SIMPLICIAL METHODS

Contents

1. Introduction

This is a minimal introduction to simplicial methods. We just add here whenever something is needed later on. A general reference to this material is perhaps [GJ99]. An example of the things you can do is the paper by Quillen on Homotopical Algebra, see [Qui67] or the paper on Étale Homotopy by Artin and Mazur, see [AM69].

2. The category of finite ordered sets

The category Δ is the category with

(1) objects $[0], [1], [2], \ldots$ with $[n] = \{0, 1, 2, \ldots, n\}$ and
(2) a morphism $[n] \to [m]$ is a nondecreasing map $\{0, 1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, m\}$ between the corresponding sets.

Here *nondecreasing* for a map $\varphi : [n] \to [m]$ means by definition that $\varphi(i) \geq \varphi(j)$ if $i \geq j$. In other words, Δ is a category equivalent to the "big" category of nonempty finite totally ordered sets and nondecreasing maps. There are exactly $n + 1$ morphisms $[0] \to [n]$ and there is exactly 1 morphism $[n] \to [0]$. There are exactly $(n+1)(n+2)/2$ morphisms $[1] \to [n]$ and there are exactly $n+2$ morphisms $[n] \to [1]$. And so on and so forth.

Definition 2.1. For any integer $n \geq 1$, and any $0 \leq j \leq n$ we let $\delta_j^n : [n-1] \to [n]$ denote the injective order preserving map skipping j. For any integer $n \geq 0$, and any $0 \leq j \leq n$ we denote $\sigma_j^n : [n+1] \to [n]$ the surjective order preserving map with $(\sigma_j^n)^{-1}(\{j\}) = \{j, j+1\}$.

Lemma 2.2. *Any morphism in Δ can be written as a composition of the morphisms δ_j^n and σ_j^n.*

Proof. Let $\varphi : [n] \to [m]$ be a morphism of Δ. If $j \notin \mathrm{Im}(\varphi)$, then we can write φ as $\delta_j^m \circ \psi$ for some morphism $\psi : [n] \to [m-1]$. If $\varphi(j) = \varphi(j+1)$ then we can write φ as $\psi \circ \sigma_j^{n-1}$ for some morphism $\psi : [n-1] \to [m]$. The result follows because each replacement as above lowers $n+m$ and hence at some point φ is both injective and surjective, hence an identity morphism. $\square$

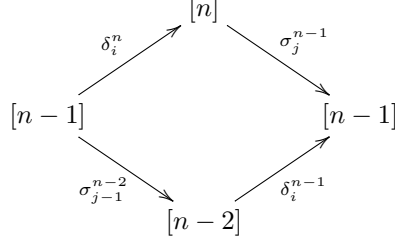**Lemma 2.3.** *The morphisms δ_j^n and σ_j^n satisfy the following relations.*

(1) *If $0 \leq i < j \leq n+1$, then $\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n$. In other words the diagram*
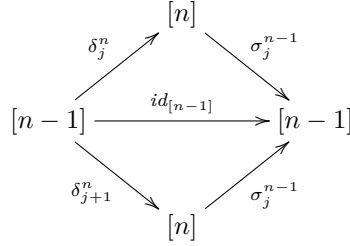

commutes.

(2) If $0 \leq i < j \leq n - 1$, then $\sigma_j^{n-1} \circ \delta_i^n = \delta_i^{n-1} \circ \sigma_{j-1}^{n-2}$. In other words the diagram

$$
\begin{array}{ccc}
 & [n] & \\
\delta_i^n \nearrow & & \searrow \sigma_j^{n-1} \\
[n-1] & & [n-1] \\
\sigma_{j-1}^{n-2} \searrow & & \nearrow \delta_i^{n-1} \\
 & [n-2] &
\end{array}
$$

commutes.

(3) If $0 \leq j \leq n - 1$, then $\sigma_j^{n-1} \circ \delta_j^n = id_{[n-1]}$ and $\sigma_j^{n-1} \circ \delta_{j+1}^n = id_{[n-1]}$. In other words the diagram

$$
\begin{array}{ccc}
 & [n] & \\
\delta_j^n \nearrow & & \searrow \sigma_j^{n-1} \\
[n-1] & \xrightarrow{\ id_{[n-1]}\ } & [n-1] \\
\delta_{j+1}^n \searrow & & \nearrow \sigma_j^{n-1} \\
 & [n] &
\end{array}
$$

commutes.

(4) If $0 < j + 1 < i \leq n$, then $\sigma_j^{n-1} \circ \delta_i^n = \delta_{i-1}^{n-1} \circ \sigma_j^{n-2}$. In other words the diagram

$$
\begin{array}{ccc}
 & [n] & \\
\delta_i^n \nearrow & & \searrow \sigma_j^{n-1} \\
[n-1] & & [n-1] \\
\sigma_j^{n-2} \searrow & & \nearrow \delta_{i-1}^{n-1} \\
 & [n-2] &
\end{array}
$$
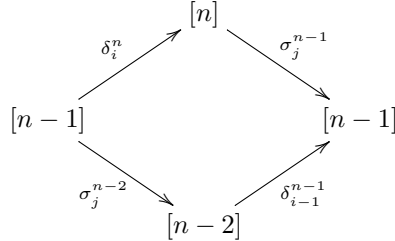
commutes.

(5) If $0 \leq i \leq j \leq n - 1$, then $\sigma_j^{n-1} \circ \sigma_i^n = \sigma_i^{n-1} \circ \sigma_{j+1}^n$. In other words the diagram

$$
\begin{array}{ccc}
 & [n] & \\
\sigma_i^n \nearrow & & \searrow \sigma_j^{n-1} \\
[n+1] & & [n-1] \\
\sigma_{j+1}^n \searrow & & \nearrow \sigma_i^{n-1} \\
 & [n] &
\end{array}
$$
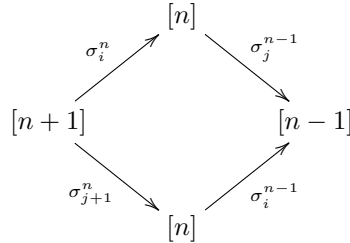
commutes.

Proof. Omitted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.4. *The category Δ is the universal category with objects $[n]$, $n \geq 0$ and morphisms δ_j^n and σ_j^n such that (a) every morphism is a composition of these*

morphisms, (b) the relations listed in Lemma 2.3 are satisfied, and (c) any relation among the morphisms is a consequence of those relations.

Proof. Omitted. $\square$

3. Simplicial objects

Definition 3.1. Let $\mathcal{C}$ be a category.

(1) A *simplicial object U of $\mathcal{C}$* is a contravariant functor U from Δ to $\mathcal{C}$, in a formula:
$$U : \Delta^{opp} \longrightarrow \mathcal{C}$$
(2) If $\mathcal{C}$ is the category of sets, then we call U a *simplicial set.*
(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a *simplicial abelian group.*
(4) A *morphism of simplicial objects $U \to U'$* is a transformation of functors.
(5) The *category of simplicial objects of $\mathcal{C}$* is denoted $\mathrm{Simp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \dots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([n]) \to U([m])$, satisfying $U(\varphi \circ \psi) = U(\psi) \circ U(\varphi)$.

In particular there is a unique morphism $U([0]) \to U([n])$ and there are exactly $n+1$ morphisms $U([n]) \to U([0])$ corresponding to the $n+1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 2 above.

Lemma 3.2. *Let $\mathcal{C}$ be a category.*

(1) *Given a simplicial object U in $\mathcal{C}$ we obtain a sequence of objects $U_n = U([n])$ endowed with the morphisms $d_j^n = U(\delta_j^n) : U_n \to U_{n-1}$ and $s_j^n = U(\sigma_j^n) : U_n \to U_{n+1}$. These morphisms satisfy the opposites of the relations displayed in Lemma 2.3, namely*
    (a) *If $0 \le i < j \le n+1$, then $d_i^n \circ d_j^{n+1} = d_{j-1}^n \circ d_i^{n+1}$.*
    (b) *If $0 \le i < j \le n-1$, then $d_i^n \circ s_j^{n-1} = s_{j-1}^{n-2} \circ d_i^{n-1}$.*
    (c) *If $0 \le j \le n-1$, then $id = d_j^n \circ s_j^{n-1} = d_{j+1}^n \circ s_j^{n-1}$.*
    (d) *If $0 < j+1 < i \le n$, then $d_i^n \circ s_j^{n-1} = s_j^{n-2} \circ d_{i-1}^{n-1}$.*
    (e) *If $0 \le i \le j \le n-1$, then $s_i^n \circ s_j^{n-1} = s_{j+1}^n \circ s_i^{n-1}$.*
(2) *Conversely, given a sequence of objects U_n and morphisms d_j^n, s_j^n satisfying (1)(a) – (e) there exists a unique simplicial object U in $\mathcal{C}$ such that $U_n = U([n])$, $d_j^n = U(\delta_j^n)$, and $s_j^n = U(\sigma_j^n)$.*
(3) *A morphism between simplicial objects U and U' is given by a family of morphisms $U_n \to U'_n$ commuting with the morphisms d_j^n and s_j^n.*

Proof. This follows from Lemma 2.4. $\square$

Remark 3.3. By abuse of notation we sometimes write $d_i : U_n \to U_{n-1}$ instead of d_i^n, and similarly for $s_i : U_n \to U_{n+1}$. The relations among the morphisms d_i^n and s_i^n may be expressed as follows:

(1) If $i < j$, then $d_i \circ d_j = d_{j-1} \circ d_i$.
(2) If $i < j$, then $d_i \circ s_j = s_{j-1} \circ d_i$.
(3) We have $id = d_j \circ s_j = d_{j+1} \circ s_j$.
(4) If $i > j+1$, then $d_i \circ s_j = s_j \circ d_{i-1}$.

(5) If $i \leq j$, then $s_i \circ s_j = s_{j+1} \circ s_i$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $s_0^0 = U(\sigma_0^0) : U_0 \to U_1$ and two morphisms $d_0^1 = U(\delta_0^1)$, and $d_1^1 = U(\delta_1^1)$ which are morphisms $U_1 \to U_0$. There are two morphisms $s_0^1 = U(\sigma_0^1)$, $s_1^1 = U(\sigma_1^1)$ which are morphisms $U_1 \to U_2$. Three morphisms $d_0^2 = U(\delta_0^2)$, $d_1^2 = U(\delta_1^2)$, $d_2^2 = U(\delta_2^2)$ which are morphisms $U_3 \to U_2$. And so on.

Pictorially we think of U as follows:

$$U_2 \underset{\longrightarrow}{\overset{\longrightarrow}{\rightleftarrows}} U_1 \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} U_0$$

Here the d-morphisms are the arrows pointing right and the s-morphisms are the arrows pointing left.

Example 3.4. The simplest example is the *constant* simplicial object with value $X \in \mathrm{Ob}(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

Example 3.5. Suppose that $Y \to X$ is a morphism of $\mathcal{C}$ such that all the fibred products $Y \times_X Y \times_X \ldots \times_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold fibre product, and we let $\varphi : [n] \to [m]$ correspond to the map (on "coordinates") $(y_0, \ldots, y_m) \mapsto (y_{\varphi(0)}, \ldots, y_{\varphi(n)})$. In other words, the map $U_0 = Y \to U_1 = Y \times_X Y$ is the diagonal map. The two maps $U_1 = Y \times_X Y \to U_0 = Y$ are the projection maps.

Geometrically Example 3.5 above is an important example. It tells us that it is a good idea to think of the maps $d_j^n : U_n \to U_{n-1}$ as projection maps (forgetting the jth component), and to think of the maps $s_j^n : U_n \to U_{n+1}$ as diagonal maps (repeating the jth coordinate). We will return to this in the sections below.

Lemma 3.6. *Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$. Each of the morphisms $s_i^n : U_n \to U_{n+1}$ has a left inverse. In particular s_i^n is a monomorphism.*

Proof. This is true because $d_i^{n+1} \circ s_i^n = \mathrm{id}_{U_n}$. $\square$

4. Simplicial objects as presheaves

Another observation is that we may think of a simplicial object of $\mathcal{C}$ as a presheaf with values in $\mathcal{C}$ over Δ. See Sites, Definition 2.2. And in fact, if U, U' are simplicial objects of $\mathcal{C}$, then we have

(4.0.1) $$\mathrm{Mor}(U, U') = \mathrm{Mor}_{PSh(\Delta)}(U, U').$$

Some of the material below could be replaced by the more general constructions in the chapter on sites. However, it seems a clearer picture arises from the arguments specific to simplicial objects.

5. Cosimplicial objects

A cosimplicial object of a category $\mathcal{C}$ could be defined simply as a simplicial object of the opposite category $\mathcal{C}^{opp}$. This is not really how the human brain works, so we introduce them separately here and point out some simple properties.

Definition 5.1. Let $\mathcal{C}$ be a category.

(1) A *cosimplicial object U of $\mathcal{C}$* is a covariant functor U from Δ to $\mathcal{C}$, in a formula:
$$U : \Delta \longrightarrow \mathcal{C}$$

(2) If $\mathcal{C}$ is the category of sets, then we call U a *cosimplicial set*.

(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a *cosimplicial abelian group*.

(4) A *morphism of cosimplicial objects $U \to U'$* is a transformation of functors.

(5) The *category of cosimplicial objects of $\mathcal{C}$* is denoted $\mathrm{CoSimp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \ldots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([m]) \to U([n])$, satisfying $U(\varphi \circ \psi) = U(\varphi) \circ U(\psi)$.

In particular there is a unique morphism $U([n]) \to U([0])$ and there are exactly $n+1$ morphisms $U([0]) \to U([n])$ corresponding to the $n+1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 2 above.

Lemma 5.2. *Let $\mathcal{C}$ be a category.*

(1) *Given a cosimplicial object U in $\mathcal{C}$ we obtain a sequence of objects $U_n = U([n])$ endowed with the morphisms $\delta_j^n = U(\delta_j^n) : U_{n-1} \to U_n$ and $\sigma_j^n = U(\sigma_j^n) : U_{n+1} \to U_n$. These morphisms satisfy the relations displayed in Lemma 2.3.*

(2) *Conversely, given a sequence of objects U_n and morphisms δ_j^n, σ_j^n satisfying these relations there exists a unique cosimplicial object U in $\mathcal{C}$ such that $U_n = U([n])$, $\delta_j^n = U(\delta_j^n)$, and $\sigma_j^n = U(\sigma_j^n)$.*

(3) *A morphism between cosimplicial objects U and U' is given by a family of morphisms $U_n \to U'_n$ commuting with the morphisms δ_j^n and σ_j^n.*

Proof. This follows from Lemma 2.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.3. By abuse of notation we sometimes write $\delta_i : U_{n-1} \to U_n$ instead of δ_i^n, and similarly for $\sigma_i : U_{n+1} \to U_n$. The relations among the morphisms δ_i^n and σ_i^n may be expressed as follows:

(1) If $i < j$, then $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$.

(2) If $i < j$, then $\sigma_j \circ \delta_i = \delta_i \circ \sigma_{j-1}$.

(3) We have $\mathrm{id} = \sigma_j \circ \delta_j = \sigma_j \circ \delta_{j+1}$.

(4) If $i > j + 1$, then $\sigma_j \circ \delta_i = \delta_{i-1} \circ \sigma_j$.

(5) If $i \leq j$, then $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $\sigma_0^0 = U(\sigma_0^0) : U_1 \to U_0$ and two morphisms $\delta_0^1 = U(\delta_0^1)$, and $\delta_1^1 = U(\delta_1^1)$ which are morphisms $U_0 \to U_1$. There are two morphisms $\sigma_0^1 = U(\sigma_0^1)$, $\sigma_1^1 = U(\sigma_1^1)$ which are morphisms $U_2 \to U_1$. Three morphisms $\delta_0^2 = U(\delta_0^2)$, $\delta_1^2 = U(\delta_1^2)$, $\delta_2^2 = U(\delta_2^2)$ which are morphisms $U_2 \to U_3$. And so on.

Pictorially we think of U as follows:
$$U_0 \overset{\longrightarrow}{\underset{\longrightarrow}{\rightleftarrows}} U_1 \mathrel{\substack{\longrightarrow \\ \rightleftarrows \\ \longrightarrow}} U_2$$

Here the δ-morphisms are the arrows pointing right and the σ-morphisms are the arrows pointing left.

Example 5.4. The simplest example is the *constant* cosimplicial object with value $X \in \mathrm{Ob}(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

Example 5.5. Suppose that $X \to Y$ is a morphism of C such that all the pushouts $Y \amalg_X Y \amalg_X \ldots \amalg_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold pushout, and we let $\varphi : [n] \to [m]$ correspond to the map

$$(y \text{ in } i\text{th component}) \mapsto (y \text{ in } \varphi(i)\text{th component})$$

on "coordinates". In other words, the map $U_1 = Y \amalg_X Y \to U_0 = Y$ is the identity on each component. The two maps $U_0 = Y \to U_1 = Y \amalg_X Y$ are the two coprojections.

Example 5.6. For every $n \geq 0$ we denote $C[n]$ the cosimplicial set

$$\Delta \longrightarrow Sets, \quad [k] \longmapsto \mathrm{Mor}_\Delta([n], [k])$$

This example is dual to Example 11.2.

Lemma 5.7. *Let $\mathcal{C}$ be a category. Let U be a cosimplicial object of $\mathcal{C}$. Each of the morphisms $\delta_i^n : U_{n-1} \to U_n$ has a left inverse. In particular δ_i^n is a monomorphism.*

Proof. This is true because $\sigma_i^{n-1} \circ \delta_i^n = \mathrm{id}_{U_n}$ for $j < n$. $\square$

6. Products of simplicial objects

Of course we should define the product of simplicial objects as the product in the category of simplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

Definition 6.1. Let $\mathcal{C}$ be a category. Let U and V be simplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The *product of U and V* is the simplicial object $U \times V$ defined as follows:

(1) $(U \times V)_n = U_n \times V_n$,
(2) $d_i^n = (d_i^n, d_i^m)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $U \times V$ is the product of the presheaves U and V on Δ.

Lemma 6.2. *If U and V are simplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have*

$$\mathrm{Mor}(W, U \times V) = \mathrm{Mor}(W, U) \times \mathrm{Mor}(W, V)$$

for any third simplicial object W of $\mathcal{C}$.

Proof. Omitted. $\square$

7. Fibre products of simplicial objects

Of course we should define the fibre product of simplicial objects as the fibre product in the category of simplicial objects. This may lead to the potentially confusing situation where the fibre product exists but is not described as below. To avoid this we define the fibre product directly as follows.

Definition 7.1. Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : V \to U$, $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The *fibre product of V and W over U* is the simplicial object $V \times_U W$ defined as follows:

(1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \times_U W$ is the fibre product of the presheaves V and W over the presheaf U on Δ.

Lemma 7.2. *If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have*

$$\mathrm{Mor}(T, V \times_U W) = \mathrm{Mor}(T, V) \times_{\mathrm{Mor}(T,U)} \mathrm{Mor}(T, W)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. $\square$

8. Pushouts of simplicial objects

Of course we should define the pushout of simplicial objects as the pushout in the category of simplicial objects. This may lead to the potentially confusing situation where the pushouts exist but are not as described below. To avoid this we define the pushout directly as follows.

Definition 8.1. Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : U \to V$, $b : U \to W$ be morphisms. Assume the pushouts $V_n \amalg_{U_n} W_n$ exist in $\mathcal{C}$. The *pushout of V and W over U* is the simplicial object $V \amalg_U W$ defined as follows:

(1) $(V \amalg_U W)_n = V_n \amalg_{U_n} W_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \amalg_U W$ is the pushout of the presheaves V and W over the presheaf U on Δ.

Lemma 8.2. *If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : U \to V$, $b : U \to W$ are morphisms and if $V \amalg_U W$ exists, then we have*

$$\mathrm{Mor}(V \amalg_U W, T) = \mathrm{Mor}(V, T) \times_{\mathrm{Mor}(U,T)} \mathrm{Mor}(W, T)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. $\square$

9. Products of cosimplicial objects

Of course we should define the product of cosimplicial objects as the product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

Definition 9.1. Let $\mathcal{C}$ be a category. Let U and V be cosimplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The *product of U and V* is the cosimplicial object $U \times V$ defined as follows:

(1) $(U \times V)_n = U_n \times V_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(U \times V)(\varphi) : U_n \times V_n \to U_m \times V_m$ is the product $U(\varphi) \times V(\varphi)$.

Lemma 9.2. *If U and V are cosimplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have*

$$\mathrm{Mor}(W, U \times V) = \mathrm{Mor}(W, U) \times \mathrm{Mor}(W, V)$$

for any third cosimplicial object W of $\mathcal{C}$.

Proof. Omitted. $\square$

10. Fibre products of cosimplicial objects

Of course we should define the fibre product of cosimplicial objects as the fibre product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the fibre product directly as follows.

Definition 10.1. Let $\mathcal{C}$ be a category. Let U, V, W be cosimplicial objects of $\mathcal{C}$. Let $a : V \to U$ and $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The *fibre product of V and W over U* is the cosimplicial object $V \times_U W$ defined as follows:

(1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(V \times_U W)(\varphi) : V_n \times_{U_n} W_n \to V_m \times_{U_m} W_m$ is the product $V(\varphi) \times_{U(\varphi)} W(\varphi)$.

Lemma 10.2. *If U, V, W are cosimplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have*

$$\mathrm{Mor}(T, V \times_U W) = \mathrm{Mor}(T, V) \times_{\mathrm{Mor}(T, U)} \mathrm{Mor}(T, W)$$

for any fourth cosimplicial object T of $\mathcal{C}$.

Proof. Omitted. $\square$

11. Simplicial sets

Let U be a simplicial set. It is a good idea to think of U_0 as the 0-*simplices*, the set U_1 as the 1-*simplices*, the set U_2 as the 2-*simplices*, and so on.

We think of the maps $s_j^n : U_n \to U_{n+1}$ as the map that associates to an n-simplex A the degenerate $(n+1)$-simplex B whose $(j, j+1)$-edge is collapsed to the vertex j of A. We think of the map $d_j^n : U_n \to U_{n-1}$ as the map that associates to an n-simplex A one of the faces, namely the face that omits the vertex j. In this way it become possible to visualize the relations among the maps s_j^n and d_j^n geometrically.

Definition 11.1. Let U be a simplicial set. We say x is an *n-simplex of U* to signify that x is an element of U_n. We say that y is the $j$the *face of x* to signify that $d_j^n x = y$. We say that z is the jth *degeneracy of x* if $z = s_j^n x$. A simplex is called *degenerate* if it is the degeneracy of another simplex.

Here are a few fundamental examples.

Example 11.2. For every $n \geq 0$ we denote $\Delta[n]$ the simplicial set

$$\Delta^{opp} \longrightarrow Sets, \quad [k] \longmapsto \mathrm{Mor}_{\Delta}([k], [n])$$

We leave it to the reader to verify the following statements. Every m-simplex of $\Delta[n]$ with $m > n$ is degenerate. There is a unique nondegenerate n-simplex of $\Delta[n]$, namely $\mathrm{id}_{[n]}$.

Lemma 11.3. *Let U be a simplicial set. Let $n \geq 0$ be an integer. There is a canonical bijection*

$$\mathrm{Mor}(\Delta[n], U) \longrightarrow U_n$$

which maps a morphism φ to the value of φ on the unique nondegenerate n-simplex of $\Delta[n]$.

Proof. Omitted. $\square$

Example 11.4. Consider the category $\Delta/[n]$ of objects over $[n]$ in Δ, see Categories, Example 2.13. There is a functor $p : \Delta/[n] \to \Delta$. The fibre category of p over $[k]$, see Categories, Section 35, has as objects the set $\Delta[n]_k$ of k-simplices in $\Delta[n]$, and as morphisms only identities. For every morphism $\varphi : [k] \to [l]$ of Δ, and every object $\psi : [l] \to [n]$ in the fibre category over $[l]$ there is a unique object over $[k]$ with a morphism covering φ, namely $\psi \circ \varphi : [k] \to [n]$. Thus $\Delta/[n]$ is fibred in sets over Δ. In other words, we may think of $\Delta/[n]$ as a presheaf of sets over Δ. See also, Categories, Example 38.7. And this presheaf of sets agrees with the simplicial set $\Delta[n]$. In particular, from Equation (4.0.1) and Lemma 11.3 above we get the formula

$$\mathrm{Mor}_{PSh(\Delta)}(\Delta/[n], U) = U_n$$

for any simplicial set U.

Lemma 11.5. *Let U, V be simplicial sets. Let $a, b \geq 0$ be integers. Assume every n-simplex of U is degenerate if $n > a$. Assume every n-simplex of V is degenerate if $n > b$. Then every n-simplex of $U \times V$ is degenerate if $n > a + b$.*

Proof. Suppose $n > a + b$. Let $(u, v) \in (U \times V)_n = U_n \times V_n$. By assumption, there exists a $\alpha : [n] \to [a]$ and a $u' \in U_a$ and a $\beta : [n] \to [b]$ and a $v' \in V_b$ such that $u = U(\alpha)(u')$ and $v = V(\beta)(v')$. Because $n > a + b$, there exists an $0 \leq i \leq a + b$ such that $\alpha(i) = \alpha(i+1)$ and $\beta(i) = \beta(i+1)$. It follows immediately that (u, v) is in the image of s_i^{n-1}. $\square$

12. Truncated simplicial objects and skeleton functors

Let $\Delta_{\leq n}$ denote the full subcategory of Δ with objects $[0], [1], [2], \ldots, [n]$. Let $\mathcal{C}$ be a category.

Definition 12.1. An *n-truncated simplicial object of $\mathcal{C}$* is a contravariant functor from $\Delta_{\leq n}$ to $\mathcal{C}$. A *morphism of n-truncated simplicial objects* is a transformation of functors. We denote the category of n-truncated simplicial objects of $\mathcal{C}$ by the symbol $\mathrm{Simp}_n(\mathcal{C})$.

Given a simplicial object U of $\mathcal{C}$ the truncation $\mathrm{sk}_n U$ is the restriction of U to the subcategory $\Delta_{\leq n}$. This defines a *skeleton functor*

$$\mathrm{sk}_n : \mathrm{Simp}(\mathcal{C}) \longrightarrow \mathrm{Simp}_n(\mathcal{C})$$

from the category of simplicial objects of $\mathcal{C}$ to the category of n-truncated simplicial objects of $\mathcal{C}$. See Remark 21.6 to avoid possible confusion with other functors in the literature.

13. Products with simplicial sets

Let $\mathcal{C}$ be a category. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. We can consider the covariant functor which associates to a simplicial object W of $\mathcal{C}$ the set

(13.0.1)
$$\left\{ (f_{n,u} : V_n \to W_n)_{n \geq 0, u \in U_n} \text{ such that } \begin{array}{c} \forall \varphi : [m] \to [n] \\ f_{m,U(\varphi)(u)} \circ V(\varphi) = W(\varphi) \circ f_{n,u} \end{array} \right\}$$

If this functor is of the form $\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(Q, -)$ then we can think of Q as the product of U with V. Instead of formalizing this in this way we just directly define the product as follows.

Definition 13.1. Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. In this case we define the *product $U \times V$ of U and V* to be the simplicial object of $\mathcal{C}$ whose nth term is the object

$$(U \times V)_n = \coprod\nolimits_{u \in U_n} V_n$$

with maps for $\varphi : [m] \to [n]$ given by the morphism

$$\coprod\nolimits_{u \in U_n} V_n \longrightarrow \coprod\nolimits_{u' \in U_m} V_m$$

which maps the component V_n corresponding to u to the component V_m corresponding to $u' = U(\varphi)(u)$ via the morphism $V(\varphi)$. More loosely, if all of the coproducts displayed above exist (without assuming anything about $\mathcal{C}$) we will say that the *product $U \times V$ exists*.

Lemma 13.2. *Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. The functor $W \mapsto \mathrm{Mor}_{Simp(\mathcal{C})}(U \times V, W)$ is canonically isomorphic to the functor which maps W to the set in Equation (13.0.1).*

Proof. Omitted. $\qquad\qquad\square$

Lemma 13.3. *Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let us temporarily denote FSSets the category of simplicial sets all of whose components are finite nonempty.*

(1) *The rule $(U, V) \mapsto U \times V$ defines a functor $FSSets \times Simp(\mathcal{C}) \to Simp(\mathcal{C})$.*
(2) *For every U, V as above there is a canonical map of simplicial objects*

$$U \times V \longrightarrow V$$

defined by taking the identity on each component of $(U \times V)_n = \coprod_u V_n$.

Proof. Omitted. $\qquad\qquad\square$

We briefly study a special case of the construction above. Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $k \geq 0$ be an integer. If all coproducts $X \amalg \ldots \amalg X$ exist then according to the definition above the product

$$X \times \Delta[k]$$

exists, where we think of X as the corresponding constant simplicial object.

Lemma 13.4. *With X and k as above. For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection*

$$\mathrm{Mor}_{Simp(\mathcal{C})}(X \times \Delta[k], V) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, V_k).$$

which maps γ to the restriction of the morphism γ_k to the component corresponding to $\mathrm{id}_{[k]}$. Similarly, for any $n \geq k$, if W is an n-truncated simplicial object of $\mathcal{C}$, then we have

$$\mathrm{Mor}_{Simp_n(\mathcal{C})}(sk_n(X \times \Delta[k]), W) = \mathrm{Mor}_{\mathcal{C}}(X, W_k).$$

Proof. A morphism $\gamma : X \times \Delta[k] \to V$ is given by a family of morphisms $\gamma_\alpha : X \to V_n$ where $\alpha : [n] \to [k]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{\ \gamma_\alpha\ } & V_n \\
\Big\downarrow{\scriptstyle \mathrm{id}_X} & & \Big\downarrow{\scriptstyle V(\varphi)} \\
X & \xrightarrow{\ \gamma_{\alpha \circ \varphi}\ } & V_m
\end{array}
$$

commute. Taking $\alpha = \mathrm{id}_{[k]}$, we see that for any $\varphi : [m] \to [k]$ we have $\gamma_\varphi = V(\varphi) \circ \gamma_{\mathrm{id}_{[k]}}$. Thus the morphism γ is determined by the value of γ on the component corresponding to $\mathrm{id}_{[k]}$. Conversely, given such a morphism $f : X \to V_k$ we easily construct a morphism γ by putting $\gamma_\alpha = V(\alpha) \circ f$.

The truncated case is similar, and left to the reader. $\qquad\square$

A particular example of this is the case $k = 0$. In this case the formula of the lemma just says that

$$\mathrm{Mor}_{\mathcal{C}}(X, V_0) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X, V)$$

where on the right hand side X indicates the constant simplicial object with value X. We will use this formula without further mention in the following.

14. Hom from simplicial sets into cosimplicial objects

Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$, and let V be a cosimplicial object of $\mathcal{C}$. Then we get a cosimplicial set $\mathrm{Hom}_{\mathcal{C}}(U, V)$ as follows:

(1) we set $\mathrm{Hom}_{\mathcal{C}}(U, V)_n = \mathrm{Mor}_{\mathcal{C}}(U_n, V_n)$, and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}_{\mathcal{C}}(U, V)_m \to \mathrm{Hom}_{\mathcal{C}}(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$.

This is our motivation for the following definition.

Definition 14.1. Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object of $\mathcal{C}$. Let U be a simplicial set such that each U_n is finite nonempty. We define $\mathrm{Hom}(U, V)$ to be the cosimplicial object of $\mathcal{C}$ defined as follows:

(1) we set $\mathrm{Hom}(U, V)_n = \prod_{u \in U_n} V_n$, in other words the unique object of $\mathcal{C}$ such that its X-valued points satisfy

$$\mathrm{Mor}_{\mathcal{C}}(X, \mathrm{Hom}(U, V)_n) = \mathrm{Map}(U_n, \mathrm{Mor}_{\mathcal{C}}(X, V_n))$$

and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}(U, V)_m \to \mathrm{Hom}(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$ on X-valued points as above.

We leave it to the reader to spell out the definition in terms of maps between products. We also point out that the construction is functorial in both U (contravariantly) and V (covariantly), exactly as in Lemma 13.3 in the case of products of simplicial sets with simplicial objects.

15. Hom from cosimplicial sets into simplicial objects

Let $\mathcal{C}$ be a category. Let U be a cosimplicial object of $\mathcal{C}$, and let V be a simplicial object of $\mathcal{C}$. Then we get a simplicial set $\mathrm{Hom}_{\mathcal{C}}(U, V)$ as follows:

(1) we set $\mathrm{Hom}_{\mathcal{C}}(U, V)_n = \mathrm{Mor}_{\mathcal{C}}(U_n, V_n)$, and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}_{\mathcal{C}}(U, V)_n \to \mathrm{Hom}_{\mathcal{C}}(U, V)_m$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$.

This is our motivation for the following definition.

Definition 15.1. Let $\mathcal{C}$ be a category with finite products. Let V be a simplicial object of $\mathcal{C}$. Let U be a cosimplicial set such that each U_n is finite nonempty. We define $\mathrm{Hom}(U, V)$ to be the simplicial object of $\mathcal{C}$ defined as follows:

(1) we set $\mathrm{Hom}(U, V)_n = \prod_{u \in U_n} V_n$, in other words the unique object of $\mathcal{C}$ such that its X-valued points satisfy

$$\mathrm{Mor}_{\mathcal{C}}(X, \mathrm{Hom}(U, V)_n) = \mathrm{Map}(U_n, \mathrm{Mor}_{\mathcal{C}}(X, V_n))$$

and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}(U, V)_n \to \mathrm{Hom}(U, V)_m$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$ on X-valued points as above.

We leave it to the reader to spell out the definition in terms of maps between products. We also point out that the construction is functorial in both U (contravariantly) and V (covariantly), exactly as in Lemma 13.3 in the case of products of simplicial sets with simplicial objects.

We spell out the construction above in a special case. Let X be an object of a category $\mathcal{C}$. Assume that self products $X \times \ldots \times X$ exist. Let k be an integer. Consider the simplicial object U with terms

$$U_n = \prod_{\alpha \in \mathrm{Mor}([k], [n])} X$$

and maps given $\varphi : [m] \to [n]$

$$U(\varphi) : \prod_{\alpha \in \mathrm{Mor}([k], [n])} X \longrightarrow \prod_{\alpha' \in \mathrm{Mor}([k], [m])} X, \quad (f_\alpha)_\alpha \longmapsto (f_{\varphi \circ \alpha'})_{\alpha'}$$

In terms of "coordinates", the element $(x_\alpha)_\alpha$ is mapped to the element $(x_{\varphi \circ \alpha'})_{\alpha'}$. We claim this object is equal to $\mathrm{Hom}(C[k], X)$ where we think of X as the constant simplicial object X and where $C[k]$ is the cosimplicial set from Example 5.6.

Lemma 15.2. *With X, k and U as above.*

(1) *For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection*

$$\mathrm{Mor}_{Simp(\mathcal{C})}(V, U) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(V_k, X).$$

wich maps γ to the morphism γ_k composed with the projection onto the factor corresponding to $\mathrm{id}_{[k]}$.
(2) *Similarly, if W is an k-truncated simplicial object of $\mathcal{C}$, then we have*

$$\mathrm{Mor}_{Simp_k(\mathcal{C})}(W, sk_k U) = \mathrm{Mor}_{\mathcal{C}}(W_k, X).$$

(3) *The object U constructed above is an incarnation of* $\mathrm{Hom}(C[k], X)$ *where $C[k]$ is the cosimplicial set from Example 5.6.*

Proof. We first prove (1). Suppose that $\gamma : V \to U$ is a morphism. This is given by a family of morphisms $\gamma_\alpha : V_n \to X$ for $\alpha : [k] \to [n]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams

$$
\begin{array}{ccc}
X & \xleftarrow{\ \gamma_{\varphi \circ \alpha'}\ } & V_n \\
\Big\downarrow{\scriptstyle \mathrm{id}_X} & & \Big\downarrow{\scriptstyle V(\varphi)} \\
X & \xleftarrow{\ \gamma_{\alpha'}\ } & V_m
\end{array}
$$

commute for all $\alpha' : [k] \to [m]$. Taking $\alpha' = \mathrm{id}_{[k]}$, we see that for any $\varphi : [k] \to [n]$ we have $\gamma_\varphi = \gamma_{\mathrm{id}_{[k]}} \circ V(\varphi)$. Thus the morphism γ is determined by the component of γ_k corresponding to $\mathrm{id}_{[k]}$. Conversely, given such a morphism $f : V_k \to X$ we easily construct a morphism γ by putting $\gamma_\alpha = f \circ V(\alpha)$.

The truncated case is similar, and left to the reader.

Part (3) is immediate from the construction of U and the fact that $C[k]_n = \mathrm{Mor}([k], [n])$ which are the index sets used in the construction of U_n. $\qquad\square$

16. Internal Hom

Let $\mathcal{C}$ be a category with finite nonempty products. Let U, V be simplicial objects $\mathcal{C}$. In some cases the functor

$$
\mathrm{Simp}(\mathcal{C})^{opp} \longrightarrow Sets, \quad W \longmapsto \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times V, U)
$$

is representable. In this case we denote $\mathcal{H}om(V, U)$ the resulting simplicial object of $\mathcal{C}$, and we say that the *internal hom of V into U exists*. Moreover, in this case, given X in $\mathcal{C}$, we would have

$$
\begin{aligned}
\mathrm{Mor}_{\mathcal{C}}(X, \mathcal{H}om(V, U)_n) &= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times \Delta[n], \mathcal{H}om(V, U)) \\
&= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times \Delta[n] \times V, U) \\
&= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X, \mathcal{H}om(\Delta[n] \times V, U)) \\
&= \mathrm{Mor}_{\mathcal{C}}(X, \mathcal{H}om(\Delta[n] \times V, U)_0)
\end{aligned}
$$

provided that $\mathcal{H}om(\Delta[n] \times V, U)$ exists also. The first and last equalities follow from Lemma 13.4.

The lesson we learn from this is that, given U and V, if we want to construct the internal hom then we should try to construct the objects

$$
\mathcal{H}om(\Delta[n] \times V, U)_0
$$

because these should be the nth term of $\mathcal{H}om(V, U)$. In the next section we study a construction of simplicial objects "$\mathrm{Hom}(\Delta[n], U)$".

17. Hom from simplicial sets into simplicial objects

Motivated by the discussion on internal hom we define what should be the simplicial object classifying morphisms from a simplicial set into a given simplicial object of the category $\mathcal{C}$.

Definition 17.1. Let $\mathcal{C}$ be a category such that the coproduct of any two objects exists. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. We denote $\mathrm{Hom}(U, V)$ any simplicial object of $\mathcal{C}$ such that

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W, \mathrm{Hom}(U, V)) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times U, V)$$

functorially in the simplicial object W of $\mathcal{C}$.

Of course $\mathrm{Hom}(U, V)$ need not exist. Also, by the discussion in Section 16 we expect that if it does exist, then $\mathrm{Hom}(U, V)_n = \mathrm{Hom}(U \times \Delta[n], V)_0$. We do not use the italic notation for these Hom objects since $\mathrm{Hom}(U, V)$ is not an internal hom.

Lemma 17.2. *Assume the category $\mathcal{C}$ has coproducts of any two objects and countable limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor*

$$\begin{aligned} \mathcal{C}^{opp} &\longrightarrow Sets \\ X &\longmapsto \mathrm{Mor}_{Simp(\mathcal{C})}(X \times U, V) \end{aligned}$$

is representable.

Proof. A morphism from $X \times U$ into V is given by a collection of morphisms $f_u : X \to V_n$ with $n \geq 0$ and $u \in U_n$. And such a collection actually defines a morphism if and only if for all $\varphi : [m] \to [n]$ all the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\ f_u\ } & V_n \\ {\scriptstyle \mathrm{id}_X} \downarrow & & \downarrow {\scriptstyle V(\varphi)} \\ X & \xrightarrow[f_{U(\varphi)(u)}]{} & V_m \end{array}$$

commute. Thus it is natural to introduce a category $\mathcal{U}$ and a functor $\mathcal{V} : \mathcal{U}^{opp} \to \mathcal{C}$ as follows:

(1) The set of objects of $\mathcal{U}$ is $\coprod_{n \geq 0} U_n$,
(2) a morphism from $u' \in U_m$ to $u \in U_n$ is a $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$
(3) for $u \in U_n$ we set $\mathcal{V}(u) = V_n$, and
(4) for $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$ we set $\mathcal{V}(\varphi) = V(\varphi) : V_n \to V_m$.

At this point it is clear that our functor is nothing but the functor defining

$$\lim_{\mathcal{U}^{opp}} \mathcal{V}$$

Thus if $\mathcal{C}$ has countable limits then this limit and hence an object representing the functor of the lemma exist. $\square$

Lemma 17.3. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor*

$$\begin{aligned} \mathcal{C}^{opp} &\longrightarrow Sets \\ X &\longmapsto \mathrm{Mor}_{Simp(\mathcal{C})}(X \times U, V) \end{aligned}$$

is representable.

Proof. We have to show that the category $\mathcal{U}$ described in the proof of Lemma 17.2 has a finite subcategory $\mathcal{U}'$ such that the limit of $\mathcal{V}$ over $\mathcal{U}'$ is the same as the limit of $\mathcal{V}$ over $\mathcal{U}$. We will use Categories, Lemma 17.4. For $m > 0$ let $\mathcal{U}_{\leq m}$ denote the full subcategory with objects $\coprod_{0 \leq n \leq m} U_m$. Let m_0 be an integer such that every n-simplex of the simplicial set U is degenerate if $n > m_0$. For any $m \geq m_0$ large enough, the subcategory $\mathcal{U}_{\leq m}$ satisfies property (1) of Categories, Definition 17.3.

Suppose that $u \in U_n$ and $u' \in U_{n'}$ with $n, n' \leq m_0$ and suppose that $\varphi : [k] \to [n]$, $\varphi' : [k] \to [n']$ are morphisms such that $U(\varphi)(u) = U(\varphi')(u')$. A simple combinatorial argument shows that if $k > 2m_0$, then there exists an index $0 \leq i \leq 2m_0$ such that $\varphi(i) = \varphi(i+1)$ and $\varphi'(i) = \varphi'(i+1)$. (The pigeon hole principle would tell you this works if $k > m_0^2$ which is good enough for the argument below anyways.) Hence, if $k > 2m_0$, we may write $\varphi = \psi \circ \sigma_i^{k-1}$ and $\varphi' = \psi' \circ \sigma_i^{k-1}$ for some $\psi : [k-1] \to [n]$ and some $\psi' : [k-1] \to [n']$. Since $s_i^{k-1} : U_{k-1} \to U_k$ is injective, see Lemma 3.6, we conclude that $U(\psi)(u) = U(\psi')(u')$ also. Continuing in this fashion we conclude that given morphisms $u \to z$ and $u' \to z$ of $\mathcal{U}$ with $u, u' \in \mathcal{U}_{\leq m_0}$, there exists a commutative diagram
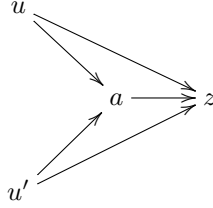
with $a \in \mathcal{U}_{\leq 2m_0}$.

It is easy to deduce from this that the finite subcategory $\mathcal{U}_{\leq 2m_0}$ works. Namely, suppose given $x' \in U_n$ and $x'' \in U_{n'}$ with $n, n' \leq 2m_0$ as well as morphisms $x' \to x$ and $x'' \to x$ of $\mathcal{U}$ with the same target. By our choice of m_0 we can find objects u, u' of $\mathcal{U}_{\leq m_0}$ and morphisms $u \to x'$, $u' \to x''$. By the above we can find $a \in \mathcal{U}_{\leq 2m_0}$ and morphisms $u \to a$, $u' \to a$ such that
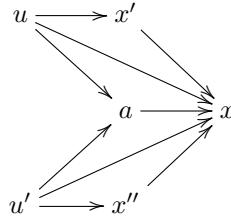
is commutative. Turning this diagram 90 degrees clockwise we get the desired diagram as in (2) of Categories, Definition 17.3. $\qquad\square$

Lemma 17.4. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then $\mathrm{Hom}(U, V)$ exists, moreover we have the expected equalities*

$$\mathrm{Hom}(U, V)_n = \mathrm{Hom}(U \times \Delta[n], V)_0.$$

Proof. We construct this simplicial object as follows. For $n \geq 0$ let $\text{Hom}(U, V)_n$ denote the object of $\mathcal{C}$ representing the functor

$$X \longmapsto \text{Mor}_{\text{Simp}(\mathcal{C})}(X \times U \times \Delta[n], V)$$

This exists by Lemma 17.3 because $U \times \Delta[n]$ is a simplicial set with finite sets of simplices and no nondegenerate simplices in high enough degree, see Lemma 11.5. For $\varphi : [m] \to [n]$ we obtain an induced map of simplicial sets $\varphi : \Delta[m] \to \Delta[n]$. Hence we obtain a morphism $X \times U \times \Delta[m] \to X \times U \times \Delta[n]$ functorial in X, and hence a transformation of functors, which in turn gives

$$\text{Hom}(U, V)(\varphi) : \text{Hom}(U, V)_n \longrightarrow \text{Hom}(U, V)_m.$$

Clearly this defines a contravariant functor $\text{Hom}(U, V)$ from Δ into the category $\mathcal{C}$. In other words, we have a simplicial object of $\mathcal{C}$.

We have to show that $\text{Hom}(U, V)$ satisfies the desired universal property

$$\text{Mor}_{\text{Simp}(\mathcal{C})}(W, \text{Hom}(U, V)) = \text{Mor}_{\text{Simp}(\mathcal{C})}(W \times U, V)$$

To see this, let $f : W \to \text{Hom}(U, V)$ be given. We want to construct the element $f' : W \times U \to V$ of the right hand side. By construction, each $f_n : W_n \to \text{Hom}(U, V)_n$ corresponds to a morphism $f_n : W_n \times U \times \Delta[n] \to V$. Further, for every morphism $\varphi : [m] \to [n]$ the diagram

$$
\begin{array}{ccc}
W_n \times U \times \Delta[m] & \xrightarrow{\;\;W(\varphi) \times \text{id} \times \text{id}\;\;} & W_m \times U \times \Delta[m] \\
{\scriptstyle \text{id} \times \text{id} \times \varphi} \downarrow & & \downarrow {\scriptstyle f_m} \\
W_n \times U \times \Delta[n] & \xrightarrow{\;\;\;\;\;\;\;f_n\;\;\;\;\;\;\;} & V
\end{array}
$$

is commutative. For $\psi : [n] \to [k]$ in $(\Delta[n])_k$ we denote $(f_n)_{k,\psi} : W_n \times U_k \to V_k$ the component of $(f_n)_k$ corresponding to the element ψ. We define $f'_n : W_n \times U_n \to V_n$ as $f'_n = (f_n)_{n,\text{id}}$, in other words, as the restriction of $(f_n)_n : W_n \times U_n \times (\Delta[n])_n \to V_n$ to $W_n \times U_n \times \text{id}_{[n]}$. To see that the collection (f'_n) defines a morphism of simplicial objects, we have to show for any $\varphi : [m] \to [n]$ that $V(\varphi) \circ f'_n = f'_m \circ W(\varphi) \times U(\varphi)$. The commutative diagram above says that $(f_n)_{m,\varphi} : W_n \times U_m \to V_m$ is equal to $(f_m)_{m,\text{id}} \circ W(\varphi) : W_n \times U_m \to V_m$. But then the fact that f_n is a morphism of simplicial objects implies that the diagram

$$
\begin{array}{ccc}
W_n \times U_n \times (\Delta[n])_n & \xrightarrow{\;\;(f_n)_n\;\;} & V_n \\
{\scriptstyle \text{id} \times U(\varphi) \times \varphi} \downarrow & & \downarrow {\scriptstyle V(\varphi)} \\
W_n \times U_m \times (\Delta[n])_m & \xrightarrow{\;\;(f_n)_m\;\;} & V_m
\end{array}
$$

is commutative. And this implies that $(f_n)_{m,\varphi} \circ U(\varphi)$ is equal to $V(\varphi) \circ (f_n)_{n,\text{id}}$. Altogether we obtain $V(\varphi) \circ (f_n)_{n,\text{id}} = (f_n)_{m,\varphi} \circ U(\varphi) = (f_m)_{m,\text{id}} \circ W(\varphi) \circ U(\varphi) = (f_m)_{m,\text{id}} \circ W(\varphi) \times U(\varphi)$ as desired.

On the other hand, given a morphism $f' : W \times U \to V$ we define a morphism $f : W \to \text{Hom}(U, V)$ as follows. By Lemma 13.4 the morphisms $\text{id} : W_n \to W_n$ corresponds to a unique morphism $c_n : W_n \times \Delta[n] \to W$. Hence we can consider the composition

$$W_n \times \Delta[n] \times U \xrightarrow{\;c_n\;} W \times U \xrightarrow{\;f'\;} V.$$

By construction this corresponds to a unique morphism $f_n : W_n \to \mathrm{Hom}(U, V)_n$. We leave it to the reader to see that these define a morphism of simplicial sets as desired.

We also leave it to the reader to see that $f \mapsto f'$ and $f' \mapsto f$ are mutually inverse operations. $\qquad\square$

Lemma 17.5. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let $a : U \to V$, $b : U \to W$ be morphisms of simplicial sets. Assume U_n, V_n, W_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U, V, W are degenerate for all $n \gg 0$. Let T be a simplicial object of $\mathcal{C}$. Then*

$$\mathrm{Hom}(V, T) \times_{\mathrm{Hom}(U,T)} \mathrm{Hom}(W, T) = \mathrm{Hom}(V \amalg_U W, T)$$

In other words, the fibre product on the left hand side is represented by the Hom object on the right hand side.

Proof. By Lemma 17.4 all the required Hom objects exist and satisfy the correct functorial properties. Now we can identify the nth term on the left hand side as the object representing the functor that associates to X the first set of the following sequence of functorial equalities

$$\mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(V, T) \times_{\mathrm{Hom}(U,T)} \mathrm{Hom}(W, T))$$
$$= \mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(V, T)) \times_{\mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(U,T))} \mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(W, T))$$
$$= \mathrm{Mor}(X \times \Delta[n] \times V, T) \times_{\mathrm{Mor}(X \times \Delta[n] \times U, T)} \mathrm{Mor}(X \times \Delta[n] \times W, T)$$
$$= \mathrm{Mor}(X \times \Delta[n] \times (V \amalg_U W), T))$$

Here we have used the fact that

$$(X \times \Delta[n] \times V) \times_{X \times \Delta[n] \times U} (X \times \Delta[n] \times W) = X \times \Delta[n] \times (V \amalg_U W)$$

which is easy to verify term by term. The result of the lemma follows as the last term in the displayed sequence of equalities corresponds to $\mathrm{Hom}(V \amalg_U W, T)_n$. $\quad\square$

18. Splitting simplicial objects

A subobject N of an object X of the category $\mathcal{C}$ is an object N of $\mathcal{C}$ together with a monomorphism $N \to X$. Of course we say (by abuse of notation) that the subobjects N, N' are equal if there exists an isomorphism $N \to N'$ compatible with the morphisms to X. The collection of subobjects forms a partially ordered set. (Because of our conventions on categories; not true for category of spaces up to homotopy for example.)

Definition 18.1. Let $\mathcal{C}$ be a category which admits finite nonempty coproducts. We say a simplicial object U of $\mathcal{C}$ is *split* if there exist subobjects $N(U_m)$ of U_m, $m \geq 0$ with the property that

$$(18.1.1) \qquad\qquad \coprod_{\varphi:[n]\to[m] \text{ surjective}} N(U_m) \longrightarrow U_n$$

is an isomorphism for all $n \geq 0$. If U is an r-truncated simplicial object of $\mathcal{C}$ then we say U is *split* if there exist subobjects $N(U_m)$ of U_m, $r \geq m \geq 0$ with the property that (18.1.1) is an isomorphism for $r \geq n \geq 0$.

If this is the case, then $N(U_0) = U_0$. Next, we have $U_1 = U_0 \amalg N(U_1)$. Second we have

$$U_2 = U_0 \amalg N(U_1) \amalg N(U_1) \amalg N(U_2).$$

It turns out that in many categories $\mathcal{C}$ every simplicial object is split.

Lemma 18.2. *Let U be a simplicial set. Then U has a unique splitting with $N(U_m)$ equal to the set of nondegenerate m-simplices.*

Proof. From the definition it follows immediately, that if there is a splitting then $N(U_m)$ has to be the set of nondegenerate simplices. Let $x \in U_n$. Suppose that there are surjections $\varphi : [n] \to [k]$ and $\psi : [n] \to [l]$ and nondegenerate simplices $y \in U_k$, $z \in U_l$ such that $x = U(\varphi)(y)$ and $x = U(\psi)(z)$. Choose a right inverse $\xi : [l] \to [n]$ of ψ, i.e., $\psi \circ \xi = \text{id}_{[l]}$. Then $z = U(\xi)(x)$. Hence $z = U(\xi)(x) = U(\varphi \circ \xi)(y)$. Since z is nondegenerate we conclude that $\varphi \circ \xi : [l] \to [k]$ is surjective, and hence $l \geq k$. Similarly $k \geq l$. Hence we see that $\varphi \circ \xi : [l] \to [k]$ has to be the identity map for any choice of right inverse ξ of ψ. This easily implies that $\psi = \varphi$. $\square$

Of course it can happen that a map of simplicial sets maps a nondegenerate n-simplex to a degenerate n-simplex. Thus the splitting of Lemma 18.2 is not functorial. Here is a case where it is functorial.

Lemma 18.3. *Let $f : U \to V$ be a morphism of simplicial sets. Suppose that (a) the image of every nondegenerate simplex of U is a nondegenerate simplex of V and (b) the restriction of f to a map from the set of nondegenerate simplices of U to the set of nondegenerate simplices of V is injective. Then f_n is injective for all n. Same holds with "injective" replaced by "surjective" or "bijective".*

Proof. Under hypothesis (a) we see that the map f preserves the disjoint union decompositions of the splitting of Lemma 18.2, in other words that we get commutative diagrams

$$
\begin{array}{ccc}
\coprod_{\varphi:[n]\to[m] \text{ surjective}} N(U_m) & \longrightarrow & U_n \\
\downarrow & & \downarrow \\
\coprod_{\varphi:[n]\to[m] \text{ surjective}} N(V_m) & \longrightarrow & V_n.
\end{array}
$$

And then (b) clearly shows that the left vertical arrow is injective (resp. surjective, resp. bijective). $\square$

Lemma 18.4. *Let U be a simplicial set. Let $n \geq 0$ be an integer. The rule*

$$U'_m = \bigcup_{\varphi:[m]\to[i],\ i\leq n} \text{Im}(U(\varphi))$$

defines a sub simplicial set $U' \subset U$ with $U'_i = U_i$ for $i \leq n$. Moreover, all m-simplices of U' are degenerate for all $m > n$.

Proof. If $x \in U_m$ and $x = U(\varphi)(y)$ for some $y \in U_i$, $i \leq n$ and some $\varphi : [m] \to [i]$ then any image $U(\psi)(x)$ for any $\psi : [m'] \to [m]$ is equal to $U(\varphi \circ \psi)(y)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial set. By construction all simplices in dimension $n + 1$ and higher are degenerate. $\square$

Lemma 18.5. *Let U be a simplicial abelian group. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \geq 1$ taking*

$$N(U_m) = \bigcap_{i=0}^{m-1} \mathrm{Ker}(d_i^m).$$

Moreover, this splitting is functorial on the category of simplicial abelian groups.

Proof. By induction on n we will show that the choice of $N(U_m)$ in the lemma guarantees that (18.1.1) is an isomorphism for $m \leq n$. This is clear for $n = 0$. In the rest of this proof we are going to drop the superscripts from the maps d_i and s_i in order to improve readability. We will also repeatedly use the relations from Remark 3.3.

First we make a general remark. For $0 \leq i \leq m$ and $z \in U_m$ we have $d_i(s_i(z)) = z$. Hence we can write any $x \in U_{m+1}$ uniquely as $x = x' + x''$ with $d_i(x') = 0$ and $x'' \in \mathrm{Im}(s_i)$ by taking $x' = (x - s_i(d_i(x)))$ and $x'' = s_i(d_i(x))$. Moreover, the element $z \in U_m$ such that $x'' = s_i(z)$ is unique because s_i is injective.

Here is a procedure for decomposing any $x \in U_{n+1}$. First, write $x = x_0 + s_0(z_0)$ with $d_0(x_0) = 0$. Next, write $x_0 = x_1 + s_1(z_1)$ with $d_n(x_1) = 0$. Continue like this to get

$$
\begin{aligned}
x &= x_0 + s_0(z_0), \\
x_0 &= x_1 + s_1(z_1), \\
x_1 &= x_2 + s_2(z_2), \\
\ldots \quad &\ldots \quad \ldots \\
x_{n-1} &= x_n + s_n(z_n)
\end{aligned}
$$

where $d_i(x_i) = 0$ for all $i = n, \ldots, 0$. By our general remark above all of the x_i and z_i are determined uniquely by x. We claim that $x_i \in \mathrm{Ker}(d_0) \cap \mathrm{Ker}(d_1) \cap \ldots \cap \mathrm{Ker}(d_i)$ and $z_i \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$ for $i = n, \ldots, 0$. Here and in the following an empty intersection of kernels indicates the whole space; i.e., the notation $z_0 \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$ when $i = 0$ means $z_0 \in U_n$ with no restriction.

We prove this by ascending induction on i. It is clear for $i = 0$ by construction of x_0 and z_0. Let us prove it for $0 < i \leq n$ assuming the result for $i-1$. First of all we have $d_i(x_i) = 0$ by construction. So pick a j with $0 \leq j < i$. We have $d_j(x_{i-1}) = 0$ by induction. Hence

$$0 = d_j(x_{i-1}) = d_j(x_i) + d_j(s_i(z_i)) = d_j(x_i) + s_{i-1}(d_j(z_i)).$$

The last equality by the relations of Remark 3.3. These relations also imply that $d_{i-1}(d_j(x_i)) = d_j(d_i(x_i)) = 0$ because $d_i(x_i) = 0$ by construction. Then the uniqueness in the general remark above shows the equality $0 = x' + x'' = d_j(x_i) + s_{i-1}(d_j(z_i))$ can only hold if both terms are zero. We conclude that $d_j(x_i) = 0$ and by injectivity of s_{i-1} we also conclude that $d_j(z_i) = 0$. This proves the claim.

The claim implies we can uniquely write

$$x = s_0(z_0) + s_1(z_1) + \ldots + s_n(z_n) + x_0$$

with $x_0 \in N(U_{n+1})$ and $z_i \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$. We can reformulate this as saying that we have found a direct sum decomposition

$$U_{n+1} = N(U_{n+1}) \oplus \bigoplus_{i=0}^{i=n} s_i\Big(\mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1}) \Big)$$

with the property that

$$\mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_j) = N(U_{n+1}) \oplus \bigoplus_{i=j+1}^{i=n} s_i\Big(\mathrm{Ker}(d_n) \cap \ldots \cap \mathrm{Ker}(d_{i-1}) \Big)$$

for $j = 0, \ldots, n$. The result follows from this statement as follows. Each of the z_i in the expression for x can be written uniquely as

$$z_i = s_i(z'_{i,i}) + \ldots + s_{n-1}(z'_{i,n-1}) + z_{i,0}$$

with $z_{i,0} \in N(U_n)$ and $z'_{i,j} \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{j-1})$. The first few steps in the decomposition of z_i are zero because z_i already is in the kernel of $d_0, \ldots, d_i$. This in turn uniquely gives

$$x = x_0 + s_0(z_{0,0}) + s_1(z_{1,0}) + \ldots + s_n(z_{n,0}) + \sum_{0 \le i \le j \le n-1} s_i(s_j(z'_{i,j})).$$

Continuing in this fashion we see that we in the end obtain a decomposition of x as a sum of terms of the form

$$s_{i_1} s_{i_2} \ldots s_{i_k}(z)$$

with $0 \le i_1 \le i_2 \le \ldots \le i_k \le n - k + 1$ and $z \in N(U_{n+1-k})$. This is exactly the required decomposition, because any surjective map $[n+1] \to [n+1-k]$ can be uniquely expressed in the form

$$\sigma_{i_k}^{n-k} \ldots \sigma_{i_2}^{n-1} \sigma_{i_1}^{n}$$

with $0 \le i_1 \le i_2 \le \ldots \le i_k \le n - k + 1$. $\qquad\square$

Lemma 18.6. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \ge 1$ taking*

$$N(U_m) = \bigcap_{i=0}^{m-1} \mathrm{Ker}(d_i^m).$$

Moreover, this splitting is functorial on the category of simplicial objects of $\mathcal{A}$.

Proof. For any object A of $\mathcal{A}$ we obtain a simplicial abelian group $\mathrm{Mor}_{\mathcal{A}}(A, U)$. Each of these are canonically split by Lemma 18.5. Moreover,

$$N(\mathrm{Mor}_{\mathcal{A}}(A, U_m)) = \bigcap_{i=0}^{m-1} \mathrm{Ker}(d_i^m) = \mathrm{Mor}_{\mathcal{A}}(A, N(U_m)).$$

Hence we see that the morphism (18.1.1) becomes an isomorphism after applying the functor $\mathrm{Mor}_{\mathcal{A}}(A, -)$ for any object of $\mathcal{A}$. Hence it is an isomorphism by the Yoneda lemma. $\qquad\square$

Lemma 18.7. *Let $\mathcal{A}$ be an abelian category. Let $f : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If the induced morphisms $N(f)_i : N(U)_i \to N(V)_i$ are injective for all i, then f_i is injective for all i. Same holds with "injective" replaced with "surjective", or "isomorphism".*

Proof. This is clear from Lemma 18.6 and the definition of a splitting. $\qquad\square$

Lemma 18.8. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Let $N(U_m)$ as in Lemma 18.6 above. Then $d_m^m(N(U_m)) \subset N(U_{m-1})$.*

Proof. For $j = 0, \ldots, m-2$ we have $d_j^{m-1} d_m^m = d_{m-1}^{m-1} d_j^m$ by the relations in Remark 3.3. The result follows. $\square$

Lemma 18.9. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \geq 0$ be an integer. The rule*

$$U_m' = \sum\nolimits_{\varphi:[m]\to[i],\ i\leq n} \mathrm{Im}(U(\varphi))$$

defines a sub simplicial object $U' \subset U$ with $U_i' = U_i$ for $i \leq n$. Moreover, $N(U_m') = 0$ for all $m > n$.

Proof. Pick m, $i \leq n$ and some $\varphi : [m] \to [i]$. The image under $U(\psi)$ of $\mathrm{Im}(U(\varphi))$ for any $\psi : [m'] \to [m]$ is equal to the image of $U(\varphi \circ \psi)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial object. Pick $m > n$. We have to show $N(U_m') = 0$. By definition of $N(U_m)$ and $N(U_m')$ we have $N(U_m') = U_m' \cap N(U_m)$ (intersection of subobjects). Since U is split by Lemma 18.6, it suffices to show that U_m' is contained in the sum

$$\sum\nolimits_{\varphi:[m]\to[m']\ \mathrm{surjective},\ m'<m} \mathrm{Im}(U(\varphi)|_{N(U_{m'})}).$$

By the splitting each $U_{m'}$ is the sum of images of $N(U_{m''})$ via $U(\psi)$ for surjective maps $\psi : [m'] \to [m'']$. Hence the displayed sum above is the same as

$$\sum\nolimits_{\varphi:[m]\to[m']\ \mathrm{surjective},\ m'<m} \mathrm{Im}(U(\varphi)).$$

Clearly U_m' is contained in this by the simple fact that any $\varphi : [m] \to [i]$, $i \leq n$ occurring in the definition of U_m' may be factored as $[m] \to [m'] \to [i]$ with $[m] \to [m']$ surjective and $m' < m$ as in the last displayed sum above. $\square$

19. Coskeleton functors

Let $\mathcal{C}$ be a category. The *coskeleton functor* (if it exists) is a functor

$$\mathrm{cosk}_n : \mathrm{Simp}_n(\mathcal{C}) \longrightarrow \mathrm{Simp}(\mathcal{C})$$

which is right adjoint to the skeleton functor. In a formula

(19.0.1) $$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(U, \mathrm{cosk}_n V) = \mathrm{Mor}_{\mathrm{Simp}_n(\mathcal{C})}(\mathrm{sk}_n U, V)$$

Given a n-truncated simplicial object V we say that *$\mathrm{cosk}_n V$ exists* if there exists a $\mathrm{cosk}_n V \in \mathrm{Ob}(\mathrm{Simp}(\mathcal{C}))$ and a morphism $\mathrm{sk}_n \mathrm{cosk}_n V \to V$ such that the displayed formula holds, in other words if the functor $U \mapsto \mathrm{Mor}_{\mathrm{Simp}_n(\mathcal{C})}(\mathrm{sk}_n U, V)$ is representable. If it exists it is unique up to unique isomorphism by the Yoneda lemma. See Categories, Section 3.

Example 19.1. Suppose the category $\mathcal{C}$ has finite nonempty self products. A 0-truncated simplicial object of $\mathcal{C}$ is the same as an object X of $\mathcal{C}$. In this case we claim that $\mathrm{cosk}_0(X)$ is the simplicial object U with $U_n = X^{n+1}$ the $(n+1)$-fold self product of X, and structure of simplicial object as in Example 3.5. Namely, a morphism $V \to U$ where V is a simplicial object is given by morphisms $V_n \to X^{n+1}$, such that all the diagrams

$$
\begin{array}{ccc}
V_n & \longrightarrow & X^{n+1} \\
{\scriptstyle V([0]\to[n],0\mapsto i)} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_i} \\
V_0 & \longrightarrow & X
\end{array}
$$

commute. Clearly this means that the map determines and is determined by a unique morphism $V_0 \to X$. This proves that formula (19.0.1) holds.

Recall the category $\Delta/[n]$, see Example 11.4. We let $(\Delta/[n])_{\leq m}$ denote the full subcategory of $\Delta/[n]$ consisting of objects $[k] \to [n]$ of $\Delta/[n]$ with $k \leq m$. In other words we have the following commutative diagram of categories and functors

$$
\begin{array}{ccc}
(\Delta/[n])_{\leq m} & \longrightarrow & \Delta/[n] \\
\downarrow & & \downarrow \\
\Delta_{\leq m} & \longrightarrow & \Delta
\end{array}
$$

Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : (\Delta/[n])_{\leq m}^{opp} \longrightarrow \mathcal{C}$$

by the rules

$$
\begin{aligned}
([k] \to [n]) & \longmapsto & U_k \\
\psi : ([k'] \to [n]) \to ([k] \to [n]) & \longmapsto & U(\psi) : U_k \to U_{k'}
\end{aligned}
$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$\overline{\varphi} : (\Delta/[n])_{\leq m} \longrightarrow (\Delta/[n'])_{\leq m}$$

which maps $\alpha : [k] \to [n]$ to $\varphi \circ \alpha : [k] \to [n']$. The composition $U(n') \circ \overline{\varphi}$ is equal to the functor $U(n)$.

Lemma 19.2. *If the category $\mathcal{C}$ has finite limits, then cosk_m functors exist for all m. Moreover, for any m-truncated simplicial object U the simplicial object $\mathrm{cosk}_m U$ is described by the formula*

$$(\mathrm{cosk}_m U)_n = \lim_{(\Delta/[n])_{\leq m}^{opp}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $\mathrm{cosk}_m U(\varphi)$ comes from the identification $U(n') \circ \overline{\varphi} = U(n)$ above via Categories, Lemma 14.9.

Proof. During the proof of this lemma we denote $\mathrm{cosk}_m U$ the simplicial object with $(\mathrm{cosk}_m U)_n$ equal to $\lim_{(\Delta/[n])_{\leq m}^{opp}} U(n)$. We will conclude at the end of the proof that it does satisfy the required mapping property.

Suppose that V is a simplicial object. A morphism $\gamma : V \to \mathrm{cosk}_m U$ is given by a sequence of morphisms $\gamma_n : V_n \to (\mathrm{cosk}_m U)_n$. By definition of a limit, this is given by a collection of morphisms $\gamma(\alpha) : V_n \to U_k$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$. These morphisms then also satisfy the rules that

$$
\begin{array}{ccc}
V_n & \xrightarrow{\gamma(\alpha)} & U_k \\
V(\varphi) \uparrow & & \uparrow U(\psi) \\
V_{n'} & \xrightarrow{\gamma(\alpha')} & U_{k'}
\end{array}
$$

are commutative, given any $0 \leq k, k' \leq m$, $0 \leq n, n'$ and any $\psi : [k] \to [k']$, $\varphi : [n] \to [n']$, $\alpha : [k] \to [n]$ and $\alpha' : [k'] \to [n']$ in Δ such that $\varphi \circ \alpha = \alpha' \circ \psi$. Taking $n = k = k'$, $\varphi = \alpha'$, and $\alpha = \psi = \mathrm{id}_{[k]}$ we deduce that $\gamma(\alpha') = \gamma(\mathrm{id}_{[k]}) \circ V(\alpha')$. In other words, the morphisms $\gamma(\mathrm{id}_{[k]})$, $k \leq m$ determine the morphism γ. And it is easy to see that these morphisms form a morphism $\mathrm{sk}_m V \to U$.

Conversely, given a morphism $\gamma : \mathrm{sk}_m V \to U$, we obtain a family of morphisms $\gamma(\alpha)$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$ by setting $\gamma(\alpha) = \gamma(\mathrm{id}_{[k]}) \circ V(\alpha)$. These morphisms satisfy all the displayed commutativity restraints pictured above, and hence give rise to a morphism $V \to \mathrm{cosk}_m U$. $\qquad \square$

Lemma 19.3. *Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For $n \leq m$ the limit $\lim_{(\Delta/[n])_{\leq m}^{opp}} U(n)$ exists and is canonically isomorphic to U_n.*

Proof. This is true because the category $(\Delta/[n])_{\leq m}$ has an final object in this case, namely the identity map $[n] \to [n]$. $\qquad \square$

Lemma 19.4. *Let $\mathcal{C}$ be a category with finite limits. Let U be an n-truncated simplicial object of $\mathcal{C}$. The morphism $\mathrm{sk}_n \mathrm{cosk}_n U \to U$ is an isomorphism.*

Proof. Combine Lemmas 19.2 and 19.3. $\qquad \square$

Let us describe a particular instance of the coskeleton functor in more detail. By abuse of notation we will denote sk_n also the restriction functor $\mathrm{Simp}_{n'}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C})$ for any $n' \geq n$. We are going to describe a right adjoint of the functor $\mathrm{sk}_n :$ $\mathrm{Simp}_{n+1}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C})$. For $n \geq 1$, $0 \leq i < j \leq n+1$ define $\delta_{i,j}^{n+1} : [n-1] \to [n+1]$ to be the increasing map omitting i and j. Note that $\delta_{i,j}^{n+1} = \delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n$, see Lemma 2.3. This motivates the following lemma.

Lemma 19.5. *Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set*

$$\{(f_0, \ldots, f_{n+1}) \in \mathrm{Mor}_{\mathcal{C}}(T, U_n) \mid d_{j-1}^n \circ f_i = d_i^n \circ f_j \ \forall \ 0 \leq i < j \leq n+1\}$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then

$$U_{n+1} = \lim_{(\Delta/[n+1])_{\leq n}^{opp}} U(n)$$

Proof. The limit, if it exists, represents the functor that associates to an object T the set

$$\{(f_\alpha)_{\alpha:[k]\to[n+1], k\leq n} \mid f_{\alpha\circ\psi} = U(\psi) \circ f_\alpha \ \forall \ \psi : [k'] \to [k], \alpha : [k] \to [n+1]\}.$$

In fact we will show this functor is isomorphic to the one displayed in the lemma. The map in one direction is given by the rule

$$(f_\alpha)_\alpha \longmapsto (f_{\delta_0^{n+1}}, \ldots, f_{\delta_{n+1}^{n+1}}).$$

This satisfies the conditions of the lemma because

$$d_{j-1}^n \circ f_{\delta_i^{n+1}} = f_{\delta_i^{n+1} \circ \delta_{j-1}^n} = f_{\delta_j^{n+1} \circ \delta_i^n} = d_i^n \circ f_{\delta_j^{n+1}}$$

by the relations we recalled above the lemma. To construct a map in the other direction we have to associate to a system $(f_0, \ldots, f_{n+1})$ as in the displayed formula of the lemma a system of maps f_α. Let $\alpha : [k] \to [n+1]$ be given. Since $k \leq n$ the map α is not surjective. Hence we can write $\alpha = \delta_i^{n+1} \circ \psi$ for some $0 \leq i \leq n+1$ and some $\psi : [k] \to [n]$. We have no choice but to define

$$f_\alpha = U(\psi) \circ f_i.$$

Of course we have to check that this is independent of the choice of the pair (i, ψ). First, observe that given i there is a unique ψ which works. Second, suppose that (j, ϕ) is another pair. Then $i \neq j$ and we may assume $i < j$. Since both i, j are

not in the image of α we may actually write $\alpha = \delta_{i,j}^{n+1} \circ \xi$ and then we see that $\psi = \delta_{j-1}^n \circ \xi$ and $\phi = \delta_i^n \circ \xi$. Thus

$$
\begin{aligned}
U(\psi) \circ f_i &= U(\delta_{j-1}^n \circ \xi) \circ f_i \\
&= U(\xi) \circ d_{j-1}^n \circ f_i \\
&= U(\xi) \circ d_i^n \circ f_j \\
&= U(\delta_i^n \circ \xi) \circ f_j \\
&= U(\phi) \circ f_j
\end{aligned}
$$

as desired. We still have to verify that the maps f_α so defined satisfy the rules of a system of maps $(f_\alpha)_\alpha$. To see this suppose that $\psi : [k'] \to [k]$, $\alpha : [k] \to [n+1]$ with $k, k' \leq n$. Set $\alpha' = \alpha \circ \psi$. Choose i not in the image of α. Then clearly i is not in the image of α' also. Write $\alpha = \delta_i^{n+1} \circ \phi$ (we cannot use the letter ψ here because we've already used it). Then obviously $\alpha' = \delta_i^{n+1} \circ \phi \circ \psi$. By construction above we then have

$$
U(\psi) \circ f_\alpha = U(\psi) \circ U(\phi) \circ f_i = U(\phi \circ \psi) \circ f_i = f_{\alpha \circ \psi} = f_{\alpha'}
$$

as desired. We leave to the reader the pleasant task of verifying that our constructions are mutually inverse bijections, and are functorial in T. $\qquad\square$

Lemma 19.6. *Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set*

$$
\{(f_0, \ldots, f_{n+1}) \in \mathrm{Mor}_\mathcal{C}(T, U_n) \mid d_{j-1}^n \circ f_i = d_i^n \circ f_j \ \forall \ 0 \leq i < j \leq n+1\}
$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then there exists an $(n+1)$-truncated simplicial object $\tilde{U}$, with $\mathrm{sk}_n\tilde{U} = U$ and $\tilde{U}_{n+1} = U_{n+1}$ such that the following adjointness holds

$$
\mathrm{Mor}_{Simp_{n+1}(\mathcal{C})}(V, \tilde{U}) = \mathrm{Mor}_{Simp_n(\mathcal{C})}(\mathrm{sk}_n V, U)
$$

Proof. By Lemma 19.3 there are identifications

$$
U_i = \lim_{(\Delta/[i])_{\leq n}^{opp}} U(i)
$$

for $0 \leq i \leq n$. By Lemma 19.5 we have

$$
U_{n+1} = \lim_{(\Delta/[n+1])_{\leq n}^{opp}} U(n).
$$

Thus we may define for any $\varphi : [i] \to [j]$ with $i, j \leq n + 1$ the corresponding map $\tilde{U}(\varphi) : \tilde{U}_j \to \tilde{U}_i$ exactly as in Lemma 19.2. This defines an $(n + 1)$-truncated simplicial object $\tilde{U}$ with $\mathrm{sk}_n\tilde{U} = U$.

To see the adjointness we argue as follows. Given any element $\gamma : \mathrm{sk}_n V \to U$ of the right hand side of the formula consider the morphisms $f_i = \gamma_n \circ d_i^{n+1} : V_{n+1} \to V_n \to U_n$. These clearly satisfy the relations $d_{j-1}^n \circ f_i = d_i^n \circ f_j$ and hence define a unique morphism $V_{n+1} \to U_{n+1}$ by our choice of U_{n+1}. Conversely, given a morphism $\gamma' : V \to \tilde{U}$ of the left hand side we can simply restrict to $\Delta_{\leq n}$ to get an element of the right hand side. We leave it to the reader to show these are mutually inverse constructions. $\qquad\square$

Remark 19.7. Let U, and U_{n+1} be as in Lemma 19.6. On T-valued points we can easily describe the face and degeneracy maps of $\tilde{U}$. Explicitly, the maps $d_i^{n+1} : U_{n+1} \to U_n$ are given by

$$(f_0, \ldots, f_{n+1}) \longmapsto f_i.$$

And the maps $s_j^n : U_n \to U_{n+1}$ are given by

$$
\begin{aligned}
f \quad \longmapsto \quad & (s_{j-1}^{n-1} \circ d_0^{n-1} \circ f, \\
& s_{j-1}^{n-1} \circ d_1^{n-1} \circ f, \\
& \ldots \\
& s_{j-1}^{n-1} \circ d_{j-1}^{n-1} \circ f, \\
& f, \\
& f, \\
& s_j^{n-1} \circ d_{j+1}^{n-1} \circ f, \\
& s_j^{n-1} \circ d_{j+2}^{n-1} \circ f, \\
& \ldots \\
& s_j^{n-1} \circ d_n^{n-1} \circ f)
\end{aligned}
$$

where we leave it to the reader to verify that the RHS is an element of the displayed set of Lemma 19.6. For $n = 0$ there is one map, namely $f \mapsto (f, f)$. For $n = 1$ there are two maps, namely $f \mapsto (f, f, s_0 d_1 f)$ and $f \mapsto (s_0 d_0 f, f, f)$. For $n = 2$ there are three maps, namely $f \mapsto (f, f, s_0 d_1 f, s_0 d_2 f)$, $f \mapsto (s_0 d_0 f, f, f, s_1 d_2 f)$, and $f \mapsto (s_1 d_0 f, s_1 d_1 f, f, f)$. And so on and so forth.

Remark 19.8. The construction of Lemma 19.6 above in the case of simplicial sets is the following. Given an n-truncated simplicial set U, we make a canonical $(n + 1)$-truncated simplicial set $\tilde{U}$ as follows. We add a set of $(n + 1)$-simplices U_{n+1} by the formula of the lemma. Namely, an element of U_{n+1} is a numbered collection of $(f_0, \ldots, f_{n+1})$ of n-simplices, with the property that they glue as they would in a $(n + 1)$-simplex. In other words, the ith face of f_j is the $(j - 1)$st face of f_i for $i < j$. Geometrically it is obvious how to define the face and degeneracy maps for $\tilde{U}$. If V is an $(n + 1)$-truncated simplicial set, then its $(n + 1)$-simplices give rise to compatible collections of n-simplices $(f_0, \ldots, f_{n+1})$ with $f_i \in V_n$. Hence there is a natural map $\mathrm{Mor}(\mathrm{sk}_n V, U) \to \mathrm{Mor}(V, \tilde{U})$ which is inverse to the canonical restriction mapping the other way.

Also, it is enough to do the combinatorics of the construction in the case of truncated simplicial sets. Namely, for any object T of the category $\mathcal{C}$, and any n-truncated simplicial object U of $\mathcal{C}$ we can consider the n-truncated simplicial set $\mathrm{Mor}(T, U)$. We may apply the construction to this, and take its set of $(n + 1)$-simplices, and require this to be representable. This is a good way to think about the result of Lemma 19.6.

Remark 19.9. *Inductive construction of coskeleta.* Suppose that $\mathcal{C}$ is a category with finite limits. Suppose that U is an m-truncated simplicial object in $\mathcal{C}$. Then we can inductively construct n-truncated objects U^n as follows:

(1) To start, set $U^m = U$.
(2) Given U^n for $n \geq m$ set $U^{n+1} = \tilde{U}^n$, where $\tilde{U}^n$ is constructed from U^n as in Lemma 19.6.

Since the construction of Lemma 19.6 has the property that it leaves the n-skeleton of U^n unchanged, we can then define $\text{cosk}_m U$ to be the simplicial object with $(\text{cosk}_m U)_n = U_n^n = U_n^{n+1} = \ldots$. And it follows formally from Lemma 19.6 that U^n satisfies the formula

$$\text{Mor}_{\text{Simp}_n(\mathcal{C})}(V, U^n) = \text{Mor}_{\text{Simp}_m(\mathcal{C})}(\text{sk}_m V, U)$$

for all $n \geq m$. It also then follows formally from this that

$$\text{Mor}_{\text{Simp}(\mathcal{C})}(V, \text{cosk}_m U) = \text{Mor}_{\text{Simp}_m(\mathcal{C})}(\text{sk}_m V, U)$$

with $\text{cosk}_m U$ chosen as above.

Lemma 19.10. *Let $\mathcal{C}$ be a category which has finite limits.*

(1) *For every n the functor $sk_n : Simp(\mathcal{C}) \to Simp_n(\mathcal{C})$ has a right adjoint $cosk_n$.*
(2) *For every $n' \geq n$ the functor $sk_n : Simp_{n'}(\mathcal{C}) \to Simp_n(\mathcal{C})$ has a right adjoint, namely $sk_{n'} cosk_n$.*
(3) *For every $m \geq n \geq 0$ and every n-truncated simplicial object U of $\mathcal{C}$ we have $cosk_m sk_m cosk_n U = cosk_n U$.*
(4) *If U is a simplicial object of $\mathcal{C}$ such that the canonical map $U \to cosk_n sk_n U$ is an isomorphism for some $n \geq 0$, then the canonical map $U \to cosk_m sk_m U$ is an isomorphism for all $m \geq n$.*

Proof. The existence in (1) follows from Lemma 19.2 above. Parts (2) and (3) follow from the discussion in Remark 19.9. After this (4) is obvious. $\square$

Remark 19.11. We do not need all finite limits in order to be able to define the coskeleton functors. Here are some remarks

(1) We have seen in Example 19.1 that if $\mathcal{C}$ has products of pairs of objects then cosk_0 exists.
(2) For $k > 0$ the functor cosk_k exists if $\mathcal{C}$ has finite connected limits.

This is clear from the inductive procedure of constructing coskeleta (Remarks 19.8 and 19.9) but it also follows from the fact that the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k + 1$ used in Lemma 19.2 are connected. Observe that we do not need the categories for $n \leq k$ by Lemma 19.3 or Lemma 19.4. (As k gets higher the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k + 1$ are more and more connected in a topological sense.)

Lemma 19.12. *Let U, V be n-truncated simplicial objects of a category $\mathcal{C}$. Then*

$$cosk_n(U \times V) = cosk_n U \times cosk_n V$$

whenever the left and right hand sides exist.

Proof. Let W be a simplicial object. We have

$$
\begin{aligned}
\text{Mor}(W, \text{cosk}_n(U \times V)) &= \text{Mor}(\text{sk}_n W, U \times V) \\
&= \text{Mor}(\text{sk}_n W, U) \times \text{Mor}(\text{sk}_n W, V) \\
&= \text{Mor}(W, \text{cosk}_n U) \times \text{Mor}(W, \text{cosk}_n V) \\
&= \text{Mor}(W, \text{cosk}_n U \times \text{cosk}_n V)
\end{aligned}
$$

The lemma follows. $\square$

Lemma 19.13. *Assume $\mathcal{C}$ has fibre products. Let $U \to V$ and $W \to V$ be morphisms of n-truncated simplicial objects of the category $\mathcal{C}$. Then*

$$cosk_n(U \times_V W) = cosk_n U \times_{cosk_n V} cosk_n W$$

whenever the left and right hand side exist.

Proof. Omitted, but very similar to the proof of Lemma 19.12 above. $\qquad\square$

Lemma 19.14. *Let $\mathcal{C}$ be a category with finite limits. Let $X \in \mathrm{Ob}(\mathcal{C})$. The functor $\mathcal{C}/X \to \mathcal{C}$ commutes with the coskeleton functors $cosk_k$ for $k \geq 1$.*

Proof. The statement means that if U is a simplicial object of $\mathcal{C}/X$ which we can think of as a simplicial object of $\mathcal{C}$ with a morphism towards the constant simplicial object X, then $cosk_k U$ computed in $\mathcal{C}/X$ is the same as computed in $\mathcal{C}$. This follows for example from Categories, Lemma 16.2 because the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k+1$ used in Lemma 19.2 are connected. Observe that we do not need the categories for $n \leq k$ by Lemma 19.3 or Lemma 19.4. $\qquad\square$

Lemma 19.15. *The canonical map $\Delta[n] \to cosk_1 sk_1 \Delta[n]$ is an isomorphism.*

Proof. Consider a simplicial set U and a morphism $f : U \to \Delta[n]$. This is a rule that associates to each $u \in U_i$ a map $f_u : [i] \to [n]$ in Δ. Furthermore, these maps should have the property that $f_u \circ \varphi = f_{U(\varphi)(u)}$ for any $\varphi : [j] \to [i]$. Denote $\epsilon_j^i : [0] \to [i]$ the map which maps 0 to j. Denote $F : U_0 \to [n]$ the map $u \mapsto f_u(0)$. Then we see that

$$f_u(j) = F(\epsilon_j^i(u))$$

for all $0 \leq j \leq i$ and $u \in U_i$. In particular, if we know the function F then we know the maps f_u for all $u \in U_i$ all i. Conversely, given a map $F : U_0 \to [n]$, we can set for any i, and any $u \in U_i$ and any $0 \leq j \leq i$

$$f_u(j) = F(\epsilon_j^i(u))$$

This does not in general define a morphism f of simplicial sets as above. Namely, the condition is that all the maps f_u are nondecreasing. This clearly is equivalent to the condition that $F(\epsilon_j^i(u)) \leq F(\epsilon_{j'}^i(u))$ whenever $0 \leq j \leq j' \leq i$ and $u \in U_i$. But in this case the morphisms

$$\epsilon_j^i, \epsilon_{j'}^i : [0] \to [i]$$

both factor through the map $\epsilon_{j,j'}^i : [1] \to [i]$ defined by the rules $0 \mapsto j$, $1 \mapsto j'$. In other words, it is enough to check the inequalities for $i = 1$ and $u \in X_1$. In other words, we have

$$\mathrm{Mor}(U, \Delta[n]) = \mathrm{Mor}(sk_1 U, sk_1 \Delta[n])$$

as desired. $\qquad\square$

20. Augmentations

Definition 20.1. Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$. An *augmentation $\epsilon : U \to X$ of U towards an object X of $\mathcal{C}$* is a morphism from U into the constant simplicial object X.

Lemma 20.2. *Let $\mathcal{C}$ be a category. Let $X \in \mathrm{Ob}(\mathcal{C})$. Let U be a simplicial object of $\mathcal{C}$. To give an augmentation of U towards X is the same as giving a morphism $\epsilon_0 : U_0 \to X$ such that $\epsilon_0 \circ d_0^1 = \epsilon_0 \circ d_1^1$.*

Proof. Given a morphism $\epsilon : U \to X$ we certainly obtain an ϵ_0 as in the lemma. Conversely, given ϵ_0 as in the lemma, define $\epsilon_n : U_n \to X$ by choosing any morphism $\alpha : [0] \to [n]$ and taking $\epsilon_n = \epsilon_0 \circ U(\alpha)$. Namely, if $\beta : [0] \to [n]$ is another choice, then there exists a morphism $\gamma : [1] \to [n]$ such that α and β both factor as $[0] \to [1] \to [n]$. Hence the condition on ϵ_0 shows that ϵ_n is well defined. Then it is easy to show that $(\epsilon_n) : U \to X$ is a morphism of simplicial objects. $\square$

Lemma 20.3. *Let $\mathcal{C}$ be a category with fibred products. Let $f : Y \to X$ be a morphism of $\mathcal{C}$. Let U be the simplicial object of $\mathcal{C}$ whose nth term is the $(n+1)$fold fibred product $Y \times_X Y \times_X \ldots \times_X Y$. See Example 3.5. For any simplicial object V of $\mathcal{C}$ we have*

$$\mathrm{Mor}_{Simp(\mathcal{C})}(V, U) = \mathrm{Mor}_{Simp_1(\mathcal{C})}(sk_1 V, sk_1 U)$$
$$= \{g_0 : V_0 \to Y \mid f \circ g_0 \circ d_0^1 = f \circ g_0 \circ d_1^1\}$$

In particular we have $U = cosk_1 sk_1 U$.

Proof. Suppose that $g : sk_1 V \to sk_1 U$ is a morphism of 1-truncated simplicial objects. Then the diagram

is commutative, which proves that the relation shown in the lemma holds. We have to show that, conversely, given a morphism g_0 satisfying the relation $f \circ g_0 \circ d_0^1 = f \circ g_0 \circ d_1^1$ we get a unique morphism of simplicial objects $g : V \to U$. This is done as follows. For any $n \geq 1$ let $g_{n,i} = g_0 \circ V([0] \to [n], 0 \mapsto i) : V_n \to Y$. The equality above implies that $f \circ g_{n,i} = f \circ g_{n,i+1}$ because of the commutative diagram

Hence we get $(g_{n,0}, \ldots, g_{n,n}) : V_n \to Y \times_X \ldots \times_X Y = U_n$. We leave it to the reader to see that this is a morphism of simplicial objects. The last assertion of the lemma is equivalent to the first equality in the displayed formula of the lemma. $\square$

Remark 20.4. Let $\mathcal{C}$ be a category with fibre products. Let V be a simplicial object. Let $\epsilon : V \to X$ be an augmentation. Let U be the simplicial object whose nth term is the $(n+1)$st fibred product of V_0 over X. By a simple combination of Lemmas 20.2 and 20.3 we obtain a canonical morphism $V \to U$.

21. Left adjoints to the skeleton functors

In this section we construct a left adjoint $i_{m!}$ of the skeleton functor sk_m in certain cases. The adjointness formula is

$$\mathrm{Mor}_{\mathrm{Simp}_m(\mathcal{C})}(U, \mathrm{sk}_m V) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(i_{m!} U, V).$$

It turns out that this left adjoint exists when the category $\mathcal{C}$ has finite colimits.

We use a similar construction as in Section 12. Recall the category $[n]/\Delta$ of objects under $[n]$, see Categories, Example 2.14. Its objects are morphisms $\alpha : [n] \to [k]$ and its morphisms are commutative triangles. We let $([n]/\Delta)_{\leq m}$ denote the full subcategory of $[n]/\Delta$ consisting of objects $[n] \to [k]$ with $k \leq m$. Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : ([n]/\Delta)_{\leq m}^{opp} \longrightarrow \mathcal{C}$$

by the rules

$$
\begin{aligned}
([n] \to [k]) &\longmapsto U_k \\
\psi : ([n] \to [k']) \to ([n] \to [k]) &\longmapsto U(\psi) : U_k \to U_{k'}
\end{aligned}
$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$\underline{\varphi} : ([n']/\Delta)_{\leq m} \longrightarrow ([n]/\Delta)_{\leq m}$$

which maps $\alpha : [n'] \to [k]$ to $\varphi \circ \alpha : [n] \to [k]$. The composition $U(n) \circ \underline{\varphi}$ is equal to the functor $U(n')$.

Lemma 21.1. *Let $\mathcal{C}$ be a category which has finite colimits. The functors $i_{m!}$ exist for all m. Let U be an m-truncated simplicial object of $\mathcal{C}$. The simplicial object $i_{m!}U$ is described by the formula*

$$(i_{m!}U)_n = \mathrm{colim}_{([n]/\Delta)_{\leq m}^{opp}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $i_{m!}U(\varphi)$ comes from the identification $U(n) \circ \underline{\varphi} = U(n')$ above via Categories, Lemma 14.8.

Proof. In this proof we denote $i_{m!}U$ the simplicial object whose nth term is given by the displayed formula of the lemma. We will show it satisfies the adjointness property.

Let V be a simplicial object of $\mathcal{C}$. Let $\gamma : U \to \mathrm{sk}_m V$ be given. A morphism

$$\mathrm{colim}_{([n]/\Delta)_{\leq m}^{opp}} U(n) \to T$$

is given by a compatible system of morphisms $f_\alpha : U_k \to T$ where $\alpha : [n] \to [k]$ with $k \leq m$. Certainly, we have such a system of morphisms by taking the compositions

$$U_k \xrightarrow{\gamma_k} V_k \xrightarrow{V(\alpha)} V_n.$$

Hence we get an induced morphism $(i_{m!}U)_n \to V_n$. We leave it to the reader to see that these form a morphism of simplicial objects $\gamma' : i_{m!}U \to V$.

Conversely, given a morphism $\gamma' : i_{m!}U \to V$ we obtain a morphism $\gamma : U \to \mathrm{sk}_m V$ by setting $\gamma_i : U_i \to V_i$ equal to the composition

$$U_i \xrightarrow{\mathrm{id}_{[i]}} \mathrm{colim}_{([i]/\Delta)_{\leq m}^{opp}} U(i) \xrightarrow{\gamma'_i} V_i$$

for $0 \leq i \leq n$. We leave it to the reader to see that this is the inverse of the construction above. $\square$

Lemma 21.2. *Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For any $n \leq m$ the colimit*

$$\mathrm{colim}_{([n]/\Delta)_{\leq m}^{opp}} U(n)$$

exists and is equal to U_n.

Proof. This is so because the category $([n]/\Delta)_{\leq m}$ has an initial object, namely $\mathrm{id} : [n] \to [n]$. $\square$

Lemma 21.3. *Let $\mathcal{C}$ be a category which has finite colimits. Let U be an m-truncated simplicial object of $\mathcal{C}$. The map $U \to sk_m i_{m!}U$ is an isomorphism.*

Proof. Combine Lemmas 21.1 and 21.2. $\square$

Lemma 21.4. *If U is an m-truncated simplicial set and $n > m$ then all n-simplices of $i_{m!}U$ are degenerate.*

Proof. This can be seen from the construction of $i_{m!}U$ in Lemma 21.1, but we can also argue directly as follows. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subset with $V_i' = V_i$ for $i \leq m$ and all i simplices degenerate for $i > m$, see Lemma 18.4. By the adjunction formula, since $\mathrm{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. $\square$

Lemma 21.5. *Let U be a simplicial set. Let $n \geq 0$ be an integer. The morphism $i_{n!}sk_n U \to U$ identifies $i_{n!}sk_n U$ with the simplicial set $U' \subset U$ defined in Lemma 18.4.*

Proof. By Lemma 21.4 the only nondegenerate simplices of $i_{n!}\mathrm{sk}_n U$ are in degrees $\leq n$. The map $i_{n!}\mathrm{sk}_n U \to U$ is an isomorphism in degrees $\leq n$. Combined we conclude that the map $i_{n!}\mathrm{sk}_n U \to U$ maps nondegenerate simplices to nondegenerate simplices and no two nondegenerate simplices have the same image. Hence Lemma 18.3 applies. Thus $i_{n!}\mathrm{sk}_n U \to U$ is injective. The result follows easily from this. $\square$

Remark 21.6. In some texts the composite functor

$$\mathrm{Simp}(\mathcal{C}) \xrightarrow{\mathrm{sk}_m} \mathrm{Simp}_m(\mathcal{C}) \xrightarrow{i_{m!}} \mathrm{Simp}(\mathcal{C})$$

is denoted sk_m. This makes sense for simplicial sets, because then Lemma 21.5 says that $i_{m!}\mathrm{sk}_m V$ is just the sub simplicial set of V consisting of all i-simplices of V, $i \leq m$ and their degeneracies. In those texts it is also customary to denote the composition

$$\mathrm{Simp}(\mathcal{C}) \xrightarrow{\mathrm{sk}_m} \mathrm{Simp}_m(\mathcal{C}) \xrightarrow{\mathrm{cosk}_m} \mathrm{Simp}(\mathcal{C})$$

by cosk_m.

Lemma 21.7. *Let $U \subset V$ be simplicial sets. Suppose $n \geq 0$ and $x \in V_n$, $x \notin U_n$ are such that*

(1) *$V_i = U_i$ for $i < n$,*
(2) *$V_n = U_n \cup \{x\}$,*
(3) *any $z \in V_j$, $z \notin U_j$ for $j > n$ is degenerate.*

Let $\Delta[n] \to V$ be the unique morphism mapping the nondegenerate n-simplex of $\Delta[n]$ to x. In this case the diagram

$$
\begin{array}{ccc}
\Delta[n] & \longrightarrow & V \\
\uparrow & & \uparrow \\
i_{(n-1)!}sk_{n-1}\Delta[n] & \longrightarrow & U
\end{array}
$$

is a pushout diagram.

Proof. Let us denote $\partial\Delta[n] = i_{(n-1)!}\mathrm{sk}_{n-1}\Delta[n]$ for convenience. There is a natural map $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$. We have to show that it is bijective in degree j for all j. This is clear for $j \leq n$. Let $j > n$. The third condition means that any $z \in V_j$, $z \notin U_j$ is a degenerate simplex, say $z = s_i^{j-1}(z')$. Of course $z' \notin U_{j-1}$. By induction it follows that z' is a degeneracy of x. Thus we conclude that all j-simplices of V are either in U or degeneracies of x. This implies that the map $U\amalg_{\partial\Delta[n]}\Delta[n] \to V$ is surjective. Note that a nondegenerate simplex of $U\amalg_{\partial\Delta[n]}\Delta[n]$ is either the image of a nondegenerate simplex of U, or the image of the (unique) nondegenerate n-simplex of $\Delta[n]$. Since clearly x is nondegenerate we deduce that $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$ maps nondegenerate simplices to nondegenerate simplices and is injective on nondegenerate simplices. Hence it is injective, by Lemma 18.3. $\square$

Lemma 21.8. *Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U and V have finitely many nondegenerate simplices. Then there exists a sequence of sub simplicial sets*

$$U = W^0 \subset W^1 \subset W^2 \subset \dots W^r = V$$

such that Lemma 21.7 applies to each of the inclusions $W^i \subset W^{i+1}$.

Proof. Let n be the smallest integer such that V has a nondegenerate simplex that does not belong to U. Let $x \in V_n$, $x \notin U_n$ be such a nondegenerate simplex. Let $W \subset V$ be the set of elements which are either in U, or are a (repeated) degeneracy of x (in other words, are of the form $V(\varphi)(x)$ with $\varphi : [m] \to [n]$ surjective). It is easy to see that W is a simplicial set. The inclusion $U \subset W$ satisfies the conditions of Lemma 21.7. Moreover the number of nondegenerate simplices of V which are not contained in W is exactly one less than the number of nondegenerate simplices of V which are not contained in U. Hence we win by induction on this number. $\square$

Lemma 21.9. *Let $\mathcal{A}$ be an abelian category Let U be an m-truncated simplicial object of $\mathcal{A}$. For $n > m$ we have $N(i_{m!}U)_n = 0$.*

Proof. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subobject of V with $V_i' = V_i$ for $i \leq m$ and $N(V_i') = 0$ for $i > m$, see Lemma 18.9. By the adjunction formula, since $\mathrm{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. $\square$

Lemma 21.10. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \geq 0$ be an integer. The morphism $i_{n!}sk_nU \to U$ identifies $i_{n!}sk_nU$ with the simplicial subobject $U' \subset U$ defined in Lemma 18.9.*

Proof. By Lemma 21.9 we have $N(i_{n!}\mathrm{sk}_nU)_i = 0$ for $i > n$. The map $i_{n!}\mathrm{sk}_nU \to U$ is an isomorphism in degrees $\leq n$, see Lemma 21.3. Combined we conclude that the map $i_{n!}\mathrm{sk}_nU \to U$ induces injective maps $N(i_{n!}\mathrm{sk}_nU)_i \to N(U)_i$ for all i. Hence

Lemma 18.7 applies. Thus $i_{n!}\mathrm{sk}_n U \to U$ is injective. The result follows easily from this. $\square$

Here is another way to think about the coskeleton functor using the material above.

Lemma 21.11. *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let V be a simplicial object of $\mathcal{C}$. In this case*

$$(cosk_n\, sk_n V)_{n+1} = \mathrm{Hom}(i_{n!}sk_n\Delta[n+1], V)_0.$$

Proof. By Lemma 13.4 the object on the left represents the functor which assigns to X the first set of the following equalities

$$
\begin{aligned}
\mathrm{Mor}(X \times \Delta[n+1], \mathrm{cosk}_n\mathrm{sk}_n V) &= \mathrm{Mor}(X \times \mathrm{sk}_n\Delta[n+1], \mathrm{sk}_n V) \\
&= \mathrm{Mor}(X \times i_{n!}\mathrm{sk}_n\Delta[n+1], V).
\end{aligned}
$$

The object on the right in the formula of the lemma is represented by the functor which assigns to X the last set in the sequence of equalities. This proves the result.

In the sequence of equalities we have used that $\mathrm{sk}_n(X \times \Delta[n+1]) = X \times \mathrm{sk}_n\Delta[n+1]$ and that $i_{n!}(X \times \mathrm{sk}_n\Delta[n+1]) = X \times i_{n!}\mathrm{sk}_n\Delta[n+1]$. The first equality is obvious. For any (possibly truncated) simplicial object W of $\mathcal{C}$ and any object X of $\mathcal{C}$ denote temporarily $\mathrm{Mor}_{\mathcal{C}}(X, W)$ the (possibly truncated) simplicial set $[n] \mapsto \mathrm{Mor}_{\mathcal{C}}(X, W_n)$. From the definitions it follows that $\mathrm{Mor}(U \times X, W) = \mathrm{Mor}(U, \mathrm{Mor}_{\mathcal{C}}(X, W))$ for any (possibly truncated) simplicial set U. Hence

$$
\begin{aligned}
\mathrm{Mor}(X \times i_{n!}\mathrm{sk}_n\Delta[n+1], W) &= \mathrm{Mor}(i_{n!}\mathrm{sk}_n\Delta[n+1], \mathrm{Mor}_{\mathcal{C}}(X, W)) \\
&= \mathrm{Mor}(\mathrm{sk}_n\Delta[n+1], \mathrm{sk}_n\,\mathrm{Mor}_{\mathcal{C}}(X, W)) \\
&= \mathrm{Mor}(X \times \mathrm{sk}_n\Delta[n+1], \mathrm{sk}_n W) \\
&= \mathrm{Mor}(i_{n!}(X \times \mathrm{sk}_n\Delta[n+1]), W).
\end{aligned}
$$

This proves the second equality used, and ends the proof of the lemma. $\square$

22. Simplicial objects in abelian categories

Recall that an abelian category is defined in Homology, Section 5.

Lemma 22.1. *Let $\mathcal{A}$ be an abelian category.*

(1) *The categories $Simp(\mathcal{A})$ and $CoSimp(\mathcal{A})$ are abelian.*
(2) *A morphism of (co)simplicial objects $f : A \to B$ is injective if and only if each $f_n : A_n \to B_n$ is injective.*
(3) *A morphism of (co)simplicial objects $f : A \to B$ is surjective if and only if each $f_n : A_n \to B_n$ is surjective.*
(4) *A sequence of (co)simplicial objects*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at B if and only if each sequence

$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$

is exact at B_i.

Proof. Pre-additivity is easy. A final object is given by $U_n = 0$ in all degrees. Existence of direct products we saw in Lemmas 6.2 and 9.2. Kernels and cokernels are obtained by taking termwise kernels and cokernels. $\square$

For an object A of $\mathcal{A}$ and an integer k consider the k-truncated simplicial object U with

(1) $U_i = 0$ for $i < k$,
(2) $U_k = A$,
(3) all morphisms $U(\varphi)$ equal to zero, except $U(\mathrm{id}_{[k]}) = \mathrm{id}_A$.

Since $\mathcal{A}$ has both finite limits and finite colimits we see that both $\mathrm{cosk}_k U$ and $i_{k!} U$ exist. We will describe both of these and the canonical map $i_{k!} U \to \mathrm{cosk}_k U$.

Lemma 22.2. *With A, k and U as above, so $U_i = 0$, $i < k$ and $U_k = A$.*

(1) *Given a k-truncated simplicial object V we have*
$$\mathrm{Mor}(U, V) = \{f : A \to V_k \mid d_i^k \circ f = 0, \ i = 0, \ldots, k\}$$
and
$$\mathrm{Mor}(V, U) = \{f : V_k \to A \mid f \circ s_i^{k-1} = 0, \ i = 0, \ldots, k - 1\}.$$

(2) *The object $i_{k!} U$ has nth term equal to $\bigoplus_\alpha A$ where α runs over all surjective morphisms $\alpha : [n] \to [k]$.*

(3) *For any $\varphi : [m] \to [n]$ the map $i_{k!} U(\varphi)$ is described as the mapping $\bigoplus_\alpha A \to \bigoplus_{\alpha'} A$ which maps to component corresponding to $\alpha : [n] \to [k]$ to zero if $\alpha \circ \varphi$ is not surjective and by the identity to the component corresponding to $\alpha \circ \varphi$ if it is surjective.*

(4) *The object $\mathrm{cosk}_k U$ has nth term equal to $\bigoplus_\beta A$, where β runs over all injective morphisms $\beta : [k] \to [n]$.*

(5) *For any $\varphi : [m] \to [n]$ the map $\mathrm{cosk}_k U(\varphi)$ is described as the mapping $\bigoplus_\beta A \to \bigoplus_{\beta'} A$ which maps to component corresponding to $\beta : [k] \to [n]$ to zero if β does not factor through φ and by the identity to each of the components corresponding to β' such that $\beta = \varphi \circ \beta'$ if it does.*

(6) *The canonical map $c : i_{k!} U \to \mathrm{cosk}_k U$ in degree n has (α, β) coefficient $A \to A$ equal to zero if $\alpha \circ \beta$ is not the identity and equal to id_A if it is.*

(7) *The canonical map $c : i_{k!} U \to \mathrm{cosk}_k U$ is injective.*

Proof. The proof of (1) is left to the reader.

Let us take the rules of (2) and (3) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $i_{k!} U$. This will prove (2), (3) at the same time. We have to show that given a morphism $f : U \to \mathrm{sk}_k V$ there exists a unique morphism $\tilde{f} : \tilde{U} \to V$ which recovers f upon taking the k-skeleton. From (1) we see that f corresponds with a morphism $f_k : A \to V_k$ which maps into the kernel of d_i^k for all i. For any surjective $\alpha : [n] \to [k]$ we set $\tilde{f}_\alpha : A \to V_n$ equal to the composition $\tilde{f}_\alpha = V(\alpha) \circ f_k : A \to V_n$. We define $\tilde{f}_n : \tilde{U}_n \to V_n$ as the sum of the $\tilde{f}_\alpha$ over $\alpha : [n] \to [k]$ surjective. Such a collection of $\tilde{f}_\alpha$ defines a morphism of simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$
\begin{array}{ccc}
\bigoplus_{\alpha : [n] \to [k] \text{ surjective}} A & \xrightarrow{\ \tilde{f}_n\ } & V_n \\
{\scriptstyle (3)} \Big\downarrow & & \Big\downarrow {\scriptstyle V(\varphi)} \\
\bigoplus_{\alpha' : [m] \to [k] \text{ surjective}} A & \xrightarrow{\ \tilde{f}_m\ } & V_m
\end{array}
$$

is commutative. Choosing $\varphi = \alpha$ shows our choice of $\tilde{f}_\alpha$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the left

upper corner separately. It is clear for the summands corresponding to α where $\alpha \circ \varphi$ is surjective, because those get mapped by id_A to the summand with $\alpha' = \alpha \circ \varphi$, and we have $\tilde{f}_{\alpha'} = V(\alpha') \circ f_k = V(\alpha \circ \varphi) \circ f_k = V(\varphi) \circ \tilde{f}_\alpha$. For those where $\alpha \circ \varphi$ is not surjective, we have to show that $V(\varphi) \circ \tilde{f}_\alpha = 0$. By definition this is equal to $V(\varphi) \circ V(\alpha) \circ f_k = V(\alpha \circ \varphi) \circ f_k$. Since $\alpha \circ \varphi$ is not surjective we can write it as $\delta_i^k \circ \psi$, and we deduce that $V(\varphi) \circ V(\alpha) \circ f_k = V(\psi) \circ d_i^k \circ f_k = 0$ see above.

Let us take the rules of (4) and (5) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $\mathrm{cosk}_k U$. This will prove (4), (5) at the same time. The argument is completely dual to the proof of (2), (3) above, but we give it anyway. We have to show that given a morphism $f : \mathrm{sk}_k V \to U$ there exists a unique morphism $\tilde{f} : V \to \tilde{U}$ which recovers f upon taking the k-skeleton. From (1) we see that f corresponds with a morphism $f_k : V_k \to A$ which is zero on the image of s_i^{k-1} for all i. For any injective $\beta : [k] \to [n]$ we set $\tilde{f}_\beta : V_n \to A$ equal to the composition $\tilde{f}_\beta = f_k \circ V(\beta) : V_n \to A$. We define $\tilde{f}_n : V_n \to \tilde{U}_n$ as the sum of the $\tilde{f}_\beta$ over $\beta : [k] \to [n]$ injective. Such a collection of $\tilde{f}_\beta$ defines a morphism of simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$
\begin{array}{ccc}
V_n & \xrightarrow{\ \tilde{f}_n\ } & \bigoplus_{\beta:[k]\to[n]\ \text{injective}} A \\
{\scriptstyle V(\varphi)}\downarrow & & \downarrow{\scriptstyle (5)} \\
V_m & \xrightarrow{\ \tilde{f}_m\ } & \bigoplus_{\beta':[k]\to[m]\ \text{injective}} A
\end{array}
$$

is commutative. Choosing $\varphi = \beta$ shows our choice of $\tilde{f}_\beta$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the right lower corner separately. It is clear for the summands corresponding to β' where $\varphi \circ \beta'$ is injective, because these summands get mapped into by exactly the summand with $\beta = \varphi \circ \beta'$ and we have in that case $\tilde{f}_{\beta'} \circ V(\varphi) = f_k \circ V(\beta') \circ V(\varphi) = f_k \circ V(\beta) = \tilde{f}_\beta$. For those where $\varphi \circ \beta'$ is not injective, we have to show that $\tilde{f}_{\beta'} \circ V(\varphi) = 0$. By definition this is equal to $f_k \circ V(\beta') \circ V(\varphi) = f_k \circ V(\varphi \circ \beta')$. Since $\varphi \circ \beta'$ is not injective we can write it as $\psi \circ \sigma_i^{k-1}$, and we deduce that $f_k \circ V(\beta') \circ V(\varphi) = f_k \circ s_i^{k-1} \circ V(\psi) = 0$ see above.

The composition $i_{k!} U \to \mathrm{cosk}_k U$ is the unique map of simplicial objects which is the identity on $A = U_k = (i_{k!}U)_k = (\mathrm{cosk}_k U)_k$. Hence it suffices to check that the proposed rule defines a morphism of simplicial objects. To see this we have to show that for any $\varphi : [m] \to [n]$ the diagram

$$
\begin{array}{ccc}
\bigoplus_{\alpha:[n]\to[k]\ \text{surjective}} A & \xrightarrow[\ (6)\ ]{} & \bigoplus_{\beta:[k]\to[n]\ \text{injective}} A \\
{\scriptstyle (3)}\downarrow & & \downarrow{\scriptstyle (5)} \\
\bigoplus_{\alpha':[m]\to[k]\ \text{surjective}} A & \xrightarrow{\ (6)\ } & \bigoplus_{\beta':[k]\to[m]\ \text{injective}} A
\end{array}
$$

is commutative. Now we can think of this in terms of matrices filled with only 0's and 1's as follows: The matrix of (3) has a nonzero (α', α) entry if and only if $\alpha' = \alpha \circ \varphi$. Likewise the matrix of (5) has a nonzero (β', β) entry if and only if $\beta = \varphi \circ \beta'$. The upper matrix of (6) has a nonzero (α, β) entry if and only if $\alpha \circ \beta = \mathrm{id}_{[k]}$. Similarly for the lower matrix of (6). The commutativity of the diagram then comes down to computing the (α, β') entry for both compositions

and seeing they are equal. This comes down to the following equality

$$\#\left\{\beta \mid \beta = \varphi \circ \beta' \text{ and } \alpha \circ \beta = \mathrm{id}_{[k]}\right\} = \#\left\{\alpha' \mid \alpha' = \alpha \circ \varphi \text{ and } \alpha' \circ \beta' = \mathrm{id}_{[k]}\right\}$$

whose proof may safely be left to the reader.

Finally, we prove (7). This follows directly from Lemmas 18.7, 19.4, 21.3 and 21.9. $\qquad\square$

Definition 22.3. Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. The *Eilenberg-Maclane object* $K(A, k)$ is given by the object $K(A, k) = i_{k!}U$ which is described in Lemma 22.2 above.

Lemma 22.4. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. Consider the simplicial object E defined by the following rules*

   (1) *$E_n = \bigoplus_\alpha A$, where the sum is over $\alpha : [n] \to [k+1]$ whose image is either $[k]$ or $[k+1]$.*
   (2) *Given $\varphi : [m] \to [n]$ the map $E_n \to E_m$ maps the summand corresponding to α via id_A to the summand corresponding to $\alpha \circ \varphi$, provided $\mathrm{Im}(\alpha \circ \varphi)$ is equal to $[k]$ or $[k+1]$.*

Then there exists a short exact sequence

$$0 \to K(A, k) \to E \to K(A, k+1) \to 0$$

which is term by term split exact.

Proof. The maps $K(A, k)_n \to E_n$ resp. $E_n \to K(A, k+1)_n$ are given by the inclusion of direct sums, resp. projection of direct sums which is obvious from the inclusions of index sets. It is clear that these are maps of simplicial objects. $\qquad\square$

Lemma 22.5. *Let $\mathcal{A}$ be an abelian category. For any simplicial object V of $\mathcal{A}$ we have*

$$V = \mathrm{colim}_n\, i_{n!} sk_n V$$

where all the transition maps are injections.

Proof. This is true simply because each V_m is equal to $(i_{n!}\mathrm{sk}_n V)_m$ as soon as $n \geq m$. See also Lemma 21.10 for the transition maps. $\qquad\square$

23. Simplicial objects and chain complexes

Let $\mathcal{A}$ be an abelian category. See Homology, Section 13 for conventions and notation regarding chain complexes. Let U be a simplicial object of $\mathcal{A}$. The *associated chain complex $s(U)$ of U*, sometimes called the *Moore complex*, is the chain complex

$$\ldots \to U_2 \to U_1 \to U_0 \to 0 \to 0 \to \ldots$$

with boundary maps $d_n : U_n \to U_{n-1}$ given by the formula

$$d_n = \sum_{i=0}^n (-1)^i d_i^n.$$

This is a complex because, by the relations listed in Remark 3.3, we have

$$
\begin{aligned}
d_n \circ d_{n+1} &= \left(\sum_{i=0}^n (-1)^i d_i^n\right) \circ \left(\sum_{j=0}^{n+1} (-1)^j d_j^{n+1}\right) \\
&= \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} d_{j-1}^n \circ d_i^{n+1} + \sum_{n \geq i \geq j \geq 0} (-1)^{i+j} d_i^n \circ d_j^{n+1} \\
&= 0.
\end{aligned}
$$

The signs cancel! We denote the associated chain complex $s(U)$. Clearly, the construction is functorial and hence defines a functor

$$s : \mathrm{Simp}(\mathcal{A}) \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A}).$$

Thus we have the confusing but correct formula $s(U)_n = U_n$.

Lemma 23.1. *The functor s is exact.*

Proof. Clear from Lemma 22.1. $\square$

Lemma 23.2. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. Let E be the object described in Lemma 22.4. Then the complex $s(E)$ is acyclic.*

Proof. For a morphism $\alpha : [n] \to [k+1]$ we define $\alpha' : [n+1] \to [k+1]$ to be the map such that $\alpha'|_{[n]} = \alpha$ and $\alpha'(n+1) = k+1$. Note that if the image of α is $[k]$ or $[k+1]$, then the image of α' is $[k+1]$. Consider the family of maps $h_n : E_n \to E_{n+1}$ which maps the summand corresponding to α to the summand corresponding to α' via the identity on A. Let us compute $d_{n+1} \circ h_n - h_{n-1} \circ d_n$. We will first do this in case the category $\mathcal{A}$ is the category of abelian groups. Let us use the notation x_α to indicate the element $x \in A$ in the summand of E_n corresponding to the map α occurring in the index set. Let us also adopt the convention that x_α designates the zero element of E_n whenever $\mathrm{Im}(\alpha)$ is not $[k]$ or $[k+1]$. With these conventions we see that

$$d_{n+1}(h_n(x_\alpha)) = \sum_{i=0}^{n+1} (-1)^i x_{\alpha' \circ \delta_i^{n+1}}$$

and

$$h_{n-1}(d_n(x_\alpha)) = \sum_{i=0}^{n} (-1)^i x_{(\alpha \circ \delta_i^n)'}$$

It is easy to see that $\alpha' \circ \delta_i^{n+1} = (\alpha \circ \delta_i^n)'$ for $i = 0, \dots, n$. It is also easy to see that $\alpha' \circ \delta_{n+1}^{n+1} = \alpha$. Thus we see that

$$(d_{n+1} \circ h_n - h_{n-1} \circ d_n)(x_\alpha) = (-1)^{n+1} x_\alpha$$

These identities continue to hold if $\mathcal{A}$ is any abelian category because they hold in the simplicial abelian group $[n] \mapsto \mathrm{Hom}(A, E_n)$; details left to the reader. We conclude that the identity map on E is homotopic to zero, with homotopy given by the system of maps $h'_n = (-1)^{n+1} h_n : E_n \to E_{n+1}$. Hence we see that E is acyclic, for example by Homology, Lemma 13.5. $\square$

Lemma 23.3. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $H_i(s(K(A, k))) = A$ if $i = k$ and 0 else.*

Proof. First, let us prove this if $k = 0$. In this case we have $K(A, 0)_n = A$ for all n. Furthermore, all the maps in this simplicial abelian group are id_A, in other words $K(A, 0)$ is the constant simplicial object with value A. The boundary maps $d_n = \sum_{i=0}^n (-1)^i \mathrm{id}_A = 0$ if n odd and $= \mathrm{id}_A$ if n is even. Thus $s(K(A, 0))$ looks like this

$$\dots \to A \xrightarrow{0} A \xrightarrow{1} A \xrightarrow{0} A \to 0$$

and the result is clear.

Next, we prove the result for all k by induction. Given the result for k consider the short exact sequence

$$0 \to K(A, k) \to E \to K(A, k+1) \to 0$$

from Lemma 22.4. By Lemma 22.1 the associated sequence of chain complexes is exact. By Lemma 23.2 we see that $s(E)$ is acyclic. Hence the result for $k+1$ follows from the long exact sequence of homology, see Homology, Lemma 13.6. $\qquad\square$

There is a second chain complex we can associate to a simplicial object of $\mathcal{A}$. Recall that by Lemma 18.6 any simplicial object U of $\mathcal{A}$ is canonically split with $N(U_m) = \bigcap_{i=0}^{m-1} \operatorname{Ker}(d_i^m)$. We define the *normalized chain complex $N(U)$* to be the chain complex

$$\ldots \to N(U_2) \to N(U_1) \to N(U_0) \to 0 \to 0 \to \ldots$$

with boundary map $d_n : N(U_n) \to N(U_{n-1})$ given by the restriction of $(-1)^n d_n^n$ to the direct summand $N(U_n)$ of U_n. Note that Lemma 18.8 implies that $d_n^n(N(U_n)) \subset N(U_{n-1})$. It is a complex because $d_n^n \circ d_{n+1}^{n+1} = d_n^n \circ d_n^{n+1}$ and d_n^{n+1} is zero on $N(U_{n+1})$ by definition. Thus we obtain a second functor

$$N : \operatorname{Simp}(\mathcal{A}) \longrightarrow \operatorname{Ch}_{\geq 0}(\mathcal{A}).$$

Here is the reason for the sign in the differential.

Lemma 23.4. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical map $N(U_n) \to U_n$ gives rise to a morphism of complexes $N(U) \to s(U)$.*

Proof. This is clear because the differential on $s(U)_n = U_n$ is $\sum (-1)^i d_i^n$ and the maps d_i^n, $i < n$ are zero on $N(U_n)$, whereas the restriction of $(-1)^n d_n^n$ is the boundary map of $N(U)$ by definition. $\qquad\square$

Lemma 23.5. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $N(K(A,k))_i = A$ if $i = k$ and 0 else.*

Proof. It is clear that $N(K(A,k))_i = 0$ when $i < k$ because $K(A,k)_i = 0$ in that case. It is clear that $N(K(A,k))_k = A$ since $K(A,k)_{k-1} = 0$ and $K(A,k)_k = A$. For $i > k$ we have $N(K(A,k))_i = 0$ by Lemma 21.9 and the definition of $K(A,k)$, see Definition 22.3. $\qquad\square$

Lemma 23.6. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical morphism of chain complexes $N(U) \to s(U)$ is split. In fact,*

$$s(U) = N(U) \oplus D(U)$$

for some complex $D(U)$. The construction $U \mapsto D(U)$ is functorial.

Proof. Define $D(U)_n$ to be the image of

$$\bigoplus_{\varphi:[n]\to[m] \text{ surjective, } m<n} N(U_m) \xrightarrow{\bigoplus U(\varphi)} U_n$$

which is a subobject of U_n complementary to $N(U_n)$ according to Lemma 18.6 and Definition 18.1. We show that $D(U)$ is a subcomplex. Pick a surjective map $\varphi : [n] \to [m]$ with $m < n$ and consider the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

This composition is the sum of the maps

$$N(U_m) \xrightarrow{U(\varphi \circ \delta_i^n)} U_{n-1}$$

with sign $(-1)^i$, $i = 0, \ldots, n$.

First we will prove by ascending induction on m, $0 \leq m < n-1$ that all the maps $U(\varphi \circ \delta_i^n)$ map $N(U_m)$ into $D(U)_{n-1}$. (The case $m = n-1$ is treated below.) Whenever the map $\varphi \circ \delta_i^n : [n-1] \to [m]$ is surjective then the image of $N(U_m)$ under $U(\varphi \circ \delta_i^n)$ is contained in $D(U)_{n-1}$ by definition. If $\varphi \circ \delta_i^n : [n-1] \to [m]$ is not surjective, set $j = \varphi(i)$ and observe that i is the unique index whose image under φ is j. We may write $\varphi \circ \delta_i^n = \delta_j^m \circ \psi \circ \delta_i^n$ for some $\psi : [n-1] \to [m-1]$. Hence $U(\varphi \circ \delta_i^n) = U(\psi \circ \delta_i^n) \circ d_j^m$ which is zero on $N(U_m)$ unless $j = m$. If $j = m$, then $d_m^m(N(U_m)) \subset N(U_{m-1})$ and hence $U(\varphi \circ \delta_i^n)(N(U_m)) \subset U(\psi \circ \delta_i^n)(N(U_{m-1}))$ and we win by induction hypothesis.

To finish proving that $D(U)$ is a subcomplex we still have to deal with the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

in case $m = n-1$. In this case $\varphi = \sigma_j^{n-1}$ for some $0 \leq j \leq n-1$ and $U(\varphi) = s_j^{n-1}$. Thus the composition is given by the sum

$$\sum (-1)^i d_i^n \circ s_j^{n-1}$$

Recall from Remark 3.3 that $d_j^n \circ s_j^{n-1} = d_{j+1}^n \circ s_j^{n-1} = \mathrm{id}$ and these drop out because the corresponding terms have opposite signs. The map $d_n^n \circ s_j^{n-1}$, if $j < n-1$, is equal to $s_j^{n-2} \circ d_{n-1}^{n-1}$. Since d_{n-1}^{n-1} maps $N(U_{n-1})$ into $N(U_{n-2})$, we see that the image $d_n^n(s_j^{n-1}(N(U_{n-1})))$ is contained in $s_j^{n-2}(N(U_{n-2}))$ which is contained in $D(U_{n-1})$ by definition. For all other combinations of (i, j) we have either $d_i^n \circ s_j^{n-1} = s_{j-1}^{n-2} \circ d_i^{n-1}$ (if $i < j$), or $d_i^n \circ s_j^{n-1} = s_j^{n-2} \circ d_{i-1}^{n-1}$ (if $n > i > j+1$) and in these cases the map is zero because of the definition of $N(U_{n-1})$. $\square$

Remark 23.7. In the situation of Lemma 23.6 the subcomplex $D(U) \subset s(U)$ can also be defined as the subcomplex with terms

$$D(U)_n = \mathrm{Im}\left(\bigoplus_{\varphi:[n]\to[m] \text{ surjective, } m<n} U_m \xrightarrow{\bigoplus U(\varphi)} U_n\right)$$

Namely, since U_m is the direct sum of the subobject $N(U_m)$ and the images of $N(U_k)$ for surjections $[m] \to [k]$ with $k < m$ this is clearly the same as the definition of $D(U)_n$ given in the proof of Lemma 23.6. Thus we see that if U is a simplicial abelian group, then elements of $D(U)_n$ are exactly the sums of degenerate n-simplices.

Lemma 23.8. *The functor N is exact.*

Proof. By Lemma 23.1 and the functorial decomposition of Lemma 23.6. $\square$

Lemma 23.9. *Let $\mathcal{A}$ be an abelian category. Let V be a simplicial object of $\mathcal{A}$. The canonical morphism of chain complexes $N(V) \to s(V)$ is a quasi-isomorphism. In other words, the complex $D(V)$ of Lemma 23.6 is acyclic.*

Proof. Note that the result holds for $K(A, k)$ for any object A and any $k \geq 0$, by Lemmas 23.3 and 23.5. Consider the hypothesis $IH_{n,m}$: for all V such that $V_j = 0$ for $j \leq m$ and all $i \leq n$ the map $N(V) \to s(V)$ induces an isomorphism $H_i(N(V)) \to H_i(s(V))$.

To start of the induction, note that $IH_{n,n}$ is trivially true, because in that case $N(V)_n = 0$ and $s(V)_n = 0$.

Assume $IH_{n,m}$, with $m \leq n$. Pick a simplicial object V such that $V_j = 0$ for $j < m$. By Lemma 22.2 and Definition 22.3 we have $K(V_m, m) = i_{m!}\mathrm{sk}_m V$. By Lemma 21.10 the natural morphism

$$K(V_m, m) = i_{m!}\mathrm{sk}_m V \to V$$

is injective. Thus we get a short exact sequence

$$0 \to K(V_m, m) \to V \to W \to 0$$

for some W with $W_i = 0$ for $i = 0, \ldots, m$. This short exact sequence induces a morphism of short exact sequence of associated complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N(K(V_m, m)) & \longrightarrow & N(V) & \longrightarrow & N(W) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & s(K(V_m, m)) & \longrightarrow & s(V) & \longrightarrow & s(W) & \longrightarrow & 0
\end{array}
$$

see Lemmas 23.1 and 23.8. Hence we deduce the result for V from the result on the ends. $\square$

24. Dold-Kan

In this section we prove the Dold-Kan theorem relating simplicial objects in an abelian category with chain complexes.

Lemma 24.1. *Let $\mathcal{A}$ be an abelian category. The functor N is faithful, and reflects isomorphisms, injections and surjections.*

Proof. The faithfulness is immediate from the canonical splitting of Lemma 18.6. The statement on reflecting injections, surjections, and isomorphisms follows from Lemma 18.7. $\square$

Lemma 24.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $N : \mathcal{A} \to \mathcal{B}$, and $S : \mathcal{B} \to \mathcal{A}$ be functors. Suppose that*

    (1) *the functors S and N are exact,*
    (2) *there is an isomorphism $g : N \circ S \to id_{\mathcal{B}}$ to the identity functor of $\mathcal{B}$,*
    (3) *N is faithful, and*
    (4) *S is essentially surjective.*

Then S and N are quasi-inverse equivalences of categories.

Proof. It suffices to construct a functorial isomorphism $S(N(A)) \cong A$. To do this choose B and an isomorphism $f : A \to S(B)$. Consider the map

$$f^{-1} \circ g_{S(B)} \circ S(N(f)) : S(N(A)) \to S(N(S(B))) \to S(B) \to A.$$

It is easy to show this does not depend on the choice of f, B and gives the desired isomorphism $S \circ N \to \mathrm{id}_{\mathcal{A}}$. $\square$

Theorem 24.3. *Let $\mathcal{A}$ be an abelian category. The functor N induces an equivalence of categories*

$$N : Simp(\mathcal{A}) \longrightarrow Ch_{\geq 0}(\mathcal{A})$$

Proof. We will describe a functor in the reverse direction inspired by the construction of Lemma 22.4 (except that we throw in a sign to get the boundaries right). Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For each $n \geq 0$ denote

$$I_n = \Big\{ \alpha : [n] \to \{0, 1, 2, \ldots\} \mid \mathrm{Im}(\alpha) = [k] \text{ for some } k \Big\}.$$

For $\alpha \in I_n$ we denote $k(\alpha)$ the unique integer such that $\mathrm{Im}(\alpha) = [k]$. We define a simplicial object $S(A_\bullet)$ as follows:

(1) $S(A_\bullet)_n = \bigoplus_{\alpha \in I_n} A_{k(\alpha)}$, which we will write as $\bigoplus_{\alpha \in I_n} A_{k(\alpha)} \cdot \alpha$ to suggest thinking of "α" as a basis vector for the summand corresponding to it,

(2) given $\varphi : [m] \to [n]$ we define $S(A_\bullet)(\varphi)$ by its restriction to the direct summand $A_{k(\alpha)} \cdot \alpha$ of $S(A_\bullet)_n$ as follows

   (a) $\alpha \circ \varphi \notin I_m$ then we set it equal to zero,

   (b) $\alpha \circ \varphi \in I_m$ but $k(\alpha \circ \varphi)$ not equal to either $k(\alpha)$ or $k(\alpha) - 1$ then we set it equal to zero as well,

   (c) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha)$ then we use the identity map to the summand $A_{k(\alpha \circ \varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$, and

   (d) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha) - 1$ then we use $(-1)^{k(\alpha)} d_{A,k(\alpha)}$ to the summand $A_{k(\alpha \circ \varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$.

Let us show that $S(A_\bullet)$ is a simplicial object of $\mathcal{A}$. To do this, assume we have maps $\varphi : [m] \to [n]$ and $\psi : [n] \to [p]$. We will show that $S(A_\bullet)(\varphi) \circ S(A_\bullet)(\psi) = S(A_\bullet)(\psi \circ \varphi)$. Choose $\beta \in I_p$ and set $\alpha = \beta \circ \psi$ and $\gamma = \alpha \circ \varphi$ viewed as maps $\alpha : [n] \to \{0, 1, 2, \ldots\}$ and $\gamma : [m] \to \{0, 1, 2, \ldots\}$. Picture

$$
\begin{array}{ccccc}
[m] & \xrightarrow{\ \varphi\ } & [n] & \xrightarrow{\ \psi\ } & [p] \\
{\scriptstyle \gamma}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\
\mathrm{Im}(\gamma) & \longrightarrow & \mathrm{Im}(\alpha) & \longrightarrow & [k(\beta)]
\end{array}
$$

We will show that the restriction of the maps $S(A_\bullet)(\varphi) \circ S(A_\bullet)(\psi)$ and $S(A_\bullet)(\psi \circ \varphi)$. to the summand $A_{k(\beta)} \cdot \beta$ agree. There are several cases to consider

(1) Say $\alpha \notin I_n$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is zero. Then either $\gamma \notin I_m$ or we have $[k(\gamma)] = \mathrm{Im}(\gamma) \subset \mathrm{Im}(\alpha) \subset [k(\beta)]$ and the subset $\mathrm{Im}(\alpha)$ of $[k(\beta)]$ has a gap so $k(\gamma) < k(\beta) - 1$. In both cases we see that the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to $A_{k(\beta)} \cdot \beta$ is zero as well.

(2) Say $\alpha \in I_n$ and $k(\alpha) < k(\beta) - 1$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is zero. Then either $\gamma \notin I_m$ or we have $[k(\gamma)] \subset [k(\alpha)] \subset [k(\beta)]$ and it follows that $k(\gamma) < k(\beta) - 1$. In both cases we see that the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to $A_{k(\beta)} \cdot \beta$ is zero as well.

(3) Say $\alpha \in I_n$ and $k(\alpha) = k(\beta)$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is the identity map from $A_{k(\beta)} \cdot \beta$ to $A_{k(\alpha)} \cdot \alpha$. In this case because $\mathrm{Im}(\alpha) = [k(\beta)]$ the rule describing the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ is exactly the same as the rule describing the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ and hence agreement holds.

(4) Say $\alpha \in I_n$ and $k(\alpha) = k(\beta) - 1$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is given by $(-1)^{k(\beta)} d_{A,k(\beta)}$ to $A_{k(\alpha)} \cdot \alpha$. Subcases

   (a) If $\gamma \notin I_m$, then both the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ and the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ are zero and we get agreement.

(b) If $\gamma \in I_m$ but $k(\gamma) < k(\alpha) - 1$, then again both restrictions are zero and we get agreement.

(c) If $\gamma \in I_m$ and $k(\gamma) = k(\alpha)$ then $\mathrm{Im}(\gamma) = \mathrm{Im}(\alpha)$. In this case the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ is given by $(-1)^{k(\beta)} d_{A,k(\beta)}$ to $A_{k(\gamma)} \cdot \gamma$ and the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ is the identity map $A_{k(\alpha)} \cdot \alpha \to A_{k(\gamma)} \cdot \gamma$. Hence agreement holds.

(d) Finally, if $\gamma \in I_m$ and $k(\gamma) = k(\alpha) - 1$ then the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ is given by $(-1)^{k(\alpha)} d_{A,k(\alpha)}$ as a map $A_{k(\alpha)} \cdot \alpha \to A_{k(\beta)} \cdot \beta$. Since $A_\bullet$ is a complex we see that the composition $A_{k(\beta)} \cdot \beta \to A_{k(\alpha)} \cdot \alpha \to A_{k(\gamma)} \cdot \gamma$ is zero which matches what we get for the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ because $k(\gamma) = k(\beta) - 2 < k(\beta) - 1$.

Thus $S(A_\bullet)$ is a simplicial object of $\mathcal{A}$.

Let us construct an isomorphism $A_\bullet \to N(S(A_\bullet))$ functorial in $A_\bullet$. Recall that

$$S(A_\bullet) = N(S(A_\bullet)) \oplus D(S(A_\bullet))$$

as chain complexes by Lemma 23.6. On the other hand it follows from Remark 23.7 and the construction of $S(A_\bullet)$ that

$$D(S(A_\bullet))_n = \bigoplus\nolimits_{\alpha \in I_n,\ k(\alpha) < n} A_{k(\alpha)} \cdot \alpha \subset \bigoplus\nolimits_{\alpha \in I_n} A_{k(\alpha)} \cdot \alpha$$

However, if $\alpha \in I_n$ then we have $k(\alpha) \geq n \Leftrightarrow \alpha = \mathrm{id}_{[n]} : [n] \to [n]$. Thus the summand $A_n \cdot \mathrm{id}_{[n]}$ of $S(A_\bullet)_n$ is a complement to the summand $D(S(A_\bullet))_n$. All the maps $d_i^n : S(A_\bullet)_n \to S(A_\bullet)_n$ restrict to zero on the summand $A_n \cdot \mathrm{id}_{[n]}$ except for d_n^n which produces $(-1)^n d_{A,n}$ from $A_n \cdot \mathrm{id}_{[n]}$ to $A_{n-1} \cdot \mathrm{id}_{[n-1]}$. We conclude that $A_n \cdot \mathrm{id}_{[n]}$ must be equal to the summand $N(S(A_\bullet))_n$ and moreover the restriction of the differential $d_n = \sum (-1)^i d_i^n : S(A_\bullet)_n \to S(A_\bullet)_{n-1}$ to the summand $A_n \cdot \mathrm{id}_{[n]}$ gives what we want!

Finally, we have to show that $S \circ N$ is isomorphic to the identity functor. Let U be a simplicial object of $\mathcal{A}$. Then we can define an obvious map

$$S(N(U))_n = \bigoplus\nolimits_{\alpha \in I_n} N(U)_{k(\alpha)} \cdot \alpha \longrightarrow U_n$$

by using $U(\alpha) : N(U)_{k(\alpha)} \to U_n$ on the summand corresponding to α. By Definition 18.1 this is an isomorphism. To finish the proof we have to show that this is compatible with the maps in the simplicial objects. Thus let $\varphi : [m] \to [n]$ and let $\alpha \in I_n$. Set $\beta = \alpha \circ \varphi$. Picture

$$
\begin{array}{ccc}
[m] & \xrightarrow{\ \varphi\ } & [n] \\
{\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\
\mathrm{Im}(\beta) & \longrightarrow & [k(\alpha)]
\end{array}
$$

There are several cases to consider

(1) Say $\beta \notin I_m$. Then there exists an index $0 \leq j < k(\alpha)$ with $j \notin \mathrm{Im}(\alpha \circ \varphi)$ and hence we can choose a factorization $\alpha \circ \varphi = \delta_j^{k(\alpha)} \circ \psi$ for some $\psi : [m] \to [k(\alpha) - 1]$. It follows that $U(\varphi)$ is zero on the image of the summand $N(U)_{k(\alpha)} \cdot \alpha$ because $U(\varphi) \circ U(\alpha) = U(\alpha \circ \varphi) = U(\psi) \circ d_j^{k(\alpha)}$ is zero on

$N(U)_{k(\alpha)}$ by construction of N. This matches our rule for $S(N(U))$ given above.

(2) Say $\beta \in I_m$ and $k(\beta) < k(\alpha) - 1$. Here we argue exactly as in case (1) with $j = k(\alpha) - 1$.

(3) Say $\beta \in I_m$ and $k(\beta) = k(\alpha)$. Here the summand $N(U)_{k(\alpha)} \cdot \alpha$ is mapped by the identity to the summand $N(U)_{k(\beta)} \cdot \beta$. This is the same as the effect of $U(\varphi)$ since in this case $U(\varphi) \circ U(\alpha) = U(\beta)$.

(4) Say $\beta \in I_m$ and $k(\beta) = k(\alpha) - 1$. Here we use the differential $(-1)^{k(\alpha)} d_{N(U),k(\alpha)}$ to map the summand $N(U)_{k(\alpha)} \cdot \alpha$ to the summand $N(U)_{k(\beta)} \cdot \beta$. On the other hand, since $\mathrm{Im}(\beta) = [k(\beta)]$ in this case we get $\alpha \circ \varphi = \delta_{k(\alpha)}^{k(\alpha)} \circ \beta$. Thus we see that $U(\varphi)$ composed with the restriction of $U(\alpha)$ to $N(U)_{k(\alpha)}$ is equal to $U(\beta)$ precomposed with $d_{k(\alpha)}^{k(\alpha)}$ restricted to $N(U)_{k(\alpha)}$. Since $d_{N(U),k(\alpha)} = \sum (-1)^i d_i^{k(\alpha)}$ and since $d_i^{k(\alpha)}$ restricts to zero on $N(U)_{k(\alpha)}$ for $i < k(\alpha)$ we see that equality holds.

This finishes the proof of the theorem. $\square$

25. Dold-Kan for cosimplicial objects

Let $\mathcal{A}$ be an abelian category. According to Homology, Lemma 5.2 also $\mathcal{A}^{opp}$ is abelian. It follows formally from the definitions that

$$\mathrm{CoSimp}(\mathcal{A}) = \mathrm{Simp}(\mathcal{A}^{opp})^{opp}.$$

Thus Dold-Kan (Theorem 24.3) implies that $\mathrm{CoSimp}(\mathcal{A})$ is equivalent to the category $\mathrm{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}$. And it follows formally from the definitions that

$$\mathrm{CoCh}_{\geq 0}(\mathcal{A}) = \mathrm{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}.$$

Putting these arrows together we obtain an equivalence

$$Q : \mathrm{CoSimp}(\mathcal{A}) \longrightarrow \mathrm{CoCh}_{\geq 0}(\mathcal{A}).$$

In this section we describe Q.

First we define the *cochain complex $s(U)$ associated to a cosimplicial object U*. It is the cochain complex with terms zero in negative degrees, and $s(U)^n = U_n$ for $n \geq 0$. As differentials we use the maps $d^n : s(U)^n \to s(U)^{n+1}$ defined by $d^n = \sum_{i=0}^{n+1} (-1)^i \delta_i^{n+1}$. In other words the complex $s(U)$ looks like

$$0 \longrightarrow U_0 \xrightarrow{\delta_0^1 - \delta_1^1} U_1 \xrightarrow{\delta_0^2 - \delta_1^2 + \delta_2^2} U_2 \longrightarrow \ldots$$

This is sometimes also called the *Moore complex* associated to U.

On the other hand, given a cosimplicial object U of $\mathcal{A}$ set $Q(U)^0 = U_0$ and

$$Q(U)^n = \mathrm{Coker}\left(\bigoplus_{i=0}^{n-1} U_{n-1} \xrightarrow{\delta_i^n} U_n \right).$$

The differential $d^n : Q(U)^n \to Q(U)^{n+1}$ is induced by $(-1)^{n+1} \delta_{n+1}^{n+1}$, i.e., by fitting the morphism $(-1)^{n+1} \delta_{n+1}^{n+1}$ into a commutative diagram

$$
\begin{array}{ccc}
U_n & \xrightarrow{(-1)^{n+1}\delta_{n+1}^{n+1}} & U_{n+1} \\
\downarrow & & \downarrow \\
Q(U)^n & \xrightarrow{d_n} & Q(U)^{n+1}.
\end{array}
$$

We leave it to the reader to show that this diagram makes sense, i.e., that the image of δ_i^n maps into the kernel of the right vertical arrow for $i = 0, \ldots, n-1$. (This is dual to Lemma 18.8.) Thus our cochain complex $Q(U)$ looks like this

$$0 \to Q(U)^0 \to Q(U)^1 \to Q(U)^2 \to \ldots$$

This is called the *normalized cochain complex associated to U*. The dual to the Dold-Kan Theorem 24.3 is the following.

Lemma 25.1. *Let $\mathcal{A}$ be an abelian category.*

(1) *The functor $s : CoSimp(\mathcal{A}) \to CoCh_{\geq 0}(\mathcal{A})$ is exact.*
(2) *The maps $s(U)^n \to Q(U)^n$ define a morphism of cochain complexes.*
(3) *There exists a functorial direct sum decomposition $s(U) = D(U) \oplus Q(U)$ in $CoCh_{\geq 0}(\mathcal{A})$.*
(4) *The functor Q is exact.*
(5) *The morphism of complexes $s(U) \to Q(U)$ is a quasi-isomorphism.*
(6) *The functor $U \mapsto Q(U)^\bullet$ defines an equivalence of categories $CoSimp(\mathcal{A}) \to CoCh_{\geq 0}(\mathcal{A})$.*

Proof. Omitted. But the results are the exact dual statements to Lemmas 23.1, 23.4, 23.6, 23.8, 23.9, and Theorem 24.3. $\qquad\square$

26. Homotopies

Consider the simplicial sets $\Delta[0]$ and $\Delta[1]$. Recall that there are two morphisms

$$e_0, e_1 : \Delta[0] \longrightarrow \Delta[1],$$

coming from the morphisms $[0] \to [1]$ mapping 0 to an element of $[1] = \{0, 1\}$. Recall also that each set $\Delta[1]_k$ is finite. Hence, if the category $\mathcal{C}$ has finite coproducts, then we can form the product

$$U \times \Delta[1]$$

for any simplicial object U of $\mathcal{C}$, see Definition 13.1. Note that $\Delta[0]$ has the property that $\Delta[0]_k = \{*\}$ is a singleton for all $k \geq 0$. Hence $U \times \Delta[0] = U$. Thus e_0, e_1 above gives rise to morphisms

$$e_0, e_1 : U \to U \times \Delta[1].$$

Definition 26.1. Let $\mathcal{C}$ be a category having finite coproducts. Suppose that U and V are two simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be two morphisms.

(1) We say a morphism

$$h : U \times \Delta[1] \longrightarrow V$$

is a *homotopy from a to b* if $a = h \circ e_0$ and $b = h \circ e_1$.
(2) We say the morphisms a and b are *homotopic* or are *in the same homotopy class* if there exists a sequence of morphisms $a = a_0, a_1, \ldots, a_n = b$ from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_{i-1} to a_i or there exists a homotopy from a_i to a_{i-1}.

The relation "there is a homotopy from a to b" is in general not transitive or symmetric; we will see it is reflexive in Example 26.3. Of course, "being homotopic" is an equivalence relation on the set $\text{Mor}(U, V)$ and it is the equivalence relation generated by the relation "there is a homotopy from a to b". It turns out we can define homotopies between pairs of maps of simplicial objects in any category. We

will do this in Remark 26.4 after we work out in some detail what it means to have a morphism $h : U \times \Delta[1] \to V$.

Let $\mathcal{C}$ be a category with finite coproducts. Let U, V be simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy from a to b. For every $n \geq 0$ let us write

$$\Delta[1]_n = \{\alpha_0^n, \ldots, \alpha_{n+1}^n\}$$

where $\alpha_i^n : [n] \to [1]$ is the map such that

$$\alpha_i^n(j) = \begin{cases} 0 & \text{if } j < i \\ 1 & \text{if } j \geq i \end{cases}$$

Thus

$$h_n : (U \times \Delta[1])_n = \coprod U_n \cdot \alpha_i^n \longrightarrow V_n$$

has a component $h_{n,i} : U_n \to V_n$ which is the restriction to the summand corresponding to α_i^n for all $i = 0, \ldots, n+1$.

Lemma 26.2. *In the situation above, we have the following relations:*

(1) *We have $h_{n,0} = b_n$ and $h_{n,n+1} = a_n$.*
(2) *We have $d_j^n \circ h_{n,i} = h_{n-1,i-1} \circ d_j^n$ for $i > j$.*
(3) *We have $d_j^n \circ h_{n,i} = h_{n-1,i} \circ d_j^n$ for $i \leq j$.*
(4) *We have $s_j^n \circ h_{n,i} = h_{n+1,i+1} \circ s_j^n$ for $i > j$.*
(5) *We have $s_j^n \circ h_{n,i} = h_{n+1,i} \circ s_j^n$ for $i \leq j$.*

Conversely, given a system of maps $h_{n,i}$ satisfying the properties listed above, then these define a morphism h which is a homotopy from a to b.

Proof. Omitted. You can prove the last statement using the fact, see Lemma 2.4 that to give a morphism of simplicial objects is the same as giving a sequence of morphisms h_n commuting with all d_j^n and s_j^n. $\square$

Example 26.3. Suppose in the situation above $a = b$. Then there is a *trivial* homotopy from a to b, namely the one with $h_{n,i} = a_n = b_n$.

Remark 26.4. Let $\mathcal{C}$ be any category (no assumptions whatsoever). Let U and V be simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{C}$. A *homotopy from a to b* is given by morphisms[1] $h_{n,i} : U_n \to V_n$, for $n \geq 0$, $i = 0, \ldots, n+1$ satisfying the relations of Lemma 26.2. As in Definition 26.1 we say the morphisms a and b are *homotopic* if there exists a sequence of morphisms $a = a_0, a_1, \ldots, a_n = b$ from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_{i-1} to a_i or there exists a homotopy from a_i to a_{i-1}. Clearly, if $F : \mathcal{C} \to \mathcal{C}'$ is any functor and $\{h_{n,i}\}$ is a homotopy from a to b, then $\{F(h_{n,i})\}$ is a homotopy from $F(a)$ to $F(b)$. Similarly, if a and b are homotopic, then $F(a)$ and $F(b)$ are homotopic. Since the lemma says that the newer notion is the same as the old one in case finite coproduct exist, we deduce in particular that functors preserve the original notion whenever both categories have finite coproducts.

Remark 26.5. Let $\mathcal{C}$ be any category. Suppose two morphisms $a, a' : U \to V$ of simplicial objects are homotopic. Then for any morphism $b : V \to W$ the two maps $b \circ a, b \circ a' : U \to W$ are homotopic. Similarly, for any morphism $c : X \to U$ the two

---

[1]In the literature, often the maps $h_{n+1,i} \circ s_i : U_n \to V_{n+1}$ are used instead of the maps $h_{n,i}$. Of course the relations these maps satisfy are different from the ones in Lemma 26.2.

maps $a \circ c, a' \circ c : X \to V$ are homotopic. In fact the maps $b \circ a \circ c, b \circ a' \circ c : X \to W$ are homotopic. Namely, if the maps $h_{n,i} : U_n \to V_n$ define a homotopy from a to a' then the maps $b \circ h_{n,i} \circ c$ define a homotopy from $b \circ a \circ c$ to $b \circ a' \circ c$. In this way we see that we obtain a new category $\mathrm{hSimp}(\mathcal{C})$ with the same objects as $\mathrm{Simp}(\mathcal{C})$ but whose morphisms are homotopy classes of of morphisms of $\mathrm{Simp}(\mathcal{C})$. Thus there is a canonical functor

$$\mathrm{Simp}(\mathcal{C}) \longrightarrow \mathrm{hSimp}(\mathcal{C})$$

which is essentially surjective and surjective on sets of morphisms.

Definition 26.6. Let U and V be two simplicial objects of a category $\mathcal{C}$. We say a morphism $a : U \to V$ is a *homotopy equivalence* if there exists a morphism $b : V \to U$ such that $a \circ b$ is homotopic to id_V and $b \circ a$ is homotopic to id_U. We say U and V are *homotopy equivalent* if there exists a homotopy equivalence $a : U \to V$.

Example 26.7. The simplicial set $\Delta[m]$ is homotopy equivalent to $\Delta[0]$. Namely, consider the unique morphism $f : \Delta[m] \to \Delta[0]$ and the morphism $g : \Delta[0] \to \Delta[m]$ given by the inclusion of the last 0-simplex of $\Delta[m]$. We have $f \circ g = \mathrm{id}$. We will give a homotopy $h : \Delta[m] \times \Delta[1] \to \Delta[m]$ from $\mathrm{id}_{\Delta[m]}$ to $g \circ f$. Namely h is given by the maps

$$\mathrm{Mor}_\Delta([n], [m]) \times \mathrm{Mor}_\Delta([n], [1]) \to \mathrm{Mor}_\Delta([n], [m])$$

which send (φ, α) to

$$k \mapsto \begin{cases} \varphi(k) & \text{if} \quad \alpha(k) = 0 \\ m & \text{if} \quad \alpha(k) = 1 \end{cases}$$

Note that this only works because we took g to be the inclusion of the last 0-simplex. If we took g to be the inclusion of the first 0-simplex we could find a homotopy from $g \circ f$ to $\mathrm{id}_{\Delta[m]}$. This is an illustration of the asymmetry inherent in homotopies in the category of simplicial sets.

The following lemma says that $U \times \Delta[1]$ is homotopy equivalent to U.

Lemma 26.8. *Let $\mathcal{C}$ be a category with finite coproducts. Let U be a simplicial object of $\mathcal{C}$. Consider the maps $e_1, e_0 : U \to U \times \Delta[1]$, and $\pi : U \times \Delta[1] \to U$, see Lemma 13.3.*

1. *We have $\pi \circ e_1 = \pi \circ e_0 = \mathrm{id}_U$, and*
2. *The morphisms $\mathrm{id}_{U \times \Delta[1]}$, and $e_0 \circ \pi$ are homotopic.*
3. *The morphisms $\mathrm{id}_{U \times \Delta[1]}$, and $e_1 \circ \pi$ are homotopic.*

Proof. The first assertion is trivial. For the second, consider the map of simplicial sets $\Delta[1] \times \Delta[1] \longrightarrow \Delta[1]$ which in degree n assigns to a pair (β_1, β_2), $\beta_i : [n] \to [1]$ the morphism $\beta : [n] \to [1]$ defined by the rule

$$\beta(i) = \max\{\beta_1(i), \beta_2(i)\}.$$

It is a morphism of simplicial sets, because the action $\Delta[1](\varphi) : \Delta[1]_n \to \Delta[1]_m$ of $\varphi : [m] \to [n]$ is by precomposing. Clearly, using notation from Section 26, we have $\beta = \beta_1$ if $\beta_2 = \alpha_0^n$ and $\beta = \alpha_{n+1}^n$ if $\beta_2 = \alpha_{n+1}^n$. This implies easily that the induced morphism

$$U \times \Delta[1] \times \Delta[1] \longrightarrow U \times \Delta[1]$$

of Lemma 13.3 is a homotopy from $\mathrm{id}_{U \times \Delta[1]}$ to $e_0 \circ \pi$. Similarly for $e_1 \circ \pi$ (use minimum instead of maximum). $\square$

Lemma 26.9. *Let $f : Y \to X$ be a morphism of a category $\mathcal{C}$ with fibre products. Assume f has a section s. Consider the simplicial object U constructed in Example 3.5 starting with f. The morphism $U \to U$ which in each degree is the self map $(s \circ f)^{n+1}$ of $Y \times_X \dots \times_X Y$ given by $s \circ f$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant simplicial object X.*

Proof. Set $g^0 = \mathrm{id}_Y$ and $g^1 = s \circ f$. We use the morphisms

$$Y \times_X \dots \times_X Y \times \mathrm{Mor}([n],[1]) \quad \to \quad Y \times_X \dots \times_X Y$$
$$(y_0, \dots, y_n) \times \alpha \quad \mapsto \quad (g^{\alpha(0)}(y_0), \dots, g^{\alpha(n)}(y_n))$$

where we use the functor of points point of view to define the maps. Another way to say this is to say that $h_{n,0} = \mathrm{id}$, $h_{n,n+1} = (s \circ f)^{n+1}$ and $h_{n,i} = \mathrm{id}_Y^{i+1} \times (s \circ f)^{n+1-i}$. We leave it to the reader to show that these satisfy the relations of Lemma 26.2. Hence they define the desired homotopy. See also Remark 26.4 which shows that we do not need to assume anything else on the category $\mathcal{C}$. $\qquad\square$

Lemma 26.10. *Let $\mathcal{C}$ be a category. Let T be a set. For $t \in T$ let X_t, Y_t be simplicial objects of $\mathcal{C}$. Assume $X = \prod_{t \in T} X_t$ and $Y = \prod_{t \in T} Y_t$ exist.*

(1) *If X_t and Y_t are homotopy equivalent for all $t \in T$ and T is finite, then X and Y are homotopy equivalent.*

For $t \in T$ let $a_t, b_t : X_t \to Y_t$ be morphisms. Set $a = \prod a_t : X \to Y$ and $b = \prod b_t : X \to Y$.

(2) *If there exists a homotopy from a_t to b_t for all $t \in T$, then there exists a homotopy from a to b.*

(3) *If T is finite and $a_t, b_t : X_t \to Y_t$ for $t \in T$ are homotopic, then a and b are homotopic.*

Proof. If $h_t = (h_{t,n,i})$ is a homotopy from a_t to b_t (see Remark 26.4), then $h = (\prod_t h_{t,n,i})$ is a homotopy from $\prod a_t$ to $\prod b_t$. This proves (2).

Proof of (3). Choose $t \in T$. There exists an integer $n \geq 0$ and a chain $a_t = a_{t,0}, a_{t,1}, \dots, a_{t,n} = b_t$ such that for every $1 \leq i \leq n$ either there is a homotopy from $a_{t,i-1}$ to $a_{t,i}$ or there is a homotopy from $a_{t,i}$ to $a_{t,i-1}$. If $n = 0$, then we pick another t. (We're done if $a_t = b_t$ for all $t \in T$.) So assume $n > 0$. By Example 26.3 there are is a homotopy from $b_{t'}$ to $b_{t'}$ for all $t' \in T \setminus \{t\}$. Thus by (2) there is a homotopy from $a_{t,n-1} \times \prod_{t'} b_{t'}$ to b or there is a homotopy from b to $a_{t,n-1} \times \prod_{t'} b_{t'}$. In this way we can decrease n by 1. This proves (3).

Part (1) follows from part (3) and the definitions. $\qquad\square$

27. Homotopies in abelian categories

Let $\mathcal{A}$ be an additive category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy from a to b. Let us prove the two morphisms of chain complexes $s(a), s(b) : s(U) \longrightarrow s(V)$ are homotopic in the sense of Homology, Section 13. Using the notation introduced in Section 26 we define

$$s(h)_n : U_n \longrightarrow V_{n+1}$$

by the formula

(27.0.1) $$s(h)_n = \sum\nolimits_{i=0}^{n} (-1)^{i+1} h_{n+1,i+1} \circ s_i^n.$$

Let us compute $d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n$. We first compute

$$
\begin{aligned}
d_{n+1} \circ s(h)_n &= \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{j+i+1} d_j^{n+1} \circ h_{n+1,i+1} \circ s_i^n \\
&= \sum_{1 \le i+1 \le j \le n+1} (-1)^{j+i+1} h_{n,i+1} \circ d_j^{n+1} \circ s_i^n \\
&\quad + \sum_{n \ge i \ge j \ge 0} (-1)^{i+j+1} h_{n,i} \circ d_j^{n+1} \circ s_i^n \\
&= \sum_{1 \le i+1 < j \le n+1} (-1)^{j+i+1} h_{n,i+1} \circ s_i^{n-1} \circ d_{j-1}^n \\
&\quad + \sum_{1 \le i+1 = j \le n+1} (-1)^{j+i+1} h_{n,i+1} \\
&\quad + \sum_{n \ge i = j \ge 0} (-1)^{i+j+1} h_{n,i} \\
&\quad + \sum_{n \ge i > j \ge 0} (-1)^{i+j+1} h_{n,i} \circ s_{i-1}^{n-1} \circ d_j^n
\end{aligned}
$$

We leave it to the reader to see that the first and the last of the four sums cancel exactly against all the terms of

$$
s(h)_{n-1} \circ d_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+1+j} h_{n,i+1} \circ s_i^{n-1} \circ d_j^n.
$$

Hence we obtain

$$
\begin{aligned}
d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n &= \sum_{j=1}^{n+1} (-1)^{2j} h_{n,j} + \sum_{i=0}^{n} (-1)^{2i+1} h_{n,i} \\
&= h_{n,n+1} - h_{n,0} \\
&= a_n - b_n
\end{aligned}
$$

as desired.

Lemma 27.1. *Let $\mathcal{A}$ be an additive category. Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$ are homotopic maps of chain complexes. If $\mathcal{A}$ is abelian, then also $N(a), N(b) : N(U) \to N(V)$ are homotopic maps of chain complexes.*

Proof. We may choose a sequence $a = a_0, a_1, \ldots, a_n = b$ of morphisms from U to V such that for each $i = 1, \ldots, n$ either there is a homotopy from a_i to a_{i-1} or there is a homotopy from a_{i-1} to a_i. The calculation above shows that in this case either $s(a_i)$ is homotopic to $s(a_{i-1})$ as a map of chain complexes or $s(a_{i-1})$ is homotopic to $s(a_i)$ as a map of chain complexes. Of course, these things are equivalent and moreover being homotopic is an equivalence relation on the set of maps of chain complexes, see Homology, Section 13. This proves that $s(a)$ and $s(b)$ are homotopic as maps of chain complexes.

Next, we turn to $N(a)$ and $N(b)$. It follows from Lemma 23.6 that $N(a)$, $N(b)$ are compositions

$$
N(U) \to s(U) \to s(V) \to N(V)
$$

where we use $s(a)$, $s(b)$ in the middle. Hence the assertion follows from Homology, Lemma 13.1. $\qquad \square$

Lemma 27.2. *Let $\mathcal{A}$ be an additive category. Let $a : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If a is a homotopy equivalence, then $s(a) : s(U) \to s(V)$ is*

a homotopy equivalence of chain complexes. If in addition $\mathcal{A}$ is abelian, then also $N(a) : N(U) \to N(V)$ is a homotopy equivalence of chain complexes.

Proof. Omitted. See Lemma 27.1 above. $\square$

28. Homotopies and cosimplicial objects

Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object and consider $\mathrm{Hom}(\Delta[1], V)$, see Section 14. The morphisms $e_0, e_1 : \Delta[0] \to \Delta[1]$ produce two morphisms $e_0, e_1 : \mathrm{Hom}(\Delta[1], V) \to V$.

Definition 28.1. Let $\mathcal{C}$ be a category having finite products. Let U and V be two cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be two morphisms of cosimplicial objects of $\mathcal{C}$.

(1) We say a morphism

$$h : U \longrightarrow \mathrm{Hom}(\Delta[1], V)$$

such that $a = e_0 \circ h$ and $b = e_1 \circ h$ is a *homotopy from a to b.*
(2) We say a and b are *homotopic* or are *in the same homotopy class* if there exists a sequence $a = a_0, a_1, \ldots, a_n = b$ of morphisms from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_i to a_{i-1} or there exists a homotopy from a_{i-1} to a_i.

This is dual to the notion we introduced for simplicial objects in Section 26. To explain this, consider a homotopy $h : U \to \mathrm{Hom}(\Delta[1], V)$ from a to b as in the definition. Recall that $\Delta[1]_n$ is a finite set. The degree n component of h is a morphism

$$h_n = (h_{n,\alpha}) : U \longrightarrow \mathrm{Hom}(\Delta[1], V)_n = \prod_{\alpha \in \Delta[1]_n} V_n$$

The morphisms $h_{n,\alpha} : U_n \to V_n$ of $\mathcal{C}$ have the property that for every morphism $f : [n] \to [m]$ of Δ we have

(28.1.1) $$h_{m,\alpha} \circ U(f) = V(f) \circ h_{n,\alpha \circ f}$$

Moreover, the condition that $a = e_0 \circ h$ means that $a_n = h_{n,0:[n] \to [1]}$ where $0 : [n] \to [1]$ is the constant map with value 0. Similarly, the condition that $b = e_1 \circ h$ means that $b_n = h_{n,1:[n] \to [1]}$ where $1 : [n] \to [1]$ is the constant map with value 1. Conversly, given a family of morphisms $\{h_{n,\alpha}\}$ such that (28.1.1) holds for all morphisms f of Δ and such that $a_n = h_{n,0:[n] \to [1]}$ and $b_n = h_{n,1:[n] \to [1]}$ for all $n \geq 0$, then we obtain a homotopy h from a to b by setting $h = \prod_{\alpha \in \Delta[1]_n} h_{n,\alpha}$.

Remark 28.2. Let $\mathcal{C}$ be any category (no assumptions whatsoever). Let U and V be cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. A *homotopy from a to b* is given by morphisms $h_{n,\alpha} : U_n \to V_n$, for $n \geq 0$, $\alpha \in \Delta[1]_n$ satisfying (28.1.1) for all morphisms f of Δ and such that $a_n = h_{n,0:[n] \to [1]}$ and $b_n = h_{n,1:[n] \to [1]}$ for all $n \geq 0$. As in Definition 28.1 we say the morphisms a and b are *homotopic* if there exists a sequence of morphisms $a = a_0, a_1, \ldots, a_n = b$ from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_{i-1} to a_i or there exists a homotopy from a_i to a_{i-1}. Clearly, if $F : \mathcal{C} \to \mathcal{C}'$ is any functor and $\{h_{n,i}\}$ is a homotopy from a to b, then $\{F(h_{n,i})\}$ is a homotopy from $F(a)$ to $F(b)$. Similarly, if a and b are homotopic, then $F(a)$ and $F(b)$ are homotopic. This new notion is the same as the old one in case finite

products exist. We deduce in particular that functors preserve the original notion whenever both categories have finite products.

Lemma 28.3. *Let $\mathcal{C}$ be a category. Suppose that U and V are two cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of cosimplicial objects. Recall that U, V correspond to simplicial objects U', V' of $\mathcal{C}^{opp}$. Moreover a, b correspond to morphisms $a', b' : V' \to U'$. The following are equivalent*

    (1) *There exists a homotopy $h = \{h_{n,\alpha}\}$ from a to b as in Remark 28.2.*
    (2) *There exists a homotopy $h = \{h_{n,i}\}$ from a' to b' as in Remark 26.4.*

Thus a is homotopic to b as in Remark 28.2 if and only if a' is homotopic to b' as in Remark 26.4.

Proof. In case $\mathcal{C}$ has finite products, then $\mathcal{C}^{opp}$ has finite coproducts and we may use Definitions 28.1 and 26.1 instead of Remarks 28.2 and 26.4. In this case $h : U \to \mathrm{Hom}(\Delta[1], V)$ is the same as a morphism $h' : \mathrm{Hom}(\Delta[1], V)' \to U'$. Since products and coproducts get switched too, it is immediate that $(\mathrm{Hom}(\Delta[1], V))' = V' \times \Delta[1]$. Moreover, the "primed" version of the morphisms $e_0, e_1 : \mathrm{Hom}(\Delta[1], V) \to V$ are the morphisms $e_0, e_1 : V' \to \Delta[1] \times V$. Thus $e_0 \circ h = a$ translates into $h' \circ e_0 = a'$ and similarly $e_1 \circ h = b$ translates into $h' \circ e_1 = b'$. This proves the lemma in this case.

In the general case, one needs to translate the relations given by (28.1.1) into the relations given in Lemma 26.2. We omit the details.

The final assertion is formal from the equivalence of (1) and (2). $\qquad\square$

Lemma 28.4. *Let $\mathcal{C}, \mathcal{C}', \mathcal{D}, \mathcal{D}'$ be categories. With terminology as in Remarks 28.2 and 26.4.*

    (1) *Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{D}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of simplicial objects.*
    (2) *Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of cosimplicial objects.*
    (3) *Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{C}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of cosimplicial objects.*
    (4) *Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of simplicial objects.*

Proof. By Lemma 28.3 above, we can turn F into a covariant functor between a pair of categories, and we have to show that the functor preserves homotopic pairs of maps. This is explained in Remark 26.4. $\qquad\square$

Lemma 28.5. *Let $f : X \to Y$ be a morphism of a category $\mathcal{C}$ with pushouts. Assume there is a morphism $s : Y \to X$ with $s \circ f = id_X$. Consider the cosimplicial object U constructed in Example 5.5 starting with f. The morphism $U \to U$ which in each degree is the self map of $Y \amalg_X \ldots \amalg_X Y$ given by $f \circ s$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant cosimplicial object X.*

Proof. This lemma is dual to Lemma 26.9. Hence this lemma follows on applying Lemma 28.3. $\square$

Lemma 28.6. *Let $\mathcal{A}$ be an additive category. Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$ are homotopic maps of cochain complexes. If in addition $\mathcal{A}$ is abelian, then $Q(a), Q(b) : Q(U) \to Q(V)$ are homotopic maps of cochain complexes.*

Proof. Let $(-)' : \mathcal{A} \to \mathcal{A}^{opp}$ be the contravariant functor $A \mapsto A$. By Lemma 28.5 the maps a' and b' are homotopic. By Lemma 27.1 we see that $s(a')$ and $s(b')$ are homotopic maps of chain complexes. Since $s(a') = (s(a))'$ and $s(b') = (s(b))'$ we conclude that also $s(a)$ and $s(b)$ are homotopic by applying the additive contravariant functor $(-)'' : \mathcal{A}^{opp} \to \mathcal{A}$. The result for the Q-complexes follows in the same manner using that $Q(U)' = N(U')$. $\square$

Lemma 28.7. *Let $\mathcal{A}$ be an additive category. Let $a : U \to V$ be a morphism of cosimplicial objects of $\mathcal{A}$. If a is a homotopy equivalence, then $s(a) : s(U) \to s(V)$ is a homotopy equivalence of chain complexes. If in addition $\mathcal{A}$ is abelian, then also $Q(a) : Q(U) \to Q(V)$ is a homotopy equivalence of chain complexes.*

Proof. Omitted. See Lemma 28.6 above. $\square$

29. More homotopies in abelian categories

Let $\mathcal{A}$ be an abelian category. In this section we show that a homotopy between morphisms in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ always comes from a morphism $U \times \Delta[1] \to V$ in the category of simplicial objects. In some sense this will provide a converse to Lemma 27.1. We first develop some material on homotopies between morphisms of chain complexes.

Lemma 29.1. *Let $\mathcal{A}$ be an abelian category. Let A be a chain complex. Consider the covariant functor*

$$B \longmapsto \{(a, b, h) \mid a, b : A \to B \text{ and } h \text{ a homotopy between } a, b\}$$

There exists a chain complex $\diamond A$ such that $\mathrm{Mor}_{Ch(\mathcal{A})}(\diamond A, -)$ is isomorphic to the displayed functor. The construction $A \mapsto \diamond A$ is functorial.

Proof. We set $\diamond A_n = A_n \oplus A_n \oplus A_{n-1}$, and we define $d_{\diamond A, n}$ by the matrix

$$d_{\diamond A,n} = \begin{pmatrix} d_{A,n} & 0 & \mathrm{id}_{A_{n-1}} \\ 0 & d_{A,n} & -\mathrm{id}_{A_{n-1}} \\ 0 & 0 & -d_{A,n-1} \end{pmatrix} : A_n \oplus A_n \oplus A_{n-1} \to A_{n-1} \oplus A_{n-1} \oplus A_{n-2}$$

If $\mathcal{A}$ is the category of abelian groups, and $(x, y, z) \in A_n \oplus A_n \oplus A_{n-1}$ then $d_{\diamond A,n}(x, y, z) = (d_n(x) + z, d_n(y) - z, -d_{n-1}(z))$. It is easy to verify that $d^2 = 0$. Clearly, there are two maps $\diamond a, \diamond b : A \to \diamond A$ (first summand and second summand), and a map $\diamond A \to A[-1]$ which give a short exact sequence

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

which is termwise split. Moreover, there is a sequence of maps $\diamond h_n : A_n \to \diamond A_{n+1}$, namely the identity from A_n to the summand A_n of $\diamond A_{n+1}$, such that $\diamond h$ is a homotopy between $\diamond a$ and $\diamond b$.

We conclude that any morphism $f : \diamond A \to B$ gives rise to a triple (a, b, h) by setting $a = f \circ \diamond a$, $b = f \circ \diamond b$ and $h_n = f_{n+1} \circ \diamond h_n$. Conversely, given a triple (a, b, h) we get a morphism $f : \diamond A \to B$ by taking

$$f_n = (a_n, b_n, h_{n-1}).$$

To see that this is a morphism of chain complexes you have to do a calculation. We only do this in case $\mathcal{A}$ is the category of abelian groups: Say $(x, y, z) \in \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$. Then

$$
\begin{aligned}
f_{n-1}(d_n(x, y, z)) &= f_{n-1}(d_n(x) + z, d_n(y) - z, -d_{n-1}(z)) \\
&= a_n(d_n(x)) + a_n(z) + b_n(d_n(y)) - b_n(z) - h_{n-2}(d_{n-1}(z))
\end{aligned}
$$

and

$$
\begin{aligned}
d_n(f_n(x, y, z) &= d_n(a_n(x) + b_n(y) + h_{n-1}(z)) \\
&= d_n(a_n(x)) + d_n(b_n(y)) + d_n(h_{n-1}(z))
\end{aligned}
$$

which are the same by definition of a homotopy. $\square$

Note that the extension

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

comes with sections of the morphisms $\diamond A_n \to A[-1]_n$ with the property that the associated morphism $\delta : A[-1] \to (A \oplus A)[-1]$, see Homology, Lemma 14.4 equals the morphism $(1, -1) : A[-1] \to A[-1] \oplus A[-1]$.

Lemma 29.2. *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to A \oplus A \to B \to C \to 0$$

be a short exact sequence of chain complexes of $\mathcal{A}$. Suppose given in addition morphisms $s_n : C_n \to B_n$ splitting the associated short exact sequence in degree n. Let $\delta(s) : C \to (A \oplus A)[-1] = A[-1] \oplus A[-1]$ be the associated morphism of complexes, see Homology, Lemma 14.4. If $\delta(s)$ factors through the morphism $(1, -1) : A[-1] \to A[-1] \oplus A[-1]$, then there is a unique morphism $B \to \diamond A$ fitting into a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A \oplus A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A \oplus A & \longrightarrow & \diamond A & \longrightarrow & A[-1] & \longrightarrow & 0
\end{array}
$$

where the vertical maps are compatible with the splittings s_n and the splittings of $\diamond A_n \to A[-1]_n$ as well.

Proof. Denote $(p_n, q_n) : B_n \to A_n \oplus A_n$ the morphism π_n of Homology, Lemma 14.4. Also write $(a, b) : A \oplus A \to B$, and $r : B \to C$ for the maps in the short exact sequence. Write the factorization of $\delta(s)$ as $\delta(s) = (1, -1) \circ f$. This means that $p_{n-1} \circ d_{B,n} \circ s_n = f_n$, and $q_{n-1} \circ d_{B,n} \circ s_n = -f_n$, and Set $B_n \to \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$ equal to $(p_n, q_n, f_n \circ r_n)$.

Now we have to check that this actually defines a morphism of complexes. We will only do this in the case of abelian groups. Pick $x \in B_n$. Then $x = a_n(x_1) + b_n(x_2) + s_n(x_3)$ and it suffices to show that our definition commutes with differential for each term separately. For the term $a_n(x_1)$ we have $(p_n, q_n, f_n \circ r_n)(a_n(x_1)) = (x_1, 0, 0)$

and the result is obvious. Similarly for the term $b_n(x_2)$. For the term $s_n(x_3)$ we have

$$
\begin{aligned}
(p_n, q_n, f_n \circ r_n)(d_n(s_n(x_3))) &= (p_n, q_n, f_n \circ r_n)(\\
&\qquad a_n(f_n(x_3)) - b_n(f_n(x_3)) + s_n(d_n(x_3))) \\
&= (f_n(x_3), -f_n(x_3), f_n(d_n(x_3)))
\end{aligned}
$$

by definition of f_n. And

$$
\begin{aligned}
d_n(p_n, q_n, f_n \circ r_n)(s_n(x_3)) &= d_n(0, 0, f_n(x_3)) \\
&= (f_n(x_3), -f_n(x_3), d_{A[-1],n}(f_n(x_3)))
\end{aligned}
$$

The result follows as f is a morphism of complexes. $\qquad\square$

Lemma 29.3. *Let $\mathcal{A}$ be an abelian category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be a pair of morphisms. Assume the corresponding maps of chain complexes $N(a), N(b) : N(U) \to N(V)$ are homotopic by a homotopy $\{N_n : N(U)_n \to N(V)_{n+1}\}$. Then there exists a homotopy from a to b as in Definition 26.1. Moreover, one can choose the homotopy $h : U \times \Delta[1] \to V$ such that $N_n = N(h)_n$ where $N(h)$ is the homotopy coming from h as in Section 27.*

Proof. Let $(\diamond N(U), \diamond a, \diamond b, \diamond h)$ be as in Lemma 29.1 and its proof. By that lemma there exists a morphism $\diamond N(U) \to N(V)$ representing the triple $(N(a), N(b), \{N_n\})$. We will show there exists a morphism $\psi : N(U \times \Delta[1]) \to \diamond N(U)$ such that $\diamond a = \psi \circ N(e_0)$, and $\diamond b = \psi \circ N(e_1)$. Moreover, we will show that the homotopy between $N(e_0), N(e_1) : N(U) \to N(U \times \Delta[1])$ coming from (27.0.1) and Lemma 27.1 with $h = \mathrm{id}_{U \times \Delta[1]}$ is mapped via ψ to the canonical homotopy $\diamond h$ between the two maps $\diamond a, \diamond b : N(U) \to \diamond N(U)$. Certainly this will imply the lemma.

Note that $N : \mathrm{Simp}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$ as a functor is a direct summand of the functor $s : \mathrm{Simp}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$. Also, the functor $\diamond$ is compatible with direct sums. Thus it suffices instead to construct a morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$ with the corresponding properties. This is what we do below.

By Definition 26.1 the morphisms $e_0 : U \to U \times \Delta[1]$ and $e_1 : U \to U \times \Delta[1]$ are homotopic with homotopy $\mathrm{id}_{U \times \Delta[1]}$. By Lemma 27.1 we get an explicit homotopy $\{h_n : s(U)_n \to s(U \times \Delta[1])_{n+1}\}$ between the morphisms of chain complexes $s(e_0) : s(U) \to s(U \times \Delta[1])$ and $s(e_1) : s(U) \to s(U \times \Delta[1])$. By Lemma 29.2 above we get a corresponding morphism

$$
\Phi : \diamond s(U) \to s(U \times \Delta[1])
$$

According to the construction, Φ_n restricted to the summand $s(U)[-1]_n = s(U)_{n-1}$ of $\diamond s(U)_n$ is equal to h_{n-1}. And

$$
h_{n-1} = \sum_{i=0}^{n-1} (-1)^{i+1} s_i^n \cdot \alpha_{i+1}^n : U_{n-1} \to \bigoplus_j U_n \cdot \alpha_j^n.
$$

with obvious notation.

On the other hand, the morphisms $e_i : U \to U \times \Delta[1]$ induce a morphism $(e_0, e_1) : U \oplus U \to U \times \Delta[1]$. Denote W the cokernel. Note that, if we write $(U \times \Delta[1])_n = \bigoplus_{\alpha:[n]\to[1]} U_n \cdot \alpha$, then we may identify $W_n = \bigoplus_{i=1}^n U_n \cdot \alpha_i^n$ with α_i^n as in Section

26. We have a commutative diagram

$$0 \longrightarrow U \oplus U \longrightarrow U \times \Delta[1] \longrightarrow W \longrightarrow 0$$

with a diagonal arrow labeled $(1,1)$ from $U \oplus U$ and a vertical arrow π from $U \times \Delta[1]$ both going to U.

This implies we have a similar commutative diagram after applying the functor s. Next, we choose the splittings $\sigma_n : s(W)_n \to s(U \times \Delta[1])_n$ by mapping the summand $U_n \cdot \alpha_i^n \subset W_n$ via $(-1,1)$ to the summands $U_n \cdot \alpha_0^n \oplus U_n \cdot \alpha_i^n \subset (U \times \Delta[1])_n$. Note that $s(\pi)_n \circ \sigma_n = 0$. It follows that $(1,1) \circ \delta(\sigma)_n = 0$. Hence $\delta(\sigma)$ factors as in Lemma 29.2. By that lemma we obtain a canonical morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$.

To compute Ψ we first compute the morphism $\delta(\sigma) : s(W) \to s(U)[-1] \oplus s(U)[-1]$. According to Homology, Lemma 14.4 and its proof, to do this we have compute

$$d_{s(U \times \delta[1]),n} \circ \sigma_n - \sigma_{n-1} \circ d_{s(W),n}$$

and write it as a morphism into $U_{n-1} \cdot \alpha_0^{n-1} \oplus U_{n-1} \cdot \alpha_n^{n-1}$. We only do this in case $\mathcal{A}$ is the category of abelian groups. We use the short hand notation x_α for $x \in U_n$ to denote the element x in the summand $U_n \cdot \alpha$ of $(U \times \Delta[1])_n$. Recall that

$$d_{s(U \times \delta[1]),n} = \sum_{i=0}^{n} (-1)^i d_i^n$$

where d_i^n maps the summand $U_n \cdot \alpha$ to the summand $U_{n-1} \cdot (\alpha \circ \delta_i^n)$ via the morphism d_i^n of the simplicial object U. In terms of the notation above this means

$$d_{s(U \times \delta[1]),n}(x_\alpha) = \sum_{i=0}^{n} (-1)^i (d_i^n(x))_{\alpha \circ \delta_i^n}$$

Starting with $x_\alpha \in W_n$, in other words $\alpha = \alpha_j^n$ for some $j \in \{1, \ldots, n\}$, we see that $\sigma_n(x_\alpha) = x_\alpha - x_{\alpha_0^n}$ and hence

$$(d_{s(U \times \delta[1]),n} \circ \sigma_n)(x_\alpha) = \sum_{i=0}^{n} (-1)^i (d_i^n(x))_{\alpha \circ \delta_i^n} - \sum_{i=0}^{n} (-1)^i (d_i^n(x))_{\alpha_0^n \circ \delta_i^n}$$

To compute $d_{s(W),n}(x_\alpha)$, we have to omit all terms where $\alpha \circ \delta_i^n = \alpha_0^{n-1}, \alpha_n^{n-1}$. Hence we get

$$(\sigma_{n-1} \circ d_{s(W),n})(x_\alpha) =$$
$$\sum_{i=0,\ldots,n \text{ and } \alpha \circ \delta_i^n \neq \alpha_0^{n-1} \text{ or } \alpha_n^{n-1}} \left((-1)^i (d_i^n(x))_{\alpha \circ \delta_i^n} - (-1)^i (d_i^n(x))_{\alpha_0^{n-1}} \right)$$

Clearly the difference of the two terms is the sum

$$\sum_{i=0,\ldots,n \text{ and } \alpha \circ \delta_i^n = \alpha_0^{n-1} \text{ or } \alpha_n^{n-1}} \left((-1)^i (d_i^n(x))_{\alpha \circ \delta_i^n} - (-1)^i (d_i^n(x))_{\alpha_0^{n-1}} \right)$$

Of course, if $\alpha \circ \delta_i^n = \alpha_0^{n-1}$ then the term drops out. Recall that $\alpha = \alpha_j^n$ for some $j \in \{1, \ldots, n\}$. The only way $\alpha_j^n \circ \delta_i^n = \alpha_n^{n-1}$ is if $j = n$ and $i = n$. Thus we actually get 0 unless $j = n$ and in that case we get $(-1)^n (d_n^n(x))_{\alpha_n^{n-1}} - (-1)^n (d_n^n(x))_{\alpha_0^{n-1}}$. In other words, we conclude the morphism

$$\delta(\sigma)_n : W_n \to (s(U)[-1] \oplus s(U)[-1])_n = U_{n-1} \oplus U_{n-1}$$

is zero on all summands except $U_n \cdot \alpha_n^n$ and on that summand it is equal to $((-1)^n d_n^n, -(-1)^n d_n^n)$. (Namely, the first summand of the two corresponds to the factor with α_n^{n-1} because that is the map $[n-1] \to [1]$ which maps everybody to 0, and hence corresponds to e_0.)

We obtain a canonical diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & \diamond s(U) & \longrightarrow & s(U)[-1] & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle \Phi} & & \downarrow & & \\
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & s(U \times \Delta[1]) & \longrightarrow & s(W) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle \Psi} & & \downarrow & & \\
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & \diamond s(U) & \longrightarrow & s(U)[-1] & \longrightarrow & 0
\end{array}
$$

We claim that $\Phi \circ \Psi$ is the identity. To see this it is enough to prove that the composition of Φ and $\delta(\sigma)$ as a map $s(U)[-1] \to s(W) \to s(U)[-1] \oplus s(U)[-1]$ is the identity in the first factor and minus identity in the second. By the computations above it is $((-1)^n d_0^n, -(-1)^n d_0^n) \circ (-1)^n s_n^n = (1, -1)$ as desired. $\qquad\square$

30. Trivial Kan fibrations

Recall that for $n \geq 0$ the simplicial set $\Delta[n]$ is given by the rule $[k] \mapsto \mathrm{Mor}_\Delta([k], [n])$, see Example 11.2. Recall that $\Delta[n]$ has a unique nondegenerate n-simplex and all nondegenerate simplices are faces of this n-simplex. In fact, the nondegenerate simplices of $\Delta[n]$ correspond exactly to injective morphisms $[k] \to [n]$, which we may identify with subsets of $[n]$. Moreover, recall that $\mathrm{Mor}(\Delta[n], X) = X_n$ for any simplicial set X (Lemma 11.3). We set

$$
\partial \Delta[n] = i_{(n-1)!} \mathrm{sk}_{n-1} \Delta[n]
$$

and we call it the *boundary* of $\Delta[n]$. From Lemma 21.5 we see that $\partial \Delta[n] \subset \Delta[n]$ is the simplicial subset having the same nondegenerate simplices in degrees $\leq n-1$ but not containing the nondegenerate n-simplex.

Definition 30.1. A map $X \to Y$ of simplicial sets is called a *trivial Kan fibration* if $X_0 \to Y_0$ is surjective and for all $n \geq 1$ and any commutative solid diagram

$$
\begin{array}{ccc}
\partial \Delta[n] & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\Delta[n] & \longrightarrow & Y
\end{array}
$$

a dotted arrow exists making the diagram commute.

A trivial Kan fibration satisfies a very general lifting property.

Lemma 30.2. *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. For any solid commutative diagram*

$$
\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow & {\scriptstyle b}\nearrow & \downarrow \\
W & \xrightarrow{\ a\ } & Y
\end{array}
$$

of simplicial sets with $Z \to W$ (termwise) injective a dotted arrow exists making the diagram commute.

Proof. Suppose that $Z \neq W$. Let n be the smallest integer such that $Z_n \neq W_n$. Let $x \in W_n$, $x \notin Z_n$. Denote $Z' \subset W$ the simplicial subset containing Z, x, and all degeneracies of x. Let $\varphi : \Delta[n] \to Z'$ be the morphism corresponding to x (Lemma 11.3). Then $\varphi|_{\partial\Delta[n]}$ maps into Z as all the nondegenerate simplices of $\partial\Delta[n]$ end up in Z. By assumption we can extend $b \circ \varphi|_{\partial\Delta[n]}$ to $\beta : \Delta[n] \to X$. By Lemma 21.7 the simplicial set Z' is the pushout of $\Delta[n]$ and Z along $\partial\Delta[n]$. Hence b and β define a morphism $b' : Z' \to X$. In other words, we have extended the morphism b to a bigger simplicial subset of Z.

The proof is finished by an application of Zorn's lemma (omitted). $\qquad\square$

Lemma 30.3. *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Let $Y' \to Y$ be a morphism of simplicial sets. Then $X \times_Y Y' \to Y'$ is a trivial Kan fibration.*

Proof. This follows immediately from the functorial properties of the fibre product (Lemma 7.2) and the definitions. $\qquad\square$

Lemma 30.4. *The composition of two trivial Kan fibrations is a trivial Kan fibration.*

Proof. Omitted. $\qquad\square$

Lemma 30.5. *Let $\ldots \to U^2 \to U^1 \to U^0$ be a sequence of trivial Kan fibrations. Let $U = \lim U^t$ defined by taking $U_n = \lim U_n^t$. Then $U \to U^0$ is a trivial Kan fibration.*

Proof. Omitted. Hint: use that for a countable sequence of surjections of sets the inverse limit is nonempty. $\qquad\square$

Lemma 30.6. *Let $X_i \to Y_i$ be a set of trivial Kan fibrations. Then $\prod X_i \to \prod Y_i$ is a trivial Kan fibration.*

Proof. Omitted. $\qquad\square$

Lemma 30.7. *A filtered colimit of trivial Kan fibrations is a trivial Kan fibration.*

Proof. Omitted. Hint: See description of filtered colimits of sets in Categories, Section 19. $\qquad\square$

Lemma 30.8. *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Then f is a homotopy equivalence.*

Proof. By Lemma 30.2 we can choose an right inverse $g : Y \to X$ to f. Consider the diagram

$$
\begin{array}{ccc}
\partial\Delta[1] \times X & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\Delta[1] \times X & \longrightarrow & Y
\end{array}
$$

Here the top horizontal arrow is given by id_X and $g \circ f$ where we use that $(\partial\Delta[1] \times X)_n = X_n \amalg X_n$ for all $n \geq 0$. The bottom horizontal arrow is given by the map $\Delta[1] \to \Delta[0]$ and $f : X \to Y$. The diagram commutes as $f \circ g \circ f = f$. By Lemma 30.2 we can fill in the dotted arrow and we win. $\qquad\square$

31. Kan fibrations

Let n, k be integers with $0 \leq k \leq n$ and $1 \leq n$. Let $\sigma_0, \ldots, \sigma_n$ be the $n+1$ faces of the unique nondegenerate n-simplex σ of $\Delta[n]$, i.e., $\sigma_i = d_i\sigma$. We let

$$\Lambda_k[n] \subset \Delta[n]$$

be the *kth horn* of the n-simplex $\Delta[n]$. It is the simplicial subset of $\Delta[n]$ generated by $\sigma_0, \ldots, \hat{\sigma}_k, \ldots, \sigma_n$. In other words, the image of the displayed inclusion contains all the nondegenerate simplices of $\Delta[n]$ except for σ and σ_k.

Definition 31.1. A map $X \to Y$ of simplicial sets is called a *Kan fibration* if for all k, n with $1 \leq n$, $0 \leq k \leq n$ and any commutative solid diagram

a dotted arrow exists making the diagram commute. A *Kan complex* is a simplicial set X such that $X \to *$ is a Kan fibration, where $*$ is the constant simplicial set on a singleton.

Note that $\Lambda_k[n]$ is always nonempty. Thus a morphism from the empty simplicial set to any simplicial set is always a Kan fibration. It follows from Lemma 30.2 that a trivial Kan fibration is a Kan fibration.

Lemma 31.2. *Let $f : X \to Y$ be a Kan fibration of simplicial sets. Let $Y' \to Y$ be a morphism of simplicial sets. Then $X \times_Y Y' \to Y'$ is a Kan fibration.*

Proof. This follows immediately from the functorial properties of the fibre product (Lemma 7.2) and the definitions. $\square$

Lemma 31.3. *The composition of two Kan fibrations is a Kan fibration.*

Proof. Omitted. $\square$

Lemma 31.4. *Let $\ldots \to U^2 \to U^1 \to U^0$ be a sequence of Kan fibrations. Let $U = \lim U^t$ defined by taking $U_n = \lim U_n^t$. Then $U \to U^0$ is a Kan fibration.*

Proof. Omitted. Hint: use that for a countable sequence of surjections of sets the inverse limit is nonempty. $\square$

Lemma 31.5. *Let $X_i \to Y_i$ be a set of Kan fibrations. Then $\prod X_i \to \prod Y_i$ is a Kan fibration.*

Proof. Omitted. $\square$

The following lemma is due to J.C. Moore, see [Moo55].

Lemma 31.6. *Let X be a simplicial group. Then X is a Kan complex.*

Proof. The following proof is basically just a translation into English of the proof in the reference mentioned above. Using the terminology as explained in the introduction to this section, suppose $f : \Lambda_k[n] \to X$ is a morphism from a horn. Set $x_i = f(\sigma_i) \in X_{n-1}$ for $i = 0, \ldots, \hat{k}, \ldots, n$. This means that for $i < j$ we have $d_i x_j = d_{j-1} x_i$ whenever $i, j \neq k$. We have to find an $x \in X_n$ such that $x_i = d_i x$ for $i = 0, \ldots, \hat{k}, \ldots, n$.

We first prove there exists a $u \in X_n$ such that $d_i u = x_i$ for $i < k$. This is trivial for $k = 0$. If $k > 0$, one defines by induction an element $u^r \in X_n$ such that $d_i u^r = x_i$ for $0 \leq i \leq r$. Start with $u^0 = s_0 x_0$. If $r < k - 1$, we set

$$y^r = s_{r+1}((d_{r+1} u^r)^{-1} x_{r+1}), \quad u^{r+1} = u^r y^r.$$

An easy calculation shows that $d_i y^r = 1$ (unit element of the group X_{n-1}) for $i \leq r$ and $d_{r+1} y^r = (d_{r+1} u^r)^{-1} x_{r+1}$. It follows that $d_i u^{r+1} = x_i$ for $i \leq r + 1$. Finally, take $u = u^{k-1}$ to get u as promised.

Next we prove, by induction on the integer r, $0 \leq r \leq n - k$, there exists a $x^r \in X_n$ such that
$$d_i x^r = x_i \quad \text{for } i < k \text{ and } i > n - r.$$
Start with $x^0 = u$ for $r = 0$. Having defined x^r for $r \leq n - k - 1$ we set

$$z^r = s_{n-r-1}((d_{n-r} x^r)^{-1} x_{n-r}), \quad x^{r+1} = x^r z^r$$

A simple calculation, using the given relations, shows that $d_i z^r = 1$ for $i < k$ and $i > n - r$ and that $d_{n-r}(z^r) = (d_{n-r} x^r)^{-1} x_{n-r}$. It follows that $d_i x^{r+1} = x_i$ for $i < k$ and $i > n - r - 1$. Finally, we take $x = x^{n-k}$ which finishes the proof. $\square$

Lemma 31.7. *Let $f : X \to Y$ be a homomorphism of simplicial abelian groups which is termwise surjective. Then f is a Kan fibration of simplicial sets.*

Proof. Consider a commutative solid diagram

as in Definition 31.1. The map a corresponds to $x_0, \ldots, \hat{x}_k, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_{j-1} x_i$ for $i < j$, $i, j \neq k$. The map b corresponds to an element $y \in Y_n$ such that $d_i y = f(x_i)$ for $i \neq k$. Our task is to produce an $x \in X_n$ such that $d_i x = x_i$ for $i \neq k$ and $f(x) = y$.

Since f is termwise surjective we can find $x \in X_n$ with $f(x) = y$. Replace y by $0 = y - f(x)$ and x_i by $x_i - d_i x$ for $i \neq k$. Then we see that we may assume $y = 0$. In particular $f(x_i) = 0$. In other words, we can replace X by $\text{Ker}(f) \subset X$ and Y by 0. In this case the statement become Lemma 31.6. $\square$

Lemma 31.8. *Let $f : X \to Y$ be a homomorphism of simplicial abelian groups which is termwise surjective and induces a quasi-isomorphism on associated chain complexes. Then f is a trivial Kan fibration of simplicial sets.*

Proof. Consider a commutative solid diagram

as in Definition 30.1. The map a corresponds to $x_0, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_{j-1} x_i$ for $i < j$. The map b corresponds to an element $y \in Y_n$ such that $d_i y = f(x_i)$. Our task is to produce an $x \in X_n$ such that $d_i x = x_i$ and $f(x) = y$.

Since f is termwise surjective we can find $x \in X_n$ with $f(x) = y$. Replace y by $0 = y - f(x)$ and x_i by $x_i - d_i x$. Then we see that we may assume $y = 0$. In particular $f(x_i) = 0$. In other words, we can replace X by $\mathrm{Ker}(f) \subset X$ and Y by 0. This works, because by Homology, Lemma 13.6 the homology of the chain complex associated to $\mathrm{Ker}(f)$ is zero and hence $\mathrm{Ker}(f) \to 0$ induces a quasi-isomorphism on associated chain complexes.

Since X is a Kan complex (Lemma 31.6) we can find $x \in X_n$ with $d_i x = x_i$ for $i = 0, \ldots, n - 1$. After replacing x_i by $x_i - d_i x$ for $i = 0, \ldots, n$ we may assume that $x_0 = x_1 = \ldots = x_{n-1} = 0$. In this case we see that $d_i x_n = 0$ for $i = 0, \ldots, n - 1$. Thus $x_n \in N(X)_{n-1}$ and lies in the kernel of the differential $N(X)_{n-1} \to N(X)_{n-2}$. Here $N(X)$ is the normalized chain complex associated to X, see Section 23. Since $N(X)$ is quasi-isomorphic to $s(X)$ (Lemma 23.9) and thus acyclic we find $x \in N(X_n)$ whose differential is x_n. This x answers the question posed by the lemma and we are done. $\square$

Lemma 31.9. *Let $f : X \to Y$ be a map of simplicial abelian groups. If f is a homotopy equivalence of simplicial sets, then f induces a quasi-isomorphism of associated chain complexes.*

Proof. In this proof we will write $H_n(Z) = H_n(s(Z)) = H_n(N(Z))$ when Z is a simplicial abelian group, with s and N as in Section 23. Let $\mathbf{Z}[X]$ denote the free abelian group on X viewed as a simplicial set and similarly for $\mathbf{Z}[Y]$. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathbf{Z}[X] & \xrightarrow{\ g\ } & \mathbf{Z}[Y] \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

of simplicial abelian groups. Since taking the free abelian group on a set is a functor, we see that the horizontal arrow is a homotopy equivalence of simplicial abelian groups, see Lemma 28.4. By Lemma 27.2 we see that $H_n(g) : H_n(\mathbf{Z}[X]) \to H_n(\mathbf{Z}[Y])$ is bijective for all $n \geq 0$.

Let $\xi \in H_n(Y)$. By definition of $N(Y)$ we can represent ξ by an element $y \in N(Y_n)$ whose boundary is zero. This means $y \in Y_n$ with $d_0^n(y) = \ldots = d_{n-1}^n(y) = 0$ because $y \in N(Y_n)$ and $d_n^n(y) = 0$ because the boundary of y is zero. Denote $0_n \in Y_n$ the zero element. Then we see that

$$
\tilde{y} = [y] - [0_n] \in (\mathbf{Z}[Y])_n
$$

is an element with $d_0^n(\tilde{y}) = \ldots = d_{n-1}^n(\tilde{y}) = 0$ and $d_n^n(\tilde{y}) = 0$. Thus $\tilde{y}$ is in $N(\mathbf{Z}[Y])_n$ has boundary 0, i.e., $\tilde{y}$ determines a class $\tilde{\xi} \in H_n(\mathbf{Z}[Y])$ mapping to ξ. Because $H_n(\mathbf{Z}[X]) \to H_n(\mathbf{Z}[Y])$ is bijective we can lift $\tilde{\xi}$ to a class in $H_n(\mathbf{Z}[X])$. Looking at the commutative diagram above we see that ξ is in the image of $H_n(X) \to H_n(Y)$.

Let $\xi \in H_n(X)$ be an element mapping to zero in $H_n(Y)$. Exactly as in the previous parapgraph we can represent ξ by an element $x \in N(X_n)$ whose boundary is zero, i.e., $d_0^n(x) = \ldots = d_{n-1}^n(x) = d_n^n(x) = 0$. In particular, we see that $[x] - [0_n]$ is an element of $N(\mathbf{Z}[X])_n$ whose boundary is zero, whence defines a lift $\tilde{\xi} \in H_n(\mathbf{Z}[x])$ of ξ. The fact that ξ maps to zero in $H_n(Y)$ means there exists a $y \in N(Y_{n+1})$ whose

boundary is $f_n(x)$. This means $d_0^{n+1}(y) = \ldots = d_n^{n+1}(y) = 0$ and $d_{n+1}^{n+1}(y) = f(x)$. However, this means exactly that $z = [y] - [0_{n+1}]$ is in $N(\mathbf{Z}[y])_{n+1}$ and

$$g([x] - [0_n]) = [f(x)] - [0_n] = \text{boundary of } z$$

This proves that $\tilde{\xi}$ maps to zero in $H_n(\mathbf{Z}[y])$. As $H_n(\mathbf{Z}[X]) \to H_n(\mathbf{Z}[Y])$ is bijective we conclude $\tilde{\xi} = 0$ and hence $\xi = 0$. $\square$

32. A homotopy equivalence

Suppose that A, B are sets, and that $f : A \to B$ is a map. Consider the associated map of simplicial sets

$$\text{cosk}_0(A) = \left(\ldots A \times A \times A \rightrightarrows A \times A \rightrightarrows A \right)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\text{cosk}_0(B) = \left(\ldots B \times B \times B \rightrightarrows B \times B \rightrightarrows B \right)$$

See Example 19.1. The case $n = 0$ of the following lemma says that this map of simplicial sets is a trivial Kan fibration if f is surjective.

Lemma 32.1. *Let $f : V \to U$ be a morphism of simplicial sets. Let $n \geq 0$ be an integer. Assume*

(1) *The map $f_i : V_i \to U_i$ is a bijection for $i < n$.*
(2) *The map $f_n : V_n \to U_n$ is a surjection.*
(3) *The canonical morphism $U \to \text{cosk}_n\text{sk}_n U$ is an isomorphism.*
(4) *The canonical morphism $V \to \text{cosk}_n\text{sk}_n V$ is an isomorphism.*

Then f is a trivial Kan fibration.

Proof. Consider a solid diagram

$$\begin{array}{ccc} \partial\Delta[k] & \longrightarrow & V \\ \downarrow & \nearrow & \downarrow \\ \Delta[k] & \longrightarrow & U \end{array}$$

as in Definition 30.1. Let $x \in U_k$ be the k-simplex corresponding to the lower horizontal arrow. If $k \leq n$ then the dotted arrow is the one corresponding to a lift $y \in V_k$ of x; the diagram will commute as the other nondegenerate simplices of $\Delta[k]$ are in degrees $< k$ where f is an isomorphism. If $k > n$, then by conditions (3) and (4) we have (using adjointness of skeleton and coskeleton functors)

$$\text{Mor}(\Delta[k], U) = \text{Mor}(\text{sk}_n\Delta[k], \text{sk}_n U) = \text{Mor}(\text{sk}_n\partial\Delta[k], \text{sk}_n U) = \text{Mor}(\partial\Delta[k], U)$$

and similarly for V because $\text{sk}_n\Delta[k] = \text{sk}_n\partial\Delta[k]$ for $k > n$. Thus we obtain a unique dotted arrow fitting into the diagram in this case also. $\square$

Let A, B be sets. Let $f^0, f^1 : A \to B$ be maps of sets. Consider the induced maps $f^0, f^1 : \text{cosk}_0(A) \to \text{cosk}_0(B)$ abusively denoted by the same symbols. The following lemma for $n = 0$ says that f^0 is homotopic to f^1. In fact, there is a homotopy $h : \text{cosk}_0(A) \times \Delta[1] \to \text{cosk}_0(A)$ from f^0 to f^1 with components

$$h_m : A \times \ldots \times A \times \text{Mor}_\Delta([m], [1]) \longrightarrow B \times \ldots \times B,$$
$$(a_0, \ldots, a_m, \alpha) \longmapsto (f^{\alpha(0)}(a_0), \ldots, f^{\alpha(m)}(a_m))$$

To check that this works, note that for a map $\varphi : [k] \to [m]$ the induced maps are $(a_0, \ldots, a_m) \mapsto (a_{\varphi(0)}, \ldots, a_{\varphi(k)})$ and $\alpha \mapsto \alpha \circ \varphi$. Thus $h = (h_m)_{m \geq 0}$ is clearly a map of simplicial sets as desired.

Lemma 32.2. *Let $f^0, f^1 : V \to U$ be maps of simplicial sets. Let $n \geq 0$ be an integer. Assume*

(1) *The maps $f_i^j : V_i \to U_i$, $j = 0, 1$ are equal for $i < n$.*
(2) *The canonical morphism $U \to cosk_n sk_n U$ is an isomorphism.*
(3) *The canonical morphism $V \to cosk_n sk_n V$ is an isomorphism.*

Then f^0 is homotopic to f^1.

First proof. Let W be the n-truncated simplicial set with $W_i = U_i$ for $i < n$ and $W_n = U_n / \sim$ where $\sim$ is the equivalence relation generated by $f^0(y) \sim f^1(y)$ for $y \in V_n$. This makes sense as the morphisms $U(\varphi) : U_n \to U_i$ corresponding to $\varphi : [i] \to [n]$ for $i < n$ factor through the quotient map $U_n \to W_n$ because f^0 and f^1 are morphisms of simplicial sets and equal in degrees $< n$. Next, we upgrade W to a simplicial set by taking $cosk_n W$. By Lemma 32.1 the morphism $g : U \to W$ is a trivial Kan fibration. Observe that $g \circ f^0 = g \circ f^1$ by construction and denote this morphism $f : V \to W$. Consider the diagram

$$
\begin{array}{ccc}
\partial\Delta[1] \times V & \xrightarrow{\quad f^0, f^1 \quad} & U \\
\downarrow & \nearrow & \downarrow \\
\Delta[1] \times V & \xrightarrow{\quad f \quad} & W
\end{array}
$$

By Lemma 30.2 the dotted arrow exists and the proof is done. $\qquad\square$

Second proof. We have to construct a morphism of simplicial sets $h : V \times \Delta[1] \to U$ which recovers f^i on composing with e_i. The case $n = 0$ was dealt with above the lemma. Thus we may assume that $n \geq 1$. The map $\Delta[1] \to cosk_1 sk_1 \Delta[1]$ is an isomorphism, see Lemma 19.15. Thus we see that $\Delta[1] \to cosk_n sk_n \Delta[1]$ is an isomorphism as $n \geq 1$, see Lemma 19.10. And hence $V \times \Delta[1] \to cosk_n sk_n(V \times \Delta[1])$ is an isomorphism too, see Lemma 19.12. In other words, in order to construct the homotopy it suffices to construct a suitable morphism of n-truncated simplicial sets $h : sk_n V \times sk_n \Delta[1] \to sk_n U$.

For $k = 0, \ldots, n-1$ we define h_k by the formula $h_k(v, \alpha) = f^0(v) = f^1(v)$. The map $h_n : V_n \times \mathrm{Mor}_\Delta([k], [1]) \to U_n$ is defined as follows. Pick $v \in V_n$ and $\alpha : [n] \to [1]$:

(1) If $\mathrm{Im}(\alpha) = \{0\}$, then we set $h_n(v, \alpha) = f^0(v)$.
(2) If $\mathrm{Im}(\alpha) = \{0, 1\}$, then we set $h_n(v, \alpha) = f^0(v)$.
(3) If $\mathrm{Im}(\alpha) = \{1\}$, then we set $h_n(v, \alpha) = f^1(v)$.

Let $\varphi : [k] \to [l]$ be a morphism of $\Delta_{\leq n}$. We will show that the diagram

$$
\begin{array}{ccc}
V_l \times \mathrm{Mor}([l], [1]) & \longrightarrow & U_l \\
\downarrow & & \downarrow \\
V_k \times \mathrm{Mor}([k], [1]) & \longrightarrow & U_k
\end{array}
$$

commutes. Pick $v \in V_l$ and $\alpha : [l] \to [1]$. The commutativity means that

$$
h_k(V(\varphi)(v), \alpha \circ \varphi) = U(\varphi)(h_l(v, \alpha)).
$$

In almost every case this holds because $h_k(V(\varphi)(v), \alpha \circ \varphi) = f^0(V(\varphi)(v))$ and $U(\varphi)(h_l(v, \alpha)) = U(\varphi)(f^0(v))$, combined with the fact that f^0 is a morphism of simplicial sets. The only cases where this does not hold is when either (A) $\text{Im}(\alpha) = \{1\}$ and $l = n$ or (B) $\text{Im}(\alpha \circ \varphi) = \{1\}$ and $k = n$. Observe moreover that necessarily $f^0(v) = f^1(v)$ for any degenerate n-simplex of V. Thus we can narrow the cases above down even further to the cases (A) $\text{Im}(\alpha) = \{1\}$, $l = n$ and v nondegenerate, and (B) $\text{Im}(\alpha \circ \varphi) = \{1\}$, $k = n$ and $V(\varphi)(v)$ nondegenerate.

In case (A), we see that also $\text{Im}(\alpha \circ \varphi) = \{1\}$. Hence we see that not only $h_l(v, \alpha) = f^1(v)$ but also $h_k(V(\varphi)(v), \alpha \circ \varphi) = f^1(V(\varphi)(v))$. Thus we see that the relation holds because f^1 is a morphism of simplicial sets.

In case (B) we conclude that $l = k = n$ and φ is bijective, since otherwise $V(\varphi)(v)$ is degenerate. Thus $\varphi = \text{id}_{[n]}$, which is a trivial case. $\qquad \square$

Lemma 32.3. *Let A, B be sets, and that $f : A \to B$ is a map. Consider the simplicial set U with n-simplices*

$$A \times_B A \times_B \ldots \times_B A \ (n+1 \ factors).$$

see Example 3.5. If f is surjective, the morphism $U \to B$ where B indicates the constant simplicial set with value B is a trivial Kan fibration.

Proof. Observe that U fits into a cartesian square

$$
\begin{array}{ccc}
U & \longrightarrow & \text{cosk}_0(A) \\
\downarrow & & \downarrow \\
B & \longrightarrow & \text{cosk}_0(B)
\end{array}
$$

Since the right vertical arrow is a trivial Kan fibration by Lemma 32.1, so is the left by Lemma 30.3. $\qquad \square$

33. Preparation for standard resolutions

The material in this section can be found in [God73, Appendix 1]

Example 33.1. Let $Y : \mathcal{C} \to \mathcal{C}$ be a functor from a category to itself and suppose given transformations of functors

$$d : Y \longrightarrow \text{id}_{\mathcal{C}} \quad \text{and} \quad s : Y \longrightarrow Y \circ Y$$

Using these transformations we can construct something that looks like a simplicial object. Namely, for $n \geq 0$ we define

$$X_n = Y \circ \ldots \circ Y \quad (n+1 \ \text{compositions})$$

Observe that $X_{n+m+1} = X_n \circ X_m$ for $n, m \geq 0$. Next, for $n \geq 0$ and $0 \leq j \leq n$ we define using notation as in Categories, Section 28

$$d_j^n = 1_{X_{j-1}} \star d \star 1_{X_{n-j-1}} : X_n \to X_{n-1} \quad \text{and} \quad s_j^n = 1_{X_{j-1}} \star s \star 1_{X_{n-j-1}} : X_n \to X_{n+1}$$

So d_j^n, resp. s_j^n is the natural transformation using d, resp. s on the jth Y (counted from the left) in the composition defining X_n.

Lemma 33.2. *In Example 33.1 if*

$$1_Y = (d \star 1_Y) \circ s = (1_Y \star d) \circ s \quad and \quad (s \star 1) \circ s = (1 \star s) \circ s$$

then $X = (X_n, d_j^n, s_j^n)$ is a simplicial object in the category of endofunctors of $\mathcal{C}$ and $d : X_0 = Y \to id_{\mathcal{C}}$ defines an augmentation.

Proof. To see that we obtain a simplicial object we have to check that the relations $(1)(a) - (e)$ of Lemma 3.2 are satisfied. We will use the short hand notation

$$1_a = 1_{X_{a-1}} = 1_Y \star \ldots \star 1_Y \quad (a \text{ factors})$$

for $a \geq 0$. With this notation we have

$$d_j^n = 1_j \star d \star 1_{n-j} \quad \text{and} \quad s_j^n = 1_j \star s \star 1_{n-j}$$

We are repeatedly going to use the rule that for transformations of funtors a, a', b, b' we have $(a' \circ a) \star (b' \circ b) = (a' \star b') \circ (a \star b)$ provided that the $\star$ and $\circ$ compositions in this formula make sense, see Categories, Lemma 28.2.

Condition $(1)(a)$ always holds (no conditions needed on d and s). Namely, let $0 \leq i < j \leq n+1$. We have to show that $d_i^n \circ d_j^{n+1} = d_{j-1}^n \circ d_i^{n+1}$, i.e.,

$$(1_i \star d \star 1_{n-i}) \circ (1_j \star d \star 1_{n+1-j}) = (1_{j-1} \star d \star 1_{n+1-j}) \circ (1_i \star d \star 1_{n+1-i})$$

We can rewrite the left hand side as

$$(1_i \star d \star 1_{j-i-1} \star 1_{n+1-j}) \circ (1_i \star 1_1 \star 1_{j-i-1} \star d \star 1_{n+1-j})$$
$$= 1_i \star ((d \star 1_{j-i-1}) \circ (1_1 \star 1_{j-i-1} \star d)) \star 1_{n+1-j}$$
$$= 1_i \star d \star 1_{j-i-1} \star d \star 1_{n+1-j}$$

The second equality is true because $d \circ 1_1 = d$ and $1_{j-i} \circ (1_{j-i-1} \star d) = 1_{j-i-1} \star d$. A similar computation gives the same result for the right hand side.

We check condition $(1)(b)$. Let $0 \leq i < j \leq n-1$. We have to show that $d_i^n \circ s_j^{n-1} = s_{j-1}^{n-2} \circ d_i^{n-1}$, i.e.,

$$(1_i \star d \star 1_{n-i}) \circ (1_j \star s \star 1_{n-1-j}) = (1_{j-1} \star s \star 1_{n-1-j}) \circ (1_i \star d \star 1_{n-1-i})$$

By the same kind of calculus as in case $(1)(a)$ both sides simplify to $1_i \star d \star 1_{j-i-1} \star s \star 1_{n-j-1}$.

We check condition $(1)(c)$. Let $0 \leq j \leq n-1$. We have to show id $= d_j^n \circ s_j^{n-1} = d_{j+1}^n \circ s_j^{n-1}$, i.e.,

$$1_n = (1_j \star d \star 1_{n-j}) \circ (1_j \star s \star 1_{n-1-j}) = (1_{j+1} \star d \star 1_{n-j-1}) \circ (1_j \star s \star 1_{n-1-j})$$

This is easily seen to be implied by the first assumption of the lemma.

We check condition $(1)(d)$. Let $0 < j+1 < i \leq n$. We have to show $d_i^n \circ s_j^{n-1} = s_j^{n-2} \circ d_{i-1}^{n-1}$, i.e.,

$$(1_i \star d \star 1_{n-i}) \circ (1_j \star s \star 1_{n-1-j}) = (1_j \star s \star 1_{n-2-j}) \circ (1_{i-1} \star d \star 1_{n-i})$$

By the same kind of calculus as in case $(1)(a)$ both sides simplify to $1_j \star s \star 1_{i-j-2} \star d \star 1_{n-i}$.

We check condition $(1)(e)$. Let $0 \leq i \leq j \leq n-1$. We have to show that $s_i^n \circ s_j^{n-1} = s_{j+1}^n \circ s_i^{n-1}$, i.e.,

$$(1_i \star s \star 1_{n-i}) \circ (1_j \star s \star 1_{n-1-j}) = (1_{j+1} \star s \star 1_{n-1-j}) \circ (1_i \star s \star 1_{n-1-i})$$

By the same kind of calculus as in case (1)(a) this reduces to

$$(s \star 1_{j-i+1}) \circ (1_{j-i} \star s) = (1_{j-i+1} \star s) \circ (s \star 1_{j-i})$$

If $j = i$ this is exactly one of the two assumptions of the lemma. For $j > i$ left and right hand side both reduce to the equality $s \star 1_{j-i-1} \star s$ by calculations similar to those we did in case (1)(a).

Finally, in order to show that d defines an augmentation we have to show that $d \circ (1_1 \star d) = d \circ (d \star 1_1)$ which is true because both sides are equal to $d \star d$. $\square$

Example 33.3. Let $\mathcal{C}$, Y, d, s be as in Example 33.1 satisfying the equations of Lemma 33.2. Given functors $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{B}$ we obtain a simplicial object $G \circ X \circ F$ in the category of functors from $\mathcal{A}$ to $\mathcal{B}$ which comes with an augmentation to $G \circ F$.

Lemma 33.4. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, Y, d, s, F, G be as in Example 33.3. Given a transformation of functors $h_0 : G \circ F \to G \circ Y \circ F$ such that*

$$1_{G \circ F} = (1_G \star d \star 1_F) \circ h_0$$

Then there is a morphism $h : G \circ F \to G \circ X \circ F$ of simplicial objects such that $\epsilon \circ h = \mathrm{id}$ where $\epsilon : G \circ X \circ F \to G \circ F$ is the augmentation.

Proof. Denote $u_n : Y = X_0 \to X_n$ the map of the simplicial object X corresponding to the unique morphism $[n] \to [0]$ in Δ. Set $h_n : G \circ F \to G \circ X_n \circ F$ equal to $(1_G \star u_n \star 1_F) \circ h_0$.

For any simplicial object $X = (X_n)$ in any category $u = (u_n) : X_0 \to X$ is a morphism from the constant simplicial object on X_0 to X. Hence h is a morphism of simplicial objects because it is the composition of $1_G \star u \star 1_F$ and h_0.

Let us check that $\epsilon \circ h = \mathrm{id}$. We compute

$$\epsilon_n \circ (1_G \star u_n \star 1_F) \circ h_0 = \epsilon_0 \circ h_0 = \mathrm{id}$$

The first equality because ϵ is a morphism of simplicial objects and the second equality because $\epsilon_0 = (1_G \star d \star 1_F)$ and we can apply the assumption in the statement of the lemma. $\square$

Lemma 33.5. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, Y, d, s, F, G be as in Example 33.3. Let $F' : \mathcal{A} \to \mathcal{C}$ and $G' : \mathcal{C} \to \mathcal{B}$ be two functors. Let $(a_n) : G \circ X \to G' \circ X$ be a morphism of simplicial objects compatible via augmentations with $a : G \to G'$. Let $(b_n) : X \circ F \to X \circ F'$ be a morphism of simplicial objects compatible via augmentations with $b : F \to F'$. Then the two maps*

$$a \star (b_n), (a_n) \star b : G \circ X \circ F \to G' \circ X \circ F'$$

are homotopic.

Proof. To show the morphisms are homotopic we construct morphisms

$$h_{n,i} : G \circ X_n \circ F \to G' \circ X_n \circ F'$$

for $n \geq 0$ and $0 \leq i \leq n + 1$ satisfying the relations described in Lemma 26.2. See also Remark 26.4. To satisfy condition (1) of Lemma 26.2 we are forced to set $h_{n,0} = a \star b_n$ and $h_{n,n+1} = a_n \star b$. Thus a logical choice is

$$h_{n,i} = a_{i-1} \star b_{n-i}$$

for $1 \le i \le n$. Setting $a = a_{-1}$ and $b = b_{-1}$ we see the displayed formular holds for $0 \le i \le n+1$.

Recall that
$$d_j^n = 1_G \star 1_j \star d \star 1_{n-j} \star 1_F$$
on $G \circ X \circ F$ where we use the notation $1_a = 1_{Y \circ \ldots \circ Y}$ introduced in the proof of Lemma 33.2. We are going to use below that we can rewrite this as
$$d_j^n = d_j^j \star 1_{n-j} = d_j^{j+1} \star 1_{n-j} = \ldots = d_j^{n-1} \star 1_1$$
$$= 1_j \star d_0^{n-j} = 1_{j-1} \star d_1^{n-j+1} = \ldots = 1_1 \star d_{j-1}^{n-1}$$
Of course we have the analogous formulae for d_j^n on $G' \circ X \circ F'$.

We check condition (2) of Lemma 26.2. Let $i > j$. We have to show
$$d_j^n \circ (a_{i-1} \star b_{n-i}) = (a_{i-2} \star b_{n-i}) \circ d_j^n$$
Since $i - 1 \ge j$ we can use one of the possible descriptions of d_j^n to rewrite the left hand side as
$$(d_j^{i-1} \star 1_{n-i+1}) \circ (a_{i-1} \star b_{n-i}) = (d_j^{i-1} \circ a_{i-1}) \star b_{n-i} = (a_{i-2} \circ d_j^{i-1}) \star b_{n-i}$$
Similarly the right hand side becomes
$$(a_{i-2} \star b_{n-i}) \circ (d_j^{i-1} \star 1_{n-i+1}) = (a_{i-2} \circ d_j^{i-1}) \star b_{n-i}$$
Thus we obtain the same result and (2) is checked.

We check condition (3) of Lemma 26.2. Let $i \le j$. We have to show
$$d_j^n \circ (a_{i-1} \star b_{n-i}) = (a_{i-1} \star b_{n-1-i}) \circ d_j^n$$
Since $j \ge i$ we may rewrite the left hand side as
$$(1_i \star d_{j-i}^{n-i}) \circ (a_{i-1} \star b_{n-i}) = a_{i-1} \star (b_{n-1-i} \circ d_{j-i}^{n-i})$$
A similar manipulation shows this agrees with the right hand side.

Recall that
$$s_j^n = 1_G \star 1_j \star s \star 1_{n-j} \star 1_F$$
on $G \circ X \circ F$. We are going to use below that we can rewrite this as
$$s_j^n = s_j^j \star 1_{n-j} = s_j^{j+1} \star 1_{n-j-1} = \ldots = s_j^{n-1} \star 1_1$$
$$= 1_j \star s_0^{n-j} = 1_{j-1} \star s_1^{n-j+1} = \ldots = 1_1 \star s_{j-1}^{n-1}$$
Of course we have the analogous formulae for s_j^n on $G' \circ X \circ F'$.

We check condition (4) of Lemma 26.2. Let $i > j$. We have to show
$$s_j^n \circ (a_{i-1} \star b_{n-i}) = (a_i \star b_{n-i}) \circ s_j^n$$
Since $i - 1 \ge j$ we can rewrite the left hand side as
$$(s_j^{i-1} \star 1_{n-i+1}) \circ (a_{i-1} \star b_{n-i}) = (s_j^{i-1} \circ a_{i-1}) \star b_{n-i} = (a_i \circ s_j^{i-1}) \star b_{n-i}$$
Similarly the right hand side becomes
$$(a_i \star b_{n-i}) \circ (s_j^{i-1} \star 1_{n-i+1}) = (a_i \circ s_j^{i-1}) \star b_{n-i}$$
as desired.

We check condition (5) of Lemma 26.2. Let $i \le j$. We have to show
$$s_j^n \circ (a_{i-1} \star b_{n-i}) = (a_{i-1} \star b_{n+1-i}) \circ s_j^n$$

This equality holds because both sides evaluate to $a_{i-1} \star (s^{n-i}_{j-i} \circ b_{n-i}) = a_{i-1} \star (b_{n+1-i} \circ s^{n-i}_{j-i})$ by exactly the same arguments as above. $\square$

Lemma 33.6. *Let $\mathcal{C}$, Y, d, s be as in Example 33.1 satisfying the equations of Lemma 33.2. Let $f : \mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}$ be an endomorphism of the identity functor. Then $f \star 1_X, 1_X \star f : X \to X$ are maps of simplicial objects compatible with f via the augmentation $\epsilon : X \to \mathrm{id}_\mathcal{C}$. Moreover, $f \star 1_X$ and $1_X \star f$ are homotopic.*

Proof. The map $f \star 1_X$ is the map with components

$$X_n = \mathrm{id}_\mathcal{C} \circ X_n \xrightarrow{f \star 1_{X_n}} \mathrm{id}_\mathcal{C} \circ X_n = X_n$$

For a transformation $a : F \to G$ of endofunctors of $\mathcal{C}$ we have $a \circ (f \star 1_F) = f \star a = (f \star 1_G) \circ a$. Thus $f \star 1_X$ is indeed a morphism of simplicial objects. Similarly for $1_X \star f$.

To show the morphisms are homotopic we construct morphisms $h_{n,i} : X_n \to X_n$ for $n \geq 0$ and $0 \leq i \leq n+1$ satisfying the relations described in Lemma 26.2. See also Remark 26.4. It turns out we can take

$$h_{n,i} = 1_i \star f \star 1_{n+1-i}$$

where 1_i is the identity transformation on $Y \circ \ldots \circ Y$ as in the proof of Lemma 33.2. We have $h_{n,0} = f \star 1_{X_n}$ and $h_{n,n+1} = 1_{X_n} \star f$ which checks the first condition. In checking the other conditions we use the comments made in the proof of Lemma 33.5 about the maps d^n_j and s^n_j.

We check condition (2) of Lemma 26.2. Let $i > j$. We have to show

$$d^n_j \circ (1_i \star f \star 1_{n+1-i}) = (1_{i-1} \star f \star 1_{n+1-i}) \circ d^n_j$$

Since $i - 1 \geq j$ we can use one of the possible descriptions of d^n_j to rewrite the left hand side as

$$(d^{i-1}_j \star 1_{n-i+1}) \circ (1_i \star f \star 1_{n+1-i}) = d^{i-1}_j \star f \star 1_{n+1-i}$$

Similarly the right hand side becomes

$$(1_{i-1} \star f \star 1_{n+1-i}) \circ (d^{i-1}_j \star 1_{n-i+1}) = d^{i-1}_j \star f \star 1_{n+1-i}$$

Thus we obtain the same result and (2) is checked.

The conditions (3), (4), and (5) of Lemma 26.2 are checked in exactly the same manner using the strategy of the proof of Lemma 33.5. We omit the details[2]. $\square$

34. Standard resolutions

Some of the material in this section can be found in [God73, Appendix 1] and [Ill72, I 1.5].

Situation 34.1. Let $\mathcal{A}$, $\mathcal{S}$ be categories and let $V : \mathcal{A} \to \mathcal{S}$ be a functor with a left adjoint $U : \mathcal{S} \to \mathcal{A}$.

---

[2]When f is invertible it suffices to prove that $(a_n) = 1_X$ and $(b_n) = f^{-1} \star 1_X \star f$ are homotopic. But this follows from Lemma 33.5 because in this case $a = b = 1_{\mathrm{id}_\mathcal{C}}$.

In this very general situation we will construct a simplicial object X in the category of functors from $\mathcal{A}$ to $\mathcal{A}$. We suggest looking at the examples presented later on before reading the text of this section.

For the construction we will use the horizontal composition as defined in Categories, Section 28. The definition of the adjunction morphisms[3]

$$d : U \circ V \to \mathrm{id}_{\mathcal{A}} \quad (\text{counit}) \quad \text{and} \quad \eta : \mathrm{id}_{\mathcal{S}} \to V \circ U \quad (\text{unit})$$

in Categories, Section 24 shows that the compositions

(34.1.1) $\quad V \xrightarrow{\eta \star 1_V} V \circ U \circ V \xrightarrow{1_V \star d} V \quad \text{and} \quad U \xrightarrow{1_U \star \eta} U \circ V \circ U \xrightarrow{d \star 1_U} U$

are the identity morphisms. Here to define the morphism $\eta \star 1_V$ we silently identify V with $\mathrm{id}_{\mathcal{S}} \circ V$ and 1_V stands for $\mathrm{id}_V : V \to V$. We will use this notation and these relations repeatedly in what follows. For $n \geq 0$ we set

$$X_n = (U \circ V)^{\circ(n+1)} = U \circ V \circ U \circ \ldots \circ U \circ V$$

In other words, X_n is the $(n+1)$-fold composition of $U \circ V$ with itself. We also set $X_{-1} = \mathrm{id}_{\mathcal{A}}$. We have $X_{n+m+1} = X_n \circ X_m$ for all $n, m \geq -1$. We will endow this sequence of functors with the structure of a simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ by constructing the morphisms of functors

$$d_j^n : X_n \to X_{n-1}, \quad s_j^n : X_n \to X_{n+1}$$

satisfying the relations displayed in Lemma 2.3. Namely, we set

$$d_j^n = 1_{X_{j-1}} \star d \star 1_{X_{n-j-1}} \quad \text{and} \quad s_j^n = 1_{X_{j-1} \circ U} \star \eta \star 1_{V \circ X_{n-j-1}}$$

Finally, write $\epsilon_0 = d : X_0 \to X_{-1}$.

Lemma 34.2. *In Situation 34.1 the system $X = (X_n, d_j^n, s_j^n)$ is a simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ and ϵ_0 defines an augmentation ϵ from X to the constant simplicial object with value $X_{-1} = \mathrm{id}_{\mathcal{A}}$.*

Proof. Consider $Y = U \circ V : \mathcal{A} \to \mathcal{A}$. We already have the transformation $d : Y = U \circ V \to \mathrm{id}_{\mathcal{A}}$. Let us denote

$$s = 1_U \star \eta \star 1_V : Y = U \circ \mathrm{id}_{\mathcal{S}} \circ V \longrightarrow U \circ V \circ U \circ V = Y \circ Y$$

This places us in the sitation of Example 33.1. It is immediate from the formulas that the X, d_i^n, s_i^n constructed above and the X, s_i^n, s_i^n constructed from Y, d, s in Example 33.1 agree. Thus, according to Lemma 33.2 it suffices to prove that

$$1_Y = (d \star 1_Y) \circ s = (1_Y \star d) \circ s \quad \text{and} \quad (s \star 1) \circ s = (1 \star s) \circ s$$

The first equal sign translates into the equality

$$1_U \star 1_V = (d \star 1_U \star 1_V) \circ (1_U \star \eta \star 1_V)$$

which holds if we have $1_U = (d \star 1_U) \circ (1_U \star \eta)$ which in turn holds by (34.1.1). Similarly for the second equal sign. For the last equation we need to prove

$$(1_U \star \eta \star 1_V \star 1_U \star 1_V) \circ (1_U \star \eta \star 1_V) = (1_U \star 1_V \star 1_U \star \eta \star 1_V) \circ (1_U \star \eta \star 1_V)$$

For this it suffices to prove $(\eta \star 1_V \star 1_U) \circ \eta = (1_V \star 1_U \star \eta) \circ \eta$ which is true because both sides are the same as $\eta \star \eta$. $\quad\square$

---

[3]We can't use ϵ for the counit of the adjunction because we want to use ϵ for the augmentation of our simplicial obejct.

Before reading the proof of the following lemma, we advise the reader to look at the example discussed in Example 34.8 in order to understand the purpose of the lemma.

Lemma 34.3. *In Situation 34.1 the maps*

$$1_V \star \epsilon : V \circ X \to V, \quad and \quad \epsilon \star 1_U : X \circ U \to U$$

are homotopy equivalences.

Proof. As in the proof of Lemma 34.2 we set $Y = U \circ V$ so that we are in the sitation of Example 33.1.

Proof of the first homotopy equivalence. By Lemma 33.4 to construct a map $h : V \to V \circ X$ right inverse to $1_V \star \epsilon$ it suffices to construct a map $h_0 : V \to V \circ Y = V \circ U \circ V$ such that $1_V = (1_V \star d) \circ h_0$. Of course we take $h_0 = \eta \star 1_V$ and the equality holds by (34.1.1). To finish the proof we need to show the two maps

$$(1_V \star \epsilon) \circ h, 1_V \star \mathrm{id}_X : V \circ X \longrightarrow V \circ X$$

are homotopic. This follows immediately from Lemma 33.5 (with $G = G' = V$ and $F = F' = \mathrm{id}_\mathcal{S}$).

The proof of the second homotopy equivalence. By Lemma 33.4 to construct a map $h : U \to X \circ U$ right inverse to $\epsilon \star 1_U$ it suffices to construct a map $h_0 : U \to Y \circ U = U \circ V \circ U$ such that $1_U = (d \star 1_U) \circ h_0$. Of course we take $h_0 = 1_U \star \eta$ and the equality holds by (34.1.1). To finish the proof we need to show the two maps

$$(\epsilon \star 1_U) \circ h, \mathrm{id}_X \star 1_U : X \circ U \longrightarrow X \circ U$$

are homotopic. This follows immediately from Lemma 33.5 (with $G = G' = \mathrm{id}_\mathcal{A}$ and $F = F' = U$). $\square$

Example 34.4. Let R be a ring. As an example of the above we can take $i : \mathrm{Mod}_R \to Sets$ to be the forgetful functor and $F : Sets \to \mathrm{Mod}_R$ to be the functor that associates to a set E the free R-module $R[E]$ on E. For an R-module M the simplicial R-module $X(M)$ will have the following shape

$$X(M) = \left(\ldots \ R[R[R[M]]] \ \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \ R[R[M]] \ \substack{\longrightarrow \\ \longrightarrow} \ R[M] \ \right)$$

which comes with an augmentation towards M. We will also show this augmentation is a homotopy equivalence of sets. By Lemmas 30.8, 31.9, and 31.8 this is equivalent to asking M to be the only nonzero cohomology group of the chain complex associated to the simplicial module $X(M)$.

Example 34.5. Let A be a ring. Let Alg_A be the category of commutative A-algebras. As an example of the above we can take $i : Alg_A \to Sets$ to be the forgetful functor and $F : Sets \to Alg_A$ to be the functor that associates to a set E the polynomial algebra $A[E]$ on E over A. (We apologize for the overlap in notation between this example and Example 34.4.) For an A-algebra B the simplicial A-algebra $X(B)$ will have the following shape

$$X(B) = \left(\ldots \ A[A[A[B]]] \ \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \ A[A[B]] \ \substack{\longrightarrow \\ \longrightarrow} \ A[B] \ \right)$$

which comes with an augmentation towards B. We will also show this augmentation is a homotopy equivalence of sets. By Lemmas 30.8, 31.9, and 31.8 this is equivalent to asking B to be the only nonzero cohomology group of the chain complex of A-modules associated to $X(B)$ viewed as a simplicial A-module.

Example 34.6. In Example 34.4 we have $X_n(M) = R[R[\dots[M]\dots]]$ with $n+1$ brackets. We describe the maps constructed above using a typical element

$$\xi = \sum_i r_i \left[\sum_j r_{ij}[m_{ij}]\right]$$

of $X_1(M)$. The maps $d_0, d_1 : R[R[M]] \to R[M]$ are given by

$$d_0(\xi) = \sum_{i,j} r_i r_{ij}[m_{ij}] \quad \text{and} \quad d_1(\xi) = \sum_i r_i \left[\sum_j r_{ij} m_{ij}\right].$$

The maps $s_0, s_1 : R[R[M]] \to R[R[R[M]]]$ are given by

$$s_0(\xi) = \sum_i r_i \left[\left[\sum_j r_{ij}[m_{ij}]\right]\right] \quad \text{and} \quad s_1(\xi) = \sum_i r_i \left[\sum_j r_{ij}[[m_{ij}]]\right].$$

Example 34.7. In Example 34.5 we have $X_n(B) = A[A[\dots[B]\dots]]$ with $n+1$ brackets. We describe the maps constructed above using a typical element

$$\xi = \sum_i a_i[x_{i,1}]\dots[x_{i,m_i}] \in A[A[B]] = X_1(B)$$

where for each i, j we can write

$$x_{i,j} = \sum a_{i,j,k}[b_{i,j,k,1}]\dots[b_{i,j,k,n_{i,j,k}}] \in A[B]$$

Obviously this is horrendous! To ease the notation, to see what the A-algebra maps $d_0, d_1 : A[A[B]] \to A[B]$ are doing it suffices to see what happens to the variables $[x]$ where

$$x = \sum a_k[b_{k,1}]\dots[b_{k,n_k}] \in A[B]$$

is a general element. For these we get

$$d_0([x]) = x = \sum a_k[b_{k,1}]\dots[b_{k,n_k}] \quad \text{and} \quad d_1([x]) = \left[\sum a_k b_{k,1}\dots b_{k,n_k}\right]$$

The maps $s_0, s_1 : A[A[B]] \to A[A[A[B]]]$ are given by

$$s_0([x]) = \left[\left[\sum a_k[b_{k,1}]\dots[b_{k,n_k}]\right]\right] \quad \text{and} \quad s_1([x]) = \left[\sum a_k[[b_{k,1}]]\dots[[b_{k,n_k}]]\right]$$

Example 34.8. Going back to the example discussed in Example 34.5 our Lemma 34.3 signifies that for any ring map $A \to B$ the map of simplicial rings

$$A[A[A[B]]] \rightrightarrows A[A[B]] \rightrightarrows A[B]$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$B \rightrightarrows B \rightrightarrows B$$

is a homotopy equivalence on underlying simplicial sets. Moreover, the inverse map constructed in Lemma 34.3 is in degree n given by

$$b \longmapsto [\dots[b]\dots]$$

with obvious notation. In the other direction the lemma tells us that for every set E there is a homotopy equivalence

$$A[A[A[A[E]]]] \rightrightarrows A[A[A[E]]] \rightrightarrows A[A[E]]$$
$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$A[E] \rightrightarrows A[E] \rightrightarrows A[E]$$

of rings. The inverse map constructed in the lemma is in degree n given by the ring map

$$\sum a_{e_1,\ldots,e_p}[e_1][e_2]\ldots[e_p] \longmapsto \sum a_{e_1,\ldots,e_p}[\ldots[e_1]\ldots][\ldots[e_2]\ldots]\ldots[\ldots[e_p]\ldots]$$

(with obvious notation).

35. Other chapters

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