SHEAVES ON SPACES

Contents

1. Introduction

Basic properties of sheaves on topological spaces will be explained in this document. A reference is [God73].

This will be superseded by the discussion of sheaves over sites later in the documents. But perhaps it makes sense to briefly define some of the notions here.

2. Basic notions

The following is a list of basic notions in topology.

(1) Let X be a topological space. The phrase: "Let $U = \bigcup_{i \in I} U_i$ be an open covering" means the following: I is a set and for each $i \in I$ we are given an open subset $U_i \subset X$ such that U is the union of the U_i. It is allowed to have $I = \emptyset$ in which case there are no U_i and $U = \emptyset$. It is also allowed, in case $I \neq \emptyset$ to have any or all of the U_i be empty.
(2) etc, etc.

3. Presheaves

Definition 3.1. Let X be a topological space.

(1) A *presheaf $\mathcal{F}$ of sets on X* is a rule which assigns to each open $U \subset X$ a set $\mathcal{F}(U)$ and to each inclusion $V \subset U$ a map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho^U_U = \text{id}_{\mathcal{F}(U)}$ and whenever $W \subset V \subset U$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on X* is a rule which assigns to each open $U \subset X$ a map of sets $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps, i.e., whenever $V \subset U \subset X$ are open the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi"] \arrow[d, "\rho^U_V"] & \mathcal{G}(U) \arrow[d, "\rho^U_V"] \\ \mathcal{F}(V) \arrow[r, "\varphi"] & \mathcal{G}(V) \end{tikzcd}$$

commutes.
(3) The category of presheaves of sets on X will be denoted $PSh(X)$.

The elements of the set $\mathcal{F}(U)$ are called the *sections* of $\mathcal{F}$ over U. For every $V \subset U$ the map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the *restriction map*. We will use the notation $s|_V := \rho^U_V(s)$ if $s \in \mathcal{F}(U)$. This notation is consistent with the notion of restriction of functions from topology because if $W \subset V \subset U$ and s is a section of $\mathcal{F}$ over U then $s|_W = (s|_V)|_W$ by the property of the restriction maps expressed in the definition above.

Another notation that is often used is to indicate sections over an open U by the symbol $\Gamma(U, -)$ or by $H^0(U, -)$. In other words, the following equalities are tautological

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U) = H^0(U, \mathcal{F}).$$

In this chapter we will not use this notation, but in others we will.

Definition 3.2. Let X be a topological space. Let A be a set. The *constant presheaf with value A* is the presheaf that assigns the set A to every open $U \subset X$, and such that all restriction mappings are id_A.

4. Abelian presheaves

In this section we briefly point out some features of the category of presheaves that allow one to define presheaves of abelian groups.

Example 4.1. Let X be a topological space. Consider a rule $\mathcal{F}$ that associates to every open subset of X a singleton set. Since every set has a unique map into a singleton set, there exist unique restriction maps ρ^U_V. The resulting structure is a presheaf of sets on X. It is a final object in the category of presheaves of sets on X, by the property of singleton sets mentioned above. Hence it is also unique up to unique isomorphism. We will sometimes write $*$ for this presheaf.

Lemma 4.2. *Let X be a topological space. The category of presheaves of sets on X has products (see Categories, Definition 14.6). Moreover, the set of sections of the product $\mathcal{F} \times \mathcal{G}$ over an open U is the product of the sets of sections of $\mathcal{F}$ and $\mathcal{G}$ over U.*

Proof. Namely, suppose $\mathcal{F}$ and $\mathcal{G}$ are presheaves of sets on the topological space X. Consider the rule $U \mapsto \mathcal{F}(U) \times \mathcal{G}(U)$, denoted $\mathcal{F} \times \mathcal{G}$. If $V \subset U \subset X$ are open then define the restriction mapping

$$(\mathcal{F} \times \mathcal{G})(U) \longrightarrow (\mathcal{F} \times \mathcal{G})(V)$$

by mapping $(s, t) \mapsto (s|_V, t|_V)$. Then it is immediately clear that $\mathcal{F} \times \mathcal{G}$ is a presheaf. Also, there are projection maps $p : \mathcal{F} \times \mathcal{G} \to \mathcal{F}$ and $q : \mathcal{F} \times \mathcal{G} \to \mathcal{G}$. We leave it to the reader to show that for any third presheaf $\mathcal{H}$ we have $\mathrm{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G}) = \mathrm{Mor}(\mathcal{H}, \mathcal{F}) \times \mathrm{Mor}(\mathcal{H}, \mathcal{G})$. $\square$

Recall that if $(A, + : A \times A \to A, - : A \to A, 0 \in A)$ is an abelian group, then the zero and the negation maps are uniquely determined by the addition law. In other words, it makes sense to say "let $(A, +)$ be an abelian group".

Lemma 4.3. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets. Consider the following types of structure on $\mathcal{F}$:*

1. *For every open U the structure of an abelian group on $\mathcal{F}(U)$ such that all restriction maps are abelian group homomorphisms.*
2. *A map of presheaves $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, a map of presheaves $- : \mathcal{F} \to \mathcal{F}$ and a map $0 : * \to \mathcal{F}$ (see Example 4.1) satisfying all the axioms of $+, -, 0$ in a usual abelian group.*
3. *A map of presheaves $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, a map of presheaves $- : \mathcal{F} \to \mathcal{F}$ and a map $0 : * \to \mathcal{F}$ such that for each open $U \subset X$ the quadruple $(\mathcal{F}(U), +, -, 0)$ is an abelian group,*
4. *A map of presheaves $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ such that for every open $U \subset X$ the map $+ : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an abelian group.*

There are natural bijections between the collections of types of data (1) - (4) above.

Proof. Omitted. $\square$

The lemma says that to give an abelian group object $\mathcal{F}$ in the category of presheaves is the same as giving a presheaf of sets $\mathcal{F}$ such that all the sets $\mathcal{F}(U)$ are endowed with the structure of an abelian group and such that all the restriction mappings are group homomorphisms. For most algebra structures we will take this approach to (pre)sheaves of such objects, i.e., we will define a (pre)sheaf of such objects to be a (pre)sheaf $\mathcal{F}$ of sets all of whose sets of sections $\mathcal{F}(U)$ are endowed with this structure compatibly with the restriction mappings.

Definition 4.4. Let X be a topological space.

(1) A *presheaf of abelian groups on* X or an *abelian presheaf over* X is a presheaf of sets $\mathcal{F}$ such that for each open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an abelian group, and such that all restriction maps ρ^U_V are homomorphisms of abelian groups, see Lemma 4.3 above.
(2) A *morphism of abelian presheaves over* X $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets which induces a homomorphism of abelian groups $\mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subset X$.
(3) The category of presheaves of abelian groups on X is denoted $PAb(X)$.

Example 4.5. Let X be a topological space. For each $x \in X$ suppose given an abelian group M_x. For $U \subset X$ open we set

$$\mathcal{F}(U) = \bigoplus\nolimits_{x \in U} M_x.$$

We denote a typical element in this abelian group by $\sum_{i=1}^n m_{x_i}$, where $x_i \in U$ and $m_{x_i} \in M_{x_i}$. (Of course we may always choose our representation such that $x_1, \ldots, x_n$ are pairwise distinct.) We define for $V \subset U \subset X$ open a restriction mapping $\mathcal{F}(U) \to \mathcal{F}(V)$ by mapping an element $s = \sum_{i=1}^n m_{x_i}$ to the element $s|_V = \sum_{x_i \in V} m_{x_i}$. We leave it to the reader to verify that this is a presheaf of abelian groups.

5. Presheaves of algebraic structures

Let us clarify the definition of presheaves of algebraic structures. Suppose that $\mathcal{C}$ is a category and that $F : \mathcal{C} \to \mathit{Sets}$ is a faithful functor. Typically F is a "forgetful" functor. For an object $M \in \mathrm{Ob}(\mathcal{C})$ we often call $F(M)$ the *underlying set* of the object M. If $M \to M'$ is a morphism in $\mathcal{C}$ we call $F(M) \to F(M')$ the *underlying map of sets*. In fact, we will often not distinguish between an object and its underlying set, and similarly for morphisms. So we will say a map of sets $F(M) \to F(M')$ is a *morphism of algebraic structures*, if it is equal to $F(f)$ for some morphism $f : M \to M'$ in $\mathcal{C}$.

In analogy with Definition 4.4 above a "presheaf of objects of $\mathcal{C}$" could be defined by the following data:

(1) a presheaf of sets $\mathcal{F}$, and
(2) for every open $U \subset X$ a choice of an object $A(U) \in \mathrm{Ob}(\mathcal{C})$

subject to the following conditions (using the phraseology above)

(1) for every open $U \subset X$ the set $\mathcal{F}(U)$ is the underlying set of $A(U)$, and
(2) for every $V \subset U \subset X$ open the map of sets $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is a morphism of algebraic structures.

In other words, for every $V \subset U$ open in X the restriction mappings ρ^U_V is the image $F(\alpha^U_V)$ for some unique morphism $\alpha^U_V : A(U) \to A(V)$ in the category $\mathcal{C}$. The uniqueness is forced by the condition that F is faithful; it also implies that $\alpha^U_W = \alpha^V_W \circ \alpha^U_V$ whenever $W \subset V \subset U$ are open in X. The system $(A(-), \alpha^U_V)$ is what we will define as a presheaf with values in $\mathcal{C}$ on X, compare Sites, Definition 2.2. We recover our presheaf of sets $(\mathcal{F}, \rho^U_V)$ via the rules $\mathcal{F}(U) = F(A(U))$ and $\rho^U_V = F(\alpha^U_V)$.

Definition 5.1. Let X be a topological space. Let $\mathcal{C}$ be a category.

(1) A *presheaf $\mathcal{F}$ on X with values in $\mathcal{C}$* is given by a rule which assigns to every open $U \subset X$ an object $\mathcal{F}(U)$ of $\mathcal{C}$ and to each inclusion $V \subset U$ a morphism $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ such that whenever $W \subset V \subset U$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with value in $\mathcal{C}$* is given by a morphism $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathcal{C}$ compatible with restriction morphisms.

Definition 5.2. Let X be a topological space. Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to$ *Sets* be a faithful functor. Let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{C}$. The presheaf of sets $U \mapsto F(\mathcal{F}(U))$ is called the *underlying presheaf of sets of $\mathcal{F}$*.

It is customary to use the same letter $\mathcal{F}$ to denote the underlying presheaf of sets, and this makes sense according to our discussion preceding Definition 5.1. In particular, the phrase "let $s \in \mathcal{F}(U)$" or "let s be a section of $\mathcal{F}$ over U" signifies that $s \in F(\mathcal{F}(U))$.

This notation and these definitions apply in particular to: *Presheaves of (not necessarily abelian) groups, rings, modules over a fixed ring, vector spaces over a fixed field,* etc and *morphisms between these.*

6. Presheaves of modules

Suppose that $\mathcal{O}$ is a presheaf of rings on X. We would like to define the notion of a presheaf of $\mathcal{O}$-modules over X. In analogy with Definition 4.4 we are tempted to define this as a presheaf of sets $\mathcal{F}$ such that for every open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an $\mathcal{O}(U)$-module compatible with restriction mappings (of $\mathcal{F}$ and $\mathcal{O}$). However, it is customary (and equivalent) to define it as in the following definition.

Definition 6.1. Let X be a topological space, and let $\mathcal{O}$ be a presheaf of rings on X.

(1) A *presheaf of $\mathcal{O}$-modules* is given by an abelian presheaf $\mathcal{F}$ together with a map of presheaves of sets
$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$
such that for every open $U \subset X$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module structure on the abelian group $\mathcal{F}(U)$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules* is a morphism of abelian presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that the diagram
$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}\times\varphi"'] & \mathcal{F} \arrow[d, "\varphi"] \\ \mathcal{O} \times \mathcal{G} \arrow[r] & \mathcal{G} \end{tikzcd}$$
commutes.
(3) The set of $\mathcal{O}$-module morphisms as above is denoted $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.
(4) The category of presheaves of $\mathcal{O}$-modules is denoted *PMod*$(\mathcal{O})$.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on X. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition

$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$

We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the *restriction of* $\mathcal{F}$. We obtain the restriction functor

$$\textit{PMod}(\mathcal{O}_2) \longrightarrow \textit{PMod}(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule

$$(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

The index p stands for "presheaf" and not "point". This presheaf is called the tensor product presheaf. We obtain the *change of rings* functor

$$\textit{PMod}(\mathcal{O}_1) \longrightarrow \textit{PMod}(\mathcal{O}_2)$$

Lemma 6.2. *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. $\square$

7. Sheaves

In this section we explain the sheaf condition.

Definition 7.1. Let X be a topological space.

(1) A *sheaf $\mathcal{F}$ of sets on X* is a presheaf of sets which satisfies the following additional property: Given any open covering $U = \bigcup_{i \in I} U_i$ and any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

(2) A *morphism of sheaves of sets* is simply a morphism of presheaves of sets.

(3) The category of sheaves of sets on X is denoted $\textit{Sh}(X)$.

Remark 7.2. There is always a bit of confusion as to whether it is necessary to say something about the set of sections of a sheaf over the empty set $\emptyset \subset X$. It is necessary, and we already did if you read the definition right. Namely, note that the empty set is covered by the empty open covering, and hence the "collection of sections s_i" from the definition above actually form an element of the empty product which is the final object of the category the sheaf has values in. In other words, if you read the definition right you automatically deduce that $\mathcal{F}(\emptyset) =$ *a final object*, which in the case of a sheaf of sets is a singleton. If you do not like this argument, then you can just require that $\mathcal{F}(\emptyset) = \{*\}$.

In particular, this condition will then ensure that if $U, V \subset X$ are open and *disjoint* then

$$\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V).$$

(Because the fibre product over a final object is a product.)

Example 7.3. Let X, Y be topological spaces. Consider the rule $\mathcal{F}$ which associates to the open $U \subset X$ the set

$$\mathcal{F}(U) = \{f : U \to Y \mid f \text{ is continuous}\}$$

with the obvious restriction mappings. We claim that $\mathcal{F}$ is a sheaf. To see this suppose that $U = \bigcup_{i \in I} U_i$ is an open covering, and $f_i \in \mathcal{F}(U_i)$, $i \in I$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. In this case define $f : U \to Y$ by setting $f(u)$ equal to the value of $f_i(u)$ for any $i \in I$ such that $u \in U_i$. This is well defined by assumption. Moreover, $f : U \to Y$ is a map such that its restriction to U_i agrees with the continuous map f_i. Hence clearly f is continuous!

We can use the result of the example to define constant sheaves. Namely, suppose that A is a set. Endow A with the discrete topology. Let $U \subset X$ be an open subset. Then we have

$$\{f : U \to A \mid f \text{ continuous}\} = \{f : U \to A \mid f \text{ locally constant}\}.$$

Thus the rule which assigns to an open all locally constant maps into A is a sheaf.

Definition 7.4. Let X be a topological space. Let A be a set. The *constant sheaf with value* A denoted $\underline{A}$, or $\underline{A}_X$ is the sheaf that assigns to an open $U \subset X$ the set of all locally constant maps $U \to A$ with restriction mappings given by restrictions of functions.

Example 7.5. Let X be a topological space. Let $(A_x)_{x \in X}$ be a family of sets A_x indexed by points $x \in X$. We are going to construct a sheaf of sets Π from this data. For $U \subset X$ open set

$$\Pi(U) = \prod\nolimits_{x \in U} A_x.$$

For $V \subset U \subset X$ open define a restriction mapping by the following rule: An element $s = (a_x)_{x \in U} \in \Pi(U)$ restricts to $s|_V = (a_x)_{x \in V}$. It is obvious that this defines a presheaf of sets. We claim this is a sheaf. Namely, let $U = \bigcup U_i$ be an open covering. Suppose that $s_i \in \Pi(U_i)$ are such that s_i and s_j agree over $U_i \cap U_j$. Write $s_i = (a_{i,x})_{x \in U_i}$. The compatibility condition implies that $a_{i,x} = a_{j,x}$ in the set A_x whenever $x \in U_i \cap U_j$. Hence there exists a unique element $s = (a_x)_{x \in U}$ in $\Pi(U) = \prod_{x \in U} A_x$ with the property that $a_x = a_{i,x}$ whenever $x \in U_i$ for some i. Of course this element s has the property that $s|_{U_i} = s_i$ for all i.

Example 7.6. Let X be a topological space. Suppose for each $x \in X$ we are given an abelian group M_x. Consider the presheaf $\mathcal{F} : U \mapsto \bigoplus_{x \in U} M_x$ defined in Example 4.5. This is not a sheaf in general. For example, if X is an infinite set with the discrete topology, then the sheaf condition would imply that $\mathcal{F}(X) = \prod_{x \in X} \mathcal{F}(\{x\})$ but by definition we have $\mathcal{F}(X) = \bigoplus_{x \in X} M_x = \bigoplus_{x \in X} \mathcal{F}(\{x\})$. And an infinite direct sum is in general different from an infinite direct product.

However, if X is a topological space such that every open of X is quasi-compact, then $\mathcal{F}$ *is* a sheaf. This is left as an exercise to the reader.

8. Abelian sheaves

Definition 8.1. Let X be a topological space.

(1) An *abelian sheaf on* X or *sheaf of abelian groups on* X is an abelian presheaf on X such that the underlying presheaf of sets is a sheaf.
(2) The category of sheaves of abelian groups is denoted $Ab(X)$.

Let X be a topological space. In the case of an abelian presheaf $\mathcal{F}$ the sheaf condition with regards to an open covering $U = \bigcup U_i$ is often expressed by saying that the complex of abelian groups

$$0 \to \mathcal{F}(U) \to \prod\nolimits_i \mathcal{F}(U_i) \to \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is exact. The first map is the usual one, whereas the second maps the element $(s_i)_{i\in I}$ to the element

$$(s_{i_0}|_{U_{i_0}\cap U_{i_1}} - s_{i_1}|_{U_{i_0}\cap U_{i_1}})_{(i_0,i_1)} \in \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

9. Sheaves of algebraic structures

Let us clarify the definition of sheaves of certain types of structures. First, let us reformulate the sheaf condition. Namely, suppose that $\mathcal{F}$ is a presheaf of sets on the topological space X. The sheaf condition can be reformulated as follows. Let $U = \bigcup_{i\in I} U_i$ be an open covering. Consider the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

Here the left map is defined by the rule $s \mapsto \prod_{i\in I} s|_{U_i}$. The two maps on the right are the maps

$$\prod\nolimits_i s_i \mapsto \prod\nolimits_{(i_0,i_1)} s_{i_0}|_{U_{i_0}\cap U_{i_1}} \text{ resp. } \prod\nolimits_i s_i \mapsto \prod\nolimits_{(i_0,i_1)} s_{i_1}|_{U_{i_0}\cap U_{i_1}}.$$

The sheaf condition exactly says that the left arrow is the equalizer of the right two. This generalizes immediately to the case of presheaves with values in a category as long as the category has products.

Definition 9.1. Let X be a topological space. Let $\mathcal{C}$ be a category with products. A presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on X is a *sheaf* if for every open covering the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is an equalizer diagram in the category $\mathcal{C}$.

Suppose that $\mathcal{C}$ is a category and that $F : \mathcal{C} \to \textit{Sets}$ is a faithful functor. A good example to keep in mind is the case where $\mathcal{C}$ is the category of abelian groups and F is the forgetful functor. Consider a presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on X. We would like to reformulate the condition above in terms of the underlying presheaf of sets (Definition 5.2). Note that the underlying presheaf of sets is a sheaf of sets if and only if all the diagrams

$$F(\mathcal{F}(U)) \longrightarrow \prod\nolimits_{i\in I} F(\mathcal{F}(U_i)) \rightrightarrows \prod\nolimits_{(i_0,i_1)\in I\times I} F(\mathcal{F}(U_{i_0} \cap U_{i_1}))$$

of sets – after applying the forgetful functor F – are equalizer diagrams! Thus we would like $\mathcal{C}$ to have products and equalizers and we would like F to commute with them. This is equivalent to the condition that $\mathcal{C}$ has limits and that F commutes with them, see Categories, Lemma 14.11. But this is not yet good enough (see Example 9.4); we also need F to *reflect isomorphisms*. This property means that given a morphism $f : A \to A'$ in $\mathcal{C}$, then f is an isomorphism if (and only if) $F(f)$ is a bijection.

Lemma 9.2. *Suppose the category $\mathcal{C}$ and the functor $F : \mathcal{C} \to \textit{Sets}$ have the following properties:*

(1) *F is faithful,*
(2) *$\mathcal{C}$ has limits and F commutes with them, and*
(3) *the functor F reflects isomorphisms.*

Let X be a topological space. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$. Then $\mathcal{F}$ is a sheaf if and only if the underlying presheaf of sets is a sheaf.

Proof. Assume that $\mathcal{F}$ is a sheaf. Then $\mathcal{F}(U)$ is the equalizer of the diagram above and by assumption we see $F(\mathcal{F}(U))$ is the equalizer of the corresponding diagram of sets. Hence $F(\mathcal{F})$ is a sheaf of sets.

Assume that $F(\mathcal{F})$ is a sheaf. Let $E \in \mathrm{Ob}(\mathcal{C})$ be the equalizer of the two parallel arrows in Definition 9.1. We get a canonical morphism $\mathcal{F}(U) \to E$, simply because $\mathcal{F}$ is a presheaf. By assumption, the induced map $F(\mathcal{F}(U)) \to F(E)$ is an isomorphism, because $F(E)$ is the equalizer of the corresponding diagram of sets. Hence we see $\mathcal{F}(U) \to E$ is an isomorphism by condition (3) of the lemma. $\square$

The lemma in particular applies to *sheaves of groups, rings, algebras over a fixed ring, modules over a fixed ring, vector spaces over a fixed field,* etc. In other words, these are presheaves of groups, rings, modules over a fixed ring, vector spaces over a fixed field, etc such that the underlying presheaf of sets is a sheaf.

Example 9.3. Let X be a topological space. For each open $U \subset X$ consider the $\mathbf{R}$-algebra $\mathcal{C}^0(U) = \{f : U \to \mathbf{R} \mid f \text{ is continuous}\}$. There are obvious restriction mappings that turn this into a presheaf of $\mathbf{R}$-algebras over X. By Example 7.3 it is a sheaf of sets. Hence by the Lemma 9.2 it is a sheaf of $\mathbf{R}$-algebras over X.

Example 9.4. Consider the category of topological spaces *Top*. There is a natural faithful functor *Top* $\to$ *Sets* which commutes with products and equalizers. But it does not reflect isomorphisms. And, in fact it turns out that the analogue of Lemma 9.2 is wrong. Namely, suppose $X = \mathbf{N}$ with the discrete topology. Let A_i, for $i \in \mathbf{N}$ be a discrete topological space. For any subset $U \subset \mathbf{N}$ define $\mathcal{F}(U) = \prod_{i \in U} A_i$ with the discrete topology. Then this is a presheaf of topological spaces whose underlying presheaf of sets is a sheaf, see Example 7.5. However, if each A_i has at least two elements, then this is not a sheaf of topological spaces according to Definition 9.1. The reader may check that putting the *product topology* on each $\mathcal{F}(U) = \prod_{i \in U} A_i$ does lead to a sheaf of topological spaces over X.

10. Sheaves of modules

Definition 10.1. Let X be a topological space. Let $\mathcal{O}$ be a sheaf of rings on X.

(1) A *sheaf of $\mathcal{O}$-modules* is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, see Definition 6.1, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.
(2) A *morphism of sheaves of $\mathcal{O}$-modules* is a morphism of presheaves of $\mathcal{O}$-modules.
(3) Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules.
(4) The category of sheaves of $\mathcal{O}$-modules is denoted *Mod*$(\mathcal{O})$.

This definition kind of makes sense even if $\mathcal{O}$ is just a presheaf of rings, although we do not know any examples where this is useful, and we will avoid using the terminology "sheaves of $\mathcal{O}$-modules" in case $\mathcal{O}$ is not a sheaf of rings.

11. Stalks

Let X be a topological space. Let $x \in X$ be a point. Let $\mathcal{F}$ be a presheaf of sets on X. The *stalk of $\mathcal{F}$ at x* is the set

$$\mathcal{F}_x = \mathrm{colim}_{x \in U} \mathcal{F}(U)$$

where the colimit is over the set of open neighbourhoods U of x in X. The set of open neighbourhoods is partially ordered by (reverse) inclusion: We say $U \geq U' \Leftrightarrow U \subset U'$. The transition maps in the system are given by the restriction maps of $\mathcal{F}$. See Categories, Section 21 for notation and terminology regarding (co)limits over systems. Note that the colimit is a directed colimit. Thus it is easy to describe $\mathcal{F}_x$. Namely,

$$\mathcal{F}_x = \{(U, s) \mid x \in U, s \in \mathcal{F}(U)\}/\sim$$

with equivalence relation given by $(U, s) \sim (U', s')$ if and only if there exists an open $U'' \subset U \cap U'$ with $x \in U''$ and $s|_{U''} = s'|_{U''}$. By abuse of notation we will often denote (U, s), s_x, or even s the corresponding element in $\mathcal{F}_x$. Also we will say $s = s'$ in $\mathcal{F}_x$ for two local sections of $\mathcal{F}$ defined in an open neighbourhood of x to denote that they have the same image in $\mathcal{F}_x$.

An obvious consequence of this definition is that for any open $U \subset X$ there is a canonical map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{x \in U} \mathcal{F}_x$$

defined by $s \mapsto \prod_{x \in U}(U, s)$. Think about it!

Lemma 11.1. *Let $\mathcal{F}$ be a sheaf of sets on the topological space X. For every open $U \subset X$ the map*

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{x \in U} \mathcal{F}_x$$

is injective.

Proof. Suppose that $s, s' \in \mathcal{F}(U)$ map to the same element in every stalk $\mathcal{F}_x$ for all $x \in U$. This means that for every $x \in U$, there exists an open $V^x \subset U$, $x \in V^x$ such that $s|_{V^x} = s'|_{V^x}$. But then $U = \bigcup_{x \in U} V^x$ is an open covering. Thus by the uniqueness in the sheaf condition we see that $s = s'$. $\square$

Definition 11.2. Let X be a topological space. A presheaf of sets $\mathcal{F}$ on X is *separated* if for every open $U \subset X$ the map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective.

Another observation is that the construction of the stalk $\mathcal{F}_x$ is functorial in the presheaf $\mathcal{F}$. In other words, it gives a functor

$$PSh(X) \longrightarrow Sets, \ \mathcal{F} \longmapsto \mathcal{F}_x.$$

This functor is called the *stalk functor*. Namely, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then we define $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ by the rule $(U, s) \mapsto (U, \varphi(s))$. To see that this works we have to check that if $(U, s) = (U', s')$ in $\mathcal{F}_x$ then also $(U, \varphi(s)) = (U', \varphi(s'))$ in $\mathcal{G}_x$. This is clear since φ is compatible with the restriction mappings.

Example 11.3. Let X be a topological space. Let A be a set. Denote temporarily A_p the constant presheaf with value A (p for presheaf – not for point). There is a canonical map of presheaves $A_p \to \underline{A}$ into the constant sheaf with value A. For every point we have canonical bijections $A = (A_p)_x = \underline{A}_x$, where the second map is induced by functoriality from the map $A_p \to \underline{A}$.

Example 11.4. Suppose $X = \mathbf{R}^n$ with the Euclidean topology. Consider the presheaf of $\mathcal{C}^\infty$ functions on X, denoted $\mathcal{C}^\infty_{\mathbf{R}^n}$. In other words, $\mathcal{C}^\infty_{\mathbf{R}^n}(U)$ is the set of $\mathcal{C}^\infty$-functions $f : U \to \mathbf{R}$. As in Example 7.3 it is easy to show that this is a sheaf. In fact it is a sheaf of $\mathbf{R}$-vector spaces.

Next, let $x \in X = \mathbf{R}^n$ be a point. How do we think of an element in the stalk $\mathcal{C}^\infty_{\mathbf{R}^n,x}$? Such an element is given by a $\mathcal{C}^\infty$-function f whose domain contains x. And a pair of such functions f, g determine the same element of the stalk if they agree in a neighbourhood of x. In other words, an element if $\mathcal{C}^\infty_{\mathbf{R}^n,x}$ is the same thing as what is sometimes called a *germ of a $\mathcal{C}^\infty$-function at x*.

Example 11.5. Let X be a topological space. Let A_x be a set for each $x \in X$. Consider the sheaf $\mathcal{F} : U \mapsto \prod_{x\in U} A_x$ of Example 7.5. We would just like to point out here that the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x is in general *not* equal to the set A_x. Of course there is a map $\mathcal{F}_x \to A_x$, but that is in general the best you can say. For example, suppose $x = \lim x_n$ with $x_n \neq x_m$ for all $n \neq m$ and suppose that $A_y = \{0, 1\}$ for all $y \in X$. Then $\mathcal{F}_x$ maps onto the (infinite) set of tails of sequences of 0s and 1s. Namely, every open neighbourhood of x contains almost all of the x_n. On the other hand, if every neighbourhood of x contains a point y such that $A_y = \emptyset$, then $\mathcal{F}_x = \emptyset$.

12. Stalks of abelian presheaves

We first deal with the case of abelian groups as a model for the general case.

Lemma 12.1. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of abelian groups on X. There exists a unique structure of an abelian group on $\mathcal{F}_x$ such that for every $U \subset X$ open, $x \in U$ the map $\mathcal{F}(U) \to \mathcal{F}_x$ is a group homomorphism. Moreover,*

$$\mathcal{F}_x = \operatorname{colim}_{x\in U} \mathcal{F}(U)$$

holds in the category of abelian groups.

Proof. We define addition of a pair of elements (U, s) and (V, t) as the pair $(U \cap V, s|_{U\cap V} + t|_{U\cap V})$. The rest is easy to check. $\square$

What is crucial in the proof above is that the partially ordered set of open neighbourhoods is a directed set (Categories, Definition 21.1). Namely, the coproduct of two abelian groups A, B is the direct sum $A \oplus B$, whereas the coproduct in the category of sets is the disjoint union $A \amalg B$, showing that colimits in the category of abelian groups do not agree with colimits in the category of sets in general.

13. Stalks of presheaves of algebraic structures

The proof of Lemma 12.1 will work for any type of algebraic structure such that directed colimits commute with the forgetful functor.

Lemma 13.1. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to Sets$ be a functor. Assume that*

1. *F is faithful, and*
2. *directed colimits exist in $\mathcal{C}$ and F commutes with them.*

Let X be a topological space. Let $x \in X$. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$. Then

$$\mathcal{F}_x = \operatorname{colim}_{x\in U} \mathcal{F}(U)$$

exists in $\mathcal{C}$. Its underlying set is equal to the stalk of the underlying presheaf of sets of $\mathcal{F}$. Furthermore, the construction $\mathcal{F} \mapsto \mathcal{F}_x$ is a functor from the category of presheaves with values in $\mathcal{C}$ to $\mathcal{C}$.

Proof. Omitted. $\square$

By the very definition, all the morphisms $\mathcal{F}(U) \to \mathcal{F}_x$ are morphisms in the category $\mathcal{C}$ which (after applying the forgetful functor F) turn into the corresponding maps for the underlying sheaf of sets. As usual we will not distinguish between the morphism in $\mathcal{C}$ and the underlying map of sets, which is permitted since F is faithful.

This lemma applies in particular to: *Presheaves of (not necessarily abelian) groups, rings, modules over a fixed ring, vector spaces over a fixed field.*

14. Stalks of presheaves of modules

Lemma 14.1. *Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $x \in X$. The canonical map $\mathcal{O}_x \times \mathcal{F}_x \to \mathcal{F}_x$ coming from the multiplication map $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ defines a $\mathcal{O}_x$-module structure on the abelian group $\mathcal{F}_x$.*

Proof. Omitted. $\square$

Lemma 14.2. *Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of presheaves of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $x \in X$. We have*

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{O}')_x$$

as $\mathcal{O}'_x$-modules.

Proof. Omitted. $\square$

15. Algebraic structures

In this section we mildly formalize the notions we have encountered in the sections above.

Definition 15.1. A *type of algebraic structure* is given by a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathit{Sets}$ with the following properties

1. F is faithful,
2. $\mathcal{C}$ has limits and F commutes with limits,
3. $\mathcal{C}$ has filtered colimits and F commutes with them, and
4. F reflects isomorphisms.

We make this definition to point out the properties we will use in a number of arguments below. But we will not actually study this notion in any great detail, since we are prohibited from studying "big" categories by convention, except for those listed in Categories, Remark 2.2. Among those the following have the required properties.

Lemma 15.2. *The following categories, endowed with the obvious forgetful functor, define types of algebraic structures:*

1. *The category of pointed sets.*
2. *The category of abelian groups.*

(3) *The category of groups.*
(4) *The category of monoids.*
(5) *The category of rings.*
(6) *The category of R-modules for a fixed ring R.*
(7) *The category of Lie algebras over a fixed field.*

Proof. Omitted. $\square$

From now on we will think of a (pre)sheaf of algebraic structures and their stalks, in terms of the underlying (pre)sheaf of sets. This is allowable by Lemmas 9.2 and 13.1.

In the rest of this section we point out some results on algebraic structures that will be useful in the future.

Lemma 15.3. *Let $(\mathcal{C}, F)$ be a type of algebraic structure.*

(1) *$\mathcal{C}$ has a final object 0 and $F(0) = \{*\}$.*
(2) *$\mathcal{C}$ has products and $F(\prod A_i) = \prod F(A_i)$.*
(3) *$\mathcal{C}$ has fibre products and $F(A \times_B C) = F(A) \times_{F(B)} F(C)$.*
(4) *$\mathcal{C}$ has equalizers, and if $E \to A$ is the equalizer of $a, b : A \to B$, then $F(E) \to F(A)$ is the equalizer of $F(a), F(b) : F(A) \to F(B)$.*
(5) *$A \to B$ is a monomorphism if and only if $F(A) \to F(B)$ is injective.*
(6) *if $F(a) : F(A) \to F(B)$ is surjective, then a is an epimorphism.*
(7) *given $A_1 \to A_2 \to A_3 \to \ldots$, then $\operatorname{colim} A_i$ exists and $F(\operatorname{colim} A_i) = \operatorname{colim} F(A_i)$, and more generally for any filtered colimit.*

Proof. Omitted. The only interesting statement is (5) which follows because $A \to B$ is a monomorphism if and only if $A \to A \times_B A$ is an isomorphism, and then applying the fact that F reflects isomorphisms. $\square$

Lemma 15.4. *Let $(\mathcal{C}, F)$ be a type of algebraic structure. Suppose that $A, B, C \in \mathrm{Ob}(\mathcal{C})$. Let $f : A \to B$ and $g : C \to B$ be morphisms of $\mathcal{C}$. If $F(g)$ is injective, and $\mathrm{Im}(F(f)) \subset \mathrm{Im}(F(g))$, then f factors as $f = g \circ t$ for some morphism $t : A \to C$.*

Proof. Consider $A \times_B C$. The assumptions imply that $F(A \times_B C) = F(A) \times_{F(B)} F(C) = F(A)$. Hence $A = A \times_B C$ because F reflects isomorphisms. The result follows. $\square$

Example 15.5. The lemma will be applied often to the following situation. Suppose that we have a diagram

$$\begin{tikzcd} A \arrow[r] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathcal{C}$. Suppose $C \to D$ is injective on underlying sets, and suppose that the composition $A \to B \to D$ has image on underlying sets in the image of $C \to D$. Then we get a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathcal{C}$.

Example 15.6. Let $F : \mathcal{C} \to \textit{Sets}$ be a type of algebraic structures. Let X be a topological space. Suppose that for every $x \in X$ we are given an object $A_x \in \mathrm{Ob}(\mathcal{C})$. Consider the presheaf Π with values in $\mathcal{C}$ on X defined by the rule $\Pi(U) = \prod_{x\in U} A_x$ (with obvious restriction mappings). Note that the associated presheaf of sets $U \mapsto F(\Pi(U)) = \prod_{x\in U} F(A_x)$ is a sheaf by Example 7.5. Hence Π is a sheaf of algebraic structures of type $(\mathcal{C}, F)$. This gives many examples of sheaves of abelian groups, groups, rings, etc.

16. Exactness and points

In any category we have the notion of epimorphism, monomorphism, isomorphism, etc.

Lemma 16.1. *Let X be a topological space. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of sets on X.*

(1) *The map φ is a monomorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective.*
(2) *The map φ is an epimorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective.*
(3) *The map φ is an isomorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective.*

Proof. Omitted. $\square$

It follows that in the category of sheaves of sets the notions epimorphism and monomorphism can be described as follows.

Definition 16.2. Let X be a topological space.

(1) A presheaf $\mathcal{F}$ is called a *subpresheaf* of a presheaf $\mathcal{G}$ if $\mathcal{F}(U) \subset \mathcal{G}(U)$ for all open $U \subset X$ such that the restriction maps of $\mathcal{G}$ induce the restriction maps of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\mathcal{F}$ is called a *subsheaf* of $\mathcal{G}$. We sometimes indicate this by the notation $\mathcal{F} \subset \mathcal{G}$.
(2) A morphism of presheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *injective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U open in X.
(3) A morphism of presheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *surjective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all U open in X.
(4) A morphism of sheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *injective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U open in X.
(5) A morphism of sheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *surjective* if and only if for every open U of X and every section s of $\mathcal{G}(U)$ there exists an open covering $U = \bigcup U_i$ such that $s|_{U_i}$ is in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$ for all i.

Lemma 16.3. *Let X be a topological space.*

(1) *Epimorphisms (resp. monomorphisms) in the category of presheaves are exactly the surjective (resp. injective) maps of presheaves.*
(2) *Epimorphisms (resp. monomorphisms) in the category of sheaves are exactly the surjective (resp. injective) maps of sheaves, and are exactly those maps which are surjective (resp. injective) on all the stalks.*
(3) *The sheafification of a surjective (resp. injective) morphism of presheaves of sets is surjective (resp. injective).*

Proof. Omitted. $\square$

Lemma 16.4. *let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Suppose that $\mathcal{F}$, $\mathcal{G}$ are sheaves on X with values in $\mathcal{C}$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of the underlying sheaves of sets. If for all points $x \in X$ the map $\mathcal{F}_x \to \mathcal{G}_x$ is a morphism of algebraic structures, then φ is a morphism of sheaves of algebraic structures.*

Proof. Let U be an open subset of X. Consider the diagram of (underlying) sets

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] & \prod_{x \in U} \mathcal{F}_x \arrow[d] \\ \mathcal{G}(U) \arrow[r] & \prod_{x \in U} \mathcal{G}_x \end{tikzcd}$$

By assumption, and previous results, all but the left vertical arrow are morphisms of algebraic structures. In addition the bottom horizontal arrow is injective, see Lemma 11.1. Hence we conclude by Lemma 15.4, see also Example 15.5 $\square$

Short exact sequences of abelian sheaves, etc will be discussed in the chapter on sheaves of modules. See Modules, Section 3.

17. Sheafification

In this section we explain how to get the sheafification of a presheaf on a topological space. We will use stalks to describe the sheafification in this case. This is different from the general procedure described in Sites, Section 10, and perhaps somewhat easier to understand.

The basic construction is the following. Let $\mathcal{F}$ be a presheaf of sets on a topological space X. For every open $U \subset X$ we define

$$\mathcal{F}^\#(U) = \{(s_u) \in \prod\nolimits_{u \in U} \mathcal{F}_u \text{ such that } (*)\}$$

where $(*)$ is the property:

(*) For every $u \in U$, there exists an open neighbourhood $u \in V \subset U$, and a section $\sigma \in \mathcal{F}(V)$ such that for all $v \in V$ we have $s_v = (V, \sigma)$ in $\mathcal{F}_v$.

Note that $(*)$ is a condition for each $u \in U$, and that given $u \in U$ the truth of this condition depends only on the values s_v for v in any open neighbourhood of u. Thus it is clear that, if $V \subset U \subset X$ are open, the projection maps

$$\prod\nolimits_{u \in U} \mathcal{F}_u \longrightarrow \prod\nolimits_{v \in V} \mathcal{F}_v$$

maps elements of $\mathcal{F}^\#(U)$ into $\mathcal{F}^\#(V)$. Using these maps as the restriction mappings, we turn $\mathcal{F}^\#$ into a presheaf of sets on X.

Furthermore, the map $\mathcal{F}(U) \to \prod_{u \in U} \mathcal{F}_u$ described in Section 11 clearly has image in $\mathcal{F}^\#(U)$. In addition, if $V \subset U \subset X$ are open then we have the following commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}^\#(U) \arrow[r] \arrow[d] & \prod_{u \in U} \mathcal{F}_u \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{F}^\#(V) \arrow[r] & \prod_{v \in V} \mathcal{F}_v \end{tikzcd}$$

where the vertical maps are induced from the restriction mappings. Thus we see that there is a canonical morphism of presheaves $\mathcal{F} \to \mathcal{F}^\#$.

In Example 7.5 we saw that the rule $\Pi(\mathcal{F}) : U \mapsto \prod_{u \in U} \mathcal{F}_u$ is a sheaf, with obvious restriction mappings. And by construction $\mathcal{F}^\#$ is a subpresheaf of this. In other words, we have morphisms of presheaves

$$\mathcal{F} \to \mathcal{F}^\# \to \Pi(\mathcal{F}).$$

In addition the rule that associates to $\mathcal{F}$ the sequence above is clearly functorial in the presheaf $\mathcal{F}$. This notation will be used in the proofs of the lemmas below.

Lemma 17.1. *The presheaf $\mathcal{F}^\#$ is a sheaf.*

Proof. It is probably better for the reader to find their own explanation of this than to read the proof here. In fact the lemma is true for the same reason as why the presheaf of continuous function is a sheaf, see Example 7.3 (and this analogy can be made precise using the "espace étalé").

Anyway, let $U = \bigcup U_i$ be an open covering. Suppose that $s_i = (s_{i,u})_{u \in U_i} \in \mathcal{F}^\#(U_i)$ such that s_i and s_j agree over $U_i \cap U_j$. Because $\Pi(\mathcal{F})$ is a sheaf, we find an element $s = (s_u)_{u \in U}$ in $\prod_{u \in U} \mathcal{F}_u$ restricting to s_i on U_i. We have to check property $(*)$. Pick $u \in U$. Then $u \in U_i$ for some i. Hence by $(*)$ for s_i, there exists a V open, $u \in V \subset U_i$ and a $\sigma \in \mathcal{F}(V)$ such that $s_{i,v} = (V, \sigma)$ in $\mathcal{F}_v$ for all $v \in V$. Since $s_{i,v} = s_v$ we get $(*)$ for s. □

Lemma 17.2. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets on X. Let $x \in X$. Then $\mathcal{F}_x = \mathcal{F}^\#_x$.*

Proof. The map $\mathcal{F}_x \to \mathcal{F}^\#_x$ is injective, since already the map $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ is injective. Namely, there is a canonical map $\Pi(\mathcal{F})_x \to \mathcal{F}_x$ which is a left inverse to the map $\mathcal{F}_x \to \Pi(\mathcal{F})_x$, see Example 11.5. To show that it is surjective, suppose that $\overline{s} \in \mathcal{F}^\#_x$. We can find an open neighbourhood U of x such that $\overline{s}$ is the equivalence class of (U, s) with $s \in \mathcal{F}^\#(U)$. By definition, this means there exists an open neighbourhood $V \subset U$ of x and a section $\sigma \in \mathcal{F}(V)$ such that $s|_V$ is the image of σ in $\Pi(\mathcal{F})(V)$. Clearly the class of (V, σ) defines an element of $\mathcal{F}_x$ mapping to $\overline{s}$. □

Lemma 17.3. *Let $\mathcal{F}$ be a presheaf of sets on X. Any map $\mathcal{F} \to \mathcal{G}$ into a sheaf of sets factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$.*

Proof. Clearly, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^\# \arrow[r] \arrow[d] & \Pi(\mathcal{F}) \arrow[d] \\
\mathcal{G} \arrow[r] & \mathcal{G}^\# \arrow[r] & \Pi(\mathcal{G})
\end{tikzcd}$$

So it suffices to prove that $\mathcal{G} = \mathcal{G}^\#$. To see this it suffices to prove, for every point $x \in X$ the map $\mathcal{G}_x \to \mathcal{G}^\#_x$ is bijective, by Lemma 16.1. And this is Lemma 17.2 above. □

This lemma really says that there is an adjoint pair of functors: $i : \mathit{Sh}(X) \to \mathit{PSh}(X)$ (inclusion) and $\# : \mathit{PSh}(X) \to \mathit{Sh}(X)$ (sheafification). The formula is that

$$\mathrm{Mor}_{\mathit{PSh}(X)}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{\mathit{Sh}(X)}(\mathcal{F}^\#, \mathcal{G})$$

which says that sheafification is a left adjoint of the inclusion functor. See Categories, Section 24.

Example 17.4. See Example 11.3 for notation. The map $A_p \to \underline{A}$ induces a map $A_p^\# \to \underline{A}$. It is easy to see that this is an isomorphism. In words: The sheafification of the constant presheaf with value A is the constant sheaf with value A.

Lemma 17.5. *Let X be a topological space. A presheaf $\mathcal{F}$ is separated (see Definition 11.2) if and only if the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is injective.*

Proof. This is clear from the construction of $\mathcal{F}^\#$ in this section. □

18. Sheafification of abelian presheaves

The following strange looking lemma is likely unnecessary, but very convenient to deal with sheafification of presheaves of algebraic structures.

Lemma 18.1. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets on X. Let $U \subset X$ be open. There is a canonical fibre product diagram*

$$\begin{tikzcd} \mathcal{F}^\#(U) \arrow[r] \arrow[d] & \Pi(\mathcal{F})(U) \arrow[d] \\ \prod_{x \in U} \mathcal{F}_x \arrow[r] & \prod_{x \in U} \Pi(\mathcal{F})_x \end{tikzcd}$$

where the maps are the following:

1. *The left vertical map has components $\mathcal{F}^\#(U) \to \mathcal{F}^\#_x = \mathcal{F}_x$ where the equality is Lemma 17.2.*
2. *The top horizontal map comes from the map of presheaves $\mathcal{F} \to \Pi(\mathcal{F})$ described in Section 17.*
3. *The right vertical map has obvious component maps $\Pi(\mathcal{F})(U) \to \Pi(\mathcal{F})_x$.*
4. *The bottom horizontal map has components $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ which come from the map of presheaves $\mathcal{F} \to \Pi(\mathcal{F})$ described in Section 17.*

Proof. It is clear that the diagram commutes. We have to show it is a fibre product diagram. The bottom horizontal arrow is injective since all the maps $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ are injective (see beginning proof of Lemma 17.2). A section $s \in \Pi(\mathcal{F})(U)$ is in $\mathcal{F}^\#$ if and only if $(*)$ holds. But $(*)$ says that around every point the section s comes from a section of $\mathcal{F}$. By definition of the stalk functors, this is equivalent to saying that the value of s in every stalk $\Pi(\mathcal{F})_x$ comes from an element of the stalk $\mathcal{F}_x$. Hence the lemma. □

Lemma 18.2. *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. Then there exists a unique structure of abelian sheaf on $\mathcal{F}^\#$ such that $\mathcal{F} \to \mathcal{F}^\#$ is a morphism of abelian presheaves. Moreover, the following adjointness property holds*

$$\mathrm{Mor}_{PAb(X)}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{Ab(X)}(\mathcal{F}^\#, \mathcal{G}).$$

Proof. Recall the sheaf of sets $\Pi(\mathcal{F})$ defined in Section 17. All the stalks $\mathcal{F}_x$ are abelian groups, see Lemma 12.1. Hence $\Pi(\mathcal{F})$ is a sheaf of abelian groups by Example 15.6. Also, it is clear that the map $\mathcal{F} \to \Pi(\mathcal{F})$ is a morphism of abelian presheaves. If we show that condition $(*)$ of Section 17 defines a subgroup of $\Pi(\mathcal{F})(U)$ for all open subsets $U \subset X$, then $\mathcal{F}^\#$ canonically inherits the structure of abelian sheaf. This is quite easy to do by hand, and we leave it to the reader to find a good simple argument. The argument we use here, which generalizes to presheaves of algebraic structures is the following: Lemma 18.1 show that $\mathcal{F}^\#(U)$ is the fibre product of a diagram of abelian groups. Thus $\mathcal{F}^\#$ is an abelian subgroup as desired.

Note that at this point $\mathcal{F}_x^\#$ is an abelian group by Lemma 12.1 and that $\mathcal{F}_x \to \mathcal{F}_x^\#$ is a bijection (Lemma 17.2) and a homomorphism of abelian groups. Hence $\mathcal{F}_x \to \mathcal{F}_x^\#$ is an isomorphism of abelian groups. This will be used below without further mention.

To prove the adjointness property we use the adjointness property of sheafification of presheaves of sets. For example if $\psi : \mathcal{F} \to i(\mathcal{G})$ is morphism of presheaves then we obtain a morphism of sheaves $\psi' : \mathcal{F}^\# \to \mathcal{G}$. What we have to do is to check that this is a morphism of abelian sheaves. We may do this for example by noting that it is true on stalks, by Lemma 17.2, and then using Lemma 16.4 above. □

19. Sheafification of presheaves of algebraic structures

Lemma 19.1. *Let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$ on X. Then there exists a sheaf $\mathcal{F}^\#$ with values in $\mathcal{C}$ and a morphism $\mathcal{F} \to \mathcal{F}^\#$ of presheaves with values in $\mathcal{C}$ with the following properties:*

1. *The map $\mathcal{F} \to \mathcal{F}^\#$ identifies the underlying sheaf of sets of $\mathcal{F}^\#$ with the sheafification of the underlying presheaf of sets of $\mathcal{F}$.*
2. *For any morphism $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf with values in $\mathcal{C}$ there exists a unique factorization $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$.*

Proof. The proof is the same as the proof of Lemma 18.2, with repeated application of Lemma 15.4 (see also Example 15.5). The main idea however, is to define $\mathcal{F}^\#(U)$ as the fibre product in $\mathcal{C}$ of the diagram

$$\begin{tikzcd} & \Pi(\mathcal{F})(U) \arrow[d] \\ \prod_{x\in U} \mathcal{F}_x \arrow[r] & \prod_{x \in U} \Pi(\mathcal{F})_x \end{tikzcd}$$

compare Lemma 18.1. □

20. Sheafification of presheaves of modules

Lemma 20.1. *Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $\mathcal{O}^\#$ be the sheafification of $\mathcal{O}$. Let $\mathcal{F}^\#$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a map of sheaves of sets*

$$\mathcal{O}^\# \times \mathcal{F}^\# \longrightarrow \mathcal{F}^\#$$

which makes the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{O}^\# \times \mathcal{F}^\# \arrow[r] & \mathcal{F}^\# \end{tikzcd}$$

commute and which makes $\mathcal{F}^\#$ into a sheaf of $\mathcal{O}^\#$-modules. In addition, if $\mathcal{G}$ is a sheaf of $\mathcal{O}^\#$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ (into the restriction of $\mathcal{G}$ to a $\mathcal{O}$-module) factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ where $\mathcal{F}^\# \to \mathcal{G}$ is a morphism of $\mathcal{O}^\#$-modules.

Proof. Omitted. □

This actually means that the functor $i : \mathit{Mod}(\mathcal{O}^\#) \to \mathit{PMod}(\mathcal{O})$ (combining restriction and including sheaves into presheaves) and the sheafification functor of the lemma $^\# : \mathit{PMod}(\mathcal{O}) \to \mathit{Mod}(\mathcal{O}^\#)$ are adjoint. In a formula

$$\mathrm{Mor}_{\mathit{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \mathrm{Mor}_{\mathit{Mod}(\mathcal{O}^\#)}(\mathcal{F}^\#, \mathcal{G})$$

Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on X. In Section 6 we defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_1}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$\mathit{Mod}(\mathcal{O}_2) \longrightarrow \mathit{Mod}(\mathcal{O}_1)$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the *tensor product sheaf* $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^\#$$

as the sheafification of our construction for presheaves. We obtain the *change of rings* functor

$$\mathit{Mod}(\mathcal{O}_1) \longrightarrow \mathit{Mod}(\mathcal{O}_2)$$

Lemma 20.2. *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Lemma 6.2 and the fact that $\mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$ because $\mathcal{F}$ is a sheaf. □

Lemma 20.3. *Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf $\mathcal{O}$-modules. Let $x \in X$. We have*

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}')_x$$

as $\mathcal{O}'_x$-modules.

Proof. Follows directly from Lemma 14.2 and the fact that taking stalks commutes with sheafification. □

21. Continuous maps and sheaves

Let $f : X \to Y$ be a continuous map of topological spaces. We will define the pushforward and pullback functors for presheaves and sheaves.

Let $\mathcal{F}$ be a presheaf of sets on X. We define the *pushforward* of $\mathcal{F}$ by the rule

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

for any open $V \subset Y$. Given $V_1 \subset V_2 \subset Y$ open the restriction map is given by the commutativity of the diagram

$$\begin{tikzcd} f_*\mathcal{F}(V_2) \arrow[r, equal] \arrow[d] & \mathcal{F}(f^{-1}(V_2)) \arrow[d, "\text{restriction for } \mathcal{F}"] \\ f_*\mathcal{F}(V_1) \arrow[r, equal] & \mathcal{F}(f^{-1}(V_1)) \end{tikzcd}$$

It is clear that this defines a presheaf of sets. The construction is clearly functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$f_* : PSh(X) \longrightarrow PSh(Y).$$

Lemma 21.1. *Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X. Then $f_*\mathcal{F}$ is a sheaf on Y.*

Proof. This immediately follows from the fact that if $V = \bigcup V_j$ is an open covering in Y, then $f^{-1}(V) = \bigcup f^{-1}(V_j)$ is an open covering in X. $\square$

As a consequence we obtain a functor

$$f_* : Sh(X) \longrightarrow Sh(Y).$$

This is compatible with composition in the following strong sense.

Lemma 21.2. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal (on both presheaves and sheaves of sets).*

Proof. This is because $(g \circ f)_*\mathcal{F}(W) = \mathcal{F}((g \circ f)^{-1}W)$ and $(g_* \circ f_*)\mathcal{F}(W) = \mathcal{F}(f^{-1}g^{-1}W)$ and $(g \circ f)^{-1}W = f^{-1}g^{-1}W$. $\square$

Let $\mathcal{G}$ be a presheaf of sets on Y. The *pullback presheaf* $f_p\mathcal{G}$ of a given presheaf $\mathcal{G}$ is defined as the left adjoint of the pushforward f_* on presheaves. In other words it should be a presheaf $f_p\mathcal{G}$ on X such that

$$\mathrm{Mor}_{PSh(X)}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

By the Yoneda lemma this determines the pullback uniquely. It turns out that it actually exists.

Lemma 21.3. *Let $f : X \to Y$ be a continuous map. There exists a functor $f_p : PSh(Y) \to PSh(X)$ which is left adjoint to f_*. For a presheaf $\mathcal{G}$ it is determined by the rule*

$$f_p\mathcal{G}(U) = \mathrm{colim}_{f(U) \subset V}\, \mathcal{G}(V)$$

where the colimit is over the collection of open neighbourhoods V of $f(U)$ in Y. The colimits are over directed partially ordered sets. (The restriction mappings of $f_p\mathcal{G}$ are explained in the proof.)

Proof. The colimit is over the partially ordered set consisting of open subsets $V \subset Y$ which contain $f(U)$ with ordering by reverse inclusion. This is a directed partially ordered set, since if V, V' are in it then so is $V \cap V'$. Furthermore, if $U_1 \subset U_2$, then every open neighbourhood of $f(U_2)$ is an open neighbourhood of $f(U_1)$. Hence the system defining $f_p\mathcal{G}(U_2)$ is a subsystem of the one defining $f_p\mathcal{G}(U_1)$ and we obtain a restriction map (for example by applying the generalities in Categories, Lemma 14.8).

Note that the construction of the colimit is clearly functorial in $\mathcal{G}$, and similarly for the restriction mappings. Hence we have defined f_p as a functor.

A small useful remark is that there exists a canonical map $\mathcal{G}(U) \to f_p\mathcal{G}(f^{-1}(U))$, because the system of open neighbourhoods of $f(f^{-1}(U))$ contains the element U. This is compatible with restriction mappings. In other words, there is a canonical map $i_\mathcal{G} : \mathcal{G} \to f_* f_p \mathcal{G}$.

Let $\mathcal{F}$ be a presheaf of sets on X. Suppose that $\psi : f_p\mathcal{G} \to \mathcal{F}$ is a map of presheaves of sets. The corresponding map $\mathcal{G} \to f_*\mathcal{F}$ is the map $f_*\psi \circ i_\mathcal{G} : \mathcal{G} \to f_* f_p\mathcal{G} \to f_*\mathcal{F}$.

Another small useful remark is that there exists a canonical map $c_\mathcal{F} : f_p f_* \mathcal{F} \to \mathcal{F}$. Namely, let $U \subset X$ open. For every open neighbourhood $V \supset f(U)$ in Y there exists a map $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$, namely the restriction map on $\mathcal{F}$. And this is compatible with the restriction mappings between values of $\mathcal{F}$ on f^{-1} of varying opens containing $f(U)$. Thus we obtain a canonical map $f_p f_* \mathcal{F}(U) \to \mathcal{F}(U)$. Another trivial verification shows that these maps are compatible with restriction maps and define a map $c_\mathcal{F}$ of presheaves of sets.

Suppose that $\varphi : \mathcal{G} \to f_*\mathcal{F}$ is a map of presheaves of sets. Consider $f_p\varphi : f_p\mathcal{G} \to f_p f_*\mathcal{F}$. Postcomposing with $c_\mathcal{F}$ gives the desired map $c_\mathcal{F} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F}$. We omit the verification that this construction is inverse to the construction in the other direction given above. □

Lemma 21.4. *Let $f : X \to Y$ be a continuous map. Let $x \in X$. Let $\mathcal{G}$ be a presheaf of sets on Y. There is a canonical bijection of stalks $(f_p\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

Proof. This you can see as follows

$$\begin{aligned}
(f_p\mathcal{G})_x &= \mathrm{colim}_{x \in U}\, f_p\mathcal{G}(U) \\
&= \mathrm{colim}_{x \in U}\, \mathrm{colim}_{f(U) \subset V}\, \mathcal{G}(V) \\
&= \mathrm{colim}_{f(x) \in V}\, \mathcal{G}(V) \\
&= \mathcal{G}_{f(x)}
\end{aligned}$$

Here we have used Categories, Lemma 14.10, and the fact that any V open in Y containing $f(x)$ occurs in the third description above. Details omitted. □

Let $\mathcal{G}$ be a sheaf of sets on Y. The *pullback sheaf* $f^{-1}\mathcal{G}$ is defined by the formula

$$f^{-1}\mathcal{G} = (f_p\mathcal{G})^\#.$$

The pullback f^{-1} is a left adjoint of pushforward on sheaves. In other words,

$$\mathrm{Mor}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

Namely, we have

$$\begin{aligned}\mathrm{Mor}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{PSh(X)}(f_p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F})\end{aligned}$$

For the first equality we use that sheafification is a left adjoint to the inclusion of sheaves in presheaves. For the second equality we use that f_p is a left adjoint to f_* on presheaves. We will return to this statement in the proof of Lemma 21.8.

Lemma 21.5. *Let $x \in X$. Let $\mathcal{G}$ be a sheaf of sets on Y. There is a canonical bijection of stalks $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

Proof. This is a combination of Lemmas 17.2 and 21.4. □

Lemma 21.6. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are canonically isomorphic. Similarly $(g \circ f)_p \cong f_p \circ g_p$ on presheaves.*

Proof. To see this use that adjoint functors are unique up to unique isomorphism, and Lemma 21.2. □

Definition 21.7. Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X and let $\mathcal{G}$ be a sheaf of sets on Y. An *f-map* $\xi : \mathcal{G} \to \mathcal{F}$ is a collection of maps $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ indexed by open subsets $V \subset Y$ such that

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r, "\xi_V"'] \arrow[d, "\text{restriction of } \mathcal{G}"'] & \mathcal{F}(f^{-1}V) \arrow[d, "\text{restriction of } \mathcal{F}"] \\ \mathcal{G}(V') \arrow[r, "\xi_{V'}"] & \mathcal{F}(f^{-1}V') \end{tikzcd}$$

commutes for all $V' \subset V \subset Y$ open.

In the literature we sometimes find this defined alternatively as in part (4) of Lemma 21.8 but as the lemma shows there is really no difference.

Lemma 21.8. *Let $f : X \to Y$ be a continuous map. There are bijections between the following four sets*

1. *the set of maps $\mathcal{G} \to f_*\mathcal{F}$,*
2. *the set of maps $f^{-1}\mathcal{G} \to \mathcal{F}$,*
3. *the set of f-maps $\xi : \mathcal{G} \to \mathcal{F}$, and*
4. *the set of all collections of maps $\xi_{U,V} : \mathcal{G}(V) \to \mathcal{F}(U)$ for all $U \subset X$ and $V \subset Y$ open such that $f(U) \subset V$ compatible with all restriction maps,*

functorially in $\mathcal{F} \in Sh(X)$ and $\mathcal{G} \in Sh(Y)$.

Proof. A map of sheaves $a : \mathcal{G} \to f_*\mathcal{F}$ is by definition a rule which to each open V of Y assigns a map $a_V : \mathcal{G}(V) \to f_*\mathcal{F}(V)$ and we have $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$. Thus at least the "data" corresponds exactly to what you need for an f-map ξ from $\mathcal{G}$ to $\mathcal{F}$. To show that the sets (1) and (3) are in bijection we observe that a is a map of sheaves if and only if corresponding family of maps a_V satisfy the condition in Definition 21.7.

Recall that $f^{-1}\mathcal{G}$ is the sheafification of $f_p\mathcal{G}$. By the universal property of sheafification a map of sheaves $b : f^{-1}\mathcal{G} \to \mathcal{F}$ is the same thing as a map of presheaves

$b_p : f_p\mathcal{G} \to \mathcal{F}$ where f_p is the functor defined earlier in the section. To give such a map b_p you need to specify for each open U of X a map

$$b_{p,U} : \mathrm{colim}_{f(U)\subset V}\, \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

compatible with restriction mappings. We may and do view $b_{p,U}$ as a collection of maps $b_{p,U,V} : \mathcal{G}(V) \to \mathcal{F}(U)$ for all V open in Y with $f(U) \subset V$. These maps have to be compatible with all possible restriction mappings you can think of. In other words, we see that b_p corresponds to a collection of maps as in (4). Of course, conversely such a collection defines a map b_p and in turn a map $b : f^{-1}\mathcal{G} \to \mathcal{F}$.

To finish the proof of the lemma you have to show that by "forgetting structure" the rule that to a collection $\xi_{U,V}$ as in (4) associates the f-map ξ with $\xi_V = \xi_{f^{-1}(V),V}$ is bijective. To do this, if ξ is a usual f-map then we just define $\tilde{\xi}_{U,V}$ to be the composition of $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ by the restruction map $\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$ which makes sense exactly because $f(U) \subset V$, i.e., $U \subset f^{-1}(V)$. This finishes the proof. $\square$

It is sometimes convenient to think about f-maps instead of maps between sheaves either on X or on Y. We define composition of f-maps as follows.

Definition 21.9. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. Suppose that $\mathcal{F}$ is a sheaf on X, $\mathcal{G}$ is a sheaf on Y, and $\mathcal{H}$ is a sheaf on Z. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map. Let $\psi : \mathcal{H} \to \mathcal{G}$ be an g-map. The *composition of* φ *and* ψ is the $(g \circ f)$-map $\varphi \circ \psi$ defined by the commutativity of the diagrams

$$\begin{tikzcd} \mathcal{H}(W) \arrow[rr, "(\varphi\circ\psi)_W"'] \arrow[dr, "\psi_W"'] & & \mathcal{F}(f^{-1}g^{-1}W) \\ & \mathcal{G}(g^{-1}W) \arrow[ur, "\varphi_{g^{-1}W}"'] & \end{tikzcd}$$

We leave it to the reader to verify that this works. Another way to think about this is to think of $\varphi \circ \psi$ as the composition

$$\mathcal{H} \xrightarrow{\psi} g_*\mathcal{G} \xrightarrow{g_*\varphi} g_*f_*\mathcal{F} = (g \circ f)_*\mathcal{F}$$

Now, doesn't it seem that thinking about f-maps is somehow easier?

Finally, given a continuous map $f : X \to Y$, and an f-map $\varphi : \mathcal{G} \to \mathcal{F}$ there is a natural map on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

for all $x \in X$. The image of a representative (V, s) of an element in $\mathcal{G}_{f(x)}$ is mapped to the element in $\mathcal{F}_x$ with representative $(f^{-1}V, \varphi_V(s))$. We leave it to the reader to see that this is well defined. Another way to state it is that it is the unique map such that all diagrams

$$\begin{tikzcd} \mathcal{F}(f^{-1}V) \arrow[r] & \mathcal{F}_x \\ \mathcal{G}(V) \arrow[r] \arrow[u, "\varphi_V"] & \mathcal{G}_{f(x)} \arrow[u, "\varphi_x"'] \end{tikzcd}$$

(for $f(x) \in V \subset Y$ open) commute.

Lemma 21.10. *Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. Suppose that $\mathcal{F}$ is a sheaf on X, $\mathcal{G}$ is a sheaf on Y, and $\mathcal{H}$ is a sheaf on Z. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map. Let $\psi : \mathcal{H} \to \mathcal{G}$ be an g-map. Let $x \in X$ be a point. The map on stalks $(\varphi \circ \psi)_x : \mathcal{H}_{g(f(x))} \to \mathcal{F}_x$ is the composition*

$$\mathcal{H}_{g(f(x))} \xrightarrow{\psi_{f(x)}} \mathcal{G}_{f(x)} \xrightarrow{\varphi_x} \mathcal{F}_x$$

Proof. Immediate from Definition 21.9 and the definition of the map on stalks above. $\square$

22. Continuous maps and abelian sheaves

Let $f : X \to Y$ be a continuous map. We claim there are functors

$$\begin{aligned} f_* : PAb(X) &\longrightarrow PAb(Y) \\ f_* : Ab(X) &\longrightarrow Ab(Y) \\ f_p : PAb(Y) &\longrightarrow PAb(X) \\ f^{-1} : Ab(Y) &\longrightarrow Ab(X) \end{aligned}$$

with similar properties to their counterparts in Section 21. To see this we argue in the following way.

Each of the functors will be constructed in the same way as the corresponding functor in Section 21. This works because all the colimits in that section are directed colimits (but we will work through it below).

First off, given an abelian presheaf $\mathcal{F}$ on X and an abelian presheaf $\mathcal{G}$ on Y we define

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}(V)) \\ f_p\mathcal{G}(U) &= \operatorname{colim}_{f(U) \subset V} \mathcal{G}(V) \end{aligned}$$

as abelian groups. The restriction mappings are the same as the restriction mappings for presheaves of sets (and they are all homomorphisms of abelian groups).

The assignments $\mathcal{F} \mapsto f_*\mathcal{F}$ and $\mathcal{G} \to f_p\mathcal{G}$ are functors on the categories of presheaves of abelian groups. This is clear, as (for example) a map of abelian presheaves $\mathcal{G}_1 \to \mathcal{G}_2$ gives rise to a map of directed systems $\{\mathcal{G}_1(V)\}_{f(U) \subset V} \to \{\mathcal{G}_2(V)\}_{f(U) \subset V}$ all of whose maps are homomorphisms and hence gives rise to a homomorphism of abelian groups $f_p\mathcal{G}_1(U) \to f_p\mathcal{G}_2(U)$.

The functors f_* and f_p are adjoint on the category of presheaves of abelian groups, i.e., we have

$$\operatorname{Mor}_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) = \operatorname{Mor}_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

To prove this, note that the map $i_\mathcal{G} : \mathcal{G} \to f_*f_p\mathcal{G}$ from the proof of Lemma 21.3 is a map of abelian presheaves. Hence if $\psi : f_p\mathcal{G} \to \mathcal{F}$ is a map of abelian presheaves, then the corresponding map $\mathcal{G} \to f_*\mathcal{F}$ is the map $f_*\psi \circ i_\mathcal{G} : \mathcal{G} \to f_*f_p\mathcal{G} \to f_*\mathcal{F}$ is also a map of abelian presheaves. For the other direction we point out that the map $c_\mathcal{F} : f_pf_*\mathcal{F} \to \mathcal{F}$ from the proof of Lemma 21.3 is a map of abelian presheaves as well (since it is made out of restriction mappings of $\mathcal{F}$ which are all homomorphisms). Hence given a map of abelian presheaves $\varphi : \mathcal{G} \to f_*\mathcal{F}$ the map $c_\mathcal{F} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F}$ is a map of abelian presheaves as well. Since these constructions $\psi \mapsto f_*\psi$ and $\varphi \mapsto c_\mathcal{F} \circ f_p\varphi$ are inverse to each other as constructions on maps of presheaves of sets we see they are also inverse to each other on maps of abelian presheaves.

If $\mathcal{F}$ is an abelian sheaf on Y, then $f_*\mathcal{F}$ is an abelian sheaf on X. This is true because of the definition of an abelian sheaf and because this is true for sheaves of sets, see Lemma 21.1. This defines the functor f_* on the category of abelian sheaves.

We define $f^{-1}\mathcal{G} = (f_p\mathcal{G})^\#$ as before. Adjointness of f_* and f^{-1} follows formally as in the case of presheaves of sets. Here is the argument:

$$\begin{aligned} \mathrm{Mor}_{Ab(X)}(f^{-1}\mathcal{G}, \mathcal{F}) & = \mathrm{Mor}_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) \\ & = \mathrm{Mor}_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ & = \mathrm{Mor}_{Ab(Y)}(\mathcal{G}, f_*\mathcal{F}) \end{aligned}$$

Lemma 22.1. *Let $f : X \to Y$ be a continuous map.*

(1) *Let $\mathcal{G}$ be an abelian presheaf on Y. Let $x \in X$. The bijection $\mathcal{G}_{f(x)} \to (f_p\mathcal{G})_x$ of Lemma 21.4 is an isomorphism of abelian groups.*
(2) *Let $\mathcal{G}$ be an abelian sheaf on Y. Let $x \in X$. The bijection $\mathcal{G}_{f(x)} \to (f^{-1}\mathcal{G})_x$ of Lemma 21.5 is an isomorphism of abelian groups.*

Proof. Omitted. □

Given a continuous map $f : X \to Y$ and sheaves of abelian groups $\mathcal{F}$ on X, $\mathcal{G}$ on Y, the notion of an *f-map $\mathcal{G} \to \mathcal{F}$ of sheaves of abelian groups* makes sense. We can just define it exactly as in Definition 21.7 (replacing maps of sets with homomorphisms of abelian groups) or we can simply say that it is the same as a map of abelian sheaves $\mathcal{G} \to f_*\mathcal{F}$. We will use this notion freely in the following. The group of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the groups $\mathrm{Mor}_{Ab(X)}(f^{-1}\mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}_{Ab(Y)}(\mathcal{G}, f_*\mathcal{F})$.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, the induced maps on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

are abelian group homomorphisms.

23. Continuous maps and sheaves of algebraic structures

Let $(\mathcal{C}, F)$ be a type of algebraic structure. For a topological space X let us introduce the notation:

(1) $\mathit{PSh}(X, \mathcal{C})$ will be the category of presheaves with values in $\mathcal{C}$.
(2) $\mathit{Sh}(X, \mathcal{C})$ will be the category of sheaves with values in $\mathcal{C}$.

Let $f : X \to Y$ be a continuous map of topological spaces. The same arguments as in the previous section show there are functors

$$\begin{aligned} f_* : \mathit{PSh}(X, \mathcal{C}) & \longrightarrow \mathit{PSh}(Y, \mathcal{C}) \\ f_* : \mathit{Sh}(X, \mathcal{C}) & \longrightarrow \mathit{Sh}(Y, \mathcal{C}) \\ f_p : \mathit{PSh}(Y, \mathcal{C}) & \longrightarrow \mathit{PSh}(X, \mathcal{C}) \\ f^{-1} : \mathit{Sh}(Y, \mathcal{C}) & \longrightarrow \mathit{Sh}(X, \mathcal{C}) \end{aligned}$$

constructed in the same manner and with the same properties as the functors constructed for abelian (pre)sheaves. In particular there are commutative diagrams

$$\begin{tikzcd} PSh(X,\mathcal{C}) \arrow[r, "f_*"] \arrow[d, "F"] & PSh(Y,\mathcal{C}) \arrow[d, "F"] \\ PSh(X) \arrow[r, "f_*"] & PSh(Y) \end{tikzcd} \qquad \begin{tikzcd} Sh(X,\mathcal{C}) \arrow[r, "f_*"] \arrow[d, "F"] & Sh(Y,\mathcal{C}) \arrow[d, "F"] \\ Sh(X) \arrow[r, "f_*"] & Sh(Y) \end{tikzcd}$$

$$\begin{tikzcd} PSh(Y,\mathcal{C}) \arrow[r, "f_p"] \arrow[d, "F"] & PSh(X,\mathcal{C}) \arrow[d, "F"] \\ PSh(Y) \arrow[r, "f_p"] & PSh(X) \end{tikzcd} \qquad \begin{tikzcd} Sh(Y,\mathcal{C}) \arrow[r, "f^{-1}"] \arrow[d, "F"] & Sh(X,\mathcal{C}) \arrow[d, "F"] \\ Sh(Y) \arrow[r, "f^{-1}"] & Sh(X) \end{tikzcd}$$

The main formulas to keep in mind are the following

$$\begin{array}{rcl} f_*\mathcal{F}(V) & = & \mathcal{F}(f^{-1}(V)) \\ f_p\mathcal{G}(U) & = & \mathrm{colim}_{f(U)\subset V}\, \mathcal{G}(V) \\ f^{-1}\mathcal{G} & = & (f_p\mathcal{G})^\# \\ (f_p\mathcal{G})_x & = & \mathcal{G}_{f(x)} \\ (f^{-1}\mathcal{G})_x & = & \mathcal{G}_{f(x)} \end{array}$$

Each of these formulas has the property that they hold in the category $\mathcal{C}$ and that upon taking underlying sets we get the corresponding formula for presheaves of sets. In addition we have the adjointness properties

$$\begin{array}{rcl} \mathrm{Mor}_{PSh(X,\mathcal{C})}(f_p\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{PSh(Y,\mathcal{C})}(\mathcal{G}, f_*\mathcal{F}) \\ \mathrm{Mor}_{Sh(X,\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{Sh(Y,\mathcal{C})}(\mathcal{G}, f_*\mathcal{F}). \end{array}$$

To prove these, the main step is to construct the maps

$$i_\mathcal{G} : \mathcal{G} \longrightarrow f_*f_p\mathcal{G}$$

and

$$c_\mathcal{F} : f_pf_*\mathcal{F} \longrightarrow \mathcal{F}$$

which occur in the proof of Lemma 21.3 as morphisms of presheaves with values in $\mathcal{C}$. This may be safely left to the reader since the constructions are exactly the same as in the case of presheaves of sets.

Given a continuous map $f : X \to Y$ and sheaves of algebraic structures $\mathcal{F}$ on X, $\mathcal{G}$ on Y, the notion of an *f-map $\mathcal{G} \to \mathcal{F}$ of sheaves of algebraic structures* makes sense. We can just define it exactly as in Definition 21.7 (replacing maps of sets with morphisms in $\mathcal{C}$) or we can simply say that it is the same as a map of sheaves of algebraic structures $\mathcal{G} \to f_*\mathcal{F}$. We will use this notion freely in the following. The set of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the sets $\mathrm{Mor}_{Sh(X,\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}_{Sh(Y,\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, the induced maps on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

are homomorphisms of algebraic structures.

Lemma 23.1. *Let $f : X \to Y$ be a continuous map of topological spaces. Suppose given sheaves of algebraic structures $\mathcal{F}$ on X, $\mathcal{G}$ on Y. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map of underlying sheaves of sets. If for every $V \subset Y$ open the map of sets $\varphi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$ is the effect of a morphism in $\mathcal{C}$ on underlying sets, then φ comes from a unique f-morphism between sheaves of algebraic structures.*

Proof. Omitted. $\square$

24. Continuous maps and sheaves of modules

The case of sheaves of modules is more complicated. The reason is that the natural setting for defining the pullback and pushforward functors, is the setting of ringed spaces, which we will define below. First we state a few obvious lemmas.

Lemma 24.1. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a presheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Let $V \subset Y$ is open. We define the map of the lemma to be the map

$$f_*\mathcal{O}(V) \times f_*\mathcal{F}(V) = \mathcal{O}(f^{-1}V) \times \mathcal{F}(f^{-1}V) \to \mathcal{F}(f^{-1}V) = f_*\mathcal{F}(V).$$

Here the arrow in the middle is the multiplication map on X. We leave it to the reader to see this is compatible with restriction mappings and defines a structure of $f_*\mathcal{O}$-module on $f_*\mathcal{F}$. $\square$

Lemma 24.2. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f_p\mathcal{O} \times f_p\mathcal{G} \longrightarrow f_p\mathcal{G}$$

which turns $f_p\mathcal{G}$ into a presheaf of $f_p\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Let $U \subset X$ is open. We define the map of the lemma to be the map

$$\begin{aligned} f_p\mathcal{O}(U) \times f_p\mathcal{G}(U) &= \operatorname{colim}_{f(U)\subset V} \mathcal{O}(V) \times \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \\ &= \operatorname{colim}_{f(U)\subset V} (\mathcal{O}(V) \times \mathcal{G}(V)) \\ &\to \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \\ &= f_p\mathcal{G}(U). \end{aligned}$$

Here the arrow in the middle is the multiplication map on Y. The second equality holds because directed colimits commute with finite limits, see Categories, Lemma 19.2. We leave it to the reader to see this is compatible with restriction mappings and defines a structure of $f_p\mathcal{O}$-module on $f_p\mathcal{G}$. $\square$

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a presheaf of rings on X and let $\mathcal{O}_Y$ be a presheaf of rings on Y. So at the moment we have defined functors

$$\begin{array}{rcl} f_* : PMod(\mathcal{O}_X) & \longrightarrow & PMod(f_*\mathcal{O}_X) \\ f_p : PMod(\mathcal{O}_Y) & \longrightarrow & PMod(f_p\mathcal{O}_Y) \end{array}$$

These satisfy some compatibilities as follows.

Lemma 24.3. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a presheaf of $f_p\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{PMod(f_p\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 24.2 and 24.1, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module via the map $i_\mathcal{O} : \mathcal{O} \to f_*f_p\mathcal{O}$ (defined first in the proof of Lemma 21.3).

Proof. Note that we have

$$\mathrm{Mor}_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

according to Section 22. So what we have to prove is that under this correspondence, the subsets of module maps correspond. In addition, the correspondence is determined by the rule

$$(\psi : f_p\mathcal{G} \to \mathcal{F}) \longmapsto (f_*\psi \circ i_\mathcal{G} : \mathcal{G} \to f_*\mathcal{F})$$

and in the other direction by the rule

$$(\varphi : \mathcal{G} \to f_*\mathcal{F}) \longmapsto (c_\mathcal{F} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F})$$

where $i_\mathcal{G}$ and $c_\mathcal{F}$ are as in Section 22. Hence, using the functoriality of f_* and f_p we see that it suffices to check that the maps $i_\mathcal{G} : \mathcal{G} \to f_*f_p\mathcal{G}$ and $c_\mathcal{F} : f_pf_*\mathcal{F} \to \mathcal{F}$ are compatible with module structures, which we leave to the reader. $\square$

Lemma 24.4. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a presheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{O} \otimes_{p, f_pf_*\mathcal{O}} f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PMod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 24.2 and 24.1, and we use the map $c_\mathcal{O} : f_pf_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{array}{rcl} \mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{O} \otimes_{p, f_pf_*\mathcal{O}} f_p\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{PMod(f_pf_*\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}_{f_pf_*\mathcal{O}}) \\ & = & \mathrm{Mor}_{PMod(f_*\mathcal{O})}(\mathcal{G}, f_*(\mathcal{F}_{f_pf_*\mathcal{O}})) \\ & = & \mathrm{Mor}_{PMod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}). \end{array}$$

The first equality is Lemma 6.2. The second equality is Lemma 24.3. The third equality is given by the equality $f_*(\mathcal{F}_{f_pf_*\mathcal{O}}) = f_*\mathcal{F}$ of abelian sheaves which is $f_*\mathcal{O}$-linear. Namely, $\mathrm{id}_{f_*\mathcal{O}}$ corresponds to $c_\mathcal{O}$ under the adjunction described in the proof of Lemma 21.3 and thus $\mathrm{id}_{f_*\mathcal{O}} = f_*c_\mathcal{O} \circ i_{f_*\mathcal{O}}$. $\square$

Lemma 24.5. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. The pushforward $f_*\mathcal{F}$, as defined in Lemma 24.1 is a sheaf of $f_*\mathcal{O}$-modules.*

Proof. Obvious from the definition and Lemma 21.1. $\square$

Lemma 24.6. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules.

Proof. Recall that f^{-1} is defined as the composition of the functor f_p and sheafification. Thus the lemma is a combination of Lemma 24.2 and Lemma 20.1. ☐

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a sheaf of rings on X and let $\mathcal{O}_Y$ be a sheaf of rings on Y. So now we have defined functors

$$\begin{array}{rcl} f_* : Mod(\mathcal{O}_X) & \longrightarrow & Mod(f_*\mathcal{O}_X) \\ f^{-1} : Mod(\mathcal{O}_Y) & \longrightarrow & Mod(f^{-1}\mathcal{O}_Y) \end{array}$$

These satisfy some compatibilities as follows.

Lemma 24.7. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 24.6 and 24.5, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O} \to f_*f^{-1}\mathcal{O}$.

Proof. Argue by the equalities

$$\begin{array}{rcl} \mathrm{Mor}_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{Mod(f_p\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}) \\ & = & \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}). \end{array}$$

where the second is Lemmas 24.3 and the first is by Lemma 20.1. ☐

Lemma 24.8. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 24.6 and 24.5, and we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{array}{rcl} \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{Mod(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}}) \\ & = & \mathrm{Mor}_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}). \end{array}$$

which are a combination of Lemma 20.2 and 24.7. ☐

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a (pre)sheaf of rings on X and let $\mathcal{O}_Y$ be a (pre)sheaf of rings on Y. So at the moment we have defined functors

$$\begin{array}{rcl} f_* : PMod(\mathcal{O}_X) & \longrightarrow & PMod(f_*\mathcal{O}_X) \\ f_* : Mod(\mathcal{O}_X) & \longrightarrow & Mod(f_*\mathcal{O}_X) \\ f_p : PMod(\mathcal{O}_Y) & \longrightarrow & PMod(f_p\mathcal{O}_Y) \\ f^{-1} : Mod(\mathcal{O}_Y) & \longrightarrow & Mod(f^{-1}\mathcal{O}_Y) \end{array}$$

Clearly, usually the pair of functors (f_*, f^{-1}) on sheaves of modules are not adjoint, because their target categories do not match. Namely, as we saw above, it works only if by some miracle the sheaves of rings $\mathcal{O}_X, \mathcal{O}_Y$ satisfy the relations $\mathcal{O}_X = f^{-1}\mathcal{O}_Y$ and $\mathcal{O}_Y = f_*\mathcal{O}_X$. This is almost never true in practice. We interrupt the discussion to define the correct notion of morphism for which a suitable adjoint pair of functors on sheaves of modules exists.

25. Ringed spaces

Let X be a topological space and let $\mathcal{O}_X$ be a sheaf of rings on X. We are supposed to think of the sheaf of rings $\mathcal{O}_X$ as a sheaf of functions on X. And if $f : X \to Y$ is a "suitable" map, then by composition a function on Y turns into a function on X. Thus there should be a natural f-map from $\mathcal{O}_Y$ to $\mathcal{O}_X$, see Definition 21.7 and Lemma 21.8. For a precise example, see Example 25.2 below. Here is the relevant abstract definition.

Definition 25.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair consisting of a continuous map $f : X \to Y$ and an f-map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$.

Example 25.2. Let $f : X \to Y$ be a continuous map of topological spaces. Consider the sheaves of continuous real valued functions $\mathcal{C}^0_X$ on X and $\mathcal{C}^0_Y$ on Y, see Example 9.3. We claim that there is a natural f-map $f^\sharp : \mathcal{C}^0_Y \to \mathcal{C}^0_X$ associated to f. Namely, we simply define it by the rule

$$\begin{array}{ccc} \mathcal{C}^0_Y(V) & \longrightarrow & \mathcal{C}^0_X(f^{-1}V) \\ h & \longmapsto & h \circ f \end{array}$$

Strictly speaking we should write $f^\sharp(h) = h \circ f|_{f^{-1}(V)}$. It is clear that this is a family of maps as in Definition 21.7 and compatible with the $\mathbf{R}$-algebra structures. Hence it is an f-map of sheaves of $\mathbf{R}$-algebras, see Lemma 23.1.

Of course there are lots of other situations where there is a canonical morphism of ringed spaces associated to a geometrical type of morphism. For example, if M, N are $\mathcal{C}^\infty$-manifolds and $f : M \to N$ is a infinitely differentiable map, then f induces a canonical morphism of ringed spaces $(M, \mathcal{C}^\infty_M) \to (N, \mathcal{C}^\infty_N)$. The construction (which is identical to the above) is left to the reader.

It may not be completely obvious how to compose morphisms of ringed spaces hence we spell it out here.

Definition 25.3. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\sharp) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. Then we define the *composition of morphisms of ringed spaces* by the rule

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Here we use composition of f-maps defined in Definition 21.9.

26. Morphisms of ringed spaces and modules

We have now introduced enough notation so that we are able to define the pullback and pushforward of modules along a morphism of ringed spaces.

Definition 26.1. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the *pushforward* of $\mathcal{F}$ as the sheaf of $\mathcal{O}_Y$-modules which as a sheaf of abelian groups equals $f_*\mathcal{F}$ and with module structure given by the restriction via $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of the module structure given in Lemma 24.5.
(2) Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. We define the *pullback* $f^*\mathcal{G}$ to be the sheaf of $\mathcal{O}_X$-modules defined by the formula
$$f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$$
where the ring map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is the map corresponding to $f^\sharp$, and where the module structure is given by Lemma 24.6.

Thus we have defined functors
$$\begin{array}{rcl} f_* : \mathit{Mod}(\mathcal{O}_X) & \longrightarrow & \mathit{Mod}(\mathcal{O}_Y) \\ f^* : \mathit{Mod}(\mathcal{O}_Y) & \longrightarrow & \mathit{Mod}(\mathcal{O}_X) \end{array}$$
The final result on these functors is that they are indeed adjoint as expected.

Lemma 26.2. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical bijection*
$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$
In other words: the functor f^* is the left adjoint to f_*.

Proof. This follows from the work we did before:
$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{\mathit{Mod}(\mathcal{O}_X)}(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}, \mathcal{F}) \\ & = & \mathrm{Mor}_{\mathit{Mod}(f^{-1}\mathcal{O}_Y)}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}\mathcal{O}_Y}) \\ & = & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}). \end{array}$$
Here we use Lemmas 20.2 and 24.7. □

Lemma 26.3. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. The functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal. There is a canonical isomorphism of functors $(g \circ f)^* \cong f^* \circ g^*$.*

Proof. The result on pushforwards is a consequence of Lemma 21.2 and our definitions. The result on pullbacks follows from this by the same argument as in the proof of Lemma 21.6. □

Given a morphism of ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, and a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, a sheaf of $\mathcal{O}_Y$-modules $\mathcal{G}$ on Y, the notion of an *f-map $\varphi : \mathcal{G} \to \mathcal{F}$ of sheaves of modules* makes sense. We can just define it as an f-map $\varphi : \mathcal{G} \to \mathcal{F}$ of abelian sheaves (see Definition 21.7 and Lemma 21.8) such that for all open $V \subset Y$ the map
$$\mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$
is an $\mathcal{O}_Y(V)$-module map. Here we think of $\mathcal{F}(f^{-1}V)$ as an $\mathcal{O}_Y(V)$-module via the map $f^\sharp_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$. The set of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the sets $\mathrm{Mor}_{\mathit{Mod}(\mathcal{O}_X)}(f^*\mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}_{\mathit{Mod}(\mathcal{O}_Y)}(\mathcal{G}, f_*\mathcal{F})$. See above.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, and $x \in X$ the induced map on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

is an $\mathcal{O}_{Y,f(x)}$-module map where the $\mathcal{O}_{Y,f(x)}$-module structure on $\mathcal{F}_x$ comes from the $\mathcal{O}_{X,x}$-module structure via the map $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Here is a related lemma.

Lemma 26.4. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $x \in X$. Then*

$$(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$$

as $\mathcal{O}_{X,x}$-modules where the tensor product on the right uses $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$.

Proof. This follows from Lemma 20.3 and the identification of the stalks of pullback sheaves at x with the corresponding stalks at $f(x)$. See the formulae in Section 23 for example. ☐

27. Skyscraper sheaves and stalks

Definition 27.1. Let X be a topological space.

(1) Let $x \in X$ be a point. Denote $i_x : \{x\} \to X$ the inclusion map. Let A be a set and think of A as a sheaf on the one point space $\{x\}$. We call $i_{x,*}A$ the *skyscraper sheaf at x with value A*.
(2) If in (1) above A is an abelian group then we think of $i_{x,*}A$ as a sheaf of abelian groups on X.
(3) If in (1) above A is an algebraic structure then we think of $i_{x,*}A$ as a sheaf of algebraic structures.
(4) If $(X, \mathcal{O}_X)$ is a ringed space, then we think of $i_x : \{x\} \to X$ as a morphism of ringed spaces $(\{x\}, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$ and if A is a $\mathcal{O}_{X,x}$-module, then we think of $i_{x,*}A$ as a sheaf of $\mathcal{O}_X$-modules.
(5) We say a sheaf of sets $\mathcal{F}$ is a *skyscraper sheaf* if there exists a point x of X and a set A such that $\mathcal{F} \cong i_{x,*}A$.
(6) We say a sheaf of abelian groups $\mathcal{F}$ is a *skyscraper sheaf* if there exists a point x of X and an abelian group A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of abelian groups.
(7) We say a sheaf of algebraic structures $\mathcal{F}$ is a *skyscraper sheaf* if there exists a point x of X and an algebraic structure A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of algebraic structures.
(8) If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then we say $\mathcal{F}$ is a *skyscraper sheaf* if there exists a point $x \in X$ and a $\mathcal{O}_{X,x}$-module A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of $\mathcal{O}_X$-modules.

Lemma 27.2. *Let X be a topological space, $x \in X$ a point, and A a set. For any point $x' \in X$ the stalk of the skyscraper sheaf at x with value A at x' is*

$$(i_{x,*}A)_{x'} = \begin{cases} A & \text{if} \quad x' \in \overline{\{x\}} \\ \{*\} & \text{if} \quad x' \notin \overline{\{x\}} \end{cases}$$

A similar description holds for the case of abelian groups, algebraic structures and sheaves of modules.

Proof. Omitted. □

Lemma 27.3. *Let X be a topological space, and let $x \in X$ a point. The functors $\mathcal{F} \mapsto \mathcal{F}_x$ and $A \mapsto i_{x,*}A$ are adjoint. In a formula*

$$\mathrm{Mor}_{Sets}(\mathcal{F}_x, A) = \mathrm{Mor}_{Sh(X)}(\mathcal{F}, i_{x,*}A).$$

A similar statement holds for the case of abelian groups, algebraic structures. In the case of sheaves of modules we have

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, A) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_{x,*}A).$$

Proof. Omitted. Hint: The stalk functor can be seen as the pullback functor for the morphism $i_x : \{x\} \to X$. Then the adjointness follows from adjointness of i_x^{-1} and $i_{x,*}$ (resp. i_x^* and $i_{x,*}$ in the case of sheaves of modules). □

28. Limits and colimits of presheaves

Let X be a topological space. Let $\mathcal{I} \to PSh(X)$, $i \mapsto \mathcal{F}_i$ be a diagram.

(1) Both $\lim_i \mathcal{F}_i$ and $\mathrm{colim}_i \mathcal{F}_i$ exist.
(2) For any open $U \subset X$ we have
$$(\lim_i \mathcal{F}_i)(U) = \lim_i \mathcal{F}_i(U)$$
and
$$(\mathrm{colim}_i \mathcal{F}_i)(U) = \mathrm{colim}_i \mathcal{F}_i(U).$$
(3) Let $x \in X$ be a point. In general the stalk of $\lim_i \mathcal{F}_i$ at x is not equal to the limit of the stalks. But if the index category is finite then it is the case. In other words, the stalk functor is left exact (see Categories, Definition 23.1).
(4) Let $x \in X$. We always have
$$(\mathrm{colim}_i \mathcal{F}_i)_x = \mathrm{colim}_i \mathcal{F}_{i,x}.$$

The proofs are all easy.

29. Limits and colimits of sheaves

Let X be a topological space. Let $\mathcal{I} \to Sh(X)$, $i \mapsto \mathcal{F}_i$ be a diagram.

(1) Both $\lim_i \mathcal{F}_i$ and $\mathrm{colim}_i \mathcal{F}_i$ exist.
(2) The inclusion functor $i : Sh(X) \to PSh(X)$ commutes with limits. In other words, we may compute the limit in the category of sheaves as the limit in the category of presheaves. In particular, for any open $U \subset X$ we have
$$(\lim_i \mathcal{F}_i)(U) = \lim_i \mathcal{F}_i(U).$$
(3) The inclusion functor $i : Sh(X) \to PSh(X)$ does not commute with colimits in general (not even with finite colimits – think surjections). The colimit is computed as the sheafification of the colimit in the category of presheaves:
$$\mathrm{colim}_i \mathcal{F}_i = \Big(U \mapsto \mathrm{colim}_i \mathcal{F}_i(U)\Big)^\#.$$
(4) Let $x \in X$ be a point. In general the stalk of $\lim_i \mathcal{F}_i$ at x is not equal to the limit of the stalks. But if the index category is finite then it is the case. In other words, the stalk functor is left exact.
(5) Let $x \in X$. We always have
$$(\mathrm{colim}_i \mathcal{F}_i)_x = \mathrm{colim}_i \mathcal{F}_{i,x}.$$

(6) The sheafification functor $^\#: PSh(X) \to Sh(X)$ commutes with all colimits, and with finite limits. But it does not commute with all limits.

The proofs are all easy. Here is an example of what is true for directed colimits of sheaves.

Lemma 29.1. *Let X be a topological space. Let I be a directed set. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a system of sheaves of sets over I, see Categories, Section 21. Let $U \subset X$ be an open subset. Consider the canonical map*

$$\Psi : \operatorname{colim}_i \mathcal{F}_i(U) \longrightarrow (\operatorname{colim}_i \mathcal{F}_i)(U)$$

1. *If all the transition maps are injective then Ψ is injective for any open U.*
2. *If U is quasi-compact, then Ψ is injective.*
3. *If U is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.*
4. *If U has a cofinal system of open coverings $\mathcal{U} : U = \bigcup_{j\in J} U_j$ with J finite and $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$, then Ψ is bijective.*

Proof. Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto \operatorname{colim}_i \mathcal{F}_i(V)$ is separated (see Definition 11.2). By the discussion above we have $(\mathcal{F}')^\# = \operatorname{colim}_i \mathcal{F}_i$. By Lemma 17.5 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Assume U is quasi-compact. Suppose that $s \in \mathcal{F}_i(U)$ and $s' \in \mathcal{F}_{i'}(U)$ give rise to elements on the left hand side which have the same image under Ψ. Since U is quasi-compact this means there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$ and for each j an index $i_j \in I$, $i_j \geq i$, $i_j \geq i'$ such that $\varphi_{ii_j}(s) = \varphi_{i'i_j}(s')$. Let $i'' \in I$ be $\geq$ than all of the i_j. We conclude that $\varphi_{ii''}(s)$ and $\varphi_{i'i''}(s)$ agree on the opens U_j for all j and hence that $\varphi_{ii''}(s) = \varphi_{i'i''}(s)$. This proves (2).

Assume U is quasi-compact and all transition maps injective. Let s be an element of the target of Ψ. Since U is quasi-compact there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$, for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ comes from s_j for all j. Pick $i \in I$ which is $\geq$ than all of the i_j. By (1) the sections $\varphi_{i_j i}(s_j)$ agree over the overlaps $U_j \cap U_{j'}$. Hence they glue to a section $s' \in \mathcal{F}_i(U)$ which maps to s under Ψ. This proves (3).

Assume the hypothesis of (4). In particular we see that U is quasi-compact and hence by (2) we have injectivity of Ψ. Let s be an element of the target of Ψ. By assumption there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$, with $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$ and for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ is the image of s_j for all j. Since $U_j \cap U_{j'}$ is quasi-compact we can apply (2) and we see that there exists an $i_{jj'} \in I$, $i_{jj'} \geq i_j$, $i_{jj'} \geq i_{j'}$ such that $\varphi_{i_j i_{jj'}}(s_j)$ and $\varphi_{i_{j'} i_{jj'}}(s_{j'})$ agree over $U_j \cap U_{j'}$. Choose an index $i \in I$ wich is bigger or equal than all the $i_{jj'}$. Then we see that the sections $\varphi_{i_j i}(s_j)$ of $\mathcal{F}_i$ glue to a section of $\mathcal{F}_i$ over U. This section is mapped to the element s as desired. □

Example 29.2. Let $X = \{s_1, s_2, \xi_1, \xi_2, \xi_3, \ldots\}$ as a set. Declare a subset $U \subset X$ to be open if $s_1 \in U$ or $s_2 \in U$ implies U contains all of the ξ_i. Let $U_n = \{\xi_n, \xi_{n+1}, \ldots\}$, and let $j_n : U_n \to X$ be the inclusion map. Set $\mathcal{F}_n = j_{n,*}\underline{\mathbf{Z}}$. There are transition maps $\mathcal{F}_n \to \mathcal{F}_{n+1}$. Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_n$. Note that $\mathcal{F}_{n,\xi_m} = 0$ if $m < n$ because $\{\xi_m\}$

is an open subset of X which misses U_n. Hence we see that $\mathcal{F}_{\xi_n} = 0$ for all n. On the other hand the stalk $\mathcal{F}_{s_i}$, $i = 1, 2$ is the colimit

$$M = \text{colim}_n \prod\nolimits_{m \geq n} \mathbf{Z}$$

which is not zero. We conclude that the sheaf $\mathcal{F}$ is the direct sum of the skyscraper sheaves with value M at the closed points s_1 and s_2. Hence $\Gamma(X, \mathcal{F}) = M \oplus M$. On the other hand, the reader can verify that $\text{colim}_n \Gamma(X, \mathcal{F}_n) = M$. Hence some condition is necessary in part (4) of Lemma 29.1 above.

There is a version of the previous lemma dealing with sheaves on a diagram of spectral spaces. To state it we introduce some notation. Let $\mathcal{I}$ be a cofiltered index category. Let $i \mapsto X_i$ be a diagram of spectral spaces over $\mathcal{I}$ such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Set $X = \lim X_i$ and denote $p_i : X \to X_i$ the projection.

Lemma 29.3. *In the situation described above, let $i \in \text{Ob}(\mathcal{I})$ and let $\mathcal{G}$ be a sheaf on X_i. For $U_i \subset X_i$ quasi-compact open we have*

$$p_i^{-1}\mathcal{G}(p_i^{-1}(U_i)) = \text{colim}_{a:j\to i} f_a^{-1}\mathcal{G}(f_a^{-1}(U_i))$$

Proof. Let us prove the canonical map $\text{colim}_{a:j\to i} f_a^{-1}\mathcal{G}(f_a^{-1}(U_i)) \to p_i^{-1}\mathcal{G}(p_i^{-1}(U_i))$ is injective. Let s, s' be sections of $f_a^{-1}\mathcal{G}$ over $f_a^{-1}(U_i)$ for some $a : j \to i$. For $b : k \to j$ let $Z_k \subset f_{a\circ b}^{-1}(U_i)$ be the closed subset of points x such that the image of s and s' in the stalk $(f_{a\circ b}^{-1}\mathcal{G})_x$ are different. If Z_k is nonempty for all $b : k \to j$, then by Topology, Lemma 24.2 we see that $\lim_{b:k\to j} Z_k$ is nonempty too. Then for $x \in \lim_{b:k\to j} Z_k \subset X$ (observe that $\mathcal{I}/j \to \mathcal{I}$ is initial) we see that the image of s and s' in the stalk of $p_i^{-1}\mathcal{G}$ at x are different too since $(p_i^{-1}\mathcal{G})_x = (f_{b\circ a}^{-1}\mathcal{G})_{p_k(x)}$ for all $b : k \to j$ as above. Thus if the images of s and s' in $p_i^{-1}\mathcal{G}(p_i^{-1}(U_i))$ are the same, then Z_k is empty for some $b : k \to j$. This proves injectivity.

Surjectivity. Let s be a section of $p_i^{-1}\mathcal{G}$ over $p_i^{-1}(U_i)$. By Topology, Lemma 24.5 the set $p_i^{-1}(U_i)$ is a quasi-compact open of the spectral space X. By construction of the pullback sheaf, we can find an open covering $p_i^{-1}(U_i) = \bigcup_{l\in L} W_l$, opens $V_{l,i} \subset X_i$, sections $s_{l,i} \in \mathcal{G}(V_{l,i})$ such that $p_i(W_l) \subset V_{l,i}$ and $p_i^{-1}s_{l,i}|_{W_l} = s|_{W_l}$. Because X and X_i are spectral and $p_i^{-1}(U_i)$ is quasi-compact open, we may assume L is finite and W_l and $V_{l,i}$ quasi-compact open for all l. Then we can apply Topology, Lemma 24.6 to find $a : j \to i$ and open covering $f_a^{-1}(U_i) = \bigcup_{l\in L} W_{l,j}$ by quasi-compact opens whose pullback to X is the covering $p_i^{-1}(U_i) = \bigcup_{l\in L} W_l$ and such that moreover $W_{l,j} \subset f_a^{-1}(V_{l,i})$. Write $s_{l,j}$ the restriction of the pullback of $s_{l,i}$ by f_a to $W_{l,j}$. Then we see that $s_{l,j}$ and $s_{l',j}$ restrict to elements of $(f_a^{-1}\mathcal{G})(W_{l,j} \cap W_{l',j})$ which pullback to the same element $(p_i^{-1}\mathcal{G})(W_l \cap W_{l'})$, namely, the restriction of s. Hence by injectivity, we can find $b : k \to j$ such that the sections $f_b^{-1}s_{l,j}$ glue to a section over $f_{a\circ b}^{-1}(U_i)$ as desired. $\square$

Next, in addition to the cofiltered system X_i of spectral spaces, assume given

(1) a sheaf $\mathcal{F}_i$ on X_i for all $i \in \text{Ob}(\mathcal{I})$,
(2) for $a : j \to i$ an f_a-map $\varphi_a : \mathcal{F}_i \to \mathcal{F}_j$

such that $\varphi_c = \varphi_b \circ \varphi_a$ whenever $c = a \circ b$. Set $\mathcal{F} = \text{colim}\, p_i^{-1}\mathcal{F}_i$ on X.

Lemma 29.4. *In the situation described above, let $i \in \mathrm{Ob}(\mathcal{I})$ and let $U_i \subset X_i$ be a quasi-compact open. Then*
$$\mathrm{colim}_{a:j\to i}\, \mathcal{F}_j(f_a^{-1}(U_i)) = \mathcal{F}(p_i^{-1}(U_i))$$

Proof. Recall that $p_i^{-1}(U_i)$ is a quasi-compact open of the spectral space X, see Topology, Lemma 24.5. Hence Lemma 29.1 applies and we have
$$\mathcal{F}(p_i^{-1}(U_i)) = \mathrm{colim}_{a:j\to i}\, p_j^{-1}\mathcal{F}_j(p_i^{-1}(U_i)).$$
A formal argument shows that
$$\mathrm{colim}_{a:j\to i}\, \mathcal{F}_j(f_a^{-1}(U_i)) = \mathrm{colim}_{a:j\to i}\, \mathrm{colim}_{b:k\to j}\, f_b^{-1}\mathcal{F}_j(f_{a\circ b}^{-1}(U_i))$$
Thus it suffices to show that
$$p_j^{-1}\mathcal{F}_j(p_i^{-1}(U_i)) = \mathrm{colim}_{b:k\to j}\, f_b^{-1}\mathcal{F}_j(f_{a\circ b}^{-1}(U_i))$$
This is Lemma 29.3 applied to $\mathcal{F}_j$ and the quasi-compact open $f_a^{-1}(U_i)$. □

30. Bases and sheaves

Sometimes there exists a basis for the topology consisting of opens that are easier to work with than general opens. For convenience we give here some definitions and simple lemmas in order to facilitate working with (pre)sheaves in such a situation.

Definition 30.1. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X.

1. A *presheaf $\mathcal{F}$ of sets on $\mathcal{B}$* is a rule which assigns to each $U \in \mathcal{B}$ a set $\mathcal{F}(U)$ and to each inclusion $V \subset U$ of elements of $\mathcal{B}$ a map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho^U_U = \mathrm{id}_{\mathcal{F}(U)}$ for all $U \in \mathcal{B}$ whenever $W \subset V \subset U$ in $\mathcal{B}$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
2. A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on $\mathcal{B}$* is a rule which assigns to each element $U \in \mathcal{B}$ a map of sets $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps.

As in the case of usual presheaves we use the terminology of sections, restrictions of sections, etc. In particular, we may define the *stalk* of $\mathcal{F}$ at a point $x \in X$ by the colimit
$$\mathcal{F}_x = \mathrm{colim}_{U\in\mathcal{B}, x\in U}\, \mathcal{F}(U).$$
As in the case of the stalk of a presheaf on X this limit is directed. The reason is that the collection of $U \in \mathcal{B}$, $x \in U$ is a fundamental system of open neighbourhoods of x.

It is easy to make examples to show that the notion of a presheaf on X is very different from the notion of a presheaf on a basis for the topology on X. This does not happen in the case of sheaves. A much more useful notion therefore, is the following.

Definition 30.2. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X.

1. A *sheaf $\mathcal{F}$ of sets on $\mathcal{B}$* is a presheaf of sets on $\mathcal{B}$ which satisfies the following additional property: Given any $U \in \mathcal{B}$, and any covering $U = \bigcup_{i\in I} U_i$ with $U_i \in \mathcal{B}$, and any coverings $U_i \cap U_j = \bigcup_{k\in I_{ij}} U_{ijk}$ with $U_{ijk} \in \mathcal{B}$ the sheaf condition holds:

(**) For any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$, $\forall k \in I_{ij}$
$$s_i|_{U_{ijk}} = s_j|_{U_{ijk}}$$
there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

(2) A *morphism of sheaves of sets on $\mathcal{B}$* is simply a morphism of presheaves of sets.

First we explain that it suffices to check the sheaf condition $(**)$ on a cofinal system of coverings. In the situation of the definition, suppose $U \in \mathcal{B}$. Let us temporarily denote $\mathrm{Cov}_{\mathcal{B}}(U)$ the set of all coverings of U by elements of $\mathcal{B}$. Note that $\mathrm{Cov}_{\mathcal{B}}(U)$ is preordered by refinement. A subset $C \subset \mathrm{Cov}_{\mathcal{B}}(U)$ is a cofinal system, if for every $\mathcal{U} \in \mathrm{Cov}_{\mathcal{B}}(U)$ there exists a covering $\mathcal{V} \in C$ which refines $\mathcal{U}$.

Lemma 30.3. *With notation as above. For each $U \in \mathcal{B}$, let $C(U) \subset Cov_{\mathcal{B}}(U)$ be a cofinal system. For each $U \in \mathcal{B}$, and each $\mathcal{U} : U = \bigcup U_i$ in $C(U)$, let coverings $\mathcal{U}_{ij} : U_i \cap U_j = \bigcup U_{ijk}$, $U_{ijk} \in \mathcal{B}$ be given. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent*

(1) *The presheaf $\mathcal{F}$ is a sheaf on $\mathcal{B}$.*
(2) *For every $U \in \mathcal{B}$ and every covering $\mathcal{U} : U = \bigcup U_i$ in $C(U)$ the sheaf condition $(**)$ holds (for the given coverings $\mathcal{U}_{ij}$).*

Proof. We have to show that (2) implies (1). Suppose that $U \in \mathcal{B}$, and that $\mathcal{U} : U = \bigcup_{i \in I} U_i$ is an arbitrary covering by elements of $\mathcal{B}$. Because the system $C(U)$ is cofinal we can find an element $\mathcal{V} : U = \bigcup_{j \in J} V_j$ in $C(U)$ which refines $\mathcal{U}$. This means there exists a map $\alpha : J \to I$ such that $V_j \subset U_{\alpha(j)}$.

Note that if $s, s' \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = s'|_{U_i}$, then
$$s|_{V_j} = (s|_{U_{\alpha(j)}})|_{V_j} = (s'|_{U_{\alpha(j)}})|_{V_j} = s'|_{V_j}$$
for all j. Hence by the uniqueness in $(**)$ for the covering $\mathcal{V}$ we conclude that $s = s'$. Thus we have proved the uniqueness part of $(**)$ for our arbitrary covering $\mathcal{U}$.

Suppose furthermore that $U_i \cap U_{i'} = \bigcup_{k \in I_{ii'}} U_{ii'k}$ are arbitrary coverings by $U_{ii'k} \in \mathcal{B}$. Let us try to prove the existence part of $(**)$ for the system $(\mathcal{U}, \mathcal{U}_{ij})$. Thus let $s_i \in \mathcal{F}(U_i)$ and suppose we have
$$s_i|_{U_{ii'k}} = s_{i'}|_{U_{ii'k}}$$
for all i, i', k. Set $t_j = s_{\alpha(j)}|_{V_j}$, where $\mathcal{V}$ and α are as above.

There is one small kink in the argument here. Namely, let $\mathcal{V}_{jj'} : V_j \cap V_{j'} = \bigcup_{l \in J_{jj'}} V_{jj'l}$ be the covering given to us by the statement of the lemma. It is not a priori clear that
$$t_j|_{V_{jj'l}} = t_{j'}|_{V_{jj'l}}$$
for all j, j', l. To see this, note that we do have
$$t_j|_W = t_{j'}|_W \text{ for all } W \in \mathcal{B}, W \subset V_{jj'l} \cap U_{\alpha(j)\alpha(j')k}$$
for all $k \in I_{\alpha(j)\alpha(j')}$, by our assumption on the family of elements s_i. And since $V_j \cap V_{j'} \subset U_{\alpha(j)} \cap U_{\alpha(j')}$ we see that $t_j|_{V_{jj'l}}$ and $t_{j'}|_{V_{jj'l}}$ agree on the members of a covering of $V_{jj'l}$ by elements of $\mathcal{B}$. Hence by the uniqueness part proved above we finally deduce the desired equality of $t_j|_{V_{jj'l}}$ and $t_{j'}|_{V_{jj'l}}$. Then we get the existence of an element $t \in \mathcal{F}(U)$ by property $(**)$ for $(\mathcal{V}, \mathcal{V}_{jj'})$.

Again there is a small snag. We know that t restricts to t_j on V_j but we do not yet know that t restricts to s_i on U_i. To conclude this note that the sets $U_i \cap V_j$, $j \in J$ cover U_i. Hence also the sets $U_{i\alpha(j)k} \cap V_j$, $j \in J$, $k \in I_{i\alpha(j)}$ cover U_i. We leave it to the reader to see that t and s_i restrict to the same section of $\mathcal{F}$ on any $W \in \mathcal{B}$ which is contained in one of the open sets $U_{i\alpha(j)k} \cap V_j$, $j \in J$, $k \in I_{i\alpha(j)}$. Hence by the uniqueness part seen above we win. □

Lemma 30.4. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Assume that for every triple $U, U', U'' \in \mathcal{B}$ with $U' \subset U$ and $U'' \subset U$ we have $U' \cap U'' \in \mathcal{B}$. For each $U \in \mathcal{B}$, let $C(U) \subset Cov_{\mathcal{B}}(U)$ be a cofinal system. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent*

(1) *The presheaf $\mathcal{F}$ is a sheaf on $\mathcal{B}$.*
(2) *For every $U \in \mathcal{B}$ and every covering $\mathcal{U} : U = \bigcup U_i$ in $C(U)$ and for every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ there exists a unique section $s \in \mathcal{F}(U)$ which restricts to s_i on U_i.*

Proof. This is a reformulation of Lemma 30.3 above in the special case where the coverings $\mathcal{U}_{ij}$ each consist of a single element. But also this case is much easier and is an easy exercise to do directly. □

Lemma 30.5. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $U \in \mathcal{B}$. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{B}$. The map*

$$\mathcal{F}(U) \to \prod\nolimits_{x \in U} \mathcal{F}_x$$

identifies $\mathcal{F}(U)$ with the elements $(s_x)_{x \in U}$ with the property

(*) *For any $x \in U$ there exists a $V \in \mathcal{B}$, with $x \in V \subset U$ and a section $\sigma \in \mathcal{F}(V)$ such that for all $y \in V$ we have $s_y = (V, \sigma)$ in $\mathcal{F}_y$.*

Proof. First note that the map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective by the uniqueness in the sheaf condition of Definition 30.2. Let (s_x) be any element on the right hand side which satisfies $(*)$. Clearly this means we can find a covering $U = \bigcup U_i$, $U_i \in \mathcal{B}$ such that $(s_x)_{x \in U_i}$ comes from certain $\sigma_i \in \mathcal{F}(U_i)$. For every $y \in U_i \cap U_j$ the sections σ_i and σ_j agree in the stalk $\mathcal{F}_y$. Hence there exists an element $V_{ijy} \in \mathcal{B}$, $y \in V_{ijy}$ such that $\sigma_i|_{V_{ijy}} = \sigma_j|_{V_{ijy}}$. Thus the sheaf condition $(**)$ of Definition 30.2 applies to the system of σ_i and we obtain a section $s \in \mathcal{F}(U)$ with the desired property. □

Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. There is a natural restriction functor from the category of sheaves of sets on X to the category of sheaves of sets on $\mathcal{B}$. It turns out that this is an equivalence of categories. In down to earth terms this means the following.

Lemma 30.6. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{B}$. There exists a unique sheaf of sets $\mathcal{F}^{ext}$ on X such that $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$ compatibly with the restriction mappings.*

Proof. We first construct a presheaf $\mathcal{F}^{ext}$ with the desired property. Namely, for an arbitrary open $U \subset X$ we define $\mathcal{F}^{ext}(U)$ as the set of elements $(s_x)_{x \in U}$ such that $(*)$ of Lemma 30.5 holds. It is clear that there are restriction mappings that turn $\mathcal{F}^{ext}$ into a presheaf of sets. Also, by Lemma 30.5 we see that $\mathcal{F}(U) = \mathcal{F}^{ext}(U)$ whenever U is an element of the basis $\mathcal{B}$. To see $\mathcal{F}^{ext}$ is a sheaf one may argue as in the proof of Lemma 17.1. □

Note that we have

$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$

in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

Lemma 30.7. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Denote $Sh(\mathcal{B})$ the category of sheaves on $\mathcal{B}$. There is an equivalence of categories*

$$Sh(X) \longrightarrow Sh(\mathcal{B})$$

which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 30.6 above. Checking the obvious functorialities is left to the reader. □

This ends the discussion of sheaves of sets on a basis $\mathcal{B}$. Let $(\mathcal{C}, F)$ be a type of algebraic structure. At the end of this section we would like to point out that the constructions above work for sheaves with values in $\mathcal{C}$. Let us briefly define the relevant notions.

Definition 30.8. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $(\mathcal{C}, F)$ be a type of algebraic structure.

(1) A *presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$* is a rule which assigns to each $U \in \mathcal{B}$ an object $\mathcal{F}(U)$ of $\mathcal{C}$ and to each inclusion $V \subset U$ of elements of $\mathcal{B}$ a morphism $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ such that $\rho^U_U = \mathrm{id}_{\mathcal{F}(U)}$ for all $U \in \mathcal{B}$ and whenever $W \subset V \subset U$ in $\mathcal{B}$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with values in $\mathcal{C}$ on $\mathcal{B}$* is a rule which assigns to each element $U \in \mathcal{B}$ a morphism of algebraic structures $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps.
(3) Given a presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$ we say that $U \mapsto F(\mathcal{F}(U))$ is the underlying presheaf of sets.
(4) A *sheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$* is a presheaf with values in $\mathcal{C}$ on $\mathcal{B}$ whose underlying presheaf of sets is a sheaf.

At this point we can define the *stalk* at $x \in X$ of a presheaf with values in $\mathcal{C}$ on $\mathcal{B}$ as the directed colimit

$$\mathcal{F}_x = \mathrm{colim}_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

It exists as an object of $\mathcal{C}$ because of our assumptions on $\mathcal{C}$. Also, we see that the underlying set of $\mathcal{F}_x$ is the stalk of the underlying presheaf of sets on $\mathcal{B}$.

Note that Lemmas 30.3, 30.4 and 30.5 refer to the sheaf property which we have defined in terms of the associated presheaf of sets. Hence they generalize without change to the notion of a presheaf with values in $\mathcal{C}$. The analogue of Lemma 30.6 need some care. Here it is.

Lemma 30.9. *Let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on $\mathcal{B}$. There exists a unique sheaf $\mathcal{F}^{ext}$ with values in $\mathcal{C}$ on X such that $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$ compatibly with the restriction mappings.*

Proof. By the conditions imposed on the pair $(\mathcal{C}, F)$ it suffices to come up with a presheaf $\mathcal{F}^{ext}$ which does the correct thing on the level of underlying presheaves of sets. Thus our first task is to construct a suitable object $\mathcal{F}^{ext}(U)$ for all open

$U \subset X$. We could do this by imitating Lemma 18.1 in the setting of presheaves on $\mathcal{B}$. However, a slightly different method (but basically equivalent) is the following: Define it as the directed colimit

$$\mathcal{F}^{ext}(U) := \text{colim}_{\mathcal{U}} FIB(\mathcal{U})$$

over all coverings $\mathcal{U} : U = \bigcup_{i \in I} U_i$ by $U_i \in \mathcal{B}$ of the fibre product

$$\begin{tikzcd} FIB(\mathcal{U}) \arrow[r] \arrow[d] & \prod_{x \in U} \mathcal{F}_x \arrow[d] \\ \prod_{i \in I} \mathcal{F}(U_i) \arrow[r] & \prod_{i \in I} \prod_{x \in U_i} \mathcal{F}_x \end{tikzcd}$$

By the usual arguments, see Lemma 15.4 and Example 15.5 it suffices to show that this construction on underlying sets is the same as the definition using $(**)$ above. Details left to the reader. $\square$

Note that we have

$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$

as objects in $\mathcal{C}$ in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

Lemma 30.10. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Denote $Sh(\mathcal{B}, \mathcal{C})$ the category of sheaves with values in $\mathcal{C}$ on $\mathcal{B}$. There is an equivalence of categories*

$$Sh(X, \mathcal{C}) \longrightarrow Sh(\mathcal{B}, \mathcal{C})$$

which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 30.9 above. Checking the obvious functorialities is left to the reader. $\square$

Finally we come to the case of (pre)sheaves of modules on a basis. We will use the easy fact that the category of presheaves of sets on a basis has products and that they are described by taking products of values on elements of the bases.

Definition 30.11. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{B}$.

(1) A *presheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{B}$* is a presheaf of abelian groups on $\mathcal{B}$ together with a morphism of presheaves of sets $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ such that for all $U \in \mathcal{B}$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ turns the group $\mathcal{F}(U)$ into an $\mathcal{O}(U)$-module.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules on $\mathcal{B}$* is a morphism of abelian presheaves on $\mathcal{B}$ which induces an $\mathcal{O}(U)$-module homomorphism $\mathcal{F}(U) \to \mathcal{G}(U)$ for every $U \in \mathcal{B}$.
(3) Suppose that $\mathcal{O}$ is a sheaf of rings on $\mathcal{B}$. A *sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on $\mathcal{B}$* is a presheaf of $\mathcal{O}$-modules on $\mathcal{B}$ whose underlying presheaf of abelian groups is a sheaf.

We can define the *stalk* at $x \in X$ of a presheaf of $\mathcal{O}$-modules on $\mathcal{B}$ as the directed colimit

$$\mathcal{F}_x = \text{colim}_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

It is a $\mathcal{O}_x$-module.

Note that Lemmas 30.3, 30.4 and 30.5 refer to the sheaf property which we have defined in terms of the associated presheaf of sets. Hence they generalize without change to the notion of a presheaf of $\mathcal{O}$-modules. The analogue of Lemma 30.6 is as follows.

Lemma 30.12. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{B}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{B}$. Let $\mathcal{O}^{ext}$ be the sheaf of rings on X extending $\mathcal{O}$ and let $\mathcal{F}^{ext}$ be the abelian sheaf on X extending $\mathcal{F}$, see Lemma 30.9. There exists a canonical map*

$$\mathcal{O}^{ext} \times \mathcal{F}^{ext} \longrightarrow \mathcal{F}^{ext}$$

which agrees with the given map over elements of $\mathcal{B}$ and which endows $\mathcal{F}^{ext}$ with the structure of an $\mathcal{O}^{ext}$-module.

Proof. It suffices to construct the multiplication map on the level of presheaves of sets. Perhaps the easiest way to see this is to prove directly that if $(f_x)_{x \in U}$, $f_x \in \mathcal{O}_x$ and $(m_x)_{x \in U}$, $m_x \in \mathcal{F}_x$ satisfy $(*)$, then the element $(f_x m_x)_{x \in U}$ also satisfies $(*)$. Then we get the desired result, because in the proof of Lemma 30.6 we construct the extension in terms of families of elements of stalks satisfying $(*)$. $\square$

Note that we have

$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$

as $\mathcal{O}_x$-modules in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x, or simply because it is true on the underlying sets.

Lemma 30.13. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a sheaf of rings on X. Denote $Mod(\mathcal{O}|_\mathcal{B})$ the category of sheaves of $\mathcal{O}|_\mathcal{B}$-modules on $\mathcal{B}$. There is an equivalence of categories*

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}|_\mathcal{B})$$

which assigns to a sheaf of $\mathcal{O}$-modules on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 30.12 above. Checking the obvious functorialities is left to the reader. $\square$

Finally, we address the question of the relationship of this with continuous maps. This is now very easy thanks to the work above. First we do the case where there is a basis on the target given.

Lemma 30.14. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{C}, F)$ be a type of algebraic structures. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on X. Let $\mathcal{G}$ be a sheaf with values in $\mathcal{C}$ on Y. Let $\mathcal{B}$ be a basis for the topology on Y. Suppose given for every $V \in \mathcal{B}$ a morphism*

$$\varphi_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

of $\mathcal{C}$ compatible with restriction mappings. Then there is a unique f-map (see Definition 21.7 and discussion of f-maps in Section 23) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V for $V \in \mathcal{B}$.

Proof. This is trivial because the collection of maps amounts to a morphism between the restrictions of $\mathcal{G}$ and $f_*\mathcal{F}$ to $\mathcal{B}$. By Lemma 30.10 this is the same as giving a morphism from $\mathcal{G}$ to $f_*\mathcal{F}$, which by Lemma 21.8 is the same as an f-map. See also Lemma 23.1 and the discussion preceding it for how to deal with the case of sheaves of algebraic structures. $\square$

Here is the analogue for ringed spaces.

Lemma 30.15. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $\mathcal{B}$ be a basis for the topology on Y. Suppose given for every $V \in \mathcal{B}$ a $\mathcal{O}_Y(V)$-module map*

$$\varphi_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

(where $\mathcal{F}(f^{-1}V)$ has a module structure using $f_V^\sharp : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$) compatible with restriction mappings. Then there is a unique f-map (see discussion of f-maps in Section 26) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V for $V \in \mathcal{B}$.

Proof. Same as the proof of the corresponding lemma for sheaves of algebraic structures above. $\square$

Lemma 30.16. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{C}, F)$ be a type of algebraic structures. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on X. Let $\mathcal{G}$ be a sheaf with values in $\mathcal{C}$ on Y. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Let $\mathcal{B}_X$ be a basis for the topology on X. Suppose given for every $V \in \mathcal{B}_Y$, and $U \in \mathcal{B}_X$ such that $f(U) \subset V$ a morphism*

$$\varphi_V^U : \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

of $\mathcal{C}$ compatible with restriction mappings. Then there is a unique f-map (see Definition 21.7 and the discussion of f-maps in Section 23) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V^U as the composition

$$\mathcal{G}(V) \xrightarrow{\varphi_V} \mathcal{F}(f^{-1}(V)) \xrightarrow{restr.} \mathcal{F}(U)$$

for every pair (U, V) as above.

Proof. Let us first proves this for sheaves of sets. Fix $V \subset Y$ open. Pick $s \in \mathcal{G}(V)$. We are going to construct an element $\varphi_V(s) \in \mathcal{F}(f^{-1}V)$. We can define a value $\varphi(s)_x$ in the stalk $\mathcal{F}_x$ for every $x \in f^{-1}V$ by picking a $U \in \mathcal{B}_X$ with $x \in U \subset f^{-1}V$ and setting $\varphi(s)_x$ equal to the equivalence class of $(U, \varphi_V^U(s))$ in the stalk. Clearly, the family $(\varphi(s)_x)_{x \in f^{-1}V}$ satisfies condition $(*)$ because the maps φ_V^U for varying U are compatible with restrictions in the sheaf $\mathcal{F}$. Thus, by the proof of Lemma 30.6 we see that $(\varphi(s)_x)_{x \in f^{-1}V}$ corresponds to a unique element $\varphi_V(s)$ of $\mathcal{F}(f^{-1}V)$. Thus we have defined a set map $\varphi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$. The compatibility between φ_V and φ_V^U follows from Lemma 30.5.

We leave it to the reader to show that the construction of φ_V is compatible with restriction mappings as we vary $V \in \mathcal{B}_Y$. Thus we may apply Lemma 30.14 above to "glue" them to the desired f-map.

Finally, we note that the map of sheaves of sets so constructed satisfies the property that the map on stalks

$$\mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

is the colimit of the system of maps φ_V^U as $V \in \mathcal{B}_Y$ varies over those elements that contain $f(x)$ and $U \in \mathcal{B}_X$ varies over those elements that contain x. In particular, if $\mathcal{G}$ and $\mathcal{F}$ are the underlying sheaves of sets of sheaves of algebraic structures, then we see that the maps on stalks is a morphism of algebraic structures. Hence we conclude that the associated map of sheaves of underlying sets $f^{-1}\mathcal{G} \to \mathcal{F}$ satisfies the assumptions of Lemma 23.1. We conclude that $f^{-1}\mathcal{G} \to \mathcal{F}$ is a morphism of sheaves with values in $\mathcal{C}$. And by adjointness this means that φ is an f-map of sheaves of algebraic structures. $\square$

Lemma 30.17. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Let $\mathcal{B}_X$ be a basis for the topology on X. Suppose given for every $V \in \mathcal{B}_Y$, and $U \in \mathcal{B}_X$ such that $f(U) \subset V$ a $\mathcal{O}_Y(V)$-module map*

$$\varphi_V^U : \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

compatible with restriction mappings. Here the $\mathcal{O}_Y(V)$-module structure on $\mathcal{F}(U)$ comes from the $\mathcal{O}_X(U)$-module structure via the map $f_V^\sharp : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V) \to \mathcal{O}_X(U)$. Then there is a unique f-map of sheaves of modules (see Definition 21.7 and the discussion of f-maps in Section 26) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V^U as the composition

$$\mathcal{G}(V) \xrightarrow{\varphi_V} \mathcal{F}(f^{-1}(V)) \xrightarrow{restr.} \mathcal{F}(U)$$

for every pair (U, V) as above.

Proof. Similar to the above and omitted. $\square$

31. Open immersions and (pre)sheaves

Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset U into X. In Section 21 we have defined functors j_* and j^{-1} such that j_* is right adjoint to j^{-1}. It turns out that for an open immersion there is a left adjoint for j^{-1}, which we will denote $j_!$. First we point out that j^{-1} has a particularly simple description in the case of an open immersion.

Lemma 31.1. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset U into X.*

1. *Let $\mathcal{G}$ be a presheaf of sets on X. The presheaf $j_p\mathcal{G}$ (see Section 21) is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open.*
2. *Let $\mathcal{G}$ be a sheaf of sets on X. The sheaf $j^{-1}\mathcal{G}$ is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open.*
3. *For any point $u \in U$ and any sheaf $\mathcal{G}$ on X we have a canonical identification of stalks*
$$j^{-1}\mathcal{G}_u = (\mathcal{G}|_U)_u = \mathcal{G}_u.$$
4. *On the category of presheaves of U we have $j_p j_* = id$.*
5. *On the category of sheaves of U we have $j^{-1} j_* = id$.*

The same description holds for (pre)sheaves of abelian groups, (pre)sheaves of algebraic structures, and (pre)sheaves of modules.

Proof. The colimit in the definition of $j_p\mathcal{G}(V)$ is over collection of all $W \subset X$ open such that $V \subset W$ ordered by reverse inclusion. Hence this has a largest element, namely V. This proves (1). And (2) follows because the assignment $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open is clearly a sheaf if $\mathcal{G}$ is a sheaf. Assertion (3) follows from (2) since the collection of open neighbourhoods of u which are contained in U is cofinal in the collection of all open neighbourhoods of u in X. Parts (4) and (5) follow by computing $j^{-1}j_*\mathcal{F}(V) = j_*\mathcal{F}(V) = \mathcal{F}(V)$.

The exact same arguments work for (pre)sheaves of abelian groups and (pre)sheaves of algebraic structures. □

Definition 31.2. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{G}$ be a presheaf of sets, abelian groups or algebraic structures on X. The presheaf $j_p\mathcal{G}$ described in Lemma 31.1 is called the *restriction of $\mathcal{G}$ to U* and denoted $\mathcal{G}|_U$.
(2) Let $\mathcal{G}$ be a sheaf of sets on X, abelian groups or algebraic structures on X. The sheaf $j^{-1}\mathcal{G}$ is called the *restriction of $\mathcal{G}$ to U* and denoted $\mathcal{G}|_U$.
(3) If $(X, \mathcal{O})$ is a ringed space, then the pair $(U, \mathcal{O}|_U)$ is called the *open subspace of $(X, \mathcal{O})$ associated to U*.
(4) If $\mathcal{G}$ is a presheaf of $\mathcal{O}$-modules then $\mathcal{G}|_U$ together with the multiplication map $\mathcal{O}|_U \times \mathcal{G}|_U \to \mathcal{G}|_U$ (see Lemma 24.6) is called the *restriction of $\mathcal{G}$ to U*.

We leave a definition of the restriction of presheaves of modules to the reader. Ok, so in this section we will discuss a left adjoint to the restriction functor. Here is the definition in the case of (pre)sheaves of sets.

Definition 31.3. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{F}$ be a presheaf of sets on U. We define the *extension of $\mathcal{F}$ by the empty set $j_{p!}\mathcal{F}$* to be the presheaf of sets on X defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} \emptyset & \text{if } V \not\subset U \\ \mathcal{F}(V) & \text{if } V \subset U \end{cases}$$
with obvious restriction mappings.
(2) Let $\mathcal{F}$ be a sheaf of sets on U. We define the *extension of $\mathcal{F}$ by the empty set $j_!\mathcal{F}$* to be the sheafification of the presheaf $j_{p!}\mathcal{F}$.

Lemma 31.4. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 31.1).*
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Sh(X)}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{Sh(U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be a sheaf of sets on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*
$$j_!\mathcal{F}_x = \begin{cases} \emptyset & \text{if } x \notin U \\ \mathcal{F}_x & \text{if } x \in U \end{cases}$$
(4) *On the category of presheaves of U we have $j_pj_{p!} = id$.*
(5) *On the category of sheaves of U we have $j^{-1}j_! = id$.*

Proof. To map $j_{p!}\mathcal{F}$ into $\mathcal{G}$ it is enough to map $\mathcal{F}(V) \to \mathcal{G}(V)$ whenever $V \subset U$ compatibly with restriction mappings. And by Lemma 31.1 the same description holds for maps $\mathcal{F} \to \mathcal{G}|_U$. The adjointness of $j_!$ and restriction follows from this and the properties of sheafification. The identification of stalks is obvious from the definition of the extension by the empty set and the definition of a stalk. Statements (4) and (5) follow by computing the value of the sheaf on any open of U. □

Note that if $\mathcal{F}$ is a sheaf of abelian groups on U, then in general $j_!\mathcal{F}$ as defined above, is not a sheaf of abelian groups, for example because some of its stalks are empty (hence not abelian groups for sure). Thus we need to modify the definition of $j_!$ depending on the type of sheaves we consider. The reason for choosing the empty set in the definition of the extension by the empty set, is that it is the initial object in the category of sets. Thus in the case of abelian groups we use 0 (and more generally for sheaves with values in any abelian category).

Definition 31.5. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{F}$ be an abelian presheaf on U. We define the *extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by* 0 to be the abelian presheaf on X defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} 0 & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$
with obvious restriction mappings.

(2) Let $\mathcal{F}$ be an abelian sheaf on U. We define the *extension $j_!\mathcal{F}$ of $\mathcal{F}$ by* 0 to be the sheafification of the abelian presheaf $j_{p!}\mathcal{F}$.

(3) Let $\mathcal{C}$ be a category having an initial object e. Let $\mathcal{F}$ be a presheaf on U with values in $\mathcal{C}$. We define the *extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by e* to be the presheaf on X with values in $\mathcal{C}$ defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} e & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$
with obvious restriction mappings.

(4) Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. Let $\mathcal{F}$ be a sheaf of algebraic structures on U (of the give type). We define the *extension $j_!\mathcal{F}$ of $\mathcal{F}$ by e* to be the sheafification of the presheaf $j_{p!}\mathcal{F}$ defined above.

(5) Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}|_U$-modules. In this case we define the *extension by* 0 to be the presheaf of $\mathcal{O}$-modules which is equal to $j_{p!}\mathcal{F}$ as an abelian presheaf endowed with the multiplication map $\mathcal{O} \times j_{p!}\mathcal{F} \to j_{p!}\mathcal{F}$.

(6) Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}|_U$-modules. In this case we define the *extension by* 0 to be the $\mathcal{O}$-module which is equal to $j_!\mathcal{F}$ as an abelian sheaf endowed with the multiplication map $\mathcal{O} \times j_!\mathcal{F} \to j_!\mathcal{F}$.

It is true that one can define $j_!$ in the setting of sheaves of algebraic structures (see below). However, it depends on the type of algebraic structures involved what the resulting object is. For example, if $\mathcal{O}$ is a sheaf of rings on U, then $j_{!,rings}\mathcal{O} \neq j_{!,abelian}\mathcal{O}$ since the initial object in the category of rings is $\mathbf{Z}$ and the initial object in the category of abelian groups is 0. In particular the functor $j_!$ *does not commute with taking underlying sheaves of sets*, in contrast to what we have seen so far! We separate out the case of (pre)sheaves of abelian groups, (pre)sheaves of algebraic structures and (pre)sheaves of modules as usual.

Lemma 31.6. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Consider the functors of restriction and extension by 0 for abelian (pre)sheaves.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 31.1).*
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Ab(X)}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Ab(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{Ab(U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be an abelian sheaf on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*
$$j_!\mathcal{F}_x = \begin{cases} 0 & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$
(4) *On the category of abelian presheaves of U we have $j_p j_{p!} = id$.*
(5) *On the category of abelian sheaves of U we have $j^{-1}j_! = id$.*

Proof. Omitted. □

Lemma 31.7. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. Consider the functors of restriction and extension by e for (pre)sheaves of algebraic structure defined above.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 31.1).*
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Sh(X,\mathcal{C})}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(U,\mathcal{C})}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{Sh(U,\mathcal{C})}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be a sheaf on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*
$$j_!\mathcal{F}_x = \begin{cases} e & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$
(4) *On the category of presheaves of algebraic structures on U we have $j_p j_{p!} = id$.*
(5) *On the category of sheaves of algebraic structures on U we have $j^{-1}j_! = id$.*

Proof. Omitted. □

Lemma 31.8. *Let $(X, \mathcal{O})$ be a ringed space. Let $j : (U, \mathcal{O}|_U) \to (X, \mathcal{O})$ be an open subspace. Consider the functors of restriction and extension by 0 for (pre)sheaves of modules defined above.*

(1) *The functor $j_{p!}$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{PMod(\mathcal{O})}(j_{p!}\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{PMod(\mathcal{O}|_U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Mod(\mathcal{O})}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Mod(\mathcal{O}|_U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

(3) *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*

$$j_!\mathcal{F}_x = \begin{cases} 0 & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$

(4) *On the category of sheaves of $\mathcal{O}|_U$-modules on U we have $j^{-1}j_! = id$.*

Proof. Omitted. □

Note that by the lemmas above, both the functors j_* and $j_!$ are fully faithful embeddings of the category of sheaves on U into the category of sheaves on X. It is only true for the functor $j_!$ that one can easily describe the essential image of this functor.

Lemma 31.9. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. The functor*

$$j_! : Sh(U) \longrightarrow Sh(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = \emptyset$ for all $x \in X \setminus U$.

Proof. Fully faithfulness follows formally from $j^{-1}j_! = \text{id}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathcal{G}$ has the indicated property. Then it is easy to check that

$$j_!j^{-1}\mathcal{G} \to \mathcal{G}$$

is an isomorphism on all stalks and hence an isomorphism. □

Lemma 31.10. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. The functor*

$$j_! : Ab(U) \longrightarrow Ab(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus U$.

Proof. Omitted. □

Lemma 31.11. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. The functor*

$$j_! : Sh(U, \mathcal{C}) \longrightarrow Sh(X, \mathcal{C})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = e$ for all $x \in X \setminus U$.

Proof. Omitted. □

Lemma 31.12. *Let $(X, \mathcal{O})$ be a ringed space. Let $j : (U, \mathcal{O}|_U) \to (X, \mathcal{O})$ be an open subspace. The functor*

$$j_! : Mod(\mathcal{O}|_U) \longrightarrow Mod(\mathcal{O})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus U$.

Proof. Omitted. □

Remark 31.13. Let $j : U \to X$ be an open immersion of topological spaces as above. Let $x \in X$, $x \notin U$. Let $\mathcal{F}$ be a sheaf of sets on U. Then $j_!\mathcal{F}_x = \emptyset$ by Lemma 31.4. Hence $j_!$ does not transform a final object of $Sh(U)$ into a final object of $Sh(X)$ unless $U = X$. According to our conventions in Categories, Section 23 this means that the functor $j_!$ is not left exact as a functor between the categories of sheaves of sets. It will be shown later that $j_!$ on abelian sheaves is exact, see Modules, Lemma 3.4.

32. Closed immersions and (pre)sheaves

Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset Z into X. In Section 21 we have defined functors i_* and i^{-1} such that i_* is right adjoint to i^{-1}.

Lemma 32.1. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset Z into X. Let $\mathcal{F}$ be a sheaf of sets on Z. The stalks of $i_*\mathcal{F}$ are described as follows*

$$i_*\mathcal{F}_x = \begin{cases} \{*\} & \text{if} \quad x \notin Z \\ \mathcal{F}_x & \text{if} \quad x \in Z \end{cases}$$

where $\{*\}$ denotes a singleton set. Moreover, $i^{-1}i_* = id$ on the category of sheaves of sets on Z. Moreover, the same holds for abelian sheaves on Z, resp. sheaves of algebraic structures on Z where $\{*\}$ has to be replaced by 0, resp. a final object of the category of algebraic structures.

Proof. If $x \notin Z$, then there exist arbitrarily small open neighbourhoods U of x which do not meet Z. Because $\mathcal{F}$ is a sheaf we have $\mathcal{F}(i^{-1}(U)) = \{*\}$ for any such U, see Remark 7.2. This proves the first case. The second case comes from the fact that for $z \in Z$ any open neighbourhood of z is of the form $Z \cap U$ for some open U of X. For the statement that $i^{-1}i_* = \text{id}$ consider the canonical map $i^{-1}i_*\mathcal{F} \to \mathcal{F}$. This is an isomorphism on stalks (see above) and hence an isomorphism.

For sheaves of abelian groups, and sheaves of algebraic structures you argue in the same manner. ☐

Lemma 32.2. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor*

$$i_* : Sh(Z) \longrightarrow Sh(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = \{*\}$ for all $x \in X \setminus Z$.

Proof. Fully faithfulness follows formally from $i^{-1}i_* = \text{id}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathcal{G}$ has the indicated property. Then it is easy to check that

$$\mathcal{G} \to i_*i^{-1}\mathcal{G}$$

is an isomorphism on all stalks and hence an isomorphism. ☐

Lemma 32.3. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor*

$$i_* : Ab(Z) \longrightarrow Ab(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus Z$.

Proof. Omitted. $\square$

Lemma 32.4. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure with final object 0. The functor*

$$i_* : Sh(Z, \mathcal{C}) \longrightarrow Sh(X, \mathcal{C})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus Z$.

Proof. Omitted. $\square$

Remark 32.5. Let $i : Z \to X$ be a closed immersion of topological spaces as above. Let $x \in X$, $x \notin Z$. Let $\mathcal{F}$ be a sheaf of sets on Z. Then $(i_*\mathcal{F})_x = \{*\}$ by Lemma 32.1. Hence if $\mathcal{F} = * \amalg *$, where $*$ is the singleton sheaf, then $i_*\mathcal{F}_x = \{*\} \neq i_*(*)_x \amalg i_*(*)_x$ because the latter is a two point set. According to our conventions in Categories, Section 23 this means that the functor i_* is not right exact as a functor between the categories of sheaves of sets. In particular, it cannot have a right adjoint, see Categories, Lemma 24.6.

On the other hand, we will see later (see Modules, Lemma 6.3) that i_* on abelian sheaves is exact, and does have a right adjoint, namely the functor that associates to an abelian sheaf on X the sheaf of sections supported in Z.

Remark 32.6. We have not discussed the relationship between closed immersions and ringed spaces. This is because the notion of a closed immersion of ringed spaces is best discussed in the setting of quasi-coherent sheaves, see Modules, Section 13.

33. Glueing sheaves

In this section we glue sheaves defined on the members of a covering of X. We first deal with maps.

Lemma 33.1. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of sets on X. Given a collection*

$$\varphi_i : \mathcal{F}|_{U_i} \longrightarrow \mathcal{G}|_{U_i}$$

of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\mathcal{F}|_{U_i \cap U_j} \to \mathcal{G}|_{U_i \cap U_j}$ then there exists a unique map of sheaves

$$\varphi : \mathcal{F} \longrightarrow \mathcal{G}$$

whose restriction to each U_i agrees with φ_i.

Proof. For each open subset $U \subset X$ define

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U), \quad s \mapsto \varphi_U(s)$$

where $\varphi_U(s)$ is the unique section verifying

$$(\varphi_U(s))|_{U \cap U_i} = (\varphi_i)_{U \cap U_i}(s|_{U \cap U_i}).$$

Existence and uniqueness of such a section follows from the sheaf axioms due to the fact that

$$\begin{aligned}((\varphi_i)_{U\cap U_i}(s|_{U\cap U_i}))|_{U\cap U_i\cap U_j} &= (\varphi_i)_{U\cap U_i\cap U_j}(s|_{U\cap U_i\cap U_j})\\ &= (\varphi_j)_{U\cap U_i\cap U_j}(s|_{U\cap U_i\cap U_j})\\ &= ((\varphi_j)_{U\cap U_j}(s|_{U\cap U_j}))|_{U\cap U_i\cap U_j}.\end{aligned}$$

This family of maps gives us indeed a map of sheaves: Let $V \subset U \subset X$ be open subsets then

$$(\varphi_U(s))|_V = \varphi_V(s|_V)$$

since for each $i \in I$ the following holds

$$\begin{aligned}(\varphi_U(s))|_{V\cap U_i} &= ((\varphi_U(s))|_{U\cap U_i})|_{V\cap U_i}\\ &= ((\varphi_i)_{U\cap U_i}(s|_{U\cap U_i}))|_{V\cap U_i}\\ &= (\varphi_i)_{V\cap U_i}(s|_{V\cap U_i})\\ &= \varphi_V(s_V)|_{V\cap U_i}.\end{aligned}$$

Furthermore, its restriction to each U_i agrees with φ_i since given $U \subset X$ open subset and $s \in \mathcal{F}(U\cap U_i)$ then

$$\begin{aligned}\varphi_{U\cap U_i}(s) &= \varphi_{U\cap U_i}(s)|_{U\cap U_i}\\ &= (\varphi_i)_{U\cap U_i}(s|_{U\cap U_i})\\ &= (\varphi_i)_{U\cap U_i}(s).\end{aligned}$$

□

The previous lemma implies that given two sheaves $\mathcal{F}$, $\mathcal{G}$ on the topological space X the rule

$$U \longmapsto \mathrm{Mor}_{Sh(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$$

defines a sheaf. This is a kind of *internal hom sheaf*. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules, see Modules, Section 22.

Let X be a topological space. Let $X = \bigcup_{i\in I} U_i$ be an open covering. For each $i \in I$ let $\mathcal{F}_i$ be a sheaf of sets on U_i. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathcal{F}_i|_{U_i\cap U_j} \longrightarrow \mathcal{F}_j|_{U_i\cap U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in I$ the following diagram is commutative

$$\begin{tikzcd} \mathcal{F}_i|_{U_i\cap U_j\cap U_k} \arrow[rr, "\varphi_{ik}"'] \arrow[dr, "\varphi_{ij}"'] & & \mathcal{F}_k|_{U_i\cap U_j\cap U_k} \\ & \mathcal{F}_j|_{U_i\cap U_j\cap U_k} \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

We will call such a collection of data $(\mathcal{F}_i, \varphi_{ij})$ a *glueing data for sheaves of sets with respect to the covering* $X = \bigcup U_i$.

Lemma 33.2. *Let X be a topological space. Let $X = \bigcup_{i\in I} U_i$ be an open covering. Given any glueing data $(\mathcal{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering $X = \bigcup U_i$ there exists a sheaf of sets $\mathcal{F}$ on X together with isomorphisms*

$$\varphi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$$

such that the diagrams

$$\begin{array}{ccc} \mathcal{F}|_{U_i \cap U_j} & \xrightarrow{\varphi_i} & \mathcal{F}_i|_{U_i \cap U_j} \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \varphi_{ij}} \\ \mathcal{F}|_{U_i \cap U_j} & \xrightarrow{\varphi_j} & \mathcal{F}_j|_{U_i \cap U_j} \end{array}$$

are commutative.

Proof. First proof. In this proof we give a formula for the set of sections of $\mathcal{F}$ over an open $W \subset X$. Namely, we define

$$\mathcal{F}(W) = \{(s_i)_{i \in I} \mid s_i \in \mathcal{F}_i(W \cap U_i), \varphi_{ij}(s_i|_{W \cap U_i \cap U_j}) = s_j|_{W \cap U_i \cap U_j}\}.$$

Restriction mappings for $W' \subset W$ are defined by the restricting each of the s_i to $W' \cap U_i$. The sheaf condition for $\mathcal{F}$ follows immediately from the sheaf condition for each of the $\mathcal{F}_i$.

We still have to prove that $\mathcal{F}|_{U_i}$ maps isomorphically to $\mathcal{F}_i$. Let $W \subset U_i$. In this case the condition in the definition of $\mathcal{F}(W)$ implies that $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$. And the commutativity of the diagrams in the definition of a glueing data assures that we may start with *any* section $s \in \mathcal{F}_i(W)$ and obtain a compatible collection by setting $s_i = s$ and $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$.

Second proof (sketch). Let $\mathcal{B}$ be the set of opens $U \subset X$ such that $U \subset U_i$ for some $i \in I$. Then $\mathcal{B}$ is a base for the topology on X. For $U \in \mathcal{B}$ we pick $i \in I$ with $U \subset U_i$ and we set $\mathcal{F}(U) = \mathcal{F}_i(U)$. Using the isomorphisms φ_{ij} we see that this prescription is "independent of the choice of i". Using the restriction mappings of $\mathcal{F}_i$ we find that $\mathcal{F}$ is a sheaf on $\mathcal{B}$. Finally, use Lemma 30.6 to extend $\mathcal{F}$ to a unique sheaf $\mathcal{F}$ on X. $\square$

Lemma 33.3. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $(\mathcal{F}_i, \varphi_{ij})$ be a glueing data of sheaves of abelian groups, resp. sheaves of algebraic structures, resp. sheaves of $\mathcal{O}$-modules for some sheaf of rings $\mathcal{O}$ on X. Then the construction in the proof of Lemma 33.2 above leads to a sheaf of abelian groups, resp. sheaf of algebraic structures, resp. sheaf of $\mathcal{O}$-modules.*

Proof. This is true because in the construction the set of sections $\mathcal{F}(W)$ over an open W is given as the equalizer of the maps

$$\prod\nolimits_{i \in I} \mathcal{F}_i(W \cap U_i) \rightrightarrows \prod\nolimits_{i,j \in I} \mathcal{F}_i(W \cap U_i \cap U_j)$$

And in each of the cases envisioned this equalizer gives an object in the relevant category whose underlying set is the object considered in the cited lemma. $\square$

Lemma 33.4. *Let X be a topological space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. The functor which associates to a sheaf of sets $\mathcal{F}$ the following collection of glueing data*

$$(\mathcal{F}|_{U_i}, (\mathcal{F}|_{U_i})|_{U_i \cap U_j} \to (\mathcal{F}|_{U_j})|_{U_i \cap U_j})$$

with respect to the covering $X = \bigcup U_i$ defines an equivalence of categories between $Sh(X)$ and the category of glueing data. A similar statement holds for abelian sheaves, resp. sheaves of algebraic structures, resp. sheaves of $\mathcal{O}$-modules.

Proof. The functor is fully faithful by Lemma 33.1 and essentially surjective (via an explicitly given quasi-inverse functor) by Lemma 33.2. $\square$

This lemma means that if the sheaf $\mathcal{F}$ was constructed from the glueing data $(\mathcal{F}_i, \varphi_{ij})$ and if $\mathcal{G}$ is a sheaf on X, then a morphism $f : \mathcal{F} \to \mathcal{G}$ is given by a collection of morphisms of sheaves

$$f_i : \mathcal{F}_i \longrightarrow \mathcal{G}|_{U_i}$$

compatible with the glueing maps φ_{ij}. Similarly, to give a morphism of sheaves $g : \mathcal{G} \to \mathcal{F}$ is the same as giving a collection of morphisms of sheaves

$$g_i : \mathcal{G}|_{U_i} \longrightarrow \mathcal{F}_i$$

compatible with the glueing maps φ_{ij}.

34. Other chapters

References

[God73] Roger Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1973, Troisième édition revue et corrigée, Publications de l'Institut de Mathématique de l'Université de Strasbourg, XIII, Actualités Scientifiques et Industrielles, No. 1252.