MORE ON ALGEBRA

Contents

1. Introduction

In this chapter we prove some results in commutative algebra which are less elementary than those in the first chapter on commutative algebra, see Algebra, Section 1. A reference is [Mat70].

2. Advice for the reader

More than in the chapter on commutative algebra, each of the sections in this chapter stands on its own. Starting with Section 56 we freely use the (unbounded) derived category of modules over rings and all the machinery that comes with it.

3. Stably free modules

Here is what seems to be the generally accepted definition.

Definition 3.1. Let R be a ring.

(1) Two modules M, N over R are said to be *stably isomorphic* if there exist $n, m \geq 0$ such that $M \oplus R^{\oplus m} \cong N \oplus R^{\oplus n}$ as R-modules.
(2) A module M is *stably free* if it is stably isomorphic to a free module.

Observe that a stably free module is projective.

Lemma 3.2. *Let R be a ring. Let $0 \to P' \to P \to P'' \to 0$ be a short exact sequence of finite projective R-modules. If 2 out of 3 of these modules are stably free, then so is the third.*

Proof. Since the modules are projective, the sequence is split. Thus we can choose an isomorphism $P = P' \oplus P''$. If $P' \oplus R^{\oplus n}$ and $P'' \oplus R^{\oplus m}$ are free, then we see that $P \oplus R^{\oplus n+m}$ is free. Suppose that P' and P are stably free, say $P \oplus R^{\oplus n}$ is free and $P' \oplus R^{\oplus m}$ is free. Then

$$P'' \oplus (P' \oplus R^{\oplus m}) \oplus R^{\oplus n} = (P'' \oplus P') \oplus R^{\oplus m} \oplus R^{\oplus n} = (P \oplus R^{\oplus n}) \oplus R^{\oplus m}$$

is free. Thus P'' is stably free. By symmetry we get the last of the three cases. □

Lemma 3.3. *Let R be a ring. Let $I \subset R$ be an ideal. Assume that every element of $1 + I$ is a unit (in other words I is contained in the Jacobson radical of R). For every finite stably free R/I-module E there exists a finite stably free R-module M such that $M/IM \cong E$.*

Proof. Choose a n and m and an isomorphism $E \oplus (R/I)^{\oplus n} \cong (R/I)^{\oplus m}$. Choose R-linear maps $\varphi : R^{\oplus m} \to R^{\oplus n}$ and $\psi : R^{\oplus n} \to R^{\oplus m}$ lifting the projection $(R/I)^{\oplus m} \to (R/I)^{\oplus n}$ and injection $(R/I)^{\oplus n} \to (R/I)^{\oplus m}$. Then $\varphi \circ \psi : R^{\oplus n} \to R^{\oplus n}$ reduces to the identity modulo I. Thus the determinant of this map is invertible by our assumption on I. Hence $P = \text{Ker}(\varphi)$ is stably free and lifts E. □

Lemma 3.4. *Let R be a ring. Let $I \subset R$ be an ideal. Assume that every element of $1 + I$ is a unit (in other words I is contained in the Jacobson radical of R). Let M be a finite flat R-module such that M/IM is a projective R/I-module. Then M is a finite projective R-module.*

Proof. By Algebra, Lemma 78.5 we see that $M_\mathfrak{p}$ is finite free for all prime ideals $\mathfrak{p} \subset R$. By Algebra, Lemma 78.2 it suffices to show that the function $\rho_M : \mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$ is locally constant on $\text{Spec}(R)$. Because M/IM is finite projective, this is true on $V(I) \subset \text{Spec}(R)$. Since every closed point of $\text{Spec}(R)$ is in $V(I)$

and since $\rho_M(\mathfrak{p}) = \rho_M(\mathfrak{q})$ whenever $\mathfrak{p} \subset \mathfrak{q} \subset R$ are prime ideals, we conclude by an elementary argument on topological spaces which we omit. $\square$

The lift of Lemma 3.3 is unique up to isomorphism by the following lemma.

Lemma 3.5. *Let R be a ring. Let $I \subset R$ be an ideal. Assume that every element of $1 + I$ is a unit (in other words I is contained in the Jacobson radical of R). If P and P' are finite projective R-modules, then*

1. *if $\varphi : P \to P'$ is an R-module map inducing an isomorphism $\overline{\varphi} : P/IP \to P'/IP'$, then φ is an isomorphism,*
2. *if $P/IP \cong P'/IP'$, then $P \cong P'$.*

Proof. Proof of (1). As P' is projective as an R-module we may choose a lift $\psi : P' \to P$ of the map $P' \to P'/IP' \xrightarrow{\overline{\varphi}^{-1}} P/IP$. By Nakayama's lemma (Algebra, Lemma 20.1) $\psi \circ \varphi$ and $\varphi \circ \psi$ are surjective. Hence these maps are isomorphisms (Algebra, Lemma 16.4). Thus φ is an isomorphism.

Proof of (2). Choose an isomorphism $P/IP \cong P'/IP'$. Since P is projective we can choose a lift $\varphi : P \to P'$ of the map $P \to P/IP \to P'/IP'$. Then φ is an isomorphism by (1). $\square$

4. A comment on the Artin-Rees property

Some of this material is taken from [CdJ02]. A general discussion with additional references can be found in [EH05, Section 1].

Let A be a Noetherian ring and let $I \subset A$ be an ideal. Given a homomorphism $f : M \to N$ of finite A-modules there exists a $c \geq 0$ such that

$$f(M) \cap I^n N \subset f(I^{n-c} M)$$

for all $n \geq c$, see Algebra, Lemma 51.3. In this situation we will say *c works for f in the Artin-Rees lemma.*

Lemma 4.1. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal contained in the Jacobson radical of A. Let*

$$S : L \xrightarrow{f} M \xrightarrow{g} N \quad \text{and} \quad S' : L \xrightarrow{f'} M \xrightarrow{g'} N$$

be two complexes of finite A-modules as shown. Assume that

1. *c works in the Artin-Rees lemma for f and g,*
2. *the complex S is exact, and*
3. *$f' = f \bmod I^{c+1}M$ and $g' = g \bmod I^{c+1}N$.*

Then c works in the Artin-Rees lemma for g' and the complex S' is exact.

Proof. We first show that $g'(M) \cap I^n N \subset g'(I^{n-c}M)$ for $n \geq c$. Let a be an element of M such that $g'(a) \in I^n N$. We want to adjust a by an element of $f'(L)$, i.e, without changing $g'(a)$, so that $a \in I^{n-c}M$. Assume that $a \in I^r M$, where $r < n - c$. Then

$$g(a) = g'(a) + (g - g')(a) \in I^n N + I^{r+c+1} N = I^{r+c+1} N.$$

By Artin-Rees for g we have $g(a) \in g(I^{r+1}M)$. Say $g(a) = g(a_1)$ with $a_1 \in I^{r+1}M$. Since the sequence S is exact, $a - a_1 \in f(L)$. Accordingly, we write $a = f(b) + a_1$ for some $b \in L$. Then $f(b) = a - a_1 \in I^r M$. Artin-Rees for f shows that if $r \geq c$, we may replace b by an element of $I^{r-c}L$. Then in all cases, $a = f'(b) + a_2$, where

$a_2 = (f - f')(b) + a_1 \in I^{r+1}M$. (Namely, either $c \geq r$ and $(f - f')(b) \in I^{r+1}M$ by assumption, or $c < r$ and $b \in I^{r-c}$, whence again $(f - f')(b) \in I^{c+1}I^{r-c}M = I^{r+1}M$.) So we can adjust a by the element $f'(b) \in f'(L)$ to increase r by 1.

In fact, the argument above shows that $(g')^{-1}(I^nN) \subset f'(L) + I^{n-c}M$ for all $n \geq c$. Hence S' is exact because

$$(g')^{-1}(0) = (g')^{-1}(\bigcap I^nN) \subset \bigcap f'(L) + I^{n-c}M = f'(L)$$

as I is contained in the Jacobson radical of A, see Algebra, Lemma 51.5. □

Given an ideal $I \subset A$ of a ring A and an A-module M we set

$$\text{Gr}_I(M) = \bigoplus I^nM/I^{n+1}M.$$

We think of this as a graded $\text{Gr}_I(A)$-module.

Lemma 4.2. *Assumptions as in Lemma 4.1. Let $Q = \text{Coker}(g)$ and $Q' = \text{Coker}(g')$. Then $Gr_I(Q) \cong Gr_I(Q')$ as graded $Gr_I(A)$-modules.*

Proof. In degree n we have $\text{Gr}_I(Q)_n = I^nN/(I^{n+1}N + g(M) \cap I^nN)$ and similarly for Q'. We claim that

$$g(M) \cap I^nN \subset I^{n+1}N + g'(M) \cap I^nN.$$

By symmetry (the proof of the claim will only use that c works for g which also holds for g' by the lemma) this will imply that

$$I^{n+1}N + g(M) \cap I^nN = I^{n+1}N + g'(M) \cap I^nN$$

whence $\text{Gr}_I(Q)_n$ and $\text{Gr}_I(Q')_n$ agree as subquotients of N, implying the lemma. Observe that the claim is clear for $n \leq c$ as $f = f' \bmod I^{c+1}N$. If $n > c$, then suppose $b \in g(M) \cap I^nN$. Write $b = g(a)$ for $a \in I^{n-c}M$. Set $b' = g'(a)$. We have $b - b' = (g - g')(a) \in I^{n+1}N$ as desired. □

Lemma 4.3. *Let $A \to B$ be a flat map of Noetherian rings. Let $I \subset A$ be an ideal. Let $f : M \to N$ be a homomorphism of finite A-modules. Assume that c works for f in the Artin-Rees lemma. Then c works for $f \otimes 1 : M \otimes_A B \to N \otimes_A B$ in the Artin-Rees lemma for the ideal IB.*

Proof. Note that

$$(f \otimes 1)(M) \cap I^nN \otimes_A B = (f \otimes 1)\left((f \otimes 1)^{-1}(I^nN \otimes_A B)\right)$$

On the other hand,

$$\begin{aligned}(f \otimes 1)^{-1}(I^nN \otimes_A B) &= \text{Ker}(M \otimes_A B \to N \otimes_A B/(I^nN \otimes_A B)) \\ &= \text{Ker}(M \otimes_A B \to (N/I^nN) \otimes_A B)\end{aligned}$$

As $A \to B$ is flat taking kernels and cokernels commutes with tensoring with B, whence this is equal to $f^{-1}(I^nN) \otimes_A B$. By assumption $f^{-1}(I^nN)$ is contained in $\text{Ker}(f) + I^{n-c}M$. Thus the lemma holds. □

5. Fibre products of rings, I

Fibre products of rings have to do with pushouts of schemes. Some cases of pushouts of schemes are discussed in More on Morphisms, Section 14.

Lemma 5.1. *Let R be a ring. Let $A \to B$ and $C \to B$ be R-algebra maps. Assume*

1. *R is Noetherian,*
2. *A, B, C are of finite type over R,*
3. *$A \to B$ is surjective, and*
4. *B is finite over C.*

Then $A \times_B C$ is of finite type over R.

Proof. Set $D = A \times_B C$. There is a commutative diagram

$$\begin{tikzcd}
0 & B \arrow[l] & A \arrow[l] & I \arrow[l] & 0 \arrow[l] \\
0 & C \arrow[l] \arrow[u] & D \arrow[l] \arrow[u] & I \arrow[l] \arrow[u] & 0 \arrow[l]
\end{tikzcd}$$

with exact rows. Choose $y_1, \ldots, y_n \in B$ which are generators for B as a C-module. Choose $x_i \in A$ mapping to y_i. Then $1, x_1, \ldots, x_n$ are generators for A as a D-module. The map $D \to A \times C$ is injective, and the ring $A \times C$ is finite as a D-module (because it is the direct sum of the finite D-modules A and C). Hence the lemma follows from the Artin-Tate lemma (Algebra, Lemma 51.7). $\square$

Lemma 5.2. *Let R be a Noetherian ring. Let I be a finite set. Suppose given a cartesian diagram*

$$\begin{tikzcd}
\prod B_i & \prod A_i \arrow[l, "\prod \varphi_i"'] \\
Q \arrow[u, "\prod \psi_i"] & P \arrow[l] \arrow[u]
\end{tikzcd}$$

with ψ_i and φ_i surjective, and Q, A_i, B_i of finite type over R. Then P is of finite type over R.

Proof. Follows from Lemma 5.1 and induction on the size of I. Namely, let $I = I' \amalg \{i_0\}$. Let P' be the ring defined by the diagram of the lemma using I'. Then P' is of finite type by induction hypothesis. Finally, P sits in a fibre product diagram

$$\begin{tikzcd}
B_{i_0} & A_{i_0} \arrow[l] \\
P' \arrow[u] & P \arrow[l] \arrow[u]
\end{tikzcd}$$

to which the lemma applies. $\square$

Lemma 5.3. *Suppose given a cartesian diagram of rings*

$$\begin{tikzcd}
R & R' \arrow[l, "t"'] \\
B \arrow[u, "s"] & B' \arrow[l] \arrow[u]
\end{tikzcd}$$

i.e., $B' = B \times_R R'$. If $h \in B'$ corresponds to $g \in B$ and $f \in R'$ such that $s(g) = t(f)$, then the diagram

$$\begin{tikzcd} R_{s(g)} = R_{t(f)} & (R')_f \arrow[l, "t"'] \\ B_g \arrow[u, "s"] & (B')_h \arrow[l] \arrow[u] \end{tikzcd}$$

is cartesian too.

Proof. The equality $B' = B \times_R R'$ tells us that

$$0 \to B' \to B \oplus R' \xrightarrow{s,-t} R$$

is an exact sequence of B'-modules. We have $B_g = B_h$, $R'_f = R'_h$, and $R_{s(g)} = R_{t(f)} = R_h$ as B'-modules. By exactness of localization (Algebra, Proposition 9.12) we find that

$$0 \to B'_h \to B_g \oplus R'_f \xrightarrow{s,-t} R_{s(g)} = R_{t(f)}$$

is an exact sequence. This proves the lemma. ☐

Consider a commutative diagram of rings

$$\begin{tikzcd} R & R' \arrow[l] \\ B \arrow[u] & B' \arrow[l] \arrow[u] \end{tikzcd}$$

Consider the functor (where the fibre product of categories is as constructed in Categories, Example 31.3)

$$\text{(5.3.1)} \qquad \text{Mod}_{B'} \longrightarrow \text{Mod}_B \times_{\text{Mod}_R} \text{Mod}_{R'}, \quad L' \longmapsto (L' \otimes_{B'} B, L' \otimes_{B'} R', can)$$

where *can* is the canonical identification $L' \otimes_{B'} B \otimes_B R = L' \otimes_{B'} R' \otimes_{R'} R$. In the following we will write (N, M', φ) for an object of the right hand side, i.e., N is a B-module, M' is an R'-module and $\varphi : N \otimes_B R \to M' \otimes_{R'} R$ is an isomorphism.

Lemma 5.4. *Given a commutative diagram of rings*

$$\begin{tikzcd} R & R' \arrow[l] \\ B \arrow[u] & B' \arrow[l] \arrow[u] \end{tikzcd}$$

the functor (5.3.1) has a right adjoint, namely the functor

$$F : (N, M', \varphi) \longmapsto N \times_\varphi M'$$

(see proof for elucidation).

Proof. Given an object (N, M', φ) of the category $\text{Mod}_B \times_{\text{Mod}_R} \text{Mod}_{R'}$ we set

$$N \times_\varphi M' = \{(n, m') \in N \times M' \mid \varphi(n \otimes 1) = m' \otimes 1 \text{ in } M' \otimes_{R'} R\}$$

viewed as a B'-module. The adjointness statement is that for a B'-module L' and a triple (N, M', φ) we have

$$\text{Hom}_{B'}(L', N \times_\varphi M') = \text{Hom}_B(L' \otimes_{B'} B, N) \times_{\text{Hom}_R(L' \otimes_{B'} R, M' \otimes_{R'} R)} \text{Hom}_{R'}(L' \otimes_{B'} R', M')$$

By Algebra, Lemma 14.3 the right hand side is equal to

$$\text{Hom}_{B'}(L', N) \times_{\text{Hom}_{B'}(L', M' \otimes_{R'} R)} \text{Hom}_{B'}(L', M')$$

Thus it is clear that for a pair (g, f') of elements of this fibre product we get an B'-linear map $L' \to N \times_\varphi M'$, $l' \mapsto (g(l'), f'(l'))$. Conversely, given a B' linear map $g' : L' \to N \times_\varphi M'$ we can set g equal to the composition $L' \to N \times_\varphi M' \to N$ and f' equal to the composition $L' \to N \times_\varphi M' \to M'$. These constructions are mutually inverse to each other and define the desired isomorphism. $\square$

6. Fibre products of rings, II

In this section we discuss fibre products in the following situation.

Situation 6.1. In the following we will consider ring maps

$$B \longrightarrow A \longleftarrow A'$$

where we assume $A' \to A$ is surjective with kernel I. In this situation we set $B' = B \times_A A'$ to obtain a cartesian square

$$\begin{tikzcd} A & A' \arrow[l] \\ B \arrow[u] & B' \arrow[l] \arrow[u] \end{tikzcd}$$

Lemma 6.2. *In Situation 6.1 we have*

$$\mathrm{Spec}(B') = \mathrm{Spec}(B) \amalg_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$$

as topological spaces.

Proof. Since $B' = B \times_A A'$ we obtain a commutative square of spectra, which induces a continuous map

$$can : \mathrm{Spec}(B) \amalg_{\mathrm{Spec}(A)} \mathrm{Spec}(A') \longrightarrow \mathrm{Spec}(B')$$

as the source is a pushout in the category of topological spaces (which exists by Topology, Section 29).

To show the map can is surjective, let $\mathfrak{q}' \subset B'$ be a prime ideal. If $I \subset \mathfrak{q}'$ (here and below we take the liberty of considering I as an ideal of B' as well as an ideal of A'), then $\mathfrak{q}'$ corresponds to a prime ideal of B and is in the image. If not, then pick $h \in I$, $h \notin \mathfrak{q}'$. In this case $B_h = A_h = 0$ and the ring map $B'_h \to A'_h$ is an isomorphism, see Lemma 5.3. Thus we see that $\mathfrak{q}'$ corresponds to a unique prime ideal $\mathfrak{p}' \subset A'$ which does not contain I.

Since $B' \to B$ is surjective, we see that can is injective on the summand $\mathrm{Spec}(B)$. We have seen above that $\mathrm{Spec}(A') \to \mathrm{Spec}(B')$ is injective on the complement of $V(I) \subset \mathrm{Spec}(A')$. Since $V(I) \subset \mathrm{Spec}(A')$ is exactly the image of $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ a trivial set theoretic argument shows that can is injective.

To finish the proof we have to show that can is open. To do this, observe that an open of the pushout is of the form $V \amalg U'$ where $V \subset \mathrm{Spec}(B)$ and $U' \subset \mathrm{Spec}(A')$ are opens whose inverse images in $\mathrm{Spec}(A)$ agree. Let $v \in V$. We can find a $g \in B$ such that $v \in D(g) \subset V$. Let $f \in A$ be the image. Pick $f' \in A'$ mapping to f. Then $D(f') \cap U' \cap V(I) = D(f') \cap V(I)$. Hence $V(I) \cap D(f')$ and $D(f') \cap (U')^c$ are disjoint closed subsets of $D(f') = \mathrm{Spec}(A'_{f'})$. Write $(U')^c = V(J)$ for some ideal $J \subset A'$. Since $A'_{f'} \to A'_{f'}/IA'_{f'} \times A'_{f'}/JA'_{f'}$ is surjective by the disjointness just shown, we can find an $a'' \in A'_{f'}$ mapping to 1 in $A'_{f'}/IA'_{f'}$ and mapping to zero in $A'_{f'}/JA'_{f'}$. Clearing denominators, we find an element $a' \in J$ mapping to f^n in A.

Then $D(a'f') \subset U'$. Let $h' = (g^{n+1}, a'f') \in B'$. Since $B'_{h'} = B_{g^{n+1}} \times_{A_{f^{n+1}}} A'_{a'f'}$ by a previously cited lemma, we see that $D(h')$ pulls back to an open neighbourhood of v in the pushout, i.e., the image of $V \amalg U'$ contains an open neighbourhood of the image of v. We omit the (easier) proof that the same thing is true for $u' \in U'$ with $u' \notin V(I)$. $\square$

Lemma 6.3. *In Situation 6.1 if $B \to A$ is integral, then $B' \to A'$ is integral.*

Proof. Let $a' \in A'$ with image $a \in A$. Let $x^d + b_1 x^{d-1} + \ldots + b_d$ be a monical polynomial with coefficients in B satisfied by a. Choose $b'_i \in B'$ mapping to $b_i \in B$ (possible). Then $(a')^d + b'_1(a')^{d-1} + \ldots + b'_d$ is in the kernel of $A' \to A$. Since $\mathrm{Ker}(B' \to B) = \mathrm{Ker}(A' \to A)$ we can modify our choice of b'_d to get $(a')^d + b'_1(a')^{d-1} + \ldots + b'_d = 0$ as desired. $\square$

In Situation 6.1 we'd like to understand B'-modules in terms of modules over A', A, and B. In order to do this we consider the functor (where the fibre product of categories as constructed in Categories, Example 31.3)

$$\mathrm{Mod}_{B'} \longrightarrow \mathrm{Mod}_B \times_{\mathrm{Mod}_A} \mathrm{Mod}_{A'}, \quad L' \longmapsto (L' \otimes_{B'} B, L' \otimes_{B'} A', can) \tag{6.3.1}$$

where can is the canonical identification $L' \otimes_{B'} B \otimes_B A = L' \otimes_{B'} A' \otimes_{A'} A$. In the following we will write (N, M', φ) for an object of the right hand side, i.e., N is a B-module, M' is an A'-module and $\varphi : N \otimes_B A \to M' \otimes_{A'} A$ is an isomorphism. However, it is often more convenient think of φ as a B-linear map $\varphi : N \to M'/IM'$ which induces an isomorphism $N \otimes_B A \to M' \otimes_{A'} A = M'/IM'$.

Lemma 6.4. *In Situation 6.1 the functor (6.3.1) has a right adjoint, namely the functor*

$$F : (N, M', \varphi) \longmapsto N \times_{\varphi, M} M'$$

where $M = M'/IM'$. Moreover, the composition of F with (6.3.1) is the identity functor on $Mod_B \times_{Mod_A} Mod_{A'}$. In other words, setting $N' = N \times_{\varphi, M} M'$ we have $N' \otimes_{B'} B = N$ and $N' \otimes_{B'} A' = M'$.

Proof. The adjointness statement follows from the more general Lemma 5.4. To prove the final assertion, recall that $B' = B \times_A A'$ and $N' = N \times_{\varphi, M} M'$ and extend these equalities to

$$\begin{tikzcd} A & A' \arrow[l] & I \arrow[l] \\ B \arrow[u] & B' \arrow[l] \arrow[u] & J \arrow[l] \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M & M' \arrow[l] & K \arrow[l] \\ N \arrow[u, "\varphi"] & N' \arrow[l] \arrow[u] & L \arrow[l] \arrow[u] \end{tikzcd}$$

where I, J, K, L are the kernels of the horizontal maps of the original diagrams. We present the proof as a sequence of observations:

(1) $K = IM'$ (see statement lemma),
(2) $B' \to B$ is surjective with kernel J and $J \to I$ is bijective,
(3) $N' \to N$ is surjective with kernel L and $L \to K$ is bijective,
(4) $JN' \subset L$,
(5) $\mathrm{Im}(N \to M)$ generates M as an A-module (because $N \otimes_B A = M$),
(6) $\mathrm{Im}(N' \to M')$ generates M' as an A'-module (because it holds modulo K and L maps isomorphically to K),
(7) $JN' = L$ (because $L \cong K = IM'$ is generated by images of elements xn' with $x \in I$ and $n' \in N'$ by the previous statement),

(8) $N' \otimes_{B'} B = N$ (because $N = N'/L$, $B = B'/J$, and the previous statement),
(9) there is a map $\gamma : N' \otimes_{B'} A' \to M'$,
(10) γ is surjective (see above),
(11) the kernel of the composition $N' \otimes_{B'} A' \to M' \to M$ is generated by elements $l \otimes 1$ and $n' \otimes x$ with $l \in K$, $n' \in N'$, $x \in I$ (because $M = N \otimes_B A$ by assumption and because $N' \to N$ and $A' \to A$ are surjective with kernels L and I),
(12) any element of $N' \otimes_{B'} A'$ in the submodule generated by the elements $l \otimes 1$ and $n' \otimes x$ with $l \in L$, $n' \in N'$, $x \in I$ can be written as $l \otimes 1$ for some $l \in L$ (because J maps isomorphically to I we see that $n' \otimes x = n'x \otimes 1$ in $N' \otimes_{B'} A'$; similarly $xn' \otimes a' = n' \otimes xa' = n'(xa') \otimes 1$ in $N' \otimes_{B'} A'$ when $n' \in N'$, $x \in J$ and $a' \in A'$; since we have seen that $JN' = L$ this proves the assertion),
(13) the kernel of γ is zero (because by (10) and (11) any element of the kernel is of the form $l \otimes 1$ with $l \in L$ which is mapped to $l \in K \subset M'$ by γ).

This finishes the proof. $\square$

Lemma 6.5. *In the situation of Lemma 6.4 for a B'-module L' the adjunction map*

$$L' \longrightarrow (L' \otimes_{B'} B) \times_{(L' \otimes_{B'} A)} (L' \otimes_{B'} A')$$

is surjective but in general not injective.

Proof. As in the proof of Lemma 6.4 let $J \subset B'$ be the kernel of the map $B' \to B$. Then $L' \otimes_{B'} B = L'/JL'$. Hence to prove surjectivity it suffices to show that elements of the form $(0, z)$ of the fibre product are in the image of the map of the lemma. The kernel of the map $L' \otimes_{B'} A' \to L' \otimes_{B'} A$ is the image of $L' \otimes_{B'} I \to L' \otimes_{B'} A'$. Since the map $J \to I$ induced by $B' \to A'$ is an isomorphism the composition

$$L' \otimes_{B'} J \to L' \to (L' \otimes_{B'} B) \times_{(L' \otimes_{B'} A)} (L' \otimes_{B'} A')$$

induces a surjection of $L' \otimes_{B'} J$ onto the set of elements of the form $(0, z)$. To see the map is not injective in general we present a simple example. Namely, take a field k, set $B' = k[x, y]/(xy)$, $A' = B'/(x)$, $B = B'/(y)$, $A = B'/(x, y)$ and $L' = B'/(x - y)$. In that case the class of x in L' is nonzero but is mapped to zero under the displayed arrow. $\square$

Lemma 6.6. *In Situation 6.1 let $(N_1, M_1', \varphi_1) \to (N_2, M_2', \varphi_2)$ be a morphism of $Mod_B \times_{Mod_A} Mod_{A'}$ with $N_1 \to N_2$ and $M_1' \to M_2'$ surjective. Then*

$$N_1 \times_{\varphi_1, M_1} M_1' \to N_2 \times_{\varphi_2, M_2} M_2'$$

where $M_1 = M_1'/IM_1'$ and $M_2 = M_2'/IM_2'$ is surjective.

Proof. Pick $(x_2, y_2) \in N_2 \times_{\varphi_2, M_2} M_2'$. Choose $x_1 \in N_1$ mapping to x_2. Since $M_1' \to M_1$ is surjective we can find $y_1 \in M_1'$ mapping to $\varphi_1(x_1)$. Then (x_1, y_1) maps to (x_2, y_2') in $N_2 \times_{\varphi_2, M_2} M_2'$. Thus it suffices to show that elements of the form $(0, y_2)$ are in the image of the map. Here we see that $y_2 \in IM_2'$. Write $y_2 = \sum t_i y_{2,i}$ with $t_i \in I$. Choose $y_{1,i} \in M_1'$ mapping to $y_{2,i}$. Then $y_1 = \sum t_i y_{1,i} \in IM_1'$ and the element $(0, y_1)$ does the job. $\square$

Lemma 6.7. *Let $A, A', B, B', I, M, M', N, \varphi$ be as in Lemma 6.4. If N finite over B and M' finite over A', then $N' = N \times_{\varphi, M} M'$ is finite over B'.*

Proof. We will use the results of Lemma 6.4 without further mention. Choose generators $y_1, \ldots, y_r$ of N over B and generators $x_1, \ldots, x_s$ of M' over A'. Using that $N = N' \otimes_{B'} B$ and $B' \to B$ is surjective we can find $u_1, \ldots, u_r \in N'$ mapping to $y_1, \ldots, y_r$ in N. Using that $M' = N' \otimes_{B'} A'$ we can find $v_1, \ldots, v_t \in N'$ such that $x_i = \sum v_j \otimes a'_{ij}$ for some $a'_{ij} \in A'$. In particular we see that the images $\overline{v}_j \in M'$ of the v_j generate M' over A'. We claim that $u_1, \ldots, u_r, v_1, \ldots, v_t$ generate N' as a B'-module. Namely, pick $\xi \in N'$. We first choose $b'_1, \ldots, b'_r \in B'$ such that ξ and $\sum b'_i u_i$ map to the same element of N. This is possible because $B' \to B$ is surjective and $y_1, \ldots, y_r$ generate N over B. The difference $\xi - \sum b'_i u_i$ is of the form $(0, \theta)$ for some θ in IM'. Say θ is $\sum t_j \overline{v}_j$ with $t_j \in I$. As $J = \text{Ker}(B' \to B)$ maps isomorphically to I we can choose $s_j \in J \subset B'$ mapping to t_j. Because $N' = N \times_{\varphi, M} M'$ it follows that $\xi = \sum b'_i u_i + \sum s_j v_j$ as desired. $\square$

Lemma 6.8. *With A, A', B, B', I as in Situation 6.1.*

(1) *Let (N, M', φ) be an object of $Mod_B \times_{Mod_A} Mod_{A'}$. If M' is flat over A' and N is flat over B, then $N' = N \times_{\varphi, M} M'$ is flat over B'.*
(2) *If L' is a flat B'-module, then $L' = (L \otimes_{B'} B) \times_{(L \otimes_{B'} A)} (L \otimes_{B'} A')$.*
(3) *The category of flat B'-modules is equivalent to the full subcategory of $Mod_B \times_{Mod_A} Mod_{A'}$ consisting of triples (N, M', φ) with N flat over B and M' flat over A'.*

Proof. In the proof we will use Lemma 6.4 without further mention.

Proof of (1). Set $J = \text{Ker}(B' \to B)$. This is an ideal of B' mapping isomorphically to $I = \text{Ker}(A' \to A)$. Let $\mathfrak{b}' \subset B'$ be an ideal. We have to show that $\mathfrak{b}' \otimes_{B'} N' \to N'$ is injective, see Algebra, Lemma 39.5. We know that

$$\mathfrak{b}'/(\mathfrak{b}' \cap J) \otimes_{B'} N' = \mathfrak{b}'/(\mathfrak{b}' \cap J) \otimes_B N \to N$$

is injective as N is flat over B. As $\mathfrak{b}' \cap J \to \mathfrak{b}' \to \mathfrak{b}'/(\mathfrak{b}' \cap J) \to 0$ is exact, we conclude that it suffices to show that $(\mathfrak{b}' \cap J) \otimes_{B'} N' \to N'$ is injective. Thus we may assume that $\mathfrak{b}' \subset J$. Next, since $J \to I$ is an isomorphism we have

$$J \otimes_{B'} N' = I \otimes_{A'} A' \otimes_{B'} N' = I \otimes_{A'} M'$$

which maps injectively into M' as M' is a flat A'-module. Hence $J \otimes_{B'} N' \to N'$ is injective and we conclude that $\text{Tor}_1^{B'}(B'/J, N') = 0$, see Algebra, Remark 75.9. Thus we may apply Algebra, Lemma 99.8 to N' over B' and the ideal J. Going back to our ideal $\mathfrak{b}' \subset J$, let $\mathfrak{b}' \subset \mathfrak{b}'' \subset J$ be the smallest ideal whose image in I is an A'-submodule of I. In other words, we have $\mathfrak{b}'' = A'\mathfrak{b}'$ if we view $J = I$ as A'-module. Then $\mathfrak{b}''/\mathfrak{b}'$ is killed by J and we get a short exact sequence

$$0 \to \mathfrak{b}' \otimes_{B'} N' \to \mathfrak{b}'' \otimes_{B'} N' \to \mathfrak{b}''/\mathfrak{b}' \otimes_{B'} N' \to 0$$

by the vanishing of $\text{Tor}_1^{B'}(\mathfrak{b}''/\mathfrak{b}', N')$ we get from the application of the lemma. Thus we may replace $\mathfrak{b}'$ by $\mathfrak{b}''$. In particular we may assume $\mathfrak{b}'$ is an A'-module and maps to an ideal of A'. Then

$$\mathfrak{b}' \otimes_{B'} N' = \mathfrak{b}' \otimes_{A'} A' \otimes_{B'} N' = \mathfrak{b}' \otimes_{A'} M'$$

This tensor product maps injectively into M' by our assumption that M' is flat over A'. We conclude that $\mathfrak{b}' \otimes_{B'} N' \to N' \to M'$ is injective and hence the first map is injective as desired.

Proof of (2). This follows by tensoring the short exact sequence $0 \to B' \to B \oplus A' \to A \to 0$ with L' over B'.

Proof of (3). Immediate consequence of (1) and (2). $\square$

Lemma 6.9. *Let A, A', B, B', I be as in Situation 6.1. The category of finite projective B'-modules is equivalent to the full subcategory of $Mod_B \times_{Mod_A} Mod_{A'}$ consisting of triples (N, M', φ) with N finite projective over B and M' finite projective over A'.*

Proof. Recall that a module is finite projective if and only if it is finitely presented and flat, see Algebra, Lemma 78.2. Using Lemmas 6.8 and 6.7 we reduce to showing that $N' = N \times_{\varphi, M} M'$ is a B'-module of finite presentation if N finite projective over B and M' finite projective over A'.

By Lemma 6.7 the module N' is finite over B'. Choose a surjection $(B')^{\oplus n} \to N'$ with kernel K'. By base change we obtain maps $B^{\oplus n} \to N$, $(A')^{\oplus n} \to M'$, and $A^{\oplus n} \to M$ with kernels K_B, $K_{A'}$, and K_A. There is a canonical map

$$K' \longrightarrow K_B \times_{K_A} K_{A'}$$

On the other hand, since $N' = N \times_{\varphi, M} M'$ and $B' = B \times_A A'$ there is also a canonical map $K_B \times_{K_A} K_{A'} \to K'$ inverse to the displayed arrow. Hence the displayed map is an isomorphism. By Algebra, Lemma 5.3 the modules K_B and $K_{A'}$ are finite. We conclude from Lemma 6.7 that K' is a finite B'-module provided that $K_B \to K_A$ and $K_{A'} \to K_A$ induce isomorphisms $K_B \otimes_B A = K_A = K_{A'} \otimes_{A'} A$. This is true because the flatness assumptions implies the sequences

$$0 \to K_B \to B^{\oplus n} \to N \to 0 \quad \text{and} \quad 0 \to K_{A'} \to (A')^{\oplus n} \to M' \to 0$$

stay exact upon tensoring, see Algebra, Lemma 39.12. $\square$

7. Fibre products of rings, III

In this section we discuss fibre products in the following situation.

Situation 7.1. Let A, A', B, B', I be as in Situation 6.1. Let $B' \to D'$ be a ring map. Set $D = D' \otimes_{B'} B$, $C' = D' \otimes_{B'} A'$, and $C = D' \otimes_{B'} A$. This leads to a big commutative diagram

$$\begin{tikzcd}
C & & & C' \arrow[lll] \\
& A \arrow[ul] & A' \arrow[l] \arrow[ur] & \\
& B \arrow[u] \arrow[dl] & B' \arrow[l] \arrow[u] \arrow[dr] & \\
D \arrow[uuu] & & & D' \arrow[lll] \arrow[uuu]
\end{tikzcd}$$

of rings. Observe that we do **not** assume that the map $D' \to D \times_C C'$ is an isomorphism[1]. In this situation we have the functor

$$(7.1.1) \qquad \mathrm{Mod}_{D'} \longrightarrow \mathrm{Mod}_D \times_{\mathrm{Mod}_C} \mathrm{Mod}_{C'}, \quad L' \longmapsto (L' \otimes_{D'} D, L' \otimes_{D'} C', can)$$

[1]But $D' \to D \times_C C'$ is surjective by Lemma 6.5.

analogous to (6.3.1). Note that $L' \otimes_{D'} D = L \otimes_{D'} (D' \otimes_{B'} B) = L \otimes_{B'} B$ and similarly $L' \otimes_{D'} C' = L \otimes_{D'} (D' \otimes_{B'} A') = L \otimes_{B'} A'$ hence the diagram

$$\begin{tikzcd} \mathrm{Mod}_{D'} \arrow[r] \arrow[d] & \mathrm{Mod}_D \times_{\mathrm{Mod}_C} \mathrm{Mod}_{C'} \arrow[d] \\ \mathrm{Mod}_{B'} \arrow[r] & \mathrm{Mod}_B \times_{\mathrm{Mod}_A} \mathrm{Mod}_{A'} \end{tikzcd}$$

is commutative. In the following we will write (N, M', φ) for an object of $\mathrm{Mod}_D \times_{\mathrm{Mod}_C} \mathrm{Mod}_{C'}$, i.e., N is a D-module, M' is an C'-module and $\varphi : N \otimes_B A \to M' \otimes_{A'} A$ is an isomorphism of C-modules. However, it is often more convenient think of φ as a D-linear map $\varphi : N \to M'/IM'$ which induces an isomorphism $N \otimes_B A \to M' \otimes_{A'} A = M'/IM'$.

Lemma 7.2. *In Situation 7.1 the functor (7.1.1) has a right adjoint, namely the functor*

$$F : (N, M', \varphi) \longmapsto N \times_{\varphi, M} M'$$

where $M = M'/IM'$. Moreover, the composition of F with (7.1.1) is the identity functor on $Mod_D \times_{Mod_C} Mod_{C'}$. In other words, setting $N' = N \times_{\varphi, M} M'$ we have $N' \otimes_{D'} D = N$ and $N' \otimes_{D'} C' = M'$.

Proof. The adjointness statement follows from the more general Lemma 5.4. The final assertion follows from the corresponding assertion of Lemma 6.4 because $N' \otimes_{D'} D = N' \otimes_{D'} D' \otimes_{B'} B = N' \otimes_{B'} B$ and $N' \otimes_{D'} C' = N' \otimes_{D'} D' \otimes_{B'} A' = N' \otimes_{B'} A'$. □

Lemma 7.3. *In Situation 7.1 the map $JD' \to IC'$ is surjective where $J = \mathrm{Ker}(B' \to B)$.*

Proof. Since $C' = D' \otimes_{B'} A'$ we have that IC' is the image of $D' \otimes_{B'} I = C' \otimes_{A'} I \to C'$. As the ring map $B' \to A'$ induces an isomorphism $J \to I$ the lemma follows. □

Lemma 7.4. *Let $A, A', B, B', C, C', D, D', I, M', M, N, \varphi$ be as in Lemma 7.2. If N finite over D and M' finite over C', then $N' = N \times_{\varphi, M} M'$ is finite over D'.*

Proof. Recall that $D' \to D \times_C C'$ is surjective by Lemma 6.5. Observe that $N' = N \times_{\varphi, M} M'$ is a module over $D \times_C C'$. We can apply Lemma 6.7 to the data $C, C', D, D', IC', M', M, N, \varphi$ to see that $N' = N \times_{\varphi, M} M'$ is finite over $D \times_C C'$. Thus it is finite over D'. □

Lemma 7.5. *With $A, A', B, B', C, C', D, D', I$ as in Situation 7.1.*

1. *Let (N, M', φ) be an object of $Mod_D \times_{Mod_C} Mod_{C'}$. If M' is flat over A' and N is flat over B, then $N' = N \times_{\varphi, M} M'$ is flat over B'.*
2. *If L' is a D'-module flat over B', then $L' = (L \otimes_{D'} D) \times_{(L \otimes_{D'} C)} (L \otimes_{D'} C')$.*
3. *The category of D'-modules flat over B' is equivalent to the categories of objects (N, M', φ) of $Mod_D \times_{Mod_C} Mod_{C'}$ with N flat over B and M' flat over A'.*

Proof. Part (1) follows from part (1) of Lemma 6.8.

Part (2) follows from part (2) of Lemma 6.8 using that $L' \otimes_{D'} D = L' \otimes_{B'} B$, $L' \otimes_{D'} C' = L' \otimes_{B'} A'$, and $L' \otimes_{D'} C = L' \otimes_{B'} A$, see discussion in Situation 7.1.

Part (3) is an immediate consequence of (1) and (2). □

The following lemma is a good deal more interesting than its counter part in the absolute case (Lemma 6.9), although the proof is essentially the same.

Lemma 7.6. *Let $A, A', B, B', C, C', D, D', I, M', M, N, \varphi$ be as in Lemma 7.2. If*

(1) *N is finitely presented over D and flat over B,*
(2) *M' finitely presented over C' and flat over A', and*
(3) *the ring map $B' \to D'$ factors as $B' \to D'' \to D'$ with $B' \to D''$ flat and $D'' \to D'$ of finite presentation,*

then $N' = N \times_M M'$ is finitely presented over D'.

Proof. Choose a surjection $D''' = D''[x_1, \ldots, x_n] \to D'$ with finitely generated kernel J. By Algebra, Lemma 36.23 it suffices to show that N' is finitely presented as a D'''-module. Moreover, $D''' \otimes_{B'} B \to D' \otimes_{B'} B = D$ and $D''' \otimes_{B'} A' \to D' \otimes_{B'} A' = C'$ are surjections whose kernels are generated by the image of J, hence N is a finitely presented $D''' \otimes_{B'} B$-module and M' is a finitely presented $D''' \otimes_{B'} A'$-module by Algebra, Lemma 36.23 again. Thus we may replace D' by D''' and D by $D''' \otimes_{B'} B$, etc. Since D''' is flat over B', it follows that we may assume that $B' \to D'$ is flat.

Assume $B' \to D'$ is flat. By Lemma 7.4 the module N' is finite over D'. Choose a surjection $(D')^{\oplus n} \to N'$ with kernel K'. By base change we obtain maps $D^{\oplus n} \to N$, $(C')^{\oplus n} \to M'$, and $C^{\oplus n} \to M$ with kernels K_D, $K_{C'}$, and K_C. There is a canonical map

$$K' \longrightarrow K_D \times_{K_C} K_{C'}$$

On the other hand, since $N' = N \times_M M'$ and $D' = D \times_C C'$ (by Lemma 6.8; applied to the flat B'-module D') there is also a canonical map $K_D \times_{K_C} K_{C'} \to K'$ inverse to the displayed arrow. Hence the displayed map is an isomorphism. By Algebra, Lemma 5.3 the modules K_D and $K_{C'}$ are finite. We conclude from Lemma 7.4 that K' is a finite D'-module provided that $K_D \to K_C$ and $K_{C'} \to K_C$ induce isomorphisms $K_D \otimes_B A = K_C = K_{C'} \otimes_{A'} A$. This is true because the flatness assumptions implies the sequences

$$0 \to K_D \to D^{\oplus n} \to N \to 0 \quad \text{and} \quad 0 \to K_{C'} \to (C')^{\oplus n} \to M' \to 0$$

stay exact upon tensoring, see Algebra, Lemma 39.12. $\square$

Lemma 7.7. *Let A, A', B, B', I be as in Situation 6.1. Let (D, C', φ) be a system consisting of an B-algebra D, a A'-algebra C' and an isomorphism $D \otimes_B A \to C'/IC' = C$. Set $D' = D \times_C C'$ (as in Lemma 6.4). Then*

(1) *$B' \to D'$ is finite type if and only if $B \to D$ and $A' \to C'$ are finite type,*
(2) *$B' \to D'$ is flat if and only if $B \to D$ and $A' \to C'$ are flat,*
(3) *$B' \to D'$ is flat and of finite presentation if and only if $B \to D$ and $A' \to C'$ are flat and of finite presentation,*
(4) *$B' \to D'$ is smooth if and only if $B \to D$ and $A' \to C'$ are smooth,*
(5) *$B' \to D'$ is étale if and only if $B \to D$ and $A' \to C'$ are étale.*

Moreover, if D' is a flat B'-algebra, then $D' \to (D' \otimes_{B'} B) \times_{(D' \otimes_{B'} A)} (D' \otimes_{B'} A')$ is an isomorphism. In this way the category of flat B'-algebras is equivalent to the categories of systems (D, C', φ) as above with D flat over B and C' flat over A'.

Proof. The implication "⇒" follows from Algebra, Lemmas 14.2, 39.7, 137.4, and 143.3 because we have $D' \otimes_{B'} B = D$ and $D' \otimes_{B'} A' = C'$ by Lemma 6.4. Thus it suffices to prove the implications in the other direction.

Ad (1). Assume D of finite type over B and C' of finite type over A'. We will use the results of Lemma 6.4 without further mention. Choose generators $x_1, \ldots, x_r$ of D over B and generators $y_1, \ldots, y_s$ of C' over A'. Using that $D = D' \otimes_{B'} B$ and $B' \to B$ is surjective we can find $u_1, \ldots, u_r \in D'$ mapping to $x_1, \ldots, x_r$ in D. Using that $C' = D' \otimes_{B'} A'$ we can find $v_1, \ldots, v_t \in D'$ such that $y_i = \sum v_j \otimes a'_{ij}$ for some $a'_{ij} \in A'$. In particular, the images of v_j in C' generate C' as an A'-algebra. Set $N = r + t$ and consider the cube of rings

$$\begin{tikzcd}
A[x_1, \ldots, x_N] & & A'[x_1, \ldots, x_N] \arrow[ll] & \\
& A \arrow[ul] & & A' \arrow[ll] \arrow[ul] \\
B[x_1, \ldots, x_N] \arrow[uu] & & B'[x_1, \ldots, x_N] \arrow[ll] \arrow[uu] & \\
& B \arrow[ul] \arrow[uu] & & B' \arrow[ll] \arrow[ul] \arrow[uu]
\end{tikzcd}$$

Observe that the back square is cartesian as well. Consider the ring map

$$B'[x_1, \ldots, x_N] \to D', \quad x_i \mapsto u_i \quad \text{and} \quad x_{r+j} \mapsto v_j.$$

Then we see that the induced maps $B[x_1, \ldots, x_N] \to D$ and $A'[x_1, \ldots, x_N] \to C'$ are surjective, in particular finite. We conclude from Lemma 7.4 that $B'[x_1, \ldots, x_N] \to D'$ is finite, which implies that D' is of finite type over B' for example by Algebra, Lemma 6.2.

Ad (2). The implication "$\Leftarrow$" follows from Lemma 7.5. Moreover, the final statement follows from the final statement of Lemma 7.5.

Ad (3). Assume $B \to D$ and $A' \to C'$ are flat and of finite presentation. The flatness of $B' \to D'$ we've seen in (2). We know $B' \to D'$ is of finite type by (1). Choose a surjection $B'[x_1, \ldots, x_N] \to D'$. By Algebra, Lemma 6.3 the ring D is of finite presentation as a $B[x_1, \ldots, x_N]$-module and the ring C' is of finite presentation as a $A'[x_1, \ldots, x_N]$-module. By Lemma 7.6 we see that D' is of finite presentation as a $B'[x_1, \ldots, x_N]$-module, i.e., $B' \to D'$ is of finite presentation.

Ad (4). Assume $B \to D$ and $A' \to C'$ smooth. By (3) we see that $B' \to D'$ is flat and of finite presentation. By Algebra, Lemma 137.17 it suffices to check that $D' \otimes_{B'} k$ is smooth for any field k over B'. If the composition $J \to B' \to k$ is zero, then $B' \to k$ factors as $B' \to B \to k$ and we see that

$$D' \otimes_{B'} k = D' \otimes_{B'} B \otimes_B k = D \otimes_B k$$

is smooth as $B \to D$ is smooth. If the composition $J \to B' \to k$ is nonzero, then there exists an $h \in J$ which does not map to zero in k. Then $B' \to k$ factors as $B' \to B'_h \to k$. Observe that h maps to zero in B, hence $B_h = 0$. Thus by Lemma 5.3 we have $B'_h = A'_h$ and we get

$$D' \otimes_{B'} k = D' \otimes_{B'} B'_h \otimes_{B'_h} k = C'_h \otimes_{A'_h} k$$

is smooth as $A' \to C'$ is smooth.

Ad (5). Assume $B \to D$ and $A' \to C'$ are étale. By (4) we see that $B' \to D'$ is smooth. As we can read off whether or not a smooth map is étale from the dimension of fibres we see that (5) holds (argue as in the proof of (4) to identify fibres – some details omitted). ☐

Remark 7.8. In Situation 7.1. Assume $B' \to D'$ is of finite presentation and suppose we are given a D'-module L'. We claim there is a bijective correspondence between

1. surjections of D'-modules $L' \to Q'$ with Q' of finite presentation over D' and flat over B', and
2. pairs of surjections of modules $(L' \otimes_{D'} D \to Q_1, L' \otimes_{D'} C' \to Q_2)$ with
   (a) Q_1 of finite presentation over D and flat over B,
   (b) Q_2 of finite presentation over C' and flat over A',
   (c) $Q_1 \otimes_D C = Q_2 \otimes_{C'} C$ as quotients of $L' \otimes_{D'} C$.

The correspondence between these is given by $Q \mapsto (Q_1, Q_2)$ with $Q_1 = Q \otimes_{D'} D$ and $Q_2 = Q \otimes_{D'} C'$. And for the converse we use $Q = Q_1 \times_{Q_{12}} Q_2$ where Q_{12} the common quotient $Q_1 \otimes_D C = Q_2 \otimes_{C'} C$ of $L' \otimes_{D'} C$. As quotient map we use

$$L' \longrightarrow (L' \otimes_{D'} D) \times_{(L' \otimes_{D'} C)} (L' \otimes_{D'} C') \longrightarrow Q_1 \times_{Q_{12}} Q_2 = Q$$

where the first arrow is surjective by Lemma 6.5 and the second by Lemma 6.6. The claim follows by Lemmas 7.5 and 7.6.

8. Fitting ideals

The Fitting ideals of a finite module are the ideals determined by the construction of Lemma 8.2.

Lemma 8.1. *Let R be a ring. Let A be an $n \times m$ matrix with coefficients in R. Let $I_r(A)$ be the ideal generated by the $r \times r$-minors of A with the convention that $I_0(A) = R$ and $I_r(A) = 0$ if $r > \min(n, m)$. Then*

1. *$I_0(A) \supset I_1(A) \supset I_2(A) \supset \ldots$,*
2. *if B is an $(n + n') \times m$ matrix, and A is the first n rows of B, then $I_{r+n'}(B) \subset I_r(A)$,*
3. *if C is an $n \times n$ matrix then $I_r(CA) \subset I_r(A)$.*
4. *If A is a block matrix*
$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$
then $I_r(A) = \sum_{r_1+r_2=r} I_{r_1}(A_1)I_{r_2}(A_2)$.
5. *Add more here.*

Proof. Omitted. (Hint: Use that a determinant can be computed by expanding along a column or a row.) ☐

Lemma 8.2. *Let R be a ring. Let M be a finite R-module. Choose a presentation*

$$\bigoplus\nolimits_{j \in J} R \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0.$$

of M. Let $A = (a_{ij})_{i=1,\ldots,n, j \in J}$ be the matrix of the map $\bigoplus_{j \in J} R \to R^{\oplus n}$. The ideal $Fit_k(M)$ generated by the $(n-k) \times (n-k)$ minors of A is independent of the choice of the presentation.

Proof. Let $K \subset R^{\oplus n}$ be the kernel of the surjection $R^{\oplus n} \to M$. Pick $z_1, \ldots, z_{n-k} \in K$ and write $z_j = (z_{1j}, \ldots, z_{nj})$. Another description of the ideal $\text{Fit}_k(M)$ is that it is the ideal generated by the $(n-k) \times (n-k)$ minors of all the matrices (z_{ij}) we obtain in this way.

Suppose we change the surjection into the surjection $R^{\oplus n+n'} \to M$ with kernel K' where we use the original map on the first n standard basis elements of $R^{\oplus n+n'}$ and 0 on the last n' basis vectors. Then the corresponding ideals are the same. Namely, if $z_1, \ldots, z_{n-k} \in K$ as above, let $z'_j = (z_{1j}, \ldots, z_{nj}, 0, \ldots, 0) \in K'$ for $j = 1, \ldots, n-k$ and $z'_{n+j'} = (0, \ldots, 0, 1, 0, \ldots, 0) \in K'$. Then we see that the ideal of $(n-k) \times (n-k)$ minors of (z_{ij}) agrees with the ideal of $(n+n'-k) \times (n+n'-k)$ minors of (z'_{ij}). This gives one of the inclusions. Conversely, given $z'_1, \ldots, z'_{n+n'-k}$ in K' we can project these to $R^{\oplus n}$ to get $z_1, \ldots, z_{n+n'-k}$ in K. By Lemma 8.1 we see that the ideal generated by the $(n+n'-k) \times (n+n'-k)$ minors of (z'_{ij}) is contained in the ideal generated by the $(n-k) \times (n-k)$ minors of (z_{ij}). This gives the other inclusion.

Let $R^{\oplus m} \to M$ be another surjection with kernel L. By Schanuel's lemma (Algebra, Lemma 109.1) and the results of the previous paragraph, we may assume $m = n$ and that there is an isomorphism $R^{\oplus n} \to R^{\oplus m}$ commuting with the surjections to M. Let $C = (c_{li})$ be the (invertible) matrix of this map (it is a square matrix as $n = m$). Then given $z'_1, \ldots, z'_{n-k} \in L$ as above we can find $z_1, \ldots, z_{n-k} \in K$ with $z'_1 = Cz_1, \ldots, z'_{n-k} = Cz_{n-k}$. By Lemma 8.1 we get one of the inclusions. By symmetry we get the other. $\square$

Definition 8.3. Let R be a ring. Let M be a finite R-module. Let $k \geq 0$. The *kth Fitting ideal* of M is the ideal $\text{Fit}_k(M)$ constructed in Lemma 8.2. Set $\text{Fit}_{-1}(M) = 0$.

Since the Fitting ideals are the ideals of minors of a big matrix (numbered in reverse ordering from the ordering in Lemma 8.1) we see that

$$0 = \text{Fit}_{-1}(M) \subset \text{Fit}_0(M) \subset \text{Fit}_1(M) \subset \ldots \subset \text{Fit}_t(M) = R$$

for some $t \gg 0$. Here are some basic properties of Fitting ideals.

Lemma 8.4. *Let R be a ring. Let M be a finite R-module.*

(1) *If M can be generated by n elements, then $Fit_n(M) = R$.*
(2) *Given a second finite R-module M' we have*
$$Fit_l(M \oplus M') = \sum\nolimits_{k+k'=l} Fit_k(M) Fit_{k'}(M')$$
(3) *If $R \to R'$ is a ring map, then $Fit_k(M \otimes_R R')$ is the ideal of R' generated by the image of $Fit_k(M)$.*
(4) *If M is of finite presentation, then $Fit_k(M)$ is a finitely generated ideal.*
(5) *If $M \to M'$ is a surjection, then $Fit_k(M) \subset Fit_k(M')$.*
(6) *We have $Fit_0(M) \subset Ann_R(M)$.*
(7) *We have $V(Fit_0(M)) = Supp(M)$.*
(8) *Add more here.*

Proof. Part (1) follows from the fact that $I_0(A) = R$ in Lemma 8.1.

Part (2) follows form the corresponding statement in Lemma 8.1.

Part (3) follows from the fact that $\otimes_R R'$ is right exact, so the base change of a presentation of M is a presentation of $M \otimes_R R'$.

Proof of (4). Let $R^{\oplus m} \xrightarrow{A} R^{\oplus n} \to M \to 0$ be a presentation. Then $\text{Fit}_k(M)$ is the ideal generated by the $n-k \times n-k$ minors of the matrix A.

Part (5) is immediate from the definition.

Proof of (6). Choose a presentation of M with matrix A as in Lemma 8.2. Let $J' \subset J$ be a subset of cardinality n. It suffices to show that $f = \det(a_{ij})_{i=1,\ldots,n,j\in J'}$ annihilates M. This is clear because the cokernel of

$$R^{\oplus n} \xrightarrow{A'=(a_{ij})_{i=1,\ldots,n,j\in J'}} R^{\oplus n} \to M \to 0$$

is killed by f as there is a matrix B with $A'B = f1_{n\times n}$.

Proof of (7). Choose a presentation of M with matrix A as in Lemma 8.2. By Nakayama's lemma (Algebra, Lemma 20.1) we have

$$M_{\mathfrak p} \neq 0 \Leftrightarrow M \otimes_R \kappa(\mathfrak p) \neq 0 \Leftrightarrow \text{rank}(\text{image } A \text{ in } \kappa(\mathfrak p)) < n$$

Clearly $\text{Fit}_0(M)$ exactly cuts out the set of primes with this property. ☐

Example 8.5. Let R be a ring. The Fitting ideals of the finite free module $M = R^{\oplus n}$ are $\text{Fit}_k(M) = 0$ for $k < n$ and $\text{Fit}_k(M) = R$ for $k \geq n$.

Lemma 8.6. *Let R be a ring. Let M be a finite R-module. Let $k \geq 0$. Let $\mathfrak p \subset R$ be a prime ideal. The following are equivalent*

1. *$Fit_k(M) \not\subset \mathfrak p$,*
2. *$\dim_{\kappa(\mathfrak p)} M \otimes_R \kappa(\mathfrak p) \leq k$,*
3. *$M_{\mathfrak p}$ can be generated by k elements over $R_{\mathfrak p}$, and*
4. *M_f can be generated by k elements over R_f for some $f \in R$, $f \not\in \mathfrak p$.*

Proof. By Nakayama's lemma (Algebra, Lemma 20.1) we see that M_f can be generated by k elements over R_f for some $f \in R$, $f \not\in \mathfrak p$ if $M \otimes_R \kappa(\mathfrak p)$ can be generated by k elements. Hence (2), (3), and (4) are equivalent. Using Lemma 8.4 part (3) this reduces the problem to the case where R is a field and $\mathfrak p = (0)$. In this case the result follows from Example 8.5. ☐

Lemma 8.7. *Let R be a ring. Let M be a finite R-module. Let $r \geq 0$. The following are equivalent*

1. *M is finite locally free of rank r (Algebra, Definition 78.1),*
2. *$Fit_{r-1}(M) = 0$ and $Fit_r(M) = R$, and*
3. *$Fit_k(M) = 0$ for $k < r$ and $Fit_k(M) = R$ for $k \geq r$.*

Proof. It is immediate that (2) is equivalent to (3) because the Fitting ideals form an increasing sequence of ideals. Since the formation of $\text{Fit}_k(M)$ commutes with base change (Lemma 8.4) we see that (1) implies (2) by Example 8.5 and glueing results (Algebra, Section 23). Conversely, assume (2). By Lemma 8.6 we may assume that M is generated by r elements. Thus a presentation $\bigoplus_{j\in J} R \to R^{\oplus r} \to M \to 0$. But now the assumption that $\text{Fit}_{r-1}(M) = 0$ implies that all entries of the matrix of the map $\bigoplus_{j\in J} R \to R^{\oplus r}$ are zero. Thus M is free. ☐

Lemma 8.8. *Let R be a local ring. Let M be a finite R-module. Let $k \geq 0$. Assume that $Fit_k(M) = (f)$ for some $f \in R$. Let M' be the quotient of M by $\{x \in M \mid fx = 0\}$. Then M' can be generated by k elements.*

Proof. Choose generators $x_1, \ldots, x_n \in M$ corresponding to the surjection $R^{\oplus n} \to M$. Since R is local if a set of elements $E \subset (f)$ generates (f), then some $e \in E$ generates (f), see Algebra, Lemma 20.1. Hence we may pick $z_1, \ldots, z_{n-k}$ in the kernel of $R^{\oplus n} \to M$ such that some $(n-k) \times (n-k)$ minor of the $n \times (n-k)$ matrix $A = (z_{ij})$ generates (f). After renumbering the x_i we may assume the first minor $\det(z_{ij})_{1 \leq i,j \leq n-k}$ generates (f), i.e., $\det(z_{ij})_{1 \leq i,j \leq n-k} = uf$ for some unit $u \in R$. Every other minor is a multiple of f. By Algebra, Lemma 15.6 there exists a $n-k \times n-k$ matrix B such that

$$AB = f \begin{pmatrix} u1_{n-k \times n-k} \\ C \end{pmatrix}$$

for some matrix C with coefficients in R. This implies that for every $i \leq n-k$ the element $y_i = ux_i + \sum_j c_{ji}x_j$ is annihilated by f. Since $M/\sum Ry_i$ is generated by the images of $x_{n-k+1}, \ldots, x_n$ we win. $\square$

Lemma 8.9. *Let R be a ring. Let M be a finitely presented R-module. Let $k \geq 0$. Assume that $Fit_k(M) = (f)$ for some nonzerodivisor $f \in R$ and $Fit_{k-1}(M) = 0$. Then*

1. *M has projective dimension ≤ 1,*
2. *$M' = \mathrm{Ker}(f : M \to M)$ is the f-power torsion submodule of M,*
3. *M' has projective dimension ≤ 1,*
4. *M/M' is finite locally free of rank k, and*
5. *$M \cong M/M' \oplus M'$.*

Proof. Choose a presentation

$$R^{\oplus m} \xrightarrow{A} R^{\oplus n} \to M \to 0$$

for some matrix A with coefficients in R.

We first prove the lemma when R is local. Set $M' = \{x \in M \mid fx = 0\}$ as in the statement. By Lemma 8.8 we can choose $x_1, \ldots, x_k \in M$ which generate M/M'. Then $x_1, \ldots, x_k$ generate $M_f = (M/M')_f$. Hence, if there is a relation $\sum a_i x_i = 0$ in M, then we see that $a_1, \ldots, a_k$ map to zero in R_f since otherwise $\mathrm{Fit}_{k-1}(M)R_f = \mathrm{Fit}_{k-1}(M_f)$ would be nonzero. Since f is a nonzerodivisor, we conclude $a_1 = \ldots = a_k = 0$. Thus $M \cong R^{\oplus k} \oplus M'$. After a change of basis in our presentation above, we may assume the first $n-k$ basis vectors of $R^{\oplus n}$ map into the summand M' of M and the last k-basis vectors of $R^{\oplus n}$ map to basis elements of the summand $R^{\oplus k}$ of M. Having done so, the last k rows of the matrix A vanish. In this way we see that, replacing M by M', k by 0, n by $n-k$, and A by the submatrix where we delete the last k rows, we reduce to the case discussed in the next paragraph.

Assume R is local, $k = 0$, and M annihilated by f. Now the 0th Fitting ideal of M is (f) and is generated by the $n \times n$ minors of the matrix A of size $n \times m$. (This in particular implies $m \geq n$.) Since R is local, some $n \times n$ minor of A is uf for a unit $u \in R$. After renumbering we may assume this minor is the first one. Moreover, we know all other $n \times n$ minors of A are divisible by f. Write $A = (A_1 A_2)$ in block form where A_1 is an $n \times n$ matrix and A_2 is an $n \times (m-n)$ matrix. By Algebra, Lemma 15.6 applied to the transpose of A (!) we find there exists an $n \times n$ matrix B such that

$$BA = B(A_1 A_2) = f \begin{pmatrix} u1_{n \times n} & C \end{pmatrix}$$

for some $n \times (m-n)$ matrix C with coefficients in R. Then we first conclude $BA_1 = fu1_{n \times n}$. Thus

$$BA_2 = fC = u^{-1}fuC = u^{-1}BA_1C$$

Since the determinant of B is a nonzerodivisor we conclude that $A_2 = u^{-1}A_1C$. Therefore the image of A is equal to the image of A_1 which is isomorphic to $R^{\oplus n}$ because the determinant of A_1 is a nonzerodivisor. Hence M has projective dimension ≤ 1.

We return to the case of a general ring R. By the local case we see that M/M' is a finite locally free module of rank k, see Algebra, Lemma 78.2. Hence the extension $0 \to M' \to M \to M/M' \to 0$ splits. It follows that M' is a finitely presented module. Choose a short exact sequence $0 \to K \to R^{\oplus a} \to M' \to 0$. Then K is a finite R-module, see Algebra, Lemma 5.3. By the local case we see that $K_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus a}$ for all primes. Hence by Algebra, Lemma 78.2 again we see that K is finite locally free of rank a. It follows that M' has projective dimension ≤ 1 and the lemma is proved. □

9. Lifting

In this section we collection some lemmas concerning lifting statements of the following kind: If A is a ring and $I \subset A$ is an ideal, and $\overline{\xi}$ is some kind of structure over A/I, then we can lift $\overline{\xi}$ to a similar kind of structure ξ over A or over some étale extension of A. Here are some types of structure for which we have already proved some results:

(1) idempotents, see Algebra, Lemmas 32.6 and 32.7,
(2) projective modules, see Algebra, Lemmas 77.5 and 77.6,
(3) finite stably free modules, see Lemma 3.3,
(4) basis elements, see Algebra, Lemmas 101.1 and 101.3,
(5) ring maps, i.e., proving certain algebras are formally smooth, see Algebra, Lemma 138.4, Proposition 138.13, and Lemma 138.17,
(6) syntomic ring maps, see Algebra, Lemma 136.18,
(7) smooth ring maps, see Algebra, Lemma 137.20,
(8) étale ring maps, see Algebra, Lemma 143.10,
(9) factoring polynomials, see Algebra, Lemma 143.13, and
(10) Algebra, Section 153 discusses henselian local rings.

The interested reader will find more results of this nature in Smoothing Ring Maps, Section 3 in particular Smoothing Ring Maps, Proposition 3.2.

Let A be a ring and let $I \subset A$ be an ideal. Let $\overline{\xi}$ be some kind of structure over A/I. In the following lemmas we look for étale ring maps $A \to A'$ which induce isomorphisms $A/I \to A'/IA'$ and objects ξ' over A' lifting $\overline{\xi}$. A general remark is that given étale ring maps $A \to A' \to A''$ such that $A/I \cong A'/IA'$ and $A'/IA' \cong A''/IA''$ the composition $A \to A''$ is also étale (Algebra, Lemma 143.3) and also satisfies $A/I \cong A''/IA''$. We will frequently use this in the following lemmas without further mention. Here is a trivial example of the type of result we are looking for.

Lemma 9.1. *Let A be a ring, let $I \subset A$ be an ideal, let $\overline{u} \in A/I$ be an invertible element. There exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an invertible element $u' \in A'$ lifting $\overline{u}$.*

Proof. Choose any lift $f \in A$ of $\overline{u}$ and set $A' = A_f$ and u the image of f in A'. $\square$

Lemma 9.2. *Let A be a ring, let $I \subset A$ be an ideal, let $\overline{e} \in A/I$ be an idempotent. There exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an idempotent $e' \in A'$ lifting $\overline{e}$.*

Proof. Choose any lift $x \in A$ of $\overline{e}$. Set

$$A' = A[t]/(t^2 - t)\left[\frac{1}{t - 1 + x}\right].$$

The ring map $A \to A'$ is étale because $(2t-1)\mathrm{d}t = 0$ and $(2t-1)(2t-1) = 1$ which is invertible. We have $A'/IA' = A/I[t]/(t^2 - t)[\frac{1}{t-1+\overline{e}}] \cong A/I$ the last map sending t to $\overline{e}$ which works as $\overline{e}$ is a root of $t^2 - t$. This also shows that setting e' equal to the class of t in A' works. $\square$

Lemma 9.3. *Let A be a ring, let $I \subset A$ be an ideal. Let $\mathrm{Spec}(A/I) = \coprod_{j \in J} \overline{U}_j$ be a finite disjoint open covering. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite disjoint open covering $\mathrm{Spec}(A') = \coprod_{j \in J} U'_j$ lifting the given covering.*

Proof. This follows from Lemma 9.2 and the fact that open and closed subsets of Spectra correspond to idempotents, see Algebra, Lemma 21.3. $\square$

Lemma 9.4. *Let $A \to B$ be a ring map and $J \subset B$ an ideal. If $A \to B$ is étale at every prime of $V(J)$, then there exists a $g \in B$ mapping to an invertible element of B/J such that $A' = B_g$ is étale over A.*

Proof. The set of points of $\mathrm{Spec}(B)$ where $A \to B$ is not étale is a closed subset of $\mathrm{Spec}(B)$, see Algebra, Definition 143.1. Write this as $V(J')$ for some ideal $J' \subset B$. Then $V(J') \cap V(J) = \emptyset$ hence $J + J' = B$ by Algebra, Lemma 17.2. Write $1 = f + g$ with $f \in J$ and $g \in J'$. Then g works. $\square$

Next we have three lemmas saying we can lift factorizations of polynomials.

Lemma 9.5. *Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ such that $\overline{g}$ and $\overline{h}$ are monic and generate the unit ideal in $A/I[x]$. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ with g', h' monic lifting the given factorization over A/I.*

Proof. We will deduce this from results on the universal factorization proved earlier; however, we encourage the reader to find their own proof not using this trick. Say $\deg(\overline{g}) = n$ and $\deg(\overline{h}) = m$ so that $\deg(f) = n + m$. Write $f = x^{n+m} + \sum \alpha_i x^{n+m-i}$ for some $\alpha_1, \ldots, \alpha_{n+m} \in A$. Consider the ring map

$$R = \mathbf{Z}[a_1, \ldots, a_{n+m}] \longrightarrow S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$$

of Algebra, Example 143.12. Let $R \to A$ be the ring map which sends a_i to α_i. Set

$$B = A \otimes_R S$$

By construction the image f_B of f in $B[x]$ factors, say $f_B = g_B h_B$ with $g_B = x^n + \sum (1 \otimes b_i) x^{n-i}$ and similarly for h_B. Write $\overline{g} = x^n + \sum \overline{\beta}_i x^{n-i}$ and $\overline{h} = x^m + \sum \overline{\gamma}_i x^{m-i}$. The A-algebra map

$$B \longrightarrow A/I, \quad 1 \otimes b_i \mapsto \overline{\beta}_i, \quad 1 \otimes c_i \mapsto \overline{\gamma}_i$$

maps g_B and h_B to $\overline{g}$ and $\overline{h}$ in $A/I[x]$. The displayed map is surjective; denote $J \subset B$ its kernel. From the discussion in Algebra, Example 143.12 it is clear that $A \to B$ is etale at all points of $V(J) \subset \text{Spec}(B)$. Choose $g \in B$ as in Lemma 9.4 and consider the A-algebra B_g. Since g maps to a unit in $B/J = A/I$ we obtain also a map $B_g/IB_g \to A/I$ of A/I-algebras. Since $A/I \to B_g/IB_g$ is étale, also $B_g/IB_g \to A/I$ is étale (Algebra, Lemma 143.8). Hence there exists an idempotent $e \in B_g/IB_g$ such that $A/I = (B_g/IB_g)_e$ (Algebra, Lemma 143.9). Choose a lift $h \in B_g$ of e. Then $A \to A' = (B_g)_h$ with factorization given by the image of the factorization $f_B = g_B h_B$ in A' is a solution to the problem posed by the lemma. $\square$

The assumption on the leading coefficient in the following lemma will be removed in Lemma 9.7.

Lemma 9.6. *Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ and assume*

(1) *the leading coefficient of $\overline{g}$ is an invertible element of A/I, and*
(2) *$\overline{g}$, $\overline{h}$ generate the unit ideal in $A/I[x]$.*

Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ lifting the given factorization over A/I.

Proof. Applying Lemma 9.1 we may assume that the leading coefficient of $\overline{g}$ is the reduction of an invertible element $u \in A$. Then we may replace $\overline{g}$ by $\overline{u}^{-1}\overline{g}$ and $\overline{h}$ by $\overline{u}\overline{h}$. Thus we may assume that $\overline{g}$ is monic. Since f is monic we conclude that $\overline{h}$ is monic too. In this case the result follows from Lemma 9.5. $\square$

Lemma 9.7. *Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ and assume that $\overline{g}$, $\overline{h}$ generate the unit ideal in $A/I[x]$. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ lifting the given factorization over A/I.*

Proof. Say $f = x^d + a_1 x^{d-1} + \ldots + a_d$ has degree d. Write $\overline{g} = \sum \overline{b}_j x^j$ and $\overline{h} = \sum \overline{c}_j x^j$. Then we see that $1 = \sum \overline{b}_j \overline{c}_{d-j}$. It follows that $\text{Spec}(A/I)$ is covered by the standard opens $D(\overline{b}_j \overline{c}_{d-j})$. However, each point $\mathfrak{p}$ of $\text{Spec}(A/I)$ is contained in at most one of these as by looking at the induced factorization of f over the field $\kappa(\mathfrak{p})$ we see that $\deg(\overline{g} \bmod \mathfrak{p}) + \deg(\overline{h} \bmod \mathfrak{p}) = d$. Hence our open covering is a disjoint open covering. Applying Lemma 9.3 (and replacing A by A') we see that we may assume there is a corresponding disjoint open covering of $\text{Spec}(A)$. This disjoint open covering corresponds to a product decomposition of A, see Algebra, Lemma 24.3. It follows that

$$A = A_0 \times \ldots \times A_d, \quad I = I_0 \times \ldots \times I_d,$$

where the image of $\overline{g}$, resp. $\overline{h}$ in A_j/I_j has degree j, resp. $d-j$ with invertible leading coefficient. Clearly, it suffices to prove the result for each factor A_j separatedly. Hence the lemma follows from Lemma 9.6. $\square$

Lemma 9.8. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal of R and let $J \subset S$ be an ideal of S. If the closure of the image of $V(J)$ in $\text{Spec}(R)$ is disjoint from $V(I)$, then there exists an element $f \in R$ which maps to 1 in R/I and to an element of J in S.*

Proof. Let $I' \subset R$ be an ideal such that $V(I')$ is the closure of the image of $V(J)$. Then $V(I) \cap V(I') = \emptyset$ by assumption and hence $I + I' = R$ by Algebra, Lemma 17.2. Write $1 = g + f$ with $g \in I$ and $f \in I'$. We have $V(f') \supset V(J)$ where f' is the image of f in S. Hence $(f')^n \in J$ for some n, see Algebra, Lemma 17.2. Replacing f by f^n we win. $\square$

Lemma 9.9. *Let I be an ideal of a ring A. Let $A \to B$ be an integral ring map. Let $b \in B$ map to an idempotent in B/IB. Then there exists a monic $f \in A[x]$ with $f(b) = 0$ and $f \bmod I = x^d(x-1)^d$ for some $d \geq 1$.*

Proof. Observe that $z = b^2 - b$ is an element of IB. By Algebra, Lemma 38.4 there exist a monic polynomial $g(x) = x^d + \sum a_j x^j$ of degree d with $a_j \in I$ such that $g(z) = 0$ in B. Hence $f(x) = g(x^2 - x) \in A[x]$ is a monic polynomial such that $f(x) \equiv x^d(x-1)^d \bmod I$ and such that $f(b) = 0$ in B. $\square$

Lemma 9.10. *Let A be a ring, let $I \subset A$ be an ideal. Let $A \to B$ be an integral ring map. Let $\overline{e} \in B/IB$ be an idempotent. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an idempotent $e' \in B \otimes_A A'$ lifting $\overline{e}$.*

Proof. Choose an element $y \in B$ lifting $\overline{e}$. Choose $f \in A[x]$ as in Lemma 9.9 for y. By Lemma 9.6 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and such that $f = gh$ in $A[x]$ with $g(x) = x^d \bmod IA'$ and $h(x) = (x-1)^d \bmod IA'$. After replacing A by A' we may assume that the factorization is defined over A. In that case we see that $b_1 = g(y) \in B$ is a lift of $\overline{e}^d = \overline{e}$ and $b_2 = h(y) \in B$ is a lift of $(\overline{e} - 1)^d = (-1)^d(1 - \overline{e})^d = (-1)^d(1 - \overline{e})$ and moreover $b_1 b_2 = 0$. Thus $(b_1, b_2)B/IB = B/IB$ and $V(b_1, b_2) \subset \text{Spec}(B)$ is disjoint from $V(IB)$. Since $\text{Spec}(B) \to \text{Spec}(A)$ is closed (see Algebra, Lemmas 36.22 and 41.6) we can find an $a \in A$ which maps to an invertible element of A/I whose image in B lies in (b_1, b_2), see Lemma 9.8. After replacing A by the localization A_a we get that $(b_1, b_2) = B$. Then $\text{Spec}(B) = D(b_1) \amalg D(b_2)$; disjoint union because $b_1 b_2 = 0$ and covers $\text{Spec}(B)$ because $(b_1, b_2) = B$. Let $e \in B$ be the idempotent corresponding to the open and closed subset $D(b_1)$, see Algebra, Lemma 21.3. Since b_1 is a lift of $\overline{e}$ and b_2 is a lift of $\pm(1 - \overline{e})$ we conclude that e is a lift of $\overline{e}$ by the uniqueness statement in Algebra, Lemma 21.3. $\square$

Lemma 9.11. *Let A be a ring, let $I \subset A$ be an ideal. Let $\overline{P}$ be a finite projective A/I-module. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite projective A'-module P' lifting $\overline{P}$.*

Proof. We can choose an integer n and a direct sum decomposition $(A/I)^{\oplus n} = \overline{P} \oplus \overline{K}$ for some R/I-module $\overline{K}$. Choose a lift $\varphi : A^{\oplus n} \to A^{\oplus n}$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$. Let $f \in A[x]$ be the characteristic polynomial of φ. Set $B = A[x]/(f)$. By Cayley-Hamilton (Algebra, Lemma 16.1) there is a map $B \to \text{End}_A(A^{\oplus n})$ mapping x to φ. For every prime $\mathfrak{p} \supset I$ the image of f in $\kappa(\mathfrak{p})$ is $(x-1)^r x^{n-r}$ where r is the dimension of $\overline{P} \otimes_{A/I} \kappa(\mathfrak{p})$. Hence $(x-1)^n x^n$ maps to zero in $B \otimes_A \kappa(\mathfrak{p})$ for all $\mathfrak{p} \supset I$. Thus $x(1-x)$ is contained in every prime ideal of B/IB. Hence $x^N(1-x)^N$ is contained in IB for some $N \geq 1$. It follows that $x^N + (1-x)^N$ is a unit in B/IB and that

$$\overline{e} = \text{image of } \frac{x^N}{x^N + (1-x)^N} \text{ in } B/IB$$

is an idempotent as both assertions hold in $\mathbf{Z}[x]/(x^N(x-1)^N)$. The image of $\overline{e}$ in $\text{End}_{A/I}((A/I)^{\oplus n})$ is

$$\frac{\overline{p}^N}{\overline{p}^N + (1-\overline{p})^N} = \overline{p}$$

as $\overline{p}$ is an idempotent. After replacing A by an étale extension A' as in the lemma, we may assume there exists an idempotent $e \in B$ which maps to $\overline{e}$ in B/IB, see Lemma 9.10. Then the image of e under the map

$$B = A[x]/(f) \longrightarrow \text{End}_A(A^{\oplus n}).$$

is an idempotent element p which lifts $\overline{p}$. Setting $P = \text{Im}(p)$ we win. □

Lemma 9.12. *Let A be a ring. Let $0 \to K \to A^{\oplus m} \to M \to 0$ be a sequence of A-modules. Consider the A-algebra $C = \text{Sym}^*_A(M)$ with its presentation $\alpha : A[y_1, \ldots, y_m] \to C$ coming from the surjection $A^{\oplus m} \to M$. Then*

$$NL(\alpha) = (K \otimes_A C \to \bigoplus\nolimits_{j=1,\ldots,m} C\,dy_j)$$

(see Algebra, Section 134) in particular $\Omega_{C/A} = M \otimes_A C$.

Proof. Let $J = \text{Ker}(\alpha)$. The lemma asserts that $J/J^2 \cong K \otimes_A C$. Note that α is a homomorphism of graded algebras. We will prove that in degree d we have $(J/J^2)_d = K \otimes_A C_{d-1}$. Note that

$$J_d = \text{Ker}(\text{Sym}^d_A(A^{\oplus m}) \to \text{Sym}^d_A(M)) = \text{Im}(K \otimes_A \text{Sym}^{d-1}_A(A^{\oplus m}) \to \text{Sym}^d_A(A^{\oplus m})),$$

see Algebra, Lemma 13.2. It follows that $(J^2)_d = \sum_{a+b=d} J_a \cdot J_b$ is the image of

$$K \otimes_A K \otimes_A \text{Sym}^{d-2}_A(A^{\otimes m}) \to \text{Sym}^d_A(A^{\oplus m}).$$

The cokernel of the map $K \otimes_A \text{Sym}^{d-2}_A(A^{\otimes m}) \to \text{Sym}^{d-1}_A(A^{\oplus m})$ is $\text{Sym}^{d-1}_A(M)$ by the lemma referenced above. Hence it is clear that $(J/J^2)_d = J_d/(J^2)_d$ is equal to

$$\begin{aligned}\text{Coker}(K \otimes_A K \otimes_A \text{Sym}^{d-2}_A(A^{\otimes m}) \to K \otimes_A \text{Sym}^{d-1}_A(A^{\otimes m})) &= K \otimes_A \text{Sym}^{d-1}_A(M) \\ &= K \otimes_A C_{d-1}\end{aligned}$$

as desired. □

Lemma 9.13. *Let A be a ring. Let M be an A-module. Then $C = \text{Sym}^*_A(M)$ is smooth over A if and only if M is a finite projective A-module.*

Proof. Let $\sigma : C \to A$ be the projection onto the degree 0 part of C. Then $J = \text{Ker}(\sigma)$ is the part of degree > 0 and we see that $J/J^2 = M$ as an A-module. Hence if $A \to C$ is smooth then M is a finite projective A-module by Algebra, Lemma 139.4.

Conversely, assume that M is finite projective and choose a surjection $A^{\oplus n} \to M$ with kernel K. Of course the sequence $0 \to K \to A^{\oplus n} \to M \to 0$ is split as M is projective. In particular we see that K is a finite A-module and hence C is of finite presentation over A as C is a quotient of $A[x_1, \ldots, x_n]$ by the ideal generated by $K \subset \bigoplus Ax_i$. The computation of Lemma 9.12 shows that $NL_{C/A}$ is homotopy equivalent to $(K \to M) \otimes_A C$. Hence $NL_{C/A}$ is quasi-isomorphic to $C \otimes_A M$ placed in degree 0 which means that C is smooth over A by Algebra, Definition 137.1. □

Lemma 9.14. *Let A be a ring, let $I \subset A$ be an ideal. Consider a commutative diagram*

$$\begin{tikzcd} B \arrow[rd] & \\ A \arrow[u] \arrow[r] & A/I \end{tikzcd}$$

where B is a smooth A-algebra. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an A-algebra map $B \to A'$ lifting the ring map $B \to A/I$.

Proof. Let $J \subset B$ be the kernel of $B \to A/I$ so that $B/J = A/I$. By Algebra, Lemma 139.3 the sequence

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

is split exact. Thus $\overline{P} = J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J$ is a finite projective A/I-module. Choose an integer n and a direct sum decomposition $A/I^{\oplus n} = \overline{P} \oplus \overline{K}$. By Lemma 9.11 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite projective A-module K which lifts $\overline{K}$. We may and do replace A by A'. Set $B' = B \otimes_A \text{Sym}^*_A(K)$. Since $A \to \text{Sym}^*_A(K)$ is smooth by Lemma 9.13 we see that $B \to B'$ is smooth which in turn implies that $A \to B'$ is smooth (see Algebra, Lemmas 137.4 and 137.13). Moreover the section $\text{Sym}^*_A(K) \to A$ determines a section $B' \to B$ and we let $B' \to A/I$ be the composition $B' \to B \to A/I$. Let $J' \subset B'$ be the kernel of $B' \to A/I$. We have $JB' \subset J'$ and $B \otimes_A K \subset J'$. These maps combine to give an isomorphism

$$(A/I)^{\oplus n} \cong J/J^2 \oplus \overline{K} \longrightarrow J'/((J')^2 + IB')$$

Thus, after replacing B by B' we may assume that $J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J$ is a free A/I-module of rank n.

In this case, choose $f_1, \ldots, f_n \in J$ which map to a basis of $J/(J^2 + IB)$. Consider the finitely presented A-algebra $C = B/(f_1, \ldots, f_n)$. Note that we have an exact sequence

$$0 \to H_1(L_{C/A}) \to (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

see Algebra, Lemma 134.4 (note that $H_1(L_{B/A}) = 0$ and that $\Omega_{B/A}$ is finite projective, in particular flat so the Tor group vanishes). For any prime $\mathfrak q \supset J$ of B the module $\Omega_{B/A,\mathfrak q}$ is free of rank n because $\Omega_{B/A}$ is finite projective and because $\Omega_{B/A} \otimes_B B/J$ is free of rank n (see Algebra, Lemma 78.2). By our choice of $f_1, \ldots, f_n$ the map

$$\left((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2\right)_{\mathfrak q} \to \Omega_{B/A,\mathfrak q}$$

is surjective modulo J. Hence we see that this map of modules over the local ring $C_{\mathfrak q}$ has to be an isomorphism (this is because by Nakayama's Algebra, Lemma 20.1 the map is surjective and then for example by Algebra, Lemma 16.4 because $((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2)_{\mathfrak q}$ is generated by n elements the map is injective). Thus $H_1(L_{C/A})_{\mathfrak q} = 0$ and $\Omega_{C/A,\mathfrak q} = 0$. By Algebra, Lemma 137.12 we see that $A \to C$ is smooth at the prime $\overline{\mathfrak q}$ of C corresponding to $\mathfrak q$. Since $\Omega_{C/A,\mathfrak q} = 0$ it is actually étale at $\overline{\mathfrak q}$. Thus $A \to C$ is étale at all primes of C containing JC. By Lemma 9.4 we can find an $f \in C$ mapping to an invertible element of C/JC such that $A \to C_f$ is étale. By our choice of f it is still true that $C_f/JC_f = A/I$. The map $C_f/IC_f \to A/I$ is surjective and étale by Algebra, Lemma 143.8. Hence A/I is isomorphic to the localization of C_f/IC_f at some element $g \in C$, see Algebra, Lemma 143.9. Set $A' = C_{fg}$ to conclude the proof. $\square$

10. Zariski pairs

In this section and the next a *pair* is a pair (A, I) where A is a ring and $I \subset A$ is an ideal. A *morphism of pairs* $(A, I) \to (B, J)$ is a ring map $\varphi : A \to B$ with $\varphi(I) \subset J$.

Definition 10.1. A *Zariski pair* is a pair (A, I) such that I is contained in the Jacobson radical of A.

Lemma 10.2. *Let (A, I) be a Zariski pair. Then the map from idempotents of A to idempotents of A/I is injective.*

Proof. An idempotent of a local ring is either 0 or 1. Thus an idempotent is determined by the set of maximal ideals where it vanishes, by Algebra, Lemma 23.1. $\square$

Lemma 10.3. *Let (A, I) be a Zariski pair. Let $A \to B$ be a flat, integral, finitely presented ring map such that $A/I \to B/IB$ is an isomorphism. Then $A \to B$ is an isomorphism.*

Proof. The ring map $A \to B$ is finite by Algebra, Lemma 36.5. Hence B is finitely presented as an A-module by Algebra, Lemma 36.23. Hence B is a finite locally free A-module by Algebra, Lemma 78.2. Since the module B has rank 1 along $V(I)$ (see rank function described in Algebra, Lemma 78.2), and as (A, I) is a Zariski pair, we conclude that the rank is 1 everywhere. It follows that $A \to B$ is an isomorphism: it is a pleasant exercise to show that a ring map $R \to S$ such that S is a locally free R-module of rank 1 is an isomorphism (hint: look at local rings). $\square$

Lemma 10.4. *Let (A, I) be a Zariski pair. Let $A \to B$ be a finite ring map. Assume*

1. *$B/IB = B_1 \times B_2$ is a product of A/I-algebras*
2. *$A/I \to B_1/IB_1$ is surjective,*
3. *$b \in B$ maps to $(1, 0)$ in the product.*

Then there exists a monic $f \in A[x]$ with $f(b) = 0$ and $f \bmod I = (x-1)x^d$ for some $d \geq 1$.

Proof. By Lemma 9.10 we can find an étale ring map $A \to A'$ inducing an isomorphism $A/I \to A'/IA'$ such that $B' = B \otimes_A A'$ contains an idempotent e' lifting the image of b in B'/IB'. Consider the corresponding A'-algebra decomposition

$$B' = B'_1 \times B'_2$$

which is compatible with the one given in the lemma upon reduction modulo I. The map $A' \to B'_1$ is surjective modulo IA'. By Nakayama's lemma (Algebra, Lemma 20.1) we can find $i \in IA'$ such that after replacing A' by A'_{1+i} the map $A' \to B'_1$ is surjective. Observe that the image $b'_1 \in B'_1$ of b satisfies $b'_1 - 1 \in IB'_1$. Thus we may pick $a' \in IA'$ mapping to $b'_1 - 1$. On the other hand, the image $b'_2 \in B'_2$ of b is in IB'_2. By Algebra, Lemma 38.4 there exist a monic polynomial $g(x) = x^d + \sum a'_j x^j$ of degree d with $a'_j \in IA'$ such that $g(b'_2) = 0$ in B'_2. Thus the image $b' = (b'_1, b'_2) \in B'$ of b is a root of the polynomial $(x - 1 - a')g(x)$. We conclude that

$$(b' - 1)(b')^d \in \sum\nolimits_{j=0,\ldots,d} IA' \cdot (b')^j$$

We claim that this implies

$$(b - 1)b^d \in \sum\nolimits_{j=0,\ldots,d} I \cdot b^j$$

in B. For this it is enough to see that the ring map $A \to A'$ is faithfully flat, because the condition is that the image of $(b-1)b^d$ is zero in $B/\sum_{j=0,\ldots,d} Ib^j$ (use Algebra, Lemma 82.11). The map $A \to A'$ flat because it is étale (Algebra, Lemma 143.3). On the other hand, the induced map on spectra is open (see Algebra, Proposition 41.8 and use previous lemma referenced) and the image contains $V(I)$. Since I is contained in the Jacobson radical of A we conclude. $\square$

Lemma 10.5. *Let (A, I) be a Zariski pair with A Noetherian. Let $f \in I$. Then A_f is a Jacobson ring.*

Proof. We will use the criterion of Algebra, Lemma 61.4. Let $\mathfrak{p} \subset A$ be a prime ideal such that $\mathfrak{p}_f = \mathfrak{p}A_f$ is prime and not maximal. We have to show that $A_f/\mathfrak{p}_f = (A/\mathfrak{p})_f$ has infinitely many prime ideals. After replacing A by $A/\mathfrak{p}$ we may assume A is a domain, $\dim A_f > 0$, and our goal is to show that $\text{Spec}(A_f)$ is infinite. Since $\dim A_f > 0$ we can find a nonzero prime ideal $\mathfrak{q} \subset A$ not containing f. Choose a maximal ideal $\mathfrak{m} \subset A$ containing $\mathfrak{q}$. Since (A, I) is a Zariski pair, we see $I \subset \mathfrak{m}$. Hence $\mathfrak{m} \neq \mathfrak{q}$ and $\dim(A_\mathfrak{m}) > 1$. Hence $\text{Spec}((A_\mathfrak{m})_f) \subset \text{Spec}(A_f)$ is infinite by Algebra, Lemma 61.1 and we win. $\square$

11. Henselian pairs

Some of the results of Section 9 may be viewed as results about henselian pairs. In this section a *pair* is a pair (A, I) where A is a ring and $I \subset A$ is an ideal. A *morphism of pairs* $(A, I) \to (B, J)$ is a ring map $\varphi : A \to B$ with $\varphi(I) \subset J$. As in Section 9 given an object ξ over A we denote $\overline{\xi}$ the "base change" of ξ to an object over A/I (provided this makes sense).

Definition 11.1. A *henselian pair* is a pair (A, I) satisfying

1. I is contained in the Jacobson radical of A, and
2. for any monic polynomial $f \in A[T]$ and factorization $\overline{f} = g_0 h_0$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$, there exists a factorization $f = gh$ in $A[T]$ with g, h monic and $g_0 = \overline{g}$ and $h_0 = \overline{h}$.

Observe that if A is a local ring and $I = \mathfrak{m}$ is the maximal ideal, then (A, I) is a henselian pair if and only if A is a henselian local ring, see Algebra, Lemma 153.3. In Lemma 11.6 we give a number of equivalent characterizations of henselian pairs (and we will add more as time goes on).

Lemma 11.2. *Let (A, I) be a pair with I locally nilpotent. Then the functor $B \mapsto B/IB$ induces an equivalence between the category of étale algebras over A and the category of étale algebras over A/I. Moreover, the pair is henselian.*

Proof. Essential surjectivity holds by Algebra, Lemma 143.10. If B, B' are étale over A and $B/IB \to B'/IB'$ is a morphism of A/I-algebras, then we can lift this by Algebra, Lemma 138.17. Finally, suppose that $f, g : B \to B'$ are two A-algebra maps with $f \bmod I = g \bmod I$. Choose an idempotent $e \in B \otimes_A B$ generating the kernel of the multiplication map $B \otimes_A B \to B$, see Algebra, Lemmas 151.4 and 151.3 (to see that étale is unramified). Then $(f \otimes g)(e) \in IB'$. Since IB' is locally nilpotent (Algebra, Lemma 32.3) this implies $(f \otimes g)(e) = 0$ by Algebra, Lemma 32.6. Thus $f = g$.

It is clear that I is contained in the Jacobson radical of A. Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_0 h_0$ be a factorization of $\overline{f} = f \bmod I$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. By Lemma 9.5 there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ such that the factorization lifts to a factorization into monic polynomials over A'. By the above we have $A = A'$ and the factorization is over A. □

Lemma 11.3. *Let $A = \lim A_n$ where (A_n) is an inverse system of rings whose transition maps are surjective and have locally nilpotent kernels. Then (A, I_n) is a henselian pair, where $I_n = \mathrm{Ker}(A \to A_n)$.*

Proof. Fix n. Let $a \in A$ be an element which maps to 1 in A_n. By Algebra, Lemma 32.4 we see that a maps to a unit in A_m for all $m \geq n$. Hence a is a unit in A. Thus by Algebra, Lemma 19.1 the ideal I_n is contained in the Jacobson radical of A. Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_n h_n$ be a factorization of $\overline{f} = f \bmod I_n$ with $g_n, h_n \in A_n[T]$ monic generating the unit ideal in $A_n[T]$. By Lemma 11.2 we can successively lift this factorization to $f \bmod I_m = g_m h_m$ with g_m, h_m monic in $A_m[T]$ for all $m \geq n$. At each step we have to verify that our lifts g_m, h_m generate the unit ideal in $A_n[T]$; this follows from the corresponding fact for g_n, h_n and the fact that $\mathrm{Spec}(A_n[T]) = \mathrm{Spec}(A_m[T])$ because the kernel of $A_m \to A_n$ is locally nilpotent. As $A = \lim A_m$ this finishes the proof. □

Lemma 11.4. *Let (A, I) be a pair. If A is I-adically complete, then the pair is henselian.*

Proof. By Algebra, Lemma 96.6 the ideal I is contained in the Jacobson radical of A. Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_0 h_0$ be a factorization of $\overline{f} = f \bmod I$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. By Lemma 11.2 we can successively lift this factorization to $f \bmod I^n = g_n h_n$ with g_n, h_n monic in $A/I^n[T]$ for all $n \geq 1$. As $A = \lim A/I^n$ this finishes the proof. □

Lemma 11.5. *Let (A, I) be a pair. Let $A \to B$ be a finite type ring map such that $B/IB = C_1 \times C_2$ with $A/I \to C_1$ finite. Let B' be the integral closure of A in B. Then we can write $B'/IB' = C_1 \times C'_2$ such that the map $B'/IB' \to B/IB$ preserves product decompositions and there exists a $g \in B'$ mapping to $(1, 0)$ in $C_1 \times C'_2$ with $B'_g \to B_g$ an isomorphism.*

Proof. Observe that $A \to B$ is quasi-finite at every prime of the closed subset $T = \mathrm{Spec}(C_1) \subset \mathrm{Spec}(B)$ (this follows by looking at fibre rings, see Algebra, Definition 122.3). Consider the diagram of topological spaces

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & \mathrm{Spec}(B') \arrow[dl, "\psi'"] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

By Algebra, Theorem 123.12 for every $\mathfrak{p} \in T$ there is a $h_\mathfrak{p} \in B'$, $h_\mathfrak{p} \not\in \mathfrak{p}$ such that $B'_h \to B_h$ is an isomorphism. The union $U = \bigcup D(h_\mathfrak{p})$ gives an open $U \subset \text{Spec}(B')$ such that $\phi^{-1}(U) \to U$ is a homeomorphism and $T \subset \phi^{-1}(U)$. Since T is open in $\psi^{-1}(V(I))$ we conclude that $\phi(T)$ is open in $U \cap (\psi')^{-1}(V(I))$. Thus $\phi(T)$ is open in $(\psi')^{-1}(V(I))$. On the other hand, since C_1 is finite over A/I it is finite over B'. Hence $\phi(T)$ is a closed subset of $\text{Spec}(B')$ by Algebra, Lemmas 41.6 and 36.22. We conclude that $\text{Spec}(B'/IB') \supset \phi(T)$ is open and closed. By Algebra, Lemma 24.3 we get a corresponding product decomposition $B'/IB' = C'_1 \times C'_2$. The map $B'/IB' \to B/IB$ maps C'_1 into C_1 and C'_2 into C_2 as one sees by looking at what happens on spectra (hint: the inverse image of $\phi(T)$ is exactly T; some details omitted). Pick a $g \in B'$ mapping to $(1, 0)$ in $C'_1 \times C'_2$ such that $D(g) \subset U$; this is possible because $\text{Spec}(C'_1)$ and $\text{Spec}(C'_2)$ are disjoint and closed in $\text{Spec}(B')$ and $\text{Spec}(C'_1)$ is contained in U. Then $B'_g \to B_g$ defines a homeomorphism on spectra and an isomorphism on local rings (by our choice of U above). Hence it is an isomorphism, as follows for example from Algebra, Lemma 23.1. Finally, it follows that $C'_1 = C_1$ and the proof is complete. □

Lemma 11.6. *Let (A, I) be a pair. The following are equivalent*

1. *(A, I) is a henselian pair,*
2. *given an étale ring map $A \to A'$ and an A-algebra map $\sigma : A' \to A/I$, there exists an A-algebra map $A' \to A$ lifting σ,*
3. *for any finite A-algebra B the map $B \to B/IB$ induces a bijection on idempotents,*
4. *for any integral A-algebra B the map $B \to B/IB$ induces a bijection on idempotents, and*
5. *(Gabber) I is contained in the Jacobson radical of A and every monic polynomial $f(T) \in A[T]$ of the form*
$$f(T) = T^n(T - 1) + a_nT^n + \ldots + a_1T + a_0$$
with $a_n, \ldots, a_0 \in I$ and $n \geq 1$ has a root $\alpha \in 1 + I$.

Moreover, in part (5) the root is unique.

Proof. Assume (2) holds. Then I is contained in the Jacobson radical of A, since otherwise there would be a nonunit $f \in A$ congruent to 1 modulo I and the map $A \to A_f$ would contradict (2). Hence $IB \subset B$ is contained in the Jacobson radical of B for B integral over A because $\text{Spec}(B) \to \text{Spec}(A)$ is closed by Algebra, Lemmas 41.6 and 36.22. Thus the map from idempotents of B to idempotents of B/IB is injective by Lemma 10.2. On the other hand, since (2) holds, every idempotent of B/IB lifts to an idempotent of B by Lemma 9.10. In this way we see that (2) implies (4).

The implication (4) ⇒ (3) is trivial.

Assume (3). Let $\mathfrak{m}$ be a maximal ideal and consider the finite map $A \to B = A/(I \cap \mathfrak{m})$. The condition that $B \to B/IB$ induces a bijection on idempotents implies that $I \subset \mathfrak{m}$ (if not, then $B = A/I \times A/\mathfrak{m}$ and $B/IB = A/I$). Thus we see that I is contained in the Jacobson radical of A. Let $f \in A[T]$ be monic and suppose given a factorization $\overline{f} = g_0h_0$ with $g_0, h_0 \in A/I[T]$ monic. Set $B = A[T]/(f)$. Let $\overline{e}$ be the idempotent of B/IB corresponding to the decomposition
$$B/IB = A/I[T]/(g_0) \times A[T]/(h_0)$$

of A-algebras. Let $e \in B$ be an idempotent lifting $\overline{e}$ which exists as we assumed (3). This gives a product decomposition

$$B = eB \times (1 - e)B$$

Note that B is free of rank $\deg(f)$ as an A-module. Hence eB and $(1 - e)B$ are finite locally free A-modules. However, since eB and $(1 - e)B$ have constant rank $\deg(g_0)$ and $\deg(h_0)$ over A/I we find that the same is true over $\text{Spec}(A)$. We conclude that

$$\begin{aligned} f &= \text{CharPol}_A(T : B \to B) \\ &= \text{CharPol}_A(T : eB \to eB)\text{CharPol}_A(T : (1 - e)B \to (1 - e)B) \end{aligned}$$

is a factorization into monic polynomials reducing to the given factorization modulo I. Here CharPol_A denotes the characteristic polynomial of an endomorphism of a finite locally free module over A. If the module is free the CharPol_A is defined as the characteristic polynomial of the corresponding matrix and in general one uses Algebra, Lemma 24.2 to glue. Details omitted. Thus (3) implies (1).

Assume (1). Let f be as in (5). The factorization of $f \bmod I$ as T^n times $T - 1$ lifts to a factorization $f = gh$ with g and h monic by Definition 11.1. Then h has to have degree 1 and we see that f has a root reducing to 1 modulo 1. Finally, I is contained in the Jacobson radical by the definition of a henselian pair. Thus (1) implies (5).

Before we give the proof of the last step, let us show that the root α in (5), if it exists, is unique. Namely, Due to the explicit shape of $f(T)$, we have $f'(\alpha) \in 1 + I$ where f' is the derivative of f with respect to T. An elementary argument shows that

$$f(T) = f(\alpha + T - \alpha) = f(\alpha) + f'(\alpha) \cdot (T - \alpha) \bmod (T - \alpha)^2 A[T]$$

This shows that any other root $\alpha' \in 1 + I$ of $f(T)$ satisfies $0 = f(\alpha') - f(\alpha) = (\alpha' - \alpha)(1 + i)$ for some $i \in I$, so that, since $1 + i$ is a unit in A, we have $\alpha = \alpha'$.

Assume (5). We will show that (2) holds, in other words, that for every étale map $A \to A'$, every section $\sigma : A' \to A/I$ modulo I lifts to a section $A' \to A$. Since $A \to A'$ is étale, the section σ determines a decomposition

$$A'/IA' \cong A/I \times C \tag{11.6.1}$$

of A/I-algebras. Namely, the surjective ring map $A'/IA' \to A/I$ is étale by Algebra, Lemma 143.8 and then we get the desired idempotent by Algebra, Lemma 143.9. We will show that this decomposition lifts to a decomposition

$$A' \cong A'_1 \times A'_2 \tag{11.6.2}$$

of A-algebras with A'_1 integral over A. Then $A \to A'_1$ is integral and étale and $A/I \to A'_1/IA'_1$ is an isomorphism, thus $A \to A'_1$ is an isomorphism by Lemma 10.3 (here we also use that an étale ring map is flat and of finite presentation, see Algebra, Lemma 143.3).

Let B' be the integral closure of A in A'. By Lemma 11.5 we may decompose

$$B'/IB' \cong A/I \times C' \tag{11.6.3}$$

as A/I-algebras compatibly with (11.6.1) and we may find $b \in B'$ that lifts $(1, 0)$ such that $B'_b \to A'_b$ is an isomorphism. If the decomposition (11.6.3) lifts to a decomposition

$$B' \cong B'_1 \times B'_2 \tag{11.6.4}$$

of A-algebras, then the induced decomposition $A' = A'_1 \times A'_2$ will give the desired (11.6.2): indeed, since b is a unit in B'_1 (details omitted), we will have $B'_1 \cong A'_1$, so that A'_1 will be integral over A.

Choose a finite A-subalgebra $B'' \subset B'$ containing b (observe that any finitely generated A-subalgebra of B' is finite over A). After enlarging B'' we may assume b maps to an idempotent in B''/IB'' producing

$$B''/IB'' \cong C''_1 \times C''_2 \tag{11.6.5}$$

Since $B'_b \cong A'_b$ we see that B'_b is of finite type over A. Say B'_b is generated by $b_1/b^n, \ldots, b_t/b^n$ over A and enlarge B'' so that $b_1, \ldots, b_t \in B''$. Then $B''_b \to B'_b$ is surjective as well as injective, hence an isomorphism. In particular, we see that $C''_1 = A/I$! Therefore $A/I \to C''_1$ is an isomorphism, in particular surjective. By Lemma 10.4 we can find an $f(T) \in A[T]$ of the form

$$f(T) = T^n(T-1) + a_nT^n + \ldots + a_1T + a_0$$

with $a_n, \ldots, a_0 \in I$ and $n \geq 1$ such that $f(b) = 0$. In particular, we find that B' is a $A[T]/(f)$-algebra. By (5) we deduce there is a root $a \in 1 + I$ of f. This produces a product decomposition $A[T]/(f) = A[T]/(T-a) \times D$ compatible with the splitting (11.6.3) of B'/IB'. The induced splitting of B' is then a desired (11.6.4). □

Lemma 11.7. *Let A be a ring. Let $I, J \subset A$ be ideals with $V(I) = V(J)$. Then (A, I) is henselian if and only if (A, J) is henselian.*

Proof. For any integral ring map $A \to B$ we see that $V(IB) = V(JB)$. Hence idempotents of B/IB and B/JB are in bijective correspondence (Algebra, Lemma 21.3). It follows that $B \to B/IB$ induces a bijection on sets of idempotents if and only if $B \to B/JB$ induces a bijection on sets of idempotents. Thus we conclude by Lemma 11.6. □

Lemma 11.8. *Let (A, I) be a henselian pair and let $A \to B$ be an integral ring map. Then (B, IB) is a henselian pair.*

Proof. Immediate from the fourth characterization of henselian pairs in Lemma 11.6 and the fact that the composition of integral ring maps is integral. □

Lemma 11.9. *Let $I \subset J \subset A$ be ideals of a ring A. The following are equivalent*

1. *(A, I) and $(A/I, J/I)$ are henselian pairs, and*
2. *(A, J) is an henselian pair.*

Proof. Assume (1). Let B be an integral A-algebra. Consider the ring maps

$$B \to B/IB \to B/JB$$

By Lemma 11.6 we find that both arrows induce bijections on idempotents. Hence so does the composition. Whence (A, J) is a henselian pair by Lemma 11.6.

Conversely, assume (2) holds. Then $(A/I, J/I)$ is a henselian pair by Lemma 11.8. Let B be an integral A-algebra. Consider the ring maps

$$B \to B/IB \to B/JB$$

By Lemma 11.6 we find that the composition and the second arrow induce bijections on idempotents. Hence so does the first arrow. It follows that (A, I) is a henselian pair (by the lemma again). □

Lemma 11.10. *Let A be a ring and let (A, I) and (A, I') be henselian pairs. Then $(A, I + I')$ is an henselian pair.*

Proof. By Lemma 11.8 the pair $(A/I, (I' + I)/I)$ is henselian. Thus we get the conclusion from Lemma 11.9. □

Lemma 11.11. *Let J be a set and let $\{(A_j, I_j)\}_{j\in J}$ be a collection of pairs. Then $(\prod_{j\in J} A_j, \prod_{j\in J} I_j)$ is Henselian if and only if so is each (A_j, I_j).*

Proof. For every $j \in J$, the projection $\prod_{j\in J} A_j \to A_j$ is an integral ring map, so Lemma 11.8 proves that each (A_j, I_j) is Henselian if $(\prod_{j\in J} A_j, \prod_{j\in J} I_j)$ is Henselian.

Conversely, suppose that each (A_j, I_j) is a Henselian pair. Then every $1 + x$ with $x \in \prod_{j\in J} I_j$ is a unit in $\prod_{j\in J} A_j$ because it is so componentwise by Algebra, Lemma 19.1 and Definition 11.1. Thus, by Algebra, Lemma 19.1 again, $\prod_{j\in J} I_j$ is contained in the Jacobson radical of $\prod_{j\in J} A_j$. Continuing to work componentwise, it likewise follows that for every monic $f \in (\prod_{j\in J} A_j)[T]$ and every factorization $\overline{f} = g_0h_0$ with monic $g_0, h_0 \in (\prod_{j\in J} A_j/\prod_{j\in J} I_j)[T] = (\prod_{j\in J} A_j/I_j)[T]$ that generate the unit ideal in $(\prod_{j\in J} A_j/\prod_{j\in J} I_j)[T]$, there exists a factorization $f = gh$ in $(\prod_{j\in J} A_j)[T]$ with g, h monic and reducing to g_0, h_0. In conclusion, according to Definition 11.1 $(\prod_{j\in J} A_j, \prod_{j\in J} I_j)$ is a Henselian pair. □

Lemma 11.12. *The property of being Henselian is preserved under limits of pairs. More precisely, let J be a preordered set and let (A_j, I_j) be an inverse system of henselian pairs over J. Then $A = \lim A_j$ equipped with the ideal $I = \lim I_j$ is a henselian pair (A, I).*

Proof. By Categories, Lemma 14.11, we only need to consider products and equalizers. For products, the claim follows from Lemma 11.11. Thus, consider an equalizer diagram

$$\begin{tikzcd} (A, I) \arrow[r] & (A', I') \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & (A'', I'') \end{tikzcd}$$

in which the pairs (A', I') and (A'', I'') are henselian. To check that the pair (A, I) is also henselian, we will use the Gabber's criterion in Lemma 11.6. Every element of $1 + I$ is a unit in A because, due to the uniqueness of the inverses of units, this may be checked in (A', I'). Thus I is contained in the Jacobson radical of A, see Algebra, Lemma 19.1. Thus, let

$$f(T) = T^{N-1}(T - 1) + a_{N-1}T^{N-1} + \cdots + a_1T + a_0$$

be a polynomial in $A[T]$ with $a_{N-1}, \ldots, a_0 \in I$ and $N \geq 1$. The image of $f(T)$ in $A'[T]$ has a unique root $\alpha' \in 1 + I'$ and likewise for the further image in $A''[T]$. Thus, due to the uniqueness, $\varphi(\alpha') = \psi(\alpha')$, to the effect that α' defines a root of $f(T)$ in $1 + I$, as desired. □

Lemma 11.13. *The property of being Henselian is preserved under filtered colimits of pairs. More precisely, let J be a directed set and let (A_j, I_j) be a system of* *henselian pairs over J. Then $A = \mathrm{colim}\, A_j$ equipped with the ideal $I = \mathrm{colim}\, I_j$ is a henselian pair (A, I).*

Proof. If $u \in 1+I$ then for some $j \in J$ we see that u is the image of some $u_j \in 1+I_j$. Then u_j is invertible in A_j by Algebra, Lemma 19.1 and the assumption that I_j is contained in the Jacobson radical of A_j. Hence u is invertible in A. Thus I is contained in the Jacobson radical of A (by the lemma).

Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_0 h_0$ be a factorization with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. Write $1 = g_0 g_0' + h_0 h_0'$ for some $g_0', h_0' \in A/I[T]$. Since $A = \mathrm{colim}\, A_j$ and $A/I = \mathrm{colim}\, A_j/I_j$ are filterd colimits we can find a $j \in J$ and $f_j \in A_j$ and a factorization $\overline{f}_j = g_{j,0} h_{j,0}$ with $g_{j,0}, h_{j,0} \in A_j/I_j[T]$ monic and $1 = g_{j,0} g_{j,0}' + h_{j,0} h_{j,0}'$ for some $g_{j,0}', h_{j,0}' \in A_j/I_j[T]$ with $f_j, g_{j,0}, h_{j,0}, g_{j,0}', h_{j,0}'$ mapping to f, g_0, h_0, g_0', h_0'. Since (A_j, I_j) is a henselian pair, we can lift $\overline{f}_j = g_{j,0} h_{j,0}$ to a factorization over A_j and taking the image in A we obtain a corresponding factorization in A. Hence (A, I) is henselian. $\square$

Example 11.14 (Moret-Bailly)**.** Lemma 11.13 is wrong if the colimit isn't filtered. For example, if we take the coproduct of the henselian pairs $(\mathbf{Z}_p, (p))$ and $(\mathbf{Z}_p, (p))$, then we obtain (A, pA) with $A = \mathbf{Z}_p \otimes_{\mathbf{Z}} \mathbf{Z}_p$. This isn't a henselian pair: $A/pA = \mathbf{F}_p$ hence if (A, pA) where henselian, then A would have to be local. However, $\mathrm{Spec}(A)$ is disconnected; for example for odd primes p we have the nontrivial idempotent

$$(1/2 \otimes 1)\left(1 \otimes 1 - (1+p)^{-1} u \otimes u\right)$$

where $u \in \mathbf{Z}_p$ is a square root of $1 + p$. Some details omitted.

Lemma 11.15. *Let A be a ring. There exists a largest ideal $I \subset A$ such that (A, I) is a henselian pair.*

Proof. Combine Lemmas 11.9, 11.10, and 11.13. $\square$

Lemma 11.16. *Let (A, I) be a henselian pair. Let $\mathfrak{p} \subset A$ be a prime ideal. Then $V(\mathfrak{p} + I)$ is connected.*

Proof. By Lemma 11.8 we see that $(A/\mathfrak{p}, I + \mathfrak{p}/\mathfrak{p})$ is a henselian pair. Thus it suffices to prove: If (A, I) is a henselian pair and A is a domain, then $\mathrm{Spec}(A/I) = V(I)$ is connected. If not, then A/I has a nontrivial idempotent by Algebra, Lemma 21.4. By Lemma 11.6 this would imply A has a nontrivial idempotent. This is a contradiction. $\square$

12. Henselization of pairs

We continue the discussion started in Section 11.

Lemma 12.1. *The inclusion functor*

$$\text{category of henselian pairs} \longrightarrow \text{category of pairs}$$

has a left adjoint $(A, I) \mapsto (A^h, I^h)$.

Proof. Let (A, I) be a pair. Consider the category $\mathcal{C}$ consisting of étale ring maps $A \to B$ such that $A/I \to B/IB$ is an isomorphism. We will show that the category $\mathcal{C}$ is directed and that $A^h = \mathrm{colim}_{B \in \mathcal{C}}\, B$ with ideal $I^h = IA^h$ gives the desired adjoint.

We first prove that $\mathcal{C}$ is directed (Categories, Definition 19.1). It is nonempty because $\text{id} : A \to A$ is an object. If B and B' are two objects of $\mathcal{C}$, then $B'' = B \otimes_A B'$ is an object of $\mathcal{C}$ (use Algebra, Lemma 143.3) and there are morphisms $B \to B''$ and $B' \to B''$. Suppose that $f, g : B \to B'$ are two maps between objects of $\mathcal{C}$. Then a coequalizer is

$$(B' \otimes_{f,B,g} B') \otimes_{(B' \otimes_A B')} B'$$

which is étale over A by Algebra, Lemmas 143.3 and 143.8. Thus the category $\mathcal{C}$ is directed.

Since $B/IB = A/I$ for all objects B of $\mathcal{C}$ we see that $A^h/I^h = A^h/IA^h = \operatorname{colim} B/IB = \operatorname{colim} A/I = A/I$.

Next, we show that $A^h = \operatorname{colim}_{B \in \mathcal{C}} B$ with $I^h = IA^h$ is a henselian pair. To do this we will verify condition (2) of Lemma 11.6. Namely, suppose given an étale ring map $A^h \to A'$ and A^h-algebra map $\sigma : A' \to A^h/I^h$. Then there exists a $B \in \mathcal{C}$ and an étale ring map $B \to B'$ such that $A' = B' \otimes_B A^h$. See Algebra, Lemma 143.3. Since $A^h/I^h = A/IB$, the map σ induces an A-algebra map $s : B' \to A/I$. Then $B'/IB' = A/I \times C$ as A/I-algebra, where C is the kernel of the map $B'/IB' \to A/I$ induced by s. Let $g \in B'$ map to $(1, 0) \in A/I \times C$. Then $B \to B'_g$ is étale and $A/I \to B'_g/IB'_g$ is an isomorphism, i.e., B'_g is an object of $\mathcal{C}$. Thus we obtain a canonical map $B'_g \to A^h$ such that

$$\begin{tikzcd} B'_g \arrow[r] & A^h \\ B \arrow[u] \arrow[ur] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B' \arrow[r] \arrow[drr, "s"'] & B'_g \arrow[r] & A^h \arrow[d] \\ & & A/I \end{tikzcd}$$

commute. This induces a map $A' = B' \otimes_B A^h \to A^h$ compatible with σ as desired.

Let $(A, I) \to (A', I')$ be a morphism of pairs with (A', I') henselian. We will show there is a unique factorization $A \to A^h \to A'$ which will finish the proof. Namely, for each $A \to B$ in $\mathcal{C}$ the ring map $A' \to B' = A' \otimes_A B$ is étale and induces an isomorphism $A'/I' \to B'/I'B'$. Hence there is a section $\sigma_B : B' \to A'$ by Lemma 11.6. Given a morphism $B_1 \to B_2$ in $\mathcal{C}$ we claim the diagram

$$\begin{tikzcd} B'_1 \arrow[rr] \arrow[dr, "\sigma_{B_1}"'] & & B'_2 \arrow[dl, "\sigma_{B_2}"] \\ & A' & \end{tikzcd}$$

commutes. This follows once we prove that for every B in $\mathcal{C}$ the section σ_B is the unique A'-algebra map $B' \to A'$. We have $B' \otimes_{A'} B' = B' \times R$ for some ring R, see Algebra, Lemma 151.4. In our case $R/I'R = 0$ as $B'/I'B' = A'/I'$. Thus given two A'-algebra maps $\sigma_B, \sigma'_B : B' \to A'$ then $e = (\sigma_B \otimes \sigma'_B)(0, 1) \in A'$ is an idempotent contained in I'. We conclude that $e = 0$ by Lemma 10.2. Hence $\sigma_B = \sigma'_B$ as desired. Using the commutativity we obtain

$$A^h = \operatorname{colim}_{B \in \mathcal{C}} B \to \operatorname{colim}_{B \in \mathcal{C}} A' \otimes_A B \xrightarrow{\operatorname{colim} \sigma_B} A'$$

as desired. The uniqueness of the maps σ_B also guarantees that this map is unique. Hence $(A, I) \mapsto (A^h, I^h)$ is the desired adjoint. $\square$

Lemma 12.2. *Let (A, I) be a pair. Let (A^h, I^h) be as in Lemma 12.1. Then $A \to A^h$ is flat, $I^h = IA^h$ and $A/I^n \to A^h/I^nA^h$ is an isomorphism for all n.*

Proof. In the proof of Lemma 12.1 we have seen that A^h is a filtered colimit of étale A-algebras B such that $A/I \to B/IB$ is an isomorphism and we have seen that $I^h = IA^h$. As an étale ring map is flat (Algebra, Lemma 143.3) we conclude that $A \to A^h$ is flat by Algebra, Lemma 39.3. Since each $A \to B$ is flat we find that the maps $A/I^n \to B/I^nB$ are isomorphisms as well (for example by Algebra, Lemma 101.3). Taking the colimit we find that $A/I^n = A^h/I^nA^h$ as desired. □

Lemma 12.3. *The functor of Lemma 12.1 associates to a local ring $(A, \mathfrak{m})$ its henselization.*

Proof. Let $(A^h, \mathfrak{m}^h)$ be the henselization of the pair $(A, \mathfrak{m})$ constructed in Lemma 12.1. Then $\mathfrak{m}^h = \mathfrak{m}A^h$ is a maximal ideal by Lemma 12.2 and since it is contained in the Jacobson radical, we conclude A^h is local with maximal ideal $\mathfrak{m}^h$. Having said this there are two ways to finish the proof.

First proof: observe that the construction in the proof of Algebra, Lemma 155.1 as a colimit is the same as the colimit used to construct A^h in Lemma 12.1. Second proof: Both the henselization $A \to S$ and $A \to A^h$ of Lemma 12.1 are local ring homomorphisms, both S and A^h are filtered colimits of étale A-algebras, both S and A^h are henselian local rings, and both S and A^h have residue fields equal to $\kappa(\mathfrak{m})$ (by Lemma 12.2 for the second case). Hence they are canonically isomorphic by Algebra, Lemma 154.7. □

Lemma 12.4. *Let (A, I) be a pair with A Noetherian. Let (A^h, I^h) be as in Lemma 12.1. Then the map of I-adic completions*

$$A^\wedge \to (A^h)^\wedge$$

is an isomorphism. Moreover, A^h is Noetherian, the maps $A \to A^h \to A^\wedge$ are flat, and $A^h \to A^\wedge$ is faithfully flat.

Proof. The first statement is an immediate consequence of Lemma 12.2 and in fact holds without assuming A is Noetherian. In the proof of Lemma 12.1 we have seen that A^h is a filtered colimit of étale A-algebras B such that $A/I \to B/IB$ is an isomorphism. For each such $A \to B$ the induced map $A^\wedge \to B^\wedge$ is an isomorphism (see proof of Lemma 12.2). By Algebra, Lemma 97.2 the ring map $B \to A^\wedge = B^\wedge = (A^h)^\wedge$ is flat for each B. Thus $A^h \to A^\wedge = (A^h)^\wedge$ is flat by Algebra, Lemma 39.6. Since $I^h = IA^h$ is contained in the Jacobson radical of A^h and since $A^h \to A^\wedge$ induces an isomorphism $A^h/I^h \to A/I$ we see that $A^h \to A^\wedge$ is faithfully flat by Algebra, Lemma 39.15. By Algebra, Lemma 97.6 the ring $A^\wedge$ is Noetherian. Hence we conclude that A^h is Noetherian by Algebra, Lemma 164.1. □

Lemma 12.5. *Let $(A, I) = \operatorname{colim}(A_i, I_i)$ be a filtered colimit of pairs. The functor of Lemma 12.1 gives $A^h = \operatorname{colim} A_i^h$ and $I^h = \operatorname{colim} I_i^h$.*

This lemma is false for non-filtered colimits, see Example 11.14.

Proof. By Categories, Lemma 24.5 we see that (A^h, I^h) is the colimit of the system (A_i^h, I_i^h) in the category of henselian pairs. Thus for a henselian pair (B, J) we have

$$\operatorname{Mor}((A^h, I^h), (B, J)) = \lim \operatorname{Mor}((A_i^h, I_i^h), (B, J)) = \operatorname{Mor}(\operatorname{colim}(A_i^h, I_i^h), (B, J))$$

Here the colimit is in the category of pairs. Since the colimit is filtered we obtain $\operatorname{colim}(A_i^h, I_i^h) = (\operatorname{colim} A_i^h, \operatorname{colim} I_i^h)$ in the category of pairs; details omitted. Again using the colimit is filtered, this is a henselian pair (Lemma 11.13). Hence by the Yoneda lemma we find $(A^h, I^h) = (\operatorname{colim} A_i^h, \operatorname{colim} I_i^h)$. □

Lemma 12.6. *Let A be a ring with ideals I and J. If $V(I) = V(J)$ then the functor of Lemma 12.1 produces the same ring for the pair (A, I) as for the pair (A, J).*

Proof. Let (A', IA') be the pair produced by Lemma 12.1 starting with the pair (A, I), see Lemma 12.2. Let (A'', JA'') be the pair produced by Lemma 12.1 starting with the pair (A, J). By Lemma 11.7 we see that (A', JA') is a henselian pair and (A'', IA'') is a henselian pair. By the universal property of the construction we obtain unique A-algebra maps $A'' \to A'$ and $A' \to A''$. The uniqueness shows that these are mutually inverse. □

Lemma 12.7. *Let $(A, I) \to (B, J)$ be a map of pairs such that $V(J) = V(IB)$. Let $(A^h, I^h) \to (B^h, J^h)$ be the induced map on henselizations (Lemma 12.1). If $A \to B$ is integral, then the induced map $A^h \otimes_A B \to B^h$ is an isomorphism.*

Proof. By Lemma 12.6 we may assume $J = IB$. By Lemma 11.8 the pair $(A^h \otimes_A B, I^h(A^h \otimes_A B))$ is henselian. By the universal property of (B^h, IB^h) we obtain a map $B^h \to A^h \otimes_A B$. We omit the proof that this map is the inverse of the map in the lemma. □

13. Lifting and henselian pairs

In this section we mostly combine results from Sections 9 and 11.

Lemma 13.1. *Let (R, I) be a henselian pair. The map*

$$P \longrightarrow P/IP$$

induces a bijection between the sets of isomorphism classes of finite projective R-modules and finite projective R/I-modules. In particular, any finite projective R/I-module is isomorphic to P/IP for some finite projective R-module P.

Proof. We first prove the final statement. Let $\overline{P}$ be a finite projective R/I-module. We can find a finite projective module P' over some R' étale over R with $R/I = R'/IR'$ such that P'/IP' is isomorphic to $\overline{P}$, see Lemma 9.11. Then, since (R, I) is a henselian pair, the étale ring map $R \to R'$ has a section $\tau : R' \to R$ (Lemma 11.6). Setting $P = P' \otimes_{R', \tau} R$ we conclude that P/IP is isomorphic to $\overline{P}$. Of course, this tells us that the map in the statement of the lemma is surjective.

Injectivity. Suppose that P_1 and P_2 are finite projective R-modules such that $P_1/IP_1 \cong P_2/IP_2$ as R/I-modules. Since P_1 is projective, we can find an R-module map $u : P_1 \to P_2$ lifting the given isomorphism. Then u is surjective by Nakayama's lemma (Algebra, Lemma 20.1). We similarly find a surjection $v : P_2 \to P_1$. By Algebra, Lemma 16.4 the map $v \circ u$ is an isomorphism and we conclude u is an isomorphism. □

Lemma 13.2. *Let (A, I) be a henselian pair. The functor $B \to B/IB$ determines an equivalence between finite étale A-algebras and finite étale A/I-algebras.*

Proof. Let B, B' be two A-algebras finite étale over A. Then $B' \to B'' = B \otimes_A B'$ is finite étale as well (Algebra, Lemmas 143.3 and 36.13). Now we have 1-to-1 correspondences between

(1) A-algebra maps $B \to B'$,
(2) sections of $B' \to B''$, and
(3) idempotents e of B'' such that $B' \to B'' \to eB''$ is an isomorphism.

The bijection between (2) and (3) sends $\sigma : B'' \to B'$ to e such that $(1-e)$ is the idempotent that generates the kernel of σ which exists by Algebra, Lemmas 143.8 and 143.9. There is a similar correspondence between A/I-algebra maps $B/IB \to B'/IB'$ and idempotents $\overline{e}$ of B''/IB'' such that $B'/IB' \to B''/IB'' \to \overline{e}(B''/IB'')$ is an isomorphism. However every idempotent $\overline{e}$ of B''/IB'' lifts uniquely to an idempotent e of B'' (Lemma 11.6). Moreover, if $B'/IB' \to \overline{e}(B''/IB'')$ is an isomorphism, then $B' \to eB''$ is an isomorphism too by Nakayama's lemma (Algebra, Lemma 20.1). In this way we see that the functor is fully faithful.

Essential surjectivity. Let $A/I \to C$ be a finite étale map. By Algebra, Lemma 143.10 there exists an étale map $A \to B$ such that $B/IB \cong C$. Let B' be the integral closure of A in B. By Lemma 11.5 we have $B'/IB' = C \times C'$ for some ring C' and $B'_g \cong B_g$ for some $g \in B'$ mapping to $(1,0) \in C \times C'$. Since idempotents lift (Lemma 11.6) we get $B' = B'_1 \times B'_2$ with $C = B'_1/IB'_1$ and $C' = B'_2/IB'_2$. The image of g in B'_1 is invertible. Then $B_g = B'_g = B'_1 \times (B_2)_g$ and this implies that $A \to B'_1$ is étale. We conclude that B'_1 is finite étale over A (integral étale implies finite étale by Algebra, Lemma 36.5 for example) and the proof is done. □

Lemma 13.3. *Let $A = \lim A_n$ be a limit of an inverse system (A_n) of rings. Suppose given A_n-modules M_n and A_{n+1}-module maps $M_{n+1} \to M_n$. Assume*

(1) *the transition maps $A_{n+1} \to A_n$ are surjective with locally nilpotent kernels,*
(2) *M_1 is a finite projective A_1-module,*
(3) *M_n is a finite flat A_n-module, and*
(4) *the maps induce isomorphisms $M_{n+1} \otimes_{A_{n+1}} A_n \to M_n$.*

Then $M = \lim M_n$ is a finite projective A-module and $M \otimes_A A_n \to M_n$ is an isomorphism for all n.

Proof. By Lemma 11.3 the pair $(A, \mathrm{Ker}(A \to A_1))$ is henselian. By Lemma 13.1 we can choose a finite projective A-module P and an isomorphism $P \otimes_A A_1 \to M_1$. Since P is projective, we can successively lift the A-module map $P \to M_1$ to A-module maps $P \to M_2$, $P \to M_3$, and so on. Thus we obtain a map

$$P \longrightarrow M$$

Since P is finite projective, we can write $A^{\oplus m} = P \oplus Q$ for some $m \geq 0$ and A-module Q. Since $A = \lim A_n$ we conclude that $P = \lim P \otimes_A A_n$. Hence, in order to show that the displayed A-module map is an isomorphism, it suffices to show that the maps $P \otimes_A A_n \to M_n$ are isomorphisms. From Lemma 3.4 we see that M_n is a finite projective module. By Lemma 3.5 the maps $P \otimes_A A_n \to M_n$ are isomorphisms. □

14. Absolute integral closure

Here is our definition.

Definition 14.1. A ring A is *absolutely integrally closed* if every monic $f \in A[T]$ is a product of linear factors.

Be careful: it may be possible to write f as a product of linear factors in many different ways.

Lemma 14.2. *Let A be a ring. The following are equivalent*

(1) *A is absolutely integrally closed, and*
(2) *any monic $f \in A[T]$ has a root in A.*

Proof. Omitted. □

Lemma 14.3. *Let A be absolutely integrally closed.*

(1) *Any quotient ring A/I of A is absolutely integrally closed.*
(2) *Any localization $S^{-1}A$ is absolutely integrally closed.*

Proof. Omitted. □

Lemma 14.4. *Let A be a ring. Let $S \subset A$ be a multiplicative subset consisting of nonzerodivisors. If $S^{-1}A$ is absolutely integrally closed and $A \subset S^{-1}A$ is integrally closed in $S^{-1}A$, then A is absolutely integrally closed.*

Proof. Omitted. □

Lemma 14.5. *Let A be a normal domain. Then A is absolutely integrally closed if and only if its fraction field is algebraically closed.*

Proof. Observe that a field is algebraically closed if and only if it is absolutely integrally closed as a ring. Hence the lemma follows from Lemmas 14.3 and 14.4. □

Lemma 14.6. *For any ring A there exists an extension $A \subset B$ such that*

(1) *B is a filtered colimit of finite free A-algebras,*
(2) *B is free as an A-module, and*
(3) *B is absolutely integrally closed.*

Proof. Let I be the set of monic polynomials over A. For $i \in I$ denote x_i a variable and P_i the corresponding monic polynomial in the variable x_i. Then we set

$$F(A) = A[x_i; i \in I]/(P_i; i \in I)$$

As the notation suggests F is a functor from the category of rings to itself. Note that $A \subset F(A)$, that $F(A)$ is free as an A-module, and that $F(A)$ is a filtered colimit of finite free A-algebras. Then we take

$$B = \operatorname{colim} F^n(A)$$

where the transition maps are the inclusions $F^n(A) \subset F(F^n(A)) = F^{n+1}(A)$. Any monic polynomial with coefficients in B actually has coefficients in $F^n(A)$ for some n and hence has a solution in $F^{n+1}(A)$ by construction. This implies that B is absolutely integrally closed by Lemma 14.2. We omit the proof of the other properties. □

Lemma 14.7. *Let A be absolutely integrally closed. Let $\mathfrak{p} \subset A$ be a prime. Then the local ring $A_{\mathfrak{p}}$ is strictly henselian.*

Proof. By Lemma 14.3 we may assume A is a local ring and $\mathfrak{p}$ is its maximal ideal. The residue field is algebraically closed by Lemma 14.3. Every monic polynomial decomposes completely into linear factors hence Algebra, Definition 153.1 applies directly. $\square$

Lemma 14.8. *Let A be absolutely integrally closed. Let $I \subset A$ be an ideal. Then (A, I) is a henselian pair if (and only if) the following conditions hold*

(1) *I is contained in the Jacobson radical of A,*
(2) *$A \to A/I$ induces a bijection on idempotents.*

Proof. Let $f \in A[T]$ be a monic polynomial and let $f \bmod I = g_0 h_0$ be a factorization over A/I with g_0, h_0 monic such that g_0 and h_0 generate the unit ideal of $A/I[T]$. This means that

$$A/I[T]/(f) = A/I[T]/(g_0) \times A/I[T]/(h_0)$$

Denote $e \in A/I[T]/(f)$ the element correspoing to the idempotent $(1, 0)$ in the ring on the right. Write $f = (T - a_1) \ldots (T - a_d)$ with $a_i \in A$. For each $i \in \{1, \ldots, d\}$ we obtain an A-algebra map $\varphi_i : A[T]/(f) \to A$, $T \mapsto a_i$ which induces a similar A/I-algebra map $\overline{\varphi}_i : A/I[T]/(f) \to A/I$. Denote $e_i = \overline{\varphi}_i(e) \in A/I$. These are idempotents. By our assumption (2) we can lift e_i to an idempotent in A. This means we can write $A = \prod A_j$ as a finite product of rings such that in A_j/IA_j each e_i is either 0 or 1. Some details omitted. Observe that A_j is absolutely integrally closed as a factor ring of A. It suffices to lift the factorization of f over A_j/IA_j to A_j. This reduces us to the situation discussed in the next paragraph.

Assume $e_i = 1$ for $i = 1, \ldots, r$ and $e_i = 0$ for $i = r + 1, \ldots, d$. From $(g_0, h_0) = A/I[T]$ we have that there are $k_0, l_0 \in A/I[T]$ such that $g_0 k_0 + h_0 l_0 = 1$. We see that $e = h_0 l_0$ and $e_i = h_0(a_i) l_0(a_i)$. We conclude that $h_0(a_i)$ is a unit for $i = 1, \ldots, r$. Since $f(a_i) = 0$ we find $0 = h_0(a_i) g_0(a_i)$ and we conclude that $g_0(a_i) = 0$ for $i = 1, \ldots, r$. Thus $(T - a_1)$ divides g_0 in $A/I[T]$, say $g_0 = (T - a_1) g_0'$. Set $f' = (T - a_2) \ldots (T - a_d)$ and $h_0' = h_0$. By induction on d we can lift the factorization $f' \bmod I = g_0' h_0'$ to a factorization of $f' = g' h'$ over over A which gives the factorization $f = (T - a_1) g' h'$ lifting the factorization $f \bmod I = g_0 h_0$ as desired. $\square$

15. Auto-associated rings

Some of this material is in [Laz69].

Definition 15.1. A ring R is said to be *auto-associated* if R is local and its maximal ideal $\mathfrak{m}$ is weakly associated to R.

Lemma 15.2. *An auto-associated ring R has the following property: (P) Every proper finitely generated ideal $I \subset R$ has a nonzero annihilator.*

Proof. By assumption there exists a nonzero element $x \in R$ such that for every $f \in \mathfrak{m}$ we have $f^n x = 0$. Say $I = (f_1, \ldots, f_r)$. Then x is in the kernel of $R \to \bigoplus R_{f_i}$. Hence we see that there exists a nonzero $y \in R$ such that $f_i y = 0$ for all i, see Algebra, Lemma 24.4. As $y \in \text{Ann}_R(I)$ we win. $\square$

Lemma 15.3. *Let R be a ring having property (P) of Lemma 15.2. Let $u : N \to M$ be a homomorphism of projective R-modules. Then u is universally injective if and only if u is injective.*

Proof. Assume u is injective. Our goal is to show u is universally injective. First we choose a module Q such that $N \oplus Q$ is free. On considering the map $N \oplus Q \to M \oplus Q$ we see that it suffices to prove the lemma in case N is free. In this case N is a directed colimit of finite free R-modules. Thus we reduce to the case that N is a finite free R-module, say $N = R^{\oplus n}$. We prove the lemma by induction on n. The case $n = 0$ is trivial.

Let $u : R^{\oplus n} \to M$ be an injective module map with M projective. Choose an R-module Q such that $M \oplus Q$ is free. After replacing u by the composition $R^{\oplus n} \to M \to M \oplus Q$ we see that we may assume that M is free. Then we can find a direct summand $R^{\oplus m} \subset M$ such that $u(R^{\oplus n}) \subset R^{\oplus m}$. Hence we may assume that $M = R^{\oplus m}$. In this case u is given by a matrix $A = (a_{ij})$ so that $u(x_1, \ldots, x_n) = (\sum x_i a_{i1}, \ldots, \sum x_i a_{im})$. As u is injective, in particular $u(x, 0, \ldots, 0) = (xa_{11}, xa_{12}, \ldots, xa_{1m}) \neq 0$ if $x \neq 0$, and as R has property (P) we see that $a_{11}R + a_{12}R + \ldots + a_{1m}R = R$. Hence see that $R(a_{11}, \ldots, a_{1m}) \subset R^{\oplus m}$ is a direct summand of $R^{\oplus m}$, in particular $R^{\oplus m}/R(a_{11}, \ldots, a_{1m})$ is a projective R-module. We get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d, "1"] & R^{\oplus n} \arrow[r] \arrow[d, "u"] & R^{\oplus n-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & R \arrow[r, "{(a_{11}, \ldots, a_{1m})}"] & R^{\oplus m} \arrow[r] & R^{\oplus m}/R(a_{11}, \ldots, a_{1m}) \arrow[r] & 0 \end{tikzcd}$$

with split exact rows. Thus the right vertical arrow is injective and we may apply the induction hypothesis to conclude that the right vertical arrow is universally injective. It follows that the middle vertical arrow is universally injective. □

Lemma 15.4. *Let R be a ring. The following are equivalent*

1. *R has property (P) of Lemma 15.2,*
2. *any injective map of projective R-modules is universally injective,*
3. *if $u : N \to M$ is injective and N, M are finite projective R-modules then* $\text{Coker}(u)$ *is a finite projective R-module,*
4. *if $N \subset M$ and N, M are finite projective as R-modules, then N is a direct summand of M, and*
5. *any injective map $R \to R^{\oplus n}$ is a split injection.*

Proof. The implication (1) $\Rightarrow$ (2) is Lemma 15.3. It is clear that (3) and (4) are equivalent. We have (2) $\Rightarrow$ (3), (4) by Algebra, Lemma 82.4. Part (5) is a special case of (4). Assume (5). Let $I = (a_1, \ldots, a_n)$ be a proper finitely generated ideal of R. As $I \neq R$ we see that $R \to R^{\oplus n}$, $x \mapsto (xa_1, \ldots, xa_n)$ is not a split injection. Hence it has a nonzero kernel and we conclude that $\text{Ann}_R(I) \neq 0$. Thus (1) holds. □

Example 15.5. If the equivalent conditions of Lemma 15.4 hold, then it is not always the case that every injective map of free R-modules is a split injection. For example suppose that $R = k[x_1, x_2, x_3, \ldots]/(x_i^2)$. This is an auto-associated ring. Consider the map of free R-modules

$$u : \bigoplus\nolimits_{i \geq 1} Re_i \longrightarrow \bigoplus\nolimits_{i \geq 1} Rf_i, \quad e_i \longmapsto f_i - x_i f_{i+1}.$$

For any integer n the restriction of u to $\bigoplus_{i=1,\ldots,n} Re_i$ is injective as the images $u(e_1), \ldots, u(e_n)$ are R-linearly independent. Hence u is injective and hence universally injective by the lemma. Since $u \otimes \mathrm{id}_k$ is bijective we see that if u were a split injection then u would be surjective. But u is not surjective because the inverse image of f_1 would be the element

$$\sum\nolimits_{i \geq 0} x_1 \ldots x_i e_{i+1} = e_1 + x_1 e_2 + x_1 x_2 e_3 + \ldots$$

which is not an element of the direct sum. A side remark is that $\mathrm{Coker}(u)$ is a flat (because u is universally injective), countably generated R-module which is not projective (as u is not split), hence not Mittag-Leffler (see Algebra, Lemma 93.1).

The following lemma is a special case of Algebra, Proposition 102.9 in case the local ring is Noetherian.

Lemma 15.6. *Let $(R, \mathfrak m)$ be a local ring. Suppose that $\varphi : R^m \to R^n$ is a map of finite free modules. The following are equivalent*

(1) *φ is injective,*
(2) *the rank of φ is m and the annihilator of $I(\varphi)$ in R is zero.*

If R is Noetherian these are also equivalent to

(3) *the rank of φ is m and either $I(\varphi) = R$ or it contains a nonzerodivisor.*

Here the rank of φ and $I(\varphi)$ are defined as in Algebra, Definition 102.5.

Proof. If any matrix coefficient of φ is not in $\mathfrak m$, then we apply Algebra, Lemma 102.2 to write φ as the sum of $1 : R \to R$ and a map $\varphi' : R^{m-1} \to R^{n-1}$. It is easy to see that the lemma for φ' implies the lemma for φ. Thus we may assume from the outset that all the matrix coefficients of φ are in $\mathfrak m$.

Suppose φ is injective. We may assume $m > 0$. Let $\mathfrak q \in \mathrm{WeakAss}(R)$ so that $R_{\mathfrak q}$ is an auto-associated ring. Then φ induces a injective map $R_{\mathfrak q}^m \to R_{\mathfrak q}^n$ which is universally injective by Lemmas 15.2 and 15.3. Thus $\varphi : \kappa(\mathfrak q)^m \to \kappa(\mathfrak q)^n$ is injective. Hence the rank of $\varphi \bmod \mathfrak q$ is m and $I(\varphi \otimes \kappa(\mathfrak q))$ is not the zero ideal. Since m is the maximum rank φ can have, we conclude that φ has rank m as well (ranks of matrices can only drop after base change). Hence $I(\varphi) \cdot \kappa(\mathfrak q) = I(\varphi \otimes \kappa(\mathfrak q))$ is not zero. Thus $I(\varphi)$ is not contained in $\mathfrak q$. Thus none of the weakly associated primes of R are weakly associated primes of the R-module $\mathrm{Ann}_R I(\varphi)$. Thus $\mathrm{Ann}_R I(\varphi)$ has no weakly associated primes, see Algebra, Lemma 66.4. It follows from Algebra, Lemma 66.5 that $\mathrm{Ann}_R I(\varphi)$ is zero.

Conversely, assume (2). The rank being m implies $n \geq m$. Write $I(\varphi) = (f_1, \ldots, f_r)$ which is possible as $I(\varphi)$ is finitely generated. By Algebra, Lemma 15.5 we can find maps $\psi_i : R^n \to R^m$ such that $\psi \circ \varphi = f_i \mathrm{id}_{R^m}$. Thus $\varphi(x) = 0$ implies $f_i x = 0$ for $i = 1, \ldots, r$. This implies $x = 0$ and hence φ is injective.

For the equivalence of (1) and (3) in the Noetherian local case we refer to Algebra, Proposition 102.9. If the ring R is Noetherian but not local, then the reader can deduce it from the local case; details omitted. Another option is to redo the argument above using associated primes, using that there are finitely many of these, using prime avoidance, and using the characterization of nonzerodivisors as elements of a Noetherian ring not contained in any associated prime. $\square$

Lemma 15.7. *Let R be a ring. Suppose that $\varphi : R^n \to R^n$ be an injective map of finite free modules of the same rank. Then $\mathrm{Hom}_R(\mathrm{Coker}(\varphi), R) = 0$.*

Proof. Let $\varphi^t : R^n \to R^n$ be the transpose of φ. The lemma claims that φ^t is injective. With notation as in Lemma 15.6 we see that the rank of φ^t is n and that $I(\varphi) = I(\varphi^t)$. Thus we conclude by the equivalence of (1) and (2) of the lemma. $\square$

16. Flattening stratification

Let $R \to S$ be a ring map and let M be an S-module. For any R-algebra R' we can consider the base changes $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We say $R \to R'$ *flattens* M if the module M' is flat over R'. We would like to understand the structure of the collection of ring maps $R \to R'$ which flatten M. In particular we would like to know if there exists a *universal flattening* $R \to R_{univ}$ *of* M, i.e., a ring map $R \to R_{univ}$ which flattens M and has the property that any ring map $R \to R'$ which flattens M factors through $R \to R_{univ}$. It turns out that such a universal solution usually does not exist.

We will discuss *universal flattenings* and *flattening stratifications* in a scheme theoretic setting $\mathcal{F}/X/S$ in More on Flatness, Section 21. If the universal flattening $R \to R_{univ}$ exists then the morphism of schemes $\text{Spec}(R_{univ}) \to \text{Spec}(R)$ is the universal flattening of the quasi-coherent module $\widetilde{M}$ on $\text{Spec}(S)$.

In this and the next few sections we prove some basic algebra facts related to this. The most basic result is perhaps the following.

Lemma 16.1. *Let R be a ring. Let M be an R-module. Let I_1, I_2 be ideals of R. If M/I_1M is flat over R/I_1 and M/I_2M is flat over R/I_2, then $M/(I_1 \cap I_2)M$ is flat over $R/(I_1 \cap I_2)$.*

Proof. By replacing R with $R/(I_1 \cap I_2)$ and M by $M/(I_1 \cap I_2)M$ we may assume that $I_1 \cap I_2 = 0$. Let $J \subset R$ be an ideal. To prove that M is flat over R we have to show that $J \otimes_R M \to M$ is injective, see Algebra, Lemma 39.5. By flatness of M/I_1M over R/I_1 the map

$$J/(J \cap I_1) \otimes_R M = (J + I_1)/I_1 \otimes_{R/I_1} M/I_1M \longrightarrow M/I_1M$$

is injective. As $0 \to (J \cap I_1) \to J \to J/(J \cap I_1) \to 0$ is exact we obtain a diagram

$$\begin{tikzcd}
(J \cap I_1) \otimes_R M \arrow[r] \arrow[d] & J \otimes_R M \arrow[r] \arrow[d] & J/(J \cap I_1) \otimes_R M \arrow[r] \arrow[d] & 0 \\
M \arrow[r, equal] & M \arrow[r] & M/I_1M &
\end{tikzcd}$$

hence it suffices to show that $(J \cap I_1) \otimes_R M \to M$ is injective. Since $I_1 \cap I_2 = 0$ the ideal $J \cap I_1$ maps isomorphically to an ideal $J' \subset R/I_2$ and we see that $(J \cap I_1) \otimes_R M = J' \otimes_{R/I_2} M/I_2M$. By flatness of M/I_2M over R/I_2 the map $J' \otimes_{R/I_2} M/I_2M \to M/I_2M$ is injective, which clearly implies that $(J \cap I_1) \otimes_R M \to M$ is injective. $\square$

17. Flattening over an Artinian ring

A universal flattening exists when the base ring is an Artinian local ring. It exists for an arbitrary module. Hence, as we will see later, a flatting stratification exists when the base scheme is the spectrum of an Artinian local ring.

Lemma 17.1. *Let R be an Artinian ring. Let M be an R-module. Then there exists a smallest ideal $I \subset R$ such that M/IM is flat over R/I.*

Proof. This follows directly from Lemma 16.1 and the Artinian property. □

This ideal has the following universal property.

Lemma 17.2. *Let R be an Artinian ring. Let M be an R-module. Let $I \subset R$ be the smallest ideal $I \subset R$ such that M/IM is flat over R/I. Then I has the following universal property: For every ring map $\varphi : R \to R'$ we have*

$$R' \otimes_R M \text{ is flat over } R' \Leftrightarrow \text{we have } \varphi(I) = 0.$$

Proof. Note that I exists by Lemma 17.1. The implication $\Rightarrow$ follows from Algebra, Lemma 39.7. Let $\varphi : R \to R'$ be such that $M \otimes_R R'$ is flat over R'. Let $J = \text{Ker}(\varphi)$. By Algebra, Lemma 101.7 and as $R' \otimes_R M = R' \otimes_{R/J} M/JM$ is flat over R' we conclude that M/JM is flat over R/J. Hence $I \subset J$ as desired. □

18. Flattening over a closed subset of the base

Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. In the following we will consider the following condition

$$\forall \mathfrak q \in V(IS) \subset \text{Spec}(S) : M_{\mathfrak q} \text{ is flat over } R. \tag{18.0.1}$$

Geometrically, this means that M is flat over R along the inverse image of $V(I)$ in $\text{Spec}(S)$. If R and S are Noetherian rings and M is a finite S-module, then (18.0.1) is equivalent to the condition that M/I^nM is flat over R/I^n for all $n \geq 1$, see Algebra, Lemma 99.11.

Lemma 18.1. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal. If (18.0.1) holds for $(R \to S, I, M)$, then (18.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$.*

Proof. Assume (18.0.1) holds for $(R \to S, I \subset R, M)$. Let $I'(S \otimes_R R') \subset \mathfrak q'$ be a prime of $S \otimes_R R'$. Let $\mathfrak q \subset S$ be the corresponding prime of S. Then $IS \subset \mathfrak q$. Note that $(M \otimes_R R')_{\mathfrak q'}$ is a localization of the base change $M_{\mathfrak q} \otimes_R R'$. Hence $(M \otimes_R R')_{\mathfrak q'}$ is flat over R' as a localization of a flat module, see Algebra, Lemmas 39.7 and 39.18. □

Lemma 18.2. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal such that*

1. *the map $V(I') \to V(I)$ induced by $\text{Spec}(R') \to \text{Spec}(R)$ is surjective, and*
2. *$R'_{\mathfrak p'}$ is flat over R for all primes $\mathfrak p' \in V(I')$.*

If (18.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then (18.0.1) holds for $(R \to S, I, M)$.

Proof. Assume (18.0.1) holds for $(R' \to S \otimes_R R', IR', M \otimes_R R')$. Pick a prime $IS \subset \mathfrak q \subset S$. Let $I \subset \mathfrak p \subset R$ be the corresponding prime of R. By assumption there exists a prime $\mathfrak p' \in V(I')$ of R' lying over $\mathfrak p$ and $R_{\mathfrak p} \to R'_{\mathfrak p'}$ is flat. Choose a prime $\overline{\mathfrak q}' \subset \kappa(\mathfrak q) \otimes_{\kappa(\mathfrak p)} \kappa(\mathfrak p')$ which corresponds to a prime $\mathfrak q' \subset S \otimes_R R'$ which lies over $\mathfrak q$ and over $\mathfrak p'$. Note that $(S \otimes_R R')_{\mathfrak q'}$ is a localization of $S_{\mathfrak q} \otimes_{R_{\mathfrak p}} R'_{\mathfrak p'}$. By assumption the module $(M \otimes_R R')_{\mathfrak q'}$ is flat over $R'_{\mathfrak p'}$. Hence Algebra, Lemma 100.1 implies that $M_{\mathfrak q}$ is flat over $R_{\mathfrak p}$ which is what we wanted to prove. □

Lemma 18.3. *Let $R \to S$ be a ring map of finite presentation. Let M be an S-module of finite presentation. Let $R' = \text{colim}_{\lambda \in \Lambda} R_\lambda$ be a directed colimit of R-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set*

$I' = \mathrm{colim}_\lambda I_\lambda$. If (18.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then there exists a $\lambda \in \Lambda$ such that (18.0.1) holds for $(R_\lambda \to S \otimes_R R_\lambda, I_\lambda, M \otimes_R R_\lambda)$.

Proof. We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change S' is of finite presentation over R' and M' is of finite presentation over S' and similarly for the versions with subscript λ, see Algebra, Lemma 14.2. By Algebra, Theorem 129.4 the set

$$U' = \{\mathfrak q' \in \mathrm{Spec}(S') \mid M'_{\mathfrak q'} \text{ is flat over } R'\}$$

is open in $\mathrm{Spec}(S')$. Note that $V(I'S')$ is a quasi-compact space which is contained in U' by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over R'.

We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j,\lambda} \in S_\lambda$ mapping to g'_j. The inclusion $V(I'S') \subset \bigcup D(g'_j)$ means that $I'S' + (g'_1, \ldots, g'_m) = S'$ which can be expressed as $1 = \sum z_s h_s + \sum f_j g'_j$ for some $z_s \in I'$, $h_s, f_j \in S'$. After increasing λ we may assume such an equation holds in S_λ. Hence we may assume that $V(I_\lambda S_\lambda) \subset \bigcup D(g_{j,\lambda})$. By Algebra, Lemma 168.1 we see that for some sufficiently large λ the modules $(M_\lambda)_{g_{j,\lambda}}$ are flat over R_λ. In particular the module M_λ is flat over R_λ at all the primes lying over the ideal I_λ. □

19. Flattening over a closed subsets of source and base

In this section we slightly generalize the discussion in Section 18. We strongly suggest the reader first read and understand that section.

Situation 19.1. Let $R \to S$ be a ring map. Let $J \subset S$ be an ideal. Let M be an S-module.

In this situation, given an R-algebra R' and an ideal $I' \subset R'$ we set $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We will consider the condition

$$\forall \mathfrak q' \in V(I'S' + JS') \subset \mathrm{Spec}(S') : M'_{\mathfrak q'} \text{ is flat over } R'. \tag{19.1.1}$$

Geometrically, this means that M' is flat over R' along the intersection of the inverse image of $V(I')$ with the inverse image of $V(J)$. Since $(R \to S, J, M)$ are fixed, condition (19.1.1) only depends on the pair (R', I') where R' is viewed as an R-algebra.

Lemma 19.2. *In Situation 19.1 let $R' \to R''$ be an R-algebra map. Let $I' \subset R'$ and $I'R'' \subset I'' \subset R''$ be ideals. If (19.1.1) holds for (R', I'), then (19.1.1) holds for (R'', I'').*

Proof. Assume (19.1.1) holds for (R', I'). Let $I''S'' + JS'' \subset \mathfrak q''$ be a prime of S''. Let $\mathfrak q' \subset S'$ be the corresponding prime of S'. Then both $I'S' \subset \mathfrak q'$ and $JS' \subset \mathfrak q'$ because the corresponding conditions hold for $\mathfrak q''$. Note that $(M'')_{\mathfrak q''}$ is a localization of the base change $M'_{\mathfrak q'} \otimes_R R''$. Hence $(M'')_{\mathfrak q''}$ is flat over R'' as a localization of a flat module, see Algebra, Lemmas 39.7 and 39.18. □

Lemma 19.3. *In Situation 19.1 let $R' \to R''$ be an R-algebra map. Let $I' \subset R'$ and $I'R'' \subset I'' \subset R''$ be ideals. Assume*

(1) *the map $V(I'') \to V(I')$ induced by $\mathrm{Spec}(R'') \to \mathrm{Spec}(R')$ is surjective, and*

(2) $R''_{\mathfrak p''}$ *is flat over* R' *for all primes* $\mathfrak p'' \in V(I'')$.

If (19.1.1) holds for (R'', I'')*, then (19.1.1) holds for* (R', I').

Proof. Assume (19.1.1) holds for (R'', I''). Pick a prime $I'S' + JS' \subset \mathfrak q' \subset S'$. Let $I' \subset \mathfrak p' \subset R'$ be the corresponding prime of R'. By assumption there exists a prime $\mathfrak p'' \in V(I'')$ of R'' lying over $\mathfrak p'$ and $R'_{\mathfrak p'} \to R''_{\mathfrak p''}$ is flat. Choose a prime $\overline{\mathfrak q}'' \subset \kappa(\mathfrak q') \otimes_{\kappa(\mathfrak p')} \kappa(\mathfrak p'')$. This corresponds to a prime $\mathfrak q'' \subset S'' = S' \otimes_{R'} R''$ which lies over $\mathfrak q'$ and over $\mathfrak p''$. In particular we see that $I''S'' \subset \mathfrak q''$ and that $JS'' \subset \mathfrak q''$. Note that $(S' \otimes_{R'} R'')_{\mathfrak q''}$ is a localization of $S'_{\mathfrak q'} \otimes_{R'_{\mathfrak p'}} R''_{\mathfrak p''}$. By assumption the module $(M' \otimes_{R'} R'')_{\mathfrak q''}$ is flat over $R''_{\mathfrak p''}$. Hence Algebra, Lemma 100.1 implies that $M'_{\mathfrak q'}$ is flat over $R'_{\mathfrak p'}$ which is what we wanted to prove. □

Lemma 19.4. *In Situation 19.1 assume $R \to S$ is essentially of finite presentation and M is an S-module of finite presentation. Let $R' = \operatorname{colim}_{\lambda \in \Lambda} R_\lambda$ be a directed colimit of R-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set $I' = \operatorname{colim}_\lambda I_\lambda$. If (19.1.1) holds for (R', I'), then there exists a $\lambda \in \Lambda$ such that (19.1.1) holds for (R_λ, I_λ).*

Proof. We first prove the lemma in case $R \to S$ is of finite presentation and then we explain what needs to be changed in the general case. We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change S' is of finite presentation over R' and M' is of finite presentation over S' and similarly for the versions with subscript λ, see Algebra, Lemma 14.2. By Algebra, Theorem 129.4 the set

$$U' = \{\mathfrak q' \in \operatorname{Spec}(S') \mid M'_{\mathfrak q'} \text{ is flat over } R'\}$$

is open in $\operatorname{Spec}(S')$. Note that $V(I'S' + JS')$ is a quasi-compact space which is contained in U' by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S' + JS') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over R'.

We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j,\lambda} \in S_\lambda$ mapping to g'_j. The inclusion $V(I'S' + JS') \subset \bigcup D(g'_j)$ means that $I'S' + JS' + (g'_1, \ldots, g'_m) = S'$ which can be expressed as

$$1 = \sum y_t k_t + \sum z_s h_s + \sum f_j g'_j$$

for some $z_s \in I'$, $y_t \in J$, $k_t, h_s, f_j \in S'$. After increasing λ we may assume such an equation holds in S_λ. Hence we may assume that $V(I_\lambda S_\lambda + JS_\lambda) \subset \bigcup D(g_{j,\lambda})$. By Algebra, Lemma 168.1 we see that for some sufficiently large λ the modules $(M_\lambda)_{g_{j,\lambda}}$ are flat over R_λ. In particular the module M_λ is flat over R_λ at all the primes corresponding to points of $V(I_\lambda S_\lambda + JS_\lambda)$.

In the case that S is essentially of finite presentation, we can write $S = \Sigma^{-1}C$ where $R \to C$ is of finite presentation and $\Sigma \subset C$ is a multiplicative subset. We can also write $M = \Sigma^{-1}N$ for some finitely presented C-module N, see Algebra, Lemma 126.3. At this point we introduce C_λ, C', N_λ, N'. Then in the discussion above we obtain an open $U' \subset \operatorname{Spec}(C')$ over which N' is flat over R'. The assumption that (19.1.1) is true means that $V(I'S' + JS')$ maps into U', because for a prime $\mathfrak q' \subset S'$, corresponding to a prime $\mathfrak r' \subset C'$ we have $M'_{\mathfrak q'} = N'_{\mathfrak r'}$. Thus we can find $g'_j \in C'$ such that $\bigcup D(g'_j)$ contains the image of $V(I'S' + JS')$. The rest of the proof is exactly the same as before. □

Lemma 19.5. *In Situation 19.1. Let $I \subset R$ be an ideal. Assume*

1. *R is a Noetherian ring,*
2. *S is a Noetherian ring,*
3. *M is a finite S-module, and*
4. *for each $n \geq 1$ and any prime $\mathfrak{q} \in V(J + IS)$ the module $(M/I^nM)_{\mathfrak{q}}$ is flat over R/I^n.*

Then (19.1.1) holds for (R, I), i.e., for every prime $\mathfrak{q} \in V(J + IS)$ the localization $M_{\mathfrak{q}}$ is flat over R.

Proof. Let $\mathfrak{q} \in V(J + IS)$. Then Algebra, Lemma 99.11 applied to $R \to S_{\mathfrak{q}}$ and $M_{\mathfrak{q}}$ implies that $M_{\mathfrak{q}}$ is flat over R. $\square$

20. Flattening over a Noetherian complete local ring

The following three lemmas give a completely algebraic proof of the existence of the "local" flattening stratification when the base is a complete local Noetherian ring R and the given module is finite over a finite type R-algebra S.

Lemma 20.1. *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

1. *$(R, \mathfrak{m})$ is a complete local Noetherian ring,*
2. *S is a Noetherian ring, and*
3. *M is finite over S.*

Then there exists an ideal $I \subset \mathfrak{m}$ such that

1. *$(M/IM)_{\mathfrak{q}}$ is flat over R/I for all primes $\mathfrak{q}$ of S/IS lying over $\mathfrak{m}$, and*
2. *if $J \subset R$ is an ideal such that $(M/JM)_{\mathfrak{q}}$ is flat over R/J for all primes $\mathfrak{q}$ lying over $\mathfrak{m}$, then $I \subset J$.*

In other words, I is the smallest ideal of R such that (18.0.1) holds for $(\overline{R} \to \overline{S}, \overline{\mathfrak{m}}, \overline{M})$ where $\overline{R} = R/I$, $\overline{S} = S/IS$, $\overline{\mathfrak{m}} = \mathfrak{m}/I$ and $\overline{M} = M/IM$.

Proof. Let $J \subset R$ be an ideal. Apply Algebra, Lemma 99.11 to the module M/JM over the ring R/J. Then we see that $(M/JM)_{\mathfrak{q}}$ is flat over R/J for all primes $\mathfrak{q}$ of S/JS if and only if $M/(J + \mathfrak{m}^n)M$ is flat over $R/(J + \mathfrak{m}^n)$ for all $n \geq 1$. We will use this remark below.

For every $n \geq 1$ the local ring $R/\mathfrak{m}^n$ is Artinian. Hence, by Lemma 17.1 there exists a smallest ideal $I_n \supset \mathfrak{m}^n$ such that M/I_nM is flat over R/I_n. It is clear that $I_{n+1} + \mathfrak{m}^n$ is contains I_n and applying Lemma 16.1 we see that $I_n = I_{n+1} + \mathfrak{m}^n$. Since $R = \lim_n R/\mathfrak{m}^n$ we see that $I = \lim_n I_n/\mathfrak{m}^n$ is an ideal in R such that $I_n = I + \mathfrak{m}^n$ for all $n \geq 1$. By the initial remarks of the proof we see that I verifies (1) and (2). Some details omitted. $\square$

Lemma 20.2. *With notation $R \to S$, M, and I and assumptions as in Lemma 20.1. Consider a local homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ such that R' is Noetherian. Then the following are equivalent*

1. *condition (18.0.1) holds for $(R' \to S \otimes_R R', \mathfrak{m}', M \otimes_R R')$, and*
2. *$\varphi(I) = 0$.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Lemma 18.1. Let $\varphi : R \to R'$ be as in the lemma satisfying (1). We have to show that $\varphi(I) = 0$. This is equivalent to the condition that $\varphi(I)R' = 0$. By Artin-Rees in the Noetherian local ring R' (see Algebra, Lemma 51.4) this is equivalent to the condition that

$\varphi(I)R' + (\mathfrak{m}')^n = (\mathfrak{m}')^n$ for all $n > 0$. Hence this is equivalent to the condition that the composition $\varphi_n : R \to R' \to R'/(\mathfrak{m}')^n$ annihilates I for each n. Now assumption (1) for φ implies assumption (1) for φ_n by Lemma 18.1. This reduces us to the case where R' is Artinian local.

Assume R' Artinian. Let $J = \text{Ker}(\varphi)$. We have to show that $I \subset J$. By the construction of I in Lemma 20.1 it suffices to show that $(M/JM)_\mathfrak{q}$ is flat over R/J for every prime $\mathfrak{q}$ of S/JS lying over $\mathfrak{m}$. As R' is Artinian, condition (1) signifies that $M \otimes_R R'$ is flat over R'. As R' is Artinian and $R/J \to R'$ is a local injective ring map, it follows that R/J is Artinian too. Hence the flatness of $M \otimes_R R' = M/JM \otimes_{R/J} R'$ over R' implies that M/JM is flat over R/J by Algebra, Lemma 101.7. This concludes the proof. □

Lemma 20.3. *With notation $R \to S$, M, and I and assumptions as in Lemma 20.1. In addition assume that $R \to S$ is of finite type. Then for any local homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ the following are equivalent*

1. *condition (18.0.1) holds for $(R' \to S \otimes_R R', \mathfrak{m}', M \otimes_R R')$, and*
2. *$\varphi(I) = 0$.*

Proof. The implication (2) ⇒ (1) follows from Lemma 18.1. Let $\varphi : R \to R'$ be as in the lemma satisfying (1). As R is Noetherian we see that $R \to S$ is of finite presentation and M is an S-module of finite presentation. Write $R' = \text{colim}_\lambda R_\lambda$ as a directed colimit of local R-subalgebras $R_\lambda \subset R'$, with maximal ideals $\mathfrak{m}_\lambda = R_\lambda \cap \mathfrak{m}'$ such that each R_λ is essentially of finite type over R. By Lemma 18.3 we see that condition (18.0.1) holds for $(R_\lambda \to S \otimes_R R_\lambda, \mathfrak{m}_\lambda, M \otimes_R R_\lambda)$ for some λ. Hence Lemma 20.2 applies to the ring map $R \to R_\lambda$ and we see that I maps to zero in R_λ, a fortiori it maps to zero in R'. □

21. Descent of flatness along integral maps

First a few simple lemmas.

Lemma 21.1. *Let R be a ring. Let $P(T)$ be a monic polynomial with coefficients in R. Let $\alpha \in R$ be such that $P(\alpha) = 0$. Then $P(T) = (T - \alpha)Q(T)$ for some monic polynomial $Q(T) \in R[T]$.*

Proof. By induction on the degree of P. If $\deg(P) = 1$, then $P(T) = T - \alpha$ and the result is true. If $\deg(P) > 1$, then we can write $P(T) = (T - \alpha)Q(T) + r$ for some polynomial $Q \in R[T]$ of degree $< \deg(P)$ and some $r \in R$ by long division. By assumption $0 = P(\alpha) = (\alpha - \alpha)Q(\alpha) + r = r$ and we conclude that $r = 0$ as desired. □

Lemma 21.2. *Let R be a ring. Let $P(T)$ be a monic polynomial with coefficients in R. There exists a finite free ring map $R \to R'$ such that $P(T) = (T - \alpha)Q(T)$ for some $\alpha \in R'$ and some monic polynomial $Q(T) \in R'[T]$.*

Proof. Write $P(T) = T^d + a_1 T^{d-1} + \ldots + a_0$. Set $R' = R[x]/(x^d + a_1 x^{d-1} + \ldots + a_0)$. Set α equal to the congruence class of x. Then it is clear that $P(\alpha) = 0$. Thus we win by Lemma 21.1. □

Lemma 21.3. *Let $R \to S$ be a finite ring map. There exists a finite free ring extension $R \subset R'$ such that $S \otimes_R R'$ is a quotient of a ring of the form*

$$R'[T_1, \ldots, T_n]/(P_1(T_1), \ldots, P_n(T_n))$$

with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ *for some* $\alpha_{ij} \in R'$.

Proof. Let $x_1, \ldots, x_n \in S$ be generators of S over R. For each i we can choose a monic polynomial $P_i(T) \in R[T]$ such that $P_i(x_i) = 0$ in S, see Algebra, Lemma 36.3. Say $\deg(P_i) = d_i$. By Lemma 21.2 (applied $\sum d_i$ times) there exists a finite free ring extension $R \subset R'$ such that each P_i splits completely:

$$P_i(T) = \prod\nolimits_{j=1,\ldots,d_i}(T - \alpha_{ij})$$

for certain $\alpha_{ik} \in R'$. Let $R'[T_1, \ldots, T_n] \to S \otimes_R R'$ be the R'-algebra map which maps T_i to $x_i \otimes 1$. As this maps $P_i(T_i)$ to zero, this induces the desired surjection. $\square$

Lemma 21.4. *Let R be a ring. Let $S = R[T_1, \ldots, T_n]/J$. Assume J contains elements of the form $P_i(T_i)$ with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R$. For $\underline{k} = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq d_i$ consider the ring map*

$$\Phi_{\underline{k}} : R[T_1, \ldots, T_n] \to R, \quad T_i \longmapsto \alpha_{ik_i}$$

Set $J_{\underline{k}} = \Phi_{\underline{k}}(J)$. Then the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is equal to $V(\bigcap J_{\underline{k}})$.

Proof. This lemma proves itself. Hint: $V(\bigcap J_{\underline{k}}) = \bigcup V(J_{\underline{k}})$. $\square$

The following result is due to Ferrand, see [Fer69].

Lemma 21.5. *Let $R \to S$ be a finite injective homomorphism of Noetherian rings. Let M be an R-module. If $M \otimes_R S$ is a flat S-module, then M is a flat R-module.*

Proof. Let M be an R-module such that $M \otimes_R S$ is flat over S. By Algebra, Lemma 39.8 in order to prove that M is flat we may replace R by any faithfully flat ring extension. By Lemma 21.3 we can find a finite locally free ring extension $R \subset R'$ such that $S' = S \otimes_R R' = R'[T_1, \ldots, T_n]/J$ for some ideal $J \subset R'[T_1, \ldots, T_n]$ which contains the elements of the form $P_i(T_i)$ with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R'$. Note that R' is Noetherian and that $R' \subset S'$ is a finite extension of rings. Hence we may replace R by R' and assume that S has a presentation as in Lemma 21.4. Note that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective, see Algebra, Lemma 36.17. Thus, using Lemma 21.4 we conclude that $I = \bigcap J_{\underline{k}}$ is an ideal such that $V(I) = \mathrm{Spec}(R)$. This means that $I \subset \sqrt{(0)}$, and since R is Noetherian that I is nilpotent. The maps $\Phi_{\underline{k}}$ induce commutative diagrams

$$\begin{tikzcd}
S \arrow[rr] & & R/J_{\underline{k}} \\
& R \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

from which we conclude that $M/J_{\underline{k}}M$ is flat over $R/J_{\underline{k}}$. By Lemma 16.1 we see that M/IM is flat over R/I. Finally, applying Algebra, Lemma 101.5 we conclude that M is flat over R. $\square$

Lemma 21.6. *Let $R \to S$ be an injective integral ring map. Let M be a finitely presented module over $R[x_1, \ldots, x_n]$. If $M \otimes_R S$ is flat over S, then M is flat over R.*

Proof. Choose a presentation

$$R[x_1, \ldots, x_n]^{\oplus t} \to R[x_1, \ldots, x_n]^{\oplus r} \to M \to 0.$$

Let's say that the first map is given by the $r \times t$-matrix $T = (f_{ij})$ with $f_{ij} \in R[x_1, \ldots, x_n]$. Write $f_{ij} = \sum f_{ij,I} x^I$ with $f_{ij,I} \in R$ (multi-index notation). Consider diagrams

$$\begin{tikzcd} R \arrow[r] & S \\ R_\lambda \arrow[u] \arrow[r] & S_\lambda \arrow[u] \end{tikzcd}$$

where R_λ is a finitely generated $\mathbf{Z}$-subalgebra of R containing all $f_{ij,I}$ and S_λ is a finite R_λ-subalgebra of S. Let M_λ be the finite $R_\lambda[x_1, \ldots, x_n]$-module defined by a presentation as above, using the same matrix T but now viewed as a matrix over $R_\lambda[x_1, \ldots, x_n]$. Note that S is the directed colimit of the S_λ (details omitted). By Algebra, Lemma 168.1 we see that for some λ the module $M_\lambda \otimes_{R_\lambda} S_\lambda$ is flat over S_λ. By Lemma 21.5 we conclude that M_λ is flat over R_λ. Since $M = M_\lambda \otimes_{R_\lambda} R$ we win by Algebra, Lemma 39.7. □

22. Torsion free modules

In this section we discuss torsion free modules and the relationship with flatness (especially over dimension 1 rings).

Definition 22.1. Let R be a domain. Let M be an R-module.

(1) We say an element $x \in M$ is *torsion* if there exists a nonzero $f \in R$ such that $fx = 0$.
(2) We say M is *torsion free* if the only torsion element of M is 0.

Let R be a domain and let $S = R \setminus \{0\}$ be the multiplicative set of nonzero elements of R. Then an R-module M is torsion free if and only if $M \to S^{-1}M$ is injective. In other words, if and only if the map $M \to M \otimes_R K$ is injective where $K = S^{-1}R$ is the fraction field of R.

Lemma 22.2. *Let R be a domain. Let M be an R-module. The set of torsion elements of M forms a submodule $M_{tors} \subset M$. The quotient module M/M_{tors} is torsion free.*

Proof. Omitted. □

Lemma 22.3. *Let R be a domain. Let M be a torsion free R-module. For any multiplicative set $S \subset R$ the module $S^{-1}M$ is a torsion free $S^{-1}R$-module.*

Proof. Omitted. □

Lemma 22.4. *Let $R \to R'$ be a flat homomorphism of domains. If M is a torsion free R-module, then $M \otimes_R R'$ is a torsion free R'-module.*

Proof. If M is torsion free, then $M \subset M \otimes_R K$ is injective where K is the fraction field of R. Since R' is flat over R we see that $M \otimes_R R' \to (M \otimes_R K) \otimes_R R'$ is injective. Since $M \otimes_R K$ is isomorphic to a direct sum of copies of K, it suffices to see that $K \otimes_R R'$ is torsion free. This is true because it is a localization of R'. □

Lemma 22.5. *Let R be a domain. Let $0 \to M \to M' \to M'' \to 0$ be a short exact sequence of R-modules. If M and M'' are torsion free, then M' is torsion free.*

Proof. Omitted. $\square$

Lemma 22.6. *Let R be a domain. Let M be an R-module. Then M is torsion free if and only if $M_{\mathfrak m}$ is a torsion free $R_{\mathfrak m}$-module for all maximal ideals $\mathfrak m$ of R.*

Proof. Omitted. Hint: Use Lemma 22.3 and Algebra, Lemma 23.1. $\square$

Lemma 22.7. *Let R be a domain. Let M be a finite R-module. Then M is torsion free if and only if M is a submodule of a finite free module.*

Proof. If M is a submodule of $R^{\oplus n}$, then M is torsion free. For the converse, assume M is torsion free. Let K be the fraction field of R. Then $M \otimes_R K$ is a finite dimensional K-vector space. Choose a basis $e_1, \ldots, e_r$ for this vector space. Let $x_1, \ldots, x_n$ be generators of M. Write $x_i = \sum (a_{ij}/b_{ij})e_j$ for some $a_{ij}, b_{ij} \in R$ with $b_{ij} \neq 0$. Set $b = \prod_{i,j} b_{ij}$. Since M is torsion free the map $M \to M \otimes_R K$ is injective and the image is contained in $R^{\oplus r} = Re_1/b \oplus \ldots \oplus Re_r/b$. $\square$

Lemma 22.8. *Let R be a Noetherian domain. Let M be a nonzero finite R-module. The following are equivalent*
1. *M is torsion free,*
2. *M is a submodule of a finite free module,*
3. *(0) is the only associated prime of M,*
4. *(0) is in the support of M and M has property (S_1), and*
5. *(0) is in the support of M and M has no embedded associated prime.*

Proof. We have seen the equivalence of (1) and (2) in Lemma 22.7. We have seen the equivalence of (4) and (5) in Algebra, Lemma 157.2. The equivalence between (3) and (5) is immediate from the definition. A localization of a torsion free module is torsion free (Lemma 22.3), hence it is clear that a M has no associated primes different from (0). Thus (1) implies (5). Conversely, assume (5). If M has torsion, then there exists an embedding $R/I \subset M$ for some nonzero ideal I of R. Hence M has an associated prime different from (0) (see Algebra, Lemmas 63.3 and 63.7). This is an embedded associated prime which contradicts the assumption. $\square$

Lemma 22.9. *Let R be a domain. Any flat R-module is torsion free.*

Proof. If $x \in R$ is nonzero, then $x : R \to R$ is injective, and hence if M is flat over R, then $x : M \to M$ is injective. Thus if M is flat over R, then M is torsion free. $\square$

Lemma 22.10. *Let A be a valuation ring. An A-module M is flat over A if and only if M is torsion free.*

Proof. The implication "flat $\Rightarrow$ torsion free" is Lemma 22.9. For the converse, assume M is torsion free. By the equational criterion of flatness (see Algebra, Lemma 39.11) we have to show that every relation in M is trivial. To do this assume that $\sum_{i=1,\ldots,n} a_i x_i = 0$ with $x_i \in M$ and $a_i \in A$. After renumbering we may assume that $v(a_1) \leq v(a_i)$ for all i. Hence we can write $a_i = a'_i a_1$ for some $a'_i \in A$. Note that $a'_1 = 1$. As M is torsion free we see that $x_1 = -\sum_{i \geq 2} a'_i x_i$. Thus, if we choose $y_i = x_i$, $i = 2, \ldots, n$ then

$$x_1 = \sum\nolimits_{j \geq 2} -a'_j y_j, \quad x_i = y_i, (i \geq 2) \quad 0 = a_1 \cdot (-a'_j) + a_j \cdot 1 (j \geq 2)$$

shows that the relation was trivial (to be explicit the elements a_{ij} are defined by setting $a_{11} = 0$, $a_{1j} = -a'_j$ for $j > 1$, and $a_{ij} = \delta_{ij}$ for $i, j \geq 2$). $\square$

Lemma 22.11. *Let A be a Dedekind domain (for example a discrete valuation ring or more generally a PID).*

(1) *An A-module is flat if and only if it is torsion free.*
(2) *A finite torsion free A-module is finite locally free.*
(3) *A finite torsion free A-module is finite free if A is a PID.*

Proof. (For the parenthetical remark in the statement of the lemma, see Algebra, Lemma 120.15.) Proof of (1). By Lemma 22.6 and Algebra, Lemma 39.18 it suffices to check the statement over $A_{\mathfrak m}$ for $\mathfrak m \subset A$ maximal. Since $A_{\mathfrak m}$ is a discrete valuation ring (Algebra, Lemma 120.17) we win by Lemma 22.10.

Proof of (2). Follows from Algebra, Lemma 78.2 and (1).

Proof of (3). Let A be a PID and let M be a finite torsion free module. By Lemma 22.7 we see that $M \subset A^{\oplus n}$ for some n. We argue that M is free by induction on n. The case $n = 1$ expresses exactly the fact that A is a PID. If $n > 1$ let $M' \subset R^{\oplus n-1}$ be the image of the projection onto the last $n-1$ summands of $R^{\oplus n}$. Then we obtain a short exact sequence $0 \to I \to M \to M' \to 0$ where I is the intersection of M with the first summand R of $R^{\oplus n}$. By induction we see that M is an extension of finite free R-modules, whence finite free. □

Lemma 22.12. *Let R be a domain. Let M, N be R-modules. If N is torsion free, so is* $\mathrm{Hom}_R(M, N)$.

Proof. Choose a surjection $\bigoplus_{i \in I} R \to M$. Then $\mathrm{Hom}_R(M, N) \subset \prod_{i \in I} N$. □

23. Reflexive modules

Here is our definition.

Definition 23.1. Let R be a domain. We say an R-module M is *reflexive* if the natural map

$$j : M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$$

which sends $m \in M$ to the map sending $\varphi \in \mathrm{Hom}_R(M, R)$ to $\varphi(m) \in R$ is an isomorphism.

We can make this definition for more general rings, but already the definition above has drawbacks. It would be wise to restrict to Noetherian domains and finite torsion free modules and (perhaps) impose some regularity conditions on R (e.g., R is normal).

Lemma 23.2. *Let R be a domain and let M be an R-module.*

(1) *If M is reflexive, then M is torsion free.*
(2) *If M is finite, then $j : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is injective if and only if M is torsion free*

Proof. Follows immediately from Lemmas 22.12 and 22.7. □

Lemma 23.3. *Let R be a discrete valuation ring and let M be a finite R-module. Then the map $j : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is surjective.*

Proof. Let $M_{tors} \subset M$ be the torsion submodule. Then we have $\mathrm{Hom}_R(M, R) = \mathrm{Hom}_R(M/M_{tors}, R)$ (holds over any domain). Hence we may assume that M is torsion free. Then M is free by Lemma 22.11 and the lemma is clear. □

Lemma 23.4. *Let R be a Noetherian domain. Let M be a finite R-module. The following are equivalent:*

1. *M is reflexive,*
2. *$M_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module for all primes $\mathfrak{p} \subset R$, and*
3. *$M_\mathfrak{m}$ is a reflexive $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of R.*

Proof. The localization of $j : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ at a prime $\mathfrak{p}$ is the corresponding map for the module $M_\mathfrak{p}$ over the Noetherian local domain $R_\mathfrak{p}$. See Algebra, Lemma 10.2. Thus the lemma holds by Algebra, Lemma 23.1. $\square$

Lemma 23.5. *Let R be a Noetherian domain. Let $0 \to M \to M' \to M''$ an exact sequence of finite R-modules. If M' is reflexive and M'' is torsion free, then M is reflexive.*

Proof. We will use without further mention that $\mathrm{Hom}_R(N, N')$ is a finite R-module for any finite R-modules N and N', see Algebra, Lemma 71.9. We take duals to get a sequence

$$\mathrm{Hom}_R(M, R) \leftarrow \mathrm{Hom}_R(M', R) \leftarrow \mathrm{Hom}_R(M'', R)$$

Dualizing again we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) \arrow[r, "j"'] & \mathrm{Hom}_R(\mathrm{Hom}_R(M', R), R) \arrow[r] & \mathrm{Hom}_R(\mathrm{Hom}_R(M'', R), R) \\
M \arrow[u] \arrow[r] & M' \arrow[u] \arrow[r] & M'' \arrow[u]
\end{tikzcd}$$

We do not know the top row is exact. But, by assumption the middle vertical arrow is an isomorphism and the right vertical arrow is injective (Lemma 23.2). We claim j is injective. Assuming the claim a diagram chase shows that the left vertical arrow is an isomorphism, i.e., M is reflexive.

Proof of the claim. Consider the exact sequence $\mathrm{Hom}_R(M', R) \to \mathrm{Hom}_R(M, R) \to Q \to 0$ defining Q. One applies Algebra, Lemma 10.2 to obtain

$$\mathrm{Hom}_K(M' \otimes_R K, K) \to \mathrm{Hom}_K(M \otimes_R K, K) \to Q \otimes_R K \to 0$$

But $M \otimes_R K \to M' \otimes_R K$ is an injective map of vector spaces, hence split injective, so $Q \otimes_R K = 0$, that is, Q is torsion. Then one gets the exact sequence

$$0 \to \mathrm{Hom}_R(Q, R) \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) \to \mathrm{Hom}_R(\mathrm{Hom}_R(M', R), R)$$

and $\mathrm{Hom}_R(Q, R) = 0$ because Q is torsion. $\square$

Lemma 23.6. *Let R be a Noetherian domain. Let M be a finite R-module. The following are equivalent*

1. *M is reflexive,*
2. *there exists a short exact sequence $0 \to M \to F \to N \to 0$ with F finite free and N torsion free.*

Proof. Observe that a finite free module is reflexive. By Lemma 23.5 we see that (2) implies (1). Assume M is reflexive. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to \mathrm{Hom}_R(M, R) \to 0$. Dualizing we get an exact sequence

$$0 \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) \to R^{\oplus n} \to N \to 0$$

with $N = \mathrm{Im}(R^{\oplus n} \to R^{\oplus m})$ a torsion free module. As $M = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ we get an exact sequence as in (2). $\square$

Lemma 23.7. *Let $R \to R'$ be a flat homomorphism of Noetherian domains. If M is a finite reflexive R-module, then $M \otimes_R R'$ is a finite reflexive R'-module.*

Proof. Choose a short exact sequence $0 \to M \to F \to N \to 0$ with F finite free and N torsion free, see Lemma 23.6. Since $R \to R'$ is flat we obtain a short exact sequence $0 \to M \otimes_R R' \to F \otimes_R R' \to N \otimes_R R' \to 0$ with $F \otimes_R R'$ finite free and $N \otimes_R R'$ torsion free (Lemma 22.4). Thus $M \otimes_R R'$ is reflexive by Lemma 23.6. $\square$

Lemma 23.8. *Let R be a Noetherian domain. Let M be a finite R-module. Let N be a finite reflexive R-module. Then $\mathrm{Hom}_R(M, N)$ is reflexive.*

Proof. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Then we obtain

$$0 \to \mathrm{Hom}_R(M, N) \to N^{\oplus n} \to N' \to 0$$

with $N' = \mathrm{Im}(N^{\oplus n} \to N^{\oplus m})$ torsion free. We conclude by Lemma 23.5. $\square$

Definition 23.9. Let R be a Noetherian domain. Let M be a finite R-module. The module $M^{**} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is called the *reflexive hull* of M.

This makes sense because the reflexive hull is reflexive by Lemma 23.8. The assignment $M \mapsto M^{**}$ is a functor. If $\varphi : M \to N$ is an R-module map into a reflexive R-module N, then φ factors $M \to M^{**} \to N$ through the reflexive hull of M. Another way to say this is that taking the reflexive hull is the left adjoint to the inclusion functor

$$\text{finite reflexive modules} \subset \text{finite modules}$$

over a Noetherian domain R.

Lemma 23.10. *Let R be a Noetherian local ring. Let M, N be finite R-modules.*

1. *If N has depth ≥ 1, then $\mathrm{Hom}_R(M, N)$ has depth ≥ 1.*
2. *If N has depth ≥ 2, then $\mathrm{Hom}_R(M, N)$ has depth ≥ 2.*

Proof. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Dualizing we get an exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to N^{\oplus n} \to N' \to 0$$

with $N' = \mathrm{Im}(N^{\oplus n} \to N^{\oplus m})$. A submodule of a module with depth ≥ 1 has depth ≥ 1; this follows immediately from the definition. Thus part (1) is clear. For (2) note that here the assumption and the previous remark implies N' has depth ≥ 1. The module $N^{\oplus n}$ has depth ≥ 2. From Algebra, Lemma 72.6 we conclude $\mathrm{Hom}_R(M, N)$ has depth ≥ 2. $\square$

Lemma 23.11. *Let R be a Noetherian ring. Let M, N be finite R-modules.*

1. *If N has property (S_1), then $\mathrm{Hom}_R(M, N)$ has property (S_1).*
2. *If N has property (S_2), then $\mathrm{Hom}_R(M, N)$ has property (S_2).*
3. *If R is a domain, N is torsion free and (S_2), then $\mathrm{Hom}_R(M, N)$ is torsion free and has property (S_2).*

Proof. Since localizing at primes commutes with taking Hom_R for finite R-modules (Algebra, Lemma 71.9) parts (1) and (2) follow immediately from Lemma 23.10. Part (3) follows from (2) and Lemma 22.12. $\square$

Lemma 23.12. *Let R be a Noetherian ring. Let $\varphi : M \to N$ be a map of R-modules. Assume that for every prime $\mathfrak{p}$ of R at least one of the following happens*

(1) $M_{\mathfrak p} \to N_{\mathfrak p}$ *is injective, or*
(2) $\mathfrak p \not\in Ass(M)$.

Then φ is injective.

Proof. Let $\mathfrak p$ be an associated prime of $\text{Ker}(\varphi)$. Then there exists an element $x \in M_{\mathfrak p}$ which is in the kernel of $M_{\mathfrak p} \to N_{\mathfrak p}$ and is annihilated by $\mathfrak pR_{\mathfrak p}$ (Algebra, Lemma 63.15). This is impossible in both cases. Hence $\text{Ass}(\text{Ker}(\varphi)) = \emptyset$ and we conclude $\text{Ker}(\varphi) = 0$ by Algebra, Lemma 63.7. □

Lemma 23.13. *Let R be a Noetherian ring. Let $\varphi : M \to N$ be a map of R-modules. Assume M is finite and that for every prime $\mathfrak p$ of R one of the following happens*

(1) $M_{\mathfrak p} \to N_{\mathfrak p}$ *is an isomorphism, or*
(2) $depth(M_{\mathfrak p}) \geq 2$ *and* $\mathfrak p \not\in Ass(N)$.

Then φ is an isomorphism.

Proof. By Lemma 23.12 we see that φ is injective. Let $N' \subset N$ be an finitely generated R-module containing the image of M. Then $\text{Ass}(N_{\mathfrak p}) = \emptyset$ implies $\text{Ass}(N'_{\mathfrak p}) = \emptyset$. Hence the assumptions of the lemma hold for $M \to N'$. In order to prove that φ is an isomorphism, it suffices to prove the same thing for every such $N' \subset N$. Thus we may assume N is a finite R-module. In this case, $\mathfrak p \not\in \text{Ass}(N) \Rightarrow \text{depth}(N_{\mathfrak p}) \geq 1$, see Algebra, Lemma 63.18. Consider the short exact sequence

$$0 \to M \to N \to Q \to 0$$

defining Q. Looking at the conditions we see that either $Q_{\mathfrak p} = 0$ in case (1) or $\text{depth}(Q_{\mathfrak p}) \geq 1$ in case (2) by Algebra, Lemma 72.6. This implies that Q does not have any associated primes, hence $Q = 0$ by Algebra, Lemma 63.7. □

Lemma 23.14. *Let R be a Noetherian domain. Let $\varphi : M \to N$ be a map of R-modules. Assume M is finite, N is torsion free, and that for every prime $\mathfrak p$ of R one of the following happens*

(1) $M_{\mathfrak p} \to N_{\mathfrak p}$ *is an isomorphism, or*
(2) $depth(M_{\mathfrak p}) \geq 2$.

Then φ is an isomorphism.

Proof. This is a special case of Lemma 23.13. □

Lemma 23.15. *Let R be a Noetherian domain. Let M be a finite R-module. The following are equivalent*

(1) *M is reflexive,*
(2) *for every prime $\mathfrak p$ of R one of the following happens*
    (a) *$M_{\mathfrak p}$ is a reflexive $R_{\mathfrak p}$-module, or*
    (b) $depth(M_{\mathfrak p}) \geq 2$.

Proof. If (1) is true, then $M_{\mathfrak p}$ is a reflexive module for all primes of $\mathfrak p$ by Lemma 23.4. Thus (1) $\Rightarrow$ (2). Assume (2). Set $N = \text{Hom}_R(\text{Hom}_R(M, R), R)$ so that

$$N_{\mathfrak p} = \text{Hom}_{R_{\mathfrak p}}(\text{Hom}_{R_{\mathfrak p}}(M_{\mathfrak p}, R_{\mathfrak p}), R_{\mathfrak p})$$

for every prime $\mathfrak p$ of R. See Algebra, Lemma 10.2. We apply Lemma 23.14 to the map $j : M \to N$. This is allowed because M is finite and N is torsion free by Lemma 22.12. In case (2)(a) the map $M_{\mathfrak p} \to N_{\mathfrak p}$ is an isomorphism and in case (2)(b) we have $\text{depth}(M_{\mathfrak p}) \geq 2$. □

Lemma 23.16. *Let R be a Noetherian domain. Let M be a finite reflexive R-module. Let $\mathfrak p \subset R$ be a prime ideal.*

1. *If $depth(R_{\mathfrak p}) \geq 2$, then $depth(M_{\mathfrak p}) \geq 2$.*
2. *If R is (S_2), then M is (S_2).*

Proof. Since formation of reflexive hull $\mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ commutes with localization (Algebra, Lemma 10.2) part (1) follows from Lemma 23.10. Part (2) is immediate from Lemma 23.11. □

Example 23.17. The results above and below suggest reflexivity is related to the (S_2) condition; here is an example to prevent too optimistic conjectures. Let k be a field. Let R be the k-subalgebra of $k[x, y]$ generated by $1, y, x^2, xy, x^3$. Then R is not (S_2). So R as an R-module is an example of a reflexive R-module which is not (S_2). Let $M = k[x, y]$ viewed as an R-module. Then M is a reflexive R-module because

$$\mathrm{Hom}_R(M, R) = \mathfrak m = (y, x^2, xy, x^3) \quad \text{and} \quad \mathrm{Hom}_R(\mathfrak m, R) = M$$

and M is (S_2) as an R-module (computations omitted). Thus R is a Noetherian domain possessing a reflexive (S_2) module but R is not (S_2) itself.

Lemma 23.18. *Let R be a Noetherian normal domain with fraction field K. Let M be a finite R-module. The following are equivalent*

1. *M is reflexive,*
2. *M is torsion free and has property (S_2),*
3. *M is torsion free and $M = \bigcap_{height(\mathfrak p)=1} M_{\mathfrak p}$ where the intersection happens in $M_K = M \otimes_R K$.*

Proof. By Algebra, Lemma 157.4 we see that R satisfies (R_1) and (S_2).

Assume (1). Then M is torsion free by Lemma 23.2 and satisfies (S_2) by Lemma 23.16. Thus (2) holds.

Assume (2). By definition $M' = \bigcap_{\text{height}(\mathfrak p)=1} M_{\mathfrak p}$ is the kernel of the map

$$M_K \longrightarrow \bigoplus\nolimits_{\text{height}(\mathfrak p)=1} M_K/M_{\mathfrak p} \subset \prod\nolimits_{\text{height}(\mathfrak p)=1} M_K/M_{\mathfrak p}$$

Observe that our map indeed factors through the direct sum as indicated since given $a/b \in K$ there are at most finitely many height 1 primes $\mathfrak p$ with $b \in \mathfrak p$. Let $\mathfrak p_0$ be a prime of height 1. Then $(M_K/M_{\mathfrak p})_{\mathfrak p_0} = 0$ unless $\mathfrak p = \mathfrak p_0$ in which case we get $(M_K/M_{\mathfrak p})_{\mathfrak p_0} = M_K/M_{\mathfrak p_0}$. Thus by exactness of localization and the fact that localization commutes with direct sums, we see that $M'_{\mathfrak p_0} = M_{\mathfrak p_0}$. Since M has depth ≥ 2 at primes of height > 1, we see that $M \to M'$ is an isomorphism by Lemma 23.14. Hence (3) holds.

Assume (3). Let $\mathfrak p$ be a prime of height 1. Then $R_{\mathfrak p}$ is a discrete valuation ring by (R_1). By Lemma 22.11 we see that $M_{\mathfrak p}$ is finite free, in particular reflexive. Hence the map $M \to M^{**}$ induces an isomorphism at all the primes $\mathfrak p$ of height 1. Thus the condition $M = \bigcap_{\text{height}(\mathfrak p)=1} M_{\mathfrak p}$ implies that $M = M^{**}$ and (1) holds. □

Lemma 23.19. *Let R be a Noetherian normal domain. Let M be a finite R-module. Then the reflexive hull of M is the intersection*

$$M^{**} = \bigcap\nolimits_{height(\mathfrak p)=1} M_{\mathfrak p}/(M_{\mathfrak p})_{tors} = \bigcap\nolimits_{height(\mathfrak p)=1} (M/M_{tors})_{\mathfrak p}$$

taken in $M \otimes_R K$.

Proof. Let $\mathfrak p$ be a prime of height 1. The kernel of $M_\mathfrak p \to M \otimes_R K$ is the torsion submodule $(M_\mathfrak p)_{tors}$ of $M_\mathfrak p$. Moreover, we have $(M/M_{tors})_\mathfrak p = M_\mathfrak p/(M_\mathfrak p)_{tors}$ and this is a finite free module over the discrete valuation ring $R_\mathfrak p$ (Lemma 22.11). Then $M_\mathfrak p/(M_\mathfrak p)_{tors} \to (M_\mathfrak p)^{**} = (M^{**})_\mathfrak p$ is an isomorphism, hence the lemma is a consequence of Lemma 23.18. $\square$

Lemma 23.20. *Let A be a Noetherian normal domain with fraction field K. Let L be a finite extension of K. If the integral closure B of A in L is finite over A, then B is reflexive as an A-module.*

Proof. It suffices to show that $B = \bigcap B_\mathfrak p$ where the intersection is over height 1 primes $\mathfrak p \subset A$, see Lemma 23.18. Let $b \in \bigcap B_\mathfrak p$. Let $x^d + a_1 x^{d-1} + \ldots + a_d$ be the minimal polynomial of b over K. We want to show $a_i \in A$. By Algebra, Lemma 38.6 we see that $a_i \in A_\mathfrak p$ for all i and all height one primes $\mathfrak p$. Hence we get what we want from Algebra, Lemma 157.6 (or the lemma already cited as A is a reflexive module over itself). $\square$

24. Content ideals

The definition may not be what you expect.

Definition 24.1. Let A be a ring. Let M be a flat A-module. Let $x \in M$. If the set of ideals I in A such that $x \in IM$ has a smallest element, we call it the *content ideal of x*.

Note that since M is flat over A, for a pair of ideals I, I' of A we have $IM \cap I'M = (I \cap I')M$ as can be seen by tensoring the exact sequence $0 \to I \cap I' \to I \oplus I' \to I + I' \to 0$ by M.

Lemma 24.2. *Let A be a ring. Let M be a flat A-module. Let $x \in M$. The content ideal of x, if it exists, is finitely generated.*

Proof. Say $x \in IM$. Then we can write $x = \sum_{i=1,\ldots,n} f_i x_i$ with $f_i \in I$ and $x_i \in M$. Hence $x \in I'M$ with $I' = (f_1, \ldots, f_n)$. $\square$

Lemma 24.3. *Let $(A, \mathfrak m)$ be a local ring. Let $u : M \to N$ be a map of flat A-modules such that $\overline{u} : M/\mathfrak m M \to N/\mathfrak m N$ is injective. If $x \in M$ has content ideal I, then $u(x)$ has content ideal I as well.*

Proof. It is clear that $u(x) \in IN$. If $u(x) \in I'N$, then $u(x) \in (I' \cap I)N$, see discussion following Definition 24.1. Hence it suffices to show: if $x \in I'N$ and $I' \subset I$, $I' \neq I$, then $u(x) \notin I'N$. Since I/I' is a nonzero finite A-module (Lemma 24.2) there is a nonzero map $\chi : I/I' \to A/\mathfrak m$ of A-modules by Nakayama's lemma (Algebra, Lemma 20.1). Since I is the content ideal of x we see that $x \notin I''M$ where $I'' = \text{Ker}(\chi)$. Hence x is not in the kernel of the map

$$IM = I \otimes_A M \xrightarrow{\chi \otimes 1} A/\mathfrak m \otimes M \cong M/\mathfrak m M$$

Applying our hypothesis on $\overline{u}$ we conclude that $u(x)$ does not map to zero under the map

$$IN = I \otimes_A N \xrightarrow{\chi \otimes 1} A/\mathfrak m \otimes N \cong N/\mathfrak m N$$

and we conclude. $\square$

Lemma 24.4. *Let A be a ring. Let M be a flat Mittag-Leffler module. Then every element of M has a content ideal.*

Proof. This is a special case of Algebra, Lemma 91.2. □

25. Flatness and finiteness conditions

In this section we discuss some implications of the type "flat + finite type ⇒ finite presentation". We will revisit this result in the chapter on flatness, see More on Flatness, Section 1. A first result of this type was proved in Algebra, Lemma 108.6.

Lemma 25.1. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]$ be a polynomial ring over R. Let M be an S-module. Assume*

1. *there exist finitely many primes $\mathfrak p_1, \ldots, \mathfrak p_m$ of R such that the map $R \to \prod R_{\mathfrak p_j}$ is injective,*
2. *M is a finite S-module,*
3. *M flat over R, and*
4. *for every prime $\mathfrak p$ of R the module $M_{\mathfrak p}$ is of finite presentation over $S_{\mathfrak p}$.*

Then M is of finite presentation over S.

Proof. Choose a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0$$

of M as an S-module. Let $\mathfrak q$ be a prime ideal of S lying over a prime $\mathfrak p$ of R. By assumption there exist finitely many elements $k_1, \ldots, k_t \in K$ such that if we set $K' = \sum S k_j \subset K$ then $K'_{\mathfrak p} = K_{\mathfrak p}$ and $K'_{\mathfrak p_j} = K_{\mathfrak p_j}$ for $j = 1, \ldots, m$. Setting $M' = S^{\oplus r}/K'$ we deduce that in particular $M'_{\mathfrak q} = M_{\mathfrak q}$. By openness of flatness, see Algebra, Theorem 129.4 we conclude that there exists a $g \in S$, $g \notin \mathfrak q$ such that M'_g is flat over R. Thus $M'_g \to M_g$ is a surjective map of flat R-modules. Consider the commutative diagram

$$\begin{tikzcd} M'_g \arrow[r] \arrow[d] & M_g \arrow[d] \\ \prod (M'_g)_{\mathfrak p_j} \arrow[r] & \prod (M_g)_{\mathfrak p_j} \end{tikzcd}$$

The bottom arrow is an isomorphism by choice of $k_1, \ldots, k_t$. The left vertical arrow is an injective map as $R \to \prod R_{\mathfrak p_j}$ is injective and M'_g is flat over R. Hence the top horizontal arrow is injective, hence an isomorphism. This proves that M_g is of finite presentation over S_g. We conclude by applying Algebra, Lemma 23.2. □

Lemma 25.2. *Let $R \to S$ be a ring homomorphism. Assume*

1. *there exist finitely many primes $\mathfrak p_1, \ldots, \mathfrak p_m$ of R such that the map $R \to \prod R_{\mathfrak p_j}$ is injective,*
2. *$R \to S$ is of finite type,*
3. *S flat over R, and*
4. *for every prime $\mathfrak p$ of R the ring $S_{\mathfrak p}$ is of finite presentation over $R_{\mathfrak p}$.*

Then S is of finite presentation over R.

Proof. By assumption S is a quotient of a polynomial ring over R. Thus the result follows directly from Lemma 25.1. □

Lemma 25.3. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]$ be a graded polynomial algebra over R, i.e., $\deg(x_i) > 0$ but not necessarily equal to 1. Let M be a graded S-module. Assume*

(1) *R is a local ring,*
(2) *M is a finite S-module, and*
(3) *M is flat over R.*

Then M is finitely presented as an S-module.

Proof. Let $M = \bigoplus M_d$ be the grading on M. Pick homogeneous generators $m_1, \ldots, m_r \in M$ of M. Say $\deg(m_i) = d_i \in \mathbf{Z}$. This gives us a presentation

$$0 \to K \to \bigoplus\nolimits_{i=1,\ldots,r} S(-d_i) \to M \to 0$$

which in each degree d leads to the short exact sequence

$$0 \to K_d \to \bigoplus\nolimits_{i=1,\ldots,r} S_{d-d_i} \to M_d \to 0.$$

By assumption each M_d is a finite flat R-module. By Algebra, Lemma 78.5 this implies each M_d is a finite free R-module. Hence we see each K_d is a finite R-module. Also each K_d is flat over R by Algebra, Lemma 39.13. Hence we conclude that each K_d is finite free by Algebra, Lemma 78.5 again.

Let $\mathfrak{m}$ be the maximal ideal of R. By the flatness of M over R the short exact sequences above remain short exact after tensoring with $\kappa = \kappa(\mathfrak{m})$. As the ring $S \otimes_R \kappa$ is Noetherian we see that there exist homogeneous elements $k_1, \ldots, k_t \in K$ such that the images $\overline{k}_j$ generate $K \otimes_R \kappa$ over $S \otimes_R \kappa$. Say $\deg(k_j) = e_j$. Thus for any d the map

$$\bigoplus\nolimits_{j=1,\ldots,t} S_{d-e_j} \longrightarrow K_d$$

becomes surjective after tensoring with κ. By Nakayama's lemma (Algebra, Lemma 20.1) this implies the map is surjective over R. Hence K is generated by $k_1, \ldots, k_t$ over S and we win. □

Lemma 25.4. *Let R be a ring. Let $S = \bigoplus_{n \geq 0} S_n$ be a graded R-algebra. Let $M = \bigoplus_{d \in \mathbf{Z}} M_d$ be a graded S-module. Assume S is finitely generated as an R-algebra, assume S_0 is a finite R-algebra, and assume there exist finitely many primes $\mathfrak{p}_j$, $i = 1, \ldots, m$ such that $R \to \prod R_{\mathfrak{p}_j}$ is injective.*

(1) *If S is flat over R, then S is a finitely presented R-algebra.*
(2) *If M is flat as an R-module and finite as an S-module, then M is finitely presented as an S-module.*

Proof. As S is finitely generated as an R-algebra, it is finitely generated as an S_0 algebra, say by homogeneous elements $t_1, \ldots, t_n \in S$ of degrees $d_1, \ldots, d_n > 0$. Set $P = R[x_1, \ldots, x_n]$ with $\deg(x_i) = d_i$. The ring map $P \to S$, $x_i \to t_i$ is finite as S_0 is a finite R-module. To prove (1) it suffices to prove that S is a finitely presented P-module. To prove (2) it suffices to prove that M is a finitely presented P-module. Thus it suffices to prove that if $S = P$ is a graded polynomial ring and M is a finite S-module flat over R, then M is finitely presented as an S-module. By Lemma 25.3 we see $M_{\mathfrak{p}}$ is a finitely presented $S_{\mathfrak{p}}$-module for every prime $\mathfrak{p}$ of R. Thus the result follows from Lemma 25.1. □

Remark 25.5. Let R be a ring. When does R satisfy the condition mentioned in Lemmas 25.1, 25.2, and 25.4? This holds if

(1) R is local,
(2) R is Noetherian,
(3) R is a domain,

(4) R is a reduced ring with finitely many minimal primes, or
(5) R has finitely many weakly associated primes, see Algebra, Lemma 66.17.

Thus these lemmas hold in all cases listed above.

The following lemma will be improved on in More on Flatness, Proposition 13.10.

Lemma 25.6. *Let A be a valuation ring. Let $A \to B$ be a ring map of finite type. Let M be a finite B-module.*

(1) *If B is flat over A, then B is a finitely presented A-algebra.*
(2) *If M is flat as an A-module, then M is finitely presented as a B-module.*

Proof. We are going to use that an A-module is flat if and only if it is torsion free, see Lemma 22.10. By Algebra, Lemma 57.10 we can find a graded A-algebra S with $S_0 = A$ and generated by finitely many elements in degree 1, an element $f \in S_1$ and a finite graded S-module N such that $B \cong S_{(f)}$ and $M \cong N_{(f)}$. If M is torsion free, then we can take N torsion free by replacing it by N/N_{tors}, see Lemma 22.2. Similarly, if B is torsion free, then we can take S torsion free by replacing it by S/S_{tors}. Hence in case (1), we may apply Lemma 25.4 to see that S is a finitely presented A-algebra, which implies that $B = S_{(f)}$ is a finitely presented A-algebra. To see (2) we may first replace S by a graded polynomial ring, and then we may apply Lemma 25.3 to conclude. $\square$

Lemma 25.7. *Let A be a valuation ring. Let $A \to B$ be a local homomorphism which is essentially of finite type. Let M be a finite B-module.*

(1) *If B is flat over A, then B is essentially of finite presentation over A.*
(2) *If M is flat as an A-module, then M is finitely presented as a B-module.*

Proof. By assumption we can write B as a quotient of the localization of a polynomial algebra $P = A[x_1, \ldots, x_n]$ at a prime ideal $\mathfrak{q}$. In case (1) we consider $M = B$ as a finite module over $P_\mathfrak{q}$ and in case (2) we consider M as a finite module over $P_\mathfrak{q}$. In both cases, we have to show that this is a finitely presented $P_\mathfrak{q}$-module, see Algebra, Lemma 6.4 for case (2).

Choose a presentation $0 \to K \to P_\mathfrak{q}^{\oplus r} \to M \to 0$ which is possible because M is finite over $P_\mathfrak{q}$. Let $L = P^{\oplus r} \cap K$. Then $K = L_\mathfrak{q}$, see Algebra, Lemma 9.15. Then $N = P^{\oplus r}/L$ is a submodule of M and hence flat by Lemma 22.10. Since also N is a finite P-module, we see that N is finitely presented as a P-module by Lemma 25.6. Since localization is exact (Algebra, Proposition 9.12) we see that $N_\mathfrak{q} = M$ and we conclude. $\square$

26. Blowing up and flatness

In this section we begin our discussion of results of the form: "After a blowup the strict transform becomes flat". More results of this type may be found in Divisors, Section 35 and More on Flatness, Section 30.

Definition 26.1. Let R be a ring. Let $I \subset R$ be an ideal and $a \in I$. Let $R[\frac{I}{a}]$ be the affine blowup algebra, see Algebra, Definition 70.1. Let M be an R-module. The *strict transform of M along $R \to R[\frac{I}{a}]$* is the $R[\frac{I}{a}]$-module

$$M' = \left(M \otimes_R R[\tfrac{I}{a}]\right) /a\text{-power torsion}$$

The following is a very weak version of flattening by blowing up, but it is already sometimes a useful result.

Lemma 26.2. *Let $(R, \mathfrak m)$ be a local domain with fraction field K. Let S be a finite type R-algebra. Let M be a finite S-module. For every valuation ring $A \subset K$ dominating R there exists an ideal $I \subset \mathfrak m$ and a nonzero element $a \in I$ such that*

1. *I is finitely generated,*
2. *A has center on $R[\frac{I}{a}]$,*
3. *the fibre ring of $R \to R[\frac{I}{a}]$ at $\mathfrak m$ is not zero, and*
4. *the strict transform $S_{I,a}$ of S along $R \to R[\frac{I}{a}]$ is flat and of finite presentation over R, and the strict transform $M_{I,a}$ of M along $R \to R[\frac{I}{a}]$ is flat over R and finitely presented over $S_{I,a}$.*

Proof. Write $S = R[x_1, \ldots, x_n]/J$ and denote $N = S \oplus M$ viewed as a module over $P = R[x_1, \ldots, x_n]$. If we can prove the lemma in case S is a polynomial algebra over R, then we can find I, a satisfying (1), (2), (3) such that the strict transform $N_{I,a}$ of N along $R \to R[\frac{I}{a}]$ is flat over R and finitely presented as a module over the strict transform $P_{I,a}]$ of P. Since $P_{I,a} = R[\frac{I}{a}][x_1, \ldots, x_n]$ (small detail omitted) we find that the summand $S_{I,a} \subset N_{I,a}$ is flat over R and finitely presented as a module over $R[\frac{I}{a}][x_1, \ldots, x_n]$. Hence $S_{I,a}$ is finitely presented as an $R[\frac{I}{a}]$-algebra. Moreover, the summand $M_{I,a} \subset N_{I,a}$ is flat over R and finitely presented as a module over $P_{I,a}$ hence also finitely presented as a module over $S_{I,a}$, see Algebra, Lemma 6.4. This reduces us to the case discussed in the next paragraph.

Assume $S = R[x_1, \ldots, x_n]$. Choose a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0.$$

Let M_A be the quotient of $M \otimes_R A$ by its torsion submodule, see Lemma 22.2. Then M_A is a finite module over $S_A = A[x_1, \ldots, x_n]$. By Lemma 22.10 we see that M_A is flat over A. By Lemma 25.6 we see that M_A is finitely presented. Hence there exist finitely many elements $k_1, \ldots, k_t \in S_A^{\oplus r}$ which generate the kernel of the presentation $S_A^{\oplus r} \to M_A$ as an S_A-module. For any choice of $a \in I \subset \mathfrak m$ satisfying (1), (2), and (3) we denote $M_{I,a}$ the strict transform of M along $R \to R[\frac{I}{a}]$. It is a finite module over $S_{I,a} = R[\frac{I}{a}][x_1, \ldots, x_n]$. By Algebra, Lemma 70.12 we have $A = \mathrm{colim}_{I,a} R[\frac{I}{a}]$. This implies that $S_A = \mathrm{colim}\, S_{I,a}$ and

$$\mathrm{colim}\, M \otimes_R R[\tfrac{I}{a}] = M \otimes_R A$$

Choose I, a and lifts $k_1, \ldots, k_t \in S_{I,a}^{\oplus r}$. Since M_A is the quotient of $M \otimes_R A$ by torsion, we see that the images of $k_1, \ldots, k_t$ in $M \otimes_R A$ are annihilated by a nonzero element $\alpha \in A$. After replacing I, a by a different pair (recall that the colimit is filtered), we may assume $\alpha = x/a^n$ for some $x \in I^n$ nonzero. Then we find that $xk_1, \ldots, xk_t$ map to zero in $M \otimes_R A$. Hence after replacing I, a by a different pair we may assume $xk_1, \ldots, xk_t$ map to zero in $M \otimes_R R[\frac{I}{a}]$ for some nonzero $x \in R$. Then finally replacing I, a by xI, xa we find that we may assume $k_1, \ldots, k_t$ map to a-power torsion elements of $M \otimes_R R[\frac{I}{a}]$. For any such pair (I, a) we set

$$M'_{I,a} = S_{I,a}^{\oplus r} / \sum S_{I,a} k_j.$$

Since $M_A = S_A^{\oplus r}/\sum S_A k_j$ we see that $M_A = \mathrm{colim}_{I,a} M'_{I,a}$. At this point we finally apply Algebra, Lemma 168.1 (3) to conclude that $M'_{I,a}$ is flat for some pair (I, a) as above. This lemma does not apply a priori to the system of strict transforms

$$M_{I,a} = (M \otimes_R R[\tfrac{I}{a}])/a\text{-power torsion}$$

as the transition maps may not satisfy the assumptions of the lemma. But now, since flatness implies torsion free (Lemma 22.9) and since $M_{I,a}$ is the quotient of $M'_{I,a}$ (because we arranged it so the elements $k_1, \ldots, k_t$ map to zero in $M_{I,a}$) by the a-power torsion submodule we also conclude that $M'_{I,a} = M_{I,a}$ for such a pair and we win. $\square$

Lemma 26.3. *Let R be a ring. Let M be a finite R-module. Let $k \geq 0$ and $I = Fit_k(M)$. For every $a \in I$ with $R' = R[\frac{I}{a}]$ the strict transform*

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

has $Fit_k(M') = R'$.

Proof. First observe that $\text{Fit}_k(M \otimes_R R') = IR' = aR'$. The first equality by Lemma 8.4 part (3) and the second equality by Algebra, Lemma 70.2. From Lemma 8.8 and exactness of localization we see that $M'_{\mathfrak{p}'}$ can be generated by $\leq k$ elements for every prime $\mathfrak{p}'$ of R'. Then $\text{Fit}_k(M') = R'$ for example by Lemma 8.6. $\square$

Lemma 26.4. *Let R be a ring. Let M be a finite R-module. Let $k \geq 0$ and $I = Fit_k(M)$. Asssume that $M_{\mathfrak{p}}$ is free of rank k for every $\mathfrak{p} \notin V(I)$. Then for every $a \in I$ with $R' = R[\frac{I}{a}]$ the strict transform*

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

is locally free of rank k.

Proof. By Lemma 26.3 we have $\text{Fit}_k(M') = R'$. By Lemma 8.7 it suffices to show that $\text{Fit}_{k-1}(M') = 0$. Recall that $R' \subset R'_a = R_a$, see Algebra, Lemma 70.2. Hence it suffices to prove that $\text{Fit}_{k-1}(M')$ maps to zero in $R'_a = R_a$. Since clearly $(M')_a = M_a$ this reduces us to showing that $\text{Fit}_{k-1}(M_a) = 0$ because formation of Fitting ideals commutes with base change according to Lemma 8.4 part (3). This is true by our assumption that M_a is finite locally free of rank k (see Algebra, Lemma 78.2) and the already cited Lemma 8.7. $\square$

Lemma 26.5. *Let R be a ring. Let M be a finite R-module. Let $f \in R$ be an element such that M_f is finite locally free of rank r. Then there exists a finitely generated ideal $I \subset R$ with $V(f) = V(I)$ such that for all $a \in I$ with $R' = R[\frac{I}{a}]$ the strict transform*

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

is locally free of rank r.

Proof. Choose a surjection $R^{\oplus n} \to M$. Choose a finite submodule $K \subset \text{Ker}(R^{\oplus n} \to M)$ such that $R^{\oplus n}/K \to M$ becomes an isomorphism after inverting f. This is possible because M_f is of finite presentation for example by Algebra, Lemma 78.2. Set $M_1 = R^{\oplus n}/K$ and suppose we can prove the lemma for M_1. Say $I \subset R$ is the corresponding ideal. Then for $a \in I$ the map

$$M'_1 = (M_1 \otimes_R R')/a\text{-power torsion} \longrightarrow M' = (M \otimes_R R')/a\text{-power torsion}$$

is surjective. It is also an isomorphism after inverting a in R' as $R'_a = R_f$, see Algebra, Lemma 70.7. But a is a nonzerodivisor on M'_1, whence the displayed map is an isomorphism. Thus it suffices to prove the lemma in case M is a finitely presented R-module.

Assume M is a finitely presented R-module. Then $J = \text{Fit}_r(M) \subset S$ is a finitely generated ideal. We claim that $I = fJ$ works.

We first check that $V(f) = V(I)$. The inclusion $V(f) \subset V(I)$ is clear. Conversely, if $f \not\in \mathfrak p$, then $\mathfrak p$ is not an element of $V(J)$ by Lemma 8.6. Thus $\mathfrak p \not\in V(fJ) = V(I)$.

Let $a \in I$ and set $R' = R[\frac{I}{a}]$. We may write $a = fb$ for some $b \in J$. By Algebra, Lemmas 70.2 and 70.8 we see that $JR' = bR'$ and b is a nonzerodivisor in R'. Let $\mathfrak p' \subset R' = R[\frac{I}{a}]$ be a prime ideal. Then $JR'_{\mathfrak p'}$ is generated by b. It follows from Lemma 8.8 that $M'_{\mathfrak p'}$ can be generated by r elements. Since M' is finite, there exist $m_1, \ldots, m_r \in M'$ and $g \in R'$, $g \not\in \mathfrak p'$ such that the corresponding map $(R')^{\oplus r} \to M'$ becomes surjective after inverting g.

Finally, consider the ideal $J' = \text{Fit}_{k-1}(M')$. Note that $J'R'_g$ is generated by the coefficients of relations between $m_1, \ldots, m_r$ (compatibility of Fitting ideal with base change). Thus it suffices to show that $J' = 0$, see Lemma 8.7. Since $R'_a = R_f$ (Algebra, Lemma 70.7) and $M'_a = M_f$ is free of rank r we see that $J'_a = 0$. Since a is a nonzerodivisor in R' we conclude that $J' = 0$ and we win. □

27. Completion and flatness

In this section we discuss when the completion of a "big" flat module is flat.

Lemma 27.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian and complete with respect to I. There is a canonical map*

$$\left(\bigoplus\nolimits_{\alpha \in A} R\right)^\wedge \longrightarrow \prod\nolimits_{\alpha \in A} R$$

from the I-adic completion of the direct sum into the product which is universally injective.

Proof. By definition an element x of the left hand side is $x = (x_n)$ where $x_n = (x_{n,\alpha}) \in \bigoplus_{\alpha \in A} R/I^n$ such that $x_{n,\alpha} = x_{n+1,\alpha} \bmod I^n$. As $R = R^\wedge$ we see that for any α there exists a $y_\alpha \in R$ such that $x_{n,\alpha} = y_\alpha \bmod I^n$. Note that for each n there are only finitely many α such that the elements $x_{n,\alpha}$ are nonzero. Conversely, given $(y_\alpha) \in \prod_\alpha R$ such that for each n there are only finitely many α such that $y_\alpha \bmod I^n$ is nonzero, then this defines an element of the left hand side. Hence we can think of an element of the left hand side as infinite "convergent sums" $\sum_\alpha y_\alpha$ with $y_\alpha \in R$ such that for each n there are only finitely many y_α which are nonzero modulo I^n. The displayed map maps this element to the element to (y_α) in the product. In particular the map is injective.

Let Q be a finite R-module. We have to show that the map

$$Q \otimes_R \left(\bigoplus\nolimits_{\alpha \in A} R\right)^\wedge \longrightarrow Q \otimes_R \left(\prod\nolimits_{\alpha \in A} R\right)$$

is injective, see Algebra, Theorem 82.3. Choose a presentation $R^{\oplus k} \to R^{\oplus m} \to Q \to 0$ and denote $q_1, \ldots, q_m \in Q$ the corresponding generators for Q. By Artin-Rees (Algebra, Lemma 51.2) there exists a constant c such that $\text{Im}(R^{\oplus k} \to R^{\oplus m}) \cap (I^N)^{\oplus m} \subset \text{Im}((I^{N-c})^{\oplus k} \to R^{\oplus m})$. Let us contemplate the diagram

$$\begin{tikzcd}
\bigoplus_{l=1}^k \left(\bigoplus_{\alpha \in A} R\right)^\wedge \arrow[r] \arrow[d] & \bigoplus_{j=1}^m \left(\bigoplus_{\alpha \in A} R\right)^\wedge \arrow[r] \arrow[d] & Q \otimes_R \left(\bigoplus_{\alpha \in A} R\right)^\wedge \arrow[r] \arrow[d] & 0 \\
\bigoplus_{l=1}^k \left(\prod_{\alpha \in A} R\right) \arrow[r] & \bigoplus_{j=1}^m \left(\prod_{\alpha \in A} R\right) \arrow[r] & Q \otimes_R \left(\prod_{\alpha \in A} R\right) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Pick an element $\sum_j \sum_\alpha y_{j,\alpha}$ of $\bigoplus_{j=1,\ldots,m} \left(\bigoplus_{\alpha \in A} R\right)^\wedge$. If this element maps to zero in the module $Q \otimes_R \left(\prod_{\alpha \in A} R\right)$, then we see in particular that $\sum_j q_j \otimes y_{j,\alpha} = 0$ in Q for each α. Thus we can find an element $(z_{1,\alpha}, \ldots, z_{k,\alpha}) \in \bigoplus_{l=1,\ldots,k} R$ which maps to $(y_{1,\alpha}, \ldots, y_{m,\alpha}) \in \bigoplus_{j=1,\ldots,m} R$. Moreover, if $y_{j,\alpha} \in I^{N_\alpha}$ for $j = 1, \ldots, m$, then we may assume that $z_{l,\alpha} \in I^{N_\alpha - c}$ for $l = 1, \ldots, k$. Hence the sum $\sum_l \sum_\alpha z_{l,\alpha}$ is "convergent" and defines an element of $\bigoplus_{l=1,\ldots,k} \left(\bigoplus_{\alpha \in A} R\right)^\wedge$ which maps to the element $\sum_j \sum_\alpha y_{j,\alpha}$ we started out with. Thus the right vertical arrow is injective and we win. $\square$

The following lemma can also be deduced from Lemma 27.4 below.

Lemma 27.2. *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian. The completion $(\bigoplus_{\alpha \in A} R)^\wedge$ is a flat R-module.*

Proof. Denote $R^\wedge$ the completion of R with respect to I. As $R \to R^\wedge$ is flat by Algebra, Lemma 97.2 it suffices to prove that $(\bigoplus_{\alpha \in A} R)^\wedge$ is a flat $R^\wedge$-module (use Algebra, Lemma 39.4). Since

$$(\bigoplus\nolimits_{\alpha \in A} R)^\wedge = (\bigoplus\nolimits_{\alpha \in A} R^\wedge)^\wedge$$

we may replace R by $R^\wedge$ and assume that R is complete with respect to I (see Algebra, Lemma 97.4). In this case Lemma 27.1 tells us the map $(\bigoplus_{\alpha \in A} R)^\wedge \to \prod_{\alpha \in A} R$ is universally injective. Thus, by Algebra, Lemma 82.7 it suffices to show that $\prod_{\alpha \in A} R$ is flat. By Algebra, Proposition 90.6 (and Algebra, Lemma 90.5) we see that $\prod_{\alpha \in A} R$ is flat. $\square$

Lemma 27.3. *Let A be a Noetherian ring. Let I be an ideal of A. Let M be a finite A-module. For every $p > 0$ there exists a $c > 0$ such that $Tor_p^A(M, A/I^n) \to Tor_p^A(M, A/I^{n-c})$ is zero for all $n \geq c$.*

Proof. Proof for $p = 1$. Choose a short exact sequence $0 \to K \to A^{\oplus t} \to M \to 0$. Then $\text{Tor}_1^A(M, A/I^n) = K \cap (I^n)^{\oplus t}/I^n K$. By Artin-Rees (Algebra, Lemma 51.2) there is a constant $c \geq 0$ such that $K \cap (I^n)^{\oplus t} \subset I^{n-c}K$ for $n \geq c$. Thus the result for $p = 1$. For $p > 1$ we have $\text{Tor}_p^A(M, A/I^n) = \text{Tor}_{p-1}^A(K, A/I^n)$. Thus the lemma follows by induction. $\square$

Lemma 27.4. *Let A be a Noetherian ring. Let I be an ideal of A. Let (M_n) be an inverse system of A-modules such that*

1. *M_n is a flat A/I^n-module,*
2. *$M_{n+1} \to M_n$ is surjective.*

Then $M = \lim M_n$ is a flat A-module and $Q \otimes_A M = \lim Q \otimes_A M_n$ for every finite A-module Q.

Proof. We first show that $Q \otimes_A M = \lim Q \otimes_A M_n$ for every finite A-module Q. Choose a resolution $F_2 \to F_1 \to F_0 \to Q \to 0$ by finite free A-modules F_i. Then

$$F_2 \otimes_A M_n \to F_1 \otimes_A M_n \to F_0 \otimes_A M_n$$

is a chain complex whose homology in degree 0 is $Q \otimes_A M_n$ and whose homology in degree 1 is

$$\text{Tor}_1^A(Q, M_n) = \text{Tor}_1^A(Q, A/I^n) \otimes_{A/I^n} M_n$$

as M_n is flat over A/I^n. By Lemma 27.3 we see that this system is essentially constant (with value 0). It follows from Homology, Lemma 31.7 that $\lim Q \otimes_A$

$A/I^n = \text{Coker}(\lim F_1 \otimes_A M_n \to \lim F_0 \otimes_A M_n)$. Since F_i is finite free this equals $\text{Coker}(F_1 \otimes_A M \to F_0 \otimes_A M) = Q \otimes_A M$.

Next, let $Q \to Q'$ be an injective map of finite A-modules. We have to show that $Q \otimes_A M \to Q' \otimes_A M$ is injective (Algebra, Lemma 39.5). By the above we see

$$\text{Ker}(Q \otimes_A M \to Q' \otimes_A M) = \text{Ker}(\lim Q \otimes_A M_n \to \lim Q' \otimes_A M_n).$$

For each n we have an exact sequence

$$\text{Tor}_1^A(Q', M_n) \to \text{Tor}_1^A(Q'', M_n) \to Q \otimes_A M_n \to Q' \otimes_A M_n$$

where $Q'' = \text{Coker}(Q \to Q')$. Above we have seen that the inverse systems of Tor's are essentially constant with value 0. It follows from Homology, Lemma 31.7 that the inverse limit of the right most maps is injective. $\square$

Lemma 27.5. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

1. *I is finitely generated,*
2. *R/I is Noetherian,*
3. *M/IM is flat over R/I,*
4. *$Tor_1^R(M, R/I) = 0$.*

Then the I-adic completion $R^\wedge$ is a Noetherian ring and $M^\wedge$ is flat over $R^\wedge$.

Proof. By Algebra, Lemma 99.8 the modules M/I^nM are flat over R/I^n for all n. By Algebra, Lemma 96.3 we have (a) $R^\wedge$ and $M^\wedge$ are I-adically complete and (b) $R/I^n = R^\wedge/I^nR^\wedge$ for all n. By Algebra, Lemma 97.5 the ring $R^\wedge$ is Noetherian. Applying Lemma 27.4 we conclude that $M^\wedge = \lim M/I^nM$ is flat as an $R^\wedge$-module. $\square$

28. The Koszul complex

We define the Koszul complex as follows.

Definition 28.1. Let R be a ring. Let $\varphi : E \to R$ be an R-module map. The *Koszul complex* $K_\bullet(\varphi)$ associated to φ is the commutative differential graded algebra defined as follows:

1. the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(E)$,
2. the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all $e \in E = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is one of the generators of degree n in $K_\bullet(\varphi)$, then

$$d(e_1 \wedge \ldots \wedge e_n) = \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n.$$

It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \otimes e$ and hence factors through the exterior algebra.

We often assume that E is a finite free module, say $E = R^{\oplus n}$. In this case the map φ is given by a sequence of elements $f_1, \ldots, f_n \in R$.

Definition 28.2. Let R be a ring and let $f_1, \ldots, f_r \in R$. The *Koszul complex on $f_1, \ldots, f_r$* is the Koszul complex associated to the map $(f_1, \ldots, f_r) : R^{\oplus r} \to R$. Notation $K_\bullet(f_\bullet)$, $K_\bullet(f_1, \ldots, f_r)$, $K_\bullet(R, f_1, \ldots, f_r)$, or $K_\bullet(R, f_\bullet)$.

Of course, if E is finite locally free, then $K_\bullet(\varphi)$ is locally on $\text{Spec}(R)$ isomorphic to a Koszul complex $K_\bullet(f_1, \ldots, f_r)$. This complex has many interesting formal properties.

Lemma 28.3. *Let $\varphi : E \to R$ and $\varphi' : E' \to R$ be R-module maps. Let $\psi : E \to E'$ be an R-module map such that $\varphi' \circ \psi = \varphi$. Then ψ induces a homomorphism of differential graded algebras $K_\bullet(\varphi) \to K_\bullet(\varphi')$.*

Proof. This is immediate from the definitions. $\square$

Lemma 28.4. *Let $f_1, \ldots, f_r \in R$ be a sequence. Let (x_{ij}) be an invertible $r \times r$-matrix with coefficients in R. Then the complexes $K_\bullet(f_\bullet)$ and*

$$K_\bullet(\sum x_{1j}f_j, \sum x_{2j}f_j, \ldots, \sum x_{rj}f_j)$$

are isomorphic.

Proof. Set $g_i = \sum x_{ij}f_j$. The matrix (x_{ji}) gives an isomorphism $x : R^{\oplus r} \to R^{\oplus r}$ such that $(g_1, \ldots, g_r) = (f_1, \ldots, f_r) \circ x$. Hence this follows from the functoriality of the Koszul complex described in Lemma 28.3. $\square$

Lemma 28.5. *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $e \in E$ with image $f = \varphi(e)$ in R. Then*

$$f = de + ed$$

as endomorphisms of $K_\bullet(\varphi)$.

Proof. This is true because $d(ea) = d(e)a - ed(a) = fa - ed(a)$. $\square$

Lemma 28.6. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be a sequence. Multiplication by f_i on $K_\bullet(f_\bullet)$ is homotopic to zero, and in particular the cohomology modules $H_i(K_\bullet(f_\bullet))$ are annihilated by the ideal $(f_1, \ldots, f_r)$.*

Proof. Special case of Lemma 28.5. $\square$

In Derived Categories, Section 9 we defined the cone of a morphism of cochain complexes. The cone $C(f)_\bullet$ of a morphism of chain complexes $f : A_\bullet \to B_\bullet$ is the complex $C(f)_\bullet$ given by $C(f)_n = B_n \oplus A_{n-1}$ and differential

$$d_{C(f),n} = \begin{pmatrix} d_{B,n} & f_{n-1} \\ 0 & -d_{A,n-1} \end{pmatrix} \tag{28.6.1}$$

It comes equipped with canonical morphisms of complexes $i : B_\bullet \to C(f)_\bullet$ and $p : C(f)_\bullet \to A_\bullet[-1]$ induced by the obvious maps $B_n \to C(f)_n \to A_{n-1}$.

Lemma 28.7. *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $f \in R$. Set $E' = E \oplus R$ and define $\varphi' : E' \to R$ by φ on E and multiplication by f on R. The complex $K_\bullet(\varphi')$ is isomorphic to the cone of the map of complexes*

$$f : K_\bullet(\varphi) \longrightarrow K_\bullet(\varphi).$$

Proof. Denote $e_0 \in E'$ the element $1 \in R \subset R \oplus E$. By our definition of the cone above we see that

$$C(f)_n = K_n(\varphi) \oplus K_{n-1}(\varphi) = \wedge^n(E) \oplus \wedge^{n-1}(E) = \wedge^n(E')$$

where in the last $=$ we map $(0, e_1 \wedge \ldots \wedge e_{n-1})$ to $e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1}$ in $\wedge^n(E')$. A computation shows that this isomorphism is compatible with differentials. Namely, this is clear for elements of the first summand as $\varphi'|_E = \varphi$ and $d_{C(f)}$ restricted to the first summand is just $d_{K_\bullet(\varphi)}$. On the other hand, if $e_1 \wedge \ldots \wedge e_{n-1}$ is in the second summand, then

$$d_{C(f)}(0, e_1 \wedge \ldots \wedge e_{n-1}) = fe_1 \wedge \ldots \wedge e_{n-1} - d_{K_\bullet(\varphi)}(e_1 \wedge \ldots \wedge e_{n-1})$$

and on the other hand

$$\begin{aligned} & d_{K_\bullet(\varphi')}(0, e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1}) \\ & = \sum\nolimits_{i=0,\ldots,n-1} (-1)^i \varphi'(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \\ & = fe_1 \wedge \ldots \wedge e_{n-1} + \sum\nolimits_{i=1,\ldots,n-1} (-1)^i \varphi(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \\ & = fe_1 \wedge \ldots \wedge e_{n-1} - e_0 \left(\sum\nolimits_{i=1,\ldots,n-1} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \right) \end{aligned}$$

which is the image of the result of the previous computation. □

Lemma 28.8. *Let R be a ring. Let $f_1, \ldots, f_r$ be a sequence of elements of R. The complex $K_\bullet(f_1, \ldots, f_r)$ is isomorphic to the cone of the map of complexes*

$$f_r : K_\bullet(f_1, \ldots, f_{r-1}) \longrightarrow K_\bullet(f_1, \ldots, f_{r-1}).$$

Proof. Special case of Lemma 28.7. □

Lemma 28.9. *Let R be a ring. Let $A_\bullet$ be a complex of R-modules. Let $f, g \in R$. Let $C(f)_\bullet$ be the cone of $f : A_\bullet \to A_\bullet$. Define similarly $C(g)_\bullet$ and $C(fg)_\bullet$. Then $C(fg)_\bullet$ is homotopy equivalent to the cone of a map*

$$C(f)_\bullet[1] \longrightarrow C(g)_\bullet$$

Proof. We first prove this if $A_\bullet$ is the complex consisting of R placed in degree 0. In this case the complex $C(f)_\bullet$ is the complex

$$\ldots \to 0 \to R \xrightarrow{f} R \to 0 \to \ldots$$

with R placed in (homological) degrees 1 and 0. The map of complexes we use is

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R \arrow[r, "f"] \arrow[d, "1"] & R \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & R \arrow[r, "g"] & R \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

The cone of this is the chain complex consisting of $R^{\oplus 2}$ placed in degrees 1 and 0 and differential (28.6.1)

$$\begin{pmatrix} g & 1 \\ 0 & -f \end{pmatrix} : R^{\oplus 2} \longrightarrow R^{\oplus 2}$$

To see this chain complex is homotopic to $C(fg)_\bullet$, i.e., to $R \xrightarrow{fg} R$, consider the maps of complexes

$$\begin{tikzcd} R \arrow[r, "fg"'] \arrow[d, "{(1,-g)}"'] & R \arrow[d, "{(0,1)}"] \\ R^{\oplus 2} \arrow[r] & R^{\oplus 2} \end{tikzcd} \qquad \begin{tikzcd} R^{\oplus 2} \arrow[r] \arrow[d, "{(1,0)}"'] & R^{\oplus 2} \arrow[d, "{(f,1)}"] \\ R \arrow[r, "fg"] & R \end{tikzcd}$$

with obvious notation. The composition of these two maps in one direction is the identity on $C(fg)_\bullet$, but in the other direction it isn't the identity. We omit writing out the required homotopy.

To see the result holds in general, we use that we have a functor $K_\bullet \mapsto \text{Tot}(A_\bullet \otimes_R K_\bullet)$ on the category of complexes which is compatible with homotopies and cones. Then we write $C(f)_\bullet$ and $C(g)_\bullet$ as the total complex of the double complexes

$$(R \xrightarrow{f} R) \otimes_R A_\bullet \quad \text{and} \quad (R \xrightarrow{g} R) \otimes_R A_\bullet$$

and in this way we deduce the result from the special case discussed above. Some details omitted. $\square$

Lemma 28.10. *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $f, g \in R$. Set $E' = E \oplus R$ and define $\varphi'_f, \varphi'_g, \varphi'_{fg} : E' \to R$ by φ on E and multiplication by f, g, fg on R. The complex $K_\bullet(\varphi'_{fg})$ is homotopy equivalent to the cone of a map of complexes*

$$K_\bullet(\varphi'_f)[1] \longrightarrow K_\bullet(\varphi'_g).$$

Proof. By Lemma 28.7 the complex $K_\bullet(\varphi'_f)$ is isomorphic to the cone of multiplication by f on $K_\bullet(\varphi)$ and similarly for the other two cases. Hence the lemma follows from Lemma 28.9. $\square$

Lemma 28.11. *Let R be a ring. Let $f_1, \ldots, f_{r-1}$ be a sequence of elements of R. Let $f, g \in R$. The complex $K_\bullet(f_1, \ldots, f_{r-1}, fg)$ is homotopy equivalent to the cone of a map of complexes*

$$K_\bullet(f_1, \ldots, f_{r-1}, f)[1] \longrightarrow K_\bullet(f_1, \ldots, f_{r-1}, g)$$

Proof. Special case of Lemma 28.10. $\square$

Lemma 28.12. *Let R be a ring. Let $f_1, \ldots, f_r$, $g_1, \ldots, g_s$ be elements of R. Then there is an isomorphism of Koszul complexes*

$$K_\bullet(R, f_1, \ldots, f_r, g_1, \ldots, g_s) = \mathit{Tot}(K_\bullet(R, f_1, \ldots, f_r) \otimes_R K_\bullet(R, g_1, \ldots, g_s)).$$

Proof. Omitted. Hint: If $K_\bullet(R, f_1, \ldots, f_r)$ is generated as a differential graded algebra by $x_1, \ldots, x_r$ with $\text{d}(x_i) = f_i$ and $K_\bullet(R, g_1, \ldots, g_s)$ is generated as a differential graded algebra by $y_1, \ldots, y_s$ with $\text{d}(y_j) = g_j$, then we can think of $K_\bullet(R, f_1, \ldots, f_r, g_1, \ldots, g_s)$ as the differential graded algebra generated by the sequence of elements $x_1, \ldots, x_r, y_1, \ldots, y_s$ with $\text{d}(x_i) = f_i$ and $\text{d}(y_j) = g_j$. $\square$

29. The extended alternating Čech complex

Let R be a ring. Let $f_1, \ldots, f_r \in R$. The extended alternating Čech complex of R is the cochain complex

$$R \to \bigoplus\nolimits_{i_0} R_{f_{i_0}} \to \bigoplus\nolimits_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r}$$

where R is in degree 0, the term $\bigoplus_{i_0} R_{f_{i_0}}$ is in degre 1, and so on. The maps are defined as follows

(1) The map $R \to \bigoplus_{i_0} R_{f_{i_0}}$ is given by the canonical maps $R \to R_{f_{i_0}}$.
(2) Given $1 \leq i_0 < \ldots < i_{p+1} \leq r$ and $0 \leq j \leq p+1$ we have the canonical localization map

$$R_{f_{i_0} \ldots \hat{f}_{i_j} \ldots f_{i_{p+1}}} \to R_{f_{i_0} \ldots f_{i_{p+1}}}$$

(3) The differentials use the canonical maps of (2) with sign $(-1)^j$.

If M is any R-module, the extended alternating Čech complex of M is the similarly constructed cochain complex

$$M \to \bigoplus\nolimits_{i_0} M_{f_{i_0}} \to \bigoplus\nolimits_{i_0<i_1} M_{f_{i_0}f_{i_1}} \to \ldots \to M_{f_1\ldots f_r}$$

where M is in degree 0 as before.

Lemma 29.1. *The extended alternating Čech complexes defined above are complexes of R-modules.*

Proof. Omitted. ☐

Lemma 29.2. *Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. The extended alternating Čech complex of M is the tensor product over R of M with the extended alternating Čech complex of R.*

Proof. Omitted. ☐

Lemma 29.3. *Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. Let $R \to S$ be a ring map, denote $g_1, \ldots, g_r \in S$ the images of $f_1, \ldots, f_r$, and set $N = M \otimes_R S$. The extended alternating Čech complex constructed using S, $g_1, \ldots, g_r$, and N is the tensor product of the extended alternating Čech complex of M with S over R.*

Proof. Omitted. ☐

Lemma 29.4. *Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. If there exists an $i \in \{1, \ldots, r\}$ such that f_i is a unit, then the extended alternating Čech complex of M is homotopy equivalent to 0.*

Proof. We will use the following notation: a cochain x of degree $p+1$ in the extended alternating Čech complex of M is $x = (x_{i_0\ldots i_p})$ where $x_{i_0\ldots i_p}$ is in $M_{f_{i_0}\ldots f_{i_p}}$. With this notation we have

$$d(x)_{i_0\ldots i_{p+1}} = \sum\nolimits_j (-1)^j x_{i_0\ldots \hat{i}_j \ldots i_{p+1}}$$

As homotopy we use the maps

$$h : \text{cochains of degree } p+2 \to \text{cochains of degree } p+1$$

given by the rule

$$h(x)_{i_0\ldots i_p} = 0 \text{ if } i \in \{i_0, \ldots, i_p\} \text{ and } h(x)_{i_0\ldots i_p} = (-1)^j x_{i_0\ldots i_j i i_{j+1}\ldots i_p} \text{ if not}$$

Here j is the unique index such that $i_j < i < i_{j+1}$ in the second case; also, since f_i is a unit we have the equality

$$M_{f_{i_0}\ldots f_{i_p}} = M_{f_{i_0}\ldots f_{i_j} f_i f_{i_{j+1}}\ldots f_{i_p}}$$

which we can use to make sense of thinking of $(-1)^j x_{i_0\ldots i_j i i_{j+1}\ldots i_p}$ as an element of $M_{f_{i_0}\ldots f_{i_p}}$. We will show by a computation that $dh + hd$ equals the negative of the identity map which finishes the proof. To do this fix x a cochain of degree $p+1$ and let $1 \leq i_0 < \ldots < i_p \leq r$.

Case I: $i \in \{i_0, \ldots, i_p\}$. Say $i = i_t$. Then we have $h(d(x))_{i_0\ldots i_p} = 0$. On the other hand we have

$$d(h(x))_{i_0\ldots i_p} = \sum (-1)^j h(x)_{i_0\ldots \hat{i}_j \ldots i_p} = (-1)^t h(x)_{i_0\ldots \hat{i} \ldots i_p} = (-1)^t (-1)^{t-1} x_{i_0\ldots i_p}$$

Thus $(dh + hd)(x)_{i_0\ldots i_p} = -x_{i_0\ldots i_p}$ as desired.

Case II: $i \not\in \{i_0, \ldots, i_p\}$. Let j be such that $i_j < i < i_{j+1}$. Then we see that

$$\begin{aligned} h(d(x))_{i_0\ldots i_p} &= (-1)^j d(x)_{i_0\ldots i_j i i_{j+1}\ldots i_p} \\ &= \sum\nolimits_{j' \leq j} (-1)^{j+j'} x_{i_0\ldots \hat{i}_{j'} \ldots i_j i i_{j+1}\ldots i_p} - x_{i_0\ldots i_p} \\ &+ \sum\nolimits_{j' > j} (-1)^{j+j'+1} x_{i_0\ldots i_j i i_{j+1}\ldots \hat{i}_{j'} \ldots i_p} \end{aligned}$$

On the other hand we have

$$\begin{aligned} d(h(x))_{i_0\ldots i_p} &= \sum\nolimits_{j'} (-1)^{j'} h(x)_{i_0\ldots \hat{i}_{j'} \ldots i_p} \\ &= \sum\nolimits_{j' \leq j} (-1)^{j'+j-1} x_{i_0\ldots \hat{i}_{j'} \ldots i_j i i_{j+1}\ldots i_p} \\ &+ \sum\nolimits_{j' > j} (-1)^{j'+j} x_{i_0\ldots i_j i i_{j+1}\ldots \hat{i}_{j'} \ldots i_p} \end{aligned}$$

Adding these up we obtain $(dh + hd)(x)_{i_0\ldots i_p} = -x_{i_0\ldots i_p}$ as desired. ☐

Lemma 29.5. *Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. Let H^q be the qth cohomology module of the extended alternation Čech complex of M. Then*

1. *$H^q = 0$ if $q \not\in [0, r]$,*
2. *for $x \in H^i$ there exists an $n \geq 1$ such that $f_i^n x = 0$ for $i = 1, \ldots, r$,*
3. *the support of H^q is contained in $V(f_1, \ldots, f_r)$,*
4. *if there is an $f \in (f_1, \ldots, f_r)$ which acts invertibly on M, then $H^q = 0$.*

Proof. Part (1) follows from the fact that the extended alternating Čech complex is zero in degrees < 0 and $> r$. To prove (2) it suffices to show that for each i there exists an $n \geq 1$ such that $f_i^n x = 0$. To see this it suffices to show that $(H^q)_{f_i} = 0$. Since localization is exact, $(H^q)_{f_i}$ is the qth cohomology module of the localization of the extended alternating complex of M at f_i. By Lemma 29.3 this localization is the extended alternating Čech complex of M_{f_i} over R_{f_i} with respect to the images of $f_1, \ldots, f_r$ in R_{f_i}. Thus we reduce to showing that H^q is zero if f_i is invertible, which follows from Lemma 29.4. Part (3) follows from the observation that $(H^q)_{f_i} = 0$ for all i that we just proved. To see part (4) note that in this case f acts invertibly on H^q and H^q is supported on $V(f)$ by (3). This forces H^q to be zero (small detail omitted). ☐

Lemma 29.6. *Let R be a ring. Let $f_1, \ldots, f_r \in R$. The extended alternating Čech complex*

$$R \to \bigoplus\nolimits_{i_0} R_{f_{i_0}} \to \bigoplus\nolimits_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r}$$

is a colimit of the Koszul complexes $K(R, f_1^n, \ldots, f_r^n)$; see proof for a precise statement.

Proof. We urge the reader to prove this for themselves. Denote $K(R, f_1^n, \ldots, f_r^n)$ the Koszul complex of Definition 28.2 viewed as a cochain complex sitting in degrees $0, \ldots, r$. Thus we have

$$K(R, f_1^n, \ldots, f_r^n) : 0 \to \wedge^r(R^{\oplus r}) \to \wedge^{r-1}(R^{\oplus r}) \to \ldots \to R^{\oplus r} \to R \to 0$$

with the term $\wedge^r(R^{\oplus r})$ sitting in degree 0. Let $e_1^n, \ldots, e_r^n$ be the standard basis of $R^{\oplus r}$. Then the elements $e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n$ for $1 \leq j_1 < \ldots < j_{r-p} \leq r$ form a basis for the term in degree p of the Koszul complex. Further, observe that

$$d(e^n_{j_1} \wedge \ldots \wedge e^n_{j_{r-p}}) = \sum (-1)^{a+1} f^n_{j_a} e^n_{j_1} \wedge \ldots \wedge \hat{e}^n_{j_a} \wedge \ldots \wedge e^n_{j_{r-p}}$$

by our construction of the Koszul complex in Section 28. The transition maps of our system

$$K(R, f_1^n, \ldots, f_r^n) \to K(R, f_1^{n+1}, \ldots, f_r^{n+1})$$

are given by the rule

$$e^n_{j_1} \wedge \ldots \wedge e^n_{j_{r-p}} \longmapsto f_{i_0} \ldots f_{i_{p-1}} e^{n+1}_{j_1} \wedge \ldots \wedge e^{n+1}_{j_{r-p}}$$

where the indices $1 \leq i_0 < \ldots < i_{p-1} \leq r$ are such that $\{1, \ldots r\} = \{i_0, \ldots, i_{p-1}\} \amalg \{j_1, \ldots, j_{r-p}\}$. We omit the short computation that shows this is compatible with differentials. Observe that the transition maps are always 1 in degree 0 and equal to $f_1 \ldots f_r$ in degree r.

Denote $K^p(R, f_1^n, \ldots, f_r^n)$ the term of degree p in the Koszul complex. Observe that for any $f \in R$ we have

$$R_f = \text{colim}(R \xrightarrow{f} R \xrightarrow{f} R \to \ldots)$$

Hence we see that in degree p we obtain

$$\text{colim}\, K^p(R, f_1^n, \ldots f_r^n) = \bigoplus\nolimits_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0} \ldots f_{i_{p-1}}}$$

Here the element $e^n_{j_1} \wedge \ldots \wedge e^n_{j_{r-p}}$ of the Koszul complex above maps in the colimit to the element $(f_{i_0} \ldots f_{i_{p-1}})^{-n}$ in the summand $R_{f_{i_0} \ldots f_{i_{p-1}}}$ where the indices are chosen such that $\{1, \ldots r\} = \{i_0, \ldots, i_{p-1}\} \amalg \{j_1, \ldots, j_{r-p}\}$. Thus the differential on this complex is given by

$$d(1 \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}}) = \sum\nolimits_{i \not\in \{i_0, \ldots, i_{p-1}\}} (-1)^{i-t} \text{ in } R_{f_{i_0} \ldots f_{i_t} f_i f_{i_{t+1}} \ldots f_{i_{p-1}}}$$

Thus if we consider the map of complexes given in degree p by the map

$$\bigoplus\nolimits_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0} \ldots f_{i_{p-1}}} \longrightarrow \bigoplus\nolimits_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0} \ldots f_{i_{p-1}}}$$

determined by the rule

$$1 \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}} \longmapsto (-1)^{i_0 + \ldots + i_{p-1} + p} \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}}$$

then we get an isomorphism of complexes from $\text{colim}\, K(R, f_1^n, \ldots, f_r^n)$ to the extended alternating Čech complex defined in this section. We omit the verification that the signs work out. $\square$

30. Koszul regular sequences

Please take a look at Algebra, Sections 68, 69, and 72 before looking at this one.

Definition 30.1. Let R be a ring. Let $r \geq 0$ and let $f_1, \ldots, f_r \in R$ be a sequence of elements. Let M be an R-module. The sequence $f_1, \ldots, f_r$ is called

(1) *M-Koszul-regular* if $H_i(K_\bullet(f_1, \ldots, f_r) \otimes_R M) = 0$ for all $i \neq 0$,
(2) *M-H_1-regular* if $H_1(K_\bullet(f_1, \ldots, f_r) \otimes_R M) = 0$,
(3) *Koszul-regular* if $H_i(K_\bullet(f_1, \ldots, f_r)) = 0$ for all $i \neq 0$, and
(4) *H_1-regular* if $H_1(K_\bullet(f_1, \ldots, f_r)) = 0$.

We will see in Lemmas 30.2, 30.3, and 30.6 that for elements $f_1, \ldots, f_r$ of a ring R we have the following implications

$$\begin{aligned} f_1, \ldots, f_r \text{ is a regular sequence} &\Rightarrow f_1, \ldots, f_r \text{ is a Koszul-regular sequence} \\ &\Rightarrow f_1, \ldots, f_r \text{ is an } H_1\text{-regular sequence} \\ &\Rightarrow f_1, \ldots, f_r \text{ is a quasi-regular sequence.} \end{aligned}$$

In general none of these implications can be reversed, but if R is a Noetherian local ring and $f_1, \ldots, f_r \in \mathfrak{m}_R$, then the four conditions are all equivalent (Lemma 30.7). If $f = f_1 \in R$ is a length 1 sequence and f is not a unit of R then it is clear that the following are all equivalent

1. (1) f is a regular sequence of length one,
2. (2) f is a Koszul-regular sequence of length one, and
3. (3) f is a H_1-regular sequence of length one.

It is also clear that these imply that f is a quasi-regular sequence of length one. But there do exist quasi-regular sequences of length 1 which are not regular sequences. Namely, let

$$R = k[x, y_0, y_1, \ldots]/(xy_0, xy_1 - y_0, xy_2 - y_1, \ldots)$$

and let f be the image of x in R. Then f is a zerodivisor, but $\bigoplus_{n \geq 0}(f^n)/(f^{n+1}) \cong k[x]$ is a polynomial ring.

Lemma 30.2. *An M-regular sequence is M-Koszul-regular. A regular sequence is Koszul-regular.*

Proof. Let R be a ring and let M be an R-module. It is immediate that an M-regular sequence of length 1 is M-Koszul-regular. Let $f_1, \ldots, f_r$ be an M-regular sequence. Then f_1 is a nonzerodivisor on M. Hence

$$0 \to K_\bullet(f_2, \ldots, f_r) \otimes M \xrightarrow{f_1} K_\bullet(f_2, \ldots, f_r) \otimes M \to K_\bullet(\overline{f}_2, \ldots, \overline{f}_r) \otimes M/f_1M \to 0$$

is a short exact sequence of complexes where $\overline{f}_i$ is the image of f_i in $R/(f_1)$. By Lemma 28.8 the complex $K_\bullet(R, f_1, \ldots, f_r)$ is isomorphic to the cone of multiplication by f_1 on $K_\bullet(f_2, \ldots, f_r)$. Thus $K_\bullet(R, f_1, \ldots, f_r) \otimes M$ is isomorphic to the cone on the first map. Hence $K_\bullet(\overline{f}_2, \ldots, \overline{f}_r) \otimes M/f_1M$ is quasi-isomorphic to $K_\bullet(f_1, \ldots, f_r) \otimes M$. As $\overline{f}_2, \ldots, \overline{f}_r$ is an M/f_1M-regular sequence in $R/(f_1)$ the result follows from the case $r = 1$ and induction. $\square$

Lemma 30.3. *A M-Koszul-regular sequence is M-H_1-regular. A Koszul-regular sequence is H_1-regular.*

Proof. This is immediate from the definition. $\square$

Lemma 30.4. *Let $f_1, \ldots, f_{r-1} \in R$ be a sequence and $f, g \in R$. Let M be an R-module.*

1. (1) *If $f_1, \ldots, f_{r-1}, f$ and $f_1, \ldots, f_{r-1}, g$ are M-H_1-regular then $f_1, \ldots, f_{r-1}, fg$ is M-H_1-regular too.*
2. (2) *If $f_1, \ldots, f_{r-1}, f$ and $f_1, \ldots, f_{r-1}, g$ are M-Koszul-regular then $f_1, \ldots, f_{r-1}, fg$ is M-Koszul-regular too.*

Proof. By Lemma 28.11 we have exact sequences

$$H_i(K_\bullet(f_1, \ldots, f_{r-1}, f) \otimes M) \to H_i(K_\bullet(f_1, \ldots, f_{r-1}, fg) \otimes M) \to H_i(K_\bullet(f_1, \ldots, f_{r-1}, g) \otimes M)$$

for all i. $\square$

Lemma 30.5. *Let $\varphi : R \to S$ be a flat ring map. Let $f_1, \ldots, f_r \in R$. Let M be an R-module and set $N = M \otimes_R S$.*

1. *If $f_1, \ldots, f_r$ in R is an M-H_1-regular sequence, then $\varphi(f_1), \ldots, \varphi(f_r)$ is an N-H_1-regular sequence in S.*
2. *If $f_1, \ldots, f_r$ is an M-Koszul-regular sequence in R, then $\varphi(f_1), \ldots, \varphi(f_r)$ is an N-Koszul-regular sequence in S.*

Proof. This is true because $K_\bullet(f_1, \ldots, f_r) \otimes_R S = K_\bullet(\varphi(f_1), \ldots, \varphi(f_r))$ and therefore $(K_\bullet(f_1, \ldots, f_r) \otimes_R M) \otimes_R S = K_\bullet(\varphi(f_1), \ldots, \varphi(f_r)) \otimes_S N$. $\square$

Lemma 30.6. *An M-H_1-regular sequence is M-quasi-regular.*

Proof. Let R be a ring and let M be an R-module. Let $f_1, \ldots, f_r$ be an M-H_1-regular sequence. Denote $J = (f_1, \ldots, f_r)$. The assumption means that we have an exact sequence

$$\wedge^2(R^r) \otimes M \to R^{\oplus r} \otimes M \to JM \to 0$$

where the first arrow is given by $e_i \wedge e_j \otimes m \mapsto (f_i e_j - f_j e_i) \otimes m$. Tensoring the sequence with R/J we see that

$$JM/J^2M = (R/J)^{\oplus r} \otimes_R M = (M/JM)^{\oplus r}$$

is a finite free module. To finish the proof we have to prove for every $n \geq 2$ the following: if

$$\xi = \sum\nolimits_{|I|=n, I=(i_1,\ldots,i_r)} m_I f_1^{i_1} \ldots f_r^{i_r} \in J^{n+1}M$$

then $m_I \in JM$ for all I. In the next paragraph, we prove $m_I \in JM$ for $I = (0, \ldots, 0, n)$ and in the last paragraph we deduce the general case from this special case.

Let $I = (0, \ldots, 0, n)$. Let ξ be as above. We can write $\xi = m_1 f_1 + \ldots + m_{r-1} f_{r-1} + m_I f_r^n$. As we have assumed $\xi \in J^{n+1}M$, we can also write $\xi = \sum_{1 \leq i \leq j \leq r-1} m_{ij} f_i f_j + \sum_{1 \leq i \leq r-1} m'_i f_i f_r^n + m'' f_r^{n+1}$. Then we see that

$$\begin{gathered}(m_1 - m_{11} f_1 - m'_1 f_r^n) f_1 + \\ (m_2 - m_{12} f_1 - m_{22} f_2 - m'_2 f_r^n) f_2 + \\ \ldots + \\ (m_{r-1} - m_{1r-1} f_1 - \ldots - m_{r-1r-1} f_{r-1} - m'_{r-1} f_r^n) f_{r-1} + \\ (m_I - m'' f_r) f_r^n = 0\end{gathered}$$

Since $f_1, \ldots, f_{r-1}, f_r^n$ is M-H_1-regular by Lemma 30.4 we see that $m_I - m'' f_r$ is in the submodule $f_1 M + \ldots + f_{r-1} M + f_r^n M$. Thus $m_I \in f_1 M + \ldots + f_r M$.

Let $S = R[x_1, x_2, \ldots, x_r, 1/x_r]$. The ring map $R \to S$ is faithfully flat, hence $f_1, \ldots, f_r$ is an M-H_1-regular sequence in S, see Lemma 30.5. By Lemma 28.4 we see that

$$g_1 = f_1 - \frac{x_1}{x_r} f_r, \ \ldots, \ g_{r-1} = f_{r-1} - \frac{x_{r-1}}{x_r} f_r, \ g_r = \frac{1}{x_r} f_r$$

is an M-H_1-regular sequence in S. Finally, note that our element ξ can be rewritten

$$\xi = \sum\nolimits_{|I|=n, I=(i_1,\ldots,i_r)} m_I (g_1 + x_i g_r)^{i_1} \ldots (g_{r-1} + x_i g_r)^{i_{r-1}} (x_r g_r)^{i_r}$$

and the coefficient of g_r^n in this expression is

$$\sum m_I x_1^{i_1} \ldots x_r^{i_r}$$

By the case discussed in the previous paragraph this sum is in $J(M \otimes_R S)$. Since the monomials $x_1^{i_1} \ldots x_r^{i_r}$ form part of an R-basis of S over R we conclude that $m_I \in J$ for all I as desired. $\square$

For nonzero finite modules over Noetherian local rings all of the types of regular sequences introduced so far are equivalent.

Lemma 30.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a nonzero finite R-module. Let $f_1, \ldots, f_r \in \mathfrak{m}$. The following are equivalent*

1. *$f_1, \ldots, f_r$ is an M-regular sequence,*
2. *$f_1, \ldots, f_r$ is a M-Koszul-regular sequence,*
3. *$f_1, \ldots, f_r$ is an M-H_1-regular sequence,*
4. *$f_1, \ldots, f_r$ is an M-quasi-regular sequence.*

In particular the sequence $f_1, \ldots, f_r$ is a regular sequence in R if and only if it is a Koszul regular sequence, if and only if it is a H_1-regular sequence, if and only if it is a quasi-regular sequence.

Proof. The implication (1) $\Rightarrow$ (2) is Lemma 30.2. The implication (2) $\Rightarrow$ (3) is Lemma 30.3. The implication (3) $\Rightarrow$ (4) is Lemma 30.6. The implication (4) $\Rightarrow$ (1) is Algebra, Lemma 69.6. $\square$

Lemma 30.8. *Let A be a ring. Let $I \subset A$ be an ideal. Let $g_1, \ldots, g_m$ be a sequence in A whose image in A/I is H_1-regular. Then $I \cap (g_1, \ldots, g_m) = I(g_1, \ldots, g_m)$.*

Proof. Consider the exact sequence of complexes

$$0 \to I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A/I, g_1, \ldots, g_m) \to 0$$

Since the complex on the right has $H_1 = 0$ by assumption we see that

$$\text{Coker}(I^{\oplus m} \to I) \longrightarrow \text{Coker}(A^{\oplus m} \to A)$$

is injective. This is equivalent to the assertion of the lemma. $\square$

Lemma 30.9. *Let A be a ring. Let $I \subset J \subset A$ be ideals. Assume that $J/I \subset A/I$ is generated by an H_1-regular sequence. Then $I \cap J^2 = IJ$.*

Proof. To prove this choose $g_1, \ldots, g_m \in J$ whose images in A/I form a H_1-regular sequence which generates J/I. In particular $J = I + (g_1, \ldots, g_m)$. Suppose that $x \in I \cap J^2$. Because $x \in J^2$ can write

$$x = \sum a_{ij} g_i g_j + \sum a_j g_j + a$$

with $a_{ij} \in A$, $a_j \in I$ and $a \in I^2$. Then $\sum a_{ij} g_i g_j \in I \cap (g_1, \ldots, g_m)$ hence by Lemma 30.8 we see that $\sum a_{ij} g_i g_j \in I(g_1, \ldots, g_m)$. Thus $x \in IJ$ as desired. $\square$

Lemma 30.10. *Let A be a ring. Let I be an ideal generated by a quasi-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a quasi-regular sequence in A.*

Proof. We claim that $g_1, \ldots, g_m$ forms an H_1-regular sequence in A/I^d for every d. By induction assume that this holds in A/I^{d-1}. We have a short exact sequence of complexes

$$0 \to K_\bullet(A, g_\bullet) \otimes_A I^{d-1}/I^d \to K_\bullet(A/I^d, g_\bullet) \to K_\bullet(A/I^{d-1}, g_\bullet) \to 0$$

Since $f_1, \ldots, f_n$ is quasi-regular we see that the first complex is a direct sum of copies of $K_\bullet(A/I, g_1, \ldots, g_m)$ hence acyclic in degree 1. By induction hypothesis the last complex is acyclic in degree 1. Hence also the middle complex is. In particular, the sequence $g_1, \ldots, g_m$ forms a quasi-regular sequence in A/I^d for every $d \geq 1$, see Lemma 30.6. Now we are ready to prove that $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a quasi-regular sequence in A. Namely, set $J = (f_1, \ldots, f_n, g_1, \ldots, g_m)$ and suppose that (with multinomial notation)

$$\sum\nolimits_{|N|+|M|=d} a_{N,M} f^N g^M \in J^{d+1}$$

for some $a_{N,M} \in A$. We have to show that $a_{N,M} \in J$ for all N, M. Let $e \in \{0, 1, \ldots, d\}$. Then

$$\sum\nolimits_{|N|=d-e,\ |M|=e} a_{N,M} f^N g^M \in (g_1, \ldots, g_m)^{e+1} + I^{d-e+1}$$

Because $g_1, \ldots, g_m$ is a quasi-regular sequence in A/I^{d-e+1} we deduce

$$\sum\nolimits_{|N|=d-e} a_{N,M} f^N \in (g_1, \ldots, g_m) + I^{d-e+1}$$

for each M with $|M| = e$. By Lemma 30.8 applied to I^{d-e}/I^{d-e+1} in the ring A/I^{d-e+1} this implies $\sum_{|N|=d-e} a_{N,M} f^N \in I^{d-e}(g_1, \ldots, g_m)$. Since $f_1, \ldots, f_n$ is quasi-regular in A this implies that $a_{N,M} \in J$ for each N, M with $|N| = d - e$ and $|M| = e$. This proves the lemma. $\square$

Lemma 30.11. *Let A be a ring. Let I be an ideal generated by an H_1-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is an H_1-regular sequence in A.*

Proof. We have to show that $H_1(A, f_1, \ldots, f_n, g_1, \ldots, g_m) = 0$. To do this consider the commutative diagram

$$\begin{tikzcd}
\wedge^2(A^{\oplus n+m}) \arrow[r] \arrow[d] & A^{\oplus n+m} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
\wedge^2(A/I^{\oplus m}) \arrow[r] & A/I^{\oplus m} \arrow[r] & A/I \arrow[r] & 0
\end{tikzcd}$$

Consider an element $(a_1, \ldots, a_{n+m}) \in A^{\oplus n+m}$ which maps to zero in A. Because $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I we see that $(\overline{a}_{n+1}, \ldots, \overline{a}_{n+m})$ is the image of some element $\overline{\alpha}$ of $\wedge^2(A/I^{\oplus m})$. We can lift $\overline{\alpha}$ to an element $\alpha \in \wedge^2(A^{\oplus n+m})$ and substract the image of it in $A^{\oplus n+m}$ from our element $(a_1, \ldots, a_{n+m})$. Thus we may assume that $a_{n+1}, \ldots, a_{n+m} \in I$. Since $I = (f_1, \ldots, f_n)$ we can modify our element $(a_1, \ldots, a_{n+m})$ by linear combinations of the elements

$$(0, \ldots, g_j, 0, \ldots, 0, f_i, 0, \ldots, 0)$$

in the image of the top left horizontal arrow to reduce to the case that $a_{n+1}, \ldots, a_{n+m}$ are zero. In this case $(a_1, \ldots, a_n, 0, \ldots, 0)$ defines an element of $H_1(A, f_1, \ldots, f_n)$ which we assumed to be zero. $\square$

Lemma 30.12. *Let A be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$ be an H_1-regular sequence. Then the images $\overline{g}_1, \ldots, \overline{g}_m$ in $A/(f_1, \ldots, f_n)$ form an H_1-regular sequence.*

Proof. Set $I = (f_1, \ldots, f_n)$. We have to show that any relation $\sum_{j=1,\ldots,m} \overline{a}_j\overline{g}_j$ in A/I is a linear combination of trivial relations. Because $I = (f_1, \ldots, f_n)$ we can lift this relation to a relation

$$\sum\nolimits_{j=1,\ldots,m} a_jg_j + \sum\nolimits_{i=1,\ldots,n} b_if_i = 0$$

in A. By assumption this relation in A is a linear combination of trivial relations. Taking the image in A/I we obtain what we want. □

Lemma 30.13. *Let A be a ring. Let I be an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form a Koszul-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a Koszul-regular sequence in A.*

Proof. Our assumptions say that $K_\bullet(A, f_1, \ldots, f_n)$ is a finite free resolution of A/I and $K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m)$ is a finite free resolution of $A/(f_i, g_j)$ over A/I. Then

$$\begin{aligned} K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) &= \text{Tot}(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &\cong A/I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \\ &= K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m) \\ &\cong A/(f_i, g_j) \end{aligned}$$

The first equality by Lemma 28.12. The first quasi-isomorphism $\cong$ by (the dual of) Homology, Lemma 25.4 as the qth row of the double complex $K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)$ is a resolution of $A/I \otimes_A K_q(A, g_1, \ldots, g_m)$. The second equality is clear. The last quasi-isomorphism by assumption. Hence we win. □

To conclude in the following lemma it is necessary to assume that both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular. A counter example to dropping the assumption that $f_1, \ldots, f_n$ is Koszul-regular is Examples, Lemma 14.1.

Lemma 30.14. *Let A be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$. If both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular sequences in A, then $\overline{g}_1, \ldots, \overline{g}_m$ in $A/(f_1, \ldots, f_n)$ form a Koszul-regular sequence.*

Proof. Set $I = (f_1, \ldots, f_n)$. Our assumptions say that $K_\bullet(A, f_1, \ldots, f_n)$ is a finite free resolution of A/I and $K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m)$ is a finite free resolution of $A/(f_i, g_j)$ over A. Then

$$\begin{aligned} A/(f_i, g_j) &\cong K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) \\ &= \text{Tot}(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &\cong A/I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \\ &= K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m) \end{aligned}$$

The first quasi-isomorphism $\cong$ by assumption. The first equality by Lemma 28.12. The second quasi-isomorphism by (the dual of) Homology, Lemma 25.4 as the qth row of the double complex $K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)$ is a resolution of $A/I \otimes_A K_q(A, g_1, \ldots, g_m)$. The second equality is clear. Hence we win. □

Lemma 30.15. *Let R be a ring. Let I be an ideal generated by $f_1, \ldots, f_r \in R$.*

1. *If I can be generated by a quasi-regular sequence of length r, then $f_1, \ldots, f_r$ is a quasi-regular sequence.*

(2) *If I can be generated by an H_1-regular sequence of length r, then $f_1, \ldots, f_r$ is an H_1-regular sequence.*
(3) *If I can be generated by a Koszul-regular sequence of length r, then $f_1, \ldots, f_r$ is a Koszul-regular sequence.*

Proof. If I can be generated by a quasi-regular sequence of length r, then I/I^2 is free of rank r over R/I. Since $f_1, \ldots, f_r$ generate by assumption we see that the images $\overline{f}_i$ form a basis of I/I^2 over R/I. It follows that $f_1, \ldots, f_r$ is a quasi-regular sequence as all this means, besides the freeness of I/I^2, is that the maps $\text{Sym}^n_{R/I}(I/I^2) \to I^n/I^{n+1}$ are isomorphisms.

We continue to assume that I can be generated by a quasi-regular sequence, say $g_1, \ldots, g_r$. Write $g_j = \sum a_{ij} f_i$. As $f_1, \ldots, f_r$ is quasi-regular according to the previous paragraph, we see that $\det(a_{ij})$ is invertible mod I. The matrix a_{ij} gives a map $R^{\oplus r} \to R^{\oplus r}$ which induces a map of Koszul complexes $\alpha : K_\bullet(R, f_1, \ldots, f_r) \to K_\bullet(R, g_1, \ldots, g_r)$, see Lemma 28.3. This map becomes an isomorphism on inverting $\det(a_{ij})$. Since the cohomology modules of both $K_\bullet(R, f_1, \ldots, f_r)$ and $K_\bullet(R, g_1, \ldots, g_r)$ are annihilated by I, see Lemma 28.6, we see that α is a quasi-isomorphism.

Now assume that $g_1, \ldots, g_r$ is a H_1-regular sequence generating I. Then $g_1, \ldots, g_r$ is a quasi-regular sequence by Lemma 30.6. By the previous paragraph we conclude that $f_1, \ldots, f_r$ is a H_1-regular sequence. Similarly for Koszul-regular sequences. □

Lemma 30.16. *Let R be a ring. Let $a_1, \ldots, a_n \in R$ be elements such that $R \to R^{\oplus n}$, $x \mapsto (xa_1, \ldots, xa_n)$ is injective. Then the element $\sum a_i t_i$ of the polynomial ring $R[t_1, \ldots, t_n]$ is a nonzerodivisor.*

Proof. If one of the a_i is a unit this is just the statement that any element of the form $t_1 + a_2 t_2 + \ldots + a_n t_n$ is a nonzerodivisor in the polynomial ring over R.

Case I: R is Noetherian. Let $\mathfrak{q}_j$, $j = 1, \ldots, m$ be the associated primes of R. We have to show that each of the maps

$$\sum a_i t_i : \text{Sym}^d(R^{\oplus n}) \longrightarrow \text{Sym}^{d+1}(R^{\oplus n})$$

is injective. As $\text{Sym}^d(R^{\oplus n})$ is a free R-module its associated primes are $\mathfrak{q}_j$, $j = 1, \ldots, m$. For each j there exists an $i = i(j)$ such that $a_i \notin \mathfrak{q}_j$ because there exists an $x \in R$ with $\mathfrak{q}_j x = 0$ but $a_i x \neq 0$ for some i by assumption. Hence a_i is a unit in $R_{\mathfrak{q}_j}$ and the map is injective after localizing at $\mathfrak{q}_j$. Thus the map is injective, see Algebra, Lemma 63.19.

Case II: R general. We can write R as the union of Noetherian rings R_λ with $a_1, \ldots, a_n \in R_\lambda$. For each R_λ the result holds, hence the result holds for R. □

Lemma 30.17. *Let R be a ring. Let $f_1, \ldots, f_n$ be a Koszul-regular sequence in R such that $(f_1, \ldots, f_n) \neq R$. Consider the faithfully flat, smooth ring map*

$$R \longrightarrow S = R[\{t_{ij}\}_{i \leq j}, t_{11}^{-1}, t_{22}^{-1}, \ldots, t_{nn}^{-1}]$$

For $1 \leq i \leq n$ set

$$g_i = \sum\nolimits_{i \leq j} t_{ij} f_j \in S.$$

Then $g_1, \ldots, g_n$ is a regular sequence in S and $(f_1, \ldots, f_n)S = (g_1, \ldots, g_n)$.

Proof. The equality of ideals is obvious as the matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & \ldots \\ 0 & t_{22} & t_{23} & \ldots \\ 0 & 0 & t_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

is invertible in S. Because $f_1, \ldots, f_n$ is a Koszul-regular sequence we see that the kernel of $R \to R^{\oplus n}$, $x \mapsto (xf_1, \ldots, xf_n)$ is zero (as it computes the $n$the Koszul homology of R w.r.t. $f_1, \ldots, f_n$). Hence by Lemma 30.16 we see that $g_1 = f_1 t_{11} + \ldots + f_n t_{1n}$ is a nonzerodivisor in $S' = R[t_{11}, t_{12}, \ldots, t_{1n}, t_{11}^{-1}]$. We see that $g_1, f_2, \ldots, f_n$ is a Koszul-sequence in S' by Lemma 30.5 and 30.15. We conclude that $\overline{f}_2, \ldots, \overline{f}_n$ is a Koszul-regular sequence in $S'/(g_1)$ by Lemma 30.14. Hence by induction on n we see that the images $\overline{g}_2, \ldots, \overline{g}_n$ of $g_2, \ldots, g_n$ in $S'/(g_1)[\{t_{ij}\}_{2 \leq i \leq j}, t_{22}^{-1}, \ldots, t_{nn}^{-1}]$ form a regular sequence. This in turn means that $g_1, \ldots, g_n$ forms a regular sequence in S. □

31. More on Koszul regular sequences

We continue the discussion from Section 30.

Lemma 31.1. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be an Koszul-regular sequence. Then the extended alternating Čech complex $R \to \bigoplus_{i_0} R_{f_{i_0}} \to \bigoplus_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r}$ from Section 29 only has cohomology in degree r.*

Proof. By Lemma 30.4 and induction the sequence $f_1, \ldots, f_{r-1}, f_r^n$ is Koszul regular for all $n \geq 1$. By Lemma 28.4 any permutation of a Koszul regular sequence is a Koszul regular sequence. Hence we see that we may replace any (or all) f_i by its nth power and still have a Koszul regular sequence. Thus $K_\bullet(R, f_1^n, \ldots, f_r^n)$ has nonzero cohomology only in homological degree 0. This implies what we want by Lemma 29.6. □

Lemma 31.2. *Let $a, a_2, \ldots, a_r$ be an H_1-regular sequence in a ring R (for example a Koszul regular sequence or a regular sequence, see Lemmas 30.2 and 30.3). With $I = (a, a_2, \ldots, a_r)$ the blowup algebra $R' = R[\frac{I}{a}]$ is isomorphic to $R'' = R[y_2, \ldots, y_r]/(ay_i - a_i)$.*

Proof. By Algebra, Lemma 70.6 it suffices to show that R'' is a-torsion free.

We claim $a, ay_2 - a_2, \ldots, ay_n - a_r$ is a H_1-regular sequence in $R[y_2, \ldots, y_r]$. Namely, the map

$$(a, ay_2 - a_2, \ldots, ay_n - a_r) : R[y_2, \ldots, y_r]^{\oplus r} \longrightarrow R[y_2, \ldots, y_r]$$

used to define the Koszul complex on $a, ay_2 - a_2, \ldots, ay_n - a_r$ is isomorphic to the map

$$(a, a_2, \ldots, a_r) : R[y_2, \ldots, y_r]^{\oplus r} \longrightarrow R[y_2, \ldots, y_r]$$

used to the define the Koszul complex on $a, a_2, \ldots, a_r$ via the isomorphism

$$R[y_2, \ldots, y_r]^{\oplus r} \longrightarrow R[y_2, \ldots, y_r]^{\oplus r}$$

sending $(b_1, \ldots, b_r)$ to $(b_1 - b_2 y_2 \ldots - b_r y_r, -b_2, \ldots, -b_r)$. By Lemma 28.3 these Koszul complexes are isomorphic. By Lemma 30.5 applied to the flat ring map $R \to R[y_2, \ldots, y_r]$ we conclude our claim is true. By Lemma 28.8 we see that the Koszul complex K on $a, ay_2 - a_2, \ldots, ay_n - a_r$ is the cone on $a : L \to L$ where L is the Koszul complex on $ay_2 - a_2, \ldots, ay_n - a_r$. Since $H_1(K) = 0$ by the claim, we conclude that $a : H_0(L) \to H_0(L)$ is injective, in other words that $R'' = R[y_2, \ldots, y_r]/(ay_i - a_i)$ has no nonzero a-torsion elements as desired. $\square$

Lemma 31.3. *Let $A \to B$ be a ring map. Let $f_1, \ldots, f_r$ be a sequence in B such that $B/(f_1, \ldots, f_r)$ is A-flat. Let $A \to A'$ be a ring map. Then the canonical map*

$$H_1(K_\bullet(B, f_1, \ldots, f_r)) \otimes_A A' \longrightarrow H_1(K_\bullet(B', f'_1, \ldots, f'_r))$$

is surjective. Here $B' = B \otimes_A A'$ and $f'_i \in B'$ is the image of f_i.

Proof. The sequence

$$\wedge^2(B^{\oplus r}) \to B^{\oplus r} \to B \to B/J \to 0$$

is a complex of A-modules with B/J flat over A and cohomology group $H_1 = H_1(K_\bullet(B, f_1, \ldots, f_r))$ in the spot $B^{\oplus r}$. If we tensor this with A' we obtain a complex

$$\wedge^2((B')^{\oplus r}) \to (B')^{\oplus r} \to B' \to B'/J' \to 0$$

which is exact at B' and B'/J'. In order to compute its cohomology group $H'_1 = H_1(K_\bullet(B', f'_1, \ldots, f'_r))$ at $(B')^{\oplus r}$ we split the first sequence above into the exact sequences $0 \to J \to B \to B/J \to 0$, $0 \to K \to B^{\oplus r} \to J \to 0$, and $\wedge^2(B^{\oplus r}) \to K \to H_1 \to 0$. Tensoring over A with A' we obtain the exact sequences

$$\begin{gathered} 0 \to J \otimes_A A' \to B \otimes_A A' \to (B/J) \otimes_A A' \to 0 \\ K \otimes_A A' \to B^{\oplus r} \otimes_A A' \to J \otimes_A A' \to 0 \\ \wedge^2(B^{\oplus r}) \otimes_A A' \to K \otimes_A A' \to H_1 \otimes_A A' \to 0 \end{gathered}$$

where the first one is exact as B/J is flat over A, see Algebra, Lemma 39.12. We conclude that $J' = J \otimes_A A' \subset B'$ and that $K \otimes_A A' \to \mathrm{Ker}((B')^{\oplus r} \to B')$ is surjective. Thus

$$\begin{aligned} H_1 \otimes_A A' &= \mathrm{Coker}\left(\wedge^2(B^{\oplus r}) \otimes_A A' \to K \otimes_A A'\right) \\ &\to \mathrm{Coker}\left(\wedge^2((B')^{\oplus r}) \to \mathrm{Ker}((B')^{\oplus r} \to B')\right) = H'_1 \end{aligned}$$

is surjective too. $\square$

Lemma 31.4. *Let $A \to B$ and $A \to A'$ be ring maps. Set $B' = B \otimes_A A'$. Let $f_1, \ldots, f_r \in B$. Assume $B/(f_1, \ldots, f_r)B$ is flat over A*

1. *If $f_1, \ldots, f_r$ is a quasi-regular sequence, then the image in B' is a quasi-regular sequence.*
2. *If $f_1, \ldots, f_r$ is a H_1-regular sequence, then the image in B' is a H_1-regular sequence.*

Proof. Assume $f_1, \ldots, f_r$ is quasi-regular. Set $J = (f_1, \ldots, f_r)$. By assumption J^n/J^{n+1} is isomorphic to a direct sum of copies of B/J hence flat over A. By induction and Algebra, Lemma 39.13 we conclude that B/J^n is flat over A. The ideal $(J')^n$ is equal to $J^n \otimes_A A'$, see Algebra, Lemma 39.12. Hence $(J')^n/(J')^{n+1} = J^n/J^{n+1} \otimes_A A'$ which clearly implies that $f_1, \ldots, f_r$ is a quasi-regular sequence in B'.

Assume $f_1, \ldots, f_r$ is H_1-regular. By Lemma 31.3 the vanishing of the Koszul homology group $H_1(K_\bullet(B, f_1, \ldots, f_r))$ implies the vanishing of $H_1(K_\bullet(B', f'_1, \ldots, f'_r))$ and we win. $\square$

Lemma 31.5. *Let $A' \to B'$ be a ring map. Let $I \subset A'$ be an ideal. Set $A = A'/I$ and $B = B'/IB'$. Let $f'_1, \ldots, f'_r \in B'$. Assume*

(1) *$A' \to B'$ is flat and of finite presentation,*
(2) *I is locally nilpotent,*
(3) *the images $f_1, \ldots, f_r \in B$ form a quasi-regular sequence,*
(4) *$B/(f_1, \ldots, f_r)$ is flat over A.*

Then $B'/(f'_1, \ldots, f'_r)$ is flat over A'.

Proof. Set $C' = B'/(f'_1, \ldots, f'_r)$. We have to show $A' \to C'$ is flat. Let $\mathfrak{r}' \subset C'$ be a prime ideal lying over $\mathfrak{p}' \subset A'$. We let $\mathfrak{q}' \subset B'$ be the inverse image of $\mathfrak{r}'$. By Algebra, Lemma 39.18 it suffices to show that $A'_{\mathfrak{p}'} \to C'_{\mathfrak{q}'}$ is flat. Algebra, Lemma 128.6 tells us it suffices to show that $f'_1, \ldots, f'_r$ map to a regular sequence in

$$B'_{\mathfrak{q}'}/\mathfrak{p}'B'_{\mathfrak{q}'} = B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} = (B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{q}}$$

with obvious notation. What we know is that $f_1, \ldots, f_r$ is a quasi-regular sequence in B and that $B/(f_1, \ldots, f_r)$ is flat over A. By Lemma 31.4 the images $\overline{f}_1, \ldots, \overline{f}_r$ of $f'_1, \ldots, f'_r$ in $B \otimes_A \kappa(\mathfrak{p})$ form a quasi-regular sequence. Since $(B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{q}}$ is a Noetherian local ring, we conclude by Lemma 30.7. ☐

Lemma 31.6. *Let $A' \to B'$ be a ring map. Let $I \subset A'$ be an ideal. Set $A = A'/I$ and $B = B'/IB'$. Let $f'_1, \ldots, f'_r \in B'$. Assume*

(1) *$A' \to B'$ is flat and of finite presentation (for example smooth),*
(2) *I is locally nilpotent,*
(3) *the images $f_1, \ldots, f_r \in B$ form a quasi-regular sequence,*
(4) *$B/(f_1, \ldots, f_r)$ is smooth over A.*

Then $B'/(f'_1, \ldots, f'_r)$ is smooth over A'.

Proof. Set $C' = B'/(f'_1, \ldots, f'_r)$ and $C = B/(f_1, \ldots, f_r)$. Then $A' \to C'$ is of finite presentation. By Lemma 31.5 we see that $A' \to C'$ is flat. The fibre rings of $A' \to C'$ are equal to the fibre rings of $A \to C$ and hence smooth by assumption (4). It follows that $A' \to C'$ is smooth by Algebra, Lemma 137.17. ☐

32. Regular ideals

We will discuss the notion of a regular ideal sheaf in great generality in Divisors, Section 20. Here we define the corresponding notion in the affine case, i.e., in the case of an ideal in a ring.

Definition 32.1. Let R be a ring and let $I \subset R$ be an ideal.

(1) We say I is a *regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(2) We say I is a *Koszul-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a Koszul-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(3) We say I is a *H_1-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and an H_1-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(4) We say I is a *quasi-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a quasi-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.

It is clear that given $I \subset R$ we have the implications

$$\begin{aligned} I \text{ is a regular ideal} &\Rightarrow I \text{ is a Koszul-regular ideal} \\ &\Rightarrow I \text{ is a } H_1\text{-regular ideal} \\ &\Rightarrow I \text{ is a quasi-regular ideal} \end{aligned}$$

see Lemmas 30.2, 30.3, and 30.6. Such an ideal is always finitely generated.

Lemma 32.2. *A quasi-regular ideal is finitely generated.*

Proof. Let $I \subset R$ be a quasi-regular ideal. Since $V(I)$ is quasi-compact, there exist $g_1, \ldots, g_m \in R$ such that $V(I) \subset D(g_1) \cup \ldots \cup D(g_m)$ and such that I_{g_j} is generated by a quasi-regular sequence $g_{j1}, \ldots, g_{jr_j} \in R_{g_j}$. Write $g_{ji} = g'_{ji}/g_j^{e_{ij}}$ for some $g'_{ij} \in I$. Write $1 + x = \sum g_j h_j$ for some $x \in I$ which is possible as $V(I) \subset D(g_1) \cup \ldots \cup D(g_m)$. Note that $\text{Spec}(R) = D(g_1) \cup \ldots \cup D(g_m) \bigcup D(x)$ Then I is generated by the elements g'_{ij} and x as these generate on each of the pieces of the cover, see Algebra, Lemma 23.2. □

Lemma 32.3. *Let $I \subset R$ be a quasi-regular ideal of a ring. Then I/I^2 is a finite projective R/I-module.*

Proof. This follows from Algebra, Lemma 78.2 and the definitions. □

We prove flat descent for Koszul-regular, H_1-regular, quasi-regular ideals.

Lemma 32.4. *Let $A \to B$ be a faithfully flat ring map. Let $I \subset A$ be an ideal. If IB is a Koszul-regular (resp. H_1-regular, resp. quasi-regular) ideal in B, then I is a Koszul-regular (resp. H_1-regular, resp. quasi-regular) ideal in A.*

Proof. We fix the prime $\mathfrak{p} \supset I$ throughout the proof. Assume IB is quasi-regular. By Lemma 32.2 IB is a finite module, hence I is a finite A-module by Algebra, Lemma 83.2. As $A \to B$ is flat we see that

$$I/I^2 \otimes_{A/I} B/IB = I/I^2 \otimes_A B = IB/(IB)^2.$$

As IB is quasi-regular, the B/IB-module $IB/(IB)^2$ is finite locally free. Hence I/I^2 is finite projective, see Algebra, Proposition 83.3. In particular, after replacing A by A_f for some $f \in A$, $f \notin \mathfrak{p}$ we may assume that I/I^2 is free of rank r. Pick $f_1, \ldots, f_r \in I$ which give a basis of I/I^2. By Nakayama's lemma (see Algebra, Lemma 20.1) we see that, after another replacement $A \leadsto A_f$ as above, I is generated by $f_1, \ldots, f_r$.

Proof of the "quasi-regular" case. Above we have seen that I/I^2 is free on the r-generators $f_1, \ldots, f_r$. To finish the proof in this case we have to show that the maps $\text{Sym}^d(I/I^2) \to I^d/I^{d+1}$ are isomorphisms for each $d \geq 2$. This is clear as the faithfully flat base changes $\text{Sym}^d(IB/(IB)^2) \to (IB)^d/(IB)^{d+1}$ are isomorphisms locally on B by assumption. Details omitted.

Proof of the "H_1-regular" and "Koszul-regular" case. Consider the sequence of elements $f_1, \ldots, f_r$ generating I we constructed above. By Lemma 30.15 we see that $f_1, \ldots, f_r$ map to a H_1-regular or Koszul-regular sequence in B_g for any $g \in B$ such that IB is generated by an H_1-regular or Koszul-regular sequence. Hence $K_\bullet(A, f_1, \ldots, f_r) \otimes_A B_g$ has vanishing H_1 or H_i, $i > 0$. Since the homology of $K_\bullet(B, f_1, \ldots, f_r) = K_\bullet(A, f_1, \ldots, f_r) \otimes_A B$ is annihilated by IB (see Lemma 28.6) and since $V(IB) \subset \bigcup_{g \text{ as above}} D(g)$ we conclude that $K_\bullet(A, f_1, \ldots, f_r) \otimes_A B$ has vanishing homology in degree 1 or all positive degrees. Using that $A \to B$ is faithfully flat we conclude that the same is true for $K_\bullet(A, f_1, \ldots, f_r)$. $\square$

Lemma 32.5. *Let A be a ring. Let $I \subset J \subset A$ be ideals. Assume that $J/I \subset A/I$ is a H_1-regular ideal. Then $I \cap J^2 = IJ$.*

Proof. Follows immediately from Lemma 30.9 by localizing. $\square$

33. Local complete intersection maps

We can use the material above to define a local complete intersection map between rings using presentations by (finite) polynomial algebras.

Lemma 33.1. *Let $A \to B$ be a finite type ring map. If for some presentation $\alpha : A[x_1, \ldots, x_n] \to B$ the kernel I is a Koszul-regular ideal then for any presentation $\beta : A[y_1, \ldots, y_m] \to B$ the kernel J is a Koszul-regular ideal.*

Proof. Choose $f_j \in A[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in A[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{tikzcd}
A[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & A[x_1, \ldots, x_n] \arrow[d] \\
A[y_1, \ldots, y_m] \arrow[r] & B
\end{tikzcd}$$

Note that the kernel K of $A[x_i, y_j] \to B$ is equal to $K = (I, y_j - f_j) = (J, x_i - f_i)$. In particular, as I is finitely generated by Lemma 32.2 we see that $J = K/(x_i - f_i)$ is finitely generated too.

Pick a prime $\mathfrak{q} \subset B$. Since $I/I^2 \oplus B^{\oplus m} = J/J^2 \oplus B^{\oplus n}$ (Algebra, Lemma 134.15) we see that

$$\dim J/J^2 \otimes_B \kappa(\mathfrak{q}) + n = \dim I/I^2 \otimes_B \kappa(\mathfrak{q}) + m.$$

Pick $p_1, \ldots, p_t \in I$ which map to a basis of $I/I^2 \otimes \kappa(\mathfrak{q}) = I \otimes_{A[x_i]} \kappa(\mathfrak{q})$. Pick $q_1, \ldots, q_s \in J$ which map to a basis of $J/J^2 \otimes \kappa(\mathfrak{q}) = J \otimes_{A[y_j]} \kappa(\mathfrak{q})$. So $s + n = t + m$. By Nakayama's lemma there exist $h \in A[x_i]$ and $h' \in A[y_j]$ both mapping to a nonzero element of $\kappa(\mathfrak{q})$ such that $I_h = (p_1, \ldots, p_t)$ in $A[x_i, 1/h]$ and $J_{h'} = (q_1, \ldots, q_s)$ in $A[y_j, 1/h']$. As I is Koszul-regular we may also assume that I_h is generated by a Koszul regular sequence. This sequence must necessarily have length $t = \dim I/I^2 \otimes_B \kappa(\mathfrak{q})$, hence we see that $p_1, \ldots, p_t$ is a Koszul-regular sequence by Lemma 30.15. As also $y_1 - f_1, \ldots, y_m - f_m$ is a regular sequence we conclude

$$y_1 - f_1, \ldots, y_m - f_m, p_1, \ldots, p_t$$

is a Koszul-regular sequence in $A[x_i, y_j, 1/h]$ (see Lemma 30.13). This sequence generates the ideal K_h. Hence the ideal $K_{hh'}$ is generated by a Koszul-regular sequence of length $m + t = n + s$. But it is also generated by the sequence

$$x_1 - g_1, \ldots, x_n - g_n, q_1, \ldots, q_s$$

of the same length which is thus a Koszul-regular sequence by Lemma 30.15. Finally, by Lemma 30.14 we conclude that the images of $q_1, \ldots, q_s$ in

$$A[x_i, y_j, 1/hh']/(x_1 - g_1, \ldots, x_n - g_n) \cong A[y_j, 1/h'']$$

form a Koszul-regular sequence generating $J_{h''}$. Since h'' is the image of hh' it doesn't map to zero in $\kappa(\mathfrak{q})$ and we win. $\square$

This lemma allows us to make the following definition.

Definition 33.2. A ring map $A \to B$ is called a *local complete intersection* if it is of finite type and for some (equivalently any) presentation $B = A[x_1, \ldots, x_n]/I$ the ideal I is Koszul-regular.

This notion is local.

Lemma 33.3. *Let $R \to S$ be a ring map. Let $g_1, \ldots, g_m \in S$ generate the unit ideal. If each $R \to S_{g_j}$ is a local complete intersection so is $R \to S$.*

Proof. Let $S = R[x_1, \ldots, x_n]/I$ be a presentation. Pick $h_j \in R[x_1, \ldots, x_n]$ mapping to g_j in S. Then $R[x_1, \ldots, x_n, x_{n+1}]/(I, x_{n+1}h_j - 1)$ is a presentation of S_{g_j}. Hence $I_j = (I, x_{n+1}h_j - 1)$ is a Koszul-regular ideal in $R[x_1, \ldots, x_n, x_{n+1}]$. Pick a prime $I \subset \mathfrak q \subset R[x_1, \ldots, x_n]$. Then $h_j \not\in \mathfrak q$ for some j and $\mathfrak q_j = (\mathfrak q, x_{n+1}h_j - 1)$ is a prime ideal of $V(I_j)$ lying over $\mathfrak q$. Pick $f_1, \ldots, f_r \in I$ which map to a basis of $I/I^2 \otimes \kappa(\mathfrak q)$. Then $x_{n+1}h_j - 1, f_1, \ldots, f_r$ is a sequence of elements of I_j which map to a basis of $I_j \otimes \kappa(\mathfrak q_j)$. By Nakayama's lemma there exists an $h \in R[x_1, \ldots, x_n, x_{n+1}]$ such that $(I_j)_h$ is generated by $x_{n+1}h_j - 1, f_1, \ldots, f_r$. We may also assume that $(I_j)_h$ is generated by a Koszul regular sequence of some length e. Looking at the dimension of $I_j \otimes \kappa(\mathfrak q_j)$ we see that $e = r + 1$. Hence by Lemma 30.15 we see that $x_{n+1}h_j - 1, f_1, \ldots, f_r$ is a Koszul-regular sequence generating $(I_j)_h$ for some $h \in R[x_1, \ldots, x_n, x_{n+1}]$, $h \not\in \mathfrak q_j$. By Lemma 30.14 we see that $I_{h'}$ is generated by a Koszul-regular sequence for some $h' \in R[x_1, \ldots, x_n]$, $h' \not\in \mathfrak q$ as desired. □

Lemma 33.4. *Let R be a ring. If $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection, then $f_1, \ldots, f_c$ is a Koszul regular sequence.*

Proof. Recall that the homology groups $H_i(K_\bullet(f_\bullet))$ are annihilated by the ideal $(f_1, \ldots, f_c)$. Hence it suffices to show that $H_i(K_\bullet(f_\bullet))_\mathfrak q$ is zero for all primes $\mathfrak q \subset R[x_1, \ldots, x_n]$ containing $(f_1, \ldots, f_c)$. This follows from Algebra, Lemma 136.12 and the fact that a regular sequence is Koszul regular (Lemma 30.2). □

Lemma 33.5. *Let $R \to S$ be a ring map. The following are equivalent*

1. *$R \to S$ is syntomic (Algebra, Definition 136.1), and*
2. *$R \to S$ is flat and a local complete intersection.*

Proof. Assume (1). Then $R \to S$ is flat by definition. By Algebra, Lemma 136.15 and Lemma 33.3 we see that it suffices to show a relative global complete intersection is a local complete intersection homomorphism which is Lemma 33.4.

Assume (2). A local complete intersection is of finite presentation because a Koszul-regular ideal is finitely generated. Let $R \to k$ be a map to a field. It suffices to show that $S' = S \otimes_R k$ is a local complete intersection over k, see Algebra, Definition 135.1. Choose a prime $\mathfrak q' \subset S'$. Write $S = R[x_1, \ldots, x_n]/I$. Then $S' = k[x_1, \ldots, x_n]/I'$ where $I' \subset k[x_1, \ldots, x_n]$ is the image of I. Let $\mathfrak p' \subset k[x_1, \ldots, x_n]$, $\mathfrak q \subset S$, and $\mathfrak p \subset R[x_1, \ldots, x_n]$ be the corresponding primes. By Definition 32.1 exists an $g \in R[x_1, \ldots, x_n]$, $g \not\in \mathfrak p$ and $f_1, \ldots, f_r \in R[x_1, \ldots, x_n]_g$ which form a Koszul-regular sequence generating I_g. Since S and hence S_g is flat over R we see that the images $f'_1, \ldots, f'_r$ in $k[x_1, \ldots, x_n]_g$ form a H_1-regular sequence generating I'_g, see Lemma 31.4. Thus $f'_1, \ldots, f'_r$ map to a regular sequence in $k[x_1, \ldots, x_n]_{\mathfrak p'}$ generating $I'_{\mathfrak p'}$ by Lemma 30.7. Applying Algebra, Lemma 135.4 we conclude $S'_{gg'}$ for some $g' \in S$, $g' \not\in \mathfrak q'$ is a global complete intersection over k as desired. □

For a local complete intersection $R \to S$ we have $H_n(L_{S/R}) = 0$ for $n \geq 2$. Since we haven't (yet) defined the full cotangent complex we can't state and prove this, but we can deduce one of the consequences.

Lemma 33.6. *Let $A \to B \to C$ be ring maps. Assume $B \to C$ is a local complete intersection homomorphism. Choose a presentation $\alpha : A[x_s, s \in S] \to B$ with kernel I. Choose a presentation $\beta : B[y_1, \ldots, y_m] \to C$ with kernel J. Let $\gamma : A[x_s, y_t] \to C$ be the induced presentation of C with kernel K. Then we get a canonical commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \Omega_{A[x_s]/A} \otimes C \arrow[r] & \Omega_{A[x_s,y_t]/A} \otimes C \arrow[r] & \Omega_{B[y_t]/B} \otimes C \arrow[r] & 0 \\ 0 \arrow[r] & I/I^2 \otimes C \arrow[r] \arrow[u] & K/K^2 \arrow[r] \arrow[u] & J/J^2 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. In particular, the six term exact sequence of Algebra, Lemma 134.4 can be completed with a zero on the left, i.e., the sequence

$$0 \to H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

is exact.

Proof. The only thing to prove is the injectivity of the map $I/I^2 \otimes C \to K/K^2$. By assumption the ideal J is Koszul-regular. Hence we have $IA[x_s, y_j] \cap K^2 = IK$ by Lemma 32.5. This means that the kernel of $K/K^2 \to J/J^2$ is isomorphic to $IA[x_s, y_j]/IK$. Since $I/I^2 \otimes_A C = IA[x_s, y_j]/IK$ by right exactness of tensor product, this provides us with the desired injectivity of $I/I^2 \otimes_A C \to K/K^2$. $\square$

Lemma 33.7. *Let $A \to B \to C$ be ring maps. If $B \to C$ is a filtered colimit of local complete intersection homomorphisms then the conclusion of Lemma 33.6 remains valid.*

Proof. Follows from Lemma 33.6 and Algebra, Lemma 134.9. $\square$

Lemma 33.8. *Let $A \to B$ be a local homomorphism of local rings. Let $A^h \to B^h$, resp. $A^{sh} \to B^{sh}$ be the induced map on henselizations, resp. strict henselizations (Algebra, Lemma 155.6, resp. Lemma 155.10). Then $NL_{B/A} \otimes_B B^h \to NL_{B^h/A^h}$ and $NL_{B/A} \otimes_B B^{sh} \to NL_{B^{sh}/A^{sh}}$ induce isomorphisms on cohomology groups.*

Proof. Since A^h is a filtered colimit of étale algebras over A we see that $NL_{A^h/A}$ is an acyclic complex by Algebra, Lemma 134.9 and Algebra, Definition 143.1. The same is true for B^h/B. Using the Jacobi-Zariski sequence (Algebra, Lemma 134.4) for $A \to A^h \to B^h$ we find that $NL_{B^h/A} \to NL_{B^h/A^h}$ induces isomorphisms on cohomology groups. Moreover, an étale ring map is a local complete intersection as it is even a global complete intersection, see Algebra, Lemma 143.2. By Lemma 33.7 we get a six term exact Jacobi-Zariski sequence associated to $A \to B \to B^h$ which proves that $NL_{B/A} \otimes_B B^h \to NL_{B^h/A}$ induces isomorphisms on cohomology groups. This finishes the proof in the case of the map on henselizations. The case of strict henselization is proved in exactly the same manner. $\square$

34. Cartier's equality and geometric regularity

A reference for this section and the next is [Mat70, Section 39]. In order to comfortably read this section the reader should be familiar with the naive cotangent complex and its properties, see Algebra, Section 134.

Lemma 34.1 (Cartier equality)**.** *Let K/k be a finitely generated field extension. Then $\Omega_{K/k}$ and $H_1(L_{K/k})$ are finite dimensional and $trdeg_k(K) = \dim_K \Omega_{K/k} - \dim_K H_1(L_{K/k})$.*

Proof. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over k such that K is isomorphic to the fraction field of A, see Algebra, Lemma 158.11 and its proof. In this case we see that $NL_{K/k}$ is homotopy equivalent to the complex

$$\bigoplus\nolimits_{j=1,\ldots,c} K \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} K \mathrm{d}x_i$$

by Algebra, Lemmas 134.2 and 134.13. The transcendence degree of K over k is the dimension of A (by Algebra, Lemma 116.1) which is $n - c$ and we win. $\square$

Lemma 34.2. *Let $M/L/K$ be field extensions. Then the Jacobi-Zariski sequence*

$$0 \to H_1(L_{L/K}) \otimes_L M \to H_1(L_{M/K}) \to H_1(L_{M/L}) \to \Omega_{L/K} \otimes_L M \to \Omega_{M/K} \to \Omega_{M/L} \to 0$$

is exact.

Proof. Combine Lemma 33.7 with Algebra, Lemma 158.11. $\square$

Lemma 34.3. *Given a commutative diagram of fields*

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

with k'/k and K'/K finitely generated field extensions the kernel and cokernel of the maps

$$\alpha : \Omega_{K/k} \otimes_K K' \to \Omega_{K'/k'} \quad \text{and} \quad \beta : H_1(L_{K/k}) \otimes_K K' \to H_1(L_{K'/k'})$$

are finite dimensional and

$$\dim \mathrm{Ker}(\alpha) - \dim \mathrm{Coker}(\alpha) - \dim \mathrm{Ker}(\beta) + \dim \mathrm{Coker}(\beta) = trdeg_k(k') - trdeg_K(K')$$

Proof. The Jacobi-Zariski sequences for $k \subset k' \subset K'$ and $k \subset K \subset K'$ are

$$0 \to H_1(L_{k'/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/k'}) \to \Omega_{k'/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/k'} \to 0$$

and

$$0 \to H_1(L_{K/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/K}) \to \Omega_{K/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/K} \to 0$$

By Lemma 34.1 the vector spaces $\Omega_{k'/k}$, $\Omega_{K'/K}$, $H_1(L_{K'/K})$, and $H_1(L_{k'/k})$ are finite dimensional and the alternating sum of their dimensions is $\mathrm{trdeg}_k(k') - \mathrm{trdeg}_K(K')$. The lemma follows. $\square$

35. Geometric regularity

Let k be a field. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. The Jacobi-Zariski sequence (Algebra, Lemma 134.4) is a canonical exact sequence

$$H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A K \to \Omega_{K/k} \to 0$$

because $H_1(L_{K/A}) = \mathfrak{m}/\mathfrak{m}^2$ by Algebra, Lemma 134.6. We will show that exactness on the left of this sequence characterizes whether or not a regular local ring A is geometrically regular over k. We will link this to the notion of formal smoothness in Section 40.

Proposition 35.1. *Let k be a field of characteristic $p > 0$. Let $(A, \mathfrak m, K)$ be a Noetherian local k-algebra. The following are equivalent*

1. *A is geometrically regular over k,*
2. *for all $k \subset k' \subset k^{1/p}$ finite over k the ring $A \otimes_k k'$ is regular,*
3. *A is regular and the canonical map $H_1(L_{K/k}) \to \mathfrak m/\mathfrak m^2$ is injective, and*
4. *A is regular and the map $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ is injective.*

Proof. Proof of (3) $\Rightarrow$ (1). Assume (3). Let k'/k be a finite purely inseparable extension. Set $A' = A \otimes_k k'$. This is a local ring with maximal ideal $\mathfrak m'$. Set $K' = A'/\mathfrak m'$. We get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_1(L_{K/k}) \otimes K' \arrow[r] \arrow[d, "\beta"'] & \mathfrak m/\mathfrak m^2 \otimes K' \arrow[r] \arrow[d] & \Omega_{A/k} \otimes_A K' \arrow[r] \arrow[d, "\cong"'] & \Omega_{K/k} \otimes K' \arrow[r] \arrow[d, "\alpha"'] & 0 \\
 & H_1(L_{K'/k'}) \arrow[r] & \mathfrak m'/(\mathfrak m')^2 \arrow[r] & \Omega_{A'/k'} \otimes_{A'} K' \arrow[r] & \Omega_{K'/k'} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The third vertical arrow is an isomorphism by base change for modules of differentials (Algebra, Lemma 131.12). Thus α is surjective. By Lemma 34.3 we have

$$\dim \mathrm{Ker}(\alpha) - \dim \mathrm{Ker}(\beta) + \dim \mathrm{Coker}(\beta) = 0$$

(and these dimensions are all finite). A diagram chase shows that $\dim \mathfrak m'/(\mathfrak m')^2 \leq \dim \mathfrak m/\mathfrak m^2$. However, since $A \to A'$ is finite flat we see that $\dim(A) = \dim(A')$, see Algebra, Lemma 112.6. Hence A' is regular by definition.

Equivalence of (3) and (4). Consider the Jacobi-Zariski sequences for rows of the commutative diagram

$$\begin{tikzcd}
\mathbf{F}_p \arrow[r] & A \arrow[r] & K \\
\mathbf{F}_p \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & K \arrow[u]
\end{tikzcd}$$

to get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak m/\mathfrak m^2 \arrow[r] & \Omega_{A/\mathbf{F}_p} \otimes_A K \arrow[r] & \Omega_{K/\mathbf{F}_p} \arrow[r] & 0 \\
0 \arrow[r] & H_1(L_{K/k}) \arrow[r] \arrow[u] & \Omega_{k/\mathbf{F}_p} \otimes_k K \arrow[r] \arrow[u] & \Omega_{K/\mathbf{F}_p} \arrow[r] \arrow[u] & \Omega_{K/k} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with exact rows. We have used that $H_1(L_{K/A}) = \mathfrak m/\mathfrak m^2$ and that $H_1(L_{K/\mathbf{F}_p}) = 0$ as $K/\mathbf{F}_p$ is separable, see Algebra, Proposition 158.9. Thus it is clear that the kernels of $H_1(L_{K/k}) \to \mathfrak m/\mathfrak m^2$ and $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ have the same dimension.

Proof of (2) $\Rightarrow$ (4) following Faltings, see [Fal78]. Let $a_1, \ldots, a_n \in k$ be elements such that $\mathrm{d}a_1, \ldots, \mathrm{d}a_n$ are linearly independent in $\Omega_{k/\mathbf{F}_p}$. Consider the field extension $k' = k(a_1^{1/p}, \ldots, a_n^{1/p})$. By Algebra, Lemma 158.3 we see that $k' = k[x_1, \ldots, x_n]/(x_1^p - a_1, \ldots, x_n^p - a_n)$. In particular we see that the naive cotangent complex of k'/k is homotopic to the complex $\bigoplus_{j=1,\ldots,n} k' \to \bigoplus_{i=1,\ldots,n} k'$ with the zero differential as $\mathrm{d}(x_j^p - a_j) = 0$ in $\Omega_{k[x_1,\ldots,x_n]/k}$. Set $A' = A \otimes_k k'$ and $K' = A'/\mathfrak m'$ as above. By Algebra, Lemma 134.8 we see that $NL_{A'/A}$ is homotopy equivalent to the complex $\bigoplus_{j=1,\ldots,n} A' \to \bigoplus_{i=1,\ldots,n} A'$ with the zero differential, i.e., $H_1(L_{A'/A})$ and $\Omega_{A'/A}$ are free of rank n. The Jacobi-Zariski sequence for $\mathbf{F}_p \to A \to A'$ is

$$H_1(L_{A'/A}) \to \Omega_{A/\mathbf{F}_p} \otimes_A A' \to \Omega_{A'/\mathbf{F}_p} \to \Omega_{A'/A} \to 0$$

Using the presentation $A[x_1, \ldots, x_n] \to A'$ with kernel $(x_j^p - a_j)$ we see, unwinding the maps in Algebra, Lemma 134.4, that the jth basis vector of $H_1(L_{A'/A})$ maps to $\mathrm{d}a_j \otimes 1$ in $\Omega_{A/\mathbf{F}_p} \otimes A'$. As $\Omega_{A'/A}$ is free (hence flat) we get on tensoring with K' an exact sequence

$$K'^{\oplus n} \to \Omega_{A/\mathbf{F}_p} \otimes_A K' \xrightarrow{\beta} \Omega_{A'/\mathbf{F}_p} \otimes_{A'} K' \to K'^{\oplus n} \to 0$$

We conclude that the elements $\mathrm{d}a_j \otimes 1$ generate $\mathrm{Ker}(\beta)$ and we have to show that are linearly independent, i.e., we have to show $\dim(\mathrm{Ker}(\beta)) = n$. Consider the following big diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}'/(\mathfrak{m}')^2 \arrow[r] & \Omega_{A'/\mathbf{F}_p} \otimes K' \arrow[r] & \Omega_{K'/\mathbf{F}_p} \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \otimes K' \arrow[r] \arrow[u, "\alpha"] & \Omega_{A/\mathbf{F}_p} \otimes K' \arrow[r] \arrow[u, "\beta"] & \Omega_{K/\mathbf{F}_p} \otimes K' \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

By Lemma 34.1 and the Jacobi-Zariski sequence for $\mathbf{F}_p \to K \to K'$ we see that the kernel and cokernel of γ have the same finite dimension. By assumption A' is regular (and of the same dimension as A, see above) hence the kernel and cokernel of α have the same dimension. It follows that the kernel and cokernel of β have the same dimension which is what we wanted to show.

The implication (1) $\Rightarrow$ (2) is trivial. This finishes the proof of the proposition. $\square$

Lemma 35.2. *Let k be a field of characteristic $p > 0$. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. Assume A is geometrically regular over k. Let $K/F/k$ be a finitely generated subextension. Let $\varphi : k[y_1, \ldots, y_m] \to A$ be a k-algebra map such that y_i maps to an element of F in K and such that $dy_1, \ldots, dy_m$ map to a basis of $\Omega_{F/k}$. Set $\mathfrak{p} = \varphi^{-1}(\mathfrak{m})$. Then*

$$k[y_1, \ldots, y_m]_{\mathfrak{p}} \to A$$

is flat and $A/\mathfrak{p}A$ is regular.

Proof. Set $A_0 = k[y_1, \ldots, y_m]_{\mathfrak{p}}$ with maximal ideal $\mathfrak{m}_0$ and residue field K_0. Note that $\Omega_{A_0/k}$ is free of rank m and $\Omega_{A_0/k} \otimes K_0 \to \Omega_{K_0/k}$ is an isomorphism. It is clear that A_0 is geometrically regular over k. Hence $H_1(L_{K_0/k}) \to \mathfrak{m}_0/\mathfrak{m}_0^2$ is an isomorphism, see Proposition 35.1. Now consider

$$\begin{tikzcd}
H_1(L_{K_0/k}) \otimes K \arrow[r] \arrow[d] & \mathfrak{m}_0/\mathfrak{m}_0^2 \otimes K \arrow[d] \\
H_1(L_{K/k}) \arrow[r] & \mathfrak{m}/\mathfrak{m}^2
\end{tikzcd}$$

Since the left vertical arrow is injective by Lemma 34.2 and the lower horizontal by Proposition 35.1 we conclude that the right vertical one is too. Hence a regular system of parameters in A_0 maps to part of a regular system of parameters in A. We win by Algebra, Lemmas 128.2 and 106.3. $\square$

36. Topological rings and modules

Let's quickly discuss some properties of topological abelian groups. An abelian group M is a *topological abelian group* if M is endowed with a topology such that addition $M \times M \to M$, $(x, y) \mapsto x+y$ and inverse $M \to M$, $x \mapsto -x$ are continuous. A *homomorphism of topological abelian groups* is just a homomorphism of abelian groups which is continuous. The category of commutative topological groups is additive and has kernels and cokernels, but is not abelian (as the axiom $\mathrm{Im} = \mathrm{Coim}$ doesn't hold). If $N \subset M$ is a subgroup, then we think of N and M/N as topological groups also, namely using the induced topology on N and the quotient topology on M/N (i.e., such that $M \to M/N$ is submersive). Note that if $N \subset M$ is an open subgroup, then the topology on M/N is discrete.

We say the topology on M is *linear* if there exists a fundamental system of neighbourhoods of 0 consisting of subgroups. If so then these subgroups are also open. An example is the following. Let I be a directed set and let G_i be an inverse system of (discrete) abelian groups over I. Then

$$G = \lim_{i \in I} G_i$$

with the inverse limit topology is linearly topologized with a fundamental system of neighbourhoods of 0 given by $\mathrm{Ker}(G \to G_i)$. Conversely, let M be a linearly topologized abelian group. Choose any fundamental system of open subgroups $U_i \subset M$, $i \in I$ (i.e., the U_i form a fundamental system of open neighbourhoods and each U_i is a subgroup of M). Setting $i \geq i' \Leftrightarrow U_i \subset U_{i'}$ we see that I is a directed set. We obtain a homomorphism of linearly topologized abelian groups

$$c : M \longrightarrow \lim_{i \in I} M/U_i.$$

It is clear that M is *separated* (as a topological space) if and only if c is injective. We say that M is *complete* if c is an isomorphism[2]. We leave it to the reader to check that this condition is independent of the choice of fundamental system of open subgroups $\{U_i\}_{i \in I}$ chosen above. In fact the topological abelian group $M^\wedge = \lim_{i \in I} M/U_i$ is independent of this choice and is sometimes called the *completion* of M. Any $G = \lim G_i$ as above is complete, in particular, the completion $M^\wedge$ is always complete.

Definition 36.1 (Topological rings)**.** Let R be a ring and let M be an R-module.

(1) We say R is a *topological ring* if R is endowed with a topology such that both addition and multiplication are continuous as maps $R \times R \to R$ where $R \times R$ has the product topology. In this case we say M is a *topological module* if M is endowed with a topology such that addition $M \times M \to M$ and scalar multiplication $R \times M \to M$ are continuous.
(2) A *homomorphism of topological modules* is just a continuous R-module map. A *homomorphism of topological rings* is a ring homomorphism which is continuous for the given topologies.
(3) We say M is *linearly topologized* if 0 has a fundamental system of neighbourhoods consisting of submodules. We say R is *linearly topologized* if 0 has a fundamental system of neighbourhoods consisting of ideals.

[2]We include being separated as part of being complete as we'd like to have a unique limits in complete groups. There is a definition of completeness for any topological group, agreeing, modulo the separation issue, with this one in our special case.

(4) If R is linearly topologized, we say that $I \subset R$ is an *ideal of definition* if I is open and if every neighbourhood of 0 contains I^n for some n.
(5) If R is linearly topologized, we say that R is *pre-admissible* if R has an ideal of definition.
(6) If R is linearly topologized, we say that R is *admissible* if it is pre-admissible and complete[3].
(7) If R is linearly topologized, we say that R is *pre-adic* if there exists an ideal of definition I such that $\{I^n\}_{n\geq 0}$ forms a fundamental system of neighbourhoods of 0.
(8) If R is linearly topologized, we say that R is *adic* if R is pre-adic and complete.

Note that a (pre)adic topological ring is the same thing as a (pre)admissible topological ring which has an ideal of definition I such that I^n is open for all $n \geq 1$.

Let R be a ring and let M be an R-module. Let $I \subset R$ be an ideal. Then we can consider the linear topology on R which has $\{I^n\}_{n\geq 0}$ as a fundamental system of neighbourhoods of 0. This topology is called the I-adic topology; R is a pre-adic topological ring in the I-adic topology[4]. Moreover, the linear topology on M which has $\{I^nM\}_{n\geq 0}$ as a fundamental system of open neighbourhoods of 0 turns M into a topological R-module. This is called the *I-adic topology* on M. We see that M is I-adically complete (as defined in Algebra, Definition 96.2) if and only if M is complete in the I-adic topology[5]. In particular, we see that R is I-adically complete if and only if R is an adic topological ring in the I-adic topology.

As a special case, note that the discrete topology is the 0-adic topology and that any ring in the discrete topology is adic.

Lemma 36.2. *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ and $J \subset S$ be ideals and endow R with the I-adic topology and S with the J-adic topology. Then φ is a homomorphism of topological rings if and only if $\varphi(I^n) \subset J$ for some $n \geq 1$.*

Proof. Omitted. $\square$

Lemma 36.3 (Baire category theorem)**.** *Let M be a topological abelian group. Assume M is linearly topologized, complete, and has a countable fundamental system of neighbourhoods of 0. If $U_n \subset M$, $n \geq 1$ are open dense subsets, then $\bigcap_{n\geq 1} U_n$ is dense.*

Proof. Let U_n be as in the statement of the lemma. After replacing U_n by $U_1 \cap \ldots \cap U_n$, we may assume that $U_1 \supset U_2 \supset \ldots$. Let M_n, $n \in \mathbf{N}$ be a fundamental system of neighbourhoods of 0. We may assume that $M_{n+1} \subset M_n$. Pick $x \in M$. We will show that for every $k \geq 1$ there exists a $y \in \bigcap_{n\geq 1} U_n$ with $x - y \in M_k$.

To construct y we argue as follows. First, we pick a $y_1 \in U_1$ with $y_1 \in x + M_k$. This is possible because U_1 is dense and $x + M_k$ is open. Then we pick a $k_1 > k$ such that $y_1 + M_{k_1} \subset U_1$. This is possible because U_1 is open. Next, we pick a $y_2 \in U_2$ with $y_2 \in y_1 + M_{k_1}$. This is possible because U_2 is dense and $y_2 + M_{k_1}$ is

[3]By our conventions this includes separated.

[4]Thus the I-adic topology is sometimes called the I-pre-adic topology.

[5]It may happen that the I-adic completion $M^\wedge$ is not I-adically complete, even though $M^\wedge$ is always complete with respect to the limit topology. If I is finitely generated then the I-adic topology and the limit topology on $M^\wedge$ agree, see Algebra, Lemma 96.3 and its proof.

open. Then we pick a $k_2 > k_1$ such that $y_2 + M_{k_2} \subset U_2$. This is possible because U_2 is open.

Continuing in this fashion we get a converging sequence y_i of elements of M with limit y. By construction $x - y \in M_k$. Since

$$y - y_i = (y_{i+1} - y_i) + (y_{i+2} - y_{i+1}) + \ldots$$

is in M_{k_i} we see that $y \in y_i + M_{k_i} \subset U_i$ for all i as desired. $\square$

Lemma 36.4. *With same assumptions as Lemma 36.3 if $M = \bigcup_{n \geq 1} N_n$ for some closed subgroups N_n, then N_n is open for some n.*

Proof. If not, then $U_n = M \setminus N_n$ is dense for all n and we get a contradiction with Lemma 36.3. $\square$

Lemma 36.5 (Open mapping lemma)**.** *Let $u : N \to M$ be a continuous map of linearly topologized abelian groups. Assume that N is complete, M separated, and N has a countable fundamental system of neighbourhoods of* 0*. Then exactly one of the following holds*

1. *u is open, or*
2. *for some open subgroup $N' \subset N$ the image $u(N')$ is nowhere dense in M.*

Proof. Let N_n, $n \in \mathbf{N}$ be a fundamental system of neighbourhoods of 0. We may assume that $N_{n+1} \subset N_n$. If (2) does not hold, then the closure M_n of $u(N_n)$ is an open subgroup for $n = 1, 2, 3, \ldots$. Since u is continuous, we see that M_n, $n \in \mathbf{N}$ must be a fundamental system of open neighbourhoods of 0 in M. Also, since M_n is the closure of $u(N_n)$ we see that

$$u(N_n) + M_{n+1} = M_n$$

for all $n \geq 1$. Pick $x_1 \in M_1$. Then we can inductively choose $y_i \in N_i$ and $x_{i+1} \in M_{i+1}$ such that

$$u(y_i) + x_{i+1} = x_i$$

The element $y = y_1 + y_2 + y_3 + \ldots$ of N exists because N is complete. Whereupon we see that $x = u(y)$ because M is separated. Thus $M_1 = u(N_1)$. In exactly the same way the reader shows that $M_i = u(N_i)$ for all $i \geq 2$ and we see that u is open. $\square$

37. Formally smooth maps of topological rings

There is a version of formal smoothness which applies to homomorphisms of topological rings.

Definition 37.1. Let $R \to S$ be a homomorphism of topological rings with R and S linearly topologized. We say S is *formally smooth over R* if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

of homomorphisms of topological rings where A is a discrete ring and $J \subset A$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

We will mostly use this notion when given ideals $\mathfrak{m} \subset R$ and $\mathfrak{n} \subset S$ and we endow R with the $\mathfrak{m}$-adic topology and S with the $\mathfrak{n}$-adic topology. Continuity of $\varphi : R \to S$ holds if and only if $\varphi(\mathfrak{m}^m) \subset \mathfrak{n}$ for some $m \geq 1$, see Lemma 36.2. It turns out that in this case only the topology on S is relevant.

Lemma 37.2. *Let $\varphi : R \to S$ be a ring map.*

(1) *If $R \to S$ is formally smooth in the sense of Algebra, Definition 138.1, then $R \to S$ is formally smooth for any linear topology on R and any pre-adic topology on S such that $R \to S$ is continuous.*
(2) *Let $\mathfrak{n} \subset S$ and $\mathfrak{m} \subset R$ ideals such that φ is continuous for the $\mathfrak{m}$-adic topology on R and the $\mathfrak{n}$-adic topology on S. Then the following are equivalent*
    (a) *φ is formally smooth for the $\mathfrak{m}$-adic topology on R and the $\mathfrak{n}$-adic topology on S, and*
    (b) *φ is formally smooth for the discrete topology on R and the $\mathfrak{n}$-adic topology on S.*

Proof. Assume $R \to S$ is formally smooth in the sense of Algebra, Definition 138.1. If S has a pre-adic topology, then there exists an ideal $\mathfrak{n} \subset S$ such that S has the $\mathfrak{n}$-adic topology. Suppose given a solid commutative diagram as in Definition 37.1. Continuity of $S \to A/J$ means that $\mathfrak{n}^k$ maps to zero in A/J for some $k \geq 1$, see Lemma 36.2. We obtain a ring map $\psi : S \to A$ from the assumed formal smoothness of S over R. Then $\psi(\mathfrak{n}^k) \subset J$ hence $\psi(\mathfrak{n}^{2k}) = 0$ as $J^2 = 0$. Hence ψ is continuous by Lemma 36.2. This proves (1).

The proof of (2)(b) $\Rightarrow$ (2)(a) is the same as the proof of (1). Assume (2)(a). Suppose given a solid commutative diagram as in Definition 37.1 where we use the discrete topology on R. Since φ is continuous we see that $\varphi(\mathfrak{m}^n) \subset \mathfrak{n}$ for some $n \geq 1$. As $S \to A/J$ is continuous we see that $\mathfrak{n}^k$ maps to zero in A/J for some $k \geq 1$. Hence $\mathfrak{m}^{nk}$ maps into J under the map $R \to A$. Thus $\mathfrak{m}^{2nk}$ maps to zero in A and we see that $R \to A$ is continuous in the $\mathfrak{m}$-adic topology. Thus (2)(a) gives a dotted arrow as desired. $\square$

Definition 37.3. Let $R \to S$ be a ring map. Let $\mathfrak{n} \subset S$ be an ideal. If the equivalent conditions (2)(a) and (2)(b) of Lemma 37.2 hold, then we say $R \to S$ is *formally smooth for the $\mathfrak{n}$-adic topology.*

This property is inherited by the completions.

Lemma 37.4. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be rings endowed with finitely generated ideals. Endow R and S with the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic topologies. Let $R \to S$ be a homomorphism of topological rings. The following are equivalent*

(1) *$R \to S$ is formally smooth for the $\mathfrak{n}$-adic topology,*
(2) *$R \to S^\wedge$ is formally smooth for the $\mathfrak{n}^\wedge$-adic topology,*
(3) *$R^\wedge \to S^\wedge$ is formally smooth for the $\mathfrak{n}^\wedge$-adic topology.*

Here $R^\wedge$ and $S^\wedge$ are the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic completions of R and S.

Proof. The assumption that $\mathfrak{m}$ is finitely generated implies that $R^\wedge$ is $\mathfrak{m}R^\wedge$-adically complete, that $\mathfrak{m}R^\wedge = \mathfrak{m}^\wedge$ and that $R^\wedge/\mathfrak{m}^n R^\wedge = R/\mathfrak{m}^n$, see Algebra, Lemma 96.3 and its proof. Similarly for $(S, \mathfrak{n})$. Thus it is clear that diagrams as in Definition 37.1 for the cases (1), (2), and (3) are in 1-to-1 correspondence. $\square$

The advantage of working with adic rings is that one gets a stronger lifting property.

Lemma 37.5. *Let $R \to S$ be a ring map. Let $\mathfrak{n}$ be an ideal of S. Assume that $R \to S$ is formally smooth in the $\mathfrak{n}$-adic topology. Consider a solid commutative diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed, "\psi"] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

of homomorphisms of topological rings where A is adic and A/J is the quotient (as topological ring) of A by a closed ideal $J \subset A$ such that J^t is contained in an ideal of definition of A for some $t \geq 1$. Then there exists a dotted arrow in the category of topological rings which makes the diagram commute.

Proof. Let $I \subset A$ be an ideal of definition so that $I \supset J^t$ for some n. Then $A = \lim A/I^n$ and $A/J = \lim A/J + I^n$ because J is assumed closed. Consider the following diagram of discrete R algebras $A_{n,m} = A/J^n + I^m$:

$$\begin{tikzcd} A/J^3 + I^3 \arrow[r] \arrow[d] & A/J^2 + I^3 \arrow[r] \arrow[d] & A/J + I^3 \arrow[d] \\ A/J^3 + I^2 \arrow[r] \arrow[d] & A/J^2 + I^2 \arrow[r] \arrow[d] & A/J + I^2 \arrow[d] \\ A/J^3 + I \arrow[r] & A/J^2 + I \arrow[r] & A/J + I \end{tikzcd}$$

Note that each of the commutative squares defines a surjection

$$A_{n+1,m+1} \longrightarrow A_{n+1,m} \times_{A_{n,m}} A_{n,m+1}$$

of R-algebras whose kernel has square zero. We will inductively construct R-algebra maps $\varphi_{n,m} : S \to A_{n,m}$. Namely, we have the maps $\varphi_{1,m} = \psi \bmod J + I^m$. Note that each of these maps is continuous as ψ is. We can inductively choose the maps $\varphi_{n,1}$ by starting with our choice of $\varphi_{1,1}$ and lifting up, using the formal smoothness of S over R, along the right column of the diagram above. We construct the remaining maps $\varphi_{n,m}$ by induction on $n + m$. Namely, we choose $\varphi_{n+1,m+1}$ by lifting the pair $(\varphi_{n+1,m}, \varphi_{n,m+1})$ along the displayed surjection above (again using the formal smoothness of S over R). In this way all of the maps $\varphi_{n,m}$ are compatible with the transition maps of the system. As $J^t \subset I$ we see that for example $\varphi_n = \varphi_{nt,n} \bmod I^n$ induces a map $S \to A/I^n$. Taking the limit $\varphi = \lim \varphi_n$ we obtain a map $S \to A = \lim A/I^n$. The composition into A/J agrees with ψ as we have seen that $A/J = \lim A/J + I^n$. Finally we show that φ is continuous. Namely, we know that $\psi(\mathfrak{n}^r) \subset J + I/J$ for some $r \geq 1$ by our assumption that ψ is a morphism of topological rings, see Lemma 36.2. Hence $\varphi(\mathfrak{n}^r) \subset J + I$ hence $\varphi(\mathfrak{n}^{rt}) \subset I$ as desired. ☐

Lemma 37.6. *Let $R \to S$ be a ring map. Let $\mathfrak{n} \subset \mathfrak{n}' \subset S$ be ideals. If $R \to S$ is formally smooth for the $\mathfrak{n}$-adic topology, then $R \to S$ is formally smooth for the $\mathfrak{n}'$-adic topology.*

Proof. Omitted. ☐

Lemma 37.7. *A composition of formally smooth continuous homomorphisms of linearly topologized rings is formally smooth.*

Proof. Omitted. (Hint: This is completely formal, and follows from considering a suitable diagram.) □

Lemma 37.8. *Let R, S be rings. Let $\mathfrak n \subset S$ be an ideal. Let $R \to S$ be formally smooth for the $\mathfrak n$-adic topology. Let $R \to R'$ be any ring map. Then $R' \to S' = S \otimes_R R'$ is formally smooth in the $\mathfrak n' = \mathfrak n S'$-adic topology.*

Proof. Let a solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rrd, dashed] & S' \arrow[r] \arrow[rd, dashed] & A/J \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Definition 37.1 be given. Then the composition $S \to S' \to A/J$ is continuous. By assumption the longer dotted arrow exists. By the universal property of tensor product we obtain the shorter dotted arrow. □

We have seen descent for formal smoothness along faithfully flat ring maps in Algebra, Lemma 138.16. Something similar holds in the current setting of topological rings. However, here we just prove the following very simple and easy to prove version which is already quite useful.

Lemma 37.9. *Let R, S be rings. Let $\mathfrak n \subset S$ be an ideal. Let $R \to R'$ be a ring map. Set $S' = S \otimes_R R'$ and $\mathfrak n' = \mathfrak n S$. If*

1. *the map $R \to R'$ embeds R as a direct summand of R' as an R-module, and*
2. *$R' \to S'$ is formally smooth for the $\mathfrak n'$-adic topology,*

then $R \to S$ is formally smooth in the $\mathfrak n$-adic topology.

Proof. Let a solid diagram

$$\begin{tikzcd} S \arrow[r] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Definition 37.1 be given. Set $A' = A \otimes_R R'$ and $J' = \mathrm{Im}(J \otimes_R R' \to A')$. The base change of the diagram above is the diagram

$$\begin{tikzcd} S' \arrow[r] \arrow[rd, dashed, "\psi'"] & A'/J' \\ R' \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

with continuous arrows. By condition (2) we obtain the dotted arrow $\psi' : S' \to A'$. Using condition (1) choose a direct summand decomposition $R' = R \oplus C$ as R-modules. (Warning: C isn't an ideal in R'.) Then $A' = A \oplus A \otimes_R C$. Set

$$J'' = \mathrm{Im}(J \otimes_R C \to A \otimes_R C) \subset J' \subset A'.$$

Then $J' = J \oplus J''$ as A-modules. The image of the composition $\psi : S \to A'$ of ψ' with $S \to S'$ is contained in $A + J' = A \oplus J''$. However, in the ring $A + J' = A \oplus J''$ the A-submodule J'' is an ideal! (Use that $J^2 = 0$.) Hence the composition $S \to A + J' \to (A + J')/J'' = A$ is the arrow we were looking for. □

38. Formally smooth maps of local rings

In the case of a local homomorphism of local rings one can limit the diagrams for which the lifting property has to be checked. Please compare with Algebra, Lemma 141.2.

Lemma 38.1. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of local rings. The following are equivalent*

(1) *$R \to S$ is formally smooth in the $\mathfrak{n}$-adic topology,*
(2) *for every solid commutative diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

of local homomorphisms of local rings where $J \subset A$ is an ideal of square zero, $\mathfrak{m}_A^n = 0$ for some $n > 0$, and $S \to A/J$ induces an isomorphism on residue fields, a dotted arrow exists which makes the diagram commute.

If S is Noetherian these conditions are also equivalent to

(3) *same as in (2) but only for diagrams where in addition $A \to A/J$ is a small extension (Algebra, Definition 141.1).*

Proof. The implication (1) $\Rightarrow$ (2) follows from the definitions. Consider a diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Definition 37.1 for the $\mathfrak{m}$-adic topology on R and the $\mathfrak{n}$-adic topology on S. Pick $m > 0$ with $\mathfrak{n}^m(A/J) = 0$ (possible by continuity of maps in diagram). Consider the subring A' of A which is the inverse image of the image of S in A/J. Set $J' = J$ viewed as an ideal in A'. Then J' is an ideal of square zero in A' and A'/J' is a quotient of $S/\mathfrak{n}^m$. Hence A' is local and $\mathfrak{m}_{A'}^{2m} = 0$. Thus we get a diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & A'/J' \\ R \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

as in (2). If we can construct the dotted arrow in this diagram, then we obtain the dotted arrow in the original one by composing with $A' \to A$. In this way we see that (2) implies (1).

Assume S Noetherian. The implication (1) $\Rightarrow$ (3) is immediate. Assume (3) and suppose a diagram as in (2) is given. Then $\mathfrak{m}_A^n J = 0$ for some $n > 0$. Considering the maps

$$A \to A/\mathfrak{m}_A^{n-1}J \to \ldots \to A/\mathfrak{m}J \to A/J$$

we see that it suffices to produce the lifting if $\mathfrak{m}_A J = 0$. Assume $\mathfrak{m}_A J = 0$ and let $A' \subset A$ be the ring constructed above. Then A'/J' is Artinian as a quotient of the Artinian local ring $S/\mathfrak{n}^m$. Thus it suffices to show that given property (3) we can find the dotted arrow in diagrams as in (2) with A/J Artinian and $\mathfrak{m}_A J = 0$.

Let κ be the common residue field of A, A/J, and S. By (3), if $J_0 \subset J$ is an ideal with $\dim_\kappa(J/J_0) = 1$, then we can produce a dotted arrow $S \to A/J_0$. Taking the product we obtain

$$S \longrightarrow \prod\nolimits_{J_0 \text{ as above}} A/J_0$$

Clearly the image of this arrow is contained in the sub R-algebra A' of elements which map into the small diagonal $A/J \subset \prod_{J_0} A/J$. Let $J' \subset A'$ be the elements mapping to zero in A/J. Then J' is an ideal of square zero and as κ-vector space equal to

$$J' = \prod\nolimits_{J_0 \text{ as above}} J/J_0$$

Thus the map $J \to J'$ is injective. By the theory of vector spaces we can choose a splitting $J' = J \oplus M$. It follows that

$$A' = A \oplus M$$

as an R-algebra. Hence the map $S \to A'$ can be composed with the projection $A' \to A$ to give the desired dotted arrow thereby finishing the proof of the lemma. $\square$

The following lemma will be improved on in Section 40.

Lemma 38.2. *Let k be a field and let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. If $k \to A$ is formally smooth for the $\mathfrak{m}$-adic topology, then A is a regular local ring.*

Proof. Let $k_0 \subset k$ be the prime field. Then k_0 is perfect, hence k/k_0 is separable, hence formally smooth by Algebra, Lemma 158.7. By Lemmas 37.2 and 37.7 we see that $k_0 \to A$ is formally smooth for the $\mathfrak{m}$-adic topology on A. Hence we may assume $k = \mathbf{Q}$ or $k = \mathbf{F}_p$.

By Algebra, Lemmas 97.3 and 110.9 it suffices to prove the completion $A^\wedge$ is regular. By Lemma 37.4 we may replace A by $A^\wedge$. Thus we may assume that A is a Noetherian complete local ring. By the Cohen structure theorem (Algebra, Theorem 160.8) there exist a map $K \to A$. As k is the prime field we see that $K \to A$ is a k-algebra map.

Let $x_1, \ldots, x_n \in \mathfrak{m}$ be elements whose images form a basis of $\mathfrak{m}/\mathfrak{m}^2$. Set $T = K[[X_1, \ldots, X_n]]$. Note that

$$A/\mathfrak{m}^2 \cong K[x_1, \ldots, x_n]/(x_i x_j)$$

and

$$T/\mathfrak{m}_T^2 \cong K[X_1, \ldots, X_n]/(X_i X_j).$$

Let $A/\mathfrak{m}^2 \to T/m_T^2$ be the local K-algebra isomorphism given by mapping the class of x_i to the class of X_i. Denote $f_1 : A \to T/\mathfrak{m}_T^2$ the composition of this isomorphism with the quotient map $A \to A/\mathfrak{m}^2$. The assumption that $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology means we can lift f_1 to a map $f_2 : A \to T/\mathfrak{m}_T^3$, then to a map $f_3 : A \to T/\mathfrak{m}_T^4$, and so on, for all $n \geq 1$. Warning: the maps f_n are continuous k-algebra maps and may not be K-algebra maps. We get an induced map $f : A \to T = \lim T/\mathfrak{m}_T^n$ of local k-algebras. By our choice of f_1, the map f induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_T/\mathfrak{m}_T^2$ hence each f_n is surjective and we conclude f is surjective as A is complete. This implies $\dim(A) \geq \dim(T) = n$. Hence A is regular by definition. (It also follows that f is an isomorphism.) $\square$

Lemma 38.3. *Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a complete local k-algebra. If κ/k is separable, then there exists a k-algebra map $\kappa \to A$ such that $\kappa \to A \to \kappa$ is id_κ.*

Proof. By Algebra, Proposition 158.9 the extension κ/k is formally smooth. By Lemma 37.2 $k \to \kappa$ is formally smooth in the sense of Definition 37.1. Then we get $\kappa \to A$ from Lemma 37.5. $\square$

Lemma 38.4. *Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a complete local k-algebra. If κ/k is separable and A regular, then there exists an isomorphism of $A \cong \kappa[[t_1, \ldots, t_d]]$ as k-algebras.*

Proof. Choose $\kappa \to A$ as in Lemma 38.3 and apply Algebra, Lemma 160.10. $\square$

The following result will be improved on in Section 40

Lemma 38.5. *Let k be a field. Let $(A, \mathfrak{m}, K)$ be a regular local k-algebra such that K/k is separable. Then $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*

Proof. It suffices to prove that the completion of A is formally smooth over k, see Lemma 37.4. Hence we may assume that A is a complete local regular k-algebra with residue field K separable over k. By Lemma 38.4 we see that $A = K[[x_1, \ldots, x_n]]$.

The power series ring $K[[x_1, \ldots, x_n]]$ is formally smooth over k. Namely, K is formally smooth over k and $K[x_1, \ldots, x_n]$ is formally smooth over K as a polynomial algebra. Hence $K[x_1, \ldots, x_n]$ is formally smooth over k by Algebra, Lemma 138.3. It follows that $k \to K[x_1, \ldots, x_n]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 37.2. Finally, it follows that $k \to K[[x_1, \ldots, x_n]]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 37.4. $\square$

Lemma 38.6. *Let $A \to B$ be a finite type ring map with A Noetherian. Let $\mathfrak{q} \subset B$ be a prime ideal lying over $\mathfrak{p} \subset A$. The following are equivalent*

1. *$A \to B$ is smooth at $\mathfrak{q}$, and*
2. *$A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth in the $\mathfrak{q}$-adic topology.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Algebra, Lemma 141.2. Conversely, if $A \to B$ is smooth at $\mathfrak{q}$, then $A \to B_g$ is smooth for some $g \in B$, $g \notin \mathfrak{q}$. Then $A \to B_g$ is formally smooth by Algebra, Proposition 138.13. Hence $A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth as localization preserves formal smoothness (for example by the criterion of Algebra, Proposition 138.8 and the fact that the cotangent complex behaves well with respect to localization, see Algebra, Lemmas 134.11 and 134.13). Finally, Lemma 37.2 implies that $A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth in the $\mathfrak{q}$-adic topology. $\square$

39. Some results on power series rings

Questions on formally smooth maps between Noetherian local rings can often be reduced to questions on maps between power series rings. In this section we prove some helper lemmas to facilitate this kind of argument.

Lemma 39.1. *Let K be a field of characteristic 0 and $A = K[[x_1, \ldots, x_n]]$. Let L be a field of characteristic $p > 0$ and $B = L[[x_1, \ldots, x_n]]$. Let Λ be a Cohen ring. Let $C = \Lambda[[x_1, \ldots, x_n]]$.*

1. *$\mathbf{Q} \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*
2. *$\mathbf{F}_p \to B$ is formally smooth in the $\mathfrak{m}$-adic topology.*
3. *$\mathbf{Z} \to C$ is formally smooth in the $\mathfrak{m}$-adic topology.*

Proof. By the universal property of power series rings it suffices to prove:

(1) $\mathbf{Q} \to K$ is formally smooth.
(2) $\mathbf{F}_p \to L$ is formally smooth.
(3) $\mathbf{Z} \to \Lambda$ is formally smooth in the $\mathfrak{m}$-adic topology.

The first two are Algebra, Proposition 158.9. The third follows from Algebra, Lemma 160.7 since for any test diagram as in Definition 37.1 some power of p will be zero in A/J and hence some power of p will be zero in A. $\square$

Lemma 39.2. *Let K be a field and $A = K[[x_1, \ldots, x_n]]$. Let Λ be a Cohen ring and let $B = \Lambda[[x_1, \ldots, x_n]]$.*
(1) *If $y_1, \ldots, y_n \in A$ is a regular system of parameters then $K[[y_1, \ldots, y_n]] \to A$ is an isomorphism.*
(2) *If $z_1, \ldots, z_r \in A$ form part of a regular system of parameters for A, then $r \leq n$ and $A/(z_1, \ldots, z_r) \cong K[[y_1, \ldots, y_{n-r}]]$.*
(3) *If $p, y_1, \ldots, y_n \in B$ is a regular system of parameters then $\Lambda[[y_1, \ldots, y_n]] \to B$ is an isomorphism.*
(4) *If $p, z_1, \ldots, z_r \in B$ form part of a regular system of parameters for B, then $r \leq n$ and $B/(z_1, \ldots, z_r) \cong \Lambda[[y_1, \ldots, y_{n-r}]]$.*

Proof. Proof of (1). Set $A' = K[[y_1, \ldots, y_n]]$. It is clear that the map $A' \to A$ induces an isomorphism $A'/\mathfrak{m}_{A'}^n \to A/\mathfrak{m}_A^n$ for all $n \geq 1$. Since A and A' are both complete we deduce that $A' \to A$ is an isomorphism. Proof of (2). Extend $z_1, \ldots, z_r$ to a regular system of parameters $z_1, \ldots, z_r, y_1, \ldots, y_{n-r}$ of A. Consider the map $A' = K[[z_1, \ldots, z_r, y_1, \ldots, y_{n-r}]] \to A$. This is an isomorphism by (1). Hence (2) follows as it is clear that $A'/(z_1, \ldots, z_r) \cong K[[y_1, \ldots, y_{n-r}]]$. The proofs of (3) and (4) are exactly the same as the proofs of (1) and (2). $\square$

Lemma 39.3. *Let $A \to B$ be a local homomorphism of Noetherian complete local rings. Then there exists a commutative diagram*

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

with the following properties:
(1) *the horizontal arrows are surjective,*
(2) *if the characteristic of $A/\mathfrak{m}_A$ is zero, then S and R are power series rings over fields,*
(3) *if the characteristic of $A/\mathfrak{m}_A$ is $p > 0$, then S and R are power series rings over Cohen rings, and*
(4) *$R \to S$ maps a regular system of parameters of R to part of a regular system of parameters of S.*

In particular $R \to S$ is flat (see Algebra, Lemma 128.2) with regular fibre $S/\mathfrak{m}_R S$ (see Algebra, Lemma 106.3).

Proof. Use the Cohen structure theorem (Algebra, Theorem 160.8) to choose a surjection $S \to B$ as in the statement of the lemma where we choose S to be a power series over a Cohen ring if the residue characteristic is $p > 0$ and a power series over a field else. Let $J \subset S$ be the kernel of $S \to B$. Next, choose a surjection $R = \Lambda[[x_1, \ldots, x_n]] \to A$ where we choose Λ to be a Cohen ring if the residue characteristic of A is $p > 0$ and Λ equal to the residue field of A otherwise.

We lift the composition $\Lambda[[x_1, \ldots, x_n]] \to A \to B$ to a map $\varphi : R \to S$. This is possible because $\Lambda[[x_1, \ldots, x_n]]$ is formally smooth over $\mathbf{Z}$ in the $\mathfrak{m}$-adic topology (see Lemma 39.1) by an application of Lemma 37.5. Finally, we replace φ by the map $\varphi' : R = \Lambda[[x_1, \ldots, x_n]] \to S' = S[[y_1, \ldots, y_n]]$ with $\varphi'|_\Lambda = \varphi|_\Lambda$ and $\varphi'(x_i) = \varphi(x_i) + y_i$. We also replace $S \to B$ by the map $S' \to B$ which maps y_i to zero. After this replacement it is clear that a regular system of parameters of R maps to part of a regular sequence in S' and we win. □

There should be an elementary proof of the following lemma.

Lemma 39.4. *Let $S \to R$ and $S' \to R$ be surjective maps of complete Noetherian local rings. Then $S \times_R S'$ is a complete Noetherian local ring.*

Proof. Let k be the residue field of R. If the characteristic of k is $p > 0$, then we denote Λ a Cohen ring (Algebra, Definition 160.5) with residue field k (Algebra, Lemma 160.6). If the characteristic of k is 0 we set $\Lambda = k$. Choose a surjection $\Lambda[[x_1, \ldots, x_n]] \to R$ (as in the Cohen structure theorem, see Algebra, Theorem 160.8) and lift this to maps $\Lambda[[x_1, \ldots, x_n]] \to S$ and $\varphi : \Lambda[[x_1, \ldots, x_n]] \to S$ and $\varphi' : \Lambda[[x_1, \ldots, x_n]] \to S'$ using Lemmas 39.1 and 37.5. Next, choose $f_1, \ldots, f_m \in S$ generating the kernel of $S \to R$ and $f'_1, \ldots, f'_{m'} \in S'$ generating the kernel of $S' \to R$. Then the map

$$\Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_{m'}]] \longrightarrow S \times_R S,$$

which sends x_i to $(\varphi(x_i), \varphi'(x_i))$ and y_j to $(f_j, 0)$ and $z_{j'}$ to $(0, f'_j)$ is surjective. Thus $S \times_R S'$ is a quotient of a complete local ring, whence complete. □

40. Geometric regularity and formal smoothness

In this section we combine the results of the previous sections to prove the following characterization of geometrically regular local rings over fields. We then recycle some of our arguments to prove a characterization of formally smooth maps in the $\mathfrak{m}$-adic topology between Noetherian local rings.

Theorem 40.1. *Let k be a field. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. If the characteristic of k is zero then the following are equivalent*

1. *A is a regular local ring, and*
2. *$k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*

If the characteristic of k is $p > 0$ then the following are equivalent

1. *A is geometrically regular over k,*
2. *$k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*
3. *for all $k \subset k' \subset k^{1/p}$ finite over k the ring $A \otimes_k k'$ is regular,*
4. *A is regular and the canonical map $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ is injective, and*
5. *A is regular and the map $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ is injective.*

Proof. If the characteristic of k is zero, then the equivalence of (1) and (2) follows from Lemmas 38.2 and 38.5.

If the characteristic of k is $p > 0$, then it follows from Proposition 35.1 that (1), (3), (4), and (5) are equivalent. Assume (2) holds. By Lemma 37.8 we see that $k' \to A' = A \otimes_k k'$ is formally smooth for the $\mathfrak{m}' = \mathfrak{m}A'$-adic topology. Hence if $k \subset k'$ is finite purely inseparable, then A' is a regular local ring by Lemma 38.2. Thus we see that (1) holds.

Finally, we will prove that (5) implies (2). Choose a solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed, "\bar\psi"] & B/J \\ k \arrow[u, "i"] \arrow[r, "\varphi"] & B \arrow[u, "\pi"'] \end{tikzcd}$$

as in Definition 37.1. As $J^2 = 0$ we see that J has a canonical B/J module structure and via $\bar\psi$ an A-module structure. As $\bar\psi$ is continuous for the $\mathfrak m$-adic topology we see that $\mathfrak m^n J = 0$ for some n. Hence we can filter J by B/J-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient J_{t+1}/J_t is annihilated by $\mathfrak m$. Considering the sequence of ring maps $B \to B/J_1 \to B/J_2 \to \ldots \to B/J$ we see that it suffices to prove the existence of the dotted arrow when J is annihilated by $\mathfrak m$, i.e., when J is a K-vector space.

Assume given a diagram as above such that J is annihilated by $\mathfrak m$. By Lemma 38.5 we see that $\mathbf{F}_p \to A$ is formally smooth in the $\mathfrak m$-adic topology. Hence we can find a ring map $\psi : A \to B$ such that $\pi \circ \psi = \bar\psi$. Then $\psi \circ i, \varphi : k \to B$ are two maps whose compositions with π are equal. Hence $D = \psi \circ i - \varphi : k \to J$ is a derivation. By Algebra, Lemma 131.3 we can write $D = \xi \circ \mathrm{d}$ for some k-linear map $\xi : \Omega_{k/\mathbf{F}_p} \to J$. Using the K-vector space structure on J we extend ξ to a K-linear map $\xi' : \Omega_{k/\mathbf{F}_p} \otimes_k K \to J$. Using (5) we can find a K-linear map $\xi'' : \Omega_{A/\mathbf{F}_p} \otimes_A K$ whose restriction to $\Omega_{k/\mathbf{F}_p} \otimes_k K$ is ξ'. Write

$$D' : A \xrightarrow{\mathrm{d}} \Omega_{A/\mathbf{F}_p} \to \Omega_{A/\mathbf{F}_p} \otimes_A K \xrightarrow{\xi''} J.$$

Finally, set $\psi' = \psi - D' : A \to B$. The reader verifies that ψ' is a ring map such that $\pi \circ \psi' = \bar\psi$ and such that $\psi' \circ i = \varphi$ as desired. $\square$

Example 40.2. Let k be a field of characteristic $p > 0$. Suppose that $a \in k$ is an element which is not a pth power. A standard example of a geometrically regular local k-algebra whose residue field is purely inseparable over k is the ring

$$A = k[x, y]_{(x, y^p - a)}/(y^p - a - x)$$

Namely, A is a localization of a smooth algebra over k hence $k \to A$ is formally smooth, hence $k \to A$ is formally smooth for the $\mathfrak m$-adic topology. A closely related example is the following. Let $k = \mathbf{F}_p(s)$ and $K = \mathbf{F}_p(t)^{perf}$. We claim the ring map

$$k \longrightarrow A = K[[x]], \quad s \longmapsto t + x$$

is formally smooth for the (x)-adic topology on A. Namely, $\Omega_{k/\mathbf{F}_p}$ is 1-dimensional with basis $\mathrm{d}s$. It maps to the element $\mathrm{d}x + \mathrm{d}t = \mathrm{d}x$ in $\Omega_{A/\mathbf{F}_p}$. We leave it to the reader to show that $\Omega_{A/\mathbf{F}_p}$ is free on $\mathrm{d}x$ as an A-module. Hence we see that condition (5) of Theorem 40.1 holds and we conclude that $k \to A$ is formally smooth in the (x)-adic topology.

Lemma 40.3. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Assume $A \to B$ is formally smooth in the $\mathfrak m_B$-adic topology. Then $A \to B$ is flat.*

Proof. We may assume that A and B a Noetherian complete local rings by Lemma 37.4 and Algebra, Lemma 97.6 (this also uses Algebra, Lemma 39.9 and 97.3 to see that flatness of the map on completions implies flatness of $A \to B$). Choose a commutative diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Lemma 39.3 with $R \to S$ flat. Let $I \subset R$ be the kernel of $R \to A$. Because B is formally smooth over A we see that the A-algebra map

$$S/IS \longrightarrow B$$

has a section, see Lemma 37.5. Hence B is a direct summand of the flat A-module S/IS (by base change of flatness, see Algebra, Lemma 39.7), whence flat. $\square$

Lemma 40.4. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Assume $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Let K be the residue field of B. Then the Jacobi-Zariski sequence for $A \to B \to K$ gives an exact sequence*

$$0 \to H_1(NL_{K/A}) \to \mathfrak{m}_B/\mathfrak{m}_B^2 \to \Omega_{B/A} \otimes_B K \to \Omega_{K/A} \to 0$$

Proof. Observe that $\mathfrak{m}_B/\mathfrak{m}_B^2 = H_1(NL_{K/B})$ by Algebra, Lemma 134.6. By Algebra, Lemma 134.4 it remains to show injectivity of $H_1(NL_{K/A}) \to \mathfrak{m}_B/\mathfrak{m}_B^2$. With k the residue field of A, the Jacobi-Zariski sequence for $A \to k \to K$ gives $\Omega_{K/A} = \Omega_{K/k}$ and an exact sequence

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K \to H_1(NL_{K/A}) \to H_1(NL_{K/k}) \to 0$$

Set $\overline{B} = B \otimes_A k$. Since $\overline{B}$ is regular the ideal $\mathfrak{m}_{\overline{B}}$ is generated by a regular sequence. Applying Lemmas 30.9 and 30.7 to $\mathfrak{m}_A B \subset \mathfrak{m}_B$ we find $\mathfrak{m}_A B/(\mathfrak{m}_A B \cap \mathfrak{m}_B^2) = \mathfrak{m}_A B/\mathfrak{m}_A \mathfrak{m}_B$ which is equal to $\mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K$ as $A \to B$ is flat by Lemma 40.3. Thus we obtain a short exact sequence

$$0 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K \to \mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2 \to 0$$

Functoriality of the Jacobi-Zariski sequences shows that we obtain a commutative diagram

$$\begin{tikzcd} & \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K \arrow[r] \arrow[d] & H_1(NL_{K/A}) \arrow[r] \arrow[d] & H_1(NL_{K/k}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K \arrow[r] & \mathfrak{m}_B/\mathfrak{m}_B^2 \arrow[r] & \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2 \arrow[r] & 0 \end{tikzcd}$$

The left vertical arrow is injective by Theorem 40.1 as $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology by Lemma 37.8. This finishes the proof by the snake lemma. $\square$

Proposition 40.5. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Let k be the residue field of A and $\overline{B} = B \otimes_A k$ the special fibre. The following are equivalent*

1. *$A \to B$ is flat and $\overline{B}$ is geometrically regular over k,*
2. *$A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology, and*
3. *$A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology.*

Proof. The equivalence of (1) and (2) follows from Theorem 40.1.

Assume (3). By Lemma 40.3 we see that $A \to B$ is flat. By Lemma 37.8 we see that $k \to \overline{B}$ is formally smooth in the $\mathfrak m_{\overline{B}}$-adic topology. Thus (2) holds.

Assume (2). Lemma 37.4 tells us formal smoothness is preserved under completion. The same is true for flatness by Algebra, Lemma 97.3. Hence we may replace A and B by their respective completions and assume that A and B are Noetherian complete local rings. In this case choose a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Lemma 39.3. We will use all of the properties of this diagram without further mention. Fix a regular system of parameters $t_1, \ldots, t_d$ of R with $t_1 = p$ in case the characteristic of k is $p > 0$. Set $\overline{S} = S \otimes_R k$. Consider the short exact sequence

$$0 \to J \to S \to B \to 0$$

As $\overline{B}$ and $\overline{S}$ are regular, the kernel of $\overline{S} \to \overline{B}$ is generated by elements $\overline{x}_1, \ldots, \overline{x}_r$ which form part of a regular system of parameters of $\overline{S}$, see Algebra, Lemma 106.4. Lift these elements to $x_1, \ldots, x_r \in J$. Then $t_1, \ldots, t_d, x_1, \ldots, x_r$ is part of a regular system of parameters for S. Hence $S/(x_1, \ldots, x_r)$ is a power series ring over a field (if the characteristic of k is zero) or a power series ring over a Cohen ring (if the characteristic of k is $p > 0$), see Lemma 39.2. Moreover, it is still the case that $R \to S/(x_1, \ldots, x_r)$ maps $t_1, \ldots, t_d$ to a part of a regular system of parameters of $S/(x_1, \ldots, x_r)$. In other words, we may replace S by $S/(x_1, \ldots, x_r)$ and assume we have a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Lemma 39.3 with moreover $\overline{S} = \overline{B}$. In this case the map

$$S \otimes_R A \longrightarrow B$$

is an isomorphism as it is surjective, an isomorphism on special fibres, and source and target are flat over A (for example use Algebra, Lemma 99.1 or use that tensoring the short exact sequence $0 \to I \to S \otimes_R A \to B \to 0$ over A with k we find $I \otimes_A k = 0$ hence $I = 0$ by Nakayama). Thus by Lemma 37.8 it suffices to show that $R \to S$ is formally smooth in the $\mathfrak m_S$-adic topology. Of course, since $\overline{S} = \overline{B}$, we have that $\overline{S}$ is formally smooth over $k = R/\mathfrak m_R$.

Choose elements $y_1, \ldots, y_m \in S$ such that $t_1, \ldots, t_d, y_1, \ldots, y_m$ is a regular system of parameters for S. If the characteristic of k is zero, choose a coefficient field $K \subset S$ and if the characteristic of k is $p > 0$ choose a Cohen ring $\Lambda \subset S$ with residue field K. At this point the map $K[[t_1, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic zero case) or $\Lambda[[t_2, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic $p > 0$ case) is an isomorphism, see Lemma 39.2. From now on we think of S as the above power series ring.

The rest of the proof is analogous to the argument in the proof of Theorem 40.1. Choose a solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed, "\bar{\psi}"] & N/J \\ R \arrow[u, "i"] \arrow[r, "\varphi"] & N \arrow[u, "\pi"'] \end{tikzcd}$$

as in Definition 37.1. As $J^2 = 0$ we see that J has a canonical N/J module structure and via $\bar{\psi}$ a S-module structure. As $\bar{\psi}$ is continuous for the $\mathfrak{m}_S$-adic topology we see that $\mathfrak{m}_S^n J = 0$ for some n. Hence we can filter J by N/J-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient J_{t+1}/J_t is annihilated by $\mathfrak{m}_S$. Considering the sequence of ring maps $N \to N/J_1 \to N/J_2 \to \ldots \to N/J$ we see that it suffices to prove the existence of the dotted arrow when J is annihilated by $\mathfrak{m}_S$, i.e., when J is a K-vector space.

Assume given a diagram as above such that J is annihilated by $\mathfrak{m}_S$. As $\mathbf{Q} \to S$ (characteristic zero case) or $\mathbf{Z} \to S$ (characteristic $p > 0$ case) is formally smooth in the $\mathfrak{m}_S$-adic topology (see Lemma 39.1), we can find a ring map $\psi : S \to N$ such that $\pi \circ \psi = \bar{\psi}$. Since S is a power series ring in $t_1, \ldots, t_d$ (characteristic zero) or $t_2, \ldots, t_d$ (characteristic $p > 0$) over a subring, it follows from the universal property of power series rings that we can change our choice of ψ so that $\psi(t_i)$ equals $\varphi(t_i)$ (automatic for $t_1 = p$ in the characteristic p case). Then $\psi \circ i$ and $\varphi : R \to N$ are two maps whose compositions with π are equal and which agree on $t_1, \ldots, t_d$. Hence $D = \psi \circ i - \varphi : R \to J$ is a derivation which annihilates $t_1, \ldots, t_d$. By Algebra, Lemma 131.3 we can write $D = \xi \circ \mathrm{d}$ for some R-linear map $\xi : \Omega_{R/\mathbf{Z}} \to J$ which annihilates $\mathrm{d}t_1, \ldots, \mathrm{d}t_d$ (by construction) and $\mathfrak{m}_R \Omega_{R/\mathbf{Z}}$ (as J is annihilated by $\mathfrak{m}_R$). Hence ξ factors as a composition

$$\Omega_{R/\mathbf{Z}} \to \Omega_{k/\mathbf{Z}} \xrightarrow{\xi'} J$$

where ξ' is k-linear. Using the K-vector space structure on J we extend ξ' to a K-linear map

$$\xi'' : \Omega_{k/\mathbf{Z}} \otimes_k K \longrightarrow J.$$

Using that $\overline{S}/k$ is formally smooth we see that

$$\Omega_{k/\mathbf{Z}} \otimes_k K \to \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K$$

is injective by Theorem 40.1 (this is true also in the characteristic zero case as it is even true that $\Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective in characteristic zero, see Algebra, Proposition 158.9). Hence we can find a K-linear map $\xi''' : \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K \to J$ whose restriction to $\Omega_{k/\mathbf{Z}} \otimes_k K$ is ξ''. Write

$$D' : S \xrightarrow{\mathrm{d}} \Omega_{S/\mathbf{Z}} \to \Omega_{\overline{S}/\mathbf{Z}} \to \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K \xrightarrow{\xi'''} J.$$

Finally, set $\psi' = \psi - D' : S \to N$. The reader verifies that ψ' is a ring map such that $\pi \circ \psi' = \bar{\psi}$ and such that $\psi' \circ i = \varphi$ as desired. $\square$

As an application of the result above we prove that deformations of formally smooth algebras are unobstructed.

Lemma 40.6. *Let A be a Noetherian complete local ring with residue field k. Let B be a Noetherian complete local k-algebra. Assume $k \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then there exists a Noetherian complete local ring C and a* *local homomorphism $A \to C$ which is formally smooth in the $\mathfrak m_C$-adic topology such that $C \otimes_A k \cong B$.*

Proof. Choose a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Lemma 39.3. Let $t_1, \ldots, t_d$ be a regular system of parameters for R with $t_1 = p$ in case the characteristic of k is $p > 0$. As B and $\overline{S} = S \otimes_R k$ are regular we see that $\text{Ker}(\overline{S} \to B)$ is generated by elements $\overline{x}_1, \ldots, \overline{x}_r$ which form part of a regular system of parameters of $\overline{S}$, see Algebra, Lemma 106.4. Lift these elements to $x_1, \ldots, x_r \in S$. Then $t_1, \ldots, t_d, x_1, \ldots, x_r$ is part of a regular system of parameters for S. Hence $S/(x_1, \ldots, x_r)$ is a power series ring over a field (if the characteristic of k is zero) or a power series ring over a Cohen ring (if the characteristic of k is $p > 0$), see Lemma 39.2. Moreover, it is still the case that $R \to S/(x_1, \ldots, x_r)$ maps $t_1, \ldots, t_d$ to a part of a regular system of parameters of $S/(x_1, \ldots, x_r)$. In other words, we may replace S by $S/(x_1, \ldots, x_r)$ and assume we have a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Lemma 39.3 with moreover $\overline{S} = B$. In this case $R \to S$ is formally smooth in the $\mathfrak m_S$-adic topology by Proposition 40.5. Hence the base change $C = S \otimes_R A$ is formally smooth over A in the $\mathfrak m_C$-adic topology by Lemma 37.8. $\square$

Remark 40.7. The assertion of Lemma 40.6 is quite strong. Namely, suppose that we have a diagram

$$\begin{tikzcd} & B \\ A \arrow[r] & A' \arrow[u] \end{tikzcd}$$

of local homomorphisms of Noetherian complete local rings where $A \to A'$ induces an isomorphism of residue fields $k = A/\mathfrak m_A = A'/\mathfrak m_{A'}$ and with $B \otimes_{A'} k$ formally smooth over k. Then we can extend this to a commutative diagram

$$\begin{tikzcd} C \arrow[r] & B \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

of local homomorphisms of Noetherian complete local rings where $A \to C$ is formally smooth in the $\mathfrak m_C$-adic topology and where $C \otimes_A k \cong B \otimes_{A'} k$. Namely, pick $A \to C$ as in Lemma 40.6 lifting $B \otimes_{A'} k$ over k. By formal smoothness we can find the arrow $C \to B$, see Lemma 37.5. Denote $C \otimes_A^\wedge A'$ the completion of $C \otimes_A A'$ with respect to the ideal $C \otimes_A \mathfrak m_{A'}$. Note that $C \otimes_A^\wedge A'$ is a Noetherian complete local ring (see Algebra, Lemma 97.5) which is flat over A' (see Algebra, Lemma 99.11). We have moreover

(1) $C \otimes_A^\wedge A' \to B$ is surjective,

(2) if $A \to A'$ is surjective, then $C \to B$ is surjective,
(3) if $A \to A'$ is finite, then $C \to B$ is finite, and
(4) if $A' \to B$ is flat, then $C \otimes_A^\wedge A' \cong B$.

Namely, by Nakayama's lemma for nilpotent ideals (see Algebra, Lemma 20.1) we see that $C \otimes_A k \cong B \otimes_{A'} k$ implies that $C \otimes_A A'/\mathfrak{m}_{A'}^n \to B/\mathfrak{m}_{A'}^n B$ is surjective for all n. This proves (1). Parts (2) and (3) follow from part (1). Part (4) follows from Algebra, Lemma 99.1.

41. Regular ring maps

Let k be a field. Recall that a Noetherian k-algebra A is said to be *geometrically regular* over k if and only if $A \otimes_k k'$ is regular for all finite purely inseparable extensions k' of k, see Algebra, Definition 166.2. Moreover, if this is the case then $A \otimes_k k'$ is regular for every finitely generated field extension k'/k, see Algebra, Lemma 166.1. We use this notion in the following definition.

Definition 41.1. A ring map $R \to \Lambda$ is *regular* if it is flat and for every prime $\mathfrak{p} \subset R$ the fibre ring

$$\Lambda \otimes_R \kappa(\mathfrak{p}) = \Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}$$

is Noetherian and geometrically regular over $\kappa(\mathfrak{p})$.

If $R \to \Lambda$ is a ring map with Λ Noetherian, then the fibre rings are always Noetherian.

Lemma 41.2 (Regular is a local property)**.** *Let $R \to \Lambda$ be a ring map with Λ Noetherian. The following are equivalent*

(1) *$R \to \Lambda$ is regular,*
(2) *$R_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is regular for all $\mathfrak{q} \subset \Lambda$ lying over $\mathfrak{p} \subset R$, and*
(3) *$R_\mathfrak{m} \to \Lambda_{\mathfrak{m}'}$ is regular for all maximal ideals $\mathfrak{m}' \subset \Lambda$ lying over $\mathfrak{m}$ in R.*

Proof. This is true because a Noetherian ring is regular if and only if all the local rings are regular local rings, see Algebra, Definition 110.7 and a ring map is flat if and only if all the induced maps of local rings are flat, see Algebra, Lemma 39.18. ☐

Lemma 41.3 (Regular maps and base change)**.** *Let $R \to \Lambda$ be a regular ring map. For any finite type ring map $R \to R'$ the base change $R' \to \Lambda \otimes_R R'$ is regular too.*

Proof. Flatness is preserved under any base change, see Algebra, Lemma 39.7. Consider a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p} \subset R$. The residue field extension $\kappa(\mathfrak{p}')/\kappa(\mathfrak{p})$ is finitely generated as R' is of finite type over R. Hence the fibre ring

$$(\Lambda \otimes_R R') \otimes_{R'} \kappa(\mathfrak{p}') = \Lambda \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

is Noetherian by Algebra, Lemma 31.8 and the assumption on the fibre rings of $R \to \Lambda$. Geometric regularity of the fibres is preserved by Algebra, Lemma 166.1. ☐

Lemma 41.4 (Composition of regular maps)**.** *Let $A \to B$ and $B \to C$ be regular ring maps. If the fibre rings of $A \to C$ are Noetherian, then $A \to C$ is regular.*

Proof. Let $\mathfrak{p} \subset A$ be a prime. Let $\kappa(\mathfrak{p}) \subset k$ be a finite purely inseparable extension. We have to show that $C \otimes_A k$ is regular. By Lemma 41.3 we may assume that $A = k$ and we reduce to proving that C is regular. The assumption is that B is regular and that $B \to C$ is flat with regular fibres. Then C is regular by Algebra, Lemma 112.8. Some details omitted. ☐

Lemma 41.5. *Let R be a ring. Let $(A_i, \varphi_{ii'})$ be a directed system of smooth R-algebras. Set $\Lambda = \operatorname{colim} A_i$. If the fibre rings $\Lambda \otimes_R \kappa(\mathfrak p)$ are Noetherian for all $\mathfrak p \subset R$, then $R \to \Lambda$ is regular.*

Proof. Note that Λ is flat over R by Algebra, Lemmas 39.3 and 137.10. Let $\kappa(\mathfrak p) \subset k$ be a finite purely inseparable extension. Note that

$$\Lambda \otimes_R \kappa(\mathfrak p) \otimes_{\kappa(\mathfrak p)} k = \Lambda \otimes_R k = \operatorname{colim} A_i \otimes_R k$$

is a colimit of smooth k-algebras, see Algebra, Lemma 137.4. Since each local ring of a smooth k-algebra is regular by Algebra, Lemma 140.3 we conclude that all local rings of $\Lambda \otimes_R k$ are regular by Algebra, Lemma 106.8. This proves the lemma. $\square$

Let's see when a field extension defines a regular ring map.

Lemma 41.6. *Let K/k be a field extension. Then $k \to K$ is a regular ring map if and only if K is a separable field extension of k.*

Proof. If $k \to K$ is regular, then K is geometrically reduced over k, hence K is separable over k by Algebra, Proposition 158.9. Conversely, if K/k is separable, then K is a colimit of smooth k-algebras, see Algebra, Lemma 158.11 hence is regular by Lemma 41.5. $\square$

Lemma 41.7. *Let $A \to B \to C$ be ring maps. If $A \to C$ is regular and $B \to C$ is flat and surjective on spectra, then $A \to B$ is regular.*

Proof. By Algebra, Lemma 39.10 we see that $A \to B$ is flat. Let $\mathfrak p \subset A$ be a prime. The ring map $B \otimes_A \kappa(\mathfrak p) \to C \otimes_A \kappa(\mathfrak p)$ is flat and surjective on spectra. Hence $B \otimes_A \kappa(\mathfrak p)$ is geometrically regular by Algebra, Lemma 166.3. $\square$

42. Ascending properties along regular ring maps

This section is the analogue of Algebra, Section 163 but where the ring map $R \to S$ is regular.

Lemma 42.1. *Let $\varphi : R \to S$ be a ring map. Assume*

1. *φ is regular,*
2. *S is Noetherian, and*
3. *R is Noetherian and reduced.*

Then S is reduced.

Proof. For Noetherian rings being reduced is the same as having properties (S_1) and (R_0), see Algebra, Lemma 157.3. Hence we may apply Algebra, Lemmas 163.4 and 163.5. $\square$

Lemma 42.2. *Let $\varphi : R \to S$ be a ring map. Assume*

1. *φ is regular,*
2. *S is Noetherian, and*
3. *R is Noetherian and normal.*

Then S is normal.

Proof. For Noetherian rings being normal is the same as having properties (S_2) and (R_1), see Algebra, Lemma 157.4. Hence we may apply Algebra, Lemmas 163.4 and 163.5. $\square$

43. Permanence of properties under completion

Given a Noetherian local ring $(A, \mathfrak{m})$ we denote $A^\wedge$ the completion of A with respect to $\mathfrak{m}$. We will use without further mention that $A^\wedge$ is a Noetherian complete local ring with maximal ideal $\mathfrak{m}^\wedge = \mathfrak{m}A^\wedge$ and that $A \to A^\wedge$ is faithfully flat. See Algebra, Lemmas 97.6, 97.4, and 97.3.

Lemma 43.1. *Let A be a Noetherian local ring. Then $\dim(A) = \dim(A^\wedge)$.*

Proof. By Algebra, Lemma 97.4 the map $A \to A^\wedge$ induces isomorphisms $A/\mathfrak{m}^n = A^\wedge/(\mathfrak{m}^\wedge)^n$ for $n \geq 1$. By Algebra, Lemma 52.12 this implies that

$$\text{length}_A(A/\mathfrak{m}^n) = \text{length}_{A^\wedge}(A^\wedge/(\mathfrak{m}^\wedge)^n)$$

for all $n \geq 1$. Thus $d(A) = d(A^\wedge)$ and we conclude by Algebra, Proposition 60.9. An alternative proof is to use Algebra, Lemma 112.7. $\square$

Lemma 43.2. *Let A be a Noetherian local ring. Then depth(A) = depth$(A^\wedge)$.*

Proof. See Algebra, Lemma 163.2. $\square$

Lemma 43.3. *Let A be a Noetherian local ring. Then A is Cohen-Macaulay if and only if $A^\wedge$ is so.*

Proof. A local ring A is Cohen-Macaulay if and only if $\dim(A) = \text{depth}(A)$. As both of these invariants are preserved under completion (Lemmas 43.1 and 43.2) the claim follows. $\square$

Lemma 43.4. *Let A be a Noetherian local ring. Then A is regular if and only if $A^\wedge$ is so.*

Proof. If $A^\wedge$ is regular, then A is regular by Algebra, Lemma 110.9. Assume A is regular. Let $\mathfrak{m}$ be the maximal ideal of A. Then $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = \dim(A) = \dim(A^\wedge)$ (Lemma 43.1). On the other hand, $\mathfrak{m}A^\wedge$ is the maximal ideal of $A^\wedge$ and hence $\mathfrak{m}_{A^\wedge}$ is generated by at most $\dim(A^\wedge)$ elements. Thus $A^\wedge$ is regular. (You can also use Algebra, Lemma 112.8.) $\square$

Lemma 43.5. *Let A be a Noetherian local ring. Then A is a discrete valuation ring if and only if $A^\wedge$ is so.*

Proof. This follows from Lemmas 43.1 and 43.4 and Algebra, Lemma 119.7. $\square$

Lemma 43.6. *Let A be a Noetherian local ring.*

(1) *If $A^\wedge$ is reduced, then so is A.*
(2) *In general A reduced does not imply $A^\wedge$ is reduced.*
(3) *If A is Nagata, then A is reduced if and only if $A^\wedge$ is reduced.*

Proof. As $A \to A^\wedge$ is faithfully flat we have (1) by Algebra, Lemma 164.2. For (2) see Algebra, Example 119.5 (there are also examples in characteristic zero, see Algebra, Remark 119.6). For (3) see Algebra, Lemmas 162.13 and 162.10. $\square$

Lemma 43.7. *Let A be a Noetherian local ring. If $A^\wedge$ is normal, then so is A.*

Proof. As $A \to A^\wedge$ is faithfully flat this follows from Algebra, Lemma 164.3. $\square$

Lemma 43.8. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Then the induced map of completions $A^\wedge \to B^\wedge$ is flat if and only if $A \to B$ is flat.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} A^\wedge \arrow[r] & B^\wedge \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

The vertical arrows are faithfully flat. Assume that $A^\wedge \to B^\wedge$ is flat. Then $A \to B^\wedge$ is flat. Hence B is flat over A by Algebra, Lemma 39.9.

Assume that $A \to B$ is flat. Then $A \to B^\wedge$ is flat. Hence $B^\wedge/\mathfrak{m}_A^n B^\wedge$ is flat over $A/\mathfrak{m}_A^n$ for all $n \geq 1$. Note that $\mathfrak{m}_A^n A^\wedge$ is the nth power of the maximal ideal $\mathfrak{m}_A^\wedge$ of $A^\wedge$ and $A/\mathfrak{m}_A^n = A^\wedge/(\mathfrak{m}_A^\wedge)^n$. Thus we see that $B^\wedge$ is flat over $A^\wedge$ by applying Algebra, Lemma 99.11 (with $R = A^\wedge$, $I = \mathfrak{m}_A^\wedge$, $S = B^\wedge$, $M = S$). $\square$

Lemma 43.9. *Let $A \to B$ be a flat local homomorphism of Noetherian local rings such that $\mathfrak{m}_A B = \mathfrak{m}_B$ and $\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$. Then $A \to B$ induces an isomorphism $A^\wedge \to B^\wedge$ of completions.*

Proof. By Algebra, Lemma 97.7 we see that $B^\wedge$ is the $\mathfrak{m}_A$-adic completion of B and that $A^\wedge \to B^\wedge$ is finite. Since $A \to B$ is flat we have $\text{Tor}_1^A(B, \kappa(\mathfrak{m}_A)) = 0$. Hence we see that $B^\wedge$ is flat over $A^\wedge$ by Lemma 27.5. Thus $B^\wedge$ is a free $A^\wedge$-module by Algebra, Lemma 78.5. Since $A^\wedge \to B^\wedge$ induces an isomorphism $\kappa(\mathfrak{m}_A) = A^\wedge/\mathfrak{m}_A A^\wedge \to B^\wedge/\mathfrak{m}_A B^\wedge = B^\wedge/\mathfrak{m}_B B^\wedge = \kappa(\mathfrak{m}_B)$ by our assumptions (and Algebra, Lemma 96.3), we see that $B^\wedge$ is free of rank 1. Thus $A^\wedge \to B^\wedge$ is an isomorphism. $\square$

44. Permanence of properties under étale maps

In this section we consider an étale ring map $\varphi : A \to B$ and we study which properties of A are inherited by B and which properties of the local ring of B at $\mathfrak{q}$ are inherited by the local ring of A at $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Basically, this section reviews and collects earlier results and does not add any new material.

We will use without further mention that an étale ring map is flat (Algebra, Lemma 143.3) and that a flat local homomorphism of local rings is faithfully flat (Algebra, Lemma 39.17).

Lemma 44.1. *If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $A_\mathfrak{p}$ is Noetherian if and only if $B_\mathfrak{q}$ is Noetherian.*

Proof. Since $A_\mathfrak{p} \to B_\mathfrak{q}$ is faithfully flat we see that $B_\mathfrak{q}$ Noetherian implies that $A_\mathfrak{p}$ is Noetherian, see Algebra, Lemma 164.1. Conversely, if $A_\mathfrak{p}$ is Noetherian, then $B_\mathfrak{q}$ is Noetherian as it is a localization of a finite type $A_\mathfrak{p}$-algebra. $\square$

Lemma 44.2. *If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then* $\dim(A_\mathfrak{p}) = \dim(B_\mathfrak{q})$.

Proof. Namely, because $A_\mathfrak{p} \to B_\mathfrak{q}$ is flat we have going down, and hence the inequality $\dim(A_\mathfrak{p}) \leq \dim(B_\mathfrak{q})$, see Algebra, Lemma 112.1. On the other hand, suppose that $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_n$ is a chain of primes in $B_\mathfrak{q}$. Then the corresponding sequence of primes $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ (with $\mathfrak{p}_i = \mathfrak{q}_i \cap A_\mathfrak{p}$) is chain also (i.e., no equalities in the sequence) as an étale ring map is quasi-finite (see Algebra, Lemma 143.6) and a quasi-finite ring map induces a map of spectra with discrete fibres (by definition). This means that $\dim(A_\mathfrak{p}) \geq \dim(B_\mathfrak{q})$ as desired. $\square$

Lemma 44.3. *If $A \to B$ is an étale ring map and $\mathfrak q$ is a prime of B lying over $\mathfrak p \subset A$, then $A_\mathfrak p$ is regular if and only if $B_\mathfrak q$ is regular.*

Proof. By Lemma 44.1 we may assume both $A_\mathfrak p$ and $B_\mathfrak q$ are Noetherian in order to prove the equivalence. Let $x_1, \ldots, x_t \in \mathfrak pA_\mathfrak p$ be a minimal set of generators. As $A_\mathfrak p \to B_\mathfrak q$ is faithfully flat we see that the images $y_1, \ldots, y_t$ in $B_\mathfrak q$ form a minimal system of generators for $\mathfrak pB_\mathfrak q = \mathfrak qB_\mathfrak q$ (Algebra, Lemma 143.5). Regularity of $A_\mathfrak p$ by definition means $t = \dim(A_\mathfrak p)$ and similarly for $B_\mathfrak q$. Hence the lemma follows from the equality $\dim(A_\mathfrak p) = \dim(B_\mathfrak q)$ of Lemma 44.2. $\square$

Lemma 44.4. *If $A \to B$ is an étale ring map and A is a Dedekind domain, then B is a finite product of Dedekind domains. In particular, the localizations $B_\mathfrak q$ for $\mathfrak q \subset B$ maximal are discrete valuation rings.*

Proof. The statement on the local rings follows from Lemmas 44.2 and 44.3 and Algebra, Lemma 119.7. It follows that B is a Noetherian normal ring of dimension 1. By Algebra, Lemma 37.16 we conclude that B is a finite product of normal domains of dimension 1. These are Dedekind domains by Algebra, Lemma 120.17. $\square$

45. Permanence of properties under henselization

Given a local ring R we denote R^h, resp. R^{sh} the henselization, resp. strict henselization of R, see Algebra, Definition 155.3. Many of the properties of R are reflected in R^h and R^{sh} as we will show in this section.

Lemma 45.1. *Let $(R, \mathfrak m, \kappa)$ be a local ring. Then we have the following*
(1) *$R \to R^h \to R^{sh}$ are faithfully flat ring maps,*
(2) *$\mathfrak mR^h = \mathfrak m^h$ and $\mathfrak mR^{sh} = \mathfrak m^hR^{sh} = \mathfrak m^{sh}$,*
(3) *$R/\mathfrak m^n = R^h/\mathfrak m^nR^h$ for all n,*
(4) *there exist elements $x_i \in R^{sh}$ such that $R^{sh}/\mathfrak m^nR^{sh}$ is a free $R/\mathfrak m^n$-module on $x_i \bmod \mathfrak m^nR^{sh}$.*

Proof. By construction R^h is a colimit of étale R-algebras, see Algebra, Lemma 155.1. Since étale ring maps are flat (Algebra, Lemma 143.3) we see that R^h is flat over R by Algebra, Lemma 39.3. As a flat local ring homomorphism is faithfully flat (Algebra, Lemma 39.17) we see that $R \to R^h$ is faithfully flat. The ring map $R^h \to R^{sh}$ is a colimit of finite étale ring maps, see proof of Algebra, Lemma 155.2. Hence the same arguments as above show that $R^h \to R^{sh}$ is faithfully flat.

Part (2) follows from Algebra, Lemmas 155.1 and 155.2. Part (3) follows from Algebra, Lemma 101.1 because $R/\mathfrak m \to R^h/\mathfrak mR^h$ is an isomorphism and $R/\mathfrak m^n \to R^h/\mathfrak m^nR^h$ is flat as a base change of the flat ring map $R \to R^h$ (Algebra, Lemma 39.7). Let κ^{sep} be the residue field of R^{sh} (it is a separable algebraic closure of κ). Choose $x_i \in R^{sh}$ mapping to a basis of κ^{sep} as a κ-vector space. Then (4) follows from Algebra, Lemma 101.1 in exactly the same way as above. $\square$

Lemma 45.2. *Let $(R, \mathfrak m, \kappa)$ be a local ring. Then*
(1) *$R \to R^h$, $R^h \to R^{sh}$, and $R \to R^{sh}$ are formally étale,*
(2) *$R \to R^h$, $R^h \to R^{sh}$, resp. $R \to R^{sh}$ are formally smooth in the $\mathfrak m^h$, $\mathfrak m^{sh}$, resp. $\mathfrak m^{sh}$-topology.*

Proof. Part (1) follows from the fact that R^h and R^{sh} are directed colimits of étale algebras (by construction), that étale algebras are formally étale (Algebra, Lemma

150.2), and that colimits of formally étale algebras are formally étale (Algebra, Lemma 150.3). Part (2) follows from the fact that a formally étale ring map is formally smooth and Lemma 37.2. □

Lemma 45.3. *Let R be a local ring. The following are equivalent*

(1) *R is Noetherian,*
(2) *R^h is Noetherian, and*
(3) *R^{sh} is Noetherian.*

In this case we have

(a) *$(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian complete local rings,*
(b) *$R^\wedge \to (R^h)^\wedge$ is an isomorphism,*
(c) *$R^h \to (R^h)^\wedge$ and $R^{sh} \to (R^{sh})^\wedge$ are flat,*
(d) *$R^\wedge \to (R^{sh})^\wedge$ is formally smooth in the $\mathfrak{m}_{(R^{sh})^\wedge}$-adic topology,*
(e) *$(R^\wedge)^{sh} = R^\wedge \otimes_{R^h} R^{sh}$, and*
(f) *$((R^\wedge)^{sh})^\wedge = (R^{sh})^\wedge$.*

Proof. Since $R \to R^h \to R^{sh}$ are faithfully flat (Lemma 45.1), we see that R^h or R^{sh} being Noetherian implies that R is Noetherian, see Algebra, Lemma 164.1. In the rest of the proof we assume R is Noetherian.

As $\mathfrak{m} \subset R$ is finitely generated it follows that $\mathfrak{m}^h = \mathfrak{m}R^h$ and $\mathfrak{m}^{sh} = \mathfrak{m}R^{sh}$ are finitely generated, see Lemma 45.1. Hence $(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian by Algebra, Lemma 160.3. This proves (a).

Note that (b) is immediate from Lemma 45.1. In particular we see that $(R^h)^\wedge$ is flat over R, see Algebra, Lemma 97.3.

Next, we show that $R^h \to (R^h)^\wedge$ is flat. Write $R^h = \text{colim}_i R_i$ as a directed colimit of localizations of étale R-algebras. By Algebra, Lemma 39.6 if $(R^h)^\wedge$ is flat over each R_i, then $R^h \to (R^h)^\wedge$ is flat. Note that $R^h = R_i^h$ (by construction). Hence $R_i^\wedge = (R^h)^\wedge$ by part (b) is flat over R_i as desired. To finish the proof of (c) we show that $R^{sh} \to (R^{sh})^\wedge$ is flat. To do this, by a limit argument as above, it suffices to show that $(R^{sh})^\wedge$ is flat over R. Note that it follows from Lemma 45.1 that $(R^{sh})^\wedge$ is the completion of a free R-module. By Lemma 27.2 we see this is flat over R as desired. This finishes the proof of (c).

At this point we know (c) is true and that $(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian. It follows from Algebra, Lemma 164.1 that R^h and R^{sh} are Noetherian.

Part (d) follows from Lemma 45.2 and Lemma 37.4.

Part (e) follows from Algebra, Lemma 155.13 and the fact that $R^\wedge$ is henselian by Algebra, Lemma 153.9.

Proof of (f). Using (e) there is a map $R^{sh} \to (R^\wedge)^{sh}$ which induces a map $(R^{sh})^\wedge \to ((R^\wedge)^{sh})^\wedge$ upon completion. Using (e) there is a map $R^\wedge \to (R^{sh})^\wedge$. Since $(R^{sh})^\wedge$ is strictly henselian (see above) this map induces a map $(R^\wedge)^{sh} \to (R^{sh})^\wedge$ by Algebra, Lemma 155.10. Completing we obtain a map $((R^\wedge)^{sh})^\wedge \to (R^{sh})^\wedge$. We omit the verification that these two maps are mutually inverse. □

Lemma 45.4. *Let R be a local ring. The following are equivalent: R is reduced, the henselization R^h of R is reduced, and the strict henselization R^{sh} of R is reduced.*

Proof. The ring maps $R \to R^h \to R^{sh}$ are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma 164.2. Conversely, assume R is reduced. Since R^h and R^{sh} are filtered colimits of étale, hence smooth R-algebras, the result follows from Algebra, Lemma 163.7. $\square$

Lemma 45.5. *Let R be a local ring. Let $nil(R)$ denote the ideal of nilpotent elements of R. Then $nil(R)R^h = nil(R^h)$ and $nil(R)R^{sh} = nil(R^{sh})$.*

Proof. Note that $nil(R)$ is the biggest ideal consisting of nilpotent elements such that the quotient $R/nil(R)$ is reduced. Note that $nil(R)R^h$ consists of nilpotent elements by Algebra, Lemma 32.3. Also, note that $R^h/nil(R)R^h$ is the henselization of $R/nil(R)$ by Algebra, Lemma 156.2. Hence $R^h/nil(R)R^h$ is reduced by Lemma 45.4. We conclude that $nil(R)R^h = nil(R^h)$ as desired. Similarly for the strict henselization but using Algebra, Lemma 156.4. $\square$

Lemma 45.6. *Let R be a local ring. The following are equivalent: R is a normal domain, the henselization R^h of R is a normal domain, and the strict henselization R^{sh} of R is a normal domain.*

Proof. A preliminary remark is that a local ring is normal if and only if it is a normal domain (see Algebra, Definition 37.11). The ring maps $R \to R^h \to R^{sh}$ are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma 164.3. Conversely, assume R is normal. Since R^h and R^{sh} are filtered colimits of étale hence smooth R-algebras, the result follows from Algebra, Lemmas 163.9 and 37.17. $\square$

Lemma 45.7. *Given any local ring R we have $\dim(R) = \dim(R^h) = \dim(R^{sh})$.*

Proof. Since $R \to R^{sh}$ is faithfully flat (Lemma 45.1) we see that $\dim(R^{sh}) \geq \dim(R)$ by going down, see Algebra, Lemma 112.1. For the converse, we write $R^{sh} = \operatorname{colim} R_i$ as a directed colimit of local rings R_i each of which is a localization of an étale R-algebra. Now if $\mathfrak q_0 \subset \mathfrak q_1 \subset \ldots \subset \mathfrak q_n$ is a chain of prime ideals in R^{sh}, then for some sufficiently large i the sequence

$$R_i \cap \mathfrak q_0 \subset R_i \cap \mathfrak q_1 \subset \ldots \subset R_i \cap \mathfrak q_n$$

is a chain of primes in R_i. Thus we see that $\dim(R^{sh}) \leq \sup_i \dim(R_i)$. But by the result of Lemma 44.2 we have $\dim(R_i) = \dim(R)$ for each i and we win. $\square$

Lemma 45.8. *Given a Noetherian local ring R we have $depth(R) = depth(R^h) = depth(R^{sh})$.*

Proof. By Lemma 45.3 we know that R^h and R^{sh} are Noetherian. Hence the lemma follows from Algebra, Lemma 163.2. $\square$

Lemma 45.9. *Let R be a Noetherian local ring. The following are equivalent: R is Cohen-Macaulay, the henselization R^h of R is Cohen-Macaulay, and the strict henselization R^{sh} of R is Cohen-Macaulay.*

Proof. By Lemma 45.3 we know that R^h and R^{sh} are Noetherian, hence the lemma makes sense. Since we have $\text{depth}(R) = \text{depth}(R^h) = \text{depth}(R^{sh})$ and $\dim(R) = \dim(R^h) = \dim(R^{sh})$ by Lemmas 45.8 and 45.7 we conclude. $\square$

Lemma 45.10. *Let R be a Noetherian local ring. The following are equivalent: R is a regular local ring, the henselization R^h of R is a regular local ring, and the strict henselization R^{sh} of R is a regular local ring.*

Proof. By Lemma 45.3 we know that R^h and R^{sh} are Noetherian, hence the lemma makes sense. Let $\mathfrak m$ be the maximal ideal of R. Let $x_1, \ldots, x_t \in \mathfrak m$ be a minimal system of generators of $\mathfrak m$, i.e., such that the images in $\mathfrak m/\mathfrak m^2$ form a basis over $\kappa = R/\mathfrak m$. Because $R \to R^h$ and $R \to R^{sh}$ are faithfully flat, it follows that the images $x_1^h, \ldots, x_t^h$ in R^h, resp. $x_1^{sh}, \ldots, x_t^{sh}$ in R^{sh} are a minimal system of generators for $\mathfrak m^h = \mathfrak m R^h$, resp. $\mathfrak m^{sh} = \mathfrak m R^{sh}$. Regularity of R by definition means $t = \dim(R)$ and similarly for R^h and R^{sh}. Hence the lemma follows from the equality of dimensions $\dim(R) = \dim(R^h) = \dim(R^{sh})$ of Lemma 45.7 $\square$

Lemma 45.11. *Let R be a Noetherian local ring. Then R is a discrete valuation ring if and only if R^h is a discrete valuation ring if and only if R^{sh} is a discrete valuation ring.*

Proof. This follows from Lemmas 45.7 and 45.10 and Algebra, Lemma 119.7. $\square$

Lemma 45.12. *Let A be a ring. Let B be a filtered colimit of étale A-algebras. Let $\mathfrak p$ be a prime of A. If B is Noetherian, then there are finitely many primes $\mathfrak q_1, \ldots, \mathfrak q_r$ lying over $\mathfrak p$, we have $B \otimes_A \kappa(\mathfrak p) = \prod \kappa(\mathfrak q_i)$, and each of the field extensions $\kappa(\mathfrak q_i)/\kappa(\mathfrak p)$ is separable algebraic.*

Proof. Write B as a filtered colimit $B = \operatorname{colim} B_i$ with $A \to B_i$ étale. Then on the one hand $B \otimes_A \kappa(\mathfrak p) = \operatorname{colim} B_i \otimes_A \kappa(\mathfrak p)$ is a filtered colimit of étale $\kappa(\mathfrak p)$-algebras, and on the other hand it is Noetherian. An étale $\kappa(\mathfrak p)$-algebra is a finite product of finite separable field extensions (Algebra, Lemma 143.4). Hence there are no nontrivial specializations between the primes (which are all maximal and minimal primes) of the algebras $B_i \otimes_A \kappa(\mathfrak p)$ and hence there are no nontrivial specializations between the primes of $B \otimes_A \kappa(\mathfrak p)$. Thus $B \otimes_A \kappa(\mathfrak p)$ is reduced and has finitely many primes which all minimal. Thus it is a finite product of fields (use Algebra, Lemma 25.4 or Algebra, Proposition 60.7). Each of these fields is a colimit of finite separable extensions and hence the final statement of the lemma follows. $\square$

Lemma 45.13. *Let R be a Noetherian local ring. Let $\mathfrak p \subset R$ be a prime. Then*

$$R^h \otimes_R \kappa(\mathfrak p) = \prod\nolimits_{i=1,\ldots,t} \kappa(\mathfrak q_i) \quad \textit{resp.} \quad R^{sh} \otimes_R \kappa(\mathfrak p) = \prod\nolimits_{i=1,\ldots,s} \kappa(\mathfrak r_i)$$

where $\mathfrak q_1, \ldots, \mathfrak q_t$, resp. $\mathfrak r_1, \ldots, \mathfrak r_s$ are the prime of R^h, resp. R^{sh} lying over $\mathfrak p$. Moreover, the field extensions $\kappa(\mathfrak q_i)/\kappa(\mathfrak p)$ resp. $\kappa(\mathfrak r_i)/\kappa(\mathfrak p)$ are separable algebraic.

Proof. This can be deduced from the more general Lemma 45.12 using that the henselization and strict henselization are Noetherian (as we've seen above). But we also give a direct proof as follows.

We will use without further mention the results of Lemmas 45.1 and 45.3. Note that $R^h/\mathfrak p R^h$, resp. $R^{sh}/\mathfrak p R^{sh}$ is the henselization, resp. strict henselization of $R/\mathfrak p$, see Algebra, Lemma 156.2 resp. Algebra, Lemma 156.4. Hence we may replace R by $R/\mathfrak p$ and assume that R is a Noetherian local domain and that $\mathfrak p = (0)$. Since R^h, resp. R^{sh} is Noetherian, it has finitely many minimal primes $\mathfrak q_1, \ldots, \mathfrak q_t$, resp. $\mathfrak r_1, \ldots, \mathfrak r_s$. Since $R \to R^h$, resp. $R \to R^{sh}$ is flat these are exactly the primes lying over $\mathfrak p = (0)$ (by going down). Finally, as R is a domain, we see that R^h, resp. R^{sh} is reduced, see Lemma 45.4. Thus we see that $R^h \otimes_R \kappa(\mathfrak p)$ resp. $R^{sh} \otimes_R \kappa(\mathfrak p)$ is a reduced Noetherian ring with finitely many primes, all of which are minimal (and hence maximal). Thus these rings are Artinian and are products of their localizations at maximal ideals, each necessarily a field (see Algebra, Proposition 60.7 and Algebra, Lemma 25.1).

The final statement follows from the fact that $R \to R^h$, resp. $R \to R^{sh}$ is a colimit of étale ring maps and hence the induced residue field extensions are colimits of finite separable extensions, see Algebra, Lemma 143.5. $\square$

46. Field extensions, revisited

In this section we study some peculiarities of field extensions in characteristic $p > 0$.

Definition 46.1. Let p be a prime number. Let $k \to K$ be an extension of fields of characteristic p. Denote kK^p the compositum of k and K^p in K.

1. A subset $\{x_i\} \subset K$ is called *p-independent over k* if the elements $x^E = \prod x_i^{e_i}$ where $0 \leq e_i < p$ are linearly independent over kK^p.
2. A subset $\{x_i\}$ of K is called a *p-basis of K over k* if the elements x^E form a basis of K over kK^p.

This is related to the notion of a p-basis of a $\mathbf{F}_p$-algebra which we will discuss later (insert future reference here).

Lemma 46.2. *Let K/k be a field extension. Assume k has characteristic $p > 0$. Let $\{x_i\}$ be a subset of K. The following are equivalent*

1. *the elements $\{x_i\}$ are p-independent over k, and*
2. *the elements dx_i are K-linearly independent in $\Omega_{K/k}$.*

Any p-independent collection can be extended to a p-basis of K over k. In particular, the field K has a p-basis over k. Moreover, the following are equivalent:

(a) *$\{x_i\}$ is a p-basis of K over k, and*
(b) *dx_i is a basis of the K-vector space $\Omega_{K/k}$.*

Proof. Assume (2) and suppose that $\sum a_E x^E = 0$ is a linear relation with $a_E \in kK^p$. Let $\theta_i : K \to K$ be a k-derivation such that $\theta_i(x_j) = \delta_{ij}$ (Kronecker delta). Note that any k-derivation of K annihilates kK^p. Applying θ_i to the given relation we obtain new relations

$$\sum\nolimits_{E, e_i > 0} e_i a_E x_1^{e_1} \ldots x_i^{e_i - 1} \ldots x_n^{e_n} = 0$$

Hence if we pick $\sum a_E x^E$ as the relation with minimal total degree $|E| = \sum e_i$ for some $a_E \neq 0$, then we get a contradiction. Hence (1) holds.

If $\{x_i\}$ is a p-basis for K over k, then $K \cong kK^p[X_i]/(X_i^p - x_i^p)$. Hence we see that $\text{d}x_i$ forms a basis for $\Omega_{K/k}$ over K. Thus (a) implies (b).

Let $\{x_i\}$ be a p-independent subset of K over k. An application of Zorn's lemma shows that we can enlarge this to a maximal p-independent subset of K over k. We claim that any maximal p-independent subset $\{x_i\}$ of K is a p-basis of K over k. The claim will imply that (1) implies (2) and establish the existence of p-bases. To prove the claim let L be the subfield of K generated by kK^p and the x_i. We have to show that $L = K$. If $x \in K$ but $x \notin L$, then $x^p \in L$ and $L(x) \cong L[z]/(z^p - x)$. Hence $\{x_i\} \cup \{x\}$ is p-independent over k, a contradiction.

Finally, we have to show that (b) implies (a). By the equivalence of (1) and (2) we see that $\{x_i\}$ is a maximal p-independent subset of K over k. Hence by the claim above it is a p-basis. $\square$

Lemma 46.3. *Let K/k be a field extension. Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of subfields of K with the following properties*

1. *$k \subset K_\alpha$ for all $\alpha \in A$,*
2. *$k = \bigcap_{\alpha \in A} K_\alpha$,*
3. *for $\alpha, \alpha' \in A$ there exists an $\alpha'' \in A$ such that $K_{\alpha''} \subset K_\alpha \cap K_{\alpha'}$.*

Then for $n \geq 1$ and $V \subset K^{\oplus n}$ a K-vector space we have $V \cap k^{\oplus n} \neq 0$ if and only if $V \cap K_\alpha^{\oplus n} \neq 0$ for all $\alpha \in A$.

Proof. By induction on n. The case $n = 1$ follows from the assumptions. Assume the result proven for subspaces of $K^{\oplus n-1}$. Assume that $V \subset K^{\oplus n}$ has nonzero intersection with $K_\alpha^{\oplus n}$ for all $\alpha \in A$. If $V \cap 0 \oplus k^{\oplus n-1}$ is nonzero then we win. Hence we may assume this is not the case. By induction hypothesis we can find an α such that $V \cap 0 \oplus K_\alpha^{\oplus n-1}$ is zero. Let $v = (x_1, \ldots, x_n) \in V \cap K_\alpha^{\oplus n}$ be a nonzero element. By our choice of α we see that x_1 is not zero. Replace v by $x_1^{-1}v$ so that $v = (1, x_2, \ldots, x_n)$. Note that if $v' = (x_1', \ldots, x_n') \in V \cap K_\alpha$, then $v' - x_1'v = 0$ by our choice of α. Hence we see that $V \cap K_\alpha^{\oplus n} = K_\alpha v$. If we choose some α' such that $K_{\alpha'} \subset K_\alpha$, then we see that necessarily $v \in V \cap K_{\alpha'}^{\oplus n}$ (by the same arguments applied to α'). Hence

$$x_2, \ldots, x_n \in \bigcap\nolimits_{\alpha' \in A, K_{\alpha'} \subset K_\alpha} K_{\alpha'}$$

which equals k by (2) and (3). □

Lemma 46.4. *Let K be a field of characteristic p. Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of subfields of K with the following properties*

1. *$K^p \subset K_\alpha$ for all $\alpha \in A$,*
2. *$K^p = \bigcap_{\alpha \in A} K_\alpha$,*
3. *for $\alpha, \alpha' \in A$ there exists an $\alpha'' \in A$ such that $K_{\alpha''} \subset K_\alpha \cap K_{\alpha'}$.*

Then

1. *the intersection of the kernels of the maps $\Omega_{K/\mathbf{F}_p} \to \Omega_{K/K_\alpha}$ is zero,*
2. *for any finite extension L/K we have $L^p = \bigcap_{\alpha \in A} L^p K_\alpha$.*

Proof. Proof of (1). Choose a p-basis $\{x_i\}$ for K over $\mathbf{F}_p$. Suppose that $\eta = \sum_{i \in I'} y_i \text{d}x_i$ maps to zero in Ω_{K/K_α} for every $\alpha \in A$. Here the index set I' is finite. By Lemma 46.2 this means that for every α there exists a relation

$$\sum\nolimits_E a_{E,\alpha} x^E, \quad a_{E,\alpha} \in K_\alpha$$

where E runs over multi-indices $E = (e_i)_{i \in I'}$ with $0 \leq e_i < p$. On the other hand, Lemma 46.2 guarantees there is no such relation $\sum a_E x^E = 0$ with $a_E \in K^p$. This is a contradiction by Lemma 46.3.

Proof of (2). Suppose that we have a tower $L/M/K$ of finite extensions of fields. Set $M_\alpha = M^p K_\alpha$ and $L_\alpha = L^p K_\alpha = L^p M_\alpha$. Then we can first prove that $M^p = \bigcap_{\alpha \in A} M_\alpha$, and after that prove that $L^p = \bigcap_{\alpha \in A} L_\alpha$. Hence it suffices to prove (2) for primitive field extensions having no nontrivial subfields. First, assume that $L = K(\theta)$ is separable over K. Then L is generated by θ^p over K, hence we may assume that $\theta \in L^p$. In this case we see that

$$L^p = K^p \oplus K^p\theta \oplus \ldots K^p\theta^{d-1} \quad \text{and} \quad L^pK_\alpha = K_\alpha \oplus K_\alpha\theta \oplus \ldots K_\alpha\theta^{d-1}$$

where $d = [L : K]$. Thus the conclusion is clear in this case. The other case is where $L = K(\theta)$ with $\theta^p = t \in K$, $t \not\in K^p$. In this case we have

$$L^p = K^p \oplus K^p t \oplus \ldots K^p t^{p-1} \quad \text{and} \quad L^p K_\alpha = K_\alpha \oplus K_\alpha t \oplus \ldots K_\alpha t^{p-1}$$

Again the result is clear. $\square$

Lemma 46.5. *Let k be a field of characteristic $p > 0$. Let $n, m \geq 0$. Let K be the fraction field of $k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$. As k' ranges through all subfields $k/k'/k^p$ with $[k : k'] < \infty$ the subfields*

$$\text{fraction field of } k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p] \subset K$$

form a family of subfields as in Lemma 46.4. Moreover, each of the ring extensions $k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p] \subset k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$ is finite.

Proof. Write $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$ and $A' = k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]$. We also denote K' the fraction field of A'. The ring extension $k'[[x_1^p, \ldots, x_d^p]] \subset k[[x_1, \ldots, x_d]]$ is finite by Algebra, Lemma 97.7 which implies that $A' \to A$ is finite. For $f \in A$ we see that $f^p \in A'$. Hence $K^p \subset K'$. Any element of K' can be written as a/b^p with $a \in A'$ and $b \in A$ nonzero. Suppose that $f/g^p \in K$, $f, g \in A$, $g \neq 0$ is contained in K' for every choice of k'. Fix a choice of k' for the moment. By the above we see $f/g^p = a/b^p$ for some $a \in A'$ and some nonzero $b \in A$. Hence $b^p f \in A'$. For any A'-derivation $D : A \to A$ we see that $0 = D(b^p f) = b^p D(f)$ hence $D(f) = 0$ as A is a domain. Taking $D = \partial_{x_i}$ and $D = \partial_{y_j}$ we conclude that $f \in k[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_d^p]$. Applying a k'-derivation $\theta : k \to k$ we similarly conclude that all coefficients of f are in k', i.e., $f \in A'$. Since it is clear that $A^p = \bigcap_{k'} A'$ where k' ranges over all subfields as in the lemma we win. $\square$

47. The singular locus

Let R be a Noetherian ring. The *regular locus* $\text{Reg}(X)$ of $X = \text{Spec}(R)$ is the set of primes $\mathfrak{p}$ such that $R_\mathfrak{p}$ is a regular local ring. The *singular locus* $\text{Sing}(X)$ of $X = \text{Spec}(R)$ is the complement $X \setminus \text{Reg}(X)$, i.e., the set of primes $\mathfrak{p}$ such that $R_\mathfrak{p}$ is not a regular local ring. By the discussion preceding Algebra, Definition 110.7 we see that $\text{Reg}(X)$ is stable under generalization. In this section we study conditions that guarantee that $\text{Reg}(X)$ is open.

Definition 47.1. Let R be a Noetherian ring. Let $X = \text{Spec}(R)$.

(1) We say R is *J-0* if $\text{Reg}(X)$ contains a nonempty open.
(2) We say R is *J-1* if $\text{Reg}(X)$ is open.
(3) We say R is *J-2* if any finite type R-algebra is J-1.

The ring $\mathbf{Q}[x]/(x^2)$ does not satisfy J-0, but it does satisfy J-1. On the other hand, J-1 implies J-0 for Noetherian domains and more generally nonzero reduced Noetherian rings as such a ring is regular at the minimal primes. Here is a characterization of the J-1 property.

Lemma 47.2. *Let R be a Noetherian ring. Let $X = \text{Spec}(R)$. The ring R is J-1 if and only if $V(\mathfrak{p}) \cap \text{Reg}(X)$ contains a nonempty open subset of $V(\mathfrak{p})$ for all $\mathfrak{p} \in \text{Reg}(X)$.*

Proof. This follows from Topology, Lemma 16.5 and the fact that $\text{Reg}(X)$ is stable under generalization by Algebra, Lemma 110.6. $\square$

Lemma 47.3. *Let R be a Noetherian ring. Let $X = \mathrm{Spec}(R)$. Assume that for all primes $\mathfrak p \subset R$ the ring $R/\mathfrak p$ is J-0. Then R is J-1.*

Proof. We will show that the criterion of Lemma 47.2 applies. Let $\mathfrak p \in \mathrm{Reg}(X)$ be a prime of height r. Pick $f_1, \ldots, f_r \in \mathfrak p$ which map to generators of $\mathfrak pR_{\mathfrak p}$. Since $\mathfrak p \in \mathrm{Reg}(X)$ we see that $f_1, \ldots, f_r$ maps to a regular sequence in $R_{\mathfrak p}$, see Algebra, Lemma 106.3. Thus by Algebra, Lemma 68.6 we see that after replacing R by R_g for some $g \in R$, $g \notin \mathfrak p$ the sequence $f_1, \ldots, f_r$ is a regular sequence in R. After another replacement we may also assume $f_1, \ldots, f_r$ generate $\mathfrak p$. Next, let $\mathfrak p \subset \mathfrak q$ be a prime ideal such that $(R/\mathfrak p)_{\mathfrak q}$ is a regular local ring. By the assumption of the lemma there exists a non-empty open subset of $V(\mathfrak p)$ consisting of such primes, hence it suffices to prove $R_{\mathfrak q}$ is regular. Note that $f_1, \ldots, f_r$ is a regular sequence in $R_{\mathfrak q}$ such that $R_{\mathfrak q}/(f_1, \ldots, f_r)R_{\mathfrak q}$ is regular. Hence $R_{\mathfrak q}$ is regular by Algebra, Lemma 106.7. ☐

Lemma 47.4. *Let $R \to S$ be a ring map. Assume that*

1. *R is a Noetherian domain,*
2. *$R \to S$ is injective and of finite type, and*
3. *S is a domain and J-0.*

Then R is J-0.

Proof. After replacing S by S_g for some nonzero $g \in S$ we may assume that S is a regular ring. By generic flatness we may assume that also $R \to S$ is faithfully flat, see Algebra, Lemma 118.1. Then R is regular by Algebra, Lemma 164.4. ☐

Lemma 47.5. *Let $R \to S$ be a ring map. Assume that*

1. *R is a Noetherian domain and J-0,*
2. *$R \to S$ is injective and of finite type, and*
3. *S is a domain, and*
4. *the induced extension of fraction fields is separable.*

Then S is J-0.

Proof. We may replace R by a principal localization and assume R is a regular ring. By Algebra, Lemma 140.9 the ring map $R \to S$ is smooth at (0). Hence after replacing S by a principal localization we may assume that S is smooth over R. Then S is regular too, see Algebra, Lemma 163.10. ☐

Lemma 47.6. *Let R be a Noetherian ring. The following are equivalent*

1. *R is J-2,*
2. *every finite type R-algebra which is a domain is J-0,*
3. *every finite R-algebra is J-1,*
4. *for every prime $\mathfrak p$ and every finite purely inseparable extension $L/\kappa(\mathfrak p)$ there exists a finite R-algebra R' which is a domain, which is J-0, and whose field of fractions is L.*

Proof. It is clear that we have the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4). Recall that a domain which is J-1 is J-0. Hence we also have the implications (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4).

Let $R \to S$ be a finite type ring map and let's try to show S is J-1. By Lemma 47.3 it suffices to prove that $S/\mathfrak q$ is J-0 for every prime $\mathfrak q$ of S. In this way we see (2) $\Rightarrow$ (1).

Assume (4). We will show that (2) holds which will finish the proof. Let $R \to S$ be a finite type ring map with S a domain. Let $\mathfrak{p} = \mathrm{Ker}(R \to S)$. Let K be the fraction field of S. There exists a diagram of fields

$$\begin{tikzcd} K \arrow[r] & K' \\ \kappa(\mathfrak{p}) \arrow[u] \arrow[r] & L \arrow[u] \end{tikzcd}$$

where the horizontal arrows are finite purely inseparable field extensions and where K'/L is separable, see Algebra, Lemma 42.4. Choose $R' \subset L$ as in (4) and let S' be the image of the map $S \otimes_R R' \to K'$. Then S' is a domain whose fraction field is K', hence S' is J-0 by Lemma 47.5 and our choice of R'. Then we apply Lemma 47.4 to see that S is J-0 as desired. □

48. Regularity and derivations

Let $R \to S$ be a ring map. Let $D : R \to R$ be a derivation. We say that D *extends to S* if there exists a derivation $D' : S \to S$ such that

$$\begin{tikzcd} S \arrow[r, "D'"'] & S \\ R \arrow[u] \arrow[r, "D"] & R \arrow[u] \end{tikzcd}$$

is commutative.

Lemma 48.1. *Let R be a ring. Let $D : R \to R$ be a derivation.*

(1) *For any ideal $I \subset R$ the derivation D extends canonically to a derivation $D^\wedge : R^\wedge \to R^\wedge$ on the I-adic completion.*
(2) *For any multiplicative subset $S \subset R$ the derivation D extends uniquely to the localization $S^{-1}R$ of R.*

If $R \subset R'$ is a finite type extension of rings such that $R_g \cong R'_g$ for some $g \in R$ which is a nonzerodivisor in R', then $g^N D$ extends to R' for some $N \geq 0$.

Proof. Proof of (1). For $n \geq 2$ we have $D(I^n) \subset I^{n-1}$ by the Leibniz rule. Hence D induces maps $D_n : R/I^n \to R/I^{n-1}$. Taking the limit we obtain $D^\wedge$. We omit the verification that $D^\wedge$ is a derivation.

Proof of (2). To extend D to $S^{-1}R$ just set $D(r/s) = D(r)/s - rD(s)/s^2$ and check the axioms.

Proof of the final statement. Let $x_1, \ldots, x_n \in R'$ be generators of R' over R. Choose an N such that $g^N x_i \in R$. Consider $g^{N+1}D$. By (2) this extends to R_g. Moreover, by the Leibniz rule and our construction of the extension above we have

$$g^{N+1}D(x_i) = g^{N+1}D(g^{-N}g^N x_i) = -Ng^N x_i D(g) + gD(g^N x_i)$$

and both terms are in R. This implies that

$$g^{N+1}D(x_1^{e_1} \ldots x_n^{e_n}) = \sum e_i x_1^{e_1} \ldots x_i^{e_i - 1} \ldots x_n^{e_n} g^{N+1} D(x_i)$$

is an element of R'. Hence every element of R' (which can be written as a sum of monomials in the x_i with coefficients in R) is mapped to an element of R' by $g^{N+1}D$ and we win. □

Lemma 48.2. *Let R be a regular ring. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of $R/(f)$. Then $R/(f)$ is regular.*

Proof. It suffices to prove this when R is a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. In this case it suffices to prove that $f \not\in \mathfrak{m}^2$, see Algebra, Lemma 106.3. However, if $f \in \mathfrak{m}^2$ then $D(f) \in \mathfrak{m}$ by the Leibniz rule, a contradiction. $\square$

Lemma 48.3. *Let $(R, \mathfrak{m}, \kappa)$ be a regular local ring. Let $m \geq 1$. Let $f_1, \ldots, f_m \in \mathfrak{m}$. Assume there exist derivations $D_1, \ldots, D_m : R \to R$ such that $\det_{1 \leq i,j \leq m}(D_i(f_j))$ is a unit of R. Then $R/(f_1, \ldots, f_m)$ is regular and $f_1, \ldots, f_m$ is a regular sequence.*

Proof. It suffices to prove that $f_1, \ldots, f_m$ are κ-linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, see Algebra, Lemma 106.3. However, if there is a nontrivial linear relation the we get $\sum a_i f_i \in \mathfrak{m}^2$ for some $a_i \in R$ but not all $a_i \in \mathfrak{m}$. Observe that $D_i(\mathfrak{m}^2) \subset \mathfrak{m}$ and $D_i(a_j f_j) \equiv a_j D_i(f_j) \bmod \mathfrak{m}$ by the Leibniz rule for derivations. Hence this would imply

$$\sum a_j D_i(f_j) \in \mathfrak{m}$$

which would contradict the assumption on the determinant. $\square$

Lemma 48.4. *Let R be a regular ring. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of R. Then $R[z]/(z^n - f)$ is regular for any integer $n \geq 1$. More generally, $R[z]/(p(z) - f)$ is regular for any $p \in \mathbf{Z}[z]$.*

Proof. By Algebra, Lemma 163.10 we see that $R[z]$ is a regular ring. Apply Lemma 48.2 to the extension of D to $R[z]$ which maps z to zero. This works because D annihilates any polynomial with integer coefficients and sends f to a unit. $\square$

Lemma 48.5. *Let p be a prime number. Let B be a domain with $p = 0$ in B. Let $f \in B$ be an element which is not a pth power in the fraction field of B. If B is of finite type over a Noetherian complete local ring, then there exists a derivation $D : B \to B$ such that $D(f)$ is not zero.*

Proof. Let R be a Noetherian complete local ring such that there exists a finite type ring map $R \to B$. Of course we may replace R by its image in B, hence we may assume R is a domain of characteristic $p > 0$ (as well as Noetherian complete local). By Algebra, Lemma 160.11 we can write R as a finite extension of $k[[x_1, \ldots, x_n]]$ for some field k and integer n. Hence we may replace R by $k[[x_1, \ldots, x_n]]$. Next, we use Algebra, Lemma 115.7 to factor $R \to B$ as

$$R \subset R[y_1, \ldots, y_d] \subset B' \subset B$$

with B' finite over $R[y_1, \ldots, y_d]$ and $B'_g \cong B_g$ for some nonzero $g \in R$. Note that $f' = g^{pN} f \in B'$ for some large integer N. It is clear that f' is not a pth power in the fraction field of B'. If we can find a derivation $D' : B' \to B'$ with $D'(f') \neq 0$, then Lemma 48.1 guarantees that $D = g^M D'$ extends to B for some $M > 0$. Then $D(f) = g^N D'(f) = g^M D'(g^{-pN} f') = g^{M - pN} D'(f')$ is nonzero. Thus it suffices to prove the lemma in case B is a finite extension of $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$.

Assume B is a finite extension of $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$. Denote L the fraction field of B. Note that $\text{d}f$ is not zero in $\Omega_{L/\mathbf{F}_p}$, see Algebra, Lemma 158.2. We apply Lemma 46.5 to find a subfield $k' \subset k$ of finite index such that with $A' = k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]$ the element $\text{d}f$ does not map to zero in $\Omega_{L/K'}$ where K' is the fraction field of A'. Thus we can choose a K'-derivation $D' : L \to L$

with $D'(f) \neq 0$. Since $A' \subset A$ and $A \subset B$ are finite by construction we see that $A' \subset B$ is finite. Choose $b_1, \ldots, b_t \in B$ which generate B as an A'-module. Then $D'(b_i) = f_i/g_i$ for some $f_i, g_i \in B$ with $g_i \neq 0$. Setting $D = g_1 \ldots g_t D'$ we win. $\square$

Lemma 48.6. *Let A be a Noetherian complete local domain. Then A is J-0.*

Proof. By Algebra, Lemma 160.11 we can find a regular subring $A_0 \subset A$ with A finite over A_0. The induced extension K/K_0 of fraction fields is finite. If K/K_0 is separable, then we are done by Lemma 47.5. If not, then A_0 and A have characteristic $p > 0$. For any subextension $K/M/K_0$ there exists a finite subextension $A_0 \subset B \subset A$ whose fraction field is M. Hence, arguing by induction on $[K : K_0]$ we may assume there exists $A_0 \subset B \subset A$ such that B is J-0 and K/M has no nontrivial subextensions. In this case, if K/M is separable, then we see that A is J-0 by Lemma 47.5. If not, then $K = M[z]/(z^p - b_1/b_2)$ for some $b_1, b_2 \in B$ with $b_2 \neq 0$ and b_1/b_2 not a pth power in M. Choose $a \in A$ nonzero such that $az \in A$. After replacing z by $b_2 a^p z$ we obtain $K = M[z]/(z^p - b)$ with $z \in A$ and $b \in B$ not a pth power in M. By Lemma 48.5 we can find a derivation $D : B \to B$ with $D(b) \neq 0$. Applying Lemma 48.4 we see that $A_\mathfrak{p}$ is regular for any prime $\mathfrak{p}$ of A lying over a regular prime of B and not containing $D(b)$. As B is J-0 we conclude A is too. $\square$

Proposition 48.7. *The following types of rings are J-2:*

1. *fields,*
2. *Noetherian complete local rings,*
3. $\mathbf{Z}$*,*
4. *Noetherian local rings of dimension* 1*,*
5. *Nagata rings of dimension* 1*,*
6. *Dedekind domains with fraction field of characteristic zero,*
7. *finite type ring extensions of any of the above.*

Proof. For cases (1), (3), (5), and (6) this is proved by checking condition (4) of Lemma 47.6. We will only do this in case R is a Nagata ring of dimension 1. Let $\mathfrak{p} \subset R$ be a prime ideal and let $L/\kappa(\mathfrak{p})$ be a finite purely inseparable extension. If $\mathfrak{p} \subset R$ is a maximal ideal, then $R \to L$ is finite and L is a regular ring and we've checked the condition. If $\mathfrak{p} \subset R$ is a minimal prime, then the Nagata condition insures that the integral closure $R' \subset L$ of R in L is finite over R. Then R' is a normal domain of dimension 1 (Algebra, Lemma 112.3) hence regular (Algebra, Lemma 157.4) and we've checked the condition in this case as well.

For case (2), we will use condition (3) of Lemma 47.6. Let R be a Noetherian complete local ring. Note that if $R \to R'$ is finite, then R' is a product of Noetherian complete local rings, see Algebra, Lemma 160.2. Hence it suffices to prove that a Noetherian complete local ring which is a domain is J-0, which is Lemma 48.6.

For case (4), we also use condition (3) of Lemma 47.6. Namely, if R is a local Noetherian ring of dimension 1 and $R \to R'$ is finite, then $\text{Spec}(R')$ is finite. Since the regular locus is stable under generalization, we see that R' is J-1. $\square$

49. Formal smoothness and regularity

The title of this section refers to Proposition 49.2.

Lemma 49.1. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Let $D : A \to A$ be a derivation. Assume that B is complete and $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then there exists an extension $D' : B \to B$ of D.*

Proof. Denote $B[\epsilon] = B[x]/(x^2)$ the ring of dual numbers over B. Consider the ring map $\psi : A \to B[\epsilon]$, $a \mapsto a + \epsilon D(a)$. Consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "1"'] & B \\ A \arrow[u] \arrow[r, "\psi"] & B[\epsilon] \arrow[u] \end{tikzcd}$$

By Lemma 37.5 and the assumption of formal smoothness of B/A we find a map $\varphi : B \to B[\epsilon]$ fitting into the diagram. Write $\varphi(b) = b + \epsilon D'(b)$. Then $D' : B \to B$ is the desired extension. $\square$

Proposition 49.2. *Let $A \to B$ be a local homomorphism of Noetherian complete local rings. Let k be the residue field of A and $\overline{B} = B \otimes_A k$ the special fibre. The following are equivalent*

(1) *$A \to B$ is regular,*
(2) *$A \to B$ is flat and $\overline{B}$ is geometrically regular over k,*
(3) *$A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology, and*
(4) *$A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology.*

Proof. We have seen the equivalence of (2), (3), and (4) in Proposition 40.5. It is clear that (1) implies (2). Thus we assume the equivalent conditions (2), (3), and (4) hold and we prove (1).

Let $\mathfrak{p}$ be a prime of A. We will show that $B \otimes_A \kappa(\mathfrak{p})$ is geometrically regular over $\kappa(\mathfrak{p})$. By Lemma 37.8 we may replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$. Thus we may assume that A is a domain and that $\mathfrak{p} = (0)$.

Choose $A_0 \subset A$ as in Algebra, Lemma 160.11. We will use all the properties stated in that lemma without further mention. As $A_0 \to A$ induces an isomorphism on residue fields, and as $B/\mathfrak{m}_A B$ is geometrically regular over $A/\mathfrak{m}_A$ we can find a diagram

$$\begin{tikzcd} C \arrow[r] & B \\ A_0 \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

with $A_0 \to C$ formally smooth in the $\mathfrak{m}_C$-adic topology such that $B = C \otimes_{A_0} A$, see Remark 40.7. (Completion in the tensor product is not needed as $A_0 \to A$ is finite, see Algebra, Lemma 97.1.) Hence it suffices to show that $C \otimes_{A_0} K_0$ is a geometrically regular algebra over the fraction field K_0 of A_0.

The upshot of the preceding paragraph is that we may assume that $A = k[[x_1, \ldots, x_n]]$ where k is a field or $A = \Lambda[[x_1, \ldots, x_n]]$ where Λ is a Cohen ring. In this case B is a regular ring, see Algebra, Lemma 112.8. Hence $B \otimes_A K$ is a regular ring too (where K is the fraction field of A) and we win if the characteristic of K is zero.

Thus we are left with the case where $A = k[[x_1, \ldots, x_n]]$ and k is a field of characteristic $p > 0$. Let L/K be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $B \otimes_A L$ is regular. The base case is $L = K$ which we've seen above. Let $K \subset M \subset L$ be a subfield such that L is a degree p extension of M obtained by adjoining a pth root of an element $f \in M$. Let A' be a finite A-subalgebra of M with fraction field M. Clearing denominators, we may and do assume $f \in A'$. Set $A'' = A'[z]/(z^p - f)$ and note that $A' \subset A''$ is finite and that the fraction field of A'' is L. By induction we know that $B \otimes_A M$ ring is regular. We have

$$B \otimes_A L = B \otimes_A M[z]/(z^p - f)$$

By Lemma 48.5 we know there exists a derivation $D : A' \to A'$ such that $D(f) \neq 0$. As $A' \to B \otimes_A A'$ is formally smooth in the $\mathfrak{m}$-adic topology by Lemma 37.9 we can use Lemma 49.1 to extend D to a derivation $D' : B \otimes_A A' \to B \otimes_A A'$. Note that $D'(f) = D(f)$ is a unit in $B \otimes_A M$ as $D(f)$ is not zero in $A' \subset M$. Hence $B \otimes_A L$ is regular by Lemma 48.4 and we win. $\square$

50. G-rings

Let A be a Noetherian local ring A. In Section 43 we have seen that some but not all properties of A are reflected in the completion $A^\wedge$ of A. To study this further we introduce some terminology. For a prime $\mathfrak{q}$ of A the fibre ring

$$A^\wedge \otimes_A \kappa(\mathfrak{q}) = (A^\wedge)_\mathfrak{q}/\mathfrak{q}(A^\wedge)_\mathfrak{q} = (A/\mathfrak{q})^\wedge \otimes_{A/q} \kappa(\mathfrak{q})$$

is called a *formal fibre* of A. We think of the formal fibre as an algebra over $\kappa(\mathfrak{q})$. Thus $A \to A^\wedge$ is a regular ring homomorphism if and only if all the formal fibres are geometrically regular algebras.

Definition 50.1. A ring R is called a *G-ring* if R is Noetherian and for every prime $\mathfrak{p}$ of R the ring map $R_\mathfrak{p} \to (R_\mathfrak{p})^\wedge$ is regular.

By the discussion above we see that R is a G-ring if and only if every local ring $R_\mathfrak{p}$ has geometrically regular formal fibres. Note that if $\mathbf{Q} \subset R$, then it suffices to check the formal fibres are regular. Another way to express the G-ring condition is described in the following lemma.

Lemma 50.2. *Let R be a Noetherian ring. Then R is a G-ring if and only if for every pair of primes $\mathfrak{q} \subset \mathfrak{p} \subset R$ the algebra*

$$(R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

is geometrically regular over $\kappa(\mathfrak{q})$.

Proof. This follows from the fact that

$$R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) = (R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

as algebras over $\kappa(\mathfrak{q})$. $\square$

Lemma 50.3. *Let $R \to R'$ be a finite type map of Noetherian rings and let*

$$\begin{tikzcd}
\mathfrak{q}' \arrow[r] \arrow[d, dash] & \mathfrak{p}' \arrow[r] \arrow[d, dash] & R' \\
\mathfrak{q} \arrow[r] & \mathfrak{p} \arrow[r] & R \arrow[u]
\end{tikzcd}$$

be primes. Assume $R \to R'$ is quasi-finite at $\mathfrak{p}'$.

(1) *If the formal fibre $R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$, then the formal fibre $R'_{\mathfrak{p}'} \otimes_{R'} \kappa(\mathfrak{q}')$ is geometrically regular over $\kappa(\mathfrak{q}')$.*

(2) *If the formal fibres of $R_\mathfrak{p}$ are geometrically regular, then the formal fibres of $R'_{\mathfrak{p}'}$ are geometrically regular.*
(3) *If $R \to R'$ is quasi-finite and R is a G-ring, then R' is a G-ring.*

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume $R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$. By Algebra, Lemma 124.3 we see that

$$R_\mathfrak{p}^\wedge \otimes_R R' = (R'_{\mathfrak{p}'})^\wedge \times B$$

for some $R_\mathfrak{p}^\wedge$-algebra B. Hence $R'_{\mathfrak{p}'} \to (R'_{\mathfrak{p}'})^\wedge$ is a factor of a base change of the map $R_\mathfrak{p} \to R_\mathfrak{p}^\wedge$. It follows that $(R'_{\mathfrak{p}'})^\wedge \otimes_{R'} \kappa(\mathfrak{q}')$ is a factor of

$$R_\mathfrak{p}^\wedge \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}').$$

Thus the result follows as extension of base field preserves geometric regularity, see Algebra, Lemma 166.1. $\square$

Lemma 50.4. *Let R be a Noetherian ring. Then R is a G-ring if and only if for every finite free ring map $R \to S$ the formal fibres of S are regular rings.*

Proof. Assume that for any finite free ring map $R \to S$ the ring S has regular formal fibres. Let $\mathfrak{q} \subset \mathfrak{p} \subset R$ be primes and let $\kappa(\mathfrak{q}) \subset L$ be a finite purely inseparable extension. To show that R is a G-ring it suffices to show that

$$R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} L$$

is a regular ring. Choose a finite free extension $R \to R'$ such that $\mathfrak{q}' = \mathfrak{q}R'$ is a prime and such that $\kappa(\mathfrak{q}')$ is isomorphic to L over $\kappa(\mathfrak{q})$, see Algebra, Lemma 159.3. By Algebra, Lemma 97.8 we have

$$R_\mathfrak{p}^\wedge \otimes_R R' = \prod (R'_{\mathfrak{p}'_i})^\wedge$$

where $\mathfrak{p}'_i$ are the primes of R' lying over $\mathfrak{p}$. Thus we have

$$R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} L = R_\mathfrak{p}^\wedge \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = \prod (R'_{\mathfrak{p}'_i})^\wedge \otimes_{R'_{\mathfrak{p}'_i}} \kappa(\mathfrak{q}')$$

Our assumption is that the rings on the right are regular, hence the ring on the left is regular too. Thus R is a G-ring. The converse follows from Lemma 50.3. $\square$

Lemma 50.5. *Let k be a field of characteristic p. Let $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_n]$ and denote K the fraction field of A. Let $\mathfrak{p} \subset A$ be a prime. Then $A_\mathfrak{p}^\wedge \otimes_A K$ is geometrically regular over K.*

Proof. Let L/K be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $A_\mathfrak{p}^\wedge \otimes L$ is regular. The base case is $L = K$: as A is regular, $A_\mathfrak{p}^\wedge$ is regular (Lemma 43.4), hence the localization $A_\mathfrak{p}^\wedge \otimes K$ is regular. Let $K \subset M \subset L$ be a subfield such that L is a degree p extension of M obtained by adjoining a pth root of an element $f \in M$. Let B be a finite A-subalgebra of M with fraction field M. Clearing denominators, we may and do assume $f \in B$. Set $C = B[z]/(z^p - f)$ and note that $B \subset C$ is finite and that the fraction field of C is L. Since $A \subset B \subset C$ are finite and $L/M/K$ are purely inseparable we see that for every element of B or C some power of it lies in A. Hence there is a unique prime $\mathfrak{r} \subset B$, resp. $\mathfrak{q} \subset C$ lying over $\mathfrak{p}$. Note that

$$A_\mathfrak{p}^\wedge \otimes_A M = B_\mathfrak{r}^\wedge \otimes_B M$$

see Algebra, Lemma 97.8. By induction we know that this ring is regular. In the same manner we have

$$A_{\mathfrak p}^\wedge \otimes_A L = C_{\mathfrak r}^\wedge \otimes_C L = B_{\mathfrak r}^\wedge \otimes_B M[z]/(z^p - f)$$

the last equality because the completion of $C = B[z]/(z^p - f)$ equals $B_{\mathfrak r}^\wedge[z]/(z^p - f)$. By Lemma 48.5 we know there exists a derivation $D : B \to B$ such that $D(f) \neq 0$. In other words, $g = D(f)$ is a unit in M! By Lemma 48.1 D extends to a derivation of $B_{\mathfrak r}$, $B_{\mathfrak r}^\wedge$ and $B_{\mathfrak r}^\wedge \otimes_B M$ (successively extending through a localization, a completion, and a localization). Since it is an extension we end up with a derivation of $B_{\mathfrak r}^\wedge \otimes_B M$ which maps f to g and g is a unit of the ring $B_{\mathfrak r}^\wedge \otimes_B M$. Hence $A_{\mathfrak p}^\wedge \otimes_A L$ is regular by Lemma 48.4 and we win. □

Proposition 50.6. *A Noetherian complete local ring is a G-ring.*

Proof. Let A be a Noetherian complete local ring. By Lemma 50.2 it suffices to check that $B = A/\mathfrak q$ has geometrically regular formal fibres over the minimal prime (0) of B. Thus we may assume that A is a domain and it suffices to check the condition for the formal fibres over the minimal prime (0) of A. Let K be the fraction field of A.

We can choose a subring $A_0 \subset A$ which is a regular complete local ring such that A is finite over A_0, see Algebra, Lemma 160.11. Moreover, we may assume that A_0 is a power series ring over a field or a Cohen ring. By Lemma 50.3 we see that it suffices to prove the result for A_0.

Assume that A is a power series ring over a field or a Cohen ring. Since A is regular the localizations $A_{\mathfrak p}$ are regular (see Algebra, Definition 110.7 and the discussion preceding it). Hence the completions $A_{\mathfrak p}^\wedge$ are regular, see Lemma 43.4. Hence the fibre $A_{\mathfrak p}^\wedge \otimes_A K$ is, as a localization of $A_{\mathfrak p}^\wedge$, also regular. Thus we are done if the characteristic of K is 0. The positive characteristic case is the case $A = k[[x_1, \ldots, x_d]]$ which is a special case of Lemma 50.5. □

Lemma 50.7. *Let R be a Noetherian ring. Then R is a G-ring if and only if $R_{\mathfrak m}$ has geometrically regular formal fibres for every maximal ideal $\mathfrak m$ of R.*

Proof. Assume $R_{\mathfrak m} \to R_{\mathfrak m}^\wedge$ is regular for every maximal ideal $\mathfrak m$ of R. Let $\mathfrak p$ be a prime of R and choose a maximal ideal $\mathfrak p \subset \mathfrak m$. Since $R_{\mathfrak m} \to R_{\mathfrak m}^\wedge$ is faithfully flat we can choose a prime $\mathfrak p'$ if $R_{\mathfrak m}^\wedge$ lying over $\mathfrak p R_{\mathfrak m}$. Consider the commutative diagram

$$\begin{tikzcd} R_{\mathfrak m}^\wedge \arrow[r] & (R_{\mathfrak m}^\wedge)_{\mathfrak p'} \arrow[r] & (R_{\mathfrak m}^\wedge)_{\mathfrak p'}^\wedge \\ R_{\mathfrak m} \arrow[r] \arrow[u] & R_{\mathfrak p} \arrow[r] \arrow[u] & R_{\mathfrak p}^\wedge \arrow[u] \end{tikzcd}$$

By assumption the ring map $R_{\mathfrak m} \to R_{\mathfrak m}^\wedge$ is regular. By Proposition 50.6 $(R_{\mathfrak m}^\wedge)_{\mathfrak p'} \to (R_{\mathfrak m}^\wedge)_{\mathfrak p'}^\wedge$ is regular. The localization $R_{\mathfrak m}^\wedge \to (R_{\mathfrak m}^\wedge)_{\mathfrak p'}$ is regular. Hence $R_{\mathfrak m} \to (R_{\mathfrak m}^\wedge)_{\mathfrak p'}^\wedge$ is regular by Lemma 41.4. Since it factors through the localization $R_{\mathfrak p}$, also the ring map $R_{\mathfrak p} \to (R_{\mathfrak m}^\wedge)_{\mathfrak p'}^\wedge$ is regular. Thus we may apply Lemma 41.7 to see that $R_{\mathfrak p} \to R_{\mathfrak p}^\wedge$ is regular. □

Lemma 50.8. *Let R be a Noetherian local ring which is a G-ring. Then the henselization R^h and the strict henselization R^{sh} are G-rings.*

Proof. We will use the criterion of Lemma 50.7. Let $\mathfrak q \subset R^h$ be a prime and set $\mathfrak p = R \cap \mathfrak q$. Set $\mathfrak q_1 = \mathfrak q$ and let $\mathfrak q_2, \ldots, \mathfrak q_t$ be the other primes of R^h lying over $\mathfrak p$, so that $R^h \otimes_R \kappa(\mathfrak p) = \prod_{i=1,\ldots,t} \kappa(\mathfrak q_i)$, see Lemma 45.13. Using that $(R^h)^\wedge = R^\wedge$ (Lemma 45.3) we see

$$\prod\nolimits_{i=1,\ldots,t} (R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak q_i) = (R^h)^\wedge \otimes_{R^h} (R^h \otimes_R \kappa(\mathfrak p)) = R^\wedge \otimes_R \kappa(\mathfrak p)$$

Hence $(R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak q_i)$ is geometrically regular over $\kappa(\mathfrak p)$ by assumption. Since $\kappa(\mathfrak q_i)$ is separable algebraic over $\kappa(\mathfrak p)$ it follows from Algebra, Lemma 166.6 that $(R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak q_i)$ is geometrically regular over $\kappa(\mathfrak q_i)$.

Let $\mathfrak r \subset R^{sh}$ be a prime and set $\mathfrak p = R \cap \mathfrak r$. Set $\mathfrak r_1 = \mathfrak r$ and let $\mathfrak r_2, \ldots, \mathfrak r_s$ be the other primes of R^{sh} lying over $\mathfrak p$, so that $R^{sh} \otimes_R \kappa(\mathfrak p) = \prod_{i=1,\ldots,s} \kappa(\mathfrak r_i)$, see Lemma 45.13. Then we see that

$$\prod\nolimits_{i=1,\ldots,s} (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak r_i) = (R^{sh})^\wedge \otimes_{R^{sh}} (R^{sh} \otimes_R \kappa(\mathfrak p)) = (R^{sh})^\wedge \otimes_R \kappa(\mathfrak p)$$

Note that $R^\wedge \to (R^{sh})^\wedge$ is formally smooth in the $\mathfrak m_{(R^{sh})^\wedge}$-adic topology, see Lemma 45.3. Hence $R^\wedge \to (R^{sh})^\wedge$ is regular by Proposition 49.2. We conclude that $(R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak r_i)$ is regular over $\kappa(\mathfrak p)$ by Lemma 41.4 as $R^\wedge \otimes_R \kappa(\mathfrak p)$ is regular over $\kappa(\mathfrak p)$ by assumption. Since $\kappa(\mathfrak r_i)$ is separable algebraic over $\kappa(\mathfrak p)$ it follows from Algebra, Lemma 166.6 that $(R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak r_i)$ is geometrically regular over $\kappa(\mathfrak r_i)$. □

Lemma 50.9. *Let p be a prime number. Let A be a Noetherian complete local domain with fraction field K of characteristic p. Let $\mathfrak q \subset A[x]$ be a maximal ideal lying over the maximal ideal of A and let $(0) \neq \mathfrak r \subset \mathfrak q$ be a prime lying over $(0) \subset A$. Then $A[x]^\wedge_{\mathfrak q} \otimes_{A[x]} \kappa(\mathfrak r)$ is geometrically regular over $\kappa(\mathfrak r)$.*

Proof. Note that $K \subset \kappa(\mathfrak r)$ is finite. Hence, given a finite purely inseparable extension $L/\kappa(\mathfrak r)$ there exists a finite extension of Noetherian complete local domains $A \subset B$ such that $\kappa(\mathfrak r) \otimes_A B$ surjects onto L. Namely, you take $B \subset L$ a finite A-subalgebra whose field of fractions is L. Denote $\mathfrak r' \subset B[x]$ the kernel of the map $B[x] = A[x] \otimes_A B \to \kappa(\mathfrak r) \otimes_A B \to L$ so that $\kappa(\mathfrak r') = L$. Then

$$A[x]^\wedge_{\mathfrak q} \otimes_{A[x]} L = A[x]^\wedge_{\mathfrak q} \otimes_{A[x]} B[x] \otimes_{B[x]} \kappa(\mathfrak r') = \prod B[x]^\wedge_{\mathfrak q_i} \otimes_{B[x]} \kappa(\mathfrak r')$$

where $\mathfrak q_1, \ldots, \mathfrak q_t$ are the primes of $B[x]$ lying over $\mathfrak q$, see Algebra, Lemma 97.8. Thus we see that it suffices to prove the rings $B[x]^\wedge_{\mathfrak q_i} \otimes_{B[x]} \kappa(\mathfrak r')$ are regular. This reduces us to showing that $A[x]^\wedge_{\mathfrak q} \otimes_{A[x]} \kappa(\mathfrak r)$ is regular in the special case that $K = \kappa(\mathfrak r)$.

Assume $K = \kappa(\mathfrak r)$. In this case we see that $\mathfrak r K[x]$ is generated by $x - f$ for some $f \in K$ and

$$A[x]^\wedge_{\mathfrak q} \otimes_{A[x]} \kappa(\mathfrak r) = (A[x]^\wedge_{\mathfrak q} \otimes_A K)/(x - f)$$

The derivation $D = \mathrm{d}/\mathrm{d}x$ of $A[x]$ extends to $K[x]$ and maps $x - f$ to a unit of $K[x]$. Moreover D extends to $A[x]^\wedge_{\mathfrak q} \otimes_A K$ by Lemma 48.1. As $A \to A[x]^\wedge_{\mathfrak q}$ is formally smooth (see Lemmas 37.2 and 37.4) the ring $A[x]^\wedge_{\mathfrak q} \otimes_A K$ is regular by Proposition 49.2 (the arguments of the proof of that proposition simplify significantly in this particular case). We conclude by Lemma 48.2. □

Proposition 50.10. *Let R be a G-ring. If $R \to S$ is essentially of finite type then S is a G-ring.*

Proof. Since being a G-ring is a property of the local rings it is clear that a localization of a G-ring is a G-ring. Conversely, if every localization at a prime is a G-ring, then the ring is a G-ring. Thus it suffices to show that $S_{\mathfrak q}$ is a G-ring for every finite type R-algebra S and every prime $\mathfrak q$ of S. Writing S as a quotient of $R[x_1, \ldots, x_n]$ we see from Lemma 50.3 that it suffices to prove that $R[x_1, \ldots, x_n]$ is a G-ring. By induction on n it suffices to prove that $R[x]$ is a G-ring. Let $\mathfrak q \subset R[x]$ be a maximal ideal. By Lemma 50.7 it suffices to show that

$$R[x]_{\mathfrak q} \longrightarrow R[x]_{\mathfrak q}^\wedge$$

is regular. If $\mathfrak q$ lies over $\mathfrak p \subset R$, then we may replace R by $R_{\mathfrak p}$. Hence we may assume that R is a Noetherian local G-ring with maximal ideal $\mathfrak m$ and that $\mathfrak q \subset R[x]$ lies over $\mathfrak m$. Note that there is a unique prime $\mathfrak q' \subset R^\wedge[x]$ lying over $\mathfrak q$. Consider the diagram

$$\begin{tikzcd} R[x]_{\mathfrak q}^\wedge \arrow[r] & (R^\wedge[x]_{\mathfrak q'})^\wedge \\ R[x]_{\mathfrak q} \arrow[r] \arrow[u] & R^\wedge[x]_{\mathfrak q'} \arrow[u] \end{tikzcd}$$

Since R is a G-ring the lower horizontal arrow is regular (as a localization of a base change of the regular ring map $R \to R^\wedge$). Suppose we can prove the right vertical arrow is regular. Then it follows that the composition $R[x]_{\mathfrak q} \to (R^\wedge[x]_{\mathfrak q'})^\wedge$ is regular, and hence the left vertical arrow is regular by Lemma 41.7. Hence we see that we may assume R is a Noetherian complete local ring and $\mathfrak q$ a prime lying over the maximal ideal of R.

Let R be a Noetherian complete local ring and let $\mathfrak q \subset R[x]$ be a maximal ideal lying over the maximal ideal of R. Let $\mathfrak r \subset \mathfrak q$ be a prime ideal. We want to show that $R[x]_{\mathfrak q}^\wedge \otimes_{R[x]} \kappa(\mathfrak r)$ is a geometrically regular algebra over $\kappa(\mathfrak r)$. Set $\mathfrak p = R \cap \mathfrak r$. Then we can replace R by $R/\mathfrak p$ and $\mathfrak q$ and $\mathfrak r$ by their images in $R/\mathfrak p[x]$, see Lemma 50.2. Hence we may assume that R is a domain and that $\mathfrak r \cap R = (0)$.

By Algebra, Lemma 160.11 we can find $R_0 \subset R$ which is regular and such that R is finite over R_0. Applying Lemma 50.3 we see that it suffices to prove $R[x]_{\mathfrak q}^\wedge \otimes_{R[x]} \kappa(\mathfrak r)$ is geometrically regular over $\kappa(r)$ when, in addition to the above, R is a regular complete local ring.

Now R is a regular complete local ring, we have $\mathfrak q \subset \mathfrak r \subset R[x]$, we have $(0) = R \cap \mathfrak r$ and $\mathfrak q$ is a maximal ideal lying over the maximal ideal of R. Since R is regular the ring $R[x]$ is regular (Algebra, Lemma 163.10). Hence the localization $R[x]_{\mathfrak q}$ is regular. Hence the completions $R[x]_{\mathfrak q}^\wedge$ are regular, see Lemma 43.4. Hence the fibre $R[x]_{\mathfrak q}^\wedge \otimes_{R[x]} \kappa(\mathfrak r)$ is, as a localization of $R[x]_{\mathfrak q}^\wedge$, also regular. Thus we are done if the characteristic of the fraction field of R is 0.

If the characteristic of R is positive, then $R = k[[x_1, \ldots, x_n]]$. In this case we split the argument in two subcases:

(1) The case $\mathfrak r = (0)$. The result is a direct consequence of Lemma 50.5.
(2) The case $\mathfrak r \neq (0)$. This is Lemma 50.9.

$\square$

Remark 50.11. Let R be a G-ring and let $I \subset R$ be an ideal. In general it is not the case that the I-adic completion $R^\wedge$ is a G-ring. An example was given by Nishimura in [Nis81]. A generalization and, in some sense, clarification of this example can be found in the last section of [Dum00].

Proposition 50.12. *The following types of rings are G-rings:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) **Z**,
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. For fields, **Z** and Dedekind domains of characteristic zero this follows immediately from the definition and the fact that the completion of a discrete valuation ring is a discrete valuation ring. A Noetherian complete local ring is a G-ring by Proposition 50.6. The statement on finite type overrings is Proposition 50.10. □

Lemma 50.13. *Let $(A, \mathfrak m)$ be a henselian local ring. Then A is a filtered colimit of a system of henselian local G-rings with local transition maps.*

Proof. Write $A = \mathrm{colim}\, A_i$ as a filtered colimit of finite type **Z**-algebras. Let $\mathfrak p_i$ be the prime ideal of A_i lying under $\mathfrak m$. We may replace A_i by the localization of A_i at $\mathfrak p_i$. Then A_i is a Noetherian local G-ring (Proposition 50.12). By Lemma 12.5 we see that $A = \mathrm{colim}\, A_i^h$. By Lemma 50.8 the rings A_i^h are G-rings. □

Lemma 50.14. *Let A be a G-ring. Let $I \subset A$ be an ideal and let $A^\wedge$ be the completion of A with respect to I. Then $A \to A^\wedge$ is regular.*

Proof. The ring map $A \to A^\wedge$ is flat by Algebra, Lemma 97.2. The ring $A^\wedge$ is Noetherian by Algebra, Lemma 97.6. Thus it suffices to check the third condition of Lemma 41.2. Let $\mathfrak m' \subset A^\wedge$ be a maximal ideal lying over $\mathfrak m \subset A$. By Algebra, Lemma 96.6 we have $IA^\wedge \subset \mathfrak m'$. Since $A^\wedge/IA^\wedge = A/I$ we see that $I \subset \mathfrak m$, $\mathfrak m/I = \mathfrak m'/IA^\wedge$, and $A/\mathfrak m = A^\wedge/\mathfrak m'$. Since $A^\wedge/\mathfrak m'$ is a field, we conclude that $\mathfrak m$ is a maximal ideal as well. Then $A_\mathfrak m \to A^\wedge_{\mathfrak m'}$ is a flat local ring homomorphism of Noetherian local rings which identifies residue fields and such that $\mathfrak m A^\wedge_{\mathfrak m'} = \mathfrak m' A^\wedge_{\mathfrak m'}$. Thus it induces an isomorphism on complete local rings, see Lemma 43.9. Let $(A_\mathfrak m)^\wedge$ be the completion of $A_\mathfrak m$ with respect to its maximal ideal. The ring map

$$(A^\wedge)_{\mathfrak m'} \to ((A^\wedge)_{\mathfrak m'})^\wedge = (A_\mathfrak m)^\wedge$$

is faithfully flat (Algebra, Lemma 97.3). Thus we can apply Lemma 41.7 to the ring maps

$$A_\mathfrak m \to (A^\wedge)_{\mathfrak m'} \to (A_\mathfrak m)^\wedge$$

to conclude because $A_\mathfrak m \to (A_\mathfrak m)^\wedge$ is regular as A is a G-ring. □

Lemma 50.15. *Let A be a G-ring. Let $I \subset A$ be an ideal. Let (A^h, I^h) be the henselization of the pair (A, I), see Lemma 12.1. Then A^h is a G-ring.*

Proof. Let $\mathfrak m^h \subset A^h$ be a maximal ideal. We have to show that the map from $A^h_{\mathfrak m^h}$ to its completion has geometrically regular fibres, see Lemma 50.7. Let $\mathfrak m$ be the inverse image of $\mathfrak m^h$ in A. Note that $I^h \subset \mathfrak m^h$ and hence $I \subset \mathfrak m$ as (A^h, I^h) is a henselian pair. Recall that A^h is Noetherian, $I^h = IA^h$, and that $A \to A^h$ induces an isomorphism on I-adic completions, see Lemma 12.4. Then the local homomorphism of Noetherian local rings

$$A_\mathfrak m \to A^h_{\mathfrak m^h}$$

induces an isomorphism on completions at maximal ideals by Lemma 43.9 (details omitted). Let $\mathfrak q^h$ be a prime of $A^h_{\mathfrak m^h}$ lying over $\mathfrak q \subset A_\mathfrak m$. Set $\mathfrak q_1 = \mathfrak q^h$ and let $\mathfrak q_2, \ldots, \mathfrak q_t$ be the other primes of A^h lying over $\mathfrak q$, so that $A^h \otimes_A \kappa(\mathfrak q) = \prod_{i=1,\ldots,t} \kappa(\mathfrak q_i)$, see Lemma 45.12. Using that $(A^h)^\wedge_{\mathfrak m^h} = (A_\mathfrak m)^\wedge$ as discussed above we see

$$\prod\nolimits_{i=1,\ldots,t} (A^h_{\mathfrak m^h})^\wedge \otimes_{A^h_{\mathfrak m^h}} \kappa(\mathfrak q_i) = (A^h_{\mathfrak m^h})^\wedge \otimes_{A^h_{\mathfrak m^h}} (A^h_{\mathfrak m^h} \otimes_{A_\mathfrak m} \kappa(\mathfrak q)) = (A_\mathfrak m)^\wedge \otimes_{A_\mathfrak m} \kappa(\mathfrak q)$$

Hence, as one of the components, the ring

$$(A^h_{\mathfrak m^h})^\wedge \otimes_{A^h_{\mathfrak m^h}} \kappa(\mathfrak q^h)$$

is geometrically regular over $\kappa(\mathfrak q)$ by assumption on A. Since $\kappa(\mathfrak q^h)$ is separable algebraic over $\kappa(\mathfrak q)$ it follows from Algebra, Lemma 166.6 that

$$(A^h_{\mathfrak m^h})^\wedge \otimes_{A^h_{\mathfrak m^h}} \kappa(\mathfrak q^h)$$

is geometrically regular over $\kappa(\mathfrak q^h)$ as desired. □

51. Properties of formal fibres

In this section we redo some of the arguments of Section 50 for to be able to talk intelligently about properties of the formal fibres of Noetherian rings.

Let P be a property of ring maps $k \to R$ where k is a field and R is Noetherian. We say P holds for the fibres of a ring homomorphism $A \to B$ with B Noetherian if P holds for $\kappa(\mathfrak q) \to B \otimes_A \kappa(\mathfrak q)$ for all primes $\mathfrak q$ of A. In the following we will use the following assertions

(A) $P(k \to R) \Rightarrow P(k' \to R \otimes_k k')$ for finitely generated field extensions k'/k,
(B) $P(k \to R_\mathfrak p),\ \forall \mathfrak p \in \operatorname{Spec}(R) \Leftrightarrow P(k \to R)$,
(C) given flat maps $A \to B \to C$ of Noetherian rings, if the fibres of $A \to B$ have P and $B \to C$ is regular, then the fibres of $A \to C$ have P,
(D) given flat maps $A \to B \to C$ of Noetherian rings if the fibres of $A \to C$ have P and $B \to C$ is faithfully flat, then the fibres of $A \to B$ have P,
(E) given $k \to k' \to R$ with R Noetherian if k'/k is separable algebraic and $P(k \to R)$, then $P(k' \to R)$, and
(F) add more here.

Given a Noetherian local ring A we say "*the formal fibres of A have P*" if P holds for the fibres of $A \to A^\wedge$. We say that R *is a P-ring* if R is Noetherian and for all primes $\mathfrak p$ of R the formal fibres of $R_\mathfrak p$ have P.

Lemma 51.1. *Let R be a Noetherian ring. Let P be a property as above. Then R is a P-ring if and only if for every pair of primes $\mathfrak q \subset \mathfrak p \subset R$ the $\kappa(\mathfrak q)$-algebra*

$$(R/\mathfrak q)^\wedge_\mathfrak p \otimes_{R/\mathfrak q} \kappa(\mathfrak q)$$

has property P.

Proof. This follows from the fact that

$$R^\wedge_\mathfrak p \otimes_R \kappa(\mathfrak q) = (R/\mathfrak q)^\wedge_\mathfrak p \otimes_{R/\mathfrak q} \kappa(\mathfrak q)$$

as algebras over $\kappa(\mathfrak q)$. □

Lemma 51.2. *Let $R \to \Lambda$ be a homomorphism of Noetherian rings. Assume P has property (B). The following are equivalent*

(1) *the fibres of $R \to \Lambda$ have P,*

(2) *the fibres of* $R_\mathfrak{p} \to \Lambda_\mathfrak{q}$ *have* P *for all* $\mathfrak{q} \subset \Lambda$ *lying over* $\mathfrak{p} \subset R$*, and*
(3) *the fibres of* $R_\mathfrak{m} \to \Lambda_{\mathfrak{m}'}$ *have* P *for all maximal ideals* $\mathfrak{m}' \subset \Lambda$ *lying over* $\mathfrak{m}$ *in* R*.*

Proof. Let $\mathfrak{p} \subset R$ be a prime. Then the fibre over $\mathfrak{p}$ is the ring $\Lambda \otimes_R \kappa(\mathfrak{p})$ whose spectrum maps bijectively onto the subset of $\mathrm{Spec}(\Lambda)$ consisting of primes $\mathfrak{q}$ lying over $\mathfrak{p}$, see Algebra, Remark 17.8. For such a prime $\mathfrak{q}$ choose a maximal ideal $\mathfrak{q} \subset \mathfrak{m}'$ and set $\mathfrak{m} = R \cap \mathfrak{m}'$. Then $\mathfrak{p} \subset \mathfrak{m}$ and we have

$$(\Lambda \otimes_R \kappa(\mathfrak{p}))_\mathfrak{q} \cong (\Lambda_{\mathfrak{m}'} \otimes_{R_\mathfrak{m}} \kappa(\mathfrak{p}))_\mathfrak{q}$$

as $\kappa(\mathfrak{q})$-algebras. Thus (1), (2), and (3) are equivalent because by (B) we can check property P on local rings. □

Lemma 51.3. *Let* $R \to R'$ *be a finite type map of Noetherian rings and let*

$$\begin{tikzcd} \mathfrak{q}' \arrow[r] \arrow[d, dash] & \mathfrak{p}' \arrow[r] \arrow[d, dash] & R' \\ \mathfrak{q} \arrow[r] & \mathfrak{p} \arrow[r] & R \arrow[u] \end{tikzcd}$$

be primes. Assume $R \to R'$ *is quasi-finite at* $\mathfrak{p}'$*. Assume* P *satisfies (A) and (B).*
(1) *If* $\kappa(\mathfrak{q}) \to R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q})$ *has* P*, then* $\kappa(\mathfrak{q}') \to R'^\wedge_{\mathfrak{p}'} \otimes_{R'} \kappa(\mathfrak{q}')$ *has* P*.*
(2) *If the formal fibres of* $R_\mathfrak{p}$ *have* P*, then the formal fibres of* $R'_{\mathfrak{p}'}$ *have* P*.*
(3) *If* $R \to R'$ *is quasi-finite and* R *is a* P*-ring, then* R' *is a* P*-ring.*

Proof. It is clear that (1) ⇒ (2) ⇒ (3). Assume P holds for $\kappa(\mathfrak{q}) \to R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q})$. By Algebra, Lemma 124.3 we see that

$$R_\mathfrak{p}^\wedge \otimes_R R' = (R'_{\mathfrak{p}'})^\wedge \times B$$

for some $R_\mathfrak{p}^\wedge$-algebra B. Hence $R'_{\mathfrak{p}'} \to (R'_{\mathfrak{p}'})^\wedge$ is a factor of a base change of the map $R_\mathfrak{p} \to R_\mathfrak{p}^\wedge$. It follows that $(R'_{\mathfrak{p}'})^\wedge \otimes_{R'} \kappa(\mathfrak{q}')$ is a factor of

$$R_\mathfrak{p}^\wedge \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}').$$

Thus the result follows from the assumptions on P. □

Lemma 51.4. *Let* R *be a Noetherian ring. Assume* P *satisfies (C) and (D). Then* R *is a* P*-ring if and only if the formal fibres of* $R_\mathfrak{m}$ *have* P *for every maximal ideal* $\mathfrak{m}$ *of* R*.*

Proof. Assume the formal fibres of $R_\mathfrak{m}$ have P for all maximal ideals $\mathfrak{m}$ of R. Let $\mathfrak{p}$ be a prime of R and choose a maximal ideal $\mathfrak{p} \subset \mathfrak{m}$. Since $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ is faithfully flat we can choose a prime $\mathfrak{p}'$ if $R_\mathfrak{m}^\wedge$ lying over $\mathfrak{p}R_\mathfrak{m}$. Consider the commutative diagram

$$\begin{tikzcd} R_\mathfrak{m}^\wedge \arrow[r] & (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'} \arrow[r] & (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge \\ R_\mathfrak{m} \arrow[r] \arrow[u] & R_\mathfrak{p} \arrow[r] \arrow[u] & R_\mathfrak{p}^\wedge \arrow[u] \end{tikzcd}$$

By assumption the fibres of the ring map $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ have P. By Proposition 50.6 $(R_\mathfrak{m}^\wedge)_{\mathfrak{p}'} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ is regular. The localization $R_\mathfrak{m}^\wedge \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}$ is regular. Hence $R_\mathfrak{m}^\wedge \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ is regular by Lemma 41.4. Hence the fibres of $R_\mathfrak{m} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ have P by (C). Since $R_\mathfrak{m} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ factors through the localization $R_\mathfrak{p}$, also the fibres of

$R_\mathfrak{p} \to (R^\wedge_\mathfrak{m})^\wedge_{\mathfrak{p}'}$ have P. Thus we may apply (D) to see that the fibres of $R_\mathfrak{p} \to R^\wedge_\mathfrak{p}$ have P. $\square$

Proposition 51.5. *Let R be a P-ring where P satisfies (A), (B), (C), and (D). If $R \to S$ is essentially of finite type then S is a P-ring.*

Proof. Since being a P-ring is a property of the local rings it is clear that a localization of a P-ring is a P-ring. Conversely, if every localization at a prime is a P-ring, then the ring is a P-ring. Thus it suffices to show that $S_\mathfrak{q}$ is a P-ring for every finite type R-algebra S and every prime $\mathfrak{q}$ of S. Writing S as a quotient of $R[x_1, \ldots, x_n]$ we see from Lemma 51.3 that it suffices to prove that $R[x_1, \ldots, x_n]$ is a P-ring. By induction on n it suffices to prove that $R[x]$ is a P-ring. Let $\mathfrak{q} \subset R[x]$ be a maximal ideal. By Lemma 51.4 it suffices to show that the fibres of

$$R[x]_\mathfrak{q} \longrightarrow R[x]^\wedge_\mathfrak{q}$$

have P. If $\mathfrak{q}$ lies over $\mathfrak{p} \subset R$, then we may replace R by $R_\mathfrak{p}$. Hence we may assume that R is a Noetherian local P-ring with maximal ideal $\mathfrak{m}$ and that $\mathfrak{q} \subset R[x]$ lies over $\mathfrak{m}$. Note that there is a unique prime $\mathfrak{q}' \subset R^\wedge[x]$ lying over $\mathfrak{q}$. Consider the diagram

$$\begin{array}{ccc} R[x]^\wedge_\mathfrak{q} & \longrightarrow & (R^\wedge[x]_{\mathfrak{q}'})^\wedge \\ \uparrow & & \uparrow \\ R[x]_\mathfrak{q} & \longrightarrow & R^\wedge[x]_{\mathfrak{q}'} \end{array}$$

Since R is a P-ring the fibres of $R[x] \to R^\wedge[x]$ have P because they are base changes of the fibres of $R \to R^\wedge$ by a finitely generated field extension so (A) applies. Hence the fibres of the lower horizontal arrow have P for example by Lemma 51.2. The right vertical arrow is regular because $R^\wedge$ is a G-ring (Propositions 50.6 and 50.10). It follows that the fibres of the composition $R[x]_\mathfrak{q} \to (R^\wedge[x]_{\mathfrak{q}'})^\wedge$ have P by (C). Hence the fibres of the left vertical arrow have P by (D) and the proof is complete. $\square$

Lemma 51.6. *Let A be a P-ring where P satisfies (B) and (D). Let $I \subset A$ be an ideal and let $A^\wedge$ be the completion of A with respect to I. Then the fibres of $A \to A^\wedge$ have P.*

Proof. The ring map $A \to A^\wedge$ is flat by Algebra, Lemma 97.2. The ring $A^\wedge$ is Noetherian by Algebra, Lemma 97.6. Thus it suffices to check the third condition of Lemma 51.2. Let $\mathfrak{m}' \subset A^\wedge$ be a maximal ideal lying over $\mathfrak{m} \subset A$. By Algebra, Lemma 96.6 we have $IA^\wedge \subset \mathfrak{m}'$. Since $A^\wedge/IA^\wedge = A/I$ we see that $I \subset \mathfrak{m}$, $\mathfrak{m}/I = \mathfrak{m}'/IA^\wedge$, and $A/\mathfrak{m} = A^\wedge/\mathfrak{m}'$. Since $A^\wedge/\mathfrak{m}'$ is a field, we conclude that $\mathfrak{m}$ is a maximal ideal as well. Then $A_\mathfrak{m} \to A^\wedge_{\mathfrak{m}'}$ is a flat local ring homomorphism of Noetherian local rings which identifies residue fields and such that $\mathfrak{m}A^\wedge_{\mathfrak{m}'} = \mathfrak{m}'A^\wedge_{\mathfrak{m}'}$. Thus it induces an isomorphism on complete local rings, see Lemma 43.9. Let $(A_\mathfrak{m})^\wedge$ be the completion of $A_\mathfrak{m}$ with respect to its maximal ideal. The ring map

$$(A^\wedge)_{\mathfrak{m}'} \to ((A^\wedge)_{\mathfrak{m}'})^\wedge = (A_\mathfrak{m})^\wedge$$

is faithfully flat (Algebra, Lemma 97.3). Thus we can apply (D) to the ring maps

$$A_\mathfrak{m} \to (A^\wedge)_{\mathfrak{m}'} \to (A_\mathfrak{m})^\wedge$$

to conclude because the fibres of $A_\mathfrak{m} \to (A_\mathfrak{m})^\wedge$ have P as A is a P-ring. $\square$

Lemma 51.7. *Let A be a P-ring where P satisfies (B), (C), (D), and (E). Let $I \subset A$ be an ideal. Let (A^h, I^h) be the henselization of the pair (A, I), see Lemma 12.1. Then A^h is a P-ring.*

Proof. Let $\mathfrak{m}^h \subset A^h$ be a maximal ideal. We have to show that the fibres of $A^h_{\mathfrak{m}^h} \to (A^h_{\mathfrak{m}^h})^\wedge$ have P, see Lemma 51.4. Let $\mathfrak{m}$ be the inverse image of $\mathfrak{m}^h$ in A. Note that $I^h \subset \mathfrak{m}^h$ and hence $I \subset \mathfrak{m}$ as (A^h, I^h) is a henselian pair. Recall that A^h is Noetherian, $I^h = IA^h$, and that $A \to A^h$ induces an isomorphism on I-adic completions, see Lemma 12.4. Then the local homomorphism of Noetherian local rings

$$A_{\mathfrak{m}} \to A^h_{\mathfrak{m}^h}$$

induces an isomorphism on completions at maximal ideals by Lemma 43.9 (details omitted). Let $\mathfrak{q}^h$ be a prime of $A^h_{\mathfrak{m}^h}$ lying over $\mathfrak{q} \subset A_{\mathfrak{m}}$. Set $\mathfrak{q}_1 = \mathfrak{q}^h$ and let $\mathfrak{q}_2, \ldots, \mathfrak{q}_t$ be the other primes of A^h lying over $\mathfrak{q}$, so that $A^h \otimes_A \kappa(\mathfrak{q}) = \prod_{i=1,\ldots,t} \kappa(\mathfrak{q}_i)$, see Lemma 45.12. Using that $(A^h)^\wedge_{\mathfrak{m}^h} = (A_{\mathfrak{m}})^\wedge$ as discussed above we see

$$\prod\nolimits_{i=1,\ldots,t} (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}_i) = (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} (A^h_{\mathfrak{m}^h} \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{q})) = (A_{\mathfrak{m}})^\wedge \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{q})$$

Hence, looking at local rings and using (B), we see that

$$\kappa(\mathfrak{q}) \longrightarrow (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}^h)$$

has P as $\kappa(\mathfrak{q}) \to (A_{\mathfrak{m}})^\wedge \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{q})$ does by assumption on A. Since $\kappa(\mathfrak{q}^h)/\kappa(\mathfrak{q})$ is separable algebraic, by (E) we find that $\kappa(\mathfrak{q}^h) \to (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}^h)$ has P as desired. $\square$

Lemma 51.8. *Let R be a Noetherian local ring which is a P-ring where P satisfies (B), (C), (D), and (E). Then the henselization R^h and the strict henselization R^{sh} are P-rings.*

Proof. We have seen this for the henselization in Lemma 51.7. To prove it for the strict henselization, it suffices to show that the formal fibres of R^{sh} have P, see Lemma 51.4. Let $\mathfrak{r} \subset R^{sh}$ be a prime and set $\mathfrak{p} = R \cap \mathfrak{r}$. Set $\mathfrak{r}_1 = \mathfrak{r}$ and let $\mathfrak{r}_2, \ldots, \mathfrak{r}_s$ be the other primes of R^{sh} lying over $\mathfrak{p}$, so that $R^{sh} \otimes_R \kappa(\mathfrak{p}) = \prod_{i=1,\ldots,s} \kappa(\mathfrak{r}_i)$, see Lemma 45.13. Then we see that

$$\prod\nolimits_{i=1,\ldots,t} (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r}_i) = (R^{sh})^\wedge \otimes_{R^{sh}} (R^{sh} \otimes_R \kappa(\mathfrak{p})) = (R^{sh})^\wedge \otimes_R \kappa(\mathfrak{p})$$

Note that $R^\wedge \to (R^{sh})^\wedge$ is formally smooth in the $\mathfrak{m}_{(R^{sh})^\wedge}$-adic topology, see Lemma 45.3. Hence $R^\wedge \to (R^{sh})^\wedge$ is regular by Proposition 49.2. We conclude that property P holds for $\kappa(\mathfrak{p}) \to (R^{sh})^\wedge \otimes_R \kappa(\mathfrak{p})$ by (C) and our assumption on R. Using property (B), using the decomposition above, and looking at local rings we conclude that property P holds for $\kappa(\mathfrak{p}) \to (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r})$. Since $\kappa(\mathfrak{r})/\kappa(\mathfrak{p})$ is separable algebraic, it follows from (E) that P holds for $\kappa(\mathfrak{r}) \to (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r})$. $\square$

Lemma 51.9. *Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R is geometrically reduced over k".*

Proof. Part (A) follows from the definition of geometrically reduced algebras (Algebra, Definition 43.1). Part (B) follows too: a ring is reduced if and only if all local rings are reduced. Part (C). This follows from Lemma 42.1. Part (D). This follows from Algebra, Lemma 164.2. Part (E). This follows from Algebra, Lemma 43.9. $\square$

Lemma 51.10. *Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R is geometrically normal over k".*

Proof. Part (A) follows from the definition of geometrically normal algebras (Algebra, Definition 165.2). Part (B) follows too: a ring is normal if and only if all of its local rings are normal. Part (C). This follows from Lemma 42.2. Part (D). This follows from Algebra, Lemma 164.3. Part (E). This follows from Algebra, Lemma 165.6. ☐

Lemma 51.11. *Fix $n \geq 1$. Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R has (S_n)".*

Proof. Let $k \to R$ be a ring map where k is a field and R a Noetherian ring. Let k'/k be a finitely generated field extension. Then the fibres of the ring map $R \to R \otimes_k k'$ are Cohen-Macaulay by Algebra, Lemma 167.1. Hence we may apply Algebra, Lemma 163.4 to the ring map $R \to R \otimes_k k'$ to see that if R has (S_n) so does $R \otimes_k k'$. This proves (A). Part (B) follows too: a Noetherian rings has (S_n) if and only if all of its local rings have (S_n). Part (C). This follows from Algebra, Lemma 163.4 as the fibres of a regular homomorphism are regular and in particular Cohen-Macaulay. Part (D). This follows from Algebra, Lemma 164.5. Part (E). This is immediate as the condition does not refer to the ground field. ☐

Lemma 51.12. *Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R is Cohen-Macaulay".*

Proof. Follows immediately from Lemma 51.11 and the fact that a Noetherian ring is Cohen-Macaulay if and only if it satisfies conditions (S_n) for all n. ☐

Lemma 51.13. *Fix $n \geq 0$. Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"$R \otimes_k k'$ has (R_n) for all finite extensions k'/k".*

Proof. Let $k \to R$ be a ring map where k is a field and R a Noetherian ring. Assume $P(k \to R)$ is true. Let K/k be a finitely generated field extension. By Algebra, Lemma 45.3 we can find a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is separable. By Algebra, Lemma 158.10 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $R \otimes_k k'$ satisfies (S_n) because we assumed P for $k \to R$. Step 2: $R \otimes_k k' \to R \otimes_k k' \otimes_{k'} B$ is a smooth ring map (Algebra, Lemma 137.4) and we conclude $R \otimes_k k' \otimes_{k'} B$ satisfies (S_n) by Algebra, Lemma 163.5 (and using Algebra, Lemma 140.3 to see that the hypotheses are satisfied). Step 3. $R \otimes_k k' \otimes_{k'} K' = R \otimes_k K'$ satisfies (R_n) as it is a localization of a ring having (R_n). Step 4. Finally $R \otimes_k K$ satisfies (R_n) by descent of (R_n) along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Algebra, Lemma 164.6). This proves (A). Part (B) follows too: a Noetherian ring has (R_n) if and only if all of its local rings have (R_n). Part (C). This follows from Algebra, Lemma 163.5 as the fibres of a regular homomorphism are regular (small detail omitted). Part (D). This follows from Algebra, Lemma 164.6 (small detail omitted).

Part (E). Let l/k be a separable algebraic extension of fields and let $l \to R$ be a ring map with R Noetherian. Assume that $k \to R$ has P. We have to show that $l \to R$ has P. Let l'/l be a finite extension. First observe that there exists a finite subextension $l/m/k$ and a finite extension m'/m such that $l' = l \otimes_m m'$. Then $R \otimes_l l' = R \otimes_m m'$. Hence it suffices to prove that $m \to R$ has property P, i.e., we may assume that l/k is finite. If l/k is finite, then l'/k is finite and we see that

$$l' \otimes_l R = (l' \otimes_k R) \otimes_{l \otimes_k l} l$$

is a localization (by Algebra, Lemma 43.8) of the Noetherian ring $l' \otimes_k R$ which has property (R_n) by assumption P for $k \to R$. This proves that $l' \otimes_l R$ has property (R_n) as desired. □

52. Excellent rings

In this section we discuss Grothendieck's notion of excellent rings. For the definitions of G-rings, J-2 rings, and universally catenary rings we refer to Definition 50.1, Definition 47.1, and Algebra, Definition 105.3.

Definition 52.1. Let R be a ring.

(1) We say R is *quasi-excellent* if R is Noetherian, a G-ring, and J-2.
(2) We say R is *excellent* if R is quasi-excellent and universally catenary.

Thus a Noetherian ring is quasi-excellent if it has geometrically regular formal fibres and if any finite type algebra over it has closed singular set. For such a ring to be excellent we require in addition that there exists (locally) a good dimension function. We will see later (Section 109) that to be universally catenary can be formulated as a condition on the maps $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ for maximal ideals $\mathfrak{m}$ of R.

Lemma 52.2. *Any localization of a finite type ring over a (quasi-)excellent ring is (quasi-)excellent.*

Proof. For finite type algebras this follows from the definitions for the properties J-2 and universally catenary. For G-rings, see Proposition 50.10. We omit the proof that localization preserves (quasi-)excellency. □

Proposition 52.3. *The following types of rings are excellent:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. See Propositions 50.12 and 48.7 to see that these rings are G-rings and have J-2. Any Cohen-Macaulay ring is universally catenary, see Algebra, Lemma 105.9. In particular fields, Dedekind rings, and more generally regular rings are universally catenary. Via the Cohen structure theorem we see that complete local rings are universally catenary, see Algebra, Remark 160.9. □

The material developed above has some consequences for Nagata rings.

Lemma 52.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. The following are equivalent*

(1) *A is Nagata, and*
(2) *the formal fibres of A are geometrically reduced.*

Proof. Assume (2). By Algebra, Lemma 162.14 we have to show that if $A \to B$ is finite, B is a domain, and $\mathfrak{m}' \subset B$ is a maximal ideal, then $B_{\mathfrak{m}'}$ is analytically unramified. Combining Lemmas 51.9 and 51.4 and Proposition 51.5 we see that the formal fibres of $B_{\mathfrak{m}'}$ are geometrically reduced. In particular $B_{\mathfrak{m}'}^\wedge \otimes_B L$ is reduced where L is the fraction field of B. It follows that $B_{\mathfrak{m}'}^\wedge$ is reduced, i.e., $B_{\mathfrak{m}'}$ is analytically unramified.

Assume (1). Let $\mathfrak{q} \subset A$ be a prime ideal and let $K/\kappa(\mathfrak{q})$ be a finite extension. We have to show that $A^\wedge \otimes_A K$ is reduced. Let $A/\mathfrak{q} \subset B \subset K$ be a local subring finite over A whose fraction field is K. To construct B choose $x_1, \ldots, x_n \in K$ which generate K over $\kappa(\mathfrak{q})$ and which satisfy monic polynomials $P_i(T) = T^{d_i} + a_{i,1}T^{d_i-1} + \ldots + a_{i,d_i} = 0$ with $a_{i,j} \in \mathfrak{m}$. Then let B be the A-subalgebra of K generated by $x_1, \ldots, x_n$. (For more details see the proof of Algebra, Lemma 162.14.) Then

$$A^\wedge \otimes_A K = (A^\wedge \otimes_A B)_\mathfrak{q} = B^\wedge_\mathfrak{q}$$

Since $B^\wedge$ is reduced by Algebra, Lemma 162.14 the proof is complete. □

Lemma 52.5. *A quasi-excellent ring is Nagata.*

Proof. Let R be quasi-excellent. Using that a finite type algebra over R is quasi-excellent (Lemma 52.2) we see that it suffices to show that any quasi-excellent domain is N-1, see Algebra, Lemma 162.3. Applying Algebra, Lemma 161.15 (and using that a quasi-excellent ring is J-2) we reduce to showing that a quasi-excellent local domain R is N-1. As $R \to R^\wedge$ is regular we see that $R^\wedge$ is reduced by Lemma 42.1. In other words, R is analytically unramified. Hence R is N-1 by Algebra, Lemma 162.10. □

Lemma 52.6. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. If A is normal and the formal fibres of A are normal (for example if A is excellent or quasi-excellent), then $A^\wedge$ is normal.*

Proof. Follows immediately from Algebra, Lemma 163.8. □

53. Abelian categories of modules

Let R be a ring. The category Mod_R of R-modules is an abelian category. Here are some examples of subcategories of Mod_R which are abelian (we use the terminology introduced in Homology, Definition 10.1 as well as Homology, Lemmas 10.2 and 10.3):

1. The category of coherent R-modules is a weak Serre subcategory of Mod_R. This follows from Algebra, Lemma 90.3.
2. Let $S \subset R$ be a multiplicative subset. The full subcategory consisting of R-modules M such that multiplication by $s \in S$ is an isomorphism on M is a Serre subcategory of Mod_R. This follows from Algebra, Lemma 9.5.
3. Let $I \subset R$ be a finitely generated ideal. The full subcategory of I-power torsion modules is a Serre subcategory of Mod_R. See Lemma 88.5.
4. In some texts a *torsion module* is defined as a module M such that for all $x \in M$ there exists a nonzerodivisor $f \in R$ such that $fx = 0$. The full subcategory of torsion modules is a Serre subcategory of Mod_R.
5. If R is not Noetherian, then the category Mod_R^{fg} of finitely generated R-modules is **not** abelian. Namely, if $I \subset R$ is a non-finitely generated ideal, then the map $R \to R/I$ does not have a kernel in Mod_R^{fg}.

(6) If R is Noetherian, then coherent R-modules agree with finitely generated (i.e., finite) R-modules, see Algebra, Lemmas 90.5, 90.4, and 31.4. Hence Mod_R^{fg} is abelian by (1) above, but in fact,in this case the category Mod_R^{fg} is a (strong) Serre subcategory of Mod_R.

54. Injective abelian groups

In this section we show the category of abelian groups has enough injectives. Recall that an abelian group M is *divisible* if and only if for every $x \in M$ and every $n \in \mathbf{N}$ there exists a $y \in M$ such that $ny = x$.

Lemma 54.1. *An abelian group J is an injective object in the category of abelian groups if and only if J is divisible.*

Proof. Suppose that J is not divisible. Then there exists an $x \in J$ and $n \in \mathbf{N}$ such that there is no $y \in J$ with $ny = x$. Then the morphism $\mathbf{Z} \to J$, $m \mapsto mx$ does not extend to $\frac{1}{n}\mathbf{Z} \supset \mathbf{Z}$. Hence J is not injective.

Let $A \subset B$ be abelian groups. Assume that J is a divisible abelian group. Let $\varphi : A \to J$ be a morphism. Consider the set of homomorphisms $\varphi' : A' \to J$ with $A \subset A' \subset B$ and $\varphi'|_A = \varphi$. Define $(A', \varphi') \geq (A'', \varphi'')$ if and only if $A' \supset A''$ and $\varphi'|_{A''} = \varphi''$. If $(A_i, \varphi_i)_{i \in I}$ is a totally ordered collection of such pairs, then we obtain a map $\bigcup_{i \in I} A_i \to J$ defined by $a \in A_i$ maps to $\varphi_i(a)$. Thus Zorn's lemma applies. To conclude we have to show that if the pair (A', φ') is maximal then $A' = B$. In other words, it suffices to show, given any subgroup $A \subset B$, $A \neq B$ and any $\varphi : A \to J$, then we can find $\varphi' : A' \to J$ with $A \subset A' \subset B$ such that (a) the inclusion $A \subset A'$ is strict, and (b) the morphism φ' extends φ.

To prove this, pick $x \in B$, $x \notin A$. If there exists no $n \in \mathbf{N}$ such that $nx \in A$, then $A \oplus \mathbf{Z} \cong A + \mathbf{Z}x$. Hence we can extend φ to $A' = A + \mathbf{Z}x$ by using φ on A and mapping x to zero for example. If there does exist an $n \in \mathbf{N}$ such that $nx \in A$, then let n be the minimal such integer. Let $z \in J$ be an element such that $nz = \varphi(nx)$. Define a morphism $\tilde{\varphi} : A \oplus \mathbf{Z} \to J$ by $(a, m) \mapsto \varphi(a) + mz$. By our choice of z the kernel of $\tilde{\varphi}$ contains the kernel of the map $A \oplus \mathbf{Z} \to B$, $(a, m) \mapsto a + mx$. Hence $\tilde{\varphi}$ factors through the image $A' = A + \mathbf{Z}x$, and this extends the morphism φ. $\square$

We can use this lemma to show that every abelian group can be embedded in a injective abelian group. But this is a special case of the result of the following section.

55. Injective modules

Some lemmas on injective modules.

Definition 55.1. Let R be a ring. An R-module J is *injective* if and only if the functor $\text{Hom}_R(-, J) : \text{Mod}_R \to \text{Mod}_R$ is an exact functor.

The functor $\text{Hom}_R(-, M)$ is left exact for any R-module M, see Algebra, Lemma 10.1. Hence the condition for J to be injective really signifies that given an injection of R-modules $M \to M'$ the map $\text{Hom}_R(M', J) \to \text{Hom}_R(M, J)$ is surjective.

Before we reformulate this in terms of *Ext*-modules we discuss the relationship between $\text{Ext}^1_R(M, N)$ and extensions as in Homology, Section 6.

Lemma 55.2. *Let R be a ring. Let $\mathcal{A}$ be the abelian category of R-modules. There is a canonical isomorphism $\text{Ext}_{\mathcal{A}}(M, N) = \text{Ext}^1_R(M, N)$ compatible with the long exact sequences of Algebra, Lemmas 71.6 and 71.7 and the 6-term exact sequences of Homology, Lemma 6.4.*

Proof. Omitted. $\square$

Lemma 55.3. *Let R be a ring. Let J be an R-module. The following are equivalent*

1. *J is injective,*
2. *$\text{Ext}^1_R(M, J) = 0$ for every R-module M.*

Proof. Let $0 \to M'' \to M' \to M \to 0$ be a short exact sequence of R-modules. Consider the long exact sequence

$$\begin{aligned} 0 \to \text{Hom}_R(M, J) \to \text{Hom}_R(M', J) \to \text{Hom}_R(M'', J) \\ \to \text{Ext}^1_R(M, J) \to \text{Ext}^1_R(M', J) \to \text{Ext}^1_R(M'', J) \to \ldots \end{aligned}$$

of Algebra, Lemma 71.7. Thus we see that (2) implies (1). Conversely, if J is injective then the Ext-group is zero by Homology, Lemma 27.2 and Lemma 55.2. $\square$

Lemma 55.4. *Let R be a ring. Let J be an R-module. The following are equivalent*

1. *J is injective,*
2. *$\text{Ext}^1_R(R/I, J) = 0$ for every ideal $I \subset R$, and*
3. *for an ideal $I \subset R$ and module map $I \to J$ there exists an extension $R \to J$.*

Proof. If $I \subset R$ is an ideal, then the short exact sequence $0 \to I \to R \to R/I \to 0$ gives an exact sequence

$$\text{Hom}_R(R, J) \to \text{Hom}_R(I, J) \to \text{Ext}^1_R(R/I, J) \to 0$$

by Algebra, Lemma 71.7 and the fact that $\text{Ext}^1_R(R, J) = 0$ as R is projective (Algebra, Lemma 77.2). Thus (2) and (3) are equivalent. In this proof we will show that (1) $\Leftrightarrow$ (3) which is known as Baer's criterion.

Assume (1). Given a module map $I \to J$ as in (3) we find the extension $R \to J$ because the map $\text{Hom}_R(R, J) \to \text{Hom}_R(I, J)$ is surjective by definition.

Assume (3). Let $M \subset N$ be an inclusion of R-modules. Let $\varphi : M \to J$ be a homomorphism. We will show that φ extends to N which finishes the proof of the lemma. Consider the set of homomorphisms $\varphi' : M' \to J$ with $M \subset M' \subset N$ and $\varphi'|_M = \varphi$. Define $(M', \varphi') \geq (M'', \varphi'')$ if and only if $M' \supset M''$ and $\varphi'|_{M''} = \varphi''$. If $(M_i, \varphi_i)_{i \in I}$ is a totally ordered collection of such pairs, then we obtain a map $\bigcup_{i \in I} M_i \to J$ defined by $a \in M_i$ maps to $\varphi_i(a)$. Thus Zorn's lemma applies. To conclude we have to show that if the pair (M', φ') is maximal then $M' = N$. In other words, it suffices to show, given any subgroup $M \subset N$, $M \neq N$ and any $\varphi : M \to J$, then we can find $\varphi' : M' \to J$ with $M \subset M' \subset N$ such that (a) the inclusion $M \subset M'$ is strict, and (b) the morphism φ' extends φ.

To prove this, pick $x \in N$, $x \notin M$. Let $I = \{f \in R \mid fx \in M\}$. This is an ideal of R. Define a homomorphism $\psi : I \to J$ by $f \mapsto \varphi(fx)$. Extend to a map $\tilde{\psi} : R \to J$ which is possible by assumption (3). By our choice of I the kernel of $M \oplus R \to J$, $(y, f) \mapsto y - \tilde{\psi}(f)$ contains the kernel of the map $M \oplus R \to N$, $(y, f) \mapsto y + fx$. Hence this homomorphism factors through the image $M' = M + Rx$ and this extends the given homomorphism as desired. $\square$

In the rest of this section we prove that there are enough injective modules over a ring R. We start with the fact that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. This follows from Lemma 54.1.

Definition 55.5. Let R be a ring.

(1) For any R-module M over R we denote $M^\vee = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$ with its natural R-module structure. We think of $M \mapsto M^\vee$ as a contravariant functor from the category of R-modules to itself.
(2) For any R-module M we denote

$$F(M) = \bigoplus\nolimits_{m \in M} R[m]$$

the *free module* with basis given by the elements $[m]$ with $m \in M$. We let $F(M) \to M$, $\sum f_i[m_i] \mapsto \sum f_i m_i$ be the natural surjection of R-modules. We think of $M \mapsto (F(M) \to M)$ as a functor from the category of R-modules to the category of arrows in R-modules.

Lemma 55.6. *Let R be a ring. The functor $M \mapsto M^\vee$ is exact.*

Proof. This because $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group by Lemma 54.1. $\square$

There is a canonical map $ev : M \to (M^\vee)^\vee$ given by evaluation: given $x \in M$ we let $ev(x) \in (M^\vee)^\vee = \mathrm{Hom}(M^\vee, \mathbf{Q}/\mathbf{Z})$ be the map $\varphi \mapsto \varphi(x)$.

Lemma 55.7. *For any R-module M the evaluation map $ev : M \to (M^\vee)^\vee$ is injective.*

Proof. You can check this using that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. Namely, if $x \in M$ is not zero, then let $M' \subset M$ be the cyclic group it generates. There exists a nonzero map $M' \to \mathbf{Q}/\mathbf{Z}$ which necessarily does not annihilate x. This extends to a map $\varphi : M \to \mathbf{Q}/\mathbf{Z}$ and then $ev(x)(\varphi) = \varphi(x) \neq 0$. $\square$

The canonical surjection $F(M) \to M$ of R-modules turns into a canonical injection, see above, of R-modules

$$(M^\vee)^\vee \longrightarrow (F(M^\vee))^\vee.$$

Set $J(M) = (F(M^\vee))^\vee$. The composition of ev with this the displayed map gives $M \to J(M)$ functorially in M.

Lemma 55.8. *Let R be a ring. For every R-module M the R-module $J(M)$ is injective.*

Proof. Note that $J(M) \cong \prod_{\varphi \in M^\vee} R^\vee$ as an R-module. As the product of injective modules is injective, it suffices to show that $R^\vee$ is injective. For this we use that

$$\mathrm{Hom}_R(N, R^\vee) = \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})) = N^\vee$$

and the fact that $(-)^\vee$ is an exact functor by Lemma 55.6. $\square$

Lemma 55.9. *Let R be a ring. The construction above defines a covariant functor $M \mapsto (M \to J(M))$ from the category of R-modules to the category of arrows of R-modules such that for every module M the output $M \to J(M)$ is an injective map of M into an injective R-module $J(M)$.*

Proof. Follows from the above. $\square$

In particular, for any map of R-modules $M \to N$ there is an associated morphism $J(M) \to J(N)$ making the following diagram commute:

$$
\begin{tikzcd}
M \arrow[r] \arrow[d] & N \arrow[d] \\
J(M) \arrow[r] & J(N)
\end{tikzcd}
$$

This is the kind of construction we would like to have in general. In Homology, Section 27 we introduced terminology to express this. Namely, we say this means that the category of R-modules has functorial injective embeddings.

56. Derived categories of modules

In this section we put some generalities concerning the derived category of modules over a ring.

Let A be a ring. The category of A-modules is denoted Mod_A. We will use the symbol $K(A)$ to denote the homotopy category of complexes of A-modules, i.e., we set $K(A) = K(\text{Mod}_A)$ as a category, see Derived Categories, Section 8. The bounded versions are $K^+(A)$, $K^-(A)$, and $K^b(A)$. We view $K(A)$ as a triangulated category as in Derived Categories, Section 10. The *derived category* of A, denoted $D(A)$, is the category obtained from $K(A)$ by inverting quasi-isomorphisms, i.e., we set $D(A) = D(\text{Mod}_A)$, see Derived Categories, Section 11[6]. The bounded versions are $D^+(A)$, $D^-(A)$, and $D^b(A)$.

Let A be a ring. The category of A-modules has products and products are exact. The category of A-modules has enough injectives by Lemma 55.9. Hence every complex of A-modules is quasi-isomorphic to a K-injective complex (Derived Categories, Lemma 34.5). It follows that $D(A)$ has countable products (Derived Categories, Lemma 34.2) and in fact arbitrary products (Injectives, Lemma 13.4). This implies that every inverse system of objects of $D(A)$ has a derived limit (well defined up to isomorphism), see Derived Categories, Section 34.

Lemma 56.1. *Let $R \to S$ be a flat ring map. If $I^\bullet$ is a K-injective complex of S-modules, then $I^\bullet$ is K-injective as a complex of R-modules.*

Proof. This is true because $\text{Hom}_{K(R)}(M^\bullet, I^\bullet) = \text{Hom}_{K(S)}(M^\bullet \otimes_R S, I^\bullet)$ by Algebra, Lemma 14.3 and the fact that tensoring with S is exact. ☐

Lemma 56.2. *Let $R \to S$ be an epimorphism of rings. Let $I^\bullet$ be a complex of S-modules. If $I^\bullet$ is K-injective as a complex of R-modules, then $I^\bullet$ is a K-injective complex of S-modules.*

Proof. This is true because $\text{Hom}_{K(R)}(N^\bullet, I^\bullet) = \text{Hom}_{K(S)}(N^\bullet, I^\bullet)$ for any complex of S-modules $N^\bullet$, see Algebra, Lemma 107.14. ☐

Lemma 56.3. *Let $A \to B$ be a ring map. If $I^\bullet$ is a K-injective complex of A-modules, then $\text{Hom}_A(B, I^\bullet)$ is a K-injective complex of B-modules.*

Proof. This is true because $\text{Hom}_{K(B)}(N^\bullet, \text{Hom}_A(B, I^\bullet)) = \text{Hom}_{K(A)}(N^\bullet, I^\bullet)$ by Algebra, Lemma 14.4. ☐

[6] See also Injectives, Remark 13.3.

57. Computing Tor

Let R be a ring. We denote $D(R)$ the derived category of the abelian category Mod_R of R-modules. Note that Mod_R has enough projectives as every free R-module is projective. Thus we can define the left derived functors of any additive functor from Mod_R to any abelian category.

This applies in particular to the functor $- \otimes_R M : \text{Mod}_R \to \text{Mod}_R$ whose left derived functors are the Tor functors $\text{Tor}_i^R(-, M)$, see Algebra, Section 75. There is also a total left derived functor

$$- \otimes_R^{\mathbf{L}} M : D^-(R) \longrightarrow D^-(R) \tag{57.0.1}$$

which is denoted $- \otimes_R^{\mathbf{L}} M$. Its satellites are the Tor modules, i.e., we have

$$H^{-p}(N \otimes_R^{\mathbf{L}} M) = \text{Tor}_p^R(N, M).$$

A special situation occurs when we consider the tensor product with an R-algebra A. In this case we think of $- \otimes_R A$ as a functor from Mod_R to Mod_A. Hence the total right derived functor

$$- \otimes_R^{\mathbf{L}} A : D^-(R) \longrightarrow D^-(A) \tag{57.0.2}$$

which is denoted $- \otimes_R^{\mathbf{L}} A$. Its satellites are the tor groups, i.e., we have

$$H^{-p}(N \otimes_R^{\mathbf{L}} A) = \text{Tor}_p^R(N, A).$$

In particular these Tor groups naturally have the structure of A-modules.

We will generalize the material in this section to unbounded complexes in the next few sections.

58. Tensor products of complexes

Let R be a ring. The category $\text{Comp}(R)$ of complexes of R-modules has a symmetric monoidal structure. Namely, suppose that we have two complexes of R-modules $L^\bullet$ and $M^\bullet$. Using Homology, Example 18.2 and Homology, Definition 18.3 we obtain a third complex of R-modules, namely

$$\text{Tot}(L^\bullet \otimes_R M^\bullet)$$

Clearly this construction is functorial in both $L^\bullet$ and $M^\bullet$. The associativity constraint will be the canonical isomorphism of complexes

$$\text{Tot}(\text{Tot}(K^\bullet \otimes_R L^\bullet) \otimes_R M^\bullet) \longrightarrow \text{Tot}(K^\bullet \otimes_R \text{Tot}(L^\bullet \otimes_R M^\bullet))$$

constructed in Homology, Remark 18.4 from the triple complex $K^\bullet \otimes_R L^\bullet \otimes_R M^\bullet$. The commutativity constraint is the canonical isomorphism

$$\text{Tot}(L^\bullet \otimes_R M^\bullet) \to \text{Tot}(M^\bullet \otimes_R L^\bullet)$$

which uses the sign $(-1)^{pq}$ on the summand $L^p \otimes_R M^q$. To see that it is a map of complexes we compute for $x \in L^p$ and $y \in M^q$ that

$$\text{d}(x \otimes y) = \text{d}_L(x) \otimes y + (-1)^p x \otimes \text{d}_M(y)$$

Our rule says the right hand side is mapped to

$$(-1)^{(p+1)q} y \otimes \text{d}_L(x) + (-1)^{p+p(q+1)} \text{d}_M(y) \otimes x$$

On the other hand, we see that

$$\text{d}((-1)^{pq} y \otimes x) = (-1)^{pq} \text{d}_M(y) \otimes x + (-1)^{pq+q} y \otimes \text{d}_L(x)$$

These two expressions agree by inspection as desired.

Lemma 58.1. *Let R be a ring. The category $Comp(R)$ of complexes of R-modules endowed with the functor $(L^\bullet, M^\bullet) \mapsto Tot(L^\bullet \otimes_R M^\bullet)$ and associativity and commutativity constraints as above is a symmetric monoidal category.*

Proof. Omitted. Hints: as unit $\mathbf{1}$ we take the complex having R in degree 0 and zero in other degrees with obvious isomorphisms $\text{Tot}(\mathbf{1} \otimes_R M^\bullet) = M^\bullet$ and $\text{Tot}(K^\bullet \otimes_R \mathbf{1}) = K^\bullet$. to prove the lemma you have to check the commutativity of various diagrams, see Categories, Definitions 43.1 and 43.9. The verifications are straightforward in each case. $\square$

Lemma 58.2. *Let R be a ring. Let $P^\bullet$ be a complex of R-modules. Let $\alpha, \beta : L^\bullet \to M^\bullet$ be homotopic maps of complexes. Then α and β induce homotopic maps*

$$Tot(\alpha \otimes id_P), Tot(\beta \otimes id_P) : Tot(L^\bullet \otimes_R P^\bullet) \longrightarrow Tot(M^\bullet \otimes_R P^\bullet).$$

In particular the construction $L^\bullet \mapsto Tot(L^\bullet \otimes_R P^\bullet)$ defines an endo-functor of the homotopy category of complexes.

Proof. Say $\alpha = \beta + dh + hd$ for some homotopy h defined by $h^n : L^n \to M^{n-1}$. Set

$$H^n = \bigoplus\nolimits_{a+b=n} h^a \otimes \text{id}_{P^b} : \bigoplus\nolimits_{a+b=n} L^a \otimes_R P^b \longrightarrow \bigoplus\nolimits_{a+b=n} M^{a-1} \otimes_R P^b$$

Then a straightforward computation shows that

$$\text{Tot}(\alpha \otimes \text{id}_P) = \text{Tot}(\beta \otimes \text{id}_P) + dH + Hd$$

as maps $\text{Tot}(L^\bullet \otimes_R P^\bullet) \to \text{Tot}(M^\bullet \otimes_R P^\bullet)$. $\square$

Lemma 58.3. *Let R be a ring. The homotopy category $K(R)$ of complexes of R-modules endowed with the functor $(L^\bullet, M^\bullet) \mapsto Tot(L^\bullet \otimes_R M^\bullet)$ and associativity and commutativity constraints as above is a symmetric monoidal category.*

Proof. This follows from Lemmas 58.1 and 58.2. Details omitted. $\square$

Lemma 58.4. *Let R be a ring. Let $P^\bullet$ be a complex of R-modules. The functors*

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto Tot(P^\bullet \otimes_R L^\bullet)$$

and

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto Tot(L^\bullet \otimes_R P^\bullet)$$

are exact functors of triangulated categories.

Proof. This follows from Derived Categories, Remark 10.9. $\square$

59. Derived tensor product

We can construct the derived tensor product in greater generality. In fact, it turns out that the boundedness assumptions are not necessary, provided we choose K-flat resolutions.

Definition 59.1. Let R be a ring. A complex $K^\bullet$ is called *K-flat* if for every acyclic complex $M^\bullet$ the total complex $\text{Tot}(M^\bullet \otimes_R K^\bullet)$ is acyclic.

Lemma 59.2. *Let R be a ring. Let $K^\bullet$ be a K-flat complex. Then the functor*

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto Tot(L^\bullet \otimes_R K^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 58.4 and the fact that quasi-isomorphisms in $K(R)$ are characterized by having acyclic cones. $\square$

Lemma 59.3. *Let $R \to R'$ be a ring map. If $K^\bullet$ is a K-flat complex of R-modules, then $K^\bullet \otimes_R R'$ is a K-flat complex of R'-modules.*

Proof. Follows from the definitions and the fact that $(K^\bullet \otimes_R R') \otimes_{R'} L^\bullet = K^\bullet \otimes_R L^\bullet$ for any complex $L^\bullet$ of R'-modules. $\square$

Lemma 59.4. *Let R be a ring. If $K^\bullet$, $L^\bullet$ are K-flat complexes of R-modules, then $Tot(K^\bullet \otimes_R L^\bullet)$ is a K-flat complex of R-modules.*

Proof. Follows from the isomorphism

$$\mathrm{Tot}(M^\bullet \otimes_R \mathrm{Tot}(K^\bullet \otimes_R L^\bullet)) = \mathrm{Tot}(\mathrm{Tot}(M^\bullet \otimes_R K^\bullet) \otimes_R L^\bullet)$$

and the definition. $\square$

Lemma 59.5. *Let R be a ring. Let $(K_1^\bullet, K_2^\bullet, K_3^\bullet)$ be a distinguished triangle in $K(R)$. If two out of three of $K_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 58.4 and the fact that in a distinguished triangle in $K(R)$ if two out of three are acyclic, so is the third. $\square$

Lemma 59.6. *Let R be a ring. Let $0 \to K_1^\bullet \to K_2^\bullet \to K_3^\bullet \to 0$ be a short exact sequence of complexes. If K_3^n is flat for all $n \in \mathbf{Z}$ and two out of three of $K_i^\bullet$ are K-flat, so is the third.*

Proof. Let $L^\bullet$ be a complex of R-modules. Then

$$0 \to \mathrm{Tot}(L^\bullet \otimes_R K_1^\bullet) \to \mathrm{Tot}(L^\bullet \otimes_R K_2^\bullet) \to \mathrm{Tot}(L^\bullet \otimes_R K_3^\bullet) \to 0$$

is a short exact sequence of complexes. Namely, for each n, m the sequence of modules $0 \to L^n \otimes_R K_1^m \to L^n \otimes_R K_2^m \to L^n \otimes_R K_3^m \to 0$ is exact by Algebra, Lemma 39.12 and the sequence of complexes is a direct sum of these. Thus the lemma follows from this and the fact that in a short exact sequence of complexes if two out of three are acyclic, so is the third. $\square$

Lemma 59.7. *Let R be a ring. Let $P^\bullet$ be a bounded above complex of flat R-modules. Then $P^\bullet$ is K-flat.*

Proof. Let $L^\bullet$ be an acyclic complex of R-modules. Let $\xi \in H^n(\mathrm{Tot}(L^\bullet \otimes_R P^\bullet))$. We have to show that $\xi = 0$. Since $\mathrm{Tot}^n(L^\bullet \otimes_R P^\bullet)$ is a direct sum with terms $L^a \otimes_R P^b$ we see that ξ comes from an element in $H^n(\mathrm{Tot}(\tau_{\leq m} L^\bullet \otimes_R P^\bullet))$ for some $m \in \mathbf{Z}$. Since $\tau_{\leq m} L^\bullet$ is also acyclic we may replace $L^\bullet$ by $\tau_{\leq m} L^\bullet$. Hence we may assume that $L^\bullet$ is bounded above. In this case the spectral sequence of Homology, Lemma 25.3 has

$$'E_1^{p,q} = H^p(L^\bullet \otimes_R P^q)$$

which is zero as P^q is flat and $L^\bullet$ acyclic. Hence $H^*(\mathrm{Tot}(L^\bullet \otimes_R P^\bullet)) = 0$. $\square$

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

Lemma 59.8. *Let R be a ring. Let $K_1^\bullet \to K_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\mathrm{colim}_i K_i^\bullet$ is K-flat. More generally any filtered colimit of K-flat complexes is K-flat.*

Proof. Because we are taking termwise colimits we have

$$\mathrm{colim}_i \mathrm{Tot}(M^\bullet \otimes_R K_i^\bullet) = \mathrm{Tot}(M^\bullet \otimes_R \mathrm{colim}_i K_i^\bullet)$$

by Algebra, Lemma 12.9. Hence the lemma follows from the fact that filtered colimits are exact, see Algebra, Lemma 8.8. □

Lemma 59.9. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. If $K^\bullet \otimes_R M$ is acyclic for all finitely presented R-modules M, then $K^\bullet$ is K-flat.*

Proof. We will use repeatedly that tensor product commute with colimits (Algebra, Lemma 12.9). Thus we see that $K^\bullet \otimes_R M$ is acyclic for any R-module M, because any R-module is a filtered colimit of finitely presented R-modules M, see Algebra, Lemma 11.3. Let $M^\bullet$ be an acyclic complex of R-modules. We have to show that $\mathrm{Tot}(M^\bullet \otimes_R K^\bullet)$ is acyclic. Since $M^\bullet = \mathrm{colim}\, \tau_{\leq n} M^\bullet$ (termwise colimit) we have

$$\mathrm{Tot}(M^\bullet \otimes_R K^\bullet) = \mathrm{colim}\, \mathrm{Tot}(\tau_{\leq n} M^\bullet \otimes_R K^\bullet)$$

with truncations as in Homology, Section 15. As filtered colimits are exact (Algebra, Lemma 8.8) we may replace $M^\bullet$ by $\tau_{\leq n} M^\bullet$ and assume that $M^\bullet$ is bounded above. In the bounded above case, we can write $M^\bullet = \mathrm{colim}\, \sigma_{\geq -n} M^\bullet$ where the complexes $\sigma_{\geq -n} M^\bullet$ are bounded but possibly no longer acyclic. Arguing as above we reduce to the case where $M^\bullet$ is a bounded complex. Finally, for a bounded complex $M^a \to \ldots \to M^b$ we can argue by induction on the length $b - a$ of the complex. The case $b - a = 1$ we have seen above. For $b - a > 1$ we consider the split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1} M^\bullet \to M^\bullet \to M^a[-a] \to 0$$

and we apply Lemma 58.4 to do the induction step. Some details omitted. □

Lemma 59.10. *Let R be a ring. For any complex $M^\bullet$ there exists a K-flat complex $K^\bullet$ whose terms are flat R-modules and a quasi-isomorphism $K^\bullet \to M^\bullet$ which is termwise surjective.*

Proof. Let $\mathcal{P} \subset \mathrm{Ob}(\mathrm{Mod}_R)$ be the class of flat R-modules. By Derived Categories, Lemma 29.1 there exists a system $K_1^\bullet \to K_2^\bullet \to \ldots$ and a diagram

$$\begin{tikzcd}
K_1^\bullet \arrow[r] \arrow[d] & K_2^\bullet \arrow[r] \arrow[d] & \ldots \\
\tau_{\leq 1} M^\bullet \arrow[r] & \tau_{\leq 2} M^\bullet \arrow[r] & \ldots
\end{tikzcd}$$

with the properties (1), (2), (3) listed in that lemma. These properties imply each complex $K_i^\bullet$ is a bounded above complex of flat modules. Hence $K_i^\bullet$ is K-flat by Lemma 59.7. The induced map $\mathrm{colim}_i K_i^\bullet \to M^\bullet$ is a quasi-isomorphism and termwise surjective by construction. The complex $\mathrm{colim}_i K_i^\bullet$ is K-flat by Lemma 59.8. The terms $\mathrm{colim}\, K_i^n$ are flat because filtered colimits of flat modules are flat, see Algebra, Lemma 39.3. □

Remark 59.11. In fact, we can do better than Lemma 59.10. Namely, we can find a quasi-isomorphism $P^\bullet \to M^\bullet$ where $P^\bullet$ is a complex of R-modules endowed with a filtration

$$0 = F_{-1} P^\bullet \subset F_0 P^\bullet \subset F_1 P^\bullet \subset \ldots \subset P^\bullet$$

by subcomplexes such that

(1) $P^\bullet = \bigcup F_p P^\bullet$,
(2) the inclusions $F_i P^\bullet \to F_{i+1} P^\bullet$ are termwise split injections,
(3) the quotients $F_{i+1} P^\bullet / F_i P^\bullet$ are isomorphic to direct sums of shifts $R[k]$ (as complexes, so differentials are zero).

This will be shown in Differential Graded Algebra, Lemma 20.4. Moreover, given such a complex we obtain a distinguished triangle

$$\bigoplus F_i P^\bullet \to \bigoplus F_i P^\bullet \to M^\bullet \to \bigoplus F_i P^\bullet[1]$$

in $D(R)$. Using this we can sometimes reduce statements about general complexes to statements about $R[k]$ (this of course only works if the statement is preserved under taking direct sums). More precisely, let T be a property of objects of $D(R)$. Suppose that

(1) if $K_i \in D(R)$, $i \in I$ is a family of objects with $T(K_i)$ for all $i \in I$, then $T(\bigoplus K_i)$,
(2) if $K \to L \to M \to K[1]$ is a distinguished triangle and T holds for two, then T holds for the third object,
(3) $T(R[k])$ holds for all k.

Then T holds for all objects of $D(R)$.

Lemma 59.12. *Let R be a ring. Let $\alpha : P^\bullet \to Q^\bullet$ be a quasi-isomorphism of K-flat complexes of R-modules. For every complex $L^\bullet$ of R-modules the induced map*

$$Tot(id_L \otimes \alpha) : Tot(L^\bullet \otimes_R P^\bullet) \longrightarrow Tot(L^\bullet \otimes_R Q^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $K^\bullet \to L^\bullet$ with $K^\bullet$ a K-flat complex, see Lemma 59.10. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Tot}(K^\bullet \otimes_R P^\bullet) \arrow[r] \arrow[d] & \mathrm{Tot}(K^\bullet \otimes_R Q^\bullet) \arrow[d] \\ \mathrm{Tot}(L^\bullet \otimes_R P^\bullet) \arrow[r] & \mathrm{Tot}(L^\bullet \otimes_R Q^\bullet) \end{tikzcd}$$

The result follows as by Lemma 59.2 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. $\square$

Let R be a ring. Let $M^\bullet$ be an object of $D(R)$. Choose a K-flat resolution $K^\bullet \to M^\bullet$, see Lemma 59.10. By Lemmas 58.2 and 58.4 we obtain an exact functor of triangulated categories

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto \mathrm{Tot}(L^\bullet \otimes_R K^\bullet)$$

By Lemma 59.2 this functor induces a functor $D(R) \to D(R)$ simply because $D(R)$ is the localization of $K(R)$ at quasi-isomorphism. By Lemma 59.12 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

Definition 59.13. Let R be a ring. Let $M^\bullet$ be an object of $D(R)$. The *derived tensor product*

$$- \otimes_R^{\mathbf{L}} M^\bullet : D(R) \longrightarrow D(R)$$

is the exact functor of triangulated categories described above.

This functor extends the functor (57.0.1). It is clear from our explicit constructions that there is an isomorphism (involving a choice of signs, see below)

$$M^\bullet \otimes_R^{\mathbf{L}} L^\bullet \cong L^\bullet \otimes_R^{\mathbf{L}} M^\bullet$$

whenever both $L^\bullet$ and $M^\bullet$ are in $D(R)$. Hence when we write $M^\bullet \otimes_R^{\mathbf{L}} L^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

Lemma 59.14. *Let R be a ring. Let $K^\bullet, L^\bullet$ be complexes of R-modules. There is a canonical isomorphism*

$$K^\bullet \otimes_R^{\mathbf{L}} L^\bullet \longrightarrow L^\bullet \otimes_R^{\mathbf{L}} K^\bullet$$

functorial in both complexes which uses a sign of $(-1)^{pq}$ for the map $K^p \otimes_R L^q \to L^q \otimes_R K^p$ (see proof for explanation).

Proof. We may and do replace the complexes by K-flat complexes $K^\bullet$ and $L^\bullet$ and then we use the commutativity constraint discussed in Section 58. $\square$

Lemma 59.15. *Let R be a ring. Let $K^\bullet, L^\bullet, M^\bullet$ be complexes of R-modules. There is a canonical isomorphism*

$$(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet) \otimes_R^{\mathbf{L}} M^\bullet = K^\bullet \otimes_R^{\mathbf{L}} (L^\bullet \otimes_R^{\mathbf{L}} M^\bullet)$$

functorial in all three complexes.

Proof. Replace the complexes by K-flat complexes and use the associativity constraint in Section 58. $\square$

Lemma 59.16. *Let R be a ring. Let $a : K^\bullet \to L^\bullet$ be a map of complexes of R-modules. If $K^\bullet$ is K-flat, then there exist a complex $N^\bullet$ and maps of complexes $b : K^\bullet \to N^\bullet$ and $c : N^\bullet \to L^\bullet$ such that*

1. *$N^\bullet$ is K-flat,*
2. *c is a quasi-isomorphism,*
3. *a is homotopic to $c \circ b$.*

If the terms of $K^\bullet$ are flat, then we may choose $N^\bullet$, b, and c such that the same is true for $N^\bullet$.

Proof. We will use that the homotopy category $K(R)$ is a triangulated category, see Derived Categories, Proposition 10.3. Choose a distinguished triangle $K^\bullet \to L^\bullet \to C^\bullet \to K^\bullet[1]$. Choose a quasi-isomorphism $M^\bullet \to C^\bullet$ with $M^\bullet$ K-flat with flat terms, see Lemma 59.10. By the axioms of triangulated categories, we may fit the composition $M^\bullet \to C^\bullet \to K^\bullet[1]$ into a distinguished triangle $K^\bullet \to N^\bullet \to M^\bullet \to K^\bullet[1]$. By Lemma 59.5 we see that $N^\bullet$ is K-flat. Again using the axioms of triangulated categories, we can choose a map $N^\bullet \to L^\bullet$ fitting into the following morphism of distinghuised triangles

$$\begin{tikzcd}
K^\bullet \arrow[r] \arrow[d] & N^\bullet \arrow[r] \arrow[d] & M^\bullet \arrow[r] \arrow[d] & K^\bullet[1] \arrow[d] \\
K^\bullet \arrow[r] & L^\bullet \arrow[r] & C^\bullet \arrow[r] & K^\bullet[1]
\end{tikzcd}$$

Since two out of three of the arrows are quasi-isomorphisms, so is the third arrow $N^\bullet \to L^\bullet$ by the long exact sequences of cohomology associated to these distinguished triangles (or you can look at the image of this diagram in $D(R)$ and use

Derived Categories, Lemma 4.3 if you like). This finishes the proof of (1), (2), and (3). To prove the final assertion, we may choose $N^\bullet$ such that $N^n \cong M^n \oplus K^n$, see Derived Categories, Lemma 10.7. Hence we get the desired flatness if the terms of $K^\bullet$ are flat. $\square$

60. Derived change of rings

Let $R \to A$ be a ring map. Let $N^\bullet$ be a complex of A-modules. We can also use K-flat resolutions to define a functor

$$- \otimes_R^{\mathbf{L}} N^\bullet : D(R) \to D(A)$$

as the left derived functor of the functor $K(R) \to K(A)$, $M^\bullet \mapsto \text{Tot}(M^\bullet \otimes_R N^\bullet)$. In particular, taking $N^\bullet = A[0]$ we obtain a derived base change functor

$$- \otimes_R^{\mathbf{L}} A : D(R) \to D(A)$$

extending the functor (57.0.2). Namely, for every complex of R-modules $M^\bullet$ we can choose a K-flat resolution $K^\bullet \to M^\bullet$ and set

$$M^\bullet \otimes_R^{\mathbf{L}} N^\bullet = \text{Tot}(K^\bullet \otimes_R N^\bullet).$$

You can use Lemmas 59.10 and 59.12 to see that this is well defined. However, to cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

Lemma 60.1. *The construction above is independent of choices and defines an exact functor of triangulated categories $- \otimes_R^{\mathbf{L}} N^\bullet : D(R) \to D(A)$. There is a functorial isomorphism*

$$E^\bullet \otimes_R^{\mathbf{L}} N^\bullet = (E^\bullet \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} N^\bullet$$

for $E^\bullet$ in $D(R)$.

Proof. To prove the existence of the derived functor $- \otimes_R^{\mathbf{L}} N^\bullet$ we use the general theory developed in Derived Categories, Section 14. Set $\mathcal{D} = K(R)$ and $\mathcal{D}' = D(A)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(M^\bullet) = \text{Tot}(M^\bullet \otimes_R N^\bullet)$. To prove the stated properties of F use Lemmas 58.2 and 58.4. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(R)$. This gives a situation as in Derived Categories, Situation 14.1 so that Derived Categories, Definition 14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 14.15 with $\mathcal{P} \subset \text{Ob}(\mathcal{D})$ the collection of K-flat complexes: (1) follows from Lemma 59.10 and (2) follows from Lemma 59.12. Thus we obtain a derived functor

$$LF : D(R) = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(A)$$

see Derived Categories, Equation (14.9.1). Finally, Derived Categories, Lemma 14.15 guarantees that $LF(K^\bullet) = F(K^\bullet) = \text{Tot}(K^\bullet \otimes_R N^\bullet)$ when $K^\bullet$ is K-flat, i.e., LF is indeed computed in the way described above. Moreover, by Lemma 59.3 the complex $K^\bullet \otimes_R A$ is a K-flat complex of A-modules. Hence

$$(K^\bullet \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} N^\bullet = \text{Tot}((K^\bullet \otimes_R A) \otimes_A N^\bullet) = \text{Tot}(K^\bullet \otimes_A N^\bullet) = K^\bullet \otimes_A^{\mathbf{L}} N^\bullet$$

which proves the final statement of the lemma. $\square$

Lemma 60.2. *Let $R \to A$ be a ring map. Let $f : L^\bullet \to N^\bullet$ be a map of complexes of A-modules. Then f induces a transformation of functors*

$$1 \otimes f : - \otimes_A^{\mathbf{L}} L^\bullet \longrightarrow - \otimes_A^{\mathbf{L}} N^\bullet$$

If f is a quasi-isomorphism, then $1 \otimes f$ is an isomorphism of functors.

Proof. Since the functors are computing by evaluating on K-flat complexes $K^\bullet$ we can simply use the functoriality

$$\mathrm{Tot}(K^\bullet \otimes_R L^\bullet) \to \mathrm{Tot}(K^\bullet \otimes_R N^\bullet)$$

to define the transformation. The last statement follows from Lemma 59.2. □

Lemma 60.3. *Let $R \to A$ be a ring map. The functor $D(R) \to D(A)$, $E \mapsto E \otimes_R^{\mathbf{L}} A$ of Lemma 60.1 is left adjoint to the restriction functor $D(A) \to D(R)$.*

Proof. This follows from Derived Categories, Lemma 30.1 and the fact that $- \otimes_R A$ and restriction are adjoint by Algebra, Lemma 14.3. □

Remark 60.4 (Warning)**.** Let $R \to A$ be a ring map, and let N and N' be A-modules. Denote N_R and N'_R the restriction of N and N' to R-modules, see Algebra, Section 14. In this situation, the objects $N_R \otimes_R^{\mathbf{L}} N'$ and $N \otimes_R^{\mathbf{L}} N'_R$ of $D(A)$ are in general not isomorphic! In other words, one has to pay careful attention as to which of the two sides is being used to provide the A-module structure.

For a specific example, set $R = k[x, y]$, $A = R/(xy)$, $N = R/(x)$ and $N' = A = R/(xy)$. The resolution $0 \to R \xrightarrow{xy} R \to N'_R \to 0$ shows that $N \otimes_R^{\mathbf{L}} N'_R = N[1] \oplus N$ in $D(A)$. The resolution $0 \to R \xrightarrow{x} R \to N_R \to 0$ shows that $N_R \otimes_R^{\mathbf{L}} N'$ is represented by the complex $A \xrightarrow{x} A$. To see these two complexes are not isomorphic, one can show that the second complex is not isomorphic in $D(A)$ to the direct sum of its cohomology groups, or one can show that the first complex is not a perfect object of $D(A)$ whereas the second one is. Some details omitted.

Lemma 60.5. *Let $A \to B \to C$ be ring maps. Let $N^\bullet$ be a complex of B-modules and $K^\bullet$ a complex of C-modules. The compositions of the functors*

$$D(A) \xrightarrow{- \otimes_A^{\mathbf{L}} N^\bullet} D(B) \xrightarrow{- \otimes_B^{\mathbf{L}} K^\bullet} D(C)$$

is the functor $- \otimes_A^{\mathbf{L}} (N^\bullet \otimes_B^{\mathbf{L}} K^\bullet) : D(A) \to D(C)$. If M, N, K are modules over A, B, C, then we have

$$(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} K = M \otimes_A^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} K) = (M \otimes_A^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} K)$$

in $D(C)$. We also have a canonical isomorphism

$$(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} K \longrightarrow (M \otimes_A^{\mathbf{L}} K) \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} C)$$

using signs. Similar results holds for complexes.

Proof. Choose a K-flat complex $P^\bullet$ of B-modules and a quasi-isomorphism $P^\bullet \to N^\bullet$ (Lemma 59.10). Let $M^\bullet$ be a K-flat complex of A-modules representing an arbitrary object of $D(A)$. Then we see that

$$(M^\bullet \otimes_A^{\mathbf{L}} P^\bullet) \otimes_B^{\mathbf{L}} K^\bullet \longrightarrow (M^\bullet \otimes_A^{\mathbf{L}} N^\bullet) \otimes_B^{\mathbf{L}} K^\bullet$$

is an isomorphism by Lemma 60.2 applied to the material inside the brackets. By Lemmas 59.3 and 59.4 the complex

$$\mathrm{Tot}(M^\bullet \otimes_A P^\bullet) = \mathrm{Tot}((M^\bullet \otimes_R A) \otimes_A P^\bullet$$

is K-flat as a complex of B-modules and it represents the derived tensor product in $D(B)$ by construction. Hence we see that $(M^\bullet \otimes_A^{\mathbf{L}} P^\bullet) \otimes_B^{\mathbf{L}} K^\bullet$ is represented by the complex

$$\text{Tot}(\text{Tot}(M^\bullet \otimes_A P^\bullet) \otimes_B K^\bullet) = \text{Tot}(M^\bullet \otimes_A \text{Tot}(P^\bullet \otimes_B K^\bullet))$$

of C-modules. Equality by Homology, Remark 18.4. Going back the way we came we see that this is equal to

$$M^\bullet \otimes_A^{\mathbf{L}} (P^\bullet \otimes_B^{\mathbf{L}} K^\bullet) \longleftarrow M^\bullet \otimes_A^{\mathbf{L}} (N^\bullet \otimes_B^{\mathbf{L}} K^\bullet)$$

The arrow is an isomorphism by definition of the functor $- \otimes_B^{\mathbf{L}} K^\bullet$. All of these constructions are functorial in the complex $M^\bullet$ and hence we obtain our isomorphism of functors.

By the above we have the first equality in

$$(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} K = M \otimes_A^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} K) = (M \otimes_A^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} K)$$

The second equality follows from the final statement of Lemma 60.1. The same thing allows us to write $N \otimes_B^{\mathbf{L}} K = (N \otimes_B^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} K$ and substituting we get

$$\begin{aligned}
(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} K &= (M \otimes_A^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} ((N \otimes_B^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} K) \\
&= (M \otimes_A^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} (K \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} C)) \\
&= ((M \otimes_A^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} K) \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} C)) \\
&= (M \otimes_C^{\mathbf{L}} K) \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} C)
\end{aligned}$$

by Lemmas 59.14 and 59.15 as well as the previously mentioned lemma. ☐

61. Tor independence

Consider a commutative diagram

$$\begin{tikzcd}
A \arrow[r] & A' \\
R \arrow[u] \arrow[r] & R' \arrow[u]
\end{tikzcd}$$

of rings. Given an object K of $D(A)$ we can consider its derived base change $K \otimes_A^{\mathbf{L}} A'$ to an object of $D(A')$. Or we can take the restriction of K to an object of $D(R)$ and consider the derived base change of this to an object of $D(R')$, denoted $K \otimes_R^{\mathbf{L}} R'$ We claim there is a functorial comparison map

$$K \otimes_R^{\mathbf{L}} R' \longrightarrow K \otimes_A^{\mathbf{L}} A' \qquad (61.0.1)$$

in $D(R')$. To construct this comparison map choose a K-flat complex $K^\bullet$ of A-modules representing K. Next, choose a quasi-isomorphism $E^\bullet \to K^\bullet$ where $E^\bullet$ is a K-flat complex of R-modules. The map above is the map

$$K \otimes_R^{\mathbf{L}} R' = E^\bullet \otimes_R R' \longrightarrow K^\bullet \otimes_A A' = K \otimes_A^{\mathbf{L}} A'$$

In general there is no chance that this map is an isomorphism.

However, we often encounter the situation where the diagram above is a "base change" diagram of rings, i.e., $A' = A \otimes_R R'$. In this situation, for any A-module M we have $M \otimes_A A' = M \otimes_R R'$. Thus $- \otimes_R R'$ is equal to $- \otimes_A A'$ as a functor $\text{Mod}_A \to \text{Mod}_{A'}$. In general this equality **does not extend** to derived tensor products. In other words, the comparison map is not an isomorphism. A simple example is to take $R = k[x]$, $A = R' = A' = k[x]/(x) = k$ and $K^\bullet = A[0]$. Clearly, a necessary condition is that $\text{Tor}_p^R(A, R') = 0$ for all $p > 0$.

Definition 61.1. Let R be a ring. Let A, B be R-algebras. We say A and B are *Tor independent over R* if $\text{Tor}_p^R(A, B) = 0$ for all $p > 0$.

Lemma 61.2. *The comparison map (61.0.1) is an isomorphism if $A' = A \otimes_R R'$ and A and R' are Tor independent over R.*

Proof. To prove this we choose a free resolution $F^\bullet \to R'$ of R' as an R-module. Because A and R' are Tor independent over R we see that $F^\bullet \otimes_R A$ is a free A-module resolution of A' over A. By our general construction of the derived tensor product above we see that

$$K^\bullet \otimes_A A' \cong \text{Tot}(K^\bullet \otimes_A (F^\bullet \otimes_R A)) = \text{Tot}(K^\bullet \otimes_R F^\bullet) \cong \text{Tot}(E^\bullet \otimes_R F^\bullet) \cong E^\bullet \otimes_R R'$$

as desired. □

Lemma 61.3. *Consider a commutative diagram of rings*

$$\begin{tikzcd}
A' & R' \arrow[l] \arrow[r] & B' \\
A \arrow[u] & R \arrow[l] \arrow[r] \arrow[u] & B \arrow[u]
\end{tikzcd}$$

Assume that R' is flat over R and A' is flat over $A \otimes_R R'$ and B' is flat over $R' \otimes_R B$. Then

$$Tor_i^R(A, B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B') = Tor_i^{R'}(A', B')$$

Proof. By Algebra, Section 76 there are canonical maps

$$\text{Tor}_i^R(A, B) \longrightarrow \text{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R') \longrightarrow \text{Tor}_i^{R'}(A', B')$$

These induce a map from left to right in the formula of the lemma.

Take a free resolution $F_\bullet \to A$ of A as an R-module. Then we see that $F_\bullet \otimes_R R'$ is a resolution of $A \otimes_R R'$. Hence $\text{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R')$ is computed by $F_\bullet \otimes_R B \otimes_R R'$. By our assumption that R' is flat over R, this computes $\text{Tor}_i^R(A, B) \otimes_R R'$. Thus $\text{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R') = \text{Tor}_i^R(A, B) \otimes_R R'$ (uses only flatness of R' over R).

By Lazard's theorem (Algebra, Theorem 81.4) we can write A', resp. B' as a filtered colimit of finite free $A \otimes_R R'$, resp. $B \otimes_R R'$-modules. Say $A' = \text{colim}\, M_i$ and $B' = \text{colim}\, N_j$. The result above gives

$$\text{Tor}_i^{R'}(M_i, N_j) = \text{Tor}_i^R(A, B) \otimes_{A \otimes_R B} (M_i \otimes_{R'} N_j)$$

as one can see by writing everything out in terms of bases. Taking the colimit we get the result of the lemma. □

Lemma 61.4. *Let $R \to A$ and $R \to B$ be ring maps. Let $R \to R'$ be a ring map and set $A' = A \otimes_R R'$ and $B' = B \otimes_R R'$. If A and B are tor independent over R and $R \to R'$ is flat, then A' and B' are tor independent over R'.*

Proof. Follows immediately from Lemma 61.3 and Definition 61.1. □

Lemma 61.5. *Assumptions as in Lemma 61.3. For $M \in D(A)$ there are canonical isomorphisms*

$$H^i((M \otimes_A^{\mathbf{L}} A') \otimes_{R'}^{\mathbf{L}} B') = H^i(M \otimes_R^{\mathbf{L}} B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B')$$

of $A' \otimes_{R'} B'$-modules.

Proof. Let us elucidate the two sides of the equation. On the left hand side we have the composition of the functors $D(A) \to D(A') \to D(R') \to D(B')$ with the functor $H^i : D(B') \to \text{Mod}_{B'}$. Since there is a map from A' to the endomorphisms of the object $(M \otimes_A^{\mathbf{L}} A') \otimes_{R'}^{\mathbf{L}} B'$ in $D(B')$, we see that the left hand side is indeed an $A' \otimes_{R'} B'$-module. By the same arguments we see that $H^i(M \otimes_R^{\mathbf{L}} B)$ has an $A \otimes_R B$-module structure.

We first prove the result in case $B' = R' \otimes_R B$. In this case we choose a resolution $F^\bullet \to B$ by free R-modules. We also choose a K-flat complex $M^\bullet$ of A-modules representing M. Then the left hand side is represented by

$$\begin{aligned} H^i(\text{Tot}((M^\bullet \otimes_A A') \otimes_{R'} (R' \otimes_R F^\bullet))) &= H^i(\text{Tot}(M^\bullet \otimes_A A' \otimes_R F^\bullet)) \\ &= H^i(\text{Tot}(M^\bullet \otimes_R F^\bullet) \otimes_A A') \\ &= H^i(M \otimes_R^{\mathbf{L}} B) \otimes_A A' \end{aligned}$$

The final equality because $A \to A'$ is flat. The final module is the desired module because $A' \otimes_{R'} B' = A' \otimes_R B$ since we've assumed $B' = R' \otimes_R B$ in this paragraph.

General case. Suppose that $B' \to B''$ is a flat ring map. Then it is easy to see that

$$H^i((M \otimes_A^{\mathbf{L}} A') \otimes_{R'}^{\mathbf{L}} B'') = H^i((M \otimes_A^{\mathbf{L}} A') \otimes_{R'}^{\mathbf{L}} B') \otimes_{B'} B''$$

and

$$H^i(M \otimes_R^{\mathbf{L}} B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B'') = \left(H^i(M \otimes_R^{\mathbf{L}} B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B')\right) \otimes_{B'} B''$$

Thus the result for B' implies the result for B''. Since we've proven the result for $R' \otimes_R B$ in the previous paragraph, this implies the result in general. □

Lemma 61.6. *Let R be a ring. Let A, B be R-algebras. The following are equivalent*

1. *A and B are Tor independent over R,*
2. *for every pair of primes $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$ lying over the same prime $\mathfrak{r} \subset R$ the rings $A_\mathfrak{p}$ and $B_\mathfrak{q}$ are Tor independent over $R_\mathfrak{r}$, and*
3. *For every prime $\mathfrak{s}$ of $A \otimes_R B$ the module*

$$Tor_i^R(A, B)_\mathfrak{s} = Tor_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q})_\mathfrak{s}$$

(where $\mathfrak{p} = A \cap \mathfrak{s}$, $\mathfrak{q} = B \cap \mathfrak{s}$ and $\mathfrak{r} = R \cap \mathfrak{s}$) is zero.

Proof. Let $\mathfrak{s}$ be a prime of $A \otimes_R B$ as in (3). The equality

$$\text{Tor}_i^R(A, B)_\mathfrak{s} = \text{Tor}_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q})_\mathfrak{s}$$

where $\mathfrak{p} = A \cap \mathfrak{s}$, $\mathfrak{q} = B \cap \mathfrak{s}$ and $\mathfrak{r} = R \cap \mathfrak{s}$ follows from Lemma 61.3. Hence (2) implies (3). Since we can test the vanishing of modules by localizing at primes (Algebra, Lemma 23.1) we conclude that (3) implies (1). For (1) $\Rightarrow$ (2) we use that

$$\text{Tor}_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q}) = \text{Tor}_i^R(A, B) \otimes_{(A \otimes_R B)} (A_\mathfrak{p} \otimes_{R_\mathfrak{r}} B_\mathfrak{q})$$

again by Lemma 61.3. □

62. Spectral sequences for Tor

In this section we collect various spectral sequences that come up when considering the Tor functors.

Example 62.1. Let R be a ring. Let $K_\bullet$ be a chain complex of R-modules with $K_n = 0$ for $n \ll 0$. Let M be an R-module. Choose a resolution $P_\bullet \to M$ of M by free R-modules. We obtain a double chain complex $K_\bullet \otimes_R P_\bullet$. Applying the material in Homology, Section 25 (especially Homology, Lemma 25.3) translated into the language of chain complexes we find two spectral sequences converging to $H_*(K_\bullet \otimes_R^{\mathbf{L}} M)$. Namely, on the one hand a spectral sequence with E_2-page

$$(E_2)_{i,j} = \text{Tor}_j^R(H_i(K_\bullet), M) \Rightarrow H_{i+j}(K_\bullet \otimes_R^{\mathbf{L}} M)$$

and differential d_2 given by maps $\text{Tor}_j^R(H_i(K_\bullet), M) \to \text{Tor}_{j-2}^R(H_{i+1}(K_\bullet), M)$. Another spectral sequence with E_1-page

$$(E_1)_{i,j} = \text{Tor}_j^R(K_i, M) \Rightarrow H_{i+j}(K_\bullet \otimes_R^{\mathbf{L}} M)$$

with differential d_1 given by maps $\text{Tor}_j^R(K_i, M) \to \text{Tor}_j^R(K_{i-1}, M)$ induced by $K_i \to K_{i-1}$.

Example 62.2. Let $R \to S$ be a ring map. Let M be an R-module and let N be an S-module. Then there is a spectral sequence

$$\text{Tor}_n^S(\text{Tor}_m^R(M, S), N) \Rightarrow \text{Tor}_{n+m}^R(M, N).$$

To construct it choose a R-free resolution $P_\bullet$ of M. Then we have

$$M \otimes_R^{\mathbf{L}} N = P^\bullet \otimes_R N = (P^\bullet \otimes_R S) \otimes_S N$$

and then apply the first spectral sequence of Example 62.1.

Example 62.3. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & B' = B \otimes_A A' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

and B-modules M, N. Set $M' = M \otimes_A A' = M \otimes_B B'$ and $N' = N \otimes_A A' = N \otimes_B B'$. *Assume that $A \to B$ is flat and that M and N are A-flat.* Then there is a spectral sequence

$$\text{Tor}_i^A(\text{Tor}_j^B(M, N), A') \Rightarrow \text{Tor}_{i+j}^{B'}(M', N')$$

The reason is as follows. Choose free resolution $F_\bullet \to M$ as a B-module. As B and M are A-flat we see that $F_\bullet \otimes_A A'$ is a free B'-resolution of M'. Hence we see that the groups $\text{Tor}_n^{B'}(M', N')$ are computed by the complex

$$(F_\bullet \otimes_A A') \otimes_{B'} N' = (F_\bullet \otimes_B N) \otimes_A A' = (F_\bullet \otimes_B N) \otimes_A^{\mathbf{L}} A'$$

the last equality because $F_\bullet \otimes_B N$ is a complex of flat A-modules as N is flat over A. Hence we obtain the spectral sequence by applying the spectral sequence of Example 62.1.

Example 62.4. Let $K^\bullet, L^\bullet$ be objects of $D^-(R)$. Then there are spectral sequences

$$E_2^{p,q} = H^p(K^\bullet \otimes_R^{\mathbf{L}} H^q(L^\bullet)) \Rightarrow H^{p+q}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet)$$

with $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ and

$$H^q(H^p(K^\bullet) \otimes_R^{\mathbf{L}} L^\bullet) \Rightarrow H^{p+q}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet)$$

After replacing $K^\bullet$ and $L^\bullet$ by bounded above complexes of projectives, these spectral sequences are simply the two spectral sequences for computing the cohomology of $\text{Tot}(K^\bullet \otimes L^\bullet)$ discussed in Homology, Section 25.

63. Products and Tor

The simplest example of the product maps comes from the following situation. Suppose that $K^\bullet, L^\bullet \in D(R)$. Then there are maps

$$H^i(K^\bullet) \otimes_R H^j(L^\bullet) \longrightarrow H^{i+j}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet) \tag{63.0.1}$$

Namely, to define these maps we may assume that one of $K^\bullet, L^\bullet$ is a K-flat complex of R-modules (for example a bounded above complex of free or projective R-modules). In that case $K^\bullet \otimes_R^{\mathbf{L}} L^\bullet$ is represented by the complex $\text{Tot}(K^\bullet \otimes_R L^\bullet)$, see Section 59 (or Section 57). Next, suppose that $\xi \in H^i(K^\bullet)$ and $\zeta \in H^j(L^\bullet)$. Choose $k \in \text{Ker}(K^i \to K^{i+1})$ and $l \in \text{Ker}(L^j \to L^{j+1})$ representing ξ and ζ. Then we set

$$\xi \cup \zeta = \text{class of } k \otimes l \text{ in } H^{i+j}(\text{Tot}(K^\bullet \otimes_R L^\bullet)).$$

This make sense because the formula (see Homology, Definition 18.3) for the differential d on the total complex shows that $k \otimes l$ is a cocycle. Moreover, if $k' = d_K(k'')$ for some $k'' \in K^{i-1}$, then $k' \otimes l = \text{d}(k'' \otimes l)$ because l is a cocycle. Similarly, altering the choice of l representing ζ does not change the class of $k \otimes l$. It is equally clear that $\cup$ is bilinear, and hence to a general element of $H^i(K^\bullet) \otimes_R H^j(L^\bullet)$ we assign

$$\sum \xi_i \otimes \zeta_i \longmapsto \sum \xi_i \cup \zeta_i$$

in $H^{i+j}(\text{Tot}(K^\bullet \otimes_R L^\bullet))$.

Let $R \to A$ be a ring map. Let $K^\bullet, L^\bullet \in D(R)$. Then we have a canonical identification

$$(K^\bullet \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} (L^\bullet \otimes_R^{\mathbf{L}} A) = (K^\bullet \otimes_R^{\mathbf{L}} L^\bullet) \otimes_R^{\mathbf{L}} A \tag{63.0.2}$$

in $D(A)$. It is constructed as follows. First, choose K-flat resolutions $P^\bullet \to K^\bullet$ and $Q^\bullet \to L^\bullet$ over R. Then the left hand side is represented by the complex $\text{Tot}((P^\bullet \otimes_R A) \otimes_A (Q^\bullet \otimes_R A))$ and the right hand side by the complex $\text{Tot}(P^\bullet \otimes_R Q^\bullet) \otimes_R A$. These complexes are canonically isomorphic. Thus the construction above induces products

$$\text{Tor}_n^R(K^\bullet, A) \otimes_A \text{Tor}_m^R(L^\bullet, A) \longrightarrow \text{Tor}_{n+m}^R(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet, A)$$

which are occasionally useful.

Let M, N be R-modules. Using the general construction above, the canonical map $M \otimes_R^{\mathbf{L}} N \to M \otimes_R N$ and functoriality of Tor we obtain canonical maps

$$\text{Tor}_n^R(M, A) \otimes_A \text{Tor}_m^R(N, A) \longrightarrow \text{Tor}_{n+m}^R(M \otimes_R N, A) \tag{63.0.3}$$

Here is a direct construction using projective resolutions. First, choose projective resolutions

$$P_\bullet \to M, \quad Q_\bullet \to N, \quad T_\bullet \to M \otimes_R N$$

over R. We have $H_0(\text{Tot}(P_\bullet \otimes_R Q_\bullet)) = M \otimes_R N$ by right exactness of $\otimes_R$. Hence Derived Categories, Lemmas 19.6 and 19.7 guarantee the existence and uniqueness of a map of complexes $\mu : \text{Tot}(P_\bullet \otimes_R Q_\bullet) \to T_\bullet$ such that $H_0(\mu) = \text{id}_{M \otimes_R N}$. This induces a canonical map

$$\begin{aligned}(M \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} (N \otimes_R^{\mathbf{L}} A) &= \text{Tot}((P_\bullet \otimes_R A) \otimes_A (Q_\bullet \otimes_R A)) \\ &= \text{Tot}(P_\bullet \otimes_R Q_\bullet) \otimes_R A \\ &\to T_\bullet \otimes_R A \\ &= (M \otimes_R N) \otimes_R^{\mathbf{L}} A\end{aligned}$$

in $D(A)$. Hence the products (63.0.3) above are constructed using (63.0.1) over A to construct

$$\text{Tor}_n^R(M, A) \otimes_A \text{Tor}_m^R(N, A) \to H^{-n-m}((M \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} (N \otimes_R^{\mathbf{L}} A))$$

and then composing by the displayed map above to end up in $\text{Tor}_{n+m}^R(M \otimes_R N, A)$.

An interesting special case of the above occurs when $M = N = B$ where B is an R-algebra. In this case we obtain maps

$$\text{Tor}_n^R(B, A) \otimes_A \text{Tor}_m^R(B, A) \longrightarrow \text{Tor}_{n+m}^R(B \otimes_R B, A) \longrightarrow \text{Tor}_{n+m}^R(B, A)$$

the second arrow being induced by the multiplication map $B \otimes_R B \to B$ via functoriality for Tor. In other words we obtain an A-algebra structure on $\text{Tor}_\star^R(B, A)$. This algebra structure has many intriguing properties (associativity, graded commutative, B-algebra structure, divided powers in some case, etc) which we will discuss elsewhere (insert future reference here).

Lemma 63.1. *Let R be a ring. Let A, B, C be R-algebras and let $B \to C$ be an R-algebra map. Then the induced map*

$$Tor_\star^R(B, A) \longrightarrow Tor_\star^R(C, A)$$

is an A-algebra homomorphism.

Proof. Omitted. Hint: You can prove this by working through the definitions, writing all the complexes explicitly. ☐

64. Pseudo-coherent modules, I

Suppose that R is a ring. Recall that an R-module M is of finite type if there exists a surjection $R^{\oplus a} \to M$ and of finite presentation if there exists a presentation $R^{\oplus a_1} \to R^{\oplus a_0} \to M \to 0$. Similarly, we can consider those R-modules for which there exists a length n resolution

$$R^{\oplus a_n} \to R^{\oplus a_{n-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0 \tag{64.0.1}$$

by finite free R-modules. A module is called *pseudo-coherent* if we can find such a resolution for every n. Here is the formal definition.

Definition 64.1. Let R be a ring. Denote $D(R)$ its derived category. Let $m \in \mathbf{Z}$.

(1) An object $K^\bullet$ of $D(R)$ is *m-pseudo-coherent* if there exists a bounded complex $E^\bullet$ of finite free R-modules and a morphism $\alpha : E^\bullet \to K^\bullet$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and $H^m(\alpha)$ is surjective.
(2) An object $K^\bullet$ of $D(R)$ is *pseudo-coherent* if it is quasi-isomorphic to a bounded above complex of finite free R-modules.
(3) An R-module M is called *m-pseudo-coherent* if $M[0]$ is an m-pseudo-coherent object of $D(R)$.

(4) An R-module M is called *pseudo-coherent*[7] if $M[0]$ is a pseudo-coherent object of $D(R)$.

As usual we apply this terminology also to complexes of R-modules. Since any morphism $E^\bullet \to K^\bullet$ in $D(R)$ is represented by an actual map of complexes, see Derived Categories, Lemma 19.8, there is no ambiguity. It turns out that $K^\bullet$ is pseudo-coherent if and only if $K^\bullet$ is m-pseudo-coherent for all $m \in \mathbf{Z}$, see Lemma 64.5. Also, if the ring is Noetherian the condition can be understood as a finite generation condition on the cohomology, see Lemma 64.17. Let us first relate this to the informal discussion above.

Lemma 64.2. *Let R be a ring and $m \in \mathbf{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$.*

1. *If $K^\bullet$ is $(m+1)$-pseudo-coherent and $L^\bullet$ is m-pseudo-coherent then $M^\bullet$ is m-pseudo-coherent.*
2. *If $K^\bullet, M^\bullet$ are m-pseudo-coherent, then $L^\bullet$ is m-pseudo-coherent.*
3. *If $L^\bullet$ is $(m+1)$-pseudo-coherent and $M^\bullet$ is m-pseudo-coherent, then $K^\bullet$ is $(m+1)$-pseudo-coherent.*

Proof. Proof of (1). Choose $\alpha : P^\bullet \to K^\bullet$ with $P^\bullet$ a bounded complex of finite free modules such that $H^i(\alpha)$ is an isomorphism for $i > m+1$ and surjective for $i = m+1$. We may replace $P^\bullet$ by $\sigma_{\geq m+1}P^\bullet$ and hence we may assume that $P^i = 0$ for $i < m+1$. Choose $\beta : E^\bullet \to L^\bullet$ with $E^\bullet$ a bounded complex of finite free modules such that $H^i(\beta)$ is an isomorphism for $i > m$ and surjective for $i = m$. By Derived Categories, Lemma 19.11 we can find a map $\gamma : P^\bullet \to E^\bullet$ such that the diagram

$$\begin{tikzcd} K^\bullet \arrow[r] & L^\bullet \\ P^\bullet \arrow[u] \arrow[r, "\gamma"] & E^\bullet \arrow[u, "\beta"'] \end{tikzcd}$$

is commutative in $D(R)$. The cone $C(\gamma)^\bullet$ is a bounded complex of finite free R-modules, and the commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(P^\bullet, E^\bullet, C(\gamma)^\bullet) \longrightarrow (K^\bullet, L^\bullet, M^\bullet).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 5.19 and 5.20 that $C(\gamma)^\bullet \to M^\bullet$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence $M^\bullet$ is m-pseudo-coherent.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. $\square$

Lemma 64.3. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $m \in \mathbf{Z}$.*

1. *If $K^\bullet$ is m-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m$, then $H^m(K^\bullet)$ is a finite type R-module.*
2. *If $K^\bullet$ is m-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m+1$, then $H^{m+1}(K^\bullet)$ is a finitely presented R-module.*

Proof. Proof of (1). Choose a bounded complex $E^\bullet$ of finite projective R-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It is clear that it suffices to prove the result

[7] This clashes with what is meant by a pseudo-coherent module in [Bou61].

for $E^\bullet$. Let n be the largest integer such that $E^n \neq 0$. If $n = m$, then the result is clear. If $n > m$, then $E^{n-1} \to E^n$ is surjective as $H^n(E^\bullet) = 0$. As E^n is finite projective we see that $E^{n-1} = E' \oplus E^n$. Hence it suffices to prove the result for the complex $(E')^\bullet$ which is the same as $E^\bullet$ except has E' in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). Choose a bounded complex $E^\bullet$ of finite projective R-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $E^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K^\bullet) \cong H^{m+1}(E^\bullet) = \mathrm{Coker}(E^m \to E^{m+1})$ which is of finite presentation. □

Lemma 64.4. *Let R be a ring. Let M be an R-module. Then*

1. *M is 0-pseudo-coherent if and only if M is a finite R-module,*
2. *M is (-1)-pseudo-coherent if and only if M is a finitely presented R-module,*
3. *M is $(-d)$-pseudo-coherent if and only if there exists a resolution*
$$R^{\oplus a_d} \to R^{\oplus a_{d-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0$$
of length d, and
4. *M is pseudo-coherent if and only if there exists an infinite resolution*
$$\ldots \to R^{\oplus a_1} \to R^{\oplus a_0} \to M \to 0$$
by finite free R-modules.

Proof. If M is of finite type (resp. of finite presentation), then M is 0-pseudo-coherent (resp. (-1)-pseudo-coherent) as follows from the discussion preceding Definition 64.1. Conversely, if M is 0-pseudo-coherent, then $M = H^0(M[0])$ is of finite type by Lemma 64.3. If M is (-1)-pseudo-coherent, then it is 0-pseudo-coherent hence of finite type. Choose a surjection $R^{\oplus a} \to M$ and denote $K = \mathrm{Ker}(R^{\oplus a} \to M)$. By Lemma 64.2 we see that K is 0-pseudo-coherent, hence of finite type, whence M is of finite presentation.

To prove the third and fourth statement use induction and an argument similar to the above (details omitted). □

Lemma 64.5. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. The following are equivalent*

1. *$K^\bullet$ is pseudo-coherent,*
2. *$K^\bullet$ is m-pseudo-coherent for every $m \in \mathbf{Z}$, and*
3. *$K^\bullet$ is quasi-isomorphic to a bounded above complex of finite projective R-modules.*

If (1), (2), and (3) hold and $H^i(K^\bullet) = 0$ for $i > b$, then we can find a quasi-isomorphism $F^\bullet \to K^\bullet$ with F^i finite free R-modules and $F^i = 0$ for $i > b$.

Proof. We see that (1) ⇒ (3) as a finite free module is a finite projective R-module. Conversely, suppose $P^\bullet$ is a bounded above complex of finite projective R-modules. Say $P^i = 0$ for $i > n_0$. We choose a direct sum decompositions $F^{n_0} = P^{n_0} \oplus C^{n_0}$ with F^{n_0} a finite free R-module, and inductively
$$F^{n-1} = P^{n-1} \oplus C^n \oplus C^{n-1}$$

for $n \leq n_0$ with F^{n_0} a finite free R-module. As a complex $F^\bullet$ has maps $F^{n-1} \to F^n$ which agree with $P^{n-1} \to P^n$, induce the identity $C^n \to C^n$, and are zero on C^{n-1}. The map $F^\bullet \to P^\bullet$ is a quasi-isomorphism (even a homotopy equivalence) and hence (3) implies (1).

Assume (1). Let $E^\bullet$ be a bounded above complex of finite free R-modules and let $E^\bullet \to K^\bullet$ be a quasi-isomorphism. Then the induced maps $\sigma_{\geq m} E^\bullet \to K^\bullet$ from the stupid truncation of $E^\bullet$ to $K^\bullet$ show that $K^\bullet$ is m-pseudo-coherent. Hence (1) implies (2).

Assume (2). Since $K^\bullet$ is 0-pseudo-coherent we see in particular that $K^\bullet$ is bounded above. Let b be an integer such that $H^i(K^\bullet) = 0$ for $i > b$. By descending induction on $n \in \mathbf{Z}$ we are going to construct finite free R-modules F^i for $i \geq n$, differentials $d^i : F^i \to F^{i+1}$ for $i \geq n$, maps $\alpha : F^i \to K^i$ compatible with differentials, such that (1) $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$, and (2) $F^i = 0$ for $i > b$. Picture

$$\begin{tikzcd}
 & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\
K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

The base case is $n = b + 1$ where we can take $F^i = 0$ for all i. Induction step. Let $C^\bullet$ be the cone on α (Derived Categories, Definition 9.1). The long exact sequence of cohomology shows that $H^i(C^\bullet) = 0$ for $i \geq n$. By Lemma 64.2 we see that $C^\bullet$ is $(n-1)$-pseudo-coherent. By Lemma 64.3 we see that $H^{n-1}(C^\bullet)$ is a finite R-module. Choose a finite free R-module F^{n-1} and a map $\beta : F^{n-1} \to C^{n-1}$ such that the composition $F^{n-1} \to C^{n-1} \to C^n$ is zero and such that F^{n-1} surjects onto $H^{n-1}(C^\bullet)$. Since $C^{n-1} = K^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$ implies these maps fit into a morphism of complexes

$$\begin{tikzcd}
 & F^{n-1} \arrow[r, "d^{n-1}"'] \arrow[d, "\alpha^{n-1}"] & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\
\ldots \arrow[r] & K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{tikzcd}
(F^n \to \ldots) \arrow[r] \arrow[d] & (F^{n-1} \to \ldots) \arrow[r] \arrow[d] & F^{n-1} \arrow[r] \arrow[d, "\beta"'] & (F^n \to \ldots)[1] \arrow[d] \\
(F^n \to \ldots) \arrow[r] & K^\bullet \arrow[r] & C^\bullet \arrow[r] & (F^n \to \ldots)[1]
\end{tikzcd}$$

Hence our choice of β implies that the map of complexes $(F^{n-1} \to \ldots) \to K^\bullet$ induces an isomorphism on cohomology in degrees $\geq n$ and a surjection in degree $n - 1$. This finishes the proof of the lemma. $\square$

Lemma 64.6. *Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. If two out of three of $K^\bullet, L^\bullet, M^\bullet$ are pseudo-coherent then the third is also pseudo-coherent.*

Proof. Combine Lemmas 64.2 and 64.5. $\square$

Lemma 64.7. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $m \in \mathbf{Z}$.*

1. *If $H^i(K^\bullet) = 0$ for all $i \geq m$, then $K^\bullet$ is m-pseudo-coherent.*
2. *If $H^i(K^\bullet) = 0$ for $i > m$ and $H^m(K^\bullet)$ is a finite R-module, then $K^\bullet$ is m-pseudo-coherent.*
3. *If $H^i(K^\bullet) = 0$ for $i > m+1$, the module $H^{m+1}(K^\bullet)$ is of finite presentation, and $H^m(K^\bullet)$ is of finite type, then $K^\bullet$ is m-pseudo-coherent.*

Proof. It suffices to prove (3). Set $M = H^{m+1}(K^\bullet)$. Note that $\tau_{\geq m+1}K^\bullet$ is quasi-isomorphic to $M[-m-1]$. By Lemma 64.4 we see that $M[-m-1]$ is m-pseudo-coherent. Since we have the distinguished triangle

$$(\tau_{\leq m}K^\bullet, K^\bullet, \tau_{\geq m+1}K^\bullet)$$

(Derived Categories, Remark 12.4) by Lemma 64.2 it suffices to prove that $\tau_{\leq m}K^\bullet$ is pseudo-coherent. By assumption $H^m(\tau_{\leq m}K^\bullet)$ is a finite type R-module. Hence we can find a finite free R-module E and a map $E \to \mathrm{Ker}(d_K^m)$ such that the composition $E \to \mathrm{Ker}(d_K^m) \to H^m(\tau_{\leq m}K^\bullet)$ is surjective. Then $E[-m] \to \tau_{\leq m}K^\bullet$ witnesses the fact that $\tau_{\leq m}K^\bullet$ is m-pseudo-coherent. $\square$

Lemma 64.8. *Let R be a ring. Let $m \in \mathbf{Z}$. If $K^\bullet \oplus L^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) so are $K^\bullet$ and $L^\bullet$.*

Proof. In this proof we drop the superscript $^\bullet$. Assume that $K \oplus L$ is m-pseudo-coherent. It is clear that $K, L \in D^-(R)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 4.10. By Lemma 64.2 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. By Lemma 64.7 we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n], L[n-1], \ldots, L$ are m-pseudo-coherent as desired. The pseudo-coherent case follows from this and Lemma 64.5. $\square$

Lemma 64.9. *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a bounded above complex of R-modules such that K^i is $(m-i)$-pseudo-coherent for all i. Then $K^\bullet$ is m-pseudo-coherent. In particular, if $K^\bullet$ is a bounded above complex of pseudo-coherent R-modules, then $K^\bullet$ is pseudo-coherent.*

Proof. We may replace $K^\bullet$ by $\sigma_{\geq m-1}K^\bullet$ (for example) and hence assume that $K^\bullet$ is bounded. Then the complex $K^\bullet$ is m-pseudo-coherent as each $K^i[-i]$ is m-pseudo-coherent by induction on the length of the complex: use Lemma 64.2 and the stupid truncations. For the final statement, it suffices to prove that $K^\bullet$ is m-pseudo-coherent for all $m \in \mathbf{Z}$, see Lemma 64.5. This follows from the first part. $\square$

Lemma 64.10. *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet \in D^-(R)$ such that $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent (resp. pseudo-coherent) for all i. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. Assume $K^\bullet$ is an object of $D^-(R)$ such that each $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent. Let n be the largest integer such that $H^n(K^\bullet)$ is nonzero. We will prove the lemma by induction on n. If $n < m$, then $K^\bullet$ is m-pseudo-coherent by Lemma 64.7. If $n \geq m$, then we have the distinguished triangle

$$(\tau_{\leq n-1}K^\bullet, K^\bullet, H^n(K^\bullet)[-n])$$

(Derived Categories, Remark 12.4) Since $H^n(K^\bullet)[-n]$ is m-pseudo-coherent by assumption, we can use Lemma 64.2 to see that it suffices to prove that $\tau_{\leq n-1}K^\bullet$ is m-pseudo-coherent. By induction on n we win. (The pseudo-coherent case follows from this and Lemma 64.5.) □

Lemma 64.11. *Let $A \to B$ be a ring map. Assume that B is pseudo-coherent as an A-module. Let $K^\bullet$ be a complex of B-modules. The following are equivalent*

(1) *$K^\bullet$ is m-pseudo-coherent as a complex of B-modules, and*
(2) *$K^\bullet$ is m-pseudo-coherent as a complex of A-modules.*

The same equivalence holds for pseudo-coherence.

Proof. Assume (1). Choose a bounded complex of finite free B-modules $E^\bullet$ and a map $\alpha : E^\bullet \to K^\bullet$ which is an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Consider the distinguished triangle $(E^\bullet, K^\bullet, C(\alpha)^\bullet)$. By Lemma 64.7 $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of A-modules. Hence it suffices to prove that $E^\bullet$ is pseudo-coherent as a complex of A-modules, which follows from Lemma 64.9. The pseudo-coherent case of (1) $\Rightarrow$ (2) follows from this and Lemma 64.5.

Assume (2). Let n be the largest integer such that $H^n(K^\bullet) \neq 0$. We will prove that $K^\bullet$ is m-pseudo-coherent as a complex of B-modules by induction on $n - m$. The case $n < m$ follows from Lemma 64.7. Choose a bounded complex of finite free A-modules $E^\bullet$ and a map $\alpha : E^\bullet \to K^\bullet$ which is an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Consider the induced map of complexes

$$\alpha \otimes 1 : E^\bullet \otimes_A B \to K^\bullet.$$

Note that $C(\alpha\otimes 1)^\bullet$ is acyclic in degrees $\geq n$ as $H^n(E) \to H^n(E^\bullet \otimes_A B) \to H^n(K^\bullet)$ is surjective by construction and since $H^i(E^\bullet \otimes_A B) = 0$ for $i > n$ by the spectral sequence of Example 62.4. On the other hand, $C(\alpha \otimes 1)^\bullet$ is m-pseudo-coherent as a complex of A-modules because both $K^\bullet$ and $E^\bullet \otimes_A B$ (see Lemma 64.9) are so, see Lemma 64.2. Hence by induction we see that $C(\alpha \otimes 1)^\bullet$ is m-pseudo-coherent as a complex of B-modules. Finally another application of Lemma 64.2 shows that $K^\bullet$ is m-pseudo-coherent as a complex of B-modules (as clearly $E^\bullet \otimes_A B$ is pseudo-coherent as a complex of B-modules). The pseudo-coherent case of (2) $\Rightarrow$ (1) follows from this and Lemma 64.5. □

Lemma 64.12. *Let $A \to B$ be a ring map. Let $K^\bullet$ be an m-pseudo-coherent (resp. pseudo-coherent) complex of A-modules. Then $K^\bullet \otimes_A^{\mathbf{L}} B$ is an m-pseudo-coherent (resp. pseudo-coherent) complex of B-modules.*

Proof. First we note that the statement of the lemma makes sense as $K^\bullet$ is bounded above and hence $K^\bullet \otimes_A^{\mathbf{L}} B$ is defined by Equation (57.0.2). Having said this, choose a bounded complex $E^\bullet$ of finite free A-modules and $\alpha : E^\bullet \to K^\bullet$ with $H^i(\alpha)$ an isomorphism for $i > m$ and surjective for $i = m$. Then the cone $C(\alpha)^\bullet$

is acyclic in degrees $\geq m$. Since $- \otimes_A^{\mathbf{L}} B$ is an exact functor we get a distinguished triangle

$$(E^\bullet \otimes_A^{\mathbf{L}} B, K^\bullet \otimes_A^{\mathbf{L}} B, C(\alpha)^\bullet \otimes_A^{\mathbf{L}} B)$$

of complexes of B-modules. By the dual to Derived Categories, Lemma 16.1 we see that $H^i(C(\alpha)^\bullet \otimes_A^{\mathbf{L}} B) = 0$ for $i \geq m$. Since $E^\bullet$ is a complex of projective R-modules we see that $E^\bullet \otimes_A^{\mathbf{L}} B = E^\bullet \otimes_A B$ and hence

$$E^\bullet \otimes_A B \longrightarrow K^\bullet \otimes_A^{\mathbf{L}} B$$

is a morphism of complexes of B-modules that witnesses the fact that $K^\bullet \otimes_A^{\mathbf{L}} B$ is m-pseudo-coherent. The case of pseudo-coherent complexes follows from the case of m-pseudo-coherent complexes via Lemma 64.5. $\square$

Lemma 64.13. *Let $A \to B$ be a flat ring map. Let M be an m-pseudo-coherent (resp. pseudo-coherent) A-module. Then $M \otimes_A B$ is an m-pseudo-coherent (resp. pseudo-coherent) B-module.*

Proof. Immediate consequence of Lemma 64.12 and the fact that $M \otimes_A^{\mathbf{L}} B = M \otimes_A B$ because B is flat over A. $\square$

The following lemma also follows from the stronger Lemma 64.15.

Lemma 64.14. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ is m-pseudo-coherent (resp. pseudo-coherent), then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. We will use without further mention that $- \otimes_R R_{f_i}$ is an exact functor and that therefore

$$H^i(K^\bullet)_{f_i} = H^i(K^\bullet) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i}).$$

Assume $K^\bullet \otimes_R R_{f_i}$ is m-pseudo-coherent for $i = 1, \ldots, r$. Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(K^\bullet \otimes_R R_{f_i})$ is nonzero for some i. This implies in particular that $H^i(K^\bullet) = 0$ for $i > n$ (and that $H^n(K^\bullet) \neq 0$) see Algebra, Lemma 23.2. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is true by Lemma 64.7. If $n \geq m$, then $H^n(K^\bullet)_{f_i}$ is a finite R_{f_i}-module for each i, see Lemma 64.3. Hence $H^n(K^\bullet)$ is a finite R-module, see Algebra, Lemma 23.2. Choose a finite free R-module E and a surjection $E \to H^n(K^\bullet)$. As E is projective we can lift this to a map of complexes $\alpha : E[-n] \to K^\bullet$. Then the cone $C(\alpha)^\bullet$ has vanishing cohomology in degrees $\geq n$. On the other hand, the complexes $C(\alpha)^\bullet \otimes_R R_{f_i}$ are m-pseudo-coherent for each i, see Lemma 64.2. Hence by induction we see that $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of R-modules. Applying Lemma 64.2 once more we conclude. $\square$

Lemma 64.15. *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ is m-pseudo-coherent (resp. pseudo-coherent), then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. We will use without further mention that $- \otimes_R R'$ is an exact functor and that therefore

$$H^i(K^\bullet) \otimes_R R' = H^i(K^\bullet \otimes_R R').$$

Assume $K^\bullet \otimes_R R'$ is m-pseudo-coherent. Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(K^\bullet)$ is nonzero; then n is also the largest integer such that $H^n(K^\bullet \otimes_R R')$ is nonzero. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is true by Lemma 64.7. If $n \geq m$, then $H^n(K^\bullet) \otimes_R R'$ is a finite R'-module, see Lemma 64.3. Hence $H^n(K^\bullet)$ is a finite R-module, see Algebra, Lemma 83.2. Choose a finite free R-module E and a surjection $E \to H^n(K^\bullet)$. As E is projective we can lift this to a map of complexes $\alpha : E[-n] \to K^\bullet$. Then the cone $C(\alpha)^\bullet$ has vanishing cohomology in degrees $\geq n$. On the other hand, the complex $C(\alpha)^\bullet \otimes_R R'$ is m-pseudo-coherent, see Lemma 64.2. Hence by induction we see that $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of R-modules. Applying Lemma 64.2 once more we conclude. $\square$

Lemma 64.16. *Let R be a ring. Let K, L be objects of $D(R)$.*

1. *If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes_R^{\mathbf{L}} L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.*
2. *If K and L are pseudo-coherent, then $K \otimes_R^{\mathbf{L}} L$ is pseudo-coherent.*

Proof. Proof of (1). We may assume there exist bounded complexes $K^\bullet$ and $L^\bullet$ of finite free R-modules and maps $\alpha : K^\bullet \to K$ and $\beta : L^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \mathrm{Tot}(K^\bullet \otimes_R L^\bullet) \to K \otimes_R^{\mathbf{L}} L$$

induces isomorphisms on cohomology in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted). Part (2) follows from part (1) and Lemma 64.5. $\square$

Lemma 64.17. *Let R be a Noetherian ring. Then*

1. *A complex of R-modules $K^\bullet$ is m-pseudo-coherent if and only if $K^\bullet \in D^-(R)$ and $H^i(K^\bullet)$ is a finite R-module for $i \geq m$.*
2. *A complex of R-modules $K^\bullet$ is pseudo-coherent if and only if $K^\bullet \in D^-(R)$ and $H^i(K^\bullet)$ is a finite R-module for all i.*
3. *An R-module is pseudo-coherent if and only if it is finite.*

Proof. In Algebra, Lemma 71.1 we have seen that any finite R-module is pseudo-coherent. On the other hand, a pseudo-coherent module is finite, see Lemma 64.4. Hence (3) holds. Suppose that $K^\bullet$ is an m-pseudo-coherent complex. Then there exists a bounded complex of finite free R-modules $E^\bullet$ such that $H^i(K^\bullet)$ is isomorphic to $H^i(E^\bullet)$ for $i > m$ and such that $H^m(K^\bullet)$ is a quotient of $H^m(E^\bullet)$. Thus it is clear that each $H^i(K^\bullet)$, $i \geq m$ is a finite module. The converse implication in (1) follows from Lemma 64.10 and part (3). Part (2) follows from (1) and Lemma 64.5. $\square$

Lemma 64.18. *Let R be a coherent ring (Algebra, Definition 90.1). Let $K \in D^-(R)$. The following are equivalent*

1. *K is m-pseudo-coherent,*
2. *$H^m(K)$ is a finite R-module and $H^i(K)$ is coherent for $i > m$, and*
3. *$H^m(K)$ is a finite R-module and $H^i(K)$ is finitely presented for $i > m$.*

Thus K is pseudo-coherent if and only if $H^i(K)$ is a coherent module for all i.

Proof. Recall that an R-module M is coherent if and only if it is of finite presentation (Algebra, Lemma 90.4). This explains the equivalence of (2) and (3). If so and if we choose an exact sequence $0 \to N \to R^{\oplus m} \to M \to 0$, then N is coherent by Algebra, Lemma 90.3. Thus in this case, repeating this procedure with N we find a resolution

$$\ldots \to R^{\oplus n} \to R^{\oplus m} \to M \to 0$$

by finite free R-modules. In other words, M is pseudo-coherent. The equivalence of (1) and (2) follows from this and Lemmas 64.10 and 64.4. The final assertion follows from the equivalence of (1) and (2) combined with Lemma 64.5. $\square$

65. Pseudo-coherent modules, II

We continue the discussion started in Section 64.

Lemma 65.1. *Let R be a ring. Let $M = \operatorname{colim} M_i$ be a filtered colimit of R-modules. Let $K \in D(R)$ be m-pseudo-coherent. Then $\operatorname{colim} \operatorname{Ext}^n_R(K, M_i) = \operatorname{Ext}^n_R(K, M)$ for $n < -m$ and $\operatorname{colim} \operatorname{Ext}^{-m}_R(K, M_i) \to \operatorname{Ext}^{-m}_R(K, M)$ is injective.*

Proof. By definition we can find a distinguished triangle

$$E \to K \to L \to E[1]$$

in $D(R)$ such that E is represented by a bounded complex of finite free R-modules and such that $H^i(L) = 0$ for $i \geq m$. Then $\operatorname{Ext}^n_R(L, N) = 0$ for any R-module N and $n \leq -m$, see Derived Categories, Lemma 27.3. By the long exact sequence of Ext associated to the distinguished triangle we see that $\operatorname{Ext}^n_R(K, N) \to \operatorname{Ext}^n_R(E, N)$ is an isomorphism for $n < -m$ and injective for $n = -m$. Thus it suffices to prove that $M \mapsto \operatorname{Ext}^n_R(E, M)$ commutes with filtered colimits when E can be represented by a bounded complex of finite free R-modules $E^\bullet$. The modules $\operatorname{Ext}^n_R(E, M)$ are computed by the complex $\operatorname{Hom}_R(E^\bullet, M)$, see Derived Categories, Lemma 19.8. The functor $M \mapsto \operatorname{Hom}_R(E^p, M)$ commutes with filtered colimits as E^p is finite free. Thus $\operatorname{Hom}_R(E^\bullet, M) = \operatorname{colim} \operatorname{Hom}_R(E^\bullet, M_i)$ as complexes. Since filtered colimits are exact (Algebra, Lemma 8.8) we conclude. $\square$

Lemma 65.2. *Let R be a ring. Let $K \in D^-(R)$. Let $m \in \mathbf{Z}$. Then K is m-pseudo-coherent if and only if for any filtered colimit $M = \operatorname{colim} M_i$ of R-modules we have $\operatorname{colim} \operatorname{Ext}^n_R(K, M_i) = \operatorname{Ext}^n_R(K, M)$ for $n < -m$ and $\operatorname{colim} \operatorname{Ext}^{-m}_R(K, M_i) \to \operatorname{Ext}^{-m}_R(K, M)$ is injective.*

Proof. One implication was shown in Lemma 65.1. Assume for any filtered colimit $M = \operatorname{colim} M_i$ of R-modules we have $\operatorname{colim} \operatorname{Ext}^n_R(K, M_i) = \operatorname{Ext}^n_R(K, M)$ for $n < -m$ and $\operatorname{colim} \operatorname{Ext}^{-m}_R(K, M_i) \to \operatorname{Ext}^{-m}_R(K, M)$ is injective. We will show K is m-pseudo-coherent.

Let t be the maximal integer such that $H^t(K)$ is nonzero. We will use induction on t. If $t < m$, then K is m-pseudo-coherent by Lemma 64.7. If $t \geq m$, then since $\operatorname{Hom}_R(H^t(K), M) = \operatorname{Ext}^{-t}_R(K, M)$ we conclude that $\operatorname{colim} \operatorname{Hom}_R(H^t(K), M_i) \to \operatorname{Hom}_R(H^t(K), M)$ is injective for any filtered colimit $M = \operatorname{colim} M_i$. This implies that $H^t(K)$ is a finite R-module by Algebra, Lemma 11.1. Choose a finite free R-module F and a surjection $F \to H^t(K)$. We can lift this to a morphism $F[-t] \to K$ in $D(R)$ and choose a distinguished triangle

$$F[-t] \to K \to L \to F[-t+1]$$

in $D(R)$. Then $H^i(L) = 0$ for $i \geq t$. Moreover, the long exact sequence of Ext associated to this distinguished triangle shows that L inherts the assumption we made on K by a small argument we omit. By induction on t we conclude that L is m-pseudo-coherent. Hence K is m-pseudo-coherent by Lemma 64.2. □

Lemma 65.3. *Let R be a ring. Let L, M, N be R-modules.*

(1) *If M is finitely presented and L is flat, then the canonical map $\mathrm{Hom}_R(M, N) \otimes_R L \to \mathrm{Hom}_R(M, N \otimes_R L)$ is an isomorphism.*
(2) *If M is $(-m)$-pseudo-coherent and L is flat, then the canonical map $\mathrm{Ext}^i_R(M, N) \otimes_R L \to \mathrm{Ext}^i_R(M, N \otimes_R L)$ is an isomorphism for $i < m$.*

Proof. Choose a resolution $F_\bullet \to M$ whose terms are free R-modules, see Algebra, Lemma 71.1. The complex $\mathrm{Hom}_R(F_\bullet, N)$ computes $\mathrm{Ext}^i_R(M, N)$ and the complex $\mathrm{Hom}_R(F_\bullet, N \otimes_R L)$ computes $\mathrm{Ext}^i_R(M, N \otimes_R L)$. There always is a map of cochain complexes

$$\mathrm{Hom}_R(F_\bullet, N) \otimes_R L \longrightarrow \mathrm{Hom}_R(F_\bullet, N \otimes_R L)$$

which induces canonical maps $\mathrm{Ext}^i_R(M, N) \otimes_R L \to \mathrm{Ext}^i_R(M, N \otimes_R L)$ for all $i \geq 0$ (canonical for example in the sense that these maps do not depend on the choice of the resolution $F_\bullet$). If L is flat, then the complex $\mathrm{Hom}_R(F_\bullet, N) \otimes_R L$ computes $\mathrm{Ext}^i_R(M, N) \otimes_R L$ since taking cohomology commutes with tensoring by L.

Having said all of the above, if M is $(-m)$-pseudo-coherent, then we may choose $F_\bullet$ such that F_i is finite free for $i = 0, \ldots, m$. Then the map of cochain complexes displayed above is an isomorphism in degrees $\leq m$ and hence an isomorphism on cohomology groups in degrees $< m$. This proves (2). If M is finitely presented, then M is (-1)-pseudo-coherent by Lemma 64.4 and we get the result because $\mathrm{Hom} = \mathrm{Ext}^0$. □

Lemma 65.4. *Let $R \to R'$ be a flat ring map. Let M, N be R-modules.*

(1) *If M is a finitely presented R-module, then $\mathrm{Hom}_R(M, N) \otimes_R R' = \mathrm{Hom}_{R'}(M \otimes_R R', N \otimes_R R')$.*
(2) *If M is $(-m)$-pseudo-coherent, then $\mathrm{Ext}^i_R(M, N) \otimes_R R' = \mathrm{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R')$ for $i < m$.*

In particular if R is Noetherian and M is a finite module this holds for all i.

Proof. By Algebra, Lemma 73.1 we have $\mathrm{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R') = \mathrm{Ext}^i_R(M, N \otimes_R R')$. Combined with Lemma 65.3 we conclude (1) and (2) holds. The final statement follows from this and Lemma 64.17. □

Lemma 65.5. *Let R be a ring. Let $K \in D^-(R)$. The following are equivalent:*

(1) *K is pseudo-coherent,*
(2) *for every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map*

$$\alpha : K \otimes_R^{\mathbf{L}} \left(\prod\nolimits_\alpha Q_\alpha\right) \longrightarrow \prod\nolimits_\alpha (K \otimes_R^{\mathbf{L}} Q_\alpha)$$

is an isomorphism in $D(R)$,
(3) *for every R-module Q and every set A, the canonical map*

$$\beta : K \otimes_R^{\mathbf{L}} Q^A \longrightarrow (K \otimes_R^{\mathbf{L}} Q)^A$$

is an isomorphism in $D(R)$, and

(4) *for every set A, the canonical map*

$$\gamma : K \otimes_R^{\mathbf{L}} R^A \longrightarrow K^A$$

is an isomorphism in $D(R)$.

Given $m \in \mathbf{Z}$ the following are equivalent

(a) *K is m-pseudo-coherent,*
(b) *for every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, with α as above $H^i(\alpha)$ is an isomorphism for $i > m$ and surjective for $i = m$,*
(c) *for every R-module Q and every set A, with β as above $H^i(\beta)$ is an isomorphism for $i > m$ and surjective for $i = m$,*
(d) *for every set A, with γ as above $H^i(\gamma)$ is an isomorphism for $i > m$ and surjective for $i = m$.*

Proof. If K is pseudo-coherent, then K can be represented by a bounded above complex of finite free R-modules. Then the derived tensor products are computed by tensoring with this complex. Also, products in $D(R)$ are given by taking products of any choices of representative complexes. Hence (1) implies (2), (3), (4) by the corresponding fact for modules, see Algebra, Proposition 89.3.

In the same way (using the tensor product is right exact) the reader shows that (a) implies (b), (c), and (d).

Assume (4) holds. To show that K is pseudo-coherent it suffices to show that K is m-pseudo-coherent for all m (Lemma 64.5). Hence to finish then proof it suffices to prove that (d) implies (a).

Assume (d). Let i be the largest integer such that $H^i(K)$ is nonzero. If $i < m$, then we are done. If not, then from (d) and the description of products in $D(R)$ given above we find that $H^i(K) \otimes_R R^A \to H^i(K)^A$ is surjective. Hence $H^i(K)$ is a finitely generated R-module by Algebra, Proposition 89.2. Thus we may choose a complex L consisting of a single finite free module sitting in degree i and a map of complexes $L \to K$ such that $H^i(L) \to H^i(K)$ is surjective. In particular L satisfies (1), (2), (3), and (4). Choose a distinguished triangle

$$L \to K \to M \to L[1]$$

Then we see that $H^j(M) = 0$ for $j \geq i$. On the other hand, M still has property (d) by a small argument which we omit. By induction on i we find that M is m-pseudo-coherent. Hence K is m-pseudo-coherent by Lemma 64.2. $\square$

Lemma 65.6. *Let R be a ring. Let $K \in D(R)$ be pseudo-coherent. Let $i \in \mathbf{Z}$. There exists a finitely presented R-module M and a map $K \to M[-i]$ in $D(R)$ which induces an injection $H^i(K) \to M$.*

Proof. By Definition 64.1 we may represent K by a complex $P^\bullet$ of finite free R-modules. Set $M = \mathrm{Coker}(P^{i-1} \to P^i)$. $\square$

Lemma 65.7. *Let A be a Noetherian ring. Let $K \in D(A)$ be pseudo-coherent, i.e., $K \in D^-(A)$ with finite cohomology modules. Let $\mathfrak{m}$ be a maximal ideal of A. If $H^i(K)/\mathfrak{m}H^i(K) \neq 0$, then there exists a finite A-module E annihilated by a power of $\mathfrak{m}$ and a map $K \to E[-i]$ which is nonzero on $H^i(K)$.*

Proof. (The equivalent formulation of pseudo-coherence in the statement of the lemma is Lemma 64.17.) Choose $K \to M[-i]$ as in Lemma 65.6. By Artin-Rees (Algebra, Lemma 51.2) we can find an n such that $H^i(K) \cap \mathfrak{m}^n M \subset \mathfrak{m} H^i(K)$. Take $E = M/\mathfrak{m}^n M$. $\square$

66. Tor dimension

Instead of resolving by projective modules we can look at resolutions by flat modules. This leads to the following concept.

Definition 66.1. Let R be a ring. Denote $D(R)$ its derived category. Let $a, b \in \mathbf{Z}$.

(1) An object $K^\bullet$ of $D(R)$ has *tor-amplitude in* $[a, b]$ if $H^i(K^\bullet \otimes_R^{\mathbf{L}} M) = 0$ for all R-modules M and all $i \not\in [a, b]$.
(2) An object $K^\bullet$ of $D(R)$ has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some a, b.
(3) An R-module M has *tor dimension* $\leq d$ if $M[0]$ as an object of $D(R)$ has tor-amplitude in $[-d, 0]$.
(4) An R-module M has *finite tor dimension* if $M[0]$ as an object of $D(R)$ has finite tor dimension.

We observe that if $K^\bullet$ has finite tor dimension, then $K^\bullet \in D^b(R)$.

Lemma 66.2. *Let R be a ring. Let $K^\bullet$ be a bounded above complex of flat R-modules with tor-amplitude in $[a, b]$. Then $\text{Coker}(d_K^{a-1})$ is a flat R-module.*

Proof. As $K^\bullet$ is a bounded above complex of flat modules we see that $K^\bullet \otimes_R M = K^\bullet \otimes_R^{\mathbf{L}} M$. Hence for every R-module M the sequence

$$K^{a-2} \otimes_R M \to K^{a-1} \otimes_R M \to K^a \otimes_R M$$

is exact in the middle. Since $K^{a-2} \to K^{a-1} \to K^a \to \text{Coker}(d_K^{a-1}) \to 0$ is a flat resolution this implies that $\text{Tor}_1^R(\text{Coker}(d_K^{a-1}), M) = 0$ for all R-modules M. This means that $\text{Coker}(d_K^{a-1})$ is flat, see Algebra, Lemma 75.8. $\square$

Lemma 66.3. *Let R be a ring. Let $K^\bullet$ be an object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

(1) *$K^\bullet$ has tor-amplitude in $[a, b]$.*
(2) *$K^\bullet$ is quasi-isomorphic to a complex $E^\bullet$ of flat R-modules with $E^i = 0$ for $i \not\in [a, b]$.*

Proof. If (2) holds, then we may compute $K^\bullet \otimes_R^{\mathbf{L}} M = E^\bullet \otimes_R M$ and it is clear that (1) holds. Assume that (1) holds. We may replace $K^\bullet$ by a projective resolution with $K^i = 0$ for $i > b$. See Derived Categories, Lemma 19.3. Set $E^\bullet = \tau_{\geq a} K^\bullet$. Everything is clear except that E^a is flat which follows immediately from Lemma 66.2 and the definitions. $\square$

Lemma 66.4. *Let R be a ring. Let $a \in \mathbf{Z}$ and let K be an object of $D(R)$. The following are equivalent*

(1) *K has tor-amplitude in $[a, \infty]$, and*
(2) *K is quasi-isomorphic to a K-flat complex $E^\bullet$ whose terms are flat R-modules with $E^i = 0$ for $i \not\in [a, \infty]$.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1) holds. First we choose a K-flat complex $K^\bullet$ with flat terms representing K, see Lemma 59.10. For any R-module M the cohomology of

$$K^{n-1} \otimes_R M \to K^n \otimes_R M \to K^{n+1} \otimes_R M$$

computes $H^n(K \otimes_R^{\mathbf{L}} M)$. This is always zero for $n < a$. Hence if we apply Lemma 66.2 to the complex $\ldots \to K^{a-1} \to K^a \to K^{a+1}$ we conclude that $N = \text{Coker}(K^{a-1} \to K^a)$ is a flat R-module. We set

$$E^\bullet = \tau_{\geq a} K^\bullet = (\ldots \to 0 \to N \to K^{a+1} \to \ldots)$$

The kernel $L^\bullet$ of $K^\bullet \to E^\bullet$ is the complex

$$L^\bullet = (\ldots \to K^{a-1} \to I \to 0 \to \ldots)$$

where $I \subset K^a$ is the image of $K^{a-1} \to K^a$. Since we have the short exact sequence $0 \to I \to K^a \to N \to 0$ we see that I is a flat R-module. Thus $L^\bullet$ is a bounded above complex of flat modules, hence K-flat by Lemma 59.7. It follows that $E^\bullet$ is K-flat by Lemma 59.6. $\square$

Lemma 66.5. *Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. Let $a, b \in \mathbf{Z}$.*

(1) *If $K^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $L^\bullet$ has tor-amplitude in $[a, b]$ then $M^\bullet$ has tor-amplitude in $[a, b]$.*
(2) *If $K^\bullet, M^\bullet$ have tor-amplitude in $[a, b]$, then $L^\bullet$ has tor-amplitude in $[a, b]$.*
(3) *If $L^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $M^\bullet$ has tor-amplitude in $[a, b]$, then $K^\bullet$ has tor-amplitude in $[a+1, b+1]$.*

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes_R^{\mathbf{L}} M$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. $\square$

Lemma 66.6. *Let R be a ring. Let M be an R-module. Let $d \geq 0$. The following are equivalent*

(1) *M has tor dimension $\leq d$, and*
(2) *there exists a resolution*

$$0 \to F_d \to \ldots \to F_1 \to F_0 \to M \to 0$$

with F_i a flat R-module.

In particular an R-module has tor dimension 0 if and only if it is a flat R-module.

Proof. Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to M. Hence the Tor dimension of M is at most d by Lemma 66.3. Conversely, assume (1). Let $P^\bullet \to M$ be a projective resolution of M. By Lemma 66.2 we see that $\tau_{\geq -d} P^\bullet$ is a flat resolution of M of length d, i.e., (2) holds. $\square$

Lemma 66.7. *Let R be a ring. Let $a, b \in \mathbf{Z}$. If $K^\bullet \oplus L^\bullet$ has tor amplitude in $[a, b]$ so do $K^\bullet$ and $L^\bullet$.*

Proof. Clear from the fact that the Tor functors are additive. $\square$

Lemma 66.8. *Let R be a ring. Let $K^\bullet$ be a bounded complex of R-modules such that K^i has tor amplitude in $[a-i, b-i]$ for all i. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet$ is a finite complex of R-modules of finite tor dimension, then $K^\bullet$ has finite tor dimension.*

Proof. Follows by induction on the length of the finite complex: use Lemma 66.5 and the stupid truncations. $\square$

Lemma 66.9. *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet \in D^b(R)$ such that $H^i(K^\bullet)$ has tor amplitude in $[a-i, b-i]$ for all i. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet \in D^b(R)$ and all its cohomology groups have finite tor dimension then $K^\bullet$ has finite tor dimension.*

Proof. Follows by induction on the length of the finite complex: use Lemma 66.5 and the canonical truncations. $\square$

Lemma 66.10. *Let $A \to B$ be a ring map. Let $K^\bullet$ and $L^\bullet$ be complexes of B-modules. Let $a, b, c, d \in \mathbf{Z}$. If*

(1) *$K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$,*
(2) *$L^\bullet$ as a complex of A-modules has tor amplitude in $[c, d]$,*

then $K^\bullet \otimes_B^{\mathbf{L}} L^\bullet$ as a complex of A-modules has tor amplitude in $[a+c, b+d]$.

Proof. We may assume that $K^\bullet$ is a complex of flat B-modules with $K^i = 0$ for $i \not\in [a, b]$, see Lemma 66.3. Let M be an A-module. Choose a free resolution $F^\bullet \to M$. Then

$$(K^\bullet \otimes_B^{\mathbf{L}} L^\bullet) \otimes_A^{\mathbf{L}} M = \text{Tot}(\text{Tot}(K^\bullet \otimes_B L^\bullet) \otimes_A F^\bullet) = \text{Tot}(K^\bullet \otimes_B \text{Tot}(L^\bullet \otimes_A F^\bullet))$$

see Homology, Remark 18.4 for the second equality. By assumption (2) the complex $\text{Tot}(L^\bullet \otimes_A F^\bullet)$ has nonzero cohomology only in degrees $[c, d]$. Hence the spectral sequence of Homology, Lemma 25.1 for the double complex $K^\bullet \otimes_B \text{Tot}(L^\bullet \otimes_A F^\bullet)$ proves that $(K^\bullet \otimes_B^{\mathbf{L}} L^\bullet) \otimes_A^{\mathbf{L}} M$ has nonzero cohomology only in degrees $[a+c, b+d]$. $\square$

Lemma 66.11. *Let $A \to B$ be a ring map. Assume that B is flat as an A-module. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. If $K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of A-modules has tor amplitude in $[a, b]$.*

Proof. This is a special case of Lemma 66.10, but can also be seen directly as follows. We have $K^\bullet \otimes_A^{\mathbf{L}} M = K^\bullet \otimes_B^{\mathbf{L}} (M \otimes_A B)$ since any projective resolution of $K^\bullet$ as a complex of B-modules is a flat resolution of $K^\bullet$ as a complex of A-modules and can be used to compute $K^\bullet \otimes_A^{\mathbf{L}} M$. $\square$

Lemma 66.12. *Let $A \to B$ be a ring map. Assume that B has tor dimension $\leq d$ as an A-module. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. If $K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of A-modules has tor amplitude in $[a-d, b]$.*

Proof. This is a special case of Lemma 66.10, but can also be seen directly as follows. Let M be an A-module. Choose a free resolution $F^\bullet \to M$. Then

$$K^\bullet \otimes_A^{\mathbf{L}} M = \text{Tot}(K^\bullet \otimes_A F^\bullet) = \text{Tot}(K^\bullet \otimes_B (F^\bullet \otimes_A B)) = K^\bullet \otimes_B^{\mathbf{L}} (M \otimes_A^{\mathbf{L}} B).$$

By our assumption on B as an A-module we see that $M \otimes_A^{\mathbf{L}} B$ has cohomology only in degrees $-d, -d+1, \ldots, 0$. Because $K^\bullet$ has tor amplitude in $[a, b]$ we see from the spectral sequence in Example 62.4 that $K^\bullet \otimes_B^{\mathbf{L}} (M \otimes_A^{\mathbf{L}} B)$ has cohomology only in degrees $[-d+a, b]$ as desired. $\square$

Lemma 66.13. *Let $A \to B$ be a ring map. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules with tor amplitude in $[a, b]$. Then $K^\bullet \otimes_A^{\mathbf{L}} B$ as a complex of B-modules has tor amplitude in $[a, b]$.*

Proof. By Lemma 66.3 we can find a quasi-isomorphism $E^\bullet \to K^\bullet$ where $E^\bullet$ is a complex of flat A-modules with $E^i = 0$ for $i \notin [a, b]$. Then $E^\bullet \otimes_A B$ computes $K^\bullet \otimes_A^{\mathbf{L}} B$ by construction and each $E^i \otimes_A B$ is a flat B-module by Algebra, Lemma 39.7. Hence we conclude by Lemma 66.3. $\square$

Lemma 66.14. *Let $A \to B$ be a flat ring map. Let $d \geq 0$. Let M be an A-module of tor dimension $\leq d$. Then $M \otimes_A B$ is a B-module of tor dimension $\leq d$.*

Proof. Immediate consequence of Lemma 66.13 and the fact that $M \otimes_A^{\mathbf{L}} B = M \otimes_A B$ because B is flat over A. $\square$

Lemma 66.15. *Let $A \to B$ be a ring map. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. The following are equivalent*

1. *$K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules,*
2. *$K^\bullet_{\mathfrak q}$ has tor amplitude in $[a, b]$ as a complex of $A_{\mathfrak p}$-modules for every prime $\mathfrak q \subset B$ with $\mathfrak p = A \cap \mathfrak q$,*
3. *$K^\bullet_{\mathfrak m}$ has tor amplitude in $[a, b]$ as a complex of $A_{\mathfrak p}$-modules for every maximal ideal $\mathfrak m \subset B$ with $\mathfrak p = A \cap \mathfrak m$.*

Proof. Assume (3) and let M be an A-module. Then $H^i = H^i(K^\bullet \otimes_A^{\mathbf{L}} M)$ is a B-module and $(H^i)_{\mathfrak m} = H^i(K^\bullet_{\mathfrak m} \otimes_{A_{\mathfrak p}}^{\mathbf{L}} M_{\mathfrak p})$. Hence $H^i = 0$ for $i \notin [a, b]$ by Algebra, Lemma 23.1. Thus (3) $\Rightarrow$ (1). We omit the proofs of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). $\square$

Lemma 66.16. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.*

Proof. This follows immediately from Lemma 66.15 but can also be seen directly as follows. Note that $- \otimes_R R_{f_i}$ is an exact functor and that therefore

$$H^i(K^\bullet)_{f_i} = H^i(K^\bullet) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i}).$$

and similarly for every R-module M we have

$$H^i(K^\bullet \otimes_R^{\mathbf{L}} M)_{f_i} = H^i(K^\bullet \otimes_R^{\mathbf{L}} M) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i} \otimes_{R_{f_i}}^{\mathbf{L}} M_{f_i}).$$

Hence the result follows from the fact that an R-module N is zero if and only if N_{f_i} is zero for each i, see Algebra, Lemma 23.2. $\square$

Lemma 66.17. *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.*

Proof. Let M be an R-module. Since $R \to R'$ is flat we see that

$$(M \otimes_R^{\mathbf{L}} K^\bullet) \otimes_R R' = ((M \otimes_R R') \otimes_{R'}^{\mathbf{L}} (K^\bullet \otimes_R R')$$

and taking cohomology commutes with tensoring with R'. Hence $\text{Tor}_i^R(M, K^\bullet) \otimes_R R' = \text{Tor}_i^{R'}(M \otimes_R R', K^\bullet \otimes_R R')$. Since $R \to R'$ is faithfully flat, the vanishing of $\text{Tor}_i^{R'}(M \otimes_R R', K^\bullet \otimes_R R')$ for $i \notin [a, b]$ implies the same thing for $\text{Tor}_i^R(M, K^\bullet)$. $\square$

Lemma 66.18. *Given ring maps $R \to A \to B$ with $A \to B$ faithfully flat and $K \in D(A)$ the tor amplitude of K over R is the same as the tor amplitude of $K \otimes_A^{\mathbf{L}} B$ over R.*

Proof. This is true because for an R-module M we have $H^i(K \otimes_R^{\mathbf{L}} M) \otimes_A B = H^i((K \otimes_A^{\mathbf{L}} B) \otimes_R^{\mathbf{L}} M)$ for all i. Namely, represent K by a complex $K^\bullet$ of A-modules and choose a free resolution $F^\bullet \to M$. Then we have the equality

$$\mathrm{Tot}(K^\bullet \otimes_A B \otimes_R F^\bullet) = \mathrm{Tot}(K^\bullet \otimes_R F^\bullet) \otimes_A B$$

The cohomology groups of the left hand side are $H^i((K \otimes_A^{\mathbf{L}} B) \otimes_R^{\mathbf{L}} M)$ and on the right hand side we obtain $H^i(K \otimes_R^{\mathbf{L}} M) \otimes_A B$. $\square$

Lemma 66.19. *Let R be a ring of finite global dimension d. Then*

1. *every module has tor dimension $\leq d$,*
2. *a complex of R-modules $K^\bullet$ with $H^i(K^\bullet) \neq 0$ only if $i \in [a, b]$ has tor amplitude in $[a - d, b]$, and*
3. *a complex of R-modules $K^\bullet$ has finite tor dimension if and only if $K^\bullet \in D^b(R)$.*

Proof. The assumption on R means that every module has a finite projective resolution of length at most d, in particular every module has tor dimension $\leq d$. The second statement follows from Lemma 66.9 and the definitions. The third statement is a rephrasing of the second. $\square$

67. Spectral sequences for Ext

In this section we collect various spectral sequences that come up when considering the Ext functors. For any pair of objects L, K of the derived category $D(R)$ of a ring R we denote

$$\mathrm{Ext}^n_R(L, K) = \mathrm{Hom}_{D(R)}(L, K[n])$$

according to our general conventions in Derived Categories, Section 27.

For M an R-module and $K \in D^+(R)$ there is a spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_R(M, H^j(K)) \Rightarrow \mathrm{Ext}^{i+j}_R(M, K) \tag{67.0.1}$$

and if K is represented by the bounded below complex $K^\bullet$ of R-modules there is a spectral sequence

$$E_1^{i,j} = \mathrm{Ext}^j_R(M, K^i) \Rightarrow \mathrm{Ext}^{i+j}_R(M, K) \tag{67.0.2}$$

These spectral sequences come from applying Derived Categories, Lemma 21.3 to the functor $\mathrm{Hom}_R(M, -)$.

68. Projective dimension

We defined the projective dimension of a module in Algebra, Definition 109.2.

Definition 68.1. Let R be a ring. Let K be an object of $D(R)$. We say K has *finite projective dimension* if K can be represented by a bounded complex of projective modules. We say K has *projective-amplitude in* $[a, b]$ if K is quasi-isomorphic to a complex

$$\ldots \to 0 \to P^a \to P^{a+1} \to \ldots \to P^{b-1} \to P^b \to 0 \to \ldots$$

where P^i is a projective R-module for all $i \in \mathbf{Z}$.

Clearly, K has finite projective dimension if and only if K has projective-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Furthermore, if K has finite projective dimension, then K is bounded. Here is a lemma to detect such objects of $D(R)$.

Lemma 68.2. *Let R be a ring. Let K be an object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

(1) *K has projective-amplitude in $[a, b]$,*
(2) *$\mathrm{Ext}^i_R(K, N) = 0$ for all R-modules N and all $i \notin [-b, -a]$,*
(3) *$H^n(K) = 0$ for $n > b$ and $\mathrm{Ext}^i_R(K, N) = 0$ for all R-modules N and all $i > -a$, and*
(4) *$H^n(K) = 0$ for $n \notin [a-1, b]$ and $\mathrm{Ext}^{-a+1}_R(K, N) = 0$ for all R-modules N.*

Proof. Assume (1). We may assume K is the complex

$$\ldots \to 0 \to P^a \to P^{a+1} \to \ldots \to P^{b-1} \to P^b \to 0 \to \ldots$$

where P^i is a projective R-module for all $i \in \mathbf{Z}$. In this case we can compute the ext groups by the complex

$$\ldots \to 0 \to \mathrm{Hom}_R(P^b, N) \to \ldots \to \mathrm{Hom}_R(P^a, N) \to 0 \to \ldots$$

and we obtain (2).

Assume (2) holds. Choose an injection $H^n(K) \to I$ where I is an injective R-module. Since $\mathrm{Hom}_R(-, I)$ is an exact functor, we see that $\mathrm{Ext}^{-n}(K, I) = \mathrm{Hom}_R(H^n(K), I)$. We conclude in particular that $H^n(K)$ is zero for $n > b$. Thus (2) implies (3).

By the same argument as in (2) implies (3) gives that (3) implies (4).

Assume (4). The same argument as in (2) implies (3) shows that $H^{a-1}(K) = 0$, i.e., we have $H^i(K) = 0$ unless $i \in [a, b]$. In particular, K is bounded above and we can choose a a complex $P^\bullet$ representing K with P^i projective (for example free) for all $i \in \mathbf{Z}$ and $P^i = 0$ for $i > b$. See Derived Categories, Lemma 15.4. Let $Q = \mathrm{Coker}(P^{a-1} \to P^a)$. Then K is quasi-isomorphic to the complex

$$\ldots \to 0 \to Q \to P^{a+1} \to \ldots \to P^b \to 0 \to \ldots$$

as $H^i(K) = 0$ for $i < a$. Denote $K' = (P^{a+1} \to \ldots \to P^b)$ the corresponding object of $D(R)$. We obtain a distinguished triangle

$$K' \to K \to Q[-a] \to K'[1]$$

in $D(R)$. Thus for every R-module N an exact sequence

$$\mathrm{Ext}^{-a}(K', N) \to \mathrm{Ext}^1(Q, N) \to \mathrm{Ext}^{1-a}(K, N)$$

By assumption the term on the right vanishes. By the implication (1) $\Rightarrow$ (2) the term on the left vanishes. Thus Q is a projective R-module by Algebra, Lemma 77.2. Hence (1) holds and the proof is complete. $\square$

Example 68.3. Let k be a field and let R be the ring of dual numbers over k, i.e., $R = k[x]/(x^2)$. Denote $\epsilon \in R$ the class of x. Let $M = R/(\epsilon)$. Then M is quasi-isomorphic to the complex

$$R \xrightarrow{\epsilon} R \xrightarrow{\epsilon} R \to \ldots$$

but M does not have finite projective dimension as defined in Algebra, Definition 109.2. This explains why we consider bounded (in both directions) complexes of projective modules in our definition of finite projective dimension of objects of $D(R)$.

69. Injective dimension

This section is the dual of the section on projective dimension.

Definition 69.1. Let R be a ring. Let K be an object of $D(R)$. We say K has *finite injective dimension* if K can be represented by a finite complex of injective R-modules. We say K has *injective-amplitude in* $[a, b]$ if K is isomorphic to a complex

$$\ldots \to 0 \to I^a \to I^{a+1} \to \ldots \to I^{b-1} \to I^b \to 0 \to \ldots$$

with I^i an injective R-module for all $i \in \mathbf{Z}$.

Clearly, K has bounded injective dimension if and only if K has injective-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Furthermore, if K has bounded injective dimension, then K is bounded. Here is the obligatory lemma.

Lemma 69.2. *Let R be a ring. Let K be an object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

(1) *K has injective-amplitude in $[a, b]$,*
(2) *$\mathrm{Ext}^i_R(N, K) = 0$ for all R-modules N and all $i \notin [a, b]$,*
(3) *$\mathrm{Ext}^i(R/I, K) = 0$ for all ideals $I \subset R$ and all $i \notin [a, b]$.*

Proof. Assume (1). We may assume K is the complex

$$\ldots \to 0 \to I^a \to I^{a+1} \to \ldots \to I^{b-1} \to I^b \to 0 \to \ldots$$

where I^i is a injective R-module for all $i \in \mathbf{Z}$. In this case we can compute the ext groups by the complex

$$\ldots \to 0 \to \mathrm{Hom}_R(N, I^a) \to \ldots \to \mathrm{Hom}_R(N, I^b) \to 0 \to \ldots$$

and we obtain (2). It is clear that (2) implies (3).

Assume (3) holds. Choose a nonzero map $R \to H^n(K)$. Since $\mathrm{Hom}_R(R, -)$ is an exact functor, we see that $\mathrm{Ext}^n_R(R, K) = \mathrm{Hom}_R(R, H^n(K)) = H^n(K)$. We conclude that $H^n(K)$ is zero for $n \notin [a, b]$. In particular, K is bounded below and we can choose a quasi-isomorphism

$$K \to I^\bullet$$

with I^i injective for all $i \in \mathbf{Z}$ and $I^i = 0$ for $i < a$. See Derived Categories, Lemma 15.5. Let $J = \mathrm{Ker}(I^b \to I^{b+1})$. Then K is quasi-isomorphic to the complex

$$\ldots \to 0 \to I^a \to \ldots \to I^{b-1} \to J \to 0 \to \ldots$$

Denote $K' = (I^a \to \ldots \to I^{b-1})$ the corresponding object of $D(R)$. We obtain a distinguished triangle

$$J[-b] \to K \to K' \to J[1 - b]$$

in $D(R)$. Thus for every ideal $I \subset R$ an exact sequence

$$\mathrm{Ext}^b(R/I, K') \to \mathrm{Ext}^1(R/I, J) \to \mathrm{Ext}^{1+b}(R/I, K)$$

By assumption the term on the right vanishes. By the implication (1) $\Rightarrow$ (2) the term on the left vanishes. Thus J is a injective R-module by Lemma 55.4. $\square$

Example 69.3. Let R be a Dedekind domain. Then every nonzero ideal I is a finite projective module, see Lemma 22.11. Thus R/I has projective dimension 1. Hence every R-module M has injective dimension ≤ 1 by Lemma 69.2. Thus $\mathrm{Ext}^i_R(M, N) = 0$ for $i \geq 2$ and any pair of R-modules M, N. It follows that any object K in $D^b(R)$ is isomorphic to the direct sum of its cohomologies: $K \cong \bigoplus H^i(K)[-i]$, see Derived Categories, Lemma 27.10.

Example 69.4. Let k be a field and let R be the ring of dual numbers over k, i.e., $R = k[x]/(x^2)$. Denote $\epsilon \in R$ the class of x. Let $M = R/(\epsilon)$. Then M is quasi-isomorphic to the complex

$$\ldots \to R \xrightarrow{\epsilon} R \xrightarrow{\epsilon} R$$

and R is an injective R-module. However one usually does not consider M to have finite injective dimension in this situation. This explains why we consider bounded (in both directions) complexes of injective modules in our definition of bounded injective dimension of objects of $D(R)$.

Lemma 69.5. *Let R be a ring. Let $K \in D(R)$.*

1. *If K is in $D^b(R)$ and $H^i(K)$ has finite injective dimension for all i, then K has finite injective dimension.*
2. *If $K^\bullet$ represents K, is a bounded complex of R-modules, and K^i has finite injective dimension for all i, then K has finite injective dimension.*

Proof. Omitted. Hint: Apply the spectral sequences of Derived Categories, Lemma 21.3 to the functor $F = \mathrm{Hom}_R(N, -)$ to get a computation of $\mathrm{Ext}^i_A(N, K)$ and use the criterion of Lemma 69.2. $\square$

Lemma 69.6. *Let R be a Noetherian ring. Let $I \subset R$ be an ideal contained in the Jacobson radical of R. Let $K \in D^+(R)$ have finite cohomology modules. Then the following are equivalent*

1. *K has finite injective dimension, and*
2. *there exists a b such that $\mathrm{Ext}^i_R(R/J, K) = 0$ for $i > b$ and any ideal $J \supset I$.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2). Say $H^i(K) = 0$ for $i < a$. Then $\mathrm{Ext}^i(M, K) = 0$ for $i < a$ and all R-modules M. Thus it suffices to show that $\mathrm{Ext}^i(M, K) = 0$ for $i > b$ any finite R-module M, see Lemma 69.2. By Algebra, Lemma 62.1 the module M has a finite filtration whose successive quotients are of the form $R/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal. If $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence and $\mathrm{Ext}^i(M_j, K) = 0$ for $i > b$ and $j = 1, 2$, then $\mathrm{Ext}^i(M, K) = 0$ for $i > b$. Thus we may assume $M = R/\mathfrak{p}$. If $I \subset \mathfrak{p}$, then the vanishing follows from the assumption. If not, then choose $f \in I$, $f \notin \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{f} R/\mathfrak{p} \to R/(\mathfrak{p}, f) \to 0$$

The R-module $R/(\mathfrak{p}, f)$ has a filtration whose successive quotients are $R/\mathfrak{q}$ with $(\mathfrak{p}, f) \subset \mathfrak{q}$. Thus by Noetherian induction and the argument above we may assume the vanishing holds for $R/(\mathfrak{p}, f)$. On the other hand, the modules $E^i = \mathrm{Ext}^i(R/\mathfrak{p}, K)$ are finite by our assumption on K (bounded below with finite cohomology modules), the spectral sequence (67.0.1), and Algebra, Lemma 71.9. Thus E^i for $i > b$ is a finite R-module such that $E^i/fE^i = 0$. We conclude by Nakayama's lemma (Algebra, Lemma 20.1) that E^i is zero. $\square$

Lemma 69.7. *Let $(R, \mathfrak{m}, \kappa)$ be a local Noetherian ring. Let $K \in D^+(R)$ have finite cohomology modules. Then the following are equivalent*

(1) *K has finite injective dimension, and*
(2) *$\text{Ext}^i_R(\kappa, K) = 0$ for $i \gg 0$.*

Proof. This is a special case of Lemma 69.6. $\square$

70. Modules which are close to being projective

There seem to be many different of definitions in the literature of "almost projective modules". In this section we discuss just one of the many possibilities.

Lemma 70.1. *Let R be a ring. Let M, N be R-modules.*

(1) *Given an R-module map $\varphi : M \to N$ the following are equivalent: (a) φ factors through a projective R-module, and (b) φ factors through a free R-module.*
(2) *The set of $\varphi : M \to N$ satisfying the equivalent conditions of (1) is an R-submodule of $\text{Hom}_R(M, N)$.*
(3) *Given maps $\psi : M' \to M$ and $\xi : N \to N'$, if $\varphi : M \to N$ satisfies the equivalent conditions of (1), then $\xi \circ \varphi \circ \psi : M' \to N'$ does too.*

Proof. The equivalence of (1)(a) and (1)(b) follows from Algebra, Lemma 77.2. If $\varphi : M \to N$ and $\varphi' : M \to N$ factor through the modules P and P' then $\varphi + \varphi'$ factors through $P \oplus P'$ and $\lambda\varphi$ factors through P for all $\lambda \in R$. This proves (2). If $\varphi : M \to N$ factors through the module P and ψ and ξ are as in (3), then $\xi \circ \varphi \circ \psi$ factors through P. This proves (3). $\square$

Lemma 70.2. *Let R be a ring. Let $\varphi : M \to N$ be an R-module map. If φ factors through a projective module and M is a finite R-module, then φ factors through a finite projective module.*

Proof. By Lemma 70.1 we can factor $\varphi = \tau \circ \sigma$ where the target of σ is $\bigoplus_{i \in I} R$ for some set I. Choose generators $x_1, \ldots, x_n$ for M. Write $\sigma(x_j) = (a_{ji})_{i \in I}$. For each j only a finite number of a_{ij} are nonzero. Hence σ has image contained in a finite free R-module and we conclude. $\square$

Let R be a ring. Observe that an R-module is projective if and only if the identity on R factors through a projective module.

Lemma 70.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. The following conditions are equivalent*

(1) *for every $a \in I$ the map $a : M \to M$ factors through a projective R-module,*
(2) *for every $a \in I$ the map $a : M \to M$ factors through a free R-module, and*
(3) *$\text{Ext}^1_R(M, N)$ is annihilated by I for every R-module N.*

Proof. The equivalence of (1) and (2) follows from Lemma 70.1. If (1) holds, then (3) holds because $\text{Ext}^1_R(P, N)$ for any N and any projective module P. Conversely, assume (3) holds. Choose a short exact sequence $0 \to N \to P \to M \to 0$ with P projective (or even free). By assumption the corresponding element of $\text{Ext}^1_R(M, N)$ is annihilated by I. Hence for every $a \in I$ the map $a : M \to M$ can be factored through the surjection $P \to M$ and we conclude (1) holds. $\square$

In order to comfortably talk about modules satisfying the equivalent conditions of Lemma 70.3 we give the property a name.

Definition 70.4. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. We say M is *I-projective*[8] if the equivalent conditions of Lemma 70.3 hold.

Modules annihilated by I are I-projective.

Lemma 70.5. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. If M is annihilated by I, then M is I-projective.*

Proof. Immediate from the definition and the fact that the zero module is projective. $\square$

Lemma 70.6. *Let R be a ring. Let $I \subset R$ be an ideal. Let*

$$0 \to K \to P \to M \to 0$$

be a short exact sequence of R-modules. If M is I-projective and P is projective, then K is I-projective.

Proof. The element $\text{id}_K \in \text{Hom}_R(K, K)$ maps to the class of the given extension in $\text{Ext}^1_R(M, K)$. Since by assumption this class is annihilated by any $a \in I$ we see that $a : K \to K$ factors through $K \to P$ and we conclude. $\square$

Lemma 70.7. *Let R be a ring. Let $I \subset R$ be an ideal. If M is a finite, I-projective R-module, then $M^\vee = \text{Hom}_R(M, R)$ is I-projective.*

Proof. Assume M is finite and I-projective. Choose a short exact sequence $0 \to K \to R^{\oplus r} \to M \to 0$. This produces an injection $M^\vee \to R^{\oplus r} = (R^{\oplus r})^\vee$. Since the extension class in $\text{Ext}^1_R(M, K)$ corresponding to the short exact sequence is annihilated by I, we see that for any $a \in I$ we can find a map $M \to R^{\oplus r}$ such that the composition with the given map $R^{\oplus r} \to M$ is equal to $a : M \to M$. Taking duals we find that $a : M^\vee \to M^\vee$ factors through the map $M^\vee \to R^{\oplus r}$ given above and we conclude. $\square$

71. Hom complexes

Let R be a ring. Let $L^\bullet$ and $M^\bullet$ be two complexes of R-modules. We construct a complex $\text{Hom}^\bullet(L^\bullet, M^\bullet)$. Namely, for each n we set

$$\text{Hom}^n(L^\bullet, M^\bullet) = \prod\nolimits_{n=p+q} \text{Hom}_R(L^{-q}, M^p)$$

It is a good idea to think of Hom^n as the R-module of all R-linear maps from $L^\bullet$ to $M^\bullet$ (viewed as graded modules) which are homogenous of degree n. In this terminology, we define the differential by the rule

$$\text{d}(f) = \text{d}_M \circ f - (-1)^n f \circ \text{d}_L$$

for $f \in \text{Hom}^n(L^\bullet, M^\bullet)$. We omit the verification that $\text{d}^2 = 0$. See Section 72 for sign rules. This construction is a special case of Differential Graded Algebra, Example 26.6. It follows immediately from the construction that we have

$$H^n(\text{Hom}^\bullet(L^\bullet, M^\bullet)) = \text{Hom}_{K(R)}(L^\bullet, M^\bullet[n]) \tag{71.0.1}$$

for all $n \in \mathbf{Z}$.

[8] This is nonstandard notation.

Lemma 71.1. *Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical isomorphism*

$$\mathrm{Hom}^\bullet(K^\bullet, \mathrm{Hom}^\bullet(L^\bullet, M^\bullet)) = \mathrm{Hom}^\bullet(\mathit{Tot}(K^\bullet \otimes_R L^\bullet), M^\bullet)$$

of complexes of R-modules.

Proof. Let α be an element of degree n on the left hand side. Thus

$$\alpha = (\alpha^{p,q}) \in \prod\nolimits_{p+q=n} \mathrm{Hom}_R(K^{-q}, \mathrm{Hom}^p(L^\bullet, M^\bullet))$$

Each $\alpha^{p,q}$ is an element

$$\alpha^{p,q} = (\alpha^{r,s,q}) \in \prod\nolimits_{r+s+q=n} \mathrm{Hom}_R(K^{-q}, \mathrm{Hom}_R(L^{-s}, M^r))$$

If we make the identifications

$$\mathrm{Hom}_R(K^{-q}, \mathrm{Hom}_R(L^{-s}, M^r)) = \mathrm{Hom}_R(K^{-q} \otimes_R L^{-s}, M^r) \tag{71.1.1}$$

then by our sign rules we get

$$\begin{aligned} \mathrm{d}(\alpha^{r,s,q}) &= \mathrm{d}_{\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)} \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_K \\ &= \mathrm{d}_M \circ \alpha^{r,s,q} - (-1)^{r+s} \alpha^{r,s,q} \circ \mathrm{d}_L - (-1)^{r+s+q} \alpha^{r,s,q} \circ \mathrm{d}_K \end{aligned}$$

On the other hand, if β is an element of degree n of the right hand side, then

$$\beta = (\beta^{r,s,q}) \in \prod\nolimits_{r+s+q=n} \mathrm{Hom}_R(K^{-q} \otimes_R L^{-s}, M^r)$$

and by our sign rule (Homology, Definition 18.3) we get

$$\begin{aligned} \mathrm{d}(\beta^{r,s,q}) &= \mathrm{d}_M \circ \beta^{r,s,q} - (-1)^n \beta^{r,s,q} \circ \mathrm{d}_{\mathrm{Tot}(K^\bullet \otimes L^\bullet)} \\ &= \mathrm{d}_M \circ \beta^{r,s,q} - (-1)^{r+s+q} \left(\beta^{r,s,q} \circ \mathrm{d}_K + (-1)^{-q} \beta^{r,s,q} \circ \mathrm{d}_L\right) \end{aligned}$$

Thus we see that the map induced by the identifications (71.1.1) indeed is a morphism of complexes. $\square$

Remark 71.2. Let R be a ring. The category $\mathrm{Comp}(R)$ of complexes of R-modules is a symmetric monoidal category with tensor product given by $\mathrm{Tot}(- \otimes_R -)$, see Lemma 58.1. Given $L^\bullet$ and $M^\bullet$ in $\mathrm{Comp}(R)$ an element $f \in \mathrm{Hom}^0(L^\bullet, M^\bullet)$ defines a map of complexes $f : L^\bullet \to M^\bullet$ if and only if $\mathrm{d}(f) = 0$. Hence Lemma 71.1 also tells us that

$$\mathrm{Mor}_{\mathrm{Comp}(R)}(K^\bullet, \mathrm{Hom}^\bullet(L^\bullet, M^\bullet)) = \mathrm{Mor}_{\mathrm{Comp}(R)}(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet), M^\bullet)$$

functorially in $K^\bullet, L^\bullet, M^\bullet$ in $\mathrm{Comp}(R)$. This means that $\mathrm{Hom}^\bullet(-,-)$ is an internal hom for the symmetric monoidal category $\mathrm{Comp}(R)$ as discussed in Categories, Remark 43.12.

Lemma 71.3. *Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical morphism*

$$\mathit{Tot}(\mathrm{Hom}^\bullet(L^\bullet, M^\bullet) \otimes_R \mathrm{Hom}^\bullet(K^\bullet, L^\bullet)) \longrightarrow \mathrm{Hom}^\bullet(K^\bullet, M^\bullet)$$

of complexes of R-modules.

Proof. Via the discussion in Remark 71.2 the existence of such a canonical map follows from Categories, Remark 43.12. We also give a direct construction.

An element α of degree n of the left hand side is

$$\alpha = (\alpha^{p,q}) \in \bigoplus\nolimits_{p+q=n} \mathrm{Hom}^p(L^\bullet, M^\bullet) \otimes_R \mathrm{Hom}^q(K^\bullet, L^\bullet)$$

The element $\alpha^{p,q}$ is a finite sum $\alpha^{p,q} = \sum \beta_i^p \otimes \gamma_i^q$ with

$$\beta_i^p = (\beta_i^{r,s}) \in \prod\nolimits_{r+s=p} \mathrm{Hom}_R(L^{-s}, M^r)$$

and

$$\gamma_i^q = (\gamma_i^{u,v}) \in \prod\nolimits_{u+v=q} \mathrm{Hom}_R(K^{-v}, L^u)$$

The map is given by sending α to $\delta = (\delta^{r,v})$ with

$$\delta^{r,v} = \sum\nolimits_{i,s} \beta_i^{r,s} \circ \gamma_i^{-s,v} \in \mathrm{Hom}_R(K^{-v}, M^r)$$

For given $r + v = n$ this sum is finite as there are only finitely many nonzero $\alpha^{p,q}$, hence only finitely many nonzero β_i^p and γ_i^q. By our sign rules we have

$$\begin{aligned} \mathrm{d}(\alpha^{p,q}) &= \mathrm{d}_{\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)}(\alpha^{p,q}) + (-1)^p \mathrm{d}_{\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)}(\alpha^{p,q}) \\ &= \sum \left(\mathrm{d}_M \circ \beta_i^p \circ \gamma_i^q - (-1)^p \beta_i^p \circ \mathrm{d}_L \circ \gamma_i^q\right) \\ &\quad + (-1)^p \sum \left(\beta_i^p \circ \mathrm{d}_L \circ \gamma_i^q - (-1)^q \beta_i^p \circ \gamma_i^q \circ \mathrm{d}_K\right) \\ &= \sum \left(\mathrm{d}_M \circ \beta_i^p \circ \gamma_i^q - (-1)^n \beta_i^p \circ \gamma_i^q \circ \mathrm{d}_K\right) \end{aligned}$$

It follows that the rules $\alpha \mapsto \delta$ is compatible with differentials and the lemma is proved. □

Lemma 71.4. *Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical morphism*

$$\mathit{Tot}(K^\bullet \otimes_R \mathrm{Hom}^\bullet(M^\bullet, L^\bullet)) \longrightarrow \mathrm{Hom}^\bullet(M^\bullet, \mathit{Tot}(K^\bullet \otimes_R L^\bullet))$$

of complexes of R-modules functorial in all three complexes.

Proof. Via the discussion in Remark 71.2 the existence of such a canonical map follows from Categories, Remark 43.12. We also give a direct construction.

Let α be an element of degree n of the right hand side. Thus

$$\alpha = (\alpha^{p,q}) \in \prod\nolimits_{p+q=n} \mathrm{Hom}_R(M^{-q}, \mathrm{Tot}^p(K^\bullet \otimes_R L^\bullet))$$

Each $\alpha^{p,q}$ is an element

$$\alpha^{p,q} = (\alpha^{r,s,q}) \in \mathrm{Hom}_R(M^{-q}, \bigoplus\nolimits_{r+s+q=n} K^r \otimes_R L^s)$$

where we think of $\alpha^{r,s,q}$ as a family of maps such that for every $x \in M^{-q}$ only a finite number of $\alpha^{r,s,q}(x)$ are nonzero. By our sign rules we get

$$\begin{aligned} \mathrm{d}(\alpha^{r,s,q}) &= \mathrm{d}_{\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)} \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_M \\ &= \mathrm{d}_K \circ \alpha^{r,s,q} + (-1)^r \mathrm{d}_L \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_M \end{aligned}$$

On the other hand, if β is an element of degree n of the left hand side, then

$$\beta = (\beta^{p,q}) \in \bigoplus\nolimits_{p+q=n} K^p \otimes_R \mathrm{Hom}^q(M^\bullet, L^\bullet)$$

and we can write $\beta^{p,q} = \sum \gamma_i^p \otimes \delta_i^q$ with $\gamma_i^p \in K^p$ and

$$\delta_i^q = (\delta_i^{r,s}) \in \prod\nolimits_{r+s=q} \mathrm{Hom}_R(M^{-s}, L^r)$$

By our sign rules we have

$$\begin{aligned}\mathrm{d}(\beta^{p,q}) &= \mathrm{d}_K(\beta^{p,q}) + (-1)^p \mathrm{d}_{\mathrm{Hom}^\bullet(M^\bullet, L^\bullet)}(\beta^{p,q}) \\ &= \sum \mathrm{d}_K(\gamma_i^p) \otimes \delta_i^q + (-1)^p \sum \gamma_i^p \otimes (\mathrm{d}_L \circ \delta_i^q - (-1)^q \delta_i^q \circ \mathrm{d}_M)\end{aligned}$$

We send the element β to α with

$$\alpha^{r,s,q} = c^{r,s,q}(\sum \gamma_i^r \otimes \delta_i^{s,q})$$

where $c^{r,s,q} : K^r \otimes_R \mathrm{Hom}_R(M^{-q}, L^s) \to \mathrm{Hom}_R(M^{-q}, K^r \otimes_R L^s)$ is the canonical map. For a given β and r there are only finitely many nonzero γ_i^r hence only finitely many nonzero $\alpha^{r,s,q}$ are nonzero (for a given r). Thus this family of maps satisfies the conditions above and the map is well defined. Comparing signs we see that this is compatible with differentials. □

Lemma 71.5. *Let R be a ring. Given complexes $K^\bullet, L^\bullet$ of R-modules there is a canonical morphism*

$$K^\bullet \longrightarrow \mathrm{Hom}^\bullet(L^\bullet, \mathit{Tot}(K^\bullet \otimes_R L^\bullet))$$

of complexes of R-modules functorial in both complexes.

Proof. Via the discussion in Remark 71.2 the existence of such a canonical map follows from Categories, Remark 43.12. We also give a direct construction.

Let α be an element of degree n of the right hand side. Thus

$$\alpha = (\alpha^{p,q}) \in \prod\nolimits_{p+q=n} \mathrm{Hom}_R(L^{-q}, \mathrm{Tot}^p(K^\bullet \otimes_R L^\bullet))$$

Each $\alpha^{p,q}$ is an element

$$\alpha^{p,q} = (\alpha^{r,s,q}) \in \mathrm{Hom}_R(L^{-q}, \bigoplus\nolimits_{r+s+q=n} K^r \otimes_R L^s)$$

where we think of $\alpha^{r,s,q}$ as a family of maps such that for every $x \in L^{-q}$ only a finite number of $\alpha^{r,s,q}(x)$ are nonzero. By our sign rules we get

$$\begin{aligned}\mathrm{d}(\alpha^{r,s,q}) &= \mathrm{d}_{\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)} \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_L \\ &= \mathrm{d}_K \circ \alpha^{r,s,q} + (-1)^r \mathrm{d}_L \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_L\end{aligned}$$

Now an element $\beta \in K^n$ we send to α with $\alpha^{n,-q,q} = \beta \otimes \mathrm{id}_{L^{-q}}$ and $\alpha^{r,s,q} = 0$ if $r \neq n$. This is indeed an element as above, as for fixed q there is only one nonzero $\alpha^{r,s,q}$. The description of the differential shows this is compatible with differentials. □

Lemma 71.6. *Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical morphism*

$$\mathit{Tot}(\mathrm{Hom}^\bullet(L^\bullet, M^\bullet) \otimes_R K^\bullet) \longrightarrow \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(K^\bullet, L^\bullet), M^\bullet)$$

of complexes of R-modules functorial in all three complexes.

Proof. Via the discussion in Remark 71.2 the existence of such a canonical map follows from Categories, Remark 43.12. We also give a direct construction.

Consider an element β of degree n of the right hand side. Then

$$\beta = (\beta^{p,s}) \in \prod\nolimits_{p+s=n} \mathrm{Hom}_R(\mathrm{Hom}^{-s}(K^\bullet, L^\bullet), M^p)$$

Our sign rules tell us that

$$\mathrm{d}(\beta^{p,s}) = \mathrm{d}_M \circ \beta^{p,s} - (-1)^n \beta^{p,s} \circ \mathrm{d}_{\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)}$$

We can describe the last term as follows

$$(\beta^{p,s} \circ \mathrm{d}_{\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)})(f) = \beta^{p,s}(\mathrm{d}_L \circ f - (-1)^{s+1} f \circ \mathrm{d}_K)$$

if $f \in \mathrm{Hom}^{-s-1}(K^\bullet, L^\bullet)$. We conclude that in some unspecified sense $\mathrm{d}(\beta^{p,s})$ is a sum of three terms with signs as follows

$$\mathrm{d}(\beta^{p,s}) = \mathrm{d}_M(\beta^{p,s}) - (-1)^n \mathrm{d}_L(\beta^{p,s}) + (-1)^{p+1} \mathrm{d}_K(\beta^{p,s}) \tag{71.6.1}$$

Next, we consider an element α of degree n of the left hand side. We can write it like so

$$\alpha = (\alpha^{t,r}) \in \bigoplus\nolimits_{t+r=n} \mathrm{Hom}^t(L^\bullet, M^\bullet) \otimes K^r$$

Each $\alpha^{t,r}$ maps to an element

$$\alpha^{t,r} \mapsto (\alpha^{p,q,r}) \in \prod\nolimits_{p+q=t} \mathrm{Hom}_R(L^{-q}, M^p) \otimes_R K^r$$

Our sign rules tell us that

$$\mathrm{d}(\alpha^{p,q,r}) = \mathrm{d}_{\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)}(\alpha^{p,q,r}) + (-1)^{p+q} \mathrm{d}_K(\alpha^{p,q,r})$$

where if we further write $\alpha^{p,q,r} = \sum g_i^{p,q} \otimes k_i^r$ then we have

$$\mathrm{d}_{\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)}(\alpha^{p,q,r}) = \sum (\mathrm{d}_M \circ g_i^{p,q}) \otimes k_i^r - (-1)^{p+q} \sum (g_i^{p,q} \circ \mathrm{d}_L) \otimes k_i^r$$

We conclude that in some unspecified sense $\mathrm{d}(\alpha^{p,q,r})$ is a sum of three terms with signs as follows

$$\mathrm{d}(\alpha^{p,q,r}) = \mathrm{d}_M(\alpha^{p,q,r}) - (-1)^{p+q} \mathrm{d}_L(\alpha^{p,q,r}) + (-1)^{p+q} \mathrm{d}_K(\alpha^{p,q,r}) \tag{71.6.2}$$

To define our map we will use the canonical maps

$$c_{p,q,r} : \mathrm{Hom}_R(L^{-q}, M^p) \otimes_R K^r \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(K^r, L^{-q}), M^p)$$

which sends $\varphi \otimes k$ to the map $\psi \mapsto \varphi(\psi(k))$. This is functorial in all three variables. With $s = q + r$ there is an inclusion

$$\mathrm{Hom}_R(\mathrm{Hom}_R(K^r, L^{-q}), M^p) \subset \mathrm{Hom}_R(\mathrm{Hom}^{-s}(K^\bullet, L^\bullet), M^p)$$

coming from the projection $\mathrm{Hom}^{-s}(K^\bullet, L^\bullet) \to \mathrm{Hom}_R(K^r, L^{-q})$. Since $\alpha^{p,q,r}$ is nonzero only for a finite number of r we see that for a given s there is only a finite number of q, r with $q + r = s$. Thus we can send α to the element β with

$$\beta^{p,s} = \sum\nolimits_{q+r=s} \epsilon_{p,q,r} c_{p,q,r}(\alpha^{p,q,r})$$

where where the sum uses the inclusions given above and where $\epsilon_{p,q,r} \in \{\pm 1\}$. Comparing signs in the equations (71.6.1) and (71.6.2) we see that

(1) $\epsilon_{p,q,r} = \epsilon_{p+1,q,r}$
(2) $-(-1)^n \epsilon_{p,q,r} = -(-1)^{p+q} \epsilon_{p,q-1,r}$ or equivalently $\epsilon_{p,q,r} = (-1)^r \epsilon_{p,q-1,r}$
(3) $(-1)^{p+1} \epsilon_{p,q,r} = (-1)^{p+q} \epsilon_{p,q,r+1}$ or equivalently $(-1)^{q+1} \epsilon_{p,q,r} = \epsilon_{p,q,r+1}$.

A good solution is to take

$$\epsilon_{p,r,s} = (-1)^{r+qr}$$

The choice of this sign is explained in the remark following the proof. $\square$

Remark 71.7. Let us explain why the sign used in the direct construction in the proof of Lemma 71.6 agrees with the sign we get from the construction using the discussion in Remark 71.2 and Categories, Remark 43.12. Denote $- \otimes - = \text{Tot}(- \otimes_R -)$ and $hom(-, -) = \text{Hom}^\bullet(-, -)$. The construction using monoidal category language tells us to use the arrow

$$hom(L^\bullet, M^\bullet) \otimes K^\bullet \longrightarrow hom(hom(K^\bullet, L^\bullet), M^\bullet)$$

in $\text{Comp}(R)$ corresponding to the arrow

$$hom(L^\bullet, M^\bullet) \otimes K^\bullet \otimes hom(K^\bullet, L^\bullet) \longrightarrow M^\bullet$$

gotten by swapping the order of the last two tensor products and then using the evaluation maps $hom(K^\bullet, L^\bullet) \otimes K^\bullet \to L^\bullet$ and $hom(L^\bullet, K^\bullet) \otimes L^\bullet \to M^\bullet$. Only in swapping does a sign intervene. Namely, in the isomorphism

$$K^\bullet \otimes hom(K^\bullet, L^\bullet) \to hom(K^\bullet, L^\bullet) \otimes K^\bullet$$

there is a sign $(-1)^{r(q+r')}$ on $K^r \otimes_R \text{Hom}_R(K^{-r'}, L^q)$, see Section 72 item (9). The reader can convince themselves that, because of the correspondence we are using to describe maps into an internal hom, this sign only matters if $r = r'$ and in this case we obtain $(-1)^{r(q+r)} = (-1)^{r+qr}$ as in the direct proof.

72. Sign rules

In this section we review the sign rules used so far and we discuss some of their ramifications. It also seems appropriate to discuss these issues in the setting of the category of complexes of modules over a ring, as most interesting phenomena already occur in this case. We sincerely hope the reader will not need to use the more esoteric aspects of this section.

For the rest of this section, we fix a ring R and we denote $M^\bullet$ a complex of R-modules with differentials $d_M^n : M^n \to M^{n+1}$.

(1) The kth shifted complex $M^\bullet[k]$ has terms $(M^\bullet[k])^n = M^{n+k}$ and differentials $d^n_{M[k]} = (-1)^k d^{n+k}_M$, see Homology, Definition 14.7.
(2) Given a map $f : M^\bullet \to N^\bullet$ of complexes, we define $f[k] : M^\bullet[k] \to N^\bullet[k]$ without the intervention of signs, see Homology, Definition 14.7.
(3) We identify $H^n(M^\bullet[k])$ with $H^{n+k}(M^\bullet)$ without the intervention of signs, see Homology, Definition 14.8.
(4) The boundary map of a short exact sequence of complexes is defined as in the snake lemma without the intervention of signs, see Homology, Lemma 13.12.
(5) The distinguished triangle associated to a termwise split short exact sequence $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ of complexes is given by

$$K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$$

where $M^n \to K^{n+1}$ is the map $\pi^{n+1} \circ d^n_L \circ s^n$ if s and π are compatible termwise splittings. In other words, without the intervention of signs. See Derived Categories, Definitions 10.1 and 9.9.
(6) The total complex $\text{Tot}(M^\bullet \otimes_R N^\bullet)$ has differential d satisfying the Leibniz rule $d(x \otimes y) = d(x) \otimes y + (-1)^{\deg(x)} x \otimes d(y)$. See Homology, Example 18.2 and Homology, Definition 18.3.

(7) There is a canonical isomorphism
$$\mathrm{Tot}(M^\bullet \otimes_R N^\bullet)[a+b] \to \mathrm{Tot}(M^\bullet[a] \otimes_R N^\bullet[b])$$
which uses the sign $(-1)^{pb}$ on the summand $M^p \otimes_R N^q$, see Homology, Remark 18.5. It is often more convenient to consider the corresponding shifted map $\mathrm{Tot}(M^\bullet \otimes_R N^\bullet) \to \mathrm{Tot}(M^\bullet[a] \otimes_R N^\bullet[b])[-a-b]$.

(8) There is a canonical isomorphism of complexes
$$\mathrm{Tot}(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet) \otimes_R M^\bullet) \to \mathrm{Tot}(K^\bullet \otimes_R \mathrm{Tot}(L^\bullet \otimes_R M^\bullet))$$
defined without the intervention of signs. See Section 58.

(9) There is a canonical isomorphism
$$\mathrm{Tot}(L^\bullet \otimes_R M^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_R L^\bullet)$$
which uses the sign $(-1)^{pq}$ on the summand $L^p \otimes_R M^q$. See Section 58.

Before we get into a discussion of the sign conventions regarding Hom-complexes, we construct the dual of a complex with respect to the conventions above.

Lemma 72.1. *Let R be a ring. Let M be an R-module. Let N, η, ϵ be a left dual of M in the monoidal category of R-modules, see Categories, Definition 43.5. Then*

(1) *M and N are finite projective R-modules,*
(2) *the map $e : \mathrm{Hom}_R(M, R) \to N$, $\lambda \mapsto (\lambda \otimes 1)(\eta)$ is an isomorphism,*
(3) *we have $\epsilon(n, m) = e^{-1}(n)(m)$ for $n \in N$ and $m \in M$.*

Proof. The assumptions mean that
$$M \xrightarrow{\eta\otimes 1} M \otimes_R N \otimes_R M \xrightarrow{1\otimes\epsilon} M \quad \text{and} \quad N \xrightarrow{1\otimes\eta} N \otimes_R M \otimes_R N \xrightarrow{\epsilon\otimes 1} N$$
are the identity map. We can choose a finite free module F, an R-module map $F \to M$, and a lift $\tilde{\eta} : R \to F \otimes_R N$ of η. We obtain a commutative diagram
$$\begin{tikzcd} M \arrow[r, "\eta\otimes 1"'] \arrow[dr, "\tilde\eta\otimes 1"'] & M \otimes_R N \otimes_R M \arrow[r, "1\otimes\epsilon"'] & M \\ & F \otimes_R N \otimes_R M \arrow[u] \arrow[r, "1\otimes\epsilon"] & F \arrow[u] \end{tikzcd}$$
This shows that the identity on M factors through a finite free module and hence M is finite projective. By symmetry we see that N is finite projective. This proves part (1). Part (2) follows from Categories, Lemma 43.6 and its proof. Part (3) follows from the first equality of the proof. $\square$

Lemma 72.2. *Let R be a ring. Let $M^\bullet$ be a complex of R-modules. Let $N^\bullet, \eta, \epsilon$ be a left dual of $M^\bullet$ in the monoidal category of complexes of R-modules. Then*

(1) *$M^\bullet$ and $N^\bullet$ are bounded,*
(2) *M^n and N^n are finite projective R-modules,*
(3) *writing $\epsilon = \sum \epsilon_n$ with $\epsilon_n : N^{-n} \otimes_R M^n \to R$ and $\eta = \sum \eta_n$ with $\eta_n : R \to M^n \otimes_R N^{-n}$ then $(N^{-n}, \eta_n, \epsilon_n)$ is the left dual of M^n as in Lemma 72.1,*
(4) *the differential $d_N^n : N^n \to N^{n+1}$ is equal to $-(-1)^n$ times the map*
$$N^n = \mathrm{Hom}_R(M^{-n}, R) \xrightarrow{d_M^{-n-1}} \mathrm{Hom}_R(M^{-n-1}, R) = N^{n+1}$$
where the equality signs are the identifications from Lemma 72.1 part (2).

Conversely, given a bounded complex $M^\bullet$ of finite projective R-modules, setting $N^n = \mathrm{Hom}_R(M^{-n}, R)$ with differentials as above, setting $\epsilon = \sum \epsilon_n$ with $\epsilon_n : N^{-n} \otimes_R M^n \to R$ given by evaluation, and setting $\eta = \sum \eta_n$ with $\eta_n : R \to M^n \otimes_R N^{-n}$ mapping 1 to id_{M_n} we obtain a left dual of $M^\bullet$ in the monoidal category of complexes of R-modules.

Proof. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{M^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{N^\bullet}$ by Categories, Definition 43.5 we see immediately that we have $(1 \otimes \epsilon_n) \circ (\eta_n \otimes 1) = \mathrm{id}_{M^n}$ and $(\epsilon_n \otimes 1) \circ (1 \otimes \eta_n) = \mathrm{id}_{N^{-n}}$ which proves (3). By Lemma 72.1 we have (2). Since the sum $\eta = \sum \eta_n$ is finite, we get (1). Since $\eta = \sum \eta_n$ is a map of complexes $R \to \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$ we see that

$$(d_M^{-n-1} \otimes 1) \circ \eta_{-n-1} + (-1)^n (1 \otimes d_N^{-n}) \circ \eta_{-n} = 0$$

by our choice of signs for the differential on $\mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$. Unwinding definitions, this proves (4). To see the final statement of the lemma one reads the above backwards. □

We will use the description of the left dual of a complex in Lemma 72.2 as a motivation for our sign rule on the Hom-complex. Namelly, we choose the signs such that (11) holds. We continue with the discussion of various sign rules as above

(10) Given complexes $K^\bullet$, $M^\bullet$ we let $\mathrm{Hom}^\bullet(M^\bullet, K^\bullet)$ be the complex with terms

$$\mathrm{Hom}^n(M^\bullet, K^\bullet) = \prod\nolimits_{n=p+q} \mathrm{Hom}_R(M^{-q}, K^p)$$

and differential given by the rule

$$d(f) = d_K \circ f - (-1)^n f \circ d_M$$

(11) The choice above is such that if $M^\bullet$ has a left dual $N^\bullet$ as in Lemma 72.2, then we have a canonical isomorphism

$$\mathrm{Tot}(K^\bullet \otimes_R N^\bullet) \longrightarrow \mathrm{Hom}^\bullet(M^\bullet, K^\bullet)$$

defined without the intervention of signs sending the summand $K^p \otimes_R N^q$ to the summand $\mathrm{Hom}_R(M^{-q}, K^p)$ via $N^q = \mathrm{Hom}_R(M^{-q}, R)$ and the canonical map $K^p \otimes_R \mathrm{Hom}_R(M^{-q}, R) \to \mathrm{Hom}_R(M^{-q}, K^p)$.

(12) There is a composition

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(L^\bullet, K^\bullet) \otimes_R \mathrm{Hom}^\bullet(M^\bullet, L^\bullet)) \longrightarrow \mathrm{Hom}^\bullet(M^\bullet, K^\bullet)$$

defined without the intervention of signs, see Lemma 71.3.

(13) There is a canonical isomorphism

$$\mathrm{Hom}^\bullet(K^\bullet, \mathrm{Hom}^\bullet(L^\bullet, M^\bullet)) = \mathrm{Hom}^\bullet(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet), M^\bullet)$$

defined without the intervention of signs, see Lemma 71.1.

(14) There is a canonical map

$$\mathrm{Tot}(K^\bullet \otimes_R \mathrm{Hom}^\bullet(M^\bullet, L^\bullet)) \longrightarrow \mathrm{Hom}^\bullet(M^\bullet, \mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$$

defined without the intervention of signs, see Lemma 71.4.

(15) There is a canonical map

$$K^\bullet \longrightarrow \mathrm{Hom}^\bullet(L^\bullet, \mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$$

defined without the intervention of signs, see Lemma 71.5.

(16) By Lemma 71.6 is a canonical map
$$\text{Tot}(\text{Hom}^\bullet(L^\bullet, M^\bullet) \otimes_R K^\bullet) \longrightarrow \text{Hom}^\bullet(\text{Hom}^\bullet(K^\bullet, L^\bullet), M^\bullet)$$
which uses a sign $(-1)^{r+qr}$ on the module $\text{Hom}_R(L^{-q}, M^p) \otimes_R K^r$ whose reason is explained in Remark 71.7.

(17) Taking $L^\bullet = M^\bullet$ and using $R \to \text{Hom}^\bullet(M^\bullet, M^\bullet)$ the map from the previous item becomes the evaluation map
$$ev : K^\bullet \longrightarrow \text{Hom}^\bullet(\text{Hom}^\bullet(K^\bullet, M^\bullet), M^\bullet)$$
It sends $x \in K^n$ to the map which sends $f \in \text{Hom}^m(K^\bullet, M^\bullet)$ to $(-1)^{nm} f(x)$.

(18) There is a canonical identification
$$\text{Hom}^\bullet(M^\bullet, K^\bullet)[a-b] \to \text{Hom}^\bullet(M^\bullet[b], K^\bullet[a])$$
which uses signs. It is defined as the map whose corresponding shifted map
$$\text{Hom}^\bullet(M^\bullet, K^\bullet) \to \text{Hom}^\bullet(M^\bullet[b], K^\bullet[a])[b-a]$$
uses the sign $(-1)^{nb}$ on the module $\text{Hom}_R(M^{-q}, K^p)$ with $p + q = n$. Namely, if $f \in \text{Hom}^n(M^\bullet, K^\bullet)$ then
$$d(f) = d_K \circ f - (-1)^n f \circ d_M$$
on the source, whereas on the target f lies in $(\text{Hom}^\bullet(M^\bullet[b], K^\bullet[a])[b-a])^n = \text{Hom}^{n+b-a}(M^\bullet[b], K^\bullet[a])$ and hence we get
$$\begin{aligned} d(f) &= (-1)^{b-a}\left(d_{K[a]} \circ f - (-1)^{n+b-a} f \circ d_{M[b]}\right) \\ &= (-1)^{b-a}\left((-1)^a d_K \circ f - (-1)^{n+b-a} f \circ (-1)^b d_M\right) \\ &= (-1)^b d_K \circ f - (-1)^{n+b} f \circ d_M \end{aligned}$$
and one sees that the chosen sign of $(-1)^{nb}$ in degree n produces a map of complexes for these differentials.

73. Derived hom

Let R be a ring. The derived hom we will define in this section is a functor
$$D(R)^{opp} \times D(R) \longrightarrow D(R), \quad (K, L) \longmapsto R\,\text{Hom}_R(K, L)$$
This is an internal hom in the derived category of R-modules in the sense that it is characterized by the formula

(73.0.1) $$\text{Hom}_{D(R)}(K, R\,\text{Hom}_R(L, M)) = \text{Hom}_{D(R)}(K \otimes_R^{\mathbf{L}} L, M)$$

for objects K, L, M of $D(R)$. Note that this formula characterizes the objects up to unique isomorphism by the Yoneda lemma. A construction can be given as follows. Choose a K-injective complex $I^\bullet$ of R-modules representing M, choose a complex $L^\bullet$ representing L, and set
$$R\,\text{Hom}_R(L, M) = \text{Hom}^\bullet(L^\bullet, I^\bullet)$$
with notation as in Section 71. A generalization of this construction is discussed in Differential Graded Algebra, Section 31. From (71.0.1) and Derived Categories, Lemma 31.2 that we have

(73.0.2) $$H^n(R\,\text{Hom}_R(L, M)) = \text{Hom}_{D(R)}(L, M[n])$$

for all $n \in \mathbf{Z}$. In particular, the object $R\,\text{Hom}_R(L, M)$ of $D(R)$ is well defined, i.e., independent of the choice of the K-injective complex $I^\bullet$.

Lemma 73.1. *Let R be a ring. Let K, L, M be objects of $D(R)$. There is a canonical isomorphism*

$$R\operatorname{Hom}_R(K, R\operatorname{Hom}_R(L, M)) = R\operatorname{Hom}_R(K \otimes_R^{\mathbf{L}} L, M)$$

in $D(R)$ functorial in K, L, M which recovers (73.0.1) by taking H^0.

Proof. Choose a K-injective complex $I^\bullet$ representing M and a K-flat complex of R-modules $L^\bullet$ representing L. For any complex of R-modules $K^\bullet$ we have

$$\operatorname{Hom}^\bullet(K^\bullet, \operatorname{Hom}^\bullet(L^\bullet, I^\bullet)) = \operatorname{Hom}^\bullet(\operatorname{Tot}(K^\bullet \otimes_R L^\bullet), I^\bullet)$$

by Lemma 71.1. The lemma follows by the definition of $R\operatorname{Hom}$ and because $\operatorname{Tot}(K^\bullet \otimes_R L^\bullet)$ represents the derived tensor product. $\square$

Lemma 73.2. *Let R be a ring. Let $P^\bullet$ be a bounded above complex of projective R-modules. Let $L^\bullet$ be a complex of R-modules. Then $R\operatorname{Hom}_R(P^\bullet, L^\bullet)$ is represented by the complex $\operatorname{Hom}^\bullet(P^\bullet, L^\bullet)$.*

Proof. By (71.0.1) and Derived Categories, Lemma 19.8 the cohomology groups of the complex are "correct". Hence if we choose a quasi-isomorphism $L^\bullet \to I^\bullet$ with $I^\bullet$ a K-injective complex of R-modules then the induced map

$$\operatorname{Hom}^\bullet(P^\bullet, L^\bullet) \longrightarrow \operatorname{Hom}^\bullet(P^\bullet, I^\bullet)$$

is a quasi-isomorphism. As the right hand side is our definition of $R\operatorname{Hom}_R(P^\bullet, L^\bullet)$ we win. $\square$

Lemma 73.3. *Let R be a ring. Let K, L, M be objects of $D(R)$. There is a canonical morphism*

$$R\operatorname{Hom}_R(L, M) \otimes_R^{\mathbf{L}} K \longrightarrow R\operatorname{Hom}_R(R\operatorname{Hom}_R(K, L), M)$$

in $D(R)$ functorial in K, L, M.

Proof. Choose a K-injective complex $I^\bullet$ representing M, a K-injective complex $J^\bullet$ representing L, and a K-flat complex $K^\bullet$ representing K. The map is defined using the map

$$\operatorname{Tot}(\operatorname{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R K^\bullet) \longrightarrow \operatorname{Hom}^\bullet(\operatorname{Hom}^\bullet(K^\bullet, J^\bullet), I^\bullet)$$

of Lemma 71.6. We omit the proof that this is functorial in all three objects of $D(R)$. $\square$

Lemma 73.4. *Let R be a ring. Given K, L, M in $D(R)$ there is a canonical morphism*

$$R\operatorname{Hom}_R(L, M) \otimes_R^{\mathbf{L}} R\operatorname{Hom}_R(K, L) \longrightarrow R\operatorname{Hom}_R(K, M)$$

in $D(R)$ functorial in K, L, M.

Proof. Choose a K-injective complex $I^\bullet$ representing M, a K-injective complex $J^\bullet$ representing L, and any complex of R-modules $K^\bullet$ representing K. By Lemma 71.3 there is a map of complexes

$$\operatorname{Tot}\left(\operatorname{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R \operatorname{Hom}^\bullet(K^\bullet, J^\bullet)\right) \longrightarrow \operatorname{Hom}^\bullet(K^\bullet, I^\bullet)$$

The complexes of R-modules $\operatorname{Hom}^\bullet(J^\bullet, I^\bullet)$, $\operatorname{Hom}^\bullet(K^\bullet, J^\bullet)$, and $\operatorname{Hom}^\bullet(K^\bullet, I^\bullet)$ represent $R\operatorname{Hom}_R(L, M)$, $R\operatorname{Hom}_R(K, L)$, and $R\operatorname{Hom}_R(K, M)$. If we choose a K-flat complex $H^\bullet$ and a quasi-isomorphism $H^\bullet \to \operatorname{Hom}^\bullet(K^\bullet, J^\bullet)$, then there is a map

$$\operatorname{Tot}\left(\operatorname{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R H^\bullet\right) \longrightarrow \operatorname{Tot}\left(\operatorname{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R \operatorname{Hom}^\bullet(K^\bullet, J^\bullet)\right)$$

whose source represents $R\operatorname{Hom}_R(L, M) \otimes_R^{\mathbf{L}} R\operatorname{Hom}_R(K, L)$. Composing the two displayed arrows gives the desired map. We omit the proof that the construction is functorial. $\square$

Lemma 73.5. *Let R be a ring. Given complexes K, L, M in $D(R)$ there is a canonical morphism*

$$K \otimes_R^{\mathbf{L}} R\operatorname{Hom}_R(M, L) \longrightarrow R\operatorname{Hom}_R(M, K \otimes_R^{\mathbf{L}} L)$$

in $D(R)$ functorial in K, L, M.

Proof. Choose a K-flat complex $K^\bullet$ representing K, and a K-injective complex $I^\bullet$ representing L, and choose any complex $M^\bullet$ representing M. Choose a quasi-isomorphism $\operatorname{Tot}(K^\bullet \otimes_R I^\bullet) \to J^\bullet$ where $J^\bullet$ is K-injective. Then we use the map

$$\operatorname{Tot}\left(K^\bullet \otimes_R \operatorname{Hom}^\bullet(M^\bullet, I^\bullet)\right) \to \operatorname{Hom}^\bullet(M^\bullet, \operatorname{Tot}(K^\bullet \otimes_R I^\bullet)) \to \operatorname{Hom}^\bullet(M^\bullet, J^\bullet)$$

where the first map is the map from Lemma 71.4. $\square$

Lemma 73.6. *Let R be a ring. Given complexes K, L in $D(R)$ there is a canonical morphism*

$$K \longrightarrow R\operatorname{Hom}_R(L, K \otimes_R^{\mathbf{L}} L)$$

in $D(R)$ functorial in both K and L.

Proof. This is a special case of Lemma 73.5 but we will also prove it directly. Choose a K-flat complex $K^\bullet$ representing K and any complex $L^\bullet$ representing L. Choose a quasi-isomorphism $\operatorname{Tot}(K^\bullet \otimes_R L^\bullet) \to J^\bullet$ where $J^\bullet$ is K-injective. Then we use the map

$$K^\bullet \to \operatorname{Hom}^\bullet(L^\bullet, \operatorname{Tot}(K^\bullet \otimes_R L^\bullet)) \to \operatorname{Hom}^\bullet(L^\bullet, J^\bullet)$$

where the first map is the map from Lemma 71.5. $\square$

74. Perfect complexes

A perfect complex is a pseudo-coherent complex of finite tor dimension. We will not use this as the definition, but define perfect complexes over a ring directly as follows.

Definition 74.1. Let R be a ring. Denote $D(R)$ the derived category of the abelian category of R-modules.

1. An object K of $D(R)$ is *perfect* if it is quasi-isomorphic to a bounded complex of finite projective R-modules.
2. An R-module M is *perfect* if $M[0]$ is a perfect object in $D(R)$.

For example, over a Noetherian ring a finite module is perfect if and only if it has finite projective dimension, see Lemma 74.3 and Algebra, Definition 109.2.

Lemma 74.2. *Let $K^\bullet$ be an object of $D(R)$. The following are equivalent*

1. *$K^\bullet$ is perfect, and*
2. *$K^\bullet$ is pseudo-coherent and has finite tor dimension.*

If (1) and (2) hold and $K^\bullet$ has tor-amplitude in $[a, b]$, then $K^\bullet$ is quasi-isomorphic to a complex $E^\bullet$ of finite projective R-modules with $E^i = 0$ for $i \notin [a, b]$.

Proof. It is clear that (1) implies (2), see Lemmas 64.5 and 66.3. Assume (2) holds and that $K^\bullet$ has tor-amplitude in $[a, b]$. In particular, $H^i(K^\bullet) = 0$ for $i > b$. Choose a complex $F^\bullet$ of finite free R-modules with $F^i = 0$ for $i > b$ and a quasi-isomorphism $F^\bullet \to K^\bullet$ (Lemma 64.5). Set $E^\bullet = \tau_{\geq a} F^\bullet$. Note that E^i is finite free except E^a which is a finitely presented R-module. By Lemma 66.2 E^a is flat. Hence by Algebra, Lemma 78.2 we see that E^a is finite projective. □

Lemma 74.3. *Let M be a module over a ring R. The following are equivalent*

(1) *M is a perfect module, and*
(2) *there exists a resolution*

$$0 \to F_d \to \ldots \to F_1 \to F_0 \to M \to 0$$

with each F_i a finite projective R-module.

Proof. Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to $M[0]$. Hence M is perfect. Conversely, assume (1). By Lemmas 74.2 and 64.4 we can find resolution $E^\bullet \to M$ with E^{-i} a finite free R-module. By Lemma 66.2 we see that $F_d = \text{Coker}(E^{d-1} \to E^d)$ is flat for some d sufficiently large. By Algebra, Lemma 78.2 we see that F_d is finite projective. Hence

$$0 \to F_d \to E^{-d+1} \to \ldots \to E^0 \to M \to 0$$

is the desired resolution. □

Lemma 74.4. *Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. If two out of three of $K^\bullet, L^\bullet, M^\bullet$ are perfect then the third is also perfect.*

Proof. Combine Lemmas 74.2, 64.6, and 66.5. □

Lemma 74.5. *Let R be a ring. If $K^\bullet \oplus L^\bullet$ is perfect, then so are $K^\bullet$ and $L^\bullet$.*

Proof. Follows from Lemmas 74.2, 64.8, and 66.7. □

Lemma 74.6. *Let R be a ring. Let $K^\bullet$ be a bounded complex of perfect R-modules. Then $K^\bullet$ is a perfect complex.*

Proof. Follows by induction on the length of the finite complex: use Lemma 74.4 and the stupid truncations. □

Lemma 74.7. *Let R be a ring. If $K^\bullet \in D^b(R)$ and all its cohomology modules are perfect, then $K^\bullet$ is perfect.*

Proof. Follows by induction on the length of the finite complex: use Lemma 74.4 and the canonical truncations. □

Lemma 74.8. *Let $A \to B$ be a ring map. Assume that B is perfect as an A-module. Let $K^\bullet$ be a perfect complex of B-modules. Then $K^\bullet$ is perfect as a complex of A-modules.*

Proof. Using Lemma 74.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 66.12 and Lemma 64.11 for those results. □

Lemma 74.9. *Let $A \to B$ be a ring map. Let $K^\bullet$ be a perfect complex of A-modules. Then $K^\bullet \otimes_A^{\mathbf{L}} B$ is a perfect complex of B-modules.*

Proof. Using Lemma 74.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 66.13 and Lemma 64.12 for those results. $\square$

Lemma 74.10. *Let $A \to B$ be a flat ring map. Let M be a perfect A-module. Then $M \otimes_A B$ is a perfect B-module.*

Proof. By Lemma 74.3 the assumption implies that M has a finite resolution $F_\bullet$ by finite projective R-modules. As $A \to B$ is flat the complex $F_\bullet \otimes_A B$ is a finite length resolution of $M \otimes_A B$ by finite projective modules over B. Hence $M \otimes_A B$ is perfect. $\square$

Lemma 74.11. *Let R be a ring. If K and L are perfect objects of $D(R)$, then $K \otimes_R^{\mathbf{L}} L$ is a perfect object too.*

Proof. We can prove this using the definition as follows. We may represent K, resp. L by a bounded complex $K^\bullet$, resp. $L^\bullet$ of finite projective R-modules. Then $K \otimes_R^{\mathbf{L}} L$ is represented by the bounded complex $\text{Tot}(K^\bullet \otimes_R L^\bullet)$. The terms of this complex are direct sums of the modules $M^a \otimes_R L^b$. Since M^a and L^b are direct summands of finite free R-modules, so is $M^a \otimes_R L^b$. Hence we conclude the terms of the complex $\text{Tot}(K^\bullet \otimes_R L^\bullet)$ are finite projective.

Another proof can be given using the characterization of perfect complexes in Lemma 74.2 and the corresponding lemmas for pseudo-coherent complexes (Lemma 64.16) and for tor amplitude (Lemma 66.10 used with $A = B = R$). $\square$

Lemma 74.12. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ is perfect, then $K^\bullet$ is perfect.*

Proof. Using Lemma 74.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 66.16 and Lemma 64.14 for those results. $\square$

Lemma 74.13. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ is perfect, then $K^\bullet$ is perfect.*

Proof. Using Lemma 74.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 66.17 and Lemma 64.15 for those results. $\square$

Lemma 74.14. *Let R be a regular ring. Then*

(1) *an R-module is perfect if and only if it is a finite R-module, and*
(2) *a complex of R-modules $K^\bullet$ is perfect if and only if $K^\bullet \in D^b(R)$ and each $H^i(K^\bullet)$ is a finite R-module.*

Proof. Any perfect R-module is finite by definition. Conversely, let M be a finite R-module. Choose a resolution

$$\ldots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to M \to 0$$

with F_i finite free R-modules (Algebra, Lemma 71.1). Set $M_i = \text{Ker}(d_i)$. Denote $U_i \subset \text{Spec}(R)$ the set of primes $\mathfrak{p}$ such that $M_{i,\mathfrak{p}}$ is free; U_i is open by Algebra, Lemma 79.3. We have a exact sequence $0 \to M_{i+1} \to F_{i+1} \to M_i \to 0$. If $\mathfrak{p} \in U_i$, then $0 \to M_{i+1,\mathfrak{p}} \to F_{i+1,\mathfrak{p}} \to M_{i,\mathfrak{p}} \to 0$ splits. Thus $M_{i+1,\mathfrak{p}}$ is finite projective, hence free (Algebra, Lemma 78.2). This shows that $U_i \subset U_{i+1}$. We claim that $\text{Spec}(R) = \bigcup U_i$. Namely, for every prime ideal $\mathfrak{p}$ the regular local ring $R_\mathfrak{p}$ has finite global dimension by Algebra, Proposition 110.1. It follows that $M_{i,\mathfrak{p}}$ is finite projective (hence free) for $i \gg 0$ for example by Algebra, Lemma 109.3. Since the spectrum of R is Noetherian (Algebra, Lemma 31.5) we conclude that $U_n = \text{Spec}(R)$ for some n. Then M_n is a projective R-module by Algebra, Lemma 78.2. Thus

$$0 \to M_n \to F_n \to \ldots \to F_1 \to M \to 0$$

is a bounded resolution by finite projective modules and hence M is perfect. This proves part (1).

Let $K^\bullet$ be a complex of R-modules. If $K^\bullet$ is perfect, then it is in $D^b(R)$ and it is quasi-isomorphic to a finite complex of finite projective R-modules so certainly each $H^i(K^\bullet)$ is a finite R-module (as R is Noetherian). Conversely, suppose that $K^\bullet$ is in $D^b(R)$ and each $H^i(K^\bullet)$ is a finite R-module. Then by (1) each $H^i(K^\bullet)$ is a perfect R-module, whence $K^\bullet$ is perfect by Lemma 74.7 $\square$

Lemma 74.15. *Let A be a ring. Let $K \in D(A)$ be perfect. Then $K^\vee = R\operatorname{Hom}_A(K, A)$ is a perfect complex and $K \cong (K^\vee)^\vee$. There are functorial isomorphisms*

$$L \otimes_A^{\mathbf{L}} K^\vee = R\operatorname{Hom}_A(K, L) \quad \textit{and} \quad H^0(L \otimes_A^{\mathbf{L}} K^\vee) = \operatorname{Ext}^0_A(K, L)$$

for $L \in D(A)$.

Proof. We can represent K by a complex $K^\bullet$ of finite projective A-modules. By Lemma 73.2 the object $K^\vee$ is represented by the complex $E^\bullet = \operatorname{Hom}^\bullet(K^\bullet, A)$. Note that $E^n = \operatorname{Hom}_A(K^{-n}, A)$ and the differentials of $E^\bullet$ are the transpose of the differentials of $K^\bullet$ up to sign. Observe that $E^\bullet$ is the left dual of $K^\bullet$ in the symmetric monoidal category of complexes of R-modules, see Lemma 72.2. There is a canonical map

$$K^\bullet = \operatorname{Tot}(\operatorname{Hom}^\bullet(A, A) \otimes_A K^\bullet) \longrightarrow \operatorname{Hom}^\bullet(\operatorname{Hom}^\bullet(K^\bullet, A), A)$$

which up to sign uses the evaluation map in each degree, see Lemma 71.6. (For sign rules see Section 72.) Thus this map defines a canonical isomorphism $(K^\vee)^\vee \cong K$ as the double dual of a finite projective module is itself.

The second equality follows from the first by Lemma 73.1 and Derived Categories, Lemma 19.8 as well as the definition of Ext groups, see Derived Categories, Section 27. Let $L^\bullet$ be a complex of A-modules representing L. By Section 72 item (11) there is a canonical isomorphism

$$\operatorname{Tot}(L^\bullet \otimes_A E^\bullet) \longrightarrow \operatorname{Hom}^\bullet(K^\bullet, L^\bullet)$$

of complexes of A-modules. This proves the first displayed equality and the proof is complete. $\square$

Lemma 74.16. *Let A be a ring. Let $(K_n)_{n \in \mathbf{N}}$ be a system of perfect objects of $D(A)$. Let $K = hocolim K_n$ be the derived colimit (Derived Categories, Definition 33.1). Then for any object E of $D(A)$ we have*

$$R\operatorname{Hom}_A(K, E) = R\lim E \otimes_A^{\mathbf{L}} K_n^\vee$$

where $(K_n^\vee)$ is the inverse system of dual perfect complexes.

Proof. By Lemma 74.15 we have $R\lim E \otimes_A^{\mathbf{L}} K_n^\vee = R\lim R\operatorname{Hom}_A(K_n, E)$ which fits into the distinguished triangle

$$R\lim R\operatorname{Hom}_A(K_n, E) \to \prod R\operatorname{Hom}_A(K_n, E) \to \prod R\operatorname{Hom}_A(K_n, E)$$

Because K similarly fits into the distinguished triangle $\bigoplus K_n \to \bigoplus K_n \to K$ it suffices to show that $\prod R\operatorname{Hom}_A(K_n, E) = R\operatorname{Hom}_A(\bigoplus K_n, E)$. This is a formal consequence of (73.0.1) and the fact that derived tensor product commutes with direct sums. ☐

Lemma 74.17. *Let $R = \operatorname{colim}_{i \in I} R_i$ be a filtered colimit of rings.*

1. *Given a perfect K in $D(R)$ there exists an $i \in I$ and a perfect K_i in $D(R_i)$ such that $K \cong K_i \otimes_{R_i}^{\mathbf{L}} R$ in $D(R)$.*
2. *Given $0 \in I$ and $K_0, L_0 \in D(R_0)$ with K_0 perfect, we have*

$$\operatorname{Hom}_{D(R)}(K_0 \otimes_{R_0}^{\mathbf{L}} R, L_0 \otimes_{R_0}^{\mathbf{L}} R) = \operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(R_i)}(K_0 \otimes_{R_0}^{\mathbf{L}} R_i, L_0 \otimes_{R_0}^{\mathbf{L}} R_i)$$

In other words, the triangulated category of perfect complexes over R is the colimit of the triangulated categories of perfect complexes over R_i.

Proof. We will use the results of Algebra, Lemmas 127.5 and 127.6 without further mention. These lemmas in particular say that the category of finitely presented R-modules is the colimit of the categories of finitely presented R_i-modules. Since finite projective modules can be characterized as summands of finite free modules (Algebra, Lemma 78.2) we see that the same is true for the category of finite projective modules. This proves (1) by our definition of perfect objects of $D(R)$.

To prove (2) we may represent K_0 by a bounded complex $K_0^\bullet$ of finite projective R_0-modules. We may represent L_0 by a K-flat complex $L_0^\bullet$ (Lemma 59.10). Then we have

$$\operatorname{Hom}_{D(R)}(K_0 \otimes_{R_0}^{\mathbf{L}} R, L_0 \otimes_{R_0}^{\mathbf{L}} R) = \operatorname{Hom}_{K(R)}(K_0^\bullet \otimes_{R_0} R, L_0^\bullet \otimes_{R_0} R)$$

by Derived Categories, Lemma 19.8. Similarly for the Hom with R replaced by R_i. Since in the right hand side only a finite number of terms are involved, since

$$\operatorname{Hom}_R(K_0^p \otimes_{R_0} R, L_0^q \otimes_{R_0} R) = \operatorname{colim}_{i \geq 0} \operatorname{Hom}_{R_i}(K_0^p \otimes_{R_0} R_i, L_0^q \otimes_{R_0} R_i)$$

by the lemmas cited at the beginning of the proof, and since filtered colimits are exact (Algebra, Lemma 8.8) we conclude that (2) holds as well. ☐

75. Lifting complexes

Let R be a ring. Let $I \subset R$ be an ideal. The lifting problem we will consider is the following. Suppose given an object K of $D(R)$ and a complex $E^\bullet$ of R/I-modules such that $E^\bullet$ represents $K \otimes_R^{\mathbf{L}} R/I$ in $D(R)$. Question: Does there exist a complex of R-modules $P^\bullet$ lifting $E^\bullet$ representing K in $D(R)$? In general the answer to this question is no, but in good cases something can be done. We first discuss lifting acyclic complexes.

Lemma 75.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let $\mathcal{P}$ be a class of R-modules. Assume*

1. *each $P \in \mathcal{P}$ is a projective R-module,*
2. *if $P_1 \in \mathcal{P}$ and $P_1 \oplus P_2 \in \mathcal{P}$, then $P_2 \in \mathcal{P}$, and*
3. *if $f : P_1 \to P_2$, $P_1, P_2 \in \mathcal{P}$ is surjective modulo I, then f is surjective.*

Then given any bounded above acyclic complex $E^\bullet$ whose terms are of the form P/IP for $P \in \mathcal{P}$ there exists a bounded above acyclic complex $P^\bullet$ whose terms are in $\mathcal{P}$ lifting $E^\bullet$.

Proof. Say $E^i = 0$ for $i > b$. Assume given n and a morphism of complexes

$$\begin{tikzcd} & & P^n \arrow[r] \arrow[d] & P^{n+1} \arrow[r] \arrow[d] & \ldots \arrow[r] & P^b \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & E^{n-1} \arrow[r] & E^n \arrow[r] & E^{n+1} \arrow[r] & \ldots \arrow[r] & E^b \arrow[r] & 0 \arrow[r] & \ldots \end{tikzcd}$$

with $P^i \in \mathcal{P}$, with $P^n \to P^{n+1} \to \ldots \to P^b$ acyclic in degrees $\geq n+1$, and with vertical maps inducing isomorphisms $P^i/IP^i \to E^i$. In this situation one can inductively choose isomorphisms $P^i = Z^i \oplus Z^{i+1}$ such that the maps $P^i \to P^{i+1}$ are given by $Z^i \oplus Z^{i+1} \to Z^{i+1} \to Z^{i+1} \oplus Z^{i+2}$. By property (2) and arguing inductively we see that $Z^i \in \mathcal{P}$. Choose $P^{n-1} \in \mathcal{P}$ and an isomorphism $P^{n-1}/IP^{n-1} \to E^{n-1}$. Since P^{n-1} is projective and since $Z^n/IZ^n = \text{Im}(E^{n-1} \to E^n)$, we can lift the map $P^{n-1} \to E^{n-1} \to E^n$ to a map $P^{n-1} \to Z^n$. By property (3) the map $P^{n-1} \to Z^n$ is surjective. Thus we obtain an extension of the diagram by adding P^{n-1} and the maps just constructed to the left of P^n. Since a diagram of the desired form exists for $n > b$ we conclude by induction on n. $\square$

Lemma 75.2. *Let R be a ring. Let $I \subset R$ be an ideal. Let $\mathcal{P}$ be a class of R-modules. Let $K \in D(R)$ and let $E^\bullet$ be a complex of R/I-modules representing $K \otimes_R^{\mathbf{L}} R/I$. Assume*

1. *each $P \in \mathcal{P}$ is a projective R-module,*
2. *$P_1 \in \mathcal{P}$ and $P_1 \oplus P_2 \in \mathcal{P}$ if and only if $P_1, P_2 \in \mathcal{P}$,*
3. *if $f : P_1 \to P_2$, $P_1, P_2 \in \mathcal{P}$ is surjective modulo I, then f is surjective,*
4. *$E^\bullet$ is bounded above and E^i is of the form P/IP for $P \in \mathcal{P}$, and*
5. *K can be represented by a bounded above complex whose terms are in $\mathcal{P}$.*

Then there exists a bounded above complex $P^\bullet$ whose terms are in $\mathcal{P}$ with $P^\bullet/IP^\bullet$ isomorphic to $E^\bullet$ and representing K in $D(R)$.

Proof. By assumption (5) we can represent K by a bounded above complex $K^\bullet$ whose terms are in $\mathcal{P}$. Then $K \otimes_R^{\mathbf{L}} R/I$ is represented by $K^\bullet/IK^\bullet$. Since $E^\bullet$ is a bounded above complex of projective R/I-modules by (4), we can choose a quasi-isomorphism $\delta : E^\bullet \to K^\bullet/IK^\bullet$ (Derived Categories, Lemma 19.8). Let $C^\bullet$ be cone on δ (Derived Categories, Definition 9.1). The module C^i is the direct sum $K^i/IK^i \oplus E^{i+1}$ hence is of the form P/IP for some $P \in \mathcal{P}$ as (2) says in particular that $\mathcal{P}$ is preserved under taking sums. Since $C^\bullet$ is acyclic, we can apply Lemma 75.1 and find a acyclic lift $A^\bullet$ of $C^\bullet$. The complex $A^\bullet$ is bounded above and has terms in $\mathcal{P}$. In

$$\begin{tikzcd} K^\bullet \arrow[r, dotted] \arrow[d] & A^\bullet \arrow[d] & \\ K^\bullet/IK^\bullet \arrow[r] & C^\bullet \arrow[r] & E^\bullet[1] \end{tikzcd}$$

we can find the dotted arrow making the diagram commute by Derived Categories, Lemma 19.6. We will show below that it follows from (1), (2), (3) that $K^i \to A^i$ is the inclusion of a direct summand for every i. By property (2) we see that

$P^i = \mathrm{Coker}(K^i \to A^i)$ is in $\mathcal{P}$. Thus we can take $P^\bullet = \mathrm{Coker}(K^\bullet \to A^\bullet)[-1]$ to conclude.

To finish the proof we have to show the following: Let $f : P_1 \to P_2$, $P_1, P_2 \in \mathcal{P}$ and $P_1/IP_1 \to P_2/IP_2$ is split injective with cokernel of the form P_3/IP_3 for some $P_3 \in \mathcal{P}$, then f is split injective. Write $E_i = P_i/IP_i$. Then $E_2 = E_1 \oplus E_3$. Since P_2 is projective we can choose a map $g : P_2 \to P_3$ lifting the map $E_2 \to E_3$. By condition (3) the map g is surjective, hence split as P_3 is projective. Set $P_1' = \mathrm{Ker}(g)$ and choose a splitting $P_2 = P_1' \oplus P_3$. Then $P_1' \in \mathcal{P}$ by (2). We do not know that $g \circ f = 0$, but we can consider the map

$$P_1 \xrightarrow{f} P_2 \xrightarrow{projection} P_1'$$

The composition modulo I is an isomorphism. Since P_1' is projective we can split $P_1 = T \oplus P_1'$. If $T = 0$, then we are done, because then $P_2 \to P_1'$ is a splitting of f. We see that $T \in \mathcal{P}$ by (2). Calculating modulo I we see that $T/IT = 0$. Since $0 \in \mathcal{P}$ (as the summand of any P in $\mathcal{P}$) we see the map $0 \to T$ is surjective and we conclude that $T = 0$ as desired. $\square$

Lemma 75.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let $E^\bullet$ be a complex of R/I-modules. Let K be an object of $D(R)$. Assume that*

1. *$E^\bullet$ is a bounded above complex of projective R/I-modules,*
2. *$K \otimes_R^{\mathbf{L}} R/I$ is represented by $E^\bullet$ in $D(R/I)$, and*
3. *I is a nilpotent ideal.*

Then there exists a bounded above complex $P^\bullet$ of projective R-modules representing K in $D(R)$ such that $P^\bullet \otimes_R R/I$ is isomorphic to $E^\bullet$.

Proof. We apply Lemma 75.2 using the class $\mathcal{P}$ of all projective R-modules. Properties (1) and (2) of the lemma are immediate. Property (3) follows from Nakayama's lemma (Algebra, Lemma 20.1). Property (4) follows from the fact that we can lift projective R/I-modules to projective R-modules, see Algebra, Lemma 77.5. To see that (5) holds it suffices to show that K is in $D^-(R)$. We are given that $K \otimes_R^{\mathbf{L}} R/I$ is in $D^-(R/I)$ (because $E^\bullet$ is bounded above). We will show by induction on n that $K \otimes_R^{\mathbf{L}} R/I^n$ is in $D^-(R/I^n)$. This will finish the proof because I being nilpotent exactly means that $I^n = 0$ for some n. We may represent K by a K-flat complex $K^\bullet$ with flat terms (Lemma 59.10). Then derived tensor products are represented by usual tensor products. Thus we consider the exact sequence

$$0 \to K^\bullet \otimes_R I^n/I^{n+1} \to K^\bullet \otimes_R R/I^{n+1} \to K^\bullet \otimes_R R/I^n \to 0$$

Thus the cohomology of $K \otimes_R^{\mathbf{L}} R/I^{n+1}$ sits in a long exact sequence with the cohomology of $K \otimes_R^{\mathbf{L}} R/I^n$ and the cohomology of

$$K \otimes_R^{\mathbf{L}} I^n/I^{n+1} = K \otimes_R^{\mathbf{L}} R/I \otimes_{R/I}^{\mathbf{L}} I^n/I^{n+1}$$

The first cohomologies vanish above a certain degree by induction assumption and the second cohomologies vanish above a certain degree because $K^\bullet \otimes_R^{\mathbf{L}} R/I$ is bounded above and I^n/I^{n+1} is in degree 0. $\square$

Lemma 75.4. *Let R be a ring. Let $I \subset R$ be an ideal. Let $E^\bullet$ be a complex of R/I-modules. Let K be an object of $D(R)$. Assume that*

1. *$E^\bullet$ is a bounded above complex of finite stably free R/I-modules,*
2. *$K \otimes_R^{\mathbf{L}} R/I$ is represented by $E^\bullet$ in $D(R/I)$,*
3. *$K^\bullet$ is pseudo-coherent, and*

(4) *every element of* $1 + I$ *is invertible.*

Then there exists a bounded above complex $P^\bullet$ *of finite stably free* R*-modules representing* K *in* $D(R)$ *such that* $P^\bullet \otimes_R R/I$ *is isomorphic to* $E^\bullet$*. Moreover, if* E^i *is free, then* P^i *is free.*

Proof. We apply Lemma 75.2 using the class $\mathcal{P}$ of all finite stably free R-modules. Property (1) of the lemma is immediate. Property (2) follows from Lemma 3.2. Property (3) follows from Nakayama's lemma (Algebra, Lemma 20.1). Property (4) follows from the fact that we can lift finite stably free R/I-modules to finite stably free R-modules, see Lemma 3.3. Part (5) holds because a pseudo-coherent complex can be represented by a bounded above complex of finite free R-modules. The final assertion of the lemma follows from Lemma 3.5. $\square$

Lemma 75.5. *Let* $(R, \mathfrak{m}, \kappa)$ *be a local ring. Let* $K \in D(R)$ *be pseudo-coherent. Set* $d_i = \dim_\kappa H^i(K \otimes_R^{\mathbf{L}} \kappa)$*. Then* $d_i < \infty$ *and for some* $b \in \mathbf{Z}$ *we have* $d_i = 0$ *for* $i > b$*. Then there exists a complex*

$$\ldots \to R^{\oplus d_{b-2}} \to R^{\oplus d_{b-1}} \to R^{\oplus d_b} \to 0 \to \ldots$$

representing K *in* $D(R)$*. Moreover, this complex is unique up to isomorphism(!).*

Proof. Observe that $K \otimes_R^{\mathbf{L}} \kappa$ is pseudo-coherent as an object of $D(\kappa)$, see Lemma 64.12. Hence the cohomology spaces are finite dimensional and vanish above some cutoff. Every object of $D(\kappa)$ is isomorphic in $D(\kappa)$ to a complex $E^\bullet$ with zero differentials. In particular $E^i \cong \kappa^{\oplus d_i}$ is finite free. Applying Lemma 75.4 we obtain the existence.

If we have two complexes $F^\bullet$ and $G^\bullet$ with F^i and G^i free of rank d_i representing K. Then we may choose a map of complexes $\beta : F^\bullet \to G^\bullet$ representing the isomorphism $F^\bullet \cong K \cong G^\bullet$, see Derived Categories, Lemma 19.8. The induced map of complexes $\beta \otimes 1 : F^\bullet \otimes_R^{\mathbf{L}} \kappa \to G^\bullet \otimes_R^{\mathbf{L}} \kappa$ must be an isomorphism of complexes as the differentials in $F^\bullet \otimes_R^{\mathbf{L}} \kappa$ and $G^\bullet \otimes_R^{\mathbf{L}} \kappa$ are zero. Thus $\beta^i : F^i \to G^i$ is a map of finite free R-modules whose reduction modulo $\mathfrak{m}$ is an isomorphism. Hence β^i is an isomorphism and we win. $\square$

Lemma 75.6. *Let* R *be a ring. Let* $\mathfrak{p} \subset R$ *be a prime. Let* $K \in D(R)$ *be perfect. Set* $d_i = \dim_{\kappa(\mathfrak{p})} H^i(K \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p}))$*. Then* $d_i < \infty$ *and only a finite number are nonzero. Then there exists an* $f \in R$*,* $f \notin \mathfrak{p}$ *and a complex*

$$\ldots \to 0 \to R_f^{\oplus d_a} \to R_f^{\oplus d_{a+1}} \to \ldots \to R_f^{\oplus d_{b-1}} \to R_f^{\oplus d_b} \to 0 \to \ldots$$

representing $K \otimes_R^{\mathbf{L}} R_f$ *in* $D(R_f)$*.*

Proof. Observe that $K \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})$ is perfect as an object of $D(\kappa(\mathfrak{p}))$, see Lemma 74.9. Hence only a finite number of d_i are nonzero and they are all finite. Applying Lemma 75.5 we get a complex representing K having the desired shape over the local ring $R_\mathfrak{p}$. We have $R_\mathfrak{p} = \text{colim}\, R_f$ for $f \in R$, $f \notin \mathfrak{p}$ (Algebra, Lemma 9.9). We conclude by Lemma 74.17. Some details omitted. $\square$

Lemma 75.7. *Let* R *be a ring. Let* $\mathfrak{p} \subset R$ *be a prime. Let* $M^\bullet$ *and* $N^\bullet$ *be bounded complexes of finite projective* R*-modules representing the same object of* $D(R)$*. Then there exists an* $f \in R$*,* $f \notin \mathfrak{p}$ *such that there is an isomorphism (!) of complexes*

$$M_f^\bullet \oplus P^\bullet \cong N_f^\bullet \oplus Q^\bullet$$

where $P^\bullet$ and $Q^\bullet$ are finite direct sums of trivial complexes, i.e., complexes of the form the form $\ldots \to 0 \to R_f \xrightarrow{1} R_f \to 0 \to \ldots$ (placed in arbitrary degrees).

Proof. If we have an isomorphism of the type described over the localization $R_\mathfrak{p}$, then using that $R_\mathfrak{p} = \operatorname{colim} R_f$ (Algebra, Lemma 9.9) we can descend the isomorphism to an isomorphism over R_f for some f. Thus we may assume R is local and $\mathfrak{p}$ is the maximal ideal. In this case the result follows from the uniqueness of a "minimal" complex representing a perfect object, see Lemma 75.5, and the fact that any complex is a direct sum of a trivial complex and a minimal one (Algebra, Lemma 102.2). $\square$

Lemma 75.8. *Let R be a ring. Let $I \subset R$ be an ideal. Let $E^\bullet$ be a complex of R/I-modules. Let K be an object of $D(R)$. Assume that*

1. *$E^\bullet$ is a bounded above complex of finite projective R/I-modules,*
2. *$K \otimes_R^{\mathbf{L}} R/I$ is represented by $E^\bullet$ in $D(R/I)$,*
3. *K is pseudo-coherent, and*
4. *(R, I) is a henselian pair.*

Then there exists a bounded above complex $P^\bullet$ of finite projective R-modules representing K in $D(R)$ such that $P^\bullet \otimes_R R/I$ is isomorphic to $E^\bullet$. Moreover, if E^i is free, then P^i is free.

Proof. We apply Lemma 75.2 using the class $\mathcal{P}$ of all finite projective R-modules. Properties (1) and (2) of the lemma are immediate. Property (3) follows from Nakayama's lemma (Algebra, Lemma 20.1). Property (4) follows from the fact that we can lift finite projective R/I-modules to finite projective R-modules, see Lemma 13.1. Property (5) holds because a pseudo-coherent complex can be represented by a bounded above complex of finite free R-modules. Thus Lemma 75.2 applies and we find $P^\bullet$ as desired. The final assertion of the lemma follows from Lemma 3.5. $\square$

76. Splitting complexes

In this section we discuss conditions which imply an object of the derived category of a ring is a direct sum of its truncations. Our method is to use the following lemma (under suitable hypotheses) to split the canonical distinguished triangles

$$\tau_{\leq i}K \to K \to \tau_{\geq i+1}K \to (\tau_{\leq i}K)[1]$$

in $D(R)$, see Derived Categories, Remark 12.4.

Lemma 76.1. *Let R be a ring. Let K and L be objects of $D(R)$. Assume L has projective-amplitude in $[a, b]$, for example if L is perfect of tor-amplitude in $[a, b]$.*

1. *If $H^i(K) = 0$ for $i \geq a$, then $\operatorname{Hom}_{D(R)}(L, K) = 0$.*
2. *If $H^i(K) = 0$ for $i \geq a + 1$, then given any distinguished triangle $K \to M \to L \to K[1]$ there is an isomorphism $M \cong K \oplus L$ in $D(R)$ compatible with the maps in the distinguished triangle.*
3. *If $H^i(K) = 0$ for $i \geq a$, then the isomorphism in (2) exists and is unique.*

Proof. The assumption that L has projective-amplitude in $[a, b]$ means we can represent L by a complex $L^\bullet$ of projective R-modules with $L^i = 0$ for $i \notin [a, b]$, see Definition 68.1. If L is perfect of tor-amplitude in $[a, b]$, then we can represent L by a complex $L^\bullet$ of finite projective R-modules with $L^i = 0$ for $i \notin [a, b]$, see Lemma 74.2. If $H^i(K) = 0$ for $i \geq a$, then K is quasi-isomorphic to $\tau_{\leq a-1}K$. Hence we can represent K by a complex $K^\bullet$ of R-modules with $K^i = 0$ for $i \geq a$. Then we obtain

$$\mathrm{Hom}_{D(R)}(L, K) = \mathrm{Hom}_{K(R)}(L^\bullet, K^\bullet) = 0$$

by Derived Categories, Lemma 19.8. This proves (1). Under the hypotheses of (2) we see that $\mathrm{Hom}_{D(R)}(L, K[1]) = 0$ by (1), hence the distinguished triangle is split by Derived Categories, Lemma 4.11. The uniqueness of (3) follows from (1). $\square$

Lemma 76.2. *Let R be a ring. Let $\mathfrak p \subset R$ be a prime ideal. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. Assume that for some $i \in \mathbf{Z}$ the map*

$$H^i(K^\bullet) \otimes_R \kappa(\mathfrak p) \longrightarrow H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak p))$$

is surjective. Then there exists an $f \in R$, $f \notin \mathfrak p$ such that $\tau_{\geq i+1}(K^\bullet \otimes_R R_f)$ is a perfect object of $D(R_f)$ with tor amplitude in $[i+1, \infty]$ and a canonical isomorphism

$$K^\bullet \otimes_R R_f \cong \tau_{\leq i}(K^\bullet \otimes_R R_f) \oplus \tau_{\geq i+1}(K^\bullet \otimes_R R_f)$$

in $D(R_f)$.

Proof. In this proof all tensor products are over R and we write $\kappa = \kappa(\mathfrak p)$. We may assume that $K^\bullet$ is a bounded above complex of finite free R-modules. Let us inspect what is happening in degree i:

$$\ldots \to K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \to \ldots$$

Let $0 \subset V \subset W \subset K^i \otimes \kappa$ be defined by the formulas

$$V = \mathrm{Im}\left(K^{i-1} \otimes \kappa \to K^i \otimes \kappa\right) \quad \text{and} \quad W = \mathrm{Ker}\left(K^i \otimes \kappa \to K^{i+1} \otimes \kappa\right)$$

Set $\dim(V) = r$, $\dim(W/V) = s$, and $\dim(K^i \otimes \kappa / W) = t$. We can pick $x_1, \ldots, x_r \in K^{i-1}$ which map by d^{i-1} to a basis of V. By our assumption we can pick $y_1, \ldots, y_s \in \mathrm{Ker}(d^i)$ mapping to a basis of W/V. Finally, choose $z_1, \ldots, z_t \in K^i$ mapping to a basis of $K^i \otimes \kappa / W$. Then we see that the elements $d^i(z_1), \ldots, d^i(z_t) \in K^{i+1}$ are linearly independent in $K^{i+1} \otimes_R \kappa$. By Algebra, Lemma 79.4 we may after replacing R by R_f for some $f \in R$, $f \notin \mathfrak p$ assume that

1. $d^i(x_a), y_b, z_c$ is an R-basis of K^i,
2. $d^i(z_1), \ldots, d^i(z_t)$ are R-linearly independent in K^{i+1}, and
3. the quotient $E^{i+1} = K^{i+1} / \sum R d^i(z_c)$ is finite projective.

Since d^i annihilates $d^{i-1}(x_a)$ and y_b, we deduce from condition (2) that $E^{i+1} = \mathrm{Coker}(d^i : K^i \to K^{i+1})$. Thus we see that

$$\tau_{\geq i+1} K^\bullet = (\ldots \to 0 \to E^{i+1} \to K^{i+2} \to \ldots)$$

is a bounded complex of finite projective modules sitting in degrees $[i+1, b]$ for some b. Thus $\tau_{\geq i+1} K^\bullet$ is perfect of amplitude $[i+1, b]$. Since $\tau_{\leq i} K^\bullet$ has no cohomology in degrees $> i$, we may apply Lemma 76.1 to the distinguished triangle

$$\tau_{\leq i} K^\bullet \to K^\bullet \to \tau_{\geq i+1} K^\bullet \to (\tau_{\leq i} K^\bullet)[1]$$

(Derived Categories, Remark 12.4) to conclude. $\square$

Lemma 76.3. *Let R be a ring. Let $\mathfrak p \subset R$ be a prime ideal. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. Assume that for some $i \in \mathbf{Z}$ the maps*

$$H^i(K^\bullet) \otimes_R \kappa(\mathfrak p) \longrightarrow H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak p)) \quad \textit{and} \quad H^{i-1}(K^\bullet) \otimes_R \kappa(\mathfrak p) \longrightarrow H^{i-1}(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak p))$$

are surjective. Then there exists an $f \in R$, $f \notin \mathfrak p$ such that

1. *$\tau_{\geq i+1}(K^\bullet \otimes_R R_f)$ is a perfect object of $D(R_f)$ with tor amplitude in $[i+1, \infty]$,*

(2) *$H^i(K^\bullet)_f$ is a finite free R_f-module, and*
(3) *there is a canonical direct sum decomposition*
$$K^\bullet \otimes_R R_f \cong \tau_{\leq i-1}(K^\bullet \otimes_R R_f) \oplus H^i(K^\bullet)_f[-i] \oplus \tau_{\geq i+1}(K^\bullet \otimes_R R_f)$$
in $D(R_f)$.

Proof. We get (1) from Lemma 76.2 as well as a splitting $K^\bullet \otimes_R R_f = \tau_{\leq i}K^\bullet \otimes_R R_f \oplus \tau_{\geq i+1}K^\bullet \otimes_R R_f$ in $D(R_f)$. Applying Lemma 76.2 once more to $\tau_{\leq i}K^\bullet \otimes_R R_f$ we obtain (after suitably choosing f) a splitting $\tau_{\leq i}K^\bullet \otimes_R R_f = \tau_{\leq i-1}K^\bullet \otimes_R R_f \oplus H^i(K^\bullet)_f$ in $D(R_f)$ as well as the conclusion that $H^i(K)_f$ is a flat perfect module, i.e., finite projective. ☐

Lemma 76.4. *Let R be a ring. Let $\mathfrak p \subset R$ be a prime ideal. Let $i \in \mathbf{Z}$. Let $K^\bullet$ be a pseudo-coherent complex of R-modules such that $H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak p)) = 0$. Then there exists an $f \in R$, $f \notin \mathfrak p$ and a canonical direct sum decomposition*
$$K^\bullet \otimes_R R_f = \tau_{\geq i+1}(K^\bullet \otimes_R R_f) \oplus \tau_{\leq i-1}(K^\bullet \otimes_R R_f)$$
in $D(R_f)$ with $\tau_{\geq i+1}(K^\bullet \otimes_R R_f)$ a perfect complex with tor-amplitude in $[i+1, \infty]$.

Proof. This is an often used special case of Lemma 76.2. A direct proof is as follows. We may assume that $K^\bullet$ is a bounded above complex of finite free R-modules. Let us inspect what is happening in degree i:
$$\ldots \to K^{i-2} \to R^{\oplus l} \to R^{\oplus m} \to R^{\oplus n} \to K^{i+2} \to \ldots$$
Let A be the $m \times l$ matrix corresponding to $K^{i-1} \to K^i$ and let B be the $n \times m$ matrix corresponding to $K^i \to K^{i+1}$. The assumption is that $A \bmod \mathfrak p$ has rank r and that $B \bmod \mathfrak p$ has rank $m - r$. In other words, there is some $r \times r$ minor a of A which is not in $\mathfrak p$ and there is some $(m-r) \times (m-r)$-minor b of B which is not in $\mathfrak p$. Set $f = ab$. Then after inverting f we can find direct sum decompositions $K^{i-1} = R^{\oplus l-r} \oplus R^{\oplus r}$, $K^i = R^{\oplus r} \oplus R^{\oplus m-r}$, $K^{i+1} = R^{\oplus m-r} \oplus R^{\oplus n-m+r}$ such that the module map $K^{i-1} \to K^i$ kills of $R^{\oplus l-r}$ and induces an isomorphism of $R^{\oplus r}$ onto the corresponding summand of K^i and such that the module map $K^i \to K^{i+1}$ kills of $R^{\oplus r}$ and induces an isomorphism of $R^{\oplus m-r}$ onto the corresponding summand of K^{i+1}. Thus $K^\bullet$ becomes quasi-isomorphic to
$$\ldots \to K^{i-2} \to R^{\oplus l-r} \to 0 \to R^{\oplus n-m+r} \to K^{i+2} \to \ldots$$
and everything is clear. ☐

Lemma 76.5. *Let R be a ring. Let $K \in D^-(R)$. Let $a \in \mathbf{Z}$. Assume that for any injective R-module map $M \to M'$ the map $\mathrm{Ext}_R^{-a}(K, M) \to \mathrm{Ext}_R^{-a}(K, M')$ is injective. Then there is a unique direct sum decomposition $K \cong \tau_{\leq a}K \oplus \tau_{\geq a+1}K$ and $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b.*

Proof. Consider the distinguished triangle
$$\tau_{\leq a}K \to K \to \tau_{\geq a+1}K \to (\tau_{\leq a}K)[1]$$
in $D(R)$, see Derived Categories, Remark 12.4. Observe that $\mathrm{Ext}_R^{-a}(\tau_{\leq a}K, M) = \mathrm{Hom}_R(H^a(K), M)$ and $\mathrm{Ext}_R^{-a-1}(\tau_{\leq a}K, M) = 0$, see Derived Categories, Lemma 27.3. Thus the long exact sequence of Ext gives an exact sequence
$$0 \to \mathrm{Ext}_R^{-a}(\tau_{\geq a+1}K, M) \to \mathrm{Ext}_R^{-a}(K, M) \to \mathrm{Hom}_R(H^a(K), M)$$
functorial in the R-module M. Now if I is an injective R-module, then $\mathrm{Ext}_R^{-a}(\tau_{\geq a+1}K, I) = 0$ for example by Derived Categories, Lemma 27.2. Since every module injects into

an injective module, we conclude that $\mathrm{Ext}^{-a}_R(\tau_{\geq a+1}K, M) = 0$ for every R-module M. By Lemma 68.2 we conclude that $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b (this is where we use that K is bounded above). We obtain the splitting by Lemma 76.1. $\square$

Lemma 76.6. *Let R be a ring. Let $K \in D^-(R)$. Let $a \in \mathbf{Z}$. Assume $\mathrm{Ext}^{-a}_R(K, M) = 0$ for any R-module M. Then there is a unique direct sum decomposition $K \cong \tau_{\leq a-1}K \oplus \tau_{\geq a+1}K$ and $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b.*

Proof. By Lemma 76.5 we have a direct sum decomposition $K \cong \tau_{\leq a}K \oplus \tau_{\geq a+1}K$ and $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b. Clearly, we must have $H^a(K) = 0$ and we conclude that $\tau_{\leq a}K = \tau_{\leq a-1}K$ in $D(R)$. $\square$

77. Recognizing perfect complexes

Some lemmas that allow us to prove certain complexes are perfect.

Lemma 77.1. *Let R be a ring and let $\mathfrak{p} \subset R$ be a prime. Let K be pseudo-coherent and bounded below. Set $d_i = \dim_{\kappa(\mathfrak{p})} H^i(K \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}))$. If there exists an $a \in \mathbf{Z}$ such that $d_i = 0$ for $i < a$, then there exists an $f \in R$, $f \not\in \mathfrak{p}$ and a complex*

$$\ldots \to 0 \to R_f^{\oplus d_a} \to R_f^{\oplus d_{a+1}} \to \ldots \to R_f^{\oplus d_{b-1}} \to R_f^{\oplus d_b} \to 0 \to \ldots$$

representing $K \otimes^{\mathbf{L}}_R R_f$ in $D(R_f)$. In particular $K \otimes^{\mathbf{L}}_R R_f$ is perfect.

Proof. After decreasing a we may assume that also $H^i(K^\bullet) = 0$ for $i < a$. By Lemma 76.4 after replacing R by R_f for some $f \in R$, $f \not\in \mathfrak{p}$ we can write $K^\bullet = \tau_{\leq a-1}K^\bullet \oplus \tau_{\geq a}K^\bullet$ in $D(R)$ with $\tau_{\geq a}K^\bullet$ perfect. Since $H^i(K^\bullet) = 0$ for $i < a$ we see that $\tau_{\leq a-1}K^\bullet = 0$ in $D(R)$. Hence $K^\bullet$ is perfect. Then we can conclude using Lemma 75.6. $\square$

Lemma 77.2. *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. The following are equivalent*

1. *$K^\bullet$ is perfect with tor amplitude in $[a, b]$,*
2. *for every prime $\mathfrak{p}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})) = 0$ for all $i \not\in [a, b]$, and*
3. *for every maximal ideal $\mathfrak{m}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{m})) = 0$ for all $i \not\in [a, b]$.*

Proof. We omit the proof of the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume (3). Let $i \in \mathbf{Z}$ with $i \not\in [a, b]$. By Lemma 76.4 we see that the assumption implies that $H^i(K^\bullet)_\mathfrak{m} = 0$ for all maximal ideals of R. Hence $H^i(K^\bullet) = 0$, see Algebra, Lemma 23.1. Moreover, Lemma 76.4 now also implies that for every maximal ideal $\mathfrak{m}$ there exists an element $f \in R$, $f \not\in \mathfrak{m}$ such that $K^\bullet \otimes_R R_f$ is perfect with tor amplitude in $[a, b]$. Hence we conclude by appealing to Lemmas 74.12 and 66.16. $\square$

Lemma 77.3. *Let R be a ring. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. Consider the following conditions*

1. *$K^\bullet$ is perfect,*
2. *for every prime ideal $\mathfrak{p}$ the complex $K^\bullet \otimes_R R_\mathfrak{p}$ is perfect,*
3. *for every maximal ideal $\mathfrak{m}$ the complex $K^\bullet \otimes_R R_\mathfrak{m}$ is perfect,*
4. *for every prime $\mathfrak{p}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})) = 0$ for all $i \ll 0$,*
5. *for every maximal ideal $\mathfrak{m}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{m})) = 0$ for all $i \ll 0$.*

We always have the implications

$$(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$$

If $K^\bullet$ is bounded below, then all conditions are equivalent.

Proof. By Lemma 74.9 we see that (1) implies (2). It is immediate that (2) $\Rightarrow$ (3). Since every prime $\mathfrak{p}$ is contained in a maximal ideal $\mathfrak{m}$, we can apply Lemma 74.9 to the map $R_\mathfrak{m} \to R_\mathfrak{p}$ to see that (3) implies (2). Applying Lemma 74.9 to the residue maps $R_\mathfrak{p} \to \kappa(\mathfrak{p})$ and $R_\mathfrak{m} \to \kappa(\mathfrak{m})$ we see that (2) implies (4) and (3) implies (5).

Assume R is local with maximal ideal $\mathfrak{m}$ and residue field κ. We will show that if $H^i(K^\bullet \otimes^\mathbf{L} \kappa) = 0$ for $i < a$ for some a, then K is perfect. This will show that (4) implies (2) and (5) implies (3) whence the first part of the lemma. First we apply Lemma 76.4 with $i = a - 1$ to see that $K^\bullet = \tau_{\leq a-1}K^\bullet \oplus \tau_{\geq a}K^\bullet$ in $D(R)$ with $\tau_{\geq a}K^\bullet$ perfect of tor-amplitude contained in $[a, \infty]$. To finish we need to show that $\tau_{\leq a-1}K$ is zero, i.e., that its cohomology groups are zero. If not let i be the largest index such that $M = H^i(\tau_{\leq a-1}K)$ is not zero. Then M is a finite R-module because $\tau_{\leq a-1}K^\bullet$ is pseudo-coherent (Lemmas 64.3 and 64.8). Thus by Nakayama's lemma (Algebra, Lemma 20.1) we find that $M \otimes_R \kappa$ is nonzero. This implies that

$$H^i((\tau_{\leq a-1}K^\bullet) \otimes_R^\mathbf{L} \kappa) = H^i(K^\bullet \otimes_R^\mathbf{L} \kappa)$$

is nonzero which is a contradiction.

Assume the equivalent conditions (2) – (5) hold and that $K^\bullet$ is bounded below. Say $H^i(K^\bullet) = 0$ for $i < a$. Pick a maximal ideal $\mathfrak{m}$ of R. It suffices to show there exists an $f \in R$, $f \notin \mathfrak{m}$ such that $K^\bullet \otimes_R^\mathbf{L} R_f$ is perfect (Lemma 74.12 and Algebra, Lemma 17.10). This follows from Lemma 77.1. □

Lemma 77.4. *Let R be a ring. Let K be a pseudo-coherent object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

1. *K has projective-amplitude in $[a, b]$,*
2. *K is perfect of tor-amplitude in $[a, b]$,*
3. *$\text{Ext}^i_R(K, N) = 0$ for all finitely presented R-modules N and all $i \notin [-b, -a]$,*
4. *$H^n(K) = 0$ for $n > b$ and $\text{Ext}^i_R(K, N) = 0$ for all finitely presented R-modules N and all $i > -a$, and*
5. *$H^n(K) = 0$ for $n \notin [a - 1, b]$ and $\text{Ext}^{-a+1}_R(K, N) = 0$ for all finitely presented R-modules N.*

Proof. From the final statement of Lemma 74.2 we see that (2) implies (1). If (1) holds, then K can be represented by a complex of projective modules P^i with $P^i = 0$ for $i \notin [a, b]$. Since projective modules are flat (as summands of free modules), we see that K has tor-amplitude in $[a, b]$, see Lemma 66.3. Thus by Lemma 74.2 we see that (2) holds.

In conditions (3), (4), (5) the assumed vanishing of ext groups $\text{Ext}^i_R(K, M)$ for M of finite presentation is equivalent to the vanishing for all R-modules M by Lemma 65.1 and Algebra, Lemma 11.3. Thus the equivalence of (1), (3), (4), and (5) follows from Lemma 68.2. □

The following lemma useful in order to find perfect complexes over a polynomial ring $B = A[x_1, \ldots, x_d]$.

Lemma 77.5. *Let $A \to B$ be a ring map. Let $a, b \in \mathbf{Z}$. Let $d \geq 0$. Let $K^\bullet$ be a complex of B-modules. Assume*

1. *the ring map $A \to B$ is flat,*
2. *for every prime $\mathfrak{p} \subset A$ the ring $B \otimes_A \kappa(\mathfrak{p})$ has finite global dimension $\leq d$,*
3. *$K^\bullet$ is pseudo-coherent as a complex of B-modules, and*
4. *$K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules.*

Then $K^\bullet$ is perfect as a complex of B-modules with tor amplitude in $[a - d, b]$.

Proof. We may assume that $K^\bullet$ is a bounded above complex of finite free B-modules. In particular, $K^\bullet$ is flat as a complex of A-modules and $K^\bullet \otimes_A M = K^\bullet \otimes_A^{\mathbf{L}} M$ for any A-module M. For every prime $\mathfrak{p}$ of A the complex

$$K^\bullet \otimes_A \kappa(\mathfrak{p})$$

is a bounded above complex of finite free modules over $B \otimes_A \kappa(\mathfrak{p})$ with vanishing H^i except for $i \in [a, b]$. As $B \otimes_A \kappa(\mathfrak{p})$ has global dimension d we see from Lemma 66.19 that $K^\bullet \otimes_A \kappa(\mathfrak{p})$ has tor amplitude in $[a - d, b]$. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p}$. Since $K^\bullet \otimes_A \kappa(\mathfrak{p})$ is a bounded above complex of free $B \otimes_A \kappa(\mathfrak{p})$-modules we see that

$$\begin{aligned} K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{q}) &= K^\bullet \otimes_B \kappa(\mathfrak{q}) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{p})) \otimes_{B \otimes_A \kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{p})) \otimes_{B \otimes_A \kappa(\mathfrak{p})}^{\mathbf{L}} \kappa(\mathfrak{q}) \end{aligned}$$

Hence the arguments above imply that $H^i(K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{q})) = 0$ for $i \notin [a - d, b]$. We conclude by Lemma 77.2. $\square$

The following lemma is a local version of Lemma 77.5. It can be used to find perfect complexes over regular local rings.

Lemma 77.6. *Let $A \to B$ be a local ring homomorphism. Let $a, b \in \mathbf{Z}$. Let $d \geq 0$. Let $K^\bullet$ be a complex of B-modules. Assume*

1. *the ring map $A \to B$ is flat,*
2. *the ring $B/\mathfrak{m}_A B$ is regular of dimension d,*
3. *$K^\bullet$ is pseudo-coherent as a complex of B-modules, and*
4. *$K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules, in fact it suffices if $H^i(K^\bullet \otimes_A^{\mathbf{L}} \kappa(\mathfrak{m}_A))$ is nonzero only for $i \in [a, b]$.*

Then $K^\bullet$ is perfect as a complex of B-modules with tor amplitude in $[a - d, b]$.

Proof. By (3) we may assume that $K^\bullet$ is a bounded above complex of finite free B-modules. We compute

$$\begin{aligned} K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{m}_B) &= K^\bullet \otimes_B \kappa(\mathfrak{m}_B) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{m}_A)) \otimes_{B/\mathfrak{m}_A B} \kappa(\mathfrak{m}_B) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{m}_A)) \otimes_{B/\mathfrak{m}_A B}^{\mathbf{L}} \kappa(\mathfrak{m}_B) \end{aligned}$$

The first equality because $K^\bullet$ is a bounded above complex of flat B-modules. The second equality follows from basic properties of the tensor product. The third equality holds because $K^\bullet \otimes_A \kappa(\mathfrak{m}_A) = K^\bullet/\mathfrak{m}_A K^\bullet$ is a bounded above complex of flat $B/\mathfrak{m}_A B$-modules. Since $K^\bullet$ is a bounded above complex of flat A-modules by (1), the cohomology modules H^i of the complex $K^\bullet \otimes_A \kappa(\mathfrak{m}_A)$ are nonzero only for $i \in [a, b]$ by assumption (4). Thus the spectral sequence of Example 62.1 and the fact that $B/\mathfrak{m}_A B$ has finite global dimension d (by (2) and Algebra, Proposition 110.1) shows that $H^j(K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{m}_B))$ is zero for $j \notin [a-d, b]$. This finishes the proof by Lemma 77.2. $\square$

78. Characterizing perfect complexes

In this section we prove that the perfect complexes are exactly the compact objects of the derived category of a ring. First we show the following.

Lemma 78.1. *Let R be a ring. The full subcategory $D_{perf}(R) \subset D(R)$ of perfect objects is the smallest strictly full, saturated, triangulated subcategory containing $R = R[0]$. In other words $D_{perf}(R) = \langle R \rangle$. In particular, R is a classical generator for $D_{perf}(R)$.*

Proof. To see what the statement means, please look at Derived Categories, Definitions 6.1 and 36.3. It was shown in Lemmas 74.4 and 74.5 that $D_{perf}(R) \subset D(R)$ is a strictly full, saturated, triangulated subcategory of $D(R)$. Of course $R \in D_{perf}(R)$.

Recall that $\langle R \rangle = \bigcup \langle R \rangle_n$. To finish the proof we will show that if $M \in D_{perf}(R)$ is represented by

$$\ldots \to 0 \to M^a \to M^{a+1} \to \ldots \to M^b \to 0 \to \ldots$$

with M^i finite projective, then $M \in \langle R \rangle_{b-a+1}$. The proof is by induction on $b-a$. By definition $\langle R \rangle_1$ contains any finite projective R-module placed in any degree; this deals with the base case $b-a=0$ of the induction. In general, we consider the distinguished triangle

$$M_b[-b] \to M^\bullet \to \sigma_{\leq b-1} M^\bullet \to M_b[-b+1]$$

By induction the truncated complex $\sigma_{\leq b-1} M^\bullet$ is in $\langle R \rangle_{b-a}$ and $M_b[-b]$ is in $\langle R \rangle_1$. Hence $M^\bullet \in \langle R \rangle_{b-a+1}$ by definition. $\square$

Let R be a ring. Recall that $D(R)$ has direct sums which are given simply by taking direct sums of complexes, see Derived Categories, Lemma 33.5. We will use this in the lemmas of this section without further mention.

Lemma 78.2. *Let R be a ring. Let $K \in D(R)$ be an object such that for every countable set of objects $E_n \in D(R)$ the canonical map*

$$\bigoplus \mathrm{Hom}_{D(R)}(K, E_n) \longrightarrow \mathrm{Hom}_{D(R)}(K, \bigoplus E_n)$$

is a bijection. Then, given any system $L_n^\bullet$ of complexes over $\mathbf{N}$ we have that

$$\mathrm{colim}\, \mathrm{Hom}_{D(R)}(K, L_n^\bullet) \longrightarrow \mathrm{Hom}_{D(R)}(K, L^\bullet)$$

is a bijection, where $L^\bullet$ is the termwise colimit, i.e., $L^m = \mathrm{colim}\, L_n^m$ for all $m \in \mathbf{Z}$.

Proof. Consider the short exact sequence of complexes

$$0 \to \bigoplus L_n^\bullet \to \bigoplus L_n^\bullet \to L^\bullet \to 0$$

where the first map is given by $1 - t_n$ in degree n where $t_n : L_n^\bullet \to L_{n+1}^\bullet$ is the transition map. By Derived Categories, Lemma 12.1 this is a distinguished triangle in $D(R)$. Apply the homological functor $\mathrm{Hom}_{D(R)}(K, -)$, see Derived Categories, Lemma 4.2. Thus a long exact cohomology sequence

$$\begin{tikzcd} & \ldots \arrow[r] & \mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet[-1]) \arrow[dll] \\ \mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet) \arrow[r] & \mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet) \arrow[r] & \mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet) \arrow[dll] \\ \mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet[1]) \arrow[r] & \ldots & \end{tikzcd}$$

Since we have assumed that $\mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet)$ is equal to $\bigoplus \mathrm{Hom}_{D(R)}(K, L_n^\bullet)$ we see that the first map on every row of the diagram is injective (by the explicit description of this map as the sum of the maps induced by $1 - t_n$). Hence we conclude that $\mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet)$ is the cokernel of the first map of the middle row in the diagram above which is what we had to show. $\square$

The following proposition, characterizing perfect complexes as the compact objects (Derived Categories, Definition 37.1) of the derived category, shows up in various places. See for example [Ric89, proof of Proposition 6.3] (this treats the bounded case), [TT90, Theorem 2.4.3] (the statement doesn't match exactly), and [BN93, Proposition 6.4] (watch out for horrendous notational conventions).

Proposition 78.3. *Let R be a ring. For an object K of $D(R)$ the following are equivalent*

(1) *K is perfect, and*
(2) *K is a compact object of $D(R)$.*

Proof. Assume K is perfect, i.e., K is quasi-isomorphic to a bounded complex $P^\bullet$ of finite projective modules, see Definition 74.1. If E_i is represented by the complex $E_i^\bullet$, then $\bigoplus E_i$ is represented by the complex whose degree n term is $\bigoplus E_i^n$. On the other hand, as P^n is projective for all n we have $\mathrm{Hom}_{D(R)}(P^\bullet, K^\bullet) = \mathrm{Hom}_{K(R)}(P^\bullet, K^\bullet)$ for every complex of R-modules $K^\bullet$, see Derived Categories, Lemma 19.8. Thus $\mathrm{Hom}_{D(R)}(P^\bullet, E^\bullet)$ is the cohomology of the complex

$$\prod \mathrm{Hom}_R(P^n, E^{n-1}) \to \prod \mathrm{Hom}_R(P^n, E^n) \to \prod \mathrm{Hom}_R(P^n, E^{n+1}).$$

Since $P^\bullet$ is bounded we see that we may replace the $\prod$ signs by $\bigoplus$ signs in the complex above. Since each P^n is a finite R-module we see that $\mathrm{Hom}_R(P^n, \bigoplus_i E_i^m) = \bigoplus_i \mathrm{Hom}_R(P^n, E_i^m)$ for all n, m. Combining these remarks we see that the map of Derived Categories, Definition 37.1 is a bijection.

Conversely, assume K is compact. Represent K by a complex $K^\bullet$ and consider the map

$$K^\bullet \longrightarrow \bigoplus\nolimits_{n \geq 0} \tau_{\geq n} K^\bullet$$

where we have used the canonical truncations, see Homology, Section 15. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that $K \to \tau_{\geq n} K$ is zero for at least one n, i.e., K is in $D^-(R)$.

Since $K \in D^-(R)$ and since every R-module is a quotient of a free module, we may represent K by a bounded above complex $K^\bullet$ of free R-modules, see Derived Categories, Lemma 15.4. Note that we have

$$K^\bullet = \bigcup\nolimits_{n \leq 0} \sigma_{\geq n} K^\bullet$$

where we have used the stupid truncations, see Homology, Section 15. Hence by Lemma 78.2 we see that $1 : K^\bullet \to K^\bullet$ factors through $\sigma_{\geq n} K^\bullet \to K^\bullet$ in $D(R)$. Thus we see that $1 : K^\bullet \to K^\bullet$ factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in $D(R)$ for some complex $L^\bullet$ which is bounded and whose terms are free R-modules. Say $L^i = 0$ for $i \not\in [a, b]$. Fix a, b from now on. Let c be the largest integer $\leq b + 1$ such that we can find a factorization of $1_{K^\bullet}$ as above with L^i is finite free for $i < c$. We will show by induction that $c = b + 1$. Namely, write $L^c = \bigoplus_{\lambda \in \Lambda} R$. Since L^{c-1} is finite free we can find a finite subset $\Lambda' \subset \Lambda$ such that $L^{c-1} \to L^c$ factors through $\bigoplus_{\lambda \in \Lambda'} R \subset L^c$. Consider the map of complexes

$$\pi : L^\bullet \longrightarrow (\bigoplus\nolimits_{\lambda \in \Lambda \setminus \Lambda'} R)[-c]$$

given by the projection onto the factors corresponding to $\Lambda \setminus \Lambda'$ in degree i. By our assumption on K we see that, after possibly replacing Λ' by a larger finite subset, we may assume that $\pi \circ \varphi = 0$ in $D(R)$. Let $(L')^\bullet \subset L^\bullet$ be the kernel of π. Since π is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in $D(R)$ (see Derived Categories, Lemma 12.1). Since $\mathrm{Hom}_{D(R)}(K, -)$ is homological (see Derived Categories, Lemma 4.2) and $\pi \circ \varphi = 0$, we can find a morphism $\varphi' : K^\bullet \to (L')^\bullet$ in $D(R)$ whose composition with $(L')^\bullet \to L^\bullet$ gives φ. Setting ψ' equal to the composition of ψ with $(L')^\bullet \to L^\bullet$ we obtain a new factorization. Since $(L')^\bullet$ agrees with $L^\bullet$ except in degree c and since $(L')^c = \bigoplus_{\lambda \in \Lambda'} R$ the induction step is proved.

The conclusion of the discussion of the preceding paragraph is that $1_K : K \to K$ factors as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in $D(R)$ where L can be represented by a finite complex of free R-modules. In particular we see that L is perfect. Note that $e = \varphi \circ \psi \in \mathrm{End}_{D(R)}(L)$ is an idempotent. By Derived Categories, Lemma 4.14 we see that $L = \mathrm{Ker}(e) \oplus \mathrm{Ker}(1 - e)$. The map $\varphi : K \to L$ induces an isomorphism with $\mathrm{Ker}(1 - e)$ in $D(R)$. Hence we finally conclude that K is perfect by Lemma 74.5. $\square$

Lemma 78.4. *Let R be a ring. Let $I \subset R$ be an ideal. Let K be an object of $D(R)$. Assume that*

(1) *$K \otimes_R^{\mathbf{L}} R/I$ is perfect in $D(R/I)$, and*
(2) *I is a nilpotent ideal.*

Then K is perfect in $D(R)$.

Proof. Choose a finite complex $\overline{P}^\bullet$ of finite projective R/I-modules representing $K \otimes_R^{\mathbf{L}} R/I$, see Definition 74.1. By Lemma 75.3 there exists a complex $P^\bullet$ of projective R-modules representing K such that $\overline{P}^\bullet = P^\bullet / IP^\bullet$. It follows from Nakayama's lemma (Algebra, Lemma 20.1) that $P^\bullet$ is a finite complex of finite projective R-modules. $\square$

Lemma 78.5. *Let R be a ring. Let $I, J \subset R$ be ideals. Let K be an object of $D(R)$. Assume that*

(1) *$K \otimes_R^{\mathbf{L}} R/I$ is perfect in $D(R/I)$, and*
(2) *$K \otimes_R^{\mathbf{L}} R/J$ is perfect in $D(R/J)$.*

Then $K \otimes_R^{\mathbf{L}} R/IJ$ is perfect in $D(R/IJ)$.

Proof. It is clear that we may assume replace R by R/IJ and K by $K \otimes_R^{\mathbf{L}} R/IJ$. Then $R \to R/(I \cap J)$ is a surjection whose kernel has square zero. Hence by Lemma 78.4 it suffices to prove that $K \otimes_R^{\mathbf{L}} R/(I \cap J)$ is perfect. Thus we may assume that $I \cap J = 0$.

We prove the lemma in case $I \cap J = 0$. First, we may represent K by a K-flat complex $K^\bullet$ with all K^n flat, see Lemma 59.10. Then we see that we have a short exact sequence of complexes

$$0 \to K^\bullet \to K^\bullet/IK^\bullet \oplus K^\bullet/JK^\bullet \to K^\bullet/(I + J)K^\bullet \to 0$$

Note that $K^\bullet/IK^\bullet$ represents $K \otimes_R^{\mathbf{L}} R/I$ by construction of the derived tensor product. Similarly for $K^\bullet/JK^\bullet$ and $K^\bullet/(I + J)K^\bullet$. Note that $K^\bullet/(I + J)K^\bullet$ is a perfect complex of $R/(I + J)$-modules, see Lemma 74.9. Hence the complexes $K^\bullet/IK^\bullet$, and $K^\bullet/JK^\bullet$ and $K^\bullet/(I+J)K^\bullet$ have finitely many nonzero cohomology groups (since a perfect complex has finite Tor-amplitude, see Lemma 74.2). We conclude that $K \in D^b(R)$ by the long exact cohomology sequence associated to short exact sequence of complexes displayed above. In particular we assume $K^\bullet$ is a bounded above complex of free R-modules (see Derived Categories, Lemma 15.4).

We will now show that K is perfect using the criterion of Proposition 78.3. Thus we let $E_j \in D(R)$ be a family of objects parametrized by a set J. We choose complexes $E_j^\bullet$ with flat terms representing E_j, see for example Lemma 59.10. It is clear that

$$0 \to E_j^\bullet \to E_j^\bullet/IE_j^\bullet \oplus E_j^\bullet/JE_j^\bullet \to E_j^\bullet/(I + J)E_j^\bullet \to 0$$

is a short exact sequence of complexes. Taking direct sums we obtain a similar short exact sequence

$$0 \to \bigoplus E_j^\bullet \to \bigoplus E_j^\bullet/IE_j^\bullet \oplus E_j^\bullet/JE_j^\bullet \to \bigoplus E_j^\bullet/(I + J)E_j^\bullet \to 0$$

(Note that $- \otimes_R R/I$ commutes with direct sums.) This short exact sequence determines a distinguished triangle in $D(R)$, see Derived Categories, Lemma 12.1. Apply the homological functor $\mathrm{Hom}_{D(R)}(K, -)$ (see Derived Categories, Lemma

4.2) to get a commutative diagram

$$\begin{tikzcd}
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/(I+J))[-1] \arrow[r] \arrow[d] & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/(I+J))[-1] \arrow[d] \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/I \oplus E_j^\bullet/J)[-1] \arrow[r] \arrow[d] & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/I \oplus E_j^\bullet/J)[-1] \arrow[d] \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet) \arrow[r] \arrow[d] & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet) \arrow[d] \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/I \oplus E_j^\bullet/J) \arrow[r] \arrow[d] & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/I \oplus E_j^\bullet/J) \arrow[d] \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/(I+J)) \arrow[r] & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/(I+J))
\end{tikzcd}$$

with exact columns. It is clear that, for any complex $E^\bullet$ of R-modules we have

$$\begin{aligned}
\mathrm{Hom}_{D(R)}(K^\bullet, E^\bullet/I) &= \mathrm{Hom}_{K(R)}(K^\bullet, E^\bullet/I) \\
&= \mathrm{Hom}_{K(R/I)}(K^\bullet/IK^\bullet, E^\bullet/I) \\
&= \mathrm{Hom}_{D(R/I)}(K^\bullet/IK^\bullet, E^\bullet/I)
\end{aligned}$$

and similarly for when dividing by J or $I+J$, see Derived Categories, Lemma 19.8. Derived Categories. Thus all the horizontal arrows, except for possibly the middle one, are isomorphisms as the complexes $K^\bullet/IK^\bullet$, $K^\bullet/JK^\bullet$, $K^\bullet/(I+J)K^\bullet$ are perfect complexes of R/I, R/J, $R/(I+J)$-modules, see Proposition 78.3. It follows from the 5-lemma (Homology, Lemma 5.20) that the middle map is an isomorphism and the lemma follows by Proposition 78.3. $\square$

79. Strong generators and regular rings

Let R be a ring. Denote $D(R)_c$ the saturated full triangulated subcategory of $D(R)$. We already know that

$$\langle R \rangle = D_{perf}(R) = D(R)_c$$

See Lemma 78.1 and Proposition 78.3. It turns out that if R is regular, then R is a strong generator (Derived Categories, Definition 36.3).

Lemma 79.1. *Let R be a ring. Let $n \geq 1$. Let $K \in \langle R \rangle_n$ with notation as in Derived Categories, Section 36. Consider maps*

$$K \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \xrightarrow{f_3} \ldots \xrightarrow{f_n} K_n$$

in $D(R)$. If $H^i(f_j) = 0$ for all i, j, then $f_n \circ \ldots \circ f_1 = 0$.

Proof. If $n = 1$, then K is a direct summand in $D(R)$ of a bounded complex $P^\bullet$ whose terms are finite free R-modules and whose differentials are zero. Thus it suffices to show any morphism $f : P^\bullet \to K_1$ in $D(R)$ with $H^i(f) = 0$ for all i is zero. Since $P^\bullet$ is a finite direct sum $P^\bullet = \bigoplus R[m_j]$ it suffices to show any morphism $g : R[m] \to K_1$ with $H^{-m}(g) = 0$ in $D(R)$ is zero. This follows from the fact that $\mathrm{Hom}_{D(R)}(R[-m], K) = H^m(K)$.

For $n > 1$ we proceed by induction on n. Namely, we know that K is a summand in $D(R)$ of an object P which sits in a distinguished triangle

$$P' \xrightarrow{i} P \xrightarrow{p} P'' \to P'[1]$$

with $P' \in \langle R \rangle_1$ and $P'' \in \langle R \rangle_{n-1}$. As above we may replace K by P and assume that we have

$$P \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \xrightarrow{f_3} \ldots \xrightarrow{f_n} K_n$$

in $D(R)$ with f_j zero on cohomology. By the case $n = 1$ the composition $f_1 \circ i$ is zero. Hence by Derived Categories, Lemma 4.2 we can find a morphism $h : P'' \to K_1$ such that $f_1 = h \circ p$. Observe that $f_2 \circ h$ is zero on cohomology. Hence by induction we find that $f_n \circ \ldots \circ f_2 \circ h = 0$ which implies $f_n \circ \ldots \circ f_1 = f_n \circ \ldots \circ f_2 \circ h \circ p = 0$ as desired. □

Lemma 79.2. *Let R be a Noetherian ring. If R is a strong generator for $D_{perf}(R)$, then R is regular of finite dimension.*

Proof. Assume $D_{perf}(R) = \langle R \rangle_n$ for some $n \geq 1$. For any finite R-module M we can choose a complex

$$P = (P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \xrightarrow{d^1} \ldots \xrightarrow{d^{-1}} P^0)$$

of finite free R-modules with $H^i(P) = 0$ for $i = -n, \ldots, -1$ and $M \cong \text{Coker}(d^{-1})$. Note that P is in $D_{perf}(R)$. For any R-module N we can compute $\text{Ext}^n_R(M, N)$ the finite free resolution P of M, see Algebra, Section 71 and compare with Derived Categories, Section 27. In particular, the sequence above defines an element

$$\xi \in \text{Ext}^n_R(\text{Coker}(d^{-1}), \text{Coker}(d^{-n-1})) = \text{Ext}^n_R(M, \text{Coker}(d^{-n-1}))$$

and for any element $\overline{\xi}$ in $\text{Ext}^n_R(M, N)$ there is a R-module map $\varphi : \text{Coker}(d^{-n-1}) \to N$ such that φ maps ξ to $\overline{\xi}$. For $j = 1, \ldots, n-1$ consider the complexes

$$K_j = (\text{Coker}(d^{-n-1}) \to P^{-n+1} \to \ldots \to P^{-j})$$

with $\text{Coker}(d^{-n-1})$ in degree $-n$ and P^t in degree t. We also set $K_n = \text{Coker}(d^{-n-1})[n]$. Then we have maps

$$P \to K_1 \to K_2 \to \ldots \to K_n$$

which induce vanishing maps on cohomology. By Lemma 79.1 since $P \in D_{perf}(R) = \langle R \rangle_n$ we find that the composition of this maps is zero in $D(R)$. Since $\text{Hom}_{D(R)}(P, K_n) = \text{Hom}_{K(R)}(P, K_n)$ by Derived Categories, Lemma 19.8 we conclude $\xi = 0$. Hence $\text{Ext}^n_R(M, N) = 0$ for all R-modules N, see discussion above. It follows that M has projective dimension $\leq n - 1$ by Algebra, Lemma 109.8. Since this holds for all finite R-modules M we conclude that R has finite global dimension, see Algebra, Lemma 109.12. We finally conclude by Algebra, Lemma 110.8. □

Lemma 79.3. *Let R be a Noetherian regular ring of dimension $d < \infty$. Let $K, L \in D^-(R)$. Assume there exists an k such that $H^i(K) = 0$ for $i \leq k$ and $H^i(L) = 0$ for $i \geq k - d + 1$. Then $\text{Hom}_{D(R)}(K, L) = 0$.*

Proof. Let $K^\bullet$ be a bounded above complex representing K, say $K^i = 0$ for $i \geq n + 1$. After replacing $K^\bullet$ by $\tau_{\geq k+1} K^\bullet$ we may assume $K^i = 0$ for $i \leq k$. Then we may use the distinguished triangle

$$K^n[-n] \to K^\bullet \to \sigma_{\leq n-1} K^\bullet$$

to see it suffices to prove the lemma for $K^n[-n]$ and $\sigma_{\leq n-1}K^\bullet$. By induction on n, we conclude that it suffices to prove the lemma in case K is represented by the complex $M[-m]$ for some R-module M and some $m \geq k+1$. Since R has global dimension d by Algebra, Lemma 110.8 we see that M has a projective resolution $0 \to P_d \to \ldots \to P_0 \to M \to 0$. Then the complex $P^\bullet$ having P_i in degree $m-i$ is a bounded complex of projectives representing $M[-m]$. On the other hand, we can choose a complex $L^\bullet$ representing L with $L^i = 0$ for $i \geq k-d+1$. Hence any map of complexes $P^\bullet \to L^\bullet$ is zero. This implies the lemma by Derived Categories, Lemma 19.8. ☐

Lemma 79.4. *Let R be a Noetherian regular ring of dimension $1 \leq d < \infty$. Let $K \in D(R)$ be perfect and let $k \in \mathbf{Z}$ such that $H^i(K) = 0$ for $i = k-d+2, \ldots, k$ (empty condition if $d = 1$). Then $K = \tau_{\leq k-d+1}K \oplus \tau_{\geq k+1}K$.*

Proof. The vanishing of cohomology shows that we have a distinguished triangle

$$\tau_{\leq k-d+1}K \to K \to \tau_{\geq k+1}K \to (\tau_{\leq k-d+1}K)[1]$$

By Derived Categories, Lemma 4.11 it suffices to show that the third arrow is zero. Thus it suffices to show that $\mathrm{Hom}_{D(R)}(\tau_{\geq k+1}K, (\tau_{\leq k-d+1}K)[1]) = 0$ which follows from Lemma 79.3. ☐

Lemma 79.5. *Let R be a Noetherian regular ring of finite dimension. Then R is a strong generator for the full subcategory $D_{perf}(R) \subset D(R)$ of perfect objects.*

Proof. We will use that an object K of $D(R)$ is perfect if and only if K is bounded and has finite cohomology modules, see Lemma 74.14. Strong generators of triangulated categories are defined in Derived Categories, Definition 36.3. Let $d = \dim(R)$.

Let $K \in D_{perf}(R)$. We will show $K \in \langle R \rangle_{d+1}$. By Algebra, Lemma 110.8 every finite R-module has projective dimension $\leq d$. We will show by induction on $0 \leq i \leq d$ that if $H^n(K)$ has projective dimension $\leq i$ for all $n \in \mathbf{Z}$, then K is in $\langle R \rangle_{i+1}$.

Base case $i = 0$. In this case $H^n(K)$ is a finite R-module of projective dimension 0. In other words, each cohomology is a projective R-module. Thus $\mathrm{Ext}^i_R(H^n(K), H^m(K)) = 0$ for all $i > 0$ and $m, n \in \mathbf{Z}$. By Derived Categories, Lemma 27.9 we find that K is isomorphic to the direct sum of the shifts of its cohomology modules. Since each cohomology module is a finite projective R-module, it is a direct summand of a direct sum of copies of R. Hence by definition we see that K is contained in $\langle R \rangle_1$.

Induction step. Assume the claim holds for $i < d$ and let $K \in D_{perf}(R)$ have the property that $H^n(K)$ has projective dimension $\leq i+1$ for all $n \in \mathbf{Z}$. Choose $a \leq b$ such that $H^n(K)$ is zero for $n \notin [a, b]$. For each $n \in [a, b]$ choose a surjection $F^n \to H^n(K)$ where F^n is a finite free R-module. Since F^n is projective, we can lift $F^n \to H^n(K)$ to a map $F^n[-n] \to K$ in $D(R)$ (small detail omitted). Thus we obtain a morphism $\bigoplus_{a \leq n \leq b} F^n[-n] \to K$ which is surjective on cohomology modules. Choose a distinguished triangle

$$K' \to \bigoplus\nolimits_{a \leq n \leq b} F^n[-n] \to K \to K'[1]$$

in $D(R)$. Of course, the object K' is bounded and has finite cohomology modules. The long exact sequence of cohomology breaks into short exact sequences

$$0 \to H^n(K') \to F^n \to H^n(K) \to 0$$

by the choices we made. By Algebra, Lemma 109.9 we see that the projective dimension of $H^n(K')$ is $\leq \max(0, i)$. Thus $K' \in \langle R \rangle_{i+1}$. By definition this means that K is in $\langle R \rangle_{i+1+1}$ as desired. $\square$

Proposition 79.6. *Let R be a Noetherian ring. The following are equivalent*

(1) *R is regular of finite dimension,*
(2) *$D_{perf}(R)$ has a strong generator, and*
(3) *R is a strong generator for $D_{perf}(R)$.*

Proof. This is a formal consequence of Lemmas 78.1, 79.2, and 79.5 as well as Derived Categories, Lemma 36.6. $\square$

80. Relatively finitely presented modules

Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let M be a finite B-module. In this case it is **not true** that

$$M \text{ of finite presentation over } B \Leftrightarrow M \text{ of finite presentation over } A$$

A counter example is $R = k[x_1, x_2, x_3, \ldots]$, $A = R$, $B = R/(x_i)$, and $M = B$. To "fix" this we introduce a relative notion of finite presentation.

Lemma 80.1. *Let $R \to A$ be a ring map of finite type. Let M be an A-module. The following are equivalent*

(1) *for some presentation $\alpha : R[x_1, \ldots, x_n] \to A$ the module M is a finitely presented $R[x_1, \ldots, x_n]$-module,*
(2) *for all presentations $\alpha : R[x_1, \ldots, x_n] \to A$ the module M is a finitely presented $R[x_1, \ldots, x_n]$-module, and*
(3) *for any surjection $A' \to A$ where A' is a finitely presented R-algebra, the module M is finitely presented as A'-module.*

In this case M is a finitely presented A-module.

Proof. If $\alpha : R[x_1, \ldots, x_n] \to A$ and $\beta : R[y_1, \ldots, y_m] \to A$ are presentations. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{tikzcd} R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & R[x_1, \ldots, x_n] \arrow[d] \\ R[y_1, \ldots, y_m] \arrow[r] & A \end{tikzcd}$$

Hence the equivalence of (1) and (2) follows by applying Algebra, Lemmas 6.4 and 36.23. The equivalence of (2) and (3) follows by choosing a presentation $A' = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and using Algebra, Lemma 36.23 to show that M is finitely presented as A'-module if and only if M is finitely presented as a $R[x_1, \ldots, x_n]$-module. $\square$

Definition 80.2. Let $R \to A$ be a finite type ring map. Let M be an A-module. We say M is an A-module *finitely presented relative to R* if the equivalent conditions of Lemma 80.1 hold.

Note that if $R \to A$ is of finite presentation, then M is an A-module finitely presented relative to R if and only if M is a finitely presented A-module. It is equally clear that A as an A-module is finitely presented relative to R if and only if A is of finite presentation over R. If R is Noetherian the notion is uninteresting. Now we can formulate the result we were looking for.

Lemma 80.3. *Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let M be a B-module. Then M is an A-module finitely presented relative to R if and only if M is a B-module finitely presented relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose $y_1, \ldots, y_m \in B$ which generate B over A. As $A \to B$ is finite each y_i satisfies a monic equation with coefficients in A. Hence we can find monic polynomials $P_j(T) \in R[x_1, \ldots, x_n][T]$ such that $P_j(y_j) = 0$ in B. Then we get a commutative diagram

$$\begin{tikzcd} R[x_1, \ldots, x_n] \arrow[r] \arrow[d] & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

Since the top arrow is a finite and finitely presented ring map we conclude by Algebra, Lemma 36.23 and the definition. □

With this result in hand we see that the relative notion makes sense and behaves well with regards to finite maps of rings of finite type over R. It is also stable under localization, stable under base change, and "glues" well.

Lemma 80.4. *Let R be a ring, $f \in R$ an element, $R_f \to A$ is a finite type ring map, $g \in A$, and M an A-module. If M of finite presentation relative to R_f, then M_g is an A_g-module of finite presentation relative to R.*

Proof. Choose a presentation $R_f[x_1, \ldots, x_n] \to A$. We write $R_f = R[x_0]/(fx_0 - 1)$. Consider the presentation $R[x_0, x_1, \ldots, x_n, x_{n+1}] \to A_g$ which extends the given map, maps x_0 to the image of $1/f$, and maps x_{n+1} to $1/g$. Choose $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to g (this is possible). Suppose that

$$R_f[x_1, \ldots, x_n]^{\oplus s} \to R_f[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

is a presentation of M given by a matrix (h_{ij}). Pick $h'_{ij} \in R[x_0, x_1, \ldots, x_n]$ which map to h_{ij}. Then

$$R[x_0, x_1, \ldots, x_n, x_{n+1}]^{\oplus s+2t} \to R[x_0, x_1, \ldots, x_n, x_{n+1}]^{\oplus t} \to M_g \to 0$$

is a presentation of M_f. Here the $t \times (s + 2t)$ matrix defining the map has a first $t \times s$ block consisting of the matrix h'_{ij}, a second $t \times t$ block which is $(x_0 f -)I_t$, and a third block which is $(x_{n+1}g' - 1)I_t$. □

Lemma 80.5. *Let $R \to A$ be a finite type ring map. Let M be an A-module finitely presented relative to R. For any ring map $R \to R'$ the $A \otimes_R R'$-module*

$$M \otimes_A A' = M \otimes_R R'$$

is finitely presented relative to R'.

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation

$$R[x_1, \ldots, x_n]^{\oplus s} \to R[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

Then

$$R'[x_1, \ldots, x_n]^{\oplus s} \to R'[x_1, \ldots, x_n]^{\oplus t} \to M \otimes_R R' \to 0$$

is a presentation of the base change and we win. □

Lemma 80.6. *Let $R \to A$ be a finite type ring map. Let M be an A-module finitely presented relative to R. Let $A \to A'$ be a ring map of finite presentation. The A'-module $M \otimes_A A'$ is finitely presented relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation $A' = A[y_1, \ldots, y_m]/(g_1, \ldots, g_l)$. Pick $g_i' \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ mapping to g_i. Say

$$R[x_1, \ldots, x_n]^{\oplus s} \to R[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

is a presentation of M given by a matrix (h_{ij}). Then

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m]^{\oplus s+tl} \to R[x_0, x_1, \ldots, x_n, y_1, \ldots, y_m]^{\oplus t} \to M \otimes_A A' \to 0$$

is a presentation of $M \otimes_A A'$. Here the $t \times (s + lt)$ matrix defining the map has a first $t \times s$ block consisting of the matrix h_{ij}, followed by l blocks of size $t \times t$ which are $g_i' I_t$. □

Lemma 80.7. *Let $R \to A \to B$ be finite type ring maps. Let M be a B-module. If M is finitely presented relative to A and A is of finite presentation over R, then M is finitely presented relative to R.*

Proof. Choose a surjection $A[x_1, \ldots, x_n] \to B$. Choose a presentation

$$A[x_1, \ldots, x_n]^{\oplus s} \to A[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

given by a matrix (h_{ij}). Choose a presentation

$$A = R[y_1, \ldots, y_m]/(g_1, \ldots, g_u).$$

Choose $h'_{ij} \in R[y_1, \ldots, y_m, x_1, \ldots, x_n]$ mapping to h_{ij}. Then we obtain the presentation

$$R[y_1, \ldots, y_m, x_1, \ldots, x_n]^{\oplus s+tu} \to R[y_1, \ldots, y_m, x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

where the $t \times (s+tu)$-matrix is given by a first $t \times s$ block consisting of h'_{ij} followed by u blocks of size $t \times t$ given by $g_i I_t$, $i = 1, \ldots, u$. □

Lemma 80.8. *Let $R \to A$ be a finite type ring map. Let M be an A-module. Let $f_1, \ldots, f_r \in A$ generate the unit ideal. The following are equivalent*

(1) *each M_{f_i} is finitely presented relative to R, and*
(2) *M is finitely presented relative to R.*

Proof. The implication (2) ⇒ (1) is in Lemma 80.4. Assume (1). Write $1 = \sum f_i g_i$ in A. Choose a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A$. such that y_i maps to f_i and z_i maps to g_i. Then we see that there exists a surjection

$$P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \longrightarrow A.$$

By Lemma 80.1 we see that M_{f_i} is a finitely presented A_{f_i}-module, hence by Algebra, Lemma 23.2 we see that M is a finitely presented A-module. Hence M is a finite P-module (with P as above). Choose a surjection $P^{\oplus t} \to M$. We have to show that the kernel K of this map is a finite P-module. Since P_{y_i} surjects onto

A_{f_i} we see by Lemma 80.1 and Algebra, Lemma 5.3 that the localization K_{y_i} is a finitely generated P_{y_i}-module. Choose elements $k_{i,j} \in K$, $i = 1, \ldots, r$, $j = 1, \ldots, s_i$ such that the images of $k_{i,j}$ in K_{y_i} generate. Set $K' \subset K$ equal to the P-module generated by the elements $k_{i,j}$. Then K/K' is a module whose localization at y_i is zero for all i. Since $(y_1, \ldots, y_r) = P$ we see that $K/K' = 0$ as desired. □

Lemma 80.9. *Let $R \to A$ be a finite type ring map. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules.*

1. *If M', M'' are finitely presented relative to R, then so is M.*
2. *If M' is a finite type A-module and M is finitely presented relative to R, then M'' is finitely presented relative to R.*

Proof. Follows immediately from Algebra, Lemma 5.3. □

Lemma 80.10. *Let $R \to A$ be a finite type ring map. Let M, M' be A-modules. If $M \oplus M'$ is finitely presented relative to R, then so are M and M'.*

Proof. Omitted. □

81. Relatively pseudo-coherent modules

This section is the analogue of Section 80 for pseudo-coherence.

Lemma 81.1. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Consider the R-algebra map $R[x] \to R$ which maps x to zero. Then*

$$K^\bullet \otimes_{R[x]}^{\mathbf{L}} R \cong K^\bullet \oplus K^\bullet[1]$$

in $D(R)$.

Proof. Choose a K-flat resolution $P^\bullet \to K^\bullet$ over R such that P^n is a flat R-module for all n, see Lemma 59.10. Then $P^\bullet \otimes_R R[x]$ is a K-flat complex of $R[x]$-modules whose terms are flat $R[x]$-modules, see Lemma 59.3 and Algebra, Lemma 39.7. In particular $x : P^n \otimes_R R[x] \to P^n \otimes_R R[x]$ is injective with cokernel isomorphic to P^n. Thus

$$P^\bullet \otimes_R R[x] \xrightarrow{x} P^\bullet \otimes_R R[x]$$

is a double complex of $R[x]$-modules whose associated total complex is quasi-isomorphic to $P^\bullet$ and hence $K^\bullet$. Moreover, this associated total complex is a K-flat complex of $R[x]$-modules for example by Lemma 59.4 or by Lemma 59.5. Hence

$$\begin{aligned} K^\bullet \otimes_{R[x]}^{\mathbf{L}} R &\cong \text{Tot}(P^\bullet \otimes_R R[x] \xrightarrow{x} P^\bullet \otimes_R R[x]) \otimes_{R[x]} R = \text{Tot}(P^\bullet \xrightarrow{0} P^\bullet) \\ &= P^\bullet \oplus P^\bullet[1] \cong K^\bullet \oplus K^\bullet[1] \end{aligned}$$

as desired. □

Lemma 81.2. *Let R be a ring and $K^\bullet$ a complex of R-modules. Let $m \in \mathbf{Z}$. Consider the R-algebra map $R[x] \to R$ which maps x to zero. Then $K^\bullet$ is m-pseudo-coherent as a complex of R-modules if and only if $K^\bullet$ is m-pseudo-coherent as a complex of $R[x]$-modules.*

Proof. This is a special case of Lemma 64.11. We also prove it in another way as follows.

Note that $0 \to R[x] \to R[x] \to R \to 0$ is exact. Hence R is pseudo-coherent as an $R[x]$-module. Thus one implication of the lemma follows from Lemma 64.11. To prove the other implication, assume that $K^\bullet$ is m-pseudo-coherent as a complex of $R[x]$-modules. By Lemma 64.12 we see that $K^\bullet \otimes_{R[x]}^{\mathbf{L}} R$ is m-pseudo-coherent as a complex of R-modules. By Lemma 81.1 we see that $K^\bullet \oplus K^\bullet[1]$ is m-pseudo-coherent as a complex of R-modules. Finally, we conclude that $K^\bullet$ is m-pseudo-coherent as a complex of R-modules from Lemma 64.8. $\square$

Lemma 81.3. *Let $R \to A$ be a ring map of finite type. Let $K^\bullet$ be a complex of A-modules. Let $m \in \mathbf{Z}$. The following are equivalent*

(1) *for some presentation $\alpha : R[x_1, \ldots, x_n] \to A$ the complex $K^\bullet$ is an m-pseudo-coherent complex of $R[x_1, \ldots, x_n]$-modules,*
(2) *for all presentations $\alpha : R[x_1, \ldots, x_n] \to A$ the complex $K^\bullet$ is an m-pseudo-coherent complex of $R[x_1, \ldots, x_n]$-modules.*

In particular the same equivalence holds for pseudo-coherence.

Proof. If $\alpha : R[x_1, \ldots, x_n] \to A$ and $\beta : R[y_1, \ldots, y_m] \to A$ are presentations. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{tikzcd} R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & R[x_1, \ldots, x_n] \arrow[d] \\ R[y_1, \ldots, y_m] \arrow[r] & A \end{tikzcd}$$

After a change of coordinates the ring homomorphism $R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to R[x_1, \ldots, x_n]$ is isomorphic to the ring homomorphism which maps each y_i to zero. Similarly for the left vertical map in the diagram. Hence, by induction on the number of variables this lemma follows from Lemma 81.2. The pseudo-coherent case follows from this and Lemma 64.5. $\square$

Definition 81.4. Let $R \to A$ be a finite type ring map. Let $K^\bullet$ be a complex of A-modules. Let M be an A-module. Let $m \in \mathbf{Z}$.

(1) We say $K^\bullet$ is *m-pseudo-coherent relative to R* if the equivalent conditions of Lemma 81.3 hold.
(2) We say $K^\bullet$ is *pseudo-coherent relative to R* if $K^\bullet$ is m-pseudo-coherent relative to R for all $m \in \mathbf{Z}$.
(3) We say M is *m-pseudo-coherent relative to R* if $M[0]$ is m-pseudo-coherent relative to R.
(4) We say M is *pseudo-coherent relative to R* if $M[0]$ is pseudo-coherent relative to R.

Part (2) means that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n]$-modules for any surjection $R[y_1, \ldots, y_m] \to A$, see Lemma 64.5. This definition has the following pleasing property.

Lemma 81.5. *Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of B-modules. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R if and only if $K^\bullet$ seen as a complex of A-modules is m-pseudo-coherent (pseudo-coherent) relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose $y_1, \ldots, y_m \in B$ which generate B over A. As $A \to B$ is finite each y_i satisfies a monic equation with coefficients in A. Hence we can find monic polynomials $P_j(T) \in R[x_1, \ldots, x_n][T]$ such that $P_j(y_j) = 0$ in B. Then we get a commutative diagram

$$\begin{tikzcd} & R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[d] \\ R[x_1, \ldots, x_n] \arrow[r] \arrow[d] & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

The top horizontal arrow and the top right vertical arrow satisfy the assumptions of Lemma 64.11. Hence $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n]$-modules if and only if $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$-modules. $\square$

Lemma 81.6. *Let R be a ring. Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(A)$.*

1. *If $K^\bullet$ is $(m+1)$-pseudo-coherent relative to R and $L^\bullet$ is m-pseudo-coherent relative to R then $M^\bullet$ is m-pseudo-coherent relative to R.*
2. *If $K^\bullet, M^\bullet$ are m-pseudo-coherent relative to R, then $L^\bullet$ is m-pseudo-coherent relative to R.*
3. *If $L^\bullet$ is $(m+1)$-pseudo-coherent relative to R and $M^\bullet$ is m-pseudo-coherent relative to R, then $K^\bullet$ is $(m+1)$-pseudo-coherent relative to R.*

Moreover, if two out of three of $K^\bullet, L^\bullet, M^\bullet$ are pseudo-coherent relative to R, the so is the third.

Proof. Follows immediately from Lemma 64.2 and the definitions. $\square$

Lemma 81.7. *Let $R \to A$ be a finite type ring map. Let M be an A-module. Then*

1. *M is 0-pseudo-coherent relative to R if and only if M is a finite type A-module,*
2. *M is (-1)-pseudo-coherent relative to R if and only if M is a finitely presented relative to R,*
3. *M is $(-d)$-pseudo-coherent relative to R if and only if for every surjection $R[x_1, \ldots, x_n] \to A$ there exists a resolution*
$$R[x_1, \ldots, x_n]^{\oplus a_d} \to R[x_1, \ldots, x_n]^{\oplus a_{d-1}} \to \ldots \to R[x_1, \ldots, x_n]^{\oplus a_0} \to M \to 0$$
of length d, and
4. *M is pseudo-coherent relative to R if and only if for every presentation $R[x_1, \ldots, x_n] \to A$ there exists an infinite resolution*
$$\ldots \to R[x_1, \ldots, x_n]^{\oplus a_1} \to R[x_1, \ldots, x_n]^{\oplus a_0} \to M \to 0$$
by finite free $R[x_1, \ldots, x_n]$-modules.

Proof. Follows immediately from Lemma 64.4 and the definitions. $\square$

Lemma 81.8. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet, L^\bullet \in D(A)$. If $K^\bullet \oplus L^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R so are $K^\bullet$ and $L^\bullet$.*

Proof. Immediate from Lemma 64.8 and the definitions. $\square$

Lemma 81.9. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a bounded above complex of A-modules such that K^i is $(m-i)$-pseudo-coherent relative to R for all i. Then $K^\bullet$ is m-pseudo-coherent relative to R. In particular, if $K^\bullet$ is a bounded above complex of A-modules pseudo-coherent relative to R, then $K^\bullet$ is pseudo-coherent relative to R.*

Proof. Immediate from Lemma 64.9 and the definitions. □

Lemma 81.10. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet \in D^-(A)$ such that $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent (resp. pseudo-coherent) relative to R for all i. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Immediate from Lemma 64.10 and the definitions. □

Lemma 81.11. *Let R be a ring, $f \in R$ an element, $R_f \to A$ is a finite type ring map, $g \in A$, and $K^\bullet$ a complex of A-modules. If $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R_f, then $K^\bullet \otimes_A A_g$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. First we show that $K^\bullet$ is m-pseudo-coherent relative to R. Namely, suppose $R_f[x_1, \ldots, x_n] \to A$ is surjective. Write $R_f = R[x_0]/(fx_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Hence by Lemma 64.11 we see that $K^\bullet$ is m-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n]$-modules.

Choose an element $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. By Lemma 64.12 we see that

$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_{R[x_0,x_1,\ldots,x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] &= K^\bullet \otimes_{R[x_0,x_1,\ldots,x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] \\ &= K^\bullet \otimes_A A_f \end{aligned}$$

is m-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n, \frac{1}{g'}]$-modules. write

$$R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] = R[x_0, \ldots, x_n, x_{n+1}]/(x_{n+1}g' - 1).$$

As $R[x_0, x_1, \ldots, x_n, \frac{1}{g'}]$ is pseudo-coherent as a $R[x_0, \ldots, x_n, x_{n+1}]$-module we conclude (see Lemma 64.11) that $K^\bullet \otimes_A A_g$ is m-pseudo-coherent as a complex of $R[x_0, \ldots, x_n, x_{n+1}]$-modules as desired. □

Lemma 81.12. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules which is m-pseudo-coherent (resp. pseudo-coherent) relative to R. Let $R \to R'$ be a ring map such that A and R' are Tor independent over R. Set $A' = A \otimes_R R'$. Then $K^\bullet \otimes^{\mathbf{L}}_A A'$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R'.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Note that

$$K^\bullet \otimes^{\mathbf{L}}_A A' = K^\bullet \otimes^{\mathbf{L}}_R R' = K^\bullet \otimes^{\mathbf{L}}_{R[x_1,\ldots,x_n]} R'[x_1, \ldots, x_n]$$

by Lemma 61.2 applied twice. Hence we win by Lemma 64.12. □

Lemma 81.13. *Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules. Assume B as a B-module is pseudo-coherent relative to A. If $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R, then $K^\bullet \otimes^{\mathbf{L}}_A B$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Combined we get a surjection $R[x_1, \ldots, x_n, y_1, \ldots y_m] \to B$. Choose a resolution $E^\bullet \to B$ of B by a complex of finite free $A[y_1, \ldots, y_n]$-modules (which is possible by our assumption on the ring map $A \to B$). We may assume that $K^\bullet$ is a bounded above complex of flat A-modules. Then

$$\begin{aligned} K^\bullet \otimes_A^{\mathbf{L}} B &= \mathrm{Tot}(K^\bullet \otimes_A B[0]) \\ &= \mathrm{Tot}(K^\bullet \otimes_A A[y_1, \ldots, y_m] \otimes_{A[y_1, \ldots, y_m]} B[0]) \\ &\cong \mathrm{Tot}\left((K^\bullet \otimes_A A[y_1, \ldots, y_m]) \otimes_{A[y_1, \ldots, y_m]} E^\bullet\right) \\ &= \mathrm{Tot}(K^\bullet \otimes_A E^\bullet) \end{aligned}$$

in $D(A[y_1, \ldots, y_m])$. The quasi-isomorphism $\cong$ comes from an application of Lemma 59.7. Thus we have to show that $\mathrm{Tot}(K^\bullet \otimes_A E^\bullet)$ is m-pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots y_m]$-modules. Note that $\mathrm{Tot}(K^\bullet \otimes_A E^\bullet)$ has a filtration by subcomplexes with successive quotients the complexes $K^\bullet \otimes_A E^i[-i]$. Note that for $i \ll 0$ the complexes $K^\bullet \otimes_A E^i[-i]$ have zero cohomology in degrees $\leq m$ and hence are m-pseudo-coherent (over any ring). Hence, applying Lemma 81.6 and induction, it suffices to show that $K^\bullet \otimes_A E^i[-i]$ is pseudo-coherent relative to R for all i. Note that $E^i = 0$ for $i > 0$. Since also E^i is finite free this reduces to proving that $K^\bullet \otimes_A A[y_1, \ldots, y_m]$ is m-pseudo-coherent relative to R which follows from Lemma 81.12 for instance. □

Lemma 81.14. *Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbf{Z}$. Let M be an A-module. Assume B is flat over A and B as a B-module is pseudo-coherent relative to A. If M is m-pseudo-coherent (resp. pseudo-coherent) relative to R, then $M \otimes_A B$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Immediate from Lemma 81.13. □

Lemma 81.15. *Let R be a ring. Let $A \to B$ be a map of finite type R-algebras. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of B-modules. Assume A is pseudo-coherent relative to R. Then the following are equivalent*

1. *$K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to A, and*
2. *$K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Then we get a surjection

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to A[y_1, \ldots, y_m]$$

which is a flat base change of $R[x_1, \ldots, x_n] \to A$. By assumption A is a pseudo-coherent module over $R[x_1, \ldots, x_n]$ hence by Lemma 64.13 we see that $A[y_1, \ldots, y_m]$ is pseudo-coherent over $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Thus the lemma follows from Lemma 64.11 and the definitions. □

Lemma 81.16. *Let $R \to A$ be a finite type ring map. Let $K^\bullet$ be a complex of A-modules. Let $m \in \mathbf{Z}$. Let $f_1, \ldots, f_r \in A$ generate the unit ideal. The following are equivalent*

1. *each $K^\bullet \otimes_A A_{f_i}$ is m-pseudo-coherent relative to R, and*
2. *$K^\bullet$ is m-pseudo-coherent relative to R.*

The same equivalence holds for pseudo-coherence relative to R.

Proof. The implication (2) $\Rightarrow$ (1) is in Lemma 81.11. Assume (1). Write $1 = \sum f_i g_i$ in A. Choose a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A$. such that y_i maps to f_i and z_i maps to g_i. Then we see that there exists a surjection

$$P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \longrightarrow A.$$

Note that P is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-module and that $P[1/y_i]$ is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r, 1/y_i]$-module. Hence by Lemma 64.11 we see that $K^\bullet \otimes_A A_{f_i}$ is an m-pseudo-coherent complex of $P[1/y_i]$-modules for each i. Thus by Lemma 64.14 we see that $K^\bullet$ is pseudo-coherent as a complex of P-modules, and Lemma 64.11 shows that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-modules. $\square$

Lemma 81.17. *Let R be a Noetherian ring. Let $R \to A$ be a finite type ring map. Then*

1. *A complex of A-modules $K^\bullet$ is m-pseudo-coherent relative to R if and only if $K^\bullet \in D^-(A)$ and $H^i(K^\bullet)$ is a finite A-module for $i \geq m$.*
2. *A complex of A-modules $K^\bullet$ is pseudo-coherent relative to R if and only if $K^\bullet \in D^-(A)$ and $H^i(K^\bullet)$ is a finite A-module for all i.*
3. *An A-module is pseudo-coherent relative to R if and only if it is finite.*

Proof. Immediate consequence of Lemma 64.17 and the definitions. $\square$

82. Pseudo-coherent and perfect ring maps

We can define these types of ring maps as follows.

Definition 82.1. Let $A \to B$ be a ring map.

1. We say $A \to B$ is a *pseudo-coherent ring map* if it is of finite type and B, as a B-module, is pseudo-coherent relative to A.
2. We say $A \to B$ is a *perfect ring map* if it is a pseudo-coherent ring map such that B as an A-module has finite tor dimension.

This terminology may be nonstandard. Using Lemma 81.7 we see that $A \to B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules. The motivation for the definition of a perfect ring map is Lemma 74.2. The following lemmas gives a more useful and intuitive characterization of a perfect ring map.

Lemma 82.2. *A ring map $A \to B$ is perfect if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules.*

Proof. If $A \to B$ is perfect, then $B = A[x_1, \ldots, x_n]/I$ and B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module and has finite tor dimension as an A-module. Hence Lemma 77.5 implies that B is perfect as a $A[x_1, \ldots, x_n]$-module, i.e., it has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules (Lemma 74.3). Conversely, if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules then B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module, hence $A \to B$ is pseudo-coherent. Moreover, the given resolution over $A[x_1, \ldots, x_n]$ is a finite resolution by flat A-modules and hence B has finite tor dimension as an A-module. $\square$

Lots of the results of the preceding sections can be reformulated in terms of this terminology. We also refer to More on Morphisms, Sections 60 and 61 for the corresponding discussion concerning morphisms of schemes.

Lemma 82.3. *A finite type ring map of Noetherian rings is pseudo-coherent.*

Proof. See Lemma 81.17. ☐

Lemma 82.4. *A ring map which is flat and of finite presentation is perfect.*

Proof. Let $A \to B$ be a ring map which is flat and of finite presentation. It is clear that B has finite tor dimension. By Algebra, Lemma 168.1 there exists a finite type $\mathbf{Z}$-algebra $A_0 \subset A$ and a flat finite type ring map $A_0 \to B_0$ such that $B = B_0 \otimes_{A_0} A$. By Lemma 81.17 we see that $A_0 \to B_0$ is pseudo-coherent. As $A_0 \to B_0$ is flat we see that B_0 and A are tor independent over A_0, hence we may use Lemma 81.12 to conclude that $A \to B$ is pseudo-coherent. ☐

Lemma 82.5. *Let $A \to B$ be a finite type ring map with A a regular ring of finite dimension. Then $A \to B$ is perfect.*

Proof. By Algebra, Lemma 110.8 the assumption on A means that A has finite global dimension. Hence every module has finite tor dimension, see Lemma 66.19, in particular B does. By Lemma 82.3 the map is pseudo-coherent. ☐

Lemma 82.6. *A local complete intersection homomorphism is perfect.*

Proof. Let $A \to B$ be a local complete intersection homomorphism. By Definition 33.2 this means that $B = A[x_1, \ldots, x_n]/I$ where I is a Koszul ideal in $A[x_1, \ldots, x_n]$. By Lemmas 82.2 and 74.3 it suffices to show that I is a perfect module over $A[x_1, \ldots, x_n]$. By Lemma 74.12 this is a local question. Hence we may assume that I is generated by a Koszul-regular sequence (by Definition 32.1). Of course this means that I has a finite free resolution and we win. ☐

Lemma 82.7. *Let $R \to A$ be a pseudo-coherent ring map. Let $K \in D(A)$. The following are equivalent*

(1) *K is m-pseudo-coherent (resp. pseudo-coherent) relative to R, and*
(2) *K is m-pseudo-coherent (resp. pseudo-coherent) in $D(A)$.*

Proof. Reformulation of a special case of Lemma 81.15. ☐

Lemma 82.8. *Let $R \to B \to A$ be ring maps with $\varphi : B \to A$ surjective and $R \to B$ and $R \to A$ flat and of finite presentation. For $K \in D(A)$ denote $\varphi_*K \in D(B)$ the restriction. The following are equivalent*

(1) *K is pseudo-coherent,*
(2) *K is pseudo-coherent relative to R,*
(3) *K is pseudo-coherent relative to A,*
(4) *φ_*K is pseudo-coherent,*
(5) *φ_*K is pseudo-coherent relative to R.*

Similar holds for m-pseudo-coherence.

Proof. Observe that $R \to A$ and $R \to B$ are perfect ring maps (Lemma 82.4) hence a fortiori pseudo-coherent ring maps. Thus (1) ⇔ (2) and (4) ⇔ (5) by Lemma 82.7.

Using that A is pseudo-coherent relative to R we use Lemma 81.15 to see that (2) $\Leftrightarrow$ (3). However, since $A \to B$ is surjective, we see directly from Definition 81.4 that (3) is equivalent with (4). $\square$

83. Relatively perfect modules

This section is the analogue of Section 81 for perfect objects of the derived category. we only define this notion in a limited generality as we are not sure what the correct definition is in general. See Derived Categories of Schemes, Remark 35.14 for a discussion.

Definition 83.1. Let $R \to A$ be a flat ring map of finite presentation. An object K of $D(A)$ is *R-perfect* or *perfect relative to R* if K is pseudo-coherent (Definition 64.1) and has finite tor dimension over R (Definition 66.1).

By Lemma 82.8 it would have been the same thing to ask K to be pseudo-coherent relative to R. Here are some obligatory lemmas.

Lemma 83.2. *Let $R \to A$ be a flat ring map of finite presentation. The R-perfect objects of $D(A)$ form a saturated*[9] *triangulated strictly full subcategory.*

Proof. This follows from Lemmas 64.2, 64.8, 66.5, and 66.7. $\square$

Lemma 83.3. *Let $R \to A$ be a flat ring map of finite presentation. A perfect object of $D(A)$ is R-perfect. If $K, M \in D(A)$ then $K \otimes_A^{\mathbf{L}} M$ is R-perfect if K is perfect and M is R-perfect.*

Proof. The first statement follows from the second by taking $M = A$. The second statement follows from Lemmas 74.2, 66.10, and 64.16. $\square$

Lemma 83.4. *Let $R \to A$ be a flat ring map of finite presentation. Let $K \in D(A)$. The following are equivalent*

1. *K is R-perfect, and*
2. *K is isomorphic to a finite complex of R-flat, finitely presented A-modules.*

Proof. To prove (2) implies (1) it suffices by Lemma 83.2 to show that an R-flat, finitely presented A-module M defines an R-perfect object of $D(A)$. Since M has finite tor dimension over R, it suffices to show that M is pseudo-coherent. By Algebra, Lemma 168.1 there exists a finite type $\mathbf{Z}$-algebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ and a finite A_0-module M_0 flat over R_0 such that $A = A_0 \otimes_{R_0} R$ and $M = M_0 \otimes_{R_0} R$. By Lemma 64.17 we see that M_0 is pseudo-coherent A_0-module. Choose a resolution $P_0^\bullet \to M_0$ by finite free A_0-modules P_0^n. Since A_0 is flat over R_0, this is a flat resolution. Since M_0 is flat over R_0 we find that $P^\bullet = P_0^\bullet \otimes_{R_0} R$ still resolves $M = M_0 \otimes_{R_0} R$. (You can use Lemma 61.2 to see this.) Hence $P^\bullet$ is a finite free resolution of M over A and we conclude that M is pseudo-coherent.

Assume (1). We can represent K by a bounded above complex $P^\bullet$ of finite free A-modules. Assume that K viewed as an object of $D(R)$ has tor amplitude in $[a, b]$. By Lemma 66.2 we see that $\tau_{\geq a} P^\bullet$ is a complex of R-flat, finitely presented A-modules representing K. $\square$

[9]Derived Categories, Definition 6.1.

Lemma 83.5. *Let $R \to A$ be a flat ring map of finite presentation. Let $R \to R'$ be a ring map and set $A' = A \otimes_R R'$. If $K \in D(A)$ is R-perfect, then $K \otimes_A^{\mathbf{L}} A'$ is R'-perfect.*

Proof. By Lemma 64.12 we see that $K \otimes_A^{\mathbf{L}} A'$ is pseudo-coherent. By Lemma 61.2 we see that $K \otimes_A^{\mathbf{L}} A'$ is equal to $K \otimes_R^{\mathbf{L}} R'$ in $D(R')$. Then we can apply Lemma 66.13 to see that $K \otimes_R^{\mathbf{L}} R'$ in $D(R')$ has finite tor dimension. □

Lemma 83.6. *Let $R \to A$ be a flat ring map. Let $K, L \in D(A)$ with K pseudo-coherent and L finite tor dimension over R. We may choose*

(1) *a bounded above complex $P^\bullet$ of finite free A-modules representing K, and*
(2) *a bounded complex of R-flat A-modules $F^\bullet$ representing L.*

Given these choices we have

(a) *$E^\bullet = \mathrm{Hom}^\bullet(P^\bullet, F^\bullet)$ is a bounded below complex of R-flat A-modules representing $R\mathrm{Hom}_A(K, L)$,*
(b) *for any ring map $R \to R'$ with $A' = A \otimes_R R'$ the complex $E^\bullet \otimes_R R'$ represents $R\mathrm{Hom}_{A'}(K \otimes_A^{\mathbf{L}} A', L \otimes_A^{\mathbf{L}} A')$.*

If in addition $R \to A$ is of finite presentation and L is R-perfect, then we may choose F^p to be finitely presented A-modules and consequently E^n will be finitely presented A-modules as well.

Proof. The existence of $P^\bullet$ is the definition of a pseudo-coherent complex. We first represent L by a bounded above complex $F^\bullet$ of free A-modules (this is possible because bounded tor dimension in particular implies bounded). Next, say L viewed as an object of $D(R)$ has tor amplitude in $[a, b]$. Then, after replacing $F^\bullet$ by $\tau_{\geq a} F^\bullet$, we get a complex as in (2). This follows from Lemma 66.2.

Proof of (a). Since $F^\bullet$ is bounded an since $P^\bullet$ is bounded above, we see that $E^n = 0$ for $n \ll 0$ and that E^n is a finite (!) direct sum

$$E^n = \bigoplus\nolimits_{p+q=n} \mathrm{Hom}_A(P^{-q}, F^p)$$

and since P^{-q} is finite free, this is indeed an R-flat A-module. The fact that $E^\bullet$ represents $R\mathrm{Hom}_A(K, L)$ follows from Lemma 73.2.

Proof of (b). Let $R \to R'$ be a ring map and $A' = A \otimes_R R'$. By Lemma 61.2 the object $L \otimes_A^{\mathbf{L}} A'$ is represented by $F^\bullet \otimes_R R'$ viewed as a complex of A'-modules (by flatness of F^p over R). Similarly for $P^\bullet \otimes_R R'$. As above $R\mathrm{Hom}_{A'}(K \otimes_A^{\mathbf{L}} A', L \otimes_A^{\mathbf{L}} A')$ is represented by

$$\mathrm{Hom}^\bullet(P^\bullet \otimes_R R', F^\bullet \otimes_R R') = E^\bullet \otimes_R R'$$

The equality holds by looking at the terms of the complex individually and using that $\mathrm{Hom}_{A'}(P^{-q} \otimes_R R', F^p \otimes_R R') = \mathrm{Hom}_A(P^{-q}, F^p) \otimes_R R'$. □

Lemma 83.7. *Let $R = \mathrm{colim}_{i \in I} R_i$ be a filtered colimit of rings. Let $0 \in I$ and $R_0 \to A_0$ be a flat ring map of finite presentation. For $i \geq 0$ set $A_i = R_i \otimes_{R_0} A_0$ and set $A = R \otimes_{R_0} A_0$.*

(1) *Given an R-perfect K in $D(A)$ there exists an $i \in I$ and an R_i-perfect K_i in $D(A_i)$ such that $K \cong K_i \otimes_{A_i}^{\mathbf{L}} A$ in $D(A)$.*
(2) *Given $K_0, L_0 \in D(A_0)$ with K_0 pseudo-coherent and L_0 finite tor dimension over R_0, then we have*

$$\mathrm{Hom}_{D(A)}(K_0 \otimes_{A_0}^{\mathbf{L}} A, L_0 \otimes_{A_0}^{\mathbf{L}} A) = \mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(A_i)}(K_0 \otimes_{A_0}^{\mathbf{L}} A_i, L_0 \otimes_{A_0}^{\mathbf{L}} A_i)$$

In particular, the triangulated category of R-perfect complexes over A is the colimit of the triangulated categories of R_i-perfect complexes over A_i.

Proof. By Algebra, Lemma 127.6 the category of finitely presented A-modules is the colimit of the categories of finitely presented A_i-modules. Given this, Algebra, Lemma 168.1 tells us that category of R-flat, finitely presented A-modules is the colimit of the categories of R_i-flat, finitely presented A_i-modules. Thus the characterization in Lemma 83.4 proves that (1) is true.

To prove (2) we choose $P_0^\bullet$ representing K_0 and $F_0^\bullet$ representing L_0 as in Lemma 83.6. Then $E_0^\bullet = \mathrm{Hom}^\bullet(P_0^\bullet, F_0^\bullet)$ satisfies

$$H^0(E_0^\bullet \otimes_{R_0} R_i) = \mathrm{Hom}_{D(A_i)}(K_0 \otimes_{A_0}^{\mathbf{L}} A_i, L_0 \otimes_{A_0}^{\mathbf{L}} A_i)$$

and

$$H^0(E_0^\bullet \otimes_{R_0} R) = \mathrm{Hom}_{D(A)}(K_0 \otimes_{A_0}^{\mathbf{L}} A, L_0 \otimes_{A_0}^{\mathbf{L}} A)$$

by the lemma. Thus the result because tensor product commutes with colimits and filtered colimits are exact (Algebra, Lemma 8.8). □

Lemma 83.8. *Let $R' \to A'$ be a flat ring map of finite presentation. Let $R' \to R$ be a surjective ring map whose kernel is a nilpotent ideal. Set $A = A' \otimes_{R'} R$. Let $K' \in D(A')$ and set $K = K' \otimes_{A'}^{\mathbf{L}} A$ in $D(A)$. If K is R-perfect, then K' is R'-perfect.*

Proof. We can represent K by a bounded above complex of finite free A-modules $E^\bullet$, see Lemma 64.5. By Lemma 75.3 we conclude that K' is pseudo-coherent because it can be represented by a bounded above complex $P^\bullet$ of finite free A'-modules with $P^\bullet \otimes_{A'} A = E^\bullet$. Observe that this also means $P^\bullet \otimes_{R'} R = E^\bullet$ (since $A = A' \otimes_{R'} R$).

Let $I = \mathrm{Ker}(R' \to R)$. Then $I^n = 0$ for some n. Choose $[a, b]$ such that K has tor amplitude in $[a, b]$ as a complex of R-modules. We will show K' has tor amplitude in $[a, b]$. To do this, let M' be an R'-module. If $IM' = 0$, then

$$K' \otimes_{R'}^{\mathbf{L}} M' = P^\bullet \otimes_{R'} M' = E^\bullet \otimes_R M' = K \otimes_R^{\mathbf{L}} M'$$

(because A' is flat over R' and A is flat over R) which has nonzero cohomology only for degrees in $[a, b]$ by choice of a, b. If $I^{t+1}M' = 0$, then we consider the short exact sequence

$$0 \to IM' \to M' \to M'/IM' \to 0$$

with $M = M'/IM'$. By induction on t we have that both $K' \otimes_{R'}^{\mathbf{L}} IM'$ and $K' \otimes_{R'}^{\mathbf{L}} M'/IM'$ have nonzero cohomology only for degrees in $[a, b]$. Then the distinguished triangle

$$K' \otimes_{R'}^{\mathbf{L}} IM' \to K' \otimes_{R'}^{\mathbf{L}} M' \to K' \otimes_{R'}^{\mathbf{L}} M'/IM' \to (K' \otimes_{R'}^{\mathbf{L}} IM')[1]$$

proves the same is true for $K' \otimes_{R'}^{\mathbf{L}} M'$. This proves the desired bound for all M' and hence the desired bound on the tor amplitude of K'. □

Lemma 83.9. *Let R be a ring. Let $A = R[x_1, \ldots, x_d]/I$ be flat and of finite presentation over R. Let $\mathfrak{q} \subset A$ be a prime ideal lying over $\mathfrak{p} \subset R$. Let $K \in D(A)$ be pseudo-coherent. Let $a, b \in \mathbf{Z}$. If $H^i(K_\mathfrak{q} \otimes_{R_\mathfrak{p}}^{\mathbf{L}} \kappa(\mathfrak{p}))$ is nonzero only for $i \in [a, b]$, then $K_\mathfrak{q}$ has tor amplitude in $[a - d, b]$ over R.*

Proof. By Lemma 82.8 K is pseudo-coherent as a complex of $R[x_1, \ldots, x_d]$-modules. Therefore we may assume $A = R[x_1, \ldots, x_d]$. Applying Lemma 77.6 to $R_\mathfrak{p} \to A_\mathfrak{q}$ and the complex $K_\mathfrak{q}$ using our assumption, we find that $K_\mathfrak{q}$ is perfect in $D(A_\mathfrak{q})$ with tor amplitude in $[a - d, b]$. Since $R_\mathfrak{p} \to A_\mathfrak{q}$ is flat, we conclude by Lemma 66.11. $\square$

Lemma 83.10. *Let $R \to A$ be a ring map which is flat and of finite presentation. Let $K \in D(A)$ be pseudo-coherent. The following are equivalent*

1. *K is R-perfect, and*
2. *K is bounded below and for every prime ideal $\mathfrak{p} \subset R$ the object $K \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})$ is bounded below.*

Proof. Observe that (1) implies (2) as an R-perfect complex has bounded tor dimension as a complex of R-modules by definition. Let us prove the other implication.

Write $A = R[x_1, \ldots, x_d]/I$. Denote L in $D(R[x_1, \ldots, x_d])$ the restriction of K. By Lemma 82.8 we see that L is pseudo-coherent. Since L and K have the same image in $D(R)$ we see that L is R-perfect if and only if K is R-perfect. Also $L \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})$ and $K \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})$ are the same objects of $D(\kappa(\mathfrak{p}))$. This reduces us to the case $A = R[x_1, \ldots, x_d]$.

Say $A = R[x_1, \ldots, x_d]$ and K satisfies (2). Let $\mathfrak{q} \subset A$ be a prime lying over a prime $\mathfrak{p} \subset R$. By Lemma 77.6 applied to $R_\mathfrak{p} \to A_\mathfrak{q}$ and the complex $K_\mathfrak{q}$ using our assumption, we find that $K_\mathfrak{q}$ is perfect in $D(A_\mathfrak{q})$. Since K is bounded below, we see that K is perfect in $D(A)$ by Lemma 77.3. This implies that K is R-perfect by Lemma 83.3 and the proof is complete. $\square$

84. Two term complexes

In this section we prove some results on two term complexes of modules which will help us understand conditions on the naive cotangent complex.

Lemma 84.1. *Let R be a ring. Let $K \in D(R)$ with $H^i(K) = 0$ for $i \not\in \{-1, 0\}$. The following are equivalent*

1. *$H^{-1}(K) = 0$ and $H^0(K)$ is a projective module and*
2. *$\text{Ext}^1_R(K, M) = 0$ for every R-module M.*

If R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then these are also equivalent to

3. *$\text{Ext}^1_R(K, M) = 0$ for every finite R-module M.*

Proof. The equivalence of (1) and (2) follows from Lemma 68.2. If R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then K is pseudo-coherent, see Lemma 64.17. Thus the equivalence of (1) and (3) follows from Lemma 77.4. $\square$

Remark 84.2. The following two statements follow from Lemma 84.1, Algebra, Definition 137.1, and Algebra, Proposition 138.8.

1. A ring map $A \to B$ is smooth if and only if $A \to B$ is of finite presentation and $\text{Ext}^1_B(NL_{B/A}, N) = 0$ for every B-module N.
2. A ring map $A \to B$ is formally smooth if and only if $\text{Ext}^1_B(NL_{B/A}, N) = 0$ for every B-module N.

Lemma 84.3. *Let R be a ring. Let K be an object of $D(R)$ with $H^i(K) = 0$ for $i \notin \{-1, 0\}$. Then*

1. *K can be represented by a two term complex $K^{-1} \to K^0$ with K^0 a free module, and*
2. *if R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then K can be represented by a two term complex $K^{-1} \to K^0$ with K^0 a finite free module and K^{-1} finite.*

Proof. Proof of (1). Suppose K is given by the complex of modules $M^\bullet$. We may first replace $M^\bullet$ by $\tau_{\leq 0}M^\bullet$. Thus we may assume $M^i = 0$ for $i > 0$, Next, we may choose a free resolution $P^\bullet \to M^\bullet$ with $P^i = 0$ for $i > 0$, see Derived Categories, Lemma 15.4. Finally, we can set $K^\bullet = \tau_{\geq -1}P^\bullet$.

Proof of (2). Assume R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$. By Lemma 64.5 we can choose a quasi-isomorphism $F^\bullet \to M^\bullet$ with $F^i = 0$ for $i > 0$ and F^i finite free. Then we can set $K^\bullet = \tau_{\geq -1}F^\bullet$. $\square$

Maps in the derived category out of the naive cotangent complex $NL_{B/A}$ or $NL(\alpha)$ (see Algebra, Section 134) are easy to understand by the result of the following lemma.

Lemma 84.4. *Let R be a ring. Let $M^\bullet$ be a complex of modules over R with $M^i = 0$ for $i > 0$ and M^0 a projective R-module. Let $K^\bullet$ be a second complex.*

1. *Assume $K^i = 0$ for $i \leq -2$. Then $\mathrm{Hom}_{D(R)}(M^\bullet, K^\bullet) = \mathrm{Hom}_{K(R)}(M^\bullet, K^\bullet)$.*
2. *Assume $K^i = 0$ for $i \notin [-1, 0]$ and K^0 a projective R-module. Then for a map of complexes $a^\bullet : M^\bullet \to K^\bullet$, the following are equivalent*
   (a) *$a^\bullet$ induces the zero map $\mathrm{Ext}^1_R(K^\bullet, N) \to \mathrm{Ext}^1_R(M^\bullet, N)$ for all R-modules N, and*
   (b) *there is a map $h^0 : M^0 \to K^{-1}$ such that $a^{-1} + h^0 \circ d_K^{-1} = 0$.*
3. *Assume $K^i = 0$ for $i \leq -3$. Let $\alpha \in \mathrm{Hom}_{D(R)}(M^\bullet, K^\bullet)$. If the composition of α with $K^\bullet \to K^{-2}[2]$ comes from an R-module map $a : M^{-2} \to K^{-2}$ with $a \circ d_M^{-3} = 0$, then α can be represented by a map of complexes $a^\bullet : M^\bullet \to K^\bullet$ with $a^{-2} = a$.*
4. *In (2) for any second map of complexes $(a')^\bullet : M^\bullet \to K^\bullet$ representing α with $a = (a')^{-2}$ there exist $h^i : M^i \to K^{i-1}$ for $i = 0, -1$ such that*

$$h^{-1} \circ d_M^{-2} = 0, \quad (a')^{-1} = a^{-1} + d_K^{-2} \circ h^{-1} + h^0 \circ d_M^{-1}, \quad (a')^0 = a^0 + d_K^{-1} \circ h^0$$

Proof. Set $F^0 = M^0$. Choose a free R-module F^{-1} and a surjection $F^{-1} \to M^{-1}$. Choose a free R-module F^{-2} and a surjection $F^{-2} \to M^{-2} \times_{M^{-1}} F^{-1}$. Continuing in this way we obtain a quasi-isomorphism $p^\bullet : F^\bullet \to M^\bullet$ which is termwise surjective and with F^i projective for all i.

Proof of (1). By Derived Categories, Lemma 19.8 we have

$$\mathrm{Hom}_{D(R)}(M^\bullet, K^\bullet) = \mathrm{Hom}_{K(R)}(F^\bullet, K^\bullet)$$

If $K^i = 0$ for $i \leq -2$, then any morphism of complexes $F^\bullet \to K^\bullet$ factors through $p^\bullet$. Similarly, any homotopy $\{h^i : F^i \to K^{i-1}\}$ factors through $p^\bullet$. Thus (1) holds.

Proof of (2). If (2)(b) holds, then $a^\bullet$ is homotopic to a map of complexes $(a')^\bullet : M^\bullet \to K^\bullet$ which is zero in degree -1. On the other hand, let $N \to I^\bullet$ be an injective resolution. We have

$$\mathrm{Ext}^1_R(K^\bullet, N) = \mathrm{Hom}_{D(R)}(K^\bullet, I^\bullet[1]) = \mathrm{Hom}_{K(R)}(K^\bullet, I^\bullet[1])$$

by Derived Categories, Lemma 18.8. Let $b^\bullet : K^\bullet \to I^\bullet[1]$ be a map of complexes. Since $K^1 = 0$ the map $b^0 : K^0 \to I^1$ maps into the kernel of $I^1 \to I^2$ which is the image of $I^0 \to I^1$. Since K^0 is projective we can lift b^0 to a map $h : K^0 \to I^0$. Thus we see that $b^\bullet$ is homotopic to a map of complexes $(b')^\bullet$ with $(b')^0 = 0$. Since $K^i = 0$ for $i \not\in [-1, 0]$ it follows that $(b')^\bullet \circ (a')^\bullet = 0$ as a map of complexes. Hence the map $\text{Ext}^1_R(K^\bullet, N) \to \text{Ext}^1_R(M^\bullet, N)$ is zero. In this way we see that (2)(b) implies (2)(a). Conversely, assume (2)(a). We see that the canonical element in $\text{Ext}^1_R(K^\bullet, K^{-1})$ maps to zero in $\text{Ext}^1_R(M^\bullet, K^{-1})$. Using (1) we see immediately that we get a map h^0 as in (2)(b).

Proof of (3). Choose $b^\bullet : F^\bullet \to K^\bullet$ representing α. The composition of α with $K^\bullet \to K^{-2}[2]$ is represented by $b^{-2} : F^{-2} \to K^{-2}$. As this is homotopic to $a \circ p^{-2} : F^{-2} \to M^{-2} \to K^{-2}$, there is a map $h : F^{-1} \to K^{-2}$ such that $b^{-2} = a \circ p^{-2} + h \circ d_F^{-2}$. Adjusting $b^\bullet$ by h viewed as a homotopy from $F^\bullet$ to $K^\bullet$, we find that $b^{-2} = a \circ p^{-2}$. Hence b^{-2} factors through p^{-2}. Since $F^0 = M^0$ the kernel of p^{-2} surjects onto the kernel of p^{-1} (for example because the kernel of $p^\bullet$ is an acyclic complex or by a diagram chase). Hence b^{-1} necessarily factors through p^{-1} as well and we see that (3) holds for these factorizations and $a^0 = b^0$.

Proof of (4) is omitted. Hint: There is a homotopy between $a^\bullet \circ p^\bullet$ and $(a')^\bullet \circ p^\bullet$ and we argue as before that this homotopy factors through $p^\bullet$. □

Let $A \to B$ be a finitely presented ring map. Given an ideal $I \subset B$ we can consider the condition

(*) $\text{Ext}^1_B(NL_{B/A}, N)$ is annihilated by I for all B-modules N.

This condition is one possible precise mathematical formulation of the notion "the singular locus of $A \to B$ is scheme theoretically contained in $V(I)$". Please compare with Remark 84.2 and the following lemmas.

Lemma 84.5. *Let R be a ring and let $I \subset R$ be an ideal. Let $K \in D(R)$. Assume $H^i(K) = 0$ for $i \not\in \{-1, 0\}$. The following are equivalent*

1. $\text{Ext}^1_R(K, N)$ *is annihilated by I for all R-modules N,*
2. *K can be represented by a complex $K^{-1} \to K^0$ with K^0 free such that for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ factors through $d_K^{-1} : K^{-1} \to K^0$,*
3. *whenever K is represented by a two term complex $K^{-1} \to K^0$ with K^0 projective, then for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ factors through $d_K^{-1} : K^{-1} \to K^0$.*

If R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then these are also equivalent to

4. $\text{Ext}^1_R(K, N)$ *is annihilated by I for every finite R-module N,*
5. *K can be represented by a complex $K^{-1} \to K^0$ with K^0 finite free and K^{-1} finite such that for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ factors through $d_K^{-1} : K^{-1} \to K^0$.*

Proof. Assume (1) and let $K^{-1} \to K^0$ be a two term complex representing K with K^0 projective. We will use the description of maps in $D(R)$ out of $K^\bullet$ given in Lemma 84.4 without further mention. Choosing $N = K^{-1}$ consider the element ξ of $\text{Ext}^1_R(K, N)$ given by $\text{id}_{K^{-1}} : K^{-1} \to K^{-1}$. Since is annihilated by $a \in I$ we see that we get the dotted arrow fitting into the following commutative diagram

$$\begin{tikzcd} K^{-1} \arrow[r, "d_K^{-1}"'] \arrow[d, "a"] & K^0 \arrow[dl, dotted, "h"] \\ K^{-1} & \end{tikzcd}$$

This proves that (3) holds. Part (3) implies (2) in view of Lemma 84.3 part (1). Assume $K^\bullet$ is as in (2) and N is an arbitrary R-module. Any element ξ of $\text{Ext}^1_R(K, N)$ is given as the class of a map $\varphi : K^{-1} \to N$. Then for $a \in I$ by assumption we may choose a map h as in the diagram above and we see that $a\varphi = \varphi \circ a = \varphi \circ h \circ \mathrm{d}_K^{-1}$ which proves that $a\xi$ is zero in $\text{Ext}^1_R(K, N)$. Thus (1), (2), and (3) are equivalent.

Assume R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$. Part (3) implies (5) in view of Lemma 84.3 part (2). It is clear that (5) implies (2). Trivially (1) implies (4). Thus to finish the proof it suffices to show that (4) implies any of the other conditions. Let $K^{-1} \to K^0$ be a complex representing K with K^0 finite free and K^{-1} finite as in Lemma 84.3 part (2). The argument given in the proof of (2) $\Rightarrow$ (1) shows that if $\text{Ext}^1_R(K, K^{-1})$ is annihilated by I, then (1) holds. In this way we see that (4) implies (1) and the proof is complete. $\square$

Lemma 84.6. *Let R be a ring. Let K be an object of $D(R)$ with $H^i(K) = 0$ for $i \notin \{-1, 0\}$. Let $K^{-1} \to K^0$ be a two term complex of R-modules representing K such that K^0 is a flat R-module (for example projective or free). Let $R \to R'$ be a ring map. Then the complex $K^\bullet \otimes_R R'$ represents $\tau_{\geq -1}(K \otimes_R^{\mathbf{L}} R')$.*

Proof. We have a distinguished triangle

$$K^0 \to K^\bullet \to K^{-1}[1] \to K^0[1]$$

in $D(R)$. This determines a map of distinguished triangles

$$\begin{tikzcd} K^0 \otimes_R^{\mathbf{L}} R' \arrow[r] \arrow[d] & K^\bullet \otimes_R^{\mathbf{L}} R' \arrow[r] \arrow[d] & K^{-1} \otimes_R^{\mathbf{L}} R'[1] \arrow[r] \arrow[d] & K^0 \otimes_R^{\mathbf{L}} R'[1] \arrow[d] \\ K^0 \otimes_R R' \arrow[r] & K^\bullet \otimes_R R' \arrow[r] & K^{-1} \otimes_R R'[1] \arrow[r] & K^0 \otimes_R R'[1] \end{tikzcd}$$

The left and right vertical arrows are isomorphisms as K^0 is flat. Since $K^{-1} \otimes_R^{\mathbf{L}} R' \to K^{-1} \otimes_R R'$ is an isomorphism on cohomology in degree 0 we conclude. $\square$

Lemma 84.7. *Let I be an ideal of a ring R. Let K be an object of $D(R)$ with $H^i(K) = 0$ for $i \notin \{-1, 0\}$. Let $R \to R'$ be a ring map. If K satisfies the equivalent conditions (1), (2), and (3) of Lemma 84.5 with respect to (R, I), then $\tau_{\geq -1}(K \otimes_R^{\mathbf{L}} R')$ satisfies the equivalent conditions (1), (2), and (3) of Lemma 84.5 with respect to (R', IR')*

Proof. We may assume K is represented by a two term complex $K^{-1} \to K^0$ with K^0 free such that for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ is equal to $h_a \circ d_K^{-1}$ for some map $h_a : K^0 \to K^{-1}$. By Lemma 84.6 we see that $\tau_{\geq -1}(K \otimes_R^{\mathbf{L}} R')$ is represented by $K^\bullet \otimes_R R'$. Then of course for every $a \in I$ we see that $a \otimes 1 : K^{-1} \otimes_R R' \to K^{-1} \otimes_R R'$ is equal to $(h_a \otimes 1) \circ (d_K^{-1} \otimes 1)$. Since the collection of maps $K^{-1} \otimes_R R' \to K^{-1} \otimes_R R'$ which factor through $\mathrm{d}_K^{-1} \otimes 1$ forms an R'-module we conclude. $\square$

Lemma 84.8. *Let R be a ring. Let $\alpha : K \to K'$ be a morphism of $D(R)$. Assume*

(1) $H^i(K) = H^i(K') = 0$ *for* $i \not\in \{-1, 0\}$
(2) $H^0(\alpha)$ *is an isomorphism and* $H^{-1}(\alpha)$ *is surjective.*

For any $f \in R$ *if* $f : K \to K$ *is* 0*, then* $f : K' \to K'$ *is* 0*.*

Proof. Set $M = \text{Ker}(H^{-1}(\alpha))$. Then α fits into a distinguished triangle

$$M[1] \to K \to K' \to M[2]$$

Since $K \to K' \xrightarrow{f} K'$ is zero by our assumption, we see that $f : K' \to K'$ factors over a map $M[2] \to K'$. However $\text{Hom}(M[2], K') = 0$ for example by Derived Categories, Lemma 27.3. □

Lemma 84.9. *Let* I *be an ideal of a ring* R*. Let* $\alpha : K \to K'$ *be a morphism of* $D(R)$*. Assume*

(1) $H^i(K) = H^i(K') = 0$ *for* $i \not\in \{-1, 0\}$
(2) $H^0(\alpha)$ *is an isomorphism and* $H^{-1}(\alpha)$ *is surjective.*

If K *satisfies the equivalent conditions (1), (2), and (3) of Lemma 84.5, then* K' *does too.*

Proof. Set $M = \text{Ker}(H^{-1}(\alpha))$. Then α fits into a distinguished trangle

$$M[1] \to K \to K' \to M[2]$$

For any R-module N this determines an exact sequence

$$\text{Ext}^0_R(M[1], N) \to \text{Ext}^1_R(K', N) \to \text{Ext}^1_R(K, N)$$

Since $\text{Ext}^0_R(M[1], N) = \text{Ext}^{-1}_R(M, N) = 0$ we see that $\text{Ext}^1_R(K', N)$ is a submodule of $\text{Ext}^1_R(K, N)$. Hence if $\text{Ext}^1_R(K, N)$ is annihilated by I so is $\text{Ext}^1_R(K', N)$. □

Lemma 84.10. *Let* R *be ring and let* $I \subset R$ *be an ideal. Let* $K \in D(R)$ *with* $H^i(K) = 0$ *for* $i \not\in \{-1, 0\}$*. The following are equivalent*

(1) *there exists a* $c \geq 0$ *such that the equivalent conditions (1), (2), (3) of Lemma 84.5 hold for* K *and the ideal* I^c*,*
(2) *there exists a* $c \geq 0$ *such that (a)* I^c *annihilates* $H^{-1}(K)$ *and (b)* $H^0(K)$ *is an* I^c*-projective module (see Section 70).*

If R *is Noetherian and* $H^i(K)$ *is a finite* R*-module for* $i = -1, 0$*, then these are also equivalent to*

(3) *there exists a* $c \geq 0$ *such that the equivalent conditions (4), (5) of Lemma 84.5 hold for* K *and the ideal* I^c*,*
(4) $H^{-1}(K)$ *is* I*-power torsion and there exist* $f_1, \ldots, f_s \in R$ *with* $V(f_1, \ldots, f_s) \subset V(I)$ *such that the localizations* $H^0(K)_{f_i}$ *are projective* R_{f_i}*-modules,*
(5) $H^{-1}(K)$ *is* I*-power torsion and there exist* $f_1, \ldots, f_s \in I$ *with* $V(f_1, \ldots, f_s) = V(I)$ *such that the localizations* $H^0(K)_{f_i}$ *are projective* R_{f_i}*-modules.*

Proof. The distinguished triangle $H^{-1}(K)[1] \to K \to H^0(K)[0] \to H^{-1}(K)[2]$ determines an exact sequence

$$0 \to \text{Ext}^1_R(H^0(K), N) \to \text{Ext}^1_R(K, N) \to \text{Hom}_R(H^{-1}(K), N) \to \text{Ext}^2_R(H^0(K), N)$$

Thus (2) implies that I^{2c} annihilates $\text{Ext}^1_R(K, N)$ for every R-module N. Assuming (1) we immediately see that $H^0(K)$ is I^c-projective. On the other hand, we may choose an injective map $H^{-1}(K) \to N$ for some injective R-module N. Then this map is the image of an element of $\text{Ext}^1_R(K, N)$ by the vanishing of the Ext^2 in the sequence and we conclude $H^{-1}(K)$ is annihilated by I^c.

Assume R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$. By Lemma 84.5 we see that (3) is equivalent to (1) and (2). Also, if (3) holds then for $f \in I$ the multiplication by f on $H^0(K)$ factors through a projective module, which implies that $H^0(K)_f$ is a summand of a projective R_f-module and hence itself a projective R_f-module. Choosing $f_1, \ldots, f_s$ to be generators of I we find the equivalent conditions (1), (2), and (3) imply (5). Of course (5) trivially implies (4).

Assume (4). Since $H^{-1}(K)$ is a finite R-module and I-power torsion we see that I^{c_1} annihilates $H^{-1}(K)$ for some $c_1 \geq 0$. Choose a short exact sequence

$$0 \to M \to R^{\oplus r} \to H^0(K) \to 0$$

which determines an element $\xi \in \text{Ext}^1_R(H^0(K), M)$. For any $f \in I$ we have $\text{Ext}^1_R(H^0(K), M)_f = \text{Ext}^1_{R_f}(H^0(K)_f, M_f)$ by Lemma 65.4. Hence if $H^0(K)_f$ is projective, then a power of f annihilates ξ. We conclude that ξ is annihilated by $(f_1, \ldots, f_s)^{c_2}$ for some $c_2 \geq 0$. Since $V(f_1, \ldots, f_s) \subset V(I)$ we have $\sqrt{I} \subset (f_1, \ldots, f_s)$ (Algebra, Lemma 17.2). Since R is Noetherian we find $I^{c_3} \subset (f_1, \ldots, f_s)$ for some $c_3 \geq 0$ (Algebra, Lemma 32.5). Hence I^{c2c3} annihilates ξ. This in turn says that $H^0(K)$ is $I^{c_2c_3}$-projective (as multiplication by $a \in I$ which annihilate ξ factor through $R^{\oplus r}$). Hence taking $c = \max(c_1, c_2c_3)$ we see that (2) holds. $\square$

Lemma 84.11. *Let R be a ring. Let $K_j \in D(R)$, $j = 1, 2, 3$ with $H^i(K_j) = 0$ for $i \notin \{-1, 0\}$. Let $\varphi : K_1 \to K_2$ and $\psi : K_2 \to K_3$ be maps in $D(R)$. If $H^0(\varphi) = 0$ and $H^{-1}(\psi) = 0$, then $\varphi \circ \psi = 0$.*

Proof. Apply Derived Categories, Lemma 12.5 to see that $\varphi \circ \psi$ factors through $\tau_{\leq -2}K_2 = 0$. $\square$

Lemma 84.12. *Let R be a ring. Let $K \in D(R)$ be given by a two term complex of the form $R^{\oplus n} \to R^{\oplus n}$. Denote $A \in Mat(n \times n, R)$ the matrix of the differential. Then* $\det(a) : K \to K$ *is zero in $D(R)$.*

Proof. Omitted. Good exercise. $\square$

85. The naive cotangent complex

In this section we continue the discussion started in Algebra, Section 134. We begin with a discussion of base change. The first lemma shows that taking the naive tensor product of the naive cotangent complex with a ring extension isn't quite as naive as one might think.

Lemma 85.1. *Let $R \to S$ and $S \to S'$ be ring maps. The canonical map $NL_{S/R} \otimes_S^{\mathbf{L}} S' \to NL_{S/R} \otimes_S S'$ induces an isomorphism $\tau_{\geq -1}(NL_{S/R} \otimes_S^{\mathbf{L}} S') \to NL_{S/R} \otimes_S S'$ in $D(S')$. Similarly, given a presentation α of S over R the canonical map $NL(\alpha) \otimes_S^{\mathbf{L}} S' \to NL(\alpha) \otimes_S S'$ induces an isomorphism $\tau_{\geq -1}(NL(\alpha) \otimes_S^{\mathbf{L}} S') \to NL(\alpha) \otimes_S S'$ in $D(S')$.*

Proof. Special case of Lemma 84.6. $\square$

Lemma 85.2. *Let $R \to S$ and $R \to R'$ be ring maps. Let $\alpha : P \to S$ be a presentation of S over R. Then $\alpha' : P \otimes_R R' \to S \otimes_R R'$ is a presentation of $S' = S \otimes_R R'$ over R'. The canonical map*

$$NL(\alpha) \otimes_S S' \to NL(\alpha')$$

is an isomorphism on H^0 and surjective on H^{-1}. In particular, the canonical map

$$NL_{S/R} \otimes_S S' \to NL_{S'/R'}$$

is an isomorphism on H^0 and surjective on H^{-1}.

Proof. Denote $I = \mathrm{Ker}(P \to S)$. Denote $P' = P \otimes_R R'$ and $I' = \mathrm{Ker}(P' \to S')$. Suppose P is a polynomial algebra on x_j for $j \in J$. The map displayed in the lemma becomes

$$\begin{tikzcd} \bigoplus_{j\in J} S'\mathrm{d}x_j \arrow[r] & \bigoplus_{j\in J} S'\mathrm{d}x_j \\ I/I^2 \otimes_S S' \arrow[u] \arrow[r] & I'/(I')^2 \arrow[u] \end{tikzcd}$$

where the left column is $NL(\alpha) \otimes_S S'$ and the right column is $NL(\alpha')$. By right exactness of tensor product we see that $I \otimes_R R' \to I'$ is surjective. Hence the bottom arrow is a surjection. This proves the first statement of the lemma. The statement for $NL_{S/R} \otimes_S S' \to NL_{S'/R'}$ follows as these complexes are homotopic to $NL(\alpha) \otimes_S S'$ and $NL(\alpha')$. $\square$

Lemma 85.3. *Consider a cocartesian diagram of rings*

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

If B is flat over A, then the canonical map $NL_{B/A} \otimes_B B' \to NL_{B'/A'}$ is a quasi-isomorphism. If in addition $NL_{B/A}$ has tor-amplitude in $[-1, 0]$ then $NL_{B/A} \otimes^{\mathbf{L}}_B B' \to NL_{B'/A'}$ is a quasi-isomorphism too.

Proof. Choose a presentation $\alpha : P \to B$ as in Algebra, Section 134. Let $I = \mathrm{Ker}(\alpha)$. Set $P' = P \otimes_A A'$ and denote $\alpha' : P' \to B'$ the corresponding presentation of B' over A'. As B is flat over A we see that $I' = \mathrm{Ker}(\alpha')$ is equal to $I \otimes_A A'$. Hence

$$I'/(I')^2 = \mathrm{Coker}(I^2 \otimes_A A' \to I \otimes_A A') = I/I^2 \otimes_A A' = I/I^2 \otimes_B B'$$

We have $\Omega_{P'/A'} = \Omega_{P/A} \otimes_A A'$ because both sides have the same basis. It follows that $\Omega_{P'/A'} \otimes_{P'} B' = \Omega_{P/A} \otimes_P B \otimes_B B'$. This proves that $NL(\alpha) \otimes_B B' \to NL(\alpha')$ is an isomorphism of complexes and hence the first statement holds.

We have

$$NL(\alpha) = I/I^2 \longrightarrow \Omega_{P/A} \otimes_P B$$

as a complex of B-modules with I/I^2 placed in degree -1. Since the term in degree 0 is free, this complex has tor-amplitude in $[-1, 0]$ if and only if I/I^2 is a flat B-module, see Lemma 66.2. If this holds, then $NL(\alpha) \otimes^{\mathbf{L}}_B B' = NL(\alpha) \otimes_B B'$ and we get the second statement. $\square$

Lemma 85.4. *Let $A \to B$ be a local complete intersection as in Definition 33.2. Then $NL_{B/A}$ is a perfect object of $D(B)$ with tor amplitude in $[-1, 0]$.*

Proof. Write $B = A[x_1, \ldots, x_n]/I$. Then $NL_{B/A}$ is represented by the complex

$$I/I^2 \longrightarrow \bigoplus B\mathrm{d}x_i$$

of B-modules with I/I^2 placed in degree -1. Since the term in degree 0 is finite free, this complex has tor-amplitude in $[-1, 0]$ if and only if I/I^2 is a flat B-module, see Lemma 66.2. By definition I is a Koszul regular ideal and hence a quasi-regular ideal, see Section 32. Thus I/I^2 is a finite projective B-module (Lemma 32.3) and we conclude both that $NL_{B/A}$ is perfect and that it has tor amplitude in $[-1, 0]$. □

Lemma 85.5. *Consider a cocartesian diagram of rings*

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

If $A \to B$ and $A' \to B'$ are local complete intersections as in Definition 33.2, then the kernel of $H^{-1}(NL_{B/A} \otimes_B B') \to H^{-1}(NL_{B'/A'})$ is a finite projective B'-module.

Proof. By Lemma 85.4 the complexes $NL_{B/A}$ and $NL_{B'/A'}$ are perfect of tor-amplitude in $[-1, 0]$. Combining Lemmas 85.1, 74.9, and 66.13 we have $NL_{B/A} \otimes_B B' = NL_{B/A} \otimes_B^{\mathbf{L}} B'$ and this complex is also perfect of tor-amplitude in $[-1, 0]$. Choose a distinguished triangle

$$C \to NL_{B/A} \otimes_B B' \to NL_{B'/A'} \to C[1]$$

in $D(B')$. By Lemmas 74.4 and 66.5 we conclude that C is perfect with tor-amplitude in $[-1, 1]$. By Lemma 85.2 the complex C has only one nonzero cohomology module, namely the module of the lemma sitting in degree -1. This module is of finite presentation (Lemma 64.4) and flat (Lemma 66.6). Hence it is finite projective by Algebra, Lemma 78.2. □

86. Rlim of abelian groups

We briefly discuss $R\lim$ on abelian groups. In this section we will denote $Ab(\mathbf{N})$ the abelian category of inverse systems of abelian groups. The notation is compatible with the notation for sheaves of abelian groups on a site, as an inverse system of abelian groups is the same thing as a sheaf of groups on the category $\mathbf{N}$ (with a unique morphism $i \to j$ if $i \leq j$), see Remark 86.6. Many of the arguments in this section duplicate the arguments used to construct the cohomological machinery for sheaves of abelian groups on sites.

Lemma 86.1. *The functor* $\lim : Ab(\mathbf{N}) \to Ab$ *has a right derived functor*

$$R\lim : D(Ab(\mathbf{N})) \longrightarrow D(Ab) \quad (86.1.1)$$

As usual we set $R^p \lim(K) = H^p(R\lim(K))$. Moreover, we have

1. *for any (A_n) in $Ab(\mathbf{N})$ we have $R^p \lim A_n = 0$ for $p > 1$,*
2. *the object $R\lim A_n$ of $D(Ab)$ is represented by the complex*

   $$\prod A_n \to \prod A_n, \quad (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))$$

   sitting in degrees 0 *and* 1*,*
3. *if (A_n) is ML, then $R^1 \lim A_n = 0$, i.e., (A_n) is right acyclic for* $\lim$*,*

(4) *every* $K^\bullet \in D(Ab(\mathbf{N}))$ *is quasi-isomorphic to a complex whose terms are right acyclic for* lim, *and*
(5) *if each* $K^p = (K_n^p)$ *is right acyclic for* lim, *i.e., of* $R^1 \lim_n K_n^p = 0$, *then* $R\lim K$ *is represented by the complex whose term in degree* p *is* $\lim_n K_n^p$.

Proof. Let (A_n) be an arbitrary inverse system. Let (B_n) be the inverse system with

$$B_n = A_n \oplus A_{n-1} \oplus \ldots \oplus A_1$$

and transition maps given by projections. Let $A_n \to B_n$ be given by $(1, f_n, f_{n-1} \circ f_n, \ldots, f_2 \circ \ldots \circ f_n$ where $f_i : A_i \to A_{i-1}$ are the transition maps. In this way we see that every inverse system is a subobject of a ML system (Homology, Section 31). It follows from Derived Categories, Lemma 15.6 using Homology, Lemma 31.3 that every ML system is right acyclic for lim, i.e., (3) holds. This already implies that RF is defined on $D^+(Ab(\mathbf{N}))$, see Derived Categories, Proposition 16.8. Set $C_n = A_{n-1} \oplus \ldots \oplus A_1$ for $n > 1$ and $C_1 = 0$ with transition maps given by projections as well. Then there is a short exact sequence of inverse systems $0 \to (A_n) \to (B_n) \to (C_n) \to 0$ where $B_n \to C_n$ is given by $(x_i) \mapsto (x_i - f_{i+1}(x_{i+1}))$. Since (C_n) is ML as well, we conclude that (2) holds (by proposition reference above) which also implies (1). Finally, this implies by Derived Categories, Lemma 32.2 that $R\lim$ is in fact defined on all of $D(Ab(\mathbf{N}))$. In fact, the proof of Derived Categories, Lemma 32.2 proceeds by proving assertions (4) and (5). $\square$

Lemma 86.2. *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups. Then there is an associated 6 *term exact sequence* $0 \to \lim A_i \to \lim B_i \to \lim C_i \to R^1 \lim A_i \to R^1 \lim B_i \to R^1 \lim C_i \to 0$.

Proof. Follows from the vanishing in Lemma 86.1. $\square$

Here is the "correct" formulation of Homology, Lemma 31.7.

Lemma 86.3. *Let*

$$(A_n^{-2} \to A_n^{-1} \to A_n^0 \to A_n^1)$$

be an inverse system of complexes of abelian groups and denote $A^{-2} \to A^{-1} \to A^0 \to A^1$ *its limit. Denote* (H_n^{-1}), (H_n^0) *the inverse systems of cohomologies, and denote* H^{-1}, H^0 *the cohomologies of* $A^{-2} \to A^{-1} \to A^0 \to A^1$. *If*

(1) (A_n^{-2}) *and* (A_n^{-1}) *have vanishing* $R^1 \lim$,
(2) (H_n^{-1}) *has vanishing* $R^1 \lim$,

then $H^0 = \lim H_n^0$.

Proof. Let $K \in D(Ab(\mathbf{N}))$ be the object represented by the system of complexes whose nth constituent is the complex $A_n^{-2} \to A_n^{-1} \to A_n^0 \to A_n^1$. We will compute $H^0(R\lim K)$ using both spectral sequences[10] of Derived Categories, Lemma 21.3. The first has E_1-page

$$\begin{array}{cccc} 0 & 0 & R^1 \lim A_n^0 & R^1 \lim A_n^1 \\ A^{-2} & A^{-1} & A^0 & A^1 \end{array}$$

[10] To use these spectral sequences we have to show that $Ab(\mathbf{N})$ has enough injectives. A inverse system (I_n) of abelian groups is injective if and only if each I_n is an injective abelian group and the transition maps are split surjections. Every system embeds in one of these. Details omitted.

with horizontal differentials and all higher differentials are zero. The second has E_2 page

$$\begin{matrix} R^1\lim H_n^{-2} & 0 & R^1\lim H_n^0 & R^1\lim H_n^1 \\ \lim H_n^{-2} & \lim H_n^{-1} & \lim H_n^0 & \lim H_n^1 \end{matrix}$$

and degenerates at this point. The result follows. □

Lemma 86.4. *Let $\mathcal{D}$ be a triangulated category. Let (K_n) be an inverse system of objects of $\mathcal{D}$. Let K be a derived limit of the system (K_n). Then for every L in $\mathcal{D}$ we have a short exact sequence*

$$0 \to R^1\lim \mathrm{Hom}_\mathcal{D}(L, K_n[-1]) \to \mathrm{Hom}_\mathcal{D}(L, K) \to \lim \mathrm{Hom}_\mathcal{D}(L, K_n) \to 0$$

Proof. This follows from Derived Categories, Definition 34.1 and Lemma 4.2, and the description of lim and $R^1\lim$ in Lemma 86.1 above. □

Lemma 86.5. *Let $\mathcal{D}$ be a triangulated category. Let (K_n) be a system of objects of $\mathcal{D}$. Let K be a derived colimit of the system (K_n). Then for every L in $\mathcal{D}$ we have a short exact sequence*

$$0 \to R^1\lim \mathrm{Hom}_\mathcal{D}(K_n, L[-1]) \to \mathrm{Hom}_\mathcal{D}(K, L) \to \lim \mathrm{Hom}_\mathcal{D}(K_n, L) \to 0$$

Proof. This follows from Derived Categories, Definition 33.1 and Lemma 4.2, and the description of lim and $R^1\lim$ in Lemma 86.1 above. □

Remark 86.6 (Rlim as cohomology)**.** Consider the category $\mathbf{N}$ whose objects are natural numbers and whose morphisms are unique arrows $i \to j$ if $j \geq i$. Endow $\mathbf{N}$ with the chaotic topology (Sites, Example 6.6) so that a sheaf $\mathcal{F}$ is the same thing as an inverse system

$$\mathcal{F}_1 \leftarrow \mathcal{F}_2 \leftarrow \mathcal{F}_3 \leftarrow \ldots$$

of sets over $\mathbf{N}$. Note that $\Gamma(\mathbf{N}, \mathcal{F}) = \lim \mathcal{F}_n$. For an inverse system of abelian groups $\mathcal{F}_n$ we have

$$R^p\lim \mathcal{F}_n = H^p(\mathbf{N}, \mathcal{F})$$

because both sides are the higher right derived functors of $\mathcal{F} \mapsto \lim \mathcal{F}_n = H^0(\mathbf{N}, \mathcal{F})$. Thus the existence of $R\lim$ also follows from the general material in Cohomology on Sites, Sections 2 and 19.

The products in the following lemma can be seen as termwise products of complexes or as products in the derived category $D(Ab)$, see Derived Categories, Lemma 34.2.

Lemma 86.7. *Let $K = (K_n^\bullet)$ be an object of $D(Ab(\mathbf{N}))$. There exists a canonical distinguished triangle*

$$R\lim K \to \prod\nolimits_n K_n^\bullet \to \prod\nolimits_n K_n^\bullet \to R\lim K[1]$$

in $D(Ab)$. In other words, $R\lim K$ is a derived limit of the inverse system $(K_n^\bullet)$ of $D(Ab)$, see Derived Categories, Definition 34.1.

Proof. Suppose that for each p the inverse system (K_n^p) is right acyclic for lim. By Lemma 86.1 this gives a short exact sequence

$$0 \to \lim_n K_n^p \to \prod\nolimits_n K_n^p \to \prod\nolimits_n K_n^p \to 0$$

for each p. Since the complex consisting of $\lim_n K_n^p$ computes $R\lim K$ by Lemma 86.1 we see that the lemma holds in this case.

Next, assume $K = (K_n^\bullet)$ is general. By Lemma 86.1 there is a quasi-isomorphism $K \to L$ in $D(Ab(\mathbf{N}))$ such that (L_n^p) is acyclic for each p. Then $\prod K_n^\bullet$ is quasi-isomorphic to $\prod L_n^\bullet$ as products are exact in Ab, whence the result for L (proved above) implies the result for K. $\square$

Lemma 86.8. *With notation as in Lemma 86.7 the long exact cohomology sequence associated to the distinguished triangle breaks up into short exact sequences*

$$0 \to R^1 \lim_n H^{p-1}(K_n^\bullet) \to H^p(R\lim K) \to \lim_n H^p(K_n^\bullet) \to 0$$

Proof. The long exact sequence of the distinguished triangle is

$$\ldots \to H^p(R\lim K) \to \prod\nolimits_n H^p(K_n^\bullet) \to \prod\nolimits_n H^p(K_n^\bullet) \to H^{p+1}(R\lim K) \to \ldots$$

The map in the middle has kernel $\lim_n H^p(K_n^\bullet)$ by its explicit description given in the lemma. The cokernel of this map is $R^1 \lim_n H^p(K_n^\bullet)$ by Lemma 86.1. $\square$

Warning. An object of $D(Ab(\mathbf{N}))$ is a complex of inverse systems of abelian groups. You can also think of this as an inverse system $(K_n^\bullet)$ of complexes. However, this is **not** the same thing as an inverse system of objects of $D(Ab)$; the following lemma and remark explain the difference.

Lemma 86.9. *Let (K_n) be an inverse system of objects of $D(Ab)$. Then there exists an object $M = (M_n^\bullet)$ of $D(Ab(\mathbf{N}))$ and isomorphisms $M_n^\bullet \to K_n$ in $D(Ab)$ such that the diagrams*

$$\begin{tikzcd} M_{n+1}^\bullet \arrow[r] \arrow[d] & M_n^\bullet \arrow[d] \\ K_{n+1} \arrow[r] & K_n \end{tikzcd}$$

commute in $D(Ab)$.

Proof. Namely, let $M_1^\bullet$ be a complex of abelian groups representing K_1. Suppose we have constructed $M_e^\bullet \to M_{e-1}^\bullet \to \ldots \to M_1^\bullet$ and maps $\psi_i : M_i^\bullet \to K_i$ such that the diagrams in the statement of the lemma commute for all $n < e$. Then we consider the diagram

$$\begin{tikzcd} & M_n^\bullet \arrow[d, "\psi_n"] \\ K_{n+1} \arrow[r] & K_n \end{tikzcd}$$

in $D(Ab)$. By the definition of morphisms in $D(Ab)$ we can find a complex $M_{n+1}^\bullet$ of abelian groups, an isomorphism $M_{n+1}^\bullet \to K_{n+1}$ in $D(Ab)$, and a morphism of complexes $M_{n+1}^\bullet \to M_n^\bullet$ representing the composition

$$K_{n+1} \to K_n \xrightarrow{\psi_n^{-1}} M_n^\bullet$$

in $D(Ab)$. Thus the lemma holds by induction. $\square$

Remark 86.10. Let (K_n) be an inverse system of objects of $D(Ab)$. Let $K = R\lim K_n$ be a derived limit of this system (see Derived Categories, Section 34). Such a derived limit exists because $D(Ab)$ has countable products (Derived Categories, Lemma 34.2). By Lemma 86.9 we can also lift (K_n) to an object M of $D(\mathbf{N})$. Then $K \cong R\lim M$ where $R\lim$ is the functor (86.1.1) because $R\lim M$ is also a derived limit of the system (K_n) by Lemma 86.7. Thus, although there may be many isomorphism classes of lifts M of the system (K_n), the isomorphism type of $R\lim M$ is independent of the choice because it is isomorphic to the derived limit $K = R\lim K_n$ of the system. Thus we may apply results on $R\lim$ proved in this section to derived limits. For example, for every $p \in \mathbf{Z}$ there is a canonical short exact sequence

$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(K) \to \lim H^p(K_n) \to 0$$

because we may apply Lemma 86.7 to M. This can also been seen directly, without invoking the existence of M, by applying the argument of the proof of Lemma 86.7 to the (defining) distinguished triangle $K \to \prod K_n \to \prod K_n \to K[1]$.

Lemma 86.11. *Let $E \to D$ be a morphism of $D(Ab(\mathbf{N}))$. Let (E_n), resp. (D_n) be the system of objects of $D(Ab)$ associated to E, resp. D. If $(E_n) \to (D_n)$ is an isomorphism of pro-objects, then $R\lim E \to R\lim D$ is an isomorphism in $D(Ab)$.*

Proof. The assumption in particular implies that the pro-objects $H^p(E_n)$ and $H^p(D_n)$ are isomorphic. By the short exact sequences of Lemma 86.8 it suffices to show that given a map $(A_n) \to (B_n)$ of inverse systems of abelian groupsc which induces an isomorphism of pro-objects, then $\lim A_n \cong \lim B_n$ and $R^1 \lim A_n \cong R^1 \lim B_n$.

The assumption implies there are $1 \leq m_1 < m_2 < m_3 < \ldots$ and maps $\varphi_n : B_{m_n} \to A_n$ such that $(\varphi_n) : (B_{m_n}) \to (A_n)$ is a map of systems which is inverse to the given map $\psi = (\psi_n) : (A_n) \to (B_n)$ as a morphism of pro-objects. What this means is that (after possibly replacing m_n by larger integers) we may assume that the compositions $A_{m_n} \to B_{m_n} \to A_n$ and $B_{m_n} \to A_n \to B_n$ are equal to the transition maps of the inverse systems. Now, if $(b_n) \in \lim B_n$ we can set $a_n = \varphi_{m_n}(b_{m_n})$. This defines an inverse $\lim B_n \to \lim A_n$ (computation omitted). Let us use the cokernel of the map

$$\prod B_n \longrightarrow \prod B_n$$

as an avatar of $R^1 \lim B_n$ (Lemma 86.1). Any element in this cokernel can be represented by an element (b_i) with $b_i = 0$ if $i \neq m_n$ for some n (computation omitted). We can define a map $R^1 \lim B_n \to R^1 \lim A_n$ by mapping the class of such a special element (b_n) to the class of $(\varphi_n(b_{m_n}))$. We omit the verification this map is inverse to the map $R^1 \lim A_n \to R^1 \lim B_n$. □

Lemma 86.12 (Emmanouil)**.** *Let (A_n) be an inverse system of abelian groups. The following are equivalent*

1. *(A_n) is Mittag-Leffler,*
2. *$R^1 \lim A_n = 0$ and the same holds for $\bigoplus_{i \in \mathbf{N}}(A_n)$.*

Proof. Set $B = \bigoplus_{i \in \mathbf{N}}(A_n)$ and hence $B = (B_n)$ with $B_n = \bigoplus_{i \in \mathbf{N}} A_n$. If (A_n) is ML, then B is ML and hence $R^1 \lim A_n = 0$ and $R^1 \lim B_n = 0$ by Lemma 86.1.

Conversely, assume (A_n) is not ML. Then we can pick an m and a sequence of integers $m < m_1 < m_2 < \ldots$ and elements $x_i \in A_{m_i}$ whose image $y_i \in A_m$ is not in the image of $A_{m_i+1} \to A_m$. We will use the elements x_i and y_i to show that $R^1 \lim B_n \neq 0$ in two ways. This will finish the proof of the lemma.

First proof. Set $C = (C_n)$ with $C_n = \prod_{i \in \mathbf{N}} A_n$. There is a canonical injective map $B_n \to C_n$ with cokernel Q_n. Set $Q = (Q_n)$. We may and do think of elements q_n of

Q_n as sequences of elements $q_n = (q_{n,1}, q_{n,2}, \ldots)$ with $q_{n,i} \in A_n$ modulo sequences whose tail is zero (in other words, we identify sequences which differ in finitely many places). We have a short exact sequence of inverse systems

$$0 \to (B_n) \to (C_n) \to (Q_n) \to 0$$

Consider the element $q_n \in Q_n$ given by

$$q_{n,i} = \begin{cases} \text{image of } x_i & \text{if} \quad m_i \geq n \\ 0 & \text{else} \end{cases}$$

Then it is clear that q_{n+1} maps to q_n. Hence we obtain $q = (q_n) \in \lim Q_n$. On the other hand, we claim that q is not in the image of $\lim C_n \to \lim Q_n$. Namely, say that $c = (c_n)$ maps to q. Then we can write $c_n = (c_{n,i})$ and since $c_{n',i} \mapsto c_{n,i}$ for $n' \geq n$, we see that $c_{n,i} \in \mathrm{Im}(C_{n'} \to C_n)$ for all $n, i, n' \geq n$. In particular, the image of $c_{m,i}$ in A_m is in $\mathrm{Im}(A_{m_i+1} \to A_m)$ whence cannot be equal to y_i. Thus c_m and $q_m = (y_1, y_2, y_3, \ldots)$ differ in infinitely many spots, which is a contradiction. Considering the long exact cohomology sequence

$$0 \to \lim B_n \to \lim C_n \to \lim Q_n \to R^1 \lim B_n$$

we conclude that the last group is nonzero as desired.

Second proof. For $n' \geq n$ we denote $A_{n,n'} = \mathrm{Im}(A_{n'} \to A_n)$. Then we have $y_i \in A_m$, $y_i \notin A_{m,m_i+1}$. Let $\xi = (\xi_n) \in \prod B_n$ be the element with $\xi_n = 0$ unless $n = m_i$ and $\xi_{m_i} = (0, \ldots, 0, x_i, 0, \ldots)$ with x_i placed in the ith summand. We claim that ξ is not in the image of the map $\prod B_n \to \prod B_n$ of Lemma 86.1. This shows that $R^1 \lim B_n$ is nonzero and finishes the proof. Namely, suppose that ξ is the image of $\eta = (z_1, z_2, \ldots)$ with $z_n = \sum z_{n,i} \in \bigoplus_i A_n$. Observe that $x_i = z_{m_i,i} \bmod A_{m_i,m_i+1}$. Then $z_{m_i-1,i}$ is the image of $z_{m_i,i}$ under $A_{m_i} \to A_{m_i-1}$, and so on, and we conclude that $z_{m,i}$ is the image of $z_{m_i,i}$ under $A_{m_i} \to A_m$. We conclude that $z_{m,i}$ is congruent to y_i modulo A_{m,m_i+1}. In particular $z_{m,i} \neq 0$. This is impossible as $\sum z_{m,i} \in \bigoplus_i A_m$ hence only a finite number of $z_{m,i}$ can be nonzero. ☐

Lemma 86.13. *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups. If (A_i) and (C_i) are ML, then so is (B_i).

Proof. This follows from Lemma 86.12, the fact that taking infinite direct sums is exact, and the long exact sequence of cohomology associated to $R\lim$. ☐

Lemma 86.14. *Let (A_n) be an inverse system of abelian groups. The following are equivalent*

1. *(A_n) is zero as a pro-object,*
2. *$\lim A_n = 0$ and $R^1 \lim A_n = 0$ and the same holds for $\bigoplus_{i \in \mathbf{N}} (A_n)$.*

Proof. It follows from Lemma 86.11 that (1) implies (2). Assume (2). Then (A_n) is ML by Lemma 86.12. For $m \geq n$ let $A_{n,m} = \mathrm{Im}(A_m \to A_n)$ so that $A_n = A_{n,n} \supset A_{n,n+1} \supset \ldots$. Note that (A_n) is zero as a pro-object if and only if for every n there is an $m \geq n$ such that $A_{n,m} = 0$. Note that (A_n) is ML if and only if for every n there is an $m_n \geq n$ such that $A_{n,m} = A_{n,m+1} = \ldots$. In the ML case it is clear that $\lim A_n = 0$ implies that $A_{n,m_n} = 0$ because the maps $A_{n+1,m_{n+1}} \to A_{n,m}$ are surjective. This finishes the proof. ☐

87. Rlim of modules

We briefly discuss $R\lim$ on modules. Many of the arguments in this section duplicate the arguments used to construct the cohomological machinery for modules on ringed sites.

Let (A_n) be an inverse system of rings. We will denote $\textit{Mod}(\mathbf{N}, (A_n))$ the category of inverse systems (M_n) of abelian groups such that each M_n is given the structure of a A_n-module and the transition maps $M_{n+1} \to M_n$ are A_{n+1}-module maps. This is an abelian category. Set $A = \lim A_n$. Given an object (M_n) of $\textit{Mod}(\mathbf{N}, (A_n))$ the limit $\lim M_n$ is an A-module.

Lemma 87.1. *In the situation above. The functor* $\lim : \textit{Mod}(\mathbf{N}, (A_n)) \to \textit{Mod}_A$ *has a right derived functor*

$$R\lim : D(\textit{Mod}(\mathbf{N}, (A_n))) \longrightarrow D(A)$$

As usual we set $R^p\lim(K) = H^p(R\lim(K))$*. Moreover, we have*

1. *for any* (M_n) *in* $\textit{Mod}(\mathbf{N}, (A_n))$ *we have* $R^p\lim M_n = 0$ *for* $p > 1$,
2. *the object* $R\lim M_n$ *of* $D(\textit{Mod}_A)$ *is represented by the complex*
$$\prod M_n \to \prod M_n, \quad (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))$$
sitting in degrees 0 *and* 1,
3. *if* (M_n) *is ML, then* $R^1\lim M_n = 0$*, i.e.,* (M_n) *is right acyclic for* $\lim$,
4. *every* $K^\bullet \in D(\textit{Mod}(\mathbf{N}, (A_n)))$ *is quasi-isomorphic to a complex whose terms are right acyclic for* $\lim$*, and*
5. *if each* $K^p = (K_n^p)$ *is right acyclic for* $\lim$*, i.e., of* $R^1\lim_n K_n^p = 0$*, then* $R\lim K$ *is represented by the complex whose term in degree* p *is* $\lim_n K_n^p$.

Proof. The proof of this is word for word the same as the proof of Lemma 86.1. □

Remark 87.2. This remark is a continuation of Remark 86.6. A sheaf of rings on $\mathbf{N}$ is just an inverse system of rings (A_n). A sheaf of modules over (A_n) is exactly the same thing as an object of the category $\textit{Mod}(\mathbf{N}, (A_n))$ defined above. The derived functor $R\lim$ of Lemma 87.1 is simply $R\Gamma(\mathbf{N}, -)$ from the derived category of modules to the derived category of modules over the global sections of the structure sheaf. It is true in general that cohomology of groups and modules agree, see Cohomology on Sites, Lemma 12.4.

The products in the following lemma can be seen as termwise products of complexes or as products in the derived category $D(A)$, see Derived Categories, Lemma 34.2.

Lemma 87.3. *Let* $K = (K_n^\bullet)$ *be an object of* $D(\textit{Mod}(\mathbf{N}, (A_n)))$*. There exists a canonical distinguished triangle*

$$R\lim K \to \prod\nolimits_n K_n^\bullet \to \prod\nolimits_n K_n^\bullet \to R\lim K[1]$$

in $D(A)$*. In other words,* $R\lim K$ *is a derived limit of the inverse system* $(K_n^\bullet)$ *of* $D(A)$*, see Derived Categories, Definition 34.1.*

Proof. The proof is exactly the same as the proof of Lemma 86.7 using Lemma 87.1 in stead of Lemma 86.1. □

Lemma 87.4. *With notation as in Lemma 87.3 the long exact cohomology sequence associated to the distinguished triangle breaks up into short exact sequences*

$$0 \to R^1 \lim_n H^{p-1}(K_n^\bullet) \to H^p(R\lim K) \to \lim_n H^p(K_n^\bullet) \to 0$$

of A-modules.

Proof. The proof is exactly the same as the proof of Lemma 86.8 using Lemma 87.1 in stead of Lemma 86.1. □

Warning. As in the case of abelian groups an object $M = (M_n^\bullet)$ of $D(\mathit{Mod}(\mathbf{N}, (A_n)))$ is an inverse system of complexes of modules, which is **not** the same thing as an inverse system of objects in the derived categories. In the following lemma we show how an inverse system of objects in derived categories always lifts to an object of $D(\mathit{Mod}(\mathbf{N}, (A_n)))$.

Lemma 87.5. *Let (A_n) be an inverse system of rings. Suppose that we are given*

(1) *for every n an object K_n of $D(A_n)$, and*
(2) *for every n a map $\varphi_n : K_{n+1} \to K_n$ of $D(A_{n+1})$ where we think of K_n as an object of $D(A_{n+1})$ by restriction via $A_{n+1} \to A_n$.*

There exists an object $M = (M_n^\bullet) \in D(\mathit{Mod}(\mathbf{N}, (A_n)))$ and isomorphisms $\psi_n : M_n^\bullet \to K_n$ in $D(A_n)$ such that the diagrams

$$\begin{tikzcd} M_{n+1}^\bullet \arrow[r] \arrow[d, "\psi_{n+1}"'] & M_n^\bullet \arrow[d, "\psi_n"] \\ K_{n+1} \arrow[r, "\varphi_n"] & K_n \end{tikzcd}$$

commute in $D(A_{n+1})$.

Proof. We write out the proof in detail. For an A_n-module T we write $T_{A_{n+1}}$ for the same module viewd as an A_{n+1}-module. Suppose that $K_n^\bullet$ is a complex of A_n-modules representing K_n. Then $K_{n,A_{n+1}}^\bullet$ is the same complex, but viewed as a complex of A_{n+1}-modules. By the construction of the derived category, the map ψ_n can be given as

$$\psi_n = \tau_n \circ \sigma_n^{-1}$$

where $\sigma_n : L_{n+1}^\bullet \to K_{n+1}^\bullet$ is a quasi-isomorphism of complexes of A_{n+1}-modules and $\tau_n : L_{n+1}^\bullet \to K_{n,A_{n+1}}^\bullet$ is a map of complexes of A_{n+1}-modules.

Now we construct the complexes $M_n^\bullet$ by induction. As base case we let $M_1^\bullet = K_1^\bullet$. Suppose we have already constructed $M_e^\bullet \to M_{e-1}^\bullet \to \ldots \to M_1^\bullet$ and maps of complexes $\psi_i : M_i^\bullet \to K_i^\bullet$ such that the diagrams

$$\begin{tikzcd} M_{n+1}^\bullet \arrow[rr] \arrow[d, "\psi_{n+1}"'] & & M_{n,A_{n+1}}^\bullet \arrow[d, "\psi_{n,A_{n+1}}"] \\ K_{n+1}^\bullet & L_{n+1}^\bullet \arrow[l, "\sigma_n"'] \arrow[r, "\tau_n"] & K_{n,A_{n+1}}^\bullet \end{tikzcd}$$

above commute in $D(A_{n+1})$ for all $n < e$. Then we consider the diagram

$$\begin{tikzcd} & & M^\bullet_{e, A_{e+1}} \arrow[d, "\psi_{e, A_{e+1}}"] \\ K^\bullet_{e+1} & L^\bullet_{e+1} \arrow[l, "\sigma_e"'] \arrow[r, "\tau_e"] & K^\bullet_{e, A_{e+1}} \end{tikzcd}$$

in $D(A_{e+1})$. Because ψ_e is a quasi-isomorphism, we see that $\psi_{e, A_{e+1}}$ is a quasi-isomorphism too. By the definition of morphisms in $D(A_{e+1})$ we can find a quasi-isomorphism $\psi_{e+1} : M^\bullet_{e+1} \to K^\bullet_{e+1}$ of complexes of A_{e+1}-modules such that there exists a morphism of complexes $M^\bullet_{e+1} \to M^\bullet_{e, A_{e+1}}$ of A_{e+1}-modules representing the composition $\psi_{e, A_{e+1}}^{-1} \circ \tau_e \circ \sigma_e^{-1}$ in $D(A_{e+1})$. Thus the lemma holds by induction. $\square$

Remark 87.6. With assumptions as in Lemma 87.5. A priori there are many isomorphism classes of objects M of $D(Mod(\mathbf{N}, (A_n)))$ which give rise to the system (K_n, φ_n) of the lemma. For each such M we can consider the complex $R\lim M \in D(A)$ where $A = \lim A_n$. By Lemma 87.3 we see that $R\lim M$ is a derived limit of the inverse system (K_n) of $D(A)$. Hence we see that the isomorphism class of $R\lim M$ in $D(A)$ is independent of the choices made in constructing M. In particular, we may apply results on $R\lim$ proved in this section to derived limits of inverse systems in $D(A)$. For example, for every $p \in \mathbf{Z}$ there is a canonical short exact sequence

$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(R\lim K_n) \to \lim H^p(K_n) \to 0$$

because we may apply Lemma 87.3 to M. This can also been seen directly, without invoking the existence of M, by applying the argument of the proof of Lemma 87.3 to the (defining) distinguished triangle $R\lim K_n \to \prod K_n \to \prod K_n \to (R\lim K_n)[1]$ of the derived limit.

Lemma 87.7. *Let (A_n) be an inverse system of rings. Every $K \in D(Mod(\mathbf{N}, (A_n)))$ can be represented by a system of complexes $(M_n^\bullet)$ such that all the transition maps $M_{n+1}^\bullet \to M_n^\bullet$ are surjective.*

Proof. Let K be represented by the system $(K_n^\bullet)$. Set $M_1^\bullet = K_1^\bullet$. Suppose we have constructed surjective maps of complexes $M_n^\bullet \to M_{n-1}^\bullet \to \ldots \to M_1^\bullet$ and homotopy equivalences $\psi_e : K_e^\bullet \to M_e^\bullet$ such that the diagrams

$$\begin{tikzcd} K^\bullet_{e+1} \arrow[r] \arrow[d] & K^\bullet_e \arrow[d] \\ M^\bullet_{e+1} \arrow[r] & M^\bullet_e \end{tikzcd}$$

commute for all $e < n$. Then we consider the diagram

$$\begin{tikzcd} K^\bullet_{n+1} \arrow[r] & K^\bullet_n \arrow[d] \\ & M^\bullet_n \end{tikzcd}$$

By Derived Categories, Lemma 9.8 we can factor the composition $K^\bullet_{n+1} \to M^\bullet_n$ as $K^\bullet_{n+1} \to M^\bullet_{n+1} \to M^\bullet_n$ such that the first arrow is a homotopy equivalence and the second a termwise split surjection. The lemma follows from this and induction. $\square$

Lemma 87.8. *Let (A_n) be an inverse system of rings. Every $K \in D(Mod(\mathbf{N}, (A_n)))$ can be represented by a system of complexes $(K_n^\bullet)$ such that each $K_n^\bullet$ is K-flat.*

Proof. First use Lemma 87.7 to represent K by a system of complexes $(M_n^\bullet)$ such that all the transition maps $M_{n+1}^\bullet \to M_n^\bullet$ are surjective. Next, let $K_1^\bullet \to M_1^\bullet$ be a quasi-isomorphism with $K_1^\bullet$ a K-flat complex of A_1-modules (Lemma 59.10). Suppose we have constructed $K_n^\bullet \to K_{n-1}^\bullet \to \ldots \to K_1^\bullet$ and maps of complexes $\psi_e : K_e^\bullet \to M_e^\bullet$ such that

$$\begin{tikzcd} K_{e+1}^\bullet \arrow[r] \arrow[d] & K_e^\bullet \arrow[d] \\ M_{e+1}^\bullet \arrow[r] & M_e^\bullet \end{tikzcd}$$

commutes for all $e < n$. Then we consider the diagram

$$\begin{tikzcd} C^\bullet \arrow[r, dotted] \arrow[d, dotted] & K_n^\bullet \arrow[d, "\psi_n"] \\ M_{n+1}^\bullet \arrow[r, "\varphi_n"] & M_n^\bullet \end{tikzcd}$$

in $D(A_{n+1})$. As $M_{n+1}^\bullet \to M_n^\bullet$ is termwise surjective, the complex $C^\bullet$ fitting into the left upper corner with terms

$$C^p = M_{n+1}^p \times_{M_n^p} K_n^p$$

is quasi-isomorphic to $M_{n+1}^\bullet$ (details omitted). Choose a quasi-isomorphism $K_{n+1}^\bullet \to C^\bullet$ with $K_{n+1}^\bullet$ K-flat. Thus the lemma holds by induction. $\square$

Lemma 87.9. *Let (A_n) be an inverse system of rings. Given $K, L \in D(Mod(\mathbf{N}, (A_n)))$ there is a canonical derived tensor product $K \otimes^{\mathbf{L}} L$ in $D(\mathbf{N}, (A_n))$ compatible with the maps to $D(A_n)$. The construction is symmetric in K and L and an exact functor of triangulated categories in each variable.*

Proof. Choose a representative $(K_n^\bullet)$ for K such that each $K_n^\bullet$ is a K-flat complex (Lemma 87.8). Then you can define $K \otimes^{\mathbf{L}} L$ as the object represented by the system of complexes

$$(\text{Tot}(K_n^\bullet \otimes_{A_n} L_n^\bullet))$$

for any choice of representative $(L_n^\bullet)$ for L. This is well defined in both variables by Lemmas 59.2 and 59.12. Compatibility with the map to $D(A_n)$ is clear. Exactness follows exactly as in Lemma 58.4. $\square$

Remark 87.10. Let A be a ring. Let (E_n) be an inverse system of objects of $D(A)$. We've seen above that a derived limit $R\lim E_n$ exists. Thus for every object K of $D(A)$ also the derived limit $R\lim(K \otimes_A^{\mathbf{L}} E_n)$ exists. It turns out that we can construct these derived limits functorially in K and obtain an exact functor

$$R\lim(- \otimes_A^{\mathbf{L}} E_n) : D(A) \longrightarrow D(A)$$

of triangulated categories. Namely, we first lift (E_n) to an object E of $D(\mathbf{N}, A)$, see Lemma 87.5. (The functor will depend on the choice of this lift.) Next, observe that there is a "diagonal" or "constant" functor

$$\Delta : D(A) \longrightarrow D(\mathbf{N}, A)$$

mapping the complex $K^\bullet$ to the constant inverse system of complexes with value $K^\bullet$. Then we simply define

$$R\lim(K \otimes_A^{\mathbf{L}} E_n) = R\lim(\Delta(K) \otimes^{\mathbf{L}} E)$$

where on the right hand side we use the functor $R\lim$ of Lemma 87.1 and the functor $- \otimes^{\mathbf{L}} -$ of Lemma 87.9.

Lemma 87.11. *Let A be a ring. Let $E \to D \to F \to E[1]$ be a distinguished triangle of $D(\mathbf{N}, A)$. Let (E_n), resp. (D_n), resp. (F_n) be the system of objects of $D(A)$ associated to E, resp. D, resp. F. Then for every $K \in D(A)$ there is a canonical distinguished triangle*

$$R\lim(K \otimes_A^{\mathbf{L}} E_n) \to R\lim(K \otimes_A^{\mathbf{L}} D_n) \to R\lim(K \otimes_A^{\mathbf{L}} F_n) \to R\lim(K \otimes_A^{\mathbf{L}} E_n)[1]$$

in $D(A)$ with notation as in Remark 87.10.

Proof. This is clear from the construction in Remark 87.10 and the fact that $\Delta : D(A) \to D(\mathbf{N}, A)$, $- \otimes^{\mathbf{L}} -$, and $R\lim$ are exact functors of triangulated categories. $\square$

Lemma 87.12. *Let A be a ring. Let $E \to D$ be a morphism of $D(\mathbf{N}, A)$. Let (E_n), resp. (D_n) be the system of objects of $D(A)$ associated to E, resp. D. If $(E_n) \to (D_n)$ is an isomorphism of pro-objects, then for every $K \in D(A)$ the corresponding map*

$$R\lim(K \otimes_A^{\mathbf{L}} E_n) \longrightarrow R\lim(K \otimes_A^{\mathbf{L}} D_n)$$

in $D(A)$ is an isomorphism (notation as in Remark 87.10).

Proof. Follows from the definitions and Lemma 86.11. $\square$

88. Torsion modules

In this section "torsion modules" will refer to modules supported on a given closed subset $V(I)$ of an affine scheme $\text{Spec}(R)$. This is different, but analogous to, the notion of a torsion module over a domain (Definition 22.1).

Definition 88.1. Let R be a ring. Let M be an R-module.

(1) Let $I \subset R$ be an ideal. We say M is an *I-power torsion module* if for every $m \in M$ there exists an $n > 0$ such that $I^n m = 0$.
(2) Let $f \in R$. We say M is *an f-power torsion module* if for each $m \in M$, there exists an $n > 0$ such that $f^n m = 0$.

Thus an f-power torsion module is the same thing as an I-power torsion module for $I = (f)$. We will use the notation

$$M[I^n] = \{m \in M \mid I^n m = 0\}$$

and

$$M[I^\infty] = \bigcup M[I^n]$$

for an R-module M. Thus M is I-power torsion if and only if $M = M[I^\infty]$ if and only if $M = \bigcup M[I^n]$.

Lemma 88.2. *Let R be a ring. Let I be an ideal of R. Let M be an I-power torsion module. Then M admits a resolution*

$$\ldots \to K_2 \to K_1 \to K_0 \to M \to 0$$

with each K_i a direct sum of copies of R/I^n for n variable.

Proof. There is a canonical surjection

$$\oplus_{m \in M} R/I^{n_m} \to M \to 0$$

where n_m is the smallest positive integer such that $I^{n_m} \cdot m = 0$. The kernel of the preceding surjection is also an I-power torsion module. Proceeding inductively, we construct the desired resolution of M. $\square$

Lemma 88.3. *Let R be a ring. Let I be an ideal of R. For any R-module M set $M[I^n] = \{m \in M \mid I^n m = 0\}$. If I is finitely generated then the following are equivalent*

1. *$M[I] = 0$,*
2. *$M[I^n] = 0$ for all $n \geq 1$, and*
3. *if $I = (f_1, \ldots, f_t)$, then the map $M \to \bigoplus M_{f_i}$ is injective.*

Proof. This follows from Algebra, Lemma 24.4. $\square$

Lemma 88.4. *Let R be a ring. Let I be a finitely generated ideal of R.*

1. *For any R-module M we have $(M/M[I^\infty])[I] = 0$.*
2. *An extension of I-power torsion modules is I-power torsion.*

Proof. Let $m \in M$. If m maps to an element of $(M/M[I^\infty])[I]$ then $Im \subset M[I^\infty]$. Write $I = (f_1, \ldots, f_t)$. Then we see that $f_i m \in M[I^\infty]$, i.e., $I^{n_i} f_i m = 0$ for some $n_i > 0$. Thus we see that $I^N m = 0$ with $N = \sum n_i + 2$. Hence m maps to zero in $(M/M[I^\infty])$ which proves the first statement of the lemma.

For the second, suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules with M' and M'' both I-power torsion modules. Then $M[I^\infty] \supset M'$ and hence $M/M[I^\infty]$ is a quotient of M'' and therefore I-power torsion. Combined with the first statement and Lemma 88.3 this implies that it is zero $\square$

Lemma 88.5. *Let I be a finitely generated ideal of a ring R. The I-power torsion modules form a Serre subcategory of the abelian category Mod_R, see Homology, Definition 10.1.*

Proof. It is clear that a submodule and a quotient module of an I-power torsion module is I-power torsion. Moreover, the extension of two I-power torsion modules is I-power torsion by Lemma 88.4. Hence the statement of the lemma by Homology, Lemma 10.2. $\square$

Lemma 88.6. *Let R be a ring and let $I \subset R$ be a finitely generated ideal. The subcategory I^∞-torsion $\subset Mod_R$ depends only on the closed subset $Z = V(I) \subset$ $\mathrm{Spec}(R)$. In fact, an R-module M is I-power torsion if and only if its support is contained in Z.*

Proof. Let M be an R-module. Let $x \in M$. If $x \in M[I^\infty]$, then x maps to zero in M_f for all $f \in I$. Hence x maps to zero in $M_\mathfrak{p}$ for all $\mathfrak{p} \not\supset I$. Conversely, if x maps to zero in $M_\mathfrak{p}$ for all $\mathfrak{p} \not\supset I$, then x maps to zero in M_f for all $f \in I$. Hence if $I = (f_1, \ldots, f_r)$, then $f_i^{n_i} x = 0$ for some $n_i \geq 1$. It follows that $x \in M[I^{\sum n_i}]$.

Thus $M[I^\infty]$ is the kernel of $M \to \prod_{\mathfrak p \not\in Z} M_{\mathfrak p}$. The second statement of the lemma follows and it implies the first. $\square$

The next lemma should probably go somewhere else.

Lemma 88.7. *Let R be a ring. Let $I \subset R$ be an ideal. Let K be an object of $D(R)$ such hat $K \otimes_R^{\mathbf{L}} R/I = 0$ in $D(R)$. Then*

(1) *$K \otimes_R^{\mathbf{L}} R/I^n = 0$ for all $n \geq 1$,*
(2) *$K \otimes_R^{\mathbf{L}} N = 0$ for any I-power torsion R-module N,*
(3) *$K \otimes_R^{\mathbf{L}} M = 0$ for any $M \in D^b(R)$ whose cohomology modules are I-power torsion.*

Proof. Proof of (2). We can write $N = \bigcup N[I^n]$. We have $K \otimes_R^{\mathbf{L}} N = \text{hocolim}_n K \otimes_R^{\mathbf{L}} N[I^n]$ as tensor products commute with colimits (details omitted; hint: represent K by a K-flat complex and compute directly). Hence we may assume N is annihilated by I^n. Consider the R-algebra $R' = R/I^n \oplus N$ where N is an ideal of square zero. It suffices to show that $K' = K \otimes_R^{\mathbf{L}} R'$ is 0 in $D(R')$. We have a surjection $R' \to R/I$ of R-algebras whose kernel J is nilpotent (any product of n elements in the kernel is zero). We have

$$0 = K \otimes_R^{\mathbf{L}} R/I = (K \otimes_R^{\mathbf{L}} R') \otimes_{R'}^{\mathbf{L}} R/I = K' \otimes_{R'}^{\mathbf{L}} R/I$$

by Lemma 60.5. Hence by Lemma 78.4 we find that K' is a perfect complex of R'-modules. In particular K' is bounded above and if $H^b(K')$ is the rightmost nonvanishing cohomology module (if it exists), then $H^b(K')$ is a finite R'-module (use Lemmas 74.2 and 64.3) with $H^b(K') \otimes_{R'} R'/J = H^b(K')/JH^b(K') = 0$ (because $K' \otimes_{R'}^{\mathbf{L}} R'/J = 0$). By Nakayama's lemma (Algebra, Lemma 20.1) we find $H^b(K') = 0$, i.e., $K' = 0$ as desired.

Part (1) follows trivially from part (2). Part (3) follows from part (2), induction on the number of nonzero cohomology modules of M, and the distinguished triangles of truncation from Derived Categories, Remark 12.4. Details omitted. $\square$

89. Formal glueing of module categories

Fix a Noetherian scheme X, and a closed subscheme Z with complement U. Our goal is to explain how coherent sheaves on X can be constructed (uniquely) from coherent sheaves on the formal completion of X along Z, and those on U with a suitable compatibility on the overlap. We first do this using only commutative algebra (this section) and later we explain this in the setting of algebraic spaces (Pushouts of Spaces, Section 10).

Here are some references treating some of the material in this section: [Art70, Section 2], [FR70, Appendix], [BL95], [MB96], and [dJ95, Section 4.6].

Lemma 89.1. *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. The following are equivalent*

(1) *φ is flat and $R/I \to S/IS$ is faithfully flat,*
(2) *φ is flat, and the map $\text{Spec}(S/IS) \to \text{Spec}(R/I)$ is surjective.*
(3) *φ is flat, and the base change functor $M \mapsto M \otimes_R S$ is faithful on modules annihilated by I, and*
(4) *φ is flat, and the base change functor $M \mapsto M \otimes_R S$ is faithful on I-power torsion modules.*

Proof. If $R \to S$ is flat, then $R/I^n \to S/I^nS$ is flat for every n, see Algebra, Lemma 39.7. Hence (1) and (2) are equivalent by Algebra, Lemma 39.16. The equivalence of (1) with (3) follows by identifying I-torsion R-modules with R/I-modules, using that

$$M \otimes_R S = M \otimes_{R/I} S/IS$$

for R-modules M annihilated by I, and Algebra, Lemma 39.14. The implication (4) $\Rightarrow$ (3) is immediate. Assume (3). We have seen above that $R/I^n \to S/I^nS$ is flat, and by assumption it induces a surjection on spectra, as $\text{Spec}(R/I^n) = \text{Spec}(R/I)$ and similarly for S. Hence the base change functor is faithful on modules annihilated by I^n. Since any I-power torsion module M is the union $M = \bigcup M_n$ where M_n is annihilated by I^n we see that the base change functor is faithful on the category of all I-power torsion modules (as tensor product commutes with colimits). $\square$

Lemma 89.2. *Assume $(\varphi : R \to S, I)$ satisfies the equivalent conditions of Lemma 89.1. The following are equivalent*

1. *for any I-power torsion module M, the natural map $M \to M \otimes_R S$ is an isomorphism, and*
2. *$R/I \to S/IS$ is an isomorphism.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2). First assume that M is annihilated by I. In this case, M is an R/I-module. Hence, we have an isomorphism

$$M \otimes_R S = M \otimes_{R/I} S/IS = M \otimes_{R/I} R/I = M$$

proving the claim. Next we prove by induction that $M \to M \otimes_R S$ is an isomorphism for any module M is annihilated by I^n. Assume the induction hypothesis holds for n and assume M is annihilated by I^{n+1}. Then we have a short exact sequence

$$0 \to I^nM \to M \to M/I^nM \to 0$$

and as $R \to S$ is flat this gives rise to a short exact sequence

$$0 \to I^nM \otimes_R S \to M \otimes_R S \to M/I^nM \otimes_R S \to 0$$

Using that the canonical map is an isomorphism for $M' = I^nM$ and $M'' = M/I^nM$ (by induction hypothesis) we conclude the same thing is true for M. Finally, suppose that M is a general I-power torsion module. Then $M = \bigcup M_n$ where M_n is annihilated by I^n and we conclude using that tensor products commute with colimits. $\square$

Lemma 89.3. *Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Then*

1. *for any R-module M the map $M \to M \otimes_R S$ induces an isomorphism $M[I^\infty] \to (M \otimes_R S)[(IS)^\infty]$ of I-power torsion submodules,*
2. *the natural map*
$$\text{Hom}_R(M, N) \longrightarrow \text{Hom}_S(M \otimes_R S, N \otimes_R S)$$
is an isomorphism if either M or N is I-power torsion, and
3. *the base change functor $M \mapsto M \otimes_R S$ defines an equivalence of categories between I-power torsion modules and IS-power torsion modules.*

Proof. Note that the equivalent conditions of both Lemma 89.1 and Lemma 89.2 are satisfied. We will use these without further mention. We first prove (1). Let M be any R-module. Set $M' = M/M[I^\infty]$ and consider the exact sequence

$$0 \to M[I^\infty] \to M \to M' \to 0$$

As $M[I^\infty] = M[I^\infty] \otimes_R S$ we see that it suffices to show that $(M' \otimes_R S)[(IS)^\infty] = 0$. Write $I = (f_1, \ldots, f_t)$. By Lemma 88.4 we see that $M'[I^\infty] = 0$. Hence for every $n > 0$ the map

$$M' \longrightarrow \bigoplus\nolimits_{i=1,\ldots t} M', \quad x \longmapsto (f_1^n x, \ldots, f_t^n x)$$

is injective. As S is flat over R also the corresponding map $M' \otimes_R S \to \bigoplus_{i=1,\ldots t} M' \otimes_R S$ is injective. This means that $(M' \otimes_R S)[I^n] = 0$ as desired.

Next we prove (2). If N is I-power torsion, then $N \otimes_R S = N$ and the displayed map of (2) is an isomorphism by Algebra, Lemma 14.3. If M is I-power torsion, then the image of any map $M \to N$ factors through $M[I^\infty]$ and the image of any map $M \otimes_R S \to N \otimes_R S$ factors through $(N \otimes_R S)[(IS)^\infty]$. Hence in this case part (1) guarantees that we may replace N by $N[I^\infty]$ and the result follows from the case where N is I-power torsion we just discussed.

Next we prove (3). The functor is fully faithful by (2). For essential surjectivity, we simply note that for any IS-power torsion S-module N, the natural map $N \otimes_R S \to N$ is an isomorphism. $\square$

Lemma 89.4. *Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. For any $f_1, \ldots, f_r \in R$ such that $V(f_1, \ldots, f_r) = V(I)$*

1. *the map of Koszul complexes $K(R, f_1, \ldots, f_r) \to K(S, f_1, \ldots, f_r)$ is a quasi-isomorphism, and*
2. *The map of extended alternating Čech complexes*

$$\begin{array}{c} R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r} \\ \downarrow \\ S \to \prod_{i_0} S_{f_{i_0}} \to \prod_{i_0 < i_1} S_{f_{i_0} f_{i_1}} \to \ldots \to S_{f_1 \ldots f_r} \end{array}$$

is a quasi-isomorphism.

Proof. In both cases we have a complex $K_\bullet$ of R modules and we want to show that $K_\bullet \to K_\bullet \otimes_R S$ is a quasi-isomorphism. By Lemma 89.2 and the flatness of $R \to S$ this will hold as soon as all homology groups of K are I-power torsion. This is true for the Koszul complex by Lemma 28.6 and for the extended alternating Čech complex by Lemma 29.5. $\square$

Lemma 89.5. *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. Let M be the R-module generated by elements $e_1, \ldots, e_n$ subject to the relations $f_i e_j - f_j e_i = 0$. There exists a short exact sequence*

$$0 \to K \to M \to I \to 0$$

such that K is annihilated by I.

Proof. This is just a truncation of the Koszul complex. The map $M \to I$ is determined by the rule $e_i \mapsto f_i$. If $m = \sum a_i e_i$ is in the kernel of $M \to I$, i.e., $\sum a_i f_i = 0$, then $f_j m = \sum f_j a_i e_i = (\sum f_i a_i) e_j = 0$. $\square$

Lemma 89.6. *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. For any R-module N set*

$$H_1(N, f_\bullet) = \frac{\{(x_1, \ldots, x_n) \in N^{\oplus n} \mid f_i x_j = f_j x_i\}}{\{f_1 x, \ldots, f_n x) \mid x \in N\}}$$

For any R-module N there exists a canonical short exact sequence

$$0 \to \mathrm{Ext}_R(R/I, N) \to H_1(N, f_\bullet) \to \mathrm{Hom}_R(K, N)$$

where K is as in Lemma 89.5.

Proof. The notation above indicates the Ext-groups in Mod_R as defined in Homology, Section 6. These are denoted $\mathrm{Ext}_R(M, N)$. Using the long exact sequence of Homology, Lemma 6.4 associated to the short exact sequence $0 \to I \to R \to R/I \to 0$ and the fact that $\mathrm{Ext}_R(R, N) = 0$ we see that

$$\mathrm{Ext}_R(R/I, N) = \mathrm{Coker}(N \longrightarrow \mathrm{Hom}(I, N))$$

Using the short exact sequence of Lemma 89.5 we see that we get a complex

$$N \to \mathrm{Hom}(M, N) \to \mathrm{Hom}_R(K, N)$$

whose homology in the middle is canonically isomorphic to $\mathrm{Ext}_R(R/I, N)$. The proof of the lemma is now complete as the cokernel of the first map is canonically isomorphic to $H_1(N, f_\bullet)$. $\square$

Lemma 89.7. *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. For any R-module N the Koszul homology group $H_1(N, f_\bullet)$ defined in Lemma 89.6 is annihilated by I.*

Proof. Let $(x_1, \ldots, x_n) \in N^{\oplus n}$ with $f_i x_j = f_j x_i$. Then we have $f_i(x_1, \ldots, x_n) = (f_i x_i, \ldots, f_i x_n)$. In other words f_i annihilates $H_1(N, f_\bullet)$. $\square$

We can improve on the full faithfulness of Lemma 89.3 by showing that Ext-groups whose source is I-power torsion are insensitive to passing to S as well. See Dualizing Complexes, Lemma 9.8 for a derived version of the following lemma.

Lemma 89.8. *Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Let M, N be R-modules. Assume M is I-power torsion. Given an short exact sequence*

$$0 \to N \otimes_R S \to \tilde{E} \to M \otimes_R S \to 0$$

there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \otimes_R S \arrow[r] & \tilde{E} \arrow[r] & M \otimes_R S \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

Proof. As M is I-power torsion we see that $M \otimes_R S = M$, see Lemma 89.2. We will use this identification without further mention. As $R \to S$ is flat, the base change functor is exact and we obtain a functorial map of Ext-groups

$$\mathrm{Ext}_R(M, N) \longrightarrow \mathrm{Ext}_S(M \otimes_R S, N \otimes_R S),$$

see Homology, Lemma 7.3. The claim of the lemma is that this map is surjective when M is I-power torsion. In fact we will show that it is an isomorphism. By Lemma 88.2 we can find a surjection $M' \to M$ with M' a direct sum of modules of the form R/I^n. Using the long exact sequence of Homology, Lemma 6.4 we see that it suffices to prove the lemma for M'. Using compatibility of Ext with direct sums (details omitted) we reduce to the case where $M = R/I^n$ for some n.

Let $f_1, \ldots, f_t$ be generators for I^n. By Lemma 89.6 we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}_R(R/I^n, N) \arrow[r] \arrow[d] & H_1(N, f_\bullet) \arrow[r] \arrow[d] & \mathrm{Hom}_R(K, N) \arrow[d] \\
0 \arrow[r] & \mathrm{Ext}_S(S/I^nS, N \otimes S) \arrow[r] & H_1(N \otimes S, f_\bullet) \arrow[r] & \mathrm{Hom}_S(K \otimes S, N \otimes S)
\end{tikzcd}$$

with exact rows where K is as in Lemma 89.5. Hence it suffices to prove that the two right vertical arrows are isomorphisms. Since K is annihilated by I^n we see that $\mathrm{Hom}_R(K, N) = \mathrm{Hom}_S(K \otimes_R S, N \otimes_R S)$ by Lemma 89.3. As $R \to S$ is flat we have $H_1(N, f_\bullet) \otimes_R S = H_1(N \otimes_R S, f_\bullet)$. As $H_1(N, f_\bullet)$ is annihilated by I^n, see Lemma 89.7 we have $H_1(N, f_\bullet) \otimes_R S = H_1(N, f_\bullet)$ by Lemma 89.2. □

Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Then for any R-module M we can define a complex

$$0 \to M \xrightarrow{\alpha} M \otimes_R S \times \prod M_{f_i} \xrightarrow{\beta} \prod (M \otimes_R S)_{f_i} \times \prod M_{f_i f_j} \tag{89.8.1}$$

where $\alpha(m) = (m \otimes 1, m/1, \ldots, m/1)$ and

$\beta(m', m_1, \ldots, m_t) = ((m'/1 - m_1 \otimes 1, \ldots, m'/1 - m_t \otimes 1), (m_1 - m_2, \ldots, m_{t-1} - m_t)$.

We would like to know when this complex is exact.

Lemma 89.9. *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Let M be an R-module. Then the complex (89.8.1) is exact.*

Proof. First proof. Denote $\check{\mathcal{C}}_R \to \check{\mathcal{C}}_S$ the quasi-isomorphism of extended alternating Čech complexes of Lemma 89.4. Since these complexes are bounded with flat terms, we see that $M \otimes_R \check{\mathcal{C}}_R \to M \otimes_R \check{\mathcal{C}}_S$ is a quasi-isomorphism too (Lemmas 59.7 and 59.12). Now the complex (89.8.1) is a truncation of the cone of the map $M \otimes_R \check{\mathcal{C}}_R \to M \otimes_R \check{\mathcal{C}}_S$ and we win.

Second computational proof. Let $m \in M$. If $\alpha(m) = 0$, then $m \in M[I^\infty]$, see Lemma 88.3. Pick n such that $I^n m = 0$ and consider the map $\varphi : R/I^n \to M$. If $m \otimes 1 = 0$, then $\varphi \otimes 1_S = 0$, hence $\varphi = 0$ (see Lemma 89.3) hence $m = 0$. In this way we see that α is injective.

Let $(m', m'_1, \ldots, m'_t) \in \mathrm{Ker}(\beta)$. Write $m'_i = m_i / f_i^n$ for some $n > 0$ and $m_i \in M$. We may, after possibly enlarging n assume that $f_i^n m' = m_i \otimes 1$ in $M \otimes_R S$ and $f_j^n m_i - f_i^n m_j = 0$ in M. In particular we see that $(m_1, \ldots, m_t)$ defines an element

ξ of $H_1(M, (f_1^n, \ldots, f_t^n))$. Since $H_1(M, (f_1^n, \ldots, f_t^n))$ is annihilated by I^{tn+1} (see Lemma 89.7) and since $R \to S$ is flat we see that

$$H_1(M, (f_1^n, \ldots, f_t^n)) = H_1(M, (f_1^n, \ldots, f_t^n)) \otimes_R S = H_1(M \otimes_R S, (f_1^n, \ldots, f_t^n))$$

by Lemma 89.2 The existence of m' implies that ξ maps to zero in the last group, i.e., the element ξ is zero. Thus there exists an $m \in M$ such that $m_i = f_i^n m$. Then $(m', m_1', \ldots, m_t') - \alpha(m) = (m'', 0, \ldots, 0)$ for some $m'' \in (M \otimes_R S)[(IS)^\infty]$. By Lemma 89.3 we conclude that $m'' \in M[I^\infty]$ and we win. $\square$

Remark 89.10. In this remark we define a category of glueing data. Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Consider the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ as the category whose

1. objects are systems $(M', M_i, \alpha_i, \alpha_{ij})$, where M' is an S-module, M_i is an R_{f_i}-module, $\alpha_i : (M')_{f_i} \to M_i \otimes_R S$ is an isomorphism, and $\alpha_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ are isomorphisms such that
   (a) $\alpha_{ij} \circ \alpha_i = \alpha_j$ as maps $(M')_{f_i f_j} \to (M_j)_{f_i}$, and
   (b) $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ as maps $(M_i)_{f_j f_k} \to (M_k)_{f_i f_j}$ (cocycle condition).
2. morphisms $(M', M_i, \alpha_i, \alpha_{ij}) \to (N', N_i, \beta_i, \beta_{ij})$ are given by maps $\varphi' : M' \to N'$ and $\varphi_i : M_i \to N_i$ compatible with the given maps $\alpha_i, \beta_i, \alpha_{ij}, \beta_{ij}$.

There is a canonical functor

$$\text{Can} : \text{Mod}_R \longrightarrow \text{Glue}(R \to S, f_1, \ldots, f_t), \quad M \longmapsto (M \otimes_R S, M_{f_i}, \text{can}_i, \text{can}_{ij})$$

where $\text{can}_i : (M \otimes_R S)_{f_i} \to M_{f_i} \otimes_R S$ and $\text{can}_{ij} : (M_{f_i})_{f_j} \to (M_{f_j})_{f_i}$ are the canonical isomorphisms. For any object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ we define

$$H^0(\mathbf{M}) = \{(m', m_i) \mid \alpha_i(m') = m_i \otimes 1, \alpha_{ij}(m_i) = m_j\}$$

in other words defined by the exact sequence

$$0 \to H^0(\mathbf{M}) \to M' \times \prod M_i \to \prod M'_{f_i} \times \prod (M_i)_{f_j}$$

similar to (89.8.1). We think of $H^0(\mathbf{M})$ as an R-module. Thus we also get a functor

$$H^0 : \text{Glue}(R \to S, f_1, \ldots, f_t) \longrightarrow \text{Mod}_R$$

Our next goal is to show that the functors Can and H^0 are sometimes quasi-inverse to each other.

Lemma 89.11. *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then the functor H^0 is a left quasi-inverse to the functor Can of Remark 89.10.*

Proof. This is a reformulation of Lemma 89.9. $\square$

Lemma 89.12. *Assume $\varphi : R \to S$ is a flat ring map and let $I = (f_1, \ldots, f_t) \subset R$ be an ideal. Then $\text{Glue}(R \to S, f_1, \ldots, f_t)$ is an abelian category, and the functor Can is exact and commutes with arbitrary colimits.*

Proof. Given a morphism $(\varphi', \varphi_i) : (M', M_i, \alpha_i, \alpha_{ij}) \to (N', N_i, \beta_i, \beta_{ij})$ of the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ we see that its kernel exists and is equal to the object $(\text{Ker}(\varphi'), \text{Ker}(\varphi_i), \alpha_i, \alpha_{ij})$ and its cokernel exists and is equal to the object $(\text{Coker}(\varphi'), \text{Coker}(\varphi_i), \beta_i, \beta_{ij})$. This works because $R \to S$ is flat, hence taking kernels/cokernels commutes with $- \otimes_R S$. Details omitted. The exactness follows from the R-flatness of R_{f_i} and S, while commuting with colimits follows as tensor products commute with colimits. $\square$

Lemma 89.13. *Let $\varphi : R \to S$ be a flat ring map and $(f_1, \ldots, f_t) = R$. Then Can and H^0 are quasi-inverse equivalences of categories*

$$Mod_R = Glue(R \to S, f_1, \ldots, f_t)$$

Proof. Consider an object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$. By Algebra, Lemma 24.5 there exists a unique module M and isomorphisms $M_{f_i} \to M_i$ which recover the glueing data α_{ij}. Then both M' and $M \otimes_R S$ are S-modules which recover the modules $M_i \otimes_R S$ upon localizing at f_i. Whence there is a canonical isomorphism $M \otimes_R S \to M'$. This shows that $\mathbf{M}$ is in the essential image of Can. Combined with Lemma 89.11 the lemma follows. $\square$

Lemma 89.14. *Let $\varphi : R \to S$ be a flat ring map and $I = (f_1, \ldots, f_t)$ and ideal. Let $R \to R'$ be a flat ring map, and set $S' = S \otimes_R R'$. Then we obtain a commutative diagram of categories and functors*

$$\begin{tikzcd}
Mod_R \arrow[r, "Can"'] \arrow[d, "-\otimes_R R'"'] & Glue(R \to S, f_1, \ldots, f_t) \arrow[r, "H^0"'] \arrow[d, "-\otimes_R R'"] & Mod_R \arrow[d, "-\otimes_R R'"] \\
Mod_{R'} \arrow[r, "Can"] & Glue(R' \to S', f_1, \ldots, f_t) \arrow[r, "H^0"] & Mod_{R'}
\end{tikzcd}$$

Proof. Omitted. $\square$

Proposition 89.15. *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then Can and H^0 are quasi-inverse equivalences of categories*

$$Mod_R = Glue(R \to S, f_1, \ldots, f_t)$$

Proof. We have already seen that $H^0 \circ \mathrm{Can}$ is isomorphic to the identity functor, see Lemma 89.11. Consider an object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$. We get a natural morphism

$$\Psi : (H^0(\mathbf{M}) \otimes_R S, H^0(\mathbf{M})_{f_i}, \mathrm{can}_i, \mathrm{can}_{ij}) \longrightarrow (M', M_i, \alpha_i, \alpha_{ij}).$$

Namely, by definition $H^0(\mathbf{M})$ comes equipped with compatible R-module maps $H^0(\mathbf{M}) \to M'$ and $H^0(\mathbf{M}) \to M_i$. We have to show that this map is an isomorphism.

Pick an index i and set $R' = R_{f_i}$. Combining Lemmas 89.14 and 89.13 we see that $\Psi \otimes_R R'$ is an isomorphism. Hence the kernel, resp. cokernel of Ψ is a system of the form $(K, 0, 0, 0)$, resp. $(Q, 0, 0, 0)$. Note that $H^0((K, 0, 0, 0)) = K$, that H^0 is left exact, and that by construction $H^0(\Psi)$ is bijective. Hence we see $K = 0$, i.e., the kernel of Ψ is zero.

The conclusion of the above is that we obtain a short exact sequence

$$0 \to H^0(\mathbf{M}) \otimes_R S \to M' \to Q \to 0$$

and that $M_i = H^0(\mathbf{M})_{f_i}$. Note that we may think of Q as an R-module which is I-power torsion so that $Q = Q \otimes_R S$. By Lemma 89.8 we see that there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbf{M}) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(\mathbf{M}) \otimes_R S \arrow[r] & M' \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

with exact rows. This clearly determines an isomorphism $\text{Can}(E) \to (M', M_i, \alpha_i, \alpha_{ij})$ in the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ and we win. (Of course, a posteriori we have $Q = 0$.) ☐

Lemma 89.16. *Let $\varphi : R \to S$ be a flat ring map and let $I \subset R$ be a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism.*

(1) *Given an R-module N, an S-module M' and an S-module map $\varphi : M' \to N \otimes_R S$ whose kernel and cokernel are I-power torsion, there exists an R-module map $\psi : M \to N$ and an isomorphism $M \otimes_R S = M'$ compatible with φ and ψ.*
(2) *Given an R-module M, an S-module N' and an S-module map $\varphi : M \otimes_R S \to N'$ whose kernel and cokernel are I-power torsion, there exists an R-module map $\psi : M \to N$ and an isomorphism $N \otimes_R S = N'$ compatible with φ and ψ.*

In both cases we have $\text{Ker}(\varphi) \cong \text{Ker}(\psi)$ *and* $\text{Coker}(\varphi) \cong \text{Coker}(\psi)$.

Proof. Proof of (1). Say $I = (f_1, \ldots, f_t)$. It is clear that the localization φ_{f_i} is an isomorphism. Thus we see that $(M', N_{f_i}, \varphi_{f_i}, can_{ij})$ is an object of $\text{Glue}(R \to S, f_1, \ldots, f_t)$, see Remark 89.10. By Proposition 89.15 we conclude that there exists an R-module M such that $M' = M \otimes_R S$ and $N_{f_i} = M_{f_i}$ compatibly with the isomorphisms φ_{f_i} and can_{ij}. There is a morphism

$$(M \otimes_R S, M_{f_i}, can_i, can_{ij}) = (M', N_{f_i}, \varphi_{f_i}, can_{ij}) \to (N \otimes_R S, N_{f_i}, can_i, can_{ij})$$

of $\text{Glue}(R \to S, f_1, \ldots, f_t)$ which uses φ in the first component. This corresponds to an R-module map $\psi : M \to N$ (by the equivalence of categories of Proposition 89.15). The composition of the base change of $M \to N$ with the isomorphism $M' \cong M \otimes_R S$ is φ, in other words $M \to N$ is compatible with φ.

Proof of (2). This is just the dual of the argument above. Namely, the localization φ_{f_i} is an isomorphism. Thus we see that $(N', M_{f_i}, \varphi_{f_i}^{-1}, can_{ij})$ is an object of $\text{Glue}(R \to S, f_1, \ldots, f_t)$, see Remark 89.10. By Proposition 89.15 we conclude that there exists an R-module N such that $N' = N \otimes_R S$ and $N_{f_i} = M_{f_i}$ compatibly with the isomorphisms $\varphi_{f_i}^{-1}$ and can_{ij}. There is a morphism

$$(M \otimes_R S, M_{f_i}, can_i, can_{ij}) \to (N', M_{f_i}, \varphi_{f_i}, can_{ij}) = (N \otimes_R S, N_{f_i}, can_i, can_{ij})$$

of $\text{Glue}(R \to S, f_1, \ldots, f_t)$ which uses φ in the first component. This corresponds to an R-module map $\psi : M \to N$ (by the equivalence of categories of Proposition 89.15). The composition of the base change of $M \to N$ with the isomorphism $N' \cong N \otimes_R S$ is φ, in other words $M \to N$ is compatible with φ.

The final statement follows for example from Lemma 89.3. ☐

Next, we specialize Proposition 89.15 to get something more useable. Namely, if $I = (f)$ is a principal ideal then the objects of $\text{Glue}(R \to S, f)$ are simply triples (M', M_1, α_1) and there is *no* cocycle condition to check!

Theorem 89.17. *Let R be a ring, and let $f \in R$. Let $\varphi : R \to S$ be a flat ring map inducing an isomorphism $R/fR \to S/fS$. Then the functor*

$$Mod_R \longrightarrow Mod_S \times_{Mod_{S_f}} Mod_{R_f}, \quad M \longmapsto (M \otimes_R S, M_f, can)$$

is an equivalence.

Proof. The category appearing on the right side of the arrow is the category of triples (M', M_1, α_1) where M' is an S-module, M_1 is a R_f-module, and $\alpha_1 : M'_f \to M_1 \otimes_R S$ is a S_f-isomorphism, see Categories, Example 31.3. Hence this theorem is a special case of Proposition 89.15. $\square$

A useful special case of Theorem 89.17 is when R is Noetherian, and S is a completion of R at an element f. The completion $R \to S$ is flat, and the functor $M \mapsto M \otimes_R S$ can be identified with the f-adic completion functor when M is finitely generated. To state this more precisely, let Mod_R^{fg} denote the category of finitely generated R-modules.

Proposition 89.18. *Let R be a Noetherian ring. Let $f \in R$ be an element. Let $R^\wedge$ be the f-adic completion of R. Then the functor $M \mapsto (M^\wedge, M_f, can)$ defines an equivalence*

$$Mod_R^{fg} \longrightarrow Mod_{R^\wedge}^{fg} \times_{Mod_{(R^\wedge)_f}^{fg}} Mod_{R_f}^{fg}$$

Proof. The ring map $R \to R^\wedge$ is flat by Algebra, Lemma 97.2. It is clear that $R/fR = R^\wedge/fR^\wedge$. By Algebra, Lemma 97.1 the completion of a finite R-module M is equal to $M \otimes_R R^\wedge$. Hence the displayed functor of the proposition is equal to the functor occurring in Theorem 89.17. In particular it is fully faithful. Let (M_1, M_2, ψ) be an object of the right hand side. By Theorem 89.17 there exists an R-module M such that $M_1 = M \otimes_R R^\wedge$ and $M_2 = M_f$. As $R \to R^\wedge \times R_f$ is faithfully flat we conclude from Algebra, Lemma 23.2 that M is finitely generated, i.e., $M \in \text{Mod}_R^{fg}$. This proves the proposition. $\square$

Remark 89.19. The equivalences of Proposition 89.15, Theorem 89.17, and Proposition 89.18 preserve properties of modules. For example if M corresponds to $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ then M is finite, or finitely presented, or flat, or projective over R if and only if M' and M_i have the corresponding property over S and R_{f_i}. This follows from the fact that $R \to S \times \prod R_{f_i}$ is faithfully flat and descend and ascent of these properties along faithfully flat maps, see Algebra, Lemma 83.2 and Theorem 95.6. These functors also preserve the $\otimes$-structures on either side. Thus, it defines equivalences of various categories built out of the pair $(\text{Mod}_R, \otimes)$, such as the category of algebras.

Remark 89.20. Given a differential manifold X with a compact closed submanifold Z having complement U, specifying a sheaf on X is the same as specifying a sheaf on U, a sheaf on an unspecified tubular neighbourhood T of Z in X, and an isomorphism between the two resulting sheaves along $T \cap U$. Tubular neighbourhoods do not exist in algebraic geometry as such, but results such as Proposition 89.15, Theorem 89.17, and Proposition 89.18 allow us to work with formal neighbourhoods instead.

90. The Beauville-Laszlo theorem

Let R be a ring and let f be an element of R. Denote $R^\wedge = \lim R/f^nR$ the f-adic completion of R. In this section we discuss and slightly generalize a theorem of Beauville and Laszlo, see [BL95]. The theorem asserts that under suitable conditions, a module over R can be constructed by "glueing together" modules over $R^\wedge$ and R_f along an isomorphism between the base extensions to $(R^\wedge)_f$.

In [BL95] it is assumed that f is a nonzerodivisor on both R and M. In fact, one only needs to assume that

$$R[f^\infty] \longrightarrow R^\wedge[f^\infty]$$

is bijective and that

$$M[f^\infty] \longrightarrow M \otimes_R R^\wedge$$

is injective. This optimization was partly inspired by an alternate approach to glueing introduced in [KL15, §1.3] for use in the theory of nonarchimedean analytic spaces.

In fact, we will establish the Beauville-Laszlo theorem in the more general setting of a ring map

$$R \longrightarrow R'$$

which induces isomorphisms $R/f^nR \to R'/f^nR'$ for every $n > 0$ and an isomorphism $R[f^\infty] \to R'[f^\infty]$. This is better suited for globalizing and does not formally follow from the case when R' is the completion of R because, for instance, the condition that $R[f^\infty] \to R'[f^\infty]$ is a bijection does not imply that $R[f^\infty] \to R^\wedge[f^\infty]$ is a bijection.

The theorem of Beauville and Laszlo as proved in this section can be viewed as a non-flat version of Theorem 89.17 and in the case where $R' = R^\wedge$ can be viewed as a non-Noetherian version of Proposition 89.18. For a comparison with flat descent, please see Remark 90.6.

One can establish even stronger results (without imposing restrictions on M for example) but for this one must work at the level of derived categories. See [Bha16, §5] for more details.

Lemma 90.1. *Let R be a ring and let $f \in R$. For every positive integer n the map $R/f^nR \to R^\wedge/f^nR^\wedge$ is an isomorphism.*

Proof. This is a special case of Algebra, Lemma 96.3. □

We will use the notation introduced in Section 88. Thus for an R-module M, we denote $M[f^n]$ the submodule of M annihilated by f^n and we put

$$M[f^\infty] = \bigcup\nolimits_{n=1}^{\infty} M[f^n] = \mathrm{Ker}(M \to M_f).$$

If $M = M[f^\infty]$, we say that M is an f-power torsion module.

Lemma 90.2. *Let R be a ring, let $f \in R$ be an element, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. For any f-power torsion R-module M the map $M \to M \otimes_R R'$ is an isomorphism. For example, we have $M \cong M \otimes_R R^\wedge$.*

Proof. If M is annihilated by f^n, then

$$M \otimes_R R' \cong M \otimes_{R/f^nR} R'/f^nR' \cong M \otimes_{R/f^nR} R/f^nR \cong M.$$

Since $M = \bigcup M[f^n]$ and since tensor products commute with direct limits (Algebra, Lemma 12.9), we obtain the desired isomorphism. The last statement is a special case of the first statement by Lemma 90.1. $\square$

Lemma 90.3. *Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. The R-module $R' \oplus R_f$ is faithful: for every nonzero R-module M, the module $M \otimes_R (R' \oplus R_f)$ is also nonzero. For example, if M is nonzero, then $M \otimes_R (R^\wedge \oplus R_f)$ is nonzero.*

However, the map $M \to M \otimes_R (R' \oplus R_f)$ need not be injective; see Example 90.10.

Proof. If $M \neq 0$ but $M \otimes_R R_f = 0$, then M is f-power torsion. By Lemma 90.2 we find that $M \otimes_R R' \cong M \neq 0$. The last statement is a special case of the first statement by Lemma 90.1. $\square$

Lemma 90.4. *Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces an isomorphism $R/fR \to R'/fR'$. The map $\mathrm{Spec}(R') \amalg \mathrm{Spec}(R_f) \to \mathrm{Spec}(R)$ is surjective. For example, the map $\mathrm{Spec}(R^\wedge) \amalg \mathrm{Spec}(R_f) \to \mathrm{Spec}(R)$ is surjective.*

Proof. Recall that $\mathrm{Spec}(R) = V(f) \amalg D(f)$ where $V(f) = \mathrm{Spec}(R/fR)$ and $D(f) = \mathrm{Spec}(R_f)$, see Algebra, Section 17 and especially Lemmas 17.7 and 17.6. Thus the lemma follows as the map $R \to R/fR$ factors through R'. The last statement is a special case of the first statement by Lemma 90.1. $\square$

Lemma 90.5. *Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. An R-module M is finitely generated if and only if the $(R' \oplus R_f)$-module $M \otimes_R (R' \oplus R_f)$ is finitely generated. For example, if $M \otimes_R (R^\wedge \oplus R_f)$ is finitely generated as a module over $R^\wedge \oplus R_f$, then M is a finitely generated R-module.*

Proof. The 'only if' is clear, so we assume that $M \otimes_R (R' \oplus R_f)$ is finitely generated. In this case, by writing each generator as a sum of simple tensors, $M \otimes_R (R' \oplus R_f)$ admits a finite generating set consisting of elements of M. That is, there exists a morphism from a finite free R-module to M whose cokernel is killed by tensoring with $R' \oplus R_f$; we may thus deduce M is finite generated by applying Lemma 90.3 to this cokernel. The last statement is a special case of the first statement by Lemma 90.1. $\square$

Remark 90.6. While $R \to R_f$ is always flat, $R \to R^\wedge$ is typically not flat unless R is Noetherian (see Algebra, Lemma 97.2 and the discussion in Examples, Section 12). Consequently, we cannot in general apply faithfully flat descent as discussed in Descent, Section 3 to the morphism $R \to R^\wedge \oplus R_f$. Moreover, even in the Noetherian case, the usual definition of a descent datum for this morphism refers to the ring $R^\wedge \otimes_R R^\wedge$, which we will avoid considering in this section.

Glueing pairs. Let $R \to R'$ be a ring map that induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. Consider the sequence

$$0 \to R \to R' \oplus R_f \to R'_f \to 0, \tag{90.6.1}$$

in which the map on the right is the difference between the two canonical homomorphisms. If this sequence is exact, then we say that $(R \to R', f)$ is a *glueing* *pair*. We will say that (R, f) is a *glueing pair* if $(R \to R^\wedge, f)$ is a glueing pair; this makes sense by Lemma 90.1. Thus (R, f) is a glueing pair if and only if the sequence

$$0 \to R \to R^\wedge \oplus R_f \to (R^\wedge)_f \to 0, \tag{90.6.2}$$

is exact.

Lemma 90.7. *Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. The sequence (90.6.1) is*

1. *exact on the right,*
2. *exact on the left if and only if $R[f^\infty] \to R'[f^\infty]$ is injective, and*
3. *exact in the middle if and only if $R[f^\infty] \to R'[f^\infty]$ is surjective.*

In particular, $(R \to R', f)$ is a glueing pair if and only if $R[f^\infty] \to R'[f^\infty]$ is bijective. For example, (R, f) is a glueing pair if and only if $R[f^\infty] \to R^\wedge[f^\infty]$ is bijective.

Proof. Let $x \in R'_f$. Write $x = x'/f^n$ with $x' \in R'$. Write $x' = x'' + f^ny$ with $x'' \in R$ and $y \in R'$. Then we see that $(y, -x''/f^n)$ maps to x. Thus (1) holds.

Part (2) follows from the fact that $\mathrm{Ker}(R \to R_f) = R[f^\infty]$.

If the sequence is exact in the middle, then elements of the form $(x, 0)$ with $x \in R'[f^\infty]$ are in the image of the first arrow. This implies that $R[f^\infty] \to R'[f^\infty]$ is surjective. Conversely, assume that $R[f^\infty] \to R'[f^\infty]$ is surjective. Let (x, y) be an element in the middle which maps to zero on the right. Write $y = y'/f^n$ for some $y' \in R$. Then we see that $f^nx - y'$ is annihilated by some power of f in R'. By assumption we can write $f^nx - y' = z$ for some $z \in R[f^\infty]$. Then $y = y''/f^n$ where $y'' = y' + z$ is in the kernel of $R \to R/f^nR$. Hence we see that y can be represented as $y'''/1$ for some $y''' \in R$. Then $x - y'''$ is in $R'[f^\infty]$. Thus $x - y''' = z' \in R[f^\infty]$. Then $(x, y'''/1) = (y''' + z', (y''' + z')/1)$ as desired.

The last statement of the lemma is a special case of the penultimate statement by Lemma 90.1. $\square$

Remark 90.8. Suppose that f is a nonzerodivisor. Then Algebra, Lemma 96.4 shows that f is a nonzerodivisor in $R^\wedge$. Hence (R, f) is a glueing pair.

Remark 90.9. If $R \to R^\wedge$ is flat, then for each positive integer n tensoring the sequence $0 \to R[f^n] \to R \to R$ with $R^\wedge$ gives the sequence $0 \to R[f^n] \otimes_R R^\wedge \to R^\wedge \to R^\wedge$. Combined with Lemma 90.2 we conclude that $R[f^n] \to R^\wedge[f^n]$ is an isomorphism. Thus (R, f) is a glueing pair. This holds in particular if R is Noetherian, see Algebra, Lemma 97.2.

Example 90.10. Let k be a field and put

$$R = k[f, T_1, T_2, \ldots]/(fT_1, fT_2 - T_1, fT_3 - T_2, \ldots).$$

Then (R, f) is not a glueing pair because the map $R[f^\infty] \to R^\wedge[f^\infty]$ is not injective as the image of T_1 is f-divisible in $R^\wedge$. For

$$R = k[f, T_1, T_2, \ldots]/(fT_1, f^2T_2, \ldots),$$

the map $R[f^\infty] \to R^\wedge[f^\infty]$ is not surjective as the element $T_1 + fT_2 + f^2T_3 + \ldots$ is not in the image. In particular, by Remark 90.9, these are both examples where $R \to R^\wedge$ is not flat.

Glueable modules. Let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. For any R-module M, we may tensor (90.6.1) with M to obtain a sequence

$$0 \to M \to (M \otimes_R R') \oplus (M \otimes_R R_f) \to M \otimes_R R'_f \to 0 \quad (90.10.1)$$

Observe that $M \otimes_R R_f = M_f$ and that $M \otimes_R R'_f = (M \otimes_R R')_f$. If this sequence is exact, we say that M is *glueable for* $(R \to R', f)$. If R is a ring and $f \in R$, then we say an R-module is *glueable* if M is glueable for $(R \to R^\wedge, f)$. Thus M is glueable if and only if the sequence

$$0 \to M \to (M \otimes_R R^\wedge) \oplus (M \otimes_R R_f) \to M \otimes_R (R^\wedge)_f \to 0 \quad (90.10.2)$$

is exact.

Lemma 90.11. *Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. The sequence (90.10.1) is*

1. *exact on the right,*
2. *exact on the left if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is injective, and*
3. *exact in the middle if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is surjective.*

Thus M is glueable for $(R \to R', f)$ if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is bijective. If $(R \to R', f)$ is a glueing pair, then M is glueable for $(R \to R', f)$ if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is injective. For example, if (R, f) is a glueing pair, then M is glueable if and only if $M[f^\infty] \to (M \otimes_R R^\wedge)[f^\infty]$ is injective.

Proof. We will use the results of Lemma 90.7 without further mention. The functor $M \otimes_R -$ is right exact (Algebra, Lemma 12.10) hence we get (1).

The kernel of $M \to M \otimes_R R_f = M_f$ is $M[f^\infty]$. Thus (2) follows.

If the sequence is exact in the middle, then elements of the form $(x, 0)$ with $x \in (M \otimes_R R')[f^\infty]$ are in the image of the first arrow. This implies that $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is surjective. Conversely, assume that $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is surjective. Let (x, y) be an element in the middle which maps to zero on the right. Write $y = y'/f^n$ for some $y' \in M$. Then we see that $f^n x - y'$ is annihilated by some power of f in $M \otimes_R R'$. By assumption we can write $f^n x - y' = z$ for some $z \in M[f^\infty]$. Then $y = y''/f^n$ where $y'' = y' + z$ is in the kernel of $M \to M/f^nM$. Hence we see that y can be represented as $y'''/1$ for some $y''' \in M$. Then $x - y'''$ is in $(M \otimes_R R')[f^\infty]$. Thus $x - y''' = z' \in M[f^\infty]$. Then $(x, y'''/1) = (y''' + z', (y''' + z')/1)$ as desired.

If $(R \to R', f)$ is a glueing pair, then (90.10.1) is exact in the middle for any M by Algebra, Lemma 12.10. This gives the penultimate statement of the lemma. The final statement of the lemma follows from this and the fact that (R, f) is a glueing pair if and only if $(R \to R^\wedge, f)$ is a glueing pair. □

Remark 90.12. Let $(R \to R', f)$ be a glueing pair and let M be an R-module. Here are some observations which can be used to determine whether M is glueable for $(R \to R', f)$.

1. By Lemma 90.11 we see that M is glueable for $(R \to R^\wedge, f)$ if and only if $M[f^\infty] \to M \otimes_R R^\wedge$ is injective. This holds if $M[f] \to M^\wedge$ is injective, i.e., when $M[f] \cap \bigcap_{n=1}^{\infty} f^n M = 0$.

(2) If $\mathrm{Tor}_1^R(M, R'_f) = 0$, then M is glueable for $(R \to R', f)$ (use Algebra, Lemma 75.2). This is equivalent to saying that $\mathrm{Tor}_1^R(M, R')$ is f-power torsion. In particular, any flat R-module is glueable for $(R \to R', f)$.
(3) If $R \to R'$ is flat, then $\mathrm{Tor}_1^R(M, R') = 0$ for every R-module so every R-module is glueable for $(R \to R', f)$. This holds in particular when R is Noetherian and $R' = R^\wedge$, see Algebra, Lemma 97.2

Example 90.13 (Non glueable module)**.** Let R be the ring of germs at 0 of C^∞ functions on $\mathbf{R}$. Let $f \in R$ be the function $f(x) = x$. Then f is a nonzerodivisor in R, so (R, f) is a glueing pair and $R^\wedge \cong \mathbf{R}[[x]]$. Let $\varphi \in R$ be the function $\varphi(x) = \exp(-1/x^2)$. Then φ has zero Taylor series, so $\varphi \in \mathrm{Ker}(R \to R^\wedge)$. Since $\varphi(x) \neq 0$ for $x \neq 0$, we see that φ is a nonzerodivisor in R. The function φ/f also has zero Taylor series, so its image in $M = R/\varphi R$ is a nonzero element of $M[f]$ which maps to zero in $M \otimes_R R^\wedge = R^\wedge/\varphi R^\wedge = R^\wedge$. Hence M is not glueable.

We next make some calculations of Tor groups.

Lemma 90.14. *Let $(R \to R', f)$ be a glueing pair. Then $Tor_1^R(R', f^n R) = 0$ for each $n > 0$.*

Proof. From the exact sequence $0 \to R[f^n] \to R \to f^n R \to 0$ we see that it suffices to check that $R[f^n] \otimes_R R' \to R'$ is injective. By Lemma 90.2 we have $R[f^n] \otimes_R R' = R[f^n]$ and by Lemma 90.7 we see that $R[f^n] \to R'$ is injective as $(R \to R', f)$ is a glueing pair. $\square$

Lemma 90.15. *Let $(R \to R', f)$ be a glueing pair. Then $Tor_1^R(R', R/R[f^\infty]) = 0$.*

Proof. We have $R/R[f^\infty] = \mathrm{colim}\, R/R[f^n] = \mathrm{colim}\, f^n R$. As formation of Tor groups commutes with filtered colimits (Algebra, Lemma 76.2) we may apply Lemma 90.14. $\square$

Lemma 90.16. *Let $(R \to R', f)$ be a glueing pair. For every R-module M, we have $Tor_1^R(R', \mathrm{Coker}(M \to M_f)) = 0$.*

Proof. Set $\overline{M} = M/M[f^\infty]$. Then $\mathrm{Coker}(M \to M_f) \cong \mathrm{Coker}(\overline{M} \to \overline{M}_f)$ hence we may and do assume that f is a nonzerodivisor on M. In this case $M \subset M_f$ and $M_f/M = \mathrm{colim}\, M/f^n M$ where the transition maps are given by multiplication by f. Since formation of Tor groups commutes with colimits (Algebra, Lemma 76.2) it suffices to show that $\mathrm{Tor}_1^R(R', M/f^n M) = 0$.

We first treat the case $M = R/R[f^\infty]$. By Lemma 90.7 we have $M \otimes_R R' = R'/R'[f^\infty]$. From the short exact sequence $0 \to M \to M \to M/f^n M \to 0$ we obtain the exact sequence

$$\begin{tikzcd} \mathrm{Tor}_1^R(R', R/R[f^\infty]) \arrow[r] & \mathrm{Tor}_1^R(R', M/f^n M) \arrow[r] & R'/R'[f^\infty] \arrow[dll, "f^n"'] \\ R'/R'[f^\infty] \arrow[r] & (R'/R'[f^\infty])/(f^n(R'/R'[f^\infty])) \arrow[r] & 0 \end{tikzcd}$$

by Algebra, Lemma 75.2. Here the diagonal arrow is injective. Since the first group $\mathrm{Tor}_1^R(R', R/R[f^\infty])$ is zero by Lemma 90.15, we deduce that $\mathrm{Tor}_1^R(R', M/f^n M) = 0$ as desired.

To treat the general case, choose a surjection $F \to M$ with F a free $R/R[f^\infty]$-module, and form an exact sequence

$$0 \to N \to F/f^nF \to M/f^nM \to 0.$$

By Lemma 90.2 this sequence remains unchanged, and hence exact, upon tensoring with R'. Since $\text{Tor}_1^R(R', F/f^nF) = 0$ by the previous paragraph, we deduce that $\text{Tor}_1^R(R', M/f^nM) = 0$ as desired. □

Let $(R \to R', f)$ be a glueing pair. This means that $R/f^nR \to R'/f^nR'$ is an isomorphism for $n > 0$ and the sequence

$$0 \to R \to R' \oplus R_f \to R'_f \to 0$$

is exact. Consider the category $\text{Glue}(R \to R', f)$ introduced in Remark 89.10. We will call an object (M', M_1, α_1) of $\text{Glue}(R \to R', f)$ a *glueing datum*. It consists of an R'-module M', an R_f-module M_1, and an isomorphism $\alpha_1 : (M')_f \to M_1 \otimes_R R'$. There is an obvious functor

$$\text{Can} : \text{Mod}_R \longrightarrow \text{Glue}(R \to R', f), \quad M \longmapsto (M \otimes_R R', M_f, \text{can}),$$

and there is a functor

$$H^0 : \text{Glue}(R \to R', f) \longrightarrow \text{Mod}_R, \quad (M', M_1, \alpha_1) \longmapsto \text{Ker}(M' \oplus M_1 \to (M')_f)$$

in the reverse direction, see Remark 89.10 for the precise definition.

Theorem 90.17. *Let $(R \to R', f)$ be a glueing pair. The functor Can : $\text{Mod}_R \longrightarrow$ Glue$(R \to R', f)$ determines an equivalence of the category of R-modules glueable for $(R \to R', f)$ and the category Glue$(R \to R', f)$ of glueing data.*

Proof. The functor is fully faithful due to the exactness of (90.10.1) for glueable modules, which tells us exactly that $H^0 \circ \text{Can} = \text{id}$ on the full subcategory of glueable modules. Hence it suffices to check essential surjectivity. That is, we must show that an arbitrary glueing datum (M', M_1, α_1) arises from some glueable R-module.

We first check that the map $\text{d} : M' \oplus M_1 \to (M')_f$ used in the definition of the functor H^0 is surjective. Observe that $(x, y) \in M' \oplus M_1$ maps to $\text{d}(x, y) = x/1 - \alpha_1^{-1}(y \otimes 1)$ in $(M')_f$. If $z \in (M')_f$, then we can write $\alpha_1(z) = \sum y_i \otimes g_i$ with $g_i \in R'$ and $y_i \in M_1$. Write $\alpha_1^{-1}(y_i \otimes 1) = y'_i/f^n$ for some $y'_i \in M'$ and $n \geq 0$ (we can pick the same n for all i). Write $g_i = a_i + f^nb_i$ with $a_i \in R$ and $b_i \in R'$. Then with $y = \sum a_iy_i \in M_1$ and $x = \sum b_iy'_i \in M'$ we have $\text{d}(x, -y) = z$ as desired.

Put $M = H^0((M', M_1, \alpha_1)) = \text{Ker}(\text{d})$. We obtain an exact sequence of R-modules

$$0 \to M \to M' \oplus M_1 \to (M')_f \to 0. \tag{90.17.1}$$

We will prove that the maps $M \to M'$ and $M \to M_1$ induce isomorphisms $M \otimes_R R' \to M'$ and $M \otimes_R R_f \to M_1$. This will imply that M is glueable for $(R \to R', f)$ and gives rise to the original glueing datum.

Since f is a nonzerodivisor on M_1, we have $M[f^\infty] \cong M'[f^\infty]$. This yields an exact sequence

$$0 \to M/M[f^\infty] \to M_1 \to (M')_f/M' \to 0. \tag{90.17.2}$$

Since $R \to R_f$ is flat, we may tensor this exact sequence with R_f to deduce that $M \otimes_R R_f = (M/M[f^\infty]) \otimes_R R_f \to M_1$ is an isomorphism.

By Lemma 90.16 we have $\mathrm{Tor}_1^R(R', \mathrm{Coker}(M' \to (M')_f)) = 0$. The sequence (90.17.2) thus remains exact upon tensoring over R with R'. Using α_1 and Lemma 90.2 the resulting exact sequence can be written as

$$0 \to (M/M[f^\infty]) \otimes_R R' \to (M')_f \to (M')_f/M' \to 0 \tag{90.17.3}$$

This yields an isomorphism $(M/M[f^\infty]) \otimes_R R' \cong M'/M'[f^\infty]$. This implies that in the diagram

$$\begin{tikzcd}
 & M[f^\infty] \otimes_R R' \arrow[r] \arrow[d] & M \otimes_R R' \arrow[r] \arrow[d] & (M/M[f^\infty]) \otimes_R R' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'[f^\infty] \arrow[r] & M' \arrow[r] & M'/M'[f^\infty] \arrow[r] & 0,
\end{tikzcd}$$

the third vertical arrow is an isomorphism. Since the rows are exact and the first vertical arrow is an isomorphism by Lemma 90.2 and $M[f^\infty] = M'[f^\infty]$, the five lemma implies that $M \otimes_R R' \to M'$ is an isomorphism. This completes the proof. $\square$

Remark 90.18. Let $(R \to R', f)$ be a glueing pair. Let M be an R-module that is not necessarily glueable for $(R \to R', f)$. Setting $M' = M \otimes_R R'$ and $M_1 = M_f$ we obtain the glueing datum $\mathrm{Can}(M) = (M', M_1, \mathrm{can})$. Then $\tilde{M} = H^0(M', M_1, \mathrm{can})$ is an R-module that is glueable for $(R \to R', f)$ and the canonical map $M \to \tilde{M}$ gives isomorphisms $M \otimes_R R' \to \tilde{M} \otimes_R R'$ and $M_f \to \tilde{M}_f$, see Theorem 90.17. From the exactness of the sequences

$$M \to (M \otimes_R R') \oplus M_f \to M \otimes_R (R')_f \to 0$$

and

$$0 \to \tilde{M} \to (\tilde{M} \otimes_R R') \oplus \tilde{M}_f \to \tilde{M} \otimes_R (R')_f \to 0$$

we conclude that the map $M \to \tilde{M}$ is surjective.

Recall that flat R-modules over a glueing pair $(R \to R', f)$ are glueable (Remark 90.12). Hence the following lemma shows that Theorem 90.17 determines an equivalence between the category of flat R-modules and the category of glueing data (M', M_1, α_1) where M' and M_1 are flat over R' and R_f.

Lemma 90.19. *Let $(R \to R', f)$ be a glueing pair. Let M be an R-module which is not necessarily glueable for $(R \to R', f)$. Then M is flat over R if and only if $M \otimes_R R'$ is flat over R' and M_f is flat over R_f.*

Proof. One direction of the lemma follows from Algebra, Lemma 39.7. For the other direction, assume $M \otimes_R R'$ is flat over R' and M_f is flat over R_f. Let $\tilde{M}$ be as in Remark 90.18. If $\tilde{M}$ is flat over R, then applying Algebra, Lemma 39.12 to the short exact sequence $0 \to \mathrm{Ker}(M \to \tilde{M}) \to M \to \tilde{M} \to 0$ we find that $\mathrm{Ker}(M \to \tilde{M}) \otimes_R (R' \oplus R_f)$ is zero. Hence $M = \tilde{M}$ by Lemma 90.3 and we conclude. In other words, we may replace M by $\tilde{M}$ and assume M is glueable for $(R \to R', f)$. Let N be a second R-module. It suffices to prove that $\mathrm{Tor}_1^R(M, N) = 0$, see Algebra, Lemma 75.8.

The long the exact sequence of Tors associated to the short exact sequence $0 \to R \to R' \oplus R_f \to (R')_f \to 0$ and N gives an exact sequence

$$0 \to \mathrm{Tor}_1^R(R', N) \to \mathrm{Tor}_1^R((R')_f, N)$$

and isomorphisms $\text{Tor}_i^R(R', N) = \text{Tor}_i^R((R')_f, N)$ for $i \geq 2$. Since $\text{Tor}_i^R((R')_f, N) = \text{Tor}_i^R(R', N)_f$ we conclude that f is a nonzerodivisor on $\text{Tor}_1^R(R', N)$ and invertible on $\text{Tor}_i^R(R', N)$ for $i \geq 2$. Since $M \otimes_R R'$ is flat over R' we have

$$\text{Tor}_i^R(M \otimes_R R', N) = (M \otimes_R R') \otimes_{R'} \text{Tor}_i^R(R', N)$$

by the spectral sequence of Example 62.2. Writing $M \otimes_R R'$ as a filtered colimit of finite free R'-modules (Algebra, Theorem 81.4) we conclude that f is a nonzerodivisor on $\text{Tor}_1^R(M \otimes_R R', N)$ and invertible on $\text{Tor}_i^R(M \otimes_R R', N)$. Next, we consider the exact sequence $0 \to M \to M \otimes_R R' \oplus M_f \to M \otimes_R (R')_f \to 0$ coming from the fact that M is glueable and the associated long exact sequence of Tor. The relevant part is

$$\begin{tikzcd}
\text{Tor}_1^R(M, N) \arrow[r] & \text{Tor}_1^R(M \otimes_R R', N) \arrow[r] & \text{Tor}_1^R(M \otimes_R (R')_f, N) \\
& \text{Tor}_2^R(M \otimes_R R', N) \arrow[r] & \text{Tor}_2^R(M \otimes_R (R')_f, N) \arrow[ull]
\end{tikzcd}$$

We conclude that $\text{Tor}_1^R(M, N) = 0$ by our remarks above on the action on f on $\text{Tor}_i^R(M \otimes_R R', N)$. ☐

Observe that we have seen the result of the following lemma for "finitely generated" in Lemma 90.5.

Lemma 90.20. *Let $(R \to R', f)$ be a glueing pair. Let M be an R-module which is not necessarily glueable for $(R \to R', f)$. Then M is a finite projective R-module if and only if $M \otimes_R R'$ is finite projective over R' and M_f is finite projective over R_f.*

Proof. Assume that $M \otimes_R R'$ is a finite projective module over R' and that M_f is a finite projective module over R_f. Our task is to prove that M is finite projective over R. We will use Algebra, Lemma 78.2 without further mention. By Lemma 90.19 we see that M is flat. By Lemma 90.5 we see that M is finite. Choose a short exact sequence $0 \to K \to R^{\oplus n} \to M \to 0$. Since a finite projective module is of finite presentation and since the sequence remains exact after tensoring with R' (by Algebra, Lemma 39.12) and R_f, we conclude that $K \otimes_R R'$ and K_f are finite modules. Using the lemma above we conclude that K is finitely generated. Hence M is finitely presented and hence finite projective. ☐

Remark 90.21. In [BL95] it is assumed that f is a nonzerodivisor in R and $R' = R^\wedge$, which gives a glueing pair by Lemma 90.7. Even in this setting Theorem 90.17 says something new: the results of [BL95] only apply to modules on which f is a nonzerodivisor (and hence glueable in our sense, see Lemma 90.11). Lemma 90.20 also provides a slight extension of the results of [BL95]: not only can we allow M to have nonzero f-power torsion, we do not even require it to be glueable.

91. Derived Completion

Some references for the material in this section are [DG02], [GM92], [PSY14], [Lur11] (especially Chapter 4). Our exposition follows [BS13]. The analogue (or "dual") of this section for torsion modules is Dualizing Complexes, Section 9. The relationship between the derived category of complexes with torsion cohomology and derived complete complexes can be found in Dualizing Complexes, Section 12.

Let $K \in D(A)$. Let $f \in A$. We denote $T(K, f)$ a derived limit of the system

$$\ldots \to K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(A)$.

Lemma 91.1. *Let A be a ring. Let $f \in A$. Let $K \in D(A)$. The following are equivalent*

1. $\mathrm{Ext}^n_A(A_f, K) = 0$ *for all n,*
2. $\mathrm{Hom}_{D(A)}(E, K) = 0$ *for all E in $D(A_f)$,*
3. $T(K, f) = 0$,
4. *for every $p \in \mathbf{Z}$ we have $T(H^p(K), f) = 0$,*
5. *for every $p \in \mathbf{Z}$ we have $\mathrm{Hom}_A(A_f, H^p(K)) = 0$ and $\mathrm{Ext}^1_A(A_f, H^p(K)) = 0$,*
6. $R\,\mathrm{Hom}_A(A_f, K) = 0$,
7. *the map $\prod_{n \geq 0} K \to \prod_{n \geq 0} K$, $(x_0, x_1, \ldots) \mapsto (x_0 - fx_1, x_1 - fx_2, \ldots)$ is an isomorphism in $D(A)$, and*
8. *add more here.*

Proof. It is clear that (2) implies (1) and that (1) is equivalent to (6). Assume (1). Let $I^\bullet$ be a K-injective complex of A-modules representing K. Condition (1) signifies that $\mathrm{Hom}_A(A_f, I^\bullet)$ is acyclic. Let $M^\bullet$ be a complex of A_f-modules representing E. Then

$$\mathrm{Hom}_{D(A)}(E, K) = \mathrm{Hom}_{K(A)}(M^\bullet, I^\bullet) = \mathrm{Hom}_{K(A_f)}(M^\bullet, \mathrm{Hom}_A(A_f, I^\bullet))$$

by Algebra, Lemma 14.4. As $\mathrm{Hom}_A(A_f, I^\bullet)$ is a K-injective complex of A_f-modules by Lemma 56.3 the fact that it is acyclic implies that it is homotopy equivalent to zero (Derived Categories, Lemma 31.2). Thus we get (2).

A free resolution of the A-module A_f is given by

$$0 \to \bigoplus\nolimits_{n \in \mathbf{N}} A \to \bigoplus\nolimits_{n \in \mathbf{N}} A \to A_f \to 0$$

where the first map sends the $(a_0, a_1, a_2, \ldots)$ to $(a_0, a_1 - fa_0, a_2 - fa_1, \ldots)$ and the second map sends $(a_0, a_1, a_2, \ldots)$ to $a_0 + a_1/f + a_2/f^2 + \ldots$. Applying $\mathrm{Hom}_A(-, I^\bullet)$ we get

$$0 \to \mathrm{Hom}_A(A_f, I^\bullet) \to \prod I^\bullet \to \prod I^\bullet \to 0$$

Since $\prod I^\bullet$ represents $\prod_{n \geq 0} K$ this proves the equivalence of (1) and (7). On the other hand, by construction of derived limits in Derived Categories, Section 34 the displayed exact sequence shows the object $T(K, f)$ is a representative of $R\,\mathrm{Hom}_A(A_f, K)$ in $D(A)$. Thus the equivalence of (1) and (3).

There is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_A(A_f, H^q(K)) \Rightarrow \mathrm{Ext}^{p+q}_A(A_f, K)$$

See Equation (67.0.1). This spectral sequence degenerates at E_2 because A_f has a length 1 resolution by projective A-modules (see above) hence the E_2-page has only 2 nonzero columns. Thus we obtain short exact sequences

$$0 \to \mathrm{Ext}^1_A(A_f, H^{p-1}(K)) \to \mathrm{Ext}^p_A(A_f, K) \to \mathrm{Hom}_A(A_f, H^p(K)) \to 0$$

This proves (4) and (5) are equivalent to (1). $\square$

Lemma 91.2. *Let A be a ring. Let $K \in D(A)$. The set I of $f \in A$ such that $T(K, f) = 0$ is a radical ideal of A.*

Proof. We will use the results of Lemma 91.1 without further mention. If $f \in I$, and $g \in A$, then A_{gf} is an A_f-module hence $\mathrm{Ext}^n_A(A_{gf}, K) = 0$ for all n, hence $gf \in I$. Suppose $f, g \in I$. Then there is a short exact sequence

$$0 \to A_{f+g} \to A_{f(f+g)} \oplus A_{g(f+g)} \to A_{gf(f+g)} \to 0$$

because f, g generate the unit ideal in A_{f+g}. This follows from Algebra, Lemma 24.2 and the easy fact that the last arrow is surjective. From the long exact sequence of Ext and the vanishing of $\mathrm{Ext}^n_A(A_{f(f+g)}, K)$, $\mathrm{Ext}^n_A(A_{g(f+g)}, K)$, and $\mathrm{Ext}^n_A(A_{gf(f+g)}, K)$ for all n we deduce the vanishing of $\mathrm{Ext}^n_A(A_{f+g}, K)$ for all n. Finally, if $f^n \in I$ for some $n > 0$, then $f \in I$ because $T(K, f) = T(K, f^n)$ or because $A_f \cong A_{f^n}$. □

Lemma 91.3. *Let A be a ring. Let $I \subset A$ be an ideal. Let M be an A-module.*

(1) *If M is I-adically complete, then $T(M, f) = 0$ for all $f \in I$.*
(2) *Conversely, if $T(M, f) = 0$ for all $f \in I$ and I is finitely generated, then $M \to \lim M/I^nM$ is surjective.*

Proof. Proof of (1). Assume M is I-adically complete. By Lemma 91.1 it suffices to prove $\mathrm{Ext}^1_A(A_f, M) = 0$ and $\mathrm{Hom}_A(A_f, M) = 0$. Since $M = \lim M/I^nM$ and since $\mathrm{Hom}_A(A_f, M/I^nM) = 0$ it follows that $\mathrm{Hom}_A(A_f, M) = 0$. Suppose we have an extension

$$0 \to M \to E \to A_f \to 0$$

For $n \geq 0$ pick $e_n \in E$ mapping to $1/f^n$. Set $\delta_n = fe_{n+1} - e_n \in M$ for $n \geq 0$. Replace e_n by

$$e'_n = e_n + \delta_n + f\delta_{n+1} + f^2\delta_{n+2} + \ldots$$

The infinite sum exists as M is complete with respect to I and $f \in I$. A simple calculation shows that $fe'_{n+1} = e'_n$. Thus we get a splitting of the extension by mapping $1/f^n$ to e'_n.

Proof of (2). Assume that $I = (f_1, \ldots, f_r)$ and that $T(M, f_i) = 0$ for $i = 1, \ldots, r$. By Algebra, Lemma 96.7 we may assume $I = (f)$ and $T(M, f) = 0$. Let $x_n \in M$ for $n \geq 0$. Consider the extension

$$0 \to M \to E \to A_f \to 0$$

given by

$$E = M \oplus \bigoplus Ae_n \Big/ \langle x_n - fe_{n+1} + e_n \rangle$$

mapping e_n to $1/f^n$ in A_f (see above). By assumption and Lemma 91.1 this extension is split, hence we obtain an element $x + e_0$ which generates a copy of A_f in E. Then

$$x + e_0 = x - x_0 + fe_1 = x - x_0 - fx_1 + f^2e_2 = \ldots$$

Since $M/f^nM = E/f^nE$ by the snake lemma, we see that $x = x_0 + fx_1 + \ldots + f^{n-1}x_{n-1}$ modulo f^nM. In other words, the map $M \to \lim M/f^nM$ is surjective as desired. □

Motivated by the results above we make the following definition.

Definition 91.4. Let A be a ring. Let $K \in D(A)$. Let $I \subset A$ be an ideal. We say K is *derived complete with respect to* I if for every $f \in I$ we have $T(K, f) = 0$. If M is an A-module, then we say M is *derived complete with respect to* I if $M[0] \in D(A)$ is derived complete with respect to I.

The full subcategory $D_{comp}(A) = D_{comp}(A, I) \subset D(A)$ consisting of derived complete objects is a strictly full, saturated triangulated subcategory, see Derived Categories, Definitions 3.4 and 6.1. By Lemma 91.2 the subcategory $D_{comp}(A, I)$ depends only on the radical $\sqrt{I}$ of I, in other words it depends only on the closed subset $Z = V(I)$ of $\text{Spec}(A)$. The subcategory $D_{comp}(A, I)$ is preserved under products and homotopy limits in $D(A)$. But it is not preserved under countable direct sums in general. We will often simply say M is a derived complete module if the choice of the ideal I is clear from the context.

Proposition 91.5. *Let $I \subset A$ be a finitely generated ideal of a ring A. Let M be an A-module. The following are equivalent*

1. *M is I-adically complete, and*
2. *M is derived complete with respect to I and $\bigcap I^n M = 0$.*

Proof. This is clear from the results of Lemma 91.3. □

The next lemma shows that the category $\mathcal{C}$ of derived complete modules is abelian. It turns out that $\mathcal{C}$ is not a Grothendieck abelian category, see Examples, Section 11.

Lemma 91.6. *Let I be an ideal of a ring A.*

1. *The derived complete A-modules form a weak Serre subcategory $\mathcal{C}$ of Mod_A.*
2. *$D_{\mathcal{C}}(A) \subset D(A)$ is the full subcategory of derived complete objects.*

Proof. Part (2) is immediate from Lemma 91.1 and the definitions. For part (1), suppose that $M \to N$ is a map of derived complete modules. Denote $K = (M \to N)$ the corresponding object of $D(A)$. Pick $f \in I$. Then $\text{Ext}^n_A(A_f, K)$ is zero for all n because $\text{Ext}^n_A(A_f, M)$ and $\text{Ext}^n_A(A_f, N)$ are zero for all n. Hence K is derived complete. By (2) we see that $\text{Ker}(M \to N)$ and $\text{Coker}(M \to N)$ are objects of $\mathcal{C}$. Finally, suppose that $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of A-modules and M_1, M_3 are derived complete. Then it follows from the long exact sequence of Ext's that M_2 is derived complete. Thus $\mathcal{C}$ is a weak Serre subcategory by Homology, Lemma 10.3. □

We will generalize the following lemma in Lemma 91.19.

Lemma 91.7. *Let I be a finitely generated ideal of a ring A. Let M be a derived complete A-module. If $M/IM = 0$, then $M = 0$.*

Proof. Assume that M/IM is zero. Let $I = (f_1, \ldots, f_r)$. Let $i < r$ be the largest integer such that $N = M/(f_1, \ldots, f_i)M$ is nonzero. If i does not exist, then $M = 0$ which is what we want to show. Then N is derived complete as a cokernel of a map between derived complete modules, see Lemma 91.6. By our choice of i we have that $f_{i+1} : N \to N$ is surjective. Hence

$$\lim(\ldots \to N \xrightarrow{f_{i+1}} N \xrightarrow{f_{i+1}} N)$$

is nonzero, contradicting the derived completeness of N. □

If the ring is I-adically complete, then one obtains an ample supply of derived complete complexes.

Lemma 91.8. *Let A be a ring and $I \subset A$ an ideal. If A is derived complete (eg. I-adically complete) then any pseudo-coherent object of $D(A)$ is derived complete.*

Proof. (Lemma 91.3 explains the parenthetical statement of the lemma.) Let K be a pseudo-coherent object of $D(A)$. By definition this means K is represented by a bounded above complex $K^\bullet$ of finite free A-modules. Since A is derived complete it follows that $H^n(K)$ is derived complete for all n, by part (1) of Lemma 91.6. This in turn implies that K is derived complete by part (2) of the same lemma. □

Lemma 91.9. *Let A be a ring. Let $f, g \in A$. Then for $K \in D(A)$ we have $R\operatorname{Hom}_A(A_f, R\operatorname{Hom}_A(A_g, K)) = R\operatorname{Hom}_A(A_{fg}, K)$.*

Proof. This follows from Lemma 73.1. □

Lemma 91.10. *Let I be a finitely generated ideal of a ring A. The inclusion functor $D_{comp}(A, I) \to D(A)$ has a left adjoint, i.e., given any object K of $D(A)$ there exists a map $K \to K^\wedge$ of K into a derived complete object of $D(A)$ such that the map*

$$\operatorname{Hom}_{D(A)}(K^\wedge, E) \longrightarrow \operatorname{Hom}_{D(A)}(K, E)$$

is bijective whenever E is a derived complete object of $D(A)$. In fact, if I is generated by $f_1, \ldots, f_r \in A$, then we have

$$K^\wedge = R\operatorname{Hom}\left((A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}), K\right)$$

functorially in K.

Proof. Define $K^\wedge$ by the last displayed formula of the lemma. There is a map of complexes

$$(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \longrightarrow A$$

which induces a map $K \to K^\wedge$. It suffices to prove that $K^\wedge$ is derived complete and that $K \to K^\wedge$ is an isomorphism if K is derived complete.

Let $f \in A$. By Lemma 91.9 the object $R\operatorname{Hom}_A(A_f, K^\wedge)$ is equal to

$$R\operatorname{Hom}\left((A_f \to \prod\nolimits_{i_0} A_{f f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f f_{i_0} f_{i_1}} \to \ldots \to A_{f f_1 \ldots f_r}), K\right)$$

If $f \in I$, then $f_1, \ldots, f_r$ generate the unit ideal in A_f, hence the extended alternating Čech complex

$$A_f \to \prod\nolimits_{i_0} A_{f f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f f_{i_0} f_{i_1}} \to \ldots \to A_{f f_1 \ldots f_r}$$

is zero in $D(A)$ by Lemma 29.5. (In fact, if $f = f_i$ for some i, then this complex is homotopic to zero by Lemma 29.4; this is the only case we need.) Hence $R\operatorname{Hom}_A(A_f, K^\wedge) = 0$ and we conclude that $K^\wedge$ is derived complete by Lemma 91.1.

Conversely, if K is derived complete, then $R\operatorname{Hom}_A(A_f, K)$ is zero for all $f = f_{i_0} \ldots f_{i_p}$, $p \geq 0$. Thus $K \to K^\wedge$ is an isomorphism in $D(A)$. □

Remark 91.11. Let A be a ring and let $I \subset A$ be a finitely generated ideal. The left adjoint to the inclusion functor $D_{comp}(A, I) \to D(A)$ which exists by Lemma 91.10 is called the *derived completion*. To indicate this we will say "let $K^\wedge$ be the derived completion of K". Please keep in mind that the unit of the adjunction is a functorial map $K \to K^\wedge$.

Lemma 91.12. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of A_f-modules. Then the derived completion of $K^\bullet$ is zero.*

Proof. Indeed, in this case the $R\operatorname{Hom}_A(K, L)$ is zero for any derived complete complex L, see Lemma 91.1. Hence $K^\wedge$ is zero by the universal property in Lemma 91.10. □

Lemma 91.13. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K, L \in D(A)$. Then*

$$R\operatorname{Hom}_A(K, L)^\wedge = R\operatorname{Hom}_A(K, L^\wedge) = R\operatorname{Hom}_A(K^\wedge, L^\wedge)$$

Proof. By Lemma 91.10 we know that derived completion is given by $R\operatorname{Hom}_A(C, -)$ for some $C \in D(A)$. Then

$$\begin{aligned} R\operatorname{Hom}_A(C, R\operatorname{Hom}_A(K, L)) &= R\operatorname{Hom}_A(C \otimes_A^{\mathbf{L}} K, L) \\ &= R\operatorname{Hom}_A(K, R\operatorname{Hom}_A(C, L)) \end{aligned}$$

by Lemma 73.1. This proves the first equation. The map $K \to K^\wedge$ induces a map

$$R\operatorname{Hom}_A(K^\wedge, L^\wedge) \to R\operatorname{Hom}_A(K, L^\wedge)$$

which is an isomorphism in $D(A)$ by definition of the derived completion as the left adjoint to the inclusion functor. □

Lemma 91.14. *Let A be a ring and let $I \subset A$ be an ideal. Let (K_n) be an inverse system of objects of $D(A)$ such that for all $f \in I$ and n there exists an $e = e(n, f)$ such that f^e is zero on K_n. Then for $K \in D(A)$ the object $K' = R\lim(K \otimes_A^{\mathbf{L}} K_n)$ is derived complete with respect to I.*

Proof. Since the category of derived complete objects is preserved under $R\lim$ it suffices to show that each $K \otimes_A^{\mathbf{L}} K_n$ is derived complete. By assumption for all $f \in I$ there is an e such that f^e is zero on $K \otimes_A^{\mathbf{L}} K_n$. Of course this implies that $T(K \otimes_A^{\mathbf{L}} K_n, f) = 0$ and we win. □

Situation 91.15. Let A be a ring. Let $I = (f_1, \ldots, f_r) \subset A$. Let $K_n^\bullet = K_\bullet(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex on $f_1^n, \ldots, f_r^n$ viewed as a cochain complex in degrees $-r, -r+1, \ldots, 0$. Using the functoriality of Lemma 28.3 we obtain an inverse system

$$\ldots \to K_3^\bullet \to K_2^\bullet \to K_1^\bullet$$

compatible with the inverse system $H^0(K_n^\bullet) = A/(f_1^n, \ldots, f_r^n)$ and compatible with the maps $A \to K_n^\bullet$.

A key feature of the discussion below will use that for $m > n$ the map

$$K_m^{-p} = \wedge^p(A^{\oplus r}) \to \wedge^p(A^{\oplus r}) = K_n^{-p}$$

is given by multiplication by $f_{i_1}^{m-n} \ldots f_{i_p}^{m-n}$ on the basis element $e_{i_1} \wedge \ldots \wedge e_{i_p}$.

Lemma 91.16. *In Situation 91.15. For $K \in D(A)$ the object $K' = R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet)$ is derived complete with respect to I.*

Proof. This is a special case of Lemma 91.14 because f_i^n acts by an endomorphism of $K_n^\bullet$ which is homotopic to zero by Lemma 28.6. $\square$

Lemma 91.17. *In Situation 91.15. Let $K \in D(A)$. The following are equivalent*

(1) *K is derived complete with respect to I, and*
(2) *the canonical map $K \to R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet)$ is an isomorphism of $D(A)$.*

Proof. If (2) holds, then K is derived complete with respect to I by Lemma 91.16. Conversely, assume that K is derived complete with respect to I. Consider the filtrations

$$K_n^\bullet \supset \sigma_{\geq -r+1} K_n^\bullet \supset \sigma_{\geq -r+2} K_n^\bullet \supset \ldots \supset \sigma_{\geq -1} K_n^\bullet \supset \sigma_{\geq 0} K_n^\bullet = A$$

by stupid truncations (Homology, Section 15). Because the construction $R\lim(K \otimes E)$ is exact in the second variable (Lemma 87.11) we see that it suffices to show

$$R\lim\left(K \otimes_A^{\mathbf{L}} (\sigma_{\geq p} K_n^\bullet / \sigma_{\geq p+1} K_n^\bullet)\right) = 0$$

for $p < 0$. The explicit description of the Koszul complexes above shows that

$$R\lim\left(K \otimes_A^{\mathbf{L}} (\sigma_{\geq p} K_n^\bullet / \sigma_{\geq p+1} K_n^\bullet)\right) = \bigoplus\nolimits_{i_1, \ldots, i_{-p}} T(K, f_{i_1} \ldots f_{i_{-p}})$$

which is zero for $p < 0$ by assumption on K. $\square$

Lemma 91.18. *In Situation 91.15. The functor which sends $K \in D(A)$ to the derived limit $K' = R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet)$ is the left adjoint to the inclusion functor $D_{comp}(A) \to D(A)$ constructed in Lemma 91.10.*

First proof. The assignment $K \rightsquigarrow K'$ is a functor and K' is derived complete with respect to I by Lemma 91.16. By a formal argument (omitted) we see that it suffices to show $K \to K'$ is an isomorphism if K is derived complete with respect to I. This is Lemma 91.17. $\square$

Second proof. Denote $K \mapsto K^\wedge$ the adjoint constructed in Lemma 91.10. By that lemma we have

$$K^\wedge = R\operatorname{Hom}\left((A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}), K\right)$$

In Lemma 29.6 we have seen that the extended alternating Čech complex

$$A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}$$

is a colimit of the Koszul complexes $K^n = K(A, f_1^n, \ldots, f_r^n)$ sitting in degrees $0, \ldots, r$. Note that K^n is a finite chain complex of finite free A-modules with dual (as in Lemma 74.15) $R\operatorname{Hom}_A(K^n, A) = K_n$ where K_n is the Koszul cochain complex sitting in degrees $-r, \ldots, 0$ (as usual). Thus it suffices to show that

$$R\operatorname{Hom}_A(\operatorname{hocolim} K^n, K) = R\lim(K \otimes_A^{\mathbf{L}} K_n)$$

This follows from Lemma 74.16. $\square$

Lemma 91.19. *Let I be a finitely generated ideal of a ring A. Let K be a derived complete object of $D(A)$. If $K \otimes_A^{\mathbf{L}} A/I = 0$, then $K = 0$.*

Proof. Choose generators $f_1, \ldots, f_r$ of I. Denote K_n the Koszul complex on $f_1^n, \ldots, f_r^n$ over A. Recall that K_n is bounded and that the cohomology modules of K_n are annihilated by $f_1^n, \ldots, f_r^n$ and hence by I^{nr}. By Lemma 88.7 we see that $K \otimes_A^{\mathbf{L}} K_n = 0$. Since K is derived complete by Lemma 91.18 we have $K = R\lim K \otimes_A^{\mathbf{L}} K_n = 0$ as desired. $\square$

As an application of the relationship with the Koszul complex we obtain that derived completion has finite cohomological dimension.

Lemma 91.20. *Let A be a ring and let $I \subset A$ be an ideal which can be generated by r elements. Then derived completion has finite cohomological dimension:*

1. *Let $K \to L$ be a morphism in $D(A)$ such that $H^i(K) \to H^i(L)$ is an isomorphism for $i \geq 1$ and surjective for $i = 0$. Then $H^i(K^\wedge) \to H^i(L^\wedge)$ is an isomorphism for $i \geq 1$ and surjective for $i = 0$.*
2. *Let $K \to L$ be a morphism of $D(A)$ such that $H^i(K) \to H^i(L)$ is an isomorphism for $i \leq -1$ and injective for $i = 0$. Then $H^i(K^\wedge) \to H^i(L^\wedge)$ is an isomorphism for $i \leq -r-1$ and injective for $i = -r$.*

Proof. Say I is generated by $f_1, \ldots, f_r$. For any $K \in D(A)$ by Lemma 91.18 we have $K^\wedge = R\lim K \otimes_A^{\mathbf{L}} K_n$ where K_n is the Koszul complex on $f_1^n, \ldots, f_r^n$ and hence we obtain a short exact sequence

$$0 \to R^1 \lim H^{i-1}(K \otimes_A^{\mathbf{L}} K_n) \to H^i(K^\wedge) \to \lim H^i(K \otimes_A^{\mathbf{L}} K_n) \to 0$$

by Lemma 87.4.

Proof of (1). Pick a distinguished triangle $K \to L \to C \to K[1]$. Then $H^i(C) = 0$ for $i \geq 0$. Since K_n is sitting in degrees ≤ 0 we see that $H^i(C \otimes_A^{\mathbf{L}} K_n) = 0$ for $i \geq 0$ and that $H^{-1}(C \otimes_A^{\mathbf{L}} K_n) = H^{-1}(C) \otimes_A A/(f_1^n, \ldots, f_r^n)$ is a system with surjective transition maps. The displayed equation above shows that $H^i(C^\wedge) = 0$ for $i \geq 0$. Applying the distinguished triangle $K^\wedge \to L^\wedge \to C^\wedge \to K^\wedge[1]$ we get (1).

Proof of (2). Pick a distinguished triangle $K \to L \to C \to K[1]$. Then $H^i(C) = 0$ for $i < 0$. Since K_n is sitting in degrees $-r, \ldots, 0$ we see that $H^i(C \otimes_A^{\mathbf{L}} K_n) = 0$ for $i < -r$. The displayed equation above shows that $H^i(C^\wedge) = 0$ for $i < r$. Applying the distinguished triangle $K^\wedge \to L^\wedge \to C^\wedge \to K^\wedge[1]$ we get (2). $\square$

Lemma 91.21. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a filtered complex of A-modules. There exists a canonical spectral sequence $(E_r, d_r)_{r \geq 1}$ of bigraded derived complete A-modules with d_r of bidegree $(r, -r+1)$ and with*

$$E_1^{p,q} = H^{p+q}((gr^p K^\bullet)^\wedge)$$

If the filtration on each K^n is finite, then the spectral sequence is bounded and converges to $H^*((K^\bullet)^\wedge)$.

Proof. By Lemma 91.10 we know that derived completion is given by $R\operatorname{Hom}_A(C, -)$ for some $C \in D^b(A)$. By Lemmas 91.20 and 68.2 we see that C has finite projective dimension. Thus we may choose a bounded complex of projective modules $P^\bullet$ representing C. Then

$$M^\bullet = \operatorname{Hom}^\bullet(P^\bullet, K^\bullet)$$

is a complex of A-modules representing $(K^\bullet)^\wedge$. It comes with a filtration given by $F^pM^\bullet = \operatorname{Hom}^\bullet(P^\bullet, F^pK^\bullet)$. We see that $F^pM^\bullet$ represents $(F^pK^\bullet)^\wedge$ and hence $\operatorname{gr}^pM^\bullet$ represents $(\operatorname{gr}K^\bullet)^\wedge$. Thus we find our spectral sequence by taking the spectral sequence of the filtered complex $M^\bullet$, see Homology, Section 24. If the filtration on each K^n is finite, then the filtration on each M^n is finite because $P^\bullet$ is a bounded complex. Hence the final statement follows from Homology, Lemma 24.11. $\square$

Example 91.22. Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules. We can apply Lemma 91.21 with $F^pK^\bullet = \tau_{\leq -p}K^\bullet$. Then we get a bounded spectral sequence

$$E_1^{p,q} = H^{p+q}(H^{-p}(K^\bullet)^\wedge[p]) = H^{2p+q}(H^{-p}(K^\bullet)^\wedge)$$

converging to $H^{p+q}((K^\bullet)^\wedge)$. After renumbering $p = -j$ and $q = i + 2j$ we find that for any $K \in D(A)$ there is a bounded spectral sequence $(E'_r, d'_r)_{r\geq 2}$ of bigraded derived complete modules with d'_r of bidegree $(r, -r+1)$, with

$$(E'_2)^{i,j} = H^i(H^j(K)^\wedge)$$

and converging to $H^{i+j}(K^\wedge)$.

Lemma 91.23. *Let $A \to B$ be a ring map. Let $I \subset A$ be an ideal. The inverse image of $D_{comp}(A, I)$ under the restriction functor $D(B) \to D(A)$ is $D_{comp}(B, IB)$.*

Proof. Using Lemma 91.2 we see that $L \in D(B)$ is in $D_{comp}(B, IB)$ if and only if $T(L, f)$ is zero for every local section $f \in I$. Observe that the cohomology of $T(L, f)$ is computed in the category of abelian groups, so it doesn't matter whether we think of f as an element of A or take the image of f in B. The lemma follows immediately from this and the definition of derived complete objects. $\square$

Lemma 91.24. *Let $A \to B$ be a ring map. Let $I \subset A$ be a finitely generated ideal. If $A \to B$ is flat and $A/I \cong B/IB$, then the restriction functor $D(B) \to D(A)$ induces an equivalence $D_{comp}(B, IB) \to D_{comp}(A, I)$.*

Proof. Choose generators $f_1, \ldots, f_r$ of I. Denote $\check{\mathcal{C}}_A^\bullet \to \check{\mathcal{C}}_B^\bullet$ the quasi-isomorphism of extended alternating Čech complexes of Lemma 89.4. Let $K \in D_{comp}(A, I)$. Let $I^\bullet$ be a K-injective complex of A-modules representing K. Since $\mathrm{Ext}^n_A(A_f, K)$ and $\mathrm{Ext}^n_A(B_f, K)$ are zero for all $f \in I$ and $n \in \mathbf{Z}$ (Lemma 91.1) we conclude that $\check{\mathcal{C}}_A^\bullet \to A$ and $\check{\mathcal{C}}_B^\bullet \to B$ induce quasi-isomorphisms

$$I^\bullet = \mathrm{Hom}_A(A, I^\bullet) \longrightarrow \mathrm{Tot}(\mathrm{Hom}_A(\check{\mathcal{C}}_A^\bullet, I^\bullet))$$

and

$$\mathrm{Hom}_A(B, I^\bullet) \longrightarrow \mathrm{Tot}(\mathrm{Hom}_A(\check{\mathcal{C}}_B^\bullet, I^\bullet))$$

Some details omitted. Since $\check{\mathcal{C}}_A^\bullet \to \check{\mathcal{C}}_B^\bullet$ is a quasi-isomorphism and $I^\bullet$ is K-injective we conclude that $\mathrm{Hom}_A(B, I^\bullet) \to I^\bullet$ is a quasi-isomorphism. As the complex $\mathrm{Hom}_A(B, I^\bullet)$ is a complex of B-modules we conclude that K is in the image of the restriction map, i.e., the functor is essentially surjective

In fact, the argument shows that $F : D_{comp}(A, I) \to D_{comp}(B, IB)$, $K \mapsto \mathrm{Hom}_A(B, I^\bullet)$ is a left inverse to restriction. Finally, suppose that $L \in D_{comp}(B, IB)$. Represent L by a K-injective complex $J^\bullet$ of B-modules. Then $J^\bullet$ is also K-injective as a complex of A-modules (Lemma 56.1) hence $F(\text{restriction of } L) = \mathrm{Hom}_A(B, J^\bullet)$. There is a map $J^\bullet \to \mathrm{Hom}_A(B, J^\bullet)$ of complexes of B-modules, whose composition with $\mathrm{Hom}_A(B, J^\bullet) \to J^\bullet$ is the identity. We conclude that F is also a right inverse to restriction and the proof is finished. $\square$

92. The category of derived complete modules

Let A be a ring and let I be an ideal. Denote $\mathcal{C}$ the category of derived complete modules, see Definition 91.4. In this section we discuss some properties of this category. In Examples, Section 11 we show that $\mathcal{C}$ isn't a Grothendieck abelian category in general.

By Lemma 91.6 the category $\mathcal{C}$ is abelian and the inclusion functor $\mathcal{C} \to \text{Mod}_A$ is exact.

Since $D_{comp}(A) \subset D(A)$ is closed under products (see discussion following Definition 91.4) and since products in $D(A)$ are computed on the level of complexes, we see that $\mathcal{C}$ has products which agree with products in Mod_A. Thus $\mathcal{C}$ in fact has arbitrary limits and the inclusion functor $\mathcal{C} \to \text{Mod}_A$ commutes with them, see Categories, Lemma 14.11.

Assume I is finitely generated. Let $^\wedge : D(A) \to D(A)$ denote the derived completion functor of Lemma 91.10. Let us show the functor

$$\text{Mod}_A \longrightarrow \mathcal{C}, \quad M \longmapsto H^0(M^\wedge)$$

is a left adjoint to the inclusion functor $\mathcal{C} \to \text{Mod}_A$. Note that $H^i(M^\wedge) = 0$ for $i > 0$ for example by Lemma 91.20. Hence, if N is a derived complete A-module, then we have

$$\begin{aligned} \text{Hom}_\mathcal{C}(H^0(M^\wedge), N) &= \text{Hom}_{D_{comp}(A)}(M^\wedge, N) \\ &= \text{Hom}_{D(A)}(M, N) \\ &= \text{Hom}_A(M, N) \end{aligned}$$

as desired.

Let T be a preordered set and let $t \mapsto M_t$ be a system of derived complete A-modules, i.e., a system over T in $\mathcal{C}$, see Categories, Section 21. Denote $\text{colim}_{t \in T} M_t$ the colimit of the system in Mod_A. It follows formally from the above that

$$H^0((\text{colim}_{t \in T} M_t)^\wedge)$$

is the colimit of the system in $\mathcal{C}$. In this way we see that $\mathcal{C}$ has all colimits. In general the inclusion functor $\mathcal{C} \to \text{Mod}_A$ will not commute with colimits, see Examples, Section 11.

Lemma 92.1. *Let A be a ring and let $I \subset A$ be an ideal. The category $\mathcal{C}$ of derived complete modules is abelian, has arbitrary limits, and the inclusion functor $F : \mathcal{C} \to Mod_A$ is exact and commutes with limits. If I is finitely generated, then $\mathcal{C}$ has arbitrary colimits and F has a left adjoint*

Proof. This summarizes the discussion above. □

93. Derived completion for a principal ideal

In this section we discuss what happens with derived completion when the ideal is generated by a single element.

Lemma 93.1. *Let A be a ring. Let $f \in A$. If there exists an integer $c \geq 1$ such that $A[f^c] = A[f^{c+1}] = A[f^{c+2}] = \ldots$ (for example if A is Noetherian), then for all $n \geq 1$ there exist maps*

$$(A \xrightarrow{f^n} A) \longrightarrow A/(f^n), \quad \text{and} \quad A/(f^{n+c}) \longrightarrow (A \xrightarrow{f^n} A)$$

in $D(A)$ inducing an isomorphism of the pro-objects $\{A/(f^n)\}$ and $\{(f^n : A \to A)\}$ in $D(A)$.

Proof. The first displayed arrow is obvious. We can define the second arrow of the lemma by the diagram

$$\begin{tikzcd} A/A[f^c] \arrow[r, "f^{n+c}"'] \arrow[d, "f^c"'] & A \arrow[d, "1"] \\ A \arrow[r, "f^n"] & A \end{tikzcd}$$

Since the top horizontal arrow is injective the complex in the top row is quasi-isomorphic to $A/f^{n+c}A$. We omit the calculation of compositions needed to show the statement on pro objects. $\square$

Lemma 93.2. *Let A be a ring and $f \in A$. Set $I = (f)$. In this situation we have the naive derived completion $K \mapsto K' = R\lim(K \otimes_A^{\mathbf{L}} A/f^nA)$ and the derived completion*

$$K \mapsto K^\wedge = R\lim(K \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A))$$

of Lemma 91.18. The natural transformation of functors $K^\wedge \to K'$ is an isomorphism if and only if the f-power torsion of A is bounded.

Proof. If the f-power torsion is bounded, then the pro-objects $\{(f^n : A \to A)\}$ and $\{A/f^nA\}$ are isomorphic by Lemma 93.1. Hence the functors are isomorphic by Lemma 86.11. Conversely, we see from Lemma 87.11 that the condition is exactly that

$$R\lim(K \otimes_A^{\mathbf{L}} A[f^n])$$

is zero for all $K \in D(A)$. Here the maps of the system $(A[f^n])$ are given by multiplication by f. Taking $K = A$ and $K = \bigoplus_{i \in \mathbf{N}} A$ we see from Lemma 86.14 this implies $(A[f^n])$ is zero as a pro-object, i.e., $f^{n-1}A[f^n] = 0$ for some n, i.e., $A[f^{n-1}] = A[f^n]$, i.e., the f-power torsion is bounded. $\square$

Example 93.3. Let A be a ring. Let $f \in A$ be a nonzerodivisor. An example to keep in mind is $A = \mathbf{Z}_p$ and $f = p$. Let M be an A-module. Claim: M is derived complete with respect to f if and only if there exists a short exact sequence

$$0 \to K \to L \to M \to 0$$

where K, L are f-adically complete modules whose f-torsion is zero. Namely, if there is a such a short exact sequence, then

$$M \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A) = (K/f^nK \to L/f^nL)$$

because f is a nonzerodivisor on K and L and we conclude that $R\lim(M \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A))$ is quasi-isomorphic to $K \to L$, i.e., M. This shows that M is derived complete by Lemma 91.17. Conversely, suppose that M is derived complete. Choose a surjection $F \to M$ where F is a free A-module. Since f is a nonzerodivisor on F the derived completion of F is $L = \lim F/f^nF$. Note that L is f-torsion free: if (x_n) with $x_n \in F$ represents an element ξ of L and $f\xi = 0$, then $x_n = x_{n+1} + f^nz_n$ and $fx_n = f^ny_n$ for some $z_n, y_n \in F$. Then $f^ny_n = fx_n = fx_{n+1} + f^{n+1}z_n = f^{n+1}y_{n+1} + f^{n+1}z_n$ and since f is a nonzerodivisor on F we see that $y_n \in fF$ which implies that $x_n \in f^nF$, i.e., $\xi = 0$. Since L is the derived completion, the universal property gives a map $L \to M$ factoring $F \to M$. Let $K = \mathrm{Ker}(L \to M)$ be the kernel. Again K is f-torsion free, hence the derived completion of K is $\lim K/f^nK$. On the other hand, both M and L are derived complete, hence K is too by Lemma 91.6. It follows that $K = \lim K/f^nK$ and the claim is proved.

Example 93.4. Let p be a prime number. Consider the map $\mathbf{Z}_p[x] \to \mathbf{Z}_p[y]$ of polynomial algebras sending x to py. Consider the cokernel $M = \mathrm{Coker}(\mathbf{Z}_p[x]^\wedge \to \mathbf{Z}_p[y]^\wedge)$ of the induced map on (ordinary) p-adic completions. Then M is a derived complete $\mathbf{Z}_p$-module by Proposition 91.5 and Lemma 91.6; see also discussion in Example 93.3. However, M is not p-adically complete as $1 + py + p^2y^2 + \ldots$ maps to a nonzero element of M which is contained in $\bigcap p^nM$.

Example 93.5. Let A be a ring and let $f \in A$. Denote $K \mapsto K^\wedge$ the derived completion with respect to (f). Let M be an A-module. Using that

$$M^\wedge = R\lim(M \xrightarrow{f^n} M)$$

by Lemma 91.18 and using Lemma 87.4 we obtain

$$H^{-1}(M^\wedge) = \lim M[f^n] = T_f(M)$$

the *f-adic Tate module of M*. Here the maps $M[f^n] \to M[f^{n-1}]$ are given by multiplication by f. Then there is a short exact sequence

$$0 \to R^1\lim M[f^n] \to H^0(M^\wedge) \to \lim M/f^nM \to 0$$

describing $H^0(M^\wedge)$. We have $H^1(M^\wedge) = R^1\lim M/f^nM = 0$ as the transition maps are surjective (Lemma 87.1). All the other cohomologies of $M^\wedge$ are zero for trivial reasons. Finally, for $K \in D(A)$ and $p \in \mathbf{Z}$ there is a short exact sequence

$$0 \to H^0(H^p(K)^\wedge) \to H^p(K^\wedge) \to T_f(H^{p+1}(K)) \to 0$$

This follows from the spectral sequence of Example 91.22 because it degenerates at E_2 (as only $i = -1, 0$ give nonzero terms); the next lemma gives more information.

Lemma 93.6. *Let A be a ring and let $f \in A$. Let K be an object of $D(A)$. Denote $K_n = K \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A)$. For all $p \in \mathbf{Z}$ there is a commutative diagram*

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \widehat{H^p(K)} \arrow[r] \arrow[u] & \lim H^p(K_n) \arrow[r] \arrow[u] & T_f(H^{p+1}(K)) \arrow[r] & 0 \\
0 \arrow[r] & H^0(H^p(K)^\wedge) \arrow[r] \arrow[u] & H^p(K^\wedge) \arrow[r] \arrow[u] & T_f(H^{p+1}(K)) \arrow[r] \arrow[u, equal] & 0 \\
 & R^1\lim H^p(K)[f^n] \arrow[r, "\cong"] \arrow[u] & R^1\lim H^{p-1}(K_n) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

with exact rows and columns where $\widehat{H^p(K)} = \lim H^p(K)/f^nH^p(K)$ is the usual f-adic completion. The left vertical short exact sequence and the middle horizontal *short exact sequence are taken from Example 93.5 The middle vertical short exact sequence is the one from Lemma 87.4.*

Proof. To construct the top horizontal short exact sequence, observe that we have the following inverse system short exact sequences

$$0 \to H^p(K)/f^nH^p(K) \to H^p(K_n) \to H^{p+1}(K)[f^n] \to 0$$

coming from the construction of K_n as a shift of the cone on $f^n : K \to K$. Taking the inverse limit of these we obtain the top horizontal short exact sequence, see Homology, Lemma 31.3.

Let us prove that we have a commutative diagram as in the lemma. We consider the map $L = \tau_{\leq p}K \to K$. Setting $L_n = L \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A)$ we obtain a map $(L_n) \to (K_n)$ of inverse systems which induces a map of short exact sequences

$$\begin{tikzcd}
0 & 0 \\
\lim H^p(L_n) \arrow[u] \arrow[r] & \lim H^p(K_n) \arrow[u] \\
H^p(L^\wedge) \arrow[u] \arrow[r] & H^p(K^\wedge) \arrow[u] \\
R^1 \lim H^{p-1}(L_n) \arrow[u] \arrow[r] & R^1 \lim H^{p-1}(K_n) \arrow[u] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Since $H^i(L) = 0$ for $i > p$ and $H^p(L) = H^p(K)$, a computation using the references in the statement of the lemma shows that $H^p(L^\wedge) = H^0(H^p(K)^\wedge)$ and that $H^p(L_n) = H^p(K)/f^nH^p(K)$. On the other hand, we have $H^{p-1}(L_n) = H^{p-1}(K_n)$ and hence we see that we get the isomorphism as indicated in the statement of the lemma since we already know the kernel of $H^0(H^p(K)^\wedge) \to \widehat{H^p(K)}$ is equal to $R^1 \lim H^p(K)[f^n]$. We omit the verification that the rightmost square in the diagram commutes if we define the top row by the construction in the first paragraph of the proof. $\square$

Remark 93.7. With notation as in Lemma 93.6 we also see that the inverse system $H^p(K_n)$ has ML if and only if the inverse system $H^{p+1}(K)[f^n]$ has ML. This follows from the inverse system of short exact sequences $0 \to H^p(K)/f^nH^p(K) \to H^p(K_n) \to H^{p+1}(K)[f^n] \to 0$ (see proof of the lemma) combined with Homology, Lemma 31.3 and Lemma 86.13.

Lemma 93.8 (Bhatt)**.** *Let I be a finitely generated ideal in a ring A. Let M be a derived complete A-module. If M is an I-power torsion module, then $I^nM = 0$ for some n.*

Proof. Say $I = (f_1, \ldots, f_r)$. It suffices to show that for each i there is an n_i such that $f_i^{n_i}M = 0$. Hence we may assume that $I = (f)$ is a principal ideal. Let $B = \mathbf{Z}[x] \to A$ be the ring map sending x to f. By Lemma 91.23 we see that M

is derived complete as a B-module with respect to the ideal (x). After replacing A by B, we may assume that f is a nonzerodivisor in A.

Assume $I = (f)$ with $f \in A$ a nonzerodivisor. According to Example 93.3 there exists a short exact sequence

$$0 \to K \xrightarrow{u} L \to M \to 0$$

where K and L are I-adically complete A-modules whose f-torsion is zero[11]. Consider K and L as topological modules with the I-adic topology. Then u is continuous. Let

$$L_n = \{x \in L \mid f^n x \in u(K)\}$$

Since M is f-power torsion we see that $L = \bigcup L_n$. Let N_n be the closure of L_n in L. By Lemma 36.4 we see that N_n is open in L for some n. Fix such an n. Since $f^{n+m} : L \to L$ is a continuous open map, and since $f^{n+m}L_n \subset u(f^m K)$ we conclude that the closure of $u(f^m K)$ is open for all $m \geq 1$. Thus by Lemma 36.5 we conclude that u is open. Hence $f^t L \subset \mathrm{Im}(u)$ for some t and we conclude that f^t annihilates M as desired. $\square$

Lemma 93.9. *Let $f \in A$ be an element of a ring. Set $J = \bigcap f^n A$. Let M be an A-module derived complete with respect to f. Then $JM' = 0$ where $M' = \mathrm{Ker}(M \to \lim M/f^n M)$. In particular, if A is derived complete then J is an ideal of square zero.*

Proof. Take $x \in M'$ and $g \in J$. For every $n \geq 1$ we may write $x = f^n x_n$. Since g is in $f^n A$ we see that the element $y_n = g x_n$ in M' is independent of the choice of x_n. In particular, we may take $x_n = f x_{n+1}$ and we find that $y_n = f y_{n+1}$. Thus we obtain a map $A_f \to M$ sending $1/f^n$ to y_n. This map has to be zero as M is derived complete (Lemma 91.1) and hence $y_n = 0$ for all n. Since $gx = gfx_1 = fy_1$ this completes the proof. $\square$

Lemma 93.10. *Let A be a ring derived complete with respect to an ideal I. Then (A, I) is a henselian pair.*

Proof. Let $f \in I$. By Lemma 11.15 it suffices to show that (A, fA) is a henselian pair. Observe that A is derived complete with respect to fA (follows immediately from Definition 91.4). By Lemma 91.3 the map from A to the f-adic completion A' of A is surjective. By Lemma 11.4 the pair (A', fA') is henselian. Thus it suffices to show that $(A, \bigcap f^n A)$ is a henselian pair, see Lemma 11.9. This follows from Lemmas 93.9 and 11.2. $\square$

Lemma 93.11. *Let A be a ring derived complete with respect to an ideal I. Set $J = \bigcap I^n$. If I can be generated by r elements then $J^N = 0$ where $N = 2^r$.*

Proof. When $r = 1$ this is Lemma 93.9. Say $I = (f_1, \ldots, f_r)$ with $r > 1$. By Lemma 91.6 the ring $A_t = A/f_r^t A$ is derived complete with respect to I and hence a fortiori derived complete with respect to $I_t = (f_1, \ldots, f_{r-1})A_t$. Observe that $A \to A_t$ sends J into $J_t = \bigcap I_t^n$. By induction $J_t^{N/2} = 0$ with $N = 2^r$. The ideal

[11] For the proof it is enough to show that there exists a sequence $K \xrightarrow{u} L \to M \to 0$ where K and L are I-adically complete A-modules. This can be shown by choosing a presentation $F_1 \to F_0 \to M \to 0$ with F_i free and then setting K and L equal to the f-adic completions of F_1 and F_0. Namely, as f is a nonzerodivisor these completions will be the derived completions and the sequence will remain exact.

$\bigcap \operatorname{Ker}(A \to A_t) = \bigcap f_r^t A$ has square zero by the case $r = 1$. This finishes the proof. □

Lemma 93.12. *Let A be a reduced ring derived complete with respect to a finitely generated ideal I. Then A is I-adically complete.*

Proof. Follows from Lemma 93.11 and Proposition 91.5. □

94. Derived completion for Noetherian rings

Let A be a ring and let $I \subset A$ be an ideal. For any $K \in D(A)$ we can consider the derived limit

$$K' = R\lim(K \otimes_A^{\mathbf{L}} A/I^n)$$

This is a functor in K, see Remark 87.10. The system of maps $A \to A/I^n$ induces a map $K \to K'$ and K' is derived complete with respect to I (Lemma 91.14). This "naive" derived completion construction does not agree with the adjoint of Lemma 91.10 in general. For example, if $A = \mathbf{Z}_p \oplus \mathbf{Q}_p/\mathbf{Z}_p$ with the second summand an ideal of square zero, $K = A[0]$, and $I = (p)$, then the naive derived completion gives $\mathbf{Z}_p[0]$, but the construction of Lemma 91.10 gives $K^\wedge \cong \mathbf{Z}_p[1] \oplus \mathbf{Z}_p[0]$ (computation omitted). Lemma 93.2 characterizes when the two functors agree in the case I is generated by a single element.

The main goal of this section is the show that the naive derived completion is equal to derived completion if A is Noetherian.

Lemma 94.1. *In Situation 91.15. If A is Noetherian, then the pro-objects $\{K_n^\bullet\}$ and $\{A/(f_1^n, \ldots, f_r^n)\}$ of $D(A)$ are isomorphic*[12].

Proof. We have an inverse system of distinguished triangles

$$\tau_{\leq -1} K_n^\bullet \to K_n^\bullet \to A/(f_1^m, \ldots, f_r^m) \to (\tau_{\leq -1} K_n^\bullet)[1]$$

See Derived Categories, Remark 12.4. By Derived Categories, Lemma 42.4 it suffices to show that the inverse system $\tau_{\leq -1} K_n^\bullet$ is pro-zero. Recall that $K_n^\bullet$ has nonzero terms only in degrees i with $-r \leq i \leq 0$. Thus by Derived Categories, Lemma 42.3 it suffices to show that $H^p(K_n^\bullet)$ is pro-zero for $p \leq -1$. In other words, for every $n \in \mathbf{N}$ we have to show there exists an $m \geq n$ such that $H^p(K_m^\bullet) \to H^p(K_n^\bullet)$ is zero. Since A is Noetherian, we see that

$$H^p(K_n^\bullet) = \frac{\operatorname{Ker}(K_n^p \to K_n^{p+1})}{\operatorname{Im}(K_n^{p-1} \to K_n^p)}$$

is a finite A-module. Moreover, the map $K_m^p \to K_n^p$ is given by a diagonal matrix whose entries are in the ideal $(f_1^{m-n}, \ldots, f_r^{m-n})$ as $p < 0$. Note that $H^p(K_n^\bullet)$ is annihilated by $J = (f_1^n, \ldots, f_r^n)$, see Lemma 28.6. Now $(f_1^{m-n}, \ldots, f_r^{m-n}) \subset J^t$ for $m - n \geq tn$. Thus by Algebra, Lemma 51.2 (Artin-Rees) applied to the ideal J and the module $M = K_n^p$ with submodule $N = \operatorname{Ker}(K_n^p \to K_n^{p+1})$ for m large enough the image of $K_m^p \to K_n^p$ intersected with $\operatorname{Ker}(K_n^p \to K_n^{p+1})$ is contained in $J \operatorname{Ker}(K_n^p \to K_n^{p+1})$. For such m we get the zero map. □

[12] In particular, for every n there exists an $m \geq n$ such that $K_m^\bullet \to K_n^\bullet$ factors through the map $K_m^\bullet \to A/(f_1^m, \ldots, f_r^m)$.

Proposition 94.2. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. The functor which sends $K \in D(A)$ to the derived limit $K' = R\lim(K \otimes_A^{\mathbf{L}} A/I^n)$ is the left adjoint to the inclusion functor $D_{comp}(A) \to D(A)$ constructed in Lemma 91.10.*

Proof. Say $(f_1, \ldots, f_r) = I$ and let $K_n^\bullet$ be the Koszul complex with respect to $f_1^n, \ldots, f_r^n$. By Lemma 91.18 it suffices to prove that

$$R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet) = R\lim(K \otimes_A^{\mathbf{L}} A/(f_1^n, \ldots, f_r^n)) = R\lim(K \otimes_A^{\mathbf{L}} A/I^n).$$

By Lemma 94.1 the pro-objects $\{K_n^\bullet\}$ and $\{A/(f_1^n, \ldots, f_r^n)\}$ of $D(A)$ are isomorphic. It is clear that the pro-objects $\{A/(f_1^n, \ldots, f_r^n)\}$ and $\{A/I^n\}$ are isomorphic. Thus the map from left to right is an isomorphism by Lemma 87.12. □

Lemma 94.3. *Let I be an ideal of a Noetherian ring A. Let M be an A-module with derived completion $M^\wedge$. Then there are short exact sequences*

$$0 \to R^1 \lim Tor_{i+1}^A(M, A/I^n) \to H^{-i}(M^\wedge) \to \lim Tor_i^A(M, A/I^n) \to 0$$

A similar result holds for $M \in D^-(A)$.

Proof. Immediate consequence of Proposition 94.2 and Lemma 87.4. □

As an application of the proposition above we identify the derived completion in the Noetherian case for pseudo-coherent complexes.

Lemma 94.4. *Let A be a Noetherian ring and $I \subset A$ an ideal. Let K be an object of $D(A)$ such that $H^n(K)$ a finite A-module for all $n \in \mathbf{Z}$. Then the cohomology modules $H^n(K^\wedge)$ of the derived completion are the I-adic completions of the cohomology modules $H^n(K)$.*

Proof. The complex $\tau_{\leq m}K$ is pseudo-coherent for all m by Lemma 64.17. Thus $\tau_{\leq m}K$ is represented by a bounded above complex $P^\bullet$ of finite free A-modules. Then $\tau_{\leq m}K \otimes_A^{\mathbf{L}} A/I^n = P^\bullet/I^nP^\bullet$. Hence $(\tau_{\leq m}K)^\wedge = R\lim P^\bullet/I^nP^\bullet$ (Proposition 94.2) and since the $R\lim$ is just given by termwise lim (Lemma 87.1) and since I-adic completion is an exact functor on finite A-modules (Algebra, Lemma 97.2) we conclude the result holds for $\tau_{\leq m}K$. Hence the result holds for K as derived completion has finite cohomological dimension, see Lemma 91.20. □

Lemma 94.5. *Let I be an ideal of a Noetherian ring A. Let M be a derived complete A-module. If M/IM is a finite A/I-module, then $M = \lim M/I^nM$ and M is a finite $A^\wedge$-module.*

Proof. Assume M/IM is finite. Pick $x_1, \ldots, x_t \in M$ which map to generators of M/IM. We obtain a map $A^{\oplus t} \to M$ mapping the ith basis vector to x_i. By Proposition 94.2 the derived completion of A is $A^\wedge = \lim A/I^n$. As M is derived complete, we see that our map factors through a map $q : (A^\wedge)^{\oplus t} \to M$. The module $\text{Coker}(q)$ is zero by Lemma 91.7. Thus M is a finite $A^\wedge$-module. Since $A^\wedge$ is Noetherian and complete with respect to $IA^\wedge$, it follows that M is I-adically complete (use Algebra, Lemmas 97.5, 96.11, and 51.2). □

Lemma 94.6. *Let I be an ideal in a Noetherian ring A.*

1. *If M is a finite A-module and N is a flat A-module, then the derived I-adic completion of $M \otimes_A N$ is the usual I-adic completion of $M \otimes_A N$.*
2. *If M is a finite A-module and $f \in A$, then the derived I-adic completion of M_f is the usual I-adic completion of M_f.*

Proof. For an A-module M denote $M^\wedge$ the derived completion and $\lim M/I^nM$ the usual completion. Assume M is finite. The system $\text{Tor}_i^A(M, A/I^n)$ is pro-zero for $i > 0$, see Lemma 27.3. Since $\text{Tor}_i^A(M \otimes_A N, A/I^n) = \text{Tor}_i^A(M, A/I^n) \otimes_A N$ as N is flat, the same is true for the system $\text{Tor}_i^A(M \otimes_A N, A/I^n)$. By Lemma 94.3 we conclude $R\lim(M \otimes_A N) \otimes_A^{\mathbf{L}} A/I^n$ only has cohomology in degree 0 given by the usual completion $\lim M \otimes_A N/I^n(M \otimes_A N)$. This proves (1). Part (2) follows from (1) and the fact that $M_f = M \otimes_A A_f$. $\square$

Lemma 94.7. *Let I be an ideal in a Noetherian ring A. Let $^\wedge$ denote derived completion with respect to I. Let $K \in D^-(A)$.*

1. *If M is a finite A-module, then $(K \otimes_A^{\mathbf{L}} M)^\wedge = K^\wedge \otimes_A^{\mathbf{L}} M$.*
2. *If $L \in D(A)$ is pseudo-coherent, then $(K \otimes_A^{\mathbf{L}} L)^\wedge = K^\wedge \otimes_A^{\mathbf{L}} L$.*

Proof. Let L be as in (2). We may represent K by a bounded above complex $P^\bullet$ of free A-modules. We may represent L by a bounded above complex $F^\bullet$ of finite free A-modules. Since $\text{Tot}(P^\bullet \otimes_A F^\bullet)$ represents $K \otimes_A^{\mathbf{L}} L$ we see that $(K \otimes_A^{\mathbf{L}} L)^\wedge$ is represented by

$$\text{Tot}((P^\bullet)^\wedge \otimes_A F^\bullet)$$

where $(P^\bullet)^\wedge$ is the complex whose terms are the usual = derived completions $(P^n)^\wedge$, see for example Proposition 94.2 and Lemma 94.6. This proves (2). Part (1) is a special case of (2). $\square$

95. An operator introduced by Berthelot and Ogus

In this section we discuss a construction introduced in [BO78, Section 8] and generalized in [BMS18, Section 6]. We urge the reader to look at the original papers discussing this notion.

Let A be a ring and let $f \in A$ be a nonzerodivisor. If M is a A-module then by Lemma 88.3 following are equivalent

1. f is a nonzerodivisor on M,
2. $M[f] = 0$,
3. $M[f^n] = 0$ for all $n \geq 1$, and
4. the map $M \to M_f$ is injective.

If these equivalent conditions hold, then (in this section) we will say M *is f-torsion free*. If so, then we denote $f^iM \subset M_f$ the submodule consisting of elements of the form f^ix with $x \in M$. Of course f^iM is isomorphic to M as an A-module. Let $M^\bullet$ be a complex of f-torsion free A-modules with differentials $d^i : M^i \to M^{i+1}$. In this case we define $\eta_f M^\bullet$ to be the complex with terms

$$(\eta_f M)^i = \{x \in f^iM^i \mid d^i(x) \in f^{i+1}M^{i+1}\}$$

and differential induced by d^i. Observe that $\eta_f M^\bullet$ is another complex of f-torsion free A-modules. If $a^\bullet : M^\bullet \to N^\bullet$ is a map of complexes of f-torsion free A-modules, then we obtain a map of complexes

$$\eta_f a^\bullet : \eta_f M^\bullet \longrightarrow \eta_f N^\bullet$$

induced by the maps $f^iM^i \to f^iN^i$. The reader checks that we obtain an endofunctor on the category of complexes of f-torsion free A-modules. If $a^\bullet, b^\bullet : M^\bullet \to N^\bullet$ are two maps of complexes of f-torsion free A-modules and $h = \{h^i : M^i \to N^{i-1}\}$ is a homotopy between $a^\bullet$ and $b^\bullet$, then we define $\eta_f h$ to be the family of maps $(\eta_f h)^i : (\eta_f M)^i \to (\eta_f N)^{i-1}$ which sends x to $h^i(x)$; this makes sense as $x \in f^i M^i$ implies $h^i(x) \in f^i N^{i-1}$ which is certainly contained in $(\eta_f N)^{i-1}$. The reader checks that $\eta_f h$ is a homotopy between $\eta_f a^\bullet$ and $\eta_f b^\bullet$. All in all we see that we obtain a functor

$$\eta_f : K(f\text{-torsion free } A\text{-modules}) \longrightarrow K(f\text{-torsion free } A\text{-modules})$$

on the homotopy category (Derived Categories, Section 8) of the additive category of f-torsion free A-modules. There is no sense in which η_f is an exact functor of triangulated categories, see Example 95.1.

Example 95.1. Let A be a ring. Let $f \in A$ be a nonzerodivisor. Consider the functor $\eta_f : K(f\text{-torsion free } A\text{-modules}) \to K(f\text{-torsion free } A\text{-modules})$. Let $M^\bullet$ be a complex of f-torsion free A-modules. Multiplication by f defines an isomorphism $\eta_f(M^\bullet[1]) \to (\eta_f M^\bullet)[1]$, so in this sense η_f is compatible with shifts. However, consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"'] & A \arrow[r, "1"'] & A \arrow[r] & 0 \\ 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & A \arrow[r, "-1"] \arrow[u, "f"] & A \arrow[u] \end{tikzcd}$$

Think of each column as a complex of f-torsion free A-modules with the module on top in degree 1 and the module under it in degree 0. Then this diagram provides us with a distinguished triangle in $K(f\text{-torsion free } A\text{-modules})$ with triangulated structure as given in Derived Categories, Section 10. Namely the third complex is the cone of the map between the first two complexes. However, applying η_f to each column we obtain

$$\begin{tikzcd} fA \arrow[r, "f"'] & fA \arrow[r, "1"'] & fA \arrow[r] & 0 \\ 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & A \arrow[r, "-1"] \arrow[u, "f"] & A \arrow[u] \end{tikzcd}$$

However, the third complex is acyclic and even homotopic to zero. Hence if this were a distinguished triangle, then the first arrow would have to be an isomorphism in the homotopy category, which is not true unless f is a unit.

Lemma 95.2. *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. There is a canonical isomorphism*

$$f^i : H^i(M^\bullet)/H^i(M^\bullet)[f] \longrightarrow H^i(\eta_f M^\bullet)$$

given by multiplication by f^i.

Proof. Observe that $\text{Ker}(d^i : (\eta_f M)^i \to (\eta_f M)^{i+1})$ is equal to $\text{Ker}(d^i : f^i M^i \to f^i M^{i+1}) = f^i \text{Ker}(d^i : M^i \to M^{i+1})$. This we get a surjection $f^i : H^i(M^\bullet) \to H^i(\eta_f M^\bullet)$ by sending the class of $z \in \text{Ker}(d^i : M^i \to M^{i+1})$ to the class of $f^i z$. If we obtain the zero class in $H^i(\eta_f M^\bullet)$ then we see that $f^i z = d^{i-1}(f^{i-1} y)$ for some $y \in M^{i-1}$. Since f is a nonzerodivisor on all the modules involved, this means $fz = d^{i-1}(y)$ which exactly means that the class of z is f-torsion as desired. $\square$

Lemma 95.3. *Let A be a ring and let $f \in A$ be a nonzerodivisor. If $M^\bullet \to N^\bullet$ is a quasi-isomorphism of complexes of f-torsion free A-modules, then the induced map $\eta_f M^\bullet \to \eta_f N^\bullet$ is a quasi-isomorphism too.*

Proof. This is true because the isomorphisms of Lemma 95.2 are compatible with maps of complexes. $\square$

Lemma 95.4. *Let A be a ring and let $f \in A$ be a nonzerodivisor. There is an additive functor*[13] *$L\eta_f : D(A) \to D(A)$ such that if $M \in D(A)$ is represented by a complex $M^\bullet$ of f-torsion free A-modules, then $L\eta_f M = \eta_f M^\bullet$ and similarly for morphisms.*

Proof. Denote $\mathcal{T} \subset \text{Mod}_A$ the full subcategory of f-torsion free A-modules. We have a corresponding inclusion

$$K(\mathcal{T}) \quad \subset \quad K(\text{Mod}_A) = K(A)$$

of $K(\mathcal{T})$ as a full triangulated subcategory of $K(A)$. Let $S \subset \text{Arrows}(K(\mathcal{T}))$ be the quasi-isomorphisms. We will apply Derived Categories, Lemma 5.8 to show that the map

$$S^{-1}K(\mathcal{T}) \longrightarrow D(A)$$

is an equivalence of triangulated categories. The lemma shows that it suffices to prove: given a complex $M^\bullet$ of A-modules, there exists a quasi-isomorphism $K^\bullet \to M^\bullet$ with $K^\bullet$ a complex of f-torsion free modules. By Lemma 59.10 we can find a quasi-isomorphism $K^\bullet \to M^\bullet$ such that the complex $K^\bullet$ is K-flat (we won't use this) and consists of flat A-modules K^i. In particular, f is a nonzerodivisor on K^i for all i as desired.

With these preliminaries out of the way we can define $L\eta_f$. Namely, by the discussion at the start of this section we have already a well defined functor

$$K(\mathcal{T}) \xrightarrow{\eta_f} K(\mathcal{T}) \to K(A) \to D(A)$$

which according to Lemma 95.3 sends quasi-isomorphisms to quasi-isomorphisms. Hence this functor factors over $S^{-1}K(\mathcal{T}) = D(A)$ by Categories, Lemma 27.8. $\square$

Remark 95.5. Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. For every i set $\overline{M}^i = M^i/fM^i$. Denote $B^i \subset Z^i \subset \overline{M}^i$ the boundaries and cocycles for the differentials on the complex $\overline{M}^\bullet = M^\bullet \otimes_A A/fA$. We claim that there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B^{i+1} \arrow[r] \arrow[d, equal] & B^{i+1} \oplus B^i \arrow[r] \arrow[d, "{s,s'}"] & B^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^{i+1} \arrow[r, "s"] & (\eta_f M)^i/f(\eta_f M)^i \arrow[r, "t"] & Z^i \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Here are the constructions of the maps

1. If $x \in (\eta_f M)^i$ then $x = f^i x'$ with $d^i(x') = 0$ in $\overline{M}^{i+1}$. Hence we can define the map t by sending x to the class of x'.
2. If $y \in M^{i+1}$ has class $\overline{y}$ in $B^{i+1} \subset \overline{M}^{i+1}$ then we can write $y = fy' + d^i(x)$ for $y' \in M^{i+1}$ and $x \in M^i$. Hence we can define the map s sending $\overline{y}$ to the class of $f^{i+1}x$ in $(\eta_f M)^i/f(\eta_f M)^i$; we omit the verification that this is well defined.

[13]Beware that this functor isn't exact, i.e., does not transform distinguished triangles into distinguished triangles. See Example 95.1.

(3) If $x \in M^i$ has class $\overline{x}$ in $B^i \subset \overline{M}^i$ then we can write $x = fx' + d^{i-1}(z)$ for $x' \in M^i$ and $z \in M^{i-1}$. We define the map s' by sending $\overline{x}$ to the class of $f^i d^{i-1}(z)$ in $(\eta_f M)^i/f(\eta_f M)^i$. This is well defined because if $fx' + d^{i-1}(z) = 0$, then $f^i x'$ is in $(\eta_f M)^i$ and consequently $f^i d^{i-1}(z)$ is in $f(\eta_f M)^i$.

We omit the verification that the lower row in the displayed diagram is a short exact sequence of modules. It is immediately clear from these constructions that we have commutative diagrams

$$\begin{tikzcd} B^{i+1} \oplus B^i \arrow[r] \arrow[d, "{s,s'}"] & B^{i+2} \oplus B^{i+1} \arrow[d, "{s,s'}"] \\ (\eta_f M)^i/f(\eta_f M)^i \arrow[r] & (\eta_f M)^{i+1}/f(\eta_f M)^{i+1} \end{tikzcd}$$

where the upper horizontal arrow is given by the identification of the summands B^{i+1} in source and target. In other words, we have found an acyclic subcomplex of $\eta_f M^\bullet/f(\eta_f M^\bullet) = \eta_f M^\bullet \otimes_A A/fA$ and the quotient by this subcomplex is a complex whose terms Z^i/B^i are the cohomology modules of the complex $\overline{M}^\bullet = M^\bullet \otimes_A A/fA$.

To explain the phenomenon observed in Remark 95.5 in a more canonical manner, we are going to construct the Bockstein operators. Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. For every $i \in \mathbf{Z}$ there is a commutative diagram (with tensor products over A)

$$\begin{tikzcd} 0 \arrow[r] & M^\bullet \otimes f^{i+1}A \arrow[r] \arrow[d] & M^\bullet \otimes f^iA \arrow[r] \arrow[d] & M^\bullet \otimes f^iA/f^{i+1}A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M^\bullet \otimes f^{i+1}A/f^{i+2}A \arrow[r] & M^\bullet \otimes f^iA/f^{i+2}A \arrow[r] & M^\bullet \otimes f^iA/f^{i+1}A \arrow[r] & 0 \end{tikzcd}$$

whose rows are short exact sequences of complexes. Of course these short exact sequences for different i are all isomorphic to each other by suitably multiplying with powers of f. The long exact sequence of cohomology of the bottom sequence in particular determines the *Bockstein operator*

$$\beta = \beta^i : H^i(M^\bullet \otimes f^iA/f^{i+1}A) \to H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A)$$

for all $i \in \mathbf{Z}$. For later use we record here that by the commutative diagram above there is a factorization

(95.5.1)
$$\begin{tikzcd} H^i(M^\bullet \otimes f^iA/f^{i+1}A) \arrow[r, "\delta"'] \arrow[dr, "\beta"'] & H^{i+1}(M^\bullet \otimes f^{i+1}A) \arrow[d] \\ & H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A) \end{tikzcd}$$

of the Bockstein operator where δ is the boundary operator coming from the top row in the commutative diagram above. Let us show that we obtain a complex

$$H^\bullet(M^\bullet/f) = \left[\begin{matrix} \ldots \\ \downarrow \\ H^{i-1}(M^\bullet \otimes f^{i-1}A/f^iA) \\ \downarrow \beta \\ H^i(M^\bullet \otimes f^iA/f^{i+1}A) \\ \downarrow \beta \\ H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A) \\ \downarrow \\ \ldots \end{matrix}\right] \tag{95.5.2}$$

i.e., that $\beta \circ \beta = 0$[14]. Namely, using the factorization (95.5.1) we see that it suffices to show that

$$H^{i+1}(M^\bullet \otimes f^{i+1}A) \to H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A) \xrightarrow{\beta^{i+1}} H^{i+2}(M^\bullet \otimes f^{i+2}A/f^{i+3}A)$$

is zero. This is true because the kernel of β^{i+1} consists of the cohomology classes which can be lifted to $H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+3}A)$ and those in the image of the first map certainly can!

Lemma 95.6. *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. There is a canonical map of complexes*

$$\eta_f M^\bullet \otimes_A A/fA \longrightarrow H^\bullet(M^\bullet/f)$$

which is a quasi-isomorphism where the right hand side is the complex (95.5.2).

Proof. Let $x \in (\eta_f M)^i$. Then $x = f^i x' \in f^i M$ and $d^i(x) = f^{i+1}y \in f^{i+1}M^{i+1}$. Thus d^i maps $x' \otimes f^i$ to zero in $M^{i+1} \otimes f^iA/f^{i+1}A$. All tensor products are over A in this proof. Hence we may map x to the class of $x' \otimes f^i$ in $H^i(M^\bullet \otimes f^iA/f^{i+1}A)$. It is clear that this rule defines a map

$$(\eta_f M)^i \otimes A/fA \longrightarrow H^i(M^\bullet \otimes f^iA/f^{i+1}A)$$

of A/fA-modules. Observe that in the situation above, we may view $x' \otimes f^i$ as an element of $M^i \otimes f^iA/f^{i+2}A$ with differential $d^i(x' \otimes f^i) = y \otimes f^{i+1}$. By the construction of β above we find that $\beta(x' \otimes f^i) = y \otimes f^{i+1}$ and we conclude that our maps are compatible with differentials, i.e., we have a map of complexes.

To finish the proof, we observe that the construction given in the previous paragraph agrees with the maps $(\eta_f M)^i \otimes A/fA \to Z^i/B^i$ discussed in Remark 95.5. Since we have seen that the kernel of these maps is an acyclic subcomplex of $\eta_f M^\bullet \otimes A/fA$, the lemma is proved. $\square$

[14]An alternative is to argue that β occurs as the differential for the spectral sequence for the complex $(M^\bullet)_f$ filtered by the subcomplexes $f^iM^\bullet$. Yet another argument, which proves something stronger, is to first consider the case $M^\bullet = A$. Here the short exact sequences $0 \to f^{i+1}A/f^{i+2}A \to f^iA/f^{i+2}A \to f^iA/f^{i+1}A \to 0$ define maps $\beta^i : f^iA/f^{i+1}A \to f^{i+1}A/f^{i+2}A[1]$ in $D(A)$. Then one computes (arguing similarly to the text) that the composition $f^iA/f^{i+1}A \to f^{i+1}A/f^{i+2}A[1] \to f^{i+2}A/f^{i+3}A[2]$ is zero in $D(A)$. Since $M^\bullet \otimes f^iA/f^{i+1}A = M^\bullet \otimes^{\mathbf{L}} f^iA/f^{i+1}A$ by our assumption on $M^\bullet$ having f-torsion free terms, we conclude the compostion

$$(M^\bullet \otimes f^iA/f^{i+1}A) \to (M^\bullet \otimes f^{i+1}A/f^{i+2}A)[1] \to (M^\bullet \otimes f^{i+2}A/f^{i+3}A)[2]$$

in $D(A)$ is zero as well.

Lemma 95.7. *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. For $i \in \mathbf{Z}$ the following are equivalent*

(1) $\text{Ker}(d^i \bmod f^2)$ *surjects onto* $\text{Ker}(d^i \bmod f)$,
(2) $\beta : H^i(M^\bullet \otimes_A f^iA/f^{i+1}A) \to H^{i+1}(M^\bullet \otimes_A f^{i+1}A/f^{i+2}A)$ *is zero.*

These equivalent conditions are implied by the condition $H^{i+1}(M^\bullet)[f] = 0$.

Proof. The equivalence of (1) and (2) follows from the definition of β as the boundary map on cohomology of a short exact sequence of complexes isomorphic to the short exact sequence of complexes $0 \to fM^\bullet/f^2M^\bullet \to M^\bullet/f^2M^\bullet \to M^\bullet/fM^\bullet \to 0$. If $\beta \neq 0$, then $H^{i+1}(M^\bullet)[f] \neq 0$ because of the factorization (95.5.1). $\square$

Lemma 95.8. *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. If* $\text{Ker}(d^i \bmod f^2)$ *surjects onto* $\text{Ker}(d^i \bmod f)$*, then the canonical map*

$$(1, d^i) : (\eta_f M)^i/f(\eta_f M)^i \longrightarrow f^iM^i/f^{i+1}M^i \oplus f^{i+1}M^{i+1}/f^{i+2}M^{i+1}$$

identifies the left hand side with a direct sum of submodules of the right hand side.

Proof. With notation as in Remark 95.5 we define a map $t^{-1} : Z^i \to (\eta_f M)^i/f(\eta_f M)^i$. Namely, for $x \in M^i$ with $d^i(x) = f^2y$ we send the class of x in Z^i to the class of f^ix in $(\eta_f M)^i/f(\eta_f M)^i$. We omit the verification that this is well defined; the assumption of the lemma exactly signifies that the domain of this operation is all of Z^i. Then $t \circ t^{-1} = \text{id}_{Z^i}$. Hence t^{-1} defines a splitting of the short exact sequence in Remark 95.5 and the resulting direct sum decomposition

$$(\eta_f M)^i/f(\eta_f M)^i = Z^i \oplus B^{i+1}$$

is compatible with the map displayed in the lemma. $\square$

Lemma 95.9. *Let A be a ring and let $f, g \in A$ be nonzerodivisors. Let $M^\bullet$ be a complex of A-modules such that fg is a nonzerodivisor on all M^i. Then $\eta_f \eta_g M^\bullet = \eta_{fg} M^\bullet$.*

Proof. The statement means that in degree i we obtain the same submodule of the localization $M^i_{fg} = (M^i_g)_f$. We omit the details. $\square$

Lemma 95.10. *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $A \to B$ be a flat ring map and let $g \in B$ the image of f. Let $M^\bullet$ be a complex of f-torsion free A-modules. Then g is a nonzerodivisor, $M^\bullet \otimes_A B$ is a complex of g-torsion free modules, and $\eta_f M^\bullet \otimes_A B = \eta_g(M^\bullet \otimes_A B)$.*

Proof. Omitted. $\square$

96. Perfect complexes and the eta operator

In this section we do some algebra to prepare for our version of Macpherson's graph construction, see More on Flatness, Section 44. We will use the η_f operator introduced in Section 95.

Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. For each i let r_i be the rank of M^i and set

$$I_i(M^\bullet, f) = \text{ideal generated by the } r_i \times r_i\text{-minors of } (f, d^i) : M^i \to M^i \oplus M^{i+1}$$

Observe that $f^{r_i} \in I_i(M^\bullet, f)$.

Lemma 96.1. *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ and $N^\bullet$ be two bounded complexes of finite free A-modules representing the same object of $D(A)$. Then*

$$f^m I_i(M^\bullet, f) = f^n I_i(N^\bullet, f)$$

as ideals of A for integers $n, m \geq 0$ such that

$$m + \sum\nolimits_{j \geq i} (-1)^{j-i} rk(M^j) = n + \sum\nolimits_{j \geq i} (-1)^{j-i} rk(N^j)$$

Proof. It suffices to prove the equality after localization at every prime ideal of A. Thus by Lemma 75.7 and an induction argument we omit we may assume $N^\bullet = M^\bullet \oplus Q^\bullet$ for some trivial complex $Q^\bullet$, i.e.,

$$Q^\bullet = \ldots \to 0 \to A \xrightarrow{1} A \to 0 \to \ldots$$

where A is placed in degree j and $j+1$. If $j \neq i-1, i, i+1$ then we clearly have equality $I_i(M^\bullet, f) = I_i(N^\bullet, f)$ and $m = n$ and we have the desired equality. If $j = i+1$ then the maps

$$(f, d^i) : M^i \to M^i \oplus M^{i+1} \quad \text{and} \quad (f, d^i, 0) : M^i \to M^i \oplus M^{i+1} \oplus A$$

have the same nonzero minors hence in this case we also have $I_i(M^\bullet, f) = I_i(N^\bullet, f)$ and $m = n$. If $j = i$, then $I_i(M^\bullet, f)$ is the ideal generated by the $r_i \times r_i$-minors of

$$(f, d^i) : M^i \to M^i \oplus M^{i+1}$$

and $I_i(N^\bullet, f)$ is the ideal generated by the $(r_i+1) \times (r_i+1)$-minors of

$$(f \oplus f, d^i \oplus 1) : (M^i \oplus A) \to (M^i \oplus A) \oplus (M^{i+1} \oplus A)$$

With suitable choice of coordinates we see that the matrix of the second map is in block form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad T_1 = \text{matrix of first map}, \quad T_2 = \begin{pmatrix} f \\ 1 \end{pmatrix}$$

With notation as in Lemma 8.1 we have $I_0(T_2) = A$, $I_1(T_2) = A$, $I_p(T_2) = 0$ for $p \geq 2$ and hence $I_{r_i+1}(T) = I_{r_i+1}(T_1) + I_{r_i}(T_1) = I_{r_i}(T_1)$ which means that $I_i(M^\bullet, f) = I_i(N^\bullet, f)$. We also have $m = n$ so this finishes the case $j = i$. Finally, say $j = i-1$. Then we see that $m = n+1$, thus we have to show that $fI_i(M^\bullet, f) = I_i(N^\bullet, f)$. In this case $I_i(M^\bullet, f)$ is the ideal generated by the $r_i \times r_i$-minors of

$$(f, d^i) : M^i \to M^i \oplus M^{i+1}$$

and $I_i(N^\bullet, f)$ is the ideal generated by the $(r_i+1) \times (r_i+1)$-minors of

$$(f \oplus f, d^i) : (M^i \oplus A) \to (M^i \oplus A) \oplus M^{i+1}$$

With suitable choice of coordinates we see that the matrix of the second map is in block form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad T_1 = \text{matrix of first map}, \quad T_2 = \begin{pmatrix} f \end{pmatrix}$$

Arguing as above we find that indeed $fI_i(M^\bullet, f) = I_i(N^\bullet, f)$. □

Lemma 96.2. *Let $f \in A$ be a nonzerodivisor of a ring A. Let $u \in A$ be a unit. Let $M^\bullet$ be a bounded complex of finite free A-modules. Then $I_i(M^\bullet, f) = I_i(M^\bullet, uf)$.*

Proof. Omitted. □

Lemma 96.3. *Let $A \to B$ be a ring map. Let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume f maps to a nonzerodivisor g in B. Then $I_i(M^\bullet, f)B = I_i(M^\bullet \otimes_A B, g)$.*

Proof. The minors of $(f, d^i) : M^i \to M^i \oplus M^{i+1}$ map to the corresponding minors of $(g, d^i) : M^i \otimes_A B \to M^i \otimes_A B \oplus M^{i+1} \otimes_A B$. $\square$

Lemma 96.4. *Let A be a ring, let $\mathfrak{p} \subset A$ be a prime ideal, and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. If $H^i(M^\bullet)_\mathfrak{p}$ is free for all i, then $I_i(M^\bullet, f)_\mathfrak{p}$ is a principal ideal and in fact generated by a power of f for all i.*

Proof. We may assume A is local with maximal ideal $\mathfrak{p}$ by Lemma 96.3. We may also replace $M^\bullet$ with a quasi-isomorphic complex by Lemma 96.1. By our assumption on the freeness of cohomology modules we see that $M^\bullet$ is quasi-isomorphic to the complex whose term in degree i is $H^i(M^\bullet)$ with vanishing differentials, see for example Derived Categories, Lemma 27.9. In other words, we may assume the differentials in the complex $M^\bullet$ are all zero. In this case it is clear that $I_i(M^\bullet, f) = (f^{r_i})$ is principal. $\square$

Lemma 96.5. *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume $I_i(M^\bullet, f)$ is a principal ideal. Then $(\eta_f M)^i$ is locally free of rank r_i and the map $(1, d^i) : (\eta_f M)^i \to f^i M^i \oplus f^{i+1} M^{i+1}$ is the inclusion of a direct summand.*

Proof. Choose a generator g for $I_i(M^\bullet, f)$. Since $f^{r_i} \in I_i(M^\bullet, f)$ we see that g divides a power of f. In particular g is a nonzerodivisor in A. The $r_i \times r_i$-minors of the map $(f, d^i) : M^i \to M^i \oplus M^{i+1}$ generate the ideal $I_i(M^\bullet, f)$ and the $(r_i + 1) \times (r_i + 1)$-minors of (f, d^i) are zero: we may check this after localizing at f where the rank of the map is equal to r_i. Consider the surjection

$$M^i \oplus M^{i+1} \longrightarrow Q = \text{Coker}(f, d^i)/g\text{-torsion}$$

By Lemma 8.9 the module Q is finite locally free of rank r_{i+1}. Hence Q is f-torsion free and we conclude the cokernel of (f, d^i) modulo f-power torsion is Q as well.

Consider the complex of finite free A-modules

$$0 \to f^{i+1} M^i \xrightarrow{1, d^i} f^i M^i \oplus f^{i+1} M^{i+1} \xrightarrow{d^i, -1} f^i M^{i+1} \to 0$$

which becomes split exact after localizing at f. The map $(1, d^i) : f^{i+1} M^i \to f^i M^i \oplus f^{i+1} M^{i+1}$ is isomorphic to the map $(f, d^i) : M^i \to M^i \oplus M^{i+1}$ we studied above. Hence the image

$$Q' = \text{Im}(f^i M^i \oplus f^{i+1} M^{i+1} \xrightarrow{d^i, -1} f^i M^{i+1})$$

is isomorphic to Q in particular projective. On the other hand, by construction of η_f in Section 95 the image of the injective map $(1, d^i) : (\eta_f M)^i \to f^i M^i \oplus f^{i+1} M^{i+1}$ is the kernel of $(d^i, -1)$. We conclude that we obtain an isomorphism $(\eta_f M)^i \oplus Q' = f^i M^i \oplus f^{i+1} M^{i+1}$ and we see that indeed $\eta_f M^i$ is finite locally free of rank r_i and that $(1, d^i)$ is the inclusion of a direct summand. $\square$

Lemma 96.6. *Let $A \to B$ be a ring map. Let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume f maps to a nonzerodivisor g in B and $I_i(M^\bullet, f)$ is a principal ideal for all $i \in \mathbf{Z}$. Then there is a canonical isomorphism $\eta_f M^\bullet \otimes_A B = \eta_g(M^\bullet \otimes_A B)$.*

Proof. Set $N^i = M^i \otimes_A B$. Observe that $f^i M^i \otimes_A B = g^i N^i$ as submodules of $(N^i)_g$. The maps

$$(\eta_f M)^i \otimes_A B \to g^i N^i \otimes g^{i+1} N^{i+1} \quad \text{and} \quad (\eta_g N)^i \to g^i N^i \otimes g^{i+1} N^{i+1}$$

are inclusions of direct summands by Lemma 96.5. Since their images agree after localizing at g we conclude. □

Lemma 96.7. *Let A be a ring. Let M, N_1, N_2 be finite projective A-modules. Let $s : M \to N_1 \oplus N_2$ be a split injection. There exists a finitely generated ideal $J \subset A$ with the following property: a ring map $A \to B$ factors through A/J if and only if $s \otimes id_B$ identifies $M \otimes_A B$ with a direct sum of submodules of $N_1 \otimes_A B \oplus N_2 \otimes_A B$.*

Proof. Choose a splitting $\pi : N_1 \oplus N_2 \to M$ of s. Denote $q_i : N_1 \oplus N_2 \to N_1 \oplus N_2$ the projector onto N_i. Set $p_i = \pi \circ q_i \circ s$. Observe that $p_1 + p_2 = \text{id}_M$. We claim M is a direct sum of submodules of $N_1 \oplus N_2$ if and only if p_1 and p_2 are orthogonal projectors. Thus J is the smallest ideal of A such that $p_1 \circ p_1 - p_1$, $p_2 \circ p_2 - p_2$, $p_1 \circ p_2$, and $p_2 \circ p_1$ are contained in $J \otimes_A \text{End}_A(M)$. Some details omitted. □

Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume the ideals $I_i(M^\bullet, f)$ are principal for all $i \in \mathbf{Z}$. Then the maps

$$(1, d^i) : (\eta_f M)^i / f(\eta_f M)^i \longrightarrow f^i M^i / f^{i+1} M^i \oplus f^{i+1} M^{i+1} / f^{i+2} M^{i+1}$$

are split injections by Lemma 96.5. Denote $J_i(M^\bullet, f) \subset A/fA$ the finitely generated ideal of Lemma 96.7 corresponding to the split injection $(1, d^i)$ displayed above.

Lemma 96.8. *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ and $N^\bullet$ be two bounded complexes of finite free A-modules representing the same object in $D(A)$. Assume $I_i(M^\bullet, f)$ is a principal ideal for all $i \in \mathbf{Z}$. Then $J_i(M^\bullet, f) = J_i(N^\bullet, f)$ as ideals in A/fA.*

Proof. Observe that the fact that $I_i(M^\bullet, f)$ is a principal ideal implies that $I_i(M^\bullet, f)$ is a principal ideal by Lemma 96.1 and hence the statement makes sense. As in the proof of Lemma 96.1 we may assume $N^\bullet = M^\bullet \oplus Q^\bullet$ for some trivial complex $Q^\bullet$, i.e.,

$$Q^\bullet = \ldots \to 0 \to A \xrightarrow{1} A \to 0 \to \ldots$$

where A is placed in degree j and $j + 1$. Since η_f is compatible with direct sums, we see that the map

$$(1, d^i) : (\eta_f N)^i / f(\eta_f N)^i \longrightarrow f^i N^i / f^{i+1} N^i \oplus f^{i+1} N^{i+1} / f^{i+2} N^{i+1}$$

is the direct sum of the corresponding map for $M^\bullet$ and for $Q^\bullet$. By the universal property defining the ideals in question, we conclude that $J_i(N^\bullet, f) = J_i(M^\bullet, f) + J_i(Q^\bullet, f)$. Hence it suffices to show that $J_i(Q^\bullet, f) = 0$ for all i. This is a computation that we omit. □

Lemma 96.9. *Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume $I_i(M^\bullet, f)$ is a principal ideal for* *all $i \in \mathbf{Z}$. Consider the ideal $J(M^\bullet, f) = \sum_i J_i(M^\bullet, f)$ of A/fA. Consider the set of prime ideals*

$$\begin{aligned} E &= \{f \in \mathfrak{p} \subset A \mid \mathrm{Ker}(d^i \bmod f^2)_\mathfrak{p} \textit{ surjects onto } \mathrm{Ker}(d^i \bmod f)_\mathfrak{p} \textit{ for all } i \in \mathbf{Z}\} \\ &= \{f \in \mathfrak{p} \subset A \mid \textit{the localizations } \beta_\mathfrak{p} \textit{ of the Bockstein operators are zero}\} \end{aligned}$$

Then we have

1. *$J(M^\bullet, f)$ is finitely generated,*
2. *$A/fA \to C = (A/fA)/J(M^\bullet, f)$ is surjective of finite presentation,*
3. *$J(M^\bullet, f)_\mathfrak{p} = 0$ for $\mathfrak{p} \in E$,*
4. *if $f \in \mathfrak{p}$ and $H^i(M^\bullet)_\mathfrak{p}$ is free for all $i \in \mathbf{Z}$, then $\mathfrak{p} \in E$, and*
5. *the cohomology modules of $\eta_f M^\bullet \otimes_A C$ are finite locally free C-modules.*

Proof. The equality in the definition of E follows from Lemma 95.7 and in addition the final statement of that lemma implies part (4).

Part (1) is true because the ideals $J_i(M^\bullet, f)$ are finitely generated and because $M^\bullet$ is bounded and hence $J_i(M^\bullet, f)$ is zero for almost all i. Part (2) is just a reformulation of part (1).

Proof of (3). By Lemma 96.5 we find that $(\eta_f M)^i$ is finite locally free of rank r_i for all i. Consider the map

$$(1, d^i) : (\eta_f M)^i / f(\eta_f M)^i \longrightarrow f^i M^i / f^{i+1} M^i \oplus f^{i+1} M^{i+1} / f^{i+2} M^{i+1}$$

Pick $\mathfrak{p} \in E$. By Lemma 95.8 and the local freeness of the modules $(\eta_f M)^i$ we may write

$$\left((\eta_f M)^i / f(\eta_f M)^i\right)_\mathfrak{p} = (A/fA)_\mathfrak{p}^{\oplus m_i} \oplus (A/fA)_\mathfrak{p}^{\oplus n_i}$$

compatible with the arrow $(1, d^i)$ above. By the universal property of the ideal $J_i(M^\bullet, f)$ we conclude that $J_i(M^\bullet, f)_\mathfrak{p} = 0$. Hence $I_\mathfrak{p} = fA_\mathfrak{p}$ for $\mathfrak{p} \in E$.

Proof of (5). Observe that the differential on $\eta_f M^\bullet$ fits into a commutative diagram

$$\begin{tikzcd} (\eta_f M)^i \arrow[r] \arrow[d] & f^i M^i \oplus f^{i+1} M^{i+1} \arrow[d, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] \\ (\eta_f M)^{i+1} \arrow[r] & f^{i+1} M^i \oplus f^{i+2} M^{i+2} \end{tikzcd}$$

By construction, after tensoring with C, the modules on the left are direct sums of direct summands of the summands on the right. Picture

$$\begin{tikzcd} (\eta_f M)^i \otimes_A C \arrow[r, equal] \arrow[d] & K^i \oplus L^i \arrow[r] \arrow[d] & f^i M^i \otimes_A C \oplus f^{i+1} M^{i+1} \otimes_A C \arrow[d, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] \\ (\eta_f M)^{i+1} \otimes_A C \arrow[r, equal] & K^{i+1} \oplus L^{i+1} \arrow[r] & f^{i+1} M^i \otimes_A C \oplus f^{i+2} M^{i+2} \otimes_A C \end{tikzcd}$$

where the horizontal arrows are compatible with direct sum decompositions as well as inclusions of direct summands. It follows that the differential identifies L^i with a direct summand of K^{i+1} and we conclude that the cohomology of $\eta_f M^\bullet \otimes_A C$ in degree i is the module K^{i+1}/L^i which is finite projective as desired. $\square$

97. Taking limits of complexes

In this section we discuss what happens when we have a "formal deformation" of a complex and we take its limit. We will consider two cases

(1) we have a limit $A = \lim A_n$ of an inverse system of rings whose transition maps are surjective with locally nilpotent kernels and objects $K_n \in D(A_n)$ which fit together in the sense that $K_n = K_{n+1} \otimes^{\mathbf{L}}_{A_{n+1}} A_n$, or
(2) we have a ring A, an ideal I, and objects $K_n \in D(A/I^n)$ which fit together in the sense that $K_n = K_{n+1} \otimes^{\mathbf{L}}_{A/I^{n+1}} A/I^n$.

Under additional hypotheses we can show that $K = R\lim K_n$ reproduces the system in the sense that $K_n = K \otimes^{\mathbf{L}}_A A_n$ or $K_n = K \otimes^{\mathbf{L}}_A A/I^n$.

Lemma 97.1. *Let $A = \lim A_n$ be a limit of an inverse system (A_n) of rings. Suppose given $K_n \in D(A_n)$ and maps $K_{n+1} \to K_n$ in $D(A_{n+1})$. Assume*

(1) *the transition maps $A_{n+1} \to A_n$ are surjective with locally nilpotent kernels,*
(2) *K_1 is pseudo-coherent, and*
(3) *the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A_{n+1}} A_n \to K_n$.*

Then $K = R\lim K_n$ is a pseudo-coherent object of $D(A)$ and $K \otimes^{\mathbf{L}}_A A_n \to K_n$ is an isomorphism for all n.

Proof. By assumption we can find a bounded above complex of finite free A_1-modules $P_1^\bullet$ representing K_1, see Definition 64.1. By Lemma 75.4 we can, by induction on $n > 1$, find complexes $P_n^\bullet$ of finite free A_n-modules representing K_n and maps $P_n^\bullet \to P_{n-1}^\bullet$ representing the maps $K_n \to K_{n-1}$ inducing isomorphisms (!) of complexes $P_n^\bullet \otimes_{A_n} A_{n-1} \to P_{n-1}^\bullet$. Thus $K = R\lim K_n$ is represented by $P^\bullet = \lim P_n^\bullet$, see Lemma 87.1 and Remark 87.6. Since P_n^i is a finite free A_n-module for each n and $A = \lim A_n$ we see that P^i is finite free of the same rank as P_1^i for each i. This means that K is pseudo-coherent. It also follows that $K \otimes^{\mathbf{L}}_A A_n$ is represented by $P^\bullet \otimes_A A_n = P_n^\bullet$ which proves the final assertion. □

Lemma 97.2. *Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. Assume*

(1) *A is I-adically complete,*
(2) *K_1 is pseudo-coherent, and*
(3) *the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A/I^{n+1}} A/I^n \to K_n$.*

Then $K = R\lim K_n$ is a pseudo-coherent, derived complete object of $D(A)$ and $K \otimes^{\mathbf{L}}_A A/I^n \to K_n$ is an isomorphism for all n.

Proof. We already know that K is pseudo-coherent and that $K \otimes^{\mathbf{L}}_A A/I^n \to K_n$ is an isomorphism for all n, see Lemma 97.1. Finally, K is derived complete by Lemma 91.14. □

Lemma 97.3. *Let $A = \lim A_n$ be a limit of an inverse system (A_n) of rings. Suppose given $K_n \in D(A_n)$ and maps $K_{n+1} \to K_n$ in $D(A_{n+1})$. Assume*

(1) *the transition maps $A_{n+1} \to A_n$ are surjective with locally nilpotent kernels,*
(2) *K_1 is a perfect object, and*
(3) *the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A_{n+1}} A_n \to K_n$.*

Then $K = R\lim K_n$ is a perfect object of $D(A)$ and $K \otimes^{\mathbf{L}}_A A_n \to K_n$ is an isomorphism for all n.

Proof. We already know that K is pseudo-coherent and that $K \otimes_A^{\mathbf{L}} A_n \to K_n$ is an isomorphism for all n by Lemma 97.1. Thus it suffices to show that $H^i(K \otimes_A^{\mathbf{L}} \kappa) = 0$ for $i \ll 0$ and every surjective map $A \to \kappa$ whose kernel is a maximal ideal $\mathfrak{m}$, see Lemma 77.3. Any element of A which maps to a unit in A_1 is a unit in A by Algebra, Lemma 32.4 and hence $\text{Ker}(A \to A_1)$ is contained in the Jacobson radical of A by Algebra, Lemma 19.1. Hence $A \to \kappa$ factors as $A \to A_1 \to \kappa$. Hence

$$K \otimes_A^{\mathbf{L}} \kappa = K \otimes_A^{\mathbf{L}} A_1 \otimes_{A_1}^{\mathbf{L}} \kappa = K_1 \otimes_{A_1}^{\mathbf{L}} \kappa$$

and we get what we want as K_1 has finite tor dimension by Lemma 74.2. $\square$

Lemma 97.4. *Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. Assume*

1. *A is I-adically complete,*
2. *K_1 is a perfect object, and*
3. *the maps induce isomorphisms $K_{n+1} \otimes_{A/I^{n+1}}^{\mathbf{L}} A/I^n \to K_n$.*

Then $K = R\lim K_n$ is a perfect, derived complete object of $D(A)$ and $K \otimes_A^{\mathbf{L}} A/I^n \to K_n$ is an isomorphism for all n.

Proof. Combine Lemmas 97.3 and 97.2 (to get derived completeness). $\square$

We do not know if the following lemma holds for unbounded complexes.

Lemma 97.5. *Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. If*

1. *A is Noetherian,*
2. *K_1 is bounded above, and*
3. *the maps induce isomorphisms $K_{n+1} \otimes_{A/I^{n+1}}^{\mathbf{L}} A/I^n \to K_n$,*

then $K = R\lim K_n$ is a derived complete object of $D^-(A)$ and $K \otimes_A^{\mathbf{L}} A/I^n \to K_n$ is an isomorphism for all n.

Proof. The object K of $D(A)$ is derived complete by Lemma 91.14.

Suppose that $H^i(K_1) = 0$ for $i > b$. Then we can find a complex of free A/I-modules $P_1^\bullet$ representing K_1 with $P_1^i = 0$ for $i > b$. By Lemma 75.3 we can, by induction on $n > 1$, find complexes $P_n^\bullet$ of free A/I^n-modules representing K_n and maps $P_n^\bullet \to P_{n-1}^\bullet$ representing the maps $K_n \to K_{n-1}$ inducing isomorphisms (!) of complexes $P_n^\bullet / I^{n-1} P_n^\bullet \to P_{n-1}^\bullet$.

Thus we have arrived at the situation where $R\lim K_n$ is represented by $P^\bullet = \lim P_n^\bullet$, see Lemma 87.1 and Remark 87.6. The complexes $P_n^\bullet$ are uniformly bounded above complexes of flat A/I^n-modules and the transition maps are termwise surjective. Then $P^\bullet$ is a bounded above complex of flat A-modules by Lemma 27.4. It follows that $K \otimes_A^{\mathbf{L}} A/I^t$ is represented by $P^\bullet \otimes_A A/I^t$. We have $P^\bullet \otimes_A A/I^t = \lim P_n^\bullet \otimes_A A/I^t$ termwise by Lemma 27.4. The transition maps $P_{n+1}^\bullet \otimes_A A/I^t \to P_n^\bullet \otimes_A A/I^t$ are isomorphisms for $n \geq t$ by our choice of $P_n^\bullet$, hence we have $\lim P_n^\bullet \otimes_A A/I^t = P_t^\bullet \otimes_A A/I^t = P_t^\bullet$. Since $P_t^\bullet$ represents K_t, we see that $K \otimes_A^{\mathbf{L}} A/I^t \to K_t$ is an isomorphism. $\square$

Here is a different type of result.

Lemma 97.6 (Kollár-Kovács)**.** *Let I be an ideal of a Noetherian ring A. Let $K \in D(A)$. Set $K_n = K \otimes_A^{\mathbf{L}} A/I^n$. Assume for all $i \in \mathbf{Z}$ we have*

(1) $H^i(K)$ *is a finite* A*-module, and*
(2) *the system* $H^i(K_n)$ *satisfies Mittag-Leffler.*

Then $\lim H^i(K)/I^nH^i(K)$ *is equal to* $\lim H^i(K_n)$ *for all* $i \in \mathbf{Z}$.

Proof. Recall that $K^\wedge = R\lim K_n$ is the derived completion of K, see Proposition 94.2. By Lemma 94.4 we have $H^i(K^\wedge) = \lim H^i(K)/I^nH^i(K)$. By Lemma 87.4 we get short exact sequences

$$0 \to R^1\lim H^{i-1}(K_n) \to H^i(K^\wedge) \to \lim H^i(K_n) \to 0$$

The Mittag-Leffler condition guarantees that the left terms are zero (Lemma 87.1) and we conclude the lemma is true. □

98. Some evaluation maps

In this section we prove that certain canonical maps of $R\operatorname{Hom}$'s are isomorphisms for suitable types of complexes.

Lemma 98.1. *Let R be a ring. Let K, L, M be objects of $D(R)$. the map*

$$R\operatorname{Hom}_R(L, M) \otimes_R^{\mathbf{L}} K \longrightarrow R\operatorname{Hom}_R(R\operatorname{Hom}_R(K, L), M)$$

of Lemma 73.3 is an isomorphism in the following two cases

(1) K *perfect, or*
(2) K *is pseudo-coherent,* $L \in D^+(R)$*, and* M *finite injective dimension.*

Proof. Choose a K-injective complex $I^\bullet$ representing M, a K-injective complex $J^\bullet$ representing L, and a bounded above complex of finite projective modules $K^\bullet$ representing K. Consider the map of complexes

$$\operatorname{Tot}(\operatorname{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R K^\bullet) \longrightarrow \operatorname{Hom}^\bullet(\operatorname{Hom}^\bullet(K^\bullet, J^\bullet), I^\bullet)$$

of Lemma 71.6. Note that

$$\left(\prod\nolimits_{p+r=t} \operatorname{Hom}_R(J^{-r}, I^p)\right) \otimes_R K^s = \prod\nolimits_{p+r=t} \operatorname{Hom}_R(J^{-r}, I^p) \otimes_R K^s$$

because K^s is finite projective. The map is given by the maps

$$c_{p,r,s} : \operatorname{Hom}_R(J^{-r}, I^p) \otimes_R K^s \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(K^s, J^{-r}), I^p)$$

which are isomorphisms as K^s is finite projective. For every element $\alpha = (\alpha^{p,r,s})$ of degree n of the left hand side, there are only finitely many values of s such that $\alpha^{p,r,s}$ is nonzero (for some p, r with $n = p+r+s$). Hence our map is an isomorphism if the same vanishing condition is forced on the elements $\beta = (\beta^{p,r,s})$ of the right hand side. If $K^\bullet$ is a bounded complex of finite projective modules, this is clear. On the other hand, if we can choose $I^\bullet$ bounded and $J^\bullet$ bounded below, then $\beta^{p,r,s}$ is zero for p outside a fixed range, for $s \gg 0$, and for $r \gg 0$. Hence among solutions of $n = p + r + s$ with $\beta^{p,r,s}$ nonzero only a finite number of s values occur. □

Lemma 98.2. *Let R be a ring. Let K, L, M be objects of $D(R)$. the map*

$$R\operatorname{Hom}_R(L, M) \otimes_R^{\mathbf{L}} K \longrightarrow R\operatorname{Hom}_R(R\operatorname{Hom}_R(K, L), M)$$

of Lemma 73.3 is an isomorphism if the following three conditions are satisfied

(1) L, M *have finite injective dimension,*
(2) $R\operatorname{Hom}_R(L, M)$ *has finite tor dimension,*
(3) *for every* $n \in \mathbf{Z}$ *the truncation* $\tau_{\leq n}K$ *is pseudo-coherent*

Proof. Pick an integer n and consider the distinguished triangle

$$\tau_{\leq n}K \to K \to \tau_{\geq n+1}K \to \tau_{\leq n}K[1]$$

see Derived Categories, Remark 12.4. By assumption (3) and Lemma 98.1 the map is an isomorphism for $\tau_{\leq n}K$. Hence it suffices to show that both

$$R\operatorname{Hom}_R(L, M) \otimes_R^{\mathbf{L}} \tau_{\geq n+1}K \quad \text{and} \quad R\operatorname{Hom}_R(R\operatorname{Hom}_R(\tau_{\geq n+1}K, L), M)$$

have vanishing cohomology in degrees $\leq n - c$ for some c. This follows immediately from assumptions (2) and (1). $\square$

Lemma 98.3. *Let R be a ring. Let K, L, M be objects of $D(R)$. The map*

$$K \otimes_R^{\mathbf{L}} R\operatorname{Hom}_R(M, L) \longrightarrow R\operatorname{Hom}_R(M, K \otimes_R^{\mathbf{L}} L)$$

of Lemma 73.5 is an isomorphism in the following cases

1. *M perfect, or*
2. *K is perfect, or*
3. *M is pseudo-coherent, $L \in D^+(R)$, and K has tor amplitude in $[a, \infty]$.*

Proof. Proof in case M is perfect. Note that both sides of the arrow transform distinguished triangles in M into distinguished triangles and commute with direct sums. Hence it suffices to check it holds when $M = R[n]$, see Derived Categories, Remark 36.7 and Lemma 78.1. In this case the result is obvious.

Proof in case K is perfect. Same argument as in the previous case.

Proof in case (3). We may represent K and L by bounded below complexes of R-modules $K^\bullet$ and $L^\bullet$. We may assume that $K^\bullet$ is a K-flat complex consisting of flat R-modules, see Lemma 66.4. We may represent M by a bounded above complex $M^\bullet$ of finite free R-modules, see Definition 64.1. Then the object on the LHS is represented by

$$\operatorname{Tot}(K^\bullet \otimes_R \operatorname{Hom}^\bullet(M^\bullet, L^\bullet))$$

and the object on the RHS by

$$\operatorname{Hom}^\bullet(M^\bullet, \operatorname{Tot}(K^\bullet \otimes_R L^\bullet))$$

This uses Lemma 73.2. Both complexes have in degree n the module

$$\bigoplus\nolimits_{p+q+r=n} K^p \otimes \operatorname{Hom}_R(M^{-r}, L^q) = \bigoplus\nolimits_{p+q+r=n} \operatorname{Hom}_R(M^{-r}, K^p \otimes_R L^q)$$

because M^{-r} is finite free (as well these are finite direct sums). The map defined in Lemma 73.5 comes from the map of complexes defined in Lemma 71.4 which uses the canonical isomorphisms between these modules. $\square$

Lemma 98.4. *Let R be a ring. Let $P^\bullet$ be a bounded above complex of projective R-modules. Let $K^\bullet$ be a K-flat complex of R-modules. If $P^\bullet$ is a perfect object of $D(R)$, then $\operatorname{Hom}^\bullet(P^\bullet, K^\bullet)$ is K-flat and represents $R\operatorname{Hom}_R(P^\bullet, K^\bullet)$.*

Proof. The last statement is Lemma 73.2. Since $P^\bullet$ represents a perfect object, there exists a finite complex of finite projective R-modules $F^\bullet$ such that $P^\bullet$ and $F^\bullet$ are isomorphic in $D(R)$, see Definition 74.1. Then $P^\bullet$ and $F^\bullet$ are homotopy equivalent, see Derived Categories, Lemma 19.8. Then $\operatorname{Hom}^\bullet(P^\bullet, K^\bullet)$ and $\operatorname{Hom}^\bullet(F^\bullet, K^\bullet)$ are homotopy equivalent. Hence the first is K-flat if and only if the second is (follows from Definition 59.1 and Lemma 58.2). It is clear that

$$\operatorname{Hom}^\bullet(F^\bullet, K^\bullet) = \operatorname{Tot}(E^\bullet \otimes_R K^\bullet)$$

where $E^\bullet$ is the dual complex to $F^\bullet$ with terms $E^n = \mathrm{Hom}_R(F^{-n}, R)$, see Lemma 74.15 and its proof. Since $E^\bullet$ is a bounded complex of projectives we find that it is K-flat by Lemma 59.7. Then we conclude by Lemma 59.4. □

99. Base change for derived hom

We have already seen some material discussing this in Lemma 65.4 and in Algebra, Section 73.

Lemma 99.1. *Let $R \to R'$ be a ring map. For $K \in D(R)$ and $M \in D(R')$ there is a canonical isomorphism*

$$R\,\mathrm{Hom}_R(K, M) = R\,\mathrm{Hom}_{R'}(K \otimes_R^{\mathbf{L}} R', M)$$

Proof. Choose a K-injective complex of R'-modules $J^\bullet$ representing M. Choose a quasi-isomorphism $J^\bullet \to I^\bullet$ where $I^\bullet$ is a K-injective complex of R-modules. Choose a K-flat complex $K^\bullet$ of R-modules representing K. Consider the map

$$\mathrm{Hom}^\bullet(K^\bullet \otimes_R R', J^\bullet) \longrightarrow \mathrm{Hom}^\bullet(K^\bullet, I^\bullet)$$

The map on degree n terms is given by the map

$$\prod\nolimits_{n=p+q} \mathrm{Hom}_{R'}(K^{-q} \otimes_R R', J^p) \longrightarrow \prod\nolimits_{n=p+q} \mathrm{Hom}_R(K^{-q}, I^p)$$

coming from precomposing by $K^{-q} \to K^{-q} \otimes_R R'$ and postcomposing by $J^p \to I^p$. To finish the proof it suffices to show that we get isomorphisms on cohomology groups:

$$\mathrm{Hom}_{D(R)}(K, M) = \mathrm{Hom}_{D(R')}(K \otimes_R^{\mathbf{L}} R', M)$$

which is true because base change $- \otimes_R^{\mathbf{L}} R' : D(R) \to D(R')$ is left adjoint to the restriction functor $D(R') \to D(R)$ by Lemma 60.3. □

Let $R \to R'$ be a ring map. There is a base change map

$$R\,\mathrm{Hom}_R(K, M) \otimes_R^{\mathbf{L}} R' \longrightarrow R\,\mathrm{Hom}_{R'}(K \otimes_R^{\mathbf{L}} R', M \otimes_R^{\mathbf{L}} R') \tag{99.1.1}$$

in $D(R')$ functorial in $K, M \in D(R)$. Namely, by adjointness of $- \otimes_R^{\mathbf{L}} R' : D(R) \to D(R')$ and the restriction functor $D(R') \to D(R)$, this is the same thing as a map

$$R\,\mathrm{Hom}_R(K, M) \longrightarrow R\,\mathrm{Hom}_{R'}(K \otimes_R^{\mathbf{L}} R', M \otimes_R^{\mathbf{L}} R') = R\,\mathrm{Hom}_R(K, M \otimes_R^{\mathbf{L}} R')$$

(equality by Lemma 99.1) for which we can use the canonical map $M \to M \otimes_R^{\mathbf{L}} R'$ (unit of the adjunction).

Lemma 99.2. *Let $R \to R'$ be a ring map. Let $K, M \in D(R)$. The map (99.1.1)*

$$R\,\mathrm{Hom}_R(K, M) \otimes_R^{\mathbf{L}} R' \longrightarrow R\,\mathrm{Hom}_{R'}(K \otimes_R^{\mathbf{L}} R', M \otimes_R^{\mathbf{L}} R')$$

is an isomorphism in $D(R')$ in the following cases

1. *K is perfect,*
2. *R' is perfect as an R-module,*
3. *$R \to R'$ is flat, K is pseudo-coherent, and $M \in D^+(R)$, or*
4. *R' has finite tor dimension as an R-module, K is pseudo-coherent, and $M \in D^+(R)$*

Proof. We may check the map is an isomorphism after applying the restriction functor $D(R') \to D(R)$. After applying this functor our map becomes the map

$$R\operatorname{Hom}_R(K, L) \otimes_R^{\mathbf{L}} R' \longrightarrow R\operatorname{Hom}_R(K, L \otimes_R^{\mathbf{L}} R')$$

of Lemma 73.5. See discussion above the lemma to match the left and right hand sides; in particular, this uses Lemma 99.1. Thus we conclude by Lemma 98.3. $\square$

100. Systems of modules

Let I be an ideal of a Noetherian ring A. In this section we add to our knowledge of the relationship between finite modules over A and systems of finite A/I^n-modules.

Lemma 100.1. *Let I be an ideal of a Noetherian ring A. Let $K \xrightarrow{\alpha} L \xrightarrow{\beta} M$ be a complex of finite A-modules. Set $H = \operatorname{Ker}(\beta)/\operatorname{Im}(\alpha)$. For $n \geq 0$ let*

$$K/I^nK \xrightarrow{\alpha_n} L/I^nL \xrightarrow{\beta_n} M/I^nM$$

be the induced complex. Set $H_n = \operatorname{Ker}(\beta_n)/\operatorname{Im}(\alpha_n)$. Then there are canonical A-module maps giving a commutative diagram

$$\begin{tikzcd} & & & H \arrow[dll] \arrow[dl] \arrow[d] \\ \ldots \arrow[r] & H_3 \arrow[r] & H_2 \arrow[r] & H_1 \end{tikzcd}$$

Moreover, there exists a $c > 0$ and canonical A-module maps $H_n \to H/I^{n-c}H$ for $n \geq c$ such that the compositions

$$H/I^nH \to H_n \to H/I^{n-c}H \quad \text{and} \quad H_n \to H/I^{n-c}H \to H_{n-c}$$

are the canonical ones. Moreover, we have

1. *(H_n) and (H/I^nH) are isomorphic as pro-objects of Mod_A,*
2. *$\lim H_n = \lim H/I^nH$,*
3. *the inverse system (H_n) is Mittag-Leffler,*
4. *the image of $H_{n+c} \to H_n$ is equal to the image of $H \to H_n$,*
5. *the composition $I^cH_n \to H_n \to H/I^{n-c}H \to H_n/I^{n-c}H_n$ is the inclusion $I^cH_n \to H_n$ followed by the quotient map $H_n \to H_n/I^{n-c}H_n$, and*
6. *the kernel and cokernel of $H/I^nH \to H_n$ is annihilated by I^c.*

Proof. Observe that $H_n = \beta^{-1}(I^nM)/\operatorname{Im}(\alpha) + I^nL$. For $n \geq 2$ we have $\beta^{-1}(I^nM) \subset \beta^{-1}(I^{n-1}M)$ and $\operatorname{Im}(\alpha) + I^nL \subset \operatorname{Im}(\alpha) + I^{n-1}L$. Thus we obtain our canonical map $H_n \to H_{n-1}$. Similarly, we have $\operatorname{Ker}(\beta) \subset \beta^{-1}(I^nM)$ and $\operatorname{Im}(\alpha) \subset \operatorname{Im}(\alpha) + I^nL$ which produces the canonical map $H \to H_n$. We omit the verification that the diagram commutes.

By Artin-Rees we may choose $c_1, c_2 \geq 0$ such that $\beta^{-1}(I^nM) \subset \operatorname{Ker}(\beta) + I^{n-c_1}L$ for $n \geq c_1$ and $\operatorname{Ker}(\beta) \cap I^nL \subset I^{n-c_2}\operatorname{Ker}(\beta)$ for $n \geq c_2$, see Algebra, Lemmas 51.3 and 51.2. Set $c = c_1 + c_2$.

Let $n \geq c$. We define $\psi_n : H_n \to H/I^{n-c}H$ as follows. Say $x \in H_n$. Choose $y \in \beta^{-1}(I^nM)$ representing x. Write $y = z + w$ with $z \in \operatorname{Ker}(\beta)$ and $w \in I^{n-c_1}L$ (this is possible by our choice of c_1). We set $\psi_n(x)$ equal to the class of z in $H/I^{n-c}H$. To see this is well defined, suppose we have a second set of choices

y', z', w' as above for x with obvious notation. Then $y' - y \in \mathrm{Im}(\alpha) + I^nL$, say $y' - y = \alpha(v) + u$ with $v \in K$ and $u \in I^nL$. Thus

$$y' = z' + w' = \alpha(v) + u + z + w \Rightarrow z' = z + \alpha(v) + u + w - w'$$

Since $\beta(z' - z - \alpha(v)) = 0$ we find that $u + w - w' \in \mathrm{Ker}(\beta) \cap I^{n-c_1}L$ which is contained in $I^{n-c_1-c_2}\mathrm{Ker}(\beta) = I^{n-c}\mathrm{Ker}(\beta)$ by our choice of c_2. Thus z' and z have the same image in $H/I^{n-c}H$ as desired.

The composition $H/I^nH \to H_n \to H/I^{n-c}H$ is the canonical map because if $z \in \mathrm{Ker}(\beta)$ represents an element x in $H/I^nH = \mathrm{Ker}(\beta)/\mathrm{Im}(\alpha) + I^n\mathrm{Ker}(\beta)$ then it is clear from the above that x maps to the class of z in $H/I^{n-c}H$ under the maps constructed above.

Let us consider the composition $H_n \to H/I^{n-c}H \to H_{n-c}$. Given x, y, z, w as in the construction of ψ_n above, we see that x is mapped to the cass of z in H_{n-c}. On the other hand, the canonical map $H_n \to H_{n-c}$ from the first paragraph of the proof sends x to the class of y. Thus we have to show that $y - z \in \mathrm{Im}(\alpha) + I^{n-c}L$ which is the case because $y - z = w \in I^{n-c_1}L \subset I^{n-c}L$.

Statements (1) – (4) are formal consequences of what we just proved. Namely, (1) follows from the existence of the maps and the definition of morphisms of pro-objects in Categories, Remark 22.5. Part (2) holds because isomorphic pro-objects have isomorphic limits. Part (3) is immediate from part (4). Part (4) follows from the factorization $H_{n+c} \to H/I^nH \to H_n$ of the canonical map $H_{n+c} \to H_n$.

Proof of part (5). Let $x \in I^cH_n$. Write $x = \sum f_ix_i$ with $x_i \in H_n$ and $f_i \in I^c$. Choose y_i, z_i, w_i as in the construction of ψ_n for x_i. Then for the computation of ψ_n of x we may choose $y = \sum f_iy_i$, $z = \sum f_iz_i$ and $w = \sum f_iw_i$ and we see that $\psi_n(x)$ is given by the class of z. The image of this in $H_n/I^{n-c}H_n$ is equal to the class of y as $w = \sum f_iw_i$ is in I^nL. This proves (5).

Proof of part (6). Let $y \in \mathrm{Ker}(\beta)$ whose class is x in H. If x maps to zero in H_n, then $y \in I^nL + \mathrm{Im}(\alpha)$. Hence $y - \alpha(v) \in \mathrm{Ker}(\beta) \cap I^nL$ for some $v \in K$. Then $y - \alpha(v) \in I^{n-c_2}\mathrm{Ker}(\beta)$ and hence the class of y in H/I^nH is annihilated by I^{c_2}. Finally, let $x \in H_n$ be the class of $y \in \beta^{-1}(I^nM)$. Then we write $y = z + w$ with $z \in \mathrm{Ker}(\beta)$ and $w \in I^{n-c_1}L$ as above. Clearly, if $f \in I^{c_1}$ then fx is the class of $fy + fw \equiv fy$ modulo $\mathrm{Im}(\alpha) + I^nL$ and hence fx is the image of the class of fy in H as desired. $\square$

Lemma 100.2. *Let I be an ideal of a Noetherian ring A. Let $K \in D(A)$ be pseudo-coherent. Set $K_n = K \otimes_A^{\mathbf{L}} A/I^n$. Then for all $i \in \mathbf{Z}$ the system $H^i(K_n)$ satisfies Mittag-Leffler and $\lim H^i(K)/I^nH^i(K)$ is equal to $\lim H^i(K_n)$.*

Proof. We may represent K by a bounded above complex $P^\bullet$ of finite free A-modules. Then K_n is represented by $P^\bullet/I^nP^\bullet$. Hence the Mittag-Leffler property by Lemma 100.1. The final statement follows then from Lemma 97.6. $\square$

Lemma 100.3. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $M^\bullet$ be a bounded complex of finite A-modules. The inverse system of maps*

$$M^\bullet \otimes_A^{\mathbf{L}} A/I^n \longrightarrow M^\bullet/I^nM^\bullet$$

defines an isomorphism of pro-objects of $D(A)$.

Proof. Say $I = (f_1, \ldots, f_r)$. Let $K_n \in D(A)$ be the object represented by the Koszul complex on $f_1^n, \ldots, f_r^n$. Recall that we have maps $K_n \to A/I^n$ which induce a pro-isomorphism of inverse systems, see Lemma 94.1. Hence it suffices to show that

$$M^\bullet \otimes_A^{\mathbf{L}} K_n \longrightarrow M^\bullet/I^nM^\bullet$$

defines an isomorphism of pro-objects of $D(A)$. Since K_n is represented by a complex of finite free A-modules sitting in degrees $-r, \ldots, 0$ there exist $a, b \in \mathbf{Z}$ such that the source and target of the displayed arrow have vanishing cohomology in degrees outside $[a, b]$ for all n. Thus we may apply Derived Categories, Lemma 42.5 and we find that it suffices to show that the maps

$$H^i(M^\bullet \otimes_A^{\mathbf{L}} A/I^n) \to H^i(M^\bullet/I^nM^\bullet)$$

define isomorphisms of pro-systems of A-modules for any $i \in \mathbf{Z}$. To see this choose a quasi-isomorphism $P^\bullet \to M^\bullet$ where $P^\bullet$ is a bounded above complex of finite free A-modules. The arrows above are given by the maps

$$H^i(P^\bullet/I^nP^\bullet) \to H^i(M^\bullet/I^nM^\bullet)$$

These define an isomorphism of pro-systems by Lemma 100.1. Namely, the lemma shows both are isomorphic to the pro-system H^i/I^nH^i with $H^i = H^i(M^\bullet) = H^i(P^\bullet)$. $\square$

Lemma 100.4. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be finite A-modules. Set $M_n = M/I^nM$ and $N_n = N/I^nN$. Then*

1. *the systems $(\mathrm{Hom}_A(M_n, N_n))$ and $(\mathit{Isom}_A(M_n, N_n))$ are Mittag-Leffler,*
2. *there exists a $c \geq 0$ such that the kernels and cokernels of*
$$\mathrm{Hom}_A(M, N)/I^n \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M_n, N_n)$$
are killed by I^c for all n,
3. *we have $\lim \mathrm{Hom}_A(M_n, N_n) = \mathrm{Hom}_A(M, N)^\wedge = \mathrm{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$*
4. *$\lim \mathit{Isom}_A(M_n, N_n) = \mathit{Isom}_{A^\wedge}(M^\wedge, N^\wedge)$.*

Here $^\wedge$ denotes usual I-adic completion.

Proof. Note that $\mathrm{Hom}_A(M_n, N_n) = \mathrm{Hom}_A(M, N_n)$. Choose a presentation

$$A^{\oplus t} \to A^{\oplus s} \to M \to 0$$

Applying the right exact functor $\mathrm{Hom}_A(-, N)$ we obtain a complex

$$0 \xrightarrow{\alpha} N^{\oplus s} \xrightarrow{\beta} N^{\oplus t}$$

whose cohomology in the middle is $\mathrm{Hom}_A(M, N)$ and such that for $n \geq 0$ the cohomology of

$$0 \xrightarrow{\alpha_n} N_n^{\oplus s} \xrightarrow{\beta_n} N_n^{\oplus t}$$

is $\mathrm{Hom}_A(M_n, N_n)$. Let $c \geq 0$ be as in Lemma 100.1 for this A, I, α, and β. By part (3) of the lemma we deduce the Mittag-Leffler property for $(\mathrm{Hom}_A(M_n, N_n))$. The kernel and cokernel of the maps $\mathrm{Hom}_A(M, N)/I^n \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M_n, N_n)$ are killed by I^c by [art part (6) of the lemma. We find that $\lim \mathrm{Hom}_A(M_n, N_n) = \mathrm{Hom}_A(M, N)^\wedge$ by part (2) of the lemma. The equality

$$\mathrm{Hom}_{A^\wedge}(M^\wedge, N^\wedge) = \lim \mathrm{Hom}_A(M_n, N_n)$$

follows formally from the fact that $M^\wedge = \lim M_n$ and $M_n = M^\wedge/I^nM^\wedge$ and the corresponding facts for N, see Algebra, Lemma 97.4.

The result for isomorphisms follows from the case of homomorphisms applied to both $(\mathrm{Hom}(M_n, N_n))$ and $(\mathrm{Hom}(N_n, M_n))$ and the following fact: for $n > m > 0$, if we have maps $\alpha : M_n \to N_n$ and $\beta : N_n \to M_n$ which induce an isomorphisms $M_m \to N_m$ and $N_m \to M_m$, then α and β are isomorphisms. Namely, then $\alpha \circ \beta$ is surjective by Nakayama's lemma (Algebra, Lemma 20.1) hence $\alpha \circ \beta$ is an isomorphism by Algebra, Lemma 16.4. $\square$

Lemma 100.5. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be finite A-modules. Set $M_n = M/I^nM$ and $N_n = N/I^nN$. If $M_n \cong N_n$ for all n, then $M^\wedge \cong N^\wedge$ as $A^\wedge$-modules.*

Proof. By Lemma 100.4 the system $(\mathrm{Isom}_A(M_n, N_n))$ is Mittag-Leffler. By assumption each of the sets $\mathrm{Isom}_A(M_n, N_n)$ is nonempty. Hence $\lim \mathrm{Isom}_A(M_n, N_n)$ is nonempty. Since $\lim \mathrm{Isom}_A(M_n, N_n) = \mathrm{Isom}_{A^\wedge}(M^\wedge, N^\wedge)$ we obtain an isomorphism. $\square$

Remark 100.6. Let I be an ideal of a Noetherian ring A. Set $A_n = A/I^n$ for $n \geq 1$. Consider the following category:

(1) An object is a sequence $\{E_n\}_{n \geq 1}$ where E_n is a finite A_n-module.
(2) A morphism $\{E_n\} \to \{E'_n\}$ is given by maps
$$\varphi_n : I^cE_n \longrightarrow E'_n/E'_n[I^c] \quad \text{for } n \geq c$$
where $E'_n[I^c]$ is the torsion submodule (Section 88) up to equivalence: we say (c, φ_n) is the same as $(c+1, \overline{\varphi}_n)$ where $\overline{\varphi}_n : I^{c+1}E_n \longrightarrow E'_n/E'_n[I^{c+1}]$ is the induced map.

Composition of $(c, \varphi_n) : \{E_n\} \to \{E'_n\}$ and $(c', \varphi'_n) : \{E'_n\} \to \{E''_n\}$ is defined by the obvious compositions
$$I^{c+c'}E_n \to I^{c'}E'_n/E'_n[I^c] \to E''_n/E''_n[I^{c+c'}]$$
for $n \geq c + c'$. We omit the verification that this is a category.

Lemma 100.7. *A morphism (c, φ_n) of the category of Remark 100.6 is an isomorphism if and only if there exists a $c' \geq 0$ such that $\mathrm{Ker}(\varphi_n)$ and $\mathrm{Coker}(\varphi_n)$ are $I^{c'}$-torsion for all $n \gg 0$.*

Proof. We may and do assume $c' \geq c$ and that the $\mathrm{Ker}(\varphi_n)$ and $\mathrm{Coker}(\varphi_n)$ are $I^{c'}$-torsion for all n. For $n \geq c'$ and $x \in I^{c'}E'_n$ we can choose $y \in I^cE_n$ with $x = \varphi_n(y) \bmod E'_n[I^c]$ as $\mathrm{Coker}(\varphi_n)$ is annihilated by $I^{c'}$. Set $\psi_n(x)$ equal to the class of y in $E_n/E_n[I^{c'}]$. For a different choice $y' \in I^cE_n$ with $x = \varphi_n(y') \bmod E'_n[I^c]$ the difference $y - y'$ maps to zero in $E'_n/E'_n[I^c]$ and hence is annihilated by $I^{c'}$ in I^cE_n. Thus the maps $\psi_n : I^{c'}E'_n \to E_n/E_n[I^{c'}]$ are well defined. We omit the verification that (c', ψ_n) is the inverse of (c, φ_n) in the category. $\square$

Lemma 100.8. *Let I be an ideal of the Noetherian ring A. Let M and N be finite A-modules. Write $A_n = A/I^n$, $M_n = M/I^nM$, and $N_n = N/I^nN$. For every $i \geq 0$ the objects*
$$\{\mathrm{Ext}^i_A(M, N)/I^n \mathrm{Ext}^i_A(M, N)\}_{n \geq 1} \quad \text{and} \quad \{\mathrm{Ext}^i_{A_n}(M_n, N_n)\}_{n \geq 1}$$
are isomorphic in the category $\mathcal{C}$ of Remark 100.6.

Proof. Choose a short exact sequence
$$0 \to K \to A^{\oplus r} \to M \to 0$$
and set $K_n = K/I^nK$. For $n \geq 1$ define $K(n) = \mathrm{Ker}(A_n^{\oplus r} \to M_n)$ so that we have exact sequences
$$0 \to K(n) \to A_n^{\oplus r} \to M_n \to 0$$
and surjections $K_n \to K(n)$. In fact, by Lemma 100.1 there is a $c \geq 0$ and maps $K(n) \to K_n/I^{n-c}K_n$ which are "almost inverse". Since $I^{n-c}K_n \subset K_n[I^c]$ these maps which witness the fact that the systems $\{K(n)\}_{n\geq 1}$ and $\{K_n\}_{n\geq 1}$ are isomorphic in $\mathcal{C}$.

We claim the systems
$$\{\mathrm{Ext}^i_{A_n}(K(n), N_n)\}_{n\geq 1} \quad \text{and} \quad \{\mathrm{Ext}^i_{A_n}(K_n, N_n)\}_{n\geq 1}$$
are isomorphic in the category $\mathcal{C}$. Namely, the surjective maps $K_n \to K(n)$ have kernels annihilated by I^c and therefore determine maps
$$\mathrm{Ext}^i_{A_n}(K(n), N_n) \to \mathrm{Ext}^i_{A_n}(K_n, N_n)$$
whose kernel and cokernel are annihilated by I^c. Hence the claim by Lemma 100.7.

For $i \geq 2$ we have isomorphisms
$$\mathrm{Ext}^{i-1}_A(K, N) = \mathrm{Ext}^i_A(M, N) \quad \text{and} \quad \mathrm{Ext}^{i-1}_{A_n}(K(n), N_n) = \mathrm{Ext}^i_{A_n}(M_n, N_n)$$
In this way we see that it suffices to prove the lemma for $i = 0, 1$.

For $i = 0, 1$ we consider the commutative diagram
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(M, N) \arrow[r] \arrow[dd] & N^{\oplus r} \arrow[r, "\varphi"'] \arrow[dd] & \mathrm{Hom}(K, N) \arrow[r] \arrow[d] & \mathrm{Ext}^1(M, N) \arrow[r] & 0 \\
 & & & \mathrm{Hom}(K_n, N_n) & & \\
0 \arrow[r] & \mathrm{Hom}(M_n, N_n) \arrow[r] & N_n^{\oplus r} \arrow[r] & \mathrm{Hom}(K(n), N_n) \arrow[r] \arrow[u] & \mathrm{Ext}^1(M_n, N_n) \arrow[r] & 0
\end{tikzcd}$$
By Lemma 100.4 we see that the kernel and cokernel of $\mathrm{Hom}(M, N)/I^n\,\mathrm{Hom}(M, N) \to \mathrm{Hom}(M_n, N_n)$ and $\mathrm{Hom}(K, N)/I^n\,\mathrm{Hom}(K, N) \to \mathrm{Hom}(K_n, N_n)$ and are I^c-torsion for some $c \geq 0$ independent of n. Above we have seen the cokernel of the injective maps $\mathrm{Hom}(K(n), N_n) \to \mathrm{Hom}(K_n, N_n)$ are annihilated by I^c after possibly increasing c. For such a c we obtain maps $\delta_n : I^c\,\mathrm{Hom}(K, N)/I^n\,\mathrm{Hom}(K, N) \to \mathrm{Hom}(K(n), N_n)$ fitting into the diagram (precise formulation omitted). The kernel and cokernel of δ_n are annihilated by I^c after possibly increasing c since we know that the same thing is true for $\mathrm{Hom}(K, N)/I^n\,\mathrm{Hom}(K, N) \to \mathrm{Hom}(K_n, N_n)$ and $\mathrm{Hom}(K(n), N_n) \to \mathrm{Hom}(K_n, N_n)$. Then we can use commutativity of the solid diagram
$$\begin{tikzcd}
\varphi^{-1}(I^c\,\mathrm{Hom}(K, N)) \arrow[r, "\varphi"'] \arrow[d] & I^c\,\mathrm{Hom}(K, N)/I^n\,\mathrm{Hom}(K, N) \arrow[r] \arrow[d, "\delta_n"] & I^c\,\mathrm{Ext}^1(M, N)/I^n\,\mathrm{Ext}^1(M, N) \arrow[r] \arrow[d, dotted] & 0 \\
N_n^{\oplus r} \arrow[r] & \mathrm{Hom}(K(n), N_n) \arrow[r] & \mathrm{Ext}^1(M_n, N_n) \arrow[r] & 0
\end{tikzcd}$$

to define the dotted arrow. A straightforward diagram chase (omitted) shows that the kernel and cokernel of the dotted arrow are annihilated buy I^c after possibly increasing c one final time. $\square$

Remark 100.9. The awkwardness in the statement of Lemma 100.8 is partly due to the fact that there are no obvious maps between the modules $\text{Ext}^i_{A_n}(M_n, N_n)$ for varying n. What we may conclude from the lemma is that there exists a $c \geq 0$ such that for $m \gg n \gg 0$ there are (canonical) maps

$$I^c \text{Ext}^i_{A_n}(M_m, N_m)/I^n \text{Ext}^i_{A_n}(M_m, N_m) \to \text{Ext}^i_{A_n}(M_n, N_n)/\text{Ext}^i_{A_n}(M_n, N_n)[I^c]$$

whose kernel and cokernel are annihilated by I^c. This is the (weak) sense in which we get a system of modules.

Example 100.10. Let k be a field. Let $A = k[[x, y]]/(xy)$. By abuse of notation we denote x and y the images of x and y in A. Let $I = (x)$. Let $M = A/(y)$. There is a free resolution

$$\ldots \to A \xrightarrow{y} A \xrightarrow{x} A \xrightarrow{y} A \to M \to 0$$

We conclude that

$$\text{Ext}^2_A(M, N) = N[y]/xN$$

where $N[y] = \text{Ker}(y : N \to N)$. We denote $A_n = A/I^n$, $M_n = M/I^nM$, and $N_n = N/I^nN$. For each n we have a free resolution

$$\ldots \to A_n^{\oplus 2} \xrightarrow{y, x^{n-1}} A_n \xrightarrow{x} A_n \xrightarrow{y} A_n \to M_n \to 0$$

We conclude that

$$\text{Ext}^2_{A_n}(M_n, N_n) = (N_n[y] \cap N_n[x^{n-1}])/xN_n$$

where $N_n[y] = \text{Ker}(y : N_n \to N_n)$ and $N[x^{n-1}] = \text{Ker}(x^{n-1} : N_n \to N_n)$. Take $N = A/(y)$. Then we see that

$$\text{Ext}^2_A(M, N) = N[y]/xN = N/xN \cong k$$

but

$$\text{Ext}^2_{A_n}(M_n, N_n) = (N_n[y] \cap N_n[x^{n-1}])/xN_n = N_n[x^{n-1}]/xN_n = 0$$

for all r because $N_n = k[x]/(x^n)$ and the sequence

$$N_n \xrightarrow{x} N_n \xrightarrow{x^{n-1}} N_n$$

is exact. Thus ignoring some kind of I-power torsion is necessary to get a result as in Lemma 100.8.

Lemma 100.11. *Let $A \to B$ be a flat homomorphism of Noetherian rings. Let $I \subset A$ be an ideal. Let M, N be A-modules. Set $B_n = B/I^nB$, $M_n = M/I^nM$, $N_n = N/I^nN$. If M is flat over A, then we have*

$$\lim \text{Ext}^i_B(M, N)/I^n \text{Ext}^i_B(M, N) = \lim \text{Ext}^i_{B_n}(M_n, N_n)$$

for all $i \in \mathbf{Z}$.

Proof. Choose a resolution

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$$

by finite free B-modues P_i. Set $P_{i,n} = P_i/I^nP_i$. Since M and B are flat over A, the sequence

$$\ldots \to P_{2,n} \to P_{1,n} \to P_{0,n} \to M_n \to 0$$

is exact. We see that on the one hand the complex

$$\mathrm{Hom}_B(P_0, N) \to \mathrm{Hom}_B(P_1, N) \to \mathrm{Hom}_B(P_2, N) \to \ldots$$

computes the modules $\mathrm{Ext}^i_B(M, N)$ and on the other hand the complex

$$\mathrm{Hom}_{B_n}(P_{0,n}, N_n) \to \mathrm{Hom}_{B_n}(P_{1,n}, N_n) \to \mathrm{Hom}_{B_n}(P_{2,n}, N_n) \to \ldots$$

computes the modules $\mathrm{Ext}^i_{B_n}(M_n, N_n)$. Since

$$\mathrm{Hom}_{B_n}(P_{i,n}, N_n) = \mathrm{Hom}_B(P_i, N)/I^n \mathrm{Hom}_B(P_i, N)$$

we obtain the result from Lemma 100.1 part (2). □

101. Systems of modules, bis

Let I be an ideal of a Noetherian ring A. In Section 100 we considered what happens when considering systems of the form M/I^nM for finite A-modules M. In this section we consider the systems I^nM instead.

Lemma 101.1. *Let I be an ideal of a Noetherian ring A. Let $K \xrightarrow{\alpha} L \xrightarrow{\beta} M$ be a complex of finite A-modules. Set $H = \mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$. For $n \geq 0$ let*

$$I^nK \xrightarrow{\alpha_n} I^nL \xrightarrow{\beta_n} I^nM$$

be the induced complex. Set $H_n = \mathrm{Ker}(\beta_n)/\mathrm{Im}(\alpha_n)$. Then there are canonical A-module maps

$$\ldots \to H_3 \to H_2 \to H_1 \to H$$

There exists a $c > 0$ such that for $n \geq c$ the image of $H_n \to H$ is contained in $I^{n-c}H$ and there is a canonical A-module map $I^nH \to H_{n-c}$ such that the compositions

$$I^nH \to H_{n-c} \to I^{n-2c}H \quad \text{and} \quad H_n \to I^{n-c}H \to H_{n-2c}$$

are the canonical ones. In particular, the inverse systems (H_n) and (I^nH) are isomorphic as pro-objects of Mod_A.

Proof. We have $H_n = \mathrm{Ker}(\beta) \cap I^nL/\alpha(I^nK)$. Since $\mathrm{Ker}(\beta) \cap I^nL \subset \mathrm{Ker}(\beta) \cap I^{n-1}L$ and $\alpha(I^nK) \subset \alpha(I^{n-1}K)$ we get the maps $H_n \to H_{n-1}$. Similarly for the map $H_1 \to H$.

By Artin-Rees we may choose $c_1, c_2 \geq 0$ such that $\mathrm{Im}(\alpha) \cap I^nL \subset \alpha(I^{n-c_1}K)$ for $n \geq c_1$ and $\mathrm{Ker}(\beta) \cap I^nL \subset I^{n-c_2}\mathrm{Ker}(\beta)$ for $n \geq c_2$, see Algebra, Lemmas 51.3 and 51.2. Set $c = c_1 + c_2$.

It follows immediately from our choice of $c \geq c_2$ that for $n \geq c$ the image of $H_n \to H$ is contained in $I^{n-c}H$.

Let $n \geq c$. We define $\psi_n : I^nH \to H_{n-c}$ as follows. Say $x \in I^nH$. Choose $y \in I^n \mathrm{Ker}(\beta)$ representing x. We set $\psi_n(x)$ equal to the class of y in H_{n-c}. To see this is well defined, suppose we have a second choice y' as above for x. Then $y' - y \in \mathrm{Im}(\alpha)$. By our choice of $c \geq c_1$ we conclude that $y' - y \in \alpha(I^{n-c}K)$ which implies that y and y' represent the same element of H_{n-c}. Thus ψ_n is well defined.

The statements on the compositions $I^nH \to H_{n-c} \to I^{n-2c}H$ and $H_n \to I^{n-c}H \to H_{n-2c}$ follow immediately from our definitions. □

Lemma 101.2. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be A-modules with M finite. For each $p > 0$ there exists a $c \geq 0$ such that for $n \geq c$ the map $\mathrm{Ext}^p_A(M, N) \to \mathrm{Ext}^p_A(I^nM, N)$ factors through $\mathrm{Ext}^p_A(I^nM, I^{n-c}N) \to \mathrm{Ext}^p_A(I^nM, N)$.*

Proof. For $p = 0$, if $\varphi : M \to N$ is an A-linear map, then $\varphi(\sum f_i m_i) = \sum f_i \varphi(m_i)$ for $f_i \in A$ and $m_i \in M$. Hence φ induces a map $I^nM \to I^nN$ for all n and the result is true with $c = 0$.

Choose a short exact sequence $0 \to K \to A^{\oplus t} \to M \to 0$. For each n we pick a short exact sequence $0 \to L_n \to A^{\oplus s_n} \to I^nM \to 0$. It is clear that we can construct a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L_n \arrow[r] \arrow[d] & A^{\oplus s_n} \arrow[r] \arrow[d] & I^nM \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & A^{\oplus t} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

such that $A^{\oplus s_n} \to A^{\oplus t}$ has image in $(I^n)^{\oplus t}$. By Artin-Rees (Algebra, Lemma 51.2) there exists a $c \geq 0$ such that $L_n \to K$ factors through $I^{n-c}K$ if $n \geq c$.

For $p = 1$ our choices above induce a solid commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(A^{\oplus s_n}, N) \arrow[r] & \mathrm{Hom}_A(L_n, N) \arrow[r] & \mathrm{Ext}^1_A(I^nM, N) \arrow[r] & 0 \\
\mathrm{Hom}_A((I^n)^{\oplus t}, I^{n-c}N) \arrow[r] \arrow[u] & \mathrm{Hom}_A(K \cap (I^n)^{\oplus t}, I^{n-c}N) \arrow[r] \arrow[u] & \mathrm{Ext}^1_A(I^nM, I^{n-c}N) \arrow[u] & \\
\mathrm{Hom}_A(A^{\oplus t}, N) \arrow[r] \arrow[u] & \mathrm{Hom}_A(K, N) \arrow[r] \arrow[u] & \mathrm{Ext}^1_A(M, N) \arrow[r] \arrow[u, dotted] & 0
\end{tikzcd}$$

whose horizontal arrows are exact. The lower middle vertical arrow arises because $K \cap (I^n)^{\oplus t} \subset I^{n-c}K$ and hence any A-linear map $K \to N$ induces an A-linear map $(I^n)^{\oplus t} \to I^{n-c}N$ by the argument of the first paragraph. Thus we obtain the dotted arrow as desired.

For $p > 1$ we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^{p-1}_A(I^{n-c}K, N) \arrow[r] & \mathrm{Ext}^{p-1}_A(L_n, N) \arrow[r] & \mathrm{Ext}^p_A(I^nM, N) \\
\mathrm{Ext}^{p-1}_A(K, N) \arrow[rr] \arrow[u] & & \mathrm{Ext}^p_A(M, N) \arrow[u]
\end{tikzcd}$$

whose bottom horizontal arrow is an isomorphism. By induction on p the left vertical map factors through $\mathrm{Ext}^{p-1}_A(I^{n-c}K, I^{n-c-c'}N)$ for some $c' \geq 0$ and all $n \geq c + c'$. Using the composition $\mathrm{Ext}^{p-1}_A(I^{n-c}K, I^{n-c-c'}N) \to \mathrm{Ext}^{p-1}_A(L_n, I^{n-c-c'}N) \to \mathrm{Ext}^p_A(I^nM, I^{n-c-c'}N)$ we obtain the desired factorization (for $n \geq c + c'$ and with c replaced by $c + c'$). $\square$

Lemma 101.3. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be A-modules with M finite and N annihilated by a power of I. For each $p > 0$ there exists an n such that the map $\mathrm{Ext}^p_A(M, N) \to \mathrm{Ext}^p_A(I^nM, N)$ is zero.*

Proof. Immediate consequence of Lemma 101.2 and the fact that $I^m N = 0$ for some $m > 0$. $\square$

Lemma 101.4. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $K \in D(A)$ be pseudo-coherent and let M be a finite A-module. For each $p \in \mathbf{Z}$ there exists an c such that the image of $\text{Ext}^p_A(K, I^n M) \to \text{Ext}^p_A(K, M)$ is contained in $I^{n-c}\text{Ext}^p_A(K, M)$ for $n \geq c$.*

Proof. Choose a bounded above complex $P^\bullet$ of finite free A-modules representing K. Then $\text{Ext}^p_A(K, M)$ is the cohomology of

$$\text{Hom}_A(F^{-p+1}, M) \xrightarrow{a} \text{Hom}_A(F^{-p}, M) \xrightarrow{b} \text{Hom}_A(F^{-p-1}, M)$$

and $\text{Ext}^p_A(K, I^n M)$ is computed by replacing these finite A-modules by I^n times themselves. Thus the result by Lemma 101.1 (and much more is true). $\square$

In Situation 91.15 we define complexes $I_n^\bullet$ such that we have distinguished triangles

$$I_n^\bullet \to A \to K_n^\bullet \to I_n^\bullet[1]$$

in the triangulated category $K(A)$ of complexes of A-modules up to homotopy. Namely, we set $I_n^\bullet = \sigma_{\leq -1} K_n^\bullet[-1]$. We have termwise split short exact sequences of complexes

$$0 \to A \to K_n^\bullet \to I_n^\bullet[1] \to 0$$

defining distinguished triangles by definition of the triangulated structure on $K(A)$. Their rotations determine the desired distinguished triangles above. Note that $I_n^0 = A^{\oplus r} \to A$ is given by multiplication by f_i^n on the ith factor. Hence $I_n^\bullet \to A$ factors as

$$I_n^\bullet \to (f_1^n, \ldots, f_r^n) \to A$$

In fact, there is a short exact sequence

$$0 \to H^{-1}(K_n^\bullet) \to H^0(I_n^\bullet) \to (f_1^n, \ldots, f_r^n) \to 0$$

and for every $i < 0$ we have $H^i(I_n^\bullet) = H^{i-1}(K_n^\bullet$. The maps $K_{n+1}^\bullet \to K_n^\bullet$ induce maps $I_{n+1}^\bullet \to I_n^\bullet$ and we obtain a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & I_3^\bullet \arrow[r] \arrow[d] & I_2^\bullet \arrow[r] \arrow[d] & I_1^\bullet \arrow[d] \\
\ldots \arrow[r] & (f_1^3, \ldots, f_r^3) \arrow[r] & (f_1^2, \ldots, f_r^2) \arrow[r] & (f_1, \ldots, f_r)
\end{tikzcd}$$

in $K(A)$.

Lemma 101.5. *In Situation 91.15 assume A is Noetherian. With notation as above, the inverse system (I^n) is pro-isomorphic in $D(A)$ to the inverse system $(I_n^\bullet)$.*

Proof. It is elementary to show that the inverse system I^n is pro-isomorphic to the inverse system $(f_1^n, \ldots, f_r^n)$ in the category of A-modules. Consider the inverse system of distinguished triangles

$$I_n^\bullet \to (f_1^n, \ldots, f_r^n) \to C_n^\bullet \to I_n^\bullet[1]$$

where $C_n^\bullet$ is the cone of the first arrow. By Derived Categories, Lemma 42.4 it suffices to show that the inverse system $C_n^\bullet$ is pro-zero. The complex $I_n^\bullet$ has nonzero terms only in degrees i with $-r + 1 \leq i \leq 0$ hence $C_n^\bullet$ is bounded similarly. Thus by Derived Categories, Lemma 42.3 it suffices to show that $H^p(C_n^\bullet)$ is pro-zero. By the discussion above we have $H^p(C_n^\bullet) = H^p(K_n^\bullet)$ for $p \leq -1$ and $H^p(C_n^\bullet) = 0$ for $p \geq 0$. The fact that the inverse systems $H^p(K_n^\bullet)$ are pro-zero was shown in the proof of Lemma 94.1 (and this is where the assumption that A is Noetherian is used). $\square$

Lemma 101.6. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $M^\bullet$ be a bounded complex of finite A-modules. The inverse system of maps*

$$I^n \otimes_A^{\mathbf{L}} M^\bullet \longrightarrow I^n M^\bullet$$

defines an isomorphism of pro-objects of $D(A)$.

Proof. Choose generators $f_1, \ldots, f_r \in I$ of I. The inverse system I^n is pro-isomorphic to the inverse system $(f_1^n, \ldots, f_r^n)$ in the category of A-modules. With notation as in Lemma 101.5 we find that it suffices to prove the inverse system of maps

$$I_n^\bullet \otimes_A^{\mathbf{L}} M^\bullet \longrightarrow (f_1^n, \ldots, f_r^n)M^\bullet$$

defines an isomorphism of pro-objects of $D(A)$. Say we have $a \leq b$ such that $M^i = 0$ if $i \notin [a, b]$. Then source and target of the arrows above have cohomology only in degrees $[-r + a, b]$. Thus it suffices to show that for any $p \in \mathbf{Z}$ the inverse system of maps

$$H^p(I_n^\bullet \otimes_A^{\mathbf{L}} M^\bullet) \longrightarrow H^p((f_1^n, \ldots, f_r^n)M^\bullet)$$

defines an isomorphism of pro-objects of A-modules, see Derived Categories, Lemma 42.5. Using the pro-isomorphism between $I_n^\bullet \otimes_A^{\mathbf{L}} M^\bullet$ and $I^n \otimes_A^{\mathbf{L}} M^\bullet$ and the pro-isomorphism between $(f_1^n, \ldots, f_r^n)M^\bullet$ and $I^n M^\bullet$ this is equivalent to showing that the inverse system of maps

$$H^p(I^n \otimes_A^{\mathbf{L}} M^\bullet) \longrightarrow H^p(I^n M^\bullet)$$

defines an isomorphism of pro-objects of A-modules Choose a bounded above complex of finite free A-modules $P^\bullet$ and a quasi-isomorphism $P^\bullet \to M^\bullet$. Then it suffices to show that the inverse system of maps

$$H^p(I^n P^\bullet) \longrightarrow H^p(I^n M^\bullet)$$

is a pro-isomorphism. This follows from Lemma 101.1 as $H^p(P^\bullet) = H^p(M^\bullet)$. $\square$

Lemma 101.7. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. There exists an integer $n > 0$ such that $I^n M \to M$ factors through the map $I \otimes_A^{\mathbf{L}} M \to M$ in $D(A)$.*

Proof. This follows from Lemma 101.6. It can also been seen directly as follows. Consider the distinguished triangle

$$I \otimes_A^{\mathbf{L}} M \to M \to A/I \otimes_A^{\mathbf{L}} M \to I \otimes_A^{\mathbf{L}} M[1]$$

By the axioms of a triangulated category it suffices to prove that $I^n M \to A/I \otimes_A^{\mathbf{L}} M$ is zero in $D(A)$ for some n. Choose generators $f_1, \ldots, f_r$ of I and let $K = K_\bullet(A, f_1, \ldots, f_r)$ be the Koszul complex and consider the factorization $A \to K \to A/I$ of the quotient map. Then we see that it suffices to show that $I^n M \to K \otimes_A M$ is zero in $D(A)$ for some $n > 0$. Suppose that we have found an $n > 0$ such that $I^n M \to K \otimes_A M$ factors through $\tau_{\geq t}(K \otimes_A M)$ in $D(A)$. Then the obstruction to factoring through $\tau_{\geq t+1}(K \otimes_A M)$ is an element in $\text{Ext}^t(I^n M, H_t(K \otimes_A M))$. The finite A-module $H_t(K \otimes_A M)$ is annihilated by I. Then by Lemma 101.3

we can after increasing n assume this obstruction element is zero. Repeating this a finite number of times we find n such that $I^nM \to K \otimes_A M$ factors through $0 = \tau_{\geq r+1}(K \otimes_A M)$ in $D(A)$ and we win. $\square$

102. Miscellany

Some results which do not fit anywhere else.

Lemma 102.1. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $K \in D(A)$ be pseudo-coherent. Let $a \in \mathbf{Z}$. Assume that for every finite A-module M the modules $\text{Ext}^i_A(K, M)$ are I-power torsion for $i \geq a$. Then for $i \geq a$ and M finite the system $\text{Ext}^i_A(K, M/I^nM)$ is essentially constant with value*

$$\text{Ext}^i_A(K, M) = \lim \text{Ext}^i_A(K, M/I^nM)$$

Proof. Let M be a finite A-module. Since K is pseudo-coherent we see that $\text{Ext}^i_A(K, M)$ is a finite A-module. Thus for $i \geq a$ it is annihilated by I^t for some $t \geq 0$. By Lemma 101.4 we see that the image of $\text{Ext}^i_A(K, I^nM) \to \text{Ext}^i_A(K, M)$ is zero for some $n > 0$. The short exact sequence $0 \to I^nM \to M \to M/I^nM \to 0$ gives a long exact sequence

$$\text{Ext}^i_A(K, I^nM) \to \text{Ext}^i_A(K, M) \to \text{Ext}^i_A(K, M/I^nM) \to \text{Ext}^{i+1}_A(K, I^nM)$$

The systems $\text{Ext}^i_A(K, I^nM)$ and $\text{Ext}^{i+1}_A(K, I^nM)$ are essentially constant with value 0 by what we just said (applied to the finite A-modules I^mM). A diagram chase shows $\text{Ext}^i_A(K, M/I^nM)$ is essentially constant with value $\text{Ext}^i_A(K, M)$. $\square$

Lemma 102.2. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let N be an A-module annihilated by I. There exists an integer $n > 0$ such that $Tor^A_p(I^nM, N) \to Tor^A_p(M, N)$ is zero for all $p \geq 0$.*

Proof. By Lemma 101.7 we can factor $I^nM \to M$ as $I^nM \to M \otimes^{\mathbf{L}}_A I \to M$. We claim the composition

$$I^nM \otimes^{\mathbf{L}}_A N \to (M \otimes^{\mathbf{L}}_A I) \otimes^{\mathbf{L}}_A N \to M \otimes^{\mathbf{L}}_A N$$

is zero. Namely, the diagram

$$\begin{tikzcd}
(M \otimes^{\mathbf{L}}_A I) \otimes^{\mathbf{L}}_A N \arrow[rr] \arrow[dr] & & M \otimes^{\mathbf{L}}_A (I \otimes^{\mathbf{L}}_A N) \arrow[dl] \\
& M \otimes^{\mathbf{L}}_A N &
\end{tikzcd}$$

commutes (details omitted) and the map $I \otimes^{\mathbf{L}}_A N \to N$ is zero as N is annihilated by I. $\square$

Lemma 102.3. *Let R be a ring. Let $K \in D(R)$ be pseudo-coherent. Let (M_n) be an inverse system of R-modules. Then $R\lim K \otimes^{\mathbf{L}}_R M_n = K \otimes^{\mathbf{L}}_R R\lim M_n$.*

Proof. Consider the defining distinguished triangle

$$R\lim M_n \to \prod M_n \to \prod M_n \to R\lim M_n[1]$$

and apply Lemma 65.5. $\square$

Lemma 102.4. *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal and let E be a nonzero module over R/I. If R/I has finite projective dimension and E has finite projective dimension over R/I, then E has finite projective dimension over R and*

$$pd_R(E) = pd_R(R/I) + pd_{R/I}(E)$$

Proof. We will use that, for a finite module, having finite projective dimension over R, resp. R/I is the same as being a perfect module, see discussion following Definition 74.1. We see that E has finite projective dimension over R by Lemma 74.7. Thus we can apply Auslander-Buchsbaum (Algebra, Proposition 111.1) to see that

$$\mathrm{pd}_R(E) + \mathrm{depth}(E) = \mathrm{depth}(R), \quad \mathrm{pd}_{R/I}(E) + \mathrm{depth}(E) = \mathrm{depth}(R/I),$$

and

$$\mathrm{pd}_R(R/I) + \mathrm{depth}(R/I) = \mathrm{depth}(R)$$

Note that in the first equation we take the depth of E as an R-module and in the second as an R/I-module. However these depths are the same (this is trivial but also follows from Algebra, Lemma 72.11). This concludes the proof. $\square$

Lemma 102.5. *Let $A \to B$ be a ring map. There exists a cardinal $\kappa = \kappa(A \to B)$ with the following property: Let $M^\bullet$, resp. $N^\bullet$ be a complex of A-modules, resp. B-modules. Let $a : M^\bullet \to N^\bullet$ be a map of complexes of A-modules which induces an isomorphism $M^\bullet \otimes_A^{\mathbf{L}} B \to N^\bullet$ in $D(B)$. Let $M_1^\bullet \subset M^\bullet$, resp. $N_1^\bullet \subset N^\bullet$ be a subcomplex of A-modules, resp. B-modules such that $a(M_1^\bullet) \subset N_1^\bullet$. Then there exist subcomplexes*

$$M_1^\bullet \subset M_2^\bullet \subset M^\bullet \quad \textit{and} \quad N_1^\bullet \subset N_2^\bullet \subset N^\bullet$$

such that $a(M_2^\bullet) \subset N_2^\bullet$ with the following properties:

1. $\mathrm{Ker}(H^i(M_1^\bullet \otimes_A^{\mathbf{L}} B) \to H^i(N_1^\bullet))$ *maps to zero in* $H^i(M_2^\bullet \otimes_A^{\mathbf{L}} B)$*,*
2. $\mathrm{Im}(H^i(N_1^\bullet) \to H^i(N_2^\bullet))$ *is contained in* $\mathrm{Im}(H^i(M_2^\bullet \otimes_A^{\mathbf{L}} B) \to H^2(N_2^\bullet))$*,*
3. $|\bigcup M_2^i \cup \bigcup N_2^i| \leq \max(\kappa, |\bigcup M_1^i \cup \bigcup N_1^i|)$*.*

Proof. Let $\kappa = \max(|A|, |B|, \aleph_0)$. Set $|M^\bullet| = |\bigcup M^i|$ and similarly for other complexes. With this notation we have

$$\max(\kappa, |\bigcup M_1^i \cup \bigcup N_1^i|) = \max(\kappa, |M_1^\bullet|, |M_2^\bullet|)$$

for the quantity used in the statement of the lemma. We are going to use this and other observations coming from arithmetic of cardinals without further mention.

First, let us show that there are plenty of "small" subcomplexes. For every pair of collections $E = \{E^i\}$ and $F = \{F^i\}$ of finite subsets $E^i \subset M^i$, $i \in \mathbf{Z}$ and $F^i \subset N^i$, $i \in \mathbf{Z}$ we can let

$$M_1^\bullet \subset M_1(E, F)^\bullet \subset M^\bullet \quad \text{and} \quad N_1^\bullet \subset N_1(E, F)^\bullet \subset N^\bullet$$

be the smallest subcomplexes of A and B-modules such that $a(M_1(E, F)^\bullet) \subset N_1(E, F)^\bullet$ and such that $E^i \subset M_1(E, F)^i$ and $F^i \subset M_2(E, F)^i$. Then it is easy to see that

$$|M_1(E, F)^\bullet| \leq \max(\kappa, |M_1^\bullet|) \quad \text{and} \quad |M_2(E, F)^\bullet| \leq \max(\kappa, |M_2^\bullet|)$$

Details omitted. It is clear that we have

$$M^\bullet = \mathrm{colim}_{(E,F)} M_1(E, F)^\bullet \quad \text{and} \quad N^\bullet = \mathrm{colim}_{(E,F)} N_1(E, F)^\bullet$$

and the colimits are (termwise) filtered colimits.

There exists a resolution $\ldots \to F^{-1} \to F^0 \to B$ by free A-modules F_i with $|F_i| \leq \kappa$ (details omitted). The cohomology modules of $M_1^\bullet \otimes_A^{\mathbf{L}} B$ are computed by $\text{Tot}(M_1^\bullet \otimes_A F^\bullet)$. It follows that $|H^i(M_1^\bullet \otimes_A^{\mathbf{L}} B)| \leq \max(\kappa, |M_1^\bullet|)$.

Let $i \in \mathbf{Z}$ and let $\xi \in H^i(M_1^\bullet \otimes_A^{\mathbf{L}} B)$ be an element which maps to zero in $H^i(N_1^\bullet)$. Then ξ maps to zero in $H^i(N^\bullet)$ and hence ξ maps to zero in $H^i(M^\bullet \otimes_A^{\mathbf{L}} B)$. Since derived tensor product commutes with filtered colimits, we can find finite collections E_ξ and F_ξ as above such that ξ maps to zero in $H^i(M_1(E_\xi, F_\xi)^\bullet \otimes_A^{\mathbf{L}} B)$.

Let $i \in \mathbf{Z}$ and let $\eta \in H^i(N_1^\bullet)$. Then the image of η in $H^i(N^\bullet)$ is in the image of $H^i(M^\bullet \otimes_A^{\mathbf{L}} B) \to H^i(N^\bullet)$. Hence as before, we can find finite collections E_η and F_η as above such that η maps to an element of $H^i(N_1(E_\eta, F_\eta)$ which is in the image of the map $H^i(M_1(E_\eta, F_\eta)^\bullet \otimes_A^{\mathbf{L}} B) \to H^i(N_1(E_\eta, F_\eta)$.

Now we simply define

$$M_2^\bullet = \sum\nolimits_\xi M_1(E_\xi, F_\xi)^\bullet + \sum\nolimits_\eta M_1(E_\eta, F_\eta)^\bullet$$

where the sum is over ξ and η as in the previous two paragraphs and the sum is taken inside $M^\bullet$. Similiarly we set

$$N_2^\bullet = \sum\nolimits_\xi N_1(E_\xi, F_\xi)^\bullet + \sum\nolimits_\eta N_1(E_\eta, F_\eta)^\bullet$$

where the sum is taken inside $N^\bullet$. By construction we will have properties (1) and (2) with these choices. The bound (3) also follows as the set of ξ and η has cardinality at most $\max(\kappa, |M_1^\bullet|, |N_1^\bullet|)$. ☐

103. Tricks with double complexes

This section continues the discussion in Homology, Section 26.

Lemma 103.1. *Let $A_0^\bullet \to A_1^\bullet \to A_2^\bullet \to \ldots$ be a complex of complexes of abelian groups. Assume $H^{-p}(A_p^\bullet) = 0$ for all $p \geq 0$. Set $A^{p,q} = A_p^q$ and view $A^{\bullet,\bullet}$ as a double complex. Then $H^0(\text{Tot}_\pi(A^{\bullet,\bullet})) = 0$.*

Proof. Denote $f_p : A_p^\bullet \to A_{p+1}^\bullet$ the given maps of complexes. Recall that the differential on $\text{Tot}_\pi(A^{\bullet,\bullet})$ is given by

$$\prod\nolimits_{p+q=n} (f_p^q + (-1)^p \text{d}^q_{A_p^\bullet})$$

on elements in degree n. Let $\xi \in H^0(\text{Tot}_\pi(A^{\bullet,\bullet}))$ be a cohomology class. We will show ξ is zero. Represent ξ as the class of an cocycle $x = (x_p) \in \prod A^{p,-p}$. Since $\text{d}(x) = 0$ we find that $\text{d}_{A_0^\bullet}(x_0) = 0$. Since $H^0(A_0^\bullet) = 0$ there exists a $y_{-1} \in A^{0,-1}$ with $\text{d}_{A_0^\bullet}(y_{-1}) = x_0$. Then we see that $\text{d}_{A_1^\bullet}(x_1 + f_0(y_{-1})) = 0$. Since $H^{-1}(A_1^\bullet) = 0$ we can find a $y_{-2} \in A^{1,-2}$ such that $-\text{d}_{A_1^\bullet}(y_{-2}) = x_1 + f_0(y_{-1})$. By induction we can find $y_{-p-1} \in A^{p,-p-1}$ such that

$$(-1)^p \text{d}_{A_p^\bullet}(y_{-p-1}) = x_p + f_{p-1}(y_{-p})$$

This implies that $\text{d}(y) = x$ where $y = (y_{-p-1})$. ☐

Lemma 103.2. *Let*

$$(A_0^\bullet \to A_1^\bullet \to A_2^\bullet \to \ldots) \longrightarrow (B_0^\bullet \to B_1^\bullet \to B_2^\bullet \to \ldots)$$

be a map between two complexes of complexes of abelian groups. Set $A^{p,q} = A_p^q$, $B^{p,q} = B_p^q$ to obtain double complexes. Let $Tot_\pi(A^{\bullet,\bullet})$ and $Tot_\pi(B^{\bullet,\bullet})$ be the product total complexes associated to the double complexes. If each $A_p^\bullet \to B_p^\bullet$ is a quasi-isomorphism, then $Tot_\pi(A^{\bullet,\bullet}) \to Tot_\pi(B^{\bullet,\bullet})$ is a quasi-isomorphism.

Proof. Recall that $\mathrm{Tot}_\pi(A^{\bullet,\bullet})$ in degree n is given by $\prod_{p+q=n} A^{p,q} = \prod_{p+1=n} A_p^q$. Let $C_p^\bullet$ be the cone on the map $A_p^\bullet \to B_p^\bullet$, see Derived Categories, Section 9. By the functoriality of the cone construction we obtain a complex of complexes

$$C_0^\bullet \to C_1^\bullet \to C_2^\bullet \to \ldots$$

Then we see $\mathrm{Tot}_\pi(C^{\bullet,\bullet})$ in degree n is given by

$$\prod_{p+q=n} C^{p,q} = \prod_{p+q=n} C_p^q = \prod_{p+q=n} (B_p^q \oplus A_p^{q+1}) = \prod_{p+q=n} B_p^q \oplus \prod_{p+q=n} A_p^{q+1}$$

We conclude that $\mathrm{Tot}_\pi(C^{\bullet,\bullet})$ is the cone of the map $\mathrm{Tot}_\pi(A^{\bullet,\bullet}) \to \mathrm{Tot}_\pi(B^{\bullet,\bullet})$ (We omit the verification that the differentials agree.) Thus it suffices to show $\mathrm{Tot}_\pi(A^{\bullet,\bullet})$ is acyclic if each $A_p^\bullet$ is acyclic. This follows from Lemma 103.1. $\square$

104. Weakly étale ring maps

Most of the results in this section are from the paper [Oli83] by Olivier. See also the related paper [Fer67].

Definition 104.1. A ring A is called *absolutely flat* if every A-module is flat over A. A ring map $A \to B$ is *weakly étale* or *absolutely flat* if both $A \to B$ and $B \otimes_A B \to B$ are flat.

Absolutely flat rings are sometimes called von Neumann regular rings (often in the setting of noncommutative rings). A localization is a weakly étale ring map. An étale ring map is weakly étale. Here is a simple, yet key property.

Lemma 104.2. *Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat. Let N be a B-module. If N is flat as an A-module, then N is flat as a B-module.*

Proof. Assume N is a flat as an A-module. Then the functor

$$\mathrm{Mod}_B \longrightarrow \mathrm{Mod}_{B \otimes_A B}, \quad N' \mapsto N \otimes_A N'$$

is exact. As $B \otimes_A B \to B$ is flat we conclude that the functor

$$\mathrm{Mod}_B \longrightarrow \mathrm{Mod}_B, \quad N' \mapsto (N \otimes_A N') \otimes_{B \otimes_A B} B = N \otimes_B N'$$

is exact, hence N is flat over B. $\square$

Definition 104.3. Let A be a ring. Let $d \geq 0$ be an integer. We say that A has *weak dimension* $\leq d$ if every A-module has tor dimension $\leq d$.

Lemma 104.4. *Let $A \to B$ be a weakly étale ring map. If A has weak dimension at most d, then so does B.*

Proof. Let N be a B-module. If $d = 0$, then N is flat as an A-module, hence flat as a B-module by Lemma 104.2. Assume $d > 0$. Choose a resolution $F_\bullet \to N$ by free B-modules. Our assumption implies that $K = \mathrm{Im}(F_d \to F_{d-1})$ is A-flat, see Lemma 66.2. Hence it is B-flat by Lemma 104.2. Thus $0 \to K \to F_{d-1} \to \ldots \to F_0 \to N \to 0$ is a flat resolution of length d and we see that N has tor dimension at most d. $\square$

Lemma 104.5. *Let A be a ring. The following are equivalent*

1. *A has weak dimension ≤ 0,*
2. *A is absolutely flat, and*
3. *A is reduced and every prime is maximal.*

In this case every local ring of A is a field.

Proof. The equivalence of (1) and (2) is immediate. Assume A is absolutely flat. This implies every ideal of A is pure, see Algebra, Definition 108.1. Hence every finitely generated ideal is generated by an idempotent by Algebra, Lemma 108.5. If $f \in A$, then $(f) = (e)$ for some idempotent $e \in A$ and $D(f) = D(e)$ is open and closed (Algebra, Lemma 21.1). This already implies every ideal of A is maximal for example by Algebra, Lemma 26.5. Moreover, if f is nilpotent, then $e = 0$ hence $f = 0$. Thus A is reduced.

Assume A is reduced and every prime of A is maximal. Let M be an A-module. Our goal is to show that M is flat. We may write M as a filtered colimit of finite A-modules, hence we may assume M is finite (Algebra, Lemma 39.3). There is a finite filtration of M by modules of the form A/I (Algebra, Lemma 5.4), hence we may assume that $M = A/I$ (Algebra, Lemma 39.13). Thus it suffices to show every ideal of A is pure. Since every local ring of A is a field (by Algebra, Lemma 25.1 and the fact that every prime of A is minimal), we see that every ideal $I \subset A$ is radical. Note that every closed subset of $\text{Spec}(A)$ is closed under generalization. Thus every (radical) ideal of A is pure by Algebra, Lemma 108.4. $\square$

Lemma 104.6. *A product of fields is an absolutely flat ring.*

Proof. Let K_i be a family of fields. If $f = (f_i) \in \prod K_i$, then the ideal generated by f is the same as the ideal generated by the idempotent $e = (e_i)$ with $e_i = 0, 1$ according to whether f_i is 0 or not. Thus $D(f) = D(e)$ is open and closed and we conclude by Lemma 104.5 and Algebra, Lemma 26.5. $\square$

Lemma 104.7. *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B.*

1. *If $B \otimes_A B \to B$ is flat, then $B' \otimes_{A'} B' \to B'$ is flat.*
2. *If $A \to B$ is weakly étale, then $A' \to B'$ is weakly étale.*

Proof. Assume $B \otimes_A B \to B$ is flat. The ring map $B' \otimes_{A'} B' \to B'$ is the base change of $B \otimes_A B \to B$ by $A \to A'$. Hence it is flat by Algebra, Lemma 39.7. This proves (1). Part (2) follows from (1) and the fact (just used) that the base change of a flat ring map is flat. $\square$

Lemma 104.8. *Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat.*

1. *If A is an absolutely flat ring, then so is B.*
2. *If A is reduced and $A \to B$ is weakly étale, then B is reduced.*

Proof. Part (1) follows immediately from Lemma 104.2 and the definitions. If A is reduced, then there exists an injection $A \to A' = \prod_{\mathfrak p \subset A \text{ minimal}} A_{\mathfrak p}$ of A into an absolutely flat ring (Algebra, Lemma 25.2 and Lemma 104.6). If $A \to B$ is flat, then the induced map $B \to B' = B \otimes_A A'$ is injective too. By Lemma 104.7 the ring map $A' \to B'$ is weakly étale. By part (1) we see that B' is absolutely flat. By Lemma 104.5 the ring B' is reduced. Hence B is reduced. $\square$

Lemma 104.9. *Let $A \to B$ and $B \to C$ be ring maps.*

(1) *If $B \otimes_A B \to B$ and $C \otimes_B C \to C$ are flat, then $C \otimes_A C \to C$ is flat.*
(2) *If $A \to B$ and $B \to C$ are weakly étale, then $A \to C$ is weakly étale.*

Proof. Part (1) follows from the factorization

$$C \otimes_A C \longrightarrow C \otimes_B C \longrightarrow C$$

of the multiplication map, the fact that

$$C \otimes_B C = (C \otimes_A C) \otimes_{B \otimes_A B} B,$$

the fact that a base change of a flat map is flat, and the fact that the composition of flat ring maps is flat. See Algebra, Lemmas 39.7 and 39.4. Part (2) follows from (1) and the fact (just used) that the composition of flat ring maps is flat. $\square$

Lemma 104.10. *Let $A \to B \to C$ be ring maps.*

(1) *If $B \to C$ is faithfully flat and $C \otimes_A C \to C$ is flat, then $B \otimes_A B \to B$ is flat.*
(2) *If $B \to C$ is faithfully flat and $A \to C$ is weakly étale, then $A \to B$ is weakly étale.*

Proof. Assume $B \to C$ is faithfully flat and $C \otimes_A C \to C$ is flat. Consider the commutative diagram

$$\begin{tikzcd} C \otimes_A C \arrow[r] & C \\ B \otimes_A B \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

The vertical arrows are flat, the top horizontal arrow is flat. Hence C is flat as a $B \otimes_A B$-module. The map $B \to C$ is faithfully flat and $C = B \otimes_B C$. Hence B is flat as a $B \otimes_A B$-module by Algebra, Lemma 39.9. This proves (1). Part (2) follows from (1) and the fact that $A \to B$ is flat if $A \to C$ is flat and $B \to C$ is faithfully flat (Algebra, Lemma 39.9). $\square$

Lemma 104.11. *Let A be a ring. Let $B \to C$ be an A-algebra map of weakly étale A-algebras. Then $B \to C$ is weakly étale.*

Proof. The ring map $B \to C$ is flat by Lemma 104.2. The ring map $C \otimes_A C \to C \otimes_B C$ is surjective, hence an epimorphism. Thus Lemma 104.2 implies, that since C is flat over $C \otimes_A C$ also C is flat over $C \otimes_B C$. $\square$

Lemma 104.12. *Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat. Then $\Omega_{B/A} = 0$, i.e., B is formally unramified over A.*

Proof. Let $I \subset B \otimes_A B$ be the kernel of the flat surjective map $B \otimes_A B \to B$. Then I is a pure ideal (Algebra, Definition 108.1), so $I^2 = I$ (Algebra, Lemma 108.2). Since $\Omega_{B/A} = I/I^2$ (Algebra, Lemma 131.13) we obtain the vanishing. This means B is formally unramified over A by Algebra, Lemma 148.2. $\square$

Lemma 104.13. *Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat.*

(1) *If $A \to B$ is of finite type, then $A \to B$ is unramified.*
(2) *If $A \to B$ is of finite presentation and flat, then $A \to B$ is étale.*

In particular a weakly étale ring map of finite presentation is étale.

Proof. Part (1) follows from Lemma 104.12 and Algebra, Definition 151.1. Part (2) follows from part (1) and Algebra, Lemma 151.8. $\square$

Lemma 104.14. *Let $A \to B$ be a ring map. Then $A \to B$ is weakly étale in each of the following cases*

1. *$B = S^{-1}A$ is a localization of A,*
2. *$A \to B$ is étale,*
3. *B is a filtered colimit of weakly étale A-algebras.*

Proof. An étale ring map is flat and the map $B \otimes_A B \to B$ is also étale as a map between étale A-algebras (Algebra, Lemma 143.8). This proves (2).

Let B_i be a directed system of weakly étale A-algebras. Then $B = \operatorname{colim} B_i$ is flat over A by Algebra, Lemma 39.3. Note that the transition maps $B_i \to B_{i'}$ are flat by Lemma 104.11. Hence B is flat over B_i for each i, and we see that B is flat over $B_i \otimes_A B_i$ by Algebra, Lemma 39.4. Thus B is flat over $B \otimes_A B = \operatorname{colim} B_i \otimes_A B_i$ by Algebra, Lemma 39.6.

Part (1) can be proved directly, but also follows by combining (2) and (3). $\square$

Lemma 104.15. *Let L/K be an extension of fields. If $L \otimes_K L \to L$ is flat, then L is an algebraic separable extension of K.*

Proof. By Lemma 104.10 we see that any subfield $K \subset L' \subset L$ the map $L' \otimes_K L' \to L'$ is flat. Thus we may assume L is a finitely generated field extension of K. In this case the fact that L/K is formally unramified (Lemma 104.12) implies that L/K is finite separable, see Algebra, Lemma 158.1. $\square$

Lemma 104.16. *Let B be an algebra over a field K. The following are equivalent*

1. *$B \otimes_K B \to B$ is flat,*
2. *$K \to B$ is weakly étale, and*
3. *B is a filtered colimit of étale K-algebras.*

Moreover, every finitely generated K-subalgebra of B is étale over K.

Proof. Parts (1) and (2) are equivalent because every K-algebra is flat over K. Part (3) implies (1) and (2) by Lemma 104.14

Assume (1) and (2) hold. We will prove (3) and the finite statement of the lemma. A field is absolutely flat ring, hence B is a absolutely flat ring by Lemma 104.8. Hence B is reduced and every local ring is a field, see Lemma 104.5.

Let $\mathfrak{q} \subset B$ be a prime. The ring map $B \to B_\mathfrak{q}$ is weakly étale, hence $B_\mathfrak{q}$ is weakly étale over K (Lemma 104.9). Thus $B_\mathfrak{q}$ is a separable algebraic extension of K by Lemma 104.15.

Let $K \subset A \subset B$ be a finitely generated K-sub algebra. We will show that A is étale over K which will finish the proof of the lemma. Then every minimal prime $\mathfrak{p} \subset A$ is the image of a prime $\mathfrak{q}$ of B, see Algebra, Lemma 30.5. Thus $\kappa(\mathfrak{p})$ as a subfield of $B_\mathfrak{q} = \kappa(\mathfrak{q})$ is separable algebraic over K. Hence every generic point of $\operatorname{Spec}(A)$ is closed (Algebra, Lemma 35.9). Thus $\dim(A) = 0$. Then A is the product of its local rings, e.g., by Algebra, Proposition 60.7. Moreover, since A is reduced, all local rings are equal to their residue fields wich are finite separable over K. This means that A is étale over K by Algebra, Lemma 143.4 and finishes the proof. $\square$

Lemma 104.17. *Let $A \to B$ be a ring map. If $A \to B$ is weakly étale, then $A \to B$ induces separable algebraic residue field extensions.*

Proof. Let $\mathfrak p$ be a prime of A. Then $\kappa(\mathfrak p) \to B \otimes_A \kappa(\mathfrak p)$ is weakly étale by Lemma 104.7. Hence $B \otimes_A \kappa(\mathfrak p)$ is a filtered colimit of étale $\kappa(\mathfrak p)$-algebras by Lemma 104.16. Hence for $\mathfrak q \subset B$ lying over $\mathfrak p$ the extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is a filtered colimit of finite separable extensions by Algebra, Lemma 143.4. $\square$

Lemma 104.18. *Let A be a ring. The following are equivalent*

(1) *A has weak dimension ≤ 1,*
(2) *every ideal of A is flat,*
(3) *every finitely generated ideal of A is flat,*
(4) *every submodule of a flat A-module is flat, and*
(5) *every local ring of A is a valuation ring.*

Proof. If A has weak dimension ≤ 1, then the resolution $0 \to I \to A \to A/I \to 0$ shows that every ideal I is flat by Lemma 66.2. Hence (1) $\Rightarrow$ (2).

Assume (4). Let M be an A-module. Choose a surjection $F \to M$ where F is a free A-module. Then $\text{Ker}(F \to M)$ is flat by assumption, and we see that M has tor dimension ≤ 1 by Lemma 66.6. Hence (4) $\Rightarrow$ (1).

Every ideal is the union of the finitely generated ideals contained in it. Hence (3) implies (2) by Algebra, Lemma 39.3. Thus (3) $\Leftrightarrow$ (2).

Assume (2). Suppose that $N \subset M$ with M a flat A-module. We will prove that N is flat. We can write $M = \text{colim}\, M_i$ with each M_i finite free, see Algebra, Theorem 81.4. Setting $N_i \subset M_i$ the inverse image of N we see that $N = \text{colim}\, N_i$. By Algebra, Lemma 39.3. it suffices to prove N_i is flat and we reduce to the case $M = R^{\oplus n}$. In this case the module N has a finite filtration by the submodules $R^{\oplus j} \cap N$ whose subquotients are ideals. By (2) these ideals are flat and hence N is flat by Algebra, Lemma 39.13. Thus (2) $\Rightarrow$ (4).

Assume A satisfies (1) and let $\mathfrak p \subset A$ be a prime ideal. By Lemmas 104.14 and 104.4 we see that $A_{\mathfrak p}$ satisfies (1). We will show A is a valuation ring if A is a local ring satisfying (3). Let $f \in \mathfrak m$ be a nonzero element. Then (f) is a flat nonzero module generated by one element. Hence it is a free A-module by Algebra, Lemma 78.5. It follows that f is a nonzerodivisor and A is a domain. If $I \subset A$ is a finitely generated ideal, then we similarly see that I is a finite free A-module, hence (by considering the rank) free of rank 1 and I is a principal ideal. Thus A is a valuation ring by Algebra, Lemma 50.15. Thus (1) $\Rightarrow$ (5).

Assume (5). Let $I \subset A$ be a finitely generated ideal. Then $I_{\mathfrak p} \subset A_{\mathfrak p}$ is a finitely generated ideal in a valuation ring, hence principal (Algebra, Lemma 50.15), hence flat. Thus I is flat by Algebra, Lemma 39.18. Thus (5) $\Rightarrow$ (3). This finishes the proof of the lemma. $\square$

Lemma 104.19. *Let J be a set. For each $j \in J$ let A_j be a valuation ring with fraction field K_j. Set $A = \prod A_j$ and $K = \prod K_j$. Then A has weak dimension at most 1 and $A \to K$ is a localization.*

Proof. Let $I \subset A$ be a finitely generated ideal. By Lemma 104.18 it suffices to show that I is a flat A-module. Let $I_j \subset A_j$ be the image of I. Observe that $I_j = I \otimes_A A_j$, hence $I \to \prod I_j$ is surjective by Algebra, Proposition 89.2. Thus $I = \prod I_j$. Since A_j is a valuation ring, the ideal I_j is generated by a single element (Algebra, Lemma 50.15). Say $I_j = (f_j)$. Then I is generated by the element $f = (f_j)$. Let $e \in A$ be the idempotent which has a 0 or 1 in A_j depending on whether f_j is 0 or not. Then $f = ge$ for some nonzerodivisor $g \in A$: take $g = (g_j)$ with $g_j = 1$ if $f_j = 0$ and $g_j = f_j$ else. Thus $I \cong (e)$ as a module. We conclude I is flat as (e) is a direct summand of A. The final statement is true because $K = S^{-1}A$ where $S = \prod(A_j \setminus \{0\})$. $\square$

Lemma 104.20. *Let A be a normal domain with fraction field K. There exists a cartesian diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & K \arrow[d] \\ V \arrow[r] & L \end{tikzcd}$$

of rings where V has weak dimension at most 1 and $V \to L$ is a flat, injective, epimorphism of rings.

Proof. For every $x \in K$, $x \notin A$ pick $V_x \subset K$ as in Algebra, Lemma 50.11. Set $V = \prod_{x \in K \setminus A} V_x$ and $L = \prod_{x \in K \setminus A} K$. The ring V has weak dimension at most 1 by Lemma 104.19 which also shows that $V \to L$ is a localization. A localization is flat and an epimorphism, see Algebra, Lemmas 39.18 and 107.5. $\square$

Lemma 104.21. *Let A be a ring of weak dimension at most 1. If $A \to B$ is a flat, injective, epimorphism of rings, then A is integrally closed in B.*

Proof. Let $x \in B$ be integral over A. Let $A' = A[x] \subset B$. Then A' is a finite ring extension of A by Algebra, Lemma 36.5. To show $A = A'$ it suffices to show $A \to A'$ is an epimorphism by Algebra, Lemma 107.6. Note that A' is flat over A by assumption on A and the fact that B is flat over A (Lemma 104.18). Hence the composition

$$A' \otimes_A A' \to B \otimes_A A' \to B \otimes_A B \to B$$

is injective, i.e., $A' \otimes_A A' \cong A'$ and the lemma is proved. $\square$

Lemma 104.22. *Let A be a normal domain with fraction field K. Let $A \to B$ be weakly étale. Then B is integrally closed in $B \otimes_A K$.*

Proof. Choose a diagram as in Lemma 104.20. As $A \to B$ is flat, the base change gives a cartesian diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B \otimes_A K \arrow[d] \\ B \otimes_A V \arrow[r] & B \otimes_A L \end{tikzcd}$$

of rings. Note that $V \to B \otimes_A V$ is weakly étale (Lemma 104.7), hence $B \otimes_A V$ has weak dimension at most 1 by Lemma 104.4. Note that $B \otimes_A V \to B \otimes_A L$ is a flat, injective, epimorphism of rings as a flat base change of such (Algebra, Lemmas 39.7 and 107.3). By Lemma 104.21 we see that $B \otimes_A V$ is integrally closed in $B \otimes_A L$. It follows from the cartesian property of the diagram that B is integrally closed in $B \otimes_A K$. $\square$

Lemma 104.23. *Let $A \to B$ be a ring homomorphism. Assume*

1. *A is a henselian local ring,*
2. *$A \to B$ is integral,*
3. *B is a domain.*

Then B is a henselian local ring and $A \to B$ is a local homomorphism. If A is strictly henselian, then B is a strictly henselian local ring and the extension $\kappa(\mathfrak{m}_B)/\kappa(\mathfrak{m}_A)$ of residue fields is purely inseparable.

Proof. Write B as a filtered colimit $B = \operatorname{colim} B_i$ of finite A-sub algebras. If we prove the results for each B_i, then the result follows for B. See Algebra, Lemma 154.8. If $A \to B$ is finite, then B is a product of local henselian rings by Algebra, Lemma 153.4. Since B is a domain we see that B is a local ring. The maximal ideal of B lies over the maximal ideal of A by going up for $A \to B$ (Algebra, Lemma 36.22). If A is strictly henselian, then the field extension $\kappa(\mathfrak{m}_B)/\kappa(\mathfrak{m}_A)$ being algebraic, has to be purely inseparable. Of course, then $\kappa(\mathfrak{m}_B)$ is separably algebraically closed and B is strictly henselian. □

Theorem 104.24 (Olivier)**.** *Let $A \to B$ be a local homomorphism of local rings. If A is strictly henselian and $A \to B$ is weakly étale, then $A = B$.*

Proof. We will show that for all $\mathfrak{p} \subset A$ there is a unique prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$ and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. This implies that $B \otimes_A B \to B$ is bijective on spectra as well as surjective and flat. Hence it is an isomorphism for example by the description of pure ideals in Algebra, Lemma 108.4. Hence $A \to B$ is a faithfully flat epimorphism of rings. We get $A = B$ by Algebra, Lemma 107.7.

Note that the fibre ring $B \otimes_A \kappa(\mathfrak{p})$ is a colimit of étale extensions of $\kappa(\mathfrak{p})$ by Lemmas 104.7 and 104.16. Hence, if there exists more than one prime lying over $\mathfrak{p}$ or if $\kappa(\mathfrak{p}) \neq \kappa(\mathfrak{q})$ for some $\mathfrak{q}$, then $B \otimes_A L$ has a nontrivial idempotent for some (separable) algebraic field extension $L/\kappa(\mathfrak{p})$.

Let $L/\kappa(\mathfrak{p})$ be an algebraic field extension. Let $A' \subset L$ be the integral closure of $A/\mathfrak{p}$ in L. By Lemma 104.23 we see that A' is a strictly henselian local ring whose residue field is a purely inseparable extension of the residue field of A. Thus $B \otimes_A A'$ is a local ring by Algebra, Lemma 156.5. On the other hand, $B \otimes_A A'$ is integrally closed in $B \otimes_A L$ by Lemma 104.22. Since $B \otimes_A A'$ is local, it follows that the ring $B \otimes_A L$ does not have nontrivial idempotents which is what we wanted to prove. □

105. Weakly étale algebras over fields

If K is a field, then an algebra B is weakly étale over K if and only if it is a filtered colimit of étale K-algebras. This is Lemma 104.16.

Lemma 105.1. *Let K be a field. If B is weakly étale over K, then*
1. *B is reduced,*
2. *B is integral over K,*
3. *any finitely generated K-subalgebra of B is a finite product of finite separable extensions of K,*
4. *B is a field if and only if B does not have nontrivial idempotents and in this case it is a separable algebraic extension of K,*
5. *any sub or quotient K-algebra of B is weakly étale over K,*
6. *if B' is weakly étale over K, then $B \otimes_K B'$ is weakly étale over K.*

Proof. Part (1) follows from Lemma 104.8 but of course it follows from part (3) as well. Part (3) follows from Lemma 104.16 and the fact that étale K-algebras are finite products of finite separable extensions of K, see Algebra, Lemma 143.4. Part

(3) implies (2). Part (4) follows from (3) as a product of fields is a field if and only if it has no nontrivial idempotents.

If $S \subset B$ is a subalgebra, then it is the filtered colimit of its finitely generated subalgebras which are all étale over K by the above and hence S is weakly étale over K by Lemma 104.16. If $B \to Q$ is a quotient algebra, then Q is the filtered colimit of K-algebra quotients of finite products $\prod_{i \in I} L_i$ of finite separable extensions L_i/K. Such a quotient is of the form $\prod_{i \in J} L_i$ for some subset $J \subset I$ and hence the result holds for quotients by the same reasoning.

The statement on tensor products follows in a similar manner or by combining Lemmas 104.7 and 104.9. $\square$

Lemma 105.2. *Let K be a field. Let A be a K-algebra. There exists a maximal weakly étale K-subalgebra $B_{max} \subset A$.*

Proof. Let $B_1, B_2 \subset A$ be weakly étale K-subalgebras. Then $B_1 \otimes_K B_2$ is weakly étale over K and so is the image of $B_1 \otimes_K B_2 \to A$ (Lemma 105.1). Thus the collection $\mathcal{B}$ of weakly étale K-subalgebras $B \subset A$ is directed and the colimit $B_{max} = \text{colim}_{B \in \mathcal{B}} B$ is a weakly étale K-algebra by Lemma 104.14. Hence the image of $B_{max} \to A$ is weakly étale over K (previous lemma cited). It follows that this image is in $\mathcal{B}$ and hence $\mathcal{B}$ has a maximal element (and the image is the same as B_{max}). $\square$

Lemma 105.3. *Let K be a field. For a K-algebra A denote $B_{max}(A)$ the maximal weakly étale K-subalgebra of A as in Lemma 105.2. Then*

(1) *any K-algebra map $A' \to A$ induces a K-algebra map $B_{max}(A') \to B_{max}(A)$,*
(2) *if $A' \subset A$, then $B_{max}(A') = B_{max}(A) \cap A'$,*
(3) *if $A = \text{colim}\, A_i$ is a filtered colimit, then $B_{max}(A) = \text{colim}\, B_{max}(A_i)$,*
(4) *the map $B_{max}(A) \to B_{max}(A_{red})$ is an isomorphism,*
(5) *$B_{max}(A_1 \times \ldots \times A_n) = B_{max}(A_1) \times \ldots \times B_{max}(A_n)$,*
(6) *if A has no nontrivial idempotents, then $B_{max}(A)$ is a field and a separable algebraic extension of K,*
(7) *add more here.*

Proof. Proof of (1). This is true because the image of $B_{max}(A') \to A$ is weakly étale over K by Lemma 105.1.

Proof of (2). By (1) we have $B_{max}(A') \subset B_{max}(A)$. Conversely, $B_{max}(A) \cap A'$ is a weakly étale K-algebra by Lemma 105.1 and hence contained in $B_{max}(A')$.

Proof of (3). By (1) there is a map $\text{colim}\, B_{max}(A_i) \to A$ which is injective because the system is filtered and $B_{max}(A_i) \subset A_i$. The colimit $\text{colim}\, B_{max}(A_i)$ is weakly étale over K by Lemma 104.14. Hence we get an injective map $\text{colim}\, B_{max}(A_i) \to B_{max}(A)$. Suppose that $a \in B_{max}(A)$. Then a generates a finitely presented K-subalgebra $B \subset B_{max}(A)$. By Algebra, Lemma 127.3 there is an i and a K-algebra map $f : B \to A_i$ lifting the given map $B \to A$. Since B is weakly étale by Lemma 105.1, we see that $f(B) \subset B_{max}(A_i)$ and we conclude that a is in the image of $\text{colim}\, B_{max}(A_i) \to B_{max}(A)$.

Proof of (4). Write $B_{max}(A_{red}) = \text{colim}\, B_i$ as a filtered colimit of étale K-algebras (Lemma 104.16). By Algebra, Lemma 138.17 for each i there is a K-algebra map $f_i : B_i \to A$ lifting the given map $B_i \to A_{red}$. It follows that the canonical map

$B_{max}(A_{red}) \to B_{max}(A)$ is surjective. The kernel consists of nilpotent elements and hence is zero as $B_{max}(A_{red})$ is reduced (Lemma 105.1).

Proof of (5). Omitted.

Proof of (6). Follows from Lemma 105.1 part (4). □

Lemma 105.4. *Let L/K be an extension of fields. Let A be a K-algebra. Let $B \subset A$ be the maximal weakly étale K-subalgebra of A as in Lemma 105.2. Then $B \otimes_K L$ is the maximal weakly étale L-subalgebra of $A \otimes_K L$.*

Proof. For an algebra A over K we write $B_{max}(A/K)$ for the maximal weakly étale K-subalgebra of A. Similarly we write $B_{max}(A'/L)$ for the maximal weakly étale L-subalgebra of A' if A' is an L-algebra. Since $B_{max}(A/K) \otimes_K L$ is weakly étale over L (Lemma 104.7) and since $B_{max}(A/K) \otimes_K L \subset A \otimes_K L$ we obtain a canonical injective map

$$B_{max}(A/K) \otimes_K L \to B_{max}((A \otimes_K L)/L)$$

The lemma states that this map is an isomorphism.

To prove the lemma for L and our K-algebra A, it suffices to prove the lemma for any field extension L' of L. Namely, we have the factorization

$$B_{max}(A/K) \otimes_K L' \to B_{max}((A \otimes_K L)/L) \otimes_L L' \to B_{max}((A \otimes_K L')/L')$$

hence the composition cannot be surjective without $B_{max}(A/K) \otimes_K L \to B_{max}((A \otimes_K L)/L)$ being surjective. Thus we may assume L is algebraically closed.

Reduction to finite type K-algebra. We may write A is the filtered colimit of its finite type K-subalgebras. Using Lemma 105.3 we see that it suffices to prove the lemma for finite type K-algebras.

Assume A is a finite type K-algebra. Since the kernel of $A \to A_{red}$ is nilpotent, the same is true for $A \otimes_K L \to A_{red} \otimes_K L$. Then

$$B_{max}((A \otimes_K L)/L) \to B_{max}((A_{red} \otimes_K L)/L)$$

is injective because the kernel is nilpotent and the weakly étale L-algebra $B_{max}((A \otimes_K L)/L)$ is reduced (Lemma 105.1). Since $B_{max}(A/K) = B_{max}(A_{red}/K)$ by Lemma 105.3 we conclude that it suffices to prove the lemma for A_{red}.

Assume A is a reduced finite type K-algebra. Let $Q = Q(A)$ be the total quotient ring of A. Then $A \subset Q$ and $A \otimes_K L \subset Q \otimes_A L$ and hence

$$B_{max}(A/K) = A \cap B_{max}(Q/K)$$

and

$$B_{max}((A \otimes_K L)/L) = (A \otimes_K L) \cap B_{max}((Q \otimes_K L)/L)$$

by Lemma 105.3. Since $- \otimes_K L$ is an exact functor, it follows that if we prove the result for Q, then the result follows for A. Since Q is a finite product of fields (Algebra, Lemmas 25.4, 25.1, 31.6, and 31.1) and since B_{max} commutes with products (Lemma 105.3) it suffices to prove the lemma when A is a field.

Assume A is a field. We reduce to A being finitely generated over K by the argument in the third paragraph of the proof. (In fact the way we reduced to the case of a field produces a finitely generated field extension of K.)

Assume A is a finitely generated field extension of K. Then $K' = B_{max}(A/K)$ is a field separable algebraic over K by Lemma 105.3 part (6). Hence K' is a finite separable field extension of K and A is geometrically irreducible over K' by Algebra, Lemma 47.13. Since L is algebraically closed and K'/K finite separable we see that

$$K' \otimes_K L \to \prod\nolimits_{\sigma \in \mathrm{Hom}_K(K', L)} L, \quad \alpha \otimes \beta \mapsto (\sigma(\alpha)\beta)_\sigma$$

is an isomorphism (Fields, Lemma 13.4). We conclude

$$A \otimes_K L = A \otimes_{K'} (K' \otimes_K L) = \prod\nolimits_{\sigma \in \mathrm{Hom}_K(K', L)} A \otimes_{K', \sigma} L$$

Since A is geometrically irreducible over K' we see that $A \otimes_{K', \sigma} L$ has a unique minimal prime. Since L is algebraically closed it follows that $B_{max}((A \otimes_{K', \sigma} L)/L) = L$ because this L-algebra is a field algebraic over L by Lemma 105.3 part (6). It follows that the maximal weakly étale $K' \otimes_K L$-subalgebra of $A \otimes_K L$ is $K' \otimes_K L$ because we can decompose these subalgebras into products as above. Hence the inclusion $K' \otimes_K L \subset B_{max}((A \otimes_K L)/L)$ is an equality: the ring map $K' \otimes_K L \to B_{max}((A \otimes_K L)/L)$ is weakly étale by Lemma 104.11. □

106. Local irreducibility

The following definition seems to be the generally accepted one. To parse it, observe that if $A \subset B$ is an integral extension of local domains, then $A \to B$ is a local ring homomorphism by going up (Algebra, Lemma 36.22).

Definition 106.1. Let A be a local ring. We say A is *unibranch* if the reduction A_{red} is a domain and if the integral closure A' of A_{red} in its field of fractions is local. We say A is *geometrically unibranch* if A is unibranch and moreover the residue field of A' is purely inseparable over the residue field of A.

Let A be a local ring. Here is an equivalent formulation

(1) A is unibranch if A has a unique minimal prime $\mathfrak{p}$ and the integral closure of $A/\mathfrak{p}$ in its fraction field is a local ring, and
(2) A is geometrically unibranch if A has a unique minimal prime $\mathfrak{p}$ and the integral closure of $A/\mathfrak{p}$ in its fraction field is a local ring whose residue field is purely inseparable over the residue field of A.

A local ring which is normal is geometrically unibranch (follows from Definition 106.1 and Algebra, Definition 37.11). Lemmas 106.3 and 106.5 suggest that being (geometrically) unibranch is a reasonable property to look at.

Lemma 106.2. *Let A be a local ring. Assume A has finitely many minimal prime ideals. Let A' be the integral closure of A in the total ring of fractions of A_{red}. Let A^h be the henselization of A. Consider the maps*

$$\mathrm{Spec}(A') \leftarrow \mathrm{Spec}((A')^h) \to \mathrm{Spec}(A^h)$$

where $(A')^h = A' \otimes_A A^h$. Then

(1) *the left arrow is bijective on maximal ideals,*
(2) *the right arrow is bijective on minimal primes,*
(3) *every minimal prime of $(A')^h$ is contained in a unique maximal ideal and every maximal ideal contains exactly one minimal prime.*

Proof. Let $I \subset A$ be the ideal of nilpotents. We have $(A/I)^h = A^h/IA^h$ by (Algebra, Lemma 156.2). The spectra of A, A^h, A', and $(A')^h$ are the same as the spectra of A/I, A^h/IA^h, A', and $(A')^h = A' \otimes_{A/I} A^h/IA^h$. Thus we may replace

A by $A_{red} = A/I$ and assume A is reduced. Then $A \subset A'$ which we will use below without further mention.

Proof of (1). As A' is integral over A we see that $(A')^h$ is integral over A^h. By going up (Algebra, Lemma 36.22) every maximal ideal of A', resp. $(A')^h$ lies over the maximal ideal $\mathfrak{m}$, resp. $\mathfrak{m}^h$ of A, resp. A^h. Thus (1) follows from the isomorphism

$$(A')^h \otimes_{A^h} \kappa^h = A' \otimes_A A^h \otimes_{A^h} \kappa^h = A' \otimes_A \kappa$$

because the residue field extension κ^h/κ induced by $A \to A^h$ is trivial. We will use below that the displayed ring is integral over a field hence spectrum of this ring is a profinite space, see Algebra, Lemmas 36.19 and 26.5.

Proof of (3). The ring A' is a normal ring and in fact a finite product of normal domains, see Algebra, Lemma 37.16. Since A^h is a filtered colimit of étale A-algebras, $(A')^h$ is filtered colimit of étale A'-algebras hence $(A')^h$ is a normal ring by Algebra, Lemmas 163.9 and 37.17. Thus every local ring of $(A')^h$ is a normal domain and we see that every maximal ideal contains a unique minimal prime. By Lemma 11.8 applied to $A^h \to (A')^h$ we see that $((A')^h, \mathfrak{m}(A')^h)$ is a henselian pair. If $\mathfrak{q} \subset (A')^h$ is a minimal prime (or any prime), then the intersection of $V(\mathfrak{q})$ with $V(\mathfrak{m}(A')^h)$ is connected by Lemma 11.16 Since $V(\mathfrak{m}(A')^h) = \text{Spec}((A')^h \otimes \kappa^h)$ is a profinite space by we see there is a unique maximal ideal containing $\mathfrak{q}$.

Proof of (2). The minimal primes of A' are exactly the primes lying over a minimal prime of A (by construction). Since $A' \to (A')^h$ is flat by going down (Algebra, Lemma 39.19) every minimal prime of $(A')^h$ lies over a minimal prime of A'. Conversely, any prime of $(A')^h$ lying over a minimal prime of A' is minimal because $(A')^h$ is a filtered colimit of étale hence quasi-finite algebras over A' (small detail omitted). We conclude that the minimal primes of $(A')^h$ are exactly the primes which lie over a minimal prime of A. Similarly, the minimal primes of A^h are exactly the primes lying over minimal primes of A. By construction we have $A' \otimes_A Q(A) = Q(A)$ where $Q(A)$ is the total fraction ring of our reduced local ring A. Of course $Q(A)$ is the finite product of residue fields of the minimal primes of A. It follows that

$$(A')^h \otimes_A Q(A) = A^h \otimes_A A' \otimes_A Q(A) = A^h \otimes_A Q(A)$$

Our discussion above shows the spectrum of the ring on the left is the set of minimal primes of $(A')^h$ and the spectrum of the ring on the right is the is the set of minimal primes of A^h. This finishes the proof. $\square$

Lemma 106.3. *Let A be a local ring. Let A^h be the henselization of A. The following are equivalent*

(1) *A is unibranch, and*
(2) *A^h has a unique minimal prime.*

Proof. This follows from Lemma 106.2 but we will also give a direct proof. Denote $\mathfrak{m}$ the maximal ideal of the ring A. Recall that the residue field $\kappa = A/\mathfrak{m}$ is the same as the residue field of A^h.

Assume (2). Let $\mathfrak{p}^h$ be the unique minimal prime of A^h. The flatness of $A \to A^h$ implies that $\mathfrak{p} = A \cap \mathfrak{p}^h$ is the unique minimal prime of A (by going down, see Algebra, Lemma 39.19). Also, since $A^h/\mathfrak{p}A^h = (A/\mathfrak{p})^h$ (see Algebra, Lemma 156.2) is reduced by Lemma 45.4 we see that $\mathfrak{p}^h = \mathfrak{p}A^h$. Let A' be the integral closure of $A/\mathfrak p$ in its fraction field. We have to show that A' is local. Since $A \to A'$ is integral, every maximal ideal of A' lies over $\mathfrak m$ (by going up for integral ring maps, see Algebra, Lemma 36.22). If A' is not local, then we can find distinct maximal ideals $\mathfrak m_1$, $\mathfrak m_2$. Choose elements $f_1, f_2 \in A'$ with $f_i \in \mathfrak m_i$ and $f_i \not\in \mathfrak m_{3-i}$. We find a finite subalgebra $B = A[f_1, f_2] \subset A'$ with distinct maximal ideals $B \cap \mathfrak m_i$, $i = 1, 2$. Note that the inclusions

$$A/\mathfrak p \subset B \subset \kappa(\mathfrak p)$$

give, on tensoring with the flat ring map $A \to A^h$ the inclusions

$$A^h/\mathfrak p^h \subset B \otimes_A A^h \subset \kappa(\mathfrak p) \otimes_A A^h \subset \kappa(\mathfrak p^h)$$

the last inclusion because $\kappa(\mathfrak p) \otimes_A A^h = \kappa(\mathfrak p) \otimes_{A/\mathfrak p} A^h/\mathfrak p^h$ is a localization of the domain $A^h/\mathfrak p^h$. Note that $B \otimes_A \kappa$ has at least two maximal ideals because $B/\mathfrak m B$ has two maximal ideals. Hence, as A^h is henselian we see that $B \otimes_A A^h$ is a product of ≥ 2 local rings, see Algebra, Lemma 153.5. But we've just seen that $B \otimes_A A^h$ is a subring of a domain and we get a contradiction.

Assume (1). Let $\mathfrak p \subset A$ be the unique minimal prime and let A' be the integral closure of $A/\mathfrak p$ in its fraction field. Let $A \to B$ be a local map of local rings inducing an isomorphism of residue fields which is a localization of an étale A-algebra. In particular $\mathfrak m_B$ is the unique prime containing $\mathfrak m B$. Then $B' = A' \otimes_A B$ is integral over B and the assumption that $A \to A'$ is local implies that B' is local (Algebra, Lemma 156.5). On the other hand, $A' \to B'$ is the localization of an étale ring map, hence B' is normal, see Algebra, Lemma 163.9. Thus B' is a (local) normal domain. Finally, we have

$$B/\mathfrak p B \subset B \otimes_A \kappa(\mathfrak p) = B' \otimes_{A'} (\text{fraction field of } A') \subset \text{fraction field of } B'$$

Hence $B/\mathfrak p B$ is a domain, which implies that B has a unique minimal prime (since by flatness of $A \to B$ these all have to lie over $\mathfrak p$). Since A^h is a filtered colimit of the local rings B it follows that A^h has a unique minimal prime. Namely, if $fg = 0$ in A^h for some non-nilpotent elements f, g, then we can find a B as above containing both f and g which leads to a contradiction. $\square$

Lemma 106.4. *Let $(A, \mathfrak m, \kappa)$ be a local ring. Assume A has finitely many minimal prime ideals. Let A' be the integral closure of A in the total ring of fractions of A_{red}. Choose an algebraic closure $\overline{\kappa}$ of κ and denote $\kappa^{sep} \subset \overline{\kappa}$ the separable algebraic closure of κ. Let A^{sh} be the strict henselization of A with respect to κ^{sep}. Consider the maps*

$$\text{Spec}(A') \xleftarrow{c} \text{Spec}((A')^{sh}) \xrightarrow{e} \text{Spec}(A^{sh})$$

where $(A')^{sh} = A' \otimes_A A^{sh}$. Then

1. *for $\mathfrak m' \subset A'$ maximal the residue field κ' is algebraic over κ and the fibre of c over $\mathfrak m'$ can be canonically identified with $\text{Hom}_\kappa(\kappa', \overline{\kappa})$,*
2. *the right arrow is bijective on minimal primes,*
3. *every minimal prime of $(A')^{sh}$ is contained in a unique maximal ideal and every maximal ideal contains a unique minimal prime.*

Proof. The proof is almost exactly the same as for Lemma 106.2. Let $I \subset A$ be the ideal of nilpotents. We have $(A/I)^{sh} = A^{sh}/IA^{sh}$ by (Algebra, Lemma 156.2). The spectra of A, A^{sh}, A', and $(A')^h$ are the same as the spectra of A/I, A^{sh}/IA^{sh}, A', and $(A')^{sh} = A' \otimes_{A/I} A^{sh}/IA^{sh}$. Thus we may replace A by $A_{red} = A/I$

and assume A is reduced. Then $A \subset A'$ which we will use below without further mention.

Proof of (1). The field extension κ'/κ is algebraic because A' is integral over A. Since A' is integral over A, we see that $(A')^{sh}$ is integral over A^{sh}. By going up (Algebra, Lemma 36.22) every maximal ideal of A', resp. $(A')^{sh}$ lies over the maximal ideal $\mathfrak{m}$, resp. $\mathfrak{m}^{sh}$ of A, resp. A^h. We have

$$(A')^{sh} \otimes_{A^{sh}} \kappa^{sep} = A' \otimes_A A^h \otimes_{A^h} \kappa^{sep} = (A' \otimes_A \kappa) \otimes_\kappa \kappa^{sep}$$

because the residue field of A^{sh} is κ^{sep}. Thus the fibre of c over $\mathfrak{m}'$ is the spectrum of $\kappa' \otimes_\kappa \kappa^{sep}$. We conclude (1) is true because there is a bijection

$$\mathrm{Hom}_\kappa(\kappa', \overline{\kappa}) \to \mathrm{Spec}(\kappa' \otimes_\kappa \kappa^{sep}), \quad \sigma \mapsto \mathrm{Ker}(\sigma \otimes 1 : \kappa' \otimes_\kappa \kappa^{sep} \to \overline{\kappa})$$

We will use below that the displayed ring is integral over a field hence spectrum of this ring is a profinite space, see Algebra, Lemmas 36.19 and 26.5.

Proof of (3). The ring A' is a normal ring and in fact a finite product of normal domains, see Algebra, Lemma 37.16. Since A^{sh} is a filtered colimit of étale A-algebras, $(A')^{sh}$ is filtered colimit of étale A'-algebras hence $(A')^{sh}$ is a normal ring by Algebra, Lemmas 163.9 and 37.17. Thus every local ring of $(A')^{sh}$ is a normal domain and we see that every maximal ideal contains a unique minimal prime. By Lemma 11.8 applied to $A^{sh} \to (A')^{sh}$ to see that $((A')^{sh}, \mathfrak{m}(A')^{sh})$ is a henselian pair. If $\mathfrak{q} \subset (A')^{sh}$ is a minimal prime (or any prime), then the intersection of $V(\mathfrak{q})$ with $V(\mathfrak{m}(A')^{sh})$ is connected by Lemma 11.16 Since $V(\mathfrak{m}(A')^{sh}) = \mathrm{Spec}((A')^{sh} \otimes \kappa^{sh})$ is a profinite space by we see there is a unique maximal ideal containing $\mathfrak{q}$.

Proof of (2). The minimal primes of A' are exactly the primes lying over a minimal prime of A (by construction). Since $A' \to (A')^{sh}$ is flat by going down (Algebra, Lemma 39.19) every minimal prime of $(A')^{sh}$ lies over a minimal prime of A'. Conversely, any prime of $(A')^{sh}$ lying over a minimal prime of A' is minimal because $(A')^{sh}$ is a filtered colimit of étale hence quasi-finite algebras over A' (small detail omitted). We conclude that the minimal primes of $(A')^{sh}$ are exactly the primes which lie over a minimal prime of A. Similarly, the minimal primes of A^{sh} are exactly the primes lying over minimal primes of A. By construction we have $A' \otimes_A Q(A) = Q(A)$ where $Q(A)$ is the total fraction ring of our reduced local ring A. Of course $Q(A)$ is the finite product of residue fields of the minimal primes of A. It follows that

$$(A')^{sh} \otimes_A Q(A) = A^{sh} \otimes_A A' \otimes_A Q(A) = A^{sh} \otimes_A Q(A)$$

Our discussion above shows the spectrum of the ring on the left is the set of minimal primes of $(A')^{sh}$ and the spectrum of the ring on the right is the is the set of minimal primes of A^{sh}. This finishes the proof. □

Lemma 106.5. *Let A be a local ring. Let A^{sh} be a strict henselization of A. The following are equivalent*

(1) *A is geometrically unibranch, and*
(2) *A^{sh} has a unique minimal prime.*

Proof. This follows from Lemma 106.4 but we will also give a direct proof; this direct proof is almost exactly the same as the direct proof of Lemma 106.3. Denote $\mathfrak{m}$ the maximal ideal of the ring A. Denote κ, κ^{sh} the residue field of A, A^{sh}.

Assume (2). Let $\mathfrak p^{sh}$ be the unique minimal prime of A^{sh}. The flatness of $A \to A^{sh}$ implies that $\mathfrak p = A \cap \mathfrak p^{sh}$ is the unique minimal prime of A (by going down, see Algebra, Lemma 39.19). Also, since $A^{sh}/\mathfrak p A^{sh} = (A/\mathfrak p)^{sh}$ (see Algebra, Lemma 156.4) is reduced by Lemma 45.4 we see that $\mathfrak p^{sh} = \mathfrak p A^{sh}$. Let A' be the integral closure of $A/\mathfrak p$ in its fraction field. We have to show that A' is local and that its residue field is purely inseparable over κ. Since $A \to A'$ is integral, every maximal ideal of A' lies over $\mathfrak m$ (by going up for integral ring maps, see Algebra, Lemma 36.22). If A' is not local, then we can find distinct maximal ideals $\mathfrak m_1$, $\mathfrak m_2$. Choosing elements $f_1, f_2 \in A'$ with $f_i \in \mathfrak m_i, f_i \not\in \mathfrak m_{3-i}$ we find a finite subalgebra $B = A[f_1, f_2] \subset A'$ with distinct maximal ideals $B \cap \mathfrak m_i$, $i = 1, 2$. If A' is local with maximal ideal $\mathfrak m'$, but $A/\mathfrak m \subset A'/\mathfrak m'$ is not purely inseparable, then we can find $f \in A'$ whose image in $A'/\mathfrak m'$ generates a finite, not purely inseparable extension of $A/\mathfrak m$ and we find a finite local subalgebra $B = A[f] \subset A'$ whose residue field is not a purely inseparable extension of $A/\mathfrak m$. Note that the inclusions

$$A/\mathfrak p \subset B \subset \kappa(\mathfrak p)$$

give, on tensoring with the flat ring map $A \to A^{sh}$ the inclusions

$$A^{sh}/\mathfrak p^{sh} \subset B \otimes_A A^{sh} \subset \kappa(\mathfrak p) \otimes_A A^{sh} \subset \kappa(\mathfrak p^{sh})$$

the last inclusion because $\kappa(\mathfrak p) \otimes_A A^{sh} = \kappa(\mathfrak p) \otimes_{A/\mathfrak p} A^{sh}/\mathfrak p^{sh}$ is a localization of the domain $A^{sh}/\mathfrak p^{sh}$. Note that $B \otimes_A \kappa^{sh}$ has at least two maximal ideals because $B/\mathfrak m B$ either has two maximal ideals or one whose residue field is not purely inseparable over κ, and because κ^{sh} is separably algebraically closed. Hence, as A^{sh} is strictly henselian we see that $B \otimes_A A^{sh}$ is a product of ≥ 2 local rings, see Algebra, Lemma 153.6. But we've just seen that $B \otimes_A A^{sh}$ is a subring of a domain and we get a contradiction.

Assume (1). Let $\mathfrak p \subset A$ be the unique minimal prime and let A' be the integral closure of $A/\mathfrak p$ in its fraction field. Let $A \to B$ be a local map of local rings which is a localization of an étale A-algebra. In particular $\mathfrak m_B$ is the unique prime containing $\mathfrak m_A B$. Then $B' = A' \otimes_A B$ is integral over B and the assumption that $A \to A'$ is local with purely inseparable residue field extension implies that B' is local (Algebra, Lemma 156.5). On the other hand, $A' \to B'$ is the localization of an étale ring map, hence B' is normal, see Algebra, Lemma 163.9. Thus B' is a (local) normal domain. Finally, we have

$$B/\mathfrak p B \subset B \otimes_A \kappa(\mathfrak p) = B' \otimes_{A'} (\text{fraction field of } A') \subset \text{fraction field of } B'$$

Hence $B/\mathfrak p B$ is a domain, which implies that B has a unique minimal prime (since by flatness of $A \to B$ these all have to lie over $\mathfrak p$). Since A^{sh} is a filtered colimit of the local rings B it follows that A^{sh} has a unique minimal prime. Namely, if $fg = 0$ in A^{sh} for some non-nilpotent elements f, g, then we can find a B as above containing both f and g which leads to a contradiction. □

Definition 106.6. Let A be a local ring with henselization A^h and strict henselization A^{sh}. The *number of branches of* A is the number of minimal primes of A^h if finite and ∞ otherwise. The *number of geometric branches of* A is the number of minimal primes of A^{sh} if finite and ∞ otherwise.

We spell out the relationship with Definition 106.1.

Lemma 106.7. *Let $(A, \mathfrak m, \kappa)$ be a local ring.*

(1) *If A has infinitely many minimal prime ideals, then the number of (geometric) branches of A is ∞.*
(2) *The number of branches of A is 1 if and only if A is unibranch.*
(3) *The number of geometric branches of A is 1 if and only if A is geometrically unibranch.*

Assume A has finitely many minimal primes and let A' be the integral closure of A in the total ring of fractions of A_{red}. Then

(4) *the number of branches of A is the number of maximal ideals $\mathfrak m'$ of A',*
(5) *to get the number of geometric branches of A we have to count each maximal ideal $\mathfrak m'$ of A' with multiplicity given by the separable degree of $\kappa(\mathfrak m')/\kappa$.*

Proof. This lemma follows immediately from the definitions, Lemma 106.2, Lemma 106.4, and Fields, Lemma 14.8. □

Lemma 106.8. *Let $A \to B$ be a local homomorphism of local rings which is the localization of a smooth ring map.*

(1) *The number of geometric branches of A is equal to the number of geometric branches of B.*
(2) *If $A \to B$ induces a purely inseparable extension of residue fields, then the number of branches of A is the number of branches of B.*

Proof. We will use that smooth ring maps are flat (Algebra, Lemma 137.10), that localizations are flat (Algebra, Lemma 39.18), that compositions of flat ring maps are flat (Algebra, Lemma 39.4), that base change of a flat ring map is flat (Algebra, Lemma 39.7), that flat local homomorphisms are faithfully flat (Algebra, Lemma 39.17), that (strict) henselization is flat (Lemma 45.1), and Going down for flat ring maps (Algebra, Lemma 39.19).

Proof of (2). Let A^h, B^h be the henselizations of A, B. Then B^h is the henselization of $A^h \otimes_A B$ at the unique maximal ideal lying over $\mathfrak m_B$, see Algebra, Lemma 155.8. Thus we may and do assume A is henselian. Since $A \to B \to B^h$ is flat, every minimal prime of B^h lies over a minimal prime of A and since $A \to B^h$ is faithfully flat, every minimal prime of A does lie under a minimal prime of B^h; in both cases use going down for flat ring maps. Therefore it suffices to show that given a minimal prime $\mathfrak p \subset A$, there is at most one minimal prime of B^h lying over $\mathfrak p$. After replacing A by $A/\mathfrak p$ and B by $B/\mathfrak pB$ we may assume that A is a domain; the A is still henselian by Algebra, Lemma 156.2. By Lemma 106.3 we see that the integral closure A' of A in its field of fractions is a local domain. Of course A' is a normal domain. By Algebra, Lemma 163.9 we see that $A' \otimes_A B^h$ is a normal ring (the lemma just gives it for $A' \otimes_A B$, to go up to $A' \otimes_A B^h$ use that B^h is a colimit of étale B-algebras and use Algebra, Lemma 37.17). By Algebra, Lemma 156.5 we see that $A' \otimes_A B^h$ is local (this is where we use the assumption on the residue fields of A and B). Hence $A' \otimes_A B^h$ is a local normal ring, hence a local domain. Since $B^h \subset A' \otimes_A B^h$ by flatness of $A \to B^h$ we conclude that B^h is a domain as desired.

Proof of (1). Let A^{sh}, B^{sh} be strict henselizations of A, B. Then B^{sh} is a strict henselization of $A^h \otimes_A B$ at a maximal ideal lying over $\mathfrak m_B$ and $\mathfrak m_{A^h}$, see Algebra, Lemma 155.12. Thus we may and do assume A is strictly henselian. Since $A \to B \to B^{sh}$ is flat, every minimal prime of B^{sh} lies over a minimal prime of A and since $A \to B^{sh}$ is faithfully flat, every minimal prime of A does lie under a minimal prime of B^{sh}; in both cases use going down for flat ring maps. Therefore it suffices to show that given a minimal prime $\mathfrak{p} \subset A$, there is at most one minimal prime of B^{sh} lying over $\mathfrak{p}$. After replacing A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$ we may assume that A is a domain; then A is still strictly henselian by Algebra, Lemma 156.4. By Lemma 106.5 we see that the integral closure A' of A in its field of fractions is a local domain whose residue field is a purely inseparable extension of the residue field of A. Of course A' is a normal domain. By Algebra, Lemma 163.9 we see that $A' \otimes_A B^{sh}$ is a normal ring (the lemma just gives it for $A' \otimes_A B$, to go up to $A' \otimes_A B^{sh}$ use that B^{sh} is a colimit of étale B-algebras and use Algebra, Lemma 37.17). By Algebra, Lemma 156.5 we see that $A' \otimes_A B^{sh}$ is local (since $A \subset A'$ induces a purely inseparable residue field extension). Hence $A' \otimes_A B^{sh}$ is a local normal ring, hence a local domain. Since $B^{sh} \subset A' \otimes_A B^{sh}$ by flatness of $A \to B^{sh}$ we conclude that B^{sh} is a domain as desired. $\square$

107. Miscellaneous on branches

Some results related to branches of local rings as defined in Section 106.

Lemma 107.1. *Let A and B be domains and let $A \to B$ be a ring map. Assume $A \to B$ has additionally at least one of the following properties*

1. *it is the localization of an étale ring map,*
2. *it is flat and the localization of an unramified ring map,*
3. *it is flat and the localization of a quasi-finite ring map,*
4. *it is flat and the localization of an integral ring map,*
5. *it is flat and there are no nontrivial specializations between points of fibres of* $\text{Spec}(B) \to \text{Spec}(A)$*,*
6. $\text{Spec}(B) \to \text{Spec}(A)$ *maps the generic point to the generic point and there are no nontrivial specializations between points of fibres, or*
7. *exactly one point of* $\text{Spec}(B)$ *is mapped to the generic point of* $\text{Spec}(A)$*.*

Then $A \cap J$ is nonzero for every nonzero ideal J of B.

Proof. Proof in case (7). Let K, resp. L be the fraction field of A, resp. B. By Algebra, Lemma 30.7 we see that the unique point of $\text{Spec}(B)$ which maps to the generic point $(0) \in \text{Spec}(A)$ is $(0) \in \text{Spec}(B)$. We conclude that $B \otimes_A K$ is a ring with a unique prime ideal whose residue field is L (in fact it is equal to L but we do not need this). Choose $b \in J$ nonzero. Then b maps to a unit of L. Hence b maps to a unit of $B \otimes_A K$ (Algebra, Lemma 19.2). Since $B \otimes_A K = \text{colim}_{f \in A \setminus \{0\}} B_f$ we see that b maps to a unit of B_f for some $f \in A$ nonzero. This means that $bb' = f^n$ for some $b' \in B$ and $n \geq 1$. Thus $f^n \in A \cap J$ as desired.

In the rest of the proof, we show that each of the other assumptions imply (7). Under assumptions (1) – (5), the ring map $A \to B$ is flat and hence $A \to B$ is injective (since flat local homomorphisms are faithfully flat by Algebra, Lemma 39.17). Hence the generic point of $\text{Spec}(B)$ maps to the generic point of $\text{Spec}(A)$. Now, if there are no nontrivial specializations between points of fibres of $\text{Spec}(B) \to \text{Spec}(A)$, then of course this generic point of $\text{Spec}(B)$ has to be the unique point mapping to the generic point of $\text{Spec}(A)$. So (6) implies (7). Finally, to finish we show that in cases (1) – (5) there are no nontrivial specializations between the points of fibres of $\text{Spec}(B) \to \text{Spec}(A)$. Namely, see Algebra, Lemma 36.20 for the integral case, Algebra, Definition 122.3 for the quasi-finite case, and use that unramified and étale ring maps are quasi-finite (Algebra, Lemmas 151.6 and 143.6). $\square$

Lemma 107.2. *Let $A \to B$ be a ring map. Let $\mathfrak q \subset B$ be a prime ideal lying over the prime $\mathfrak p \subset A$. Assume*

1. *A is a domain,*
2. *$A_{\mathfrak p}$ is geometrically unibranch,*
3. *$A \to B$ is unramified at $\mathfrak q$, and*
4. *$A_{\mathfrak p} \to B_{\mathfrak q}$ is injective.*

Then there exists a $g \in B$, $g \notin \mathfrak q$ such that B_g is étale over A.

Proof. By Algebra, Proposition 152.1 after replacing B by a principal localization, we can find a standard étale ring map $A \to B'$ and a surjection $B' \to B$. Denote $\mathfrak q' \subset B'$ the inverse image of $\mathfrak q$. We will show that $B' \to B$ is injective after possibly replacing B' by a principal localization.

In this paragraph we reduce to the case that B' is a domain. Since A is a domain, the ring B' is reduced, see Algebra, Lemma 42.1. Let K be the fraction field of A. Then $B' \otimes_A K$ is étale over a field, hence is a finite product of fields, see Algebra, Lemma 143.4. Since $A \to B'$ is étale (hence flat) the minimal primes of B' are lie over $(0) \subset A$ (by going down for flat ring maps). We conclude that B' has finitely many minimal primes, say $\mathfrak r_1, \ldots, \mathfrak r_r \subset B'$. Since $A_{\mathfrak p}$ is geometrically unibranch and $A \to B'$ étale, the ring $B'_{\mathfrak q'}$ is a domain, see Lemmas 106.8 and 106.7. Hence $\mathfrak q' \supset \mathfrak r_i$ for exactly one $i = i_0$. Choose $g' \in B'$, $g' \notin \mathfrak r_{i_0}$ but $g' \in \mathfrak r_i$ for $i \neq i_0$, see Algebra, Lemma 15.2. After replacing B' and B by $B'_{g'}$ and $B_{g'}$ we obtain that B' is a domain.

Assume B' is a domain, in particular $B' \subset B'_{\mathfrak q'}$. If $B' \to B$ is not injective, then $J = \text{Ker}(B'_{\mathfrak q'} \to B_{\mathfrak q})$ is nonzero. By Lemma 107.1 applied to $A_{\mathfrak p} \to B'_{\mathfrak q'}$ we find a nonzero element $a \in A_{\mathfrak p}$ mapping to zero in $B_{\mathfrak q}$ contradicting assumption (4). This finishes the proof. □

Lemma 107.3. *Let $(A, \mathfrak m)$ be a geometrically unibranch local domain. Let $A \to B$ be an injective local homomorphism of local rings, which is essentially of finite type. If $\mathfrak m B$ is the maximal ideal of B and the induced extension of residue fields is separable, then $A \to B$ is the localization of an étale ring map.*

Proof. We may write $B = C_{\mathfrak q}$ where $A \to C$ is a finite type ring map and $\mathfrak q \subset C$ is a prime ideal lying over $\mathfrak m$. By Algebra, Lemma 151.7 the ring map $A \to C$ is unramified at $\mathfrak q$. By Algebra, Proposition 152.1 after replacing C by a principal localization, we can find a standard étale ring map $A \to C'$ and a surjection $C' \to C$. Denote $\mathfrak q' \subset C'$ the inverse image of $\mathfrak q$ and set $B' = C'_{\mathfrak q'}$. Then $B' \to B$ is surjective. It suffices to show that $B' \to B$ is also injective.

Since A is a domain, the rings C' and B' are reduced, see Algebra, Lemma 42.1. Since A is geometrically unibranch, the ring B' is a domain, see by Lemmas 106.8 and 106.7. If $B' \to B$ is not injective, then $A \cap \text{Ker}(B' \to B)$ is nonzero by Lemma 107.1 which contradicts the assumption that $A \to B$ is injective. □

Lemma 107.4. *Let k be an algebraically closed field. Let A, B be strictly henselian local k-algebras with residue field equal to k. Let C be the strict henselization of $A \otimes_k B$ at the maximal ideal $\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B$. Then the minimal primes of C correspond* 1*-to-*1 *to pairs of minimal primes of A and B.*

Proof. First note that a minimal prime $\mathfrak{r}$ of C maps to a minimal prime $\mathfrak{p}$ in A and to a minimal prime $\mathfrak{q}$ of B because the ring maps $A \to C$ and $B \to C$ are flat (by going down for flat ring map Algebra, Lemma 39.19). Hence it suffices to show that the strict henselization of $(A/\mathfrak{p} \otimes_k B/\mathfrak{q})_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$ has a unique minimal prime ideal. By Algebra, Lemma 156.4 the rings $A/\mathfrak{p}$, $B/\mathfrak{q}$ are strictly henselian. Hence we may assume that A and B are strictly henselian local domains and our goal is to show that C has a unique minimal prime. By Lemma 106.5 the integral closure A' of A in its fraction field is a normal local domain with residue field k. Similarly for the integral closure B' of B into its fraction field. By Algebra, Lemma 165.5 we see that $A' \otimes_k B'$ is a normal ring. Hence its localization

$$R = (A' \otimes_k B')_{\mathfrak{m}_{A'} \otimes_k B' + A' \otimes_k \mathfrak{m}_{B'}}$$

is a normal local domain. Note that $A \otimes_k B \to A' \otimes_k B'$ is integral (hence gong up holds – Algebra, Lemma 36.22) and that $\mathfrak{m}_{A'} \otimes_k B' + A' \otimes_k \mathfrak{m}_{B'}$ is the unique maximal ideal of $A' \otimes_k B'$ lying over $\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B$. Hence we see that

$$R = (A' \otimes_k B')_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$$

by Algebra, Lemma 41.11. It follows that

$$(A \otimes_k B)_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B} \longrightarrow R$$

is integral. We conclude that R is the integral closure of $(A \otimes_k B)_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$ in its fraction field, and by Lemma 106.5 once again we conclude that C has a unique prime ideal. ☐

108. Branches of the completion

Let $(A, \mathfrak{m})$ be a Noetherian local ring. Consider the maps $A \to A^h \to A^\wedge$. In general the map $A^h \to A^\wedge$ need not induce a bijection on minimal primes, see Examples, Section 19. In other words, the number of branches of A (as defined in Definition 106.6) may be different from the number of branches of $A^\wedge$. However, under some conditions the number of branches is the same, for example if the dimension of A is 1.

Lemma 108.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring.*

1. *The map $A^h \to A^\wedge$ defines a surjective map from minimal primes of $A^\wedge$ to minimal primes of A^h.*
2. *The number of branches of A is at most the number of branches of $A^\wedge$.*
3. *The number of geometric branches of A is at most the number of geometric branches of $A^\wedge$.*

Proof. By Lemma 45.3 the map $A^h \to A^\wedge$ is flat and injective. Combining going down (Algebra, Lemma 39.19) and Algebra, Lemma 30.5 we see that part (1) holds. Part (2) follows from this, Definition 106.6, and the fact that $A^\wedge$ is henselian (Algebra, Lemma 153.9). By Lemma 45.3 we have $(A^\wedge)^{sh} = A^{sh} \otimes_{A^h} A^\wedge$. Thus we can repeat the arguments above using the flat injective map $A^{sh} \to (A^\wedge)^{sh}$ to prove (3). ☐

Lemma 108.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. The number of branches of A is the same as the number of branches of $A^\wedge$ if and only if $\sqrt{\mathfrak{q}A^\wedge}$ is prime for every minimal prime $\mathfrak{q} \subset A^h$ of the henselization.*

Proof. Follows from Lemma 108.1 and the fact that there are only a finite number of branches for both A and $A^\wedge$ by Algebra, Lemma 31.6 and the fact that A^h and $A^\wedge$ are Noetherian (Lemma 45.3). □

A simple glueing lemma.

Lemma 108.3. *Let A be a ring and let I be a finitely generated ideal. Let $A \to C$ be a ring map such that for all $f \in I$ the ring map $A_f \to C_f$ is localization at an idempotent. Then there exists a surjection $A \to C'$ such that $A_f \to (C \times C')_f$ is an isomorphism for all $f \in I$.*

Proof. Choose generators $f_1, \ldots, f_r$ of I. Write

$$C_{f_i} = (A_{f_i})_{e_i}$$

for some idempotent $e_i \in A_{f_i}$. Write $e_i = a_i/f_i^n$ for some $a_i \in A$ and $n \geq 0$; we may use the same n for all $i = 1, \ldots, r$. After replacing a_i by $f_i^m a_i$ and n by $n+m$ for a suitable $m \gg 0$, we may assume $a_i^2 = f_i^n a_i$ for all i. Since e_i maps to 1 in $C_{f_i f_j} = (A_{f_i f_j})_{e_j} = A_{f_i f_j a_j}$ we see that

$$(f_i f_j a_j)^N (f_j^n a_i - f_i^n a_j) = 0$$

for some N (we can pick the same N for all pairs i, j). Using $a_j^2 = f_j^n a_j$ this gives

$$f_i^{N+n} f_j^{N+nN} a_j = f_i^N f_j^{N+n} a_i a_j^N$$

After increasing n to $n + N + nN$ and replacing a_i by $f_i^{N+nN} a_i$ we see that $f_i^n a_j$ is in the ideal of a_i for all pairs i, j. Let $C' = A/(a_1, \ldots, a_r)$. Then

$$C'_{f_i} = A_{f_i}/(a_i) = A_{f_i}/(e_i)$$

because a_j is in the ideal generated by a_i after inverting f_i. Since for an idempotent e of a ring B we have $B = B_e \times B/(e)$ we see that the conclusion of the lemma holds for f equal to one of $f_1, \ldots, f_r$. Using glueing of functions, in the form of Algebra, Lemma 23.2, we conclude that the result holds for all $f \in I$. Namely, for $f \in I$ the elements $f_1, \ldots, f_r$ generate the unit ideal in A_f so $A_f \to (C \times C')_f$ is an isomorphism if and only if this is the case after localizing at $f_1, \ldots, f_r$. □

Lemma 108.4 can be used to construct finite type extensions from given finite type extensions of the formal completion. We will generalize this lemma in Algebraization of Formal Spaces, Lemma 10.3.

Lemma 108.4. *Let A be a Noetherian ring and I an ideal. Let B be a finite type A-algebra. Let $B^\wedge \to C$ be a surjective ring map with kernel J where $B^\wedge$ is the I-adic completion. If J/J^2 is annihilated by I^c for some $c \geq 0$, then C is isomorphic to the completion of a finite type A-algebra.*

Proof. Let $f \in I$. Since $B^\wedge$ is Noetherian (Algebra, Lemma 97.6), we see that J is a finitely generated ideal. Hence we conclude from Algebra, Lemma 21.5 that

$$C_f = ((B^\wedge)_f)_e$$

for some idempotent $e \in (B^\wedge)_f$. By Lemma 108.3 we can find a surjection $B^\wedge \to C'$ such that $B^\wedge \to C \times C'$ becomes an isomorphism after inverting any $f \in I$. Observe that $C \times C'$ is a finite $B^\wedge$-algebra.

Choose generators $f_1, \ldots, f_r \in I$. Denote $\alpha_i : (C \times C')_{f_i} \to B_{f_i} \otimes_B B^\wedge$ the inverse of the isomorphism of $(B^\wedge)_{f_i}$-algebras we obtained above. Denote $\alpha_{ij} : (B_{f_i})_{f_j} \to (B_{f_j})_{f_i}$ the obvious B-algebra isomorphism. Consider the object

$$(C \times C', B_{f_i}, \alpha_i, \alpha_{ij})$$

of the category $\text{Glue}(B \to B^\wedge, f_1, \ldots, f_r)$ introduced in Remark 89.10. We omit the verification of conditions (1)(a) and (1)(b). Since $B \to B^\wedge$ is a flat map (Algebra, Lemma 97.2) inducing an isomorphism $B/IB \to B^\wedge/IB^\wedge$ we may apply Proposition 89.15 and Remark 89.19. We conclude that $C \times C'$ is isomorphic to $D \otimes_B B^\wedge$ for some finite B-algebra D. Then $D/ID \cong C/IC \times C'/IC'$. Let $\overline{e} \in D/ID$ be the idempotent corresponding to the factor C/IC. By Lemma 9.10 there exists an étale ring map $B \to B'$ which induces an isomorphism $B/IB \to B'/IB'$ such that $D' = D \otimes_B B'$ contains an idempotent e lifting $\overline{e}$. Since $C \times C'$ is I-adically complete the pair $(C \times C', IC \times IC')$ is henselian (Lemma 11.4). Thus we can factor the map $B \to C \times C'$ through B'. Doing so we may replace B by B' and D by D'. Then we find that $D = D_e \times D_{1-e} = D/(1-e) \times D/(e)$ is a product of finite type A-algebras and the completion of the first part is C and the completion of the second part is C'. □

Lemma 108.5. *Let $(A, \mathfrak m)$ be a Noetherian local ring with henselization A^h. Let $\mathfrak q \subset A^\wedge$ be a minimal prime with $\dim(A^\wedge/\mathfrak q) = 1$. Then there exists a minimal prime $\mathfrak q^h$ of A^h such that $\mathfrak q = \sqrt{\mathfrak q^h A^\wedge}$.*

Proof. Since the completion of A and A^h are the same, we may assume that A is henselian (Lemma 45.3). We will apply Lemma 108.4 to $A^\wedge \to A^\wedge/J$ where $J = \text{Ker}(A^\wedge \to (A^\wedge)_{\mathfrak q})$. Since $\dim((A^\wedge)_{\mathfrak q}) = 0$ we see that $\mathfrak q^n \subset J$ for some n. Hence J/J^2 is annihilated by $\mathfrak q^n$. On the other hand $(J/J^2)_{\mathfrak q} = 0$ because $J_{\mathfrak q} = 0$. Hence $\mathfrak m$ is the only associated prime of J/J^2 and we find that a power of $\mathfrak m$ annihilates J/J^2. Thus the lemma applies and we find that $A^\wedge/J = C^\wedge$ for some finite type A-algebra C.

Then $C/\mathfrak m C = A/\mathfrak m$ because $A^\wedge/J$ has the same property. Hence $\mathfrak m_C = \mathfrak m C$ is a maximal ideal and $A \to C$ is unramified at $\mathfrak m_C$ (Algebra, Lemma 151.7). After replacing C by a principal localization we may assume that C is a quotient of an étale A-algebra B, see Algebra, Proposition 152.1. However, since the residue field extension of $A \to C_{\mathfrak m_C}$ is trivial and A is henselian, we conclude that $B = A$ again after a localization. Thus $C = A/I$ for some ideal $I \subset A$ and it follows that $J = IA^\wedge$ (because completion is exact in our situation by Algebra, Lemma 97.2) and $I = J \cap A$ (by flatness of $A \to A^\wedge$). Since $\mathfrak q^n \subset J \subset \mathfrak q$ we see that $\mathfrak p = \mathfrak q \cap A$ satisfies $\mathfrak p^n \subset I \subset \mathfrak p$. Then $\sqrt{\mathfrak p A^\wedge} = \mathfrak q$ and the proof is complete. □

Lemma 108.6. *Let $(A, \mathfrak m)$ be a Noetherian local ring. The punctured spectrum of $A^\wedge$ is disconnected if and only if the punctured spectrum of A^h is disconnected.*

Proof. Since the completion of A and A^h are the same, we may assume that A is henselian (Lemma 45.3).

Since $A \to A^\wedge$ is faithfully flat (see reference just given) the map from the punctured spectrum of $A^\wedge$ to the punctured spectrum of A is surjective (see Algebra, Lemma 39.16). Hence if the punctured spectrum of A is disconnected, then the same is true for $A^\wedge$.

Assume the punctured spectrum of $A^\wedge$ is disconnected. This means that

$$\mathrm{Spec}(A^\wedge) \setminus \{\mathfrak{m}^\wedge\} = Z \amalg Z'$$

with Z and Z' closed. Let $\overline{Z}, \overline{Z}' \subset \mathrm{Spec}(A^\wedge)$ be the closures. Say $\overline{Z} = V(J)$, $\overline{Z}' = V(J')$ for some ideals $J, J' \subset A^\wedge$. Then $V(J+J') = \{\mathfrak{m}^\wedge\}$ and $V(JJ') = \mathrm{Spec}(A^\wedge)$. The first equality means that $\mathfrak{m}^\wedge = \sqrt{J+J'}$ which implies $(\mathfrak{m}^\wedge)^e \subset J + J'$ for some $e \geq 1$. The second equality implies every element of JJ' is nilpotent hence $(JJ')^n = 0$ for some $n \geq 1$. Combined this means that J^n/J^{2n} is annihilated by J^n and $(J')^n$ and hence by $(\mathfrak{m}^\wedge)^{2en}$. Thus we may apply Lemma 108.4 to see that there is a finite type A-algebra C and an isomorphism $A^\wedge/J^n = C^\wedge$.

The rest of the proof is exactly the same as the second part of the proof of Lemma 108.5; of course that lemma is a special case of this one! We have $C/\mathfrak{m}C = A/\mathfrak{m}$ because $A^\wedge/J^n$ has the same property. Hence $\mathfrak{m}_C = \mathfrak{m}C$ is a maximal ideal and $A \to C$ is unramified at $\mathfrak{m}_C$ (Algebra, Lemma 151.7). After replacing C by a principal localization we may assume that C is a quotient of an étale A-algebra B, see Algebra, Proposition 152.1. However, since the residue field extension of $A \to C_{\mathfrak{m}_C}$ is trivial and A is henselian, we conclude that $B = A$ again after a localization. Thus $C = A/I$ for some ideal $I \subset A$ and it follows that $J^n = IA^\wedge$ (because completion is exact in our situation by Algebra, Lemma 97.2) and $I = J^n \cap A$ (by flatness of $A \to A^\wedge$). By symmetry $I' = (J')^n \cap A$ satisfies $(J')^n = I'A^\wedge$. Then $\mathfrak{m}^e \subset I + I'$ and $II' = 0$ and we conclude that $V(I)$ and $V(I')$ are closed subschemes which give the desired disjoint union decomposition of the punctured spectrum of A. $\square$

Lemma 108.7. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Then the number of (geometric) branches of A and $A^\wedge$ is the same.*

Proof. To see this for the number of branches, combine Lemmas 108.1, 108.2, and 108.5 and use that the dimension of $A^\wedge$ is one, see Lemma 43.1. To see this is true for the number of geometric branches we use the result for branches, the fact that the dimension does not change under strict henselization (Lemma 45.7), and the fact that $(A^{sh})^\wedge = ((A^\wedge)^{sh})^\wedge$ by Lemma 45.3. $\square$

Lemma 108.8. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. If the formal fibres of A are geometrically normal (for example if A is excellent or quasi-excellent), then A is Nagata and the number of (geometric) branches of A and $A^\wedge$ is the same.*

Proof. Since a normal ring is reduced, we see that A is Nagata by Lemma 52.4. In the rest of the proof we will use Lemma 51.10, Proposition 51.5, and Lemma 51.4. This tells us that A is a P-ring where $P(k \to R) =$"R is geometrically normal over k" and the same is true for any (essentially of) finite type A-algebra.

Let $\mathfrak{q} \subset A$ be a minimal prime. Then $A^\wedge/\mathfrak{q}A^\wedge = (A/\mathfrak{q})^\wedge$ and $A^h/\mathfrak{q}A^h = (A/\mathfrak{q})^h$ (Algebra, Lemma 156.2). Hence the number of branches of A is the sum of the number of branches of the rings $A/\mathfrak{q}$ and similarly for $A^\wedge$. In this way we reduce to the case that A is a domain.

Assume A is a domain. Let A' be the integral closure of A in the fraction field K of A. Since A is Nagata, we see that $A \to A'$ is finite. Recall that the number of branches of A is the number of maximal ideals $\mathfrak{m}'$ of A' (Lemma 106.2). Also, recall that

$$(A')^\wedge = A' \otimes_A A^\wedge = \prod\nolimits_{\mathfrak{m}' \subset A'} (A'_{\mathfrak{m}'})^\wedge$$

by Algebra, Lemma 97.8. Because $A'_{\mathfrak m'}$ is a local ring whose formal fibres are geometrically normal, we see that $(A'_{\mathfrak m'})^\wedge$ is normal (Lemma 52.6). Hence the minimal primes of $A' \otimes_A A^\wedge$ are in 1-to-1 correspondence with the factors in the decomposition above. By flatness of $A \to A^\wedge$ we have

$$A^\wedge \subset A' \otimes_A A^\wedge \subset K \otimes_A A^\wedge$$

Since the left and the right ring have the same set of minimal primes, the same is true for the ring in the middle (small detail omitted) and this finishes the proof.

To see this is true for the number of geometric branches we use the result for branches, the fact that the formal fibres of A^{sh} are geometrically normal (Lemmas 51.10 and 51.8) and the fact that $(A^{sh})^\wedge = ((A^\wedge)^{sh})^\wedge$ by Lemma 45.3. $\square$

109. Formally catenary rings

In this section we prove a theorem of Ratliff [Rat71] that a Noetherian local ring is universally catenary if and only if it is formally catenary.

Definition 109.1. A Noetherian local ring A is *formally catenary* if for every minimal prime $\mathfrak p \subset A$ the spectrum of $A^\wedge/\mathfrak p A^\wedge$ is equidimensional.

Let A be a Noetherian local ring which is formally catenary. By Ratliff's result (Proposition 109.5) we see that any quotient of A is also formally catenary (because the class of universally catenary rings is stable under quotients). We conclude that the spectrum of $A^\wedge/\mathfrak p A^\wedge$ is equidimensional for every prime ideal $\mathfrak p$ of A.

Lemma 109.2. *Let $(A, \mathfrak m)$ be a Noetherian local ring which is not formally catenary. Then A is not universally catenary.*

Proof. By assumption there exists a minimal prime $\mathfrak p \subset A$ such that the spectrum of $A^\wedge/\mathfrak p A^\wedge$ is not equidimensional. After replacing A by $A/\mathfrak p$ we may assume that A is a domain and that the spectrum of $A^\wedge$ is not equidimensional. Let $\mathfrak q$ be a minimal prime of $A^\wedge$ such that $d = \dim(A^\wedge/\mathfrak q)$ is minimal and hence $0 < d < \dim(A)$. We prove the lemma by induction on d.

The case $d = 1$. In this case $\dim(A^\wedge_{\mathfrak q}) = 0$. Hence $A^\wedge_{\mathfrak q}$ is Artinian local and we see that for some $n > 0$ the ideal $J = \mathfrak q^n$ maps to zero in $A^\wedge_{\mathfrak q}$. It follows that $\mathfrak m$ is the only associated prime of J/J^2, whence $\mathfrak m^m$ annihilates J/J^2 for some $m > 0$. Thus we can use Lemma 108.4 to find $A \to B$ of finite type such that $B^\wedge \cong A^\wedge/J$. It follows that $\mathfrak m_B = \sqrt{\mathfrak m B}$ is a maximal ideal with the same residue field as $\mathfrak m$ and $B^\wedge$ is the $\mathfrak m_B$-adic completion (Algebra, Lemma 97.7). Then

$$\dim(B_{\mathfrak m_B}) = \dim(B^\wedge) = 1 = d.$$

Since we have the factorization $A \to B \to A^\wedge/J$ the inverse image of $\mathfrak q/J$ is a prime $\mathfrak q' \subset \mathfrak m_B$ lying over (0) in A. Thus, if A were universally catenary, the dimension formula (Algebra, Lemma 113.1) would give

$$\begin{aligned}\dim(B_{\mathfrak m_B}) &\geq \dim((B/\mathfrak q')_{\mathfrak m_B}) \\ &= \dim(A) + \text{trdeg}_A(B/\mathfrak q') - \text{trdeg}_{\kappa(\mathfrak m)}(\kappa(\mathfrak m_B)) \\ &= \dim(A) + \text{trdeg}_A(B/\mathfrak q')\end{aligned}$$

This contradiction finishes the argument in case $d = 1$.

Assume $d > 1$. Let $Z \subset \text{Spec}(A^\wedge)$ be the union of the irreducible components distinct from $V(\mathfrak{q})$. Let $\mathfrak{r}_1, \ldots, \mathfrak{r}_m \subset A^\wedge$ be the prime ideals corresponding to irreducible components of $V(\mathfrak{q}) \cap Z$ of dimension > 0. Choose $f \in \mathfrak{m}$, $f \notin A \cap \mathfrak{r}_j$ using prime avoidance (Algebra, Lemma 15.2). Then $\dim(A/fA) = \dim(A) - 1$ and there is some irreducible component of $V(\mathfrak{q}, f)$ of dimension $d-1$. Thus A/fA is not formally catenary and the invariant d has decreased. By induction A/fA is not universally catenary, hence A is not universally catenary. $\square$

Lemma 109.3. *Let $A \to B$ be a flat local ring map of local Noetherian rings. Assume B is catenary and is* $\text{Spec}(B)$ *equidimensional. Then*

(1) $\text{Spec}(B/\mathfrak{p}B)$ *is equidimensional for all* $\mathfrak{p} \subset A$ *and*
(2) A *is catenary and* $\text{Spec}(A)$ *is equidimensional.*

Proof. Let $\mathfrak{p} \subset A$ be a prime ideal. Let $\mathfrak{q} \subset B$ be a prime minimal over $\mathfrak{p}B$. Then $\mathfrak{q} \cap A = \mathfrak{p}$ by going down for $A \to B$ (Algebra, Lemma 39.19). Hence $A_\mathfrak{p} \to B_\mathfrak{q}$ is a flat local ring map with special fibre of dimension 0 and hence

$$\dim(A_\mathfrak{p}) = \dim(B_\mathfrak{q}) = \dim(B) - \dim(B/\mathfrak{q})$$

(Algebra, Lemma 112.7). The second equality because $\text{Spec}(B)$ is equidimensional and B is catenary. Thus $\dim(B/\mathfrak{q})$ is independent of the choice of $\mathfrak{q}$ and we conclude that $\text{Spec}(B/\mathfrak{p}B)$ is equidimensional of dimension $\dim(B) - \dim(A_\mathfrak{p})$. On the other hand, we have $\dim(B/\mathfrak{p}B) = \dim(A/\mathfrak{p}) + \dim(B/\mathfrak{m}_A B)$ and $\dim(B) = \dim(A) + \dim(B/\mathfrak{m}_A B)$ by flatness (see lemma cited above) and we get

$$\dim(A_\mathfrak{p}) = \dim(A) - \dim(A/\mathfrak{p})$$

for all $\mathfrak{p}$ in A. Applying this to all minimal primes in A we see that A is equidimensional. If $\mathfrak{p} \subset \mathfrak{p}'$ is a strict inclusion with no primes in between, then we may apply the above to the prime $\mathfrak{p}'/\mathfrak{p}$ in $A/\mathfrak{p}$ because $A/\mathfrak{p} \to B/\mathfrak{p}B$ is flat and $\text{Spec}(B/\mathfrak{p}B)$ is equidimensional, to get

$$1 = \dim((A/\mathfrak{p})_{\mathfrak{p}'}) = \dim(A/\mathfrak{p}) - \dim(A/\mathfrak{p}')$$

Thus $\mathfrak{p} \mapsto \dim(A/\mathfrak{p})$ is a dimension function and we conclude that A is catenary. $\square$

Lemma 109.4. *Let A be a formally catenary Noetherian local ring. Then A is universally catenary.*

Proof. We may replace A by $A/\mathfrak{p}$ where $\mathfrak{p}$ is a minimal prime of A, see Algebra, Lemma 105.8. Thus we may assume that the spectrum of $A^\wedge$ is equidimensional. It suffices to show that every local ring essentially of finite type over A is catenary (see for example Algebra, Lemma 105.6). Hence it suffices to show that $A[x_1, \ldots, x_n]_\mathfrak{m}$ is catenary where $\mathfrak{m} \subset A[x_1, \ldots, x_n]$ is a maximal ideal lying over $\mathfrak{m}_A$, see Algebra, Lemma 54.5 (and Algebra, Lemmas 105.7 and 105.4). Let $\mathfrak{m}' \subset A^\wedge[x_1, \ldots, x_n]$ be the unique maximal ideal lying over $\mathfrak{m}$. Then

$$A[x_1, \ldots, x_n]_\mathfrak{m} \to A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}$$

is local and flat (Algebra, Lemma 97.2). Hence it suffices to show that the ring on the right hand side catenary with equidimensional spectrum, see Lemma 109.3. It is catenary because complete local rings are universally catenary (Algebra, Remark 160.9). Pick any minimal prime $\mathfrak{q}$ of $A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}$. Then $\mathfrak{q} = \mathfrak{p}A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}$ for some minimal prime $\mathfrak{p}$ of $A^\wedge$ (small detail omitted). Hence

$$\dim(A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}/\mathfrak{q}) = \dim(A^\wedge/\mathfrak{p}) + n = \dim(A^\wedge) + n$$

the first equality by Algebra, Lemma 112.7 and the second because the spectrum of $A^\wedge$ is equidimensional. This finishes the proof. □

Proposition 109.5 (Ratliff)**.** *A Noetherian local ring is universally catenary if and only if it is formally catenary.*

Proof. Combine Lemmas 109.2 and 109.4. □

Lemma 109.6. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with geometrically normal formal fibres. Then*

1. *A^h is universally catenary, and*
2. *if A is unibranch (for example normal), then A is universally catenary.*

Proof. By Lemma 108.8 the number of branches of A and $A^\wedge$ are the same, hence Lemma 108.2 applies. Then for any minimal prime $\mathfrak{q} \subset A^h$ we see that $A^\wedge/\mathfrak{q}A^\wedge$ has a unique minimal prime. Thus A^h is formally catenary (by definition) and hence universally catenary by Proposition 109.5. If A is unibranch, then A^h has a unique minimal prime, hence $A^\wedge$ has a unique minimal prime, hence A is formally catenary and we conclude in the same way. □

110. Group actions and integral closure

This section is in some sense a continuation of Algebra, Section 38. More material of a similar kind can be found in Fundamental Groups, Section 12

Lemma 110.1. *Let $\varphi : A \to B$ be a surjection of rings. Let G be a finite group of order n acting on $\varphi : A \to B$. If $b \in B^G$, then there exists a monic polynomial $P \in A^G[T]$ which maps to $(T - b)^n$ in $B^G[T]$.*

Proof. Choose $a \in A$ lifting b and set $P = \prod_{\sigma \in G}(T - \sigma(a))$. □

Lemma 110.2. *Let R be a ring. Let G be a finite group acting on R. Let $I \subset R$ be an ideal such that $\sigma(I) \subset I$ for all $\sigma \in G$. Then $R^G/I^G \subset (R/I)^G$ is an integral extension of rings which induces a homeomorphism on spectra and purely inseparable extensions of residue fields.*

Proof. Since $I^G = R^G \cap I$ it is clear that the map is injective. Lemma 110.1 shows that Algebra, Lemma 46.11 applies. □

Lemma 110.3. *Let G be a finite group of order n acting on a ring R. Let $J \subset R^G$ be an ideal. For $x \in JR$ we have $\prod_{\sigma \in G}(T - \sigma(x)) = T^n + a_1T^{n-1} + \ldots + a_n$ with $a_i \in J$.*

Proof. Observe that the polynomial is indeed monic and has coefficients in R^G. We can write $x = f_1b_1 + \ldots + f_mb_m$ with $f_j \in J$ and $b_j \in R$. Thus, arguing by induction on m, we may assume that $x = y - fb$ with $f \in J$, $b \in R$, and $y \in JR$ such that the result holds for y. Then we see that

$$\prod\nolimits_{\sigma \in G}(T - \sigma(x)) = \prod\nolimits_{\sigma \in G}(T - \sigma(y) + f\sigma(b)) = \prod\nolimits_{\sigma \in G}(T - \sigma(y)) + \sum_{i=1,\ldots,n} f^i a_i$$

where we have

$$a_i = \sum\nolimits_{S \subset G,\ |S| = i} \prod\nolimits_{\sigma \in S} \sigma(b) \prod\nolimits_{\sigma \not\in S}(T - \sigma(y))$$

A computation we omit shows that $a_i \in R^G$ (hint: the given expression is symmetric). Thus the polynomial of the statement of the lemma for x is congruent modulo J to the polynomial for y and this proves the induction step. $\square$

Lemma 110.4. *Let R be a ring. Let G be a finite group of order n acting on R. Let $J \subset R^G$ be an ideal. Then $R^G/J \to (R/JR)^G$ is ring map such that*

(1) *for $b \in (R/JR)^G$ there is a monic polynomial $P \in R^G/J[T]$ whose image in $(R/JR)^G[T]$ is $(T-b)^n$,*
(2) *for $a \in \mathrm{Ker}(R^G/J \to (R/JR)^G)$ we have $(T-a)^n = T^n$ in $R^G/J[T]$.*

In particular, $R^G/J \to (R/JR)^G$ is an integral ring map which induces homeomorphisms on spectra and purely inseparable extensions of residue fields.

Proof. Part (1) follow from Lemma 110.1 with $I = JR$. If a is as in part (2), then a is the image of $x \in R^G \cap JR$. Hence $(T-x)^n = \prod_{\sigma \in G}(T - \sigma(x))$ is congruent to T^n modulo J by Lemma 110.3. This proves part (2). To see the final statement we may apply Algebra, Lemma 46.11. $\square$

Remark 110.5. In Lemma 110.4 we see that the map $R^G/J \to (R/JR)^G$ is an isomorphism if n is invertible in R.

Lemma 110.6. *Let R be a ring. Let G be a finite group of order n acting on R. Let A be an R^G-algebra.*

(1) *for $b \in (A \otimes_{R^G} R)^G$ there exists a monic polynomial $P \in A[T]$ whose image in $(A \otimes_{R^G} R)^G[T]$ is $(T-b)^n$,*
(2) *for $a \in \mathrm{Ker}(A \to (A \otimes_{R^G} R)^G)$ we have $(T-a)^n = T^n$ in $A[T]$.*

Proof. Choose a surjection $E \to A$ where E is a polynomial algebra over R^G. Then $(E \otimes_{R^G} R)^G = E$ because E is free as an R^G-module. Denote $J = \mathrm{Ker}(E \to A)$. Since tensor product is right exact we see that $A \otimes_{R^G} R$ is the quotient of $E \otimes_{R^G} R$ by the ideal generated by J. In this way we see that our lemma is a special case of Lemma 110.4. $\square$

Lemma 110.7. *Let R be a ring. Let G be a finite group acting on R. Let $R^G \to A$ be a ring map. The map*

$$A \to (A \otimes_{R^G} R)^G$$

is an isomorphism if $R^G \to A$ is flat. In general the map is integral, induces a homeomorphism on spectra, and induces purely inseparable residue field extensions.

Proof. To see the first statement consider the exact sequence $0 \to R^G \to R \to \bigoplus_{\sigma \in G} R$ where the second map sends x to $(\sigma(x) - x)_{\sigma \in G}$. Tensoring with A the sequence remains exact if $R^G \to A$ is flat. Thus A is the G-invariants in $(A \otimes_{R^G} R)^G$.

The second statement follows from Lemma 110.6 and Algebra, Lemma 46.11. $\square$

Lemma 110.8. *Let G be a finite group acting on a ring R. For any two primes $\mathfrak{q}, \mathfrak{q}' \subset R$ lying over the same prime in R^G there exists a $\sigma \in G$ with $\sigma(\mathfrak{q}) = \mathfrak{q}'$.*

Proof. The extension $R^G \subset R$ is integral because every $x \in R$ is a root of the monic polynomial $\prod_{\sigma \in G}(T - \sigma(x))$ in $R^G[T]$. Thus there are no inclusion relations among the primes lying over a given prime $\mathfrak{p}$ (Algebra, Lemma 36.20). If the lemma is wrong, then we can choose $x \in \mathfrak{q}'$, $x \notin \sigma(\mathfrak{q})$ for all $\sigma \in G$. See Algebra, Lemma 15.2. Then $y = \prod_{\sigma \in G} \sigma(x)$ is in R^G and in $\mathfrak{p} = R^G \cap \mathfrak{q}'$. On the other hand, $x \notin \sigma(\mathfrak{q})$ for all σ means $\sigma(x) \notin \mathfrak{q}$ for all σ. Hence $y \notin \mathfrak{q}$ as $\mathfrak{q}$ is a prime ideal. This is impossible as $y \in \mathfrak{p} \subset \mathfrak{q}$. $\square$

Lemma 110.9. *Let G be a finite group acting on a ring R. Let $\mathfrak q \subset R$ be a prime lying over $\mathfrak p \subset R^G$. Then $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is an algebraic normal extension and the map*

$$D = \{\sigma \in G \mid \sigma(\mathfrak q) = \mathfrak q\} \longrightarrow Aut(\kappa(\mathfrak q)/\kappa(\mathfrak p))$$

is surjective[15].

Proof. With $A = (R^G)_{\mathfrak p}$ and $B = A \otimes_{R^G} R$ we see that $A = B^G$ as localization is flat, see Lemma 110.7. Observe that $\mathfrak pA$ and $\mathfrak qB$ are prime ideals, D is the stabilizer of $\mathfrak qB$, and $\kappa(\mathfrak p) = \kappa(\mathfrak pA)$ and $\kappa(\mathfrak q) = \kappa(\mathfrak qB)$. Thus we may replace R by B and assume that $\mathfrak p$ is a maximal ideal. Since $R^G \subset R$ is an integral ring extension, we find that the maximal ideals of R are exactly the primes lying over $\mathfrak p$ (follows from Algebra, Lemmas 36.20 and 36.22). By Lemma 110.8 there are finitely many of them $\mathfrak q = \mathfrak q_1, \mathfrak q_2, \ldots, \mathfrak q_m$ and they form a single orbit for G. By the Chinese remainder theorem (Algebra, Lemma 15.4) the map $R \to \prod_{j=1,\ldots,m} R/\mathfrak q_j$ is surjective.

First we prove that the extension is normal. Pick an element $\alpha \in \kappa(\mathfrak q)$. We have to show that the minimal polynomial P of α over $\kappa(\mathfrak p)$ splits completely. By the above we can choose $a \in \mathfrak q_2 \cap \ldots \cap \mathfrak q_m$ mapping to α in $\kappa(\mathfrak q)$. Consider the polynomial $Q = \prod_{\sigma \in G}(T - \sigma(a))$ in $R^G[T]$. The image of Q in $R[T]$ splits completely into linear factors, hence the same is true for its image in $\kappa(\mathfrak q)[T]$. Since P divides the image of Q in $\kappa(\mathfrak p)[T]$ we conclude that P splits completely into linear factors over $\kappa(\mathfrak q)$ as desired.

Since $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is normal we may assume $\kappa(\mathfrak q) = \kappa_1 \otimes_{\kappa(\mathfrak p)} \kappa_2$ with $\kappa_1/\kappa(\mathfrak p)$ purely inseparable and $\kappa_2/\kappa(\mathfrak p)$ Galois, see Fields, Lemma 27.3. Pick $\alpha \in \kappa_2$ which generates κ_2 over $\kappa(\mathfrak p)$ if it is finite and a subfield of degree $> |G|$ if it is infinite (to get a contradiction). This is possible by Fields, Lemma 19.1. Pick a, P, and Q as in the previous paragraph. If $\alpha' \in \kappa_2$ is a Galois conjugate of α over $\kappa(\mathfrak p)$, then the fact that P divides the image of P in $\kappa(\mathfrak p)[T]$ shows there exists a $\sigma \in G$ such that $\sigma(a)$ maps to α'. By our choice of a (vanishing at other maximal ideals) this implies $\sigma \in D$ and that the image of σ in $\text{Aut}(\kappa(\mathfrak q)/\kappa(\mathfrak p))$ maps α to α'. Hence the surjectivity or the desired absurdity in case α has degree $> |G|$ over $\kappa(\mathfrak p)$. $\square$

Lemma 110.10. *Let A be a normal domain with fraction field K. Let L/K be a (possibly infinite) Galois extension. Let $G = Gal(L/K)$ and let B be the integral closure of A in L.*

(1) *For any two primes $\mathfrak q, \mathfrak q' \subset B$ lying over the same prime in A there exists a $\sigma \in G$ with $\sigma(\mathfrak q) = \mathfrak q'$.*
(2) *Let $\mathfrak q \subset B$ be a prime lying over $\mathfrak p \subset A$. Then $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is an algebraic normal extension and the map*

$$D = \{\sigma \in G \mid \sigma(\mathfrak q) = \mathfrak q\} \longrightarrow Aut(\kappa(\mathfrak q)/\kappa(\mathfrak p))$$

is surjective.

Proof. Proof of (1). Consider pairs (M, σ) where $K \subset M \subset L$ is a subfield such that M/K is Galois, $\sigma \in \text{Gal}(M/K)$ with $\sigma(\mathfrak q \cap M) = \mathfrak q' \cap M$. We say $(M', \sigma') \geq (M, \sigma)$ if and only if $M \subset M'$ and $\sigma'|_M = \sigma$. Observe that (K, id_K) is such a pair as $A = K \cap B$ since A is a normal domain. The collection of these pairs satisfies the hypotheses of Zorn's lemma, hence there exists a maximal pair (M, σ).

[15]Recall that we use the notation Gal only in the case of Galois extensions.

If $M \neq L$, then we can find $M \subset M' \subset L$ with M'/M nontrivial and finite and M'/K Galois (Fields, Lemma 16.5). Choose $\sigma' \in \text{Gal}(M'/K)$ whose restriction to M is σ (Fields, Lemma 22.2). Then the primes $\sigma'(\mathfrak q \cap M')$ and $\mathfrak q' \cap M'$ restrict to the same prime of $B \cap M$. Since $B \cap M = (B \cap M')^{\text{Gal}(M'/M)}$ we can use Lemma 110.8 to find $\tau \in \text{Gal}(M'/M)$ with $\tau(\sigma'(\mathfrak q \cap M')) = \mathfrak q' \cap M'$. Hence $(M', \tau \circ \sigma') > (M, \sigma)$ contradicting the maximality of (M, σ).

Part (2) is proved in exactly the same manner as part (1). We write out the details. Pick $\overline{\sigma} \in \text{Aut}(\kappa(\mathfrak q)/\kappa(\mathfrak p))$. Consider pairs (M, σ) where $K \subset M \subset L$ is a subfield such that M/K is Galois, $\sigma \in \text{Gal}(M/K)$ with $\sigma(\mathfrak q \cap M) = \mathfrak q \cap M$ and

$$\begin{tikzcd} \kappa(\mathfrak q \cap M) \arrow[r] \arrow[d, "\sigma"'] & \kappa(\mathfrak q) \arrow[d, "\overline{\sigma}"'] \\ \kappa(\mathfrak q \cap M) \arrow[r] & \kappa(\mathfrak q) \end{tikzcd}$$

commutes. We say $(M', \sigma') \geq (M, \sigma)$ if and only if $M \subset M'$ and $\sigma'|_M = \sigma$. As above (K, id_K) is such a pair. The collection of these pairs satisfies the hypotheses of Zorn's lemma, hence there exists a maximal pair (M, σ). If $M \neq L$, then we can find $M \subset M' \subset L$ with M'/M finite and M'/K Galois (Fields, Lemma 16.5). Choose $\sigma' \in \text{Gal}(M'/K)$ whose restriction to M is σ (Fields, Lemma 22.2). Then the primes $\sigma'(\mathfrak q \cap M')$ and $\mathfrak q \cap M'$ restrict to the same prime of $B \cap M$. Adjusting the choice of σ' as in the first paragraph, we may assume that $\sigma'(\mathfrak q \cap M') = \mathfrak q \cap M'$. Then σ' and $\overline{\sigma}$ define maps $\kappa(\mathfrak q \cap M') \to \kappa(\mathfrak q)$ which agree on $\kappa(\mathfrak q \cap M)$. Since $B \cap M = (B \cap M')^{\text{Gal}(M'/M)}$ we can use Lemma 110.9 to find $\tau \in \text{Gal}(M'/M)$ with $\tau(\mathfrak q \cap M') = \mathfrak q \cap M'$ such that $\tau \circ \sigma$ and $\overline{\sigma}$ induce the same map on $\kappa(\mathfrak q \cap M')$. There is a small detail here in that the lemma first guarantees that $\kappa(\mathfrak q \cap M')/\kappa(\mathfrak q \cap M)$ is normal, which then tells us that the difference between the maps is an automorphism of this extension (Fields, Lemma 15.10), to which we can apply the lemma to get τ. Hence $(M', \tau \circ \sigma') > (M, \sigma)$ contradicting the maximality of (M, σ). $\square$

Lemma 110.11. *Let A be a normal domain with fraction field K. Let $M/L/K$ be a tower of (possibly infinite) Galois extensions of K. Let $H = Gal(M/K)$ and $G = Gal(L/K)$ and let C and B be the integral closure of A in M and L. Let $\mathfrak r \subset C$ and $\mathfrak q = B \cap \mathfrak r$. Set $D_{\mathfrak r} = \{\tau \in H \mid \tau(\mathfrak r) = \mathfrak r\}$ and $I_{\mathfrak r} = \{\tau \in D_{\mathfrak r} \mid \tau \bmod \mathfrak r = id_{\kappa(\mathfrak r)}\}$ and similarly for $D_{\mathfrak q}$ and $I_{\mathfrak q}$. Under the map $H \to G$ the induced maps $D_{\mathfrak r} \to D_{\mathfrak q}$ and $I_{\mathfrak r} \to I_{\mathfrak q}$ are surjective.*

Proof. Let $\sigma \in D_{\mathfrak q}$. Pick $\tau \in H$ mapping to σ. This is possible by Fields, Lemma 22.2. Then $\tau(\mathfrak r)$ and $\mathfrak r$ both lie over $\mathfrak q$. Hence by Lemma 110.10 there exists a $\sigma' \in \text{Gal}(M/L)$ with $\sigma'(\tau(\mathfrak r)) = \mathfrak r$. Hence $\sigma'\tau \in D_{\mathfrak r}$ maps to σ. The case of inertia groups is proved in exactly the same way using surjectivity onto automorphism groups. $\square$

111. Extensions of discrete valuation rings

In this section and the next few we use the following definitions.

Definition 111.1. We say that $A \to B$ or $A \subset B$ is an *extension of discrete valuation rings* if A and B are discrete valuation rings and $A \to B$ is injective and local. In particular, if π_A and π_B are uniformizers of A and B, then $\pi_A = u\pi_B^e$ for some $e \geq 1$ and unit u of B. The integer e does not depend on the choice of the uniformizers as it is also the unique integer ≥ 1 such that

$$\mathfrak{m}_A B = \mathfrak{m}_B^e$$

The integer e is called the *ramification index* of B over A. We say that B is *weakly unramified* over A if $e = 1$. If the extension of residue fields $\kappa_A = A/\mathfrak{m}_A \subset \kappa_B = B/\mathfrak{m}_B$ is finite, then we set $f = [\kappa_B : \kappa_A]$ and we call it the *residual degree* or *residue degree* of the extension $A \subset B$.

Note that we do not require the extension of fraction fields to be finite.

Lemma 111.2. *Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If the extension L/K is finite, then the residue field extension is finite and we have $ef \leq [L : K]$.*

Proof. Finiteness of the residue field extension is Algebra, Lemma 119.10. The inequality follows from Algebra, Lemmas 119.9 and 52.12. $\square$

Lemma 111.3. *Let $A \subset B \subset C$ be extensions of discrete valuation rings. Then the ramification indices of B/A and C/B multiply to give the ramification index of C/A. In a formula $e_{C/A} = e_{B/A}e_{C/B}$. Similarly for the residual degrees in case they are finite.*

Proof. This is immediate from the definitions and Fields, Lemma 7.7. $\square$

Lemma 111.4. *Let $A \subset B$ be an extension of discrete valuation rings inducing the field extension $K \subset L$. If the characteristic of K is $p > 0$ and L is purely inseparable over K, then the ramification index e is a power of p.*

Proof. Write $\pi_A = u\pi_B^e$ for some $u \in B^*$. On the other hand, we have $\pi_B^q \in K$ for some p-power q. Write $\pi_B^q = v\pi_A^k$ for some $v \in A^*$ and $k \in \mathbf{Z}$. Then $\pi_A^q = u^q\pi_B^{qe} = u^q v^e \pi_A^{ke}$. Taking valuations in B we conclude that $ke = q$. $\square$

In the following lemma we discuss what it means for an extension $A \subset B$ of discrete valuation rings to be "unramified", i.e., have ramification index 1 and separable (possibly nonalgebraic) extension of residue fields. However, we cannot use the term "unramified" itself because there already exists a notion of an unramified ring map, see Algebra, Section 151.

Lemma 111.5. *Let $A \subset B$ be an extension of discrete valuation rings. The following are equivalent*

1. *$A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology, and*
2. *$A \to B$ is weakly unramified and κ_B/κ_A is a separable field extension.*

Proof. This follows from Proposition 40.5 and Algebra, Proposition 158.9. $\square$

Remark 111.6. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite separable field extension. Let $B \subset L$ be the integral closure of A in L. Picture:

$$\begin{tikzcd} B \arrow[r] & L \\ A \arrow[u] \arrow[r] & K \arrow[u] \end{tikzcd}$$

By Algebra, Lemma 161.8 the ring extension $A \subset B$ is finite, hence B is Noetherian. By Algebra, Lemma 112.4 the dimension of B is 1, hence B is a Dedekind domain, see Algebra, Lemma 120.17. Let $\mathfrak m_1, \ldots, \mathfrak m_n$ be the maximal ideals of B (i.e., the primes lying over $\mathfrak m_A$). We obtain extensions of discrete valuation rings

$$A \subset B_{\mathfrak m_i}$$

and hence ramification indices e_i and residue degrees f_i. We have

$$[L : K] = \sum\nolimits_{i = 1, \ldots, n} e_i f_i$$

by Algebra, Lemma 121.8 applied to a uniformizer in A. We observe that $n = 1$ if A is henselian (by Algebra, Lemma 153.4), e.g. if A is complete.

Definition 111.7. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite separable extension. With B and $\mathfrak m_i$, $i = 1, \ldots, n$ as in Remark 111.6 we say the extension L/K is

1. *unramified with respect to* A if $e_i = 1$ and the extension $\kappa(\mathfrak m_i)/\kappa_A$ is separable for all i,
2. *tamely ramified with respect to* A if either the characteristic of κ_A is 0 or the characteristic of κ_A is $p > 0$, the field extensions $\kappa(\mathfrak m_i)/\kappa_A$ are separable, and the ramification indices e_i are prime to p, and
3. *totally ramified with respect to* A if $n = 1$ and the residue field extension $\kappa(\mathfrak m_1)/\kappa_A$ is trivial.

If the discrete valuation ring A is clear from context, then we sometimes say L/K is unramified, totally ramified, or tamely ramified for short.

For unramified extensions we have the following basic lemma.

Lemma 111.8. *Let A be a discrete valuation ring with fraction field K.*

1. *If $M/L/K$ are finite separable extensions and M is unramified with respect to A, then L is unramified with respect to A.*
2. *If L/K is a finite separable extension which is unramified with respect to A, then there exists a Galois extension M/K containing L which is unramified with respect to A.*
3. *If L_1/K, L_2/K are finite separable extensions which are unramified with respect to A, then there exists a a finite separable extension L/K which is unramified with respect to A containing L_1 and L_2.*

Proof. We will use the results of the discussion in Remark 111.6 without further mention.

Proof of (1). Let $C/B/A$ be the integral closures of A in $M/L/K$. Since C is a finite ring extension of B, we see that $\text{Spec}(C) \to \text{Spec}(B)$ is surjective. Hence for ever maximal ideal $\mathfrak m \subset B$ there is a maximal ideal $\mathfrak m' \subset C$ lying over $\mathfrak m$. By the multiplicativity of ramification indices (Lemma 111.3) and the assumption, we conclude that the ramification index of $B_\mathfrak m$ over A is 1. Since $\kappa(\mathfrak m')/\kappa_A$ is finite separable, the same is true for $\kappa(\mathfrak m)/\kappa_A$.

Proof of (2). Let M be the normal closure of L over K, see Fields, Definition 16.4. Then M/K is Galois by Fields, Lemma 21.5. On the other hand, there is a surjection

$$L \otimes_K \ldots \otimes_K L \longrightarrow M$$

of K-algebras, see Fields, Lemma 16.6. Let B be the integral closure of A in L as in Remark 111.6. The condition that L is unramified with respect to A exactly means that $A \to B$ is an étale ring map, see Algebra, Lemma 143.7. By permanence properties of étale ring maps we see that

$$B \otimes_A \ldots \otimes_A B$$

is étale over A, see Algebra, Lemma 143.3. Hence the displayed ring is a product of Dedekind domains, see Lemma 44.4. We conclude that M is the fraction field of a Dedekind domain finite étale over A. This means that M is unramified with respect to A as desired.

Proof of (3). Let $B_i \subset L_i$ be the integral closure of A. Argue in the same manner as above to show that $B_1 \otimes_A B_2$ is finite étale over A. Details omitted. $\square$

Lemma 111.9. *Let A be a discrete valuation ring with fraction field K. Let $M/L/K$ be finite separable extensions. Let B be the integral closure of A in L. If L/K is unramified with respect to A and M/L is unramified with respect to $B_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of B, then M/K is unramified with respect to A.*

Proof. Let C be the integral closure of A in M. Every maximal ideal $\mathfrak{m}'$ of C lies over a maximal ideal $\mathfrak{m}$ of B. Then the lemma follows from the multiplicativity of ramification indices (Lemma 111.3) and the fact that we have the tower $\kappa(\mathfrak{m}')/\kappa(\mathfrak{m})/\kappa_A$ of finite extensions of fields. $\square$

112. Galois extensions and ramification

In the case of Galois extensions, we can elaborate on the discussion in Section 111.

Lemma 112.1. *Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension with Galois group G. Then G acts on the ring B of Remark 111.6 and acts transitively on the set of maximal ideals of B.*

Proof. Observe that $A = B^G$ as A is integrally closed in K and $K = L^G$. Hence this lemma is a special case of Lemma 110.8. $\square$

Lemma 112.2. *Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension. Then there are $e \geq 1$ and $f \geq 1$ such that $e_i = e$ and $f_i = f$ for all i (notation as in Remark 111.6). In particular $[L : K] = nef$.*

Proof. Immediate consequence of Lemma 112.1 and the definitions. $\square$

Definition 112.3. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak{m} \subset B$ be a maximal ideal.

1. The *decomposition group of* $\mathfrak{m}$ is the subgroup $D = \{\sigma \in G \mid \sigma(\mathfrak{m}) = \mathfrak{m}\}$.
2. The *inertia group of* $\mathfrak{m}$ is the kernel I of the map $D \to \mathrm{Aut}(\kappa(\mathfrak{m})/\kappa_A)$.

Note that the field $\kappa(\mathfrak{m})$ may be inseparable over κ_A. In particular the field extension $\kappa(\mathfrak{m})/\kappa_A$ need not be Galois. If κ_A is perfect, then it is.

Lemma 112.4. *Let A be a discrete valuation ring with fraction field K and residue field κ. Let L/K be a finite Galois extension with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak{m}$ be a maximal ideal of B. Then*

1. *the field extension $\kappa(\mathfrak{m})/\kappa$ is normal, and*

(2) $D \to Aut(\kappa(\mathfrak{m})/\kappa)$ *is surjective.*

If for some (equivalently all) maximal ideal(s) $\mathfrak{m} \subset B$ *the field extension* $\kappa(\mathfrak{m})/\kappa$ *is separable, then*

(3) $\kappa(\mathfrak{m})/\kappa$ *is Galois, and*
(4) $D \to Gal(\kappa(\mathfrak{m})/\kappa)$ *is surjective.*

Here $D \subset G$ *is the decomposition group of* $\mathfrak{m}$.

Proof. Observe that $A = B^G$ as A is integrally closed in K and $K = L^G$. Thus parts (1) and (2) follow from Lemma 110.9. The "equivalently all" part of the lemma follows from Lemma 112.1. Assume $\kappa(\mathfrak{m})/\kappa$ is separable. Then parts (3) and (4) follow immediately from (1) and (2). $\square$

Lemma 112.5. *Let* A *be a discrete valuation ring with fraction field* K. *Let* L/K *be a finite Galois extension with Galois group* G. *Let* B *be the integral closure of* A *in* L. *Let* $\mathfrak{m} \subset B$ *be a maximal ideal. The inertia group* I *of* $\mathfrak{m}$ *sits in a canonical exact sequence*

$$1 \to P \to I \to I_t \to 1$$

such that

(1) $P = \{\sigma \in D \mid \sigma|_{B/\mathfrak{m}^2} = id_{B/\mathfrak{m}^2}\}$ *where* D *is the decomposition group,*
(2) P *is a normal subgroup of* D,
(3) P *is a* p*-group if the characteristic of* κ_A *is* $p > 0$ *and* $P = \{1\}$ *if the characteristic of* κ_A *is zero,*
(4) I_t *is cyclic of order the prime to* p *part of the integer* e, *and*
(5) *there is a canonical isomorphism* $\theta : I_t \to \mu_e(\kappa(\mathfrak{m}))$.

Here e *is the integer of Lemma 112.2.*

Proof. Recall that $|G| = [L : K] = nef$, see Lemma 112.2. Since G acts transitively on the set $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ of maximal ideals of B (Lemma 112.1) and since D is the stabilizer of an element we see that $|D| = ef$. By Lemma 112.4 we have

$$ef = |D| = |I| \cdot |\mathrm{Aut}(\kappa(\mathfrak{m})/\kappa)|$$

where κ is the residue field of A. As $\kappa(\mathfrak{m})$ is normal over κ the order of $\mathrm{Aut}(\kappa(\mathfrak{m})/\kappa)$ differs from f by a power of p (see Fields, Lemma 15.9 and discussion following Fields, Definition 14.7). Hence the prime to p part of $|I|$ is equal to the prime to p part of e.

Set $C = B_\mathfrak{m}$. Then I acts on C over A and trivially on the residue field of C. Let $\pi_A \in A$ and $\pi_C \in C$ be uniformizers. Write $\pi_A = u\pi_C^e$ for some unit u in C. For $\sigma \in I$ write $\sigma(\pi_C) = \theta_\sigma \pi_C$ for some unit θ_σ in C. Then we have

$$\pi_A = \sigma(\pi_A) = \sigma(u)(\theta_\sigma \pi_C)^e = \sigma(u)\theta_\sigma^e \pi_C^e = \frac{\sigma(u)}{u}\theta_\sigma^e \pi_A$$

Since $\sigma(u) \equiv u \bmod \mathfrak{m}_C$ as $\sigma \in I$ we see that the image $\overline{\theta}_\sigma$ of θ_σ in $\kappa_C = \kappa(\mathfrak{m})$ is an eth root of unity. We obtain a map

$$\theta : I \longrightarrow \mu_e(\kappa(\mathfrak{m})), \quad \sigma \mapsto \overline{\theta}_\sigma \tag{112.5.1}$$

We claim that θ is a homomorphism of groups and independent of the choice of uniformizer π_C. Namely, if τ is a second element of I, then $\tau(\sigma(\pi_C)) = \tau(\theta_\sigma \pi_C) = \tau(\theta_\sigma)\theta_\tau \pi_C$, hence $\theta_{\tau\sigma} = \tau(\theta_\sigma)\theta_\tau$ and since $\tau \in I$ we conclude that $\overline{\theta}_{\tau\sigma} = \overline{\theta}_\sigma \overline{\theta}_\tau$. If π'_C is a second uniformizer, then we see that $\pi'_C = w\pi_C$ for some unit w of C and

$\sigma(\pi'_C) = w^{-1}\sigma(w)\theta_\sigma\pi'_C$, hence $\theta'_\sigma = w^{-1}\sigma(w)\theta_\sigma$, hence θ'_σ and θ_σ map to the same element of the residue field as before.

Since $\kappa(\mathfrak{m})$ has characteristic p, the group $\mu_e(\kappa(\mathfrak{m}))$ is cyclic of order at most the prime to p part of e (see Fields, Section 17).

Let $P = \text{Ker}(\theta)$. The elements of P are exactly the elements of D acting trivially on $C/\pi_C^2 C \cong B/\mathfrak{m}^2$. Thus (a) is true. This implies (b) as P is the kernel of the map $D \to \text{Aut}(B/\mathfrak{m}^2)$. If we can prove (c), then parts (d) and (e) will follow as I_t will be isomorphic to $\mu_e(\kappa(\mathfrak{m}))$ as the arguments above show that $|I_t| \geq |\mu_e(\kappa(\mathfrak{m}))|$.

Thus it suffices to prove that the kernel P of θ is a p-group. Let σ be a nontrivial element of the kernel. Then $\sigma - \text{id}$ sends $\mathfrak{m}_C^i$ into $\mathfrak{m}_C^{i+1}$ for all i. Let m be the order of σ. Pick $c \in C$ such that $\sigma(c) \neq c$. Then $\sigma(c) - c \in \mathfrak{m}_C^i$, $\sigma(c) - c \notin \mathfrak{m}_C^{i+1}$ for some i and we have

$$\begin{aligned} 0 &= \sigma^m(c) - c \\ &= \sigma^m(c) - \sigma^{m-1}(c) + \ldots + \sigma(c) - c \\ &= \sum\nolimits_{j=0,\ldots,m-1} \sigma^j(\sigma(c) - c) \\ &\equiv m(\sigma(c) - c) \bmod \mathfrak{m}_C^{i+1} \end{aligned}$$

It follows that $p|m$ (or $m = 0$ if $p = 1$). Thus every element of the kernel of θ has order divisible by p, i.e., $\text{Ker}(\theta)$ is a p-group. □

Definition 112.6. With assumptions and notation as in Lemma 112.5.

(1) The *wild inertia group of* $\mathfrak{m}$ is the subgroup P.
(2) The *tame inertia group of* $\mathfrak{m}$ is the quotient $I \to I_t$.

We denote $\theta : I \to \mu_e(\kappa(\mathfrak{m}))$ the surjective map (112.5.1) whose kernel is P and which induces the isomorphism $I_t \to \mu_e(\kappa(\mathfrak{m}))$.

Lemma 112.7. *With assumptions and notation as in Lemma 112.5. The inertia character $\theta : I \to \mu_e(\kappa(\mathfrak{m}))$ satisfies the following property*

$$\theta(\tau\sigma\tau^{-1}) = \tau(\theta(\sigma))$$

for $\tau \in D$ and $\sigma \in I$.

Proof. The formula makes sense as I is a normal subgroup of D and as τ acts on $\kappa(\mathfrak{m})$ via the map $D \to \text{Aut}(\kappa(\mathfrak{m}))$ discussed in Lemma 112.4 for example. Recall the construction of θ. Choose a uniformizer π of $B_\mathfrak{m}$ and for $\sigma \in I$ write $\sigma(\pi) = \theta_\sigma\pi$. Then $\theta(\sigma)$ is the image $\overline{\theta}_\sigma$ of θ_σ in the residue field. For any $\tau \in D$ we can write $\tau(\pi) = \theta_\tau\pi$ for some unit θ_τ. Then $\theta_{\tau^{-1}} = \tau^{-1}(\theta_\tau^{-1})$. We compute

$$\begin{aligned} \theta_{\tau\sigma\tau^{-1}} &= \tau(\sigma(\tau^{-1}(\pi)))/\pi \\ &= \tau(\sigma(\tau^{-1}(\theta_\tau^{-1})\pi))/\pi \\ &= \tau(\sigma(\tau^{-1}(\theta_\tau^{-1}))\theta_\sigma\pi)/\pi \\ &= \tau(\sigma(\tau^{-1}(\theta_\tau^{-1})))\tau(\theta_\sigma)\theta_\tau \end{aligned}$$

However, since σ acts trivially modulo π we see that the product $\tau(\sigma(\tau^{-1}(\theta_\tau^{-1})))\theta_\tau$ maps to 1 in the residue field. This proves the lemma. □

We will generalize the following lemma in Fundamental Groups, Lemma 12.5.

Lemma 112.8. *Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension. Let $\mathfrak m \subset B$ be a maximal ideal of the integral closure of A in L. Let $I \subset G$ be the inertia group of $\mathfrak m$. Then B^I is the integral closure of A in L^I and $A \to (B^I)_{B^I \cap \mathfrak m}$ is étale.*

Proof. Write $B' = B^I$. It follows from the definitions that $B' = B^I$ is the integral closure of A in L^I. Write $\mathfrak m' = B^I \cap \mathfrak m = B' \cap \mathfrak m \subset B'$. By Lemma 110.8 the maximal ideal $\mathfrak m$ is the unique prime ideal of B lying over $\mathfrak m'$. As I acts trivially on $\kappa(\mathfrak m)$ we see from Lemma 110.2 that the extension $\kappa(\mathfrak m)/\kappa(\mathfrak m')$ is purely inseparable (perhaps an easier alternative is to apply the result of Lemma 110.9). Since D/I acts faithfully on $\kappa(\mathfrak m')$, we conclude that D/I acts faithfully on $\kappa(\mathfrak m)$. Of course the elements of the residue field κ of A are fixed by this action. By Galois theory we see that $[\kappa(\mathfrak m') : \kappa] \geq |D/I|$, see Fields, Lemma 21.6.

Let π be the uniformizer of A. Since $\text{Norm}_{L/K}(\pi) = \pi^{[L:K]}$ we see from Algebra, Lemma 121.8 that

$$|G| = [L : K] = [L : K] \text{ ord}_A(\pi) = |G/D| \; [\kappa(\mathfrak m) : \kappa] \text{ ord}_{B_\mathfrak m}(\pi)$$

as there are $n = |G/D|$ maximal ideals of B which are all conjugate under G, see Remark 111.6 and Lemma 112.1. Applying the same reasoning to the finite extension the finite extension L/L^I of degree $|I|$ we find

$$|I| \text{ ord}_{B'_{\mathfrak m'}}(\pi) = [\kappa(\mathfrak m) : \kappa(\mathfrak m')] \text{ ord}_{B_\mathfrak m}(\pi)$$

We conclude that

$$\text{ord}_{B'_{\mathfrak m'}}(\pi) = \frac{|D/I|}{[\kappa(\mathfrak m') : \kappa]}$$

Since the left hand side is a positive integer and since the right hand side is ≤ 1 by the above, we conclude that we have equality, $\text{ord}_{B'_{\mathfrak m'}}(\pi) = 1$ and $\kappa(\mathfrak m')/\kappa$ has degree $|D/I|$. Thus $\pi B'_{\mathfrak m'} = \mathfrak m' B'_\mathfrak m$ and $\kappa(\mathfrak m')$ is Galois over κ with Galois group D/I, in particular separable, see Fields, Lemma 21.2. By Algebra, Lemma 143.7 we find that $A \to B'_{\mathfrak m'}$ is étale as desired. □

Remark 112.9. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension. Let $\mathfrak m \subset B$ be a maximal ideal of the integral closure of A in L. Let

$$P \subset I \subset D \subset G$$

be the wild inertia, inertia, decomposition group of $\mathfrak m$. Consider the diagram

$$\begin{tikzcd}
\mathfrak m \arrow[r, dash] \arrow[d, dash] & \mathfrak m^P \arrow[r, dash] \arrow[d, dash] & \mathfrak m^I \arrow[r, dash] \arrow[d, dash] & \mathfrak m^D \arrow[r, dash] \arrow[d, dash] & A \cap \mathfrak m \arrow[d, dash] \\
B & B^P \arrow[l] & B^I \arrow[l] & B^D \arrow[l] & A \arrow[l]
\end{tikzcd}$$

Observe that B^P, B^I, B^D are the integral closures of A in the fields L^P, L^I, L^D. Thus we also see that B^P is the integral closure of B^I in L^P and so on. Observe that $\mathfrak m^P = \mathfrak m \cap B^P$, $\mathfrak m^I = \mathfrak m \cap B^I$, and $\mathfrak m^D = \mathfrak m \cap B^D$. Hence the top line of the diagram corresponds to the images of $\mathfrak m \in \text{Spec}(B)$ under the induced maps of spectra. Having said all of this we have the following

1. the extension L^I/L^D is Galois with group D/I,
2. the extension L^P/L^I is Galois with group $I_t = I/P$,
3. the extension L^P/L^D is Galois with group D/P,

(4) $\mathfrak{m}^I$ is the unique prime of B^I lying over $\mathfrak{m}^D$,
(5) $\mathfrak{m}^P$ is the unique prime of B^P lying over $\mathfrak{m}^I$,
(6) $\mathfrak{m}$ is the unique prime of B lying over $\mathfrak{m}^P$,
(7) $\mathfrak{m}^P$ is the unique prime of B^P lying over $\mathfrak{m}^D$,
(8) $\mathfrak{m}$ is the unique prime of B lying over $\mathfrak{m}^I$,
(9) $\mathfrak{m}$ is the unique prime of B lying over $\mathfrak{m}^D$,
(10) $A \to B^D_{\mathfrak{m}^D}$ is étale and induces a trivial residue field extension,
(11) $B^D_{\mathfrak{m}^D} \to B^I_{\mathfrak{m}^I}$ is étale and induces a Galois extension of residue fields with Galois group D/I,
(12) $A \to B^I_{\mathfrak{m}^I}$ is étale,
(13) $B^I_{\mathfrak{m}^I} \to B^P_{\mathfrak{m}^P}$ has ramification index $|I/P|$ prime to p and induces a trivial residue field extension,
(14) $B^D_{\mathfrak{m}^D} \to B^P_{\mathfrak{m}^P}$ has ramification index $|I/P|$ prime to p and induces a separable residue field extension,
(15) $A \to B^P_{\mathfrak{m}^P}$ has ramification index $|I/P|$ prime to p and induces a separable residue field extension.

Statements (1), (2), and (3) are immediate from Galois theory (Fields, Section 21) and Lemma 112.5. Statements (4) – (9) are clear from Lemma 112.1. Part (12) is Lemma 112.8. Since we have the factorization $A \to B^D_{\mathfrak{m}^D} \to B^I_{\mathfrak{m}^I}$ we obtain the étaleness in (10) and (11) as a consequence. The residue field extension in (10) must be trivial because it is separable and D/I maps onto $\text{Aut}(\kappa(\mathfrak{m})/\kappa_A)$ as shown in Lemma 112.4. The same argument provides the proof of the statement on residue fields in (11). To see (13), (14), and (15) it suffices to prove (13). By the above, the extension L^P/L^I is Galois with a cyclic Galois group of order prime to p, the prime $\mathfrak{m}^P$ is the unique prime lying over $\mathfrak{m}^I$ and the action of I/P on the residue field is trivial. Thus we can apply Lemma 112.5 to this extension and the discrete valuation ring $B^I_{\mathfrak{m}^I}$ to see that (13) holds.

Lemma 112.10. *Let A be a discrete valuation ring with fraction field K. Let $M/L/K$ be a tower with M/K and L/K finite Galois. Let C, B be the integral closure of A in M, L. Let $\mathfrak{m}' \subset C$ be a maximal ideal and set $\mathfrak{m} = \mathfrak{m}' \cap B$. Let*

$$P \subset I \subset D \subset \mathit{Gal}(L/K) \quad \textit{and} \quad P' \subset I' \subset D' \subset \mathit{Gal}(M/K)$$

be the wild inertia, inertia, decomposition group of $\mathfrak{m}$ and $\mathfrak{m}'$. Then the canonical surjection $\mathit{Gal}(M/K) \to \mathit{Gal}(L/K)$ induces surjections $P' \to P$, $I' \to I$, and $D' \to D$. Moreover these fit into commutative diagrams

$$\begin{tikzcd} D' \arrow[r] \arrow[d] & Aut(\kappa(\mathfrak{m}')/\kappa_A) \arrow[d] \\ D \arrow[r] & Aut(\kappa(\mathfrak{m})/\kappa_A) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} I' \arrow[r, "\theta'"] \arrow[d] & \mu_{e'}(\kappa(\mathfrak{m}')) \arrow[d, "(-)^{e'/e}"] \\ I \arrow[r, "\theta"] & \mu_e(\kappa(\mathfrak{m})) \end{tikzcd}$$

where e' and e are the ramification indices of $A \to C_{\mathfrak{m}'}$ and $A \to B_{\mathfrak{m}}$.

Proof. The fact that under the map $\text{Gal}(M/K) \to \text{Gal}(L/K)$ the groups P', I', D' map into P, I, D is immediate from the definitions of these groups. The commutativity of the first diagram is clear (observe that since $\kappa(\mathfrak{m})/\kappa_A$ is normal every automorphism of $\kappa(\mathfrak{m}')$ over κ_A indeed induces an automorphism of $\kappa(\mathfrak{m})$ over κ_A and hence we obtain the right vertical arrow in the first diagram, see Lemma 112.4 and Fields, Lemma 15.7).

The maps $I' \to I$ and $D' \to D$ are surjective by Lemma 110.11. The surjectivity of $P' \to P$ follows as P' and P are p-Sylow subgroups of I' and I.

To see the commutativity of the second diagram we choose a uniformizer π' of $C_{\mathfrak{m}'}$ and a uniformizer π of $B_\mathfrak{m}$. Then $\pi = c'(\pi')^{e'/e}$ for some unit c' of $C_{\mathfrak{m}'}$. For $\sigma' \in I'$ the image $\sigma \in I$ is simply the restriction of σ' to L. Write $\sigma'(\pi') = c\pi'$ for a unit $c \in C_{\mathfrak{m}'}$ and write $\sigma(\pi) = b\pi$ for a unit b of $B_\mathfrak{m}$. Then $\sigma'(\pi) = b\pi$ and we obtain

$$b\pi = \sigma'(\pi) = \sigma'(c'(\pi')^{e'/e}) = \sigma'(c')c^{e'/e}(\pi')^{e'/e} = \frac{\sigma'(c')}{c'}c^{e'/e}\pi$$

As $\sigma' \in I'$ we see that b and $c^{e'/e}$ have the same image in the residue field which proves what we want. □

Remark 112.11. In order to use the inertia character $\theta : I \to \mu_e(\kappa(\mathfrak{m}))$ for infinite Galois extensions, it is convenient to scale it. Let $A, K, L, B, \mathfrak{m}, G, P, I, D, e, \theta$ be as in Lemma 112.5 and Definition 112.6. Then $e = q|I_t|$ with q is a power of the characteristic p of $\kappa(\mathfrak{m})$ if positive or 1 if zero. Note that $\mu_e(\kappa(\mathfrak{m})) = \mu_{|I_t|}(\kappa(\mathfrak{m}))$ because the characteristic of $\kappa(\mathfrak{m})$ is p. Consider the map

$$\theta_{can} = q\theta : I \longrightarrow \mu_{|I_t|}(\kappa(\mathfrak{m}))$$

This map induces an isomorphism $\theta_{can} : I_t \to \mu_{|I_t|}(\kappa(\mathfrak{m}))$. We have $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$ and $\sigma \in I$ by Lemma 112.7. Finally, if M/L is an extension such that M/K is Galois and $\mathfrak{m}'$ is a prime of the integral closure of A in M lying over $\mathfrak{m}$, then we get the commutative diagram

$$\begin{tikzcd} I' \arrow[r, "\theta'_{can}"'] \arrow[d] & \mu_{|I'_t|}(\kappa(\mathfrak{m}')) \arrow[d, "(-)^{|I'_t|/|I_t|}"] \\ I \arrow[r, "\theta_{can}"] & \mu_{|I_t|}(\kappa(\mathfrak{m})) \end{tikzcd}$$

by Lemma 112.10.

113. Krasner's lemma

Here is Krasner's lemma in the case of discretely valued fields.

Lemma 113.1 (Krasner's lemma)**.** *Let A be a complete local domain of dimension 1. Let $P(t) \in A[t]$ be a polynomial with coefficients in A. Let $\alpha \in A$ be a root of P but not a root of the derivative $P' = dP/dt$. For every $c \geq 0$ there exists an integer n such that for any $Q \in A[t]$ whose coefficients are in $\mathfrak{m}_A^n$ the polynomial $P + Q$ has a root $\beta \in A$ with $\beta - \alpha \in \mathfrak{m}_A^c$.*

Proof. Choose a nonzero $\pi \in \mathfrak{m}$. Since the dimension of A is 1 we have $\mathfrak{m} = \sqrt{(\pi)}$. By assumption we may write $P'(\alpha)^{-1} = \pi^{-m}a$ for some $m \geq 0$ and $a \in A$. We may and do assume that $c \geq m + 1$. Pick n such that $\mathfrak{m}_A^n \subset (\pi^{c+m})$. Pick any Q as in the statement. For later use we observe that we can write

$$P(x + y) = P(x) + P'(x)y + R(x, y)y^2$$

for some $R(x, y) \in A[x, y]$. We will show by induction that we can find a sequence $\alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots$ such that

(1) $\alpha_k \equiv \alpha \bmod \pi^c$,
(2) $\alpha_{k+1} - \alpha_k \in (\pi^k)$, and
(3) $(P + Q)(\alpha_k) \in (\pi^{m+k})$.

Setting $\beta = \lim \alpha_k$ will finish the proof.

Base case. Since the coefficients of Q are in (π^{c+m}) we have $(P+Q)(\alpha) \in (\pi^{c+m})$. Hence $\alpha_m = \alpha$ works. This choice guarantees that $\alpha_k \equiv \alpha \bmod \pi^c$ for all $k \geq m$.

Induction step. Given α_k we write $\alpha_{k+1} = \alpha_k + \delta$ for some $\delta \in (\pi^k)$. Then we have

$$(P+Q)(\alpha_{k+1}) = P(\alpha_k + \delta) + Q(\alpha_k + \delta)$$

Because the coefficients of Q are in (π^{c+m}) we see that $Q(\alpha_k + \delta) \equiv Q(\alpha_k) \bmod \pi^{c+m+k}$. On the other hand we have

$$P(\alpha_k + \delta) = P(\alpha_k) + P'(\alpha_k)\delta + R(\alpha_k, \delta)\delta^2$$

Note that $P'(\alpha_k) \equiv P'(\alpha) \bmod (\pi^{m+1})$ as $\alpha_k \equiv \alpha \bmod \pi^{m+1}$. Hence we obtain

$$P(\alpha_k + \delta) \equiv P(\alpha_k) + P'(\alpha)\delta \bmod \pi^{k+m+1}$$

Recombining the two terms we see that

$$(P+Q)(\alpha_{k+1}) \equiv (P+Q)(\alpha_k) + P'(\alpha)\delta \bmod \pi^{k+m+1}$$

Thus a solution is to take $\delta = -P'(\alpha)^{-1}(P+Q)(\alpha_k) = -\pi^{-m}a(P+Q)(\alpha_k)$ which is contained in (π^k) by induction assumption. $\square$

Lemma 113.2. *Let A be a discrete valuation ring with field of fractions K. Let $A^\wedge$ be the completion of A with fraction field $K^\wedge$. If $M/K^\wedge$ is a finite separable extension, then there exists a finite separable extension L/K such that $M = K^\wedge \otimes_K L$.*

Proof. Note that $A^\wedge$ is a discrete valuation ring too (by Lemmas 43.4 and 43.1). In particular $A^\wedge$ is a domain. The proof will work more generally for Noetherian local rings A such that $A^\wedge$ is a local domain of dimension 1.

Let $\theta \in M$ be an element that generates M over $K^\wedge$. (Theorem of the primitive element.) Let $P(t) \in K^\wedge[t]$ be the minimal polynomial of θ over $K^\wedge$. Let $\pi \in \mathfrak{m}_A$ be a nonzero element. After replacing θ by $\pi^n\theta$ we may assume that the coefficients of $P(t)$ are in $A^\wedge$. Let $B = A^\wedge[\theta] = A^\wedge[t]/(P(t))$. Note that B is a complete local domain of dimension 1 because it is finite over A and contained in M. Since M is separable over K the element θ is not a root of the derivative of P. For any integer n we can find a monic polynomial $P_1 \in A[t]$ such that $P - P_1$ has coefficients in $\pi^n A^\wedge[t]$. By Krasner's lemma (Lemma 113.1) we see that P_1 has a root β in B for n sufficiently large. Moreover, we may assume (if n is chosen large enough) that $\theta - \beta \in \pi B$. Consider the map $\Phi : A^\wedge[t]/(P_1) \to B$ of $A^\wedge$-algebras which maps t to β. Since $B = \pi B + \sum_{i < \deg(P)} A^\wedge \theta^i$, the map Φ is surjective by Nakayama's lemma. As $\deg(P_1) = \deg(P)$ it follows that Φ is an isomorphism. We conclude that the ring extension $L = K[t]/(P_1(t))$ satisfies $K^\wedge \otimes_K L \cong M$. This implies that L is a field and the proof is complete. $\square$

Definition 113.3. Let A be a discrete valuation ring. We say A has *mixed characteristic* if the characteristic of the residue field of A is $p > 0$ and the characteristic of the fraction field of A is 0. In this case we obtain an extension of discrete valuation rings $\mathbf{Z}_{(p)} \subset A$ and the *absolute ramification index* of A is the ramification index of this extension.

114. Abhyankar's lemma and tame ramification

In this section we prove what we think is the most general version of Abhyankar's lemma for discrete valuation rings. After doing so, we apply this to prove some results about tamely ramified extensions of the fraction field of a discrete valuation ring.

Remark 114.1. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Let K_1/K be a finite extension of fields. Let $A_1 \subset K_1$ be the integral closure of A in K_1. On the other hand, let $L_1 = (L \otimes_K K_1)_{red}$. Then L_1 is a nonempty finite product of finite field extensions of L. Let B_1 be the integral closure of B in L_1. We obtain compatible commutative diagrams

$$\begin{tikzcd} L \arrow[r] & L_1 \\ K \arrow[u] \arrow[r] & K_1 \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r] & B_1 \\ A \arrow[u] \arrow[r] & A_1 \arrow[u] \end{tikzcd}$$

In this situation we have the following

(1) By Algebra, Lemma 120.18 the ring A_1 is a Dedekind domain and B_1 is a finite product of Dedekind domains.
(2) Note that $L \otimes_K K_1 = (B \otimes_A A_1)_\pi$ where $\pi \in A$ is a uniformizer and that π is a nonzerodivisor on $B \otimes_A A_1$. Thus the ring map $B \otimes_A A_1 \to B_1$ is integral with kernel consisting of nilpotent elements. Hence $\text{Spec}(B_1) \to \text{Spec}(B \otimes_A A_1)$ is surjective on spectra (Algebra, Lemma 36.17). The map $\text{Spec}(B \otimes_A A_1) \to \text{Spec}(A_1)$ is surjective as $A_1/\mathfrak{m}_A A_1 \to B/\mathfrak{m}_A B \otimes_{\kappa_A} A_1/\mathfrak{m}_A A_1$ is an injective ring map with $A_1/\mathfrak{m}_A A_1$ Artinian. We conclude that $\text{Spec}(B_1) \to \text{Spec}(A_1)$ is surjective.
(3) Let $\mathfrak{m}_i$, $i = 1, \ldots n$ with $n \geq 1$ be the maximal ideals of A_1. For each $i = 1, \ldots, n$ let $\mathfrak{m}_{ij}$, $j = 1, \ldots, m_i$ with $m_i \geq 1$ be the maximal ideals of B_1 lying over $\mathfrak{m}_i$. We obtain diagrams

$$\begin{tikzcd} B \arrow[r] & (B_1)_{\mathfrak{m}_{ij}} \\ A \arrow[u] \arrow[r] & (A_1)_{\mathfrak{m}_i} \arrow[u] \end{tikzcd}$$

of extensions of discrete valuation rings.
(4) If A is henselian (for example complete), then A_1 is a discrete valuation ring, i.e., $n = 1$. Namely, A_1 is a union of finite extensions of A which are domains, hence local by Algebra, Lemma 153.4.
(5) If B is henselian (for example complete), then B_1 is a product of discrete valuation rings, i.e., $m_i = 1$ for $i = 1, \ldots, n$.
(6) If $K \subset K_1$ is purely inseparable, then A_1 and B_1 are both discrete valuation rings, i.e., $n = 1$ and $m_1 = 1$. This is true because for every $b \in B_1$ a p-power power of b is in B, hence B_1 can only have one maximal ideal.
(7) If $K \subset K_1$ is finite separable, then $L_1 = L \otimes_K K_1$ and is a finite product of finite separable extensions too. Hence $A \subset A_1$ and $B \subset B_1$ are finite by Algebra, Lemma 161.8.
(8) If A is Nagata, then $A \subset A_1$ is finite.
(9) If B is Nagata, then $B \subset B_1$ is finite.

Lemma 114.2. *Let A be a discrete valuation ring with uniformizer π. Let $n \geq 2$. Then $K_1 = K[\pi^{1/n}]$ is a degree n extension of K and the integral closure A_1 of A in K_1 is the ring $A[\pi^{1/n}]$ which is a discrete valuation ring with ramification index n over A.*

Proof. This lemma proves itself. □

Lemma 114.3. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume that $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then for any finite extension K_1/K we have $L_1 = L \otimes_K K_1$, $B_1 = B \otimes_A A_1$, and each extension $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ (see Remark 114.1) is formally smooth in the $\mathfrak{m}_{ij}$-adic topology.*

Proof. We will use the equivalence of Lemma 111.5 without further mention. Let $\pi \in A$ and $\pi_i \in (A_1)_{\mathfrak{m}_i}$ be uniformizers. As $\kappa_A \subset \kappa_B$ is separable, the ring

$$(B \otimes_A (A_1)_{\mathfrak{m}_i})/\pi_i(B \otimes_A (A_1)_{\mathfrak{m}_i}) = B/\pi B \otimes_{A/\pi A} (A_1)_{\mathfrak{m}_i}/\pi_i(A_1)_{\mathfrak{m}_i}$$

is a product of fields each separable over $\kappa_{\mathfrak{m}_i}$. Hence the element π_i in $B \otimes_A (A_1)_{\mathfrak{m}_i}$ is a nonzerodivisor and the quotient by this element is a product of fields. It follows that $B \otimes_A A_1$ is a Dedekind domain in particular reduced. Thus $B \otimes_A A_1 \subset B_1$ is an equality. □

The following lemma is our version of Abhyankar's lemma for discrete valuation rings. Observe that κ_B/κ_A is not assumed to be an algebraic extension of fields.

Lemma 114.4 (Abhyankar's lemma)**.** *Let $A \subset B$ be an extension of discrete valuation rings. Assume that either the residue characteristic of A is 0 or it is p, the ramification index e is prime to p, and κ_B/κ_A is a separable field extension. Let K_1/K be a finite extension. Using the notation of Remark 114.1 assume e divides the ramification index of $A \subset (A_1)_{\mathfrak{m}_i}$ for some i. Then $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ is formally smooth in the $\mathfrak{m}_{ij}$-adic topology for all $j = 1, \ldots, m_i$.*

Proof. Let $\pi \in A$ be a uniformizer. Let π_1 be a uniformizer of $(A_1)_{\mathfrak{m}_i}$. Write $\pi = u\pi_1^{e_1}$ with u a unit of $(A_1)_{\mathfrak{m}_i}$ and e_1 the ramification index of $A \subset (A_1)_{\mathfrak{m}_i}$.

Claim: we may assume that u is an eth power in K_1. Namely, let K_2 be an extension of K_1 obtained by adjoining a root of $x^e = u$; thus K_2 is a factor of $K_1[x]/(x^e - u)$. Then K_2/K_1 is a finite separable extension (by our assumption on e) and hence $A_1 \subset A_2$ is finite. Since $(A_1)_{\mathfrak{m}_i} \to (A_1)_{\mathfrak{m}_i}[x]/(x^e - u)$ is finite étale (as e is prime to the residue characteristic and u a unit) we conclude that $(A_2)_{\mathfrak{m}_i}$ is a factor of a finite étale extension of $(A_1)_{\mathfrak{m}_i}$ hence finite étale over $(A_1)_{\mathfrak{m}_i}$ itself. The same reasoning shows that $B_1 \subset B_2$ induces finite étale extensions $(B_1)_{\mathfrak{m}_{ij}} \subset (B_2)_{\mathfrak{m}_{ij}}$. Pick a maximal ideal $\mathfrak{m}'_{ij} \subset B_2$ lying over $\mathfrak{m}_{ij} \subset B_1$ (of course there may be more than one) and consider

$$\begin{tikzcd}
(B_1)_{\mathfrak{m}_{ij}} \arrow[r] & (B_2)_{\mathfrak{m}'_{ij}} \\
(A_1)_{\mathfrak{m}_i} \arrow[r] \arrow[u] & (A_2)_{\mathfrak{m}'_i} \arrow[u]
\end{tikzcd}$$

where $\mathfrak{m}'_i \subset A_2$ is the image. Now the horizontal arrows have ramification index 1 and induce finite separable residue field extensions. Thus, using the equivalence of

Lemma 111.5, we see that it suffices to show that the right vertical arrow is formally smooth in the $\mathfrak{m}'_{ij}$-adic topology. Since u has a eth root in K_2 we obtain the claim.

Assume u has an eth root in K_1. Since $e|e_1$ and since u has a eth root in K_1 we see that $\pi = \theta^e$ for some $\theta \in K_1$. Let $K'_1 = K[\theta] \subset K_1$ be the subfield generated by θ. By Lemma 114.2 the integral closure A'_1 of A in $K[\theta]$ is the discrete valuation ring $A'_1 = A[\theta]$ which has ramification index e over A. If we can prove the lemma for the extension K'_1/K, then we conclude by Lemma 114.3 applied to the diagram

$$\begin{tikzcd} (B'_1)_{B'_1 \cap \mathfrak{m}_{ij}} \arrow[r] & (B_1)_{\mathfrak{m}_{ij}} \\ A'_1 \arrow[r] \arrow[u] & (A_1)_{\mathfrak{m}_i} \arrow[u] \end{tikzcd}$$

for all $j = 1, \ldots, m_i$. This reduces us to the case discussed in the next paragraph.

Assume $K_1 = K[\pi^{1/e}]$ and set $\theta = \pi^{1/e}$. Let π_B be a uniformizer for B and write $\pi = w\pi_B^e$ for some unit w of B. Then we see that $L_1 = L \otimes_K K_1$ is obtained by adjoining π_B/θ which is an eth root of the unit w. Thus $B \subset B_1$ is finite étale. Thus for any maximal ideal $\mathfrak{m} \subset B_1$ consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "1"'] & (B_1)_{\mathfrak{m}} \\ A \arrow[r, "e"] \arrow[u, "e"] & A_1 \arrow[u, "e_{\mathfrak{m}}"'] \end{tikzcd}$$

Here the numbers along the arrows are the ramification indices. By multiplicativity of ramification indices (Lemma 111.3) we conclude $e_{\mathfrak{m}} = 1$. Looking at the residue field extensions we find that $\kappa(\mathfrak{m})$ is a finite separable extension of κ_B which is separable over κ_A. Therefore $\kappa(\mathfrak{m})$ is separable over κ_A which is equal to the residue field of A_1 and we win by Lemma 111.5. $\square$

Lemma 114.5. *Let A be a discrete valuation ring with fraction field K. Let $M/L/K$ be finite separable extensions. Let B be the integral closure of A in L. If L/K is tamely ramified with respect to A and M/L is tamely ramified with respect to $B_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of B, then M/K is tamely ramified with respect to A.*

Proof. Let C be the integral closure of A in M. Every maximal ideal $\mathfrak{m}'$ of C lies over a maximal ideal $\mathfrak{m}$ of B. Then the lemma follows from the multiplicativity of ramification indices (Lemma 111.3) and the fact that we have the tower $\kappa(\mathfrak{m}')/\kappa(\mathfrak{m})/\kappa_A$ of finite extensions of fields. $\square$

Lemma 114.6. *Let A be a discrete valuation ring with fraction field K. If $M/L/K$ are finite separable extensions and M is tamely ramified with respect to A, then L is tamely ramified with respect to A.*

Proof. We will use the results of the discussion in Remark 111.6 without further mention. Let $C/B/A$ be the integral closures of A in $M/L/K$. Since C is a finite ring extension of B, we see that $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ is surjective. Hence for ever maximal ideal $\mathfrak{m} \subset B$ there is a maximal ideal $\mathfrak{m}' \subset C$ lying over $\mathfrak{m}$. By the multiplicativity of ramification indices (Lemma 111.3) and the assumption, we conclude that the ramification index of $B_{\mathfrak m}$ over A is prime to the residue characteristic. Since $\kappa(\mathfrak m')/\kappa_A$ is finite separable, the same is true for $\kappa(\mathfrak m)/\kappa_A$. □

Lemma 114.7. *Let A be a discrete valuation ring with fraction field K. Let $\pi \in A$ be a uniformizer. Let L/K be a finite separable extension. The following are equivalent*

1. *L is tamely ramified with respect to A,*
2. *there exists an $e \geq 1$ invertible in κ_A and an extension $L'/K' = K[\pi^{1/e}]$ unramified with respect to $A' = A[\pi^{1/e}]$ such that L is contained in L', and*
3. *there exists an $e_0 \geq 1$ invertible in κ_A such that for every $d \geq 1$ invertible in κ_A (2) holds with $e = de_0$.*

Proof. Observe that A' is a discrete valuation ring with fraction field K', see Lemma 114.2. Of course the ramification index of A' over A is e. Thus if (2) holds, then L' is tamely ramified with respect to A by Lemma 114.5. Hence L is tamely ramified with respect to A by Lemma 114.6.

The implication (3) ⇒ (2) is immediate.

Assume that (1) holds. Let B be the integral closure of A in L and let $\mathfrak m_1, \ldots, \mathfrak m_n$ be its maximal ideals. Denote e_i the ramification index of $A \to B_{\mathfrak m_i}$. Let e_0 be the least common multiple of $e_1, \ldots, e_r$. This is invertible in κ_A by our assumption (1). Let $e = de_0$ as in (3). Set $A' = A[\pi^{1/e}]$. Then $A \to A'$ is an extension of discrete valuation rings with fraction field $K' = K[\pi^{1/e}]$, see Lemma 114.2. Choose a product decomposition

$$L \otimes_K K' = \prod L'_j$$

where L'_j are fields. Let B'_j be the integral closure of A in L'_j. Let $\mathfrak m_{ijk}$ be the maximal ideals of B'_j lying over $\mathfrak m_i$. Observe that $(B'_j)_{\mathfrak m_i}$ is the integral closure of $B_{\mathfrak m_i}$ in L'_j. By Abhyankar's lemma (Lemma 114.4) applied to $A \subset B_{\mathfrak m_i}$ and the extension K'/K we see that $A' \to (B'_j)_{\mathfrak m_{ijk}}$ is formally smooth in the $\mathfrak m_{ijk}$-adic topology. This implies that the ramification index is 1 and that the residue field extension is separable (Lemma 111.5). In this way we see that L'_j is unramified with respect to A'. This finishes the proof: we take $L' = L'_j$ for some j. □

Lemma 114.8. *Let A be a discrete valuation ring with fraction field K.*

1. *If L/K is a finite separable extension which is tamely ramified with respect to A, then there exists a Galois extension M/K containing L which is tamely ramified with respect to A.*
2. *If L_1/K, L_2/K are finite separable extensions which are tamely ramified with respect to A, then there exists a a finite separable extension L/K which is tamely ramified with respect to A containing L_1 and L_2.*

Proof. Proof of (2). Choose a uniformizer $\pi \in A$. We can choose an integer e invertible in κ_A and extensions $L'_i/K' = K[\pi^{1/e}]$ unramified with respect to $A' = A[\pi^{1/e}]$ with L'_i/L_i as extensions of K, see Lemma 114.7. By Lemma 111.8 we can find an extension L'/K' which is unramified with respect to A' such that L'_i/K is isomorphic to a subextension of L'/K' for $i = 1, 2$. This finishes the proof of (3) as L'/K is tamely ramified (use same lemma as above).

Proof of (1). We may first replace L by a larger extension and assume that L is an extension of $K' = K[\pi^{1/e}]$ unramified with respect to $A' = A[\pi^{1/e}]$ where e is invertible in κ_A, see Lemma 114.7. Let M be the normal closure of L over K, see Fields, Definition 16.4. Then M/K is Galois by Fields, Lemma 21.5. On the other hand, there is a surjection

$$L \otimes_K \ldots \otimes_K L \longrightarrow M$$

of K-algebras, see Fields, Lemma 16.6. Let B be the integral closure of A in L as in Remark 111.6. The condition that L is unramified with respect to $A' = A[\pi^{1/e}]$ exactly means that $A' \to B$ is an étale ring map, see Algebra, Lemma 143.7. Claim:

$$K' \otimes_K \ldots \otimes_K K' = \prod K'_i$$

is a product of field extensions K'_i/K tamely ramified with respect to A. Then if A'_i is the integral closure of A in K'_i we see that

$$\prod A'_i \otimes_{(A' \otimes_A \ldots \otimes_A A')} (B \otimes_A \ldots \otimes_A B)$$

is finite étale over $\prod A'_i$ and hence a product of Dedekind domains (Lemma 44.4). We conclude that M is the fraction field of one of these Dedekind domains which is finite étale over A'_i for some i. It follows that M/K'_i is unramified with respect to every maximal ideal of A'_i and hence M/K is tamely ramified by Lemma 114.5.

It remains the prove the claim. For this we write $A' = A[x]/(x^e - \pi)$ and we see that

$$A' \otimes_A \ldots \otimes_A A' = A'[x_1, \ldots, x_r]/(x_1^e - \pi, \ldots, x_r^e - \pi)$$

The normalization of this ring certainly contains the elements $y_i = x_i/x_1$ for $i = 2, \ldots, r$ subject to the relations $y_i^e - 1 = 0$ and we obtain

$$A[x_1, y_2, \ldots, y_r]/(x_1^e - \pi, y_2^e - 1, \ldots, y_r - 1) = A'[y_2, \ldots, y_r]/(y_2^e - 1, \ldots, y_r^e - 1)$$

This ring is finite étale over A' because e is invertible in A'. Hence it is a product of Dedekind domains each unramified over A' as desired (see references given above in case of confusion). ☐

Lemma 114.9. *Let $A \subset B$ be an extension of discrete valuation rings. Denote L/K the corresponding extension of fraction fields. Let K'/K be a finite separable extension. Then*

$$K' \otimes_K L = \prod L'_i$$

is a finite product of fields and the following is true

1. *If K' is unramified with respect to A, then each L'_i is unramified with respect to B.*
2. *If K' is tamely ramified with respect to A, then each L'_i is tamely ramified with respect to B.*

Proof. The algebra $K' \otimes_K L$ is a finite product of fields as it is a finite étale algebra over L. Let A' be the integral closure of A in K'.

In case (1) the ring map $A \to A'$ is finite étale. Hence $B' = B \otimes_A A'$ is finite étale over B and is a finite product of Dedekind domains (Lemma 44.4). Hence B' is the integral closure of B in $K' \otimes_K L$. It follows immediately that each L'_i is unramified with respect to B.

Choose a uniformizer $\pi \in A$. To prove (2) we may replace K' by a larger extension tame ramified with respect to A (details omitted; hint: use Lemma 114.6). Thus by Lemma 114.7 we may assume there exists some $e \geq 1$ invertible in κ_A such that K'

contains $K[\pi^{1/e}]$ and such that K' is unramified with respect to $A[\pi^{1/e}]$. Choose a product decomposition

$$K[\pi^{1/e}] \otimes_K L = \prod L_{e,j}$$

For every i there exists a j_i such that $L'_i/L_{e,j_i}$ is a finite separable extension. Let $B_{e,j}$ be the integral closure of B in $L_{e,j}$. By (1) applied to $K'/K[\pi^{1/e}]$ and $A[\pi^{1/e}] \subset (B_{e,j_i})_\mathfrak{m}$ we see that L'_i is unramified with respect to $(B_{e,j_i})_\mathfrak{m}$ for every maximal ideal $\mathfrak{m} \subset B_{e,j_i}$. Hence the proof will be complete if we can show that $L_{e,j}$ is tamely ramified with respect to B, see Lemma 114.5.

Choose a uniformizer θ in B. Write $\pi = u\theta^t$ where u is a unit of B and $t \geq 1$. Then we have

$$A[\pi^{1/e}] \otimes_A B = B[x]/(x^e - u\theta^t) \subset B[y,z]/(y^{e'} - \theta, z^e - u)$$

where $e' = e/\gcd(e,t)$. The map sends x to $zy^{t/\gcd(e,t)}$. Since the right hand side is a product of Dedekind domains each tamely ramified over B the proof is complete (details omitted). ☐

115. Eliminating ramification

In this section we discuss a result of Helmut Epp, see [Epp73]. We strongly encourage the reader to read the original. Our approach is slightly different as we try to handle the mixed and equicharacteristic cases by the same method. For related results, see also [Pon98], [Pon99], [Kuh03], and [ZK99].

Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. The goal in this section is to find a finite extension K_1/K such that with

$$\begin{tikzcd} L \arrow[r] & L_1 \\ K \arrow[u] \arrow[r] & K_1 \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r] & B_1 \arrow[r] & (B_1)_{\mathfrak{m}_{ij}} \\ A \arrow[u] \arrow[r] & A_1 \arrow[u] \arrow[r] & (A_1)_{\mathfrak{m}_i} \arrow[u] \end{tikzcd}$$

as in Remark 114.1 the extensions $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ are all weakly unramified or even formally smooth in the relevant adic topologies. The simplest (but nontrivial) example of this is Abhyankar's lemma, see Lemma 114.4.

Definition 115.1. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$.

1. We say a finite field extension K_1/K is a *weak solution for* $A \subset B$ if all the extensions $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ of Remark 114.1 are weakly unramified.
2. We say a finite field extension K_1/K is a *solution for* $A \subset B$ if each extension $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ of Remark 114.1 is formally smooth in the $\mathfrak{m}_{ij}$-adic topology.

We say a solution K_1/K is a *separable solution* if K_1/K is separable.

In general (weak) solutions do not exist; there is an example in [Epp73]. Under a mild hypothesis on the residue field extension, we will prove the existence of weak solutions in Theorem 115.18 following [Epp73]. In the next section, we will deduce the existence of solutions and sometimes separable solutions in geometrically meaningful cases, see Proposition 116.8 and Lemma 116.9. However, the following example shows that in general one needs inseparable extensions to get even a weak solution.

Example 115.2. Let k be a perfect field of characteristic $p > 0$. Let $A = k[[x]]$ and $K = k((x))$. Let $B = A[x^{1/p}]$. Any weak solution K_1/K for $A \to B$ is inseparable (and any finite inseparable extension of K is a solution). We omit the proof.

Solutions are stable under further extensions, see Lemma 116.1. This may not be true for weak solutions. Weak solutions are in some sense stable under totally ramified extensions, see Lemma 115.3.

Lemma 115.3. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume that $A \to B$ is weakly unramified. Then for any finite separable extension K_1/K totally ramified with respect to A we have that $L_1 = L \otimes_K K_1$ is a field, A_1 and $B_1 = B \otimes_A A_1$ are discrete valuation rings, and the extension $A_1 \subset B_1$ (see Remark 114.1) is weakly unramified.*

Proof. Let $\pi \in A$ and $\pi_1 \in A_1$ be uniformizers. As K_1/K is totally ramified with respect to A we have $\pi_1^e = u_1\pi$ for some unit u_1 in A_1. Hence A_1 is generated by π_1 over A and the minimal polynomial $P(t)$ of π_1 over K has the form

$$P(t) = t^e + a_{e-1}t^{e-1} + \ldots + a_0$$

with $a_i \in (\pi)$ and $a_0 = u\pi$ for some unit u of A. Note that $e = [K_1 : K]$ as well. Since $A \to B$ is weakly unramified we see that π is a uniformizer of B and hence $B_1 = B[t]/(P(t))$ is a discrete valuation ring with uniformizer the class of t. Thus the lemma is clear. $\square$

Lemma 115.4. *Let $A \to B \to C$ be extensions of discrete valuation rings with fraction fields $K \subset L \subset M$. Let K_1/K be a finite extension.*

1. *If K_1 is a (weak) solution for $A \to C$, then K_1 is a (weak) solution for $A \to B$.*
2. *If K_1 is a (weak) solution for $A \to B$ and $L_1 = (L \otimes_K K_1)_{red}$ is a product of fields which are (weak) solutions for $B \to C$, then K_1 is a (weak) solution for $A \to C$.*

Proof. Let $L_1 = (L \otimes_K K_1)_{red}$ and $M_1 = (M \otimes_K K_1)_{red}$ and let $B_1 \subset L_1$ and $C_1 \subset M_1$ be the integral closure of B and C. Note that $M_1 = (M \otimes_L L_1)_{red}$ and that L_1 is a (nonempty) finite product of finite extensions of L. Hence the ring map $B_1 \to C_1$ is a finite product of ring maps of the form discussed in Remark 114.1. In particular, the map $\operatorname{Spec}(C_1) \to \operatorname{Spec}(B_1)$ is surjective. Choose a maximal ideal $\mathfrak{m} \subset C_1$ and consider the extensions of discrete valuation rings

$$(A_1)_{A_1 \cap \mathfrak{m}} \to (B_1)_{B_1 \cap \mathfrak{m}} \to (C_1)_{\mathfrak{m}}$$

If the composition is weakly unramified, so is the map $(A_1)_{A_1 \cap \mathfrak{m}} \to (B_1)_{B_1 \cap \mathfrak{m}}$. If the residue field extension $\kappa_{A_1 \cap \mathfrak{m}} \to \kappa_{\mathfrak{m}}$ is separable, so is the subextension $\kappa_{A_1 \cap \mathfrak{m}} \to \kappa_{B_1 \cap \mathfrak{m}}$. Taking into account Lemma 111.5 this proves (1). A similar argument works for (2). $\square$

Lemma 115.5. *Let $A \to B$ be an extension of discrete valuation rings. There exists a commutative diagram*

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

of extensions of discrete valuation rings such that

(1) *the extensions K'/K and L'/L of fraction fields are separable algebraic,*
(2) *the residue fields of A' and B' are separable algebraic closures of the residue fields of A and B, and*
(3) *if a solution, weak solution, or separable solution exists for $A' \to B'$, then a solution, weak solution, or separable solution exists for $A \to B$.*

Proof. By Algebra, Lemma 159.2 there exists an extension $A \subset A'$ which is a filtered colimit of finite étale extensions such that the residue field of A' is a separable algebraic closure of the residue field of A. Then $A \subset A'$ is an extension of discrete valuation rings such that the induced extension K'/K of fraction fields is separable algebraic.

Let $B \subset B'$ be a strict henselization of B. Then $B \subset B'$ is an extension of discrete valuation rings whose fraction field extension is separable algebraic. By Algebra, Lemma 155.9 there exists a commutative diagram as in the statement of the lemma. Parts (1) and (2) of the lemma are clear.

Let K_1'/K' be a (weak) solution for $A' \to B'$. Since A' is a colimit, we can find a finite étale extension $A \subset A_1'$ and a finite extension K_1 of the fraction field F of A_1' such that $K_1' = K' \otimes_F K_1$. As $A \subset A_1'$ is finite étale and B' strictly henselian, it follows that $B' \otimes_A A_1'$ is a finite product of rings isomorphic to B'. Hence

$$L' \otimes_K K_1 = L' \otimes_K F \otimes_F K_1$$

is a finite product of rings isomorphic to $L' \otimes_{K'} K_1'$. Thus we see that K_1/K is a (weak) solution for $A \to B'$. Hence it is also a (weak) solution for $A \to B$ by Lemma 115.4. □

Lemma 115.6. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Let K_1/K be a normal extension. Say $G = Aut(K_1/K)$. Then G acts on the rings K_1, L_1, A_1 and B_1 of Remark 114.1 and acts transitively on the set of maximal ideals of B_1.*

Proof. Everything is clear apart from the last assertion. If there are two or more orbits of the action, then we can find an element $b \in B_1$ which vanishes at all the maximal ideals of one orbit and has residue 1 at all the maximal ideals in another orbit. Then $b' = \prod_{\sigma \in G} \sigma(b)$ is a G-invariant element of $B_1 \subset L_1 = (L \otimes_K K_1)_{red}$ which is in some maximal ideals of B_1 but not in all maximal ideals of B_1. Lifting it to an element of $L \otimes_K K_1$ and raising to a high power we obtain a G-invariant element b'' of $L \otimes_K K_1$ mapping to $(b')^N$ for some $N > 0$; in fact, we only need to do this in case the characteristic is $p > 0$ and in this case raising to a suitably large p-power q defines a canonical map $(L \otimes_K K_1)_{red} \to L \otimes_K K_1$. Since $K = (K_1)^G$ we conclude that $b'' \in L$. Since b'' maps to an element of B_1 we see that $b'' \in B$ (as B is normal). Then on the one hand it must be true that $b'' \in \mathfrak{m}_B$ as b' is in some maximal ideal of B_1 and on the other hand it must be true that $b'' \notin \mathfrak{m}_B$ as b' is not in all maximal ideals of B_1. This contradiction finishes the proof of the lemma. □

Lemma 115.7. *Let A be a discrete valuation ring with uniformizer π. If the residue characteristic of A is $p > 0$, then for every $n > 1$ and p-power q there exists a degree q separable extension L/K totally ramified with respect to A such that the integral closure B of A in L has ramification index q and a uniformizer π_B such that $\pi_B^q = \pi + \pi^n b$ and $\pi_B^q = \pi + (\pi_B)^{nq} b'$ for some $b, b' \in B$.*

Proof. If the characteristic of K is zero, then we can take the extension given by $\pi_B^q = \pi$, see Lemma 114.2. If the characteristic of K is $p > 0$, then we can take the extension of K given by $z^q - \pi^n z = \pi^{1-q}$. Namely, then we see that $y^q - \pi^{n+q-1}y = \pi$ where $y = \pi z$. Taking $\pi_B = y$ we obtain the desired result. □

Lemma 115.8. *Let A be a discrete valuation ring. Assume the reside field κ_A has characteristic $p > 0$ and that $a \in A$ is an element whose residue class in κ_A is not a pth power. Then a is not a pth power in K and the integral closure of A in $K[a^{1/p}]$ is the ring $A[a^{1/p}]$ which is a discrete valuation ring weakly unramified over A.*

Proof. This lemma proves itself. □

Lemma 115.9. *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Let $\pi \in A$ be a uniformizer. Assume*

1. *B is a Nagata ring,*
2. *$A \subset B$ is weakly unramified,*
3. *M is a degree p purely inseparable extension of L.*

Then either

1. *$A \to C$ is weakly unramified, or*
2. *$C = B[\pi^{1/p}]$, or*
3. *there exists a degree p separable extension K_1/K totally ramified with respect to A such that $L_1 = L \otimes_K K_1$ and $M_1 = M \otimes_K K_1$ are fields and the maps of integral closures $A_1 \to B_1 \to C_1$ are weakly unramified extensions of discrete valuation rings.*

Proof. Let e be the ramification index of C over B. If $e = 1$, then we are done. If not, then $e = p$ by Lemmas 111.2 and 111.4. This in turn implies that the residue fields of B and C agree. Choose a uniformizer π_C of C. Write $\pi_C^p = u\pi$ for some unit u of C. Since $\pi_C^p \in L$, we see that $u \in B^*$. Also $M = L[\pi_C]$.

Suppose there exists an integer $m \geq 0$ such that

$$u = \sum\nolimits_{0 \leq i < m} b_i^p \pi^i + b\pi^m$$

with $b_i \in B$ and with $b \in B$ an element whose image in κ_B is not a pth power. Choose an extension K_1/K as in Lemma 115.7 with $n = m + 2$ and denote π' the uniformizer of the integral closure A_1 of A in K_1 such that $\pi = (\pi')^p + (\pi')^{np}a$ for some $a \in A_1$. Let B_1 be the integral closure of B in $L \otimes_K K_1$. Observe that $A_1 \to B_1$ is weakly unramified by Lemma 115.3. In B_1 we have

$$u\pi = \left(\sum\nolimits_{0 \leq i < m} b_i (\pi')^{i+1}\right)^p + b(\pi')^{(m+1)p} + (\pi')^{np} b_1$$

for some $b_1 \in B_1$ (computation omitted). We conclude that M_1 is obtained from L_1 by adjoining a pth root of

$$b + (\pi')^{n-m-1} b_1$$

Since the residue field of B_1 equals the residue field of B we see from Lemma 115.8 that M_1/L_1 has degree p and the integral closure C_1 of B_1 is weakly unramified over B_1. Thus we conclude in this case.

If there does not exist an integer m as in the preceding paragraph, then u is a pth power in the π-adic completion of B_1. Since B is Nagata, this means that u is a pth power in B_1 by Algebra, Lemma 162.18. Whence the second case of the statement of the lemma holds. $\square$

Lemma 115.10. *Let A be a local ring annihilated by a prime p whose maximal ideal is nilpotent. There exists a ring map $\sigma : \kappa_A \to A$ which is a section to the residue map $A \to \kappa_A$. If $A \to A'$ is a local homomorphism of local rings, then we can choose a similar ring map $\sigma' : \kappa_{A'} \to A'$ compatible with σ provided that the extension $\kappa_{A'}/\kappa_A$ is separable.*

Proof. Separable extensions are formally smooth by Algebra, Proposition 158.9. Thus the existence of σ follows from the fact that $\mathbf{F}_p \to \kappa_A$ is separable. Similarly for the existence of σ' compatible with σ. $\square$

Lemma 115.11. *Let A be a discrete valuation ring with fraction field K of characteristic $p > 0$. Let $\xi \in K$. Let L be an extension of K obtained by adjoining a root of $z^p - z = \xi$. Then L/K is Galois and one of the following happens*

1. *$L = K$,*
2. *L/K is unramified with respect to A of degree p,*
3. *L/K is totally ramified with respect to A with ramification index p, and*
4. *the integral closure B of A in L is a discrete valuation ring, $A \subset B$ is weakly unramified, and $A \to B$ induces a purely inseparable residue field extension of degree p.*

Let π be a uniformizer of A. We have the following implications:

- (A) *If $\xi \in A$, then we are in case (1) or (2).*
- (B) *If $\xi = \pi^{-n}a$ where $n > 0$ is not divisible by p and a is a unit in A, then we are in case (3)*
- (C) *If $\xi = \pi^{-n}a$ where $n > 0$ is divisible by p and the image of a in κ_A is not a pth power, then we are in case (4).*

Proof. The extension is Galois of order dividing p by the discussion in Fields, Section 25. It immediately follows from the discussion in Section 112 that we are in one of the cases (1) – (4) listed in the lemma.

Case (A). Here we see that $A \to A[x]/(x^p - x - \xi)$ is a finite étale ring extension. Hence we are in cases (1) or (2).

Case (B). Write $\xi = \pi^{-n}a$ where p does not divide n. Let $B \subset L$ be the integral closure of A in L. If $C = B_\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, then it is clear that $p\text{ord}_C(z) = -n\text{ord}_C(\pi)$. In particular $A \subset C$ has ramification index divisible by p. It follows that it is p and that $B = C$.

Case (C). Set $k = n/p$. Then we can rewrite the equation as

$$(\pi^k z)^p - \pi^{n-k}(\pi^k z) = a$$

Since $A[y]/(y^p - \pi^{n-k}y - a)$ is a discrete valuation ring weakly unramified over A, the lemma follows. $\square$

Lemma 115.12. *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume*

1. *$A \subset B$ weakly unramified,*
2. *the characteristic of K is p,*
3. *M is a degree p Galois extension of L, and*

(4) $\kappa_A = \bigcap_{n \geq 1} \kappa_B^{p^n}$.

Then there exists a finite Galois extension K_1/K totally ramified with respect to A which is a weak solution for $A \to C$.

Proof. Since the characteristic of L is p we know that M is an Artin-Schreier extension of L (Fields, Lemma 25.1). Thus we may pick $z \in M$, $z \notin L$ such that $\xi = z^p - z \in L$. Choose $n \geq 0$ such that $\pi^n \xi \in B$. We pick z such that n is minimal. If $n = 0$, then M/L is unramified with respect to B (Lemma 115.11) and we are done. Thus we have $n > 0$.

Assumption (4) implies that κ_A is perfect. Thus we may choose compatible ring maps $\overline{\sigma} : \kappa_A \to A/\pi^n A$ and $\overline{\sigma} : \kappa_B \to B/\pi^n B$ as in Lemma 115.10. We lift the second of these to a map of sets $\sigma : \kappa_B \to B$[16]. Then we can write

$$\xi = \sum\nolimits_{i=n,\ldots,1} \sigma(\lambda_i)\pi^{-i} + b$$

for some $\lambda_i \in \kappa_B$ and $b \in B$. Let

$$I = \{i \in \{n, \ldots, 1\} \mid \lambda_i \in \kappa_A\}$$

and

$$J = \{j \in \{n, \ldots, 1\} \mid \lambda_i \notin \kappa_A\}$$

We will argue by induction on the size of the finite set J.

The case $J = \emptyset$. Here for all $i \in \{n, \ldots, 1\}$ we have $\sigma(\lambda_i) = a_i + \pi^n b_i$ for some $a_i \in A$ and $b_i \in B$ by our choice of σ. Thus $\xi = \pi^{-n} a + b$ for some $a \in A$ and $b \in B$. If $p|n$, then we write $a = a_0^p + \pi a_1$ for some $a_0, a_1 \in A$ (as the residue field of A is perfect). We compute

$$(z - \pi^{-n/p} a_0)^p - (z - \pi^{-n/p} a_0) = \pi^{-(n-1)}(a_1 + \pi^{n-1-n/p} a_0) + b'$$

for some $b' \in B$. This would contradict the minimality of n. Thus p does not divide n. Consider the degree p extension K_1 of K given by $w^p - w = \pi^{-n} a$. By Lemma 115.11 this extension is Galois and totally ramified with respect to A. Thus $L_1 = L \otimes_K K_1$ is a field and $A_1 \subset B_1$ is weakly unramified (Lemma 115.3). By Lemma 115.11 the ring $M_1 = M \otimes_K K_1$ is either a product of p copies of L_1 (in which case we are done) or a field extension of L_1 of degree p. Moreover, in the second case, either C_1 is weakly unramified over B_1 (in which case we are done) or M_1/L_1 is degree p, Galois, and totally ramified with respect to B_1. In this last case the extension M_1/L_1 is generated by the element $z - w$ and

$$(z - w)^p - (z - w) = z^p - z - (w^p - w) = b$$

with $b \in B$ (see above). Thus by Lemma 115.11 once more the extension M_1/L_1 is unramified with respect to B_1 and we conclude that K_1 is a weak solution for $A \to C$. From now on we assume $J \neq \emptyset$.

Suppose that $j', j \in J$ such that $j' = p^r j$ for some $r > 0$. Then we change our choice of z into

$$z' = z - (\sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^p)\pi^{-pj} + \ldots + \sigma(\lambda_j^{p^{r-1}})\pi^{-p^{r-1}j})$$

Then ξ changes into $\xi' = (z')^p - (z')$ as follows

$$\xi' = \xi - \sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^{p^r})\pi^{-j'} + \text{something in } B$$

[16]If B is complete, then we can choose σ to be a ring map. If A is also complete and σ is a ring map, then σ maps κ_A into A.

Writing $\xi' = \sum_{i=n,\ldots,1} \sigma(\lambda'_i)\pi^{-i} + b'$ as before we find that $\lambda'_i = \lambda_i$ for $i \neq j, j'$ and $\lambda'_j = 0$. Thus the set J has gotten smaller. By induction on the size of J we may assume no such pair j, j' exists. (Please observe that in this procedure we may get thrown back into the case that $J = \emptyset$ we treated above.)

For $j \in J$ write $\lambda_j = \mu_j^{p^{r_j}}$ for some $r_j \geq 0$ and $\mu_j \in \kappa_B$ which is not a pth power. This is possible by our assumption (4). Let $j \in J$ be the unique index such that jp^{-r_j} is maximal. (The index is unique by the result of the preceding paragraph.) Choose $r > \max(r_j + 1)$ and such that $jp^{r-r_j} > n$ for $j \in J$. Choose a separable extension K_1/K totally ramified with respect to A of degree p^r such that the corresponding discrete valuation ring $A_1 \subset K_1$ has uniformizer π' with $(\pi')^{p^r} = \pi + \pi^{n+1}a$ for some $a \in A_1$ (Lemma 115.7). Observe that $L_1 = L \otimes_K K_1$ is a field and that L_1/L is totally ramified with respect to B (Lemma 115.3). Computing in the integral closure B_1 we get

$$\xi = \sum\nolimits_{i\in I} \sigma(\lambda_i)(\pi')^{-ip^r} + \sum\nolimits_{j\in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + b_1$$

for some $b_1 \in B_1$. Note that $\sigma(\lambda_i)$ for $i \in I$ is a qth power modulo π^n, i.e., modulo $(\pi')^{np^r}$. Hence we can rewrite the above as

$$\xi = \sum\nolimits_{i\in I} x_i^{p^r}(\pi')^{-ip^r} + \sum\nolimits_{j\in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + b_1$$

As in the previous paragraph we change our choice of z into

$$\begin{aligned} z' = {} & z \\ & - \sum\nolimits_{i\in I} \left(x_i(\pi')^{-i} + \ldots + x_i^{p^{r-1}}(\pi')^{-ip^{r-1}}\right) \\ & - \sum\nolimits_{j\in J} \left(\sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + \ldots + \sigma(\mu_j)^{p^{r_j-1}}(\pi')^{-jp^{r-1}}\right) \end{aligned}$$

to obtain

$$(z')^p - z' = \sum\nolimits_{i\in I} x_i(\pi')^{-i} + \sum\nolimits_{j\in J} \sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + b'_1$$

for some $b'_1 \in B_1$. Since there is a unique j such that jp^{r-r_j} is maximal and since jp^{r-r_j} is bigger than $i \in I$ and divisible by p, we see that M_1/L_1 falls into case (C) of Lemma 115.11. This finishes the proof. $\square$

Lemma 115.13. *Let A be a ring which contains a primitive pth root of unity ζ. Set $w = 1 - \zeta$. Then*

$$P(z) = \frac{(1+wz)^p - 1}{w^p} = z^p - z + \sum\nolimits_{0<i<p} a_i z^i$$

is an element of $A[z]$ and in fact $a_i \in (w)$. Moreover, we have

$$P(z_1 + z_2 + wz_1z_2) = P(z_1) + P(z_2) + w^p P(z_1)P(z_2)$$

in the polynomial ring $A[z_1, z_2]$.

Proof. It suffices to prove this when

$$A = \mathbf{Z}[\zeta] = \mathbf{Z}[x]/(x^{p-1} + \ldots + x + 1)$$

is the ring of integers of the cyclotomic field. The polynomial identity $t^p - 1 = (t-1)(t-\zeta)\ldots(t-\zeta^{p-1})$ (which is proved by looking at the roots on both sides) shows that $t^{p-1} + \ldots + t + 1 = (t-\zeta)\ldots(t-\zeta^{p-1})$. Substituting $t = 1$ we obtain $p = (1-\zeta)(1-\zeta^2)\ldots(1-\zeta^{p-1})$. The maximal ideal $(p, w) = (w)$ is the unique prime ideal of A lying over p (as fields of characteristic p do not have nontrivial pth roots of 1). It follows that $p = uw^{p-1}$ for some unit u. This implies that

$$a_i = \frac{1}{p}\binom{p}{i}uw^{i-1}$$

for $p > i > 1$ and $-1+a_1 = pw/w^p = u$. Since $P(-1) = 0$ we see that $0 = (-1)^p - u$ modulo (w). Hence $a_1 \in (w)$ and the proof if the first part is done. The second part follows from a direct computation we omit. □

Lemma 115.14. *Let A be a discrete valuation ring of mixed characteristic $(0, p)$ which contains a primitive pth root of 1. Let $P(t) \in A[t]$ be the polynomial of Lemma 115.13. Let $\xi \in K$. Let L be an extension of K obtained by adjoining a root of $P(z) = \xi$. Then L/K is Galois and one of the following happens*

1. *$L = K$,*
2. *L/K is unramified with respect to A of degree p,*
3. *L/K is totally ramified with respect to A with ramification index p, and*
4. *the integral closure B of A in L is a discrete valuation ring, $A \subset B$ is weakly unramified, and $A \to B$ induces a purely inseparable residue field extension of degree p.*

Let π be a uniformizer of A. We have the following implications:

- (A) *If $\xi \in A$, then we are in case (1) or (2).*
- (B) *If $\xi = \pi^{-n}a$ where $n > 0$ is not divisible by p and a is a unit in A, then we are in case (3)*
- (C) *If $\xi = \pi^{-n}a$ where $n > 0$ is divisible by p and the image of a in κ_A is not a pth power, then we are in case (4).*

Proof. Adjoining a root of $P(z) = \xi$ is the same thing as adjoining a root of $y^p = w^p(1 + \xi)$. Since K contains a primitive pth root of 1 the extension is Galois of order dividing p by the discussion in Fields, Section 24. It immediately follows from the discussion in Section 112 that we are in one of the cases (1) – (4) listed in the lemma.

Case (A). Here we see that $A \to A[x]/(P(x) - \xi)$ is a finite étale ring extension. Hence we are in cases (1) or (2).

Case (B). Write $\xi = \pi^{-n}a$ where p does not divide n. Let $B \subset L$ be the integral closure of A in L. If $C = B_\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, then it is clear that $p\text{ord}_C(z) = -n\text{ord}_C(\pi)$. In particular $A \subset C$ has ramification index divisible by p. It follows that it is p and that $B = C$.

Case (C). Set $k = n/p$. Then we can rewrite the equation as

$$(\pi^k z)^p - \pi^{n-k}(\pi^k z) + \sum a_i \pi^{n-ik}(\pi^k z)^i = a$$

Since $A[y]/(y^p - \pi^{n-k}y - \sum a_i\pi^{n-ik}y^i - a)$ is a discrete valuation ring weakly unramified over A, the lemma follows. □

Let A be a discrete valuation ring of mixed characteristic $(0, p)$ containing a primitive pth root of 1. Let $w \in A$ and $P(t) \in A[t]$ be as in Lemma 115.13. Let L be a finite extension of K. We say L/K is a *degree p extension of finite level* if L is a degree p extension of K obtained by adjoining a root of the equation $P(z) = \xi$ where $\xi \in K$ is an element with $w^p\xi \in \mathfrak{m}_A$.

This definition is relevant to the discussion in this section due to the following straightforward lemma.

Lemma 115.15. *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume that*

(1) *A has mixed characteristic $(0, p)$,*
(2) *$A \subset B$ is weakly unramified,*
(3) *B contains a primitive pth root of 1, and*
(4) *M/L is Galois of degree p.*

Then there exists a finite Galois extension K_1/K totally ramified with respect to A which is either a weak solution for $A \to C$ or is such that M_1/L_1 is a degree p extension of finite level.

Proof. Let $\pi \in A$ be a uniformizer. By Kummer theory (Fields, Lemma 24.1) M is obtained from L by adjoining the root of $y^p = b$ for some $b \in L$.

If $\text{ord}_B(b)$ is prime to p, then we choose a degree p separable extension K_1/K totally ramified with respect to A (for example using Lemma 115.7). Let A_1 be the integral closure of A in K_1. By Lemma 115.3 the integral closure B_1 of B in $L_1 = L \otimes_K K_1$ is a discrete valuation ring weakly unramified over A_1. If K_1/K is not a weak solution for $A \to C$, then the integral closure C_1 of C in $M_1 = M \otimes_K K_1$ is a discrete valuation ring and $B_1 \to C_1$ has ramification index p. In this case, the field M_1 is obtained from L_1 by adjoining the pth root of b with $\text{ord}_{B_1}(b)$ divisible by p. Replacing A by A_1, etc we may assume that $b = \pi^n u$ where $u \in B$ is a unit and n is divisible by p. Of course, in this case the extension M is obtained from L by adjoining the pth root of a unit.

Suppose M is obtained from L by adjoining the root of $y^p = u$ for some unit u of B. If the residue class of u in κ_B is not a pth power, then $B \subset C$ is weakly unramified (Lemma 115.8) and we are done. Otherwise, we can replace our choice of y by y/v where v^p and u have the same image in κ_B. After such a replacement we have

$$y^p = 1 + \pi b$$

for some $b \in B$. Then we see that $P(z) = \pi b / w^p$ where $z = (y - 1)/w$. Thus we see that the extension is a degree p extension of finite level with $\xi = \pi b / w^p$. $\square$

Let A be a discrete valuation ring of mixed characteristic $(0, p)$ containing a primitive pth root of 1. Let $w \in A$ and $P(t) \in A[t]$ be as in Lemma 115.13. Let L be a degree p extension of K of finite level. Choose $z \in L$ generating L over K with $\xi = P(z) \in K$. Choose a uniformizer π for A and write $w = u\pi^{e_1}$ for some integer $e_1 = \text{ord}_A(w)$ and unit $u \in A$. Finally, pick $n \geq 0$ such that

$$\pi^n \xi \in A$$

The *level* of L/K is the smallest value of the quantity n/e_1 taking over all z generating L/K with $\xi = P(z) \in K$.

We make a couple of remarks. Since the extension is of finite level we know that we can choose z such that $n < pe_1$. Thus the level is a rational number contained in $[0, p)$. If the level is zero then L/K is unramified with respect to A by Lemma 115.14. Our next goal is to lower the level.

Lemma 115.16. *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume*

(1) *A has mixed characteristic $(0, p)$,*
(2) *$A \subset B$ weakly unramified,*
(3) *B contains a primitive pth root of 1,*
(4) *M/L is a degree p extension of finite level $l > 0$,*
(5) *$\kappa_A = \bigcap_{n \geq 1} \kappa_B^{p^n}$.*

Then there exists a finite separable extension K_1 of K totally ramified with respect to A such that either K_1 is a weak solution for $A \to C$, or the extension M_1/L_1 is a degree p extension of finite level $\leq \max(0, l-1, 2l-p)$.

Proof. Let $\pi \in A$ be a uniformizer. Let $w \in B$ and $P \in B[t]$ be as in Lemma 115.13 (for B). Set $e_1 = \text{ord}_B(w)$, so that w and π^{e_1} are associates in B. Pick $z \in M$ generating M over L with $\xi = P(z) \in K$ and n such that $\pi^n \xi \in B$ as in the definition of the level of M over L, i.e., $l = n/e_1$.

The proof of this lemma is completely similar to the proof of Lemma 115.12. To explain what is going on, observe that

$$P(z) \equiv z^p - z \bmod \pi^{-n+e_1}B \tag{115.16.1}$$

for any $z \in L$ such that $\pi^{-n}P(z) \in B$ (use that z has valuation at worst $-n/p$ and the shape of the polynomial P). Moreover, we have

$$\xi_1 + \xi_2 + w^p\xi_1\xi_2 \equiv \xi_1 + \xi_2 \bmod \pi^{-2n+pe_1}B \tag{115.16.2}$$

for $\xi_1, \xi_2 \in \pi^{-n}B$. Finally, observe that $n - e_1 = (l-1)/e_1$ and $-2n + pe_1 = -(2l-p)e_1$. Write $m = n - e_1 \max(0, l-1, 2l-p)$. The above shows that doing calculations in $\pi^{-n}B/\pi^{-n+m}B$ the polynomial P behaves exactly as the polynomial $z^p - z$. This explains why the lemma is true but we also give the details below.

Assumption (4) implies that κ_A is perfect. Observe that $m \leq e_1$ and hence A/π^m is annihilated by w and hence p. Thus we may choose compatible ring maps $\overline{\sigma} : \kappa_A \to A/\pi^m A$ and $\overline{\sigma} : \kappa_B \to B/\pi^m B$ as in Lemma 115.10. We lift the second of these to a map of sets $\sigma : \kappa_B \to B$. Then we can write

$$\xi = \sum\nolimits_{i=n,\ldots,n-m+1} \sigma(\lambda_i)\pi^{-i} + \pi^{-n+m)}b$$

for some $\lambda_i \in \kappa_B$ and $b \in B$. Let

$$I = \{i \in \{n, \ldots, n-m+1\} \mid \lambda_i \in \kappa_A\}$$

and

$$J = \{j \in \{n, \ldots, n-m+1\} \mid \lambda_i \notin \kappa_A\}$$

We will argue by induction on the size of the finite set J.

The case $J = \emptyset$. Here for all $i \in \{n, \ldots, n-m+1\}$ we have $\sigma(\lambda_i) = a_i + \pi^{n-m}b_i$ for some $a_i \in A$ and $b_i \in B$ by our choice of $\overline{\sigma}$. Thus $\xi = \pi^{-n}a + \pi^{-n+m}b$ for some $a \in A$ and $b \in B$. If $p|n$, then we write $a = a_0^p + \pi a_1$ for some $a_0, a_1 \in A$ (as the residue field of A is perfect). Set $z_1 = -\pi^{-n/p}a_0$. Note that $P(z_1) \in \pi^{-n}B$ and that $z + z_1 + wzz_1$ is an element generating M over L (note that $wz_1 \neq -1$ as $n < pe_1$). Moreover, by Lemma 115.13 we have

$$P(z + z_1 + wzz_1) = P(z) + P(z_1) + w^pP(z)P(z_1) \in K$$

and by equations (115.16.1) and (115.16.2) we have

$$P(z) + P(z_1) + w^pP(z)P(z_1) \equiv \xi + z_1^p - z_1 \bmod \pi^{-n+m}B$$

for some $b' \in B$. This contradict the minimality of n! Thus p does not divide n. Consider the degree p extension K_1 of K given by $P(y) = -\pi^{-n}a$. By Lemma 115.14 this extension is separable and totally ramified with respect to A. Thus $L_1 = L \otimes_K K_1$ is a field and $A_1 \subset B_1$ is weakly unramified (Lemma 115.3). By Lemma 115.14 the ring $M_1 = M \otimes_K K_1$ is either a product of p copies of L_1 (in which case we are done) or a field extension of L_1 of degree p. Moreover, in the second case, either C_1 is weakly unramified over B_1 (in which case we are done) or M_1/L_1 is degree p, Galois, totally ramified with respect to B_1. In this last case the extension M_1/L_1 is generated by the element $z + y + wzy$ and we see that $P(z + y + wzy) \in L_1$ and

$$\begin{aligned} P(z+y+wzy) &= P(z) + P(y) + w^p P(z)P(y) \\ &\equiv \xi - \pi^{-n}a \bmod \pi^{-n+m}B_1 \\ &\equiv 0 \bmod \pi^{-n+m}B_1 \end{aligned}$$

in exactly the same manner as above. By our choice of m this means exactly that M_1/L_1 has level at most $\max(0, l-1, 2l-p)$. From now on we assume that $J \neq \emptyset$.

Suppose that $j', j \in J$ such that $j' = p^r j$ for some $r > 0$. Then we set

$$z_1 = -\sigma(\lambda_j)\pi^{-j} - \sigma(\lambda_j^p)\pi^{-pj} - \ldots - \sigma(\lambda_j^{p^{r-1}})\pi^{-p^{r-1}j}$$

and we change z into $z' = z + z_1 + wzz_1$. Observe that $z' \in M$ generates M over L and that we have $\xi' = P(z') = P(z) + P(z_1) + wP(z)P(z_1) \in L$ with

$$\xi' \equiv \xi - \sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^{p^r})\pi^{-j'} \bmod \pi^{-n+m}B$$

by using equations (115.16.1) and (115.16.2) as above. Writing

$$\xi' = \sum\nolimits_{i=n,\ldots,n-m+1} \sigma(\lambda'_i)\pi^{-i} + \pi^{-n+m}b'$$

as before we find that $\lambda'_i = \lambda_i$ for $i \neq j, j'$ and $\lambda'_j = 0$. Thus the set J has gotten smaller. By induction on the size of J we may assume there is no pair j, j' of J such that j'/j is a power of p. (Please observe that in this procedure we may get thrown back into the case that $J = \emptyset$ we treated above.)

For $j \in J$ write $\lambda_j = \mu_j^{p^{r_j}}$ for some $r_j \geq 0$ and $\mu_j \in \kappa_B$ which is not a pth power. This is possible by our assumption (4). Let $j \in J$ be the unique index such that jp^{-r_j} is maximal. (The index is unique by the result of the preceding paragraph.) Choose $r > \max(r_j + 1)$ and such that $jp^{r-r_j} > n$ for $j \in J$. Let K_1/K be the extension of degree p^r, totally ramified with respect to A, defined by $(\pi')^{p^r} = \pi$. Observe that π' is the uniformizer of the corresponding discrete valuation ring $A_1 \subset K_1$. Observe that $L_1 = L \otimes_K K_1$ is a field and L_1/L is totally ramified with respect to B (Lemma 115.3). Computing in the integral closure B_1 we get

$$\xi = \sum\nolimits_{i\in I} \sigma(\lambda_i)(\pi')^{-ip^r} + \sum\nolimits_{j\in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + \pi^{-n+m}b_1$$

for some $b_1 \in B_1$. Note that $\sigma(\lambda_i)$ for $i \in I$ is a qth power modulo π^m, i.e., modulo $(\pi')^{mp^r}$. Hence we can rewrite the above as

$$\xi = \sum\nolimits_{i\in I} x_i^{p^r}(\pi')^{-ip^r} + \sum\nolimits_{j\in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + \pi^{-n+m}b_1$$

Similar to our choice in the previous paragraph we set

$$\begin{aligned} z_1 - \sum\nolimits_{i \in I} \left(x_i (\pi')^{-i} + \ldots + x_i^{p^{r-1}} (\pi')^{-ip^{r-1}} \right) \\ - \sum\nolimits_{j \in J} \left(\sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + \ldots + \sigma(\mu_j)^{p^{r_j - 1}} (\pi')^{-jp^{r-1}} \right) \end{aligned}$$

and we change our choice of z into $z' = z + z_1 + wzz_1$. Then z' generates M_1 over L_1 and $\xi' = P(z') = P(z) + P(z_1) + w^p P(z) P(z_1) \in L_1$ and a calculation shows that

$$\xi' \equiv \sum\nolimits_{i \in I} x_i (\pi')^{-i} + \sum\nolimits_{j \in J} \sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + (\pi')^{(-n+m)p^r} b'_1$$

for some $b'_1 \in B_1$. There is a unique j such that jp^{r-r_j} is maximal and jp^{r-r_j} is bigger than $i \in I$. If $jp^{r-r_j} \leq (n-m)p^r$ then the level of the extension M_1/L_1 is less than $\max(0, l-1, 2l-p)$. If not, then, as p divides jp^{r-r_j}, we see that M_1/L_1 falls into case (C) of Lemma 115.14. This finishes the proof. $\square$

Lemma 115.17. *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fraction fields $K \subset L \subset M$. Assume*

1. *the residue field k of A is algebraically closed of characteristic $p > 0$,*
2. *A and B are complete,*
3. *$A \to B$ is weakly unramified,*
4. *M is a finite extension of L,*
5. *$k = \bigcap_{n \geq 1} \kappa_B^{p^n}$*

Then there exists a finite extension K_1/K which is a weak solution for $A \to C$.

Proof. Let M' be any finite extension of L and consider the integral closure C' of B in M'. Then C' is finite over B as B is Nagata by Algebra, Lemma 162.8. Moreover, C' is a discrete valuation ring, see discussion in Remark 114.1. Moreover C' is complete as a B-module, hence complete as a discrete valuation ring, see Algebra, Section 96. It follows in particular that C is the integral closure of B in M (by definition of valuation rings as maximal for the relation of domination).

Let $M \subset M'$ be a finite extension and let $C' \subset M'$ be the integral closure of B as above. By Lemma 115.4 it suffices to prove the result for $A \to B \to C'$. Hence we may assume that M/L is normal, see Fields, Lemma 16.3.

If M/L is normal, we can find a chain of finite extensions

$$L = L^0 \subset L^1 \subset L^2 \subset \ldots \subset L^r = M$$

such that each extension L^{j+1}/L^j is either:

(a) purely inseparable of degree p,
(b) totally ramified with respect to B^j and Galois of degree p,
(c) totally ramified with respect to B^j and Galois cyclic of order prime to p,
(d) Galois and unramified with respect to B^j.

Here B^j is the integral closure of B in L^j. Namely, since M/L is normal we can write it as a compositum of a Galois extension and a purely inseparable extension (Fields, Lemma 27.3). For the purely inseparable extension the existence of the filtration is clear. In the Galois case, note that G is "the" decomposition group and let $I \subset G$ be the inertia group. Then on the one hand I is solvable by Lemma 112.5 and on the other hand the extension M^I/L is unramified with respect to B by Lemma 112.8. This proves we have a filtration as stated.

We are going to argue by induction on the integer r. Suppose that we can find a finite extension K_1/K which is a weak solution for $A \to B^1$ where B^1 is the integral closure of B in L^1. Let K_1' be the normal closure of K_1/K (Fields, Lemma 16.3). Since A is complete and the residue field of A is algebraically closed we see that K_1'/K_1 is separable and totally ramified with respect to A_1 (some details omitted). Hence K_1'/K is a weak solution for $A \to B^1$ as well by Lemma 115.3. In other words, we may and do assume that K_1 is a normal extension of K. Having done so we consider the sequence

$$L_1^0 = (L^0 \otimes_K K_1)_{red} \subset L_1^1 = (L^1 \otimes_K K_1)_{red} \subset \ldots \subset L_1^r = (L^r \otimes_K K_1)_{red}$$

and the corresponding integral closures B_1^i. Note that $C_1 = B_1^r$ is a product of discrete valuation rings which are transitively permuted by $G = \mathrm{Aut}(K_1/K)$ by Lemma 115.6. In particular all the extensions of discrete valuation rings $A_1 \to (C_1)_\mathfrak{m}$ are isomorphic and a weak solution for one will be a weak solution for all of them. We can apply the induction hypothesis to the sequence

$$A_1 \to (B_1^1)_{B_1^1 \cap \mathfrak{m}} \to (B_1^2)_{B_1^2 \cap \mathfrak{m}} \to \ldots \to (B_1^r)_{B_1^r \cap \mathfrak{m}} = (C_1)_\mathfrak{m}$$

to get a weak solution K_2/K_1 for $A_1 \to (C_1)_\mathfrak{m}$. The extension K_2/K will then be a weak solution for $A \to C$ by what we said before. Note that the induction hypothesis applies: the ring map $A_1 \to (B_1^1)_{B_1^1 \cap \mathfrak{m}}$ is weakly unramified by our choice of K_1 and the sequence of fraction field extensions each still have one of the properties (a), (b), (c), or (d) listed above. Moreover, observe that for any finite extension $\kappa_B \subset \kappa$ we still have $k = \bigcap \kappa^{p^n}$.

Thus everything boils down to finding a weak solution for $A \subset C$ when the field extension M/L satisfies one of the properties (a), (b), (c), or (d).

Case (d). This case is trivial as here $B \to C$ is unramified already.

Case (c). Say M/L is cyclic of order n prime to p. Because M/L is totally ramified with respect to B we see that the ramification index of $B \subset C$ is n and hence the ramification index of $A \subset C$ is n as well. Choose a uniformizer $\pi \in A$ and set $K_1 = K[\pi^{1/n}]$. Then K_1/K is a solution for $A \subset C$ by Abhyankar's lemma (Lemma 114.4).

Case (b). We divide this case into the mixed characteristic case and the equicharacteristic case. In the equicharacteristic case this is Lemma 115.12. In the mixed characteristic case, we first replace K by a finite extension to get to the situation where M/L is a degree p extension of finite level using Lemma 115.15. Then the level is a rational number $l \in [0, p)$, see discussion preceding Lemma 115.16. If the level is 0, then $B \to C$ is weakly unramified and we're done. If not, then we can replacing the field K by a finite extension to obtain a new situation with level $l' \leq \max(0, l-1, 2l-p)$ by Lemma 115.16. If $l = p - \epsilon$ for $\epsilon < 1$ then we see that $l' \leq p - 2\epsilon$. Hence after a finite number of replacements we obtain a case with level $\leq p-1$. Then after at most $p-1$ more such replacements we reach the situation where the level is zero.

Case (a) is Lemma 115.9. This is the only case where we possibly need a purely inseparable extension of K, namely, in case (2) of the statement of the lemma we win by adjoining a pth power of the element π. This finishes the proof of the lemma. $\square$

At this point we have collected all the lemmas we need to prove the main result of this section.

Theorem 115.18 (Epp)**.** *Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If the characteristic of κ_A is $p > 0$, assume that every element of*

$$\bigcap\nolimits_{n \geq 1} \kappa_B^{p^n}$$

is separable algebraic over κ_A. Then there exists a finite extension K_1/K which is a weak solution for $A \to B$ as defined in Definition 115.1.

Proof. If the characteristic of κ_A is zero or if the residue characteristic is p, the ramification index is prime to p, and the residue field extension is separable, then this follows from Abhyankar's lemma (Lemma 114.4). Namely, suppose the ramification index is e. Choose a uniformizer $\pi \in A$. Let K_1/K be the extension obtained by adjoining an eth root of π. By Lemma 114.2 we see that the integral closure A_1 of A in K_1 is a discrete valuation ring with ramification index over A. Thus $A_1 \to (B_1)_\mathfrak{m}$ is formally smooth in the $\mathfrak{m}$-adic topology for all maximal ideals $\mathfrak{m}$ of B_1 by Lemma 114.4 and a fortiori these are weakly unramified extensions of discrete valuation rings.

From now on we let p be a prime number and we assume that κ_A has characteristic p. We first apply Lemma 115.5 to reduce to the case that A and B have separably closed residue fields. Since κ_A and κ_B are replaced by their separable algebraic closures by this procedure we see that we obtain

$$\kappa_A \supset \bigcap\nolimits_{n \geq 1} \kappa_B^{p^n}$$

from the condition of the theorem.

Let $\pi \in A$ be a uniformizer. Let $A^\wedge$ and $B^\wedge$ be the completions of A and B. We have a commutative diagram

$$\begin{tikzcd} B \arrow[r] & B^\wedge \\ A \arrow[u] \arrow[r] & A^\wedge \arrow[u] \end{tikzcd}$$

of extensions of discrete valuation rings. Let $K^\wedge$ be the fraction field of $A^\wedge$. Suppose that we can find a finite extension $M/K^\wedge$ which is (a) a weak solution for $A^\wedge \to B^\wedge$ and (b) a compositum of a separable extension and an extension obtained by adjoining a p-power root of π. Then by Lemma 113.2 we can find a finite extension K_1/K such that $K^\wedge \otimes_K K_1 = M$. Let A_1, resp. $A_1^\wedge$ be the integral closure of A, resp. $A^\wedge$ in K_1, resp. M. Since $A \to A^\wedge$ is formally smooth in the $\mathfrak{m}^\wedge$-adic topology (Lemma 111.5) we see that $A_1 \to A_1^\wedge$ is formally smooth in the $\mathfrak{m}_1^\wedge$-adic topology (Lemma 114.3 and A_1 and $A_1^\wedge$ are discrete valuation rings by discussion in Remark 114.1). We conclude from Lemma 115.4 part (2) that K_1/K is a weak solution for $A \to B^\wedge$. Applying Lemma 115.4 part (1) we see that K_1/K is a weak solution for $A \to B$.

Thus we may assume A and B are complete discrete valuation rings with separably closed residue fields of characteristic p and with $\kappa_A \supset \bigcap_{n \geq 1} \kappa_B^{p^n}$. We are also given a uniformizer $\pi \in A$ and we have to find a weak solution for $A \to B$ which is a compositum of a separable extension and a field obtained by taking p-power roots of π. Note that the second condition is automatic if A has mixed characteristic.

Set $k = \bigcap_{n \geq 1} \kappa_B^{p^n}$. Observe that k is an algebraically closed field of characteristic p. If A has mixed characteristic let Λ be a Cohen ring for k and in the equicharacteristic case set $\Lambda = k[[t]]$. We can choose a ring map $\Lambda \to A$ which maps t to π in the equicharacteristic case. In the equicharacteristic case this follows from the Cohen structure theorem (Algebra, Theorem 160.8) and in the mixed characteristic case this follows as $\mathbf{Z}_p \to \Lambda$ is formally smooth in the adic topology (Lemmas 111.5 and 37.5). Applying Lemma 115.4 we see that it suffices to prove the existence of a weak solution for $\Lambda \to B$ which in the equicharacteristic p case is a compositum of a separable extension and a field obtained by taking p-power roots of t. However, since $\Lambda = k[[t]]$ in the equicharacteristic case and any extension of $k((t))$ is such a compositum, we can now drop this requirement!

Thus we arrive at the situation where A and B are complete, the residue field k of A is algebraically closed of characteristic $p > 0$, we have $k = \bigcap \kappa_B^{p^n}$, and in the mixed characteristic case p is a uniformizer of A (i.e., A is a Cohen ring for k). If A has mixed characteristic choose a Cohen ring Λ for κ_B and in the equicharacteristic case set $\Lambda = \kappa_B[[t]]$. Arguing as above we may choose a ring map $A \to \Lambda$ lifting $k \to \kappa_B$ and mapping a uniformizer to a uniformizer. Since $k \subset \kappa_B$ is separable the ring map $A \to \Lambda$ is formally smooth in the adic topology (Lemma 111.5). Hence we can find a ring map $\Lambda \to B$ such that the composition $A \to \Lambda \to B$ is the given ring map $A \to B$ (see Lemma 37.5). Since Λ and B are complete discrete valuation rings with the same residue field, B is finite over Λ (Algebra, Lemma 96.12). This reduces us to the special case discussed in Lemma 115.17. $\square$

116. Eliminating ramification, II

In this section we use the results of Section 115 to obtain (separable) solutions in some cases.

Lemma 116.1. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If K_1/K is a solution for $A \subset B$, then for any finite extension K_2/K_1 the extension K_2/K is a solution for $A \subset B$.*

Proof. This follows from Lemma 114.3. Details omitted. $\square$

Lemma 116.2. *Let $A \subset B$ be an extension of discrete valuation rings. If B is Nagata and the extension L/K of fraction fields is separable, then A is Nagata.*

Proof. A discrete valuation ring is Nagata if and only if it is N-2. Let K_1/K be a finite purely inseparable field extension. We have to show that the integral closure A_1 of A in K_1 is finite over A, see Algebra, Lemma 161.12. Since L/K is separable and K_1/K is purely inseparable, the algebra $L \otimes_K K_1$ is a field (by Algebra, Lemmas 43.6 and 46.10). Let B_1 be the integral closure of B in $L \otimes_K K_1$. Since B is Nagata, B_1 is finite over B. Since $B \otimes_A A_1 \subset B_1$ and B is Noetherian, we see that $B \otimes_A A_1$ is finite over B. As $A \to B$ is faithfully flat, this implies A_1 is finite over A, see Algebra, Lemma 83.2. $\square$

Lemma 116.3. *Let $A' \subset A$ be an extension of rings. Let $f \in A'$. Assume that (a) A is finite over A', (b) f is a nonzerodivisor on A, and (c) $A'_f = A_f$. Then there exists an integer $n_0 > 0$ such that for all $n \geq n_0$ the following is true: given a ring*

B', a nonzerodivisor $g \in B'$, and an isomorphism $\varphi' : A'/f^nA' \to B'/g^nB'$ with $\varphi'(f) \equiv g$, there is a finite extension $B' \subset B$ and an isomorphism $\varphi : A/fA \to B/gB$ compatible with φ'.

Proof. Since A is finite over A' and since $A'_f = A_f$ we can Cchoose $t > 0$ such that $f^tA \subset A'$. Set $n_0 = 2t$. Given n, B', g, φ' as in the statement of the lemma, denote $N \subset B'$ the set of elements $b \in B'$ such that $b \bmod g^nB' \in \varphi'(f^tA)$. Set $B = g^{-t}N$. As $f^tA' \subset f^tA$ and φ' sends f to g we have $g^tB' \subset N$, hence $B' \subset B$. Since $f^tA \cdot f^tA \subset f^t \cdot f^tA$ and φ' sends f to g, we see that $N \cdot N \subset g^tN$. Hence we obtain a multiplication on B extending the multiplication of B'. We have an isomorphism of A'/f^nA'-modules

$$A/f^tA' \xrightarrow{f^t} f^tA/f^nA' \xrightarrow{\varphi'} g^tB/g^nB' \xrightarrow{g^{-t}} B/g^tB'$$

where the module structures on the right are defined using φ'. Since A/f^tA' is a finite A'-module, we conclude that B/g^tB' is a finite B'-module and hence we see that $B' \to B$ is finite. Finally, we leave it to the reader to see that the displayed isomorphism of modules sends fA into gB and induces an isomorphism of rings $\varphi : A/fA \to B/gB$ compatible with φ' (it even induces an isomorphism $A/f^tA \to B/g^tB$ but we don't need this). $\square$

Remark 116.4. The construction in Lemma 116.3 satisfies the following "functoriality". Suppose we have a commutative diagram

$$\begin{tikzcd} A'_2 \arrow[r] & A_2 \\ A'_1 \arrow[r] \arrow[u] & A_1 \arrow[u] \end{tikzcd}$$

with injective horizontal arrows. Suppose given an element $f \in A'_1$ such that $(A'_1 \subset A_1, f)$ and $(A'_2 \subset A_2, f)$ satisfy properties (a), (b), (c) of Lemma 116.3. Let $n_{0,1}$ and $n_{0,2}$ be the integers found in the lemma for these two situations. Finally, let $B'_1 \to B'_2$ be a ring map, let $g \in B'_1$ be a nonzerodivisor on B_1 and B_2, let $n \geq \max(n_{0,1}, n_{0,2})$, and let a commutative diagram

$$\begin{tikzcd} A'_2/f^nA'_2 \arrow[r, "\varphi'_2"'] & B'_2/g^nB'_2 \\ A'_1/f^nA'_1 \arrow[r, "\varphi'_1"] \arrow[u] & B'_2/g^nB'_2 \arrow[u] \end{tikzcd}$$

be given whose horizontal arrows are isomorphisms and where $\varphi'_1(f) \equiv g$. Then we obtain commutative diagrams

$$\begin{tikzcd} B'_2 \arrow[r] & B_2 \\ B'_1 \arrow[r] \arrow[u] & B_1 \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_2/fA_2 \arrow[r, "\varphi_2"'] & B_2/gB_2 \\ A_1/fA_1 \arrow[r, "\varphi_1"] \arrow[u] & B_2/gB_2 \arrow[u] \end{tikzcd}$$

where $(B'_1 \subset B_1, \varphi_1)$ and $(B'_2 \subset B_2, \varphi_2)$ are constructed as in the proof of Lemma 116.3. We omit the detailed verification.

Lemma 116.5. *Let p be a prime number. Let $A \subset B$ be an extension of discrete valuation rings with fraction field extension L/K. Let $K_2/K_1/K$ be a tower of finite field extensions. Assume*

(1) *K has characteristic p,*
(2) *L/K is separable,*
(3) *B is Nagata,*
(4) *K_2 is a solution for $A \subset B$,*
(5) *K_2/K_1 is purely inseparable of degree p.*

Then there exists a separable extension K_3/K_1 which is a solution for $A \subset B$.

Proof. Let us use notation as in Remark 114.1; we will use all the observations made there. Since L/K is separable, the algebra $L_1 = L \otimes_K K_1$ is reduced (Algebra, Lemma 43.6). Since B is Nagata, the ring extension $B \subset B_1$ is finite where B_1 is the integral closure of B in L_1 and B_1 is a Nagata ring. Similarly, the ring A is Nagata by Lemma 116.2 hence $A \subset A_1$ is finite and A_1 is a Nagata ring too. Moreover, the same assertions are true for K_2, i.e., $L_2 = L \otimes_K K_2$ is reduced, the ring extensions $A_1 \subset A_2$ and $B_1 \subset B_2$ are finite where A_2, resp. B_2 is the integral closure of A, resp. B in K_2, resp. L_2.

Let $\pi \in A$ be a uniformizer. Observe that π is a nonzerodivisor on K_1, K_2, A_1, A_2, L_1, L_2, B_1, and B_2 and we have $K_1 = (A_1)_\pi$, $K_2 = (A_2)_\pi$, $L_1 = (B_1)_\pi$, and $L_2 = (B_2)_\pi$. We may write $K_2 = K_1(\alpha)$ where $\alpha^p = a_1 \in K_1$, see Fields, Lemma 14.5. After multiplying α by a power of π we may and do assume $a_1 \in A_1$. For the rest of the proof it is convenient to write $K_2 = K_1[x]/(x^p - a_1)$ and $L_2 = L_1[x]/(x^p - a_1)$. Consider the extensions of rings

$$A_2' = A_1[x]/(x^p - a_1) \subset A_2 \quad \text{and} \quad B_2' = B_1[x]/(x^p - a_1) \subset B_2$$

We may apply Lemma 116.3 to $A_2' \subset A_2$ and $f = \pi^2$ and to $B_2' \subset B_2$ and $f = \pi^2$. Choose an integer n large enough which works for both of these.

Consider the algebras

$$K_3 = K_1[x]/(x^p - \pi^{2n}x - a_1) \quad \text{and} \quad L_3 = L_1[x]/(x^p - \pi^{2n}x - a_1)$$

Observe that K_3/K_1 and L_3/L_1 are finite étale algebra extensions of degree p. Consider the subrings

$$A_3' = A_1[x]/(x^p - \pi^n x - a_1) \quad \text{and} \quad B_3' = B_1[x]/(x^p - \pi^n x - a_1)$$

of $K_3 = (A_2')_\pi$ and $L_3 = (B_3')_\pi$. We are going to construct a commutative diagram

$$\begin{tikzcd}
B_2'/\pi^{2n}B_2' \arrow[r, "\psi'"'] & B_3'/\pi^{2n}B_3' \\
A_2'/\pi^{2n}A_2' \arrow[u] \arrow[r, "\varphi'"] & A_3'/\pi^{2n}A_3' \arrow[u]
\end{tikzcd}$$

Namely, φ' is the unique A_1-algebra isomorphism sending the class of x to the class of x. Simiarly, ψ' is the unique B_1-algebra isomorphism sending the class of x to the class of x. By our choice of n we obtain, via Lemma 116.3 and Remark 116.4 finite ring extensions $A_3' \subset A_3$ and $B_3' \subset B_3$ such that $A_3' \to B_3'$ extends to a ring map $A_3 \to B_3$ and a commutative diagram

$$\begin{tikzcd} B_2/\pi^2 B_2 \arrow[r, "\psi"'] & B_3/\pi^2 B_3 \\ A_2/\pi^2 A_2 \arrow[r, "\varphi"] \arrow[u] & A_3/\pi^2 A_3 \arrow[u] \end{tikzcd}$$

with all the properties asserted in the references mentioned above (in particular φ and ψ are isomorphisms).

With all of this data in hand, we can finish the proof. Namely, we first observe that A_3 and B_3 are finite products of Dedekind domains with π contained in all of the maximal ideals. Namely, if $\mathfrak p \subset A_3$ is a maximal ideal, then $\pi \in \mathfrak p$ as $A \to A_3$ is finite. Then $\mathfrak p/\pi^2 A_3$ corresponds via φ to a maximal ideal in $A_2/\pi^2 A_2$ which is principal as A_2 is a finite product of Dedekind domains. We conclude that $\mathfrak p/\pi^2 A_3$ is principal and hence by Nakayama we see that $\mathfrak p(A_3)_{\mathfrak p}$ is principal. The same argument works for B_3. We conclude that A_3 is the integral closure of A in K_3 and that B_3 is the integral closure of B in L_3. Let $\mathfrak q \subset B_3$ be a maximal ideal lying over $\mathfrak p \subset A_3$. To finish the proof we have to show that $(A_3)_{\mathfrak p} \to (B_3)_{\mathfrak q}$ is formally smooth in the $\mathfrak q$-adic topology. By the criterion of Lemma 111.5 it suffices to show that $\mathfrak p(B_3)_{\mathfrak q} = \mathfrak q(B_3)_{\mathfrak q}$ and that the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is separable. This is true because we may check both assertions by looking at the ring map $A_3/\pi^2 A_3 \to B_3/\pi^2 B_3$ and this is isomorphic to the ring map $A_2/\pi^2 A_2 \to B_2/\pi^2 B_2$ where the corresponding statement holds by our assumption that K_2 is a solution for $A \subset B$. Some details omitted. □

Lemma 116.6. *Let $A \subset B$ be an extension of discrete valuation rings. Assume*

1. *the extension L/K of fraction fields is separable,*
2. *B is Nagata, and*
3. *there exists a solution for $A \subset B$.*

Then there exists a separable solution for $A \subset B$.

Proof. The lemma is trivial if the characteristic of K is zero; thus we may and do assume that the characteristic of K is $p > 0$.

Let K_2/K be a solution for $A \to B$. We will use induction on the inseparable degree $[K_2 : K]_i$ (Fields, Definition 14.7) of K_2/K. If $[K_2 : K]_i = 1$, then K_2 is separable over K and we are done. If not, then there exists a subfield $K_2/K_1/K$ such that K_2/K_1 is purely inseparable of degree p (Fields, Lemmas 14.6 and 14.5). By Lemma 116.5 there exists a separable extension K_3/K_1 which is a solution for $A \subset B$. Then $[K_3 : K]_i = [K_1 : K]_i = [K_2 : K]_i/p$ (Fields, Lemma 14.9) is smaller and we conclude by induction. □

Lemma 116.7. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume B is essentially of finite type over A. Let K'/K be an algebraic extension of fields such that the integral closure A' of A in K' is Noetherian. Then the integral closure B' of B in $L' = (L \otimes_K K')_{red}$ is Noetherian as well. Moreover, the map $\mathrm{Spec}(B') \to \mathrm{Spec}(A')$ is surjective and the corresponding residue field extensions are finitely generated field extensions.*

Proof. Let $A \to C$ be a finite type ring map such that B is a localization of C at a prime $\mathfrak p$. Then $C' = C \otimes_A A'$ is a finite type A'-algebra, in particular Noetherian.

Since $A \to A'$ is integral, so is $C \to C'$. Thus $B = C_{\mathfrak p} \subset C'_{\mathfrak p}$ is integral too. It follows that the dimension of $C'_{\mathfrak p}$ is 1 (Algebra, Lemma 112.4). Of course $C'_{\mathfrak p}$ is Noetherian. Let $\mathfrak q_1, \ldots, \mathfrak q_n$ be the minimal primes of $C'_{\mathfrak p}$. Let B'_i be the integral closure of $B = C_{\mathfrak p}$, or equivalently by the above of $C'_{\mathfrak p}$ in the field of fractions of $C'_{\mathfrak p'}/\mathfrak q_i$. It follows from Krull-Akizuki (Algebra, Lemma 119.12 applied to the finitely many localizations of $C'_{\mathfrak p}$ at its maximal ideals) that each B'_i is Noetherian. Moreover the residue field extensions in $C'_{\mathfrak p} \to B'_i$ are finite by Algebra, Lemma 119.10. Finally, we observe that $B' = \prod B'_i$ is the integral closure of B in $L' = (L \otimes_K K')_{red}$. $\square$

Proposition 116.8. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If B is essentially of finite type over A, then there exists a finite extension K_1/K which is a solution for $A \to B$ as defined in Definition 115.1.*

Proof. Observe that a weak solution is a solution if the residue field of A is perfect, see Lemma 111.5. Thus the proposition follows immediately from Theorem 115.18 if the residue characteristic of A is 0 (and in fact we do not need the assumption that $A \to B$ is essentially of finite type). If the residue characteristic of A is $p > 0$ we will also deduce it from Epp's theorem.

Let $x_i \in A$, $i \in I$ be a set of elements mapping to a p-base of the residue field κ of A. Set

$$A' = \bigcup\nolimits_{n \geq 1} A[t_{i,n}]/(t_{i,n}^{p^n} - x_i)$$

where the transition maps send $t_{i,n+1}$ to $t_{i,n}^p$. Observe that A' is a filtered colimit of weakly unramified finite extensions of discrete valuation rings over A. Thus A' is a discrete valuation ring and $A \to A'$ is weakly unramified. By construction the residue field $\kappa' = A'/\mathfrak m_A A'$ is the perfection of κ.

Let K' be the fraction field of A'. We may apply Lemma 116.7 to the extension K'/K. Thus B' is a finite product of Dedekind domains. Let $\mathfrak m_1, \ldots, \mathfrak m_n$ be the maximal ideals of B'. Using Epp's theorem (Theorem 115.18) we find a weak solution K'_i/K' for each of the extensions $A' \subset B'_{\mathfrak m_i}$. Since the residue field of A' is perfect, these are actually solutions. Let K'_1/K' be a finite extension which contains each K'_i. Then K'_1/K' is still a solution for each $A' \subset B'_{\mathfrak m_i}$ by Lemma 116.1.

Let A'_1 be the integral closure of A in K'_1. Note that A'_1 is a Dedekind domain by the discussion in Remark 114.1 applied to $K' \subset K'_1$. Thus Lemma 116.7 applies to K'_1/K. Therefore the integral closure B'_1 of B in $L'_1 = (L \otimes_K K'_1)_{red}$ is a Dedekind domain and because K'_1/K' is a solution for each $A' \subset B'_{\mathfrak m_i}$ we see that $(A'_1)_{A'_1 \cap \mathfrak m} \to (B'_1)_{\mathfrak m}$ is formally smooth in the $\mathfrak m$-adic topology for each maximal ideal $\mathfrak m \subset B'_1$.

By construction, the field K'_1 is a filtered colimit of finite extensions of K. Say $K'_1 = \mathrm{colim}_{i \in I} K_i$. For each i let A_i, resp. B_i be the integral closure of A, resp. B in K_i, resp. $L_i = (L \otimes_K K_i)_{red}$. Then it is clear that

$$A'_1 = \mathrm{colim}\, A_i \quad \text{and} \quad B'_1 = \mathrm{colim}\, B_i$$

Since the ring maps $A_i \to A'_1$ and $B_i \to B'_1$ are injective integral ring maps and since A'_1 and B'_1 have finite spectra, we see that for all i large enough the ring maps $A_i \to A'_1$ and $B_i \to B'_1$ are bijective on spectra. Once this is true, for all i

large enough the maps $A_i \to A_1'$ and $B_i \to B_1'$ will be weakly unramified (once the uniformizer is in the image). It follows from multiplicativity of ramification indices that $A_i \to B_i$ induces weakly unramified maps on all localizations at maximal ideals of B_i for such i. Increasing i a bit more we see that

$$B_i \otimes_{A_i} A_1' \longrightarrow B_1'$$

induces surjective maps on residue fields (because the residue fields of B_1' are finitely generated over those of A_1' by Lemma 116.7). Picture of residue fields at maximal ideals lying under a chosen maximal ideal of B_1':

$$\begin{tikzcd} \kappa_{B_i} \arrow[r] & \kappa_{B_{i'}} \arrow[r] & \ldots & \kappa_{B_1'} \\ \kappa_{A_i} \arrow[u] \arrow[r] & \kappa_{A_{i'}} \arrow[u] \arrow[r] & \ldots & \kappa_{A_1'} \arrow[u] \end{tikzcd}$$

Thus κ_{B_i} is a finitely generated extension of κ_{A_i} such that the compositum of κ_{B_i} and $\kappa_{A_1'}$ in $\kappa_{B_1'}$ is separable over $\kappa_{A_1'}$. Then that happens already at a finite stage: for example, say $\kappa_{B_1'}$ is finite separable over $\kappa_{A_1'}(x_1, \ldots, x_n)$, then just increase i such that $x_1, \ldots, x_n$ are in κ_{B_i} and such that all generators satisfy separable polynomial equations over $\kappa_{A_i}(x_1, \ldots, x_n)$. This means that $A_i \to (B_i)_\mathfrak{m}$ is formally smooth in the $\mathfrak{m}$-adic topology for all maximal ideals $\mathfrak{m}$ of B_i and the proof is complete. □

Lemma 116.9. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume*

1. *B is essentially of finite type over A,*
2. *either A or B is a Nagata ring, and*
3. *L/K is separable.*

Then there exists a separable solution for $A \to B$ (Definition 115.1).

Proof. Observe that if A is Nagata, then so is B (Algebra, Lemma 162.6 and Proposition 162.15). Thus the lemma follows on combining Proposition 116.8 and Lemma 116.6. □

117. Picard groups of rings

We first define invertible modules as follows.

Definition 117.1. Let R be a ring. An R-module M is *invertible* if the functor

$$\text{Mod}_R \longrightarrow \text{Mod}_R, \quad N \longmapsto M \otimes_R N$$

is an equivalence of categories. An invertible R-module is said to be *trivial* if it is isomorphic to R as an R-module.

Lemma 117.2. *Let R be a ring. Let M be an R-module. Equivalent are*

1. *M is finite locally free module of rank 1,*
2. *M is invertible, and*
3. *there exists an R-module N such that $M \otimes_R N \cong R$.*

Moreover, in this case the module N in (3) is isomorphic to $\text{Hom}_R(M, R)$.

Proof. Assume (1). Consider the module $N = \mathrm{Hom}_R(M, R)$ and the evaluation map $M \otimes_R N = M \otimes_R \mathrm{Hom}_R(M, R) \to R$. If $f \in R$ such that $M_f \cong R_f$, then the evaluation map becomes an isomorphism after localization at f (details omitted). Thus we see the evaluation map is an isomorphism by Algebra, Lemma 23.2. Thus (1) $\Rightarrow$ (3).

Assume (3). Then the functor $K \mapsto K \otimes_R N$ is a quasi-inverse to the functor $K \mapsto K \otimes_R M$. Thus (3) $\Rightarrow$ (2). Conversely, if (2) holds, then $K \mapsto K \otimes_R M$ is essentially surjective and we see that (3) holds.

Assume the equivalent conditions (2) and (3) hold. Denote $\psi : M \otimes_R N \to R$ the isomorphism from (3). Choose an element $\xi = \sum_{i=1,\ldots,n} x_i \otimes y_i$ such that $\psi(\xi) = 1$. Consider the isomorphisms

$$M \to M \otimes_R M \otimes_R N \to M$$

where the first arrow sends x to $\sum x_i \otimes x \otimes y_i$ and the second arrow sends $x \otimes x' \otimes y$ to $\psi(x' \otimes y)x$. We conclude that $x \mapsto \sum \psi(x \otimes y_i)x_i$ is an automorphism of M. This automorphism factors as

$$M \to R^{\oplus n} \to M$$

where the first arrow is given by $x \mapsto (\psi(x \otimes y_1), \ldots, \psi(x \otimes y_n))$ and the second arrow by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$. In this way we conclude that M is a direct summand of a finite free R-module. This means that M is finite locally free (Algebra, Lemma 78.2). Since the same is true for N by symmetry and since $M \otimes_R N \cong R$, we see that M and N both have to have rank 1. □

The set of isomorphism classes of these modules is often called the *class group* or *Picard group* of R. The group structure is determined by assigning to the isomorphism classes of the invertible modules L and L' the isomorphism class of $L \otimes_R L'$. The inverse of an invertible module L is the module

$$L^{\otimes -1} = \mathrm{Hom}_R(L, R),$$

because as seen in the proof of Lemma 117.2 the evaluation map $L \otimes_R L^{\otimes -1} \to R$ is an isomorphism. Let us denote the Picard group of R by $\mathrm{Pic}(R)$.

Lemma 117.3. *Let R be a UFD. Then* $\mathrm{Pic}(R)$ *is trivial.*

Proof. Let L be an invertible R-module. By Lemma 117.2 we see that L is a finite locally free R-module. In particular L is torsion free and finite over R. Pick a nonzero element $\varphi \in \mathrm{Hom}_R(L, R)$ of the dual invertible module. Then $I = \varphi(L) \subset R$ is an ideal which is an invertible module. Pick a nonzero $f \in I$ and let

$$f = u p_1^{e_1} \ldots p_r^{e_r}$$

be the factorization into prime elements with p_i pairwise distinct. Since L is finite locally free there exist $a_i \in R$, $a_i \notin (p_i)$ such that $I_{a_i} = (g_i)$ for some $g_i \in R_{a_i}$. Then p_i is still a prime element of the UFD R_{a_i} and we can write $g_i = p_i^{c_i} g_i'$ for some $g_i' \in R_{a_i}$ not divisible by p_i. Since $f \in I_{a_i}$ we see that $e_i \geq c_i$. We claim that I is generated by $h = p_1^{c_1} \ldots p_r^{c_r}$ which finishes the proof.

To prove the claim it suffices to show that I_a is generated by h for any $a \in R$ such that I_a is a principal ideal (Algebra, Lemma 23.2). Say $I_a = (g)$. Let $J \subset \{1, \ldots, r\}$ be the set of i such that p_i is a nonunit (and hence a prime element) in R_a. Because

$f \in I_a = (g)$ we find the prime factorization $g = v\prod_{i\in J} p_j^{b_j}$ with v a unit and $b_j \leq e_j$. For each $j \in J$ we have $I_{aa_j} = gR_{aa_j} = g_jR_{aa_j}$, in other words g and g_j map to associates in R_{aa_j}. By uniqueness of factorization this implies that $b_j = c_j$ and the proof is complete. $\square$

118. Determinants

Let R be a ring. Let M be a finite projective R-module. There exists a product decomposition $R = R_0 \times \ldots \times R_t$ such that in the corresponding decomposition $M = M_0 \times \ldots \times M_t$ of M we have that M_i is finite locally free of rank i over R_i. This follows from Algebra, Lemma 78.2 (to see that the rank is locally constant) and Algebra, Lemmas 21.3 and 24.3 (to decompose R into a product). In this situation we define

$$\det(M) = \wedge^0_{R_0}(M_0) \times \ldots \times \wedge^t_{R_t}(M_t)$$

as an R-module. This is a finite locally free module of rank 1 as each term is finite locally free of rank 1. If $\varphi : M \to N$ is an isomorphism of finite projective R-modules, then we obtain a canonical isomorphism

$$\det(\varphi) : \det(M) \longrightarrow \det(N)$$

of locally free modules of rank 1. More generally, if for all primes $\mathfrak p$ of R the ranks of the free modules $M_{\mathfrak p}$ and $N_{\mathfrak p}$ are the same, then any R-module homomorphism $\varphi : M \to N$ induces an R-module map $\det(\varphi) : \det(M) \to \det(N)$. Finally, if $M = N$ then $\det(\varphi) : \det(M) \to \det(M)$ is an endomorphism of an invertible R-module. Since $R = \mathrm{Hom}_R(L, L)$ for an invertible R-module we may and do view $\det(\varphi)$ as an element of R. In this way we obtain the *determinant*

$$\det : \mathrm{Hom}_R(M, M) \longrightarrow R$$

which is a multiplicative map.

Remark 118.1. Let R be a ring. Let M be a finite projective R-module. Then we can consider the graded commutative R-algebra exterior algebra $\wedge^*_R(M)$ on M over R. A formula for $\det(M)$ is that $\det(M) \subset \wedge^*_R(M)$ is the annihilator of $M \subset \wedge^*_R(M)$. This is sometimes useful as it does not refer to the decomposition of R into a product. Of course, to prove this satisfies the desired properties one has to either decompose R into a product (as above), or one has to look at the localizations at primes of R.

Next, we consider what happens to the determinant give a short exact sequence of finite projective modules.

Lemma 118.2. *Let R be a ring. Let*

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of finite projective R-modules. Then there is a canonical isomorphism

$$\gamma : \det(M') \otimes \det(M'') \longrightarrow \det(M)$$

First proof. First proof. Decompose R into a product of rings R_{ij} such that $M' = \prod M'_{ij}$ and $M'' = \prod M''_{ij}$ where M'_{ij} has rank i and M''_{ij} has rank j. Of course then $M = \prod M_{ij}$ and M_{ij} has rank $i + j$. This reduces us to the case where

M' and M'' have constant rank say i and j. In this case we have to construct a canonical map

$$\wedge^i(M') \otimes \wedge^j(M'') \longrightarrow \wedge^{i+j}(M)$$

To do this choose $m'_1, \ldots, m'_i$ in M' and $m''_1, \ldots, m''_j$ in M''. Denote $m_1, \ldots, m_i \in M$ the images of $m'_1, \ldots, m'_i$ and denote $m_{i+1}, \ldots, m_{i+j} \in M$ elements mapping to $m''_1, \ldots, m''_j$ in M''. Our rule will be that

$$m'_1 \wedge \ldots \wedge m'_i \otimes m''_1 \wedge \ldots \wedge m''_j \longmapsto m_1 \wedge \ldots \wedge m_{i+j}$$

We omit the detailed proof that this is well defined and an isomorphism. □

Second proof. We will use the description of $\det(M)$, $\det(M')$, and $\det(M'')$ given in Remark 118.1. Consider the R-algebra maps $\wedge^*_R(M') \to \wedge^*_R(M)$ and $\wedge^*_R(M) \to \wedge^*_R(M'')$. The first is injective and the second is surjective. Take an element $x' \in \det(M') \subset \wedge^*_R(M')$ and an element $x'' \in \det(M'') \subset \wedge^*_R(M'')$. Choose an element $y'' \in \wedge^*(M)$ mapping to x'' and set

$$\gamma(x' \otimes x'') = x' \wedge y'' \in \det(M) \subset \wedge^*_R(M)$$

The reader verifies easily by looking at localizations at primes that this well defined and an isomorphism. Moreover, this construction gives the same map as the construction given in the first proof. □

Lemma 118.3. *Let R be a ring. Let*

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "u"] & M \arrow[r] \arrow[d, "v"] & M'' \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & K' \arrow[r] & K \arrow[r] & K'' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of finite projective R-modules whose vertical arrows are isomorphisms. Then we get a commutative diagram of isomorphisms

$$\begin{tikzcd} \det(M') \otimes \det(M'') \arrow[r, "\gamma"'] \arrow[d, "\det(u)\otimes\det(w)"'] & \det(M) \arrow[d, "\det(v)"] \\ \det(K') \otimes \det(K'') \arrow[r, "\gamma"] & \det(K) \end{tikzcd}$$

where the horizontal arrows are the ones constructed in Lemma 118.2.

Proof. Omitted. Hint: use the second construction of the maps γ in Lemma 118.2. □

Lemma 118.4. *Let R be a ring. Let*

$$K \subset L \subset M$$

be R-modules such that K, L/K, and M/L are finite projective R-modules. Then the diagram

$$\begin{tikzcd} \det(K) \otimes \det(L/K) \otimes \det(M/L) \arrow[r] \arrow[d] & \det(L) \otimes \det(M/L) \arrow[d] \\ \det(K) \otimes \det(M/K) \arrow[r] & \det(M) \end{tikzcd}$$

commutes where the maps are those of Lemma 118.2.

Proof. Omitted. Hint: after localizing at a prime of R we can assume $K \subset L \subset M$ is isomorphic to $R^{\oplus a} \subset R^{\oplus a+b} \subset R^{\oplus a+b+c}$ and in this case the result is an evident computation. □

Lemma 118.5. *Let R be a ring. Let M' and M'' be two finite projective R-modules. Then the diagram*

$$\begin{tikzcd}
\det(M') \otimes \det(M'') \arrow[r] \arrow[d, "\epsilon\cdot(switch\ tensors)"'] & \det(M' \oplus M'') \arrow[d, "\det(swith\ summands)"] \\
\det(M'') \otimes \det(M') \arrow[r] & \det(M'' \oplus M')
\end{tikzcd}$$

commutes where $\epsilon = \det(-id_{M'\otimes M''}) \in R^*$ and the horizontal arrows are those of Lemma 118.2.

Proof. Omitted. □

Lemma 118.6. *Let R be a ring. Let M, N be finite projective R-modules. Let $a : M \to N$ and $b : N \to M$ be R-linear maps. Then*

$$\det(id + a \circ b) = \det(id + b \circ a)$$

as elements of R.

Proof. It suffices to prove the assertion after replacing R by a localization at a prime ideal. Thus we may assume R is local and M and N are finite free. In this case we have to prove the equality

$$\det(I_n + AB) = \det(I_m + BA)$$

of usual determinants of matrices where A has size $n \times m$ and B has size $m \times n$. This reduces to the case of the ring $R = \mathbf{Z}[a_{ij}, b_{ji}; 1 \leq i \leq n, 1 \leq j \leq m]$ where a_{ij} and b_{ij} are variables and the entries of the matrices A and B. Taking the fraction field, this reduces to the case of a field of characteristic zero. In characteristic zero there is a universal polynomial expressing the determinant of a matrix of size $\leq N$ in the traces of the powers of said matrix. Hence it suffices to prove

$$\text{Trace}((I_n + AB)^k) = \text{Trace}((I_m + BA)^k)$$

for all $k \geq 1$. Expanding we see that it suffices to prove $\text{Trace}((AB)^k) = \text{Trace}((BA)^k)$ for all $k \geq 0$. For $k = 1$ this is the well known fact that $\text{Trace}(AB) = \text{Trace}(BA)$. For $k > 1$ it follows from this by writing $(AB)^k = A(BA)^{k-1}B$ and $(BA)^k = (BA)^{k-1}AB$. □

Recall that we have defined in Algebra, Section 55 a group $K_0(R)$ as the free group on isomorphism classes of finite projective R-modules modulo the relations $[M'] + [M''] = [M' \oplus M'']$.

Lemma 118.7. *Let R be a ring. There is a map*

$$\det : K_0(R) \longrightarrow \text{Pic}(R)$$

which maps $[M]$ to the class of the invertible module $\wedge^n(M)$ if M is a finite locally free module of rank n.

Proof. This follows immediately from the constructions above and in particular Lemma 118.2 to see that the relations are mapped to 0. □

119. Perfect complexes and K-groups

We quickly show that the zeroth K-group of the derived category of perfect complexes of a ring R is the same as $K_0(R)$ defined in Algebra, Section 55.

Lemma 119.1. *Let R be a ring. There is a map*

$$c : \textit{perfect complexes over } R \longrightarrow K_0(R)$$

with the following properties

1. *$c(K[n]) = (-1)^n c(K)$ for a perfect complex K,*
2. *if $K \to L \to M \to K[1]$ is a distinguished triangle of perfect complexes, then $c(L) = c(K) + c(M)$,*
3. *if K is represented by a finite complex $M^\bullet$ consisting of finite projective modules, then $c(K) = \sum (-1)^i [M_i]$.*

Proof. Let K be a perfect object of $D(R)$. By definition we can represent K by a finite complex $M^\bullet$ of finite projective R-modules. We define c by setting

$$c(K) = \sum (-1)^n [M^n]$$

in $K_0(R)$. Of course we have to show that this is well defined, but once it is well defined, then (1) and (3) are immediate. For the moment we view the map c as defined on complexes of finite projective R-modules.

Suppose that $L^\bullet \to M^\bullet$ is a surjective map of finite complexes of finite projective R-modules. Let $K^\bullet$ be the kernel. Then we obtain short exact sequences of R-modules

$$0 \to K^n \to L^n \to M^n \to 0$$

which are split because M^n is projective. Hence $K^\bullet$ is also a finite complex of finite projective R-modules and $c(L^\bullet) = c(K^\bullet) + c(M^\bullet)$ in $K_0(R)$.

Suppose given finite complex $M^\bullet$ of finite projective R-modules which is acyclic. Say $M^n = 0$ for $n \notin [a, b]$. Then we can break $M^\bullet$ into short exact sequences

$$\begin{gathered} 0 \to M^a \to M^{a+1} \to N^{a+1} \to 0, \\ 0 \to N^{a+1} \to M^{a+2} \to N^{a+3} \to 0, \\ \ldots \\ 0 \to N^{b-3} \to M^{b-2} \to N^{b-2} \to 0, \\ 0 \to N^{b-2} \to M^{b-1} \to M^b \to 0 \end{gathered}$$

Arguing by descending induction we see that $N^{b-2}, \ldots, N^{a+1}$ are finite projective R-modules, the sequences are split exact, and

$$c(M^\bullet) = \sum (-1)[M^n] = \sum (-1)^n([N^{n-1}] + [N^n]) = 0$$

Thus our construction gives zero on acyclic complexes.

It follows formally from the results of the preceding two paragraphs that c is well defined and satisfies (2). Namely, suppose the finite complexes $M^\bullet$ and $L^\bullet$ of finite projective R-modules represent the same object of $D(R)$. Then we can represent the isomorphism by a map $f : M^\bullet \to L^\bullet$ of complexes, see Derived Categories, Lemma 19.8. We obtain a short exact sequence of complexes

$$0 \to L^\bullet \to C(f)^\bullet \to K^\bullet[1] \to 0$$

see Derived Categories, Definition 9.1. Since f is a quasi-isomorphism, the cone $C(f)^\bullet$ is acyclic (this follows for example from the discussion in Derived Categories, Section 12). Hence

$$0 = c(C(f)^\bullet) = c(L^\bullet) + c(K^\bullet[1]) = c(L^\bullet) - c(K^\bullet)$$

as desired. We omit the proof of (2) which is similar. □

The following lemma shows that $K_0(R)$ is equal to $K_0(D_{perf}(R))$.

Lemma 119.2. *Let R be a ring. Let $D_{perf}(R)$ be the derived category of perfect objects, see Lemma 78.1. The map c of Lemma 119.1 gives an isomorphism $K_0(D_{perf}(R)) = K_0(R)$.*

Proof. It follows from the definition of $K_0(D_{perf}(R))$ (Derived Categories, Definition 28.1) that c induces a homomorphism $K_0(D_{perf}(R)) \to K_0(R)$.

Given a finite projective module M over R let us denote $M[0]$ the perfect complex over R which has M sitting in degree 0 and zero in other degrees. Given a short exact sequence $0 \to M \to M' \to M'' \to 0$ of finite projective modules we obtain a distinguished triangle $M[0] \to M'[0] \to M''[0] \to M[1]$, see Derived Categories, Section 12. This shows that we obtain a map $K_0(R) \to K_0(D_{perf}(R))$ by sending $[M]$ to $[M[0]]$ with apologies for the horrendous notation.

It is clear that $K_0(R) \to K_0(D_{perf}(R)) \to K_0(R)$ is the identity. On the other hand, if $M^\bullet$ is a bounded complex of finite projective R-modules, then the the existence of the distinguished triangles of "stupid truncations" (see Homology, Section 15)

$$\sigma_{\geq n} M^\bullet \to \sigma_{\geq n-1} M^\bullet \to M^{n-1}[-n+1] \to (\sigma_{\geq n} M^\bullet)[1]$$

and induction show that

$$[M^\bullet] = \sum (-1)^i [M^i[0]]$$

in $K_0(D_{perf}(R))$ (with again apologies for the notation). Hence the map $K_0(R) \to K_0(D_{perf}(R))$ is surjective which finishes the proof. □

120. Determinants of endomorphisms of finite length modules

Let $(R, \mathfrak m, \kappa)$ be a local ring. Consider the category of pairs (M, φ) consisting of a finite length R-module and an endomorphism $\varphi : M \to M$. This category is abelian and every object is Artinian as well as Noetherian. See Homology, Section 9 for definitions.

If (M, φ) is a simple object of this category, then M is annihilated by $\mathfrak m$ since otherwise $(\mathfrak m M, \varphi|_{\mathfrak m M})$ would be a nontrivial suboject. Also $\dim_\kappa(M) = \text{length}_R(M)$ is finite. Thus we may define the determinant and the trace

$$\det\nolimits_\kappa(\varphi), \quad \text{Trace}_\kappa(\varphi)$$

as elements of κ using linear algebra. Simlarly for the characteristic polynomial of φ in this case.

By Homology, Lemma 9.6 for an arbitrary object (M, φ) of our category we have a finite filtration

$$0 \subset M_1 \subset \ldots \subset M_n = M$$

by submodules stable under φ such that $(M_i/M_{i-1}, \varphi_i)$ is a simple object of the category where $\varphi_i : M_i/M_{i-1} \to M_i/M_{i-1}$ is the induced map. We define the *determinant* of (M, φ) over κ as

$$\det_\kappa(\varphi) = \prod \det_\kappa(\varphi_i)$$

with $\det_\kappa(\varphi_i)$ as defined in the previous paragraph. We define the *trace* of (M, φ) over κ as

$$\text{Trace}_\kappa(\varphi) = \sum \text{Trace}_\kappa(\varphi_i)$$

with $\text{Trace}_\kappa(\varphi_i)$ as defined in the previous paragraph. We can similarly define the characteristic polynomial of φ over κ as the product of the characteristic polynomials of φ_i as defined in the previous paragraph. By Jordan-Hölder (Homology, Lemma 9.7) this is well defined.

Lemma 120.1. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $0 \to (M, \varphi) \to (M', \varphi') \to (M'', \varphi'') \to 0$ be a short exact sequence in the category discussed above. Then*

$$\det_\kappa(\varphi') = \det_\kappa(\varphi)\det_\kappa(\varphi''), \quad \mathit{Trace}_\kappa(\varphi') = \mathit{Trace}_\kappa(\varphi) + \mathit{Trace}_\kappa(\varphi'')$$

Also, the characteristic polynomial of φ' over κ is the product of the characteristic polynomials of φ and φ''.

Proof. Left as an exercise. □

Lemma 120.2. *Let $(R, \mathfrak{m}, \kappa) \to (R', \mathfrak{m}', \kappa')$ be a local homomorphism of local rings. Assume that κ'/κ is a finite extension. Let $u \in R'$. Then for any finite length R'-module M' we have*

$$\det_\kappa(u : M' \to M') = \mathit{Norm}_{\kappa'/\kappa}(u \bmod \mathfrak{m}')^m$$

where $m = \mathit{length}_{R'}(M')$.

Proof. Observe that the statement makes sense as $\text{length}_R(M') = \text{length}_{R'}(M')[\kappa' : \kappa]$. If $M' = \kappa'$, then the equality holds by definition of the norm as the determinant of the linear operator given by multiplication by u. In general one reduces to this case by chosing a suitable filtration and using the multiplicativity of Lemma 120.1. Some details omitted. □

Lemma 120.3. *Let $(R, \mathfrak{m}, \kappa) \to (R', \mathfrak{m}', \kappa')$ be a flat local homomorphism of local rings such that $m = \mathit{length}_{R'}(R'/\mathfrak{m}R') < \infty$. For any (M, φ) as above, the element $\det_\kappa(\varphi)^m$ maps to $\det_{\kappa'}(\varphi \otimes 1 : M \otimes_R R' \to M \otimes_R R')$ in κ'.*

Proof. The flatness of $R \to R'$ assures us that short exact sequences as in Lemma 120.1 base change to short exact sequences over R'. Hence by the multiplicativity of Lemma 120.1 we may assume that (M, φ) is a simple object of our category (see introduction to this section). In the simple case M is annihilated by $\mathfrak{m}$. Choose a filtration

$$0 \subset I_1 \subset I_2 \subset \ldots \subset I_{m-1} \subset R'/\mathfrak{m}R'$$

whose successive quotients are isomorphic to κ' as R'-modules. Then we obtain the filtration

$$0 \subset M \otimes_\kappa I_1 \subset M \otimes_\kappa I_2 \subset \ldots \subset M \otimes_\kappa I_{m-1} \subset M \otimes_\kappa R'/\mathfrak{m}R' = M \otimes_R R'$$

whose successive quotients are isomorphic to $M \otimes_\kappa \kappa'$. Also, these submodules are invariant under $\varphi \otimes 1$. By Lemma 120.1 we find

$$\det_{\kappa'}(\varphi \otimes 1 : M \otimes_R R' \to M \otimes_R R') = \det_{\kappa'}(\varphi \otimes 1 : M \otimes_\kappa \kappa' \to M \otimes_\kappa \kappa')^m = \det_\kappa(\varphi)^m$$

The last equality holds by the compatibility of determinants of linear maps with field extensions. This proves the lemma. ☐

121. A regular local ring is a UFD

We prove the result mentioned in the section title.

Lemma 121.1. *Let R be a regular local ring. Let $f \in R$. Then* $\text{Pic}(R_f) = 0$.

Proof. Let L be an invertible R_f-module. In particular L is a finite R_f-module. There exists a finite R-module M such that $M_f \cong L$, see Algebra, Lemma 126.3. By Algebra, Proposition 110.1 we see that M has a finite free resolution $F_\bullet$ over R. It follows that L is quasi-isomorphic to a finite complex of *free* R_f-modules. Hence by Lemma 119.1 we see that $[L] = n[R_f]$ in $K_0(R)$ for some $n \in \mathbf{Z}$. Applying the map of Lemma 118.7 we see that L is trivial. ☐

Lemma 121.2. *A regular local ring is a UFD.*

Proof. Recall that a regular local ring is a domain, see Algebra, Lemma 106.2. We will prove the unique factorization property by induction on the dimension of the regular local ring R. If $\dim(R) = 0$, then R is a field and in particular a UFD. Assume $\dim(R) > 0$. Let $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$. Then $R/(x)$ is regular by Algebra, Lemma 106.3, hence a domain by Algebra, Lemma 106.2, hence x is a prime element. Let $\mathfrak{p} \subset R$ be a height 1 prime. We have to show that $\mathfrak{p}$ is principal, see Algebra, Lemma 120.6. We may assume $x \notin \mathfrak{p}$, since if $x \in \mathfrak{p}$, then $\mathfrak{p} = (x)$ and we are done. For every nonmaximal prime $\mathfrak{q} \subset R$ the local ring $R_\mathfrak{q}$ is a regular local ring, see Algebra, Lemma 110.6. By induction we see that $\mathfrak{p}R_\mathfrak{q}$ is principal. In particular, the R_x-module $\mathfrak{p}_x = \mathfrak{p}R_x \subset R_x$ is a finitely presented R_x-module whose localization at any prime is free of rank 1. By Algebra, Lemma 78.2 we see that $\mathfrak{p}_x$ is an invertible R_x-module. By Lemma 121.1 we see that $\mathfrak{p}_x = (y)$ for some $y \in R_x$. We can write $y = x^e f$ for some $f \in \mathfrak{p}$ and $e \in \mathbf{Z}$. Factor $f = a_1 \ldots a_r$ into irreducible elements of R (Algebra, Lemma 120.3). Since $\mathfrak{p}$ is prime, we see that $a_i \in \mathfrak{p}$ for some i. Since $\mathfrak{p}_x = (y)$ is prime and $a_i | y$ in R_x, it follows that $\mathfrak{p}_x$ is generated by a_i in R_x, i.e., the image of a_i in R_x is prime. As x is a prime element, we find that a_i is prime in R by Algebra, Lemma 120.7. Since $(a_i) \subset \mathfrak{p}$ and $\mathfrak{p}$ has height 1 we conclude that $(a_i) = \mathfrak{p}$ as desired. ☐

Lemma 121.3. *Let R be a valuation ring with fraction field K and residue field κ. Let $R \to A$ be a homomorphism of rings such that*

1. *A is local and $R \to A$ is local,*
2. *A is flat and essentially of finite type over R,*
3. *$A \otimes_R \kappa$ regular.*

Then $\text{Pic}(A \otimes_R K) = 0$.

Proof. Let L be an invertible $A \otimes_R K$-module. In particular L is a finite module. There exists a finite A-module M such that $M \otimes_R K \cong L$, see Algebra, Lemma 126.3. We may assume M is torsion free as an R-module. Thus M is flat as an R-module (Lemma 22.10). From Lemma 25.6 we deduce that M is of finite presentation as an A-module and A is essentially of finite presentation as an R-algebra. By Lemma 83.4 we see that M is perfect relative to R, in particular M is pseudo-coherent as an A-module. By Lemma 77.6 we see that M is perfect, hence M has a finite free resolution $F_\bullet$ over A. It follows that L is quasi-isomorphic to a finite complex of *free* $A \otimes_R K$-modules. Hence by Lemma 119.1 we see that $[L] = n[A \otimes_R K]$ in $K_0(A \otimes_R K)$ for some $n \in \mathbf{Z}$. Applying the map of Lemma 118.7 we see that L is trivial. $\square$

122. Determinants of complexes

In Section 119 we have seen how to a perfect complex K over a ring R there is associated an isomorphism class of invertible R-modules, i.e., an element of $\text{Pic}(R)$. In fact, analogously to Section 118 it turns out there is a functor

$$\det : \left\{ \begin{matrix} \text{category of perfect complexes} \\ \text{morphisms are isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{category of invertible modules} \\ \text{morphisms are isomorphisms} \end{matrix} \right\}$$

Moreover, given an object (L, F) of the filtered derived category $DF(R)$ of R whose filtration is finite and whose graded parts are perfect complexes, there is a canonical isomorphism $\det(\text{gr}L) \to \det(L)$. See [KM76] for the original exposition. We will add this material later (insert future reference).

For the moment we will present an ad hoc construction in the case of perfect objects L in $D(R)$ of tor-amplitude in $[-1, 0]$. Such an object may be represented by a complex

$$L^\bullet = \ldots \to 0 \to L^{-1} \to L^0 \to 0 \to \ldots$$

with L^{-1} and L^0 finite projective R-modules, see Lemma 74.2. In this case we set

$$\det(L^\bullet) = \det(L^0) \otimes_R \det(L^{-1})^{\otimes -1} = \text{Hom}_R(\det(L^{-1}), \det(L^0))$$

Let us say a complex of this form has *rank* 0 if $L^{-1}_\mathfrak{p}$ and $L^0_\mathfrak{p}$ have the same rank for all primes of R. If $L^\bullet$ has rank 0, then we have seen in Section 118 that there is a canonical element

$$\delta(L^\bullet) \in \det(L^\bullet)$$

which is simply the determininant of $d : L^{-1} \to L^0$. Note that $\delta(L^\bullet)$ is a trivialization of $\det(L^\bullet)$ if and only if $L^\bullet$ is acyclic.

Consider a map of complexes $a^\bullet : K^\bullet \to L^\bullet$ such that

(1) $a^\bullet$ is a quasi-isomorphism,
(2) $a^n : K^n \to L^n$ is surjective for all n,
(3) K^n, L^n are finite projective R-modules, nonzero only for $n \in \{-1, 0\}$.

In this situation we will construct an isomorphism

$$\det(a^\bullet) : \det(K^\bullet) \longrightarrow \det(L^\bullet)$$

Using the exact sequences $0 \to \text{Ker}(a^i) \to K^i \to L^i \to 0$ we obtain isomorphisms

$$\gamma^i : \det(\text{Ker}(a^i)) \otimes \det(L^i) \to \det(K^i)$$

for $i = -1, 0$ by Lemma 118.2. Since $a^\bullet$ is a quasi-isomorphism the complex $\text{Ker}(a^\bullet)$ is acyclic and has rank 0. Hence the canonical element $\delta(\text{Ker}(a^\bullet))$ is a trivialization of the invertible R-module $\det(\text{Ker}(a^\bullet))$, see above. We define $\det(a^\bullet) : \det(K^\bullet) \to \det(L^\bullet)$ as the unique isomorphism such that the diagram

$$\begin{tikzcd} \det(K^\bullet) \arrow[rr, "\det(a^\bullet)"'] \arrow[dr, "\delta(\text{Ker}(a^\bullet))"'] & & \det(L^\bullet) \\ & \det(K^\bullet) \otimes \det(\text{Ker}(a^\bullet)) \arrow[ur, "\gamma^0 \otimes (\gamma^{-1})^{\otimes -1}"'] & \end{tikzcd}$$

commutes.

Lemma 122.1. *Let R be a ring. Let $a^\bullet : K^\bullet \to L^\bullet$ be a map of complexes of R-modules satisfying (1), (2), (3) above. If $L^\bullet$ has rank 0, then $\det(a^\bullet)$ maps the canonical element $\delta(K^\bullet)$ to $\delta(L^\bullet)$.*

Proof. Write $M^i = \mathrm{Ker}(a^i)$. Thus we have a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & M^{-1} \arrow[r] \arrow[d, "d_M"'] & K^{-1} \arrow[r] \arrow[d, "d_K"'] & L^{-1} \arrow[r] \arrow[d, "d_L"'] & 0 \\ 0 \arrow[r] & M^0 \arrow[r] & K^0 \arrow[r] & L^0 \arrow[r] & 0 \end{tikzcd}$$

By Lemma 118.3 we know that $\det(d_K)$ corresponds to $\det(d_M) \otimes \det(d_L)$ as maps. Unwinding the definitions this gives the required equality. $\square$

Lemma 122.2. *Let R be a ring. Let $a^\bullet : K^\bullet \to L^\bullet$ be a map of complexes of R-modules satisfying (1), (2), (3) above. Let $h : K^0 \to L^{-1}$ be a map such that $b^0 = a^0 + d \circ h$ and $b^{-1} = a^{-1} + h \circ d$ are surjective. Then $\det(a^\bullet) = \det(b^\bullet)$ as maps $\det(K^\bullet) \to \det(L^\bullet)$.*

Proof. Suppose there exists a map $\tilde{h} : K^0 \to K^{-1}$ such that $h = a^{-1} \circ \tilde{h}$ and such that $k^0 = \mathrm{id} + d \circ \tilde{h} : K^0 \to K^0$ and $k^1 = \mathrm{id} + \tilde{h} \circ d : K^{-1} \to K^{-1}$ are isomorphisms. Then we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(b^\bullet) \arrow[r] \arrow[d, "c^\bullet"'] & K^\bullet \arrow[r, "b^\bullet"'] \arrow[d, "k^\bullet"'] & L^\bullet \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(a^\bullet) \arrow[r] & K^\bullet \arrow[r, "a^\bullet"] & L^\bullet \arrow[r] & 0 \end{tikzcd}$$

of complexes, where $c^\bullet$ is the induced isomorphism of kernels. Using Lemma 118.3 we see that

$$\begin{tikzcd} \det(\mathrm{Ker}(b^i)) \otimes \det(L^i) \arrow[r] \arrow[d, "\det(c^i)\otimes 1"'] & \det(K^i) \arrow[d, "\det(k^i)"] \\ \det(\mathrm{Ker}(a^i)) \otimes \det(L^i) \arrow[r] & \det(K^i) \end{tikzcd}$$

commutes. Since $\det(c^\bullet)$ maps the canonical trivialization of $\det(\mathrm{Ker}(a^\bullet))$ to the canonical trivializatio of $\mathrm{Ker}(b^\bullet)$ (Lemma 122.1) we see that we conclude if (and only if)

$$\det(k^0) = \det(k^{-1})$$

as elements of R which follows from Lemma 118.6.

Suppose there exists a direct summand $U \subset K^{-1}$ such that both $a^{-1}|_U : U \to L^{-1}$ and $b^{-1}|_U : U \to L^{-1}$ are isomorphisms. Define $\tilde{h}$ as the composition of h with the inverse of $a^{-1}|_U$. We claim that $\tilde{h}$ is a map as in the first paragraph of the proof. Namely, we have $h = a^{-1} \circ \tilde{h}$ by construction. To show that $k^{-1} : K^{-1} \to K^{-1}$ is an isomorphism it suffices to show that it is surjective (Algebra, Lemma 16.4). Let $u \in U$. We may choose $u' \in U$ such that $b^{-1}(u') = a^{-1}(u)$. Then $u = k^{-1}(u')$. Namely, both u and $k^{-1}(u')$ are in U and $a^{-1}(u) = a^{-1}(k^{-1}(u'))$ by a calculation[17] Since $a^{-1}|_U$ is an isomorphism we get the equality. Thus $U \subset \mathrm{Im}(k^{-1})$. On the other hand, if $x \in \mathrm{Ker}(a^{-1})$ then $x = k^{-1}(x) \bmod U$. Since $K^{-1} = \mathrm{Ker}(a^{-1}) + U$ we conclude k^{-1} is surjective. Finally, we show that $k^0 : K^0 \to K^0$ is surjective.

[17] $a^{-1}(k^{-1}(u')) = a^{-1}(u') + a^{-1}(\tilde{h}(d(u'))) = a^{-1}(u') + h(d(u')) = b^{-1}(u') = a^{-1}(u)$

First, since $a^0 \circ k^0 = b^0$ we see that $a^0 \circ k^0$ is surjective. If $x \in \mathrm{Ker}(a^0)$, then $x = d(y)$ for some $y \in \mathrm{Ker}(a^{-1})$. We may write $y = k^{-1}(z)$ for some $z \in K^{-1}$ by the above. Then $x = k^0(d(z))$ and we conclude.

Final step of the proof. It suffices to find U as in the preceding paragraph, but this may not always be possible. However, in order to show equality of two maps of R-modules, it suffices to do so after localization at primes of R. Hence we may assume R is local. Then we get the following problem: suppose

$$\alpha, \beta : R^{\oplus n} \longrightarrow R^{\oplus m}$$

are two surjective R-linear maps. Find a direct summand $U \subset R^{\oplus n}$ such that both $\alpha|_U$ and $\beta|_U$ are isomorphisms. If R is a field, this is possible by linear algebra. In general, one takes a solution over the residue field and lifts this to a solution over the local ring R. Some details omitted. □

Lemma 122.3. *Let R be a ring. Let $a^\bullet : K^\bullet \to L^\bullet$ and $b^\bullet : L^\bullet \to M^\bullet$ be maps of complexes of R-modules satisfying (1), (2), (3) above. Then we have $\det(b^\bullet) \circ \det(a^\bullet) = \det(b^\bullet \circ a^\bullet)$ as maps $\det(M^\bullet) \to \det(K^\bullet)$.*

Proof. Omitted. Hints: Straightforward from Lemmas 118.2, 118.3, and 118.4. □

Lemma 122.4. *Let R be a ring. The constructions above determine a functor*

$$\det : \left\{ \begin{matrix} \textit{category of perfect complexes} \\ \textit{with tor amplitude in } [-1, 0] \\ \textit{morphisms are isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \textit{category of invertible modules} \\ \textit{morphisms are isomorphisms} \end{matrix} \right\}$$

Moreover, given a rank 0 perfect object L of $D(R)$ with tor-amplitude in $[-1, 0]$ there is a canonical element $\delta(L) \in \det(L)$ such that for any isomorphism $a : L \to K$ in $D(R)$ we have $\det(a)(\delta(L)) = \delta(K)$.

Proof. By Lemma 74.2 every object of the source category may be represented by a complex

$$L^\bullet = \ldots \to 0 \to L^{-1} \to L^0 \to 0 \to \ldots$$

with L^{-1} and L^0 finite projective R-modules. Let us temporarily call a complex of this type good. By Derived Categories, Lemma 19.8 morphisms between good complexes in the derived category are homotopy classes of maps of complexes. Thus we may work with good complexes and we can use the determinant $\det(L^\bullet) = \det(L^0) \otimes \det(L^{-1})^{\otimes -1}$ we investigated above.

Let $a^\bullet : L^\bullet \to K^\bullet$ be a morphism of good complexes which is an isomorphism in $D(R)$, i.e., a quasi-isomorphism. We say that

$$\begin{tikzcd} L^\bullet \arrow[rr, "a^\bullet"'] & & K^\bullet \\ & M^\bullet \arrow[ul, "b^\bullet"] \arrow[ur, "c^\bullet"'] & \end{tikzcd}$$

is a good diagram if it commutes up to homotopy and $b^\bullet$ and $c^\bullet$ satisfy conditions (1), (2), (3) above. Whenever we have such a diagram it makes sense to define

$$\det(a^\bullet) = \det(c^\bullet) \circ \det(b^\bullet)^{-1}$$

where $\det(c^\bullet)$ and $\det(b^\bullet)$ are the isomorphisms constructed in the text above. We will show that good diagrams always exist and that the resulting map $\det(a^\bullet)$ is independent of the choice of good diagram.

Existence of good diagrams for a quasi-isomorphism $a^\bullet : L^\bullet \to K^\bullet$ of good complexes. Choose a surjection $p : R^{\oplus n} \to K^{-1}$. Then we can consider the new good complex

$$M^\bullet = \ldots \to 0 \to L^{-1} \oplus R^{\oplus n} \xrightarrow{d \oplus 1} L^0 \oplus R^{\oplus n} \to 0 \to \ldots$$

with the projection map $b^\bullet : M^\bullet \to L^\bullet$ and the map $c^\bullet : M^\bullet \to K^\bullet$ using $a^{-1} \oplus p$ in degree -1 and using $a^0 \oplus d \circ p$ in degree 0. The maps $b^\bullet : M^\bullet \to L^\bullet$ and $c^\bullet : M^\bullet \to K^\bullet$ satisfy conditions (1), (2), (3) above and we get a good diagram.

Suppose that we have a good diagram

$$\begin{tikzcd} L^\bullet \arrow[rr, "\mathrm{id}^\bullet"'] & & L^\bullet \\ & M^\bullet \arrow[ul, "b^\bullet"] \arrow[ur, "c^\bullet"'] & \end{tikzcd}$$

Then by Lemma 122.2 we see that $\det(c^\bullet) = \det(b^\bullet)$. Thus we see that $\det(\mathrm{id}^\bullet) = \mathrm{id}$ is independent of the choice of good diagram.

Before we prove independence in general, we think about composition. Suppose we have quasi-isomorphisms $L_1^\bullet \to L_2^\bullet$ and $L_2^\bullet \to L_3^\bullet$ of good complexes and good diagrams

$$\begin{tikzcd} L_1^\bullet \arrow[rr] & & L_2^\bullet \\ & M_{12}^\bullet \arrow[ul] \arrow[ur] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L_2^\bullet \arrow[rr] & & L_3^\bullet \\ & M_{23}^\bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We can extend this to a diagram

$$\begin{tikzcd} L_1^\bullet \arrow[rr] & & L_2^\bullet \arrow[rr] & & L_3^\bullet \\ & M_{12}^\bullet \arrow[ul] \arrow[ur] & & M_{23}^\bullet \arrow[ul] \arrow[ur] & \\ & & M_{123}^\bullet \arrow[ul] \arrow[ur] & & \end{tikzcd}$$

where $M_{123}^\bullet \to M_{12}^\bullet$ and $M_{123}^\bullet \to M_{23}^\bullet$ have properties (1), (2), (3) and the square in the diagram commutes: we can just take $M_{123}^n = M_{12}^n \times_{L_2^n} M_{23}^n$. Then Lemma 122.3 shows that

$$\begin{tikzcd} \det(L_2^\bullet) & \det(M_{23}^\bullet) \arrow[l] \\ \det(M_{12}^\bullet) \arrow[u] & \det(M_{123}^\bullet) \arrow[l] \arrow[u] \end{tikzcd}$$

commutes. A diagram chase shows that the composition $\det(L_1^\bullet) \to \det(L_2^\bullet) \to \det(L_3^\bullet)$ of the maps associated to the two good diagrams using $M_{12}^\bullet$ and $M_{23}^\bullet$ is equal to the map associated to the good diagram

$$\begin{tikzcd} L_1^\bullet \arrow[rr] & & L_3^\bullet \\ & M_{123}^\bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Thus if we can show that these maps are independent of choices, then the composition law is satisfied too and we obtain our functor.

Independence. Let a quasi-isomorphism $a^\bullet : L^\bullet \to K^\bullet$ of good complexes be given. Choose an inverse quasi-isomorphism $b^\bullet : K^\bullet \to L^\bullet$. Setting $L_1^\bullet = L$, $L_2^\bullet = K^\bullet$ and $L_3^\bullet = L^\bullet$ may fix our choice of good diagram for $b^\bullet$ and consider varying good diagrams for $a^\bullet$. Then the result of the previous paragraphs is that no matter what choices, the composition always equals the identity map on $\det(L^\bullet)$. This clearly proves indepence of those choices.

The statement on canonical elements follows immediately from Lemma 122.1 and our construction. $\square$

123. Extensions of valuation rings

This section is the analogue of Section 111 for general valuation rings.

Definition 123.1. We say that $A \to B$ or $A \subset B$ is an *extension of valuation rings* if A and B are valuation rings and $A \to B$ is injective and local. Such an extension induces a commutative diagram

$$\begin{tikzcd} A \setminus \{0\} \arrow[r] \arrow[d, "v"'] & B \setminus \{0\} \arrow[d, "v"] \\ \Gamma_A \arrow[r] & \Gamma_B \end{tikzcd}$$

where Γ_A and Γ_B are the value groups. We say that B is *weakly unramified* over A if the lower horizontal arrow is a bijection. If the extension of residue fields $\kappa_A = A/\mathfrak{m}_A \subset \kappa_B = B/\mathfrak{m}_B$ is finite, then we set $f = [\kappa_B : \kappa_A]$ and we call it the *residual degree* or *residue degree* of the extension $A \subset B$.

Note that $\Gamma_A \to \Gamma_B$ is injective, because the units of A are the inverse of the units of B under the map $A \to B$. Note also, that we do not require the extension of fraction fields to be finite.

Lemma 123.2. *Let $A \subset B$ be an extension of valuation rings with fraction fields $K \subset L$. If the extension L/K is finite, then the residue field extension is finite, the index of Γ_A in Γ_B is finite, and*

$$[\Gamma_B : \Gamma_A][\kappa_B : \kappa_A] \leq [L : K].$$

Proof. Let $b_1, \ldots, b_n \in B$ be units whose images in κ_B are linearly independent over κ_A. Let $c_1, \ldots, c_m \in B$ be nonzero elements whose images in Γ_B/Γ_A are pairwise distinct. We claim that $b_i c_j$ are K-linearly independent in L. Namely, we claim a sum

$$\sum a_{ij} b_i c_j$$

with $a_{ij} \in K$ not all zero cannot be zero. Choose (i_0, j_0) with $v(a_{i_0j_0}b_{i_0}c_{j_0})$ minimal. Replace a_{ij} by $a_{ij}/a_{i_0j_0}$, so that $a_{i_0j_0} = 1$. Let

$$P = \{(i,j) \mid v(a_{ij}b_ic_j) = v(a_{i_0j_0}b_{i_0}c_{j_0})\}$$

By our choice of $c_1, \ldots, c_m$ we see that $(i,j) \in P$ implies $j = j_0$. Hence if $(i,j) \in P$, then $v(a_{ij}) = v(a_{i_0j_0}) = 0$, i.e., a_{ij} is a unit. By our choice of $b_1, \ldots, b_n$ we see that

$$\sum\nolimits_{(i,j)\in P} a_{ij}b_i$$

is a unit in B. Thus the valuation of $\sum_{(i,j)\in P} a_{ij}b_ic_j$ is $v(c_{j_0}) = v(a_{i_0j_0}b_{i_0}c_{j_0})$. Since the terms with $(i,j) \notin P$ in the first displayed sum have strictly bigger valuation, we conclude that this sum cannot be zero, thereby proving the lemma. □

Lemma 123.3. *Let A be a valuation ring with fraction field K of characteristic $p > 0$. Let L/K be a purely inseparable extension. Then the integral closure B of A in L is a valuation ring with fraction field L and $A \subset B$ is an extension of valuation rings.*

Proof. Omitted. Hints: use Algebra, Lemmas 50.5 and 36.17 for example. □

Lemma 123.4. *Let $A \to B$ be a flat local homomorphism of Noetherian local normal domains. Let $f \in A$ and $h \in B$ such that $f = wh^n$ for some $n > 1$ and some unit w of B. Assume that for every height 1 prime $\mathfrak{p} \subset A$ there is a height 1 prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$ such that the extension $A_\mathfrak{p} \subset B_\mathfrak{q}$ is weakly unramified. Then $f = ug^n$ for some $g \in A$ and unit u of A.*

Proof. The local rings of A and B at height 1 primes are discrete valuation rings (Algebra, Lemma 119.7). Thus the assumption makes sense (via Definition 111.1). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the primes of A minimal over f. These have height 1 by Algebra, Lemma 60.11. For each i let $\mathfrak{q}_{i,j} \subset B$, $j = 1, \ldots, r_i$ be the height 1 primes of B lying over $\mathfrak{p}_i$. Say we number them so that $A_{\mathfrak{p}_i} \to B_{\mathfrak{q}_{i,1}}$ is weakly unramified. Since f maps to an nth power times a unit in $B_{\mathfrak{q}_{i,1}}$ we see that the valuation v_i of f in $A_{\mathfrak{p}_i}$ is divisible by n. Say $v_i = nw_i$ for some $w_i \geq 0$. Consider the exact sequence

$$0 \to I \to A \to \prod\nolimits_{i=1,\ldots,r} A_{\mathfrak{p}_i}/\mathfrak{p}_i^{w_i}A_{\mathfrak{p}_i}$$

defining the ideal I. Applying the exact functor $- \otimes_A B$ we obtain an exact sequence

$$0 \to I \otimes_A B \to B \to \prod\nolimits_{i=1,\ldots,r} (A_{\mathfrak{p}_i}/\mathfrak{p}_i^{w_i}A_{\mathfrak{p}_i}) \otimes_A B$$

Fix i. We claim that the canonical map

$$(A_{\mathfrak{p}_i}/\mathfrak{p}_i^{w_i}A_{\mathfrak{p}_i}) \otimes_A B \to \prod\nolimits_{j=1,\ldots,r_i} B_{\mathfrak{q}_{i,j}}/\mathfrak{q}_{i,j}^{e_{i,j}w_i}B_{\mathfrak{q}_{i,j}}$$

is injective. Here $e_{i,j}$ is the ramification index of $A_{\mathfrak{p}_i} \to B_{\mathfrak{q}_{i,j}}$. The claim asserts that $\mathfrak{p}_i^{w_i}B_{\mathfrak{p}_i}$ is equal to the set of elements b of $B_{\mathfrak{p}_i}$ whose valuation at $\mathfrak{q}_{i,j}$ is $\geq e_{i,j}w_i$. Choose a generator $a \in A_{\mathfrak{p}_i}$ of the principal ideal $\mathfrak{p}_i^{w_i}$. Then the valuation of a at $\mathfrak{q}_{i,j}$ is equal to $e_{i,j}w_i$. Hence, as $B_{\mathfrak{p}_i}$ is a normal domain whose height one primes are the primes $\mathfrak{q}_{i,j}$, $j = 1, \ldots, r_i$, we see that, for b as above, we have $b/a \in B_{\mathfrak{p}_i}$ by Algebra, Lemma 157.6. Thus the claim.

The claim combined with the second exact sequence above determines an exact sequence

$$0 \to I \otimes_A B \to B \to \prod\nolimits_{i=1,\ldots,r} \prod\nolimits_{j=1,\ldots,r_i} B_{\mathfrak{q}_{i,j}}/\mathfrak{q}_{i,j}^{e_{i,j}w_i}B_{\mathfrak{q}_{i,j}}$$

It follows that $I \otimes_A B$ is the set of elements h' of B which have valuation $\geq e_{i,j}w_i$ at $\mathfrak{q}_{i,j}$. Since $f = wh^n$ in B we see that h has valuation $e_{i,j}w_i$ at $\mathfrak{q}_{i,j}$. Thus $h'/h \in B$ by Algebra, Lemma 157.6. It follows that $I \otimes_A B$ is a free B-module of rank 1 (generated by h). Therefore I is a free A-module of rank 1, see Algebra, Lemma 78.6. Let $g \in I$ be a generator. Then we see that g and h differ by a unit in B. Working backwards we conclude that the valuation of g in $A_{\mathfrak{p}_i}$ is $w_i = v_i/n$. Hence g^n and f differ by a unit in A (by Algebra, Lemma 157.6) as desired. □

Lemma 123.5. *Let A be a valuation ring. Let $A \to B$ be an étale ring map and let $\mathfrak{m} \subset B$ be a prime lying over the maximal ideal of A. Then $A \subset B_\mathfrak{m}$ is an extension of valuation rings which is weakly unramified.*

Proof. The ring A has weak dimension ≤ 1 by Lemma 104.18. Then B has weak dimension ≤ 1 by Lemmas 104.4 and 104.14. hence the local ring $B_\mathfrak{m}$ is a valuation ring by Lemma 104.18. Since the extension $A \subset B_\mathfrak{m}$ induces a finite extension of fraction fields, we see that the Γ_A has finite index in the value group of $B_\mathfrak{m}$. Thus for every $h \in B_\mathfrak{m}$ there exists an $n > 0$, an element $f \in A$, and a unit $w \in B_\mathfrak{m}$ such that $f = wh^n$ in $B_\mathfrak{m}$. We will show that this implies $f = ug^n$ for some $g \in A$ and unit $u \in A$; this will show that the value groups of A and $B_\mathfrak{m}$ agree, as claimed in the lemma.

Write $A = \operatorname{colim} A_i$ as the colimit of its local subrings which are essentially of finite type over $\mathbf{Z}$. Since A is a normal domain (Algebra, Lemma 50.3), we may assume that each A_i is normal (here we use that taking normalizations the local rings remain essentially of finite type over $\mathbf{Z}$ by Algebra, Proposition 162.16). For some i we can find an étale extension $A_i \to B_i$ such that $B = A \otimes_{A_i} B_i$, see Algebra, Lemma 143.3. Let $\mathfrak{m}_i$ be the intersection of B_i with $\mathfrak{m}$. Then we may apply Lemma 123.4 to the ring map $A_i \to (B_i)_{\mathfrak{m}_i}$ to conclude. The hypotheses of the lemma are satisfied because:

(1) A_i and $(B_i)_{\mathfrak{m}_i}$ are Noetherian as they are essentially of finite type over $\mathbf{Z}$,
(2) $A_i \to (B_i)_{\mathfrak{m}_i}$ is flat as $A_i \to B_i$ is étale,
(3) B_i is normal as $A_i \to B_i$ is étale, see Algebra, Lemma 163.9,
(4) for every height 1 prime of A_i there exists a height 1 prime of $(B_i)_{\mathfrak{m}_i}$ lying over it by Algebra, Lemma 113.2 and the fact that $\operatorname{Spec}((B_i)_{\mathfrak{m}_i}) \to \operatorname{Spec}(A_i)$ is surjective,
(5) the induced extensions $(A_i)_\mathfrak{p} \to (B_i)_\mathfrak{q}$ are unramified for every prime $\mathfrak{q}$ lying over a prime $\mathfrak{p}$ as $A_i \to B_i$ is étale.

This concludes the proof of the lemma. □

Lemma 123.6. *Let A be a valuation ring. Let A^h, resp. A^{sh} be its henselization, resp. strict henselization. Then*

$$A \subset A^h \subset A^{sh}$$

are extensions of valuation rings which induce bijections on value groups, i.e., which are weakly unramified.

Proof. Write $A^h = \operatorname{colim}(B_i)_{\mathfrak{q}_i}$ where $A \to B_i$ is étale and $\mathfrak{q}_i \subset B_i$ is a prime ideal lying over $\mathfrak{m}_A$, see Algebra, Lemma 155.7. Then Lemma 123.5 tells us that $(B_i)_{\mathfrak{q}_i}$ is a valuation ring and that the induced map

$$(A \setminus \{0\})/A^* \longrightarrow ((B_i)_{\mathfrak{q}_i} \setminus \{0\})/(B_i)^*_{\mathfrak{q}_i}$$

is bijective. By Algebra, Lemma 50.6 we conclude that A^h is a valuation ring. It also follows that $(A \setminus \{0\})/A^* \to (A^h \setminus \{0\})/(A^h)^*$ is bijective. This proves the lemma for the inclusion $A \subset A^h$. To prove it for $A \subset A^{sh}$ we can use exactly the same argument except we replace Algebra, Lemma 155.7 by Algebra, Lemma 155.11. Since $A^{sh} = (A^h)^{sh}$ we see that this also proves the assertions of the lemma for the inclusion $A^h \subset A^{sh}$. □

124. Structure of modules over a PID

We work a little bit more generally (following the papers [War69] and [War70] by Warfield) so that the proofs work over valuation rings.

Lemma 124.1. *Let P be a module over a ring R. The following are equivalent*

1. *P is a direct summand of a direct sum of modules of the form R/fR, for $f \in R$ varying.*
2. *for every short exact sequence $0 \to A \to B \to C \to 0$ of R-modules such that $fA = A \cap fB$ for all $f \in R$ the map $\mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C)$ is surjective.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence as in (2). To prove that (1) implies (2) it suffices to prove that $\mathrm{Hom}_R(R/fR, B) \to \mathrm{Hom}_R(R/fR, C)$ is surjective for every $f \in R$. Let $\psi : R/fR \to C$ be a map. Say $\psi(1)$ is the image of $b \in B$. Then $fb \in A$. Hence there exists an $a \in A$ such that $fa = fb$. Then $f(b - a) = 0$ hence we get a morphism $\varphi : R/fR \to B$ mapping 1 to $b - a$ which lifts ψ.

Conversely, assume that (2) holds. Let I be the set of pairs (f, φ) where $f \in R$ and $\varphi : R/fR \to P$. For $i \in I$ denote (f_i, φ_i) the corresponding pair. Consider the map

$$B = \bigoplus\nolimits_{i \in I} R/f_iR \longrightarrow P$$

which sends the element r in the summand R/f_iR to $\varphi_i(r)$ in P. Let $A = \mathrm{Ker}(B \to P)$. Then we see that (1) is true if the sequence

$$0 \to A \to B \to P \to 0$$

is an exact sequence as in (2). To see this suppose $f \in R$ and $a \in A$ maps to fb in B. Write $b = (r_i)_{i \in I}$ with almost all $r_i = 0$. Then we see that

$$f \sum \varphi_i(r_i) = 0$$

in P. Hence there is an $i_0 \in I$ such that $f_{i_0} = f$ and $\varphi_{i_0}(1) = \sum \varphi_i(r_i)$. Let $x_{i_0} \in R/f_{i_0}R$ be the class of 1. Then we see that

$$a' = (r_i)_{i \in I} - (0, \ldots, 0, x_{i_0}, 0, \ldots)$$

is an element of A and $fa' = a$ as desired. □

Lemma 124.2 (Generalized valuation rings)**.** *Let R be a nonzero ring. The following are equivalent*

1. *For $a, b \in R$ either a divides b or b divides a.*
2. *Every finitely generated ideal is principal and R is local.*
3. *The set of ideals of R is linearly ordered by inclusion.*

This holds in particular if R is a valuation ring.

Proof. Assume (2) and let $a, b \in R$. Then $(a, b) = (c)$. If $c = 0$, then $a = b = 0$ and a divides b. Assume $c \neq 0$. Write $c = ua + vb$ and $a = wc$ and $b = zc$. Then $c(1 - uw - vz) = 0$. Since R is local, this implies that $1 - uw - vz \in \mathfrak{m}$. Hence either w or z is a unit, so either a divides b or b divides a. Thus (2) implies (1).

Assume (1). If R has two maximal ideals $\mathfrak{m}_i$ we can choose $a \in \mathfrak{m}_1$ with $a \notin \mathfrak{m}_2$ and $b \in \mathfrak{m}_2$ with $b \notin \mathfrak{m}_1$. Then a does not divide b and b does not divide a. Hence R has a unique maximal ideal and is local. It follows easily from condition (1) and induction that every finitely generated ideal is principal. Thus (1) implies (2).

It is straightforward to prove that (1) and (3) are equivalent. The final statement is Algebra, Lemma 50.4. ☐

Lemma 124.3. *Let R be a ring satisfying the equivalent conditions of Lemma 124.2. Then every finitely presented R-module is isomorphic to a finite direct sum of modules of the form R/fR.*

Proof. Let M be a finitely presented R-module. We will use all the equivalent properties of R from Lemma 124.2 without further mention. Denote $\mathfrak{m} \subset R$ the maximal ideal and $\kappa = R/\mathfrak{m}$ the residue field. Let $I \subset R$ be the annihilator of M. Choose a basis $y_1, \ldots, y_n$ of the finite dimensional κ-vector space $M/\mathfrak{m}M$. We will argue by induction on n.

By Nakayama's lemma any collection of elements $x_1, \ldots, x_n \in M$ lifting the elements $y_1, \ldots, y_n$ in $M/\mathfrak{m}M$ generate M, see Algebra, Lemma 20.1. This immediately proves the base case $n = 0$ of the induction.

We claim there exists an index i such that for any choice of $x_i \in M$ mapping to y_i the annihilator of x_i is I. Namely, if not, then we can choose $x_1, \ldots, x_n$ such that $I_i = \text{Ann}(x_i) \neq I$ for all i. But as $I \subset I_i$ for all i, ideals being totally ordered implies I_i is strictly bigger than I for $i = 1, \ldots, n$, and by total ordering once more we would see that $\text{Ann}(M) = I_1 \cap \ldots \cap I_n$ is bigger than I which is a contradiction. After renumbering we may assume that y_1 has the property: for any $x_1 \in M$ lifting y_1 the annihilator of x_1 is I.

We set $A = Rx_1 \subset M$. Consider the exact sequence $0 \to A \to M \to M/A \to 0$. Since A is finite, we see that M/A is a finitely presented R-module (Algebra, Lemma 5.3) with fewer generators. Hence $M/A \cong \bigoplus_{j=1,\ldots,m} R/f_jR$ by induction. On the other hand, we claim that $A \to M$ satisfies the property: if $f \in R$, then $fA = A \cap fM$. The inclusion $fA \subset A \cap fM$ is trivial. Conversely, if $x \in A \cap fM$, then $x = gx_1 = fy$ for some $g \in R$ and $y \in M$. If f divides g, then $x \in fA$ as desired. If not, then we can write $f = hg$ for some $h \in \mathfrak{m}$. The element $x'_1 = x_1 - hy$ has annihilator I by the previous paragraph. Thus $g \in I$ and we see that $x = 0$ as desired. The claim and Lemma 124.1 imply the sequence $0 \to A \to M \to M/A \to 0$ is split and we find $M \cong A \oplus \bigoplus_{j=1,\ldots,m} R/f_jR$. Then $A = R/I$ is finitely presented (as a summand of M) and hence I is finitely generated, hence principal. This finishes the proof. ☐

Lemma 124.4. *Let R be a ring such that every local ring of R at a maximal ideal satisfies the equivalent conditions of Lemma 124.2. Then every finitely presented R-module is a summand of a finite direct sum of modules of the form R/fR for f in R varying.*

Proof. Let M be a finitely presented R-module. We first show that M is a summand of a direct sum of modules of the form R/fR and at the end we argue the direct sum can be taken to be finite. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules such that $fA = A \cap fB$ for all $f \in R$. By Lemma 124.1 we have to show that $\mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$ is surjective. It suffices to prove this after localization at maximal ideals $\mathfrak{m}$, see Algebra, Lemma 23.1. Note that the localized sequences $0 \to A_\mathfrak{m} \to B_\mathfrak{m} \to C_\mathfrak{m} \to 0$ satisfy the condition that $fA_\mathfrak{m} = A_\mathfrak{m} \cap fB_\mathfrak{m}$ for all $f \in R_\mathfrak{m}$ (because we can write $f = uf'$ with $u \in R_\mathfrak{m}$ a unit and $f' \in R$ and because localization is exact). Since M is finitely presented, we see that

$$\mathrm{Hom}_R(M, B)_\mathfrak{m} = \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, B_\mathfrak{m}) \quad \text{and} \quad \mathrm{Hom}_R(M, C)_\mathfrak{m} = \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, C_\mathfrak{m})$$

by Algebra, Lemma 10.2. The module $M_\mathfrak{m}$ is a finitely presented $R_\mathfrak{m}$-module. By Lemma 124.3 we see that $M_\mathfrak{m}$ is a direct sum of modules of the form $R_\mathfrak{m}/fR_\mathfrak{m}$. Thus we conclude by Lemma 124.1 that the map on localizations is surjective.

At this point we know that M is a summand of $\bigoplus_{i \in I} R/f_iR$. Consider the map $M \to \bigoplus_{i \in I} R/f_iR$. Since M is a finite R-module, the image is contained in $\bigoplus_{i \in I'} R/f_iR$ for some finite subset $I' \subset I$. This finishes the proof. $\square$

Definition 124.5. Let R be a domain.

(1) We say R is a *Bézout domain* if every finitely generated ideal of R is principal.
(2) We say R is an *elementary divisor domain* if for all $n, m \geq 1$ and every $n \times m$ matrix A, there exist invertible matrices U, V of size $n \times n, m \times m$ such that
$$UAV = \begin{pmatrix} f_1 & 0 & 0 & \ldots \\ 0 & f_2 & 0 & \ldots \\ 0 & 0 & f_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$
with $f_1, \ldots, f_{\min(n,m)} \in R$ and $f_1 | f_2 | \ldots$.

It is apparently still an open question as to whether every Bézout domain R is an elementary divisor domain (or not). This is equivalent to the question of whether every finitely presented module over R is a direct sum of cyclic modules. The converse implication is true.

Lemma 124.6. *An elementary divisor domain is Bézout.*

Proof. Let $a, b \in R$ be nonzero. Consider the 1×2 matrix $A = (a\ b)$. Then we see that $u(a\ b)V = (f\ 0)$ with $u \in R$ invertible and $V = (g_{ij})$ an invertible 2×2 matrix. Then $f = uag_{11} + ubg_{21}$ and $(g_{11}, g_{21}) = R$. It follows that $(a, b) = (f)$. An induction argument (omitted) then shows any finitely generated ideal in R is generated by one element. $\square$

Lemma 124.7. *The localization of a Bézout domain is Bézout. Every local ring of a Bézout domain is a valuation ring. A local domain is Bézout if and only if it is a valuation ring.*

Proof. We omit the proof of the statement on localizations. The final statement is Algebra, Lemma 50.15. The second statement follows from the other two. $\square$

Lemma 124.8. *Let R be a Bézout domain.*

1. *Every finite submodule of a free module is finite free.*
2. *Every finitely presented R-module M is a direct sum of a finite free module and a torsion module M_{tors} which is a summand of a module of the form $\bigoplus_{i=1,\ldots,n} R/f_iR$ with $f_1, \ldots, f_n \in R$ nonzero.*

Proof. Proof of (1). Let $M \subset F$ be a finite submodule of a free module F. Since M is finite, we may assume F is a finite free module (details omitted). Say $F = R^{\oplus n}$. We argue by induction on n. If $n = 1$, then M is a finitely generated ideal, hence principal by our assumption that R is Bézout. If $n > 1$, then we consider the image I of M under the projection $R^{\oplus n} \to R$ onto the last summand. If $I = (0)$, then $M \subset R^{\oplus n-1}$ and we are done by induction. If $I \neq 0$, then $I = (f) \cong R$. Hence $M \cong R \oplus \text{Ker}(M \to I)$ and we are done by induction as well.

Let M be a finitely presented R-module. Since the localizations of R are maximal ideals are valuation rings (Lemma 124.7) we may apply Lemma 124.4. Thus M is a summand of a module of the form $R^{\oplus r} \oplus \bigoplus_{i=1,\ldots,n} R/f_iR$ with $f_i \neq 0$. Since taking the torsion submodule is a functor we see that M_{tors} is a summand of the module $\bigoplus_{i=1,\ldots,n} R/f_iR$ and M/M_{tors} is a summand of $R^{\oplus r}$. By the first part of the proof we see that M/M_{tors} is finite free. Hence $M \cong M_{tors} \oplus M/M_{tors}$ as desired. □

Lemma 124.9. *Let R be a PID. Every finite R-module M is of isomorphic to a module of the form*

$$R^{\oplus r} \oplus \bigoplus\nolimits_{i=1,\ldots,n} R/f_iR$$

for some $r, n \geq 0$ and $f_1, \ldots, f_n \in R$ nonzero.

Proof. A PID is a Noetherian Bézout ring. By Lemma 124.8 it suffices to prove the result if M is torsion. Since M is finite, this means that the annihilator of M is nonzero. Say $fM = 0$ for some $f \in R$ nonzero. Then we can think of M as a module over R/fR. Since R/fR is Noetherian of dimension 0 (small detail omitted) we see that $R/fR = \prod R_j$ is a finite product of Artinian local rings R_i (Algebra, Proposition 60.7). Each R_i, being a local ring and a quotient of a PID, is a generalized valuation ring in the sense of Lemma 124.2 (small detail omitted). Write $M = \prod M_j$ with $M_j = e_jM$ where $e_j \in R/fR$ is the idempotent corresponding to the factor R_j. By Lemma 124.3 we see that $M_j = \bigoplus_{i=1,\ldots,n_j} R_j/\overline{f}_{ji}R_j$ for some $\overline{f}_{ji} \in R_j$. Choose lifts $f_{ji} \in R$ and choose $g_{ji} \in R$ with $(g_{ji}) = (f_j, f_{ji})$. Then we conclude that

$$M \cong \bigoplus R/g_{ji}R$$

as an R-module which finishes the proof. □

One can also prove that a PID is a elementary divisor domain (insert future reference here), by proving lemmas similar to the following.

Lemma 124.10. *Let R be a Bézout domain. Let $n \geq 1$ and $f_1, \ldots, f_n \in R$ generate the unit ideal. There exists an invertible $n \times n$ matrix in R whose first row is $f_1 \ldots f_n$.*

Proof. This follows from Lemma 124.8 but we can also prove it directly as follows. By induction on n. The result holds for $n = 1$. Assume $n > 1$. We may assume $f_1 \neq 0$ after renumbering. Choose $f \in R$ such that $(f) = (f_1, \ldots, f_{n-1})$. Let A be an $(n-1) \times (n-1)$ matrix whose first row is $f_1/f, \ldots, f_{n-1}/f$. Choose $a, b \in R$

such that $af - bf_n = 1$ which is possible because $1 \in (f_1, \ldots, f_n) = (f, f_n)$. Then a solution is the matrix

$$\begin{pmatrix} f & 0 & \ldots & 0 & f_n \\ 0 & 1 & \ldots & 0 & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & 1 & 0 \\ b & 0 & \ldots & 0 & a \end{pmatrix} \begin{pmatrix} & & & 0 \\ & A & & \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

Observe that the left matrix is invertible because it has determinant 1. $\square$

125. Principal radical ideals

In this section we prove that a catenary Noetherian normal local domain there exists a nontrivial principal radical ideal. This result can be found in [Art86].

Lemma 125.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension one, and let $x \in \mathfrak{m}$ be an element not contained in any minimal prime of R. Then*

1. *the function $P : n \mapsto \text{length}_R(R/x^nR)$ satisfies $P(n) \leq nP(1)$ for $n \geq 0$,*
2. *if x is a nonzerodivisor, then $P(n) = nP(1)$ for $n \geq 0$.*

Proof. Since $\dim(R) = 1$, we have $\dim(R/x^nR) = 0$ and so $\text{length}_R(R/x^nR)$ is finite for each n (Algebra, Lemma 62.3). To show the lemma we will induct on n. Since $x^0R = R$, we have that $P(0) = \text{length}_R(R/x^0R) = \text{length}_R 0 = 0$. The statement also holds for $n = 1$. Now let $n \geq 2$ and suppose the statement holds for $n - 1$. The following sequence is exact

$$R/x^{n-1}R \xrightarrow{x} R/x^nR \to R/xR \to 0$$

where x denotes the multiplication by x map. Since length is additive (Algebra, Lemma 52.3), we have that $P(n) \leq P(n-1) + P(1)$. By induction $P(n-1) \leq (n-1)P(1)$, whence $P(n) \leq nP(1)$. This proves the induction step.

If x is a nonzerodivisor, then the displayed exact sequence above is exact on the left also. Hence we get $P(n) = P(n-1) + P(1)$ for all $n \geq 1$. $\square$

Lemma 125.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $x \in \mathfrak{m}$ be an element not contained in any minimal prime of R. Let t be the number of minimal prime ideals of R. Then $t \leq \text{length}_R(R/xR)$.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal prime ideals of R. Set $R' = R/\sqrt{0} = R/(\bigcap_{i=1}^t \mathfrak{p}_i)$. We claim it suffices to prove the lemma for R'. Namely, it is clear that R' has t minimal primes too and $\text{length}_{R'}(R'/xR') = \text{length}_R(R'/xR')$ is less than $\text{length}_R(R/xR)$ as there is a surjection $R/xR \to R'/xR'$. Thus we may assume R is reduced.

Assume R is reduced with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. This means there is an exact sequence

$$0 \to R \to \prod\nolimits_{i=1}^t R/\mathfrak{p}_i \to Q \to 0$$

Here Q is the cokernel of the first map. Write $M = \prod_{i=1}^t R/\mathfrak{p}_i$. Localizing at $\mathfrak{p}_j$ we see that

$$R_{\mathfrak{p}_j} \to M_{\mathfrak{p}_j} = \left(\prod\nolimits_{i=1}^t R/\mathfrak{p}_i\right)_{\mathfrak{p}_j} = (R/\mathfrak{p}_j)_{\mathfrak{p}_j}$$

is surjective. Thus $Q_{\mathfrak{p}_j} = 0$ for all j. We conclude that $\text{Supp}(Q) = \{\mathfrak{m}\}$ as $\mathfrak{m}$ is the only prime of R different from the $\mathfrak{p}_i$. It follows that Q has finite length (Algebra, Lemma 62.3). Since $\text{Supp}(Q) = \{\mathfrak{m}\}$ we can pick an $n \gg 0$ such that x^n acts as 0 on Q (Algebra, Lemma 62.4). Now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "x^n"] & M \arrow[r] \arrow[d, "x^n"] & Q \arrow[r] \arrow[d, "x^n"] & 0 \\
0 \arrow[r] & R \arrow[r] & M \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are multiplication by x^n. This is injective on R and on M since x is not contained in any of the $\mathfrak{p}_i$. By the snake lemma (Algebra, Lemma 4.1), the following sequence is exact:

$$0 \to Q \to R/x^nR \to M/x^nM \to Q \to 0$$

Hence we find that $\text{length}_R(R/x^nR) = \text{length}_R(M/x^nM)$ for large enough n. Writing $R_i = R/\mathfrak{p}_i$ we see that $\text{length}(M/x^nM) = \sum_{i=1}^t \text{length}_R(R_i/x^nR_i)$. Applying Lemma 125.1 and the fact that x is a nonzerodivisor on R and R_i, we conclude that

$$n\text{length}_R(R/xR) = \sum\nolimits_{i=1}^t n\text{length}_{R_i}(R_i/xR_i)$$

Since $\text{length}_{R_i}(R_i/xR_i) \geq 1$ the lemma is proved. □

Lemma 125.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 1$, let $f \in \mathfrak{m}$ be an element not contained in any minimal prime ideal of R, and let $k \in \mathbf{N}$. Then there exist elements $g_1, \ldots, g_{d-1} \in \mathfrak{m}^k$ such that $f, g_1, \ldots, g_{d-1}$ is a system of parameters.*

Proof. We have $\dim(R/fR) = d - 1$ by Algebra, Lemma 60.13. Choose a system of parameters $\overline{g}_1, \ldots, \overline{g}_{d-1}$ in R/fR (Algebra, Proposition 60.9) and take lifts $g_1, \ldots, g_{d-1}$ in R. It is straightforward to see that $f, g_1, \ldots, g_{d-1}$ is a system of parameters in R. Then $f, g_1^k, \ldots, g_{d-1}^k$ is also a system of parameters and the proof is complete. □

Lemma 125.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension two, and let $f \in \mathfrak{m}$ be an element not contained in any minimal prime ideal of R. Then there exist $g \in \mathfrak{m}$ and $N \in \mathbf{N}$ such that*

(a) *f, g form a system of parameters for R.*
(b) *If $h \in \mathfrak{m}^N$, then $f + h, g$ is a system of parameters and $\text{length}_R(R/(f, g)) = \text{length}_R(R/(f + h, g))$.*

Proof. By Lemma 125.3 there exists a $g \in \mathfrak{m}$ such that f, g is a system of parameters for R. Then $\mathfrak{m} = \sqrt{(f, g)}$. Thus there exists an n such that $\mathfrak{m}^n \subset (f, g)$, see Algebra, Lemma 32.5. We claim that $N = n + 1$ works. Namely, let $h \in \mathfrak{m}^N$. By our choice of N we can write $h = af + bg$ with $a, b \in \mathfrak{m}$. Thus

$$(f + h, g) = (f + af + bg, g) = ((1 + a)f, g) = (f, g)$$

because $1 + a$ is a unit in R. This proves the equality of lengths and the fact that $f + h, g$ is a system of parameters. □

Lemma 125.5. *Let R be a Noetherian local normal domain of dimension 2. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be pairwise distinct primes of height 1. There exists a nonzero element $f \in \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ such that R/fR is reduced.*

Proof. Let $f \in \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ be a nonzero element. We will modify f slightly to obtain an element that generates a radical ideal. The localization $R_\mathfrak{p}$ of R at each height one prime ideal $\mathfrak{p}$ is a discrete valuation ring, see Algebra, Lemma 119.7 or Algebra, Lemma 157.4. We denote by $\text{ord}_\mathfrak{p}(f)$ the corresponding valuation of f in $R_\mathfrak{p}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be the distinct height one prime ideals containing f. Write $\text{ord}_{\mathfrak{q}_j}(f) = m_j \geq 1$ for each j. Then we define $\text{div}(f) = \sum_{j=1}^{s} m_j \mathfrak{q}_j$ as a formal linear combination of height one primes with integer coefficients. Note for later use that each of the primes $\mathfrak{p}_i$ occurs among the primes $\mathfrak{q}_j$. The ring R/fR is reduced if and only if $m_j = 1$ for $j = 1, \ldots, s$. Namely, if m_j is 1 then $(R/fR)\mathfrak{q}_j$ is reduced and $R/fR \subset \prod (R/fR)_{\mathfrak{q}_j}$ as $\mathfrak{q}_1, \ldots, \mathfrak{q}_j$ are the associated primes of R/fR, see Algebra, Lemmas 63.19 and 157.6.

Choose and fix g and N as in Lemma 125.4. For a nonzero $y \in R$ denote $t(y)$ the number of primes minimal over y. Since R is a normal domain, these primes are height one and correspond 1-to-1 to the minimal primes of R/yR (Algebra, Lemmas 60.11 and 157.6). For example $t(f) = s$ is the number of primes $\mathfrak{q}_j$ occurring in $\text{div}(f)$. Let $h \in \mathfrak{m}^N$. By Lemma 125.2 we have

$$\begin{aligned} t(f+h) &\leq \text{length}_{R/(f+h)}(R/(f+h, g)) \\ &= \text{length}_R(R/(f+h, g)) \\ &= \text{length}_R(R/(f, g)) \end{aligned}$$

see Algebra, Lemma 52.5 for the first equality. Therefore we see that $t(f+h)$ is bounded independent of $h \in \mathfrak{m}^N$.

By the boundedness proved above we may pick $h \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ such that $t(f+h)$ is maximal among such h. Set $f' = f + h$. Given $h' \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ we see that the number $t(f' + h') \leq t(f+h)$. Thus after replacing f by f' we may assume that for every $h \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ we have $t(f+h) \leq s$.

Next, assume that we can find an element $h \in \mathfrak{m}^N$ such that for each j we have $\text{ord}_{\mathfrak{q}_j}(h) \geq 1$ and $\text{ord}_{\mathfrak{q}_j}(h) = 1 \Leftrightarrow m_j > 1$. Observe that $h \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$. Then $\text{ord}_{\mathfrak{q}_j}(f+h) = 1$ for every j by elementary properties of valuations. Thus

$$\text{div}(f+h) = \sum\nolimits_{j=1}^{s} \mathfrak{q}_j + \sum\nolimits_{k=1}^{v} e_k \mathfrak{r}_k$$

for some pairwise distinct height one prime ideals $\mathfrak{r}_1, \ldots, \mathfrak{r}_v$ and $e_k \geq 1$. However, since $s = t(f) \geq t(f+h)$ we see that $v = 0$ and we have found the desired element.

Now we will pick h that satisfies the above criteria. By prime avoidance (Algebra, Lemma 15.2) for each $1 \leq j \leq s$ we can find an element $a_j \in \mathfrak{q}_j$ such that $a_j \notin \mathfrak{q}_{j'}$ for $j' \neq j$ and $a_j \notin \mathfrak{q}_j^{(2)}$. Here $\mathfrak{q}_j^{(2)} = \{x \in R \mid \text{ord}_{\mathfrak{q}_j}(x) \geq 2\}$ is the second symbolic power of $\mathfrak{q}_j$. Then we take

$$h = \prod\nolimits_{m_j = 1} a_j^2 \times \prod\nolimits_{m_j > 1} a_j$$

Then h clearly satisfies the conditions on valuations imposed above. If $h \notin \mathfrak{m}^N$, then we multiply by an element of $\mathfrak{m}^N$ which is not contained in $\mathfrak{q}_j$ for all j. $\square$

Lemma 125.6. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian normal local domain of dimension 2. If $a \in \mathfrak{m}$ is nonzero, then there exists an element $c \in A$ such that A/cA is reduced and such that a divides c^n for some n.*

Proof. Let $\text{div}(a) = \sum_{i=1,\ldots,r} n_i\mathfrak{p}_i$ with notation as in the proof of Lemma 125.5. Choose $c \in \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ with A/cA reduced, see Lemma 125.5. For $n \geq \max(n_i)$ we see that $-\text{div}(a) + \text{div}(c^n)$ is an effective divisor (all coefficients nonnegative). Thus $c^n/a \in A$ by Algebra, Lemma 157.6. $\square$

In the rest of this section we prove the result in dimension > 2.

Lemma 125.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, let $g_1, \ldots, g_d$ be a system of parameters, and let $I = (g_1, \ldots, g_d)$. If $e_I/d!$ is the leading coefficient of the numerical polynomial $n \mapsto \text{length}_R(R/I^{n+1})$, then $e_I \leq \text{length}_R(R/I)$.*

Proof. The function is a numerical polynomial by Algebra, Proposition 59.5. It has degree d by Algebra, Proposition 60.9. If $d = 0$, then the result is trivial. If $d = 1$, then the result is Lemma 125.1. To prove it in general, observe that there is a surjection

$$\bigoplus\nolimits_{i_1,\ldots,i_d \geq 0,\ \sum i_j = n} R/I \longrightarrow I^n/I^{n+1}$$

sending the basis element corresponding to $i_1, \ldots, i_d$ to the class of $g_1^{i_1} \ldots g_d^{i_d}$ in I^n/I^{n+1}. Thus we see that

$$\text{length}_R(R/I^{n+1}) - \text{length}_R(R/I^n) \leq \text{length}_R(R/I)\binom{n+d-1}{d-1}$$

Since $d \geq 2$ the numerical polynomial on the left has degree $d-1$ with leading coefficient $e_I/(d-1)!$. The polynomial on the right has degree $d-1$ and its leading coefficient is $\text{length}_R(R/I)/(d-1)!$. This proves the lemma. $\square$

Lemma 125.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, let t be the number of minimal prime ideals of R of dimension d, and let $(g_1, \ldots, g_d)$ be a system of parameters. Then $t \leq \text{length}_R(R/(g_1, \ldots, g_n))$.*

Proof. If $d = 0$ the lemma is trivial. If $d = 1$ the lemma is Lemma 125.2. Thus we may assume $d > 1$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals of R where the first t have dimension d, and denote $I = (g_1, \ldots, g_n)$. Arguing in exactly the same way as in the proof of Lemma 125.2 we can assume R is reduced.

Assume R is reduced with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. This means there is an exact sequence

$$0 \to R \to \prod\nolimits_{i=1}^t R/\mathfrak{p}_i \to Q \to 0$$

Here Q is the cokernel of the first map. Write $M = \prod_{i=1}^t R/\mathfrak{p}_i$. Localizing at $\mathfrak{p}_j$ we see that

$$R_{\mathfrak{p}_j} \to M_{\mathfrak{p}_j} = \left(\prod\nolimits_{i=1}^t R/\mathfrak{p}_i\right)_{\mathfrak{p}_j} = (R/\mathfrak{p}_j)_{\mathfrak{p}_j}$$

is surjective. Thus $Q_{\mathfrak{p}_j} = 0$ for all j. Therefore no height 0 prime of R is in the support of Q. It follows that the degree of the numerical polynomial $n \mapsto \text{length}_R(Q/I^nQ)$ equals $\dim(\text{Supp}(Q)) < d$, see Algebra, Lemma 62.6. By Algebra, Lemma 59.10 (which applies as R does not have finite length) the polynomial

$$n \longmapsto \text{length}_R(M/I^nM) - \text{length}_R(R/I^n) - \text{length}_R(Q/I^nQ)$$

has degree $< d$. Since $M = \prod R/\mathfrak{p}_i$ and since $n \to \text{length}_R(R/\mathfrak{p}_i + I^n)$ is a numerical polynomial of degree exactly(!) d for $i = 1, \ldots, t$ (by Algebra, Lemma 62.6) we see that the leading coefficient of $n \mapsto \text{length}_R(M/I^nM)$ is at least $t/d!$. Thus we conclude by Lemma 125.7. $\square$

Lemma 125.9. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and let $f \in \mathfrak{m}$ be an element not contained in any minimal prime ideal of R. Then there exist elements $g_1, \ldots, g_{d-1} \in \mathfrak{m}$ and $N \in \mathbf{N}$ such that*

(1) *$f, g_1, \ldots, g_{d-1}$ form a system of parameters for R*
(2) *If $h \in \mathfrak{m}^N$, then $f + h, g_1, \ldots, g_{d-1}$ is a system of parameters and we have $\text{length}_R R/(f, g_1, \ldots, g_{d-1}) = \text{length}_R R/(f + h, g_1, \ldots, g_{d-1})$.*

Proof. By Lemma 125.3 there exist $g_1, \ldots, g_{d-1} \in \mathfrak{m}$ such that $f, g_1, \ldots, g_{d-1}$ is a system of parameters for R. Then $\mathfrak{m} = \sqrt{(f, g_1, \ldots, g_{d-1})}$. Thus there exists an n such that $\mathfrak{m}^n \subset (f, g)$, see Algebra, Lemma 32.5. We claim that $N = n + 1$ works. Namely, let $h \in \mathfrak{m}^N$. By our choice of N we can write $h = af + \sum b_i g_i$ with $a, b_i \in \mathfrak{m}$. Thus

$$
\begin{aligned}
(f + h, g_1, \ldots, g_{d-1}) &= (f + af + \sum b_i g_i, g_1, \ldots, g_{d-1}) \\
&= ((1 + a)f, g_1, \ldots, g_{d-1}) \\
&= (f, g_1, \ldots, g_{d-1})
\end{aligned}
$$

because $1 + a$ is a unit in R. This proves the equality of lengths and the fact that $f + h, g_1, \ldots, g_{d-1}$ is a system of parameters. ☐

Proposition 125.10. *Let R be a catenary Noetherian local normal domain. Let $J \subset R$ be a radical ideal. Then there exists a nonzero element $f \in J$ such that R/fR is reduced.*

Proof. The proof is the same as that of Lemma 125.5, using Lemma 125.8 instead of Lemma 125.2 and Lemma 125.9 instead of Lemma 125.4. We can use Lemma 125.8 because R is a catenary domain, so every height one prime ideal of R has dimension $d - 1$, and hence the spectrum of $R/(f + h)$ is equidimensional. For the convenience of the reader we write out the details.

Let $f \in J$ be a nonzero element. We will modify f slightly to obtain an element that generates a radical ideal. The localization $R_\mathfrak{p}$ of R at each height one prime ideal $\mathfrak{p}$ is a discrete valuation ring, see Algebra, Lemma 119.7 or Algebra, Lemma 157.4. We denote by $\text{ord}_\mathfrak{p}(f)$ the corresponding valuation of f in $R_\mathfrak{p}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be the distinct height one prime ideals containing f. Write $\text{ord}_{\mathfrak{q}_j}(f) = m_j \geq 1$ for each j. Then we define $\text{div}(f) = \sum_{j=1}^{s} m_j \mathfrak{q}_j$ as a formal linear combination of height one primes with integer coefficients. The ring R/fR is reduced if and only if $m_j = 1$ for $j = 1, \ldots, s$. Namely, if m_j is 1 then $(R/fR)\mathfrak{q}_j$ is reduced and $R/fR \subset \prod (R/fR)_{\mathfrak{q}_j}$ as $\mathfrak{q}_1, \ldots, \mathfrak{q}_j$ are the associated primes of R/fR, see Algebra, Lemmas 63.19 and 157.6.

Choose and fix $g_2, \ldots, g_{d-1}$ and N as in Lemma 125.9. For a nonzero $y \in R$ denote $t(y)$ the number of primes minimal over y. Since R is a normal domain, these primes are height one and correspond 1-to-1 to the minimal primes of R/yR (Algebra, Lemmas 60.11 and 157.6). For example $t(f) = s$ is the number of primes $\mathfrak{q}_j$ occurring in $\text{div}(f)$. Let $h \in \mathfrak{m}^N$. Because R is catenary, for each height one prime $\mathfrak{p}$ of R we have $\dim(R/\mathfrak{p}) = d$. Hence by Lemma 125.8 we have

$$
\begin{aligned}
t(f + h) &\leq \text{length}_{R/(f+h)}(R/(f + h, g_1, \ldots, g_{d-1})) \\
&= \text{length}_R(R/(f + h, g_1, \ldots, g_{d-1})) \\
&= \text{length}_R(R/(f, g_1, \ldots, g_{d-1}))
\end{aligned}
$$

see Algebra, Lemma 52.5 for the first equality. Therefore we see that $t(f+h)$ is bounded independent of $h \in \mathfrak{m}^N$.

By the boundedness proved above we may pick $h \in \mathfrak{m}^N \cap J$ such that $t(f+h)$ is maximal among such h. Set $f' = f + h$. Given $h' \in \mathfrak{m}^N \cap J$ we see that the number $t(f' + h') \leq t(f+h)$. Thus after replacing f by f' we may assume that for every $h \in \mathfrak{m}^N \cap J$ we have $t(f+h) \leq s$.

Next, assume that we can find an element $h \in \mathfrak{m}^N \cap J$ such that for each j we have $\text{ord}_{\mathfrak{q}_j}(h) \geq 1$ and $\text{ord}_{\mathfrak{q}_j}(h) = 1 \Leftrightarrow m_j > 1$. Then $\text{ord}_{\mathfrak{q}_j}(f+h) = 1$ for every j by elementary properties of valuations. Thus

$$\text{div}(f+h) = \sum\nolimits_{j=1}^{s} \mathfrak{q}_j + \sum\nolimits_{k=1}^{v} e_k \mathfrak{r}_k$$

for some pairwise distinct height one prime ideals $\mathfrak{r}_1, \ldots, \mathfrak{r}_v$ and $e_k \geq 1$. However, since $s = t(f) \geq t(f+h)$ we see that $v = 0$ and we have found the desired element.

Now we will pick h that satisfies the above criteria. By prime avoidance (Algebra, Lemma 15.2) for each $1 \leq j \leq s$ we can find an element $a_j \in \mathfrak{q}_j \cap J$ such that $a_j \not\in \mathfrak{q}_{j'}$ for $j' \neq j$. Next, we can pick $b_j \in J \cap \mathfrak{q}_1 \cap \ldots \cap q_s$ with $b_j \not\in \mathfrak{q}_j^{(2)}$. Here $\mathfrak{q}_j^{(2)} = \{x \in R \mid \text{ord}_{\mathfrak{q}_j}(x) \geq 2\}$ is the second symbolic power of $\mathfrak{q}_j$. Prime avoidance applies because the ideal $J' = J \cap \mathfrak{q}_1 \cap \ldots \cap q_s$ is radical, hence R/J' is reduced, hence $(R/J')_{\mathfrak{q}_j}$ is reduced, hence J' contains an element x with $\text{ord}_{\mathfrak{q}_j}(x) = 1$, hence $J' \not\subset \mathfrak{q}_j^{(2)}$. Then the element

$$c = \sum\nolimits_{j=1,\ldots,s} b_j \times \prod\nolimits_{j' \neq j} a_{j'}$$

is an element of J with $\text{ord}_{\mathfrak{q}_j}(c) = 1$ for all $j = 1, \ldots, s$ by elementary properties of valuations. Finally, we let

$$h = c \times \prod\nolimits_{m_j = 1} a_j \times y$$

where $y \in \mathfrak{m}^N$ is an element which is not contained in $\mathfrak{q}_j$ for all j. □

126. Invertible objects in the derived category

We characterize invertible objects in the derived category of a ring.

Lemma 126.1. *Let R be a ring. The derived category $D(R)$ of R is a symmetric monoidal category with tensor product given by derived tensor product and associativity and commutativity constraints as in Section 72.*

Proof. Omitted. Hints: The associativity constraint is the isomorphism of Lemma 59.15 and the commutativity constraint is the isomorphism of Lemma 59.14. Having said this the commutativity of various diagrams follows from the corresponding result for the category of complexes of R-modules, see Section 58. □

Thus we know what it means for an object of $D(R)$ to have a (left) dual or to be invertible. Before we can work out what this amounts to we need a simple lemma.

Lemma 126.2. *Let R be a ring. Let $F^\bullet$ be a bounded above complex of free R-modules. Given pairs (n_i, f_i), $i = 1, \ldots, N$ with $n_i \in \mathbf{Z}$ and $f_i \in F^{n_i}$ there exists a subcomplex $G^\bullet \subset F^\bullet$ containing all f_i which is bounded and consists of finite free R-modules.*

Proof. By descending induction on $a = \min(n_i; i = 1, \ldots, N)$. If $F^n = 0$ for $n \geq a$, then the result is true with $G^\bullet$ equal to the zero complex. In general, after renumbering we may assume there exists an $1 \leq r \leq N$ such that $n_1 = \ldots = n_r = a$ and $n_i > a$ for $i > r$. Choose a basis $b_j, j \in J$ for F^a. We can choose a finite subset $J' \subset J$ such that $f_i \in \bigoplus_{j \in J'} Rb_j$ for $i = 1, \ldots, r$. Choose a basis $c_k, k \in K$ for F^{a+1}. We can choose a finite subset $K' \subset K$ such that $\mathrm{d}_F^a(b_j) \in \bigoplus_{k \in K'} Rc_k$ for $j \in J'$. Then we can apply the induction hypothesis to find a subcomplex $H^\bullet \subset F^\bullet$ containing $c_k \in F^{a+1}$ for $k \in K'$ and $f_i \in F^{n_i}$ for $i > r$. Take $G^\bullet$ equal to $H^\bullet$ in degrees $> a$ and equal to $\bigoplus_{j \in J'} Rb_j$ in degree a. $\square$

Lemma 126.3. *Let R be a ring. Let M be an object of $D(R)$. The following are equivalent*

1. *M has a left dual in $D(R)$ as in Categories, Definition 43.5,*
2. *M is a perfect object of $D(R)$.*

Moreover, in this case the left dual of M is the object $M^\vee$ of Lemma 74.15.

Proof. If M is perfect, then we can represent M by a bounded complex $M^\bullet$ of finite projective R-modules. In this case $M^\bullet$ has a left dual in the category of complexes by Lemma 72.2 which is a fortiori a left dual in $D(R)$.

Assume (1). Say N, $\eta : R \to M \otimes_R^{\mathbf{L}} N$, and $\epsilon : M \otimes_R^{\mathbf{L}} N \to R$ is a left dual as in Categories, Definition 43.5. Choose a complex $M^\bullet$ representing M. Choose a K-flat complexes $N^\bullet$ with flat terms representing N, see Lemma 59.10. Then η is given by a map of complexes

$$\eta : R \longrightarrow \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$$

We can write the image of 1 as a finite sum

$$\eta(1) = \sum\nolimits_n \sum\nolimits_i m_{n,i} \otimes n_{-n,i}$$

with $m_{n,i} \in M^n$ and $n_{-n,i} \in N^{-n}$. Let $K^\bullet \subset M^\bullet$ be the subcomplex generated by all the elements $m_{n,i}$ and $\mathrm{d}(m_{n,i})$. By our choice of $N^\bullet$ we find that $\mathrm{Tot}(K^\bullet \otimes_R N^\bullet) \subset \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$ and $\eta(1)$ is in the subcomplex by our choice above. Denote K the object of $D(R)$ represented by $K^\bullet$. Then we see that η factors over a map $\tilde{\eta} : R \longrightarrow K \otimes_R^{\mathbf{L}} N$. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_M$ we conclude that the identity on M factors through K by the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\eta\otimes 1"'] \arrow[rd, "\tilde{\eta}\otimes 1"'] & M \otimes_R^{\mathbf{L}} N \otimes_R^{\mathbf{L}} M \arrow[r, "1\otimes\epsilon"'] & M \\
 & K \otimes_R^{\mathbf{L}} N \otimes_R^{\mathbf{L}} M \arrow[r, "1\otimes\epsilon"] \arrow[u] & K \arrow[u]
\end{tikzcd}$$

Since K is bounded above it follows that $M \in D^-(R)$. Thus we can represent M by a bounded above complex $M^\bullet$ of free R-modules, see for example Derived Categories, Lemma 15.4. Write $\eta(1) = \sum_n \sum_i m_{n,i} \otimes n_{-n,i}$ as before. By Lemma 126.2 we can find a subcomplex $K^\bullet \subset M^\bullet$ containing all the elements $m_{n,i}$ which is bounded and consists of finite free R-modules. As above we find that the identity on M factors through K. Since K is perfect we conclude M is perfect too, see Lemma 74.5. $\square$

Lemma 126.4. *Let R be a ring. Let M be an object of $D(R)$. The following are equivalent*

(1) *M is invertible in $D(R)$, see Categories, Definition 43.4, and*
(2) *for every prime ideal $\mathfrak{p} \subset R$ there exists an $f \in R$, $f \not\in \mathfrak{p}$ such that $M_f \cong R_f[-n]$ for some $n \in \mathbf{Z}$.*

Moreover, in this case

(a) *M is a perfect object of $D(R)$,*
(b) *$M = \bigoplus H^n(M)[-n]$ in $D(R)$,*
(c) *each $H^n(M)$ is a finite projective R-module,*
(d) *we can write $R = \prod_{a \leq n \leq b} R_n$ such that $H^n(M)$ corresponds to an invertible R_n-module.*

Proof. Assume (2). Consider the object $R\operatorname{Hom}_R(M, R)$ and the composition map

$$R\operatorname{Hom}(M, R) \otimes_R^{\mathbf{L}} M \to R$$

Checking locally we see that this is an isomorphism; we omit the details. Because $D(R)$ is symmetric monoidal we see that M is invertible.

Assume (1). Observe that an invertible object of a monoidal category has a left dual, namely, its inverse. Thus M is perfect by Lemma 126.3. Consider a prime ideal $\mathfrak{p} \subset R$ with residue field κ. Then we see that $M \otimes_R^{\mathbf{L}} \kappa$ is an invertible object of $D(\kappa)$. Clearly this implies that $\dim H^i(M \otimes_R^{\mathbf{L}} \kappa)$ is nonzero exactly for one i and equal to 1 in that case. By Lemma 75.6 this gives (2).

In the proof above we have seen that (a) holds. Let $U_n \subset \operatorname{Spec}(R)$ be the union of the opens of the form $D(f)$ such that $M_f \cong R_f[-n]$. Clearly, $U_n \cap U_{n'} = \emptyset$ if $n \neq n'$. If M has tor amplitude in $[a, b]$, then $U_n = \emptyset$ if $n \not\in [a, b]$. Hence we see that we have a product decomposition $R = \prod_{a \leq n \leq b} R_n$ as in (d) such that U_n corresponds to $\operatorname{Spec}(R_n)$, see Algebra, Lemma 24.3. Since $D(R) = \prod_{a \leq n \leq b} D(R_n)$ and similary for the category of modules parts (b), (c), and (d) follow immediately. □

127. Splitting off a free module

The arguments in this section are due to Serre, see [Ser58].

Situation 127.1. Here R is a ring and M is a finitely presented R-module. Denote $\Omega \subset \operatorname{Spec}(R)$ the set of closed points with the induced topology. For $x \in \Omega$ denote $M(x) = M/xM$ the fibre of M at x. This is a finite dimensional vector space over the residue field $\kappa(x)$ at x. Given $s \in M$ we denote $s(x)$ the image of s in $M(x)$.

Lemma 127.2. *In Situation 127.1 let $x \in \Omega$. There exists a canonical short exact sequence*

$$0 \to B(x) \to M(x) \to V(x) \to 0$$

of $\kappa(x)$-vector spaces which the following property: for $s_1, \ldots, s_r \in M$ the following are equivalent

(1) *there exists an $f \in R$, $f \not\in x$ such that the map $s_1, \ldots, s_r : R^{\oplus r} \to M$ becomes the inclusion of a direct summand after inverting f, and*
(2) *$s_1(x), \ldots, s_r(x)$ map to linearly independent elements of $V(x)$.*

Proof. Define $B(x) \subset M(x)$ as the perpendicular of the image of the map

$$\operatorname{Hom}_R(M, R) \to \operatorname{Hom}_{\kappa(x)}(M(x), \kappa(x))$$

and set $V(x) = M(x)/B(x)$. Then any R-linear map $\varphi : M \to R$ induces a map $\overline{\varphi} : V(x) \to \kappa(x)$ and conversely any $\kappa(x)$-linear map $\lambda : V(x) \to \kappa(x)$ is equal to $\overline{\varphi}$ for some φ. Let $s_1, \ldots, s_r \in M$.

Suppose $s_1, \ldots, s_r$ map to linearly independent elements of $V(x)$. Then we can find $\varphi_1, \ldots, \varphi_r \in \mathrm{Hom}_R(M, R)$ such that $\varphi_i(s_j)$ maps to δ_{ij}[18] in $\kappa(x)$. Hence the matrix of the composition

$$R^{\oplus r} \xrightarrow{s_1, \ldots, s_r} M \xrightarrow{\varphi_1, \ldots, \varphi_r} R^{\oplus r}$$

has a determinant $f \in R$ which maps to 1 in $\kappa(x)$ Clearly, this implies that $s_1, \ldots, s_r : R^{\oplus r} \to M$ is the inclusion of a direct summand after inverting f.

Conversely, suppose that we have an $f \in R$, $f \notin x$ such that $s_1, \ldots, s_r : R^{\oplus r} \to M$ is the inclusion of a direct summand after inverting f. Hence we can find R_f-linear maps $\varphi_i : M_f \to R_f$ such that $\varphi_i(s_j) = \delta_{ij} \in R_f$. Since $\mathrm{Hom}_R(M, R)_f = \mathrm{Hom}_{R_f}(M_f, R_f)$ by Algebra, Lemma 10.2 we conclude that we can find $n \geq 0$ and $\varphi'_i \in \mathrm{Hom}_R(M, R)$ such that $\varphi'_i(s_j) = f^n \delta_{ij} \in R$. It follows that $s_1, \ldots, s_r$ map to linearly independent elements of $V(x)$ as $\overline{\varphi}'_i(s_j) = f^n \delta_{ij}$. $\square$

In Situation 127.1 given $s_1, \ldots, s_r \in M$ we denote $Z(s_1, \ldots, s_r) \subset \Omega$ the set of $x \in \Omega$ such that $s_1(x), \ldots, s_r(x)$ map to linearly dependent elements of $V(x)$. By the lemma this is a closed subset of Ω.

Lemma 127.3. *In Situation 127.1 let $x_1, \ldots, x_n \in \Omega$ be pairwise distinct. Let $v_i \in V(x_i)$. Then there exists an $s \in M$ such that $s(x_i)$ maps to v_i for $i = 1, \ldots, n$.*

Proof. Since x_i is a maximal ideal of R we may use Algebra, Lemma 15.4 to see that $M(x_1) \oplus \ldots \oplus M(x_n)$ is a quotient of M. $\square$

Proposition 127.4. *In Situation 127.1 assume Ω is a Noetherian topological space. Let $s_1, \ldots, s_h \in M$. Let $Z(s_1, \ldots, s_h) \subset F \subset \Omega$ be closed. Let $x_1, \ldots, x_n \in F$ be pairwise distinct. Let $v_i \in V(x_i)$. Let $k \geq 0$ be an integer such that*

$$(*) \quad h + k \leq \dim_{\kappa(x)} V(x) \text{ for all } x \in \Omega$$

Then there exist $s \in M$ and $F' \subset \Omega$ closed such that

(a) *$s(x_i)$ maps to v_i,*
(b) *$Z(s_1, \ldots, s_h, s) \subset F \cup F'$, and*
(c) *every irreducible component of F' has codimension $\geq k$ in Ω.*

Proof. We note that codimension was defined in Topology, Section 11 and that we will use some results on Noetherian topological spaces contained in Topology, Section 9.

The proof is by induction on k. If $k = 0$, then we choose $s \in M$ as in Lemma 127.3 and we choose $F' = \Omega$.

Assume $k > 0$. By our induction hypothesis we may choose $u \in M$ and $G \subset \Omega$ closed satisfying (a), (b), (c) for $s_1, \ldots, s_h$, F, $x_1, \ldots, x_n$, $v_1, \ldots, v_n$, and $k - 1$.

Let $G = G_1 \cup \ldots \cup G_m$ be the decomposition of G into its irreducible components. If $G_j \subset F$, then we can remove it from the list. Thus we may assume G_j is not contained in F for $j = 1, \ldots, m$. For $j = 1, \ldots, m$ choose $y_j \in G_j$ with $y_j \notin F$ and $y_j \notin G_{j'}$ for $j' \neq j$. This is possible as there are no inclusions among the irreducible components of G. Choose $w_j \in V(y_j)$ not contained in the span of the images of $s_1(y_j), \ldots, s_h(y_j)$; this is possible because $h + k \leq \dim V(y_j)$ and $k > 0$.

[18]Kronecker delta.

Apply the induction hypothesis to the $h+1$ sections $s_1, \ldots, s_h, u$, the closed set $F \cup G$, the points $x_1, \ldots, x_n, y_1, \ldots, y_m \in F \cup G$, the elements $0 \in V(x_i)$ and $w_j \in V(y_j)$, and the integer $k-1$. Note that we have increased h by 1 and decreased k by 1 hence the assumption $(*)$ of the proposition remains valid. This produces $t \in M$ and $H \subset \Omega$ closed satisfying (a), (b), (c) for $s_1, \ldots, s_h, u$, $F \cup G$, $x_1, \ldots, x_n, y_1, \ldots, y_m, 0, \ldots, 0, w_1, \ldots, w_m$, and $k-1$.

Let $H_1, \ldots, H_p \subset H$ be the irreducible components of H which are not contained in $F \cup G$. As before pick $z_l \in H_l$, $z_l \notin F \cup G$ and $z_l \notin H_{l'}$ for $l' \neq l$. Using Algebra, Lemma 15.4 we may choose $f \in R$ such that $f(y_j) = 1$, $j = 1, \ldots, m$ and $f(z_l) = 0$, $l = 1, \ldots, p$. Claim: the element $s = u + ft$ works.

First, the value $s(x_i)$ agrees with $u(x_i)$ because $t(x_i) = 0$ and hence we see that $s(x_i)$ maps to v_i. This proves (a). To finish the proof it suffices to show that every irreducible component Z of $Z(s_1, \ldots, s_h, s)$ not contained in F has codimension $\geq k$ in Ω. Namely, then we can set F' equal to the union of these and we get (b) and (c). We can see that irreducible components Z of $Z(s_1, \ldots, s_h, s)$ of codimension $\leq k-1$ do not exist as follows:

(1) Observe that $Z(s_1, \ldots, s_h, s) \subset Z(s_1, \ldots, s_h, u, t) = F \cup H$ as $s = u + ft$. Hence $Z \subset H$.
(2) The irreducible components of H have codimension $\geq k-1$. Hence Z is equal to an irreducible component of H as Z has codimension $\leq k-1$. Hence $Z = H_l$ for some $l \in \{1, \ldots, p\}$ or $Z = G_j$ for some $j \in \{1, \ldots m\}$.
(3) But $Z = G_j$ is impossible as $s_1(y_j), \ldots, s_h(y_j)$ map to linearly independent elements of $V(y_j)$ and $s(y_j) = u(y_j) + f(y_j)t(y_j) = u(y_j) + t(y_j)$ maps to an element of the form

$$\text{linear combination images of } s_i(y_j) + w_j$$

which is linearly independent of the images of $s_1(y_j), \ldots, s_h(y_j)$ in $V(y_j)$ by our choice of w_j.
(4) Also $Z = Z_l$ is impossible. Namely, again $s_1(z_l), \ldots, s_h(z_l)$ map to linearly independent elements of $V(z_l)$ and $s(z_l) = u(z_l) + f(z_l)t(z_l) = u(z_l)$ maps to an element of $V(z_l)$ linearly independent of those as $z_l \notin F \cup G$.

This finishes the proof. □

Theorem 127.5. *Let R be a ring whose max spectrum $\Omega \subset \mathrm{Spec}(R)$ is a Noetherian topological space of dimension $d < \infty$. Let M be a finitely presented R-module such that for all $\mathfrak{m} \in \Omega$ the $R_\mathfrak{m}$-module $M_\mathfrak{m}$ has a free direct summand of rank $> d$. Then $M \cong R \oplus M'$.*

Proof. For $\mathfrak{m} \in \Omega$ suppose that $R_\mathfrak{m}^{\oplus r}$ is a direct summand of $M_\mathfrak{m}$. Then by Algebra, Lemmas 9.9 and 127.6 we see that $R_f^{\oplus r}$ is a direct summand of M_f for some $f \in R$, $f \notin \mathfrak{m}$. Hence the assumption means that $\dim V(x) > d$ for all $x \in \Omega$ where $V(x)$ is as in Lemma 127.2. By Proposition 127.4 applied with $F = \emptyset$, $h = 0$ and no s_i, $n = 0$ and no x_i, v_i, and $k = d+1$ we find an $s \in M$ and $F' \subset \Omega$ such that every irreducible component of F' has codimension $\geq d+1$ and $Z(s) \subset F'$. Since $d = \dim(\Omega)$ this forces $F' = \emptyset$. Hence $s : R \to M$ is the inclusion of a direct summand at all maximal ideals. It follows that s is universally injective, see Algebra, Lemma 82.12. Then s is split injective by Algebra, Lemma 82.4. □

128. Big projective modules are free

In this section we discuss one of the results of [Bas63]; we suggest the reader look at the original paper. Our argument will use the slightly simplified proof given in the papers [Aka70] and [Hin63].

Lemma 128.1 (Eilenberg's lemma)**.** *If $P \oplus Q \cong F$ with F a nonfinitely generated free module, then $P \oplus F \cong F$.*

Proof.

$$F \cong F \oplus F \oplus \ldots \cong P \oplus Q \oplus P \oplus Q \oplus \ldots \cong P \oplus F \oplus F \oplus \ldots \cong P \oplus F$$

$\square$

Lemma 128.2. *Let R be a ring. Let P be a projective module. There exists a free module F such that $P \oplus F$ is free.*

Proof. Since P is projective we see that $F_0 = P \oplus Q$ is a free module for some module Q. Set $F = \bigoplus_{n \geq 1} F_0$. Then $P \oplus F \cong F$ by Lemma 128.1. $\square$

Lemma 128.3. *Let R be a ring. Let P be a projective module. Let $s \in P$. There exists a finite free module F and a finite free direct summand $K \subset F \oplus P$ with $(0, s) \in K$.*

Proof. By Lemma 128.2 we can find a (possibly infinite) free module F such that $F \oplus P$ is free. Then of course $(0, s)$ is contained in a finite free direct summand $K \subset F \oplus P$. In turn K is contained in $F' \oplus P$ where $F' \subset F$ is a finite free direct summand. $\square$

Lemma 128.4. *Let R be a ring with Jacobson radical J such that R/J is Noetherian. Let P be a projective R-module such that $P_\mathfrak{m}$ has infinite rank for all maximal ideals $\mathfrak{m}$ of R. Let $s \in P$ and $M \subset P$ such that $Rs + M = P$. Then we can find $m \in M$ such that $R(s + m)$ is a free direct summand of P.*

Proof. The statement makes sense as $P_\mathfrak{m}$ is free by Algebra, Theorem 85.4.

Denote $M' \subset P/JP$ the image of M and $s' \in P/JP$ the image of s. Observe that $R/Js' + M' = P/JP$. Suppose we can find $m' \in M'$ such that $R/J(s' + m')$ is a free direct summand of M'. Choose $\varphi' : P/JP \to R/J$ which gives a splitting, i.e., we have $\varphi'(s' + m') = 1$ in R/J. Then since P is a projective R-module we can find a lift $\varphi : P \to R$ of φ'. Choose $m \in M$ mapping to m'. Then $\varphi(s + m) \in R$ is congruent to 1 modulo J and hence a unit in R (Algebra, Lemma 19.1). Whence $R(s + m)$ is a free direct summand of P. This reduces us to the case discussed in the next paragraph.

Assume R is Noetherian. Let $m \in M$ be an element and let $\varphi_1, \ldots, \varphi_n : P \to R$ be R-linear maps. Denote

$$Z(s + m, \varphi_1, \ldots, \varphi_n) \subset \mathrm{Spec}(R)$$

the vanishing locus of $\varphi_1(s + m), \ldots, \varphi_n(s + m) \in R$.

Suppose $\mathfrak{m}$ is a maximal ideal of R and $\mathfrak{m} \in Z(s, \varphi_1, \ldots, \varphi_n)$. Set $K = M \cap \bigcap \mathrm{Ker}(\varphi_i)$. We claim the image of

$$K/\mathfrak{m}K \to P/\mathfrak{m}P$$

has infinite dimension. Namely, the quotient P/K is a finite R-module as it is isomorphic to a submodule of $P/M \oplus R^{\oplus n}$. Thus we see that the kernel of the displayed arrow is a quotient of $\text{Tor}_1^R(P/K, \kappa(\mathfrak{m}))$ which is finite by Algebra, Lemma 75.7. Combined with the fact that $P/\mathfrak{m}P$ has infinite dimension we obtain our claim. Thus we can find a $t \in K$ which maps to a nonzero element $\overline{t}$ of the vector space $P/\mathfrak{m}P$. By linear algebra, we find an R-linear map $\overline{\varphi} : P \to \kappa(\mathfrak{m})$ such that $\overline{\varphi}(\overline{t}) = 1$. Since P is projective, we can find an R-linear map $\varphi : P \to R$ lifting $\overline{\varphi}$. Then we see that the vanishing locus $Z(s + m + t, \varphi_1, \ldots, \varphi_n, \varphi)$ is contained in $Z(s + m, \varphi_1, \ldots, \varphi_n)$ but does not contain $\mathfrak{m}$, i.e., it is strictly smaller than $Z(s + m, \varphi_1, \ldots, \varphi_n)$.

Since $\text{Spec}(R)$ is a Noetherian topological space, we see from the arguments above that we may find $m \in M$ and $\varphi_1, \ldots, \varphi_n : P \to R$ such that the closed subset $Z(s + m, \varphi_1, \ldots, \varphi_n)$ does not contain any closed points of $\text{Spec}(R)$. Hence $Z(s + m, \varphi_1, \ldots, \varphi_n) = \emptyset$. Hence we can find $r_1, \ldots, r_n \in R$ such that $\sum r_i \varphi_i(s + m) = 1$. Hence

$$R \xrightarrow{s+m} P \xrightarrow{\sum r_i \varphi_i} R$$

is the desired splitting. □

Lemma 128.5. *Let R be a ring with Jacobson radical J such that R/J is Noetherian. Let P be a projective R-module such that $P_\mathfrak{m}$ has infinite rank for all maximal ideals $\mathfrak{m}$ of R. Let $s \in P$. Then we can find a finite stably free direct summand $M \subset P$ such that $s \in M$.*

Proof. By Lemma 128.3 we can find a finite free module F and a finite free direct summand $K \subset F \oplus P$ such that $(0, s) \in K$. By induction on the rank of F we reduce to the case discussed in the next paragraph.

Assume there exists a finite stably free direct summand $K \subset R \oplus P$ such that $(0, s) \in K$. Choose a complement K' of K, i.e., such that $R \oplus P = K \oplus K'$. The projection $\pi : R \oplus P \to K'$ is surjectve, hence by Lemma 128.4 we find a $p \in P$ such that $\pi(1, p) \in K'$ generates a free direct summand. Accordingly we write $K' = R\pi(1, p) \oplus K''$. We see that

$$R \oplus P = K \oplus K' = K \oplus R\pi(1, p) \oplus K''$$

The projection $\pi' : P \to K''$ is surjective[19] and hence split (as K'' is projective). Thus $\text{Ker}(\pi') \subset P$ is a direct summand containing s. Finally, by construction we have an isomorphism

$$R \oplus \text{Ker}(\pi') \cong K \oplus R\pi(1, p)$$

and hence since K is finite and stably free, so is $\text{Ker}(\pi')$. □

Theorem 128.6. *Let R be a ring with Jacobson radical J such that R/J is Noetherian. Let P be a countably generated projective R-module such that $P_\mathfrak{m}$ has infinite rank for all maximal ideals $\mathfrak{m}$ of R. Then P is free.*

[19] Namely, if $k'' \in K''$ then k'' viewed as an element of K' can be written as $k'' = \lambda\pi(1, 0) + \pi(0, q)$ for some $\lambda \in R$ and $q \in P$. This means $k'' = \lambda\pi(1, p) + \pi(0, q - \lambda p)$. This in turn means that $q - \lambda p$ maps to k'' by the composition $P \to R \oplus P \xrightarrow{\pi} K' \to K''$ since $K' \to K''$ annihilates $\pi(1, p)$.

Proof. We first prove that P is a countable direct sum of finite stably free modules. Let $x_1, x_2, \ldots$ be a countable set of generators for P. We inductively construct finite stably free direct summands $F_1, F_2, \ldots$ of P such that for all n we have that $F_1 \oplus \ldots \oplus F_n$ is a direct summand of P which contains $x_1, \ldots, x_n$. Namely, given $F_1, \ldots, F_n$ with the desired properties, write

$$P = F_1 \oplus \ldots \oplus F_n \oplus P'$$

and let $s \in P'$ be the image of x_{n+1}. By Lemma 128.5 we can find a finite stably free direct summand $F_{n+1} \subset P'$ containing s. Then $P = \bigoplus_{i=1}^{\infty} F_i$.

Assume that P is an infinite direct sum $P = \bigoplus_{i=1}^{\infty} F_i$ of nonzero finite stably free modules. The stable freeness of the modules F_i will be used in the following manner: the rank of each F_i is constant (and positive). Hence we see that $P_\mathfrak{m}$ is free of countably infinite rank for each maximal ideal $\mathfrak{m}$ of R. By Lemma 128.4 applied with $s = 0$ and $M = P$, we can find a $t_1 \in P$ such that Rt_1 is a free direct summand of P. Then t_1 is contained in $F_1 \oplus \ldots \oplus F_{n_1}$ for some $n_1 > n_0 = 0$. The same reasoning applied to $\bigoplus_{n > n_1} F_n$ produces an $n_1 < n_2$ and $t_2 \in F_{n_1+1} \oplus \ldots \oplus F_{n_2}$ which generates a free direct summand. Continuing in this fashion we obtain a free direct summand

$$\bigoplus\nolimits_{i \geq 1} t_i : \bigoplus\nolimits_{i \geq 1} R \longrightarrow \bigoplus\nolimits_{i \geq 1} \bigoplus\nolimits_{n_i \geq n > n_{i-1}} F_n = P$$

of infinite rank. Thus we see that $P \cong Q \oplus F$ for some free R-module F of countable rank. Since Q is countably generated it follows that $Q \oplus Q' \cong F$ for some module Q'. Then the Eilenberg swindle (Lemma 128.1) implies that $Q \oplus F \cong F$ and P is free. ☐

129. Other chapters

Preliminaries

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Schemes

(26) Schemes
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(36) Derived Categories of Schemes
(37) More on Morphisms
(38) More on Flatness
(39) Groupoid Schemes
(40) More on Groupoid Schemes

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