. *If $\mathcal{J}$ is Koszul-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an Koszul-regular sequence generating $\mathcal{J}|_U$.*

Proof. First assume that $\mathcal{J}$ is quasi-regular. We may choose an open neighbourhood $U \subset X$ of x and a quasi-regular sequence $g_1, \ldots, g_s \in \mathcal{O}_X(U)$ which generates $\mathcal{J}|_U$. Note that this implies that $\mathcal{J}/\mathcal{J}^2$ is free of rank s over $\mathcal{O}_U/\mathcal{J}|_U$ (see Lemma 20.4 and its proof) and hence $r = s$. We may shrink U and assume $f_1, \ldots, f_r \in \mathcal{J}(U)$. Thus we may write

$$f_i = \sum a_{ij} g_j$$

for some $a_{ij} \in \mathcal{O}_X(U)$. By assumption the matrix $A = (a_{ij})$ maps to an invertible matrix over $\kappa(x)$. Hence, after shrinking U once more, we may assume that (a_{ij}) is invertible. Thus we see that $f_1, \ldots, f_r$ give a basis for $(\mathcal{J}/\mathcal{J}^2)|_U$ which proves that $f_1, \ldots, f_r$ is a quasi-regular sequence over U.

Note that in order to prove (2) and (3) we may, because the assumptions of (2) and (3) are stronger than the assumption in (1), already assume that $f_1, \ldots, f_r \in \mathcal{J}(U)$ and $f_i = \sum a_{ij} g_j$ with (a_{ij}) invertible as above, where now $g_1, \ldots, g_r$ is a H_1-regular or Koszul-regular sequence. Since the Koszul complex on $f_1, \ldots, f_r$ is isomorphic to the Koszul complex on $g_1, \ldots, g_r$ via the matrix (a_{ij}) (see More on Algebra, Lemma 28.4) we conclude that $f_1, \ldots, f_r$ is H_1-regular or Koszul-regular as desired. □

Lemma 20.6. *Any regular, Koszul-regular, H_1-regular, or quasi-regular sheaf of ideals on a scheme is a finite type quasi-coherent sheaf of ideals.*

Proof. This follows as such a sheaf of ideals is locally generated by finitely many sections. And any sheaf of ideals locally generated by sections on a scheme is quasi-coherent, see Schemes, Lemma 10.1. □

Lemma 20.7. *Let X be a scheme. Let $\mathcal{J}$ be a sheaf of ideals. Then $\mathcal{J}$ is regular (resp. Koszul-regular, H_1-regular, quasi-regular) if and only if for every $x \in$ Supp$(\mathcal{O}_X/\mathcal{J})$ there exists an affine open neighbourhood $x \in U \subset X$, $U = \text{Spec}(A)$ such that $\mathcal{J}|_U = \widetilde{I}$ and such that I is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in A$.*

Proof. By assumption we can find an open neighbourhood U of x over which $\mathcal{J}$ is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$. After shrinking U we may assume that U is affine, say $U = \operatorname{Spec}(A)$. Since $\mathcal{J}$ is quasi-coherent by Lemma 20.6 we see that $\mathcal{J}|_U = \widetilde{I}$ for some ideal $I \subset A$. Now we can use the fact that

$$\widetilde{\ }: \mathrm{Mod}_A \longrightarrow QCoh(\mathcal{O}_U)$$

is an equivalence of categories which preserves exactness. For example the fact that the functions f_i generate $\mathcal{J}$ means that the f_i, seen as elements of A generate I. The fact that (20.0.1) is injective (resp. (20.0.2) is exact, (20.0.2) is exact in degree 1, (20.0.3) is an isomorphism) implies the corresponding property of the map $A/(f_1, \ldots, f_{i-1}) \to A/(f_1, \ldots, f_{i-1})$ (resp. the complex $K_\bullet(A, f_1, \ldots, f_r)$, the map $A/I[T_1, \ldots, T_r] \to \bigoplus I^n/I^{n+1}$). Thus $f_1, \ldots, f_r \in A$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence of the ring A. □

Lemma 20.8. *Let X be a locally Noetherian scheme. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let x be a point of the support of $\mathcal{O}_X/\mathcal{J}$. The following are equivalent*

(1) *$\mathcal{J}_x$ is generated by a regular sequence in $\mathcal{O}_{X,x}$,*
(2) *$\mathcal{J}_x$ is generated by a Koszul-regular sequence in $\mathcal{O}_{X,x}$,*
(3) *$\mathcal{J}_x$ is generated by an H_1-regular sequence in $\mathcal{O}_{X,x}$,*
(4) *$\mathcal{J}_x$ is generated by a quasi-regular sequence in $\mathcal{O}_{X,x}$,*
(5) *there exists an affine neighbourhood $U = \operatorname{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a regular sequence in A, and*
(6) *there exists an affine neighbourhood $U = \operatorname{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a Koszul-regular sequence in A, and*
(7) *there exists an affine neighbourhood $U = \operatorname{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by an H_1-regular sequence in A, and*
(8) *there exists an affine neighbourhood $U = \operatorname{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a quasi-regular sequence in A,*
(9) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is regular, and*
(10) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is Koszul-regular, and*
(11) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is H_1-regular, and*
(12) *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is quasi-regular.*

In particular, on a locally Noetherian scheme the notions of regular, Koszul-regular, H_1-regular, or quasi-regular ideal sheaf all agree.

Proof. It follows from Lemma 20.7 that (5) $\Leftrightarrow$ (9), (6) $\Leftrightarrow$ (10), (7) $\Leftrightarrow$ (11), and (8) $\Leftrightarrow$ (12). It is clear that (5) $\Rightarrow$ (1), (6) $\Rightarrow$ (2), (7) $\Rightarrow$ (3), and (8) $\Rightarrow$ (4). We have (1) $\Rightarrow$ (5) by Algebra, Lemma 68.6. We have (9) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (12) by Lemma 20.3. Finally, (4) $\Rightarrow$ (1) by Algebra, Lemma 69.6. Now all 12 statements are equivalent. □

21. Regular immersions

Let $i : Z \to X$ be an immersion of schemes. By definition this means there exists an open subscheme $U \subset X$ such that Z is identified with a closed subscheme of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the corresponding quasi-coherent sheaf of ideals. Suppose $U' \subset X$ is a second such open subscheme, and denote $\mathcal{I}' \subset \mathcal{O}_{U'}$ the corresponding quasi-coherent sheaf of ideals. Then $\mathcal{I}|_{U \cap U'} = \mathcal{I}'|_{U \cap U'}$. Moreover, the support of $\mathcal{O}_U/\mathcal{I}$

is Z which is contained in $U \cap U'$ and is also the support of $\mathcal{O}_{U'}/\mathcal{I}'$. Hence it follows from Definition 20.2 that $\mathcal{I}$ is a regular ideal if and only if $\mathcal{I}'$ is a regular ideal. Similarly for being Koszul-regular, H_1-regular, or quasi-regular.

Definition 21.1. Let $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U and denote $\mathcal{I} \subset \mathcal{O}_U$ the corresponding quasi-coherent sheaf of ideals.

(1) We say i is a *regular immersion* if $\mathcal{I}$ is regular.
(2) We say i is a *Koszul-regular immersion* if $\mathcal{I}$ is Koszul-regular.
(3) We say i is a *H_1-regular immersion* if $\mathcal{I}$ is H_1-regular.
(4) We say i is a *quasi-regular immersion* if $\mathcal{I}$ is quasi-regular.

The discussion above shows that this is independent of the choice of U. The conditions are listed in decreasing order of strength, see Lemma 21.2. A Koszul-regular closed immersion is smooth locally a regular immersion, see Lemma 21.11. In the locally Noetherian case all four notions agree, see Lemma 20.8.

Lemma 21.2. *Let $i : Z \to X$ be an immersion of schemes. We have the following implications: i is regular $\Rightarrow$ i is Koszul-regular $\Rightarrow$ i is H_1-regular $\Rightarrow$ i is quasi-regular.*

Proof. The lemma immediately reduces to Lemma 20.3. ☐

Lemma 21.3. *Let $i : Z \to X$ be an immersion of schemes. Assume X is locally Noetherian. Then i is regular $\Leftrightarrow$ i is Koszul-regular $\Leftrightarrow$ i is H_1-regular $\Leftrightarrow$ i is quasi-regular.*

Proof. Follows immediately from Lemma 21.2 and Lemma 20.8. ☐

Lemma 21.4. *Let $i : Z \to X$ be a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion. Let $X' \to X$ be a flat morphism. Then the base change $i' : Z \times_X X' \to X'$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion.*

Proof. Via Lemma 20.7 this translates into the algebraic statements in Algebra, Lemmas 68.5 and 69.3 and More on Algebra, Lemma 30.5. ☐

Lemma 21.5. *Let $i : Z \to X$ be an immersion of schemes. Then i is a quasi-regular immersion if and only if the following conditions are satisfied*

(1) *i is locally of finite presentation,*
(2) *the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and*
(3) *the map (19.1.2) is an isomorphism.*

Proof. An open immersion is locally of finite presentation. Hence we may replace X by an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U, i.e., we may assume that i is a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Recall, see Morphisms, Lemma 21.7 that $\mathcal{I}$ is of finite type if and only if i is locally of finite presentation. Hence the equivalence follows from Lemma 20.4 and unwinding the definitions. ☐

Lemma 21.6. *Let $Z \to Y \to X$ be immersions of schemes. Assume that $Z \to Y$ is H_1-regular. Then the canonical sequence of Morphisms, Lemma 31.5*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and locally split.

Proof. Since $\mathcal{C}_{Z/Y}$ is finite locally free (see Lemma 21.5 and Lemma 20.3) it suffices to prove that the sequence is exact. By what was proven in Morphisms, Lemma 31.5 it suffices to show that the first map is injective. Working affine locally this reduces to the following question: Suppose that we have a ring A and ideals $I \subset J \subset A$. Assume that $J/I \subset A/I$ is generated by an H_1-regular sequence. Does this imply that $I/I^2 \otimes_A A/J \to J/J^2$ is injective? Note that $I/I^2 \otimes_A A/J = I/IJ$. Hence we are trying to prove that $I \cap J^2 = IJ$. This is the result of More on Algebra, Lemma 30.9. □

A composition of quasi-regular immersions may not be quasi-regular, see Algebra, Remark 69.8. The other types of regular immersions are preserved under composition.

Lemma 21.7. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes.*

1. *If i and j are regular immersions, so is $j \circ i$.*
2. *If i and j are Koszul-regular immersions, so is $j \circ i$.*
3. *If i and j are H_1-regular immersions, so is $j \circ i$.*
4. *If i is an H_1-regular immersion and j is a quasi-regular immersion, then $j \circ i$ is a quasi-regular immersion.*

Proof. The algebraic version of (1) is Algebra, Lemma 68.7. The algebraic version of (2) is More on Algebra, Lemma 30.13. The algebraic version of (3) is More on Algebra, Lemma 30.11. The algebraic version of (4) is More on Algebra, Lemma 30.10. □

Lemma 21.8. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume that the sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Morphisms, Lemma 31.5 is exact and locally split.

1. *If $j \circ i$ is a quasi-regular immersion, so is i.*
2. *If $j \circ i$ is a H_1-regular immersion, so is i.*
3. *If both j and $j \circ i$ are Koszul-regular immersions, so is i.*

Proof. After shrinking Y and X we may assume that i and j are closed immersions. Denote $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of Y and $\mathcal{J} \subset \mathcal{O}_X$ the ideal sheaf of Z. The conormal sequence is $0 \to \mathcal{I}/\mathcal{I}\mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{J}/(\mathcal{I} + \mathcal{J}^2) \to 0$. Let $z \in Z$ and set $y = i(z)$, $x = j(y) = j(i(z))$. Choose $f_1, \ldots, f_n \in \mathcal{I}_x$ which map to a basis of $\mathcal{I}_x/\mathfrak{m}_z\mathcal{I}_x$. Extend this to $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathcal{J}_x$ which map to a basis of $\mathcal{J}_x/\mathfrak{m}_z\mathcal{J}_x$. This is possible as we have assumed that the sequence of conormal sheaves is split in a neighbourhood of z, hence $\mathcal{I}_x/\mathfrak{m}_x\mathcal{I}_x \to \mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$ is injective.

Proof of (1). By Lemma 20.5 we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a quasi-regular sequence generating $\mathcal{J}$. Hence by Algebra, Lemma 69.5 we see that $g_1, \ldots, g_m$ induces a quasi-regular sequence on $Y \cap U$ cutting out Z.

Proof of (2). Exactly the same as the proof of (1) except using More on Algebra, Lemma 30.12.

Proof of (3). By Lemma 20.5 (applied twice) we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n$ forms a Koszul-regular sequence generating $\mathcal{I}$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a Koszul-regular sequence generating $\mathcal{J}$. Hence by More on Algebra, Lemma 30.14 we see that $g_1, \ldots, g_m$ induces a Koszul-regular sequence on $Y \cap U$ cutting out Z. $\square$

Lemma 21.9. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Pick $z \in Z$ and denote $y \in Y$, $x \in X$ the corresponding points. Assume X is locally Noetherian. The following are equivalent*

1. *i is a regular immersion in a neighbourhood of z and j is a regular immersion in a neighbourhood of y,*
2. *i and $j \circ i$ are regular immersions in a neighbourhood of z,*
3. *$j \circ i$ is a regular immersion in a neighbourhood of z and the conormal sequence*
$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$
is split exact in a neighbourhood of z.

Proof. Since X (and hence Y) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Lemma 20.8. The implication (1) $\Rightarrow$ (2) is Lemma 21.7. The implication (2) $\Rightarrow$ (3) is Lemma 21.6. Thus it suffices to prove that (3) implies (1).

Assume (3). Set $A = \mathcal{O}_{X,x}$. Denote $I \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ and denote $J \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$. Note that any minimal sequence of elements generating J in A is a quasi-regular hence regular sequence, see Lemma 20.5. By assumption the conormal sequence
$$0 \to I/IJ \to J/J^2 \to J/(I + J^2) \to 0$$
is split exact as a sequence of A/J-modules. Hence we can pick a minimal system of generators $f_1, \ldots, f_n, g_1, \ldots, g_m$ of J with $f_1, \ldots, f_n \in I$ a minimal system of generators of I. As pointed out above $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a regular sequence in A. It follows directly from the definition of a regular sequence that $f_1, \ldots, f_n$ is a regular sequence in A and $\overline{g}_1, \ldots, \overline{g}_m$ is a regular sequence in A/I. Thus j is a regular immersion at y and i is a regular immersion at z. $\square$

Remark 21.10. In the situation of Lemma 21.9 parts (1), (2), (3) are **not** equivalent to "$j \circ i$ and j are regular immersions at z and y". An example is $X = \mathbf{A}^1_k = \text{Spec}(k[x])$, $Y = \text{Spec}(k[x]/(x^2))$ and $Z = \text{Spec}(k[x]/(x))$.

Lemma 21.11. *Let $i : Z \to X$ be a Koszul regular closed immersion. Then there exists a surjective smooth morphism $X' \to X$ such that the base change $i' : Z \times_X X' \to X'$ of i is a regular immersion.*

Proof. We may assume that X is affine and the ideal of Z generated by a Koszul-regular sequence by replacing X by the members of a suitable affine open covering (affine opens as in Lemma 20.7). The affine case is More on Algebra, Lemma 30.17. $\square$

Lemma 21.12. *Let $i : Z \to X$ be an immersion. If Z and X are regular schemes, then i is a regular immersion.*

Proof. Let $z \in Z$. By Lemma 20.8 it suffices to show that the kernel of $\mathcal{O}_{X,z} \to \mathcal{O}_{Z,z}$ is generated by a regular sequence. This follows from Algebra, Lemmas 106.4 and 106.3. $\square$

22. Relative regular immersions

In this section we consider the base change property for regular immersions. The following lemma does not hold for regular immersions or for Koszul immersions, see Examples, Lemma 14.2.

Lemma 22.1. *Let $f : X \to S$ be a morphism of schemes. Let $i : Z \subset X$ be an immersion. Assume*

1. *i is an H_1-regular (resp. quasi-regular) immersion, and*
2. *$Z \to S$ is a flat morphism.*

Then for every morphism of schemes $g : S' \to S$ the base change $Z' = S' \times_S Z \to X' = S' \times_S X$ is an H_1-regular (resp. quasi-regular) immersion.

Proof. Unwinding the definitions and using Lemma 20.7 this translates into More on Algebra, Lemma 31.4. □

This lemma is the motivation for the following definition.

Definition 22.2. Let $f : X \to S$ be a morphism of schemes. Let $i : Z \to X$ be an immersion.

1. We say i is a *relative quasi-regular immersion* if $Z \to S$ is flat and i is a quasi-regular immersion.
2. We say i is a *relative H_1-regular immersion* if $Z \to S$ is flat and i is an H_1-regular immersion.

We warn the reader that this may be nonstandard notation. Lemma 22.1 guarantees that relative quasi-regular (resp. H_1-regular) immersions are preserved under any base change. A relative H_1-regular immersion is a relative quasi-regular immersion, see Lemma 21.2. Please take a look at Lemma 22.6 (or Lemma 22.4) which shows that if $Z \to X$ is a relative H_1-regular (or quasi-regular) immersion and the ambient scheme is (flat and) locally of finite presentation over S, then $Z \to X$ is actually a regular immersion and the same remains true after any base change.

Lemma 22.3. *Let $f : X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative quasi-regular immersion. If $x \in Z$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.*

Proof. Let $f_1, \ldots, f_r \in \mathcal{O}_{X,x}$ be a quasi-regular sequence cutting out the ideal of Z at x. By Algebra, Lemma 69.6 we know that $f_1, \ldots, f_r$ is a regular sequence. Hence f_r is a nonzerodivisor on $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ such that the quotient is a flat $\mathcal{O}_{S,f(x)}$-module. By Lemma 18.5 we conclude that $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ is a flat $\mathcal{O}_{S,f(x)}$-module. Continuing by induction we find that $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module. □

Lemma 22.4. *Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be an immersion. Assume*

1. *$X \to S$ is flat and locally of finite presentation,*
2. *$Z \to X$ is a relative quasi-regular immersion.*

Then $Z \to X$ is a regular immersion and the same remains true after any base change.

Proof. Pick $x \in Z$ with image $s \in S$. To prove this it suffices to find an affine neighbourhood of x contained in U such that the result holds on that affine open. Hence we may assume that X is affine and there exist a quasi-regular sequence

$f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ such that $Z = V(f_1, \ldots, f_r)$. By More on Algebra, Lemma 31.4 the sequence $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a quasi-regular sequence in $\Gamma(X_s, \mathcal{O}_{X_s})$. Since X_s is Noetherian, this implies, possibly after shrinking X a bit, that $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a regular sequence, see Algebra, Lemmas 69.6 and 68.6. By Lemma 18.9 it follows that $Z_1 = V(f_1) \subset X$ is a relative effective Cartier divisor, again after possibly shrinking X a bit. Applying the same lemma again, but now to $Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until on reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Since being a relative effective Cartier divisor is preserved under arbitrary base change, see Lemma 18.1, we also see that the final statement of the lemma holds. □

Remark 22.5. The codimension of a relative quasi-regular immersion, if it is constant, does not change after a base change. In fact, if we have a ring map $A \to B$ and a quasi-regular sequence $f_1, \ldots, f_r \in B$ such that $B/(f_1, \ldots, f_r)$ is flat over A, then for any ring map $A \to A'$ we have a quasi-regular sequence $f_1 \otimes 1, \ldots, f_r \otimes 1$ in $B' = B \otimes_A A'$ by More on Algebra, Lemma 31.4 (which was used in the proof of Lemma 22.1 above). Now the proof of Lemma 22.4 shows that if $A \to B$ is flat and locally of finite presentation, then for every prime ideal $\mathfrak{q}' \subset B'$ the sequence $f_1 \otimes 1, \ldots, f_r \otimes 1$ is even a regular sequence in the local ring $B'_{\mathfrak{q}'}$.

Lemma 22.6. *Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative H_1-regular immersion. Assume $X \to S$ is locally of finite presentation. Then*

(1) *there exists an open subscheme $U \subset X$ such that $Z \subset U$ and such that $U \to S$ is flat, and*
(2) *$Z \to X$ is a regular immersion and the same remains true after any base change.*

Proof. Pick $x \in Z$. To prove (1) suffices to find an open neighbourhood $U \subset X$ of x such that $U \to S$ is flat. Hence the lemma reduces to the case that $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine and that Z is given by an H_1-regular sequence $f_1, \ldots, f_r \in B$. By assumption B is a finitely presented A-algebra and $B/(f_1, \ldots, f_r)B$ is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume f_i is the image of $f'_i \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $f'_1, \ldots, f'_r, g_1, \ldots, g_m$ are in A_0. We set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and we denote $f_{i,0}$ the image of f'_i in B_0. Then $B = B_0 \otimes_{A_0} A$ and

$$B/(f_1, \ldots, f_r) = B_0/(f_{0,1}, \ldots, f_{0,r}) \otimes_{A_0} A.$$

By Algebra, Lemma 168.1 we may, after enlarging A_0, assume that $B_0/(f_{0,1}, \ldots, f_{0,r})$ is flat over A_0. It may not be the case at this point that the Koszul cohomology group $H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ is zero. On the other hand, as B_0 is Noetherian, it is a finitely generated B_0-module. Let $\xi_1, \ldots, \xi_n \in H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ be generators. Let $A_0 \subset A_1 \subset A$ be a larger finite type $\mathbf{Z}$-subalgebra of A. Denote $f_{1,i}$ the image of $f_{0,i}$ in $B_1 = B_0 \otimes_{A_0} A_1$. By More on Algebra, Lemma 31.3 the map

$$H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r})) \otimes_{A_0} A_1 \longrightarrow H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$$

is surjective. Furthermore, it is clear that the colimit (over all choices of A_1 as above) of the complexes $K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})$ is the complex $K_\bullet(B, f_1, \ldots, f_r)$

which is acyclic in degree 1. Hence

$$\mathrm{colim}_{A_0 \subset A_1 \subset A} H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})) = 0$$

by Algebra, Lemma 8.8. Thus we can find a choice of A_1 such that $\xi_1, \ldots, \xi_n$ all map to zero in $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$. In other words, the Koszul cohomology group $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$ is zero.

Consider the morphism of affine schemes $X_1 \to S_1$ equal to Spec of the ring map $A_1 \to B_1$ and $Z_1 = \mathrm{Spec}(B_1/(f_{1,1}, \ldots, f_{1,r}))$. Since $B = B_1 \otimes_{A_1} A$, i.e., $X = X_1 \times_{S_1} S$, and similarly $Z = Z_1 \times_S S_1$, it now suffices to prove (1) for $X_1 \to S_1$ and the relative H_1-regular immersion $Z_1 \to X_1$, see Morphisms, Lemma 25.7. Hence we have reduced to the case where $X \to S$ is a finite type morphism of Noetherian schemes. In this case we know that $X \to S$ is flat at every point of Z by Lemma 22.3. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 129.4 we see that (1) holds. Part (2) then follows from an application of Lemma 22.4. □

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative quasi-regular immersion in terms of its fibres.

Lemma 22.7. *Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $T \subset X$ be a closed subscheme. Let $x \in T$ with image $s \in S$.*

1. *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, then there exists an open $U \subset X$ and a relative quasi-regular immersion $Z \subset U$ such that $Z_s = T_s \cap U_s$ and $T \cap U \subset Z$.*
2. *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, the morphism $T \to X$ is of finite presentation, and $T \to S$ is flat at x, then we can choose U and Z as in (1) such that $T \cap U = Z$.*
3. *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, and T is cut out by c equations in a neighbourhood of x, where $c = \dim_x(X_s) - \dim_x(T_s)$, then we can choose U and Z as in (1) such that $T \cap U = Z$.*

In each case $Z \to U$ is a regular immersion by Lemma 22.4. In particular, if $T \to S$ is locally of finite presentation and flat and all fibres $T_s \subset X_s$ are quasi-regular immersions, then $T \to X$ is a relative quasi-regular immersion.

Proof. Choose affine open neighbourhoods $\mathrm{Spec}(A)$ of s and $\mathrm{Spec}(B)$ of x such that $\varphi(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to T. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $\mathrm{Spec}(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a quasi-regular sequence of elements. After possibly localizing B at some $g \in B$, $g \notin \mathfrak{q}$ we may assume there exist $f_1, \ldots, f_r \in I$ which map to a quasi-regular sequence in $B \otimes_A \kappa(\mathfrak{p})$ which generates $I(B \otimes_A \kappa(\mathfrak{p}))$. By Algebra, Lemmas 69.6 and 68.6 we may assume after another localization that $f_1, \ldots, f_r \in I$ form a regular sequence in $B \otimes_A \kappa(\mathfrak{p})$. By Lemma 18.9 it follows that $Z_1 = V(f_1) \subset \mathrm{Spec}(B)$ is a relative effective Cartier divisor, again after possibly localizing B. Applying the same lemma again, but now to $Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until one reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Then $Z \to \mathrm{Spec}(B)$ is a regular immersion and Z is flat over S, in particular $Z \to \mathrm{Spec}(B)$ is a relative quasi-regular immersion over $\mathrm{Spec}(A)$. This proves (1).

To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 128.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 126.6.

To see (3), after possibly shrinking U we may assume that the ideal of Z is generated by a regular sequence $f_1, \ldots, f_r$ (see our construction of Z above) and the ideal of T is generated by $g_1, \ldots, g_c$. We claim that $c = r$. Namely,

$$\begin{aligned} \dim_x(X_s) &= \dim(\mathcal{O}_{X_s,x}) + \mathrm{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim_x(T_s) &= \dim(\mathcal{O}_{T_s,x}) + \mathrm{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim(\mathcal{O}_{X_s,x}) &= \dim(\mathcal{O}_{T_s,x}) + r \end{aligned}$$

the first two equalities by Algebra, Lemma 116.3 and the second by r times applying Algebra, Lemma 60.13. As $T \subset Z$ we see that $f_i = \sum b_{ij} g_j$. But the ideals of Z and T cut out the same quasi-regular closed subscheme of X_s in a neighbourhood of x. Hence the matrix (b_{ij}) mod $\mathfrak{m}_x$ is invertible (some details omitted). Hence (b_{ij}) is invertible in an open neighbourhood of x. In other words, $T \cap U = Z$ after shrinking U.

The final statements of the lemma follow immediately from part (2), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 21.3 and 21.11. $\square$

The following lemma is an enhancement of Morphisms, Lemma 34.20.

Lemma 22.8. *Let $f : X \to S$ be a smooth morphism of schemes. Let $\sigma : S \to X$ be a section of f. Then σ is a regular immersion.*

Proof. By Schemes, Lemma 21.10 the morphism σ is an immersion. After replacing X by an open neighbourhood of $\sigma(S)$ we may assume that σ is a closed immersion. Let $T = \sigma(S)$ be the corresponding closed subscheme of X. Since $T \to S$ is an isomorphism it is flat and of finite presentation. Also a smooth morphism is flat and locally of finite presentation, see Morphisms, Lemmas 34.9 and 34.8. Thus, according to Lemma 22.7, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. This follows immediately from Morphisms, Lemma 34.20 but we can also see it directly as follows. Let k be a field and let A be a smooth k-algebra. Let $\mathfrak{m} \subset A$ be a maximal ideal whose residue field is k. Then $\mathfrak{m}$ is generated by a quasi-regular sequence, possibly after replacing A by A_g for some $g \in A$, $g \notin \mathfrak{m}$. In Algebra, Lemma 140.3 we proved that $A_{\mathfrak{m}}$ is a regular local ring, hence $\mathfrak{m}A_{\mathfrak{m}}$ is generated by a regular sequence. This does indeed imply that $\mathfrak{m}$ is generated by a regular sequence (after replacing A by A_g for some $g \in A$, $g \notin \mathfrak{m}$), see Algebra, Lemma 68.6. $\square$

The following lemma has a kind of converse, see Lemma 22.12.

Lemma 22.9. *Let*

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr, "j"'] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth, and i, j immersions. If j is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, then so is i.

Proof. We can write i as the composition

$$Y \to Y \times_S X \to X$$

By Lemma 22.8 the first arrow is a regular immersion. The second arrow is a flat base change of $Y \to S$, hence is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, see Lemma 21.4. We conclude by an application of Lemma 21.7. $\square$

Lemma 22.10. *Let*

$$\begin{tikzcd} Y \arrow[rr, "i"'] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is syntomic, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. After replacing X by an open neighbourhood of $i(Y)$ we may assume that i is a closed immersion. Let $T = i(Y)$ be the corresponding closed subscheme of X. Since $T \cong Y$ the morphism $T \to S$ is flat and of finite presentation (Morphisms, Lemmas 30.6 and 30.7). Also a smooth morphism is flat and locally of finite presentation (Morphisms, Lemmas 34.9 and 34.8). Thus, according to Lemma 22.7, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. As X_s is locally of finite type over a field, it is Noetherian (Morphisms, Lemma 15.6). Thus we can check that $T_s \subset X_s$ is a quasi-regular immersion at points, see Lemma 20.8. Take $t \in T_s$. By Morphisms, Lemma 30.9 the local ring $\mathcal{O}_{T_s,t}$ is a local complete intersection over $\kappa(s)$. The local ring $\mathcal{O}_{X_s,t}$ is regular, see Algebra, Lemma 140.3. By Algebra, Lemma 135.7 we see that the kernel of the surjection $\mathcal{O}_{X_s,t} \to \mathcal{O}_{T_s,t}$ is generated by a regular sequence, which is what we had to show. $\square$

Lemma 22.11. *Let*

$$\begin{tikzcd} Y \arrow[rr, "i"'] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is smooth, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. This is a special case of Lemma 22.10 because a smooth morphism is syntomic, see Morphisms, Lemma 34.7. $\square$

Lemma 22.12. *Let*

$$\begin{tikzcd} Y \arrow[rr, "i"'] \arrow[dr, "j"'] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth and i and j immersions. If i is a Koszul-regular (resp. H_1-regular, quasi-regular) immersion, then so is j.

Proof. We will use Lemma 21.2 without further mention. Let $y \in Y$ be any point. Set $x = i(y)$ and set $s = j(y)$. It suffices to prove the result after replacing X and S by open neighbourhoods U and V of x and s and Y by an open neighbourhood of y in $i^{-1}(U) \cap j^{-1}(V)$.

We first prove the result for $X = \mathbf{A}^n_S$. After replacing S by an affine open V and replacing Y by $j^{-1}(V)$ we may assume that j is a closed immersions and S is affine. Write $S = \text{Spec}(A)$. Then $j : Y \to S$ defines an isomorphism of Y to the closed subscheme $\text{Spec}(A/I)$ for some ideal $I \subset A$. The map $i : Y = \text{Spec}(A/I) \to \mathbf{A}^n_S = \text{Spec}(A[x_1, \ldots, x_n])$ corresponds to an A-algebra homomorphism $i^\sharp : A[x_1, \ldots, x_n] \to A/I$. Choose $a_i \in A$ which map to $i^\sharp(x_i)$ in A/I. Observe that the ideal of the closed immersion i is

$$J = (x_1 - a_1, \ldots, x_n - a_n) + IA[x_1, \ldots, x_n].$$

Set $K = (x_1 - a_1, \ldots, x_n - a_n)$. We claim the sequence

$$0 \to K/KJ \to J/J^2 \to J/(K + J^2) \to 0$$

is split exact. To see this note that K/K^2 is free with basis $x_i - a_i$ over the ring $A[x_1, \ldots, x_n]/K \cong A$. Hence K/KJ is free with the same basis over the ring $A[x_1, \ldots, x_n]/J \cong A/I$. On the other hand, taking derivatives gives a map

$$\text{d}_{A[x_1,\ldots,x_n]/A} : J/J^2 \longrightarrow \Omega_{A[x_1,\ldots,x_n]/A} \otimes_{A[x_1,\ldots,x_n]} A[x_1, \ldots, x_n]/J$$

which maps the generators $x_i - a_i$ to the basis elements $\text{d}x_i$ of the free module on the right. The claim follows. Moreover, note that $x_1 - a_1, \ldots, x_n - a_n$ is a regular sequence in $A[x_1, \ldots, x_n]$ with quotient ring $A[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n) \cong A$. Thus we have a factorization

$$Y \to V(x_1 - a_1, \ldots, x_n - a_n) \to \mathbf{A}^n_S$$

of our closed immersion i where the composition is Koszul-regular (resp. H_1-regular, quasi-regular), the second arrow is a regular immersion, and the associated conormal sequence is split. Now the result follows from Lemma 21.8.

Next, we prove the result holds if i is H_1-regular or quasi-regular. Namely, shrinking as in the first paragraph of the proof, we may assume that Y, X, and S are affine. In this case we can choose a closed immersion $h : X \to \mathbf{A}^n_S$ over S for some n. Note that h is a regular immersion by Lemma 22.11. Hence $h \circ i$ is a H_1-regular or quasi-regular immersion, see Lemma 21.7 (note that this step does not work in the "quasi-regular case"). Thus we reduce to the case $X = \mathbf{A}^n_S$ and S affine we proved above.

Finally, assume i is quasi-regular. After shrinking as in the first paragraph of the proof, we may use Morphisms, Lemma 36.20 to factor f as $X \to \mathbf{A}^n_S \to S$ where the first morphism $X \to \mathbf{A}^n_S$ is étale. This reduces the problem to the the two cases (a) $X = \mathbf{A}^n_S$ and (b) f is étale. Case (a) was handled in the second paragraph of the proof. Case (b) is handled by the next paragraph.

Assume f is étale. After shrinking we may assume X, Y, and S affine i and j closed immersions (small detail omitted). Say $S = \text{Spec}(A)$, $X = \text{Spec}(B)$ and $Y = \text{Spec}(B/J) = \text{Spec}(A/I)$. Shrinking further we may assume J is generated by a quasi-regular sequence. The ring map $A \to B$ is étale, hence formally étale (Algebra, Lemma 150.2). Thus $\bigoplus I^n/I^{n+1} \cong \bigoplus J^n/J^{n+1}$ by Algebra, Lemma

150.5. Since J is generated by a quasi-regular sequence, so is I. This finishes the proof. $\square$

23. Meromorphic functions and sections

This section contains only the general definitions and some elementary results. See [Kle79] for some possible pitfalls[3].

Let $(X, \mathcal{O}_X)$ be a locally ringed space. For any open $U \subset X$ we have defined the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ of regular sections of $\mathcal{O}_X$ over U, see Definition 14.6. The restriction of a regular section to a smaller open is regular. Hence $\mathcal{S} : U \mapsto \mathcal{S}(U)$ is a subsheaf (of sets) of $\mathcal{O}_X$. We sometimes denote $\mathcal{S} = \mathcal{S}_X$ if we want to indicate the dependence on X. Moreover, $\mathcal{S}(U)$ is a multiplicative subset of the ring $\mathcal{O}_X(U)$ for each U. Hence we may consider the presheaf of rings

$$U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U),$$

see Modules, Lemma 27.1.

Definition 23.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space. The *sheaf of meromorphic functions on* X is the sheaf $\mathcal{K}_X$ associated to the presheaf displayed above. A *meromorphic function* on X is a global section of $\mathcal{K}_X$.

Since each element of each $\mathcal{S}(U)$ is a nonzerodivisor on $\mathcal{O}_X(U)$ we see that the natural map of sheaves of rings $\mathcal{O}_X \to \mathcal{K}_X$ is injective.

Example 23.2. Let $A = \mathbf{C}[x, \{y_\alpha\}_{\alpha \in \mathbf{C}}]/((x-\alpha)y_\alpha, y_\alpha y_\beta)$. Any element of A can be written uniquely as $f(x) + \sum \lambda_\alpha y_\alpha$ with $f(x) \in \mathbf{C}[x]$ and $\lambda_\alpha \in \mathbf{C}$. Let $X = \operatorname{Spec}(A)$. In this case $\mathcal{O}_X = \mathcal{K}_X$, since on any affine open $D(f)$ the ring A_f any nonzerodivisor is a unit (proof omitted).

Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Consider the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Its sheafification is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$, see Modules, Lemma 27.2.

Definition 23.3. Let X be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We denote $\mathcal{K}_X(\mathcal{F})$ the sheaf of $\mathcal{K}_X$-modules which is the sheafification of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Equivalently $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ (see above).
(2) A *meromorphic section of* $\mathcal{F}$ is a global section of $\mathcal{K}_X(\mathcal{F})$.

In particular we have

$$\mathcal{K}_X(\mathcal{F})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_{X,x} = \mathcal{S}_x^{-1}\mathcal{F}_x$$

for any point $x \in X$. However, one has to be careful since it may not be the case that $\mathcal{S}_x$ is the set of nonzerodivisors in the local ring $\mathcal{O}_{X,x}$. Namely, there is always an injective map

$$\mathcal{K}_{X,x} \longrightarrow Q(\mathcal{O}_{X,x})$$

to the total quotient ring. It is also surjective if and only if $\mathcal{S}_x$ is the set of nonzerodivisors in $\mathcal{O}_{X,x}$. The sheaves of meromorphic sections aren't quasi-coherent modules in general, but they do have some properties in common with quasi-coherent modules.

[3]Danger, Will Robinson!

Definition 23.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. We say that *pullbacks of meromorphic functions are defined for f* if for every pair of open $U \subset X$, $V \subset Y$ such that $f(U) \subset V$, and any section $s \in \Gamma(V, \mathcal{S}_Y)$ the pullback $f^\sharp(s) \in \Gamma(U, \mathcal{O}_X)$ is an element of $\Gamma(U, \mathcal{S}_X)$.

In this case there is an induced map $f^\sharp : f^{-1}\mathcal{K}_Y \to \mathcal{K}_X$, in other words we obtain a commutative diagram of morphisms of ringed spaces

$$\begin{tikzcd}
(X, \mathcal{K}_X) \arrow[r] \arrow[d, "f"] & (X, \mathcal{O}_X) \arrow[d, "f"] \\
(Y, \mathcal{K}_Y) \arrow[r] & (Y, \mathcal{O}_Y)
\end{tikzcd}$$

We sometimes denote $f^*(s) = f^\sharp(s)$ for a section $s \in \Gamma(Y, \mathcal{K}_Y)$.

Lemma 23.5. *Let $f : X \to Y$ be a morphism of schemes. In each of the following cases pullbacks of meromorphic functions are defined.*

(1) *every weakly associated point of X maps to a generic point of an irreducible component of Y,*
(2) *X, Y are integral and f is dominant,*
(3) *X is integral and the generic point of X maps to a generic point of an irreducible component of Y,*
(4) *X is reduced and every generic point of every irreducible component of X maps to the generic point of an irreducible component of Y,*
(5) *X is locally Noetherian, and any associated point of X maps to a generic point of an irreducible component of Y,*
(6) *X is locally Noetherian, has no embedded points and any generic point of an irreducible component of X maps to the generic point of an irreducible component of Y, and*
(7) *f is flat.*

Proof. The question is local on X and Y. Hence we reduce to the case where $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(R)$ and f is given by a ring map $\varphi : R \to A$. By the characterization of regular sections of the structure sheaf in Lemma 14.7 we have to show that $R \to A$ maps nonzerodivisors to nonzerodivisors. Let $t \in R$ be a nonzerodivisor.

If $R \to A$ is flat, then $t : R \to R$ being injective shows that $t : A \to A$ is injective. This proves (7).

In the other cases we note that t is not contained in any of the minimal primes of R (because every element of a minimal prime in a ring is a zerodivisor). Hence in case (1) we see that $\varphi(t)$ is not contained in any weakly associated prime of A. Thus this case follows from Algebra, Lemma 66.7. Case (5) is a special case of (1) by Lemma 5.8. Case (6) follows from (5) and the definitions. Case (4) is a special case of (1) by Lemma 5.12. Cases (2) and (3) are special cases of (4). $\square$

Lemma 23.6. *Let X be a scheme such that*

(a) *every weakly associated point of X is a generic point of an irreducible component of X, and*
(b) *any quasi-compact open has a finite number of irreducible components.*

Let X^0 be the set of generic points of irreducible components of X. Then we have

$$\mathcal{K}_X = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta} = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta}$$

where $j_\eta : \mathrm{Spec}(\mathcal{O}_{X,\eta}) \to X$ is the canonical map of Schemes, Section 13. Moreover

(1) *$\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras,*
(2) *for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf*
$$\mathcal{K}_X(\mathcal{F}) = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta$$
of meromorphic sections of $\mathcal{F}$ is quasi-coherent,
(3) *$\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors for any $x \in X$,*
(4) *$\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$ for any $x \in X$,*
(5) *$\mathcal{K}_X(U)$ equals the total quotient ring of $\mathcal{O}_X(U)$ for any affine open $U \subset X$,*
(6) *the ring of rational functions of X (Morphisms, Definition 49.3) is the ring of meromorphic functions on X, in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.*

Proof. Observe that a locally finite direct sum of sheaves of modules is equal to the product since you can check this on stalks for example. Then since $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ we see that (2) follows from the other statements. Also, observe that part (6) follows from the initial statement of the lemma and Morphisms, Lemma 49.5 when X^0 is finite; the general case of (6) follows from this by glueing (argument omitted).

Let $j : Y = \coprod_{\eta \in X^0} \mathrm{Spec}(\mathcal{O}_{X,\eta}) \to X$ be the product of the morphisms j_η. We have to show that $\mathcal{K}_X = j_*\mathcal{O}_Y$. First note that $\mathcal{K}_Y = \mathcal{O}_Y$ as Y is a disjoint union of spectra of local rings of dimension 0: in a local ring of dimension zero any nonzerodivisor is a unit. Next, note that pullbacks of meromorphic functions are defined for j by Lemma 23.5. This gives a map

$$\mathcal{K}_X \longrightarrow j_*\mathcal{O}_Y.$$

Let $\mathrm{Spec}(A) = U \subset X$ be an affine open. Then A is a ring with finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ and every weakly associated prime of A is one of the $\mathfrak{q}_i$. We obtain $Q(A) = \prod A_{\mathfrak{q}_i}$ by Algebra, Lemmas 25.4 and 66.7. In other words, already the value of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U)$ agrees with $j_*\mathcal{O}_Y(U)$ on our affine open U. Hence the displayed map is an isomorphism which proves the first displayed equality in the statement of the lemma.

Finally, we prove (1), (3), (4), and (5). Part (5) we saw during the course of the proof that $\mathcal{K}_X = j_*\mathcal{O}_Y$. The morphism j is quasi-compact by our assumption that the set of irreducible components of X is locally finite. Hence j is quasi-compact and quasi-separated (as Y is separated). By Schemes, Lemma 24.1 $j_*\mathcal{O}_Y$ is quasi-coherent. This proves (1). Let $x \in X$. We may choose an affine open neighbourhood $U = \mathrm{Spec}(A)$ of x all of whose irreducible components pass through x. Then $A \subset A_\mathfrak{p}$ because every weakly associated prime of A is contained in $\mathfrak{p}$ hence elements of $A \setminus \mathfrak{p}$ are nonzerodivisors by Algebra, Lemma 66.7. It follows easily that any nonzerodivisor of $A_\mathfrak{p}$ is the image of a nonzerodivisor on a (possibly smaller) affine open neighbourhood of x. This proves (3). Part (4) follows from part (3) by computing stalks. $\square$

Definition 23.7. Let X be a locally ringed space. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A meromorphic section s of $\mathcal{L}$ is said to be *regular* if the induced map

$\mathcal{K}_X \to \mathcal{K}_X(\mathcal{L})$ is injective. In other words, s is a regular section of the invertible $\mathcal{K}_X$-module $\mathcal{K}_X(\mathcal{L})$, see Definition 14.6.

Let us spell out when (regular) meromorphic sections can be pulled back.

Lemma 23.8. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Assume that pullbacks of meromorphic functions are defined for f (see Definition 23.4).*

1. *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical pullback map $f^* : \Gamma(Y, \mathcal{K}_Y(\mathcal{F})) \to \Gamma(X, \mathcal{K}_X(f^*\mathcal{F}))$ for meromorphic sections of $\mathcal{F}$.*
2. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A regular meromorphic section s of $\mathcal{L}$ pulls back to a regular meromorphic section f^*s of $f^*\mathcal{L}$.*

Proof. Omitted. □

Lemma 23.9. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. Let us denote $\mathcal{I} \subset \mathcal{O}_X$ the sheaf of ideals defined by the rule*

$$\mathcal{I}(V) = \{f \in \mathcal{O}_X(V) \mid fs \in \mathcal{L}(V)\}.$$

The formula makes sense since $\mathcal{L}(V) \subset \mathcal{K}_X(\mathcal{L})(V)$. Then $\mathcal{I}$ is a quasi-coherent sheaf of ideals and we have injective maps

$$1 : \mathcal{I} \longrightarrow \mathcal{O}_X, \quad s : \mathcal{I} \longrightarrow \mathcal{L}$$

whose cokernels are supported on closed nowhere dense subsets of X.

Proof. The question is local on X. Hence we may assume that $X = \mathrm{Spec}(A)$, and $\mathcal{L} = \mathcal{O}_X$. After shrinking further we may assume that $s = a/b$ with $a, b \in A$ *both* nonzerodivisors in A. Set $I = \{x \in A \mid x(a/b) \in A\}$.

To show that $\mathcal{I}$ is quasi-coherent we have to show that $I_f = \{x \in A_f \mid x(a/b) \in A_f\}$ for every $f \in A$. If $c/f^n \in A_f$, $(c/f^n)(a/b) \in A_f$, then we see that $f^m c(a/b) \in A$ for some m, hence $c/f^n \in I_f$. Conversely it is easy to see that I_f is contained in $\{x \in A_f \mid x(a/b) \in A_f\}$. This proves quasi-coherence.

Let us prove the final statement. It is clear that $(b) \subset I$. Hence $V(I) \subset V(b)$ is a nowhere dense subset as b is a nonzerodivisor. Thus the cokernel of 1 is supported in a nowhere dense closed set. The same argument works for the cokernel of s since $s(b) = (a) \subset sI \subset A$. □

Definition 23.10. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. The sheaf of ideals $\mathcal{I}$ constructed in Lemma 23.9 is called the *ideal sheaf of denominators of s*.

24. Meromorphic functions and sections; Noetherian case

For locally Noetherian schemes we can prove some results about the sheaf of meromorphic functions. However, there is an example in [Kle79] showing that $\mathcal{K}_X$ need not be quasi-coherent for a Noetherian scheme X.

Lemma 24.1. *Let X be a quasi-compact scheme. Let $h \in \Gamma(X, \mathcal{O}_X)$ and $f \in \Gamma(X, \mathcal{K}_X)$ such that f restricts to zero on X_h. Then $h^n f = 0$ for some $n \gg 0$.*

Proof. We can find a covering of X by affine opens U such that $f|_U = s^{-1}a$ with $a \in \mathcal{O}_X(U)$ and $s \in \mathcal{S}(U)$. Since X is quasi-compact we can cover it by finitely many affine opens of this form. Thus it suffices to prove the lemma when $X = \mathrm{Spec}(A)$ and $f = s^{-1}a$. Note that $s \in A$ is a nonzerodivisor hence it suffices to prove the result when $f = a$. The condition $f|_{X_h} = 0$ implies that a maps to zero in $A_h = \mathcal{O}_X(X_h)$ as $\mathcal{O}_X \subset \mathcal{K}_X$. Thus $h^n a = 0$ for some $n > 0$ as desired. □

Lemma 24.2. *Let X be a locally Noetherian scheme.*

1. *For any $x \in X$ we have $\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors, and hence $\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$.*
2. *For any affine open $U \subset X$ the ring $\mathcal{K}_X(U)$ equals the total quotient ring of $\mathcal{O}_X(U)$.*

Proof. To prove this lemma we may assume X is the spectrum of a Noetherian ring A. Say $x \in X$ corresponds to $\mathfrak{p} \subset A$.

Proof of (1). It is clear that $\mathcal{S}_x$ is contained in the set of nonzerodivisors of $\mathcal{O}_{X,x} = A_\mathfrak{p}$. For the converse, let $f, g \in A$, $g \notin \mathfrak{p}$ and assume f/g is a nonzerodivisor in $A_\mathfrak{p}$. Let $I = \{a \in A \mid af = 0\}$. Then we see that $I_\mathfrak{p} = 0$ by exactness of localization. Since A is Noetherian we see that I is finitely generated and hence that $g'I = 0$ for some $g' \in A$, $g' \notin \mathfrak{p}$. Hence f is a nonzerodivisor in $A_{g'}$, i.e., in a Zariski open neighbourhood of $\mathfrak{p}$. Thus f/g is an element of $\mathcal{S}_x$.

Proof of (2). Let $f \in \Gamma(X, \mathcal{K}_X)$ be a meromorphic function. Set $I = \{a \in A \mid af \in A\}$. Fix a prime $\mathfrak{p} \subset A$ corresponding to the point $x \in X$. By (1) we can write the image of f in the stalk at $\mathfrak{p}$ as a/b, $a, b \in A_\mathfrak{p}$ with $b \in A_\mathfrak{p}$ not a zerodivisor. Write $b = c/d$ with $c, d \in A$, $d \notin \mathfrak{p}$. Then $ad - cf$ is a section of $\mathcal{K}_X$ which vanishes in an open neighbourhood of x. Say it vanishes on $D(e)$ with $e \in A$, $e \notin \mathfrak{p}$. Then $e^n(ad - cf) = 0$ for some $n \gg 0$ by Lemma 24.1. Thus $e^n c \in I$ and $e^n c$ maps to a nonzerodivisor in $A_\mathfrak{p}$. Let $\text{Ass}(A) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ be the associated primes of A. By looking at $IA_{\mathfrak{q}_i}$ and using Algebra, Lemma 63.15 the above says that $I \not\subset \mathfrak{q}_i$ for each i. By Algebra, Lemma 15.2 there exists an element $x \in I$, $x \notin \bigcup \mathfrak{q}_i$. By Algebra, Lemma 63.9 we see that x is not a zerodivisor on A. Hence $f = (xf)/x$ is an element of the total ring of fractions of A. This proves (2). □

Lemma 24.3. *Let X be a locally Noetherian scheme having no embedded points. Let X^0 be the set of generic points of irreducible components of X. Then we have*

$$\mathcal{K}_X = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta} = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta}$$

where $j_\eta : \text{Spec}(\mathcal{O}_{X,\eta}) \to X$ is the canonical map of Schemes, Section 13. Moreover

1. *$\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras,*
2. *for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf*
$$\mathcal{K}_X(\mathcal{F}) = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta$$
of meromorphic sections of $\mathcal{F}$ is quasi-coherent, and
3. *the ring of rational functions of X is the ring of meromorphic functions on X, in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.*

Proof. This lemma is a special case of Lemma 23.6 because in the locally Noetherian case weakly associated points are the same thing as associated points by Lemma 5.8. □

Lemma 24.4. *Let X be a locally Noetherian scheme having no embedded points. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ has a regular meromorphic section.*

Proof. For each generic point η of X pick a generator s_η of the free rank 1 module $\mathcal{L}_\eta$ over the artinian local ring $\mathcal{O}_{X,\eta}$. It follows immediately from the description of $\mathcal{K}_X$ and $\mathcal{K}_X(\mathcal{L})$ in Lemma 24.3 that $s = \prod s_\eta$ is a regular meromorphic section of $\mathcal{L}$. $\square$

Lemma 24.5. *Suppose given*

1. *X a locally Noetherian scheme,*
2. *$\mathcal{L}$ an invertible $\mathcal{O}_X$-module,*
3. *s a regular meromorphic section of $\mathcal{L}$, and*
4. *$\mathcal{F}$ coherent on X without embedded associated points and Supp$(\mathcal{F}) = X$.*

Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s. Let $T \subset X$ be the union of the supports of $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{L}/s(\mathcal{I})$ which is a nowhere dense closed subset $T \subset X$ according to Lemma 23.9. Then there are canonical injective maps

$$1 : \mathcal{I}\mathcal{F} \to \mathcal{F}, \quad s : \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$$

whose cokernels are supported on T.

Proof. Reduce to the affine case with $\mathcal{L} \cong \mathcal{O}_X$, and $s = a/b$ with $a, b \in A$ both nonzerodivisors. Proof of reduction step omitted. Write $\mathcal{F} = \widetilde{M}$. Let $I = \{x \in A \mid x(a/b) \in A\}$ so that $\mathcal{I} = \widetilde{I}$ (see proof of Lemma 23.9). Note that $T = V(I) \cup V((a/b)I)$. For any A-module M consider the map $1 : IM \to M$; this is the map that gives rise to the map 1 of the lemma. Consider on the other hand the map $\sigma : IM \to M_b, x \mapsto ax/b$. Since b is not a zerodivisor in A, and since M has support Spec(A) and no embedded primes we see that b is a nonzerodivisor on M also. Hence $M \subset M_b$. By definition of I we have $\sigma(IM) \subset M$ as submodules of M_b. Hence we get an A-module map $s : IM \to M$ (namely the unique map such that $s(z)/1 = \sigma(z)$ in M_b for all $z \in IM$). It is injective because a is a nonzerodivisor also (on both A and M). It is clear that M/IM is annihilated by I and that $M/s(IM)$ is annihilated by $(a/b)I$. Thus the lemma follows. $\square$

25. Meromorphic functions and sections; reduced case

For a scheme which is reduced and which locally has finitely many irreducible components, the sheaf of meromorphic functions is quasi-coherent.

Lemma 25.1. *Let X be a reduced scheme such that any quasi-compact open has a finite number of irreducible components. Let X^0 be the set of generic points of irreducible components of X. Then we have*

$$\mathcal{K}_X = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\kappa(\eta) = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\kappa(\eta)$$

where $j_\eta : \mathrm{Spec}(\kappa(\eta)) \to X$ is the canonical map of Schemes, Section 13. Moreover

1. *$\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras,*
2. *for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf*

$$\mathcal{K}_X(\mathcal{F}) = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta$$

   of meromorphic sections of $\mathcal{F}$ is quasi-coherent,

3. *$\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors for any $x \in X$,*
4. *$\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$ for any $x \in X$,*
5. *$\mathcal{K}_X(U)$ equals the total quotient ring of $\mathcal{O}_X(U)$ for any affine open $U \subset X$,*

(6) *the ring of rational functions of X is the ring of meromorphic functions on X, in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.*

Proof. This lemma is a special case of Lemma 23.6 because on a reduced scheme the weakly associated points are the generic points by Lemma 5.12. $\square$

Lemma 25.2. *Let X be a scheme. Assume X is reduced and any quasi-compact open $U \subset X$ has a finite number of irreducible components. Then the normalization morphism $\nu : X^\nu \to X$ is the morphism*

$$\underline{\mathrm{Spec}}_X(\mathcal{O}') \longrightarrow X$$

where $\mathcal{O}' \subset \mathcal{K}_X$ is the integral closure of $\mathcal{O}_X$ in the sheaf of meromorphic functions.

Proof. Compare the definition of the normalization morphism $\nu : X^\nu \to X$ (see Morphisms, Definition 54.1) with the description of $\mathcal{K}_X$ in Lemma 25.1 above. $\square$

Lemma 25.3. *Let X be an integral scheme with generic point η. We have*

(1) *the sheaf of meromorphic functions is isomorphic to the constant sheaf with value the function field (see Morphisms, Definition 49.6) of X.*
(2) *for any quasi-coherent sheaf $\mathcal{F}$ on X the sheaf $\mathcal{K}_X(\mathcal{F})$ is isomorphic to the constant sheaf with value $\mathcal{F}_\eta$.*

Proof. Omitted. $\square$

In some cases we can show regular meromorphic sections exist.

Lemma 25.4. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. In each of the following cases $\mathcal{L}$ has a regular meromorphic section:*

(1) *X is integral,*
(2) *X is reduced and any quasi-compact open has a finite number of irreducible components,*
(3) *X is locally Noetherian and has no embedded points.*

Proof. In case (1) let $\eta \in X$ be the generic point. We have seen in Lemma 25.3 that $\mathcal{K}_X$, resp. $\mathcal{K}_X(\mathcal{L})$ is the constant sheaf with value $\kappa(\eta)$, resp. $\mathcal{L}_\eta$. Since $\dim_{\kappa(\eta)} \mathcal{L}_\eta = 1$ we can pick a nonzero element $s \in \mathcal{L}_\eta$. Clearly s is a regular meromorphic section of $\mathcal{L}$. In case (2) pick $s_\eta \in \mathcal{L}_\eta$ nonzero for all generic points η of X; this is possible as $\mathcal{L}_\eta$ is a 1-dimensional vector space over $\kappa(\eta)$. It follows immediately from the description of $\mathcal{K}_X$ and $\mathcal{K}_X(\mathcal{L})$ in Lemma 25.1 that $s = \prod s_\eta$ is a regular meromorphic section of $\mathcal{L}$. Case (3) is Lemma 24.4. $\square$

26. Weil divisors

We will introduce Weil divisors and rational equivalence of Weil divisors for locally Noetherian integral schemes. Since we are not assuming our schemes are quasi-compact we have to be a little careful when defining Weil divisors. We have to allow infinite sums of prime divisors because a rational function may have infinitely many poles for example. For quasi-compact schemes our Weil divisors are finite sums as usual. Here is a basic lemma we will often use to prove collections of closed subschemes are locally finite.

Lemma 26.1. *Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. The collection of irreducible components of Z is locally finite in X.*

Proof. Let $U \subset X$ be a quasi-compact open subscheme. Then U is a Noetherian scheme, and hence has a Noetherian underlying topological space (Properties, Lemma 5.5). Hence every subspace is Noetherian and has finitely many irreducible components (see Topology, Lemma 9.2). □

Recall that if Z is an irreducible closed subset of a scheme X, then the codimension of Z in X is equal to the dimension of the local ring $\mathcal{O}_{X,\xi}$, where $\xi \in Z$ is the generic point. See Properties, Lemma 10.3.

Definition 26.2. Let X be a locally Noetherian integral scheme.

(1) A *prime divisor* is an integral closed subscheme $Z \subset X$ of codimension 1.
(2) A *Weil divisor* is a formal sum $D = \sum n_Z Z$ where the sum is over prime divisors of X and the collection $\{Z \mid n_Z \neq 0\}$ is locally finite (Topology, Definition 28.4).

The group of all Weil divisors on X is denoted $\text{Div}(X)$.

Our next task is to define the Weil divisor associated to a rational function. In order to do this we use the order of vanishing of a rational function along a prime divisor which is defined as follows.

Definition 26.3. Let X be a locally Noetherian integral scheme. Let $f \in R(X)^*$. For every prime divisor $Z \subset X$ we define the *order of vanishing of f along Z* as the integer

$$\text{ord}_Z(f) = \text{ord}_{\mathcal{O}_{X,\xi}}(f)$$

where the right hand side is the notion of Algebra, Definition 121.2 and ξ is the generic point of Z.

Note that for $f, g \in R(X)^*$ we have

$$\text{ord}_Z(fg) = \text{ord}_Z(f) + \text{ord}_Z(g).$$

Of course it can happen that $\text{ord}_Z(f) < 0$. In this case we say that f has a *pole* along Z and that $-\text{ord}_Z(f) > 0$ is the *order of pole of f along Z*. It is important to note that the condition $\text{ord}_Z(f) \geq 0$ is **not** equivalent to the condition $f \in \mathcal{O}_{X,\xi}$ unless the local ring $\mathcal{O}_{X,\xi}$ is a discrete valuation ring.

Lemma 26.4. *Let X be a locally Noetherian integral scheme. Let $f \in R(X)^*$. Then the collections*

$$\{Z \subset X \mid Z \textit{ a prime divisor with generic point } \xi \textit{ and } f \textit{ not in } \mathcal{O}_{X,\xi}\}$$

and

$$\{Z \subset X \mid Z \textit{ a prime divisor and } \mathit{ord}_Z(f) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subscheme $U \subset X$ such that f corresponds to a section of $\Gamma(U, \mathcal{O}_X^*)$. Hence the prime divisors which can occur in the sets of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 26.1 gives the desired result. □

This lemma allows us to make the following definition.

Definition 26.5. Let X be a locally Noetherian integral scheme. Let $f \in R(X)^*$. The *principal Weil divisor associated to* f is the Weil divisor

$$\mathrm{div}(f) = \mathrm{div}_X(f) = \sum \mathrm{ord}_Z(f)[Z]$$

where the sum is over prime divisors and $\mathrm{ord}_Z(f)$ is as in Definition 26.3. This makes sense by Lemma 26.4.

Lemma 26.6. *Let X be a locally Noetherian integral scheme. Let $f, g \in R(X)^*$. Then*

$$div_X(fg) = div_X(f) + div_X(g)$$

as Weil divisors on X.

Proof. This is clear from the additivity of the ord functions. □

We see from the lemma above that the collection of principal Weil divisors form a subgroup of the group of all Weil divisors. This leads to the following definition.

Definition 26.7. Let X be a locally Noetherian integral scheme. The *Weil divisor class group* of X is the quotient of the group of Weil divisors by the subgroup of principal Weil divisors. Notation: $\mathrm{Cl}(X)$.

By construction we obtain an exact complex

$$R(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \to \mathrm{Cl}(X) \to 0 \tag{26.7.1}$$

which we can think of as a presentation of $\mathrm{Cl}(X)$. Our next task is to relate the Weil divisor class group to the Picard group.

27. The Weil divisor class associated to an invertible module

In this section we go through exactly the same progression as in Section 26 to define a canonical map $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ on a locally Noetherian integral scheme.

Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\xi \in X$ be a point. If $s_\xi, s'_\xi \in \mathcal{L}_\xi$ generate $\mathcal{L}_\xi$ as $\mathcal{O}_{X,\xi}$-module, then there exists a unit $u \in \mathcal{O}_{X,\xi}^*$ such that $s_\xi = us'_\xi$. The stalk of the sheaf of meromorphic sections $\mathcal{K}_X(\mathcal{L})$ of $\mathcal{L}$ at x is equal to $\mathcal{K}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x$. Thus the image of any meromorphic section s of $\mathcal{L}$ in the stalk at x can be written as $s = fs_\xi$ with $f \in \mathcal{K}_{X,x}$. Below we will abbreviate this by saying $f = s/s_\xi$. Also, if X is integral we have $\mathcal{K}_{X,x} = R(X)$ is equal to the function field of X, so $s/s_\xi \in R(X)$. If s is a regular meromorphic section, then actually $s/s_\xi \in R(X)^*$. On an integral scheme a regular meromorphic section is the same thing as a nonzero meromorphic section. Finally, we see that s/s_ξ is independent of the choice of s_ξ up to multiplication by a unit of the local ring $\mathcal{O}_{X,x}$. Putting everything together we see the following definition makes sense.

Definition 27.1. Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a regular meromorphic section of $\mathcal{L}$. For every prime divisor $Z \subset X$ we define the *order of vanishing of s along Z* as the integer

$$\mathrm{ord}_{Z,\mathcal{L}}(s) = \mathrm{ord}_{\mathcal{O}_{X,\xi}}(s/s_\xi)$$

where the right hand side is the notion of Algebra, Definition 121.2, $\xi \in Z$ is the generic point, and $s_\xi \in \mathcal{L}_\xi$ is a generator.

As in the case of principal divisors we have the following lemma.

Lemma 27.2. *Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \mathcal{K}_X(\mathcal{L})$ be a regular (i.e., nonzero) meromorphic section of $\mathcal{L}$. Then the sets*

$$\{Z \subset X \mid Z \text{ a prime divisor with generic point } \xi \text{ and } s \text{ not in } \mathcal{L}_\xi\}$$

and

$$\{Z \subset X \mid Z \text{ is a prime divisor and } ord_{Z,\mathcal{L}}(s) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subscheme $U \subset X$ such that s corresponds to a section of $\Gamma(U, \mathcal{L})$ which generates $\mathcal{L}$ over U. Hence the prime divisors which can occur in the sets of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 26.1. gives the desired result. $\square$

Lemma 27.3. *Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s, s' \in \mathcal{K}_X(\mathcal{L})$ be nonzero meromorphic sections of $\mathcal{L}$. Then $f = s/s'$ is an element of $R(X)^*$ and we have*

$$\sum ord_{Z,\mathcal{L}}(s)[Z] = \sum ord_{Z,\mathcal{L}}(s')[Z] + div(f)$$

as Weil divisors.

Proof. This is clear from the definitions. Note that Lemma 27.2 guarantees that the sums are indeed Weil divisors. $\square$

Definition 27.4. Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the *Weil divisor associated to s* as

$$\mathrm{div}_{\mathcal{L}}(s) = \sum \mathrm{ord}_{Z,\mathcal{L}}(s)[Z] \in \mathrm{Div}(X)$$

where the sum is over prime divisors.

(2) We define *Weil divisor class associated to* $\mathcal{L}$ as the image of $\mathrm{div}_{\mathcal{L}}(s)$ in $\mathrm{Cl}(X)$ where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Lemma 27.3.

As expected this construction is additive in the invertible module.

Lemma 27.5. *Let X be a locally Noetherian integral scheme. Let $\mathcal{L}, \mathcal{N}$ be invertible $\mathcal{O}_X$-modules. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$, and*

$$div_{\mathcal{L}\otimes\mathcal{N}}(st) = div_{\mathcal{L}}(s) + div_{\mathcal{N}}(t)$$

in $Div(X)$. In particular, the Weil divisor class of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sum of the Weil divisor classes of $\mathcal{L}$ and $\mathcal{N}$.

Proof. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$. Let $Z \subset X$ be a prime divisor. Let $\xi \in Z$ be its generic point. Choose generators $s_\xi \in \mathcal{L}_\xi$, and $t_\xi \in \mathcal{N}_\xi$. Then $s_\xi t_\xi$ is a generator for $(\mathcal{L} \otimes \mathcal{N})_\xi$. So $st/(s_\xi t_\xi) = (s/s_\xi)(t/t_\xi)$. Hence we see that

$$\mathrm{div}_{\mathcal{L}\otimes\mathcal{N},Z}(st) = \mathrm{div}_{\mathcal{L},Z}(s) + \mathrm{div}_{\mathcal{N},Z}(t)$$

by the additivity of the ord_Z function. $\square$

In this way we obtain a homomorphism of abelian groups

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X) \tag{27.5.1}$$

which assigns to an invertible module its Weil divisor class.

Lemma 27.6. *Let X be a locally Noetherian integral scheme. If X is normal, then the map (27.5.1) $\mathrm{Pic}(X) \to Cl(X)$ is injective.*

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose associated Weil divisor class is trivial. Let s be a regular meromorphic section of $\mathcal{L}$. The assumption means that $\mathrm{div}_\mathcal{L}(s) = \mathrm{div}(f)$ for some $f \in R(X)^*$. Then we see that $t = f^{-1}s$ is a regular meromorphic section of $\mathcal{L}$ with $\mathrm{div}_\mathcal{L}(t) = 0$, see Lemma 27.3. We will show that t defines a trivialization of $\mathcal{L}$ which finishes the proof of the lemma. In order to prove this we may work locally on X. Hence we may assume that $X = \mathrm{Spec}(A)$ is affine and that $\mathcal{L}$ is trivial. Then A is a Noetherian normal domain and t is an element of its fraction field such that $\mathrm{ord}_{A_\mathfrak{p}}(t) = 0$ for all height 1 primes $\mathfrak{p}$ of A. Our goal is to show that t is a unit of A. Since $A_\mathfrak{p}$ is a discrete valuation ring for height one primes of A (Algebra, Lemma 157.4), the condition signifies that $t \in A_\mathfrak{p}^*$ for all primes $\mathfrak{p}$ of height 1. This implies $t \in A$ and $t^{-1} \in A$ by Algebra, Lemma 157.6 and the proof is complete. $\square$

Lemma 27.7. *Let X be a locally Noetherian integral scheme. Consider the map (27.5.1) $\mathrm{Pic}(X) \to Cl(X)$. The following are equivalent*

1. *the local rings of X are UFDs, and*
2. *X is normal and $\mathrm{Pic}(X) \to Cl(X)$ is surjective.*

In this case $\mathrm{Pic}(X) \to Cl(X)$ is an isomorphism.

Proof. If (1) holds, then X is normal by Algebra, Lemma 120.11. Hence the map (27.5.1) is injective by Lemma 27.6. Moreover, every prime divisor $D \subset X$ is an effective Cartier divisor by Lemma 15.7. In this case the canonical section 1_D of $\mathcal{O}_X(D)$ (Definition 14.1) vanishes exactly along D and we see that the class of D is the image of $\mathcal{O}_X(D)$ under the map (27.5.1). Thus the map is surjective as well.

Assume (2) holds. Pick a prime divisor $D \subset X$. Since (27.5.1) is surjective there exists an invertible sheaf $\mathcal{L}$, a regular meromorphic section s, and $f \in R(X)^*$ such that $\mathrm{div}_\mathcal{L}(s) + \mathrm{div}(f) = [D]$. In other words, $\mathrm{div}_\mathcal{L}(fs) = [D]$. Let $x \in X$ and let $A = \mathcal{O}_{X,x}$. Thus A is a Noetherian local normal domain with fraction field $K = R(X)$. Every height 1 prime of A corresponds to a prime divisor on X and every invertible $\mathcal{O}_X$-module restricts to the trivial invertible module on $\mathrm{Spec}(A)$. It follows that for every height 1 prime $\mathfrak{p} \subset A$ there exists an element $f \in K$ such that $\mathrm{ord}_{A_\mathfrak{p}}(f) = 1$ and $\mathrm{ord}_{A_{\mathfrak{p}'}}(f) = 0$ for every other height one prime $\mathfrak{p}'$. Then $f \in A$ by Algebra, Lemma 157.6. Arguing in the same fashion we see that every element $g \in \mathfrak{p}$ is of the form $g = af$ for some $a \in A$. Thus we see that every height one prime ideal of A is principal and A is a UFD by Algebra, Lemma 120.6. $\square$

28. More on invertible modules

In this section we discuss some properties of invertible modules.

Lemma 28.1. *Let $\varphi : X \to Y$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that*

1. *X is locally Noetherian,*

(2) Y *is locally Noetherian, integral, and normal,*
(3) φ *is flat with integral (hence nonempty) fibres,*
(4) φ *is either quasi-compact or locally of finite type,*
(5) $\mathcal{L}$ *is trivial when restricted to the generic fibre of* φ.

Then $\mathcal{L} \cong \varphi^*\mathcal{N}$ *for some invertible* $\mathcal{O}_Y$*-module* $\mathcal{N}$.

Proof. Let $\xi \in Y$ be the generic point. Let X_ξ be the scheme theoretic fibre of φ over ξ. Denote $\mathcal{L}_\xi$ the pullback of $\mathcal{L}$ to X_ξ. Assumption (5) means that $\mathcal{L}_\xi$ is trivial. Choose a trivializing section $s \in \Gamma(X_\xi, \mathcal{L}_\xi)$. Observe that X is integral by Lemma 11.7. Hence we can think of s as a regular meromorphic section of $\mathcal{L}$. Pullbacks of meromorphic functions are defined for φ by Lemma 23.5. Let $\mathcal{N} \subset \mathcal{K}_Y$ be the $\mathcal{O}_Y$-module whose sections over an open $V \subset Y$ are those meromorphic functions $g \in \mathcal{K}_Y(V)$ such that $\varphi^*(g)s \in \mathcal{L}(\varphi^{-1}V)$. A priori $\varphi^*(g)s$ is a section of $\mathcal{K}_X(\mathcal{L})$ over $\varphi^{-1}V$. We claim that $\mathcal{N}$ is an invertible $\mathcal{O}_Y$-module and that the map

$$\varphi^*\mathcal{N} \longrightarrow \mathcal{L}, \quad g \longmapsto gs$$

is an isomorphism.

We first prove the claim in the following situation: X and Y are affine and $\mathcal{L}$ trivial. Say $Y = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$ and s given by the element $s \in A \otimes_R K$ where K is the fraction field of R. We can write $s = a/r$ for some nonzero $r \in R$ and $a \in A$. Since s generates $\mathcal{L}$ on the generic fibre we see that there exists an $s' \in A \otimes_R K$ such that $ss' = 1$. Thus we see that $s = r'/a'$ for some nonzero $r' \in R$ and $a' \in A$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset R$ be the minimal primes over rr'. Each $R_{\mathfrak{p}_i}$ is a discrete valuation ring (Algebra, Lemmas 60.11 and 157.4). By assumption $\mathfrak{q}_i = \mathfrak{p}_i A$ is a prime. Hence $\mathfrak{q}_i A_{\mathfrak{q}_i}$ is generated by a single element and we find that $A_{\mathfrak{q}_i}$ is a discrete valuation ring as well (Algebra, Lemma 119.7). Of course $R_{\mathfrak{p}_i} \to A_{\mathfrak{q}_i}$ has ramification index 1. Let $e_i, e_i' \geq 0$ be the valuation of a, a' in $A_{\mathfrak{q}_i}$. Then $e_i + e_i'$ is the valuation of rr' in $R_{\mathfrak{p}_i}$. Note that

$$\mathfrak{p}_1^{(e_1+e'_1)} \cap \ldots \cap \mathfrak{p}_i^{(e_n+e'_n)} = (rr')$$

in R by Algebra, Lemma 157.6. Set

$$I = \mathfrak{p}_1^{(e_1)} \cap \ldots \cap \mathfrak{p}_i^{(e_n)} \quad \text{and} \quad I' = \mathfrak{p}_1^{(e'_1)} \cap \ldots \cap \mathfrak{p}_i^{(e'_n)}$$

so that $II' \subset (rr')$. Observe that

$$IA = (\mathfrak{p}_1^{(e_1)} \cap \ldots \cap \mathfrak{p}_i^{(e_n)})A = (\mathfrak{p}_1 A)^{(e_1)} \cap \ldots \cap (\mathfrak{p}_i A)^{(e_n)}$$

by Algebra, Lemmas 64.3 and 39.2. Similarly for $I'A$. Hence $a \in IA$ and $a' \in I'A$. We conclude that $IA \otimes_A I'A \to rr'A$ is surjective. By faithful flatness of $R \to A$ we find that $I \otimes_R I' \to (rr')$ is surjective as well. It follows that $II' = (rr')$ and I and I' are finite locally free of rank 1, see Algebra, Lemma 120.16. Thus Zariski locally on R we can write $I = (g)$ and $I' = (g')$ with $gg' = rr'$. Then $a = ug$ and $a' = u'g'$ for some $u, u' \in A$. We conclude that u, u' are units. Thus Zariski locally on R we have $s = ug/r$ and the claim follows in this case.

Let $y \in Y$ be a point. Pick $x \in X$ mapping to y. We may apply the result of the previous paragraph to $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$. We conclude there exists an element $g \in R(Y)^*$ well defined up to multiplication by an element of $\mathcal{O}_{Y,y}^*$ such that $\varphi^*(g)s$ generates $\mathcal{L}_x$. Hence $\varphi^*(g)s$ generates $\mathcal{L}$ in a neighbourhood U of x. Suppose x' is a second point lying over y and $g' \in R(Y)^*$ is such that $\varphi^*(g')s$ generates $\mathcal{L}$ in an open neighbourhood U' of x'. Then we can choose a point x'' in

$U \cap U' \cap \varphi^{-1}(\{y\})$ because the fibre is irreducible. By the uniqueness for the ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x''}$ we find that g and g' differ (multiplicatively) by an element in $\mathcal{O}_{Y,y}^*$. Hence we see that $\varphi^*(g)s$ is a generator for $\mathcal{L}$ on an open neighbourhood of $\varphi^{-1}(y)$. Let $Z \subset X$ be the set of points $z \in X$ such that $\varphi^*(g)s$ does not generate $\mathcal{L}_z$. The arguments above show that Z is closed and that $Z = \varphi^{-1}(T)$ for some subset $T \subset Y$ with $y \notin T$. If we can show that T is closed, then g will be a generator for $\mathcal{N}$ as an $\mathcal{O}_Y$-module in the open neighbourhood $Y \setminus T$ of y thereby finishing the proof (some details omitted).

If φ is quasi-compact, then T is closed by Morphisms, Lemma 25.12. If φ is locally of finite type, then φ is open by Morphisms, Lemma 25.10. Then $Y \setminus T$ is open as the image of the open $X \setminus Z$. $\square$

Lemma 28.2. *Let X be a locally Noetherian scheme. Let $U \subset X$ be an open and let $D \subset U$ be an effective Cartier divisor. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an effective Cartier divisor $D' \subset X$ with $D = U \cap D'$.*

Proof. Let $D' \subset X$ be the scheme theoretic image of the morphism $D \to X$. Since X is locally Noetherian the morphism $D \to X$ is quasi-compact, see Properties, Lemma 5.3. Hence the formation of D' commutes with passing to opens in X by Morphisms, Lemma 6.3. Thus we may assume $X = \mathrm{Spec}(A)$ is affine. Let $I \subset A$ be the ideal corresponding to D'. Let $\mathfrak{p} \subset A$ be a prime ideal corresponding to a point of $X \setminus U$. To finish the proof it is enough to show that $I_\mathfrak{p}$ is generated by one element, see Lemma 15.2. Thus we may replace X by $\mathrm{Spec}(A_\mathfrak{p})$, see Morphisms, Lemma 25.16. In other words, we may assume that X is the spectrum of a local UFD A. Then all local rings of A are UFD's. It follows that $D = \sum a_i D_i$ with $D_i \subset U$ an integral effective Cartier divisor, see Lemma 15.11. The generic points ξ_i of D_i correspond to prime ideals $\mathfrak{p}_i \subset A$ of height 1, see Lemma 15.3. Then $\mathfrak{p}_i = (f_i)$ for some prime element $f_i \in A$ and we conclude that D' is cut out by $\prod f_i^{a_i}$ as desired. $\square$

Lemma 28.3. *Let X be a locally Noetherian scheme. Let $U \subset X$ be an open and let $\mathcal{L}$ be an invertible $\mathcal{O}_U$-module. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}'$ with $\mathcal{L} \cong \mathcal{L}'|_U$.*

Proof. Choose $x \in X$, $x \notin U$. We will show there exists an affine open neighbourhood $W \subset X$, such that $\mathcal{L}|_{W \cap U}$ extends to an invertible sheaf on W. This implies by glueing of sheaves (Sheaves, Section 33) that we can extend $\mathcal{L}$ to the strictly bigger open $U \cup W$. Let $W = \mathrm{Spec}(A)$ be an affine open neighbourhood. Since $U \cap W$ is quasi-affine, we see that we can write $\mathcal{L}|_{W \cap U}$ as $\mathcal{O}(D_1) \otimes \mathcal{O}(D_2)^{\otimes -1}$ for some effective Cartier divisors $D_1, D_2 \subset W \cap U$, see Lemma 15.12. Then D_1 and D_2 extend to effective Cartier divisors of W by Lemma 28.2 which gives us the extension of the invertible sheaf.

If X is Noetherian (which is the case most used in practice), the above combined with Noetherian induction finishes the proof. In the general case we argue as follows. First, because every local ring of a point outside of U is a domain and X is locally Noetherian, we see that the closure of U in X is open. Thus we may assume that $U \subset X$ is dense and schematically dense. Now we consider the set T of triples $(U', \mathcal{L}', \alpha)$ where $U \subset U' \subset X$ is an open subscheme, $\mathcal{L}'$ is an invertible $\mathcal{O}_{U'}$-module, and $\alpha : \mathcal{L}'|_U \to \mathcal{L}$ is an isomorphism. We endow T with a partial ordering $\leq$ defined by the rule $(U', \mathcal{L}', \alpha) \leq (U'', \mathcal{L}'', \alpha')$ if and only if $U' \subset U''$ and there exists an isomorphism $\beta : \mathcal{L}''|_{U'} \to \mathcal{L}'$ compatible with α and α'. Observe that β is unique (if it exists) because $U \subset X$ is dense. The first part of the proof shows that for any element $t = (U', \mathcal{L}', \alpha)$ of T with $U' \neq X$ there exists a $t' \in T$ with $t' > t$. Hence to finish the proof it suffices to show that Zorn's lemma applies. Thus consider a totally ordered subset $I \subset T$. If $i \in I$ corresponds to the triple $(U_i, \mathcal{L}_i, \alpha_i)$, then we can construct an invertible module $\mathcal{L}'$ on $U' = \bigcup U_i$ as follows. For $W \subset U'$ open and quasi-compact we see that $W \subset U_i$ for some i and we set

$$\mathcal{L}'(W) = \mathcal{L}_i(W)$$

For the transition maps we use the β's (which are unique and hence compose correctly). This defines an invertible $\mathcal{O}$-module $\mathcal{L}'$ on the basis of quasi-compact opens of U' which is sufficient to define an invertible module (Sheaves, Section 30). We omit the details. □

Lemma 28.4. *Let R be a UFD. The Picard groups of the following are trivial.*

(1) $\mathrm{Spec}(R)$ *and any open subscheme of it.*
(2) $\mathbf{A}^n_R = \mathrm{Spec}(R[x_1, \ldots, x_n])$ *and any open subscheme of it.*

In particular, the Picard group of any open subscheme of affine n-space $\mathbf{A}^n_k$ over a field k is trivial.

Proof. Since R is a UFD so is any localization of it and any polynomial ring over it (Algebra, Lemma 120.10). Thus if $U \subset \mathbf{A}^n_R$ is open, then the map $\mathrm{Pic}(\mathbf{A}^n_R) \to \mathrm{Pic}(U)$ is surjective by Lemma 28.3. The vanishing of $\mathrm{Pic}(\mathbf{A}^n_R)$ is equivalent to the vanishing of the picard group of the UFD $R[x_1, \ldots, x_n]$ which is proved in More on Algebra, Lemma 117.3. □

Lemma 28.5. *Let R be a UFD. The Picard group of $\mathbf{P}^n_R$ is $\mathbf{Z}$. More precisely, there is an isomorphism*

$$\mathbf{Z} \longrightarrow \mathrm{Pic}(\mathbf{P}^n_R), \quad m \longmapsto \mathcal{O}_{\mathbf{P}^n_R}(m)$$

In particular, the Picard group of $\mathbf{P}^n_k$ of projective space over a field k is $\mathbf{Z}$.

Proof. Observe that the local rings of $X = \mathbf{P}^n_R$ are UFDs because X is covered by affine pieces isomorphic to $\mathbf{A}^n_R$ and $R[x_1, \ldots, x_n]$ is a UFD (Algebra, Lemma 120.10). Hence X is an integral Noetherian scheme all of whose local rings are UFDs and we see that $\mathrm{Pic}(X) = \mathrm{Cl}(X)$ by Lemma 27.7.

The displayed map is a group homomorphism by Constructions, Lemma 10.3. The map is injective because H^0 of $\mathcal{O}_X$ and $\mathcal{O}_X(m)$ are non-isomorphic R-modules if $m > 0$, see Cohomology of Schemes, Lemma 8.1. Let $\mathcal{L}$ be an invertible module on X. Consider the open $U = D_+(T_0) \cong \mathbf{A}^n_R$. The complement $H = X \setminus U$ is a prime divisor because it is isomorphic to $\mathrm{Proj}(R[T_1, \ldots, T_n])$ which is integral by the discussion in the previous paragraph. In fact H is the zero scheme of the regular global section T_0 of $\mathcal{O}_X(1)$ hence $\mathcal{O}_X(1)$ maps to the class of H in $\mathrm{Cl}(X)$. By Lemma 28.4 we see that $\mathcal{L}|_U \cong \mathcal{O}_U$. Let $s \in \mathcal{L}(U)$ be a trivializing section. Then we can think of s as a regular meromorphic section of $\mathcal{L}$ and we see that necessarily $\mathrm{div}_\mathcal{L}(s) = m[H]$ for some $m \in \mathbf{Z}$ as H is the only prime divisor of X not meeting U. In other words, we see that $\mathcal{L}$ and $\mathcal{O}_X(m)$ map to the same element of $\mathrm{Cl}(X)$ and hence $\mathcal{L} \cong \mathcal{O}_X(m)$ as desired. □

29. Weil divisors on normal schemes

First we discuss properties of reflexive modules.

Lemma 29.1. *Let X be an integral locally Noetherian normal scheme. For $\mathcal{F}$ and $\mathcal{G}$ coherent reflexive $\mathcal{O}_X$-modules the map*

$$(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{G})^{**} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

is an isomorphism. The rule $\mathcal{F}, \mathcal{G} \mapsto (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{**}$ defines an abelian group law on the set of isomorphism classes of rank 1 coherent reflexive $\mathcal{O}_X$-modules.

Proof. Although not strictly necessary, we recommend reading Remark 12.9 before proceeding with the proof. Choose an open subscheme $j : U \to X$ such that every irreducible component of $X \setminus U$ has codimension ≥ 2 in X and such that $j^*\mathcal{F}$ and $j^*\mathcal{G}$ are finite locally free, see Lemma 12.13. The map

$$\mathcal{H}om_{\mathcal{O}_U}(j^*\mathcal{F}, \mathcal{O}_U) \otimes_{\mathcal{O}_U} j^*\mathcal{G} \to \mathcal{H}om_{\mathcal{O}_U}(j^*\mathcal{F}, j^*\mathcal{G})$$

is an isomorphism, because we may check it locally and it is clear when the modules are finite free. Observe that j^* applied to the displayed arrow of the lemma gives the arrow we've just shown is an isomorphism (small detail omitted). Since j^* defines an equivalence between coherent reflexive modules on U and coherent reflexive modules on X (by Lemma 12.12 and Serre's criterion Properties, Lemma 12.5), we conclude that the arrow of the lemma is an isomorphism too. If $\mathcal{F}$ has rank 1, then $j^*\mathcal{F}$ is an invertible $\mathcal{O}_U$-module and the reflexive module $\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ restricts to its inverse. It follows in the same manner as before that $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee)^{**} = \mathcal{O}_X$. In this way we see that we have inverses for the group law given in the statement of the lemma. $\square$

Lemma 29.2. *Let X be an integral locally Noetherian normal scheme. The group of rank 1 coherent reflexive $\mathcal{O}_X$-modules is isomorphic to the Weil divisor class group $Cl(X)$ of X.*

Proof. Let $\mathcal{F}$ be a rank 1 coherent reflexive $\mathcal{O}_X$-module. Choose an open $U \subset X$ such that every irreducible component of $X \setminus U$ has codimension ≥ 2 in X and such that $\mathcal{F}|_U$ is invertible, see Lemma 12.13. Observe that $\mathrm{Cl}(U) = \mathrm{Cl}(X)$ as the Weil divisor class group of X only depends on its field of rational functions and the points of codimension 1 and their local rings. Thus we can define the Weil divisor class of $\mathcal{F}$ to be the Weil divisor class of $\mathcal{F}|_U$ in $\mathrm{Cl}(U)$. We omit the verification that this is independent of the choice of U.

Denote $\mathrm{Cl}'(X)$ the set of isomorphism classes of rank 1 coherent reflexive $\mathcal{O}_X$-modules. The construction above gives a group homorphism

$$\mathrm{Cl}'(X) \longrightarrow \mathrm{Cl}(X)$$

because for any pair $\mathcal{F}, \mathcal{G}$ of elements of $\mathrm{Cl}'(X)$ we can choose a U which works for both and the assignment (27.5.1) sending an invertible module to its Weil divisor class is a homorphism. If $\mathcal{F}$ is in the kernel of this map, then we find that $\mathcal{F}|_U$ is trivial (Lemma 27.6) and hence $\mathcal{F}$ is trivial too by Lemma 12.12 and Serre's criterion Properties, Lemma 12.5. To finish the proof it suffices to check the map is surjective.

Let $D = \sum n_Z Z$ be a Weil divisor on X. We claim that there is an open $U \subset X$ such that every irreducible component of $X \setminus U$ has codimension ≥ 2 in X and such that $Z|_U$ is an effective Cartier divisor for $n_Z \neq 0$. To prove the claim we may assume X is affine. Then we may assume $D = n_1 Z_1 + \ldots + n_r Z_r$ is a finite sum with $Z_1, \ldots, Z_r$ pairwise distinct. After throwing out $Z_i \cap Z_j$ for $i \neq j$ we may assume $Z_1, \ldots, Z_r$ are pairwise disjoint. This reduces us to the case of a single prime divisor Z on X. As X is (R_1) by Properties, Lemma 12.5 the local ring $\mathcal{O}_{X,\xi}$ at the generic point ξ of Z is a discrete valuation ring. Let $f \in \mathcal{O}_{X,\xi}$ be a uniformizer. Let $V \subset X$ be an open neighbourhood of ξ such that f is the image of an element $f \in \mathcal{O}_X(V)$. After shrinking V we may assume that $Z \cap V = V(f)$ scheme theoretically, since this is true in the local ring at ξ. In this case taking

$$U = X \setminus (Z \setminus V) = (X \setminus Z) \cup V$$

gives the desired open, thereby proving the claim.

In order to show that the divisor class of D is in the image, we may write $D = \sum_{n_Z < 0} n_Z Z - \sum_{n_Z > 0} (-n_Z) Z$. By additivity of the map constructed above, we may and do assume $n_Z \leq 0$ for all prime divisors Z (this step may be avoided if the reader so desires). Let $U \subset X$ be as in the claim above. If U is quasi-compact, then we write $D|_U = -n_1 Z_1 - \ldots - n_r Z_r$ for pairwise distinct prime divisors Z_i and $n_i > 0$ and we consider the invertible $\mathcal{O}_U$-module

$$\mathcal{L} = \mathcal{I}_1^{n_1} \ldots \mathcal{I}_r^{n_r} \subset \mathcal{O}_U$$

where $\mathcal{I}_i$ is the ideal sheaf of Z_i. This is invertible by our choice of U and Lemma 13.7. Also $\text{div}_{\mathcal{L}}(1) = D|_U$. Since $\mathcal{L} = \mathcal{F}|_U$ for some rank 1 coherent reflexive $\mathcal{O}_X$-module $\mathcal{F}$ by Lemma 12.12 we find that D is in the image of our map.

If U is not quasi-compact, then we define $\mathcal{L} \subset \mathcal{O}_U$ locally by the displayed formula above. The reader shows that the construction glues and finishes the proof exactly as before. Details omitted. □

Lemma 29.3. *Let X be an integral locally Noetherian normal scheme. Let $\mathcal{F}$ be a rank 1 coherent reflexive $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{F})$. Let*

$$U = \{x \in X \mid s : \mathcal{O}_{X,x} \to \mathcal{F}_x \textit{ is an isomorphism}\}$$

Then $j : U \to X$ is an open subscheme of X and

$$j_*\mathcal{O}_U = \text{colim}(\mathcal{O}_X \xrightarrow{s} \mathcal{F} \xrightarrow{s} \mathcal{F}^{[2]} \xrightarrow{s} \mathcal{F}^{[3]} \xrightarrow{s} \ldots)$$

where $\mathcal{F}^{[1]} = \mathcal{F}$ and inductively $\mathcal{F}^{[n+1]} = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^{[n]})^{**}$.

Proof. The set U is open by Modules, Lemmas 9.4 and 12.6. Observe that j is quasi-compact by Properties, Lemma 5.3. To prove the final statement it suffices to show for every quasi-compact open $W \subset X$ there is an isomorphism

$$\text{colim}\, \Gamma(W, \mathcal{F}^{[n]}) \longrightarrow \Gamma(U \cap W, \mathcal{O}_U)$$

of $\mathcal{O}_X(W)$-modules compatible with restriction maps. We will omit the verification of compatibilities. After replacing X by W and rewriting the above in terms of homs, we see that it suffices to construct an isomorphism

$$\text{colim}\, \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}^{[n]}) \longrightarrow \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U)$$

Choose an open $V \subset X$ such that every irreducible component of $X \setminus V$ has codimension ≥ 2 in X and such that $\mathcal{F}|_V$ is invertible, see Lemma 12.13. Then restriction defines an equivalence of categories between rank 1 coherent reflexive modules on X and V and between rank 1 coherent reflexive modules on U and $V \cap U$. See Lemma

12.12 and Serre's criterion Properties, Lemma 12.5. Thus it suffices to construct an isomorphism

$$\operatorname{colim} \Gamma(V, (\mathcal{F}|_V)^{\otimes n}) \longrightarrow \Gamma(V \cap U, \mathcal{O}_U)$$

Since $\mathcal{F}|_V$ is invertible and since $U \cap V$ is equal to the set of points where $s|_V$ generates this invertible module, this is a special case of Properties, Lemma 17.2 (there is an explicit formula for the map as well). $\square$

Lemma 29.4. *Assumptions and notation as in Lemma 29.3. If s is nonzero, then every irreducible component of $X \setminus U$ has codimension 1 in X.*

Proof. Let $\xi \in X$ be a generic point of an irreducible component Z of $X \setminus U$. After replacing X by an open neighbourhood of ξ we may assume that $Z = X \setminus U$ is irreducible. Since $s : \mathcal{O}_U \to \mathcal{F}|_U$ is an isomorphism, if the codimension of Z in X is ≥ 2, then $s : \mathcal{O}_X \to \mathcal{F}$ is an isomorphism by Lemma 12.12 and Serre's criterion Properties, Lemma 12.5. This would mean that $Z = \emptyset$, a contradiction. $\square$

Remark 29.5. Let A be a Noetherian normal domain. Let M be a rank 1 finite reflexive A-module. Let $s \in M$ be nonzero. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the height 1 primes of A in the support of M/As. Then the open U of Lemma 29.3 is

$$U = \operatorname{Spec}(A) \setminus (V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_r))$$

by Lemma 29.4. Moreover, if $M^{[n]}$ denotes the reflexive hull of $M \otimes_A \ldots \otimes_A M$ (n-factors), then

$$\Gamma(U, \mathcal{O}_U) = \operatorname{colim} M^{[n]}$$

according to Lemma 29.3.

Lemma 29.6. *Assumptions and notation as in Lemma 29.3. The following are equivalent*

1. *the inclusion morphism $j : U \to X$ is affine, and*
2. *for every $x \in X \setminus U$ there is an $n > 0$ such that $s^n \in \mathfrak{m}_x \mathcal{F}_x^{[n]}$.*

Proof. Assume (1). Then for $x \in X \setminus U$ the inverse image U_x of U under the canonical morphism $f_x : \operatorname{Spec}(\mathcal{O}_{X,x}) \to X$ is affine and does not contain x. Thus $\mathfrak{m}_x \Gamma(U_x, \mathcal{O}_{U_x})$ is the unit ideal. In particular, we see that we can write

$$1 = \sum f_i g_i$$

with $f_i \in \mathfrak{m}_x$ and $g_i \in \Gamma(U_x, \mathcal{O}_{U_x})$. By Lemma 29.3 we have $\Gamma(U_x, \mathcal{O}_{U_x}) = \operatorname{colim} \mathcal{F}_x^{[n]}$ with transition maps given by multiplication by s. Hence for some $n > 0$ we have

$$s^n = \sum f_i t_i$$

for some $t_i = s^n g_i \in \mathcal{F}_x^{[n]}$. Thus (2) holds.

Conversely, assume that (2) holds. To prove j is affine is local on X, see Morphisms, Lemma 11.3. Thus we may and do assume that X is affine. Our goal is to show that U is affine. By Cohomology of Schemes, Lemma 17.8 it suffices to show that $H^p(U, \mathcal{O}_U) = 0$ for $p > 0$. Since $H^p(U, \mathcal{O}_U) = H^0(X, R^p j_* \mathcal{O}_U)$ (Cohomology of Schemes, Lemma 4.6) and since $R^p j_* \mathcal{O}_U$ is quasi-coherent (Cohomology of Schemes, Lemma 4.5) it is enough to show the stalk $(R^p j_* \mathcal{O}_U)_x$ at a point $x \in X$ is zero. Consider the base change diagram

$$\begin{tikzcd} U_x \arrow[r] \arrow[d, "j_x"'] & U \arrow[d, "j"] \\ \mathrm{Spec}(\mathcal{O}_{X,x}) \arrow[r] & X \end{tikzcd}$$

By Cohomology of Schemes, Lemma 5.2 we have $(R^p j_* \mathcal{O}_U)_x = R^p j_{x,*} \mathcal{O}_{U_x}$. Hence we may assume X is local with closed point x and we have to show U is affine (because this is equivalent to the desired vanishing by the reference given above). In particular $d = \dim(X)$ is finite (Algebra, Proposition 60.9). If $x \in U$, then $U = X$ and the result is clear. If $d = 0$ and $x \notin U$, then $U = \emptyset$ and the result is clear. Now assume $d > 0$ and $x \notin U$. Since $j_* \mathcal{O}_U = \mathrm{colim}\, \mathcal{F}^{[n]}$ our assumption means that we can write

$$1 = \sum f_i g_i$$

for some $n > 0$, $f_i \in \mathfrak{m}_x$, and $g_i \in \mathcal{O}(U)$. By induction on d we know that $D(f_i) \cap U$ is affine for all i: going through the whole argument just given with X replaced by $D(f_i)$ we end up with Noetherian local rings whose dimension is strictly smaller than d. Hence U is affine by Properties, Lemma 27.3 as desired. $\square$

30. Relative Proj

Some results on relative Proj. First some very basic results. Recall that a relative Proj is always separated over the base, see Constructions, Lemma 16.9.

Lemma 30.1. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If one of the following holds*

1. *$\mathcal{A}$ is of finite type as a sheaf of $\mathcal{A}_0$-algebras,*
2. *$\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra and $\mathcal{A}_1$ is a finite type $\mathcal{A}_0$-module,*
3. *there exists a finite type quasi-coherent $\mathcal{A}_0$-submodule $\mathcal{F} \subset \mathcal{A}_+$ such that $\mathcal{A}_+/\mathcal{F}\mathcal{A}$ is a locally nilpotent sheaf of ideals of $\mathcal{A}/\mathcal{F}\mathcal{A}$,*

then p is quasi-compact.

Proof. The question is local on the base, see Schemes, Lemma 19.2. Thus we may assume S is affine. Say $S = \mathrm{Spec}(R)$ and $\mathcal{A}$ corresponds to the graded R-algebra A. Then $X = \mathrm{Proj}(A)$, see Constructions, Section 15. In case (1) we may after possibly localizing more assume that A is generated by homogeneous elements $f_1, \ldots, f_n \in A_+$ over A_0. Then $A_+ = (f_1, \ldots, f_n)$ by Algebra, Lemma 58.1. In case (3) we see that $\mathcal{F} = \widetilde{M}$ for some finite type A_0-module $M \subset A_+$. Say $M = \sum A_0 f_i$. Say $f_i = \sum f_{i,j}$ is the decomposition into homogeneous pieces. The condition in (3) signifies that $A_+ \subset \sqrt{(f_{i,j})}$. Thus in both cases we conclude that $\mathrm{Proj}(A)$ is quasi-compact by Constructions, Lemma 8.9. Finally, (2) follows from (1). $\square$

Lemma 30.2. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}$ is of finite type as a sheaf of $\mathcal{O}_S$-algebras, then p is of finite type and $\mathcal{O}_X(d)$ is a finite type $\mathcal{O}_X$-module.*

Proof. The assumption implies that p is quasi-compact, see Lemma 30.1. Hence it suffices to show that p is locally of finite type. Thus the question is local on the base and target, see Morphisms, Lemma 15.2. Say $S = \mathrm{Spec}(R)$ and $\mathcal{A}$ corresponds to the graded R-algebra A. After further localizing on S we may assume that A is a finite type R-algebra. The scheme X is constructed out of glueing the spectra of the rings $A_{(f)}$ for $f \in A_+$ homogeneous. Each of these is of finite type over R by Algebra, Lemma 57.9 part (1). Thus $\text{Proj}(A)$ is of finite type over R. To see the statement on $\mathcal{O}_X(d)$ use part (2) of Algebra, Lemma 57.9. $\square$

Lemma 30.3. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{O}_S \to \mathcal{A}_0$ is an integral algebra map*[4] *and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra, then p is universally closed.*

Proof. The question is local on the base. Thus we may assume that $X = \text{Spec}(R)$ is affine. Let $\mathcal{A}$ be the quasi-coherent $\mathcal{O}_X$-algebra associated to the graded R-algebra A. The assumption is that $R \to A_0$ is integral and A is of finite type over A_0. Write $X \to \text{Spec}(R)$ as the composition $X \to \text{Spec}(A_0) \to \text{Spec}(R)$. Since $R \to A_0$ is an integral ring map, we see that $\text{Spec}(A_0) \to \text{Spec}(R)$ is universally closed, see Morphisms, Lemma 44.7. The quasi-compact (see Constructions, Lemma 8.9) morphism

$$X = \text{Proj}(A) \to \text{Spec}(A_0)$$

satisfies the existence part of the valuative criterion by Constructions, Lemma 8.11 and hence it is universally closed by Schemes, Proposition 20.6. Thus $X \to \text{Spec}(R)$ is universally closed as a composition of universally closed morphisms. $\square$

Lemma 30.4. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. The following conditions are equivalent*

1. *$\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra,*
2. *$\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module and $\mathcal{A}$ is of finite type as an $\mathcal{O}_S$-algebra*

If these conditions hold, then p is locally projective and in particular proper.

Proof. Assume that $\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module. Choose an affine open $U = \text{Spec}(R) \subset X$ such that $\mathcal{A}$ corresponds to a graded R-algebra A with A_0 a finite R-module. Condition (1) means that (after possibly localizing further on S) that A is a finite type A_0-algebra and condition (2) means that (after possibly localizing further on S) that A is a finite type R-algebra. Thus these conditions imply each other by Algebra, Lemma 6.2.

A locally projective morphism is proper, see Morphisms, Lemma 43.5. Thus we may now assume that $S = \text{Spec}(R)$ and $X = \text{Proj}(A)$ and that A_0 is finite over R and A of finite type over R. We will show that $X = \text{Proj}(A) \to \text{Spec}(R)$ is projective. We urge the reader to prove this for themselves, by directly constructing a closed immersion of X into a projective space over R, instead of reading the argument we give below.

By Lemma 30.2 we see that X is of finite type over $\text{Spec}(R)$. Constructions, Lemma 10.6 tells us that $\mathcal{O}_X(d)$ is ample on X for some $d \geq 1$ (see Properties, Section 26). Hence $X \to \text{Spec}(R)$ is quasi-projective (by Morphisms, Definition 40.1). By Morphisms, Lemma 43.12 we conclude that X is isomorphic to an open subscheme of a scheme projective over $\text{Spec}(R)$. Therefore, to finish the proof, it suffices to show that $X \to \text{Spec}(R)$ is universally closed (use Morphisms, Lemma 41.7). This follows from Lemma 30.3. $\square$

[4]In other words, the integral closure of $\mathcal{O}_S$ in $\mathcal{A}_0$, see Morphisms, Definition 53.2, equals $\mathcal{A}_0$.

Lemma 30.5. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$ and $\mathcal{A}_1$ is a finite type $\mathcal{O}_S$-module, then p is projective.*

Proof. Namely, the morphism associated to the graded $\mathcal{O}_S$-algebra map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion $X \to \mathbf{P}(\mathcal{A}_1)$, see Constructions, Lemma 18.5. $\square$

Lemma 30.6. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}_d$ is a flat $\mathcal{O}_S$-module for $d \gg 0$, then p is flat and $\mathcal{O}_X(d)$ is flat over S.*

Proof. Affine locally flatness of X over S reduces to the following statement: Let R be a ring, let A be a graded R-algebra with A_d flat over R for $d \gg 0$, let $f \in A_d$ for some $d > 0$, then $A_{(f)}$ is flat over R. Since $A_{(f)} = \mathrm{colim}\, A_{nd}$ where the transition maps are given by multiplication by f, this follows from Algebra, Lemma 39.3. Argue similarly to get flatness of $\mathcal{O}_X(d)$ over S. $\square$

Lemma 30.7. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}$ is a finitely presented $\mathcal{O}_S$-algebra, then p is of finite presentation and $\mathcal{O}_X(d)$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Affine locally this reduces to the following statement: Let R be a ring and let A be a finitely presented graded R-algebra. Then $\mathrm{Proj}(A) \to \mathrm{Spec}(R)$ is of finite presentation and $\mathcal{O}_{\mathrm{Proj}(A)}(d)$ is a $\mathcal{O}_{\mathrm{Proj}(A)}$-module of finite presentation. The finite presentation condition implies we can choose a presentation

$$A = R[X_1, \ldots, X_n]/(F_1, \ldots, F_m)$$

where $R[X_1, \ldots, X_n]$ is a polynomial ring graded by giving weights d_i to X_i and $F_1, \ldots, F_m$ are homogeneous polynomials of degree e_j. Let $R_0 \subset R$ be the subring generated by the coefficients of the polynomials $F_1, \ldots, F_m$. Then we set $A_0 = R_0[X_1, \ldots, X_n]/(F_1, \ldots, F_m)$. By construction $A = A_0 \otimes_{R_0} R$. Thus by Constructions, Lemma 11.6 it suffices to prove the result for $X_0 = \mathrm{Proj}(A_0)$ over R_0. By Lemma 30.2 we know X_0 is of finite type over R_0 and $\mathcal{O}_{X_0}(d)$ is a quasi-coherent $\mathcal{O}_{X_0}$-module of finite type. Since R_0 is Noetherian (as a finitely generated $\mathbf{Z}$-algebra) we see that X_0 is of finite presentation over R_0 (Morphisms, Lemma 21.9) and $\mathcal{O}_{X_0}(d)$ is of finite presentation by Cohomology of Schemes, Lemma 9.1. This finishes the proof. $\square$

31. Closed subschemes of relative proj

Some auxiliary lemmas about closed subschemes of relative proj.

Lemma 31.1. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. Denote $\mathcal{I} \subset \mathcal{A}$ the kernel of the canonical map*

$$\mathcal{A} \longrightarrow \bigoplus\nolimits_{d \geq 0} p_*\left((i_*\mathcal{O}_Z)(d)\right).$$

If p is quasi-compact, then there is an isomorphism $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{I})$.

Proof. The morphism p is separated by Constructions, Lemma 16.9. As p is quasi-compact, p_* transforms quasi-coherent modules into quasi-coherent modules, see Schemes, Lemma 24.1. Hence $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_S$-module. In particular, $\mathcal{B} = \mathcal{A}/\mathcal{I}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. The functoriality morphism $Z' = \underline{\text{Proj}}_S(\mathcal{B}) \to \underline{\text{Proj}}_S(\mathcal{A})$ is everywhere defined and a closed immersion, see Constructions, Lemma 18.3. Hence it suffices to prove $Z = Z'$ as closed subschemes of X.

Having said this, the question is local on the base and we may assume that $S = \text{Spec}(R)$ and that $X = \text{Proj}(A)$ for some graded R-algebra A. Assume $\mathcal{I} = \widetilde{I}$ for $I \subset A$ a graded ideal. By Constructions, Lemma 8.9 there exist $f_0, \ldots, f_n \in A_+$ such that $A_+ \subset \sqrt{(f_0, \ldots, f_n)}$ in other words $X = \bigcup D_+(f_i)$. Therefore, it suffices to check that $Z \cap D_+(f_i) = Z' \cap D_+(f_i)$ for each i. By renumbering we may assume $i = 0$. Say $Z \cap D_+(f_0)$, resp. $Z' \cap D_+(f_0)$ is cut out by the ideal J, resp. J' of $A_{(f_0)}$.

The inclusion $J' \subset J$. Let d be the least common multiple of $\deg(f_0), \ldots, \deg(f_n)$. Note that each of the twists $\mathcal{O}_X(nd)$ is invertible, trivialized by $f_i^{nd/\deg(f_i)}$ over $D_+(f_i)$, and that for any quasi-coherent module $\mathcal{F}$ on X the multiplication maps $\mathcal{O}_X(nd) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \to \mathcal{F}(nd + m)$ are isomorphisms, see Constructions, Lemma 10.2. Observe that J' is the ideal generated by the elements g/f_0^e where $g \in I$ is homogeneous of degree $e\deg(f_0)$ (see proof of Constructions, Lemma 11.3). Of course, by replacing g by $f_0^l g$ for suitable l we may always assume that $d|e$. Then, since g vanishes as a section of $\mathcal{O}_X(e\deg(f_0))$ restricted to Z we see that g/f_0^d is an element of J. Thus $J' \subset J$.

Conversely, suppose that $g/f_0^e \in J$. Again we may assume $d|e$. Pick $i \in \{1, \ldots, n\}$. Then $Z \cap D_+(f_i)$ is cut out by some ideal $J_i \subset A_{(f_i)}$. Moreover,

$$J \cdot A_{(f_0 f_i)} = J_i \cdot A_{(f_0 f_i)}.$$

The right hand side is the localization of J_i with respect to $f_0^{\deg(f_i)}/f_i^{\deg(f_0)}$. It follows that

$$f_0^{e_i} g / f_i^{(e_i + e)\deg(f_0)/\deg(f_i)} \in J_i$$

for some $e_i \gg 0$ sufficiently divisible. This proves that $f_0^{\max(e_i)} g$ is an element of I, because its restriction to each affine open $D_+(f_i)$ vanishes on the closed subscheme $Z \cap D_+(f_i)$. Hence $g/f_0^e \in J'$ and we conclude $J \subset J'$ as desired. $\square$

Example 31.2. Let A be a graded ring. Let $X = \text{Proj}(A)$ and $S = \text{Spec}(A_0)$. Given a graded ideal $I \subset A$ we obtain a closed subscheme $V_+(I) = \text{Proj}(A/I) \to X$ by Constructions, Lemma 11.3. Translating the result of Lemma 31.1 we see that if X is quasi-compact, then any closed subscheme Z is of the form $V_+(I(Z))$ where the graded ideal $I(Z) \subset A$ is given by the rule

$$I(Z) = \text{Ker}(A \longrightarrow \bigoplus\nolimits_{n \geq 0} \Gamma(Z, \mathcal{O}_Z(n)))$$

Then we can ask the following two natural questions:

(1) Which ideals I are of the form $I(Z)$?
(2) Can we describe the operation $I \mapsto I(V_+(I))$?

We will answer this when A is Noetherian.

First, assume that A is generated by A_1 over A_0. In this case, for any ideal $I \subset A$ the kernel of the map $A/I \to \bigoplus \Gamma(\text{Proj}(A/I), \mathcal{O})$ is the set of torsion elements of A/I, see Cohomology of Schemes, Proposition 14.4. Hence we conclude that

$$I(V_+(I)) = \{x \in A \mid A_n x \subset I \text{ for some } n \geq 0\}$$

The ideal on the right is sometimes called the saturation of I. This answers (2) and the answer to (1) is that an ideal is of the form $I(Z)$ if and only if it is saturated, i.e., equal to its own saturation.

If A is a general Noetherian graded ring, then we use Cohomology of Schemes, Proposition 15.3. Thus we see that for d equal to the lcm of the degrees of generators of A over A_0 we get

$$I(V_+(I)) = \{x \in A \mid (Ax)_{nd} \subset I \text{ for all } n \gg 0\}$$

This can be different from the saturation of I if $d \neq 1$. For example, suppose that $A = \mathbf{Q}[x, y]$ with $\deg(x) = 2$ and $\deg(y) = 3$. Then $d = 6$. Let $I = (y^2)$. Then we see $y \in I(V_+(I))$ because for any homogeneous $f \in A$ such that $6|\deg(fy)$ we have $y|f$, hence $fy \in I$. It follows that $I(V_+(I)) = (y)$ but $x^n y \notin I$ for all n hence $I(V_+(I))$ is not equal to the saturation.

Lemma 31.3. *Let R be a UFD. Let $Z \subset \mathbf{P}^n_R$ be a closed subscheme which has no embedded points such that every irreducible component of Z has codimension 1 in $\mathbf{P}^n_R$. Then the ideal $I(Z) \subset R[T_0, \ldots, T_n]$ corresponding to Z is principal.*

Proof. Observe that the local rings of $X = \mathbf{P}^n_R$ are UFDs because X is covered by affine pieces isomorphic to $\mathbf{A}^n_R$ and $R[x_1, \ldots, x_n]$ is a UFD (Algebra, Lemma 120.10). Thus Z is an effective Cartier divisor by Lemma 15.9. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to Z. Choose an isomorphism $\mathcal{O}(m) \to \mathcal{I}$ for some $m \in \mathbf{Z}$, see Lemma 28.5. Then the composition

$$\mathcal{O}_X(m) \to \mathcal{I} \to \mathcal{O}_X$$

is nonzero. We conclude that $m \leq 0$ and that the corresponding section of $\mathcal{O}_X(m)^{\otimes -1} = \mathcal{O}_X(-m)$ is given by some $F \in R[T_0, \ldots, T_n]$ of degree $-m$, see Cohomology of Schemes, Lemma 8.1. Thus on the ith standard open $U_i = D_+(T_i)$ the closed subscheme $Z \cap U_i$ is cut out by the ideal

$$(F(T_0/T_i, \ldots, T_n/T_i)) \subset R[T_0/T_i, \ldots, T_n/T_i]$$

Thus the homogeneous elements of the graded ideal $I(Z) = \text{Ker}(R[T_0, \ldots, T_n] \to \bigoplus \Gamma(\mathcal{O}_Z(m)))$ is the set of homogeneous polynomials G such that

$$G(T_0/T_i, \ldots, T_n/T_i) \in (F(T_0/T_i, \ldots, T_n/T_i))$$

for $i = 0, \ldots, n$. Clearing denominators, we see there exist $e_i \geq 0$ such that

$$T_i^{e_i} G \in (F)$$

for $i = 0, \ldots, n$. As R is a UFD, so is $R[T_0, \ldots, T_n]$. Then $F|T_0^{e_0}G$ and $F|T_1^{e_1}G$ implies $F|G$ as $T_0^{e_0}$ and $T_1^{e_1}$ have no factor in common. Thus $I(Z) = (F)$. $\square$

In case the closed subscheme is locally cut out by finitely many equations we can define it by a finite type ideal sheaf of $\mathcal{A}$.

Lemma 31.4. *Let S be a quasi-compact and quasi-separated scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. If p is quasi-compact and i of finite presentation, then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{A}_d$ such that $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{F}\mathcal{A})$.*

Proof. By Lemma 31.1 we know there exists a quasi-coherent graded sheaf of ideals $\mathcal{I} \subset \mathcal{A}$ such that $Z = \underline{\text{Proj}}(\mathcal{A}/\mathcal{I})$. Since S is quasi-compact we can choose a finite affine open covering $S = U_1 \cup \ldots \cup U_n$. Say $U_i = \text{Spec}(R_i)$. Let $\mathcal{A}|_{U_i}$ correspond to the graded R_i-algebra A_i and $\mathcal{I}|_{U_i}$ to the graded ideal $I_i \subset A_i$. Note that $p^{-1}(U_i) = \text{Proj}(A_i)$ as schemes over R_i. Since p is quasi-compact we can choose finitely many homogeneous elements $f_{i,j} \in A_{i,+}$ such that $p^{-1}(U_i) = D_+(f_{i,j})$. The condition on $Z \to X$ means that the ideal sheaf of Z in $\mathcal{O}_X$ is of finite type, see Morphisms, Lemma 21.7. Hence we can find finitely many homogeneous elements $h_{i,j,k} \in I_i \cap A_{i,+}$ such that the ideal of $Z \cap D_+(f_{i,j})$ is generated by the elements $h_{i,j,k}/f_{i,j}^{e_{i,j,k}}$. Choose $d > 0$ to be a common multiple of all the integers $\deg(f_{i,j})$ and $\deg(h_{i,j,k})$. By Properties, Lemma 22.3 there exists a finite type quasi-coherent $\mathcal{F} \subset \mathcal{I}_d$ such that all the local sections

$$h_{i,j,k} f_{i,j}^{(d-\deg(h_{i,j,k}))/\deg(f_{i,j})}$$

are sections of $\mathcal{F}$. By construction $\mathcal{F}$ is a solution. □

The following version of Lemma 31.4 will be used in the proof of Lemma 34.2.

Lemma 31.5. *Let S be a quasi-compact and quasi-separated scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. Let $U \subset X$ be an open. Assume that*

1. *p is quasi-compact,*
2. *i of finite presentation,*
3. *$U \cap p(i(Z)) = \emptyset$,*
4. *U is quasi-compact,*
5. *$\mathcal{A}_n$ is a finite type $\mathcal{O}_S$-module for all n.*

Then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{A}_d$ with (a) $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{F}\mathcal{A})$ and (b) the support of $\mathcal{A}_d/\mathcal{F}$ is disjoint from U.

Proof. Let $\mathcal{I} \subset \mathcal{A}$ be the sheaf of quasi-coherent graded ideals constructed in Lemma 31.1. Let U_i, R_i, A_i, I_i, $f_{i,j}$, $h_{i,j,k}$, and d be as constructed in the proof of Lemma 31.4. Since $U \cap p(i(Z)) = \emptyset$ we see that $\mathcal{I}_d|_U = \mathcal{A}_d|_U$ (by our construction of $\mathcal{I}$ as a kernel). Since U is quasi-compact we can choose a finite affine open covering $U = W_1 \cup \ldots \cup W_m$. Since $\mathcal{A}_d$ is of finite type we can find finitely many sections $g_{t,s} \in \mathcal{A}_d(W_t)$ which generate $\mathcal{A}_d|_{W_t} = \mathcal{I}_d|_{W_t}$ as an $\mathcal{O}_{W_t}$-module. To finish the proof, note that by Properties, Lemma 22.3 there exists a finite type $\mathcal{F} \subset \mathcal{I}_d$ such that all the local sections

$$h_{i,j,k} f_{i,j}^{(d-\deg(h_{i,j,k}))/\deg(f_{i,j})} \quad \text{and} \quad g_{t,s}$$

are sections of $\mathcal{F}$. By construction $\mathcal{F}$ is a solution. □

Lemma 31.6. *Let X be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. There is a bijection*

$$\left\{ \begin{matrix} \textit{sections } \sigma \textit{ of the} \\ \textit{morphism } \mathbf{P}(\mathcal{E}) \to X \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{surjections } \mathcal{E} \to \mathcal{L} \textit{ where} \\ \mathcal{L} \textit{ is an invertible } \mathcal{O}_X\textit{-module} \end{matrix} \right\}$$

In this case σ is a closed immersion and there is a canonical isomorphism

$$\mathrm{Ker}(\mathcal{E} \to \mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \longrightarrow \mathcal{C}_{\sigma(X)/\mathbf{P}(\mathcal{E})}$$

Both the bijection and isomorphism are compatible with base change.

Proof. Recall that $\pi : \mathbf{P}(\mathcal{E}) \to X$ is the relative proj of the symmetric algebra on $\mathcal{E}$, see Constructions, Definition 21.1. Hence the descriptions of sections σ follows immediately from the description of the functor of points of $\mathbf{P}(\mathcal{E})$ in Constructions, Lemma 16.11. Since π is separated, any section is a closed immersion (Constructions, Lemma 16.9 and Schemes, Lemma 21.11). Let $U \subset X$ be an affine open and $k \in \mathcal{E}(U)$ and $s \in \mathcal{E}(U)$ be local sections such that k maps to zero in $\mathcal{L}$ and s maps to a generator $\overline{s}$ of $\mathcal{L}$. Then $f = k/s$ is a section of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}$ defined in an open neighbourhood $D_+(s)$ of $s(U)$ in $\pi^{-1}(U)$. Moreover, since k maps to zero in $\mathcal{L}$ we see that f is a section of the ideal sheaf of $s(U)$ in $\pi^{-1}(U)$. Thus we can take the image $\overline{f}$ of f in $\mathcal{C}_{\sigma(X)/\mathbf{P}(\mathcal{E})}(U)$. We claim (1) that the image $\overline{f}$ depends only on the sections k and $\overline{s}$ and not on the choice of s and (2) that we get an isomorphism over U in this manner (see below). However, once (1) and (2) are established, we see that the construction is compatible with base change by $U' \to U$ where U' is affine, which proves that these local maps glue and are compatible with arbitrary base change.

To prove (1) and (2) we make explicit what is going on. Namely, say $U = \mathrm{Spec}(A)$ and say $\mathcal{E} \to \mathcal{L}$ corresponds to the map of A-modules $M \to N$. Then $k \in K = \mathrm{Ker}(M \to N)$ and $s \in M$ maps to a generator $\overline{s}$ of N. Hence $M = K \oplus As$. Thus

$$\mathrm{Sym}(M) = \mathrm{Sym}(K)[s]$$

Consider the identification $\mathrm{Sym}(K) \to \mathrm{Sym}(M)_{(s)}$ via the rule $g \mapsto g/s^n$ for $g \in \mathrm{Sym}^n(K)$. This gives an isomorphism $D_+(s) = \mathrm{Spec}(\mathrm{Sym}(K))$ such that σ corresponds to the ring map $\mathrm{Sym}(K) \to A$ mapping K to zero. Via this isomorphism we see that the quasi-coherent module corresponding to K is identified with $\mathcal{C}_{\sigma(U)/D_+(s)}$ proving (2). Finally, suppose that $s' = k' + s$ for some $k' \in K$. Then

$$k/s' = (k/s)(s/s') = (k/s)(s'/s)^{-1} = (k/s)(1 + k'/s)^{-1}$$

in an open neighbourhood of $\sigma(U)$ in $D_+(s)$. Thus we see that s'/s restricts to 1 on $\sigma(U)$ and we see that k/s' maps to the same element of the conormal sheaf as does k/s thereby proving (1). □

32. Blowing up

Blowing up is an important tool in algebraic geometry.

Definition 32.1. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be the closed subscheme corresponding to $\mathcal{I}$, see Schemes, Definition 10.2. The *blowing up of X along Z*, or the *blowing up of X in the ideal sheaf $\mathcal{I}$* is the morphism

$$b : \underline{\mathrm{Proj}}_X \left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n\right) \longrightarrow X$$

The *exceptional divisor* of the blowup is the inverse image $b^{-1}(Z)$. Sometimes Z is called the *center* of the blowup.

We will see later that the exceptional divisor is an effective Cartier divisor. Moreover, the blowing up is characterized as the "smallest" scheme over X such that the inverse image of Z is an effective Cartier divisor.

If $b : X' \to X$ is the blowup of X in Z, then we often denote $\mathcal{O}_{X'}(n)$ the twists of the structure sheaf. Note that these are invertible $\mathcal{O}_{X'}$-modules and that $\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(1)^{\otimes n}$ because X' is the relative Proj of a quasi-coherent graded $\mathcal{O}_X$-algebra which is generated in degree 1, see Constructions, Lemma 16.11. Note that $\mathcal{O}_{X'}(1)$ is b-relatively very ample, even though b need not be of finite type or even quasi-compact, because X' comes equipped with a closed immersion into $\mathbf{P}(\mathcal{I})$, see Morphisms, Example 38.3.

Lemma 32.2. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $U = \mathrm{Spec}(A)$ be an affine open subscheme of X and let $I \subset A$ be the ideal corresponding to $\mathcal{I}|_U$. If $b : X' \to X$ is the blowup of X in $\mathcal{I}$, then there is a canonical isomorphism*

$$b^{-1}(U) = \mathit{Proj}(\bigoplus\nolimits_{d \geq 0} I^d)$$

of $b^{-1}(U)$ with the homogeneous spectrum of the Rees algebra of I in A. Moreover, $b^{-1}(U)$ has an affine open covering by spectra of the affine blowup algebras $A[\frac{I}{a}]$.

Proof. The first statement is clear from the construction of the relative Proj via glueing, see Constructions, Section 15. For $a \in I$ denote $a^{(1)}$ the element a seen as an element of degree 1 in the Rees algebra $\bigoplus_{n \geq 0} I^n$. Since these elements generate the Rees algebra over A we see that $\mathrm{Proj}(\bigoplus_{d \geq 0} I^d)$ is covered by the affine opens $D_+(a^{(1)})$. The affine scheme $D_+(a^{(1)})$ is the spectrum of the affine blowup algebra $A' = A[\frac{I}{a}]$, see Algebra, Definition 70.1. This finishes the proof. $\square$

Lemma 32.3. *Let $X_1 \to X_2$ be a flat morphism of schemes. Let $Z_2 \subset X_2$ be a closed subscheme. Let Z_1 be the inverse image of Z_2 in X_1. Let X'_i be the blowup of Z_i in X_i. Then there exists a cartesian diagram*

$$\begin{tikzcd} X'_1 \arrow[r] \arrow[d] & X'_2 \arrow[d] \\ X_1 \arrow[r] & X_2 \end{tikzcd}$$

of schemes.

Proof. Let $\mathcal{I}_2$ be the ideal sheaf of Z_2 in X_2. Denote $g : X_1 \to X_2$ the given morphism. Then the ideal sheaf $\mathcal{I}_1$ of Z_1 is the image of $g^*\mathcal{I}_2 \to \mathcal{O}_{X_1}$ (by definition of the inverse image, see Schemes, Definition 17.7). By Constructions, Lemma 16.10 we see that $X_1 \times_{X_2} X'_2$ is the relative Proj of $\bigoplus_{n \geq 0} g^*\mathcal{I}_2^n$. Because g is flat the map $g^*\mathcal{I}_2^n \to \mathcal{O}_{X_1}$ is injective with image $\mathcal{I}_1^n$. Thus we see that $X_1 \times_{X_2} X'_2 = X'_1$. $\square$

Lemma 32.4. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. The blowing up $b : X' \to X$ of Z in X has the following properties:*

1. *$b|_{b^{-1}(X \setminus Z)} : b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism,*
2. *the exceptional divisor $E = b^{-1}(Z)$ is an effective Cartier divisor on X',*
3. *there is a canonical isomorphism $\mathcal{O}_{X'}(-1) = \mathcal{O}_{X'}(E)$*

Proof. As blowing up commutes with restrictions to open subschemes (Lemma 32.3) the first statement just means that $X' = X$ if $Z = \emptyset$. In this case we are blowing up in the ideal sheaf $\mathcal{I} = \mathcal{O}_X$ and the result follows from Constructions, Example 8.14.

The second statement is local on X, hence we may assume X affine. Say $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/I)$. By Lemma 32.2 we see that X' is covered by the spectra of the affine blowup algebras $A' = A[\frac{I}{a}]$. Then $IA' = aA'$ and a maps to a nonzerodivisor in A' according to Algebra, Lemma 70.2. This proves the lemma as the inverse image of Z in $\text{Spec}(A')$ corresponds to $\text{Spec}(A'/IA') \subset \text{Spec}(A')$.

Consider the canonical map $\psi_{univ,1} : b^*\mathcal{I} \to \mathcal{O}_{X'}(1)$, see discussion following Constructions, Definition 16.7. We claim that this factors through an isomorphism $\mathcal{I}_E \to \mathcal{O}_{X'}(1)$ (which proves the final assertion). Namely, on the affine open corresponding to the blowup algebra $A' = A[\frac{I}{a}]$ mentioned above $\psi_{univ,1}$ corresponds to the A'-module map

$$I \otimes_A A' \longrightarrow \left(\left(\bigoplus\nolimits_{d \geq 0} I^d\right)_{a^{(1)}}\right)_1$$

where $a^{(1)}$ is as in Algebra, Definition 70.1. We omit the verification that this is the map $I \otimes_A A' \to IA' = aA'$. $\square$

Lemma 32.5 (Universal property blowing up)**.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{C}$ be the full subcategory of (Sch/X) consisting of $Y \to X$ such that the inverse image of Z is an effective Cartier divisor on Y. Then the blowing up $b : X' \to X$ of Z in X is a final object of $\mathcal{C}$.*

Proof. We see that $b : X' \to X$ is an object of $\mathcal{C}$ according to Lemma 32.4. Let $f : Y \to X$ be an object of $\mathcal{C}$. We have to show there exists a unique morphism $Y \to X'$ over X. Let $D = f^{-1}(Z)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z and let $\mathcal{I}_D$ be the ideal sheaf of D. Then $f^*\mathcal{I} \to \mathcal{I}_D$ is a surjection to an invertible $\mathcal{O}_Y$-module. This extends to a map $\psi : \bigoplus f^*\mathcal{I}^d \to \bigoplus \mathcal{I}_D^d$ of graded $\mathcal{O}_Y$-algebras. (We observe that $\mathcal{I}_D^d = \mathcal{I}_D^{\otimes d}$ as D is an effective Cartier divisor.) By the material in Constructions, Section 16 the triple $(1, f : Y \to X, \psi)$ defines a morphism $Y \to X'$ over X. The restriction

$$Y \setminus D \longrightarrow X' \setminus b^{-1}(Z) = X \setminus Z$$

is unique. The open $Y \setminus D$ is scheme theoretically dense in Y according to Lemma 13.4. Thus the morphism $Y \to X'$ is unique by Morphisms, Lemma 7.10 (also b is separated by Constructions, Lemma 16.9). $\square$

Lemma 32.6. *Let $b : X' \to X$ be the blowing up of the scheme X along a closed subscheme Z. Let $U = \text{Spec}(A)$ be an affine open of X and let $I \subset A$ be the ideal corresponding to $Z \cap U$. Let $a \in I$ and let $x' \in X'$ be a point mapping to a point of U. Then x' is a point of the affine open $U' = \text{Spec}(A[\frac{I}{a}])$ if and only if the image of a in $\mathcal{O}_{X',x'}$ cuts out the exceptional divisor.*

Proof. Since the exceptional divisor over U' is cut out by the image of a in $A' = A[\frac{I}{a}]$ one direction is clear. Conversely, assume that the image of a in $\mathcal{O}_{X',x'}$ cuts out E. Since every element of I maps to an element of the ideal defining E over $b^{-1}(U)$ we see that elements of I become divisible by a in $\mathcal{O}_{X',x'}$. Thus for $f \in I^n$ we can write $f = \psi(f)a^n$ for some $\psi(f) \in \mathcal{O}_{X',x'}$. Observe that since a maps to a nonzerodivisor of $\mathcal{O}_{X',x'}$ the element $\psi(f)$ is uniquely characterized by this. Then we define

$$A' \longrightarrow \mathcal{O}_{X',x'}, \quad f/a^n \longmapsto \psi(f)$$

Here we use the description of blowup algebras given following Algebra, Definition 32.1. The uniqueness mentioned above shows that this is an A-algebra homomorphism. This gives a morphism $\mathrm{Spec}(\mathcal{O}_{X',x''}) \to \mathrm{Spec}(A') = U'$. By the universal property of blowing up (Lemma 32.5) this is a morphism over X', which of course implies that $x' \in U'$. □

Lemma 32.7. *Let X be a scheme. Let $Z \subset X$ be an effective Cartier divisor. The blowup of X in Z is the identity morphism of X.*

Proof. Immediate from the universal property of blowups (Lemma 32.5). □

Lemma 32.8. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is reduced, then the blowup X' of X in $\mathcal{I}$ is reduced.*

Proof. Combine Lemma 32.2 with Algebra, Lemma 70.9. □

Lemma 32.9. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a nonzero quasi-coherent sheaf of ideals. If X is integral, then the blowup X' of X in $\mathcal{I}$ is integral.*

Proof. Combine Lemma 32.2 with Algebra, Lemma 70.10. □

Lemma 32.10. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let $b : X' \to X$ be the blowing up of X along Z. Then b induces an bijective map from the set of generic points of irreducible components of X' to the set of generic points of irreducible components of X which are not in Z.*

Proof. The exceptional divisor $E \subset X'$ is an effective Cartier divisor and $X' \setminus E \to X \setminus Z$ is an isomorphism, see Lemma 32.4. Thus it suffices to show the following: given an effective Cartier divisor $D \subset S$ of a scheme S none of the generic points of irreducible components of S are contained in D. To see this, we may replace S by the members of an affine open covering. Hence by Lemma 13.2 we may assume $S = \mathrm{Spec}(A)$ and $D = V(f)$ where $f \in A$ is a nonzerodivisor. Then we have to show f is not contained in any minimal prime ideal $\mathfrak{p} \subset A$. If so, then f would map to a nonzerodivisor contained in the maximal ideal of $R_\mathfrak{p}$ which is a contradiction with Algebra, Lemma 25.1. □

Lemma 32.11. *Let X be a scheme. Let $b : X' \to X$ be a blowup of X in a closed subscheme. The pullback $b^{-1}D$ is defined for all effective Cartier divisors $D \subset X$ and pullbacks of meromorphic functions are defined for b (Definitions 13.12 and 23.4).*

Proof. By Lemmas 32.2 and 13.2 this reduces to the following algebra fact: Let A be a ring, $I \subset A$ an ideal, $a \in I$, and $x \in A$ a nonzerodivisor. Then the image of x in $A[\frac{I}{a}]$ is a nonzerodivisor. Namely, suppose that $x(y/a^n) = 0$ in $A[\frac{I}{a}]$. Then $a^m xy = 0$ in A for some m. Hence $a^m y = 0$ as x is a nonzerodivisor. Whence y/a^n is zero in $A[\frac{I}{a}]$ as desired. □

Lemma 32.12. *Let X be a scheme. Let $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $b : X' \to X$ be the blowing up of X in $\mathcal{I}$. Let $b' : X'' \to X'$ be the blowing up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$. Then $X'' \to X$ is canonically isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$.*

Proof. Let $E \subset X'$ be the exceptional divisor of b which is an effective Cartier divisor by Lemma 32.4. Then $(b')^{-1}E$ is an effective Cartier divisor on X'' by Lemma 32.11. Let $E' \subset X''$ be the exceptional divisor of b' (also an effective Cartier divisor). Consider the effective Cartier divisor $E'' = E' + (b')^{-1}E$. By construction the ideal of E'' is $(b \circ b')^{-1}\mathcal{I}(b \circ b')^{-1}\mathcal{J}\mathcal{O}_{X''}$. Hence according to Lemma 32.5 there is a canonical morphism from X'' to the blowup $c : Y \to X$ of X in $\mathcal{I}\mathcal{J}$. Conversely, as $\mathcal{I}\mathcal{J}$ pulls back to an invertible ideal we see that $c^{-1}\mathcal{I}\mathcal{O}_Y$ defines an effective Cartier divisor, see Lemma 13.9. Thus a morphism $c' : Y \to X'$ over X by Lemma 32.5. Then $(c')^{-1}b^{-1}\mathcal{J}\mathcal{O}_Y = c^{-1}\mathcal{J}\mathcal{O}_Y$ which also defines an effective Cartier divisor. Thus a morphism $c'' : Y \to X''$ over X'. We omit the verification that this morphism is inverse to the morphism $X'' \to Y$ constructed earlier. $\square$

Lemma 32.13. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $b : X' \to X$ be the blowing up of X in the ideal sheaf $\mathcal{I}$. If $\mathcal{I}$ is of finite type, then*

1. *$b : X' \to X$ is a projective morphism, and*
2. *$\mathcal{O}_{X'}(1)$ is a b-relatively ample invertible sheaf.*

Proof. The surjection of graded $\mathcal{O}_X$-algebras

$$\text{Sym}^*_{\mathcal{O}_X}(\mathcal{I}) \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{I}^d$$

defines via Constructions, Lemma 18.5 a closed immersion

$$X' = \underline{\text{Proj}}_X(\bigoplus\nolimits_{d \geq 0} \mathcal{I}^d) \longrightarrow \mathbf{P}(\mathcal{I}).$$

Hence b is projective, see Morphisms, Definition 43.1. The second statement follows for example from the characterization of relatively ample invertible sheaves in Morphisms, Lemma 37.4. Some details omitted. $\square$

Lemma 32.14. *Let X be a quasi-compact and quasi-separated scheme. Let $Z \subset X$ be a closed subscheme of finite presentation. Let $b : X' \to X$ be the blowing up with center Z. Let $Z' \subset X'$ be a closed subscheme of finite presentation. Let $X'' \to X'$ be the blowing up with center Z'. There exists a closed subscheme $Y \subset X$ of finite presentation, such that*

1. *$Y = Z \cup b(Z')$ set theoretically, and*
2. *the composition $X'' \to X$ is isomorphic to the blowing up of X in Y.*

Proof. The condition that $Z \to X$ is of finite presentation means that Z is cut out by a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, see Morphisms, Lemma 21.7. Write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ so that $X' = \underline{\text{Proj}}(\mathcal{A})$. Note that $X \setminus Z$ is a quasi-compact open of X by Properties, Lemma 24.1. Since $b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism (Lemma 32.4) the same result shows that $b^{-1}(X \setminus Z) \setminus Z'$ is quasi-compact open in X'. Hence $U = X \setminus (Z \cup b(Z'))$ is quasi-compact open in X. By Lemma 31.5 there exist a $d > 0$ and a finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{I}^d$ such that $Z' = \underline{\text{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ and such that the support of $\mathcal{I}^d/\mathcal{F}$ is contained in $X \setminus U$.

Since $\mathcal{F} \subset \mathcal{I}^d$ is an $\mathcal{O}_X$-submodule we may think of $\mathcal{F} \subset \mathcal{I}^d \subset \mathcal{O}_X$ as a finite type quasi-coherent sheaf of ideals on X. Let's denote this $\mathcal{J} \subset \mathcal{O}_X$ to prevent confusion. Since $\mathcal{I}^d/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}^d$ are supported on $X \setminus U$ we see that $V(\mathcal{J})$ is contained in $X \setminus U$. Conversely, as $\mathcal{J} \subset \mathcal{I}^d$ we see that $Z \subset V(\mathcal{J})$. Over $X \setminus Z \cong X' \setminus b^{-1}(Z)$

the sheaf of ideals $\mathcal{J}$ cuts out Z' (see displayed formula below). Hence $V(\mathcal{J})$ equals $Z \cup b(Z')$. It follows that also $V(\mathcal{I}\mathcal{J}) = Z \cup b(Z')$ set theoretically. Moreover, $\mathcal{I}\mathcal{J}$ is an ideal of finite type as a product of two such. We claim that $X'' \to X$ is isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$ which finishes the proof of the lemma by setting $Y = V(\mathcal{I}\mathcal{J})$.

First, recall that the blowup of X in $\mathcal{I}\mathcal{J}$ is the same as the blowup of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$, see Lemma 32.12. Hence it suffices to show that the blowup of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$ agrees with the blowup of X' in Z'. We will show that

$$b^{-1}\mathcal{J}\mathcal{O}_{X'} = \mathcal{I}_E^d \mathcal{I}_{Z'}$$

as ideal sheaves on X''. This will prove what we want as $\mathcal{I}_E^d$ cuts out the effective Cartier divisor dE and we can use Lemmas 32.7 and 32.12.

To see the displayed equality of the ideals we may work locally. With notation A, I, $a \in I$ as in Lemma 32.2 we see that $\mathcal{F}$ corresponds to an R-submodule $M \subset I^d$ mapping isomorphically to an ideal $J \subset R$. The condition $Z' = \underline{\text{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ means that $Z' \cap \text{Spec}(A[\frac{I}{a}])$ is cut out by the ideal generated by the elements m/a^d, $m \in M$. Say the element $m \in M$ corresponds to the function $f \in J$. Then in the affine blowup algebra $A' = A[\frac{I}{a}]$ we see that $f = (a^d m)/a^d = a^d(m/a^d)$. Thus the equality holds. $\square$

33. Strict transform

In this section we briefly discuss strict transform under blowing up. Let S be a scheme and let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up of S in Z and denote $E \subset S'$ the exceptional divisor $E = b^{-1}Z$. In the following we will often consider a scheme X over S and form the cartesian diagram

$$\begin{tikzcd}
\text{pr}_{S'}^{-1}E \arrow[r] \arrow[d] & X \times_S S' \arrow[r, "\text{pr}_X"'] \arrow[d, "\text{pr}_{S'}"'] & X \arrow[d, "f"] \\
E \arrow[r] & S' \arrow[r] & S
\end{tikzcd}$$

Since E is an effective Cartier divisor (Lemma 32.4) we see that $\text{pr}_{S'}^{-1}E \subset X \times_S S'$ is locally principal (Lemma 13.11). Thus the complement of $\text{pr}_{S'}^{-1}E$ in $X \times_S S'$ is retrocompact (Lemma 13.3). Consequently, for a quasi-coherent $\mathcal{O}_{X \times_S S'}$-module $\mathcal{G}$ the subsheaf of sections supported on $\text{pr}_{S'}^{-1}E$ is a quasi-coherent submodule, see Properties, Lemma 24.5. If $\mathcal{G}$ is a quasi-coherent sheaf of algebras, e.g., $\mathcal{G} = \mathcal{O}_{X \times_S S'}$, then this subsheaf is an ideal of $\mathcal{G}$.

Definition 33.1. With $Z \subset S$ and $f : X \to S$ as above.

(1) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the *strict transform* of $\mathcal{F}$ with respect to the blowup of S in Z is the quotient $\mathcal{F}'$ of $\text{pr}_X^*\mathcal{F}$ by the submodule of sections supported on $\text{pr}_{S'}^{-1}E$.
(2) The *strict transform* of X is the closed subscheme $X' \subset X \times_S S'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X \times_S S'}$ supported on $\text{pr}_{S'}^{-1}E$.

Note that taking the strict transform along a blowup depends on the closed subscheme used for the blowup (and not just on the morphism $S' \to S$). This notion is often used for closed subschemes of S. It turns out that the strict transform of X is a blowup of X.

Lemma 33.2. *In the situation of Definition 33.1.*

1. *The strict transform X' of X is the blowup of X in the closed subscheme $f^{-1}Z$ of X.*
2. *For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform $\mathcal{F}'$ is canonically isomorphic to the pushforward along $X' \to X \times_S S'$ of the strict transform of $\mathcal{F}$ relative to the blowing up $X' \to X$.*

Proof. Let $X'' \to X$ be the blowup of X in $f^{-1}Z$. By the universal property of blowing up (Lemma 32.5) there exists a commutative diagram

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

whence a morphism $X'' \to X \times_S S'$. Thus the first assertion is that this morphism is a closed immersion with image X'. The question is local on X. Thus we may assume X and S are affine. Say that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and Z is cut out by the ideal $I \subset A$. Set $J = IB$. The map $B \otimes_A \bigoplus_{n \geq 0} I^n \to \bigoplus_{n \geq 0} J^n$ defines a closed immersion $X'' \to X \times_S S'$, see Constructions, Lemmas 11.6 and 11.5. We omit the verification that this morphism is the same as the one constructed above from the universal property. Pick $a \in I$ corresponding to the affine open $\mathrm{Spec}(A[\frac{I}{a}]) \subset S'$, see Lemma 32.2. The inverse image of $\mathrm{Spec}(A[\frac{I}{a}])$ in the strict transform X' of X is the spectrum of

$$B' = (B \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 24.5. On the other hand, letting $b \in J$ be the image of a we see that $\mathrm{Spec}(B[\frac{J}{b}])$ is the inverse image of $\mathrm{Spec}(A[\frac{I}{a}])$ in X''. By Algebra, Lemma 70.3 the open $\mathrm{Spec}(B[\frac{J}{b}])$ maps isomorphically to the open subscheme $\mathrm{pr}_{S'}^{-1}(\mathrm{Spec}(A[\frac{I}{a}]))$ of X'. Thus $X'' \to X'$ is an isomorphism.

In the notation above, let $\mathcal{F}$ correspond to the B-module N. The strict transform of $\mathcal{F}$ corresponds to the $B \otimes_A A[\frac{I}{a}]$-module

$$N' = (N \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 24.5. The strict transform of $\mathcal{F}$ relative to the blowup of X in $f^{-1}Z$ corresponds to the $B[\frac{J}{b}]$-module $N \otimes_B B[\frac{J}{b}]/b$-power-torsion. In exactly the same way as above one proves that these two modules are isomorphic. Details omitted. $\square$

Lemma 33.3. *In the situation of Definition 33.1.*

1. *If X is flat over S at all points lying over Z, then the strict transform of X is equal to the base change $X \times_S S'$.*
2. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over S at all points lying over Z, then the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is equal to the pullback $pr_X^*\mathcal{F}$.*

Proof. We will prove part (2) as it implies part (1) by the definition of the strict transform of a scheme over S. The question is local on X. Thus we may assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and that $\mathcal{F}$ corresponds to the B-module N. Then $\mathcal{F}'$ over the open $\mathrm{Spec}(B \otimes_A A[\frac{I}{a}])$ of $X \times_S S'$ corresponds to the module

$$N' = (N \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 24.5. Thus we have to show that the a-power-torsion of $N \otimes_A A[\frac{I}{a}]$ is zero. Let $y \in N \otimes_A A[\frac{I}{a}]$ with $a^n y = 0$. If $\mathfrak q \subset B$ is a prime and $a \not\in \mathfrak q$, then y maps to zero in $(N \otimes_A A[\frac{I}{a}])_{\mathfrak q}$. on the other hand, if $a \in \mathfrak q$, then $N_{\mathfrak q}$ is a flat A-module and we see that $N_{\mathfrak q} \otimes_A A[\frac{I}{a}] = (N \otimes_A A[\frac{I}{a}])_{\mathfrak q}$ has no a-power torsion (as $A[\frac{I}{a}]$ doesn't). Hence y maps to zero in this localization as well. We conclude that y is zero by Algebra, Lemma 23.1. $\square$

Lemma 33.4. *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up of Z in S. Let $g : X \to Y$ be an affine morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g' : X \times_S S' \to Y \times_S S'$ be the base change of g. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ relative to b. Then $g'_*\mathcal{F}'$ is the strict transform of $g_*\mathcal{F}$.*

Proof. Observe that $g'_*\text{pr}_X^*\mathcal{F} = \text{pr}_Y^*g_*\mathcal{F}$ by Cohomology of Schemes, Lemma 5.1. Let $\mathcal{K} \subset \text{pr}_X^*\mathcal{F}$ be the subsheaf of sections supported in the inverse image of Z in $X \times_S S'$. By Properties, Lemma 24.7 the pushforward $g'_*\mathcal{K}$ is the subsheaf of sections of $\text{pr}_Y^*g_*\mathcal{F}$ supported in the inverse image of Z in $Y \times_S S'$. As g' is affine (Morphisms, Lemma 11.8) we see that g'_* is exact, hence we conclude. $\square$

Lemma 33.5. *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $D \subset S$ be an effective Cartier divisor. Let $Z' \subset S$ be the closed subscheme cut out by the product of the ideal sheaves of Z and D. Let $S' \to S$ be the blowup of S in Z.*

1. *The blowup of S in Z' is isomorphic to $S' \to S$.*
2. *Let $f : X \to S$ be a morphism of schemes and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ has no nonzero local sections supported in $f^{-1}D$, then the strict transform of $\mathcal{F}$ relative to the blowing up in Z agrees with the strict transform of $\mathcal{F}$ relative to the blowing up of S in Z'.*

Proof. The first statement follows on combining Lemmas 32.12 and 32.7. Using Lemma 32.2 the second statement translates into the following algebra problem. Let A be a ring, $I \subset A$ an ideal, $x \in A$ a nonzerodivisor, and $a \in I$. Let M be an A-module whose x-torsion is zero. To show: the a-power torsion in $M \otimes_A A[\frac{I}{a}]$ is equal to the xa-power torsion. The reason for this is that the kernel and cokernel of the map $A \to A[\frac{I}{a}]$ is a-power torsion, so this map becomes an isomorphism after inverting a. Hence the kernel and cokernel of $M \to M \otimes_A A[\frac{I}{a}]$ are a-power torsion too. This implies the result. $\square$

Lemma 33.6. *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up with center Z. Let $Z' \subset S'$ be a closed subscheme. Let $S'' \to S'$ be the blowing up with center Z'. Let $Y \subset S$ be a closed subscheme such that $Y = Z \cup b(Z')$ set theoretically and the composition $S'' \to S$ is isomorphic to the blowing up of S in Y. In this situation, given any scheme X over S and $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have*

1. *the strict transform of $\mathcal{F}$ with respect to the blowing up of S in Y is equal to the strict transform with respect to the blowup $S'' \to S'$ in Z' of the strict transform of $\mathcal{F}$ with respect to the blowup $S' \to S$ of S in Z, and*
2. *the strict transform of X with respect to the blowing up of S in Y is equal to the strict transform with respect to the blowup $S'' \to S'$ in Z' of the strict transform of X with respect to the blowup $S' \to S$ of S in Z.*

Proof. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ with respect to the blowup $S' \to S$ of S in Z. Let $\mathcal{F}''$ be the strict transform of $\mathcal{F}'$ with respect to the blowup $S'' \to S'$ of S' in Z'. Let $\mathcal{G}$ be the strict transform of $\mathcal{F}$ with respect to the blowup $S'' \to S$ of S in Y. We also label the morphisms

$$\begin{tikzcd} X \times_S S'' \arrow[r, "q"'] \arrow[d, "f''"] & X \times_S S' \arrow[r, "p"'] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

By definition there is a surjection $p^*\mathcal{F} \to \mathcal{F}'$ and a surjection $q^*\mathcal{F}' \to \mathcal{F}''$ which combine by right exactness of q^* to a surjection $(p \circ q)^*\mathcal{F} \to \mathcal{F}''$. Also we have the surjection $(p \circ q)^*\mathcal{F} \to \mathcal{G}$. Thus it suffices to prove that these two surjections have the same kernel.

The kernel of the surjection $p^*\mathcal{F} \to \mathcal{F}'$ is supported on $(f \circ p)^{-1}Z$, so this map is an isomorphism at points in the complement. Hence the kernel of $q^*p^*\mathcal{F} \to q^*\mathcal{F}'$ is supported on $(f \circ p \circ q)^{-1}Z$. The kernel of $q^*\mathcal{F}' \to \mathcal{F}''$ is supported on $(f' \circ q)^{-1}Z'$. Combined we see that the kernel of $(p \circ q)^*\mathcal{F} \to \mathcal{F}''$ is supported on $(f \circ p \circ q)^{-1}Z \cup (f' \circ q)^{-1}Z' = (f \circ p \circ q)^{-1}Y$. By construction of $\mathcal{G}$ we see that we obtain a factorization $(p \circ q)^*\mathcal{F} \to \mathcal{F}'' \to \mathcal{G}$. To finish the proof it suffices to show that $\mathcal{F}''$ has no nonzero (local) sections supported on $(f \circ p \circ q)^{-1}(Y) = (f \circ p \circ q)^{-1}Z \cup (f' \circ q)^{-1}Z'$. This follows from Lemma 33.5 applied to $\mathcal{F}'$ on $X \times_S S'$ over S', the closed subscheme Z' and the effective Cartier divisor $b^{-1}Z$. $\square$

Lemma 33.7. *In the situation of Definition 33.1. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence of quasi-coherent sheaves on X which remains exact after any base change $T \to S$. Then the strict transforms of $\mathcal{F}'_i$ relative to any blowup $S' \to S$ form a short exact sequence $0 \to \mathcal{F}'_1 \to \mathcal{F}'_2 \to \mathcal{F}'_3 \to 0$ too.

Proof. We may localize on S and X and assume both are affine. Then we may push $\mathcal{F}_i$ to S, see Lemma 33.4. We may assume that our blowup is the morphism $1 : S \to S$ associated to an effective Cartier divisor $D \subset S$. Then the translation into algebra is the following: Suppose that A is a ring and $0 \to M_1 \to M_2 \to M_3 \to 0$ is a universally exact sequence of A-modules. Let $a \in A$. Then the sequence

$$0 \to M_1/a\text{-power torsion} \to M_2/a\text{-power torsion} \to M_3/a\text{-power torsion} \to 0$$

is exact too. Namely, surjectivity of the last map and injectivity of the first map are immediate. The problem is exactness in the middle. Suppose that $x \in M_2$ maps to zero in M_3/a-power torsion. Then $y = a^n x \in M_1$ for some n. Then y maps to zero in M_2/a^nM_2. Since $M_1 \to M_2$ is universally injective we see that y maps to zero in M_1/a^nM_1. Thus $y = a^nz$ for some $z \in M_1$. Thus $a^n(x - y) = 0$. Hence y maps to the class of x in M_2/a-power torsion as desired. $\square$

34. Admissible blowups

To have a bit more control over our blowups we introduce the following standard terminology.

Definition 34.1. Let X be a scheme. Let $U \subset X$ be an open subscheme. A morphism $X' \to X$ is called a *U-admissible blowup* if there exists a closed immersion $Z \to X$ of finite presentation with Z disjoint from U such that X' is isomorphic to the blowup of X in Z.

We recall that $Z \to X$ is of finite presentation if and only if the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is of finite type, see Morphisms, Lemma 21.7. In particular, a U-admissible blowup is a projective morphism, see Lemma 32.13. Note that there can be multiple centers which give rise to the same morphism. Hence the requirement is just the existence of some center disjoint from U which produces X'. Finally, as the morphism $b : X' \to X$ is an isomorphism over U (see Lemma 32.4) we will often abuse notation and think of U as an open subscheme of X' as well.

Lemma 34.2. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open subscheme. Let $b : X' \to X$ be a U-admissible blowup. Let $X'' \to X'$ be a U-admissible blowup. Then the composition $X'' \to X$ is a U-admissible blowup.*

Proof. Immediate from the more precise Lemma 32.14. $\square$

Lemma 34.3. *Let X be a quasi-compact and quasi-separated scheme. Let $U, V \subset X$ be quasi-compact open subschemes. Let $b : V' \to V$ be a $U \cap V$-admissible blowup. Then there exists a U-admissible blowup $X' \to X$ whose restriction to V is V'.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_V$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I})$ is disjoint from $U \cap V$ and such that V' is isomorphic to the blowup of V in $\mathcal{I}$. Let $\mathcal{I}' \subset \mathcal{O}_{U \cup V}$ be the quasi-coherent sheaf of ideals whose restriction to U is $\mathcal{O}_U$ and whose restriction to V is $\mathcal{I}$ (see Sheaves, Section 33). By Properties, Lemma 22.2 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ whose restriction to $U \cup V$ is $\mathcal{I}'$. The lemma follows. $\square$

Lemma 34.4. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open subscheme. Let $b_i : X_i \to X$, $i = 1, \ldots, n$ be U-admissible blowups. There exists a U-admissible blowup $b : X' \to X$ such that (a) b factors as $X' \to X_i \to X$ for $i = 1, \ldots, n$ and (b) each of the morphisms $X' \to X_i$ is a U-admissible blowup.*

Proof. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_i)$ is disjoint from U and such that X_i is isomorphic to the blowup of X in $\mathcal{I}_i$. Set $\mathcal{I} = \mathcal{I}_1 \cdot \ldots \cdot \mathcal{I}_n$ and let X' be the blowup of X in $\mathcal{I}$. Then $X' \to X$ factors through b_i by Lemma 32.12. $\square$

Lemma 34.5. *Let X be a quasi-compact and quasi-separated scheme. Let U, V be quasi-compact disjoint open subschemes of X. Then there exist a $U \cup V$-admissible blowup $b : X' \to X$ such that X' is a disjoint union of open subschemes $X' = X'_1 \amalg X'_2$ with $b^{-1}(U) \subset X'_1$ and $b^{-1}(V) \subset X'_2$.*

Proof. Choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}$, resp. $\mathcal{J}$ such that $X \setminus U = V(\mathcal{I})$, resp. $X \setminus V = V(\mathcal{J})$, see Properties, Lemma 24.1. Then $V(\mathcal{I}\mathcal{J}) = X$ set theoretically, hence $\mathcal{I}\mathcal{J}$ is a locally nilpotent sheaf of ideals. Since $\mathcal{I}$ and $\mathcal{J}$ are of finite type and X is quasi-compact there exists an $n > 0$ such that $\mathcal{I}^n\mathcal{J}^n = 0$. We may and do replace $\mathcal{I}$ by $\mathcal{I}^n$ and $\mathcal{J}$ by $\mathcal{J}^n$. Whence $\mathcal{I}\mathcal{J} = 0$. Let $b : X' \to X$ be the blowing up in $\mathcal{I} + \mathcal{J}$. This is $U \cup V$-admissible as $V(\mathcal{I} + \mathcal{J}) = X \setminus U \cup V$.

We will show that X' is a disjoint union of open subschemes $X' = X'_1 \amalg X'_2$ such that $b^{-1}\mathcal{I}|_{X'_2} = 0$ and $b^{-1}\mathcal{J}|_{X'_1} = 0$ which will prove the lemma.

We will use the description of the blowing up in Lemma 32.2. Suppose that $U = \text{Spec}(A) \subset X$ is an affine open such that $\mathcal{I}|_U$, resp. $\mathcal{J}|_U$ corresponds to the finitely generated ideal $I \subset A$, resp. $J \subset A$. Then

$$b^{-1}(U) = \text{Proj}(A \oplus (I + J) \oplus (I + J)^2 \oplus \ldots)$$

This is covered by the affine open subsets $A[\frac{I+J}{x}]$ and $A[\frac{I+J}{y}]$ with $x \in I$ and $y \in J$. Since $x \in I$ is a nonzerodivisor in $A[\frac{I+J}{x}]$ and $IJ = 0$ we see that $JA[\frac{I+J}{x}] = 0$. Since $y \in J$ is a nonzerodivisor in $A[\frac{I+J}{y}]$ and $IJ = 0$ we see that $IA[\frac{I+J}{y}] = 0$. Moreover,

$$\text{Spec}(A[\tfrac{I+J}{x}]) \cap \text{Spec}(A[\tfrac{I+J}{y}]) = \text{Spec}(A[\tfrac{I+J}{xy}]) = \emptyset$$

because xy is both a nonzerodivisor and zero. Thus $b^{-1}(U)$ is the disjoint union of the open subscheme U_1 defined as the union of the standard opens $\text{Spec}(A[\frac{I+J}{x}])$ for $x \in I$ and the open subscheme U_2 which is the union of the affine opens $\text{Spec}(A[\frac{I+J}{y}])$ for $y \in J$. We have seen that $b^{-1}\mathcal{I}\mathcal{O}_{X'}$ restricts to zero on U_2 and $b^{-1}\mathcal{I}\mathcal{O}_{X'}$ restricts to zero on U_1. We omit the verification that these open subschemes glue to global open subschemes X'_1 and X'_2. $\square$

Lemma 34.6. *Let X be a locally Noetherian scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. Let $U \subset X$ be the maximal open subscheme such that s corresponds to a section of $\mathcal{L}$ over U. The blowup $b : X' \to X$ in the ideal of denominators of s is U-admissible. There exists an effective Cartier divisor $D \subset X'$ and an isomorphism*

$$b^*\mathcal{L} = \mathcal{O}_{X'}(D - E),$$

where $E \subset X'$ is the exceptional divisor such that the meromorphic section b^*s corresponds, via the isomorphism, to the meromorphic section $1_D \otimes (1_E)^{-1}$.

Proof. From the definition of the ideal of denominators in Definition 23.10 we immediately see that b is a U-admissible blowup. For the notation 1_D, 1_E, and $\mathcal{O}_{X'}(D - E)$ please see Definition 14.1. The pullback b^*s is defined by Lemmas 32.11 and 23.8. Thus the statement of the lemma makes sense. We can reinterpret the final assertion as saying that b^*s is a global regular section of $b^*\mathcal{L}(E)$ whose zero scheme is D. This uniquely defines D hence to prove the lemma we may work affine locally on X and X'. Assume $X = \text{Spec}(A)$ is affine and $\mathcal{L} = \mathcal{O}_X$. Then s is a regular meromorphic function and shrinking further we may assume $s = a'/a$ with $a', a \in A$ nonzerodivisors. Then the ideal of denominators of s corresponds to the ideal $I = \{x \in A \mid xa' \in aA\}$. Recall that X' is covered by spectra of affine blowup algebras $A' = A[\frac{I}{x}]$ with $x \in I$ (Lemma 32.2). Fix $x \in I$ and write $xa' = aa''$ for some $a'' \in A$. The divisor $E \subset X'$ is cut out by $x \in A'$ over the spectrum of A' and hence $1/x$ is a generator of $\mathcal{O}_{X'}(E)$ over $\text{Spec}(A')$. Finally, in the total quotient ring of A' we have $a'/a = a''/x$. Hence $b^*s = a'/a$ restricts to a regular section of $\mathcal{O}_{X'}(E)$ which is over $\text{Spec}(A')$ given by a''/x. This finishes the proof. (The divisor $D \cap \text{Spec}(A')$ is cut out by the image of a'' in A'.) $\square$

35. Blowing up and flatness

We continue the discussion started in More on Algebra, Section 26. We will prove further results in More on Flatness, Section 30.

Lemma 35.1. *Let S be a scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. Let $Z_k \subset S$ be the closed subscheme cut out by $Fit_k(\mathcal{F})$, see Section 9. Let $S' \to S$ be the blowup of S in Z_k and let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$. Then $\mathcal{F}'$ can locally be generated by $\leq k$ sections.*

Proof. Recall that $\mathcal{F}'$ can locally be generated by $\leq k$ sections if and only if $\text{Fit}_k(\mathcal{F}') = \mathcal{O}_{S'}$, see Lemma 9.4. Hence this lemma is a translation of More on Algebra, Lemma 26.3. □

Lemma 35.2. *Let S be a scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. Let $Z_k \subset S$ be the closed subscheme cut out by $Fit_k(\mathcal{F})$, see Section 9. Assume that $\mathcal{F}$ is locally free of rank k on $S \setminus Z_k$. Let $S' \to S$ be the blowup of S in Z_k and let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$. Then $\mathcal{F}'$ is locally free of rank k.*

Proof. Translation of More on Algebra, Lemma 26.4. □

Lemma 35.3. *Let X be a scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $U \subset X$ be a scheme theoretically dense open such that $\mathcal{F}|_U$ is finite locally free of constant rank r. Then*

1. *the blowup $b : X' \to X$ of X in the rth Fitting ideal of $\mathcal{F}$ is U-admissible,*
2. *the strict transform $\mathcal{F}'$ of $\mathcal{F}$ with respect to b is locally free of rank r,*
3. *the kernel $\mathcal{K}$ of the surjection $b^*\mathcal{F} \to \mathcal{F}'$ is finitely presented and $\mathcal{K}|_U = 0$,*
4. *$b^*\mathcal{F}$ and $\mathcal{K}$ are perfect $\mathcal{O}_{X'}$-modules of tor dimension ≤ 1.*

Proof. The ideal $\text{Fit}_r(\mathcal{F})$ is of finite type by Lemma 9.2 and its restriction to U is equal to $\mathcal{O}_U$ by Lemma 9.5. Hence $b : X' \to X$ is U-admissible, see Definition 34.1.

By Lemma 9.5 the restriction of $\text{Fit}_{r-1}(\mathcal{F})$ to U is zero, and since U is scheme theoretically dense we conclude that $\text{Fit}_{r-1}(\mathcal{F}) = 0$ on all of X. Thus it follows from Lemma 9.5 that $\mathcal{F}$ is locally free of rank r on the complement of subscheme cut out by the rth Fitting ideal of $\mathcal{F}$ (this complement may be bigger than U which is why we had to do this step in the argument). Hence by Lemma 35.2 the strict transform

$$b^*\mathcal{F} \longrightarrow \mathcal{F}'$$

is locally free of rank r. The kernel $\mathcal{K}$ of this map is supported on the exceptional divisor of the blowup b and hence $\mathcal{K}|_U = 0$. Finally, since $\mathcal{F}'$ is finite locally free and since the displayed arrow is surjective, we can locally on X' write $b^*\mathcal{F}$ as the direct sum of $\mathcal{K}$ and $\mathcal{F}'$. Since $b^*\mathcal{F}'$ is finitely presented (Modules, Lemma 11.4) the same is true for $\mathcal{K}$.

The statement on tor dimension follows from More on Algebra, Lemma 8.9. □

36. Modifications

In this section we will collect results of the type: after a modification such and such are true. We will later see that a modification can be dominated by a blowup (More on Flatness, Lemma 31.4).

Lemma 36.1. *Let X be an integral scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. There exists a modification $f : X' \to X$ such that $f^*\mathcal{E}$ has a filtration whose successive quotients are invertible $\mathcal{O}_{X'}$-modules.*

Proof. We prove this by induction on the rank r of $\mathcal{E}$. If $r = 1$ or $r = 0$ the lemma is obvious. Assume $r > 1$. Let $P = \mathbf{P}(\mathcal{E})$ with structure morphism $\pi : P \to X$, see Constructions, Section 21. Then π is proper (Lemma 30.4). There is a canonical surjection

$$\pi^*\mathcal{E} \to \mathcal{O}_P(1)$$

whose kernel is finite locally free of rank $r-1$. Choose a nonempty open subscheme $U \subset X$ such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$. Then $P_U = \pi^{-1}(U)$ is isomorphic to $\mathbf{P}^{r-1}_U$. In particular, there exists a section $s : U \to P_U$ of π. Let $X' \subset P$ be the scheme theoretic image of the morphism $U \to P_U \to P$. Then X' is integral (Morphisms, Lemma 6.7), the morphism $f = \pi|_{X'} : X' \to X$ is proper (Morphisms, Lemmas 41.6 and 41.4), and $f^{-1}(U) \to U$ is an isomorphism. Hence f is a modification (Morphisms, Definition 51.11). By construction the pullback $f^*\mathcal{E}$ has a two step filtration whose quotient is invertible because it is equal to $\mathcal{O}_P(1)|_{X'}$ and whose sub $\mathcal{E}'$ is locally free of rank $r - 1$. By induction we can find a modification $g : X'' \to X'$ such that $g^*\mathcal{E}'$ has a filtration as in the statement of the lemma. Thus $f \circ g : X'' \to X$ is the required modification. □

Lemma 36.2. *Let S be a scheme. Let X, Y be schemes over S. Assume X is Noetherian and Y is proper over S. Given an S-rational map $f : U \to Y$ from X to Y there exists a morphism $p : X' \to X$ and an S-morphism $f' : X' \to Y$ such that*

1. *p is proper and $p^{-1}(U) \to U$ is an isomorphism,*
2. *$f'|_{p^{-1}(U)}$ is equal to $f \circ p|_{p^{-1}(U)}$.*

Proof. Denote $j : U \to X$ the inclusion morphism. Let $X' \subset Y \times_S X$ be the scheme theoretic image of $(f, j) : U \to Y \times_S X$ (Morphisms, Definition 6.2). The projection $g : Y \times_S X \to X$ is proper (Morphisms, Lemma 41.5). The composition $p : X' \to X$ of $X' \to Y \times_S X$ and g is proper (Morphisms, Lemmas 41.6 and 41.4). Since g is separated and $U \subset X$ is retrocompact (as X is Noetherian) we conclude that $p^{-1}(U) \to U$ is an isomorphism by Morphisms, Lemma 6.8. On the other hand, the composition $f' : X' \to Y$ of $X' \to Y \times_S X$ and the projection $Y \times_S X \to Y$ agrees with f on $p^{-1}(U)$. □

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