MORPHISMS OF ALGEBRAIC STACKS

Contents

1. Introduction

In this chapter we introduce some types of morphisms of algebraic stacks. A reference in the case of quasi-separated algebraic stacks with representable diagonal is [LMB00].

The goal is to extend the definition of each of the types of morphisms of algebraic spaces to morphisms of algebraic stacks. Each case is slightly different and it seems best to treat them all separately.

For morphisms of algebraic stacks which are representable by algebraic spaces we have already defined a large number of types of morphisms, see Properties of Stacks, Section 3. For each corresponding case in this chapter we have to make sure the definition in the general case is compatible with the definition given there.

2. Conventions and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2.

3. Properties of diagonals

The diagonal of an algebraic stack is closely related to the *Isom*-sheaves, see Algebraic Stacks, Lemma 10.11. By the second defining property of an algebraic stack these *Isom*-sheaves are always algebraic spaces.

Lemma 3.1. *Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then the morphism $Isom_\mathcal{X}(x, y) \to T$ is locally of finite type.*

Proof. By Algebraic Stacks, Lemma 16.2 we may assume that $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces. By Descent on Spaces, Lemma 11.9 it suffices to check the property fppf locally on T. Thus we may assume that x, y come from morphisms $x', y' : T \to U$. By Groupoids in Spaces, Lemma 22.1 we see that in this case $Isom_\mathcal{X}(x, y) = T \times_{(y', x'), U \times_S U} R$. Hence it suffices to prove that $R \to U \times_S U$ is locally of finite type. This follows from the fact that the composition

$s : R \to U \times_S U \to U$ is smooth (hence locally of finite type, see Morphisms of Spaces, Lemmas 37.5 and 28.5) and Morphisms of Spaces, Lemma 23.6. $\square$

Lemma 3.2. *Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then*

(1) *$Isom_{\mathcal{X}}(y, y)$ is a group algebraic space over T, and*
(2) *$Isom_{\mathcal{X}}(x, y)$ is a pseudo torsor for $Isom_{\mathcal{X}}(y, y)$ over T.*

Proof. See Groupoids in Spaces, Definitions 5.1 and 9.1. The lemma follows immediately from the fact that $\mathcal{X}$ is a stack in groupoids. $\square$

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The *diagonal of f* is the morphism

$$\Delta_f : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

Here are two properties that every diagonal morphism has.

Lemma 3.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

(1) *Δ_f is representable by algebraic spaces, and*
(2) *Δ_f is locally of finite type.*

Proof. Let T be a scheme and let $a : T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be a morphism. By definition of the fibre product and the 2-Yoneda lemma the morphism a is given by a triple $a = (x, x', \alpha)$ where x, x' are objects of $\mathcal{X}$ over T, and $\alpha : f(x) \to f(x')$ is a morphism in the fibre category of $\mathcal{Y}$ over T. By definition of an algebraic stack the sheaves $Isom_{\mathcal{X}}(x, x')$ and $Isom_{\mathcal{Y}}(f(x), f(x'))$ are algebraic spaces over T. In this language α defines a section of the morphism $Isom_{\mathcal{Y}}(f(x), f(x')) \to T$. A T'-valued point of $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T$ for $T' \to T$ a scheme over T is the same thing as an isomorphism $x|_{T'} \to x'|_{T'}$ whose image under f is $\alpha|_{T'}$. Thus we see that

(3.3.1)
$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T \arrow[r] \arrow[d] & Isom_{\mathcal{X}}(x, x') \arrow[d] \\ T \arrow[r, "\alpha"] & Isom_{\mathcal{Y}}(f(x), f(x')) \end{tikzcd}$$

is a fibre square of sheaves over T. In particular we see that $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T$ is an algebraic space which proves part (1) of the lemma.

To prove the second statement we have to show that the left vertical arrow of Diagram (3.3.1) is locally of finite type. By Lemma 3.1 the algebraic space $Isom_{\mathcal{X}}(x, x')$ and is locally of finite type over T. Hence the right vertical arrow of Diagram (3.3.1) is locally of finite type, see Morphisms of Spaces, Lemma 23.6. We conclude by Morphisms of Spaces, Lemma 23.3. $\square$

Lemma 3.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then*

(1) *Δ_f is representable (by schemes),*
(2) *Δ_f is locally of finite type,*
(3) *Δ_f is a monomorphism,*
(4) *Δ_f is separated, and*
(5) *Δ_f is locally quasi-finite.*

Proof. We have already seen in Lemma 3.3 that Δ_f is representable by algebraic spaces. Hence the statements (2) – (5) make sense, see Properties of Stacks, Section 3. Also Lemma 3.3 guarantees (2) holds. Let $T \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ be a morphism and contemplate Diagram (3.3.1). By Algebraic Stacks, Lemma 9.2 the right vertical arrow is injective as a map of sheaves, i.e., a monomorphism of algebraic spaces. Hence also the morphism $T \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}} \mathcal{X} \to T$ is a monomorphism. Thus (3) holds. We already know that $T \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}} \mathcal{X} \to T$ is locally of finite type. Thus Morphisms of Spaces, Lemma 27.10 allows us to conclude that $T \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}} \mathcal{X} \to T$ is locally quasi-finite and separated. This proves (4) and (5). Finally, Morphisms of Spaces, Proposition 50.2 implies that $T \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}} \mathcal{X}$ is a scheme which proves (1). □

Lemma 3.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

1. *f is separated,*
2. *Δ_f is a closed immersion,*
3. *Δ_f is proper, or*
4. *Δ_f is universally closed.*

Proof. The statements "f is separated", "Δ_f is a closed immersion", "Δ_f is universally closed", and "Δ_f is proper" refer to the notions defined in Properties of Stacks, Section 3. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_\mathcal{Y} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 31.14 and Properties of Stacks, Lemma 3.3 we see that for any property P (as in that lemma) we have: Δ_f has P if and only if $\Delta_{U/V} : U \to U \times_V U$ has P. Hence the equivalence of (2), (3) and (4) follows from Morphisms of Spaces, Lemma 40.9 applied to $U \to V$. Moreover, if (1) holds, then $U \to V$ is separated and we see that $\Delta_{U/V}$ is a closed immersion, i.e., (2) holds. Finally, assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_\mathcal{Y} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is separated. Using Categories, Lemma 31.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is a closed immersion, hence $T' \to T$ is separated as desired. □

Lemma 3.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

1. *f is quasi-separated,*
2. *Δ_f is quasi-compact, or*
3. *Δ_f is of finite type.*

Proof. The statements "f is quasi-separated", "Δ_f is quasi-compact", and "Δ_f is of finite type" refer to the notions defined in Properties of Stacks, Section 3. Note that (2) and (3) are equivalent in view of the fact that Δ_f is locally of finite type by Lemma 3.4 (and Algebraic Stacks, Lemma 10.9). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_\mathcal{Y} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 31.14 and Properties of Stacks, Lemma 3.3 we see that we have: Δ_f is quasi-compact if and only if $\Delta_{U/V} : U \to U \times_V U$ is quasi-compact. If (1) holds, then $U \to V$ is quasi-separated and we see that $\Delta_{U/V}$ is quasi-compact, i.e., (2) holds. Assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_\mathcal{Y} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is quasi-separated. Using Categories, Lemma 31.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is quasi-compact, hence $T' \to T$ is quasi-separated as desired. $\square$

Lemma 3.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is locally separated, and*
(2) *Δ_f is an immersion.*

Proof. The statements "f is locally separated", and "Δ_f is an immersion" refer to the notions defined in Properties of Stacks, Section 3. Proof omitted. Hint: Argue as in the proofs of Lemmas 3.5 and 3.6. $\square$

4. Separation axioms

Let $\mathcal{X} = [U/R]$ be a presentation of an algebraic stack. Then the properties of the diagonal of $\mathcal{X}$ over S, are the properties of the morphism $j : R \to U \times_S U$. For example, if $\mathcal{X} = [S/G]$ for some smooth group G in algebraic spaces over S then j is the structure morphism $G \to S$. Hence the diagonal is not automatically separated itself (contrary to what happens in the case of schemes and algebraic spaces). To say that $[S/G]$ is quasi-separated over S should certainly imply that $G \to S$ is quasi-compact, but we hesitate to say that $[S/G]$ is quasi-separated over S without also requiring the morphism $G \to S$ to be quasi-separated. In other words, requiring the diagonal morphism to be quasi-compact does not really agree with our intuition for a "quasi-separated algebraic stack", and we should also require the diagonal itself to be quasi-separated.

What about "separated algebraic stacks"? We have seen in Morphisms of Spaces, Lemma 40.9 that an algebraic space is separated if and only if the diagonal is proper. This is the condition that is usually used to define separated algebraic stacks too. In the example $[S/G] \to S$ above this means that $G \to S$ is a proper group scheme. This means algebraic stacks of the form $[\mathrm{Spec}(k)/E]$ are proper over k where E is an elliptic curve over k (insert future reference here). In certain situations it may be more natural to assume the diagonal is finite.

Definition 4.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *DM* if Δ_f is unramified[1].
(2) We say f is *quasi-DM* if Δ_f is locally quasi-finite[2].
(3) We say f is *separated* if Δ_f is proper.
(4) We say f is *quasi-separated* if Δ_f is quasi-compact and quasi-separated.

In this definition we are using that Δ_f is representable by algebraic spaces and we are using Properties of Stacks, Section 3 to make sense out of imposing conditions

[1]The letters DM stand for Deligne-Mumford. If f is DM then given any scheme T and any morphism $T \to \mathcal{Y}$ the fibre product $\mathcal{X}_T = \mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic stack over T whose diagonal is unramified, i.e., $\mathcal{X}_T$ is DM. This implies $\mathcal{X}_T$ is a Deligne-Mumford stack, see Theorem 21.6. In other words a DM morphism is one whose "fibres" are Deligne-Mumford stacks. This hopefully at least motivates the terminology.

[2]If f is quasi-DM, then the "fibres" $\mathcal{X}_T$ of $\mathcal{X} \to \mathcal{Y}$ are quasi-DM. An algebraic stack $\mathcal{X}$ is quasi-DM exactly if there exists a scheme U and a surjective flat morphism $U \to \mathcal{X}$ of finite presentation which is locally quasi-finite, see Theorem 21.3. Note the similarity to being Deligne-Mumford, which is defined in terms of having an étale covering by a scheme.

on Δ_f. We note that these definitions do not conflict with the already existing notions if f is representable by algebraic spaces, see Lemmas 3.6 and 3.5. There is an interesting way to characterize these conditions by looking at higher diagonals, see Lemma 6.5.

Definition 4.2. Let $\mathcal{X}$ be an algebraic stack over the base scheme S. Denote $p : \mathcal{X} \to S$ the structure morphism.

(1) We say $\mathcal{X}$ is *DM over S* if $p : \mathcal{X} \to S$ is DM.
(2) We say $\mathcal{X}$ is *quasi-DM over S* if $p : \mathcal{X} \to S$ is quasi-DM.
(3) We say $\mathcal{X}$ is *separated over S* if $p : \mathcal{X} \to S$ is separated.
(4) We say $\mathcal{X}$ is *quasi-separated over S* if $p : \mathcal{X} \to S$ is quasi-separated.
(5) We say $\mathcal{X}$ is *DM* if $\mathcal{X}$ is DM[3] over $\text{Spec}(\mathbf{Z})$.
(6) We say $\mathcal{X}$ is *quasi-DM* if $\mathcal{X}$ is quasi-DM over $\text{Spec}(\mathbf{Z})$.
(7) We say $\mathcal{X}$ is *separated* if $\mathcal{X}$ is separated over $\text{Spec}(\mathbf{Z})$.
(8) We say $\mathcal{X}$ is *quasi-separated* if $\mathcal{X}$ is quasi-separated over $\text{Spec}(\mathbf{Z})$.

In the last 4 definitions we view $\mathcal{X}$ as an algebraic stack over $\text{Spec}(\mathbf{Z})$ via Algebraic Stacks, Definition 19.2.

Thus in each case we have an absolute notion and a notion relative to our given base scheme (mention of which is usually suppressed by our abuse of notation introduced in Properties of Stacks, Section 2). We will see that (1) $\Leftrightarrow$ (5) and (2) $\Leftrightarrow$ (6) in Lemma 4.13. We spend some time proving some standard results on these notions.

Lemma 4.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If f is separated, then f is quasi-separated.*
(2) *If f is DM, then f is quasi-DM.*
(3) *If f is representable by algebraic spaces, then f is DM.*

Proof. To see (1) note that a proper morphism of algebraic spaces is quasi-compact and quasi-separated, see Morphisms of Spaces, Definition 40.1. To see (2) note that an unramified morphism of algebraic spaces is locally quasi-finite, see Morphisms of Spaces, Lemma 38.7. Finally (3) follows from Lemma 3.4. $\square$

Lemma 4.4. *All of the separation axioms listed in Definition 4.1 are stable under base change.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks. Let $f' : \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ be the base change of f by $\mathcal{Y}' \to \mathcal{Y}$. Then $\Delta_{f'}$ is the base change of Δ_f by the morphism $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}' \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, see Categories, Lemma 31.14. By the results of Properties of Stacks, Section 3 each of the properties of the diagonal used in Definition 4.1 is stable under base change. Hence the lemma is true. $\square$

Lemma 4.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ has one of the separation properties of Definition 4.1 then so does f.*

Proof. Denote $g : W \times_{\mathcal{Y}} \mathcal{X} \to W$ the base change. Then Δ_g is the base change of Δ_f by the morphism $q : W \times_{\mathcal{Y}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Since q is the base change of $W \to \mathcal{Y}$ we see that q is representable by algebraic spaces, surjective, flat, and

[3]Theorem 21.6 shows that this is equivalent to $\mathcal{X}$ being a Deligne-Mumford stack.

locally of finite presentation. Hence the result follows from Properties of Stacks, Lemma 3.4. □

Lemma 4.6. *Let S be a scheme. The property of being quasi-DM over S, quasi-separated over S, or separated over S (see Definition 4.2) is stable under change of base scheme, see Algebraic Stacks, Definition 19.3.*

Proof. Follows immediately from Lemma 4.4. □

Lemma 4.7. *Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ and $\mathcal{Z} \to \mathcal{T}$ be morphisms of algebraic stacks. Consider the induced morphism $i : \mathcal{X} \times_\mathcal{Z} \mathcal{Y} \to \mathcal{X} \times_\mathcal{T} \mathcal{Y}$. Then*

1. *i is representable by algebraic spaces and locally of finite type,*
2. *if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is quasi-separated, then i is quasi-separated,*
3. *if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is separated, then i is separated,*
4. *if $\mathcal{Z} \to \mathcal{T}$ is DM, then i is unramified,*
5. *if $\mathcal{Z} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,*
6. *if $\mathcal{Z} \to \mathcal{T}$ is separated, then i is proper, and*
7. *if $\mathcal{Z} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.*

Proof. The following diagram

$$\begin{tikzcd} \mathcal{X} \times_\mathcal{Z} \mathcal{Y} \arrow[r, "i"'] \arrow[d] & \mathcal{X} \times_\mathcal{T} \mathcal{Y} \arrow[d] \\ \mathcal{Z} \arrow[r, "\Delta_{\mathcal{Z}/\mathcal{T}}"] & \mathcal{Z} \times_\mathcal{T} \mathcal{Z} \end{tikzcd}$$

is a 2-fibre product diagram, see Categories, Lemma 31.13. Hence i is the base change of the diagonal morphism $\Delta_{\mathcal{Z}/\mathcal{T}}$. Thus the lemma follows from Lemma 3.3, and the material in Properties of Stacks, Section 3. □

Lemma 4.8. *Let $\mathcal{T}$ be an algebraic stack. Let $g : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over $\mathcal{T}$. Consider the graph $i : \mathcal{X} \to \mathcal{X} \times_\mathcal{T} \mathcal{Y}$ of g. Then*

1. *i is representable by algebraic spaces and locally of finite type,*
2. *if $\mathcal{Y} \to \mathcal{T}$ is DM, then i is unramified,*
3. *if $\mathcal{Y} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,*
4. *if $\mathcal{Y} \to \mathcal{T}$ is separated, then i is proper, and*
5. *if $\mathcal{Y} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.*

Proof. This is a special case of Lemma 4.7 applied to the morphism $\mathcal{X} = \mathcal{X} \times_\mathcal{Y} \mathcal{Y} \to \mathcal{X} \times_\mathcal{T} \mathcal{Y}$. □

Lemma 4.9. *Let $f : \mathcal{X} \to \mathcal{T}$ be a morphism of algebraic stacks. Let $s : \mathcal{T} \to \mathcal{X}$ be a morphism such that $f \circ s$ is 2-isomorphic to $\mathrm{id}_\mathcal{T}$. Then*

1. *s is representable by algebraic spaces and locally of finite type,*
2. *if f is DM, then s is unramified,*
3. *if f is quasi-DM, then s is locally quasi-finite,*
4. *if f is separated, then s is proper, and*
5. *if f is quasi-separated, then s is quasi-compact and quasi-separated.*

Proof. This is a special case of Lemma 4.8 applied to $g = s$ and $\mathcal{Y} = \mathcal{T}$ in which case $i : \mathcal{T} \to \mathcal{T} \times_\mathcal{T} \mathcal{X}$ is 2-isomorphic to s. □

Lemma 4.10. *All of the separation axioms listed in Definition 4.1 are stable under composition of morphisms.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks to which the axiom in question applies. The diagonal $\Delta_{\mathcal{X}/\mathcal{Z}}$ is the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 4.7 above we see that the second arrow also has this property. Hence the lemma follows since the composition of morphisms which are representable by algebraic spaces with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$, see our general discussion in Properties of Stacks, Section 3 and Morphisms of Spaces, Lemmas 38.3, 27.3, 40.4, 8.5, and 4.8. $\square$

Lemma 4.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.*

(1) *If $\mathcal{Y}$ is DM over S and f is DM, then $\mathcal{X}$ is DM over S.*
(2) *If $\mathcal{Y}$ is quasi-DM over S and f is quasi-DM, then $\mathcal{X}$ is quasi-DM over S.*
(3) *If $\mathcal{Y}$ is separated over S and f is separated, then $\mathcal{X}$ is separated over S.*
(4) *If $\mathcal{Y}$ is quasi-separated over S and f is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.*
(5) *If $\mathcal{Y}$ is DM and f is DM, then $\mathcal{X}$ is DM.*
(6) *If $\mathcal{Y}$ is quasi-DM and f is quasi-DM, then $\mathcal{X}$ is quasi-DM.*
(7) *If $\mathcal{Y}$ is separated and f is separated, then $\mathcal{X}$ is separated.*
(8) *If $\mathcal{Y}$ is quasi-separated and f is quasi-separated, then $\mathcal{X}$ is quasi-separated.*

Proof. Parts (1), (2), (3), and (4) follow immediately from Lemma 4.10 and Definition 4.2. For (5), (6), (7), and (8) think of $\mathcal{X}$ and $\mathcal{Y}$ as algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ and apply Lemma 4.10. Details omitted. $\square$

The following lemma is a bit different to the analogue for algebraic spaces. To compare take a look at Morphisms of Spaces, Lemma 4.10.

Lemma 4.12. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks.*

(1) *If $g \circ f$ is DM then so is f.*
(2) *If $g \circ f$ is quasi-DM then so is f.*
(3) *If $g \circ f$ is separated and Δ_g is separated, then f is separated.*
(4) *If $g \circ f$ is quasi-separated and Δ_g is quasi-separated, then f is quasi-separated.*

Proof. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces, see Lemmas 3.3 and 4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T.$$

If $g \circ f$ is DM (resp. quasi-DM), then the composition $A \to T$ is unramified (resp. locally quasi-finite). Hence (1) (resp. (2)) follows on applying Morphisms of Spaces, Lemma 38.11 (resp. Morphisms of Spaces, Lemma 27.8). This proves (1) and (2).

Proof of (4). Assume $g \circ f$ is quasi-separated and Δ_g is quasi-separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is quasi-separated, see Lemmas 3.3 and 4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_\mathcal{Z} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_\mathcal{Y} \mathcal{X}) \times_{(\mathcal{X} \times_\mathcal{Z} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is quasi-separated. The composition $A \to T$ is quasi-compact and quasi-separated as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is quasi-separated by Morphisms of Spaces, Lemma 4.10. And $A \to B$ is quasi-compact by Morphisms of Spaces, Lemma 8.9. Thus f is quasi-separated.

Proof of (3). Assume $g \circ f$ is separated and Δ_g is separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X} \to \mathcal{X} \times_\mathcal{Z} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is separated, see Lemmas 3.3 and 4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_\mathcal{Z} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_\mathcal{Y} \mathcal{X}) \times_{(\mathcal{X} \times_\mathcal{Z} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is separated. The composition $A \to T$ is proper as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is proper by Morphisms of Spaces, Lemma 40.6 which means that f is separated. □

Lemma 4.13. *Let $\mathcal{X}$ be an algebraic stack over the base scheme S.*

1. *$\mathcal{X}$ is DM $\Leftrightarrow$ $\mathcal{X}$ is DM over S.*
2. *$\mathcal{X}$ is quasi-DM $\Leftrightarrow$ $\mathcal{X}$ is quasi-DM over S.*
3. *If $\mathcal{X}$ is separated, then $\mathcal{X}$ is separated over S.*
4. *If $\mathcal{X}$ is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.*

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.

5. *If $\mathcal{X}$ is DM over S, then f is DM.*
6. *If $\mathcal{X}$ is quasi-DM over S, then f is quasi-DM.*
7. *If $\mathcal{X}$ is separated over S and $\Delta_{\mathcal{Y}/S}$ is separated, then f is separated.*
8. *If $\mathcal{X}$ is quasi-separated over S and $\Delta_{\mathcal{Y}/S}$ is quasi-separated, then f is quasi-separated.*

Proof. Parts (5), (6), (7), and (8) follow immediately from Lemma 4.12 and Spaces, Definition 13.2. To prove (3) and (4) think of X and Y as algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ and apply Lemma 4.12. Similarly, to prove (1) and (2), think of $\mathcal{X}$ as an algebraic stack over $\mathrm{Spec}(\mathbf{Z})$ consider the morphisms

$$\mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathrm{Spec}(\mathbf{Z})} \mathcal{X}$$

Both arrows are representable by algebraic spaces. The second arrow is unramified and locally quasi-finite as the base change of the immersion $\Delta_{S/\mathbf{Z}}$. Hence the composition is unramified (resp. locally quasi-finite) if and only if the first arrow is unramified (resp. locally quasi-finite), see Morphisms of Spaces, Lemmas 38.3 and 38.11 (resp. Morphisms of Spaces, Lemmas 27.3 and 27.8). □

Lemma 4.14. *Let $\mathcal{X}$ be an algebraic stack. Let W be an algebraic space, and let $f : W \to \mathcal{X}$ be a surjective, flat, locally finitely presented morphism.*

1. *If f is unramified (i.e., étale, i.e., $\mathcal{X}$ is Deligne-Mumford), then $\mathcal{X}$ is DM.*
2. *If f is locally quasi-finite, then $\mathcal{X}$ is quasi-DM.*

Proof. Note that if f is unramified, then it is étale by Morphisms of Spaces, Lemma 39.12. This explains the parenthetical remark in (1). Assume f is unramified (resp. locally quasi-finite). We have to show that $\Delta_\mathcal{X} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified (resp. locally quasi-finite). Note that $W \times W \to \mathcal{X} \times \mathcal{X}$ is also surjective, flat, and locally of finite presentation. Hence it suffices to show that

$$W \times_{\mathcal{X} \times \mathcal{X}, \Delta_\mathcal{X}} \mathcal{X} = W \times_\mathcal{X} W \longrightarrow W \times W$$

is unramified (resp. locally quasi-finite), see Properties of Stacks, Lemma 3.3. By assumption the morphism $\text{pr}_i : W \times_\mathcal{X} W \to W$ is unramified (resp. locally quasi-finite). Hence the displayed arrow is unramified (resp. locally quasi-finite) by Morphisms of Spaces, Lemma 38.11 (resp. Morphisms of Spaces, Lemma 27.8). □

Lemma 4.15. *A monomorphism of algebraic stacks is separated and DM. The same is true for immersions of algebraic stacks.*

Proof. If $f : \mathcal{X} \to \mathcal{Y}$ is a monomorphism of algebraic stacks, then Δ_f is an isomorphism, see Properties of Stacks, Lemma 8.4. Since an isomorphism of algebraic spaces is proper and unramified we see that f is separated and DM. The second assertion follows from the first as an immersion is a monomorphism, see Properties of Stacks, Lemma 9.5. □

Lemma 4.16. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Assume the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists. If $\mathcal{X}$ is DM, resp. quasi-DM, resp. separated, resp. quasi-separated, then so is $\mathcal{Z}_x$.*

Proof. This is true because $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism hence DM and separated by Lemma 4.15. Apply Lemma 4.11 to conclude. □

5. Inertia stacks

The (relative) inertia stack of a stack in groupoids is defined in Stacks, Section 7. The actual construction, in the setting of fibred categories, and some of its properties is in Categories, Section 34.

Lemma 5.1. *Let $\mathcal{X}$ be an algebraic stack. Then the inertia stack $\mathcal{I}_\mathcal{X}$ is an algebraic stack as well. The morphism*

$$\mathcal{I}_\mathcal{X} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type. More generally, let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then the relative inertia $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is an algebraic stack and the morphism

$$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type.

Proof. By Categories, Lemma 34.1 there are equivalences

$$\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_S \mathcal{X}, \Delta} \mathcal{X} \quad \text{and} \quad \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_\mathcal{Y} \mathcal{X}, \Delta} \mathcal{X}$$

which shows that the inertia stacks are algebraic stacks. Let $T \to \mathcal{X}$ be a morphism given by the object x of the fibre category of $\mathcal{X}$ over T. Then we get a 2-fibre product square

$$\begin{tikzcd} \mathit{Isom}_{\mathcal{X}}(x, x) \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}} \arrow[d] \\ T \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

This follows immediately from the definition of $\mathcal{I}_{\mathcal{X}}$. Since $\mathit{Isom}_{\mathcal{X}}(x, x)$ is always an algebraic space locally of finite type over T (see Lemma 3.1) we conclude that $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is representable by algebraic spaces and locally of finite type. Finally, for the relative inertia we get

$$\begin{tikzcd} \mathit{Isom}_{\mathcal{X}}(x, x) \arrow[d] & K \arrow[l] \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[d] \\ \mathit{Isom}_{\mathcal{Y}}(f(x), f(x)) & T \arrow[l, "e"'] \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

with both squares 2-fibre products. This follows from Categories, Lemma 34.3. The left vertical arrow is a morphism of algebraic spaces locally of finite type over T, and hence is locally of finite type, see Morphisms of Spaces, Lemma 23.6. Thus K is an algebraic space and $K \to T$ is locally of finite type. This proves the assertion on the relative inertia. □

Remark 5.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. In Properties of Stacks, Remark 3.7 we have seen that the 2-category of morphisms $\mathcal{Z} \to \mathcal{X}$ representable by algebraic spaces with target $\mathcal{X}$ forms a category. In this category the inertia stack of $\mathcal{X}/\mathcal{Y}$ is a *group object*. Recall that an object of $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is just a pair (x, α) where x is an object of $\mathcal{X}$ and α is an automorphism of x in the fibre category of $\mathcal{X}$ that x lives in with $f(\alpha) = \mathrm{id}$. The composition

$$c : \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \longrightarrow \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

is given by the rule on objects

$$((x, \alpha), (x', \alpha'), \beta) \mapsto (x, \alpha \circ \beta^{-1} \circ \alpha' \circ \beta)$$

which makes sense as $\beta : x \to x'$ is an isomorphism in the fibre category by our definition of fibre products. The neutral element $e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is given by the functor $x \mapsto (x, \mathrm{id}_x)$. We omit the proof that the axioms of a group object hold.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and let $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ be its inertia stack. Let T be a scheme and let x be an object of $\mathcal{X}$ over T. Set $y = f(x)$. We have seen in the proof of Lemma 5.1 that for any scheme T and object x of $\mathcal{X}$ over T there is an exact sequence of sheaves of groups

$$0 \to \mathit{Isom}_{\mathcal{X}/\mathcal{Y}}(x, x) \to \mathit{Isom}_{\mathcal{X}}(x, x) \to \mathit{Isom}_{\mathcal{Y}}(y, y) \tag{5.2.1}$$

The group structure on the second and third term is the one defined in Lemma 3.2 and the sequence gives a meaning to the first term. Also, there is a canonical cartesian square

$$\begin{tikzcd} \mathit{Isom}_{\mathcal{X}/\mathcal{Y}}(x, x) \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[d] \\ T \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

In fact, the group structure on $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ discussed in Remark 5.2 induces the group structure on $Isom_{\mathcal{X}/\mathcal{Y}}(x, x)$. This allows us to define the sheaf $Isom_{\mathcal{X}/\mathcal{Y}}(x, x)$ also for morphisms from algebraic spaces to $\mathcal{X}$. We formalize this in the following definition.

Definition 5.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let Z be an algebraic space.

(1) Let $x : Z \to \mathcal{X}$ be a morphism. We set

$$Isom_{\mathcal{X}/\mathcal{Y}}(x, x) = Z \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

We endow it with the structure of a group algebraic space over Z by pulling back the composition law discussed in Remark 5.2. We will sometimes refer to $Isom_{\mathcal{X}/\mathcal{Y}}(x, x)$ as the *relative sheaf of automorphisms of x*.

(2) Let $x_1, x_2 : Z \to \mathcal{X}$ be morphisms. Set $y_i = f \circ x_i$. Let $\alpha : y_1 \to y_2$ be a 2-morphism. Then α determines a morphism $\Delta^\alpha : Z \to Z \times_{y_1,\mathcal{Y},y_2} Z$ and we set

$$Isom^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1, x_2) = (Z \times_{x_1,\mathcal{X},x_2} Z) \times_{Z \times_{y_1,\mathcal{Y},y_2} Z, \Delta^\alpha} Z.$$

We will sometimes refer to $Isom^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1, x_2)$ as the *relative sheaf of isomorphisms from x_1 to x_2*.

If $\mathcal{Y} = \mathrm{Spec}(\mathbf{Z})$ or more generally when $\mathcal{Y}$ is an algebraic space, then we use the notation $Isom_{\mathcal{X}}(x, x)$ and $Isom_{\mathcal{X}}(x_1, x_2)$ and we use the terminology *sheaf of automorphisms of x* and *sheaf of isomorphisms from x_1 to x_2*.

Lemma 5.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let Z be an algebraic space and let $x_i : Z \to \mathcal{X}$, $i = 1, 2$ be morphisms. Then*

(1) *$Isom_{\mathcal{X}/\mathcal{Y}}(x_2, x_2)$ is a group algebraic space over Z,*
(2) *there is an exact sequence of groups*

$$0 \to Isom_{\mathcal{X}/\mathcal{Y}}(x_2, x_2) \to Isom_{\mathcal{X}}(x_2, x_2) \to Isom_{\mathcal{Y}}(f \circ x_2, f \circ x_2)$$

(3) *there is a map of algebraic spaces $Isom_{\mathcal{X}}(x_1, x_2) \to Isom_{\mathcal{Y}}(f \circ x_1, f \circ x_2)$ such that for any 2-morphism $\alpha : f \circ x_1 \to f \circ x_2$ we obtain a cartesian diagram*

$$\begin{tikzcd} Isom^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1, x_2) \arrow[r] \arrow[d] & Z \arrow[d, "\alpha"] \\ Isom_{\mathcal{X}}(x_1, x_2) \arrow[r] & Isom_{\mathcal{Y}}(f \circ x_1, f \circ x_2) \end{tikzcd}$$

(4) *for any 2-morphism $\alpha : f \circ x_1 \to f \circ x_2$ the algebraic space $Isom^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1, x_2)$ is a pseudo torsor for $Isom_{\mathcal{X}/\mathcal{Y}}(x_2, x_2)$ over Z.*

Proof. Part (1) follows from Definition 5.3. Part (2) comes from the exact sequence (5.2.1) étale locally on Z. Part (3) can be seen by unwinding the definitions. Locally on Z in the étale topology part (4) reduces to part (2) of Lemma 3.2. $\square$

Lemma 5.5. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ and $f : \mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks. Set $\mathcal{X}' = \mathcal{X} \times_\mathcal{Y} \mathcal{Y}'$. Then both squares in the diagram*

$$\begin{tikzcd} \mathcal{I}_{\mathcal{X}'/\mathcal{Y}'} \arrow[r] \arrow[d, "\text{Categories, Equation (34.2.3)}"'] & \mathcal{X}' \arrow[r, "\pi'"'] \arrow[d] & \mathcal{Y}' \arrow[d, "f"] \\ \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[r] & \mathcal{X} \arrow[r, "\pi"] & \mathcal{Y} \end{tikzcd}$$

are fibre product squares.

Proof. The inertia stack $\mathcal{I}_{\mathcal{X}'/\mathcal{Y}'}$ is defined as the category of pairs (x', α') where x' is an object of $\mathcal{X}'$ and α' is an automorphism of x' with $\pi'(\alpha') = \text{id}$, see Categories, Section 34. Suppose that x' lies over the scheme U and maps to the object x of $\mathcal{X}$. By the construction of the 2-fibre product in Categories, Lemma 32.3 we see that $x' = (U, x, y', \beta)$ where y' is an object of $\mathcal{Y}'$ over U and β is an isomorphism $\beta : \pi(x) \to f(y')$ in the fibre category of $\mathcal{Y}$ over U. By the very construction of the 2-fibre product the automorphism α' is a pair (α, γ) where α is an automorphism of x over U and γ is an automorphism of y' over U such that α and γ are compatible via β. The condition $\pi'(\alpha') = \text{id}$ signifies that $\gamma = \text{id}$ whereupon the condition that α, β, γ are compatible is exactly the condition $\pi(\alpha) = \text{id}$, i.e., means exactly that (x, α) is an object of $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$. In this way we see that the left square is a fibre product square (some details omitted). □

Lemma 5.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of algebraic stacks. Then the diagram*

$$\begin{tikzcd} \mathcal{I}_\mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{I}_\mathcal{Y} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

is a fibre product square.

Proof. This follows immediately from the fact that f is fully faithful (see Properties of Stacks, Lemma 8.4) and the definition of the inertia in Categories, Section 34. Namely, an object of $\mathcal{I}_\mathcal{X}$ over a scheme T is the same thing as a pair (x, α) consisting of an object x of $\mathcal{X}$ over T and a morphism $\alpha : x \to x$ in the fibre category of $\mathcal{X}$ over T. As f is fully faithful we see that α is the same thing as a morphism $\beta : f(x) \to f(x)$ in the fibre category of $\mathcal{Y}$ over T. Hence we can think of objects of $\mathcal{I}_\mathcal{X}$ over T as triples $((y, \beta), x, \gamma)$ where y is an object of $\mathcal{Y}$ over T, $\beta : y \to y$ in $\mathcal{Y}_T$ and $\gamma : y \to f(x)$ is an isomorphism over T, i.e., an object of $\mathcal{I}_\mathcal{Y} \times_\mathcal{Y} \mathcal{X}$ over T. □

Lemma 5.7. *Let $\mathcal{X}$ be an algebraic stack. Let $[U/R] \to \mathcal{X}$ be a presentation. Let G/U be the stabilizer group algebraic space associated to the groupoid (U, R, s, t, c). Then*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{I}_\mathcal{X} \arrow[r] & \mathcal{X} \end{tikzcd}$$

is a fibre product diagram.

Proof. Immediate from Groupoids in Spaces, Lemma 26.2. □

6. Higher diagonals

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. In this situation it makes sense to consider not only the diagonal

$$\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

but also the diagonal of the diagonal, i.e., the morphism

$$\Delta_{\Delta_f} : \mathcal{X} \longrightarrow \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} \mathcal{X}$$

Because of this we sometimes use the following terminology. We denote $\Delta_{f,0} = f$ the *zeroth diagonal*, we denote $\Delta_{f,1} = \Delta_f$ the *first diagonal*, and we denote $\Delta_{f,2} = \Delta_{\Delta_f}$ the *second diagonal*. Note that $\Delta_{f,1}$ is representable by algebraic spaces and locally of finite type, see Lemma 3.3. Hence $\Delta_{f,2}$ is representable, a monomorphism, locally of finite type, separated, and locally quasi-finite, see Lemma 3.4.

We can describe the second diagonal using the relative inertia stack. Namely, the fibre product $\mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} \mathcal{X}$ is equivalent to the relative inertia stack $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ by Categories, Lemma 34.1. Moreover, via this identification the second diagonal becomes the *neutral section*

$$\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

of the relative inertia stack. By analogy with what happens for groupoids in algebraic spaces (Groupoids in Spaces, Lemma 29.2) we have the following equivalences.

Lemma 6.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

1. *The following are equivalent*
   (a) *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is separated,*
   (b) *$\Delta_{f,1} = \Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is separated, and*
   (c) *$\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is a closed immersion.*
2. *The following are equivalent*
   (a) *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is quasi-separated,*
   (b) *$\Delta_{f,1} = \Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-separated, and*
   (c) *$\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is a quasi-compact.*
3. *The following are equivalent*
   (a) *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is locally separated,*
   (b) *$\Delta_{f,1} = \Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is locally separated, and*
   (c) *$\Delta_{f,2} = e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is an immersion.*
4. *The following are equivalent*
   (a) *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is unramified,*
   (b) *f is DM.*
5. *The following are equivalent*
   (a) *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is locally quasi-finite,*
   (b) *f is quasi-DM.*

Proof. Proof of (1), (2), and (3). Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X}$. Then $G = U \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is an algebraic space over U (Lemma 5.1). In fact, G is a group algebraic space over U by the group law on relative inertia constructed in Remark 5.2. Moreover, $G \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is surjective and smooth as a base change of $U \to \mathcal{X}$. Finally, the base change of $e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ by $G \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is the identity $U \to G$ of G/U. Thus the equivalence of (a) and (c) follows from Groupoids in Spaces, Lemma 6.1. Since $\Delta_{f,2}$ is the diagonal of Δ_f we have (b) $\Leftrightarrow$ (c) by definition.

Proof of (4) and (5). Recall that (4)(b) means Δ_f is unramified and (5)(b) means that Δ_f is locally quasi-finite. Choose a scheme Z and a morphism $a : Z \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$. Then $a = (x_1, x_2, \alpha)$ where $x_i : Z \to \mathcal{X}$ and $\alpha : f \circ x_1 \to f \circ x_2$ is a 2-morphism. Recall that

$$\begin{tikzcd} Isom^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1, x_2) \arrow[r] \arrow[d] & Z \arrow[d] \\ \mathcal{X} \arrow[r, "\Delta_f"] & \mathcal{X} \times_\mathcal{Y} \mathcal{X} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Isom_{\mathcal{X}/\mathcal{Y}}(x_2, x_2) \arrow[r] \arrow[d] & Z \arrow[d, "x_2"] \\ \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[r] & \mathcal{X} \end{tikzcd}$$

are cartesian squares. By Lemma 5.4 the algebraic space $Isom^\alpha_{\mathcal{X}/\mathcal{Y}}(x_1, x_2)$ is a pseudo torsor for $Isom_{\mathcal{X}/\mathcal{Y}}(x_2, x_2)$ over Z. Thus the equivalences in (4) and (5) follow from Groupoids in Spaces, Lemma 9.5. $\square$

Lemma 6.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

1. *the morphism f is representable by algebraic spaces,*
2. *the second diagonal of f is an isomorphism,*
3. *the group stack $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is trivial over $\mathcal{X}$, and*
4. *for a scheme T and a morphism $x : T \to \mathcal{X}$ the kernel of $Isom_\mathcal{X}(x, x) \to Isom_\mathcal{Y}(f(x), f(x))$ is trivial.*

Proof. We first prove the equivalence of (1) and (2). Namely, f is representable by algebraic spaces if and only if f is faithful, see Algebraic Stacks, Lemma 15.2. On the other hand, f is faithful if and only if for every object x of $\mathcal{X}$ over a scheme T the functor f induces an injection $Isom_\mathcal{X}(x, x) \to Isom_\mathcal{Y}(f(x), f(x))$, which happens if and only if the kernel K is trivial, which happens if and only if $e : T \to K$ is an isomorphism for every $x : T \to \mathcal{X}$. Since $K = T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ as discussed above, this proves the equivalence of (1) and (2). To prove the equivalence of (2) and (3), by the discussion above, it suffices to note that a group stack is trivial if and only if its identity section is an isomorphism. Finally, the equivalence of (3) and (4) follows from the definitions: in the proof of Lemma 5.1 we have seen that the kernel in (4) corresponds to the fibre product $T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ over T. $\square$

This lemma leads to the following hierarchy for morphisms of algebraic stacks.

Lemma 6.3. *A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is*

1. *a monomorphism if and only if $\Delta_{f,1}$ is an isomorphism, and*
2. *representable by algebraic spaces if and only if $\Delta_{f,1}$ is a monomorphism.*

Moreover, the second diagonal $\Delta_{f,2}$ is always a monomorphism.

Proof. Recall from Properties of Stacks, Lemma 8.4 that a morphism of algebraic stacks is a monomorphism if and only if its diagonal is an isomorphism of stacks. Thus Lemma 6.2 can be rephrased as saying that a morphism is representable by algebraic spaces if the diagonal is a monomorphism. In particular, it shows that condition (3) of Lemma 3.4 is actually an if and only if, i.e., a morphism of algebraic stacks is representable by algebraic spaces if and only if its diagonal is a monomorphism. $\square$

Lemma 6.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

1. *$\Delta_{f,1}$ separated $\Leftrightarrow \Delta_{f,2}$ closed immersion $\Leftrightarrow \Delta_{f,2}$ proper $\Leftrightarrow \Delta_{f,2}$ universally closed,*
2. *$\Delta_{f,1}$ quasi-separated $\Leftrightarrow \Delta_{f,2}$ finite type $\Leftrightarrow \Delta_{f,2}$ quasi-compact, and*
3. *$\Delta_{f,1}$ locally separated $\Leftrightarrow \Delta_{f,2}$ immersion.*

Proof. Follows from Lemmas 3.5, 3.6, and 3.7 applied to $\Delta_{f,1}$. □

The following lemma is kind of cute and it may suggest a generalization of these conditions to higher algebraic stacks.

Lemma 6.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

1. *f is separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are universally closed, and*
2. *f is quasi-separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are quasi-compact.*
3. *f is quasi-DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are locally quasi-finite.*
4. *f is DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are unramified.*

Proof. Proof of (1). Assume that $\Delta_{f,2}$ and $\Delta_{f,1}$ are universally closed. Then $\Delta_{f,1}$ is separated and universally closed by Lemma 6.4. By Morphisms of Spaces, Lemma 9.7 and Algebraic Stacks, Lemma 10.9 we see that $\Delta_{f,1}$ is quasi-compact. Hence it is quasi-compact, separated, universally closed and locally of finite type (by Lemma 3.3) so proper. This proves "$\Leftarrow$" of (1). The proof of the implication in the other direction is omitted.

Proof of (2). This follows immediately from Lemma 6.4.

Proof of (3). This follows from the fact that $\Delta_{f,2}$ is always locally quasi-finite by Lemma 3.4 applied to $\Delta_f = \Delta_{f,1}$.

Proof of (4). This follows from the fact that $\Delta_{f,2}$ is always unramified as Lemma 3.4 applied to $\Delta_f = \Delta_{f,1}$ shows that $\Delta_{f,2}$ is locally of finite type and a monomorphism. See More on Morphisms of Spaces, Lemma 14.8. □

Lemma 6.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a separated (resp. quasi-separated, resp. quasi-DM, resp. DM) morphism of algebraic stacks. Then*

1. *given algebraic spaces T_i, $i = 1, 2$ and morphisms $x_i : T_i \to \mathcal{X}$, with $y_i = f \circ x_i$ the morphism*
$$T_1 \times_{x_1, \mathcal{X}, x_2} T_2 \longrightarrow T_1 \times_{y_1, \mathcal{Y}, y_2} T_2$$
is proper (resp. quasi-compact and quasi-separated, resp. locally quasi-finite, resp. unramified),
2. *given an algebraic space T and morphisms $x_i : T \to \mathcal{X}$, $i = 1, 2$, with $y_i = f \circ x_i$ the morphism*
$$Isom_{\mathcal{X}}(x_1, x_2) \longrightarrow Isom_{\mathcal{Y}}(y_1, y_2)$$
is proper (resp. quasi-compact and quasi-separated, resp. locally quasi-finite, resp. unramified).

Proof. Proof of (1). Observe that the diagram

$$\begin{tikzcd}
T_1 \times_{x_1, \mathcal{X}, x_2} T_2 \arrow[r] \arrow[d] & T_1 \times_{y_1, \mathcal{Y}, y_2} T_2 \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}
\end{tikzcd}$$

is cartesian. Hence this follows from the fact that f is separated (resp. quasi-separated, resp. quasi-DM, resp. DM) if and only if the diagonal is proper (resp. quasi-compact and quasi-separated, resp. locally quasi-finite, resp. unramified).

Proof of (2). This is true because

$$Isom_{\mathcal{X}}(x_1, x_2) = (T \times_{x_1, \mathcal{X}, x_2} T) \times_{T \times T, \Delta_T} T$$

hence the morphism in (2) is a base change of the morphism in (1). □

7. Quasi-compact morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 3 we have defined what it means for f to be quasi-compact. Here is another characterization.

Lemma 7.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent:*

1. *f is quasi-compact (as in Properties of Stacks, Section 3), and*
2. *for every quasi-compact algebraic stack $\mathcal{Z}$ and any morphism $\mathcal{Z} \to \mathcal{Y}$ the algebraic stack $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks with $\mathcal{Z}$ quasi-compact. By Properties of Stacks, Lemma 6.2 there exists a quasi-compact scheme U and a surjective smooth morphism $U \to \mathcal{Z}$. Since f is representable by algebraic spaces and quasi-compact we see by definition that $U \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and that $U \times_{\mathcal{Y}} \mathcal{X} \to U$ is quasi-compact. Hence $U \times_Y X$ is a quasi-compact algebraic space. The morphism $U \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is smooth and surjective (as the base change of the smooth and surjective morphism $U \to \mathcal{Z}$). Hence $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact by another application of Properties of Stacks, Lemma 6.2

Assume (2). Let $Z \to \mathcal{Y}$ be a morphism, where Z is a scheme. We have to show that the morphism of algebraic spaces $p : Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ is quasi-compact. Let $U \subset Z$ be affine open. Then $p^{-1}(U) = U \times_{\mathcal{Y}} \mathcal{Z}$ and the algebraic space $U \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact by assumption (2). Hence p is quasi-compact, see Morphisms of Spaces, Lemma 8.8. □

This motivates the following definition.

Definition 7.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *quasi-compact* if for every quasi-compact algebraic stack $\mathcal{Z}$ and morphism $\mathcal{Z} \to \mathcal{Y}$ the fibre product $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.

By Lemma 7.1 above this agrees with the already existing notion for morphisms of algebraic stacks representable by algebraic spaces. In particular this notion agrees with the notions already defined for morphisms between algebraic stacks and schemes.

Lemma 7.3. *The base change of a quasi-compact morphism of algebraic stacks by any morphism of algebraic stacks is quasi-compact.*

Proof. Omitted. □

Lemma 7.4. *The composition of a pair of quasi-compact morphisms of algebraic stacks is quasi-compact.*

Proof. Omitted. $\square$

Lemma 7.5. *A closed immersion of algebraic stacks is quasi-compact.*

Proof. This follows from the fact that immersions are always representable and the corresponding fact for closed immersion of algebraic spaces. $\square$

Lemma 7.6. *Let*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{Y} \arrow[dl, "q"] \\ & \mathcal{Z} & \end{tikzcd}$$

be a 2-commutative diagram of morphisms of algebraic stacks. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $\mathcal{T}$ be a quasi-compact algebraic stack, and let $\mathcal{T} \to \mathcal{Z}$ be a morphism. By Properties of Stacks, Lemma 5.3 the morphism $\mathcal{T} \times_\mathcal{Z} \mathcal{X} \to \mathcal{T} \times_\mathcal{Z} \mathcal{Y}$ is surjective and by assumption $\mathcal{T} \times_\mathcal{Z} \mathcal{X}$ is quasi-compact. Hence $\mathcal{T} \times_\mathcal{Z} \mathcal{Y}$ is quasi-compact by Properties of Stacks, Lemma 6.2. $\square$

Lemma 7.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : \mathcal{X} \to \mathcal{X} \times_\mathcal{Z} \mathcal{Y} \to \mathcal{Y}$. The first map is quasi-compact by Lemma 4.9 because it is a section of the quasi-separated morphism $\mathcal{X} \times_\mathcal{Z} \mathcal{Y} \to \mathcal{X}$ (a base change of g, see Lemma 4.4). The second map is quasi-compact as it is the base change of f, see Lemma 7.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 7.4. $\square$

Lemma 7.8. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If $\mathcal{X}$ is quasi-compact and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact.*
(2) *If $\mathcal{X}$ is quasi-compact and quasi-separated and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact and quasi-separated.*
(3) *A fibre product of quasi-compact and quasi-separated algebraic stacks is quasi-compact and quasi-separated.*

Proof. Part (1) follows from Lemma 7.7. Part (2) follows from (1) and Lemma 4.12. For (3) let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ be morphisms of quasi-compact and quasi-separated algebraic stacks. Then $\mathcal{X} \times_\mathcal{Y} \mathcal{Z} \to \mathcal{Z}$ is quasi-compact and quasi-separated as a base change of $\mathcal{X} \to \mathcal{Y}$ using (2) and Lemmas 7.3 and 4.4. Hence $\mathcal{X} \times_\mathcal{Y} \mathcal{Z}$ is quasi-compact and quasi-separated as an algebraic stack quasi-compact and quasi-separated over $\mathcal{Z}$, see Lemmas 4.11 and 7.4. $\square$

Lemma 7.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Let $y \in |\mathcal{Y}|$ be a point in the closure of the image of $|f|$. There exists a valuation ring A with fraction field K and a commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathcal{Y} \end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to y.

Proof. Choose an affine scheme V and a point $v \in V$ and a smooth morphism $V \to \mathcal{Y}$ sending v to y. Consider the base change diagram

$$\begin{tikzcd} V \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d, "g"'] & \mathcal{X} \arrow[d, "f"] \\ V \arrow[r] & \mathcal{Y} \end{tikzcd}$$

Recall that $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V| \times_{|\mathcal{Y}|} |\mathcal{X}|$ is surjective (Properties of Stacks, Lemma 4.3). Because $|V| \to |\mathcal{Y}|$ is open (Properties of Stacks, Lemma 4.7) we conclude that v is in the closure of the image of $|g|$. Thus it suffices to prove the lemma for the quasi-compact morphism g (Lemma 7.3) which we do in the next paragraph.

Assume $\mathcal{Y} = Y$ is an affine scheme. Then $\mathcal{X}$ is quasi-compact as f is quasi-compact (Definition 7.2). Choose an affine scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Then the image of $|f|$ is the image of $W \to Y$. By Morphisms, Lemma 6.5 we can choose a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \arrow[r] & Y \end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to y. Composing with $W \to \mathcal{X}$ we obtain a solution. $\square$

Lemma 7.10. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is quasi-compact, then f is quasi-compact.*

Proof. Assume $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is quasi-compact. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism with $\mathcal{Z}$ a quasi-compact algebraic stack. Choose a scheme U and a surjective smooth morphism $U \to W \times_{\mathcal{Y}} \mathcal{Z}$. Since $U \to \mathcal{Z}$ is flat, surjective, and locally of finite presentation and $\mathcal{Z}$ is quasi-compact, we can find a quasi-compact open subscheme $U' \subset U$ such that $U' \to \mathcal{Z}$ is surjective. Then $U' \times_{\mathcal{Y}} \mathcal{X} = U' \times_W (W \times_{\mathcal{Y}} \mathcal{X})$ is quasi-compact by assumption and surjects onto $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. Hence $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact as desired. $\square$

8. Noetherian algebraic stacks

We have already defined locally Noetherian algebraic stacks in Properties of Stacks, Section 7.

Definition 8.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *Noetherian* if $\mathcal{X}$ is quasi-compact, quasi-separated and locally Noetherian.

Note that a Noetherian algebraic stack $\mathcal{X}$ is not just quasi-compact and locally Noetherian, but also quasi-separated. In the language of Section 6 if we denote $p : \mathcal{X} \to \mathrm{Spec}(\mathbf{Z})$ the "absolute" structure morphism (i.e., the structure morphism of $\mathcal{X}$ viewed as an algebraic stack over $\mathbf{Z}$), then

$$\mathcal{X} \text{ Noetherian} \Leftrightarrow \mathcal{X} \text{ locally Noetherian and } \Delta_{p,0}, \Delta_{p,1}, \Delta_{p,2} \text{ quasi-compact.}$$

This will later mean that an algebraic stack of finite type over a Noetherian algebraic stack is not automatically Noetherian.

Lemma 8.2. *Let $j : \mathcal{X} \to \mathcal{Y}$ be an immersion of algebraic stacks.*

(1) *If $\mathcal{Y}$ is locally Noetherian, then $\mathcal{X}$ is locally Noetherian and j is quasi-compact.*
(2) *If $\mathcal{Y}$ is Noetherian, then $\mathcal{X}$ is Noetherian.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Then $U = \mathcal{X} \times_{\mathcal{Y}} V$ is a scheme and $V \to U$ is an immersion, see Properties of Stacks, Definition 9.1. Recall that $\mathcal{Y}$ is locally Noetherian if and only if V is locally Noetherian. In this case U is locally Noetherian too (Morphisms, Lemmas 15.5 and 15.6) and $U \to V$ is quasi-compact (Properties, Lemma 5.3). This shows that j is quasi-compact (Lemma 7.10) and that $\mathcal{X}$ is locally Noetherian. Finally, if $\mathcal{Y}$ is Noetherian, then we see from the above that $\mathcal{X}$ is quasi-compact and locally Noetherian. To finish the proof observe that j is separated and hence $\mathcal{X}$ is quasi-separated because $\mathcal{Y}$ is so by Lemma 4.11. $\square$

Lemma 8.3. *Let $\mathcal{X}$ be an algebraic stack.*

(1) *If $\mathcal{X}$ is locally Noetherian then $|\mathcal{X}|$ is a locally Noetherian topological space.*
(2) *If $\mathcal{X}$ is quasi-compact and locally Noetherian, then $|\mathcal{X}|$ is a Noetherian topological space.*

Proof. Assume $\mathcal{X}$ is locally Noetherian. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. As $\mathcal{X}$ is locally Noetherian we see that U is locally Noetherian. By Properties, Lemma 5.5 this means that $|U|$ is a locally Noetherian topological space. Since $|U| \to |\mathcal{X}|$ is open and surjective we conclude that $|\mathcal{X}|$ is locally Noetherian by Topology, Lemma 9.3. This proves (1). If $\mathcal{X}$ is quasi-compact and locally Noetherian, then $|\mathcal{X}|$ is quasi-compact and locally Noetherian. Hence $|\mathcal{X}|$ is Noetherian by Topology, Lemma 12.14. $\square$

Lemma 8.4. *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Then $|\mathcal{X}|$ is quasi-sober (Topology, Definition 8.6).*

Proof. We have to prove that every irreducible closed subset $T \subset |\mathcal{X}|$ has a generic point. Choose an affine scheme U and a smooth morphism $f : U \to \mathcal{X}$ such that $f^{-1}(T) \subset |U|$ is nonempty. Since U is Noetherian, the closed subset $f^{-1}(T)$ has finitely many irreducible components (Topology, Lemma 9.2). Say $f^{-1}(T) = Z_1 \cup \ldots \cup Z_n$ is the decomposition into irreducible components. As f is open, the image of $f|_{f^{-1}(T)} : f^{-1}(T) \to T$ contains a nonempty open subset of T. Since T is irreducible, this means that $f(f^{-1}(T))$ is dense. Since T is irreducible, it follows that $f(Z_i)$ is dense for some i. Then if $\xi_i \in Z_i$ is the generic point we see that $f(\xi_i)$ is a generic point of T. $\square$

9. Affine morphisms

Affine morphisms of algebraic stacks are defined as follows.

Definition 9.1. A morphism of algebraic stacks is said to be *affine* if it is representable and affine in the sense of Properties of Stacks, Section 3.

For us it is a little bit more convenient to think of an affine morphism of algebraic stacks as a morphism of algebraic stacks which is representable by algebraic spaces and affine in the sense of Properties of Stacks, Section 3. (Recall that the default for "representable" in the Stacks project is representable by schemes.) Since this is clearly equivalent to the notion just defined we shall use this characterization without further mention. We prove a few simple lemmas about this notion.

Lemma 9.2. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be an affine morphism of algebraic stacks. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is an affine morphism of algebraic stacks.*

Proof. This follows from the discussion in Properties of Stacks, Section 3. $\square$

Lemma 9.3. *Compositions of affine morphisms of algebraic stacks are affine.*

Proof. This follows from the discussion in Properties of Stacks, Section 3 and Morphisms of Spaces, Lemma 20.4. $\square$

Lemma 9.4. *Let*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "a"'] & & \mathcal{Y} \arrow[dl, "b"] \\ & \mathcal{Z} & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic stacks. If a is affine and Δ_b is affine, then f is affine.

Proof. The base change $\mathrm{pr}_2 : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$ of a is affine by Lemma 9.2. The morphism $(1, f) : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is the base change of $\Delta_b : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}$ by the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}$ (see material in Categories, Section 31). Hence it is affine by Lemma 9.2. The composition $f = \mathrm{pr}_2 \circ (1, f)$ of affine morphisms is affine by Lemma 9.3 and the proof is done. $\square$

10. Integral and finite morphisms

Integral and finite morphisms of algebraic stacks are defined as follows.

Definition 10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

1. We say f is *integral* if f is representable and integral in the sense of Properties of Stacks, Section 3.
2. We say f is *finite* if f is representable and finite in the sense of Properties of Stacks, Section 3.

For us it is a little bit more convenient to think of an integral, resp. finite morphism of algebraic stacks as a morphism of algebraic stacks which is representable by algebraic spaces and integral, resp. finite in the sense of Properties of Stacks, Section 3. (Recall that the default for "representable" in the Stacks project is representable by schemes.) Since this is clearly equivalent to the notion just defined we shall use this characterization without further mention. We prove a few simple lemmas about this notion.

Lemma 10.2. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be an integral (or finite) morphism of algebraic stacks. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is an integral (or finite) morphism of algebraic stacks.*

Proof. This follows from the discussion in Properties of Stacks, Section 3. $\square$

Lemma 10.3. *Compositions of integral, resp. finite morphisms of algebraic stacks are integral, resp. finite.*

Proof. This follows from the discussion in Properties of Stacks, Section 3 and Morphisms of Spaces, Lemma 45.4. $\square$

11. Open morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 3 we have defined what it means for f to be universally open. Here is another characterization.

Lemma 11.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

(1) *f is universally open (as in Properties of Stacks, Section 3), and*
(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is open.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ of algebraic spaces is universally open, in particular the map $|V \times_\mathcal{Y} \mathcal{X}| \to |V|$ is open. By Properties of Stacks, Section 4 in the commutative diagram

$$\begin{tikzcd} {|V \times_\mathcal{Y} \mathcal{X}|} \arrow[r] \arrow[d] & {|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|} \arrow[d] \\ {|V|} \arrow[r] & {|\mathcal{Z}|} \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_\mathcal{Y} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is open it follows that the right vertical arrow is open. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. $\square$

Thus we may use the following natural definition.

Definition 11.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *open* if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is open.
(2) We say f is *universally open* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces
$$|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$$
is open, i.e., the base change $\mathcal{Z} \times_\mathcal{Y} \mathcal{X} \to \mathcal{Z}$ is open.

Lemma 11.3. *The base change of a universally open morphism of algebraic stacks by any morphism of algebraic stacks is universally open.*

Proof. This is immediate from the definition. $\square$

Lemma 11.4. *The composition of a pair of (universally) open morphisms of algebraic stacks is (universally) open.*

Proof. Omitted. $\square$

12. Submersive morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 3 we have defined what it means for f to be universally submersive. Here is another characterization.

Lemma 12.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

1. *f is universally submersive (as in Properties of Stacks, Section 3), and*
2. *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is submersive.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ of algebraic spaces is universally submersive, in particular the map $|V \times_\mathcal{Y} \mathcal{X}| \to |V|$ is submersive. By Properties of Stacks, Section 4 in the commutative diagram

$$\begin{tikzcd} {|V \times_\mathcal{Y} \mathcal{X}|} \arrow[r] \arrow[d] & {|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|} \arrow[d] \\ {|V|} \arrow[r] & {|\mathcal{Z}|} \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_\mathcal{Y} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is submersive it follows that the right vertical arrow is submersive. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. □

Thus we may use the following natural definition.

Definition 12.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

1. We say f is *submersive*[4] if the continuous map $|\mathcal{X}| \to |\mathcal{Y}|$ is submersive, see Topology, Definition 6.3.
2. We say f is *universally submersive* if for every morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the base change $\mathcal{Y}' \times_\mathcal{Y} \mathcal{X} \to \mathcal{Y}'$ is submersive.

We note that a submersive morphism is in particular surjective.

Lemma 12.3. *The base change of a universally submersive morphism of algebraic stacks by any morphism of algebraic stacks is universally submersive.*

Proof. This is immediate from the definition. □

Lemma 12.4. *The composition of a pair of (universally) submersive morphisms of algebraic stacks is (universally) submersive.*

Proof. Omitted. □

13. Universally closed morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 3 we have defined what it means for f to be universally closed. Here is another characterization.

Lemma 13.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

1. *f is universally closed (as in Properties of Stacks, Section 3), and*

[4] This is very different from the notion of a submersion of differential manifolds.

(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is closed.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ of algebraic spaces is universally closed, in particular the map $|V \times_\mathcal{Y} \mathcal{X}| \to |V|$ is closed. By Properties of Stacks, Section 4 in the commutative diagram

$$\begin{tikzcd} {|V \times_\mathcal{Y} \mathcal{X}|} \arrow[r] \arrow[d] & {|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|} \arrow[d] \\ {|V|} \arrow[r] & {|\mathcal{Z}|} \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_\mathcal{Y} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is closed it follows that the right vertical arrow is closed. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. $\square$

Thus we may use the following natural definition.

Definition 13.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *closed* if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is closed.
(2) We say f is *universally closed* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces
$$|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$$
is closed, i.e., the base change $\mathcal{Z} \times_\mathcal{Y} \mathcal{X} \to \mathcal{Z}$ is closed.

Lemma 13.3. *The base change of a universally closed morphism of algebraic stacks by any morphism of algebraic stacks is universally closed.*

Proof. This is immediate from the definition. $\square$

Lemma 13.4. *The composition of a pair of (universally) closed morphisms of algebraic stacks is (universally) closed.*

Proof. Omitted. $\square$

Lemma 13.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is universally closed,*
(2) *for every scheme Z and every morphism $Z \to \mathcal{Y}$ the projection $|Z \times_\mathcal{Y} \mathcal{X}| \to |Z|$ is closed,*
(3) *for every affine scheme Z and every morphism $Z \to \mathcal{Y}$ the projection $|Z \times_\mathcal{Y} \mathcal{X}| \to |Z|$ is closed, and*
(4) *there exists an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$ such that $V \times_\mathcal{Y} \mathcal{X} \to V$ is a universally closed morphism of algebraic stacks.*

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective smooth morphism $V \to \mathcal{Y}$. We are going to show that $V \times_\mathcal{Y} \mathcal{X} \to V$ is a universally closed morphism of algebraic stacks. Let $\mathcal{Z} \to V$ be a morphism from an algebraic stack to V. Let $W \to \mathcal{Z}$ be a surjective smooth morphism where $W = \coprod W_i$ is a disjoint union of affine schemes. Then we have the following commutative diagram

$$\begin{tikzcd} \coprod_i |W_i \times_\mathcal{Y} \mathcal{X}| \arrow[r, equal] \arrow[d] & |W \times_\mathcal{Y} \mathcal{X}| \arrow[r] \arrow[d] & |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \arrow[r, equal] \arrow[d] & |\mathcal{Z} \times_V (V \times_\mathcal{Y} \mathcal{X})| \arrow[ld] \\ \coprod |W_i| \arrow[r, equal] & |W| \arrow[r] & |\mathcal{Z}| & \end{tikzcd}$$

We have to show the south-east arrow is closed. The middle horizontal arrows are surjective and open (Properties of Stacks, Lemma 4.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are closed. Hence it follows that the right vertical arrow is closed.

Assume (4). We will show that f is universally closed. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider the diagram

$$\begin{tikzcd} |(V \times_\mathcal{Y} \mathcal{Z}) \times_V (V \times_\mathcal{Y} \mathcal{X})| \arrow[r, equal] \arrow[rd] & |V \times_\mathcal{Y} \mathcal{X}| \arrow[r] \arrow[d] & |Z \times_\mathcal{Y} \mathcal{X}| \arrow[d] \\ & |V \times_\mathcal{Y} \mathcal{Z}| \arrow[r] & |\mathcal{Z}| \end{tikzcd}$$

The south-west arrow is closed by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic stacks are surjective and smooth (see reference above). It follows that the right vertical arrow is closed. ☐

14. Universally injective morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 3 we have defined what it means for f to be universally injective. Here is another characterization.

Lemma 14.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

(1) *f is universally injective (as in Properties of Stacks, Section 3), and*
(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is injective.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ of algebraic spaces is universally injective, in particular the map $|V \times_\mathcal{Y} \mathcal{X}| \to |V|$ is injective. By Properties of Stacks, Section 4 in the commutative diagram

$$\begin{tikzcd} |V \times_\mathcal{Y} \mathcal{X}| \arrow[r] \arrow[d] & |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \arrow[d] \\ |V| \arrow[r] & |\mathcal{Z}| \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_\mathcal{Y} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is injective it follows that the right vertical arrow is injective. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. $\square$

Thus we may use the following natural definition.

Definition 14.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *universally injective* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map

$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$$

is injective.

Lemma 14.3. *The base change of a universally injective morphism of algebraic stacks by any morphism of algebraic stacks is universally injective.*

Proof. This is immediate from the definition. $\square$

Lemma 14.4. *The composition of a pair of universally injective morphisms of algebraic stacks is universally injective.*

Proof. Omitted. $\square$

Lemma 14.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is universally injective,*
(2) *$\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is surjective, and*
(3) *for an algebraically closed field, for $x_1, x_2 : \mathrm{Spec}(k) \to \mathcal{X}$, and for a 2-arrow $\beta : f \circ x_1 \to f \circ x_2$ there is a 2-arrow $\alpha : x_1 \to x_2$ with $\beta = id_f \star \alpha$.*

Proof. (1) $\Rightarrow$ (2). If f is universally injective, then the first projection $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{X}|$ is injective, which implies that $|\Delta|$ is surjective.

(2) $\Rightarrow$ (1). Assume Δ is surjective. Then any base change of Δ is surjective (see Properties of Stacks, Section 5). Since the diagonal of a base change of f is a base change of Δ, we see that it suffices to show that $|\mathcal{X}| \to |\mathcal{Y}|$ is injective. If not, then by Properties of Stacks, Lemma 4.3 we find that the first projection $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{X}|$ is not injective. Of course this means that $|\Delta|$ is not surjective.

(3) $\Rightarrow$ (2). Let $t \in |\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}|$. Then we can represent t by a morphism $t : \mathrm{Spec}(k) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ with k an algebraically closed field. By our construction of 2-fibre products we can represent t by (x_1, x_2, β) where $x_1, x_2 : \mathrm{Spec}(k) \to \mathcal{X}$ and $\beta : f \circ x_1 \to f \circ x_2$ is a 2-morphism. Then (3) implies that there is a 2-morphism $\alpha : x_1 \to x_2$ mapping to β. This exactly means that $\Delta(x_1) = (x_1, x_1, \mathrm{id})$ is isomorphic to t. Hence (2) holds.

(2) $\Rightarrow$ (3). Let $x_1, x_2 : \mathrm{Spec}(k) \to \mathcal{X}$ be morphisms with k an algebraically closed field. Let $\beta : f \circ x_1 \to f \circ x_2$ be a 2-morphism. As in the previous paragraph, we obtain a morphism $t = (x_1, x_2, \beta) : \mathrm{Spec}(k) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 3.3

$$T = \mathcal{X} \times_{\Delta, \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, t} \mathrm{Spec}(k)$$

is an algebraic space locally of finite type over $\mathrm{Spec}(k)$. Condition (2) implies that T is nonempty. Then since k is algebraically closed, there is a k-point in T. Unwinding the definitions this means there is a morphism $\alpha : x_1 \to x_2$ in $\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})$ such that $\beta = \mathrm{id}_f \star \alpha$. $\square$

Lemma 14.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a universally injective morphism of algebraic stacks. Let $y : \mathrm{Spec}(k) \to \mathcal{Y}$ be a morphism where k is an algebraically closed field. If y is in the image of $|\mathcal{X}| \to |\mathcal{Y}|$, then there is a morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ with $y = f \circ x$.*

Proof. We first remark this lemma is not a triviality, because the assumption that y is in the image of $|f|$ means only that we can lift y to a morphism into $\mathcal{X}$ after possibly replacing k by an extension field. To prove the lemma we may base change f by y, hence we may assume we have a nonempty algebraic stack $\mathcal{X}$ and a universally injective morphism $\mathcal{X} \to \mathrm{Spec}(k)$ and we want to find a k-valued point of $\mathcal{X}$. We may replace $\mathcal{X}$ by its reduction. We may choose a field k' and a surjective, flat, locally finite type morphism $\mathrm{Spec}(k') \to \mathcal{X}$, see Properties of Stacks, Lemma 11.2. Since $\mathcal{X} \to \mathrm{Spec}(k)$ is universally injective, we find that

$$\mathrm{Spec}(k') \times_{\mathcal{X}} \mathrm{Spec}(k') \to \mathrm{Spec}(k' \otimes_k k')$$

is surjective as the base change of the surjective morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathrm{Spec}(k)} \mathcal{X}$ (Lemma 14.5). Since k is algebraically closed $k' \otimes_k k'$ is a domain (Algebra, Lemma 49.4). Let $\xi \in \mathrm{Spec}(k') \times_{\mathcal{X}} \mathrm{Spec}(k')$ be a point mapping to the generic point of $\mathrm{Spec}(k' \otimes_k k')$. Let U be the reduced induced closed subscheme structure on the connected component of $\mathrm{Spec}(k') \times_{\mathcal{X}} \mathrm{Spec}(k')$ containing ξ. Then the two projections $U \to \mathrm{Spec}(k')$ are locally of finite type, as this was true for the projections $\mathrm{Spec}(k') \times_{\mathcal{X}} \mathrm{Spec}(k') \to \mathrm{Spec}(k')$ as base changes of the morphism $\mathrm{Spec}(k') \to \mathcal{X}$. Applying Varieties, Proposition 31.1 we find that the integral closures of the two images of k' in $\Gamma(U, \mathcal{O}_U)$ are equal. Looking in $\kappa(\xi)$ means that any element of the form $\lambda \otimes 1$ is algebraically dependend on the subfield

$$1 \otimes k' \subset (\text{fraction field of } k' \otimes_k k') \subset \kappa(\xi).$$

Since k is algebraically closed, this is only possible if $k' = k$ and the proof is complete. □

Lemma 14.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

(1) *f is universally injective,*
(2) *for every affine scheme Z and any morphism $Z \to \mathcal{Y}$ the morphism $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ is universally injective, and*
(3) *add more here.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2) holds. We will show that $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is surjective, which implies (1) by Lemma 14.5. Consider an affine scheme V and a smooth morphism $V \to \mathcal{Y}$. Since $g : V \times_{\mathcal{Y}} \mathcal{X} \to V$ is universally injective by (2), we see that Δ_g is surjective. However, Δ_g is the base change of Δ_f by the smooth morphism $V \to \mathcal{Y}$. Since the collection of these morphisms $V \to \mathcal{Y}$ are jointly surjective, we conclude Δ_f is surjective. □

Lemma 14.8. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is universally injective, then f is universally injective.*

Proof. Observe that the diagonal Δ_g of the morphism $g : W \times_{\mathcal{Y}} \mathcal{X} \to W$ is the base change of Δ_f by $W \to \mathcal{Y}$. Hence if Δ_g is surjective, then so is Δ_f by Properties of Stacks, Lemma 3.3. Thus the lemma follows from the characterization (2) in Lemma 14.5. □

15. Universal homeomorphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 3 we have defined what it means for f to be a universal homeomorphism. Here is another characterization.

Lemma 15.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

1. *f is a universal homeomorphism (Properties of Stacks, Section 3), and*
2. *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map of topological spaces $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is a homeomorphism.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ of algebraic spaces is a universal homeomorphism, in particular the map $|V \times_\mathcal{Y} \mathcal{X}| \to |V|$ is a homeomorphism. By Properties of Stacks, Section 4 in the commutative diagram

$$\begin{tikzcd} |V \times_\mathcal{Y} \mathcal{X}| \arrow[r] \arrow[d] & |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \arrow[d] \\ |V| \arrow[r] & |\mathcal{Z}| \end{tikzcd}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_\mathcal{Y} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is a homeomorphism it follows that the right vertical arrow is a homeomorphism. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. □

Thus we may use the following natural definition.

Definition 15.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is a *universal homeomorphism* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the map of topological spaces

$$|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$$

is a homeomorphism.

Lemma 15.3. *The base change of a universal homeomorphism of algebraic stacks by any morphism of algebraic stacks is a universal homeomorphism.*

Proof. This is immediate from the definition. □

Lemma 15.4. *The composition of a pair of universal homeomorphisms of algebraic stacks is a universal homeomorphism.*

Proof. Omitted. □

Lemma 15.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_\mathcal{Y} \mathcal{X} \to W$ is a universal homeomorphism, then f is a universal homeomorphism.*

Proof. Assume $g : W \times_\mathcal{Y} \mathcal{X} \to W$ is a universal homeomorphism. Then g is universally injective, hence f is universally injective by Lemma 14.8. On the other hand, let $\mathcal{Z} \to \mathcal{Y}$ be a morphism with $\mathcal{Z}$ an algebraic stack. Choose a scheme U and a surjective smooth morphism $U \to W \times_\mathcal{Y} \mathcal{Z}$. Consider the diagram

$$\begin{tikzcd} W \times_\mathcal{Y} \mathcal{X} \arrow[d, "g"] & U \times_\mathcal{Y} \mathcal{X} \arrow[l] \arrow[r] \arrow[d] & \mathcal{Z} \times_\mathcal{Y} \mathcal{X} \arrow[d] \\ W & U \arrow[l] \arrow[r] & \mathcal{Z} \end{tikzcd}$$

The middle vertical arrow induces a homeomorphism on topological space by assumption on g. The morphism $U \to \mathcal{Z}$ and $U \times_\mathcal{Y} \mathcal{X} \to \mathcal{Z} \times_\mathcal{Y} \mathcal{X}$ are surjective, flat, and locally of finite presentation hence induce open maps on topological spaces. We conclude that $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is open. Surjectivity is easy to prove; we omit the proof. □

16. Types of morphisms smooth local on source-and-target

Given a property of morphisms of algebraic spaces which is *smooth local on the source-and-target*, see Descent on Spaces, Definition 20.1 we may use it to define a corresponding property of morphisms of algebraic stacks, namely by imposing either of the equivalent conditions of the lemma below.

Lemma 16.1. *Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider commutative diagrams*

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

where U and V are algebraic spaces and the vertical arrows are smooth. The following are equivalent

1. *for any diagram as above such that in addition $U \to \mathcal{X} \times_\mathcal{Y} V$ is smooth the morphism h has property $\mathcal{P}$, and*
2. *for some diagram as above with $a : U \to \mathcal{X}$ surjective the morphism h has property $\mathcal{P}$.*

If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces, then this is also equivalent to f (as a morphism of algebraic spaces) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for morphisms f which are representable by algebraic spaces this is also equivalent to f having property $\mathcal{P}$ in the sense of Properties of Stacks, Section 3.

Proof. Let us prove the implication (1) $\Rightarrow$ (2). Pick an algebraic space V and a surjective and smooth morphism $V \to \mathcal{Y}$. Pick an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_\mathcal{Y} V$. Note that $U \to \mathcal{X}$ is surjective and smooth as well, as a composition of the base change $\mathcal{X} \times_\mathcal{Y} V \to \mathcal{X}$ and the chosen map $U \to \mathcal{X} \times_\mathcal{Y} V$. Hence we obtain a diagram as in (1). Thus if (1) holds, then $h : U \to V$ has property $\mathcal{P}$, which means that (2) holds as $U \to \mathcal{X}$ is surjective.

Conversely, assume (2) holds and let U, V, a, b, h be as in (2). Next, let U', V', a', b', h' be any diagram as in (1). Picture

$$
\begin{tikzcd}
U \arrow[r, "h"'] \arrow[d] & V \arrow[d] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
U' \arrow[r, "h'"'] \arrow[d] & V' \arrow[d] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd}
$$

To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the commutative diagram

$$
\begin{tikzcd}
U \arrow[dd, "h"] & U \times_{\mathcal{X}} U' \arrow[l] \arrow[r] \arrow[d] \arrow[dd, bend left, "{(h,h')}"] & U' \arrow[dd, "h'"] \\
& U \times_{\mathcal{Y}} V' \arrow[ul] \arrow[d] & \\
V & V \times_{\mathcal{Y}} V' \arrow[l] \arrow[r] & V'
\end{tikzcd}
$$

of algebraic spaces. Note that the horizontal arrows are smooth as base changes of the smooth morphisms $V \to \mathcal{Y}$, $V' \to \mathcal{Y}$, $U \to \mathcal{X}$, and $U' \to \mathcal{X}$. Note that

$$
\begin{tikzcd}
U \times_{\mathcal{X}} U' \arrow[r] \arrow[d] & U' \arrow[d] \\
U \times_{\mathcal{Y}} V' \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} V'
\end{tikzcd}
$$

is cartesian, hence the left vertical arrow is smooth as U', V', a', b', h' is as in (1). Since $\mathcal{P}$ is smooth local on the target by Descent on Spaces, Lemma 20.2 part (2) we see that the base change $U \times_{\mathcal{Y}} V' \to V \times_{\mathcal{Y}} V'$ has $\mathcal{P}$. Since $\mathcal{P}$ is smooth local on the source by Descent on Spaces, Lemma 20.2 part (1) we can precompose by the smooth morphism $U \times_{\mathcal{X}} U' \to U \times_{\mathcal{Y}} V'$ and conclude (h, h') has $\mathcal{P}$. Since $V \times_{\mathcal{Y}} V' \to V'$ is smooth we conclude $U \times_{\mathcal{X}} U' \to V'$ has $\mathcal{P}$ by Descent on Spaces, Lemma 20.2 part (3). Finally, since $U \times_X U' \to U'$ is surjective and smooth and $\mathcal{P}$ is smooth local on the source (same lemma) we conclude that h' has $\mathcal{P}$. This finishes the proof of the equivalence of (1) and (2).

If $\mathcal{X}$ and $\mathcal{Y}$ are representable, then Descent on Spaces, Lemma 20.3 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to \mathcal{X}$ the base change $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. Consider the diagram

$$
\begin{tikzcd}
Z \times_{\mathcal{Y}} U \arrow[r] \arrow[d] & Z \times_{\mathcal{Y}} V \arrow[d] \\
Z \times_{\mathcal{Y}} \mathcal{X} \arrow[r] & Z
\end{tikzcd}
$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is smooth and surjective and the right vertical arrow is smooth. Thus Descent on Spaces, Lemma 20.3 kicks in and shows that $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. $\square$

Definition 16.2. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. We say a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks *has property* $\mathcal{P}$ if the equivalent conditions of Lemma 16.1 hold.

Remark 16.3. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under composition. Then the property of morphisms of algebraic stacks defined in Definition 16.2 is stable under composition. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks having property $\mathcal{P}$. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Finally, choose an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Then the morphisms $V \to W$ and $U \to V$ have property $\mathcal{P}$ by definition. Whence $U \to W$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under composition. Thus, by definition again, we see that $g \circ f : \mathcal{X} \to \mathcal{Z}$ has property $\mathcal{P}$.

Remark 16.4. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under base change. Then the property of morphisms of algebraic stacks defined in Definition 16.2 is stable under base change. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks and assume f has property $\mathcal{P}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Finally, choose an algebraic space V' and a surjective and smooth morphism $V' \to \mathcal{Y}' \times_{\mathcal{Y}} V$. Then the morphism $U \to V$ has property $\mathcal{P}$ by definition. Whence $V' \times_V U \to V'$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under base change. Considering the diagram

$$\begin{tikzcd} V' \times_V U \arrow[r] \arrow[d] & \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ V' \arrow[r] & \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

we see that the left top horizontal arrow is smooth and surjective, whence by definition we see that the projection $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property $\mathcal{P}$.

Remark 16.5. Let $\mathcal{P}, \mathcal{P}'$ be properties of morphisms of algebraic spaces which are smooth local on the source-and-target. Suppose that we have $\mathcal{P} \Rightarrow \mathcal{P}'$ for morphisms of algebraic spaces. Then we also have $\mathcal{P} \Rightarrow \mathcal{P}'$ for the properties of morphisms of algebraic stacks defined in Definition 16.2 using $\mathcal{P}$ and $\mathcal{P}'$. This is clear from the definition.

17. Morphisms of finite type

The property "locally of finite type" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 23.3 and Descent on Spaces, Lemma 11.9. Hence, by Lemma 16.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite type as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 3 when the morphism is representable by algebraic spaces.

Definition 17.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f *locally of finite type* if the equivalent conditions of Lemma 16.1 hold with $\mathcal{P}$ = locally of finite type.
(2) We say f is *of finite type* if it is locally of finite type and quasi-compact.

Lemma 17.2. *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. Combine Remark 16.3 with Morphisms of Spaces, Lemma 23.2. $\square$

Lemma 17.3. *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. Combine Remark 16.4 with Morphisms of Spaces, Lemma 23.3. $\square$

Lemma 17.4. *An immersion is locally of finite type.*

Proof. Combine Remark 16.5 with Morphisms of Spaces, Lemma 23.7. $\square$

Lemma 17.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and $\mathcal{Y}$ is locally Noetherian, then $\mathcal{X}$ is locally Noetherian.*

Proof. Let

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a commutative diagram where U, V are schemes, $V \to \mathcal{Y}$ is surjective and smooth, and $U \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective and smooth. Then $U \to V$ is locally of finite type. If $\mathcal{Y}$ is locally Noetherian, then V is locally Noetherian. By Morphisms, Lemma 15.6 we see that U is locally Noetherian, which means that $\mathcal{X}$ is locally Noetherian. $\square$

The following two lemmas will be improved on later (after we have discussed morphisms of algebraic stacks which are locally of finite presentation).

Lemma 17.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is locally of finite type, then f is locally of finite type.*

Proof. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is locally of finite presentation. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of locally finite type morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is locally of finite type, we see that $U' \to V'$ is locally of finite type. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is locally of finite type by Descent on Spaces, Lemma 11.9 and we win. $\square$

Lemma 17.7. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume $\mathcal{X} \to \mathcal{Z}$ is locally of finite type and that $\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite type.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Set $U = V \times_{\mathcal{Y}} \mathcal{X}$ which is an algebraic space. We know that $U \to V$ is surjective, flat, and locally of finite presentation and that $U \to W$ is locally of finite type. Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 16.2. □

Lemma 17.8. *Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f : \mathcal{X} \to \mathcal{Z}$ is locally of finite type, then $f : \mathcal{X} \to \mathcal{Y}$ is locally of finite type.*

Proof. We can find a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

where U, V, W are schemes, the vertical arrow $W \to \mathcal{Z}$ is surjective and smooth, the arrow $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ is surjective and smooth, and the arrow $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is surjective and smooth. Then also $U \to \mathcal{X} \times_{\mathcal{Z}} V$ is surjective and smooth (as a composition of a surjective and smooth morphism with a base change of such). By definition we see that $U \to W$ is locally of finite type. Hence $U \to V$ is locally of finite type by Morphisms, Lemma 15.8 which in turn means (by definition) that $\mathcal{X} \to \mathcal{Y}$ is locally of finite type. □

18. Points of finite type

Let $\mathcal{X}$ be an algebraic stack. A finite type point $x \in |\mathcal{X}|$ is a point which can be represented by a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ which is locally of finite type. Finite type points are a suitable replacement of closed points for algebraic spaces and algebraic stacks. There are always "enough of them" for example.

Lemma 18.1. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:*

(1) *There exists a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ which is locally of finite type and represents x.*
(2) *There exists a scheme U, a closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ such that $\varphi(u) = x$.*

Proof. Let $u \in U$ and $U \to \mathcal{X}$ be as in (2). Then $\mathrm{Spec}(\kappa(u)) \to U$ is of finite type, and $U \to \mathcal{X}$ is representable and locally of finite type (by Morphisms of Spaces, Lemmas 39.8 and 28.5). Hence we see (1) holds by Lemma 17.2.

Conversely, assume $\mathrm{Spec}(k) \to \mathcal{X}$ is locally of finite type and represents x. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. By assumption $U \times_{\mathcal{X}} \mathrm{Spec}(k) \to U$ is a morphism of algebraic spaces which is locally of finite type. Pick a finite type point v of $U \times_{\mathcal{X}} \mathrm{Spec}(k)$ (there exists at least one, see Morphisms of Spaces, Lemma 25.3). By Morphisms of Spaces, Lemma 25.4 the image $u \in U$

of v is a finite type point of U. Hence by Morphisms, Lemma 16.4 after shrinking U we may assume that u is a closed point of U, i.e., (2) holds. $\square$

Definition 18.2. Let $\mathcal{X}$ be an algebraic stack. We say a point $x \in |\mathcal{X}|$ is a *finite type point*[5] if the equivalent conditions of Lemma 18.1 are satisfied. We denote $\mathcal{X}_{\text{ft-pts}}$ the set of finite type points of $\mathcal{X}$.

We can describe the set of finite type points as follows.

Lemma 18.3. *Let $\mathcal{X}$ be an algebraic stack. We have*

$$\mathcal{X}_{\textit{ft-pts}} = \bigcup\nolimits_{\varphi : U \to \mathcal{X} \ \textit{smooth}} |\varphi|(U_0)$$

where U_0 is the set of closed points of U. Here we may let U range over all schemes smooth over $\mathcal{X}$ or over all affine schemes smooth over $\mathcal{X}$.

Proof. Immediate from Lemma 18.1. $\square$

Lemma 18.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type, then $f(\mathcal{X}_{\textit{ft-pts}}) \subset \mathcal{Y}_{\textit{ft-pts}}$.*

Proof. Take $x \in \mathcal{X}_{\text{ft-pts}}$. Represent x by a locally finite type morphism $x : \text{Spec}(k) \to \mathcal{X}$. Then $f \circ x$ is locally of finite type by Lemma 17.2. Hence $f(x) \in \mathcal{Y}_{\text{ft-pts}}$. $\square$

Lemma 18.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and surjective, then $f(\mathcal{X}_{\textit{ft-pts}}) = \mathcal{Y}_{\textit{ft-pts}}$.*

Proof. We have $f(\mathcal{X}_{\text{ft-pts}}) \subset \mathcal{Y}_{\text{ft-pts}}$ by Lemma 18.4. Let $y \in |\mathcal{Y}|$ be a finite type point. Represent y by a morphism $\text{Spec}(k) \to \mathcal{Y}$ which is locally of finite type. As f is surjective the algebraic stack $\mathcal{X}_k = \text{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}$ is nonempty, therefore has a finite type point $x \in |\mathcal{X}_k|$ by Lemma 18.3. Now $\mathcal{X}_k \to \mathcal{X}$ is a morphism which is locally of finite type as a base change of $\text{Spec}(k) \to \mathcal{Y}$ (Lemma 17.3). Hence the image of x in $\mathcal{X}$ is a finite type point by Lemma 18.4 which maps to y by construction. $\square$

Lemma 18.6. *Let $\mathcal{X}$ be an algebraic stack. For any locally closed subset $T \subset |\mathcal{X}|$ we have*

$$T \neq \emptyset \Rightarrow T \cap \mathcal{X}_{\textit{ft-pts}} \neq \emptyset.$$

In particular, for any closed subset $T \subset |\mathcal{X}|$ we see that $T \cap \mathcal{X}_{\textit{ft-pts}}$ is dense in T.

Proof. Let $i : \mathcal{Z} \to \mathcal{X}$ be the reduced induced substack structure on T, see Properties of Stacks, Remark 10.5. An immersion is locally of finite type, see Lemma 17.4. Hence by Lemma 18.4 we see $\mathcal{Z}_{\text{ft-pts}} \subset \mathcal{X}_{\text{ft-pts}} \cap T$. Finally, any nonempty affine scheme U with a smooth morphism towards $\mathcal{Z}$ has at least one closed point, hence $\mathcal{Z}$ has at least one finite type point by Lemma 18.3. The lemma follows. $\square$

Here is another, more technical, characterization of a finite type point on an algebraic stack. It tells us in particular that the residual gerbe of $\mathcal{X}$ at x exists whenever x is a finite type point!

Lemma 18.7. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:*

[5]This is a slight abuse of language as it would perhaps be more correct to say "locally finite type point".

(1) *x is a finite type point,*
(2) *there exists an algebraic stack $\mathcal{Z}$ whose underlying topological space $|\mathcal{Z}|$ is a singleton, and a morphism $f : \mathcal{Z} \to \mathcal{X}$ which is locally of finite type such that $\{x\} = |f|(|\mathcal{Z}|)$, and*
(3) *the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and the inclusion morphism $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type.*

Proof. (All of the morphisms occurring in this paragraph are representable by algebraic spaces, hence the conventions and results of Properties of Stacks, Section 3 are applicable.) Assume x is a finite type point. Choose an affine scheme U, a closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ with $\varphi(u) = x$, see Lemma 18.3. Set $u = \mathrm{Spec}(\kappa(u))$ as usual. Set $R = u \times_\mathcal{X} u$ so that we obtain a groupoid in algebraic spaces (u, R, s, t, c), see Algebraic Stacks, Lemma 16.1. The projection morphisms $R \to u$ are the compositions

$$R = u \times_\mathcal{X} u \to u \times_\mathcal{X} U \to u \times_\mathcal{X} X = u$$

where the first arrow is of finite type (a base change of the closed immersion of schemes $u \to U$) and the second arrow is smooth (a base change of the smooth morphism $U \to \mathcal{X}$). Hence $s, t : R \to u$ are locally of finite type (as compositions, see Morphisms of Spaces, Lemma 23.2). Since u is the spectrum of a field, it follows that s, t are flat and locally of finite presentation (by Morphisms of Spaces, Lemma 28.7). We see that $\mathcal{Z} = [u/R]$ is an algebraic stack by Criteria for Representability, Theorem 17.2. By Algebraic Stacks, Lemma 16.1 we obtain a canonical morphism

$$f : \mathcal{Z} \longrightarrow \mathcal{X}$$

which is fully faithful. Hence this morphism is representable by algebraic spaces, see Algebraic Stacks, Lemma 15.2 and a monomorphism, see Properties of Stacks, Lemma 8.4. It follows that the residual gerbe $\mathcal{Z}_x \subset \mathcal{X}$ of $\mathcal{X}$ at x exists and that f factors through an equivalence $\mathcal{Z} \to \mathcal{Z}_x$, see Properties of Stacks, Lemma 11.12. By construction the diagram

$$\begin{tikzcd} u \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{Z} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

is commutative. By Criteria for Representability, Lemma 17.1 the left vertical arrow is surjective, flat, and locally of finite presentation. Consider

$$\begin{tikzcd} u \times_\mathcal{X} U \arrow[r] \arrow[d] & \mathcal{Z} \times_\mathcal{X} U \arrow[r] \arrow[d] & U \arrow[d] \\ u \arrow[r] & \mathcal{Z} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

As $u \to \mathcal{X}$ is locally of finite type, we see that the base change $u \times_\mathcal{X} U \to U$ is locally of finite type. Moreover, $u \times_\mathcal{X} U \to \mathcal{Z} \times_\mathcal{X} U$ is surjective, flat, and locally of finite presentation as a base change of $u \to \mathcal{Z}$. Thus $\{u \times_\mathcal{X} U \to \mathcal{Z} \times_\mathcal{X} U\}$ is an fppf covering of algebraic spaces, and we conclude that $\mathcal{Z} \times_\mathcal{X} U \to U$ is locally of finite type by Descent on Spaces, Lemma 16.1. By definition this means that f is locally of finite type (because the vertical arrow $\mathcal{Z} \times_\mathcal{X} U \to \mathcal{Z}$ is smooth as a base change of $U \to \mathcal{X}$ and surjective as $\mathcal{Z}$ has only one point). Since $\mathcal{Z} = \mathcal{Z}_x$ we see that (3) holds.

It is clear that (3) implies (2). If (2) holds then x is a finite type point of $\mathcal{X}$ by Lemma 18.4 and Lemma 18.6 to see that $\mathcal{Z}_{\text{ft-pts}}$ is nonempty, i.e., the unique point of $\mathcal{Z}$ is a finite type point of $\mathcal{Z}$. □

19. Automorphism groups

Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ correspond to $x : \mathrm{Spec}(k) \to \mathcal{X}$. In this situation we often use the phrase "let G_x/k be the automorphism group algebraic space of x". This just means that

$$G_x = \mathit{Isom}_{\mathcal{X}}(x, x) = \mathrm{Spec}(k) \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}}$$

is the group algebraic space of automorphism of x. This is a group algebraic space over $\mathrm{Spec}(k)$. If k'/k is an extension of fields then the automorphism group algebraic space of the induced morphism $x' : \mathrm{Spec}(k') \to \mathcal{X}$ is the base change of G_x to $\mathrm{Spec}(k')$.

Lemma 19.1. *In the situation above G_x is a scheme if one of the following holds*

1. $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ *is quasi-separated*
2. $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ *is locally separated,*
3. $\mathcal{X}$ *is quasi-DM,*
4. $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ *is quasi-separated,*
5. $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ *is locally separated, or*
6. $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ *is locally quasi-finite.*

Proof. Observe that (1) $\Rightarrow$ (4), (2) $\Rightarrow$ (5), and (3) $\Rightarrow$ (6) by Lemma 6.1. In case (4) we see that G_x is a quasi-separated algebraic space and in case (5) we see that G_x is a locally separated algebraic space. In both cases G_x is a decent algebraic space (Decent Spaces, Section 6 and Lemma 15.2). Then G_x is separated by More on Groupoids in Spaces, Lemma 9.4 whereupon we conclude that G_x is a scheme by More on Groupoids in Spaces, Proposition 10.3. In case (6) we see that $G_x \to \mathrm{Spec}(k)$ is locally quasi-finite and hence G_x is a scheme by Spaces over Fields, Lemma 10.8. □

Lemma 19.2. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point. Let P be a property of algebraic spaces over fields which is invariant under ground field extensions; for example $P(X/k) = X \to \mathrm{Spec}(k)$ is finite. The following are equivalent*

1. *for some morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x the automorphism group algebraic space G_x/k has P, and*
2. *for any morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x the automorphism group algebraic space G_x/k has P.*

Proof. Omitted. □

Remark 19.3. Let P be a property of algebraic spaces over fields which is invariant under ground field extensions. Given an algebraic stack $\mathcal{X}$ and $x \in |\mathcal{X}|$, we say the automorphism group of $\mathcal{X}$ at x has P if the equivalent conditions of Lemma 19.2 are satisfied. For example, we say *the automorphism group of $\mathcal{X}$ at x is finite*, if $G_x \to \mathrm{Spec}(k)$ is finite whenever $x : \mathrm{Spec}(k) \to \mathcal{X}$ is a representative of x. Similarly for smooth, proper, etc. (There is clearly an abuse of language going on here, but we believe it will not cause confusion or imprecision.)

Lemma 19.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ be a point. The following are equivalent*

(1) *for some morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x setting $y = f \circ x$ the map $G_x \to G_y$ of automorphism group algebraic spaces is an isomorphism, and*
(2) *for any morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$ in the class of x setting $y = f \circ x$ the map $G_x \to G_y$ of automorphism group algebraic spaces is an isomorphism.*

Proof. This comes down to the fact that being an isomorphism is fpqc local on the target, see Descent on Spaces, Lemma 11.15. Namely, suppose that k'/k is an extension of fields and denote $x' : \mathrm{Spec}(k') \to \mathcal{X}$ the composition and set $y' = f \circ x'$. Then the morphism $G_{x'} \to G_{y'}$ is the base change of $G_x \to G_y$ by $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$. Hence $G_x \to G_y$ is an isomorphism if and only if $G_{x'} \to G_{y'}$ is an isomorphism. Thus we see that the property propagates through the equivalence class if it holds for one. $\square$

Remark 19.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ be a point. To indicate the equivalent conditions of Lemma 19.4 are satisfied for f and x in the literature the terminology f *is stabilizer preserving at* x or f *is fixed-point reflecting at* x is used. We prefer to say f *induces an isomorphism between automorphism groups at* x *and* $f(x)$.

20. Presentations and properties of algebraic stacks

Let (U, R, s, t, c) be a groupoid in algebraic spaces. If $s, t : R \to U$ are flat and locally of finite presentation, then the quotient stack $[U/R]$ is an algebraic stack, see Criteria for Representability, Theorem 17.2. In this section we study what properties of (U, R, s, t, c) imply for the algebraic stack $[U/R]$.

Lemma 20.1. *Let (U, R, s, t, c) be a groupoid in algebraic spaces such that $s, t : R \to U$ are flat and locally of finite presentation. Consider the algebraic stack $\mathcal{X} = [U/R]$ (see above).*

(1) *If $R \to U \times U$ is separated, then $\Delta_\mathcal{X}$ is separated.*
(2) *If U, R are separated, then $\Delta_\mathcal{X}$ is separated.*
(3) *If $R \to U \times U$ is locally quasi-finite, then $\mathcal{X}$ is quasi-DM.*
(4) *If $s, t : R \to U$ are locally quasi-finite, then $\mathcal{X}$ is quasi-DM.*
(5) *If $R \to U \times U$ is proper, then $\mathcal{X}$ is separated.*
(6) *If $s, t : R \to U$ are proper and U is separated, then $\mathcal{X}$ is separated.*
(7) *Add more here.*

Proof. Observe that the morphism $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 17.1. Hence the same is true for $U \times U \to \mathcal{X} \times \mathcal{X}$. We have the cartesian diagram

$$\begin{tikzcd} R = U \times_\mathcal{X} U \arrow[r] \arrow[d] & U \times U \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

(see Groupoids in Spaces, Lemma 22.2). Thus we see that $\Delta_\mathcal{X}$ has one of the properties listed in Properties of Stacks, Section 3 if and only if the morphism $R \to U \times U$ does, see Properties of Stacks, Lemma 3.3. This explains why (1), (3), and (5) are true. The condition in (2) implies $R \to U \times U$ is separated hence (2) follows from (1). The condition in (4) implies the condition in (3) hence (4) follows from (3). The condition in (6) implies the condition in (5) by Morphisms of Spaces, Lemma 40.6 hence (6) follows from (5). ☐

Lemma 20.2. *Let (U, R, s, t, c) be a groupoid in algebraic spaces such that $s, t : R \to U$ are flat and locally of finite presentation. Consider the algebraic stack $\mathcal{X} = [U/R]$ (see above). Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}|$ is the quotient of $|U|$ by this equivalence relation.*

Proof. The induced morphism $p : U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation, see Criteria for Representability, Lemma 17.1. Hence $|U| \to |\mathcal{X}|$ is surjective by Properties of Stacks, Lemma 4.4. Note that $R = U \times_{\mathcal{X}} U$, see Groupoids in Spaces, Lemma 22.2. Hence Properties of Stacks, Lemma 4.3 implies the map

$$|R| \longrightarrow |U| \times_{|\mathcal{X}|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|\mathcal{X}|$. Combining these two statements we get the result of the lemma. ☐

21. Special presentations of algebraic stacks

In this section we prove two important theorems. The first is the characterization of quasi-DM stacks $\mathcal{X}$ as the stacks of the form $\mathcal{X} = [U/R]$ with $s, t : R \to U$ locally quasi-finite (as well as flat and locally of finite presentation). The second is the statement that DM algebraic stacks are Deligne-Mumford.

The following lemma gives a criterion for when a "slice" of a presentation is still flat over the algebraic stack.

Lemma 21.1. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{tikzcd} U \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ \mathcal{X} & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $f_1, \ldots, f_r \in \Gamma(U, \mathcal{O}_U)$ and $z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\overline{z}}$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow \mathcal{X}$$

is flat and locally of finite presentation.

Proof. Choose a scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Choose an extension of fields k'/k and a morphism $w : \mathrm{Spec}(k') \to W$ such that $\mathrm{Spec}(k') \to W \to \mathcal{X}$ is 2-isomorphic to $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \to \mathcal{X}$. This is possible as $W \to \mathcal{X}$ is surjective. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[d] & U \times_{\mathcal{X}} W \arrow[l, "\mathrm{pr}_0"'] \arrow[d] & F' \arrow[l, "p'"'] \arrow[d] \\ \mathcal{X} & W \arrow[l] & \mathrm{Spec}(k') \arrow[l] \end{tikzcd}$$

both of whose squares are cartesian. By our choice of w we see that $F' = F \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$. Thus $F' \to F$ is surjective and we can choose a point $z' \in |F'|$ mapping to z. Since $F' \to F$ is flat we see that $\mathcal{O}_{F,\overline{z}} \to \mathcal{O}_{F',\overline{z}'}$ is flat, see Morphisms of Spaces, Lemma 30.8. Hence $f_1, \ldots, f_r$ map to a regular sequence in $\mathcal{O}_{F',\overline{z}'}$, see Algebra, Lemma 68.5. Note that $U \times_\mathcal{X} W \to W$ is a morphism of algebraic spaces which is flat and locally of finite presentation. Hence by More on Morphisms of Spaces, Lemma 28.1 we see that there exists an open subspace U' of $U \times_\mathcal{X} W$ containing $p(z')$ such that the intersection $U' \cap (V(f_1, \ldots, f_r) \times_\mathcal{X} W)$ is flat and locally of finite presentation over W. Note that $\mathrm{pr}_0(U')$ is an open subspace of U containing $p(z)$ as pr_0 is smooth hence open. Now we see that $U' \cap (V(f_1, \ldots, f_r) \times_\mathcal{X} W) \to \mathcal{X}$ is flat and locally of finite presentation as the composition

$$U' \cap (V(f_1, \ldots, f_r) \times_\mathcal{X} W) \to W \to \mathcal{X}.$$

Hence Properties of Stacks, Lemma 3.5 implies $\mathrm{pr}_0(U') \cap V(f_1, \ldots, f_r) \to \mathcal{X}$ is flat and locally of finite presentation as desired. □

Lemma 21.2. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{tikzcd} U \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ \mathcal{X} & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is locally of finite type. Let $z \in |F|$ be such that $\dim_z(F) = 0$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is locally quasi-finite.

Proof. Since $f : U \to \mathcal{X}$ is locally of finite type there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is locally quasi-finite, see Properties of Stacks, Remark 9.20 (2). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to \mathrm{Spec}(k)$ is locally quasi-finite at z. This follows immediately from Morphisms of Spaces, Lemma 34.6. □

A quasi-DM stack has a locally quasi-finite "covering" by a scheme.

Theorem 21.3. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is quasi-DM, and*
(2) *there exists a scheme W and a surjective, flat, locally finitely presented, locally quasi-finite morphism $W \to \mathcal{X}$.*

Proof. The implication (2) ⇒ (1) is Lemma 4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which "works" in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

Let U be an affine scheme, $U \to \mathcal{X}$ a smooth morphism, and $u \in U$ a closed point which maps to x, see Lemma 18.1. Denote $u = \mathrm{Spec}(\kappa(u))$ as usual. Consider the following commutative diagram

$$\begin{tikzcd} u & R \arrow[l] \arrow[d] \\ U \arrow[u, leftarrow] \arrow[d] & F \arrow[l, "p"] \arrow[d] \\ \mathcal{X} & u \arrow[l] \end{tikzcd}$$

with both squares fibre product squares, in particular $R = u \times_\mathcal{X} u$. In the proof of Lemma 18.7 we have seen that (u, R, s, t, c) is a groupoid in algebraic spaces with s, t locally of finite type. Let $G \to u$ be the stabilizer group algebraic space (see Groupoids in Spaces, Definition 16.2). Note that

$$G = R \times_{(u \times u)} u = (u \times_\mathcal{X} u) \times_{(u \times u)} u = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} u.$$

As $\mathcal{X}$ is quasi-DM we see that G is locally quasi-finite over u. By More on Groupoids in Spaces, Lemma 9.11 we have $\dim(R) = 0$.

Let $e : u \to R$ be the identity of the groupoid. Thus both compositions $u \to R \to u$ are equal to the identity morphism of u. Note that $R \subset F$ is a closed subspace as $u \subset U$ is a closed subscheme. Hence we can also think of e as a point of F. Consider the maps of étale local rings

$$\mathcal{O}_{U,u} \xrightarrow{p^\sharp} \mathcal{O}_{F,\overline{e}} \longrightarrow \mathcal{O}_{R,\overline{e}}$$

Note that $\mathcal{O}_{R,\overline{e}}$ has dimension 0 by the result of the first paragraph. On the other hand, the kernel of the second arrow is $p^\sharp(\mathfrak{m}_u)\mathcal{O}_{F,\overline{e}}$ as R is cut out in F by $\mathfrak{m}_u$. Thus we see that

$$\mathfrak{m}_{\overline{z}} = \sqrt{p^\sharp(\mathfrak{m}_u)\mathcal{O}_{F,\overline{e}}}$$

On the other hand, as the morphism $U \to \mathcal{X}$ is smooth we see that $F \to u$ is a smooth morphism of algebraic spaces. This means that F is a regular algebraic space (Spaces over Fields, Lemma 16.1). Hence $\mathcal{O}_{F,\overline{e}}$ is a regular local ring (Properties of Spaces, Lemma 25.1). Note that a regular local ring is Cohen-Macaulay (Algebra, Lemma 106.3). Let $d = \dim(\mathcal{O}_{F,\overline{e}})$. By Algebra, Lemma 104.10 we can find $f_1, \ldots, f_d \in \mathcal{O}_{U,u}$ whose images $\varphi(f_1), \ldots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{F,\overline{z}}$. By Lemma 21.1 after shrinking U we may assume that $Z = V(f_1, \ldots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation. Note that by construction $F_Z = Z \times_\mathcal{X} u$ is a closed subspace of $F = U \times_\mathcal{X} u$, that e is a point of this closed subspace, and that

$$\dim(\mathcal{O}_{F_Z,\overline{e}}) = 0.$$

By Morphisms of Spaces, Lemma 34.1 it follows that $\dim_e(F_Z) = 0$ because the transcendence degree of e relative to u is zero. Hence it follows from Lemma 21.2 that after possibly shrinking U the morphism $Z \to \mathcal{X}$ is locally quasi-finite.

We conclude that for every finite type point x of $\mathcal{X}$ there exists a locally quasi-finite, flat, locally finitely presented morphism $f_x : Z_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x Z_x$ and $f = \coprod f_x$. Then f is flat, locally of finite presentation, and locally quasi-finite. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 4.7. By construction the image contains all finite type points of $\mathcal{X}$, hence f is surjective by Lemma 18.6 (and Properties of Stacks, Lemma 4.4). $\square$

Lemma 21.4. *Let $\mathcal{Z}$ be a DM, locally Noetherian, reduced algebraic stack with $|\mathcal{Z}|$ a singleton. Then there exists a field k and a surjective étale morphism* $\mathrm{Spec}(k) \to \mathcal{Z}$.

Proof. By Properties of Stacks, Lemma 11.3 there exists a field k and a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Set $U = \mathrm{Spec}(k)$ and $R = U \times_{\mathcal{Z}} U$ so we obtain a groupoid in algebraic spaces (U, R, s, t, c), see Algebraic Stacks, Lemma 9.2. Note that by Algebraic Stacks, Remark 16.3 we have an equivalence

$$f_{can} : [U/R] \longrightarrow \mathcal{Z}$$

The projections $s, t : R \to U$ are locally of finite presentation. As $\mathcal{Z}$ is DM we see that the stabilizer group algebraic space

$$G = U \times_{U \times U} R = U \times_{U \times U} (U \times_{\mathcal{Z}} U) = U \times_{\mathcal{Z} \times \mathcal{Z}, \Delta_{\mathcal{Z}}} \mathcal{Z}$$

is unramified over U. In particular $\dim(G) = 0$ and by More on Groupoids in Spaces, Lemma 9.11 we have $\dim(R) = 0$. This implies that R is a scheme, see Spaces over Fields, Lemma 9.1. By Varieties, Lemma 20.2 we see that R (and also G) is the disjoint union of spectra of Artinian local rings finite over k via either s or t. Let $P = \mathrm{Spec}(A) \subset R$ be the open and closed subscheme whose underlying point is the identity e of the groupoid scheme (U, R, s, t, c). As $s \circ e = t \circ e = \mathrm{id}_{\mathrm{Spec}(k)}$ we see that A is an Artinian local ring whose residue field is identified with k via either $s^\sharp : k \to A$ or $t^\sharp : k \to A$. Note that $s, t : \mathrm{Spec}(A) \to \mathrm{Spec}(k)$ are finite (by the lemma referenced above). Since $G \to \mathrm{Spec}(k)$ is unramified we see that

$$G \cap P = P \times_{U \times U} U = \mathrm{Spec}(A \otimes_{k \otimes k} k)$$

is unramified over k. On the other hand $A \otimes_{k \otimes k} k$ is local as a quotient of A and surjects onto k. We conclude that $A \otimes_{k \otimes k} k = k$. It follows that $P \to U \times U$ is universally injective (as P has only one point with residue field k), unramified (by the computation of the fibre over the unique image point above), and of finite type (because s, t are) hence a monomorphism (see Étale Morphisms, Lemma 7.1). Thus $s|_P, t|_P : P \to U$ define a finite flat equivalence relation. Thus we may apply Groupoids, Proposition 23.9 to conclude that U/P exists and is a scheme $\overline{U}$. Moreover, $U \to \overline{U}$ is finite locally free and $P = U \times_{\overline{U}} U$. In fact $\overline{U} = \mathrm{Spec}(k_0)$ where $k_0 \subset k$ is the ring of R-invariant functions. As k is a field it follows from the definition Groupoids, Equation (23.0.1) that k_0 is a field.

We claim that

$$\mathrm{Spec}(k_0) = \overline{U} = U/P \to [U/R] = \mathcal{Z} \tag{21.4.1}$$

is the desired surjective étale morphism. It follows from Properties of Stacks, Lemma 11.1 that this morphism is surjective. Thus it suffices to show that (21.4.1) is étale[6]. Instead of proving the étaleness directly we first apply Bootstrap, Lemma 9.1 to see that there exists a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ such that (U, R, s, t, c) is the restriction of $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ via the quotient morphism $U \to \overline{U}$. (We verified all the hypothesis of the lemma above except for the assertion that $j : R \to U \times U$ is separated and locally quasi-finite which follows from the fact that R is a separated scheme locally quasi-finite over k.) Since $U \to \overline{U}$ is finite locally free we see that $[U/R] \to [\overline{U}/\overline{R}]$ is an equivalence, see Groupoids in Spaces, Lemma 25.2.

[6]We urge the reader to find his/her own proof of this fact. In fact the argument has a lot in common with the final argument of the proof of Bootstrap, Theorem 10.1 hence probably should be isolated into its own lemma somewhere.

Note that s, t are the base changes of the morphisms $\overline{s}, \overline{t}$ by $U \to \overline{U}$. As $\{U \to \overline{U}\}$ is an fppf covering we conclude $\overline{s}, \overline{t}$ are flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas 23.15, 23.11, and 23.24. Consider the commutative diagram

$$\begin{tikzcd} U \times_{\overline{U}} U \arrow[r, equal] \arrow[rd] & P \arrow[r] \arrow[d] & R \arrow[d] \\ & \overline{U} \arrow[r, "\overline{e}"] & \overline{R} \end{tikzcd}$$

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since P is open in R we conclude that $\overline{e}$ is an open immersion by Descent, Lemma 23.16.

But of course, if $\overline{e}$ is an open immersion and $\overline{s}, \overline{t}$ are flat and locally of finite presentation then the morphisms $\overline{t}, \overline{s}$ are étale. For example you can see this by applying More on Groupoids, Lemma 4.1 which shows that $\Omega_{\overline{R}/\overline{U}} = 0$ implies that $\overline{s}, \overline{t} : \overline{R} \to \overline{U}$ is unramified (see Morphisms, Lemma 35.2), which in turn implies that $\overline{s}, \overline{t}$ are étale (see Morphisms, Lemma 36.16). Hence $\mathcal{Z} = [\overline{U}/\overline{R}]$ is an étale presentation of the algebraic stack $\mathcal{Z}$ and we conclude that $\overline{U} \to \mathcal{Z}$ is étale by Properties of Stacks, Lemma 3.3. □

Lemma 21.5. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{tikzcd} U \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ \mathcal{X} & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $z \in |F|$ be such that $F \to \mathrm{Spec}(k)$ is unramified at z. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is étale.

Proof. Since $f : U \to \mathcal{X}$ is flat and locally of finite presentation there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is étale, see Properties of Stacks, Remark 9.20 (5). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to \mathrm{Spec}(k)$ is étale at z. Since it is flat and locally of finite presentation as a base change of $U \to \mathcal{X}$ and since $F \to \mathrm{Spec}(k)$ is unramified at z by assumption, this follows from Morphisms of Spaces, Lemma 39.12. □

A DM stack is a Deligne-Mumford stack.

Theorem 21.6. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

1. *$\mathcal{X}$ is DM,*
2. *$\mathcal{X}$ is Deligne-Mumford, and*
3. *there exists a scheme W and a surjective étale morphism $W \to \mathcal{X}$.*

Proof. Recall that (3) is the definition of (2), see Algebraic Stacks, Definition 12.2. The implication (3) $\Rightarrow$ (1) is Lemma 4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which "works" in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

By Lemma 18.7 the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type. By Lemma 4.16 the algebraic stack $\mathcal{Z}_x$ is DM. By Lemma 21.4 there exists a field k and a surjective étale morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}_x$. In particular the composition $x : \mathrm{Spec}(k) \to \mathcal{X}$ is locally of finite type (by Morphisms of Spaces, Lemmas 23.2 and 39.9).

Pick a scheme U and a smooth morphism $U \to \mathcal{X}$ such that x is in the image of $|U| \to |\mathcal{X}|$. Consider the following fibre square

$$\begin{tikzcd} U \arrow[d] & F \arrow[l] \arrow[d] \\ \mathcal{X} & \mathrm{Spec}(k) \arrow[l, "x"'] \end{tikzcd}$$

in other words $F = U \times_{\mathcal{X},x} \mathrm{Spec}(k)$. By Properties of Stacks, Lemma 4.3 we see that F is nonempty. As $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism we have

$$\mathrm{Spec}(k) \times_{z,\mathcal{Z}_x,z} \mathrm{Spec}(k) = \mathrm{Spec}(k) \times_{x,\mathcal{X},x} \mathrm{Spec}(k)$$

with étale projection maps to $\mathrm{Spec}(k)$ by construction of z. Since

$$F \times_U F = (\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k)) \times_{\mathrm{Spec}(k)} F$$

we see that the projections maps $F \times_U F \to F$ are étale as well. It follows that $\Delta_{F/U} : F \to F \times_U F$ is étale (see Morphisms of Spaces, Lemma 39.11). By Morphisms of Spaces, Lemma 51.2 this implies that $\Delta_{F/U}$ is an open immersion, which finally implies by Morphisms of Spaces, Lemma 38.9 that $F \to U$ is unramified.

Pick a nonempty affine scheme V and an étale morphism $V \to F$. (This could be avoided by working directly with F, but it seems easier to explain what's going on by doing so.) Picture

$$\begin{tikzcd} U \arrow[d] & F \arrow[l] \arrow[d] & V \arrow[l] \arrow[dl] \\ \mathcal{X} & \mathrm{Spec}(k) \arrow[l, "x"'] \end{tikzcd}$$

Then $V \to \mathrm{Spec}(k)$ is a smooth morphism of schemes and $V \to U$ is an unramified morphism of schemes (see Morphisms of Spaces, Lemmas 37.2 and 38.3). Pick a closed point $v \in V$ with $k \subset \kappa(v)$ finite separable, see Varieties, Lemma 25.6. Let $u \in U$ be the image point. The local ring $\mathcal{O}_{V,v}$ is regular (see Varieties, Lemma 25.3) and the local ring homomorphism

$$\varphi : \mathcal{O}_{U,u} \longrightarrow \mathcal{O}_{V,v}$$

coming from the morphism $V \to U$ is such that $\varphi(\mathfrak{m}_u)\mathcal{O}_{V,v} = \mathfrak{m}_v$, see Morphisms, Lemma 35.14. Hence we can find $f_1, \ldots, f_d \in \mathcal{O}_{U,u}$ such that the images $\varphi(f_1), \ldots, \varphi(f_d)$ form a basis for $\mathfrak{m}_v/\mathfrak{m}_v^2$ over $\kappa(v)$. Since $\mathcal{O}_{V,v}$ is a regular local ring this implies that $\varphi(f_1), \ldots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{V,v}$ (see Algebra, Lemma 106.3). After replacing U by an open neighbourhood of u we may assume $f_1, \ldots, f_d \in \Gamma(U, \mathcal{O}_U)$. After replacing U by a possibly even smaller open neighbourhood of u we may assume that $V(f_1, \ldots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation, see Lemma 21.1. By construction

$$V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k) \longleftarrow V(f_1, \ldots, f_d) \times_U V$$

is étale and $V(f_1, \ldots, f_d) \times_U V$ is the closed subscheme $T \subset V$ cut out by $f_1|_V, \ldots, f_d|_V$. Hence by construction $v \in T$ and

$$\mathcal{O}_{T,v} = \mathcal{O}_{V,v}/(\varphi(f_1), \ldots, \varphi(f_d)) = \kappa(v)$$

a finite separable extension of k. It follows that $T \to \mathrm{Spec}(k)$ is unramified at v, see Morphisms, Lemma 35.14. By definition of an unramified morphism of algebraic spaces this means that $V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is unramified at the image of v in $V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k)$. Applying Lemma 21.5 we see that on shrinking U to yet another open neighbourhood of u the morphism $V(f_1, \ldots, f_d) \to \mathcal{X}$ is étale.

We conclude that for every finite type point x of $\mathcal{X}$ there exists an étale morphism $f_x : W_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x W_x$ and $f = \coprod f_x$. Then f is étale. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 4.7. By construction the image contains all finite type points of $\mathcal{X}$, hence f is surjective by Lemma 18.6 (and Properties of Stacks, Lemma 4.4). □

Here is a useful corollary which tells us that the "fibres" of a DM morphism of algebraic stacks are Deligne-Mumford.

Lemma 21.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a DM morphism of algebraic stacks. Then*

1. *For every DM algebraic stack $\mathcal{Z}$ and morphism $\mathcal{Z} \to \mathcal{Y}$ there exists a scheme and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*
2. *For every algebraic space Z and morphism $Z \to \mathcal{Y}$ there exists a scheme and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} Z$.*

Proof. Proof of (1). As f is DM we see that the base change $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ is DM by Lemma 4.4. Since $\mathcal{Z}$ is DM this implies that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is DM by Lemma 4.11. Hence there exists a scheme U and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$, see Theorem 21.6. Part (2) is a special case of (1) since an algebraic space (when viewed as an algebraic stack) is DM by Lemma 4.3. □

22. The Deligne-Mumford locus

Every algebraic stack has a largest open substack which is a Deligne-Mumford stack; this is more or less clear but we also write out the proof below. Of course this substack may be empty, for example if $X = [\mathrm{Spec}(\mathbf{Z})/\mathbf{G}_{m,\mathbf{Z}}]$. Below we will characterize the points of the DM locus.

Lemma 22.1. *Let $\mathcal{X}$ be an algebraic stack. There exist open substacks*

$$\mathcal{X}'' \subset \mathcal{X}' \subset \mathcal{X}$$

such that $\mathcal{X}''$ is DM, $\mathcal{X}'$ is quasi-DM, and such that these are the largest open substacks with these properties.

Proof. All we are really saying here is that if $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{V} \subset \mathcal{X}$ are open substacks which are DM, then the open substack $\mathcal{W} \subset \mathcal{X}$ with $|\mathcal{W}| = |\mathcal{U}| \cup |\mathcal{V}|$ is DM as well. (Similarly for quasi-DM.) Although this is a cheat, let us use Theorem 21.6 to prove this. By that theorem we can choose schemes U and V and surjective étale morphisms $U \to \mathcal{U}$ and $V \to \mathcal{V}$. Then of course $U \amalg V \to \mathcal{W}$ is surjective and étale. The quasi-DM case is proven by exactly the same method using Theorem 21.3. $\square$

Lemma 22.2. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ correspond to $x : \mathrm{Spec}(k) \to \mathcal{X}$. Let G_x/k be the automorphism group algebraic space of x. Then*

(1) *x is in the DM locus of $\mathcal{X}$ if and only if $G_x \to \mathrm{Spec}(k)$ is unramified, and*
(2) *x is in the quasi-DM locus of $\mathcal{X}$ if and only if $G_x \to \mathrm{Spec}(k)$ is locally quasi-finite.*

Proof. Proof of (2). Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Consider the fibre product

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \mathcal{I}_\mathcal{X} \arrow[d] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

Recall that G is the automorphism group algebraic space of $U \to \mathcal{X}$. By Groupoids in Spaces, Lemma 6.3 there is a maximal open subscheme $U' \subset U$ such that $G_{U'} \to U'$ is locally quasi-finite. Moreover, formation of U' commutes with arbitrary base change. In particular the two inverse images of U' in $R = U \times_\mathcal{X} U$ are the same open subspace of R (since after all the two maps $R \to \mathcal{X}$ are isomorphic and hence have isomorphic automorphism group spaces). Hence U' is the inverse image of an open substack $\mathcal{X}' \subset \mathcal{X}$ by Properties of Stacks, Lemma 9.11 and we have a cartesian diagram

$$\begin{tikzcd} G_{U'} \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}'} \arrow[d] \\ U' \arrow[r] & \mathcal{X}' \end{tikzcd}$$

Thus the morphism $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is locally quasi-finite and we conclude that $\mathcal{X}'$ is quasi-DM by Lemma 6.1 part (5). On the other hand, if $\mathcal{W} \subset \mathcal{X}$ is an open substack which is quasi-DM, then the inverse image $W \subset U$ of $\mathcal{W}$ must be contained in U' by our construction of U' since $\mathcal{I}_\mathcal{W} = \mathcal{W} \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is locally quasi-finite over $\mathcal{W}$. Thus $\mathcal{X}'$ is the quasi-DM locus. Finally, choose a field extension K/k and a 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d, "x"] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

Then we find an isomorphism $G_x \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \cong G \times_U \mathrm{Spec}(K)$ of group algebraic spaces over K. Hence G_x is locally quasi-finite over k if and only if $\mathrm{Spec}(K) \to U$ maps into U' (use the commutation of formation of U' and Groupoids in Spaces, Lemma 6.3 applied to $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ and G_x to see this). This finishes the proof of (2). The proof of (1) is exactly the same. $\square$

23. Locally quasi-finite morphisms

The property "locally quasi-finite" of morphisms of algebraic spaces is not smooth local on the source-and-target so we cannot use the material in Section 16 to define locally quasi-finite morphisms of algebraic stacks. We do already know what it means for a morphism of algebraic stacks representable by algebraic spaces to be locally quasi-finite, see Properties of Stacks, Section 3. To find a condition suitable for general morphisms we make the following observation.

Lemma 23.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces. The following are equivalent*

(1) *f is locally quasi-finite (as in Properties of Stacks, Section 3), and*
(2) *f is locally of finite type and for every morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ where k is a field the space $|\mathrm{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}|$ is discrete.*

Proof. Assume (1). In this case the morphism of algebraic spaces $\mathcal{X}_k \to \mathrm{Spec}(k)$ is locally quasi-finite as a base change of f. Hence $|\mathcal{X}_k|$ is discrete by Morphisms of Spaces, Lemma 27.5. Conversely, assume (2). Pick a surjective smooth morphism $V \to \mathcal{Y}$ where V is a scheme. It suffices to show that the morphism of algebraic spaces $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally quasi-finite, see Properties of Stacks, Lemma 3.3. The morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally of finite type by assumption. For any morphism $\mathrm{Spec}(k) \to V$ where k is a field

$$\mathrm{Spec}(k) \times_V (V \times_{\mathcal{Y}} \mathcal{X}) = \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}$$

has a discrete space of points by assumption. Hence we conclude that $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally quasi-finite by Morphisms of Spaces, Lemma 27.5. □

A morphism of algebraic stacks which is representable by algebraic spaces is quasi-DM, see Lemma 4.3. Combined with the lemma above we see that the following definition does not conflict with the already existing notion in the case of morphisms representable by algebraic spaces.

Definition 23.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *locally quasi-finite* if f is quasi-DM, locally of finite type, and for every morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ where k is a field the space $|\mathcal{X}_k|$ is discrete.

The condition that f be quasi-DM is natural. For example, let k be a field and consider the morphism $\pi : [\mathrm{Spec}(k)/\mathbf{G}_m] \to \mathrm{Spec}(k)$ which has singleton fibres and is locally of finite type. As we will see later this morphism is smooth of relative dimension -1, and we'd like our locally quasi-finite morphisms to have relative dimension 0. Also, note that the section $\mathrm{Spec}(k) \to [\mathrm{Spec}(k)/\mathbf{G}_m]$ does not have discrete fibres, hence is not locally quasi-finite, and we'd like to have the following permanence property for locally quasi-finite morphisms: If $f : \mathcal{X} \to \mathcal{X}'$ is a morphism of algebraic stacks locally quasi-finite over the algebraic stack $\mathcal{Y}$, then f is locally quasi-finite (in fact something a bit stronger holds, see Lemma 23.8).

Another justification for the definition above is Lemma 23.7 below which characterizes being locally quasi-finite in terms of the existence of suitable "presentations" or "coverings" of $\mathcal{X}$ and $\mathcal{Y}$.

Lemma 23.3. *A base change of a locally quasi-finite morphism is locally quasi-finite.*

Proof. We have seen this for quasi-DM morphisms in Lemma 4.4 and for locally finite type morphisms in Lemma 17.3. It is immediate that the condition on fibres is inherited by a base change. □

Lemma 23.4. *Let $\mathcal{X} \to \mathrm{Spec}(k)$ be a locally quasi-finite morphism where $\mathcal{X}$ is an algebraic stack and k is a field. Let $f : V \to \mathcal{X}$ be a locally quasi-finite morphism where V is a scheme. Then $V \to \mathrm{Spec}(k)$ is locally quasi-finite.*

Proof. By Lemma 17.2 we see that $V \to \mathrm{Spec}(k)$ is locally of finite type. Assume, to get a contradiction, that $V \to \mathrm{Spec}(k)$ is not locally quasi-finite. Then there exists a nontrivial specialization $v \rightsquigarrow v'$ of points of V, see Morphisms, Lemma 20.6. In particular $\mathrm{trdeg}_k(\kappa(v)) > \mathrm{trdeg}_k(\kappa(v'))$, see Morphisms, Lemma 28.7. Because $|\mathcal{X}|$ is discrete we see that $|f|(v) = |f|(v')$. Consider $R = V \times_{\mathcal{X}} V$. Then R is an algebraic space and the projections $s, t : R \to V$ are locally quasi-finite as base changes of $V \to \mathcal{X}$ (which is representable by algebraic spaces so this follows from the discussion in Properties of Stacks, Section 3). By Properties of Stacks, Lemma 4.3 we see that there exists an $r \in |R|$ such that $s(r) = v$ and $t(r) = v'$. By Morphisms of Spaces, Lemma 33.3 we see that the transcendence degree of v/k is equal to the transcendence degree of r/k is equal to the transcendence degree of v'/k. This contradiction proves the lemma. □

Lemma 23.5. *A composition of a locally quasi-finite morphisms is locally quasi-finite.*

Proof. We have seen this for quasi-DM morphisms in Lemma 4.10 and for locally finite type morphisms in Lemma 17.2. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be locally quasi-finite. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be a morphism. It suffices to show that $|\mathcal{X}_k|$ is discrete. By Lemma 23.3 the morphisms $\mathcal{X}_k \to \mathcal{Y}_k$ and $\mathcal{Y}_k \to \mathrm{Spec}(k)$ are locally quasi-finite. In particular we see that $\mathcal{Y}_k$ is a quasi-DM algebraic stack, see Lemma 4.13. By Theorem 21.3 we can find a scheme V and a surjective, flat, locally finitely presented, locally quasi-finite morphism $V \to \mathcal{Y}_k$. By Lemma 23.4 we see that V is locally quasi-finite over k, in particular $|V|$ is discrete. The morphism $V \times_{\mathcal{Y}_k} \mathcal{X}_k \to \mathcal{X}_k$ is surjective, flat, and locally of finite presentation hence $|V \times_{\mathcal{Y}_k} \mathcal{X}_k| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to show that $|V \times_{\mathcal{Y}_k} \mathcal{X}_k|$ is discrete. Note that V is a disjoint union of spectra of Artinian local k-algebras A_i with residue fields k_i, see Varieties, Lemma 20.2. Thus it suffices to show that each

$$|\mathrm{Spec}(A_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\mathrm{Spec}(k_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\mathrm{Spec}(k_i) \times_{\mathcal{Y}} \mathcal{X}|$$

is discrete, which follows from the assumption that $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite. □

Before we characterize locally quasi-finite morphisms in terms of coverings we do it for quasi-DM morphisms.

Lemma 23.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is quasi-DM,*
(2) *for any morphism $V \to \mathcal{Y}$ with V an algebraic space there exists a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space, and*
(3) *there exist algebraic spaces U, V and a morphism $V \to \mathcal{Y}$ which is surjective, flat, and locally of finite presentation, and a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ which is surjective, flat, locally of finite presentation, and locally quasi-finite.*

Proof. The implication (2) $\Rightarrow$ (3) is immediate.

Assume (1) and let $V \to \mathcal{Y}$ be as in (2). Then $\mathcal{X} \times_\mathcal{Y} V \to V$ is quasi-DM, see Lemma 4.4. By Lemma 4.3 the algebraic space V is DM, hence quasi-DM. Thus $\mathcal{X} \times_\mathcal{Y} V$ is quasi-DM by Lemma 4.11. Hence we may apply Theorem 21.3 to get the morphism $U \to \mathcal{X} \times_\mathcal{Y} V$ as in (2).

Assume (3). Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_\mathcal{Y} V$ be as in (3). To prove that f is quasi-DM it suffices to show that $\mathcal{X} \times_\mathcal{Y} V \to V$ is quasi-DM, see Lemma 4.5. By Lemma 4.14 we see that $\mathcal{X} \times_\mathcal{Y} V$ is quasi-DM. Hence $\mathcal{X} \times_\mathcal{Y} V \to V$ is quasi-DM by Lemma 4.13 and (1) holds. This finishes the proof of the lemma. $\square$

Lemma 23.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

1. *f is locally quasi-finite,*
2. *f is quasi-DM and for any morphism $V \to \mathcal{Y}$ with V an algebraic space and any locally quasi-finite morphism $U \to \mathcal{X} \times_\mathcal{Y} V$ where U is an algebraic space the morphism $U \to V$ is locally quasi-finite,*
3. *for any morphism $V \to \mathcal{Y}$ from an algebraic space V there exists a surjective, flat, locally finitely presented, and locally quasi-finite morphism $U \to \mathcal{X} \times_\mathcal{Y} V$ where U is an algebraic space such that $U \to V$ is locally quasi-finite,*
4. *there exists algebraic spaces U, V, a surjective, flat, and locally of finite presentation morphism $V \to \mathcal{Y}$, and a morphism $U \to \mathcal{X} \times_\mathcal{Y} V$ which is surjective, flat, locally of finite presentation, and locally quasi-finite such that $U \to V$ is locally quasi-finite.*

Proof. Assume (1). Then f is quasi-DM by assumption. Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_\mathcal{Y} V$ be as in (2). By Lemma 23.5 the composition $U \to \mathcal{X} \times_\mathcal{Y} V \to V$ is locally quasi-finite. Thus (1) implies (2).

Assume (2). Let $V \to \mathcal{Y}$ be as in (3). By Lemma 23.6 we can find an algebraic space U and a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_\mathcal{Y} V$. By (2) the composition $U \to V$ is locally quasi-finite. Thus (2) implies (3).

It is immediate that (3) implies (4).

Assume (4). We will prove (1) holds, which finishes the proof. By Lemma 23.6 we see that f is quasi-DM. To prove that f is locally of finite type it suffices to prove that $g : \mathcal{X} \times_\mathcal{Y} V \to V$ is locally of finite type, see Lemma 17.6. Then it suffices to check that g precomposed with $h : U \to \mathcal{X} \times_\mathcal{Y} V$ is locally of finite type, see Lemma 17.7. Since $g \circ h : U \to V$ was assumed to be locally quasi-finite this holds, hence f is locally of finite type. Finally, let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Y}$ be a morphism. Then $V \times_\mathcal{Y} \mathrm{Spec}(k)$ is a nonempty algebraic space which is locally of finite presentation over k. Hence we can find a finite extension k'/k and a morphism $\mathrm{Spec}(k') \to V$ such that

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r] \arrow[d] & V \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \mathcal{Y} \end{tikzcd}$$

commutes (details omitted). Then $\mathcal{X}_{k'} \to \mathcal{X}_k$ is representable (by schemes), surjective, and finite locally free. In particular $|\mathcal{X}_{k'}| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to prove that $|\mathcal{X}_{k'}|$ is discrete. Since

$$U \times_V \mathrm{Spec}(k') = U \times_{\mathcal{X} \times_{\mathcal{Y}} V} \mathcal{X}_{k'}$$

we see that $U \times_V \mathrm{Spec}(k') \to \mathcal{X}_{k'}$ is surjective, flat, and locally of finite presentation (as a base change of $U \to \mathcal{X} \times_{\mathcal{Y}} V$). Hence $|U \times_V \mathrm{Spec}(k')| \to |\mathcal{X}_{k'}|$ is surjective and open. Thus it suffices to show that $|U \times_V \mathrm{Spec}(k')|$ is discrete. This follows from the fact that $U \to V$ is locally quasi-finite (either by our definition above or from the original definition for morphisms of algebraic spaces, via Morphisms of Spaces, Lemma 27.5). $\square$

Lemma 23.8. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume that $\mathcal{X} \to \mathcal{Z}$ is locally quasi-finite and $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Then $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite.*

Proof. Write $\mathcal{X} \to \mathcal{Y}$ as the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{Y}$$

The second arrow is locally quasi-finite as a base change of $\mathcal{X} \to \mathcal{Z}$, see Lemma 23.3. The first arrow is locally quasi-finite by Lemma 4.8 as $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Hence $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite by Lemma 23.5. $\square$

24. Quasi-finite morphisms

We have defined "locally quasi-finite" morphisms of algebraic stacks in Section 23 and "quasi-compact" morphisms of algebraic stacks in Section 7. Since a morphism of algebraic spaces is by definition quasi-finite if and only if it is both locally quasi-finite and quasi-compact (Morphisms of Spaces, Definition 27.1), we may define what it means for a morphism of algebraic stacks to be quasi-finite as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 3 when the morphism is representable by algebraic spaces.

Definition 24.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *quasi-finite* if f is locally quasi-finite (Definition 23.2) and quasi-compact (Definition 7.2).

Lemma 24.2. *The composition of quasi-finite morphisms is quasi-finite.*

Proof. Combine Lemmas 23.5 and 7.4. $\square$

Lemma 24.3. *A base change of a quasi-finite morphism is quasi-finite.*

Proof. Combine Lemmas 23.3 and 7.3. $\square$

Lemma 24.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f$ is quasi-finite and g is quasi-separated and quasi-DM then f is quasi-finite.*

Proof. Combine Lemmas 23.8 and 7.7. $\square$

25. Flat morphisms

The property "being flat" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 30.4 and Descent on Spaces, Lemma 11.13. Hence, by Lemma 16.1 above, we may define what it means for a morphism of algebraic spaces to be flat as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 3 when the morphism is representable by algebraic spaces.

Definition 25.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *flat* if the equivalent conditions of Lemma 16.1 hold with $\mathcal{P} =$ flat.

Lemma 25.2. *The composition of flat morphisms is flat.*

Proof. Combine Remark 16.3 with Morphisms of Spaces, Lemma 30.3. □

Lemma 25.3. *A base change of a flat morphism is flat.*

Proof. Combine Remark 16.4 with Morphisms of Spaces, Lemma 30.4. □

Lemma 25.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective flat morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is flat, then f is flat.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Then $W \to \mathcal{Z}$ is surjective and flat (Morphisms of Spaces, Lemma 37.7) hence $W \to \mathcal{Y}$ is surjective and flat (by Properties of Stacks, Lemma 5.2 and Lemma 25.2). Since the base change of $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ by $W \to \mathcal{Z}$ is a flat morphism (Lemma 25.3) we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is flat. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of flat morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is flat, we see that $U' \to V'$ is flat. Then, since $V' \to V$ is surjective as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is flat by Morphisms of Spaces, Lemma 31.3 (2) and we win. □

Lemma 25.5. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is flat and $\mathcal{X} \to \mathcal{Y}$ is surjective and flat, then $\mathcal{Y} \to \mathcal{Z}$ is flat.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We know that $U \to V$ is flat and that $U \to W$ is flat. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Morphisms of Spaces, Lemma 31.5. □

Lemma 25.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Let* $\text{Spec}(A) \to \mathcal{Y}$ *be a morphism where A is a valuation ring. If the closed point* *of* $\mathrm{Spec}(A)$ *maps to a point of* $|\mathcal{Y}|$ *in the image of* $|\mathcal{X}| \to |\mathcal{Y}|$, *then there exists a commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(A') \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathcal{Y} \end{tikzcd}$$

where $A \to A'$ *is an extension of valuation rings (More on Algebra, Definition 123.1).*

Proof. The base change $\mathcal{X}_A \to \mathrm{Spec}(A)$ is flat (Lemma 25.3) and the closed point of $\mathrm{Spec}(A)$ is in the image of $|\mathcal{X}_A| \to |\mathrm{Spec}(A)|$ (Properties of Stacks, Lemma 4.3). Thus we may assume $\mathcal{Y} = \mathrm{Spec}(A)$. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. Then we can apply Morphisms of Spaces, Lemma 42.4 to the morphism $U \to \mathrm{Spec}(A)$ to conclude. $\square$

26. Flat at a point

We still have to develop the general machinery needed to say what it means for a morphism of algebraic stacks to have a given property at a point. For the moment the following lemma is sufficient.

Lemma 26.1. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a morphism of algebraic stacks. Let* $x \in |\mathcal{X}|$. *Consider commutative diagrams*

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd} \quad \textit{with points} \quad \begin{tikzcd} u \in |U| \arrow[d] \\ x \in |\mathcal{X}| \end{tikzcd}$$

where U *and* V *are algebraic spaces,* b *is flat, and* $(a, h) : U \to \mathcal{X} \times_{\mathcal{Y}} V$ *is flat. The following are equivalent*

(1) h *is flat at* u *for one diagram as above,*
(2) h *is flat at* u *for every diagram as above.*

Proof. Suppose we are given a second diagram U', V', u', a', b', h' as in the lemma. Then we can consider

$$\begin{tikzcd} U \arrow[d] & U \times_{\mathcal{X}} U' \arrow[l] \arrow[r] \arrow[d] & U' \arrow[d] \\ V & V \times_{\mathcal{Y}} V' \arrow[l] \arrow[r] & V' \end{tikzcd}$$

By Properties of Stacks, Lemma 4.3 there is a point $u'' \in |U \times_{\mathcal{X}} U'|$ mapping to u and u'. If h is flat at u, then the base change $U \times_V (V \times_{\mathcal{Y}} V') \to V \times_{\mathcal{Y}} V'$ is flat at any point over u, see Morphisms of Spaces, Lemma 31.3. On the other hand, the morphism

$$U \times_{\mathcal{X}} U' \to U \times_{\mathcal{X}} (\mathcal{X} \times_{\mathcal{Y}} V') = U \times_{\mathcal{Y}} V' = U \times_V (V \times_{\mathcal{Y}} V')$$

is flat as a base change of (a', h'), see Lemma 25.3. Composing and using Morphisms of Spaces, Lemma 31.4 we conclude that $U \times_{\mathcal{X}} U' \to V \times_{\mathcal{Y}} V'$ is flat at u''. Then we can use composition by the flat map $V \times_{\mathcal{Y}} V' \to V'$ to conclude that $U \times_{\mathcal{X}} U' \to V'$ is flat at u''. Finally, since $U \times_{\mathcal{X}} U' \to U'$ is flat at u'' and u'' maps to u' we conclude that $U' \to V'$ is flat at u' by Morphisms of Spaces, Lemma 31.5. $\square$

Definition 26.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$. We say f is *flat at x* if the equivalent conditions of Lemma 26.1 hold.

27. Morphisms of finite presentation

The property "locally of finite presentation" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 28.3 and Descent on Spaces, Lemma 11.10. Hence, by Lemma 16.1 above, we may define what it means for a morphism of algebraic stacks to be locally of finite presentation as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 3 when the morphism is representable by algebraic spaces.

Definition 27.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f *locally of finite presentation* if the equivalent conditions of Lemma 16.1 hold with $\mathcal{P} =$ locally of finite presentation.
(2) We say f is *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation.

Lemma 27.2. *The composition of finitely presented morphisms is of finite presentation. The same holds for morphisms which are locally of finite presentation.*

Proof. Combine Remark 16.3 with Morphisms of Spaces, Lemma 28.2. ☐

Lemma 27.3. *A base change of a finitely presented morphism is of finite presentation. The same holds for morphisms which are locally of finite presentation.*

Proof. Combine Remark 16.4 with Morphisms of Spaces, Lemma 28.3. ☐

Lemma 27.4. *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Combine Remark 16.5 with Morphisms of Spaces, Lemma 28.5. ☐

Lemma 27.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If $\mathcal{Y}$ is locally Noetherian and f locally of finite type then f is locally of finite presentation.*
(2) *If $\mathcal{Y}$ is locally Noetherian and f of finite type and quasi-separated then f is of finite presentation.*

Proof. Assume $f : \mathcal{X} \to \mathcal{Y}$ locally of finite type and $\mathcal{Y}$ locally Noetherian. This means there exists a diagram as in Lemma 16.1 with h locally of finite type and surjective vertical arrow a. By Morphisms of Spaces, Lemma 28.7 h is locally of finite presentation. Hence $\mathcal{X} \to \mathcal{Y}$ is locally of finite presentation by definition. This proves (1). If f is of finite type and quasi-separated then it is also quasi-compact and quasi-separated and (2) follows immediately. ☐

Lemma 27.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks If $g \circ f$ is locally of finite presentation and g is locally of finite type, then f is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. The lemma follows upon applying Morphisms of Spaces, Lemma 28.9 to the morphisms $U \to V \to W$. $\square$

Lemma 27.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks with diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.*

Proof. Note that Δ is a morphism over $\mathcal{X}$ (via the second projection). If f is locally of finite type, then $\mathcal{X}$ is of finite presentation over $\mathcal{X}$ and $\mathrm{pr}_2 : \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is locally of finite type by Lemma 17.3. Thus the first statement holds by Lemma 27.6. The second statement follows from the first and the definitions (because f being quasi-separated means by definition that Δ_f is quasi-compact and quasi-separated). $\square$

Lemma 27.8. *An open immersion is locally of finite presentation.*

Proof. In view of Properties of Stacks, Definition 9.1 this follows from Morphisms of Spaces, Lemma 28.11. $\square$

Lemma 27.9. *Let P be a property of morphisms of algebraic spaces which is fppf local on the target and preserved by arbitrary base change. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Set $\mathcal{W} = \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. Then*

$$(f \text{ has } P) \Leftrightarrow (\text{the projection } \mathcal{W} \to \mathcal{Z} \text{ has } P).$$

For the meaning of this statement see Properties of Stacks, Section 3.

Proof. Choose an algebraic space W and a morphism $W \to \mathcal{Z}$ which is surjective, flat, and locally of finite presentation. By Properties of Stacks, Lemma 5.2 and Lemmas 25.2 and 27.2 the composition $W \to \mathcal{Y}$ is also surjective, flat, and locally of finite presentation. Denote $V = W \times_{\mathcal{Z}} \mathcal{W} = V \times_{\mathcal{Y}} \mathcal{X}$. By Properties of Stacks, Lemma 3.3 we see that f has $\mathcal{P}$ if and only if $V \to W$ does and that $\mathcal{W} \to \mathcal{Z}$ has $\mathcal{P}$ if and only if $V \to W$ does. The lemma follows. $\square$

Lemma 27.10. *Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and fppf local on the target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ has $\mathcal{P}$, then f has $\mathcal{P}$.*

Proof. Assume $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ has $\mathcal{P}$. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Observe that $W \times_{\mathcal{Z}} \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} = W \times_{\mathcal{Y}} \mathcal{X}$. Thus by the very definition of what it means for $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ to have $\mathcal{P}$ (see Definition 16.2 and Lemma 16.1) we see that $W \times_{\mathcal{Y}} \mathcal{X} \to W$ has $\mathcal{P}$. On the other hand, $W \to \mathcal{Z}$ is surjective, flat, and locally of finite presentation (Morphisms of Spaces, Lemmas 37.7 and 37.5) hence $W \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation (by Properties of Stacks, Lemma 5.2 and Lemmas 25.2 and 27.2). Thus we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_\mathcal{Y} \mathcal{X}$. We have to show that $U \to V$ has $\mathcal{P}$. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_\mathcal{Y} U$, $V' = W \times_\mathcal{Y} V$, $\mathcal{X}' = W \times_\mathcal{Y} \mathcal{X}$, and $\mathcal{Y}' = W \times_\mathcal{Y} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by Lemma 16.1 used in the definition of $\mathcal{X}' \to \mathcal{Y}' = W$ having $\mathcal{P}$ we see that $U' \to V'$ has $\mathcal{P}$. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is local in the fppf topology on the target. □

Lemma 27.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_\mathcal{Y} \mathcal{X} \to \mathcal{Z}$ is locally of finite presentation, then f is locally of finite presentation.*

Proof. The property "locally of finite presentation" satisfies the conditions of Lemma 27.10. Smooth local on the source-and-target we have seen in the introduction to this section and fppf local on the target is Descent on Spaces, Lemma 11.10. □

Lemma 27.12. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is locally of finite presentation and $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_\mathcal{Z} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_\mathcal{Y} \mathcal{X}$. We know that $U \to V$ is flat and locally of finite presentation and that $U \to W$ is locally of finite presentation. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 16.1. □

Lemma 27.13. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Then for every scheme U and object y of $\mathcal{Y}$ over U there exists an fppf covering $\{U_i \to U\}$ and objects x_i of $\mathcal{X}$ over U_i such that $f(x_i) \cong y|_{U_i}$ in $\mathcal{Y}_{U_i}$.*

Proof. We may think of y as a morphism $U \to \mathcal{Y}$. By Properties of Stacks, Lemma 5.3 and Lemmas 27.3 and 25.3 we see that $\mathcal{X} \times_\mathcal{Y} U \to U$ is surjective, flat, and locally of finite presentation. Let V be a scheme and let $V \to \mathcal{X} \times_\mathcal{Y} U$ smooth and surjective. Then $V \to \mathcal{X} \times_\mathcal{Y} U$ is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 37.7 and 37.5). Hence also $V \to U$ is surjective, flat, and locally of finite presentation, see Properties of Stacks, Lemma 5.2 and Lemmas 27.2, and 25.2. Hence $\{V \to U\}$ is the desired fppf covering and $x : V \to \mathcal{X}$ is the desired object. □

Lemma 27.14. *Let $f_j : \mathcal{X}_j \to \mathcal{X}$, $j \in J$ be a family of morphisms of algebraic stacks which are each flat and locally of finite presentation and which are jointly surjective, i.e., $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Then for every scheme U and object x of $\mathcal{X}$ over U there exists an fppf covering $\{U_i \to U\}_{i \in I}$, a map $a : I \to J$, and objects x_i of $\mathcal{X}_{a(i)}$ over U_i such that $f_{a(i)}(x_i) \cong y|_{U_i}$ in $\mathcal{X}_{U_i}$.*

Proof. Apply Lemma 27.13 to the morphism $\coprod_{j \in J} \mathcal{X}_j \to \mathcal{X}$. (There is a slight set theoretic issue here – due to our setup of things – which we ignore.) To finish, note that a morphism $x_i : U_i \to \coprod_{j \in J} \mathcal{X}_j$ is given by a disjoint union decomposition $U_i = \coprod U_{i,j}$ and morphisms $U_{i,j} \to \mathcal{X}_j$. Then the fppf covering $\{U_{i,j} \to U\}$ and the morphisms $U_{i,j} \to \mathcal{X}_j$ do the job. □

Lemma 27.15. *Let $f : \mathcal{X} \to \mathcal{Y}$ be flat and locally of finite presentation. Then $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is open.*

Proof. Choose a scheme V and a smooth surjective morphism $V \to \mathcal{Y}$. Choose a scheme U and a smooth surjective morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. By assumption the morphism of schemes $U \to V$ is flat and locally of finite presentation. Hence $U \to V$ is open by Morphisms, Lemma 25.10. By construction of the topology on $|\mathcal{Y}|$ the map $|V| \to |\mathcal{Y}|$ is open. The map $|U| \to |\mathcal{X}|$ is surjective. The result follows from these facts by elementary topology. □

Lemma 27.16. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is quasi-compact, then f is quasi-compact.*

Proof. We have to show that given $\mathcal{Y}' \to \mathcal{Y}$ with $\mathcal{Y}'$ quasi-compact, we have $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact. Denote $\mathcal{Z}' = \mathcal{Z} \times_{\mathcal{Y}} \mathcal{Y}'$. Then $|\mathcal{Z}'| \to |\mathcal{Y}'|$ is open, see Lemma 27.15. Hence we can find a quasi-compact open substack $\mathcal{W} \subset \mathcal{Z}'$ mapping onto $\mathcal{Y}'$. Because $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is quasi-compact, we know that

$$\mathcal{W} \times_{\mathcal{Z}} \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} = \mathcal{W} \times_{\mathcal{Y}} \mathcal{X}$$

is quasi-compact. And the map $\mathcal{W} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ is surjective, hence we win. Some details omitted. □

Lemma 27.17. *Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be composable morphisms of algebraic stacks with composition $h = g \circ f : \mathcal{X} \to \mathcal{Z}$. If f is surjective, flat, locally of finite presentation, and universally injective and if h is separated, then g is separated.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\Delta"'] \arrow[dr] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \times_{\mathcal{Z}} \mathcal{X} \arrow[d] \\
 & \mathcal{Y} \arrow[r] & \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}
\end{tikzcd}$$

The square is cartesian. We have to show the bottom horizontal arrow is proper. We already know that it is representable by algebraic spaces and locally of finite type (Lemma 3.3). Since the right vertical arrow is surjective, flat, and locally of finite presentation it suffices to show the top right horizontal arrow is proper (Lemma 27.9). Since h is separated, the composition of the top horizontal arrows is proper.

Since f is universally injective Δ is surjective (Lemma 14.5). Since the composition of Δ with the projection $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is the identity, we see that Δ is universally closed. By Morphisms of Spaces, Lemma 9.8 we conclude that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$ is separated as $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$ is separated. Here we use that implications between properties of morphisms of algebraic spaces can be transferred to the same implications between properties of morphisms of algebraic stacks representable by algebraic spaces; this is discussed in Properties of Stacks, Section 3. Finally, we use the same principle to conlude that $\mathcal{X} \times_\mathcal{Y} \mathcal{X} \to \mathcal{X} \times_\mathcal{Z} \mathcal{X}$ is proper from Morphisms of Spaces, Lemma 40.7. $\square$

28. Gerbes

An important type of algebraic stack are the stacks of the form $[B/G]$ where B is an algebraic space and G is a flat and locally finitely presented group algebraic space over B (acting trivially on B), see Criteria for Representability, Lemma 18.3. It turns out that an algebraic stack is a gerbe when it locally in the fppf topology is of this form, see Lemma 28.7. In this section we briefly discuss this notion and the corresponding relative notion.

Definition 28.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say $\mathcal{X}$ is a *gerbe over* $\mathcal{Y}$ if $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ as stacks in groupoids over $(Sch/S)_{fppf}$, see Stacks, Definition 11.4. We say an algebraic stack $\mathcal{X}$ is a *gerbe* if there exists a morphism $\mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X.

The condition that $\mathcal{X}$ be a gerbe over $\mathcal{Y}$ is defined purely in terms of the topology and category theory underlying the given algebraic stacks; but as we will see later this condition has geometric consequences. For example it implies that $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, see Lemma 28.8. The absolute notion is trickier to parse, because it may not be at first clear that X is well determined. Actually, it is.

Lemma 28.2. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe, then the sheafification of the presheaf*

$$(Sch/S)_{fppf}^{opp} \to Sets, \quad U \mapsto \mathrm{Ob}(\mathcal{X}_U)/\cong$$

is an algebraic space and $\mathcal{X}$ is a gerbe over it.

Proof. (In this proof the abuse of language introduced in Section 2 really pays off.) Choose a morphism $\pi : \mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X. It suffices to prove that X is the sheafification of the presheaf $\mathcal{F}$ displayed in the lemma. It is clear that there is a map $c : \mathcal{F} \to X$. We will use Stacks, Lemma 11.3 properties (2)(a) and (2)(b) to see that the map $c^\# : \mathcal{F}^\# \to X$ is surjective and injective, hence an isomorphism, see Sites, Lemma 11.2. Surjective: Let T be a scheme and let $f : T \to X$. By property (2)(a) there exists an fppf covering $\{h_i : T_i \to T\}$ and morphisms $x_i : T_i \to \mathcal{X}$ such that $f \circ h_i$ corresponds to $\pi \circ x_i$. Hence we see that $f|_{T_i}$ is in the image of c. Injective: Let T be a scheme and let $x, y : T \to \mathcal{X}$ be morphisms such that $c \circ x = c \circ y$. By (2)(b) we can find a covering $\{T_i \to T\}$ and morphisms $x|_{T_i} \to y|_{T_i}$ in the fibre category $\mathcal{X}_{T_i}$. Hence the restrictions $x|_{T_i}, y|_{T_i}$ are equal in $\mathcal{F}(T_i)$. This proves that x, y give the same section of $\mathcal{F}^\#$ over T as desired. $\square$

Lemma 28.3. *Let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a fibre product of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$.

Proof. Immediate from the definitions and Stacks, Lemma 11.5. ☐

Lemma 28.4. *Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{Y}$ is a gerbe over $\mathcal{Z}$, then $\mathcal{X}$ is a gerbe over $\mathcal{Z}$.*

Proof. Immediate from Stacks, Lemma 11.6. ☐

Lemma 28.5. *Let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a fibre product of algebraic stacks. If $\mathcal{Y}' \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation and $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$, then $\mathcal{X}$ is a gerbe over $\mathcal{Y}$.

Proof. Follows immediately from Lemma 27.13 and Stacks, Lemma 11.7. ☐

Lemma 28.6. *Let $\pi : \mathcal{X} \to U$ be a morphism from an algebraic stack to an algebraic space and let $x : U \to \mathcal{X}$ be a section of π. Set $G = Isom_{\mathcal{X}}(x, x)$, see Definition 5.3. If $\mathcal{X}$ is a gerbe over U, then*

1. *there is a canonical equivalence of stacks in groupoids*
$$x_{can} : [U/G] \longrightarrow \mathcal{X}.$$
where $[U/G]$ is the quotient stack for the trivial action of G on U,
2. *$G \to U$ is flat and locally of finite presentation, and*
3. *$U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation.*

Proof. Set $R = U \times_{x,\mathcal{X},x} U$. The morphism $R \to U \times U$ factors through the diagonal $\Delta_U : U \to U \times U$ as it factors through $U \times_U U = U$. Hence $R = G$ because

$$\begin{aligned} G &= Isom_{\mathcal{X}}(x, x) \\ &= U \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}} \\ &= U \times_{x,\mathcal{X}} (\mathcal{X} \times_{\Delta,\mathcal{X}\times_S\mathcal{X},\Delta} \mathcal{X}) \\ &= (U \times_{x,\mathcal{X},x} U) \times_{U\times U,\Delta_U} U \\ &= R \times_{U\times U,\Delta_U} U \\ &= R \end{aligned}$$

for the fourth equality use Categories, Lemma 31.12. Let $t, s : R \to U$ be the projections. The composition law $c : R \times_{s,U,t} R \to R$ constructed on R in Algebraic Stacks, Lemma 16.1 agrees with the group law on G (proof omitted). Thus Algebraic Stacks, Lemma 16.1 shows we obtain a canonical fully faithful 1-morphism

$$x_{can} : [U/G] \longrightarrow \mathcal{X}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. To see that it is an equivalence it suffices to show that it is essentially surjective. To do this it suffices to show that any object of $\mathcal{X}$ over a scheme T comes fppf locally from x via a morphism $T \to U$, see Stacks, Lemma 4.8. However, this follows the condition that π turns $\mathcal{X}$ into a gerbe over U, see property (2)(a) of Stacks, Lemma 11.3.

By Criteria for Representability, Lemma 18.3 we conclude that $G \to U$ is flat and locally of finite presentation. Finally, $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 17.1. ☐

Lemma 28.7. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *$\mathcal{X}$ is a gerbe over $\mathcal{Y}$, and*
(2) *there exists an algebraic space U, a group algebraic space G flat and locally of finite presentation over U, and a surjective, flat, and locally finitely presented morphism $U \to \mathcal{Y}$ such that $\mathcal{X} \times_{\mathcal{Y}} U \cong [U/G]$ over U.*

Proof. Assume (2). By Lemma 28.5 to prove (1) it suffices to show that $[U/G]$ is a gerbe over U. This is immediate from Groupoids in Spaces, Lemma 27.2.

Assume (1). Any base change of π is a gerbe, see Lemma 28.3. As a first step we choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Thus we may assume that $\pi : \mathcal{X} \to V$ is a gerbe over a scheme. This means that there exists an fppf covering $\{V_i \to V\}$ such that the fibre category $\mathcal{X}_{V_i}$ is nonempty, see Stacks, Lemma 11.3 (2)(a). Note that $U = \coprod V_i \to V$ is surjective, flat, and locally of finite presentation. Hence we may replace V by U and assume that $\pi : \mathcal{X} \to U$ is a gerbe over a scheme U and that there exists an object x of $\mathcal{X}$ over U. By Lemma 28.6 we see that $\mathcal{X} = [U/G]$ over U for some flat and locally finitely presented group algebraic space G over U. $\square$

Lemma 28.8. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then π is surjective, flat, and locally of finite presentation.*

Proof. By Properties of Stacks, Lemma 5.4 and Lemmas 25.4 and 27.11 it suffices to prove to the lemma after replacing π by a base change with a surjective, flat, locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$. By Lemma 28.7 we may assume $\mathcal{Y} = U$ is an algebraic space and $\mathcal{X} = [U/G]$ over U. Then $U \to [U/G]$ is surjective, flat, and locally of finite presentation, see Lemma 28.6. This implies that π is surjective, flat, and locally of finite presentation by Properties of Stacks, Lemma 5.5 and Lemmas 25.5 and 27.12. $\square$

Proposition 28.9. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is a gerbe, and*
(2) *$\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is flat and locally of finite presentation.*

Proof. Assume (1). Choose a morphism $\mathcal{X} \to X$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $X' \to X$ be a surjective, flat, locally finitely presented morphism and set $\mathcal{X}' = X' \times_X \mathcal{X}$. Note that $\mathcal{X}'$ is a gerbe over X' by Lemma 28.3. Then both squares in

$$\begin{tikzcd} \mathcal{I}_{\mathcal{X}'} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathcal{I}_{\mathcal{X}} \arrow[r] & \mathcal{X} \arrow[r] & X \end{tikzcd}$$

are fibre product squares, see Lemma 5.5. Hence to prove $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is flat and locally of finite presentation it suffices to do so after such a base change by Lemmas 25.4 and 27.11. Thus we can apply Lemma 28.7 to assume that $\mathcal{X} = [U/G]$. By Lemma 28.6 we see G is flat and locally of finite presentation over U and that $x : U \to [U/G]$ is surjective, flat, and locally of finite presentation. Moreover, the pullback of $\mathcal{I}_{\mathcal{X}}$ by x is G and we conclude that (2) holds by descent again, i.e., by Lemmas 25.4 and 27.11.

Conversely, assume (2). Choose a smooth presentation $\mathcal{X} = [U/R]$, see Algebraic Stacks, Section 16. Denote $G \to U$ the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Groupoids in Spaces, Definition 16.2. By Lemma 5.7 we see that $G \to U$ is flat and locally of finite presentation as a base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$, see Lemmas 25.3 and 27.3. Consider the following action

$$a : G \times_{U,t} R \to R, \quad (g, r) \mapsto c(g, r)$$

of G on R. This action is free on T-valued points for any scheme T as R is a groupoid. Hence $R' = R/G$ is an algebraic space and the quotient morphism $\pi : R \to R'$ is surjective, flat, and locally of finite presentation by Bootstrap, Lemma 11.7. The projections $s, t : R \to U$ are G-invariant, hence we obtain morphisms $s', t' : R' \to U$ such that $s = s' \circ \pi$ and $t = t' \circ \pi$. Since $s, t : R \to U$ are flat and locally of finite presentation we conclude that s', t' are flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 31.5 and Descent on Spaces, Lemma 16.1. Consider the morphism

$$j' = (t', s') : R' \longrightarrow U \times U.$$

We claim this is a monomorphism. Namely, suppose that T is a scheme and that $a, b : T \to R'$ are morphisms which have the same image in $U \times U$. By definition of the quotient $R' = R/G$ there exists an fppf covering $\{h_j : T_j \to T\}$ such that $a \circ h_j = \pi \circ a_j$ and $b \circ h_j = \pi \circ b_j$ for some morphisms $a_j, b_j : T_j \to R$. Since a_j, b_j have the same image in $U \times U$ we see that $g_j = c(a_j, i(b_j))$ is a T_j-valued point of G such that $c(g_j, b_j) = a_j$. In other words, a_j and b_j have the same image in R' and the claim is proved. Since $j : R \to U \times U$ is a pre-equivalence relation (see Groupoids in Spaces, Lemma 11.2) and $R \to R'$ is surjective (as a map of sheaves) we see that $j' : R' \to U \times U$ is an equivalence relation. Hence Bootstrap, Theorem 10.1 shows that $X = U/R'$ is an algebraic space. Finally, we claim that the morphism

$$\mathcal{X} = [U/R] \longrightarrow X = U/R'$$

turns $\mathcal{X}$ into a gerbe over X. This follows from Groupoids in Spaces, Lemma 27.1 as $R \to R'$ is surjective, flat, and locally of finite presentation (if needed use Bootstrap, Lemma 4.6 to see this implies the required hypothesis). □

Lemma 28.10. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which makes $\mathcal{X}$ a gerbe over $\mathcal{Y}$. Then*

1. *$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is flat and locally of finite presentation,*
2. *$\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is surjective, flat, and locally of finite presentation,*
3. *given algebraic spaces T_i, $i = 1, 2$ and morphisms $x_i : T_i \to \mathcal{X}$, with $y_i = f \circ x_i$ the morphism*

$$T_1 \times_{x_1, \mathcal{X}, x_2} T_2 \longrightarrow T_1 \times_{y_1, \mathcal{Y}, y_2} T_2$$

is surjective, flat, and locally of finite presentation,

4. *given an algebraic space T and morphisms $x_i : T \to \mathcal{X}$, $i = 1, 2$, with $y_i = f \circ x_i$ the morphism*

$$Isom_\mathcal{X}(x_1, x_2) \longrightarrow Isom_\mathcal{Y}(y_1, y_2)$$

is surjective, flat, and locally of finite presentation.

Proof. Proof of (1). Choose a scheme Y and a surjective smooth morphism $Y \to \mathcal{Y}$. Set $\mathcal{X}' = \mathcal{X} \times_\mathcal{Y} Y$. By Lemma 5.5 we obtain cartesian squares

$$\begin{tikzcd} \mathcal{I}_{\mathcal{X}'} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[r] & \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

By Lemmas 25.4 and 27.11 it suffices to prove that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is flat and locally of finite presentation. This follows from Proposition 28.9 (because $\mathcal{X}'$ is a gerbe over Y by Lemma 28.3).

Proof of (2). With notation as above, note that we may assume that $\mathcal{X}' = [Y/G]$ for some group algebraic space G flat and locally of finite presentation over Y, see Lemma 28.7. The base change of the morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ over $\mathcal{Y}$ by the morphism $Y \to \mathcal{Y}$ is the morphism $\Delta' : \mathcal{X}' \to \mathcal{X}' \times_Y \mathcal{X}'$. Hence it suffices to show that Δ' is surjective, flat, and locally of finite presentation (see Lemmas 25.4 and 27.11). In other words, we have to show that

$$[Y/G] \longrightarrow [Y/G \times_Y G]$$

is surjective, flat, and locally of finite presentation. This is true because the base change by the surjective, flat, locally finitely presented morphism $Y \to [Y/G \times_Y G]$ is the morphism $G \to Y$.

Proof of (3). Observe that the diagram

$$\begin{tikzcd} T_1 \times_{x_1, \mathcal{X}, x_2} T_2 \arrow[r] \arrow[d] & T_1 \times_{y_1, \mathcal{Y}, y_2} T_2 \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_\mathcal{Y} \mathcal{X} \end{tikzcd}$$

is cartesian. Hence (3) follows from (2).

Proof of (4). This is true because

$$Isom_\mathcal{X}(x_1, x_2) = (T \times_{x_1, \mathcal{X}, x_2} T) \times_{T \times T, \Delta_T} T$$

hence the morphism in (4) is a base change of the morphism in (3). $\square$

Proposition 28.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

1. *$\mathcal{X}$ is a gerbe over $\mathcal{Y}$, and*
2. *$f : \mathcal{X} \to \mathcal{Y}$ and $\Delta : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ are surjective, flat, and locally of finite presentation.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemmas 28.8 and 28.10.

Assume (2). It suffices to prove (1) for the base change of f by a surjective, flat, and locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$, see Lemma 28.5 (note that the base change of the diagonal of f is the diagonal of the base change). Thus we may assume $\mathcal{Y}$ is a scheme Y. In this case $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is a base change of Δ and we conclude that $\mathcal{X}$ is a gerbe by Proposition 28.9. We still have to show that $\mathcal{X}$ is a gerbe over Y. Let $\mathcal{X} \to X$ be the morphism of Lemma 28.2 turning $\mathcal{X}$ into a gerbe over the algebraic space X classifying isomorphism classes of objects of $\mathcal{X}$. It is clear that $f : \mathcal{X} \to Y$ factors as $\mathcal{X} \to X \to Y$. Since f is surjective, flat, and locally of finite presentation, we conclude that $X \to Y$ is surjective as a map of fppf sheaves (for example use Lemma 27.13). On the other hand, $X \to Y$ is injective too: for any scheme T and any two T-valued points x_1, x_2 of X which map to the same point of Y, we can first fppf locally on T lift x_1, x_2 to objects ξ_1, ξ_2 of $\mathcal{X}$ over T and second deduce that ξ_1 and ξ_2 are fppf locally isomorphic by our assumption that $\Delta : \mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}$ is surjective, flat, and locally of finite presentation. Whence $x_1 = x_2$ by construction of X. Thus $X = Y$ and the proof is complete. $\square$

At this point we have developed enough machinery to prove that residual gerbes (when they exist) are gerbes.

Lemma 28.12. *Let $\mathcal{Z}$ be a reduced, locally Noetherian algebraic stack such that $|\mathcal{Z}|$ is a singleton. Then $\mathcal{Z}$ is a gerbe over a reduced, locally Noetherian algebraic space Z with $|Z|$ a singleton.*

Proof. By Properties of Stacks, Lemma 11.3 there exists a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field. Then $\mathcal{I}_Z \times_{\mathcal{Z}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is representable by algebraic spaces and locally of finite type (as a base change of $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$, see Lemmas 5.1 and 17.3). Therefore it is locally of finite presentation, see Morphisms of Spaces, Lemma 28.7. Of course it is also flat as k is a field. Hence we may apply Lemmas 25.4 and 27.11 to see that $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$ is flat and locally of finite presentation. We conclude that $\mathcal{Z}$ is a gerbe by Proposition 28.9. Let $\pi : \mathcal{Z} \to Z$ be a morphism to an algebraic space such that $\mathcal{Z}$ is a gerbe over Z. Then π is surjective, flat, and locally of finite presentation by Lemma 28.8. Hence $\mathrm{Spec}(k) \to Z$ is surjective, flat, and locally of finite presentation as a composition, see Properties of Stacks, Lemma 5.2 and Lemmas 25.2 and 27.2. Hence by Properties of Stacks, Lemma 11.3 we see that $|Z|$ is a singleton and that Z is locally Noetherian and reduced. $\square$

Lemma 28.13. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ then f is a universal homeomorphism.*

Proof. By Lemma 28.3 the assumption on f is preserved under base change. Hence it suffices to show that the map $|\mathcal{X}| \to |\mathcal{Y}|$ is a homeomorphism of topological spaces. Let k be a field and let y be an object of $\mathcal{Y}$ over $\mathrm{Spec}(k)$. By Stacks, Lemma 11.3 property (2)(a) there exists an fppf covering $\{T_i \to \mathrm{Spec}(k)\}$ and objects x_i of $\mathcal{X}$ over T_i with $f(x_i) \cong y|_{T_i}$. Choose an i such that $T_i \neq \emptyset$. Choose a morphism $\mathrm{Spec}(K) \to T_i$ for some field K. Then $k \subset K$ and $x_i|_K$ is an object of $\mathcal{X}$ lying over $y|_K$. Thus we see that $|\mathcal{Y}| \to |\mathcal{X}|$. is surjective. The map $|\mathcal{Y}| \to |\mathcal{X}|$ is also injective. Namely, if x, x' are objects of $\mathcal{X}$ over $\mathrm{Spec}(k)$ whose images $f(x), f(x')$ become isomorphic (over an extension) in $\mathcal{Y}$, then Stacks, Lemma 11.3 property (2)(b) guarantees the existence of an extension of k over which x and x' become isomorphic (details omitted). Hence $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous and bijective and it suffices to show that it is also open. This follows from Lemmas 28.8 and 27.15. $\square$

Lemma 28.14. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks such that $\mathcal{X}$ is a gerbe over $\mathcal{Y}$. If $\Delta_{\mathcal{X}}$ is quasi-compact, so is $\Delta_{\mathcal{Y}}$.*

Proof. Consider the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \times \mathcal{X} \arrow[d] \\ & \mathcal{Y} \arrow[r] & \mathcal{Y} \times \mathcal{Y} \end{tikzcd}$$

By Proposition 28.11 we find that the arrow on the top left is surjective. Since the composition of the top horizontal arrows is quasi-compact, we conclude that the top right arrow is quasi-compact by Lemma 7.6. The square is cartesian and the right vertical arrow is surjective, flat, and locally of finite presentation. Thus we conclude by Lemma 27.16. □

The following lemma tells us that residual gerbes exist for all points on any algebraic stack which is a gerbe.

Lemma 28.15. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe then for every $x \in |\mathcal{X}|$ the residual gerbe of $\mathcal{X}$ at x exists.*

Proof. Let $\pi : \mathcal{X} \to X$ be a morphism from $\mathcal{X}$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $Z_x \to X$ be the residual space of X at x, see Decent Spaces, Definition 13.6. Let $\mathcal{Z} = \mathcal{X} \times_X Z_x$. By Lemma 28.3 the algebraic stack $\mathcal{Z}$ is a gerbe over Z_x. Hence $|\mathcal{Z}| = |Z_x|$ (Lemma 28.13) is a singleton. Since $\mathcal{Z} \to Z_x$ is locally of finite presentation as a base change of π (see Lemmas 28.8 and 27.3) we see that $\mathcal{Z}$ is locally Noetherian, see Lemma 17.5. Thus the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and is equal to $\mathcal{Z}_x = \mathcal{Z}_{red}$ the reduction of the algebraic stack $\mathcal{Z}$. Namely, we have seen above that $|\mathcal{Z}_{red}|$ is a singleton mapping to $x \in |\mathcal{X}|$, it is reduced by construction, and it is locally Noetherian (as the reduction of a locally Noetherian algebraic stack is locally Noetherian, details omitted). □

29. Stratification by gerbes

The goal of this section is to show that many algebraic stacks $\mathcal{X}$ have a "stratification" by locally closed substacks $\mathcal{X}_i \subset \mathcal{X}$ such that each $\mathcal{X}_i$ is a gerbe. This shows that in some sense gerbes are the building blocks out of which any algebraic stack is constructed. Note that by stratification we only mean that

$$|\mathcal{X}| = \bigcup\nolimits_i |\mathcal{X}_i|$$

is a stratification of the topological space associated to $\mathcal{X}$ and nothing more (in this section). Hence it is harmless to replace $\mathcal{X}$ by its reduction (see Properties of Stacks, Section 10) in order to study this stratification.

The following proposition tells us there is (almost always) a dense open substack of the reduction of $\mathcal{X}$

Proposition 29.1. *Let $\mathcal{X}$ be a reduced algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe.*

Proof. According to Proposition 28.9 it is enough to find a dense open substack $\mathcal{U}$ such that $\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation. Note that $\mathcal{I}_\mathcal{U} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{U}$, see Lemma 5.5.

Choose a presentation $\mathcal{X} = [U/R]$. Let $G \to U$ be the stabilizer group algebraic space of the groupoid R. By Lemma 5.7 we see that $G \to U$ is the base change of

$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ hence quasi-compact (by assumption) and locally of finite type (by Lemma 5.1). Let $W \subset U$ be the largest open (possibly empty) subscheme such that the restriction $G_W \to W$ is flat and locally of finite presentation (we omit the proof that W exists; hint: use that the properties are local). By Morphisms of Spaces, Proposition 32.1 we see that $W \subset U$ is dense. Note that $W \subset U$ is R-invariant by More on Groupoids in Spaces, Lemma 6.2. Hence W corresponds to an open substack $\mathcal{U} \subset \mathcal{X}$ by Properties of Stacks, Lemma 9.11. Since $|U| \to |\mathcal{X}|$ is open and $|W| \subset |U|$ is dense we conclude that $\mathcal{U}$ is dense in $\mathcal{X}$. Finally, the morphism $\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation because the base change by the surjective smooth morphism $W \to \mathcal{U}$ is the morphism $G_W \to W$ which is flat and locally of finite presentation by construction. See Lemmas 25.4 and 27.11. $\square$

The above proposition immediately implies that any point has a residual gerbe on an algebraic stack with quasi-compact inertia, as we will show in Lemma 31.1. It turns out that there doesn't always exist a finite stratification by gerbes. Here is an example.

Example 29.2. Let k be a field. Take $U = \mathrm{Spec}(k[x_0, x_1, x_2, \ldots])$ and let $\mathbf{G}_m$ act by $t(x_0, x_1, x_2, \ldots) = (tx_0, t^p x_1, t^{p^2} x_2, \ldots)$ where p is a prime number. Let $\mathcal{X} = [U/\mathbf{G}_m]$. This is an algebraic stack. There is a stratification of $\mathcal{X}$ by strata

1. $\mathcal{X}_0$ is where x_0 is not zero,
2. $\mathcal{X}_1$ is where x_0 is zero but x_1 is not zero,
3. $\mathcal{X}_2$ is where x_0, x_1 are zero, but x_2 is not zero,
4. and so on, and
5. $\mathcal{X}_\infty$ is where all the x_i are zero.

Each stratum is a gerbe over a scheme with group μ_{p^i} for $\mathcal{X}_i$ and $\mathbf{G}_m$ for $\mathcal{X}_\infty$. The strata are reduced locally closed substacks. There is no coarser stratification with the same properties.

Nonetheless, using transfinite induction we can use Proposition 29.1 find possibly infinite stratifications by gerbes...!

Lemma 29.3. *Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a well-ordered index set I and for every $i \in I$ a reduced locally closed substack $\mathcal{U}_i \subset \mathcal{X}$ such that*

1. *each $\mathcal{U}_i$ is a gerbe,*
2. *we have $|\mathcal{X}| = \bigcup_{i \in I} |\mathcal{U}_i|$,*
3. *$T_i = |\mathcal{X}| \setminus \bigcup_{i' < i} |\mathcal{U}_{i'}|$ is closed in $|\mathcal{X}|$ for all $i \in I$, and*
4. *$|\mathcal{U}_i|$ is open in T_i.*

We can moreover arrange it so that either (a) $|\mathcal{U}_i| \subset T_i$ is dense, or (b) $\mathcal{U}_i$ is quasi-compact. In case (a), if we choose $\mathcal{U}_i$ as large as possible (see proof for details), then the stratification is canonical.

Proof. Let $T \subset |\mathcal{X}|$ be a nonempty closed subset. We are going to find (resp. choose) for every such T a reduced locally closed substack $\mathcal{U}(T) \subset \mathcal{X}$ with $|\mathcal{U}(T)| \subset T$ open dense (resp. nonempty quasi-compact). Namely, by Properties of Stacks, Lemma 10.1 there exists a unique reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{X}'$ by Lemma 5.6. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 7.3. Therefore Proposition 29.1 implies there exists a dense maximal (see proof proposition) open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe.

In case (a) we set $\mathcal{U}(T) = \mathcal{U}$ (this is canonical) and in case (b) we simply choose a nonempty quasi-compact open $\mathcal{U}(T) \subset \mathcal{U}$, see Properties of Stacks, Lemma 4.9 (we can do this for all T simultaneously by the axiom of choice).

Using transfinite recursion we construct for every ordinal α a closed subset $T_\alpha \subset |\mathcal{X}|$. For $\alpha = 0$ we set $T_0 = |\mathcal{X}|$. Given T_α set

$$T_{\alpha+1} = T_\alpha \setminus |\mathcal{U}(T_\alpha)|.$$

If β is a limit ordinal we set

$$T_\beta = \bigcap\nolimits_{\alpha<\beta} T_\alpha.$$

We claim that $T_\alpha = \emptyset$ for all α large enough. Namely, assume that $T_\alpha \neq \emptyset$ for all α. Then we obtain an injective map from the class of ordinals into the set of subsets of $|\mathcal{X}|$ which is a contradiction.

The claim implies the lemma. Namely, let

$$I = \{\alpha \mid \mathcal{U}_\alpha \neq \emptyset\}.$$

This is a well-ordered set by the claim. For $i = \alpha \in I$ we set $\mathcal{U}_i = \mathcal{U}_\alpha$. So $\mathcal{U}_i$ is a reduced locally closed substack and a gerbe, i.e., (1) holds. By construction $T_i = T\alpha$ if $i = \alpha \in I$, hence (3) holds. Also, (4) and (a) or (b) hold by our choice of $\mathcal{U}(T)$ as well. Finally, to see (2) let $x \in |\mathcal{X}|$. There exists a smallest ordinal β with $x \notin T_\beta$ (because the ordinals are well-ordered). In this case β has to be a successor ordinal by the definition of T_β for limit ordinals. Hence $\beta = \alpha + 1$ and $x \in |\mathcal{U}(T_\alpha)|$ and we win. $\square$

Remark 29.4. We can wonder about the order type of the canonical stratifications which occur as output of the stratifications of type (a) constructed in Lemma 29.3. A natural guess is that the well-ordered set I has *cardinality* at most $\aleph_0$. We have no idea if this is true or false. If you do please email stacks.project@gmail.com.

30. The topological space of an algebraic stack

In this section we apply the previous results to the topological space $|\mathcal{X}|$ associated to an algebraic stack.

Lemma 30.1. *Let $\mathcal{X}$ be a quasi-compact algebraic stack whose diagonal Δ is quasi-compact. Then $|\mathcal{X}|$ is a spectral topological space.*

Proof. Choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$, see Properties of Stacks, Lemma 6.2. Then $|U| \to |\mathcal{X}|$ is continuous, open, and surjective, see Properties of Stacks, Lemma 4.7. Hence the quasi-compact opens of $|\mathcal{X}|$ form a basis for the topology. For $W_1, W_2 \subset |\mathcal{X}|$ quasi-compact open, we may choose a quasi-compact opens V_1, V_2 of U mapping to W_1 and W_2. Since Δ is quasi-compact, we see that

$$V_1 \times_{\mathcal{X}} V_2 = (V_1 \times V_2) \times_{\mathcal{X}\times\mathcal{X},\Delta} \mathcal{X}$$

is quasi-compact. Then image of $|V_1 \times_{\mathcal{X}} V_2|$ in $|\mathcal{X}|$ is $W_1 \cap W_2$ by Properties of Stacks, Lemma 4.3. Thus $W_1 \cap W_2$ is quasi-compact. To finish the proof, it suffices to show that $|\mathcal{X}|$ is sober, see Topology, Definition 23.1.

Let $T \subset |\mathcal{X}|$ be an irreducible closed subset. We have to show T has a unique generic point. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced induced closed substack corresponding to T, see Properties of Stacks, Definition 10.4. Since $\mathcal{Z} \to \mathcal{X}$ is a closed immersion, we see that $\Delta_{\mathcal{Z}}$ is quasi-compact: first show that $\mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact as the composition of $\mathcal{Z} \to \mathcal{X}$ with Δ, then write $\mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ as the composition of $\Delta_{\mathcal{Z}}$ and $\mathcal{Z} \times \mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ and use Lemma 7.7 and the fact that $\mathcal{Z} \times \mathcal{Z} \to \mathcal{X} \times \mathcal{X}$ is separated. Thus we reduce to the case discussed in the next paragraph.

Assume $\mathcal{X}$ is quasi-compact, Δ is quasi-compact, $\mathcal{X}$ is reduced, and $|\mathcal{X}|$ irreducible. We have to show $|\mathcal{X}|$ has a unique generic point. Since $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is a base change of Δ, we see that $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is quasi-compact (Lemma 7.3). Thus there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe by Proposition 29.1. In other words, $|\mathcal{U}| \subset |\mathcal{X}|$ is open dense. Thus we may assume that $\mathcal{X}$ is a gerbe. Say $\mathcal{X} \to X$ turns $\mathcal{X}$ into a gerbe over the algebraic space X. Then $|\mathcal{X}| \cong |X|$ by Lemma 28.13. In particular, X is quasi-compact. By Lemma 28.14 we see that X has quasi-compact diagonal, i.e., X is a quasi-separated algebraic space. Then $|X|$ is spectral by Properties of Spaces, Lemma 15.2 which implies what we want is true. $\square$

Lemma 30.2. *Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack. Then $|\mathcal{X}|$ is a spectral topological space.*

Proof. This is a special case of Lemma 30.1. $\square$

Lemma 30.3. *Let $\mathcal{X}$ be an algebraic stack whose diagonal is quasi-compact (for example if $\mathcal{X}$ is quasi-separated). Then there is an open covering $|\mathcal{X}| = \bigcup U_i$ with U_i spectral. In particular $|\mathcal{X}|$ is a sober topological space.*

Proof. Immediate consequence of Lemma 30.1. $\square$

31. Existence of residual gerbes

The definition of a residual gerbe of a point on an algebraic stack is Properties of Stacks, Definition 11.8. We have already shown that residual gerbes exist for finite type points (Lemma 18.7) and for any point of a gerbe (Lemma 28.15). In this section we prove that residual gerbes exist on many algebraic stacks. First, here is the promised application of Proposition 29.1.

Lemma 31.1. *Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is quasi-compact. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Let $T = \overline{\{x\}} \subset |\mathcal{X}|$ be the closure of x. By Properties of Stacks, Lemma 10.1 there exists a reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{X}'$ by Lemma 5.6. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 7.3. Therefore Proposition 29.1 implies there exists a dense open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. Note that $x \in |\mathcal{U}|$ because $\{x\} \subset T$ is a dense subset too. Hence a residual gerbe $\mathcal{Z}_x \subset \mathcal{U}$ of $\mathcal{U}$ at x exists by Lemma 28.15. It is immediate from the definitions that $\mathcal{Z}_x \to \mathcal{X}$ is a residual gerbe of $\mathcal{X}$ at x. $\square$

If the stack is quasi-DM then residual gerbes exist too. In particular, residual gerbes always exist for Deligne-Mumford stacks.

Lemma 31.2. *Let $\mathcal{X}$ be a quasi-DM algebraic stack. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Choose a scheme U and a surjective, flat, locally finite presented, and locally quasi-finite morphism $U \to \mathcal{X}$, see Theorem 21.3. Set $R = U \times_{\mathcal{X}} U$. The projections $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite as base changes of the morphism $U \to \mathcal{X}$. There is a canonical morphism $[U/R] \to \mathcal{X}$ (see Algebraic Stacks, Lemma 16.1) which is an equivalence because $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation, see Algebraic Stacks, Remark 16.3. Thus we may assume that $\mathcal{X} = [U/R]$ where (U, R, s, t, c) is a groupoid in algebraic spaces such that $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite. Set

$$U' = \coprod\nolimits_{u \in U \text{ lying over } x} \mathrm{Spec}(\kappa(u)).$$

The canonical morphism $U' \to U$ is a monomorphism. Let

$$R' = U' \times_{\mathcal{X}} U' = R \times_{(U \times U)} (U' \times U')$$

Because $U' \to U$ is a monomorphism we see that both projections $s', t' : R' \to U'$ factor as a monomorphism followed by a locally quasi-finite morphism. Hence, as U' is a disjoint union of spectra of fields, using Spaces over Fields, Lemma 10.9 we conclude that the morphisms $s', t' : R' \to U'$ are locally quasi-finite. Again since U' is a disjoint union of spectra of fields, the morphisms s', t' are also flat. Finally, s', t' locally quasi-finite implies s', t' locally of finite type, hence s', t' locally of finite presentation (because U' is a disjoint union of spectra of fields in particular locally Noetherian, so that Morphisms of Spaces, Lemma 28.7 applies). Hence $\mathcal{Z} = [U'/R']$ is an algebraic stack by Criteria for Representability, Theorem 17.2. As R' is the restriction of R by $U' \to U$ we see $\mathcal{Z} \to \mathcal{X}$ is a monomorphism by Groupoids in Spaces, Lemma 25.1 and Properties of Stacks, Lemma 8.4. Since $\mathcal{Z} \to \mathcal{X}$ is a monomorphism we see that $|\mathcal{Z}| \to |\mathcal{X}|$ is injective, see Properties of Stacks, Lemma 8.5. By Properties of Stacks, Lemma 4.3 we see that

$$|U'| = |\mathcal{Z} \times_{\mathcal{X}} U'| \longrightarrow |\mathcal{Z}| \times_{|\mathcal{X}|} |U'|$$

is surjective which implies (by our choice of U') that $|\mathcal{Z}| \to |\mathcal{X}|$ has image $\{x\}$. We conclude that $|\mathcal{Z}|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., $\mathcal{Z}$ is reduced and locally Noetherian. This means that the essential image of $\mathcal{Z} \to \mathcal{X}$ is the residual gerbe of $\mathcal{X}$ at x, see Properties of Stacks, Lemma 11.12. $\square$

Lemma 31.3. *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Choose an affine scheme U and a smooth morphism $U \to \mathcal{X}$ such that x is in the image of the open continuous map $|U| \to |\mathcal{X}|$. We may and do replace $\mathcal{X}$ with the open substack corresponding to the image of $|U| \to |\mathcal{X}|$, see Properties of Stacks, Lemma 9.12. Thus we may assume $\mathcal{X} = [U/R]$ for a smooth groupoid (U, R, s, t, c) in algebraic spaces where U is a Noetherian affine scheme, see Algebraic Stacks, Section 16.

Let $E \subset |U|$ be the inverse image of $\{x\} \subset |\mathcal{X}|$. Of course $E \neq \emptyset$. Since $|U|$ is a Noetherian topological space, we can choose an element $u \in E$ such that $\overline{\{u\}} \cap E = \{u\}$. As usual, we think of $u = \mathrm{Spec}(\kappa(u))$ as the spectrum of its residue field. Let us write

$$F = u \times_{U,t} R = u \times_{\mathcal{X}} U \quad \text{and} \quad R' = (u \times u) \times_{(U \times U),(t,s)} R = u \times_{\mathcal{X}} u$$

Furthermore, denote $Z = \overline{\{u\}} \subset U$ with the reduced induced scheme structure. Denote $p : F \to U$ the morphism induced by the second projection (using $s : R \to U$ in the first fibre product description of F). Then E is the set theoretic image of

p. The morphism $R' \to F$ is a monomorphism which factors through the inverse image $p^{-1}(Z)$ of Z. This inverse image $p^{-1}(Z) \subset F$ is a closed subscheme and the restriction $p|_{p^{-1}(Z)} : p^{-1}(Z) \to Z$ has image set theoretically contained in $\{u\} \subset Z$ by our careful choice of $u \in E$ above. Since $u = \lim W$ where the limit is over the nonempty affine open subschemes of the irreducible reduced scheme Z, we conclude that the morphism $p|_{p^{-1}(Z)} : p^{-1}(Z) \to Z$ factors through the morphism $u \to Z$. Clearly this implies that $R' = p^{-1}(Z)$. In particular the morphism $t' : R' \to u$ is locally of finite presentation as the composition of the closed immersion $p^{-1}(Z) \to F$ of locally Noetherian algebraic spaces with the smooth morphism $\text{pr}_1 : F \to u$; use Morphisms of Spaces, Lemmas 23.5, 28.12, and 28.2. Hence the restriction (u, R', s', t', c') of (U, R, s, t, c) by $u \to U$ is a groupoid in algebraic spaces where s' and t' are flat and locally of finite presentation. Therefore $\mathcal{Z} = [u/R']$ is an algebraic stack by Criteria for Representability, Theorem 17.2. As R' is the restriction of R by $u \to U$ we see $\mathcal{Z} \to \mathcal{X}$ is a monomorphism by Groupoids in Spaces, Lemma 25.1 and Properties of Stacks, Lemma 8.4. Then $\mathcal{Z}$ is (isomorphic to) the residual gerbe by the material in Properties of Stacks, Section 11. □

32. Étale local structure

In this section we start discussing what we can say about the étale local structure of an algebraic stack.

Lemma 32.1. *Let Y be an algebraic space. Let (U, R, s, t, c) be a groupoid in algebraic spaces over Y. Assume $U \to Y$ is flat and locally of finite presentation and $R \to U \times_Y U$ an open immersion. Then $X = [U/R] = U/R$ is an algebraic space and $X \to Y$ is étale.*

Proof. The quotient stack $[U/R]$ is an algebraic stacks by Criteria for Representability, Theorem 17.2. On the other hand, since $R \to U \times U$ is a monomorphism, it is an algebraic space (by our abuse of language and Algebraic Stacks, Proposition 13.3) and of course it is equal to the algebraic space U/R (of Bootstrap, Theorem 10.1). Since $U \to X$ is surjective, flat, and locally of finite presenation (Bootstrap, Lemma 11.6) we conclude that $X \to Y$ is flat and locally of finite presentation by Morphisms of Spaces, Lemma 31.5 and Descent on Spaces, Lemma 8.2. Finally, consider the cartesian diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \times_Y U \arrow[d] \\ X \arrow[r] & X \times_Y X \end{tikzcd}$$

Since the right vertical arrow is surjective, flat, and locally of finite presentation (small detail omitted), we find that $X \to X \times_Y X$ is an open immersion as the top horizonal arrow has this property by assumption (use Properties of Stacks, Lemma 3.3). Thus $X \to Y$ is unramified by Morphisms of Spaces, Lemma 38.9. We conclude by Morphisms of Spaces, Lemma 39.12. □

Lemma 32.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Let $P \subset R$ be an open subspace such that $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over S. Then*

$$[U/P] \longrightarrow [U/R]$$

is a morphism of algebraic stacks which is representable by algebraic spaces, surjective, and étale.

Proof. Since $P \subset R$ is open, we see that $s|_P$ and $t|_P$ are flat and locally of finite presentation. Thus $[U/R]$ and $[U/P]$ are algebraic stacks by Criteria for Representability, Theorem 17.2. To see that the morphism is representable by algebraic spaces, it suffices to show that $[U/P] \to [U/R]$ is faithful on fibre categories, see Algebraic Stacks, Lemma 15.2. This follows immediately from the fact that $P \to R$ is a monomorphism and the explicit description of quotient stacks given in Groupoids in Spaces, Lemma 24.1. Having said this, we know what it means for $[U/P] \to [U/R]$ to be surjective and étale by Algebraic Stacks, Definition 10.1. Surjectivity follows for example from Criteria for Representability, Lemma 7.3 and the description of objects of quotient stacks (see lemma cited above) over spectra of fields. It remains to prove that our morphism is étale.

To do this it suffices to show that $U \times_{[U/R]} [U/P] \to U$ is étale, see Properties of Stacks, Lemma 3.3. By Groupoids in Spaces, Lemma 21.2 the fibre product is equal to $[R/P \times_{s,U,t} R]$ with morphism to U induced by $s : R \to U$. The maps $s', t' : P \times_{s,U,t} R \to R$ are given by $s' : (p, r) \mapsto r$ and $t' : (p, r) \mapsto c(p, r)$. Since $P \subset R$ is open we conclude that $(t', s') : P \times_{s,U,t} R \to R \times_{s,U,s} R$ is an open immersion. Thus we may apply Lemma 32.1 to conclude. $\square$

Lemma 32.3. *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{X}$ is quasi-DM with separated diagonal (equivalently $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally quasi-finite and separated). Let $x \in |\mathcal{X}|$. Then there exists a morphism of algebraic stacks*

$$\mathcal{U} \longrightarrow \mathcal{X}$$

with the following properties

1. *there exists a point $u \in |\mathcal{U}|$ mapping to x,*
2. *$\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces and étale,*
3. *$\mathcal{U} = [U/R]$ where (U, R, s, t, c) is a groupoid scheme with U, R affine, and s, t finite, flat, and locally of finite presentation.*

Proof. (The parenthetical statement follows from the equivalences in Lemma 6.1). Choose an affine scheme U and a flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X}$ such that x is the image of some point $u \in U$. This is possible by Theorem 21.3 and the assumption that $\mathcal{X}$ is quasi-DM. Let (U, R, s, t, c) be the groupoid in algebraic spaces we obtain by setting $R = U \times_\mathcal{X} U$, see Algebraic Stacks, Lemma 16.1. Let $\mathcal{X}' \subset \mathcal{X}$ be the open substack corresponding to the open image of $|U| \to |\mathcal{X}|$ (Properties of Stacks, Lemmas 4.7 and 9.12). Clearly, we may replace $\mathcal{X}$ by the open substack $\mathcal{X}'$. Thus we may assume $U \to \mathcal{X}$ is surjective and then Algebraic Stacks, Remark 16.3 gives $\mathcal{X} = [U/R]$. Observe that $s, t : R \to U$ are flat, locally of finite presentation, and locally quasi-finite. Since $R = U \times U \times_{(\mathcal{X} \times \mathcal{X})} \mathcal{X}$ and since the diagonal of $\mathcal{X}$ is separated, we find that R is separated. Hence $s, t : R \to U$ are separated. It follows that R is a scheme by Morphisms of Spaces, Proposition 50.2 applied to $s : R \to U$.

Above we have verified all the assumptions of More on Groupoids in Spaces, Lemma 15.13 are satisfied for (U, R, s, t, c) and u. Hence we can find an elementary étale neighbourhood $(U', u') \to (U, u)$ such that the restriction R' of R to U' is quasi-split over u. Note that $R' = U' \times_\mathcal{X} U'$ (small detail omitted; hint: transitivity of fibre products). Replacing (U, R, s, t, c) by (U', R', s', t', c') and shrinking $\mathcal{X}$ as above, we may assume that (U, R, s, t, c) has a quasi-splitting over u (the point u is irrelevant from now on as can be seen from the footnote in More on Groupoids in Spaces, Definition 15.1). Let $P \subset R$ be a quasi-splitting of R over u. Apply Lemma 32.2 to see that

$$\mathcal{U} = [U/P] \longrightarrow [U/R] = \mathcal{X}$$

has all the desired properties. □

Lemma 32.4. *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{X}$ is quasi-DM with separated diagonal (equivalently $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally quasi-finite and separated). Let $x \in |\mathcal{X}|$. Assume the automorphism group of $\mathcal{X}$ at x is finite (Remark 19.3). Then there exists a morphism of algebraic stacks*

$$g : \mathcal{U} \longrightarrow \mathcal{X}$$

with the following properties

1. *there exists a point $u \in |\mathcal{U}|$ mapping to x and g induces an isomorphism between automorphism groups at u and x (Remark 19.5),*
2. *$\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces and étale,*
3. *$\mathcal{U} = [U/R]$ where (U, R, s, t, c) is a groupoid scheme with U, R affine, and s, t finite, flat, and locally of finite presentation.*

Proof. Observe that G_x is a group scheme by Lemma 19.1. The first part of the proof is **exactly** the same as the first part of the proof of Lemma 32.3. Thus we may assume $\mathcal{X} = [U/R]$ where (U, R, s, t, c) and $u \in U$ mapping to x satisfy all the assumptions of More on Groupoids in Spaces, Lemma 15.13. Our assumption on G_x implies that G_u is finite over u. Hence all the assumptions of More on Groupoids in Spaces, Lemma 15.12 are satisfied. Hence we can find an elementary étale neighbourhood $(U', u') \to (U, u)$ such that the restriction R' of R to U' is split over u. Note that $R' = U' \times_\mathcal{X} U'$ (small detail omitted; hint: transitivity of fibre products). Replacing (U, R, s, t, c) by (U', R', s', t', c') and shrinking $\mathcal{X}$ as above, we may assume that (U, R, s, t, c) has a splitting over u. Let $P \subset R$ be a splitting of R over u. Apply Lemma 32.2 to see that

$$\mathcal{U} = [U/P] \longrightarrow [U/R] = \mathcal{X}$$

is representable by algebraic spaces and étale. By construction G_u is contained in P, hence this morphism defines an isomorphism on automorphism groups at u as desired. □

Lemma 32.5. *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{X}$ is quasi-DM with separated diagonal (equivalently $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally quasi-finite and separated). Let $x \in |\mathcal{X}|$. Assume x can be represented by a quasi-compact morphism* $\text{Spec}(k) \to \mathcal{X}$. *Then there exists a morphism of algebraic stacks*

$$g : \mathcal{U} \longrightarrow \mathcal{X}$$

with the following properties

1. *there exists a point $u \in |\mathcal{U}|$ mapping to x and g induces an isomorphism between the residual gerbes at u and x,*
2. *$\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces and étale,*
3. *$\mathcal{U} = [U/R]$ where (U, R, s, t, c) is a groupoid scheme with U, R affine, and s, t finite, flat, and locally of finite presentation.*

Proof. The first part of the proof is **exactly** the same as the first part of the proof of Lemma 32.3. Thus we may assume $\mathcal{X} = [U/R]$ where (U, R, s, t, c) and $u \in U$ mapping to x satisfy all the assumptions of More on Groupoids in Spaces, Lemma 15.13. Observe that $u = \mathrm{Spec}(\kappa(u)) \to \mathcal{X}$ is quasi-compact, see Properties of Stacks, Lemma 14.1. Consider the cartesian diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & U \arrow[d] \\ u \arrow[r, "u"] & \mathcal{X} \end{tikzcd}$$

Since U is an affine scheme and $F \to U$ is quasi-compact, we see that F is quasi-compact. Since $U \to \mathcal{X}$ is locally quasi-finite, we see that $F \to u$ is locally quasi-finite. Hence $F \to u$ is quasi-finite and F is an affine scheme whose underlying topological space is finite discrete (Spaces over Fields, Lemma 10.8). Observe that we have a monomorphism $u \times_\mathcal{X} u \to F$. In particular the set $\{r \in R : s(r) = u, t(r) = u\}$ which is the image of $|u \times_\mathcal{X} u| \to |R|$ is finite. we conclude that all the assumptions of More on Groupoids in Spaces, Lemma 15.11 hold.

Thus we can find an elementary étale neighbourhood $(U', u') \to (U, u)$ such that the restriction R' of R to U' is strongly split over u'. Note that $R' = U' \times_\mathcal{X} U'$ (small detail omitted; hint: transitivity of fibre products). Replacing (U, R, s, t, c) by (U', R', s', t', c') and shrinking $\mathcal{X}$ as above, we may assume that (U, R, s, t, c) has a strong splitting over u. Let $P \subset R$ be a strong splitting of R over u. Apply Lemma 32.2 to see that

$$\mathcal{U} = [U/P] \longrightarrow [U/R] = \mathcal{X}$$

is representable by algebraic spaces and étale. Since $P \subset R$ is open and contains $\{r \in R : s(r) = u, t(r) = u\}$ by construction we see that $u \times_\mathcal{U} u \to u \times_\mathcal{X} u$ is an isomorphism. The statement on residual gerbes then follows from Properties of Stacks, Lemma 11.14 (we observe that the residual gerbes in question exist by Lemma 31.2). ☐

33. Smooth morphisms

The property "being smooth" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 20.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 37.3 and Descent on Spaces, Lemma 11.26. Hence, by Lemma 16.1 above, we may define what it means for a morphism of algebraic spaces to be smooth as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 3 when the morphism is representable by algebraic spaces.

Definition 33.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *smooth* if the equivalent conditions of Lemma 16.1 hold with $\mathcal{P} =$ smooth.

Lemma 33.2. *The composition of smooth morphisms is smooth.*

Proof. Combine Remark 16.3 with Morphisms of Spaces, Lemma 37.2. ☐

Lemma 33.3. *A base change of a smooth morphism is smooth.*

Proof. Combine Remark 16.4 with Morphisms of Spaces, Lemma 37.3. ☐

Lemma 33.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is smooth, then f is smooth.*

Proof. The property "smooth" satisfies the conditions of Lemma 27.10. Smooth local on the source-and-target we have seen in the introduction to this section and fppf local on the target is Descent on Spaces, Lemma 11.26. $\square$

Lemma 33.5. *A smooth morphism of algebraic stacks is locally of finite presentation.*

Proof. Omitted. $\square$

Lemma 33.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. There is a maximal open substack $\mathcal{U} \subset \mathcal{X}$ such that $f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{Y}$ is smooth. Moreover, formation of this open commutes with*

(1) *precomposing by smooth morphisms,*
(2) *base change by morphisms which are flat and locally of finite presentation,*
(3) *base change by flat morphisms provided f is locally of finite presentation.*

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

where U and V are algebraic spaces, the vertical arrows are smooth, and $a : U \to \mathcal{X}$ surjective. There is a maximal open subspace $U' \subset U$ such that $h_{U'} : U' \to V$ is smooth, see Morphisms of Spaces, Lemma 37.9. Let $\mathcal{U} \subset \mathcal{X}$ be the open substack corresponding to the image of $|U'| \to |\mathcal{X}|$ (Properties of Stacks, Lemmas 4.7 and 9.12). By the equivalence in Lemma 16.1 we find that $f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{Y}$ is smooth and that $\mathcal{U}$ is the largest open substack with this property.

Assertion (1) follows from the fact that being smooth is smooth local on the source (this property was used to even define smooth morphisms of algebraic stacks). Assertions (2) and (3) follow from the case of algebraic spaces, see Morphisms of Spaces, Lemma 37.9. $\square$

Lemma 33.7. *Let $X \to Y$ be a smooth morphism of algebraic spaces. Let G be a group algebraic space over Y which is flat and locally of finite presentation over Y. Let G act on X over Y. Then the quotient stack $[X/G]$ is smooth over Y.*

This holds even if G is not smooth over S!

Proof. The quotient $[X/G]$ is an algebraic stack by Criteria for Representability, Theorem 17.2. The smoothness of $[X/G]$ over Y follows from the fact that smoothness descends under fppf coverings: Choose a surjective smooth morphism $U \to [X/G]$ where U is a scheme. Smoothness of $[X/G]$ over Y is equivalent to smoothness of U over Y. Observe that $U \times_{[X/G]} X$ is smooth over X and hence smooth over Y (because compositions of smooth morphisms are smooth). On the other hand, $U \times_{[X/G]} X \to U$ is locally of finite presentation, flat, and surjective (because it is the base change of $X \to [X/G]$ which has those properties for example by Criteria for Representability, Lemma 17.1). Therefore we may apply Descent on Spaces, Lemma 8.4. $\square$

Lemma 33.8. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then π is surjective and smooth.*

Proof. We have seen surjectivity in Lemma 28.8. By Lemma 33.4 it suffices to prove to the lemma after replacing π by a base change with a surjective, flat, locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$. By Lemma 28.7 we may assume $\mathcal{Y} = U$ is an algebraic space and $\mathcal{X} = [U/G]$ over U with $G \to U$ flat and locally of finite presentation. Then we win by Lemma 33.7. $\square$

34. Types of morphisms étale-smooth local on source-and-target

Given a property of morphisms of algebraic spaces which is *étale-smooth local on the source-and-target*, see Descent on Spaces, Definition 21.1 we may use it to define a corresponding property of DM morphisms of algebraic stacks, namely by imposing either of the equivalent conditions of the lemma below.

Lemma 34.1. *Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a DM morphism of algebraic stacks. Consider commutative diagrams*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

where U and V are algebraic spaces, $V \to \mathcal{Y}$ is smooth, and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is étale. The following are equivalent

1. *for any diagram as above the morphism h has property $\mathcal{P}$, and*
2. *for some diagram as above with $a : U \to \mathcal{X}$ surjective the morphism h has property $\mathcal{P}$.*

If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces, then this is also equivalent to f (as a morphism of algebraic spaces) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for morphisms f which are representable by algebraic spaces this is also equivalent to f having property $\mathcal{P}$ in the sense of Properties of Stacks, Section 3.

Proof. Let us prove the implication (1) $\Rightarrow$ (2). Pick an algebraic space V and a surjective and smooth morphism $V \to \mathcal{Y}$. As f is DM there exists a scheme U and a surjective étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$, see Lemma 21.7. Thus we see that (2) holds. Note that $U \to \mathcal{X}$ is surjective and smooth as well, as a composition of the base change $\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X}$ and the chosen map $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Hence we obtain a diagram as in (1). Thus if (1) holds, then $h : U \to V$ has property $\mathcal{P}$, which means that (2) holds as $U \to \mathcal{X}$ is surjective.

Conversely, assume (2) holds and let U, V, a, b, h be as in (2). Next, let U', V', a', b', h' be any diagram as in (1). Picture

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd} \qquad \begin{tikzcd} U' \arrow[r, "h'"] \arrow[d] & V' \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the commutative diagram

$$\begin{tikzcd} U & U \times_{\mathcal{X}} U' \arrow[l] \arrow[r] \arrow[d, "{(h,h')}"] & U' \arrow[d, "h'"] \\ V \arrow[from=u, "h"] & V \times_{\mathcal{Y}} V' \arrow[l] \arrow[r] & V' \end{tikzcd}$$

of algebraic spaces. Note that the horizontal arrows are smooth as base changes of the smooth morphisms $V \to \mathcal{Y}$, $V' \to \mathcal{Y}$, $U \to \mathcal{X}$, and $U' \to \mathcal{X}$. Note that the squares

$$\begin{tikzcd} U \arrow[d] & U \times_{\mathcal{X}} U' \arrow[l] \arrow[d] \\ V \times_{\mathcal{Y}} \mathcal{X} & V \times_{\mathcal{Y}} U' \arrow[l] \end{tikzcd} \quad \begin{tikzcd} U \times_{\mathcal{X}} U' \arrow[r] \arrow[d] & U' \arrow[d] \\ U \times_{\mathcal{Y}} V' \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} V' \end{tikzcd}$$

are cartesian, hence the vertical arrows are étale by our assumptions on U', V', a', b', h' and U, V, a, b, h. Since $\mathcal{P}$ is smooth local on the target by Descent on Spaces, Lemma 21.2 part (2) we see that the base change $t : U \times_{\mathcal{Y}} V' \to V \times_{\mathcal{Y}} V'$ of h has $\mathcal{P}$. Since $\mathcal{P}$ is étale local on the source by Descent on Spaces, Lemma 21.2 part (1) and $s : U \times_{\mathcal{X}} U' \to U \times_{\mathcal{Y}} V'$ is étale, we see the morphism $(h, h') = t \circ s$ has $\mathcal{P}$. Consider the diagram

$$\begin{tikzcd} U \times_{\mathcal{X}} U' \arrow[r, "{(h,h')}"'] \arrow[d] & V \times_{\mathcal{Y}} V' \arrow[d] \\ U' \arrow[r, "h'"] & V' \end{tikzcd}$$

The left vertical arrow is surjective, the right vertical arrow is smooth, and the induced morphism

$$U \times_{\mathcal{X}} U' \longrightarrow U' \times_{V'} (V \times_{\mathcal{Y}} V') = V \times_{\mathcal{Y}} U'$$

is étale as seen above. Hence by Descent on Spaces, Definition 21.1 part (3) we conclude that h' has $\mathcal{P}$. This finishes the proof of the equivalence of (1) and (2).

If $\mathcal{X}$ and $\mathcal{Y}$ are representable, then Descent on Spaces, Lemma 21.3 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to \mathcal{X}$ the base change $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. Consider the diagram

$$\begin{tikzcd} Z \times_{\mathcal{Y}} U \arrow[r] \arrow[d] & Z \times_{\mathcal{Y}} V \arrow[d] \\ Z \times_{\mathcal{Y}} \mathcal{X} \arrow[r] & Z \end{tikzcd}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is surjective, the induced morphism

$$Z \times_{\mathcal{Y}} U \longrightarrow (Z \times_{\mathcal{Y}} \mathcal{X}) \times_Z (Z \times_{\mathcal{Y}} V)$$

is étale, and the right vertical arrow is smooth. Thus Descent on Spaces, Lemma 21.3 kicks in and shows that $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. $\square$

Definition 34.2. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target. We say a DM morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks *has property* $\mathcal{P}$ if the equivalent conditions of Lemma 16.1 hold.

Remark 34.3. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target and stable under composition. Then the property of DM morphisms of algebraic stacks defined in Definition 34.2 is stable under composition. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be DM morphisms of algebraic stacks having property $\mathcal{P}$. By Lemma 4.10 the composition $g \circ f$ is DM. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective étale morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ (Lemma 21.7). Choose an algebraic space U and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Then the morphisms $V \to W$ and $U \to V$ have property $\mathcal{P}$ by definition. Whence $U \to W$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under composition. Thus, by definition again, we see that $g \circ f : \mathcal{X} \to \mathcal{Z}$ has property $\mathcal{P}$.

Remark 34.4. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is étale-smooth local on the source-and-target and stable under base change. Then the property of DM morphisms of algebraic stacks defined in Definition 34.2 is stable under arbitrary base change. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ be a DM morphism of algebraic stacks and $g : \mathcal{Y}' \to \mathcal{Y}$ be a morphism of algebraic stacks and assume f has property $\mathcal{P}$. Then the base change $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ is a DM morphism by Lemma 4.4. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ (Lemma 21.7). Finally, choose an algebraic space V' and a surjective and smooth morphism $V' \to \mathcal{Y}' \times_{\mathcal{Y}} V$. Then the morphism $U \to V$ has property $\mathcal{P}$ by definition. Whence $V' \times_V U \to V'$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under base change. Considering the diagram

$$\begin{tikzcd} V' \times_V U \arrow[r] \arrow[d] & \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ V' \arrow[r] & \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

we see that the left top horizontal arrow is surjective and

$$V' \times_V U \to V' \times_{\mathcal{Y}} (\mathcal{Y}' \times_{\mathcal{Y}'} \mathcal{X}) = V' \times_V (\mathcal{X} \times_{\mathcal{Y}} V)$$

is étale as a base change of $U \to \mathcal{X} \times_{\mathcal{Y}} V$, whence by definition we see that the projection $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property $\mathcal{P}$.

Remark 34.5. Let $\mathcal{P}, \mathcal{P}'$ be properties of morphisms of algebraic spaces which are étale-smooth local on the source-and-target. Suppose that we have $\mathcal{P} \Rightarrow \mathcal{P}'$ for morphisms of algebraic spaces. Then we also have $\mathcal{P} \Rightarrow \mathcal{P}'$ for the properties of morphisms of algebraic stacks defined in Definition 34.2 using $\mathcal{P}$ and $\mathcal{P}'$. This is clear from the definition.

35. Étale morphisms

An étale morphism of algebraic stacks should not be defined as a smooth morphism of relative dimension 0. Namely, the morphism

$$[\mathbf{A}^1_k/\mathbf{G}_{m,k}] \longrightarrow \mathrm{Spec}(k)$$

is smooth of relative dimension 0 for any choice of action of the group scheme $\mathbf{G}_{m,k}$ on $\mathbf{A}^1_k$. This does not correspond to our usual idea that étale morphisms should identify tangent spaces. The example above isn't quasi-finite, but the morphism

$$\mathcal{X} = [\mathrm{Spec}(k)/\mu_{p,k}] \longrightarrow \mathrm{Spec}(k)$$

is smooth and quasi-finite (Section 23). However, if the characteristic of k is $p > 0$, then we see that the representable morphism $\mathrm{Spec}(k) \to \mathcal{X}$ isn't étale as the base change $\mu_{p,k} = \mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is a morphism from a nonreduced scheme to the spectrum of a field. Thus if we define an étale morphism as smooth and locally quasi-finite, then the analogue of Morphisms of Spaces, Lemma 39.11 would fail.

Instead, our approach will be to start with the requirements that "étaleness" should be a property preserved under base change and that if $\mathcal{X} \to X$ is an étale morphism from an algebraic stack to a scheme, then $\mathcal{X}$ should be Deligne-Mumford. In other words, we will require étale morphisms to be DM and we will use the material in Section 34 to define étale morphisms of algebraic stacks.

In Lemma 36.10 we will characterize étale morphisms of algebraic stacks as morphisms which are (a) locally of finite presentation, (b) flat, and (c) have étale diagonal.

The property "étale" of morphisms of algebraic spaces is étale-smooth local on the source-and-target, see Descent on Spaces, Remark 21.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 39.4 and Descent on Spaces, Lemma 11.28. Hence, by Lemma 34.1 above, we may define what it means for a morphism of algebraic spaces to be étale as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 3 when the morphism is representable by algebraic spaces because such a morphism is automatically DM by Lemma 4.3.

Definition 35.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *étale* if f is DM and the equivalent conditions of Lemma 34.1 hold with $\mathcal{P} = \textit{étale}$.

We will use without further mention that this agrees with the already existing notion of étale morphisms in case f is representable by algebraic spaces or if $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces.

Lemma 35.2. *The composition of étale morphisms is étale.*

Proof. Combine Remark 34.3 with Morphisms of Spaces, Lemma 39.3. $\square$

Lemma 35.3. *A base change of an étale morphism is étale.*

Proof. Combine Remark 34.4 with Morphisms of Spaces, Lemma 39.4. $\square$

Lemma 35.4. *An open immersion is étale.*

Proof. Let $j : \mathcal{U} \to \mathcal{X}$ be an open immersion of algebraic stacks. Since j is representable, it is DM by Lemma 4.3. On the other hand, if $X \to \mathcal{X}$ is a smooth and surjective morphism where X is a scheme, then $U = \mathcal{U} \times_{\mathcal{X}} X$ is an open subscheme of X. Hence $U \to X$ is étale (Morphisms, Lemma 36.9) and we conclude that j is étale from the definition. $\square$

Lemma 35.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is étale,*
(2) *f is DM and for any morphism $V \to \mathcal{Y}$ where V is an algebraic space and any étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$ where U is an algebraic space, the morphism $U \to V$ is étale,*
(3) *there exists some surjective, locally of finite presentation, and flat morphism $W \to \mathcal{Y}$ where W is an algebraic space and some surjective étale morphism $T \to W \times_{\mathcal{Y}} \mathcal{X}$ where T is an algebraic space such that the morphism $T \to W$ is étale.*

Proof. Assume (1). Then f is DM and since being étale is preserved by base change, we see that (2) holds.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to \mathcal{Y}$. Choose a scheme U and a surjective étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$ (Lemma 21.7). Thus we see that (3) holds.

Assume $W \to \mathcal{Y}$ and $T \to W \times_{\mathcal{Y}} \mathcal{X}$ are as in (3). We first check f is DM. Namely, it suffices to check $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is DM, see Lemma 4.5. By Lemma 4.12 it suffices to check $W \times_{\mathcal{Y}} \mathcal{X}$ is DM. This follows from the existence of $T \to W \times_{\mathcal{Y}} \mathcal{X}$ by (the easy direction of) Theorem 21.6.

Assume f is DM and $W \to \mathcal{Y}$ and $T \to W \times_{\mathcal{Y}} \mathcal{X}$ are as in (3). Let V be an algebraic space, let $V \to \mathcal{Y}$ be surjective smooth, let U be an algebraic space, and let $U \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective and étale (Lemma 21.7). We have to check that $U \to V$ is étale. It suffices to prove $U \times_{\mathcal{Y}} W \to V \times_{\mathcal{Y}} W$ is étale by Descent on Spaces, Lemma 11.28. We may replace $\mathcal{X}, \mathcal{Y}, W, T, U, V$ by $\mathcal{X} \times_{\mathcal{Y}} W, W, W, T, U \times_{\mathcal{Y}} W, V \times_{\mathcal{Y}} W$ (small detail omitted). Thus we may assume that $Y = \mathcal{Y}$ is an algebraic space, there exists an algebraic space T and a surjective étale morphism $T \to \mathcal{X}$ such that $T \to Y$ is étale, and U and V are as before. In this case we know that

$$U \to V \text{ is étale} \Leftrightarrow \mathcal{X} \to Y \text{ is étale} \Leftrightarrow T \to Y \text{ is étale}$$

by the equivalence of properties (1) and (2) of Lemma 34.1 and Definition 35.1. This finishes the proof. $\square$

Lemma 35.6. *Let $\mathcal{X}, \mathcal{Y}$ be algebraic stacks étale over an algebraic stack $\mathcal{Z}$. Any morphism $\mathcal{X} \to \mathcal{Y}$ over $\mathcal{Z}$ is étale.*

Proof. The morphism $\mathcal{X} \to \mathcal{Y}$ is DM by Lemma 4.12. Let $W \to \mathcal{Z}$ be a surjective smooth morphism whose source is an algebraic space. Let $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ be a surjective étale morphism whose source is an algebraic space (Lemma 21.7). Let $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be a surjective étale morphism whose source is an algebraic space (Lemma 21.7). Then

$$U \longrightarrow \mathcal{X} \times_{\mathcal{Z}} W$$

is surjective étale as the composition of $U \to \mathcal{X} \times_{\mathcal{Y}} V$ and the base change of $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ by $\mathcal{X} \times_{\mathcal{Z}} W \to \mathcal{Y} \times_{\mathcal{Z}} W$. Hence it suffices to show that $U \to W$ is étale. Since $U \to W$ and $V \to W$ are étale because $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ are étale, this follows from the version of the lemma for algebraic spaces, namely Morphisms of Spaces, Lemma 39.11. $\square$

36. Unramified morphisms

For a justification of our choice of definition of unramified morphisms we refer the reader to the discussion in the section on étale morphisms Section 35.

In Lemma 36.9 we will characterize unramified morphisms of algebraic stacks as morphisms which are locally of finite type and have étale diagonal.

The property "unramified" of morphisms of algebraic spaces is étale-smooth local on the source-and-target, see Descent on Spaces, Remark 21.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 38.4 and Descent on Spaces, Lemma 11.27. Hence, by Lemma 34.1 above, we may define what it means for a morphism of algebraic spaces to be unramified as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 3 when the morphism is representable by algebraic spaces because such a morphism is automatically DM by Lemma 4.3.

Definition 36.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *unramified* if f is DM and the equivalent conditions of Lemma 34.1 hold with $\mathcal{P}$ ="unramified".

We will use without further mention that this agrees with the already existing notion of unramified morphisms in case f is representable by algebraic spaces or if $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces.

Lemma 36.2. *The composition of unramified morphisms is unramified.*

Proof. Combine Remark 34.3 with Morphisms of Spaces, Lemma 38.3. ☐

Lemma 36.3. *A base change of an unramified morphism is unramified.*

Proof. Combine Remark 34.4 with Morphisms of Spaces, Lemma 38.4. ☐

Lemma 36.4. *An étale morphism is unramified.*

Proof. Follows from Remark 34.5 and Morphisms of Spaces, Lemma 39.10. ☐

Lemma 36.5. *An immersion is unramified.*

Proof. Let $j : \mathcal{Z} \to \mathcal{X}$ be an immersion of algebraic stacks. Since j is representable, it is DM by Lemma 4.3. On the other hand, if $X \to \mathcal{X}$ is a smooth and surjective morphism where X is a scheme, then $Z = \mathcal{Z} \times_\mathcal{X} X$ is a locally closed subscheme of X. Hence $Z \to X$ is unramified (Morphisms, Lemmas 35.7 and 35.8) and we conclude that j is unramified from the definition. ☐

Lemma 36.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

1. *f is unramified,*
2. *f is DM and for any morphism $V \to \mathcal{Y}$ where V is an algebraic space and any étale morphism $U \to V \times_\mathcal{Y} \mathcal{X}$ where U is an algebraic space, the morphism $U \to V$ is unramified,*
3. *there exists some surjective, locally of finite presentation, and flat morphism $W \to \mathcal{Y}$ where W is an algebraic space and some surjective étale morphism $T \to W \times_\mathcal{Y} \mathcal{X}$ where T is an algebraic space such that the morphism $T \to W$ is unramified.*

Proof. Assume (1). Then f is DM and since being unramified is preserved by base change, we see that (2) holds.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to \mathcal{Y}$. Choose a scheme U and a surjective étale morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$ (Lemma 21.7). Thus we see that (3) holds.

Assume $W \to \mathcal{Y}$ and $T \to W \times_{\mathcal{Y}} \mathcal{X}$ are as in (3). We first check f is DM. Namely, it suffices to check $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is DM, see Lemma 4.5. By Lemma 4.12 it suffices to check $W \times_{\mathcal{Y}} \mathcal{X}$ is DM. This follows from the existence of $T \to W \times_{\mathcal{Y}} \mathcal{X}$ by (the easy direction of) Theorem 21.6.

Assume f is DM and $W \to \mathcal{Y}$ and $T \to W \times_{\mathcal{Y}} \mathcal{X}$ are as in (3). Let V be an algebraic space, let $V \to \mathcal{Y}$ be surjective smooth, let U be an algebraic space, and let $U \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective and étale (Lemma 21.7). We have to check that $U \to V$ is unramified. It suffices to prove $U \times_{\mathcal{Y}} W \to V \times_{\mathcal{Y}} W$ is unramified by Descent on Spaces, Lemma 11.27. We may replace $\mathcal{X}, \mathcal{Y}, W, T, U, V$ by $\mathcal{X} \times_{\mathcal{Y}} W, W, W, T, U \times_{\mathcal{Y}} W, V \times_{\mathcal{Y}} W$ (small detail omitted). Thus we may assume that $Y = \mathcal{Y}$ is an algebraic space, there exists an algebraic space T and a surjective étale morphism $T \to \mathcal{X}$ such that $T \to Y$ is unramified, and U and V are as before. In this case we know that

$$U \to V \text{ is unramified} \Leftrightarrow \mathcal{X} \to Y \text{ is unramified} \Leftrightarrow T \to Y \text{ is unramified}$$

by the equivalence of properties (1) and (2) of Lemma 34.1 and Definition 36.1. This finishes the proof. $\square$

Lemma 36.7. *An unramified morphism of algebraic stacks is locally quasi-finite.*

Proof. This follows from Lemma 36.6 (characterizing unramified morphisms), Lemma 23.7 (characterizing locally quasi-finite morphisms), and Morphisms of Spaces, Lemma 38.7 (the corresponding result for algebraic spaces). $\square$

Lemma 36.8. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is unramified and $\mathcal{Y} \to \mathcal{Z}$ is DM, then $\mathcal{X} \to \mathcal{Y}$ is unramified.*

Proof. Assume $\mathcal{X} \to \mathcal{Z}$ is unramified. By Lemma 4.12 the morphism $\mathcal{X} \to \mathcal{Y}$ is DM. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

with U, V, W algebraic spaces, with $W \to \mathcal{Z}$ surjective smooth, $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ surjective étale, and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ surjective étale (see Lemma 21.7). Then also $U \to \mathcal{X} \times_{\mathcal{Z}} W$ is surjective and étale. Hence we know that $U \to W$ is unramified and we have to show that $U \to V$ is unramified. This follows from Morphisms of Spaces, Lemma 38.11. $\square$

Lemma 36.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is unramified, and*
(2) *f is locally of finite type and its diagonal is étale.*

Proof. Assume f is unramified. Then f is DM hence we can choose algebraic spaces U, V, a smooth surjective morphism $V \to \mathcal{Y}$ and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ (Lemma 21.7). Since f is unramified the induced morphism $U \to V$ is unramified. Thus $U \to V$ is locally of finite type (Morphisms of Spaces, Lemma 38.6) and we conclude that f is locally of finite type. The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is a morphism of algebraic stacks over $\mathcal{Y}$. The base change of Δ by the surjective smooth morphism $V \to \mathcal{Y}$ is the diagonal of the base change of f, i.e., of $\mathcal{X}_V = \mathcal{X} \times_{\mathcal{Y}} V \to V$. In other words, the diagram

$$\begin{tikzcd} \mathcal{X}_V \arrow[r] \arrow[d] & \mathcal{X}_V \times_V \mathcal{X}_V \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

is cartesian. Since the right vertical arrow is surjective and smooth it suffices to show that the top horizontal arrow is étale by Properties of Stacks, Lemma 3.4. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U \times_V U \arrow[d] \\ \mathcal{X}_V \arrow[r] & \mathcal{X}_V \times_V \mathcal{X}_V \end{tikzcd}$$

All arrows are representable by algebraic spaces, the vertical arrows are étale, the left one is surjective, and the top horizontal arrow is an open immersion by Morphisms of Spaces, Lemma 38.9. This implies what we want: first we see that $U \to \mathcal{X}_V \times_V \mathcal{X}_V$ is étale as a composition of étale morphisms, and then we can use Properties of Stacks, Lemma 3.5 to see that $\mathcal{X}_V \to \mathcal{X}_V \times_V \mathcal{X}_V$ is étale because being étale (for morphisms of algebraic spaces) is local on the source in the étale topology (Descent on Spaces, Lemma 19.1).

Assume f is locally of finite type and that its diagonal is étale. Then f is DM by definition (as étale morphisms of algebraic spaces are unramified). As above this means we can choose algebraic spaces U, V, a smooth surjective morphism $V \to \mathcal{Y}$ and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ (Lemma 21.7). To finish the proof we have to show that $U \to V$ is unramified. We already know that $U \to V$ is locally of finite type. Arguing as above we find a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U \times_V U \arrow[d] \\ \mathcal{X}_V \arrow[r] & \mathcal{X}_V \times_V \mathcal{X}_V \end{tikzcd}$$

where all arrows are representable by algebraic spaces, the vertical arrows are étale, and the lower horizontal one is étale as a base change of Δ. It follows that $U \to U \times_V U$ is étale for example by Lemma 35.6[7]. Thus $U \to U \times_V U$ is an étale monomorphism hence an open immersion (Morphisms of Spaces, Lemma 51.2). Then $U \to V$ is unramified by Morphisms of Spaces, Lemma 38.9. $\square$

Lemma 36.10. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

[7] It is quite easy to deduce this directly from Morphisms of Spaces, Lemma 39.11.

(1) *f is étale, and*
(2) *f is locally of finite presentation, flat, and unramified,*
(3) *f is locally of finite presentation, flat, and its diagonal is étale.*

Proof. The equivalence of (2) and (3) follows immediately from Lemma 36.9. Thus in each case the morphism f is DM. Then we can choose Then we can choose algebraic spaces U, V, a smooth surjective morphism $V \to \mathcal{Y}$ and a surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ (Lemma 21.7). To finish the proof we have to show that $U \to V$ is étale if and only if it is locally of finite presentation, flat, and unramified. This follows from Morphisms of Spaces, Lemma 39.12 (and the more trivial Morphisms of Spaces, Lemmas 39.10, 39.8, and 39.7). □

37. Proper morphisms

The notion of a proper morphism plays an important role in algebraic geometry. Here is the definition of a proper morphism of algebraic stacks.

Definition 37.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *proper* if f is separated, finite type, and universally closed.

This does not conflict with the already existing notion of a proper morphism of algebraic spaces: a morphism of algebraic spaces is proper if and only if it is separated, finite type, and universally closed (Morphisms of Spaces, Definition 40.1) and we've already checked the compatibility of these notions in Lemma 3.5, Section 17, and Lemmas 13.1. Similarly, if $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks which is representable by algebraic spaces then we have defined what it means for f to be proper in Properties of Stacks, Section 3. However, the discussion in that section shows that this is equivalent to requiring f to be separated, finite type, and universally closed and the same references as above give the compatibility.

Lemma 37.2. *A base change of a proper morphism is proper.*

Proof. See Lemmas 4.4, 17.3, and 13.3. □

Lemma 37.3. *A composition of proper morphisms is proper.*

Proof. See Lemmas 4.10, 17.2, and 13.4. □

Lemma 37.4. *A closed immersion of algebraic stacks is a proper morphism of algebraic stacks.*

Proof. A closed immersion is by definition representable (Properties of Stacks, Definition 9.1). Hence this follows from the discussion in Properties of Stacks, Section 3 and the corresponding result for morphisms of algebraic spaces, see Morphisms of Spaces, Lemma 40.5. □

Lemma 37.5. *Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[rr] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\ & \mathcal{Z} & \end{tikzcd}$$

of algebraic stacks.

(1) *If $\mathcal{X} \to \mathcal{Z}$ is universally closed and $\mathcal{Y} \to \mathcal{Z}$ is separated, then the morphism $\mathcal{X} \to \mathcal{Y}$ is universally closed. In particular, the image of $|\mathcal{X}|$ in $|\mathcal{Y}|$ is closed.*
(2) *If $\mathcal{X} \to \mathcal{Z}$ is proper and $\mathcal{Y} \to \mathcal{Z}$ is separated, then the morphism $\mathcal{X} \to \mathcal{Y}$ is proper.*

Proof. Assume $\mathcal{X} \to \mathcal{Z}$ is universally closed and $\mathcal{Y} \to \mathcal{Z}$ is separated. We factor the morphism as $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$. The first morphism is proper (Lemma 4.8) hence universally closed. The projection $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$ is the base change of a universally closed morphism and hence universally closed, see Lemma 13.3. Thus $\mathcal{X} \to \mathcal{Y}$ is universally closed as the composition of universally closed morphisms, see Lemma 13.4. This proves (1). To deduce (2) combine (1) with Lemmas 4.12, 7.7, and 17.8. $\square$

Lemma 37.6. *Let $\mathcal{Z}$ be an algebraic stack. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over $\mathcal{Z}$. If $\mathcal{X}$ is universally closed over $\mathcal{Z}$ and f is surjective then $\mathcal{Y}$ is universally closed over $\mathcal{Z}$. In particular, if also $\mathcal{Y}$ is separated and of finite type over $\mathcal{Z}$, then $\mathcal{Y}$ is proper over $\mathcal{Z}$.*

Proof. Assume $\mathcal{X}$ is universally closed and f surjective. Denote $p : \mathcal{X} \to \mathcal{Z}$, $q : \mathcal{Y} \to \mathcal{Z}$ the structure morphisms. Let $\mathcal{Z}' \to \mathcal{Z}$ be a morphism of algebraic stacks. The base change $f' : \mathcal{X}' \to \mathcal{Y}'$ of f by $\mathcal{Z}' \to \mathcal{Z}$ is surjective (Properties of Stacks, Lemma 5.3) and the base change $p' : \mathcal{X}' \to \mathcal{Z}'$ of p is closed. If $T \subset |\mathcal{Y}'|$ is closed, then $(f')^{-1}(T) \subset |\mathcal{X}'|$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. $\square$

38. Scheme theoretic image

Here is the definition.

Definition 38.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The *scheme theoretic image* of f is the smallest closed substack $\mathcal{Z} \subset \mathcal{Y}$ through which f factors[8].

We often denote $f : \mathcal{X} \to \mathcal{Z}$ the factorization of f. If the morphism f is not quasi-compact, then (in general) the construction of the scheme theoretic image does not commute with restriction to open substacks of $\mathcal{Y}$. However, if f is quasi-compact then the scheme theoretic image commutes with flat base change (Lemma 38.5).

Lemma 38.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $g : \mathcal{W} \to \mathcal{X}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Then the scheme theoretic image of f exists if and only if the scheme theoretic image of $f \circ g$ exists and if so then these scheme theoretic images are the same.*

Proof. Assume $\mathcal{Z} \subset \mathcal{Y}$ is a closed substack and $f \circ g$ factors through $\mathcal{Z}$. To prove the lemma it suffices to show that f factors through $\mathcal{Z}$. Consider a scheme T and a morphism $T \to \mathcal{X}$ given by an object x of the fibre category of $\mathcal{X}$ over T. We will show that $f(x)$ is in fact in the fibre category of $\mathcal{Z}$ over T. Namely, the projection $T \times_{\mathcal{X}} \mathcal{W} \to T$ is a surjective, flat, locally finitely presented morphism. Hence there is an fppf covering $\{T_i \to T\}$ such that $T_i \to T$ factors through $T \times_{\mathcal{X}} \mathcal{W} \to T$ for all i. Then $T_i \to \mathcal{X}$ factors through $\mathcal{W}$ and hence $T_i \to \mathcal{Y}$ factors through $\mathcal{Z}$. Thus

[8] We will see in Lemma 38.3 that the scheme theoretic image always exists.

$x|_{T_i}$ is an object of $\mathcal{Z}$. Since $\mathcal{Z}$ is a strictly full substack, we conclude that x is an object of $\mathcal{Z}$ as desired. $\square$

Lemma 38.3. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks. Then the scheme theoretic image of f exists.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. By Lemma 38.2 we may replace $\mathcal{Y}$ by V. Thus it suffices to show that if $X \to \mathcal{X}$ is a morphism from a scheme to an algebraic stack, then the scheme theoretic image exists. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Set $R = U \times_{\mathcal{X}} U$. We have $\mathcal{X} = [U/R]$ by Algebraic Stacks, Lemma 16.2. By Properties of Stacks, Lemma 9.11 the closed substacks $\mathcal{Z}$ of $\mathcal{X}$ are in 1-to-1 correspondence with R-invariant closed subschemes $Z \subset U$. Let $Z_1 \subset U$ be the scheme theoretic image of $X \times_{\mathcal{X}} U \to U$. Observe that $X \to \mathcal{X}$ factors through $\mathcal{Z}$ if and only if $X \times_{\mathcal{X}} U \to U$ factors through the corresponding R-invariant closed subscheme Z (details omitted; hint: this follows because $X \times_{\mathcal{X}} U \to X$ is surjective and smooth). Thus we have to show that there exists a smallest R-invariant closed subscheme $Z \subset U$ containing Z_1.

Let $\mathcal{I}_1 \subset \mathcal{O}_U$ be the quasi-coherent ideal sheaf corresponding to the closed subscheme $Z_1 \subset U$. Let Z_α, $\alpha \in A$ be the set of all R-invariant closed subschemes of U containing Z_1. For $\alpha \in A$, let $\mathcal{I}_\alpha \subset \mathcal{O}_U$ be the quasi-coherent ideal sheaf corresponding to the closed subscheme $Z_\alpha \subset U$. The containment $Z_1 \subset Z_\alpha$ means $\mathcal{I}_\alpha \subset \mathcal{I}_1$. The R-invariance of Z_α means that

$$s^{-1}\mathcal{I}_\alpha \cdot \mathcal{O}_R = t^{-1}\mathcal{I}_\alpha \cdot \mathcal{O}_R$$

as (quasi-coherent) ideal sheaves on (the algebraic space) R. Consider the image

$$\mathcal{I} = \mathrm{Im}\left(\bigoplus\nolimits_{\alpha \in A} \mathcal{I}_\alpha \to \mathcal{I}_1\right) = \mathrm{Im}\left(\bigoplus\nolimits_{\alpha \in A} \mathcal{I}_\alpha \to \mathcal{O}_X\right)$$

Since direct sums of quasi-coherent sheaves are quasi-coherent and since images of maps between quasi-coherent sheaves are quasi-coherent, we find that $\mathcal{I}$ is quasi-coherent. Since pull back is exact and commutes with direct sums we find

$$s^{-1}\mathcal{I} \cdot \mathcal{O}_R = t^{-1}\mathcal{I} \cdot \mathcal{O}_R$$

Hence $\mathcal{I}$ defines an R-invariant closed subscheme $Z \subset U$ which is contained in every Z_α and containes Z_1 as desired. $\square$

Lemma 38.4. *Let*

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r, "f_1"] \arrow[d] & \mathcal{Y}_1 \arrow[d] \\ \mathcal{X}_2 \arrow[r, "f_2"] & \mathcal{Y}_2 \end{tikzcd}$$

be a commutative diagram of algebraic stacks. Let $\mathcal{Z}_i \subset \mathcal{Y}_i$, $i = 1, 2$ be the scheme theoretic image of f_i. Then the morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ induces a morphism $\mathcal{Z}_1 \to \mathcal{Z}_2$ and a commutative diagram

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r] \arrow[d] & \mathcal{Z}_1 \arrow[r] \arrow[d] & \mathcal{Y}_1 \arrow[d] \\ \mathcal{X}_2 \arrow[r] & \mathcal{Z}_2 \arrow[r] & \mathcal{Y}_2 \end{tikzcd}$$

Proof. The scheme theoretic inverse image of $\mathcal{Z}_2$ in $\mathcal{Y}_1$ is a closed substack of $\mathcal{Y}_1$ through which f_1 factors. Hence $\mathcal{Z}_1$ is contained in this. This proves the lemma. $\square$

Lemma 38.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Then formation of the scheme theoretic image commutes with flat base change.*

Proof. Let $\mathcal{Y}' \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose a scheme V' and a surjective smooth morphism $V' \to \mathcal{Y}' \times_\mathcal{Y} V$. We may and do assume that $V = \coprod_{i \in I} V_i$ is a disjoint union of affine schemes and that $V' = \coprod_{i \in I} \coprod_{j \in J_i} V_{i,j}$ is a disjoint union of affine schemes with each $V_{i,j}$ mapping into V_i. Let

(1) $\mathcal{Z} \subset \mathcal{Y}$ be the scheme theoretic image of f,
(2) $\mathcal{Z}' \subset \mathcal{Y}'$ be the scheme theoretic image of the base change of f by $\mathcal{Y}' \to \mathcal{Y}$,
(3) $Z \subset V$ be the scheme theoretic image of the base change of f by $V \to \mathcal{Y}$,
(4) $Z' \subset V'$ be the scheme theoretic image of the base change of f by $V' \to \mathcal{Y}$.

If we can show that (a) $Z = V \times_\mathcal{Y} \mathcal{Z}$, (b) $Z' = V' \times_{\mathcal{Y}'} \mathcal{Z}'$, and (c) $Z' = V' \times_V Z$ then the lemma follows: the inclusion $\mathcal{Z}' \to \mathcal{Z} \times_\mathcal{Y} \mathcal{Y}'$ (Lemma 38.4) has to be an isomorphism because after base change by the surjective smooth morphism $V' \to \mathcal{Y}'$ it is.

Proof of (a). Set $R = V \times_\mathcal{Y} V$. By Properties of Stacks, Lemma 9.11 the rule $\mathcal{Z} \mapsto \mathcal{Z} \times_\mathcal{Y} V$ defines a 1-to-1 correspondence between closed substacks of $\mathcal{Y}$ and R-invariant closed subspaces of V. Moreover, $f : \mathcal{X} \to \mathcal{Y}$ factors through $\mathcal{Z}$ if and only if the base change $g : \mathcal{X} \times_\mathcal{Y} V \to V$ factors through $\mathcal{Z} \times_\mathcal{Y} V$. We claim: the scheme theoretic image $Z \subset V$ of g is R-invariant. The claim implies (a) by what we just said.

For each i the morphism $\mathcal{X} \times_\mathcal{Y} V_i \to V_i$ is quasi-compact and hence $\mathcal{X} \times_\mathcal{Y} V_i$ is quasi-compact. Thus we can choose an affine scheme W_i and a surjective smooth morphism $W_i \to \mathcal{X} \times_\mathcal{Y} V_i$. Observe that $W = \coprod W_i$ is a scheme endowed with a smooth and surjective morphism $W \to \mathcal{X} \times_\mathcal{Y} V$ such that the composition $W \to V$ with g is quasi-compact. Let $Z \to V$ be the scheme theoretic image of $W \to V$, see Morphisms, Section 6 and Morphisms of Spaces, Section 16. It follows from Lemma 38.2 that $Z \subset V$ is the scheme theoretic image of g. To show that Z is R-invariant we claim that both

$$\mathrm{pr}_0^{-1}(Z), \mathrm{pr}_1^{-1}(Z) \subset R = V \times_\mathcal{Y} V$$

are the scheme theoretic image of $\mathcal{X} \times_\mathcal{Y} R \to R$. Namely, we first use Morphisms of Spaces, Lemma 30.12 to see that $\mathrm{pr}_0^{-1}(Z)$ is the scheme theoretic image of the composition

$$W \times_{V, \mathrm{pr}_0} R = W \times_\mathcal{Y} V \to \mathcal{X} \times_\mathcal{Y} R \to R$$

Since the first arrow here is surjective and smooth we see that $\mathrm{pr}_0^{-1}(Z)$ is the scheme theoretic image of $\mathcal{X} \times_\mathcal{Y} R \to R$. The same argument applies that $\mathrm{pr}_1^{-1}(Z)$. Hence Z is R-invariant.

Statement (b) is proved in exactly the same way as one proves (a).

Proof of (c). Let $Z_i \subset V_i$ be the scheme theoretic image of $\mathcal{X} \times_\mathcal{Y} V_i \to V_i$ and let $Z_{i,j} \subset V_{i,j}$ be the scheme theoretic image of $\mathcal{X} \times_\mathcal{Y} V_{i,j} \to V_{i,j}$. Clearly it suffices to show that the inverse image of Z_i in $V_{i,j}$ is $Z_{i,j}$. Above we've seen that Z_i is the scheme theoretic image of $W_i \to V_i$ and by the same token $Z_{i,j}$ is the scheme theoretic image of $W_i \times_{V_i} V_{i,j} \to V_{i,j}$. Hence the equality follows from the case of schemes (Morphisms, Lemma 25.16) and the fact that $V_{i,j} \to V_i$ is flat. $\square$

Lemma 38.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Let $\mathcal{Z} \subset \mathcal{Y}$ be the scheme theoretic image of f. Then $|\mathcal{Z}|$ is the closure of the image of $|f|$.*

Proof. Let $z \in |\mathcal{Z}|$ be a point. Choose an affine scheme V, a point $v \in V$, and a smooth morphism $V \to \mathcal{Y}$ mapping v to z. Then $\mathcal{X} \times_\mathcal{Y} V$ is a quasi-compact algebraic stack. Hence we can find an affine scheme W and a surjective smooth morphism $W \to \mathcal{X} \times_\mathcal{Y} V$. By Lemma 38.5 the scheme theoretic image of $\mathcal{X} \times_\mathcal{Y} V \to V$ is $Z = \mathcal{Z} \times_\mathcal{Y} V$. Hence the inverse image of $|\mathcal{Z}|$ in $|V|$ is $|Z|$ by Properties of Stacks, Lemma 4.3. By Lemma 38.2 Z is the scheme theoretic image of $W \to V$. By Morphisms of Spaces, Lemma 16.3 we see that the image of $|W| \to |Z|$ is dense. Hence the image of $|\mathcal{X} \times_\mathcal{Y} V| \to |Z|$ is dense. Observe that $v \in Z$. Since $|V| \to |\mathcal{Y}|$ is open, a topology argument tells us that z is in the closure of the image of $|f|$ as desired. $\square$

Lemma 38.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces and separated. Let $\mathcal{V} \subset \mathcal{Y}$ be an open substack such that $\mathcal{V} \to \mathcal{Y}$ is quasi-compact. Let $s : \mathcal{V} \to \mathcal{X}$ be a morphism such that $f \circ s = id_\mathcal{V}$. Let $\mathcal{Y}'$ be the scheme theoretic image of s. Then $\mathcal{Y}' \to \mathcal{Y}$ is an isomorphism over $\mathcal{V}$.*

Proof. By Lemma 7.7 the morphism $s : \mathcal{V} \to \mathcal{Y}$ is quasi-compact. Hence the construction of the scheme theoretic image $\mathcal{Y}'$ of s commutes with flat base change by Lemma 38.5. Thus to prove the lemma we may assume $\mathcal{Y}$ is representable by an algebraic space and we reduce to the case of algebraic spaces which is Morphisms of Spaces, Lemma 16.7. $\square$

39. Valuative criteria

We need to be careful when stating the valuative criterion. Namely, in the formulation we need to speak about commutative diagrams but we are working in a 2-category and we need to make sure the 2-morphisms compose correctly as well!

Definition 39.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider a 2-commutative solid diagram

(39.1.1)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] \arrow[ru, dotted] & \mathcal{Y} \end{tikzcd}$$

where A is a valuation ring with field of fractions K. Let

$$\gamma : y \circ j \longrightarrow f \circ x$$

be a 2-morphism witnessing the 2-commutativity of the diagram. (Notation as in Categories, Sections 28 and 29.) Given (39.1.1) and γ a *dotted arrow* is a triple (a, α, β) consisting of a morphism $a : \mathrm{Spec}(A) \to \mathcal{X}$ and 2-arrows $\alpha : a \circ j \to x$, $\beta : y \to f \circ a$ such that $\gamma = (\mathrm{id}_f \star \alpha) \circ (\beta \star \mathrm{id}_j)$, in other words such that

$$\begin{tikzcd} & f \circ a \circ j \arrow[rd, "\mathrm{id}_f \star \alpha"] & \\ y \circ j \arrow[ru, "\beta \star \mathrm{id}_j"] \arrow[rr, "\gamma"] & & f \circ x \end{tikzcd}$$

is commutative. A *morphism of dotted arrows* $(a, \alpha, \beta) \to (a', \alpha', \beta')$ is a 2-arrow $\theta : a \to a'$ such that $\alpha = \alpha' \circ (\theta \star \mathrm{id}_j)$ and $\beta' = (\mathrm{id}_f \star \theta) \circ \beta$.

The preceding definition is a special case of Categories, Definition 44.1. The category of dotted arrows depends on γ in general. If $\mathcal{Y}$ is representable by an algebraic space (or if automorphism groups of objects over fields are trivial), then of course there is at most one γ and one does not need to check the commutativity of the triangle. More generally, we have Lemma 39.3. The commutativity of the triangle is important in the proof of compatibility with base change, see proof of Lemma 39.4.

Lemma 39.2. *In the situation of Definition 39.1 the category of dotted arrows is a groupoid. If Δ_f is separated, then it is a setoid.*

Proof. Since 2-arrows are invertible it is clear that the category of dotted arrows is a groupoid. Given a dotted arrow (a, α, β) an automorphism of (a, α, β) is a 2-morphism $\theta : a \to a$ satisfying two conditions. The first condition $\beta = (\mathrm{id}_f \star \theta) \circ \beta$ signifies that θ defines a morphism $(a, \theta) : \mathrm{Spec}(A) \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$. The second condition $\alpha = \alpha \circ (\theta \star \mathrm{id}_j)$ implies that the restriction of (a, θ) to $\mathrm{Spec}(K)$ is the identity. Picture

$$\begin{tikzcd}
\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \arrow[d] & \mathrm{Spec}(K) \arrow[l, "{(a \circ j, \mathrm{id})}"'] \arrow[d, "j"] \\
\mathcal{X} & \mathrm{Spec}(A) \arrow[l, "a"] \arrow[ul, "{(a, \theta)}"']
\end{tikzcd}$$

In other words, if $G \to \mathrm{Spec}(A)$ is the group algebraic space we get by pulling back the relative inertia by a, then θ defines a point $\theta \in G(A)$ whose image in $G(K)$ is trivial. Certainly, if the identity $e : \mathrm{Spec}(A) \to G$ is a closed immersion, then this can happen only if θ is the identity. Looking at Lemma 6.1 we obtain the result we want. $\square$

Lemma 39.3. *In Definition 39.1 assume $\mathcal{I}_{\mathcal{Y}} \to \mathcal{Y}$ is proper (for example if $\mathcal{Y}$ is separated or if $\mathcal{Y}$ is separated over an algebraic space). Then the category of dotted arrows is independent (up to noncanonical equivalence) of the choice of γ and the existence of a dotted arrow (for some and hence equivalently all γ) is equivalent to the existence of a diagram*

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r, "y"] \arrow[ur, "a"] & \mathcal{Y}
\end{tikzcd}$$

with 2-commutative triangles (without checking the 2-morphisms compose correctly).

Proof. Let $\gamma, \gamma' : y \circ j \longrightarrow f \circ x$ be two 2-morphisms. Then $\gamma^{-1} \circ \gamma'$ is an automorphism of y over $\mathrm{Spec}(K)$. Hence if $\mathit{Isom}_{\mathcal{Y}}(y, y) \to \mathrm{Spec}(A)$ is proper, then by the valuative criterion of properness (Morphisms of Spaces, Lemma 44.1) we can find $\delta : y \to y$ whose restriction to $\mathrm{Spec}(K)$ is $\gamma^{-1} \circ \gamma'$. Then we can use δ to define an equivalence between the category of dotted arrows for γ to the category of dotted arrows for γ' by sending (a, α, β) to $(a, \alpha, \beta \circ \delta)$. The final statement is clear. $\square$

Lemma 39.4. *Assume given a 2-commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x'"'] \arrow[d, "j"'] & \mathcal{X}' \arrow[r, "q"'] \arrow[d, "p"] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y'"] & \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

with the right square 2-cartesian. Choose a 2-arrow $\gamma' : y' \circ j \to p \circ x'$. Set $x = q \circ x'$, $y = g \circ y'$ and let $\gamma : y \circ j \to f \circ x$ be the composition of γ' with the 2-arrow implicit in the 2-commutativity of the right square. Then the category of dotted arrows for the left square and γ' is equivalent to the category of dotted arrows for the outer rectangle and γ.

Proof. (We do not know how to prove the analogue of this lemma if instead of the category of dotted arrows we look at the set of isomorphism classes of morphisms producing two 2-commutative triangles as in Lemma 39.3; in fact this analogue may very well be wrong.) First proof: this lemma is a special case of Categories, Lemma 44.2. Second proof: we are allowed to replace $\mathcal{X}'$ by the 2-fibre product $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as described in Categories, Lemma 32.3. Then the object x' becomes the triple $(y' \circ j, x, \gamma)$. Then we can go from a dotted arrow (a, α, β) for the outer rectangle to a dotted arrow (a', α', β') for the left square by taking $a' = (y', a, \beta)$ and $\alpha' = (\mathrm{id}_{y' \circ j}, \alpha)$ and $\beta' = \mathrm{id}_{y'}$. Details omitted. $\square$

Lemma 39.5. *Assume given a 2-commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[dd, "j"'] & \mathcal{X} \arrow[d, "f"] \\ & \mathcal{Y} \arrow[d, "g"] \\ \mathrm{Spec}(A) \arrow[r, "z"] & \mathcal{Z} \end{tikzcd}$$

Choose a 2-arrow $\gamma : z \circ j \to g \circ f \circ x$. Let $\mathcal{C}$ be the category of dotted arrows for the outer rectangle and γ. Let $\mathcal{C}'$ be the category of dotted arrows for the square

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "f \circ x"'] \arrow[d, "j"'] & \mathcal{Y} \arrow[d, "g"] \\ \mathrm{Spec}(A) \arrow[r, "z"] & \mathcal{Z} \end{tikzcd}$$

and γ. Then $\mathcal{C}$ is equivalent to a category $\mathcal{C}''$ which has the following property: there is a functor $\mathcal{C}'' \to \mathcal{C}'$ which turns $\mathcal{C}''$ into a category fibred in groupoids over $\mathcal{C}'$ and whose fibre categories are categories of dotted arrows for certain squares of the form

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

and some choices of $y \circ j \to f \circ x$.

Proof. This lemma is a special case of Categories, Lemma 44.3. $\square$

Definition 39.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f satisfies the *uniqueness part of the valuative criterion* if for every diagram (39.1.1) and γ as in Definition 39.1 the category of dotted arrows is either empty or a setoid with exactly one isomorphism class.

Lemma 39.7. *The base change of a morphism of algebraic stacks which satisfies the uniqueness part of the valuative criterion by any morphism of algebraic stacks is a morphism of algebraic stacks which satisfies the uniqueness part of the valuative criterion.*

Proof. Follows from Lemma 39.4 and the definition. $\square$

Lemma 39.8. *The composition of morphisms of algebraic stacks which satisfy the uniqueness part of the valuative criterion is another morphism of algebraic stacks which satisfies the uniqueness part of the valuative criterion.*

Proof. Follows from Lemma 39.5 and the definition. $\square$

Lemma 39.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then the following are equivalent*

(1) *f satisfies the uniqueness part of the valuative criterion,*
(2) *for every scheme T and morphism $T \to \mathcal{Y}$ the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ satisfies the uniqueness part of the valuative criterion as a morphism of algebraic spaces.*

Proof. Follows from Lemma 39.4 and the definition. $\square$

Definition 39.10. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f satisfies the *existence part of the valuative criterion* if for every diagram (39.1.1) and γ as in Definition 39.1 there exists an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of dotted arrows for the outer rectangle of the diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d, "j'"'] \arrow[rr, bend left, "x'"] & \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A') \arrow[r] \arrow[rr, bend right, "y'"'] & \mathrm{Spec}(A) \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

with induced 2-arrow $\gamma' : y' \circ j' \to f \circ x'$ is nonempty.

We have already seen in the case of morphisms of algebraic spaces, that it is necessary to allow extensions of the fractions fields in order to get the correct notion of the valuative criterion. See Morphisms of Spaces, Example 41.6. Still, for morphisms between separated algebraic spaces, such an extension is not needed (Morphisms of Spaces, Lemma 41.5). However, for morphisms between algebraic stacks, an extension may be needed even if $\mathcal{X}$ and $\mathcal{Y}$ are both separated. For example consider the morphism of algebraic stacks

$$[\mathrm{Spec}(\mathbf{C})/G] \to \mathrm{Spec}(\mathbf{C})$$

over the base scheme $\mathrm{Spec}(\mathbf{C})$ where G is a group of order 2. Both source and target are separated algebraic stacks and the morphism is proper. Whence it satisfies the uniqueness and existence parts of the valuative criterion (see Lemma 43.1). But on the other hand, there is a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & {[\mathrm{Spec}(\mathbf{C})/G]} \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(\mathbf{C}) \end{tikzcd}$$

where no dotted arrow exists with $A = \mathbf{C}[[t]]$ and $K = \mathbf{C}((t))$. Namely, the top horizontal arrow is given by a G-torsor over the spectrum of $K = \mathbf{C}((t))$. Since any G-torsor over the strictly henselian local ring $A = \mathbf{C}[[t]]$ is trivial, we see that if a dotted arrow always exists, then every G-torsor over K is trivial. This is not true because $G = \{+1, -1\}$ and by Kummer theory G-torsors over K are classified by $K^*/(K^*)^2$ which is nontrivial.

Lemma 39.11. *The base change of a morphism of algebraic stacks which satisfies the existence part of the valuative criterion by any morphism of algebraic stacks is a morphism of algebraic stacks which satisfies the existence part of the valuative criterion.*

Proof. Follows from Lemma 39.4 and the definition. $\square$

Lemma 39.12. *The composition of morphisms of algebraic stacks which satisfy the existence part of the valuative criterion is another morphism of algebraic stacks which satisfies the existence part of the valuative criterion.*

Proof. Follows from Lemma 39.5 and the definition. $\square$

Lemma 39.13. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then the following are equivalent*

(1) *f satisfies the existence part of the valuative criterion,*
(2) *for every scheme T and morphism $T \to \mathcal{Y}$ the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ satisfies the existence part of the valuative criterion as a morphism of algebraic spaces.*

Proof. Follows from Lemma 39.4 and the definition. $\square$

Lemma 39.14. *A closed immersion of algebraic stacks satisfies both the existence and uniqueness part of the valuative criterion.*

Proof. Omitted. Hint: reduce to the case of a closed immersion of schemes by Lemmas 39.9 and 39.13. $\square$

40. Valuative criterion for second diagonal

The converse statement has already been proved in Lemma 39.2. The criterion itself is the following.

Lemma 40.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If Δ_f is quasi-separated and if for every diagram (39.1.1) and choice of γ as in Definition 39.1 the category of dotted arrows is a setoid, then Δ_f is separated.*

Proof. We are going to write out a detailed proof, but we strongly urge the reader to find their own proof, inspired by reading the argument given in the proof of Lemma 39.2.

Assume Δ_f is quasi-separated and for every diagram (39.1.1) and choice of γ as in Definition 39.1 the category of dotted arrows is a setoid. By Lemma 6.1 it suffices to show that $e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is a closed immersion. By Lemma 6.4 it in fact suffices to show that $e = \Delta_{f,2}$ is universally closed. Either of these lemmas tells us that $e = \Delta_{f,2}$ is quasi-compact by our assumption that Δ_f is quasi-separated.

In this paragraph we will show that e satisfies the existence part of the valuative criterion. Consider a 2-commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "e"] \\ \mathrm{Spec}(A) \arrow[r, "(a,\theta)"] & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \end{tikzcd}$$

and let $\alpha : (a, \theta) \circ j \to e \circ x$ be any 2-morphism witnessing the 2-commutativity of the diagram (we use α instead of the letter γ used in Definition 39.1). Note that $f \circ \theta = \mathrm{id}$; we will use this below. Observe that $e \circ x = (x, \mathrm{id}_x)$ and $(a, \theta) \circ j = (a \circ j, \theta \star \mathrm{id}_j)$. Thus we see that α is a 2-arrow $\alpha : a \circ j \to x$ compatible with $\theta \star \mathrm{id}_j$ and id_x. Set $y = f \circ x$ and $\beta = \mathrm{id}_{f \circ a}$. Reading the arguments given in the proof of Lemma 39.2 backwards, we see that θ is an automorphism of the dotted arrow (a, α, β) with

$$\gamma : y \circ j \to f \circ x \quad \text{equal to} \quad \mathrm{id}_f \star \alpha : f \circ a \circ j \to f \circ x$$

On the other hand, id_a is an automorphism too, hence we conclude $\theta = \mathrm{id}_a$ from the assumption on f. Then we can take as dotted arrow for the displayed diagram above the morphism $a : \mathrm{Spec}(A) \to \mathcal{X}$ with 2-morphisms $(a, \mathrm{id}_a) \circ j \to (x, \mathrm{id}_x)$ given by α and $(a, \theta) \to e \circ a$ given by id_a.

By Lemma 39.11 any base change of e satisfies the existence part of the valuative criterion. Since e is representable by algebraic spaces, it suffices to show that e is universally closed after a base change by a morphism $I \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ which is surjective and smooth and with I an algebraic space (see Properties of Stacks, Section 3). This base change $e' : X' \to I'$ is a quasi-compact morphism of algebraic spaces which satisfies the existence part of the valuative criterion and hence is universally closed by Morphisms of Spaces, Lemma 42.1. $\square$

41. Valuative criterion for the diagonal

The result is Lemma 41.2. We first state and prove a formal helper lemma.

Lemma 41.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider a 2-commutative solid diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[rr, "x"'] \arrow[d, "j"'] & & \mathcal{X} \arrow[d, "\Delta_f"] \\ \mathrm{Spec}(A) \arrow[rr, "{(a_1,a_2,\varphi)}"] \arrow[urr, dotted] & & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

where A is a valuation ring with field of fractions K. Let $\gamma : (a_1, a_2, \varphi) \circ j \longrightarrow \Delta_f \circ x$ be a 2-morphism witnessing the 2-commutativity of the diagram. Then

(1) *Writing $\gamma = (\alpha_1, \alpha_2)$ with $\alpha_i : a_i \circ j \to x$ we obtain two dotted arrows (a_1, α_1, id) and (a_2, α_2, φ) in the diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "f\circ a_1"] \arrow[ru, dotted] & \mathcal{Y} \end{tikzcd}$$

(2) *The category of dotted arrows for the original diagram and γ is a setoid whose set of isomorphism classes of objects equal to the set of morphisms $(a_1, \alpha_1, id) \to (a_2, \alpha_2, \varphi)$ in the category of dotted arrows.*

Proof. Since Δ_f is representable by algebraic spaces (hence the diagonal of Δ_f is separated), we see that the category of dotted arrows in the first commutative diagram of the lemma is a setoid by Lemma 39.2. All the other statements of the lemma are consequences of 2-diagramatic computations which we omit. □

Lemma 41.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is quasi-separated. If f satisfies the uniqueness part of the valuative criterion, then f is separated.*

Proof. The assumption on f means that Δ_f is quasi-compact and quasi-separated (Definition 4.1). We have to show that Δ_f is proper. Lemma 40.1 says that Δ_f is separated. By Lemma 3.3 we know that Δ_f is locally of finite type. To finish the proof we have to show that Δ_f is universally closed. A formal argument (see Lemma 41.1) shows that the uniqueness part of the valuative criterion implies that we have the existence of a dotted arrow in any solid diagram like so:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta_f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dotted] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

Using that this property is preserved by any base change we conclude that any base change by Δ_f by an algebraic space mapping into $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ has the existence part of the valuative criterion and we conclude is universally closed by the valuative criterion for morphisms of algebraic spaces, see Morphisms of Spaces, Lemma 42.1. □

Here is a converse.

Lemma 41.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is separated, then f satisfies the uniqueness part of the valuative criterion.*

Proof. Since f is separated we see that all categories of dotted arrows are setoids by Lemma 39.2. Consider a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] \arrow[ru, dotted] & \mathcal{Y} \end{tikzcd}$$

and a 2-morphism $\gamma : y \circ j \to f \circ x$ as in Definition 39.1. Consider two objects (a, α, β) and (a', β', α') of the category of dotted arrows. To finish the proof we have to show these objects are isomorphic. The isomorphism

$$f \circ a \xrightarrow{\beta^{-1}} y \xrightarrow{\beta'} f \circ a'$$

means that $(a, a', \beta' \circ \beta^{-1})$ is a morphism $\mathrm{Spec}(A) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. On the other hand, α and α' define a 2-arrow

$$(a, a', \beta' \circ \beta^{-1}) \circ j = (a \circ j, a' \circ j, (\beta' \star \mathrm{id}_j) \circ (\beta \star \mathrm{id}_j)^{-1}) \xrightarrow{(\alpha, \alpha')} (x, x, \mathrm{id}) = \Delta_f \circ x$$

Here we use that both (a, α, β) and (a', α', β') are dotted arrows with respect to γ. We obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "\Delta_f"] \\ \mathrm{Spec}(A) \arrow[r, "(a, a', \beta' \circ \beta^{-1})"] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

with 2-commutativity witnessed by (α, α'). Now Δ_f is representable by algebraic spaces (Lemma 3.3) and proper as f is separated. Hence by Lemma 39.13 and the valuative criterion for properness for algebraic spaces (Morphisms of Spaces, Lemma 44.1) we see that there exists a dotted arrow. Unwinding the construction, we see that this means (a, α, β) and (a', α', β') are isomorphic in the category of dotted arrows as desired. □

42. Valuative criterion for universal closedness

Here is a statement.

Lemma 42.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume*

(1) *f is quasi-compact, and*
(2) *f satisfies the existence part of the valuative criterion.*

Then f is universally closed.

Proof. By Lemmas 7.3 and 39.11 properties (1) and (2) are preserved under any base change. By Lemma 13.5 we only have to show that $|T \times_{\mathcal{Y}} \mathcal{X}| \to |T|$ is closed, whenever T is an affine scheme mapping into $\mathcal{Y}$. Hence it suffices to show: if $f : \mathcal{X} \to Y$ is a quasi-compact morphism from an algebraic stack to an affine scheme satisfying the existence part of the valuative criterion, then $|f|$ is closed. Let $T \subset |\mathcal{X}|$ be a closed subset. We have to show that $f(T)$ is closed to finish the proof.

Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced induced algebraic stack structure on T (Properties of Stacks, Definition 10.4). Then $i : \mathcal{Z} \to \mathcal{X}$ is a closed immersion and we have to show that the image of $|\mathcal{Z}| \to |Y|$ is closed. Since closed immersions are quasi-compact (Lemma 7.5) and satisfies the existence part of the valuative criterion (Lemma 39.14) and since compositions of quasi-compact morphisms are quasi-compact (Lemma 7.4) and since compositions preserve the property of satisfying the existence part of the valuative criterion (Lemma 39.12) we conclude that it suffices to show: if $f : \mathcal{X} \to Y$ is a quasi-compact morphism from an algebraic stack to an affine scheme satisfying the existence part of the valuative criterion, then $|f|(|\mathcal{X}|)$ is closed.

Since $\mathcal{X}$ is quasi-compact (being quasi-compact over the affine Y), we can choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$ (Properties of Stacks, Lemma 6.2). Suppose that $y \in Y$ is in the closure of the image of $U \to Y$ (in other words, in the closure of the image of $|f|$). Then by Morphisms, Lemma 6.5 we can find a valuation ring A with fraction field K and a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & Y
\end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to y. By assumption we get an extension K'/K and a valuation ring $A' \subset K'$ dominating A and the dotted arrow in the following diagram

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
\mathrm{Spec}(A') \arrow[r] \arrow[rrru, dotted] & \mathrm{Spec}(A) \arrow[r] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Thus y is in the image of $|f|$ and we win. □

Here is a converse.

Lemma 42.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume*

(1) *f is quasi-separated, and*
(2) *f is universally closed.*

Then f satisfies the existence part of the valuative criterion.

Proof. Consider a solid diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "x"] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r, "y"] \arrow[ru, dotted] & \mathcal{Y}
\end{tikzcd}$$

where A is a valuation ring with field of fractions K and $\gamma : y \circ j \longrightarrow f \circ x$ as in Definition 39.1. By Lemma 39.4 in order to find a dotted arrow (after possibly replacing K by an extension and A by a valuation ring dominating it) we may replace $\mathcal{Y}$ by $\mathrm{Spec}(A)$ and $\mathcal{X}$ by $\mathrm{Spec}(A) \times_{\mathcal{Y}} \mathcal{X}$. Of course we use here that being quasi-separated and universally closed are preserved under base change. Thus we reduce to the case discussed in the next paragraph.

Consider a solid diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "x"] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r, equal] \arrow[ru, dotted] & \mathrm{Spec}(A)
\end{tikzcd}$$

where A is a valuation ring with field of fractions K as in Definition 39.1. By Lemma 7.7 and the fact that f is quasi-separated we have that the morphism x is quasi-compact. Since f is universally closed, we have in particular that $|f|(\overline{\{x\}})$

is closed in $\mathrm{Spec}(A)$. Since this image contains the generic point of $\mathrm{Spec}(A)$ there exists a point $x' \in |\mathcal{X}|$ in the closure of x mapping to the closed point of $\mathrm{Spec}(A)$. By Lemma 7.9 we can find a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] & \mathcal{X} \end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A')$ maps to $x' \in |\mathcal{X}|$. It follows that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ maps the closed point to the closed point, i.e., A' dominates A and this finishes the proof. ☐

43. Valuative criterion for properness

Here is the statement.

Lemma 43.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is of finite type and quasi-separated. Then the following are equivalent*

1. *f is proper, and*
2. *f satisfies both the uniqueness and existence parts of the valuative criterion.*

Proof. A proper morphism is the same thing as a separated, finite type, and universally closed morphism. Thus this lemma follows from Lemmas 41.2, 41.3, 42.1, and 42.2. ☐

44. Local complete intersection morphisms

The property "being a local complete intersection morphism" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Lemma 20.4 and More on Morphisms of Spaces, Lemmas 48.9 and 48.10. By Lemma 16.1 above, we may define what it means for a morphism of algebraic spaces to be a local complete intersection morphism as follows and it agrees with the already existing notion defined in More on Morphisms of Spaces, Section 48 when both source and target are algebraic spaces.

Definition 44.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is a *local complete intersection morphism* or *Koszul* if the equivalent conditions of Lemma 16.1 hold with $\mathcal{P}$ = local complete intersection.

Lemma 44.2. *The composition of local complete intersection morphisms is a local complete intersection.*

Proof. Combine Remark 16.3 with More on Morphisms of Spaces, Lemma 48.5. ☐

Lemma 44.3. *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: Argue exactly as in Remark 16.4 (but only for flat $\mathcal{Y}' \to \mathcal{Y}$) using More on Morphisms of Spaces, Lemma 48.4. ☐

Lemma 44.4. *Let*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\ & \mathcal{Z} & \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic stacks. Assume $\mathcal{Y} \to \mathcal{Z}$ is smooth and $\mathcal{X} \to \mathcal{Z}$ is a local complete intersection morphism. Then $f : \mathcal{X} \to \mathcal{Y}$ is a local complete intersection morphism.

Proof. Choose a scheme W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose a scheme V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose a scheme U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. Then $U \to W$ is a local complete intersection morphism of schemes and $V \to W$ is a smooth morphism of schemes. By the result for schemes (More on Morphisms, Lemma 62.10) we conclude that $U \to V$ is a local complete intersection morphism. By definition this means that f is a local complete intersection morphism. $\square$

45. Stabilizer preserving morphisms

In the literature a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is said to be *stabilizer preserving* or *fixed-point reflecting* if the induced morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism. Such a morphism induces an isomorphism between automorphism groups (Remark 19.5) in every point of $\mathcal{X}$. In this section we prove some simple lemmas around this concept.

Lemma 45.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism, then f is representable by algebraic spaces.*

Proof. Immediate from Lemma 6.2. $\square$

Remark 45.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $U \to \mathcal{X}$ be a morphism whose source is an algebraic space. Let $G \to H$ be the pullback of the morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ to U. If Δ_f is unramified, étale, etc, so is $G \to H$. This is true because

$$\begin{tikzcd} U \times_\mathcal{X} U \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta_f"] \\ U \times_\mathcal{Y} U \arrow[r] & \mathcal{X} \times_\mathcal{Y} \mathcal{X} \end{tikzcd}$$

is cartesian and the morphism $G \to H$ is the base change of the left vertical arrow by the diagonal $U \to U \times U$. Compare with the proof of Lemma 6.6.

Lemma 45.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be an unramified morphism of algebraic stacks. The following are equivalent*

1. *$\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism, and*
2. *f induces an isomorphism between automorphism groups at x and $f(x)$ (Remark 19.5) for all $x \in |\mathcal{X}|$.*

Proof. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Denote $G \to H$ the pullback of the morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ to U. By Remark 45.2 and Lemma 36.9 the morphism $G \to H$ is étale. Condition (1) is equivalent to the condition that $G \to H$ is an isomorphism (this follows for example by applying Properties of Stacks, Lemma 3.3). Condition (2) is equivalent to the condition that for every $u \in U$ the morphism $G_u \to H_u$ of fibres is an isomorphism. Thus (1) $\Rightarrow$ (2) is trivial. If (2) holds, then $G \to H$ is a surjective, universally injective, étale morphism of algebraic spaces. Such a morphism is an isomorphism by Morphisms of Spaces, Lemma 51.2. $\square$

Lemma 45.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume*

1. *f is representable by algebraic spaces and unramified, and*
2. *$\mathcal{I}_\mathcal{Y} \to \mathcal{Y}$ is proper.*

Then the set of $x \in |\mathcal{X}|$ such that f induces an isomorphism between automorphism groups at x and $f(x)$ (Remark 19.5) is open. Letting $\mathcal{U} \subset \mathcal{X}$ be the corresponding open substack, the morphism $\mathcal{I}_\mathcal{U} \to \mathcal{U} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism.

Proof. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Denote $G \to H$ the pullback of the morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ to U. By Remark 45.2 and Lemma 36.9 the morphism $G \to H$ is étale. Since f is representable by algebraic spaces, we see that $G \to H$ is a monomorphism. Hence $G \to H$ is an open immersion, see Morphisms of Spaces, Lemma 51.2. By assumption $H \to U$ is proper.

With these preparations out of the way, we can prove the lemma as follows. The inverse image of the subset of $|\mathcal{X}|$ of the lemma is clearly the set of $u \in U$ such that $G_u \to H_u$ is an isomorphism (since after all G_u is an open sub group algebraic space of H_u). This is an open subset because the complement is the image of the closed subset $|H| \setminus |G|$ and $|H| \to |U|$ is closed. By Properties of Stacks, Lemma 9.12 we can consider the corresponding open substack $\mathcal{U}$ of $\mathcal{X}$. The final statement of the lemma follows from applying Lemma 45.3 to $\mathcal{U} \to \mathcal{Y}$. $\square$

Lemma 45.5. *Let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a cartesian diagram of algebraic stacks.

1. *Let $x' \in |\mathcal{X}'|$ with image $x \in |\mathcal{X}|$. If f induces an isomorphism between automorphism groups at x and $f(x)$ (Remark 19.5), then f' induces an isomorphism between automorphism groups at x' and $f(x')$.*
2. *If $\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{I}_\mathcal{Y}$ is an isomorphism, then $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{I}_{\mathcal{Y}'}$ is an isomorphism.*

Proof. Omitted. $\square$

Lemma 45.6. *Let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

be a cartesian diagram of algebraic stacks. If f induces an isomorphism between automorphism groups at points (Remark 19.5), then

$$\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}') \longrightarrow \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{Y}') \times \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})$$

is injective on isomorphism classes for any field k.

Proof. We have to show that given (y', x) there is at most one x' mapping to it. By our construction of 2-fibre products, a morphism x' is given by a triple (x, y', α) where $\alpha : g \circ y' \to f \circ x$ is a 2-morphism. Now, suppose we have a second such triple (x, y', β). Then α and β differ by a k-valued point ϵ of the automorphism group algebraic space $G_{f(x)}$. Since f induces an isomorphism $G_x \to G_{f(x)}$ by assumption, this means we can lift ϵ to a k-valued point γ of G_x. Then $(\gamma, \mathrm{id}) : (x, y', \alpha) \to (x, y', \beta)$ is an isomorphism as desired. $\square$

Lemma 45.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is étale, f induces an isomorphism between automorphism groups at points (Remark 19.5), and for every algebraically closed field k the functor*

$$f : \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}) \longrightarrow \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{Y})$$

is an equivalence. Then f is an isomorphism.

Proof. By Lemma 14.5 we see that f is universally injective. Combining Lemmas 45.1 and 45.3 we see that f is representable by algebraic spaces. Hence f is an open immersion by Morphisms of Spaces, Lemma 51.2. To finish we remark that the condition in the lemma also guarantees that f is surjective. $\square$

46. Normalization

This section is the analogue of Morphisms of Spaces, Section 49.

Lemma 46.1. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

1. *there is a surjective smooth morphism $U \to \mathcal{X}$ where U is a scheme such that every quasi-compact open of U has finitely many irreducible components,*
2. *for every scheme U and every smooth morphism $U \to \mathcal{X}$ every quasi-compact open of U has finitely many irreducible components,*
3. *for every algebraic space Y and smooth morphism $Y \to \mathcal{X}$ the space Y satisfies the equivalent conditions of Morphisms of Spaces, Lemma 49.1, and*
4. *for every quasi-compact algebraic stack $\mathcal{Y}$ smooth over $\mathcal{X}$ the space $|\mathcal{Y}|$ has finitely many irreducible components.*

Proof. The equivalence of (1), (2), and (3) follow from Descent, Lemma 16.3, Properties of Stacks, Lemma 7.1, and Morphisms of Spaces, Lemma 49.1. It is also clear from these references that condition (4) implies condition (1). Conversely, assume the equivalent conditions (1), (2), and (3) hold and let $\mathcal{Y} \to \mathcal{X}$ be a smooth morphism of algebraic stacks with $\mathcal{Y}$ quasi-compact. Then we can choose an affine scheme V and a surjective smooth morphism $V \to \mathcal{Y}$ by Properties of Stacks, Lemma 6.2. Since V has finitely many irreducible components by (2) and since $|V| \to |\mathcal{Y}|$ is surjective and continuous, we conclude that $|\mathcal{Y}|$ has finitely many irreducible components by Topology, Lemma 8.5. $\square$

Lemma 46.2. *Let $\mathcal{X}$ be an algebraic stack satisfying the equivalent conditions of Lemma 46.1. Then there exists an integral morphism of algebraic stacks*

$$\mathcal{X}^\nu \longrightarrow \mathcal{X}$$

such that for every scheme U and smooth morphism $U \to \mathcal{X}$ the fibre product $\mathcal{X}^\nu \times_\mathcal{X} U$ is the normalization of U.

Proof. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. Set $R = U \times_{\mathcal{X}} U$. Recall that we obtain a smooth groupoid (U, R, s, t, c) in algebraic spaces and a presentation $\mathcal{X} = [U/R]$ of $\mathcal{X}$, see Algebraic Stacks, Lemmas 16.1 and 16.2 and Definition 16.5. The assumption on $\mathcal{X}$ means that the normalization U^ν of U is defined, see Morphisms, Definition 54.1. By Morphisms of Spaces, Lemma 49.5 taking normalization commutes with smooth morphisms of algebraic spaces. Thus we see that the normalization R^ν of R is isomorphic to both $R \times_{s,U} U^\nu$ and $U^\nu \times_{U,t} R$. Thus we obtain two smooth morphisms $s^\nu : R^\nu \to U^\nu$ and $t^\nu : R^\nu \to U^\nu$ of algebraic spaces. A formal computation with fibre products shows that $R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu$ is the normalization of $R \times_{s,U,t} R$. Hence the smooth morphism $c : R \times_{s,U,t} R \to R$ extends to c^ν as well. Similarly, the inverse $i : R \to R$ (an isomorphism) induces an isomorphism $i^\nu : R^\nu \to R^\nu$. Finally, the identity $e : U \to R$ lifts to $e^\nu : U^\nu \to R^\nu$ for example because e is a section of s and $R^\nu = R \times_{U,s} U^\nu$. We claim that $(U^\nu, R^\nu, s^\nu, t^\nu, c^\nu)$ is a smooth groupoid in algebraic spaces. To see this involves checking the axioms (1), (2)(a), (2)(b), (3)(a), and (3)(b) of Groupoids, Section 13 for $(U^\nu, R^\nu, s^\nu, t^\nu, c^\nu, e^\nu, i^\nu)$. For example, for (1) we have to see that the two morphisms $a, b : R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu \to R^\nu$ we obtain are the same. This holds because we know that the corresponding pair of morphisms $R \times_{s,U,t} R \times_{s,U,t} R \to R$ are the same and the morphisms a and b are the unique extensions of this morphism to the normalizations. Similarly for the other axioms.

Consider the algebraic stack $\mathcal{X}^\nu = [U^\nu/R^\nu]$ (Algebraic Stacks, Theorem 17.3). Since we have a morphism $(U^\nu, R^\nu, s^\nu, t^\nu, c^\nu) \to (U, R, s, t, c)$ of groupoids in algebraic spaces, we obtain a morphism $\nu : \mathcal{X}^\nu \to \mathcal{X}$ of algebraic stacks. Since $R^\nu = R \times_{s,U} U^\nu$ we see that $U^\nu = \mathcal{X}^\nu \times_{\mathcal{X}} U$ by Groupoids in Spaces, Lemma 25.3. In particular, as $U^\nu \to U$ is integral, we see that ν is integral. We omit the verification that the base change property stated in the lemma holds for every smooth morphism from a scheme to $\mathcal{X}$. $\square$

This leads us to the following definition.

Definition 46.3. Let $\mathcal{X}$ be an algebraic stack satisfying the equivalent conditions of Lemma 46.1. We define the *normalization* of $\mathcal{X}$ as the morphism

$$\nu : \mathcal{X}^\nu \longrightarrow \mathcal{X}$$

constructed in Lemma 46.2.

47. Points and specializations

This section is the analogue of Decent Spaces, Section 7.

Lemma 47.1. *Let $\mathcal{X}$ be an algebraic stack. Let $f : U \to \mathcal{X}$ be a smooth morphism where U is an algebraic space. Let $x' \leadsto x$ be a specialization of points of $|\mathcal{X}|$. Let $u \in |U|$ with $f(u) = x$. If $(\mathcal{X}, x')$ satisfy the equivalent conditions of Properties of Stacks, Lemma 14.1, then there exists a specialization $u' \leadsto u$ in $|U|$ with $f(u') = x'$.*

Proof. Choose an étale morphism $(U_1, u_1) \to (U, u)$ where U_1 is an affine scheme. Then we may and do replace U by U_1. Thus we may assume U is an affine scheme. Consider the algebraic space $R = U \times_{\mathcal{X}} U$ with smooth projections $t, s : R \to U$. Choose a point $w \in U$ mapping to x'; this is possible as $f : |U| \to |\mathcal{X}|$ is open. By our assumption on x' the fibre $F' = t^{-1}(w) = R \times_{t,U} w$ of $t : R \to U$ over w is a quasi-compact algebraic space. Choose an affine scheme T and a surjective étale morphism $T \to F'$. The fact that $x' \rightsquigarrow x$ means that u is in the closure of the image of the morphism

$$T \to F' \to R \xrightarrow{s} U$$

Namely, this image is the fibre of $|U| \to |\mathcal{X}'|$ over x'; if some $u \in V \subset |U|$ open is disjoint from this fibre, then $f(V)$ is an open neighbourhood of x not containing x'; contradiction. Thus by Morphisms, Lemma 6.5 we see that there exists $u' \in |U|$ in the fibre of $|U| \to |\mathcal{X}|$ over x' which specializes to u. $\square$

48. Decent algebraic stacks

This section is the analogue of Decent Spaces, Section 6. In particular, the following definition is compatible with the notion of a decent algebraic space defined there.

Definition 48.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *decent* if for every $x \in |\mathcal{X}|$ the equivalent conditions of Properties of Stacks, Lemma 14.1 are satisfied.

Some people would rephrase this definition by saying that every point of $\mathcal{X}$ is quasi-compact. A slightly stronger condition would be to ask that any morphism $\mathrm{Spec}(k) \to \mathcal{X}$ in the equivalence class of x is quasi-separated as well as quasi-compact.

Lemma 48.2. *A quasi-separated algebraic stack $\mathcal{X}$ is decent. More generally, if $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact, then $\mathcal{X}$ is decent.*

Proof. Namely, if $\mathcal{X}$ is quasi-separated, then any morphism $f : T \to \mathcal{X}$ whose source is a quasi-compact scheme T, is quasi-compact, see Lemma 7.7. If Δ is on known to be quasi-compact, then one uses the description

$$T \times_{f,\mathcal{X},f'} T' = (T \times T') \times_{(f,f'),\mathcal{X}\times\mathcal{X},\Delta} \mathcal{X}$$

to prove this. Details omitted. $\square$

Lemma 48.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume Y is decent and f is representable (by schemes) or f is representable by algebraic spaces and quasi-separated. Then $\mathcal{X}$ is decent.*

Proof. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. Choose a morphism $y : \mathrm{Spec}(k) \to \mathcal{Y}$ in the equivalence class defining y. Set $\mathcal{X}_y = \mathrm{Spec}(k) \times_{y,\mathcal{Y}} \mathcal{X}$. Choose a point $x' \in |\mathcal{X}_y|$ mapping to x, see Properties of Stacks, Lemma 4.3. Choose a morphism $x' : \mathrm{Spec}(k') \to \mathcal{X}_y$ in the equivalence class of x'. Diagram

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r, "x'"'] & \mathcal{X}_y \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ & \mathrm{Spec}(k) \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

The morphism y is quasi-compact if $\mathcal{Y}$ is decent. Hence $\mathcal{X}_y \to \mathcal{X}$ is quasi-compact as a base change (Lemma 7.3). Thus to conclude it suffices to prove that x' is quasi-compact (Lemma 7.4). If f is representable, then $\mathcal{X}_y$ is a scheme and x' is quasi-compact. If f is representable by algebraic spaces and quasi-separated, then $\mathcal{X}_y$ is a quasi-separated algebraic space and hence decent (Decent Spaces, Lemma 17.2). $\square$

Lemma 48.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is quasi-compact and surjective and $\mathcal{X}$ is decent, then $\mathcal{Y}$ is decent.*

Proof. Let $x : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism where k is a field and denote $y = f \circ x$. Since f is surjective, every point of $|\mathcal{Y}|$ arises in this manner, see Properties of Stacks, Lemma 4.4. Consider an affine scheme T and morphism $T \to \mathcal{Y}$. It suffices to show that $T \times_{\mathcal{Y},y} \mathrm{Spec}(k)$ is quasi-compact, see Lemma 7.10. We have

$$(T \times_{\mathcal{Y}} \mathcal{X}) \times_{\mathcal{X},x} \mathrm{Spec}(k) = T \times_{\mathcal{Y},y} \mathrm{Spec}(k)$$

The morphism $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is quasi-compact hence $T \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact. Since x is a quasi-compact morphism as $\mathcal{X}$ is decent we see that the displayed fibre product is quasi-compact. $\square$

Lemma 48.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{X}$ is decent, then $\mathcal{Y}$ is decent.*

Proof. Assume $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{X}$ is decent. Note that f is a universal homeomorphism by Lemma 28.13. Thus the lemma follows from Lemma 48.4. $\square$

49. Points on decent stacks

This section is the analogue of Decent Spaces, Section 12. We do not know whether or not the topological space associated to a decent algebraic stack is always sober; see Proposition 49.3 for a slightly weaker result.

Lemma 49.1. *Let $\mathcal{X}$ be a decent algebraic stack. Then $|\mathcal{X}|$ is Kolmogorov (see Topology, Definition 8.6).*

Proof. Let $x_1, x_2 \in |\mathcal{X}|$ with $x_1 \rightsquigarrow x_2$ and $x_2 \rightsquigarrow x_1$. We have to show that $x_1 = x_2$. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced closed substack with $|\mathcal{Z}|$ equal to $\overline{\{x_1\}} = \overline{\{x_2\}}$. By Lemma 48.3 we see that $\mathcal{Z}$ is decent. After replacing $\mathcal{X}$ by $\mathcal{Z}$ we reduce to the case discussed in the next paragraph.

Assume $|\mathcal{X}|$ is irreducible with generic points x_1 and x_2. Pick an affine scheme U and $u_1, u_2 \in U$ and a smooth morphism $f : U \to \mathcal{X}$ such that $f(u_i) = x_i$. Then we find a third point $u_3 \in U$ which is the generic point of an irreducible component of U whose image $x_3 \in |\mathcal{X}|$ is *also* a generic point of $|\mathcal{X}|$. Namely, we can simply choose u_3 any generic point of an irreducible component passing through u_1 (or u_2 if you like). In the next paragraph we will show that $x_1 = x_3$ and $x_2 = x_3$ which will prove what we want.

By symmetry it suffices to prove that $x_1 = x_3$. Since x_1 is a generic point of $|\mathcal{X}|$ we have a specialization $x_1 \rightsquigarrow x_3$. By Lemma 47.1 we can find a specialization $u'_1 \rightsquigarrow u_3$ in U (!) mapping to $x_1 \rightsquigarrow x_3$. However, u_3 is the generic point of an irreducible component and hence $u'_1 = u_3$ as desired. $\square$

Lemma 49.2. *Let $\mathcal{X}$ be a decent, locally Noetherian algebraic stack. Then $|\mathcal{X}|$ is a sober locally Noetherian topological space.*

Proof. By Lemma 8.3 the topological space $|\mathcal{X}|$ is locally Noetherian. By Lemma 49.1 the topological space $|\mathcal{X}|$ is Kolmogorov. By Lemma 8.4 the topological space $|\mathcal{X}|$ is quasi-sober. This finishes the proof, see Topology, Definition 8.6. $\square$

Proposition 49.3. *Let $\mathcal{X}$ be a decent algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then $|\mathcal{X}|$ is sober.*

Proof. By Lemma 49.1 we know that $|\mathcal{X}|$ is Kolmogorov (in fact we will reprove this). Let $T \subset |\mathcal{X}|$ be an irreducible closed subset. We have to show T has a generic point. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced induced closed substack corresponding to T, see Properties of Stacks, Definition 10.4. Since $\mathcal{Z} \to \mathcal{X}$ is a closed immersion, we see that $\mathcal{Z}$ is a decent algebraic stack, see Lemma 48.3. Also, the morphism $\mathcal{I}_\mathcal{Z} \to \mathcal{Z}$ is the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ (Lemma 5.6). Hence $\mathcal{I}_\mathcal{Z} \to \mathcal{Z}$ is quasi-compact (Lemma 7.3). Thus we reduce to the case discussed in the next paragraph.

Assume $\mathcal{X}$ is decent, $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact, $\mathcal{X}$ is reduced, and $|\mathcal{X}|$ irreducible. We have to show $|\mathcal{X}|$ has a generic point. By Proposition 29.1. there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe. In other words, $|\mathcal{U}| \subset |\mathcal{X}|$ is open dense. Thus we may assume that $\mathcal{X}$ is a gerbe in addition to all the other properties. Say $\mathcal{X} \to X$ turns $\mathcal{X}$ into a gerbe over the algebraic space X. Then $|\mathcal{X}| \cong |X|$ by Lemma 28.13. In particular, X is quasi-compact and $|X|$ is irreducible. Also, by Lemma 48.5 we see that X is a decent algebraic space. Then $|\mathcal{X}| = |X|$ is sober by Decent Spaces, Proposition 12.4 and hence has a (unique) generic point. $\square$

50. Integral algebraic stacks

This section is the analogue of Spaces over Fields, Section 4. Motivated by the considerations in that section and by the result of Proposition 49.3 we define an integral algebraic stack as follows (and it does not conflict with the already existing definitions of integral schemes and integral algebraic spaces).

Definition 50.1. We say an algebraic stack $\mathcal{X}$ is *integral* if it is reduced, decent, $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact, and $|\mathcal{X}|$ is irreducible.

Note that if $\mathcal{X}$ is quasi-separated, then for it to be integral, it suffices that $\mathcal{X}$ is reduced and that $|\mathcal{X}|$ is irreducible, see Lemma 50.3.

Lemma 50.2. *Let $\mathcal{X}$ be an integral algebraic stack. Then*
1. *$|\mathcal{X}|$ is sober, irreducible, and has a unique generic point,*
2. *there exists an open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe over an integral scheme U.*

Proof. Proposition 49.3 tells us that $|\mathcal{X}|$ is sober. Of course it is also irreducible and hence has a unique generic point x (by the definition of sobriety). Proposition 29.1 shows the existence of a dense open $\mathcal{U} \subset \mathcal{X}$ which is a gerbe over an algebraic space U. Then U is a decent algebraic space by Lemma 48.5 (and the fact that $\mathcal{U}$ is decent by Lemma 48.3). Since $|U| = |\mathcal{U}|$ we see that $|U|$ is irreducible. Finally, since $\mathcal{U}$ is reduced the morphism $\mathcal{U} \to U$ factors through U_{red}, see Properties of Stacks, Lemma 10.3. Now since $\mathcal{U} \to U$ is flat, locally of finite presentation, and surjective (Lemma 28.8), this implies that $U = U_{red}$, i.e., U is reduced (small detail omitted). It follows that U is an integral algebraic space, see Spaces over Fields, Definition 4.1. Then finally, we may replace U (and correspondingly $\mathcal{U}$) by an open subspace and assume that U is an integral scheme, see discussion in Spaces over Fields, Section 4. $\square$

Lemma 50.3. *Let $\mathcal{X}$ be an algebraic stack which is reduced and quasi-separated and whose associated topological space $|\mathcal{X}|$ is irreducible. Then $\mathcal{X}$ is integral.*

Proof. If $\mathcal{X}$ is quasi-separated, then $\mathcal{X}$ is decent by Lemma 48.2. If $\mathcal{X}$ is quasi-separated, then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact, hence $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact as the base change of Δ by Δ, see Lemma 7.3. Thus we see that all the hypotheses of Definition 50.1 hold (and we also see that we may replace "quasi-separated" by "$\Delta_\mathcal{X}$ is quasi-compact"). $\square$

Lemma 50.4. *Let $\mathcal{X}$ be a decent algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. There are canonical bijections between the following sets:*

1. *the set of points of $\mathcal{X}$, i.e., $|\mathcal{X}|$,*
2. *the set of irreducible closed subsets of $|\mathcal{X}|$,*
3. *the set of integral closed substacks of $\mathcal{X}$.*

The bijection from (1) to (2) sends x to $\overline{\{x\}}$. The bijection from (3) to (2) sends $\mathcal{Z}$ to $|\mathcal{Z}|$.

Proof. Our map defines a bijection between (1) and (2) as $|\mathcal{X}|$ is sober by Proposition 49.3. Given $T \subset |\mathcal{X}|$ closed and irreducible, there is a unique reduced closed substack $\mathcal{Z} \subset \mathcal{X}$ such that $|\mathcal{Z}| = T$, namely, $\mathcal{Z}$ is the reduced induced subspace structure on T, see Properties of Stacks, Definition 10.4. Then $\mathcal{Z}$ is an integral algebraic stack because it is decent (Lemma 48.3), the morphism $\mathcal{I}_\mathcal{Z} \to \mathcal{Z}$ is quasi-compact (as the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$, see Lemma 5.6), $\mathcal{Z}$ is reduced, and $|\mathcal{Z}|$ is irreducible. $\square$

51. Residual gerbes

This section is the continuation of Properties of Stacks, Section 11.

Lemma 51.1. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ with image $y \in |\mathcal{Y}|$. Assume the residual gerbe $\mathcal{Z}_y \subset \mathcal{Y}$ of $\mathcal{Y}$ at y exists and that $\mathcal{X}$ is a gerbe over $\mathcal{Y}$. Then $\mathcal{Z}_x = \mathcal{Z}_y \times_\mathcal{Y} \mathcal{X}$ is the residual gerbe of $\mathcal{X}$ at x.*

Proof. The morphism $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism as the base change of the monomorphism $\mathcal{Z}_y \to \mathcal{Y}$. The morphism π is a univeral homeomorphism by Lemma 28.13 and hence $|\mathcal{Z}_x| = \{x\}$. Finally, the morphism $\mathcal{Z}_x \to \mathcal{Z}_y$ is smooth as a base change of the smooth morphism π, see Lemma 33.8. Hence as $\mathcal{Z}_y$ is reduced and locally Noetherian, so is $\mathcal{Z}_x$ (details omitted). Thus $\mathcal{Z}_x$ is the residual gerbe of $\mathcal{X}$ at x by Properties of Stacks, Definition 11.8. $\square$

Lemma 51.2. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks. Let $x \in |\mathcal{X}|$ be a point. Assume*

1. *$\mathcal{X}$ is decent or locally Noetherian (or both),*
2. *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact,*
3. *$|f|(|\mathcal{Y}|)$ is contained in $\{x\} \subset |\mathcal{X}|$, and*
4. *$\mathcal{Y}$ is reduced.*

Then f factors through the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x (whose existence is guaranteed by Lemma 31.1 or 31.3).

Proof. Let $T = \overline{\{x\}} \subset |\mathcal{X}|$ be the closure of x. By Properties of Stacks, Lemma 10.1 there exists a reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. By Properties of Stacks, Lemma 10.3 the morphism f factors through $\mathcal{X}'$. If $\mathcal{X}$ is decent, then by Lemma 48.3 the stack $\mathcal{X}'$ is decent. If $\mathcal{X}$ is locally Noetherian, then $\mathcal{X}'$ is locally Noetherian (details omitted). Note that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ by Lemma 5.6 we see that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact by Lemma 7.3. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{X}$ is reduced and $x \in |\mathcal{X}|$ is a generic point. By Proposition 29.1 implies there exists a dense open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. Note that $x \in |\mathcal{U}|$. Repeating the arguments above we reduce to the case discussed in the next paragraph.

Assume $\mathcal{X} \to X$ is a gerbe over the algebraic space X. If $\mathcal{X}$ is decent, then by Lemmas 28.13 and 48.4 the space X is decent. If $\mathcal{X}$ is locally Noetherian, then X is locally Noetherian by fppf descent (details omitted). Hence the corresponding result holds for X, see Decent Spaces, Lemma 13.10 or 13.9 (small detail omitted). Applying Lemma 51.1 we conclude that the result holds for $\mathcal{X}$ as well. $\square$

Remark 51.3. We do not know whether Lemma 51.2 holds if we only assume $\mathcal{X}$ is locally Noetherian, i.e., we drop the assumption on the inertia being quasi-compact. In this case, if x is a closed point, this is certainly true as follows from the following much simpler lemma.

Lemma 51.4. *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $x \in |\mathcal{X}|$ with residual gerbe $\mathcal{Z}_x \subset \mathcal{X}$ (Lemma 31.3). Then x is a closed point of $|\mathcal{X}|$ if and only if the morphism $\mathcal{Z}_x \to \mathcal{X}$ is a closed immersion.*

Proof. If $\mathcal{Z}_x \to \mathcal{X}$ is a closed immersion, then x is a closed point of $|\mathcal{X}|$, see for example Lemma 37.4. Conversely, assume x is a closed point of $|\mathcal{X}|$. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced closed substack with $|Z| = \{x\}$ (Properties of Stacks, Lemma 10.1). Then $\mathcal{Z}$ is a locally Noetherian algebraic stack by Lemmas 17.4 and 17.5. Since also $\mathcal{Z}$ is reduced and $|\mathcal{Z}| = \{x\}$ it follows that $\mathcal{Z} = \mathcal{Z}_x$ is the residual gerbe by definition. $\square$

52. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
(34) Topologies on Schemes
(35) Descent
(36) Derived Categories of Schemes
(37) More on Morphisms
(38) More on Flatness
(39) Groupoid Schemes
(40) More on Groupoid Schemes
(41) Étale Morphisms of Schemes

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