

# ALGEBRAIC SPACES

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## 1. Introduction

Algebraic spaces were first introduced by Michael Artin, see [Art69b], [Art70], [Art73], [Art71b], [Art71a], [Art69a], [Art69c], and [Art74]. Some of the foundational material was developed jointly with Knutson, who produced the book [Knu71]. Artin defined (see [Art69c, Definition 1.3]) an algebraic space as a sheaf for the étale topology which is locally in the étale topology representable. In most of Artin's work the categories of schemes considered are schemes locally of finite type over a fixed excellent Noetherian base.

Our definition is slightly different from Artin's original definition. Namely, our algebraic spaces are sheaves for the fppf topology whose diagonal is representable and which have an étale "cover" by a scheme. Working with the fppf topology instead of the étale topology is just a technical point and scarcely makes any difference; we will show in Bootstrap, Section 12 that we would have gotten the same category of algebraic spaces if we had worked with the étale topology. In that same chapter we will prove that the condition on the diagonal can in some sense be removed, see Bootstrap, Section 6.

After defining algebraic spaces we make some foundational observations. The main result in this chapter is that with our definitions an algebraic space is the same thing

as an étale equivalence relation, see the discussion in Section 9 and Theorem 10.5. The analogue of this theorem in Artin's setting is [Art69c, Theorem 1.5], or [Knu71, Proposition II.1.7]. In other words, the sheaf defined by an étale equivalence relation has a representable diagonal. It follows that our definition agrees with Artin's original definition in a broad sense. It also means that one can give examples of algebraic spaces by simply writing down an étale equivalence relation.

In Section 13 we introduce various separation axioms on algebraic spaces that we have found in the literature. Finally in Section 14 we give some weird and not so weird examples of algebraic spaces.

## 2. General remarks

We work in a suitable big fppf site  $Sch_{fppf}$  as in Topologies, Definition 7.6. So, if not explicitly stated otherwise all schemes will be objects of  $Sch_{fppf}$ . In Section 15 we discuss what changes if you change the big fppf site.

We will always work relative to a base  $S$  contained in  $Sch_{fppf}$ . And we will then work with the big fppf site  $(Sch/S)_{fppf}$ , see Topologies, Definition 7.8. The absolute case can be recovered by taking  $S = \text{Spec}(\mathbf{Z})$ .

If  $U, T$  are schemes over  $S$ , then we denote  $U(T)$  for the set of  $T$ -valued points *over*  $S$ . In a formula:  $U(T) = \text{Mor}_S(T, U)$ .

Note that any fpqc covering is a universal effective epimorphism, see Descent, Lemma 13.7. Hence the topology on  $Sch_{fppf}$  is weaker than the canonical topology and all representable presheaves are sheaves.

## 3. Representable morphisms of presheaves

Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F, G : (Sch/S)_{fppf}^{opp} \rightarrow \text{Sets}$ . Let  $a : F \rightarrow G$  be a representable transformation of functors, see Categories, Definition 8.2. This means that for every  $U \in \text{Ob}((Sch/S)_{fppf})$  and any  $\xi \in G(U)$  the fiber product  $h_U \times_{\xi, G} F$  is representable. Choose a representing object  $V_\xi$  and an isomorphism  $h_{V_\xi} \rightarrow h_U \times_G F$ . By the Yoneda lemma, see Categories, Lemma 3.5, the projection  $h_{V_\xi} \rightarrow h_U \times_G F \rightarrow h_U$  comes from a unique morphism of schemes  $a_\xi : V_\xi \rightarrow U$ . Suggestively we could represent this by the diagram

$$\begin{array}{ccccc} V_\xi & \rightsquigarrow & h_{V_\xi} & \longrightarrow & F \\ a_\xi \downarrow & & \downarrow & & \downarrow a \\ U & \rightsquigarrow & h_U & \xrightarrow{\xi} & G \end{array}$$

where the squiggly arrows represent the Yoneda embedding. Here are some lemmas about this notion that work in great generality.

**Lemma 3.1.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$  and let  $X, Y$  be objects of  $(Sch/S)_{fppf}$ . Let  $f : X \rightarrow Y$  be a morphism of schemes. Then*

$$h_f : h_X \longrightarrow h_Y$$

*is a representable transformation of functors.*

**Proof.** This is formal and relies only on the fact that the category  $(Sch/S)_{fppf}$  has fibre products.  $\square$

**Lemma 3.2.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F, G, H : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ . Let  $a : F \rightarrow G, b : G \rightarrow H$  be representable transformations of functors. Then*

$$b \circ a : F \longrightarrow H$$

*is a representable transformation of functors.*

**Proof.** This is entirely formal and works in any category.  $\square$

**Lemma 3.3.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F, G, H : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ . Let  $a : F \rightarrow G$  be a representable transformation of functors. Let  $b : H \rightarrow G$  be any transformation of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b, G, a} F & \xrightarrow{b'} & F \\ a' \downarrow & & \downarrow a \\ H & \xrightarrow{b} & G \end{array}$$

*Then the base change  $a'$  is a representable transformation of functors.*

**Proof.** This is entirely formal and works in any category.  $\square$

**Lemma 3.4.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F_i, G_i : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ ,  $i = 1, 2$ . Let  $a_i : F_i \rightarrow G_i$ ,  $i = 1, 2$  be representable transformations of functors. Then*

$$a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$$

*is a representable transformation of functors.*

**Proof.** Write  $a_1 \times a_2$  as the composition  $F_1 \times F_2 \rightarrow G_1 \times F_2 \rightarrow G_1 \times G_2$ . The first arrow is the base change of  $a_1$  by the map  $G_1 \times F_2 \rightarrow G_1$ , and the second arrow is the base change of  $a_2$  by the map  $G_1 \times G_2 \rightarrow G_2$ . Hence this lemma is a formal consequence of Lemmas 3.2 and 3.3.  $\square$

**Lemma 3.5.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F, G : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ . Let  $a : F \rightarrow G$  be a representable transformation of functors. If  $G$  is a sheaf, then so is  $F$ .*

**Proof.** Let  $\{\varphi_i : T_i \rightarrow T\}$  be a covering of the site  $(Sch/S)_{fppf}$ . Let  $s_i \in F(T_i)$  which satisfy the sheaf condition. Then  $\sigma_i = a(s_i) \in G(T_i)$  satisfy the sheaf condition also. Hence there exists a unique  $\sigma \in G(T)$  such that  $\sigma_i = \sigma|_{T_i}$ . By assumption  $F' = h_T \times_{\sigma, G, a} F$  is a representable presheaf and hence (see remarks in Section 2) a sheaf. Note that  $(\varphi_i, s_i) \in F'(T_i)$  satisfy the sheaf condition also, and hence come from some unique  $(\text{id}_T, s) \in F'(T)$ . Clearly  $s$  is the section of  $F$  we are looking for.  $\square$

**Lemma 3.6.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F, G : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ . Let  $a : F \rightarrow G$  be a representable transformation of functors. Then  $\Delta_{F/G} : F \rightarrow F \times_G F$  is representable.*

**Proof.** Let  $U \in \text{Ob}((Sch/S)_{fppf})$ . Let  $\xi = (\xi_1, \xi_2) \in (F \times_G F)(U)$ . Set  $\xi' = a(\xi_1) = a(\xi_2) \in G(U)$ . By assumption there exist a scheme  $V$  and a morphism  $V \rightarrow U$  representing the fibre product  $h_U \times_{\xi', G} F$ . In particular, the elements  $\xi_1, \xi_2$  give morphisms  $f_1, f_2 : U \rightarrow V$  over  $U$ . Because  $V$  represents the fibre product

$h_U \times_{\xi', G} F$  and because  $\xi' = a \circ \xi_1 = a \circ \xi_2$  we see that if  $g : U' \rightarrow U$  is a morphism then

$$g^* \xi_1 = g^* \xi_2 \Leftrightarrow f_1 \circ g = f_2 \circ g.$$

In other words, we see that  $h_U \times_{\xi, F \times_G F} F$  is represented by  $V \times_{\Delta, V \times V, (f_1, f_2)} U$  which is a scheme.  $\square$

#### 4. Lists of useful properties of morphisms of schemes

For ease of reference we list in the following remarks the properties of morphisms which possess some of the properties required of them in later results.

**Remark 4.1.** Here is a list of properties/types of morphisms which are *stable under arbitrary base change*:

- (1) closed, open, and locally closed immersions, see Schemes, Lemma 18.2,
- (2) quasi-compact, see Schemes, Lemma 19.3,
- (3) universally closed, see Schemes, Definition 20.1,
- (4) (quasi-)separated, see Schemes, Lemma 21.12,
- (5) monomorphism, see Schemes, Lemma 23.5
- (6) surjective, see Morphisms, Lemma 9.4,
- (7) universally injective, see Morphisms, Lemma 10.2,
- (8) affine, see Morphisms, Lemma 11.8,
- (9) quasi-affine, see Morphisms, Lemma 13.5,
- (10) (locally) of finite type, see Morphisms, Lemma 15.4,
- (11) (locally) quasi-finite, see Morphisms, Lemma 20.13,
- (12) (locally) of finite presentation, see Morphisms, Lemma 21.4,
- (13) locally of finite type of relative dimension  $d$ , see Morphisms, Lemma 29.2,
- (14) universally open, see Morphisms, Definition 23.1,
- (15) flat, see Morphisms, Lemma 25.8,
- (16) syntomic, see Morphisms, Lemma 30.4,
- (17) smooth, see Morphisms, Lemma 34.5,
- (18) unramified (resp.  $G$ -unramified), see Morphisms, Lemma 35.5,
- (19) étale, see Morphisms, Lemma 36.4,
- (20) proper, see Morphisms, Lemma 41.5,
- (21)  $H$ -projective, see Morphisms, Lemma 43.8,
- (22) (locally) projective, see Morphisms, Lemma 43.9,
- (23) finite or integral, see Morphisms, Lemma 44.6,
- (24) finite locally free, see Morphisms, Lemma 48.4,
- (25) universally submersive, see Morphisms, Lemma 24.2,
- (26) universal homeomorphism, see Morphisms, Lemma 45.2.

Add more as needed.

**Remark 4.2.** Of the properties of morphisms which are stable under base change (as listed in Remark 4.1) the following are also *stable under compositions*:

- (1) closed, open and locally closed immersions, see Schemes, Lemma 24.3,
- (2) quasi-compact, see Schemes, Lemma 19.4,
- (3) universally closed, see Morphisms, Lemma 41.4,
- (4) (quasi-)separated, see Schemes, Lemma 21.12,
- (5) monomorphism, see Schemes, Lemma 23.4,
- (6) surjective, see Morphisms, Lemma 9.2,
- (7) universally injective, see Morphisms, Lemma 10.5,

- (8) affine, see Morphisms, Lemma 11.7,
- (9) quasi-affine, see Morphisms, Lemma 13.4,
- (10) (locally) of finite type, see Morphisms, Lemma 15.3,
- (11) (locally) quasi-finite, see Morphisms, Lemma 20.12,
- (12) (locally) of finite presentation, see Morphisms, Lemma 21.3,
- (13) universally open, see Morphisms, Lemma 23.3,
- (14) flat, see Morphisms, Lemma 25.6,
- (15) syntomic, see Morphisms, Lemma 30.3,
- (16) smooth, see Morphisms, Lemma 34.4,
- (17) unramified (resp.  $G$ -unramified), see Morphisms, Lemma 35.4,
- (18) étale, see Morphisms, Lemma 36.3,
- (19) proper, see Morphisms, Lemma 41.4,
- (20)  $H$ -projective, see Morphisms, Lemma 43.7,
- (21) finite or integral, see Morphisms, Lemma 44.5,
- (22) finite locally free, see Morphisms, Lemma 48.3,
- (23) universally submersive, see Morphisms, Lemma 24.3,
- (24) universal homeomorphism, see Morphisms, Lemma 45.3.

Add more as needed.

**Remark 4.3.** Of the properties mentioned which are stable under base change (as listed in Remark 4.1) the following are also *fppc local on the base* (and a fortiori *fppf local on the base*):

- (1) for immersions we have this for
  - (a) closed immersions, see Descent, Lemma 23.19,
  - (b) open immersions, see Descent, Lemma 23.16, and
  - (c) quasi-compact immersions, see Descent, Lemma 23.21,
- (2) quasi-compact, see Descent, Lemma 23.1,
- (3) universally closed, see Descent, Lemma 23.3,
- (4) (quasi-)separated, see Descent, Lemmas 23.2, and 23.6,
- (5) monomorphism, see Descent, Lemma 23.31,
- (6) surjective, see Descent, Lemma 23.7,
- (7) universally injective, see Descent, Lemma 23.8,
- (8) affine, see Descent, Lemma 23.18,
- (9) quasi-affine, see Descent, Lemma 23.20,
- (10) (locally) of finite type, see Descent, Lemmas 23.10, and 23.12,
- (11) (locally) quasi-finite, see Descent, Lemma 23.24,
- (12) (locally) of finite presentation, see Descent, Lemmas 23.11, and 23.13,
- (13) locally of finite type of relative dimension  $d$ , see Descent, Lemma 23.25,
- (14) universally open, see Descent, Lemma 23.4,
- (15) flat, see Descent, Lemma 23.15,
- (16) syntomic, see Descent, Lemma 23.26,
- (17) smooth, see Descent, Lemma 23.27,
- (18) unramified (resp.  $G$ -unramified), see Descent, Lemma 23.28,
- (19) étale, see Descent, Lemma 23.29,
- (20) proper, see Descent, Lemma 23.14,
- (21) finite or integral, see Descent, Lemma 23.23,
- (22) finite locally free, see Descent, Lemma 23.30,
- (23) universally submersive, see Descent, Lemma 23.5,
- (24) universal homeomorphism, see Descent, Lemma 23.9.

Note that the property of being an “immersion” may not be fpqc local on the base, but in Descent, Lemma 24.1 we proved that it is fppf local on the base.

### 5. Properties of representable morphisms of presheaves

Here is the definition that makes this work.

**Definition 5.1.** With  $S$ , and  $a : F \rightarrow G$  representable as above. Let  $\mathcal{P}$  be a property of morphisms of schemes which

- (1) is preserved under any base change, see Schemes, Definition 18.3, and
- (2) is fppf local on the base, see Descent, Definition 22.1.

In this case we say that  $a$  has *property  $\mathcal{P}$*  if for every  $U \in \text{Ob}((\text{Sch}/S)_{\text{fppf}})$  and any  $\xi \in G(U)$  the resulting morphism of schemes  $V_\xi \rightarrow U$  has property  $\mathcal{P}$ .

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn’t make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

**Remark 5.2.** Consider the property  $\mathcal{P}$  = “surjective”. In this case there could be some ambiguity if we say “let  $F \rightarrow G$  be a surjective map”. Namely, we could mean the notion defined in Definition 5.1 above, or we could mean a surjective map of presheaves, see Sites, Definition 3.1, or, if both  $F$  and  $G$  are sheaves, we could mean a surjective map of sheaves, see Sites, Definition 11.1. If not mentioned otherwise when discussing morphisms of algebraic spaces we will always mean the first. See Lemma 5.9 for a case where surjectivity implies surjectivity as a map of sheaves.

Here is a sanity check.

**Lemma 5.3.** *Let  $S, X, Y$  be objects of  $\text{Sch}_{\text{fppf}}$ . Let  $f : X \rightarrow Y$  be a morphism of schemes. Let  $\mathcal{P}$  be as in Definition 5.1. Then  $h_X \rightarrow h_Y$  has property  $\mathcal{P}$  if and only if  $f$  has property  $\mathcal{P}$ .*

**Proof.** Note that the lemma makes sense by Lemma 3.1. Proof omitted.  $\square$

**Lemma 5.4.** *Let  $S$  be a scheme contained in  $\text{Sch}_{\text{fppf}}$ . Let  $F, G, H : (\text{Sch}/S)_{\text{fppf}}^{\text{opp}} \rightarrow \text{Sets}$ . Let  $\mathcal{P}$  be a property as in Definition 5.1 which is stable under composition. Let  $a : F \rightarrow G, b : G \rightarrow H$  be representable transformations of functors. If  $a$  and  $b$  have property  $\mathcal{P}$  so does  $b \circ a : F \rightarrow H$ .*

**Proof.** Note that the lemma makes sense by Lemma 3.2. Proof omitted.  $\square$

**Lemma 5.5.** *Let  $S$  be a scheme contained in  $\text{Sch}_{\text{fppf}}$ . Let  $F, G, H : (\text{Sch}/S)_{\text{fppf}}^{\text{opp}} \rightarrow \text{Sets}$ . Let  $\mathcal{P}$  be a property as in Definition 5.1. Let  $a : F \rightarrow G$  be a representable transformation of functors. Let  $b : H \rightarrow G$  be any transformation of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b,G,a} F & \xrightarrow{\quad b' \quad} & F \\ a' \downarrow & & \downarrow a \\ H & \xrightarrow{\quad b \quad} & G \end{array}$$

*If  $a$  has property  $\mathcal{P}$  then also the base change  $a'$  has property  $\mathcal{P}$ .*

**Proof.** Note that the lemma makes sense by Lemma 3.3. Proof omitted.  $\square$

**Lemma 5.6.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F, G, H : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ . Let  $\mathcal{P}$  be a property as in Definition 5.1. Let  $a : F \rightarrow G$  be a representable transformation of functors. Let  $b : H \rightarrow G$  be any transformation of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b, G, a} F & \xrightarrow{b'} & F \\ a' \downarrow & & \downarrow a \\ H & \xrightarrow{b} & G \end{array}$$

*Assume that  $b$  induces a surjective map of fppf sheaves  $H^\# \rightarrow G^\#$ . In this case, if  $a'$  has property  $\mathcal{P}$ , then also  $a$  has property  $\mathcal{P}$ .*

**Proof.** First we remark that by Lemma 3.3 the transformation  $a'$  is representable. Let  $U \in \text{Ob}((Sch/S)_{fppf})$ , and let  $\xi \in G(U)$ . By assumption there exists an fppf covering  $\{U_i \rightarrow U\}_{i \in I}$  and elements  $\xi_i \in H(U_i)$  mapping to  $\xi|_{U_i}$  via  $b$ . From general category theory it follows that for each  $i$  we have a fibre product diagram

$$\begin{array}{ccc} U_i \times_{\xi_i, H, a'} (H \times_{b, G, a} F) & \longrightarrow & U \times_{\xi, G, a} F \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & U \end{array}$$

By assumption the left vertical arrow is a morphism of schemes which has property  $\mathcal{P}$ . Since  $\mathcal{P}$  is local in the fppf topology this implies that also the right vertical arrow has property  $\mathcal{P}$  as desired.  $\square$

**Lemma 5.7.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F_i, G_i : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ ,  $i = 1, 2$ . Let  $a_i : F_i \rightarrow G_i$ ,  $i = 1, 2$  be representable transformations of functors. Let  $\mathcal{P}$  be a property as in Definition 5.1 which is stable under composition. If  $a_1$  and  $a_2$  have property  $\mathcal{P}$  so does  $a_1 \times a_2 : F_1 \times F_2 \rightarrow G_1 \times G_2$ .*

**Proof.** Note that the lemma makes sense by Lemma 3.4. Proof omitted.  $\square$

**Lemma 5.8.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F, G : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ . Let  $a : F \rightarrow G$  be a representable transformation of functors. Let  $\mathcal{P}, \mathcal{P}'$  be properties as in Definition 5.1. Suppose that for any morphism of schemes  $f : X \rightarrow Y$  we have  $\mathcal{P}(f) \Rightarrow \mathcal{P}'(f)$ . If  $a$  has property  $\mathcal{P}$  then  $a$  has property  $\mathcal{P}'$ .*

**Proof.** Formal.  $\square$

**Lemma 5.9.** *Let  $S$  be a scheme. Let  $F, G : (Sch/S)_{fppf}^{opp} \rightarrow Sets$  be sheaves. Let  $a : F \rightarrow G$  be representable, flat, locally of finite presentation, and surjective. Then  $a : F \rightarrow G$  is surjective as a map of sheaves.*

**Proof.** Let  $T$  be a scheme over  $S$  and let  $g : T \rightarrow G$  be a  $T$ -valued point of  $G$ . By assumption  $T' = F \times_G T$  is (representable by) a scheme and the morphism  $T' \rightarrow T$  is a flat, locally of finite presentation, and surjective. Hence  $\{T' \rightarrow T\}$  is an fppf covering such that  $g|_{T'} \in G(T')$  comes from an element of  $F(T')$ , namely the map  $T' \rightarrow F$ . This proves the map is surjective as a map of sheaves, see Sites, Definition 11.1.  $\square$

Here is a characterization of those functors for which the diagonal is representable.

**Lemma 5.10.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F$  be a presheaf of sets on  $(Sch/S)_{fppf}$ . The following are equivalent:*

- (1) *the diagonal  $F \rightarrow F \times F$  is representable,*
- (2) *for  $U \in \text{Ob}((Sch/S)_{fppf})$  and any  $a \in F(U)$  the map  $a : h_U \rightarrow F$  is representable,*
- (3) *for every pair  $U, V \in \text{Ob}((Sch/S)_{fppf})$  and any  $a \in F(U)$ ,  $b \in F(V)$  the fibre product  $h_U \times_{a, F, b} h_V$  is representable.*

**Proof.** This is completely formal, see Categories, Lemma 8.4. It depends only on the fact that the category  $(Sch/S)_{fppf}$  has products of pairs of objects and fibre products, see Topologies, Lemma 7.10.  $\square$

In the situation of the lemma, for any morphism  $\xi : h_U \rightarrow F$  as in the lemma, it makes sense to say that  $\xi$  has property  $\mathcal{P}$ , for any property as in Definition 5.1. In particular this holds for  $\mathcal{P} = \text{“surjective”}$  and  $\mathcal{P} = \text{“étale”}$ , see Remark 4.3 above. We will use this remark in the definition of algebraic spaces below.

**Lemma 5.11.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F$  be a presheaf of sets on  $(Sch/S)_{fppf}$ . Let  $\mathcal{P}$  be a property as in Definition 5.1. If for every  $U, V \in \text{Ob}((Sch/S)_{fppf})$  and  $a \in F(U)$ ,  $b \in F(V)$  we have*

- (1)  *$h_U \times_{a, F, b} h_V$  is representable, say by the scheme  $W$ , and*
- (2) *the morphism  $W \rightarrow U \times_S V$  corresponding to  $h_U \times_{a, F, b} h_V \rightarrow h_U \times h_V$  has property  $\mathcal{P}$ ,*

*then  $\Delta : F \rightarrow F \times F$  is representable and has property  $\mathcal{P}$ .*

**Proof.** Observe that  $\Delta$  is representable by Lemma 5.10. We can formulate condition (2) as saying that the transformation  $h_U \times_{a, F, b} h_V \rightarrow h_{U \times_S V}$  has property  $\mathcal{P}$ , see Lemma 5.3. Consider  $T \in \text{Ob}((Sch/S)_{fppf})$  and  $(a, b) \in (F \times F)(T)$ . Observe that we have the commutative diagram

$$\begin{array}{ccc}
 F \times_{\Delta, F \times F, (a, b)} h_T & \longrightarrow & h_T \\
 \downarrow & & \downarrow \Delta_{T/S} \\
 h_T \times_{a, F, b} h_T & \longrightarrow & h_{T \times_S T} \\
 \downarrow & & \downarrow (a, b) \\
 F & \xrightarrow{\Delta} & F \times F
 \end{array}$$

both of whose squares are cartesian. In this way we see that the morphism  $F \times_{F \times F} h_T \rightarrow h_T$  is the base change of a morphism having property  $\mathcal{P}$  by  $\Delta_{T/S}$ . Since  $\mathcal{P}$  is preserved under base change this finishes the proof.  $\square$

## 6. Algebraic spaces

Here is the definition.

**Definition 6.1.** Let  $S$  be a scheme contained in  $Sch_{fppf}$ . An *algebraic space over  $S$*  is a presheaf

$$F : (Sch/S)_{fppf}^{opp} \longrightarrow \text{Sets}$$

with the following properties

- (1) The presheaf  $F$  is a sheaf.
- (2) The diagonal morphism  $F \rightarrow F \times F$  is representable.



- (3) There exists a scheme  $U \in \text{Ob}((\text{Sch}/S)_{\text{fppf}})$  and a map  $h_U \rightarrow F$  which is surjective, and étale.

There are two differences with the “usual” definition, for example the definition in Knutson’s book [Knu71].

The first is that we require  $F$  to be a sheaf in the fppf topology. One reason for doing this is that many natural examples of algebraic spaces satisfy the sheaf condition for the fppf coverings (and even for fpqc coverings). Also, one of the reasons that algebraic spaces have been so useful is via Michael Artin’s results on algebraic spaces. Built into his method is a condition which guarantees the result is locally of finite presentation over  $S$ . Combined it somehow seems to us that the fppf topology is the natural topology to work with. In the end the category of algebraic spaces ends up being the same. See Bootstrap, Section 12.

The second is that we only require the diagonal map for  $F$  to be representable, whereas in [Knu71] it is required that it also be quasi-compact. If  $F = h_U$  for some scheme  $U$  over  $S$  this corresponds to the condition that  $U$  be quasi-separated. Our point of view is to try to prove a certain number of the results that follow only assuming that the diagonal of  $F$  be representable, and simply add an additional hypothesis wherever this is necessary. In any case it has the pleasing consequence that the following lemma is true.

**Lemma 6.2.** *A scheme is an algebraic space. More precisely, given a scheme  $T \in \text{Ob}((\text{Sch}/S)_{\text{fppf}})$  the representable functor  $h_T$  is an algebraic space.*

**Proof.** The functor  $h_T$  is a sheaf by our remarks in Section 2. The diagonal  $h_T \rightarrow h_T \times h_T = h_{T \times T}$  is representable because  $(\text{Sch}/S)_{\text{fppf}}$  has fibre products. The identity map  $h_T \rightarrow h_T$  is surjective étale.  $\square$

**Definition 6.3.** Let  $F, F'$  be algebraic spaces over  $S$ . A *morphism*  $f : F \rightarrow F'$  of algebraic spaces over  $S$  is a transformation of functors from  $F$  to  $F'$ .

The category of algebraic spaces over  $S$  contains the category  $(\text{Sch}/S)_{\text{fppf}}$  as a full subcategory via the Yoneda embedding  $T/S \mapsto h_T$ . From now on we no longer distinguish between a scheme  $T/S$  and the algebraic space it represents. Thus when we say “Let  $f : T \rightarrow F$  be a morphism from the scheme  $T$  to the algebraic space  $F$ ”, we mean that  $T \in \text{Ob}((\text{Sch}/S)_{\text{fppf}})$ , that  $F$  is an algebraic space over  $S$ , and that  $f : h_T \rightarrow F$  is a morphism of algebraic spaces over  $S$ .

## 7. Fibre products of algebraic spaces

The category of algebraic spaces over  $S$  has both products and fibre products.

**Lemma 7.1.** *Let  $S$  be a scheme contained in  $\text{Sch}_{\text{fppf}}$ . Let  $F, G$  be algebraic spaces over  $S$ . Then  $F \times G$  is an algebraic space, and is a product in the category of algebraic spaces over  $S$ .*

**Proof.** It is clear that  $H = F \times G$  is a sheaf. The diagonal of  $H$  is simply the product of the diagonals of  $F$  and  $G$ . Hence it is representable by Lemma 3.4. Finally, if  $U \rightarrow F$  and  $V \rightarrow G$  are surjective étale morphisms, with  $U, V \in \text{Ob}((\text{Sch}/S)_{\text{fppf}})$ , then  $U \times V \rightarrow F \times G$  is surjective étale by Lemma 5.7.  $\square$

**Lemma 7.2.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $H$  be a sheaf on  $(Sch/S)_{fppf}$  whose diagonal is representable. Let  $F, G$  be algebraic spaces over  $S$ . Let  $F \rightarrow H, G \rightarrow H$  be maps of sheaves. Then  $F \times_H G$  is an algebraic space.*

**Proof.** We check the 3 conditions of Definition 6.1. A fibre product of sheaves is a sheaf, hence  $F \times_H G$  is a sheaf. The diagonal of  $F \times_H G$  is the left vertical arrow in

$$\begin{array}{ccc} F \times_H G & \longrightarrow & F \times G \\ \Delta \downarrow & & \downarrow \Delta_F \times \Delta_G \\ (F \times F) \times_{(H \times H)} (G \times G) & \longrightarrow & (F \times F) \times (G \times G) \end{array}$$

which is cartesian. Hence  $\Delta$  is representable as the base change of the morphism on the right which is representable, see Lemmas 3.4 and 3.3. Finally, let  $U, V \in \text{Ob}((Sch/S)_{fppf})$  and  $a : U \rightarrow F, b : V \rightarrow G$  be surjective and étale. As  $\Delta_H$  is representable, we see that  $U \times_H V$  is a scheme. The morphism

$$U \times_H V \longrightarrow F \times_H G$$

is surjective and étale as a composition of the base changes  $U \times_H V \rightarrow U \times_H G$  and  $U \times_H G \rightarrow F \times_H G$  of the étale surjective morphisms  $U \rightarrow F$  and  $V \rightarrow G$ , see Lemmas 3.2 and 3.3. This proves the last condition of Definition 6.1 holds and we conclude that  $F \times_H G$  is an algebraic space.  $\square$

**Lemma 7.3.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F \rightarrow H, G \rightarrow H$  be morphisms of algebraic spaces over  $S$ . Then  $F \times_H G$  is an algebraic space, and is a fibre product in the category of algebraic spaces over  $S$ .*

**Proof.** It follows from the stronger Lemma 7.2 that  $F \times_H G$  is an algebraic space. It is clear that  $F \times_H G$  is a fibre product in the category of algebraic spaces over  $S$  since that is a full subcategory of the category of (pre)sheaves of sets on  $(Sch/S)_{fppf}$ .  $\square$

## 8. Glueing algebraic spaces

In this section we really start abusing notation and not distinguish between schemes and the spaces they represent.

**Lemma 8.1.** *Let  $S \in \text{Ob}(Sch_{fppf})$ . Let  $F$  and  $G$  be sheaves on  $(Sch/S)_{fppf}^{opp}$  and denote  $F \amalg G$  the coproduct in the category of sheaves. The map  $F \rightarrow F \amalg G$  is representable by open and closed immersions.*

**Proof.** Let  $U$  be a scheme and let  $\xi \in (F \amalg G)(U)$ . Recall the coproduct in the category of sheaves is the sheafification of the coproduct presheaf (Sites, Lemma 10.13). Thus there exists an fppf covering  $\{g_i : U_i \rightarrow U\}_{i \in I}$  and a disjoint union decomposition  $I = I' \amalg I''$  such that  $U_i \rightarrow U \rightarrow F \amalg G$  factors through  $F$ , resp.  $G$  if and only if  $i \in I'$ , resp.  $i \in I''$ . Since  $F$  and  $G$  have empty intersection in  $F \amalg G$  we conclude that  $U_i \times_U U_j$  is empty if  $i \in I'$  and  $j \in I''$ . Hence  $U' = \bigcup_{i \in I'} g_i(U_i)$  and  $U'' = \bigcup_{i \in I''} g_i(U_i)$  are disjoint open (Morphisms, Lemma 25.10) subschemes of  $U$  with  $U = U' \amalg U''$ . We omit the verification that  $U' = U \times_{F \amalg G} F$ .  $\square$

**Lemma 8.2.** *Let  $S \in \text{Ob}(Sch_{fppf})$ . Let  $U \in \text{Ob}((Sch/S)_{fppf})$ . Given a set  $I$  and sheaves  $F_i$  on  $\text{Ob}((Sch/S)_{fppf})$ , if  $U \cong \coprod_{i \in I} F_i$  as sheaves, then each  $F_i$  is representable by an open and closed subscheme  $U_i$  and  $U \cong \coprod U_i$  as schemes.*

**Proof.** By Lemma 8.1 the map  $F_i \rightarrow U$  is representable by open and closed immersions. Hence  $F_i$  is representable by an open and closed subscheme  $U_i$  of  $U$ . We have  $U = \coprod U_i$  because we have  $U \cong \coprod F_i$  as sheaves and we can test the equality on points.  $\square$

**Lemma 8.3.** *Let  $S \in \text{Ob}(\text{Sch}_{fppf})$ . Let  $F$  be an algebraic space over  $S$ . Given a set  $I$  and sheaves  $F_i$  on  $\text{Ob}((\text{Sch}/S)_{fppf})$ , if  $F \cong \coprod_{i \in I} F_i$  as sheaves, then each  $F_i$  is an algebraic space over  $S$ .*

**Proof.** The representability of  $F \rightarrow F \times F$  implies that each diagonal morphism  $F_i \rightarrow F_i \times F_i$  is representable (immediate from the definitions and the fact that  $F \times_{(F \times F)} (F_i \times F_i) = F_i$ ). Choose a scheme  $U$  in  $(\text{Sch}/S)_{fppf}$  and a surjective étale morphism  $U \rightarrow F$  (this exist by hypothesis). The base change  $U \times_F F_i \rightarrow F_i$  is surjective and étale by Lemma 5.5. On the other hand,  $U \times_F F_i$  is a scheme by Lemma 8.1. Thus we have verified all the conditions in Definition 6.1 and  $F_i$  is an algebraic space.  $\square$

The condition on the size of  $I$  and the  $F_i$  in the following lemma may be ignored by those not worried about set theoretic questions.

**Lemma 8.4.** *Let  $S \in \text{Ob}(\text{Sch}_{fppf})$ . Suppose given a set  $I$  and algebraic spaces  $F_i$ ,  $i \in I$ . Then  $F = \coprod_{i \in I} F_i$  is an algebraic space provided  $I$ , and the  $F_i$  are not too “large”: for example if we can choose surjective étale morphisms  $U_i \rightarrow F_i$  such that  $\coprod_{i \in I} U_i$  is isomorphic to an object of  $(\text{Sch}/S)_{fppf}$ , then  $F$  is an algebraic space.*

**Proof.** By construction  $F$  is a sheaf. We omit the verification that the diagonal morphism of  $F$  is representable. Finally, if  $U$  is an object of  $(\text{Sch}/S)_{fppf}$  isomorphic to  $\coprod_{i \in I} U_i$  then it is straightforward to verify that the resulting map  $U \rightarrow \coprod F_i$  is surjective and étale.  $\square$

Here is the analogue of Schemes, Lemma 15.4.

**Lemma 8.5.** *Let  $S \in \text{Ob}(\text{Sch}_{fppf})$ . Let  $F$  be a presheaf of sets on  $(\text{Sch}/S)_{fppf}$ . Assume*

- (1)  $F$  is a sheaf,
- (2) *there exists an index set  $I$  and subfunctors  $F_i \subset F$  such that*
  - (a) *each  $F_i$  is an algebraic space,*
  - (b) *each  $F_i \rightarrow F$  is representable,*
  - (c) *each  $F_i \rightarrow F$  is an open immersion (see Definition 5.1),*
  - (d) *the map  $\coprod F_i \rightarrow F$  is surjective as a map of sheaves, and*
  - (e)  *$\coprod F_i$  is an algebraic space (set theoretic condition, see Lemma 8.4).*

*Then  $F$  is an algebraic space.*

**Proof.** Let  $T$  be an object of  $(\text{Sch}/S)_{fppf}$ . Let  $T \rightarrow F$  be a morphism. By assumption (2)(b) and (2)(c) the fibre product  $F_i \times_F T$  is representable by an open subscheme  $V_i \subset T$ . It follows that  $(\coprod F_i) \times_F T$  is represented by the scheme  $\coprod V_i$  over  $T$ . By assumption (2)(d) there exists an fppf covering  $\{T_j \rightarrow T\}_{j \in J}$  such that  $T_j \rightarrow T \rightarrow F$  factors through  $F_i$ ,  $i = i(j)$ . Hence  $T_j \rightarrow T$  factors through the open subscheme  $V_{i(j)} \subset T$ . Since  $\{T_j \rightarrow T\}$  is jointly surjective, it follows that  $T = \bigcup V_i$  is an open covering. In particular, the transformation of functors  $\coprod F_i \rightarrow F$  is representable and surjective in the sense of Definition 5.1 (see Remark 5.2 for a discussion).

Next, let  $T' \rightarrow F$  be a second morphism from an object in  $(Sch/S)_{fppf}$ . Write as above  $T' = \bigcup V'_i$  with  $V'_i = T' \times_F F_i$ . To show that the diagonal  $F \rightarrow F \times F$  is representable we have to show that  $G = T \times_F T'$  is representable, see Lemma 5.10. Consider the subfunctors  $G_i = G \times_F F_i$ . Note that  $G_i = V_i \times_{F_i} V'_i$ , and hence is representable as  $F_i$  is an algebraic space. By the above the  $G_i$  form a Zariski covering of  $G$ . Hence by Schemes, Lemma 15.4 we see  $G$  is representable.

Choose a scheme  $U \in \text{Ob}((Sch/S)_{fppf})$  and a surjective étale morphism  $U \rightarrow \coprod F_i$  (this exists by hypothesis). We may write  $U = \coprod U_i$  with  $U_i$  the inverse image of  $F_i$ , see Lemma 8.2. We claim that  $U \rightarrow F$  is surjective and étale. Surjectivity follows as  $\coprod F_i \rightarrow F$  is surjective (see first paragraph of the proof) by applying Lemma 5.4. Consider the fibre product  $U \times_F T$  where  $T \rightarrow F$  is as above. We have to show that  $U \times_F T \rightarrow T$  is étale. Since  $U \times_F T = \coprod U_i \times_F T$  it suffices to show each  $U_i \times_F T \rightarrow T$  is étale. Since  $U_i \times_F T = U_i \times_{F_i} V_i$  this follows from the fact that  $U_i \rightarrow F_i$  is étale and  $V_i \rightarrow T$  is an open immersion (and Morphisms, Lemmas 36.9 and 36.3).  $\square$

### 9. Presentations of algebraic spaces

Given an algebraic space we can find a “presentation” of it.

**Lemma 9.1.** *Let  $F$  be an algebraic space over  $S$ . Let  $f : U \rightarrow F$  be a surjective étale morphism from a scheme to  $F$ . Set  $R = U \times_F U$ . Then*

- (1)  $j : R \rightarrow U \times_S U$  defines an equivalence relation on  $U$  over  $S$  (see Groupoids, Definition 3.1).
- (2) the morphisms  $s, t : R \rightarrow U$  are étale, and
- (3) the diagram

$$R \rightrightarrows U \longrightarrow F$$

is a coequalizer diagram in  $Sh((Sch/S)_{fppf})$ .

**Proof.** Let  $T/S$  be an object of  $(Sch/S)_{fppf}$ . Then  $R(T) = \{(a, b) \in U(T) \times U(T) \mid f \circ a = f \circ b\}$  which defines an equivalence relation on  $U(T)$ . The morphisms  $s, t : R \rightarrow U$  are étale because the morphism  $U \rightarrow F$  is étale.

To prove (3) we first show that  $U \rightarrow F$  is a surjection of sheaves, see Sites, Definition 11.1. Let  $\xi \in F(T)$  with  $T$  as above. Let  $V = T \times_{\xi, F, f} U$ . By assumption  $V$  is a scheme and  $V \rightarrow T$  is surjective étale. Hence  $\{V \rightarrow T\}$  is a covering for the fppf topology. Since  $\xi|_V$  factors through  $U$  by construction we conclude  $U \rightarrow F$  is surjective. Surjectivity implies that  $F$  is the coequalizer of the diagram by Sites, Lemma 11.3.  $\square$

This lemma suggests the following definitions.

**Definition 9.2.** Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ . An *étale equivalence relation* on  $U$  over  $S$  is an equivalence relation  $j : R \rightarrow U \times_S U$  such that  $s, t : R \rightarrow U$  are étale morphisms of schemes.

**Definition 9.3.** Let  $F$  be an algebraic space over  $S$ . A *presentation* of  $F$  is given by a scheme  $U$  over  $S$  and an étale equivalence relation  $R$  on  $U$  over  $S$ , and a surjective étale morphism  $U \rightarrow F$  such that  $R = U \times_F U$ .

Equivalently we could ask for the existence of an isomorphism

$$U/R \cong F$$

where the quotient  $U/R$  is as defined in Groupoids, Section 20. To construct algebraic spaces we will study the converse question, namely, for which equivalence relations the quotient sheaf  $U/R$  is an algebraic space. It will finally turn out this is always the case if  $R$  is an étale equivalence relation on  $U$  over  $S$ , see Theorem 10.5.

## 10. Algebraic spaces and equivalence relations

Suppose given a scheme  $U$  over  $S$  and an étale equivalence relation  $R$  on  $U$  over  $S$ . We would like to show this defines an algebraic space. We will produce a series of lemmas that prove the quotient sheaf  $U/R$  (see Groupoids, Definition 20.1) has all the properties required of it in Definition 6.1.

**Lemma 10.1.** *Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ . Let  $j = (s, t) : R \rightarrow U \times_S U$  be an étale equivalence relation on  $U$  over  $S$ . Let  $U' \rightarrow U$  be an étale morphism. Let  $R'$  be the restriction of  $R$  to  $U'$ , see Groupoids, Definition 3.3. Then  $j' : R' \rightarrow U' \times_S U'$  is an étale equivalence relation also.*

**Proof.** It is clear from the description of  $s', t'$  in Groupoids, Lemma 18.1 that  $s', t' : R' \rightarrow U'$  are étale as compositions of base changes of étale morphisms (see Morphisms, Lemma 36.4 and 36.3).  $\square$

We will often use the following lemma to find open subspaces of algebraic spaces. A slight improvement (with more general hypotheses) of this lemma is Bootstrap, Lemma 7.1.

**Lemma 10.2.** *Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ . Let  $j = (s, t) : R \rightarrow U \times_S U$  be a pre-relation. Let  $g : U' \rightarrow U$  be a morphism. Assume*

- (1)  *$j$  is an equivalence relation,*
- (2)  *$s, t : R \rightarrow U$  are surjective, flat and locally of finite presentation,*
- (3)  *$g$  is flat and locally of finite presentation.*

*Let  $R' = R|_{U'}$  be the restriction of  $R$  to  $U'$ . Then  $U'/R' \rightarrow U/R$  is representable, and is an open immersion.*

**Proof.** By Groupoids, Lemma 3.2 the morphism  $j' = (s', t') : R' \rightarrow U' \times_S U'$  defines an equivalence relation. Since  $g$  is flat and locally of finite presentation we see that  $g$  is universally open as well (Morphisms, Lemma 25.10). For the same reason  $s, t$  are universally open as well. Let  $W^1 = g(U') \subset U$ , and let  $W = t(s^{-1}(W^1))$ . Then  $W^1$  and  $W$  are open in  $U$ . Moreover, as  $j$  is an equivalence relation we have  $t(s^{-1}(W)) = W$  (see Groupoids, Lemma 19.2 for example).

By Groupoids, Lemma 20.5 the map of sheaves  $F' = U'/R' \rightarrow F = U/R$  is injective. Let  $a : T \rightarrow F$  be a morphism from a scheme into  $U/R$ . We have to show that  $T \times_F F'$  is representable by an open subscheme of  $T$ .

The morphism  $a$  is given by the following data: an fppf covering  $\{\varphi_j : T_j \rightarrow T\}_{j \in J}$  of  $T$  and morphisms  $a_j : T_j \rightarrow U$  such that the maps

$$a_j \times a_{j'} : T_j \times_T T_{j'} \longrightarrow U \times_S U$$

factor through  $j : R \rightarrow U \times_S U$  via some (unique) maps  $r_{jj'} : T_j \times_T T_{j'} \rightarrow R$ . The system  $(a_j)$  corresponds to  $a$  in the sense that the diagrams

$$\begin{array}{ccc} T_j & \xrightarrow{\quad} & U \\ \downarrow & & \downarrow \\ T & \xrightarrow{\quad a \quad} & F \end{array}$$

commute.

Consider the open subsets  $W_j = a_j^{-1}(W) \subset T_j$ . Since  $t(s^{-1}(W)) = W$  we see that

$$W_j \times_T T_{j'} = r_{jj'}^{-1}(t^{-1}(W)) = r_{jj'}^{-1}(s^{-1}(W)) = T_j \times_T W_{j'}.$$

By Descent, Lemma 13.6 this means there exists an open  $W_T \subset T$  such that  $\varphi_j^{-1}(W_T) = W_j$  for all  $j \in J$ . We claim that  $W_T \rightarrow T$  represents  $T \times_F F' \rightarrow T$ .

First, let us show that  $W_T \rightarrow T \rightarrow F$  is an element of  $F'(W_T)$ . Since  $\{W_j \rightarrow W_T\}_{j \in J}$  is an fppf covering of  $W_T$ , it is enough to show that each  $W_j \rightarrow U \rightarrow F$  is an element of  $F'(W_j)$  (as  $F'$  is a sheaf for the fppf topology). Consider the commutative diagram

$$\begin{array}{ccccc} W'_j & \xrightarrow{\quad} & U' & & \\ \downarrow & \searrow & \downarrow g & & \\ & s^{-1}(W^1) & \xrightarrow{s} & W^1 & \\ & \downarrow t & & \downarrow & \\ W_j & \xrightarrow{a_j|_{W_j}} & W & \xrightarrow{\quad} & F \end{array}$$

where  $W'_j = W_j \times_W s^{-1}(W^1) \times_{W^1} U'$ . Since  $t$  and  $g$  are surjective, flat and locally of finite presentation, so is  $W'_j \rightarrow W_j$ . Hence the restriction of the element  $W_j \rightarrow U \rightarrow F$  to  $W'_j$  is an element of  $F'$  as desired.

Suppose that  $f : T' \rightarrow T$  is a morphism of schemes such that  $a|_{T'} \in F'(T')$ . We have to show that  $f$  factors through the open  $W_T$ . Since  $\{T' \times_T T_j \rightarrow T'\}$  is an fppf covering of  $T'$  it is enough to show each  $T' \times_T T_j \rightarrow T$  factors through  $W_T$ . Hence we may assume  $f$  factors as  $\varphi_j \circ f_j : T' \rightarrow T_j \rightarrow T$  for some  $j$ . In this case the condition  $a|_{T'} \in F'(T')$  means that there exists some fppf covering  $\{\psi_i : T'_i \rightarrow T'\}_{i \in I}$  and some morphisms  $b_i : T'_i \rightarrow U'$  such that

$$\begin{array}{ccccc} T'_i & \xrightarrow{\quad b_i \quad} & U' & \xrightarrow{\quad g \quad} & U \\ f_j \circ \psi_i \downarrow & & & & \downarrow \\ T_j & \xrightarrow{\quad a_j \quad} & U & \xrightarrow{\quad} & F \end{array}$$

is commutative. This commutativity means that there exists a morphism  $r'_i : T'_i \rightarrow R$  such that  $t \circ r'_i = a_j \circ f_j \circ \psi_i$ , and  $s \circ r'_i = g \circ b_i$ . This implies that  $\text{Im}(f_j \circ \psi_i) \subset W_j$  and we win.  $\square$

The following lemma is not completely trivial although it looks like it should be trivial.

**Lemma 10.3.** *Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ . Let  $j = (s, t) : R \rightarrow U \times_S U$  be an étale equivalence relation on  $U$  over  $S$ . If the quotient  $U/R$  is an algebraic space, then  $U \rightarrow U/R$  is étale and surjective. Hence  $(U, R, U \rightarrow U/R)$  is a presentation of the algebraic space  $U/R$ .*

**Proof.** Denote  $c : U \rightarrow U/R$  the morphism in question. Let  $T$  be a scheme and let  $a : T \rightarrow U/R$  be a morphism. We have to show that the morphism (of schemes)  $\pi : T \times_{a, U/R, c} U \rightarrow T$  is étale and surjective. The morphism  $a$  corresponds to an fppf covering  $\{\varphi_i : T_i \rightarrow T\}$  and morphisms  $a_i : T_i \rightarrow U$  such that  $a_i \times a_{i'} : T_i \times_T T_{i'} \rightarrow U \times_S U$  factors through  $R$ , and such that  $c \circ a_i = a \circ \varphi_i$ . Hence

$$T_i \times_{\varphi_i, T} T \times_{a, U/R, c} U = T_i \times_{c \circ a_i, U/R, c} U = T_i \times_{a_i, U} U \times_{c, U/R, c} U = T_i \times_{a_i, U, t} R.$$

Since  $t$  is étale and surjective we conclude that the base change of  $\pi$  to  $T_i$  is surjective and étale. Since the property of being surjective and étale is local on the base in the fpqc topology (see Remark 4.3) we win.  $\square$

**Lemma 10.4.** *Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ . Let  $j = (s, t) : R \rightarrow U \times_S U$  be an étale equivalence relation on  $U$  over  $S$ . Assume that  $U$  is affine. Then the quotient  $F = U/R$  is an algebraic space, and  $U \rightarrow F$  is étale and surjective.*

**Proof.** Since  $j : R \rightarrow U \times_S U$  is a monomorphism we see that  $j$  is separated (see Schemes, Lemma 23.3). Since  $U$  is affine we see that  $U \times_S U$  (which comes equipped with a monomorphism into the affine scheme  $U \times U$ ) is separated. Hence we see that  $R$  is separated. In particular the morphisms  $s, t$  are separated as well as étale.

Since the composition  $R \rightarrow U \times_S U \rightarrow U$  is locally of finite type we conclude that  $j$  is locally of finite type (see Morphisms, Lemma 15.8). As  $j$  is also a monomorphism it has finite fibres and we see that  $j$  is locally quasi-finite by Morphisms, Lemma 20.7. Altogether we see that  $j$  is separated and locally quasi-finite.

Our first step is to show that the quotient map  $c : U \rightarrow F$  is representable. Consider a scheme  $T$  and a morphism  $a : T \rightarrow F$ . We have to show that the sheaf  $G = T \times_{a, F, c} U$  is representable. As seen in the proofs of Lemmas 10.2 and 10.3 there exists an fppf covering  $\{\varphi_i : T_i \rightarrow T\}_{i \in I}$  and morphisms  $a_i : T_i \rightarrow U$  such that  $a_i \times a_{i'} : T_i \times_T T_{i'} \rightarrow U \times_S U$  factors through  $R$ , and such that  $c \circ a_i = a \circ \varphi_i$ . As in the proof of Lemma 10.3 we see that

$$\begin{aligned} T_i \times_{\varphi_i, T} G &= T_i \times_{\varphi_i, T} T \times_{a, U/R, c} U \\ &= T_i \times_{c \circ a_i, U/R, c} U \\ &= T_i \times_{a_i, U} U \times_{c, U/R, c} U \\ &= T_i \times_{a_i, U, t} R \end{aligned}$$

Since  $t$  is separated and étale, and in particular separated and locally quasi-finite (by Morphisms, Lemmas 35.10 and 36.16) we see that the restriction of  $G$  to each  $T_i$  is representable by a morphism of schemes  $X_i \rightarrow T_i$  which is separated and locally quasi-finite. By Descent, Lemma 39.1 we obtain a descent datum  $(X_i, \varphi_{ii'})$  relative to the fppf-covering  $\{T_i \rightarrow T\}$ . Since each  $X_i \rightarrow T_i$  is separated and locally quasi-finite we see by More on Morphisms, Lemma 57.1 that this descent datum is effective. Hence by Descent, Lemma 39.1 (2) we conclude that  $G$  is representable as desired.

The second step of the proof is to show that  $U \rightarrow F$  is surjective and étale. This is clear from the above since in the first step above we saw that  $G = T \times_{a,F,c} U$  is a scheme over  $T$  which base changes to schemes  $X_i \rightarrow T_i$  which are surjective and étale. Thus  $G \rightarrow T$  is surjective and étale (see Remark 4.3). Alternatively one can reread the proof of Lemma 10.3 in the current situation.

The third and final step is to show that the diagonal map  $F \rightarrow F \times F$  is representable. We first observe that the diagram

$$\begin{array}{ccc} R & \longrightarrow & F \\ j \downarrow & & \downarrow \Delta \\ U \times_S U & \longrightarrow & F \times F \end{array}$$

is a fibre product square. By Lemma 3.4 the morphism  $U \times_S U \rightarrow F \times F$  is representable (note that  $h_U \times h_U = h_{U \times_S U}$ ). Moreover, by Lemma 5.7 the morphism  $U \times_S U \rightarrow F \times F$  is surjective and étale (note also that étale and surjective occur in the lists of Remarks 4.3 and 4.2). It follows either from Lemma 3.3 and the diagram above, or by writing  $R \rightarrow F$  as  $R \rightarrow U \rightarrow F$  and Lemmas 3.1 and 3.2 that  $R \rightarrow F$  is representable as well. Let  $T$  be a scheme and let  $a : T \rightarrow F \times F$  be a morphism. We have to show that  $G = T \times_{a,F \times F, \Delta} F$  is representable. By what was said above the morphism (of schemes)

$$T' = (U \times_S U) \times_{F \times F, a} T \longrightarrow T$$

is surjective and étale. Hence  $\{T' \rightarrow T\}$  is an étale covering of  $T$ . Note also that

$$T' \times_T G = T' \times_{U \times_S U, j} R$$

as can be seen contemplating the following cube

$$\begin{array}{ccccc} & & R & \longrightarrow & F \\ & \nearrow & \downarrow & & \downarrow \\ T' \times_T G & \longrightarrow & G & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ & \nearrow & U \times_S U & \longrightarrow & F \times F \\ T' & \longrightarrow & T & \longrightarrow & F \end{array}$$

Hence we see that the restriction of  $G$  to  $T'$  is representable by a scheme  $X$ , and moreover that the morphism  $X \rightarrow T'$  is a base change of the morphism  $j$ . Hence  $X \rightarrow T'$  is separated and locally quasi-finite (see second paragraph of the proof). By Descent, Lemma 39.1 we obtain a descent datum  $(X, \varphi)$  relative to the fppf-covering  $\{T' \rightarrow T\}$ . Since  $X \rightarrow T'$  is separated and locally quasi-finite we see by More on Morphisms, Lemma 57.1 that this descent datum is effective. Hence by Descent, Lemma 39.1 (2) we conclude that  $G$  is representable as desired.  $\square$

**Theorem 10.5.** *Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ . Let  $j = (s, t) : R \rightarrow U \times_S U$  be an étale equivalence relation on  $U$  over  $S$ . Then the quotient  $U/R$  is an algebraic space, and  $U \rightarrow U/R$  is étale and surjective, in other words  $(U, R, U \rightarrow U/R)$  is a presentation of  $U/R$ .*



**Proof.** By Lemma 10.3 it suffices to prove that  $U/R$  is an algebraic space. Let  $U' \rightarrow U$  be a surjective, étale morphism. Then  $\{U' \rightarrow U\}$  is in particular an fppf covering. Let  $R'$  be the restriction of  $R$  to  $U'$ , see Groupoids, Definition 3.3. According to Groupoids, Lemma 20.6 we see that  $U/R \cong U'/R'$ . By Lemma 10.1  $R'$  is an étale equivalence relation on  $U'$ . Thus we may replace  $U$  by  $U'$ .

We apply the previous remark to  $U' = \coprod U_i$ , where  $U = \bigcup U_i$  is an affine open covering of  $U$ . Hence we may and do assume that  $U = \coprod U_i$  where each  $U_i$  is an affine scheme.

Consider the restriction  $R_i$  of  $R$  to  $U_i$ . By Lemma 10.1 this is an étale equivalence relation. Set  $F_i = U_i/R_i$  and  $F = U/R$ . It is clear that  $\coprod F_i \rightarrow F$  is surjective. By Lemma 10.2 each  $F_i \rightarrow F$  is representable, and an open immersion. By Lemma 10.4 applied to  $(U_i, R_i)$  we see that  $F_i$  is an algebraic space. Then by Lemma 10.3 we see that  $U_i \rightarrow F_i$  is étale and surjective. From Lemma 8.4 it follows that  $\coprod F_i$  is an algebraic space. Finally, we have verified all hypotheses of Lemma 8.5 and it follows that  $F = U/R$  is an algebraic space.  $\square$

## 11. Algebraic spaces, retrofitted

We start building our arsenal of lemmas dealing with algebraic spaces. The first result says that in Definition 6.1 we can weaken the condition on the diagonal as follows.

**Lemma 11.1.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F$  be a sheaf on  $(Sch/S)_{fppf}$  such that there exists  $U \in \text{Ob}((Sch/S)_{fppf})$  and a map  $U \rightarrow F$  which is representable, surjective, and étale. Then  $F$  is an algebraic space.*

**Proof.** Set  $R = U \times_F U$ . This is a scheme as  $U \rightarrow F$  is assumed representable. The projections  $s, t : R \rightarrow U$  are étale as  $U \rightarrow F$  is assumed étale. The map  $j = (t, s) : R \rightarrow U \times_S U$  is a monomorphism and an equivalence relation as  $R = U \times_F U$ . By Theorem 10.5 the quotient sheaf  $F' = U/R$  is an algebraic space and  $U \rightarrow F'$  is surjective and étale. Again since  $R = U \times_F U$  we obtain a canonical factorization  $U \rightarrow F' \rightarrow F$  and  $F' \rightarrow F$  is an injective map of sheaves. On the other hand,  $U \rightarrow F$  is surjective as a map of sheaves by Lemma 5.9. Thus  $F' \rightarrow F$  is also surjective and we conclude  $F' = F$  is an algebraic space.  $\square$

**Lemma 11.2.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $G$  be an algebraic space over  $S$ , let  $F$  be a sheaf on  $(Sch/S)_{fppf}$ , and let  $G \rightarrow F$  be a representable transformation of functors which is surjective and étale. Then  $F$  is an algebraic space.*

**Proof.** Pick a scheme  $U$  and a surjective étale morphism  $U \rightarrow G$ . Since  $G$  is an algebraic space  $U \rightarrow G$  is representable. Hence the composition  $U \rightarrow G \rightarrow F$  is representable, surjective, and étale. See Lemmas 3.2 and 5.4. Thus  $F$  is an algebraic space by Lemma 11.1.  $\square$

**Lemma 11.3.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F$  be an algebraic space over  $S$ . Let  $G \rightarrow F$  be a representable transformation of functors. Then  $G$  is an algebraic space.*

**Proof.** By Lemma 3.5 we see that  $G$  is a sheaf. The diagram

$$\begin{array}{ccc} G \times_F G & \longrightarrow & F \\ \downarrow & & \downarrow \Delta_F \\ G \times G & \longrightarrow & F \times F \end{array}$$

is cartesian. Hence we see that  $G \times_F G \rightarrow G \times G$  is representable by Lemma 3.3. By Lemma 3.6 we see that  $G \rightarrow G \times_F G$  is representable. Hence  $\Delta_G : G \rightarrow G \times G$  is representable as a composition of representable transformations, see Lemma 3.2. Finally, let  $U$  be an object of  $(Sch/S)_{fppf}$  and let  $U \rightarrow F$  be surjective and étale. By assumption  $U \times_F G$  is representable by a scheme  $U'$ . By Lemma 5.5 the morphism  $U' \rightarrow G$  is surjective and étale. This verifies the final condition of Definition 6.1 and we win.  $\square$

**Lemma 11.4.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F, G$  be algebraic spaces over  $S$ . Let  $G \rightarrow F$  be a representable morphism. Let  $U \in \text{Ob}((Sch/S)_{fppf})$ , and  $q : U \rightarrow F$  surjective and étale. Set  $V = G \times_F U$ . Finally, let  $\mathcal{P}$  be a property of morphisms of schemes as in Definition 5.1. Then  $G \rightarrow F$  has property  $\mathcal{P}$  if and only if  $V \rightarrow U$  has property  $\mathcal{P}$ .*

**Proof.** (This lemma follows from Lemmas 5.5 and 5.6, but we give a direct proof here also.) It is clear from the definitions that if  $G \rightarrow F$  has property  $\mathcal{P}$ , then  $V \rightarrow U$  has property  $\mathcal{P}$ . Conversely, assume  $V \rightarrow U$  has property  $\mathcal{P}$ . Let  $T \rightarrow F$  be a morphism from a scheme to  $F$ . Let  $T' = T \times_F G$  which is a scheme since  $G \rightarrow F$  is representable. We have to show that  $T' \rightarrow T$  has property  $\mathcal{P}$ . Consider the commutative diagram of schemes

$$\begin{array}{ccccc} V & \longleftarrow & T \times_F V & \longrightarrow & T \times_F G = T' \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & T \times_F U & \longrightarrow & T \end{array}$$

where both squares are fibre product squares. Hence we conclude the middle arrow has property  $\mathcal{P}$  as a base change of  $V \rightarrow U$ . Finally,  $\{T \times_F U \rightarrow T\}$  is a fppf covering as it is surjective étale, and hence we conclude that  $T' \rightarrow T$  has property  $\mathcal{P}$  as it is local on the base in the fppf topology.  $\square$

**Lemma 11.5.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $G \rightarrow F$  be a transformation of presheaves on  $(Sch/S)_{fppf}$ . Let  $\mathcal{P}$  be a property of morphisms of schemes. Assume*

- (1)  $\mathcal{P}$  is preserved under any base change, fppf local on the base, and morphisms of type  $\mathcal{P}$  satisfy descent for fppf coverings, see Descent, Definition 36.1,
- (2)  $G$  is a sheaf,
- (3)  $F$  is an algebraic space,
- (4) there exists a  $U \in \text{Ob}((Sch/S)_{fppf})$  and a surjective étale morphism  $U \rightarrow F$  such that  $V = G \times_F U$  is representable, and
- (5)  $V \rightarrow U$  has  $\mathcal{P}$ .

*Then  $G$  is an algebraic space,  $G \rightarrow F$  is representable and has property  $\mathcal{P}$ .*

**Proof.** Let  $R = U \times_F U$ , and denote  $t, s : R \rightarrow U$  the projection morphisms as usual. Let  $T$  be a scheme and let  $T \rightarrow F$  be a morphism. Then  $U \times_F T \rightarrow T$  is

surjective étale, hence  $\{U \times_F T \rightarrow T\}$  is a covering for the étale topology. Consider

$$W = G \times_F (U \times_F T) = V \times_F T = V \times_U (U \times_F T).$$

It is a scheme since  $F$  is an algebraic space. The morphism  $W \rightarrow U \times_F T$  has property  $\mathcal{P}$  since it is a base change of  $V \rightarrow U$ . There is an isomorphism

$$\begin{aligned} W \times_T (U \times_F T) &= (G \times_F (U \times_F T)) \times_T (U \times_F T) \\ &= (U \times_F T) \times_T (G \times_F (U \times_F T)) \\ &= (U \times_F T) \times_T W \end{aligned}$$

over  $(U \times_F T) \times_T (U \times_F T)$ . The middle equality maps  $((g, (u_1, t)), (u_2, t))$  to  $((u_1, t), (g, (u_2, t)))$ . This defines a descent datum for  $W/U \times_F T/T$ , see Descent, Definition 34.1. This follows from Descent, Lemma 39.1. Namely we have a sheaf  $G \times_F T$ , whose base change to  $U \times_F T$  is represented by  $W$  and the isomorphism above is the one from the proof of Descent, Lemma 39.1. By assumption on  $\mathcal{P}$  the descent datum above is representable. Hence by the last statement of Descent, Lemma 39.1 we see that  $G \times_F T$  is representable. This proves that  $G \rightarrow F$  is a representable transformation of functors.

As  $G \rightarrow F$  is representable, we see that  $G$  is an algebraic space by Lemma 11.3. The fact that  $G \rightarrow F$  has property  $\mathcal{P}$  now follows from Lemma 11.4.  $\square$

**Lemma 11.6.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F, G$  be algebraic spaces over  $S$ . Let  $a : F \rightarrow G$  be a morphism. Given any  $V \in \text{Ob}((Sch/S)_{fppf})$  and a surjective étale morphism  $q : V \rightarrow G$  there exists a  $U \in \text{Ob}((Sch/S)_{fppf})$  and a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ p \downarrow & & \downarrow q \\ F & \xrightarrow{a} & G \end{array}$$

with  $p$  surjective and étale.

**Proof.** First choose  $W \in \text{Ob}((Sch/S)_{fppf})$  with surjective étale morphism  $W \rightarrow F$ . Next, put  $U = W \times_G V$ . Since  $G$  is an algebraic space we see that  $U$  is isomorphic to an object of  $(Sch/S)_{fppf}$ . As  $q$  is surjective étale, we see that  $U \rightarrow W$  is surjective étale (see Lemma 5.5). Thus  $U \rightarrow F$  is surjective étale as a composition of surjective étale morphisms (see Lemma 5.4).  $\square$

## 12. Immersions and Zariski coverings of algebraic spaces

At this point an interesting phenomenon occurs. We have already defined the notion of an open immersion of algebraic spaces (through Definition 5.1) but we have yet to define the notion of a *point*<sup>1</sup>. Thus the *Zariski topology* of an algebraic space has already been defined, but there is no space yet!

Perhaps superfluously we formally introduce immersions as follows.

**Definition 12.1.** Let  $S \in \text{Ob}(Sch_{fppf})$  be a scheme. Let  $F$  be an algebraic space over  $S$ .

<sup>1</sup>We will associate a topological space to an algebraic space in Properties of Spaces, Section 4, and its opens will correspond exactly to the open subspaces defined below.

- (1) A morphism of algebraic spaces over  $S$  is called an *open immersion* if it is representable, and an open immersion in the sense of Definition 5.1.
- (2) An *open subspace* of  $F$  is a subfunctor  $F' \subset F$  such that  $F'$  is an algebraic space and  $F' \rightarrow F$  is an open immersion.
- (3) A morphism of algebraic spaces over  $S$  is called a *closed immersion* if it is representable, and a closed immersion in the sense of Definition 5.1.
- (4) A *closed subspace* of  $F$  is a subfunctor  $F' \subset F$  such that  $F'$  is an algebraic space and  $F' \rightarrow F$  is a closed immersion.
- (5) A morphism of algebraic spaces over  $S$  is called an *immersion* if it is representable, and an immersion in the sense of Definition 5.1.
- (6) A *locally closed subspace* of  $F$  is a subfunctor  $F' \subset F$  such that  $F'$  is an algebraic space and  $F' \rightarrow F$  is an immersion.

We note that these definitions make sense since an immersion is in particular a monomorphism (see Schemes, Lemma 23.8 and Lemma 5.8), and hence the image of an immersion  $G \rightarrow F$  of algebraic spaces is a subfunctor  $F' \subset F$  which is (canonically) isomorphic to  $G$ . Thus some of the discussion of Schemes, Section 10 carries over to the setting of algebraic spaces.

**Lemma 12.2.** *Let  $S \in \text{Ob}(\text{Sch}_{fppf})$  be a scheme. A composition of (closed, resp. open) immersions of algebraic spaces over  $S$  is a (closed, resp. open) immersion of algebraic spaces over  $S$ .*

**Proof.** See Lemma 5.4 and Remarks 4.3 (see very last line of that remark) and 4.2.  $\square$

**Lemma 12.3.** *Let  $S \in \text{Ob}(\text{Sch}_{fppf})$  be a scheme. A base change of a (closed, resp. open) immersion of algebraic spaces over  $S$  is a (closed, resp. open) immersion of algebraic spaces over  $S$ .*

**Proof.** See Lemma 5.5 and Remark 4.3 (see very last line of that remark).  $\square$

**Lemma 12.4.** *Let  $S \in \text{Ob}(\text{Sch}_{fppf})$  be a scheme. Let  $F$  be an algebraic space over  $S$ . Let  $F_1, F_2$  be locally closed subspaces of  $F$ . If  $F_1 \subset F_2$  as subfunctors of  $F$ , then  $F_1$  is a locally closed subspace of  $F_2$ . Similarly for closed and open subspaces.*

**Proof.** Let  $T \rightarrow F_2$  be a morphism with  $T$  a scheme. Since  $F_2 \rightarrow F$  is a monomorphism, we see that  $T \times_{F_2} F_1 = T \times_F F_1$ . The lemma follows formally from this.  $\square$

Let us formally define the notion of a Zariski open covering of algebraic spaces. Note that in Lemma 8.5 we have already encountered such open coverings as a method for constructing algebraic spaces.

**Definition 12.5.** Let  $S \in \text{Ob}(\text{Sch}_{fppf})$  be a scheme. Let  $F$  be an algebraic space over  $S$ . A *Zariski covering*  $\{F_i \subset F\}_{i \in I}$  of  $F$  is given by a set  $I$  and a collection of open subspaces  $F_i \subset F$  such that  $\coprod F_i \rightarrow F$  is a surjective map of sheaves.

Note that if  $T$  is a schemes, and  $a : T \rightarrow F$  is a morphism, then each of the fibre products  $T \times_F F_i$  is identified with an open subscheme  $T_i \subset T$ . The final condition of the definition signifies exactly that  $T = \bigcup_{i \in I} T_i$ .

It is clear that the collection  $F_{Zar}$  of open subspaces of  $F$  is a set (as  $(\text{Sch}/S)_{fppf}$  is a site, hence a set). Moreover, we can turn  $F_{Zar}$  into a category by letting the morphisms be inclusions of subfunctors (which are automatically open immersions

by Lemma 12.4). Finally, Definition 12.5 provides the notion of a Zariski covering  $\{F_i \rightarrow F'\}_{i \in I}$  in the category  $F_{Zar}$ . Hence, just as in the case of a topological space (see Sites, Example 6.4) by suitably choosing a set of coverings we may obtain a Zariski site of the algebraic space  $F$ .

**Definition 12.6.** Let  $S \in \text{Ob}(Sch_{fppf})$  be a scheme. Let  $F$  be an algebraic space over  $S$ . A *small Zariski site*  $F_{Zar}$  of an algebraic space  $F$  is one of the sites described above.

Hence this gives a notion of what it means for something to be true Zariski locally on an algebraic space, which is how we will use this notion. In general the Zariski topology is not fine enough for our purposes. For example we can consider the category of Zariski sheaves on an algebraic space. It will turn out that this is not the correct thing to consider, even for quasi-coherent sheaves. One only gets the desired result when using the étale or fppf site of  $F$  to define quasi-coherent sheaves.

### 13. Separation conditions on algebraic spaces

A separation condition on an algebraic space  $F$  is a condition on the diagonal morphism  $F \rightarrow F \times F$ . Let us first list the properties the diagonal has automatically. Since the diagonal is representable by definition the following lemma makes sense (through Definition 5.1).

**Lemma 13.1.** *Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F$  be an algebraic space over  $S$ . Let  $\Delta : F \rightarrow F \times F$  be the diagonal morphism. Then*

- (1)  $\Delta$  is locally of finite type,
- (2)  $\Delta$  is a monomorphism,
- (3)  $\Delta$  is separated, and
- (4)  $\Delta$  is locally quasi-finite.

**Proof.** Let  $F = U/R$  be a presentation of  $F$ . As in the proof of Lemma 10.4 the diagram

$$\begin{array}{ccc} R & \longrightarrow & F \\ j \downarrow & & \downarrow \Delta \\ U \times_S U & \longrightarrow & F \times F \end{array}$$

is cartesian. Hence according to Lemma 11.4 it suffices to show that  $j$  has the properties listed in the lemma. (Note that each of the properties (1) – (4) occur in the lists of Remarks 4.1 and 4.3.) Since  $j$  is an equivalence relation it is a monomorphism. Hence it is separated by Schemes, Lemma 23.3. As  $R$  is an étale equivalence relation we see that  $s, t : R \rightarrow U$  are étale. Hence  $s, t$  are locally of finite type. Then it follows from Morphisms, Lemma 15.8 that  $j$  is locally of finite type. Finally, as it is a monomorphism its fibres are finite. Thus we conclude that it is locally quasi-finite by Morphisms, Lemma 20.7.  $\square$

Here are some common types of separation conditions, relative to the base scheme  $S$ . There is also an absolute notion of these conditions which we will discuss in Properties of Spaces, Section 3. Moreover, we will discuss separation conditions for a morphism of algebraic spaces in Morphisms of Spaces, Section 4.

**Definition 13.2.** Let  $S$  be a scheme contained in  $Sch_{fppf}$ . Let  $F$  be an algebraic space over  $S$ . Let  $\Delta : F \rightarrow F \times F$  be the diagonal morphism.

- (1) We say  $F$  is *separated over  $S$*  if  $\Delta$  is a closed immersion.
- (2) We say  $F$  is *locally separated over  $S$* <sup>2</sup> if  $\Delta$  is an immersion.
- (3) We say  $F$  is *quasi-separated over  $S$*  if  $\Delta$  is quasi-compact.
- (4) We say  $F$  is *Zariski locally quasi-separated over  $S$* <sup>3</sup> if there exists a Zariski covering  $F = \bigcup_{i \in I} F_i$  such that each  $F_i$  is quasi-separated.

Note that if the diagonal is quasi-compact (when  $F$  is separated or quasi-separated) then the diagonal is actually quasi-finite and separated, hence quasi-affine (by More on Morphisms, Lemma 43.2).

#### 14. Examples of algebraic spaces

In this section we construct some examples of algebraic spaces. Some of these were suggested by B. Conrad. Since we do not yet have a lot of theory at our disposal the discussion is a bit awkward in some places.

**Example 14.1.** Let  $k$  be a field of characteristic  $\neq 2$ . Let  $U = \mathbf{A}_k^1$ . Set

$$j : R = \Delta \amalg \Gamma \longrightarrow U \times_k U$$

where  $\Delta = \{(x, x) \mid x \in \mathbf{A}_k^1\}$  and  $\Gamma = \{(x, -x) \mid x \in \mathbf{A}_k^1, x \neq 0\}$ . It is clear that  $s, t : R \rightarrow U$  are étale, and hence  $j$  is an étale equivalence relation. The quotient  $X = U/R$  is an algebraic space by Theorem 10.5. Since  $R$  is quasi-compact we see that  $X$  is quasi-separated. On the other hand,  $X$  is not locally separated because the morphism  $j$  is not an immersion.

**Example 14.2.** Let  $k$  be a field. Let  $k'/k$  be a degree 2 Galois extension with  $\text{Gal}(k'/k) = \{1, \sigma\}$ . Let  $S = \text{Spec}(k[x])$  and  $U = \text{Spec}(k'[x])$ . Note that

$$U \times_S U = \text{Spec}((k' \otimes_k k')[x]) = \Delta(U) \amalg \Delta'(U)$$

where  $\Delta' = (1, \sigma) : U \rightarrow U \times_S U$ . Take

$$R = \Delta(U) \amalg \Delta'(U \setminus \{0_U\})$$

where  $0_U \in U$  denotes the  $k'$ -rational point whose  $x$ -coordinate is zero. It is easy to see that  $R$  is an étale equivalence relation on  $U$  over  $S$  and hence  $X = U/R$  is an algebraic space by Theorem 10.5. Here are some properties of  $X$  (some of which will not make sense until later):

- (1)  $X \rightarrow S$  is an isomorphism over  $S \setminus \{0_S\}$ ,
- (2) the morphism  $X \rightarrow S$  is étale (see Properties of Spaces, Definition 16.2)
- (3) the fibre  $0_X$  of  $X \rightarrow S$  over  $0_S$  is isomorphic to  $\text{Spec}(k') = 0_U$ ,
- (4)  $X$  is not a scheme because if it were, then  $\mathcal{O}_{X,0_X}$  would be a local domain  $(\mathcal{O}, \mathfrak{m}, \kappa)$  with fraction field  $k(x)$ , with  $x \in \mathfrak{m}$  and residue field  $\kappa = k'$  which is impossible,
- (5)  $X$  is not separated, but it is locally separated and quasi-separated,
- (6) there exists a surjective, finite, étale morphism  $S' \rightarrow S$  such that the base change  $X' = S' \times_S X$  is a scheme (namely, if we base change to  $S' = \text{Spec}(k'[x])$  then  $U$  splits into two copies of  $S'$  and  $X'$  becomes isomorphic to the affine line with 0 doubled, see Schemes, Example 14.3), and
- (7) if we think of  $X$  as a finite type algebraic space over  $\text{Spec}(k)$ , then similarly the base change  $X_{k'}$  is a scheme but  $X$  is not a scheme.

<sup>2</sup>In the literature this often refers to quasi-separated and locally separated algebraic spaces.

<sup>3</sup>This definition was suggested by B. Conrad.

In particular, this gives an example of a descent datum for schemes relative to the covering  $\{\mathrm{Spec}(k') \rightarrow \mathrm{Spec}(k)\}$  which is not effective.

See also Examples, Lemma 65.1, which shows that descent data need not be effective even for a projective morphism of schemes. That example gives a smooth separated algebraic space of dimension 3 over  $\mathbf{C}$  which is not a scheme.

We will use the following lemma as a convenient way to construct algebraic spaces as quotients of schemes by free group actions.

**Lemma 14.3.** *Let  $U \rightarrow S$  be a morphism of  $\mathrm{Sch}_{fppf}$ . Let  $G$  be an abstract group. Let  $G \rightarrow \mathrm{Aut}_S(U)$  be a group homomorphism. Assume*

*(\*) if  $u \in U$  is a point, and  $g(u) = u$  for some non-identity element  $g \in G$ , then  $g$  induces a nontrivial automorphism of  $\kappa(u)$ .*

*Then*

$$j : R = \coprod_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

*is an étale equivalence relation and hence*

$$F = U/R$$

*is an algebraic space by Theorem 10.5.*

**Proof.** In the statement of the lemma the symbol  $\mathrm{Aut}_S(U)$  denotes the group of automorphisms of  $U$  over  $S$ . Assume (\*) holds. Let us show that

$$j : R = \coprod_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

is a monomorphism. This signifies that if  $T$  is a nonempty scheme, and  $h : T \rightarrow U$  is a  $T$ -valued point such that  $g \circ h = g' \circ h$  then  $g = g'$ . Suppose  $T \neq \emptyset$ ,  $h : T \rightarrow U$  and  $g \circ h = g' \circ h$ . Let  $t \in T$ . Consider the composition  $\mathrm{Spec}(\kappa(t)) \rightarrow \mathrm{Spec}(\kappa(h(t))) \rightarrow U$ . Then we conclude that  $g^{-1} \circ g'$  fixes  $u = h(t)$  and acts as the identity on its residue field. Hence  $g = g'$  by (\*).

Thus if (\*) holds we see that  $j$  is a relation (see Groupoids, Definition 3.1). Moreover, it is an equivalence relation since on  $T$ -valued points for a connected scheme  $T$  we see that  $R(T) = G \times U(T) \rightarrow U(T) \times U(T)$  (recall that we always work over  $S$ ). Moreover, the morphisms  $s, t : R \rightarrow U$  are étale since  $R$  is a disjoint product of copies of  $U$ . This proves that  $j : R \rightarrow U \times_S U$  is an étale equivalence relation.  $\square$

Given a scheme  $U$  and an action of a group  $G$  on  $U$  we say the action of  $G$  on  $U$  is *free* if condition (\*) of Lemma 14.3 holds. This is equivalent to the notion of a free action of the constant group scheme  $G_S$  on  $U$  as defined in Groupoids, Definition 10.2. The lemma can be interpreted as saying that quotients of schemes by free actions of groups exist in the category of algebraic spaces.

**Definition 14.4.** Notation  $U \rightarrow S$ ,  $G$ ,  $R$  as in Lemma 14.3. If the action of  $G$  on  $U$  satisfies (\*) we say  $G$  *acts freely* on the scheme  $U$ . In this case the algebraic space  $U/R$  is denoted  $U/G$  and is called the *quotient of  $U$  by  $G$* .

This notation is consistent with the notation  $U/G$  introduced in Groupoids, Definition 20.1. We will later make sense of the quotient as an algebraic stack without any assumptions on the action whatsoever; when we do this we will use the notation  $[U/G]$ . Before we discuss the examples we prove some more lemmas to facilitate the discussion. Here is a lemma discussing the various separation conditions for this quotient when  $G$  is finite.

**Lemma 14.5.** *Notation and assumptions as in Lemma 14.3. Assume  $G$  is finite. Then*

- (1) *if  $U \rightarrow S$  is quasi-separated, then  $U/G$  is quasi-separated over  $S$ , and*
- (2) *if  $U \rightarrow S$  is separated, then  $U/G$  is separated over  $S$ .*

**Proof.** In the proof of Lemma 13.1 we saw that it suffices to prove the corresponding properties for the morphism  $j : R \rightarrow U \times_S U$ . If  $U \rightarrow S$  is quasi-separated, then for every affine open  $V \subset U$  which maps into an affine of  $S$  the opens  $g(V) \cap V$  are quasi-compact. It follows that  $j$  is quasi-compact. If  $U \rightarrow S$  is separated, the diagonal  $\Delta_{U/S}$  is a closed immersion. Hence  $j : R \rightarrow U \times_S U$  is a finite coproduct of closed immersions with disjoint images. Hence  $j$  is a closed immersion.  $\square$

**Lemma 14.6.** *Notation and assumptions as in Lemma 14.3. If  $\text{Spec}(k) \rightarrow U/G$  is a morphism, then there exist*

- (1) *a finite Galois extension  $k'/k$ ,*
- (2) *a finite subgroup  $H \subset G$ ,*
- (3) *an isomorphism  $H \rightarrow \text{Gal}(k'/k)$ , and*
- (4) *an  $H$ -equivariant morphism  $\text{Spec}(k') \rightarrow U$ .*

*Conversely, such data determine a morphism  $\text{Spec}(k) \rightarrow U/G$ .*

**Proof.** Consider the fibre product  $V = \text{Spec}(k) \times_{U/G} U$ . Here is a diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & U/G \end{array}$$

Then  $V$  is a nonempty scheme étale over  $\text{Spec}(k)$  and hence is a disjoint union  $V = \coprod_{i \in I} \text{Spec}(k_i)$  of spectra of fields  $k_i$  finite separable over  $k$  (Morphisms, Lemma 36.7). We have

$$\begin{aligned} V \times_{\text{Spec}(k)} V &= (\text{Spec}(k) \times_{U/G} U) \times_{\text{Spec}(k)} (\text{Spec}(k) \times_{U/G} U) \\ &= \text{Spec}(k) \times_{U/G} U \times_{U/G} U \\ &= \text{Spec}(k) \times_{U/G} U \times G \\ &= V \times G \end{aligned}$$

The action of  $G$  on  $U$  induces an action of  $a : G \times V \rightarrow V$ . The displayed equality means that  $G \times V \rightarrow V \times_{\text{Spec}(k)} V$ ,  $(g, v) \mapsto (a(g, v), v)$  is an isomorphism. In particular we see that for every  $i$  we have an isomorphism  $H_i \times \text{Spec}(k_i) \rightarrow \text{Spec}(k_i \otimes_k k_i)$  where  $H_i \subset G$  is the subgroup of elements fixing  $i \in I$ . Thus  $H_i$  is finite and is the Galois group of  $k_i/k$ . We omit the converse construction.  $\square$

It follows from this lemma for example that if  $k'/k$  is a finite Galois extension, then  $\text{Spec}(k')/\text{Gal}(k'/k) \cong \text{Spec}(k)$ . What happens if the extension is infinite? Here is an example.

**Example 14.7.** Let  $S = \text{Spec}(\mathbf{Q})$ . Let  $U = \text{Spec}(\overline{\mathbf{Q}})$ . Let  $G = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  with obvious action on  $U$ . Then by construction property (\*) of Lemma 14.3 holds and we obtain an algebraic space

$$X = \text{Spec}(\overline{\mathbf{Q}})/G \longrightarrow S = \text{Spec}(\mathbf{Q}).$$



Of course this is totally ridiculous as an approximation of  $S$ ! Namely, by the Artin-Schreier theorem, see [Jac64, Theorem 17, page 316], the only finite subgroups of  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  are  $\{1\}$  and the conjugates of the order two group  $\text{Gal}(\overline{\mathbf{Q}}/\overline{\mathbf{Q}} \cap \mathbf{R})$ . Hence, if  $\text{Spec}(k) \rightarrow X$  is a morphism with  $k$  algebraic over  $\mathbf{Q}$ , then it follows from Lemma 14.6 and the theorem just mentioned that either  $k$  is  $\overline{\mathbf{Q}}$  or isomorphic to  $\overline{\mathbf{Q}} \cap \mathbf{R}$ .

What is wrong with the example above is that the Galois group comes equipped with a topology, and this should somehow be part of any construction of a quotient of  $\text{Spec}(\overline{\mathbf{Q}})$ . The following example is much more reasonable in my opinion and may actually occur in “nature”.

**Example 14.8.** Let  $k$  be a field of characteristic zero. Let  $U = \mathbf{A}_k^1$  and let  $G = \mathbf{Z}$ . As action we take  $n(x) = x + n$ , i.e., the action of  $\mathbf{Z}$  on the affine line by translation. The only fixed point is the generic point and it is clearly the case that  $\mathbf{Z}$  injects into the automorphism group of the field  $k(x)$ . (This is where we use the characteristic zero assumption.) Consider the morphism

$$\gamma : \text{Spec}(k(x)) \longrightarrow X = \mathbf{A}_k^1 / \mathbf{Z}$$

of the generic point of the affine line into the quotient. We claim that this morphism does not factor through any monomorphism  $\text{Spec}(L) \rightarrow X$  of the spectrum of a field to  $X$ . (Contrary to what happens for schemes, see Schemes, Section 13.) In fact, since  $\mathbf{Z}$  does not have any nontrivial finite subgroups we see from Lemma 14.6 that for any such factorization  $k(x) = L$ . Finally,  $\gamma$  is not a monomorphism since

$$\text{Spec}(k(x)) \times_{\gamma, X, \gamma} \text{Spec}(k(x)) \cong \text{Spec}(k(x)) \times \mathbf{Z}.$$

This example suggests that in order to define points of an algebraic space  $X$  we should consider equivalence classes of morphisms from spectra of fields into  $X$  and not the set of monomorphisms from spectra of fields.

We finish with a truly awful example.

**Example 14.9.** Let  $k$  be a field. Let  $A = \prod_{n \in \mathbf{N}} k$  be the infinite product. Set  $U = \text{Spec}(A)$  seen as a scheme over  $S = \text{Spec}(k)$ . Note that the projection maps  $\text{pr}_n : A \rightarrow k$  define open and closed immersions  $f_n : S \rightarrow U$ . Set

$$R = U \amalg \coprod_{(n,m) \in \mathbf{N}^2, n \neq m} S$$

with morphism  $j$  equal to  $\Delta_{U/S}$  on the component  $U$  and  $j = (f_n, f_m)$  on the component  $S$  corresponding to  $(n, m)$ . It is clear from the remark above that  $s, t$  are étale. It is also clear that  $j$  is an equivalence relation. Hence we obtain an algebraic space

$$X = U/R.$$

To see what this means we specialize to the case where the field  $k$  is finite with  $q$  elements. Let us first discuss the topological space  $|U|$  associated to the scheme  $U$  a little bit. All elements of  $A$  satisfy  $x^q = x$ . Hence every residue field of  $A$  is isomorphic to  $k$ , and all points of  $U$  are closed. But the topology on  $U$  isn't the discrete topology. Let  $u_n \in |U|$  be the point corresponding to  $f_n$ . As mentioned above the points  $u_n$  are the open points (and hence isolated). This implies there have to be other points since we know  $U$  is quasi-compact, see Algebra, Lemma

17.10 (hence not equal to an infinite discrete set). Another way to see this is because the (proper) ideal

$$I = \{x = (x_n) \in A \mid \text{all but a finite number of } x_n \text{ are zero}\}$$

is contained in a maximal ideal. Note also that every element  $x$  of  $A$  is of the form  $x = ue$  where  $u$  is a unit and  $e$  is an idempotent. Hence a basis for the topology of  $A$  consists of open and closed subsets (see Algebra, Lemma 21.1.) So the topology on  $|U|$  is totally disconnected, but nontrivial. Finally, note that  $\{u_n\}$  is dense in  $|U|$ .

We will later define a topological space  $|X|$  associated to  $X$ , see Properties of Spaces, Section 4. What can we say about  $|X|$ ? It turns out that the map  $|U| \rightarrow |X|$  is surjective and continuous. All the points  $u_n$  map to the same point  $x_0$  of  $|X|$ , and none of the other points get identified. Since  $\{u_n\}$  is dense in  $|U|$  we conclude that the closure of  $x_0$  in  $|X|$  is  $|X|$ . In other words  $|X|$  is irreducible and  $x_0$  is a generic point of  $|X|$ . This seems bizarre since also  $x_0$  is the image of a section  $S \rightarrow X$  of the structure morphism  $X \rightarrow S$  (and in the case of schemes this would imply it was a closed point, see Morphisms, Lemma 20.2).

Whatever you think is actually going on in this example, it certainly shows that some care has to be exercised when defining irreducible components, connectedness, etc of algebraic spaces.

## 15. Change of big site

In this section we briefly discuss what happens when we change big sites. The upshot is that we can always enlarge the big site at will, hence we may assume any set of schemes we want to consider is contained in the big fppf site over which we consider our algebraic space. Here is a precise statement of the result.

**Lemma 15.1.** *Suppose given big sites  $Sch_{fppf}$  and  $Sch'_{fppf}$ . Assume that  $Sch_{fppf}$  is contained in  $Sch'_{fppf}$ , see Topologies, Section 12. Let  $S$  be an object of  $Sch_{fppf}$ . Let*

$$\begin{aligned} g : Sh((Sch/S)_{fppf}) &\longrightarrow Sh((Sch'/S)_{fppf}), \\ f : Sh((Sch'/S)_{fppf}) &\longrightarrow Sh((Sch/S)_{fppf}) \end{aligned}$$

*be the morphisms of topoi of Topologies, Lemma 12.2. Let  $F$  be a sheaf of sets on  $(Sch/S)_{fppf}$ . Then*

- (1) *if  $F$  is representable by a scheme  $X \in \text{Ob}((Sch/S)_{fppf})$  over  $S$ , then  $f^{-1}F$  is representable too, in fact it is representable by the same scheme  $X$ , now viewed as an object of  $(Sch'/S)_{fppf}$ , and*
- (2) *if  $F$  is an algebraic space over  $S$ , then  $f^{-1}F$  is an algebraic space over  $S$  also.*

**Proof.** Let  $X \in \text{Ob}((Sch/S)_{fppf})$ . Let us write  $h_X$  for the representable sheaf on  $(Sch/S)_{fppf}$  associated to  $X$ , and  $h'_X$  for the representable sheaf on  $(Sch'/S)_{fppf}$  associated to  $X$ . By the description of  $f^{-1}$  in Topologies, Section 12 we see that  $f^{-1}h_X = h'_X$ . This proves (1).

Next, suppose that  $F$  is an algebraic space over  $S$ . By Lemma 9.1 this means that  $F = h_U/h_R$  for some étale equivalence relation  $R \rightarrow U \times_S U$  in  $(Sch/S)_{fppf}$ .

Since  $f^{-1}$  is an exact functor we conclude that  $f^{-1}F = h'_U/h'_R$ . Hence  $f^{-1}F$  is an algebraic space over  $S$  by Theorem 10.5.  $\square$

Note that this lemma is purely set theoretical and has virtually no content. Moreover, it is not true (in general) that the restriction of an algebraic space over the bigger site is an algebraic space over the smaller site (simply by reasons of cardinality). Hence we can only ever use a simple lemma of this kind to enlarge the base category and never to shrink it.

**Lemma 15.2.** *Suppose  $Sch_{fppf}$  is contained in  $Sch'_{fppf}$ . Let  $S$  be an object of  $Sch_{fppf}$ . Denote  $Spaces/S$  the category of algebraic spaces over  $S$  defined using  $Sch_{fppf}$ . Similarly, denote  $Spaces'/S$  the category of algebraic spaces over  $S$  defined using  $Sch'_{fppf}$ . The construction of Lemma 15.1 defines a fully faithful functor*

$$Spaces/S \longrightarrow Spaces'/S$$

*whose essential image consists of those  $X' \in \text{Ob}(Spaces'/S)$  such that there exist  $U, R \in \text{Ob}((Sch/S)_{fppf})$ <sup>4</sup> and morphisms*

$$U \longrightarrow X' \quad \text{and} \quad R \longrightarrow U \times_{X'} U$$

*in  $Sh((Sch'/S)_{fppf})$  which are surjective as maps of sheaves (for example if the displayed morphisms are surjective and étale).*

**Proof.** In Sites, Lemma 21.8 we have seen that the functor  $f^{-1} : Sh((Sch/S)_{fppf}) \rightarrow Sh((Sch'/S)_{fppf})$  is fully faithful (see discussion in Topologies, Section 12). Hence we see that the displayed functor of the lemma is fully faithful.

Suppose that  $X' \in \text{Ob}(Spaces'/S)$  such that there exists  $U \in \text{Ob}((Sch/S)_{fppf})$  and a map  $U \rightarrow X'$  in  $Sh((Sch'/S)_{fppf})$  which is surjective as a map of sheaves. Let  $U' \rightarrow X'$  be a surjective étale morphism with  $U' \in \text{Ob}((Sch'/S)_{fppf})$ . Let  $\kappa = \text{size}(U)$ , see Sets, Section 9. Then  $U$  has an affine open covering  $U = \bigcup_{i \in I} U_i$  with  $|I| \leq \kappa$ . Observe that  $U' \times_{X'} U \rightarrow U$  is étale and surjective. For each  $i$  we can pick a quasi-compact open  $U'_i \subset U'$  such that  $U'_i \times_{X'} U_i \rightarrow U_i$  is surjective (because the scheme  $U' \times_{X'} U_i$  is the union of the Zariski opens  $W \times_{X'} U_i$  for  $W \subset U'$  affine and because  $U' \times_{X'} U_i \rightarrow U_i$  is étale hence open). Then  $\coprod_{i \in I} U'_i \rightarrow X$  is surjective étale because of our assumption that  $U \rightarrow X$  and hence  $\coprod U_i \rightarrow X$  is a surjection of sheaves (details omitted). Because  $U'_i \times_{X'} U \rightarrow U'_i$  is a surjection of sheaves and because  $U'_i$  is quasi-compact, we can find a quasi-compact open  $W_i \subset U'_i \times_{X'} U$  such that  $W_i \rightarrow U'_i$  is surjective as a map of sheaves (details omitted). Then  $W_i \rightarrow U$  is étale and we conclude that  $\text{size}(W_i) \leq \text{size}(U)$ , see Sets, Lemma 9.7. By Sets, Lemma 9.11 we conclude that  $\text{size}(U'_i) \leq \text{size}(U)$ . Hence  $\coprod_{i \in I} U'_i$  is isomorphic to an object of  $(Sch/S)_{fppf}$  by Sets, Lemma 9.5.

Now let  $X', U \rightarrow X'$  and  $R \rightarrow U \times_{X'} U$  be as in the statement of the lemma. In the previous paragraph we have seen that we can find  $U' \in \text{Ob}((Sch/S)_{fppf})$  and a surjective étale morphism  $U' \rightarrow X'$  in  $Sh((Sch'/S)_{fppf})$ . Then  $U' \times_{X'} U \rightarrow U'$  is a surjection of sheaves, i.e., we can find an fppf covering  $\{U'_i \rightarrow U'\}$  such that  $U'_i \rightarrow U'$  factors through  $U' \times_{X'} U \rightarrow U'$ . By Sets, Lemma 9.12 we can find  $\tilde{U} \rightarrow U'$

<sup>4</sup>Requiring the existence of  $R$  is necessary because of our choice of the function *Bound* in Sets, Equation (9.1.1). The size of the fibre product  $U \times_{X'} U$  can grow faster than *Bound* in terms of the size of  $U$ . We can illustrate this by setting  $S = \text{Spec}(A)$ ,  $U = \text{Spec}(A[x_i, i \in I])$  and  $R = \coprod_{(\lambda_i) \in A^I} \text{Spec}(A[x_i, y_i]/(x_i - \lambda_i y_i))$ . In this case the size of  $R$  grows like  $\kappa^\kappa$  where  $\kappa$  is the size of  $U$ .

which is surjective, flat, and locally of finite presentation, with  $\text{size}(\tilde{U}) \leq \text{size}(U')$ , such that  $\tilde{U} \rightarrow U'$  factors through  $U' \times_{X'} U \rightarrow U'$ . Then we consider

$$\begin{array}{ccccc} U' \times_{X'} U' & \longleftarrow & \tilde{U} \times_{X'} \tilde{U} & \longrightarrow & U \times_{X'} U \\ \downarrow & & \downarrow & & \downarrow \\ U' \times_S U' & \longleftarrow & \tilde{U} \times_S \tilde{U} & \longrightarrow & U \times_S U \end{array}$$

The squares are cartesian. We know the objects of the bottom row are represented by objects of  $(Sch/S)_{fppf}$ . By the result of the argument of the previous paragraph, the same is true for  $U \times_{X'} U$  (as we have the surjection of sheaves  $R \rightarrow U \times_{X'} U$  by assumption). Since  $(Sch/S)_{fppf}$  is closed under fibre products (by construction), we see that  $\tilde{U} \times_{X'} \tilde{U}$  is represented by an object of  $(Sch/S)_{fppf}$ . Finally, the map  $\tilde{U} \times_{X'} \tilde{U} \rightarrow U' \times_{X'} U'$  is a surjection of fppf sheaves as  $\tilde{U} \rightarrow U'$  is so. Thus we can once more apply the result of the previous paragraph to conclude that  $R' = U' \times_{X'} U'$  is represented by an object of  $(Sch/S)_{fppf}$ . At this point Lemma 9.1 and Theorem 10.5 imply that  $X = h_{U'}/h_{R'}$  is an object of  $Spaces/S$  such that  $f^{-1}X \cong X'$  as desired.  $\square$

## 16. Change of base scheme

In this section we briefly discuss what happens when we change base schemes. The upshot is that given a morphism  $S \rightarrow S'$  of base schemes, any algebraic space over  $S$  can be viewed as an algebraic space over  $S'$ . And, given an algebraic space  $F'$  over  $S'$  there is a base change  $F'_S$  which is an algebraic space over  $S$ . We explain only what happens in case  $S \rightarrow S'$  is a morphism of the big fppf site under consideration, if only  $S$  or  $S'$  is contained in the big site, then one first enlarges the big site as in Section 15.

**Lemma 16.1.** *Suppose given a big site  $Sch_{fppf}$ . Let  $g : S \rightarrow S'$  be morphism of  $Sch_{fppf}$ . Let  $j : (Sch/S)_{fppf} \rightarrow (Sch/S')_{fppf}$  be the corresponding localization functor. Let  $F$  be a sheaf of sets on  $(Sch/S)_{fppf}$ . Then*

- (1) *for a scheme  $T'$  over  $S'$  we have  $j_! F(T'/S') = \coprod_{\varphi : T' \rightarrow S} F(T' \xrightarrow{\varphi} S)$ ,*
- (2) *if  $F$  is representable by a scheme  $X \in \text{Ob}((Sch/S)_{fppf})$ , then  $j_! F$  is representable by  $j(X)$  which is  $X$  viewed as a scheme over  $S'$ , and*
- (3) *if  $F$  is an algebraic space over  $S$ , then  $j_! F$  is an algebraic space over  $S'$ , and if  $F = U/R$  is a presentation, then  $j_! F = j(U)/j(R)$  is a presentation.*

*Let  $F'$  be a sheaf of sets on  $(Sch/S')_{fppf}$ . Then*

- (4) *for a scheme  $T$  over  $S$  we have  $j^{-1} F'(T/S') = F'(T/S)$ ,*
- (5) *if  $F'$  is representable by a scheme  $X' \in \text{Ob}((Sch/S')_{fppf})$ , then  $j^{-1} F'$  is representable, namely by  $X'_S = S \times_{S'} X'$ , and*
- (6) *if  $F'$  is an algebraic space, then  $j^{-1} F'$  is an algebraic space, and if  $F' = U'/R'$  is a presentation, then  $j^{-1} F' = U'_S/R'_S$  is a presentation.*

**Proof.** The functors  $j_!$ ,  $j_*$  and  $j^{-1}$  are defined in Sites, Lemma 25.8 where it is also shown that  $j = j_{S/S'}$  is the localization of  $(Sch/S')_{fppf}$  at the object  $S/S'$ . Hence all of the material on localization functors is available for  $j$ . The formula in (1) is Sites, Lemma 27.1. By definition  $j_!$  is the left adjoint to restriction  $j^{-1}$ , hence  $j_!$  is right exact. By Sites, Lemma 25.5 it also commutes with fibre products and equalizers. By Sites, Lemma 25.3 we see that  $j_! h_X = h_{j(X)}$  hence (2) holds. If

$F$  is an algebraic space over  $S$ , then we can write  $F = U/R$  (Lemma 9.1) and we get

$$j_!F = j(U)/j(R)$$

because  $j_!$  being right exact commutes with coequalizers, and moreover  $j(R) = j(U) \times_{j_!F} j(U)$  as  $j_!$  commutes with fibre products. Since the morphisms  $j(s), j(t) : j(R) \rightarrow j(U)$  are simply the morphisms  $s, t : R \rightarrow U$  (but viewed as morphisms of schemes over  $S'$ ), they are still étale. Thus  $(j(U), j(R), s, t)$  is an étale equivalence relation. Hence by Theorem 10.5 we conclude that  $j_!F$  is an algebraic space.

Proof of (4), (5), and (6). The description of  $j^{-1}$  is in Sites, Section 25. The restriction of the representable sheaf associated to  $X'/S'$  is the representable sheaf associated to  $X'_S = S \times_{S'} Y'$  by Sites, Lemma 27.2. The restriction functor  $j^{-1}$  is exact, hence  $j^{-1}F' = U'_S/R'_S$ . Again by exactness the sheaf  $R'_S$  is still an equivalence relation on  $U'_S$ . Finally the two maps  $R'_S \rightarrow U'_S$  are étale as base changes of the étale morphisms  $R' \rightarrow U'$ . Hence  $j^{-1}F' = U'_S/R'_S$  is an algebraic space by Theorem 10.5 and we win.  $\square$

Note how the presentation  $j_!F = j(U)/j(R)$  is just the presentation of  $F$  but viewed as a presentation by schemes over  $S'$ . Hence the following definition makes sense.

**Definition 16.2.** Let  $Sch_{fppf}$  be a big fppf site. Let  $S \rightarrow S'$  be a morphism of this site.

- (1) If  $F'$  is an algebraic space over  $S'$ , then the *base change of  $F'$  to  $S$*  is the algebraic space  $j^{-1}F'$  described in Lemma 16.1. We denote it  $F'_S$ .
- (2) If  $F$  is an algebraic space over  $S$ , then  $F$  *viewed as an algebraic space over  $S'$*  is the algebraic space  $j_!F$  over  $S'$  described in Lemma 16.1. We often simply denote this  $F$ ; if not then we will write  $j_!F$ .

The algebraic space  $j_!F$  comes equipped with a canonical morphism  $j_!F \rightarrow S$  of algebraic spaces over  $S'$ . This is true simply because the sheaf  $j_!F$  maps to  $h_S$  (see for example the explicit description in Lemma 16.1). In fact, in Sites, Lemma 25.4 we have seen that the category of sheaves on  $(Sch/S)_{fppf}$  is equivalent to the category of pairs  $(\mathcal{F}', \mathcal{F}' \rightarrow h_S)$  consisting of a sheaf on  $(Sch/S')_{fppf}$  and a map of sheaves  $\mathcal{F}' \rightarrow h_S$ . The equivalence assigns to the sheaf  $\mathcal{F}$  the pair  $(j_!\mathcal{F}, j_!\mathcal{F} \rightarrow h_S)$ . This, combined with the above, leads to the following result for categories of algebraic spaces.

**Lemma 16.3.** Let  $Sch_{fppf}$  be a big fppf site. Let  $S \rightarrow S'$  be a morphism of this site. The construction above give an equivalence of categories

$$\left\{ \begin{array}{l} \text{category of algebraic} \\ \text{spaces over } S \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{category of pairs } (F', F' \rightarrow S) \text{ consisting} \\ \text{of an algebraic space } F' \text{ over } S' \text{ and a} \\ \text{morphism } F' \rightarrow S \text{ of algebraic spaces over } S' \end{array} \right\}$$

**Proof.** Let  $F$  be an algebraic space over  $S$ . The functor from left to right assigns the pair  $(j_!F, j_!F \rightarrow S)$  of  $F$  which is an object of the right hand side by Lemma 16.1. Since this defines an equivalence of categories of sheaves by Sites, Lemma 25.4 to finish the proof it suffices to show: if  $F$  is a sheaf and  $j_!F$  is an algebraic space, then  $F$  is an algebraic space. To do this, write  $j_!F = U'/R'$  as in Lemma 9.1 with  $U', R' \in \text{Ob}((Sch/S')_{fppf})$ . Then the compositions  $U' \rightarrow j_!F \rightarrow S$  and  $R' \rightarrow j_!F \rightarrow S$  are morphisms of schemes over  $S'$ . Denote  $U, R$  the corresponding objects of  $(Sch/S)_{fppf}$ . The two morphisms  $R' \rightarrow U'$  are morphisms over  $S$  and

hence correspond to morphisms  $R \rightarrow U$ . Since these are simply the same morphisms (but viewed over  $S$ ) we see that we get an étale equivalence relation over  $S$ . As  $j_!$  defines an equivalence of categories of sheaves (see reference above) we see that  $F = U/R$  and by Theorem 10.5 we see that  $F$  is an algebraic space.  $\square$

The following lemma is a slight rephrasing of the above.

**Lemma 16.4.** *Let  $Sch_{fppf}$  be a big fppf site. Let  $S \rightarrow S'$  be a morphism of this site. Let  $F'$  be a sheaf on  $(Sch/S')_{fppf}$ . The following are equivalent:*

- (1) *The restriction  $F'|_{(Sch/S)_{fppf}}$  is an algebraic space over  $S$ , and*
- (2) *the sheaf  $h_S \times F'$  is an algebraic space over  $S'$ .*

**Proof.** The restriction and the product match under the equivalence of categories of Sites, Lemma 25.4 so that Lemma 16.3 above gives the result.  $\square$

We finish this section with a lemma on a compatibility.

**Lemma 16.5.** *Let  $Sch_{fppf}$  be a big fppf site. Let  $S \rightarrow S'$  be a morphism of this site. Let  $F$  be an algebraic space over  $S$ . Let  $T$  be a scheme over  $S$  and let  $f : T \rightarrow F$  be a morphism over  $S$ . Let  $f' : T' \rightarrow F'$  be the morphism over  $S'$  we get from  $f$  by applying the equivalence of categories described in Lemma 16.3. For any property  $\mathcal{P}$  as in Definition 5.1 we have  $\mathcal{P}(f') \Leftrightarrow \mathcal{P}(f)$ .*

**Proof.** Suppose that  $U$  is a scheme over  $S$ , and  $U \rightarrow F$  is a surjective étale morphism. Denote  $U'$  the scheme  $U$  viewed as a scheme over  $S'$ . In Lemma 16.1 we have seen that  $U' \rightarrow F'$  is surjective étale. Since

$$j(T \times_{f,F} U) = T' \times_{f',F'} U'$$

the morphism of schemes  $T \times_{f,F} U \rightarrow U$  is identified with the morphism of schemes  $T' \times_{f',F'} U' \rightarrow U'$ . It is the same morphism, just viewed over different base schemes. Hence the lemma follows from Lemma 11.4.  $\square$

## 17. Other chapters

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