MORE ON MORPHISMS OF STACKS

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1. Introduction

In this chapter we continue our study of properties of morphisms of algebraic stacks. A reference in the case of quasi-separated algebraic stacks with representable diagonal is [LMB00].

2. Conventions and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2.

3. Thickenings

The following terminology may not be completely standard, but it is convenient. If $\mathcal{Y}$ is a closed substack of an algebraic stack $\mathcal{X}$, then the morphism $\mathcal{Y} \to \mathcal{X}$ is representable.

Definition 3.1. Thickenings.

(1) We say an algebraic stack $\mathcal{X}'$ is a *thickening* of an algebraic stack $\mathcal{X}$ if $\mathcal{X}$ is a closed substack of $\mathcal{X}'$ and the associated topological spaces are equal.

(2) Given two thickenings $\mathcal{X} \subset \mathcal{X}'$ and $\mathcal{Y} \subset \mathcal{Y}'$ a *morphism of thickenings* is a morphism $f' : \mathcal{X}' \to \mathcal{Y}'$ of algebraic stacks such that $f'|_\mathcal{X}$ factors through the closed substack $\mathcal{Y}$. In this situation we set $f = f'|_\mathcal{X} : \mathcal{X} \to \mathcal{Y}$ and we say that $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ is a morphism of thickenings.
(3) Let $\mathcal{Z}$ be an algebraic stack. We similarly define *thickenings over* $\mathcal{Z}$ and *morphisms of thickenings over* $\mathcal{Z}$. This means that the algebraic stacks $\mathcal{X}'$ and $\mathcal{Y}'$ are endowed with a structure morphism to $\mathcal{Z}$ and that f' fits into a suitable 2-commutative diagram of algebraic stacks.

Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Let U' be a scheme and let $U' \to \mathcal{X}'$ be a surjective smooth morphism. Setting $U = \mathcal{X} \times_{\mathcal{X}'} U'$ we obtain a morphism of thickenings

$$(U \subset U') \longrightarrow (\mathcal{X} \subset \mathcal{X}')$$

and $U \to \mathcal{X}$ is a surjective smooth morphism. We can often deduce properties of the thickening $\mathcal{X} \subset \mathcal{X}'$ from the corresponding properties of the thickening $U \subset U'$. Sometimes, by abuse of language, we say that a morphism $\mathcal{X} \to \mathcal{X}'$ is a thickening if it is a closed immersion inducing a bijection $|\mathcal{X}| \to |\mathcal{X}'|$.

Lemma 3.2. *Let $i : \mathcal{X} \to \mathcal{X}'$ be a morphism of algebraic stacks. The following are equivalent*

(1) *i is a thickening of algebraic stacks (abuse of language as above), and*
(2) *i is representable by algebraic spaces and is a thickening in the sense of Properties of Stacks, Section 3.*

In this case i is a closed immersion and a universal homeomorphism.

Proof. By More on Morphisms of Spaces, Lemmas 9.10 and 9.8 the property P that a morphism of algebraic spaces is a (first order) thickening is fpqc local on the base and stable under base change. Thus the discussion in Properties of Stacks, Section 3 indeed applies. Having said this the equivalence of (1) and (2) follows from the fact that $P = P_1 + P_2$ where P_1 is the property of being a closed immersion and P_2 is the property of being surjective. (Strictly speaking, the reader should also consult More on Morphisms of Spaces, Definition 9.1, Properties of Stacks, Definition 9.1 and the discussion following, Morphisms of Spaces, Lemma 5.1, Properties of Stacks, Section 5 to see that all the concepts all match up.) The final assertion is clear from the foregoing. $\square$

We will use the lemma without further mention. Using the same references More on Morphisms of Spaces, Lemmas 9.10 and 9.8 as used in the lemma, allows us to define a first order thickening as follows.

Definition 3.3. We say an algebraic stack $\mathcal{X}'$ is a *first order thickening* of an algebraic stack $\mathcal{X}$ if $\mathcal{X}$ is a closed substack of $\mathcal{X}'$ and $\mathcal{X} \to \mathcal{X}'$ is a first order thickening in the sense of Properties of Stacks, Section 3.

If $(U \subset U') \to (\mathcal{X} \subset \mathcal{X}')$ is a smooth cover by a scheme as above, then this simply means that $U \subset U'$ is a first order thickening. Next we formulate the obligatory lemmas.

Lemma 3.4. *Let $\mathcal{Y} \subset \mathcal{Y}'$ be a thickening of algebraic stacks. Let $\mathcal{X}' \to \mathcal{Y}'$ be a morphism of algebraic stacks and set $\mathcal{X} = \mathcal{Y} \times_{\mathcal{Y}'} \mathcal{X}'$. Then $(\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ is a morphism of thickenings. If $\mathcal{Y} \subset \mathcal{Y}'$ is a first order thickening, then $\mathcal{X} \subset \mathcal{X}'$ is a first order thickening.*

Proof. See discussion above, Properties of Stacks, Section 3, and More on Morphisms of Spaces, Lemma 9.8. $\square$

Lemma 3.5. *If $\mathcal{X} \subset \mathcal{X}'$ and $\mathcal{X}' \subset \mathcal{X}''$ are thickenings of algebraic stacks, then so is $\mathcal{X} \subset \mathcal{X}''$.*

Proof. See discussion above, Properties of Stacks, Section 3, and More on Morphisms of Spaces, Lemma 9.9 $\square$

Example 3.6. Let $\mathcal{X}'$ be an algebraic stack. Then $\mathcal{X}'$ is a thickening of the reduction $\mathcal{X}'_{red}$, see Properties of Stacks, Definition 10.4. Moreover, if $\mathcal{X} \subset \mathcal{X}'$ is a thickening of algebraic stacks, then $\mathcal{X}'_{red} = \mathcal{X}_{red} \subset \mathcal{X}$. In other words, $\mathcal{X} = \mathcal{X}'_{red}$ if and only if $\mathcal{X}$ is a reduced algebraic stack.

Lemma 3.7. *Let $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ be a morphism of thickenings of algebraic stacks. Then $\mathcal{X} \times_\mathcal{Y} \mathcal{X} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$ is a thickening and the canonical diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Delta"'] \arrow[d] & \mathcal{X} \times_\mathcal{Y} \mathcal{X} \arrow[d] \\ \mathcal{X}' \arrow[r, "\Delta'"] & \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}' \end{tikzcd}$$

is cartesian.

Proof. Since $\mathcal{X} \to \mathcal{Y}'$ factors through the closed substack $\mathcal{Y}$ we see that $\mathcal{X} \times_\mathcal{Y} \mathcal{X} = \mathcal{X} \times_{\mathcal{Y}'} \mathcal{X}$. Hence $\mathcal{X} \times_\mathcal{Y} \mathcal{X} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$ is isomorphic to the composition

$$\mathcal{X} \times_{\mathcal{Y}'} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}'} \mathcal{X}' \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$$

both of which are thickenings as base changes of thickenings (Lemma 3.4). Hence so is the composition (Lemma 3.5). Since $\mathcal{X} \to \mathcal{X}'$ is a monomorphism, the final statement of the lemma follows from Properties of Stacks, Lemma 8.6 applied to $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}'$. $\square$

Lemma 3.8. *Let $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ be a morphism of thickenings of algebraic stacks. Let $\Delta : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ and $\Delta' : \mathcal{X}' \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}'$ be the corresponding diagonal morphisms. Then each property from the following list is satisfied by Δ if and only if it is satisfied by Δ': (a) representable by schemes, (b) affine, (c) surjective, (d) quasi-compact, (e) universally closed, (f) integral, (g) quasi-separated, (h) separated, (i) universally injective, (j) universally open, (k) locally quasi-finite, (l) finite, (m) unramified, (n) monomorphism, (o) immersion, (p) closed immersion, and (q) proper.*

Proof. Observe that

$$(\Delta, \Delta') : (\mathcal{X} \subset \mathcal{X}') \longrightarrow (\mathcal{X} \times_\mathcal{Y} \mathcal{X} \subset \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}')$$

is a morphism of thickenings (Lemma 3.7). Moreover Δ and Δ' are representable by algebraic spaces by Morphisms of Stacks, Lemma 3.3. Hence, via the discussion in Properties of Stacks, Section 3 the lemma follows for cases (a), (b), (c), (d), (e), (f), (g), (h), (i), and (j) by using More on Morphisms of Spaces, Lemma 10.1.

Lemma 3.7 tells us that $\mathcal{X} = (\mathcal{X} \times_\mathcal{Y} \mathcal{X}) \times_{(\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}')} \mathcal{X}'$. Moreover, Δ and Δ' are locally of finite type by the aforementioned Morphisms of Stacks, Lemma 3.3. Hence the result for cases (k), (l), (m), (n), (o), (p), and (q) by using More on Morphisms of Spaces, Lemma 10.3. $\square$

As a consequence we obtain the following pleasing result.

Lemma 3.9. *Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Then*

(1) *$\mathcal{X}$ is an algebraic space if and only if $\mathcal{X}'$ is an algebraic space,*
(2) *$\mathcal{X}$ is a scheme if and only if $\mathcal{X}'$ is a scheme,*
(3) *$\mathcal{X}$ is DM if and only if $\mathcal{X}'$ is DM,*
(4) *$\mathcal{X}$ is quasi-DM if and only if $\mathcal{X}'$ is quasi-DM,*
(5) *$\mathcal{X}$ is separated if and only if $\mathcal{X}'$ is separated,*
(6) *$\mathcal{X}$ is quasi-separated if and only if $\mathcal{X}'$ is quasi-separated, and*
(7) *add more here.*

Proof. In each case we reduce to a question about the diagonal and then we use Lemma 3.8 applied to the morphism of thickenings

$$(\mathcal{X} \subset \mathcal{X}') \to (\mathrm{Spec}(\mathbf{Z}) \subset \mathrm{Spec}(\mathbf{Z}))$$

We do this after viewing $\mathcal{X} \subset \mathcal{X}'$ as a thickening of algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ via Algebraic Stacks, Definition 19.2.

Case (1). An algebraic stack is an algebraic space if and only if its diagonal is a monomorphism, see Morphisms of Stacks, Lemma 6.3 (this also follows immediately from Algebraic Stacks, Proposition 13.3).

Case (2). By (1) we may assume that $\mathcal{X}$ and $\mathcal{X}'$ are algebraic spaces and then we can use More on Morphisms of Spaces, Lemma 9.5.

Case (3) – (6). Each of these cases corresponds to a condition on the diagonal, see Morphisms of Stacks, Definitions 4.1 and 4.2. $\square$

4. Morphisms of thickenings

If $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ is a morphism of thickenings of algebraic stacks, then often properties of the morphism f are inherited by f'. There are several variants.

Lemma 4.1. *Let $(f, f') : (\mathcal{X} \subset \mathcal{X}') \to (\mathcal{Y} \subset \mathcal{Y}')$ be a morphism of thickenings of algebraic stacks. Then*

(1) *f is an affine morphism if and only if f' is an affine morphism,*
(2) *f is a surjective morphism if and only if f' is a surjective morphism,*
(3) *f is quasi-compact if and only if f' quasi-compact,*
(4) *f is universally closed if and only if f' is universally closed,*
(5) *f is integral if and only if f' is integral,*
(6) *f is universally injective if and only if f' is universally injective,*
(7) *f is universally open if and only if f' is universally open,*
(8) *f is quasi-DM if and only if f' is quasi-DM,*
(9) *f is DM if and only if f' is DM,*
(10) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
(11) *f is representable if and only if f' is representable,*
(12) *f is representable by algebraic spaces if and only if f' is representable by algebraic spaces,*
(13) *add more here.*

Proof. By Lemma 3.2 the morphisms $\mathcal{X} \to \mathcal{X}'$ and $\mathcal{Y} \to \mathcal{Y}'$ are universal homeomorphisms. Thus any condition on $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is equivalent with the corresponding condition on $|f'| : |\mathcal{X}'| \to |\mathcal{Y}'|$ and the same is true after arbitrary base change by a morphism $\mathcal{Z}' \to \mathcal{Y}'$. This proves that (2), (3), (4), (6), (7) hold.

In cases (8), (9), (10), (12) we can translate the conditions on f and f' into conditions on the diagonals Δ and Δ' as in Lemma 3.8. See Morphisms of Stacks, Definition 4.1 and Lemma 6.3. Hence these cases follow from Lemma 3.8.

Proof of (11). If f' is representable, then so is f, because for a scheme T and a morphism $T \to \mathcal{Y}$ we have $\mathcal{X} \times_\mathcal{Y} T = \mathcal{X} \times_{\mathcal{X}'} (\mathcal{X}' \times_{\mathcal{Y}'} T)$ and $\mathcal{X} \to \mathcal{X}'$ is a closed immersion (hence representable). Conversely, assume f is representable, and let $T' \to \mathcal{Y}'$ be a morphism where T' is a scheme. Then

$$\mathcal{X} \times_\mathcal{Y} (\mathcal{Y} \times_{\mathcal{Y}'} T') = \mathcal{X} \times_{\mathcal{X}'} (\mathcal{X}' \times_{\mathcal{Y}'} T') \to \mathcal{X}' \times_{\mathcal{Y}'} T'$$

is a thickening (by Lemma 3.4) and the source is a scheme. Hence the target is a scheme by Lemma 3.9.

In cases (1) and (5) if either f or f' has the stated property, then both f and f' are representable by (11). In this case choose an algebraic space V' and a surjective smooth morphism $V' \to \mathcal{Y}'$. Set $V = \mathcal{Y} \times_{\mathcal{Y}'} V'$, $U' = \mathcal{X}' \times_{\mathcal{Y}'} V'$, and $U = \mathcal{X} \times_{\mathcal{Y}'} V'$. Then the desired results follow from the corresponding results for the morphism $(U \subset U') \to (V \subset V')$ of thickenings of algebraic spaces via the principle of Properties of Stacks, Lemma 3.3. See More on Morphisms of Spaces, Lemma 10.1 for the corresponding results in the case of algebraic spaces. $\square$

5. Infinitesimal deformations of algebraic stacks

This section is the analogue of More on Morphisms of Spaces, Section 18.

Lemma 5.1. *Consider a commutative diagram*

$$\begin{tikzcd} (\mathcal{X} \subset \mathcal{X}') \arrow[rr, "(f,f')"'] \arrow[dr] & & (\mathcal{Y} \subset \mathcal{Y}') \arrow[dl] \\ & (\mathcal{B} \subset \mathcal{B}') & \end{tikzcd}$$

of thickenings of algebraic stacks. Assume

1. $\mathcal{Y}' \to \mathcal{B}'$ *is locally of finite type,*
2. $\mathcal{X}' \to \mathcal{B}'$ *is flat and locally of finite presentation,*
3. f *is flat, and*
4. $\mathcal{X} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{X}'$ *and* $\mathcal{Y} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{Y}'$.

Then f' is flat and for all $y' \in |\mathcal{Y}'|$ in the image of $|f'|$ the morphism $\mathcal{Y}' \to \mathcal{B}'$ is flat at y'.

Proof. Choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$. Choose an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$. Choose an algebraic space W' and a surjective smooth morphism $W' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then flatness of f' is equivalent to flatness of $W' \to V'$ and we are given that $W \to V$ is flat. Hence we may apply the lemma in the case of algebraic spaces to the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[dr] & & (V \subset V') \arrow[dl] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of algebraic spaces. See More on Morphisms of Spaces, Lemma 18.4. The statement about flatness of $\mathcal{Y}'/\mathcal{B}'$ at points in the image of $|f'|$ follows in the same manner. □

Lemma 5.2. *Consider a commutative diagram*

$$\begin{tikzcd} (\mathcal{X} \subset \mathcal{X}') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (\mathcal{Y} \subset \mathcal{Y}') \arrow[dl] \\ & (\mathcal{B} \subset \mathcal{B}') & \end{tikzcd}$$

of thickenings of algebraic stacks. Assume $\mathcal{Y}' \to \mathcal{B}'$ locally of finite type, $\mathcal{X}' \to \mathcal{B}'$ flat and locally of finite presentation, $\mathcal{X} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{X}'$, and $\mathcal{Y} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{Y}'$. Then

(1) *f is flat if and only if f' is flat,*
(2) *f is an isomorphism if and only if f' is an isomorphism,*
(3) *f is an open immersion if and only if f' is an open immersion,*
(4) *f is a monomorphism if and only if f' is a monomorphism,*
(5) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
(6) *f is syntomic if and only if f' is syntomic,*
(7) *f is smooth if and only if f' is smooth,*
(8) *f is unramified if and only if f' is unramified,*
(9) *f is étale if and only if f' is étale,*
(10) *f is finite if and only if f' is finite, and*
(11) *add more here.*

Proof. In case (1) this follows from Lemma 5.1.

In cases (6), (7) this can be proved by the method used in the proof of Lemma 5.1. Namely, choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$. Choose an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$. Choose an algebraic space W' and a surjective smooth morphism $W' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then the property for f, resp. f' is equivalent to the property for of $W' \to V'$, resp. $W \to V$. Hence we may apply the lemma in the case of algebraic spaces to the diagram

$$\begin{tikzcd} (W \subset W') \arrow[rr] \arrow[dr] & & (V \subset V') \arrow[dl] \\ & (U \subset U') & \end{tikzcd}$$

of thickenings of algebraic spaces. See More on Morphisms of Spaces, Lemma 18.5.

In cases (8) and (9) we first see that the assumption for f or f' implies that both f and f' are DM morphisms of algebraic stacks, see Lemma 4.1. Then we can choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$. Choose an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$. Choose an algebraic space W' and a surjective étale(!) morphism $W' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then $W \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective étale as well. Hence the property for f, resp. f' is equivalent to the property for of $W' \to V'$, resp. $W \to V$. Hence we may apply the lemma in the case of algebraic spaces to the diagram

$$\begin{tikzcd}
(W \subset W') \arrow[rr] \arrow[dr] & & (V \subset V') \arrow[dl] \\
& (U \subset U') &
\end{tikzcd}$$

of thickenings of algebraic spaces. See More on Morphisms of Spaces, Lemma 18.5.

In cases (2), (3), (4), (10) we first conclude by Lemma 4.1 that f and f' are representable by algebraic spaces. Thus we may choose an algebraic space U' and a surjective smooth morphism $U' \to \mathcal{B}'$, an algebraic space V' and a surjective smooth morphism $V' \to U' \times_{\mathcal{B}'} \mathcal{Y}'$, and then $W' = V' \times_{\mathcal{Y}'} \mathcal{X}'$ will be an algebraic space. Let U, V, W be the base change of U', V', W' by $\mathcal{B} \to \mathcal{B}'$. Then $W = V \times_{\mathcal{Y}} \mathcal{X}$ as well. Then we have to see that $W' \to V'$ is an isomorphism, resp. an open immersion, resp. a monomorphism, resp. finite, if and only if $W \to V$ has the same property. See Properties of Stacks, Lemma 3.3. Thus we conclude by applying the results for algebraic spaces as above.

In the case (5) we first observe that f and f' are locally of finite type by Morphisms of Stacks, Lemma 17.8. On the other hand, the morphism f is quasi-DM if and only if f' is by Lemma 4.1. The last thing to check to see if f or f' is locally quasi-finite (Morphisms of Stacks, Definition 23.2) is a condition on underlying topological spaces which holds for f if and only if it holds for f' by the discussion in the first paragraph of the proof. $\square$

6. Lifting affines

Consider a solid diagram

$$\begin{tikzcd}
W \arrow[r, dotted] \arrow[d] & W' \arrow[d, dotted] \\
\mathcal{X} \arrow[r] & \mathcal{X}'
\end{tikzcd}$$

where $\mathcal{X} \subset \mathcal{X}'$ is a thickening of algebraic stacks, W is an affine scheme and $W \to \mathcal{X}$ is smooth. The question we address in this section is whether we can find W' and the dotted arrows so that the square is cartesian and $W' \to \mathcal{X}'$ is smooth. We do not know the answer in general, but if $\mathcal{X} \subset \mathcal{X}'$ is a first order thickening we will prove the answer is yes.

To study this problem we introduce the following category.

Remark 6.1 (Category of lifts)**.** Consider a diagram

$$\begin{tikzcd}
W \arrow[d, "x"'] & \\
\mathcal{X} \arrow[r] & \mathcal{X}'
\end{tikzcd}$$

where $\mathcal{X} \subset \mathcal{X}'$ is a thickening of algebraic stacks, W is an algebraic space, and $W \to \mathcal{X}$ is smooth. We will construct a category $\mathcal{C}$ and a functor

$$p : \mathcal{C} \longrightarrow W_{spaces,étale}$$

(see Properties of Spaces, Definition 18.2 for notation) as follows. An object of $\mathcal{C}$ will be a system $(U, U', a, i, x', \alpha)$ which forms a commutative diagram

(6.1.1)
$$\begin{tikzcd}
U \arrow[r, "i"'] \arrow[d, "a"'] & U' \arrow[dd, "x'"] \\
W \arrow[d, "x"'] & \\
\mathcal{X} \arrow[r] & \mathcal{X}'
\end{tikzcd}$$

with commutativity witnessed by the 2-morphism $\alpha : x \circ a \to x' \circ i$ such that U and U' are algebraic spaces, $a : U \to W$ is étale, $x' : U' \to \mathcal{X}'$ is smooth, and such that $U = \mathcal{X} \times_{\mathcal{X}'} U'$. In particular $U \subset U'$ is a thickening. A morphism

$$(U, U', a, i, x', \alpha) \to (V, V', b, j, y', \beta)$$

is given by (f, f', γ) where $f : U \to V$ is a morphism over W, $f' : U' \to V'$ is a morphism whose restriction to U gives f, and $\gamma : x' \circ f' \to y'$ is a 2-morphism witnessing the commutativity in right triangle of the diagram below

(6.1.2)
$$\begin{tikzcd}
& V \arrow[rr, "j"'] \arrow[dl, "f"'] \arrow[ddl, "b"] & & V' \arrow[dl, "f'"'] \arrow[dddl, "y'"] \\
U \arrow[rr, "i"'] \arrow[d, "a"'] & & U' \arrow[dd, "x'"'] & \\
W \arrow[d, "x"'] & & & \\
\mathcal{X} \arrow[rr] & & \mathcal{X}' &
\end{tikzcd}$$

Finally, we require that γ is compatible with α and β: in the calculus of 2-categories of Categories, Sections 28 and 29 this reads

$$\beta = (\gamma \star \mathrm{id}_j) \circ (\alpha \star \mathrm{id}_f)$$

(more succinctly: $\beta = j^*\gamma \circ f^*\alpha$). Another formulation is that objects are commutative diagrams (6.1.1) with some additional properties and morphisms are commutative diagrams (6.1.2) in the category *Spaces*$/\mathcal{X}'$ introduced in Properties of Stacks, Remark 3.7. This makes it clear that $\mathcal{C}$ is a category and that the rule $p : \mathcal{C} \to W_{spaces,étale}$ sending $(U, U', a, i, x', \alpha)$ to $a : U \to W$ is a functor.

Lemma 6.2. *For any morphism (6.1.2) the map $f' : V' \to U'$ is étale.*

Proof. Namely $f : V \to U$ is étale as a morphism in $W_{spaces,étale}$ and we can apply Lemma 5.2 because $U' \to \mathcal{X}'$ and $V' \to \mathcal{X}'$ are smooth and $U = \mathcal{X} \times_{\mathcal{X}'} U'$ and $V = \mathcal{X} \times_{\mathcal{X}'} V'$. $\square$

Lemma 6.3. *The category $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 6.1 is fibred in groupoids.*

Proof. We claim the fibre categories of p are groupoids. If (f, f', γ') as in (6.1.2) is a morphism such that $f : U \to V$ is an isomorphism, then f' is an isomorphism by Lemma 5.2 and hence (f, f', γ') is an isomorphism.

Consider a morphism $f : V \to U$ in $W_{spaces,étale}$ and an object $\xi = (U, U', a, i, x', \alpha)$ of $\mathcal{C}$ over U. We are going to construct the "pullback" $f^*\xi$ over V. Namely, set $b = a \circ f$. Let $f' : V' \to U'$ be the étale morphism whose restriction to V is f (More on Morphisms of Spaces, Lemma 8.2). Denote $j : V \to V'$ the corresponding thickening. Let $y' = x' \circ f'$ and $\gamma = \mathrm{id} : x' \circ f' \to y'$. Set

$$\beta = \alpha \star \mathrm{id}_f : x \circ b = x \circ a \circ f \to x' \circ i \circ f = x' \circ f' \circ j = y' \circ j$$

It is clear that $(f, f', \gamma) : (V, V', b, j, y', \beta) \to (U, U', a, i, x', \alpha)$ is a morphism as in (6.1.2). The morphisms (f, f', γ) so constructed are strongly cartesian (Categories, Definition 33.1). We omit the detailed proof, but essentially the reason is that given a morphism $(g, g', \epsilon) : (Y, Y', c, k, z', \delta) \to (U, U', a, i, x', \alpha)$ in $\mathcal{C}$ such that g factors as $g = f \circ h$ for some $h : Y \to V$, then we get a unique factorization $g' = f' \circ h'$ from More on Morphisms of Spaces, Lemma 8.2 and after that one can produce the necessary ζ such that $(h, h', \zeta) : (Y, Y', c, k, z', \delta) \to (V, V', b, j, y', \beta)$ is a morphism of $\mathcal{C}$ with $(g, g', \epsilon) = (f, f', \gamma) \circ (h, h', \zeta)$.

Therefore $p : \mathcal{C} \to W_{étale}$ is a fibred category (Categories, Definition 33.5). Combined with the fact that the fibre categories are groupoids seen above we conclude that $p : \mathcal{C} \to W_{étale}$ is fibred in groupoids by Categories, Lemma 35.2. $\square$

Lemma 6.4. *The category $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 6.1 is a stack in groupoids.*

Proof. By Lemma 6.3 we see the first condition of Stacks, Definition 5.1 holds. As is customary we check descent of objects and we leave it to the reader to check descent of morphisms. Thus suppose we have $a : U \to W$ in $W_{spaces,étale}$, a covering $\{U_k \to U\}_{k \in K}$ in $W_{spaces,étale}$, objects $\xi_k = (U_k, U'_k, a_k, i_k, x'_k, \alpha_k)$ of $\mathcal{C}$ over U_k, and morphisms

$$\varphi_{kk'} = (f_{kk'}, f'_{kk'}, \gamma_{kk'}) : \xi_k|_{U_k \times_U U_{k'}} \to \xi_{k'}|_{U_k \times_U U_{k'}}$$

between restrictions satisfying the cocycle condition. In order to prove effectivity we may first refine the covering. Hence we may assume each U_k is a scheme (even an affine scheme if you like). Let us write

$$\xi_k|_{U_k \times_U U_{k'}} = (U_k \times_U U_{k'}, U'_{kk'}, a_{kk'}, x'_{kk'}, \alpha_{kk'})$$

Then we get an étale (by Lemma 6.2) morphism $s_{kk'} : U'_{kk'} \to U'_k$ as the second component of the morphism $\xi_k|_{U_k \times_U U_{k'}} \to \xi_k$ of $\mathcal{C}$. Similarly we obtain an étale morphism $t_{kk'} : U'_{kk'} \to U'_{k'}$ by looking at the second component of the composition

$$\xi_k|_{U_k \times_U U_{k'}} \xrightarrow{\varphi_{kk'}} \xi_{k'}|_{U_k \times_U U_{k'}} \to \xi_{k'}$$

We claim that

$$j : \coprod\nolimits_{(k,k') \in K \times K} U'_{kk'} \xrightarrow{(\coprod s_{kk'}, \coprod t_{kk'})} (\coprod\nolimits_{k \in K} U'_k) \times (\coprod\nolimits_{k \in K} U'_k)$$

is an étale equivalence relation. First, we have already seen that the components s, t of the displayed morphism are étale. The base change of the morphism j by $(\coprod U_k) \times (\coprod U_k) \to (\coprod U'_k) \times (\coprod U'_k)$ is a monomorphism because it is the map

$$\coprod\nolimits_{(k,k') \in K \times K} U_k \times_U U_{k'} \longrightarrow (\coprod\nolimits_{k \in K} U_k) \times (\coprod\nolimits_{k \in K} U_k)$$

Hence j is a monomorphism by More on Morphisms, Lemma 3.4. Finally, symmetry of the relation j comes from the fact that $\varphi_{kk'}^{-1}$ is the "flip" of $\varphi_{k'k}$ (see Stacks, Remarks 3.2) and transitivity comes from the cocycle condition (details omitted). Thus the quotient of $\coprod U'_k$ by j is an algebraic space U' (Spaces, Theorem 10.5). Above we have already shown that there is a thickening $i : U \to U'$ as we saw that the restriction of j on $\coprod U_k$ gives $(\coprod U_k) \times_U (\coprod U_k)$. Finally, if we temporarily view the 1-morphisms $x'_k : U'_k \to \mathcal{X}'$ as objects of the stack $\mathcal{X}'$ over U'_k then we see that these come endowed with a descent datum with respect to the étale covering $\{U'_k \to U'\}$ given by the third component $\gamma_{kk'}$ of the morphisms $\varphi_{kk'}$ in $\mathcal{C}$. Since $\mathcal{X}'$ is a stack this descent datum is effective and translating back we obtain a 1-morphism $x' : U' \to \mathcal{X}'$ such that the compositions $U'_k \to U' \to \mathcal{X}'$ come equipped with isomorphisms to x'_k compatible with $\gamma_{kk'}$. This means that the morphisms $\alpha_k : x \circ a_k \to x'_k \circ i_k$ glue to a morphism $\alpha : x \circ a \to x' \circ i$. Then $\xi = (U, U', a, i, x', \alpha)$ is the desired object over U. $\square$

Lemma 6.5. *Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Let W be an algebraic space and let $W \to \mathcal{X}$ be a smooth morphism. There exists an étale covering $\{W_i \to W\}_{i \in I}$ and for each i a cartesian diagram*

$$\begin{tikzcd} W_i \arrow[r] \arrow[d] & W'_i \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}' \end{tikzcd}$$

with $W'_i \to \mathcal{X}'$ smooth.

Proof. Choose a scheme U' and a surjective smooth morphism $U' \to \mathcal{X}'$. As usual we set $U = \mathcal{X} \times_{\mathcal{X}'} U'$. Then $U \to \mathcal{X}$ is a surjective smooth morphism. Therefore the base change

$$V = W \times_{\mathcal{X}} U \longrightarrow W$$

is a surjective smooth morphism of algebraic spaces. By Topologies on Spaces, Lemma 4.4 we can find an étale covering $\{W_i \to W\}$ such that $W_i \to W$ factors through $V \to W$. After covering W_i by affines (Properties of Spaces, Lemma 6.1) we may assume each W_i is affine. We may and do replace W by W_i which reduces us to the situation discussed in the next paragraph.

Assume W is affine and the given morphism $W \to \mathcal{X}$ factors through U. Picture

$$W \xrightarrow{i} U \to \mathcal{X}$$

Since W and U are smooth over $\mathcal{X}$ we see that i is locally of finite type (Morphisms of Stacks, Lemma 17.8). After replacing U by $\mathbf{A}^n_U$ we may assume that i is an immersion, see Morphisms, Lemma 39.2. By Morphisms of Stacks, Lemma 44.4 the morphism i is a local complete intersection. Hence i is a Koszul-regular immersion (as defined in Divisors, Definition 21.1) by More on Morphisms, Lemma 62.3.

We may still replace W by an affine open covering. For every point $w \in W$ we can choose an affine open $U'_w \subset U'$ such that if $U_w \subset U$ is the corresponding affine open, then $w \in i^{-1}(U_w)$ and $i^{-1}(U_w) \to U_w$ is a closed immersion cut out by a Koszul-regular sequence $f_1, \ldots, f_r \in \Gamma(U_w, \mathcal{O}_{U_w})$. This follows from the definition of Koszul-regular immersions and Divisors, Lemma 20.7. Set $W_w = i^{-1}(U_w)$; this is an affine open neighbourhood of $w \in W$. Choose lifts $f'_1, \ldots, f'_r \in \Gamma(U'_w, \mathcal{O}_{U'_w})$ of $f_1, \ldots, f_r$. This is possible as $U_w \to U'_w$ is a closed immersion of affine schemes. Let

$W'_w \subset U'_w$ be the closed subscheme cut out by $f'_1, \ldots, f'_r$. We claim that $W'_w \to \mathcal{X}'$ is smooth. The claim finishes the proof as $W_w = \mathcal{X} \times_{\mathcal{X}'} W'_w$ by construction.

To check the claim it suffices to check that the base change $W'_w \times_{\mathcal{X}'} X' \to X'$ is smooth for every affine scheme X' smooth over $\mathcal{X}'$. Choose an étale morphism

$$Y' \to U'_w \times_{\mathcal{X}'} X'$$

with Y' affine. Because $U'_w \times_{\mathcal{X}'} X'$ is covered by the images of such morphisms, it is enough to show that the closed subscheme Z' of Y' cut out by $f'_1, \ldots, f'_r$ is smooth over X'. Picture

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d] & Y' \arrow[d] & \\
W'_w \times_{\mathcal{X}'} X' \arrow[r] \arrow[d] & U'_w \times_{\mathcal{X}'} X' \arrow[r] \arrow[d] & X' \\
W'_w = V(f'_1, \ldots, f'_r) \arrow[r] & U'_w &
\end{tikzcd}$$

Set $X = \mathcal{X} \times_{\mathcal{X}'} X'$, $Y = X \times_{X'} Y' = \mathcal{X} \times_{\mathcal{X}'} Y'$, and $Z = Y \times_{Y'} Z' = X \times_{X'} Z' = \mathcal{X} \times_{\mathcal{X}'} Z'$. Then $(Z \subset Z') \to (Y \subset Y') \subset (X \subset X')$ are (cartesian) morphisms of thickenings of affine schemes and we are given that $Z \to X$ and $Y' \to X'$ are smooth. Finally, the sequence of functions $f'_1, \ldots, f'_r$ map to a Koszul-regular sequence in $\Gamma(Y', \mathcal{O}_{Y'})$ by More on Algebra, Lemma 30.5 because $Y' \to U'_w$ is smooth and hence flat. By More on Algebra, Lemma 31.6 (and the fact that Koszul-regular sequences are quasi-regular sequences by More on Algebra, Lemmas 30.2, 30.3, and 30.6) we conclude that $Z' \to X'$ is smooth as desired. $\square$

Lemma 6.6. *Let $\mathcal{X} \subset \mathcal{X}'$ be a thickening of algebraic stacks. Consider a commutative diagram*

$$\begin{tikzcd}
W'' \arrow[d, "x''"'] & W \arrow[l] \arrow[r] \arrow[d, "x"] & W' \arrow[d, "x'"] \\
\mathcal{X}' & \mathcal{X} \arrow[l] \arrow[r] & \mathcal{X}'
\end{tikzcd}$$

with cartesian squares where W', W, W'' are algebraic spaces and the vertical arrows are smooth. Then there exist

(1) *an étale covering $\{f'_k : W'_k \to W'\}_{k \in K}$,*
(2) *étale morphisms $f''_k : W'_k \to W''$, and*
(3) *2-morphisms $\gamma_k : x'' \circ f''_k \to x' \circ f'_k$*

such that (a) $(f'_k)^{-1}(W) = (f''_k)^{-1}(W)$, (b) $f'_k|_{(f'_k)^{-1}(W)} = f''_k|_{(f''_k)^{-1}(W)}$, and (c) pulling back γ_k to the closed subscheme of (a) agrees with the 2-morphism given by the commutativity of the initial diagram over W.

Proof. Denote $i : W \to W'$ and $i'' : W \to W''$ the given thickenings. The commutativity of the diagram in the statement of the lemma means there is a 2-morphism $\delta : x' \circ i' \to x'' \circ i''$ This is the 2-morphism referred to in part (c) of the statement. Consider the algebraic space

$$I' = W' \times_{x', \mathcal{X}', x''} W''$$

with projections $p' : I' \to W'$ and $q' : I' \to W''$. Observe that there is a "universal" 2-morphism $\gamma : x' \circ p' \to x'' \circ q'$ (we will use this later). The choice of δ defines a morphism

$$\begin{tikzcd} W \arrow[rr, "\delta"'] & & I' \arrow[dl, "p'"] \arrow[dr, "q'"'] & \\ & W' & & W'' \end{tikzcd}$$

such that the compositions $W \to I' \to W'$ and $W \to I' \to W''$ are $i : W \to W'$ and $i' : W \to W''$. Since x'' is smooth, the morphism $p' : I' \to W'$ is smooth as a base change of x''.

Suppose we can find an étale covering $\{f'_k : W'_k \to W'\}$ and morphisms $\delta_k : W'_k \to I'$ such that the restriction of δ_k to $W_k = (f'_k)^{-1}$ is equal to $\delta \circ f_k$ where $f_k = f'_k|_{W_k}$. Picture

$$\begin{tikzcd} W_k \arrow[r, "f_k"] \arrow[d] & W \arrow[r, "\delta"] & I' \arrow[d, "p'"] \\ W'_k \arrow[rru, "\delta_k"] \arrow[rr, "f'_k"] & & W' \end{tikzcd}$$

In other words, we want to be able to extend the given section $\delta : W \to I'$ of p' to a section over W' after possibly replacing W' by an étale covering.

If this is true, then we can set $f''_k = q' \circ \delta_k$ and $\gamma_k = \gamma \star \mathrm{id}_{\delta_k}$ (more succinctly $\gamma_k = \delta_k^*\gamma$). Namely, the only thing left to show at this is that the morphism f''_k is étale. By construction the morphism $x' \circ p'$ is 2-isomorphic to $x'' \circ q'$. Hence $x'' \circ f''_k$ is 2-isomorphic to $x' \circ f'_k$. We conclude that the composition

$$W'_k \xrightarrow{f''_k} W'' \xrightarrow{x''} \mathcal{X}'$$

is smooth because $x' \circ f'_k$ is so. As f_k is étale we conclude f''_k is étale by Lemma 5.2.

If the thickening is a first order thickening, then we can choose any étale covering $\{W'_k \to W'\}$ with W'_k affine. Namely, since p' is smooth we see that p' is formally smooth by the infinitesimal lifting criterion (More on Morphisms of Spaces, Lemma 19.6). As W_k is affine and as $W_k \to W'_k$ is a first order thickening (as a base change of $\mathcal{X} \to \mathcal{X}'$, see Lemma 3.4) we get δ_k as desired.

In the general case the existence of the covering and the morphisms δ_k follows from More on Morphisms of Spaces, Lemma 19.7. ☐

Lemma 6.7. *The category $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 6.1 is a gerbe.*

Proof. In Lemma 6.4 we have seen that it is a stack in groupoids. Thus it remains to check conditions (2) and (3) of Stacks, Definition 11.1. Condition (2) follows from Lemma 6.5. Condition (3) follows from Lemma 6.6. ☐

Lemma 6.8. *In Remark 6.1 assume $\mathcal{X} \subset \mathcal{X}'$ is a first order thickening. Then*

(1) *the automorphism sheaves of objects of the gerbe $p : \mathcal{C} \to W_{spaces,étale}$ constructed in Remark 6.1 are abelian, and*
(2) *the sheaf of groups $\mathcal{G}$ constructed in Stacks, Lemma 11.8 is a quasi-coherent $\mathcal{O}_W$-module.*

Proof. We will prove both statements at the same time. Namely, given an object $\xi = (U, U', a, i, x', \alpha)$ we will endow $Aut(\xi)$ with the structure of a quasi-coherent $\mathcal{O}_U$-module on $U_{spaces,étale}$ and we will show that this structure is compatible with pullbacks. This will be sufficient by glueing of sheaves (Sites, Section 26) and the construction of $\mathcal{G}$ in the proof of Stacks, Lemma 11.8 as the glueing of the automorphism sheaves $Aut(\xi)$ and the fact that it suffices to check a module is quasi-coherent after going to an étale covering (Properties of Spaces, Lemma 29.6).

We will describe the sheaf $Aut(\xi)$ using the same method as used in the proof of Lemma 6.6. Consider the algebraic space

$$I' = U' \times_{x', \mathcal{X}', x'} U'$$

with projections $p' : I' \to U'$ and $q' : I' \to U'$. Over I' there is a universal 2-morphism $\gamma : x' \circ p' \to x' \circ q'$. The identity $x' \to x'$ defines a diagonal morphism

$$\begin{tikzcd} U' \arrow[r, "\Delta'"'] & I' \arrow[dl, "p'"] \arrow[dr, "q'"'] & \\ U' & & U' \end{tikzcd}$$

such that the compositions $U' \to I' \to U'$ and $U' \to I' \to U'$ are the identity morphisms. We will denote the base change of U', I', p', q', Δ' to $\mathcal{X}$ by U, I, p, q, Δ. Since $W' \to \mathcal{X}'$ is smooth, we see that $p' : I' \to U'$ is smooth as a base change.

A section of $Aut(\xi)$ over U is a morphism $\delta' : U' \to I'$ such that $\delta'|_U = \Delta$ and such that $p' \circ \delta' = \text{id}_{U'}$. To be explicit, $(\text{id}_U, q' \circ \delta', (\delta')^*\gamma) : \xi \to \xi$ is a formula for the corresponding automorphism. More generally, if $f : V \to U$ is an étale morphism, then there is a thickening $j : V \to V'$ and an étale morphism $f' : V' \to U'$ whose restriction to V is f and $f^*\xi$ corresponds to $(V, V', a \circ f, j, x' \circ f', f^*\alpha)$, see proof of Lemma 6.3. a section of $Aut(\xi)$ over V is a morphism $\delta' : V' \to I'$ such that $\delta'|_V = \Delta \circ f$ and $p' \circ \delta' = f'$[1].

We conclude that $Aut(\xi)$ as a sheaf of sets agrees with the sheaf defined in More on Morphisms of Spaces, Remark 17.7 for the thickenings $(U \subset U')$ and $(I \subset I')$ over $(U \subset U')$ via $\text{id}_{U'}$ and p'. The diagonal Δ' is a section of this sheaf and by acting on this section using More on Morphisms of Spaces, Lemma 17.5 we get an isomorphism

$$\mathcal{H}om_{\mathcal{O}_U}(\Delta^*\Omega_{I/U}, \mathcal{C}_{U/U'}) \longrightarrow Aut(\xi) \tag{6.8.1}$$

on $U_{spaces,étale}$. There three things left to check

(1) the construction of (6.8.1) commutes with étale localization,

[1]A formula for the corresponding automorphism is $(\text{id}_V, h', (\delta')^*\gamma)$. Here $h' : V' \to V'$ is the unique (iso)morphism such that $h'|_V = \text{id}_V$ and such that

$$\begin{tikzcd} V' \arrow[r, "h'"'] \arrow[dr, "q' \circ \delta'"'] & V' \arrow[d, "f'"] \\ & U' \end{tikzcd}$$

commutes. Uniqueness and existence of h' by topological invariance of the étale site, see More on Morphisms of Spaces, Theorem 8.1. The reader may feel we should instead look at morphisms $\delta'' : V' \to V' \times_{\mathcal{X}'} V'$ with $\delta'' \circ j = \Delta_{V'/\mathcal{X}'}$ and $\text{pr}_1 \circ \delta'' = \text{id}_{V'}$. This would be fine too: as $V' \times_{\mathcal{X}'} V' \to I'$ is étale, the same topological invariance tells us that sending δ'' to $\delta' = (V' \times_{\mathcal{X}'} V' \to I') \circ \delta''$ is a bijection between the two sets of morphisms.

(2) $\mathcal{H}om_{\mathcal{O}_U}(\Delta^*\Omega_{I/U}, \mathcal{C}_{U/U'})$ is a quasi-coherent module on U,
(3) the composition in $Aut(\xi)$ corresponds to addition of sections in this quasi-coherent module.

We will check these in order.

To see (1) we have to show that if $f : V \to U$ is étale, then (6.8.1) constructed using ξ over U, restricts to the map (6.8.1)

$$\mathcal{H}om_{\mathcal{O}_V}(\Delta_V^*\Omega_{V\times_{\mathcal{X}}V/V}, \mathcal{C}_{V/V'}) \to Aut(\xi|_V)$$

constructed using $\xi|_V$ over V on $V_{spaces,étale}$. This follows from the discussion in the footnote above and More on Morphisms of Spaces, Lemma 17.8.

Proof of (2). Since p' is smooth, the morphism $I \to U$ is smooth, and hence the relative module of differentials $\Omega_{I/U}$ is finite locally free (More on Morphisms of Spaces, Lemma 7.16). On the other hand, $\mathcal{C}_{U/U'}$ is quasi-coherent (More on Morphisms of Spaces, Definition 5.1). By Properties of Spaces, Lemma 29.7 we conclude.

Proof of (3). There exists a morphism $c' : I' \times_{p',U',q'} I' \to I'$ such that (U', I', p', q', c') is a groupoid in algebraic spaces with identity Δ'. See Algebraic Stacks, Lemma 16.1 for example. Composition in $Aut(\xi)$ is induced by the morphism c' as follows. Suppose we have two morphisms

$$\delta_1', \delta_2' : U' \longrightarrow I'$$

corresponding to sections of $Aut(\xi)$ over U as above, in other words, we have $\delta_i'|U = \Delta_U$ and $p' \circ \delta_i' = \mathrm{id}_{U'}$. Then the composition in $Aut(\xi)$ is

$$\delta_1' \circ \delta_2' = c'(\delta_1' \circ q' \circ \delta_2', \delta_2')$$

We omit the detailed verification[2]. Thus we are in the situation described in More on Groupoids in Spaces, Section 5 and the desired result follows from More on Groupoids in Spaces, Lemma 5.2. □

Proposition 6.9 (Emerton). *Let $\mathcal{X} \subset \mathcal{X}'$ be a first order thickening of algebraic stacks. Let W be an affine scheme and let $W \to \mathcal{X}$ be a smooth morphism. Then there exists a cartesian diagram*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & W' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}' \end{tikzcd}$$

with $W' \to \mathcal{X}'$ smooth and W' affine.

Proof. Consider the category $p : \mathcal{C} \to W_{spaces,étale}$ introduced in Remark 6.1. The proposition states that there exists an object of $\mathcal{C}$ lying over W. Namely, if we have such an object $(W, W', a, i, y', \alpha)$ then $W = \mathcal{X} \times_{\mathcal{X}'} W'$. Hence $W \to W'$ is a thickening of algebraic spaces so W' is affine by More on Morphisms of Spaces, Lemma 9.5 and More on Morphisms, Lemma 2.3.

Lemma 6.7 tells us $\mathcal{C}$ is a gerbe over $W_{spaces,étale}$. This means we can étale locally find a solution and these local solutions are étale locally isomorphic; this part does

[2]The reader can see immediately that it is necessary to precompose δ_1' by $q' \circ \delta_2'$ to get a well defined U'-valued point of the fibre product $I' \times_{p',U',q'} I'$.

not require the assumption that the thickening is first order. By Lemma 6.8 the automorphism sheaves of objects of our gerbe are abelian and fit together to form a quasi-coherent module $\mathcal{G}$ on $W_{spaces,étale}$. We will verify conditions (1) and (2) of Cohomology on Sites, Lemma 11.1 to conclude the existence of an object of $\mathcal{C}$ lying over W. Condition (1) is true: the étale coverings $\{W_i \to W\}$ with each W_i affine are cofinal in the collection of all coverings. For such a covering W_i and $W_i \times_W W_j$ are affine and $H^1(W_i, \mathcal{G})$ and $H^1(W_i \times_W W_j, \mathcal{G})$ are zero: the cohomology of a quasi-coherent module over an affine algebraic space is zero for example by Cohomology of Spaces, Proposition 7.2. Finally, condition (2) is that $H^2(W, \mathcal{G}) = 0$ for our quasi-coherent sheaf $\mathcal{G}$ which again follows from Cohomology of Spaces, Proposition 7.2. This finishes the proof. □

7. Infinitesimal deformations

We continue the discussion from Artin's Axioms, Section 21.

Lemma 7.1. *Let $\mathcal{X}$ be an algebraic stack over a scheme S. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally of finite presentation. Let $A \to B$ be a flat S-algebra homomorphism. Let x be an object of $\mathcal{X}$ over A and set $y = x|_B$. Then $Inf_x(M) \otimes_A B = Inf_y(M \otimes_A B)$.*

Proof. Recall that $\text{Inf}_x(M)$ is the set of automorphisms of the trivial deformation of x to $A[M]$ which induce the identity automorphism of x over A. The trivial deformation is the pullback of x to $\text{Spec}(A[M])$ via $\text{Spec}(A[M]) \to \text{Spec}(A)$. Let $G \to \text{Spec}(A)$ be the automorphism group algebraic space of x (this exists because $\mathcal{X}$ is an algebraic space). Let $e : \text{Spec}(A) \to G$ be the neutral element. The discussion in More on Morphisms of Spaces, Section 17 gives

$$\text{Inf}_x(M) = \text{Hom}_A(e^*\Omega_{G/A}, M)$$

By the same token

$$\text{Inf}_y(M \otimes_A B) = \text{Hom}_B(e_B^*\Omega_{G_B/B}, M \otimes_A B)$$

Since $G \to \text{Spec}(A)$ is locally of finite presentation by assumption, we see that $\Omega_{G/A}$ is locally of finite presentation, see More on Morphisms of Spaces, Lemma 7.15. Hence $e^*\Omega_{G/A}$ is a finitely presented A-module. Moreover, $\Omega_{G_B/B}$ is the pullback of $\Omega_{G/A}$ by More on Morphisms of Spaces, Lemma 7.12. Therefore $e_B^*\Omega_{G_B/B} = e^*\Omega_{G/A} \otimes_A B$. we conclude by More on Algebra, Lemma 65.4. □

Lemma 7.2. *Let $\mathcal{X}$ be an algebraic stack over a base scheme S. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally of finite presentation. Let $(A' \to A, x)$ be a deformation situation. Then the functor*

$$F : B' \longmapsto \{\textit{lifts of } x|_{B' \otimes_{A'} A} \textit{ to } B'\}/\textit{isomorphisms}$$

is a sheaf on the site $(Aff/\text{Spec}(A'))_{fppf}$ of Topologies, Definition 7.8.

Proof. Let $\{T'_i \to T'\}_{i=1,\ldots n}$ be a standard fppf covering of affine schemes over A'. Write $T' = \text{Spec}(B')$. As usual denote

$$T'_{i_0\ldots i_p} = T'_{i_0} \times_{T'} \ldots \times_{T'} T'_{i_p} = \text{Spec}(B'_{i_0\ldots i_p})$$

where the ring is a suitable tensor product. Set $B = B' \otimes_{A'} A$ and $B_{i_0\ldots i_p} = B'_{i_0\ldots i_p} \otimes_{A'} A$. Denote $y = x|_B$ and $y_{i_0\ldots i_p} = x|_{B_{i_0\ldots i_p}}$. Let $\gamma_i \in F(B'_i)$ such that γ_{i_0} and γ_{i_1} map to the same element of $F(B'_{i_0i_1})$. We have to find a unique $\gamma \in F(B')$ mapping to γ_i in $F(B'_i)$.

Choose an actual object y'_i of *Lift*(y_i, B'_i) in the isomorphism class γ_i. Choose isomorphisms $\varphi_{i_0i_1} : y'_{i_0}|_{B'_{i_0i_1}} \to y'_{i_1}|_{B'_{i_0i_1}}$ in the category *Lift*$(y_{i_0i_1}, B'_{i_0i_1})$. If the maps $\varphi_{i_0i_1}$ satisfy the cocycle condition, then we obtain our object γ because $\mathcal{X}$ is a stack in the fppf topology. The cocycle condition is that the composition

$$y'_{i_0}|_{B'_{i_0i_1i_2}} \xrightarrow{\varphi_{i_0i_1}|_{B'_{i_0i_1i_2}}} y'_{i_1}|_{B'_{i_0i_1i_2}} \xrightarrow{\varphi_{i_1i_2}|_{B'_{i_0i_1i_2}}} y'_{i_2}|_{B'_{i_0i_1i_2}} \xrightarrow{\varphi_{i_2i_0}|_{B'_{i_0i_1i_2}}} y'_{i_0}|_{B'_{i_0i_1i_2}}$$

is the identity. If not, then these maps give elements

$$\delta_{i_0i_1i_2} \in \mathrm{Inf}_{y_{i_0i_1i_2}}(J_{i_0i_1i_2}) = \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1i_2}$$

Here $J = \mathrm{Ker}(B' \to B)$ and $J_{i_0\ldots i_p} = \mathrm{Ker}(B'_{i_0\ldots i_p} \to B_{i_0\ldots i_p})$. The equality in the displayed equation holds by Lemma 7.1 applied to $B' \to B'_{i_0\ldots i_p}$ and y and $y_{i_0\ldots i_p}$, the flatness of the maps $B' \to B'_{i_0\ldots i_p}$ which also guarantees that $J_{i_0\ldots i_p} = J \otimes_{B'} B'_{i_0\ldots i_p}$. A computation (omitted) shows that $\delta_{i_0i_1i_2}$ gives a 2-cocycle in the Čech complex

$$\prod \mathrm{Inf}_y(J) \otimes_B B_{i_0} \to \prod \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1} \to \prod \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1i_2} \to \ldots$$

By Descent, Lemma 9.2 this complex is acyclic in positive degrees and has $H^0 = \mathrm{Inf}_y(J)$. Since $\mathrm{Inf}_{y_{i_0i_1}}(J_{i_0i_1})$ acts on morphisms (Artin's Axioms, Remark 21.4) this means we can modify our choice of $\varphi_{i_0i_1}$ to get to the case where $\delta_{i_0i_1i_2} = 0$.

Uniqueness. We still have to show there is at most one γ restricting to γ_i for all i. Suppose we have objects y', z' of *Lift*(y, B') and isomorphisms $\psi_i : y'|_{B'_i} \to z'|_{B'_i}$ in *Lift*(y_i, B'_i). Then we can consider

$$\psi_{i_1}^{-1} \circ \psi_{i_0} \in \mathrm{Inf}_{y_{i_0i_1}}(J_{i_0i_1}) = \mathrm{Inf}_y(J) \otimes_B B_{i_0i_1}$$

Arguing as before, the obstruction to finding an isomorphism between y' and z' over B' is an element in the H^1 of the Čech complex displayed above which is zero. $\square$

Lemma 7.3. *Let $\mathcal{X}$ be an algebraic stack over a scheme S whose structure morphism $\mathcal{X} \to S$ is locally of finite presentation. Let $A \to B$ be a flat S-algebra homomorphism. Let x be an object of $\mathcal{X}$ over A. Then $T_x(M) \otimes_A B = T_y(M \otimes_A B)$.*

Proof. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. We first reduce the lemma to the case where x lifts to U. Recall that $T_x(M)$ is the set of isomorphism classes of lifts of x to $A[M]$. Therefore Lemma 7.2[3] says that the rule

$$A_1 \mapsto T_{x|_{A_1}}(M \otimes_A A_1)$$

is a sheaf on the small étale site of $\mathrm{Spec}(A)$; the tensor product is needed to make $A[M] \to A_1[M \otimes_A A_1]$ a flat ring map. We may choose a faithfully flat étale ring map $A \to A_1$ such that $x|_{A_1}$ lifts to a morphism $u_1 : \mathrm{Spec}(A_1) \to U$, see for example Sheaves on Stacks, Lemma 19.10. Write $A_2 = A_1 \otimes_A A_1$ and set $B_1 = B \otimes_A A_1$

[3]This lemma applies: $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is locally of finite presentation by Morphisms of Stacks, Lemma 27.6 and the assumption that $\mathcal{X} \to S$ is locally of finite presentation. Therefore $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is locally of finite presentation as a base change of Δ.

and $B_2 = B \otimes_A A_2$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & T_y(M \otimes_A B) \arrow[r] & T_{y|_{B_1}}(M \otimes_A B_1) \arrow[r] & T_{y|_{B_2}}(M \otimes_A B_2) \\ 0 \arrow[r] & T_x(M) \arrow[r] \arrow[u] & T_{x|_{A_1}}(M \otimes_A A_1) \arrow[r] \arrow[u] & T_{x|_{A_2}}(M \otimes_A A_2) \arrow[u] \end{tikzcd}$$

The rows are exact by the sheaf condition. We have $M \otimes_A B_i = (M \otimes_A A_i) \otimes_{A_i} B_i$. Thus if we prove the result for the middle and right vertical arrow, then the result follows. This reduces us to the case discussed in the next paragraph.

Assume that x is the image of a morphism $u : \mathrm{Spec}(A) \to U$. Observe that $T_u(M) \to T_x(M)$ is surjective since $U \to \mathcal{X}$ is smooth and representable by algebraic spaces, see Criteria for Representability, Lemma 6.3 (see discussion preceding it for explanation) and More on Morphisms of Spaces, Lemma 19.6. Set $R = U \times_{\mathcal{X}} U$. Recall that we obtain a groupoid (U, R, s, t, c, e, i) in algebraic spaces with $\mathcal{X} = [U/R]$. By Artin's Axioms, Lemma 21.6 we have an exact sequence

$$T_{e \circ u}(M) \to T_u(M) \oplus T_u(M) \to T_x(M) \to 0$$

where the zero on the right was shown above. A similar sequence holds for the base change to B. Thus the result we want follows if we can prove the result of the lemma for $T_u(M)$ and $T_{e \circ u}(M)$. This reduces us to the case discussed in the next paragraph.

Assume that $\mathcal{X} = X$ is an algebraic space locally of finite presentation over S. Then we have

$$T_x(M) = \mathrm{Hom}_A(x^*\Omega_{X/S}, M)$$

by the discussion in More on Morphisms of Spaces, Section 17. By the same token

$$T_y(M \otimes_A B) = \mathrm{Hom}_B(y^*\Omega_{X/S}, M \otimes_A B)$$

Since $X \to S$ is locally of finite presentation, we see that $\Omega_{X/S}$ is locally of finite presentation, see More on Morphisms of Spaces, Lemma 7.15. Hence $x^*\Omega_{X/S}$ is a finitely presented A-module. Clearly, we have $y^*\Omega_{X/S} = x^*\Omega_{X/S} \otimes_A B$. we conclude by More on Algebra, Lemma 65.4. □

Lemma 7.4. *Let $\mathcal{X}$ be an algebraic stack over a scheme S whose structure morphism $\mathcal{X} \to S$ is locally of finite presentation. Let $(A' \to A, x)$ be a deformation situation. If there exists a faithfully flat finitely presented A'-algebra B' and an object y' of $\mathcal{X}$ over B' lifting $x|_{B' \otimes_{A'} A}$, then there exists an object x' over A' lifting x.*

Proof. Let $I = \mathrm{Ker}(A' \to A)$. Set $B'_1 = B' \otimes_{A'} B'$ and $B'_2 = B' \otimes_{A'} B' \otimes_{A'} B'$. Let $J = IB'$, $J_1 = IB'_1$, $J_2 = IB'_2$ and $B = B'/J$, $B_1 = B'_1/J_1$, $B_2 = B'_2/J_2$. Set $y = x|_B$, $y_1 = x|_{B_1}$, $y_2 = x|_{B_2}$. Let F be the fppf sheaf of Lemma 7.2 (which applies, see footnote in the proof of Lemma 7.3). Thus we have an equalizer diagram

$$\begin{tikzcd} F(A') \arrow[r] & F(B') \arrow[r, shift left] \arrow[r, shift right] & F(B'_1) \end{tikzcd}$$

On the other hand, we have $F(B') = \mathrm{Lift}(y, B')$, $F(B'_1) = \mathrm{Lift}(y_1, B'_1)$, and $F(B'_2) = \mathrm{Lift}(y_2, B'_2)$ in the terminology from Artin's Axioms, Section 21. These sets are nonempty and are (canonically) principal homogeneous spaces for $T_y(J)$,

$T_{y_1}(J_1)$, $T_{y_2}(J_2)$, see Artin's Axioms, Lemma 21.2. Thus the difference of the two images of y' in $F(B'_1)$ is an element

$$\delta_1 \in T_{y_1}(J_1) = T_x(I) \otimes_A B_1$$

The equality in the displayed equation holds by Lemma 7.3 applied to $A' \to B'_1$ and x and y_1, the flatness of $A' \to B'_1$ which also guarantees that $J_1 = I \otimes_{A'} B'_1$. We have similar equalities for B' and B'_2. A computation (omitted) shows that δ_1 gives a 1-cocycle in the Čech complex

$$T_x(I) \otimes_A B \to T_x(I) \otimes_A B_1 \to T_x(I) \otimes_A B_2 \to \ldots$$

By Descent, Lemma 9.2 this complex is acyclic in positive degrees and has $H^0 = T_x(I)$. Thus we may choose an element in $T_x(I) \otimes_A B = T_y(J)$ whose boundary is δ_1. Replacing y' by the result of this element acting on it, we find a new choice y' with $\delta_1 = 0$. Thus y' maps to the same element under the two maps $F(B') \to F(B'_1)$ and we obtain an element o $F(A')$ by the sheaf condition. $\square$

8. Formally smooth morphisms

In this section we introduce the notion of a formally smooth morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks. Such a morphism is characterized by the property that T-valued points of $\mathcal{X}$ lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 8.7. It turns out that this criterion is often easier to use than the Jacobian criterion.

Definition 8.1. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is said to be *formally smooth* if it is formally smooth on objects as a 1-morphism in categories fibred in groupoids as explained in Criteria for Representability, Section 6.

We translate the condition of the definition into the language we are currently using (see Properties of Stacks, Section 2). Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider a 2-commutative solid diagram

(8.1.1)
$$\begin{tikzcd} T \arrow[r, "x"] \arrow[d, "i"'] & \mathcal{X} \arrow[d, "f"] \\ T' \arrow[r, "y"] \arrow[ur, dotted] & \mathcal{Y} \end{tikzcd}$$

where $i : T \to T'$ is a first order thickening of affine schemes. Let

$$\gamma : y \circ i \longrightarrow f \circ x$$

be a 2-morphism witnessing the 2-commutativity of the diagram. (Notation as in Categories, Sections 28 and 29.) Given (8.1.1) and γ a *dotted arrow* is a triple (x', α, β) consisting of a morphism $x' : T' \to \mathcal{X}$ and 2-arrows $\alpha : x' \circ i \to x$, $\beta : y \to f \circ x'$ such that $\gamma = (\mathrm{id}_f \star \alpha) \circ (\beta \star \mathrm{id}_i)$, in other words such that

$$\begin{tikzcd} & f \circ x' \circ i \arrow[dr, "\mathrm{id}_f \star \alpha"] & \\ y \circ i \arrow[ur, "\beta \star \mathrm{id}_i"] \arrow[rr, "\gamma"] & & f \circ x \end{tikzcd}$$

is commutative. A *morphism of dotted arrows* $(x'_1, \alpha_1, \beta_1) \to (x'_2, \alpha_2, \beta_2)$ is a 2-arrow $\theta : x'_1 \to x'_2$ such that $\alpha_1 = \alpha_2 \circ (\theta \star \mathrm{id}_i)$ and $\beta_2 = (\mathrm{id}_f \star \theta) \circ \beta_1$.

The category of dotted arrows just described is a special case of Categories, Definition 44.1.

Lemma 8.2. *A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is formally smooth (Definition 8.1) if and only if for every diagram (8.1.1) and γ the category of dotted arrows is nonempty.*

Proof. Translation between different languages omitted. □

Lemma 8.3. *The base change of a formally smooth morphism of algebraic stacks by any morphism of algebraic stacks is formally smooth.*

Proof. Follows from Categories, Lemma 44.2 and the definition. □

Lemma 8.4. *The composition of formally smooth morphisms of algebraic stacks is formally smooth.*

Proof. Follows from Categories, Lemma 44.3 and the definition. □

Lemma 8.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then the following are equivalent*

(1) *f is formally smooth,*
(2) *for every scheme T and morphism $T \to \mathcal{Y}$ the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is formally smooth as a morphism of algebraic spaces.*

Proof. Follows from Categories, Lemma 44.2 and the definition. □

Lemma 8.6. *Let $T \to T'$ be a first order thickening of affine schemes. Let $\mathcal{X}'$ be an algebraic stack over T' whose structure morphism $\mathcal{X}' \to T'$ is smooth. Let $x : T \to \mathcal{X}'$ be a morphism over T'. Then there exists a morphsm $x' : T' \to \mathcal{X}'$ over T' with $x'|_T = x$.*

Proof. We may apply the result of Lemma 7.4. Thus it suffices to construct a smooth surjective morphism $W' \to T'$ with W' affine such that $x|_{T \times_{W'} T'}$ lifts to W'. (We urge the reader to find their own proof of this fact using the analogous result for algebraic spaces already established.) We choose a scheme U' and a surjective smooth morphism $U' \to \mathcal{X}'$. Observe that $U' \to T'$ is smooth and that the projection $T \times_{\mathcal{X}'} U' \to T$ is surjective smooth. Choose an affine scheme W and an étale morphism $W \to T \times_{\mathcal{X}'} U'$ such that $W \to T$ is surjective. Then $W \to T$ is a smooth morphism of affine schemes. After replacing W by a disjoint union of principal affine opens, we may assume there exists a smooth morphism of affines $W' \to T'$ such that $W = T \times_{T'} W'$, see Algebra, Lemma 137.20. By More on Morphisms of Spaces, Lemma 19.6 we can find a morphism $W' \to U'$ over T' lifting the given morphism $W \to U'$. This finishes the proof. □

The following lemma is the main result of this section. It implies, combined with Limits of Stacks, Proposition 3.8, that we can recognize whether a morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is smooth in terms of "simple" properties of the 1-morphism of stacks in groupoids $\mathcal{X} \to \mathcal{Y}$.

Lemma 8.7 (Infinitesimal lifting criterion)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

(1) *The morphism f is smooth.*
(2) *The morphism f is locally of finite presentation and formally smooth.*

Proof. Assume f is smooth. Then f is locally of finite presentation by Morphisms of Stacks, Lemma 33.5. Hence it suffices given a diagram (8.1.1) and a $\gamma : y \circ i \to f \circ x$ to find a dotted arrow (see Lemma 8.2). Forming fibre products we obtain

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ T' \arrow[r] & T' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

Thus we see it is sufficient to find a dotted arrow in the left square. Since $T' \times_{\mathcal{Y}} \mathcal{X} \to T'$ is smooth (Morphisms of Stacks, Lemma 33.3) existence of a dotted arrow in the left square is guaranteed by Lemma 8.6.

Conversely, suppose that f is locally of finite presentation and formally smooth. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then $a : U \to \mathcal{X}$ and $b : U \to \mathcal{Y}$ are representable by algebraic spaces and locally of finite presentation (use Morphisms of Stacks, Lemma 27.2 and the fact seen above that a smooth morphism is locally of finite presentation). We will apply the general principle of Algebraic Stacks, Lemma 10.9 with as input the equivalence of More on Morphisms of Spaces, Lemma 19.6 and simultaneously use the translation of Criteria for Representability, Lemma 6.3. We first apply this to a to see that a is formally smooth on objects. Next, we use that f is formally smooth on objects by assumption (see Lemma 8.2) and Criteria for Representability, Lemma 6.2 to see that $b = f \circ a$ is formally smooth on objects. Then we apply the principle once more to conclude that b is smooth. This means that f is smooth by the definition of smoothness for morphisms of algebraic stacks and the proof is complete. $\square$

9. Blowing up and flatness

This section quickly discusses what you can deduce from More on Morphisms of Spaces, Sections 38 and 39 for algebraic stacks over algebraic spaces.

Lemma 9.1. *Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Let $V \subset Y$ be an open subspace. Assume*

1. *Y is quasi-compact and quasi-separated,*
2. *f is of finite type and quasi-separated,*
3. *V is quasi-compact, and*
4. *$\mathcal{X}_V$ is flat and locally of finite presentation over V.*

Then there exists a V-admissible blowup $Y' \to Y$ and a closed substack $\mathcal{X}' \subset \mathcal{X}_{Y'}$ with $\mathcal{X}'_V = \mathcal{X}_V$ such that $\mathcal{X}' \to Y'$ is flat and of finite presentation.

Proof. Observe that $\mathcal{X}$ is quasi-compact. Choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Let $R = U \times_{\mathcal{X}} U$ so that we obtain a groupoid (U, R, s, t, c) in algebraic spaces over Y with $\mathcal{X} = [U/R]$ (Algebraic Stacks, Lemma 16.2). We may apply More on Morphisms of Spaces, Lemma 39.1 to $U \to Y$ and the open $V \subset Y$. Thus we obtain a V-admissible blowup $Y' \to Y$ such that the strict transform $U' \subset U_{Y'}$ is flat and of finite presentation over Y'. Let $R' \subset R_{Y'}$ be the strict transform of R. Since s and t are smooth (and in particular flat) it follows from Divisors on Spaces, Lemma 18.4 that we have cartesian diagrams

$$\begin{tikzcd} R' \arrow[r] \arrow[d] & R_{Y'} \arrow[d, "s_{Y'}"] \\ U' \arrow[r] & U_{Y'} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R' \arrow[r] \arrow[d] & R_{Y'} \arrow[d, "t_{Y'}"] \\ U' \arrow[r] & U_{Y'} \end{tikzcd}$$

In other words, U' is an $R_{Y'}$-invariant closed subspace of $U_{Y'}$. Thus U' defines a closed substack $\mathcal{X}' \subset \mathcal{X}_{Y'}$ by Properties of Stacks, Lemma 9.11. The morphism $\mathcal{X}' \to Y'$ is flat and locally of finite presentation because this is true for $U' \to Y'$. On the other hand, we already know $\mathcal{X}' \to Y'$ is quasi-compact and quasi-separated (by our assumptions on f and because this is true for closed immersions). This finishes the proof. $\square$

10. Chow's lemma for algebraic stacks

In this section we discuss Chow's lemma for algebraic stacks.

Lemma 10.1. *Let Y be a quasi-compact and quasi-separated algebraic space. Let $V \subset Y$ be a quasi-compact open. Let $f : \mathcal{X} \to V$ be surjective, flat, and locally of finite presentation. Then there exists a finite surjective morphism $g : Y' \to Y$ such that $V' = g^{-1}(V) \to Y$ factors Zariski locally through f.*

Proof. We first prove this when Y is a scheme. We may choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then $\{U \to V\}$ is an fppf covering of schemes. By More on Morphisms, Lemma 48.6 there exists a finite surjective morphism $V' \to V$ such that $V' \to V$ factors Zariski locally through U. By More on Morphisms, Lemma 48.4 we can find a finite surjective morphism $Y' \to Y$ whose restriction to V is $V' \to V$ as desired.

If Y is an algebraic space, then we see the lemma is true by first doing a finite base change by a finite surjective morphism $Y' \to Y$ where Y' is a scheme. See Limits of Spaces, Proposition 16.1. $\square$

Lemma 10.2. *Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Let $V \subset Y$ be an open subspace. Assume*

1. *f is separated and of finite type,*
2. *Y is quasi-compact and quasi-separated,*
3. *V is quasi-compact, and*
4. *$\mathcal{X}_V$ is a gerbe over V.*

Then there exists a commutative diagram

$$\begin{tikzcd} \overline{Z} \arrow[dr, "\overline{g}"'] & Z \arrow[l, "j"'] \arrow[r, "h"] \arrow[d, "g"] & \mathcal{X} \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

with j an open immersion, $\overline{g}$ and h proper, and such that $|V|$ is contained in the image of $|g|$.

Proof. Suppose we have a commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

and a quasi-compact open $V' \subset Y'$, such that $Y' \to Y$ is a proper morphism of algebraic spaces, $\mathcal{X}' \to \mathcal{X}$ is a proper morphism of algebraic stacks, $V' \subset Y'$ maps surjectively onto V, and $\mathcal{X}'_{V'}$ is a gerbe over V'. Then it suffices to prove the lemma for the pair $(f' : \mathcal{X}' \to Y', V')$. Some details omitted.

Overall strategy of the proof. We will reduce to the case where the image of f is open and f has a section over this open by repeatedly applying the above remark. Each step is straightforward, but there are quite a few of them which makes the proof a bit involved.

Using Limits of Spaces, Proposition 16.1 we reduce to the case where Y is a scheme. (Let $Y' \to Y$ be a finite surjective morphism where Y' is a scheme. Set $\mathcal{X}' = \mathcal{X}_{Y'}$ and apply the initial remark of the proof.)

Using Lemma 9.1 (and Morphisms of Stacks, Lemma 28.8 to see that a gerbe is flat and locally of finite presentation) we reduce to the case where f is flat and of finite presentation.

Since f is flat and locally of finite presentation, we see that the image of $|f|$ is an open $W \subset Y$. Since $\mathcal{X}$ is quasi-compact (as f is of finite type and Y is quasi-compact) we see that W is quasi-compact. By Lemma 10.1 we can find a finite surjective morphism $g : Y' \to Y$ such that $g^{-1}(W) \to Y$ factors Zariski locally through $\mathcal{X} \to Y$. After replacing Y by Y' and $\mathcal{X}$ by $\mathcal{X} \times_Y Y'$ we reduce to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \mathrm{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, and (c) W is the image of $|f|$. We will use induction on n. The base case is $n = 0$: this implies $V = \emptyset$ and in this case we can take $\overline{Z} = \emptyset$. If $n > 0$, then for $i = 1, \ldots, n$ consider the reduced closed subschemes Y_i with underlying topological space $Y \setminus W_i$. Consider the finite morphism

$$Y' = Y \amalg \coprod\nolimits_{i=1,\ldots,n} Y_i \longrightarrow Y$$

and the quasi-compact open

$$V' = (W_1 \cap \ldots \cap W_n \cap V) \amalg \coprod_{i=1,\ldots,n} (V \cap Y_i).$$

By the initial remark of the proof, if we can prove the lemma for the pairs

$$(\mathcal{X} \to Y, W_1 \cap \ldots \cap W_n \cap V) \quad \text{and} \quad (\mathcal{X} \times_Y Y_i \to Y_i, V \cap Y_i), \quad i = 1, \ldots, n$$

then the result is true. Here we use the settheoretic equality $V = (W_1 \cap \ldots \cap W_n \cap V) \cup \bigcup_{i=1,\ldots n}(V \cap Y_i)$. The induction hypothesis applies to the second type of pairs above. Hence we reduce to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \mathrm{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, and (d) $V \subset W_1 \cap \ldots \cap W_n$. The morphisms

$$T_{ij} = \mathit{Isom}_{\mathcal{X}}(x_i|_{W_i \cap W_j \cap V}, x_j|_{W_i \cap W_j \cap V}) \longrightarrow W_i \cap W_j \cap V$$

are surjective, flat, and locally of finite presentation (Morphisms of Stacks, Lemma 28.10). We apply Lemma 10.1 to each quasi-compact open $W_i \cap W_j \cap V$ and the morphisms $T_{ij} \to W_i \cap W_j \cap V$ to get finite surjective morphisms $Y'_{ij} \to Y$. After replacing Y by the fibre product of all of the Y'_{ij} over Y we reduce to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \mathrm{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, (d) $V \subset W_1 \cap \ldots \cap W_n$, and (e) x_i and x_j are Zariski locally isomorphic over $W_i \cap W_j \cap V$. Let $y \in V$ be arbitrary. Suppose that we can find a quasi-compact open neighbourhood $y \in V_y \subset V$ such that the lemma is true for the pair $(\mathcal{X} \to Y, V_y)$, say with solution $\overline{Z}_y, Z_y, \overline{g}_y, g_y, h_y$. Since V is quasi-compact, we can find a finite number $y_1, \ldots, y_m$ such that $V = V_{y_1} \cup \ldots \cup V_{y_m}$. Then it follows that setting

$$\overline{Z} = \coprod \overline{Z}_{y_j}, \quad Z = \coprod Z_{y_j}, \quad \overline{g} = \coprod \overline{g}_{y_j}, \quad g = \coprod g_{y_j}, \quad h = \coprod h_{y_j}$$

is a solution for the lemma. Given y by condition (e) we can choose a quasi-compact open neighbourhood $y \in V_y \subset V$ and isomorphisms $\varphi_i : x_1|_{V_y} \to x_i|_{V_y}$ for $i = 2, \ldots, n$. Set $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$. This leads us to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \mathrm{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, (d) $V \subset W_1 \cap \ldots \cap W_n$, and (f) there are isomorphisms $\varphi_{ij} : x_i|_V \to x_j|_V$ satisfying $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$. The morphisms

$$I_{ij} = \mathit{Isom}_{\mathcal{X}}(x_i|_{W_i \cap W_j}, x_j|_{W_i \cap W_j}) \longrightarrow W_i \cap W_j$$

are proper because f is separated (Morphisms of Stacks, Lemma 6.6). Observe that φ_{ij} defines a section $V \to I_{ij}$ of $I_{ij} \to W_i \cap W_j$ over V. By More on Morphisms of Spaces, Lemma 39.6 we can find V-admissible blowups $p_{ij} : Y_{ij} \to Y$ such that s_{ij} extends to $p_{ij}^{-1}(W_i \cap W_j)$. After replacing Y by the fibre product of all the Y_{ij} over Y we get to the situation described in the next paragraph.

Assume there exists $n \geq 0$, quasi-compact opens $W_i \subset Y$, $i = 1, \ldots, n$, and morphisms $x_i : W_i \to \mathcal{X}$ such that (a) $f \circ x_i = \mathrm{id}_{W_i}$, (b) $W = \bigcup_{i=1,\ldots,n} W_i$ contains V, (c) W is the image of $|f|$, (d) $V \subset W_1 \cap \ldots \cap W_n$, and (g) there are isomorphisms $\varphi_{ij} : x_i|_{W_i \cap W_j} \to x_j|_{W_i \cap W_j}$ satisfying

$$\varphi_{jk}|_V \circ \varphi_{ij}|_V = \varphi_{ik}|_V.$$

After replacing Y by another V-admissible blowup if necessary we may assume that V is dense and scheme theoretically dense in Y and hence in any open subspace of Y containing V. After such a replacement we conclude that

$$\varphi_{jk}|_{W_i \cap W_j \cap W_k} \circ \varphi_{ij}|_{W_i \cap W_j \cap W_k} = \varphi_{ik}|_{W_i \cap W_j \cap W_k}$$

by appealing to Morphisms of Spaces, Lemma 17.8 and the fact that $I_{ik} \to W_i \cap W_j$ is proper (hence separated). Of course this means that (x_i, φ_{ij}) is a desent datum and we obtain a morphism $x : W \to \mathcal{X}$ agreeing with x_i over W_i because $\mathcal{X}$ is a stack. Since x is a section of the separated morphism $\mathcal{X} \to W$ we see that x is proper (Morphisms of Stacks, Lemma 4.9). Thus the lemma now holds with $\overline{Z} = Y$, $Z = W$, $\overline{g} = \mathrm{id}_Y$, $g = \mathrm{id}_W$, $h = x$. □

Theorem 10.3 (Chow's lemma)**.** *Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Assume*

(1) *Y is quasi-compact and quasi-separated,*
(2) *f is separated of finite type.*

Then there exists a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[dr] & X \arrow[l] \arrow[r] \arrow[d] & \overline{X} \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X \to \mathcal{X}$ is proper surjective, $X \to \overline{X}$ is an open immersion, and $\overline{X} \to Y$ is proper morphism of algebraic spaces.

Proof. The rough idea is to use that $\mathcal{X}$ has a dense open which is a gerbe (Morphisms of Stacks, Proposition 29.1) and appeal to Lemma 10.2. The reason this does not work is that the open may not be quasi-compact and one runs into technical problems. Thus we first do a (standard) reduction to the Noetherian case.

First we choose a closed immersion $\mathcal{X} \to \mathcal{X}'$ where $\mathcal{X}'$ is an algebraic stack separated and of finite type over Y. See Limits of Stacks, Lemma 6.2. Clearly it suffices to prove the theorem for $\mathcal{X}'$, hence we may assume $\mathcal{X} \to Y$ is separated and of finite presentation.

Assume $\mathcal{X} \to Y$ is separated and of finite presentation. By Limits of Spaces, Proposition 8.1 we can write $Y = \lim Y_i$ as the directed limit of a system of Noetherian algebraic spaces with affine transition morphisms. By Limits of Stacks, Lemma 5.1 there is an i and a morphism $\mathcal{X}_i \to Y_i$ of finite presentation from an algebraic stack to Y_i such that $\mathcal{X} = Y \times_{Y_i} \mathcal{X}_i$. After increasing i we may assume that $\mathcal{X}_i \to Y_i$ is separated, see Limits of Stacks, Lemma 4.2. Then it suffices to prove the theorem for $\mathcal{X}_i \to Y_i$. This reduces us to the case discussed in the next paragraph.

Assume Y is Noetherian. We may replace $\mathcal{X}$ by its reduction (Properties of Stacks, Definition 10.4). This reduces us to the case discussed in the next paragraph.

Assume Y is Noetherian and $\mathcal{X}$ is reduced. Since $\mathcal{X} \to Y$ is separated and Y quasi-separated, we see that $\mathcal{X}$ is quasi-separated as an algebraic stack. Hence the inertia $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Thus by Morphisms of Stacks, Proposition 29.1 there exists a dense open substack $\mathcal{V} \subset \mathcal{X}$ which is a gerbe. Let $\mathcal{V} \to V$ be the morphism which expresses $\mathcal{V}$ as a gerbe over the algebraic space V. See Morphisms of Stacks, Lemma 28.2 for a construction of $\mathcal{V} \to V$. This construction in particular shows that the morphism $\mathcal{V} \to Y$ factors as $\mathcal{V} \to V \to Y$. Picture

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ V \arrow[r] & Y \end{tikzcd}$$

Since the morphism $\mathcal{V} \to V$ is surjective, flat, and of finite presentation (Morphisms of Stacks, Lemma 28.8) and since $\mathcal{V} \to Y$ is locally of finite presentation, it follows that $V \to Y$ is locally of finite presentation (Morphisms of Stacks, Lemma 27.12). Note that $\mathcal{V} \to V$ is a universal homeomorphism (Morphisms of Stacks, Lemma 28.13). Since $\mathcal{V}$ is quasi-compact (see Morphisms of Stacks, Lemma 8.2) we see that V is quasi-compact. Finally, since $\mathcal{V} \to Y$ is separated the same is true for $V \to Y$ by Morphisms of Stacks, Lemma 27.17 applied to $\mathcal{V} \to V \to Y$ (whose assumptions are satisfied as we've already seen).

All of the above means that the assumptions of Limits of Spaces, Lemma 13.3 apply to the morphism $V \to Y$. Thus we can find a dense open subspace $V' \subset V$ and an immersion $V' \to \mathbf{P}^n_Y$ over Y. Clearly we may replace V by V' and $\mathcal{V}$ by the inverse image of V' in $\mathcal{V}$ (recall that $|\mathcal{V}| = |V|$ as we've seen above). Thus we may assume we have a diagram

$$\begin{tikzcd} \mathcal{V} \arrow[rr] \arrow[d] & & \mathcal{X} \arrow[d] \\ V \arrow[r] & \mathbf{P}^n_Y \arrow[r] & Y \end{tikzcd}$$

where the arrow $V \to \mathbf{P}^n_Y$ is an immersion. Let $\mathcal{X}'$ be the scheme theoretic image of the morphism

$$j : \mathcal{V} \longrightarrow \mathbf{P}^n_Y \times_Y \mathcal{X}$$

and let Y' be the scheme theoretic image of the morphism $V \to \mathbf{P}^n_Y$. We obtain a commutative diagram

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[r] \arrow[d] & \mathbf{P}^n_Y \times_Y \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ V \arrow[r] & Y' \arrow[r] & \mathbf{P}^n_Y \arrow[r] & Y \end{tikzcd}$$

(See Morphisms of Stacks, Lemma 38.4). We claim that $\mathcal{V} = V \times_{Y'} \mathcal{X}'$ and that Lemma 10.2 applies to the morphism $\mathcal{X}' \to Y'$ and the open subspace $V \subset Y'$. If the claim is true, then we obtain

$$\begin{tikzcd} \overline{X} \arrow[dr, "\overline{g}"'] & X \arrow[l] \arrow[r, "h"] \arrow[d, "g"] & \mathcal{X}' \arrow[dl, "f"] \\ & Y' & \end{tikzcd}$$

with $X \to \overline{X}$ an open immersion, $\overline{g}$ and h proper, and such that $|V|$ is contained in the image of $|g|$. Then the composition $X \to \mathcal{X}' \to \mathcal{X}$ is proper (as a composition of proper morphisms) and its image contains $|\mathcal{V}|$, hence this composition is surjective. As well, $\overline{X} \to Y' \to Y$ is proper as a composition of proper morphisms.

The last step is to prove the claim. Observe that $\mathcal{X}' \to Y'$ is separated and of finite type, that Y' is quasi-compact and quasi-separated, and that V is quasi-compact (we omit checking all the details completely). Next, we observe that $b : \mathcal{X}' \to \mathcal{X}$ is an isomorphism over $\mathcal{V}$ by Morphisms of Stacks, Lemma 38.7. In particular $\mathcal{V}$ is identified with an open substack of $\mathcal{X}'$. The morphism j is quasi-compact (source is quasi-compact and target is quasi-separated), so formation of the scheme theoretic image of j commutes with flat base change by Morphisms of Stacks, Lemma 38.5.

In particular we see that $V \times_{Y'} \mathcal{X}'$ is the scheme theoretic image of $\mathcal{V} \to V \times_{Y'} \mathcal{X}'$. However, by Morphisms of Stacks, Lemma 37.5 the image of $|\mathcal{V}| \to |V \times_{Y'} \mathcal{X}'|$ is closed (use that $\mathcal{V} \to V$ is a universal homeomorphism as we've seen above and hence is universally closed). Also the image is dense (combine what we just said with Morphisms of Stacks, Lemma 38.6) we conclude $|\mathcal{V}| = |V \times_{Y'} \mathcal{X}'|$. Thus $\mathcal{V} \to V \times_{Y'} \mathcal{X}'$ is an isomorphism and the proof of the claim is complete. $\square$

11. Noetherian valuative criterion

In this section we will discuss (refined) valuative criteria for morphisms of algebraic stacks using only discrete valuation rings in the Noetherian setting. There are many different variants and we will add more here over time as needed.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks (or algebraic spaces or schemes). A *refined* valuative criterion is one where we are given a morphism $\mathcal{U} \to \mathcal{X}$ (with some properties) and we only look at existence or uniqueness of dotted arrows in solid diagrams of the form

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{U} \arrow[r] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, dotted] & & \mathcal{Y} \end{tikzcd}$$

We use this terminology below to describe the results we have obtained sofar.

Non-Noetherian valuative criteria for morphisms of algebraic stacks

(1) Morphisms of Stacks, Section 40 (for separatedness of the diagonal),
(2) Morphisms of Stacks, Section 41 (for separatedness),
(3) Morphisms of Stacks, Section 42 (for universal closedness),
(4) Morphisms of Stacks, Section 43 (for properness).

For algebraic spaces we have the following valuative criteria

(1) Morphisms of Spaces, Section 42 (for universal closedness),
(2) Morphisms of Spaces, Lemma 42.5 (refined for universal closedness)
(3) Morphisms of Spaces, Section 43 (for separatedness),
(4) Morphisms of Spaces, Section 44 (for properness),
(5) Decent Spaces, Section 16 (for universal closedness for decent spaces),
(6) Decent Spaces, Lemma 17.11 (for universal closedness for decent morphisms between algebraic spaces),
(7) Cohomology of Spaces, Section 19 contains Noetherian valuative criteria
    (a) Cohomology of Spaces, Lemma 19.1 (for separatedness using discrete valuation rings),
    (b) Cohomology of Spaces, Lemma 19.2 (for properness using discrete valuation rings),
    (c) Cohomology of Spaces, Remark 19.3 (discusses how to reduce to complete discrete valuation rings),
(8) Limits of Spaces, Section 21 discussing Noetherian valuative criteria
    (a) Limits of Spaces, Lemma 21.2 (for separatedness using discrete valuation rings and generic points)
    (b) Limits of Spaces, Lemma 21.3 (for properness using discrete valuation rings and generic points)

   (c) Limits of Spaces, Lemma 21.4 (for universal closedness using discrete valuation rings).
(9) Limits of Spaces, Section 22 discussing refined Noetherian valuative criteria
   (a) Limits of Spaces, Lemmas 22.1 and 22.3 (refined for properness using discrete valuation rings),
   (b) Limits of Spaces, Lemma 22.2 (refined for separatedness using discrete valuation rings),

For schemes we have the following valuative criteria

(1) Schemes, Section 20 (for universal closedness)
(2) Schemes, Section 22 (for separatedness),
(3) Morphisms, Section 42 (for properness)
(4) Morphisms, Lemma 42.2 (refined for universal closedness),
(5) Limits, Section 15 discussing Noetherian valuative criteria
   (a) Limits, Lemma 15.2 (for separatedness using discrete valuation rings and generic points)
   (b) Limits, Lemma 15.3 (for properness using discrete valuation rings and generic points)
   (c) Limits, Lemma 15.4 (for universal closedness using discrete valuation rings).
(6) Limits, Section 16 discussing refined Noetherian valuative criteria
   (a) Limits, Lemmas 16.1 and 16.3 (refined for properness using discrete valuation rings),
   (b) Limits, Lemma 16.2 (refined for separatedness using discrete valuation rings),
(7) Limits, Section 17 discussing valuative criteria over a Noetherian base where one can get discrete valuation rings essentially of finite type over the base.

This ends our list of previous results.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

Lemma 11.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f finite type and $\mathcal{Y}$ locally Noetherian. Let $y \in |\mathcal{Y}|$ be a point in the closure of the image of $|f|$. Then there exists a commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & \mathcal{Y} \end{tikzcd}$$

of algebraic stacks where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ *to y.*

Proof. Choose an affine scheme V, a point $v \in V$ and a smooth morphism $V \to \mathcal{Y}$ mapping v to y. The map $|V| \to |\mathcal{Y}|$ is open and by Properties of Stacks, Lemma 4.3 the image of $|\mathcal{X} \times_\mathcal{Y} V| \to |V|$ is the inverse image of the image of $|f|$. We conclude that the point v is in the closure of the image of $|\mathcal{X} \times_\mathcal{Y} V| \to |V|$. If we prove the lemma for $\mathcal{X} \times_\mathcal{Y} V \to V$ and the point v, then the lemma follows for f and y. In this way we reduce to the situation described in the next paragraph.

Assume we have $f : \mathcal{X} \to Y$ and $y \in |Y|$ as in the lemma where Y is a Noetherian affine scheme. Since f is quasi-compact, we conclude that $\mathcal{X}$ is quasi-compact. Hence we can choose an affine scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Then the image of $|f|$ is the same as the image of $|W| \to |Y|$. In this way we reduce to the case of schemes which is Limits, Lemma 15.1. $\square$

Lemma 11.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume*

(1) *$\mathcal{Y}$ is locally Noetherian,*
(2) *f is locally of finite type and quasi-separated,*
(3) *for every commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

where A is a discrete valuation ring and K its fraction field and any 2-arrow $\gamma : y \circ j \to f \circ x$ the category of dotted arrows (Morphisms of Stacks, Definition 39.1) is either empty or a setoid with exactly one isomorphism class.

Then f is separated.

Proof. To prove that f is separated we have to show that $\Delta : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is proper. We already know that Δ is representable by algebraic spaces, locally of finite type (Morphisms of Stacks, Lemma 3.3) and quasi-compact and quasi-separated (by definition of f being quasi-separated). Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$. Set

$$V = \mathcal{X} \times_{\Delta, \mathcal{X} \times_\mathcal{Y} \mathcal{X}} U$$

It suffices to show that the morphism of algebraic spaces $V \to U$ is proper (Properties of Stacks, Lemma 3.3). Observe that U is locally Noetherian (use Morphisms of Stacks, Lemma 17.5 and the fact that $U \to \mathcal{Y}$ is locally of finite type) and $V \to U$ is of finite type and quasi-separated (as the base change of Δ and properties of Δ listed above). Applying Cohomology of Spaces, Lemma 19.2 it suffices to show: Given a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "v"'] \arrow[d, "j"'] & V \arrow[r] \arrow[d, "g"] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[r, "u"] \arrow[ur, dashed] \arrow[urr, dotted] & U \arrow[r] & \mathcal{X} \times_\mathcal{Y} \mathcal{X} \end{tikzcd}$$

where A is a discrete valuation ring and K its fraction field, there is a unique dashed arrow making the diagram commute. By Morphisms of Stacks, Lemma 39.4 the categories of dashed and dotted arrows are equivalent. Assumption (3) implies there is a unique dotted arrow up to isomorphism, see Morphisms of Stacks, Lemma 41.1. We conclude there is a unique dashed arrow as desired. $\square$

Lemma 11.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $h : \mathcal{U} \to \mathcal{X}$ be morphisms of algebraic stacks. Assume that $\mathcal{Y}$ is locally Noetherian, that f and h are of finite type, that f is* *separated, and that the image of $|h| : |\mathcal{U}| \to |\mathcal{X}|$ is dense in $|\mathcal{X}|$. If given any 2-commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "u"'] \arrow[d, "j"'] & \mathcal{U} \arrow[r, "h"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr, "y"] & & \mathcal{Y} \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to f \circ h \circ u$ there exist an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of dotted arrows for the induced diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d, "j'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A') \arrow[r, "y'"] \arrow[ur, dotted, "x'"] & \mathcal{Y} \end{tikzcd}$$

with induced 2-arrow $\gamma' : y' \circ j' \to f \circ x'$ is nonempty (Morphisms of Stacks, Definition 39.1), then f is proper.

Proof. It suffices to prove that f is universally closed. Let $V \to \mathcal{Y}$ be a smooth morphism where V is an affine scheme. By Properties of Stacks, Lemma 4.3 the image I of $|\mathcal{U} \times_\mathcal{Y} V| \to |\mathcal{X} \times_\mathcal{Y} V|$ is the inverse image of the image of $|h|$. Since $|\mathcal{X} \times_\mathcal{Y} V| \to |\mathcal{X}|$ is open (Morphisms of Stacks, Lemma 27.15) we conclude that I is dense in $|\mathcal{X} \times_\mathcal{Y} V|$. Also since the category of dotted arrows behaves well with respect to base change (Morphisms of Stacks, Lemma 39.4) the assumption on existence of dotted arrows (after extension) is inherited by the morphisms $\mathcal{U} \times_\mathcal{Y} V \to \mathcal{X} \times_\mathcal{Y} V \to V$. Therefore the assumptions of the lemma are satisfied for the morphisms $\mathcal{U} \times_\mathcal{Y} V \to \mathcal{X} \times_\mathcal{Y} V \to V$. Hence we may assume $\mathcal{Y}$ is an affine scheme.

Assume $\mathcal{Y} = Y$ is an affine scheme. (From now on we no longer have to worry about the 2-arrows γ and γ', see Morphisms of Stacks, Lemma 39.3.) Then $\mathcal{U}$ is quasi-compact. Choose an affine scheme U and a surjective smooth morphism $U \to \mathcal{U}$. Then we may and do replace $\mathcal{U}$ by U. Thus we may assume that $\mathcal{U}$ is an affine scheme.

Assume $\mathcal{Y} = Y$ and $\mathcal{U} = U$ are affine schemes. By Chow's lemma (Theorem 10.3) we can choose a surjective proper morphism $X \to \mathcal{X}$ where X is an algebraic space. We will use below that $X \to Y$ is separated as a composition of separated morphisms. Consider the algebraic space $W = X \times_\mathcal{X} U$. The projection morphism $W \to X$ is of finite type. We may replace X by the scheme theoretic image of $W \to X$ and hence we may assume that the image of $|W|$ in $|X|$ is dense in $|X|$ (here we use that the image of $|h|$ is dense in $|\mathcal{X}|$, so after this replacement, the morphism $X \to \mathcal{X}$ is still surjective). We claim that for every solid commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & W \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[urr, dotted] & & Y \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute. First, it is enough to prove there exists a dotted arrow after replacing K by an extension and A by a valuation ring in this extension dominating A, see Morphisms of Spaces, Lemma 41.4. By the assumption of the lemma we get an extension K'/K and a valuation ring $A' \subset K'$ dominating A and an arrow $\mathrm{Spec}(A') \to \mathcal{X}$ lifting the composition $\mathrm{Spec}(A') \to \mathrm{Spec}(A) \to Y$ and compatible with the composition $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \to W \to X$. Because $X \to \mathcal{X}$ is proper, we can use the valuative criterion of properness (Morphisms of Stacks, Lemma 43.1) to find an extension K''/K' and a valuation ring $A'' \subset K''$ dominating A' and a morphism $\mathrm{Spec}(A'') \to X$ lifting the composition $\mathrm{Spec}(A'') \to \mathrm{Spec}(A') \to \mathcal{X}$ and compatible with the composition $\mathrm{Spec}(K'') \to \mathrm{Spec}(K') \to \mathrm{Spec}(K) \to X$. Then K''/K and $A'' \subset K''$ and the morphism $\mathrm{Spec}(A'') \to X$ is a solution to the problem posed above and the claim holds.

The claim implies the morphism $X \to Y$ is proper by the case of the lemma for algebraic spaces (Limits of Spaces, Lemma 22.1). By Morphisms of Stacks, Lemma 37.6 we conclude that $\mathcal{X} \to Y$ is proper as desired. □

Lemma 11.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $h : \mathcal{U} \to \mathcal{X}$ be morphisms of algebraic stacks. Assume that $\mathcal{Y}$ is locally Noetherian, that f is locally of finite type and quasi-separated, that h is of finite type, and that the image of $|h| : |\mathcal{U}| \to |\mathcal{X}|$ is dense in $|\mathcal{X}|$. If given any 2-commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "u"'] \arrow[d, "j"'] & \mathcal{U} \arrow[r, "h"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rru, dotted] \arrow[rr, "y"] & & \mathcal{Y} \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to f \circ h \circ u$, the category of dotted arrows is either empty or a setoid with exactly one isomorphism class, then f is separated.

Proof. We have to prove Δ is a proper morphism. Assume first that Δ is separated. Then we may apply Lemma 11.3 to the morphisms $\mathcal{U} \to \mathcal{X}$ and $\Delta : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$. Observe that Δ is quasi-compact as f is quasi-separated. Of course Δ is locally of finite type (true for any diagonal morphism, see Morphisms of Stacks, Lemma 3.3). Finally, suppose given a 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "u"'] \arrow[d, "j"'] & \mathcal{U} \arrow[r, "h"'] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[rru, dotted] \arrow[rr, "y"] & & \mathcal{X} \times_\mathcal{Y} \mathcal{X} \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to \Delta \circ h \circ u$. By Morphisms of Stacks, Lemma 41.1 and the assumption in the lemma we find there exists a unique dotted arrow. This proves the last assumption of Lemma 11.3 holds and the result follows.

In the general case, it suffices to prove Δ is separated since then we'll be back in the previous case. In fact, we claim that the assumptions of the lemma hold for

$$\mathcal{U} \to \mathcal{X} \quad \text{and} \quad \Delta : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$$

Namely, since Δ is representable by algebraic spaces, the category of dotted arrows for a diagram as in the previous paragraph is a setoid (see for example Morphisms of Stacks, Lemma 39.2). The argument in the preceding paragraph shows these categories are either empty or have one isomorphism class. Thus Δ is separated. $\square$

Lemma 11.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume that $\mathcal{Y}$ is locally Noetherian and that f is of finite type. If given any 2-commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x"'] \arrow[d, "j"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K and $\gamma : y \circ j \to f \circ x$ there exist an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of dotted arrows for the induced diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r, "x'"'] \arrow[d, "j'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A') \arrow[r, "y'"] \arrow[ru, dotted] & \mathcal{Y} \end{tikzcd}$$

with induced 2-arrow $\gamma' : y' \circ j' \to f \circ x'$ is nonempty (Morphisms of Stacks, Definition 39.1), then f is universally closed.

Proof. Let $V \to \mathcal{Y}$ be a smooth morphism where V is an affine scheme. The category of dotted arrows behaves well with respect to base change (Morphisms of Stacks, Lemma 39.4). Hence the assumption on existence of dotted arrows (after extension) is inherited by the morphism $\mathcal{X} \times_{\mathcal{Y}} V \to V$. Therefore the assumptions of the lemma are satisfied for the morphism $\mathcal{X} \times_{\mathcal{Y}} V \to V$. Hence we may assume $\mathcal{Y}$ is an affine scheme.

Assume $\mathcal{Y} = Y$ is a Noetherian affine scheme. (From now on we no longer have to worry about the 2-arrows γ and γ', see Morphisms of Stacks, Lemma 39.3.) To prove that f is universally closed it suffices to show that $|\mathcal{X} \times \mathbf{A}^n| \to |Y \times \mathbf{A}^n|$ is closed for all n by Limits of Stacks, Lemma 7.2. Since the assumption in the lemma is inherited by the product morphism $\mathcal{X} \times \mathbf{A}^n \to Y \times \mathbf{A}^n$ (details omitted) we reduce to proving that $|\mathcal{X}| \to |Y|$ is closed.

Assume Y is a Noetherian affine scheme. Let $T \subset |\mathcal{X}|$ be a closed subset. We have to show that the image of T in $|Y|$ is closed. We may replace $\mathcal{X}$ by the reduced induced closed subspace structure on T; we omit the verification that property on the existence of dotted arrows is preserved by this replacement. Thus we reduce to proving that the image of $|\mathcal{X}| \to |Y|$ is closed.

Let $y \in |Y|$ be a point in the closure of the image of $|\mathcal{X}| \to |Y|$. By Lemma 11.1 we may choose a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y. It follows immediately from the assumption in the lemma that y is in the image of $|\mathcal{X}| \to |Y|$ and the proof is complete. $\square$

12. Moduli spaces

This section discusses morphisms $f : \mathcal{X} \to Y$ from algebraic stacks to algebraic spaces. Under suitable hypotheses Y is called a *moduli space* for $\mathcal{X}$. If $\mathcal{X} = [U/R]$ is a presentation, then we obtain an R-invariant morphism $U \to Y$ and (under suitable hypotheses) Y is a *quotient* of the groupoid (U, R, s, t, c). A discussion of the different types of quotients can be found starting with Quotients of Groupoids, Section 1.

Definition 12.1. Let $\mathcal{X}$ be an algebraic stack. Let $f : \mathcal{X} \to Y$ be a morphism to an algebraic space Y.

(1) We say f is a *categorical moduli space* if any morphism $\mathcal{X} \to W$ to an algebraic space W factors uniquely through f.
(2) We say f is a *uniform categorical moduli space* if for any flat morphism $Y' \to Y$ of algebraic spaces the base change $f' : Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space.

Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces.

(3) We say f is a *categorical moduli space in* $\mathcal{C}$ if $Y \in \mathrm{Ob}(\mathcal{C})$ and any morphism $\mathcal{X} \to W$ with $W \in \mathrm{Ob}(\mathcal{C})$ factors uniquely through f.
(4) We say is a *uniform categorical moduli space in* $\mathcal{C}$ if $Y \in \mathrm{Ob}(\mathcal{C})$ and for every flat morphism $Y' \to Y$ in $\mathcal{C}$ the base change $f' : Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space in $\mathcal{C}$.

By the Yoneda lemma a categorical moduli space, if it exists, is unique. Let us match this with the language introduced for quotients.

Lemma 12.2. *Let (U, R, s, t, c) be a groupoid in algebraic spaces with $s, t : R \to U$ flat and locally of finite presentation. Consider the algebraic stack $\mathcal{X} = [U/R]$. Given an algebraic space Y there is a 1-to-1 correspondence between morphisms $f : \mathcal{X} \to Y$ and R-invariant morphisms $\phi : U \to Y$.*

Proof. Criteria for Representability, Theorem 17.2 tells us $\mathcal{X}$ is an algebraic stack. Given a morphism $f : \mathcal{X} \to Y$ we let $\phi : U \to Y$ be the composition $U \to \mathcal{X} \to Y$. Since $R = U \times_{\mathcal{X}} U$ (Groupoids in Spaces, Lemma 22.2) it is immediate that ϕ is R-invariant. Conversely, if $\phi : U \to Y$ is an R-invariant morphism towards an algebraic space, we obtain a morphism $f : \mathcal{X} \to Y$ by Groupoids in Spaces, Lemma 23.2. You can also construct f from ϕ using the explicit description of the quotient stack in Groupoids in Spaces, Lemma 24.1. $\square$

Lemma 12.3. *With assumption and notation as in Lemma 12.2. Then f is a (uniform) categorical moduli space if and only if ϕ is a (uniform) categorical quotient. Similarly for moduli spaces in a full subcategory.*

Proof. It is immediate from the 1-to-1 correspondence established in Lemma 12.2 that f is a categorical moduli space if and only if ϕ is a categorical quotient (Quotients of Groupoids, Definition 4.1). If $Y' \to Y$ is a morphism, then $U' = Y' \times_Y U \to Y' \times_Y \mathcal{X} = \mathcal{X}'$ is a surjective, flat, locally finitely presented morphism as a base change of $U \to \mathcal{X}$ (Criteria for Representability, Lemma 17.1). And $R' = Y' \times_Y R$ is equal to $U' \times_{\mathcal{X}'} U'$ by transitivity of fibre products. Hence $\mathcal{X}' = [U'/R']$, see Algebraic Stacks, Remark 16.3. Thus the base change of our situation to Y' is another situation as in the statement of the lemma. From this it immediately follows that f is a uniform categorical moduli space if and only if ϕ is a uniform categorical quotient. $\square$

Lemma 12.4. *Let $f : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. If for every affine scheme Y' and flat morphism $Y' \to Y$ the base change $f' : Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space, then f is a uniform categorical moduli space.*

Proof. Choose an étale covering $\{Y_i \to Y\}$ where Y_i is an affine scheme. For each i and j choose a affine open covering $Y_i \times_Y Y_j = \bigcup Y_{ijk}$. Set $\mathcal{X}_i = Y_i \times_Y \mathcal{X}$ and $\mathcal{X}_{ijk} = Y_{ijk} \times_Y \mathcal{X}$. Let $g : \mathcal{X} \to W$ be a morphism towards an algebraic space. Then we consider the diagram

$$\begin{tikzcd} \mathcal{X}_i \arrow[r] \arrow[d] & \mathcal{X} \arrow[r, "g"'] \arrow[d] & W \\ Y_i \arrow[r] \arrow[urr, dotted] & Y & \end{tikzcd}$$

The assumption that $\mathcal{X}_i \to Y_i$ is a categorical moduli space, produces a unique dotted arrow $h_i : Y_i \to W$. The assumption that $\mathcal{X}_{ijk} \to Y_{ijk}$ is a categorical moduli space, implies the restriction of h_i and h_j to Y_{ijk} are equal. Hence h_i and h_j agree on $Y_i \times_Y Y_j$. Since $Y = \coprod Y_i / \coprod Y_i \times_Y Y_j$ (by Spaces, Section 9) we conclude that there is a unique morphism $Y \to W$ through which g factors. Thus f is a categorical moduli space. The same argument applies after a flat base change, hence f is a uniform categorical moduli space. $\square$

13. The Keel-Mori theorem

In this section we start discussing the theorem of Keel and Mori in the setting of algebraic stacks. For a discussion of the literature, please see Guide to Literature, Subsection 5.2.

Definition 13.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *well-nigh affine* if there exists an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$.

We give this property a somewhat ridiculous name because we do not intend to use it too much.

Lemma 13.2. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is well-nigh affine, and*
(2) *there exists a groupoid scheme (U, R, s, t, c) with U and R affine and $s, t : R \to U$ finite locally free such that $\mathcal{X} = [U/R]$.*

If true then $\mathcal{X}$ is quasi-compact, quasi-DM, and separated.

Proof. Assume $\mathcal{X}$ is well-nigh affine. Choose an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$. Set $R = U \times_{\mathcal{X}} U$. Then we obtain a groupoid (U, R, s, t, c) in algebraic spaces and an isomorphism $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 16.1 and Remark 16.3. Since $s, t : R \to U$ are flat, finite, and finitely presented morphisms (as base changes of $U \to \mathcal{X}$) we see that s, t are finite locally free (Morphisms, Lemma 48.2). This implies that R is affine (as finite morphisms are affine) and hence (2) holds.

Suppose that we have a groupoid scheme (U, R, s, t, c) with U and R are affine and $s, t : R \to U$ finite locally free. Set $\mathcal{X} = [U/R]$. Then $\mathcal{X}$ is an algebraic stack by Criteria for Representability, Theorem 17.2 (strictly speaking we don't need this here, but it can't be stressed enough that this is true). The morphism $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 17.1. Thus we can check whether $U \to \mathcal{X}$ is finite by checking whether the projection $U \times_{\mathcal{X}} U \to U$ has this property, see Properties of Stacks, Lemma 3.3. Since $U \times_{\mathcal{X}} U = R$ by Groupoids in Spaces, Lemma 22.2 we see that this is true. Thus $\mathcal{X}$ is well-nigh affine.

Proof of the final statement. We see that $\mathcal{X}$ is quasi-compact by Properties of Stacks, Lemma 6.2. We see that $\mathcal{X} = [U/R]$ is quasi-DM and separated by Morphisms of Stacks, Lemma 20.1. $\square$

Lemma 13.3. *Let the algebraic stack $\mathcal{X}$ be well-nigh affine.*

(1) *If $\mathcal{X}$ is an algebraic space, then it is affine.*
(2) *If $\mathcal{X}' \to \mathcal{X}$ is an affine morphism of algebraic stacks, then $\mathcal{X}'$ is well-nigh affine.*

Proof. Part (1) follows from immediately from Limits of Spaces, Lemma 15.1. However, this is overkill, since (1) also follows from Lemma 13.2 combined with Groupoids, Proposition 23.9.

To prove (2) we choose an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$. Then $U' = \mathcal{X}' \times_{\mathcal{X}} U$ admits an affine morphism to U (Morphisms of Stacks, Lemma 9.2). Therefore U' is an affine scheme. Of course $U' \to \mathcal{X}'$ is surjective, flat, finite, and finitely presented as a base change of $U \to \mathcal{X}$. $\square$

Lemma 13.4. *Let the algebraic stack $\mathcal{X}$ be well-nigh affine. There exists a uniform categorical moduli space*

$$f : \mathcal{X} \longrightarrow M$$

in the category of affine schemes. Moreover f is separated, quasi-compact, and a universal homeomorphism.

Proof. Write $\mathcal{X} = [U/R]$ with (U, R, s, t, c) as in Lemma 13.2. Let C be the ring of R-invariant functions on U, see Groupoids, Section 23. We set $M = \mathrm{Spec}(C)$. The R-invariant morphism $U \to M$ corresponds to a morphism $f : \mathcal{X} \to M$ by Lemma 12.2. The characterization of morphisms into affine schemes given in Schemes, Lemma 6.4 immediately guarantees that $\phi : U \to M$ is a categorical quotient in the category of affine schemes. Hence f is a categorical moduli space in the category of affine schemes (Lemma 12.3).

Since $\mathcal{X}$ is separated by Lemma 13.2 we find that f is separated by Morphisms of Stacks, Lemma 4.12.

Since $U \to \mathcal{X}$ is surjective and since $U \to M$ is quasi-compact, we see that f is quasi-compact by Morphisms of Stacks, Lemma 7.6.

By Groupoids, Lemma 23.4 the composition

$$U \to \mathcal{X} \to M$$

is an integral morphism of affine schemes. In particular, it is universally closed (Morphisms, Lemma 44.7). Since $U \to \mathcal{X}$ is surjective, it follows that $\mathcal{X} \to M$ is universally closed (Morphisms of Stacks, Lemma 37.6). To conclude that $\mathcal{X} \to M$ is a universal homeomorphism, it is enough to show that it is universally bijective, i.e., surjective and universally injective.

We have $|\mathcal{X}| = |U|/|R|$ by Morphisms of Stacks, Lemma 20.2. Thus $|f|$ is surjective and even bijective by Groupoids, Lemma 23.6.

Let $C \to C'$ be a ring map. Let (U', R', s', t', c') be the base change of (U, R, s, t, c) by $M' = \mathrm{Spec}(C') \to M$. Setting $\mathcal{X}' = [U'/R']$, we observe that $M' \times_M \mathcal{X} = \mathcal{X}'$ by Quotients of Groupoids, Lemma 3.6. Let C^1 be the ring of R'-invariant functions on U'. Set $M^1 = \mathrm{Spec}(C^1)$ and consider the diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"] & \mathcal{X} \arrow[dd, "f"] \\ M^1 \arrow[d] & \\ M' \arrow[r] & M \end{tikzcd}$$

By Groupoids, Lemma 23.5 and Algebra, Lemma 46.11 the morphism $M^1 \to M'$ is a homeomorphism. On the other hand, the previous paragraph applied to (U', R', s', t', c') shows that $|f'|$ is bijective. We conclude that f induces a bijection on points after any base change by an affine scheme. Thus f is universally injective by Morphisms of Stacks, Lemma 14.7.

Finally, we still have to show that f is a uniform moduli space in the category of affine schemes. This follows from the discussion above and the fact that if the ring map $C \to C'$ is flat, then $C' \to C^1$ is an isomorphism by Groupoids, Lemma 23.5. $\square$

Lemma 13.5. *Let $h : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks. Assume $\mathcal{X}'$ and $\mathcal{X}$ are well-nigh affine, h is étale, and h induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 19.5). Then there exists a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ M' \arrow[r] & M \end{tikzcd}$$

where $M' \to M$ is étale and the vertical arrows are the moduli spaces constructed in Lemma 13.4.

Proof. Observe that h is representable by algebraic spaces by Morphisms of Stacks, Lemmas 45.3 and 45.1. Choose an affine scheme U and a surjective, flat, finite, and finitely presented morphism $U \to \mathcal{X}$. Then $U' = \mathcal{X}' \times_{\mathcal{X}} U$ is an algebraic space with a finite (in particular affine) morphism $U' \to \mathcal{X}'$. By Lemma 13.3 we conclude that U' is affine. Setting $R = U \times_{\mathcal{X}} U$ and $R' = U' \times_{\mathcal{X}'} U'$ we obtain groupoids (U, R, s, t, c) and (U', R', s', t', c') such that $\mathcal{X} = [U/R]$ and $\mathcal{X}' = [U'/R']$, see proof of Lemma 13.2. we see that the diagrams

$$
\begin{tikzcd}
R' \arrow[r, "f"'] \arrow[d, "s'"'] & R \arrow[d, "s"] \\
U' \arrow[r, "f"] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
R' \arrow[r, "f"'] \arrow[d, "t'"'] & R \arrow[d, "t"] \\
U' \arrow[r, "f"] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
G' \arrow[r, "f"'] \arrow[d] & G \arrow[d] \\
U' \arrow[r, "f"] & U
\end{tikzcd}
$$

are cartesian where G and G' are the stabilizer group schemes. This follows for the first two by transitivity of fibre products and for the last one this follows because it is the pullback of the isomorphism $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}' \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}}$ (by the already used Morphisms of Stacks, Lemma 45.3). Recall that M, resp. M' was constructed in Lemma 13.4 as the spectrum of the ring of R-invariant functions on U, resp. the ring of R'-invariant functions on U'. Thus $M' \to M$ is étale and $U' = M' \times_M U$ by Groupoids, Lemma 23.7. It follows that $R' = M' \times_M U$, in other words the groupoid (U', R', s', t', c') is the base change of (U, R, s, t, c) by $M' \to M$. This implies that the diagram in the lemma is cartesian by Quotients of Groupoids, Lemma 3.6. □

Lemma 13.6. *Let the algebraic stack $\mathcal{X}$ be well-nigh affine. The morphism*

$$f : \mathcal{X} \longrightarrow M$$

of Lemma 13.4 is a uniform categorical moduli space.

Proof. We already know that M is a uniform categorical moduli space in the category of affine schemes. By Lemma 12.4 it suffices to show that the base change $f' : M' \times_M \mathcal{X} \to M'$ is a categorical moduli space for any flat morphism $M' \to M$ of affine schemes. Observe that $\mathcal{X}' = M' \times_M \mathcal{X}$ is well-nigh affine by Lemma 13.3. This after replacing $\mathcal{X}$ by $\mathcal{X}'$ and M by M', we reduce to proving f is a categorical moduli space.

Let $g : \mathcal{X} \to Y$ be a morphism where Y is an algebraic space. We have to show that $g = h \circ f$ for a unique morphism $h : M \to Y$.

Uniqueness. Suppose we have two morphisms $h_i : M \to Y$ with $g = h_1 \circ f = h_2 \circ f$. Let $M' \subset M$ be the equalizer of h_1 and h_2. Then $M' \to M$ is a monomorphism and $f : \mathcal{X} \to M$ factors through M'. Thus $M' \to M$ is a universal homeomorphism. We conclude M' is affine (Morphisms, Lemma 45.5). But then since $f : \mathcal{X} \to M$ is a categorical moduli space in the category of affine schemes, we see $M' = M$.

Existence. Below we will show that for every $p \in M$ there exists a cartesian square

$$
\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
M' \arrow[r] & M
\end{tikzcd}
$$

with $M' \to M$ an étale morphism of affines and p in the image such that the composition $\mathcal{X}' \to \mathcal{X} \to Y$ factors through M'. This means we can construct the map $h : M \to Y$ étale locally on M. Since Y is a sheaf for the étale topology and by the uniqueness shown above, this is enough (small detail omitted).

Let $y \in |Y|$ be the image of p. Let $(V, v) \to (Y, y)$ be an étale morphism with V affine. Consider $\mathcal{X}' = V \times_Y \mathcal{X}$. Observe that $\mathcal{X}' \to \mathcal{X}$ is separated and étale as the base change of $V \to Y$. Moreover, $\mathcal{X}' \to \mathcal{X}$ induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 19.5) as this is true for $V \to$

Y, see Morphisms of Stacks, Lemma 45.5. Choose a presentation $\mathcal{X} = [U/R]$ as in Lemma 13.2. Set $U' = \mathcal{X}' \times_\mathcal{X} U = V \times_Y U$ and choose $u' \in U'$ mapping to p and v (possible by Properties of Spaces, Lemma 4.3). Since $U' \to U$ is separated and étale we see that every finite set of points of U' is contained in an affine open, see More on Morphisms, Lemma 45.1. On the other hand, the morphism $U' \to \mathcal{X}'$ is surjective, finite, flat, and locally of finite presentation. Setting $R' = U' \times_{\mathcal{X}'} U'$ we see that $s', t' : R' \to U'$ are finite locally free. By Groupoids, Lemma 24.1 there exists an R'-invariant affine open subscheme $U'' \subset U'$ containing u'. Let $\mathcal{X}'' \subset \mathcal{X}'$ be the corresponding open substack. Then $\mathcal{X}''$ is well-nigh affine. By Lemma 13.5 we obtain a cartesian square

$$\begin{tikzcd} \mathcal{X}'' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ M'' \arrow[r] & M \end{tikzcd}$$

with $M'' \to M$ étale. Since $\mathcal{X}'' \to M''$ is a categorical moduli space in the category of affine schemes we obtain a morphism $M'' \to V$ such that the composition $\mathcal{X}'' \to \mathcal{X}' \to V$ is equal to the composition $\mathcal{X}'' \to M'' \to V$. This proves our claim and finishes the proof. $\square$

Lemma 13.7. *Let $h : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks. Assume $\mathcal{X}$ is well-nigh affine, h is étale, h is separated, and h induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 19.5). Then there exists a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ M' \arrow[r] & M \end{tikzcd}$$

where $M' \to M$ is a separated étale morphism of schemes and $\mathcal{X} \to M$ is the moduli space constructed in Lemma 13.4.

Proof. Choose an affine scheme U and a surjective, flat, finite, and locally finitely presented morphism $U \to \mathcal{X}$. Since h is representable by algebraic spaces (Morphisms of Stacks, Lemmas 45.3 and 45.1) we see that $U' = \mathcal{X}' \times_\mathcal{X} U$ is an algebraic space. Since $U' \to U$ is separated and étale, we see that U' is a scheme and that every finite set of points of U' is contained in an affine open, see Morphisms of Spaces, Lemma 51.1 and More on Morphisms, Lemma 45.1. Setting $R' = U' \times_{\mathcal{X}'} U'$ we see that $s', t' : R' \to U'$ are finite locally free. By Groupoids, Lemma 24.1 there exists an open covering $U' = \bigcup U'_i$ by R'-invariant affine open subschemes $U'_i \subset U'$. Let $\mathcal{X}'_i \subset \mathcal{X}'$ be the corresponding open substacks. These are well-nigh affine as $U'_i \to \mathcal{X}'_i$ is surjective, flat, finite and of finite presentation. By Lemma 13.5 we obtain cartesian diagrams

$$\begin{tikzcd} \mathcal{X}'_i \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ M'_i \arrow[r] & M \end{tikzcd}$$

with $M'_i \to M$ an étale morphism of affine schemes and vertical arrows as in Lemma 13.4. Observe that $\mathcal{X}'_{ij} = \mathcal{X}'_i \cap \mathcal{X}'_j$ is an open subspace of $\mathcal{X}'_i$ and $\mathcal{X}'_j$. Hence we get corresponding open subschemes $V_{ij} \subset M'_i$ and $V_{ji} \subset M'_j$. By the result of Lemma

13.6 we see that both $\mathcal{X}'_{ij} \to V_{ij}$ and $\mathcal{X}'_{ji} \to V_{ji}$ are categorical moduli spaces! Thus we get a unique isomorphism $\varphi_{ij} : V_{ij} \to V_{ji}$ such that

$$\begin{tikzcd}
\mathcal{X}'_i \arrow[d] & & \mathcal{X}'_i \cap \mathcal{X}'_j \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dr] & & \mathcal{X}'_j \arrow[d] \\
M'_i & V_{ij} \arrow[l] \arrow[rr, "\varphi_{ij}"] & & V_{ji} \arrow[r] & M'_j
\end{tikzcd}$$

is commutative. These isomorphisms satisfy the cocyclce condition of Schemes, Section 14 by a computation (and another application of the previous lemma) which we omit. Thus we can glue the affine schemes in to scheme M', see Schemes, Lemma 14.1. Let us identify the M'_i with their image in M'. We claim there is a morphism $\mathcal{X}' \to M'$ fitting into cartesian diagrams

$$\begin{tikzcd}
\mathcal{X}'_i \arrow[r] \arrow[d] & \mathcal{X}' \arrow[d] \\
M'_i \arrow[r] & M'
\end{tikzcd}$$

This is clear from the description of the morphisms into the glued scheme M' in Schemes, Lemma 14.1 and the fact that to give a morphism $\mathcal{X}' \to M'$ is the same thing as given a morphism $T \to M'$ for any morphism $T \to \mathcal{X}'$. Similarly, there is a morphism $M' \to M$ restricting to the given morphisms $M'_i \to M$ on M'_i. The morphism $M' \to M$ is étale (being étale on the members of an étale covering) and the fibre product property holds as it can be checked on members of the (affine) open covering $M' = \bigcup M'_i$. Finally, $M' \to M$ is separated because the composition $U' \to \mathcal{X}' \to M'$ is surjective and universally closed and we can apply Morphisms, Lemma 41.11. $\square$

Lemma 13.8. *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite. Then there exist a set I and for $i \in I$ a morphism of algebraic stacks*

$$g_i : \mathcal{X}_i \longrightarrow \mathcal{X}$$

with the following properties

1. $|\mathcal{X}| = \bigcup |g_i|(|\mathcal{X}_i|)$,
2. $\mathcal{X}_i$ *is well-nigh affine,*
3. $\mathcal{I}_{\mathcal{X}_i} \to \mathcal{X}_i \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ *is an isomorphism, and*
4. $g_i : \mathcal{X}_i \to \mathcal{X}$ *is representable by algebraic spaces, separated, and étale,*

Proof. For any $x \in |\mathcal{X}|$ we can choose $g : \mathcal{U} \to \mathcal{X}$, $\mathcal{U} = [U/R]$, and u as in Morphisms of Stacks, Lemma 32.4. Then by Morphisms of Stacks, Lemma 45.4 we see that there exists an open substack $\mathcal{U}' \subset \mathcal{U}$ containing u such that $\mathcal{I}_{\mathcal{U}'} \to \mathcal{U}' \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is an isomorphism. Let $U' \subset U$ be the R-invariant open corresponding to the open substack $\mathcal{U}'$. Let $u' \in U'$ be a point of U' mapping to u. Observe that $t(s^{-1}(\{u'\}))$ is finite as $s : R \to U$ is finite. By Properties, Lemma 29.5 and Groupoids, Lemma 24.1 we can find an R-invariant affine open $U'' \subset U'$ containing u'. Let R'' be the restriction of R to U''. Then $\mathcal{U}'' = [U''/R'']$ is an open substack of $\mathcal{U}'$ containing u, is well-nigh affine, $\mathcal{I}_{\mathcal{U}''} \to \mathcal{U}'' \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is an isomorphism, and $\mathcal{U}'' \to \mathcal{X}$ and is representable by algebraic spaces and étale. Finally, $\mathcal{U}'' \to \mathcal{X}$ is separated as $\mathcal{U}''$ is separated (Lemma 13.2) the diagonal of $\mathcal{X}$ is separated (Morphisms of Stacks, Lemma 6.1) and separatedness follows from

Morphisms of Stacks, Lemma 4.12. Since the point $x \in |\mathcal{X}|$ is arbitrary the proof is complete. $\square$

Theorem 13.9 (Keel-Mori)**.** *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite. Then there exists a uniform categorical moduli space*

$$f : \mathcal{X} \longrightarrow M$$

and f is separated, quasi-compact, and a universal homeomorphism.

Proof. We choose a set I[4] and for $i \in I$ a morphism of algebraic stacks $g_i : \mathcal{X}_i \to \mathcal{X}$ as in Lemma 13.8; we will use all of the properties listed in this lemma without further mention. Let

$$f_i : \mathcal{X}_i \to M_i$$

be as in Lemma 13.4. Consider the stacks

$$\mathcal{X}_{ij} = \mathcal{X}_i \times_{g_i, \mathcal{X}, g_j} \mathcal{X}_j$$

for $i, j \in I$. The projections $\mathcal{X}_{ij} \to \mathcal{X}_i$ and $\mathcal{X}_{ij} \to \mathcal{X}_j$ are separated by Morphisms of Stacks, Lemma 4.4, étale by Morphisms of Stacks, Lemma 35.3, and induce isomorphisms on automorphism groups (as in Morphisms of Stacks, Remark 19.5) by Morphisms of Stacks, Lemma 45.5. Thus we may apply Lemma 13.7 to find a commutative diagram

$$\begin{tikzcd}
\mathcal{X}_i \arrow[d, "f_i"] & \mathcal{X}_{ij} \arrow[l] \arrow[r] \arrow[d, "f_{ij}"] & \mathcal{X}_j \arrow[d, "f_j"] \\
M_i & M_{ij} \arrow[l] \arrow[r] & M_j
\end{tikzcd}$$

with cartesian squares where $M_{ij} \to M_i$ and $M_{ij} \to M_j$ are separated étale morphisms of schemes; here we also use that f_i is a uniform categorical quotient by Lemma 13.6. Claim:

$$\coprod M_{ij} \longrightarrow \coprod M_i \times \coprod M_i$$

is an étale equivalence relation.

Proof of the claim. Set $R = \coprod M_{ij}$ and $U = \coprod M_i$. We have already seen that $t : R \to U$ and $s : R \to U$ are étale. Let us construct a morphism $c : R \times_{s,U,t} R \to R$ compatible with $\text{pr}_{13} : U \times U \times U \to U \times U$. Namely, for $i, j, k \in I$ we consider

$$\mathcal{X}_{ijk} = \mathcal{X}_i \times_{g_i, \mathcal{X}, g_j} \mathcal{X}_j \times_{g_j, \mathcal{X}, g_k} \mathcal{X}_k = \mathcal{X}_{ij} \times_{\mathcal{X}_j} \mathcal{X}_{jk}$$

Arguing exactly as in the previous paragraph, we find that $M_{ijk} = M_{ij} \times_{M_j} M_{jk}$ is a categorical moduli space for $\mathcal{X}_{ijk}$. In particular, there is a canonical morphism $M_{ijk} = M_{ij} \times_{M_j} M_{jk} \to M_{ik}$ coming from the projection $\mathcal{X}_{ijk} \to \mathcal{X}_{ik}$. Putting these morphisms together we obtain the morphism c. In a similar fashion we construct a morphism $e : U \to R$ compatible with $\Delta : U \to U \times U$ and $i : R \to R$ compatible with the flip $U \times U \to U \times U$. Let k be an algebraically closed field. Then

$$\text{Mor}(\text{Spec}(k), \mathcal{X}_i) \to \text{Mor}(\text{Spec}(k), M_i) = M_i(k)$$

[4]The reader who is still keeping track of set theoretic issues should make sure I is not too large.

is bijective on isomorphism classes and the same remains true after any base change by a morphism $M' \to M$. This follows from our choice of f_i and Morphisms of Stacks, Lemmas 14.5 and 14.6. By construction of 2-fibred products the diagram

$$\begin{tikzcd} \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}_{ij}) \arrow[r] \arrow[d] & \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}_j) \arrow[d] \\ \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}_i) \arrow[r] & \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}) \end{tikzcd}$$

is a fibre product of categories. By our choice of g_i the functors in this diagram induce bijections on automorphism groups. It follows that this diagram induces a fibre product diagram on sets of isomorphism classes! Thus we see that

$$R(k) = U(k) \times_{|\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})|} U(k)$$

where $|\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})|$ denotes the set of isomorphism classes. In particular, for any algebraically closed field k the map on k-valued point is an equivalence relation. We conclude the claim holds by Groupoids, Lemma 3.5.

Let $M = U/R$ be the algebraic space which is the quotient of the above étale equivalence relation, see Spaces, Theorem 10.5. There is a canonical morphism $f : \mathcal{X} \to M$ fitting into commutative diagrams

$$\begin{tikzcd} \mathcal{X}_i \arrow[r, "g_i"'] \arrow[d, "f_i"'] & \mathcal{X} \arrow[d, "f"] \\ M_i \arrow[r] & M \end{tikzcd} \tag{13.9.1}$$

Namely, such a morphism f is given by a functor

$$f : \mathrm{Mor}(T, \mathcal{X}) \longrightarrow \mathrm{Mor}(T, M)$$

for any scheme T compatible with base change. Let $a : T \to \mathcal{X}$ be an object of the left hand side. We obtain an étale covering $\{T_i \to T\}$ with $T_i = \mathcal{X}_i \times_{\mathcal{X}} T$ and morphisms $a_i : T_i \to \mathcal{X}_i$. Then we get $b_i = f_i \circ a_i : T_i \to M_i$. Since $T_i \times_T T_j = \mathcal{X}_{ij} \times_{\mathcal{X}} T$ we moreover get a morphism $a_{ij} : T_i \times_T T_j \to \mathcal{X}_{ij}$. Setting $b_{ij} = f_{ij} \circ a_{ij}$ we find that $b_i \times b_j$ factors through the monomorphism $M_{ij} \to M_i \times M_j$. Hence the morphisms

$$T_i \xrightarrow{b_i} M_i \to M$$

agree on $T_i \times_T T_j$. As M is a sheaf for the étale topology, we see that these morphisms glue to a unique morphism $b = f(a) : T \to M$. We omit the verification that this construction is compatible with base change and we omit the verification that the diagrams (13.9.1) commute.

Claim: the diagrams (13.9.1) are cartesian. To see this we study the induced morphism

$$h_i : \mathcal{X}_i \longrightarrow M_i \times_M \mathcal{X}$$

This is a morphism of stacks étale over $\mathcal{X}$ and hence h_i is étale (Morphisms of Stacks, Lemma 35.6). Since g_i is separated, we see h_i is separated (use Morphisms of Stacks, Lemma 4.12 and the fact seen above that the diagonal of $\mathcal{X}$ is separated). The morphism h_i induces isomorphisms on automorphism groups (Morphisms of

Stacks, Remark 19.5) as this is true for g_i. For an algebraically closed field k the diagram

$$\begin{tikzcd} \mathrm{Mor}(\mathrm{Spec}(k), M_i \times_M \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X}) \arrow[d] \\ M_i(k) \arrow[r] & M(k) \end{tikzcd}$$

is a catesian diagram of categories and the top arrow induces bijections on automorphism groups. On the other hand, we have

$$M(k) = U(k)/R(k) = U(k)/U(k) \times_{|\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})|} U(k) = |\mathrm{Mor}(\mathrm{Spec}(k), \mathcal{X})|$$

by what we said above. Thus the right vertical arrow in the cartesian diagram above is a bijection on isomorphism classes. We conclude that $|\mathrm{Mor}(\mathrm{Spec}(k), M_i \times_M \mathcal{X})| \to M_i(k)$ is bijective. Review: h_i is a separated, étale, induces isomorphisms on automorphism groups (as in Morphisms of Stacks, Remark 19.5), and induces an equivalence on fibre categories over algebraically closed fields. Hence it is an isomorphism by Morphisms of Stacks, Lemma 45.7.

From the claim we get in particular the following: we have a surjective étale morphism $U \to M$ such that the base change of f is separated, quasi-compact, and a universal homeomorphism. It follows that f is separated, quasi-compact, and a universal homeomorphism. See Morphisms of Stacks, Lemma 4.5, 7.10, and 15.5

To finish the proof we have to show that $f : \mathcal{X} \to M$ is a uniform categorical moduli space. To prove this it suffices to show that given a flat morphism $M' \to M$ of algebraic spaces, the base change

$$M' \times_M \mathcal{X} \longrightarrow M'$$

is a categorical moduli space. Thus we consider a morphism

$$\theta : M' \times_M \mathcal{X} \longrightarrow E$$

where E is an algebraic space. For each i we know that f_i is a uniform categorical moduli space. Hence we obtain

$$\begin{tikzcd} M' \times_M \mathcal{X}_i \arrow[r] \arrow[d] & M' \times_M \mathcal{X} \arrow[d, "\theta"] \\ M' \times_M M_i \arrow[r, "\psi_i"] & E \end{tikzcd}$$

Since $\{M' \times_M M_i \to M'\}$ is an étale covering, to obtain the desired morphism $\psi : M' \to E$ it suffices to show that ψ_i and ψ_j agree over $M' \times_M M_i \times_M M_j = M' \times_M M_{ij}$. This follows easily from the fact that $f_{ij} : \mathcal{X}_{ij} = \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_j \to M_{ij}$ is a uniform categorical quotient; details omitted. Then finally one shows that ψ fits into the commutative diagram

$$\begin{tikzcd} M' \times_M \mathcal{X} \arrow[d] \arrow[dr, "\theta"] & \\ M' \arrow[r, "\psi"] & E \end{tikzcd}$$

because "$\{M' \times_M \mathcal{X}_i \to M' \times_M \mathcal{X}\}$ is an étale covering" and the morphisms ψ_i fit into the corresponding commutative diagrams by construction. This finishes the proof of the Keel-Mori theorem. □

The following lemma emphasizes the étale local nature of the construction.

Lemma 13.10. *Let $h : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks. Assume*

1. *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite,*
2. *h is étale, separated, and induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 19.5).*

Then there exists a cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ M' \arrow[r] & M \end{tikzcd}$$

where $M' \to M$ is a separated étale morphism of algebraic spaces and the vertical arrows are the moduli spaces constructed in Theorem 13.9.

Proof. By Morphisms of Stacks, Lemma 45.3 we see that $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}' \times_\mathcal{X} \mathcal{I}_\mathcal{X}$ is an isomorphism. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is finite as a base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$. Let $f' : \mathcal{X}' \to M'$ and $f : \mathcal{X} \to M$ be as in Theorem 13.9. We obtain a commutative diagram as in the lemma because f' is categorical moduli space. Choose I and $g'_i : \mathcal{X}'_i \to \mathcal{X}'$ as in Lemma 13.8. Observe that $g_i = h \circ g'_i$ is étale, separated, and induces isomorphisms on automorphism groups (Morphisms of Stacks, Remark 19.5). Let $f'_i : \mathcal{X}'_i \to M'_i$ be as in Lemma 13.4. In the proof of Theorem 13.9 we have seen that the diagrams

$$\begin{tikzcd} \mathcal{X}'_i \arrow[r, "g'_i"'] \arrow[d, "f'_i"'] & \mathcal{X}' \arrow[d, "f'"] \\ M'_i \arrow[r] & M' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{X}'_i \arrow[r, "g_i"'] \arrow[d, "f'_i"'] & \mathcal{X} \arrow[d, "f"] \\ M'_i \arrow[r] & M \end{tikzcd}$$

are cartesian and that $M'_i \to M'$ and $M'_i \to M$ are étale (this also follows directly from the properties of the morphisms g'_i, g_i, f', f'_i, f listed sofar by arguing in exactly the same way). This first implies that $M' \to M$ is étale and second that the diagram in the lemma is cartesian. We still need to show that $M' \to M$ is separated. To do this we contemplate the diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X}' \times_\mathcal{X} \mathcal{X}' \arrow[d] \\ M' \arrow[r] & M' \times_M M' \end{tikzcd}$$

The top horizontal arrow is universally closed as $\mathcal{X}' \to \mathcal{X}$ is separated. The vertical arrows are as in Theorem 13.9 (as flat base changes of $\mathcal{X} \to M$) hence universal homeomorphisms. Thus the lower horizontal arrow is universally closed. This (combined with it being an étale monomorphism of algebraic spaces) proves it is a closed immersion as desired. $\square$

14. Properties of moduli spaces

Once the existence of a moduli space has been proven, it becomes possible (and is usually straightforward) to esthablish properties of these moduli spaces.

Lemma 14.1. *Let $p : \mathcal{X} \to Y$ be a morphism of an algebraic stack to an algebraic space. Assume*

1. *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite,*
2. *Y is locally Noetherian, and*
3. *p is locally of finite type.*

Let $f : \mathcal{X} \to M$ be the moduli space constructed in Theorem 13.9. Then $M \to Y$ is locally of finite type.

Proof. Since f is a uniform categorical moduli space we obtain the morphism $M \to Y$. It suffices to check that $M \to Y$ is locally of finite type étale locally on M and Y. Since f is a uniform categorical moduli space, we may first replace Y by an affine scheme étale over Y. Next, we may choose I and $g_i : \mathcal{X}_i \to \mathcal{X}$ as in Lemma 13.8. Then by Lemma 13.10 we reduce to the case $\mathcal{X} = \mathcal{X}_i$. In other words, we may assume $\mathcal{X}$ is well-nigh affine. In this case we have $Y = \mathrm{Spec}(A_0)$, we have $\mathcal{X} = [U/R]$ with $U = \mathrm{Spec}(A)$ and $M = \mathrm{Spec}(C)$ where $C \subset A$ is the set of R-invariant functions on U. See Lemmas 13.2 and 13.4. Then A_0 is Noetherian and $A_0 \to A$ is of finite type. Moreover A is integral over C by Groupoids, Lemma 23.4, hence finite over C (being of finite type over A_0). Thus we may finally apply Algebra, Lemma 51.7 to conclude. $\square$

Lemma 14.2. *Let $\mathcal{X}$ be an algebraic stack. Assume $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite. Let $f : \mathcal{X} \to M$ be the moduli space constructed in Theorem 13.9.*

1. *If $\mathcal{X}$ is quasi-separated, then M is quasi-separated.*
2. *If $\mathcal{X}$ is separated, then M is separated.*
3. *Add more here, for example relative versions of the above.*

Proof. To prove this consider the following diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Delta_\mathcal{X}"'] \arrow[d, "f"'] & \mathcal{X} \times \mathcal{X} \arrow[d, "f \times f"] \\ M \arrow[r, "\Delta_M"] & M \times M \end{tikzcd}$$

Since f is a universal homeomorphism, we see that $f \times f$ is a universal homeomorphism.

If $\mathcal{X}$ is separated, then $\Delta_\mathcal{X}$ is proper, hence $\Delta_\mathcal{X}$ is universally closed, hence Δ_M is universally closed, hence M is separated by Morphisms of Spaces, Lemma 40.9.

If $\mathcal{X}$ is quasi-separated, then $\Delta_\mathcal{X}$ is quasi-compact, hence Δ_M is quasi-compact, hence M is quasi-separated. $\square$

Lemma 14.3. *Let $p : \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. Assume*

1. *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is finite,*
2. *p is proper, and*
3. *Y is locally Noetherian.*

Let $f : \mathcal{X} \to M$ be the moduli space constructed in Theorem 13.9. Then $M \to Y$ is proper.

Proof. By Lemma 14.1 we see that $M \to Y$ is locally of finite type. By Lemma 14.2 we see that $M \to Y$ is separated. Of course $M \to Y$ is quasi-compact and universally closed as these are topological properties and $\mathcal{X} \to Y$ has these properties and $\mathcal{X} \to M$ is a universal homeomorphism. □

15. Stacks and fpqc coverings

Certain algebraic stacks satisfy fpqc descent. The analogue of this section for algebraic spaces is Properties of Spaces, Section 17.

Proposition 15.1. *Let $\mathcal{X}$ be an algebraic stack with quasi-affine*[5] *diagonal. Then $\mathcal{X}$ satisfies descent for fpqc coverings.*

Proof. Our conventions are that $\mathcal{X}$ is a stack in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ over the category of schemes over a base scheme S endowed with the fppf topology. The statement means the following: given an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of schemes over S the functor

$$\mathcal{X}_U \longrightarrow DD(\mathcal{U})$$

is an equivalence. Here on the left we have the category of objects of $\mathcal{X}$ over U and on the right we have the category of descent data in $\mathcal{X}$ relative to $\mathcal{U}$. See discussion in Stacks, Section 3.

Fully faithfulness. Suppose we have two objects x, y of $\mathcal{X}$ over U. Then $I = \mathit{Isom}(x, y)$ is an algebraic space over U. Hence a collection of sections of I over U_i whose restrictions to $U_i \times_U U_j$ agree, come from a unique section over U by the analogue of the proposition for algebraic spaces, see Properties of Spaces, Proposition 17.1. Thus our functor is fully faithful.

Essential surjectivity. Here we are given objects x_i over U_i and isomorphisms $\varphi_{ij} : \mathrm{pr}_0^* x_i \to \mathrm{pr}_1^* x_j$ over $U_i \times_U U_j$ satisfying the cocyle condition over $U_i \times_U U_j \times_U U_k$. Let W be an affine scheme and let $W \to \mathcal{X}$ be a morphism. For each i we can form

$$W_i = U_i \times_{x_i, \mathcal{X}} W$$

The projection $W_i \to U_i$ is quasi-affine as the diagonal of $\mathcal{X}$ is quasi-affine. For each pair $i, j \in I$ the isomorphism φ_{ij} induces an isomorphism

$$W_i \times_U U_j = (U_i \times_U U_j) \times_{x_i \circ \mathrm{pr}_0, \mathcal{X}} W \to (U_i \times_U U_j) \times_{x_j \circ \mathrm{pr}_1, \mathcal{X}} W = U_i \times_U W_j$$

Moreover, these isomorphisms satisfy the cocycle condition over $U_i \times_U U_j \times_U U_k$. In other words, these isomorphisms define a descent datum on the schemes W_i/U_i relative to $\mathcal{U}$. By Descent, Lemma 38.1 we see that this descent datum is effective[6]. We conclude that there exists a quasi-affine morphism $W' \to U$ and a commutative diagram

$$\begin{tikzcd} W' \arrow[d] & W_i \arrow[l] \arrow[r] \arrow[d] & W \arrow[d] \\ U & U_i \arrow[l] \arrow[r, "x_i"] & \mathcal{X} \end{tikzcd}$$

whose squares are cartesian. Since $\{W_i \to W'\}_{i \in I}$ is the base change of $\mathcal{U}$ by $W' \to U$ we conclude that it is an fpqc covering. Since W satisfies the sheaf condition for fpqc coverings, we obtain a unique morphism $W' \to W$ such that

[5]It suffices to assume ind-quasi-affine.

[6]Or use More on Groupoids, Lemma 15.3 in the case of ind-quasi-affine diagonal.

$W_i \to W' \to W$ is the given morphism $W_i \to W$. In other words, we have the commutative diagrams

$$\begin{tikzcd} W_i \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & W \arrow[d] \\ U_i \arrow[r] \arrow[rr, bend right, "x_i"'] & U & \mathcal{X} \end{tikzcd}$$

compatible with the isomorphisms φ_{ij} and whose square and rectangle are cartesian.

Choose a collection of affine schemes W_α, $\alpha \in A$ and smooth morphisms $W_\alpha \to \mathcal{X}$ such that $\coprod W_\alpha \to \mathcal{X}$ is surjective. By the procedure of the preceding paragraph we produce a diagram

$$\begin{tikzcd} W_{\alpha,i} \arrow[r] \arrow[d] & W'_\alpha \arrow[r] \arrow[d] & W_\alpha \arrow[d] \\ U_i \arrow[r] \arrow[rr, bend right, "x_i"'] & U & \mathcal{X} \end{tikzcd}$$

for each α. Then the morphisms $W'_\alpha \to U$ are smooth and jointly surjective.

Denote x_α the object of $\mathcal{X}$ over W'_α corresponding to $W'_\alpha \to W_\alpha \to \mathcal{X}$. Since $\mathcal{X}$ is an fppf stack and since $\{W'_\alpha \to U\}$ is an fppf covering, it suffices to show that there are isomorphisms $\text{pr}_0^*x_\alpha \to \text{pr}_1^*x_\beta$ over $W'_\alpha \times_U W'_\beta$ satisfying the cocycle condition. However, after pulling back to $W_{\alpha,i}$ we do have such isomorphisms over $W_{\alpha,i} \times_{U_i} W_{\beta,i} = U_i \times_U (W'_\alpha \times_U W'_\beta)$ since the pullback of x_α to $W_{\alpha,i}$ is isomorphic to the pullback of x_i to $W_{\alpha,i}$. Since $\{U_i \times_U (W'_\alpha \times_U W'_\beta) \to W'_\alpha \times_U W'_\beta\}_{i \in I}$ is an fpqc covering and by the aforementioned compatibility of the diagrams above with φ_{ij} these isomorphisms descend to $W'_\alpha \times_U W'_\beta$ and the proof is complete. □

16. Tensor functors

Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of Noetherian algebraic stacks. The pullback functor

$$f^* : \textit{Coh}(\mathcal{O}_\mathcal{X}) \longrightarrow \textit{Coh}(\mathcal{O}_\mathcal{Y})$$

is a right exact tensor functor: it is additive, right exact, and commutes with tensor products of coherent modules. We can ask to what extent any right exact tensor functor $F : \textit{Coh}(\mathcal{O}_\mathcal{X}) \to \textit{Coh}(\mathcal{O}_\mathcal{Y})$ comes from a morphism $f : \mathcal{Y} \to \mathcal{X}$. The reader may consult [HR19] for a very general result of this nature. The aim of this section is to give a short proof of Theorem 16.8 as an introduction to these ideas.

We begin with some lemmas.

Lemma 16.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Noetherian algebraic stacks. Any right exact tensor functor $F : \textit{Coh}(\mathcal{O}_\mathcal{X}) \to \textit{Coh}(\mathcal{O}_\mathcal{Y})$ extends uniquely to a right exact tensor functor $F : \textit{QCoh}(\mathcal{O}_\mathcal{X}) \to \textit{QCoh}(\mathcal{O}_\mathcal{Y})$ commuting with all colimits.*

Proof. The existence and uniqueness of the extension is a general fact, see Categories, Lemma 26.2. To see that the lemma applies observe that coherent modules on locally Noetherian algebraic stacks are by definition modules of finite presentation, see Cohomology of Stacks, Definition 17.2. Hence a coherent module on $\mathcal{X}$ is a categorically compact object of $\textit{QCoh}(\mathcal{O}_\mathcal{X})$ by Cohomology of Stacks, Lemma 13.5.

Finally, every quasi-coherent module is a filtered colimit of its coherent submodules by Cohomology of Stacks, Lemma 18.1.

Since F is additive, also the extension of F is additive (details omitted). Since F is a tensor functor and since colimits of modules commute with taking tensor products, also the extension of F is a tensor functor (details omitted).

In this paragraph we show the extension commutes with arbitrary direct sums. If $\mathcal{F} = \bigoplus_{j \in J} \mathcal{H}_j$ with $\mathcal{H}_j$ quasi-coherent, then $\mathcal{F} = \text{colim}_{J' \subset J \text{ finite}} \bigoplus_{j \in J'} \mathcal{H}_j$. Denoting the extension of F also by F we obtain

$$\begin{aligned} F(\mathcal{F}) &= \text{colim}_{J' \subset J \text{ finite}} F(\bigoplus\nolimits_{j \in J'} \mathcal{H}_j) \\ &= \text{colim}_{J' \subset J \text{ finite}} \bigoplus\nolimits_{j \in J'} F(\mathcal{H}_j) \\ &= \bigoplus\nolimits_{j \in J} F(\mathcal{H}_j) \end{aligned}$$

Thus F commutes with arbitrary direct sums.

In this paragraph we show that the extension is right exact. Suppose $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is a short exact sequence of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. Then we write $\mathcal{F}' = \bigcup \mathcal{F}'_i$ as the union of its coherent submodules (see reference given above). Denote $\mathcal{F}''_i \subset \mathcal{F}''$ the image of $\mathcal{F}'_i$ and denote $\mathcal{F}_i = \mathcal{F} \cap \mathcal{F}'_i = \text{Ker}(\mathcal{F}'_i \to \mathcal{F}''_i)$. Then it is clear that $\mathcal{F} = \bigcup \mathcal{F}_i$ and $\mathcal{F}'' = \bigcup \mathcal{F}''_i$ and that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{F}'_i \to \mathcal{F}''_i \to 0$$

Since the extension commutes with filtered colimits we have $F(\mathcal{F}) = \text{colim}_{i \in I} F(\mathcal{F}_i)$, $F(\mathcal{F}') = \text{colim}_{i \in I} F(\mathcal{F}'_i)$, and $F(\mathcal{F}'') = \text{colim}_{i \in I} F(\mathcal{F}''_i)$. Since filtered colimits of sheaves of modules is exact we conclude that the extension of F is right exact.

The proof is finished as a right exact functor which commutes with all coproducts commutes with all colimits, see Categories, Lemma 14.12. $\square$

Lemma 16.2. *Let $\mathcal{X}$ be an algebraic stack with affine diagonal. Let B be a ring. Let $F : QCoh(\mathcal{O}_\mathcal{X}) \to Mod_B$ be a right exact tensor functor which commutes with direct sums. Let $g : U \to \mathcal{X}$ be a morphism with $U = \text{Spec}(A)$ affine. Then*

1. *$C = F(g_{QCoh,*}\mathcal{O}_U)$ is a commutative B-algebra and*
2. *there is a ring map $A \to C$*

such that $F \circ g_{QCoh,*} : Mod_A \to Mod_B$ sends M to $M \otimes_A C$ seen as B-module.

Proof. We note that g is quasi-compact and quasi-separated, see Morphisms of Stacks, Lemma 7.8. In Cohomology of Stacks, Proposition 11.1 we have constructed the functor $g_{QCoh,*} : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_\mathcal{X})$. By Cohomology of Stacks, Remarks 11.3 and 10.6 we obtain a multiplication

$$\mu : g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U \longrightarrow g_{QCoh,*}\mathcal{O}_U$$

which turns $g_{QCoh,*}\mathcal{O}_U$ into a commutative $\mathcal{O}_\mathcal{X}$-algebra. Hence $C = F(g_{QCoh,*}\mathcal{O}_U)$ is a commutative algebra object in Mod_B, in other words, C is a commutative B-algebra. Observe that we have a map $\kappa : A \to \text{End}_{\mathcal{O}_\mathcal{X}}(g_{QCoh,*}\mathcal{O}_U)$ such that for any $a \in A$ the diagram

$$\begin{tikzcd} g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U \arrow[r, "\mu"'] \arrow[d, "\kappa(r)\otimes 1"'] & g_{QCoh,*}\mathcal{O}_U \arrow[d, "\kappa(r)"] \\ g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{X}} g_{QCoh,*}\mathcal{O}_U \arrow[r, "\mu"] & g_{QCoh,*}\mathcal{O}_U \end{tikzcd}$$

commutes. It follows that we get a map $\kappa' = F(\kappa) : A \to \text{End}_B(C)$ such that $\kappa'(a)(c)c' = \kappa'(a)(cc')$ and of course this means that $a \mapsto \kappa'(a)(1)$ is a ring map $A \to C$.

The morphism $g : U \to \mathcal{X}$ is affine, see Morphisms of Stacks, Lemma 9.4. Hence $g_{QCoh,*}$ is exact and commutes with direct sums by Cohomology of Stacks, Lemma 13.4. Thus $F \circ g_{QCoh,*} : \text{Mod}_A \to \text{Mod}_B$ is a right exact functor which commutes with direct sums and which sends A to C. By Functors and Morphisms, Lemma 3.1 we see that the functor $F \circ g_{QCoh,*}$ sends an A-module M to $M \otimes_A C$ viewed as a B-module. $\square$

Lemma 16.3. *Notation as in Lemma 16.2. Assume $\mathcal{X}$ is Noetherian and g is surjective and flat. Then $B \to C$ is universally injective.*

Proof. Consider the natural map $1 : \mathcal{O}_\mathcal{X} \to g_{QCoh,*}\mathcal{O}_U$ in $QCoh(\mathcal{O}_\mathcal{X})$. Pulling back to U and using adjunction we find that the composition

$$\mathcal{O}_U = g^*\mathcal{O}_\mathcal{X} \xrightarrow{g^*1} g^*g_{QCoh,*}\mathcal{O}_U \to \mathcal{O}_U$$

is the identity in $QCoh(\mathcal{O}_U)$. Write $g_{QCoh,*}\mathcal{O}_U = \text{colim}\,\mathcal{F}_i$ as a filtered colimit of coherent $\mathcal{O}_\mathcal{X}$-modules, see Cohomology of Stacks, Lemma 18.1. For i large enough the map $1 : \mathcal{O}_\mathcal{X} \to g_{QCoh,*}\mathcal{O}_U$ factors through $\mathcal{F}_i$, see Cohomology of Stacks, Lemma 13.5. Say $s : \mathcal{O}_\mathcal{X} \to \mathcal{F}_i$ is the factorization. Then

$$\mathcal{O}_U \xrightarrow{g^*s} g^*\mathcal{F}_i \to g^*g_{QCoh,*}\mathcal{O}_U \to \mathcal{O}_U$$

is the identity. In other words, we see that s becomes the inclusion of a direct summand upon pullback to U. Set $\mathcal{F}_i^\vee = hom(\mathcal{F}_i, \mathcal{O}_\mathcal{X})$ with notation as in Cohomology of Stacks, Lemma 10.8. In particular there is an evaluation map $ev : \mathcal{F}_i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}_i^\vee \to \mathcal{O}_\mathcal{X}$. Evaluation at s defines a map $s^\vee : \mathcal{F}_i^\vee \to \mathcal{O}_\mathcal{X}$. Dual to the statement about s we see that $g^*(s^\vee)$ is surjective, see see Cohomology of Stacks, Section 12 for compatibility of hom and $\otimes$ with restriction to U. Since g is surjective and flat, we conclude that $s^\vee$ is surjective (see locus citatus). Since F is right exact, we conclude that $F(\mathcal{F}_i^\vee) \to F(\mathcal{O}_\mathcal{X}) = B$ is surjective. Choose $\lambda \in F(\mathcal{F}_i^\vee)$ mapping to $1 \in B$. Denote $e = F(s)(1) \in F(\mathcal{F}_i)$ the image of 1 by the map $F(s) : B = F(\mathcal{O}_\mathcal{X}) \to F(\mathcal{F}_i)$. Then the map

$$F(ev) : F(\mathcal{F}_i) \otimes_B F(\mathcal{F}_i^\vee) = F(\mathcal{F}_i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}_i^\vee) \longrightarrow F(\mathcal{O}_\mathcal{X}) = B$$

sends $e \otimes \lambda$ to 1 by construction. Hence the map $B \to F(\mathcal{F}_i)$, $b \mapsto be$ is universally injective because we have the one-sided inverse $F(\mathcal{F}_i) \to B$, $\xi \mapsto F(ev)(\xi \otimes \lambda)$. Since this is true for all i large enough we conclude. $\square$

Lemma 16.4. *Let $B \to C$ be a ring map. If*

1. *the coprojections $C \to C \otimes_B C$ are flat and*
2. *$B \to C$ is universally injective,*

then $B \to C$ is faithfully flat.

Proof. The map $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ is surjective as $B \to C$ is universally injective. Thus it suffices to show that $B \to C$ is flat which follows from Descent, Theorem 4.25. $\square$

The following very simple version of Künneth should become obsoleted when we write a section on Künneth theorems for cohomology of quasi-coherent modues on algebraic stacks.

Lemma 16.5. *Let $a : \mathcal{Y} \to \mathcal{X}$ and $b : \mathcal{Z} \to \mathcal{X}$ be representable by schemes, quasi-compact, quasi-separated, and flat. Then $a_{QCoh,*}\mathcal{O}_\mathcal{Y} \otimes_{\mathcal{O}_\mathcal{X}} b_{QCoh,*}\mathcal{O}_\mathcal{Z} = f_{QCoh,*}\mathcal{O}_{\mathcal{Y}\times_\mathcal{X}\mathcal{Z}}$ where $f : \mathcal{Y} \times_\mathcal{X} \mathcal{Z} \to \mathcal{X}$ is the obvious morphism.*

Proof. We abbreviate $\mathcal{P} = \mathcal{Y} \times_\mathcal{X} \mathcal{Z}$. Since $a \circ \mathrm{pr}_1 = f$ and $b \circ \mathrm{pr}_2 = f$ we obtain maps $a_*\mathcal{O}_\mathcal{Y} \to f_*\mathcal{O}_\mathcal{P}$ and $b_*\mathcal{O}_\mathcal{Z} \to f_*\mathcal{O}_\mathcal{P}$ (using relative pullback maps, see Sites, Section 45). Hence we obtain a relative cup product

$$\mu : a_*\mathcal{O}_\mathcal{Y} \otimes_{\mathcal{O}_\mathcal{X}} b_*\mathcal{O}_\mathcal{Z} \longrightarrow f_*\mathcal{O}_{\mathcal{Y}\times_\mathcal{X}\mathcal{Z}}$$

Applying Q and its compatibility with tensor products (Cohomology of Stacks, Remark 10.6) we obtain an arrow $Q(\mu) : a_{QCoh,*}\mathcal{O}_\mathcal{Y} \otimes_{\mathcal{O}_\mathcal{X}} b_{QCoh,*}\mathcal{O}_\mathcal{Z} \to f_{QCoh,*}\mathcal{O}_{\mathcal{Y}\times_\mathcal{X}\mathcal{Z}}$ in $QCoh(\mathcal{O}_\mathcal{X})$. Next, choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. It suffices to prove the restriction of $Q(\mu)$ to $U_{étale}$ is an isomorphism, see Cohomology of Stacks, Section 12. In turn, by the material in the same section, it suffices to prove the restriction of μ to $U_{étale}$ is an isomorphism (this uses that the source and target of μ are locally quasi-coherent modules with the base change property). Moreover, we may compute pushforwards in the étale topology, see Cohomology of Stacks, Proposition 8.1. Then finally, we see that $a_*\mathcal{O}_\mathcal{Y}|_{U_{étale}} = (V \to U)_{small,*}\mathcal{O}_V$ where $V = U \times_\mathcal{X} \mathcal{Y}$. Similarly for b_* and f_*. Thus the result follows from the Künneth formula for flat, quasi-compact, quasi-separated morphisms of schemes, see Derived Categories of Schemes, Lemma 23.1. $\square$

Lemma 16.6. *Let $\mathcal{X}$ be an algebraic stack with affine diagonal. Let B be a ring. Let $f_i : \mathrm{Spec}(B) \to \mathcal{X}$, $i = 1, 2$ be two morphisms. Let $t : f_1^* \to f_2^*$ be an isomorphism of the tensor functors $f_i^* : QCoh(\mathcal{O}_\mathcal{X}) \to Mod_B$. Then there is a 2-arrow $f_1 \to f_2$ inducing t.*

Proof. Choose an affine scheme $U = \mathrm{Spec}(A)$ and a surjective smooth morphism $g : U \to \mathcal{X}$, see Properties of Stacks, Lemma 6.2. Since the diagonal of $\mathcal{X}$ is affine, we see that $U_i = \mathrm{Spec}(B) \times_{f_i,\mathcal{X},g} U$ is affine. Say $U_i = \mathrm{Spec}(C_i)$. Then C_i is the B-algebra endowed with ring map $A \to C_i$ constructed in Lemma 16.2 using the functor $F = f_i^*$. Therefore t induces an isomorphism $C_1 \to C_2$ of B-algebras, compatible with the ring maps $A \to C_1$ and $A \to C_2$. In other words, we have a commutative diagrams

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & U \arrow[d, "g"] \\ \mathrm{Spec}(B) \arrow[r, "f_i"] & \mathcal{X} \end{tikzcd} \qquad \begin{tikzcd} & U_2 \arrow[dl] \arrow[d, "\cong"] \arrow[dr] & \\ \mathrm{Spec}(B) & U_1 \arrow[l] \arrow[r] & U \end{tikzcd}$$

This already shows that the objects f_1 and f_2 of $\mathcal{X}$ over $\mathrm{Spec}(B)$ become isomorphic after the smooth covering $\{U_1 \to \mathrm{Spec}(B)\}$. To show that this descends to an isomorphism of f_1 and f_2 over $\mathrm{Spec}(B)$, we have to show that our isomorphism (which comes from the commutative diagrams above) is compatible with the descent data over $U_1 \times_{\mathrm{Spec}(B)} U_1$. For this we observe that $U \times_{\mathcal{X}} U$ is affine too, that we have the morphism $g' : U \times_{\mathcal{X}} U \to \mathcal{X}$, and that

$$U_i \times_{\mathrm{Spec}(B)} U_i = \mathrm{Spec}(B) \times_{f_i, \mathcal{X}, g'} (U \times_{\mathcal{X}} U)$$

It follows that the isomorphism $C_1 \otimes_B C_1 \to C_2 \otimes_B C_2$ coming from the isomorphism $C_1 \to C_2$ is compatible with the morphisms $U_i \times_{\mathrm{Spec}(B)} U_i \to U \times_{\mathcal{X}} U$. Some details omitted. $\square$

Lemma 16.7. *Let $\mathcal{X}$ be a Noetherian algebraic stack with affine diagonal. Let B be a ring. Let $F : QCoh(\mathcal{O}_{\mathcal{X}}) \to Mod_B$ be a right exact tensor functor which commutes with direct sums. Then F comes from a unique morphism* $\mathrm{Spec}(B) \to \mathcal{X}$.

Proof. Choose a surjective smooth morphism $g : U \to \mathcal{X}$ with $U = \mathrm{Spec}(A)$ affine, see Properties of Stacks, Lemma 6.2. Apply Lemma 16.2 to get the finite type commutative B-algebra $C = F(g_{QCoh,*}\mathcal{O}_U)$ and the ring map $A \to C$. By Lemma 16.3 the ring map $B \to C$ is universally injective. Consider the algebra

$$C \otimes_B C = F(g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_{\mathcal{X}}} g_{QCoh,*}\mathcal{O}_U)$$

Since g is flat, quasi-compact, and quasi-separated by Lemma 16.5 we have the first equality in

$$g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_{\mathcal{X}}} g_{QCoh,*}\mathcal{O}_U = f_{QCoh,*}\mathcal{O}_{U \times_{\mathcal{X}} U} = g_{QCoh,*}(\mathrm{pr}_{2,*}\mathcal{O}_{U \times_{\mathcal{X}} U})$$

where $f : U \times_{\mathcal{X}} U \to \mathcal{X}$ is the obvious morphism and $\mathrm{pr}_2 : U \times_{\mathcal{X}} U \to U$ is the second projection. The second equality follows from Cohomology of Stacks, Lemma 11.5 and $f = g \circ \mathrm{pr}_2$. Since the diagonal of $\mathcal{X}$ is affine, we see that $U \times_{\mathcal{X}} U = \mathrm{Spec}(R)$ is affine. Let us use $\mathrm{pr}_2 : A \to R$ to view R as an A-algebra. All in all we obtain

$$C \otimes_B C = F(g_{QCoh,*}\mathcal{O}_U \otimes_{\mathcal{O}_{\mathcal{X}}} g_{QCoh,*}\mathcal{O}_U) = F(g_{QCoh,*}(\mathrm{pr}_{2,*}\mathcal{O}_{U \times_{\mathcal{X}} U})) = R \otimes_A C$$

where the final equality follows from the final statement of Lemma 16.2. Since $A \to R$ is flat (because pr_2 is flat as a base change of $U \to \mathcal{X}$), we conclude that $C \otimes_B C$ is flat over C. By Lemma 16.4 we conclude that $B \to C$ is faithfully flat.

We claim there is a solid commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(C \otimes_B C) \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & U \times_{\mathcal{X}} U \arrow[d, shift right] \arrow[d, shift left] \\
\mathrm{Spec}(C) \arrow[r] \arrow[d] & U \arrow[d] \\
\mathrm{Spec}(B) \arrow[r, dotted] & \mathcal{X}
\end{tikzcd}$$

The arrow $\mathrm{Spec}(C) \to U = \mathrm{Spec}(A)$ comes from the ring map $A \to C$ in the statement of Lemma 16.2. The arrow $\mathrm{Spec}(C \otimes_B C) \to U \times_{\mathcal{X}} U$ simlarly comes from the ring map $R \to C \otimes_B C$. To verify the top square commutes use Lemma 16.6; details omitted. We conclude we get the dotted arrow $\mathrm{Spec}(B) \to \mathcal{X}$ by Proposition 15.1.

The statement that F is the functor corresponding to pullback by the dotted arrow is also clear from this and the corresponding statement in Lemma 16.2. Details omitted. $\square$

For a ring B let us denote Mod_B^{fg} the category of finitely generated B-modules (AKA finite B-modules).

Theorem 16.8. *Let $\mathcal{X}$ be a Noetherian algebraic stack with affine diagonal. Let B be a Noetherian ring. Let $F : Coh(\mathcal{O}_\mathcal{X}) \to Mod_B^{fg}$ be a right exact tensor functor. Then F comes from a unique morphism $\text{Spec}(B) \to \mathcal{X}$.*

Proof. By Lemma 16.1 we can extend F uniquely to a right exact tensor functor $F : QCoh(\mathcal{O}_\mathcal{X}) \to \text{Mod}_B$ commuting with all direct susms. Then we can apply Lemma 16.7. □

17. Other chapters

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