MODULI OF CURVES

Contents

1. Introduction

In this chapter we discuss some of the familiar moduli stacks of curves. A reference is the celebrated article of Deligne and Mumford, see [DM69].

2. Conventions and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2. Unless otherwise mentioned our base scheme will be $\mathrm{Spec}(\mathbf{Z})$.

3. The stack of curves

This section is the continuation of Quot, Section 15. Let $\mathcal{C}urves$ be the stack whose category of sections over a scheme S is the category of families of curves over S. It is somewhat important to keep in mind that a *family of curves* is a morphism $f : X \to S$ where X is an algebraic space (!) and f is flat, proper, of finite presentation and of relative dimension ≤ 1. We already know that $\mathcal{C}urves$ is an algebraic stack over $\mathbf{Z}$, see Quot, Theorem 15.11. If we did not allow algebraic spaces in the definition of our stack, then this theorem would be false.

Often base change is denoted by a subscript, but we cannot use this notation for $\mathcal{C}urves$ because $\mathcal{C}urves_S$ is our notation for the fibre category over S. This is why in Quot, Remark 15.5 we used $B\text{-}\mathcal{C}urves$ for the base change

$$B\text{-}\mathcal{C}urves = \mathcal{C}urves \times B$$

to the algebraic space B. The product on the right is over the final object, i.e., over $\mathrm{Spec}(\mathbf{Z})$. The object on the left is the stack classifying families of curves on the category of schemes over B. In particular, if k is a field, then

$$k\text{-}\mathcal{C}urves = \mathcal{C}urves \times \mathrm{Spec}(k)$$

is the moduli stack classifying families of curves on the category of schemes over k. Before we continue, here is a sanity check.

Lemma 3.1. *Let $T \to B$ be a morphism of algebraic spaces. The category*

$$\mathrm{Mor}_B(T, B\text{-}\mathcal{C}urves) = \mathrm{Mor}(T, \mathcal{C}urves)$$

is the category of families of curves over T.

Proof. A family of curves over T is a morphism $f : X \to T$ of algebraic spaces, which is flat, proper, of finite presentation, and has relative dimension ≤ 1 (Morphisms of Spaces, Definition 33.2). This is exactly the same as the definition in Quot, Situation 15.1 except that T the base is allowed to be an algebraic space. Our default base category for algebraic stacks/spaces is the category of schemes, hence the lemma does not follow immediately from the definitions. Having said this, we encourage the reader to skip the proof.

By the product description of $B\text{-}\mathcal{C}urves$ given above, it suffices to prove the lemma in the absolute case. Choose a scheme U and a surjective étale morphism $p : U \to T$. Let $R = U \times_T U$ with projections $s, t : R \to U$.

Let $v : T \to \mathcal{C}urves$ be a morphism. Then $v \circ p$ corresponds to a family of curves $X_U \to U$. The canonical 2-morphism $v \circ p \circ t \to v \circ p \circ s$ is an isomorphism $\varphi : X_U \times_{U,s} R \to X_U \times_{U,t} R$. This isomorphism satisfies the cocycle condition on $R \times_{s,t} R$. By Bootstrap, Lemma 11.3 we obtain a morphism of algebraic spaces $X \to T$ whose pullback to U is equal to X_U compatible with φ. Since $\{U \to T\}$ is an étale covering, we see that $X \to T$ is flat, proper, of finite presentation by Descent on Spaces, Lemmas 11.13, 11.19, and 11.12. Also $X \to T$ has relative dimension ≤ 1 because this is an étale local property. Hence $X \to T$ is a family of curves over T.

Conversely, let $X \to T$ be a family of curves. Then the base change X_U determines a morphism $w : U \to \mathcal{C}urves$ and the canonical isomorphism $X_U \times_{U,s} R \to X_U \times_{U,t} R$ determines a 2-arrow $w \circ s \to w \circ t$ satisfying the cocycle condition. Thus a morphism

$v : T = [U/R] \to \mathcal{C}urves$ by the universal property of the quotient $[U/R]$, see Groupoids in Spaces, Lemma 23.2. (Actually, it is much easier in this case to go back to before we introduced our abuse of language and direct construct the functor $Sch/T \to \mathcal{C}urves$ which "is" the morphism $T \to \mathcal{C}urves$.)

We omit the verification that the constructions given above extend to morphisms between objects and are mutually quasi-inverse. $\square$

4. The stack of polarized curves

In this section we work out some of the material discussed in Quot, Remark 15.13. Consider the 2-fibre product

$$\begin{tikzcd} \mathcal{C}urves \times_{\mathcal{S}paces'_{fp,flat,proper}} \mathcal{P}olarized \arrow[r] \arrow[d] & \mathcal{P}olarized \arrow[d] \\ \mathcal{C}urves \arrow[r] & \mathcal{S}paces'_{fp,flat,proper} \end{tikzcd}$$

We denote this 2-fibre product by

$$\mathcal{P}olarized\mathcal{C}urves = \mathcal{C}urves \times_{\mathcal{S}paces'_{fp,flat,proper}} \mathcal{P}olarized$$

This fibre product parametrizes polarized curves, i.e., families of curves endowed with a relatively ample invertible sheaf. More precisely, an object of $\mathcal{P}olarized\mathcal{C}urves$ is a pair $(X \to S, \mathcal{L})$ where

(1) $X \to S$ is a morphism of schemes which is proper, flat, of finite presentation, and has relative dimension ≤ 1, and
(2) $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module which is relatively ample on X/S.

A morphism $(X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ between objects of $\mathcal{P}olarized\mathcal{C}urves$ is given by a triple (f, g, φ) where $f : X' \to X$ and $g : S' \to S$ are morphisms of schemes which fit into a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

inducing an isomorphism $X' \to S' \times_S X$, in other words, the diagram is cartesian, and $\varphi : f^*\mathcal{L} \to \mathcal{L}'$ is an isomorphism. Composition is defined in the obvious manner.

Lemma 4.1. *The morphism $\mathcal{P}olarized\mathcal{C}urves \to \mathcal{P}olarized$ is an open and closed immersion.*

Proof. This is true because the 1-morphism $\mathcal{C}urves \to \mathcal{S}paces'_{fp,flat,proper}$ is representable by open and closed immersions, see Quot, Lemma 15.12. $\square$

Lemma 4.2. *The morphism $\mathcal{P}olarized\mathcal{C}urves \to \mathcal{C}urves$ is smooth and surjective.*

Proof. Surjective. Given a field k and a proper algebraic space X over k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. By Spaces over Fields, Lemma 9.3 the algebraic space X is a scheme. Hence X is a proper scheme of dimension ≤ 1 over k. By Varieties, Lemma 43.4 we see that X is H-projective over κ. In particular, there exists an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X. Then $(X, \mathcal{L})$ is an object of $\mathcal{P}olarized\mathcal{C}urves$ over k which maps to X.

Smooth. Let $X \to S$ be an object of $\mathcal{C}urves$, i.e., a morphism $S \to \mathcal{C}urves$. It is clear that

$$PolarizedCurves \times_{\mathcal{C}urves} S \subset \mathcal{P}ic_{X/S}$$

is the substack of objects $(T/S, \mathcal{L}/X_T)$ such that $\mathcal{L}$ is ample on X_T/T. This is an open substack by Descent on Spaces, Lemma 13.2. Since $\mathcal{P}ic_{X/S} \to S$ is smooth by Moduli Stacks, Lemma 8.5 we win. □

Lemma 4.3. *Let $X \to S$ be a family of curves. Then there exists an étale covering $\{S_i \to S\}$ such that $X_i = X \times_S S_i$ is a scheme. We may even assume X_i is H-projective over S_i.*

Proof. This is an immediate corollary of Lemma 4.2. Namely, unwinding the definitions, this lemma gives there is a surjective smooth morphism $S' \to S$ such that $X' = X \times_S S'$ comes endowed with an invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ which is ample on X'/S'. Then we can refine the smooth covering $\{S' \to S\}$ by an étale covering $\{S_i \to S\}$, see More on Morphisms, Lemma 38.7. After replacing S_i by a suitable open covering we may assume $X_i \to S_i$ is H-projective, see Morphisms, Lemmas 43.6 and 43.4 (this is also discussed in detail in More on Morphisms, Section 50). □

5. Properties of the stack of curves

The following lemma isn't true for moduli of surfaces, see Remark 5.2.

Lemma 5.1. *The diagonal of $\mathcal{C}urves$ is separated and of finite presentation.*

Proof. Recall that $\mathcal{C}urves$ is a limit preserving algebraic stack, see Quot, Lemma 15.6. By Limits of Stacks, Lemma 3.6 this implies that $\Delta : \mathcal{P}olarized \to \mathcal{P}olarized \times \mathcal{P}olarized$ is limit preserving. Hence Δ is locally of finite presentation by Limits of Stacks, Proposition 3.8.

Let us prove that Δ is separated. To see this, it suffices to show that given a scheme U and two objects $Y \to U$ and $X \to U$ of $\mathcal{C}urves$ over U, the algebraic space

$$Isom_U(Y, X)$$

is separated. This we have seen in Moduli Stacks, Lemmas 10.2 and 10.3 that the target is a separated algebraic space.

To finish the proof we show that Δ is quasi-compact. Since Δ is representable by algebraic spaces, it suffices to check the base change of Δ by a surjective smooth morphism $U \to \mathcal{C}urves \times \mathcal{C}urves$ is quasi-compact (see for example Properties of Stacks, Lemma 3.3). We choose $U = \coprod U_i$ to be a disjoint union of affine opens with a surjective smooth morphism

$$U \longrightarrow PolarizedCurves \times PolarizedCurves$$

Then $U \to \mathcal{C}urves \times \mathcal{C}urves$ will be surjective and smooth since $PolarizedCurves \to \mathcal{C}urves$ is surjective and smooth by Lemma 4.2. Since $PolarizedCurves$ is limit preserving (by Artin's Axioms, Lemma 11.2 and Quot, Lemmas 15.6, 14.8, and 13.6), we see that $PolarizedCurves \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, hence $U_i \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation (Limits of Stacks, Proposition 3.8 and Morphisms of Stacks, Lemmas 27.2 and 33.5). In particular, U_i is Noetherian affine. This reduces us to the case discussed in the next paragraph.

In this paragraph, given a Noetherian affine scheme U and two objects $(Y, \mathcal{N})$ and $(X, \mathcal{L})$ of *PolarizedCurves* over U, we show the algebraic space

$$\mathit{Isom}_U(Y, X)$$

is quasi-compact. Since the connected components of U are open and closed we may replace U by these. Thus we may and do assume U is connected. Let $u \in U$ be a point. Let Q, P be the Hilbert polynomials of these families, i.e.,

$$Q(n) = \chi(Y_u, \mathcal{N}_u^{\otimes n}) \quad \text{and} \quad P(n) = \chi(X_u, \mathcal{L}_u^{\otimes n})$$

see Varieties, Lemma 45.1. Since U is connected and since the functions $u \mapsto \chi(Y_u, \mathcal{N}_u^{\otimes n})$ and $u \mapsto \chi(X_u, \mathcal{L}_u^{\otimes n})$ are locally constant (see Derived Categories of Schemes, Lemma 32.2) we see that we get the same Hilbert polynomial in every point of U. Set

$$\mathcal{M} = \mathrm{pr}_1^*\mathcal{N} \otimes_{\mathcal{O}_{Y \times_U X}} \mathrm{pr}_2^*\mathcal{L}$$

on $Y \times_U X$. Given $(f, \varphi) \in \mathit{Isom}_U(Y, X)(T)$ for some scheme T over U then for every $t \in T$ we have

$$\begin{aligned} \chi(Y_t, (\mathrm{id} \times f)^*\mathcal{M}^{\otimes n}) &= \chi(Y_t, \mathcal{N}_t^{\otimes n} \otimes_{\mathcal{O}_{Y_t}} f_t^*\mathcal{L}_t^{\otimes n}) \\ &= n\deg(\mathcal{N}_t) + n\deg(f_t^*\mathcal{L}_t) + \chi(Y_t, \mathcal{O}_{Y_t}) \\ &= Q(n) + n\deg(\mathcal{L}_t) \\ &= Q(n) + P(n) - P(0) \end{aligned}$$

by Riemann-Roch for proper curves, more precisely by Varieties, Definition 44.1 and Lemma 44.7 and the fact that f_t is an isomorphism. Setting $P'(t) = Q(t) + P(t) - P(0)$ we find

$$\mathit{Isom}_U(Y, X) = \mathit{Isom}_U(Y, X) \cap \mathit{Mor}_U^{P', \mathcal{M}}(Y, X)$$

The intersection is an intersection of open subspaces of $\mathit{Mor}_U(Y, X)$, see Moduli Stacks, Lemma 10.3 and Remark 10.4. Now $\mathit{Mor}_U^{P', \mathcal{M}}(Y, X)$ is a Noetherian algebraic space as it is of finite presentation over U by Moduli Stacks, Lemma 10.5. Thus the intersection is a Noetherian algebraic space too and the proof is finished. $\square$

Remark 5.2. The boundedness argument in the proof of Lemma 5.1 does not work for moduli of surfaces and in fact, the result is wrong, for example because K3 surfaces over fields can have infinite discrete automorphism groups. The "reason" the argument does not work is that on a projective surface S over a field, given ample invertible sheaves $\mathcal{N}$ and $\mathcal{L}$ with Hilbert polynomials Q and P, there is no a priori bound on the Hilbert polynomial of $\mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{L}$. In terms of intersection theory, if H_1, H_2 are ample effective Cartier divisors on S, then there is no (upper) bound on the intersection number $H_1 \cdot H_2$ in terms of $H_1 \cdot H_1$ and $H_2 \cdot H_2$.

Lemma 5.3. *The morphism* $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ *is quasi-separated and locally of finite presentation.*

Proof. To check $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated we have to show that its diagonal is quasi-compact and quasi-separated. This is immediate from Lemma 5.1. To prove that $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation, it suffices to show that $\mathcal{C}urves$ is limit preserving, see Limits of Stacks, Proposition 3.8. This is Quot, Lemma 15.6. $\square$

6. Open substacks of the stack of curves

Below we will often characterize an open substack of $\mathcal{C}urves$ by a propery P of morphisms of algebraic spaces. To see that P defines an open substack it suffices to check

(o) given a family of curves $f : X \to S$ there exists a largest open subscheme $S' \subset S$ such that $f|_{f^{-1}(S')} : f^{-1}(S') \to S'$ has P and such that formation of S' commutes with arbitrary base change.

Namely, suppose (o) holds. Choose a scheme U and a surjective smooth morphism $m : U \to \mathcal{C}urves$. Let $R = U \times_{\mathcal{C}urves} U$ and denote $t, s : R \to U$ the projections. Recall that $\mathcal{C}urves = [U/R]$ is a presentation, see Algebraic Stacks, Lemma 16.2 and Definition 16.5. By construction of $\mathcal{C}urves$ as the stack of curves, the morphism m is the classifying morphism for a family of curves $C \to U$. The 2-commutativity of the diagram

$$\begin{tikzcd} R \arrow[r, "s"'] \arrow[d, "t"'] & U \arrow[d] \\ U \arrow[r] & \mathcal{C}urves \end{tikzcd}$$

implies that $C \times_{U,s} R \cong C \times_{U,t} R$ (isomorphism of families of curves over R). Let $W \subset U$ be the largest open subscheme such that $f|_{f^{-1}(W)} : f^{-1}(W) \to W$ has P as in (o). Since formation of W commutes with base change according to (o) and by the isomorphism above we find that $s^{-1}(W) = t^{-1}(W)$. Thus $W \subset U$ corresponds to an open substack

$$\mathcal{C}urves^P \subset \mathcal{C}urves$$

according to Properties of Stacks, Lemma 9.8.

Continuing with the setup of the previous paragrpah, we claim the open substack $\mathcal{C}urves^P$ has the following two universal properties:

(1) given a family of curves $X \to S$ the following are equivalent
   (a) the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^P$,
   (b) the morphism $X \to S$ has P,
(2) given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent
   (a) the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^P$,
   (b) the morphism $X \to \mathrm{Spec}(k)$ has P.

This follows by considering the 2-fibre product

$$\begin{tikzcd} T \arrow[r, "p"'] \arrow[d, "q"'] & U \arrow[d] \\ S \arrow[r] & \mathcal{C}urves \end{tikzcd}$$

Observe that $T \to S$ is surjective and smooth as the base change of $U \to \mathcal{C}urves$. Thus the open $S' \subset S$ given by (o) is determined by its inverse image in T. However, by the invariance under base change of these opens in (o) and because $X \times_S T \cong C \times_U T$ by the 2-commutativity, we find $q^{-1}(S') = p^{-1}(W)$ as opens of T. This immediately implies (1). Part (2) is a special case of (1).

Given two properties P and Q of morphisms of algebraic spaces, supposing we already have established $\mathcal{C}urves^Q$ is an open substack of $\mathcal{C}urves$, then we can use exactly the same method to prove openness of $\mathcal{C}urves^{Q,P} \subset \mathcal{C}urves^{Q}$. We omit a precise explanation.

7. Curves with finite reduced automorphism groups

Let X be a proper scheme over a field k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. By Lemma 5.1 the automorphism group algebraic space $\mathit{Aut}(X)$ is finite type and separated over k. In particular, $\mathit{Aut}(X)$ is a group scheme, see More on Groupoids in Spaces, Lemma 10.2. If the characteristic of k is zero, then $\mathit{Aut}(X)$ is reduced and even smooth over k (Groupoids, Lemma 8.2). However, in general $\mathit{Aut}(X)$ is not reduced, even if X is geometrically reduced.

Example 7.1 (Non-reduced automorphism group)**.** Let k be an algebraically closed field of characteristic 2. Set $Y = Z = \mathbf{P}^1_k$. Choose three pairwise distinct k-valued points a, b, c in $\mathbf{A}^1_k$. Thinking of $\mathbf{A}^1_k \subset \mathbf{P}^1_k = Y = Z$ as an open subschemes, we get a closed immersion

$$T = \mathrm{Spec}(k[t]/(t-a)^2) \amalg \mathrm{Spec}(k[t]/(t-b)^2) \amalg \mathrm{Spec}(k[t]/(t-c)^2) \longrightarrow \mathbf{P}^1_k$$

Let X be the pushout in the diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

Let $U \subset X$ be the affine open part which is the image of $\mathbf{A}^1_k \amalg \mathbf{A}^1_k$. Then we have an equalizer diagram

$$\begin{tikzcd} \mathcal{O}_X(U) \arrow[r] & k[t] \times k[t] \arrow[r, shift left] \arrow[r, shift right] & k[t]/(t-a)^2 \times k[t]/(t-b)^2 \times k[t]/(t-c)^2 \end{tikzcd}$$

Over the dual numbers $A = k[\epsilon]$ we have a nontrivial automorphism of this equalizer diagram sending t to $t + \epsilon$. We leave it to the reader to see that this automorphism extends to an automorphism of X over A. On the other hand, the reader easily shows that the automorphism group of X over k is finite. Thus $\mathit{Aut}(X)$ must be non-reduced.

Let X be a proper scheme over a field k of dimension ≤ 1, i.e., an object of $\mathcal{C}urves$ over k. If $\mathit{Aut}(X)$ is geometrically reduced, then it need not be the case that it has dimension 0, even if X is smooth and geometrically connected.

Example 7.2 (Smooth positive dimensional automorphism group)**.** Let k be an algebraically closed field. If X is a smooth genus 0, resp. 1 curve, then the automorphism group has dimension 3, resp. 1. Namely, in the genus 0 case we have $X \cong \mathbf{P}^1_k$ by Algebraic Curves, Proposition 10.4. Since

$$\mathit{Aut}(\mathbf{P}^1_k) = \mathrm{PGL}_{2,k}$$

as functors we see that the dimension is 3. On the other hand, if the genus of X is 1, then we see that the map $X = \underline{\mathrm{Hilb}}^1_{X/k} \to \underline{\mathrm{Pic}}^1_{X/k}$ is an isomorphism, see Picard Schemes of Curves, Lemma 6.7 and Algebraic Curves, Theorem 2.6. Thus X has the structure of an abelian variety (since $\underline{\mathrm{Pic}}^1_{X/k} \cong \underline{\mathrm{Pic}}^0_{X/k}$). In particular the (co)tangent bundle of X are trivial (Groupoids, Lemma 6.3). We conclude that $\dim_k H^0(X, T_X) = 1$ hence $\dim \mathit{Aut}(X) \leq 1$. On the other hand, the translations (viewing X as a group scheme) provide a 1-dimensional piece of $\mathrm{Aut}(X)$ and we conlude its dimension is indeed 1.

It turns out that there is an open substack of *Curves* parametrizing curves whose automorphism group is geometrically reduced and finite. Here is a precise statement.

Lemma 7.3. *There exist an open substack $Curves^{DM} \subset Curves$ with the following properties*

1. *$Curves^{DM} \subset Curves$ is the maximal open substack which is DM,*
2. *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to Curves$ factors through $Curves^{DM}$,*
   (b) *the group algebraic space $Aut_S(X)$ is unramified over S,*
3. *given X a proper scheme over a field k of dimension ≤ 1 the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to Curves$ factors through $Curves^{DM}$,*
   (b) *$Aut(X)$ is geometrically reduced over k and has dimension 0,*
   (c) *$Aut(X) \to \mathrm{Spec}(k)$ is unramified.*

Proof. The existence of an open substack with property (1) is Morphisms of Stacks, Lemma 22.1. The points of this open substack are characterized by (3)(c) by Morphisms of Stacks, Lemma 22.2. The equivalence of (3)(b) and (3)(c) is the statement that an algebraic space G which is locally of finite type, geometrically reduced, and of dimension 0 over a field k, is unramified over k. First, G is a scheme by Spaces over Fields, Lemma 9.1. Then we can take an affine open in G and observe that it will be proper over k and apply Varieties, Lemma 9.3. Minor details omitted.

Part (2) is true because (3) holds. Namely, the morphism $Aut_S(X) \to S$ is locally of finite type. Thus we can check whether $Aut_S(X) \to S$ is unramified at all points of $Aut_S(X)$ by checking on fibres at points of the scheme S, see Morphisms of Spaces, Lemma 38.10. But after base change to a point of S we fall back into the equivalence of (3)(a) and (3)(c). $\square$

Lemma 7.4. *Let X be a proper scheme over a field k of dimension ≤ 1. Then properties (3)(a), (b), (c) are also equivalent to $Der_k(\mathcal{O}_X, \mathcal{O}_X) = 0$.*

Proof. In the discussion above we have seen that $G = Aut(X)$ is a group scheme over $\mathrm{Spec}(k)$ which is finite type and separated; this uses Lemma 5.1 and More on Groupoids in Spaces, Lemma 10.2. Then G is unramified over k if and only if $\Omega_{G/k} = 0$ (Morphisms, Lemma 35.2). By Groupoids, Lemma 6.3 the vanishing holds if $T_{G/k,e} = 0$, where $T_{G/k,e}$ is the tangent space to G at the identity element $e \in G(k)$, see Varieties, Definition 16.3 and the formula in Varieties, Lemma 16.4. Since $\kappa(e) = k$ the tangent space is defined in terms of morphisms $\alpha : \mathrm{Spec}(k[\epsilon]) \to G = Aut(X)$ whose restriction to $\mathrm{Spec}(k)$ is e. It follows that it suffices to show any automorphism

$$\alpha : X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon]) \longrightarrow X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$$

over $\mathrm{Spec}(k[\epsilon])$ whose restriction to $\mathrm{Spec}(k)$ is id_X. Such automorphisms are called infinitesimal automorphisms.

The infinitesimal automorphisms of X correspond 1-to-1 with derivations of $\mathcal{O}_X$ over k. This follows from More on Morphisms, Lemmas 9.1 and 9.2 (we only need the first one as we don't care about the reverse direction; also, please look at More on Morphisms, Remark 9.7 for an elucidation). For a different argument proving this equality we refer the reader to Deformation Problems, Lemma 9.3. ☐

8. Cohen-Macaulay curves

There is an open substack of *Curves* parametrizing the Cohen-Macaulay "curves".

Lemma 8.1. *There exist an open substack* $\mathit{Curves}^{CM} \subset \mathit{Curves}$ *such that*

(1) *given a family of curves* $X \to S$ *the following are equivalent*
   (a) *the classifying morphism* $S \to \mathit{Curves}$ *factors through* Curves^{CM},
   (b) *the morphism* $X \to S$ *is Cohen-Macaulay,*
(2) *given a scheme* X *proper over a field* k *with* $\dim(X) \leq 1$ *the following are equivalent*
   (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathit{Curves}$ *factors through* Curves^{CM},
   (b) X *is Cohen-Macaulay.*

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 26.7 the set

$$W = \{x \in |X| : f \text{ is Cohen-Macaulay at } x\}$$

is open in $|X|$ and formation of this open commutes with arbitrary base change. Since f is proper the subset

$$S' = S \setminus f(|X| \setminus W)$$

of S is open and $X \times_S S' \to S'$ is Cohen-Macaulay. Moreover, formation of S' commutes with arbitrary base change because this is true for W Thus we get the open substack with the desired properties by the method discussed in Section 6. ☐

Lemma 8.2. *There exist an open substack* $\mathit{Curves}^{CM,1} \subset \mathit{Curves}$ *such that*

(1) *given a family of curves* $X \to S$ *the following are equivalent*
   (a) *the classifying morphism* $S \to \mathit{Curves}$ *factors through* $\mathit{Curves}^{CM,1}$,
   (b) *the morphism* $X \to S$ *is Cohen-Macaulay and has relative dimension* 1 *(Morphisms of Spaces, Definition 33.2),*
(2) *given a scheme* X *proper over a field* k *with* $\dim(X) \leq 1$ *the following are equivalent*
   (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathit{Curves}$ *factors through* $\mathit{Curves}^{CM,1}$,
   (b) X *is Cohen-Macaulay and* X *is equidimensional of dimension* 1*.*

Proof. By Lemma 8.1 it is clear that we have $\mathit{Curves}^{CM,1} \subset \mathit{Curves}^{CM}$ if it exists. Let $f : X \to S$ be a family of curves such that f is a Cohen-Macaulay morphism. By More on Morphisms of Spaces, Lemma 26.8 we have a decomposition

$$X = X_0 \amalg X_1$$

by open and closed subspaces such that $X_0 \to S$ has relative dimension 0 and $X_1 \to S$ has relative dimension 1. Since f is proper the subset

$$S' = S \setminus f(|X_0|)$$

of S is open and $X \times_S S' \to S'$ is Cohen-Macaulay and has relative dimension 1. Moreover, formation of S' commutes with arbitrary base change because this is true for the decomposition above (as relative dimension behaves well with respect to base change, see Morphisms of Spaces, Lemma 34.3). Thus we get the open substack with the desired properties by the method discussed in Section 6. ☐

9. Curves of a given genus

The convention in the Stacks project is that the genus g of a proper 1-dimensional scheme X over a field k is defined only if $H^0(X, \mathcal{O}_X) = k$. In this case $g = \dim_k H^1(X, \mathcal{O}_X)$. See Algebraic Curves, Section 8. The conditions needed to define the genus define an open substack which is then a disjoint union of open substacks, one for each genus.

Lemma 9.1. *There exist an open substack $\mathcal{C}urves^{h0,1} \subset \mathcal{C}urves$ such that*

1. *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{h0,1}$,*
   (b) *$f_*\mathcal{O}_X = \mathcal{O}_S$, this holds after arbitrary base change, and the fibres of f have dimension 1,*
2. *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{h0,1}$,*
   (b) *$H^0(X, \mathcal{O}_X) = k$ and $\dim(X) = 1$.*

Proof. Given a family of curves $X \to S$ the set of $s \in S$ where $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ is open in S by Derived Categories of Spaces, Lemma 26.2. Also, the set of points in S where the fibre has dimension 1 is open by More on Morphisms of Spaces, Lemma 31.5. Moreover, if $f : X \to S$ is a family of curves all of whose fibres have dimension 1 (and in particular f is surjective), then condition (1)(b) is equivalent to $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ for every $s \in S$, see Derived Categories of Spaces, Lemma 26.7. Thus we see that the lemma follows from the general discussion in Section 6. $\square$

Lemma 9.2. *We have $\mathcal{C}urves^{h0,1} \subset \mathcal{C}urves^{CM,1}$ as open substacks of $\mathcal{C}urves$.*

Proof. See Algebraic Curves, Lemma 6.1 and Lemmas 9.1 and 8.2. $\square$

Lemma 9.3. *Let $f : X \to S$ be a family of curves such that $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$, i.e., the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{h0,1}$ (Lemma 9.1). Then*

1. *$f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds universally,*
2. *$R^1f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module,*
3. *for any morphism $h : S' \to S$ if $f' : X' \to S'$ is the base change, then $h^*(R^1f_*\mathcal{O}_X) = R^1f'_*\mathcal{O}_{X'}$.*

Proof. We apply Derived Categories of Spaces, Lemma 26.7. This proves part (1). It also implies that locally on S we can write $Rf_*\mathcal{O}_X = \mathcal{O}_S \oplus P$ where P is perfect of tor amplitude in $[1, \infty)$. Recall that formation of $Rf_*\mathcal{O}_X$ commutes with arbitrary base change (Derived Categories of Spaces, Lemma 25.4). Thus for $s \in S$ we have

$$H^i(P \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s)) = H^i(X_s, \mathcal{O}_{X_s}) \text{ for } i \geq 1$$

This is zero unless $i = 1$ since X_s is a 1-dimensional Noetherian scheme, see Cohomology, Proposition 20.7. Then $P = H^1(P)[-1]$ and $H^1(P)$ is finite locally free for example by More on Algebra, Lemma 75.6. Since everything is compatible with base change we also see that (3) holds. $\square$

Lemma 9.4. *There is a decomposition into open and closed substacks*

$$\mathcal{C}urves^{h0,1} = \coprod\nolimits_{g \geq 0} \mathcal{C}urves_g$$

where each $\mathcal{C}urves_g$ *is characterized as follows:*

(1) *given a family of curves* $f : X \to S$ *the following are equivalent*
 (a) *the classifying morphism* $S \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves_g$*,*
 (b) $f_*\mathcal{O}_X = \mathcal{O}_S$*, this holds after arbitrary base change, the fibres of* f *have dimension* 1*, and* $R^1f_*\mathcal{O}_X$ *is a locally free* $\mathcal{O}_S$*-module of rank* g*,*

(2) *given a scheme* X *proper over a field* k *with* $\dim(X) \leq 1$ *the following are equivalent*
 (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves_g$*,*
 (b) $\dim(X) = 1$*,* $k = H^0(X, \mathcal{O}_X)$*, and the genus of* X *is* g*.*

Proof. We already have the existence of $\mathcal{C}urves^{h0,1}$ as an open substack of $\mathcal{C}urves$ characterized by the conditions of the lemma not involving R^1f_* or H^1, see Lemma 9.1. The existence of the decomposition into open and closed substacks follows immediately from the discussion in Section 6 and Lemma 9.3. This proves the characterization in (1). The characterization in (2) follows from the definition of the genus in Algebraic Curves, Definition 8.1. □

10. Geometrically reduced curves

There is an open substack of $\mathcal{C}urves$ parametrizing the geometrically reduced "curves".

Lemma 10.1. *There exist an open substack* $\mathcal{C}urves^{geomred} \subset \mathcal{C}urves$ *such that*

(1) *given a family of curves* $X \to S$ *the following are equivalent*
 (a) *the classifying morphism* $S \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{geomred}$*,*
 (b) *the fibres of the morphism* $X \to S$ *are geometrically reduced (More on Morphisms of Spaces, Definition 29.2),*

(2) *given a scheme* X *proper over a field* k *with* $\dim(X) \leq 1$ *the following are equivalent*
 (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{geomred}$*,*
 (b) X *is geometrically reduced over* k*.*

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 29.6 the set

$$E = \{s \in S : \text{the fibre of } X \to S \text{ at } s \text{ is geometrically reduced}\}$$

is open in S. Formation of this open commutes with arbitrary base change by More on Morphisms of Spaces, Lemma 29.3. Thus we get the open substack with the desired properties by the method discussed in Section 6. □

Lemma 10.2. *We have* $\mathcal{C}urves^{geomred} \subset \mathcal{C}urves^{CM}$ *as open substacks of* $\mathcal{C}urves$*.*

Proof. This is true because a reduced Noetherian scheme of dimension ≤ 1 is Cohen-Macaulay. See Algebra, Lemma 157.3. □

11. Geometrically reduced and connected curves

There is an open substack of $\mathcal{C}urves$ parametrizing the geometrically reduced and connected "curves". We will get rid of 0-dimensional objects right away.

Lemma 11.1. *There exist an open substack* $\mathcal{C}urves^{grc,1} \subset \mathcal{C}urves$ *such that*

(1) *given a family of curves* $X \to S$ *the following are equivalent*
 (a) *the classifying morphism* $S \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{grc,1}$*,*

   (b) *the geometric fibres of the morphism $X \to S$ are reduced, connected, and have dimension* 1*,*
(2) *given a scheme X proper over a field k with* $\dim(X) \leq 1$ *the following are equivalent*
   (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{grc,1}$*,*
   (b) *X is geometrically reduced, geometrically connected, and has dimension* 1*.*

Proof. By Lemmas 10.1, 10.2, 8.1, and 8.2 it is clear that we have

$$\mathcal{C}urves^{grc,1} \subset \mathcal{C}urves^{geomred} \cap \mathcal{C}urves^{CM,1}$$

if it exists. Let $f : X \to S$ be a family of curves such that f is Cohen-Macaulay, has geometrically reduced fibres, and has relative dimension 1. By More on Morphisms of Spaces, Lemma 36.9 in the Stein factorization

$$X \to T \to S$$

the morphism $T \to S$ is étale. This implies that there is an open and closed subscheme $S' \subset S$ such that $X \times_S S' \to S'$ has geometrically connected fibres (in the decomposition of Morphisms, Lemma 48.5 for the finite locally free morphism $T \to S$ this corresponds to S_1). Formation of this open commutes with arbitrary base change because the number of connected components of geometric fibres is invariant under base change (it is also true that the Stein factorization commutes with base change in our particular case but we don't need this to conclude). Thus we get the open substack with the desired properties by the method discussed in Section 6. ☐

Lemma 11.2. *We have* $\mathcal{C}urves^{grc,1} \subset \mathcal{C}urves^{h0,1}$ *as open substacks of* $\mathcal{C}urves$*. In particular, given a family of curves $f : X \to S$ whose geometric fibres are reduced, connected and of dimension* 1*, then $R^1 f_* \mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module whose formation commutes with arbitrary base change.*

Proof. This follows from Varieties, Lemma 9.3 and Lemmas 9.1 and 11.1. The final statement follows from Lemma 9.3. ☐

Lemma 11.3. *There is a decomposition into open and closed substacks*

$$\mathcal{C}urves^{grc,1} = \coprod\nolimits_{g \geq 0} \mathcal{C}urves_g^{grc,1}$$

where each $\mathcal{C}urves_g^{grc,1}$ is characterized as follows:

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through* $\mathcal{C}urves_g^{grc,1}$*,*
   (b) *the geometric fibres of the morphism $f : X \to S$ are reduced, connected, of dimension 1 and $R^1 f_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,*
(2) *given a scheme X proper over a field k with* $\dim(X) \leq 1$ *the following are equivalent*
   (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves_g^{grc,1}$*,*
   (b) *X is geometrically reduced, geometrically connected, has dimension* 1*, and has genus g.*

Proof. First proof: set $\mathcal{C}urves_g^{grc,1} = \mathcal{C}urves^{grc,1} \cap \mathcal{C}urves_g$ and combine Lemmas 11.2 and 9.4. Second proof: The existence of the decomposition into open and closed substacks follows immediately from the discussion in Section 6 and Lemma

11.2. This proves the characterization in (1). The characterization in (2) follows as well since the genus of a geometrically reduced and connected proper 1-dimensional scheme X/k is defined (Algebraic Curves, Definition 8.1 and Varieties, Lemma 9.3) and is equal to $\dim_k H^1(X, \mathcal{O}_X)$. $\square$

12. Gorenstein curves

There is an open substack of $\mathcal{C}urves$ parametrizing the Gorenstein "curves".

Lemma 12.1. *There exist an open substack $\mathcal{C}urves^{Gorenstein} \subset \mathcal{C}urves$ such that*

1. *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein}$,*
   (b) *the morphism $X \to S$ is Gorenstein,*
2. *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein}$,*
   (b) *X is Gorenstein.*

Proof. Let $f : X \to S$ be a family of curves. By More on Morphisms of Spaces, Lemma 27.7 the set

$$W = \{x \in |X| : f \text{ is Gorenstein at } x\}$$

is open in $|X|$ and formation of this open commutes with arbitrary base change. Since f is proper the subset

$$S' = S \setminus f(|X| \setminus W)$$

of S is open and $X \times_S S' \to S'$ is Gorenstein. Moreover, formation of S' commutes with arbitrary base change because this is true for W Thus we get the open substack with the desired properties by the method discussed in Section 6. $\square$

Lemma 12.2. *There exist an open substack $\mathcal{C}urves^{Gorenstein,1} \subset \mathcal{C}urves$ such that*

1. *given a family of curves $X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein,1}$,*
   (b) *the morphism $X \to S$ is Gorenstein and has relative dimension 1 (Morphisms of Spaces, Definition 33.2),*
2. *given a scheme X proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{Gorenstein,1}$,*
   (b) *X is Gorenstein and X is equidimensional of dimension 1.*

Proof. Recall that a Gorenstein scheme is Cohen-Macaulay (Duality for Schemes, Lemma 24.2) and that a Gorenstein morphism is a Cohen-Macaulay morphism (Duality for Schemes, Lemma 25.4. Thus we can set $\mathcal{C}urves^{Gorenstein,1}$ equal to the intersection of $\mathcal{C}urves^{Gorenstein}$ and $\mathcal{C}urves^{CM,1}$ inside of $\mathcal{C}urves$ and use Lemmas 12.1 and 8.2. $\square$

13. Local complete intersection curves

There is an open substack of $\mathcal{C}urves$ parametrizing the local complete intersection "curves".

Lemma 13.1. *There exist an open substack $\mathcal{C}urves^{lci} \subset \mathcal{C}urves$ such that*

1. *given a family of curves $X \to S$ the following are equivalent*

   (a) *the classifying morphism* $S \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{lci}$,
   (b) $X \to S$ *is a local complete intersection morphism, and*
   (c) $X \to S$ *is a syntomic morphism.*
(2) *given* X *a proper scheme over a field* k *of dimension* ≤ 1 *the following are equivalent*
   (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{lci}$,
   (b) X *is a local complete intersection over* k.

Proof. Recall that being a syntomic morphism is the same as being flat and a local complete intersection morphism, see More on Morphisms of Spaces, Lemma 48.6. Thus (1)(b) is equivalent to (1)(c). In Section 6 we have seen it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is a local complete intersection morphism and such that formation of S' commutes with arbitrary base change. This follows from the more general More on Morphisms of Spaces, Lemma 49.7. $\square$

14. Curves with isolated singularities

We can look at the open substack of $\mathcal{C}urves$ parametrizing "curves" with only a finite number of singular points (these may correspond to 0-dimensional components in our setup).

Lemma 14.1. *There exist an open substack* $\mathcal{C}urves^+ \subset \mathcal{C}urves$ *such that*

(1) *given a family of curves* $X \to S$ *the following are equivalent*
   (a) *the classifying morphism* $S \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^+$,
   (b) *the singular locus of* $X \to S$ *endowed with any/some closed subspace structure is finite over* S.
(2) *given* X *a proper scheme over a field* k *of dimension* ≤ 1 *the following are equivalent*
   (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^+$,
   (b) $X \to \mathrm{Spec}(k)$ *is smooth except at finitely many points.*

Proof. To prove the lemma it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that the fibre of $S' \times_S X \to S'$ have property (2). (Formation of the open will automatically commute with base change.) By definition the locus $T \subset |X|$ of points where $X \to S$ is not smooth is closed. Let $Z \subset X$ be the closed subspace given by the reduced induced algebraic space structure on T (Properties of Spaces, Definition 12.5). Now if $s \in S$ is a point where Z_s is finite, then there is an open neighbourhood $U_s \subset S$ of s such that $Z \cap f^{-1}(U_s) \to U_s$ is finite, see More on Morphisms of Spaces, Lemma 35.2. This proves the lemma. $\square$

15. The smooth locus of the stack of curves

The morphism

$$\mathcal{C}urves \longrightarrow \mathrm{Spec}(\mathbf{Z})$$

is smooth over a maximal open substack, see Morphisms of Stacks, Lemma 33.6. We want to give a criterion for when a curve is in this locus. We will do this using a bit of deformation theory.

Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Choose a Cohen ring Λ for k, see Algebra, Lemma 160.6. Then we are in the situation described in Deformation Problems, Example 9.1 and Lemma 9.2. Thus we obtain a deformation category $\mathcal{D}ef_X$ on the category $\mathcal{C}_\Lambda$ of Artinian local Λ-algebras with residue field k.

Lemma 15.1. *In the situation above the following are equivalent*

1. *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through the open where* $\mathcal{C}urves \to \mathrm{Spec}(\mathbf{Z})$ *is smooth,*
2. *the deformation category* $\mathcal{D}ef_X$ *is unobstructed.*

Proof. Since $\mathcal{C}urves \longrightarrow \mathrm{Spec}(\mathbf{Z})$ is locally of finite presentation (Lemma 5.3) formation of the open substack where $\mathcal{C}urves \longrightarrow \mathrm{Spec}(\mathbf{Z})$ is smooth commutes with flat base change (Morphisms of Stacks, Lemma 33.6). Since the Cohen ring Λ is flat over $\mathbf{Z}$, we may work over Λ. In other words, we are trying to prove that

$$\Lambda\text{-}\mathcal{C}urves \longrightarrow \mathrm{Spec}(\Lambda)$$

is smooth in an open neighbourhood of the point $x_0 : \mathrm{Spec}(k) \to \Lambda\text{-}\mathcal{C}urves$ defined by X/k if and only if $\mathcal{D}ef_X$ is unobstructed.

The lemma now follows from Geometry of Stacks, Lemma 2.7 and the equality

$$\mathcal{D}ef_X = \mathcal{F}_{\Lambda\text{-}\mathcal{C}urves,k,x_0}$$

This equality is not completely trivial to esthablish. Namely, on the left hand side we have the deformation category classifying all flat deformations $Y \to \mathrm{Spec}(A)$ of X as a scheme over $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$. On the right hand side we have the deformation category classifying all flat morphisms $Y \to \mathrm{Spec}(A)$ with special fibre X where Y is an algebraic space and $Y \to \mathrm{Spec}(A)$ is proper, of finite presentation, and of relative dimension ≤ 1. Since A is Artinian, we find that Y is a scheme for example by Spaces over Fields, Lemma 9.3. Thus it remains to show: a flat deformation $Y \to \mathrm{Spec}(A)$ of X as a scheme over an Artinian local ring A with residue field k is proper, of finite presentation, and of relative dimension ≤ 1. Relative dimension is defined in terms of fibres and hence holds automatically for Y/A since it holds for X/k. The morphism $Y \to \mathrm{Spec}(A)$ is proper and locally of finite presentation as this is true for $X \to \mathrm{Spec}(k)$, see More on Morphisms, Lemma 10.3. $\square$

Here is a "large" open of the stack of curves which is contained in the smooth locus.

Lemma 15.2. *The open substack*

$$\mathcal{C}urves^{lci+} = \mathcal{C}urves^{lci} \cap \mathcal{C}urves^{+} \subset \mathcal{C}urves$$

has the following properties

1. $\mathcal{C}urves^{lci+} \to \mathrm{Spec}(\mathbf{Z})$ *is smooth,*
2. *given a family of curves* $X \to S$ *the following are equivalent*
   (a) *the classifying morphism* $S \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{lci+}$,
   (b) $X \to S$ *is a local complete intersection morphism and the singular locus of* $X \to S$ *endowed with any/some closed subspace structure is finite over* S,
3. *given* X *a proper scheme over a field* k *of dimension* ≤ 1 *the following are equivalent*
   (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{lci+}$,
   (b) X *is a local complete intersection over* k *and* $X \to \mathrm{Spec}(k)$ *is smooth except at finitely many points.*

Proof. If we can show that there is an open substack $\mathcal{C}urves^{lci+}$ whose points are characterized by (2), then we see that (1) holds by combining Lemma 15.1 with Deformation Problems, Lemma 16.4. Since

$$\mathcal{C}urves^{lci+} = \mathcal{C}urves^{lci} \cap \mathcal{C}urves^{+}$$

inside $\mathcal{C}urves$, we conclude by Lemmas 13.1 and 14.1. □

16. Smooth curves

In this section we study open substacks of $\mathcal{C}urves$ parametrizing smooth "curves".

Lemma 16.1. *There exist an open substacks*

$$\mathcal{C}urves^{smooth,1} \subset \mathcal{C}urves^{smooth} \subset \mathcal{C}urves$$

such that

1. *given a family of curves $f : X \to S$ the following are equivalent*
   - (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth}$, resp. $\mathcal{C}urves^{smooth,1}$,*
   - (b) *f is smooth, resp. smooth of relative dimension 1,*
2. *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   - (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth}$, resp. $\mathcal{C}urves^{smooth,1}$,*
   - (b) *X is smooth over k, resp. X is smooth over k and X is equidimensional of dimension 1.*

Proof. To prove the statements regarding $\mathcal{C}urves^{smooth}$ it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is smooth and such that the formation of this open commutes with base change. We know that there is a maximal open $U \subset X$ such that $U \to S$ is smooth and that formation of U commutes with arbitrary base change, see Morphisms of Spaces, Lemma 37.9. If $T = |X| \setminus |U|$ then $f(T)$ is closed in S as f is proper. Setting $S' = S \setminus f(T)$ we obtain the desired open.

Let $f : X \to S$ be a family of curves with f smooth. Then the fibres X_s are smooth over $\kappa(s)$ and hence Cohen-Macaulay (for example you can see this using Algebra, Lemmas 137.5 and 135.3). Thus we see that we may set

$$\mathcal{C}urves^{smooth,1} = \mathcal{C}urves^{smooth} \cap \mathcal{C}urves^{CM,1}$$

and the desired equivalences follow from what we've already shown for $\mathcal{C}urves^{smooth}$ and Lemma 8.2. □

Lemma 16.2. *The morphism $\mathcal{C}urves^{smooth} \to \mathrm{Spec}(\mathbf{Z})$ is smooth.*

Proof. Follows immediately from the observation that $\mathcal{C}urves^{smooth} \subset \mathcal{C}urves^{lci+}$ and Lemma 15.2. □

Lemma 16.3. *There exist an open substack $\mathcal{C}urves^{smooth,h0} \subset \mathcal{C}urves$ such that*

1. *given a family of curves $f : X \to S$ the following are equivalent*
   - (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{smooth}$,*
   - (b) *$f_*\mathcal{O}_X = \mathcal{O}_S$, this holds after any base change, and f is smooth of relative dimension 1,*

(2) *given X a scheme proper over a field k with* $\dim(X) \leq 1$ *the following are equivalent*
  (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{smooth,h0}$,
  (b) *X is smooth,* $\dim(X) = 1$*, and* $k = H^0(X, \mathcal{O}_X)$,
  (c) *X is smooth,* $\dim(X) = 1$*, and X is geometrically connected,*
  (d) *X is smooth,* $\dim(X) = 1$*, and X is geometrically integral, and*
  (e) *$X_{\overline{k}}$ is a smooth curve.*

Proof. If we set

$$\mathcal{C}urves^{smooth,h0} = \mathcal{C}urves^{smooth} \cap \mathcal{C}urves^{h0,1}$$

then we see that (1) holds by Lemmas 9.1 and 16.1. In fact, this also gives the equivalence of (2)(a) and (2)(b). To finish the proof we have to show that (2)(b) is equivalent to each of (2)(c), (2)(d), and (2)(e).

A smooth scheme over a field is geometrically normal (Varieties, Lemma 25.4), smoothness is preserved under base change (Morphisms, Lemma 34.5), and being smooth is fpqc local on the target (Descent, Lemma 23.27). Keeping this in mind, the equivalence of (2)(b), (2)(c), 2(d), and (2)(e) follows from Varieties, Lemma 10.7. □

Definition 16.4. We denote $\mathcal{M}$ and we name it the *moduli stack of smooth proper curves* the algebraic stack $\mathcal{C}urves^{smooth,h0}$ parametrizing families of curves introduced in Lemma 16.3. For $g \geq 0$ we denote $\mathcal{M}_g$ and we name it the *moduli stack of smooth proper curves of genus g* the algebraic stack introduced in Lemma 16.5.

Here is the obligatory lemma.

Lemma 16.5. *There is a decomposition into open and closed substacks*

$$\mathcal{M} = \coprod\nolimits_{g \geq 0} \mathcal{M}_g$$

where each $\mathcal{M}_g$ is characterized as follows:

(1) *given a family of curves $f : X \to S$ the following are equivalent*
  (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{M}_g$,*
  (b) *$X \to S$ is smooth, $f_*\mathcal{O}_X = \mathcal{O}_S$, this holds after any base change, and $R^1 f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,*

(2) *given X a scheme proper over a field k with* $\dim(X) \leq 1$ *the following are equivalent*
  (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through $\mathcal{M}_g$,*
  (b) *X is smooth,* $\dim(X) = 1$*, $k = H^0(X, \mathcal{O}_X)$, and X has genus g,*
  (c) *X is smooth,* $\dim(X) = 1$*, X is geometrically connected, and X has genus g,*
  (d) *X is smooth,* $\dim(X) = 1$*, X is geometrically integral, and X has genus g, and*
  (e) *$X_{\overline{k}}$ is a smooth curve of genus g.*

Proof. Combine Lemmas 16.3 and 9.4. You can also use Lemma 11.3 instead. □

Lemma 16.6. *The morphisms $\mathcal{M} \to \mathrm{Spec}(\mathbf{Z})$ and $\mathcal{M}_g \to \mathrm{Spec}(\mathbf{Z})$ are smooth.*

Proof. Since $\mathcal{M}$ is an open substack of $\mathcal{C}urves^{lci+}$ this follows from Lemma 15.2. □

17. Density of smooth curves

The title of this section is misleading as we don't claim $\mathcal{C}urves^{smooth}$ is dense in $\mathcal{C}urves$. In fact, this is false as was shown by Mumford in [Mum75]. However, we will see that the smooth "curves" are dense in a large open.

Lemma 17.1. *The inclusion*

$$|\mathcal{C}urves^{smooth}| \subset |\mathcal{C}urves^{lci+}|$$

is that of an open dense subset.

Proof. By the very construction of the topology on $|\mathcal{C}urves^{lci+}|$ in Properties of Stacks, Section 4 we find that $|\mathcal{C}urves^{smooth}|$ is an open subset. Let $\xi \in |\mathcal{C}urves^{lci+}|$ be a point. Then there exists a field k and a scheme X over k with X proper over k, with $\dim(X) \leq 1$, with X a local complete intersection over k, and with X is smooth over k except at finitely many points, such that ξ is the equivalence class of the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves^{lci+}$ determined by X. See Lemma 15.2. By Deformation Problems, Lemma 17.6 there exists a flat projective morphism $Y \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth and whose special fibre is isomorphic to X. Consider the classifying morphism

$$\mathrm{Spec}(k[[t]]) \longrightarrow \mathcal{C}urves^{lci+}$$

determined by Y. The image of the closed point is ξ and the image of the generic point is in $|\mathcal{C}urves^{smooth}|$. Since the generic point specializes to the closed point in $|\mathrm{Spec}(k[[t]])|$ we conclude that ξ is in the closure of $|\mathcal{C}urves^{smooth}|$ as desired. $\square$

18. Nodal curves

In algebraic geometry a special role is played by nodal curves. We suggest the reader take a brief look at some of the discussion in Algebraic Curves, Sections 19 and 20 and More on Morphisms of Spaces, Section 55.

Lemma 18.1. *There exist an open substack $\mathcal{C}urves^{nodal} \subset \mathcal{C}urves$ such that*

1. *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{nodal}$,*
   (b) *f is at-worst-nodal of relative dimension 1,*
2. *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{nodal}$,*
   (b) *the singularities of X are at-worst-nodal and X is equidimensional of dimension 1.*

Proof. In fact, it suffices to show that given a family of curves $f : X \to S$, there is an open subscheme $S' \subset S$ such that $S' \times_S X \to S'$ is at-worst-nodal of relative dimension 1 and such that formation of S' commutes with arbitrary base change. By More on Morphisms of Spaces, Lemma 55.4 there is a maximal open subspace $X' \subset X$ such that $f|_{X'} : X' \to S$ is at-worst-nodal of relative dimension 1. Moreover, formation of X' commutes with base change. Hence we can take

$$S' = S \setminus |f|(|X| \setminus |X'|)$$

This is open because a proper morphism is universally closed by definition. $\square$

Lemma 18.2. *The morphism $\mathcal{C}urves^{nodal} \to \mathrm{Spec}(\mathbf{Z})$ is smooth.*

Proof. Follows immediately from the observation that $\mathcal{C}urves^{nodal} \subset \mathcal{C}urves^{lci+}$ and Lemma 15.2. $\square$

19. The relative dualizing sheaf

This section serves mainly to introduce notation in the case of families of curves. Most of the work has already been done in the chapter on duality.

Let $f : X \to S$ be a family of curves. There exists an object $\omega_{X/S}^\bullet$ in $D_{QCoh}(\mathcal{O}_X)$, called the *relative dualizing complex*, having the following property: for every base change diagram

$$\begin{tikzcd} X_U \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ U \arrow[r, "g"] & S \end{tikzcd}$$

with $U = \mathrm{Spec}(A)$ affine the complex $\omega_{X_U/U}^\bullet = L(g')^*\omega_{X/S}^\bullet$ represents the functor

$$D_{QCoh}(\mathcal{O}_{X_U}) \longrightarrow \mathrm{Mod}_A, \quad K \longmapsto \mathrm{Hom}_U(Rf_*K, \mathcal{O}_U)$$

More precisely, let $(\omega_{X/S}^\bullet, \tau)$ be the relative dualizing complex of the family as defined in Duality for Spaces, Definition 9.1. Existence is shown in Duality for Spaces, Lemma 9.5. Moreover, formation of $(\omega_{X/S}^\bullet, \tau)$ commutes with arbitrary base change (essentially by definition; a precise reference is Duality for Spaces, Lemma 9.6). From now on we will identify the base change of $\omega_{X/S}^\bullet$ with the relative dualizing complex of the base changed family without further mention.

Let $\{S_i \to S\}$ be an étale covering with S_i affine such that $X_i = X \times_S S_i$ is a scheme, see Lemma 4.3. By Duality for Spaces, Lemma 10.1 we find that $\omega_{X_i/S_i}^\bullet$ agrees with the relative dualizing complex for the proper, flat, and finitely presented morphism $f_i : X_i \to S_i$ of schemes discussed in Duality for Schemes, Remark 12.5. Thus to prove a property of $\omega_{X/S}^\bullet$ which is étale local, we may assume $X \to S$ is a morphism of schemes and use the theory developped in the chapter on duality for schemes. More generally, for any base change of X which is a scheme, the relative dualizing complex agrees with the relative dualizing complex of Duality for Schemes, Remark 12.5. From now on we will use this identification without further mention.

In particular, let $\mathrm{Spec}(k) \to S$ be a morphism where k is a field. Denote X_k the base change (this is a scheme by Spaces over Fields, Lemma 9.3). Then $\omega_{X_k/k}^\bullet$ is isomorphic to the complex $\omega_{X_k}^\bullet$ of Algebraic Curves, Lemma 4.1 (both represent the same functor and so we can use the Yoneda lemma, but really this holds because of the remarks above). We conclude that the cohomology sheaves $H^i(\omega_{X_k/k}^\bullet)$ are nonzero only for $i = 0, -1$. If X_k is Cohen-Macaulay and equidimensional of dimension 1, then we only have H^{-1} and if X_k is in addition Gorenstein, then $H^{-1}(\omega_{X_k/k})$ is invertible, see Algebraic Curves, Lemmas 4.2 and 5.2.

Lemma 19.1. *Let $X \to S$ be a family of curves with Cohen-Macaulay fibres equidimensional of dimension 1 (Lemma 8.2). Then $\omega_{X/S}^\bullet = \omega_{X/S}[1]$ where $\omega_{X/S}$ is a pseudo-coherent $\mathcal{O}_X$-module flat over S whose formation commutes with arbitrary base change.*

Proof. We urge the reader to deduce this directly from the discussion above of what happens after base change to a field. Our proof will use a somewhat cumbersome reduction to the Noetherian schemes case.

Once we show $\omega_{X/S}^\bullet = \omega_{X/S}[1]$ with $\omega_{X/S}$ flat over S, the statement on base change will follow as we already know that formation of $\omega_{X/S}^\bullet$ commutes with arbitrary base change. Moreover, the pseudo-coherence will be automatic as $\omega_{X/S}^\bullet$ is pseudo-coherent by definition. Vanishing of the other cohomology sheaves and flatness may be checked étale locally. Thus we may assume $f : X \to S$ is a morphism of schemes with S affine (see discussion above). Write $S = \lim S_i$ as a cofiltered limit of affine schemes S_i of finite type over $\mathbf{Z}$. Since $\mathcal{C}urves^{CM,1}$ is locally of finite presentation over $\mathbf{Z}$ (as an open substack of $\mathcal{C}urves$, see Lemmas 8.2 and 5.3), we can find an i and a family of curves $X_i \to S_i$ whose pullback is $X \to S$ (Limits of Stacks, Lemma 3.5). After increasing i if necessary we may assume X_i is a scheme, see Limits of Spaces, Lemma 5.11. Since formation of $\omega_{X/S}^\bullet$ commutes with arbitrary base change, we may replace S by S_i. Doing so we may and do assume S_i is Noetherian. Then f is clearly a Cohen-Macaulay morphism (More on Morphisms, Definition 22.1) by our assumption on the fibres. Also then $\omega_{X/S}^\bullet = f^!\mathcal{O}_S$ by the very construction of $f^!$ in Duality for Schemes, Section 16. Thus the lemma by Duality for Schemes, Lemma 23.3. $\square$

Definition 19.2. Let $f : X \to S$ be a family of curves with Cohen-Macaulay fibres equidimensional of dimension 1 (Lemma 8.2). Then the $\mathcal{O}_X$-module

$$\omega_{X/S} = H^{-1}(\omega_{X/S}^\bullet)$$

studied in Lemma 19.1 is called the *relative dualizing sheaf* of f.

In the situation of Definition 19.2 the relative dualizing sheaf $\omega_{X/S}$ has the following property (which moreover characterizes it locally on S): for every base change diagram

$$\begin{tikzcd} X_U \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ U \arrow[r, "g"] & S \end{tikzcd}$$

with $U = \mathrm{Spec}(A)$ affine the module $\omega_{X_U/U} = (g')^*\omega_{X/S}$ represents the functor

$$QCoh(\mathcal{O}_{X_U}) \longrightarrow \mathrm{Mod}_A, \quad \mathcal{F} \longmapsto \mathrm{Hom}_A(H^1(X, \mathcal{F}), A)$$

This follows immediately from the corresponding property of the relative dualizing complex given above. In particular, if $A = k$ is a field, then we recover the dualizing module of X_k as introduced and studied in Algebraic Curves, Lemmas 4.1, 4.2, and 5.2.

Lemma 19.3. *Let $X \to S$ be a family of curves with Gorenstein fibres equidimensional of dimension 1 (Lemma 12.2). Then the relative dualizing sheaf $\omega_{X/S}$ is an invertible $\mathcal{O}_X$-module whose formation commutes with arbitrary base change.*

Proof. This is true because the pullback of the relative dualizing module to a fibre is invertible by the discussion above. Alternatively, you can argue exactly as in the proof of Lemma 19.1 and deduce the result from Duality for Schemes, Lemma 25.10. $\square$

20. Prestable curves

The following definition is equivalent to what appears to be the generally accepted notion of a prestable family of curves.

Definition 20.1. Let $f : X \to S$ be a family of curves. We say f is a *prestable family of curves* if

(1) f is at-worst-nodal of relative dimension 1, and
(2) $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change[1].

Let X be a proper scheme over a field k with $\dim(X) \leq 1$. Then $X \to \mathrm{Spec}(k)$ is a family of curves and hence we can ask whether or not it is prestable[2] in the sense of the definition. Unwinding the definitions we see the following are equivalent

(1) X is prestable,
(2) the singularities of X are at-worst-nodal, $\dim(X) = 1$, and $k = H^0(X, \mathcal{O}_X)$,
(3) $X_{\overline{k}}$ is connected and it is smooth over $\overline{k}$ apart from a finite number of nodes (Algebraic Curves, Definition 16.2).

This shows that our definition agrees with most definitions one finds in the literature.

Lemma 20.2. *There exist an open substack $\mathcal{C}urves^{prestable} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $f : X \to S$ the following are equivalent*
    (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{prestable}$,*
    (b) *$X \to S$ is a prestable family of curves,*
(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
    (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{prestable}$,*
    (b) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, and $k = H^0(X, \mathcal{O}_X)$.*

Proof. Given a family of curves $X \to S$ we see that it is prestable if and only if the classifying morphism factors both through $\mathcal{C}urves^{nodal}$ and $\mathcal{C}urves^{h0,1}$. An alternative is to use $\mathcal{C}urves^{grc,1}$ (since a nodal curve is geometrically reduced hence has H^0 equal to the ground field if and only if it is connected). In a formula

$$\mathcal{C}urves^{prestable} = \mathcal{C}urves^{nodal} \cap \mathcal{C}urves^{h0,1} = \mathcal{C}urves^{nodal} \cap \mathcal{C}urves^{grc,1}$$

Thus the lemma follows from Lemmas 9.1 and 18.1. $\square$

For each genus $g \geq 0$ we have the algebraic stack classifying the prestable curves of genus g. In fact, from now on we will say that $X \to S$ is a *prestable family of curves of genus g* if and only if the classifying morphism $S \to \mathcal{C}urves$ factors through the open substack $\mathcal{C}urves^{prestable}_g$ of Lemma 20.3.

Lemma 20.3. *There is a decomposition into open and closed substacks*

$$\mathcal{C}urves^{prestable} = \coprod\nolimits_{g \geq 0} \mathcal{C}urves^{prestable}_g$$

where each $\mathcal{C}urves^{prestable}_g$ is characterized as follows:

---

[1] In fact, it suffices to require $f_*\mathcal{O}_X = \mathcal{O}_S$ because the Stein factorization of f is étale in this case, see More on Morphisms of Spaces, Lemma 36.9. The condition may also be replaced by asking the geometric fibres to be connected, see Lemma 11.2.

[2] We can't use the term "prestable curve" here because curve implies irreducible. See discussion in Algebraic Curves, Section 20.

(1) *given a family of curves $f : X \to S$ the following are equivalent*
  (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{prestable}$,*
  (b) *$X \to S$ is a prestable family of curves and $R^1f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,*
(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
  (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves_g^{prestable}$,*
  (b) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, and the genus of X is g.*

Proof. Since we have seen that $\mathcal{C}urves^{prestable}$ is contained in $\mathcal{C}urves^{h0,1}$, this follows from Lemmas 20.2 and 9.4. □

Lemma 20.4. *The morphisms $\mathcal{C}urves^{prestable} \to \mathrm{Spec}(\mathbf{Z})$ and $\mathcal{C}urves_g^{prestable} \to \mathrm{Spec}(\mathbf{Z})$ are smooth.*

Proof. Since $\mathcal{C}urves^{prestable}$ is an open substack of $\mathcal{C}urves^{nodal}$ this follows from Lemma 18.2. □

21. Semistable curves

The following lemma will help us understand families of semistable curves.

Lemma 21.1. *Let $f : X \to S$ be a prestable family of curves of genus $g \geq 1$. Let $s \in S$ be a point of the base scheme. Let $m \geq 2$. The following are equivalent*

(1) *X_s does not have a rational tail (Algebraic Curves, Example 22.1), and*
(2) *$f^*f_*\omega_{X/S}^{\otimes m} \to \omega_{X/S}^{\otimes m}$, is surjective over $f^{-1}(U)$ for some $s \in U \subset S$ open.*

Proof. Assume (2). Using the material in Section 19 we conclude that $\omega_{X_s}^{\otimes m}$ is globally generated. However, if $C \subset X_s$ is a rational tail, then $\deg(\omega_{X_s}|_C) < 0$ by Algebraic Curves, Lemma 22.2 hence $H^0(C, \omega_{X_s}|_C) = 0$ by Varieties, Lemma 44.12 which contradicts the fact that it is globally generated. This proves (1).

Assume (1). First assume that $g \geq 2$. Assumption (1) implies $\omega_{X_s}^{\otimes m}$ is globally generated, see Algebraic Curves, Lemma 22.6. Moreover, we have

$$\mathrm{Hom}_{\kappa(s)}(H^1(X_s, \omega_{X_s}^{\otimes m}), \kappa(s)) = H^0(X_s, \omega_{X_s}^{\otimes 1-m})$$

by duality, see Algebraic Curves, Lemma 4.2. Since $\omega_{X_s}^{\otimes m}$ is globally generated we find that the restriction to each irreducible component has nonegative degree. Hence the restriction of $\omega_{X_s}^{\otimes 1-m}$ to each irreducible component has nonpositive degree. Since $\deg(\omega_{X_s}^{\otimes 1-m}) = (1-m)(2g-2) < 0$ by Riemann-Roch (Algebraic Curves, Lemma 5.2) we conclude that the H^0 is zero by Varieties, Lemma 44.13. By cohomology and base change we conclude that

$$E = Rf_*\omega_{X/S}^{\otimes m}$$

is a perfect complex whose formation commutes with arbitrary base change (Derived Categories of Spaces, Lemma 25.4). The vanishing proved above tells us that $E \otimes^{\mathbf{L}} \kappa(s)$ is equal to $H^0(X_s, \omega_{X_s}^{\otimes m})$ placed in degree 0. After shrinking S we find $E = f_*\omega_{X/S}^{\otimes m}$ is a locally free $\mathcal{O}_S$-module placed in degree 0 (and its formation commutes with arbitrary base change as we've already said), see Derived Categories of Spaces, Lemma 26.5. The map $f^*f_*\omega_{X/S}^{\otimes m} \to \omega_{X/S}^{\otimes m}$ is surjective after restricting to X_s. Thus it is surjective in an open neighbourhood of X_s. Since f is proper, this open neighbourhood contains $f^{-1}(U)$ for some open neighbourhood U of s in S.

Assume (1) and $g = 1$. By Algebraic Curves, Lemma 22.6 the assumption (1) means that ω_{X_s} is isomorphic to $\mathcal{O}_{X_s}$. If we can show that after shrinking S the invertible sheaf $\omega_{X/S}$ because trivial, then we are done. We may assume S is affine. After shrinking S further, we can write

$$Rf_*\mathcal{O}_X = (\mathcal{O}_S \xrightarrow{0} \mathcal{O}_S)$$

sitting in degrees 0 and 1 compatibly with further base change, see Lemma 9.3. By duality this means that

$$Rf_*\omega_{X/S} = (\mathcal{O}_S \xrightarrow{0} \mathcal{O}_S)$$

sitting in degrees 0 and 1[3]. In particular we obtain an isomorphism $\mathcal{O}_S \to f_*\omega_{X/S}$ which is compatible with base change since formation of $Rf_*\omega_{X/S}$ is compatible with base change (see reference given above). By adjointness, we get a global section $\sigma \in \Gamma(X, \omega_{X/S})$. The restriction of this section to the fibre X_s is nonzero (a basis element in fact) and as ω_{X_s} is trivial on the fibres, this section is nonwhere zero on X_s. Thus it nowhere zero in an open neighbourhood of X_s. Since f is proper, this open neighbourhood contains $f^{-1}(U)$ for some open neighbourhood U of s in S. □

Motivated by Lemma 21.1 we make the following definition.

Definition 21.2. Let $f : X \to S$ be a family of curves. We say f is a *semistable family of curves* if

(1) $X \to S$ is a prestable family of curves, and
(2) X_s has genus ≥ 1 and does not have a rational tail for all $s \in S$.

In particular, a prestable family of curves of genus 0 is never semistable. Let X be a proper scheme over a field k with $\dim(X) \leq 1$. Then $X \to \text{Spec}(k)$ is a family of curves and hence we can ask whether or not it is semistable. Unwinding the definitions we see the following are equivalent

(1) X is semistable,
(2) X is prestable, has genus ≥ 1, and does not have a rational tail,
(3) $X_{\overline{k}}$ is connected, is smooth over $\overline{k}$ apart from a finite number of nodes, has genus ≥ 1, and has no irreducible component isomorphic to $\mathbf{P}^1_{\overline{k}}$ which meets the rest of $X_{\overline{k}}$ in only one point.

To see the equivalence of (2) and (3) use that X has no rational tails if and only if $X_{\overline{k}}$ has no rational tails by Algebraic Curves, Lemma 22.6. This shows that our definition agrees with most definitions one finds in the literature.

Lemma 21.3. *There exist an open substack $\mathcal{C}urves^{semistable} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $f : X \to S$ the following are equivalent*
    (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{semistable}$,*
    (b) *$X \to S$ is a semistable family of curves,*
(2) *given X a scheme proper over a field k with* $\dim(X) \leq 1$ *the following are equivalent*

[3] Use that $Rf_*\omega^\bullet_{X/S} = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X.\omega^\bullet_{X/S}) = R\mathcal{H}om_{\mathcal{O}_S}(Rf_*\mathcal{O}_X, \mathcal{O}_S)$ by Duality for Spaces, Lemma 3.3 and Remark 3.5 and then that $\omega^\bullet_{X/S} = \omega_{X/S}[1]$ by our definitions in Section 19.

(a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves^{semistable}$,
(b) *the singularities of* X *are at-worst-nodal,* $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, *the genus of* X *is* ≥ 1, *and* X *has no rational tails,*
(c) *the singularities of* X *are at-worst-nodal,* $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, *and* $\omega_{X_s}^{\otimes m}$ *is globally generated for* $m \geq 2$.

Proof. The equivalence of (2)(b) and (2)(c) is Algebraic Curves, Lemma 22.6. In the rest of the proof we will work with (2)(b) in accordance with Definition 21.2.

By the discussion in Section 6 it suffices to look at families $f : X \to S$ of prestable curves. By Lemma 21.1 we obtain the desired openness of the locus in question. Formation of this open commutes with arbitrary base change, because the (non)existence of rational tails is insensitive to ground field extensions by Algebraic Curves, Lemma 22.6. □

Lemma 21.4. *There is a decomposition into open and closed substacks*

$$\mathcal{C}urves^{semistable} = \coprod\nolimits_{g \geq 1} \mathcal{C}urves_g^{semistable}$$

where each $\mathcal{C}urves_g^{semistable}$ *is characterized as follows:*

(1) *given a family of curves* $f : X \to S$ *the following are equivalent*
  (a) *the classifying morphism* $S \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves_g^{semistable}$,
  (b) $X \to S$ *is a semistable family of curves and* $R^1 f_* \mathcal{O}_X$ *is a locally free* $\mathcal{O}_S$*-module of rank* g,
(2) *given* X *a scheme proper over a field* k *with* $\dim(X) \leq 1$ *the following are equivalent*
  (a) *the classifying morphism* $\mathrm{Spec}(k) \to \mathcal{C}urves$ *factors through* $\mathcal{C}urves_g^{semistable}$,
  (b) *the singularities of* X *are at-worst-nodal,* $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, *the genus of* X *is* g, *and* X *has no rational tail,*
  (c) *the singularities of* X *are at-worst-nodal,* $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, *the genus of* X *is* g, *and* $\omega_{X_s}^{\otimes m}$ *is globally generated for* $m \geq 2$.

Proof. Combine Lemmas 21.3 and 20.3. □

Lemma 21.5. *The morphisms* $\mathcal{C}urves^{semistable} \to \mathrm{Spec}(\mathbf{Z})$ *and* $\mathcal{C}urves_g^{semistable} \to \mathrm{Spec}(\mathbf{Z})$ *are smooth.*

Proof. Since $\mathcal{C}urves^{semistable}$ is an open substack of $\mathcal{C}urves^{nodal}$ this follows from Lemma 18.2. □

22. Stable curves

The following lemma will help us understand families of stable curves.

Lemma 22.1. *Let* $f : X \to S$ *be a prestable family of curves of genus* $g \geq 2$. *Let* $s \in S$ *be a point of the base scheme. The following are equivalent*

(1) X_s *does not have a rational tail and does not have a rational bridge (Algebraic Curves, Examples 22.1 and 23.1), and*
(2) $\omega_{X/S}$ *is ample on* $f^{-1}(U)$ *for some* $s \in U \subset S$ *open.*

Proof. Assume (2). Then ω_{X_s} is ample on X_s. By Algebraic Curves, Lemmas 22.2 and 23.2 we conclude that (1) holds (we also use the characterization of ample invertible sheaves in Varieties, Lemma 44.15).

Assume (1). Then ω_{X_s} is ample on X_s by Algebraic Curves, Lemmas 23.6. We conclude by Descent on Spaces, Lemma 13.2. $\square$

Motivated by Lemma 22.1 we make the following definition.

Definition 22.2. Let $f : X \to S$ be a family of curves. We say f is a *stable family of curves* if

(1) $X \to S$ is a prestable family of curves, and
(2) X_s has genus ≥ 2 and does not have a rational tails or bridges for all $s \in S$.

In particular, a prestable family of curves of genus 0 or 1 is never stable. Let X be a proper scheme over a field k with $\dim(X) \leq 1$. Then $X \to \mathrm{Spec}(k)$ is a family of curves and hence we can ask whether or not it is stable. Unwinding the definitions we see the following are equivalent

(1) X is stable,
(2) X is prestable, has genus ≥ 2, does not have a rational tail, and does not have a rational bridge,
(3) X is geometrically connected, is smooth over k apart from a finite number of nodes, and ω_X is ample.

To see the equivalence of (2) and (3) use Lemma 22.1 above. This shows that our definition agrees with most definitions one finds in the literature.

Lemma 22.3. *There exist an open substack $\mathcal{C}urves^{stable} \subset \mathcal{C}urves$ such that*

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{stable}$,*
   (b) *$X \to S$ is a stable family of curves,*
(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{stable}$,*
   (b) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is ≥ 2, and X has no rational tails or bridges,*
   (c) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, and ω_{X_s} is ample.*

Proof. By the discussion in Section 6 it suffices to look at families $f : X \to S$ of prestable curves. By Lemma 22.1 we obtain the desired openness of the locus in question. Formation of this open commutes with arbitrary base change, either because the (non)existence of rational tails or bridges is insensitive to ground field extensions by Algebraic Curves, Lemmas 22.6 and 23.6 or because ampleness is insenstive to base field extensions by Descent, Lemma 25.6. $\square$

Definition 22.4. We denote $\overline{\mathcal{M}}$ and we name the *moduli stack of stable curves* the algebraic stack $\mathcal{C}urves^{stable}$ parametrizing stable families of curves introduced in Lemma 22.3. For $g \geq 2$ we denote $\overline{\mathcal{M}}_g$ and we name the *moduli stack of stable curves of genus g* the algebraic stack introduced in Lemma 22.5.

Here is the obligatory lemma.

Lemma 22.5. *There is a decomposition into open and closed substacks*

$$\overline{\mathcal{M}} = \coprod\nolimits_{g \geq 2} \overline{\mathcal{M}}_g$$

where each $\overline{\mathcal{M}}_g$ is characterized as follows:

(1) *given a family of curves $f : X \to S$ the following are equivalent*
   (a) *the classifying morphism $S \to \mathcal{C}urves$ factors through $\overline{\mathcal{M}}_g$,*
   (b) *$X \to S$ is a stable family of curves and $R^1 f_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of rank g,*
(2) *given X a scheme proper over a field k with $\dim(X) \leq 1$ the following are equivalent*
   (a) *the classifying morphism $\mathrm{Spec}(k) \to \mathcal{C}urves$ factors through $\overline{\mathcal{M}}_g$,*
   (b) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is g, and X has no rational tails or bridges.*
   (c) *the singularities of X are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is g, and ω_{X_s} is ample.*

Proof. Combine Lemmas 22.3 and 20.3. $\square$

Lemma 22.6. *The morphisms $\overline{\mathcal{M}} \to \mathrm{Spec}(\mathbf{Z})$ and $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$ are smooth.*

Proof. Since $\overline{\mathcal{M}}$ is an open substack of $\mathcal{C}urves^{nodal}$ this follows from Lemma 18.2. $\square$

Lemma 22.7. *The stacks $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_g$ are open substacks of $\mathcal{C}urves^{DM}$. In particular, $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_g$ are DM (Morphisms of Stacks, Definition 4.2) as well as Deligne-Mumford stacks (Algebraic Stacks, Definition 12.2).*

Proof. Proof of the first assertion. Let X be a scheme proper over a field k whose singularities are at-worst-nodal, $\dim(X) = 1$, $k = H^0(X, \mathcal{O}_X)$, the genus of X is ≥ 2, and X has no rational tails or bridges. We have to show that the classifying morphism $\mathrm{Spec}(k) \to \overline{\mathcal{M}} \to \mathcal{C}urves$ factors through $\mathcal{C}urves^{DM}$. We may first replace k by the algebraic closure (since we already know the relevant stacks are open substacks of the algebraic stack $\mathcal{C}urves$). By Lemmas 22.3, 7.3, and 7.4 it suffices to show that $\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$. This is proven in Algebraic Curves, Lemma 25.3.

Since $\mathcal{C}urves^{DM}$ is the maximal open substack of $\mathcal{C}urves$ which is DM, we see this is true also for the open substack $\overline{\mathcal{M}}$ of $\mathcal{C}urves^{DM}$. Finally, a DM algebraic stack is Deligne-Mumford by Morphisms of Stacks, Theorem 21.6. $\square$

Lemma 22.8. *Let $g \geq 2$. The inclusion*

$$|\mathcal{M}_g| \subset |\overline{\mathcal{M}}_g|$$

is that of an open dense subset.

Proof. Since $\overline{\mathcal{M}}_g \subset \mathcal{C}urves^{lci+}$ is open and since $\mathcal{C}urves^{smooth} \cap \overline{\mathcal{M}}_g = \mathcal{M}_g$ this follows immediately from Lemma 17.1. $\square$

23. Contraction morphisms

We urge the reader to familiarize themselves with Algebraic Curves, Sections 22, 23, and 24 before continuing here. The main result of this section is the existence of a "stabilization" morphism

$$\mathcal{C}urves_g^{prestable} \longrightarrow \overline{\mathcal{M}}_g$$

See Lemma 23.5. Loosely speaking, this morphism sends the moduli point of a nodal genus g curve to the moduli point of the associated stable curve constructed in Algebraic Curves, Lemma 24.2.

Lemma 23.1. *Let S be a scheme and $s \in S$ a point. Let $f : X \to S$ and $g : Y \to S$ be families of curves. Let $c : X \to Y$ be a morphism over S. If $c_{s,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_s}$ and $R^1c_{s,*}\mathcal{O}_{X_s} = 0$, then after replacing S by an open neighbourhood of s we have $\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$ and this remains true after base change by any morphism $S' \to S$.*

Proof. Let $(U, u) \to (S, s)$ be an étale neighbourhood such that $\mathcal{O}_{Y_U} = (X_U \to Y_U)_*\mathcal{O}_{X_U}$ and $R^1(X_U \to Y_U)_*\mathcal{O}_{X_U} = 0$ and the same is true after base change by $U' \to U$. Then we replace S by the open image of $U \to S$. Given $S' \to S$ we set $U' = U \times_S S'$ and we obtain étale coverings $\{U' \to S'\}$ and $\{Y_{U'} \to Y_{S'}\}$. Thus the truth of the statement for the base change of c by $S' \to S$ follows from the truth of the statement for the base change of $X_U \to Y_U$ by $U' \to U$. In other words, the question is local in the étale topology on S. Thus by Lemma 4.3 we may assume X and Y are schemes. By More on Morphisms, Lemma 72.7 there exists an open subscheme $V \subset Y$ containing Y_s such that $c_*\mathcal{O}_X|_V = \mathcal{O}_V$ and $R^1c_*\mathcal{O}_X|_V = 0$ and such that this remains true after any base change by $S' \to S$. Since $g : Y \to S$ is proper, we can find an open neighbourhood $U \subset S$ of s such that $g^{-1}(U) \subset V$. Then U works. $\square$

Lemma 23.2. *Let S be a scheme and $s \in S$ a point. Let $f : X \to S$ and $g_i : Y_i \to S$, $i = 1, 2$ be families of curves. Let $c_i : X \to Y_i$ be morphisms over S. Assume there is an isomorphism $Y_{1,s} \cong Y_{2,s}$ of fibres compatible with $c_{1,s}$ and $c_{2,s}$. If $c_{1,s,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_{1,s}}$ and $R^1c_{1,s,*}\mathcal{O}_{X_s} = 0$, then there exist an open neighbourhood U of s and an isomorphism $Y_{1,U} \cong Y_{2,U}$ of families of curves over U compatible with the given isomorphism of fibres and with c_1 and c_2.*

Proof. Recall that $\mathcal{O}_{S,s} = \operatorname{colim} \mathcal{O}_S(U)$ where the colimit is over the system of affine neighbourhoods U of s. Thus the category of algebraic spaces of finite presentation over the local ring is the colimit of the categories of algebraic spaces of finite presentation over the affine neighbourhoods of s. See Limits of Spaces, Lemma 7.1. In this way we reduce to the case where S is the spectrum of a local ring and s is the closed point.

Assume $S = \operatorname{Spec}(A)$ where A is a local ring and s is the closed point. Write $A = \operatorname{colim} A_j$ with A_j local Noetherian (say essentially of finite type over $\mathbf{Z}$) and local transition homomorphisms. Set $S_j = \operatorname{Spec}(A_j)$ with closed point s_j. We can find a j and families of curves $X_j \to S_j$, $Y_{j,i} \to S_j$, see Lemma 5.3 and Limits of Stacks, Lemma 3.5. After possibly increasing j we can find morphisms $c_{j,i} : X_j \to Y_{j,i}$ whose base change to s is c_i, see Limits of Spaces, Lemma 7.1. Since $\kappa(s) = \operatorname{colim} \kappa(s_j)$ we can similarly assume there is an isomorphism $Y_{j,1,s_j} \cong Y_{j,2,s_j}$ compatible with $c_{j,1,s_j}$ and $c_{j,2,s_j}$. Finally, the assumptions $c_{1,s,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_{1,s}}$ and $R^1c_{1,s,*}\mathcal{O}_{X_s} = 0$ are inherited by $c_{j,1,s_j}$ because $\{s_j \to s\}$ is an fpqc covering and $c_{1,s}$ is the base of $c_{j,1,s_j}$ by this covering (details omitted). In this way we reduce the lemma to the case discussed in the next paragraph.

Assume S is the spectrum of a Noetherian local ring Λ and s is the closed point. Consider the scheme theoretic image Z of

$$(c_1, c_2) : X \longrightarrow Y_1 \times_S Y_2$$

The statement of the lemma is equivalent to the assertion that Z maps isomorphically to Y_1 and Y_2 via the projection morphisms. Since taking the scheme theoretic image of this morphism commutes with flat base change (Morphisms of Spaces, Lemma 30.12, we may replace Λ by its completion (More on Algebra, Section 43).

Assume S is the spectrum of a complete Noetherian local ring Λ. Observe that X, Y_1, Y_2 are schemes in this case (More on Morphisms of Spaces, Lemma 43.6). Denote X_n, $Y_{1,n}$, $Y_{2,n}$ the base changes of X, Y_1, Y_2 to $\text{Spec}(\Lambda/\mathfrak{m}^{n+1})$. Recall that the arrow

$$\mathcal{D}ef_{X_s \to Y_{2,s}} \cong \mathcal{D}ef_{X_s \to Y_{1,s}} \longrightarrow \mathcal{D}ef_{X_s}$$

is an equivalence, see Deformation Problems, Lemma 10.6. Thus there is an isomorphism of formal objects $(X_n \to Y_{1,n}) \cong (X_n \to Y_{2,n})$ of $\mathcal{D}ef_{X_s \to Y_{1,s}}$. Finally, by Grothendieck's algebraization theorem (Cohomology of Schemes, Lemma 28.3) this produces an isomorphism $Y_1 \to Y_2$ compatible with c_1 and c_2. ☐

Lemma 23.3. *Let $f : X \to S$ be a family of curves. Let $s \in S$ be a point. Let $h_0 : X_s \to Y_0$ be a morphism to a proper scheme Y_0 over $\kappa(s)$ such that $h_{0,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_0}$ and $R^1h_{0,*}\mathcal{O}_{X_s} = 0$. Then there exist an elementary étale neighbourhood $(U, u) \to (S, s)$, a family of curves $Y \to U$, and a morphism $h : X_U \to Y$ over U whose fibre in u is isomorphic to h_0.*

Proof. We first do some reductions; we urge the reader to skip ahead. The question is local on S, hence we may assume S is affine. Write $S = \lim S_i$ as a cofiltered limit of affine schemes S_i of finite type over $\mathbf{Z}$. For some i we can find a family of curves $X_i \to S_i$ whose base change is $X \to S$. This follows from Lemma 5.3 and Limits of Stacks, Lemma 3.5. Let $s_i \in S_i$ be the image of s. Observe that $\kappa(s) = \text{colim}\, \kappa(s_i)$ and that X_s is a scheme (Spaces over Fields, Lemma 9.3). After increasing i we may assume there exists a morphism $h_{i,0} : X_{i,s_i} \to Y_i$ of finite type schemes over $\kappa(s_i)$ whose base change to $\kappa(s)$ is h_0, see Limits, Lemma 10.1. After increasing i we may assume Y_i is proper over $\kappa(s_i)$, see Limits, Lemma 13.1. Let $g_{i,0} : Y_0 \to Y_{i,0}$ be the projection. Observe that this is a faithfully flat morphism as the base change of $\text{Spec}(\kappa(s)) \to \text{Spec}(\kappa(s_i))$. By flat base change we have

$$h_{0,*}\mathcal{O}_{X_s} = g_{i,0}^* h_{i,0,*}\mathcal{O}_{X_{i,s_i}} \quad \text{and} \quad R^1h_{0,*}\mathcal{O}_{X_s} = g_{i,0}^* Rh_{i,0,*}\mathcal{O}_{X_{i,s_i}}$$

see Cohomology of Schemes, Lemma 5.2. By faithful flatness we see that $X_i \to S_i$, $s_i \in S_i$, and $X_{i,s_i} \to Y_i$ satisfies all the assumptions of the lemma. This reduces us to the case discussed in the next paragraph.

Assume S is affine of finite type over $\mathbf{Z}$. Let $\mathcal{O}_{S,s}^h$ be the henselization of the local ring of S at s. Observe that $\mathcal{O}_{S,s}^h$ is a G-ring by More on Algebra, Lemma 50.8 and Proposition 50.12. Suppose we can construct a family of curves $Y' \to \text{Spec}(\mathcal{O}_{S,s}^h)$ and a morphism

$$h' : X \times_S \text{Spec}(\mathcal{O}_{S,s}^h) \longrightarrow Y'$$

over $\text{Spec}(\mathcal{O}_{S,s}^h)$ whose base change to the closed point is h_0. This will be enough. Namely, first we use that

$$\mathcal{O}_{S,s}^h = \text{colim}_{(U,u)}\, \mathcal{O}_U(U)$$

where the colimit is over the filtered category of elementary étale neighbourhoods (More on Morphisms, Lemma 35.5). Next, we use again that given Y' we can descend it to $Y \to U$ for some U (see references given above). Then we use Limits, Lemma 10.1 to descend h' to some h. This reduces us to the case discussed in the next paragraph.

Assume $S = \mathrm{Spec}(\Lambda)$ where $(\Lambda, \mathfrak{m}, \kappa)$ is a henselian Noetherian local G-ring and s is the closed point of S. Recall that the map

$$\mathcal{D}ef_{X_s \to Y_0} \to \mathcal{D}ef_{X_s}$$

is an equivalence, see Deformation Problems, Lemma 10.6. (This is the only important step in the proof; everything else is technique.) Denote $\Lambda^\wedge$ the $\mathfrak{m}$-adic completion. The pullbacks X_n of X to $\Lambda/\mathfrak{m}^{n+1}$ define a formal object ξ of $\mathcal{D}ef_{X_s}$ over $\Lambda^\wedge$. From the equivalence we obtain a formal object ξ' of $\mathcal{D}ef_{X_s \to Y_0}$ over $\Lambda^\wedge$. Thus we obtain a huge commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & \ldots \arrow[r] & X_s \arrow[d] \\
\ldots \arrow[r] & Y_n \arrow[r] \arrow[d] & Y_{n-1} \arrow[r] \arrow[d] & \ldots \arrow[r] & Y_0 \arrow[d] \\
\ldots \arrow[r] & \mathrm{Spec}(\Lambda/\mathfrak{m}^{n+1}) \arrow[r] & \mathrm{Spec}(\Lambda/\mathfrak{m}^n) \arrow[r] & \ldots \arrow[r] & \mathrm{Spec}(\kappa)
\end{tikzcd}$$

The formal object (Y_n) comes from a family of curves $Y' \to \mathrm{Spec}(\Lambda^\wedge)$ by Quot, Lemma 15.9. By More on Morphisms of Spaces, Lemma 43.3 we get a morphism $h' : X_{\Lambda^\wedge} \to Y'$ inducing the given morphisms $X_n \to Y_n$ for all n and in particular the given morphism $X_s \to Y_0$.

To finish we do a standard algebraization/approximation argument. First, we observe that we can find a finitely generated Λ-subalgebra $\Lambda \subset A \subset \Lambda^\wedge$, a family of curves $Y'' \to \mathrm{Spec}(A)$ and a morphism $h'' : X_A \to Y''$ over A whose base change to $\Lambda^\wedge$ is h'. This is true because $\Lambda^\wedge$ is the filtered colimit of these rings A and we can argue as before using that $\mathcal{C}urves$ is locally of finite presentation (which gives us Y'' over A by Limits of Stacks, Lemma 3.5) and using Limits of Spaces, Lemma 7.1 to descend h' to some h''. Then we can apply the approximation property for G-rings (in the form of Smoothing Ring Maps, Theorem 13.1) to find a map $A \to \Lambda$ which induces the same map $A \to \kappa$ as we obtain from $A \to \Lambda^\wedge$. Base changing h'' to Λ the proof is complete. □

Lemma 23.4. *Let $f : X \to S$ be a prestable family of curves of genus $g \geq 2$. There is a factorization $X \to Y \to S$ of f where $g : Y \to S$ is a stable family of curves and $c : X \to Y$ has the following properties*

1. *$\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$ and this remains true after base change by any morphism $S' \to S$, and*
2. *for any $s \in S$ the morphism $c_s : X_s \to Y_s$ is the contraction of rational tails and bridges discussed in Algebraic Curves, Section 24.*

Moreover $c : X \to Y$ is unique up to unique isomorphism.

Proof. Let $s \in S$. Let $c_0 : X_s \to Y_0$ be the contraction of Algebraic Curves, Section 24 (more precisely Algebraic Curves, Lemma 24.2). By Lemma 23.3 there exists an elementary étale neighbourhood (U, u) and a morphism $c : X_U \to Y$ of families of curves over U which recovers c_0 as the fibre at u. Since ω_{Y_0} is ample, after possibly shrinking U, we see that $Y \to U$ is a stable family of genus g by the openness inherent in Lemmas 22.3 and 22.5. After possibly shrinking U once more, assertion (1) of the lemma for $c : X_U \to Y$ follows from Lemma 23.1. Moreover, part (2) holds by the uniqueness in Algebraic Curves, Lemma 24.2. We conclude that a morphism c as in the lemma exists étale locally on S. More precisely, there exists an étale covering $\{U_i \to S\}$ and morphisms $c_i : X_{U_i} \to Y_i$ over U_i where $Y_i \to U_i$ is a stable family of curves having properties (1) and (2) stated in the lemma.

To finish the proof it suffices to prove uniqueness of $c : X \to Y$ (up to unique isomorphism). Namely, once this is done, then we obtain isomorphisms

$$\varphi_{ij} : Y_i \times_{U_i} (U_i \times_S U_j) \longrightarrow Y_i \times_{U_j} (U_i \times_S U_j)$$

satisfying the cocycle condition (by uniqueness) over $U_i \times U_j \times U_k$. Since $\overline{\mathcal{M}_g}$ is an algebraic stack, we have effectiveness of descent data and we obtain $Y \to S$. The morphisms c_i descend to a morphism $c : X \to Y$ over S. Finally, properties (1) and (2) for c are immediate from properties (1) and (2) for c_i.

Finally, if $c_1 : X \to Y_i$, $i = 1, 2$ are two morphisms towards stably families of curves over S satisfying (1) and (2), then we obtain a morphism $Y_1 \to Y_2$ compatible with c_1 and c_2 at least locally on S by Lemma 23.3. We omit the verification that these morphisms are unique (hint: this follows from the fact that the scheme theoretic image of c_1 is Y_1). Hence these locally given morphisms glue and the proof is complete. □

Lemma 23.5. *Let $g \geq 2$. There is a morphism of algebraic stacks over* $\mathbf{Z}$

$$\textit{stabilization} : \mathcal{C}\!\textit{urves}_g^{prestable} \longrightarrow \overline{\mathcal{M}}_g$$

which sends a prestable family of curves $X \to S$ of genus g to the stable family $Y \to S$ asssociated to it in Lemma 23.4.

Proof. To see this is true, it suffices to check that the construction of Lemma 23.4 is compatible with base change (and isomorphisms but that's immediate), see the (abuse of) language for algebraic stacks introduced in Properties of Stacks, Section 2. To see this it suffices to check properties (1) and (2) of Lemma 23.4 are stable under base change. This is immediately clear for (1). For (2) this follows either from the fact that the contractions of Algebraic Curves, Lemmas 22.6 and 23.6 are stable under ground field extensions, or because the conditions characterizing the morphisms on fibres in Algebraic Curves, Lemma 24.2 are preserved under ground field extensions. □

24. Stable reduction theorem

In the chapter on semistable reduction we have proved the celebrated theorem on semistable reduction of curves. Let K be the fraction field of a discrete valuation ring R. Let C be a projective smooth curve over K with $K = H^0(C, \mathcal{O}_C)$. According to Semistable Reduction, Definition 14.6 we say C has *semistable reduction* if either there is a prestable family of curves over R with generic fibre C, or some (equivalently any) minimal regular model of C over R is prestable. In this section we show that for curves of genus $g \geq 2$ this is also equivalent to stable reduction.

Lemma 24.1. *Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $K = H^0(C, \mathcal{O}_C)$ having genus $g \geq 2$. The following are equivalent*

1. *C has semistable reduction (Semistable Reduction, Definition 14.6), or*

(2) *there is a stable family of curves over R with generic fibre C.*

Proof. Since a stable family of curves is also prestable, it is immediate that (2) implies (1). Conversely, given a prestable family of curves over R with generic fibre C, we can contract it to a stable family of curves by Lemma 23.4. Since the generic fibre already is stable, it does not get changed by this procedure and the proof is complete. □

The following lemma tells us the stable family of curves over R promised in Lemma 24.1 is unique up to unique isomorphism.

Lemma 24.2. *Let R be a discrete valuation ring with fraction field K. Let C be a smooth proper curve over K with $K = H^0(C, \mathcal{O}_C)$ and genus g. If X and X' are models of C (Semistable Reduction, Section 8) and X and X' are stable families of genus g curves over R, then there exists an unique isomorphism $X \to X'$ of models.*

Proof. Let Y be the minimal model for C. Recall that Y exists, is unique, and is at-worst-nodal of relative dimension 1 over R, see Semistable Reduction, Proposition 8.6 and Lemmas 10.1 and 14.5 (applies because we have X). There is a contraction morphism

$$Y \longrightarrow Z$$

such that Z is a stable family of curves of genus g over R (Lemma 23.4). We claim there is a unique isomorphism of models $X \to Z$. By symmetry the same is true for X' and this will finish the proof.

By Semistable Reduction, Lemma 14.3 there exists a sequence

$$X_m \to \ldots \to X_1 \to X_0 = X$$

such that $X_{i+1} \to X_i$ is the blowing up of a closed point x_i where X_i is singular, $X_i \to \mathrm{Spec}(R)$ is at-worst-nodal of relative dimension 1, and X_m is regular. By Semistable Reduction, Lemma 8.5 there is a sequence

$$X_m = Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y_0 = Y$$

of proper regular models of C, such that each morphism is a contraction of an exceptional curve of the first kind[4]. By Semistable Reduction, Lemma 14.4 each Y_i is at-worst-nodal of relative dimension 1 over R. To prove the claim it suffices to show that there is an isomorphism $X \to Z$ compatible with the morphisms $X_m \to X$ and $X_m = Y_n \to Y \to Z$. Let $s \in \mathrm{Spec}(R)$ be the closed point. By either Lemma 23.2 or Lemma 23.4 we reduce to proving that the morphisms $X_{m,s} \to X_s$ and $X_{m,s} \to Z_s$ are both equal to the canonical morphism of Algebraic Curves, Lemma 24.2.

For a morphism $c : U \to V$ of schemes over $\kappa(s)$ we say c has property (*) if $\dim(U_v) \leq 1$ for $v \in V$, $\mathcal{O}_V = c_*\mathcal{O}_U$, and $R^1c_*\mathcal{O}_U = 0$. This property is stable under composition. Since both X_s and Z_s are stable genus g curves over $\kappa(s)$, it suffices to show that each of the morphisms $Y_s \to Z_s$, $X_{i+1,s} \to X_{i,s}$, and $Y_{i+1,s} \to Y_{i,s}$, satisfy property (*), see Algebraic Curves, Lemma 24.2.

Property (*) holds for $Y_s \to Z_s$ by construction.

---

[4]In fact we have $X_m = Y$, i.e., X_m does not contain any exceptional curves of the first kind. We encourage the reader to think this through as it simplifies the proof somewhat.

The morphisms $c : X_{i+1,s} \to X_{i,s}$ are constructed and studied in the proof of Semistable Reduction, Lemma 14.3. It suffices to check (*) étale locally on $X_{i,s}$. Hence it suffices to check (*) for the base change of the morphism "$X_1 \to X_0$" in Semistable Reduction, Example 14.1 to $R/\pi R$. We leave the explicit calculation to the reader.

The morphism $c : Y_{i+1,s} \to Y_{i,s}$ is the restriction of the blow down of an exceptional curve $E \subset Y_{i+1}$ of the first kind, i.e., $b : Y_{i+1} \to Y_i$ is a contraction of E, i.e., b is a blowing up of a regular point on the surface Y_i (Resolution of Surfaces, Section 16). Then $\mathcal{O}_{Y_i} = b_*\mathcal{O}_{Y_{i+1}}$ and $R^1b_*\mathcal{O}_{Y_{i+1}} = 0$, see for example Resolution of Surfaces, Lemma 3.4. We conclude that $\mathcal{O}_{Y_{i,s}} = c_*\mathcal{O}_{Y_{i+1,s}}$ and $R^1c_*\mathcal{O}_{Y_{i+1,s}} = 0$ by More on Morphisms, Lemmas 72.1, 72.2, and 72.4 (only gives surjectivity of $\mathcal{O}_{Y_{i,s}} \to c_*\mathcal{O}_{Y_{i+1,s}}$ but injectivity follows easily from the fact that $Y_{i,s}$ is reduced and c changes things only over one closed point). This finishes the proof. ☐

From Lemma 24.1 and Semistable Reduction, Theorem 18.1 we immediately deduce the stable reduction theorem.

Theorem 24.3. *Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and genus $g \geq 2$. Then*

(1) *there exists an extension of discrete valuation rings $R \subset R'$ inducing a finite separable extension of fraction fields K'/K and a stable family of curves $Y \to \mathrm{Spec}(R')$ of genus g with $Y_{K'} \cong C_{K'}$ over K', and*
(2) *there exists a finite separable extension L/K and a stable family of curves $Y \to \mathrm{Spec}(A)$ of genus g where $A \subset L$ is the integral closure of R in L such that $Y_L \cong C_L$ over L.*

Proof. Part (1) is an immediate consequence of Lemma 24.1 and Semistable Reduction, Theorem 18.1.

Proof of (2). Let L/K be the finite separable extension found in part (3) of Semistable Reduction, Theorem 18.1. Let $A \subset L$ be the integral closure of R. Recall that A is a Dedekind domain finite over R with finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$, see More on Algebra, Remark 111.6. Set $S = \mathrm{Spec}(A)$, $S_i = \mathrm{Spec}(A_{\mathfrak{m}_i})$, $U = \mathrm{Spec}(L)$, and $U_i = S_i \setminus \{\mathfrak{m}_i\}$. Observe that $U \cong U_i$ for $i = 1, \ldots, n$. Set $X = C_L$ viewed as a scheme over the open subscheme U of S. By our choice of L and A and Lemma 24.1 we have stable families of curves $X_i \to S_i$ and isomorphisms $X \times_U U_i \cong X_i \times_{S_i} U_i$. By Limits of Spaces, Lemma 18.4 we can find a finitely presented morphism $Y \to S$ whose base change to S_i is isomorphic to X_i for $i = 1, \ldots, n$. Alternatively, you can use that $S = \bigcup_{i=1,\ldots,n} S_i$ is an open covering of S and $S_i \cap S_j = U$ for $i \neq j$ and use $n-1$ applications of Limits of Spaces, Lemma 18.1 to get $Y \to S$ whose base change to S_i is isomorphic to X_i for $i = 1, \ldots, n$. Clearly $Y \to S$ is the stable family of curves we were looking for. ☐

25. Properties of the stack of stable curves

In this section we prove the basic structure result for $\overline{\mathcal{M}}_g$ for $g \geq 2$.

Lemma 25.1. *Let $g \geq 2$. The stack $\overline{\mathcal{M}}_g$ is separated.*

Proof. The statement means that the morphism $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$ is separated. We will prove this using the refined Noetherian valuative criterion as stated in More on Morphisms of Stacks, Lemma 11.4

Since $\overline{\mathcal{M}}_g$ is an open substack of $\mathcal{C}urves$, we see $\overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated and locally of finite presentation by Lemma 5.3. In particular the stack $\overline{\mathcal{M}}_g$ is locally Noetherian (Morphisms of Stacks, Lemma 17.5). By Lemma 22.8 the open immersion $\mathcal{M}_g \to \overline{\mathcal{M}}_g$ has dense image. Also, $\mathcal{M}_g \to \overline{\mathcal{M}}_g$ is quasi-compact (Morphisms of Stacks, Lemma 8.2), hence of finite type. Thus all the preliminary assumptions of More on Morphisms of Stacks, Lemma 11.4 are satisfied for the morphisms

$$\mathcal{M}_g \to \overline{\mathcal{M}}_g \quad \text{and} \quad \overline{\mathcal{M}}_g \to \mathrm{Spec}(\mathbf{Z})$$

and it suffices to check the following: given any 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{M}_g \arrow[r] & \overline{\mathcal{M}}_g \arrow[d] \\ \mathrm{Spec}(R) \arrow[rr] \arrow[urr, dotted] & & \mathrm{Spec}(\mathbf{Z}) \end{tikzcd}$$

where R is a discrete valuation ring with field of fractions K the category of dotted arrows is either empty or a setoid with exactly one isomorphism class. (Observe that we don't need to worry about 2-arrows too much, see Morphisms of Stacks, Lemma 39.3). Unwinding what this means using that $\mathcal{M}_g$, resp. $\overline{\mathcal{M}}_g$ are the algebraic stacks parametrizing smooth, resp. stable families of genus g curves, we find that what we have to prove is exactly the uniqueness result stated and proved in Lemma 24.2. □

Lemma 25.2. *Let $g \geq 2$. The stack $\overline{\mathcal{M}}_g$ is quasi-compact.*

Proof. We will use the notation from Section 4. Consider the subset

$$T \subset |PolarizedCurves|$$

of points ξ such that there exists a field k and a pair $(X, \mathcal{L})$ over k representing ξ with the following two properties

(1) X is a stable genus g curve, and
(2) $\mathcal{L} = \omega_X^{\otimes 3}$.

Clearly, under the continuous map

$$|PolarizedCurves| \longrightarrow |\mathcal{C}urves|$$

the image of the set T is exactly the open subset

$$|\overline{\mathcal{M}}_g| \subset |\mathcal{C}urves|$$

Thus it suffices to show that T is quasi-compact. By Lemma 4.1 we see that

$$|PolarizedCurves| \subset |\mathcal{P}olarized|$$

is an open and closed immersion. Thus it suffices to prove quasi-compactness of T as a subset of $|\mathcal{P}olarized|$. For this we use the criterion of Moduli Stacks, Lemma 11.3. First, we observe that for $(X, \mathcal{L})$ as above the Hilbert polynomial P is the function $P(t) = (6g-6)t + (1-g)$ by Riemann-Roch, see Algebraic Curves, Lemma 5.2. Next, we observe that $H^1(X, \mathcal{L}) = 0$ and $\mathcal{L}$ is very ample by Algebraic Curves, Lemma 24.3. This means exactly that with $n = P(3) - 1$ there is a closed immersion

$$i : X \longrightarrow \mathbf{P}^n_k$$

such that $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^1_k}(1)$ as desired. □

Here is the main theorem of this section.

Theorem 25.3. *Let $g \geq 2$. The algebraic stack $\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack, proper and smooth over* $\text{Spec}(\mathbf{Z})$. *Moreover, the locus $\mathcal{M}_g$ parametrizing smooth curves is a dense open substack.*

Proof. Most of the properties mentioned in the statement have already been shown. Smoothness is Lemma 22.6. Deligne-Mumford is Lemma 22.7. Openness of $\mathcal{M}_g$ is Lemma 22.8. We know that $\overline{\mathcal{M}}_g \to \text{Spec}(\mathbf{Z})$ is separated by Lemma 25.1 and we know that $\overline{\mathcal{M}}_g$ is quasi-compact by Lemma 25.2. Thus, to show that $\overline{\mathcal{M}}_g \to \text{Spec}(\mathbf{Z})$ is proper and finish the proof, we may apply More on Morphisms of Stacks, Lemma 11.3 to the morphisms $\mathcal{M}_g \to \overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_g \to \text{Spec}(\mathbf{Z})$. Thus it suffices to check the following: given any 2-commutative diagram

$$\begin{tikzcd} \text{Spec}(K) \arrow[r] \arrow[d, "j"'] & \mathcal{M}_g \arrow[r] & \overline{\mathcal{M}}_g \arrow[d] \\ \text{Spec}(A) \arrow[rr] & & \text{Spec}(\mathbf{Z}) \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exist an extension K'/K of fields, a valuation ring $A' \subset K'$ dominating A such that the category of dotted arrows for the induced diagram

$$\begin{tikzcd} \text{Spec}(K') \arrow[r] \arrow[d, "j'"'] & \overline{\mathcal{M}}_g \arrow[d] \\ \text{Spec}(A') \arrow[r] \arrow[ur, dotted] & \text{Spec}(\mathbf{Z}) \end{tikzcd}$$

is nonempty (Morphisms of Stacks, Definition 39.1). (Observe that we don't need to worry about 2-arrows too much, see Morphisms of Stacks, Lemma 39.3). Unwinding what this means using that $\mathcal{M}_g$, resp. $\overline{\mathcal{M}}_g$ are the algebraic stacks parametrizing smooth, resp. stable families of genus g curves, we find that what we have to prove is exactly the result contained in the stable reduction theorem, i.e., Theorem 24.3. $\square$

26. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes

References

[DM69] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHES **36** (1969), 75–110.

[Mum75] David Mumford, *Pathologies IV*, Amer. J. Math. **97** (1975), no. 3, 847–849.