MORE ON MORPHISMS OF SPACES

Contents

1. Introduction

In this chapter we continue our study of properties of morphisms of algebraic spaces. A fundamental reference is [Knu71].

2. Conventions

The standard assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Radicial morphisms

It turns out that a radicial morphism is not the same thing as a universally injective morphism, contrary to what happens with morphisms of schemes. In fact it is a bit stronger.

Definition 3.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *radicial* if for any morphism $\text{Spec}(K) \to Y$ where K is a field the reduction $(\text{Spec}(K) \times_Y X)_{red}$ is either empty or representable by the spectrum of a purely inseparable field extension of K.

Lemma 3.2. *A radicial morphism of algebraic spaces is universally injective.*

Proof. Let S be a scheme. Let $f : X \to Y$ be a radicial morphism of algebraic spaces over S. It is clear from the definition that given a morphism $\text{Spec}(K) \to Y$

there is at most one lift of this morphism to a morphism into X. Hence we conclude that f is universally injective by Morphisms of Spaces, Lemma 19.2. $\square$

Example 3.3. It is no longer true that universally injective is equivalent to radicial. For example the morphism

$$X = [\mathrm{Spec}(\overline{\mathbf{Q}})/\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})] \longrightarrow S = \mathrm{Spec}(\mathbf{Q})$$

of Spaces, Example 14.7 is universally injective, but is not radicial in the sense above.

Nonetheless it is often the case that the reverse implication holds.

Lemma 3.4. *Let S be a scheme. Let $f : X \to Y$ be a universally injective morphism of algebraic spaces over S.*

1. *If f is decent then f is radicial.*
2. *If f is quasi-separated then f is radicial.*
3. *If f is locally separated then f is radicial.*

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is stable under base change and composition and holds for closed immersions. Assume $f : X \to Y$ has $\mathcal{P}$ and is universally injective. Then, in the situation of Definition 3.1 the morphism $(\mathrm{Spec}(K) \times_Y X)_{red} \to \mathrm{Spec}(K)$ is universally injective and has $\mathcal{P}$. This reduces the problem of proving

$$\mathcal{P} + \text{universally injective} \Rightarrow \text{radicial}$$

to the problem of proving that any nonempty reduced algebraic space X over field whose structure morphism $X \to \mathrm{Spec}(K)$ is universally injective and $\mathcal{P}$ is representable by the spectrum of a field. Namely, then $X \to \mathrm{Spec}(K)$ will be a morphism of schemes and we conclude by the equivalence of radicial and universally injective for morphisms of schemes, see Morphisms, Lemma 10.2.

Let us prove (1). Assume f is decent and universally injective. By Decent Spaces, Lemmas 17.4, 17.6, and 17.2 (to see that an immersion is decent) we see that the discussion in the first paragraph applies. Let X be a nonempty decent reduced algebraic space universally injective over a field K. In particular we see that $|X|$ is a singleton. By Decent Spaces, Lemma 14.2 we conclude that $X \cong \mathrm{Spec}(L)$ for some extension $K \subset L$ as desired.

A quasi-separated morphism is decent, see Decent Spaces, Lemma 17.2. Hence (1) implies (2).

Let us prove (3). Recall that the separation axioms are stable under base change and composition and that closed immersions are separated, see Morphisms of Spaces, Lemmas 4.4, 4.8, and 10.7. Thus the discussion in the first paragraph of the proof applies. Let X be a reduced algebraic space universally injective and locally separated over a field K. In particular $|X|$ is a singleton hence X is quasi-compact, see Properties of Spaces, Lemma 5.2. We can find a surjective étale morphism $U \to X$ with U affine, see Properties of Spaces, Lemma 6.3. Consider the morphism of schemes

$$j : U \times_X U \longrightarrow U \times_{\mathrm{Spec}(K)} U$$

As $X \to \mathrm{Spec}(K)$ is universally injective j is surjective, and as $X \to \mathrm{Spec}(K)$ is locally separated j is an immersion. A surjective immersion is a closed immersion, see Schemes, Lemma 10.4. Hence $R = U \times_X U$ is affine as a closed subscheme of an affine scheme. In particular R is quasi-compact. It follows that $X = U/R$ is quasi-separated, and the result follows from (2). $\square$

Remark 3.5. Let $X \to Y$ be a morphism of algebraic spaces. For some applications (of radicial morphisms) it is enough to require that for every $\mathrm{Spec}(K) \to Y$ where K is a field

1. the space $|\mathrm{Spec}(K) \times_Y X|$ is a singleton,
2. there exists a monomorphism $\mathrm{Spec}(L) \to \mathrm{Spec}(K) \times_Y X$, and
3. $K \subset L$ is purely inseparable.

If needed later we will may call such a morphism *weakly radicial*. For example if $X \to Y$ is a surjective weakly radicial morphism then $X(k) \to Y(k)$ is surjective for every algebraically closed field k. Note that the base change $X_{\overline{\mathbf{Q}}} \to \mathrm{Spec}(\overline{\mathbf{Q}})$ of the morphism in Example 3.3 is weakly radicial, but not radicial. The analogue of Lemma 3.4 is that if $X \to Y$ has property (β) and is universally injective, then it is weakly radicial (proof omitted).

Lemma 3.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

1. *f is locally of finite type,*
2. *for every étale morphism $V \to Y$ the map $|X \times_Y V| \to |V|$ is injective.*

Then f is universally injective.

Proof. The question is étale local on Y by Morphisms of Spaces, Lemma 19.6. Hence we may assume that Y is a scheme. Then Y is in particular decent and by Decent Spaces, Lemma 18.9 we see that f is locally quasi-finite. Let $y \in Y$ be a point and let X_y be the scheme theoretic fibre. Assume X_y is not empty. By Spaces over Fields, Lemma 10.8 we see that X_y is a scheme which is locally quasi-finite over $\kappa(y)$. Since $|X_y| \subset |X|$ is the fibre of $|X| \to |Y|$ over y we see that X_y has a unique point x. The same is true for $X_y \times_{\mathrm{Spec}(\kappa(y))} \mathrm{Spec}(k)$ for any finite separable extension $k/\kappa(y)$ because we can realize k as the residue field at a point lying over y in an étale scheme over Y, see More on Morphisms, Lemma 35.2. Thus X_y is geometrically connected, see Varieties, Lemma 7.11. This implies that the finite extension $\kappa(x)/\kappa(y)$ is purely inseparable.

We conclude (in the case that Y is a scheme) that for every $y \in Y$ either the fibre X_y is empty, or $(X_y)_{red} = \mathrm{Spec}(\kappa(x))$ with $\kappa(y) \subset \kappa(x)$ purely inseparable. Hence f is radicial (some details omitted), whence universally injective by Lemma 3.2. $\square$

4. Monomorphisms

This section is the continuation of Morphisms of Spaces, Section 10. We would like to know whether or not every monomorphism of algebraic spaces is representable. If you can prove this is true or have a counterexample, please email stacks.project@gmail.com. For the moment this is known in the following cases

1. for monomorphisms which are locally of finite type (more generally any separated, locally quasi-finite morphism is representable by Morphisms of Spaces, Lemma 51.1 and a monomorphism which is locally of finite type is locally quasi-finite by Morphisms of Spaces, Lemma 27.10),
2. if the target is a disjoint union of spectra of zero dimensional local rings (Decent Spaces, Lemma 19.1), and

(3) for flat monomorphisms (see below).

Lemma 4.1 (David Rydh)**.** *A flat monomorphism of algebraic spaces is representable by schemes.*

Proof. Let $f : X \to Y$ be a flat monomorphism of algebraic spaces. To prove f is representable, we have to show $X \times_Y V$ is a scheme for every scheme V mapping to Y. Since being a scheme is local (Properties of Spaces, Lemma 13.1), we may assume V is affine. Thus we may assume $Y = \text{Spec}(B)$ is an affine scheme. Next, we can assume that X is quasi-compact by replacing X by a quasi-compact open. The space X is separated as $X \to X \times_{\text{Spec}(B)} X$ is an isomorphism. Applying Limits of Spaces, Lemma 17.3 we reduce to the case where B is local, $X \to \text{Spec}(B)$ is a flat monomorphism, and there exists a point $x \in X$ mapping to the closed point of $\text{Spec}(B)$. Then $X \to \text{Spec}(B)$ is surjective as generalizations lift along flat morphisms of separated algebraic spaces, see Decent Spaces, Lemma 7.4. Hence we see that $\{X \to \text{Spec}(B)\}$ is an fpqc cover. Then $X \to \text{Spec}(B)$ is a morphism which becomes an isomorphism after base change by $X \to \text{Spec}(B)$. Hence it is an isomorphism by fpqc descent, see Descent on Spaces, Lemma 11.15. $\square$

The following is (in some sense) a variant of the lemma above.

Lemma 4.2. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact monomorphism of algebraic spaces such that for every $T \to Y$ the map*

$$\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X \times_Y T}$$

is injective. Then f is an isomorphism (and hence representable by schemes).

Proof. The question is étale local on Y, hence we may assume $Y = \text{Spec}(A)$ is affine. Then X is quasi-compact and we may choose an affine scheme $U = \text{Spec}(B)$ and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 6.3). Note that $U \times_X U = \text{Spec}(B \otimes_A B)$. Hence the category of quasi-coherent $\mathcal{O}_X$-modules is equivalent to the category $DD_{B/A}$ of descent data on modules for $A \to B$. See Properties of Spaces, Proposition 32.1, Descent, Definition 3.1, and Descent, Subsection 4.14. On the other hand,

$$A \to B$$

is a universally injective ring map. Namely, given an A-module M we see that $A \oplus M \to B \otimes_A (A \oplus M)$ is injective by the assumption of the lemma. Hence $DD_{B/A}$ is equivalent to the category of A-modules by Descent, Theorem 4.22. Thus pullback along $f : X \to \text{Spec}(A)$ determines an equivalence of categories of quasi-coherent modules. In particular f^* is exact on quasi-coherent modules and we see that f is flat (small detail omitted). Moreover, it is clear that f is surjective (for example because $\text{Spec}(B) \to \text{Spec}(A)$ is surjective). Hence we see that $\{X \to \text{Spec}(A)\}$ is an fpqc cover. Then $X \to \text{Spec}(A)$ is a morphism which becomes an isomorphism after base change by $X \to \text{Spec}(A)$. Hence it is an isomorphism by fpqc descent, see Descent on Spaces, Lemma 11.15. $\square$

Lemma 4.3. *A quasi-compact flat surjective monomorphism of algebraic spaces is an isomorphism.*

Proof. Such a morphism satisfies the assumptions of Lemma 4.2. $\square$

5. Conormal sheaf of an immersion

Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals, see Morphisms of Spaces, Lemma 13.1. Consider the short exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

of quasi-coherent sheaves on X. Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by $\mathcal{I}$ it corresponds to a sheaf on Z by Morphisms of Spaces, Lemma 14.1. This quasi-coherent $\mathcal{O}_Z$-module is the *conormal sheaf of Z in X* and is often denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Morphisms of Spaces, Section 14.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, see Morphisms of Spaces, Remark 12.4. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 5.1. Let $i : Z \to X$ be an immersion. The *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X* or the *conormal sheaf of i* is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above.

In [DG67, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the *normal sheaf of the immersion*. It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

Lemma 5.2. *Let S be a scheme. Let $i : Z \to X$ be an immersion. Let $\varphi : U \to X$ be an étale morphism where U is a scheme. Set $Z_U = U \times_X Z$ which is a locally closed subscheme of U. Then*

$$\mathcal{C}_{Z/X}|_{Z_U} = \mathcal{C}_{Z_U/U}$$

canonically and functorially in U.

Proof. Let $T \subset X$ be a closed subspace such that i defines a closed immersion into $X \setminus T$. Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals on $X \setminus T$ defining Z. Then the lemma just states that $\mathcal{I}|_{U \setminus \varphi^{-1}(T)}$ is the sheaf of ideals of the immersion $Z_U \to U \setminus \varphi^{-1}(T)$. This is clear from the construction of $\mathcal{I}$ in Morphisms of Spaces, Lemma 13.1. $\square$

Lemma 5.3. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Assume i, i' immersions. There is a canonical map of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

Proof. First find open subspaces $U' \subset X'$ and $U \subset X$ such that $g(U) \subset U'$ and such that $i(Z) \subset U$ and $i(Z') \subset U'$ are closed (proof existence omitted). Replacing X by U and X' by U' we may assume that i and i' are closed immersions. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaves of ideals associated to i' and i, see Morphisms of Spaces, Lemma 13.1. Consider the composition

$$g^{-1}\mathcal{I}' \to g^{-1}\mathcal{O}_{X'} \xrightarrow{g^\sharp} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$$

Since $g(i(Z)) \subset Z'$ we conclude this composition is zero (see statement on factorizations in Morphisms of Spaces, Lemma 13.1). Thus we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] & i_*\mathcal{O}_Z \arrow[r] & 0 \\ 0 \arrow[r] & g^{-1}\mathcal{I}' \arrow[r] \arrow[u] & g^{-1}\mathcal{O}_{X'} \arrow[r] \arrow[u] & g^{-1}i'_*\mathcal{O}_{Z'} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The lower row is exact since g^{-1} is an exact functor. By exactness we also see that $(g^{-1}\mathcal{I}')^2 = g^{-1}((\mathcal{I}')^2)$. Hence the diagram induces a map $g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{I}/\mathcal{I}^2$. Pulling back (using i^{-1} for example) to Z we obtain $i^{-1}g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{C}_{Z/X}$. Since $i^{-1}g^{-1} = f^{-1}(i')^{-1}$ this gives a map $f^{-1}\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$, which induces the desired map. $\square$

Lemma 5.4. *Let S be a scheme. The conormal sheaf of Definition 5.1, and its functoriality of Lemma 5.3 satisfy the following properties:*

1. *If $Z \to X$ is an immersion of schemes over S, then the conormal sheaf agrees with the one from Morphisms, Definition 31.1.*
2. *If in Lemma 5.3 all the spaces are schemes, then the map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ is the same as the one constructed in Morphisms, Lemma 31.3.*
3. *Given a commutative diagram*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] \arrow[d, "f'"'] & X' \arrow[d, "g'"] \\ Z'' \arrow[r, "i''"] & X'' \end{tikzcd}$$

then the map $(f' \circ f)^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z/X}$ is the same as the composition of $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ with the pullback by f of $(f')^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z'/X'}$

Proof. Omitted. Note that Part (1) is a special case of Lemma 5.2. $\square$

Lemma 5.5. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S. Assume i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ of Lemma 5.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ U' \arrow[r] & X' \end{tikzcd}$$

where U, U' are schemes and the horizontal arrows are surjective and étale, see Spaces, Lemma 11.6. Then using Lemmas 5.2 and 5.4 we see that the question reduces to the case of a morphism of schemes. In the schemes case this is Morphisms, Lemma 31.4. $\square$

Lemma 5.6. *Let S be a scheme. Let $Z \to Y \to X$ be immersions of algebraic spaces. Then there is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 5.3 and $i : Z \to Y$ is the first morphism.

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. Via Lemmas 5.2 and 5.4 the exactness of the sequence translates immediately into the exactness of the corresponding sequence for the immersions of schemes $Z \times_X U \to Y \times_X U \to U$. Hence the lemma follows from Morphisms, Lemma 31.5. $\square$

6. The normal cone of an immersion

Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals, see Morphisms of Spaces, Lemma 13.1. Consider the quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. Since the sheaves $\mathcal{I}^n/\mathcal{I}^{n+1}$ are each annihilated by $\mathcal{I}$ this graded algebra corresponds to a quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras by Morphisms of Spaces, Lemma 14.1. This quasi-coherent graded $\mathcal{O}_Z$-algebra is called the *conormal algebra of Z in X* and is often simply denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ by the abuse of notation mentioned in Morphisms of Spaces, Section 14.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal algebra of i as the conormal algebra of the closed immersion $i : Z \to X \setminus \partial Z$, see Morphisms of Spaces, Remark 12.4. It is often denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 6.1. Let $i : Z \to X$ be an immersion. The *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X* or the *conormal algebra of i* is the quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ described above.

Thus $\mathcal{C}_{Z/X,1} = \mathcal{C}_{Z/X}$ is the conormal sheaf of the immersion. Also $\mathcal{C}_{Z/X,0} = \mathcal{O}_Z$ and $\mathcal{C}_{Z/X,n}$ is a quasi-coherent $\mathcal{O}_Z$-module characterized by the property

$$i_*\mathcal{C}_{Z/X,n} = \mathcal{I}^n/\mathcal{I}^{n+1} \quad (6.1.1)$$

where $i : Z \to X \setminus \partial Z$ and $\mathcal{I}$ is the ideal sheaf of i as above. Finally, note that there is a canonical surjective map

$$\mathrm{Sym}^*(\mathcal{C}_{Z/X}) \longrightarrow \mathcal{C}_{Z/X,*} \quad (6.1.2)$$

of quasi-coherent graded $\mathcal{O}_Z$-algebras which is an isomorphism in degrees 0 and 1.

Lemma 6.2. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Let $\varphi : U \to X$ be an étale morphism where U is a scheme. Set $Z_U = U \times_X Z$ which is a locally closed subscheme of U. Then*

$$\mathcal{C}_{Z/X,*}|_{Z_U} = \mathcal{C}_{Z_U/U,*}$$

canonically and functorially in U.

Proof. Let $T \subset X$ be a closed subspace such that i defines a closed immersion into $X \setminus T$. Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals on $X \setminus T$ defining Z. Then the lemma follows from the fact that $\mathcal{I}|_{U \setminus \varphi^{-1}(T)}$ is the sheaf of ideals of the immersion $Z_U \to U \setminus \varphi^{-1}(T)$. This is clear from the construction of $\mathcal{I}$ in Morphisms of Spaces, Lemma 13.1. $\square$

Lemma 6.3. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Assume i, i' immersions. There is a canonical map of graded $\mathcal{O}_Z$-algebras

$$f^*\mathcal{C}_{Z'/X',*} \longrightarrow \mathcal{C}_{Z/X,*}$$

Proof. First find open subspaces $U' \subset X'$ and $U \subset X$ such that $g(U) \subset U'$ and such that $i(Z) \subset U$ and $i(Z') \subset U'$ are closed (proof existence omitted). Replacing X by U and X' by U' we may assume that i and i' are closed immersions. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaves of ideals associated to i' and i, see Morphisms of Spaces, Lemma 13.1. Consider the composition

$$g^{-1}\mathcal{I}' \to g^{-1}\mathcal{O}_{X'} \xrightarrow{g^\sharp} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$$

Since $g(i(Z)) \subset Z'$ we conclude this composition is zero (see statement on factorizations in Morphisms of Spaces, Lemma 13.1). Thus we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] & i_*\mathcal{O}_Z \arrow[r] & 0 \\ 0 \arrow[r] & g^{-1}\mathcal{I}' \arrow[r] \arrow[u] & g^{-1}\mathcal{O}_{X'} \arrow[r] \arrow[u] & g^{-1}i'_*\mathcal{O}_{Z'} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The lower row is exact since g^{-1} is an exact functor. By exactness we also see that $(g^{-1}\mathcal{I}')^n = g^{-1}((\mathcal{I}')^n)$ for all $n \geq 1$. Hence the diagram induces a map $g^{-1}((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$. Pulling back (using i^{-1} for example) to Z we obtain $i^{-1}g^{-1}((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{C}_{Z/X,n}$. Since $i^{-1}g^{-1} = f^{-1}(i')^{-1}$ this gives maps $f^{-1}\mathcal{C}_{Z'/X',n} \to \mathcal{C}_{Z/X,n}$, which induce the desired map. $\square$

Lemma 6.4. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a cartesian square of algebraic spaces over S with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X',*} \to \mathcal{C}_{Z/X,*}$ of Lemma 6.3 is surjective. If g is flat, then it is an isomorphism.

Proof. We may check the statement after étale localizing X'. In this case we may assume $X' \to X$ is a morphism of schemes, hence Z and Z' are schemes and the result follows from the case of schemes, see Divisors, Lemma 19.4. $\square$

We use the same conventions for cones and vector bundles over algebraic spaces as we do for schemes (where we use the conventions of EGA), see Constructions, Sections 7 and 6. In particular, a vector bundle is a very general gadget (and not locally isomorphic to an affine space bundle).

Definition 6.5. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. The *normal cone* C_ZX of Z in X is

$$C_ZX = \underline{\mathrm{Spec}}_Z(\mathcal{C}_{Z/X,*})$$

see Morphisms of Spaces, Definition 20.8. The *normal bundle* of Z in X is the vector bundle

$$N_ZX = \underline{\mathrm{Spec}}_Z(\mathrm{Sym}(\mathcal{C}_{Z/X}))$$

Thus $C_ZX \to Z$ is a cone over Z and $N_ZX \to Z$ is a vector bundle over Z. Moreover, the canonical surjection (6.1.2) of graded algebras defines a canonical closed immersion

$$C_ZX \longrightarrow N_ZX \tag{6.5.1}$$

of cones over Z.

7. Sheaf of differentials of a morphism

We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 131), the corresponding section in the chapter on morphism of schemes (Morphisms, Section 32) as well as Modules on Sites, Section 33. We first show that the notion of sheaf of differentials for a morphism of schemes agrees with the corresponding morphism of small étale (ringed) sites.

To clearly state the following lemma we temporarily go back to denoting $\mathcal{F}^a$ the sheaf of $\mathcal{O}_{X_{\acute{e}tale}}$-modules associated to a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ on the scheme X, see Descent, Definition 8.2.

Lemma 7.1. *Let $f : X \to Y$ be a morphism of schemes. Let $f_{small} : X_{\acute{e}tale} \to Y_{\acute{e}tale}$ be the associated morphism of small étale sites, see Descent, Remark 8.4. Then there is a canonical isomorphism*

$$(\Omega_{X/Y})^a = \Omega_{X_{\acute{e}tale}/Y_{\acute{e}tale}}$$

compatible with universal derivations. Here the first module is the sheaf on $X_{\acute{e}tale}$ associated to the quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/Y}$, see Morphisms, Definition 32.1, and the second module is the one from Modules on Sites, Definition 33.3.

Proof. Let $h : U \to X$ be an étale morphism. In this case the natural map $h^*\Omega_{X/Y} \to \Omega_{U/Y}$ is an isomorphism, see More on Morphisms, Lemma 9.9. This means that there is a natural $\mathcal{O}_{Y_{\acute{e}tale}}$-derivation

$$\mathrm{d}^a : \mathcal{O}_{X_{\acute{e}tale}} \longrightarrow (\Omega_{X/Y})^a$$

since we have just seen that the value of $(\Omega_{X/Y})^a$ on any object U of $X_{étale}$ is canonically identified with $\Gamma(U, \Omega_{U/Y})$. By the universal property of $\mathrm{d}_{X/Y} : \mathcal{O}_{X_{étale}} \to \Omega_{X_{étale}/Y_{étale}}$ there is a unique $\mathcal{O}_{X_{étale}}$-linear map $c : \Omega_{X_{étale}/Y_{étale}} \to (\Omega_{X/Y})^a$ such that $\mathrm{d}^a = c \circ \mathrm{d}_{X/Y}$.

Conversely, suppose that $\mathcal{F}$ is an $\mathcal{O}_{X_{étale}}$-module and $D : \mathcal{O}_{X_{étale}} \to \mathcal{F}$ is a $\mathcal{O}_{Y_{étale}}$-derivation. Then we can simply restrict D to the small Zariski site X_{Zar} of X. Since sheaves on X_{Zar} agree with sheaves on X, see Descent, Remark 8.3, we see that $D|_{X_{Zar}} : \mathcal{O}_X \to \mathcal{F}|_{X_{Zar}}$ is just a "usual" Y-derivation. Hence we obtain a map $\psi : \Omega_{X/Y} \longrightarrow \mathcal{F}|_{X_{Zar}}$ such that $D|_{X_{Zar}} = \psi \circ \mathrm{d}$. In particular, if we apply this with $\mathcal{F} = \Omega_{X_{étale}/Y_{étale}}$ we obtain a map

$$c' : \Omega_{X/Y} \longrightarrow \Omega_{X_{étale}/Y_{étale}}|_{X_{Zar}}$$

Consider the morphism of ringed sites $\mathrm{id}_{small,étale,Zar} : X_{étale} \to X_{Zar}$ discussed in Descent, Remark 8.4 and Lemma 8.5. Since the restriction functor $\mathcal{F} \mapsto \mathcal{F}|_{X_{Zar}}$ is equal to $\mathrm{id}_{small,étale,Zar,*}$, since $\mathrm{id}^*_{small,étale,Zar}$ is left adjoint to $\mathrm{id}_{small,étale,Zar,*}$ and since $(\Omega_{X/Y})^a = \mathrm{id}^*_{small,étale,Zar}\Omega_{X/Y}$ we see that c' is adjoint to a map

$$c'' : (\Omega_{X/Y})^a \longrightarrow \Omega_{X_{étale}/Y_{étale}}.$$

We claim that c'' and c' are mutually inverse. This claim finishes the proof of the lemma. To see this it is enough to show that $c''(\mathrm{d}(f)) = \mathrm{d}_{X/Y}(f)$ and $c(\mathrm{d}_{X/Y}(f)) = \mathrm{d}(f)$ if f is a local section of $\mathcal{O}_X$ over an open of X. We omit the verification. $\square$

This clears the way for the following definition. For an alternative, see Remark 7.5.

Definition 7.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *sheaf of differentials* $\Omega_{X/Y}$ *of* X *over* Y is sheaf of differentials (Modules on Sites, Definition 33.10) for the morphism of ringed topoi

$$(f_{small}, f^\sharp) : (X_{étale}, \mathcal{O}_X) \to (Y_{étale}, \mathcal{O}_Y)$$

of Properties of Spaces, Lemma 21.3. The *universal Y-derivation* will be denoted $\mathrm{d}_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}$.

By Lemma 7.1 this does not conflict with the already existing notion in case X and Y are representable. From now on, if X and Y are representable, we no longer distinguish between the sheaf of differentials defined above and the one defined in Morphisms, Definition 32.1. We want to relate this to the usual modules of differentials for morphisms of schemes. Here is the key lemma.

Lemma 7.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Consider any commutative diagram*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where the vertical arrows are étale morphisms of algebraic spaces. Then

$$\Omega_{X/Y}|_{U_{étale}} = \Omega_{U/V}$$

In particular, if U, V are schemes, then this is equal to the usual sheaf of differentials of the morphism of schemes $U \to V$.

Proof. By Properties of Spaces, Lemma 18.11 and Equation (18.11.1) we may think of the restriction of a sheaf on $X_{étale}$ to $U_{étale}$ as the pullback by a_{small}. Similarly for b. By Modules on Sites, Lemma 33.6 we have

$$\Omega_{X/Y}|_{U_{étale}} = \Omega_{\mathcal{O}_{U_{étale}}/a_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}}}$$

Since $a_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}} = \psi_{small}^{-1}b_{small}^{-1}\mathcal{O}_{Y_{étale}} = \psi_{small}^{-1}\mathcal{O}_{V_{étale}}$ we see that the lemma holds. $\square$

Lemma 7.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. Choose a diagram as in Lemma 7.3 with a and b surjective and U and V schemes. Then we see that $\Omega_{X/Y}|_U = \Omega_{U/V}$ which is quasi-coherent (for example by Morphisms, Lemma 32.7). Hence we conclude that $\Omega_{X/Y}$ is quasi-coherent by Properties of Spaces, Lemma 29.6. $\square$

Remark 7.5. Now that we know that $\Omega_{X/Y}$ is quasi-coherent we can attempt to construct it in another manner. For example we can use the result of Properties of Spaces, Section 32 to construct the sheaf of differentials by glueing. For example if Y is a scheme and if $U \to X$ is a surjective étale morphism from a scheme towards X, then we see that $\Omega_{U/Y}$ is a quasi-coherent $\mathcal{O}_U$-module, and since $s, t : R \to U$ are étale we get an isomorphism

$$\alpha : s^*\Omega_{U/Y} \to \Omega_{R/Y} \to t^*\Omega_{U/Y}$$

by using Morphisms, Lemma 34.16. You check that this satisfies the cocycle condition and you're done. If Y is not a scheme, then you define $\Omega_{U/Y}$ as the cokernel of the map $(U \to Y)^*\Omega_{Y/S} \to \Omega_{U/S}$, and proceed as before. This two step process is a little bit ugly. Another possibility is to glue the sheaves $\Omega_{U/V}$ for any diagram as in Lemma 7.3 but this is not very elegant either. Both approaches will work however, and will give a slightly more elementary construction of the sheaf of differentials.

Lemma 7.6. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces. The map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*d_{X'/Y'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/Y'}$ is a Y-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/Y} \to f_*\Omega_{X'/Y'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

It is uniquely characterized by the property that $f^*d_{X/Y}(t)$ mapsto $d_{X'/Y'}(f^*t)$ for any local section t of $\mathcal{O}_X$.

Proof. This is a special case of Modules on Sites, Lemma 33.11. $\square$

Lemma 7.7. *Let S be a scheme. Let*

$$\begin{tikzcd} X'' \arrow[r, "g"'] \arrow[d] & X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ Y'' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Then we have

$$c_{f \circ g} = c_g \circ g^* c_f$$

as maps $(f \circ g)^*\Omega_{X/Y} \to \Omega_{X''/Y''}$.

Proof. Omitted. Hint: Use the characterization of $c_f, c_g, c_{f \circ g}$ in terms of the effect these maps have on local sections. □

Lemma 7.8. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to B$ be morphisms of algebraic spaces over S. Then there is a canonical exact sequence*

$$f^*\Omega_{Y/B} \to \Omega_{X/B} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 7.6.

Proof. Follows from the schemes version, see Morphisms, Lemma 32.9, of this result via étale localization, see Lemma 7.3. □

Lemma 7.9. *Let S be a scheme. If $X \to Y$ is an immersion of algebraic spaces over S then $\Omega_{X/S}$ is zero.*

Proof. Follows from the schemes version, see Morphisms, Lemma 32.14, of this result via étale localization, see Lemma 7.3. □

Lemma 7.10. *Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of algebraic spaces over B. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

where the first arrow is induced by $\mathrm{d}_{X/B}$ and the second arrow comes from Lemma 7.6.

Proof. This is the algebraic spaces version of Morphisms, Lemma 32.15 and will be a consequence of that lemma by étale localization, see Lemmas 7.3 and 5.2. However, we should make sure we can define the first arrow globally. Hence we explain the meaning of "induced by $\mathrm{d}_{X/B}$" here. Namely, we may assume that i is a closed immersion after replacing X by an open subspace. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to $Z \subset X$. Then $\mathrm{d}_{X/S} : \mathcal{I} \to \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$ which is $\mathcal{O}_X/\mathcal{I}$-linear. By Morphisms of Spaces, Lemma 14.1 this corresponds to a map $\mathcal{C}_{Z/X} \to i^*\Omega_{X/S}$ as desired. □

Lemma 7.11. *Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of algebraic spaces over B, and assume i (étale locally) has a left inverse. Then the canonical sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 7.10 is (étale locally) split exact.

Proof. Clarification: we claim that if $g : X \to Z$ is a left inverse of i over B, then i^*c_g is a right inverse of the map $i^*\Omega_{X/B} \to \Omega_{Z/B}$. Having said this, the result follows from the corresponding result for morphisms of schemes by étale localization, see Lemmas 7.3 and 5.2. $\square$

Lemma 7.12. *Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Let $X' = X_{Y'}$ be the base change of X. Denote $g' : X' \to X$ the projection. Then the map*

$$(g')^*\Omega_{X/Y} \to \Omega_{X'/Y'}$$

of Lemma 7.6 is an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 32.10 and étale localization, see Lemma 7.3. $\square$

Lemma 7.13. *Let S be a scheme. Let $f : X \to B$ and $g : Y \to B$ be morphisms of algebraic spaces over S with the same target. Let $p : X \times_B Y \to X$ and $q : X \times_B Y \to Y$ be the projection morphisms. The maps from Lemma 7.6*

$$p^*\Omega_{X/B} \oplus q^*\Omega_{Y/B} \longrightarrow \Omega_{X \times_B Y/B}$$

give an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 32.11 and étale localization, see Lemma 7.3. $\square$

Lemma 7.14. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $\Omega_{X/Y}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Follows from the schemes version, see Morphisms, Lemma 32.12 and étale localization, see Lemma 7.3. $\square$

Lemma 7.15. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite presentation, then $\Omega_{X/Y}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Follows from the schemes version, see Morphisms, Lemma 32.13 and étale localization, see Lemma 7.3. $\square$

Lemma 7.16. *Let S be a scheme. Let $f : X \to Y$ be a smooth morphism of algebraic spaces over S. Then the module of differentials $\Omega_{X/Y}$ is finite locally free.*

Proof. The statement is étale local on X and Y by Lemma 7.3. Hence this follows from the case of schemes, see Morphisms, Lemma 34.12. $\square$

8. Topological invariance of the étale site

We show that the site $X_{spaces,étale}$ is a "topological invariant". It then follows that $X_{étale}$, which consists of the representable objects in $X_{spaces,étale}$, is a topological invariant too, see Lemma 8.2.

Theorem 8.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is integral, universally injective and surjective. The functor*

$$V \longmapsto V_X = X \times_Y V$$

defines an equivalence of categories $Y_{spaces,étale} \to X_{spaces,étale}$.

Proof. The morphism f is representable and a universal homeomorphism, see Morphisms of Spaces, Section 53.

We first prove that the functor is faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $a, b : V' \to V$ are distinct morphisms over Y. Since V', V are étale over Y the equalizer

$$E = V' \times_{(a,b), V \times_Y V, \Delta_{V/Y}} V$$

of a, b is étale over Y also. Hence $E \to V'$ is an étale monomorphism (i.e., an open immersion) which is an isomorphism if and only if it is surjective. Since $X \to Y$ is a universal homeomorphism we see that this is the case if and only if $E_X = V'_X$, i.e., if and only if $a_X = b_X$.

Next, we prove that the functor is fully faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $c : V'_X \to V_X$ is a morphism over X. We want to construct a morphism $a : V' \to V$ over Y such that $a_X = c$. Let $a' : V'' \to V'$ be a surjective étale morphism such that V'' is a separated algebraic space. If we can construct a morphism $a'' : V'' \to V$ such that $a''_X = c \circ a'_X$, then the two compositions

$$V'' \times_{V'} V'' \xrightarrow{\text{pr}_i} V'' \xrightarrow{a''} V$$

will be equal by the faithfulness of the functor proved in the first paragraph. Hence a'' will factor through a unique morphism $a : V' \to V$ as V' is (as a sheaf) the quotient of V'' by the equivalence relation $V'' \times_{V'} V''$. Hence we may assume that V' is separated. In this case the graph

$$\Gamma_c \subset (V' \times_Y V)_X$$

is open and closed (details omitted). Since $X \to Y$ is a universal homeomorphism, there exists an open and closed subspace $\Gamma \subset V' \times_Y V$ such that $\Gamma_X = \Gamma_c$. The projection $\Gamma \to V'$ is an étale morphism whose base change to X is an isomorphism. Hence $\Gamma \to V'$ is étale, universally injective, and surjective, so an isomorphism by Morphisms of Spaces, Lemma 51.2. Thus Γ is the graph of a morphism $a : V' \to V$ as desired.

Finally, we prove that the functor is essentially surjective. Suppose that U is an object of $X_{spaces,étale}$. We have to find an object V of $Y_{spaces,étale}$ such that $V_X \cong U$. Let $U' \to U$ be a surjective étale morphism such that $U' \cong V'_X$ and $U' \times_U U' \cong V''_X$ for some objects V'', V' of $Y_{spaces,étale}$. Then by fully faithfulness of the functor we obtain morphisms $s, t : V'' \to V'$ with $t_X = \text{pr}_0$ and $s_X = \text{pr}_1$ as morphisms $U' \times_U U' \to U'$. Using that $(\text{pr}_0, \text{pr}_1) : U' \times_U U' \to U' \times_S U'$ is an étale equivalence relation, and that $U' \to V'$ and $U' \times_U U' \to V''$ are universally injective and surjective we deduce that $(t, s) : V'' \to V' \times_S V'$ is an étale equivalence relation. Then the quotient $V = V'/V''$ (see Spaces, Theorem 10.5) is an algebraic space V over Y. There is a morphism $V' \to V$ such that $V'' = V' \times_V V'$. Thus we obtain a morphism $V \to Y$ (see Descent on Spaces, Lemma 7.2). On base change to X we see that we have a morphism $U' \to V_X$ and a compatible isomorphism $U' \times_{V_X} U' = U' \times_U U'$, which implies that $V_X \cong U$ (by the lemma just cited once more).

Pick a scheme W and a surjective étale morphism $W \to Y$. Pick a scheme U' and a surjective étale morphism $U' \to U \times_X W_X$. Note that U' and $U' \times_U U'$ are schemes étale over X whose structure morphism to X factors through the scheme

W_X. Hence by Étale Cohomology, Theorem 45.2 there exist schemes V', V'' étale over W whose base change to W_X is isomorphic to respectively U' and $U' \times_U U'$. This finishes the proof. $\square$

Lemma 8.2. *With assumption and notation as in Theorem 8.1 the equivalence of categories $Y_{spaces,étale} \to X_{spaces,étale}$ restricts to equivalences of categories $Y_{étale} \to X_{étale}$ and $Y_{affine,étale} \to X_{affine,étale}$.*

Proof. This is just the statement that given an object $V \in Y_{spaces,étale}$ we have V is a(n affine) scheme if and only if $V \times_Y X$ is a(n affine) scheme. Since $V \times_Y X \to V$ is integral, universally injective, and surjective (as a base change of $X \to Y$) this follows from Limits of Spaces, Lemma 15.4 and Proposition 15.2. $\square$

Remark 8.3. A universal homeomorphism of algebraic spaces need not be representable, see Morphisms of Spaces, Example 53.3. In fact Theorem 8.1 does not hold for universal homeomorphisms. To see this, let k be an algebraically closed field of characteristic 0 and let

$$\mathbf{A}^1 \to X \to \mathbf{A}^1$$

be as in Morphisms of Spaces, Example 53.3. Recall that the first morphism is étale and identifies t with $-t$ for $t \in \mathbf{A}^1_k \setminus \{0\}$ and that the second morphism is our universal homeomorphism. Since $\mathbf{A}^1_k$ has no nontrivial connected finite étale coverings (because k is algebraically closed of characteristic zero; details omitted), it suffices to construct a nontrivial connected finite étale covering $Y \to X$. To do this, let Y be the affine line with zero doubled (Schemes, Example 14.3). Then $Y = Y_1 \cup Y_2$ with $Y_i = \mathbf{A}^1_k$ glued along $\mathbf{A}^1_k \setminus \{0\}$. To define the morphism $Y \to X$ we use the morphisms

$$Y_1 \xrightarrow{1} \mathbf{A}^1_k \to X \quad \text{and} \quad Y_2 \xrightarrow{-1} \mathbf{A}^1_k \to X.$$

These glue over $Y_1 \cap Y_2$ by the construction of X and hence define a morphism $Y \to X$. In fact, we claim that

$$\begin{tikzcd}
Y \arrow[d] & Y_1 \amalg Y_2 \arrow[l] \arrow[d] \\
X & \mathbf{A}^1_k \arrow[l]
\end{tikzcd}$$

is a cartesian square. We omit the details; you can use for example Groupoids, Lemma 20.7. Since $\mathbf{A}^1_k \to X$ is étale and surjective, this proves that $Y \to X$ is finite étale of degree 2 which gives the desired example.

More simply, you can argue as follows. The scheme Y has a free action of the group $G = \{+1, -1\}$ where -1 acts by swapping Y_1 and Y_2 and changing the sign of the coordinate. Then $X = Y/G$ (see Spaces, Definition 14.4) and hence $Y \to X$ is finite étale. You can also show directly that there exists a universal homeomorphism $X \to \mathbf{A}^1_k$ by using $t \mapsto t^2$ on affine spaces. In fact, this X is the same as the X above.

9. Thickenings

The following terminology may not be completely standard, but it is convenient.

Definition 9.1. Thickenings. Let S be a scheme.

(1) We say an algebraic space X' is a *thickening* of an algebraic space X if X is a closed subspace of X' and the associated topological spaces are equal.
(2) We say X' is a *first order thickening* of X if X is a closed subspace of X' and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X has square zero.
(3) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a *morphism of thickenings* is a morphism $f' : X' \to Y'$ such that $f(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subspace Y. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.
(4) Let B be an algebraic space. We similarly define *thickenings over* B, and *morphisms of thickenings over* B. This means that the spaces X, X', Y, Y' above are algebraic spaces endowed with a structure morphism to B, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over B.

The fundamental equivalence. Note that if $X \subset X'$ is a thickening, then $X \to X'$ is integral and universally bijective. This implies that

$$X_{spaces,étale} = X'_{spaces,étale} \tag{9.1.1}$$

via the pullback functor, see Theorem 8.1. Hence we may think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,étale}$. Thus a canonical equivalence of locally ringed topoi

$$(Sh(X'_{spaces,étale}), \mathcal{O}_{X'}) \cong (Sh(X_{spaces,étale}), \mathcal{O}_{X'}) \tag{9.1.2}$$

Below we will frequently combine this with the fully faithfulness result of Properties of Spaces, Theorem 28.4. For example the closed immersion $i_X : X \to X'$ corresponds to the surjective map $i_X^\sharp : \mathcal{O}_{X'} \to \mathcal{O}_X$.

Let S be a scheme, and let B be an algebraic space over S. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings over B. Note that the diagram of continuous functors

$$\begin{tikzcd} X_{spaces,étale} & Y_{spaces,étale} \arrow[l] \\ X'_{spaces,étale} \arrow[u] & Y'_{spaces,étale} \arrow[l] \arrow[u] \end{tikzcd}$$

is commutative and the vertical arrows are equivalences. Hence $f_{spaces,étale}$, f_{small}, $f'_{spaces,étale}$, and f'_{small} all define the same morphism of topoi. Thus we may think of

$$(f')^\sharp : f^{-1}_{spaces,étale}\mathcal{O}_{Y'} \longrightarrow \mathcal{O}_{X'}$$

as a map of sheaves of $\mathcal{O}_B$-algebras fitting into the commutative diagram

$$\begin{tikzcd} f^{-1}_{spaces,étale}\mathcal{O}_Y \arrow[r, "f^\sharp"'] & \mathcal{O}_X \\ f^{-1}_{spaces,étale}\mathcal{O}_{Y'} \arrow[u, "i_Y^\sharp"] \arrow[r, "(f')^\sharp"] & \mathcal{O}_{X'} \arrow[u, "i_X^\sharp"'] \end{tikzcd}$$

Here $i_X : X \to X'$ and $i_Y : Y \to Y'$ are the names of the given closed immersions.

Lemma 9.2. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be thickenings of algebraic spaces over B. Let $f : X \to Y$ be a morphism of algebraic spaces over B. Given any map of $\mathcal{O}_B$-algebras*

$$\alpha : f_{spaces,étale}^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

such that

$$\begin{tikzcd} f_{spaces,étale}^{-1}\mathcal{O}_Y \arrow[r, "f^\sharp"'] & \mathcal{O}_X \\ f_{spaces,étale}^{-1}\mathcal{O}_{Y'} \arrow[u, "i_Y^\sharp"] \arrow[r, "\alpha"] & \mathcal{O}_{X'} \arrow[u, "i_X^\sharp"'] \end{tikzcd}$$

commutes, there exists a unique morphism of (f, f') of thickenings over B such that $\alpha = (f')^\sharp$.

Proof. To find f', by Properties of Spaces, Theorem 28.4, all we have to do is show that the morphism of ringed topoi

$$(f_{spaces,étale}, \alpha) : (Sh(X_{spaces,étale}), \mathcal{O}_{X'}) \longrightarrow (Sh(Y_{spaces,étale}), \mathcal{O}_{Y'})$$

is a morphism of locally ringed topoi. This follows directly from the definition of morphisms of locally ringed topoi (Modules on Sites, Definition 40.9), the fact that $(f, f^\sharp)$ is a morphism of locally ringed topoi (Properties of Spaces, Lemma 28.1), that α fits into the given commutative diagram, and the fact that the kernels of $i_X^\sharp$ and $i_Y^\sharp$ are locally nilpotent. Finally, the fact that $f' \circ i_X = i_Y \circ f$ follows from the commutativity of the diagram and another application of Properties of Spaces, Theorem 28.4. We omit the verification that f' is a morphism over B. $\square$

Lemma 9.3. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. For any open subspace $U \subset X$ there exists a unique open subspace $U' \subset X'$ such that $U = X \times_{X'} U'$.*

Proof. Let $U' \to X'$ be the object of $X'_{spaces,étale}$ corresponding to the object $U \to X$ of $X_{spaces,étale}$ via (9.1.1). The morphism $U' \to X'$ is étale and universally injective, hence an open immersion, see Morphisms of Spaces, Lemma 51.2. $\square$

Finite order thickenings. Let $i_X : X \to X'$ be a thickening of algebraic spaces. Any local section of the kernel $\mathcal{I} = \text{Ker}(i_X^\sharp) \subset \mathcal{O}_{X'}$ is locally nilpotent. Let us say that $X \subset X'$ is a *finite order thickening* if the ideal sheaf $\mathcal{I}$ is "globally" nilpotent, i.e., if there exists an $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$. Technically the class of finite order thickenings $X \subset X'$ is much easier to handle than the general case. Namely, in this case we have a filtration

$$0 \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that X' is filtered by closed subspaces

$$X = X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over B. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

Lemma 9.4. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. Let U be an affine object of $X_{spaces,étale}$. Then*

$$\Gamma(U, \mathcal{O}_{X'}) \to \Gamma(U, \mathcal{O}_X)$$

is surjective where we think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,étale}$ via (9.1.2).

Proof. Let $U' \to X'$ be the étale morphism of algebraic spaces such that $U = X \times_{X'} U'$, see Theorem 8.1. By Limits of Spaces, Lemma 15.1 we see that U' is an affine scheme. Hence $\Gamma(U, \mathcal{O}_{X'}) = \Gamma(U', \mathcal{O}_{U'}) \to \Gamma(U, \mathcal{O}_U)$ is surjective as $U \to U'$ is a closed immersion of affine schemes. Below we give a direct proof for finite order thickenings which is the case most used in practice. □

Proof for finite order thickenings. We may assume that $X \subset X'$ is a first order thickening by the principle explained above. Denote $\mathcal{I}$ the kernel of the surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$. As $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_{X'}$-module and since $\mathcal{I}^2 = 0$ by the definition of a first order thickening we may apply Morphisms of Spaces, Lemma 14.1 to see that $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module. Hence the lemma follows from the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

and the fact that $H^1_{étale}(U, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 9.3 and Cohomology of Schemes, Lemma 2.2. □

Lemma 9.5. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. If X is (representable by) a scheme, then so is X'.*

Proof. Note that $X'_{red} = X_{red}$. Hence if X is a scheme, then X'_{red} is a scheme. Thus the result follows from Limits of Spaces, Lemma 15.3. Below we give a direct proof for finite order thickenings which is the case most often used in practice. □

Proof for finite order thickenings. It suffices to prove this when X' is a first order thickening of X. By Properties of Spaces, Lemma 13.1 there is a largest open subspace of X' which is a scheme. Thus we have to show that every point x of $|X'| = |X|$ is contained in an open subspace of X' which is a scheme. Using Lemma 9.3 we may replace $X \subset X'$ by $U \subset U'$ with $x \in U$ and U an affine scheme. Hence we may assume that X is affine. Thus we reduce to the case discussed in the next paragraph.

Assume $X \subset X'$ is a first order thickening where X is an affine scheme. Set $A = \Gamma(X, \mathcal{O}_X)$ and $A' = \Gamma(X', \mathcal{O}_{X'})$. By Lemma 9.4 the map $A \to A'$ is surjective. The kernel I is an ideal of square zero. By Properties of Spaces, Lemma 33.1 we obtain a canonical morphism $f : X' \to \mathrm{Spec}(A')$ which fits into the following commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & X' \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(A') \end{tikzcd}$$

Because the horizontal arrows are thickenings it is clear that f is universally injective and surjective. Hence it suffices to show that f is étale, since then Morphisms of Spaces, Lemma 51.2 will imply that f is an isomorphism.

To prove that f is étale choose an affine scheme U' and an étale morphism $U' \to X'$. It suffices to show that $U' \to X' \to \mathrm{Spec}(A')$ is étale, see Properties of Spaces, Definition 16.2. Write $U' = \mathrm{Spec}(B')$. Set $U = X \times_{X'} U'$. Since U is a closed subspace of U', it is a closed subscheme, hence $U = \mathrm{Spec}(B)$ with $B' \to B$ surjective. Denote $J = \mathrm{Ker}(B' \to B)$ and note that $J = \Gamma(U, \mathcal{I})$ where $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X)$ on $X_{spaces,étale}$ as in the proof of Lemma 9.4. The morphism $U' \to X' \to \mathrm{Spec}(A')$ induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now, since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module we have $\mathcal{I} = (\widetilde{I})^a$, see Descent, Definition 8.2 for notation and Descent, Proposition 8.9 for why this is true. Hence we see that $J = I \otimes_A B$. Finally, note that $A \to B$ is étale as $U \to X$ is étale as the base change of the étale morphism $U' \to X'$. We conclude that $A' \to B'$ is étale by Algebra, Lemma 143.11. $\square$

Lemma 9.6. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. The functor*

$$V' \longmapsto V = X \times_{X'} V'$$

defines an equivalence of categories $X'_{étale} \to X_{étale}$.

Proof. The functor $V' \mapsto V$ defines an equivalence of categories $X'_{spaces,étale} \to X_{spaces,étale}$, see Theorem 8.1. Thus it suffices to show that V is a scheme if and only if V' is a scheme. This is the content of Lemma 9.5. $\square$

First order thickening are described as follows.

Lemma 9.7. *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on $X_{étale}$ where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_B$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_B$-algebras, and $\mathcal{I}$ is its kernel. If

1. *$\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and*
2. *$\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module*

then there exists a first order thickening $X \subset X'$ over B and an isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ of $f^{-1}\mathcal{O}_B$-algebras compatible with the surjections to $\mathcal{O}_X$.

Proof. In this proof we redo some of the arguments used in the proofs of Lemmas 9.4 and 9.5. We first handle the case $B = S = \mathrm{Spec}(\mathbf{Z})$. Let U be an affine scheme, and let $U \to X$ be étale. Then

$$0 \to \mathcal{I}(U) \to \mathcal{A}(U) \to \mathcal{O}_X(U) \to 0$$

is exact as $H^1(U_{étale}, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 9.3 and Cohomology of Schemes, Lemma 2.2. If $V \to U$ is a morphism of affine objects of $X_{spaces,étale}$ then

$$\mathcal{I}(V) = \mathcal{I}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$$

since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module, see Descent, Proposition 8.9. Hence $\mathcal{A}(U) \to \mathcal{A}(V)$ is an étale ring map, see Algebra, Lemma 143.11. Hence we see that

$$U \longmapsto U' = \text{Spec}(\mathcal{A}(U))$$

is a functor from $X_{affine,étale}$ to the category of affine schemes and étale morphisms. In fact, we claim that this functor can be extended to a functor $U \mapsto U'$ on all of $X_{étale}$. To see this, if U is an object of $X_{étale}$, note that

$$0 \to \mathcal{I}|_{U_{Zar}} \to \mathcal{A}|_{U_{Zar}} \to \mathcal{O}_X|_{U_{Zar}} \to 0$$

and $\mathcal{I}|_{U_{Zar}}$ is a quasi-coherent sheaf on U, see Descent, Proposition 9.4. Hence by More on Morphisms, Lemma 2.2 we obtain a first order thickening $U \subset U'$ of schemes such that $\mathcal{O}_{U'}$ is isomorphic to $\mathcal{A}|_{U_{Zar}}$. It is clear that this construction is compatible with the construction for affines above.

Choose a presentation $X = U/R$, see Spaces, Definition 9.3 so that $s, t : R \to U$ define an étale equivalence relation. Applying the functor above we obtain an étale equivalence relation $s', t' : R' \to U'$ in schemes. Consider the algebraic space $X' = U'/R'$ (see Spaces, Theorem 10.5). The morphism $X = U/R \to U'/R' = X'$ is a first order thickening. Consider $\mathcal{O}_{X'}$ viewed as a sheaf on $X_{étale}$. By construction we have an isomorphism

$$\gamma : \mathcal{O}_{X'}|_{U_{étale}} \longrightarrow \mathcal{A}|_{U_{étale}}$$

such that $s^{-1}\gamma$ agrees with $t^{-1}\gamma$ on $R_{étale}$. Hence by Properties of Spaces, Lemma 18.14 this implies that γ comes from a unique isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ as desired.

To handle the case of a general base algebraic space B, we first construct X' as an algebraic space over $\mathbf{Z}$ as above. Then we use the isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ to define $f^{-1}\mathcal{O}_B \to \mathcal{O}_{X'}$. According to Lemma 9.2 this defines a morphism $X' \to B$ compatible with the given morphism $X \to B$ and we are done. □

Lemma 9.8. *Let S be a scheme. Let $Y \subset Y'$ be a thickening of algebraic spaces over S. Let $X' \to Y'$ be a morphism and set $X = Y \times_{Y'} X'$. Then $(X \subset X') \to (Y \subset Y')$ is a morphism of thickenings. If $Y \subset Y'$ is a first (resp. finite order) thickening, then $X \subset X'$ is a first (resp. finite order) thickening.*

Proof. Omitted. □

Lemma 9.9. *Let S be a scheme. If $X \subset X'$ and $X' \subset X''$ are thickenings of algebraic spaces over S, then so is $X \subset X''$.*

Proof. Omitted. □

Lemma 9.10. *The property of being a thickening is fpqc local. Similarly for first order thickenings.*

Proof. The statement means the following: Let S be a scheme and let $X \to X'$ be a morphism of algebraic spaces over S. Let $\{g_i : X'_i \to X'\}$ be an fpqc covering of algebraic spaces such that the base change $X_i \to X'_i$ is a thickening for all i. Then $X \to X'$ is a thickening. Since the morphisms g_i are jointly surjective we conclude that $X \to X'$ is surjective. By Descent on Spaces, Lemma 11.17 we conclude that $X \to X'$ is a closed immersion. Thus $X \to X'$ is a thickening. We omit the proof in the case of first order thickenings. □

10. Morphisms of thickenings

If $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings of algebraic spaces, then often properties of the morphism f are inherited by f'. There are several variants.

Lemma 10.1. *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings of algebraic spaces over S. Then*

(1) *f is an affine morphism if and only if f' is an affine morphism,*
(2) *f is a surjective morphism if and only if f' is a surjective morphism,*
(3) *f is quasi-compact if and only if f' quasi-compact,*
(4) *f is universally closed if and only if f' is universally closed,*
(5) *f is integral if and only if f' is integral,*
(6) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
(7) *f is universally injective if and only if f' is universally injective,*
(8) *f is universally open if and only if f' is universally open,*
(9) *f is representable if and only if f' is representable, and*
(10) *add more here.*

Proof. Observe that $Y \to Y'$ and $X \to X'$ are integral and universal homeomorphisms. This immediately implies parts (2), (3), (4), (7), and (8). Part (1) follows from Limits of Spaces, Proposition 15.2 which tells us that there is a 1-to-1 correspondence between affine schemes étale over X and X' and between affine schemes étale over Y and Y'. Part (5) follows from (1) and (4) by Morphisms of Spaces, Lemma 45.7. Finally, note that

$$X \times_Y X = X \times_{Y'} X \to X \times_{Y'} X' \to X' \times_{Y'} X'$$

is a thickening (the two arrows are thickenings by Lemma 9.8). Hence applying (3) and (4) to the morphism $(X \subset X') \to (X \times_Y X \to X' \times_{Y'} X')$ we obtain (6). Finally, part (9) follows from the fact that an algebraic space thickening of a scheme is again a scheme, see Lemma 9.5. $\square$

Lemma 10.2. *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings of algebraic spaces over S such that $X = Y \times_{Y'} X'$. If $X \subset X'$ is a finite order thickening, then*

(1) *f is a closed immersion if and only if f' is a closed immersion,*
(2) *f is locally of finite type if and only if f' is locally of finite type,*
(3) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
(4) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
(5) *$\Omega_{X/Y} = 0$ if and only if $\Omega_{X'/Y'} = 0$,*
(6) *f is unramified if and only if f' is unramified,*
(7) *f is proper if and only if f' is proper,*
(8) *f is a finite morphism if and only if f' is an finite morphism,*
(9) *f is a monomorphism if and only if f' is a monomorphism,*
(10) *f is an immersion if and only if f' is an immersion, and*
(11) *add more here.*

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $V = Y \times_{Y'} V'$ and $U = X \times_{X'} U'$. Then for étale local properties of morphisms we can reduce to the morphism of thickenings of schemes $(U \subset U') \to (V \subset V')$ and apply More on Morphisms, Lemma 3.3. This proves (2), (3), (4), (5), and (6).

The properties of morphisms in (1), (7), (8), (9), (10) are stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Spaces, Lemma 12.3, and Morphisms of Spaces, Lemmas 40.3, 45.5, and 10.5.

The interesting direction in (1), (7), (8), (9), (10) is to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $Y \subset Y'$ is a first order thickening, see discussion on finite order thickenings above.

Proof of (1). Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$ and $U = X \times_Y V$. Then $U \to V$ is a closed immersion, which implies that U is a scheme, which in turn implies that U' is a scheme (Lemma 9.5). Thus we can apply the lemma in the case of schemes (More on Morphisms, Lemma 3.3) to $(U \subset U') \to (V \subset V')$ to conclude.

Proof of (7). Follows by combining (2) with results of Lemma 10.1 and the fact that proper equals quasi-compact + separated + locally of finite type + universally closed.

Proof of (8). Follows by combining (2) with results of Lemma 10.1 and using the fact that finite equals integral + locally of finite type (Morphisms, Lemma 44.4).

Proof of (9). As f is a monomorphism we have $X = X \times_Y X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_Y X \subset X' \times_{Y'} X')$. We conclude $X' \to X' \times_{Y'} X'$ is a closed immersion by (1). In fact, it is a first order thickening as the ideal defining the closed immersion $X' \to X' \times_{Y'} X'$ is contained in the pullback of the ideal $\mathcal{I} \subset \mathcal{O}_{Y'}$ cutting out Y in Y'. Indeed, $X = X \times_Y X = (X' \times_{Y'} X') \times_{Y'} Y$ is contained in X'. The conormal sheaf of the closed immersion $\Delta : X' \to X' \times_{Y'} X'$ is equal to $\Omega_{X'/Y'}$ (this is the analogue of Morphisms, Lemma 32.7 for algebraic spaces and follows either by étale localization or by combining Lemmas 7.11 and 7.13; some details omitted). Thus it suffices to show that $\Omega_{X'/Y'} = 0$ which follows from (5) and the corresponding statement for X/Y.

Proof of (10). If $f : X \to Y$ is an immersion, then it factors as $X \to V \to Y$ where $V \to Y$ is an open subspace and $X \to V$ is a closed immersion, see Morphisms of Spaces, Remark 12.4. Let $V' \subset Y'$ be the open subspace whose underlying topological space $|V'|$ is the same as $|V| \subset |Y| = |Y'|$. Then $X' \to Y'$ factors through V' and we conclude that $X' \to V'$ is a closed immersion by part (1). This finishes the proof. $\square$

The following lemma is a variant on the preceding one. Rather than assume that the thickenings involved are finite order (which allows us to transfer the property of being locally of finite type from f to f'), we instead take as given that each of f and f' is locally of finite type.

Lemma 10.3. *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \to Y')$ be a morphism of thickenings of algebraic spaces over S. Assume f and f' are locally of finite type and $X = Y \times_{Y'} X'$. Then*

1. *f is locally quasi-finite if and only if f' is locally quasi-finite,*
2. *f is finite if and only if f' is finite,*

(3) *f is a closed immersion if and only if f' is a closed immersion,*
(4) *$\Omega_{X/Y} = 0$ if and only if $\Omega_{X'/Y'} = 0$,*
(5) *f is unramified if and only if f' is unramified,*
(6) *f is a monomorphism if and only if f' is a monomorphism,*
(7) *f is an immersion if and only if f' is an immersion,*
(8) *f is proper if and only if f' is proper, and*
(9) *add more here.*

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $V = Y \times_{Y'} V'$ and $U = X \times_{X'} U'$. Then for étale local properties of morphisms we can reduce to the morphism of thickenings of schemes $(U \subset U') \to (V \subset V')$ and apply More on Morphisms, Lemma 3.4. This proves (1), (4), and (5).

The properties in (2), (3), (6), (7), and (8) are stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Spaces, Lemma 12.3, and Morphisms of Spaces, Lemmas 40.3, 45.5, and 10.5. Hence in each case we need only to prove that if f has the desired property, so does f'.

Case (2) follows from case (5) of Lemma 10.1 and the fact that the finite morphisms are precisely the integral morphisms that are locally of finite type (Morphisms of Spaces, Lemma 45.6).

Case (3). This follows immediately from Limits of Spaces, Lemma 15.5.

Proof of (6). As f is a monomorphism we have $X = X \times_Y X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_Y X \subset X' \times_{Y'} X')$. We conclude $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$ is a closed immersion by (3). In fact $\Delta_{X'/Y'}$ induces a bijection $|X'| \to |X' \times_{Y'} X'|$, hence $\Delta_{X'/Y'}$ is a thickening. On the other hand $\Delta_{X'/Y'}$ is locally of finite presentation by Morphisms of Spaces, Lemma 28.10. In other words, $\Delta_{X'/Y'}(X')$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{X' \times_{Y'} X'}$ of finite type. Since $\Omega_{X'/Y'} = 0$ by (5) we see that the conormal sheaf of $X' \to X' \times_{Y'} X'$ is zero. (The conormal sheaf of the closed immersion $\Delta_{X'/Y'}$ is equal to $\Omega_{X'/Y'}$; this is the analogue of Morphisms, Lemma 32.7 for algebraic spaces and follows either by étale localization or by combining Lemmas 7.11 and 7.13; some details omitted.) In other words, $\mathcal{J}/\mathcal{J}^2 = 0$. This implies $\Delta_{X'/Y'}$ is an isomorphism, for example by Algebra, Lemma 21.5.

Proof of (7). If $f : X \to Y$ is an immersion, then it factors as $X \to V \to Y$ where $V \to Y$ is an open subspace and $X \to V$ is a closed immersion, see Morphisms of Spaces, Remark 12.4. Let $V' \subset Y'$ be the open subspace whose underlying topological space $|V'|$ is the same as $|V| \subset |Y| = |Y'|$. Then $X' \to Y'$ factors through V' and we conclude that $X' \to V'$ is a closed immersion by part (3).

Case (8) follows from Lemma 10.1 and the definition of proper morphisms as being the quasi-compact, universally closed, and separated morphisms that are locally of finite type. $\square$

11. Picard groups of thickenings

Some material on Picard groups of thickenings.

Lemma 11.1. *Let S be a scheme. Let $X \subset X'$ be a first order thickening of algebraic spaces over S with ideal sheaf $\mathcal{I}$. Then there is a canonical exact sequence*

$$\begin{array}{l} 0 \to H^0(X, \mathcal{I}) \to H^0(X', \mathcal{O}^*_{X'}) \to H^0(X, \mathcal{O}^*_X) \to \\ \to H^1(X, \mathcal{I}) \to \mathrm{Pic}(X') \to \mathrm{Pic}(X) \to \\ \to H^2(X, \mathcal{I}) \to \ldots \to \ldots \end{array}$$

of abelian groups.

Proof. Recall that $X_{\acute{e}tale} = X'_{\acute{e}tale}$, see Lemma 9.6 and more generally the discussion in Section 9. The sequence of the lemma is the long exact cohomology sequence associated to the short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{I} \to \mathcal{O}^*_{X'} \to \mathcal{O}^*_X \to 0$$

on $X_{\acute{e}tale}$ where the first map sends a local section f of $\mathcal{I}$ to the invertible section $1 + f$ of $\mathcal{O}_{X'}$. We also use the identification of the Picard group of a ringed site with the first cohomology group of the sheaf of invertible functions, see Cohomology on Sites, Lemma 6.1. $\square$

12. First order infinitesimal neighbourhood

A natural construction of first order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of algebraic spaces. Choose an open subspace $U \subset X$ such that i identifies Z with a closed subspace $Z \subset U$ (see Morphisms of Spaces, Remark 12.4). Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining Z in U, see Morphisms of Spaces, Lemma 13.1. Then we can consider the closed subspace $Z' \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

Definition 12.1. Let $i : Z \to X$ be an immersion of algebraic spaces. The *first order infinitesimal neighbourhood* of Z in X is the first order thickening $Z \subset Z'$ over X described above.

This thickening has the following universal property (which will assuage any fears that the construction above depends on the choice of the open U).

Lemma 12.2. *Let $i : Z \to X$ be an immersion of algebraic spaces. The first order infinitesimal neighbourhood Z' of Z in X has the following universal property: Given any commutative diagram*

$$\begin{tikzcd} Z \arrow[d, "i"'] & T \arrow[l, "a"'] \arrow[d] \\ X & T' \arrow[l, "b"] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over X.

Proof. Let $U \subset X$ be the open subspace used in the construction of Z', i.e., an open such that Z is identified with a closed subspace of U cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $|b|(|T'|) \subset |U|$. Hence we can think of b as a morphism into U, see Properties of Spaces, Lemma 4.9. Let $\mathcal{J} \subset \mathcal{O}_{T'}$ be the square zero quasi-coherent sheaf of ideals cutting out T. By the commutativity of the diagram we have $b|_T = i \circ a$ where $i : Z \to U$ is the closed immersion. We conclude that $b^\sharp(b^{-1}\mathcal{I}) \subset \mathcal{J}$ by Morphisms of Spaces, Lemma 13.1. As T' is a first order thickening of T we see that $\mathcal{J}^2 = 0$ hence $b^\sharp(b^{-1}(\mathcal{I}^2)) = 0$. By Morphisms of Spaces, Lemma 13.1 this implies that b factors through Z'. Letting $a' : T' \to Z'$ be this factorization we win. ☐

Lemma 12.3. *Let $i : Z \to X$ be an immersion of algebraic spaces. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of Z in X. Then the diagram*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Lemma 5.3. This map is an isomorphism.

Proof. This is clear from the construction of Z' above. ☐

13. Formally smooth, étale, unramified transformations

Recall that a ring map $R \to A$ is called *formally smooth*, resp. *formally étale*, resp. *formally unramified* (see Algebra, Definition 138.1, resp. Definition 150.1, resp. Definition 148.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed] & B/I \\ R \arrow[r] \arrow[u] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, there exists a, resp. exists a unique, resp. exists at most one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of algebraic spaces, and more generally functors.

Definition 13.1. Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Consider commutative solid diagrams of the form

$$\begin{tikzcd} F \arrow[d, "a"'] & T \arrow[l] \arrow[d, "i"] \\ G & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where T and T' are affine schemes and i is a closed immersion defined by an ideal of square zero.

(1) We say a is *formally smooth* if given any solid diagram as above there exists a dotted arrow making the diagram commute[1].

[1] This is just one possible definition that one can make here. Another slightly weaker condition would be to require that the dotted arrow exists fppf locally on T'. This weaker notion has in some sense better formal properties.

(2) We say a is *formally étale* if given any solid diagram as above there exists exactly one dotted arrow making the diagram commute.
(3) We say a is *formally unramified* if given any solid diagram as above there exists at most one dotted arrow making the diagram commute.

Lemma 13.2. *Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Then a is formally étale if and only if a is both formally smooth and formally unramified.*

Proof. Formal from the definition. $\square$

Lemma 13.3. *Composition.*

(1) *A composition of formally smooth transformations of functors is formally smooth.*
(2) *A composition of formally étale transformations of functors is formally étale.*
(3) *A composition of formally unramified transformations of functors is formally unramified.*

Proof. This is formal. $\square$

Lemma 13.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre product diagram*

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"'] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

(1) *If a is formally smooth, then the base change a' is formally smooth.*
(2) *If a is formally étale, then the base change a' is formally étale.*
(3) *If a is formally unramified, then the base change a' is formally unramified.*

Proof. This is formal. $\square$

Lemma 13.5. *Let S be a scheme. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformation of functors.*

(1) *If a is smooth then a is formally smooth.*
(2) *If a is étale, then a is formally étale.*
(3) *If a is unramified, then a is formally unramified.*

Proof. Consider a solid commutative diagram

$$\begin{tikzcd} F \arrow[d, "a"'] & T \arrow[l] \arrow[d, "i"] \\ G & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 13.1. Then $F \times_G T'$ is a scheme smooth (resp. étale, resp. unramified) over T'. Hence by More on Morphisms, Lemma 11.7 (resp. Lemma 8.9, resp. Lemma 6.8) we can fill in (resp. uniquely fill in, resp. fill in at most one way) the dotted arrow in the diagram

$$\begin{tikzcd} F \times_G T' \arrow[d] & T \arrow[l] \arrow[d, "i"] \\ T' & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

an hence we also obtain the corresponding assertion in the first diagram. □

Lemma 13.6. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume that a is representable, surjective, and étale.*

1. *If b is formally smooth, then $b \circ a$ is formally smooth.*
2. *If b is formally étale, then $b \circ a$ is formally étale.*
3. *If b is formally unramified, then $b \circ a$ is formally unramified.*

Conversely, consider a solid commutative diagram

$$\begin{tikzcd} G \arrow[d, "b"'] & T \arrow[l] \arrow[d, "i"] \\ H & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

with T' an affine scheme over S and $i : T \to T'$ a closed immersion defined by an ideal of square zero.

4. *If $b \circ a$ is formally smooth, then for every $t \in T$ there exists an étale morphism of affines $U' \to T'$ and a morphism $U' \to G$ such that*

$$\begin{tikzcd} G \arrow[d, "b"'] & T \arrow[l] & T \times_{T'} U' \arrow[l] \arrow[d] \\ H & T' \arrow[l] & U' \arrow[l] \arrow[ull] \end{tikzcd}$$

   commutes and t is in the image of $U' \to T'$.
5. *If $b \circ a$ is formally unramified, then there exists at most one dotted arrow in the diagram above, i.e., b is formally unramified.*
6. *If $b \circ a$ is formally étale, then there exists exactly one dotted arrow in the diagram above, i.e., b is formally étale.*

Proof. Assume b is formally smooth (resp. formally étale, resp. formally unramified). Since an étale morphism is both smooth and unramified we see that a is representable and smooth (resp. étale, resp. unramified). Hence parts (1), (2) and (3) follow from a combination of Lemma 13.5 and Lemma 13.3.

Assume that $b \circ a$ is formally smooth. Consider a diagram as in the statement of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Choose an affine open subscheme $U' \subset W'$ such that t is in the image of $U' \to T'$. Because $b \circ a$ is formally smooth we see that the exist morphisms $U' \to F$ such that

$$\begin{tikzcd} F & W \arrow[l] & T \times_{T'} U' \arrow[l] \arrow[d] \\ H \arrow[from=u, "b\circ a"'] & T' \arrow[l] & U' \arrow[l] \arrow[ull] \end{tikzcd}$$

commutes. Taking the composition $U' \to F \to G$ gives a map as in part (5) of the lemma.

Assume that $f, g : T' \to G$ are two dotted arrows fitting into the diagram of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Since a is formally étale the compositions

$$W' \to T' \xrightarrow{f} G \quad \text{and} \quad W' \to T' \xrightarrow{g} G$$

lift to morphisms $f', g' : W' \to F$ (lift on affine opens and glue by uniqueness). Now if $b \circ a : F \to H$ is formally unramified, then $f' = g'$ and hence $f = g$ as $W' \to T'$ is an étale covering. This proves part (6) of the lemma.

Assume that $b \circ a$ is formally étale. Then by part (4) we can étale locally on T' find a dotted arrow fitting into the diagram and by part (5) this dotted arrow is unique. Hence we may glue the local solutions to get assertion (6). Some details omitted. ☐

Remark 13.7. It is tempting to think that in the situation of Lemma 13.6 we have "b formally smooth" $\Leftrightarrow$ "$b \circ a$ formally smooth". However, this is likely not true in general.

Lemma 13.8. *Let S be a scheme. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume b is formally unramified.*

1. *If $b \circ a$ is formally unramified then a is formally unramified.*
2. *If $b \circ a$ is formally étale then a is formally étale.*
3. *If $b \circ a$ is formally smooth then a is formally smooth.*

Proof. Let $T \subset T'$ be a closed immersion of affine schemes defined by an ideal of square zero. Let $g' : T' \to G$ and $f : T \to F$ be given such that $g'|_T = a \circ f$. Because b is formally unramified, there is a one to one correspondence between

$$\{f' : T' \to F \mid f = f'|_T \text{ and } a \circ f' = g'\}$$

and

$$\{f' : T' \to F \mid f = f'|_T \text{ and } b \circ a \circ f' = b \circ g'\}.$$

From this the lemma follows formally. ☐

14. Formally unramified morphisms

In this section we work out what it means that a morphism of algebraic spaces is formally unramified.

Definition 14.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally unramified* if it is formally unramified as a transformation of functors as in Definition 13.1.

We will not restate the results proved in the more general setting of formally unramified transformations of functors in Section 13. It turns out we can characterize this property in terms of vanishing of the module of relative differentials, see Lemma 14.6.

Lemma 14.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is formally unramified,*
2. *for every diagram*
$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally unramified (as in More on Morphisms, Definition 6.1), and
3. *for one such diagram with surjective vertical arrows the morphism ψ is formally unramified.*

Proof. Assume f is formally unramified. By Lemma 13.5 the morphisms $U \to X$ and $V \to Y$ are formally unramified. Thus by Lemma 13.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 13.8 that $U \to V$ is formally unramified. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 13.5 the morphism $V \to Y$ is formally unramified. Thus by Lemma 13.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 13.6 that $X \to Y$ is formally unramified, i.e., (1) holds. □

Lemma 14.3. *Let S be a scheme. If $f : X \to Y$ is a formally unramified morphism of algebraic spaces over S, then given any solid commutative diagram*
$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$
where $T \subset T'$ is a first order thickening of algebraic spaces over S there exists at most one dotted arrow making the diagram commute. In other words, in Definition 14.1 the condition that T be an affine scheme may be dropped.

Proof. This is true because there exists a surjective étale morphism $U' \to T'$ where U' is a disjoint union of affine schemes (see Properties of Spaces, Lemma 6.1) and a morphism $T' \to X$ is determined by its restriction to U'. □

Lemma 14.4. *A composition of formally unramified morphisms is formally unramified.*

Proof. This is formal. □

Lemma 14.5. *A base change of a formally unramified morphism is formally unramified.*

Proof. This is formal. □

Lemma 14.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is formally unramified, and*
2. *$\Omega_{X/Y} = 0$.*

Proof. This is a combination of Lemma 14.2, More on Morphisms, Lemma 6.7, and Lemma 7.3. □

Lemma 14.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *The morphism f is unramified,*
2. *the morphism f is locally of finite type and $\Omega_{X/Y} = 0$, and*
3. *the morphism f is locally of finite type and formally unramified.*

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. Then we see

$$\begin{aligned} f \text{ unramified} &\Leftrightarrow \psi \text{ unramified} \\ &\Leftrightarrow \psi \text{ locally finite type and } \Omega_{U/V} = 0 \\ &\Leftrightarrow f \text{ locally finite type and } \Omega_{X/Y} = 0 \\ &\Leftrightarrow f \text{ locally finite type and formally unramified} \end{aligned}$$

Here we have used Morphisms, Lemma 35.2 and Lemma 14.6. □

Lemma 14.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is unramified and a monomorphism,*
2. *f is unramified and universally injective,*
3. *f is locally of finite type and a monomorphism,*
4. *f is universally injective, locally of finite type, and formally unramified.*

Moreover, in this case f is also representable, separated, and locally quasi-finite.

Proof. We have seen in Lemma 14.7 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_Y X$ is both an open immersion (Morphisms of Spaces, Lemma 38.9) and surjective (Morphisms of Spaces, Lemma 19.2) hence an isomorphism, i.e., f is a monomorphism. In this way we see that (2) implies (1). Finally, we see that f is representable, separated, and locally quasi-finite by Morphisms of Spaces, Lemmas 27.10 and 51.1. □

Lemma 14.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is a closed immersion,*
2. *f is universally closed, unramified, and a monomorphism,*
3. *f is universally closed, unramified, and universally injective,*

(4) *f is universally closed, locally of finite type, and a monomorphism,*
(5) *f is universally closed, universally injective, locally of finite type, and formally unramified.*

Proof. The equivalence of (2) – (5) follows immediately from Lemma 14.8. Moreover, if (2) – (5) are satisfied then f is representable. Similarly, if (1) is satisfied then f is representable. Hence the result follows from the case of schemes, see Étale Morphisms, Lemma 7.2. □

15. Universal first order thickenings

Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. A *universal first order thickening* of Z over X is a first order thickening $Z \subset Z'$ over X such that given any first order thickening $T \subset T'$ over X and a solid commutative diagram

$$\begin{tikzcd}
& Z \arrow[dl] & & T \arrow[ll, "a"'] \arrow[dr] & \\
Z' \arrow[drr] & & & & T' \arrow[llll, dotted, "a'"'] \arrow[dll, "b"] \\
& & X & &
\end{tikzcd} \tag{15.0.1}$$

there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over X. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if h is formally unramified then it does. Before we prove this, let us show that a universal first order thickening in the category of schemes is a universal first order thickening in the category of algebraic spaces.

Lemma 15.1. *Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. Let $Z \subset Z'$ be a first order thickening over X. The following are equivalent*

(1) *$Z \subset Z'$ is a universal first order thickening,*
(2) *for any diagram (15.0.1) with T' a scheme a unique dotted arrow exists making the diagram commute, and*
(3) *for any diagram (15.0.1) with T' an affine scheme a unique dotted arrow exists making the diagram commute.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are formal. Assume (3) a assume given an arbitrary diagram (15.0.1). Choose a presentation $T' = U'/R'$, see Spaces, Definition 9.3. We may assume that $U' = \coprod U'_i$ is a disjoint union of affines, so $R' = U' \times_{T'} U' = \coprod_{i,j} U'_i \times'_T U'_j$. For each pair (i, j) choose an affine open covering $U'_i \times'_T U'_j = \bigcup_k R'_{ijk}$. Denote U_i, R_{ijk} the fibre products with T over T'. Then each $U_i \subset U'_i$ and $R_{ijk} \subset R'_{ijk}$ is a first order thickening of affine schemes. Denote $a_i : U_i \to Z$, resp. $a_{ijk} : R_{ijk} \to Z$ the composition of $a : T \to Z$ with the morphism $U_i \to T$, resp. $R_{ijk} \to T$. By (3) applied to $a_i : U_i \to Z$ we obtain unique morphisms $a'_i : U'_i \to Z'$. By (3) applied to a_{ijk} we see that the two compositions $R'_{ijk} \to R'_i \to Z'$ and $R'_{ijk} \to R'_j \to Z'$ are equal. Hence

$a' = \coprod a'_i : U' = \coprod U'_i \to Z'$ descends to the quotient sheaf $T' = U'/R'$ and we win. □

Lemma 15.2. *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. If $Z \subset Z'$ is a universal first order thickening of Z over Y and $Y \to X$ is formally étale, then $Z \subset Z'$ is a universal first order thickening of Z over X.*

Proof. This is formal. Namely, by Lemma 15.1 it suffices to consider solid commutative diagrams (15.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y$ as $Y \to X$ is assumed formally étale. Hence the fact that $Z \subset Z'$ is a universal first order thickening over Y produces the desired morphism $a' : T' \to Z'$. □

Lemma 15.3. *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. Assume $Z \to Y$ is étale.*

1. *If $Y \subset Y'$ is a universal first order thickening of Y over X, then the unique étale morphism $Z' \to Y'$ such that $Z = Y \times_{Y'} Z'$ (see Theorem 8.1) is a universal first order thickening of Z over X.*
2. *If $Z \to Y$ is surjective and $(Z \subset Z') \to (Y \subset Y')$ is an étale morphism of first order thickenings over X and Z' is a universal first order thickening of Z over X, then Y' is a universal first order thickening of Y over X.*

Proof. Proof of (1). By Lemma 15.1 it suffices to consider solid commutative diagrams (15.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y'$ as Y' is the universal first order thickening. Then the fact that $Z' \to Y'$ is étale implies (see Lemma 13.5) that $T' \to Y'$ lifts to the desired morphism $a' : T' \to Z'$.

Proof of (2). Let $T \subset T'$ be a first order thickening over X and let $a : T \to Y$ be a morphism. Set $W = T \times_Y Z$ and denote $c : W \to Z$ the projection Let $W' \to T'$ be the unique étale morphism such that $W = T \times_{T'} W'$, see Theorem 8.1. Note that $W' \to T'$ is surjective as $Z \to Y$ is surjective. By assumption we obtain a unique morphism $c' : W' \to Z'$ over X restricting to c on W. By uniqueness the two restrictions of c' to $W' \times_{T'} W'$ are equal (as the two restrictions of c to $W \times_T W$ are equal). Hence c' descends to a unique morphism $a' : T' \to Y'$ and we win. □

Lemma 15.4. *Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S. There exists a universal first order thickening $Z \subset Z'$ of Z over X.*

Proof. Choose any commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

where V and U are schemes and the vertical arrows are étale. Note that $V \to U$ is a formally unramified morphism of schemes, see Lemma 14.2. Combining Lemma 15.1 and More on Morphisms, Lemma 7.1 we see that a universal first order thickening $V \subset V'$ of V over U exists. By Lemma 15.2 part (1) V' is a universal first order thickening of V over X.

Fix a scheme U and a surjective étale morphism $U \to X$. The argument above shows that for any $V \to Z$ étale with V a scheme such that $V \to Z \to X$ factors through U a universal first order thickening $V \subset V'$ of V over X exists (but does not depend on the chosen factorization of $V \to X$ through U). Now we may choose V such that $V \to Z$ is surjective étale (see Spaces, Lemma 11.6). Then $R = V \times_Z V$ a scheme étale over Z such that $R \to X$ factors through U also. Hence we obtain universal first order thickenings $V \subset V'$ and $R \subset R'$ over X. As $V \subset V'$ is a universal first order thickening, the two projections $s, t : R \to V$ lift to morphisms $s', t' : R' \to V'$. By Lemma 15.3 as R' is the universal first order thickening of R over X these morphisms are étale. Then $(t', s') : R' \to V'$ is an étale equivalence relation and we can set $Z' = V'/R'$. Since $V' \to Z'$ is surjective étale and v' is the universal first order thickening of V over X we conclude from Lemma 15.2 part (2) that Z' is a universal first order thickening of Z over X. $\square$

Definition 15.5. Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S.

(1) The *universal first order thickening* of Z over X is the thickening $Z \subset Z'$ constructed in Lemma 15.4.
(2) The *conormal sheaf of Z over X* is the conormal sheaf of Z in its universal first order thickening Z' over X.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on $Z_{\acute{e}tale}$ and $\mathcal{C}_{Z/X}$ is a quasi-coherent $\mathcal{O}_Z$-module. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

Lemma 15.6. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then*

(1) *i is formally unramified,*
(2) *the universal first order thickening of Z over X is the first order infinitesimal neighbourhood of Z in X of Definition 12.1,*
(3) *the conormal sheaf of i in the sense of Definition 5.1 agrees with the conormal sheaf of i in the sense of Definition 15.5.*

Proof. An immersion of algebraic spaces is by definition a representable morphism. Hence by Morphisms, Lemmas 35.7 and 35.8 an immersion is unramified (via the abstract principle of Spaces, Lemma 5.8). Hence it is formally unramified by Lemma 14.7. The other assertions follow by combining Lemmas 12.2 and 12.3 and the definitions. $\square$

Lemma 15.7. *Let S be a scheme. Let $Z \to X$ be a formally unramified morphism of algebraic spaces over S. Then the universal first order thickening Z' is formally unramified over X.*

Proof. Let $T \subset T'$ be a first order thickening of affine schemes over X. Let

$$\begin{tikzcd} Z' \arrow[d] & T \arrow[l, "c"'] \arrow[d] \\ X & T' \arrow[l] \arrow[lu, "{a,b}"] \end{tikzcd}$$

be a commutative diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since Z' is a first order thickening of Z, we see that T' is a first order thickening of T'_a. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As T' is a first order thickening of T it follows that T'_a is a first order thickening of T_0. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of T'_a into Z' over X which agree on T_0 as morphisms into Z. Hence by the universal property of Z' we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus a and b are morphism from the first order thickening T' of T'_a whose restrictions to T'_a agree as morphisms into Z. Thus using the universal property of Z' once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. $\square$

Lemma 15.8. *Let S be a scheme Consider a commutative diagram of algebraic spaces over S*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

with h and h' formally unramified. Let $Z \subset Z'$ be the universal first order thickening of Z over X. Let $W \subset W'$ be the universal first order thickening of W over Y. There exists a canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over Y which fits into the following commutative diagram

$$\begin{tikzcd} & & Z' \arrow[dl] \arrow[d, "f'"] \\ Z \arrow[urr] \arrow[r] \arrow[d, "f"'] & X \arrow[d] & W' \arrow[dl] \\ W \arrow[r] \arrow[urr] & Y & \end{tikzcd}$$

In particular the morphism (f, f') of thickenings induces a morphism of conormal sheaves $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$.

Proof. The first assertion is clear from the universal property of W'. The induced map on conormal sheaves is the map of Lemma 5.3 applied to $(Z \subset Z') \to (W \subset W')$. $\square$

Lemma 15.9. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified. Then h is formally unramified and if $W \subset W'$ is the universal first order thickening *of W over Y, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of Z over X. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 15.8 is surjective.*

Proof. The morphism h is formally unramified by Lemma 14.5. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 5.5 for why this implies that the map of conormal sheaves is surjective. $\square$

Lemma 15.10. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified and g flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

Proof. Flatness is preserved under base change, see Morphisms of Spaces, Lemma 30.4. Hence the first statement follows from the description of W' in Lemma 15.9. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 5.5 for why this implies that the map of conormal sheaves is an isomorphism. $\square$

Lemma 15.11. *Taking the universal first order thickenings commutes with étale localization. More precisely, let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over a base scheme S. Let*

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

be a commutative diagram with étale vertical arrows. Let Z' be the universal first order thickening of Z over X. Then $V \to U$ is formally unramified and the universal first order thickening V' of V over U is étale over Z'. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.

Proof. The first statement is Lemma 14.2. The compatibility of universal first order thickenings is a consequence of Lemmas 15.2 and 15.3. $\square$

Lemma 15.12. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Let $Z \subset Z'$ be the universal first order thickening of Z over X with structure morphism $h' : Z' \to X$. The canonical map*

$$dh' : (h')^*\Omega_{X/B} \to \Omega_{Z'/B}$$

induces an isomorphism $h^*\Omega_{X/B} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z$.

Proof. The map $c_{h'}$ is the map defined in Lemma 7.6. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the case of schemes by étale localization, see Lemma 15.11 and Lemma 7.3. In this case the result is More on Morphisms, Lemma 7.9. ☐

Lemma 15.13. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to h^*\Omega_{X/B} \to \Omega_{Z/B} \to 0.$$

The first arrow is induced by $\text{d}_{Z'/B}$ where Z' is the universal first order neighbourhood of Z over X.

Proof. We know that there is a canonical exact sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

see Lemma 7.10. Hence the result follows on applying Lemma 15.12. ☐

Lemma 15.14. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 15.8 and the second from Lemma 15.13.

Proof. Since the maps have been defined, checking the sequence is exact reduces to the case of schemes by étale localization, see Lemma 15.11 and Lemma 7.3. In this case the result is More on Morphisms, Lemma 7.11. ☐

Lemma 15.15. *Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S.*

1. *If $Z \subset Z'$ is the universal first order thickening of Z over X and $Y \subset Y'$ is the universal first order thickening of Y over X, then there is a morphism $Z' \to Y'$ and $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y.*
2. *There is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 15.8 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 15.8. The assertion that $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y is clear from the universal properties of Z' and Y'. By Lemma 5.6 we have an exact sequence

$$(i')^*\mathcal{C}_{Y \times_{Y'} Z'/Z'} \to \mathcal{C}_{Z/Z'} \to \mathcal{C}_{Z/Y \times_{Y'} Z'} \to 0$$

where $i' : Z \to Y \times_{Y'} Z'$ is the given morphism. By Lemma 5.5 there exists a surjection $h^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{Y \times_{Y'} Z'/Z'}$. Combined with the equalities $\mathcal{C}_{Y/Y'} = \mathcal{C}_{Y/X}$, $\mathcal{C}_{Z/Z'} = \mathcal{C}_{Z/X}$, and $\mathcal{C}_{Z/Y \times_{Y'} Z'} = \mathcal{C}_{Z/Y}$ this proves the lemma. ☐

16. Formally étale morphisms

In this section we work out what it means that a morphism of algebraic spaces is formally étale.

Definition 16.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally étale* if it is formally étale as a transformation of functors as in Definition 13.1.

We will not restate the results proved in the more general setting of formally étale transformations of functors in Section 13.

Lemma 16.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally étale,*
(2) *for every diagram*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally étale (as in More on Morphisms, Definition 8.1), and
(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally étale.*

Proof. Assume f is formally étale. By Lemma 13.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. Thus by Lemma 13.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 13.8 that $U \to V$ is formally étale. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 13.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 13.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 13.6 that $X \to Y$ is formally étale, i.e., (1) holds. ☐

Lemma 16.3. *Let S be a scheme. Let $f : X \to Y$ be a formally étale morphism of algebraic spaces over S. Then given any solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of algebraic spaces over Y there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 16.1 the condition that T be affine may be dropped.

Proof. Let $U' \to T'$ be a surjective étale morphism where $U' = \coprod U'_i$ is a disjoint union of affine schemes. Let $U_i = T \times_{T'} U'_i$. Then we get morphisms $a'_i : U'_i \to X$ such that $a'_i|_{U_i}$ equals the composition $U_i \to T \to X$. By uniqueness (see Lemma 14.3) we see that a'_i and a'_j agree on the fibre product $U'_i \times_{T'} U'_j$. Hence $\coprod a'_i : U' \to X$ descends to give a unique morphism $a' : T' \to X$. ☐

Lemma 16.4. *A composition of formally étale morphisms is formally étale.*

Proof. This is formal. □

Lemma 16.5. *A base change of a formally étale morphism is formally étale.*

Proof. This is formal. □

Lemma 16.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S The following are equivalent:*

1. *f is formally étale,*
2. *f is formally unramified and the universal first order thickening of X over Y is equal to X,*
3. *f is formally unramified and $\mathcal{C}_{X/Y} = 0$, and*
4. *$\Omega_{X/Y} = 0$ and $\mathcal{C}_{X/Y} = 0$.*

Proof. Actually, the last assertion only make sense because $\Omega_{X/Y} = 0$ implies that $\mathcal{C}_{X/Y}$ is defined via Lemma 14.6 and Definition 15.5. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that f is formally unramified. Hence we may assume f is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening X' of X over S and the fact that $X = X' \Leftrightarrow \mathcal{C}_{X/Y} = 0$ since after all by definition $\mathcal{C}_{X/Y} = \mathcal{C}_{X/X'}$ is the ideal sheaf of X in X'. □

Lemma 16.7. *An unramified flat morphism is formally étale.*

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 8.7 and étale localization, see Lemmas 14.2 and 16.2 and Morphisms of Spaces, Lemma 30.5. □

Lemma 16.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *The morphism f is étale, and*
2. *the morphism f is locally of finite presentation and formally étale.*

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 8.9 and étale localization, see Lemma 16.2 and Morphisms of Spaces, Lemmas 28.4 and 39.2. □

17. Infinitesimal deformations of maps

In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening X' of X can be seen as a sheaf on X, see Equations (9.1.1) and (9.1.2).

Lemma 17.1. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings of algebraic spaces over B. Let $(a, a'), (b, b') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over B. Assume that*

1. *$a = b$, and*
2. *the two maps $a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ (Lemma 5.3) are equal.*

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$\mathcal{O}_{Y'} \to \mathcal{O}_Y \xrightarrow{D} a_*\mathcal{C}_{X/X'} \to a_*\mathcal{O}_{X'}$$

where D is an $\mathcal{O}_B$-derivation.

Proof. Instead of working on Y we work on X. The advantage is that the pullback functor a^{-1} is exact. Using (1) and (2) we obtain a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}_{X/X'} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & a^{-1}\mathcal{C}_{Y/Y'} \arrow[r] \arrow[u] & a^{-1}\mathcal{O}_{Y'} \arrow[r] \arrow[u, shift left, "(a')^\sharp"'] \arrow[u, shift right, "(b')^\sharp"] & a^{-1}\mathcal{O}_Y \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now it is a general fact that in such a situation the difference of the $\mathcal{O}_B$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $\mathcal{O}_B$-derivation from $a^{-1}\mathcal{O}_Y$ to $\mathcal{C}_{X/X'}$. By adjointness of the functors a^{-1} and a_* this is the same thing as an $\mathcal{O}_B$-derivation from $\mathcal{O}_Y$ into $a_*\mathcal{C}_{X/X'}$. Some details omitted. □

Note that in the situation of the lemma above we may write D as

$$D = \mathrm{d}_{Y/B} \circ \theta \tag{17.1.1}$$

where θ is an $\mathcal{O}_Y$-linear map $\theta : \Omega_{Y/B} \to a_*\mathcal{C}_{X/X'}$. Of course, then by adjunction again we may view θ as an $\mathcal{O}_X$-linear map $\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$.

Lemma 17.2. *Let S be a scheme. Let B be an algebraic space over S. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over B. Let*

$$\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$$

be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 17.1 hold and the derivation D and θ are related by Equation (17.1.1).

Proof. Consider the map

$$\alpha = (a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where D is as in Equation (17.1.1). As D is an $\mathcal{O}_B$-derivation it follows that α is a map of sheaves of $\mathcal{O}_B$-algebras. By construction we have $i_X^\sharp \circ \alpha = a^\sharp \circ i_Y^\sharp$ where $i_X : X \to X'$ and $i_Y : Y \to Y'$ are the given closed immersions. By Lemma 9.2 we obtain a unique morphism $(a, b') : (X \subset X') \to (Y \subset Y')$ of thickenings over B such that $\alpha = (b')^\sharp$. Setting $b = a$ we win. □

Remark 17.3. Assumptions and notation as in Lemma 17.2. The action of a local section θ on a' is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation D which is related to θ by Equation (17.1.1).

Lemma 17.4. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over B. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an object U' of $(X')_{spaces,étale}$ with $U = X \times_{X'} U'$ consider morphisms $a' : U' \to Y'$ such that*

1. *a' is a morphism over B,*
2. *$a'|_U = a|_U$, and*

(3) *the induced map* $a^*\mathcal{C}_{Y/Y'}|_U \to \mathcal{C}_{X/X'}|_U$ *is the restriction of* A *to* U.

Then the rule

$$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\} \tag{17.4.1}$$

defines a sheaf of sets on $(X')_{spaces,étale}$.

Proof. Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms of algebraic spaces satisfy étale descent, see Descent on Spaces, Lemma 7.2. □

Lemma 17.5. *Same notation and assumptions as in Lemma 17.4. We identify sheaves on* X *and* X' *via (9.1.1). There is an action of the sheaf*

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$$

on the sheaf (17.4.1). Moreover, the action is simply transitive for any object U' *of* $(X')_{spaces,étale}$ *over which the sheaf (17.4.1) has a section.*

Proof. This is a combination of Lemmas 17.1, 17.2, and 17.4. □

Remark 17.6. A special case of Lemmas 17.1, 17.2, 17.4, and 17.5 is where $Y = Y'$. In this case the map A is always zero. The sheaf of Lemma 17.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } B \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$.

Remark 17.7. Another special case of Lemmas 17.1, 17.2, 17.4, and 17.5 is where B itself is a thickening $Z \subset Z' = B$ and $Y = Z \times_{Z'} Y'$. Picture

$$\begin{tikzcd}
(X \subset X') \arrow[rr, dotted, "(a,?)"'] \arrow[rd, "(g,g')"'] & & (Y \subset Y') \arrow[ld, "(h,h')"] \\
& (Z \subset Z') &
\end{tikzcd}$$

In this case the map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is determined by a: the map $h^*\mathcal{C}_{Z/Z'} \to \mathcal{C}_{Y/Y'}$ is surjective (because we assumed $Y = Z \times_{Z'} Y'$), hence the pullback $g^*\mathcal{C}_{Z/Z'} = a^*h^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'}$ is surjective, and the composition $g^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ has to be the canonical map induced by g'. Thus the sheaf of Lemma 17.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y' \text{ over } Z' \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'})$.

Lemma 17.8. *Let* S *be a scheme. Consider a commutative diagram of first order thickenings*

$$\begin{tikzcd}
(T_2 \subset T_2') \arrow[r, "(a_2,a_2')"'] \arrow[d, "(h,h')"'] & (X_2 \subset X_2') \arrow[d, "(f,f')"] \\
(T_1 \subset T_1') \arrow[r, "(a_1,a_1')"] & (X_1 \subset X_1')
\end{tikzcd} \quad \textit{and a commutative diagram} \quad \begin{tikzcd}
X_2' \arrow[r] \arrow[d] & B_2 \arrow[d] \\
X_1' \arrow[r] & B_1
\end{tikzcd}$$

of algebraic spaces over S with $X_2 \to X_1$ and $B_2 \to B_1$ étale. For any $\mathcal{O}_{T_1}$-linear map $\theta_1 : a_1^*\Omega_{X_1/B_1} \to \mathcal{C}_{T_1/T_1'}$ let θ_2 be the composition

$$a_2^*\Omega_{X_2/B_2} = h^*a_1^*\Omega_{X_1/B_1} \xrightarrow{h^*\theta_1} h^*\mathcal{C}_{T_1/T_1'} \longrightarrow \mathcal{C}_{T_2/T_2'}$$

(equality sign is explained in the proof). Then the diagram

$$\begin{tikzcd} T_2' \arrow[r, "\theta_2 \cdot a_2'"'] \arrow[d] & X_2' \arrow[d] \\ T_1' \arrow[r, "\theta_1 \cdot a_1'"] & X_1' \end{tikzcd}$$

commutes where the actions $\theta_2 \cdot a_2'$ and $\theta_1 \cdot a_1'$ are as in Remark 17.3.

Proof. The equality sign comes from the identification $f^*\Omega_{X_1/S_1} = \Omega_{X_2/S_2}$ we get as the construction of the sheaf of differentials is compatible with étale localization (both on source and target), see Lemma 7.3. Namely, using this we have $a_2^*\Omega_{X_2/S_2} = a_2^*f^*\Omega_{X_1/S_1} = h^*a_1^*\Omega_{X_1/S_1}$ because $f \circ a_2 = a_1 \circ h$. Having said this, the commutativity of the diagram may be checked on étale locally. Thus we may assume T_i', X_i', B_2, and B_1 are schemes and in this case the lemma follows from More on Morphisms, Lemma 9.10. Alternative proof: using Lemma 9.2 it suffices to show a certain diagram of sheaves of rings on X_1' is commutative; then argue exactly as in the proof of the aforementioned More on Morphisms, Lemma 9.10 to see that this is indeed the case. $\square$

18. Infinitesimal deformations of algebraic spaces

The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

Lemma 18.1. *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings of algebraic spaces over S. Assume that f is flat. Then the following are equivalent*

1. *f' is flat and $X = Y \times_{Y'} X'$, and*
2. *the canonical map $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is an isomorphism.*

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $U = X \times_{X'} U'$ and $V = Y \times_{Y'} V'$. According to our definition of a flat morphism of algebraic spaces we see that the induced map $g : U \to V$ is a flat morphism of schemes and that f' is flat if and only if the corresponding morphism $g' : U' \to V'$ is flat. Also, $X = Y \times_{Y'} X'$ if and only if $U = V \times_{V'} V'$. Finally, the map $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is an isomorphism if and only if $g^*\mathcal{C}_{V/V'} \to \mathcal{C}_{U/U'}$ is an isomorphism. Hence the lemma follows from its analogue for morphisms of schemes, see More on Morphisms, Lemma 10.1. $\square$

The following lemma is the "nilpotent" version of the "critère de platitude par fibres", see Section 23.

Lemma 18.2. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (B \subset B') & \end{tikzcd}$$

of thickenings of algebraic spaces over S. Assume

1. *X' is flat over B',*
2. *f is flat,*
3. *$B \subset B'$ is a finite order thickening, and*
4. *$X = B \times_{B'} X'$ and $Y = B \times_{B'} Y'$.*

Then f' is flat and Y' is flat over B' at all points in the image of f'.

Proof. Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Then flatness of f' is equivalent to flatness of $W' \to V'$ and we are given that $W \to V$ is flat. Hence we may apply the lemma in the case of schemes to the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[dr] & & (V \subset V') \arrow[dl] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of schemes. See More on Morphisms, Lemma 10.2. The statement about flatness of Y'/B' at points in the image of f' follows in the same manner. $\square$

Many properties of morphisms of schemes are preserved under flat deformations.

Lemma 18.3. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (B \subset B') & \end{tikzcd}$$

of thickenings of algebraic spaces over S. Assume $B \subset B'$ is a finite order thickening, X' flat over B', $X = B \times_{B'} X'$, and $Y = B \times_{B'} Y'$. Then

1. *f is representable if and only if f' is representable,*
2. *f is flat if and only if f' is flat,*
3. *f is an isomorphism if and only if f' is an isomorphism,*
4. *f is an open immersion if and only if f' is an open immersion,*
5. *f is quasi-compact if and only if f' is quasi-compact,*
6. *f is universally closed if and only if f' is universally closed,*
7. *f is (quasi-)separated if and only if f' is (quasi-)separated,*
8. *f is a monomorphism if and only if f' is a monomorphism,*
9. *f is surjective if and only if f' is surjective,*
10. *f is universally injective if and only if f' is universally injective,*
11. *f is affine if and only if f' is affine,*
12. *f is locally of finite type if and only if f' is locally of finite type,*

(13) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
(14) *f is locally of finite presentation if and only if f' is locally of finite presentation,*
(15) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
(16) *f is universally open if and only if f' is universally open,*
(17) *f is syntomic if and only if f' is syntomic,*
(18) *f is smooth if and only if f' is smooth,*
(19) *f is unramified if and only if f' is unramified,*
(20) *f is étale if and only if f' is étale,*
(21) *f is proper if and only if f' is proper,*
(22) *f is integral if and only if f' is integral,*
(23) *f is finite if and only if f' is finite,*
(24) *f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and*
(25) *add more here.*

Proof. Case (1) follows from Lemma 10.1.

Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Consider the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[rd] & & (V \subset V') \arrow[ld] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of schemes. For any of the properties which are étale local on the source-and-target the result follows immediately from the corresponding result for morphisms of thickenings of schemes applied to the diagram above. Thus cases (2), (12), (13), (14), (15), (17), (18), (19), (20) follow from the corresponding cases of More on Morphisms, Lemma 10.3.

Since $X \to X'$ and $Y \to Y'$ are universal homeomorphisms we see that any question about the topology of the maps $X \to Y$ and $X' \to Y'$ has the same answer. Thus we see that cases (5), (6), (9), (10), and (16) hold.

In each of the remaining cases we only prove the implication f has $P \Rightarrow f'$ has P since the other implication follows from the fact that P is stable under base change, see Spaces, Lemma 12.3 and Morphisms of Spaces, Lemmas 4.4, 10.5, 20.5, 40.3, 45.5, and 46.5.

The case (4). Assume f is an open immersion. Then f' is étale by (20) and universally injective by (10) hence f' is an open immersion, see Morphisms of Spaces, Lemma 51.2. You can avoid using this lemma at the cost of first using (1) to reduce to the case of schemes.

The case (3). Follows from cases (4) and (9).

The case (7). See Lemma 10.1.

The case (8). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. The base change of $\Delta_{X'/Y'}$ by $B \to B'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (3) we conclude that $\Delta_{X'/Y'}$ is an isomorphism and hence f' is a monomorphism.

The case (11). See Lemma 10.1.

The case (21). See Lemma 10.2.

The case (22). See Lemma 10.1.

The case (23). See Lemma 10.2.

The case (24). Assume f finite locally free. By (23) we see that f' is finite. By (2) we see that f' is flat. By (14) f' is locally of finite presentation. Hence f' is finite locally free by Morphisms of Spaces, Lemma 46.6. □

The following lemma is the "locally nilpotent" version of the "critère de platitude par fibres", see Section 23.

Lemma 18.4. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd}
(X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\
& (B \subset B') &
\end{tikzcd}$$

of thickenings of algebraic spaces over S. Assume

1. *$Y' \to B'$ is locally of finite type,*
2. *$X' \to B'$ is flat and locally of finite presentation,*
3. *f is flat, and*
4. *$X = B \times_{B'} X'$ and $Y = B \times_{B'} Y'$.*

Then f' is flat and for all $y' \in |Y'|$ in the image of $|f'|$ the morphism $Y' \to B'$ is flat at y'.

Proof. Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Then flatness of f' is equivalent to flatness of $W' \to V'$ and we are given that $W \to V$ is flat. Hence we may apply the lemma in the case of schemes to the diagram

$$\begin{tikzcd}
(W \subset W') \arrow[rr] \arrow[dr] & & (V \subset V') \arrow[dl] \\
& (U \subset U') &
\end{tikzcd}$$

of thickenings of schemes. See More on Morphisms, Lemma 10.4. The statement about flatness of Y'/B' at points in the image of f' follows in the same manner. □

Many properties of morphisms of schemes are preserved under flat deformations as in the lemma above.

Lemma 18.5. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (B \subset B') & \end{tikzcd}$$

of thickenings of algebraic spaces over S. Assume $Y' \to B'$ locally of finite type, $X' \to B'$ flat and locally of finite presentation, $X = B \times_{B'} X'$, and $Y = B \times_{B'} Y'$. Then

(1) *f is representable if and only if f' is representable,*
(2) *f is flat if and only if f' is flat,*
(3) *f is an isomorphism if and only if f' is an isomorphism,*
(4) *f is an open immersion if and only if f' is an open immersion,*
(5) *f is quasi-compact if and only if f' is quasi-compact,*
(6) *f is universally closed if and only if f' is universally closed,*
(7) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
(8) *f is a monomorphism if and only if f' is a monomorphism,*
(9) *f is surjective if and only if f' is surjective,*
(10) *f is universally injective if and only if f' is universally injective,*
(11) *f is affine if and only if f' is affine,*
(12) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
(13) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
(14) *f is universally open if and only if f' is universally open,*
(15) *f is syntomic if and only if f' is syntomic,*
(16) *f is smooth if and only if f' is smooth,*
(17) *f is unramified if and only if f' is unramified,*
(18) *f is étale if and only if f' is étale,*
(19) *f is proper if and only if f' is proper,*
(20) *f is finite if and only if f' is finite,*
(21) *f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and*
(22) *add more here.*

Proof. Case (1) follows from Lemma 10.1.

Choose a scheme U' and a surjective étale morphism $U' \to B'$. Choose a scheme V' and a surjective étale morphism $V' \to U' \times_{B'} Y'$. Choose a scheme W' and a surjective étale morphism $W' \to V' \times_{Y'} X'$. Let U, V, W be the base change of U', V', W' by $B \to B'$. Consider the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[dr] & & (V \subset V') \arrow[dl] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of schemes. For any of the properties which are étale local on the source-and-target the result follows immediately from the corresponding result for morphisms of thickenings of schemes applied to the diagram above. Thus cases (2), (12), (13), (15), (16), (17), (18) follow from the corresponding cases of More on Morphisms, Lemma 10.5.

Since $X \to X'$ and $Y \to Y'$ are universal homeomorphisms we see that any question about the topology of the maps $X \to Y$ and $X' \to Y'$ has the same answer. Thus we see that cases (5), (6), (9), (10), and (14) hold.

In each of the remaining cases we only prove the implication f has $P \Rightarrow f'$ has P since the other implication follows from the fact that P is stable under base change, see Spaces, Lemma 12.3 and Morphisms of Spaces, Lemmas 4.4, 10.5, 20.5, 40.3, 45.5, and 46.5.

The case (4). Assume f is an open immersion. Then f' is étale by (18) and universally injective by (10) hence f' is an open immersion, see Morphisms of Spaces, Lemma 51.2. You can avoid using this lemma at the cost of first using (1) to reduce to the case of schemes.

The case (3). Follows from cases (4) and (9).

The case (7). See Lemma 10.1.

The case (8). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. The base change of $\Delta_{X'/Y'}$ by $B \to B'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (3) we conclude that $\Delta_{X'/Y'}$ is an isomorphism and hence f' is a monomorphism.

The case (11). See Lemma 10.1.

The case (19). See Lemma 10.3.

The case (20). See Lemma 10.3.

The case (21). Assume f finite locally free. By (20) we see that f' is finite. By (2) we see that f' is flat. Also f' is locally finite presentation by Morphisms of Spaces, Lemma 28.9. Hence f' is finite locally free by Morphisms of Spaces, Lemma 46.6. $\square$

19. Formally smooth morphisms

In this section we introduce the notion of a formally smooth morphism $X \to Y$ of algebraic spaces. Such a morphism is characterized by the property that T-valued points of X lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 19.6. It turns out that this criterion is often easier to use than the Jacobian criterion.

Definition 19.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally smooth* if it is formally smooth as a transformation of functors as in Definition 13.1.

In the cases of formally unramified and formally étale morphisms the condition that T' be affine could be dropped, see Lemmas 14.3 and 16.3. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow étale locally on T'. In fact, analyzing the proof of Lemma 19.6 shows that this would be equivalent to the definition as it currently stands. It is also true that requiring the existence of the dotted arrow fppf locally on T' would be sufficient, but that is slightly more difficult to prove.

We will not restate the results proved in the more general setting of formally smooth transformations of functors in Section 13.

Lemma 19.2. *A composition of formally smooth morphisms is formally smooth.*

Proof. Omitted. $\square$

Lemma 19.3. *A base change of a formally smooth morphism is formally smooth.*

Proof. Omitted, but see Algebra, Lemma 138.2 for the algebraic version. $\square$

Lemma 19.4. *Let $f : X \to S$ be a morphism of schemes. Then f is formally étale if and only if f is formally smooth and formally unramified.*

Proof. Omitted. $\square$

Here is a helper lemma which will be superseded by Lemma 19.10.

Lemma 19.5. *Let S be a scheme. Let*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. If the vertical arrows are étale and f is formally smooth, then ψ is formally smooth.

Proof. By Lemma 13.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. By Lemma 13.3 the composition $U \to Y$ is formally smooth. By Lemma 13.8 we see $\psi : U \to V$ is formally smooth. $\square$

The following lemma is the main result of this section. It implies, combined with Limits of Spaces, Proposition 3.10, that we can recognize whether a morphism of algebraic spaces $f : X \to Y$ is smooth in terms of "simple" properties of the transformation of functors $X \to Y$.

Lemma 19.6 (Infinitesimal lifting criterion)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is smooth.*
(2) *The morphism f is locally of finite presentation, and formally smooth.*

Proof. Assume $f : X \to S$ is locally of finite presentation and formally smooth. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. By Lemma 19.5 we see $\psi : U \to V$ is formally smooth. By Morphisms of Spaces, Lemma 28.4 the morphism ψ is locally of finite presentation. Hence by the case of schemes the morphism ψ is smooth, see More on Morphisms, Lemma 11.7. Hence f is smooth, see Morphisms of Spaces, Lemma 37.4.

Conversely, assume that $f : X \to Y$ is smooth. Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 19.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 17.4 as in the special case discussed in Remark 17.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{spaces,étale}$ with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 17.5. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 4.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 4.3.

First we prove (I). To see this choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. As f is assumed smooth we see that ψ is smooth and hence formally smooth by Lemma 13.5. By the same lemma the morphism $V \to Y$ is formally étale. Thus by Lemma 13.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 13.6 part (4).

Finally we prove (II). By Lemma 7.15 we see that $\Omega_{X/S}$ is of finite presentation. Hence $a^*\Omega_{X/S}$ is of finite presentation (see Properties of Spaces, Section 30). Hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$ is quasi-coherent by Properties of Spaces, Lemma 29.7. Thus by Descent, Proposition 9.3 and Cohomology of Schemes, Lemma 2.2 we have

$$H^1(T_{spaces,étale}, \mathcal{H}) = H^1(T_{étale}, \mathcal{H}) = H^1(T, \mathcal{H}) = 0$$

as desired. □

Smooth morphisms satisfy strong local lifting property, see Lemma 19.7. If in the lemma we assume T' is affine, then we do not know if it is necessary to take an étale covering. More precisely, if we have a commutative diagram

$$\begin{tikzcd} X \arrow[d] & T \arrow[l] \arrow[d] \\ Y & T' \arrow[l] \arrow[ul, dotted] \end{tikzcd}$$

of algebraic spaces where $X \to Y$ is smooth and $T \to T'$ is a thickening of affine schemes, the does a dotted arrow making the diagram commute always exist? If you know the answer, or if you have a reference, please email stacks.project@gmail.com.

Lemma 19.7. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[d] & T \arrow[l] \arrow[d] \\ Y & T' \arrow[l] \end{tikzcd}$$

of algebraic spaces over S where $X \to Y$ is smooth and $T \to T'$ is a thickening. Then there exists an étale covering $\{T'_i \to T'\}$ such that we can find the dotted arrow in

$$\begin{tikzcd} X \arrow[d] & T \arrow[l] \arrow[d] & T \times_{T'} T'_i \arrow[l] \arrow[d] \\ Y & T' \arrow[l] & T'_i \arrow[l] \arrow[llu, dotted] \end{tikzcd}$$

making the diagram commute (for all i).

Proof. Choose an étale covering $\{Y_i \to Y\}$ with each Y_i affine. After replacing T' by the induced étale covering we may assume Y is affine.

Assume Y is affine. Choose an étale covering $\{X_i \to X\}$. This gives rise to an étale covering of T. This étale covering of T comes from an étale covering of T' (by Theorem 8.1, see discussion in Section 9). Hence we may assume X is affine.

Assume X and Y are affine. We can do one more étale covering of T' and assume T' is affine. In this case the lemma follows from Algebra, Lemma 138.17. □

We do a bit more work to show that being formally smooth is étale local on the source. To begin we show that a formally smooth morphism has a nice sheaf of differentials. The notion of a locally projective quasi-coherent module is defined in Properties of Spaces, Section 31.

Lemma 19.8. *Let S be a scheme. Let $f : X \to Y$ be a formally smooth morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is locally projective on X.*

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are affine(!) schemes and the vertical arrows are étale. By Lemma 19.5 we see $\psi : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see More on Morphisms, Lemma 11.6. Hence by Algebra, Lemma 138.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega_{\Gamma(U,\mathcal{O}_U)/\Gamma(V,\mathcal{O}_V)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 21. Since $\Omega_{X/Y}|_U = \Omega_{U/V}$ we see that $\Omega_{X/Y}$ is locally projective too. (Because we can find an étale covering of X by the affine U's fitting into diagrams as above – details omitted.) □

Lemma 19.9. *Let T be an affine scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_T$-modules on $T_{étale}$. Consider the internal hom sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}, \mathcal{G})$ on $T_{étale}$. If $\mathcal{F}$ is locally projective, then $H^1(T_{étale}, \mathcal{H}) = 0$.*

Proof. By the definition of a locally projective sheaf on an algebraic space (see Properties of Spaces, Definition 31.2) we see that $\mathcal{F}_{Zar} = \mathcal{F}|_{T_{Zar}}$ is a locally projective sheaf on the scheme T. Thus $\mathcal{F}_{Zar}$ is a direct summand of a free $\mathcal{O}_{T_{Zar}}$-module. Whereupon we conclude (as $\mathcal{F} = (\mathcal{F}_{Zar})^a$, see Descent, Proposition 8.9) that $\mathcal{F}$ is a direct summand of a free $\mathcal{O}_T$-module on $T_{étale}$. Hence we may assume that $\mathcal{F} = \bigoplus_{i \in I} \mathcal{O}_T$ is a free module. In this case $\mathcal{H} = \prod_{i \in I} \mathcal{G}$ is a product of quasi-coherent modules. By Cohomology on Sites, Lemma 12.5 we conclude that $H^1 = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme is zero, see Descent, Proposition 9.3 and Cohomology of Schemes, Lemma 2.2. $\square$

Lemma 19.10. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally smooth,*
(2) *for every diagram*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally smooth (as in More on Morphisms, Definition 6.1), and
(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally smooth.*

Proof. We have seen that (1) implies (2) and (3) in Lemma 19.5. Assume (3). The proof that f is formally smooth is entirely similar to the proof of (1) $\Rightarrow$ (2) of Lemma 19.6.

Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

as in Definition 19.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 17.4 as in the special case discussed in Remark 17.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{spaces,étale}$ with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 17.5. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 4.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 4.3.

First we prove (I). To see this consider a diagram (which exists because we are assuming (3))

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes, the vertical arrows are étale and surjective, and ψ is formally smooth. By Lemma 13.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 13.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 13.6 part (4).

Finally we prove (II). By Lemma 19.8 we see that $\Omega_{U/V}$ locally projective. Hence $\Omega_{X/Y}$ is locally projective, see Descent on Spaces, Lemma 6.5. Hence $a^*\Omega_{X/Y}$ is locally projective, see Properties of Spaces, Lemma 31.3. Hence

$$H^1(T_{étale}, \mathcal{H}) = H^1(T_{étale}, \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'}) = 0$$

by Lemma 19.9 as desired. $\square$

Lemma 19.11. *The property $\mathcal{P}(f) =$"f is formally smooth" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S. Choose an index set I and diagrams

$$\begin{tikzcd} U_i \arrow[r, "\psi_i"'] \arrow[d] & V_i \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with étale vertical arrows and U_i, V_i affine schemes. Moreover, assume that $\coprod U_i \to X$ and $\coprod V_i \to Y$ are surjective, see Properties of Spaces, Lemma 6.1. By Lemma 19.10 we see that f is formally smooth if and only if each of the morphisms ψ_i are formally smooth. Hence we reduce to the case of a morphism of affine schemes. In this case the result follows from Algebra, Lemma 138.16. Some details omitted. $\square$

Lemma 19.12. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume f is formally smooth. Then*

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

Lemma 7.8 is short exact.

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 11.11, by étale localization, see Lemmas 19.10 and 7.3. $\square$

Lemma 19.13. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Assume that Z is formally smooth over B. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to h^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 15.13 is short exact.

Proof. Let $Z \to Z'$ be the universal first order thickening of Z over X. From the proof of Lemma 15.13 we see that our sequence is identified with the sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z \to \Omega_{Z/B} \to 0.$$

Since $Z \to S$ is formally smooth we can étale locally on Z' find a left inverse $Z' \to Z$ over B to the inclusion map $Z \to Z'$. Thus the sequence is étale locally split, see Lemma 7.11. $\square$

Lemma 19.14. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[rd, "j"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified and f is formally smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 15.14 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 15.13 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"] \arrow[rd, "j'"'] & Z' \arrow[r, "a"] \arrow[d, "k"] & X \arrow[d, "f"] \\ & Z'' \arrow[r, "b"] & Y \end{tikzcd}$$

The sequence above is identified with the sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

via our definitions concerning conormal sheaves of formally unramified morphisms. Let $U'' \to Z''$ be an étale morphism with U'' affine. Denote $U \to Z$ and $U' \to Z'$ the corresponding affine schemes étale over Z and Z'. As f is formally smooth there exists a morphism $h : U'' \to X$ which agrees with i on U and such that $f \circ h$ equals $b|_{U''}$. Since Z' is the universal first order thickening we obtain a unique morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of Z'' implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence g is a left inverse to k. Picture

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Z' \arrow[d, "k"] \\ U'' \arrow[r] \arrow[ru, "g"] & Z'' \end{tikzcd}$$

Thus g induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over U. $\square$

20. Smoothness over a Noetherian base

This section is the analogue of More on Morphisms, Section 12.

Lemma 20.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Assume that Y is locally Noetherian and f locally of finite type. The following are equivalent:*

1. *f is smooth at x,*
2. *for every solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ Y & \mathrm{Spec}(B') \arrow[l, "\beta"'] \arrow[ul, dashed] \end{tikzcd}$$

   where $B' \to B$ is a surjection of local rings with $\mathrm{Ker}(B' \to B)$ of square zero, and α mapping the closed point of $\mathrm{Spec}(B)$ to x there exists a dotted arrow making the diagram commute, and
3. *same as in (2) but with $B' \to B$ ranging over small extensions (see Algebra, Definition 141.1).*

Proof. Condition (1) means there is an open subspace $X' \subset X$ such that $X' \to Y$ is smooth. Hence (1) implies conditions (2) and (3) by Lemma 19.6. Condition (2) implies condition (3) trivially. Assume (3). Choose a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] \\ Y & V \arrow[l] \end{tikzcd}$$

with U and V affine, horizontal arrows étale and such that there is a point $u \in U$ mapping to x. Next, consider a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ Y & V \arrow[l] & \mathrm{Spec}(B') \arrow[l, "\beta"'] \end{tikzcd}$$

as in (3) but for $u \in U \to V$. Let $\gamma : \mathrm{Spec}(B') \to X$ be the arrow we get from our assumption that (3) holds for X. Because $U \to X$ is étale and hence formally étale (Lemma 16.8) the morphism γ has a unique lift to U compatible with α. Then because $V \to Y$ is étale hence formally étale this lift is compatible with β. Hence (3) holds for $u \in U \to V$ and we conclude that $U \to V$ is smooth at u by More on Morphisms, Lemma 12.1. This proves that $X \to Y$ is smooth at x, thereby finishing the proof. □

Sometimes it is useful to know that one only needs to check the lifting criterion for small extensions "centered" at points of finite type (see Morphisms of Spaces, Section 25).

Lemma 20.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f locally of finite type. The following are equivalent:*

(1) *f is smooth,*
(2) *for every solid commutative diagram*

$$\begin{tikzcd} X & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ Y \arrow[from=1-1, "f"'] & \mathrm{Spec}(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

where $B' \to B$ is a small extension of Artinian local rings and β of finite type (!) there exists a dotted arrow making the diagram commute.

Proof. If f is smooth, then the infinitesimal lifting criterion (Lemma 19.6) says f is formally smooth and (2) holds.

Assume f is not smooth. The set of points $x \in X$ where f is not smooth forms a closed subset T of $|X|$. By Morphisms of Spaces, Lemma 25.6, there exists a point $x \in T \subset X$ with $x \in X_{\text{ft-pts}}$. Choose a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] & u \arrow[d, mapsto] \\ Y & V \arrow[l] & v \end{tikzcd}$$

with U and V affine, horizontal arrows étale and such that there is a point $u \in U$ mapping to x. Then u is a finite type point of U. Since $U \to V$ is not smooth at the point u, by More on Morphisms, Lemma 12.1 there is a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ Y & V \arrow[l] & \mathrm{Spec}(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

with $B' \to B$ a small extension of (Artinian) local rings such that the residue field of B is equal to $\kappa(v)$ and such that the dotted arrow does not exist. Since $U \to V$ is of finite type, we see that v is a finite type point of V. By Morphisms, Lemma 16.2 the morphism β is of finite type, hence the composition $\mathrm{Spec}(B) \to Y$ is of finite type also. Arguing exactly as in the proof of Lemma 20.1 (using that $U \to X$ and $V \to Y$ are étale hence formally étale) we see that there cannot be an arrow $\mathrm{Spec}(B) \to X$ fitting into the outer rectangle of the last displayed diagram. In other words, (2) doesn't hold and the proof is complete. ☐

Here is a useful application.

Lemma 20.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y locally Noetherian. Let $Z \subset Y$ be a closed subspace with nth infinitesimal neighbourhood $Z_n \subset Y$. Set $X_n = Z_n \times_Y X$.*

(1) *If $X_n \to Z_n$ is smooth for all n, then f is smooth at every point of $f^{-1}(Z)$.*
(2) *If $X_n \to Z_n$ is étale for all n, then f is étale at every point of $f^{-1}(Z)$.*

Proof. Assume $X_n \to Z_n$ is smooth for all n. Let $x \in X$ be a point lying over a point of Z. Given a small extension $B' \to B$ and morphisms α, β as in Lemma 20.1 part (3) the maximal ideal of B' is nilpotent (as B' is Artinian) and hence the morphism β factors through Z_n and α factors through X_n for a suitable n. Thus the lifting property for $X_n \to Z_n$ kicks in to get the desired dotted arrow in the diagram. This proves (1). Part (2) follows from (1) and the fact that a morphism is étale if and only if it is smooth of relative dimension 0. $\square$

21. The naive cotangent complex

This section is the continuation of Modules on Sites, Section 35 which in turn continues the discussion in Algebra, Section 134.

Definition 21.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *naive cotangent complex of f* is the complex defined in Modules on Sites, Definition 35.4 for the morphism of ringed topoi f_{small} between the small étale sites of X and Y, see Properties of Spaces, Lemma 21.3. Notation: NL_f or $NL_{X/Y}$.

The next lemmas show this definition is compatible with the definition for ring maps and for schemes and that $NL_{X/Y}$ is an object of $D_{QCoh}(\mathcal{O}_X)$.

Lemma 21.2. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of algebraic spaces over S with p and q étale. Then there is a canonical identification $NL_{X/Y}|_{U_{étale}} = NL_{U/V}$ in $D(\mathcal{O}_U)$.

Proof. Formation of the naive cotangent complex commutes with pullback (Modules on Sites, Lemma 35.3) and we have $p_{small}^{-1}\mathcal{O}_X = \mathcal{O}_U$ and $g_{small}^{-1}\mathcal{O}_{V_{étale}} = p_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}}$ because $q_{small}^{-1}\mathcal{O}_{Y_{étale}} = \mathcal{O}_{V_{étale}}$ by Properties of Spaces, Lemma 26.1. Tracing through the definitions we conclude that $NL_{X/Y}|_{U_{étale}} = NL_{U/V}$. $\square$

Lemma 21.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X and Y representable by schemes X_0 and Y_0. Then there is a canonical identification $NL_{X/Y} = \epsilon^* NL_{X_0/Y_0}$ in $D(\mathcal{O}_X)$ where ϵ is as in Derived Categories of Spaces, Section 4 and NL_{X_0/Y_0} is as in More on Morphisms, Definition 13.1.*

Proof. Let $f_0 : X_0 \to Y_0$ be the morphism of schemes corresponding to f. There is a canonical map $\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0} \to f_{small}^{-1}\mathcal{O}_Y$ compatible with $\epsilon^\sharp : \epsilon^{-1}\mathcal{O}_{X_0} \to \mathcal{O}_X$ because there is a commutative diagram

$$\begin{tikzcd} X_{0,Zar} \arrow[d, "f_0"'] & X_{étale} \arrow[l, "\epsilon"'] \arrow[d, "f"] \\ Y_{0,Zar} & Y_{étale} \arrow[l, "\epsilon"] \end{tikzcd}$$

see Derived Categories of Spaces, Remark 6.3. Thus we obtain a canonical map

$$\epsilon^{-1} NL_{X_0/Y_0} = \epsilon^{-1} NL_{\mathcal{O}_{X_0}/f_0^{-1}\mathcal{O}_{Y_0}} = NL_{\epsilon^{-1}\mathcal{O}_{X_0}/\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0}} \to NL_{\mathcal{O}_X/f_{small}^{-1}\mathcal{O}_Y} = NL_{X/Y}$$

by functoriality of the naive cotangent complex. To see that the induced map $\epsilon^* NL_{X_0/Y_0} \to NL_{X/Y}$ is an isomorphism in $D(\mathcal{O}_X)$ we may check on stalks at geometric points (Properties of Spaces, Theorem 19.12). Let $\overline{x} : \mathrm{Spec}(k) \to X_0$ be a geometric point lying over $x \in X_0$, with $\overline{y} = f \circ \overline{x}$ lying over $y \in Y_0$. Then

$$NL_{X/Y,\overline{x}} = NL_{\mathcal{O}_{X,\overline{x}}/\mathcal{O}_{Y,\overline{y}}}$$

This is true because taking stalks at $\overline{x}$ is the same as taking inverse image via $\overline{x} : \mathrm{Spec}(k) \to X$ and we may apply Modules on Sites, Lemma 35.3. On the other hand we have

$$(\epsilon^* NL_{X_0/Y_0})_{\overline{x}} = NL_{X_0/Y_0,x} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}} = NL_{\mathcal{O}_{X_0,x}/\mathcal{O}_{Y_0,y}} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}}$$

Some details omitted (hint: use that the stalk of a pullback is the stalk at the image point, see Sites, Lemma 34.2, as well as the corresponding result for modules, see Modules on Sites, Lemma 36.4). Observe that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X_0,x}$ and similarly for $\mathcal{O}_{Y,\overline{y}}$ (Properties of Spaces, Lemma 22.1). Thus the result follows from More on Algebra, Lemma 33.8. □

Lemma 21.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The cohomology sheaves of the complex $NL_{X/Y}$ are quasi-coherent, zero outside degrees -1, 0 and equal to $\Omega_{X/Y}$ in degree 0.*

Proof. By construction of the naive cotangent complex in Modules on Sites, Section 35 we have that $NL_{X/Y}$ is a complex sitting in degrees -1, 0 and that its cohomology in degree 0 is $\Omega_{X/Y}$ (by our construction of $\Omega_{X/Y}$ in Section 7). The sheaf of differentials is quasi-coherent (by Lemma 7.4). To finish the proof it suffices to show that $H^{-1}(NL_{X/Y})$ is quasi-coherent. This follows by checking étale locally (allowed by Lemma 21.2 and Properties of Spaces, Lemma 29.6) reducing to the case of schemes (Lemma 21.3) and finally using the result in the case of schemes (More on Morphisms, Lemma 13.3). □

Lemma 21.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite presentation, then $NL_{X/Y}$ is étale locally on X quasi-isomorphic to a complex*

$$\ldots \to 0 \to \mathcal{F}^{-1} \to \mathcal{F}^0 \to 0 \to \ldots$$

of quasi-coherent $\mathcal{O}_X$-modules with $\mathcal{F}^0$ of finite presentation and $\mathcal{F}^{-1}$ of finite type.

Proof. Formation of the naive cotangent complex commutes with étale localization by Lemma 21.2. This reduces us to the case of schemes by Lemma 21.3. The result in the case of schemes is More on Morphisms, Lemma 13.4. □

Lemma 21.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

1. *f is formally smooth,*
2. *$H^{-1}(NL_{X/Y}) = 0$ and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is locally projective.*

Proof. This follows from Lemma 19.10, Lemma 21.2, Lemma 21.3 and the case of schemes which is More on Morphisms, Lemma 13.5. □

Lemma 21.7. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

1. *f is formally étale,*
2. *$H^{-1}(NL_{X/Y}) = H^0(NL_{X/Y}) = 0$.*

Proof. Assume (1). A formally étale morphism is a formally smooth morphism. Thus $H^{-1}(NL_{X/Y}) = 0$ by Lemma 21.6. On the other hand, a formally étale morphism if formally unramified hence we have $\Omega_{X/Y} = 0$ by Lemma 14.6. Conversely, if (2) holds, then f is formally smooth by Lemma 21.6 and formally unramified by Lemma 14.6 and hence formally étale by Lemmas 19.4. $\square$

Lemma 21.8. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(1) *f is smooth, and*
(2) *f is locally of finite presentation, $H^{-1}(NL_{X/Y}) = 0$, and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is finite locally free.*

Proof. This follows from Lemma 19.10, Lemma 21.2, Lemma 21.3 and the case of schemes which is More on Morphisms, Lemma 13.7. $\square$

22. Openness of the flat locus

This section is analogue of More on Morphisms, Section 15. Note that we have defined the notion of flatness for quasi-coherent modules on algebraic spaces in Morphisms of Spaces, Section 31.

Theorem 22.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume f is locally of finite presentation and that $\mathcal{F}$ is an $\mathcal{O}_X$-module which is locally of finite presentation. Then*

$$\{x \in |X| : \mathcal{F} \text{ is flat over } Y \text{ at } x\}$$

is open in $|X|$.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

with U, V schemes and p, q surjective and étale as in Spaces, Lemma 11.6. By More on Morphisms, Theorem 15.1 the set $U' = \{u \in |U| : p^*\mathcal{F} \text{ is flat over } V \text{ at } u\}$ is open in U. By Morphisms of Spaces, Definition 31.2 the image of U' in $|X|$ is the set of the theorem. Hence we are done because the map $|U| \to |X|$ is open, see Properties of Spaces, Lemma 4.6. $\square$

Lemma 22.2. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume g is flat, f is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation. Then

$$\{x' \in |X'| : (g')^*\mathcal{F} \text{ is flat over } Y' \text{ at } x'\}$$

is the inverse image of the open subset of Theorem 22.1 under the continuous map $|g'| : |X'| \to |X|$.

Proof. This follows from Morphisms of Spaces, Lemma 31.3. □

23. Critère de platitude par fibres

Let S be a scheme. Consider a commutative diagram of algebraic spaces over S

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Given a point $x \in |X|$ we consider the question as to whether $\mathcal{F}$ is flat over Y at x. If $\mathcal{F}$ is flat over Z at x, then the theorem below states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre of $X \to Z$ over $g(x)$ is flat over the fibre of $Y \to Z$ over $g(x)$. To make sense out of this we offer the following preliminary lemma.

Lemma 23.1. *In the situation above the following are equivalent*

(1) *Pick a geometric point $\overline{x}$ of X lying over x. Set $\overline{y} = f \circ \overline{x}$ and $\overline{z} = g \circ \overline{x}$. Then the module $\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}}$ is flat over $\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}$.*
(2) *Pick a morphism $x : \mathrm{Spec}(K) \to X$ in the equivalence class of x. Set $z = g \circ x$, $X_z = \mathrm{Spec}(K) \times_{z,Z} X$, $Y_z = \mathrm{Spec}(K) \times_{z,Z} Y$, and $\mathcal{F}_z$ the pullback of $\mathcal{F}$ to X_z. Then $\mathcal{F}_z$ is flat at x over Y_z (as defined in Morphisms of Spaces, Definition 31.2).*
(3) *Pick a commutative diagram*

$$\begin{tikzcd} & & U \arrow[rr] \arrow[dll, "a"'] \arrow[drr] & & V \arrow[dll, "b"'] \arrow[d] \\ X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] & & W \arrow[dlll, "c"] \\ & Z & & & \end{tikzcd}$$

where U, V, W are schemes, and a, b, c are étale, and a point $u \in U$ mapping to x. Let $w \in W$ be the image of u. Let $\mathcal{F}_w$ be the pullback of $\mathcal{F}$ to the fibre U_w of $U \to W$ at w. Then $\mathcal{F}_w$ is flat over V_w at u.

Proof. Note that in (2) the morphism $x : \mathrm{Spec}(K) \to X$ defines a K-rational point of X_z, hence the statement makes sense. Moreover, the condition in (2) is independent of the choice of $\mathrm{Spec}(K) \to X$ in the equivalence class of x (details omitted; this will also follow from the arguments below because the other conditions do not depend on this choice). Also note that we can always choose a diagram as in (3) by: first choosing a scheme W and a surjective étale morphism $W \to Z$, then choosing a scheme V and a surjective étale morphism $V \to W \times_Z Y$, and finally choosing a scheme U and a surjective étale morphism $U \to V \times_Y X$. Having made these choices we set $U \to W$ equal to the composition $U \to V \to W$ and we can pick a point $u \in U$ mapping to x because the morphism $U \to X$ is surjective.

Suppose given both a diagram as in (3) and a geometric point $\overline{x} : \mathrm{Spec}(k) \to X$ as in (1). By Properties of Spaces, Lemma 19.4 we can choose a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u such that $\overline{x} = a \circ \overline{u}$. Denote $\overline{v} : \mathrm{Spec}(k) \to V$ and $\overline{w} : \mathrm{Spec}(k) \to W$ the induced geometric points of V and W. In this setting we know that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{U,u}^{sh}$ and similarly for Y and Z, see Properties of Spaces, Lemma 22.1. In the same vein we have

$$\mathcal{F}_{\overline{x}} = (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{U,u}^{sh}$$

see Properties of Spaces, Lemma 29.4. Note that the stalk of $\mathcal{F}_w$ at u is given by

$$(\mathcal{F}_w)_u = (a^*\mathcal{F})_u/\mathfrak{m}_w(a^*\mathcal{F})_u$$

and the local ring of V_w at v is given by

$$\mathcal{O}_{V_w,v} = \mathcal{O}_{V,v}/\mathfrak{m}_w\mathcal{O}_{V,v}.$$

Since $\mathfrak{m}_{\overline{z}} = \mathfrak{m}_w\mathcal{O}_{Z,\overline{z}} = \mathfrak{m}_w\mathcal{O}_{W,w}^{sh}$ we see that

$$\begin{aligned} \mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}} &= (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}} \\ &= (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}_{U,u}^{sh}/\mathfrak{m}_w\mathcal{O}_{U,u}^{sh} \\ &= (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}_{U_w,\overline{u}}^{sh} \\ &= (\mathcal{F}_w)_{\overline{u}} \end{aligned}$$

the penultimate equality by Algebra, Lemma 156.4 and the last equality by Properties of Spaces, Lemma 29.4. The same arguments applied to the structure sheaves of V and Y show that

$$\mathcal{O}_{V_w,\overline{v}}^{sh} = \mathcal{O}_{V,v}^{sh}/\mathfrak{m}_w\mathcal{O}_{V,v}^{sh} = \mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}.$$

OK, and now we can use Morphisms of Spaces, Lemma 31.1 to see that (1) is equivalent to (3).

Finally we prove the equivalence of (2) and (3). To do this we pick a field extension $\tilde{K}$ of K and a morphism $\tilde{x} : \mathrm{Spec}(\tilde{K}) \to U$ which lies over u (this is possible because $u \times_{X,x} \mathrm{Spec}(K)$ is a nonempty scheme). Set $\tilde{z} : \mathrm{Spec}(\tilde{K}) \to U \to W$ be the composition. We obtain a commutative diagram

$$\begin{tikzcd} & & & U_w \times_w \tilde{z} \arrow[dlll, "a"'] \arrow[rr] \arrow[dr] & & V_w \times_w \tilde{z} \arrow[dlll, "b"'] \arrow[dl] \\ X_z \arrow[rr, "f"'] \arrow[dr, "g"'] & & Y_z \arrow[dl, "h"] & & \tilde{z} \arrow[dlll, "c"'] \\ & z \end{tikzcd}$$

where $z = \mathrm{Spec}(K)$ and $w = \mathrm{Spec}(\kappa(w))$. Now it is clear that $\mathcal{F}_w$ and $\mathcal{F}_z$ pull back to the same module on $U_w \times_w \tilde{z}$. This leads to a commutative diagram

$$\begin{tikzcd} X_z \arrow[d] & U_w \times_w \tilde{z} \arrow[l] \arrow[r] \arrow[d] & U_w \arrow[d] \\ Y_z & V_w \times_w \tilde{z} \arrow[l] \arrow[r] & V_w \end{tikzcd}$$

both of whose squares are cartesian and whose bottom horizontal arrows are flat: the lower left horizontal arrow is the composition of the morphism $Y \times_Z \tilde{z} \to Y \times_Z z = Y_z$ (base change of a flat morphism), the étale morphism $V \times_Z \tilde{z} \to Y \times_Z \tilde{z}$, and the étale morphism $V \times_W \tilde{z} \to V \times_Z \tilde{z}$. Thus it follows from Morphisms of Spaces, Lemma 31.3 that

$$\mathcal{F}_z \text{ flat at } x \text{ over } Y_z \Leftrightarrow \mathcal{F}|_{U_w \times_w \tilde{z}} \text{ flat at } \tilde{x} \text{ over } V_w \times_w \tilde{z} \Leftrightarrow \mathcal{F}_w \text{ flat at } u \text{ over } V_w$$

and we win. □

Definition 23.2. Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$ be a point and denote $z \in |Z|$ its image.

(1) We say *the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z* if the equivalent conditions of Lemma 23.1 are satisfied.
(2) We say *the fibre of X over z is flat at x over the fibre of Y over z* if the equivalent conditions of Lemma 23.1 hold with $\mathcal{F} = \mathcal{O}_X$.
(3) We say *the fibre of X over z is flat over the fibre of Y over z* if for all $x \in |X|$ lying over z the fibre of X over z is flat at x over the fibre of Y over z

With this definition in hand we can state a version of the criterion as follows. The Noetherian version can be found in Section 24.

Theorem 23.3. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation, and*
(3) *Y is locally of finite type over Z.*

Let $x \in |X|$ and let $y \in |Y|$ and $z \in |Z|$ be the images of x. If $\mathcal{F}_{\overline{x}} \neq 0$, then the following are equivalent:

(1) *$\mathcal{F}$ is flat over Z at x and the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z, and*
(2) *Y is flat over Z at y and $\mathcal{F}$ is flat over Y at x.*

Moreover, the set of points x where (1) and (2) hold is open in $Supp(\mathcal{F})$.

Proof. Choose a diagram as in Lemma 23.1 part (3). It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. Thus the theorem follows from the corresponding result for schemes which is More on Morphisms, Theorem 16.2. □

Lemma 23.4. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *X is flat over Z,*
(3) *for every $z \in |Z|$ the fibre of X over z is flat over the fibre of Y over z, and*
(4) *Y is locally of finite type over Z.*

Then f is flat. If f is also surjective, then Y is flat over Z.

Proof. This is a special case of Theorem 23.3. □

Lemma 23.5. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,*
(3) *$\mathcal{F}$ is flat over Z, and*
(4) *Y is locally of finite type over Z.*

Then the set

$$A = \{x \in |X| : \mathcal{F} \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change: If $Z' \to Z$ is a morphism of algebraic spaces, and A' is the set of points of $X' = X \times_Z Z'$ where $\mathcal{F}' = \mathcal{F} \times_Z Z'$ is flat over $Y' = Y \times_Z Z'$, then A' is the inverse image of A under the continuous map $|X'| \to |X|$.

Proof. One way to prove this is to translate the proof as given in More on Morphisms, Lemma 16.4 into the category of algebraic spaces. Instead we will prove this by reducing to the case of schemes. Namely, choose a diagram as in Lemma 23.1 part (3) such that a, b, and c are surjective. It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. The only minor point to make is that given a morphism of algebraic spaces $Z' \to Z$ we choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Then we set $U' = W' \times_W U$ and $V' = W' \times_W V$. We write a', b', c' for the morphisms from U', V', W' to X', Y', Z'. In this case A, resp. A' are images of the open subsets of U, resp. U' associated to $a^*\mathcal{F}$, resp. $(a')^*\mathcal{F}'$. This indeed does reduce the lemma to More on Morphisms, Lemma 16.4. $\square$

Lemma 23.6. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *X is flat over Z, and*
(3) *Y is locally of finite type over Z.*

Then the set

$$\{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change $Z' \to Z$.

Proof. This is a special case of Lemma 23.5. $\square$

Lemma 23.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $x \in |X|$ with image $y \in |Y|$. If $\mathcal{F}$ is flat at x over Y, then the following are equivalent*

(1) *$(\mathcal{F}_{\overline{y}})_{\overline{x}}$ is a flat $\mathcal{O}_{X_{\overline{y}},\overline{x}}$-module,*
(2) *$(\mathcal{F}_{\overline{y}})_{\overline{x}}$ is a free $\mathcal{O}_{X_{\overline{y}},\overline{x}}$-module,*
(3) *$\mathcal{F}_{\overline{y}}$ is finite free in an étale neighbourhood of $\overline{x}$ in $X_{\overline{y}}$, and*
(4) *$\mathcal{F}$ is finite free in an étale neighbourhood of x in X.*

Here $\overline{x}$ is a geometric point of X lying over x and $\overline{y} = f \circ \overline{x}$.

Proof. Pick a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale such that there is a point $u \in U$ mapping to x. Let $v \in V$ be the image of u. Applying Lemma 23.1 to $\mathrm{id} : X \to X$ over Y we see that (1) translates into the condition "$\mathcal{F}|_{U_v}$ is flat over

U_v at u". In other words, (1) is equivalent to $(\mathcal{F}|_{U_v})_u$ being a flat $\mathcal{O}_{U_v,u}$-module. By the case of schemes (More on Morphisms, Lemma 16.7), we find that this implies that $\mathcal{F}|_U$ is finite free in an open neighbourhood of u. In this way we see that (1) implies (4). The implications (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) are immediate. For the implication (3) $\Rightarrow$ (2) use the description of local rings and stalks in Properties of Spaces, Lemmas 22.1 and 29.4. □

Lemma 23.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module flat over Y. Then the set*

$$\{x \in |X| : \mathcal{F} \text{ free in an étale neighbourhood of } x\}$$

is open in $|X|$ and its formation commutes with arbitrary base change $Y' \to Y$.

Proof. Openness holds trivially. Let $Y' \to Y$ be a morphism of algebraic spaces, set $X' = Y' \times_Y X$, and let $x' \in |X'|$ be a point lying over $x \in |X|$. By Lemma 23.7 we see that x is in our set if and only if $(\mathcal{F}_{\overline{y}})_{\overline{x}}$ is a flat $\mathcal{O}_{X_{\overline{y}},\overline{x}}$-module. Simiarly, x' is in the analogue of our set for the pullback $\mathcal{F}'$ of $\mathcal{F}$ to X' if and only if $(\mathcal{F}'_{\overline{y}'})_{\overline{x}'}$ is a flat $\mathcal{O}_{X'_{\overline{y}'},\overline{x}'}$-module (with obvious notation). These two assertions are equivalent by Lemma 23.1 applied to the morphism $\text{id} : X \to X$ over Y. Thus the statement on base change holds. □

24. Flatness over a Noetherian base

Here is the "Critère de platitude par fibres" in the Noetherian case.

Theorem 24.1. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X, Y, Z locally Noetherian, and*
(2) *$\mathcal{F}$ a coherent $\mathcal{O}_X$-module.*

Let $x \in |X|$ and let $y \in |Y|$ and $z \in |Z|$ be the images of x. If $\mathcal{F}_{\overline{x}} \neq 0$, then the following are equivalent:

(1) *$\mathcal{F}$ is flat over Z at x and the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z, and*
(2) *Y is flat over Z at y and $\mathcal{F}$ is flat over Y at x.*

Proof. Choose a diagram as in Lemma 23.1 part (3). It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. Thus the theorem follows from the corresponding result for schemes which is More on Morphisms, Theorem 16.1. □

Lemma 24.2. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

(1) *X, Y, Z locally Noetherian,*
(2) *X is flat over Z,*
(3) *for every $z \in |Z|$ the fibre of X over z is flat over the fibre of Y over z.*

Then f is flat. If f is also surjective, then Y is flat over Z.

Proof. This is a special case of Theorem 24.1. □

Just like for checking smoothness, if the base is Noetherian it suffices to check flatness over Artinian rings. Here is a sample statement.

Lemma 24.3. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let X be an algebraic space locally of finite presentation over $S = \mathrm{Spec}(A)$. For $n \geq 1$ set $S_n = \mathrm{Spec}(A/I^n)$ and $X_n = S_n \times_S X$. Let $\mathcal{F}$ be coherent $\mathcal{O}_X$-module. If for every $n \geq 1$ the pullback $\mathcal{F}_n$ of $\mathcal{F}$ to X is flat over S_n, then the (open) locus where $\mathcal{F}$ is flat over X contains the inverse image of $V(I)$ under $X \to S$.*

Proof. The locus where $\mathcal{F}$ is flat over S is open in $|X|$ by Theorem 22.1. The statement is insensitive to replacing X by the members of an étale covering, hence we may assume X is an affine scheme. In this case the result follows immediately from Algebra, Lemma 99.11. Some details omitted. $\square$

25. Normalization revisited

Normalization commutes with smooth base change.

Lemma 25.1. *Let S be a scheme. Let $f : Y \to X$ be a smooth morphism of algebraic spaces over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The integral closure of $\mathcal{O}_Y$ in $f^*\mathcal{A}$ is equal to $f^*\mathcal{A}'$ where $\mathcal{A}' \subset \mathcal{A}$ is the integral closure of $\mathcal{O}_X$ in $\mathcal{A}$.*

Proof. By our construction of the integral closure, see Morphisms of Spaces, Definition 48.2, this reduces immediately to the case where X and Y are affine. In this case the result is Algebra, Lemma 147.4. $\square$

Lemma 25.2 (Normalization commutes with smooth base change)**.** *Let S be a scheme. Let*

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d] & Y_1 \arrow[d, "f"] \\ X_2 \arrow[r, "\varphi"] & X_1 \end{tikzcd}$$

be a fibre square of algebraic spaces over S. Assume f is quasi-compact and quasi-separated and φ is smooth. Let $Y_i \to X_i' \to X_i$ be the normalization of X_i in Y_i. Then $X_2' \cong X_2 \times_{X_1} X_1'$.

Proof. The base change of the factorization $Y_1 \to X_1' \to X_1$ to X_2 is a factorization $Y_2 \to X_2 \times_{X_1} X_1' \to X_1$ and $X_2 \times_{X_1} X_1' \to X_1$ is integral (Morphisms of Spaces, Lemma 45.5). Hence we get a morphism $h : X_2' \to X_2 \times_{X_1} X_1'$ by the universal property of Morphisms of Spaces, Lemma 48.5. Observe that X_2' is the relative spectrum of the integral closure of $\mathcal{O}_{X_2}$ in $f_{2,*}\mathcal{O}_{Y_2}$. If $\mathcal{A}' \subset f_{1,*}\mathcal{O}_{Y_1}$ denotes the integral closure of $\mathcal{O}_{X_2}$, then $X_2 \times_{X_1} X_1'$ is the relative spectrum of $\varphi^*\mathcal{A}'$ as the construction of the relative spectrum commutes with arbitrary base change. By Cohomology of Spaces, Lemma 11.2 we know that $f_{2,*}\mathcal{O}_{Y_2} = \varphi^* f_{1,*}\mathcal{O}_{Y_1}$. Hence the result follows from Lemma 25.1. $\square$

26. Cohen-Macaulay morphisms

This is the analogue of More on Morphisms, Section 22.

Lemma 26.1. *The property of morphisms of germs of schemes*

$$\mathcal{P}((X, x) \to (S, s)) =$$
the local ring $\mathcal{O}_{X_s,x}$ *of the fibre is Noetherian and Cohen-Macaulay*

is étale local on the source-and-target (Descent, Definition 33.1).

Proof. Given a diagram as in Descent, Definition 33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Descent, Lemma 33.5. Thus the strict henselizations of the local rings $\mathcal{O}_{U'_{v'},u'}$ and $\mathcal{O}_{U_v,u}$ are the same. We conclude by More on Algebra, Lemma 45.9. $\square$

Definition 26.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian (Divisors on Spaces, Definition 4.2).

(1) Let $x \in |X|$, and $y = f(x)$. We say that f is *Cohen-Macaulay at x* if f is flat at x and the equivalent conditions of Morphisms of Spaces, Lemma 22.5 hold for the property $\mathcal{P}$ described in Lemma 26.1.
(2) We say f is a *Cohen-Macaulay morphism* if f is Cohen-Macaulay at every point of X.

Here is a translation.

Lemma 26.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. The following are equivalent*

(1) *f is Cohen-Macaulay,*
(2) *f is flat and for some surjective étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Cohen-Macaulay algebraic spaces, and*
(3) *f is flat and for any étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Cohen-Macaulay algebraic spaces.*

Given $x \in |X|$ with image $y \in |Y|$ the following are equivalent

(a) *f is Cohen-Macaulay at x, and*
(b) *$\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is flat and $\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$ is Cohen-Macaulay.*

Proof. Given an étale morphism $V \to Y$ where V is a scheme choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Let $u \in U$ with images $x \in |X|$, $y \in |Y|$, and $v \in V$. Then f is Cohen-Macaulay at x if and only if $U \to V$ is Cohen-Macaulay at u (by definition). Moreover the morphism $U_v \to X_v = (X_V)_v$ is surjective étale. Hence the scheme U_v is Cohen-Macaulay if and only if the algebraic space X_v is Cohen-Macaulay. Thus the equivalence of (1), (2), and (3) follows from the corresponding equivalence for morphisms of schemes, see More on Morphisms, Lemma 22.2 by a formal argument.

Proof of equivalence of (a) and (b). The corresponding equivalence for flatness is Morphisms of Spaces, Lemma 30.8. Thus we may assume f is flat at x when proving the equivalence. Consider a diagram and x, y, u, v as above. Then $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is equal to the map $\mathcal{O}_{V,v}^{sh} \to \mathcal{O}_{U,u}^{sh}$ on strict henselizations of local rings, see Properties of Spaces, Lemma 22.1. Thus we have

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}} = (\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})^{sh}$$

by Algebra, Lemma 156.4. Thus we have to show that the Noetherian local ring $\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u}$ is Cohen-Macaulay if and only if its strict henselization is. This is More on Algebra, Lemma 45.9. □

Lemma 26.4. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume that the fibres of f, g, and $g \circ f$ are locally Noetherian. Let $x \in |X|$ with images $y \in |Y|$ and $z \in |Z|$.*

1. *If f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$, then $g \circ f$ is Cohen-Macaulay at x.*
2. *If f and g are Cohen-Macaulay, then $g \circ f$ is Cohen-Macaulay.*
3. *If $g \circ f$ is Cohen-Macaulay at x and f is flat at x, then f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$.*
4. *If $f \circ g$ is Cohen-Macaulay and f is flat, then f is Cohen-Macaulay and g is Cohen-Macaulay at every point in the image of f.*

Proof. Working étale locally this follows from the corresponding result for schemes, see More on Morphisms, Lemma 22.4. Alternatively, we can use the equivalence of (a) and (b) in Lemma 26.3. Thus we consider the local homomorphism of Noetherian local rings

$$\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}}$$

whose fibre is

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$$

and we use Algebra, Lemma 163.3. □

Lemma 26.5. *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of locally Noetherian algebraic spaces over S. If X is Cohen-Macaulay, then f is Cohen-Macaulay and $\mathcal{O}_{Y,f(\overline{x})}$ is Cohen-Macaulay for all $x \in |X|$.*

Proof. After translating into algebra using Lemma 26.3 (compare with the proof of Lemma 26.4) this follows from Algebra, Lemma 163.3. □

Lemma 26.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in |X'|$ be a point with image $x \in |X|$.*

1. *If f is Cohen-Macaulay at x, then $f' : X' \to Y'$ is Cohen-Macaulay at x'.*
2. *If f is flat at x and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.*
3. *If $Y' \to Y$ is flat at $f'(x')$ and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.*

Proof. Denote $y \in |Y|$ and $y' \in |Y'|$ the image of x'. Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to X \times_Y V$ where U is a scheme. Choose a surjectiev étale morphism $V' \to Y' \times_Y V$ where V' is a scheme. Then $U' = U \times_V V'$ is a scheme which comes equipped with a surjective étale morphism $U' \to X'$. Choose $u' \in U'$ mapping to x'. Denote $u \in U$ the image of u'. Then the lemma follows from the lemma for $U \to V$ and its base change $U' \to V'$ and the points u' and u (this follows from the definitions). Thus the lemma follows from the case of schemes, see More on Morphisms, Lemma 22.6. $\square$

Lemma 26.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let*

$$W = \{x \in |X| : f \text{ is Cohen-Macaulay at } x\}$$

Then W is open in $|X|$ and the formation of W commutes with arbitrary base change of f: For any morphism $g : Y' \to Y$, consider the base change $f' : X' \to Y'$ of f and the projection $g' : X' \to X$. Then the corresponding set W' for the morphism f' is equal to $W' = (g')^{-1}(W)$.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with étale vertical arrows and U and V schemes. Let $u \in U$ with image $x \in |X|$. Then f is Cohen-Macaulay at x if and only if $U \to V$ is Cohen-Macaulay at u (by definition). Thus we reduce to the case of the morphism $U \to V$. See More on Morphisms, Lemma 22.7. $\square$

Lemma 26.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is locally of finite presentation and Cohen-Macaulay. Then there exist open and closed subschemes $X_d \subset X$ such that $X = \coprod_{d \geq 0} X_d$ and $f|_{X_d} : X_d \to Y$ has relative dimension d.*

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with étale vertical arrows and U and V schemes. Then $U \to V$ is locally of finite presentation and Cohen-Macaulay (immediate from our definitions). Thus we have a decomposition $U = \coprod_{d \geq 0} U_d$ into open and closed subschemes with $f|_{U_d} : U_d \to V$ of relative dimension d, see Morphisms, Lemma 29.4. Let $u \in U$ with image $x \in |X|$. Then f has relative dimension d at x if and only if $U \to V$ has relative dimension d at u (this follows from our definitions). In this way we see that U_d is the inverse image of a subset $X_d \subset |X|$ which is necessarily open and closed. Denoting X_d the corresponding open and closed algebraic subspace of X we see that the lemma is true. $\square$

27. Gorenstein morphisms

This is the analogue of Duality for Schemes, Section 25.

Lemma 27.1. *The property of morphisms of germs of schemes*

$$\begin{aligned} &\mathcal{P}((X,x) \to (S,s)) = \\ &\textit{the local ring } \mathcal{O}_{X_s,x} \textit{ of the fibre is Noetherian and Gorenstein} \end{aligned}$$

is étale local on the source-and-target (Descent, Definition 33.1).

Proof. Given a diagram as in Descent, Definition 33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Descent, Lemma 33.5. Thus $\mathcal{O}_{U_v,u} \to \mathcal{O}_{U'_{v'},u'}$ is the localization of an étale ring map. Hence the first is Noetherian if and only if the second is Noetherian, see More on Algebra, Lemma 44.1. Then, since $\mathcal{O}_{U'_{v'},u'}/\mathfrak{m}_u\mathcal{O}_{U'_{v'},u'} = \kappa(u')$ (Algebra, Lemma 143.5) is a Gorenstein ring, we see that $\mathcal{O}_{U_v,u}$ is Gorenstein if and only if $\mathcal{O}_{U'_{v'},u'}$ is Gorenstein by Dualizing Complexes, Lemma 21.8. □

Definition 27.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian (Divisors on Spaces, Definition 4.2).

(1) Let $x \in |X|$, and $y = f(x)$. We say that f is *Gorenstein at x* if f is flat at x and the equivalent conditions of Morphisms of Spaces, Lemma 22.5 hold for the property $\mathcal{P}$ described in Lemma 27.1.
(2) We say f is a *Gorenstein morphism* if f is Gorenstein at every point of X.

Here is a translation.

Lemma 27.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. The following are equivalent*

(1) *f is Gorenstein,*
(2) *f is flat and for some surjective étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Gorenstein algebraic spaces, and*
(3) *f is flat and for any étale morphism $V \to Y$ where V is a scheme, the fibres of $X_V \to V$ are Gorenstein algebraic spaces.*

Given $x \in |X|$ with image $y \in |Y|$ the following are equivalent

(a) *f is Gorenstein at x, and*
(b) *$\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is flat and $\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$ is Gorenstein.*

Proof. Given an étale morphism $V \to Y$ where V is a scheme choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Let $u \in U$ with images $x \in |X|$, $y \in |Y|$, and $v \in V$. Then f is Gorenstein at x if and only if $U \to V$ is Gorenstein at u (by definition). Moreover the morphism $U_v \to X_v = (X_V)_v$ is surjective étale. Hence the scheme U_v is Gorenstein if and only if the algebraic space X_v is Gorenstein. Thus the equivalence of (1), (2), and (3) follows from the corresponding equivalence for morphisms of schemes, see Duality for Schemes, Lemma 24.4 by a formal argument.

Proof of equivalence of (a) and (b). The corresponding equivalence for flatness is Morphisms of Spaces, Lemma 30.8. Thus we may assume f is flat at x when proving the equivalence. Consider a diagram and x, y, u, v as above. Then $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is equal to the map $\mathcal{O}_{V,v}^{sh} \to \mathcal{O}_{U,u}^{sh}$ on strict henselizations of local rings, see Properties of Spaces, Lemma 22.1. Thus we have

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}} = (\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})^{sh}$$

by Algebra, Lemma 156.4. Thus we have to show that the Noetherian local ring $\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u}$ is Gorenstein if and only if its strict henselization is. This follows immediately from Dualizing Complexes, Lemma 22.3 and the definition of a Gorenstein local ring as a Noetherian local ring which is a dualizing complex over itself. □

Lemma 27.4. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume that the fibres of f, g, and $g \circ f$ are locally Noetherian. Let $x \in |X|$ with images $y \in |Y|$ and $z \in |Z|$.*

1. *If f is Gorenstein at x and g is Gorenstein at $f(x)$, then $g \circ f$ is Gorenstein at x.*
2. *If f and g are Gorenstein, then $g \circ f$ is Gorenstein.*
3. *If $g \circ f$ is Gorenstein at x and f is flat at x, then f is Gorenstein at x and g is Gorenstein at $f(x)$.*
4. *If $f \circ g$ is Gorenstein and f is flat, then f is Gorenstein and g is Gorenstein at every point in the image of f.*

Proof. Working étale locally this follows from the corresponding result for schemes, see Duality for Schemes, Lemma 25.6. Alternatively, we can use the equivalence of (a) and (b) in Lemma 27.3. Thus we consider the local homomorphism of Noetherian local rings

$$\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}}$$

whose fibre is

$$\mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$$

and we use Dualizing Complexes, Lemma 21.8. □

Lemma 27.5. *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of locally Noetherian algebraic spaces over S. If X is Gorenstein, then f is Gorenstein and $\mathcal{O}_{Y,f(\overline{x})}$ is Gorenstein for all $x \in |X|$.*

Proof. After translating into algebra using Lemma 27.3 (compare with the proof of Lemma 27.4) this follows from Dualizing Complexes, Lemma 21.8. □

Lemma 27.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume the fibres of f are locally Noetherian. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in |X'|$ be a point with image $x \in |X|$.*

1. *If f is Gorenstein at x, then $f' : X' \to Y'$ is Gorenstein at x'.*
2. *If f is flat at x and f' is Gorenstein at x', then f is Gorenstein at x.*
3. *If $Y' \to Y$ is flat at $f'(x')$ and f' is Gorenstein at x', then f is Gorenstein at x.*

Proof. Denote $y \in |Y|$ and $y' \in |Y'|$ the image of x'. Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to X \times_Y V$ where U is a scheme. Choose a surjectiev étale morphism $V' \to Y' \times_Y V$ where V' is a scheme. Then $U' = U \times_V V'$ is a scheme which comes equipped with a surjective étale morphism $U' \to X'$. Choose $u' \in U'$ mapping to x'. Denote

$u \in U$ the image of u'. Then the lemma follows from the lemma for $U \to V$ and its base change $U' \to V'$ and the points u' and u (this follows from the definitions). Thus the lemma follows from the case of schemes, see Duality for Schemes, Lemma 25.8. ☐

Lemma 27.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let*

$$W = \{x \in |X| : f \text{ is Gorenstein at } x\}$$

Then W is open in $|X|$ and the formation of W commutes with arbitrary base change of f: For any morphism $g : Y' \to Y$, consider the base change $f' : X' \to Y'$ of f and the projection $g' : X' \to X$. Then the corresponding set W' for the morphism f' is equal to $W' = (g')^{-1}(W)$.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Let $u \in U$ with image $x \in |X|$. Then f is Gorenstein at x if and only if $U \to V$ is Gorenstein at u (by definition). Thus we reduce to the case of the morphism $U \to V$ of schemes. Openness is proven in Duality for Schemes, Lemma 25.11 and compatibility with base change in Duality for Schemes, Lemma 25.9. ☐

28. Slicing Cohen-Macaulay morphisms

Let S be a scheme. Let X be an algebraic space over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. In this case we denote $V(f_1, \ldots, f_r)$ the *closed subspace of X cut out by $f_1, \ldots, f_r$*. More precisely, we can define $V(f_1, \ldots, f_r)$ as the closed subspace of X corresponding to the quasi-coherent sheaf of ideals generated by $f_1, \ldots, f_r$, see Morphisms of Spaces, Lemma 13.1. Alternatively, we can choose a presentation $X = U/R$ and consider the closed subscheme $Z \subset U$ cut out by $f_1|U, \ldots, f_r|_U$. It is clear that Z is an R-invariant (see Groupoids, Definition 19.1) closed subscheme and we may set $V(f_1, \ldots, f_r) = Z/R_Z$.

Lemma 28.1. *Let S be a scheme. Consider a cartesian diagram*

$$\begin{tikzcd} X \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ Y & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

where $X \to Y$ is a morphism of algebraic spaces over S which is flat and locally of finite presentation, and where k is a field over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ and $z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\overline{z}}$. Then, after replacing X by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow Y$$

is flat and locally of finite presentation.

Proof. Set $Z = V(f_1, \ldots, f_r)$. It is clear that $Z \to X$ is locally of finite presentation, hence the composition $Z \to Y$ is locally of finite presentation, see Morphisms of Spaces, Lemma 28.2. Hence it suffices to show that $Z \to Y$ is flat in a neighbourhood of $p(z)$. Let k'/k be an extension field. Then $F' = F \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is surjective and flat over F, hence we can find a point $z' \in |F'|$ mapping to z and the local ring map $\mathcal{O}_{F,\overline{z}} \to \mathcal{O}_{F',\overline{z}'}$ is flat, see Morphisms of Spaces, Lemma 30.8. Hence the image of $f_1, \ldots, f_r$ in $\mathcal{O}_{F',\overline{z}'}$ is a regular sequence too, see Algebra, Lemma 68.5. Thus, during the proof we may replace k by an extension field. In particular, we may assume that $z \in |F|$ comes from a section $z : \mathrm{Spec}(k) \to F$ of the structure morphism $F \to \mathrm{Spec}(k)$.

Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. After possibly enlarging k once more we may assume that $\mathrm{Spec}(k) \to F \to X$ factors through U (as $U \to X$ is surjective). Let $u : \mathrm{Spec}(k) \to U$ be such a factorization and denote $v \in V$ the image of u. Note that the morphisms

$$U_v \times_{\mathrm{Spec}(\kappa(v))} \mathrm{Spec}(k) = U \times_V \mathrm{Spec}(k) \to U \times_Y \mathrm{Spec}(k) \to F$$

are étale (the first as the base change of $V \to V \times_Y V$ and the second as the base change of $U \to X$). Moreover, by construction the point $u : \mathrm{Spec}(k) \to U$ gives a point of the left most space which maps to z on the right. Hence the elements $f_1, \ldots, f_r$ map to a regular sequence in the local ring on the right of the following map

$$\mathcal{O}_{U_v,u} \longrightarrow \mathcal{O}_{U_v \times_{\mathrm{Spec}(\kappa(v))} \mathrm{Spec}(k),\overline{u}} = \mathcal{O}_{U \times_V \mathrm{Spec}(k),\overline{u}}.$$

But since the displayed arrow is flat (combine More on Flatness, Lemma 2.5 and Morphisms of Spaces, Lemma 30.8) we see from Algebra, Lemma 68.5 that $f_1, \ldots, f_r$ maps to a regular sequence in $\mathcal{O}_{U_v,u}$. By More on Morphisms, Lemma 23.2 we conclude that the morphism of schemes

$$V(f_1, \ldots, f_r) \times_X U = V(f_1|_U, \ldots, f_r|_U) \to V$$

is flat in an open neighbourhood U' of u. Let $X' \subset X$ be the open subspace corresponding to the image of $|U'| \to |X|$ (see Properties of Spaces, Lemmas 4.6 and 4.8). We conclude that $V(f_1, \ldots, f_r) \cap X' \to Y$ is flat (see Morphisms of Spaces, Definition 30.1) as we have the commutative diagram

$$\begin{tikzcd}
V(f_1, \ldots, f_r) \times_X U' \arrow[r] \arrow[d, "a"'] & V \arrow[d, "b"] \\
V(f_1, \ldots, f_r) \cap X' \arrow[r] & Y
\end{tikzcd}$$

with a, b étale and a surjective. $\square$

29. Reduced fibres

This section is the analogue of More on Morphisms, Section 26.

Lemma 29.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. The following are equivalent*

(1) *for some morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class of y the algebraic space X_k is geometrically reduced over k,*

(2) *for every morphism* $\text{Spec}(k) \to Y$ *in the equivalence class of* y *the algebraic space* X_k *is geometrically reduced over* k*,*
(3) *for every morphism* $\text{Spec}(k) \to Y$ *in the equivalence class of* y *the algebraic space* X_k *is reduced.*

Proof. This follows immediately from Spaces over Fields, Lemma 11.6 and the definition of the equivalence relation defining $|X|$ given in Properties of Spaces, Section 4. $\square$

Definition 29.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. We say *the fibre of* $f : X \to Y$ *at* y *is geometrically reduced* if the equivalent conditions of Lemma 29.1 hold.

Here are the obligatory lemmas.

Lemma 29.3. *Let* S *be a scheme. Let* $f : X \to Y$ *and* $g : Y' \to Y$ *be morphisms of algebraic spaces over* S*. Denote* $f' : X' \to Y'$ *the base change of* f *by* g*. Then*

$$\begin{aligned} & \{y' \in |Y'| : \textit{the fibre of } f' : X' \to Y' \textit{ at } y' \textit{ is geometrically reduced}\} \\ = \; & g^{-1}(\{y \in |Y| : \textit{the fibre of } f : X \to Y \textit{ at } y \textit{ is geometrically reduced}\}). \end{aligned}$$

Proof. For $y' \in |Y'|$ choose a morphism $\text{Spec}(k) \to Y'$ in the equivalence class of y'. Then $g(y')$ is represented by the composition $\text{Spec}(k) \to Y' \to Y$. Hence $X' \times_{Y'} \text{Spec}(k) = X \times_Y \text{Spec}(k)$ and the result follows from the definition. $\square$

Lemma 29.4. *Let* S *be a scheme. Let* $f : X \to Y$ *be a morphism of algebraic spaces over* S *which is quasi-compact and locally of finite presentation. Then the set*

$$E = \{y \in |Y| : \textit{the fibre of } f : X \to Y \textit{ at } y \textit{ is geometrically reduced}\}$$

is étale locally constructible.

Proof. Choose an affine scheme V and an étale morphism $V \to Y$. The meaning of the statement is that the inverse image of E in $|V|$ is constructible. By Lemma 29.3 we may replace Y by V, i.e., we may assume that Y is an affine scheme. Then X is quasi-compact. Choose an affine scheme U and a surjective étale morphism $U \to X$. For a morphism $\text{Spec}(k) \to Y$ the morphism between fibres $U_k \to X_k$ is surjective étale. Hence U_k is geometrically reduced over k if and only if X_k is geometrically reduced over k, see Spaces over Fields, Lemma 11.7. Thus the set E for $X \to Y$ is the same as the set E for $U \to Y$. In this way we see that the lemma follows from the case of schemes, see More on Morphisms, Lemma 26.5. $\square$

Lemma 29.5. *Let* X *be an algebraic space over a discrete valuation ring* R *whose structure morphism* $X \to \text{Spec}(R)$ *is proper and flat. If the special fibre is reduced, then both* X *and the generic fibre* X_η *are reduced.*

Proof. Choose an étale morphism $U \to X$ where U is an affine scheme. Then U is of finite type over R. Let $u \in U$ be in the special fibre. The local ring $A = \mathcal{O}_{U,u}$ is essentially of finite type over R, hence Noetherian. Let $\pi \in R$ be a uniformizer. Since X is flat over R, we see that $\pi \in \mathfrak{m}_A$ is a nonzerodivisor on A and since the special fibre of X is reduced, we have that $A/\pi A$ is reduced. If $a \in A$, $a \neq 0$ then there exists an $n \geq 0$ and an element $a' \in A$ such that $a = \pi^n a'$ and $a' \notin \pi A$. This follows from Krull intersection theorem (Algebra, Lemma 51.4). If a is nilpotent, so is a', because π is a nonzerodivisor. But a' maps to a nonzero element of the reduced ring $A/\pi A$ so this is impossible. Hence A is reduced. It follows that there exists an open neighbourhood of u in U which is reduced (small detail omitted; use that U is Noetherian). Thus we can find an étale morphism $U \to X$ with U a reduced scheme, such that every point of the special fibre of X is in the image. Since X is proper over R it follows that $U \to X$ is surjective. Hence X is reduced. Since the generic fibre of $U \to \text{Spec}(R)$ is reduced as well (on affine pieces it is computed by taking localizations), we conclude the same thing is true for the generic fibre. $\square$

Lemma 29.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is flat, proper, and of finite presentation, then the set*

$$E = \{y \in |Y| : \textit{the fibre of } f : X \to Y \textit{ at } y \textit{ is geometrically reduced}\}$$

is open in $|Y|$.

Proof. By Lemma 29.3 formation of E commutes with base change. To check a subset of $|Y|$ is open, we may replace Y by the members of an étale covering. Thus we may assume Y is affine. Then Y is a cofiltered limit of affine schemes of finite type over $\mathbf{Z}$. Hence we can assume $X \to Y$ is the base change of $X_0 \to Y_0$ where Y_0 is the spectrum of a finite type $\mathbf{Z}$-algebra and $X_0 \to Y_0$ is flat and proper. See Limits of Spaces, Lemma 7.1, 6.12, and 6.13. Since the formation of E commutes with base change (see above), we may assume the base is Noetherian.

Assume Y is Noetherian. The set is constructible by Lemma 29.4. Hence it suffices to show the set is stable under generalization (Topology, Lemma 19.10). By Properties, Lemma 5.10 we reduce to the case where $Y = \text{Spec}(R)$, R is a discrete valuation ring, and the closed fibre X_y is geometrically reduced. To show: the generic fibre X_η is geometrically reduced.

If not then there exists a finite extension L of the fraction field of R such that X_L is not reduced, see Spaces over Fields, Lemmas 11.4 (characteristic zero) and 11.5 (positive characteristic). There exists a discrete valuation ring $R' \subset L$ with fraction field L dominating R, see Algebra, Lemma 120.18. After replacing R by R' we reduce to Lemma 29.5. $\square$

30. Connected components of fibres

This section is the analogue of More on Morphisms, Section 28.

Lemma 30.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let*

$$n_{X/Y} : |Y| \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of connected components of X_k where $\text{Spec}(k) \to Y$ is in the equivalence class of y with k algebraically closed. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Let $\text{Spec}(k') \to Y'$ be in the equivalence class of y' with k' algebraically closed. Then we can choose a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[dr] & \mathrm{Spec}(k') \arrow[r] & Y' \arrow[d] \\ & \mathrm{Spec}(k) \arrow[r] & Y \end{tikzcd}$$

where K is an algebraically closed field. The result follows as the morphisms of schemes

$$X'_{k'} \longleftarrow (X'_{k'})_K = (X_k)_K \longrightarrow X_k$$

induce bijections between connected components, see Spaces over Fields, Lemma 12.4. To use this to prove the function is well defined take $Y' = Y$. ☐

31. Dimension of fibres

This section is the analogue of More on Morphisms, Section 30.

Lemma 31.1. *Let S be a scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S. Let $y \in |Y|$. The following quantities are the same*

(1) *$d = -\infty$ if y is not in the image of $|f|$ and otherwise the minimal integer d such that f has relative dimension $\leq d$ at every $x \in |X|$ mapping to y,*
(2) *the dimension of the algebraic space $X_k = \mathrm{Spec}(k) \times_Y X$ for any morphism $\mathrm{Spec}(k) \to Y$ in the equivalence class defining y.*

Proof. To parse this one has to consult Morphisms of Spaces, Definition 33.1, Properties of Spaces, Definition 9.2, Properties of Spaces, Definition 9.1. We will show that the numbers in (1) and (2) are equal for a fixed morphism $\mathrm{Spec}(k) \to Y$. Choose an étale morphism $V \to Y$ where V is an affine scheme and a point $v \in V$ mapping to y. Since $V \times_Y \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is surjective étale (by Properties of Spaces, Lemma 4.3) we can find a finite separable extension k'/k (by Morphisms, Lemma 36.7) and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r] \arrow[d] & V \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & Y \end{tikzcd}$$

We may replace $X \to Y$ by $V \times_Y X \to V$ and X_k by $X_{k'} = \mathrm{Spec}(k') \times_V (V \times_Y X)$ because this does not change the dimensions in question by Properties of Spaces, Lemma 22.5 and Morphisms of Spaces, Lemma 34.3. Thus we may assume that Y is an affine scheme. In this case we may assume that $k = \kappa(y)$ because the dimension of $X_{\kappa(y)}$ and X_k are the same by the aforementioned Morphisms of Spaces, Lemma 34.3 and the fact that for an algebraic space Z over a field K the relative dimension of Z at a point $z \in |Z|$ is the same as $\dim_z(Z)$ by definition. Assume Y is affine and $k = \kappa(y)$. Then X is quasi-compact we can choose an affine scheme U and an surjective étale morphism $U \to X$. Then $\dim(X_k) = \dim(U_k) = \max \dim_u(U_k)$ is equal to the number given in (1) by definition. ☐

Lemma 31.2. *Let S be a scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S. Let*

$$n_{X/Y} : |Y| \to \{-\infty, 0, 1, 2, 3, \ldots\}$$

be the function which associates to $y \in |Y|$ the integer discussed in Lemma 31.1. If $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ |g|$$

where $X' \to Y'$ is the base change of f.

Proof. This follows immediately from Lemma 31.1. $\square$

Lemma 31.3. *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of finite presentation of algebraic spaces over S. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 31.2. Then $n_{X/Y}$ is lower semi-continuous.*

Proof. Let $V \to Y$ be a surjective étale morphism where V is a scheme. If we can show that the composition $n_{X/Y} \circ |g|$ is lower semi-continuous, then the lemma follows as $|g|$ is open. Hence we may assume Y is a scheme. Working locally we may assume V is an affine scheme. Then we can choose an affine scheme U and a surjective étale morphism $U \to X$. Then $n_{X/Y} = n_{U/Y}$. Hence we may assume X and Y are both schemes. In this case the lemma follows from More on Morphisms, Lemma 30.4. $\square$

Lemma 31.4. *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 31.2. Then $n_{X/Y}$ is upper semi-continuous.*

Proof. Let $Z_d = \{x \in |X| : \text{the fibre of } f \text{ at } x \text{ has dimension } > d\}$. Then Z_d is a closed subset of $|X|$ by Morphisms of Spaces, Lemma 34.4. Since f is proper $f(Z_d)$ is closed in $|Y|$. Since $y \in f(Z_d) \Leftrightarrow n_{X/Y}(y) > d$ we see that the lemma is true. $\square$

Lemma 31.5. *Let S be a scheme. Let $f : X \to Y$ be a proper, flat, finitely presented morphism of algebraic spaces over S. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 31.2. Then $n_{X/Y}$ is locally constant.*

Proof. Immediate consequence of Lemmas 31.3 and 31.4. $\square$

32. Catenary algebraic spaces

This section continues the discussion started in Decent Spaces, Section 25. The following lemma will be used in the proof of the next one.

Lemma 32.1. *Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces over S. Let $y \in |Y|$ be a point which can be represented by a closed immersion $y : \mathrm{Spec}(k) \to Y$. Then there exists a factorization $X \to X' \to Y$ of f such that*

1. *$X' \to Y$ is integral,*
2. *$X \to X'$ is an isomorphism over $X' \setminus X'_y$,*
3. *X'_y has a unique point x' with $\kappa(x') = k$.*

Moreover, if f is finite and Y is locally Noetherian, then $X' \to Y$ is finite.

Proof. By Morphisms of Spaces, Lemma 11.2 the sheaves $f_*\mathcal{O}_X$, $(X_y \to Y)_*\mathcal{O}_{X_y}$, and $y_*\mathcal{O}_{\mathrm{Spec}(k)}$ are quasi-coherent sheaves of $\mathcal{O}_Y$-algebras. Consider the maps

$$f_*\mathcal{O}_Y \longrightarrow (X_y \to Y)_*\mathcal{O}_{X_y} \longleftarrow y_*\mathcal{O}_{\mathrm{Spec}(k)}$$

The fibre product is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A}'$ and we can define $X' \to Y$ as the relative spectrum of $\mathcal{A}'$ over Y, see Morphisms, Lemma 11.5. This construction commutes with arbitrary change of base. In particular, it is clear that over the open subspace $|Y| \setminus \{y\}$ the morphism $X \to X'$ is an isomorphism and over $|Y| \setminus \{y\}$ the morphism $X' \to Y$ is integral. It remains to prove the statements in a small neighbourhood of y. Choose an affine scheme $V = \mathrm{Spec}(R)$ and an étale morphism $\varphi : V \to Y$ such that y is in the image of φ. Then V_y is a closed subscheme of V étale over k, whence consists of finitely many points each with residue field separable over k (see Decent Spaces, Remark 4.1). After shrinking V we may assume there is a unique closed point $v = \mathrm{Spec}(l) \to V$ mapping to y with l/k finite separable. We may write $V \times_Y X = \mathrm{Spec}(C)$ with $R \to C$ an integral ring map. The stated compatibility with base change gives us that $U \times_X Y' = \mathrm{Spec}(C')$ where

$$C' = C \times_{C \otimes_R l} l$$

Since $R \to l$ is surjective, also $C \to C \otimes_R l$ is surjective and we see that this is a fibre product of the kind studied in More on Algebra, Situation 6.1 (with A, A', B, B' corresponding to $C \otimes_R l, C, l, C'$). Observe that C' is an R-subalgebra of C and hence is integral over R; this proves (1). Finally, More on Algebra, Lemma 6.2 shows that $V \times_X Y' = \mathrm{Spec}(C')$ has a unique point y'' lying over v with residue l (this corresponds with the obvious surjective map $C' \to l$). Thus $X_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(l)$ has a unique point with residue field l. Since l/k is finite separable, this implies X'_y has a unique point with residue field k, i.e., (3) holds.

To prove the final statement, observe that if Y is locally Noetherian, then R is a Noetherian ring and if f is finite, then $R \to C$ is finite. Then C' is a finite type R-algebra by More on Algebra, Lemma 5.1. This proves that $X' \to Y$ is finite. $\square$

Lemma 32.2. *Let S be a scheme. Let B be an algebraic space over S. Let $\delta : |B| \to \mathbf{Z}$ be a function. Assume B is decent, locally Noetherian, and universally catenary and δ is a dimension function. If X is a decent algebraic space over B whose structure morphism $f : X \to B$ is locally of finite type we define $\delta_X : |X| \to \mathbf{Z}$ by the rule*

$$\delta_X(x) = \delta(f(x)) + \textit{transcendence degreeof } x/f(x)$$

(Morphisms of Spaces, Definition 33.1). Then δ_X is a dimension function.

Proof. The problem is local on B. Thus we may assume B is quasi-compact. By Decent Spaces, Lemma 14.1 we see B is quasi-separated. By Limits of Spaces, Proposition 16.1 we can choose a finite surjective morphism $\pi : Y \to X$ where Y is a scheme. Claim: δ_Y is a dimension function.

The claim implies the lemma. With $X \to B$ as in the lemma set $Z = Y \times_B X$ with projections $p : Z \to Y$ and $q : Z \to X$. Then we have

$$\delta_Z(z) = \delta_Y(p(z)) + \text{transcendence degreeof } z/p(z)$$

and $\delta_Z(z) = \delta_X(q(z))$. This follows from Morphisms of Spaces, Lemma 34.2 and the fact that these transcendence degrees are zero for finite morphisms. By Decent Spaces, Lemma 25.2 and the claim we find that δ_Z is a dimension function. Then we find that δ_X is a dimension function by Decent Spaces, Lemma 25.6.

Proof of the claim. Consider a specialization $y \rightsquigarrow y'$, $y \neq y'$ of points of the Noetherian scheme Y. Then $\delta_Y(y) > \delta_Y(y')$ because there are no specializations between points in fibres of Y (see Decent Spaces, Lemma 18.10). Using this for a chain of specializations we find

$$\delta_Y(y) - \delta_Y(y') \geq \text{codim}(\overline{\{y'\}}, \overline{\{y\}})$$

Our task is to show equality. By Properties, Lemma 5.9 we can choose a specialization $y' \rightsquigarrow y_0$. It suffices to show $\delta_Y(y) - \delta_Y(y_0) = \text{codim}(\overline{\{y_0\}}, \overline{\{y\}})$ because this will imply the equality for both $y \rightsquigarrow y'$ and $y' \rightsquigarrow y_0$.

Choose a maximal chain $y = y_c \rightsquigarrow y_{c-1} \rightsquigarrow \ldots \rightsquigarrow y_0$ of specializations in Y. Set $b = \pi(y)$ and $b_0 = \pi(y_0)$. Choose a maximal chain $b = b_e \rightsquigarrow b_{e-1} \rightsquigarrow \ldots \rightsquigarrow b_0$ of specializations in $|B|$. We have to show $e = c$. Since π is closed (Morphisms of Spaces, Lemma 45.9) we can find a sequence of specializations $y = y'_e \rightsquigarrow y'_{e-1} \rightsquigarrow \ldots \rightsquigarrow y'_0$ mapping to $b = b_e \rightsquigarrow b_{e-1} \rightsquigarrow \ldots \rightsquigarrow b_0$. Observe that $y'_e \rightsquigarrow y'_{e-1} \rightsquigarrow \ldots \rightsquigarrow y'_0$ is a maximal chain as well. If $y_0 = y'_0$, then because Y is catenary, we conclude that $e = c$ as desired. In the next paragraph we reduce to this case by sleight of hand and we conclude in the same manner.

Since π is closed we see that b_0 is a closed point of $|B|$. By Decent Spaces, Lemma 14.6 we can represent b_0 by a closed immersion $b_0 : \text{Spec}(k) \to B$. By Lemma 32.1 we can find a factorization

$$Y \to Y' \to X$$

with $\pi' : Y' \to X$ finite and $Y \to Y'$ a morphism which map y_0 and y'_0 to the same point and is an isomorphism away from the inverse image of b_0. (Of course Y' won't be a scheme but this doesn't matter for the argument that follows.) Clearly the maximal chains of specializations $y_c \rightsquigarrow y_{c-1} \rightsquigarrow \ldots \rightsquigarrow y_0$ and $y'_e \rightsquigarrow y'_{e-1} \rightsquigarrow \ldots \rightsquigarrow y'_0$ map to maximal chains of specializations in Y' having the same start and end. Since B is universally catenary, we see that $|Y'|$ is catenary and we conclude as before. □

33. Étale localization of morphisms

The section is the analogue of More on Morphisms, Section 41.

Lemma 33.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. Let $x_1, \ldots, x_n \in |X|$ mapping to y. Assume that*

1. *f is locally of finite type,*
2. *f is separated,*
3. *f is quasi-finite at $x_1, \ldots, x_n$, and*
4. *f is quasi-compact or Y is decent.*

Then there exists an étale morphism $(U, u) \to (Y, y)$ of pointed algebraic spaces and a decomposition

$$U \times_Y X = W \amalg V$$

into open and closed subspaces such that the morphism $V \to U$ is finite, every point of the fibre of $|V| \to |U|$ over u maps to an x_i, and the fibre of $|W| \to |U|$ over u contains no point mapping to an x_i.

Proof. Let $(U, u) \to (Y, y)$ be an étale morphism of algebraic spaces and consider the set of $w \in |U \times_Y X|$ mapping to $u \in |U|$ and one of the $x_i \in |X|$. By Decent Spaces, Lemma 18.4 (if f is of finite type) or Decent Spaces, Lemma 18.5 (if Y is decent) this set is finite. It follows that we may replace f by the base change $U \times_Y X \to U$ and $x_1, \ldots, x_n$ by the set of these w. In particular we may and do assume that Y is an affine scheme, whence X is a separated algebraic space.

Choose an affine scheme Z and an étale morphism $Z \to X$ such that $x_1, \ldots, x_n$ are in the image of $|Z| \to |X|$. The fibres of $|Z| \to |X|$ are finite, see Properties of Spaces, Lemma 6.7 (or the more general discussion in Decent Spaces, Section 6). Let $\{z_1, \ldots, z_m\} \subset |Z|$ be the preimage of $\{x_1, \ldots, x_n\}$. By More on Morphisms, Lemma 41.4 there exists an étale morphism $(U, u) \to (Y, y)$ such that $U \times_Y Z = Z_1 \amalg Z_2$ with $Z_1 \to U$ finite and $(Z_1)_y = \{z_1, \ldots, z_m\}$. We may assume that U is affine and hence Z_1 is affine too.

Since f is separated, the image V of $Z_1 \to X$ is both open and closed (Morphisms of Spaces, Lemma 40.6). Set $W = X \setminus V$ to get a decomposition as in the lemma. To finish the proof we have to show that $V \to U$ is finite. As $Z_1 \to V$ is surjective and étale, V is the quotient of Z_1 by the étale equivalence relation $R = Z_1 \times_V Z_1$, see Spaces, Lemma 9.1. Since f is separated, $V \to U$ is separated and R is closed in $Z_1 \times_U Z_1$. Since $Z_1 \to U$ is finite, the projections $s, t : R \to Z_1$ are finite. Thus V is an affine scheme by Groupoids, Proposition 23.9. By Morphisms, Lemma 41.9 we conclude that $V \to U$ is proper and by Morphisms, Lemma 44.11 we conclude that $V \to U$ is finite, thereby finishing the proof. $\square$

Lemma 33.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$ with image $y \in |Y|$. Assume that*

1. *f is locally of finite type,*
2. *f is separated, and*
3. *f is quasi-finite at x.*

Then there exists an étale morphism $(U, u) \to (Y, y)$ of pointed algebraic spaces and a decomposition

$$U \times_Y X = W \amalg V$$

into open and closed subspaces such that the morphism $V \to U$ is finite and there exists a point $v \in |V|$ which maps to x in $|X|$ and u in $|U|$.

Proof. Pick a scheme U, a point $u \in U$, and an étale morphism $U \to Y$ mapping u to y. There exists a point $x' \in |U \times_Y X|$ mapping to x in $|X|$ and u in $|U|$ (Properties of Spaces, Lemma 4.3). To finish, apply Lemma 33.1 to the morphism $U \times_Y X \to U$ and the point x'. It applies because U is a scheme and hence u comes from the monomorphism $\mathrm{Spec}(\kappa(u)) \to U$. $\square$

34. Zariski's Main Theorem

In this section we apply the results of the previous section to prove Zariski's main theorem for morphisms of algebraic spaces. This section is the analogue of More on Morphisms, Section 43.

Lemma 34.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is of finite type and separated. Let Y' be the normalization of Y in X. Picture:*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & Y' \arrow[dl, "\nu"] \\ & Y & \end{tikzcd}$$

Then there exists an open subspace $U' \subset Y'$ such that

1. *$(f')^{-1}(U') \to U'$ is an isomorphism, and*

(2) $(f')^{-1}(U') \subset X$ *is the set of points at which* f *is quasi-finite.*

Proof. By Morphisms of Spaces, Lemma 34.7 there is an open subspace $U \subset X$ corresponding to the points of $|X|$ where f is quasi-finite. We have to prove

(a) the image of $|U| \to |Y'|$ is $|U'|$ for some open subspace U' of Y',
(b) $U = f^{-1}(U')$, and
(c) $U \to U'$ is an isomorphism.

Since formation of U commutes with arbitrary base change (Morphisms of Spaces, Lemma 34.7), since formation of the normalization Y' commutes with smooth base change (Lemma 25.2), since étale morphisms are open, and since "being an isomorphism" is fpqc local on the base (Descent on Spaces, Lemma 11.15), it suffices to prove (a), (b), (c) étale locally on Y (some details omitted). Thus we may assume Y is an affine scheme. This implies that Y' is an (affine) scheme as well.

Let $x \in |U|$. Claim: there exists an open neighbourhood $f'(x) \in V \subset Y'$ such that $(f')^{-1}V \to V$ is an isomorphism. We first prove the claim implies the lemma. Namely, then $(f')^{-1}V \cong V$ is a scheme (as an open of Y'), locally of finite type over Y (as an open subspace of X), and for $v \in V$ the residue field extension $\kappa(v)/\kappa(\nu(v))$ is algebraic (as $V \subset Y'$ and Y' is integral over Y). Hence the fibres of $V \to Y$ are discrete (Morphisms, Lemma 20.2) and $(f')^{-1}V \to Y$ is locally quasi-finite (Morphisms, Lemma 20.8). This implies $(f')^{-1}V \subset U$ and $V \subset U'$. Since x was arbitrary we see that (a), (b), and (c) are true.

Let $y = f(x) \in |Y|$. Let $(T, t) \to (Y, y)$ be an étale morphism of pointed schemes. Denote by a subscript $_T$ the base change to T. Let $z \in X_T$ be a point in the fibre X_t lying over x. Note that $U_T \subset X_T$ is the set of points where f_T is quasi-finite, see Morphisms of Spaces, Lemma 34.7. Note that

$$X_T \xrightarrow{f'_T} Y'_T \xrightarrow{\nu_T} T$$

is the normalization of T in X_T, see Lemma 25.2. Suppose that the claim holds for $z \in U_T \subset X_T \to Y'_T \to T$, i.e., suppose that we can find an open neighbourhood $f'_T(z) \in V' \subset Y'_T$ such that $(f'_T)^{-1}V' \to V'$ is an isomorphism. The morphism $Y'_T \to Y'$ is étale hence the image $V \subset Y'$ of V' is open. Observe that $f'(x) \in V$ as $f'_T(z) \in V'$. Observe that

$$\begin{tikzcd} (f'_T)^{-1}V' \arrow[r] \arrow[d] & (f')^{-1}(V) \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

is a fibre square (as $Y'_T \times_{Y'} X = X_T$). Since the left vertical arrow is an isomorphism and $\{V' \to V\}$ is a étale covering, we conclude that the right vertical arrow is an isomorphism by Descent on Spaces, Lemma 11.15. In other words, the claim holds for $x \in U \subset X \to Y' \to Y$.

By the result of the previous paragraph to prove the claim for $x \in |U|$, we may replace Y by an étale neighbourhood T of $y = f(x)$ and x by any point lying over x in $T \times_Y X$. Thus we may assume there is a decomposition

$$X = V \amalg W$$

into open and closed subspaces where $V \to Y$ is finite and $x \in V$, see Lemma 33.1. Since X is a disjoint union of V and W over Y and since $V \to Y$ is finite we see that the normalization of Y in X is the morphism

$$X = V \amalg W \longrightarrow V \amalg W' \longrightarrow S$$

where W' is the normalization of Y in W, see Morphisms of Spaces, Lemmas 48.8, 45.6, and 48.10. The claim follows and we win. $\square$

The following lemma is a duplicate of Morphisms of Spaces, Lemma 52.2. The reason for having two copies of the same lemma is that the proofs are somewhat different. The proof given below rests on Zariski's Main Theorem for nonrepresentable morphisms of algebraic spaces as presented above, whereas the proof of Morphisms of Spaces, Lemma 52.2 rests on Morphisms of Spaces, Proposition 50.2 to reduce to the case of morphisms of schemes.

Lemma 34.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated. Let Y' be the normalization of Y in X. Picture:*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[rd, "f"'] & & Y' \arrow[ld, "\nu"] \\ & Y & \end{tikzcd}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. This follows from Lemma 34.1. Namely, by that lemma there exists an open subspace $U' \subset Y'$ such that $(f')^{-1}(U') = X$ (!) and $X \to U'$ is an isomorphism! In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : Y' \to Y$ is separated (Morphisms of Spaces, Lemma 8.9). Hence for every affine scheme Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is a quasi-compact open subscheme of the affine scheme $Z \times_Y Y'$. Hence f is quasi-affine by definition. $\square$

Lemma 34.3 (Zariski's Main Theorem)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated and assume that Y is quasi-compact and quasi-separated. Then there exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[rd, "f"'] & & T \arrow[ld, "\pi"] \\ & Y & \end{tikzcd}$$

where j is a quasi-compact open immersion and π is finite.

Proof. Let $X \to Y' \to Y$ be as in the conclusion of Lemma 34.2. By Limits of Spaces, Lemma 9.7 we can write $\nu_*\mathcal{O}_{Y'} = \operatorname{colim}_{i \in I} \mathcal{A}_i$ as a directed colimit of finite quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_i \subset \nu_*\mathcal{O}_{Y'}$. Then $\pi_i : T_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i) \to Y$ is a finite morphism for each i. Note that the transition morphisms $T_{i'} \to T_i$ are affine and that $Y' = \lim T_i$.

By Limits of Spaces, Lemma 5.7 there exists an i and a quasi-compact open $U_i \subset T_i$ whose inverse image in Y' equals $f'(X)$. For $i' \geq i$ let $U_{i'}$ be the inverse image of

U_i in $T_{i'}$. Then $X \cong f'(X) = \lim_{i' \geq i} U_{i'}$, see Limits of Spaces, Lemma 4.1. By Limits of Spaces, Lemma 5.12 we see that $X \to U_{i'}$ is a closed immersion for some $i' \geq i$. (In fact $X \cong U_{i'}$ for sufficiently large i' but we don't need this.) Hence $X \to T_{i'}$ is an immersion. By Morphisms of Spaces, Lemma 12.6 we can factor this as $X \to T \to T_{i'}$ where the first arrow is an open immersion and the second a closed immersion. Thus we win. □

Lemma 34.4. *With notation and hypotheses as in Lemma 34.3. Assume moreover that f is locally of finite presentation. Then we can choose the factorization such that T is finite and of finite presentation over Y.*

Proof. By Limits of Spaces, Lemma 11.3 we can write $T = \lim T_i$ where all T_i are finite and of finite presentation over Y and the transition morphisms $T_{i'} \to T_i$ are closed immersions. By Limits of Spaces, Lemma 5.7 there exists an i and an open subscheme $U_i \subset T_i$ whose inverse image in T is X. By Limits of Spaces, Lemma 5.12 we see that $X \cong U_i$ for large enough i. Replacing T by T_i finishes the proof. □

35. Applications of Zariski's Main Theorem, I

A first application is the characterization of finite morphisms as proper morphisms with finite fibres.

Lemma 35.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is finite,*
2. *f is proper and locally quasi-finite,*
3. *f is proper and $|X_k|$ is a discrete space for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field,*
4. *f is universally closed, separated, locally of finite type and $|X_k|$ is a discrete space for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field.*

Proof. We have (1) ⇒ (2) by Morphisms of Spaces, Lemmas 45.9, 45.8. We have (2) ⇒ (3) by Morphisms of Spaces, Lemma 27.5. By definition (3) implies (4).

Assume (4). Since f is universally closed it is quasi-compact (Morphisms of Spaces, Lemma 9.7). Pick a point y of $|Y|$. We represent y by a morphism $\mathrm{Spec}(k) \to Y$. Note that $|X_k|$ is finite discrete as a quasi-compact discrete space. The map $|X_k| \to |X|$ surjects onto the fibre of $|X| \to |Y|$ over y (Properties of Spaces, Lemma 4.3). By Morphisms of Spaces, Lemma 34.8 we see that $X \to Y$ is quasi-finite at all the points of the fibre of $|X| \to |Y|$ over y. Choose an elementary étale neighbourhood $(U, u) \to (Y, y)$ and decomposition $X_U = V \amalg W$ as in Lemma 33.1 adapted to all the points of $|X|$ lying over y. Note that $W_u = \emptyset$ because we used all the points in the fibre of $|X| \to |Y|$ over y. Since f is universally closed we see that the image of $|W|$ in $|U|$ is a closed set not containing u. After shrinking U we may assume that $W = \emptyset$. In other words we see that $X_U = V$ is finite over U. Since $y \in |Y|$ was arbitrary this means there exists a family $\{U_i \to Y\}$ of étale morphisms whose images cover Y such that the base changes $X_{U_i} \to U_i$ are finite. We conclude that f is finite by Morphisms of Spaces, Lemma 45.3. □

As a consequence we have the following useful result.

Lemma 35.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. Assume*

1. *f is proper, and*
2. *f is quasi-finite at all $x \in |X|$ lying over y (Decent Spaces, Lemma 18.10).*

Then there exists an open neighbourhood $V \subset Y$ of y such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. By Morphisms of Spaces, Lemma 34.7 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $y \notin f(Z)$. Since f is proper the set $f(Z) \subset Y$ is closed. Choose any open neighbourhood $V \subset Y$ of y with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence $f^{-1}(V) \to V$ is finite by Lemma 35.1. $\square$

Lemma 35.3. *Let S be a scheme. Let*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphism of algebraic spaces over S. Let $b \in B$ and let $\mathrm{Spec}(k) \to B$ be a morphism in the equivalence class of b. Assume

1. *$X \to B$ is a proper morphism,*
2. *$Y \to B$ is separated and locally of finite type,*
3. *one of the following is true*
   (a) *the image of $|X_k| \to |Y_k|$ is finite,*
   (b) *the image of $|f|^{-1}(\{b\})$ in $|Y|$ is finite and B is decent.*

Then there is an open subspace $B' \subset B$ containing b such that $X_{B'} \to Y_{B'}$ factors through a closed subspace $Z \subset Y_{B'}$ finite over B'.

Proof. Let $Z \subset Y$ be the scheme theoretic image of h, see Morphisms of Spaces, Section 16. By Morphisms of Spaces, Lemma 40.8 the morphism $X \to Z$ is surjective and $Z \to B$ is proper. Thus

$$\{x \in |X| \text{ lying over } b\} \to \{z \in |Z| \text{ lying over } b\}$$

and $|X_k| \to |Z_k|$ are surjective. We see that either (3)(a) or (3)(b) imply that $Z \to B$ is quasi-finite all points of $|Z|$ lying over b by Decent Spaces, Lemma 18.10. Hence $Z \to B$ is finite in an open neighbourhood of b by Lemma 35.2. $\square$

36. Stein factorization

Stein factorization is the statement that a proper morphism $f : X \to S$ with $f_*\mathcal{O}_X = \mathcal{O}_S$ has connected fibres.

Lemma 36.1. *Let S be a scheme. Let $f : X \to Y$ be a universally closed and quasi-separated morphism of algebraic spaces over S. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & Y' \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

with the following properties:

(1) *the morphism f' is universally closed, quasi-compact, quasi-separated, and surjective,*
(2) *the morphism $\pi : Y' \to Y$ is integral,*
(3) *we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,*
(4) *we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and*
(5) *Y' is the normalization of Y in X as defined in Morphisms of Spaces, Definition 48.3.*

Formation of the factorization $f = \pi \circ f'$ commutes with flat base change.

Proof. By Morphisms of Spaces, Lemma 9.7 the morphism f is quasi-compact. We just define Y' as the normalization of Y in X, so (5) and (2) hold automatically. By Morphisms of Spaces, Lemma 48.9 we see that (4) holds. The morphism f' is universally closed by Morphisms of Spaces, Lemma 40.6. It is quasi-compact by Morphisms of Spaces, Lemma 8.9 and quasi-separated by Morphisms of Spaces, Lemma 4.10.

To show the remaining statements we may assume the base Y is affine (as taking normalization commutes with étale localization). Say $Y = \mathrm{Spec}(R)$. Then $Y' = \mathrm{Spec}(A)$ with $A = \Gamma(X, \mathcal{O}_X)$ an integral R-algebra. Thus it is clear that $f'_*\mathcal{O}_X$ is $\mathcal{O}_{Y'}$ (because $f'_*\mathcal{O}_X$ is quasi-coherent, by Morphisms of Spaces, Lemma 11.2, and hence equal to $\widetilde{A}$). This proves (3).

Let us show that f' is surjective. As f' is universally closed (see above) the image of f' is a closed subset $V(I) \subset Y' = \mathrm{Spec}(A)$. Pick $h \in I$. Then $h|_X = f^\sharp(h)$ is a global section of the structure sheaf of X which vanishes at every point. As X is quasi-compact this means that $h|_X$ is a nilpotent section, i.e., $h^n|X = 0$ for some $n > 0$. But $A = \Gamma(X, \mathcal{O}_X)$, hence $h^n = 0$. In other words I is contained in the Jacobson radical of A and we conclude that $V(I) = Y'$ as desired. □

Lemma 36.2. *In Lemma 36.1 assume in addition that f is locally of finite type and Y affine. Then for $y \in Y$ the fibre $\pi^{-1}(\{y\}) = \{y_1, \ldots, y_n\}$ is finite and the field extensions $\kappa(y_i)/\kappa(y)$ are finite.*

Proof. Recall that there are no specializations among the points of $\pi^{-1}(\{y\})$, see Algebra, Lemma 36.20. As f' is surjective, we find that $|X_y| \to \pi^{-1}(\{y\})$ is surjective. Observe that X_y is a quasi-separated algebraic space of finite type over a field (quasi-compactness was shown in the proof of the referenced lemma). Thus $|X_y|$ is a Noetherian topological space (Morphisms of Spaces, Lemma 28.6). A topological argument (omitted) now shows that $\pi^{-1}(\{y\})$ is finite. For each i we can pick a finite type point $x_i \in |X_y|$ mapping to y_i (Morphisms of Spaces, Lemma 25.6). We conclude that $\kappa(y_i)/\kappa(y)$ is finite: x_i can be represented by a morphism $\mathrm{Spec}(k_i) \to X_y$ of finite type (by our definition of finite type points) and hence $\mathrm{Spec}(k_i) \to y = \mathrm{Spec}(\kappa(y))$ is of finite type (as a composition of finite type morphisms), hence $k_i/\kappa(y)$ is finite (Morphisms, Lemma 16.1). □

Let $f : X \to Y$ be a morphism of algebraic spaces and let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. Then the *fibre of f over $\overline{y}$* is the algebraic space $X_{\overline{y}} = X \times_{Y,\overline{y}} \mathrm{Spec}(k)$ over k. If Y is a scheme and $y \in Y$ is a point, then we denote $X_y = X \times_Y \mathrm{Spec}(\kappa(y))$ the fibre as usual.

Lemma 36.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Then $X_{\overline{y}}$ is connected, if and* *only if for every étale neighbourhood $(V, \overline{v}) \to (Y, \overline{y})$ where V is a scheme the base change $X_V \to V$ has connected fibre X_v.*

Proof. Since the category of étale neighbourhoods of $\overline{y}$ is cofiltered and contains a cofinal collection of schemes (Properties of Spaces, Lemma 19.3) we may replace Y by one of these neighbourhoods and assume that Y is a scheme. Let $y \in Y$ be the point corresponding to $\overline{y}$. Then X_y is geometrically connected over $\kappa(y)$ if and only if $X_{\overline{y}}$ is connected and if and only if $(X_y)_{k'}$ is connected for every finite separable extension k' of $\kappa(y)$. See Spaces over Fields, Section 12 and especially Lemma 12.8. By More on Morphisms, Lemma 35.2 there exists an affine étale neighbourhood $(V, v) \to (Y, y)$ such that $\kappa(s) \subset \kappa(u)$ is identified with $\kappa(s) \subset k'$ any given finite separable extension. The lemma follows. $\square$

Theorem 36.4 (Stein factorization; Noetherian case)**.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y locally Noetherian. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[rd, "f"'] & & Y' \arrow[ld, "\pi"] \\ & Y & \end{tikzcd}$$

with the following properties:

1. *the morphism f' is proper with connected geometric fibres,*
2. *the morphism $\pi : Y' \to Y$ is finite,*
3. *we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,*
4. *we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and*
5. *Y' is the normalization of Y in X, see Morphisms, Definition 53.3.*

Proof. Let $f = \pi \circ f'$ be the factorization of Lemma 36.1. Note that besides the conclusions of Lemma 36.1 we also have that f' is separated (Morphisms of Spaces, Lemma 4.10) and finite type (Morphisms of Spaces, Lemma 23.6). Hence f' is proper. By Cohomology of Spaces, Lemma 20.2 we see that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module. Hence we see that π is finite, i.e., (2) holds.

This proves all but the most interesting assertion, namely that the geometric fibres of f' are connected. It is clear from the discussion above that we may replace Y by Y'. Then Y is locally Noetherian, $f : X \to Y$ is proper, and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let $\overline{y}$ be a geometric point of Y. At this point we apply the theorem on formal functions, more precisely Cohomology of Spaces, Lemma 22.7. It tells us that

$$\mathcal{O}^\wedge_{Y,\overline{y}} = \lim_n H^0(X_n, \mathcal{O}_{X_n})$$

where $X_n = \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}^n_{\overline{y}}) \times_Y X$. Note that $X_1 = X_{\overline{y}} \to X_n$ is a (finite order) thickening and hence the underlying topological space of X_n is equal to that of $X_{\overline{y}}$. Thus, if $X_{\overline{y}} = T_1 \amalg T_2$ is a disjoint union of nonempty open and closed subspaces, then similarly $X_n = T_{1,n} \amalg T_{2,n}$ for all n. And this in turn means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent $e_{1,n}$, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that $e_{1,n+1}$ restricts to $e_{1,n}$ on X_n. Hence $e_1 = \lim e_{1,n}$ is a nontrivial idempotent of the limit. This contradicts the fact that $\mathcal{O}^\wedge_{Y,\overline{y}}$ is a local ring. Thus the assumption was wrong, i.e., $X_{\overline{y}}$ is connected as desired. $\square$

Theorem 36.5 (Stein factorization; general case)**.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & Y' \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

with the following properties:

(1) *the morphism f' is proper with connected geometric fibres,*
(2) *the morphism $\pi : Y' \to Y$ is integral,*
(3) *we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,*
(4) *we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and*
(5) *Y' is the normalization of Y in X (Morphisms of Spaces, Definition 48.3).*

Proof. We may apply Lemma 36.1 to get the morphism $f' : X \to Y'$. Note that besides the conclusions of Lemma 36.1 we also have that f' is separated (Morphisms of Spaces, Lemma 4.10) and finite type (Morphisms of Spaces, Lemma 23.6). Hence f' is proper. At this point we have proved all of the statements except for the statement that f' has connected geometric fibres.

It is clear from the discussion that we may replace Y by Y'. Then $f : X \to Y$ is proper and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Note that these conditions are preserved under flat base change (Morphisms of Spaces, Lemma 40.3 and Cohomology of Spaces, Lemma 11.2). Let $\overline{y}$ be a geometric point of Y. By Lemma 36.3 and the remark just made we reduce to the case where Y is a scheme, $y \in Y$ is a point, $f : X \to Y$ is a proper algebraic space over Y with $f_*\mathcal{O}_X = \mathcal{O}_Y$, and we have to show the fibre X_y is connected. Replacing Y by an affine neighbourhood of y we may assume that $Y = \mathrm{Spec}(R)$ is affine. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ signifies that the ring map $R \to \Gamma(X, \mathcal{O}_X)$ is bijective.

By Limits of Spaces, Lemma 12.2 we can write $(X \to Y) = \lim(X_i \to Y_i)$ with $X_i \to Y_i$ proper and of finite presentation and Y_i Noetherian. For i large enough Y_i is affine (Limits of Spaces, Lemma 5.10). Say $Y_i = \mathrm{Spec}(R_i)$. Let $R'_i = \Gamma(X_i, \mathcal{O}_{X_i})$. Observe that we have ring maps $R_i \to R'_i \to R$. Namely, we have the first because X_i is an algebraic space over R_i and the second because we have $X \to X_i$ and $R = \Gamma(X, \mathcal{O}_X)$. Note that $R = \mathrm{colim}\, R'_i$ by Limits of Spaces, Lemma 5.6. Then

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[d] & \\ Y \arrow[r] & Y'_i \arrow[r] & Y_i \end{tikzcd}$$

is commutative with $Y'_i = \mathrm{Spec}(R'_i)$. Let $y'_i \in Y'_i$ be the image of y. We have $X_y = \lim X_{i,y'_i}$ because $X = \lim X_i$, $Y = \lim Y'_i$, and $\kappa(y) = \mathrm{colim}\, \kappa(y'_i)$. Now let $X_y = U \amalg V$ with U and V open and closed. Then U, V are the inverse images of opens U_i, V_i in X_{i,y'_i} (Limits of Spaces, Lemma 5.7). By Theorem 36.4 the fibres of $X_i \to Y'_i$ are connected, hence either U or V is empty. This finishes the proof. $\square$

Here is an application.

Lemma 36.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is proper,*
(2) *Y is integral (Spaces over Fields, Definition 4.1) with generic point ξ,*
(3) *Y is normal,*
(4) *X is reduced,*
(5) *every generic point of an irreducible component of $|X|$ maps to ξ,*
(6) *we have $H^0(X_\xi, \mathcal{O}) = \kappa(\xi)$.*

Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ and f has geometrically connected fibres.

Proof. Apply Theorem 36.5 to get a factorization $X \to Y' \to Y$. It is enough to show that $Y' = Y$. It suffices to show that $Y' \times_Y V \to V$ is an isomorphism, where $V \to Y$ is an étale morphism and V an affine integral scheme, see Spaces over Fields, Lemma 4.5. The formation of Y' commutes with étale base change, see Morphisms of Spaces, Lemma 48.4. The generic points of $X \times_Y V$ lie over the generic points of X (Decent Spaces, Lemma 20.1) hence map to the generic point of V by assumption (5). Moreover, condition (6) is preserved under the base change by $V \to Y$, for example by flat base change (Cohomology of Spaces, Lemma 11.2). Thus it suffices to prove the lemma in case Y is a normal integral affine scheme.

Assume Y is a normal integral affine scheme. We will show $Y' \to Y$ is an isomorphism by an application of Morphisms, Lemma 54.8. Namely, Y' is reduced because X is reduced (Morphisms of Spaces, Lemma 48.6). The morphism $Y' \to Y$ is integral by the theorem cited above. Since Y is decent and $X \to Y$ is separated, we see that X is decent too; to see this use Decent Spaces, Lemmas 17.2 and 17.5. By assumption (5), Morphisms of Spaces, Lemma 48.7, and Decent Spaces, Lemma 20.1 we see that every generic point of an irreducible component of $|Y'|$ maps to ξ. On the other hand, since Y' is the relative spectrum of $f_*\mathcal{O}_X$ we see that the scheme theoretic fibre Y'_ξ is the spectrum of $H^0(X_\xi, \mathcal{O})$ which is equal to $\kappa(\xi)$ by assumption. Hence Y' is an integral scheme with function field equal to the function field of Y. This finishes the proof. $\square$

Here is another application.

Lemma 36.7. *Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. If f is proper, flat, and of finite presentation, then the function $n_{X/Y} : |Y| \to \mathbf{Z}$ counting the number of geometric connected components of fibres of f (Lemma 30.1) is lower semi-continuous.*

Proof. The question is étale local on Y, hence we may and do assume Y is an affine scheme. Let $y \in Y$. Set $n = n_{X/S}(y)$. Note that $n < \infty$ as the geometric fibre of $X \to Y$ at y is a proper algebraic space over a field, hence Noetherian, hence has a finite number of connected components. We have to find an open neighbourhood V of y such that $n_{X/S}|_V \geq n$. Let $X \to Y' \to Y$ be the Stein factorization as in Theorem 36.5. By Lemma 36.2 there are finitely many points $y'_1, \ldots, y'_m \in Y'$ lying over y and the extensions $\kappa(y'_i)/\kappa(y)$ are finite. More on Morphisms, Lemma 42.1 tells us that after replacing Y by an étale neighbourhood of y we may assume $Y' = V_1 \amalg \ldots \amalg V_m$ as a scheme with $y'_i \in V_i$ and $\kappa(y'_i)/\kappa(y)$ purely inseparable. Then the algebraic spaces $X_{y'_i}$ are geometrically connected over $\kappa(y)$, hence $m = n$. The algebraic spaces $X_i = (f')^{-1}(V_i)$, $i = 1, \ldots, n$ are flat and of finite presentation over Y. Hence the image of $X_i \to Y$ is open (Morphisms of Spaces, Lemma 30.6). Thus in a neighbourhood of y we see that $n_{X/Y}$ is at least n. $\square$

Lemma 36.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

1. *f is proper, flat, and of finite presentation, and*
2. *the geometric fibres of f are reduced.*

Then the function $n_{X/S} : |Y| \to \mathbf{Z}$ counting the numbers of geometric connected components of fibres of f (Lemma 30.1) is locally constant.

Proof. By Lemma 36.7 the function $n_{X/Y}$ is lower semincontinuous. Thus it suffices to show it is upper semi-continuous. To do this we may work étale locally on Y, hence we may assume Y is an affine scheme. For $y \in Y$ consider the $\kappa(y)$-algebra

$$A = H^0(X_y, \mathcal{O}_{X_y})$$

By Spaces over Fields, Lemma 14.3 and the fact that X_y is geometrically reduced A is finite product of finite separable extensions of $\kappa(y)$. Hence $A \otimes_{\kappa(y)} \kappa(\overline{y})$ is a product of $\beta_0(y) = \dim_{\kappa(y)} A$ copies of $\kappa(\overline{y})$. Thus $X_{\overline{y}}$ has $\beta_0(y)$ connected components. In other words, we have $n_{X/S} = \beta_0$ as functions on Y. Thus $n_{X/Y}$ is upper semi-continuous by Derived Categories of Spaces, Lemma 26.2. This finishes the proof. $\square$

Lemma 36.9. *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. Let $X \to Y' \to Y$ be the Stein factorization of f (Theorem 36.5). If f is of finite presentation, flat, with geometrically reduced fibres (Definition 29.2), then $Y' \to Y$ is finite étale.*

Proof. Formation of the Stein factorization commutes with flat base change, see Lemma 36.1. Thus we may work étale locally on Y and we may assume Y is an affine scheme. Then Y' is an affine scheme and $Y' \to Y$ is integral.

Let $y \in Y$. Set n be the number of connected components of the geometric fibre $X_{\overline{y}}$. Note that $n < \infty$ as the geometric fibre of $X \to Y$ at y is a proper algebraic space over a field, hence Noetherian, hence has a finite number of connected components. By Lemma 36.2 there are finitely many points $y'_1, \ldots, y'_m \in Y'$ lying over y and for each i we can pick a finite type point $x_i \in |X_y|$ mapping to y'_i the extension $\kappa(y'_i)/\kappa(y)$ is finite. Thus More on Morphisms, Lemma 42.1 tells us that after replacing Y by an étale neighbourhood of y we may assume $Y' = V_1 \amalg \ldots \amalg V_m$ as a scheme with $y'_i \in V_i$ and $\kappa(y'_i)/\kappa(y)$ purely inseparable. In this case the algebraic spaces $X_{y'_i}$ are geometrically connected over $\kappa(y)$, hence $m = n$. The algebraic spaces $X_i = (f')^{-1}(V_i)$, $i = 1, \ldots, n$ are proper, flat, of finite presentation, with geometrically reduced fibres over Y. It suffices to prove the lemma for each of the morphisms $X_i \to Y$. This reduces us to the case where $X_{\overline{y}}$ is connected.

Assume that $X_{\overline{y}}$ is connected. By Lemma 36.8 we see that $X \to Y$ has geometrically connected fibres in a neighbourhood of y. Thus we may assume the fibres of $X \to Y$ are geometrically connected. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ by Derived Categories of Spaces, Lemma 26.8 which finishes the proof. $\square$

The proof of the following lemma uses Stein factorization for schemes which is why it ended up in this section.

Lemma 36.10. *Let (A, I) be a henselian pair. Let X be an algebraic space separated and of finite type over A. Set $X_0 = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. Let $Y \subset X_0$ be an open and closed subspace such that $Y \to \mathrm{Spec}(A/I)$ is proper. Then* *there exists an open and closed subspace $W \subset X$ which is proper over A with $W \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) = Y$.*

Proof. We will denote $T \mapsto T_0$ the base change by $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$. By a weak version of Chow's lemma (in the form of Cohomology of Spaces, Lemma 18.1) there exists a surjective proper morphism $\varphi : X' \to X$ such that X' admits an immersion into $\mathbf{P}^n_A$. Set $Y' = \varphi^{-1}(Y)$. This is an open and closed subscheme of X'_0. The lemma holds for (X', Y') by More on Morphisms, Lemma 53.9. Let $W' \subset X'$ be the open and closed subscheme proper over A such that $Y' = W'_0$. By Morphisms of Spaces, Lemma 40.6 $Q_1 = \varphi(|W'|) \subset |X|$ and $Q_2 = \varphi(|X' \setminus W'|) \subset |X|$ are closed subsets and by Morphisms of Spaces, Lemma 40.7 any closed subspace structure on Q_1 is proper over A. The image of $Q_1 \cap Q_2$ in $\mathrm{Spec}(A)$ is closed. Since (A, I) is henselian, if $Q_1 \cap Q_2$ is nonempty, then we find that $Q_1 \cap Q_2$ has a point lying over $\mathrm{Spec}(A/I)$. This is impossible as $W'_0 = Y' = \varphi^{-1}(Y)$. We conclude that Q_1 is open and closed in $|X|$. Let $W \subset X$ be the corresponding open and closed subspace. Then W is proper over A with $W_0 = Y$. $\square$

37. Extending properties from an open

In this section we collect a number of results of the form: If $f : X \to Y$ is a flat morphism of algebraic spaces and f satisfies some property over a dense open of Y, then f satisfies the same property over all of Y.

Lemma 37.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $V \subset Y$ be an open subspace. Assume*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is of finite type and flat over Y,*
3. *$V \to Y$ is quasi-compact and scheme theoretically dense,*
4. *$\mathcal{F}|_{f^{-1}V}$ is of finite presentation.*

Then $\mathcal{F}$ is of finite presentation.

Proof. It suffices to prove the pullback of $\mathcal{F}$ to a scheme surjective and étale over X is of finite presentation. Hence we may assume X is a scheme. Similarly, we can replace Y by a scheme surjective and étale and over Y (the inverse image of V in this scheme is scheme theoretically dense, see Morphisms of Spaces, Section 17). Thus we reduce to the case of schemes which is More on Flatness, Lemma 11.1. $\square$

Lemma 37.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. Assume*

1. *f is locally of finite type and flat,*
2. *$V \to Y$ is quasi-compact and scheme theoretically dense,*
3. *$f|_{f^{-1}V} : f^{-1}V \to V$ is locally of finite presentation.*

Then f is of locally of finite presentation.

Proof. The proof is identical to the proof of Lemma 37.1 except one uses More on Flatness, Lemma 11.2. $\square$

Lemma 37.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite type. Let $V \subset Y$ be an open subspace such that $|V| \subset |Y|$ is dense and such that $X_V \to V$ has relative dimension $\leq d$. If also either*

(1) *f is locally of finite presentation, or*
(2) *$V \to Y$ is quasi-compact,*

then $f : X \to Y$ has relative dimension $\leq d$.

Proof. We may replace Y by its reduction, hence we may assume Y is reduced. Then V is scheme theoretically dense in Y, see Morphisms of Spaces, Lemma 17.7. By definition the property of having relative dimension $\leq d$ can be checked on an étale covering, see Morphisms of Spaces, Sections 33. Thus it suffices to prove f has relative dimension $\leq d$ after replacing X by a scheme surjective and étale over X. Similarly, we can replace Y by a scheme surjective and étale and over Y. The inverse image of V in this scheme is scheme theoretically dense, see Morphisms of Spaces, Section 17. Since a scheme theoretically dense open of a scheme is in particular dense, we reduce to the case of schemes which is More on Flatness, Lemma 11.3. □

Lemma 37.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and proper. Let $V \to Y$ be an open subspace with $|V| \subset |Y|$ dense such that $X_V \to V$ is finite. If also either f is locally of finite presentation or $V \to Y$ is quasi-compact, then f is finite.*

Proof. By Lemma 37.3 the fibres of f have dimension zero. By Morphisms of Spaces, Lemma 34.6 this implies that f is locally quasi-finite. By Morphisms of Spaces, Lemma 51.1 this implies that f is representable. We can check whether f is finite étale locally on Y, hence we may assume Y is a scheme. Since f is representable, we reduce to the case of schemes which is More on Flatness, Lemma 11.4. □

Lemma 37.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. If*

(1) *f is separated, locally of finite type, and flat,*
(2) *$f^{-1}(V) \to V$ is an isomorphism, and*
(3) *$V \to Y$ is quasi-compact and scheme theoretically dense,*

then f is an open immersion.

Proof. Applying Lemma 37.2 we see that f is locally of finite presentation. Applying Lemma 37.3 we see that f has relative dimension ≤ 0. By Morphisms of Spaces, Lemma 34.6 this implies that f is locally quasi-finite. By Morphisms of Spaces, Lemma 51.1 this implies that f is representable. By Descent on Spaces, Lemma 11.14 we can check whether f is an open immersion étale locally on Y. Hence we may assume that Y is a scheme. Since f is representable, we reduce to the case of schemes which is More on Flatness, Lemma 11.5. □

38. Blowing up and flatness

Instead of redoing the work in More on Flatness, Section 30 we prove an analogue of More on Flatness, Lemma 30.5 which tells us that the problem of finding a suitable blowup is often étale local on the base.

Lemma 38.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\varphi : W \to X$ be a quasi-compact separated étale morphism. Let $U \subset X$ be a quasi-compact open subspace. Let $\mathcal{I} \subset \mathcal{O}_W$ be a finite type* *quasi-coherent sheaf of ideals such that $V(\mathcal{I}) \cap \varphi^{-1}(U) = \emptyset$. Then there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that*

(1) *$V(\mathcal{J}) \cap U = \emptyset$, and*
(2) *$\varphi^{-1}(\mathcal{J})\mathcal{O}_W = \mathcal{I}\mathcal{I}'$ for some finite type quasi-coherent ideal $\mathcal{I}' \subset \mathcal{O}_W$.*

Proof. Choose a factorization $W \to Y \to X$ where $j : W \to Y$ is a quasi-compact open immersion and $\pi : Y \to X$ is a finite morphism of finite presentation (Lemma 34.4). Let $V = j(W) \cup \pi^{-1}(U) \subset Y$. Note that $\mathcal{I}$ on $W \cong j(W)$ and $\mathcal{O}_{\pi^{-1}(U)}$ glue to a finite type quasi-coherent sheaf of ideals $\mathcal{I}_1 \subset \mathcal{O}_V$. By Limits of Spaces, Lemma 9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I}_2 \subset \mathcal{O}_Y$ such that $\mathcal{I}_2|_V = \mathcal{I}_1$. In other words, $\mathcal{I}_2 \subset \mathcal{O}_Y$ is a finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_2)$ is disjoint from $\pi^{-1}(U)$ and $j^{-1}\mathcal{I}_2 = \mathcal{I}$. Denote $i : Z \to Y$ the corresponding closed immersion which is of finite presentation (Morphisms of Spaces, Lemma 28.12). In particular the composition $\tau = \pi \circ i : Z \to X$ is finite and of finite presentation (Morphisms of Spaces, Lemmas 28.2 and 45.4).

Let $\mathcal{F} = \tau_*\mathcal{O}_Z$ which we think of as a quasi-coherent $\mathcal{O}_X$-module. By Descent on Spaces, Lemma 6.7 we see that $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{J} = \text{Fit}_0(\mathcal{F})$. (Insert reference to fitting modules on ringed topoi here.) This is a finite type quasi-coherent sheaf of ideals on X (as $\mathcal{F}$ is of finite presentation, see More on Algebra, Lemma 8.4). Part (1) of the lemma holds because $|\tau|(|Z|) \cap |U| = \emptyset$ by our choice of $\mathcal{I}_2$ and because the 0th Fitting ideal of the trivial module equals the structure sheaf. To prove (2) note that $\varphi^{-1}(\mathcal{J})\mathcal{O}_W = \text{Fit}_0(\varphi^*\mathcal{F})$ because taking Fitting ideals commutes with base change. On the other hand, as $\varphi : W \to X$ is separated and étale we see that $(1, j) : W \to W \times_X Y$ is an open and closed immersion. Hence $W \times_Y Z = V(\mathcal{I}) \amalg Z'$ for some finite and finitely presented morphism of algebraic spaces $\tau' : Z' \to W$. Thus we see that

$$\begin{aligned} \text{Fit}_0(\varphi^*\mathcal{F}) &= \text{Fit}_0((W \times_Y Z \to W)_*\mathcal{O}_{W \times_Y Z}) \\ &= \text{Fit}_0(\mathcal{O}_W/\mathcal{I}) \cdot \text{Fit}_0(\tau'_*\mathcal{O}_{Z'}) \\ &= \mathcal{I} \cdot \text{Fit}_0(\tau'_*\mathcal{O}_{Z'}) \end{aligned}$$

the second equality by More on Algebra, Lemma 8.4 translated in sheaves on ringed topoi. Setting $\mathcal{I}' = \text{Fit}_0(\tau'_*\mathcal{O}_{Z'})$ finishes the proof of the lemma. □

Theorem 38.2. *Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let X be an algebraic space over B. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset B$ be a quasi-compact open subspace. Assume*

(1) *X is quasi-compact,*
(2) *X is locally of finite presentation over B,*
(3) *$\mathcal{F}$ is a module of finite type,*
(4) *$\mathcal{F}_U$ is of finite presentation, and*
(5) *$\mathcal{F}_U$ is flat over U.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is an $\mathcal{O}_{X \times_B B'}$-module of finite presentation and flat over B'.

Proof. Choose an affine scheme V and a surjective étale morphism $V \to X$. Because strict transform commutes with étale localization (Divisors on Spaces, Lemma 18.2) it suffices to prove the result with X replaced by V. Hence we may assume that $X \to B$ is representable (in addition to the hypotheses of the lemma).

Assume that $X \to B$ is representable. Choose an affine scheme W and a surjective étale morphism $\varphi : W \to B$. Note that $X \times_B W$ is a scheme. By the case of schemes (More on Flatness, Theorem 30.7) we can find a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_W$ such that (a) $|V(\mathcal{I})| \cap |\varphi^{-1}(U)| = \emptyset$ and (b) the strict transform of $\mathcal{F}|_{X \times_B W}$ with respect to the blowing up $W' \to W$ in $\mathcal{I}$ becomes flat over W' and is a module of finite presentation. Choose a finite type sheaf of ideals $\mathcal{J} \subset \mathcal{O}_B$ as in Lemma 38.1. Let $B' \to B$ be the blowing up of $\mathcal{J}$. We claim that this blowup works. Namely, it is clear that $B' \to B$ is U-admissible by our choice of ideal $\mathcal{J}$. Moreover, the base change $B' \times_B W \to W$ is the blowup of W in $\varphi^{-1}\mathcal{J} = \mathcal{I}\mathcal{I}'$ (compatibility of blowup with flat base change, see Divisors on Spaces, Lemma 17.3). Hence there is a factorization

$$W \times_B B' \to W' \to W$$

where the first morphism is a blowup as well, see Divisors on Spaces, Lemma 17.10). The restriction of $\mathcal{F}'$ (which lives on $B' \times_B X$) to $W \times_B B' \times_B X$ is the strict transform of $\mathcal{F}|_{X \times_B W}$ (Divisors on Spaces, Lemma 18.2) and hence is the twice repeated strict transform of $\mathcal{F}|_{X \times_B W}$ by the two blowups displayed above (Divisors on Spaces, Lemma 18.7). After the first blowup our sheaf is already flat over the base and of finite presentation (by construction). Whence this holds after the second strict transform as well (since this is a pullback by Divisors on Spaces, Lemma 18.4). Thus we see that the restriction of $\mathcal{F}'$ to an étale cover of $B' \times_B X$ has the desired properties and the theorem is proved. $\square$

39. Applications

In this section we apply the result on flattening by blowing up.

Lemma 39.1. *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume*

(1) *B is quasi-compact and quasi-separated,*
(2) *U is quasi-compact,*
(3) *$f : X \to B$ is of finite type and quasi-separated, and*
(4) *$f^{-1}(U) \to U$ is flat and locally of finite presentation.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform X' of X is flat and of finite presentation over B'.

Proof. Let $B' \to B$ be a U-admissible blowup. Note that the strict transform of X is quasi-compact and quasi-separated over B' as X is quasi-compact and quasi-separated over B. Hence we only need to worry about finding a U-admissible blowup such that the strict transform becomes flat and locally of finite presentation. We cannot directly apply Theorem 38.2 because X is not locally of finite presentation over B.

Choose an affine scheme V and a surjective étale morphism $V \to X$. (This is possible as X is quasi-compact as a finite type space over the quasi-compact space B.) Then it suffices to show the result for the morphism $V \to B$ (as strict transform commutes with étale localization, see Divisors on Spaces, Lemma 18.2). Hence we may assume that $X \to B$ is separated as well as finite type. In this case we can find a closed immersion $i : X \to Y$ with $Y \to B$ separated and of finite presentation, see Limits of Spaces, Proposition 11.7.

Apply Theorem 38.2 to $\mathcal{F} = i_*\mathcal{O}_X$ on Y/B. We find a U-admissible blowup $B' \to B$ such that strict transform of $\mathcal{F}$ is flat over B' and of finite presentation. Let X' be the strict transform of X under the blowup $B' \to B$. Let $i' : X' \to Y \times_B B'$ be the induced morphism. Since taking strict transform commutes with pushforward along affine morphisms (Divisors on Spaces, Lemma 18.5), we see that $i'_*\mathcal{O}_{X'}$ is flat over B' and of finite presentation as a $\mathcal{O}_{Y \times_B B'}$-module. Thus $X' \to B'$ is flat and locally of finite presentation. This implies the lemma by our earlier remarks. $\square$

Lemma 39.2. *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume*

1. *B is quasi-compact and quasi-separated,*
2. *U is quasi-compact,*
3. *$f : X \to B$ is proper, and*
4. *$f^{-1}(U) \to U$ is finite locally free.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform X' of X is finite locally free over B'.

Proof. By Lemma 39.1 we may assume that $X \to B$ is flat and of finite presentation. After replacing B by a U-admissible blowup if necessary, we may assume that $U \subset B$ is scheme theoretically dense. Then f is finite by Lemma 37.4. Hence f is finite locally free by Morphisms of Spaces, Lemma 46.6. $\square$

Lemma 39.3. *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume*

1. *B is quasi-compact and quasi-separated,*
2. *U is quasi-compact,*
3. *$f : X \to B$ is proper, and*
4. *$f^{-1}(U) \to U$ is an isomorphism.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform X' of X maps isomorphically to B'.

Proof. By Lemma 39.1 we may assume that $X \to B$ is flat and of finite presentation. After replacing B by a U-admissible blowup if necessary, we may assume that $U \subset B$ is scheme theoretically dense. Then f is finite by Lemma 37.4 and an open immersion by Lemma 37.5. Hence f is an open immersion whose image is closed and contains the dense open U, whence f is an isomorphism. $\square$

Lemma 39.4. *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume*

1. *B quasi-compact and quasi-separated,*
2. *U is quasi-compact,*
3. *f is of finite type*
4. *$f^{-1}(U) \to U$ is an isomorphism.*

Then there exists a U-admissible blowup $B' \to B$ such that U is scheme theoretically dense in B' and such that the strict transform X' of X maps isomorphically to an open subspace of B'.

Proof. This lemma is a generalization of Lemma 39.3. As the composition of U-admissible blowups is U-admissible (Divisors on Spaces, Lemma 19.2) we can proceed in stages. Pick a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_B$ with

$|B| \setminus |U| = |V(\mathcal{I})|$. Replace B by the blowup of B in $\mathcal{I}$ and X by the strict transform of X. After this replacement $B \setminus U$ is the support of an effective Cartier divisor D (Divisors on Spaces, Lemma 17.4). In particular U is scheme theoretically dense in B (Divisors on Spaces, Lemma 6.4). Next, we do another U-admissible blowup to get to the situation where $X \to B$ is flat and of finite presentation, see Lemma 39.1. Note that U is still scheme theoretically dense in B. Hence $X \to B$ is an open immersion by Lemma 37.5. □

The following lemma says that a modification can be dominated by a blowup.

Lemma 39.5. *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $U \subset B$ be an open subspace. Assume*

(1) *B is quasi-compact and quasi-separated,*
(2) *U is quasi-compact,*
(3) *$f : X \to B$ is proper,*
(4) *$f^{-1}(U) \to U$ us an isomorphism.*

Then there exists a U-admissible blowup $B' \to B$ which dominates X, i.e., such that there exists a factorization $B' \to X \to B$ of the blowup morphism.

Proof. By Lemma 39.3 we may find a U-admissible blowup $B' \to B$ such that the strict transform X' maps isomorphically to B'. Then we can use $B' = X' \to X$ as the factorization. □

Lemma 39.6. *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $U \subset W \subset Y$ be open subspaces. Let $f : X \to W$ and let $s : U \to X$ be morphisms such that $f \circ s = id_U$. Assume*

(1) *f is proper,*
(2) *Y is quasi-compact and quasi-separated, and*
(3) *U and W are quasi-compact.*

Then there exists a U-admissible blowup $b : Y' \to Y$ and a morphism $s' : b^{-1}(W) \to X$ extending s with $f \circ s' = b|_{b^{-1}(W)}$.

Proof. We may and do replace X by the scheme theoretic image of s. Then $X \to W$ is an isomorphism over U, see Morphisms of Spaces, Lemma 16.7. By Lemma 39.5 there exists a U-admissible blowup $W' \to W$ and an extension $W' \to X$ of s. We finish the proof by applying Divisors on Spaces, Lemma 19.3 to extend $W' \to W$ to a U-admissible blowup of Y. □

40. Chow's lemma

In this section we prove Chow's lemma (Lemma 40.5). We encourage the reader to take a look at Cohomology of Spaces, Section 18 for a weak version of Chow's lemma that is easy to prove and sufficient for many applications.

Since we have yet to define projective morphisms of algebraic spaces, the statements of lemmas (see for example Lemma 40.2) will involve representable proper morphisms, rather than projective ones.

Lemma 40.1. *Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $U \to X_1$ and $U \to X_2$ be open immersions of algebraic* *spaces over Y and assume U, X_1, X_2 of finite type and separated over Y. Then there exists a commutative diagram*

$$\begin{tikzcd} X_1' \arrow[r] \arrow[d] & X & X_2' \arrow[l] \arrow[d] \\ X_1 & U \arrow[l] \arrow[r] \arrow[u] \arrow[ul] \arrow[ur] & X_2 \end{tikzcd}$$

of algebraic spaces over Y where $X_i' \to X_i$ is a U-admissible blowup, $X_i' \to X$ is an open immersion, and X is separated and finite type over Y.

Proof. Throughout the proof all the algebraic spaces will be separated of finite type over Y. This in particular implies these algebraic spaces are quasi-compact and quasi-separated and that the morphisms between them will be quasi-compact and separated. See Morphisms of Spaces, Sections 4 and 8. We will use that if $U \to W$ is an immersion of such spaces over Y, then the scheme theoretic image Z of U in W is a closed subspace of W and $U \to Z$ is an open immersion, $U \subset Z$ is scheme theoretically dense, and $|U| \subset |Z|$ is dense. See Morphisms of Spaces, Lemma 17.7.

Let $X_{12} \subset X_1 \times_Y X_2$ be the scheme theoretic image of $U \to X_1 \times_Y X_2$. The projections $p_i : X_{12} \to X_i$ induce isomorphisms $p_i^{-1}(U) \to U$ by Morphisms of Spaces, Lemma 16.7. Choose a U-admissible blowup $X_i^i \to X_i$ such that the strict transform X_{12}^i of X_{12} is isomorphic to an open subspace of X_i^i, see Lemma 39.4. Let $\mathcal{I}_i \subset \mathcal{O}_{X_i}$ be the corresponding finite type quasi-coherent sheaf of ideals. Recall that $X_{12}^i \to X_{12}$ is the blowup in $p_i^{-1}\mathcal{I}_i\mathcal{O}_{X_{12}}$, see Divisors on Spaces, Lemma 18.3. Let X_{12}' be the blowup of X_{12} in $p_1^{-1}\mathcal{I}_1 p_2^{-1}\mathcal{I}_2\mathcal{O}_{X_{12}}$, see Divisors on Spaces, Lemma 17.10 for what this entails. We obtain a commutative diagram

$$\begin{tikzcd} X_{12}' \arrow[r] \arrow[d] & X_{12}^2 \arrow[d] \\ X_{12}^1 \arrow[r] & X_{12} \end{tikzcd}$$

where all the morphisms are U-admissible blowing ups. Since $X_{12}^i \subset X_i^i$ is an open we may choose a U-admissible blowup $X_i' \to X_i^i$ restricting to $X_{12}' \to X_{12}^i$, see Divisors on Spaces, Lemma 19.3. Then $X_{12}' \subset X_i'$ is an open subspace and the diagram

$$\begin{tikzcd} X_{12}' \arrow[r] \arrow[d] & X_i' \arrow[d] \\ X_{12}^i \arrow[r] & X_i^i \end{tikzcd}$$

is commutative with vertical arrows blowing ups and horizontal arrows open immersions. Note that $X_{12}' \to X_1' \times_Y X_2'$ is an immersion and proper (use that $X_{12}' \to X_{12}$ is proper and $X_{12} \to X_1 \times_Y X_2$ is closed and $X_1' \times_Y X_2' \to X_1 \times_Y X_2$ is separated and apply Morphisms of Spaces, Lemma 40.6). Thus $X_{12}' \to X_1' \times_Y X_2'$ is a closed immersion. If we define X by glueing X_1' and X_2' along the common open subspace X_{12}', then $X \to Y$ is of finite type and separated[2]. As compositions of U-admissible

[2]Because we may check closedness of the diagonal $X \to X \times_Y X$ over the four open parts $X_i' \times_Y X_j'$ of $X \times_Y X$ where it is clear.

blowups are U-admissible blowups (Divisors on Spaces, Lemma 19.2) the lemma is proved. $\square$

Lemma 40.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $U \subset X$ be an open subspace. Assume*

(1) *U is quasi-compact,*
(2) *Y is quasi-compact and quasi-separated,*
(3) *there exists an immersion $U \to \mathbf{P}^n_Y$ over Y,*
(4) *f is of finite type and separated.*

Then there exists a commutative diagram

$$\begin{tikzcd}
& U \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
X \arrow[dr] & X' \arrow[l] \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[dl] & Z \arrow[dl] \\
& Y & \mathbf{P}^n_Y \arrow[l] &
\end{tikzcd}$$

where the arrows with source U are open immersions, $X' \to X$ is a U-admissible blowup, $X' \to Z'$ is an open immersion, $Z' \to Y$ is a proper and representable morphism of algebraic spaces. More precisely, $Z' \to Z$ is a U-admissible blowup and $Z \to \mathbf{P}^n_Y$ is a closed immersion.

Proof. Let $Z \subset \mathbf{P}^n_Y$ be the scheme theoretic image of the immersion $U \to \mathbf{P}^n_Y$. Since $U \to \mathbf{P}^n_Y$ is quasi-compact we see that $U \subset Z$ is a (scheme theoretically) dense open subspace (Morphisms of Spaces, Lemma 17.7). Apply Lemma 40.1 to find a diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & \overline{X}' & Z' \arrow[l] \arrow[d] \\
X & U \arrow[l] \arrow[r] \arrow[u] \arrow[ul] \arrow[ur] & Z
\end{tikzcd}$$

with properties as listed in the statement of that lemma. As $X' \to X$ and $Z' \to Z$ are U-admissible blowups we find that U is a scheme theoretically dense open of both X' and Z' (see Divisors on Spaces, Lemmas 17.4 and 6.4). Since $Z' \to Z \to Y$ is proper we see that $Z' \subset \overline{X}'$ is a closed subspace (see Morphisms of Spaces, Lemma 40.6). It follows that $X' \subset Z'$ (scheme theoretically), hence X' is an open subspace of Z' (small detail omitted) and the lemma is proved. $\square$

Lemma 40.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated, of finite type, and Y Noetherian. Then there exists a dense open subspace $U \subset X$ and a commutative diagram*

$$\begin{tikzcd}
& U \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
X \arrow[dr] & X' \arrow[l] \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[dl] & Z \arrow[dl] \\
& Y & \mathbf{P}^n_Y \arrow[l] &
\end{tikzcd}$$

where the arrows with source U are open immersions, $X' \to X$ is a U-admissible blowup, $X' \to Z'$ is an open immersion, $Z' \to Y$ is a proper and representable morphism of algebraic spaces. More precisely, $Z' \to Z$ is a U-admissible blowup and $Z \to \mathbf{P}^n_Y$ is a closed immersion.

Proof. By Limits of Spaces, Lemma 13.3 there exists a dense open subspace $U \subset X$ and an immersion $U \to \mathbf{A}^n_Y$ over Y. Composing with the open immersion $\mathbf{A}^n_Y \to \mathbf{P}^n_Y$ we obtain a situation as in Lemma 40.2 and the result follows. $\square$

Remark 40.4. In Lemmas 40.2 and 40.3 the morphism $g : Z' \to Y$ is a composition of projective morphisms. Presumably (by the analogue for algebraic spaces of Morphisms, Lemma 37.8) there exists a g-ample invertible sheaf on Z'. If we ever need this, then we will state and prove this here.

The following result is [Knu71, IV Theorem 3.1]. Note that the immersion $X' \to \mathbf{P}^n_Y$ is quasi-compact, hence can be factored as $X' \to Z' \to \mathbf{P}^n_Y$ where the first morphism is an open immersion and the second morphism a closed immersion (Morphisms of Spaces, Lemma 17.7).

Lemma 40.5 (Chow's lemma)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated of finite type, and Y separated and Noetherian. Then there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l] \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X' \to X$ is a U-admissible blowup for some dense open $U \subset X$ and the morphism $X' \to \mathbf{P}^n_Y$ is an immersion.

Proof. In this first paragraph of the proof we reduce the lemma to the case where Y is of finite type over $\mathrm{Spec}(\mathbf{Z})$. We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. We can write $Y = \lim Y_i$ as a directed limit of separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 8.1 and Lemma 5.9. For all i sufficiently large we can find a separated finite type morphism $X_i \to Y_i$ such that $X = Y \times_{Y_i} X_i$, see Limits of Spaces, Lemmas 7.1 and 6.9. Let $\eta_1, \ldots, \eta_n$ be the generic points of the irreducible components of $|X|$ (X is Noetherian as a finite type separated algebraic space over the Noetherian algebraic space Y and therefore $|X|$ is a Noetherian topological space). By Limits of Spaces, Lemma 5.2 we find that the images of $\eta_1, \ldots, \eta_n$ in $|X_i|$ are distinct for i large enough. We may replace X_i by the scheme theoretic image of the (quasi-compact, in fact affine) morphism $X \to X_i$. After this replacement we see that the images of $\eta_1, \ldots, \eta_n$ in $|X_i|$ are the generic points of the irreducible components of $|X_i|$, see Morphisms of Spaces, Lemma 16.3. Having said this, suppose we can find a diagram

$$\begin{tikzcd} X_i \arrow[dr] & X'_i \arrow[l] \arrow[r] \arrow[d] & \mathbf{P}^n_{Y_i} \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X'_i \to X_i$ is a U_i-admissible blowup for some dense open $U_i \subset X_i$ and the morphism $X'_i \to \mathbf{P}^n_{Y_i}$ is an immersion. Then the strict transform $X' \to X$ of X

relative to $X_i' \to X_i$ is a U-admissible blowing up where $U \subset X$ is the inverse image of U_i in X. Because of our carefuly chosen index i it follows that $\eta_1, \ldots, \eta_n \in |U|$ and $U \subset X$ is dense. Moreover, $X' \to \mathbf{P}^n_Y$ is an immersion as X' is closed in $X_i' \times_{X_i} X = X_i' \times_{Y_i} Y$ which comes with an immersion into $\mathbf{P}^n_Y$. Thus we have reduced to the situation of the following paragraph.

Assume that Y is separated of finite type over $\text{Spec}(\mathbf{Z})$. Then $X \to \text{Spec}(\mathbf{Z})$ is separated of finite type as well. We apply Lemma 40.3 to $X \to \text{Spec}(\mathbf{Z})$ to find a dense open subspace $U \subset X$ and a commutative diagram

$$\begin{tikzcd} & & U \arrow[dll] \arrow[d] \arrow[dr] \arrow[drr] & & \\ X \arrow[dr] & & X' \arrow[ll] \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[dl] & Z \arrow[dl] \\ & & \text{Spec}(\mathbf{Z}) & \mathbf{P}^n_{\mathbf{Z}} \arrow[l] & \end{tikzcd}$$

with all the properties listed in the lemma. Note that Z has an ample invertible sheaf, namely $\mathcal{O}_{\mathbf{P}^n}(1)|_Z$. Hence $Z' \to Z$ is a H-projective morphism by Morphisms, Lemma 43.16. It follows that $Z' \to \text{Spec}(\mathbf{Z})$ is H-projective by Morphisms, Lemma 43.7. Thus there exists a closed immersion $Z' \to \mathbf{P}^m_{\text{Spec}(\mathbf{Z})}$ for some $m \geq 0$. It follows that the diagonal morphism

$$X' \to Y \times \mathbf{P}^m_{\mathbf{Z}} = \mathbf{P}^m_Y$$

is an immersion (because the composition with the projection to $\mathbf{P}^m_{\mathbf{Z}}$ is an immersion) and we win. $\square$

41. Variants of Chow's Lemma

In this section we prove a number of variants of Chow's lemma dealing with morphisms between non-Noetherian algebraic spaces. The Noetherian versions are Lemma 40.3 and Lemma 40.5.

Lemma 41.1. *Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $f : X \to Y$ be a separated morphism of finite type. Then there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l] \arrow[r] \arrow[d] & \overline{X}' \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X' \to X$ is proper surjective, $X' \to \overline{X}'$ is an open immersion, and $\overline{X}' \to Y$ is proper and representable morphism of algebraic spaces.

Proof. By Limits of Spaces, Proposition 11.7 we can find a closed immersion $X \to X_1$ where X_1 is separated and of finite presentation over Y. Clearly, if we prove the assertion for $X_1 \to Y$, then the result follows for X. Hence we may assume that X is of finite presentation over Y.

We may and do replace the base scheme S by $\text{Spec}(\mathbf{Z})$. Write $Y = \lim_i Y_i$ as a directed limit of quasi-separated algebraic spaces of finite type over $\text{Spec}(\mathbf{Z})$, see

Limits of Spaces, Proposition 8.1. By Limits of Spaces, Lemma 7.1 we can find an index $i \in I$ and a scheme $X_i \to Y_i$ of finite presentation so that $X = Y \times_{Y_i} X_i$. By Limits of Spaces, Lemma 6.9 we may assume that $X_i \to Y_i$ is separated. Clearly, if we prove the assertion for X_i over Y_i, then the assertion holds for X. The case $X_i \to Y_i$ is treated by Lemma 40.3. $\square$

Lemma 41.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated of finite type, and Y separated and quasi-compact. Then there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l] \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X' \to X$ is proper surjective morphism and the morphism $X' \to \mathbf{P}^n_Y$ is an immersion.

Proof. By Limits of Spaces, Proposition 11.7 we can find a closed immersion $X \to X_1$ where X_1 is separated and of finite presentation over Y. Clearly, if we prove the assertion for $X_1 \to Y$, then the result follows for X. Hence we may assume that X is of finite presentation over Y.

We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. Write $Y = \lim_i Y_i$ as a directed limit of quasi-separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 8.1. By Limits of Spaces, Lemma 5.9 we may assume that Y_i is separated for all i. By Limits of Spaces, Lemma 7.1 we can find an index $i \in I$ and a scheme $X_i \to Y_i$ of finite presentation so that $X = Y \times_{Y_i} X_i$. By Limits of Spaces, Lemma 6.9 we may assume that $X_i \to Y_i$ is separated. Clearly, if we prove the assertion for X_i over Y_i, then the assertion holds for X. The case $X_i \to Y_i$ is treated by Lemma 40.5. $\square$

42. Grothendieck's existence theorem

In this section we discuss Grothendieck's existence theorem for algebraic spaces. Instead of developing a theory of "formal algebraic spaces" we temporarily develop a bit of language that replaces the notion of a "coherent module on a Noetherian adic formal space".

Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Below we will consider inverse systems $(\mathcal{F}_n)$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}_n$ is annihilated by $\mathcal{I}^n$, and
(2) the transition maps induce isomorphisms $\mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1} \to \mathcal{F}_n$.

A morphism $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ of such inverse systems is simply a compatible system of morphisms $\alpha_n : \mathcal{F}_n \to \mathcal{G}_n$. Let us denote the category of these inverse systems with *Coh*$(X, \mathcal{I})$. We will develop some theory regarding these systems that will parallel to the corresponding results in the case of schemes, see Cohomology of Schemes, Sections 24, 25, 27, and 28.

Functoriality. Let $f : X \to Y$ be a morphism of Noetherian algebraic spaces over a scheme S, and let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$.

In this situation there is a functor

$$f^* : Coh(Y, \mathcal{J}) \longrightarrow Coh(X, \mathcal{I})$$

which sends $(\mathcal{G}_n)$ to $(f^*\mathcal{G}_n)$. Compare with Cohomology of Schemes, Lemma 23.9. If f is étale, then we may think of this as simply the restriction of the system to X, see Properties of Spaces, Equation 26.1.1.

Étale descent. Let S be a scheme. Let $U_0 \to X$ be a surjective étale morphism of Noetherian algebraic spaces. Set $U_1 = U_0 \times_X U_0$ and $U_2 = U_0 \times_X U_0 \times_X U_0$. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Set $\mathcal{I}_i = \mathcal{I}|_{U_i}$. In this situation we obtain a diagram of categories

$$\begin{tikzcd} Coh(X, \mathcal{I}) \arrow[r] & Coh(U_0, \mathcal{I}_0) \arrow[r, shift left] \arrow[r, shift right] & Coh(U_1, \mathcal{I}_1) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Coh(U_2, \mathcal{I}_2) \end{tikzcd}$$

an the first arrow presents $Coh(X, \mathcal{I})$ as the homotopy limit of the right part of the diagram. More precisely, given a *descent datum*, i.e., a pair $((\mathcal{G}_n), \varphi)$ where $(\mathcal{G}_n)$ is an object of $Coh(U_0, \mathcal{I}_0)$ and $\varphi : \text{pr}_0^*(\mathcal{G}_n) \to \text{pr}_1^*(\mathcal{G}_n)$ is an isomorphism in $Coh(U_1, \mathcal{I}_1)$ satisfying the cocycle condition in $Coh(U_2, \mathcal{I}_2)$, then there exists a unique object $(\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$ whose associated canonical descent datum is isomorphic to $((\mathcal{G}_n), \varphi)$. Compare with Descent on Spaces, Definition 3.3. The proof of this statement follows immediately by applying Descent on Spaces, Proposition 4.1 to the descent data $(\mathcal{G}_n, \varphi_n)$ for varying n.

Lemma 42.1. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals.*

(1) *The category $Coh(X, \mathcal{I})$ is abelian.*
(2) *Exactness in $Coh(X, \mathcal{I})$ can be checked étale locally.*
(3) *For any flat morphism $f : X' \to X$ of Noetherian algebraic spaces the functor $f^* : Coh(X, \mathcal{I}) \to Coh(X', f^{-1}\mathcal{I}\mathcal{O}_{X'})$ is exact.*

Proof. Proof of (1). Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X$. Set $U_1 = U_0 \times_X U_0$ and $U_2 = U_0 \times_X U_0 \times_X U_0$ as in our discussion of étale descent above. The categories $Coh(U_i, \mathcal{I}_i)$ are abelian (Cohomology of Schemes, Lemma 23.2) and the pullback functors are exact functors $Coh(U_0, \mathcal{I}_0) \to Coh(U_1, \mathcal{I}_1)$ and $Coh(U_1, \mathcal{I}_1) \to Coh(U_2, \mathcal{I}_2)$ (Cohomology of Schemes, Lemma 23.9). The lemma then follows formally from the description of $Coh(X, \mathcal{I})$ as a category of descent data. Some details omitted; compare with the proof of Groupoids, Lemma 14.6.

Part (2) follows immediately from the discussion in the previous paragraph. In the situation of (3) choose a commutative diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

where U' and U are affine schemes and the vertical morphisms are surjective étale. Then $U' \to U$ is a flat morphism of Noetherian schemes (Morphisms of Spaces, Lemma 30.5) whence the pullback functor $Coh(U, \mathcal{I}\mathcal{O}_U) \to Coh(U', \mathcal{I}\mathcal{O}_{U'})$ is exact by Cohomology of Schemes, Lemma 23.9. Since we can check exactness in $Coh(X, \mathcal{O}_X)$ on U and similarly for X', U' the assertion follows. $\square$

Lemma 42.2. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. A map $(\mathcal{F}_n) \to (\mathcal{G}_n)$ is surjective in $Coh(X, \mathcal{I})$ if and only if $\mathcal{F}_1 \to \mathcal{G}_1$ is surjective.*

Proof. We can check on an affine étale cover of X by Lemma 42.1. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 23.3. $\square$

Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. There is a functor

$$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge \tag{42.2.1}$$

which associates to the coherent $\mathcal{O}_X$-module $\mathcal{F}$ the object $\mathcal{F}^\wedge = (\mathcal{F}/\mathcal{I}^n\mathcal{F})$ of $Coh(X, \mathcal{I})$.

Lemma 42.3. *The functor (42.2.1) is exact.*

Proof. It suffices to check this étale locally on X, see Lemma 42.1. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 23.4. $\square$

Lemma 42.4. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Then*

$$\lim H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge).$$

Proof. Since $\mathcal{H}$ is a sheaf on $X_{\acute{e}tale}$ and since we have étale descent for objects of $Coh(X, \mathcal{I})$ it suffices to prove this étale locally. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 23.5. $\square$

We introduce the setting that we will focus on throughout the rest of this section.

Situation 42.5. Here A is a Noetherian ring complete with respect to an ideal I. Also $f : X \to \mathrm{Spec}(A)$ is a finite type separated morphism of algebraic spaces and $\mathcal{I} = I\mathcal{O}_X$.

In this situation we denote

$$Coh_{\text{support proper over } A}(\mathcal{O}_X)$$

be the full subcategory of $Coh(\mathcal{O}_X)$ consisting of those coherent $\mathcal{O}_X$-modules whose support is proper over $\mathrm{Spec}(A)$, or equivalently whose scheme theoretic support is proper over $\mathrm{Spec}(A)$, see Derived Categories of Spaces, Lemma 7.7. Similarly, we let

$$Coh_{\text{support proper over } A}(X, \mathcal{I})$$

be the full subcategory of $Coh(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that the support of $\mathcal{F}_1$ is proper over $\mathrm{Spec}(A)$. Since the support of a quotient module is contained in the support of the module, it follows that (42.2.1) induces a functor

$$Coh_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow Coh_{\text{support proper over } A}(X, \mathcal{I}) \tag{42.5.1}$$

Our first result is that this functor is fully faithful.

Lemma 42.6. *In Situation 42.5. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Assume that the intersection of the supports of $\mathcal{F}$ and $\mathcal{G}$ is proper over $\mathrm{Spec}(A)$. Then the map*

$$\mathrm{Mor}_{Coh(\mathcal{O}_X)}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge)$$

coming from (42.2.1) is a bijection. In particular, (42.5.1) is fully faithful.

Proof. Let $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$. This is a coherent $\mathcal{O}_X$-module because its restriction of schemes étale over X is coherent by Modules, Lemma 22.6. By Lemma 42.4 the map

$$\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) \to \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that Z is a closed subspace such that $|Z|$ is contained in the intersection of the supports of $\mathcal{F}$ and $\mathcal{G}$. Hence $Z \to \mathrm{Spec}(A)$ is proper by assumption (see Derived Categories of Spaces, Section 7). Write $\mathcal{H} = i_*\mathcal{H}'$ for some coherent $\mathcal{O}_Z$-module $\mathcal{H}'$. We have $i_*(\mathcal{H}'/I^n\mathcal{H}') = \mathcal{H}/I^n\mathcal{H}$. Hence we obtain

$$\begin{aligned}\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) &= \lim_n H^0(Z, \mathcal{H}'/\mathcal{I}^n\mathcal{H}') \\ &= H^0(Z, \mathcal{H}') \\ &= H^0(X, \mathcal{H}) \\ &= \mathrm{Mor}_{Coh(\mathcal{O}_X)}(\mathcal{F}, \mathcal{G})\end{aligned}$$

the second equality by the theorem on formal functions (Cohomology of Spaces, Lemma 22.6). This proves the lemma. $\square$

Remark 42.7. Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $Coh(X, \mathcal{I})$. Given an affine scheme $U = \mathrm{Spec}(A)$ and a surjective étale morphism $U \to X$ denote $I, K \subset A$ the ideals corresponding to the restrictions $\mathcal{I}|_U, \mathcal{K}|_U$. Denote $\alpha_U : M \to N$ of finite $A^\wedge$-modules which corresponds to $\alpha|_U$ via Cohomology of Schemes, Lemma 23.1. We claim the following are equivalent

1. there exists an integer $t \geq 1$ such that $\mathrm{Ker}(\alpha_n)$ and $\mathrm{Coker}(\alpha_n)$ are annihilated by $\mathcal{K}^t$ for all $n \geq 1$,
2. for any (or some) affine open $\mathrm{Spec}(A) = U \subset X$ as above the modules $\mathrm{Ker}(\alpha_U)$ and $\mathrm{Coker}(\alpha_U)$ are annihilated by K^t for some integer $t \geq 1$.

If these equivalent conditions hold we will say that α is a *map whose kernel and cokernel are annihilated by a power of $\mathcal{K}$*. To see the equivalence we refer to Cohomology of Schemes, Remark 25.1.

Lemma 42.8. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module, $(\mathcal{F}_n)$ an object of $Coh(X, \mathcal{I})$, and $\alpha : (\mathcal{F}_n) \to \mathcal{G}^\wedge$ a map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$. Then there exists a unique (up to unique isomorphism) triple $(\mathcal{F}, a, \beta)$ where*

1. *$\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,*
2. *$a : \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-module map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$,*
3. *$\beta : (\mathcal{F}_n) \to \mathcal{F}^\wedge$ is an isomorphism, and*
4. *$\alpha = a^\wedge \circ \beta$.*

Proof. The uniqueness and étale descent for objects of $Coh(X, \mathcal{I})$ and $Coh(\mathcal{O}_X)$ implies it suffices to construct $(\mathcal{F}, a, \beta)$ étale locally on X. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 23.6. $\square$

Lemma 42.9. *In Situation 42.5. Let $\mathcal{K} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $X_e \subset X$ be the closed subspace cut out by $\mathcal{K}^e$. Let $\mathcal{I}_e = \mathcal{I}\mathcal{O}_{X_e}$. Let $(\mathcal{F}_n)$ be an object of $Coh_{support\ proper\ over\ A}(X, \mathcal{I})$. Assume*

(1) *the functor* $\textit{Coh}_{support\ proper\ over\ A}(\mathcal{O}_{X_e}) \to \textit{Coh}_{support\ proper\ over\ A}(X_e, \mathcal{I}_e)$ *is an equivalence for all* $e \geq 1$, *and*
(2) *there exists an object* $\mathcal{H}$ *of* $\textit{Coh}_{support\ proper\ over\ A}(\mathcal{O}_X)$ *and a map* $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ *whose kernel and cokernel are annihilated by a power of* $\mathcal{K}$.

Then $(\mathcal{F}_n)$ *is in the essential image of (42.5.1).*

Proof. During this proof we will use without further mention that for a closed immersion $i : Z \to X$ the functor i_* gives an equivalence between the category of coherent modules on Z and coherent modules on X annihilated by the ideal sheaf of Z, see Cohomology of Spaces, Lemma 12.8. In particular we think of

$$\textit{Coh}_{\text{support proper over }A}(\mathcal{O}_{X_e}) \subset \textit{Coh}_{\text{support proper over }A}(\mathcal{O}_X)$$

as the full subcategory of consisting of modules annihilated by $\mathcal{K}^e$ and

$$\textit{Coh}_{\text{support proper over }A}(X_e, \mathcal{I}_e) \subset \textit{Coh}_{\text{support proper over }A}(X, \mathcal{I})$$

as the full subcategory of objects annihilated by $\mathcal{K}^e$. Moreover (1) tells us these two categories are equivalent under the completion functor (42.5.1).

Applying this equivalence we get a coherent $\mathcal{O}_X$-module $\mathcal{G}_e$ annihilated by $\mathcal{K}^e$ corresponding to the system $(\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n)$ of $\textit{Coh}_{\text{support proper over }A}(X, \mathcal{I})$. The maps $\mathcal{F}_n/\mathcal{K}^{e+1}\mathcal{F}_n \to \mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n$ correspond to canonical maps $\mathcal{G}_{e+1} \to \mathcal{G}_e$ which induce isomorphisms $\mathcal{G}_{e+1}/\mathcal{K}^e\mathcal{G}_{e+1} \to \mathcal{G}_e$. We obtain an object $(\mathcal{G}_e)$ of the category $\textit{Coh}_{\text{support proper over }A}(X, \mathcal{K})$. The map α induces a system of maps

$$\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \longrightarrow \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$$

whence maps $\mathcal{G}_e \to \mathcal{H}/\mathcal{K}^e\mathcal{H}$ (by the equivalence of categories again). Let $t \geq 1$ be an integer, which exists by assumption (2), such that $\mathcal{K}^t$ annihilates the kernel and cokernel of all the maps $\mathcal{F}_n \to \mathcal{H}/\mathcal{I}^n\mathcal{H}$. Then $\mathcal{K}^{2t}$ annihilates the kernel and cokernel of the maps $\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \to \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$ (details omitted; see Cohomology of Schemes, Remark 25.1). Whereupon we conclude that $\mathcal{K}^{4t}$ annihilates the kernel and the cokernel of the maps

$$\mathcal{G}_e \longrightarrow \mathcal{H}/\mathcal{K}^e\mathcal{H},$$

(details omitted; see Cohomology of Schemes, Remark 25.1). We apply Lemma 42.8 to obtain a coherent $\mathcal{O}_X$-module $\mathcal{F}$, a map $a : \mathcal{F} \to \mathcal{H}$ and an isomorphism $\beta : (\mathcal{G}_e) \to (\mathcal{F}/\mathcal{K}^e\mathcal{F})$ in $\textit{Coh}(X, \mathcal{K})$. Working backwards, for a given n the triple $(\mathcal{F}/\mathcal{I}^n\mathcal{F}, a \bmod \mathcal{I}^n, \beta \bmod \mathcal{I}^n)$ is a triple as in the lemma for the morphism $\alpha_n \bmod \mathcal{K}^e : (\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n) \to (\mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H})$ of $\textit{Coh}(X, \mathcal{K})$. Thus the uniqueness in Lemma 42.8 gives a canonical isomorphism $\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$ compatible with all the morphisms in sight.

To finish the proof of the lemma we still have to show that the support of $\mathcal{F}$ is proper over A. By construction the kernel of $a : \mathcal{F} \to \mathcal{H}$ is annihilated by a power of $\mathcal{K}$. Hence the support of this kernel is contained in the support of $\mathcal{G}_1$. Since $\mathcal{G}_1$ is an object of $\textit{Coh}_{\text{support proper over }A}(\mathcal{O}_{X_1})$ we see this is proper over A. Combined with the fact that the support of $\mathcal{H}$ is proper over A we conclude that the support of $\mathcal{F}$ is proper over A by Derived Categories of Spaces, Lemma 7.6. $\square$

Lemma 42.10. *Let* S *be a scheme. Let* $f : X \to Y$ *be a representable proper morphism of Noetherian algebraic spaces over* S. *Let* $\mathcal{J}, \mathcal{K} \subset \mathcal{O}_Y$ *be quasi-coherent sheaves of ideals. Assume* f *is an isomorphism over* $V = Y \setminus V(\mathcal{K})$. *Set* $\mathcal{I} =$

$f^{-1}\mathcal{J}\mathcal{O}_X$. Let $(\mathcal{G}_n)$ be an object of $Coh(Y, \mathcal{J})$, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and let $\beta : (f^*\mathcal{G}_n) \to \mathcal{F}^\wedge$ be an isomorphism in $Coh(X, \mathcal{I})$. Then there exists a map

$$\alpha : (\mathcal{G}_n) \longrightarrow (f_*\mathcal{F})^\wedge$$

in $Coh(Y, \mathcal{J})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.

Proof. Since f is a proper morphism we see that $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module (Cohomology of Spaces, Lemma 20.2). Thus the statement of the lemma makes sense. Consider the compositions

$$\gamma_n : \mathcal{G}_n \to f_*f^*\mathcal{G}_n \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F}).$$

Here the first map is the adjunction map and the second is $f_*\beta_n$. We claim that there exists a unique α as in the lemma such that the compositions

$$\mathcal{G}_n \xrightarrow{\alpha_n} f_*\mathcal{F}/\mathcal{J}^n f_*\mathcal{F} \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F})$$

equal γ_n for all n. Because of the uniqueness and étale descent for $Coh(Y, \mathcal{J})$ it suffices to prove this étale locally on Y. Thus we may assume Y is the spectrum of a Noetherian ring. As f is representable we see that X is a scheme as well. Thus we reduce to the case of schemes, see proof of Cohomology of Schemes, Lemma 25.3. □

Theorem 42.11 (Grothendieck's existence theorem)**.** *In Situation 42.5 the functor (42.5.1) is an equivalence.*

Proof. We will use the equivalence of categories of Cohomology of Spaces, Lemma 12.8 without further mention in the proof of the theorem. By Lemma 42.6 the functor is fully faithful. Thus we need to prove the functor is essentially surjective.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that the statement holds for every object $(\mathcal{F}_n)$ of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by $\mathcal{K}$. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Cohomology of Spaces, Lemma 13.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. Let $(\mathcal{F}_n)$ be an object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$. We will show that this object is in the essential image, thereby completing the proof of the theorem.

Apply Chow's lemma (Lemma 40.5) to find a proper surjective morphism $f : Y \to X$ which is an isomorphism over a dense open $U \subset X$ such that Y is H-quasi-projective over A. Note that Y is a scheme and f representable. Choose an open immersion $j : Y \to Y'$ with Y' projective over A, see Morphisms, Lemma 43.11. Let T_n be the scheme theoretic support of $\mathcal{F}_n$. Note that $|T_n| = |T_1|$, hence T_n is proper over A for all n (Morphisms of Spaces, Lemma 40.7). Then $f^*\mathcal{F}_n$ is supported on the closed subscheme $f^{-1}T_n$ which is proper over A (by Morphisms of Spaces, Lemma 40.4 and properness of f). In particular, the composition $f^{-1}T_n \to Y \to Y'$ is closed (Morphisms, Lemma 41.7). Let $T'_n \subset Y'$ be the corresponding closed subscheme; it is contained in the open subscheme Y and equal to $f^{-1}T_n$ as a closed subscheme of Y. Let $\mathcal{F}'_n$ be the coherent $\mathcal{O}_{Y'}$-module corresponding to $f^*\mathcal{F}_n$ viewed as a coherent module on Y' via the closed immersion $f^{-1}T_n = T'_n \subset Y'$. Then $(\mathcal{F}'_n)$ is an object of $Coh(Y', I\mathcal{O}_{Y'})$. By the projective case of Grothendieck's existence theorem (Cohomology of Schemes, Lemma 24.3) there exists a coherent $\mathcal{O}_{Y'}$-module $\mathcal{F}'$ and an isomorphism $(\mathcal{F}')^\wedge \cong (\mathcal{F}'_n)$ in $Coh(Y', I\mathcal{O}_{Y'})$. Let $Z' \subset Y'$ be the scheme theoretic support of $\mathcal{F}'$. Since $\mathcal{F}'/I\mathcal{F}' = \mathcal{F}'_1$ we see that $Z' \cap V(I\mathcal{O}_{Y'}) = T'_1$ set-theoretically. The structure morphism $p' : Y' \to \text{Spec}(A)$ is proper, hence $p'(Z' \cap (Y' \setminus Y))$ is closed in $\text{Spec}(A)$. If nonempty, then it would contain a point of $V(I)$ as I is contained in the Jacobson radical of A (Algebra, Lemma 96.6). But we've seen above that $Z' \cap (p')^{-1}V(I) = T'_1 \subset Y$ hence we conclude that $Z' \subset Y$. Thus $\mathcal{F}'|_Y$ is supported on a closed subscheme of Y proper over A.

Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By Cohomology of Spaces, Lemma 20.2 the $\mathcal{O}_X$-module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 42.10 there exists a morphism $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ in the category $\textit{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. Let $Z_0 \subset X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that $|Z_0| \subset f(|Z'|)$. Hence $Z_0 \to \text{Spec}(A)$ is proper (Morphisms of Spaces, Lemma 40.7). Thus $\mathcal{H}$ is an object of $\textit{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$. Since each of the sheaves of ideals $\mathcal{K}^e$ is an element of Ξ we see that the assumptions of Lemma 42.9 are satisfied and we conclude. $\square$

Remark 42.12 (Unwinding Grothendieck's existence theorem)**.** Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Picture:

$$\begin{tikzcd}
X_1 \arrow[r, "i_1"'] \arrow[d] & X_2 \arrow[r, "i_2"'] \arrow[d] & X_3 \arrow[r] \arrow[d] & \ldots & X \arrow[d] \\
S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \ldots & S
\end{tikzcd}$$

In this situation we consider systems $(\mathcal{F}_n, \varphi_n)$ where

(1) $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module,
(2) $\varphi_n : i_n^*\mathcal{F}_{n+1} \to \mathcal{F}_n$ is an isomorphism, and
(3) $\text{Supp}(\mathcal{F}_1)$ is proper over S_1.

Theorem 42.11 says that the completion functor

$$\begin{matrix} \text{coherent } \mathcal{O}_X\text{-modules } \mathcal{F} \\ \text{with support proper over } A \end{matrix} \longrightarrow \begin{matrix} \text{systems } (\mathcal{F}_n) \\ \text{as above} \end{matrix}$$

is an equivalence of categories. In the special case that X is proper over A we can omit the conditions on the supports.

43. Grothendieck's algebraization theorem

This section is the analogue of Cohomology of Schemes, Section 28. However, this section is missing the result on algebraization of deformations of proper algebraic spaces endowed with ample invertible sheaves, as a proper algebraic space which comes with an ample invertible sheaf is already a scheme. We do have an algebraization result on proper algebraic spaces of relative dimension 1. Our first result is a translation of Grothendieck's existence theorem in terms of closed subschemes and finite morphisms.

Lemma 43.1. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic* *spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{tikzcd} Z_1 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & Z_3 \arrow[r] \arrow[d] & \ldots \\ X_1 \arrow[r, "i_1"] & X_2 \arrow[r, "i_2"] & X_3 \arrow[r] & \ldots \end{tikzcd}$$

of algebraic spaces with cartesian squares. Assume that

1. *$Z_1 \to X_1$ is a closed immersion, and*
2. *$Z_1 \to S_1$ is proper.*

Then there exists a closed immersion of algebraic spaces $Z \to X$ such that $Z_n = Z \times_S S_n$ for all $n \geq 1$. Moreover, Z is proper over S.

Proof. Let's write $j_n : Z_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of j_n to X_1 is j_1. Thus Limits of Spaces, Lemma 15.5 shows that j_n is a closed immersion. Set $\mathcal{F}_n = j_{n,*}\mathcal{O}_{Z_n}$, so that $j_n^\sharp$ is a surjection $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Again using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 42.12, tells us there exists a map $\mathcal{O}_X \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules whose restriction to X_n recovers $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is exact (Lemma 42.3) we see that $\mathcal{O}_X \to \mathcal{F}$ is surjective. Thus $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$ for some quasi-coherent sheaf of ideals $\mathcal{J}$. Setting $Z = V(\mathcal{J})$ finishes the proof. $\square$

Lemma 43.2. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & \ldots \\ X_1 \arrow[r, "i_1"] & X_2 \arrow[r, "i_2"] & X_3 \arrow[r] & \ldots \end{tikzcd}$$

of algebraic spaces with cartesian squares. Assume that

1. *$Y_1 \to X_1$ is a finite morphism, and*
2. *$Y_1 \to S_1$ is proper.*

Then there exists a finite morphism of algebraic spaces $Y \to X$ such that $Y_n = Y \times_S S_n$ for all $n \geq 1$. Moreover, Y is proper over S.

Proof. Let's write $f_n : Y_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of f_n to X_1 is f_1. Thus Lemma 10.2 shows that f_n is a finite morphism. Set $\mathcal{F}_n = f_{n,*}\mathcal{O}_{Y_n}$. Using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 42.12, tells us there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ whose restriction to X_n recovers $\mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is fully faithful (Theorem 42.11) we see that the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{F}_n \to \mathcal{F}_n$ fit together to give an algebra structure on $\mathcal{F}$. Setting $Y = \underline{\mathrm{Spec}}_X(\mathcal{F})$ finishes the proof. $\square$

Lemma 43.3. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let X, Y be algebraic spaces over S. For $n \geq 1$ we set $X_n = X \times_S S_n$ and $Y_n = Y \times_S S_n$. Suppose given a compatible system of commutative diagrams*

$$\begin{tikzcd}
 & X_{n+1} \arrow[rr, "g_{n+1}"] \arrow[drr] & & Y_{n+1} \arrow[d] \\
X_n \arrow[ur] \arrow[rr, "g_n"] \arrow[dr] & & Y_n \arrow[ur] \arrow[dl] & S_{n+1} \\
 & S_n \arrow[urr] & &
\end{tikzcd}$$

Assume that

(1) *$X \to S$ is proper, and*
(2) *$Y \to S$ is separated of finite type.*

Then there exists a unique morphism of algebraic spaces $g : X \to Y$ over S such that g_n is the base change of g to S_n.

Proof. The morphisms $(1, g_n) : X_n \to X_n \times_S Y_n$ are closed immersions because $Y_n \to S_n$ is separated (Morphisms of Spaces, Lemma 4.7). Thus by Lemma 43.1 there exists a closed subspace $Z \subset X \times_S Y$ proper over S whose base change to S_n recovers $X_n \subset X_n \times_S Y_n$. The first projection $p : Z \to X$ is a proper morphism (as Z is proper over S, see Morphisms of Spaces, Lemma 40.6) whose base change to S_n is an isomorphism for all n. In particular, $p : Z \to X$ is quasi-finite on an open subspace of Z containing every point of Z_0 for example by Morphisms of Spaces, Lemma 34.7. As Z is proper over S this open neighbourhood is all of Z. We conclude that $p : Z \to X$ is finite by Zariski's main theorem (for example apply Lemma 34.3 and use properness of Z over X to see that the immersion is a closed immersion). Applying the equivalence of Theorem 42.11 we see that $p_*\mathcal{O}_Z = \mathcal{O}_X$ as this is true modulo I^n for all n. Hence p is an isomorphism and we obtain the morphism g as the composition $X \cong Z \to Y$. We omit the proof of uniqueness. $\square$

Remark 43.4. We can ask if in Grothendieck's algebraization theorem (in the form of Lemma 43.3), we can get by with weaker separation axioms on the target. Let us be more precise. Let A, I, S, S_n, X, Y, X_n, Y_n, and g_n be as in the statement of Lemma 43.3 and assume that

(1) $X \to S$ is proper, and
(2) $Y \to S$ is locally of finite type.

Does there exist a morphism of algebraic spaces $g : X \to Y$ over S such that g_n is the base change of g to S_n? We don't know the answer in general; if you do please email stacks.project@gmail.com. If $Y \to S$ is separated, then the result holds by the lemma (there is an immediate reduction to the case where X is finite type over S, by choosing a quasi-compact open containing the image of g_1). If we only assume $Y \to S$ is quasi-separated, then the result is true as well. First, as before we may assume Y is quasi-compact as well as quasi-separated. Then we can use either [Bha16] or from [HR19] to algebraize (g_n). Namely, to apply the first reference, we use

$$D_{perf}(X) \to \lim D_{perf}(X_n) \xrightarrow{\lim Lg_n^*} \lim D_{perf}(Y_n) = D_{perf}(Y)$$

where the last step uses a Grothendieck existence result for the derived category of the proper algebraic space Y over R (compare with Flatness on Spaces, Remark 13.7). The paper cited shows that this arrow determines a morphism $Y \to X$ as desired. To apply the second reference we use the same argument with coherent modules:

$$Coh(\mathcal{O}_X) \to \lim Coh(\mathcal{O}_{X_n}) \xrightarrow{\lim g_n^*} \lim Coh(\mathcal{O}_{Y_n}) = Coh(\mathcal{O}_Y)$$

where the final equality is a consequence of Grothendieck's existence theorem (Theorem 42.11). The second reference tells us that this functor corresponds to a morphism $Y \to X$ over R. If we ever need this generalization we will precisely state and carefully prove the result here.

Lemma 43.5. *Let $(A, \mathfrak{m}, \kappa)$ be a complete local Noetherian ring. Set $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/\mathfrak{m}^n)$. Consider a commutative diagram*

$$\begin{tikzcd} X_1 \arrow[r, "i_1"'] \arrow[d] & X_2 \arrow[r, "i_2"'] \arrow[d] & X_3 \arrow[r] \arrow[d] & \ldots \\ S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \ldots \end{tikzcd}$$

of algebraic spaces with cartesian squares. If $\dim(X_1) \leq 1$, then there exists a projective morphism of schemes $X \to S$ and isomorphisms $X_n \cong X \times_S S_n$ compatible with i_n.

Proof. By Spaces over Fields, Lemma 9.3 the algebraic space X_1 is a scheme. Hence X_1 is a proper scheme of dimension ≤ 1 over κ. By Varieties, Lemma 43.4 we see that X_1 is H-projective over κ. Let $\mathcal{L}_1$ be an ample invertible sheaf on X_1.

We are going to show that $\mathcal{L}_1$ lifts to a compatible system $\{\mathcal{L}_n\}$ of invertible sheaves on $\{X_n\}$. Observe that X_n is a scheme too by Lemma 9.5. Recall that $X_1 \to X_n$ induces homeomorphisms of underlying topological spaces. In the rest of the proof we do not distinguish between sheaves on X_n and sheaves on X_1. Suppose, given a lift $\mathcal{L}_n$ to X_n. We consider the exact sequence

$$1 \to (1 + \mathfrak{m}^n\mathcal{O}_{X_{n+1}})^* \to \mathcal{O}_{X_{n+1}}^* \to \mathcal{O}_{X_n}^* \to 1$$

of sheaves on X_{n+1}. The class of $\mathcal{L}_n$ in $H^1(X_n, \mathcal{O}_{X_n}^*)$ (see Cohomology, Lemma 6.1) can be lifted to an element of $H^1(X_{n+1}, \mathcal{O}_{X_{n+1}}^*)$ if and only if the obstruction in $H^2(X_{n+1}, (1 + \mathfrak{m}^n\mathcal{O}_{X_{n+1}})^*)$ is zero. As X_1 is a Noetherian scheme of dimension ≤ 1 this cohomology group vanishes (Cohomology, Proposition 20.7).

By Grothendieck's algebraization theorem (Cohomology of Schemes, Theorem 28.4) we find a projective morphism of schemes $X \to S = \mathrm{Spec}(A)$ and a compatible system of isomorphisms $X_n = S_n \times_S X$. $\square$

Lemma 43.6. *Let $(A, \mathfrak{m}, \kappa)$ be a complete Noetherian local ring. Let X be an algebraic space over $\mathrm{Spec}(A)$. If $X \to \mathrm{Spec}(A)$ is proper and $\dim(X_\kappa) \leq 1$, then X is a scheme projective over A.*

Proof. Set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n)$. By Lemma 43.5 there exists a projective morphism $Y \to \mathrm{Spec}(A)$ and compatible isomorphisms $Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n) \cong X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n)$. By Lemma 43.3 we see that $X \cong Y$ and the proof is complete. $\square$

44. Regular immersions

This section is the analogue of Divisors, Section 21 for morphisms of algebraic spaces. The reader is encouraged to read up on regular immersions of schemes in that section first.

In Divisors, Section 21 we defined four types of regular immersions for morphisms of schemes. Of these only three are (as far as we know) local on the target for the étale topology; as usual plain old regular immersions aren't. This is why for morphisms of algebraic spaces we cannot actually define regular immersions. (These kinds of annoyances prompted Grothendieck and his school to replace original notion of a regular immersion by a Koszul-regular immersions, see [BGI71, Exposee VII, Definition 1.4].) But we can define Koszul-regular, H_1-regular, and quasi-regular immersions. Another remark is that since Koszul-regular immersions are not preserved by arbitrary base change, we cannot use the strategy of Morphisms of Spaces, Section 3 to define them. Similarly, as Koszul-regular immersions are not étale local on the source, we cannot use Morphisms of Spaces, Lemma 22.1 to define them either. We replace this lemma instead by the following.

Lemma 44.1. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the target. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Consider commutative diagrams*

$$\begin{tikzcd} X \times_Y V \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where V is a scheme and $V \to Y$ is étale. The following are equivalent

1. *for any diagram as above the projection $X \times_Y V \to V$ has property $\mathcal{P}$, and*
2. *for some diagram as above with $V \to Y$ surjective the projection $X \times_Y V \to V$ has property $\mathcal{P}$.*

If X and Y are representable, then this is also equivalent to f (as a morphism of schemes) having property $\mathcal{P}$.

Proof. Let us prove the equivalence of (1) and (2). The implication (1) $\Rightarrow$ (2) is immediate. Assume

$$\begin{tikzcd} X \times_Y V \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \times_Y V' \arrow[r] \arrow[d] & V' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

are two diagrams as in the lemma. Assume $V \to Y$ is surjective and $X \times_Y V \to V$ has property $\mathcal{P}$. To show that (2) implies (1) we have to prove that $X \times_Y V' \to V'$ has $\mathcal{P}$. To do this consider the diagram

$$\begin{tikzcd} X \times_Y V \arrow[d] & (X \times_Y V) \times_X (X \times_Y V') \arrow[l] \arrow[r] \arrow[d] & X \times_Y V' \arrow[d] \\ V & V \times_Y V' \arrow[l] \arrow[r] & V' \end{tikzcd}$$

By our assumption that $\mathcal{P}$ is étale local on the source, we see that $\mathcal{P}$ is preserved under étale base change, see Descent, Lemma 22.2. Hence if the left vertical arrow has $\mathcal{P}$ the so does the middle vertical arrow. Since $U \times_X U' \to U'$ is surjective and étale (hence defines an étale covering of U') this implies (as $\mathcal{P}$ is assumed local for the étale topology on the target) that the left vertical arrow has $\mathcal{P}$.

If X and Y are representable, then we can take $\text{id}_Y : Y \to Y$ as our étale covering to see the final statement of the lemma is true. $\square$

Note that "being a Koszul-regular (resp. H_1-regular, resp. quasi-regular) immersion" is a property of morphisms of schemes which is fpqc local on the target, see Descent, Lemma 23.32. Hence the following definition now makes sense.

Definition 44.2. Let S be a scheme. Let $i : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say i is a *Koszul-regular immersion* if i is representable and the equivalent conditions of Lemma 44.1 hold with $\mathcal{P}(f)$ ="f is a Koszul-regular immersion".
(2) We say i is an *H_1-regular immersion* if i is representable and the equivalent conditions of Lemma 44.1 hold with $\mathcal{P}(f)$ ="f is an H_1-regular immersion".
(3) We say i is a *quasi-regular immersion* if i is representable and the equivalent conditions of Lemma 44.1 hold with $\mathcal{P}(f)$ ="f is a quasi-regular immersion".

Lemma 44.3. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. We have the following implications: i is Koszul-regular $\Rightarrow$ i is H_1-regular $\Rightarrow$ i is quasi-regular.*

Proof. Via the definition this lemma immediately reduces to Divisors, Lemma 21.2. $\square$

Lemma 44.4. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Assume X is locally Noetherian. Then i is Koszul-regular $\Leftrightarrow$ i is H_1-regular $\Leftrightarrow$ i is quasi-regular.*

Proof. Via Definition 44.2 (and the definition of a locally Noetherian algebraic space in Properties of Spaces, Section 7) this immediately translates to the case of schemes which is Divisors, Lemma 21.3. $\square$

Lemma 44.5. *Let S be a scheme. Let $i : Z \to X$ be a Koszul-regular, H_1-regular, or quasi-regular immersion of algebraic spaces over S. Let $X' \to X$ be a flat morphism of algebraic spaces over S. Then the base change $i' : Z \times_X X' \to X'$ is a Koszul-regular, H_1-regular, or quasi-regular immersion.*

Proof. Via Definition 44.2 (and the definition of a flat morphism of algebraic spaces in Morphisms of Spaces, Section 30) this lemma reduces to the case of schemes, see Divisors, Lemma 21.4. $\square$

Lemma 44.6. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then i is a quasi-regular immersion if and only if the following conditions are satisfied*

(1) *i is locally of finite presentation,*
(2) *the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and*
(3) *the map (6.1.2) is an isomorphism.*

Proof. Follows from the case of schemes (Divisors, Lemma 21.5) via étale localization (use Definition 44.2 and Lemma 6.2). □

Lemma 44.7. *Let S be a scheme. Let $Z \to Y \to X$ be immersions of algebraic spaces over S. Assume that $Z \to Y$ is H_1-regular. Then the canonical sequence of Lemma 5.6*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and (étale) locally split.

Proof. Since $\mathcal{C}_{Z/Y}$ is finite locally free (see Lemma 44.6 and Lemma 44.3) it suffices to prove that the sequence is exact. It suffices to show that the first map is injective as the sequence is already right exact in general. After étale localization on X this reduces to the case of schemes, see Divisors, Lemma 21.6. □

A composition of quasi-regular immersions may not be quasi-regular, see Algebra, Remark 69.8. The other types of regular immersions are preserved under composition.

Lemma 44.8. *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S.*

1. *If i and j are Koszul-regular immersions, so is $j \circ i$.*
2. *If i and j are H_1-regular immersions, so is $j \circ i$.*
3. *If i is an H_1-regular immersion and j is a quasi-regular immersion, then $j \circ i$ is a quasi-regular immersion.*

Proof. Immediate from the case of schemes, see Divisors, Lemma 21.7. □

Lemma 44.9. *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S. Assume that the sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 5.6 is exact and locally split.

1. *If $j \circ i$ is a quasi-regular immersion, so is i.*
2. *If $j \circ i$ is a H_1-regular immersion, so is i.*
3. *If both j and $j \circ i$ are Koszul-regular immersions, so is i.*

Proof. Immediate from the case of schemes, see Divisors, Lemma 21.8. □

Lemma 44.10. *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S. Assume X is locally Noetherian. The following are equivalent*

1. *i and j are Koszul regular immersions,*
2. *i and $j \circ i$ are Koszul regular immersions,*
3. *$j \circ i$ is a Koszul regular immersion and the conormal sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and locally split.

Proof. Immediate from the case of schemes, see Divisors, Lemma 21.9. □

45. Relative pseudo-coherence

This section is the analogue of More on Morphisms, Section 59. However, in the treatment of this material for algebraic spaces we have decided to work exclusively with objects in the derived category whose cohomology sheaves are quasi-coherent. There are two reasons for this: (1) it greatly simplifies the exposition and (2) we currently have no use for the more general notion.

Remark 45.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of representable algebraic spaces over S which is locally of finite type. Let $f_0 : X_0 \to Y_0$ be a morphism of schemes representing f (awkward but temporary notation). Then f_0 is locally of finite type. If E is an object of $D_{QCoh}(\mathcal{O}_X)$, then E is the pullback of a unique object E_0 in $D_{QCoh}(\mathcal{O}_{X_0})$, see Derived Categories of Spaces, Lemma 4.2. In this situation the phrase "E is m-pseudo-coherent relative to Y" will be taken to mean "E_0 is m-pseudo-coherent relative to Y_0" as defined in More on Morphisms, Section 59.

Lemma 45.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $m \in \mathbf{Z}$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. With notation as explained in Remark 45.1 the following are equivalent:*

(1) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale, the complex $E|_U$ is m-pseudo-coherent relative to V,

(2) *for some commutative diagram as in (1) with $U \to X$ surjective, the complex $E|_U$ is m-pseudo-coherent relative to V,*

(3) *for every commutative diagram as in (1) with U and V affine the complex $R\Gamma(U, E)$ of $\mathcal{O}_X(U)$-modules is m-pseudo-coherent relative to $\mathcal{O}_Y(V)$.*

Proof. Part (1) implies (3) by More on Morphisms, Lemma 59.7.

Assume (3). Pick any commutative diagram as in (1) with $U \to X$ surjective. Choose an affine open covering $V = \bigcup V_j$ and affine open coverings $(U \to V)^{-1}(V_j) = \bigcup U_{ij}$. By (3) and More on Morphisms, Lemma 59.7 we see that $E|_U$ is m-pseudo-coherent relative to V. Thus (3) implies (2).

Assume (2). Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, the morphism $U \to X$ is surjective, and $E|_U$ is m-pseudo-coherent relative to V. Next, suppose given a second commutative diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & V' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with étale vertical arrows and U', V' schemes. We want to show that $E|_{U'}$ is m-pseudo-coherent relative to V'. The morphism $U'' = U \times_X U' \to U'$ is surjective étale and $U'' \to V'$ factors through $V'' = V' \times_Y V$ which is étale over V'. Hence it suffices to show that $E|_{U''}$ is m-pseudo-coherent relative to V'', see More on Morphisms, Lemmas 70.1 and 70.2. Using the second lemma once more it suffices to show that $E|_{U''}$ is m-pseudo-coherent relative to V. This is true by More on Morphisms, Lemma 59.16 and the fact that an étale morphism of schemes is pseudo-coherent by More on Morphisms, Lemma 60.6. □

Definition 45.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Fix $m \in \mathbf{Z}$.

(1) We say E is *m-pseudo-coherent relative to Y* if the equivalent conditions of Lemma 45.2 are satisfied.
(2) We say E is *pseudo-coherent relative to Y* if E is m-pseudo-coherent relative to Y for all $m \in \mathbf{Z}$.
(3) We say $\mathcal{F}$ is *m-pseudo-coherent relative to Y* if $\mathcal{F}$ viewed as an object of $D_{QCoh}(\mathcal{O}_X)$ is m-pseudo-coherent relative to Y.
(4) We say $\mathcal{F}$ is *pseudo-coherent relative to Y* if $\mathcal{F}$ viewed as an object of $D_{QCoh}(\mathcal{O}_X)$ is pseudo-coherent relative to Y.

Most of the properties of pseudo-coherent complexes relative to a base will follow immediately from the corresponding properties in the case of schemes. We will add the relevant lemmas here as needed.

Lemma 45.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let E in $D_{QCoh}(\mathcal{O}_X)$. If f is flat and locally of finite presentation, then the following are equivalent*

(1) *E is pseudo-coherent relative to Y, and*
(2) *E is pseudo-coherent on X.*

Proof. By étale localization and the definitions we may assume X and Y are schemes. For the case of schemes this follows from More on Morphisms, Lemma 59.18. □

46. Pseudo-coherent morphisms

This section is the analogue of More on Morphisms, Section 60 for morphisms of schemes. The reader is encouraged to read up on pseudo-coherent morphisms of schemes in that section first.

The property "pseudo-coherent" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 60.10 and 60.13 and Descent, Lemma 32.6. By Morphisms of Spaces, Lemma 22.1 we may define the notion of a pseudo-coherent morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 60 when the algebraic spaces in question are representable.

Definition 46.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *pseudo-coherent* if the equivalent conditions of Morphisms of Spaces, Lemma 22.1 hold with $\mathcal{P}$ =“pseudo-coherent”.
(2) Let $x \in |X|$. We say f is *pseudo-coherent at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is pseudo-coherent.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

Lemma 46.2. *A flat base change of a pseudo-coherent morphism is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 60.3. $\square$

Lemma 46.3. *A composition of pseudo-coherent morphisms is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 60.4. $\square$

Lemma 46.4. *A pseudo-coherent morphism is locally of finite presentation.*

Proof. Immediate from the definitions. $\square$

Lemma 46.5. *A flat morphism which is locally of finite presentation is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 60.6. $\square$

Lemma 46.6. *Let $f : X \to Y$ be a morphism of algebraic spaces pseudo-coherent over a base algebraic space B. Then f is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 60.7. $\square$

Lemma 46.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If Y is locally Noetherian, then f is pseudo-coherent if and only if f is locally of finite type.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 60.9. $\square$

47. Perfect morphisms

This section is the analogue of More on Morphisms, Section 61 for morphisms of schemes. The reader is encouraged to read up on perfect morphisms of schemes in that section first.

The property “perfect” of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 61.10 and 61.14 and Descent, Lemma 32.6. By Morphisms of Spaces, Lemma 22.1 we may define the notion of a perfect morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 61 when the algebraic spaces in question are representable.

Definition 47.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *perfect* if the equivalent conditions of Morphisms of Spaces, Lemma 22.1 hold with $\mathcal{P} =$"perfect".
(2) Let $x \in |X|$. We say f is *perfect at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is perfect.

Note that a perfect morphism is pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

Lemma 47.2. *A flat base change of a perfect morphism is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 61.3. $\square$

Lemma 47.3. *A composition of perfect morphisms is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 61.4. $\square$

Lemma 47.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is flat and perfect, and*
(2) *f is flat and locally of finite presentation.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 61.5. $\square$

Lemma 47.5. *Let S be a scheme. Let Y be a Noetherian algebraic space over S. Let $f : X \to Y$ be a perfect proper morphism of algebraic spaces. Let $E \in D(\mathcal{O}_X)$ be perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$.*

Proof. We claim that Derived Categories of Spaces, Lemma 22.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on X. Thus we may assume X and Y affine and E represented by a strictly perfect complex of $\mathcal{O}_X$-modules. Thus it suffices to show that $\mathcal{O}_X$ has finite tor dimension as a sheaf of $f^{-1}\mathcal{O}_Y$-modules on the étale site. By Derived Categories of Spaces, Lemma 13.4 it suffices to check this on the Zariski site. This is equivalent to being perfect for finite type morphisms of schemes by More on Morphisms, Lemma 61.11. $\square$

48. Local complete intersection morphisms

This section is the analogue of More on Morphisms, Section 62 for morphisms of schemes. The reader is encouraged to read up on local complete intersection morphisms of schemes in that section first.

The property "being a local complete intersection morphism" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 62.19 and 62.20 and Descent, Lemma 32.6. By Morphisms of Spaces, Lemma 22.1 we may define the notion of a local complete intersection morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 62 when the algebraic spaces in question are representable.

Definition 48.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is a *Koszul morphism*, or that f is a *local complete intersection morphism* if the equivalent conditions of Morphisms of Spaces, Lemma 22.1 hold with $\mathcal{P}(f) =$"f is a local complete intersection morphism".
(2) Let $x \in |X|$. We say f is *Koszul at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is a local complete intersection morphism.

In some sense the defining property of a local complete intersection morphism is the result of the following lemma.

Lemma 48.2. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Let P be an algebraic space smooth over Y. Let $U \to X$ be an étale morphism of algebraic spaces and let $i : U \to P$ an immersion of algebraic spaces over Y. Picture:*

$$\begin{tikzcd} X \arrow[rd] & U \arrow[l] \arrow[r, "i"] \arrow[d] & P \arrow[ld] \\ & Y & \end{tikzcd}$$

Then i is a Koszul-regular immersion of algebraic spaces.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme W and a surjective étale morphism $W \to P \times_Y V$. Set $U' = U \times_P W$, which is a scheme étale over U. We have to show that $U' \to W$ is a Koszul-regular immersion of schemes, see Definition 44.2. By Definition 48.1 above the morphism of schemes $U' \to V$ is a local complete intersection morphism. Hence the result follows from More on Morphisms, Lemma 62.3. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

Lemma 48.3. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then*

(1) *f is locally of finite presentation,*
(2) *f is pseudo-coherent, and*
(3) *f is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 62.4. □

Beware that a base change of a Koszul morphism is not Koszul in general.

Lemma 48.4. *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 62.6. □

Lemma 48.5. *A composition of local complete intersection morphisms is a local complete intersection morphism.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 62.7. $\square$

Lemma 48.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is flat and a local complete intersection morphism, and*
(2) *f is syntomic.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 62.8. $\square$

Lemma 48.7. *Let S be a scheme. A Koszul-regular immersion of algebraic spaces over S is a local complete intersection morphism.*

Proof. Let $i : X \to Y$ be a Koszul-regular immersion of algebraic spaces over S. By definition there exists a surjective étale morphism $V \to Y$ where V is a scheme such that $X \times_Y V$ is a scheme and the base change $X \times_Y V \to V$ is a Koszul-regular immersion of schemes. By More on Morphisms, Lemma 62.9 we see that $X \times_Y V \to V$ is a local complete intersection morphism. From Definition 48.1 we conclude that i is a local complete intersection morphism of algebraic spaces. $\square$

Lemma 48.8. *Let S be a scheme. Let*

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. Assume $Y \to Z$ is smooth and $X \to Z$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism.

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Then $U \to W$ is a local complete intersection morphism of schemes and $V \to W$ is a smooth morphism of schemes. By the result for schemes (More on Morphisms, Lemma 62.10) we conclude that $U \to V$ is a local complete intersection morphism. By definition this means that f is a local complete intersection morphism. $\square$

Lemma 48.9. *The property $\mathcal{P}(f) =$"f is a local complete intersection morphism" is fpqc local on the base.*

Proof. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}$ be an fpqc covering (Topologies on Spaces, Definition 9.1). Let $f_i : X_i \to Y_i$ be the base change of f by $Y_i \to Y$. If f is a local complete intersection morphism, then each f_i is a local complete intersection morphism by Lemma 48.4.

Conversely, assume each f_i is a local complete intersection morphism. We may replace the covering by a refinement (again because flat base change preserves the property of being a local complete intersection morphism). Hence we may assume Y_i is a scheme for each i, see Topologies on Spaces, Lemma 9.5. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. We have to show that $U \to V$ is a local complete intersection morphism of schemes. By Topologies on Spaces, Lemma 9.4 we have that $\{Y_i \times_Y V \to V\}$ is an fpqc covering of schemes. By the case of schemes (More on Morphisms, Lemma 62.19) it suffices to prove the base change

$$U \times_Y Y_i = U \times_V (V \times_Y Y_i) \longrightarrow V$$

of $U \to V$ by $V \times_Y Y_i \to V$ is a local complete intersection morphism. We can write this as the composition

$$U \times_Y Y_i \longrightarrow (V \times_Y X) \times_Y Y_i = V \times_Y X_i \longrightarrow V \times_Y Y_i$$

The first arrow is an étale morphism of schemes (as a base change of $U \to V \times_Y X$) and the second arrow is a local complete intersection morphism of schemes as a flat base change of f_i. The result follows as being a local complete intersection morphism is syntomic local on the source and since étale morphisms are syntomic (More on Morphisms, Lemma 62.20 and Morphisms, Lemma 36.10). ☐

Lemma 48.10. *The property $\mathcal{P}(f) =$"f is a local complete intersection morphism" is syntomic local on the source.*

Proof. This follows from Descent on Spaces, Lemma 14.3 and More on Morphisms, Lemma 62.20. ☐

Lemma 48.11. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces over S. Assume that both p and q are flat and locally of finite presentation. Then there exists an open subspace $U(f) \subset X$ such that $|U(f)| \subset |X|$ is the set of points where f is Koszul. Moreover, for any morphism of algebraic spaces $Z' \to Z$, if $f' : X' \to Y'$ is the base change of f by $Z' \to Z$, then $U(f')$ is the inverse image of $U(f)$ under the projection $X' \to X$.

Proof. This lemma is the analogue of More on Morphisms, Lemma 62.21 and in fact we will deduce the lemma from it. By Definition 48.1 the set $\{x \in |X| : f \text{ is Koszul at } x\}$ is open in $|X|$ hence by Properties of Spaces, Lemma 4.8 it corresponds to an open subspace $U(f)$ of X. Hence we only need to prove the final statement.

Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Finally, choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Set $V' = W' \times_W V$ and $U' = W' \times_W U$, so that we obtain surjective étale morphisms $V' \to Y'$ and $U' \to X'$. We will use without further mention an étale morphism of algebraic spaces induces an open map of associated topological spaces (see Properties of Spaces, Lemma 16.7). Note that by definition $U(f)$ is the image in $|X|$ of the set T of points in U where the morphism of schemes $U \to V$ is Koszul. Similarly, $U(f')$ is the image in $|X'|$ of the set T' of points in

U' where the morphism of schemes $U' \to V'$ is Koszul. Now, by construction the diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

is cartesian (in the category of schemes). Hence the aforementioned More on Morphisms, Lemma 62.21 applies to show that T' is the inverse image of T. Since $|U'| \to |X'|$ is surjective this implies the lemma. □

Lemma 48.12. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then f is unramified if and only if f is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on X.*

Proof. This follows from the corresponding result for morphisms of schemes, see More on Morphisms, Lemma 62.22, by étale localization, see Lemma 15.11. (Note that in the situation of this lemma the morphism $V \to U$ is unramified and a local complete intersection morphism by definition.) □

Lemma 48.13. *Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 5.6 is short exact.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a scheme V and a surjective étale morphism $V \to U \times_X Y$. Choose a scheme W and a surjective étale morphism $W \to V \times_Y Z$. By Lemma 15.11 the morphisms $W \to V$ and $V \to U$ are formally unramified. Moreover the sequence $i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$ restricts to the corresponding sequence $i^*\mathcal{C}_{V/U} \to \mathcal{C}_{W/U} \to \mathcal{C}_{W/V} \to 0$ for $W \to V \to U$. Hence the result follows from the result for schemes (More on Morphisms, Lemma 62.23) as by definition the morphism $W \to V$ is a local complete intersection morphism. □

49. When is a morphism an isomorphism?

More generally we can ask: "When does a morphism have property $\mathcal{P}$?" A more precise question is the following. Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Does there exist a monomorphism of algebraic spaces $W \to Z$ with the following two properties:

(1) the base change $f_W : X_W \to Y_W$ has property $\mathcal{P}$, and
(2) any morphism $Z' \to Z$ of algebraic spaces factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ has property $\mathcal{P}$.

In many cases, if $W \to Z$ exists, then it is an immersion, open immersion, or closed immersion.

The answer to this question may depend on auxiliary properties of the morphisms f, p, and q. An example is $\mathcal{P}(f) =$"f is flat" which we have discussed for morphisms of schemes in the case $Y = S$ in great detail in the chapter "More on Flatness", starting with More on Flatness, Section 20.

Lemma 49.1. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that p is locally of finite type and closed. Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified.

Proof. By Morphisms of Spaces, Lemma 38.10 there exists an open subspace $U(f) \subset X$ which is the set of points where f is unramified. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 4.8) with underlying set of points

$$|W| = |Z| \setminus |p| \left(|X| \setminus |U(f)|\right)$$

i.e., $z \in |Z|$ is a point of W if and only if f is unramified at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of $U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 4.9) that W has the desired universal property. $\square$

Lemma 49.2. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

1. *p is locally of finite type,*
2. *p is closed, and*
3. *$p_2 : X \times_Y X \to Z$ is closed.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified and universally injective.

Proof. After replacing Z by the open subspace found in Lemma 49.1 we may assume that f is already unramified; note that this does not destroy assumption (2) or (3). By Morphisms of Spaces, Lemma 38.9 we see that $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion. This remains true after any base change. Hence by Morphisms of Spaces, Lemma 19.2 we see that $f_{Z'}$ is universally injective if and only if the base change of the diagonal $X_{Z'} \to (X \times_Y X)_{Z'}$ is an isomorphism. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 4.8) with underlying set of points

$$|W| = |Z| \setminus |p_2| \left(|X \times_Y X| \setminus \mathrm{Im}(|\Delta_{X/Y}|)\right)$$

i.e., $z \in |Z|$ is a point of W if and only if the fibre of $|X \times_Y X| \to |Z|$ over z is in the image of $|X| \to |X \times_Y X|$. Then it is clear from the discussion above that the restriction $p^{-1}(W) \to q^{-1}(W)$ of f is unramified and universally injective.

Conversely, suppose that $f_{Z'}$ is unramified and universally injective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 4.9. Hence it suffices to prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. The discussion above shows that

$$|X_{Z'}| \longrightarrow |(X \times_Y X)_{Z'}|$$

is surjective. By Properties of Spaces, Lemma 4.3 in the commutative diagram

$$\begin{tikzcd} |X_{Z'}| \arrow[r] \arrow[d] & |(X \times_Y X)_{Z'}| \arrow[d] \\ |p|^{-1}(\{z\}) \arrow[r] & |p_2|^{-1}(\{z\}) \end{tikzcd}$$

the vertical arrows are surjective. It follows that $z \in |W|$ as desired. $\square$

Lemma 49.3. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

1. *p is locally of finite type,*
2. *p is universally closed, and*
3. *$q : Y \to Z$ is separated.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a closed immersion.

Proof. We will use the characterization of closed immersions as universally closed, unramified, and universally injective morphisms, see Lemma 14.9. First, note that since p is universally closed and q is separated, we see that f is universally closed, see Morphisms of Spaces, Lemma 40.6. It follows that any base change of f is universally closed, see Morphisms of Spaces, Lemma 9.3. Thus to finish the proof of the lemma it suffices to prove that the assumptions of Lemma 49.2 are satisfied. The projection $\text{pr}_0 : X \times_Y X \to X$ is universally closed as a base change of f, see Morphisms of Spaces, Lemma 9.3. Hence $X \times_Y X \to Z$ is universally closed as a composition of universally closed morphisms (see Morphisms of Spaces, Lemma 9.4). This finishes the proof of the lemma. $\square$

Lemma 49.4. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) *p is locally of finite presentation,*
(2) *p is flat,*
(3) *p is closed, and*
(4) *q is locally of finite type.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is flat.

Proof. By Lemma 23.6 the set

$$A = \{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 4.8) with underlying set of points

$$|W| = |Z| \setminus |p| \left(|X| \setminus A\right)$$

i.e., $z \in |Z|$ is a point of W if and only if the whole fibre of $|X| \to |Z|$ over z is contained in A. This is open because p is closed. Since the formation of A commutes with arbitrary base change it follows that W works. $\square$

Lemma 49.5. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) *p is locally of finite presentation,*
(2) *p is flat,*
(3) *p is closed,*
(4) *q is locally of finite type, and*
(5) *q is closed.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is surjective and flat.

Proof. By Lemma 49.4 we may assume that f is flat. Note that f is locally of finite presentation by Morphisms of Spaces, Lemma 28.9. Hence f is open, see Morphisms of Spaces, Lemma 30.6. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 4.8) with underlying set of points

$$|W| = |Z| \setminus |q| \left(|Y| \setminus |f|(|X|)\right).$$

in other words for $z \in |Z|$ we have $z \in |W|$ if and only if the whole fibre of $|Y| \to |Z|$ over z is in the image of $|X| \to |Y|$. Since q is closed this set is open in $|Z|$. The morphism $X_W \to Y_W$ is surjective by construction. Finally, suppose that $X_{Z'} \to Y_{Z'}$ is surjective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 4.9. Hence it suffices to prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. By Properties of Spaces, Lemma

4.3 in the commutative diagram

$$\begin{tikzcd} {|X_{Z'}|} \arrow[r] \arrow[d] & {|Y_{Z'}|} \arrow[d] \\ {|p|^{-1}(\{z\})} \arrow[r] & {|q|^{-1}(\{z\})} \end{tikzcd}$$

the vertical arrows are surjective. It follows that $z \in |W|$ as desired. □

Lemma 49.6. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

1. *p is locally of finite presentation,*
2. *p is flat,*
3. *p is universally closed,*
4. *q is locally of finite type,*
5. *q is closed, and*
6. *q is separated.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is an isomorphism.

Proof. By Lemma 49.5 there exists an open subspace $W_1 \subset Z$ such that $f_{Z'}$ is surjective and flat if and only if $Z' \to Z$ factors through W_1. By Lemma 49.3 there exists an open subspace $W_2 \subset Z$ such that $f_{Z'}$ is a closed immersion if and only if $Z' \to Z$ factors through W_2. We claim that $W = W_1 \cap W_2$ works. Certainly, if $f_{Z'}$ is an isomorphism, then $Z' \to Z$ factors through W. Hence it suffices to show that f_W is an isomorphism. By construction f_W is a surjective flat closed immersion. In particular f_W is representable. Since a surjective flat closed immersion of schemes is an isomorphism (see Morphisms, Lemma 26.1) we win. (Note that actually f_W is locally of finite presentation, whence open, so you can avoid the use of this lemma if you like.) □

Lemma 49.7. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

1. *p is flat and locally of finite presentation,*
2. *p is closed, and*
3. *q is flat and locally of finite presentation,*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a local complete intersection morphism.

Proof. By Lemma 48.11 there exists an open subspace $U(f) \subset X$ which is the set of points where f is Koszul. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 4.8) with underlying set of points

$$|W| = |Z| \setminus |p|\left(|X| \setminus |U(f)|\right)$$

i.e., $z \in |Z|$ is a point of W if and only if f is Koszul at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of $U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 4.9) that W has the desired universal property. □

50. Exact sequences of differentials and conormal sheaves

In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to composable morphisms of algebraic spaces.

In the sequences below each of the maps are as constructed in either Lemma 7.6 or Lemma 15.8. Let S be a scheme. Let $g : Z \to Y$ and $f : Y \to X$ be morphisms of algebraic spaces over S.

(1) There is a canonical exact sequence

$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$

see Lemma 7.8. If $g : Z \to Y$ is formally smooth, then this sequence is a short exact sequence, see Lemma 19.12.

(2) If g is formally unramified, then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$

see Lemma 15.13. If $f \circ g : Z \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 19.13.

(3) if g and $f \circ g$ are formally unramified, then there is a canonical exact sequence

$$\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to 0,$$

see Lemma 15.14. If $f : Y \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 19.14.

(4) if g and f are formally unramified, then there is a canonical exact sequence

$$g^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0.$$

see Lemma 15.15. If $g : Z \to Y$ is a local complete intersection morphism, then this sequence is a short exact sequence, see Lemma 48.13.

51. Characterizing pseudo-coherent complexes, II

In this section we discuss a characterization of pseudo-coherent complexes in terms of cohomology. Earlier material on pseudo-coherent complexes on algebraic spaces may be found in Derived Categories of Spaces, Section 13 and in Derived Categories of Spaces, Section 18. The analogue of this section for schemes is More on Morphisms, Section 69. A basic tool will be to reduce to the case of projective space using a derived version of Chow's lemma, see Lemma 51.2.

Lemma 51.1. *Let S be a scheme. Consider a commutative diagram of algebraic spaces*

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X' \arrow[r] & B' \end{tikzcd}$$

over S. Let $B \to B'$ be a morphism. Denote by X and Y the base changes of X' and Y' to B. Assume $Y' \to B'$ and $Z' \to X'$ are flat. Then $X \times_B Y$ and Z' are Tor independent over $X' \times_{B'} Y'$.

Proof. By Derived Categories of Spaces, Lemma 20.3 we may check tor independence étale locally on $X \times_B Y$ and Z'. This[3] reduces the lemma to the case of schemes which is More on Morphisms, Lemma 69.1. □

Lemma 51.2 (Derived Chow's lemma)**.** *Let A be a ring. Let X be a separated algebraic space of finite presentation over A. Let $x \in |X|$. Then there exist an $n \geq 0$, a closed subspace $Z \subset X \times_A \mathbf{P}^n_A$, a point $z \in |Z|$, an open $V \subset \mathbf{P}^n_A$, and an object E in $D(\mathcal{O}_{X \times_A \mathbf{P}^n_A})$ such that*

1. *$Z \to X \times_A \mathbf{P}^n_A$ is of finite presentation,*
2. *$c : Z \to \mathbf{P}^n_A$ is a closed immersion over V, set $W = c^{-1}(V)$,*
3. *the restriction of $b : Z \to X$ to W is étale, $z \in W$, and $b(z) = x$,*
4. *$E|_{X \times_A V} \cong (b, c)_* \mathcal{O}_Z|_{X \times_A V}$,*
5. *E is pseudo-coherent and supported on Z.*

Proof. We can find a finite type $\mathbf{Z}$-subalgebra $A' \subset A$ and an algebraic space X' separated and of finite presentation over A' whose base change to A is X. See Limits of Spaces, Lemmas 7.1 and 6.9. Let $x' \in |X'|$ be the image of x. If we can prove the lemma for $(X'/A', x')$, then the lemma follows for $(X/A, x)$. Namely, if n', Z', z', V', E' provide the solution for $(X'/A', x')$, then we can let $n = n'$, let $Z \subset X \times \mathbf{P}^n$ be the inverse image of Z', let $z \in Z$ be the unique point mapping to x, let $V \subset \mathbf{P}^n_A$ be the inverse image of V', and let E be the derived pullback of E'. Observe that E is pseudo-coherent by Cohomology on Sites, Lemma 45.3. It only remains to check (5). To see this set $W = c^{-1}(V)$ and $W' = (c')^{-1}(V')$ and consider the cartesian square

$$\begin{tikzcd} W \arrow[r] \arrow[d, "{(b,c)}"'] & W' \arrow[d, "{(b',c')}"] \\ X \times_A V \arrow[r] & X' \times_{A'} V' \end{tikzcd}$$

[3]Here is the argument in more detail. Choose a surjective étale morphism $W' \to B'$ with W' a scheme. Choose a surjective étale morphism $W \to B \times_{B'} W'$ with W a scheme. Choose a surjective étale morphism $U' \to X' \times_{B'} W'$ with U' a scheme. Choose a surjective étale morphism $V' \to Y' \times_{B'} W'$ with V' a scheme. Observe that $U' \times_{W'} V' \to X' \times_{B'} Y'$ is surjective étale. Choose a surjective étale morphism $T' \to Z' \times_{X' \times_{B'} Y'} U' \times_{W'} V'$ with T' a scheme. Denote U and V the base changes of U' and V' to W. Then the lemma says that $X \times_B Y$ and Z' are Tor independent over $X' \times_{B'} Y'$ as algebraic spaces if and only if $U \times_W V$ and T' are Tor independent over $U' \times_{W'} V'$ as schemes. Thus it suffices to prove the lemma for the square with corners T', U', V', W' and base change by $W \to W'$. The flatness of $Y' \to B'$ and $Z' \to X'$ implies flatness of $V' \to W'$ and $T' \to U'$.

By Lemma 51.1 $X \times_A V$ and W' are tor-independent over $X' \times_{A'} V'$. Thus the derived pullback of $(b', c')_*\mathcal{O}_{W'}$ to $X \times_A V$ is $(b, c)_*\mathcal{O}_W$ by Derived Categories of Spaces, Lemma 20.4. This also uses that $R(b', c')_*\mathcal{O}_{Z'} = (b', c')_*\mathcal{O}_{Z'}$ because (b', c') is a closed immersion and simiarly for $(b, c)_*\mathcal{O}_Z$. Since $E'|_{U' \times_{A'} V'} = (b', c')_*\mathcal{O}_{W'}$ we obtain $E|_{U \times_A V} = (b, c)_*\mathcal{O}_W$ and (5) holds. This reduces us to the situation described in the next paragraph.

Assume A is of finite type over $\mathbf{Z}$. Choose an étale morphism $U \to X$ where U is an affine scheme and a point $u \in U$ mapping to x. Then U is of finite type over A. Choose a closed immersion $U \to \mathbf{A}^n_A$ and denote $j : U \to \mathbf{P}^n_A$ the immersion we get by composing with the open immersion $\mathbf{A}^n_A \to \mathbf{P}^n_A$. Let Z be the scheme theoretic closure of

$$(\mathrm{id}_U, j) : U \longrightarrow X \times_A \mathbf{P}^n_A$$

Let $z \in Z$ be the image of u. Let $Y \subset \mathbf{P}^n_A$ be the scheme theoretic closure of j. Then it is clear that $Z \subset X \times_A Y$ is the scheme theoretic closure of $(\mathrm{id}_U, j) : U \to X \times_A Y$. As X is separated, the morphism $X \times_A Y \to Y$ is separated as well. Hence we see that $Z \to Y$ is an isomorphism over the open subscheme $j(U) \subset Y$ by Morphisms of Spaces, Lemma 16.7. Choose $V \subset \mathbf{P}^n_A$ open with $V \cap Y = j(U)$. Then we see that (2) holds, that $W = (\mathrm{id}_U, j)(U)$, and hence that (3) holds. Part (1) holds because A is Noetherian.

Because A is Noetherian we see that X and $X \times_A \mathbf{P}^n_A$ are Noetherian algebraic spaces. Hence we can take $E = (b, c)_*\mathcal{O}_Z$ in this case: (4) is clear and for (5) see Derived Categories of Spaces, Lemma 13.7. This finishes the proof. □

Lemma 51.3. *Let X/A, $x \in |X|$, and n, Z, z, V, E be as in Lemma 51.2. For any $K \in D_{QCoh}(\mathcal{O}_X)$ we have*

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = R(W \to V)_*K|_W$$

where $p : X \times_A \mathbf{P}^n_A \to X$ and $q : X \times_A \mathbf{P}^n_A \to \mathbf{P}^n_A$ are the projections and where the morphism $W \to V$ is the finitely presented closed immersion $c|_W : W \to V$.

Proof. Since $W = c^{-1}(V)$ and since c is a closed immersion over V, we see that $c|_W$ is a closed immersion. It is of finite presentation because W and V are of finite presentation over A, see Morphisms of Spaces, Lemma 28.9. First we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right)$$

where $q' : X \times_A V \to V$ is the projection because formation of total direct image commutes with localization. Denote $i = (b, c)|_W : W \to X \times_A V$ the given closed immersion. Then

$$Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right) = Rq'_*(Lp^*K|_{X \times_A V} \otimes^{\mathbf{L}} i_*\mathcal{O}_W)$$

by property (5). Since i is a closed immersion we have $i_*\mathcal{O}_W = Ri_*\mathcal{O}_W$. Using Derived Categories of Spaces, Lemma 20.1 we can rewrite this as

$$Rq'_*Ri_*Li^*Lp^*K|_{X \times_A V} = R(q' \circ i)_*Lb^*K|_W = R(W \to V)_*K|_W$$

which is what we want. (Note that restricting to W and derived pulling back via $W \to X$ is the same thing as W is étale over X.) □

Lemma 51.4. *Let A be a ring. Let X be an algebraic space separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent* *in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A (Definition 45.3).*

Proof. Assume $K \in D_{QCoh}(\mathcal{O}_X)$ and $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$. Let $x \in |X|$. We will show that K is pseudo-coherent relative to A in an étale neighbourhood of x. This will prove the lemma by our definition of relative pseudo-coherence.

Choose n, Z, z, V, E as in Lemma 51.2. Denote $p : X \times \mathbf{P}^n \to X$ and $q : X \times \mathbf{P}^n \to \mathbf{P}^n_A$ the projections. Then for any $i \in \mathbf{Z}$ we have

$$\begin{aligned} & R\Gamma(\mathbf{P}^n_A, Rq_*(Lp^*K \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}^n_A}(i)) \\ & = R\Gamma(X \times \mathbf{P}^n, Lp^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i)) \\ & = R\Gamma(X, K \otimes^{\mathbf{L}} Rq_*(E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i))) \end{aligned}$$

by Derived Categories of Spaces, Lemma 20.1. By Derived Categories of Spaces, Lemma 25.5 the complex $Rq_*(E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i))$ is pseudo-coherent on X. Hence the assumption tells us the expression in the displayed formula is a pseudo-coherent object of $D(A)$. By Derived Categories of Schemes, Lemma 34.2 we conclude that $Rq_*(Lp^*K \otimes^{\mathbf{L}} E)$ is pseudo-coherent on $\mathbf{P}^n_A$. By Lemma 51.3 we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V} = R(W \to V)_*K|_W$$

Since $W \to V$ is a closed immersion into an open subscheme of $\mathbf{P}^n_A$ this means $K|_W$ is pseudo-coherent relative to A for example by More on Morphisms, Lemma 59.18. $\square$

Lemma 51.5. *Let A be a ring. Let X be an algebraic space separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A.*

Proof. In view of Lemma 51.4, it suffices to show $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent $E \in D(\mathcal{O}_X)$. By Derived Categories of Spaces, Proposition 29.3 it follows that $K \in D^-_{QCoh}(\mathcal{O}_X)$. Now the result follows by Derived Categories of Spaces, Lemma 25.7. $\square$

52. Relatively perfect objects

In this section we introduce a notion from [Lie06]. This notion has been discussed for morphisms of schemes in Derived Categories of Schemes, Section 35.

Definition 52.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. An object E of $D(\mathcal{O}_X)$ is *perfect relative to* Y or Y*-perfect* if E is pseudo-coherent (Cohomology on Sites, Definition 45.1) and E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_Y)$ (Cohomology on Sites, Definition 46.1).

Please see Derived Categories of Schemes, Remark 35.14 for a discussion; here we just mention that E being pseudo-coherent is the same thing as E being pseudo-coherent relative to Y by Lemma 45.4. Moreover, pseudo-coherence of E implies $E \in D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Spaces, Lemma 13.6.

Example 52.2. Let k be a field. Let X be an algebraic space of finite presentation over k (in particular X is quasi-compact). Then an object E of $D(\mathcal{O}_X)$ is k-perfect if and only if it is bounded and pseudo-coherent (by definition), i.e., if and only if it is in $D^b_{Coh}(X)$ (by Derived Categories of Spaces, Lemma 13.7). Thus being relatively perfect does **not** mean "perfect on the fibres".

The corresponding algebra concept is studied in More on Algebra, Section 83. We can link the notion for algebraic spaces with the algebraic notion as follows.

Lemma 52.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent:*

1. *E is Y-perfect,*
2. *for every commutative diagram*
$$\begin{tikzcd} U \arrow[r, "g"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
where U, V are schemes and the vertical arrows are étale, the complex $E|_U$ is V-perfect in the sense of Derived Categories of Schemes, Definition 35.1,
3. *for some commutative diagram as in (2) with $U \to X$ surjective, the complex $E|_U$ is V-perfect in the sense of Derived Categories of Schemes, Definition 35.1,*
4. *for every commutative diagram as in (2) with U and V affine the complex $R\Gamma(U, E)$ is $\mathcal{O}_Y(V)$-perfect.*

Proof. To make sense of parts (2), (3), (4) of the lemma, observe that the object $E|_U$ of $D_{QCoh}(\mathcal{O}_U)$ corresponds to an object E_0 of $D_{QCoh}(\mathcal{O}_{U_0})$ where U_0 denotes the scheme underlying U, see Derived Categories of Spaces, Lemma 4.2. Moreover, in this case E_0 is pseudo-coherent if and only if $E|_U$ is pseudo-coherent, see Derived Categories of Spaces, Lemma 13.2. Also, $E|_U$ locally has finite tor dimension over $f^{-1}\mathcal{O}_Y|_U = g^{-1}\mathcal{O}_V$ if and only if E_0 locally has finite tor dimension over $g_0^{-1}\mathcal{O}_{V_0}$ by Derived Categories of Spaces, Lemma 13.4. Here $g_0 : U_0 \to V_0$ is the morphism of schemes representing $g : U \to V$ (notation as in Derived Categories of Spaces, Remark 6.3). Finally, observe that "being pseudo-coherent" is étale local and of course "having locally finite tor dimension" is étale local. Thus we see that it suffices to check Y-perfectness étale locally and by the above discussion we see that (1) implies (2) and (3) implies (1). Since part (4) is equivalent to (2) and (3) by Derived Categories of Schemes, Lemma 35.3 the proof is complete. □

Lemma 52.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. The full subcategory of $D(\mathcal{O}_X)$ consisting of Y-perfect objects is a saturated[4] triangulated subcategory.*

Proof. This follows from Cohomology on Sites, Lemmas 45.4, 45.6, 46.6, and 46.8. □

Lemma 52.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. A perfect object of $D(\mathcal{O}_X)$ is Y-perfect. If $K, M \in D(\mathcal{O}_X)$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$ is Y-perfect if K is perfect and M is Y-perfect.*

[4] Derived Categories, Definition 6.1.

Proof. Reduce to the case of schemes using Lemma 52.3 and then apply Derived Categories of Schemes, Lemma 35.5. ☐

Lemma 52.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Set $X' = Y' \times_Y X$ and denote $g' : X' \to X$ the projection. If $K \in D(\mathcal{O}_X)$ is Y-perfect, then $L(g')^*K$ is Y'-perfect.*

Proof. Reduce to the case of schemes using Lemma 52.3 and then apply Derived Categories of Schemes, Lemma 35.6. ☐

Situation 52.7. Let S be a scheme. Let $Y = \lim_{i \in I} Y_i$ be a limit of a directed system of algebraic spaces over S with affine transition morphisms $g_{i'i} : Y_{i'} \to Y_i$. We assume that Y_i is quasi-compact and quasi-separated for all $i \in I$. We denote $g_i : Y \to Y_i$ the projection. We fix an element $0 \in I$ and a flat morphism of finite presentation $X_0 \to Y_0$. We set $X_i = Y_i \times_{Y_0} X_0$ and $X = Y \times_{Y_0} X_0$ and we denote the transition morphisms $f_{i'i} : X_{i'} \to X_i$ and $f_i : X \to X_i$ the projections.

Lemma 52.8. *In Situation 52.7. Let K_0 and L_0 be objects of $D(\mathcal{O}_{X_0})$. Set $K_i = Lf_{i0}^*K_0$ and $L_i = Lf_{i0}^*L_0$ for $i \geq 0$ and set $K = Lf_0^*K_0$ and $L = Lf_0^*L_0$. Then the map*

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{X_i})}(K_i, L_i) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L)$$

is an isomorphism if K_0 is pseudo-coherent and $L_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ has (locally) finite tor dimension as an object of $D((X_0 \to Y_0)^{-1}\mathcal{O}_{Y_0})$

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{U_i})}(K_i|_{U_i}, L_i|_{U_i}) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U, L|_U)$$

is an isomorphism where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We will prove P holds for each U_0.

Suppose that $(U_0 \subset W_0, V_0 \to W_0)$ is an elementary distinguished square in $(X_0)_{spaces,étale}$ and P holds for $U_0, V_0, U_0 \times_{W_0} V_0$. Then P holds for W_0 by Mayer-Vietoris for hom in the derived category, see Derived Categories of Spaces, Lemma 10.4.

We first consider $U_0 = W_0 \times_{Y_0} X_0$ with W_0 a quasi-compact and quasi-separated object of $(Y_0)_{spaces,étale}$. By the induction principle of Derived Categories of Spaces, Lemma 9.3 applied to these W_0 and the previous paragraph, we find that it is enough to prove P for $U_0 = W_0 \times_{Y_0} X_0$ with W_0 affine. In other words, we have reduced to the case where Y_0 is affine. Next, we apply the induction principle again, this time to all quasi-compact and quasi-separated opens of X_0, to reduce to the case where X_0 is affine as well.

If X_0 and Y_0 are affine, then we are back in the case of schemes which is proved in Derived Categories of Schemes, Lemma 35.8. The reader may use Derived Categories of Spaces, Lemmas 13.6, 4.2, 13.2, and 13.4 to accomplish the translation of the statement into a statement involving only schemes and derived categories of modules on schemes. ☐

Lemma 52.9. *In Situation 52.7 the category of Y-perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of Y_i-perfect objects of $D(\mathcal{O}_{X_i})$.*

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that the functor

$$\operatorname{colim}_{i \geq 0} D_{Y_i\text{-perfect}}(\mathcal{O}_{U_i}) \longrightarrow D_{Y\text{-perfect}}(\mathcal{O}_U)$$

is an equivalence where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We observe that we already know this functor is fully faithful by Lemma 52.8. Thus it suffices to prove essential surjectivity.

Suppose that $(U_0 \subset W_0, V_0 \to W_0)$ is an elementary distinguished square in $(X_0)_{spaces,étale}$ and P holds for $U_0, V_0, U_0 \times_{W_0} V_0$. We claim that P holds for W_0. We will use the notation $U_i = X_i \times_{X_0} U_0$, $U = X \times_{X_0} U_0$, and similarly for V_0 and W_0. We will abusively use the symbol f_i for all the morphisms $U \to U_i$, $V \to V_i$, $U \times_W V \to U_i \times_{W_i} V_i$, and $W \to W_i$. Suppose E is an Y-perfect object of $D(\mathcal{O}_W)$. Goal: show E is in the essential image of the functor. By assumption, we can find $i \geq 0$, an Y_i-perfect object $E_{U,i}$ on U_i, an Y_i-perfect object $E_{V,i}$ on V_i, and isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. Let

$$a : E_{U,i} \to (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \times_{W_i} V_i} \to E_{V,i}|_{U_i \times_{W_i} V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U \times_W V} \to E|_{U \times_W V} \to Lf_i^*E_{V,i}|_{U \times_W V}.$$

Apply Derived Categories of Spaces, Lemma 10.8 to get an object E_i on W_i and a map $d : E_i \to Rf_{i,*}E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is Y_i-perfect (because being relatively perfect is an étale local property) and that d is adjoint to an isomorphism $Lf_i^*E_i \to E$.

By exactly the same argument as used in the proof of Lemma 52.8 using the induction principle (Derived Categories of Spaces, Lemma 9.3) we reduce to the case where both X_0 and Y_0 are affine: first work with quasi-compact and quasi-separated objects in $(Y_0)_{spaces,étale}$ to reduce to Y_0 affine, then work with quasi-compact and quasi-separated object in $(X_0)_{spaces,étale}$ to reduce to X_0 affine. In the affine case the result follows from the case of schemes which is Derived Categories of Schemes, Lemma 35.9. The translation into the case for schemes is done by Lemma 52.3. $\square$

Lemma 52.10. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat, proper, and of finite presentation. Let $E \in D(\mathcal{O}_X)$ be Y-perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.*

Proof. The statement on base change is Derived Categories of Spaces, Lemma 21.4 (with $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0). Thus it suffices to show that Rf_*E is a perfect object. We will reduce to the case where Y is Noetherian affine by a limit argument.

The question is étale local on Y, hence we may assume Y is affine. Say $Y = \operatorname{Spec}(R)$. We write $R = \operatorname{colim} R_i$ as a filtered colimit of Noetherian rings R_i. By Limits of Spaces, Lemma 7.1 there exists an i and an algebraic space X_i of finite presentation over R_i whose base change to R is X. By Limits of Spaces, Lemmas 6.13 and 6.12 we may assume X_i is proper and flat over R_i. By Lemma 52.9 we may assume there exists a R_i-perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying

Derived Categories of Spaces, Lemma 22.1 to $X_i \to \mathrm{Spec}(R_i)$ and E_i and using the base change property already shown we obtain the result. $\square$

Lemma 52.11. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $E, K \in D(\mathcal{O}_X)$. Assume*

1. *Y is quasi-compact and quasi-separated,*
2. *f is proper, flat, and of finite presentation,*
3. *E is Y-perfect,*
4. *K is pseudo-coherent.*

Then there exists a pseudo-coherent $L \in D(\mathcal{O}_Y)$ such that

$$Rf_* R\mathcal{H}om(K, E) = R\mathcal{H}om(L, \mathcal{O}_Y)$$

and the same is true after arbitrary base change: given

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

cartesian, then we have

$$Rf'_* R\mathcal{H}om(L(g')^*K, L(g')^*E) = R\mathcal{H}om(Lg^*L, \mathcal{O}_{Y'})$$

Proof. Since Y is quasi-compact and quasi-separated, the same is true for X. By Derived Categories of Spaces, Lemma 18.1 we can write $K = \mathrm{hocolim} K_n$ with K_n perfect and $K_n \to K$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $K_n^\vee$ be the dual perfect complex (Cohomology on Sites, Lemma 48.4). We obtain an inverse system $\ldots \to K_3^\vee \to K_2^\vee \to K_1^\vee$ of perfect objects. By Lemma 52.5 we see that $K_n^\vee \otimes_{\mathcal{O}_X} E$ is Y-perfect. Thus we may apply Lemma 52.10 to $K_n^\vee \otimes_{\mathcal{O}_X} E$ and we obtain an inverse system

$$\ldots \to M_3 \to M_2 \to M_1$$

of perfect complexes on Y with

$$M_n = Rf_*(K_n^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = Rf_* R\mathcal{H}om(K_n, E)$$

Moreover, the formation of these complexes commutes with any base change, namely $Lg^*M_n = Rf'_*((L(g')^*K_n)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} L(g')^*E) = Rf'_* R\mathcal{H}om(L(g')^*K_n, L(g')^*E)$.

As $K_n \to K$ induces an isomorphism on $\tau_{\geq -n}$, we see that $K_n \to K_{n+1}$ induces an isomorphism on $\tau_{\geq -n}$. It follows that $K_{n+1}^\vee \to K_n^\vee$ induces an isomorphism on $\tau_{\leq n}$ as $K_n^\vee = R\mathcal{H}om(K_n, \mathcal{O}_X)$. Suppose that E has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_Y$-modules. Then the same is true after any base change, see Derived Categories of Spaces, Lemma 20.7. We find that $K_{n+1}^\vee \otimes_{\mathcal{O}_X} E \to K_n^\vee \otimes_{\mathcal{O}_X} E$ induces an isomorphism on $\tau_{\leq n+a}$ and the same is true after any base change. Applying the right derived functor Rf_* we conclude the maps $M_{n+1} \to M_n$ induce isomorphisms on $\tau_{\leq n+a}$ and the same is true after any base change. Choose a distinguished triangle

$$M_{n+1} \to M_n \to C_n \to M_{n+1}[1]$$

Pick $y \in |Y|$. Choose an elementary étale neighbourhood $(U, u) \to (Y, y)$; this is possible by Decent Spaces, Lemma 11.4. Take Y' equal to the spectrum of the residue field at u. Pull back to see that $C_n|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \kappa(u)$ has nonzero cohomology only in degrees $\geq n + a$. By More on Algebra, Lemma 75.6 we see that the perfect complex $C_n|_U$ has tor amplitude in $[n + a, m_n]$ for some integer m_n and after possibly shrinking U. Thus C_n has tor amplitude in $[n + a, m_n]$ for some integer

m_n (because Y is quasi-compact). In particular, the dual perfect complex $C_n^\vee$ has tor amplitude in $[-m_n, -n-a]$.

Let $L_n = M_n^\vee$ be the dual perfect complex. The conclusion from the discussion in the previous paragraph is that $L_n \to L_{n+1}$ induces isomorphisms on $\tau_{\geq -n-a}$. Thus $L = \text{hocolim} L_n$ is pseudo-coherent, see Derived Categories of Spaces, Lemma 18.1. Since we have

$$R\mathcal{H}om(K, E) = R\mathcal{H}om(\text{hocolim} K_n, E) = R\lim R\mathcal{H}om(K_n, E) = R\lim K_n^\vee \otimes_{\mathcal{O}_X} E$$

(Cohomology on Sites, Lemma 48.8) and since $R\lim$ commutes with Rf_* we find that

$$Rf_* R\mathcal{H}om(K, E) = R\lim M_n = R\lim R\mathcal{H}om(L_n, \mathcal{O}_Y) = R\mathcal{H}om(L, \mathcal{O}_Y)$$

This proves the formula over Y. Since the construction of M_n is compatible with base chance, the formula continues to hold after any base change. □

Remark 52.12. The reader may have noticed the similarity between Lemma 52.11 and Derived Categories of Spaces, Lemma 23.3. Indeed, the pseudo-coherent complex L of Lemma 52.11 may be characterized as the unique pseudo-coherent complex on Y such that there are functorial isomorphisms

$$\text{Ext}^i_{\mathcal{O}_Y}(L, \mathcal{F}) \longrightarrow \text{Ext}^i_{\mathcal{O}_X}(K, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})$$

compatible with boundary maps for $\mathcal{F}$ ranging over $QCoh(\mathcal{O}_Y)$. If we ever need this we will formulate a precise result here and give a detailed proof.

Lemma 52.13. *Let S be a scheme. Let X be an algebraic space over S such that the structure morphism $f : X \to S$ is flat and locally of finite presentation. Let E be a pseudo-coherent object of $D(\mathcal{O}_X)$. The following are equivalent*

1. *E is S-perfect, and*
2. *E is locally bounded below and for every point $s \in S$ the object $L(X_s \to X)^*E$ of $D(\mathcal{O}_{X_s})$ is locally bounded below.*

Proof. Since everything is local we immediately reduce to the case that X and S are affine, see Lemma 52.3. This case is handled by Derived Categories of Schemes, Lemma 35.13. □

Lemma 52.14. *Let A be a ring. Let X be an algebraic space separated, of finite presentation, and flat over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is perfect in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is* $\text{Spec}(A)$*-perfect.*

Proof. By Lemma 51.5, K is pseudo-coherent relative to A. By Lemma 45.4, K is pseudo-coherent in $D(\mathcal{O}_X)$. By Derived Categories of Spaces, Proposition 29.4 we see that K is in $D^-(\mathcal{O}_X)$. Let $\mathfrak{p}$ be a prime ideal of A and denote $i : Y \to X$ the inclusion of the scheme theoretic fibre over $\mathfrak{p}$, i.e., Y is a scheme over $\kappa(\mathfrak{p})$. By Lemma 52.13, we will be done if we can show $Li^*(K)$ is bounded below. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Spaces, Theorem 15.4. We have

$$\begin{aligned} R\,\text{Hom}_{\mathcal{O}_Y}(Li^*(G), Li^*(K)) &= R\Gamma(Y, Li^*(G^\vee \otimes^{\mathbf{L}} K)) \\ &= R\Gamma(X, G^\vee \otimes^{\mathbf{L}} K) \otimes^{\mathbf{L}}_A \kappa(\mathfrak{p}) \end{aligned}$$

The first equality uses that Li^* preserves perfect objects and duals and Cohomology on Sites, Lemma 48.4; we omit some details. The second equality follows from

Derived Categories of Spaces, Lemma 20.4 as X is flat over A. It follows from our hypothesis that this is a perfect object of $D(\kappa(\mathfrak{p}))$. The object $Li^*(G) \in D_{perf}(\mathcal{O}_Y)$ generates $D_{QCoh}(\mathcal{O}_Y)$ by Derived Categories of Spaces, Remark 15.5. Hence Derived Categories of Spaces, Proposition 29.4 now implies that $Li^*(K)$ is bounded below and we win. $\square$

53. Theorem of the cube

This section is the analogue of More on Morphisms, Section 33. The following lemma tells us that the diagonal of the Picard functor is representable by locally closed immersions under the assumptions made in the lemma.

Lemma 53.1. *Let S be a scheme. Let $f : X \to Y$ be a flat, proper morphism of finite presentation of algebraic spaces over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. For a morphism $g : Y' \to Y$ consider the base change diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Assume $\mathcal{O}_{Y'} \to f'_*\mathcal{O}_{X'}$ is an isomorphism for all $g : Y' \to Y$. Then there exists an immersion $j : Z \to Y$ of finite presentation such that a morphism $g : Y' \to Y$ factors through Z if and only if there exists a finite locally free $\mathcal{O}_{Y'}$-module $\mathcal{N}$ with $(f')^*\mathcal{N} \cong (g')^*\mathcal{L}$.

Proof. Let $y : \mathrm{Spec}(k) \to Y$ be a field valued point. Then the fibre X_y of f at y is connected by our assumption that $H^0(X_y, \mathcal{O}_{X_y}) = k$. Thus the rank of $\mathcal{E}$ is constant on the fibres. Since f is open (Morphisms of Spaces, Lemma 30.6) and closed we conclude that there is a decomposition $Y = \coprod Y_r$ of Y into open and closed subspaces such that $\mathcal{E}$ has constant rank r on the inverse image of Y_r. Thus we may assume $\mathcal{E}$ has constant rank r. We will denote $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ the dual rank r module.

By cohomology and base change (more precisely by Derived Categories of Spaces, Lemma 25.4) we see that $E = Rf_*\mathcal{E}$ is a perfect object of the derived category of Y and that its formation commutes with arbitrary change of base. Similarly for $E' = Rf_*\mathcal{E}^\vee$. Since there is never any cohomology in degrees < 0, we see that E and E' have (locally) tor-amplitude in $[0, b]$ for some b. Observe that for any $g : Y' \to Y$ we have $f'_*((g')^*\mathcal{E}) = H^0(Lg^*E)$ and $f'_*((g')^*\mathcal{E}^\vee) = H^0(Lg^*E')$. Let $j : Z \to Y$ and $j' : Z' \to Y$ be the locally closed immersions constructed in Derived Categories of Spaces, Lemma 26.6 for E and E' with $a = 0$ and $r = r$; these are characterized by the property that $H^0(Lj^*E)$ and $H^0((j')^*E')$ are locally free modules of rank r compatible with pullback.

Let $g : Y' \to Y$ be a morphism. If there exists an $\mathcal{N}$ as in the lemma, then, using the projection formula Cohomology on Sites, Lemma 50.1, we see that the modules

$$f'_*((g')^*\mathcal{E}) \cong f'_*((f')^*\mathcal{N}) \cong \mathcal{N} \otimes_{\mathcal{O}_{Y'}} f'_*\mathcal{O}_{X'} \cong \mathcal{N} \quad \text{and similarly} \quad f'_*((g')^*\mathcal{E}^\vee) \cong \mathcal{N}^\vee$$

are locally free of rank r and remain locally free of rank r after any further base change $Y'' \to Y'$. Hence in this case $g : Y' \to Y$ factors through j and through j'. Thus we may replace Y by $Z \times_Y Z'$ and assume that $f_*\mathcal{E}$ and $f_*\mathcal{E}^\vee$ are locally free $\mathcal{O}_Y$-modules of rank r whose formation commutes with arbitrary change of base.

In this sitation if $g : Y' \to Y$ is a morphism and there exists an $\mathcal{N}$ as in the lemma, then the map (cup product in degree 0)

$$f'_*((g')^*\mathcal{E}) \otimes_{\mathcal{O}_{Y'}} f'_*((g')^*\mathcal{E}^\vee) \longrightarrow \mathcal{O}_{Y'}$$

is a perfect pairing. Conversely, if this cup product map is a perfect pairing, then we see that locally on Y' we have a basis of sections $\sigma_1, \ldots, \sigma_r$ in $f'_*((g')^*\mathcal{L})$ and $\tau_1, \ldots, \tau_r$ in $f'_*((g')^*\mathcal{E}^\vee)$ whose products satisfy $\sigma_i \tau_j = \delta_{ij}$. Thinking of σ_i as a section of $(g')^*\mathcal{L}$ on X' and τ_j as a section of $(g')^*\mathcal{L}^\vee$ on X', we conclude that

$$\sigma_1, \ldots, \sigma_r : \mathcal{O}_{X'}^{\oplus r} \longrightarrow (g')^*\mathcal{E}$$

is an isomorphism with inverse given by

$$\tau_1, \ldots, \tau_r : (g')^*\mathcal{E} \longrightarrow \mathcal{O}_{X'}^{\oplus r}$$

In other words, we see that $(f')^* f'_*(g')^*\mathcal{E} \cong (g')^*\mathcal{E}$. But the condition that the cupproduct is nondegenerate picks out a retrocompact open subscheme (namely, the locus where a suitable determinant is nonzero) and the proof is complete. $\square$

54. Descent of finiteness properties of complexes

This section is the analogue of More on Morphisms, Section 70 and Derived Categories of Schemes, Section 12.

Lemma 54.1. *Let S be a scheme. Let $\{f_i : X_i \to X\}$ be an fpqc covering of algebraic spaces over S. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent if and only if each Lf_i^*E is m-pseudo-coherent.*

Proof. Pullback always preserves m-pseudo-coherence, see Cohomology on Sites, Lemma 45.3. Thus it suffices to assume Lf_i^*E is m-pseudo-coherent and to prove that E is m-pseudo-coherent. Then first we may assume X_i is a scheme for all i, see Topologies on Spaces, Lemma 9.5. Next, choose a surjective étale morphism $U \to X$ where U is a scheme. Then $U_i = U \times_X X_i$ is a scheme and we obtain an fpqc covering $\{U_i \to U\}$ of schemes, see Topologies on Spaces, Lemma 9.4. We know the result is true for $\{U_i \to U\}_{i \in I}$ by the case for schemes, see Derived Categories of Schemes, Lemma 12.2. On the other hand, the restriction $E|_U$ comes from an object of $D_{QCoh}(\mathcal{O}_U)$ (defined using the Zariski topology and the "usual" structure sheaf of U), see Derived Categories of Spaces, Lemma 4.2. The lemma follows as the two notions of pseudo-coherent (étale and Zariski) agree by Derived Categories of Spaces, Lemma 13.2. $\square$

Lemma 54.2. *Let S be a scheme. Let $\{g_i : Y_i \to Y\}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces and set $X_i = Y_i \times_Y X$ with projections $f_i : X_i \to Y_i$ and $g'_i : X_i \to X$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$. Then the following are equivalent*

1. *E has tor amplitude in $[a, b]$ as an object of $D(f^{-1}\mathcal{O}_Y)$, and*
2. *$L(g'_i)^*E$ has tor amplitude in $[a, b]$ as a object of $D(f_i^{-1}\mathcal{O}_{Y_i})$ for all i.*

Also true if "tor amplitude in $[a, b]$" is replaced by "locally finite tor dimension".

Proof. Pullback preserves "tor amplitude in $[a, b]$" by Derived Categories of Spaces, Lemma 20.7 Observe that Y_i and X are tor independent over Y as $Y_i \to Y$ is flat. Let us assume (2) and prove (1). We can compute tor dimension at stalks, see Cohomology on Sites, Lemma 46.10 and Properties of Spaces, Theorem 19.12. Let

$\overline{x}$ be a geometric point of X. Choose an i and a geometric point $\overline{x}_i$ in X_i with image $\overline{x}$ in X. Then

$$(L(g_i')^*E)_{\overline{x}_i} = E_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{X,\overline{x}_i}$$

Let $\overline{y}_i$ in Y_i and $\overline{y}$ in Y be the image of $\overline{x}_i$ and $\overline{x}$. Since X and Y_i are tor independent over Y, we can apply More on Algebra, Lemma 61.2 to see that the right hand side of the displayed formula is equal to $E_{\overline{x}} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{Y_i,\overline{y}_i}$ in $D(\mathcal{O}_{Y_i,\overline{y}_i})$. Since we have assume the tor amplitude of this is in $[a, b]$, we conclude that the tor amplitude of $E_{\overline{x}}$ in $D(\mathcal{O}_{Y,\overline{y}})$ is in $[a, b]$ by More on Algebra, Lemma 66.17. Thus (1) follows.

Using some elementary topology the case "locally finite tor dimension" follows too. $\square$

The following lemmas do not really belong in this section.

Lemma 54.3. *Let S be a scheme. Let $i : X \to X'$ be a finite order thickening of algebraic spaces. Let $K' \in D(\mathcal{O}_{X'})$ be an object such that $K = Li^*K'$ is pseudo-coherent. Then K' is pseudo-coherent.*

Proof. We first prove K' has quasi-coherent cohomology sheaves; we urge the reader to skip this part. To do this, we may reduce to the case of a first order thickening, see Section 9. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent sheaf of ideals cutting out X. Tensoring the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to i_*\mathcal{O}_X \to 0$$

with K' we obtain a distinguished triangle

$$K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I} \to K' \to K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} i_*\mathcal{O}_X \to (K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I})[1]$$

Since $i_* = Ri_*$ and since we may view $\mathcal{I}$ as a quasi-coherent $\mathcal{O}_X$-module (as we have a first order thickening) we may rewrite this as

$$i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}) \to K' \to i_*K \to i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I})[1]$$

Please use Cohomology of Spaces, Lemma 4.4 to identify the terms. Since K is in $D_{QCoh}(\mathcal{O}_X)$ we conclude that K' is in $D_{QCoh}(\mathcal{O}_{X'})$; this uses Derived Categories of Spaces, Lemmas 13.6, 5.6, and 6.1.

Assume K' is in $D_{QCoh}(\mathcal{O}_{X'})$. The question is étale local on X' hence we may assume X' is affine. In this case the result follows from the case of schemes (More on Morphisms, Lemma 71.1). The translation into the language of schemes uses Derived Categories of Spaces, Lemmas 4.2 and 13.2 and Remark 6.3. $\square$

Lemma 54.4. *Let S be a scheme. Consider a cartesian diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

of algebraic spaces over S. Assume $X' \to Y'$ is flat and locally of finite presentation and $Y \to Y'$ is a finite order thickening. Let $E' \in D(\mathcal{O}_{X'})$. If $E = Li^*(E')$ is Y-perfect, then E' is Y'-perfect.

Proof. Recall that being Y-perfect for E means E is pseudo-coherent and locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_Y$-modules (Definition 52.1). By Lemma 54.3 we find that E' is pseudo-coherent. In particular, E' is in $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Spaces, Lemma 13.6. By Lemma 52.3 this reduces us to the case of schemes. The case of schemes is More on Morphisms, Lemma 71.2. $\square$

Lemma 54.5. *Let (R, I) be a pair consisting of a ring and an ideal I contained in the Jacobson radical. Set $S = \mathrm{Spec}(R)$ and $S_0 = \mathrm{Spec}(R/I)$. Let X be an algebraic space over R whose structure morphism $f : X \to S$ is proper, flat, and of finite presentation. Denote $X_0 = S_0 \times_S X$. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. If the derived restriction E_0 of E to X_0 is S_0-perfect, then E is S-perfect.*

Proof. Choose a surjective étale morphism $U \to X$ with U affine. Choose a closed immersion $U \to \mathbf{A}^d_S$. Set $U_0 = S_0 \times_S U$. The complex $E_0|_{U_0}$ has tor amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Let $\overline{x}$ be a geometric point of X. We will show that the tor amplitude of $E_{\overline{x}}$ over R is in $[a - d, b]$. This will finish the proof as the tor amplitude can be read off from the stalks by Cohomology on Sites, Lemma 46.10 and Properties of Spaces, Theorem 19.12.

Let $x \in |X|$ be the point determined by $\overline{x}$. Recall that $|X| \to |S|$ is closed (by definition of proper morphisms). Since I is contained in the Jacobson radical, any nonempty closed subset of S contains a point of the closed subscheme S_0. Hence we can find a specialization $x \rightsquigarrow x_0$ in $|X|$ with $x_0 \in |X_0|$. Choose $u_0 \in U_0$ mapping to x_0. By Decent Spaces, Lemma 7.4 (or by Decent Spaces, Lemma 7.3 which applies directly to étale morphisms) we find a specialization $u \rightsquigarrow u_0$ in U such that u maps to x. We may lift $\overline{x}$ to a geometric point $\overline{u}$ of U lying over u. Then we have $E_{\overline{x}} = (E|_U)_{\overline{u}}$.

Write $U = \mathrm{Spec}(A)$. Then A is a flat, finitely presented R-algebra which is a quotient of a polynomial R-algebra in d-variables. The restriction $E|_U$ corresponds (by Derived Categories of Spaces, Lemmas 13.6, 4.2, and 13.2 and Derived Categories of Schemes, Lemma 3.5 and 10.2) to a pseudo-coherent object K of $D(A)$. Observe that E_0 corresponds to $K \otimes_A^{\mathbf{L}} A/IA$. Let $\mathfrak{q} \subset \mathfrak{q}_0 \subset A$ be the prime ideals corresponding to $u \rightsquigarrow u_0$. Then

$$E_{\overline{x}} = (E|_U)_{\overline{u}} = E_u \otimes^{\mathbf{L}}_{\mathcal{O}_{U,u}} \mathcal{O}_{U,\overline{u}} = K_{\mathfrak{q}} \otimes^{\mathbf{L}}_{A_{\mathfrak{q}}} A^{sh}_{\mathfrak{q}}$$

(some details omitted). Since $A_{\mathfrak{q}} \to A^{sh}_{\mathfrak{q}}$ is flat, the tor amplitude of this as an R-module is the same as the tor amplitude of $K_{\mathfrak{q}}$ as an R-module (More on Algebra, Lemma 66.18). Also, $K_{\mathfrak{q}}$ is a localization of $K_{\mathfrak{q}_0}$. Hence it suffices to show that $K_{\mathfrak{q}_0}$ has tor amplitude in $[a - d, b]$ as a complex of R-modules.

Let $I \subset \mathfrak{p}_0 \subset R$ be the prime ideal corresponding to $f(x_0)$. Then we have

$$\begin{aligned} K \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}_0) &= (K \otimes^{\mathbf{L}}_R R/I) \otimes^{\mathbf{L}}_{R/I} \kappa(\mathfrak{p}_0) \\ &= (K \otimes^{\mathbf{L}}_A A/IA) \otimes^{\mathbf{L}}_{R/I} \kappa(\mathfrak{p}_0) \end{aligned}$$

the second equality because $R \to A$ is flat. By our choice of a, b this complex has cohomology only in degrees in the interval $[a, b]$. Thus we may finally apply More on Algebra, Lemma 83.9 to $R \to A$, $\mathfrak{q}_0$, $\mathfrak{p}_0$ and K to conclude. $\square$

55. Families of nodal curves

This section is the continuation of Algebraic Curves, Section 20. Please also see that section for our choice of terminology.

The property "at-worst-nodal of relative dimension 1" of morphisms of schemes is étale local on the source-and-target, see Descent, Lemma 32.6 and Algebraic Curves, Lemmas 20.8, 20.9, and 20.7. It is also stable under base change and fpqc local on the target, see Algebraic Curves, Lemmas 20.4 and 20.9. Hence, by Morphisms of Spaces, Lemma 22.1 we may define the notion of an at-worst-nodal morphism of relative dimension 1 for algebraic spaces as follows and it agrees with the already existing notion defined in Morphisms of Spaces, Section 3 when the morphism is representable.

Definition 55.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *at-worst-nodal of relative dimension* 1 if the equivalent conditions of Morphisms of Spaces, Lemma 22.1 hold with $\mathcal{P}$ ="at-worst-nodal of relative dimension 1".

Lemma 55.2. *The property of being at-worst-nodal of relative dimension* 1 *is preserved under base change.*

Proof. See Morphisms of Spaces, Remark 22.4 and Algebraic Curves, Lemma 20.4. ☐

Lemma 55.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is at-worst-nodal of relative dimension* 1*,*
(2) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is at-worst-nodal of relative dimension* 1*,*
(3) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is at-worst-nodal of relative dimension* 1*,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is at-worst-nodal of relative dimension* 1*,*
(5) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is at-worst-nodal of relative dimension* 1*,*
(6) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is at-worst-nodal of relative dimension 1*,*
(7) *there exists a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is at-worst-nodal of relative dimension 1*, and*

(8) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is at-worst-nodal of relative dimension* 1*.*

Proof. Omitted. □

The following lemma tells us that we can check whether a morphism is at-worst-nodal of relative dimension 1 on the fibres.

Lemma 55.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is flat and locally of finite presentation. Then there is a maximal open subspace $X' \subset X$ such that $f|_{X'} : X' \to Y$ is at-worst-nodal of relative dimension* 1*. Moreover, formation of X' commutes with arbitrary base change.*

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective. By the lemma for the case of schemes (Algebraic Curves, Lemma 20.5) we find a maximal open subscheme $U' \subset U$ such that $h|_{U'} : U' \to V$ is at-worst-nodal of relative dimension 1 and such that formation of U' commutes with base change. Let $X' \subset X$ be the open subspace whose points correspond to the open subset $\mathrm{Im}(|U'| \to |X|)$. By Lemma 55.3 we see that $X' \to Y$ is at-worst-nodal of relative dimension 1 and that X' is the largest open subspace with this property (this also implies that U' is the inverse image of X' in U, but we do not need this). Since the same is true after base change the proof is complete. □

56. The resolution property

We continue the discussion in Derived Categories of Spaces, Section 28.

Situation 56.1. Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $V \to X$ be a surjective étale morphism where V is an affine scheme (such a thing exists by Properties of Spaces, Lemma 6.3). Choose a commutative diagram

$$\begin{tikzcd} V \arrow[rr, "j"'] \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

where j is an open immersion and π is a finite morphism of algebraic spaces (such a diagram exists by Lemma 34.3). Let $\mathcal{I} \subset \mathcal{O}_Y$ be a finite type quasi-coherent sheaf of ideals on Y with $V(\mathcal{I}) = Y \setminus j(V)$ (such a sheaf of ideals exists by Limits of Spaces, Lemma 14.1).

Lemma 56.2. *In Situation 56.1, assume X is Noetherian. Then for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exist $r \geq 0$, integers $n_1, \ldots, n_r \geq 0$, and a surjection*

$$\bigoplus\nolimits_{i=1,\ldots,r} \pi_*(\mathcal{I}^{n_i}) \longrightarrow \mathcal{F}$$

of $\mathcal{O}_X$-modules.

Proof. Denote $\omega_{Y/X}$ the coherent $\mathcal{O}_Y$-module such that there is an isomorphism

$$\pi_*\omega_{Y/X} \cong \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y, \mathcal{O}_X)$$

of $\pi_*\mathcal{O}_Y$-modules, see Morphisms of Spaces, Lemma 20.10 and Descent on Spaces, Lemma 6.6. The canonical map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ produces a canonical map

$$\mathrm{Tr}_\pi : \pi_*\omega_{Y/X} \longrightarrow \mathcal{O}_X$$

Since V is Noetherian affine we may choose sections

$$s_1, \ldots, s_r \in \Gamma(V, \pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X})$$

generating the coherent module $\pi^*\mathcal{F} \otimes_{\mathcal{O}_X} \omega_{Y/X}$ over V. By Cohomology of Spaces, Lemma 13.4 we can choose integers $n_i \geq 0$ such that s_i extends to a map $s'_i : \mathcal{I}^{n_i} \to \pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$. Pushing to X we obtain maps

$$\sigma_i : \pi_*\mathcal{I}^{n_i} \xrightarrow{\pi_* s'_i} \pi_*(\pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \mathcal{F} \otimes_{\mathcal{O}_X} \pi_*\omega_{Y/X} \xrightarrow{\mathrm{Tr}_\pi} \mathcal{F}$$

where the equality sign is Cohomology of Spaces, Lemma 4.3. To finish the proof we will show that the sum of these maps is surjective.

Let $x \in |X|$ be a point of X. Let $v \in |V|$ be a point mapping to x. We may choose an étale neighbourhood $(U, u) \to (X, x)$ such that

$$U \times_X Y = W \coprod W'$$

(disjoint union of algebraic spaces) such that $W \to U$ is an isomorphism and such that the unique point $w \in W$ lying over u maps to v in $V \subset Y$. To see this is true use Lemma 33.2 and Étale Morphisms, Lemma 18.1. After shrinking U further if necessary we may assume W maps into $V \subset Y$ by the projection. Since the formation of $\omega_{Y/X}$ commutes with étale localization we see that

$$\pi_*\omega_{Y/X}|_U = (\pi|_W)_*\omega_{W/U} \oplus (\pi|_{W'})_*\omega_{W'/U}$$

We have $(\pi|_W)_*\omega_{W/U} = \mathcal{O}_U$ and this isomorphism is given by the trace map $\mathrm{Tr}_\pi|_U$ restricted to the first summand in the decomposition above. Since W maps into V we see that $\mathcal{I}^{n_i}|_W = \mathcal{O}_W$. Hence

$$\pi_*(\mathcal{I}^{n_i})|_U = \mathcal{O}_U \oplus (W' \to U)_*(\mathcal{I}^{n_i}|_{W'})$$

Chasing diagrams the reader sees (details omitted) that $\sigma_i|_U$ on the summand $\mathcal{O}_U$ is the map $\mathcal{O}_U \to \mathcal{F}$ corresponding to the section

$$s_i|_W \in \Gamma(W, \pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \Gamma(W, \mathcal{F}|_W \otimes_{\mathcal{O}_W} \omega_{W/U}) = \Gamma(U, \mathcal{F})$$

Since the sections s_i generate the module $\pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_{Y/X}$ over V and since W maps into V we conclude that the restriction of $\bigoplus \sigma_i$ to U is surjective. Since x was an arbitrary point the proof is complete. $\square$

Lemma 56.3. *In Situation 56.1, assume X is Noetherian. Then X has the resolution property if and only if $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module.*

Proof. The module $\pi_*\mathcal{I}$ is coherent by Cohomology of Spaces, Lemma 12.9. Hence if X has the resolution property then $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module. Conversely, assume given a surjection $\mathcal{E} \to \pi_*\mathcal{I}$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. Observe that for all $n \geq 1$ there is a surjection

$$\pi_*\mathcal{I} \otimes_{\mathcal{O}_X} \pi_*\mathcal{I}^n \longrightarrow \pi_*\mathcal{I}^{n+1}$$

Hence $\mathcal{E}^{\otimes n}$ surjects onto $\pi_*\mathcal{I}^n$ for all $n \geq 1$. We conclude that X has the resolution property if we combine this with the result of Lemma 56.2. $\square$

Lemma 56.4. *In Situation 56.1, the algebraic space X has the resolution property if and only if $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module.*

Proof. The pushforward $\pi_*\mathcal{G}$ of a finite type quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$ is a finite type quasi-coherent $\mathcal{O}_X$-module by Descent on Spaces, Lemma 6.6. In particular, if X has the resolution property, then $\pi_*\mathcal{I}$ is the quotient of a finite locally free $\mathcal{O}_X$-module by Derived Categories of Spaces, Definition 28.1.

Assume that we have a surjection $\mathcal{E} \to \pi_*\mathcal{I}$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. In the rest of the proof we show that X has the resolution property by reducing to the Noetherian case handled in Lemma 56.3. We suggest the reader skip the rest of the proof.

A first reduction is that we may view X as an algebraic space over $\text{Spec}(\mathbf{Z})$, see Spaces, Definition 16.2. (This doesn't affect the conditions nor the conclusion of the lemma.)

By Limits of Spaces, Lemma 11.3 we can write $Y = \lim Y_i$ with Y_i finite and of finite presentation over X and where the transition maps are closed immersions. Consider the closed subspace $Z = V(\mathcal{I})$ of Y. Since $\mathcal{I}$ is of finite type, the morphism $Z \to Y$ is of finite presentation. Hence we can find an i and a morphism $Z_i \to Y_i$ of finite presentation whose base change to Y is $Z \to Y$, see Limits of Spaces, Lemma 7.1. For $i' \geq i$ denote $Z_{i'} = Z_i \times_{Y_i} Y_{i'}$. After increasing i we may assume $Z_i \to Y_i$ is a closed immersion (of finite presentation), see Limits of Spaces, Lemma 6.8. Denote $\mathcal{I}_i \subset \mathcal{O}_{Y_i}$ the ideal sheaf of Z_i and denote $\pi_i : Y_i \to X$ the structure morphism. Similarly for $i' \geq i$. Since $Z = \lim_{i' \geq i} Z_{i'}$ we have

$$\pi_*\mathcal{I} = \text{colim}\, \pi_{i',*}\mathcal{I}_{i'}$$

The transition maps in the system are all surjective as follows from the surjectivity of the maps $\pi_{i,*}\mathcal{O}_{Y_i} \to \pi_{i',*}\mathcal{O}_{Y_{i'}}$ and the fact that $Z_{i'} = Z_i \times_{Y_i} Y_{i'}$. By Cohomology of Spaces, Lemma 5.3 for some $i' \geq i$ the map $\mathcal{E} \to \pi_*\mathcal{I}$ lifts to a map $\mathcal{E} \to \pi_{i',*}\mathcal{I}_{i'}$. After increasing i' this map $\mathcal{E} \to \pi_{i',*}\mathcal{I}_{i'}$ becomes surjective (since if not the colimit of the cokernels, having surjective transition maps, is nonzero). This reduces us to the case discussed in the next paragraph.

Assume X is an algebraic space over $\mathbf{Z}$ and that $Y \to X$ is of finite presentation. By absolute Noetherian approximation we can write $X = \lim X_i$ as a directed limit, where each X_i is a quasi-separated algebraic space of finite type over $\mathbf{Z}$ and the transition morphisms are affine, see Limits of Spaces, Proposition 8.1. Since $\pi : Y \to X$ is of finite presentation we can find an i and a morphism $\pi_i : Y_i \to X_i$ of finite presentation whose base change to X is π, see Limits of Spaces, Lemma 7.1. After increasing i we may assume π_i is finite, see Limits of Spaces, Lemma 6.7. Next, we may assume there exists a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_i$ whose pullback to X is $\mathcal{E}$, see Limits of Spaces, Lemma 7.3. We may also assume there is a map $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ whose pullback to X is the composition $\mathcal{E} \to \pi_*\mathcal{I} \to \pi_*\mathcal{O}_Y$, see Limits of Spaces, Lemma 7.2. The cokernel

$$\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i} \to \mathcal{Q}_i \to 0$$

is a coherent $\mathcal{O}_{Y_i}$-module whose pullback to X is the (finitely presented) cokernel $\mathcal{Q}$ of the map $\mathcal{E} \to \pi_*\mathcal{O}_Y$. In other words, we have $\mathcal{Q} = \pi_*(\mathcal{O}_Y/\mathcal{I})$. Consider the map

$$\mathcal{E}_i \otimes_{\mathcal{O}_{X_i}} \pi_{i,*}\mathcal{O}_{Y_i} \longrightarrow \pi_{i,*}\mathcal{O}_{Y_i} \otimes_{\mathcal{O}_{X_i}} \pi_{i,*}\mathcal{O}_{Y_i} \to \pi_{i,*}\mathcal{O}_{Y_i} \to \mathcal{Q}_i$$

where the second arrow is given by the algebra structure on $\pi_{i,*}\mathcal{O}_{Y_i}$. The pullback of this map to Y is zero because the image of $\mathcal{E} \to \pi_*\mathcal{O}_Y$ is the ideal $\pi_*\mathcal{I}$. Hence by Limits of Spaces, Lemma 7.2 after increasing i we may assume the displayed composition is zero. This exactly means that the imag of $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ is of the form $\pi_{i,*}\mathcal{I}_i$ for some coherent ideal sheaf $\mathcal{I}_i \subset \mathcal{O}_{Y_i}$. Since $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ pulls back to $\mathcal{E} \to \pi_*\mathcal{O}_Y$ we see that the pullback of $\mathcal{I}_i$ to Y generates $\mathcal{I}$. Denote $V_i \subset Y_i$ the open subspace whose complement is $V(\mathcal{I}_i) \subset Y_i$. Then V is the inverse image of V_i by the comments above. After increasing i we may assume that V_i is affine and that $\pi_i|_{V_i} : V_i \to X_i$ is étale, see Limits of Spaces, Lemmas 5.10 and 6.2. Having said all of this, we may apply Lemma 56.3 to conclude that X_i has the resolution property. Since $X \to X_i$ is affine we conclude that X has the resolution property too by Derived Categories of Spaces, Lemma 28.3. ☐

Lemma 56.5. *Let S be a scheme. Let $X = \lim X_i$ be a limit of a direct system of quasi-compact and quasi-separated algebraic spaces over S with affine transition morphisms. Then X has the resolution property if and only if X_i has the resolution properties for some i.*

Proof. If X_i has the resolution property, then X does by Derived Categories of Spaces, Lemma 28.3. Assume X has the resolution property. Choose $i \in I$. We may choose an affine scheme V_i and a surjective étale morphism $V_i \to X_i$ (Properties of Spaces, Lemma 6.3). We may choose an embedding $j : V_i \to Y_i$ with Y_i finite and finitely presented over X_i (Lemma 34.4). We may choose a finite type quasi-coherent ideal $\mathcal{I}_i \subset \mathcal{O}_{Y_i}$ such that $V_i = Y_i \setminus V(\mathcal{I}_i)$ (Limits of Spaces, Lemma 14.1). Denote $V \to Y \to X$ the base changes of $V_i \to Y_i \to X_i$ to X. Denote $\mathcal{I} \subset \mathcal{O}_Y$ the pullback of the ideal $\mathcal{I}_i$. By the easy direction of Lemma 56.4 there exists a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and a surjection $\mathcal{E} \to \pi_*\mathcal{I}$. Note that since $\pi_i : Y_i \to X_i$ is finite and of finite presentation we also have that $\pi : Y \to X$ is finite and of finite presentation and that the $\mathcal{O}_{X_i}$-modules $\pi_{i,*}\mathcal{O}_{Y_i}$ and $\pi_{i,*}(\mathcal{O}_{Y_i}/\mathcal{I}_i)$ are of finite presentation and pullback to X to give $\pi_*\mathcal{O}_Y$ and $\pi_*(\mathcal{O}_Y/\mathcal{I})$. Thus by Limits of Spaces, Lemma 7.2 after increasing i we can find a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_i$ and a map $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i}$ whose base change to X recovers the composition $\mathcal{E} \to \pi_*\mathcal{I} \to \pi_*\mathcal{O}_Y$. The pullbacks of the finitely presented $\mathcal{O}_{X_i}$-modules $\text{Coker}(\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{Y_i})$ and $\pi_{i,*}(\mathcal{O}_{Y_i}/\mathcal{I}_i)$ to X agree as quotients of $\pi_*\mathcal{O}_Y$. Hence by Limits of Spaces, Lemma 7.2 we may assume that these agree, in other words that the image of $\mathcal{E}_i \to \pi_{i,*}\mathcal{O}_{X_i}$ is equal to $\pi_{i,*}\mathcal{I}_i$. Then we conclude that X_i has the resolution property by Lemma 56.4. ☐

Lemma 56.6. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space with the resolution property. Then X has affine diagonal over $\mathbf{Z}$ (as in Properties of Spaces, Definition 3.1).*

Proof. We could prove this as in the case of schemes, but instead we will deduce the lemma from the case of schemes. First, we may and do assume $S = \text{Spec}(\mathbf{Z})$. Next, we choose a scheme Y and a surjective integral morphism $f : Y \to X$, see Decent Spaces, Lemma 9.2. Then f is affine, hence Y has the resolution property by Derived Categories of Spaces, Lemma 28.3. Hence by the case of schemes, the scheme Y has affine diagonal, see Derived Categories of Schemes, Lemma 36.10. Next, we consider the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\Delta_Y"'] \arrow[d] & Y \times_{\mathbf{Z}} Y \arrow[d] \\ X \arrow[r, "\Delta_X"] & X \times_{\mathbf{Z}} X \end{tikzcd}$$

Observe that the right vertical arrow is integral, in particular affine. Let $W \to X \times_{\mathbf{Z}} X$ be a morphism with W affine. Then we see that

$$Y \times_{X \times_{\mathbf{Z}} X} W = Y \times_{\Delta_Y, Y \times_{\mathbf{Z}} Y} (Y \times_{\mathbf{Z}} Y) \times_{X \times_{\mathbf{Z}} X} W$$

is affine. On the other hand, $Y \to X$ is integral and surjective hence

$$Y \times_{X \times_{\mathbf{Z}} X} W \longrightarrow X \times_{X \times_{\mathbf{Z}} X} W$$

is integral surjective as the base change of $Y \to X$ to W. We conclude that the target of this arrow is affine by Limits of Spaces, Proposition 15.2. It follows that Δ_X is affine as desired. $\square$

57. Blowing up and the resolution property

We prove that the resolution property is satisfied after a blowing up.

Lemma 57.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Assume that $|X|$ has finitely many irreducible components. There exists a dense quasi-compact open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that the algebraic space X' has the resolution property.*

Proof. By Limits of Spaces, Lemma 16.3 there exists a surjective, finite, and finitely presented morphism $f : Y \to X$ where Y is a scheme and a quasi-compact dense open $U \subset X$ such that $f^{-1}(U) \to U$ is finite étale. By More on Morphisms, Lemma 80.2 there is a quasi-compact dense open $V \subset Y$ and a V-admissible blowing up $Y' \to Y$ such that Y' has an ample family of invertible modules. After shrinking U we may assume that $f^{-1}(U) \subset V$ (details omitted). Hence $f' : Y' \to X$ is finite étale over U and in particular, the morphism $(f')^{-1}(U) \to U$ is finite locally free. By Lemma 39.2 there is a U-admissible blowing up $X' \to X$ such that the strict transform Y'' of Y' is finite locally free over X'. Picture

$$\begin{tikzcd} Y'' \arrow[r, "g"'] \arrow[d] & Y' \arrow[r] & Y \arrow[d] \\ X' \arrow[rr] & & X \end{tikzcd}$$

Since $g : Y'' \to Y'$ is a blowing up (Divisors on Spaces, Lemma 18.3) in the inverse image of the center of $X' \to X$, we see that $g : Y'' \to Y'$ is projective and that there exists some g-ample invertible module on Y''. Hence by More on Morphisms, Lemma 79.1 we see that Y'' has an ample family of invertible modules. Hence Y'' has the resolution property, see Derived Categories of Schemes, Lemma 36.7. We conclude that X' has the resolution property by Derived Categories of Spaces, Lemma 28.4. $\square$

Lemma 57.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. There exists a $t \geq 0$ and closed subspaces*

$$X \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_i \to X$ is of finite presentation, $Z_0 \subset X$ is a thickening, and for each $i = 0, \ldots t-1$ there exists a $(Z_i \setminus Z_{i-1})$-admissible blowing up $Z_i' \to Z_i$ such that Z_i' has the resolution property.

Proof. In this paragraph we use absolute Noetherian approximation to reduce to the case of algebraic spaces of finite presentation over $\text{Spec}(\mathbf{Z})$. We may view X as an algebraic space over $\text{Spec}(\mathbf{Z})$, see Spaces, Definition 16.2 and Properties of Spaces, Definition 3.1. Thus we may apply Limits of Spaces, Proposition 8.1. It follows that we can find an affine morphism $X \to X_0$ with X_0 of finite presentation over $\mathbf{Z}$. If we can prove the lemma for X_0, then we can pull back the stratification and the centers of the blowing ups to X and get the result for X; this uses that the resolution property goes up along affine morphisms (Derived Categories of Spaces, Lemma 28.3) and that the strict transform of an affine morphism is affine – details omitted. This reduces us to the case discussed in the next paragraph.

Assume X is of finite presentation over $\mathbf{Z}$. Then X is Noetherian and $|X|$ is a Noetherian topological space (with finitely many irreducible components) of finite dimension. Hence we may use induction on $\dim(|X|)$. By Lemma 57.1 there exists a dense open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that X' has the resolution property. Set $Z_0 = X$ and let $Z_1 \subset X$ be the reduced closed subspace with $|Z_1| = |X| \setminus |U|$. By induction we find an integer $t \geq 0$ and a filtration

$$Z_1 \supset Z_{1,0} \supset Z_{1,1} \supset \ldots \supset Z_{1,t} = \emptyset$$

by closed subspaces, where $Z_{1,0} \to Z_1$ is a thickening and there exist $(Z_{1,i} \setminus Z_{1,i+1})$-admissible blowing ups $Z_{1,i}' \to Z_{1,i}$ such that $Z_{1,i}'$ has the resolution property. Since Z_1 is reduced, we have $Z_1 = Z_{1,0}$. Hence we can set $Z_i = Z_{1,i-1}$ and $Z_i' = Z_{1,i-1}'$ for $i \geq 1$ and the lemma is proved. $\square$

58. Other chapters

Preliminaries

Schemes

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