# COHOMOLOGY OF SHEAVES

## Contents

## 1. Introduction

In this document we work out some topics on cohomology of sheaves on topological spaces. We mostly work in the generality of modules over a sheaf of rings and we work with morphisms of ringed spaces. To see what happens for sheaves on sites take a look at the chapter Cohomology on Sites, Section 1. Basic references are [God73] and [Ive86].

## 2. Cohomology of sheaves

Let $X$ be a topological space. Let $\mathcal{F}$ be an abelian sheaf. We know that the category of abelian sheaves on $X$ has enough injectives, see Injectives, Lemma 4.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

$$(2.0.1) \qquad\qquad H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet))$$

to be the *ith cohomology group of the abelian sheaf* $\mathcal{F}$. The family of functors $H^i(X, -)$ forms a universal $\delta$-functor from $Ab(X) \to Ab$.

Let $f : X \to Y$ be a continuous map of topological spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$(2.0.2) \qquad\qquad R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet)$$

to be the *ith higher direct image of* $\mathcal{F}$. The family of functors $R^i f_*$ forms a universal $\delta$-functor from $Ab(X) \to Ab(Y)$.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We know that the category of $\mathcal{O}_X$-modules on $X$ has enough injectives, see Injectives, Lemma 5.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

$$(2.0.3) \qquad\qquad H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet))$$

to be the *ith cohomology group of* $\mathcal{F}$. The family of functors $H^i(X, -)$ forms a universal $\delta$-functor from $Mod(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_X(X)}$.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$(2.0.4) \qquad\qquad R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet)$$

to be the *ith higher direct image of* $\mathcal{F}$. The family of functors $R^i f_*$ forms a universal $\delta$-functor from $Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$.

## 3. Derived functors

We briefly explain how to get right derived functors using resolution functors. For the unbounded derived functors, please see Section 28.

Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is abelian, see Modules, Lemma 3.1. In this chapter we will write

$$K(\mathcal{O}_X) = K(Mod(\mathcal{O}_X)) \quad \text{and} \quad D(\mathcal{O}_X) = D(Mod(\mathcal{O}_X)).$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 8.1 and Definition 11.3. By Derived Categories, Remark 24.3 there exists a resolution functor

$$j = j_X : K^+(Mod(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $Mod(\mathcal{O}_X)$ consisting of injective sheaves. For any left exact functor $F : Mod(\mathcal{O}_X) \to \mathcal{B}$ into any abelian category $\mathcal{B}$ we will denote $RF$ the right derived functor described in Derived Categories, Section 20 and constructed using the resolution functor $j_X$ just described:

$$(3.0.1) \qquad\qquad RF = F \circ j'_X : D^+(X) \longrightarrow D^+(\mathcal{B})$$

see Derived Categories, Lemma 25.1 for notation. Note that we may think of $RF$ as defined on $Mod(\mathcal{O}_X)$, $\mathrm{Comp}^+(Mod(\mathcal{O}_X))$, $K^+(X)$, or $D^+(X)$ depending on the situation. According to Derived Categories, Definition 16.2 we obtain the $i$th right derived functor

$$(3.0.2) \qquad\qquad R^i F = H^i \circ RF : Mod(\mathcal{O}_X) \longrightarrow \mathcal{B}$$

so that $R^0 F = F$ and $\{R^i F, \delta\}_{i \geq 0}$ is universal $\delta$-functor, see Derived Categories, Lemma 20.4.

Here are two special cases of this construction. Given a ring $R$ we write $K(R) = K(\mathrm{Mod}_R)$ and $D(R) = D(\mathrm{Mod}_R)$ and similarly for bounded versions. For any open $U \subset X$ we have a left exact functor $\Gamma(U, -) : Mod(\mathcal{O}_X) \longrightarrow \mathrm{Mod}_{\mathcal{O}_X(U)}$ which gives rise to

$$(3.0.3) \qquad\qquad R\Gamma(U, -) : D^+(X) \longrightarrow D^+(\mathcal{O}_X(U))$$

by the discussion above. We set $H^i(U, -) = R^i \Gamma(U, -)$. If $U = X$ we recover (2.0.3). If $f : X \to Y$ is a morphism of ringed spaces, then we have the left exact functor $f_* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Y)$ which gives rise to the *derived pushforward*

$$(3.0.4) \qquad\qquad Rf_* : D^+(X) \longrightarrow D^+(Y)$$

The $i$th cohomology sheaf of $Rf_* \mathcal{F}^\bullet$ is denoted $R^i f_* \mathcal{F}^\bullet$ and called the $i$th *higher direct image* in accordance with (2.0.4). The two displayed functors above are exact functors of derived categories.

**Abuse of notation:** When the functor $Rf_*$, or any other derived functor, is applied to a sheaf $\mathcal{F}$ on $X$ or a complex of sheaves it is understood that $\mathcal{F}$ has been

replaced by a suitable resolution of $\mathcal{F}$. To facilitate this kind of operation we will say, given an object $\mathcal{F}^\bullet \in D(\mathcal{O}_X)$, that a bounded below complex $\mathcal{I}^\bullet$ of injectives of $Mod(\mathcal{O}_X)$ *represents* $\mathcal{F}^\bullet$ *in the derived category* if there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. In the same vein the phrase "let $\alpha : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a morphism of $D(\mathcal{O}_X)$" does not mean that $\alpha$ is represented by a morphism of complexes. If we have an actual morphism of complexes we will say so.

## 4. First cohomology and torsors

**Definition 4.1.** Let $X$ be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $X$. A *torsor*, or more precisely a $\mathcal{G}$-*torsor*, is a sheaf of sets $\mathcal{F}$ on $X$ endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

   (1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive, and
   (2) for every $x \in X$ the stalk $\mathcal{F}_x$ is nonempty.

A *morphism of $\mathcal{G}$-torsors* $\mathcal{F} \to \mathcal{F}'$ is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. The *trivial $\mathcal{G}$-torsor* is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

**Lemma 4.2.** *Let $X$ be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $X$. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\mathcal{F}(X) \neq \emptyset$.*

**Proof.** Omitted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Lemma 4.3.** *Let $X$ be a topological space. Let $\mathcal{H}$ be an abelian sheaf on $X$. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(X, \mathcal{H})$.*

**Proof.** Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \subset X$ open the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of $\sigma$ is generated by the local section of the form $[s] - [s']$. There is a canonical map $a : \mathrm{Ker}(\sigma) \to \mathcal{H}$ which maps $[s] - [s'] \mapsto h$ where $h$ is the local section of $\mathcal{H}$ such that $h \cdot s = s'$. Consider the pushout diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(\sigma) & \longrightarrow & \mathbf{Z}[\mathcal{F}] & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{E} & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0
\end{array}
$$

Here $\mathcal{E}$ is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_\mathcal{F} \in H^1(X, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(X, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(X, \mathcal{H})$ we can associate to $\xi$ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element $\xi$ is the image of a global section $q \in H^0(X, \mathcal{Q})$ because $H^1(X, \mathcal{I}) = 0$ (see Derived Categories, Lemma 20.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to $q$ in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a torsor.

We omit the verification that the two constructions given above are mutually inverse. $\square$

## 5. First cohomology and extensions

**Lemma 5.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. There is a canonical bijection*

$$\mathrm{Ext}^1_{Mod(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F})$$

*which associates to the extension*

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O}_X \to 0$$

*the image of $1 \in \Gamma(X, \mathcal{O}_X)$ in $H^1(X, \mathcal{F})$.*

**Proof.** Let us construct the inverse of the map given in the lemma. Let $\xi \in H^1(X, \mathcal{F})$. Choose an injection $\mathcal{F} \subset \mathcal{I}$ with $\mathcal{I}$ injective in $Mod(\mathcal{O}_X)$. Set $\mathcal{Q} = \mathcal{I}/\mathcal{F}$. By the long exact sequence of cohomology, we see that $\xi$ is the image of a section $\tilde{\xi} \in \Gamma(X, \mathcal{Q}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{Q})$. Now, we just form the pullback

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow{\scriptstyle\tilde{\xi}} & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0
\end{array}
$$

see Homology, Section 6. $\square$

## 6. First cohomology and invertible sheaves

The Picard group of a ringed space is defined in Modules, Section 25.

**Lemma 6.1.** *Let $(X, \mathcal{O}_X)$ be a locally ringed space. There is a canonical isomorphism*

$$H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X).$$

*of abelian groups.*

**Proof.** Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Consider the presheaf $\mathcal{L}^*$ defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s \cdot -} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if $f \in \mathcal{O}_X^*(U)$ and $s \in \mathcal{L}^*(U)$, then clearly $fs \in \mathcal{L}^*(U)$. By the same token, if $s, s' \in \mathcal{L}^*(U)$ then there exists a unique $f \in \mathcal{O}_X^*(U)$ such that $fs = s'$. Moreover, the sheaf $\mathcal{L}^*$ has sections locally by Modules, Lemma 25.4. In other words we see that $\mathcal{L}^*$ is a $\mathcal{O}_X^*$-torsor. Thus we get a map

$$
\begin{array}{ccc}
\text{invertible sheaves on } (X, \mathcal{O}_X) & & \mathcal{O}_X^*\text{-torsors} \\
\text{up to isomorphism} & \longrightarrow & \text{up to isomorphism}
\end{array}
$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 4.3 the right hand side is canonically bijective to $H^1(X, \mathcal{O}_X^*)$. Thus we have to show this map is injective and surjective.

Injective. If the torsor $\mathcal{L}^*$ is trivial, this means by Lemma 4.2 that $\mathcal{L}^*$ has a global section. Hence this means exactly that $\mathcal{L} \cong \mathcal{O}_X$ is the neutral element in $\mathrm{Pic}(X)$.

Surjective. Let $\mathcal{F}$ be an $\mathcal{O}_X^*$-torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}_X(U))/\mathcal{O}_X^*(U)$$

where the action of $f \in \mathcal{O}_X^*(U)$ on $(s, g)$ is $(fs, f^{-1}g)$. Then $\mathcal{L}_1$ is a presheaf of $\mathcal{O}_X$-modules by setting $(s, g) + (s', g') = (s, g + (s'/s)g')$ where $s'/s$ is the local section $f$ of $\mathcal{O}_X^*$ such that $fs = s'$, and $h(s, g) = (s, hg)$ for $h$ a local section of $\mathcal{O}_X$. We omit the verification that the sheafification $\mathcal{L} = \mathcal{L}_1^\#$ is an invertible $\mathcal{O}_X$-module whose associated $\mathcal{O}_X^*$-torsor $\mathcal{L}^*$ is isomorphic to $\mathcal{F}$. $\qquad\square$

## 7. Locality of cohomology

The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an open.

**Lemma 7.1.** *Let $X$ be a ringed space. Let $U \subset X$ be an open subspace.*
 (1) *If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.*
 (2) *For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(U, \mathcal{F}|_U)$.*

**Proof.** Denote $j : U \to X$ the open immersion. Recall that the functor $j^{-1}$ of restriction to $U$ is a right adjoint to the functor $j_!$ of extension by 0, see Sheaves, Lemma 31.8. Moreover, $j_!$ is exact. Hence (1) follows from Homology, Lemma 29.1.

By definition $H^p(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Mod(\mathcal{O}_X)$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in $Mod(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\Gamma(U, \mathcal{I}^\bullet|_U))$. Of course $\Gamma(U, \mathcal{F}) = \Gamma(U, \mathcal{F}|_U)$ for any sheaf $\mathcal{F}$ on $X$. Hence the equality in (2). $\qquad\square$

Let $X$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset V \subset X$ be open subsets. Then there is a canonical *restriction mapping*

(7.1.1) $$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor $H^n$. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}_X$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 11.4.

**Lemma 7.2.** *Let $X$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subspace. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

**Proof.** Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$

is exact in degree $n$. Hence $\mathrm{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \mathrm{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since $\tilde{\xi}$ is a section of the kernel sheaf over $U$ we conclude there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\tilde{\xi}|_{U_i}$ is the image under $d$ of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$. By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. $\qquad\square$

**Lemma 7.3.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. The sheaves $R^i f_* \mathcal{F}$ are the sheaves associated to the presheaves*

$$V \longmapsto H^i(f^{-1}(V), \mathcal{F})$$

*with restriction mappings as in Equation (7.1.1). There is a similar statement for $R^i f_*$ applied to a bounded below complex $\mathcal{F}^\bullet$.*

**Proof.** Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^i f_* \mathcal{F}$ is by definition the $i$th cohomology sheaf of the complex

$$f_* \mathcal{I}^0 \to f_* \mathcal{I}^1 \to f_* \mathcal{I}^2 \to \dots$$

By definition of the abelian category structure on $\mathcal{O}_Y$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_* \mathcal{I}^i(V) \to f_* \mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_* \mathcal{I}^{i-1}(V) \to f_* \mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(f^{-1}(V)) \to \mathcal{I}^{i+1}(f^{-1}(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(f^{-1}(V)) \to \mathcal{I}^i(f^{-1}(V)))}$$

which is equal to $H^i(f^{-1}(V), \mathcal{F})$ and we win. $\qquad\square$

**Lemma 7.4.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $V \subset Y$ be an open subspace. Denote $g : f^{-1}(V) \to V$ the restriction of $f$. Then we have*

$$R^p g_*(\mathcal{F}|_{f^{-1}(V)}) = (R^p f_* \mathcal{F})|_V$$

*There is a similar statement for the derived image $Rf_* \mathcal{F}^\bullet$ where $\mathcal{F}^\bullet$ is a bounded below complex of $\mathcal{O}_X$-modules.*

**Proof.** First proof. Apply Lemmas 7.3 and 7.1 to see the displayed equality. Second proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and use that $\mathcal{F}|_{f^{-1}(V)} \to \mathcal{I}^\bullet|_{f^{-1}(V)}$ is an injective resolution also. $\qquad\square$

**Remark 7.5.** Here is a different approach to the proofs of Lemmas 7.2 and 7.3 above. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i_X : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X)$ be the inclusion functor and let $\#$ be the sheafification functor. Recall that $i_X$ is left exact and $\#$ is exact.

(1) First prove Lemma 11.4 below which says that the right derived functors of $i_X$ are given by $R^p i_X \mathcal{F} = \underline{H}^p(\mathcal{F})$. Here is another proof: The equality is clear for $p = 0$. Both $(R^p i_X)_{p \geq 0}$ and $(\underline{H}^p)_{p \geq 0}$ are delta functors vanishing on injectives, hence both are universal, hence they are isomorphic. See Homology, Section 12.

(2) A restatement of Lemma 7.2 is that $(\underline{H}^p(\mathcal{F}))^\# = 0$, $p > 0$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. To see this is true, use that $\#$ is exact so

$$(\underline{H}^p(\mathcal{F}))^\# = (R^p i_X \mathcal{F})^\# = R^p(\# \circ i_X)(\mathcal{F}) = 0$$

because $\# \circ i_X$ is the identity functor.

(3) Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The presheaf $V \mapsto H^p(f^{-1}V, \mathcal{F})$ is equal to $R^p(i_Y \circ f_*)\mathcal{F}$. You can prove this by noticing that both give universal delta functors as in the argument of (1) above. Hence Lemma 7.3 says that $R^p f_* \mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$. Again using that $\#$ is exact a that $\# \circ i_Y$ is the identity functor we see that

$$R^p f_* \mathcal{F} = R^p(\# \circ i_Y \circ f_*)\mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$$

as desired.

## 8. Mayer-Vietoris

Below will construct the Čech-to-cohomology spectral sequence, see Lemma 11.5. A special case of that spectral sequence is the Mayer-Vietoris long exact sequence. Since it is such a basic, useful and easy to understand variant of the spectral sequence we treat it here separately.

**Lemma 8.1.** *Let $X$ be a ringed space. Let $U' \subset U \subset X$ be open subspaces. For any injective $\mathcal{O}_X$-module $\mathcal{I}$ the restriction mapping $\mathcal{I}(U) \to \mathcal{I}(U')$ is surjective.*

**Proof.** Let $j : U \to X$ and $j' : U' \to X$ be the open immersions. Recall that $j_! \mathcal{O}_U$ is the extension by zero of $\mathcal{O}_U = \mathcal{O}_X|_U$, see Sheaves, Section 31. Since $j_!$ is a left adjoint to restriction we see that for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules

$$\mathrm{Hom}_{\mathcal{O}_X}(j_! \mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

see Sheaves, Lemma 31.8. Similarly, the sheaf $j'_! \mathcal{O}_{U'}$ represents the functor $\mathcal{F} \mapsto \mathcal{F}(U')$. Moreover there is an obvious canonical map of $\mathcal{O}_X$-modules

$$j'_! \mathcal{O}_{U'} \longrightarrow j_! \mathcal{O}_U$$

which corresponds to the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ via Yoneda's lemma (Categories, Lemma 3.5). By the description of the stalks of the sheaves $j'_! \mathcal{O}_{U'}$, $j_! \mathcal{O}_U$ we see that the displayed map above is injective (see lemma cited above). Hence if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then the map

$$\mathrm{Hom}_{\mathcal{O}_X}(j_! \mathcal{O}_U, \mathcal{I}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(j'_! \mathcal{O}_{U'}, \mathcal{I})$$

is surjective, see Homology, Lemma 27.2. Putting everything together we obtain the lemma. $\square$

**Lemma 8.2** (Mayer-Vietoris). *Let $X$ be a ringed space. Suppose that $X = U \cup V$ is a union of two open subsets. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact cohomology sequence*

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \dots$$

*This long exact sequence is functorial in $\mathcal{F}$.*

**Proof.** The sheaf condition says that the kernel of $(1, -1) : \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$ is equal to the image of $\mathcal{F}(X)$ by the first map for any abelian sheaf $\mathcal{F}$. Lemma 8.1 above implies that the map $(1, -1) : \mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V)$ is

surjective whenever $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. Hence if $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$, then we get a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

Taking cohomology gives the result (use Homology, Lemma 13.12). We omit the proof of the functoriality of the sequence. $\square$

**Lemma 8.3** (Relative Mayer-Vietoris). *Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \cap V} : U \cap V \to Y$. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U \cap V}) \to R^1 f_*\mathcal{F} \to \dots$$

*This long exact sequence is functorial in $\mathcal{F}$.*

**Proof.** Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. We claim that we get a short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \cap V} \to 0.$$

Namely, for any open $W \subset Y$, and for any $n \geq 0$ the corresponding sequence of groups of sections over $W$

$$0 \to \mathcal{I}^n(f^{-1}(W)) \to \mathcal{I}^n(U \cap f^{-1}(W)) \oplus \mathcal{I}^n(V \cap f^{-1}(W)) \to \mathcal{I}^n(U \cap V \cap f^{-1}(W)) \to 0$$

was shown to be short exact in the proof of Lemma 8.2. The lemma follows by taking cohomology sheaves and using the fact that $\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{F}|_U$ and similarly for $\mathcal{I}^\bullet|_V$, $\mathcal{I}^\bullet|_{U \cap V}$ see Lemma 7.1. $\square$

## 9. The Čech complex and Čech cohomology

Let $X$ be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering, see Topology, Basic notion (13). As is customary we denote $U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$ for the $(p+1)$-fold intersection of members of $\mathcal{U}$. Let $\mathcal{F}$ be an abelian presheaf on $X$. Set

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \dots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0 \dots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \dots i_p}$ its value in $\mathcal{F}(U_{i_0 \dots i_p})$. Note that if $s \in \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{F})$ and $i, j \in I$ then $s_{ij}$ and $s_{ji}$ are both elements of $\mathcal{F}(U_i \cap U_j)$ but there is no imposed relation between $s_{ij}$ and $s_{ji}$. In other words, we are *not* working with alternating cochains (these will be defined in Section 23). We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

(9.0.1) $$d(s)_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0 \dots \hat{i}_j \dots i_{p+1}}|_{U_{i_0 \dots i_{p+1}}}$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

**Definition 9.1.** Let $X$ be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on $X$. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the *Čech complex* associated to $\mathcal{F}$ and the open covering $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the *Čech cohomology groups* associated to $\mathcal{F}$ and the covering $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

**Lemma 9.2.** *Let $X$ be a topological space. Let $\mathcal{F}$ be an abelian presheaf on $X$. The following are equivalent*

(1) *$\mathcal{F}$ is an abelian sheaf and*
(2) *for every open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ the natural map*
$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$

*is bijective.*

**Proof.** This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every open covering. $\qquad\square$

**Lemma 9.3.** *Let $X$ be a topological space. Let $\mathcal{F}$ be an abelian presheaf on $X$. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. If $U_i = U$ for some $i \in I$, then the extended Čech complex*
$$\mathcal{F}(U) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$
*obtained by putting $\mathcal{F}(U)$ in degree $-1$ with differential given by the canonical map of $\mathcal{F}(U)$ into $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F})$ is homotopy equivalent to 0.*

**Proof.** Fix an element $i \in I$ with $U = U_i$. Observe that $U_{i_0 \ldots i_p} = U_{i_0 \ldots \hat{i}_j \ldots i_p}$ if $i_j = i$. Let us define a homotopy
$$h : \prod_{i_0 \ldots i_{p+1}} \mathcal{F}(U_{i_0 \ldots i_{p+1}}) \longrightarrow \prod_{i_0 \ldots i_p} \mathcal{F}(U_{i_0 \ldots i_p})$$
by the rule
$$h(s)_{i_0 \ldots i_p} = s_{i i_0 \ldots i_p}$$
In other words, $h : \prod_{i_0} \mathcal{F}(U_{i_0}) \to \mathcal{F}(U)$ is projection onto the factor $\mathcal{F}(U_i) = \mathcal{F}(U)$ and in general the map $h$ equals the projection onto the factors $\mathcal{F}(U_{i i_1 \ldots i_{p+1}}) = \mathcal{F}(U_{i_1 \ldots i_{p+1}})$. We compute
$$
\begin{aligned}
(dh + hd)(s)_{i_0 \ldots i_p} &= \sum_{j=0}^p (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + d(s)_{i i_0 \ldots i_p} \\
&= \sum_{j=0}^p (-1)^j s_{i i_0 \ldots \hat{i}_j \ldots i_p} + s_{i_0 \ldots i_p} + \sum_{j=0}^p (-1)^{j+1} s_{i i_0 \ldots \hat{i}_j \ldots i_p} \\
&= s_{i_0 \ldots i_p}
\end{aligned}
$$

This proves the identity map is homotopic to zero as desired. $\qquad\square$

## 10. Čech cohomology as a functor on presheaves

Warning: In this section we work almost exclusively with presheaves and categories of presheaves and the results are completely wrong in the setting of sheaves and categories of sheaves!

Let $X$ be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules. We have the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ just by thinking of $\mathcal{F}$ as a presheaf of abelian groups. However, each term $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ has a natural structure of a $\mathcal{O}_X(U)$-module and the differential is given by $\mathcal{O}_X(U)$-module maps. Moreover, it is clear that the construction
$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$
is functorial in $\mathcal{F}$. In fact, it is a functor

(10.0.1)          $\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : PMod(\mathcal{O}_X) \longrightarrow \mathrm{Comp}^+(\mathrm{Mod}_{\mathcal{O}_X(U)})$

see Derived Categories, Definition 8.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 13.9.

**Lemma 10.1.** *The functor given by Equation (10.0.1) is an exact functor (see Homology, Lemma 7.2).*

**Proof.** For any open $W \subset U$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $PMod(\mathcal{O}_X)$ to $\mathrm{Mod}_{\mathcal{O}_X(U)}$. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. $\square$

**Lemma 10.2.** *Let $X$ be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a $\delta$-functor from the abelian category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules (see Homology, Definition 12.1).*

**Proof.** By Lemma 10.1 a short exact sequence of presheaves of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathcal{O}_X(U)$-modules. Hence we can use Homology, Lemma 13.12 to get the boundary maps $\delta_{\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. $\square$

In the formulation of the following lemma we use the functor $j_{p!}$ of extension by 0 for presheaves of modules relative to an open immersion $j : U \to X$. See Sheaves, Section 31. For any open $W \subset X$ and any presheaf $\mathcal{G}$ of $\mathcal{O}_X|_U$-modules we have

$$(j_{p!}\mathcal{G})(W) = \begin{cases} \mathcal{G}(W) & \text{if } W \subset U \\ 0 & \text{else.} \end{cases}$$

Moreover, the functor $j_{p!}$ is a left adjoint to the restriction functor see Sheaves, Lemma 31.8. In particular we have the following formula

$$\mathrm{Hom}_{\mathcal{O}_X}(j_{p!}\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U).$$

Since the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor on the category of presheaves we conclude that the presheaf $j_{p!}\mathcal{O}_U$ is a projective object in the category $PMod(\mathcal{O}_X)$, see Homology, Lemma 28.2.

Note that if we are given open subsets $U \subset V \subset X$ with associated open immersions $j_U, j_V$, then we have a canonical map $(j_U)_{p!}\mathcal{O}_U \to (j_V)_{p!}\mathcal{O}_V$. It is the identity on sections over any open $W \subset U$ and 0 else. In terms of the identification $\mathrm{Hom}_{\mathcal{O}_X}((j_U)_{p!}\mathcal{O}_U, (j_V)_{p!}\mathcal{O}_V) = (j_V)_{p!}\mathcal{O}_V(U) = \mathcal{O}_V(U)$ it corresponds to the element $1 \in \mathcal{O}_V(U)$.

**Lemma 10.3.** *Let $X$ be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Denote $j_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to X$ the open immersion. Consider the chain complex $K(\mathcal{U})_\bullet$ of presheaves of $\mathcal{O}_X$-modules*

$$\ldots \to \bigoplus_{i_0 i_1 i_2} (j_{i_0 i_1 i_2})_{p!}\mathcal{O}_{U_{i_0 i_1 i_2}} \to \bigoplus_{i_0 i_1} (j_{i_0 i_1})_{p!}\mathcal{O}_{U_{i_0 i_1}} \to \bigoplus_{i_0} (j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to 0 \to \ldots$$

*where the last nonzero term is placed in degree $0$ and where the map*

$$(j_{i_0 \ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0 \ldots i_{p+1}}} \longrightarrow (j_{i_0 \ldots \hat{i}_j \ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}}$$

*is given by $(-1)^j$ times the canonical map. Then there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_{\bullet}, \mathcal{F}) = \check{\mathcal{C}}^{\bullet}(\mathcal{U}, \mathcal{F})$$

*functorial in $\mathcal{F} \in \mathrm{Ob}(PMod(\mathcal{O}_X))$.*

**Proof.** We saw in the discussion just above the lemma that

$$\mathrm{Hom}_{\mathcal{O}_X}((j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}, \mathcal{F}) = \mathcal{F}(U_{i_0\ldots i_p}).$$

Hence we see that it is indeed the case that the direct sum

$$\bigoplus\nolimits_{i_0\ldots i_p} (j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}$$

represents the functor

$$\mathcal{F} \longmapsto \prod\nolimits_{i_0\ldots i_p} \mathcal{F}(U_{i_0\ldots i_p}).$$

Hence by Categories, Yoneda Lemma 3.5 we see that there is a complex $K(\mathcal{U})_{\bullet}$ with terms as given. It is a simple matter to see that the maps are as given in the lemma.                                                                                        $\square$

**Lemma 10.4.** *Let $X$ be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Let $\mathcal{O}_{\mathcal{U}} \subset \mathcal{O}_X$ be the image presheaf of the map $\bigoplus j_{p!}\mathcal{O}_{U_i} \to \mathcal{O}_X$. The chain complex $K(\mathcal{U})_{\bullet}$ of presheaves of Lemma 10.3 above has homology presheaves*

$$H_i(K(\mathcal{U})_{\bullet}) = \begin{cases} 0 & if \quad i \neq 0 \\ \mathcal{O}_{\mathcal{U}} & if \quad i = 0 \end{cases}$$

**Proof.** Consider the extended complex $K_{\bullet}^{ext}$ one gets by putting $\mathcal{O}_{\mathcal{U}}$ in degree $-1$ with the obvious map $K(\mathcal{U})_0 = \bigoplus_{i_0}(j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to \mathcal{O}_{\mathcal{U}}$. It suffices to show that taking sections of this extended complex over any open $W \subset X$ leads to an acyclic complex. In fact, we claim that for every $W \subset X$ the complex $K_{\bullet}^{ext}(W)$ is homotopy equivalent to the zero complex. Write $I = I_1 \amalg I_2$ where $W \subset U_i$ if and only if $i \in I_1$.

If $I_1 = \emptyset$, then the complex $K_{\bullet}^{ext}(W) = 0$ so there is nothing to prove.

If $I_1 \neq \emptyset$, then $\mathcal{O}_{\mathcal{U}}(W) = \mathcal{O}_X(W)$ and

$$K_p^{ext}(W) = \bigoplus\nolimits_{i_0\ldots i_p \in I_1} \mathcal{O}_X(W).$$

This is true because of the simple description of the presheaves $(j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}$. Moreover, the differential of the complex $K_{\bullet}^{ext}(W)$ is given by

$$d(s)_{i_0\ldots i_p} = \sum\nolimits_{j=0,\ldots,p+1} \sum\nolimits_{i \in I_1} (-1)^j s_{i_0\ldots i_{j-1}ii_j\ldots i_p}.$$

The sum is finite as the element $s$ has finite support. Fix an element $i_{\mathrm{fix}} \in I_1$. Define a map

$$h : K_p^{ext}(W) \longrightarrow K_{p+1}^{ext}(W)$$

by the rule

$$h(s)_{i_0\ldots i_{p+1}} = \begin{cases} 0 & if \quad i_0 \neq i_{\mathrm{fix}} \\ s_{i_1\ldots i_{p+1}} & if \quad i_0 = i_{\mathrm{fix}} \end{cases}$$

We will use the shorthand $h(s)_{i_0\ldots i_{p+1}} = (i_0 = i_{\text{fix}})s_{i_1\ldots i_p}$ for this. Then we compute

$$
(dh + hd)(s)_{i_0\ldots i_p}
$$
$$
= \sum_j \sum_{i\in I_1} (-1)^j h(s)_{i_0\ldots i_{j-1}ii_j\ldots i_p} + (i = i_0)d(s)_{i_1\ldots i_p}
$$
$$
= s_{i_0\ldots i_p} + \sum_{j\geq 1}\sum_{i\in I_1}(-1)^j (i_0 = i_{\text{fix}})s_{i_1\ldots i_{j-1}ii_j\ldots i_p} + (i_0 = i_{\text{fix}})d(s)_{i_1\ldots i_p}
$$

which is equal to $s_{i_0\ldots i_p}$ as desired. $\qquad\square$

**Lemma 10.5.** *Let $X$ be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering of $U \subset X$. The Čech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a $\delta$-functor to the right derived functors of the functor*

$$
\check{H}^0(\mathcal{U}, -) : PMod(\mathcal{O}_X) \longrightarrow Mod_{\mathcal{O}_X(U)}.
$$

*Moreover, there is a functorial quasi-isomorphism*

$$
\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})
$$

*where the right hand side indicates the right derived functor*

$$
R\check{H}^0(\mathcal{U}, -) : D^+(PMod(\mathcal{O}_X)) \longrightarrow D^+(\mathcal{O}_X(U))
$$

*of the left exact functor $\check{H}^0(\mathcal{U}, -)$.*

**Proof.** Note that the category of presheaves of $\mathcal{O}_X$-modules has enough injectives, see Injectives, Proposition 8.5. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 20.

Let $\mathcal{I}$ be a injective presheaf of $\mathcal{O}_X$-modules. In this case the functor $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is exact on $PMod(\mathcal{O}_X)$. By Lemma 10.3 we have

$$
\operatorname{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).
$$

By Lemma 10.4 we have that $K(\mathcal{U})_\bullet$ is quasi-isomorphic to $\mathcal{O}_\mathcal{U}[0]$. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the $\delta$-functor $(\check{H}^n, \delta)$ (see Lemma 10.2) satisfies the assumptions of Homology, Lemma 12.4, and hence is a universal $\delta$-functor.

By Derived Categories, Lemma 20.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal $\delta$-functor. By the uniqueness of universal $\delta$-functors, see Homology, Lemma 12.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any presheaf of $\mathcal{O}_X$-modules. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $PMod(\mathcal{O}_X)$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Consider the associated total complex $\operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$, see Homology, Definition 18.3. There is a map of complexes

$$
\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))
$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$
\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow \operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))
$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 25.4. Namely, the columns of the double complex are exact in positive degrees because the Čech complex as a functor is exact (Lemma 10.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Čech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathcal{O}_X(U))$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. $\square$

## 11. Čech cohomology and cohomology

**Lemma 11.1.** *Let $X$ be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

**Proof.** An injective $\mathcal{O}_X$-module is also injective as an object in the category $PMod(\mathcal{O}_X)$ (for example since sheafification is an exact left adjoint to the inclusion functor, using Homology, Lemma 29.1). Hence we can apply Lemma 10.5 (or its proof) to see the result. $\square$

**Lemma 11.2.** *Let $X$ be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. There is a transformation*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

*of functors $Mod(\mathcal{O}_X) \to D^+(\mathcal{O}_X(U))$. In particular this provides canonical maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $Mod(\mathcal{O}_X)$.*

**Proof.** Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 25.4 to see that $\alpha$ is a quasi-isomorphism. Namely, Lemma 11.1 implies that the $q$th row of the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence $\alpha$ becomes invertible in $D^+(\mathcal{O}_X(U))$ and the transformation of the lemma is the composition of $\beta$ followed by the inverse of $\alpha$. We omit the verification that this is functorial. $\square$

**Lemma 11.3.** *Let $X$ be a topological space. Let $\mathcal{H}$ be an abelian sheaf on $X$. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. The map*

$$\check{H}^1(\mathcal{U}, \mathcal{H}) \longrightarrow H^1(X, \mathcal{H})$$

*is injective and identifies $\check{H}^1(\mathcal{U}, \mathcal{H})$ via the bijection of Lemma 4.3 with the set of isomorphism classes of $\mathcal{H}$-torsors which restrict to trivial torsors over each $U_i$.*

**Proof.** To see this we construct an inverse map. Namely, let $\mathcal{F}$ be a $\mathcal{H}$-torsor whose restriction to $U_i$ is trivial. By Lemma 4.2 this means there exists a section $s_i \in \mathcal{F}(U_i)$. On $U_{i_0} \cap U_{i_1}$ there is a unique section $s_{i_0 i_1}$ of $\mathcal{H}$ such that $s_{i_0 i_1} \cdot s_{i_0}|_{U_{i_0} \cap U_{i_1}} = s_{i_1}|_{U_{i_0} \cap U_{i_1}}$. A computation shows that $s_{i_0 i_1}$ is a Čech cocycle and

that its class is well defined (i.e., does not depend on the choice of the sections $s_i$). The inverse maps the isomorphism class of $\mathcal{F}$ to the cohomology class of the cocycle $(s_{i_0 i_1})$. We omit the verification that this map is indeed an inverse.     $\square$

**Lemma 11.4.** *Let $X$ be a ringed space. Consider the functor $i : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X)$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

*see discussion in Section 7.*

**Proof.** It is clear that $i$ is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the $p$th cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an open $U$ are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^p(U) \to \mathcal{I}^{p+1}(U))}{\mathrm{Im}(\mathcal{I}^{p-1}(U) \to \mathcal{I}^p(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$.     $\square$

**Lemma 11.5.** *Let $X$ be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

*converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.*

**Proof.** This is a Grothendieck spectral sequence (see Derived Categories, Lemma 22.2) for the functors

$$i : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : PMod(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_X(U)}.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 9.2. We have that $i(\mathcal{I})$ is Čech acyclic by Lemma 11.1. And we have that $\check{H}^p(\mathcal{U}, -) = R^p \check{H}^0(\mathcal{U}, -)$ as functors on $PMod(\mathcal{O}_X)$ by Lemma 10.5. Putting everything together gives the lemma.     $\square$

**Lemma 11.6.** *Let $X$ be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that $H^i(U_{i_0 \ldots i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$ as $\mathcal{O}_X(U)$-modules.*

**Proof.** We will use the spectral sequence of Lemma 11.5. The assumptions mean that $E_2^{p,q} = 0$ for all $(p,q)$ with $q \neq 0$. Hence the spectral sequence degenerates at $E_2$ and the result follows.     $\square$

**Lemma 11.7.** *Let $X$ be a ringed space. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

*be a short exact sequence of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subset. If there exists a cofinal system of open coverings $\mathcal{U}$ of $U$ such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.*

**Proof.** Take an element $s \in \mathcal{H}(U)$. Choose an open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0 i_1}} - s_{i_0}|_{U_{i_0 i_1}}.$$

Since $s_i$ lifts $s$ we see that $s_{i_0 i_1} \in \mathcal{F}(U_{i_0 i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0 i_1}} - t_{i_0}|_{U_{i_0 i_1}}.$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathcal{G}$ over $U$ which maps to $s$. Hence we win. $\qquad\square$

**Lemma 11.8.** *Let $X$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that*
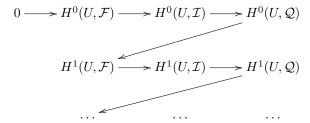
$$\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$$

*for all $p > 0$ and any open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ of an open of $X$. Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any open $U \subset X$.*

**Proof.** Let $\mathcal{F}$ be a sheaf satisfying the assumption of the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any open covering". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 11.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any open covering. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 11.7 and our assumptions this sequence is actually exact as a sequence of presheaves! In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U}$, see Lemma 10.2 for example. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all open coverings.

Next, we look at the long exact cohomology sequence

$$
\begin{array}{ccccc}
0 \longrightarrow H^0(U, \mathcal{F}) \longrightarrow H^0(U, \mathcal{I}) \longrightarrow & H^0(U, \mathcal{Q}) \\
\\
H^1(U, \mathcal{F}) \longrightarrow H^1(U, \mathcal{I}) \longrightarrow & H^1(U, \mathcal{Q}) \\
\\
\cdots \qquad\qquad \cdots \qquad\qquad \cdots
\end{array}
$$

for any open $U \subset X$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. $\qquad\square$

**Lemma 11.9.** *(Variant of Lemma 11.8.) Let $X$ be a ringed space. Let $\mathcal{B}$ be a basis for the topology on $X$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there exists a set of open coverings Cov with the following properties:*

    (1) *For every $\mathcal{U} \in$ Cov with $\mathcal{U} : U = \bigcup_{i \in I} U_i$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0 \ldots i_p} \in \mathcal{B}$.*

    (2) *For every $U \in \mathcal{B}$ the open coverings of $U$ occurring in Cov is a cofinal system of open coverings of $U$.*

    (3) *For every $\mathcal{U} \in$ Cov we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

*Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.*

**Proof.** Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 11.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 11.7 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in$ Cov we get a short exact sequence of Čech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U} \in$ Cov. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in$ Cov.

Next, we look at the long exact cohomology sequence

$$
\begin{array}{ccccc}
0 \longrightarrow H^0(U,\mathcal{F}) \longrightarrow H^0(U,\mathcal{I}) \longrightarrow & H^0(U,\mathcal{Q}) \\
\\
H^1(U,\mathcal{F}) \longrightarrow H^1(U,\mathcal{I}) \longrightarrow & H^1(U,\mathcal{Q}) \\
\\
\ldots \qquad\qquad \ldots \qquad\qquad \ldots
\end{array}
$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in$ Cov we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. $\qquad\square$

**Lemma 11.10.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

(1) *$\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ for all $p > 0$ and any open covering $\mathcal{V} : V = \bigcup_{j \in J} V_j$ of $Y$.*
(2) *$H^p(V, f_*\mathcal{I}) = 0$ for all $p > 0$ and every open $V \subset Y$.*

*In other words, $f_*\mathcal{I}$ is right acyclic for $\Gamma(V, -)$ (see Derived Categories, Definition 15.3) for any $V \subset Y$ open.*

**Proof.** Set $\mathcal{U} : f^{-1}(V) = \bigcup_{j \in J} f^{-1}(V_j)$. It is an open covering of $X$ and

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

This is true because

$$f_*\mathcal{I}(V_{j_0 \ldots j_p}) = \mathcal{I}(f^{-1}(V_{j_0 \ldots j_p})) = \mathcal{I}(f^{-1}(V_{j_0}) \cap \ldots \cap f^{-1}(V_{j_p})) = \mathcal{I}(U_{j_0 \ldots j_p}).$$

Thus the first statement of the lemma follows from Lemma 11.1. The second statement follows from the first and Lemma 11.8. $\qquad\square$

The following lemma implies in particular that $f_* : Ab(X) \to Ab(Y)$ transforms injective abelian sheaves into injective abelian sheaves.

**Lemma 11.11.** *Let $f : X \to Y$ be a morphism of ringed spaces. Assume $f$ is flat. Then $f_*\mathcal{I}$ is an injective $\mathcal{O}_Y$-module for any injective $\mathcal{O}_X$-module $\mathcal{I}$.*

**Proof.** In this case the functor $f^*$ transforms injections into injections (Modules, Lemma 20.2). Hence the result follows from Homology, Lemma 29.1. $\square$

**Lemma 11.12.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $I$ be a set. For $i \in I$ let $\mathcal{F}_i$ be an $\mathcal{O}_X$-module. Let $U \subset X$ be open. The canonical map*

$$H^p(U, \prod\nolimits_{i \in I} \mathcal{F}_i) \longrightarrow \prod\nolimits_{i \in I} H^p(U, \mathcal{F}_i)$$

*is an isomorphism for $p = 0$ and injective for $p = 1$.*

**Proof.** The statement for $p = 0$ is true because the product of sheaves is equal to the product of the underlying presheaves, see Sheaves, Section 29. Proof for $p = 1$. Set $\mathcal{F} = \prod \mathcal{F}_i$. Let $\xi \in H^1(U, \mathcal{F})$ map to zero in $\prod H^1(U, \mathcal{F}_i)$. By locality of cohomology, see Lemma 7.2, there exists an open covering $\mathcal{U} : U = \bigcup U_j$ such that $\xi|_{U_j} = 0$ for all $j$. By Lemma 11.3 this means $\xi$ comes from an element $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F})$. Since the maps $\check{H}^1(\mathcal{U}, \mathcal{F}_i) \to H^1(U, \mathcal{F}_i)$ are injective for all $i$ (by Lemma 11.3), and since the image of $\xi$ is zero in $\prod H^1(U, \mathcal{F}_i)$ we see that the image $\check{\xi}_i = 0$ in $\check{H}^1(\mathcal{U}, \mathcal{F}_i)$. However, since $\mathcal{F} = \prod \mathcal{F}_i$ we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the product of the complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_i)$, hence by Homology, Lemma 32.1 we conclude that $\check{\xi} = 0$ as desired. $\square$

## 12. Flasque sheaves

Here is the definition.

**Definition 12.1.** Let $X$ be a topological space. We say a presheaf of sets $\mathcal{F}$ is *flasque* or *flabby* if for every $U \subset V$ open in $X$ the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

We will use this terminology also for abelian sheaves and sheaves of modules if $X$ is a ringed space. Clearly it suffices to assume the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for every open $U \subset X$.

**Lemma 12.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Then any injective $\mathcal{O}_X$-module is flasque.*

**Proof.** This is a reformulation of Lemma 8.1. $\square$

**Lemma 12.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Any flasque $\mathcal{O}_X$-module is acyclic for $R\Gamma(X, -)$ as well as $R\Gamma(U, -)$ for any open $U$ of $X$.*

**Proof.** We will prove this using Derived Categories, Lemma 15.6. Since every injective module is flasque we see that we can embed every $\mathcal{O}_X$-module into a flasque module, see Injectives, Lemma 4.1. Thus it suffices to show that given a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

with $\mathcal{F}$, $\mathcal{G}$ flasque, then $\mathcal{H}$ is flasque and the sequence remains short exact after taking sections on any open of $X$. In fact, the second statement implies the first. Thus, let $U \subset X$ be an open subspace. Let $s \in \mathcal{H}(U)$. We will show that we can lift $s$ to a section of $\mathcal{G}$ over $U$. To do this consider the set $T$ of pairs $(V, t)$ where $V \subset U$ is open and $t \in \mathcal{G}(V)$ is a section mapping to $s|_V$ in $\mathcal{H}$. We put a partial

ordering on $T$ by setting $(V, t) \leq (V', t')$ if and only if $V \subset V'$ and $t'|_V = t$. If $(V_\alpha, t_\alpha)$, $\alpha \in A$ is a totally ordered subset of $T$, then $V = \bigcup V_\alpha$ is open and there is a unique section $t \in \mathcal{G}(V)$ restricting to $t_\alpha$ over $V_\alpha$ by the sheaf condition on $\mathcal{G}$. Thus by Zorn's lemma there exists a maximal element $(V, t)$ in $T$. We will show that $V = U$ thereby finishing the proof. Namely, pick any $x \in U$. We can find a small open neighbourhood $W \subset U$ of $x$ and $t' \in \mathcal{G}(W)$ mapping to $s|_W$ in $\mathcal{H}$. Then $t'|_{W \cap V} - t|_{W \cap V}$ maps to zero in $\mathcal{H}$, hence comes from some section $r' \in \mathcal{F}(W \cap V)$. Using that $\mathcal{F}$ is flasque we find a section $r \in \mathcal{F}(W)$ restricting to $r'$ over $W \cap V$. Modifying $t'$ by the image of $r$ we may assume that $t$ and $t'$ restrict to the same section over $W \cap V$. By the sheaf condition of $\mathcal{G}$ we can find a section $\tilde{t}$ of $\mathcal{G}$ over $W \cup V$ restricting to $t$ and $t'$. By maximality of $(V, t)$ we see that $V \cup W = V$. Thus $x \in V$ and we are done. $\square$

The following lemma does not hold for flasque presheaves.

**Lemma 12.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{U} : U = \bigcup U_i$ be an open covering. If $\mathcal{F}$ is flasque, then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.*

**Proof.** The presheaves $\underline{H}^q(\mathcal{F})$ used in the statement of Lemma 11.5 are zero by Lemma 12.3. Hence $\check{H}^p(U, \mathcal{F}) = H^p(U, \mathcal{F}) = 0$ by Lemma 12.3 again. $\square$

**Lemma 12.5.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is flasque, then $R^p f_* \mathcal{F} = 0$ for $p > 0$.*

**Proof.** Immediate from Lemma 7.3 and Lemma 12.3. $\square$

The following lemma can be proved by an elementary induction argument for finite coverings, compare with the discussion of Čech cohomology in [Vak].

**Lemma 12.6.** *Let $X$ be a topological space. Let $\mathcal{F}$ be an abelian sheaf on $X$. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume the restriction mappings $\mathcal{F}(U) \to \mathcal{F}(U')$ are surjective for $U'$ an arbitrary union of opens of the form $U_{i_0 \ldots i_p}$. Then $\check{H}^p(\mathcal{U}, \mathcal{F})$ vanishes for $p > 0$.*

**Proof.** Let $Y$ be the set of nonempty subsets of $I$. We will use the letters $A, B, C, \ldots$ to denote elements of $Y$, i.e., nonempty subsets of $I$. For a finite nonempty subset $J \subset I$ let

$$V_J = \{A \in Y \mid J \subset A\}$$

This means that $V_{\{i\}} = \{A \in Y \mid i \in A\}$ and $V_J = \bigcap_{j \in J} V_{\{j\}}$. Then $V_J \subset V_K$ if and only if $J \supset K$. There is a unique topology on $Y$ such that the collection of subsets $V_J$ is a basis for the topology on $Y$. Any open is of the form

$$V = \bigcup_{t \in T} V_{J_t}$$

for some family of finite subsets $J_t$. If $J_t \subset J_{t'}$ then we may remove $J_{t'}$ from the family without changing $V$. Thus we may assume there are no inclusions among the $J_t$. In this case the minimal elements of $V$ are the sets $A = J_t$. Hence we can read off the family $(J_t)_{t \in T}$ from the open $V$.

We can completely understand open coverings in $Y$. First, because the elements $A \in Y$ are nonempty subsets of $I$ we have

$$Y = \bigcup_{i \in I} V_{\{i\}}$$

To understand other coverings, let $V$ be as above and let $V_s \subset Y$ be an open corresponding to the family $(J_{s,t})_{t \in T_s}$. Then

$$V = \bigcup_{s \in S} V_s$$

if and only if for each $t \in T$ there exists an $s \in S$ and $t_s \in T_s$ such that $J_t = J_{s,t_s}$. Namely, as the family $(J_t)_{t \in T}$ is minimal, the minimal element $A = J_t$ has to be in $V_s$ for some $s$, hence $A \in V_{J_{t_s}}$ for some $t_s \in T_s$. But since $A$ is also minimal in $V_s$ we conclude that $J_{t_s} = J_t$.

Next we map the set of opens of $Y$ to opens of $X$. Namely, we send $Y$ to $U$, we use the rule

$$V_J \mapsto U_J = \bigcap_{i \in J} U_i$$

on the opens $V_J$, and we extend it to arbitrary opens $V$ by the rule

$$V = \bigcup_{t \in T} V_{J_t} \mapsto \bigcup_{t \in T} U_{J_t}$$

The classification of open coverings of $Y$ given above shows that this rule transforms open coverings into open coverings. Thus we obtain an abelian sheaf $\mathcal{G}$ on $Y$ by setting $\mathcal{G}(Y) = \mathcal{F}(U)$ and for $V = \bigcup_{t \in T} V_{J_t}$ setting

$$\mathcal{G}(V) = \mathcal{F}\left(\bigcup_{t \in T} U_{J_t}\right)$$

and using the restriction maps of $\mathcal{F}$.

With these preliminaries out of the way we can prove our lemma as follows. We have an open covering $\mathcal{V} : Y = \bigcup_{i \in I} V_{\{i\}}$ of $Y$. By construction we have an equality

$$\check{C}^\bullet(\mathcal{V}, \mathcal{G}) = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$$

of Čech complexes. Since the sheaf $\mathcal{G}$ is flasque on $Y$ (by our assumption on $\mathcal{F}$ in the statement of the lemma) the vanishing follows from Lemma 12.4.  $\square$

## 13. The Leray spectral sequence

**Lemma 13.1.** *Let $f : X \to Y$ be a morphism of ringed spaces. There is a commutative diagram*

$$
\begin{array}{ccc}
D^+(X) & \xrightarrow{\ R\Gamma(X,-)\ } & D^+(\mathcal{O}_X(X)) \\
{\scriptstyle Rf_*}\downarrow & & \downarrow{\scriptstyle restriction} \\
D^+(Y) & \xrightarrow{\ R\Gamma(Y,-)\ } & D^+(\mathcal{O}_Y(Y))
\end{array}
$$

*More generally for any $V \subset Y$ open and $U = f^{-1}(V)$ there is a commutative diagram*

$$
\begin{array}{ccc}
D^+(X) & \xrightarrow{\ R\Gamma(U,-)\ } & D^+(\mathcal{O}_X(U)) \\
{\scriptstyle Rf_*}\downarrow & & \downarrow{\scriptstyle restriction} \\
D^+(Y) & \xrightarrow{\ R\Gamma(V,-)\ } & D^+(\mathcal{O}_Y(V))
\end{array}
$$

*See also Remark 13.2 for more explanation.*

**Proof.** Let $\Gamma_{res} : Mod(\mathcal{O}_X) \to \text{Mod}_{\mathcal{O}_Y(Y)}$ be the functor which associates to an $\mathcal{O}_X$-module $\mathcal{F}$ the global sections of $\mathcal{F}$ viewed as an $\mathcal{O}_Y(Y)$-module via the map $f^\sharp :$ $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Let $restriction : \text{Mod}_{\mathcal{O}_X(X)} \to \text{Mod}_{\mathcal{O}_Y(Y)}$ be the restriction functor induced by $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Note that $restriction$ is exact so that its right derived functor is computed by simply applying the restriction functor, see Derived Categories, Lemma 16.9. It is clear that

$$\Gamma_{res} = restriction \circ \Gamma(X,-) = \Gamma(Y,-) \circ f_*$$

We claim that Derived Categories, Lemma 22.1 applies to both compositions. For the first this is clear by our remarks above. For the second, it follows from Lemma 11.10 which implies that injective $\mathcal{O}_X$-modules are mapped to $\Gamma(Y,-)$-acyclic sheaves on $Y$. $\qquad\square$

**Remark 13.2.** Here is a down-to-earth explanation of the meaning of Lemma 13.1. It says that given $f : X \to Y$ and $\mathcal{F} \in Mod(\mathcal{O}_X)$ and given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ we have

$$
\begin{array}{lcc}
R\Gamma(X,\mathcal{F}) & \text{is represented by} & \Gamma(X,\mathcal{I}^\bullet) \\
Rf_*\mathcal{F} & \text{is represented by} & f_*\mathcal{I}^\bullet \\
R\Gamma(Y,Rf_*\mathcal{F}) & \text{is represented by} & \Gamma(Y,f_*\mathcal{I}^\bullet)
\end{array}
$$

the last fact coming from Leray's acyclicity lemma (Derived Categories, Lemma 16.7) and Lemma 11.10. Finally, it combines this with the trivial observation that

$$\Gamma(X,\mathcal{I}^\bullet) = \Gamma(Y,f_*\mathcal{I}^\bullet).$$

to arrive at the commutativity of the diagram of the lemma.

**Lemma 13.3.** *Let $X$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*
  (1) *The cohomology groups $H^i(U,\mathcal{F})$ for $U \subset X$ open of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.*
  (2) *Let $f : X \to Y$ be a morphism of ringed spaces. The higher direct images $R^i f_*\mathcal{F}$ of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.*
*There are similar statements in the case of bounded below complexes of $\mathcal{O}_X$-modules.*

**Proof.** Consider the morphism of ringed spaces $(X,\mathcal{O}_X) \to (X,\underline{\mathbf{Z}}_X)$ given by the identity on the underlying topological space and by the unique map of sheaves of rings $\underline{\mathbf{Z}}_X \to \mathcal{O}_X$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{ab}$ the same sheaf seen as an $\underline{\mathbf{Z}}_X$-module, i.e., seen as a sheaf of abelian groups. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. By Remark 13.2 we see that $\Gamma(X,\mathcal{I}^\bullet)$ computes both $R\Gamma(X,\mathcal{F})$ and $R\Gamma(X,\mathcal{F}_{ab})$. This proves (1).

To prove (2) we use (1) and Lemma 7.3. The result follows immediately. $\qquad\square$

**Lemma 13.4** (Leray spectral sequence). *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. There is a spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_*(\mathcal{F}^\bullet))$$

*converging to $H^{p+q}(X,\mathcal{F}^\bullet)$.*

**Proof.** This is just the Grothendieck spectral sequence Derived Categories, Lemma 22.2 coming from the composition of functors $\Gamma_{res} = \Gamma(Y, -) \circ f_*$ where $\Gamma_{res}$ is as in the proof of Lemma 13.1. To see that the assumptions of Derived Categories, Lemma 22.2 are satisfied, see the proof of Lemma 13.1 or Remark 13.2.     □

**Remark 13.5.** The Leray spectral sequence, the way we proved it in Lemma 13.4 is a spectral sequence of $\Gamma(Y, \mathcal{O}_Y)$-modules. However, it is quite easy to see that it is in fact a spectral sequence of $\Gamma(X, \mathcal{O}_X)$-modules. For example $f$ gives rise to a morphism of ringed spaces $f' : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$. By Lemma 13.3 the terms $E_r^{p,q}$ of the Leray spectral sequence for an $\mathcal{O}_X$-module $\mathcal{F}$ and $f$ are identical with those for $\mathcal{F}$ and $f'$ at least for $r \geq 2$. Namely, they both agree with the terms of the Leray spectral sequence for $\mathcal{F}$ as an abelian sheaf. And since $(f_*\mathcal{O}_X)(Y) = \mathcal{O}_X(X)$ we see the result. It is often the case that the Leray spectral sequence carries additional structure.

**Lemma 13.6.** *Let* $f : X \to Y$ *be a morphism of ringed spaces. Let* $\mathcal{F}$ *be an* $\mathcal{O}_X$*-module.*
  (1) *If* $R^q f_* \mathcal{F} = 0$ *for* $q > 0$, *then* $H^p(X, \mathcal{F}) = H^p(Y, f_*\mathcal{F})$ *for all* $p$.
  (2) *If* $H^p(Y, R^q f_* \mathcal{F}) = 0$ *for all* $q$ *and* $p > 0$, *then* $H^q(X, \mathcal{F}) = H^0(Y, R^q f_* \mathcal{F})$ *for all* $q$.

**Proof.** These are two simple conditions that force the Leray spectral sequence to degenerate at $E_2$. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves.     □

**Lemma 13.7.** *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be morphisms of ringed spaces. In this case* $Rg_* \circ Rf_* = R(g \circ f)_*$ *as functors from* $D^+(X) \to D^+(Z)$.

**Proof.** We are going to apply Derived Categories, Lemma 22.1. It is clear that $g_* \circ f_* = (g \circ f)_*$, see Sheaves, Lemma 21.2. It remains to show that $f_*\mathcal{I}$ is $g_*$-acyclic. This follows from Lemma 11.10 and the description of the higher direct images $R^i g_*$ in Lemma 7.3.     □

**Lemma 13.8** (Relative Leray spectral sequence). *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be morphisms of ringed spaces. Let* $\mathcal{F}$ *be an* $\mathcal{O}_X$*-module. There is a spectral sequence with*

$$E_2^{p,q} = R^p g_* (R^q f_* \mathcal{F})$$

*converging to* $R^{p+q}(g \circ f)_* \mathcal{F}$. *This spectral sequence is functorial in* $\mathcal{F}$, *and there is a version for bounded below complexes of* $\mathcal{O}_X$*-modules.*

**Proof.** This is a Grothendieck spectral sequence for composition of functors and follows from Lemma 13.7 and Derived Categories, Lemma 22.2.     □

## 14. Functoriality of cohomology

**Lemma 14.1.** *Let* $f : X \to Y$ *be a morphism of ringed spaces. Let* $\mathcal{G}^\bullet$, *resp.* $\mathcal{F}^\bullet$ *be a bounded below complex of* $\mathcal{O}_Y$*-modules, resp.* $\mathcal{O}_X$*-modules. Let* $\varphi : \mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet$ *be a morphism of complexes. There is a canonical morphism*

$$\mathcal{G}^\bullet \longrightarrow Rf_*(\mathcal{F}^\bullet)$$

*in* $D^+(Y)$. *Moreover this construction is functorial in the triple* $(\mathcal{G}^\bullet, \mathcal{F}^\bullet, \varphi)$.

**Proof.** Choose an injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. By definition $Rf_*(\mathcal{F}^\bullet)$ is represented by $f_*\mathcal{I}^\bullet$ in $K^+(\mathcal{O}_Y)$. The composition

$$\mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet \to f_*\mathcal{I}^\bullet$$

is a morphism in $K^+(Y)$ which turns into the morphism of the lemma upon applying the localization functor $j_Y : K^+(Y) \to D^+(Y)$. $\qquad\square$

Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Recall that an $f$-map $\varphi$ from $\mathcal{G}$ to $\mathcal{F}$ is a map $\varphi : \mathcal{G} \to f_*\mathcal{F}$, or what is the same thing, a map $\varphi : f^*\mathcal{G} \to \mathcal{F}$. See Sheaves, Definition 21.7. Such an $f$-map gives rise to a morphism of complexes

(14.1.1) $$\varphi : R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D^+(\mathcal{O}_Y(Y))$. Namely, we use the morphism $\mathcal{G} \to Rf_*\mathcal{F}$ in $D^+(Y)$ of Lemma 14.1, and we apply $R\Gamma(Y, -)$. By Lemma 13.1 we see that $R\Gamma(X, \mathcal{F}) = R\Gamma(Y, Rf_*\mathcal{F})$ and we get the displayed arrow. We spell this out completely in Remark 14.2 below. In particular it gives rise to maps on cohomology

(14.1.2) $$\varphi : H^i(Y, \mathcal{G}) \longrightarrow H^i(X, \mathcal{F}).$$

**Remark 14.2.** Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\varphi$ be an $f$-map from $\mathcal{G}$ to $\mathcal{F}$. Choose a resolution $\mathcal{F} \to \mathcal{I}^\bullet$ by a complex of injective $\mathcal{O}_X$-modules. Choose resolutions $\mathcal{G} \to \mathcal{J}^\bullet$ and $f_*\mathcal{I}^\bullet \to (\mathcal{J}')^\bullet$ by complexes of injective $\mathcal{O}_Y$-modules. By Derived Categories, Lemma 18.6 there exists a map of complexes $\beta$ such that the diagram

(14.2.1)

$$
\begin{array}{ccccc}
\mathcal{G} & \longrightarrow & f_*\mathcal{F} & \longrightarrow & f_*\mathcal{I}^\bullet \\
\downarrow & & & & \downarrow \\
\mathcal{J}^\bullet & \xrightarrow{\quad\beta\quad} & & & (\mathcal{J}')^\bullet
\end{array}
$$

commutes. Applying global section functors we see that we get a diagram

$$
\begin{array}{ccc}
\Gamma(Y, f_*\mathcal{I}^\bullet) & =\!\!=\!\!= & \Gamma(X, \mathcal{I}^\bullet) \\
& & \downarrow{\scriptstyle qis} \\
\Gamma(Y, \mathcal{J}^\bullet) \xrightarrow{\quad\beta\quad} & & \Gamma(Y, (\mathcal{J}')^\bullet)
\end{array}
$$

The complex on the bottom left represents $R\Gamma(Y, \mathcal{G})$ and the complex on the top right represents $R\Gamma(X, \mathcal{F})$. The vertical arrow is a quasi-isomorphism by Lemma 13.1 which becomes invertible after applying the localization functor $K^+(\mathcal{O}_Y(Y)) \to D^+(\mathcal{O}_Y(Y))$. The arrow (14.1.1) is given by the composition of the horizontal map by the inverse of the vertical map.

## 15. Refinements and Čech cohomology

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : X = \bigcup_{j \in J} V_j$ be open coverings. Assume that $\mathcal{U}$ is a refinement of $\mathcal{V}$. Choose a map $c : I \to J$ such that $U_i \subset V_{c(i)}$ for all $i \in I$. This induces a map of Čech complexes

$$\gamma : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}), \quad (\xi_{j_0 \ldots j_p}) \longmapsto (\xi_{c(i_0) \ldots c(i_p)}|_{U_{i_0 \ldots i_p}})$$

functorial in the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Suppose that $c' : I \to J$ is a second map such that $U_i \subset V_{c'(i)}$ for all $i \in I$. Then the corresponding maps $\gamma$ and $\gamma'$ are

homotopic. Namely, $\gamma - \gamma' = \mathrm{d} \circ h + h \circ \mathrm{d}$ with $h : \check{C}^{p+1}(\mathcal{V}, \mathcal{F}) \to \check{C}^p(\mathcal{U}, \mathcal{F})$ given by the rule

$$h(\alpha)_{i_0 \ldots i_p} = \sum_{a=0}^{p} (-1)^a \alpha_{c(i_0) \ldots c(i_a) c'(i_a) \ldots c'(i_p)}$$

We omit the computation showing this works; please see the discussion following (25.0.2) for the proof in a more general case. In particular, the map on Čech cohomology groups is independent of the choice of $c$. Moreover, it is clear that if $\mathcal{W} : X = \bigcup_{k \in K} W_k$ is a third open covering and $\mathcal{V}$ is a refinement of $\mathcal{W}$, then the composition of the maps

$$\check{C}^{\bullet}(\mathcal{W}, \mathcal{F}) \longrightarrow \check{C}^{\bullet}(\mathcal{V}, \mathcal{F}) \longrightarrow \check{C}^{\bullet}(\mathcal{U}, \mathcal{F})$$

associated to maps $I \to J$ and $J \to K$ is the map associated to the composition $I \to K$. In particular, we can define the Čech cohomology groups

$$\check{H}^p(X, \mathcal{F}) = \mathrm{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F})$$

where the colimit is over all open coverings of $X$ preordered by refinement.

It turns out that the maps $\gamma$ defined above are compatible with the map to cohomology, in other words, the composition

$$\check{H}^p(\mathcal{V}, \mathcal{F}) \to \check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{Lemma 11.2}} H^p(X, \mathcal{F})$$

is the canonical map from the first group to cohomology of Lemma 11.2. In the lemma below we will prove this in a slightly more general setting. A consequence is that we obtain a well defined map

(15.0.1) $$\check{H}^p(X, \mathcal{F}) = \mathrm{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$$

from Čech cohomology to cohomology.

**Lemma 15.1.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\varphi : f^* \mathcal{G} \to \mathcal{F}$ be an $f$-map from an $\mathcal{O}_Y$-module $\mathcal{G}$ to an $\mathcal{O}_X$-module $\mathcal{F}$. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : Y = \bigcup_{j \in J} V_j$ be open coverings. Assume that $\mathcal{U}$ is a refinement of $f^{-1}\mathcal{V} : X = \bigcup_{j \in J} f^{-1}(V_j)$. In this case there exists a commutative diagram*

$$\begin{array}{ccc} \check{C}^{\bullet}(\mathcal{U}, \mathcal{F}) & \longrightarrow & R\Gamma(X, \mathcal{F}) \\ \gamma \uparrow & & \uparrow \\ \check{C}^{\bullet}(\mathcal{V}, \mathcal{G}) & \longrightarrow & R\Gamma(Y, \mathcal{G}) \end{array}$$

*in $D^+(\mathcal{O}_X(X))$ with horizontal arrows given by Lemma 11.2 and right vertical arrow by (14.1.1). In particular we get commutative diagrams of cohomology groups*

$$\begin{array}{ccc} \check{H}^p(\mathcal{U}, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}) \\ \gamma \uparrow & & \uparrow \\ \check{H}^p(\mathcal{V}, \mathcal{G}) & \longrightarrow & H^p(Y, \mathcal{G}) \end{array}$$

*where the right vertical arrow is (14.1.2)*

**Proof.** We first define the left vertical arrow. Namely, choose a map $c : I \to J$ such that $U_i \subset f^{-1}(V_{c(i)})$ for all $i \in I$. In degree $p$ we define the map by the rule

$$\gamma(s)_{i_0 \ldots i_p} = \varphi(s)_{c(i_0) \ldots c(i_p)}$$

This makes sense because $\varphi$ does indeed induce maps $\mathcal{G}(V_{c(i_0)\ldots c(i_p)}) \to \mathcal{F}(U_{i_0\ldots i_p})$ by assumption. It is also clear that this defines a morphism of complexes. Choose injective resolutions $\mathcal{F} \to \mathcal{I}^\bullet$ on $X$ and $\mathcal{G} \to J^\bullet$ on $Y$. According to the proof of Lemma 11.2 we introduce the double complexes $A^{\bullet,\bullet}$ and $B^{\bullet,\bullet}$ with terms

$$B^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, \mathcal{J}^q) \quad \text{and} \quad A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q).$$
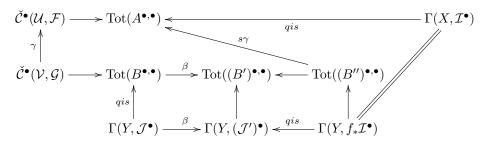
As in Remark 14.2 above we also choose an injective resolution $f_*\mathcal{I} \to (\mathcal{J}')^\bullet$ on $Y$ and a morphism of complexes $\beta : \mathcal{J} \to (\mathcal{J}')^\bullet$ making (14.2.1) commutes. We introduce some more double complexes, namely $(B')^{\bullet,\bullet}$ and $(B'')^{\bullet,\bullet}$ with

$$(B')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, (\mathcal{J}')^q) \quad \text{and} \quad (B'')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, f_*\mathcal{I}^q).$$

Note that there is an $f$-map of complexes from $f_*\mathcal{I}^\bullet$ to $\mathcal{I}^\bullet$. Hence it is clear that the same rule as above defines a morphism of double complexes

$$\gamma : (B'')^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet}.$$

Consider the diagram of complexes



The two horizontal arrows with targets $\text{Tot}(A^{\bullet,\bullet})$ and $\text{Tot}(B^{\bullet,\bullet})$ are the ones explained in Lemma 11.2. The left upper shape (a pentagon) is commutative simply because (14.2.1) is commutative. The two lower squares are trivially commutative. It is also immediate from the definitions that the right upper shape (a square) is commutative. The result of the lemma now follows from the definitions and the fact that going around the diagram on the outer sides from $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})$ to $\Gamma(X, \mathcal{I}^\bullet)$ either on top or on bottom is the same (where you have to invert any quasi-isomorphisms along the way). $\square$

## 16. Cohomology on Hausdorff quasi-compact spaces

For such a space Čech cohomology agrees with cohomology.

**Lemma 16.1.** *Let $X$ be a topological space. Let $\mathcal{F}$ be an abelian sheaf. Then the map $\check{H}^1(X, \mathcal{F}) \to H^1(X, \mathcal{F})$ defined in (15.0.1) is an isomorphism.*

**Proof.** Let $\mathcal{U}$ be an open covering of $X$. By Lemma 11.5 there is an exact sequence

$$0 \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F}))$$

Thus the map is injective. To show surjectivity it suffices to show that any element of $\check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F}))$ maps to zero after replacing $\mathcal{U}$ by a refinement. This is immediate from the definitions and the fact that $\underline{H}^1(\mathcal{F})$ is a presheaf of abelian groups whose sheafification is zero by locality of cohomology, see Lemma 7.2. $\square$

**Lemma 16.2.** *Let $X$ be a Hausdorff and quasi-compact topological space. Let $\mathcal{F}$ be an abelian sheaf on $X$. Then the map $\check{H}^n(X, \mathcal{F}) \to H^n(X, \mathcal{F})$ defined in (15.0.1) is an isomorphism for all $n$.*

**Proof.** We already know that $\check{H}^n(X, -) \to H^n(X, -)$ is an isomorphism of functors for $n = 0, 1$, see Lemma 16.1. The functors $H^n(X, -)$ form a universal $\delta$-functor, see Derived Categories, Lemma 20.4. If we show that $\check{H}^n(X, -)$ forms a universal $\delta$-functor and that $\check{H}^n(X, -) \to H^n(X, -)$ is compatible with boundary maps, then the map will automatically be an isomorphism by uniqueness of universal $\delta$-functors, see Homology, Lemma 12.5.

Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of abelian sheaves on $X$. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. This gives a complex of complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{H}) \to 0$$

which is in general not exact on the right. The sequence defines the maps

$$\check{H}^n(\mathcal{U}, \mathcal{F}) \to \check{H}^n(\mathcal{U}, \mathcal{G}) \to \check{H}^n(\mathcal{U}, \mathcal{H})$$

but isn't good enough to define a boundary operator $\delta : \check{H}^n(\mathcal{U}, \mathcal{H}) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F})$. Indeed such a thing will not exist in general. However, given an element $\overline{h} \in \check{H}^n(\mathcal{U}, \mathcal{H})$ which is the cohomology class of a cocycle $h = (h_{i_0 \ldots i_n})$ we can choose open coverings

$$U_{i_0 \ldots i_n} = \bigcup W_{i_0 \ldots i_n, k}$$

such that $h_{i_0 \ldots i_n}|_{W_{i_0 \ldots i_n, k}}$ lifts to a section of $\mathcal{G}$ over $W_{i_0 \ldots i_n, k}$. By Topology, Lemma 13.5 (this is where we use the assumption that $X$ is hausdorff and quasi-compact) we can choose an open covering $\mathcal{V} : X = \bigcup_{j \in J} V_j$ and $\alpha : J \to I$ such that $V_j \subset U_{\alpha(j)}$ (it is a refinement) and such that for all $j_0, \ldots, j_n \in J$ there is a $k$ such that $V_{j_0 \ldots j_n} \subset W_{\alpha(j_0) \ldots \alpha(j_n), k}$. We obtain maps of complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{H}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{H}) & \longrightarrow & 0
\end{array}
$$

In fact, the vertical arrows are the maps of complexes used to define the transition maps between the Čech cohomology groups. Our choice of refinement shows that we may choose

$$g_{j_0 \ldots j_n} \in \mathcal{G}(V_{j_0 \ldots j_n}), \quad g_{j_0 \ldots j_n} \longmapsto h_{\alpha(j_0) \ldots \alpha(j_n)}|_{V_{j_0 \ldots j_n}}$$

The cochain $g = (g_{j_0 \ldots j_n})$ is not a cocycle in general but we know that its Čech boundary $\mathrm{d}(g)$ maps to zero in $\check{\mathcal{C}}^{n+1}(\mathcal{V}, \mathcal{H})$ (by the commutative diagram above and the fact that $h$ is a cocycle). Hence $\mathrm{d}(g)$ is a cocycle in $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$. This allows us to define

$$\delta(\overline{h}) = \text{class of } \mathrm{d}(g) \text{ in } \check{H}^{n+1}(\mathcal{V}, \mathcal{F})$$

Now, given an element $\xi \in \check{H}^n(X, \mathcal{G})$ we choose an open covering $\mathcal{U}$ and an element $\overline{h} \in \check{H}^n(\mathcal{U}, \mathcal{G})$ mapping to $\xi$ in the colimit defining Čech cohomology. Then we choose $\mathcal{V}$ and $g$ as above and set $\delta(\xi)$ equal to the image of $\delta(\overline{h})$ in $\check{H}^n(X, \mathcal{F})$. At this point a lot of properties have to be checked, all of which are straightforward. For example, we need to check that our construction is independent of the choice of $\mathcal{U}, \overline{h}, \mathcal{V}, \alpha : J \to I, g$. The class of $\mathrm{d}(g)$ is independent of the choice of the

lifts $g_{i_0 \dots i_n}$ because the difference will be a coboundary. Independence of $\alpha$ holds[1] because a different choice of $\alpha$ determines homotopic vertical maps of complexes in the diagram above, see Section 15. For the other choices we use that given a finite collection of coverings of $X$ we can always find a covering refining all of them. We also need to check additivity which is shown in the same manner. Finally, we need to check that the maps $\check{H}^n(X, -) \to H^n(X, -)$ are compatible with boundary maps. To do this we choose injective resolutions

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{I}_1^\bullet \longrightarrow \mathcal{I}_2^\bullet \longrightarrow \mathcal{I}_3^\bullet \longrightarrow 0$$

as in Derived Categories, Lemma 18.9. This will give a commutative diagram

$$0 \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_1^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_2^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_3^\bullet)) \longrightarrow 0$$

Here $\mathcal{U}$ is an open covering as above and the vertical maps are those used to define the maps $\check{H}^n(\mathcal{U}, -) \to H^n(X, -)$, see Lemma 11.2. The bottom complex is exact as the sequence of complexes of injectives is termwise split exact. Hence the boundary map in cohomology is computed by the usual procedure for this lower exact sequence, see Homology, Lemma 13.12. The same will be true after passing to the refinement $\mathcal{V}$ where the boundary map for Čech cohomology was defined. Hence the boundary maps agree because they use the same construction (whenever the first one is defined on an element in Čech cohomology on a given covering). This finishes our discussion of the construction of the structure of a $\delta$-functor on Čech cohomology and why this structure is compatible with the given $\delta$-functor structure on usual cohomology.

Finally, we may apply Lemma 11.1 to see that higher Čech cohomology is trivial on injective sheaves. Hence we see that Čech cohomology is a universal $\delta$-functor by Homology, Lemma 12.4. $\qquad\square$

**Lemma 16.3.** *Let $X$ be a topological space. Let $Z \subset X$ be a quasi-compact subset such that any two points of $Z$ have disjoint open neighbourhoods in $X$. For every abelian sheaf $\mathcal{F}$ on $X$ the canonical map*

$$\mathrm{colim}\, H^p(U, \mathcal{F}) \longrightarrow H^p(Z, \mathcal{F}|_Z)$$

*where the colimit is over open neighbourhoods $U$ of $Z$ in $X$ is an isomorphism.*

**Proof.** We first prove this for $p = 0$. Injectivity follows from the definition of $\mathcal{F}|_Z$ and holds in general (for any subset of any topological space $X$). Next, suppose that $s \in H^0(Z, \mathcal{F}|_Z)$. Then we can find opens $U_i \subset X$ such that $Z \subset \bigcup U_i$ and such that $s|_{Z \cap U_i}$ comes from $s_i \in \mathcal{F}(U_i)$. It follows that there exist opens $W_{ij} \subset U_i \cap U_j$ with $W_{ij} \cap Z = U_i \cap U_j \cap Z$ such that $s_i|_{W_{ij}} = s_j|_{W_{ij}}$. Applying Topology, Lemma

---

[1]This is an important check because the nonuniqueness of $\alpha$ is the only thing preventing us from taking the colimit of Čech complexes over all open coverings of $X$ to get a short exact sequence of complexes computing Čech cohomology.

13.7 we find opens $V_i$ of $X$ such that $V_i \subset U_i$ and such that $V_i \cap V_j \subset W_{ij}$. Hence we see that $s_i|_{V_i}$ glue to a section of $\mathcal{F}$ over the open neighbourhood $\bigcup V_i$ of $Z$.

To finish the proof, it suffices to show that if $\mathcal{I}$ is an injective abelian sheaf on $X$, then $H^p(Z, \mathcal{I}|_Z) = 0$ for $p > 0$. This follows using short exact sequences and dimension shifting; details omitted. Thus, suppose $\bar{\xi}$ is an element of $H^p(Z, \mathcal{I}|_Z)$ for some $p > 0$. By Lemma 16.2 the element $\bar{\xi}$ comes from $\check{H}^p(\mathcal{V}, \mathcal{I}|_Z)$ for some open covering $\mathcal{V} : Z = \bigcup V_i$ of $Z$. Say $\bar{\xi}$ is the image of the class of a cocycle $\xi = (\xi_{i_0 \ldots i_p})$ in $\check{\mathcal{C}}^p(\mathcal{V}, \mathcal{I}|_Z)$.

Let $\mathcal{I}' \subset \mathcal{I}|_Z$ be the subpresheaf defined by the rule

$$\mathcal{I}'(V) = \{s \in \mathcal{I}|_Z(V) \mid \exists (U, t), \ U \subset X \text{ open}, \ t \in \mathcal{I}(U), \ V = Z \cap U, \ s = t|_{Z \cap U}\}$$

Then $\mathcal{I}|_Z$ is the sheafification of $\mathcal{I}'$. Thus for every $(p+1)$-tuple $i_0 \ldots i_p$ we can find an open covering $V_{i_0 \ldots i_p} = \bigcup W_{i_0 \ldots i_p, k}$ such that $\xi_{i_0 \ldots i_p}|_{W_{i_0 \ldots i_p, k}}$ is a section of $\mathcal{I}'$. Applying Topology, Lemma 13.5 we may after refining $\mathcal{V}$ assume that each $\xi_{i_0 \ldots i_p}$ is a section of the presheaf $\mathcal{I}'$.

Write $V_i = Z \cap U_i$ for some opens $U_i \subset X$. Since $\mathcal{I}$ is flasque (Lemma 12.2) and since $\xi_{i_0 \ldots i_p}$ is a section of $\mathcal{I}'$ for every $(p+1)$-tuple $i_0 \ldots i_p$ we can choose a section $s_{i_0 \ldots i_p} \in \mathcal{I}(U_{i_0 \ldots i_p})$ which restricts to $\xi_{i_0 \ldots i_p}$ on $V_{i_0 \ldots i_p} = Z \cap U_{i_0 \ldots i_p}$. (This appeal to injectives being flasque can be avoided by an additional application of Topology, Lemma 13.7.) Let $s = (s_{i_0 \ldots i_p})$ be the corresponding cochain for the open covering $U = \bigcup U_i$. Since $\mathrm{d}(\xi) = 0$ we see that the sections $\mathrm{d}(s)_{i_0 \ldots i_{p+1}}$ restrict to zero on $Z \cap U_{i_0 \ldots i_{p+1}}$. Hence, by the initial remarks of the proof, there exists open subsets $W_{i_0 \ldots i_{p+1}} \subset U_{i_0 \ldots i_{p+1}}$ with $Z \cap W_{i_0 \ldots i_{p+1}} = Z \cap U_{i_0 \ldots i_{p+1}}$ such that $\mathrm{d}(s)_{i_0 \ldots i_{p+1}}|_{W_{i_0 \ldots i_{p+1}}} = 0$. By Topology, Lemma 13.7 we can find $U_i' \subset U_i$ such that $Z \subset \bigcup U_i'$ and such that $U_{i_0 \ldots i_{p+1}}' \subset W_{i_0 \ldots i_{p+1}}$. Then $s' = (s_{i_0 \ldots i_p}')$ with $s_{i_0 \ldots i_p}' = s_{i_0 \ldots i_p}|_{U_{i_0 \ldots i_p}'}$ is a cocycle for $\mathcal{I}$ for the open covering $U' = \bigcup U_i'$ of an open neighbourhood of $Z$. Since $\mathcal{I}$ has trivial higher Čech cohomology groups (Lemma 11.1) we conclude that $s'$ is a coboundary. It follows that the image of $\xi$ in the Čech complex for the open covering $Z = \bigcup Z \cap U_i'$ is a coboundary and we are done. $\square$

## 17. The base change map

We will need to know how to construct the base change map in some cases. Since we have not yet discussed derived pullback we only discuss this in the case of a base change by a flat morphism of ringed spaces. Before we state the result, let us discuss flat pullback on the derived category. Namely, suppose that $g : X \to Y$ is a flat morphism of ringed spaces. By Modules, Lemma 20.2 the functor $g^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is exact. Hence it has a derived functor

$$g^* : D^+(Y) \to D^+(X)$$

which is computed by simply pulling back an representative of a given object in $D^+(Y)$, see Derived Categories, Lemma 16.9. Hence as indicated we indicate this functor by $g^*$ rather than $Lg^*$.

**Lemma 17.1.** *Let*

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
S' & \xrightarrow{g} & S
\end{array}
$$

*be a commutative diagram of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. Assume both $g$ and $g'$ are flat. Then there exists a canonical base change map*

$$g^* R f_* \mathcal{F}^\bullet \longrightarrow R(f')_* (g')^* \mathcal{F}^\bullet$$

*in $D^+(S')$.*

**Proof.** Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^* \mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 11.11 we see that $(g')_* \mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_* (g')^* \mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 18.6 and 18.7 the arrow $\beta$ in the diagram

$$
\begin{array}{ccc}
(g')_*(g')^*\mathcal{F}^\bullet & \longrightarrow & (g')_*\mathcal{J}^\bullet \\
{\scriptstyle adjunction}\Big\uparrow & & \Big\uparrow{\scriptstyle \beta} \\
\mathcal{F}^\bullet & \longrightarrow & \mathcal{I}^\bullet
\end{array}
$$

exists and is unique up to homotopy. Pushing down to $S$ we get

$$f_* \beta : f_* \mathcal{I}^\bullet \longrightarrow f_*(g')_* \mathcal{J}^\bullet = g_*(f')_* \mathcal{J}^\bullet$$

By adjunction of $g^*$ and $g_*$ we get a map of complexes $g^* f_* \mathcal{I}^\bullet \to (f')_* \mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map $\beta$ and everything was done on the level of complexes. $\square$

**Remark 17.2.** The "correct" version of the base change map is the map

$$Lg^* R f_* \mathcal{F}^\bullet \longrightarrow R(f')_* L(g')^* \mathcal{F}^\bullet.$$

The construction of this map involves unbounded complexes, see Remark 28.3.

## 18. Proper base change in topology

In this section we prove a very general version of the proper base change theorem in topology. It tells us that the stalks of the higher direct images $R^p f_*$ can be computed on the fibre.

**Lemma 18.1.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $y \in Y$. Assume that*

(1) *$f$ is closed,*
(2) *$f$ is separated, and*
(3) *$f^{-1}(y)$ is quasi-compact.*

*Then for $E$ in $D^+(\mathcal{O}_X)$ we have $(Rf_* E)_y = R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$ in $D^+(\mathcal{O}_{Y,y})$.*

**Proof.** The base change map of Lemma 17.1 gives a canonical map $(Rf_* E)_y \to R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$. To prove this map is an isomorphism, we represent $E$ by a bounded below complex of injectives $\mathcal{I}^\bullet$. Set $Z = f^{-1}(\{y\})$. The assumptions of Lemma 16.3 are satisfied, see Topology, Lemma 4.2. Hence the restrictions $\mathcal{I}^n|_Z$ are acyclic for $\Gamma(Z, -)$. Thus $R\Gamma(Z, E|_Z)$ is represented by the complex $\Gamma(Z, \mathcal{I}^\bullet|_Z)$, see Derived Categories, Lemma 16.7. In other words, we have to show the map

$$\mathrm{colim}_V \mathcal{I}^\bullet(f^{-1}(V)) \longrightarrow \Gamma(Z, \mathcal{I}^\bullet|_Z)$$

is an isomorphism. Using Lemma 16.3 we see that it suffices to show that the collection of open neighbourhoods $f^{-1}(V)$ of $Z = f^{-1}(\{y\})$ is cofinal in the system of all open neighbourhoods. If $f^{-1}(\{y\}) \subset U$ is an open neighbourhood, then as $f$ is closed the set $V = Y \setminus f(X \setminus U)$ is an open neighbourhood of $y$ with $f^{-1}(V) \subset U$. This proves the lemma. $\square$

**Theorem 18.2** (Proper base change). *Consider a cartesian square of topological spaces*

$$
\begin{array}{ccc}
X' = Y' \times_Y X & \xrightarrow{\ g'\ } & X \\
\scriptstyle f' \downarrow & & \downarrow \scriptstyle f \\
Y' & \xrightarrow{\ g\ } & Y
\end{array}
$$

*Assume that $f$ is proper. Let $E$ be an object of $D^+(X)$. Then the base change map*

$$
g^{-1} Rf_* E \longrightarrow Rf'_* (g')^{-1} E
$$

*of Lemma 17.1 is an isomorphism in $D^+(Y')$.*

**Proof.** Let $y' \in Y'$ be a point with image $y \in Y$. It suffices to show that the base change map induces an isomorphism on stalks at $y'$. As $f$ is proper it follows that $f'$ is proper, the fibres of $f$ and $f'$ are quasi-compact and $f$ and $f'$ are closed, see Topology, Theorem 17.5 and Lemma 4.4. Thus we can apply Lemma 18.1 twice to see that

$$
(Rf'_* (g')^{-1} E)_{y'} = R\Gamma((f')^{-1}(y'), (g')^{-1} E|_{(f')^{-1}(y')})
$$

and

$$
(Rf_* E)_y = R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})
$$

The induced map of fibres $(f')^{-1}(y') \to f^{-1}(y)$ is a homeomorphism of topological spaces and the pull back of $E|_{f^{-1}(y)}$ is $(g')^{-1} E|_{(f')^{-1}(y')}$. The desired result follows. $\square$

**Lemma 18.3** (Proper base change for sheaves of sets). *Consider a cartesian square of topological spaces*

$$
\begin{array}{ccc}
X' & \xrightarrow{\ g'\ } & X \\
\scriptstyle f' \downarrow & & \downarrow \scriptstyle f \\
Y' & \xrightarrow{\ g\ } & Y
\end{array}
$$

*Assume that $f$ is proper. Then $g^{-1} f_* \mathcal{F} = f'_* (g')^{-1} \mathcal{F}$ for any sheaf of sets $\mathcal{F}$ on $X$.*

**Proof.** We argue exactly as in the proof of Theorem 18.2 and we find it suffices to show $(f_* \mathcal{F})_y = \Gamma(X_y, \mathcal{F}|_{X_y})$. Then we argue as in Lemma 18.1 to reduce this to the $p = 0$ case of Lemma 16.3 for sheaves of sets. The first part of the proof of Lemma 16.3 works for sheaves of sets and this finishes the proof. Some details omitted. $\square$

## 19. Cohomology and colimits

Let $X$ be a ringed space. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a system of sheaves of $\mathcal{O}_X$-modules over the directed set $I$, see Categories, Section 21. Since for each $i$ there is a canonical map $\mathcal{F}_i \to \mathrm{colim}_i \mathcal{F}_i$ we get a canonical map

$$
\mathrm{colim}_i H^p(X, \mathcal{F}_i) \longrightarrow H^p(X, \mathrm{colim}_i \mathcal{F}_i)
$$

for every $p \geq 0$. Of course there is a similar map for every open $U \subset X$. These maps are in general not isomorphisms, even for $p = 0$. In this section we generalize the results of Sheaves, Lemma 29.1. See also Modules, Lemma 22.8 (in the special case $\mathcal{G} = \mathcal{O}_X$).

**Lemma 19.1.** *Let $X$ be a ringed space. Assume that the underlying topological space of $X$ has the following properties:*

(1) *there exists a basis of quasi-compact open subsets, and*
(2) *the intersection of any two quasi-compact opens is quasi-compact.*

*Then for any directed system $(\mathcal{F}_i, \varphi_{ii'})$ of sheaves of $\mathcal{O}_X$-modules and for any quasi-compact open $U \subset X$ the canonical map*

$$\operatorname{colim}_i H^q(U, \mathcal{F}_i) \longrightarrow H^q(U, \operatorname{colim}_i \mathcal{F}_i)$$

*is an isomorphism for every $q \geq 0$.*

**Proof.** It is important in this proof to argue for all quasi-compact opens $U \subset X$ at the same time. The result is true for $q = 0$ and any quasi-compact open $U \subset X$ by Sheaves, Lemma 29.1 (combined with Topology, Lemma 27.1). Assume that we have proved the result for all $q \leq q_0$ and let us prove the result for $q = q_0 + 1$.

By our conventions on directed systems the index set $I$ is directed, and any system of $\mathcal{O}_X$-modules $(\mathcal{F}_i, \varphi_{ii'})$ over $I$ is directed. By Injectives, Lemma 5.1 the category of $\mathcal{O}_X$-modules has functorial injective embeddings. Thus for any system $(\mathcal{F}_i, \varphi_{ii'})$ there exists a system $(\mathcal{I}_i, \varphi_{ii'})$ with each $\mathcal{I}_i$ an injective $\mathcal{O}_X$-module and a morphism of systems given by injective $\mathcal{O}_X$-module maps $\mathcal{F}_i \to \mathcal{I}_i$. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

We claim that the sequence

$$0 \to \operatorname{colim}_i \mathcal{F}_i \to \operatorname{colim}_i \mathcal{I}_i \to \operatorname{colim}_i \mathcal{Q}_i \to 0.$$

is also a short exact sequence of $\mathcal{O}_X$-modules. We may check this on stalks. By Sheaves, Sections 28 and 29 taking stalks commutes with colimits. Since a directed colimit of short exact sequences of abelian groups is short exact (see Algebra, Lemma 8.8) we deduce the result. We claim that $H^q(U, \operatorname{colim}_i \mathcal{I}_i) = 0$ for all quasi-compact open $U \subset X$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$
\begin{array}{ccccccc}
\operatorname{colim}_i H^{q_0}(U, \mathcal{I}_i) & \longrightarrow & \operatorname{colim}_i H^{q_0}(U, \mathcal{Q}_i) & \longrightarrow & \operatorname{colim}_i H^{q_0+1}(U, \mathcal{F}_i) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{q_0}(U, \operatorname{colim}_i \mathcal{I}_i) & \longrightarrow & H^{q_0}(U, \operatorname{colim}_i \mathcal{Q}_i) & \longrightarrow & H^{q_0+1}(U, \operatorname{colim}_i \mathcal{F}_i) & \longrightarrow & 0
\end{array}
$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is exact by an application of Algebra, Lemma 8.8. Hence by the snake lemma we deduce the result for $q = q_0 + 1$.

It remains to show that the claim is true. We will use Lemma 11.9. Let $\mathcal{B}$ be the collection of all quasi-compact open subsets of $X$. This is a basis for the topology on $X$ by assumption. Let Cov be the collection of finite open coverings $\mathcal{U} : U = \bigcup_{j=1,\dots,m} U_j$ with each of $U$, $U_j$ quasi-compact open in $X$. By the result for $q = 0$ we see that for $\mathcal{U} \in \mathrm{Cov}$ we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \operatorname{colim}_i \mathcal{I}_i) = \operatorname{colim}_i \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the multiple intersections $U_{j_0 \dots j_p}$ are quasi-compact. By Lemma 11.1 each of the complexes in the colimit of Čech complexes is acyclic in degree $\geq 1$. Hence by Algebra, Lemma 8.8 we see that also the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \operatorname{colim}_i \mathcal{I}_i)$

is acyclic in degrees $\geq 1$. In other words we see that $\check{H}^p(\mathcal{U}, \mathrm{colim}_i \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 11.9 are satisfied and the claim follows. $\square$

Next we formulate the analogy of Sheaves, Lemma 29.4 for cohomology. Let $X$ be a spectral space which is written as a cofiltered limit of spectral spaces $X_i$ for a diagram with spectral transition morphisms as in Topology, Lemma 24.5. Assume given

(1) an abelian sheaf $\mathcal{F}_i$ on $X_i$ for all $i \in \mathrm{Ob}(\mathcal{I})$,
(2) for $a : j \to i$ an $f_a$-map $\varphi_a : \mathcal{F}_i \to \mathcal{F}_j$ of abelian sheaves (see Sheaves, Definition 21.7)

such that $\varphi_c = \varphi_b \circ \varphi_a$ whenever $c = a \circ b$. Set $\mathcal{F} = \mathrm{colim}\, p_i^{-1}\mathcal{F}_i$ on $X$.

**Lemma 19.2.** *In the situation discussed above. Let $i \in \mathrm{Ob}(\mathcal{I})$ and let $U_i \subset X_i$ be quasi-compact open. Then*

$$\mathrm{colim}_{a:j \to i} H^p(f_a^{-1}(U_i), \mathcal{F}_j) = H^p(p_i^{-1}(U_i), \mathcal{F})$$

*for all $p \geq 0$. In particular we have $H^p(X, \mathcal{F}) = \mathrm{colim}\, H^p(X_i, \mathcal{F}_i)$.*

**Proof.** The case $p = 0$ is Sheaves, Lemma 29.4.

In this paragraph we show that we can find a map of systems $(\gamma_i) : (\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ with $\mathcal{G}_i$ an injective abelian sheaf and $\gamma_i$ injective. For each $i$ we pick an injection $\mathcal{F}_i \to \mathcal{I}_i$ where $\mathcal{I}_i$ is an injective abelian sheaf on $X_i$. Then we can consider the family of maps

$$\gamma_i : \mathcal{F}_i \longrightarrow \prod_{b:k \to i} f_{b,*}\mathcal{I}_k = \mathcal{G}_i$$

where the component maps are the maps adjoint to the maps $f_b^{-1}\mathcal{F}_i \to \mathcal{F}_k \to \mathcal{I}_k$. For $a : j \to i$ in $\mathcal{I}$ there is a canonical map

$$\psi_a : f_a^{-1}\mathcal{G}_i \to \mathcal{G}_j$$

whose components are the canonical maps $f_b^{-1}f_{a \circ b,*}\mathcal{I}_k \to f_{b,*}\mathcal{I}_k$ for $b : k \to j$. Thus we find an injection $\{\gamma_i\} : \{\mathcal{F}_i, \varphi_a\} \to (\mathcal{G}_i, \psi_a)$ of systems of abelian sheaves. Note that $\mathcal{G}_i$ is an injective sheaf of abelian groups on $X_i$, see Lemma 11.11 and Homology, Lemma 27.3. This finishes the construction.

Arguing exactly as in the proof of Lemma 19.1 we see that it suffices to prove that $H^p(X, \mathrm{colim}\, f_i^{-1}\mathcal{G}_i) = 0$ for $p > 0$.

Set $\mathcal{G} = \mathrm{colim}\, f_i^{-1}\mathcal{G}_i$. To show vanishing of cohomology of $\mathcal{G}$ on every quasi-compact open of $X$, it suffices to show that the Čech cohomology of $\mathcal{G}$ for any covering $\mathcal{U}$ of a quasi-compact open of $X$ by finitely many quasi-compact opens is zero, see Lemma 11.9. Such a covering is the inverse by $p_i$ of such a covering $\mathcal{U}_i$ on the space $X_i$ for some $i$ by Topology, Lemma 24.6. We have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) = \mathrm{colim}_{a:j \to i} \check{\mathcal{C}}^\bullet(f_a^{-1}(\mathcal{U}_i), \mathcal{G}_j)$$

by the case $p = 0$. The right hand side is a filtered colimit of complexes each of which is acyclic in positive degrees by Lemma 11.1. Thus we conclude by Algebra, Lemma 8.8. $\square$

## 20. Vanishing on Noetherian topological spaces

The aim is to prove a theorem of Grothendieck namely Proposition 20.7. See [Gro57].

**Lemma 20.1.** *Let $i : Z \to X$ be a closed immersion of topological spaces. For any abelian sheaf $\mathcal{F}$ on $Z$ we have $H^p(Z, \mathcal{F}) = H^p(X, i_*\mathcal{F})$.*

**Proof.** This is true because $i_*$ is exact (see Modules, Lemma 6.1), and hence $R^p i_* = 0$ as a functor (Derived Categories, Lemma 16.9). Thus we may apply Lemma 13.6. $\square$

**Lemma 20.2.** *Let $X$ be an irreducible topological space. Then $H^p(X, \underline{A}) = 0$ for all $p > 0$ and any abelian group $A$.*

**Proof.** Recall that $\underline{A}$ is the constant sheaf as defined in Sheaves, Definition 7.4. Since $X$ is irreducible, any nonempty open $U$ is irreducible and a fortiori connected. Hence for $U \subset X$ nonempty open we have $\underline{A}(U) = A$. We have $\underline{A}(\emptyset) = 0$. Thus $\underline{A}$ is a flasque abelian sheaf on $X$. The vanishing follows from Lemma 12.3. $\square$

**Lemma 20.3.** *Let $X$ be a topological space such that the intersection of any two quasi-compact opens is quasi-compact. Let $\mathcal{F} \subset \underline{\mathbf{Z}}$ be a subsheaf generated by finitely many sections over quasi-compact opens. Then there exists a finite filtration*

$$(0) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$

*by abelian subsheaves such that for each $0 < i \leq n$ there exists a short exact sequence*

$$0 \to j'_! \underline{\mathbf{Z}}_V \to j_! \underline{\mathbf{Z}}_U \to \mathcal{F}_i / \mathcal{F}_{i-1} \to 0$$

*with $j : U \to X$ and $j' : V \to X$ the inclusion of quasi-compact opens into $X$.*

**Proof.** Say $\mathcal{F}$ is generated by the sections $s_1, \ldots, s_t$ over the quasi-compact opens $U_1, \ldots, U_t$. Since $U_i$ is quasi-compact and $s_i$ a locally constant function to $\mathbf{Z}$ we may assume, after possibly replacing $U_i$ by the parts of a finite decomposition into open and closed subsets, that $s_i$ is a constant section. Say $s_i = n_i$ with $n_i \in \mathbf{Z}$. Of course we can remove $(U_i, n_i)$ from the list if $n_i = 0$. Flipping signs if necessary we may also assume $n_i > 0$. Next, for any subset $I \subset \{1, \ldots, t\}$ we may add $\bigcap_{i \in I} U_i$ and $\gcd(n_i, i \in I)$ to the list. After doing this we see that our list $(U_1, n_1), \ldots, (U_t, n_t)$ satisfies the following property: For $x \in X$ set $I_x = \{i \in \{1, \ldots, t\} \mid x \in U_i\}$. Then $\gcd(n_i, i \in I_x)$ is attained by $n_i$ for some $i \in I_x$.

As our filtration we take $\mathcal{F}_0 = (0)$ and $\mathcal{F}_n$ generated by the sections $n_i$ over $U_i$ for those $i$ such that $n_i \leq n$. It is clear that $\mathcal{F}_n = \mathcal{F}$ for $n \gg 0$. Moreover, the quotient $\mathcal{F}_n / \mathcal{F}_{n-1}$ is generated by the section $n$ over $U = \bigcup_{n_i \leq n} U_i$ and the kernel of the map $j_! \underline{\mathbf{Z}}_U \to \mathcal{F}_n / \mathcal{F}_{n-1}$ is generated by the section $n$ over $V = \bigcup_{n_i \leq n-1} U_i$. Thus a short exact sequence as in the statement of the lemma. $\square$

**Lemma 20.4.** *Let $X$ be a topological space. Let $d \geq 0$ be an integer. Assume*

(1) *$X$ is quasi-compact,*
(2) *the quasi-compact opens form a basis for $X$, and*
(3) *the intersection of two quasi-compact opens is quasi-compact.*
(4) *$H^p(X, j_! \underline{\mathbf{Z}}_U) = 0$ for all $p > d$ and any quasi-compact open $j : U \to X$.*

*Then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and any abelian sheaf $\mathcal{F}$ on $X$.*

**Proof.** Let $S = \coprod_{U \subset X} \mathcal{F}(U)$ where $U$ runs over the quasi-compact opens of $X$. For any finite subset $A = \{s_1, \ldots, s_n\} \subset S$, let $\mathcal{F}_A$ be the subsheaf of $\mathcal{F}$ generated by all $s_i$ (see Modules, Definition 4.5). Note that if $A \subset A'$, then $\mathcal{F}_A \subset \mathcal{F}_{A'}$. Hence $\{\mathcal{F}_A\}$ forms a system over the directed partially ordered set of finite subsets of $S$. By Modules, Lemma 4.6 it is clear that

$$\operatorname{colim}_A \mathcal{F}_A = \mathcal{F}$$

by looking at stalks. By Lemma 19.1 we have

$$H^p(X, \mathcal{F}) = \operatorname{colim}_A H^p(X, \mathcal{F}_A)$$

Hence it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}_A$. In other words, it suffices to prove the result when $\mathcal{F}$ is generated by finitely many local sections over quasi-compact opens of $X$.

Suppose that $\mathcal{F}$ is generated by the local sections $s_1, \ldots, s_n$. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf generated by $s_1, \ldots, s_{n-1}$. Then we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$$

From the long exact sequence of cohomology we see that it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}'$ and $\mathcal{F}/\mathcal{F}'$ which are generated by fewer than $n$ local sections. Hence it suffices to prove the vanishing for sheaves generated by at most one local section. These sheaves are exactly the quotients of the sheaves $j_!\underline{\mathbf{Z}}_U$ where $U$ is a quasi-compact open of $X$.

Assume now that we have a short exact sequence

$$0 \to \mathcal{K} \to j_!\underline{\mathbf{Z}}_U \to \mathcal{F} \to 0$$

with $U$ quasi-compact open in $X$. It suffices to show that $H^q(X, \mathcal{K})$ is zero for $q \geq d+1$. As above we can write $\mathcal{K}$ as the filtered colimit of subsheaves $\mathcal{K}'$ generated by finitely many sections over quasi-compact opens. Then $\mathcal{F}$ is the filtered colimit of the sheaves $j_!\underline{\mathbf{Z}}_U/\mathcal{K}'$. In this way we reduce to the case that $\mathcal{K}$ is generated by finitely many sections over quasi-compact opens. Note that $\mathcal{K}$ is a subsheaf of $\underline{\mathbf{Z}}_X$. Thus by Lemma 20.3 there exists a finite filtration of $\mathcal{K}$ whose successive quotients $\mathcal{Q}$ fit into a short exact sequence

$$0 \to j''_!\underline{\mathbf{Z}}_W \to j'_!\underline{\mathbf{Z}}_V \to \mathcal{Q} \to 0$$

with $j'' : W \to X$ and $j' : V \to X$ the inclusions of quasi-compact opens. Hence the vanishing of $H^p(X, \mathcal{Q})$ for $p > d$ follows from our assumption (in the lemma) on the vanishing of the cohomology groups of $j''_!\underline{\mathbf{Z}}_W$ and $j'_!\underline{\mathbf{Z}}_V$. Returning to $\mathcal{K}$ this, via an induction argument using the long exact cohomology sequence, implies the desired vanishing for it as well. $\qquad\square$

**Example 20.5.** Let $X = \mathbf{N}$ endowed with the topology whose opens are $\emptyset$, $X$, and $U_n = \{i \mid i \leq n\}$ for $n \geq 1$. An abelian sheaf $\mathcal{F}$ on $X$ is the same as an inverse system of abelian groups $A_n = \mathcal{F}(U_n)$ and $\Gamma(X, \mathcal{F}) = \lim A_n$. Since the inverse limit functor is not an exact functor on the category of inverse systems, we see that there is an abelian sheaf with nonzero $H^1$. Finally, the reader can check that $H^p(X, j_!\mathbf{Z}_U) = 0$, $p \geq 1$ if $j : U = U_n \to X$ is the inclusion. Thus we see that $X$ is an example of a space satisfying conditions (2), (3), and (4) of Lemma 20.4 for $d = 0$ but not the conclusion.

**Lemma 20.6.** *Let $X$ be an irreducible topological space. Let $\mathcal{H} \subset \underline{\mathbf{Z}}$ be an abelian subsheaf of the constant sheaf. Then there exists a nonempty open $U \subset X$ such that $\mathcal{H}|_U = \underline{d\mathbf{Z}}_U$ for some $d \in \mathbf{Z}$.*

**Proof.** Recall that $\underline{\mathbf{Z}}(V) = \mathbf{Z}$ for any nonempty open $V$ of $X$ (see proof of Lemma 20.2). If $\mathcal{H} = 0$, then the lemma holds with $d = 0$. If $\mathcal{H} \neq 0$, then there exists a nonempty open $U \subset X$ such that $\mathcal{H}(U) \neq 0$. Say $\mathcal{H}(U) = n\mathbf{Z}$ for some $n \geq 1$. Hence we see that $\underline{n\mathbf{Z}}_U \subset \mathcal{H}|_U \subset \underline{\mathbf{Z}}_U$. If the first inclusion is strict we can find a nonempty $U' \subset U$ and an integer $1 \leq n' < n$ such that $\underline{n'\mathbf{Z}}_{U'} \subset \mathcal{H}|_{U'} \subset \underline{\mathbf{Z}}_{U'}$. This process has to stop after a finite number of steps, and hence we get the lemma. $\square$

**Proposition 20.7** (Grothendieck)**.** *Let $X$ be a Noetherian topological space. If $\dim(X) \leq d$, then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and any abelian sheaf $\mathcal{F}$ on $X$.*

**Proof.** We prove this lemma by induction on $d$. So fix $d$ and assume the lemma holds for all Noetherian topological spaces of dimension $< d$.

Let $\mathcal{F}$ be an abelian sheaf on $X$. Suppose $U \subset X$ is an open. Let $Z \subset X$ denote the closed complement. Denote $j : U \to X$ and $i : Z \to X$ the inclusion maps. Then there is a short exact sequence

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

see Modules, Lemma 7.1. Note that $j_! j^* \mathcal{F}$ is supported on the topological closure $Z'$ of $U$, i.e., it is of the form $i'_* \mathcal{F}'$ for some abelian sheaf $\mathcal{F}'$ on $Z'$, where $i' : Z' \to X$ is the inclusion.

We can use this to reduce to the case where $X$ is irreducible. Namely, according to Topology, Lemma 9.2 $X$ has finitely many irreducible components. If $X$ has more than one irreducible component, then let $Z \subset X$ be an irreducible component of $X$ and set $U = X \setminus Z$. By the above, and the long exact sequence of cohomology, it suffices to prove the vanishing of $H^p(X, i_* i^* \mathcal{F})$ and $H^p(X, i'_* \mathcal{F}')$ for $p > d$. By Lemma 20.1 it suffices to prove $H^p(Z, i^* \mathcal{F})$ and $H^p(Z', \mathcal{F}')$ vanish for $p > d$. Since $Z'$ and $Z$ have fewer irreducible components we indeed reduce to the case of an irreducible $X$.

If $d = 0$ and $X$ is irreducible, then $X$ is the only nonempty open subset of $X$. Hence every sheaf is constant and higher cohomology groups vanish (for example by Lemma 20.2).

Suppose $X$ is irreducible of dimension $d > 0$. By Lemma 20.4 we reduce to the case where $\mathcal{F} = j_! \underline{\mathbf{Z}}_U$ for some open $U \subset X$. In this case we look at the short exact sequence

$$0 \to j_! (\underline{\mathbf{Z}}_U) \to \underline{\mathbf{Z}}_X \to i_* \underline{\mathbf{Z}}_Z \to 0$$

where $Z = X \setminus U$. By Lemma 20.2 we have the vanishing of $H^p(X, \underline{\mathbf{Z}}_X)$ for all $p \geq 1$. By induction we have $H^p(X, i_* \underline{\mathbf{Z}}_Z) = H^p(Z, \underline{\mathbf{Z}}_Z) = 0$ for $p \geq d$. Hence we win by the long exact cohomology sequence. $\square$

## 21. Cohomology with support in a closed subset

This section just discusses the bare minimum – the discussion will be continued in Section 34.

Let $X$ be a topological space and let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be an abelian sheaf on $X$. We let

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \mathrm{Supp}(s) \subset Z\}$$

be the subset of sections whose support is contained in $Z$. The support of a section is defined in Modules, Definition 5.1. Modules, Lemma 5.2 implies that $\Gamma_Z(X, \mathcal{F})$ is a subgroup of $\Gamma(X, \mathcal{F})$. The same lemma guarantees that the assignment $\mathcal{F} \mapsto \Gamma_Z(X, \mathcal{F})$ is a functor in $\mathcal{F}$. This functor is left exact but not exact in general.

Since the category of abelian sheaves has enough injectives (Injectives, Lemma 4.1) we we obtain a right derived functor

$$R\Gamma_Z(X, -) : D^+(X) \longrightarrow D^+(Ab)$$

by Derived Categories, Lemma 20.2. The value of $R\Gamma_Z(X, -)$ on an object $K$ is computed by representing $K$ by a bounded below complex $\mathcal{I}^\bullet$ of injective abelian sheaves and taking $\Gamma_Z(X, \mathcal{I}^\bullet)$, see Derived Categories, Lemma 20.1. The cohomology groups of an abelian sheaf $\mathcal{F}$ with support in $Z$ defined by $H_Z^q(X, \mathcal{F}) = R^q \Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on $X$. Let $U = X \setminus Z$. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Lemma 8.1) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D^+(X)$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D^+(Ab)$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H_Z^i(X, K) \to H^i(X, K) \to H^i(U, K) \to H_Z^{i+1}(X, K) \to \ldots$$

for any $K$ in $D^+(X)$.

For an abelian sheaf $\mathcal{F}$ on $X$ we can consider the *subsheaf of sections with support in $Z$*, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset U \cap Z\} = \Gamma_{Z \cap U}(U, \mathcal{F}|_U)$$

Using the equivalence of Modules, Lemma 6.1 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on $Z$, see Modules, Remark 6.2. Thus we obtain a functor

$$Ab(X) \longrightarrow Ab(Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as a sheaf on } Z$$

This functor is left exact, but in general not exact. Exactly as above we obtain a right derived functor

$$R\mathcal{H}_Z : D^+(X) \longrightarrow D^+(Z)$$

the derived functor. We set $\mathcal{H}_Z^q(\mathcal{F}) = R^q \mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}_Z^0(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$.

Observe that we have $\Gamma_Z(X, \mathcal{F}) = \Gamma(Z, \mathcal{H}_Z(\mathcal{F}))$ for any abelian sheaf $\mathcal{F}$. By Lemma 21.1 below the functor $\mathcal{H}_Z$ transforms injective abelian sheaves into sheaves right acyclic for $\Gamma(Z, -)$. Thus by Derived Categories, Lemma 22.2 we obtain a convergent Grothendieck spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{H}_Z^q(K)) \Rightarrow H_Z^{p+q}(X, K)$$

functorial in $K$ in $D^+(X)$.

**Lemma 21.1.** *Let $i : Z \to X$ be the inclusion of a closed subset. Let $\mathcal{I}$ be an injective abelian sheaf on $X$. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on $Z$.*

**Proof.** This follows from Homology, Lemma 29.1 as $\mathcal{H}_Z(-)$ is right adjoint to the exact functor $i_*$. See Modules, Lemmas 6.1 and 6.3. $\square$

## 22. Cohomology on spectral spaces

A key result on the cohomology of spectral spaces is Lemma 19.2 which loosely speaking says that cohomology commutes with cofiltered limits in the category of spectral spaces as defined in Topology, Definition 23.1. This can be applied to give analogues of Lemmas 16.3 and 18.1 as follows.

**Lemma 22.1.** *Let $X$ be a spectral space. Let $\mathcal{F}$ be an abelian sheaf on $X$. Let $E \subset X$ be a quasi-compact subset. Let $W \subset X$ be the set of points of $X$ which specialize to a point of $E$.*

    (1) *$H^p(W, \mathcal{F}|_W) = \operatorname{colim} H^p(U, \mathcal{F})$ where the colimit is over quasi-compact open neighbourhoods of $E$,*

    (2) *$H^p(W \setminus E, \mathcal{F}|_{W \setminus E}) = \operatorname{colim} H^p(U \setminus E, \mathcal{F}|_{U \setminus E})$ if $E$ is a constructible subset.*

**Proof.** From Topology, Lemma 24.7 we see that $W = \lim U$ where the limit is over the quasi-compact opens containing $E$. Each $U$ is a spectral space by Topology, Lemma 23.5. Thus we may apply Lemma 19.2 to conclude that (1) holds. The same proof works for part (2) except we use Topology, Lemma 24.8. $\square$

**Lemma 22.2.** *Let $f : X \to Y$ be a spectral map of spectral spaces. Let $y \in Y$. Let $E \subset Y$ be the set of points specializing to $y$. Let $\mathcal{F}$ be an abelian sheaf on $X$. Then $(R^p f_* \mathcal{F})_y = H^p(f^{-1}(E), \mathcal{F}|_{f^{-1}(E)})$.*

**Proof.** Observe that $E = \bigcap V$ where $V$ runs over the quasi-compact open neighbourhoods of $y$ in $Y$. Hence $f^{-1}(E) = \bigcap f^{-1}(V)$. This implies that $f^{-1}(E) = \lim f^{-1}(V)$ as topological spaces. Since $f$ is spectral, each $f^{-1}(V)$ is a spectral space too (Topology, Lemma 23.5). We conclude that $f^{-1}(E)$ is a spectral space and that

$$H^p(f^{-1}(E), \mathcal{F}|_{f^{-1}(E)}) = \operatorname{colim} H^p(f^{-1}(V), \mathcal{F})$$

by Lemma 19.2. On the other hand, the stalk of $R^p f_* \mathcal{F}$ at $y$ is given by the colimit on the right. $\square$

**Lemma 22.3.** *Let $X$ be a profinite topological space. Then $H^q(X, \mathcal{F}) = 0$ for all $q > 0$ and all abelian sheaves $\mathcal{F}$.*

**Proof.** Any open covering of $X$ can be refined by a finite disjoint union decomposition with open parts, see Topology, Lemma 22.4. Hence if $\mathcal{F} \to \mathcal{G}$ is a surjection of abelian sheaves on $X$, then $\mathcal{F}(X) \to \mathcal{G}(X)$ is surjective. In other words, the global sections functor is an exact functor. Therefore its higher derived functors are zero, see Derived Categories, Lemma 16.9. $\square$

The following result on cohomological vanishing improves Grothendieck's result (Proposition 20.7) and can be found in [Sch92].

**Proposition 22.4.** *Let $X$ be a spectral space of Krull dimension $d$. Let $\mathcal{F}$ be an abelian sheaf on $X$.*

    (1) *$H^q(X, \mathcal{F}) = 0$ for $q > d$,*

    (2) *$H^d(X, \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective for every quasi-compact open $U \subset X$,*

    (3) *$H^q_Z(X, \mathcal{F}) = 0$ for $q > d$ and any constructible closed subset $Z \subset X$.*

**Proof.** We prove this result by induction on $d$.

If $d = 0$, then $X$ is a profinite space, see Topology, Lemma 23.8. Thus (1) holds by Lemma 22.3. If $U \subset X$ is quasi-compact open, then $U$ is also closed as a quasi-compact subset of a Hausdorff space. Hence $X = U \amalg (X \setminus U)$ as a topological space and we see that (2) holds. Given $Z$ as in (3) we consider the long exact sequence

$$H^{q-1}(X, \mathcal{F}) \to H^{q-1}(X \setminus Z, \mathcal{F}) \to H^q_Z(X, \mathcal{F}) \to H^q(X, \mathcal{F})$$

Since $X$ and $U = X \setminus Z$ are profinite (namely $U$ is quasi-compact because $Z$ is constructible) and since we have (2) and (1) we obtain the desired vanishing of the cohomology groups with support in $Z$.

Induction step. Assume $d \geq 1$ and assume the proposition is valid for all spectral spaces of dimension $< d$. We first prove part (2) for $X$. Let $U$ be a quasi-compact open. Let $\xi \in H^d(U, \mathcal{F})$. Set $Z = X \setminus U$. Let $W \subset X$ be the set of points specializing to $Z$. By Lemma 22.1 we have

$$H^d(W \setminus Z, \mathcal{F}|_{W \setminus Z}) = \operatorname{colim}_{Z \subset V} H^d(V \setminus Z, \mathcal{F})$$

where the colimit is over the quasi-compact open neighbourhoods $V$ of $Z$ in $X$. By Topology, Lemma 24.7 we see that $W \setminus Z$ is a spectral space. Since every point of $W$ specializes to a point of $Z$, we see that $W \setminus Z$ is a spectral space of Krull dimension $< d$. By induction hypothesis we see that the image of $\xi$ in $H^d(W \setminus Z, \mathcal{F}|_{W \setminus Z})$ is zero. By the displayed formula, there exists a $Z \subset V \subset X$ quasi-compact open such that $\xi|_{V \setminus Z} = 0$. Since $V \setminus Z = V \cap U$ we conclude by the Mayer-Vietoris (Lemma 8.2) for the covering $X = U \cup V$ that there exists a $\tilde{\xi} \in H^d(X, \mathcal{F})$ which restricts to $\xi$ on $U$ and to zero on $V$. In other words, part (2) is true.

Proof of part (1) assuming (2). Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Set

$$\mathcal{G} = \operatorname{Im}(\mathcal{I}^{d-1} \to \mathcal{I}^d) = \operatorname{Ker}(\mathcal{I}^d \to \mathcal{I}^{d+1})$$

For $U \subset X$ quasi-compact open we have a map of exact sequences as follows

$$
\begin{array}{ccccccc}
\mathcal{I}^{d-1}(X) & \longrightarrow & \mathcal{G}(X) & \longrightarrow & H^d(X, \mathcal{F}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{I}^{d-1}(U) & \longrightarrow & \mathcal{G}(U) & \longrightarrow & H^d(U, \mathcal{F}) & \longrightarrow & 0
\end{array}
$$

The sheaf $\mathcal{I}^{d-1}$ is flasque by Lemma 12.2 and the fact that $d \geq 1$. By part (2) we see that the right vertical arrow is surjective. We conclude by a diagram chase that the map $\mathcal{G}(X) \to \mathcal{G}(U)$ is surjective. By Lemma 12.6 we conclude that $\check{H}^q(\mathcal{U}, \mathcal{G}) = 0$ for $q > 0$ and any finite covering $\mathcal{U} : U = U_1 \cup \ldots \cup U_n$ of a quasi-compact open by quasi-compact opens. Applying Lemma 11.9 we find that $H^q(U, \mathcal{G}) = 0$ for all $q > 0$ and all quasi-compact opens $U$ of $X$. By Leray's acyclicity lemma (Derived Categories, Lemma 16.7) we conclude that

$$H^q(X, \mathcal{F}) = H^q \left( \Gamma(X, \mathcal{I}^0) \to \ldots \to \Gamma(X, \mathcal{I}^{d-1}) \to \Gamma(X, \mathcal{G}) \right)$$

In particular the cohomology group vanishes if $q > d$.

Proof of (3). Given $Z$ as in (3) we consider the long exact sequence

$$H^{q-1}(X, \mathcal{F}) \to H^{q-1}(X \setminus Z, \mathcal{F}) \to H^q_Z(X, \mathcal{F}) \to H^q(X, \mathcal{F})$$

Since $X$ and $U = X \setminus Z$ are spectral spaces (Topology, Lemma 23.5) of dimension $\leq d$ and since we have (2) and (1) we obtain the desired vanishing. $\qquad\square$

## 23. The alternating Čech complex

This section compares the Čech complex with the alternating Čech complex and some related complexes.

Let $X$ be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. For $p \geq 0$ set

$$\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F}) = \left\{ \begin{array}{c} s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \text{ such that } s_{i_0 \ldots i_p} = 0 \text{ if } i_n = i_m \text{ for some } n \neq m \\ \text{and } s_{i_0 \ldots i_n \ldots i_m \ldots i_p} = -s_{i_0 \ldots i_m \ldots i_n \ldots i_p} \text{ in any case.} \end{array} \right\}$$

We omit the verification that the differential $d$ of Equation (9.0.1) maps $\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F})$ into $\check{\mathcal{C}}^{p+1}_{alt}(\mathcal{U}, \mathcal{F})$.

**Definition 23.1.** Let $X$ be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on $X$. The complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ is the *alternating Čech complex* associated to $\mathcal{F}$ and the open covering $\mathcal{U}$.

Hence there is a canonical morphism of complexes

$$\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

namely the inclusion of the alternating Čech complex into the usual Čech complex.

Suppose our covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ comes equipped with a total ordering $<$ on $I$. In this case, set

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \ldots, i_p) \in I^{p+1}, i_0 < \ldots < i_p} \mathcal{F}(U_{i_0 \ldots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in $\mathcal{F}(U_{i_0 \ldots i_p})$. We define

$$d : \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}_{ord}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}|_{U_{i_0 \ldots i_{p+1}}}$$

for any $i_0 < \ldots < i_{p+1}$. Note that this formula is identical to Equation (9.0.1). It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is a complex.

**Definition 23.2.** Let $X$ be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume given a total ordering on $I$. Let $\mathcal{F}$ be an abelian presheaf on $X$. The complex $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is the *ordered Čech complex* associated to $\mathcal{F}$, the open covering $\mathcal{U}$ and the given total ordering on $I$.

This complex is sometimes called the alternating Čech complex. The reason is that there is an obvious comparison map between the ordered Čech complex and the alternating Čech complex. Namely, consider the map

$$c : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

given by the rule

$$c(s)_{i_0 \ldots i_p} = \left\{ \begin{array}{ll} 0 & \text{if} \quad i_n = i_m \text{ for some } n \neq m \\ \text{sgn}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}} & \text{if} \quad i_{\sigma(0)} < i_{\sigma(1)} < \ldots < i_{\sigma(p)} \end{array} \right.$$

Here $\sigma$ denotes a permutation of $\{0, \ldots, p\}$ and $\text{sgn}(\sigma)$ denotes its sign. The alternating and ordered Čech complexes are often identified in the literature via the map $c$. Namely we have the following easy lemma.

**Lemma 23.3.** *Let $X$ be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume $I$ comes equipped with a total ordering. The map $c$ is a morphism of complexes. In fact it induces an isomorphism*

$$c : \check{\mathcal{C}}_{ord}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F})$$

*of complexes.*

**Proof.** Omitted. $\square$

There is also a map

$$\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}_{ord}^\bullet(\mathcal{U}, \mathcal{F})$$

which is described by the rule

$$\pi(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}$$

whenever $i_0 < i_1 < \ldots < i_p$.

**Lemma 23.4.** *Let $X$ be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume $I$ comes equipped with a total ordering. The map $\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}_{ord}^\bullet(\mathcal{U}, \mathcal{F})$ is a morphism of complexes. It induces an isomorphism*

$$\pi : \check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}_{ord}^\bullet(\mathcal{U}, \mathcal{F})$$

*of complexes which is a left inverse to the morphism $c$.*

**Proof.** Omitted. $\square$

**Remark 23.5.** This means that if we have two total orderings $<_1$ and $<_2$ on the index set $I$, then we get an isomorphism of complexes $\tau = \pi_2 \circ c_1 : \check{\mathcal{C}}_{ord\text{-}1}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}_{ord\text{-}2}(\mathcal{U}, \mathcal{F})$. It is clear that

$$\tau(s)_{i_0 \ldots i_p} = \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

where $i_0 <_1 i_1 <_1 \ldots <_1 i_p$ and $i_{\sigma(0)} <_2 i_{\sigma(1)} <_2 \ldots <_2 i_{\sigma(p)}$. This is the sense in which the ordered Čech complex is independent of the chosen total ordering.

**Lemma 23.6.** *Let $X$ be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume $I$ comes equipped with a total ordering. The map $c \circ \pi$ is homotopic to the identity on $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$. In particular the inclusion map $\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a homotopy equivalence.*

**Proof.** For any multi-index $(i_0, \ldots, i_p) \in I^{p+1}$ there exists a unique permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ such that

$$i_{\sigma(0)} \leq i_{\sigma(1)} \leq \ldots \leq i_{\sigma(p)} \quad \text{and} \quad \sigma(j) < \sigma(j+1) \quad \text{if} \quad i_{\sigma(j)} = i_{\sigma(j+1)}.$$

We denote this permutation $\sigma = \sigma^{i_0 \ldots i_p}$.

For any permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ and any $a$, $0 \leq a \leq p$ we denote $\sigma_a$ the permutation of $\{0, \ldots, p\}$ such that

$$\sigma_a(j) = \begin{cases} \sigma(j) & \text{if} \quad 0 \leq j < a, \\ \min\{j' \mid j' > \sigma_a(j-1), j' \neq \sigma(k), \forall k < a\} & \text{if} \quad a \leq j \end{cases}$$

So if $p = 3$ and $\sigma$, $\tau$ are given by

| id | 0 | 1 | 2 | 3 |     | id | 0 | 1 | 2 | 3 |
|----|---|---|---|---|-----|----|---|---|---|---|
| $\sigma$ | 3 | 2 | 1 | 0 | and | $\tau$ | 3 | 0 | 2 | 1 |

then we have

| id | 0 | 1 | 2 | 3 | | id | 0 | 1 | 2 | 3 |
|----|---|---|---|---|--|----|---|---|---|---|
| $\sigma_0$ | 0 | 1 | 2 | 3 | | $\tau_0$ | 0 | 1 | 2 | 3 |
| $\sigma_1$ | 3 | 0 | 1 | 2 | and | $\tau_1$ | 3 | 0 | 1 | 2 |
| $\sigma_2$ | 3 | 2 | 0 | 1 | | $\tau_2$ | 3 | 0 | 1 | 2 |
| $\sigma_3$ | 3 | 2 | 1 | 0 | | $\tau_3$ | 3 | 0 | 2 | 1 |

It is clear that always $\sigma_0 = \mathrm{id}$ and $\sigma_p = \sigma$.

Having introduced this notation we define for $s \in \check{C}^{p+1}(\mathcal{U}, \mathcal{F})$ the element $h(s) \in \check{C}^p(\mathcal{U}, \mathcal{F})$ to be the element with components

$$(23.6.1) \qquad h(s)_{i_0\ldots i_p} = \sum_{0 \leq a \leq p} (-1)^a \mathrm{sign}(\sigma_a) s_{i_{\sigma(0)}\ldots i_{\sigma(a)} i_{\sigma_a(a)}\ldots i_{\sigma_a(p)}}$$

where $\sigma = \sigma^{i_0\ldots i_p}$. The index $i_{\sigma(a)}$ occurs twice in $i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}$ once in the first group of $a + 1$ indices and once in the second group of $p - a + 1$ indices since $\sigma_a(j) = \sigma(a)$ for some $j \geq a$ by definition of $\sigma_a$. Hence the sum makes sense since each of the elements $s_{i_{\sigma(0)}\ldots i_{\sigma(a)} i_{\sigma_a(a)}\ldots i_{\sigma_a(p)}}$ is defined over the open $U_{i_0\ldots i_p}$. Note also that for $a = 0$ we get $s_{i_0\ldots i_p}$ and for $a = p$ we get $(-1)^p \mathrm{sign}(\sigma) s_{i_{\sigma(0)}\ldots i_{\sigma(p)}}$.

We claim that

$$(dh + hd)(s)_{i_0\ldots i_p} = s_{i_0\ldots i_p} - \mathrm{sign}(\sigma) s_{i_{\sigma(0)}\ldots i_{\sigma(p)}}$$

where $\sigma = \sigma^{i_0\ldots i_p}$. We omit the verification of this claim. (There is a PARI/gp script called first-homotopy.gp in the stacks-project subdirectory scripts which can be used to check finitely many instances of this claim. We wrote this script to make sure the signs are correct.) Write

$$\kappa : \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^\bullet(\mathcal{U}, \mathcal{F})$$

for the operator given by the rule

$$\kappa(s)_{i_0\ldots i_p} = \mathrm{sign}(\sigma^{i_0\ldots i_p}) s_{i_{\sigma(0)}\ldots i_{\sigma(p)}}.$$

The claim above implies that $\kappa$ is a morphism of complexes and that $\kappa$ is homotopic to the identity map of the Čech complex. This does not immediately imply the lemma since the image of the operator $\kappa$ is not the alternating subcomplex. Namely, the image of $\kappa$ is the "semi-alternating" complex $\check{C}^p_{semi\text{-}alt}(\mathcal{U}, \mathcal{F})$ where $s$ is a $p$-cochain of this complex if and only if

$$s_{i_0\ldots i_p} = \mathrm{sign}(\sigma) s_{i_{\sigma(0)}\ldots i_{\sigma(p)}}$$

for any $(i_0, \ldots, i_p) \in I^{p+1}$ with $\sigma = \sigma^{i_0\ldots i_p}$. We introduce yet another variant Čech complex, namely the semi-ordered Čech complex defined by

$$\check{C}^p_{semi\text{-}ord}(\mathcal{U}, \mathcal{F}) = \prod_{i_0 \leq i_1 \leq \ldots \leq i_p} \mathcal{F}(U_{i_0\ldots i_p})$$

It is easy to see that Equation (9.0.1) also defines a differential and hence that we get a complex. It is also clear (analogous to Lemma 23.4) that the projection map

$$\check{C}^\bullet_{semi\text{-}alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

is an isomorphism of complexes.

Hence the Lemma follows if we can show that the obvious inclusion map

$$\check{C}^p_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^p_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

is a homotopy equivalence. To see this we use the homotopy
(23.6.2)
$$h(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} & i_0 < i_1 < \ldots < i_p \\ (-1)^a s_{i_0\ldots i_{a-1}i_a i_a i_{a+1}\ldots i_p} & \text{if} & i_0 < i_1 < \ldots < i_{a-1} < i_a = i_{a+1} \end{cases}$$

We claim that
$$(dh + hd)(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} & i_0 < i_1 < \ldots < i_p \\ s_{i_0\ldots i_p} & \text{else} \end{cases}$$

We omit the verification. (There is a PARI/gp script called second-homotopy.gp
in the stacks-project subdirectory scripts which can be used to check finitely many
instances of this claim. We wrote this script to make sure the signs are correct.)
The claim clearly shows that the composition
$$\check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

of the projection with the natural inclusion is homotopic to the identity map as
desired. $\square$

**Lemma 23.7.** *Let $X$ be a topological space. Let $\mathcal{F}$ be an abelian presheaf on $X$.
Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. If $U_i = U$ for some $i \in I$, then the
extended alternating Čech complex*
$$\mathcal{F}(U) \to \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

*obtained by putting $\mathcal{F}(U)$ in degree $-1$ with differential given by the canonical map
of $\mathcal{F}(U)$ into $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F})$ is homotopy equivalent to $0$. Similarly, for any total ordering
on $I$ the extended ordered Čech complex*
$$\mathcal{F}(U) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

*is homotopy equivalent to $0$.*

**First proof.** Combine Lemmas 9.3 and 23.6. $\square$

**Second proof.** Since the alternating and ordered Čech complexes are isomorphic
it suffices to prove this for the ordered one. We will use standard notation: a cochain
$s$ of degree $p$ in the extended ordered Čech complex has the form $s = (s_{i_0\ldots i_p})$ where
$s_{i_0\ldots i_p}$ is in $\mathcal{F}(U_{i_0\ldots i_p})$ and $i_0 < \ldots < i_p$. With this notation we have
$$d(x)_{i_0\ldots i_{p+1}} = \sum_j (-1)^j x_{i_0\ldots \hat{i}_j \ldots i_p}$$

Fix an index $i \in I$ with $U = U_i$. As homotopy we use the maps
$$h : \text{cochains of degree } p + 1 \to \text{cochains of degree } p$$

given by the rule
$$h(s)_{i_0\ldots i_p} = 0 \text{ if } i \in \{i_0, \ldots, i_p\} \text{ and } h(s)_{i_0\ldots i_p} = (-1)^j s_{i_0\ldots i_j i i_{j+1}\ldots i_p} \text{ if not}$$

Here $j$ is the unique index such that $i_j < i < i_{j+1}$ in the second case; also, since
$U = U_i$ we have the equality
$$\mathcal{F}(U_{i_0\ldots i_p}) = \mathcal{F}(U_{i_0\ldots i_j i i_{j+1}\ldots i_p})$$

which we can use to make sense of thinking of $(-1)^j s_{i_0\ldots i_j i i_{j+1}\ldots i_p}$ as an element of
$\mathcal{F}(U_{i_0\ldots i_p})$. We will show by a computation that $dh + hd$ equals the negative of the
identity map which finishes the proof. To do this fix $s$ a cochain of degree $p$ and
let $i_0 < \ldots < i_p$ be elements of $I$.

Case I: $i \in \{i_0, \ldots, i_p\}$. Say $i = i_t$. Then we have $h(d(s))_{i_0 \ldots i_p} = 0$. On the other hand we have

$$d(h(s))_{i_0 \ldots i_p} = \sum (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} = (-1)^t h(s)_{i_0 \ldots \hat{i} \ldots i_p} = (-1)^t (-1)^{t-1} s_{i_0 \ldots i_p}$$

Thus $(dh + hd)(s)_{i_0 \ldots i_p} = -s_{i_0 \ldots i_p}$ as desired.

Case II: $i \notin \{i_0, \ldots, i_p\}$. Let $j$ be such that $i_j < i < i_{j+1}$. Then we see that

$$
\begin{aligned}
h(d(s))_{i_0 \ldots i_p} &= (-1)^j d(s)_{i_0 \ldots i_j i i_{j+1} \ldots i_p} \\
&= \sum_{j' \le j} (-1)^{j+j'} s_{i_0 \ldots \hat{i}_{j'} \ldots i_j i i_{j+1} \ldots i_p} - s_{i_0 \ldots i_p} \\
&+ \sum_{j' > j} (-1)^{j+j'+1} s_{i_0 \ldots i_j i i_{j+1} \ldots \hat{i}_{j'} \ldots i_p}
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
d(h(s))_{i_0 \ldots i_p} &= \sum_{j'} (-1)^{j'} h(s)_{i_0 \ldots \hat{i}_{j'} \ldots i_p} \\
&= \sum_{j' \le j} (-1)^{j'+j-1} s_{i_0 \ldots \hat{i}_{j'} \ldots i_j i i_{j+1} \ldots i_p} \\
&+ \sum_{j' > j} (-1)^{j'+j} s_{i_0 \ldots i_j i i_{j+1} \ldots \hat{i}_{j'} \ldots i_p}
\end{aligned}
$$

Adding these up we obtain $(dh + hd)(s)_{i_0 \ldots i_p} = -s_{i_0 \ldots i_p}$ as desired. $\square$

## 24. Alternative view of the Čech complex

In this section we discuss an alternative way to establish the relationship between the Čech complex and cohomology.

**Lemma 24.1.** *Let $X$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of $X$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{i_0 \ldots i_p}$ the restriction of $\mathcal{F}$ to $U_{i_0 \ldots i_p}$. There exists a complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{O}_X$-modules with*

$$\mathfrak{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 \ldots i_p} (j_{i_0 \ldots i_p})_* \mathcal{F}_{i_0 \ldots i_p}$$

*and differential $d : \mathfrak{C}^p(\mathcal{U}, \mathcal{F}) \to \mathfrak{C}^{p+1}(\mathcal{U}, \mathcal{F})$ as in Equation (9.0.1). Moreover, there exists a canonical map*

$$\mathcal{F} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$$

*which is a quasi-isomorphism, i.e., $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$.*

**Proof.** We check

$$0 \to \mathcal{F} \to \mathfrak{C}^0(\mathcal{U}, \mathcal{F}) \to \mathfrak{C}^1(\mathcal{U}, \mathcal{F}) \to \ldots$$

is exact on stalks. Let $x \in X$ and choose $i_{\mathrm{fix}} \in I$ such that $x \in U_{i_{\mathrm{fix}}}$. Then define

$$h : \mathfrak{C}^p(\mathcal{U}, \mathcal{F})_x \to \mathfrak{C}^{p-1}(\mathcal{U}, \mathcal{F})_x$$

as follows: If $s \in \mathfrak{C}^p(\mathcal{U}, \mathcal{F})_x$, take a representative

$$\tilde{s} \in \mathfrak{C}^p(\mathcal{U}, \mathcal{F})(V) = \prod_{i_0 \ldots i_p} \mathcal{F}(V \cap U_{i_0} \cap \ldots \cap U_{i_p})$$

defined on some neighborhood $V$ of $x$, and set

$$h(s)_{i_0 \ldots i_{p-1}} = \tilde{s}_{i_{\mathrm{fix}} i_0 \ldots i_{p-1}, x}.$$

By the same formula (for $p = 0$) we get a map $\mathfrak{C}^0(\mathcal{U}, \mathcal{F})_x \to \mathcal{F}_x$. We compute formally as follows:

$$
\begin{aligned}
(dh + hd)(s)_{i_0 \ldots i_p} &= \sum_{j=0}^p (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + d(s)_{i_{\text{fix}} i_0 \ldots i_p} \\
&= \sum_{j=0}^p (-1)^j s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} + s_{i_0 \ldots i_p} + \sum_{j=0}^p (-1)^{j+1} s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} \\
&= s_{i_0 \ldots i_p}
\end{aligned}
$$

This shows $h$ is a homotopy from the identity map of the extended complex

$$
0 \to \mathcal{F}_x \to \mathfrak{C}^0(\mathcal{U}, \mathcal{F})_x \to \mathfrak{C}^1(\mathcal{U}, \mathcal{F})_x \to \ldots
$$

to zero and we conclude. $\qquad\square$

With this lemma it is easy to reprove the Čech to cohomology spectral sequence of Lemma 11.5. Namely, let $X$, $\mathcal{U}$, $\mathcal{F}$ as in Lemma 24.1 and let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then we may consider the double complex

$$
A^{\bullet, \bullet} = \Gamma(X, \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).
$$

By construction we have

$$
A^{p,q} = \prod_{i_0 \ldots i_p} \mathcal{I}^q(U_{i_0 \ldots i_p})
$$

Consider the two spectral sequences of Homology, Section 25 associated to this double complex, see especially Homology, Lemma 25.1. For the spectral sequence $('E_r, 'd_r)_{r \geq 0}$ we get $'E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ because taking products is exact (Homology, Lemma 32.1). For the spectral sequence $(''E_r, ''d_r)_{r \geq 0}$ we get $''E_2^{p,q} = 0$ if $p > 0$ and $''E_2^{0,q} = H^q(X, \mathcal{F})$. Namely, for fixed $q$ the complex of sheaves $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^q)$ is a resolution (Lemma 24.1) of the injective sheaf $\mathcal{I}^q$ by injective sheaves (by Lemmas 7.1 and 11.11 and Homology, Lemma 27.3). Hence the cohomology of $\Gamma(X, \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^q))$ is zero in positive degrees and equal to $\Gamma(X, \mathcal{I}^q)$ in degree 0. Taking cohomology of the next differential we get our claim about the spectral sequence $(''E_r, ''d_r)_{r \geq 0}$. Whence the result since both spectral sequences converge to the cohomology of the associated total complex of $A^{\bullet, \bullet}$.

**Definition 24.2.** Let $X$ be a topological space. An open covering $X = \bigcup_{i \in I} U_i$ is said to be *locally finite* if for every $x \in X$ there exists an open neighbourhood $W$ of $x$ such that $\{i \in I \mid W \cap U_i \neq \emptyset\}$ is finite.

**Remark 24.3.** Let $X = \bigcup_{i \in I} U_i$ be a locally finite open covering. Denote $j_i : U_i \to X$ the inclusion map. Suppose that for each $i$ we are given an abelian sheaf $\mathcal{F}_i$ on $U_i$. Consider the abelian sheaf $\mathcal{G} = \bigoplus_{i \in I} (j_i)_* \mathcal{F}_i$. Then for $V \subset X$ open we actually have

$$
\Gamma(V, \mathcal{G}) = \prod_{i \in I} \mathcal{F}_i(V \cap U_i).
$$

In other words we have

$$
\bigoplus_{i \in I} (j_i)_* \mathcal{F}_i = \prod_{i \in I} (j_i)_* \mathcal{F}_i
$$

This seems strange until you realize that the direct sum of a collection of sheaves is the sheafification of what you think it should be. See discussion in Modules, Section 3. Thus we conclude that in this case the complex of Lemma 24.1 has terms

$$
\mathfrak{C}^p(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0 \ldots i_p} (j_{i_0 \ldots i_p})_* \mathcal{F}_{i_0 \ldots i_p}
$$

which is sometimes useful.

## 25. Čech cohomology of complexes

In general for sheaves of abelian groups $\mathcal{F}$ and $\mathcal{G}$ on $X$ there is a cup product map

$$H^i(X, \mathcal{F}) \times H^j(X, \mathcal{G}) \longrightarrow H^{i+j}(X, \mathcal{F} \otimes_{\mathbf{Z}} \mathcal{G}).$$

In this section we define it using Čech cocycles by an explicit formula for the cup product. If you are worried about the fact that cohomology may not equal Čech cohomology, then you can use hypercoverings and still use the cocycle notation. This also has the advantage that it works to define the cup product for hypercohomology on any topos (insert future reference here).

Let $\mathcal{F}^\bullet$ be a bounded below complex of presheaves of abelian groups on $X$. We can often compute $H^n(X, \mathcal{F}^\bullet)$ using Čech cocycles. Namely, let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of $X$. Since the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ (Definition 9.1) is functorial in the presheaf $\mathcal{F}$ we obtain a double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$. The associated total complex to $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ is the complex with degree $n$ term

$$\mathrm{Tot}^n(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) = \bigoplus_{p+q=n} \prod_{i_0 \ldots i_p} \mathcal{F}^q(U_{i_0 \ldots i_p})$$

see Homology, Definition 18.3. A typical element in $\mathrm{Tot}^n$ will be denoted $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ where $\alpha_{i_0 \ldots i_p} \in \mathcal{F}^q(U_{i_0 \ldots i_p})$. In other words the $\mathcal{F}$-degree of $\alpha_{i_0 \ldots i_p}$ is $q = n - p$. This notation requires us to be aware of the degree $\alpha$ lives in at all times. We indicate this situation by the formula $\deg_{\mathcal{F}}(\alpha_{i_0 \ldots i_p}) = q$. According to our conventions in Homology, Definition 18.3 the differential of an element $\alpha$ of degree $n$ is given by

$$d(\alpha)_{i_0 \ldots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_0 \ldots i_{p+1}})$$

where $d_{\mathcal{F}}$ denotes the differential on the complex $\mathcal{F}^\bullet$. The expression $\alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$ means the restriction of $\alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \in \mathcal{F}(U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}})$ to $U_{i_0 \ldots i_{p+1}}$.

The construction of $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ is functorial in $\mathcal{F}^\bullet$. As well there is a functorial transformation

(25.0.1) $$\Gamma(X, \mathcal{F}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ with $\alpha_{i_0} = s|_{U_{i_0}}$ and $\alpha_{i_0 \ldots i_p} = 0$ for $p > 0$.

Refinements. Let $\mathcal{V} = \{V_j\}_{j \in J}$ be a refinement of $\mathcal{U}$. This means there is a map $t : J \to I$ such that $V_j \subset U_{t(j)}$ for all $j \in J$. This gives rise to a functorial transformation

(25.0.2) $$T_t : \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)).$$

defined by the rule

$$T_t(\alpha)_{j_0 \ldots j_p} = \alpha_{t(j_0) \ldots t(j_p)}|_{V_{j_0 \ldots j_p}}.$$

Given two maps $t, t' : J \to I$ as above the maps $T_t$ and $T_{t'}$ constructed above are homotopic. The homotopy is given by

$$h(\alpha)_{j_0 \ldots j_p} = \sum_{a=0}^{p} (-1)^a \alpha_{t(j_0) \ldots t(j_a) t'(j_a) \ldots t'(j_p)}$$

for an element $\alpha$ of degree $n$. This works because of the following computation, again with $\alpha$ an element of degree $n$ (so $d(\alpha)$ has degree $n+1$ and $h(\alpha)$ has degree $n-1$):

$$
\begin{aligned}
(d(h(\alpha)) + h(d(\alpha)))_{j_0...j_p} &= \sum_{k=0}^{p}(-1)^k h(\alpha)_{j_0...\hat{j}_k...j_p} + \\
&\quad (-1)^p d_{\mathcal{F}}(h(\alpha)_{j_0...j_p}) + \\
&\quad \sum_{a=0}^{p}(-1)^a d(\alpha)_{t(j_0)...t(j_a)t'(j_a)...t'(j_p)} \\
&= \sum_{k=0}^{p}\sum_{a=0}^{k-1}(-1)^{k+a}\alpha_{t(j_0)...t(j_a)t'(j_a)...t'(\hat{j}_k)...t'(j_p)} + \\
&\quad \sum_{k=0}^{p}\sum_{a=k+1}^{p}(-1)^{k+a-1}\alpha_{t(j_0)...t(\hat{j}_k)...t(j_a)t'(j_a)...t'(j_p)} + \\
&\quad \sum_{a=0}^{p}(-1)^{p+a}d_{\mathcal{F}}(\alpha_{t(j_0)...t(j_a)t'(j_a)...t'(j_p)}) + \\
&\quad \sum_{a=0}^{p}\sum_{k=0}^{a}(-1)^{a+k}\alpha_{t(j_0)...t(\hat{j}_k)...t(j_a)t'(j_a)...t'(j_p)} + \\
&\quad \sum_{a=0}^{p}\sum_{k=a}^{p}(-1)^{a+k+1}\alpha_{t(j_0)...t(j_a)t'(j_a)...t'(\hat{j}_k)...t'(j_p)} + \\
&\quad \sum_{a=0}^{p}(-1)^{a+p+1}d_{\mathcal{F}}(\alpha_{t(j_0)...t(j_a)t'(j_a)...t'(j_p)}) \\
&= \alpha_{t'(j_0)...t'(j_p)} + (-1)^{2p+1}\alpha_{t(j_0)...t(j_p)} \\
&= T_{t'}(\alpha)_{j_0...j_p} - T_t(\alpha)_{j_0...j_p}
\end{aligned}
$$

We leave it to the reader to verify the cancellations. (Note that the terms having both $k$ and $a$ in the 1st, 2nd and 4th, 5th summands cancel, except the ones where $a = k$ which only occur in the 4th and 5th and these cancel against each other except for the two desired terms.) It follows that the induced map

$$H^n(T_t) : H^n(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to H^n(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)))$$

is independent of the choice of $t$. We define *Čech hypercohomology* as the limit of the Čech cohomology groups over all refinements via the maps $H^\bullet(T_t)$.

In the limit (over all open coverings of $X$) the following lemma provides a map of Čech hypercohomology into cohomology, which is often an isomorphism and is always an isomorphism if we use hypercoverings.

**Lemma 25.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. For a bounded below complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical map*

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

*functorial in $\mathcal{F}^\bullet$ and compatible with (25.0.1) and (25.0.2). There is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_2^{p,q} = H^p(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \underline{H}^q(\mathcal{F}^\bullet))))$$

*converging to $H^{p+q}(X, \mathcal{F}^\bullet)$.*

**Proof.** Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives. The map (25.0.1) for $\mathcal{I}^\bullet$ is a map $\Gamma(X, \mathcal{I}^\bullet) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$. This is a quasi-isomorphism of complexes of abelian groups as follows from Homology, Lemma 25.4 applied to the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ using Lemma 11.1. Suppose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of

$\mathcal{F}^\bullet$ into a bounded below complex of injectives. Since $R\Gamma(X, \mathcal{F}^\bullet)$ is represented by the complex $\Gamma(X, \mathcal{I}^\bullet)$ we obtain the map of the lemma using

$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

We omit the verification of functoriality and compatibilities. To construct the spectral sequence of the lemma, choose a Cartan-Eilenberg resolution $\mathcal{F}^\bullet \to \mathcal{I}^{\bullet,\bullet}$, see Derived Categories, Lemma 21.2. In this case $\mathcal{F}^\bullet \to \text{Tot}(\mathcal{I}^{\bullet,\bullet})$ is an injective resolution and hence

$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{Tot}(\mathcal{I}^{\bullet,\bullet})))$$

computes $R\Gamma(X, \mathcal{F}^\bullet)$ as we've seen above. By Homology, Remark 18.4 we can view this as the total complex associated to the triple complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^{\bullet,\bullet})$ hence, using the same remark we can view it as the total complex associate to the double complex $A^{\bullet,\bullet}$ with terms

$$A^{n,m} = \bigoplus_{p+q=n} \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^{q,m})$$

Since $\mathcal{I}^{q,\bullet}$ is an injective resolution of $\mathcal{F}^q$ we can apply the first spectral sequence associated to $A^{\bullet,\bullet}$ (Homology, Lemma 25.1) to get a spectral sequence with

$$E_1^{n,m} = \bigoplus_{p+q=n} \check{\mathcal{C}}^p(\mathcal{U}, \underline{H}^m(\mathcal{F}^q))$$

which is the $n$th term of the complex $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \underline{H}^m(\mathcal{F}^\bullet)))$. Hence we obtain $E_2$ terms as described in the lemma. Convergence by Homology, Lemma 25.3. $\square$

**Lemma 25.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. If $H^i(U_{i_0 \ldots i_p}, \mathcal{F}^q) = 0$ for all $i > 0$ and all $p, i_0, \ldots, i_p, q$, then the map $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \to R\Gamma(X, \mathcal{F}^\bullet)$ of Lemma 25.1 is an isomorphism.*

**Proof.** Immediate from the spectral sequence of Lemma 25.1. $\square$

**Remark 25.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. Let $b$ be an integer. We claim there is a commutative diagram

$$\begin{array}{ccc} \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))[b] & \longrightarrow & R\Gamma(X, \mathcal{F}^\bullet)[b] \\ \gamma \downarrow & & \downarrow \\ \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet[b])) & \longrightarrow & R\Gamma(X, \mathcal{F}^\bullet[b]) \end{array}$$

in the derived category where the map $\gamma$ is the map on complexes constructed in Homology, Remark 18.5. This makes sense because the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet[b])$ is clearly the same as the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)[0, b]$ introduced in Homology, Remark 18.5. To check that the diagram commutes, we may choose an injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ as in the proof of Lemma 25.1. Chasing diagrams, we see that it suffices to check the diagram commutes when we replace $\mathcal{F}^\bullet$ by $\mathcal{I}^\bullet$. Then we

consider the extended diagram

$$
\begin{array}{ccccc}
\Gamma(X,\mathcal{I}^\bullet)[b] & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{I}^\bullet))[b] & \longrightarrow & R\Gamma(X,\mathcal{I}^\bullet)[b] \\
\downarrow & & {\scriptstyle\gamma}\downarrow & & \downarrow \\
\Gamma(X,\mathcal{I}^\bullet[b]) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{I}^\bullet[b])) & \longrightarrow & R\Gamma(X,\mathcal{I}^\bullet[b])
\end{array}
$$

where the left horizontal arrows are (25.0.1). Since in this case the horizonal arrows are isomorphisms in the derived category (see proof of Lemma 25.1) it suffices to show that the left square commutes. This is true because the map $\gamma$ uses the sign 1 on the summands $\check{\mathcal{C}}^0(\mathcal{U},\mathcal{I}^{q+b})$, see formula in Homology, Remark 18.5.

Let $X$ be a topological space, let $\mathcal{U} : X = \bigcup_{i\in I} U_i$ be an open covering, and let $\mathcal{F}^\bullet$ be a bounded below complex of presheaves of abelian groups. Consider the map $\tau : \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}^\bullet)) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}^\bullet))$ defined by

$$
\tau(\alpha)_{i_0\dots i_p} = (-1)^{p(p+1)/2}\alpha_{i_p\dots i_0}.
$$

Then we have for an element $\alpha$ of degree $n$ that

$$
\begin{aligned}
& d(\tau(\alpha))_{i_0\dots i_{p+1}} \\
&= \sum_{j=0}^{p+1}(-1)^j\tau(\alpha)_{i_0\dots\hat{i}_j\dots i_{p+1}} + (-1)^{p+1}d_\mathcal{F}(\tau(\alpha)_{i_0\dots i_{p+1}}) \\
&= \sum_{j=0}^{p+1}(-1)^{j+\frac{p(p+1)}{2}}\alpha_{i_{p+1}\dots\hat{i}_j\dots i_0} + (-1)^{p+1+\frac{(p+1)(p+2)}{2}}d_\mathcal{F}(\alpha_{i_{p+1}\dots i_0})
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
& \tau(d(\alpha))_{i_0\dots i_{p+1}} \\
&= (-1)^{\frac{(p+1)(p+2)}{2}}d(\alpha)_{i_{p+1}\dots i_0} \\
&= (-1)^{\frac{(p+1)(p+2)}{2}}\left(\sum_{j=0}^{p+1}(-1)^j\alpha_{i_{p+1}\dots\hat{i}_{p+1-j}\dots i_0} + (-1)^{p+1}d_\mathcal{F}(\alpha_{i_{p+1}\dots i_0})\right)
\end{aligned}
$$

Thus we conclude that $d(\tau(\alpha)) = \tau(d(\alpha))$ because $p(p+1)/2 \equiv (p+1)(p+2)/2 + p+1 \bmod 2$. In other words $\tau$ is an endomorphism of the complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}^\bullet))$. Note that the diagram

$$
\begin{array}{ccc}
\Gamma(X,\mathcal{F}^\bullet) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}^\bullet)) \\
\downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\tau} \\
\Gamma(X,\mathcal{F}^\bullet) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}^\bullet))
\end{array}
$$

commutes. In addition $\tau$ is clearly compatible with refinements. This suggests that $\tau$ acts as the identity on Čech cohomology (i.e., in the limit – provided Čech hypercohomology agrees with hypercohomology, which is always the case if we use hypercoverings). We claim that $\tau$ actually is homotopic to the identity on the total Čech complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}^\bullet))$. To prove this, we use as homotopy

$$
h(\alpha)_{i_0\dots i_p} = \sum_{a=0}^{p}\epsilon_p(a)\alpha_{i_0\dots i_a i_p\dots i_a} \quad\text{with}\quad \epsilon_p(a) = (-1)^{\frac{(p-a)(p-a-1)}{2}+p}
$$

for $\alpha$ of degree $n$. As usual we omit writing $|_{U_{i_0...i_p}}$. This works because of the following computation, again with $\alpha$ an element of degree $n$:

$$
\begin{aligned}
(d(h(\alpha)) + h(d(\alpha)))_{i_0...i_p} =& \sum_{k=0}^{p}(-1)^k h(\alpha)_{i_0...\hat{i}_k...i_p} + \\
& (-1)^p d_{\mathcal{F}}(h(\alpha)_{i_0...i_p}) + \\
& \sum_{a=0}^{p} \epsilon_p(a) d(\alpha)_{i_0...i_a i_p...i_a} \\
=& \sum_{k=0}^{p} \sum_{a=0}^{k-1}(-1)^k \epsilon_{p-1}(a) \alpha_{i_0...i_a i_p...\hat{i}_k...i_a} + \\
& \sum_{k=0}^{p} \sum_{a=k+1}^{p}(-1)^k \epsilon_{p-1}(a-1) \alpha_{i_0...\hat{i}_k...i_a i_p...i_a} + \\
& \sum_{a=0}^{p}(-1)^p \epsilon_p(a) d_{\mathcal{F}}(\alpha_{i_0...i_a i_p...i_a}) + \\
& \sum_{a=0}^{p} \sum_{k=0}^{a} \epsilon_p(a)(-1)^k \alpha_{i_0...\hat{i}_k...i_a i_p...i_a} + \\
& \sum_{a=0}^{p} \sum_{k=a}^{p} \epsilon_p(a)(-1)^{p+a+1-k} \alpha_{i_0...i_a i_p...\hat{i}_k...i_a} + \\
& \sum_{a=0}^{p} \epsilon_p(a)(-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_0...i_a i_p...i_a}) \\
=& \epsilon_p(0) \alpha_{i_p...i_0} + \epsilon_p(p)(-1)^{p+1} \alpha_{i_0...i_p} \\
=& (-1)^{\frac{p(p+1)}{2}} \alpha_{i_p...i_0} - \alpha_{i_0...i_p}
\end{aligned}
$$

The cancellations follow because

$$(-1)^k \epsilon_{p-1}(a) + \epsilon_p(a)(-1)^{p+a+1-k} = 0 \quad \text{and} \quad (-1)^k \epsilon_{p-1}(a-1) + \epsilon_p(a)(-1)^k = 0$$

We leave it to the reader to verify the cancellations.

Suppose we have two bounded below complexes of abelian sheaves $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$. We define the complex $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet)$ to be to complex with terms $\bigoplus_{p+q=n} \mathcal{F}^p \otimes \mathcal{G}^q$ and differential according to the rule

$$(25.3.1) \qquad d(\alpha \otimes \beta) = d(\alpha) \otimes \beta + (-1)^{\deg(\alpha)} \alpha \otimes d(\beta)$$

when $\alpha$ and $\beta$ are homogeneous, see Homology, Definition 18.3.

Suppose that $M^\bullet$ and $N^\bullet$ are two bounded below complexes of abelian groups. Then if $m$, resp. $n$ is a cocycle for $M^\bullet$, resp. $N^\bullet$, it is immediate that $m \otimes n$ is a cocycle for $\mathrm{Tot}(M^\bullet \otimes N^\bullet)$. Hence a cup product

$$H^i(M^\bullet) \times H^j(N^\bullet) \longrightarrow H^{i+j}(Tot(M^\bullet \otimes N^\bullet)).$$

This is discussed also in More on Algebra, Section 63.

So the construction of the cup product in hypercohomology of complexes rests on a construction of a map of complexes

$$(25.3.2) \quad \mathrm{Tot}\left(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}^\bullet))\right) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{Tot}(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)))$$

This map is denoted $\cup$ and is given by the rule

$$(\alpha \cup \beta)_{i_0...i_p} = \sum_{r=0}^{p} \epsilon(n, m, p, r) \alpha_{i_0...i_r} \otimes \beta_{i_r...i_p}.$$

where $\alpha$ has degree $n$ and $\beta$ has degree $m$ and with

$$\epsilon(n, m, p, r) = (-1)^{(p+r)n+rp+r}.$$

Note that $\epsilon(n, m, p, n) = 1$. Hence if $\mathcal{F}^\bullet = \mathcal{F}[0]$ is the complex consisting in a single abelian sheaf $\mathcal{F}$ placed in degree 0, then there no signs in the formula for $\cup$ (as in that case $\alpha_{i_0\ldots i_r} = 0$ unless $r = n$). For an explanation of why there has to be a sign and how to compute it see [AGV71, Exposee XVII] by Deligne. To check (25.3.2) is a map of complexes we have to show that

$$d(\alpha \cup \beta) = d(\alpha) \cup \beta + (-1)^{\deg(\alpha)}\alpha \cup d(\beta)$$

by the definition of the differential on $\mathrm{Tot}(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}^\bullet)))$ as given in Homology, Definition 18.3. We compute first

$$d(\alpha \cup \beta)_{i_0\ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1}(-1)^j(\alpha \cup \beta)_{i_0\ldots\hat{i}_j\ldots i_{p+1}} + (-1)^{p+1}d_{\mathcal{F}\otimes\mathcal{G}}((\alpha \cup \beta)_{i_0\ldots i_{p+1}})$$

$$= \sum\nolimits_{j=0}^{p+1}\sum\nolimits_{r=0}^{j-1}(-1)^j\epsilon(n, m, p, r)\alpha_{i_0\ldots i_r} \otimes \beta_{i_r\ldots\hat{i}_j\ldots i_{p+1}} +$$

$$\sum\nolimits_{j=0}^{p+1}\sum\nolimits_{r=j+1}^{p+1}(-1)^j\epsilon(n, m, p, r-1)\alpha_{i_0\ldots\hat{i}_j\ldots i_r} \otimes \beta_{i_r\ldots i_{p+1}} +$$

$$\sum\nolimits_{r=0}^{p+1}(-1)^{p+1}\epsilon(n, m, p+1, r)d_{\mathcal{F}\otimes\mathcal{G}}(\alpha_{i_0\ldots i_r} \otimes \beta_{i_r\ldots i_{p+1}})$$

and note that the summands in the last term equal

$$(-1)^{p+1}\epsilon(n, m, p+1, r)\left(d_{\mathcal{F}}(\alpha_{i_0\ldots i_r}) \otimes \beta_{i_r\ldots i_{p+1}} + (-1)^{n-r}\alpha_{i_0\ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r\ldots i_{p+1}})\right).$$

because $\deg_{\mathcal{F}}(\alpha_{i_0\ldots i_r}) = n - r$. On the other hand

$$(d(\alpha) \cup \beta)_{i_0\ldots i_{p+1}} = \sum\nolimits_{r=0}^{p+1}\epsilon(n+1, m, p+1, r)d(\alpha)_{i_0\ldots i_r} \otimes \beta_{i_r\ldots i_{p+1}}$$

$$= \sum\nolimits_{r=0}^{p+1}\sum\nolimits_{j=0}^{r}\epsilon(n+1, m, p+1, r)(-1)^j\alpha_{i_0\ldots\hat{i}_j\ldots i_r} \otimes \beta_{i_r\ldots i_{p+1}} +$$

$$\sum\nolimits_{r=0}^{p+1}\epsilon(n+1, m, p+1, r)(-1)^r d_{\mathcal{F}}(\alpha_{i_0\ldots i_r}) \otimes \beta_{i_r\ldots i_{p+1}}$$

and

$$(\alpha \cup d(\beta))_{i_0\ldots i_{p+1}} = \sum\nolimits_{r=0}^{p+1}\epsilon(n, m+1, p+1, r)\alpha_{i_0\ldots i_r} \otimes d(\beta)_{i_r\ldots i_{p+1}}$$

$$= \sum\nolimits_{r=0}^{p+1}\sum\nolimits_{j=r}^{p+1}\epsilon(n, m+1, p+1, r)(-1)^{j-r}\alpha_{i_0\ldots i_r} \otimes \beta_{i_r\ldots\hat{i}_j\ldots i_{p+1}} +$$

$$\sum\nolimits_{r=0}^{p+1}\epsilon(n, m+1, p+1, r)(-1)^{p+1-r}\alpha_{i_0\ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r\ldots i_{p+1}})$$

The desired equality holds if we have

$$(-1)^{p+1}\epsilon(n, m, p+1, r) = \epsilon(n+1, m, p+1, r)(-1)^r$$

$$(-1)^{p+1}\epsilon(n, m, p+1, r)(-1)^{n-r} = (-1)^n\epsilon(n, m+1, p+1, r)(-1)^{p+1-r}$$

$$\epsilon(n+1, m, p+1, r)(-1)^r = (-1)^{1+n}\epsilon(n, m+1, p+1, r-1)$$

$$(-1)^j\epsilon(n, m, p, r) = (-1)^n\epsilon(n, m+1, p+1, r)(-1)^{j-r}$$

$$(-1)^j\epsilon(n, m, p, r-1) = \epsilon(n+1, m, p+1, r)(-1)^j$$

(The third equality is necessary to get the terms with $r = j$ from $d(\alpha) \cup \beta$ and $(-1)^n\alpha \cup d(\beta)$ to cancel each other.) We leave the verifications to the reader. (Alternatively, check the script signs.gp in the scripts subdirectory of the Stacks project.)

Associativity of the cup product. Suppose that $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ and $\mathcal{H}^\bullet$ are bounded below complexes of abelian groups on $X$. The obvious map (without the intervention of signs) is an isomorphism of complexes

$$\operatorname{Tot}(\operatorname{Tot}(\mathcal{F}^\bullet \otimes_\mathbf{Z} \mathcal{G}^\bullet) \otimes_\mathbf{Z} \mathcal{H}^\bullet) \longrightarrow \operatorname{Tot}(\mathcal{F}^\bullet \otimes_\mathbf{Z} \operatorname{Tot}(\mathcal{G}^\bullet \otimes_\mathbf{Z} \mathcal{H}^\bullet)).$$

Another way to say this is that the triple complex $\mathcal{F}^\bullet \otimes_\mathbf{Z} \mathcal{G}^\bullet \otimes_\mathbf{Z} \mathcal{H}^\bullet$ gives rise to a well defined total complex with differential satisfying

$$d(\alpha \otimes \beta \otimes \gamma) = d(\alpha) \otimes \beta \otimes \gamma + (-1)^{\deg(\alpha)} \alpha \otimes d(\beta) \otimes \gamma + (-1)^{\deg(\alpha)+\deg(\beta)} \alpha \otimes \beta \otimes d(\gamma)$$

for homogeneous elements. Using this map it is easy to verify that

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$$

namely, if $\alpha$ has degree $a$, $\beta$ has degree $b$ and $\gamma$ has degree $c$, then

$$
\begin{aligned}
((\alpha \cup \beta) \cup \gamma)_{i_0 \ldots i_p} &= \sum_{r=0}^p \epsilon(a+b,c,p,r)(\alpha \cup \beta)_{i_0 \ldots i_r} \otimes \gamma_{i_r \ldots i_p} \\
&= \sum_{r=0}^p \sum_{s=0}^r \epsilon(a+b,c,p,r)\epsilon(a,b,r,s)\alpha_{i_0 \ldots i_s} \otimes \beta_{i_s \ldots i_r} \otimes \gamma_{i_r \ldots i_p}
\end{aligned}
$$

and

$$
\begin{aligned}
(\alpha \cup (\beta \cup \gamma))_{i_0 \ldots i_p} &= \sum_{s=0}^p \epsilon(a,b+c,p,s)\alpha_{i_0 \ldots i_s} \otimes (\beta \cup \gamma)_{i_s \ldots i_p} \\
&= \sum_{s=0}^p \sum_{r=s}^p \epsilon(a,b+c,p,s)\epsilon(b,c,p-s,r-s)\alpha_{i_0 \ldots i_s} \otimes \beta_{i_s \ldots i_r} \otimes \gamma_{i_r \ldots i_p}
\end{aligned}
$$

and a trivial mod 2 calculation shows the signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the Stacks project.)

Finally, we indicate why the cup product preserves a graded commutative structure, at least on a cohomological level. For this we use the operator $\tau$ introduced above. Let $\mathcal{F}^\bullet$ be a bounded below complexes of abelian groups, and assume we are given a graded commutative multiplication

$$\wedge^\bullet : \operatorname{Tot}(\mathcal{F}^\bullet \otimes \mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet.$$

This means the following: For $s$ a local section of $\mathcal{F}^a$, and $t$ a local section of $\mathcal{F}^b$ we have $s \wedge t$ a local section of $\mathcal{F}^{a+b}$. Graded commutative means we have $s \wedge t = (-1)^{ab} t \wedge s$. Since $\wedge$ is a map of complexes we have $d(s \wedge t) = d(s) \wedge t + (-1)^a s \wedge d(t)$. The composition

$$\operatorname{Tot}(\operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes \operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to \operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \operatorname{Tot}(\mathcal{F}^\bullet \otimes_\mathbf{Z} \mathcal{F}^\bullet))) \to \operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

induces a cup product on cohomology

$$H^n(\operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \times H^m(\operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \longrightarrow H^{n+m}(\operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)))$$

and so in the limit also a product on Čech cohomology and therefore (using hypercoverings if needed) a product in cohomology of $\mathcal{F}^\bullet$. We claim this product (on cohomology) is graded commutative as well. To prove this we first consider an element $\alpha$ of degree $n$ in $\operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and an element $\beta$ of degree $m$ in $\operatorname{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and we compute

$$
\begin{aligned}
\wedge^\bullet(\alpha \cup \beta)_{i_0 \ldots i_p} &= \sum_{r=0}^p \epsilon(n,m,p,r)\alpha_{i_0 \ldots i_r} \wedge \beta_{i_r \ldots i_p} \\
&= \sum_{r=0}^p \epsilon(n,m,p,r)(-1)^{\deg(\alpha_{i_0 \ldots i_r})\deg(\beta_{i_r \ldots i_p})}\beta_{i_r \ldots i_p} \wedge \alpha_{i_0 \ldots i_r}
\end{aligned}
$$

because $\wedge$ is graded commutative. On the other hand we have

$$
\begin{aligned}
\tau(\wedge^\bullet(\tau(\beta) \cup \tau(\alpha)))_{i_0 \dots i_p} &= \chi(p) \sum\nolimits_{r=0}^p \epsilon(m,n,p,r) \tau(\beta)_{i_p \dots i_{p-r}} \wedge \tau(\alpha)_{i_{p-r} \dots i_0} \\
&= \chi(p) \sum\nolimits_{r=0}^p \epsilon(m,n,p,r) \chi(r) \chi(p-r) \beta_{i_{p-r} \dots i_p} \wedge \alpha_{i_0 \dots i_{p-r}} \\
&= \chi(p) \sum\nolimits_{r=0}^p \epsilon(m,n,p,p-r) \chi(r) \chi(p-r) \beta_{i_r \dots i_p} \wedge \alpha_{i_0 \dots i_r}
\end{aligned}
$$

where $\chi(t) = (-1)^{\frac{t(t+1)}{2}}$. Since we proved earlier that $\tau$ acts as the identity on cohomology we have to verify that

$$
\epsilon(n,m,p,r)(-1)^{(n-r)(m-(p-r))} = (-1)^{nm} \chi(p) \epsilon(m,n,p,p-r) \chi(r) \chi(p-r)
$$

A trivial mod 2 calculation shows these signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the Stacks project.)

Finally, we study the compatibility of cup product with boundary maps. Suppose that

$$
0 \to \mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \to 0 \quad \text{and} \quad 0 \leftarrow \mathcal{G}_1^\bullet \leftarrow \mathcal{G}_2^\bullet \leftarrow \mathcal{G}_3^\bullet \leftarrow 0
$$

are short exact sequences of bounded below complexes of abelian sheaves on $X$. Let $\mathcal{H}^\bullet$ be another bounded below complex of abelian sheaves, and suppose we have maps of complexes

$$
\gamma_i : \mathrm{Tot}(\mathcal{F}_i^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_i^\bullet) \longrightarrow \mathcal{H}^\bullet
$$

which are compatible with the maps between the complexes, namely such that the diagrams

$$
\begin{array}{ccc}
\mathrm{Tot}(\mathcal{F}_1^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_1^\bullet) & \longleftarrow & \mathrm{Tot}(\mathcal{F}_1^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) \\
{\scriptstyle \gamma_1} \downarrow & & \downarrow \\
\mathcal{H}^\bullet & \xleftarrow{\ \gamma_2\ } & \mathrm{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet)
\end{array}
$$

and

$$
\begin{array}{ccc}
\mathrm{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) & \longleftarrow & \mathrm{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_3^\bullet) \\
{\scriptstyle \gamma_2} \downarrow & & \downarrow \\
\mathcal{H}^\bullet & \xleftarrow{\ \gamma_3\ } & \mathrm{Tot}(\mathcal{F}_3^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_3^\bullet)
\end{array}
$$

are commutative.

**Lemma 25.4.** *In the situation above, assume Čech cohomology agrees with cohomology for the sheaves $\mathcal{F}_i^p$ and $\mathcal{G}_j^q$. Let $a_3 \in H^n(X, \mathcal{F}_3^\bullet)$ and $b_1 \in H^m(X, \mathcal{G}_1^\bullet)$. Then we have*

$$
\gamma_1(\partial a_3 \cup b_1) = (-1)^{n+1} \gamma_3(a_3 \cup \partial b_1)
$$

*in $H^{n+m}(X, \mathcal{H}^\bullet)$ where $\partial$ indicates the boundary map on cohomology associated to the short exact sequences of complexes above.*

**Proof.** We will use the following conventions and notation. We think of $\mathcal{F}_1^p$ as a subsheaf of $\mathcal{F}_2^p$ and we think of $\mathcal{G}_3^q$ as a subsheaf of $\mathcal{G}_2^q$. Hence if $s$ is a local section of $\mathcal{F}_1^p$ we use $s$ to denote the corresponding section of $\mathcal{F}_2^p$ as well. Similarly for local sections of $\mathcal{G}_3^q$. Furthermore, if $s$ is a local section of $\mathcal{F}_2^p$ then we denote $\bar{s}$ its image in $\mathcal{F}_3^p$. Similarly for the map $\mathcal{G}_2^q \to \mathcal{G}_1^q$. In particular if $s$ is a local section of $\mathcal{F}_2^p$ and $\bar{s} = 0$ then $s$ is a local section of $\mathcal{F}_1^p$. The commutativity of the diagrams above implies, for local sections $s$ of $\mathcal{F}_2^p$ and $t$ of $\mathcal{G}_3^q$ that $\gamma_2(s \otimes t) = \gamma_3(\bar{s} \otimes t)$ as sections of $\mathcal{H}^{p+q}$.

Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of $X$. Suppose that $\alpha_3$, resp. $\beta_1$ is a degree $n$, resp. $m$ cocycle of $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3^\bullet))$, resp. $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_1^\bullet))$ representing $a_3$, resp. $b_1$. After refining $\mathcal{U}$ if necessary, we can find cochains $\alpha_2$, resp. $\beta_2$ of degree $n$, resp. $m$ in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2^\bullet))$, resp. $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_2^\bullet))$ mapping to $\alpha_3$, resp. $\beta_1$. Then we see that

$$\overline{d(\alpha_2)} = d(\bar{\alpha}_2) = 0 \quad \text{and} \quad \overline{d(\beta_2)} = d(\bar{\beta}_2) = 0.$$

This means that $\alpha_1 = d(\alpha_2)$ is a degree $n+1$ cocycle in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_1^\bullet))$ representing $\partial a_3$. Similarly, $\beta_3 = d(\beta_2)$ is a degree $m+1$ cocycle in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_3^\bullet))$ representing $\partial b_1$. Thus we may compute

$$
\begin{aligned}
d(\gamma_2(\alpha_2 \cup \beta_2)) &= \gamma_2(d(\alpha_2 \cup \beta_2)) \\
&= \gamma_2(d(\alpha_2) \cup \beta_2 + (-1)^n \alpha_2 \cup d(\beta_2)) \\
&= \gamma_2(\alpha_1 \cup \beta_2) + (-1)^n \gamma_2(\alpha_2 \cup \beta_3) \\
&= \gamma_1(\alpha_1 \cup \beta_1) + (-1)^n \gamma_3(\alpha_3 \cup \beta_3)
\end{aligned}
$$

So this even tells us that the sign is $(-1)^{n+1}$ as indicated in the lemma[2].          $\square$

**Lemma 25.5.** *Let $X$ be a topological space. Let $\mathcal{O}' \to \mathcal{O}$ be a surjection of sheaves of rings whose kernel $\mathcal{I} \subset \mathcal{O}'$ has square zero. Then $M = H^1(X, \mathcal{I})$ is a $R = H^0(X, \mathcal{O})$-module and the boundary map $\partial : R \to M$ associated to the short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

*is a derivation (Algebra, Definition 131.1).*

**Proof.** The map $\mathcal{O}' \to \mathcal{H}om(\mathcal{I}, \mathcal{I})$ factors through $\mathcal{O}$ as $\mathcal{I} \cdot \mathcal{I} = 0$ by assumption. Hence $\mathcal{I}$ is a sheaf of $\mathcal{O}$-modules and this defines the $R$-module structure on $M$. The boundary map is additive hence it suffices to prove the Leibniz rule. Let $f \in R$. Choose an open covering $\mathcal{U} : X = \bigcup U_i$ such that there exist $f_i \in \mathcal{O}'(U_i)$ lifting $f|_{U_i} \in \mathcal{O}(U_i)$. Observe that $f_i - f_j$ is an element of $\mathcal{I}(U_i \cap U_j)$. Then $\partial(f)$ corresponds to the Čech cohomology class of the 1-cocycle $\alpha$ with $\alpha_{i_0 i_1} = f_{i_0} - f_{i_1}$. (Observe that by Lemma 11.3 the first Čech cohomology group with respect to $\mathcal{U}$ is a submodule of $M$.) Next, let $g \in R$ be a second element and assume (after possibly refining the open covering) that $g_i \in \mathcal{O}'(U_i)$ lifts $g|_{U_i} \in \mathcal{O}(U_i)$. Then we see that $\partial(g)$ is given by the cocycle $\beta$ with $\beta_{i_0 i_1} = g_{i_0} - g_{i_1}$. Since $f_i g_i \in \mathcal{O}'(U_i)$ lifts $fg|_{U_i}$ we see that $\partial(fg)$ is given by the cocycle $\gamma$ with

$$\gamma_{i_0 i_1} = f_{i_0} g_{i_0} - f_{i_1} g_{i_1} = (f_{i_0} - f_{i_1}) g_{i_0} + f_{i_1}(g_{i_0} - g_{i_1}) = \alpha_{i_0 i_1} g + f \beta_{i_0 i_1}$$

by our definition of the $\mathcal{O}$-module structure on $\mathcal{I}$. This proves the Leibniz rule and the proof is complete.          $\square$

---

[2]The sign depends on the convention for the signs in the long exact sequence in cohomology associated to a triangle in $D(X)$. The conventions in the Stacks project are (a) distinguished triangles correspond to termwise split exact sequences and (b) the boundary maps in the long exact sequence are given by the maps in the snake lemma without the intervention of signs. See Derived Categories, Section 10.

## 26. Flat resolutions

A reference for the material in this section is [Spa88]. Let $(X, \mathcal{O}_X)$ be a ringed space. By Modules, Lemma 17.6 any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module. By Derived Categories, Lemma 15.4 any bounded above complex of $\mathcal{O}_X$-modules has a left resolution by a bounded above complex of flat $\mathcal{O}_X$-modules. However, for unbounded complexes, it turns out that flat resolutions aren't good enough.

**Lemma 26.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The functors*

$$K(Mod(\mathcal{O}_X)) \longrightarrow K(Mod(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)$$

*and*

$$K(Mod(\mathcal{O}_X)) \longrightarrow K(Mod(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

*are exact functors of triangulated categories.*

**Proof.** This follows from Derived Categories, Remark 10.9. $\qquad\square$

**Definition 26.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules is called *K-flat* if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the complex

$$Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

is acyclic.

**Lemma 26.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor*

$$K(Mod(\mathcal{O}_X)) \longrightarrow K(Mod(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

*transforms quasi-isomorphisms into quasi-isomorphisms.*

**Proof.** Follows from Lemma 26.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. $\qquad\square$

**Lemma 26.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Then $\mathcal{K}^\bullet$ is K-flat if and only if for all $x \in X$ the complex $\mathcal{K}_x^\bullet$ of $\mathcal{O}_{X,x}$-modules is K-flat (More on Algebra, Definition 59.1).*

**Proof.** If $\mathcal{K}_x^\bullet$ is K-flat for all $x \in X$ then we see that $\mathcal{K}^\bullet$ is K-flat because $\otimes$ and direct sums commute with taking stalks and because we can check exactness at stalks, see Modules, Lemma 3.1. Conversely, assume $\mathcal{K}^\bullet$ is K-flat. Pick $x \in X$ $M^\bullet$ be an acyclic complex of $\mathcal{O}_{X,x}$-modules. Then $i_{x,*}M^\bullet$ is an acyclic complex of $\mathcal{O}_X$-modules. Thus $Tot(i_{x,*}M^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$ is acyclic. Taking stalks at $x$ shows that $Tot(M^\bullet \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_x^\bullet)$ is acyclic. $\qquad\square$

**Lemma 26.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}_X$-modules, then $Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}_X$-modules.*

**Proof.** Follows from the isomorphism

$$Tot(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)) = Tot(Tot(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$$

and the definition. $\qquad\square$

**Lemma 26.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$ be a distinguished triangle in $K(Mod(\mathcal{O}_X))$. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

**Proof.** Follows from Lemma 26.1 and the fact that in a distinguished triangle in $K(Mod(\mathcal{O}_X))$ if two out of three are acyclic, so is the third. $\square$

**Lemma 26.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $0 \to \mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \mathcal{K}_3^\bullet \to 0$ be a short exact sequence of complexes such that the terms of $\mathcal{K}_3^\bullet$ are flat $\mathcal{O}_X$-modules. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

**Proof.** By Modules, Lemma 17.7 for every complex $\mathcal{L}^\bullet$ we obtain a short exact sequence

$$0 \to \mathrm{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_1^\bullet) \to \mathrm{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_1^\bullet) \to \mathrm{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_1^\bullet) \to 0$$

of complexes. Hence the lemma follows from the long exact sequence of cohomology sheaves and the definition of K-flat complexes. $\square$

**Lemma 26.8.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback of a K-flat complex of $\mathcal{O}_Y$-modules is a K-flat complex of $\mathcal{O}_X$-modules.*

**Proof.** We can check this on stalks, see Lemma 26.4. Hence this follows from Sheaves, Lemma 26.4 and More on Algebra, Lemma 59.3. $\square$

**Lemma 26.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. A bounded above complex of flat $\mathcal{O}_X$-modules is K-flat.*

**Proof.** We can check this on stalks, see Lemma 26.4. Thus this lemma follows from Modules, Lemma 17.2 and More on Algebra, Lemma 59.7. $\square$

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

**Lemma 26.10.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\mathrm{colim}_i \mathcal{K}_i^\bullet$ is K-flat.*

**Proof.** Because we are taking termwise colimits it is clear that

$$\mathrm{colim}_i \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_i^\bullet) = \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathrm{colim}_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. $\square$

**Lemma 26.11.** *Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules there exists a commutative diagram of complexes of $\mathcal{O}_X$-modules*

$$
\begin{array}{ccc}
\mathcal{K}_1^\bullet & \longrightarrow & \mathcal{K}_2^\bullet \longrightarrow \ldots \\
\downarrow & & \downarrow \\
\tau_{\leq 1}\mathcal{G}^\bullet & \longrightarrow & \tau_{\leq 2}\mathcal{G}^\bullet \longrightarrow \ldots
\end{array}
$$

*with the following properties: (1) the vertical arrows are quasi-isomorphisms and termwise surjective, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $\mathrm{colim}\,\mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.*

**Proof.** The existence of the diagram and properties (1), (2), (3) follows immediately from Modules, Lemma 17.6 and Derived Categories, Lemma 29.1. The induced map $\mathrm{colim}\,\mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. $\square$

**Lemma 26.12.** *Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ there exists a K-flat complex $\mathcal{K}^\bullet$ whose terms are flat $\mathcal{O}_X$-modules and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ which is termwise surjective.*

**Proof.** Choose a diagram as in Lemma 26.11. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules, Lemma 17.5. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 26.9. Thus $\operatorname{colim} \mathcal{K}_n^\bullet$ is K-flat by Lemma 26.10. The induced map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism and termwise surjective by construction. Property (3) of Lemma 26.11 shows that $\operatorname{colim} \mathcal{K}_n^m$ is a direct sum of flat modules and hence flat which proves the final assertion. $\qquad\square$

**Lemma 26.13.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}_X$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the induced map*

$$Tot(id_{\mathcal{F}^\bullet} \otimes \alpha) : Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \longrightarrow Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet)$$

*is a quasi-isomorphism.*

**Proof.** Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 26.12. Consider the commutative diagram

$$
\begin{array}{ccc}
\operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) & \longrightarrow & \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet) \\
\downarrow & & \downarrow \\
\operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) & \longrightarrow & \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet)
\end{array}
$$

The result follows as by Lemma 26.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. $\qquad\square$

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 26.12. By Lemma 26.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}_X) \longrightarrow K(\mathcal{O}_X), \quad \mathcal{G}^\bullet \longmapsto \operatorname{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

By Lemma 26.3 this functor induces a functor $D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ simply because $D(\mathcal{O}_X)$ is the localization of $K(\mathcal{O}_X)$ at quasi-isomorphisms. By Lemma 26.13 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

**Definition 26.14.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. The *derived tensor product*

$$- \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}^\bullet : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O}_X)$. Here we use sign rules as given in More on Algebra, Section 72. Hence when we write $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

**Definition 26.15.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. The *Tor*'s of $\mathcal{F}$ and $\mathcal{G}$ are define by the formula

$$\operatorname{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H^{-p}(\mathcal{F} \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G})$$

with derived tensor product as defined above.

This definition implies that for every short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we have a long exact cohomology sequence

$$\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow 0$$

$$\operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}) \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G}) \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{G})$$

for every $\mathcal{O}_X$-module $\mathcal{G}$. This will be called the long exact sequence of Tor associated to the situation.

**Lemma 26.16.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is a flat $\mathcal{O}_X$-module, and*
(2) *$\operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{O}_X$-module $\mathcal{G}$.*

**Proof.** If $\mathcal{F}$ is flat, then $\mathcal{F} \otimes_{\mathcal{O}_X} -$ is an exact functor and the satellites vanish. Conversely assume (2) holds. Then if $\mathcal{G} \to \mathcal{H}$ is injective with cokernel $\mathcal{Q}$, the long exact sequence of Tor shows that the kernel of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}$ is a quotient of $\operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})$ which is zero by assumption. Hence $\mathcal{F}$ is flat. $\square$

**Lemma 26.17.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $a : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ be a map of complexes of $\mathcal{O}_X$-modules. If $\mathcal{K}^\bullet$ is K-flat, then there exist a complex $\mathcal{N}^\bullet$ and maps of complexes $b : \mathcal{K}^\bullet \to \mathcal{N}^\bullet$ and $c : \mathcal{N}^\bullet \to \mathcal{L}^\bullet$ such that*

(1) *$\mathcal{N}^\bullet$ is K-flat,*
(2) *$c$ is a quasi-isomorphism,*
(3) *$a$ is homotopic to $c \circ b$.*

*If the terms of $\mathcal{K}^\bullet$ are flat, then we may choose $\mathcal{N}^\bullet$, $b$, and $c$ such that the same is true for $\mathcal{N}^\bullet$.*

**Proof.** We will use that the homotopy category $K(Mod(\mathcal{O}_X))$ is a triangulated category, see Derived Categories, Proposition 10.3. Choose a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{L}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$. Choose a quasi-isomorphism $\mathcal{M}^\bullet \to \mathcal{C}^\bullet$ with $\mathcal{M}^\bullet$ K-flat with flat terms, see Lemma 26.12. By the axioms of triangulated categories, we may fit the composition $\mathcal{M}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$ into a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{N}^\bullet \to \mathcal{M}^\bullet \to \mathcal{K}^\bullet[1]$. By Lemma 26.6 we see that $\mathcal{N}^\bullet$ is K-flat. Again using the axioms of triangulated categories, we can choose a map $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ fitting into the following morphism of distinghuised triangles

$$\begin{array}{ccccccc} \mathcal{K}^\bullet & \longrightarrow & \mathcal{N}^\bullet & \longrightarrow & \mathcal{M}^\bullet & \longrightarrow & \mathcal{K}^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}^\bullet & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{C}^\bullet & \longrightarrow & \mathcal{K}^\bullet[1] \end{array}$$

Since two out of three of the arrows are quasi-isomorphisms, so is the third arrow $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ by the long exact sequences of cohomology associated to these distinguished triangles (or you can look at the image of this diagram in $D(\mathcal{O}_X)$ and use

Derived Categories, Lemma 4.3 if you like). This finishes the proof of (1), (2), and (3). To prove the final assertion, we may choose $\mathcal{N}^\bullet$ such that $\mathcal{N}^n \cong \mathcal{M}^n \oplus \mathcal{K}^n$, see Derived Categories, Lemma 10.7. Hence we get the desired flatness if the terms of $\mathcal{K}^\bullet$ are flat. $\qquad\square$

## 27. Derived pullback

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$$

Namely, for every complex of $\mathcal{O}_Y$-modules $\mathcal{G}^\bullet$ we can choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ and set $Lf^*\mathcal{G}^\bullet = f^*\mathcal{K}^\bullet$. You can use Lemmas 26.8, 26.12, and 26.13 to see that this is well defined. However, to cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

**Lemma 27.1.** *The construction above is independent of choices and defines an exact functor of triangulated categories $Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$.*

**Proof.** To see this we use the general theory developed in Derived Categories, Section 14. Set $\mathcal{D} = K(\mathcal{O}_Y)$ and $\mathcal{D}' = D(\mathcal{O}_X)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let $S$ be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}_Y)$. This gives a situation as in Derived Categories, Situation 14.1 so that Derived Categories, Definition 14.2 applies. We claim that $LF$ is everywhere defined. This follows from Derived Categories, Lemma 14.15 with $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ the collection of $K$-flat complexes: (1) follows from Lemma 26.12 and to see (2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between K-flat complexes of $\mathcal{O}_Y$-modules the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

The functor $f^{-1}$ is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. By Lemma 26.8 applied to the morphism $(X, f^{-1}\mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$ the complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}_Y$-modules. Hence Lemma 26.13 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}_Y) = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O}_X)$$

see Derived Categories, Equation (14.9.1). Finally, Derived Categories, Lemma 14.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is K-flat, i.e., $Lf^* = LF$ is indeed computed in the way described above. $\qquad\square$

**Lemma 27.2.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Then $Lf^* \circ Lg^* = L(g \circ f)^*$ as functors $D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$.*

**Proof.** Let $E$ be an object of $D(\mathcal{O}_Z)$. By construction $Lg^*E$ is computed by choosing a K-flat complex $\mathcal{K}^\bullet$ representing $E$ on $Z$ and setting $Lg^*E = g^*\mathcal{K}^\bullet$. By Lemma 26.8 we see that $g^*\mathcal{K}^\bullet$ is K-flat on $Y$. Then $Lf^*Lg^*E$ is given by $f^*g^*\mathcal{K}^\bullet = (g \circ f)^*\mathcal{K}^\bullet$ which also represents $L(g \circ f)^*E$. $\qquad\square$

**Lemma 27.3.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical bifunctorial isomorphism*

$$Lf^*(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*\mathcal{G}^\bullet$$

*for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}_Y))$.*

**Proof.** We may assume that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are K-flat complexes. In this case $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet$. By Lemma 26.5 $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$ is K-flat also. Hence the isomorphism of the lemma comes from the isomorphism

$$\mathrm{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{G}^\bullet) \longrightarrow f^*\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_{\mathcal{O}_X} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}_Y} \mathcal{G}^q)$ of Modules, Lemma 16.4. $\qquad\square$

**Lemma 27.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical bifunctorial isomorphism*

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}^\bullet$$

*for $\mathcal{F}^\bullet$ in $D(\mathcal{O}_X)$ and $\mathcal{G}^\bullet$ in $D(\mathcal{O}_Y)$.*

**Proof.** Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G} = \mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$ because $f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$. The lemma follows from this and the definitions. $\qquad\square$

**Lemma 27.5.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_Y$-modules. The diagram*

$$
\begin{array}{ccc}
Lf^*(\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{M}^\bullet) & \longrightarrow & Lf^* \mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \\
\downarrow & & \downarrow \\
Lf^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{M}^\bullet & & f^* \mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \\
\downarrow & & \downarrow \\
f^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^*\mathcal{M}^\bullet & \longrightarrow & \mathit{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{M}^\bullet)
\end{array}
$$

*commutes.*

**Proof.** We will use the existence of K-flat resolutions as in Lemma 26.8. If we choose such resolutions $\mathcal{P}^\bullet \to \mathcal{K}^\bullet$ and $\mathcal{Q}^\bullet \to \mathcal{M}^\bullet$, then we see that

$$
\begin{array}{ccc}
Lf^*\mathrm{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{Q}^\bullet) & \longrightarrow & Lf^*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \\
\downarrow & & \downarrow \\
f^*\mathrm{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{Q}^\bullet) & \longrightarrow & f^*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \\
\downarrow & & \downarrow \\
\mathrm{Tot}(f^*\mathcal{P}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{Q}^\bullet) & \longrightarrow & \mathrm{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{M}^\bullet)
\end{array}
$$

commutes. However, now the left hand side of the diagram is the left hand side of the diagram by our choice of $\mathcal{P}^\bullet$ and $\mathcal{Q}^\bullet$ and Lemma 26.5. $\qquad\square$

## 28. Cohomology of unbounded complexes

Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus_{U \subset X \text{ open}} j_{U!}\mathcal{O}_U,$$

see Modules, Section 3 and Lemmas 17.5 and 17.6. By Injectives, Theorem 12.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}_X$-modules all of whose terms are injective $\mathcal{O}_X$-modules and moreover this embedding can be chosen functorial in the complex $\mathcal{F}^\bullet$. It follows from Derived Categories, Lemma 31.7 that

(1) any exact functor $F : K(Mod(\mathcal{O}_X)) \to \mathcal{D}$ into a trianguated category $\mathcal{D}$ has a right derived functor $RF : D(\mathcal{O}_X) \to \mathcal{D}$,

(2) for any additive functor $F : Mod(\mathcal{O}_X) \to \mathcal{A}$ into an abelian category $\mathcal{A}$ we consider the exact functor $F : K(Mod(\mathcal{O}_X)) \to D(\mathcal{A})$ induced by $F$ and we obtain a right derived functor $RF : D(\mathcal{O}_X) \to K(\mathcal{A})$.

By construction we have $RF(\mathcal{F}^\bullet) = F(\mathcal{I}^\bullet)$ where $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is as above.

Here are some examples of the above:

(1) The functor $\Gamma(X, -) : Mod(\mathcal{O}_X) \to \mathrm{Mod}_{\Gamma(X, \mathcal{O}_X)}$ gives rise to

$$R\Gamma(X, -) : D(\mathcal{O}_X) \to D(\Gamma(X, \mathcal{O}_X))$$

We shall use the notation $H^i(X, K) = H^i(R\Gamma(X, K))$ for cohomology.

(2) For an open $U \subset X$ we consider the functor $\Gamma(U, -) : Mod(\mathcal{O}_X) \to \mathrm{Mod}_{\Gamma(U, \mathcal{O}_X)}$. This gives rise to

$$R\Gamma(U, -) : D(\mathcal{O}_X) \to D(\Gamma(U, \mathcal{O}_X))$$

We shall use the notation $H^i(U, K) = H^i(R\Gamma(U, K))$ for cohomology.

(3) For a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ we consider the functor $f_* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ which gives rise to the total direct image

$$Rf_* : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_Y)$$

on unbounded derived categories.

**Lemma 28.1.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The functor $Rf_*$ defined above and the functor $Lf^*$ defined in Lemma 27.1 are adjoint:*

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

*bifunctorially in $\mathcal{F}^\bullet \in \mathrm{Ob}(D(\mathcal{O}_X))$ and $\mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}_Y))$.*

**Proof.** This follows formally from the fact that $Rf_*$ and $Lf^*$ exist, see Derived Categories, Lemma 30.3. $\square$

**Lemma 28.2.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Then $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors $D(\mathcal{O}_X) \to D(\mathcal{O}_Z)$.*

**Proof.** By Lemma 28.1 we see that $Rg_* \circ Rf_*$ is adjoint to $Lf^* \circ Lg^*$. We have $Lf^* \circ Lg^* = L(g \circ f)^*$ by Lemma 27.2 and hence by uniqueness of adjoint functors we have $Rg_* \circ Rf_* = R(g \circ f)_*$. $\square$

**Remark 28.3.** The construction of unbounded derived functor $Lf^*$ and $Rf_*$ allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{array}{ccc} X' & \xrightarrow{\ g'\ } & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{\ g\ } & S \end{array}$$

is a commutative diagram of ringed spaces. Let $K$ be an object of $D(\mathcal{O}_X)$. Then there exists a canonical base change map

$$Lg^* Rf_* K \longrightarrow R(f')_* L(g')^* K$$

in $D(\mathcal{O}_{S'})$. Namely, this map is adjoint to a map $L(f')^* Lg^* Rf_* K \to L(g')^* K$ Since $L(f')^* Lg^* = L(g')^* Lf^*$ we see this is the same as a map $L(g')^* Lf^* Rf_* K \to L(g')^* K$ which we can take to be $L(g')^*$ of the adjunction map $Lf^* Rf_* K \to K$.

**Remark 28.4.** Consider a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ k\ } & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\ l\ } & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{\ m\ } & Z \end{array}$$

of ringed spaces. Then the base change maps of Remark 28.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} Lm^* \circ R(g \circ f)_* = Lm^* \circ Rg_* \circ Rf_* \\ \to Rg'_* \circ Ll^* \circ Rf_* \\ \to Rg'_* \circ Rf'_* \circ Lk^* \\ = R(g' \circ f')_* \circ Lk^* \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

**Remark 28.5.** Consider a commutative diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{\ g'\ } & X' & \xrightarrow{\ g\ } & X \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow{\ h'\ } & Y' & \xrightarrow{\ h\ } & Y \end{array}$$

of ringed spaces. Then the base change maps of Remark 28.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} L(h \circ h')^* \circ Rf_* = L(h')^* \circ Lh_* \circ Rf_* \\ \to L(h')^* \circ Rf'_* \circ Lg^* \\ \to Rf''_* \circ L(g')^* \circ Lg^* \\ = Rf''_* \circ L(g \circ g')^* \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

**Lemma 28.6.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The diagram*

$$
\begin{array}{ccc}
Lf^* f_* \mathcal{K}^\bullet & \longrightarrow & f^* f_* \mathcal{K}^\bullet \\
\downarrow & & \downarrow \\
Lf^* Rf_* \mathcal{K}^\bullet & \longrightarrow & \mathcal{K}^\bullet
\end{array}
$$

*coming from $Lf^* \to f^*$ on complexes, $f_* \to Rf_*$ on complexes, and adjunction $Lf^* \circ Rf_* \to \mathrm{id}$ commutes in $D(\mathcal{O}_X)$.*

**Proof.** We will use the existence of K-flat resolutions and K-injective resolutions, see Lemma 26.8 and the discussion above. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_X$-modules. Choose a quasi-isomorphism $\mathcal{Q}^\bullet \to f_* \mathcal{I}^\bullet$ where $\mathcal{Q}^\bullet$ is K-flat as a complex of $\mathcal{O}_Y$-modules. We can choose a K-flat complex of $\mathcal{O}_Y$-modules $\mathcal{P}^\bullet$ and a diagram of morphisms of complexes

$$
\begin{array}{ccc}
\mathcal{P}^\bullet & \longrightarrow & f_* \mathcal{K}^\bullet \\
\downarrow & & \downarrow \\
\mathcal{Q}^\bullet & \longrightarrow & f_* \mathcal{I}^\bullet
\end{array}
$$

commutative up to homotopy where the top horizontal arrow is a quasi-isomorphism. Namely, we can first choose such a diagram for some complex $\mathcal{P}^\bullet$ because the quasi-isomorphisms form a multiplicative system in the homotopy category of complexes and then we can replace $\mathcal{P}^\bullet$ by a K-flat complex. Taking pullbacks we obtain a diagram of morphisms of complexes

$$
\begin{array}{ccccc}
f^* \mathcal{P}^\bullet & \longrightarrow & f^* f_* \mathcal{K}^\bullet & \longrightarrow & \mathcal{K}^\bullet \\
\downarrow & & \downarrow & & \downarrow \\
f^* \mathcal{Q}^\bullet & \longrightarrow & f^* f_* \mathcal{I}^\bullet & \longrightarrow & \mathcal{I}^\bullet
\end{array}
$$

commutative up to homotopy. The outer rectangle witnesses the truth of the statement in the lemma. $\qquad\square$

**Remark 28.7.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The adjointness of $Lf^*$ and $Rf_*$ allows us to construct a relative cup product

$$
Rf_* K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_* L \longrightarrow Rf_* (K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)
$$

in $D(\mathcal{O}_Y)$ for all $K, L$ in $D(\mathcal{O}_X)$. Namely, this map is adjoint to a map $Lf^*(Rf_* K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_* L) \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ for which we can take the composition of the isomorphism $Lf^*(Rf_* K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_* L) = Lf^* Rf_* K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^* Rf_* L$ (Lemma 27.3) with the map $Lf^* Rf_* K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^* Rf_* L \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ coming from the counit $Lf^* \circ Rf_* \to \mathrm{id}$.

## 29. Cohomology of filtered complexes

Filtered complexes of sheaves frequently come up in a natural fashion when studying cohomology of algebraic varieties, for example the de Rham complex comes with its Hodge filtration. In this sectionwe use the very general Injectives, Lemma 13.7 to find construct spectral sequences on cohomology and we relate these to previously constructed spectral sequences.

**Lemma 29.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a filtered complex of $\mathcal{O}_X$-modules. There exists a canonical spectral sequence $(E_r, d_r)_{r \geq 1}$ of bigraded $\Gamma(X, \mathcal{O}_X)$-modules with $d_r$ of bidegree $(r, -r+1)$ and*

$$E_1^{p,q} = H^{p+q}(X, gr^p \mathcal{F}^\bullet)$$

*If for every n we have*

$$H^n(X, F^p \mathcal{F}^\bullet) = 0 \ for \ p \gg 0 \quad and \quad H^n(X, F^p \mathcal{F}^\bullet) = H^n(X, \mathcal{F}^\bullet) \ for \ p \ll 0$$

*then the spectral sequence is bounded and converges to $H^*(X, \mathcal{F}^\bullet)$.*

**Proof.** (For a proof in case the complex is a bounded below complex of modules with finite filtrations, see the remark below.) Choose an map of filtered complexes $j : \mathcal{F}^\bullet \to \mathcal{J}^\bullet$ as in Injectives, Lemma 13.7. The spectral sequence is the spectral sequence of Homology, Section 24 associated to the filtered complex

$$\Gamma(X, \mathcal{J}^\bullet) \quad with \quad F^p \Gamma(X, \mathcal{J}^\bullet) = \Gamma(X, F^p \mathcal{J}^\bullet)$$

Since cohomology is computed by evaluating on K-injective representatives we see that the $E_1$ page is as stated in the lemma. The convergence and boundedness under the stated conditions follows from Homology, Lemma 24.13.                                $\square$

**Remark 29.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a filtered complex of $\mathcal{O}_X$-modules. If $\mathcal{F}^\bullet$ is bounded from below and for each $n$ the filtration on $\mathcal{F}^n$ is finite, then there is a construction of the spectral sequence in Lemma 29.1 avoiding Injectives, Lemma 13.7. Namely, by Derived Categories, Lemma 26.9 there is a filtered quasi-isomorphism $i : \mathcal{F}^\bullet \to \mathcal{I}^\bullet$ of filtered complexes with $\mathcal{I}^\bullet$ bounded below, the filtration on $\mathcal{I}^n$ is finite for all $n$, and with each $gr^p \mathcal{I}^n$ an injective $\mathcal{O}_X$-module. Then we take the spectral sequence associated to

$$\Gamma(X, \mathcal{I}^\bullet) \quad with \quad F^p \Gamma(X, \mathcal{I}^\bullet) = \Gamma(X, F^p \mathcal{I}^\bullet)$$

Since cohomology can be computed by evaluating on bounded below complexes of injectives we see that the $E_1$ page is as stated in the lemma. The convergence and boundedness under the stated conditions follows from Homology, Lemma 24.11. In fact, this is a special case of the spectral sequence in Derived Categories, Lemma 26.14.

**Example 29.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We can apply Lemma 29.1 with $F^p \mathcal{F}^\bullet = \tau_{\leq -p} \mathcal{F}^\bullet$. (If $\mathcal{F}^\bullet$ is bounded below we can use Remark 29.2.) Then we get a spectral sequence

$$E_1^{p,q} = H^{p+q}(X, H^{-p}(\mathcal{F}^\bullet)[p]) = H^{2p+q}(X, H^{-p}(\mathcal{F}^\bullet))$$

After renumbering $p = -j$ and $q = i + 2j$ we find that for any $K \in D(\mathcal{O}_X)$ there is a spectral sequence $(E_r', d_r')_{r \geq 2}$ of bigraded modules with $d_r'$ of bidegree $(r, -r+1)$, with

$$(E_2')^{i,j} = H^i(X, H^j(K))$$

If $K$ is bounded below (for example), then this spectral sequence is bounded and converges to $H^{i+j}(X, K)$. In the bounded below case this spectral sequence is an example of the second spectral sequence of Derived Categories, Lemma 21.3 (constructed using Cartan-Eilenberg resolutions).

**Example 29.4.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We can apply Lemma 29.1 with $F^p\mathcal{F}^\bullet = \sigma_{\geq p}\mathcal{F}^\bullet$. Then we get a spectral sequence

$$E_1^{p,q} = H^{p+q}(X, \mathcal{F}^p[-p]) = H^q(X, \mathcal{F}^p)$$

If $\mathcal{F}^\bullet$ is bounded below, then

(1) we can use Remark 29.2 to construct this spectral sequence,
(2) the spectral sequence is bounded and converges to $H^{i+j}(X, \mathcal{F}^\bullet)$, and
(3) the spectral sequence is equal to the first spectral sequence of Derived Categories, Lemma 21.3 (constructed using Cartan-Eilenberg resolutions).

**Lemma 29.5.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}^\bullet$ be a filtered complex of $\mathcal{O}_X$-modules. There exists a canonical spectral sequence $(E_r, d_r)_{r \geq 1}$ of bigraded $\mathcal{O}_Y$-modules with $d_r$ of bidegree $(r, -r+1)$ and*

$$E_1^{p,q} = R^{p+q}f_* gr^p \mathcal{F}^\bullet$$

*If for every $n$ we have*

$$R^n f_* F^p \mathcal{F}^\bullet = 0 \text{ for } p \gg 0 \quad and \quad R^n f_* F^p \mathcal{F}^\bullet = R^n f_* \mathcal{F}^\bullet \text{ for } p \ll 0$$

*then the spectral sequence is bounded and converges to $Rf_*\mathcal{F}^\bullet$.*

**Proof.** The proof is exactly the same as the proof of Lemma 29.1. $\square$

## 30. Godement resolution

A reference is [God73].

Let $(X, \mathcal{O}_X)$ be a ringed space. Denote $X_{disc}$ the discrete topological space with the same points as $X$. Denote $f : X_{disc} \to X$ the obvious continuous map. Set $\mathcal{O}_{X_{disc}} = f^{-1}\mathcal{O}_X$. Then $f : (X_{disc}, \mathcal{O}_{X_{disc}}) \to (X, \mathcal{O}_X)$ is a flat morphism of ringed spaces. We can apply the *dual* of the material in Simplicial, Section 34 to the adjoint pair of functors $f^*, f_*$ on sheaves of modules. Thus we obtain an augmented cosimplicial object

$$\text{id} \longrightarrow f_*f^* \rightrightarrows f_*f^*f_*f^* \underset{\longrightarrow}{\overset{\longrightarrow}{\rightrightarrows}} f_*f^*f_*f^*f_*f^*$$

in the category of functors from $Mod(\mathcal{O}_X)$ to itself, see Simplicial, Lemma 34.2. Moreover, the augmentation

$$f^* \longrightarrow f^*f_*f^* \rightrightarrows f^*f_*f^*f_*f^* \underset{\longrightarrow}{\overset{\longrightarrow}{\rightrightarrows}} f^*f_*f^*f_*f^*f_*f^*$$

is a homotopy equivalence, see Simplicial, Lemma 34.3.

**Lemma 30.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. For every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a resolution*

$$0 \to \mathcal{F} \to f_*f^*\mathcal{F} \to f_*f^*f_*f^*\mathcal{F} \to f_*f^*f_*f^*f_*f^*\mathcal{F} \to \ldots$$

*functorial in $\mathcal{F}$ such that each term $f_*f^* \ldots f_*f^*\mathcal{F}$ is a flasque $\mathcal{O}_X$-module and such that for all $x \in X$ the map*

$$\mathcal{F}_x[0] \to \Big( (f_*f^*\mathcal{F})_x \to (f_*f^*f_*f^*\mathcal{F})_x \to (f_*f^*f_*f^*f_*f^*\mathcal{F})_x \to \ldots \Big)$$

*is a homotopy equivalence in the category of complexes of $\mathcal{O}_{X,x}$-modules.*

**Proof.** The complex $f_*f^*\mathcal{F} \to f_*f^*f_*f^*\mathcal{F} \to f_*f^*f_*f^*f_*f^*\mathcal{F} \to \ldots$ is the complex associated to the cosimplicial object with terms $f_*f^*\mathcal{F}, f_*f^*f_*f^*\mathcal{F}, f_*f^*f_*f^*f_*f^*\mathcal{F}, \ldots$ described above, see Simplicial, Section 25. The augmentation gives rise to the map $\mathcal{F} \to f_*f^*\mathcal{F}$ as indicated. For any abelian sheaf $\mathcal{H}$ on $X_{disc}$ the pushforward $f_*\mathcal{H}$ is flasque because $X_{disc}$ is a discrete space and the pushforward of a flasque sheaf is flasque. Hence the terms of the complex are flasque $\mathcal{O}_X$-modules.

If $x \in X_{disc} = X$ is a point, then $(f^*\mathcal{G})_x = \mathcal{G}_x$ for any $\mathcal{O}_X$-module $\mathcal{G}$. Hence $f^*$ is an exact functor and a complex of $\mathcal{O}_X$-modules $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3$ is exact if and only if $f^*\mathcal{G}_1 \to f^*\mathcal{G}_2 \to f^*\mathcal{G}_3$ is exact (see Modules, Lemma 3.1). The result mentioned in the introduction to this section proves the pullback by $f^*$ gives a homotopy equivalence from the constant cosimplicial object $f^*\mathcal{F}$ to the cosimplicial object with terms $f_*f^*\mathcal{F}, f_*f^*f_*f^*\mathcal{F}, f_*f^*f_*f^*f_*f^*\mathcal{F}, \ldots$. By Simplicial, Lemma 28.7 we obtain that

$$f^*\mathcal{F}[0] \to \left( f^*f_*f^*\mathcal{F} \to f^*f_*f^*f_*f^*\mathcal{F} \to f^*f_*f^*f_*f^*f_*f^*\mathcal{F} \to \ldots \right)$$

is a homotopy equivalence. This immediately implies the two remaining statements of the lemma. $\square$

**Lemma 30.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. There exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ where $\mathcal{G}^\bullet$ be a bounded below complex of flasque $\mathcal{O}_X$-modules and for all $x \in X$ the map $\mathcal{F}_x^\bullet \to \mathcal{G}_x^\bullet$ is a homotopy equivalence in the category of complexes of $\mathcal{O}_{X,x}$-modules.*

**Proof.** Let $\mathcal{A}$ be the category of complexes of $\mathcal{O}_X$-modules and let $\mathcal{B}$ be the category of complexes of $\mathcal{O}_X$-modules. Then we can apply the discussion above to the adjoint functors $f^*$ and $f_*$ between $\mathcal{A}$ and $\mathcal{B}$. Arguing exactly as in the proof of Lemma 30.1 we get a resolution

$$0 \to \mathcal{F}^\bullet \to f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*f_*f^*\mathcal{F}^\bullet \to \ldots$$

in the abelian category $\mathcal{A}$ such that each term of each $f_*f^* \ldots f_*f^*\mathcal{F}^\bullet$ is a flasque $\mathcal{O}_X$-module and such that for all $x \in X$ the map

$$\mathcal{F}_x^\bullet[0] \to \left( (f_*f^*\mathcal{F}^\bullet)_x \to (f_*f^*f_*f^*\mathcal{F}^\bullet)_x \to (f_*f^*f_*f^*f_*f^*\mathcal{F}^\bullet)_x \to \ldots \right)$$

is a homotopy equivalence in the category of complexes of complexes of $\mathcal{O}_{X,x}$-modules. Since a complex of complexes is the same thing as a double complex, we can consider the induced map

$$\mathcal{F}^\bullet \to \mathcal{G}^\bullet = \mathrm{Tot}(f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*f_*f^*\mathcal{F}^\bullet \to \ldots)$$

Since the complex $\mathcal{F}^\bullet$ is bounded below, the same is true for $\mathcal{G}^\bullet$ and in fact each term of $\mathcal{G}^\bullet$ is a finite direct sum of terms of the complexes $f_*f^* \ldots f_*f^*\mathcal{F}^\bullet$ and hence is flasque. The final assertion of the lemma now follows from Homology, Lemma 25.5. Since this in particular shows that $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism, the proof is complete. $\square$

## 31. Cup product

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, M$ be objects of $D(\mathcal{O}_X)$. Set $A = \Gamma(X, \mathcal{O}_X)$. The (global) cup product in this setting is a map

$$\mu : R\Gamma(X, K) \otimes_A^{\mathbf{L}} R\Gamma(X, M) \longrightarrow R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$$

in $D(A)$. We define it as the relative cup product for the morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (pt, A)$ as in Remark 28.7 via $D(pt, A) = D(A)$. This map in particular defines pairings

$$\cup : H^i(X, K) \times H^j(X, M) \longrightarrow H^{i+j}(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

Namely, given $\xi \in H^i(X, K) = H^i(R\Gamma(X, K))$ and $\eta \in H^j(X, M) = H^j(R\Gamma(X, M))$ we can first "tensor" them to get an element $\xi \otimes \eta$ in $H^{i+j}(R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(X, M))$, see More on Algebra, Section 63. Then we can apply $\mu$ to get the desired element $\xi \cup \eta = \mu(\xi \otimes \eta)$ of $H^{i+j}(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$.

Here is another way to think of the cup product of $\xi$ and $\eta$. Namely, we can write

$$R\Gamma(X, K) = R\operatorname{Hom}_X(\mathcal{O}_X, K) \quad \text{and} \quad R\Gamma(X, M) = R\operatorname{Hom}_X(\mathcal{O}_X, M)$$

because $\operatorname{Hom}(\mathcal{O}_X, -) = \Gamma(X, -)$. Thus $\xi$ and $\eta$ are the "same" thing as maps

$$\tilde{\xi} : \mathcal{O}_X[-i] \to K \quad \text{and} \quad \tilde{\eta} : \mathcal{O}_X[-j] \to M$$

Combining this with the functoriality of the derived tensor product we obtain

$$\mathcal{O}_X[-i-j] = \mathcal{O}_X[-i] \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X[-j] \xrightarrow{\tilde{\xi} \otimes \tilde{\eta}} K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$

which by the same token as above is an element of $H^{i+j}(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$.

**Lemma 31.1.** *This construction gives the cup product.*

**Proof.** With $f : (X, \mathcal{O}_X) \to (pt, A)$ as above we have $Rf_*(-) = R\Gamma(X, -)$ and our map $\mu$ is adjoint to the map

$$Lf^*(Rf_*K \otimes^{\mathbf{L}}_A Rf_*M) = Lf^*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*Rf_*M \xrightarrow{\epsilon_K \otimes \epsilon_M} K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$

where $\epsilon$ is the counit of the adjunction between $Lf^*$ and $Rf_*$. If we think of $\xi$ and $\eta$ as maps $\xi : A[-i] \to R\Gamma(X, K)$ and $\eta : A[-j] \to R\Gamma(X, M)$, then the tensor $\xi \otimes \eta$ corresponds to the map[3]

$$A[-i-j] = A[-i] \otimes^{\mathbf{L}}_A A[-j] \xrightarrow{\xi \otimes \eta} R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(X, M)$$

By definition the cup product $\xi \cup \eta$ is the map $A[-i-j] \to R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$ which is adjoint to

$$(\epsilon_K \otimes \epsilon_M) \circ Lf^*(\xi \otimes \eta) = (\epsilon_K \circ Lf^*\xi) \otimes (\epsilon_M \circ Lf^*\eta)$$

However, it is easy to see that $\epsilon_K \circ Lf^*\xi = \tilde{\xi}$ and $\epsilon_M \circ Lf^*\eta = \tilde{\eta}$. We conclude that $\widetilde{\xi \cup \eta} = \tilde{\xi} \otimes \tilde{\eta}$ which means we have the desired agreement.                          $\square$

**Remark 31.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, M$ be objects of $D(\mathcal{O}_X)$. Set $A = \Gamma(X, \mathcal{O}_X)$. Given $\xi \in H^i(X, K)$ we get an associated map

$$\xi = \text{``}\xi \cup -\text{''} : R\Gamma(X, M)[-i] \to R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

by representing $\xi$ as a map $\xi : A[-i] \to R\Gamma(X, K)$ as in the proof of Lemma 31.1 and then using the composition

$$R\Gamma(X, M)[-i] = A[-i] \otimes^{\mathbf{L}}_A R\Gamma(X, M) \xrightarrow{\xi \otimes 1} R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(X, M) \to R\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

---

[3]There is a sign hidden here, namely, the equality is defined by the composition

$$A[-i-j] \to (A \otimes^{\mathbf{L}}_A A)[-i-j] \to A[-i] \otimes^{\mathbf{L}}_A A[-j]$$

where in the second step we use the identification of More on Algebra, Item (7) which uses a sign in principle. Except, in this case the sign is $+1$ by our convention and even if it wasn't $+1$ it wouldn't matter since we used the same sign in the identification $\mathcal{O}_X[-i-j] = \mathcal{O}_X[-i] \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X[-j]$.

where the second arrow is the global cup product $\mu$ above. On cohomology this recovers the cup product by $\xi$ as is clear from Lemma 31.1 and its proof.

Let us formulate and prove a natural compatibility of the relative cup product. Namely, suppose that we have a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_X$-modules. There is a naive cup product

$$\mathrm{Tot}(f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_* \mathcal{M}^\bullet) \longrightarrow f_* \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)$$

We claim that this is related to the relative cup product.

**Lemma 31.3.** *In the situation above the following diagram commutes*

$$
\begin{array}{ccc}
f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_* \mathcal{M}^\bullet & \longrightarrow & Rf_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_* \mathcal{M}^\bullet \\
\downarrow & & \downarrow {\scriptstyle Remark\ 28.7} \\
Tot(f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_* \mathcal{M}^\bullet) & & Rf_*(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{M}^\bullet) \\
{\scriptstyle naive\ cup\ product}\downarrow & & \downarrow \\
f_* Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) & \longrightarrow & Rf_* Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)
\end{array}
$$

**Proof.** By the construction in Remark 28.7 we see that going around the diagram clockwise the map

$$f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_* \mathcal{M}^\bullet \longrightarrow Rf_* \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)$$

is adjoint to the map

$$
\begin{aligned}
Lf^*(f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_* \mathcal{M}^\bullet) &= Lf^* f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lf^* f_* \mathcal{M}^\bullet \\
&\to Lf^* Rf_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lf^* Rf_* \mathcal{M}^\bullet \\
&\to \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{M}^\bullet \\
&\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)
\end{aligned}
$$

By Lemma 28.6 this is also equal to

$$
\begin{aligned}
Lf^*(f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_* \mathcal{M}^\bullet) &= Lf^* f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lf^* f_* \mathcal{M}^\bullet \\
&\to f^* f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f^* f_* \mathcal{M}^\bullet \\
&\to \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{M}^\bullet \\
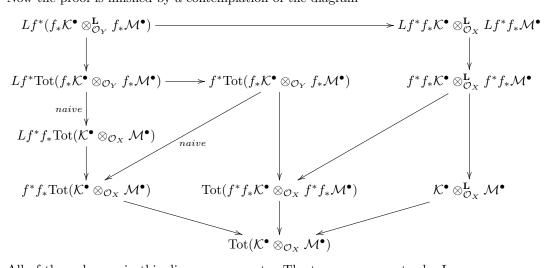&\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)
\end{aligned}
$$

Going around anti-clockwise we obtain the map adjoint to the map

$$
\begin{aligned}
Lf^*(f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_* \mathcal{M}^\bullet) &\to Lf^* \mathrm{Tot}(f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_* \mathcal{M}^\bullet) \\
&\to Lf^* f_* \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\
&\to Lf^* Rf_* \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\
&\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)
\end{aligned}
$$

By Lemma 28.6 this is also equal to

$$
\begin{aligned}
Lf^*(f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_* \mathcal{M}^\bullet) &\to Lf^* \mathrm{Tot}(f_* \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_* \mathcal{M}^\bullet) \\
&\to Lf^* f_* \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\
&\to f^* f_* \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\
&\to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)
\end{aligned}
$$

Now the proof is finished by a contemplation of the diagram

$$
\begin{array}{ccc}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) & \longrightarrow & Lf^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*f_*\mathcal{M}^\bullet \\
\downarrow & & \downarrow \\
Lf^*\mathrm{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) \longrightarrow f^*\mathrm{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) & & f^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^*f_*\mathcal{M}^\bullet \\
{\scriptstyle naive}\downarrow & & \\
Lf^*f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) & & \\
\downarrow & & \\
f^*f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \qquad \mathrm{Tot}(f^*f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} f^*f_*\mathcal{M}^\bullet) & & \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{M}^\bullet \\
& \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) &
\end{array}
$$

All of the polygons in this diagram commute. The top one commutes by Lemma 27.5. The square with the two naive cup products commutes because $Lf^* \to f^*$ is functorial in the complex of modules. Similarly with the square involving the two maps $\mathcal{A}^\bullet \otimes^{\mathbf{L}} \mathcal{B}^\bullet \to \mathrm{Tot}(\mathcal{A}^\bullet \otimes \mathcal{B}^\bullet)$. Finally, the commutativity of the remaining square is true on the level of complexes and may be viewed as the definiton of the naive cup product (by the adjointness of $f^*$ and $f_*$). The proof is finished because going around the diagram on the outside are the two maps given above. $\qquad\square$

Let $(X, \mathcal{O}_X)$ be a ring space. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_X$-modules. Then we have a "naive" cup product

$$
\mu' : \mathrm{Tot}(\Gamma(X, \mathcal{K}^\bullet) \otimes_A \Gamma(X, \mathcal{M}^\bullet)) \longrightarrow \Gamma(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))
$$

By Lemma 31.3 applied to the morphism $(X, \mathcal{O}_X) \to (pt, A)$ this naive cup product is related to the cup product $\mu$ defined in the first paragraph of this section by the following commutative diagram

$$
\begin{array}{ccc}
\Gamma(X, \mathcal{K}^\bullet) \otimes^{\mathbf{L}}_A \Gamma(X, \mathcal{M}^\bullet) & \longrightarrow & R\Gamma(X, \mathcal{K}^\bullet) \otimes^{\mathbf{L}}_A R\Gamma(X, \mathcal{M}^\bullet) \\
\downarrow & & \downarrow{\scriptstyle \mu} \\
\mathrm{Tot}(\Gamma(X, \mathcal{K}^\bullet) \otimes_A \Gamma(X, \mathcal{M}^\bullet)) & & R\Gamma(X, \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{M}^\bullet) \\
{\scriptstyle \mu'}\downarrow & & \downarrow \\
\Gamma(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) & \longrightarrow & R\Gamma(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))
\end{array}
$$

in $D(A)$. On cohomology we obtain the commutative diagram

$$
\begin{array}{ccc}
H^i(\Gamma(X, \mathcal{K}^\bullet)) \times H^j(\Gamma(X, \mathcal{M}^\bullet)) & \longrightarrow & H^{i+j}(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) \\
\downarrow & & \uparrow \\
H^i(X, \mathcal{K}^\bullet) \times H^j(X, \mathcal{M}^\bullet) & \overset{\cup}{\longrightarrow} & H^{i+j}(X, \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{M}^\bullet)
\end{array}
$$

relating the naive cup product with the actual cuproduct.

**Lemma 31.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be bounded below complexes of $\mathcal{O}_X$-modules. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering Then*

$$
\begin{array}{ccc}
Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes_A^{\mathbf{L}} Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet)) & \longrightarrow & R\Gamma(X, \mathcal{K}^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(X, \mathcal{M}^\bullet) \\
\downarrow & & \downarrow{\scriptstyle \mu} \\
Tot(\,Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes_A Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet))) & & R\Gamma(X, \mathcal{K}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{M}^\bullet) \\
\downarrow{\scriptstyle (25.3.2)} & & \downarrow \\
Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))) & \longrightarrow & R\Gamma(X, Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))
\end{array}
$$

*where the horizontal arrows are the ones in Lemma 25.1 commutes in $D(A)$.*

**Proof.** Choose quasi-isomorphisms of complexes $a : \mathcal{K}^\bullet \to \mathcal{K}_1^\bullet$ and $b : \mathcal{M}^\bullet \to \mathcal{M}_1^\bullet$ as in Lemma 30.2. Since the maps $a$ and $b$ on stalks are homotopy equivalences we see that the induced map

$$
\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \to \mathrm{Tot}(\mathcal{K}_1^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}_1^\bullet)
$$

is a homotopy equivalence on stalks too (More on Algebra, Lemma 58.2) and hence a quasi-isomorphism. Thus the targets

$$
R\Gamma(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) = R\Gamma(X, \mathrm{Tot}(\mathcal{K}_1^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}_1^\bullet))
$$

of the two diagrams are the same in $D(A)$. It follows that it suffices to prove the diagram commutes for $\mathcal{K}$ and $\mathcal{M}$ replaced by $\mathcal{K}_1$ and $\mathcal{M}_1$. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ are bounded below complexes of flasque $\mathcal{O}_X$-modules and consider the diagram relating the cup product with the cup product (25.3.2) on Čech complexes. Then we can consider the commutative diagram

$$
\begin{array}{ccc}
\Gamma(X, \mathcal{K}^\bullet) \otimes_A^{\mathbf{L}} \Gamma(X, \mathcal{M}^\bullet) & \longrightarrow & Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes_A^{\mathbf{L}} Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet)) \\
\downarrow & & \downarrow \\
\mathrm{Tot}(\Gamma(X, \mathcal{K}^\bullet) \otimes_A \Gamma(X, \mathcal{M}^\bullet)) & \longrightarrow & \mathrm{Tot}(Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes_A Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet))) \\
\downarrow & & \downarrow{\scriptstyle (25.3.2)} \\
\Gamma(X, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)))
\end{array}
$$

In this diagram the horizontal arrows are isomorphisms in $D(A)$ because for a bounded below complex of flasque modules such as $\mathcal{K}^\bullet$ we have

$$
\Gamma(X, \mathcal{K}^\bullet) = \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) = R\Gamma(X, \mathcal{K}^\bullet)
$$

in $D(A)$. This follows from Lemma 12.3, Derived Categories, Lemma 16.7, and Lemma 25.2. Hence the commutativity of the diagram of the lemma involving (25.3.2) follows from the already proven commutativity of Lemma 31.3 where $f$ is the morphism to a point (see discussion following Lemma 31.3). $\qquad\square$

**Lemma 31.5.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The relative cup product of Remark 28.7 is associative in the sense that the diagram*

$$
\begin{array}{ccc}
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*M & \longrightarrow & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*M \\
\downarrow & & \downarrow \\
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*(L \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) & \longrightarrow & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)
\end{array}
$$

*is commutative in $D(\mathcal{O}_Y)$ for all $K, L, M$ in $D(\mathcal{O}_X)$.*

**Proof.** Going around either side we obtain the map adjoint to the obvious map

$$Lf^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*M) = Lf^*(Rf_*K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*(Rf_*L) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*(Rf_*M)$$
$$\to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$

in $D(\mathcal{O}_X)$.                                                                                           $\square$

**Lemma 31.6.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The relative cup product of Remark 28.7 is commutative in the sense that the diagram*

$$
\begin{array}{ccc}
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L & \longrightarrow & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) \\
\psi \downarrow & & \downarrow Rf_*\psi \\
Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*K & \longrightarrow & Rf_*(L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K)
\end{array}
$$

*is commutative in $D(\mathcal{O}_Y)$ for all $K, L$ in $D(\mathcal{O}_X)$. Here $\psi$ is the commutativity constraint on the derived category (Lemma 50.6).*

**Proof.** Omitted.                                                                                           $\square$

**Lemma 31.7.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. The relative cup product of Remark 28.7 is compatible with compositions in the sense that the diagram*

$$
\begin{array}{ccc}
R(g \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Z} R(g \circ f)_*L & =\!\!=\!\!=\!\!=\!\!=\!\!=\!\!= & Rg_*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Z} Rg_*Rf_*L \\
\downarrow & & \downarrow \\
R(g \circ f)_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) =\!\!=\!\!= Rg_*Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) \longleftarrow Rg_*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L)
\end{array}
$$

*is commutative in $D(\mathcal{O}_Z)$ for all $K, L$ in $D(\mathcal{O}_X)$.*

**Proof.** This is true because going around the diagram either way we obtain the map adjoint to the map

$$L(g \circ f)^* \left( R(g \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Z} R(g \circ f)_*L \right)$$
$$= L(g \circ f)^*R(g \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L(g \circ f)^*R(g \circ f)_*L)$$
$$\to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$$

in $D(\mathcal{O}_X)$. To see this one uses that the composition of the counits like so

$$L(g \circ f)^*R(g \circ f)_* = Lf^*Lg^*Rg_*Rf_* \to Lf^*Rf_* \to \mathrm{id}$$

is the counit for $L(g \circ f)^*$ and $R(g \circ f)_*$. See Categories, Lemma 24.9.                 $\square$

## 32. Some properties of K-injective complexes

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be an open subset. Denote $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ the corresponding open immersion. The pullback functor $j^*$ is exact as it is just the restriction functor. Thus derived pullback $Lj^*$ is computed on any complex by simply restricting the complex. We often simply denote the corresponding functor

$$D(\mathcal{O}_X) \to D(\mathcal{O}_U), \quad E \mapsto j^*E = E|_U$$

Similarly, extension by zero $j_! : Mod(\mathcal{O}_U) \to Mod(\mathcal{O}_X)$ (see Sheaves, Section 31) is an exact functor (Modules, Lemma 3.4). Thus it induces a functor

$$j_! : D(\mathcal{O}_U) \to D(\mathcal{O}_X), \quad F \mapsto j_!F$$

by simply applying $j_!$ to any complex representing the object $F$.

**Lemma 32.1.** *Let $X$ be a ringed space. Let $U \subset X$ be an open subspace. The restriction of a K-injective complex of $\mathcal{O}_X$-modules to $U$ is a K-injective complex of $\mathcal{O}_U$-modules.*

**Proof.** Follows from Derived Categories, Lemma 31.9 and the fact that the restriction functor has the exact left adjoint $j_!$. For the construction of $j_!$ see Sheaves, Section 31 and for exactness see Modules, Lemma 3.4. $\square$

**Lemma 32.2.** *Let $X$ be a ringed space. Let $U \subset X$ be an open subspace. For $K$ in $D(\mathcal{O}_X)$ we have $H^p(U, K) = H^p(U, K|_U)$.*

**Proof.** Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}_X$-modules representing $K$. Then

$$H^q(U, K) = H^q(\Gamma(U, \mathcal{I}^\bullet)) = H^q(\Gamma(U, \mathcal{I}^\bullet|_U))$$

by construction of cohomology. By Lemma 32.1 the complex $\mathcal{I}^\bullet|_U$ is a K-injective complex representing $K|_U$ and the lemma follows. $\square$

**Lemma 32.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K$ be an object of $D(\mathcal{O}_X)$. The sheafification of*

$$U \mapsto H^q(U, K) = H^q(U, K|_U)$$

*is the qth cohomology sheaf $H^q(K)$ of $K$.*

**Proof.** The equality $H^q(U, K) = H^q(U, K|_U)$ holds by Lemma 32.2. Choose a K-injective complex $\mathcal{I}^\bullet$ representing $K$. Then

$$H^q(U, K) = \frac{\mathrm{Ker}(\mathcal{I}^q(U) \to \mathcal{I}^{q+1}(U))}{\mathrm{Im}(\mathcal{I}^{q-1}(U) \to \mathcal{I}^q(U))}.$$

by our construction of cohomology. Since $H^q(K) = \mathrm{Ker}(\mathcal{I}^q \to \mathcal{I}^{q+1})/\mathrm{Im}(\mathcal{I}^{q-1} \to \mathcal{I}^q)$ the result is clear. $\square$

**Lemma 32.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Given an open subspace $V \subset Y$, set $U = f^{-1}(V)$ and denote $g : U \to V$ the induced morphism. Then $(Rf_*E)|_V = Rg_*(E|_U)$ for $E$ in $D(\mathcal{O}_X)$.*

**Proof.** Represent $E$ by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_U) = g_*(\mathcal{I}^\bullet|_U)$ by Lemma 32.1. Since it is clear that $(f_*\mathcal{F})|_V = g_*(\mathcal{F}|_U)$ for any sheaf $\mathcal{F}$ on $X$ the result follows. $\square$

**Lemma 32.5.** *Let $f : X \to Y$ be a morphism of ringed spaces. Then $R\Gamma(Y, -) \circ Rf_* = R\Gamma(X, -)$ as functors $D(\mathcal{O}_X) \to D(\Gamma(Y, \mathcal{O}_Y))$. More generally for $V \subset Y$ open and $U = f^{-1}(V)$ we have $R\Gamma(U, -) = R\Gamma(V, -) \circ Rf_*$.*

**Proof.** Let $Z$ be the ringed space consisting of a singleton space with $\Gamma(Z, \mathcal{O}_Z) = \Gamma(Y, \mathcal{O}_Y)$. There is a canonical morphism $Y \to Z$ of ringed spaces inducing the identification on global sections of structure sheaves. Then $D(\mathcal{O}_Z) = D(\Gamma(Y, \mathcal{O}_Y))$. Hence the assertion $R\Gamma(Y, -) \circ Rf_* = R\Gamma(X, -)$ follows from Lemma 28.2 applied to $X \to Y \to Z$.

The second (more general) statement follows from the first statement after applying Lemma 32.4. $\square$

**Lemma 32.6.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $K$ be in $D(\mathcal{O}_X)$. Then $H^i(Rf_*K)$ is the sheaf associated to the presheaf*

$$V \mapsto H^i(f^{-1}(V), K) = H^i(V, Rf_*K)$$

**Proof.** The equality $H^i(f^{-1}(V), K) = H^i(V, Rf_*K)$ follows upon taking cohomology from the second statement in Lemma 32.5. Then the statement on sheafification follows from Lemma 32.3. $\square$

**Lemma 32.7.** *Let $X$ be a ringed space. Let $K$ be an object of $D(\mathcal{O}_X)$ and denote $K_{ab}$ its image in $D(\underline{\mathbf{Z}}_X)$.*

(1) *For any open $U \subset X$ there is a canonical map $R\Gamma(U, K) \to R\Gamma(U, K_{ab})$ which is an isomorphism in $D(Ab)$.*

(2) *Let $f : X \to Y$ be a morphism of ringed spaces. There is a canonical map $Rf_*K \to Rf_*(K_{ab})$ which is an isomorphism in $D(\underline{\mathbf{Z}}_Y)$.*

**Proof.** The map is constructed as follows. Choose a K-injective complex $\mathcal{I}^\bullet$ representing $K$. Choose a quasi-isomorpism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is a K-injective complex of abelian groups. Then the map in (1) is given by $\Gamma(U, \mathcal{I}^\bullet) \to \Gamma(U, \mathcal{J}^\bullet)$ and the map in (2) is given by $f_*\mathcal{I}^\bullet \to f_*\mathcal{J}^\bullet$. To show that these maps are isomorphisms, it suffices to prove they induce isomorphisms on cohomology groups and cohomology sheaves. By Lemmas 32.2 and 32.6 it suffices to show that the map

$$H^0(X, K) \longrightarrow H^0(X, K_{ab})$$

is an isomorphism. Observe that

$$H^0(X, K) = \operatorname{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X, K)$$

and similarly for the other group. Choose any complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules representing $K$. By construction of the derived category as a localization we have

$$\operatorname{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X, K) = \operatorname{colim}_{s:\mathcal{F}^\bullet \to \mathcal{O}_X} \operatorname{Hom}_{K(\mathcal{O}_X)}(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

where the colimit is over quasi-isomorphisms $s$ of complexes of $\mathcal{O}_X$-modules. Similarly, we have

$$\operatorname{Hom}_{D(\underline{\mathbf{Z}}_X)}(\underline{\mathbf{Z}}_X, K) = \operatorname{colim}_{s:\mathcal{G}^\bullet \to \underline{\mathbf{Z}}_X} \operatorname{Hom}_{K(\underline{\mathbf{Z}}_X)}(\mathcal{G}^\bullet, \mathcal{K}^\bullet)$$

Next, we observe that the quasi-isomorphisms $s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_X$ with $\mathcal{G}^\bullet$ bounded above complex of flat $\underline{\mathbf{Z}}_X$-modules is cofinal in the system. (This follows from Modules, Lemma 17.6 and Derived Categories, Lemma 15.4; see discussion in Section 26.)

Hence we can construct an inverse to the map $H^0(X, K) \longrightarrow H^0(X, K_{ab})$ by representing an element $\xi \in H^0(X, K_{ab})$ by a pair

$$(s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_X, a : \mathcal{G}^\bullet \to \mathcal{K}^\bullet)$$

with $\mathcal{G}^\bullet$ a bounded above complex of flat $\underline{\mathbf{Z}}_X$-modules and sending this to

$$(\mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_X} \mathcal{O}_X \to \mathcal{O}_X, \mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_X} \mathcal{O}_X \to \mathcal{K}^\bullet)$$

The only thing to note here is that the first arrow is a quasi-isomorphism by Lemmas 26.13 and 26.9. We omit the detailed verification that this construction is indeed an inverse. $\square$

**Lemma 32.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be an open subset. Denote $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ the corresponding open immersion. The restriction functor $D(\mathcal{O}_X) \to D(\mathcal{O}_U)$ is a right adjoint to extension by zero $j_! : D(\mathcal{O}_U) \to D(\mathcal{O}_X)$.*

**Proof.** This follows formally from the fact that $j_!$ and $j^*$ are adjoint and exact (and hence $Lj_! = j_!$ and $Rj^* = j^*$ exist), see Derived Categories, Lemma 30.3. $\square$

**Lemma 32.9.** *Let $f : X \to Y$ be a flat morphism of ringed spaces. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules, then $f_* \mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_Y$-modules.*

**Proof.** This is true because

$$\mathrm{Hom}_{K(\mathcal{O}_Y)}(\mathcal{F}^\bullet, f_* \mathcal{I}^\bullet) = \mathrm{Hom}_{K(\mathcal{O}_X)}(f^* \mathcal{F}^\bullet, \mathcal{I}^\bullet)$$

by Sheaves, Lemma 26.2 and the fact that $f^*$ is exact as $f$ is assumed to be flat. $\square$

## 33. Unbounded Mayer-Vietoris

There is a Mayer-Vietoris sequence for unbounded cohomology as well.

**Lemma 33.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces. For any object $E$ of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$j_{U \cap V!} E|_{U \cap V} \to j_{U!} E|_U \oplus j_{V!} E|_V \to E \to j_{U \cap V!} E|_{U \cap V}[1]$$

*in $D(\mathcal{O}_X)$.*

**Proof.** We have seen in Section 32 that the restriction functors and the extension by zero functors are computed by just applying the functors to any complex. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules representing $E$. The distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes of $\mathcal{O}_X$-modules

$$0 \to j_{U \cap V!} \mathcal{E}^\bullet|_{U \cap V} \to j_{U!} \mathcal{E}^\bullet|_U \oplus j_{V!} \mathcal{E}^\bullet|_V \to \mathcal{E}^\bullet \to 0$$

To see this sequence is exact one checks on stalks using Sheaves, Lemma 31.8 (computation omitted). $\square$

**Lemma 33.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces. For any object $E$ of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$E \to Rj_{U,*} E|_U \oplus Rj_{V,*} E|_V \to Rj_{U \cap V,*} E|_{U \cap V} \to E[1]$$

*in $D(\mathcal{O}_X)$.*

**Proof.** Choose a K-injective complex $\mathcal{I}^\bullet$ representing $E$ whose terms $\mathcal{I}^n$ are injective objects of $Mod(\mathcal{O}_X)$, see Injectives, Theorem 12.6. We have seen that $\mathcal{I}^\bullet|U$ is a K-injective complex as well (Lemma 32.1). Hence $Rj_{U,*}E|_U$ is represented by $j_{U,*}\mathcal{I}^\bullet|_U$. Similarly for $V$ and $U \cap V$. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \to j_{U\cap V,*}\mathcal{I}^\bullet|_{U\cap V} \to 0.$$

This sequence is exact because for any $W \subset X$ open and any $n$ the sequence

$$0 \to \mathcal{I}^n(W) \to \mathcal{I}^n(W \cap U) \oplus \mathcal{I}^n(W \cap V) \to \mathcal{I}^n(W \cap U \cap V) \to 0$$

is exact (see proof of Lemma 8.2). $\qquad\square$

**Lemma 33.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces of $X$. For objects $E$, $F$ of $D(\mathcal{O}_X)$ we have a Mayer-Vietoris sequence*

$$\ldots \longrightarrow \mathrm{Ext}^{-1}(E_{U\cap V}, F_{U\cap V})$$

$$\mathrm{Hom}(E,F) \Longrightarrow \mathrm{Hom}(E_U, F_U) \oplus \mathrm{Hom}(E_V, F_V) \longrightarrow \mathrm{Hom}(E_{U\cap V}, F_{U\cap V})$$

*where the subscripts denote restrictions to the relevant opens and the* Hom*'s and* Ext*'s are taken in the relevant derived categories.*

**Proof.** Use the distinguished triangle of Lemma 33.1 to obtain a long exact sequence of Hom's (from Derived Categories, Lemma 4.2) and use that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(j_{U!}E|_U, F) = \mathrm{Hom}_{D(\mathcal{O}_U)}(E|_U, F|_U)$$

by Lemma 32.8. $\qquad\square$

**Lemma 33.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose that $X = U \cup V$ is a union of two open subsets. For an object $E$ of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$R\Gamma(X, E) \to R\Gamma(U, E) \oplus R\Gamma(V, E) \to R\Gamma(U \cap V, E) \to R\Gamma(X, E)[1]$$

*and in particular a long exact cohomology sequence*

$$\ldots \to H^n(X, E) \to H^n(U, E) \oplus H^0(V, E) \to H^n(U \cap V, E) \to H^{n+1}(X, E) \to \ldots$$

*The construction of the distinguished triangle and the long exact sequence is functorial in $E$.*

**Proof.** Choose a K-injective complex $\mathcal{I}^\bullet$ representing $E$. We may assume $\mathcal{I}^n$ is an injective object of $Mod(\mathcal{O}_X)$ for all $n$, see Injectives, Theorem 12.6. Then $R\Gamma(X, E)$ is computed by $\Gamma(X, \mathcal{I}^\bullet)$. Similarly for $U$, $V$, and $U \cap V$ by Lemma 32.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

We have seen this is a short exact sequence in the proof of Lemma 8.2. The final statement follows from the functoriality of the construction in Injectives, Theorem 12.6. $\qquad\square$

**Lemma 33.5.** *Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \cap V} : U \cap V \to Y$. For every object $E$ of $D(\mathcal{O}_X)$ there exists a distinguished triangle*

$$Rf_*E \to Ra_*(E|_U) \oplus Rb_*(E|_V) \to Rc_*(E|_{U \cap V}) \to Rf_*E[1]$$

*This triangle is functorial in $E$.*

**Proof.** Choose a K-injective complex $\mathcal{I}^\bullet$ representing $E$. We may assume $\mathcal{I}^n$ is an injective object of $Mod(\mathcal{O}_X)$ for all $n$, see Injectives, Theorem 12.6. Then $Rf_*E$ is computed by $f_*\mathcal{I}^\bullet$. Similarly for $U$, $V$, and $U \cap V$ by Lemma 32.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \cap V} \to 0.$$

This is a short exact sequence of complexes by Lemma 8.3 and the fact that $R^1f_*\mathcal{I} = 0$ for an injective object $\mathcal{I}$ of $Mod(\mathcal{O}_X)$. The final statement follows from the functoriality of the construction in Injectives, Theorem 12.6. $\qquad\square$

**Lemma 33.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $j : U \to X$ be an open subspace. Let $T \subset X$ be a closed subset contained in $U$.*

(1) *If $E$ is an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on $T$, then $E \to Rj_*(E|_U)$ is an isomorphism.*
(2) *If $F$ is an object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on $T$, then $j_!F \to Rj_*F$ is an isomorphism.*

**Proof.** Let $V = X \setminus T$ and $W = U \cap V$. Note that $X = U \cup V$ is an open covering of $X$. Denote $j_W : W \to V$ the open immersion. Let $E$ be an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on $T$. By Lemma 32.4 we have $(Rj_*E|_U)|_V = Rj_{W,*}(E|_W) = 0$ because $E|_W = 0$ by our assumption. On the other hand, $Rj_*(E|_U)|_U = E|_U$. Thus (1) is clear. Let $F$ be an object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on $T$. By Lemma 32.4 we have $(Rj_*F)|_V = Rj_{W,*}(F|_W) = 0$ because $F|_W = 0$ by our assumption. We also have $(j_!F)|_V = j_{W!}(F|_W) = 0$ (the first equality is immediate from the definition of extension by zero). Since both $(Rj_*F)|_U = F$ and $(j_!F)|_U = F$ we see that (2) holds. $\qquad\square$

**Lemma 33.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Set $A = \Gamma(X, \mathcal{O}_X)$. Suppose that $X = U \cup V$ is a union of two open subsets. For objects $K$ and $M$ of $D(\mathcal{O}_X)$ we*

*have a map of distinguished triangles*

$$
\begin{array}{ccc}
R\Gamma(X,K) \otimes_A^{\mathbf{L}} R\Gamma(X,M) & \longrightarrow & R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M) \\
\downarrow & & \downarrow \\
R\Gamma(X,K) \otimes_A^{\mathbf{L}} (R\Gamma(U,M) \oplus R\Gamma(V,M)) \longrightarrow R\Gamma(U, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M) \oplus R\Gamma(V, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)) \\
\downarrow & & \downarrow \\
R\Gamma(X,K) \otimes_A^{\mathbf{L}} R\Gamma(U \cap V, M) & \longrightarrow & R\Gamma(U \cap V, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M) \\
\downarrow & & \downarrow \\
R\Gamma(X,K) \otimes_A^{\mathbf{L}} R\Gamma(X,M)[1] & \longrightarrow & R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)[1]
\end{array}
$$

*where*

(1) *the horizontal arrows are given by cup product,*
(2) *on the right hand side we have the distinguished triangle of Lemma 33.4 for $K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M$, and*
(3) *on the left hand side we have the exact functor $R\Gamma(X,K) \otimes_A^{\mathbf{L}} -$ applied to the distinguished triangle of Lemma 33.4 for $M$.*

**Proof.** Choose a K-flat complex $T^\bullet$ of flat $A$-modules representing $R\Gamma(X,K)$, see More on Algebra, Lemma 59.10. Denote $T^\bullet \otimes_A \mathcal{O}_X$ the pullback of $T^\bullet$ by the morphism of ringed spaces $(X, \mathcal{O}_X) \to (pt, A)$. There is a natural adjunction map $\epsilon : T^\bullet \otimes_A \mathcal{O}_X \to K$ in $D(\mathcal{O}_X)$. Observe that $T^\bullet \otimes_A \mathcal{O}_X$ is a K-flat complex of $\mathcal{O}_X$-modules with flat terms, see Lemma 26.8 and Modules, Lemma 20.2. By Lemma 26.17 we can find a morphism of complexes

$$ T^\bullet \otimes_A \mathcal{O}_X \longrightarrow \mathcal{K}^\bullet $$

of $\mathcal{O}_X$-modules representing $\epsilon$ such that $\mathcal{K}^\bullet$ is a K-flat complex with flat terms. Namely, by the construction of $D(\mathcal{O}_X)$ we can first represent $\epsilon$ by some map of complexes $e : T^\bullet \otimes_A \mathcal{O}_X \to \mathcal{L}^\bullet$ of $\mathcal{O}_X$-modules representing $\epsilon$ and then we can apply the lemma to $e$. Choose a K-injective complex $\mathcal{I}^\bullet$ whose terms are injective $\mathcal{O}_X$-modules representing $M$. Finally, choose a quasi-isomorphism

$$ \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{I}^\bullet) \longrightarrow \mathcal{J}^\bullet $$

into a K-injective complex whose terms are injective $\mathcal{O}_X$-modules. Observe that source and target of this arrow represent $K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M$ in $D(\mathcal{O}_X)$. At this point, for any open $W \subset X$ we obtain a map of complexes

$$ \mathrm{Tot}(T^\bullet \otimes_A \mathcal{I}^\bullet(W)) \to \mathrm{Tot}(\mathcal{K}^\bullet(W) \otimes_A \mathcal{I}^\bullet(W)) \to \mathcal{J}^\bullet(W) $$

of $A$-modules whose composition represents the map

$$ R\Gamma(X,K) \otimes_A^{\mathbf{L}} R\Gamma(W,M) \longrightarrow R\Gamma(W, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M) $$

in $D(A)$. Clearly, these maps are compatible with restriction mappings. OK, so now we can consider the following commutative(!) diagram of complexes of $A$-modules

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Tot}(T^\bullet \otimes_A \mathcal{I}^\bullet(X)) & \longrightarrow & \mathcal{J}^\bullet(X) \\
\downarrow & & \downarrow \\
\mathrm{Tot}(T^\bullet \otimes_A (\mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V))) & \longrightarrow & \mathcal{J}^\bullet(U) \oplus \mathcal{J}^\bullet(V) \\
\downarrow & & \downarrow \\
\mathrm{Tot}(T^\bullet \otimes_A \mathcal{I}^\bullet(U \cap V)) & \longrightarrow & \mathcal{J}^\bullet(U \cap V) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

By the proof of Lemma 8.2 the columns are exact sequences of complexes of $A$-modules (this also uses that $\mathrm{Tot}(T^\bullet \otimes_A -)$ transforms short exact sequences of complexes of $A$-modules into short exact sequences as the terms of $T^\bullet$ are flat $A$-modules). Since the distinguished triangles of Lemma 33.4 are the distinguished triangles associated to these short exact sequences of complexes, the desired result follows from the functoriality of "taking the associated distinguished triangle" discussed in Derived Categories, Section 12. $\square$

## 34. Cohomology with support in a closed subset, II

We continue the discussion started in Section 21.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. In this situation we can consider the functor $Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_X(X))$ given by $\mathcal{F} \mapsto \Gamma_Z(X, \mathcal{F})$. See Modules, Definition 5.1 and Modules, Lemma 5.2. Using K-injective resolutions, see Section 28, we obtain the right derived functor

$$R\Gamma_Z(X, -) : D(\mathcal{O}_X) \to D(\mathcal{O}_X(X))$$

Given an object $K$ in $D(\mathcal{O}_X)$ we denote $H_Z^q(X, K) = H^q(R\Gamma_Z(X, K))$ the cohomology module with support in $Z$. We will see later (Lemma 34.8) that this agrees with the construction in Section 21.

For an $\mathcal{O}_X$-module $\mathcal{F}$ we can consider the *subsheaf of sections with support in $Z$*, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset U \cap Z\} = \Gamma_{Z \cap U}(U, \mathcal{F}|_U)$$

As discussed in Modules, Remark 13.5 we may view $\mathcal{H}_Z(\mathcal{F})$ as an $\mathcal{O}_X|_Z$-module on $Z$ and we obtain a functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X|_Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as an } \mathcal{O}_X|_Z\text{-module on } Z$$

This functor is left exact, but in general not exact. Exactly as above we obtain a right derived functor

$$R\mathcal{H}_Z : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X|_Z)$$

We set $\mathcal{H}_Z^q(K) = H^q(R\mathcal{H}_Z(K))$ so that $\mathcal{H}_Z^0(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.

**Lemma 34.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset.*

    (1) *$R\mathcal{H}_Z : D(\mathcal{O}_X) \to D(\mathcal{O}_X|_Z)$ is right adjoint to $i_* : D(\mathcal{O}_X|_Z) \to D(\mathcal{O}_X)$.*
    (2) *For $K$ in $D(\mathcal{O}_X|_Z)$ we have $R\mathcal{H}_Z(i_*K) = K$.*
    (3) *Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X|_Z$-modules on $Z$. Then $\mathcal{H}_Z^p(i_*\mathcal{G}) = 0$ for $p > 0$.*

**Proof.** The functor $i_*$ is exact, so $i_* = Ri_* = Li_*$. Hence part (1) of the lemma follows from Modules, Lemma 13.6 and Derived Categories, Lemma 30.3. Let $K$ be as in (2). We can represent $K$ by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X|_Z$-modules. By Lemma 32.9 the complex $i_*\mathcal{I}^\bullet$, which represents $i_*K$, is a K-injective complex of $\mathcal{O}_X$-modules. Thus $R\mathcal{H}_Z(i_*K)$ is computed by $\mathcal{H}_Z(i_*\mathcal{I}^\bullet) = \mathcal{I}^\bullet$ which proves (2). Part (3) is a special case of (2). $\square$

Let $(X, \mathcal{O}_X)$ be a ringed space and let $Z \subset X$ be a closed subset. The category of $\mathcal{O}_X$-modules whose support is contained in $Z$ is a Serre subcategory of the category of all $\mathcal{O}_X$-modules, see Homology, Definition 10.1 and Modules, Lemma 5.2. We denote $D_Z(\mathcal{O}_X)$ the strictly full saturated triangulated subcategory of $D(\mathcal{O}_X)$ consisting of complexes whose cohomology sheaves are supported on $Z$, see Derived Categories, Section 17.

**Lemma 34.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset.*

    (1) *For $K$ in $D(\mathcal{O}_X|_Z)$ we have $i_*K$ in $D_Z(\mathcal{O}_X)$.*
    (2) *The functor $i_* : D(\mathcal{O}_X|_Z) \to D_Z(\mathcal{O}_X)$ is an equivalence with quasi-inverse $i^{-1}|_{D_Z(\mathcal{O}_X)} = R\mathcal{H}_Z|_{D_Z(\mathcal{O}_X)}$.*
    (3) *The functor $i_* \circ R\mathcal{H}_Z : D(\mathcal{O}_X) \to D_Z(\mathcal{O}_X)$ is right adjoint to the inclusion functor $D_Z(\mathcal{O}_X) \to D(\mathcal{O}_X)$.*

**Proof.** Part (1) is immediate from the definitions. Part (3) is a formal consequence of part (2) and Lemma 34.1. In the rest of the proof we prove part (2).

Let us think of $i$ as the morphism of ringed spaces $i : (Z, \mathcal{O}_X|_Z) \to (X, \mathcal{O}_X)$. Recall that $i^*$ and $i_*$ is an adjoint pair of functors. Since $i$ is a closed immersion, $i_*$ is exact. Since $i^{-1}\mathcal{O}_X = \mathcal{O}_X|_Z$ is the structure sheaf of $(Z, \mathcal{O}_X|_Z)$ we see that $i^* = i^{-1}$ is exact and we see that that $i^*i_* = i^{-1}i_*$ is isomorphic to the identify functor. See Modules, Lemmas 3.3 and 6.1. Thus $i_* : D(\mathcal{O}_X|_Z) \to D_Z(\mathcal{O}_X)$ is fully faithful and $i^{-1}$ determines a left inverse. On the other hand, suppose that $K$ is an object of $D_Z(\mathcal{O}_X)$ and consider the adjunction map $K \to i_*i^{-1}K$. Using exactness of $i_*$ and $i^{-1}$ this induces the adjunction maps $H^n(K) \to i_*i^{-1}H^n(K)$ on cohomology sheaves. Since these cohomology sheaves are supported on $Z$ we see these adjunction maps are isomorphisms and we conclude that $i_* : D(\mathcal{O}_X|_Z) \to D_Z(\mathcal{O}_X)$ is an equivalence.

To finish the proof it suffices to show that $R\mathcal{H}_Z(K) = i^{-1}K$ if $K$ is an object of $D_Z(\mathcal{O}_X)$. To do this we can use that $K = i_*i^{-1}K$ as we've just proved this is the case. Then Lemma 34.1 tells us what we want. $\square$

**Lemma 34.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules, then $\mathcal{H}_Z(\mathcal{I}^\bullet)$ is K-injective complex of $\mathcal{O}_X|_Z$-modules.*

**Proof.** Since $i_* : Mod(\mathcal{O}_X|_Z) \to Mod(\mathcal{O}_X)$ is exact and left adjoint to $\mathcal{H}_Z$ (Modules, Lemma 13.6) this follows from Derived Categories, Lemma 31.9. $\square$

**Lemma 34.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Then $R\Gamma(Z, -) \circ R\mathcal{H}_Z = R\Gamma_Z(X, -)$ as functors $D(\mathcal{O}_X) \to D(\mathcal{O}_X(X))$.*

**Proof.** Follows from the construction of right derived functors using K-injective resolutions, Lemma 34.3, and the fact that $\Gamma_Z(X, -) = \Gamma(Z, -) \circ \mathcal{H}_Z$. $\square$

**Lemma 34.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Let $U = X \setminus Z$. There is a distinguished triangle*

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

*in $D(\mathcal{O}_X(X))$ functorial for $K$ in $D(\mathcal{O}_X)$.*

**Proof.** Choose a K-injective complex $\mathcal{I}^\bullet$ all of whose terms are injective $\mathcal{O}_X$-modules representing $K$. See Section 28. Recall that $\mathcal{I}^\bullet|_U$ is a K-injective complex of $\mathcal{O}_U$-modules, see Lemma 32.1. Hence each of the derived functors in the distinguished triangle is gotten by applying the underlying functor to $\mathcal{I}^\bullet$. Hence we find that it suffices to prove that for an injective $\mathcal{O}_X$-module $\mathcal{I}$ we have a short exact sequence

$$0 \to \Gamma_Z(X, \mathcal{I}) \to \Gamma(X, \mathcal{I}) \to \Gamma(U, \mathcal{I}) \to 0$$

This follows from Lemma 8.1 and the definitions. $\square$

**Lemma 34.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Denote $j : U = X \setminus Z \to X$ the inclusion of the complement. There is a distinguished triangle*

$$i_* R\mathcal{H}_Z(K) \to K \to Rj_*(K|_U) \to i_* R\mathcal{H}_Z(K)[1]$$

*in $D(\mathcal{O}_X)$ functorial for $K$ in $D(\mathcal{O}_X)$.*

**Proof.** Choose a K-injective complex $\mathcal{I}^\bullet$ all of whose terms are injective $\mathcal{O}_X$-modules representing $K$. See Section 28. Recall that $\mathcal{I}^\bullet|_U$ is a K-injective complex of $\mathcal{O}_U$-modules, see Lemma 32.1. Hence each of the derived functors in the distinguished triangle is gotten by applying the underlying functor to $\mathcal{I}^\bullet$. Hence it suffices to prove that for an injective $\mathcal{O}_X$-module $\mathcal{I}$ we have a short exact sequence

$$0 \to i_* \mathcal{H}_Z(\mathcal{I}) \to \mathcal{I} \to j_*(\mathcal{I}|_U) \to 0$$

This follows from Lemma 8.1 and the definitions. $\square$

**Lemma 34.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. Let $j : U \to X$ be the inclusion of an open subset with $U \cap Z = \emptyset$. Then $R\mathcal{H}_Z(Rj_*K) = 0$ for all $K$ in $D(\mathcal{O}_U)$.*

**Proof.** Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_U$-modules representing $K$. Then $j_* \mathcal{I}^\bullet$ represents $Rj_* K$. By Lemma 32.9 the complex $j_* \mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules. Hence $\mathcal{H}_Z(j_* \mathcal{I}^\bullet)$ represents $R\mathcal{H}_Z(Rj_* K)$. Thus it suffices to show that $\mathcal{H}_Z(j_* \mathcal{G}) = 0$ for any abelian sheaf $\mathcal{G}$ on $U$. Thus we have to show that a section $s$ of $j_* \mathcal{G}$ over some open $W$ which is supported on $W \cap Z$ is zero. The support condition means that $s|_{W \setminus W \cap Z} = 0$. Since $j_* \mathcal{G}(W) = \mathcal{G}(U \cap W) = j_* \mathcal{G}(W \setminus W \cap Z)$ this implies that $s$ is zero as desired. $\square$

**Lemma 34.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. Let $K$ be an object of $D(\mathcal{O}_X)$ and denote $K_{ab}$ its image in $D(\underline{\mathbf{Z}}_X)$.*

(1) *There is a canonical map $R\Gamma_Z(X, K) \to R\Gamma_Z(X, K_{ab})$ which is an isomorphism in $D(Ab)$.*

(2) *There is a canonical map $R\mathcal{H}_Z(K) \to R\mathcal{H}_Z(K_{ab})$ which is an isomorphism in $D(\underline{\mathbf{Z}}_Z)$.*

**Proof.** Proof of (1). The map is constructed as follows. Choose a K-injective complex of $\mathcal{O}_X$-modules $\mathcal{I}^\bullet$ representing $K$. Choose a quasi-isomorpism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is a K-injective complex of abelian groups. Then the map in (1) is given by

$$\Gamma_Z(X, \mathcal{I}^\bullet) \to \Gamma_Z(X, \mathcal{J}^\bullet)$$

determined by the fact that $\Gamma_Z$ is a functor on abelian sheaves. An easy check shows that the resulting map combined with the canonical maps of Lemma 32.7 fit into a morphism of distinguished triangles

$$
\begin{array}{ccccc}
R\Gamma_Z(X, K) & \longrightarrow & R\Gamma(X, K) & \longrightarrow & R\Gamma(U, K) \\
\downarrow & & \downarrow & & \downarrow \\
R\Gamma_Z(X, K_{ab}) & \longrightarrow & R\Gamma(X, K_{ab}) & \longrightarrow & R\Gamma(U, K_{ab})
\end{array}
$$

of Lemma 34.5. Since two of the three arrows are isomorphisms by the lemma cited, we conclude by Derived Categories, Lemma 4.3.

The proof of (2) is omitted. Hint: use the same argument with Lemma 34.6 for the distinguished triangle. $\square$

**Remark 34.9.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Given $K$ and $M$ in $D(\mathcal{O}_X)$ there is a canonical map

$$K|_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X|_Z} R\mathcal{H}_Z(M) \longrightarrow R\mathcal{H}_Z(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

in $D(\mathcal{O}_X|_Z)$. Here $K|_Z = i^{-1}K$ is the restriction of $K$ to $Z$ viewed as an object of $D(\mathcal{O}_X|_Z)$. By adjointness of $i_*$ and $R\mathcal{H}_Z$ of Lemma 34.1 to construct this map it suffices to produce a canonical map

$$i_*\left(K|_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X|_Z} R\mathcal{H}_Z(M)\right) \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$

To construct this map, we choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules representing $M$ and a K-flat complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules representing $K$. Observe that $\mathcal{K}^\bullet|_Z$ is a K-flat complex of $\mathcal{O}_X|_Z$-modules representing $K|_Z$, see Lemma 26.8. Hence we need to produce a map of complexes

$$i_*\mathrm{Tot}\left(\mathcal{K}^\bullet|_Z \otimes_{\mathcal{O}_X|_Z} \mathcal{H}_Z(\mathcal{I}^\bullet)\right) \longrightarrow \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet)$$

of $\mathcal{O}_X$-modules. For this it suffices to produce maps

$$i_*(\mathcal{K}^a|_Z \otimes_{\mathcal{O}_X|_Z} \mathcal{H}_Z(\mathcal{I}^b)) \longrightarrow \mathcal{K}^a \otimes_{\mathcal{O}_X} \mathcal{I}^b$$

Looking at stalks (for example), we see that the left hand side of this formula is equal to $\mathcal{K}^a \otimes_{\mathcal{O}_X} i_*\mathcal{H}_Z(\mathcal{I}^b)$ and we can use the inclusion $\mathcal{H}_Z(\mathcal{I}^b) \to \mathcal{I}^b$ to get our map.

**Remark 34.10.** With notation as in Remark 34.9 we obtain a canonical cup product

$$
\begin{aligned}
H^a(X, K) \times H_Z^b(X, M) &= H^a(X, K) \times H^b(Z, R\mathcal{H}_Z(M)) \\
&\to H^a(Z, K|_Z) \times H^b(Z, R\mathcal{H}_Z(M)) \\
&\to H^{a+b}(Z, K|_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X|_Z} R\mathcal{H}_Z(M)) \\
&\to H^{a+b}(Z, R\mathcal{H}_Z(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)) \\
&= H_Z^{a+b}(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)
\end{aligned}
$$

Here the equal signs are given by Lemma 34.4, the first arrow is restriction to $Z$, the second arrow is the cup product (Section 31), and the third arrow is the map from Remark 34.9.

**Lemma 34.11.** *With notation as in Remark 34.9 the diagram*

$$
\begin{array}{ccc}
H^i(X, K) \times H_Z^j(X, M) & \longrightarrow & H_Z^{i+j}(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \\
\downarrow & & \downarrow \\
H^i(X, K) \times H^j(X, M) & \longrightarrow & H^{i+j}(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)
\end{array}
$$

*commutes where the top horizontal arrow is the cup product of Remark 34.10.*

**Proof.** Omitted. $\square$

**Remark 34.12.** Let $f : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Let $Z \subset X$ be a closed subset and $Z' = f^{-1}(Z)$. Denote $f|_{Z'} : (Z', \mathcal{O}_{X'}|_{Z'}) \to (Z, \mathcal{O}_X|Z)$ be the induced morphism of ringed spaces. For any $K$ in $D(\mathcal{O}_X)$ there is a canonical map

$$
L(f|_{Z'})^* R\mathcal{H}_Z(K) \longrightarrow R\mathcal{H}_{Z'}(Lf^*K)
$$

in $D(\mathcal{O}_{X'}|_{Z'})$. Denote $i : Z \to X$ and $i' : Z' \to X'$ the inclusion maps. By Lemma 34.2 part (2) applied to $i'$ it is the same thing to give a map

$$
i'_* L(f|_{Z'})^* R\mathcal{H}_Z(K) \longrightarrow i'_* R\mathcal{H}_{Z'}(Lf^*K)
$$

in $D_{Z'}(\mathcal{O}_{X'})$. The map of functors $Lf^* \circ i_* \to i'_* \circ L(f|_{Z'})^*$ of Remark 28.3 is an isomorphism in this case (follows by checking what happens on stalks using that $i_*$ and $i'_*$ are exact and that $\mathcal{O}_{Z,z} = \mathcal{O}_{X,z}$ and similarly for $Z'$). Hence it suffices to construct a the top horizonal arrow in the following diagram

$$
\begin{array}{ccc}
Lf^* i_* R\mathcal{H}_Z(K) & \longrightarrow & i'_* R\mathcal{H}_{Z'}(Lf^*K) \\
& \searrow \qquad \swarrow & \\
& Lf^*K &
\end{array}
$$

The complex $Lf^* i_* R\mathcal{H}_Z(K)$ is supported on $Z'$. The south-east arrow comes from the adjunction mapping $i_* R\mathcal{H}_Z(K) \to K$ (Lemma 34.1). Since the adjunction mapping $i'_* R\mathcal{H}_{Z'}(Lf^*K) \to Lf^*K$ is universal by Lemma 34.2 part (3), we find that the south-east arrow factors uniquely over the south-west arrow and we obtain the desired arrow.

**Lemma 34.13.** *With notation and assumptions as in Remark 34.12 the diagram*

$$
\begin{array}{ccc}
H_Z^p(X, K) & \longrightarrow & H_{Z'}^p(X, Lf^*K) \\
\downarrow & & \downarrow \\
H^p(X, K) & \longrightarrow & H^p(X', Lf^*K)
\end{array}
$$

*commutes. Here the top horizontal arrow comes from the identifications $H_Z^p(X, K) = H^p(Z, R\mathcal{H}_Z(K))$ and $H_{Z'}^p(X', Lf^*K) = H^p(Z', R\mathcal{H}_{Z'}(K'))$, the pullback map $H^p(Z, R\mathcal{H}_Z(K)) \to H^p(Z', L(f|_{Z'})^* R\mathcal{H}_Z(K))$, and the map constructed in Remark 34.12.*

**Proof.** Omitted. Hints: Using that $H^p(Z, R\mathcal{H}_Z(K)) = H^p(X, i_* R\mathcal{H}_Z(K))$ and similarly for $R\mathcal{H}_{Z'}(Lf^*K)$ this follows from the functoriality of the pullback maps and the commutative diagram used to define the map of Remark 34.12. $\qquad\square$

## 35. Inverse systems and cohomology, I

Let $A$ be a ring and let $I \subset A$ be an ideal. We prove some results on inverse systems of sheaves of $A/I^n$-modules.

**Lemma 35.1.** *Let $I$ be an ideal of a ring $A$. Let $X$ be a topological space. Let*

$$
\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1
$$

*be an inverse system of sheaves of $A$-modules on $X$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume*

$$
\bigoplus_{n \geq 0} H^{p+1}(X, I^n\mathcal{F}_{n+1})
$$

*satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-module. Then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[4].*

**Proof.** Set $N_n = H^{p+1}(X, I^n\mathcal{F}_{n+1})$ and let $\delta_n : M_n \to N_n$ be the boundary map on cohomology coming from the short exact sequence $0 \to I^n\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$. Then $\bigoplus \text{Im}(\delta_n) \subset \bigoplus N_n$ is a graded submodule. Namely, if $s \in M_n$ and $f \in I^m$, then we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I^n\mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & I^{n+m}\mathcal{F}_{n+m+1} & \longrightarrow & \mathcal{F}_{n+m+1} & \longrightarrow & \mathcal{F}_{n+m} & \longrightarrow & 0
\end{array}
$$

The middle vertical map is given by lifting a local section of $\mathcal{F}_{n+1}$ to a section of $\mathcal{F}_{n+m+1}$ and then multiplying by $f$; similarly for the other vertical arrows. We conclude that $\delta_{n+m}(fs) = f\delta_n(s)$. By assumption we can find $s_j \in M_{n_j}$, $j = 1, \ldots, N$ such that $\delta_{n_j}(s_j)$ generate $\bigoplus \text{Im}(\delta_n)$ as a graded module. Let $n > c = \max(n_j)$. Let $s \in M_n$. Then we can find $f_j \in I^{n-n_j}$ such that $\delta_n(s) = \sum f_j\delta_{n_j}(s_j)$. We conclude that $\delta(s - \sum f_j s_j) = 0$, i.e., we can find $s' \in M_{n+1}$ mapping to $s - \sum f_j s_j$ in $M_n$. It follows that

$$
\text{Im}(M_{n+1} \to M_{n-c}) = \text{Im}(M_n \to M_{n-c})
$$

Namely, the elements $f_j s_j$ map to zero in $M_{n-c}$. This proves the lemma. $\qquad\square$

---

[4]In fact, there exists a $c \geq 0$ such that $\text{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.

**Lemma 35.2.** *Let $I$ be an ideal of a ring $A$. Let $X$ be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

*be an inverse system of $A$-modules on $X$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Given $n$ define*

$$N_n = \bigcap\nolimits_{m \geq n} \mathrm{Im}\left(H^{p+1}(X, I^n\mathcal{F}_{m+1}) \to H^{p+1}(X, I^n\mathcal{F}_{n+1})\right)$$

*If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-module, then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[5].*

**Proof.** The proof is exactly the same as the proof of Lemma 35.1. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(X, I^n\mathcal{F}_{n+1})$ and that the boundary maps $\delta_n : M_n \to H^{p+1}(X, I^n\mathcal{F}_{n+1})$ have image contained in $N_n$.

Suppose that $\xi \in N_n$ and $f \in I^k$. Choose $m \gg n+k$. Choose $\xi' \in H^{p+1}(X, I^n\mathcal{F}_{m+1})$ lifting $\xi$. We consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I^n\mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0 \\
& & \downarrow f & & \downarrow f & & \downarrow f & & \\
0 & \longrightarrow & I^{n+k}\mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{n+k} & \longrightarrow & 0
\end{array}
$$

constructed as in the proof of Lemma 35.1. We get an induced map on cohomology and we see that $f\xi' \in H^{p+1}(X, I^{n+k}\mathcal{F}_{m+1})$ maps to $f\xi$. Since this is true for all $m \gg n+k$ we see that $f\xi$ is in $N_{n+k}$ as desired.

To see the boundary maps $\delta_n$ have image contained in $N_n$ we consider the diagrams

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I^n\mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_{m+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^n\mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0
\end{array}
$$

for $m \geq n$. Looking at the induced maps on cohomology we conclude. $\square$

**Lemma 35.3.** *Let $I$ be an ideal of a ring $A$. Let $X$ be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

*be an inverse system of sheaves of $A$-modules on $X$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume*

$$\bigoplus\nolimits_{n \geq 0} H^p(X, I^n\mathcal{F}_{n+1})$$

*satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-module. Then the limit topology on $M = \lim H^p(X, \mathcal{F}_n)$ is the $I$-adic topology.*

**Proof.** Set $F^n = \mathrm{Ker}(M \to H^p(X, \mathcal{F}_n))$ for $n \geq 1$ and $F^0 = M$. Observe that $IF^n \subset F^{n+1}$. In particular $I^n M \subset F^n$. Hence the $I$-adic topology is finer than the limit topology. For the converse, we will show that given $n$ there exists an $m \geq n$ such that $F^m \subset I^n M$[6]. We have injective maps

$$F^n/F^{n+1} \longrightarrow H^p(X, \mathcal{F}_{n+1})$$

[5]In fact, there exists a $c \geq 0$ such that $\mathrm{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.

[6]In fact, there exist a $c \geq 0$ such that $F^{n+c} \subset I^n M$ for all $n$.

whose image is contained in the image of $H^p(X, I^n \mathcal{F}_{n+1}) \to H^p(X, \mathcal{F}_{n+1})$. Denote

$$E_n \subset H^p(X, I^n \mathcal{F}_{n+1})$$

the inverse image of $F^n/F^{n+1}$. Then $\bigoplus E_n$ is a graded $\bigoplus I^n/I^{n+1}$-submodule of $\bigoplus H^p(X, I^n \mathcal{F}_{n+1})$ and $\bigoplus E_n \to \bigoplus F^n/F^{n+1}$ is a homomorphism of graded modules; details omitted. By assumption $\bigoplus E_n$ is generated by finitely many homogeneous elements over $\bigoplus I^n/I^{n+1}$. Since $E_n \to F^n/F^{n+1}$ is surjective, we see that the same thing is true of $\bigoplus F^n/F^{n+1}$. Hence we can find $r$ and $c_1, \ldots, c_r \geq 0$ and $a_i \in F^{c_i}$ whose images in $\bigoplus F^n/F^{n+1}$ generate. Set $c = \max(c_i)$.

For $n \geq c$ we claim that $IF^n = F^{n+1}$. The claim shows that $F^{n+c} = I^n F^c \subset I^n M$ as desired. To prove the claim suppose $a \in F^{n+1}$. The image of $a$ in $F^{n+1}/F^{n+2}$ is a linear combination of our $a_i$. Therefore $a - \sum f_i a_i \in F^{n+2}$ for some $f_i \in I^{n+1-c_i}$. Since $I^{n+1-c_i} = I \cdot I^{n-c_i}$ as $n \geq c_i$ we can write $f_i = \sum g_{i,j} h_{i,j}$ with $g_{i,j} \in I$ and $h_{i,j} a_i \in F^n$. Thus we see that $F^{n+1} = F^{n+2} + IF^n$. A simple induction argument gives $F^{n+1} = F^{n+e} + IF^n$ for all $e > 0$. It follows that $IF^n$ is dense in $F^{n+1}$. Choose generators $k_1, \ldots, k_r$ of $I$ and consider the continuous map

$$u : (F^n)^{\oplus r} \longrightarrow F^{n+1}, \quad (x_1, \ldots, x_r) \mapsto \sum k_i x_i$$

(in the limit topology). By the above the image of $(F^m)^{\oplus r}$ under $u$ is dense in $F^{m+1}$ for all $m \geq n$. By the open mapping lemma (More on Algebra, Lemma 36.5) we find that $u$ is open. Hence $u$ is surjective. Hence $IF^n = F^{n+1}$ for $n \geq c$. This concludes the proof. $\qquad \square$

**Lemma 35.4.** *Let $I$ be an ideal of a ring $A$. Let $X$ be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

*be an inverse system of sheaves of $A$-modules on $X$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n \mathcal{F}_{n+1}$. Let $p \geq 0$. Given $n$ define*

$$N_n = \bigcap\nolimits_{m \geq n} \mathrm{Im}\left( H^p(X, I^n \mathcal{F}_{m+1}) \to H^p(X, I^n \mathcal{F}_{n+1}) \right)$$

*If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-module, then the limit topology on $M = \lim H^p(X, \mathcal{F}_n)$ is the $I$-adic topology.*

**Proof.** The proof is exactly the same as the proof of Lemma 35.3. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(X, I^n \mathcal{F}_{n+1})$ and that $F^n/F^{n+1} \subset H^p(X, \mathcal{F}_{n+1})$ is contained in the image of $N_n \to H^p(X, \mathcal{F}_{n+1})$. In the proof of Lemma 35.2 we have seen the statement on the module structure.

Let $t \in F^n$. Choose an element $s \in H^p(X, I^n \mathcal{F}_{n+1})$ which maps to the image of $t$ in $H^p(X, \mathcal{F}_{n+1})$. We have to show that $s$ is in $N_n$. Now $F^n$ is the kernel of the map from $M \to H^p(X, \mathcal{F}_n)$ hence for all $m \geq n$ we can map $t$ to an element $t_m \in H^p(X, \mathcal{F}_{m+1})$ which maps to zero in $H^p(X, \mathcal{F}_n)$. Consider the cohomology sequence

$$H^{p-1}(X, \mathcal{F}_n) \to H^p(X, I^n \mathcal{F}_{m+1}) \to H^p(X, \mathcal{F}_{m+1}) \to H^p(X, \mathcal{F}_n)$$

coming from the short exact sequence $0 \to I^n \mathcal{F}_{m+1} \to \mathcal{F}_{m+1} \to \mathcal{F}_n \to 0$. We can choose $s_m \in H^p(X, I^n \mathcal{F}_{m+1})$ mapping to $t_m$. Comparing the sequence above with the one for $m = n$ we see that $s_m$ maps to $s$ up to an element in the image

of $H^{p-1}(X, \mathcal{F}_n) \to H^p(X, I^n \mathcal{F}_{n+1})$. However, this map factors through the map $H^p(X, I^n \mathcal{F}_{m+1}) \to H^p(X, I^n \mathcal{F}_{n+1})$ and we see that $s$ is in the image as desired. $\square$

## 36. Inverse systems and cohomology, II

This section continues the discussion in Section 35 in the setting where the ideal is principal.

**Lemma 36.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

*be inverse system of $\mathcal{O}_X$-modules. Consider the conditions*
  (1) *for all $n \geq 1$ the map $f : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_n$ to give a short exact sequence $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$,*
  (2) *for all $n \geq 1$ the map $f^n : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_1$ to give a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$*
  (3) *there exists an $\mathcal{O}_X$-module $\mathcal{G}$ which is $f$-divisible such that $\mathcal{F}_n = \mathcal{G}[f^n]$, and*
  (4) *there exists an $\mathcal{O}_X$-module $\mathcal{F}$ which is $f$-torsion free such that $\mathcal{F}_n = \mathcal{F}/f^n \mathcal{F}$.*
*Then (4) $\Rightarrow$ (3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1).*

**Proof.** We omit the proof of the equivalence of (1) and (2). We omit the proof that (3) implies (1). Given $\mathcal{F}_n$ as in (1) to prove (3) we set $\mathcal{G} = \operatorname{colim} \mathcal{F}_n$ where the maps $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \ldots$ are as in (1). The map $f : \mathcal{G} \to \mathcal{G}$ is surjective as the image of $\mathcal{F}_{n+1} \subset \mathcal{G}$ is $\mathcal{F}_n \subset \mathcal{G}$ by the short exact sequence (1). Thus $\mathcal{G}$ is an $f$-divisible $\mathcal{O}_X$-module with $\mathcal{F}_n = \mathcal{G}[f^n]$.

Assume given $\mathcal{F}$ as in (4). The map $\mathcal{F}/f^{n+1} \mathcal{F} \to \mathcal{F}/f^n \mathcal{F}$ is always surjective with kernel the image of the map $\mathcal{F}/f\mathcal{F} \to \mathcal{F}/f^{n+1}\mathcal{F}$ induced by multiplication with $f^n$. To verify (2) it suffices to see that the kernel of $f^n : \mathcal{F} \to \mathcal{F}/f^{n+1}\mathcal{F}$ is $f\mathcal{F}$. To see this it suffices to show that given sections $s, t$ of $\mathcal{F}$ over an open $U \subset X$ with $f^n s = f^{n+1} t$ we have $s = ft$. This is clear because $f : \mathcal{F} \to \mathcal{F}$ is injective as $\mathcal{F}$ is $f$-torsion free. $\square$

**Lemma 36.2.** *Suppose $X$, $f$, $(\mathcal{F}_n)$ satisfy condition (1) of Lemma 36.1. Let $p \geq 0$ and set $H^p = \lim H^p(X, \mathcal{F}_n)$. Then $f^c H^p$ is the kernel of $H^p \to H^p(X, \mathcal{F}_c)$ for all $c \geq 1$. Thus the limit topology on $H^p$ is the $f$-adic topology.*

**Proof.** Let $c \geq 1$. It is clear that $f^c H^p$ maps to zero in $H^p(X, \mathcal{F}_c)$. If $\xi = (\xi_n) \in H^p$ is small in the limit topology, then $\xi_c = 0$, and hence $\xi_n$ maps to zero in $H^p(X, \mathcal{F}_c)$ for $n \geq c$. Consider the inverse system of short exact sequences

$$0 \to \mathcal{F}_{n-c} \xrightarrow{f^c} \mathcal{F}_n \to \mathcal{F}_c \to 0$$

and the corresponding inverse system of long exact cohomology sequences

$$H^{p-1}(X, \mathcal{F}_c) \to H^p(X, \mathcal{F}_{n-c}) \to H^p(X, \mathcal{F}_n) \to H^p(X, \mathcal{F}_c)$$

Since the term $H^{p-1}(X, \mathcal{F}_c)$ is independent of $n$ we can choose a compatible sequence of elements $\xi'_n \in H^p(X, \mathcal{F}_{n-c})$ lifting $\xi_n$. Setting $\xi' = (\xi'_n)$ we see that $\xi = f^c \xi'$ as desired. $\square$

**Lemma 36.3.** *Let $A$ be a Noetherian ring complete with respect to a principal ideal $(f)$. Let $X$ be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

*be an inverse system of sheaves of A-modules. Assume*

(1) $\Gamma(X, \mathcal{F}_1)$ *is a finite A-module,*
(2) $X$, $f$, $(\mathcal{F}_n)$ *satisfy condition (1) of Lemma 36.1.*

*Then*

$$M = \lim \Gamma(X, \mathcal{F}_n)$$

*is a finite A-module, $f$ is a nonzerodivisor on $M$, and $M/fM$ is the image of $M$ in $\Gamma(X, \mathcal{F}_1)$.*

**Proof.** By Lemma 36.2 we have $M/fM \subset H^0(X, \mathcal{F}_1)$. From (1) and the Noetherian property of $A$ we get that $M/fM$ is a finite $A$-module. Observe that $\bigcap f^n M = 0$ as $f^n M$ maps to zero in $H^0(X, \mathcal{F}_n)$. By Algebra, Lemma 96.12 we conclude that $M$ is finite over $A$. Finally, suppose $s = (s_n) \in M = \lim H^0(X, \mathcal{F}_n)$ satisfies $fs = 0$. Then $s_{n+1}$ is in the kernel of $\mathcal{F}_{n+1} \to \mathcal{F}_n$ by condition (1) of Lemma 36.1. Hence $s_n = 0$. Since $n$ was arbitrary, we see $s = 0$. Thus $f$ is a nonzerodivisor on $M$.  $\square$

**Lemma 36.4.** *Let $A$ be a ring. Let $f \in A$. Let $X$ be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

*be an inverse system of sheaves of A-modules. Let $p \geq 0$. Assume*

(1) *either $H^{p+1}(X, \mathcal{F}_1)$ is an A-module of finite length or A is Noetherian and $H^{p+1}(X, \mathcal{F}_1)$ is a finite A-module,*
(2) *$X$, $f$, $(\mathcal{F}_n)$ satisfy condition (1) of Lemma 36.1.*

*Then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.*

**Proof.** Set $I = (f)$. We will use the criterion of Lemma 35.1. Observe that $f^n : \mathcal{F}_1 \to I^n \mathcal{F}_{n+1}$ is an isomorphism for all $n \geq 0$. Thus it suffices to show that

$$\bigoplus\nolimits_{n \geq 1} H^{p+1}(X, \mathcal{F}_1) \cdot f^{n+1}$$

is a graded $S = \bigoplus_{n \geq 0} A/(f) \cdot f^n$-module satisfying the ascending chain condition. If $A$ is not Noetherian, then $H^{p+1}(X, \mathcal{F}_1)$ has finite length and the result holds. If $A$ is Noetherian, then $S$ is a Noetherian ring and the result holds as the module is finite over $S$ by the assumed finiteness of $H^{p+1}(X, \mathcal{F}_1)$. Some details omitted.  $\square$

**Lemma 36.5.** *Let $A$ be a ring. Let $f \in A$. Let $X$ be a topological space. Let*

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

*be an inverse system of sheaves of A-modules. Let $p \geq 0$. Assume*

(1) *either there is an $m \geq 1$ such that the image of $H^{p+1}(X, \mathcal{F}_m) \to H^{p+1}(X, \mathcal{F}_1)$ is an A-module of finite length or A is Noetherian and the intersection of the images of $H^{p+1}(X, \mathcal{F}_m) \to H^{p+1}(X, \mathcal{F}_1)$ is a finite A-module,*
(2) *$X$, $f$, $(\mathcal{F}_n)$ satisfy condition (1) of Lemma 36.1.*

*Then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.*

**Proof.** Set $I = (f)$. We will use the criterion of Lemma 35.2 involving the modules $N_n$. For $m \geq n$ we have $I^n \mathcal{F}_{m+1} = \mathcal{F}_{m+1-n}$. Thus we see that

$$N_n = \bigcap\nolimits_{m \geq 1} \mathrm{Im}\left(H^{p+1}(X, \mathcal{F}_m) \to H^{p+1}(X, \mathcal{F}_1)\right)$$

is independent of $n$ and $\bigoplus N_n = \bigoplus N_1 \cdot f^{n+1}$. Thus we conclude exactly as in the proof of Lemma 36.4.  $\square$

**Remark 36.6.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. Let $\mathcal{F}$ be $\mathcal{O}_X$-module. If $\mathcal{F}$ is $f$-torsion free, then for every $p \geq 0$ we have a short exact sequence of inverse systems

$$0 \to \{H^p(X, \mathcal{F})/f^n H^p(X, \mathcal{F})\} \to \{H^p(X, \mathcal{F}/f^n \mathcal{F})\} \to \{H^{p+1}(X, \mathcal{F})[f^n]\} \to 0$$

Since the first inverse system has the Mittag-Leffler condition (ML) we learn three things from this:

(1) There is a short exact sequence

$$0 \to \widehat{H^p(X, \mathcal{F})} \to \lim H^p(X, \mathcal{F}/f^n \mathcal{F}) \to T_f(H^{p+1}(X, \mathcal{F})) \to 0$$

where $\widehat{\phantom{x}}$ denotes the usual $f$-adic completion and $T_f(-)$ denotes the $f$-adic Tate module from More on Algebra, Example 93.5.
(2) We have $R^1 \lim H^p(X, \mathcal{F}/f^n \mathcal{F}) = R^1 \lim H^{p+1}(X, \mathcal{F})[f^n]$.
(3) The system $\{H^{p+1}(X, \mathcal{F})[f^n]\}$ is ML if and only if $\{H^p(X, \mathcal{F}/f^n \mathcal{F})\}$ is ML.

See Homology, Lemma 31.3 and More on Algebra, Lemmas 86.2 and 86.13.

## 37. Derived limits

Let $(X, \mathcal{O}_X)$ be a ringed space. Since the triangulated category $D(\mathcal{O}_X)$ has products (Injectives, Lemma 13.4) it follows that $D(\mathcal{O}_X)$ has derived limits, see Derived Categories, Definition 34.1. If $(K_n)$ is an inverse system in $D(\mathcal{O}_X)$ then we denote $R \lim K_n$ the derived limit.

**Lemma 37.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. For $U \subset X$ open the functor $R\Gamma(U, -)$ commutes with $R \lim$. Moreover, there are short exact sequences*

$$0 \to R^1 \lim H^{m-1}(U, K_n) \to H^m(U, R \lim K_n) \to \lim H^m(U, K_n) \to 0$$

*for any inverse system $(K_n)$ in $D(\mathcal{O}_X)$ and any $m \in \mathbf{Z}$.*

**Proof.** The first statement follows from Injectives, Lemma 13.6. Then we may apply More on Algebra, Remark 86.10 to $R \lim R\Gamma(U, K_n) = R\Gamma(U, R \lim K_n)$ to get the short exact sequences. $\square$

**Lemma 37.2.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then $Rf_*$ commutes with $R \lim$, i.e., $Rf_*$ commutes with derived limits.*

**Proof.** Let $(K_n)$ be an inverse system in $D(\mathcal{O}_X)$. Consider the defining distinguished triangle

$$R \lim K_n \to \prod K_n \to \prod K_n$$

in $D(\mathcal{O}_X)$. Applying the exact functor $Rf_*$ we obtain the distinguished triangle

$$Rf_*(R \lim K_n) \to Rf_* \left( \prod K_n \right) \to Rf_* \left( \prod K_n \right)$$

in $D(\mathcal{O}_Y)$. Thus we see that it suffices to prove that $Rf_*$ commutes with products in the derived category (which are not just given by products of complexes, see Injectives, Lemma 13.4). However, since $Rf_*$ is a right adjoint by Lemma 28.1 this follows formally (see Categories, Lemma 24.5). Caution: Note that we cannot apply Categories, Lemma 24.5 directly as $R \lim K_n$ is not a limit in $D(\mathcal{O}_X)$. $\square$

**Remark 37.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(K_n)$ be an inverse system in $D(\mathcal{O}_X)$. Set $K = R \lim K_n$. For each $n$ and $m$ let $\mathcal{H}_n^m = H^m(K_n)$ be the $m$th cohomology sheaf of $K_n$ and similarly set $\mathcal{H}^m = H^m(K)$. Let us denote $\underline{\mathcal{H}}_n^m$ the presheaf

$$U \longmapsto \underline{\mathcal{H}}_n^m(U) = H^m(U, K_n)$$

Similarly we set $\underline{\mathcal{H}}^m(U) = H^m(U, K)$. By Lemma 32.3 we see that $\mathcal{H}_n^m$ is the sheafification of $\underline{\mathcal{H}}_n^m$ and $\mathcal{H}^m$ is the sheafification of $\underline{\mathcal{H}}^m$. Here is a diagram

$$
\begin{array}{ccc}
K & \underline{\mathcal{H}}^m \longrightarrow \mathcal{H}^m \\
\| & \downarrow \qquad\qquad \downarrow \\
R \lim K_n & \lim \underline{\mathcal{H}}_n^m \longrightarrow \lim \mathcal{H}_n^m
\end{array}
$$

In general it may not be the case that $\lim \mathcal{H}_n^m$ is the sheafification of $\lim \underline{\mathcal{H}}_n^m$. If $U \subset X$ is an open, then we have short exact sequences

(37.3.1) $\qquad 0 \to R^1 \lim \underline{\mathcal{H}}_n^{m-1}(U) \to \underline{\mathcal{H}}^m(U) \to \lim \underline{\mathcal{H}}_n^m(U) \to 0$

by Lemma 37.1.

The following lemma applies to an inverse system of quasi-coherent modules with surjective transition maps on a scheme.

**Lemma 37.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{F}_n)$ be an inverse system of $\mathcal{O}_X$-modules. Let $\mathcal{B}$ be a set of opens of $X$. Assume*

(1) *every open of $X$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *$H^p(U, \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$,*
(3) *the inverse system $\mathcal{F}_n(U)$ has vanishing $R^1 \lim$ for $U \in \mathcal{B}$.*

*Then $R \lim \mathcal{F}_n = \lim \mathcal{F}_n$ and we have $H^p(U, \lim \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$.*

**Proof.** Set $K_n = \mathcal{F}_n$ and $K = R \lim \mathcal{F}_n$. Using the notation of Remark 37.3 and assumption (2) we see that for $U \in \mathcal{B}$ we have $\underline{\mathcal{H}}_n^m(U) = 0$ when $m \neq 0$ and $\underline{\mathcal{H}}_n^0(U) = \mathcal{F}_n(U)$. From Equation (37.3.1) and assumption (3) we see that $\underline{\mathcal{H}}^m(U) = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n(U)$ when $m = 0$. Sheafifying using (1) we find that $\mathcal{H}^m = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n$ when $m = 0$. Hence $K = \lim \mathcal{F}_n$. Since $H^m(U, K) = \underline{\mathcal{H}}^m(U) = 0$ for $m > 0$ (see above) we see that the second assertion holds. $\qquad \square$

**Lemma 37.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(K_n)$ be an inverse system in $D(\mathcal{O}_X)$. Let $x \in X$ and $m \in \mathbf{Z}$. Assume there exist an integer $n(x)$ and a fundamental system $\mathfrak{U}_x$ of open neighbourhoods of $x$ such that for $U \in \mathfrak{U}_x$*

(1) *$R^1 \lim H^{m-1}(U, K_n) = 0$, and*
(2) *$H^m(U, K_n) \to H^m(U, K_{n(x)})$ is injective for $n \geq n(x)$.*

*Then the map on stalks $H^m(R \lim K_n)_x \to H^m(K_{n(x)})_x$ is injective.*

**Proof.** Let $\gamma$ be an element of $H^m(R \lim K_n)_x$ which maps to zero in $H^m(K_{n(x)})_x$. Since $H^m(R \lim K_n)$ is the sheafification of $U \mapsto H^m(U, R \lim K_n)$ (by Lemma 32.3) we can choose $U \in \mathfrak{U}_x$ and an element $\tilde{\gamma} \in H^m(U, R \lim K_n)$ mapping to $\gamma$. Then $\tilde{\gamma}$ maps to $\tilde{\gamma}_{n(x)} \in H^m(U, K_{n(x)})$. Using that $H^m(K_{n(x)})$ is the sheafification of $U \mapsto H^m(U, K_{n(x)})$ (by Lemma 32.3 again) we see that after shrinking $U$ we may assume that $\tilde{\gamma}_{n(x)} = 0$. For this $U$ we consider the short exact sequence

$$0 \to R^1 \lim H^{m-1}(U, K_n) \to H^m(U, R \lim K_n) \to \lim H^m(U, K_n) \to 0$$

of Lemma 37.1. By assumption (1) the group on the left is zero and by assumption (2) the group on the right maps injectively into $H^m(U, K_{n(x)})$. We conclude $\tilde{\gamma} = 0$ and hence $\gamma = 0$ as desired. $\qquad\square$

**Lemma 37.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume that for every $x \in X$ there exist a function $p(x, -) : \mathbf{Z} \to \mathbf{Z}$ and a fundamental system $\mathfrak{U}_x$ of open neighbourhoods of $x$ such that*

$$H^p(U, H^{m-p}(E)) = 0 \text{ for } U \in \mathfrak{U}_x \text{ and } p > p(x, m)$$

*Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O}_X)$.*

**Proof.** Set $K_n = \tau_{\geq -n} E$ and $K = R\lim K_n$. The canonical map $E \to K$ comes from the canonical maps $E \to K_n = \tau_{\geq -n} E$. We have to show that $E \to K$ induces an isomorphism $H^m(E) \to H^m(K)$ of cohomology sheaves. In the rest of the proof we fix $m$. If $n \geq -m$, then the map $E \to \tau_{\geq -n} E = K_n$ induces an isomorphism $H^m(E) \to H^m(K_n)$. To finish the proof it suffices to show that for every $x \in X$ there exists an integer $n(x) \geq -m$ such that the map $H^m(K)_x \to H^m(K_{n(x)})_x$ is injective. Namely, then the composition

$$H^m(E)_x \to H^m(K)_x \to H^m(K_{n(x)})_x$$

is a bijection and the second arrow is injective, hence the first arrow is bijective. Set

$$n(x) = 1 + \max\{-m, p(x, m - 1) - m, -1 + p(x, m) - m, -2 + p(x, m + 1) - m\}.$$

so that in any case $n(x) \geq -m$. Claim: the maps

$$H^{m-1}(U, K_{n+1}) \to H^{m-1}(U, K_n) \quad \text{and} \quad H^m(U, K_{n+1}) \to H^m(U, K_n)$$

are isomorphisms for $n \geq n(x)$ and $U \in \mathfrak{U}_x$. The claim implies conditions (1) and (2) of Lemma 37.5 are satisfied and hence implies the desired injectivity. Recall (Derived Categories, Remark 12.4) that we have distinguished triangles

$$H^{-n-1}(E)[n + 1] \to K_{n+1} \to K_n \to H^{-n-1}(E)[n + 2]$$

Looking at the asssociated long exact cohomology sequence the claim follows if

$$H^{m+n}(U, H^{-n-1}(E)), \quad H^{m+n+1}(U, H^{-n-1}(E)), \quad H^{m+n+2}(U, H^{-n-1}(E))$$

are zero for $n \geq n(x)$ and $U \in \mathfrak{U}_x$. This follows from our choice of $n(x)$ and the assumption in the lemma. $\qquad\square$

**Lemma 37.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume that for every $x \in X$ there exist an integer $d_x \geq 0$ and a fundamental system $\mathfrak{U}_x$ of open neighbourhoods of $x$ such that*

$$H^p(U, H^q(E)) = 0 \text{ for } U \in \mathfrak{U}_x, \ p > d_x, \ and \ q < 0$$

*Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O}_X)$.*

**Proof.** This follows from Lemma 37.6 with $p(x, m) = d_x + \max(0, m)$. $\qquad\square$

**Lemma 37.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume there exist a function $p(-) : \mathbf{Z} \to \mathbf{Z}$ and a set $\mathcal{B}$ of opens of $X$ such that*

    (1) *every open in $X$ has a covering whose members are elements of $\mathcal{B}$, and*
    (2) *$H^p(U, H^{m-p}(E)) = 0$ for $p > p(m)$ and $U \in \mathcal{B}$.*

*Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O}_X)$.*

**Proof.** Apply Lemma 37.6 with $p(x, m) = p(m)$ and $\mathfrak{U}_x = \{U \in \mathcal{B} \mid x \in U\}$.  $\square$

**Lemma 37.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume there exist an integer $d \geq 0$ and a basis $\mathcal{B}$ for the topology of $X$ such that*

$$H^p(U, H^q(E)) = 0 \text{ for } U \in \mathcal{B}, \ p > d, \text{ and } q < 0$$

*Then the canonical map $E \to R\lim \tau_{\geq -n}E$ is an isomorphism in $D(\mathcal{O}_X)$.*

**Proof.** Apply Lemma 37.7 with $d_x = d$ and $\mathfrak{U}_x = \{U \in \mathcal{B} \mid x \in U\}$.  $\square$

The lemmas above can be used to compute cohomology in certain situations.

**Lemma 37.10.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K$ be an object of $D(\mathcal{O}_X)$. Let $\mathcal{B}$ be a set of opens of $X$. Assume*

    (1) *every open of $X$ has a covering whose members are elements of $\mathcal{B}$,*
    (2) *$H^p(U, H^q(K)) = 0$ for all $p > 0$, $q \in \mathbf{Z}$, and $U \in \mathcal{B}$.*

*Then $H^q(U, K) = H^0(U, H^q(K))$ for $q \in \mathbf{Z}$ and $U \in \mathcal{B}$.*

**Proof.** Observe that $K = R\lim \tau_{\geq -n}K$ by Lemma 37.9 with $d = 0$. Let $U \in \mathcal{B}$. By Equation (37.3.1) we get a short exact sequence

$$0 \to R^1\lim H^{q-1}(U, \tau_{\geq -n}K) \to H^q(U, K) \to \lim H^q(U, \tau_{\geq -n}K) \to 0$$

Condition (2) implies $H^q(U, \tau_{\geq -n}K) = H^0(U, H^q(\tau_{\geq -n}K))$ for all $q$ by using the spectral sequence of Example 29.3. The spectral sequence converges because $\tau_{\geq -n}K$ is bounded below. If $n > -q$ then we have $H^q(\tau_{\geq -n}K) = H^q(K)$. Thus the systems on the left and the right of the displayed short exact sequence are eventually constant with values $H^0(U, H^{q-1}(K))$ and $H^0(U, H^q(K))$. The lemma follows.  $\square$

Here is another case where we can describe the derived limit.

**Lemma 37.11.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(K_n)$ be an inverse system of objects of $D(\mathcal{O}_X)$. Let $\mathcal{B}$ be a set of opens of $X$. Assume*

    (1) *every open of $X$ has a covering whose members are elements of $\mathcal{B}$,*
    (2) *for all $U \in \mathcal{B}$ and all $q \in \mathbf{Z}$ we have*
        (a) *$H^p(U, H^q(K_n)) = 0$ for $p > 0$,*
        (b) *the inverse system $H^0(U, H^q(K_n))$ has vanishing $R^1\lim$.*

*Then $H^q(R\lim K_n) = \lim H^q(K_n)$ for $q \in \mathbf{Z}$.*

**Proof.** Set $K = R\lim K_n$. We will use notation as in Remark 37.3. Let $U \in \mathcal{B}$. By Lemma 37.10 and (2)(a) we have $H^q(U, K_n) = H^0(U, H^q(K_n))$. Using that the functor $R\Gamma(U, -)$ commutes with derived limits we have

$$H^q(U, K) = H^q(R\lim R\Gamma(U, K_n)) = \lim H^0(U, H^q(K_n))$$

where the final equality follows from More on Algebra, Remark 86.10 and assumption (2)(b). Thus $H^q(U, K)$ is the inverse limit the sections of the sheaves $H^q(K_n)$ over $U$. Since $\lim H^q(K_n)$ is a sheaf we find using assumption (1) that $H^q(K)$, which is the sheafification of the presheaf $U \mapsto H^q(U, K)$, is equal to $\lim H^q(K_n)$. This proves the lemma.  $\square$

## 38. Producing K-injective resolutions

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The category $Mod(\mathcal{O}_X)$ has enough injectives, hence we can use Derived Categories, Lemma 29.3 produce a diagram

$$
\begin{array}{ccc}
\cdots \longrightarrow \tau_{\geq -2}\mathcal{F}^\bullet & \longrightarrow & \tau_{\geq -1}\mathcal{F}^\bullet \\
\downarrow & & \downarrow \\
\cdots \longrightarrow \mathcal{I}_2^\bullet & \longrightarrow & \mathcal{I}_1^\bullet
\end{array}
$$

in the category of complexes of $\mathcal{O}_X$-modules such that

(1) the vertical arrows are quasi-isomorphisms,
(2) $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,
(3) the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}_X$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 31.4 and 31.8 this is a K-injective complex. In general the canonical map

(38.0.1)                              $$\mathcal{F}^\bullet \to \mathcal{I}^\bullet$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

**Lemma 38.1.** *In the situation described above. Denote $\mathcal{H}^m = H^m(\mathcal{F}^\bullet)$ the mth cohomology sheaf. Let $\mathcal{B}$ be a set of open subsets of $X$. Let $d \in \mathbf{N}$. Assume*

(1) *every open in $X$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$ we have $H^p(U, \mathcal{H}^q) = 0$ for $p > d$ and $q < 0$[7].*

*Then (38.0.1) is a quasi-isomorphism.*

**Proof.** By Derived Categories, Lemma 34.4 it suffices to show that the canonical map $\mathcal{F}^\bullet \to R \lim \tau_{\geq -n}\mathcal{F}^\bullet$ is an isomorphism. This is Lemma 37.9.                                        □

Here is a technical lemma about the cohomology sheaves of the inverse limit of a system of complexes of sheaves. In some sense this lemma is the wrong thing to try to prove as one should take derived limits and not actual inverse limits.

**Lemma 38.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{F}_n^\bullet)$ be an inverse system of complexes of $\mathcal{O}_X$-modules. Let $m \in \mathbf{Z}$. Assume there exist a set $\mathcal{B}$ of open subsets of $X$ and an integer $n_0$ such that*

(1) *every open in $X$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$*
  (a) *the systems of abelian groups $\mathcal{F}_n^{m-2}(U)$ and $\mathcal{F}_n^{m-1}(U)$ have vanishing $R^1 \lim$ (for example these have the Mittag-Leffler condition),*
  (b) *the system of abelian groups $H^{m-1}(\mathcal{F}_n^\bullet(U))$ has vanishing $R^1 \lim$ (for example it has the Mittag-Leffler condition), and*
  (c) *we have $H^m(\mathcal{F}_n^\bullet(U)) = H^m(\mathcal{F}_{n_0}^\bullet(U))$ for all $n \geq n_0$.*

*Then the maps $H^m(\mathcal{F}^\bullet) \to \lim H^m(\mathcal{F}_n^\bullet) \to H^m(\mathcal{F}_{n_0}^\bullet)$ are isomorphisms of sheaves where $\mathcal{F}^\bullet = \lim \mathcal{F}_n^\bullet$ is the termwise inverse limit.*

---

[7]It suffices if $\forall m$, $\exists p(m)$, $H^p(U.\mathcal{H}^{m-p}) = 0$ for $p > p(m)$, see Lemma 37.8.

**Proof.** Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{F}^\bullet(U))$ is the cohomology of

$$\lim_n \mathcal{F}_n^{m-2}(U) \to \lim_n \mathcal{F}_n^{m-1}(U) \to \lim_n \mathcal{F}_n^m(U) \to \lim_n \mathcal{F}_n^{m+1}(U)$$

in the third spot from the left. By assumptions (2)(a) and (2)(b) we may apply More on Algebra, Lemma 86.3 to conclude that

$$H^m(\mathcal{F}^\bullet(U)) = \lim H^m(\mathcal{F}_n^\bullet(U))$$

By assumption (2)(c) we conclude

$$H^m(\mathcal{F}^\bullet(U)) = H^m(\mathcal{F}_n^\bullet(U))$$

for all $n \geq n_0$. By assumption (1) we conclude that the sheafification of $U \mapsto H^m(\mathcal{F}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^m(\mathcal{F}_n^\bullet(U))$ for all $n \geq n_0$. Thus the inverse system of sheaves $H^m(\mathcal{F}_n^\bullet)$ is constant for $n \geq n_0$ with value $H^m(\mathcal{F}^\bullet)$ which proves the lemma. $\square$

## 39. Inverse systems and cohomology, III

This section continues the discussion in Section 36 using derived limits.

**Lemma 39.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $A \to \Gamma(X, \mathcal{O}_X)$ be a ring map and let $f \in A$. Let $E$ be an object of $D(\mathcal{O}_X)$. Denote*

$$E_n = E \otimes_{\mathcal{O}_X} (\mathcal{O}_X \xrightarrow{f^n} \mathcal{O}_X)$$

*and set $E^\wedge = R \lim E_n$. For $p \in \mathbf{Z}$ is a canonical commutative diagram*



*with exact rows and columns where $\widehat{H^p(X, E)} = \lim H^p(X, E)/f^n H^p(X, E)$ is the usual $f$-adic completion, $H^p(X, E)^\wedge$ is the derived $f$-adic completion, and $T_f(H^{p+1}(X, E))$ is the $f$-adic Tate module, see More on Algebra, Example 93.5. Finally, we have $H^p(X, E^\wedge) = H^p(R\Gamma(X, E)^\wedge)$.*

**Proof.** Observe that $R\Gamma(X, E^\wedge) = R \lim R\Gamma(X, E_n)$ by Lemma 37.2. On the other hand, we have

$$R\Gamma(X, E_n) = R\Gamma(X, E) \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A)$$

(details omitted). We find that $R\Gamma(X, E^\wedge)$ is the derived $f$-adic completion $R\Gamma(X, E)^\wedge$. Whence the diagram by More on Algebra, Lemma 93.6. $\square$

**Lemma 39.2.** *Let $\mathcal{A}$ be an abelian category. Let $f : M \to M$ be a morphism of $\mathcal{A}$. If $M[f^n] = \text{Ker}(f^n : M \to M)$ stabilizes, then the inverse systems*

$$(M \xrightarrow{f^n} M) \quad and \quad \text{Coker}(f^n : M \to M)$$

*are pro-isomorphic in $D(\mathcal{A})$.*

**Proof.** There is clearly a map from the first inverse system to the second. Suppose that $M[f^c] = M[f^{c+1}] = M[f^{c+2}] = \ldots$. Then we can define an arrow of inverse systems in $D(\mathcal{A})$ in the other direction by the diagrams

$$
\begin{array}{ccc}
M/M[f^c] & \xrightarrow{\;f^{n+c}\;} & M \\
{\scriptstyle f^c}\downarrow & & \downarrow{\scriptstyle 1} \\
M & \xrightarrow{\;f^n\;} & M
\end{array}
$$

Since the top horizontal arrow is injective the complex in the top row is quasi-isomorphic to $\text{Coker}(f^{n+c} : M \to M)$. Some details omitted. $\square$

**Example 39.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $A \to \Gamma(X, \mathcal{O}_X)$ be a ring map and let $f \in A$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there is a $c$ such that $\mathcal{F}[f^c] = \mathcal{F}[f^n]$ for all $n \geq c$. We are going to apply Lemma 39.1 with $E = \mathcal{F}$. By Lemma 39.2 we see that the inverse system $(E_n)$ is pro-isomorphic to the inverse system $(\mathcal{F}/f^n\mathcal{F})$. We conclude that for $p \in \mathbf{Z}$ we obtain a commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & & & 0 & & \\
& & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & \widehat{H^p(X,\mathcal{F})} & \longrightarrow & \lim H^p(X, \mathcal{F}/f^n\mathcal{F}) & \longrightarrow & T_f(H^{p+1}(X,\mathcal{F})) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & H^0(H^p(X,\mathcal{F})^{\wedge}) & \longrightarrow & H^p(R\Gamma(X,\mathcal{F})^{\wedge}) & \longrightarrow & T_f(H^{p+1}(X,\mathcal{F})) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & & & \\
& & R^1 \lim H^p(X,\mathcal{F})[f^n] & \xrightarrow{\;\cong\;} & R^1 \lim H^{p-1}(X, \mathcal{F}/f^n\mathcal{F}) & & & & \\
& & \uparrow & & \uparrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

with exact rows and columns where $\widehat{H^p(X,\mathcal{F})} = \lim H^p(X,\mathcal{F})/f^n H^p(X,\mathcal{F})$ is the usual $f$-adic completion and $M^{\wedge}$ denotes derived $f$-adic completion for $M$ in $D(A)$.

## 40. Čech cohomology of unbounded complexes

The construction of Section 25 isn't the "correct" one for unbounded complexes. The problem is that in the Stacks project we use direct sums in the totalization of a double complex and we would have to replace this by a product. Instead of doing so in this section we assume the covering is finite and we use the alternating Čech complex.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^{\bullet}$ be a complex of presheaves of $\mathcal{O}_X$-modules. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a **finite** open covering of $X$. Since the alternating Čech

complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ (Section 23) is functorial in the presheaf $\mathcal{F}$ we obtain a double complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)$. In this section we work with the associated total complex. The construction of $\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$ is functorial in $\mathcal{F}^\bullet$. As well there is a functorial transformation

$$(40.0.1) \qquad\qquad \Gamma(X, \mathcal{F}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0 \dots i_p}\}$ with $\alpha_{i_0} = s|_{U_{i_0}}$ and $\alpha_{i_0 \dots i_p} = 0$ for $p > 0$.

**Lemma 40.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a finite open covering. For a complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical map*

$$Tot(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

*functorial in $\mathcal{F}^\bullet$ and compatible with (40.0.1).*

**Proof.** Let $\mathcal{I}^\bullet$ be a K-injective complex whose terms are injective $\mathcal{O}_X$-modules. The map (40.0.1) for $\mathcal{I}^\bullet$ is a map $\Gamma(X, \mathcal{I}^\bullet) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{I}^\bullet))$. This is a quasi-isomorphism of complexes of abelian groups as follows from Homology, Lemma 25.4 applied to the double complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{I}^\bullet)$ using Lemmas 11.1 and 23.6. Suppose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of $\mathcal{F}^\bullet$ into a K-injective complex whose terms are injectives (Injectives, Theorem 12.6). Since $R\Gamma(X, \mathcal{F}^\bullet)$ is represented by the complex $\Gamma(X, \mathcal{I}^\bullet)$ we obtain the map of the lemma using

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{I}^\bullet)).$$

We omit the verification of functoriality and compatibilities. $\square$

**Lemma 40.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a finite open covering. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Let $\mathcal{B}$ be a set of open subsets of $X$. Assume*

(1) *every open in $X$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *we have $U_{i_0 \dots i_p} \in \mathcal{B}$ for all $i_0, \dots, i_p \in I$,*
(3) *for every $U \in \mathcal{B}$ and $p > 0$ we have*
   (a) *$H^p(U, \mathcal{F}^q) = 0$,*
   (b) *$H^p(U, \mathrm{Coker}(\mathcal{F}^{q-1} \to \mathcal{F}^q)) = 0$, and*
   (c) *$H^p(U, H^q(\mathcal{F})) = 0$.*

*Then the map*

$$Tot(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

*of Lemma 40.1 is an isomorphism in $D(Ab)$.*

**Proof.** First assume $\mathcal{F}^\bullet$ is bounded below. In this case the map

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

is a quasi-isomorphism by Lemma 23.6. Namely, the map of double complexes $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ induces an isomorphism between the first pages of the second spectral sequences associated to these complexes (by Homology, Lemma 25.1) and these spectral sequences converge (Homology, Lemma 25.3). Thus the conclusion in this case by Lemma 25.2 and assumption (3)(a).

In general, by assumption (3)(c) we may choose a resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet = \lim \mathcal{I}^\bullet_n$ as in Lemma 38.1. Then the map of the lemma becomes

$$\lim_n \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \tau_{\geq -n} \mathcal{F}^\bullet)) \longrightarrow \Gamma(X, \mathcal{I}^\bullet) = \lim_n \Gamma(X, \mathcal{I}^\bullet_n)$$

Here the arrow is in the derived category, but the equality on the right holds on the level of complexes. Note that (3)(b) shows that $\tau_{\geq -n}\mathcal{F}^\bullet$ is a bounded below complex satisfying the hypothesis of the lemma. Thus the case of bounded below complexes shows each of the maps

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \tau_{\geq -n}\mathcal{F}^\bullet)) \longrightarrow \Gamma(X, \mathcal{I}^\bullet_n)$$

is a quasi-isomorphism. The cohomologies of the complexes on the left hand side in given degree are eventually constant (as the alternating Čech complex is finite). Hence the same is true on the right hand side. Thus the cohomology of the limit on the right hand side is this constant value by Homology, Lemma 31.7 (or the stronger More on Algebra, Lemma 86.3) and we win. $\square$

## 41. Hom complexes

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ be two complexes of $\mathcal{O}_X$-modules. We construct a complex of $\mathcal{O}_X$-modules $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. Namely, for each $n$ we set

$$\mathcal{H}om^n(\mathcal{L}^\bullet, \mathcal{M}^\bullet) = \prod_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

It is a good idea to think of $\mathcal{H}om^n$ as the sheaf of $\mathcal{O}_X$-modules of all $\mathcal{O}_X$-linear maps from $\mathcal{L}^\bullet$ to $\mathcal{M}^\bullet$ (viewed as graded $\mathcal{O}_X$-modules) which are homogenous of degree $n$. In this terminology, we define the differential by the rule

$$d(f) = d_\mathcal{M} \circ f - (-1)^n f \circ d_\mathcal{L}$$

for $f \in \mathcal{H}om^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. We omit the verification that $d^2 = 0$. This construction is a special case of Differential Graded Algebra, Example 26.6. It follows immediately from the construction that we have

(41.0.1) $$H^n(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet, \mathcal{M}^\bullet[n])$$

for all $n \in \mathbf{Z}$ and every open $U \subset X$.

**Lemma 41.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is an isomorphism*

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)) = \mathcal{H}om^\bullet(Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

*of complexes of $\mathcal{O}_X$-modules functorial in $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$.*

**Proof.** Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 71.1. $\square$

**Lemma 41.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism*

$$Tot(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$$

*of complexes of $\mathcal{O}_X$-modules.*

**Proof.** Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 71.3. $\square$

**Lemma 41.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism*

$$Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{M}^\bullet, Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet))$$

*of complexes of $\mathcal{O}_X$-modules functorial in all three complexes.*

**Proof.** Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 71.4. $\square$

**Lemma 41.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism*

$$\mathcal{K}^\bullet \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet))$$

*of complexes of $\mathcal{O}_X$-modules functorial in both complexes.*

**Proof.** Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 71.5. $\square$

**Lemma 41.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism*

$$\mathit{Tot}(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

*of complexes of $\mathcal{O}_X$-modules functorial in all three complexes.*

**Proof.** Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 71.6. $\square$

**Lemma 41.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}_X$-modules. Let $\mathcal{L}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Then*

$$H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

*for all $U \subset X$ open.*

**Proof.** We have

$$H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O}_U)}(L|_U, M|_U)$$
$$= \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

The first equality is (41.0.1). The second equality is true because $\mathcal{I}^\bullet|_U$ is K-injective by Lemma 32.1. $\square$

**Lemma 41.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{I}')^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism of K-injective complexes of $\mathcal{O}_X$-modules. Let $(\mathcal{L}')^\bullet \to \mathcal{L}^\bullet$ be a quasi-isomorphism of complexes of $\mathcal{O}_X$-modules. Then*

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet) \longrightarrow \mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet)$$

*is a quasi-isomorphism.*

**Proof.** Let $M$ be the object of $D(\mathcal{O}_X)$ represented by $\mathcal{I}^\bullet$ and $(\mathcal{I}')^\bullet$. Let $L$ be the object of $D(\mathcal{O}_X)$ represented by $\mathcal{L}^\bullet$ and $(\mathcal{L}')^\bullet$. By Lemma 41.6 we see that the sheaves

$$H^0(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet)) \quad \text{and} \quad H^0(\mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet))$$

are both equal to the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

Thus the map is a quasi-isomorphism. $\square$

**Lemma 41.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}_X$-modules. Let $\mathcal{L}^\bullet$ be a K-flat complex of $\mathcal{O}_X$-modules. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}_X$-modules.*

**Proof.** Namely, if $\mathcal{K}^\bullet$ is an acyclic complex of $\mathcal{O}_X$-modules, then

$$
\begin{aligned}
\operatorname{Hom}_{K(\mathcal{O}_X)}(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) &= H^0(\Gamma(X, \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)))) \\
&= H^0(\Gamma(X, \mathcal{H}om^\bullet(\operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet))) \\
&= \operatorname{Hom}_{K(\mathcal{O}_X)}(\operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet) \\
&= 0
\end{aligned}
$$

The first equality by (41.0.1). The second equality by Lemma 41.1. The third equality by (41.0.1). The final equality because $\operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$ is acyclic because $\mathcal{L}^\bullet$ is K-flat (Definition 26.2) and because $\mathcal{I}^\bullet$ is K-injective. $\square$

## 42. Internal hom in the derived category

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $L, M$ be objects of $D(\mathcal{O}_X)$. We would like to construct an object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O}_X)$ such that for every third object $K$ of $D(\mathcal{O}_X)$ there exists a canonical bijection

$$
(42.0.1) \qquad \operatorname{Hom}_{D(\mathcal{O}_X)}(K, R\mathcal{H}om(L, M)) = \operatorname{Hom}_{D(\mathcal{O}_X)}(K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L, M)
$$

Observe that this formula defines $R\mathcal{H}om(L, M)$ up to unique isomorphism by the Yoneda lemma (Categories, Lemma 3.5).

To construct such an object, choose a K-injective complex $\mathcal{I}^\bullet$ representing $M$ and any complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing $L$. Then we set

$$
R\mathcal{H}om(L, M) = \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)
$$

where the right hand side is the complex of $\mathcal{O}_X$-modules constructed in Section 41. This is well defined by Lemma 41.7. We get a functor

$$
D(\mathcal{O}_X)^{opp} \times D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X), \quad (K, L) \longmapsto R\mathcal{H}om(K, L)
$$

As a prelude to proving (42.0.1) we compute the cohomology groups of $R\mathcal{H}om(K, L)$.

**Lemma 42.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $L, M$ be objects of $D(\mathcal{O}_X)$. For every open $U$ we have*

$$
H^0(U, R\mathcal{H}om(L, M)) = \operatorname{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)
$$

*and in particular $H^0(X, R\mathcal{H}om(L, M)) = \operatorname{Hom}_{D(\mathcal{O}_X)}(L, M)$.*

**Proof.** Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules representing $M$ and a K-flat complex $\mathcal{L}^\bullet$ representing $L$. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is K-injective by Lemma 41.8. Hence we can compute cohomology over $U$ by simply taking sections over $U$ and the result follows from Lemma 41.6. $\square$

**Lemma 42.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L, M$ be objects of $D(\mathcal{O}_X)$. With the construction as described above there is a canonical isomorphism*

$$
R\mathcal{H}om(K, R\mathcal{H}om(L, M)) = R\mathcal{H}om(K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L, M)
$$

*in $D(\mathcal{O}_X)$ functorial in $K, L, M$ which recovers (42.0.1) by taking $H^0(X, -)$.*

**Proof.** Choose a K-injective complex $\mathcal{I}^\bullet$ representing $M$ and a K-flat complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing $L$. Let $\mathcal{K}^\bullet$ be any complex of $\mathcal{O}_X$-modules representing $K$. Then we have

$$
\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) = \mathcal{H}om^\bullet(\operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet)
$$

by Lemma 41.1. Note that the left hand side represents $R\mathcal{H}om(K, R\mathcal{H}om(L, M))$ (use Lemma 41.8) and that the right hand side represents $R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, M)$. This proves the displayed formula of the lemma. Taking global sections and using Lemma 42.1 we obtain (42.0.1). $\qquad\square$

**Lemma 42.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L$ be objects of $D(\mathcal{O}_X)$. The construction of $R\mathcal{H}om(K, L)$ commutes with restrictions to opens, i.e., for every open $U$ we have $R\mathcal{H}om(K|_U, L|_U) = R\mathcal{H}om(K, L)|_U$.*

**Proof.** This is clear from the construction and Lemma 32.1. $\qquad\square$

**Lemma 42.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. The bifunctor $R\mathcal{H}om(-, -)$ transforms distinguished triangles into distinguished triangles in both variables.*

**Proof.** This follows from the observation that the assignment

$$(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \longmapsto \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$$

transforms a termwise split short exact sequences of complexes in either variable into a termwise split short exact sequence. Details omitted. $\qquad\square$

**Lemma 42.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given $K, L, M$ in $D(\mathcal{O}_X)$ there is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, M)$$

*in $D(\mathcal{O}_X)$ functorial in $K, L, M$.*

**Proof.** Choose a K-injective complex $\mathcal{I}^\bullet$ representing $M$, a K-injective complex $\mathcal{J}^\bullet$ representing $L$, and any complex of $\mathcal{O}_X$-modules $\mathcal{K}^\bullet$ representing $K$. By Lemma 41.2 there is a map of complexes

$$\mathrm{Tot}\,(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

The complexes of $\mathcal{O}_X$-modules $\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet)$, $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, and $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$ represent $R\mathcal{H}om(L, M)$, $R\mathcal{H}om(K, L)$, and $R\mathcal{H}om(K, M)$. If we choose a K-flat complex $\mathcal{H}^\bullet$ and a quasi-isomorphism $\mathcal{H}^\bullet \to \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, then there is a map

$$\mathrm{Tot}\,(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}^\bullet) \longrightarrow \mathrm{Tot}\,(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet))$$

whose source represents $R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(K, L)$. Composing the two displayed arrows gives the desired map. We omit the proof that the construction is functorial. $\qquad\square$

**Lemma 42.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given $K, L, M$ in $D(\mathcal{O}_X)$ there is a canonical morphism*

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

*in $D(\mathcal{O}_X)$ functorial in $K, L, M$.*

**Proof.** Choose a K-flat complex $\mathcal{K}^\bullet$ representing $K$, and a K-injective complex $\mathcal{I}^\bullet$ representing $L$, and choose any complex of $\mathcal{O}_X$-modules $\mathcal{M}^\bullet$ representing $M$. Choose a quasi-isomorphism $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet) \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is K-injective. Then we use the map

$$\mathrm{Tot}\,(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{I}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{J}^\bullet)$$

where the first map is the map from Lemma 41.3. $\qquad\square$

**Lemma 42.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given $K, L$ in $D(\mathcal{O}_X)$ there is a canonical morphism*

$$K \longrightarrow R\mathcal{H}om(L, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

*in $D(\mathcal{O}_X)$ functorial in both $K$ and $L$.*

**Proof.** Choose a K-flat complex $\mathcal{K}^\bullet$ representing $K$ and any complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing $L$. Choose a K-injective complex $\mathcal{J}^\bullet$ and a quasi-isomorphism $\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet) \to \mathcal{J}^\bullet$. Then we use

$$\mathcal{K}^\bullet \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{J}^\bullet)$$

where the first map comes from Lemma 41.4. $\square$

**Lemma 42.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $L$ be an object of $D(\mathcal{O}_X)$. Set $L^\vee = R\mathcal{H}om(L, \mathcal{O}_X)$. For $M$ in $D(\mathcal{O}_X)$ there is a canonical map*

(42.8.1)           $$M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L^\vee \longrightarrow R\mathcal{H}om(L, M)$$

*which induces a canonical map*

$$H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L^\vee) \longrightarrow \text{Hom}_{D(\mathcal{O}_X)}(L, M)$$

*functorial in $M$ in $D(\mathcal{O}_X)$.*

**Proof.** The map (42.8.1) is a special case of Lemma 42.5 using the identification $M = R\mathcal{H}om(\mathcal{O}_X, M)$. $\square$

**Lemma 42.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L, M$ be objects of $D(\mathcal{O}_X)$. There is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

*in $D(\mathcal{O}_X)$ functorial in $K, L, M$.*

**Proof.** Choose a K-injective complex $\mathcal{I}^\bullet$ representing $M$, a K-injective complex $\mathcal{J}^\bullet$ representing $L$, and a K-flat complex $\mathcal{K}^\bullet$ representing $K$. The map is defined using the map

$$\text{Tot}(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet), \mathcal{I}^\bullet)$$

of Lemma 41.5. By our particular choice of complexes the left hand side represents $R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K$ and the right hand side represents $R\mathcal{H}om(R\mathcal{H}om(K, L), M)$. We omit the proof that this is functorial in all three objects of $D(\mathcal{O}_X)$. $\square$

**Remark 42.10.** Let $(X, \mathcal{O}_X)$ be a ringed space. For $K, K', M, M'$ in $D(\mathcal{O}_X)$ there is a canonical map

$$R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, M') \longrightarrow R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M, K' \otimes^{\mathbf{L}}_{\mathcal{O}_X} M')$$

Namely, by (42.0.1) is the same thing as a map

$$R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, M') \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \longrightarrow K' \otimes^{\mathbf{L}}_{\mathcal{O}_X} M'$$

For this we can first flip the middle two factors (with sign rules as in More on Algebra, Section 72) and use the maps

$$R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \to K' \quad \text{and} \quad R\mathcal{H}om(M, M') \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \to M'$$

from Lemma 42.5 when thinking of $K = R\mathcal{H}om(\mathcal{O}_X, K)$ and similarly for $K'$, $M$, and $M'$.

**Remark 42.11.** Let $f : X \to Y$ be a morphism of ringed spaces. Let $K, L$ be objects of $D(\mathcal{O}_X)$. We claim there is a canonical map

$$Rf_* R\mathcal{H}om(L, K) \longrightarrow R\mathcal{H}om(Rf_*L, Rf_*K)$$

Namely, by (42.0.1) this is the same thing as a map $Rf_* R\mathcal{H}om(L, K) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*L \to Rf_*K$. For this we can use the composition

$$Rf_* R\mathcal{H}om(L, K) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*L \to Rf_*(R\mathcal{H}om(L, K) \otimes_{\mathcal{O}_X}^{\mathbf{L}} L) \to Rf_*K$$

where the first arrow is the relative cup product (Remark 28.7) and the second arrow is $Rf_*$ applied to the canonical map $R\mathcal{H}om(L, K) \otimes_{\mathcal{O}_X}^{\mathbf{L}} L \to K$ coming from Lemma 42.5 (with $\mathcal{O}_X$ in one of the spots).

**Remark 42.12.** Let $h : X \to Y$ be a morphism of ringed spaces. Let $K, L, M$ be objects of $D(\mathcal{O}_Y)$. The diagram

$$
\begin{array}{ccc}
Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, M) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*M & \longrightarrow & Rf_* \left( R\mathcal{H}om_{\mathcal{O}_X}(K, M) \otimes_{\mathcal{O}_X}^{\mathbf{L}} M \right) \\
\downarrow & & \downarrow \\
R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, Rf_*M) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*M & \longrightarrow & Rf_*M
\end{array}
$$

is commutative. Here the left vertical arrow comes from Remark 42.11. The top horizontal arrow is Remark 28.7. The other two arrows are instances of the map in Lemma 42.5 (with one of the entries replaced with $\mathcal{O}_X$ or $\mathcal{O}_Y$).

**Remark 42.13.** Let $h : X \to Y$ be a morphism of ringed spaces. Let $K, L$ be objects of $D(\mathcal{O}_Y)$. We claim there is a canonical map

$$Lh^* R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_X)$. Namely, by (42.0.1) proved in Lemma 42.2 such a map is the same thing as a map

$$Lh^* R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} Lh^*K \longrightarrow Lh^*L$$

The source of this arrow is $Lh^*(\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K)$ by Lemma 27.3 hence it suffices to construct a canonical map

$$R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K \longrightarrow L.$$

For this we take the arrow corresponding to

$$\text{id} : R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, L)$$

via (42.0.1).

**Remark 42.14.** Suppose that

$$
\begin{array}{ccc}
X' & \xrightarrow{\ h\ } & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
S' & \xrightarrow{\ g\ } & S
\end{array}
$$

is a commutative diagram of ringed spaces. Let $K, L$ be objects of $D(\mathcal{O}_X)$. We claim there exists a canonical base change map

$$Lg^* Rf_* R\mathcal{H}om(K, L) \longrightarrow R(f')_* R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_{S'})$. Namely, we take the map adjoint to the composition

$$L(f')^* Lg^* Rf_* R\mathcal{H}om(K, L) = Lh^* Lf^* Rf_* R\mathcal{H}om(K, L)$$
$$\to Lh^* R\mathcal{H}om(K, L)$$
$$\to R\mathcal{H}om(Lh^* K, Lh^* L)$$

where the first arrow uses the adjunction mapping $Lf^* Rf_* \to \mathrm{id}$ and the second arrow is the canonical map constructed in Remark 42.13.

## 43. Ext sheaves

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L \in D(\mathcal{O}_X)$. Using the construction of the internal hom in the derived category we obtain a well defined sheaves of $\mathcal{O}_X$-modules

$$\mathcal{E}xt^n(K, L) = H^n(R\mathcal{H}om(K, L))$$

by taking the $n$th cohomology sheaf of the object $R\mathcal{H}om(K, L)$ of $D(\mathcal{O}_X)$. We will sometimes write $\mathcal{E}xt^n_{\mathcal{O}_X}(K, L)$ for this object. By Lemma 42.1 we see that this $\mathcal{E}xt^n$-sheaf is the sheafification of the rule

$$U \longmapsto \mathrm{Ext}^n_{D(\mathcal{O}_U)}(K|_U, L|_U)$$

By Example 29.3 there is always a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(K, L))$$

converging to $\mathrm{Ext}^{p+q}_{D(\mathcal{O}_X)}(K, L)$ in favorable situations (for example if $L$ is bounded below and $K$ is bounded above).

## 44. Global derived hom

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L \in D(\mathcal{O}_X)$. Using the construction of the internal hom in the derived category we obtain a well defined object

$$R\mathrm{Hom}_X(K, L) = R\Gamma(X, R\mathcal{H}om(K, L))$$

in $D(\Gamma(X, \mathcal{O}_X))$. We will sometimes write $R\mathrm{Hom}_{\mathcal{O}_X}(K, L)$ for this object. By Lemma 42.1 we have

$$H^0(R\mathrm{Hom}_X(K, L)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L), \quad H^p(R\mathrm{Hom}_X(K, L)) = \mathrm{Ext}^p_{D(\mathcal{O}_X)}(K, L)$$

If $f : Y \to X$ is a morphism of ringed spaces, then there is a canonical map

$$R\mathrm{Hom}_X(K, L) \longrightarrow R\mathrm{Hom}_Y(Lf^* K, Lf^* L)$$

in $D(\Gamma(X, \mathcal{O}_X))$ by taking global sections of the map defined in Remark 42.13.

## 45. Glueing complexes

We can glue complexes! More precisely, in certain circumstances we can glue locally given objects of the derived category to a global object. We first prove some easy cases and then we'll prove the very general [BBD82, Theorem 3.2.4] in the setting of topological spaces and open coverings.

**Lemma 45.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces of $X$. Suppose given*

    (1) *an object $A$ of $D(\mathcal{O}_U)$,*
    (2) *an object $B$ of $D(\mathcal{O}_V)$, and*
    (3) *an isomorphism $c : A|_{U \cap V} \to B|_{U \cap V}$.*

*Then there exists an object $F$ of $D(\mathcal{O}_X)$ and isomorphisms $f : F|_U \to A$, $g : F|_V \to$ $B$ such that $c = g|_{U \cap V} \circ f^{-1}|_{U \cap V}$. Moreover, given*

  (1) *an object $E$ of $D(\mathcal{O}_X)$,*
  (2) *a morphism $a : A \to E|_U$ of $D(\mathcal{O}_U)$,*
  (3) *a morphism $b : B \to E|_V$ of $D(\mathcal{O}_V)$,*

*such that*

$$a|_{U \cap V} = b|_{U \cap V} \circ c.$$

*Then there exists a morphism $F \to E$ in $D(\mathcal{O}_X)$ whose restriction to $U$ is $a \circ f$ and whose restriction to $V$ is $b \circ g$.*

**Proof.** Denote $j_U$, $j_V$, $j_{U \cap V}$ the corresponding open immersions. Choose a distinguished triangle

$$F \to Rj_{U,*}A \oplus Rj_{V,*}B \to Rj_{U \cap V,*}(B|_{U \cap V}) \to F[1]$$

where the map $Rj_{V,*}B \to Rj_{U \cap V,*}(B|_{U \cap V})$ is the obvious one and where $Rj_{U,*}A \to Rj_{U \cap V,*}(B|_{U \cap V})$ is the composition of $Rj_{U,*}A \to Rj_{U \cap V,*}(A|_{U \cap V})$ with $Rj_{U \cap V,*}c$. Restricting to $U$ we obtain

$$F|_U \to A \oplus (Rj_{V,*}B)|_U \to (Rj_{U \cap V,*}(B|_{U \cap V}))|_U \to F|_U[1]$$

Denote $j : U \cap V \to U$. Compatibility of restriction to opens and cohomology shows that both $(Rj_{V,*}B)|_U$ and $(Rj_{U \cap V,*}(B|_{U \cap V}))|_U$ are canonically isomorphic to $Rj_*(B|_{U \cap V})$. Hence the second arrow of the last displayed diagram has a section, and we conclude that the morphism $F|_U \to A$ is an isomorphism. Similarly, the morphism $F|_V \to B$ is an isomorphism. The existence of the morphism $F \to E$ follows from the Mayer-Vietoris sequence for Hom, see Lemma 33.3.                $\square$

**Lemma 45.2.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{B}$ be a basis for the topology on $Y$.*

  (1) *Assume $K$ is in $D(\mathcal{O}_X)$ such that for $V \in \mathcal{B}$ we have $H^i(f^{-1}(V), K) = 0$ for $i < 0$. Then $Rf_*K$ has vanishing cohomology sheaves in negative degrees, $H^i(f^{-1}(V), K) = 0$ for $i < 0$ for all opens $V \subset Y$, and the rule $V \mapsto H^0(f^{-1}V, K)$ is a sheaf on $Y$.*
  (2) *Assume $K, L$ are in $D(\mathcal{O}_X)$ such that for $V \in \mathcal{B}$ we have $\mathrm{Ext}^i(K|_{f^{-1}V}, L|_{f^{-1}V}) = 0$ for $i < 0$. Then $\mathrm{Ext}^i(K|_{f^{-1}V}, L|_{f^{-1}V}) = 0$ for $i < 0$ for all opens $V \subset Y$ and the rule $V \mapsto \mathrm{Hom}(K|_{f^{-1}V}, L|_{f^{-1}V})$ is a sheaf on $Y$.*

**Proof.** Lemma 32.6 tells us $H^i(Rf_*K)$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), K) = H^i(V, Rf_*K)$. The assumptions in (1) imply that $Rf_*K$ has vanishing cohomology sheaves in degrees $< 0$. We conclude that for any open $V \subset Y$ the cohomology group $H^i(V, Rf_*K)$ is zero for $i < 0$ and is equal to $H^0(V, H^0(Rf_*K))$ for $i = 0$. This proves (1).

To prove (2) apply (1) to the complex $R\mathcal{H}om(K, L)$ using Lemma 42.1 to do the translation.                $\square$

**Situation 45.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. We are given

  (1) a collection of opens $\mathcal{B}$ of $X$,
  (2) for $U \in \mathcal{B}$ an object $K_U$ in $D(\mathcal{O}_U)$,
  (3) for $V \subset U$ with $V, U \in \mathcal{B}$ an isomorphism $\rho_V^U : K_U|_V \to K_V$ in $D(\mathcal{O}_V)$,

such that whenever we have $W \subset V \subset U$ with $U, V, W$ in $\mathcal{B}$, then $\rho_W^U = \rho_W^V \circ \rho_V^U|_W$.

We won't be able to prove anything about this without making more assumptions. An interesting case is where $\mathcal{B}$ is a basis for the topology on $X$. Another is the case where we have a morphism $f : X \to Y$ of topological spaces and the elements of $\mathcal{B}$ are the inverse images of the elements of a basis for the topology of $Y$.

In Situation 45.3 a *solution* will be a pair $(K, \rho_U)$ where $K$ is an object of $D(\mathcal{O}_X)$ and $\rho_U : K|_U \to K_U$, $U \in \mathcal{B}$ are isomorphisms such that we have $\rho_V^U \circ \rho_U|_V = \rho_V$ for all $V \subset U$, $U, V \in \mathcal{B}$. In certain cases solutions are unique.

**Lemma 45.4.** *In Situation 45.3 assume*

(1) $X = \bigcup_{U \in \mathcal{B}} U$ *and for $U, V \in \mathcal{B}$ we have $U \cap V = \bigcup_{W \in \mathcal{B}, W \subset U \cap V} W$,*

(2) *for any $U \in \mathcal{B}$ we have $\mathrm{Ext}^i(K_U, K_U) = 0$ for $i < 0$.*

*If a solution $(K, \rho_U)$ exists, then it is unique up to unique isomorphism and moreover $\mathrm{Ext}^i(K, K) = 0$ for $i < 0$.*

**Proof.** Let $(K, \rho_U)$ and $(K', \rho'_U)$ be a pair of solutions. Let $f : X \to Y$ be the continuous map constructed in Topology, Lemma 5.6. Set $\mathcal{O}_Y = f_* \mathcal{O}_X$. Then $K, K'$ and $\mathcal{B}$ are as in Lemma 45.2 part (2). Hence we obtain the vanishing of negative exts for $K$ and we see that the rule

$$V \longmapsto \mathrm{Hom}(K|_{f^{-1}V}, K'|_{f^{-1}V})$$

is a sheaf on $Y$. As both $(K, \rho_U)$ and $(K', \rho'_U)$ are solutions the maps

$$(\rho'_U)^{-1} \circ \rho_U : K|_U \longrightarrow K'|_U$$

over $U = f^{-1}(f(U))$ agree on overlaps. Hence we get a unique global section of the sheaf above which defines the desired isomorphism $K \to K'$ compatible with all structure available. $\square$

**Remark 45.5.** With notation and assumptions as in Lemma 45.4. Suppose that $U, V \in \mathcal{B}$. Let $\mathcal{B}'$ be the set of elements of $\mathcal{B}$ contained in $U \cap V$. Then

$$(\{K_{U'}\}_{U' \in \mathcal{B}'}, \{\rho_{V'}^{U'}\}_{V' \subset U' \text{ with } U', V' \in \mathcal{B}'})$$

is a system on the ringed space $U \cap V$ satisfying the assumptions of Lemma 45.4. Moreover, both $(K_U|_{U \cap V}, \rho_{U'}^U)$ and $(K_V|_{U \cap V}, \rho_{U'}^V)$ are solutions to this system. By the lemma we find a unique isomorphism

$$\rho_{U,V} : K_U|_{U \cap V} \longrightarrow K_V|_{U \cap V}$$

such that for every $U' \subset U \cap V$, $U' \in \mathcal{B}$ the diagram

$$
\begin{array}{ccc}
K_U|_{U'} & \xrightarrow{\ \rho_{U,V}|_{U'}\ } & K_V|_{U'} \\
{\scriptstyle \rho_{U'}^U} \searrow & & \swarrow {\scriptstyle \rho_{U'}^V} \\
& K_{U'} &
\end{array}
$$

commutes. Pick a third element $W \in \mathcal{B}$. We obtain isomorphisms $\rho_{U,W} : K_U|_{U \cap W} \to K_W|_{U \cap W}$ and $\rho_{V,W} : K_U|_{V \cap W} \to K_W|_{V \cap W}$ satisfying similar properties to those of $\rho_{U,V}$. Finally, we have

$$\rho_{U,W}|_{U \cap V \cap W} = \rho_{V,W}|_{U \cap V \cap W} \circ \rho_{U,V}|_{U \cap V \cap W}$$

This is true by the uniqueness in the lemma because both sides of the equality are the unique isomorphism compatible with the maps $\rho_{U''}^U$ and $\rho_{U''}^W$ for $U'' \subset U \cap V \cap W$, $U'' \in \mathcal{B}$. Some minor details omitted. The collection $(K_U, \rho_{U,V})$ is a descent datum

in the derived category for the open covering $\mathcal{U} : X = \bigcup_{U \in \mathcal{B}} U$ of $X$. In this language we are looking for "effectiveness of the descent datum" when we look for the existence of a solution.

**Lemma 45.6.** *In Situation 45.3 assume*

   (1) *$X = U_1 \cup \ldots \cup U_n$ with $U_i \in \mathcal{B}$,*
   (2) *for $U, V \in \mathcal{B}$ we have $U \cap V = \bigcup_{W \in \mathcal{B}, W \subset U \cap V} W$,*
   (3) *for any $U \in \mathcal{B}$ we have $\mathrm{Ext}^i(K_U, K_U) = 0$ for $i < 0$.*

*Then a solution exists and is unique up to unique isomorphism.*

**Proof.** Uniqueness was seen in Lemma 45.4. We may prove the lemma by induction on $n$. The case $n = 1$ is immediate.

The case $n = 2$. Consider the isomorphism $\rho_{U_1, U_2} : K_{U_1}|_{U_1 \cap U_2} \to K_{U_2}|_{U_1 \cap U_2}$ constructed in Remark 45.5. By Lemma 45.1 we obtain an object $K$ in $D(\mathcal{O}_X)$ and isomorphisms $\rho_{U_1} : K|_{U_1} \to K_{U_1}$ and $\rho_{U_2} : K|_{U_2} \to K_{U_2}$ compatible with $\rho_{U_1, U_2}$. Take $U \in \mathcal{B}$. We will construct an isomorphism $\rho_U : K|_U \to K_U$ and we will leave it to the reader to verify that $(K, \rho_U)$ is a solution. Consider the set $\mathcal{B}'$ of elements of $\mathcal{B}$ contained in either $U \cap U_1$ or contained in $U \cap U_2$. Then $(K_U, \rho_{U'}^{U'})$ is a solution for the system $(\{K_{U'}\}_{U' \in \mathcal{B}'}, \{\rho_{V'}^{U'}\}_{V' \subset U' \text{ with } U', V' \in \mathcal{B}'})$ on the ringed space $U$. We claim that $(K|_U, \tau_{U'})$ is another solution where $\tau_{U'}$ for $U' \in \mathcal{B}'$ is chosen as follows: if $U' \subset U_1$ then we take the composition

$$K|_{U'} \xrightarrow{\;\rho_{U_1}|_{U'}\;} K_{U_1}|_{U'} \xrightarrow{\;\rho_{U'}^{U_1}\;} K_{U'}$$

and if $U' \subset U_2$ then we take the composition

$$K|_{U'} \xrightarrow{\;\rho_{U_2}|_{U'}\;} K_{U_2}|_{U'} \xrightarrow{\;\rho_{U'}^{U_2}\;} K_{U'}.$$

To verify this is a solution use the property of the map $\rho_{U_1, U_2}$ described in Remark 45.5 and the compatibility of $\rho_{U_1}$ and $\rho_{U_2}$ with $\rho_{U_1, U_2}$. Having said this we apply Lemma 45.4 to see that we obtain a unique isomorphism $K|_{U'} \to K_{U'}$ compatible with the maps $\tau_{U'}$ and $\rho_{U'}^U$ for $U' \in \mathcal{B}'$.

The case $n > 2$. Consider the open subspace $X' = U_1 \cup \ldots \cup U_{n-1}$ and let $\mathcal{B}'$ be the set of elements of $\mathcal{B}$ contained in $X'$. Then we find a system $(\{K_U\}_{U \in \mathcal{B}'}, \{\rho_V^U\}_{U, V \in \mathcal{B}'})$ on the ringed space $X'$ to which we may apply our induction hypothesis. We find a solution $(K_{X'}, \rho_U^{X'})$. Then we can consider the collection $\mathcal{B}^* = \mathcal{B} \cup \{X'\}$ of opens of $X$ and we see that we obtain a system $(\{K_U\}_{U \in \mathcal{B}^*}, \{\rho_V^U\}_{V \subset U \text{ with } U, V \in \mathcal{B}^*})$. Note that this new system also satisfies condition (3) by Lemma 45.4 applied to the solution $K_{X'}$. For this system we have $X = X' \cup U_n$. This reduces us to the case $n = 2$ we worked out above. $\qquad\square$

**Lemma 45.7.** *Let $X$ be a ringed space. Let $E$ be a well ordered set and let*

$$X = \bigcup_{\alpha \in E} W_\alpha$$

*be an open covering with $W_\alpha \subset W_{\alpha+1}$ and $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$ if $\alpha$ is not a successor. Let $K_\alpha$ be an object of $D(\mathcal{O}_{W_\alpha})$ with $\mathrm{Ext}^i(K_\alpha, K_\alpha) = 0$ for $i < 0$. Assume given isomorphisms $\rho_\beta^\alpha : K_\alpha|_{W_\beta} \to K_\beta$ in $D(\mathcal{O}_{W_\beta})$ for all $\beta < \alpha$ with $\rho_\gamma^\alpha = \rho_\gamma^\beta \circ \rho_\beta^\alpha|_{W_\gamma}$ for $\gamma < \beta < \alpha$. Then there exists an object $K$ in $D(\mathcal{O}_X)$ and isomorphisms $K|_{W_\alpha} \to K_\alpha$ for $\alpha \in E$ compatible with the isomorphisms $\rho_\beta^\alpha$.*

**Proof.** In this proof $\alpha, \beta, \gamma, \ldots$ represent elements of $E$. Choose a K-injective complex $I_\alpha^\bullet$ on $W_\alpha$ representing $K_\alpha$. For $\beta < \alpha$ denote $j_{\beta,\alpha} : W_\beta \to W_\alpha$ the inclusion morphism. Using transfinite recursion we will construct for all $\beta < \alpha$ a map of complexes

$$\tau_{\beta,\alpha} : (j_{\beta,\alpha})_! I_\beta^\bullet \longrightarrow I_\alpha^\bullet$$

representing the adjoint to the inverse of the isomorphism $\rho_\beta^\alpha : K_\alpha|_{W_\beta} \to K_\beta$. Moreover, we will do this in such that for $\gamma < \beta < \alpha$ we have

$$\tau_{\gamma,\alpha} = \tau_{\beta,\alpha} \circ (j_{\beta,\alpha})_! \tau_{\gamma,\beta}$$

as maps of complexes. Namely, suppose already given $\tau_{\gamma,\beta}$ composing correctly for all $\gamma < \beta < \alpha$. If $\alpha = \alpha' + 1$ is a successor, then we choose any map of complexes

$$(j_{\alpha',\alpha})_! I_{\alpha'}^\bullet \to I_\alpha^\bullet$$

which is adjoint to the inverse of the isomorphism $\rho_{\alpha'}^\alpha : K_\alpha|_{W_{\alpha'}} \to K_{\alpha'}$ (possible because $I_\alpha^\bullet$ is K-injective) and for any $\beta < \alpha'$ we set

$$\tau_{\beta,\alpha} = \tau_{\alpha',\alpha} \circ (j_{\alpha',\alpha})_! \tau_{\beta,\alpha'}$$

If $\alpha$ is not a successor, then we can consider the complex on $W_\alpha$ given by

$$C^\bullet = \mathrm{colim}_{\beta < \alpha} (j_{\beta,\alpha})_! I_\beta^\bullet$$

(termwise colimit) where the transition maps of the sequence are given by the maps $\tau_{\beta',\beta}$ for $\beta' < \beta < \alpha$. We claim that $C^\bullet$ represents $K_\alpha$. Namely, for $\beta < \alpha$ the restriction of the coprojection $(j_{\beta,\alpha})_! I_\beta^\bullet \to C^\bullet$ gives a map

$$\sigma_\beta : I_\beta^\bullet \longrightarrow C^\bullet|_{W_\beta}$$

which is a quasi-isomorphism: if $x \in W_\beta$ then looking at stalks we get

$$(C^\bullet)_x = \mathrm{colim}_{\beta' < \alpha} \left( (j_{\beta',\alpha})_! I_{\beta'}^\bullet \right)_x = \mathrm{colim}_{\beta \le \beta' < \alpha} (I_{\beta'}^\bullet)_x \longleftarrow (I_\beta^\bullet)_x$$

which is a quasi-isomorphism. Here we used that taking stalks commutes with colimits, that filtered colimits are exact, and that the maps $(I_\beta^\bullet)_x \to (I_{\beta'}^\bullet)_x$ are quasi-isomorphisms for $\beta \le \beta' < \alpha$. Hence $(C^\bullet, \sigma_\beta^{-1})$ is a solution to the system $(\{K_\beta\}_{\beta < \alpha}, \{\rho_{\beta'}^\beta\}_{\beta' < \beta < \alpha})$. Since $(K_\alpha, \rho_\beta^\alpha)$ is another solution we obtain a unique isomorphism $\sigma : K_\alpha \to C^\bullet$ in $D(\mathcal{O}_{W_\alpha})$ compatible with all our maps, see Lemma 45.6 (this is where we use the vanishing of negative ext groups). Choose a morphism $\tau : C^\bullet \to I_\alpha^\bullet$ of complexes representing $\sigma$. Then we set

$$\tau_{\beta,\alpha} = \tau|_{W_\beta} \circ \sigma_\beta$$

to get the desired maps. Finally, we take $K$ to be the object of the derived category represented by the complex

$$K^\bullet = \mathrm{colim}_{\alpha \in E} (W_\alpha \to X)_! I_\alpha^\bullet$$

where the transition maps are given by our carefully constructed maps $\tau_{\beta,\alpha}$ for $\beta < \alpha$. Arguing exactly as above we see that for all $\alpha$ the restriction of the coprojection determines an isomorphism

$$K|_{W_\alpha} \longrightarrow K_\alpha$$

compatible with the given maps $\rho_\beta^\alpha$. $\qquad\square$

Using transfinite induction we can prove the result in the general case.

**Theorem 45.8** (BBD gluing lemma). *In Situation 45.3 assume*

(1) $X = \bigcup_{U \in \mathcal{B}} U$,

(2) *for $U, V \in \mathcal{B}$ we have $U \cap V = \bigcup_{W \in \mathcal{B}, W \subset U \cap V} W$,*

(3) *for any $U \in \mathcal{B}$ we have $\mathrm{Ext}^i(K_U, K_U) = 0$ for $i < 0$.*

*Then there exists an object $K$ of $D(\mathcal{O}_X)$ and isomorphisms $\rho_U : K|_U \to K_U$ in $D(\mathcal{O}_U)$ for $U \in \mathcal{B}$ such that $\rho_V^U \circ \rho_U|_V = \rho_V$ for all $V \subset U$ with $U, V \in \mathcal{B}$. The pair $(K, \rho_U)$ is unique up to unique isomorphism.*

**Proof.** A pair $(K, \rho_U)$ is called a solution in the text above. The uniqueness follows from Lemma 45.4. If $X$ has a finite covering by elements of $\mathcal{B}$ (for example if $X$ is quasi-compact), then the theorem is a consequence of Lemma 45.6. In the general case we argue in exactly the same manner, using transfinite induction and Lemma 45.7.

First we use transfinite recursion to choose opens $W_\alpha \subset X$ for any ordinal $\alpha$. Namely, we set $W_0 = \emptyset$. If $\alpha = \beta + 1$ is a successor, then either $W_\beta = X$ and we set $W_\alpha = X$ or $W_\beta \neq X$ and we set $W_\alpha = W_\beta \cup U_\alpha$ where $U_\alpha \in \mathcal{B}$ is not contained in $W_\beta$. If $\alpha$ is a limit ordinal we set $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$. Then for large enough $\alpha$ we have $W_\alpha = X$. Observe that for every $\alpha$ the open $W_\alpha$ is a union of elements of $\mathcal{B}$. Hence if $\mathcal{B}_\alpha = \{U \in \mathcal{B}, U \subset W_\alpha\}$, then

$$S_\alpha = (\{K_U\}_{U \in \mathcal{B}_\alpha}, \{\rho_V^U\}_{V \subset U \text{ with } U, V \in \mathcal{B}_\alpha})$$

is a system as in Lemma 45.4 on the ringed space $W_\alpha$.

We will show by transfinite induction that for every $\alpha$ the system $S_\alpha$ has a solution. This will prove the theorem as this system is the system given in the theorem for large $\alpha$.

The case where $\alpha = \beta + 1$ is a successor ordinal. (This case was already treated in the proof of the lemma above but for clarity we repeat the argument.) Recall that $W_\alpha = W_\beta \cup U_\alpha$ for some $U_\alpha \in \mathcal{B}$ in this case. By induction hypothesis we have a solution $(K_{W_\beta}, \{\rho_U^{W_\beta}\}_{U \in \mathcal{B}_\beta})$ for the system $S_\beta$. Then we can consider the collection $\mathcal{B}_\alpha^* = \mathcal{B}_\alpha \cup \{W_\beta\}$ of opens of $W_\alpha$ and we see that we obtain a system $(\{K_U\}_{U \in \mathcal{B}_\alpha^*}, \{\rho_V^U\}_{V \subset U \text{ with } U, V \in \mathcal{B}_\alpha^*})$. Note that this new system also satisfies condition (3) by Lemma 45.4 applied to the solution $K_{W_\beta}$. For this system we have $W_\alpha = W_\beta \cup U_\alpha$. This reduces us to the case handled in Lemma 45.6.

The case where $\alpha$ is a limit ordinal. Recall that $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$ in this case. For $\beta < \alpha$ let $(K_{W_\beta}, \{\rho_U^{W_\beta}\}_{U \in \mathcal{B}_\beta})$ be the solution for $S_\beta$. For $\gamma < \beta < \alpha$ the restriction $K_{W_\beta}|_{W_\gamma}$ endowed with the maps $\rho_U^{W_\beta}$, $U \in \mathcal{B}_\gamma$ is a solution for $S_\gamma$. By uniqueness we get unique isomorphisms $\rho_{W_\gamma}^{W_\beta} : K_{W_\beta}|_{W_\gamma} \to K_{W_\gamma}$ compatible with the maps $\rho_U^{W_\beta}$ and $\rho_U^{W_\gamma}$ for $U \in \mathcal{B}_\gamma$. These maps compose in the correct manner, i.e., $\rho_{W_\delta}^{W_\gamma} \circ \rho_{W_\gamma}^{W_\beta}|_{W_\delta} = \rho_{W_\delta}^{W_\beta}$ for $\delta < \gamma < \beta < \alpha$. Thus we may apply Lemma 45.7 (note that the vanishing of negative exts is true for $K_{W_\beta}$ by Lemma 45.4 applied to the solution $K_{W_\beta}$) to obtain $K_{W_\alpha}$ and isomorphisms

$$\rho_{W_\beta}^{W_\alpha} : K_{W_\alpha}|_{W_\beta} \longrightarrow K_{W_\beta}$$

compatible with the maps $\rho_{W_\gamma}^{W_\beta}$ for $\gamma < \beta < \alpha$.

To show that $K_{W_\alpha}$ is a solution we still need to construct the isomorphisms $\rho_U^{W_\alpha} : K_{W_\alpha}|_U \to K_U$ for $U \in \mathcal{B}_\alpha$ satisfying certain compatibilities. We choose $\rho_U^{W_\alpha}$ to be

the unique map such that for any $\beta < \alpha$ and any $V \in \mathcal{B}_\beta$ with $V \subset U$ the diagram

$$
\begin{array}{ccc}
K_{W_\alpha}|_V & \xrightarrow{\rho_U^{W_\alpha}|_V} & K_U|_V \\
{\scriptstyle \rho_{W_\beta}^{W_\alpha}|_V} \downarrow & & \downarrow {\scriptstyle \rho_U^V} \\
K_{W_\beta} & \xrightarrow[\rho_V^{W_\beta}]{} & K_V
\end{array}
$$

commutes. This makes sense because

$$
\left( \{K_V\}_{V \subset U, V \in \mathcal{B}_\beta \text{ for some } \beta < \alpha}, \{\rho_V^{V'}\}_{V \subset V' \text{ with } V, V' \subset U \text{ and } V, V' \in \mathcal{B}_\beta \text{ for some } \beta < \alpha} \right)
$$

is a system as in Lemma 45.4 on the ringed space $U$ and because $(K_U, \rho_V^U)$ and $(K_{W_\alpha}|_U, \rho_V^{W_\beta} \circ \rho_{W_\beta}^{W_\alpha}|_V)$ are both solutions for this system. This gives existence and uniqueness. We omit the proof that these maps satisfy the desired compatibilities (it is just bookkeeping). $\square$

## 46. Strictly perfect complexes

Strictly perfect complexes of modules are used to define the notions of pseudo-coherent and perfect complexes later on. They are defined as follows.

**Definition 46.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say $\mathcal{E}^\bullet$ is *strictly perfect* if $\mathcal{E}^i$ is zero for all but finitely many $i$ and $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{O}_X$-module for all $i$.

Warning: Since we do not assume that $X$ is a locally ringed space, it may not be true that a direct summand of a finite free $\mathcal{O}_X$-module is finite locally free.

**Lemma 46.2.** *The cone on a morphism of strictly perfect complexes is strictly perfect.*

**Proof.** This is immediate from the definitions. $\square$

**Lemma 46.3.** *The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.*

**Proof.** Omitted. $\square$

**Lemma 46.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_Y$-modules, then $f^* \mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_X$-modules.*

**Proof.** The pullback of a finite free module is finite free. The functor $f^*$ is additive functor hence preserves direct summands. The lemma follows. $\square$

**Lemma 46.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given a solid diagram of $\mathcal{O}_X$-modules*

$$
\begin{array}{ccc}
\mathcal{E} & \longrightarrow & \mathcal{F} \\
& \searrow & \uparrow {\scriptstyle p} \\
& & \mathcal{G}
\end{array}
$$

*with $\mathcal{E}$ a direct summand of a finite free $\mathcal{O}_X$-module and $p$ surjective, then a dotted arrow making the diagram commute exists locally on $X$.*

**Proof.** We may assume $\mathcal{E} = \mathcal{O}_X^{\oplus n}$ for some $n$. In this case finding the dotted arrow is equivalent to lifting the images of the basis elements in $\Gamma(X, \mathcal{F})$. This is locally possible by the characterization of surjective maps of sheaves (Sheaves, Section 16). $\square$

**Lemma 46.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space.*
  (1) *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect and $\mathcal{F}^\bullet$ acyclic. Then $\alpha$ is locally on $X$ homotopic to zero.*
  (2) *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^i = 0$ for $i < a$, and $H^i(\mathcal{F}^\bullet) = 0$ for $i \geq a$. Then $\alpha$ is locally on $X$ homotopic to zero.*

**Proof.** The first statement follows from the second, hence we only prove (2). We will prove this by induction on the length of the complex $\mathcal{E}^\bullet$. If $\mathcal{E}^\bullet \cong \mathcal{E}[-n]$ for some direct summand $\mathcal{E}$ of a finite free $\mathcal{O}_X$-module and integer $n \geq a$, then the result follows from Lemma 46.5 and the fact that $\mathcal{F}^{n-1} \to \mathrm{Ker}(\mathcal{F}^n \to \mathcal{F}^{n+1})$ is surjective by the assumed vanishing of $H^n(\mathcal{F}^\bullet)$. If $\mathcal{E}^i$ is zero except for $i \in [a, b]$, then we have a split exact sequence of complexes

$$0 \to \mathcal{E}^b[-b] \to \mathcal{E}^\bullet \to \sigma_{\leq b-1}\mathcal{E}^\bullet \to 0$$

which determines a distinguished triangle in $K(\mathcal{O}_X)$. Hence an exact sequence

$$\mathrm{Hom}_{K(\mathcal{O}_X)}(\sigma_{\leq b-1}\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{E}^b[-b], \mathcal{F}^\bullet)$$

by the axioms of triangulated categories. The composition $\mathcal{E}^b[-b] \to \mathcal{F}^\bullet$ is locally homotopic to zero, whence we may assume our map comes from an element in the left hand side of the displayed exact sequence above. This element is locally zero by induction hypothesis. $\square$

**Lemma 46.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given a solid diagram of complexes of $\mathcal{O}_X$-modules*



*with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^j = 0$ for $j < a$ and $H^j(f)$ an isomorphism for $j > a$ and surjective for $j = a$, then a dotted arrow making the diagram commute up to homotopy exists locally on $X$.*

**Proof.** Our assumptions on $f$ imply the cone $C(f)^\bullet$ has vanishing cohomology sheaves in degrees $\geq a$. Hence Lemma 46.6 guarantees there is an open covering $X = \bigcup U_i$ such that the composition $\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet$ is homotopic to zero over $U_i$. Since

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet \to \mathcal{G}^\bullet[1]$$

restricts to a distinguished triangle in $K(\mathcal{O}_{U_i})$ we see that we can lift $\alpha|_{U_i}$ up to homotopy to a map $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{G}^\bullet|_{U_i}$ as desired. $\square$

**Lemma 46.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect.*
  (1) *For any element $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ there exists an open covering $X = \bigcup U_i$ such that $\alpha|_{U_i}$ is given by a morphism of complexes $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{F}^\bullet|_{U_i}$.*

(2) *Given a morphism of complexes $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ whose image in the group* $\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ *is zero, there exists an open covering $X = \bigcup U_i$ such that $\alpha|_{U_i}$ is homotopic to zero.*

**Proof.** Proof of (1). By the construction of the derived category we can find a quasi-isomorphism $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and a map of complexes $\beta : \mathcal{E}^\bullet \to \mathcal{G}^\bullet$ such that $\alpha = f^{-1}\beta$. Thus the result follows from Lemma 46.7. We omit the proof of (2). $\square$

**Lemma 46.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect. Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms*

$$\mathcal{H}^n = \bigoplus_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

*and differential as described in Section 41.*

**Proof.** Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex. Let $(\mathcal{H}')^\bullet$ be the complex with terms

$$(\mathcal{H}')^n = \prod_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

which represents $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ by the construction in Section 42. It suffices to show that the map

$$\mathcal{H}^\bullet \longrightarrow (\mathcal{H}')^\bullet$$

is a quasi-isomorphism. Given an open $U \subset X$ we have by inspection

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U) \to H^0((\mathcal{H}')^\bullet(U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U)$$

By Lemma 46.8 the sheafification of $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^0((\mathcal{H}')^\bullet(U))$. A similar argument can be given for the other cohomology sheaves. Thus $\mathcal{H}^\bullet$ is quasi-isomorphic to $(\mathcal{H}')^\bullet$ which proves the lemma. $\square$

**Lemma 46.10.** *In the situation of Lemma 46.9 if $\mathcal{F}^\bullet$ is K-flat, then $\mathcal{H}^\bullet$ is K-flat.*

**Proof.** Observe that $\mathcal{H}^\bullet$ is simply the hom complex $\mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ since the boundedness of the strictly prefect complex $\mathcal{E}^\bullet$ insures that the products in the definition of the hom complex turn into direct sums. Let $\mathcal{K}^\bullet$ be an acyclic complex of $\mathcal{O}_X$-modules. Consider the map

$$\gamma : \mathrm{Tot}(\mathcal{K}^\bullet \otimes \mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes \mathcal{F}^\bullet))$$

of Lemma 41.3. Since $\mathcal{F}^\bullet$ is K-flat, the complex $\mathrm{Tot}(\mathcal{K}^\bullet \otimes \mathcal{F}^\bullet)$ is acyclic, and hence by Lemma 46.8 (or Lemma 46.9 if you like) the target of $\gamma$ is acyclic too. Hence to prove the lemma it suffices to show that $\gamma$ is an isomorphism of complexes. To see this, we may argue by induction on the length of the complex $\mathcal{E}^\bullet$. If the length is $\leq 1$ then the $\mathcal{E}^\bullet$ is a direct summand of $\mathcal{O}_X^{\oplus n}[k]$ for some $n \geq 0$ and $k \in \mathbf{Z}$ and in this case the result follows by inspection. If the length is $> 1$, then we reduce to smaller length by considering the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1}\mathcal{E}^\bullet \to \mathcal{E}^\bullet \to \sigma_{\leq a}\mathcal{E}^\bullet \to 0$$

for a suitable $a \in \mathbf{Z}$, see Homology, Section 15. Then $\gamma$ fits into a morphism of termwise split short exact sequences of complexes. By induction $\gamma$ is an isomorphism for $\sigma_{\geq a+1}\mathcal{E}^\bullet$ and $\sigma_{\leq a}\mathcal{E}^\bullet$ and hence the result for $\mathcal{E}^\bullet$ follows. Some details omitted. $\square$

**Lemma 46.11.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with*

(1) $\mathcal{F}^n = 0$ *for* $n \ll 0$,
(2) $\mathcal{E}^n = 0$ *for* $n \gg 0$, *and*
(3) $\mathcal{E}^n$ *isomorphic to a direct summand of a finite free $\mathcal{O}_X$-module.*

*Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms*

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

*and differential as described in Section 42.*

**Proof.** Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injectives. Note that $\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 31.4). Hence the construction in Section 42 shows that $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $(\mathcal{H}')^\bullet$ with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{I}^p) = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

(equality because there are only finitely many nonzero terms). Note that $\mathcal{H}^\bullet$ is the total complex associated to the double complex with terms $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$ and similarly for $(\mathcal{H}')^\bullet$. The natural map $(\mathcal{H}')^\bullet \to \mathcal{H}^\bullet$ comes from a map of double complexes. Thus to show this map is a quasi-isomorphism, we may use the spectral sequence of a double complex (Homology, Lemma 25.3)

$$'E_1^{p,q} = H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{H}^\bullet)$ and similarly for $(\mathcal{H}')^\bullet$. To finish the proof of the lemma it suffices to show that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism

$$H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}^\bullet)) \longrightarrow H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{I}^\bullet))$$

on cohomology sheaves whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module. Since this is clear when $\mathcal{E}$ is finite free the result follows.                    $\square$

## 47. Pseudo-coherent modules

In this section we discuss pseudo-coherent complexes.

**Definition 47.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Let $m \in \mathbf{Z}$.

(1) We say $\mathcal{E}^\bullet$ is *m-pseudo-coherent* if there exists an open covering $X = \bigcup U_i$ and for each $i$ a morphism of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ where $\mathcal{E}_i$ is strictly perfect on $U_i$ and $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective.
(2) We say $\mathcal{E}^\bullet$ is *pseudo-coherent* if it is $m$-pseudo-coherent for all $m$.
(3) We say an object $E$ of $D(\mathcal{O}_X)$ is *m-pseudo-coherent* (resp. *pseudo-coherent*) if and only if it can be represented by a $m$-pseudo-coherent (resp. pseudo-coherent) complex of $\mathcal{O}_X$-modules.

If $X$ is quasi-compact, then an $m$-pseudo-coherent object of $D(\mathcal{O}_X)$ is in $D^-(\mathcal{O}_X)$. But this need not be the case if $X$ is not quasi-compact.

**Lemma 47.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E$ be an object of $D(\mathcal{O}_X)$.*

(1) *If there exists an open covering $X = \bigcup U_i$, strictly perfect complexes $\mathcal{E}_i^\bullet$ on $U_i$, and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $H^j(\alpha_i)$ an isomorphism for $j > m$ and $H^m(\alpha_i)$ surjective, then $E$ is $m$-pseudo-coherent.*

(2) *If $E$ is $m$-pseudo-coherent, then any complex representing $E$ is $m$-pseudo-coherent.*

**Proof.** Let $\mathcal{F}^\bullet$ be any complex representing $E$ and let $X = \bigcup U_i$ and $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ be as in (1). We will show that $\mathcal{F}^\bullet$ is $m$-pseudo-coherent as a complex, which will prove (1) and (2) simultaneously. By Lemma 46.8 we can after refining the open covering $X = \bigcup U_i$ represent the maps $\alpha_i$ by maps of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{F}^\bullet|_{U_i}$. By assumption $H^j(\alpha_i)$ are isomorphisms for $j > m$, and $H^m(\alpha_i)$ is surjective whence $\mathcal{F}^\bullet$ is $m$-pseudo-coherent. $\square$

**Lemma 47.3.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $E$ be an object of $D(\mathcal{O}_Y)$. If $E$ is $m$-pseudo-coherent, then $Lf^*E$ is $m$-pseudo-coherent.*

**Proof.** Represent $E$ by a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Y$-modules and choose an open covering $Y = \bigcup V_i$ and $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{V_i}$ as in Definition 47.1. Set $U_i = f^{-1}(V_i)$. By Lemma 47.2 it suffices to show that $Lf^*\mathcal{E}^\bullet|_{U_i}$ is $m$-pseudo-coherent. Choose a distinguished triangle

$$\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{V_i} \to C \to \mathcal{E}_i^\bullet[1]$$

The assumption on $\alpha_i$ means exactly that the cohomology sheaves $H^j(C)$ are zero for all $j \geq m$. Denote $f_i : U_i \to V_i$ the restriction of $f$. Note that $Lf^*\mathcal{E}^\bullet|_{U_i} = Lf_i^*(\mathcal{E}|_{V_i})$. Applying $Lf_i^*$ we obtain the distinguished triangle

$$Lf_i^*\mathcal{E}_i^\bullet \to Lf_i^*\mathcal{E}|_{V_i} \to Lf_i^*C \to Lf_i^*\mathcal{E}_i^\bullet[1]$$

By the construction of $Lf_i^*$ as a left derived functor we see that $H^j(Lf_i^*C) = 0$ for $j \geq m$ (by the dual of Derived Categories, Lemma 16.1). Hence $H^j(Lf_i^*\alpha_i)$ is an isomorphism for $j > m$ and $H^m(Lf^*\alpha_i)$ is surjective. On the other hand, $Lf_i^*\mathcal{E}_i^\bullet = f_i^*\mathcal{E}_i^\bullet$. is strictly perfect by Lemma 46.4. Thus we conclude. $\square$

**Lemma 47.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space and $m \in \mathbf{Z}$. Let $(K, L, M, f, g, h)$ be a distinguished triangle in $D(\mathcal{O}_X)$.*

(1) *If $K$ is $(m+1)$-pseudo-coherent and $L$ is $m$-pseudo-coherent then $M$ is $m$-pseudo-coherent.*

(2) *If $K$ and $M$ are $m$-pseudo-coherent, then $L$ is $m$-pseudo-coherent.*

(3) *If $L$ is $(m+1)$-pseudo-coherent and $M$ is $m$-pseudo-coherent, then $K$ is $(m+1)$-pseudo-coherent.*

**Proof.** Proof of (1). Choose an open covering $X = \bigcup U_i$ and maps $\alpha_i : \mathcal{K}_i^\bullet \to K|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{K}_i^\bullet$ strictly perfect and $H^j(\alpha_i)$ isomorphisms for $j > m + 1$ and surjective for $j = m + 1$. We may replace $\mathcal{K}_i^\bullet$ by $\sigma_{\geq m+1}\mathcal{K}_i^\bullet$ and hence we may assume that $\mathcal{K}_i^j = 0$ for $j < m + 1$. After refining the open covering we may choose maps $\beta_i : \mathcal{L}_i^\bullet \to L|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{L}_i^\bullet$ strictly perfect such that $H^j(\beta)$ is an isomorphism for $j > m$ and surjective for $j = m$. By Lemma 46.7 we can, after

refining the covering, find maps of complexes $\gamma_i : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ such that the diagrams

$$
\begin{array}{ccc}
K|_{U_i} & \longrightarrow & L|_{U_i} \\
\alpha_i \uparrow & & \uparrow \beta_i \\
\mathcal{K}_i^\bullet & \xrightarrow{\ \gamma_i\ } & \mathcal{L}_i^\bullet
\end{array}
$$

are commutative in $D(\mathcal{O}_{U_i})$ (this requires representing the maps $\alpha_i$, $\beta_i$ and $K|_{U_i} \to L|_{U_i}$ by actual maps of complexes; some details omitted). The cone $C(\gamma_i)^\bullet$ is strictly perfect (Lemma 46.2). The commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(\mathcal{K}_i^\bullet, \mathcal{L}_i^\bullet, C(\gamma_i)^\bullet) \longrightarrow (K|_{U_i}, L|_{U_i}, M|_{U_i}).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 5.19 and 5.20 that $C(\gamma_i)^\bullet \to M|_{U_i}$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree $m$. Hence $M$ is $m$-pseudo-coherent by Lemma 47.2.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. $\square$

**Lemma 47.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L$ be objects of $D(\mathcal{O}_X)$.*
  (1) *If $K$ is $n$-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and $L$ is $m$-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L$ is $t$-pseudo-coherent with $t = \max(m + a, n + b)$.*
  (2) *If $K$ and $L$ are pseudo-coherent, then $K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L$ is pseudo-coherent.*

**Proof.** Proof of (1). By replacing $X$ by the members of an open covering we may assume there exist strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ and maps $\alpha : \mathcal{K}^\bullet \to K$ and $\beta : \mathcal{L}^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet) \to K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L$$

induces isomorphisms on cohomology sheaves in degree $i$ for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted).

Proof of (2). We may first replace $X$ by the members of an open covering to reduce to the case that $K$ and $L$ are bounded above. Then the statement follows immediately from case (1). $\square$

**Lemma 47.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. If $K \oplus L$ is $m$-pseudo-coherent (resp. pseudo-coherent) in $D(\mathcal{O}_X)$ so are $K$ and $L$.*

**Proof.** Assume that $K \oplus L$ is $m$-pseudo-coherent. After replacing $X$ by the members of an open covering we may assume $K \oplus L \in D^-(\mathcal{O}_X)$, hence $L \in D^-(\mathcal{O}_X)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 4.10. By Lemma 47.4 we see that $L \oplus L[1]$ is $m$-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is $m$-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is $m$-pseudo-coherent. Since $L$ is bounded above we see that $L[n]$ is $m$-pseudo-coherent for large $n$. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n-1], L[n-2], \ldots, L$ are $m$-pseudo-coherent as desired. $\qquad\square$

**Lemma 47.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. Let $\mathcal{F}^\bullet$ be a (locally) bounded above complex of $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $(m-i)$-pseudo-coherent for all $i$. Then $\mathcal{F}^\bullet$ is $m$-pseudo-coherent.*

**Proof.** Omitted. Hint: use Lemma 47.4 and truncations as in the proof of More on Algebra, Lemma 64.9. $\qquad\square$

**Lemma 47.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. Let $E$ be an object of $D(\mathcal{O}_X)$. If $E$ is (locally) bounded above and $H^i(E)$ is $(m-i)$-pseudo-coherent for all $i$, then $E$ is $m$-pseudo-coherent.*

**Proof.** Omitted. Hint: use Lemma 47.4 and truncations as in the proof of More on Algebra, Lemma 64.10. $\qquad\square$

**Lemma 47.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K$ be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$.*
  (1) *If $K$ is $m$-pseudo-coherent and $H^i(K) = 0$ for $i > m$, then $H^m(K)$ is a finite type $\mathcal{O}_X$-module.*
  (2) *If $K$ is $m$-pseudo-coherent and $H^i(K) = 0$ for $i > m+1$, then $H^{m+1}(K)$ is a finitely presented $\mathcal{O}_X$-module.*

**Proof.** Proof of (1). We may work locally on $X$. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree $m$. It suffices to prove the result for $\mathcal{E}^\bullet$. Let $n$ be the largest integer such that $\mathcal{E}^n \neq 0$. If $n = m$, then $H^m(\mathcal{E}^\bullet)$ is a quotient of $\mathcal{E}^n$ and the result is clear. If $n > m$, then $\mathcal{E}^{n-1} \to \mathcal{E}^n$ is surjective as $H^n(E^\bullet) = 0$. By Lemma 46.5 we can locally find a section of this surjection and write $\mathcal{E}^{n-1} = \mathcal{E}' \oplus \mathcal{E}^n$. Hence it suffices to prove the result for the complex $(\mathcal{E}')^\bullet$ which is the same as $\mathcal{E}^\bullet$ except has $\mathcal{E}'$ in degree $n-1$ and $0$ in degree $n$. We win by induction on $n$.

Proof of (2). We may work locally on $X$. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree $m$. As in the proof of (1) we can reduce to the case that $\mathcal{E}^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K) \cong H^{m+1}(\mathcal{E}^\bullet) = \mathrm{Coker}(\mathcal{E}^m \to \mathcal{E}^{m+1})$ which is of finite presentation. $\qquad\square$

**Lemma 47.10.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*
  (1) *$\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is $0$-pseudo-coherent if and only if $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and*
  (2) *$\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is $(-1)$-pseudo-coherent if and only if $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

**Proof.** Use Lemma 47.9 to prove the implications in one direction and Lemma 47.8 for the other. $\qquad\square$

## 48. Tor dimension

In this section we take a closer look at resolutions by flat modules.

**Definition 48.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E$ be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$.

(1) We say $E$ has *tor-amplitude in* $[a, b]$ if $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for all $\mathcal{O}_X$-modules $\mathcal{F}$ and all $i \notin [a, b]$.

(2) We say $E$ has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some $a, b$.

(3) We say $E$ *locally has finite tor dimension* if there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ has finite tor dimension for all $i$.

An $\mathcal{O}_X$-module $\mathcal{F}$ has *tor dimension* $\le d$ if $\mathcal{F}[0]$ viewed as an object of $D(\mathcal{O}_X)$ has tor-amplitude in $[-d, 0]$.

Note that if $E$ as in the definition has finite tor dimension, then $E$ is an object of $D^b(\mathcal{O}_X)$ as can be seen by taking $\mathcal{F} = \mathcal{O}_X$ in the definition above.

**Lemma 48.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a bounded above complex of flat $\mathcal{O}_X$-modules with tor-amplitude in $[a, b]$. Then $\operatorname{Coker}(d^{a-1}_{\mathcal{E}^\bullet})$ is a flat $\mathcal{O}_X$-module.*

**Proof.** As $\mathcal{E}^\bullet$ is a bounded above complex of flat modules we see that $\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F} = \mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ for any $\mathcal{O}_X$-module $\mathcal{F}$. Hence for every $\mathcal{O}_X$-module $\mathcal{F}$ the sequence

$$\mathcal{E}^{a-2} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact in the middle. Since $\mathcal{E}^{a-2} \to \mathcal{E}^{a-1} \to \mathcal{E}^a \to \operatorname{Coker}(d^{a-1}) \to 0$ is a flat resolution this implies that $\operatorname{Tor}^{\mathcal{O}_X}_1(\operatorname{Coker}(d^{a-1}), \mathcal{F}) = 0$ for all $\mathcal{O}_X$-modules $\mathcal{F}$. This means that $\operatorname{Coker}(d^{a-1})$ is flat, see Lemma 26.16. $\square$

**Lemma 48.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E$ be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \le b$. The following are equivalent*

(1) *$E$ has tor-amplitude in $[a, b]$.*

(2) *$E$ is represented by a complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}_X$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$.*

**Proof.** If (2) holds, then we may compute $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F} = \mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}$ and it is clear that (1) holds.

Assume that (1) holds. We may represent $E$ by a bounded above complex of flat $\mathcal{O}_X$-modules $\mathcal{K}^\bullet$, see Section 26. Let $n$ be the largest integer such that $\mathcal{K}^n \neq 0$. If $n > b$, then $\mathcal{K}^{n-1} \to \mathcal{K}^n$ is surjective as $H^n(\mathcal{K}^\bullet) = 0$. As $\mathcal{K}^n$ is flat we see that $\operatorname{Ker}(\mathcal{K}^{n-1} \to \mathcal{K}^n)$ is flat (Modules, Lemma 17.8). Hence we may replace $\mathcal{K}^\bullet$ by $\tau_{\le n-1}\mathcal{K}^\bullet$. Thus, by induction on $n$, we reduce to the case that $K^\bullet$ is a complex of flat $\mathcal{O}_X$-modules with $\mathcal{K}^i = 0$ for $i > b$.

Set $\mathcal{E}^\bullet = \tau_{\ge a}\mathcal{K}^\bullet$. Everything is clear except that $\mathcal{E}^a$ is flat which follows immediately from Lemma 48.2 and the definitions. $\square$

**Lemma 48.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $E$ be an object of $D(\mathcal{O}_Y)$. If $E$ has tor amplitude in $[a, b]$, then $Lf^*E$ has tor amplitude in $[a, b]$.*

**Proof.** Assume $E$ has tor amplitude in $[a, b]$. By Lemma 48.3 we can represent $E$ by a complex of $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$. Then $Lf^*E$ is represented by $f^*\mathcal{E}^\bullet$. By Modules, Lemma 20.2 the modules $f^*\mathcal{E}^i$ are flat. Thus by Lemma 48.3 we conclude that $Lf^*E$ has tor amplitude in $[a, b]$. $\square$

**Lemma 48.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E$ be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \le b$. The following are equivalent*

  (1) $E$ has tor-amplitude in $[a, b]$.
  (2) for every $x \in X$ the object $E_x$ of $D(\mathcal{O}_{X,x})$ has tor-amplitude in $[a, b]$.

**Proof.** Taking stalks at $x$ is the same thing as pulling back by the morphism of ringed spaces $(x, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$. Hence the implication $(1) \Rightarrow (2)$ follows from Lemma 48.4. For the converse, note that taking stalks commutes with tensor products (Modules, Lemma 16.1). Hence

$$(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x = E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{F}_x$$

On the other hand, taking stalks is exact, so

$$H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x = H^i((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x) = H^i(E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{F}_x)$$

and we can check whether $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})$ is zero by checking whether all of its stalks are zero (Modules, Lemma 3.1). Thus (2) implies (1).  $\square$

**Lemma 48.6.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(K, L, M, f, g, h)$ be a distinguished triangle in $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$.

  (1) If $K$ has tor-amplitude in $[a+1, b+1]$ and $L$ has tor-amplitude in $[a, b]$ then $M$ has tor-amplitude in $[a, b]$.
  (2) If $K$ and $M$ have tor-amplitude in $[a, b]$, then $L$ has tor-amplitude in $[a, b]$.
  (3) If $L$ has tor-amplitude in $[a+1, b+1]$ and $M$ has tor-amplitude in $[a, b]$, then $K$ has tor-amplitude in $[a+1, b+1]$.

**Proof.** Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation.  $\square$

**Lemma 48.7.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L$ be objects of $D(\mathcal{O}_X)$. If $K$ has tor-amplitude in $[a, b]$ and $L$ has tor-amplitude in $[c, d]$ then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ has tor amplitude in $[a+c, b+d]$.

**Proof.** Omitted. Hint: use the spectral sequence for tors.  $\square$

**Lemma 48.8.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $a, b \in \mathbf{Z}$. For $K$, $L$ objects of $D(\mathcal{O}_X)$ if $K \oplus L$ has tor amplitude in $[a, b]$ so do $K$ and $L$.

**Proof.** Clear from the fact that the Tor functors are additive.  $\square$

## 49. Perfect complexes

In this section we discuss properties of perfect complexes on ringed spaces.

**Definition 49.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^{\bullet}$ be a complex of $\mathcal{O}_X$-modules. We say $\mathcal{E}^{\bullet}$ is *perfect* if there exists an open covering $X = \bigcup U_i$ such that for each $i$ there exists a morphism of complexes $\mathcal{E}_i^{\bullet} \to \mathcal{E}^{\bullet}|_{U_i}$ which is a quasi-isomorphism with $\mathcal{E}_i^{\bullet}$ a strictly perfect complex of $\mathcal{O}_{U_i}$-modules. An object $E$ of $D(\mathcal{O}_X)$ is *perfect* if it can be represented by a perfect complex of $\mathcal{O}_X$-modules.

**Lemma 49.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E$ be an object of $D(\mathcal{O}_X)$.

  (1) If there exists an open covering $X = \bigcup U_i$ and strictly perfect complexes $\mathcal{E}_i^{\bullet}$ on $U_i$ such that $\mathcal{E}_i^{\bullet}$ represents $E|_{U_i}$ in $D(\mathcal{O}_{U_i})$, then $E$ is perfect.
  (2) If $E$ is perfect, then any complex representing $E$ is perfect.

**Proof.** Identical to the proof of Lemma 47.2. $\qquad\square$

**Lemma 49.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E$ be an object of $D(\mathcal{O}_X)$. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Then the following are equivalent*

(1) *$E$ is perfect,*
(2) *there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ can be represented by a finite complex of finite locally free $\mathcal{O}_{U_i}$-modules, and*
(3) *there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ can be represented by a finite complex of finite free $\mathcal{O}_{U_i}$-modules.*

**Proof.** This follows from Lemma 49.2 and the fact that on $X$ every direct summand of a finite free module is finite locally free. See Modules, Lemma 14.6. $\qquad\square$

**Lemma 49.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E$ be an object of $D(\mathcal{O}_X)$. Let $a \leq b$ be integers. If $E$ has tor amplitude in $[a, b]$ and is $(a-1)$-pseudo-coherent, then $E$ is perfect.*

**Proof.** After replacing $X$ by the members of an open covering we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to E$ such that $H^i(\alpha)$ is an isomorphism for $i \geq a$. We may and do replace $\mathcal{E}^\bullet$ by $\sigma_{\geq a-1}\mathcal{E}^\bullet$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to E \to C \to \mathcal{E}^\bullet[1]$$

From the vanishing of cohomology sheaves of $E$ and $\mathcal{E}^\bullet$ and the assumption on $\alpha$ we obtain $C \cong \mathcal{K}[a-2]$ with $\mathcal{K} = \operatorname{Ker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Applying $- \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}$ the assumption that $E$ has tor amplitude in $[a, b]$ implies $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F}$ has image $\operatorname{Ker}(\mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}_X} \mathcal{F})$. It follows that $\operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{E}', \mathcal{F}) = 0$ where $\mathcal{E}' = \operatorname{Coker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Hence $\mathcal{E}'$ is flat (Lemma 26.16). Thus $\mathcal{E}'$ is locally a direct summand of a finite free module by Modules, Lemma 18.3. Thus locally the complex

$$\mathcal{E}' \to \mathcal{E}^{a-1} \to \ldots \to \mathcal{E}^b$$

is quasi-isomorphic to $E$ and $E$ is perfect. $\qquad\square$

**Lemma 49.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E$ be an object of $D(\mathcal{O}_X)$. The following are equivalent*

(1) *$E$ is perfect, and*
(2) *$E$ is pseudo-coherent and locally has finite tor dimension.*

**Proof.** Assume (1). By definition this means there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ is represented by a strictly perfect complex. Thus $E$ is pseudo-coherent (i.e., $m$-pseudo-coherent for all $m$) by Lemma 47.2. Moreover, a direct summand of a finite free module is flat, hence $E|_{U_i}$ has finite Tor dimension by Lemma 48.3. Thus (2) holds.

Assume (2). After replacing $X$ by the members of an open covering we may assume there exist integers $a \leq b$ such that $E$ has tor amplitude in $[a, b]$. Since $E$ is $m$-pseudo-coherent for all $m$ we conclude using Lemma 49.4. $\qquad\square$

**Lemma 49.6.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $E$ be an object of $D(\mathcal{O}_Y)$. If $E$ is perfect in $D(\mathcal{O}_Y)$, then $Lf^*E$ is perfect in $D(\mathcal{O}_X)$.*

**Proof.** This follows from Lemma 49.5, 48.4, and 47.3. (An alternative proof is to copy the proof of Lemma 47.3.) $\qquad\square$

**Lemma 49.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(K, L, M, f, g, h)$ be a distinguished triangle in $D(\mathcal{O}_X)$. If two out of three of $K, L, M$ are perfect then the third is also perfect.*

**Proof.** First proof: Combine Lemmas 49.5, 47.4, and 48.6. Second proof (sketch): Say $K$ and $L$ are perfect. After replacing $X$ by the members of an open covering we may assume that $K$ and $L$ are represented by strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$. After replacing $X$ by the members of an open covering we may assume the map $K \to L$ is given by a map of complexes $\alpha : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, see Lemma 46.8. Then $M$ is isomorphic to the cone of $\alpha$ which is strictly perfect by Lemma 46.2. $\square$

**Lemma 49.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. If $K, L$ are perfect objects of $D(\mathcal{O}_X)$, then so is $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$.*

**Proof.** Follows from Lemmas 49.5, 47.5, and 48.7. $\square$

**Lemma 49.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. If $K \oplus L$ is a perfect object of $D(\mathcal{O}_X)$, then so are $K$ and $L$.*

**Proof.** Follows from Lemmas 49.5, 47.6, and 48.8. $\square$

**Lemma 49.10.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $j : U \to X$ be an open subspace. Let $E$ be a perfect object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on a closed subset $T \subset U$ with $j(T)$ closed in $X$. Then $Rj_* E$ is a perfect object of $D(\mathcal{O}_X)$.*

**Proof.** Being a perfect complex is local on $X$. Thus it suffices to check that $Rj_* E$ is perfect when restricted to $U$ and $V = X \setminus j(T)$. We have $Rj_* E|_U = E$ which is perfect. We have $Rj_* E|_V = 0$ because $E|_{U \setminus T} = 0$. $\square$

**Lemma 49.11.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E$ in $D(\mathcal{O}_X)$ be perfect. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Then the set*

$$U = \{x \in X \mid H^i(E)_x \text{ is a finite free } \mathcal{O}_{X,x}\text{-module for all } i \in \mathbf{Z}\}$$

*is open in $X$ and is the maximal open set $U \subset X$ such that $H^i(E)|_U$ is finite locally free for all $i \in \mathbf{Z}$.*

**Proof.** Note that if $V \subset X$ is some open such that $H^i(E)|_V$ is finite locally free for all $i \in \mathbf{Z}$ then $V \subset U$. Let $x \in U$. We will show that an open neighbourhood of $x$ is contained in $U$ and that $H^i(E)$ is finite locally free on this neighbourhood for all $i$. This will finish the proof. During the proof we may (finitely many times) replace $X$ by an open neighbourhood of $x$. Hence we may assume $E$ is represented by a strictly perfect complex $\mathcal{E}^\bullet$. Say $\mathcal{E}^i = 0$ for $i \notin [a, b]$. We will prove the result by induction on $b - a$. The module $H^b(E) = \mathrm{Coker}(d^{b-1} : \mathcal{E}^{b-1} \to \mathcal{E}^b)$ is of finite presentation. Since $H^b(E)_x$ is finite free, we conclude $H^b(E)$ is finite free in an open neighbourhood of $x$ by Modules, Lemma 11.6. Thus after replacing $X$ by a (possibly smaller) open neighbourhood we may assume we have a direct sum decomposition $\mathcal{E}^b = \mathrm{Im}(d^{b-1}) \oplus H^b(E)$ and $H^b(E)$ is finite free, see Lemma 46.5. Doing the same argument again, we see that we may assume $\mathcal{E}^{b-1} = \mathrm{Ker}(d^{b-1}) \oplus \mathrm{Im}(d^{b-1})$. The complex $\mathcal{E}^a \to \ldots \to \mathcal{E}^{b-2} \to \mathrm{Ker}(d^{b-1})$ is a strictly perfect complex representing a perfect object $E'$ with $H^i(E) = H^i(E')$ for $i \neq b$. Hence we conclude by our induction hypothesis. $\square$

## 50. Duals

In this section we characterize the dualizable objects of the category of complexes and of the derived category. In particular, we will see that an object of $D(\mathcal{O}_X)$ has a dual if and only if it is perfect (this follows from Example 50.7 and Lemma 50.8).

**Lemma 50.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. The category of complexes of $\mathcal{O}_X$-modules with tensor product defined by $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet = \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)$ is a symmetric monoidal category (for sign rules, see More on Algebra, Section 72).*

**Proof.** Omitted. Hints: as unit $\mathbf{1}$ we take the complex having $\mathcal{O}_X$ in degree 0 and zero in other degrees with obvious isomorphisms $\mathrm{Tot}(\mathbf{1} \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet) = \mathcal{G}^\bullet$ and $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathbf{1}) = \mathcal{F}^\bullet$. to prove the lemma you have to check the commutativity of various diagrams, see Categories, Definitions 43.1 and 43.9. The verifications are straightforward in each case. $\square$

**Example 50.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a locally bounded complex of $\mathcal{O}_X$-modules such that each $\mathcal{F}^n$ is locally a direct summand of a finite free $\mathcal{O}_X$-module. In other words, there is an open covering $X = \bigcup U_i$ such that $\mathcal{F}^\bullet|_{U_i}$ is a strictly perfect complex. Consider the complex

$$\mathcal{G}^\bullet = \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{O}_X)$$

as in Section 41. Let

$$\eta : \mathcal{O}_X \to \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet) \quad \text{and} \quad \epsilon : \mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \to \mathcal{O}_X$$

be $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O}_X \to \mathcal{F}^n \otimes_{\mathcal{O}_X} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_{\mathcal{O}_X} \mathcal{F}^n \to \mathcal{O}_X$ are as in Modules, Example 18.1. Then $\mathcal{G}^\bullet, \eta, \epsilon$ is a left dual for $\mathcal{F}^\bullet$ as in Categories, Definition 43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{G}^\bullet}$. Please compare with More on Algebra, Lemma 72.2.

**Lemma 50.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. If $\mathcal{F}^\bullet$ has a left dual in the monoidal category of complexes of $\mathcal{O}_X$-modules (Categories, Definition 43.5) then $\mathcal{F}^\bullet$ is a locally bounded complex whose terms are locally direct summands of finite free $\mathcal{O}_X$-modules and the left dual is as constructed in Example 50.2.*

**Proof.** By uniqueness of left duals (Categories, Remark 43.7) we get the final statement provided we show that $\mathcal{F}^\bullet$ is as stated. Let $\mathcal{G}^\bullet, \eta, \epsilon$ be a left dual. Write $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O}_X \to \mathcal{F}^n \otimes_{\mathcal{O}_X} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_{\mathcal{O}_X} \mathcal{F}^n \to \mathcal{O}_X$. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{G}^\bullet}$ by Categories, Definition 43.5 we see immediately that we have $(1 \otimes \epsilon_n) \circ (\eta_n \otimes 1) = \mathrm{id}_{\mathcal{F}^n}$ and $(\epsilon_n \otimes 1) \circ (1 \otimes \eta_n) = \mathrm{id}_{\mathcal{G}^{-n}}$. Hence we see that $\mathcal{F}^n$ is locally a direct summand of a finite free $\mathcal{O}_X$-module by Modules, Lemma 18.2. Since the sum $\eta = \sum \eta_n$ is locally finite, we conclude that $\mathcal{F}^\bullet$ is locally bounded. $\square$

**Lemma 50.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L, M \in D(\mathcal{O}_X)$. If $K$ is perfect, then the map*

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

*of Lemma 42.9 is an isomorphism.*

**Proof.** Since the map is globally defined and since formation of the right and left hand side commute with localization (see Lemma 42.3), to prove this we may work

locally on $X$. Thus we may assume $K$ is represented by a strictly perfect complex $\mathcal{E}^\bullet$.

If $K_1 \to K_2 \to K_3$ is a distinguished triangle in $D(\mathcal{O}_X)$, then we get distinguished triangles

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_1 \to R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_2 \to R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_3$$

and

$$R\mathcal{H}om(R\mathcal{H}om(K_1, L), M) \to R\mathcal{H}om(R\mathcal{H}om(K_2, L), M) R\mathcal{H}om(R\mathcal{H}om(K_3, L), M)$$

See Section 26 and Lemma 42.4. The arrow of Lemma 42.9 is functorial in $K$ hence we get a morphism between these distinguished triangles. Thus, if the result holds for $K_1$ and $K_3$, then the result holds for $K_2$ by Derived Categories, Lemma 4.3.

Combining the remarks above with the distinguished triangles

$$\sigma_{\geq n}\mathcal{E}^\bullet \to \mathcal{E}^\bullet \to \sigma_{\leq n-1}\mathcal{E}^\bullet$$

of stupid trunctions, we reduce to the case where $K$ consists of a direct summand of a finite free $\mathcal{O}_X$-module placed in some degree. By an obvious compatibility of the problem with direct sums (similar to what was said above) and shifts this reduces us to the case where $K = \mathcal{O}_X^{\oplus n}$ for some integer $n$. This case is clear.  $\square$

**Lemma 50.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K$ be a perfect object of $D(\mathcal{O}_X)$. Then $K^\vee = R\mathcal{H}om(K, \mathcal{O}_X)$ is a perfect object too and $(K^\vee)^\vee \cong K$. There are functorial isomorphisms*

$$M \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee = R\mathcal{H}om(K, M)$$

*and*

$$H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee) = \operatorname{Hom}_{D(\mathcal{O}_X)}(K, M)$$

*for $M$ in $D(\mathcal{O}_X)$.*

**Proof.** By Lemma 42.9 there is a canonical map

$$K = R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, \mathcal{O}_X), \mathcal{O}_X) = (K^\vee)^\vee$$

which is an isomorphism by Lemma 50.4. To check the other statements we will use without further mention that formation of internal hom commutes with restriction to opens (Lemma 42.3). We may check $K^\vee$ is perfect locally on $X$. By Lemma 42.8 to see the final statement it suffices to check that the map (42.8.1)

$$M \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee \longrightarrow R\mathcal{H}om(K, M)$$

is an isomorphism. This is local on $X$ as well. Hence it suffices to prove these two statements $K$ is represented by a strictly perfect complex.

Assume $K$ is represented by the strictly perfect complex $\mathcal{E}^\bullet$. Then it follows from Lemma 46.9 that $K^\vee$ is represented by the complex whose terms are $(\mathcal{E}^{-n})^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-n}, \mathcal{O}_X)$ in degree $n$. Since $\mathcal{E}^{-n}$ is a direct summand of a finite free $\mathcal{O}_X$-module, so is $(\mathcal{E}^{-n})^\vee$. Hence $K^\vee$ is represented by a strictly perfect complex too and we see that $K^\vee$ is perfect. To see that (42.8.1) is an isomorphism, represent $M$ by a complex $\mathcal{F}^\bullet$. By Lemma 46.9 the complex $R\mathcal{H}om(K, M)$ is represented by the complex with terms

$$\bigoplus_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

On the other hand, the object $M \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee$ is represented by the complex with terms

$$\bigoplus_{n=p+q} \mathcal{F}^p \otimes_{\mathcal{O}_X} (\mathcal{E}^{-q})^\vee$$

Thus the assertion that (42.8.1) is an isomorphism reduces to the assertion that the canonical map

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$$

is an isomorphism when $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module and $\mathcal{F}$ is any $\mathcal{O}_X$-module. This follows immediately from the corresponding statement when $\mathcal{E}$ is finite free. $\square$

**Lemma 50.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. The derived category $D(\mathcal{O}_X)$ is a symmetric monoidal category with tensor product given by derived tensor product with usual associativity and commutativity constraints (for sign rules, see More on Algebra, Section 72).*

**Proof.** Omitted. Compare with Lemma 50.1. $\square$

**Example 50.7.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K$ be a perfect object of $D(\mathcal{O}_X)$. Set $K^\vee = R\mathcal{H}om(K, \mathcal{O}_X)$ as in Lemma 50.5. Then the map

$$K \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee \longrightarrow R\mathcal{H}om(K, K)$$

is an isomorphism (by the lemma). Denote

$$\eta : \mathcal{O}_X \longrightarrow K \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee$$

the map sending 1 to the section corresponding to $\mathrm{id}_K$ under the isomorphism above. Denote

$$\epsilon : K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} K \longrightarrow \mathcal{O}_X$$

the evaluation map (to construct it you can use Lemma 42.5 for example). Then $K^\vee, \eta, \epsilon$ is a left dual for $K$ as in Categories, Definition 43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_K$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{K^\vee}$.

**Lemma 50.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $M$ be an object of $D(\mathcal{O}_X)$. If $M$ has a left dual in the monoidal category $D(\mathcal{O}_X)$ (Categories, Definition 43.5) then $M$ is perfect and the left dual is as constructed in Example 50.7.*

**Proof.** Let $x \in X$. It suffices to find an open neighbourhood $U$ of $x$ such that $M$ restricts to a perfect complex over $U$. Hence during the proof we can (finitely often) replace $X$ by an open neighbourhood of $x$. Let $N, \eta, \epsilon$ be a left dual.

We are going to use the following argument several times. Choose any complex $\mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules representing $M$. Choose a K-flat complex $\mathcal{N}^\bullet$ representing $N$ whose terms are flat $\mathcal{O}_X$-modules, see Lemma 26.12. Consider the map

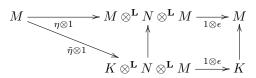$$\eta : \mathcal{O}_X \to \mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet)$$

After shrinking $X$ we can find an integer $N$ and for $i = 1, \ldots, N$ integers $n_i \in \mathbf{Z}$ and sections $f_i$ and $g_i$ of $\mathcal{M}^{n_i}$ and $\mathcal{N}^{-n_i}$ such that

$$\eta(1) = \sum_i f_i \otimes g_i$$

Let $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ be any subcomplex of $\mathcal{O}_X$-modules containing the sections $f_i$ for $i = 1, \ldots, N$. Since $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet) \subset \mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet)$ by flatness of the modules $\mathcal{N}^n$, we see that $\eta$ factors through

$$\tilde{\eta} : \mathcal{O}_X \to \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet)$$

Denoting $K$ the object of $D(\mathcal{O}_X)$ represented by $\mathcal{K}^\bullet$ we find a commutative diagram

$$\begin{array}{ccccc}
M & \xrightarrow{\eta \otimes 1} & M \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M & \xrightarrow{1 \otimes \epsilon} & M \\
& \searrow^{\tilde{\eta} \otimes 1} & \uparrow & & \uparrow \\
& & K \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M & \xrightarrow{1 \otimes \epsilon} & K
\end{array}$$

Since the composition of the upper row is the identity on $M$ we conclude that $M$ is a direct summand of $K$ in $D(\mathcal{O}_X)$.

As a first use of the argument above, we can choose the subcomplex $\mathcal{K}^\bullet = \sigma_{\geq a} \tau_{\leq b} \mathcal{M}^\bullet$ with $a < n_i < b$ for $i = 1, \ldots, N$. Thus $M$ is a direct summand in $D(\mathcal{O}_X)$ of a bounded complex and we conclude we may assume $M$ is in $D^b(\mathcal{O}_X)$. (Recall that the process above involves shrinking $X$.)

Since $M$ is in $D^b(\mathcal{O}_X)$ we may choose $\mathcal{M}^\bullet$ to be a bounded above complex of flat modules (by Modules, Lemma 17.6 and Derived Categories, Lemma 15.4). Then we can choose $\mathcal{K}^\bullet = \sigma_{\geq a} \mathcal{M}^\bullet$ with $a < n_i$ for $i = 1, \ldots, N$ in the argument above. Thus we find that we may assume $M$ is a direct summand in $D(\mathcal{O}_X)$ of a bounded complex of flat modules. In particular, $M$ has finite tor amplitude.

Say $M$ has tor amplitude in $[a, b]$. Assuming $M$ is $m$-pseudo-coherent we are going to show that (after shrinking $X$) we may assume $M$ is $(m-1)$-pseudo-coherent. This will finish the proof by Lemma 49.4 and the fact that $M$ is $(b+1)$-pseudo-coherent in any case. After shrinking $X$ we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to M$ in $D(\mathcal{O}_X)$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and surjective for $i = m$. We may and do assume that $\mathcal{E}^i = 0$ for $i < m$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to M \to L \to \mathcal{E}^\bullet[1]$$

Observe that $H^i(L) = 0$ for $i \geq m$. Thus we may represent $L$ by a complex $\mathcal{L}^\bullet$ with $\mathcal{L}^i = 0$ for $i \geq m$. The map $L \to \mathcal{E}^\bullet[1]$ is given by a map of complexes $\mathcal{L}^\bullet \to \mathcal{E}^\bullet[1]$ which is zero in all degrees except in degree $m-1$ where we obtain a map $\mathcal{L}^{m-1} \to \mathcal{E}^m$, see Derived Categories, Lemma 27.3. Then $M$ is represented by the complex
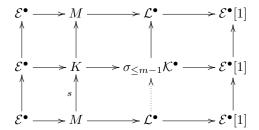
$$\mathcal{M}^\bullet : \ldots \to \mathcal{L}^{m-2} \to \mathcal{L}^{m-1} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots$$

Apply the discussion in the second paragraph to this complex to get sections $f_i$ of $\mathcal{M}^{n_i}$ for $i = 1, \ldots, N$. For $n < m$ let $\mathcal{K}^n \subset \mathcal{L}^n$ be the $\mathcal{O}_X$-submodule generated by the sections $f_i$ for $n_i = n$ and $d(f_i)$ for $n_i = n - 1$. For $n \geq m$ set $\mathcal{K}^n = \mathcal{E}^n$. Clearly, we have a morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathcal{E}^\bullet & \longrightarrow & \mathcal{M}^\bullet & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{E}^\bullet & \longrightarrow & \mathcal{K}^\bullet & \longrightarrow & \sigma_{\leq m-1} \mathcal{K}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1]
\end{array}$$

where all the morphisms are as indicated above. Denote $K$ the object of $D(\mathcal{O}_X)$ corresponding to the complex $\mathcal{K}^\bullet$. By the arguments in the second paragraph of the proof we obtain a morphism $s : M \to K$ in $D(\mathcal{O}_X)$ such that the composition $M \to K \to M$ is the identity on $M$. We don't know that the diagram

$$
\begin{array}{ccccc}
\mathcal{E}^\bullet & \longrightarrow & \mathcal{K}^\bullet & = & K \\
\text{id} \uparrow & & & & \uparrow s \\
\mathcal{E}^\bullet & \overset{i}{\longrightarrow} & \mathcal{M}^\bullet & = & M
\end{array}
$$

commutes, but we do know it commutes after composing with the map $K \to M$. By Lemma 46.8 after shrinking $X$ we may assume that $s \circ i$ is given by a map of complexes $\sigma : \mathcal{E}^\bullet \to \mathcal{K}^\bullet$. By the same lemma we may assume the composition of $\sigma$ with the inclusion $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ is homotopic to zero by some homotopy $\{h^i : \mathcal{E}^i \to \mathcal{M}^{i-1}\}$. Thus, after replacing $\mathcal{K}^{m-1}$ by $\mathcal{K}^{m-1} + \mathrm{Im}(h^m)$ (note that after doing this it is still the case that $\mathcal{K}^{m-1}$ is generated by finitely many global sections), we see that $\sigma$ itself is homotopic to zero! This means that we have a commutative solid diagram

$$
\begin{array}{ccccccc}
\mathcal{E}^\bullet & \longrightarrow & M & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{E}^\bullet & \longrightarrow & K & \longrightarrow & \sigma_{\leq m-1}\mathcal{K}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1] \\
\uparrow & & \uparrow {\scriptstyle s} & & \uparrow & & \uparrow \\
\mathcal{E}^\bullet & \longrightarrow & M & \longrightarrow & \mathcal{L}^\bullet & \longrightarrow & \mathcal{E}^\bullet[1]
\end{array}
$$

By the axioms of triangulated categories we obtain a dotted arrow fitting into the diagram. Looking at cohomology sheaves in degree $m-1$ we see that we obtain

$$
\begin{array}{ccccc}
H^{m-1}(M) & \longrightarrow & H^{m-1}(\mathcal{L}^\bullet) & \longrightarrow & H^m(\mathcal{E}^\bullet) \\
\uparrow & & \uparrow & & \uparrow \\
H^{m-1}(K) & \longrightarrow & H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) & \longrightarrow & H^m(\mathcal{E}^\bullet) \\
\uparrow & & \uparrow & & \uparrow \\
H^{m-1}(M) & \longrightarrow & H^{m-1}(\mathcal{L}^\bullet) & \longrightarrow & H^m(\mathcal{E}^\bullet)
\end{array}
$$

Since the vertical compositions are the identity in both the left and right column, we conclude the vertical composition $H^{m-1}(\mathcal{L}^\bullet) \to H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ in the middle is surjective! In particular $H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ is surjective. Using the induced map of long exact sequences of cohomology sheaves from the morphism of triangles above, a diagram chase shows this implies $H^i(K) \to H^i(M)$ is an isomorphism for $i \geq m$ and surjective for $i = m-1$. By construction we can choose an $r \geq 0$ and a surjection $\mathcal{O}_X^{\oplus r} \to \mathcal{K}^{m-1}$. Then the composition

$$(\mathcal{O}_X^{\oplus r} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots) \longrightarrow K \longrightarrow M$$

induces an isomorphism on cohomology sheaves in degrees $\geq m$ and a surjection in degree $m-1$ and the proof is complete.          $\square$

## 51. Miscellany

Some results which do not fit anywhere else.

**Lemma 51.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(K_n)_{n \in \mathbf{N}}$ be a system of perfect objects of $D(\mathcal{O}_X)$. Let $K = \mathrm{hocolim} K_n$ be the derived colimit (Derived Categories, Definition 33.1). Then for any object $E$ of $D(\mathcal{O}_X)$ we have*

$$R\mathcal{H}om(K, E) = R \lim E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_n^{\vee}$$

*where $(K_n^{\vee})$ is the inverse system of dual perfect complexes.*

**Proof.** By Lemma 50.5 we have $R \lim E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_n^{\vee} = R \lim R\mathcal{H}om(K_n, E)$ which fits into the distinguished triangle

$$R \lim R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E)$$

Because $K$ similarly fits into the distinguished triangle $\bigoplus K_n \to \bigoplus K_n \to K$ it suffices to show that $\prod R\mathcal{H}om(K_n, E) = R\mathcal{H}om(\bigoplus K_n, E)$. This is a formal consequence of (42.0.1) and the fact that derived tensor product commutes with direct sums. $\square$

**Lemma 51.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K$ and $E$ be objects of $D(\mathcal{O}_X)$ with $E$ perfect. The diagram*

$$
\begin{array}{ccc}
H^0(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} E^{\vee}) \times H^0(X, E) & \longrightarrow & H^0(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} E^{\vee} \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_X(E, K) \times H^0(X, E) & \longrightarrow & H^0(X, K)
\end{array}
$$

*commutes where the top horizontal arrow is the cup product, the right vertical arrow uses $\epsilon : E^{\vee} \otimes^{\mathbf{L}}_{\mathcal{O}_X} E \to \mathcal{O}_X$ (Example 50.7), the left vertical arrow uses Lemma 50.5, and the bottom horizontal arrow is the obvious one.*

**Proof.** We will abbreviate $\otimes = \otimes^{\mathbf{L}}_{\mathcal{O}_X}$ and $\mathcal{O} = \mathcal{O}_X$. We will identify $E$ and $K$ with $R\mathcal{H}om(\mathcal{O}, E)$ and $R\mathcal{H}om(\mathcal{O}, K)$ and we will identify $E^{\vee}$ with $R\mathcal{H}om(E, \mathcal{O})$.

Let $\xi \in H^0(X, K \otimes E^{\vee})$ and $\eta \in H^0(X, E)$. Denote $\tilde{\xi} : \mathcal{O} \to K \otimes E^{\vee}$ and $\tilde{\eta} : \mathcal{O} \to E$ the corresponding maps in $D(\mathcal{O})$. By Lemma 31.1 the cup product $\xi \cup \eta$ corresponds to $\tilde{\xi} \otimes \tilde{\eta} : \mathcal{O} \to K \otimes E^{\vee} \otimes E$.

We claim the map $\xi' : E \to K$ corresponding to $\xi$ by Lemma 50.5 is the composition

$$E = \mathcal{O} \otimes E \xrightarrow{\tilde{\xi} \otimes 1_E} K \otimes E^{\vee} \otimes E \xrightarrow{1_K \otimes \epsilon} K$$

The construction in Lemma 50.5 uses the evaluation map (42.8.1) which in turn is constructed using the identification of $E$ with $R\mathcal{H}om(\mathcal{O}, E)$ and the composition $\underline{\circ}$ constructed in Lemma 42.5. Hence $\xi'$ is the composition

$$E = \mathcal{O} \otimes R\mathcal{H}om(\mathcal{O}, E) \xrightarrow{\tilde{\xi} \otimes 1} R\mathcal{H}om(\mathcal{O}, K) \otimes R\mathcal{H}om(E, \mathcal{O}) \otimes R\mathcal{H}om(\mathcal{O}, E)$$

$$\xrightarrow{\underline{\circ} \otimes 1} R\mathcal{H}om(E, K) \otimes R\mathcal{H}om(\mathcal{O}, E)$$

$$\xrightarrow{\underline{\circ}} R\mathcal{H}om(\mathcal{O}, K) = K$$

The claim follows immediately from this and the fact that the composition $\underline{\circ}$ constructed in Lemma 42.5 is associative (insert future reference here) and the fact that $\epsilon$ is defined as the composition $\underline{\circ} : E^{\vee} \otimes E \to \mathcal{O}$ in Example 50.7.

Using the results from the previous two paragraphs, we find the statement of the lemma is that $(1_K \otimes \epsilon) \circ (\tilde{\xi} \otimes \tilde{\eta})$ is equal to $(1_K \otimes \epsilon) \circ (\tilde{\xi} \otimes 1_E) \circ (1_{\mathcal{O}} \otimes \tilde{\eta})$ which is immediate.                                                                            $\square$

**Lemma 51.3.** *Let* $h : X \to Y$ *be a morphism of ringed spaces. Let* $K, M$ *be objects of* $D(\mathcal{O}_Y)$. *The canonical map*

$$Lh^* R\mathcal{H}om(K, M) \longrightarrow R\mathcal{H}om(Lh^*K, Lh^*M)$$

*of Remark 42.13 is an isomorphism in the following cases*

(1) $K$ *is perfect,*
(2) $h$ *is flat,* $K$ *is pseudo-coherent, and* $M$ *is (locally) bounded below,*
(3) $\mathcal{O}_X$ *has finite tor dimension over* $h^{-1}\mathcal{O}_Y$, $K$ *is pseudo-coherent, and* $M$ *is (locally) bounded below,*

**Proof.** Proof of (1). The question is local on $Y$, hence we may assume that $K$ is represented by a strictly perfect complex $\mathcal{E}^\bullet$, see Section 49. Choose a K-flat complex $\mathcal{F}^\bullet$ representing $M$. Apply Lemma 46.9 to see that $R\mathcal{H}om(K, L)$ is represented by the complex $\mathcal{H}^\bullet = \mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ with terms $\mathcal{H}^n = \bigoplus_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$. By the construction of $Lh^*$ in Section 27 we see that $Lh^*K$ is represented by the strictly perfect complex $h^*\mathcal{E}^\bullet$ (Lemma 46.4). Similarly, the object $Lh^*M$ is represented by the complex $h^*\mathcal{F}^\bullet$. Finally, the object $Lh^*R\mathcal{H}om(K, M)$ is represented by $h^*\mathcal{H}^\bullet$ as $\mathcal{H}^\bullet$ is K-flat by Lemma 46.10. Thus to finish the proof it suffices to show that $h^*\mathcal{H}^\bullet = \mathcal{H}om^\bullet(h^*\mathcal{E}^\bullet, h^*\mathcal{F}^\bullet)$. For this it suffices to note that $h^*\mathcal{H}om(\mathcal{E}, \mathcal{F}) = \mathcal{H}om(h^*\mathcal{E}, \mathcal{F})$ whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module.

Proof of (2). Since $h$ is flat, we can compute $Lh^*$ by simply using $h^*$ on any complex of $\mathcal{O}_Y$-modules. In particular we have $H^i(Lh^*K) = h^*H^i(K)$ for all $i \in \mathbf{Z}$. Say $H^i(M) = 0$ for $i < a$. Let $K' \to K$ be a morphism of $D(\mathcal{O}_Y)$ which defines an isomorphism $H^i(K') \to H^i(K)$ for all $i \geq b$. Then the corresponding maps

$$R\mathcal{H}om(K, M) \to R\mathcal{H}om(K', M)$$

and

$$R\mathcal{H}om(Lh^*K, Lh^*M) \to R\mathcal{H}om(Lh^*K', Lh^*M)$$

are isomorphisms on cohomology sheaves in degrees $< a - b$ (details omitted). Thus to prove the map in the statement of the lemma induces an isomorphism on cohomology sheaves in degrees $< a - b$ it suffices to prove the result for $K'$ in those degrees. Also, as in the proof of part (1) the question is local on $Y$. Thus we may assume $K$ is represented by a strictly perfect complex, see Section 47. This reduces us to case (1).

Proof of (3). The proof is the same as the proof of (2) except one uses that $Lh^*$ has bounded cohomological dimension to get the desired vanishing. We omit the details.                                                                                            $\square$

**Lemma 51.4.** *Let* $X$ *be a ringed space. Let* $K, M$ *be objects of* $D(\mathcal{O}_X)$. *Let* $x \in X$. *The canonical map*

$$R\mathcal{H}om(K, M)_x \longrightarrow R\mathrm{Hom}_{\mathcal{O}_{X,x}}(K_x, M_x)$$

*is an isomorphism in the following cases*

(1) $K$ *is perfect,*
(2) $K$ *is pseudo-coherent and* $M$ *is (locally) bounded below.*

**Proof.** Let $Y = \{x\}$ be the singleton ringed space with structure sheaf given by $\mathcal{O}_{X,x}$. Then apply Lemma 51.3 to the flat inclusion morphism $Y \to X$. $\square$

## 52. Invertible objects in the derived category

We characterize invertible objects in the derived category of a ringed space (both in the case where the stalks of the structure sheaf are local and where not).

**Lemma 52.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. The category of $\mathcal{O}_X$-modules which are summands of finite free $\mathcal{O}_X$-modules is equivalent to the category of finite projective $R$-modules.*

**Proof.** Observe that a finite projective $R$-module is the same thing as a summand of a finite free $R$-module. The equivalence is given by the functor $\mathcal{E} \mapsto \Gamma(X, \mathcal{E})$. The inverse functor is given by the construction of Modules, Lemma 10.5. $\square$

**Lemma 52.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $M$ be an object of $D(\mathcal{O}_X)$. The following are equivalent*

(1) *$M$ is invertible in $D(\mathcal{O}_X)$, see Categories, Definition 43.4, and*
(2) *there is a locally finite direct product decomposition*

$$\mathcal{O}_X = \prod\nolimits_{n \in \mathbf{Z}} \mathcal{O}_n$$

*and for each $n$ there is an invertible $\mathcal{O}_n$-module $\mathcal{H}^n$ (Modules, Definition 25.1) and $M = \bigoplus \mathcal{H}^n[-n]$ in $D(\mathcal{O}_X)$.*

*If (1) and (2) hold, then $M$ is a perfect object of $D(\mathcal{O}_X)$. If $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$ these condition are also equivalent to*

(3) *there exists an open covering $X = \bigcup U_i$ and for each $i$ an integer $n_i$ such that $M|_{U_i}$ is represented by an invertible $\mathcal{O}_{U_i}$-module placed in degree $n_i$.*

**Proof.** Assume (2). Consider the object $R\,\mathcal{H}om(M, \mathcal{O}_X)$ and the composition map

$$R\,\mathcal{H}om(M, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \to \mathcal{O}_X$$

To prove this is an isomorphism, we may work locally. Thus we may assume $\mathcal{O}_X = \prod_{a \leq n \leq b} \mathcal{O}_n$ and $M = \bigoplus_{a \leq n \leq b} \mathcal{H}^n[-n]$. Then it suffices to show that

$$R\,\mathcal{H}om(\mathcal{H}^m, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{H}^n$$

is zero if $n \neq m$ and equal to $\mathcal{O}_n$ if $n = m$. The case $n \neq m$ follows from the fact that $\mathcal{O}_n$ and $\mathcal{O}_m$ are flat $\mathcal{O}_X$-algebras with $\mathcal{O}_n \otimes_{\mathcal{O}_X} \mathcal{O}_m = 0$. Using the local structure of invertible $\mathcal{O}_X$-modules (Modules, Lemma 25.2) and working locally the isomorphism in case $n = m$ follows in a straightforward manner; we omit the details. Because $D(\mathcal{O}_X)$ is symmetric monoidal, we conclude that $M$ is invertible.

Assume (1). The description in (2) shows that we have a candidate for $\mathcal{O}_n$, namely, $\mathcal{H}om_{\mathcal{O}_X}(H^n(M), H^n(M))$. If this is a locally finite family of sheaves of rings and if $\mathcal{O}_X = \prod \mathcal{O}_n$, then we immediately obtain the direct sum decomposition $M = \bigoplus H^n(M)[-n]$ using the idempotents in $\mathcal{O}_X$ coming from the product decomposition. This shows that in order to prove (2) we may work locally on $X$.

Choose an object $N$ of $D(\mathcal{O}_X)$ and an isomorphism $M \otimes^{\mathbf{L}}_{\mathcal{O}_X} N \cong \mathcal{O}_X$. Let $x \in X$. Then $N$ is a left dual for $M$ in the monoidal category $D(\mathcal{O}_X)$ and we conclude that $M$ is perfect by Lemma 50.8. By symmetry we see that $N$ is perfect. After replacing $X$ by an open neighbourhood of $x$, we may assume $M$ and $N$ are represented

by a strictly perfect complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$. Then $M \otimes^{\mathbf{L}}_{\mathcal{O}_X} N$ is represented by $\mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)$. After another shinking of $X$ we may assume the mutually inverse isomorphisms $\mathcal{O}_X \to M \otimes^{\mathbf{L}}_{\mathcal{O}_X} N$ and $M \otimes^{\mathbf{L}}_{\mathcal{O}_X} N \to \mathcal{O}_X$ are given by maps of complexes

$$\alpha : \mathcal{O}_X \to \mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \quad \text{and} \quad \beta : \mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \to \mathcal{O}_X$$

See Lemma 46.8. Then $\beta \circ \alpha = 1$ as maps of complexes and $\alpha \circ \beta = 1$ as a morphism in $D(\mathcal{O}_X)$. After shrinking $X$ we may assume the composition $\alpha \circ \beta$ is homotopic to 1 by some homotopy $\theta$ with components

$$\theta^n : \mathrm{Tot}^n(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \to \mathrm{Tot}^{n-1}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)$$

by the same lemma as before. Set $R = \Gamma(X, \mathcal{O}_X)$. By Lemma 52.1 we find that we obtain

(1) $M^\bullet = \Gamma(X, \mathcal{E}^\bullet)$ is a bounded complex of finite projective $R$-modules,
(2) $N^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ is a bounded complex of finite projective $R$-modules,
(3) $\alpha$ and $\beta$ correspond to maps of complexes $a : R \to \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$ and $b : \mathrm{Tot}(M^\bullet \otimes_R N^\bullet) \to R$,
(4) $\theta^n$ corresponds to a map $h^n : \mathrm{Tot}^n(M^\bullet \otimes_R N^\bullet) \to \mathrm{Tot}^{n-1}(M^\bullet \otimes_R N^\bullet)$, and
(5) $b \circ a = 1$ and $b \circ a - 1 = dh + hd$,

It follows that $M^\bullet$ and $N^\bullet$ define mutually inverse objects of $D(R)$. By More on Algebra, Lemma 126.4 we find a product decomposition $R = \prod_{a \le n \le b} R_n$ and invertible $R_n$-modules $H^n$ such that $M^\bullet \cong \bigoplus_{a \le n \le b} H^n[-n]$. This isomorphism in $D(R)$ can be lifted to an morphism

$$\bigoplus H^n[-n] \longrightarrow M^\bullet$$

of complexes because each $H^n$ is projective as an $R$-module. Correspondingly, using Lemma 52.1 again, we obtain an morphism

$$\bigoplus H^n \otimes_R \mathcal{O}_X[-n] \to \mathcal{E}^\bullet$$

which is an isomorphism in $D(\mathcal{O}_X)$. Setting $\mathcal{O}_n = R_n \otimes_R \mathcal{O}_X$ we conclude (2) is true.

If all stalks of $\mathcal{O}_X$ are local, then it is straightforward to prove the equivalence of (2) and (3). We omit the details. $\square$

## 53. Compact objects

In this section we study compact objects in the derived category of modules on a ringed space. We recall that compact objects are defined in Derived Categories, Definition 37.1. On suitable ringed spaces the perfect objects are compact.

**Lemma 53.1.** *Let $X$ be a ringed space. Let $j : U \to X$ be the inclusion of an open. The $\mathcal{O}_X$-module $j_! \mathcal{O}_U$ is a compact object of $D(\mathcal{O}_X)$ if there exists an integer $d$ such that*

(1) *$H^p(U, \mathcal{F}) = 0$ for all $p > d$, and*
(2) *the functors $\mathcal{F} \mapsto H^p(U, \mathcal{F})$ commute with direct sums.*

**Proof.** Assume (1) and (2). Since $\mathrm{Hom}(j_! \mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$ by Sheaves, Lemma 31.8 we have $\mathrm{Hom}(j_! \mathcal{O}_U, K) = R\Gamma(U, K)$ for $K$ in $D(\mathcal{O}_X)$. Thus we have to show that $R\Gamma(U, -)$ commutes with direct sums. The first assumption means that the functor $F = H^0(U, -)$ has finite cohomological dimension. Moreover, the second

assumption implies any direct sum of injective modules is acyclic for $F$. Let $K_i$ be a family of objects of $D(\mathcal{O}_X)$. Choose K-injective representatives $I_i^\bullet$ with injective terms representing $K_i$, see Injectives, Theorem 12.6. Since we may compute $RF$ by applying $F$ to any complex of acyclics (Derived Categories, Lemma 32.2) and since $\bigoplus K_i$ is represented by $\bigoplus I_i^\bullet$ (Injectives, Lemma 13.4) we conclude that $R\Gamma(U, \bigoplus K_i)$ is represented by $\bigoplus H^0(U, I_i^\bullet)$. Hence $R\Gamma(U, -)$ commutes with direct sums as desired. $\qquad\square$

**Lemma 53.2.** *Let $X$ be a ringed space. Assume that the underlying topological space of $X$ has the following properties:*

    (1) *$X$ is quasi-compact,*
    (2) *there exists a basis of quasi-compact open subsets, and*
    (3) *the intersection of any two quasi-compact opens is quasi-compact.*

*Let $K$ be a perfect object of $D(\mathcal{O}_X)$. Then*

    (a) *$K$ is a compact object of $D^+(\mathcal{O}_X)$ in the following sense: if $M = \bigoplus_{i \in I} M_i$ is bounded below, then $\mathrm{Hom}(K, M) = \bigoplus_{i \in I} \mathrm{Hom}(K, M_i)$.*
    (b) *If $X$ has finite cohomological dimension, i.e., if there exists a $d$ such that $H^i(X, \mathcal{F}) = 0$ for $i > d$, then $K$ is a compact object of $D(\mathcal{O}_X)$.*

**Proof.** Let $K^\vee$ be the dual of $K$, see Lemma 50.5. Then we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in $M$ in $D(\mathcal{O}_X)$. Since $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums it suffices to show that $R\Gamma(X, -)$ commutes with the relevant direct sums.

Proof of (b). Since $R\Gamma(X, K) = R\mathrm{Hom}(\mathcal{O}_X, K)$ and since $H^p(X, -)$ commutes with direct sums by Lemma 19.1 this is a special case of Lemma 53.1

Proof of (a). Let $\mathcal{I}_i$, $i \in I$ be a collection of injective $\mathcal{O}_X$-modules. By Lemma 19.1 we see that

$$H^p(X, \bigoplus_{i \in I} \mathcal{I}_i) = \bigoplus_{i \in I} H^p(X, \mathcal{I}_i) = 0$$

for all $p$. Now if $M = \bigoplus M_i$ is as in (a), then we see that there exists an $a \in \mathbf{Z}$ such that $H^n(M_i) = 0$ for $n < a$. Thus we can choose complexes of injective $\mathcal{O}_X$-modules $\mathcal{I}_i^\bullet$ representing $M_i$ with $\mathcal{I}_i^n = 0$ for $n < a$, see Derived Categories, Lemma 18.3. By Injectives, Lemma 13.4 we see that the direct sum complex $\bigoplus \mathcal{I}_i^\bullet$ represents $M$. By Leray acyclicity (Derived Categories, Lemma 16.7) we see that

$$R\Gamma(X, M) = \Gamma(X, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus \Gamma(X, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus R\Gamma(X, M_i)$$

as desired. $\qquad\square$

## 54. Projection formula

In this section we collect variants of the projection formula. The most basic version is Lemma 54.2. After we state and prove it, we discuss a more general version involving perfect complexes.

**Lemma 54.1.** *Let $X$ be a ringed space. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{E}$ is finite locally free on $X$, see Modules, Definition 14.1. Then $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}$ is an injective $\mathcal{O}_X$-module.*

**Proof.** This is true because under the assumptions of the lemma we have

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}) = \text{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee, \mathcal{I})$$

where $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual of $\mathcal{E}$ which is finite locally free also. Since tensoring with a finite locally free sheaf is an exact functor we win by Homology, Lemma 27.2. $\square$

**Lemma 54.2.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Assume $\mathcal{E}$ is finite locally free on $Y$, see Modules, Definition 14.1. Then there exist isomorphisms*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} R^q f_* \mathcal{F} \longrightarrow R^q f_*(f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

*for all $q \geq 0$. In fact there exists an isomorphism*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} Rf_* \mathcal{F} \longrightarrow Rf_*(f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

*in $D^+(Y)$ functorial in $\mathcal{F}$.*

**Proof.** Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ on $X$. Note that $f^* \mathcal{E}$ is finite locally free also, hence we get a resolution

$$f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet$$

which is an injective resolution by Lemma 54.1. Apply $f_*$ to see that

$$Rf_*(f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f_*(f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet).$$

Hence the lemma follows if we can show that $f_*(f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \mathcal{E} \otimes_{\mathcal{O}_Y} f_*(\mathcal{F})$ functorially in the $\mathcal{O}_X$-module $\mathcal{F}$. This is clear when $\mathcal{E} = \mathcal{O}_Y^{\oplus n}$, and follows in general by working locally on $Y$. Details omitted. $\square$

Let $f : X \to Y$ be a morphism of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. Without any further assumptions there is a map

(54.2.1) $$Rf_* E \otimes_{\mathcal{O}_Y}^{\mathbf{L}} K \longrightarrow Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* K)$$

Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_* E \otimes_{\mathcal{O}_Y}^{\mathbf{L}} K) = Lf^* Rf_* E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* K \longrightarrow E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* K$$

coming from the map $Lf^* Rf_* E \to E$ and Lemmas 27.3 and 28.1. A reasonably general version of the projection formula is the following.

**Lemma 54.3.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. If $K$ is perfect, then*

$$Rf_* E \otimes_{\mathcal{O}_Y}^{\mathbf{L}} K = Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* K)$$

*in $D(\mathcal{O}_Y)$.*

**Proof.** To check (54.2.1) is an isomorphism we may work locally on $Y$, i.e., we have to find a covering $\{V_j \to Y\}$ such that the map restricts to an isomorphism on $V_j$. By definition of perfect objects, this means we may assume $K$ is represented by a strictly perfect complex of $\mathcal{O}_Y$-modules. Note that, completely generally, the statement is true for $K = K_1 \oplus K_2$, if and only if the statement is true for $K_1$ and $K_2$. Hence we may assume $K$ is a finite complex of finite free $\mathcal{O}_Y$-modules. In this case a simple argument involving stupid truncations reduces the statement to the case where $K$ is represented by a finite free $\mathcal{O}_Y$-module. Since the statement is invariant under finite direct summands in the $K$ variable, we conclude it suffices to prove it for $K = \mathcal{O}_Y[n]$ in which case it is trivial. $\square$

Here is a case where the projection formula is true in complete generality.

**Lemma 54.4.** *Let $f : X \to Y$ be a morphism of ringed spaces such that $f$ is a homeomorphism onto a closed subset. Then (54.2.1) is an isomorphism always.*

**Proof.** Since $f$ is a homeomorphism onto a closed subset, the functor $f_*$ is exact (Modules, Lemma 6.1). Hence $Rf_*$ is computed by applying $f_*$ to any representative complex. Choose a K-flat complex $\mathcal{K}^\bullet$ of $\mathcal{O}_Y$-modules representing $K$ and choose any complex $\mathcal{E}^\bullet$ of $\mathcal{O}_X$-modules representing $E$. Then $Lf^*K$ is represented by $f^*\mathcal{K}^\bullet$ which is a K-flat complex of $\mathcal{O}_X$-modules (Lemma 26.8). Thus the right hand side of (54.2.1) is represented by

$$f_*\mathrm{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{K}^\bullet)$$

By the same reasoning we see that the left hand side is represented by

$$\mathrm{Tot}(f_*\mathcal{E}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{K}^\bullet)$$

Since $f_*$ commutes with direct sums (Modules, Lemma 6.3) it suffices to show that

$$f_*(\mathcal{E} \otimes_{\mathcal{O}_X} f^*\mathcal{K}) = f_*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{K}$$

for any $\mathcal{O}_X$-module $\mathcal{E}$ and $\mathcal{O}_Y$-module $\mathcal{K}$. We will check this by checking on stalks. Let $y \in Y$. If $y \notin f(X)$, then the stalks of both sides are zero. If $y = f(x)$, then we see that we have to show

$$\mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{F}_y) = \mathcal{E}_x \otimes_{\mathcal{O}_{Y,y}} \mathcal{F}_y$$

(using Sheaves, Lemma 32.1 and Lemma 26.4). This equality holds and therefore the lemma has been proved. $\square$

**Remark 54.5.** The map (54.2.1) is compatible with the base change map of Remark 28.3 in the following sense. Namely, suppose that

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a commutative diagram of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. Then the diagram

$$\begin{array}{ccc}
Lg^*(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K) & \xrightarrow{\ p\ } & Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K) \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle b} \\
Lg^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} Lg^*K & & Rf'_*L(g')^*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K) \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle t} \\
Rf'_*L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} Lg^*K & & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} L(g')^*Lf^*K) \\
& \searrow{\scriptstyle p} & \downarrow{\scriptstyle c} \\
& & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} L(f')^*Lg^*K)
\end{array}$$

is commutative. Here arrows labeled $t$ are gotten by an application of Lemma 27.3, arrows labeled $b$ by an application of Remark 28.3, arrows labeled $p$ by an

application of (54.2.1), and $c$ comes from $L(g')^* \circ Lf^* = L(f')^* \circ Lg^*$. We omit the verification.

## 55. An operator introduced by Berthelot and Ogus

This section continuous the discussion started in More on Algebra, Section 95. We strongly encourage the reader to read that section first.

**Lemma 55.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals. Consider the following two conditions*

(1) *for every $x \in X$ there exists an open neighbourhood $U \subset X$ of $x$ and $f \in \mathcal{I}(U)$ such that $\mathcal{I}|_U = \mathcal{O}_U \cdot f$ and $f : \mathcal{O}_U \to \mathcal{O}_U$ is injective, and*
(2) *$\mathcal{I}$ is invertible as an $\mathcal{O}_X$-module.*

*Then (1) implies (2) and the converse is true if all stalks $\mathcal{O}_{X,x}$ of the structure sheaf are local rings.*

**Proof.** Omitted. Hint: Use Modules, Lemma 25.4. $\qquad\square$

**Situation 55.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals satisfying condition (1) of Lemma 55.1[8].

**Lemma 55.3.** *In Situation 55.2 let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

(1) *the subsheaf $\mathcal{F}[\mathcal{I}] \subset \mathcal{F}$ of sections annihilated by $\mathcal{I}$ is zero,*
(2) *the subsheaf $\mathcal{F}[\mathcal{I}^n]$ is zero for all $n \geq 1$,*
(3) *the multiplication map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ is injective,*
(4) *for every open $U \subset X$ such that $\mathcal{I}|_U = \mathcal{O}_U \cdot f$ for some $f \in \mathcal{I}(U)$ the map $f : \mathcal{F}|_U \to \mathcal{F}|_U$ is injective,*
(5) *for every $x \in X$ and generator $f$ of the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ the element $f$ is a nonzerodivisor on the stalk $\mathcal{F}_x$.*

**Proof.** Omitted. $\qquad\square$

In Situation 55.2 let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If the equivalent conditions of Lemma 55.3 hold, then we will say that $\mathcal{F}$ is *$\mathcal{I}$-torsion free*. If so, then for any $i \in \mathbf{Z}$ we will denote

$$\mathcal{I}^i \mathcal{F} = \mathcal{I}^{\otimes i} \otimes_{\mathcal{O}_X} \mathcal{F}$$

so that we have inclusions

$$\ldots \subset \mathcal{I}^{i+1} \mathcal{F} \subset \mathcal{I}^i \mathcal{F} \subset \mathcal{I}^{i-1} \mathcal{F} \subset \ldots$$

The modules $\mathcal{I}^i \mathcal{F}$ are locally isomorphic to $\mathcal{F}$ as $\mathcal{O}_X$-modules, but not globally.

Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules with differentials $d^i : \mathcal{F}^i \to \mathcal{F}^{i+1}$. In this case we define $\eta_\mathcal{I} \mathcal{F}^\bullet$ to be the complex with terms

$$(\eta_\mathcal{I} \mathcal{F})^i = \mathrm{Ker}\left( d^i, -1 : \mathcal{I}^i \mathcal{F}^i \oplus \mathcal{I}^{i+1} \mathcal{F}^{i+1} \to \mathcal{I}^i \mathcal{F}^{i+1} \right)$$

$$= \mathrm{Ker}\left( d^i : \mathcal{I}^i \mathcal{F}^i \to \mathcal{I}^i \mathcal{F}^{i+1} / \mathcal{I}^{i+1} \mathcal{F}^{i+1} \right)$$

and differential induced by $d^i$. In other words, a local section $s$ of $(\eta_\mathcal{I} \mathcal{F})^i$ is the same thing as a local section $s$ of $\mathcal{I}^i \mathcal{F}^i$ such that its image $d^i(s)$ in $\mathcal{I}^i \mathcal{F}^{i+1}$ is in

---

[8]The discussion in this section can be generalized to the case where all we require is that $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module as defined in Modules, Section 25.

the subsheaf $\mathcal{I}^{i+1}\mathcal{F}^{i+1}$. Observe that $\eta_{\mathcal{I}}\mathcal{F}^{\bullet}$ is another complex of $\mathcal{I}$-torsion free modules.

Let $a^{\bullet} : \mathcal{F}^{\bullet} \to \mathcal{G}^{\bullet}$ be a map of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. Then we obtain a map of complexes

$$\eta_{\mathcal{I}}a^{\bullet} : \eta_{\mathcal{I}}\mathcal{F}^{\bullet} \longrightarrow \eta_{\mathcal{I}}\mathcal{G}^{\bullet}$$

induced by the maps $\mathcal{I}^i\mathcal{F}^i \to \mathcal{I}^i\mathcal{G}^i$. The reader checks that we obtain an endofunctor on the category of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules.

If $a^{\bullet}, b^{\bullet} : \mathcal{F}^{\bullet} \to \mathcal{G}^{\bullet}$ are two maps of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules and $h = \{h^i : \mathcal{F}^i \to \mathcal{G}^{i-1}\}$ is a homotopy between $a^{\bullet}$ and $b^{\bullet}$, then we define $\eta_{\mathcal{I}}h$ to be the family of maps $(\eta_{\mathcal{I}}h)^i : (\eta_{\mathcal{I}}\mathcal{F})^i \to (\eta_{\mathcal{I}}\mathcal{G})^{i-1}$ which sends $x$ to $h^i(x)$; this makes sense as $x$ a local section of $\mathcal{I}^i\mathcal{F}^i$ implies $h^i(x)$ is a local section of $\mathcal{I}^i\mathcal{G}^{i-1}$ which is certainly contained in $(\eta_{\mathcal{I}}\mathcal{G})^{i-1}$. The reader checks that $\eta_{\mathcal{I}}h$ is a homotopy between $\eta_{\mathcal{I}}a^{\bullet}$ and $\eta_{\mathcal{I}}b^{\bullet}$. All in all we see that we obtain a functor

$$\eta_f : K(\mathcal{I}\text{-torsion free } \mathcal{O}_X\text{-modules}) \longrightarrow K(\mathcal{I}\text{-torsion free } \mathcal{O}_X\text{-modules})$$

on the homotopy category (Derived Categories, Section 8) of the additive category of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. There is no sense in which $\eta_{\mathcal{I}}$ is an exact functor of triangulated categories; compare with More on Algebra, Example 95.1.

**Lemma 55.4.** *In Situation 55.2 let $\mathcal{F}^{\bullet}$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. For $x \in X$ choose a generator $f \in \mathcal{I}_x$. Then the stalk $(\eta_{\mathcal{I}}\mathcal{F}^{\bullet})_x$ is canonically isomorphic to the complex $\eta_f\mathcal{F}_x^{\bullet}$ constructed in More on Algebra, Section 95.*

**Proof.** Omitted. $\square$

**Lemma 55.5.** *In Situation 55.2 let $\mathcal{F}^{\bullet}$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. There is a canonical isomorphism*

$$\mathcal{I}^{\otimes i} \otimes_{\mathcal{O}_X} \left( H^i(\mathcal{F}^{\bullet})/H^i(\mathcal{F}^{\bullet})[\mathcal{I}] \right) \longrightarrow H^i(\eta_{\mathcal{I}}\mathcal{F}^{\bullet})$$

*of cohomology sheaves.*

**Proof.** We define a map

$$\mathcal{I}^{\otimes i} \otimes_{\mathcal{O}_X} H^i(\mathcal{F}^{\bullet}) \longrightarrow H^i(\eta_{\mathcal{I}}\mathcal{F}^{\bullet})$$

as follows. Let $g$ be a local section of $\mathcal{I}^{\otimes i}$ and let $\overline{s}$ be a local section of $H^i(\mathcal{F}^{\bullet})$. Then $\overline{s}$ is (locally) the class of a local section $s$ of $\mathrm{Ker}(d^i : \mathcal{F}^i \to \mathcal{F}^{i+1})$. Then we send $g \otimes \overline{s}$ to the local section $gs$ of $(\eta_{\mathcal{I}}\mathcal{F})^i \subset \mathcal{I}^i\mathcal{F}$. Of course $gs$ is in the kernel of $d^i$ on $\eta_{\mathcal{I}}\mathcal{F}^{\bullet}$ and hence defines a local section of $H^i(\eta_{\mathcal{I}}\mathcal{F}^{\bullet})$. Checking that this is well defined is without problems. We claim that this map factors through an isomorphism as given in the lemma. This we my check on stalks and hence via Lemma 55.4 this translates into the result of More on Algebra, Lemma 95.2. $\square$

**Lemma 55.6.** *In Situation 55.2 let $\mathcal{F}^{\bullet} \to \mathcal{G}^{\bullet}$ be a map of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. Then the induced map $\eta_{\mathcal{I}}\mathcal{F}^{\bullet} \to \eta_{\mathcal{I}}\mathcal{G}^{\bullet}$ is a quasi-isomorphism too.*

**Proof.** This is true because the isomorphisms of Lemma 55.5 are compatible with maps of complexes. $\square$

**Lemma 55.7.** *In Situation 55.2 there is an additive functor*[9] $L\eta_\mathcal{I} : D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ *such that if $M$ in $D(\mathcal{O}_X)$ is represented by a complex $\mathcal{F}^\bullet$ of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules, then $L\eta_\mathcal{I} M = \eta_\mathcal{I} \mathcal{F}^\bullet$. Similarly for morphisms.*

**Proof.** Denote $\mathcal{T} \subset Mod(\mathcal{O}_X)$ the full subcategory of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. We have a corresponding inclusion

$$K(\mathcal{T}) \quad \subset \quad K(Mod(\mathcal{O}_X)) = K(\mathcal{O}_X)$$

of $K(\mathcal{T})$ as a full triangulated subcategory of $K(\mathcal{O}_X)$. Let $S \subset \text{Arrows}(K(\mathcal{T}))$ be the quasi-isomorphisms. We will apply Derived Categories, Lemma 5.8 to show that the map

$$S^{-1}K(\mathcal{T}) \longrightarrow D(\mathcal{O}_X)$$

is an equivalence of triangulated categories. The lemma shows that it suffices to prove: given a complex $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules, there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ with $\mathcal{F}^\bullet$ a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. By Lemma 26.12 we can find a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ such that the complex $\mathcal{F}^\bullet$ is K-flat (we won't use this) and consists of flat $\mathcal{O}_X$-modules $\mathcal{F}^i$. By the third characterization of Lemma 55.3 we see that a flat $\mathcal{O}_X$-module is an $\mathcal{I}$-torsion free $\mathcal{O}_X$-module and we are done.

With these preliminaries out of the way we can define $L\eta_f$. Namely, by the discussion following Lemma 55.3 this section we have already a well defined functor

$$K(\mathcal{T}) \xrightarrow{\eta_f} K(\mathcal{T}) \to K(\mathcal{O}_X) \to D(\mathcal{O}_X)$$

which according to Lemma 55.6 sends quasi-isomorphisms to quasi-isomorphisms. Hence this functor factors over $S^{-1}K(\mathcal{T}) = D(\mathcal{O}_X)$ by Categories, Lemma 27.8. $\square$

In Situation 55.2 let us construct the Bockstein operators. First we observe that there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}^{i+1} & \longrightarrow & \mathcal{I}^i & \longrightarrow & \mathcal{I}^i/\mathcal{I}^{i+1} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{I}^{i+1}/\mathcal{I}^{i+2} & \longrightarrow & \mathcal{I}^i/\mathcal{I}^{i+2} & \longrightarrow & \mathcal{I}^i/\mathcal{I}^{i+1} & \longrightarrow & 0
\end{array}
$$

whose rows are short exact sequences of $\mathcal{O}_X$-modules. Let $M$ be an object of $D(\mathcal{O}_X)$. Tensoring the above diagram with $M$ gives a morphism

$$
\begin{array}{ccccc}
M \otimes^{\mathbf{L}} \mathcal{I}^{i+1} & \longrightarrow & M \otimes^{\mathbf{L}} \mathcal{I}^i & \longrightarrow & M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1} \\
\downarrow & & \downarrow & & \downarrow \text{id} \\
M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2} & \longrightarrow & M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+2} & \longrightarrow & M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1}
\end{array}
$$

of distinguished triangles. The long exact sequence of cohomology sheaves associated the bottom triangle in particular determines the *Bockstein operator*

$$\beta = \beta^i : H^i(M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1}) \longrightarrow H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2})$$

---

[9]Beware that this functor isn't exact, i.e., does not transform distinguished triangles into distinguished triangles.

for all $i \in \mathbf{Z}$. For later use we record here that by the commutative diagram above there is a factorization

(55.7.1)

$$
\begin{array}{ccc}
H^i(M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1}) & \xrightarrow{\ \delta\ } & H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}) \\
& \searrow{\scriptstyle \beta} & \downarrow \\
& & H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2})
\end{array}
$$

of the Bockstein operator where $\delta$ is the boundary operator coming from the top distinguished triangle in the commutative diagram above. We obtain a complex

(55.7.2)

$$
H^\bullet(M/\mathcal{I}) = \left[ \begin{array}{c}
\cdots \\
\downarrow \\
H^{i-1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i-1}/\mathcal{I}^i) \\
\downarrow \beta \\
H^i(M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1}) \\
\downarrow \beta \\
H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2}) \\
\downarrow \\
\cdots
\end{array} \right]
$$

i.e., that $\beta \circ \beta = 0$. Namely, we can check this on stalks and in this case we can deduce it from the corresponding result in algebra shown in More on Algebra, Section 95. Alternative proof: the short exact sequences $0 \to \mathcal{I}^{i+1}/\mathcal{I}^{i+2} \to \mathcal{I}^i/\mathcal{I}^{i+2} \to \mathcal{I}^i/\mathcal{I}^{i+1} \to 0$ define maps $b^i : \mathcal{I}^i/\mathcal{I}^{i+1} \to (\mathcal{I}^{i+1}/\mathcal{I}^{i+2})[1]$ in $D(\mathcal{O}_X)$ which induce the maps $\beta$ above by tensoring with $M$ and taking cohomology sheaves. Then one shows that the composition $b^{i+1}[1] \circ b^i : \mathcal{I}^i/\mathcal{I}^{i+1} \to (\mathcal{I}^{i+1}/\mathcal{I}^{i+2})[1] \to (\mathcal{I}^{i+2}/\mathcal{I}^{i+3})[2]$ is zero in $D(\mathcal{O}_X)$ by using the criterion in Derived Categories, Lemma 27.7 using that the module $\mathcal{I}^i/\mathcal{I}^{i+3}$ is an extension of $\mathcal{I}^{i+1}/\mathcal{I}^{i+3}$ by $\mathcal{I}^i/\mathcal{I}^{i+1}$.

**Lemma 55.8.** *In Situation 55.2 let $M$ be an object of $D(\mathcal{O}_X)$. There is a canonical isomorphism*

$$
L\eta_\mathcal{I} M \otimes^{\mathbf{L}} \mathcal{O}_X/\mathcal{I} \longrightarrow H^\bullet(M/\mathcal{I})
$$

*in $D(\mathcal{O}_X)$ where the right hand side is the complex (55.7.2).*

**Proof.** By the construction of $L\eta_\mathcal{I}$ in Lemma 55.6 we may assume $M$ is represented by a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules $\mathcal{F}^\bullet$. Then $L\eta_\mathcal{I} M$ is represented by the complex $\eta_\mathcal{I} \mathcal{F}^\bullet$ which is a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules as well. Thus $L\eta_\mathcal{I} M \otimes^{\mathbf{L}} \mathcal{O}_X/\mathcal{I}$ is represented by the complex $\eta_\mathcal{I} \mathcal{F}^\bullet \otimes \mathcal{O}_X/\mathcal{I}$. Similarly, the complex $H^\bullet(M/\mathcal{I})$ has terms $H^i(\mathcal{F}^\bullet \otimes \mathcal{I}^i/\mathcal{I}^{i+1})$.

Let $f$ be a local generator for $\mathcal{I}$. Let $s$ be a local section of $(\eta_\mathcal{I} \mathcal{F})^i$. Then we can write $s = f^i s'$ for a local section $s'$ of $\mathcal{F}^i$ and similarly $d^i(s) = f^{i+1} t$ for a local section $t$ of $\mathcal{F}^{i+1}$. Thus $d^i$ maps $f^i s'$ to zero in $\mathcal{F}^{i+1} \otimes \mathcal{I}^i/\mathcal{I}^{i+1}$. Hence we may map $s$ to the class of $f^i s'$ in $H^i(\mathcal{F}^\bullet \otimes \mathcal{I}^i/\mathcal{I}^{i+1})$. This rule defines a map

$$
(\eta_\mathcal{I} \mathcal{F})^i \otimes \mathcal{O}_X/\mathcal{I} \longrightarrow H^i(\mathcal{F}^\bullet \otimes \mathcal{I}^i/\mathcal{I}^{i+1})
$$

of $\mathcal{O}_X$-modules. A calculation shows that these maps are compatible with differentials (essentially because $\beta$ sends the class of $f^i s'$ to the class of $f^{i+1} t$), whence a map of complexes representing the arrow in the statement of the lemma.

To finish the proof, we observe that the construction given in the previous paragraph agrees on stalks with the maps constructed in More on Algebra, Lemma 95.6 hence we conclude. $\square$

**Lemma 55.9.** *In Situation 55.2 let $\mathcal{F}^\bullet$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\eta_\mathcal{I}(\mathcal{F}^\bullet \otimes \mathcal{L}) = (\eta_\mathcal{I}\mathcal{F}^\bullet) \otimes \mathcal{L}$.*

**Proof.** Immediate from the construction. $\square$

**Lemma 55.10.** *In Situation 55.2 let $M$ be an object of $D(\mathcal{O}_X)$. Let $x \in X$ with $\mathcal{O}_{X,x}$ nonzero. If $H^i(M)_x$ is finite free over $\mathcal{O}_{X,x}$, then $H^i(L\eta_\mathcal{I} M)_x$ is finite free over $\mathcal{O}_{X,x}$ of the same rank.*

**Proof.** Namely, say $f \in \mathcal{O}_{X,x}$ generates the stalk $\mathcal{I}_x$. Then $f$ is a nonzerodivisor in $\mathcal{O}_{X,x}$ and hence $H^i(M)_x[f] = 0$. Thus by Lemma 55.5 we see that $H^i(L\eta_\mathcal{I} M)_x$ is isomorphic to $\mathcal{I}_x^i \otimes_{\mathcal{O}_{X,x}} H^i(M)_x$ which is free of the same rank as desired. $\square$

## 56. Other chapters

Preliminaries

## References

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