ALGEBRAIC CURVES

Contents

1. Introduction

In this chapter we develop some of the theory of algebraic curves. A reference covering algebraic curves over the complex numbers is the book [ACGH85].

What we already know. Besides general algebraic geometry, we have already proved some specific results on algebraic curves. Here is a list.

(1) We have discussed affine opens of and ample invertible sheaves on 1 dimensional Noetherian schemes in Varieties, Section 38.
(2) We have seen a curve is either affine or projective in Varieties, Section 43.

(3) We have discussed degrees of locally free modules on proper curves in Varieties, Section 44.
(4) We have discussed the Picard scheme of a nonsingular projective curve over an algebraically closed field in Picard Schemes of Curves, Section 1.

2. Curves and function fields

In this section we elaborate on the results of Varieties, Section 4 in the case of curves.

Lemma 2.1. *Let k be a field. Let X be a curve and Y a proper variety. Let $U \subset X$ be a nonempty open and let $f : U \to Y$ be a morphism. If $x \in X$ is a closed point such that $\mathcal{O}_{X,x}$ is a discrete valuation ring, then there exist an open $U \subset U' \subset X$ containing x and a morphism of varieties $f' : U' \to Y$ extending f.*

Proof. This is a special case of Morphisms, Lemma 42.5. □

Lemma 2.2. *Let k be a field. Let X be a normal curve and Y a proper variety. The set of rational maps from X to Y is the same as the set of morphisms $X \to Y$.*

Proof. A rational map from X to Y can be extended to a morphism $X \to Y$ by Lemma 2.1 as every local ring is a discrete valuation ring (for example by Varieties, Lemma 43.8). Conversely, if two morphisms $f, g : X \to Y$ are equivalent as rational maps, then $f = g$ by Morphisms, Lemma 7.10. □

Lemma 2.3. *Let k be a field. Let $f : X \to Y$ be a nonconstant morphism of curves over k. If Y is normal, then f is flat.*

Proof. Pick $x \in X$ mapping to $y \in Y$. Then $\mathcal{O}_{Y,y}$ is either a field or a discrete valuation ring (Varieties, Lemma 43.8). Since f is nonconstant it is dominant (as it must map the generic point of X to the generic point of Y). This implies that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective (Morphisms, Lemma 8.7). Hence $\mathcal{O}_{X,x}$ is torsion free as a $\mathcal{O}_{Y,y}$-module and therefore $\mathcal{O}_{X,x}$ is flat as a $\mathcal{O}_{Y,y}$-module by More on Algebra, Lemma 22.10. □

Lemma 2.4. *Let k be a field. Let $f : X \to Y$ be a morphism of schemes over k. Assume*

(1) *Y is separated over k,*
(2) *X is proper of dimension ≤ 1 over k,*
(3) *$f(Z)$ has at least two points for every irreducible component $Z \subset X$ of dimension 1.*

Then f is finite.

Proof. The morphism f is proper by Morphisms, Lemma 41.7. Thus $f(X)$ is closed and images of closed points are closed. Let $y \in Y$ be the image of a closed point in X. Then $f^{-1}(\{y\})$ is a closed subset of X not containing any of the generic points of irreducible components of dimension 1 by condition (3). It follows that $f^{-1}(\{y\})$ is finite. Hence f is finite over an open neighbourhood of y by More on Morphisms, Lemma 44.2 (if Y is Noetherian, then you can use the easier Cohomology of Schemes, Lemma 21.2). Since we've seen above that there are enough of these points y, the proof is complete. □

Lemma 2.5. *Let k be a field. Let $X \to Y$ be a morphism of varieties with Y proper and X a curve. There exists a factorization $X \to \overline{X} \to Y$ where $X \to \overline{X}$ is an open immersion and $\overline{X}$ is a projective curve.*

Proof. This is clear from Lemma 2.1 and Varieties, Lemma 43.6. □

Here is the main theorem of this section. We will say a morphism $f : X \to Y$ of varieties is *constant* if the image $f(X)$ consists of a single point y of Y. If this happens then y is a closed point of Y (since the image of a closed point of X will be a closed point of Y).

Theorem 2.6. *Let k be a field. The following categories are canonically equivalent*

(1) *The category of finitely generated field extensions K/k of transcendence degree* 1.
(2) *The category of curves and dominant rational maps.*
(3) *The category of normal projective curves and nonconstant morphisms.*
(4) *The category of nonsingular projective curves and nonconstant morphisms.*
(5) *The category of regular projective curves and nonconstant morphisms.*
(6) *The category of normal proper curves and nonconstant morphisms.*

Proof. The equivalence between categories (1) and (2) is the restriction of the equivalence of Varieties, Theorem 4.1. Namely, a variety is a curve if and only if its function field has transcendence degree 1, see for example Varieties, Lemma 20.3.

The categories in (3), (4), (5), and (6) are the same. First of all, the terms "regular" and "nonsingular" are synonyms, see Properties, Definition 9.1. Being normal and regular are the same thing for Noetherian 1-dimensional schemes (Properties, Lemmas 9.4 and 12.6). See Varieties, Lemma 43.8 for the case of curves. Thus (3) is the same as (5). Finally, (6) is the same as (3) by Varieties, Lemma 43.4.

If $f : X \to Y$ is a nonconstant morphism of nonsingular projective curves, then f sends the generic point η of X to the generic point ξ of Y. Hence we obtain a morphism $k(Y) = \mathcal{O}_{Y,\xi} \to \mathcal{O}_{X,\eta} = k(X)$ in the category (1). If two morphisms $f, g : X \to Y$ gives the same morphism $k(Y) \to k(X)$, then by the equivalence between (1) and (2), f and g are equivalent as rational maps, so $f = g$ by Lemma 2.2. Conversely, suppose that we have a map $k(Y) \to k(X)$ in the category (1). Then we obtain a morphism $U \to Y$ for some nonempty open $U \subset X$. By Lemma 2.1 this extends to all of X and we obtain a morphism in the category (5). Thus we see that there is a fully faithful functor (5)→(1).

To finish the proof we have to show that every K/k in (1) is the function field of a normal projective curve. We already know that $K = k(X)$ for some curve X. After replacing X by its normalization (which is a variety birational to X) we may assume X is normal (Varieties, Lemma 27.1). Then we choose $X \to \overline{X}$ with $\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ as in Varieties, Lemma 43.6. Since X is normal and since each of the local rings $\mathcal{O}_{\overline{X},x_i}$ is normal we conclude that $\overline{X}$ is a normal projective curve as desired. (Remark: We can also first compactify using Varieties, Lemma 43.5 and then normalize using Varieties, Lemma 27.1. Doing it this way we avoid using the somewhat tricky Morphisms, Lemma 53.16.) □

Definition 2.7. Let k be a field. Let X be a curve. A *nonsingular projective model of* X is a pair (Y, φ) where Y is a nonsingular projective curve and $\varphi : k(X) \to k(Y)$ is an isomorphism of function fields.

A nonsingular projective model is determined up to unique isomorphism by Theorem 2.6. Thus we often say "the nonsingular projective model". We usually drop φ from the notation. Warning: it needn't be the case that Y is smooth over k but Lemma 2.8 shows this can only happen in positive characteristic.

Lemma 2.8. *Let k be a field. Let X be a curve and let Y be the nonsingular projective model of X. If k is perfect, then Y is a smooth projective curve.*

Proof. See Varieties, Lemma 43.8 for example. $\square$

Lemma 2.9. *Let k be a field. Let X be a geometrically irreducible curve over k. For a field extension K/k denote Y_K a nonsingular projective model of $(X_K)_{red}$.*

(1) *If X is proper, then Y_K is the normalization of X_K.*
(2) *There exists K/k finite purely inseparable such that Y_K is smooth.*
(3) *Whenever Y_K is smooth*[1] *we have $H^0(Y_K, \mathcal{O}_{Y_K}) = K$.*
(4) *Given a commutative diagram*

$$\begin{tikzcd} \Omega & K' \arrow[l] \\ K \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

of fields such that Y_K and $Y_{K'}$ are smooth, then $Y_\Omega = (Y_K)_\Omega = (Y_{K'})_\Omega$.

Proof. Let X' be a nonsingular projective model of X. Then X' and X have isomorphic nonempty open subschemes. In particular X' is geometrically irreducible as X is (some details omitted). Thus we may assume that X is projective.

Assume X is proper. Then X_K is proper and hence the normalization $(X_K)^\nu$ is proper as a scheme finite over a proper scheme (Varieties, Lemma 27.1 and Morphisms, Lemmas 44.11 and 41.4). On the other hand, X_K is irreducible as X is geometrically irreducible. Hence X_K^ν is proper, normal, irreducible, and birational to $(X_K)_{red}$. This proves (1) because a proper curve is projective (Varieties, Lemma 43.4).

Proof of (2). As X is proper and we have (1), we can apply Varieties, Lemma 27.4 to find K/k finite purely inseparable such that Y_K is geometrically normal. Then Y_K is geometrically regular as normal and regular are the same for curves (Properties, Lemma 12.6). Then Y is a smooth variety by Varieties, Lemma 12.6.

If Y_K is geometrically reduced, then Y_K is geometrically integral (Varieties, Lemma 9.2) and we see that $H^0(Y_K, \mathcal{O}_{Y_K}) = K$ by Varieties, Lemma 26.2. This proves (3) because a smooth variety is geometrically reduced (even geometrically regular, see Varieties, Lemma 12.6).

If Y_K is smooth, then for every extension Ω/K the base change $(Y_K)_\Omega$ is smooth over Ω (Morphisms, Lemma 34.5). Hence it is clear that $Y_\Omega = (Y_K)_\Omega$. This proves (4). $\square$

[1] Or even geometrically reduced.

3. Linear series

We deviate from the classical story (see Remark 3.6) by defining linear series in the following manner.

Definition 3.1. Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $d \geq 0$ and $r \geq 0$. A *linear series of degree d and dimension r* is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module of degree d (Varieties, Definition 44.1) and $V \subset H^0(X, \mathcal{L})$ is a k-subvector space of dimension $r+1$. We will abbreviate this by saying $(\mathcal{L}, V)$ is a $\mathfrak{g}^r_d$ on X.

We will mostly use this when X is a nonsingular proper curve. In fact, the definition above is just one way to generalize the classical definition of a $\mathfrak{g}^r_d$. For example, if X is a proper curve, then one can generalize linear series by allowing $\mathcal{L}$ to be a torsion free coherent $\mathcal{O}_X$-module of rank 1. On a nonsingular curve every torsion free coherent module is locally free, so this agrees with our notion for nonsingular proper curves.

The following lemma explains the geometric meaning of linear series for proper nonsingular curves.

Lemma 3.2. *Let k be a field. Let X be a nonsingular proper curve over k. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$ on X. Then there exists a morphism*

$$\varphi : X \longrightarrow \mathbf{P}^r_k = \mathit{Proj}(k[T_0, \ldots, T_r])$$

of varieties over k and a map $\alpha : \varphi^*\mathcal{O}_{\mathbf{P}^r_k}(1) \to \mathcal{L}$ such that $\varphi^*T_0, \ldots, \varphi^*T_r$ are sent to a basis of V by α.

Proof. Let $s_0, \ldots, s_r \in V$ be a k-basis. Since X is nonsingular the image $\mathcal{L}' \subset \mathcal{L}$ of the map $s_0, \ldots, s_r : \mathcal{O}_X^{\oplus r+1} \to \mathcal{L}$ is an invertible $\mathcal{O}_X$-module for example by Divisors, Lemma 11.11. Then we use Constructions, Lemma 13.1 to get a morphism

$$\varphi = \varphi_{(\mathcal{L}', (s_0, \ldots, s_r))} : X \longrightarrow \mathbf{P}^r_k$$

as in the statement of the lemma. □

Lemma 3.3. *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. If X has a $\mathfrak{g}^r_d$, then X has a $\mathfrak{g}^s_d$ for all $0 \leq s \leq r$.*

Proof. This is true because a vector space V of dimension $r+1$ over k has a linear subspace of dimension $s+1$ for all $0 \leq s \leq r$. □

Lemma 3.4. *Let k be a field. Let X be a nonsingular proper curve over k. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^1_d$ on X. Then the morphism $\varphi : X \to \mathbf{P}^1_k$ of Lemma 3.2 either*

1. *is nonconstant and has degree $\leq d$, or*
2. *factors through a closed point of $\mathbf{P}^1_k$ and in this case $H^0(X, \mathcal{O}_X) \neq k$.*

Proof. By Lemma 3.2 we see that $\mathcal{L}' = \varphi^*\mathcal{O}_{\mathbf{P}^1_k}(1)$ has a nonzero map $\mathcal{L}' \to \mathcal{L}$. Hence by Varieties, Lemma 44.12 we see that $0 \leq \deg(\mathcal{L}') \leq d$. If $\deg(\mathcal{L}') = 0$, then the same lemma tells us $\mathcal{L}' \cong \mathcal{O}_X$ and since we have two linearly independent sections we find we are in case (2). If $\deg(\mathcal{L}') > 0$ then φ is nonconstant (since the pullback of an invertible module by a constant morphism is trivial). Hence

$$\deg(\mathcal{L}') = \deg(X/\mathbf{P}^1_k)\deg(\mathcal{O}_{\mathbf{P}^1_k}(1))$$

by Varieties, Lemma 44.11. This finishes the proof as the degree of $\mathcal{O}_{\mathbf{P}^1_k}(1)$ is 1. □

Lemma 3.5. *Let k be a field. Let X be a proper curve over k with $H^0(X, \mathcal{O}_X) = k$. If X has a $\mathfrak{g}^r_d$, then $r \leq d$. If equality holds, then $H^1(X, \mathcal{O}_X) = 0$, i.e., the genus of X (Definition 8.1) is 0.*

Proof. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$. Since this will only increase r, we may assume $V = H^0(X, \mathcal{L})$. Choose a nonzero element $s \in V$. Then the zero scheme of s is an effective Cartier divisor $D \subset X$, we have $\mathcal{L} = \mathcal{O}_X(D)$, and we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}|_D \to 0$$

see Divisors, Lemma 14.10 and Remark 14.11. By Varieties, Lemma 44.9 we have $\deg(D) = \deg(\mathcal{L}) = d$. Since D is an Artinian scheme we have $\mathcal{L}|_D \cong \mathcal{O}_D$[2]. Thus

$$\dim_k H^0(D, \mathcal{L}|_D) = \dim_k H^0(D, \mathcal{O}_D) = \deg(D) = d$$

On the other hand, by assumption $\dim_k H^0(X, \mathcal{O}_X) = 1$ and $\dim H^0(X, \mathcal{L}) = r+1$. We conclude that $r + 1 \leq 1 + d$, i.e., $r \leq d$ as in the lemma.

Assume equality holds. Then $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_D)$ is surjective. If we knew that $H^1(X, \mathcal{L})$ was zero, then we would conclude that $H^1(X, \mathcal{O}_X)$ is zero by the long exact cohomology sequence and the proof would be complete. Our strategy will be to replace $\mathcal{L}$ by a large power which has vanishing. As $\mathcal{L}|_D$ is the trivial invertible module (see above), we can find a section t of $\mathcal{L}$ whose restriction of D generates $\mathcal{L}|_D$. Consider the multiplication map

$$\mu : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}^{\otimes 2})$$

and consider the short exact sequence

$$0 \to \mathcal{L} \xrightarrow{s} \mathcal{L}^{\otimes 2} \to \mathcal{L}^{\otimes 2}|_D \to 0$$

Since $H^0(\mathcal{L}) \to H^0(\mathcal{L}|_D)$ is surjective and since t maps to a trivialization of $\mathcal{L}|_D$ we see that $\mu(H^0(X, \mathcal{L}) \otimes t)$ gives a subspace of $H^0(X, \mathcal{L}^{\otimes 2})$ surjecting onto the global sections of $\mathcal{L}^{\otimes 2}|_D$. Thus we see that

$$\dim H^0(X, \mathcal{L}^{\otimes 2}) = r + 1 + d = 2r + 1 = \deg(\mathcal{L}^{\otimes 2}) + 1$$

Ok, so $\mathcal{L}^{\otimes 2}$ has the same property as $\mathcal{L}$, i.e., that the dimension of the space of global sections is equal to the degree plus one. Since $\mathcal{L}$ is ample (Varieties, Lemma 44.14) there exists some n_0 such that $\mathcal{L}^{\otimes n}$ has vanishing H^1 for all $n \geq n_0$ (Cohomology of Schemes, Lemma 16.1). Thus applying the argument above to $\mathcal{L}^{\otimes n}$ with $n = 2^m$ for some sufficiently large m we conclude the lemma is true. $\square$

Remark 3.6 (Classical definition)**.** Let X be a smooth projective curve over an algebraically closed field k. We say two effective Cartier divisors $D, D' \subset X$ are *linearly equivalent* if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ as $\mathcal{O}_X$-modules. Since $\text{Pic}(X) = \text{Cl}(X)$ (Divisors, Lemma 27.7) we see that D and D' are linearly equivalent if and only if the Weil divisors associated to D and D' define the same element of $\text{Cl}(X)$. Given an effective Cartier divisor $D \subset X$ of degree d the *complete linear system* or *complete linear series* $|D|$ of D is the set of effective Cartier divisors $E \subset X$ which are linearly equivalent to D. Another way to say it is that $|D|$ is the set of closed points of the fibre of the morphism

$$\gamma_d : \underline{\text{Hilb}}^d_{X/k} \longrightarrow \underline{\text{Pic}}^d_{X/k}$$

[2]In our case this follows from Divisors, Lemma 17.1 as $D \to \text{Spec}(k)$ is finite.

(Picard Schemes of Curves, Lemma 6.7) over the closed point corresponding to $\mathcal{O}_X(D)$. This gives $|D|$ a natural scheme structure and it turns out that $|D| \cong \mathbf{P}^m_k$ with $m + 1 = h^0(\mathcal{O}_X(D))$. In fact, more canonically we have

$$|D| = \mathbf{P}(H^0(X, \mathcal{O}_X(D))^\vee)$$

where $(-)^\vee$ indicates k-linear dual and $\mathbf{P}$ is as in Constructions, Example 21.2. In this language a *linear system* or a *linear series* on X is a closed subvariety $L \subset |D|$ which can be cut out by linear equations. If L has dimension r, then $L = \mathbf{P}(V^\vee)$ where $V \subset H^0(X, \mathcal{O}_X(D))$ is a linear subspace of dimension $r+1$. Thus the classical linear series $L \subset |D|$ corresponds to the linear series $(\mathcal{O}_X(D), V)$ as defined above.

4. Duality

In this section we work out the consequences of the very general material on dualizing complexes and duality for proper 1-dimensional schemes over fields. If you are interested in the analogous discussion for higher dimension proper schemes over fields, see Duality for Schemes, Section 27.

Lemma 4.1. *Let X be a proper scheme of dimension ≤ 1 over a field k. There exists a dualizing complex $\omega_X^\bullet$ with the following properties*

1. *$H^i(\omega_X^\bullet)$ is nonzero only for $i = -1, 0$,*
2. *$\omega_X = H^{-1}(\omega_X^\bullet)$ is a coherent Cohen-Macaulay module whose support is the irreducible components of dimension 1,*
3. *for $x \in X$ closed, the module $H^0(\omega_{X,x}^\bullet)$ is nonzero if and only if either*
   (a) $\dim(\mathcal{O}_{X,x}) = 0$ *or*
   (b) $\dim(\mathcal{O}_{X,x}) = 1$ *and $\mathcal{O}_{X,x}$ is not Cohen-Macaulay,*
4. *for $K \in D_{QCoh}(\mathcal{O}_X)$ there are functorial isomorphisms*[3]
$$\mathrm{Ext}^i_X(K, \omega_X^\bullet) = \mathrm{Hom}_k(H^{-i}(X, K), k)$$
   compatible with shifts and distinguished triangles,
5. *there are functorial isomorphisms* $\mathrm{Hom}(\mathcal{F}, \omega_X) = \mathrm{Hom}_k(H^1(X, \mathcal{F}), k)$ *for $\mathcal{F}$ quasi-coherent on X,*
6. *if $X \to \mathrm{Spec}(k)$ is smooth of relative dimension 1, then $\omega_X \cong \Omega_{X/k}$.*

Proof. Denote $f : X \to \mathrm{Spec}(k)$ the structure morphism. We start with the relative dualizing complex

$$\omega_X^\bullet = \omega_{X/k}^\bullet = a(\mathcal{O}_{\mathrm{Spec}(k)})$$

as described in Duality for Schemes, Remark 12.5. Then property (4) holds by construction as a is the right adjoint for $f_* : D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_{\mathrm{Spec}(k)})$. Since f is proper we have $f^!(\mathcal{O}_{\mathrm{Spec}(k)}) = a(\mathcal{O}_{\mathrm{Spec}(k)})$ by definition, see Duality for Schemes, Section 16. Hence $\omega_X^\bullet$ and ω_X are as in Duality for Schemes, Example 22.1 and as in Duality for Schemes, Example 22.2. Parts (1) and (2) follow from Duality for Schemes, Lemma 22.4. For a closed point $x \in X$ we see that $\omega_{X,x}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$, see Duality for Schemes, Lemma 21.1. Assertion (3) then follows from Dualizing Complexes, Lemma 20.2. Assertion (5) follows from Duality for Schemes, Lemma 22.5 for coherent $\mathcal{F}$ and in general by unwinding (4)

[3]This property characterizes $\omega_X^\bullet$ in $D_{QCoh}(\mathcal{O}_X)$ up to unique isomorphism by the Yoneda lemma. Since $\omega_X^\bullet$ is in $D^b_{Coh}(\mathcal{O}_X)$ in fact it suffices to consider $K \in D^b_{Coh}(\mathcal{O}_X)$.

for $K = \mathcal{F}[0]$ and $i = -1$. Assertion (6) follows from Duality for Schemes, Lemma 15.7. $\square$

Lemma 4.2. *Let X be a proper scheme over a field k which is Cohen-Macaulay and equidimensional of dimension 1. The module ω_X of Lemma 4.1 has the following properties*

1. *ω_X is a dualizing module on X (Duality for Schemes, Section 22),*
2. *ω_X is a coherent Cohen-Macaulay module whose support is X,*
3. *there are functorial isomorphisms $\text{Ext}^i_X(K, \omega_X[1]) = \text{Hom}_k(H^{-i}(X, K), k)$ compatible with shifts for $K \in D_{QCoh}(X)$,*
4. *there are functorial isomorphisms $\text{Ext}^{1+i}(\mathcal{F}, \omega_X) = \text{Hom}_k(H^{-i}(X, \mathcal{F}), k)$ for $\mathcal{F}$ quasi-coherent on X.*

Proof. Recall from the proof of Lemma 4.1 that ω_X is as in Duality for Schemes, Example 22.1 and hence is a dualizing module. The other statements follow from Lemma 4.1 and the fact that $\omega_X^\bullet = \omega_X[1]$ as X is Cohen-Macualay (Duality for Schemes, Lemma 23.1). $\square$

Remark 4.3. Let X be a proper scheme of dimension ≤ 1 over a field k. Let $\omega_X^\bullet$ and ω_X be as in Lemma 4.1. If $\mathcal{E}$ is a finite locally free $\mathcal{O}_X$-module with dual $\mathcal{E}^\vee$ then we have canonical isomorphisms

$$\text{Hom}_k(H^{-i}(X, \mathcal{E}), k) = H^i(X, \mathcal{E}^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_X^\bullet)$$

This follows from the lemma and Cohomology, Lemma 50.5. If X is Cohen-Macaulay and equidimensional of dimension 1, then we have canonical isomorphisms

$$\text{Hom}_k(H^{-i}(X, \mathcal{E}), k) = H^{1+i}(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X)$$

by Lemma 4.2. In particular if $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then we have

$$\dim_k H^0(X, \mathcal{L}) = \dim_k H^1(X, \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \omega_X)$$

and

$$\dim_k H^1(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \omega_X)$$

Here is a sanity check for the dualizing complex.

Lemma 4.4. *Let X be a proper scheme of dimension ≤ 1 over a field k. Let $\omega_X^\bullet$ and ω_X be as in Lemma 4.1.*

1. *If $X \to \text{Spec}(k)$ factors as $X \to \text{Spec}(k') \to \text{Spec}(k)$ for some field k', then $\omega_X^\bullet$ and ω_X satisfy properties (4), (5), (6) with k replaced with k'.*
2. *If K/k is a field extension, then the pullback of $\omega_X^\bullet$ and ω_X to the base change X_K are as in Lemma 4.1 for the morphism $X_K \to \text{Spec}(K)$.*

Proof. Denote $f : X \to \text{Spec}(k)$ the structure morphism. Assertion (1) really means that $\omega_X^\bullet$ and ω_X are as in Lemma 4.1 for the morphism $f' : X \to \text{Spec}(k')$. In the proof of Lemma 4.1 we took $\omega_X^\bullet = a(\mathcal{O}_{\text{Spec}(k)})$ where a be is the right adjoint of Duality for Schemes, Lemma 3.1 for f. Thus we have to show $a(\mathcal{O}_{\text{Spec}(k)}) \cong a'(\mathcal{O}_{\text{Spec}(k)})$ where a' be is the right adjoint of Duality for Schemes, Lemma 3.1 for f'. Since $k' \subset H^0(X, \mathcal{O}_X)$ we see that k'/k is a finite extension (Cohomology of Schemes, Lemma 19.2). By uniqueness of adjoints we have $a = a' \circ b$ where b is the right adjoint of Duality for Schemes, Lemma 3.1 for $g : \text{Spec}(k') \to \text{Spec}(k)$. Another way to say this: we have $f^! = (f')^! \circ g^!$. Thus it suffices to show that

$\mathrm{Hom}_k(k', k) \cong k'$ as k'-modules, see Duality for Schemes, Example 3.2. This holds because these are k'-vector spaces of the same dimension (namely dimension 1).

Proof of (2). This holds because we have base change for a by Duality for Schemes, Lemma 6.2. See discussion in Duality for Schemes, Remark 12.5. $\square$

Lemma 4.5. *Let X be a proper scheme of dimension ≤ 1 over a field k. Let $i : Y \to X$ be a closed immersion. Let $\omega_X^\bullet$, ω_X, $\omega_Y^\bullet$, ω_Y be as in Lemma 4.1. Then*

1. $\omega_Y^\bullet = R\mathcal{H}om(\mathcal{O}_Y, \omega_X^\bullet)$,
2. $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$ *and* $i_*\omega_Y = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \omega_X)$.

Proof. Denote $g : Y \to \mathrm{Spec}(k)$ and $f : X \to \mathrm{Spec}(k)$ the structure morphisms. Then $g = f \circ i$. Denote a, b, c the right adjoint of Duality for Schemes, Lemma 3.1 for f, g, i. Then $b = c \circ a$ by uniqueness of right adjoints and because $Rg_* = Rf_* \circ Ri_*$. In the proof of Lemma 4.1 we set $\omega_X^\bullet = a(\mathcal{O}_{\mathrm{Spec}(k)})$ and $\omega_Y^\bullet = b(\mathcal{O}_{\mathrm{Spec}(k)})$. Hence $\omega_Y^\bullet = c(\omega_X^\bullet)$ which implies (1) by Duality for Schemes, Lemma 9.7. Since $\omega_X = H^{-1}(\omega_X^\bullet)$ and $\omega_Y = H^{-1}(\omega_Y^\bullet)$ we conclude that $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$. This implies $i_*\omega_Y = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \omega_X)$ by Duality for Schemes, Lemma 9.3. $\square$

Lemma 4.6. *Let X be a proper scheme over a field k which is Gorenstein, reduced, and equidimensional of dimension 1. Let $i : Y \to X$ be a reduced closed subscheme equidimensional of dimension 1. Let $j : Z \to X$ be the scheme theoretic closure of $X \setminus Y$. Then*

1. *Y and Z are Cohen-Macaulay,*
2. *if $\mathcal{I} \subset \mathcal{O}_X$, resp. $\mathcal{J} \subset \mathcal{O}_X$ is the ideal sheaf of Y, resp. Z in X, then*
$$\mathcal{I} = i_*\mathcal{I}' \quad \text{and} \quad \mathcal{J} = j_*\mathcal{J}'$$
where $\mathcal{I}' \subset \mathcal{O}_Z$, resp. $\mathcal{J}' \subset \mathcal{O}_Y$ is the ideal sheaf of $Y \cap Z$ in Z, resp. Y,
3. $\omega_Y = \mathcal{J}'(i^*\omega_X)$ *and* $i_*(\omega_Y) = \mathcal{J}\omega_X$,
4. $\omega_Z = \mathcal{I}'(i^*\omega_X)$ *and* $i_*(\omega_Z) = \mathcal{I}\omega_X$,
5. *we have the following short exact sequences*
$$\begin{aligned} 0 \to \omega_X \to i_*i^*\omega_X \oplus j_*j^*\omega_X \to \mathcal{O}_{Y\cap Z} \to 0 \\ 0 \to i_*\omega_Y \to \omega_X \to j_*j^*\omega_X \to 0 \\ 0 \to j_*\omega_Z \to \omega_X \to i_*i^*\omega_X \to 0 \\ 0 \to i_*\omega_Y \oplus j_*\omega_Z \to \omega_X \to \mathcal{O}_{Y\cap Z} \to 0 \\ 0 \to \omega_Y \to i^*\omega_X \to \mathcal{O}_{Y\cap Z} \to 0 \\ 0 \to \omega_Z \to j^*\omega_X \to \mathcal{O}_{Y\cap Z} \to 0 \end{aligned}$$

Here ω_X, ω_Y, ω_Z are as in Lemma 4.1.

Proof. A reduced 1-dimensional Noetherian scheme is Cohen-Macaulay, so (1) is true. Since X is reduced, we see that $X = Y \cup Z$ scheme theoretically. With notation as in Morphisms, Lemma 4.6 and by the statement of that lemma we have a short exact sequence
$$0 \to \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Z \to \mathcal{O}_{Y\cap Z} \to 0$$
Since $\mathcal{J} = \mathrm{Ker}(\mathcal{O}_X \to \mathcal{O}_Z)$, $\mathcal{J}' = \mathrm{Ker}(\mathcal{O}_Y \to \mathcal{O}_{Y\cap Z})$, $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to \mathcal{O}_Y)$, and $\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_Z \to \mathcal{O}_{Y\cap Z})$ a diagram chase implies (2). Observe that $\mathcal{I}+\mathcal{J}$ is the ideal

sheaf of $Y \cap Z$ and that $\mathcal{I} \cap \mathcal{J} = 0$. Hence we have the following exact sequences

$$
\begin{aligned}
0 \to \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Z \to \mathcal{O}_{Y\cap Z} \to 0 \\
0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0 \\
0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \\
0 \to \mathcal{J} \oplus \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_{Y\cap Z} \to 0 \\
0 \to \mathcal{J}' \to \mathcal{O}_Y \to \mathcal{O}_{Y\cap Z} \to 0 \\
0 \to \mathcal{I}' \to \mathcal{O}_Z \to \mathcal{O}_{Y\cap Z} \to 0
\end{aligned}
$$

Since X is Gorenstein ω_X is an invertible $\mathcal{O}_X$-module (Duality for Schemes, Lemma 24.4). Since $Y \cap Z$ has dimension 0 we have $\omega_X|_{Y\cap Z} \cong \mathcal{O}_{Y\cap Z}$. Thus if we prove (3) and (4), then we obtain the short exact sequences of the lemma by tensoring the above short exact sequence with the invertible module ω_X. By symmetry it suffices to prove (3) and by (2) it suffices to prove $i_*(\omega_Y) = \mathcal{J}\omega_X$.

We have $i_*\omega_Y = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \omega_X)$ by Lemma 4.5. Again using that ω_X is invertible we finally conclude that it suffices to show $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X)$ maps isomorphically to $\mathcal{J}$ by evaluation at 1. In other words, that $\mathcal{J}$ is the annihilator of $\mathcal{I}$. This follows from the above. $\square$

5. Riemann-Roch

Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. In Varieties, Section 44 we have defined the degree of a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank by the formula

$$\deg(\mathcal{E}) = \chi(X, \mathcal{E}) - \text{rank}(\mathcal{E})\chi(X, \mathcal{O}_X) \tag{5.0.1}$$

see Varieties, Definition 44.1. In the chapter on Chow Homology we defined the first Chern class of $\mathcal{E}$ as an operation on cycles (Chow Homology, Section 38) and we proved that

$$\deg(\mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1) \tag{5.0.2}$$

see Chow Homology, Lemma 41.3. Combining (5.0.1) and (5.0.2) we obtain our first version of the Riemann-Roch formula

$$\chi(X, \mathcal{E}) = \deg(c_1(\mathcal{E}) \cap [X]_1) + \text{rank}(\mathcal{E})\chi(X, \mathcal{O}_X) \tag{5.0.3}$$

If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then we can also consider the numerical intersection $(\mathcal{L} \cdot X)$ as defined in Varieties, Definition 45.3. However, this does not give anything new as

$$(\mathcal{L} \cdot X) = \deg(\mathcal{L}) \tag{5.0.4}$$

by Varieties, Lemma 45.12. If $\mathcal{L}$ is ample, then this integer is positive and is called the degree

$$\deg_{\mathcal{L}}(X) = (\mathcal{L} \cdot X) = \deg(\mathcal{L}) \tag{5.0.5}$$

of X with respect to $\mathcal{L}$, see Varieties, Definition 45.10.

To obtain a true Riemann-Roch theorem we would like to write $\chi(X, \mathcal{O}_X)$ as the degree of a canonical zero cycle on X. We refer to [Ful98] for a fully general version of this. We will use duality to get a formula in the case where X is Gorenstein; however, in some sense this is a cheat (for example because this method cannot work in higher dimension).

We first use Lemmas 4.1 and 4.2 to get a relation between the euler characteristic of $\mathcal{O}_X$ and the euler characteristic of the dualizing complex or the dualizing module.

Lemma 5.1. *Let X be a proper scheme of dimension ≤ 1 over a field k. With $\omega_X^\bullet$ and ω_X as in Lemma 4.1 we have*

$$\chi(X, \mathcal{O}_X) = \chi(X, \omega_X^\bullet)$$

If X is Cohen-Macaulay and equidimensional of dimension 1, then

$$\chi(X, \mathcal{O}_X) = -\chi(X, \omega_X)$$

Proof. We define the right hand side of the first formula as follows:

$$\chi(X, \omega_X^\bullet) = \sum\nolimits_{i \in \mathbf{Z}} (-1)^i \dim_k H^i(X, \omega_X^\bullet)$$

This is well defined because $\omega_X^\bullet$ is in $D^b_{Coh}(\mathcal{O}_X)$, but also because

$$H^i(X, \omega_X^\bullet) = \mathrm{Ext}^i(\mathcal{O}_X, \omega_X^\bullet) = H^{-i}(X, \mathcal{O}_X)$$

which is always finite dimensional and nonzero only if $i = 0, -1$. This of course also proves the first formula. The second is a consequence of the first because $\omega_X^\bullet = \omega_X[1]$ in the CM case, see Lemma 4.2. $\square$

We will use Lemma 5.1 to get the desired formula for $\chi(X, \mathcal{O}_X)$ in the case that ω_X is invertible, i.e., that X is Gorenstein. The statement is that $-1/2$ of the first Chern class of ω_X capped with the cycle $[X]_1$ associated to X is a natural zero cycle on X with half-integer coefficients whose degree is $\chi(X, \mathcal{O}_X)$. The occurence of fractions in the statement of Riemann-Roch cannot be avoided.

Lemma 5.2 (Riemann-Roch)**.** *Let X be a proper scheme over a field k which is Gorenstein and equidimensional of dimension 1. Let ω_X be as in Lemma 4.1. Then*

1. *ω_X is an invertible $\mathcal{O}_X$-module,*
2. $\deg(\omega_X) = -2\chi(X, \mathcal{O}_X)$,
3. *for a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of constant rank we have*

$$\chi(X, \mathcal{E}) = \deg(\mathcal{E}) - \tfrac{1}{2} \mathit{rank}(\mathcal{E}) \deg(\omega_X)$$

and $\dim_k(H^i(X, \mathcal{E})) = \dim_k(H^{1-i}(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X))$ *for all* $i \in \mathbf{Z}$.

Nonsingular (normal) curves are Gorenstein, see Duality for Schemes, Lemma 24.3.

Proof. Recall that Gorenstein schemes are Cohen-Macaulay (Duality for Schemes, Lemma 24.2) and hence ω_X is a dualizing module on X, see Lemma 4.2. It follows more or less from the definition of the Gorenstein property that the dualizing sheaf is invertible, see Duality for Schemes, Section 24. By (5.0.3) applied to ω_X we have

$$\chi(X, \omega_X) = \deg(c_1(\omega_X) \cap [X]_1) + \chi(X, \mathcal{O}_X)$$

Combined with Lemma 5.1 this gives

$$2\chi(X, \mathcal{O}_X) = -\deg(c_1(\omega_X) \cap [X]_1) = -\deg(\omega_X)$$

the second equality by (5.0.2). Putting this back into (5.0.3) for $\mathcal{E}$ gives the displayed formula of the lemma. The symmetry in dimensions is a consequence of duality for X, see Remark 4.3. $\square$

6. Some vanishing results

This section contains some very weak vanishing results. Please see Section 21 for a few more and more interesting results.

Lemma 6.1. *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then X is connected, Cohen-Macaulay, and equidimensional of dimension 1.*

Proof. Since $\Gamma(X, \mathcal{O}_X) = k$ has no nontrivial idempotents, we see that X is connected. This already shows that X is equidimensional of dimension 1 (any irreducible component of dimension 0 would be a connected component). Let $\mathcal{I} \subset \mathcal{O}_X$ be the maximal coherent submodule supported in closed points. Then $\mathcal{I}$ exists (Divisors, Lemma 4.6) and is globally generated (Varieties, Lemma 33.3). Since $1 \in \Gamma(X, \mathcal{O}_X)$ is not a section of $\mathcal{I}$ we conclude that $\mathcal{I} = 0$. Thus X does not have embedded points (Divisors, Lemma 4.6). Thus X has (S_1) by Divisors, Lemma 4.3. Hence X is Cohen-Macaulay. $\square$

In this section we work in the following situation.

Situation 6.2. Here k is a field, X is a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$.

By Lemma 6.1 the scheme X is Cohen-Macaulay and equidimensional of dimension 1. The dualizing module ω_X discussed in Lemmas 4.1 and 4.2 has nonvanishing H^1 because in fact $\dim_k H^1(X, \omega_X) = \dim_k H^0(X, \mathcal{O}_X) = 1$. It turns out that anything slightly more "positive" than ω_X has vanishing H^1.

Lemma 6.3. *In Situation 6.2. Given an exact sequence*

$$\omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$$

of coherent $\mathcal{O}_X$-modules with $H^1(X, \mathcal{Q}) = 0$ (for example if $\dim(Supp(\mathcal{Q})) = 0$), then either $H^1(X, \mathcal{F}) = 0$ or $\mathcal{F} = \omega_X \oplus \mathcal{Q}$.

Proof. (The parenthetical statement follows from Cohomology of Schemes, Lemma 9.10.) Since $H^0(X, \mathcal{O}_X) = k$ is dual to $H^1(X, \omega_X)$ (see Section 5) we see that $\dim H^1(X, \omega_X) = 1$. The sheaf ω_X represents the functor $\mathcal{F} \mapsto \mathrm{Hom}_k(H^1(X, \mathcal{F}), k)$ on the category of coherent $\mathcal{O}_X$-modules (Duality for Schemes, Lemma 22.5). Consider an exact sequence as in the statement of the lemma and assume that $H^1(X, \mathcal{F}) \neq 0$. Since $H^1(X, \mathcal{Q}) = 0$ we see that $H^1(X, \omega_X) \to H^1(X, \mathcal{F})$ is an isomorphism. By the universal property of ω_X stated above, we conclude there is a map $\mathcal{F} \to \omega_X$ whose action on H^1 is the inverse of this isomorphism. The composition $\omega_X \to \mathcal{F} \to \omega_X$ is the identity (by the universal property) and the lemma is proved. $\square$

Lemma 6.4. *In Situation 6.2. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is globally generated and not isomorphic to $\mathcal{O}_X$. Then $H^1(X, \omega_X \otimes \mathcal{L}) = 0$.*

Proof. By duality as discussed in Section 5 we have to show that $H^0(X, \mathcal{L}^{\otimes -1}) = 0$. If not, then we can choose a global section t of $\mathcal{L}^{\otimes -1}$ and a global section s of $\mathcal{L}$ such that $st \neq 0$. However, then st is a constant multiple of 1, by our assumption that $H^0(X, \mathcal{O}_X) = k$. It follows that $\mathcal{L} \cong \mathcal{O}_X$, which is a contradiction. $\square$

Lemma 6.5. *In Situation 6.2. Given an exact sequence*

$$\omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$$

of coherent $\mathcal{O}_X$-modules with $\dim(Supp(\mathcal{Q})) = 0$ and $\dim_k H^0(X, \mathcal{Q}) \geq 2$ and such that there is no nonzero submodule $\mathcal{Q}' \subset \mathcal{F}$ such that $\mathcal{Q}' \to \mathcal{Q}$ is injective. Then the submodule of $\mathcal{F}$ generated by global sections surjects onto $\mathcal{Q}$.

Proof. Let $\mathcal{F}' \subset \mathcal{F}$ be the submodule generated by global sections and the image of $\omega_X \to \mathcal{F}$. Since $\dim_k H^0(X, \mathcal{Q}) \geq 2$ and $\dim_k H^1(X, \omega_X) = \dim_k H^0(X, \mathcal{O}_X) = 1$, we see that $\mathcal{F}' \to \mathcal{Q}$ is not zero and $\omega_X \to \mathcal{F}'$ is not an isomorphism. Hence $H^1(X, \mathcal{F}') = 0$ by Lemma 6.3 and our assumption on $\mathcal{F}$. Consider the short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{Q}/\operatorname{Im}(\mathcal{F}' \to \mathcal{Q}) \to 0$$

If the quotient on the right is nonzero, then we obtain a contradiction because then $H^0(X, \mathcal{F})$ is bigger than $H^0(X, \mathcal{F}')$. □

Here is an example global generation statement.

Lemma 6.6. *In Situation 6.2 assume that X is integral. Let $0 \to \omega_X \to \mathcal{F} \to \mathcal{Q} \to 0$ be a short exact sequence of coherent $\mathcal{O}_X$-modules with $\mathcal{F}$ torsion free, $\dim(Supp(\mathcal{Q})) = 0$, and $\dim_k H^0(X, \mathcal{Q}) \geq 2$. Then $\mathcal{F}$ is globally generated.*

Proof. Consider the submodule $\mathcal{F}'$ generated by the global sections. By Lemma 6.5 we see that $\mathcal{F}' \to \mathcal{Q}$ is surjective, in particular $\mathcal{F}' \neq 0$. Since X is a curve, we see that $\mathcal{F}' \subset \mathcal{F}$ is an inclusion of rank 1 sheaves, hence $\mathcal{Q}' = \mathcal{F}/\mathcal{F}'$ is supported in finitely many points. To get a contradiction, assume that $\mathcal{Q}'$ is nonzero. Then we see that $H^1(X, \mathcal{F}') \neq 0$. Then we get a nonzero map $\mathcal{F}' \to \omega_X$ by the universal property (Duality for Schemes, Lemma 22.5). The image of the composition $\mathcal{F}' \to \omega_X \to \mathcal{F}$ is generated by global sections, hence is inside of $\mathcal{F}'$. Thus we get a nonzero self map $\mathcal{F}' \to \mathcal{F}'$. Since $\mathcal{F}'$ is torsion free of rank 1 on a proper curve this has to be an automorphism (details omitted). But then this implies that $\mathcal{F}'$ is contained in $\omega_X \subset \mathcal{F}$ contradicting the surjectivity of $\mathcal{F}' \to \mathcal{Q}$. □

Lemma 6.7. *In Situation 6.2. Let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module with $\deg(\mathcal{L}) \geq 2$. Then $\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}$ is globally generated.*

Proof. Assume k is algebraically closed. Let $x \in X$ be a closed point. Let $C_i \subset X$ be the irreducible components and for each i let $x_i \in C_i$ be the generic point. By Varieties, Lemma 22.2 we can choose a section $s \in H^0(X, \mathcal{L})$ such that s vanishes at x but not at x_i for all i. The corresponding module map $s : \mathcal{O}_X \to \mathcal{L}$ is injective with cokernel $\mathcal{Q}$ supported in finitely many points and with $H^0(X, \mathcal{Q}) \geq 2$. Consider the corresponding exact sequence

$$0 \to \omega_X \to \omega_X \otimes \mathcal{L} \to \omega_X \otimes \mathcal{Q} \to 0$$

By Lemma 6.5 we see that the module generated by global sections surjects onto $\omega_X \otimes \mathcal{Q}$. Since x was arbitrary this proves the lemma. Some details omitted.

We will reduce the case where k is not algebraically closed, to the algebraically closed field case. We suggest the reader skip the rest of the proof. Choose an algebraic closure $\overline{k}$ of k and consider the base change $X_{\overline{k}}$. Let us check that $X_{\overline{k}} \to \operatorname{Spec}(\overline{k})$ is an example of Situation 6.2. By flat base change (Cohomology of Schemes, Lemma 5.2) we see that $H^0(X_{\overline{k}}, \mathcal{O}) = \overline{k}$. The scheme $X_{\overline{k}}$ is proper over $\overline{k}$ (Morphisms, Lemma 41.5) and equidimensional of dimension 1 (Morphisms, Lemma 28.3). The pullback of ω_X to $X_{\overline{k}}$ is the dualizing module of $X_{\overline{k}}$ by Lemma 4.4. The pullback of $\mathcal{L}$ to $X_{\overline{k}}$ is very ample (Morphisms, Lemma 38.8). The degree of the pullback of $\mathcal{L}$ to $X_{\overline{k}}$ is equal to the degree of $\mathcal{L}$ on X (Varieties, Lemma 44.2). Finally, we see that $\omega_X \otimes \mathcal{L}$ is globally generated if and only if its base change is so (Varieties, Lemma 22.1). In this way we see that the result follows from the result in the case of an algebraically closed ground field. $\square$

7. Very ample invertible sheaves

An often used criterion for very ampleness of an invertible module $\mathcal{L}$ on a scheme X of finite type over an algebraically closed field is: sections of $\mathcal{L}$ separate points and tangent vectors (Varieties, Section 23). Here is another criterion for curves; please compare with Varieties, Subsection 35.6.

Lemma 7.1. *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume*

1. *$\mathcal{L}$ has a regular global section,*
2. *$H^1(X, \mathcal{L}) = 0$, and*
3. *$\mathcal{L}$ is ample.*

Then $\mathcal{L}^{\otimes 6}$ is very ample on X over k.

Proof. Let s be a regular global section of $\mathcal{L}$. Let $i : Z = Z(s) \to X$ be the zero scheme of s, see Divisors, Section 14. By condition (3) we see that $Z \neq \emptyset$ (small detail omitted). Consider the short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{L} \to i_*(\mathcal{L}|_Z) \to 0$$

Tensoring with $\mathcal{L}$ we obtain

$$0 \to \mathcal{L} \to \mathcal{L}^{\otimes 2} \to i_*(\mathcal{L}^{\otimes 2}|_Z) \to 0$$

Observe that Z has dimension 0 (Divisors, Lemma 13.5) and hence is the spectrum of an Artinian ring (Varieties, Lemma 20.2) hence $\mathcal{L}|_Z \cong \mathcal{O}_Z$ (Algebra, Lemma 78.7). The short exact sequence also shows that $H^1(X, \mathcal{L}^{\otimes 2}) = 0$ (for example using Varieties, Lemma 33.3 to see vanishing in the spot on the right). Using induction on $n \geq 1$ and the sequence

$$0 \to \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{L}^{\otimes n+1} \to i_*(\mathcal{L}^{\otimes n+1}|_Z) \to 0$$

we see that $H^1(X, \mathcal{L}^{\otimes n}) = 0$ for $n > 0$ and that there exists a global section t_{n+1} of $\mathcal{L}^{\otimes n+1}$ which gives a trivialization of $\mathcal{L}^{\otimes n+1}|_Z \cong \mathcal{O}_Z$.

Consider the multiplication map

$$\mu_n : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}^{\otimes n}) \oplus H^0(X, \mathcal{L}^{\otimes 2}) \otimes_k H^0(X, \mathcal{L}^{\otimes n-1}) \longrightarrow H^0(X, \mathcal{L}^{\otimes n+1})$$

We claim this is surjective for $n \geq 3$. To see this we consider the short exact sequence

$$0 \to \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{L}^{\otimes n+1} \to i_*(\mathcal{L}^{\otimes n+1}|_Z) \to 0$$

The sections of $\mathcal{L}^{\otimes n+1}$ coming from the left in this sequence are in the image of μ_n. On the other hand, since $H^0(\mathcal{L}^{\otimes 2}) \to H^0(\mathcal{L}^{\otimes 2}|_Z)$ is surjective (see above) and since t_{n-1} maps to a trivialization of $\mathcal{L}^{\otimes n-1}|_Z$ we see that $\mu_n(H^0(X, \mathcal{L}^{\otimes 2}) \otimes t_{n-1})$ gives a subspace of $H^0(X, \mathcal{L}^{\otimes n+1})$ surjecting onto the global sections of $\mathcal{L}^{\otimes n+1}|_Z$. This proves the claim.

From the claim in the previous paragraph we conclude that the graded k-algebra

$$S = \bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$$

is generated in degrees $0, 1, 2, 3$ over k. Recall that $X = \text{Proj}(S)$, see Morphisms, Lemma 43.17. Thus $S^{(6)} = \bigoplus_n S_{6n}$ is generated in degree 1. This means that $\mathcal{L}^{\otimes 6}$ is very ample as desired. ☐

Lemma 7.2. *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume*

1. *$\mathcal{L}$ is globally generated,*
2. *$H^1(X, \mathcal{L}) = 0$, and*
3. *$\mathcal{L}$ is ample.*

Then $\mathcal{L}^{\otimes 2}$ is very ample on X over k.

Proof. Choose basis $s_0, \ldots, s_n$ of $H^0(X, \mathcal{L}^{\otimes 2})$ over k. By property (1) we see that $\mathcal{L}^{\otimes 2}$ is globally generated and we get a morphism

$$\varphi_{\mathcal{L}^{\otimes 2}, (s_0, \ldots, s_n)} : X \longrightarrow \mathbf{P}^n_k$$

See Constructions, Section 13. The lemma asserts that this morphism is a closed immersion. To check this we may replace k by its algebraic closure, see Descent, Lemma 23.19. Thus we may assume k is algebraically closed.

Assume k is algebraically closed. For each generic point $\eta_i \in X$ let $V_i \subset H^0(X, \mathcal{L})$ be the k-subvector space of sections vanishing at η_i. Since $\mathcal{L}$ is globally generated, we see that $V_i \neq H^0(X, \mathcal{L})$. Since X has only a finite number of irreducible components and k is infinite, we can find $s \in H^0(X, \mathcal{L})$ nonvanishing at η_i for all i. Then s is a regular section of $\mathcal{L}$ (because X is Cohen-Macaulay by Lemma 6.1 and hence $\mathcal{L}$ has no embedded associated points).

In particular, all of the statements given in the proof of Lemma 7.1 hold with this s. Moreover, as $\mathcal{L}$ is globally generated, we can find a global section $t \in H^0(X, \mathcal{L})$ such that $t|_Z$ is nonvanishing (argue as above using the finite number of points of Z). Then in the proof of Lemma 7.1 we can use t to see that additionally the multiplication map

$$\mu_n : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}^{\otimes 2}) \longrightarrow H^0(X, \mathcal{L}^{\otimes 3})$$

is surjective. Thus

$$S = \bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$$

is generated in degrees $0, 1, 2$ over k. Arguing as in the proof of Lemma 7.1 we find that $S^{(2)} = \bigoplus_n S_{2n}$ is generated in degree 1. This means that $\mathcal{L}^{\otimes 2}$ is very ample as desired. Some details omitted. ☐

8. The genus of a curve

If X is a smooth projective geometrically irreducible curve over a field k, then we've previously defined the genus of X as the dimension of $H^1(X, \mathcal{O}_X)$, see Picard Schemes of Curves, Definition 6.3. Observe that $H^0(X, \mathcal{O}_X) = k$ in this case, see Varieties, Lemma 26.2. Let us generalize this as follows.

Definition 8.1. Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then the *genus* of X is $g = \dim_k H^1(X, \mathcal{O}_X)$.

This is sometimes called the *arithmetic genus* of X. In the literature the arithmetic genus of a proper curve X over k is sometimes defined as

$$p_a(X) = 1 - \chi(X, \mathcal{O}_X) = 1 - \dim_k H^0(X, \mathcal{O}_X) + \dim_k H^1(X, \mathcal{O}_X)$$

This agrees with our definition when it applies because we assume $H^0(X, \mathcal{O}_X) = k$. But note that

(1) $p_a(X)$ can be negative, and
(2) $p_a(X)$ depends on the base field k and should be written $p_a(X/k)$.

For example if $k = \mathbf{Q}$ and $X = \mathbf{P}^1_{\mathbf{Q}(i)}$ then $p_a(X/\mathbf{Q}) = -1$ and $p_a(X/\mathbf{Q}(i)) = 0$.

The assumption that $H^0(X, \mathcal{O}_X) = k$ in our definition has two consequences. On the one hand, it means there is no confusion about the base field. On the other hand, it implies the scheme X is Cohen-Macaulay and equidimensional of dimension 1 (Lemma 6.1). If ω_X denotes the dualizing module as in Lemmas 4.1 and 4.2 we see that

$$g = \dim_k H^1(X, \mathcal{O}_X) = \dim_k H^0(X, \omega_X) \tag{8.1.1}$$

by duality, see Remark 4.3.

If X is proper over k of dimension ≤ 1 and $H^0(X, \mathcal{O}_X)$ is not equal to the ground field k, instead of using the arithmetic genus $p_a(X)$ given by the displayed formula above we shall use the invariant $\chi(X, \mathcal{O}_X)$. In fact, it is advocated in [Ser55, page 276] and [Hir95, Introduction] that we should call $\chi(X, \mathcal{O}_X)$ the arithmetic genus.

Lemma 8.2. *Let k'/k be a field extension. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Then $X_{k'}$ is a proper scheme over k' having dimension 1 and $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$. Moreover the genus of $X_{k'}$ is equal to the genus of X.*

Proof. The dimension of $X_{k'}$ is 1 for example by Morphisms, Lemma 28.3. The morphism $X_{k'} \to \mathrm{Spec}(k')$ is proper by Morphisms, Lemma 41.5. The equality $H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = k'$ follows from Cohomology of Schemes, Lemma 5.2. The equality of the genus follows from the same lemma. $\square$

Lemma 8.3. *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. If X is Gorenstein, then*

$$\deg(\omega_X) = 2g - 2$$

where g is the genus of X and ω_X is as in Lemma 4.1.

Proof. Immediate from Lemma 5.2. $\square$

Lemma 8.4. *Let X be a smooth proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. Then*

$$\dim_k H^0(X, \Omega_{X/k}) = g \quad \textit{and} \quad \deg(\Omega_{X/k}) = 2g - 2$$

where g is the genus of X.

Proof. By Lemma 4.1 we have $\Omega_{X/k} = \omega_X$. Hence the formulas hold by (8.1.1) and Lemma 8.3. $\square$

9. Plane curves

Let k be a field. A *plane curve* will be a curve X which is isomorphic to a closed subscheme of $\mathbf{P}^2_k$. Often the embedding $X \to \mathbf{P}^2_k$ will be considered given. By Divisors, Example 31.2 a curve is determined by the corresponding homogeneous ideal

$$I(X) = \text{Ker}\left(k[T_0, T_2, T_2] \longrightarrow \bigoplus \Gamma(X, \mathcal{O}_X(n))\right)$$

Recall that in this situation we have

$$X = \text{Proj}(k[T_0, T_2, T_2]/I)$$

as closed subschemes of $\mathbf{P}^2_k$. For more general information on these constructions we refer the reader to Divisors, Example 31.2 and the references therein. It turns out that $I(X) = (F)$ for some homogeneous polynomial $F \in k[T_0, T_1, T_2]$, see Lemma 9.1. Since X is irreducible, it follows that F is irreducible, see Lemma 9.2. Moreover, looking at the short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \xrightarrow{F} \mathcal{O}_{\mathbf{P}^2_k} \to \mathcal{O}_X \to 0$$

where $d = \deg(F)$ we find that $H^0(X, \mathcal{O}_X) = k$ and that X has genus $(d-1)(d-2)/2$, see proof of Lemma 9.3.

To find smooth plane curves it is easiest to write explicit equations. Let p denote the characteristic of k. If p does not divide d, then we can take

$$F = T_0^d + T_1^d + T_2^d$$

The corresponding curve $X = V_+(F)$ is called the *Fermat curve* of degree d. It is smooth because on each standard affine piece $D_+(T_i)$ we obtain a curve isomorphic to the affine curve

$$\text{Spec}(k[x, y]/(x^d + y^d + 1))$$

The ring map $k \to k[x, y]/(x^d + y^d + 1)$ is smooth by Algebra, Lemma 137.16 as dx^{d-1} and dy^{d-1} generate the unit ideal in $k[x, y]/(x^d + y^d + 1)$. If $p|d$ but $p \neq 3$ then you can use the equation

$$F = T_0^{d-1}T_1 + T_1^{d-1}T_2 + T_2^{d-1}T_0$$

Namely, on the affine pieces you get $x + x^{d-1}y + y^{d-1}$ with derivatives $1 - x^{d-2}y$ and $x^{d-1} - y^{d-2}$ whose common zero set (of all three) is empty[4]. We leave it to the reader to make examples in characteristic 3.

More generally for any field k and any n and d there exists a smooth hypersurface of degree d in $\mathbf{P}^n_k$, see for example [Poo05].

Of course, in this way we only find smooth curves whose genus is a triangular number. To get smooth curves of an arbitrary genus one can look for smooth curves lying on $\mathbf{P}^1 \times \mathbf{P}^1$ (insert future reference here).

Lemma 9.1. *Let $Z \subset \mathbf{P}^2_k$ be a closed subscheme which is equidimensional of dimension 1 and has no embedded points (equivalently Z is Cohen-Macaulay). Then the ideal $I(Z) \subset k[T_0, T_1, T_2]$ corresponding to Z is principal.*

---

[4]Namely, as $x^{d-1} = y^{d-2}$, then $0 = x + x^{d-1}y + y^{d-1} = x + 2x^{d-1}y$. Since $x \neq 0$ because $1 = x^{d-2}y$ we get $0 = 1 + 2x^{d-2}y = 3$ which is absurd unless $3 = 0$.

Proof. This is a special case of Divisors, Lemma 31.3 (see also Varieties, Lemma 34.4). The parenthetical statement follows from the fact that a 1 dimensional Noetherian scheme is Cohen-Macaulay if and only if it has no embedded points, see Divisors, Lemma 4.4. □

Lemma 9.2. *Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 9.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. Then Z is a curve if and only if F is irreducible.*

Proof. If F is reducible, say $F = F'F''$ then let Z' be the closed subscheme of $\mathbf{P}^2_k$ defined by F'. It is clear that $Z' \subset Z$ and that $Z' \neq Z$. Since Z' has dimension 1 as well, we conclude that either Z is not reduced, or that Z is not irreducible. Conversely, write $Z = \sum a_i D_i$ where D_i are the irreducible components of Z, see Divisors, Lemmas 15.8 and 15.9. Let $F_i \in k[T_0, T_1, T_2]$ be the homogeneous polynomial generating the ideal of D_i. Then it is clear that F and $\prod F_i^{a_i}$ cut out the same closed subscheme of $\mathbf{P}^2_k$. Hence $F = \lambda \prod F_i^{a_i}$ for some $\lambda \in k^*$ because both generate the ideal of Z. Thus we see that if F is irreducible, then Z is a prime divisor, i.e., a curve. □

Lemma 9.3. *Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 9.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. Then $H^0(Z, \mathcal{O}_Z) = k$ and the genus of Z is $(d-1)(d-2)/2$ where $d = \deg(F)$.*

Proof. Let $S = k[T_0, T_1, T_2]$. There is an exact sequence of graded modules

$$0 \to S(-d) \xrightarrow{F} S \to S/(F) \to 0$$

Denote $i : Z \to \mathbf{P}^2_k$ the given closed immersion. Applying the exact functor $\widetilde{\ }$ (Constructions, Lemma 8.4) we obtain

$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \to \mathcal{O}_{\mathbf{P}^2_k} \to i_*\mathcal{O}_Z \to 0$$

because F generates the ideal of Z. Note that the cohomology groups of $\mathcal{O}_{\mathbf{P}^2_k}(-d)$ and $\mathcal{O}_{\mathbf{P}^2_k}$ are given in Cohomology of Schemes, Lemma 8.1. On the other hand, we have $H^q(Z, \mathcal{O}_Z) = H^q(\mathbf{P}^2_k, i_*\mathcal{O}_Z)$ by Cohomology of Schemes, Lemma 2.4. Applying the long exact cohomology sequence we first obtain that

$$k = H^0(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}) \longrightarrow H^0(Z, \mathcal{O}_Z)$$

is an isomorphism and next that the boundary map

$$H^1(Z, \mathcal{O}_Z) \longrightarrow H^2(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(-d)) \cong k[T_0, T_1, T_2]_{d-3}$$

is an isomorphism. Since it is easy to see that the dimension of this is $(d-1)(d-2)/2$ the proof is finished. □

Lemma 9.4. *Let $Z \subset \mathbf{P}^2_k$ be as in Lemma 9.1 and let $I(Z) = (F)$ for some $F \in k[T_0, T_1, T_2]$. If $Z \to \mathrm{Spec}(k)$ is smooth in at least one point and k is infinite, then there exists a closed point $z \in Z$ contained in the smooth locus such that $\kappa(z)/k$ is finite separable of degree at most d.*

Proof. Suppose that $z' \in Z$ is a point where $Z \to \mathrm{Spec}(k)$ is smooth. After renumbering the coordinates if necessary we may assume z' is contained in $D_+(T_0)$. Set $f = F(1, x, y) \in k[x, y]$. Then $Z \cap D_+(X_0)$ is isomorphic to the spectrum of $k[x, y]/(f)$. Let f_x, f_y be the partial derivatives of f with respect to x, y. Since z' is a smooth point of Z/k we see that either f_x or f_y is nonzero in z' (see discussion in Algebra, Section 137). After renumbering the coordinates we may assume f_y is not zero at z'. Hence there is a nonempty open subscheme $V \subset Z \cap D_+(X_0)$ such that the projection

$$p : V \longrightarrow \mathrm{Spec}(k[x])$$

is étale. Because the degree of f as a polynomial in y is at most d, we see that the degrees of the fibres of the projection p are at most d (see discussion in Morphisms, Section 57). Moreover, as p is étale the image of p is an open $U \subset \mathrm{Spec}(k[x])$. Finally, since k is infinite, the set of k-rational points $U(k)$ of U is infinite, in particular not empty. Pick any $t \in U(k)$ and let $z \in V$ be a point mapping to t. Then z works. □

10. Curves of genus zero

Later we will need to know what a proper genus zero curve looks like. It turns out that a Gorenstein proper genus zero curve is a plane curve of degree 2, i.e., a conic, see Lemma 10.3. A general proper genus zero curve is obtained from a nonsingular one (over a bigger field) by a pushout procedure, see Lemma 10.5. Since a nonsingular curve is Gorenstein, these two results cover all possible cases.

Lemma 10.1. *Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. If X has genus 0, then every invertible $\mathcal{O}_X$-module $\mathcal{L}$ of degree 0 is trivial.*

Proof. Namely, we have $\dim_k H^0(X, \mathcal{L}) \geq 0+1-0 = 1$ by Riemann-Roch (Lemma 5.2), hence $\mathcal{L}$ has a nonzero section, hence $\mathcal{L} \cong \mathcal{O}_X$ by Varieties, Lemma 44.12. □

Lemma 10.2. *Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. Assume X has genus 0. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module of degree $d > 0$. Then we have*

1. $\dim_k H^0(X, \mathcal{L}) = d + 1$ *and* $\dim_k H^1(X, \mathcal{L}) = 0$,
2. $\mathcal{L}$ *is very ample and defines a closed immersion into* $\mathbf{P}^d_k$.

Proof. By definition of degree and genus we have

$$\dim_k H^0(X, \mathcal{L}) - \dim_k H^1(X, \mathcal{L}) = d + 1$$

Let s be a nonzero section of $\mathcal{L}$. Then the zero scheme of s is an effective Cartier divisor $D \subset X$, we have $\mathcal{L} = \mathcal{O}_X(D)$ and we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}|_D \to 0$$

see Divisors, Lemma 14.10 and Remark 14.11. Since $H^1(X, \mathcal{O}_X) = 0$ by assumption, we see that $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_D)$ is surjective. As $\mathcal{L}|_D$ is generated by global sections (because $\dim(D) = 0$, see Varieties, Lemma 33.3) we conclude that the invertible module $\mathcal{L}$ is generated by global sections. In fact, since D is an Artinian scheme we have $\mathcal{L}|_D \cong \mathcal{O}_D$[5] and hence we can find a section t of $\mathcal{L}$ whose restriction of D generates $\mathcal{L}|_D$. The short exact sequence also shows that $H^1(X, \mathcal{L}) = 0$.

For $n \geq 1$ consider the multiplication map

$$\mu_n : H^0(X, \mathcal{L}) \otimes_k H^0(X, \mathcal{L}^{\otimes n}) \longrightarrow H^0(X, \mathcal{L}^{\otimes n+1})$$

We claim this is surjective. To see this we consider the short exact sequence

$$0 \to \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{L}^{\otimes n+1} \to \mathcal{L}^{\otimes n+1}|_D \to 0$$

---

[5] In our case this follows from Divisors, Lemma 17.1 as $D \to \mathrm{Spec}(k)$ is finite.

The sections of $\mathcal{L}^{\otimes n+1}$ coming from the left in this sequence are in the image of μ_n. On the other hand, since $H^0(\mathcal{L}) \to H^0(\mathcal{L}|_D)$ is surjective and since t^n maps to a trivialization of $\mathcal{L}^{\otimes n}|_D$ we see that $\mu_n(H^0(X, \mathcal{L}) \otimes t^n)$ gives a subspace of $H^0(X, \mathcal{L}^{\otimes n+1})$ surjecting onto the global sections of $\mathcal{L}^{\otimes n+1}|_D$. This proves the claim.

Observe that $\mathcal{L}$ is ample by Varieties, Lemma 44.14. Hence Morphisms, Lemma 43.17 gives an isomorphism

$$X \longrightarrow \text{Proj}\left(\bigoplus\nolimits_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})\right)$$

Since the maps μ_n are surjective for all $n \geq 1$ we see that the graded algebra on the right hand side is a quotient of the symmetric algebra on $H^0(X, \mathcal{L})$. Choosing a k-basis $s_0, \ldots, s_d$ of $H^0(X, \mathcal{L})$ we see that it is a quotient of a polynomial algebra in $d+1$ variables. Since quotients of graded rings correspond to closed immersions of Proj (Constructions, Lemma 11.5) we find a closed immersion $X \to \mathbf{P}^d_k$. We omit the verification that this morphism is the morphism of Constructions, Lemma 13.1 associated to the sections $s_0, \ldots, s_d$ of $\mathcal{L}$. $\square$

Lemma 10.3. *Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. If X is Gorenstein and has genus 0, then X is isomorphic to a plane curve of degree 2.*

Proof. Consider the invertible sheaf $\mathcal{L} = \omega_X^{\otimes -1}$ where ω_X is as in Lemma 4.1. Then $\deg(\omega_X) = -2$ by Lemma 8.3 and hence $\deg(\mathcal{L}) = 2$. By Lemma 10.2 we conclude that choosing a basis s_0, s_1, s_2 of the k-vector space of global sections of $\mathcal{L}$ we obtain a closed immersion

$$\varphi_{(\mathcal{L},(s_0,s_1,s_2))} : X \longrightarrow \mathbf{P}^2_k$$

Thus X is a plane curve of some degree d. Let $F \in k[T_0, T_1, T_2]_d$ be its equation (Lemma 9.1). Because the genus of X is 0 we see that d is 1 or 2 (Lemma 9.3). Observe that F restricts to the zero section on $\varphi(X)$ and hence $F(s_0, s_1, s_2)$ is the zero section of $\mathcal{L}^{\otimes 2}$. Because s_0, s_1, s_2 are linearly independent we see that F cannot be linear, i.e., $d = \deg(F) \geq 2$. Thus $d = 2$ and the proof is complete. $\square$

Proposition 10.4 (Characterization of the projective line)**.** *Let k be a field. Let X be a proper curve over k. The following are equivalent*

1. $X \cong \mathbf{P}^1_k$,
2. *X is smooth and geometrically irreducible over k, X has genus 0, and X has an invertible module of odd degree,*
3. *X is geometrically integral over k, X has genus 0, X is Gorenstein, and X has an invertible sheaf of odd degree,*
4. *$H^0(X, \mathcal{O}_X) = k$, X has genus 0, X is Gorenstein, and X has an invertible sheaf of odd degree,*
5. *X is geometrically integral over k, X has genus 0, and X has an invertible $\mathcal{O}_X$-module of degree 1,*
6. *$H^0(X, \mathcal{O}_X) = k$, X has genus 0, and X has an invertible $\mathcal{O}_X$-module of degree 1,*
7. *$H^1(X, \mathcal{O}_X) = 0$ and X has an invertible $\mathcal{O}_X$-module of degree 1,*
8. *$H^1(X, \mathcal{O}_X) = 0$ and X has closed points $x_1, \ldots, x_n$ such that $\mathcal{O}_{X,x_i}$ is normal and $\gcd([\kappa(x_i) : k]) = 1$, and*
9. *add more here.*

Proof. We will prove that each condition (2) – (8) implies (1) and we omit the verification that (1) implies (2) – (8).

Assume (2). A smooth scheme over k is geometrically reduced (Varieties, Lemma 25.4) and regular (Varieties, Lemma 25.3). Hence X is Gorenstein (Duality for Schemes, Lemma 24.3). Thus we reduce to (3).

Assume (3). Since X is geometrically integral over k we have $H^0(X, \mathcal{O}_X) = k$ by Varieties, Lemma 26.2. and we reduce to (4).

Assume (4). Since X is Gorenstein the dualizing module ω_X as in Lemma 4.1 has degree $\deg(\omega_X) = -2$ by Lemma 8.3. Combined with the assumed existence of an odd degree invertible module, we conclude there exists an invertible module of degree 1. In this way we reduce to (6).

Assume (5). Since X is geometrically integral over k we have $H^0(X, \mathcal{O}_X) = k$ by Varieties, Lemma 26.2. and we reduce to (6).

Assume (6). Then $X \cong \mathbf{P}^1_k$ by Lemma 10.2.

Assume (7). Observe that $\kappa = H^0(X, \mathcal{O}_X)$ is a field finite over k by Varieties, Lemma 26.2. If $d = [\kappa : k] > 1$, then every invertible sheaf has degree divisible by d and there cannot be an invertible sheaf of degree 1. Hence $d = 1$ and we reduce to case (6).

Assume (8). Observe that $\kappa = H^0(X, \mathcal{O}_X)$ is a field finite over k by Varieties, Lemma 26.2. Since $\kappa \subset \kappa(x_i)$ we see that $k = \kappa$ by the assumption on the gcd of the degrees. The same condition allows us to find integers a_i such that $1 = \sum a_i[\kappa(x_i) : k]$. Because x_i defines an effective Cartier divisor on X by Varieties, Lemma 43.8 we can consider the invertible module $\mathcal{O}_X(\sum a_i x_i)$. By our choice of a_i the degree of $\mathcal{L}$ is 1. Thus $X \cong \mathbf{P}^1_k$ by Lemma 10.2. $\square$

Lemma 10.5. *Let X be a proper curve over a field k with $H^0(X, \mathcal{O}_X) = k$. Assume X is singular and has genus 0. Then there exists a diagram*

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\nu"] & \mathrm{Spec}(k') \arrow[d] \\ x \arrow[r] & X \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

where

1. *k'/k is a nontrivial finite extension,*
2. *$X' \cong \mathbf{P}^1_{k'}$,*
3. *x' is a k'-rational point of X',*
4. *x is a k-rational point of X,*
5. *$X' \setminus \{x'\} \to X \setminus \{x\}$ is an isomorphism,*
6. *$0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_{X'} \to k'/k \to 0$ is a short exact sequence where $k'/k = \kappa(x')/\kappa(x)$ indicates the skyscraper sheaf on the point x.*

Proof. Let $\nu : X' \to X$ be the normalization of X, see Varieties, Sections 27 and 41. Since X is singular ν is not an isomorphism. Then $k' = H^0(X', \mathcal{O}_{X'})$ is a finite extension of k (Varieties, Lemma 26.2). The short exact sequence

$$0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_{X'} \to \mathcal{Q} \to 0$$

and the fact that $\mathcal{Q}$ is supported in finitely many closed points give us that

(1) $H^1(X', \mathcal{O}_{X'}) = 0$, i.e., X' has genus 0 as a curve over k',
(2) there is a short exact sequence $0 \to k \to k' \to H^0(X, \mathcal{Q}) \to 0$.

In particular k'/k is a nontrivial extension.

Next, we consider what is often called the *conductor ideal*

$$\mathcal{I} = \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X'}, \mathcal{O}_X)$$

This is a quasi-coherent $\mathcal{O}_X$-module. We view $\mathcal{I}$ as an ideal in $\mathcal{O}_X$ via the map $\varphi \mapsto \varphi(1)$. Thus $\mathcal{I}(U)$ is the set of $f \in \mathcal{O}_X(U)$ such that $f(\nu_*\mathcal{O}_{X'}(U)) \subset \mathcal{O}_X(U)$. In other words, the condition is that f annihilates $\mathcal{Q}$. In other words, there is a defining exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{Q})$$

Let $U \subset X$ be an affine open containing the support of $\mathcal{Q}$. Then $V = \mathcal{Q}(U) = H^0(X, \mathcal{Q})$ is a k-vector space of dimension $n-1$. The image of $\mathcal{O}_X(U) \to \mathrm{Hom}_k(V, V)$ is a commutative subalgebra, hence has dimension $\leq n-1$ over k (this is a property of commutative subalgebras of matrix algebras; details omitted). We conclude that we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{A} \to 0$$

where $\mathrm{Supp}(\mathcal{A}) = \mathrm{Supp}(\mathcal{Q})$ and $\dim_k H^0(X, \mathcal{A}) \leq n-1$. On the other hand, the description $\mathcal{I} = \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X'}, \mathcal{O}_X)$ provides $\mathcal{I}$ with a $\nu_*\mathcal{O}_{X'}$-module structure such that the inclusion map $\mathcal{I} \to \nu_*\mathcal{O}_{X'}$ is a $\nu_*\mathcal{O}_{X'}$-module map. We conclude that $\mathcal{I} = \nu_*\mathcal{I}'$ for some quasi-coherent sheaf of ideals $\mathcal{I}' \subset \mathcal{O}_{X'}$, see Morphisms, Lemma 11.6. Define $\mathcal{A}'$ as the cokernel:

$$0 \to \mathcal{I}' \to \mathcal{O}_{X'} \to \mathcal{A}' \to 0$$

Combining the exact sequences so far we obtain a short exact sequence $0 \to \mathcal{A} \to \nu_*\mathcal{A}' \to \mathcal{Q} \to 0$. Using the estimate above, combined with $\dim_k H^0(X, \mathcal{Q}) = n-1$, gives

$$\dim_k H^0(X', \mathcal{A}') = \dim_k H^0(X, \mathcal{A}) + \dim_k H^0(X, \mathcal{Q}) \leq 2n-2$$

However, since X' is a curve over k' we see that the left hand side is divisible by n (Varieties, Lemma 44.10). As $\mathcal{A}$ and $\mathcal{A}'$ cannot be zero, we conclude that $\dim_k H^0(X', \mathcal{A}') = n$ which means that $\mathcal{I}'$ is the ideal sheaf of a k'-rational point x'. By Proposition 10.4 we find $X' \cong \mathbf{P}^1_{k'}$. Going back to the equalities above, we conclude that $\dim_k H^0(X, \mathcal{A}) = 1$. This means that $\mathcal{I}$ is the ideal sheaf of a k-rational point x. Then $\mathcal{A} = \kappa(x) = k$ and $\mathcal{A}' = \kappa(x') = k'$ as skyscraper sheaves. Comparing the exact sequences given above, this immediately implies the result on structure sheaves as stated in the lemma. □

Example 10.6. In fact, the situation described in Lemma 10.5 occurs for any nontrivial finite extension k'/k. Namely, we can consider

$$A = \{f \in k'[x] \mid f(0) \in k\}$$

The spectrum of A is an affine curve, which we can glue to the spectrum of $B = k'[y]$ using the isomorphism $A_x \cong B_y$ sending x^{-1} to y. The result is a proper curve X with $H^0(X, \mathcal{O}_X) = k$ and singular point x corresponding to the maximal ideal $A \cap (x)$. The normalization of X is $\mathbf{P}^1_{k'}$ exactly as in the lemma.

11. Geometric genus

If X is a proper and **smooth** curve over k with $H^0(X, \mathcal{O}_X) = k$, then

$$p_g(X) = \dim_k H^0(X, \Omega_{X/k})$$

is called the *geometric genus* of X. By Lemma 8.4 the geometric genus of X agrees with the (arithmetic) genus. However, in higher dimensions there is a difference between the geometric genus and the arithmetic genus, see Remark 11.2.

For singular curves, we will define the geometric genus as follows.

Definition 11.1. Let k be a field. Let X be a geometrically irreducible curve over k. The *geometric genus* of X is the genus of a smooth projective model of X possibly defined over an extension field of k as in Lemma 2.9.

If k is perfect, then the nonsingular projective model Y of X is smooth (Lemma 2.8) and the geometric genus of X is just the genus of Y. But if k is not perfect, this may not be true. In this case we choose an extension K/k such that the nonsingular projective model Y_K of $(X_K)_{red}$ is a smooth projective curve and we define the geometric genus of X to be the genus of Y_K. This is well defined by Lemmas 2.9 and 8.2.

Remark 11.2. Suppose that X is a d-dimensional proper smooth variety over an algebraically closed field k. Then the *arithmetic genus* is often defined as $p_a(X) = (-1)^d(\chi(X, \mathcal{O}_X) - 1)$ and the *geometric genus* as $p_g(X) = \dim_k H^0(X, \Omega^d_{X/k})$. In this situation the arithmetic genus and the geometric genus no longer agree even though it is still true that $\omega_X \cong \Omega^d_{X/k}$. For example, if $d = 2$, then we have

$$\begin{aligned}
p_a(X) - p_g(X) &= h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) - 1 - h^0(X, \Omega^2_{X/k}) \\
&= -h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) - h^0(X, \omega_X) \\
&= -h^1(X, \mathcal{O}_X)
\end{aligned}$$

where $h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})$ and where the last equality follows from duality. Hence for a surface the difference $p_g(X) - p_a(X)$ is always nonnegative; it is sometimes called the irregularity of the surface. If $X = C_1 \times C_2$ is a product of smooth projective curves of genus g_1 and g_2, then the irregularity is $g_1 + g_2$.

12. Riemann-Hurwitz

Let k be a field. Let $f : X \to Y$ be a morphism of smooth curves over k. Then we obtain a canonical exact sequence

$$f^*\Omega_{Y/k} \xrightarrow{\mathrm{d}f} \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

by Morphisms, Lemma 32.9. Since X and Y are smooth, the sheaves $\Omega_{X/k}$ and $\Omega_{Y/k}$ are invertible modules, see Morphisms, Lemma 34.12. Assume the first map is nonzero, i.e., assume f is generically étale, see Lemma 12.1. Let $R \subset X$ be the closed subscheme cut out by the different $\mathfrak{D}_f$ of f. By Discriminants, Lemma 12.6 this is the same as the vanishing locus of $\mathrm{d}f$, it is an effective Cartier divisor, and we get

$$f^*\Omega_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(R) = \Omega_{X/k}$$

In particular, if X, Y are projective with $k = H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ and X, Y have genus g_X, g_Y, then we get the Riemann-Hurwitz formula

$$\begin{aligned} 2g_X - 2 &= \deg(\Omega_{X/k}) \\ &= \deg(f^*\Omega_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(R)) \\ &= \deg(f)\deg(\Omega_{Y/k}) + \deg(R) \\ &= \deg(f)(2g_Y - 2) + \deg(R) \end{aligned}$$

The first and last equality by Lemma 8.4. The second equality by the isomorphism of invertible sheaves given above. The third equality by additivity of degrees (Varieties, Lemma 44.7), the formula for the degree of a pullback (Varieties, Lemma 44.11), and finally the formula for the degree of $\mathcal{O}_X(R)$ (Varieties, Lemma 44.9).

To use the Riemann-Hurwitz formula we need to compute $\deg(R) = \dim_k \Gamma(R, \mathcal{O}_R)$. By the structure of zero dimensional schemes over k (see for example Varieties, Lemma 20.2), we see that R is a finite disjoint union of spectra of Artinian local rings $R = \coprod_{x \in R} \text{Spec}(\mathcal{O}_{R,x})$ with each $\mathcal{O}_{R,x}$ of finite dimension over k. Thus

$$\deg(R) = \sum\nolimits_{x \in R} \dim_k \mathcal{O}_{R,x} = \sum\nolimits_{x \in R} d_x[\kappa(x) : k]$$

with

$$d_x = \text{length}_{\mathcal{O}_{R,x}} \mathcal{O}_{R,x} = \text{length}_{\mathcal{O}_{X,x}} \mathcal{O}_{R,x}$$

the multiplicity of x in R (see Algebra, Lemma 52.12). Let $x \in X$ be a closed point with image $y \in Y$. Looking at stalks we obtain an exact sequence

$$\Omega_{Y/k,y} \to \Omega_{X/k,x} \to \Omega_{X/Y,x} \to 0$$

Choosing local generators η_x and η_y of the (free rank 1) modules $\Omega_{X/k,x}$ and $\Omega_{Y/k,y}$ we see that $\eta_y \mapsto h\eta_x$ for some nonzero $h \in \mathcal{O}_{X,x}$. By the exact sequence we see that $\Omega_{X/Y,x} \cong \mathcal{O}_{X,x}/h\mathcal{O}_{X,x}$ as $\mathcal{O}_{X,x}$-modules. Since the divisor R is cut out by h (see above) we have $\mathcal{O}_{R,x} = \mathcal{O}_{X,x}/h\mathcal{O}_{X,x}$. Thus we find the following equalities

$$\begin{aligned} d_x &= \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{R,x}) \\ &= \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/h\mathcal{O}_{X,x}) \\ &= \text{length}_{\mathcal{O}_{X,x}}(\Omega_{X/Y,x}) \\ &= \text{ord}_{\mathcal{O}_{X,x}}(h) \\ &= \text{ord}_{\mathcal{O}_{X,x}}(\text{“}\eta_y/\eta_x\text{”}) \end{aligned}$$

The first equality by our definition of d_x. The second and third we saw above. The fourth equality is the definition of ord, see Algebra, Definition 121.2. Note that since $\mathcal{O}_{X,x}$ is a discrete valuation ring, the integer $\text{ord}_{\mathcal{O}_{X,x}}(h)$ just the valuation of h. The fifth equality is a mnemonic.

Here is a case where one can “calculate” the multiplicity d_x in terms of other invariants. Namely, if $\kappa(x)$ is separable over k, then we may choose $\eta_x = \text{d}s$ and $\eta_y = \text{d}t$ where s and t are uniformizers in $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ (Lemma 12.3). Then $t \mapsto us^{e_x}$ for some unit $u \in \mathcal{O}_{X,x}$ where e_x is the ramification index of the extension $\mathcal{O}_{Y,y} \subset \mathcal{O}_{X,x}$. Hence we get

$$\eta_y = \text{d}t = \text{d}(us^{e_x}) = es^{e_x - 1}u\text{d}s + s^{e_x}\text{d}u$$

Writing $\text{d}u = w\text{d}s$ for some $w \in \mathcal{O}_{X,x}$ we see that

$$\text{“}\eta_y/\eta_x\text{”} = es^{e_x-1}u + s^{e_x}w = (e_xu + sw)s^{e_x-1}$$

We conclude that the order of vanishing of this is $e_x - 1$ unless the characteristic of $\kappa(x)$ is $p > 0$ and p divides e_x in which case the order of vanishing is $> e_x - 1$.

Combining all of the above we find that if k has characteristic zero, then

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \sum\nolimits_{x \in X}(e_x - 1)[\kappa(x) : k]$$

where e_x is the ramification index of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,f(x)}$. This precise formula will hold if and only if all the ramification is tame, i.e., when the residue field extensions $\kappa(x)/\kappa(y)$ are separable and e_x is prime to the characteristic of k, although the arguments above are insufficient to prove this. We refer the reader to Lemma 12.4 and its proof.

Lemma 12.1. *Let k be a field. Let $f : X \to Y$ be a morphism of smooth curves over k. The following are equivalent*

(1) *$df : f^*\Omega_{Y/k} \to \Omega_{X/k}$ is nonzero,*
(2) *$\Omega_{X/Y}$ is supported on a proper closed subset of X,*
(3) *there exists a nonempty open $U \subset X$ such that $f|_U : U \to Y$ is unramified,*
(4) *there exists a nonempty open $U \subset X$ such that $f|_U : U \to Y$ is étale,*
(5) *the extension $k(X)/k(Y)$ of function fields is finite separable.*

Proof. Since X and Y are smooth, the sheaves $\Omega_{X/k}$ and $\Omega_{Y/k}$ are invertible modules, see Morphisms, Lemma 34.12. Using the exact sequence

$$f^*\Omega_{Y/k} \longrightarrow \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

of Morphisms, Lemma 32.9 we see that (1) and (2) are equivalent and equivalent to the condition that $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is nonzero in the generic point. The equivalence of (2) and (3) follows from Morphisms, Lemma 35.2. The equivalence between (3) and (4) follows from Morphisms, Lemma 36.16 and the fact that flatness is automatic (Lemma 2.3). To see the equivalence of (5) and (4) use Algebra, Lemma 140.9. Some details omitted. $\square$

Lemma 12.2. *Let $f : X \to Y$ be a morphism of smooth proper curves over a field k which satisfies the equivalent conditions of Lemma 12.1. If $k = H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ and X and Y have genus g_X and g_Y, then*

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \deg(R)$$

where $R \subset X$ is the effective Cartier divisor cut out by the different of f.

Proof. See discussion above; we used Discriminants, Lemma 12.6, Lemma 8.4, and Varieties, Lemmas 44.7 and 44.11. $\square$

Lemma 12.3. *Let $X \to \mathrm{Spec}(k)$ be smooth of relative dimension 1 at a closed point $x \in X$. If $\kappa(x)$ is separable over k, then for any uniformizer s in the discrete valuation ring $\mathcal{O}_{X,x}$ the element ds freely generates $\Omega_{X/k,x}$ over $\mathcal{O}_{X,x}$.*

Proof. The ring $\mathcal{O}_{X,x}$ is a discrete valuation ring by Algebra, Lemma 140.3. Since x is closed $\kappa(x)$ is finite over k. Hence if $\kappa(x)/k$ is separable, then any uniformizer s maps to a nonzero element of $\Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ by Algebra, Lemma 140.4. Since $\Omega_{X/k,x}$ is free of rank 1 over $\mathcal{O}_{X,x}$ the result follows. $\square$

Lemma 12.4. *Notation and assumptions as in Lemma 12.2. For a closed point $x \in X$ let d_x be the multiplicity of x in R. Then*

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \sum d_x[\kappa(x) : k]$$

Moreover, we have the following results

(1) $d_x = length_{\mathcal{O}_{X,x}}(\Omega_{X/Y,x})$,
(2) $d_x \geq e_x - 1$ *where* e_x *is the ramification index of* $\mathcal{O}_{X,x}$ *over* $\mathcal{O}_{Y,y}$,
(3) $d_x = e_x - 1$ *if and only if* $\mathcal{O}_{X,x}$ *is tamely ramified over* $\mathcal{O}_{Y,y}$.

Proof. By Lemma 12.2 and the discussion above (which used Varieties, Lemma 20.2 and Algebra, Lemma 52.12) it suffices to prove the results on the multiplicity d_x of x in R. Part (1) was proved in the discussion above. In the discussion above we proved (2) and (3) only in the case where $\kappa(x)$ is separable over k. In the rest of the proof we give a uniform treatment of (2) and (3) using material on differents of quasi-finite Gorenstein morphisms.

First, observe that f is a quasi-finite Gorenstein morphism. This is true for example because f is a flat quasi-finite morphism and X is Gorenstein (see Duality for Schemes, Lemma 25.7) or because it was shown in the proof of Discriminants, Lemma 12.6 (which we used above). Thus $\omega_{X/Y}$ is invertible by Discriminants, Lemma 16.1 and the same remains true after replacing X by opens and after performing a base change by some $Y' \to Y$. We will use this below without further mention.

Choose affine opens $U \subset X$ and $V \subset Y$ such that $x \in U$, $y \in V$, $f(U) \subset V$, and x is the only point of U lying over y. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$. Then $R \cap U$ is the different of $f|_U : U \to V$. By Discriminants, Lemma 9.4 formation of the different commutes with arbitrary base change in our case. By our choice of U and V we have

$$A \otimes_B \kappa(y) = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y) = \mathcal{O}_{X,x}/(s^{e_x})$$

where e_x is the ramification index as in the statement of the lemma. Let $C = \mathcal{O}_{X,x}/(s^{e_x})$ viewed as a finite algebra over $\kappa(y)$. Let $\mathfrak{D}_{C/\kappa(y)}$ be the different of C over $\kappa(y)$ in the sense of Discriminants, Definition 9.1. It suffices to show: $\mathfrak{D}_{C/\kappa(y)}$ is nonzero if and only if the extension $\mathcal{O}_{Y,y} \subset \mathcal{O}_{X,x}$ is tamely ramified and in the tamely ramified case $\mathfrak{D}_{C/\kappa(y)}$ is equal to the ideal generated by $s^{e_x - 1}$ in C. Recall that tame ramification means exactly that $\kappa(x)/\kappa(y)$ is separable and that the characteristic of $\kappa(y)$ does not divide e_x. On the other hand, the different of $C/\kappa(y)$ is nonzero if and only if $\tau_{C/\kappa(y)} \in \omega_{C/\kappa(y)}$ is nonzero. Namely, since $\omega_{C/\kappa(y)}$ is an invertible C-module (as the base change of $\omega_{A/B}$) it is free of rank 1, say with generator λ. Write $\tau_{C/\kappa(y)} = h\lambda$ for some $h \in C$. Then $\mathfrak{D}_{C/\kappa(y)} = (h) \subset C$ whence the claim. By Discriminants, Lemma 4.8 we have $\tau_{C/\kappa(y)} \neq 0$ if and only if $\kappa(x)/\kappa(y)$ is separable and e_x is prime to the characteristic. Finally, even if $\tau_{C/\kappa(y)}$ is nonzero, then it is still the case that $s\tau_{C/\kappa(y)} = 0$ because $s\tau_{C/\kappa(y)} : C \to \kappa(y)$ sends c to the trace of the nilpotent operator sc which is zero. Hence $sh = 0$, hence $h \in (s^{e_x - 1})$ which proves that $\mathfrak{D}_{C/\kappa(y)} \subset (s^{e_x-1})$ always. Since $(s^{e_x-1}) \subset C$ is the smallest nonzero ideal, we have proved the final assertion. $\square$

13. Inseparable maps

Some remarks on the behaviour of the genus under inseparable maps.

Lemma 13.1. *Let k be a field. Let $f : X \to Y$ be a surjective morphism of curves over k. If X is smooth over k and Y is normal, then Y is smooth over k.*

Proof. Let $y \in Y$. Pick $x \in X$ mapping to y. By Varieties, Lemma 25.9 it suffices to show that f is flat at x. This follows from Lemma 2.3. $\square$

Lemma 13.2. *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. If the extension $k(X)/k(Y)$ of function fields is purely inseparable, then there exists a factorization*

$$X = X_0 \to X_1 \to \ldots \to X_n = Y$$

such that each X_i is a proper nonsingular curve and $X_i \to X_{i+1}$ is a degree p morphism with $k(X_{i+1}) \subset k(X_i)$ inseparable.

Proof. This follows from Theorem 2.6 and the fact that a finite purely inseparable extension of fields can always be gotten as a sequence of (inseparable) extensions of degree p, see Fields, Lemma 14.5. $\square$

Lemma 13.3. *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. If X is smooth and $k(Y) \subset k(X)$ is inseparable of degree p, then there is a unique isomorphism $Y = X^{(p)}$ such that f is $F_{X/k}$.*

Proof. The relative frobenius morphism $F_{X/k} : X \to X^{(p)}$ is constructed in Varieties, Section 36. Observe that $X^{(p)}$ is a smooth proper curve over k as a base change of X. The morphism $F_{X/k}$ has degree p by Varieties, Lemma 36.10. Thus $k(X^{(p)})$ and $k(Y)$ are both subfields of $k(X)$ with $[k(X) : k(Y)] = [k(X) : k(X^{(p)})] = p$. To prove the lemma it suffices to show that $k(Y) = k(X^{(p)})$ inside $k(X)$. See Theorem 2.6.

Write $K = k(X)$. Consider the map $\text{d} : K \to \Omega_{K/k}$. It follows from Lemma 12.1 that both $k(Y)$ is contained in the kernel of d. By Varieties, Lemma 36.7 we see that $k(X^{(p)})$ is in the kernel of d. Since X is a smooth curve we know that $\Omega_{K/k}$ is a vector space of dimension 1 over K. Then More on Algebra, Lemma 46.2. implies that $\text{Ker}(\text{d}) = kK^p$ and that $[K : kK^p] = p$. Thus $k(Y) = kK^p = k(X^{(p)})$ for reasons of degree. $\square$

Lemma 13.4. *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. If X is smooth and $k(Y) \subset k(X)$ is purely inseparable, then there is a unique $n \geq 0$ and a unique isomorphism $Y = X^{(p^n)}$ such that f is the n-fold relative Frobenius of X/k.*

Proof. The n-fold relative Frobenius of X/k is defined in Varieties, Remark 36.11. The lemma follows by combining Lemmas 13.3 and 13.2. $\square$

Lemma 13.5. *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper nonsingular curves over k. Assume*

(1) *X is smooth,*
(2) *$H^0(X, \mathcal{O}_X) = k$,*
(3) *$k(X)/k(Y)$ is purely inseparable.*

Then Y is smooth, $H^0(Y, \mathcal{O}_Y) = k$, and the genus of Y is equal to the genus of X.

Proof. By Lemma 13.4 we see that $Y = X^{(p^n)}$ is the base change of X by $F^n_{\text{Spec}(k)}$. Thus Y is smooth and the result on the cohomology and genus follows from Lemma 8.2. $\square$

Example 13.6. This example will show that the genus can change under a purely inseparable morphism of nonsingular projective curves. Let k be a field of characteristic 3. Assume there exists an element $a \in k$ which is not a 3rd power. For example $k = \mathbf{F}_3(a)$ would work. Let X be the plane curve with homogeneous equation

$$F = T_1^2 T_0 - T_2^3 + aT_0^3$$

as in Section 9. On the affine piece $D_+(T_0)$ using coordinates $x = T_1/T_0$ and $y = T_2/T_0$ we obtain $x^2 - y^3 + a = 0$ which defines a nonsingular affine curve. Moreover, the point at infinity $(0 : 1 : 0)$ is a smooth point. Hence X is a nonsingular projective curve of genus 1 (Lemma 9.3). On the other hand, consider the morphism $f : X \to \mathbf{P}^1_k$ which on $D_+(T_0)$ sends (x, y) to $x \in \mathbf{A}^1_k \subset \mathbf{P}^1_k$. Then f is a morphism of proper nonsingular curves over k inducing an inseparable function field extension of degree $p = 3$ but the genus of X is 1 and the genus of $\mathbf{P}^1_k$ is 0.

Proposition 13.7. *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a nonconstant morphism of proper smooth curves over k. Then we can factor f as*

$$X \longrightarrow X^{(p^n)} \longrightarrow Y$$

where $X^{(p^n)} \to Y$ is a nonconstant morphism of proper smooth curves inducing a separable field extension $k(X^{(p^n)})/k(Y)$, we have

$$X^{(p^n)} = X \times_{\mathrm{Spec}(k), F^n_{\mathrm{Spec}(k)}} \mathrm{Spec}(k),$$

and $X \to X^{(p^n)}$ is the n-fold relative frobenius of X.

Proof. By Fields, Lemma 14.6 there is a subextension $k(X)/E/k(Y)$ such that $k(X)/E$ is purely inseparable and $E/k(Y)$ is separable. By Theorem 2.6 this corresponds to a factorization $X \to Z \to Y$ of f with Z a nonsingular proper curve. Apply Lemma 13.4 to the morphism $X \to Z$ to conclude. $\square$

Lemma 13.8. *Let k be a field of characteristic $p > 0$. Let X be a smooth proper curve over k. Let $(\mathcal{L}, V)$ be a $\mathfrak{g}^r_d$ with $r \geq 1$. Then one of the following two is true*

1. *there exists a $\mathfrak{g}^1_d$ whose corresponding morphism $X \to \mathbf{P}^1_k$ (Lemma 3.2) is generically étale (i.e., is as in Lemma 12.1), or*
2. *there exists a $\mathfrak{g}^r_{d'}$ on $X^{(p)}$ where $d' \leq d/p$.*

Proof. Pick two k-linearly independent elements $s, t \in V$. Then $f = s/t$ is the rational function defining the morphism $X \to \mathbf{P}^1_k$ corresponding to the linear series $(\mathcal{L}, ks + kt)$. If this morphism is not generically étale, then $f \in k(X^{(p)})$ by Proposition 13.7. Now choose a basis $s_0, \ldots, s_r$ of V and let $\mathcal{L}' \subset \mathcal{L}$ be the invertible sheaf generated by $s_0, \ldots, s_r$. Set $f_i = s_i/s_0$ in $k(X)$. If for each pair (s_0, s_i) we have $f_i \in k(X^{(p)})$, then the morphism

$$\varphi = \varphi_{(\mathcal{L}', (s_0, \ldots, s_r))} : X \longrightarrow \mathbf{P}^r_k = \mathrm{Proj}(k[T_0, \ldots, T_r])$$

factors through $X^{(p)}$ as this is true over the affine open $D_+(T_0)$ and we can extend the morphism over the affine part to the whole of the smooth curve $X^{(p)}$ by Lemma 2.2. Introducing notation, say we have the factorization

$$X \xrightarrow{F_{X/k}} X^{(p)} \xrightarrow{\psi} \mathbf{P}^r_k$$

of φ. Then $\mathcal{N} = \psi^*\mathcal{O}_{\mathbf{P}^1_k}(1)$ is an invertible $\mathcal{O}_{X^{(p)}}$-module with $\mathcal{L}' = F_{X/k}^*\mathcal{N}$ and with $\psi^*T_0, \ldots, \psi^*T_r$ k-linearly independent (as they pullback to $s_0, \ldots, s_r$ on X). Finally, we have

$$d = \deg(\mathcal{L}) \geq \deg(\mathcal{L}') = \deg(F_{X/k})\deg(\mathcal{N}) = p\deg(\mathcal{N})$$

as desired. Here we used Varieties, Lemmas 44.12, 44.11, and 36.10. ☐

Lemma 13.9. *Let k be a field. Let X be a smooth proper curve over k with $H^0(X, \mathcal{O}_X) = k$ and genus $g \geq 2$. Then there exists a closed point $x \in X$ with $\kappa(x)/k$ separable of degree $\leq 2g - 2$.*

Proof. Set $\omega = \Omega_{X/k}$. By Lemma 8.4 this has degree $2g - 2$ and has g global sections. Thus we have a $\mathfrak{g}^{g-1}_{2g-2}$. By the trivial Lemma 3.3 there exists a $\mathfrak{g}^1_{2g-2}$ and by Lemma 3.4 we obtain a morphism

$$\varphi : X \longrightarrow \mathbf{P}^1_k$$

of some degree $d \leq 2g - 2$. Since φ is flat (Lemma 2.3) and finite (Lemma 2.4) it is finite locally free of degree d (Morphisms, Lemma 48.2). Pick any rational point $t \in \mathbf{P}^1_k$ and any point $x \in X$ with $\varphi(x) = t$. Then

$$d \geq [\kappa(x) : \kappa(t)] = [\kappa(x) : k]$$

for example by Morphisms, Lemmas 57.3 and 57.2. Thus if k is perfect (for example has characteristic zero or is finite) then the lemma is proved. Thus we reduce to the case discussed in the next paragraph.

Assume that k is an infinite field of characteristic $p > 0$. As above we will use that X has a $\mathfrak{g}^{g-1}_{2g-2}$. The smooth proper curve $X^{(p)}$ has the same genus as X. Hence its genus is > 0. We conclude that $X^{(p)}$ does not have a $\mathfrak{g}^{g-1}_d$ for any $d \leq g - 1$ by Lemma 3.5. Applying Lemma 13.8 to our $\mathfrak{g}^{g-1}_{2g-2}$ (and noting that $2g - 2/p \leq g - 1$) we conclude that possibility (2) does not occur. Hence we obtain a morphism

$$\varphi : X \longrightarrow \mathbf{P}^1_k$$

which is generically étale (in the sense of the lemma) and has degree $\leq 2g - 2$. Let $U \subset X$ be the nonempty open subscheme where φ is étale. Then $\varphi(U) \subset \mathbf{P}^1_k$ is a nonempty Zariski open and we can pick a k-rational point $t \in \varphi(U)$ as k is infinite. Let $u \in U$ be a point with $\varphi(u) = t$. Then $\kappa(u)/\kappa(t)$ is separable (Morphisms, Lemma 36.7), $\kappa(t) = k$, and $[\kappa(u) : k] \leq 2g - 2$ as before. ☐

The following lemma does not really belong in this section but we don't know a good place for it elsewhere.

Lemma 13.10. *Let X be a smooth curve over a field k. Let $\overline{x} \in X_{\overline{k}}$ be a closed point with image $x \in X$. The ramification index of $\mathcal{O}_{X,x} \subset \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is the inseparable degree of $\kappa(x)/k$.*

Proof. After shrinking X we may assume there is an étale morphism $\pi : X \to \mathbf{A}^1_k$, see Morphisms, Lemma 36.20. Then we can consider the diagram of local rings

$$\begin{tikzcd}
\mathcal{O}_{X_{\overline{k}},\overline{x}} & \mathcal{O}_{\mathbf{A}^1_{\overline{k}},\pi(\overline{x})} \arrow[l] \\
\mathcal{O}_{X,x} \arrow[u] & \mathcal{O}_{\mathbf{A}^1_k,\pi(x)} \arrow[l] \arrow[u]
\end{tikzcd}$$

The horizontal arrows have ramification index 1 as they correspond to étale morphisms. Moreover, the extension $\kappa(x)/\kappa(\pi(x))$ is separable hence $\kappa(x)$ and $\kappa(\pi(x))$ have the same inseparable degree over k. By multiplicativity of ramification indices it suffices to prove the result when x is a point of the affine line.

Assume $X = \mathbf{A}^1_k$. In this case, the local ring of X at x looks like

$$\mathcal{O}_{X,x} = k[t]_{(P)}$$

where P is an irreducible monic polynomial over k. Then $P(t) = Q(t^q)$ for some separable polynomial $Q \in k[t]$, see Fields, Lemma 12.1. Observe that $\kappa(x) = k[t]/(P)$ has inseparable degree q over k. On the other hand, over $\overline{k}$ we can factor $Q(t) = \prod(t - \alpha_i)$ with α_i pairwise distinct. Write $\alpha_i = \beta_i^q$ for some unique $\beta_i \in \overline{k}$. Then our point $\overline{x}$ corresponds to one of the β_i and we conclude because the ramification index of

$$k[t]_{(P)} \longrightarrow \overline{k}[t]_{(t-\beta_i)}$$

is indeed equal to q as the uniformizer P maps to $(t - \beta_i)^q$ times a unit. $\square$

14. Pushouts

Let k be a field. Consider a solid diagram

$$\begin{tikzcd} Z' \arrow[r, "i'"'] \arrow[d] & X' \arrow[d, dotted, "a"] \\ Z \arrow[r, dotted, "i"] & X \end{tikzcd}$$

of schemes over k satisfying

(a) X' is separated of finite type over k of dimension ≤ 1,
(b) $i : Z' \to X'$ is a closed immersion,
(c) Z' and Z are finite over $\text{Spec}(k)$, and
(d) $Z' \to Z$ is surjective.

In this situation every finite set of points of X' are contained in an affine open, see Varieties, Proposition 42.7. Thus the assumptions of More on Morphisms, Proposition 67.3 are satisfied and we obtain the following

(1) the pushout $X = Z \amalg_{Z'} X'$ exists in the category of schemes,
(2) $i : Z \to X$ is a closed immersion,
(3) $a : X' \to X$ is integral surjective,
(4) $X \to \text{Spec}(k)$ is separated by More on Morphisms, Lemma 67.4
(5) $X \to \text{Spec}(k)$ is of finite type by More on Morphisms, Lemmas 67.5,
(6) thus $a : X' \to X$ is finite by Morphisms, Lemmas 44.4 and 15.8,
(7) if $X' \to \text{Spec}(k)$ is proper, then $X \to \text{Spec}(k)$ is proper by Morphisms, Lemma 41.9.

The following lemma can be generalized significantly.

Lemma 14.1. *In the situation above, let $Z = \text{Spec}(k')$ where k' is a field and $Z' = \text{Spec}(k'_1 \times \ldots \times k'_n)$ with k'_i/k' finite extensions of fields. Let $x \in X$ be the image of $Z \to X$ and $x'_i \in X'$ the image of $\text{Spec}(k'_i) \to X'$. Then we have a fibre* *product diagram*

$$\begin{tikzcd} \prod_{i=1,\ldots,n} k'_i & \prod_{i=1,\ldots,n} \mathcal{O}^\wedge_{X',x'_i} \arrow[l] \\ k' \arrow[u] & \mathcal{O}^\wedge_{X,x} \arrow[l] \arrow[u] \end{tikzcd}$$

where the horizontal arrows are given by the maps to the residue fields.

Proof. Choose an affine open neighbourhood $\mathrm{Spec}(A)$ of x in X. Let $\mathrm{Spec}(A') \subset X'$ be the inverse image. By construction we have a fibre product diagram

$$\begin{tikzcd} \prod_{i=1,\ldots,n} k'_i & A' \arrow[l] \\ k' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

Since everything is finite over A we see that the diagram remains a fibre product diagram after completion with respect to the maximal ideal $\mathfrak{m} \subset A$ corresponding to x (Algebra, Lemma 97.2). Finally, apply Algebra, Lemma 97.8 to identify the completion of A'. $\square$

15. Glueing and squishing

Below we will indicate $k[\epsilon]$ the algebra of dual numbers over k as defined in Varieties, Definition 16.1.

Lemma 15.1. *Let k be an algebraically closed field. Let $k \subset A$ be a ring extension such that A has exactly two k-sub algebras, then either $A = k \times k$ or $A = k[\epsilon]$.*

Proof. The assumption means $k \neq A$ and any subring $k \subset C \subset A$ is equal to either k or A. Let $t \in A$, $t \notin k$. Then A is generated by t over k. Hence $A = k[x]/I$ for some ideal I. If $I = (0)$, then we have the subalgebra $k[x^2]$ which is not allowed. Otherwise I is generated by a monic polynomial P. Write $P = \prod_{i=1}^{d}(t - a_i)$. If $d > 2$, then the subalgebra generated by $(t-a_1)(t-a_2)$ gives a contradiction. Thus $d = 2$. If $a_1 \neq a_2$, then $A = k \times k$, if $a_1 = a_2$, then $A = k[\epsilon]$. $\square$

Example 15.2 (Glueing points)**.** Let k be an algebraically closed field. Let $f : X' \to X$ be a morphism of algebraic k-schemes. We say X is obtained by glueing a and b in X' if the following are true:

1. $a, b \in X'(k)$ are distinct points which map to the same point $x \in X(k)$,
2. f is finite and $f^{-1}(X \setminus \{x\}) \to X \setminus \{x\}$ is an isomorphism,
3. there is a short exact sequence
$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \xrightarrow{a-b} x_*k \to 0$$
where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}$ to the difference $h(a) - h(b) \in k$.

If this is the case, then there also is a short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{ab^{-1}} x_*k^* \to 0$$

where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}^*$ to the multiplicative difference $h(a)h(b)^{-1} \in k^*$.

Example 15.3 (Squishing a tangent vector)**.** Let k be an algebraically closed field. Let $f : X' \to X$ be a morphism of algebraic k-schemes. We say X is obtained by squishing the tangent vector ϑ in X' if the following are true:

(1) $\vartheta : \mathrm{Spec}(k[\epsilon]) \to X'$ is a closed immersion over k such that $f \circ \vartheta$ factors through a point $x \in X(k)$,
(2) f is finite and $f^{-1}(X \setminus \{x\}) \to X \setminus \{x\}$ is an isomorphism,
(3) there is a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \xrightarrow{\vartheta} x_*k \to 0$$

where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}$ to the coefficient of ϵ in $\vartheta^\sharp(h) \in k[\epsilon]$.

If this is the case, then there also is a short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{\vartheta} x_*k \to 0$$

where arrow on the right sends a local section h of $f_*\mathcal{O}_{X'}^*$ to $\mathrm{d}\log(\vartheta^\sharp(h))$ where $\mathrm{d}\log : k[\epsilon]^* \to k$ is the homomorphism of abelian groups sending $a + b\epsilon$ to $b/a \in k$.

Lemma 15.4. *Let k be an algebraically closed field. Let $f : X' \to X$ be a finite morphism algebraic k-schemes such that $\mathcal{O}_X \subset f_*\mathcal{O}_{X'}$ and such that f is an isomorphism away from a finite set of points. Then there is a factorization*

$$X' = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

such that each $X_i \to X_{i-1}$ is either the glueing of two points or the squishing of a tangent vector (see Examples 15.2 and 15.3).

Proof. Let $U \subset X$ be the maximal open set over which f is an isomorphism. Then $X \setminus U = \{x_1, \ldots, x_n\}$ with $x_i \in X(k)$. We will consider factorizations $X' \to Y \to X$ of f such that both morphisms are finite and

$$\mathcal{O}_X \subset g_*\mathcal{O}_Y \subset f_*\mathcal{O}_{X'}$$

where $g : Y \to X$ is the given morphism. By assumption $\mathcal{O}_{X,x} \to (f_*\mathcal{O}_{X'})_x$ is an isomorphism onless $x = x_i$ for some i. Hence the cokernel

$$f_*\mathcal{O}_{X'}/\mathcal{O}_X = \bigoplus \mathcal{Q}_i$$

is a direct sum of skyscraper sheaves $\mathcal{Q}_i$ supported at $x_1, \ldots, x_n$. Because the displayed quotient is a coherent $\mathcal{O}_X$-module, we conclude that $\mathcal{Q}_i$ has finite length over $\mathcal{O}_{X,x_i}$. Hence we can argue by induction on the sum of these lengths, i.e., the length of the whole cokernel.

If $n > 1$, then we can define an $\mathcal{O}_X$-subalgebra $\mathcal{A} \subset f_*\mathcal{O}_{X'}$ by taking the inverse image of $\mathcal{Q}_1$. This will give a nontrivial factorization and we win by induction.

Assume $n = 1$. We abbreviate $x = x_1$. Consider the finite k-algebra extension

$$A = \mathcal{O}_{X,x} \subset (f_*\mathcal{O}_{X'})_x = B$$

Note that $\mathcal{Q} = \mathcal{Q}_1$ is the skyscraper sheaf with value B/A. We have a k-subalgebra $A \subset A + \mathfrak{m}_A B \subset B$. If both inclusions are strict, then we obtain a nontrivial factorization and we win by induction as above. If $A + \mathfrak{m}_A B = B$, then $A = B$ by Nakayama, then f is an isomorphism and there is nothing to prove. We conclude that we may assume $B = A + \mathfrak{m}_A B$. Set $C = B/\mathfrak{m}_A B$. If C has more than 2 k-subalgebras, then we obtain a subalgebra between A and B by taking the inverse image in B. Thus we may assume C has exactly 2 k-subalgebras. Thus $C = k \times k$ or $C = k[\epsilon]$ by Lemma 15.1. In this case f is correspondingly the glueing two points or the squishing of a tangent vector. $\square$

Lemma 15.5. *Let k be an algebraically closed field. If $f : X' \to X$ is the glueing of two points a, b as in Example 15.2, then there is an exact sequence*

$$k^* \to \mathrm{Pic}(X) \to \mathrm{Pic}(X') \to 0$$

The first map is zero if a and b are on different connected components of X' and injective if X' is proper and a and b are on the same connected component of X'.

Proof. The map $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is surjective by Varieties, Lemma 38.7. Using the short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{ab^{-1}} x_*k^* \to 0$$

we obtain

$$H^0(X', \mathcal{O}_{X'}^*) \xrightarrow{ab^{-1}} k^* \to H^1(X, \mathcal{O}_X^*) \to H^1(X, f_*\mathcal{O}_{X'}^*)$$

We have $H^1(X, f_*\mathcal{O}_{X'}^*) \subset H^1(X', \mathcal{O}_{X'}^*)$ (for example by the Leray spectral sequence, see Cohomology, Lemma 13.4). Hence the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is the cokernel of $ab^{-1} : H^0(X', \mathcal{O}_{X'}^*) \to k^*$. If a and b are on different connected components of X', then ab^{-1} is surjective. Because k is algebraically closed any regular function on a reduced connected proper scheme over k comes from an element of k, see Varieties, Lemma 9.3. Thus ab^{-1} is zero if X' is proper and a and b are on the same connected component. $\square$

Lemma 15.6. *Let k be an algebraically closed field. If $f : X' \to X$ is the squishing of a tangent vector ϑ as in Example 15.3, then there is an exact sequence*

$$(k, +) \to \mathrm{Pic}(X) \to \mathrm{Pic}(X') \to 0$$

and the first map is injective if X' is proper and reduced.

Proof. The map $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is surjective by Varieties, Lemma 38.7. Using the short exact sequence

$$0 \to \mathcal{O}_X^* \to f_*\mathcal{O}_{X'}^* \xrightarrow{\vartheta} x_*k \to 0$$

of Example 15.3 we obtain

$$H^0(X', \mathcal{O}_{X'}^*) \xrightarrow{\vartheta} k \to H^1(X, \mathcal{O}_X^*) \to H^1(X, f_*\mathcal{O}_{X'}^*)$$

We have $H^1(X, f_*\mathcal{O}_{X'}^*) \subset H^1(X', \mathcal{O}_{X'}^*)$ (for example by the Leray spectral sequence, see Cohomology, Lemma 13.4). Hence the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is the cokernel of the map $\vartheta : H^0(X', \mathcal{O}_{X'}^*) \to k$. Because k is algebraically closed any regular function on a reduced connected proper scheme over k comes from an element of k, see Varieties, Lemma 9.3. Thus the final statement of the lemma. $\square$

16. Multicross and nodal singularities

In this section we discuss the simplest possible curve singularities.

Let k be a field. Consider the complete local k-algebra

$$A = \{(f_1, \ldots, f_n) \in k[[t]] \times \ldots \times k[[t]] \mid f_1(0) = \ldots = f_n(0)\} \tag{16.0.1}$$

In the language introduced in Varieties, Definition 40.4 we see that A is a wedge of n copies of the power series ring in 1 variable over k. Observe that $k[[t]] \times \ldots \times k[[t]]$ is the integral closure of A in its total ring of fractions. Hence the δ-invariant of A is $n - 1$. There is an isomorphism

$$k[[x_1, \ldots, x_n]]/(\{x_i x_j\}_{i \neq j}) \longrightarrow A$$

obtained by sending x_i to $(0, \ldots, 0, t, 0, \ldots, 0)$ in A. It follows that $\dim(A) = 1$ and $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$. In particular, A is regular if and only if $n = 1$.

Lemma 16.1. *Let k be a separably closed field. Let A be a 1-dimensional reduced Nagata local k-algebra with residue field k. Then*

$$\delta\text{-invariant } A \geq \text{number of branches of } A - 1$$

If equality holds, then $A^\wedge$ is as in (16.0.1).

Proof. Since the residue field of A is separably closed, the number of branches of A is equal to the number of geometric branches of A, see More on Algebra, Definition 106.6. The inequality holds by Varieties, Lemma 40.6. Assume equality holds. We may replace A by the completion of A; this does not change the number of branches or the δ-invariant, see More on Algebra, Lemma 108.7 and Varieties, Lemma 39.6. Then A is strictly henselian, see Algebra, Lemma 153.9. By Varieties, Lemma 40.5 we see that A is a wedge of complete discrete valuation rings. Each of these is isomorphic to $k[[t]]$ by Algebra, Lemma 160.10. Hence A is as in (16.0.1). $\square$

Definition 16.2. Let k be an algebraically closed field. Let X be an algebraic 1-dimensional k-scheme. Let $x \in X$ be a closed point. We say x defines a *multicross singularity* if the completion $\mathcal{O}_{X,x}^\wedge$ is isomorphic to (16.0.1) for some $n \geq 2$. We say x is a *node*, or an *ordinary double point*, or *defines a nodal singularity* if $n = 2$.

These singularities are in some sense the simplest kind of singularities one can have on a curve over an algebraically closed field.

Lemma 16.3. *Let k be an algebraically closed field. Let X be a reduced algebraic 1-dimensional k-scheme. Let $x \in X$. The following are equivalent*

1. *x defines a multicross singularity,*
2. *the δ-invariant of X at x is the number of branches of X at x minus 1,*
3. *there is a sequence of morphisms $U_n \to U_{n-1} \to \ldots \to U_0 = U \subset X$ where U is an open neighbourhood of x, where U_n is nonsingular, and where each $U_i \to U_{i-1}$ is the glueing of two points as in Example 15.2.*

Proof. The equivalence of (1) and (2) is Lemma 16.1.

Assume (3). We will argue by descending induction on i that all singularities of U_i are multicross. This is true for U_n as U_n has no singular points. If U_i is gotten from U_{i+1} by glueing $a, b \in U_{i+1}$ to a point $c \in U_i$, then we see that

$$\mathcal{O}_{U_i,c}^\wedge \subset \mathcal{O}_{U_{i+1},a}^\wedge \times \mathcal{O}_{U_{i+1},b}^\wedge$$

is the set of elements having the same residue classes in k. Thus the number of branches at c is the sum of the number of branches at a and b, and the δ-invariant at c is the sum of the δ-invariants at a and b plus 1 (because the displayed inclusion has codimension 1). This proves that (2) holds as desired.

Assume the equivalent conditions (1) and (2). We may choose an open $U \subset X$ such that x is the only singular point of U. Then we apply Lemma 15.4 to the normalization morphism

$$U^\nu = U_n \to U_{n-1} \to \ldots \to U_1 \to U_0 = U$$

All we have to do is show that in none of the steps we are squishing a tangent vector. Suppose $U_{i+1} \to U_i$ is the smallest i such that this is the squishing of a tangent vector θ at $u' \in U_{i+1}$ lying over $u \in U_i$. Arguing as above, we see that u_i is a multicross singularity (because the maps $U_i \to \ldots \to U_0$ are glueing of pairs of points). But now the number of branches at u' and u is the same and the δ-invariant of U_i at u is 1 bigger than the δ-invariant of U_{i+1} at u'. By Lemma 16.1 this implies that u cannot be a multicross singularity which is a contradiction. $\square$

Lemma 16.4. *Let k be an algebraically closed field. Let X be a reduced algebraic 1-dimensional k-scheme. Let $x \in X$ be a multicross singularity (Definition 16.2). If X is Gorenstein, then x is a node.*

Proof. The map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}^\wedge$ is flat and unramified in the sense that $\kappa(x) = \mathcal{O}_{X,x}^\wedge/\mathfrak{m}_x\mathcal{O}_{X,x}^\wedge$. (See More on Algebra, Section 43.) Thus X is Gorenstein implies $\mathcal{O}_{X,x}$ is Gorenstein, implies $\mathcal{O}_{X,x}^\wedge$ is Gorenstein by Dualizing Complexes, Lemma 21.8. Thus it suffices to show that the ring A in (16.0.1) with $n \geq 2$ is Gorenstein if and only if $n = 2$.

If $n = 2$, then $A = k[[x,y]]/(xy)$ is a complete intersection and hence Gorenstein. For example this follows from Duality for Schemes, Lemma 24.5 applied to $k[[x,y]] \to A$ and the fact that the regular local ring $k[[x,y]]$ is Gorenstein by Dualizing Complexes, Lemma 21.3.

Assume $n > 2$. If A where Gorenstein, then A would be a dualizing complex over A (Duality for Schemes, Definition 24.1). Then $R\operatorname{Hom}(k, A)$ would be equal to $k[n]$ for some $n \in \mathbf{Z}$, see Dualizing Complexes, Lemma 15.12. It would follow that $\operatorname{Ext}^1_A(k, A) \cong k$ or $\operatorname{Ext}^1_A(k, A) = 0$ (depending on the value of n; in fact n has to be -1 but it doesn't matter to us here). Using the exact sequence

$$0 \to \mathfrak{m}_A \to A \to k \to 0$$

we find that

$$\operatorname{Ext}^1_A(k, A) = \operatorname{Hom}_A(\mathfrak{m}_A, A)/A$$

where $A \to \operatorname{Hom}_A(\mathfrak{m}_A, A)$ is given by $a \mapsto (a' \mapsto aa')$. Let $e_i \in \operatorname{Hom}_A(\mathfrak{m}_A, A)$ be the element that sends $(f_1, \ldots, f_n) \in \mathfrak{m}_A$ to $(0, \ldots, 0, f_i, 0, \ldots, 0)$. The reader verifies easily that $e_1, \ldots, e_{n-1}$ are k-linearly independent in $\operatorname{Hom}_A(\mathfrak{m}_A, A)/A$. Thus $\dim_k \operatorname{Ext}^1_A(k, A) \geq n - 1 \geq 2$ which finishes the proof. (Observe that $e_1 + \ldots + e_n$ is the image of 1 under the map $A \to \operatorname{Hom}_A(\mathfrak{m}_A, A)$.) $\square$

17. Torsion in the Picard group

In this section we bound the torsion in the Picard group of a 1-dimensional proper scheme over a field. We will use this in our study of semistable reduction for curves.

There does not seem to be an elementary way to obtain the result of Lemma 17.1. Analyzing the proof there are two key ingredients: (1) there is an abelian variety classifying degree zero invertible sheaves on a smooth projective curve and (2) the structure of torsion points on an abelian variety can be determined.

Lemma 17.1. *Let k be an algebraically closed field. Let X be a smooth projective curve of genus g over k.*

1. *If $n \geq 1$ is invertible in k, then $\mathrm{Pic}(X)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$.*
2. *If the characteristic of k is $p > 0$, then there exists an integer $0 \leq f \leq g$ such that $\mathrm{Pic}(X)[p^m] \cong (\mathbf{Z}/p^m\mathbf{Z})^{\oplus f}$ for all $m \geq 1$.*

Proof. Let $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ denote the subgroup of invertible sheaves of degree 0. In other words, there is a short exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbf{Z} \to 0.$$

The group $\mathrm{Pic}^0(X)$ is the k-points of the group scheme $\underline{\mathrm{Pic}}^0_{X/k}$, see Picard Schemes of Curves, Lemma 6.7. The same lemma tells us that $\underline{\mathrm{Pic}}^0_{X/k}$ is a g-dimensional abelian variety over k as defined in Groupoids, Definition 9.1. Thus we conclude by the results of Groupoids, Proposition 9.11. $\square$

Lemma 17.2. *Let k be a field. Let n be prime to the characteristic of k. Let X be a smooth proper curve over k with $H^0(X, \mathcal{O}_X) = k$ and of genus g.*

1. *If $g = 1$ then there exists a finite separable extension k'/k such that $X_{k'}$ has a k'-rational point and $\mathrm{Pic}(X_{k'})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2}$.*
2. *If $g \geq 2$ then there exists a finite separable extension k'/k with $[k' : k] \leq (2g-2)(n^{2g})!$ such that $X_{k'}$ has a k'-rational point and $\mathrm{Pic}(X_{k'})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$.*

Proof. Assume $g \geq 2$. First we may choose a finite separable extension of degree at most $2g-2$ such that X acquires a rational point, see Lemma 13.9. Thus we may assume X has a k-rational point $x \in X(k)$ but now we have to prove the lemma with $[k' : k] \leq (n^{2g})!$. Let $k \subset k^{sep} \subset \overline{k}$ be a separable algebraic closure inside an algebraic closure. By Lemma 17.1 we have

$$\mathrm{Pic}(X_{\overline{k}})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$$

By Picard Schemes of Curves, Lemma 7.2 we conclude that

$$\mathrm{Pic}(X_{k^{sep}})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2g}$$

By Picard Schemes of Curves, Lemma 7.2 there is a continuous action

$$\mathrm{Gal}(k^{sep}/k) \longrightarrow \mathrm{Aut}(\mathrm{Pic}(X_{k^{sep}})[n]$$

and the lemma is true for the fixed field k' of the kernel of this map. The kernel is open because the action is continuous which implies that k'/k is finite. By Galois theory $\mathrm{Gal}(k'/k)$ is the image of the displayed arrow. Since the permutation group of a set of cardinality n^{2g} has cardinality $(n^{2g})!$ we conclude by Galois theory that $[k' : k] \leq (n^{2g})!$. (Of course this proves the lemma with the bound $|\mathrm{GL}_{2g}(\mathbf{Z}/n\mathbf{Z})|$, but all we want here is that there is some bound.)

If the genus is 1, then there is no upper bound on the degree of a finite separable field extension over which X acquires a rational point (details omitted). Still, there is such an extension for example by Varieties, Lemma 25.6. The rest of the proof is the same as in the case of $g \geq 2$. $\square$

Proposition 17.3. *Let k be an algebraically closed field. Let X be a proper scheme over k which is reduced, connected, and has dimension 1. Let g be the genus of X and let g_{geom} be the sum of the geometric genera of the irreducible components of X. For any prime ℓ different from the characteristic of k we have*

$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(X)[\ell] \leq g + g_{geom}$$

and equality holds if and only if all the singularities of X are multicross.

Proof. Let $\nu : X^\nu \to X$ be the normalization (Varieties, Lemma 41.2). Choose a factorization

$$X^\nu = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

as in Lemma 15.4. Let us denote $h_i^0 = \dim_k H^0(X_i, \mathcal{O}_{X_i})$ and $h_i^1 = \dim_k H^1(X_i, \mathcal{O}_{X_i})$. By Lemmas 15.5 and 15.6 for each $n > i \geq 0$ we have one of the following there possibilities

1. X_i is obtained by glueing $a, b \in X_{i+1}$ which are on different connected components: in this case $\mathrm{Pic}(X_i) = \mathrm{Pic}(X_{i+1})$, $h_{i+1}^0 = h_i^0 + 1$, $h_{i+1}^1 = h_i^1$,
2. X_i is obtained by glueing $a, b \in X_{i+1}$ which are on the same connected component: in this case there is a short exact sequence
$$0 \to k^* \to \mathrm{Pic}(X_i) \to \mathrm{Pic}(X_{i+1}) \to 0,$$
and $h_{i+1}^0 = h_i^0$, $h_{i+1}^1 = h_i^1 - 1$,
3. X_i is obtained by squishing a tangent vector in X_{i+1}: in this case there is a short exact sequence
$$0 \to (k, +) \to \mathrm{Pic}(X_i) \to \mathrm{Pic}(X_{i+1}) \to 0,$$
and $h_{i+1}^0 = h_i^0$, $h_{i+1}^1 = h_i^1 - 1$.

To prove the statements on dimensions of cohomology groups of the structure sheaf, use the exact sequences in Examples 15.2 and 15.3. Since k is algebraically closed of characteristic prime to ℓ we see that $(k, +)$ and k^* are ℓ-divisible and with ℓ-torsion $(k, +)[\ell] = 0$ and $k^*[\ell] \cong \mathbf{F}_\ell$. Hence

$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(X_{i+1})[\ell] - \dim_{\mathbf{F}_\ell} \mathrm{Pic}(X_i)[\ell]$$

is zero, except in case (2) where it is equal to -1. At the end of this process we get the normalization $X^\nu = X_n$ which is a disjoint union of smooth projective curves over k. Hence we have

1. $h_n^1 = g_{geom}$ and
2. $\dim_{\mathbf{F}_\ell} \mathrm{Pic}(X_n)[\ell] = 2g_{geom}$.

The last equality by Lemma 17.1. Since $g = h_0^1$ we see that the number of steps of type (2) and (3) is at most $h_0^1 - h_n^1 = g - g_{geom}$. By our comptation of the differences in ranks we conclude that

$$\dim_{\mathbf{F}_\ell} \mathrm{Pic}(X)[\ell] \leq g - g_{geom} + 2g_{geom} = g + g_{geom}$$

and equality holds if and only if no steps of type (3) occur. This indeed means that all singularities of X are multicross by Lemma 16.3. Conversely, if all the singularities are multicross, then Lemma 16.3 guarantees that we can find a sequence

$X^\nu = X_n \to \ldots \to X_0 = X$ as above such that no steps of type (3) occur in the sequence and we find equality holds in the lemma (just glue the local sequences for each point to find one that works for all singular points of x; some details omitted). $\square$

18. Genus versus geometric genus

Let k be a field with algebraic closure $\overline{k}$. Let X be a proper scheme of dimension ≤ 1 over k. We define $g_{geom}(X/k)$ to be the sum of the geometric genera of the irreducible components of $X_{\overline{k}}$ which have dimension 1.

Lemma 18.1. *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Then*

$$g_{geom}(X/k) = \sum\nolimits_{C \subset X} g_{geom}(C/k)$$

where the sum is over irreducible components $C \subset X$ of dimension 1*.*

Proof. This is immediate from the definition and the fact that an irreducible component $\overline{Z}$ of $X_{\overline{k}}$ maps onto an irreducible component Z of X (Varieties, Lemma 8.10) of the same dimension (Morphisms, Lemma 28.3 applied to the generic point of $\overline{Z}$). $\square$

Lemma 18.2. *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Then*

1. *We have $g_{geom}(X/k) = g_{geom}(X_{red}/k)$.*
2. *If $X' \to X$ is a birational proper morphism, then $g_{geom}(X'/k) = g_{geom}(X/k)$.*
3. *If $X^\nu \to X$ is the normalization morphism, then $g_{geom}(X^\nu/k) = g_{geom}(X/k)$.*

Proof. Part (1) is immediate from Lemma 18.1. If $X' \to X$ is proper birational, then it is finite and an isomorphism over a dense open (see Varieties, Lemmas 17.2 and 17.3). Hence $X'_{\overline{k}} \to X_{\overline{k}}$ is an isomorphism over a dense open. Thus the irreducible components of $X'_{\overline{k}}$ and $X_{\overline{k}}$ are in bijective correspondence and the corresponding components have isomorphic function fields. In particular these components have isomorphic nonsingular projective models and hence have the same geometric genera. This proves (2). Part (3) follows from (1) and (2) and the fact that $X^\nu \to X_{red}$ is birational (Morphisms, Lemma 54.7). $\square$

Lemma 18.3. *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $f : Y \to X$ be a finite morphism such that there exists a dense open $U \subset X$ over which f is a closed immersion. Then*

$$\dim_k H^1(X, \mathcal{O}_X) \geq \dim_k H^1(Y, \mathcal{O}_Y)$$

Proof. Consider the exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to \mathcal{F} \to 0$$

of coherent sheaves on X. By assumption $\mathcal{F}$ is supported in finitely many closed points and hence has vanishing higher cohomology (Varieties, Lemma 33.3). On the other hand, we have $H^2(X, \mathcal{G}) = 0$ by Cohomology, Proposition 20.7. It follows formally that the induced map $H^1(X, \mathcal{O}_X) \to H^1(X, f_*\mathcal{O}_Y)$ is surjective. Since $H^1(X, f_*\mathcal{O}_Y) = H^1(Y, \mathcal{O}_Y)$ (Cohomology of Schemes, Lemma 2.4) we conclude the lemma holds. $\square$

Lemma 18.4. *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. If $X' \to X$ is a birational proper morphism, then*

$$\dim_k H^1(X, \mathcal{O}_X) \geq \dim_k H^1(X', \mathcal{O}_{X'})$$

If X is reduced, $H^0(X, \mathcal{O}_X) \to H^0(X', \mathcal{O}_{X'})$ is surjective, and equality holds, then $X' = X$.

Proof. If $f : X' \to X$ is proper birational, then it is finite and an isomorphism over a dense open (see Varieties, Lemmas 17.2 and 17.3). Thus the inequality by Lemma 18.3. Assume X is reduced. Then $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ is injective and we obtain a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \to \mathcal{F} \to 0$$

Under the assumptions given in the second statement, we conclude from the long exact cohomology sequence that $H^0(X, \mathcal{F}) = 0$. Then $\mathcal{F} = 0$ because $\mathcal{F}$ is generated by global sections (Varieties, Lemma 33.3). and $\mathcal{O}_X = f_*\mathcal{O}_{X'}$. Since f is affine this implies $X = X'$. □

Lemma 18.5. *Let k be a field. Let C be a proper curve over k. Set $\kappa = H^0(C, \mathcal{O}_C)$. Then*

$$[\kappa : k]_s \dim_\kappa H^1(C, \mathcal{O}_C) \geq g_{geom}(C/k)$$

Proof. Varieties, Lemma 26.2 implies κ is a field and a finite extension of k. By Fields, Lemma 14.8 we have $[\kappa : k]_s = |\operatorname{Mor}_k(\kappa, \overline{k})|$ and hence $\operatorname{Spec}(\kappa \otimes_k \overline{k})$ has $[\kappa : k]_s$ points each with residue field $\overline{k}$. Thus

$$C_{\overline{k}} = \bigcup\nolimits_{t \in \operatorname{Spec}(\kappa \otimes_k \overline{k})} C_t$$

(set theoretic union). Here $C_t = C \times_{\operatorname{Spec}(\kappa), t} \operatorname{Spec}(\overline{k})$ where we view t as a k-algebra map $t : \kappa \to \overline{k}$. The conclusion is that $g_{geom}(C/k) = \sum_t g_{geom}(C_t/\overline{k})$ and the sum is over an index set of size $[\kappa : k]_s$. We have

$$H^0(C_t, \mathcal{O}_{C_t}) = \overline{k} \quad \text{and} \quad \dim_{\overline{k}} H^1(C_t, \mathcal{O}_{C_t}) = \dim_\kappa H^1(C, \mathcal{O}_C)$$

by cohomology and base change (Cohomology of Schemes, Lemma 5.2). Observe that the normalization C_t^ν is the disjoint union of the nonsingular projective models of the irreducible components of C_t (Morphisms, Lemma 54.6). Hence $\dim_{\overline{k}} H^1(C_t^\nu, \mathcal{O}_{C_t^\nu})$ is equal to $g_{geom}(C_t/\overline{k})$. By Lemma 18.3 we have

$$\dim_{\overline{k}} H^1(C_t, \mathcal{O}_{C_t}) \geq \dim_{\overline{k}} H^1(C_t^\nu, \mathcal{O}_{C_t^\nu})$$

and this finishes the proof. □

Lemma 18.6. *Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let ℓ be a prime number invertible in k. Then*

$$\dim_{\mathbf{F}_\ell} \operatorname{Pic}(X)[\ell] \leq \dim_k H^1(X, \mathcal{O}_X) + g_{geom}(X/k)$$

where $g_{geom}(X/k)$ is as defined above.

Proof. The map $\operatorname{Pic}(X) \to \operatorname{Pic}(X_{\overline{k}})$ is injective by Varieties, Lemma 30.3. By Cohomology of Schemes, Lemma 5.2 $\dim_k H^1(X, \mathcal{O}_X)$ equals $\dim_{\overline{k}} H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$. Hence we may assume k is algebraically closed.

Let X_{red} be the reduction of X. Then the surjection $\mathcal{O}_X \to \mathcal{O}_{X_{red}}$ induces a surjection $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_{X_{red}})$ because cohomology of quasi-coherent sheaves vanishes in degrees ≥ 2 by Cohomology, Proposition 20.7. Since $X_{red} \to X$ induces an isomorphism on irreducible components over $\overline{k}$ and an isomorphism on ℓ-torsion in Picard groups (Picard Schemes of Curves, Lemma 7.2) we may replace X by X_{red}. In this way we reduce to Proposition 17.3. $\square$

19. Nodal curves

We have already defined ordinary double points over algebraically closed fields, see Definition 16.2. Namely, if $x \in X$ is a closed point of a 1-dimensional algebraic scheme over an algebraically closed field k, then x is an ordinary double point if and only if

$$\mathcal{O}^\wedge_{X,x} \cong k[[x, y]]/(xy)$$

See discussion following (16.0.1) in Section 16.

Definition 19.1. Let k be a field. Let X be a 1-dimensional locally algebraic k-scheme.

(1) We say a closed point $x \in X$ is a *node*, or an *ordinary double point*, or *defines a nodal singularity* if there exists an ordinary double point $\overline{x} \in X_{\overline{k}}$ mapping to x.
(2) We say the *singularities of X are at-worst-nodal* if all closed points of X are either in the smooth locus of the structure morphism $X \to \mathrm{Spec}(k)$ or are ordinary double points.

Often a 1-dimensional algebraic scheme X is called a *nodal curve* if the singularities of X are at worst nodal. Sometimes a nodal curve is required to be proper. Since a nodal curve so defined need not be irreducible, this conflicts with our earlier definition of a curve as a variety of dimension 1.

Lemma 19.2. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension 2. Let $I \subset \mathfrak{m}$ be an ideal.*

(1) *If A/I is reduced, then $I = (0)$, $I = \mathfrak{m}$, or $I = (f)$ for some nonzero $f \in \mathfrak{m}$.*
(2) *If A/I has depth 1, then $I = (f)$ for some nonzero $f \in \mathfrak{m}$.*

Proof. Assume $I \neq 0$. Write $I = (f_1, \ldots, f_r)$. As A is a UFD (More on Algebra, Lemma 121.2) we can write $f_i = fg_i$ where f is the gcd of $f_1, \ldots, f_r$. Thus the gcd of $g_1, \ldots, g_r$ is 1 which means that there is no height 1 prime ideal over $g_1, \ldots, g_r$. Then either $(g_1, \ldots, g_r) = A$ which implies $I = (f)$ or if not, then $\dim(A) = 2$ implies that $V(g_1, \ldots, g_r) = \{\mathfrak{m}\}$, i.e., $\mathfrak{m} = \sqrt{(g_1, \ldots, g_r)}$.

Assume A/I reduced, i.e., I radical. If f is a unit, then since I is radical we see that $I = \mathfrak{m}$. If $f \in \mathfrak{m}$, then we see that f^n maps to zero in A/I. Hence $f \in I$ by reducedness and we conclude $I = (f)$.

Assume A/I has depth 1. Then $\mathfrak{m}$ is not an associated prime of A/I. Since the class of f modulo I is annihilated by $g_1, \ldots, g_r$, this implies that the class of f is zero in A/I. Thus $I = (f)$ as desired. $\square$

Let κ be a field and let V be a vector space over κ. We will say $q \in \mathrm{Sym}^2_\kappa(V)$ is *nondegenerate* if the induced κ-linear map $V^\vee \to V$ is an isomorphism. If $q =$

$\sum_{i \leq j} a_{ij} x_i x_j$ for some κ-basis $x_1, \ldots, x_n$ of V, then this means that the determinant of the matrix

$$\begin{pmatrix} 2a_{11} & a_{12} & \ldots \\ a_{12} & 2a_{22} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

is nonzero. This is equivalent to the condition that the partial derivatives of q with respect to the x_i cut out 0 scheme theoretically.

Lemma 19.3. *Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local k-algebra. The following are equivalent*

1. *κ/k is separable, A is reduced, $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 2$, and there exists a nondegenerate $q \in Sym^2_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ which maps to zero in $\mathfrak{m}^2/\mathfrak{m}^3$,*
2. *κ/k is separable, $depth(A) = 1$, $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 2$, and there exists a nondegenerate $q \in Sym^2_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ which maps to zero in $\mathfrak{m}^2/\mathfrak{m}^3$,*
3. *κ/k is separable, $A^\wedge \cong \kappa[[x, y]]/(ax^2 + bxy + cy^2)$ as a k-algebra where $ax^2 + bxy + cy^2$ is a nondegenerate quadratic form over κ.*

Proof. Assume (3). Then $A^\wedge$ is reduced because $ax^2+bxy+cy^2$ is either irreducible or a product of two nonassociated prime elements. Hence $A \subset A^\wedge$ is reduced. It follows that (1) is true.

Assume (1). Then A cannot be Artinian, since it would not be reduced because $\mathfrak{m} \neq (0)$. Hence $\dim(A) \geq 1$, hence $\text{depth}(A) \geq 1$ by Algebra, Lemma 157.3. On the other hand $\dim(A) = 2$ implies A is regular which contradicts the existence of q by Algebra, Lemma 106.1. Thus $\dim(A) \leq 1$ and we conclude $\text{depth}(A) = 1$ by Algebra, Lemma 72.3. It follows that (2) is true.

Assume (2). Since the depth of A is the same as the depth of $A^\wedge$ (More on Algebra, Lemma 43.2) and since the other conditions are insensitive to completion, we may assume that A is complete. Choose $\kappa \to A$ as in More on Algebra, Lemma 38.3. Since $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 2$ we can choose $x_0, y_0 \in \mathfrak{m}$ which map to a basis. We obtain a continuous κ-algebra map

$$\kappa[[x, y]] \longrightarrow A$$

by the rules $x \mapsto x_0$ and $y \mapsto y_0$. Let q be the class of $ax_0^2 + bx_0y_0 + cy_0^2$ in $\text{Sym}^2_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. Write $Q(x, y) = ax^2 + bxy + cy^2$ viewed as a polynomial in two variables. Then we see that

$$Q(x_0, y_0) = ax_0^2 + bx_0y_0 + cy_0^2 = \sum\nolimits_{i+j=3} a_{ij} x_0^i y_0^j$$

for some a_{ij} in A. We want to prove that we can increase the order of vanishing by changing our choice of x_0, y_0. Suppose that $x_1, y_1 \in \mathfrak{m}^2$. Then

$$Q(x_0 + x_1, y_0 + y_1) = Q(x_0, y_0) + (2ax_0 + by_0)x_1 + (bx_0 + 2cy_0)y_1 \bmod \mathfrak{m}^4$$

Nondegeneracy of Q means exactly that $2ax_0 + by_0$ and $bx_0 + 2cy_0$ are a κ-basis for $\mathfrak{m}/\mathfrak{m}^2$, see discussion preceding the lemma. Hence we can certainly choose $x_1, y_1 \in \mathfrak{m}^2$ such that $Q(x_0 + x_1, y_0 + y_1) \in \mathfrak{m}^4$. Continuing in this fashion by induction we can find $x_i, y_i \in \mathfrak{m}^{i+1}$ such that

$$Q(x_0 + x_1 + \ldots + x_n, y_0 + y_1 + \ldots + y_n) \in \mathfrak{m}^{n+3}$$

Since A is complete we can set $x_\infty = \sum x_i$ and $y_\infty = \sum y_i$ and we can consider the map $\kappa[[x, y]] \longrightarrow A$ sending x to x_∞ and y to y_∞. This map induces a surjection $\kappa[[x, y]]/(Q) \longrightarrow A$ by Algebra, Lemma 96.1. By Lemma 19.2 the kernel of

$k[[x, y]] \to A$ is principal. But the kernel cannot contain a proper divisor of Q as such a divisor would have degree 1 in x, y and this would contradict $\dim(\mathfrak{m}/\mathfrak{m}^2) = 2$. Hence Q generates the kernel as desired. $\square$

Lemma 19.4. *Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a Nagata local k-algebra. The following are equivalent*

1. *$k \to A$ is as in Lemma 19.3,*
2. *κ/k is separable, A is reduced of dimension 1, the δ-invariant of A is 1, and A has 2 geometric branches.*

If this holds, then the integral closure A' of A in its total ring of fractions has either 1 or 2 maximal ideals $\mathfrak{m}'$ and the extensions $\kappa(\mathfrak{m}')/k$ are separable.

Proof. In both cases A and $A^\wedge$ are reduced. In case (2) because the completion of a reduced local Nagata ring is reduced (More on Algebra, Lemma 43.6). In both cases A and $A^\wedge$ have dimension 1 (More on Algebra, Lemma 43.1). The δ-invariant and the number of geometric branches of A and $A^\wedge$ agree by Varieties, Lemma 39.6 and More on Algebra, Lemma 108.7. Let A' be the integral closure of A in its total ring of fractions as in Varieties, Lemma 39.2. By Varieties, Lemma 39.5 we see that $A' \otimes_A A^\wedge$ plays the same role for $A^\wedge$. Thus we may replace A by $A^\wedge$ and assume A is complete.

Assume (1) holds. It suffices to show that A has two geometric branches and δ-invariant 1. We may assume $A = \kappa[[x, y]]/(ax^2 + bxy + cy^2)$ with $q = ax^2 + bxy + cy^2$ nondegenerate. There are two cases.

Case I: q splits over κ. In this case we may after changing coordinates assume that $q = xy$. Then we see that

$$A' = \kappa[[x, y]]/(x) \times \kappa[[x, y]]/(y)$$

Case II: q does not split. In this case $c \neq 0$ and nondegenerate means $b^2 - 4ac \neq 0$. Hence $\kappa' = \kappa[t]/(a + bt + ct^2)$ is a degree 2 separable extension of κ. Then $t = y/x$ is integral over A and we conclude that

$$A' = \kappa'[[x]]$$

with y mapping to tx on the right hand side.

In both cases one verifies by hand that the δ-invariant is 1 and the number of geometric branches is 2. In this way we see that (1) implies (2). Moreover we conclude that the final statement of the lemma holds.

Assume (2) holds. More on Algebra, Lemma 106.7 implies A' either has two maximal ideals or A' has one maximal ideal and $[\kappa(\mathfrak{m}') : \kappa]_s = 2$.

Case I: A' has two maximal ideals $\mathfrak{m}'_1$, $\mathfrak{m}'_2$ with residue fields κ_1, κ_2. Since the δ-invariant is the length of A'/A and since there is a surjection $A'/A \to (\kappa_1 \times \kappa_2)/\kappa$ we see that $\kappa = \kappa_1 = \kappa_2$. Since A is complete (and henselian by Algebra, Lemma 153.9) and A' is finite over A we see that $A' = A_1 \times A_2$ (by Algebra, Lemma 153.4). Since A' is a normal ring it follows that A_1 and A_2 are discrete valuation rings. Hence A_1 and A_2 are isomorphic to $\kappa[[t]]$ (as k-algebras) by More on Algebra, Lemma 38.4. Since the δ-invariant is 1 we conclude that A is the wedge of A_1 and A_2 (Varieties, Definition 40.4). It follows easily that $A \cong \kappa[[x, y]]/(xy)$.

Case II: A' has a single maximal ideal $\mathfrak{m}'$ with residue field κ' and $[\kappa' : \kappa]_s = 2$. Arguing exactly as in Case I we see that $[\kappa' : \kappa] = 2$ and κ' is separable over κ. Since A' is normal we see that A' is isomorphic to $\kappa'[[t]]$ (see reference above). Since A'/A has length 1 we conclude that

$$A = \{f \in \kappa'[[t]] \mid f(0) \in \kappa\}$$

Then a simple computation shows that A as in case (1). □

Lemma 19.5. *Let k be a field. Let $A = k[[x_1, \ldots, x_n]]$. Let $I = (f_1, \ldots, f_m) \subset A$ be an ideal. For any $r \geq 0$ the ideal in A/I generated by the $r \times r$-minors of the matrix $(\partial f_j/\partial x_i)$ is independent of the choice of the generators of I or the regular system of parameters $x_1, \ldots, x_n$ of A.*

Proof. The "correct" proof of this lemma is to prove that this ideal is the $(n-r)$th Fitting ideal of a module of continuous differentials of A/I over k. Here is a direct proof. If $g_1, \ldots g_l$ is a second set of generators of I, then we can write $g_s = \sum a_{sj} f_j$ and we have the equality of matrices

$$(\partial g_s/\partial x_i) = (a_{sj})(\partial f_j/\partial x_i) + (\partial a_{sj}/\partial x_i f_j)$$

The final term is zero in A/I. By the Cauchy-Binet formula we see that the ideal of minors for the g_s is contained in the ideal for the f_j. By symmetry these ideals are the same. If $y_1, \ldots, y_n \in \mathfrak{m}_A$ is a second regular system of parameters, then the matrix $(\partial y_j/\partial x_i)$ is invertible and we can use the chain rule for differentiation. Some details omitted. □

Lemma 19.6. *Let k be a field. Let $A = k[[x_1, \ldots, x_n]]$. Let $I = (f_1, \ldots, f_m) \subset \mathfrak{m}_A$ be an ideal. The following are equivalent*

(1) *$k \to A/I$ is as in Lemma 19.3,*
(2) *A/I is reduced and the $(n-1) \times (n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I + \mathfrak{m}_A$,*
(3) *depth$(A/I) = 1$ and the $(n-1) \times (n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I + \mathfrak{m}_A$.*

Proof. By Lemma 19.5 we may change our system of coordinates and the choice of generators during the proof.

If (1) holds, then we may change coordinates such that $x_1, \ldots, x_{n-2}$ map to zero in A/I and $A/I = k[[x_{n-1}, x_n]]/(ax_{n-1}^2 + bx_{n-1}x_n + cx_n^2)$ for some nondegenerate quadric $ax_{n-1}^2 + bx_{n-1}x_n + cx_n^2$. Then we can explicitly compute to show that both (2) and (3) are true.

Assume the $(n-1)\times(n-1)$ minors of the matrix $(\partial f_j/\partial x_i)$ generate $I+\mathfrak{m}_A$. Suppose that for some i and j the partial derivative $\partial f_j/\partial x_i$ is a unit in A. Then we may use the system of parameters $f_j, x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n$ and the generators $f_j, f_1, \ldots, f_{j-1}, \hat{f}_j, f_{j+1}, \ldots, f_m$ of I. Then we get a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, f_2, \ldots, f_m$ of I. Next, we look for an $i \geq 2$ and $j \geq 2$ such that $\partial f_j/\partial x_i$ is a unit in A. If such a pair exists, then we can make a replacement as above and assume that we have a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, x_2, f_3, \ldots, f_m$ of I. Continuing, in finitely many steps we reach the situation where we have a regular system of parameters $x_1, \ldots, x_n$ and generators $x_1, \ldots, x_t, f_{t+1}, \ldots, f_m$ of I such that $\partial f_j/\partial x_i \in \mathfrak{m}_A$ for all $i, j \geq t+1$.

In this case the matrix of partial derivatives has the following block shape

$$\begin{pmatrix} I_{t\times t} & * \\ 0 & \mathfrak{m}_A \end{pmatrix}$$

Hence every $(n-1)\times(n-1)$-minor is in $\mathfrak{m}_A^{n-1-t}$. Note that $I \neq \mathfrak{m}_A$ otherwise the ideal of minors would contain 1. It follows that $n-1-t \leq 1$ because there is an element of $\mathfrak{m}_A \setminus \mathfrak{m}_A^2 + I$ (otherwise $I = \mathfrak{m}_A$ by Nakayama). Thus $t \geq n-2$. We have seen that $t \neq n$ above and similarly if $t = n-1$, then there is an invertible $(n-1)\times(n-1)$-minor which is disallowed as well. Hence $t = n-2$. Then A/I is a quotient of $k[[x_{n-1}, x_n]]$ and Lemma 19.2 implies in both cases (2) and (3) that I is generated by $x_1, \ldots, x_{n-2}, f$ for some $f = f(x_{n-1}, x_n)$. In this case the condition on the minors exactly says that the quadratic term in f is nondegenerate, i.e., A/I is as in Lemma 19.3. □

Lemma 19.7. *Let k be a field. Let X be a 1-dimensional algebraic k-scheme. Let $x \in X$ be a closed point. The following are equivalent*

(1) *x is a node,*
(2) *$k \to \mathcal{O}_{X,x}$ is as in Lemma 19.3,*
(3) *any $\overline{x} \in X_{\overline{k}}$ mapping to x defines a nodal singularity,*
(4) *$\kappa(x)/k$ is separable, $\mathcal{O}_{X,x}$ is reduced, and the first Fitting ideal of $\Omega_{X/k}$ generates $\mathfrak{m}_x$ in $\mathcal{O}_{X,x}$,*
(5) *$\kappa(x)/k$ is separable, depth$(\mathcal{O}_{X,x}) = 1$, and the first Fitting ideal of $\Omega_{X/k}$ generates $\mathfrak{m}_x$ in $\mathcal{O}_{X,x}$,*
(6) *$\kappa(x)/k$ is separable and $\mathcal{O}_{X,x}$ is reduced, has δ-invariant 1, and has 2 geometric branches.*

Proof. First assume that k is algebraically closed. In this case the equivalence of (1) and (3) is trivial. The equivalence of (1) and (3) with (2) holds because the only nondegenerate quadric in two variables is xy up to change in coordinates. The equivalence of (1) and (6) is Lemma 16.1. After replacing X by an affine neighbourhood of x, we may assume there is a closed immersion $X \to \mathbf{A}^n_k$ mapping x to 0. Let $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ be generators for the ideal I of X in $\mathbf{A}^n_k$. Then $\Omega_{X/k}$ corresponds to the $R = k[x_1, \ldots, x_n]/I$-module $\Omega_{R/k}$ which has a presentation

$$R^{\oplus m} \xrightarrow{(\partial f_j/\partial x_i)} R^{\oplus n} \to \Omega_{R/k} \to 0$$

(See Algebra, Sections 131 and 134.) The first Fitting ideal of $\Omega_{R/k}$ is thus the ideal generated by the $(n-1)\times(n-1)$-minors of the matrix $(\partial f_j/\partial x_i)$. Hence (2), (4), (5) are equivalent by Lemma 19.6 applied to the completion of $k[x_1, \ldots, x_n] \to R$ at the maximal ideal $(x_1, \ldots, x_n)$.

Now assume k is an arbitrary field. In cases (2), (4), (5), (6) the residue field $\kappa(x)$ is separable over k. Let us show this holds as well in cases (1) and (3). Namely, let $Z \subset X$ be the closed subscheme of X defined by the first Fitting ideal of $\Omega_{X/k}$. The formation of Z commutes with field extension (Divisors, Lemma 10.1). If (1) or (3) is true, then there exists a point $\overline{x}$ of $X_{\overline{k}}$ such that $\overline{x}$ is an isolated point of multiplicity 1 of $Z_{\overline{k}}$ (as we have the equivalence of the conditions of the lemma over $\overline{k}$). In particular $Z_{\overline{x}}$ is geometrically reduced at $\overline{x}$ (because $\overline{k}$ is algebraically closed). Hence Z is geometrically reduced at x (Varieties, Lemma 6.6). In particular, Z is reduced at x, hence $Z = \mathrm{Spec}(\kappa(x))$ in a neighbourhood of

x and $\kappa(x)$ is geometrically reduced over k. This means that $\kappa(x)/k$ is separable (Algebra, Lemma 44.1).

The argument of the previous paragraph shows that if (1) or (3) holds, then the first Fitting ideal of $\Omega_{X/k}$ generates $\mathfrak{m}_x$. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is flat and since $\mathcal{O}_{X_{\overline{k}},\overline{x}}$ is reduced and has depth 1, we see that (4) and (5) hold (use Algebra, Lemmas 164.2 and 163.2). Conversely, (4) implies (5) by Algebra, Lemma 157.3. If (5) holds, then Z is geometrically reduced at x (because $\kappa(x)/k$ separable and Z is x in a neighbourhood). Hence $Z_{\overline{k}}$ is reduced at any point $\overline{x}$ of $X_{\overline{k}}$ lying over x. In other words, the first fitting ideal of $\Omega_{X_{\overline{k}}/\overline{k}}$ generates $\mathfrak{m}_{\overline{x}}$ in $\mathcal{O}_{X_{\overline{k}},\overline{x}}$. Moreover, since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_{\overline{k}},\overline{x}}$ is flat we see that $\text{depth}(\mathcal{O}_{X_{\overline{k}},\overline{x}}) = 1$ (see reference above). Hence (5) holds for $\overline{x} \in X_{\overline{k}}$ and we conclude that (3) holds (because of the equivalence over algebraically closed fields). In this way we see that (1), (3), (4), (5) are equivalent.

The equivalence of (2) and (6) follows from Lemma 19.4.

Finally, we prove the equivalence of (2) = (6) with (1) = (3) = (4) = (5). First we note that the geometric number of branches of X at x and the geometric number of branches of $X_{\overline{k}}$ at $\overline{x}$ are equal by Varieties, Lemma 40.2. We conclude from the information available to us at this point that in all cases this number is equal to 2. On the other hand, in case (1) it is clear that X is geometrically reduced at x, and hence

$$\delta\text{-invariant of } X \text{ at } x \leq \delta\text{-invariant of } X_{\overline{k}} \text{ at } \overline{x}$$

by Varieties, Lemma 39.8. Since in case (1) the right hand side is 1, this forces the δ-invariant of X at x to be 1 (because if it were zero, then $\mathcal{O}_{X,x}$ would be a discrete valuation ring by Varieties, Lemma 39.4 which is unibranch, a contradiction). Thus (5) holds. Conversely, if (2) = (5) is true, then assumptions (a), (b), (c) of Varieties, Lemma 27.6 hold for $x \in X$ by Lemma 19.4. Thus Varieties, Lemma 39.9 applies and shows that we have equality in the above displayed inequality. We conclude that (5) holds for $\overline{x} \in X_{\overline{k}}$ and we are back in case (1) by the equivalence of the conditions over an algebraically closed field. □

Remark 19.8 (The quadratic extension associated to a node)**.** Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local k-algebra. Assume that either $(A, \mathfrak{m}, \kappa)$ is as in Lemma 19.3, or A is Nagata as in Lemma 19.4, or A is complete and as in Lemma 19.6. Then A defines canonically a degree 2 separable κ-algebra κ' as follows

1. let $q = ax^2 + bxy + cy^2$ be a nondegenerate quadric as in Lemma 19.3 with coordinates x, y chosen such that $a \neq 0$ and set $\kappa' = \kappa[x]/(ax^2 + bx + c)$,
2. let $A' \supset A$ be the integral closure of A in its total ring of fractions and set $\kappa' = A'/\mathfrak{m}A'$, or
3. let κ' be the κ-algebra such that $\text{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}) = \text{Spec}(\kappa')$.

The equivalence of (1) and (2) was shown in the proof of Lemma 19.4. We omit the equivalence of this with (3). If X is a locally Noetherian k-scheme and $x \in X$ is a point such that $\mathcal{O}_{X,x} = A$, then (3) shows that $\text{Spec}(\kappa') = X^\nu \times_X \text{Spec}(\kappa)$ where $\nu : X^\nu \to X$ is the normalization morphism.

Remark 19.9 (Trivial quadratic extension)**.** Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be as in Remark 19.8 and let κ'/κ be the associated separable algebra of degree 2. Then the following are equivalent

1. $\kappa' \cong \kappa \times \kappa$ as κ-algebra,

(2) the form q of Lemma 19.3 can be chosen to be xy,
(3) A has two branches,
(4) the extension A'/A of Lemma 19.4 has two maximal ideals, and
(5) $A^\wedge \cong \kappa[[x, y]]/(xy)$ as a k-algebra.

The equivalence between these conditions has been shown in the proof of Lemma 19.4. If X is a locally Noetherian k-scheme and $x \in X$ is a point such that $\mathcal{O}_{X,x} = A$, then this means exactly that there are two points x_1, x_2 of the normalization X^ν lying over x and that $\kappa(x) = \kappa(x_1) = \kappa(x_2)$.

Definition 19.10. Let k be a field. Let X be a 1-dimensional algebraic k-scheme. Let $x \in X$ be a closed point. We say x is a *split node* if x is a node, $\kappa(x) = k$, and the equivalent assertions of Remark 19.9 hold for $A = \mathcal{O}_{X,x}$.

We formulate the obligatory lemma stating what we already know about this concept.

Lemma 19.11. *Let k be a field. Let X be a 1-dimensional algebraic k-scheme. Let $x \in X$ be a closed point. The following are equivalent*

(1) *x is a split node,*
(2) *x is a node and there are exactly two points x_1, x_2 of the normalization X^ν lying over x with $k = \kappa(x_1) = \kappa(x_2)$,*
(3) *$\mathcal{O}_{X,x}^\wedge \cong k[[x, y]]/(xy)$ as a k-algebra, and*
(4) *add more here.*

Proof. This follows from the discussion in Remark 19.9 and Lemma 19.7. $\square$

Lemma 19.12. *Let K/k be an extension of fields. Let X be a locally algebraic k-scheme of dimension 1. Let $y \in X_K$ be a point with image $x \in X$. The following are equivalent*

(1) *x is a closed point of X and a node, and*
(2) *y is a closed point of Y and a node.*

Proof. If x is a closed point of X, then y is too (look at residue fields). But conversely, this need not be the case, i.e., it can happen that a closed point of Y maps to a nonclosed point of X. However, in this case y cannot be a node. Namely, then X would be geometrically unibranch at x (because x would be a generic point of X and $\mathcal{O}_{X,x}$ would be Artinian and any Artinian local ring is geometrically unibranch), hence Y is geometrically unibranch at y (Varieties, Lemma 40.3), which means that y cannot be a node by Lemma 19.7. Thus we may and do assume that both x and y are closed points.

Choose algebraic closures $\overline{k}$, $\overline{K}$ and a map $\overline{k} \to \overline{K}$ extending the given map $k \to K$. Using the equivalence of (1) and (3) in Lemma 19.7 we reduce to the case where k and K are algebraically closed. In this case we can argue as in the proof of Lemma 19.7 that the geometric number of branches and δ-invariants of X at x and Y at y are the same. Another argument can be given by choosing an isomorphism $k[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \to \mathcal{O}_{X,x}^\wedge$ of k-algebras as in Varieties, Lemma 21.1. By Varieties, Lemma 21.2 this gives an isomorphism $K[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \to \mathcal{O}_{Y,y}^\wedge$ of K-algebras. By definition we have to show that

$$k[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \cong k[[s, t]]/(st)$$

if and only if

$$K[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \cong K[[s, t]]/(st)$$

We encourage the reader to prove this for themselves. Since k and K are algebraically closed fields, this is the same as asking these rings to be as in Lemma 19.3. Via Lemma 19.6 this translates into a statement about the $(n-1) \times (n-1)$-minors of the matrix $(\partial g_j / \partial x_i)$ which is clearly independent of the field used. We omit the details. □

Lemma 19.13. *Let k be a field. Let X be a locally algebraic k-scheme of dimension 1. Let $Y \to X$ be an étale morphism. Let $y \in Y$ be a point with image $x \in X$. The following are equivalent*

1. *x is a closed point of X and a node, and*
2. *y is a closed point of Y and a node.*

Proof. By Lemma 19.12 we may base change to the algebraic closure of k. Then the residue fields of x and y are k. Hence the map $\mathcal{O}_{X,x}^\wedge \to \mathcal{O}_{Y,y}^\wedge$ is an isomorphism (for example by Étale Morphisms, Lemma 11.3 or More on Algebra, Lemma 43.9). Thus the lemma is clear. □

Lemma 19.14. *Let k'/k be a finite separable field extension. Let X be a locally algebraic k'-scheme of dimension 1. Let $x \in X$ be a closed point. The following are equivalent*

1. *x is a node, and*
2. *x is a node when X viewed as a locally algebraic k-scheme.*

Proof. Follows immediately from the characterization of nodes in Lemma 19.7. □

Lemma 19.15. *Let k be a field. Let X be a locally algebraic k-scheme equidimensional of dimension 1. The following are equivalent*

1. *the singularities of X are at-worst-nodal, and*
2. *X is a local complete intersection over k and the closed subscheme $Z \subset X$ cut out by the first fitting ideal of $\Omega_{X/k}$ is unramified over k.*

Proof. We urge the reader to find their own proof of this lemma; what follows is just putting together earlier results and may hide what is really going on.

Assume (2). Since $Z \to \mathrm{Spec}(k)$ is quasi-finite (Morphisms, Lemma 35.10) we see that the residue fields of points $x \in Z$ are finite over k (as well as separable) by Morphisms, Lemma 20.5. Hence each $x \in Z$ is a closed point of X by Morphisms, Lemma 20.2. The local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay by Algebra, Lemma 135.3. Since $\dim(\mathcal{O}_{X,x}) = 1$ by dimension theory (Varieties, Section 20), we conclude that $\mathrm{depth}(\mathcal{O}_{X,x})) = 1$. Thus x is a node by Lemma 19.7. If $x \in X$, $x \notin Z$, then $X \to \mathrm{Spec}(k)$ is smooth at x by Divisors, Lemma 10.3.

Assume (1). Under this assumption X is geometrically reduced at every closed point (see Varieties, Lemma 6.6). Hence $X \to \mathrm{Spec}(k)$ is smooth on a dense open by Varieties, Lemma 25.7. Thus Z is closed and consists of closed points. By Divisors, Lemma 10.3 the morphism $X \setminus Z \to \mathrm{Spec}(k)$ is smooth. Hence $X \setminus Z$ is a local complete intersection by Morphisms, Lemma 34.7 and the definition of a local complete intersection in Morphisms, Definition 30.1. By Lemma 19.7 for every point $x \in Z$ the local ring $\mathcal{O}_{Z,x}$ is equal to $\kappa(x)$ and $\kappa(x)$ is separable over k. Thus $Z \to \mathrm{Spec}(k)$ is unramified (Morphisms, Lemma 35.11). Finally, Lemma

19.7 via part (3) of Lemma 19.3, shows that $\mathcal{O}_{X,x}$ is a complete intersection in the sense of Divided Power Algebra, Definition 8.5. However, Divided Power Algebra, Lemma 8.8 and Morphisms, Lemma 30.9 show that this agrees with the notion used to define a local complete intersection scheme over a field and the proof is complete. $\square$

Lemma 19.16. *Let k be a field. Let X be a locally algebraic k-scheme equidimensional of dimension 1 whose singularities are at-worst-nodal. Then X is Gorenstein and geometrically reduced.*

Proof. The Gorenstein assertion follows from Lemma 19.15 and Duality for Schemes, Lemma 24.5. Or you can use that it suffices to check after passing to the algebraic closure (Duality for Schemes, Lemma 25.1), then use that a Noetherian local ring is Gorenstein if and only if its completion is so (by Dualizing Complexes, Lemma 21.8), and then prove that the local rings $k[[t]]$ and $k[[x, y]]/(xy)$ are Gorenstein by hand.

To see that X is geometrically reduced, it suffices to show that $X_{\overline{k}}$ is reduced (Varieties, Lemmas 6.3 and 6.4). But $X_{\overline{k}}$ is a nodal curve over an algebraically closed field. Thus the complete local rings of $X_{\overline{k}}$ are isomorphic to either $\overline{k}[[t]]$ or $\overline{k}[[x, y]]/(xy)$ which are reduced as desired. $\square$

Lemma 19.17. *Let k be a field. Let X be a locally algebraic k-scheme equidimensional of dimension 1 whose singularities are at-worst-nodal. If $Y \subset X$ is a reduced closed subscheme equidimensional of dimension 1, then*

(1) *the singularities of Y are at-worst-nodal, and*
(2) *if $Z \subset X$ is the scheme theoretic closure of $X \setminus Y$, then*
    (a) *the scheme theoretic intersection $Y \cap Z$ is the disjoint union of spectra of finite separable extensions of k,*
    (b) *each point of $Y \cap Z$ is a node of X, and*
    (c) *$Y \to \operatorname{Spec}(k)$ is smooth at every point of $Y \cap Z$.*

Proof. Since X and Y are reduced and equidimensional of dimension 1, we see that Y is the scheme theoretic union of a subset of the irreducible components of X (in a reduced ring (0) is the intersection of the minimal primes). Let $y \in Y$ be a closed point. If y is in the smooth locus of $X \to \operatorname{Spec}(k)$, then y is on a unique irreducible component of X and we see that Y and X agree in an open neighbourhood of y. Hence $Y \to \operatorname{Spec}(k)$ is smooth at y. If y is a node of X but still lies on a unique irreducible component of X, then y is a node on Y by the same argument. Suppose that y lies on more than 1 irreducible component of X. Since the number of geometric branches of X at y is 2 by Lemma 19.7, there can be at most 2 irreducible components passing through y by Properties, Lemma 15.5. If Y contains both of these, then again $Y = X$ in an open neighbourhood of y and y is a node of Y. Finally, assume Y contains only one of the irreducible components. After replacing X by an open neighbourhood of x we may assume Y is one of the two irreducble components and Z is the other. By Properties, Lemma 15.5 again we see that X has two branches at y, i.e., the local ring $\mathcal{O}_{X,y}$ has two branches and that these branches come from $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,y}$. Write $\mathcal{O}_{X,y}^\wedge \cong \kappa(y)[[u, v]]/(uv)$ as in Remark 19.9. The field $\kappa(y)$ is finite separable over k by Lemma 19.7 for example. Thus, after possibly switching the roles of u and v, the completion of the map $\mathcal{O}_{X,y} \to \mathcal{O}_{Y,Y}$ corresponds to $\kappa(y)[[u, v]]/(uv) \to \kappa(y)[[u]]$ and the completion of the map $\mathcal{O}_{X,y} \to \mathcal{O}_{Y,Y}$ corresponds to $\kappa(y)[[u,v]]/(uv) \to \kappa(y)[[v]]$. The scheme theoretic intersection of $Y \cap Z$ is cut out by the sum of their ideas which in the completion is (u,v), i.e., the maximal ideal. Thus (2)(a) and (2)(b) are clear. Finally, (2)(c) holds: the completion of $\mathcal{O}_{Y,y}$ is regular, hence $\mathcal{O}_{Y,y}$ is regular (More on Algebra, Lemma 43.4) and $\kappa(y)/k$ is separable, hence smoothness in an open neighbourhood by Algebra, Lemma 140.5. $\square$

20. Families of nodal curves

In the Stacks project curves are irreducible varieties of dimension 1, but in the literature a "semi-stable curve" or a "nodal curve" is usually not irreducible and often assumed to be proper, especially when used in a phrase such as "family of semistable curves" or "family of nodal curves", or "nodal family". Thus it is a bit difficult for us to choose a terminology which is consistent with the literature as well as internally consistent. Moreover, we really want to first study the notion introduced in the following lemma (which is local on the source).

Lemma 20.1. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *f is flat, locally of finite presentation, every nonempty fibre X_s is equidimensional of dimension 1, and X_s has at-worst-nodal singularities, and*
2. *f is syntomic of relative dimension 1 and the closed subscheme $Sing(f) \subset X$ defined by the first Fitting ideal of $\Omega_{X/S}$ is unramified over S.*

Proof. Recall that the formation of $\text{Sing}(f)$ commutes with base change, see Divisors, Lemma 10.1. Thus the lemma follows from Lemma 19.15, Morphisms, Lemma 30.11, and Morphisms, Lemma 35.12. (We also use the trivial Morphisms, Lemmas 30.6 and 30.7.) $\square$

Definition 20.2. Let $f : X \to S$ be a morphism of schemes. We say f is *at-worst-nodal of relative dimension* 1 if f satisfies the equivalent conditions of Lemma 20.1.

Here are some reasons for the cumbersome terminology[6]. First, we want to make sure this notion is not confused with any of the other notions in the literature (see introduction to this section). Second, we can imagine several generalizations of this notion to morphisms of higher relative dimension (for example, one can ask for morphisms which are étale locally compositions of at-worst-nodal morphisms or one can ask for morphisms whose fibres are higher dimensional but have at worst ordinary double points).

Lemma 20.3. *A smooth morphism of relative dimension 1 is at-worst-nodal of relative dimension 1.*

Proof. Omitted. $\square$

Lemma 20.4. *Let $f : X \to S$ be at-worst-nodal of relative dimension 1. Then the same is true for any base change of f.*

Proof. This is true because the base change of a syntomic morphism is syntomic (Morphisms, Lemma 30.4), the base change of a morphism of relative dimension 1 has relative dimension 1 (Morphisms, Lemma 29.2), the formation of $\text{Sing}(f)$

[6]But please email the maintainer of the Stacks project if you have a better suggestion.

commutes with base change (Divisors, Lemma 10.1), and the base change of an unramified morphism is unramified (Morphisms, Lemma 35.5). ☐

The following lemma tells us that we can check whether a morphism is at-worst-nodal of relative dimension 1 on the fibres.

Lemma 20.5. *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Then there is a maximal open subscheme $U \subset X$ such that $f|_U : U \to S$ is at-worst-nodal of relative dimension 1. Moreover, formation of U commutes with arbitrary base change.*

Proof. By Morphisms, Lemma 30.12 we find that there is such an open where f is syntomic. Hence we may assume that f is a syntomic morphism. In particular f is a Cohen-Macaulay morphism (Duality for Schemes, Lemmas 25.5 and 25.4). Thus X is a disjoint union of open and closed subschemes on which f has given relative dimension, see Morphisms, Lemma 29.4. This decomposition is preserved by arbitrary base change, see Morphisms, Lemma 29.2. Discarding all but one piece we may assume f is syntomic of relative dimension 1. Let $\mathrm{Sing}(f) \subset X$ be the closed subscheem defined by the first fitting ideal of $\Omega_{X/S}$. There is a maximal open subscheme $W \subset \mathrm{Sing}(f)$ such that $W \to S$ is unramified and its formation commutes with base change (Morphisms, Lemma 35.15). Since also formation of $\mathrm{Sing}(f)$ commutes with base change (Divisors, Lemma 10.1), we see that

$$U = (X \setminus \mathrm{Sing}(f)) \cup W$$

is the maximal open subscheme of X such that $f|_U : U \to S$ is at-worst-nodal of relative dimension 1 and that formation of U commutes with base change. ☐

Lemma 20.6. *Let $f : X \to S$ be at-worst-nodal of relative dimension 1. If $Y \to X$ is an étale morphism, then the composition $g : Y \to S$ is at-worst-nodal of relative dimension 1.*

Proof. Observe that g is flat and locally of finite presentation as a composition of morphisms which are flat and locally of finite presentation (use Morphisms, Lemmas 36.11, 36.12, 21.3, and 25.6). Thus it suffices to prove the fibres have at-worst-nodal singularities. This follows from Lemma 19.13 (and the fact that the composition of an étale morphism and a smooth morphism is smooth by Morphisms, Lemmas 36.5 and 34.4). ☐

Lemma 20.7. *Let $S' \to S$ be an étale morphism of schemes. Let $f : X \to S'$ be at-worst-nodal of relative dimension 1. Then the composition $g : X \to S$ is at-worst-nodal of relative dimension 1.*

Proof. Observe that g is flat and locally of finite presentation as a composition of morphisms which are flat and locally of finite presentation (use Morphisms, Lemmas 36.11, 36.12, 21.3, and 25.6). Thus it suffices to prove the fibres of g have at-worst-nodal singularities. This follows from Lemma 19.14 and the analogous result for smooth points. ☐

Lemma 20.8. *Let $f : X \to S$ be a morphism of schemes. Let $\{U_i \to X\}$ be an étale covering. The following are equivalent*

(1) *f is at-worst-nodal of relative dimension 1,*
(2) *each $U_i \to S$ is at-worst-nodal of relative dimension 1.*

In other words, being at-worst-nodal of relative dimension 1 *is étale local on the source.*

Proof. One direction we have seen in Lemma 20.6. For the other direction, observe that being locally of finite presentation, flat, or to have relative dimension 1 is étale local on the source (Descent, Lemmas 28.1, 27.1, and 33.8). Taking fibres we reduce to the case where S is the spectrum of a field. In this case the result follows from Lemma 19.13 (and the fact that being smooth is étale local on the source by Descent, Lemma 30.1). ☐

Lemma 20.9. *Let $f : X \to S$ be a morphism of schemes. Let $\{U_i \to S\}$ be an fpqc covering. The following are equivalent*

1. *f is at-worst-nodal of relative dimension* 1*,*
2. *each $X \times_S U_i \to U_i$ is at-worst-nodal of relative dimension* 1*.*

In other words, being at-worst-nodal of relative dimension 1 *is fpqc local on the target.*

Proof. One direction we have seen in Lemma 20.4. For the other direction, observe that being locally of finite presentation, flat, or to have relative dimension 1 is fpqc local on the target (Descent, Lemmas 23.11, 23.15, and Morphisms, Lemma 28.3). Taking fibres we reduce to the case where S is the spectrum of a field. In this case the result follows from Lemma 19.12 (and the fact that being smooth is fpqc local on the target by Descent, Lemma 23.27). ☐

Lemma 20.10. *Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S. If*

1. *f is at-worst-nodal of relative dimension* 1*, and*
2. *f_0 is locally of finite presentation,*

then there exists an $i \geq 0$ such that f_i is at-worst-nodal of relative dimension 1*.*

Proof. By Limits, Lemma 8.16 there exists an i such that f_i is syntomic. Then $X_i = \coprod_{d \geq 0} X_{i,d}$ is a disjoint union of open and closed subschemes such that $X_{i,d} \to Y_i$ has relative dimension d, see Morphisms, Lemma 30.14. Because of the behaviour of dimensions of fibres under base change given in Morphisms, Lemma 28.3 we see that $X \to X_i$ maps into $X_{i,1}$. Then there exists an $i' \geq i$ such that $X_{i'} \to X_i$ maps into $X_{i,1}$, see Limits, Lemma 4.10. Thus $f_{i'} : X_{i'} \to Y_{i'}$ is syntomic of relative dimension 1 (by Morphisms, Lemma 28.3 again). Consider the morphism $\text{Sing}(f_{i'}) \to Y_{i'}$. We know that the base change to Y is an unramified morphism. Hence by Limits, Lemma 8.4 we see that after increasing i' the morphism $\text{Sing}(f_{i'}) \to Y_{i'}$ becomes unramified. This finishes the proof. ☐

Lemma 20.11. *Let $f : T \to S$ be a morphism of schemes. Let $t \in T$ with image $s \in S$. Assume*

1. *f is flat at t,*
2. *$\mathcal{O}_{S,s}$ is Noetherian,*
3. *f is locally of finite type,*
4. *t is a split node of the fibre T_s.*

Then there exists an $h \in \mathfrak{m}_s^\wedge$ *and an isomorphism*

$$\mathcal{O}_{T,t}^\wedge \cong \mathcal{O}_{S,s}^\wedge[[x,y]]/(xy-h)$$

of $\mathcal{O}_{S,s}^\wedge$*-algebras.*

Proof. We replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ and T by the base change to $\mathrm{Spec}(\mathcal{O}_{S,s})$. Then T is locally Noetherian and hence $\mathcal{O}_{T,t}$ is Noetherian. Set $A = \mathcal{O}_{S,s}^\wedge$, $\mathfrak{m} = \mathfrak{m}_A$, and $B = \mathcal{O}_{T,t}^\wedge$. By More on Algebra, Lemma 43.8 we see that $A \to B$ is flat. Since $\mathcal{O}_{T,t}/\mathfrak{m}_s\mathcal{O}_{T,t} = \mathcal{O}_{T_s,t}$ we see that $B/\mathfrak{m}B = \mathcal{O}_{T_s,t}^\wedge$. By assumption (4) and Lemma 19.11 we conclude there exist $\overline{u}, \overline{v} \in B/\mathfrak{m}B$ such that the map

$$(A/\mathfrak{m})[[x,y]] \longrightarrow B/\mathfrak{m}B, \quad x \longmapsto \overline{u}, x \longmapsto \overline{v}$$

is surjective with kernel (xy).

Assume we have $n \geq 1$ and $u, v \in B$ mapping to $\overline{u}, \overline{v}$ such that

$$uv = h + \delta$$

for some $h \in A$ and $\delta \in \mathfrak{m}^n B$. We claim that there exist $u', v' \in B$ with $u - u', v - v' \in \mathfrak{m}^n B$ such that

$$u'v' = h' + \delta'$$

for some $h' \in A$ and $\delta' \in \mathfrak{m}^{n+1}B$. To see this, write $\delta = \sum f_i b_i$ with $f_i \in \mathfrak{m}^n$ and $b_i \in B$. Then write $b_i = a_i + ub_{i,1} + vb_{i,2} + \delta_i$ with $a_i \in A$, $b_{i,1}, b_{i,2} \in B$ and $\delta_i \in \mathfrak{m}B$. This is possible because the residue field of B agrees with the residue field of A and the images of u and v in $B/\mathfrak{m}B$ generate the maximal ideal. Then we set

$$u' = u - \sum b_{i,2} f_i, \quad v' = v - \sum b_{i,1} f_i$$

and we obtain

$$u'v' = h + \delta - \sum (b_{i,1}u + b_{i,2}v) f_i + \sum c_{ij} f_i f_j = h + \sum a_i f_i + \sum f_i \delta_i + \sum c_{ij} f_i f_j$$

for some $c_{i,j} \in B$. Thus we get a formula as above with $h' = h + \sum a_i f_i$ and $\delta' = \sum f_i \delta_i + \sum c_{ij} f_i f_j$.

Arguing by induction and starting with any lifts $u_1, v_1 \in B$ of $\overline{u}, \overline{v}$ the result of the previous paragraph shows that we find a sequence of elements $u_n, v_n \in B$ and $h_n \in A$ such that $u_n - u_{n+1} \in \mathfrak{m}^n B$, $v_n - v_{n+1} \in \mathfrak{m}^n B$, $h_n - h_{n+1} \in \mathfrak{m}^n$, and such that $u_n v_n - h_n \in \mathfrak{m}^n B$. Since A and B are complete we can set $u_\infty = \lim u_n$, $v_\infty = \lim v_n$, and $h_\infty = \lim h_n$, and then we obtain $u_\infty v_\infty = h_\infty$ in B. Thus we have an A-algebra map

$$A[[x,y]]/(xy - h_\infty) \longrightarrow B$$

sending x to u_∞ and v to v_∞. This is a map of flat A-algebras which is an isomorphism after dividing by $\mathfrak{m}$. It is surjective modulo $\mathfrak{m}$ and hence surjective by completeness and Algebra, Lemma 96.1. Then we can apply Algebra, Lemma 99.1 to conclude it is an isomorphism. $\square$

Consider the morphism of schemes

$$\mathrm{Spec}(\mathbf{Z}[u,v,a]/(uv-a)) \longrightarrow \mathrm{Spec}(\mathbf{Z}[a])$$

The next lemma shows that this morphism is a model for the étale local structure of a nodal family of curves. If you know a proof of this lemma avoiding the use of Artin approximation, then please email stacks.project@gmail.com.

Lemma 20.12. *Let $f : X \to S$ be a morphism of schemes. Assume that f is at-worst-nodal of relative dimension 1. Let $x \in X$ be a point which is a singular point of the fibre X_s. Then there exists a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[rr] \arrow[dr] & & W \arrow[r] \arrow[dl] & \mathrm{Spec}(\mathbf{Z}[u,v,a]/(uv-a)) \arrow[d] \\ S & & V \arrow[ll] \arrow[rr] & & \mathrm{Spec}(\mathbf{Z}[a]) \end{tikzcd}$$

with $X \leftarrow U$, $S \leftarrow V$, and $U \to W$ étale morphisms, and with the right hand square cartesian, such that there exists a point $u \in U$ mapping to x in X.

Proof. We first use absolute Noetherian approximation to reduce to the case of schemes of finite type over $\mathbf{Z}$. The question is local on X and S. Hence we may assume that X and S are affine. Then we can write $S = \mathrm{Spec}(R)$ and write R as a filtered colimit $R = \mathrm{colim}\, R_i$ of finite type $\mathbf{Z}$-algebras. Using Limits, Lemma 10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(R_i)$ whose base change to S is f. After increasing i we may assume that f_i is at-worst-nodal of relative dimension 1, see Lemma 20.10. The image $x_i \in X_i$ of x will be a singular point of its fibre, for example because the formation of $\mathrm{Sing}(f)$ commutes with base change (Divisors, Lemma 10.1). If we can prove the lemma for $f_i : X_i \to S_i$ and x_i, then the lemma follows for $f : X \to S$ by base change. Thus we reduce to the case studied in the next paragraph.

Assume S is of finite type over $\mathbf{Z}$. Let $s \in S$ be the image of x. Recall that $\kappa(x)$ is a finite separable extension of $\kappa(s)$, for example because $\mathrm{Sing}(f) \to S$ is unramified or because x is a node of the fibre X_s and we can apply Lemma 19.7. Furthermore, let $\kappa'/\kappa(x)$ be the degree 2 separable algebra associated to $\mathcal{O}_{X_s,x}$ in Remark 19.8. By More on Morphisms, Lemma 35.2 we can choose an étale neighbourhood $(V, v) \to (S, s)$ such that the extension $\kappa(v)/\kappa(s)$ realizes either the extension $\kappa(x)/\kappa(s)$ in case $\kappa' \cong \kappa(x) \times \kappa(x)$ or the extension $\kappa'/\kappa(s)$ if κ' is a field. After replacing X by $X \times_S V$ and S by V we reduce to the situation described in the next paragraph.

Assume S is of finite type over $\mathbf{Z}$ and $x \in X_s$ is a split node, see Definition 19.10. By Lemma 20.11 we see that there exists an $\mathcal{O}_{S,s}$-algebra isomorphism

$$\mathcal{O}^\wedge_{X,x} \cong \mathcal{O}^\wedge_{S,s}[[s,t]]/(st-h)$$

for some $h \in \mathfrak{m}^\wedge_s \subset \mathcal{O}^\wedge_{S,s}$. In other words, if we consider the homomorphism

$$\sigma : \mathbf{Z}[a] \longrightarrow \mathcal{O}^\wedge_{S,s}$$

sending a to h, then there exists an $\mathcal{O}_{S,s}$-algebra isomorphism

$$\mathcal{O}^\wedge_{X,x} \longrightarrow \mathcal{O}^\wedge_{Y_\sigma,y_\sigma}$$

where

$$Y_\sigma = \mathrm{Spec}(\mathbf{Z}[u,v,t]/(uv-a)) \times_{\mathrm{Spec}(\mathbf{Z}[a]),\sigma} \mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$$

and y_σ is the point of Y_σ lying over the closed point of $\mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$ and having coordinates u, v equal to zero. Since $\mathcal{O}_{S,s}$ is a G-ring by More on Algebra, Proposition 50.12 we may apply More on Morphisms, Lemma 39.3 to conclude. □

Lemma 20.13. *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is proper,*

(2) *f is at-worst-nodal of relative dimension 1, and*
(3) *the geometric fibres of f are connected.*

Then (a) $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change, (b) $R^1f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module whose formation commutes with any base change, and (c) $R^qf_*\mathcal{O}_X = 0$ for $q \geq 2$.

Proof. Part (a) follows from Derived Categories of Schemes, Lemma 32.6. By Derived Categories of Schemes, Lemma 32.5 locally on S we can write $Rf_*\mathcal{O}_X = \mathcal{O}_S \oplus P$ where P is perfect of tor amplitude in $[1, \infty)$. Recall that formation of $Rf_*\mathcal{O}_X$ commutes with arbitrary base change (Derived Categories of Schemes, Lemma 30.4). Thus for $s \in S$ we have

$$H^i(P \otimes_{\mathcal{O}_S}^{\mathbf{L}} \kappa(s)) = H^i(X_s, \mathcal{O}_{X_s}) \text{ for } i \geq 1$$

This is zero unless $i = 1$ since X_s is a 1-dimensional Noetherian scheme, see Cohomology, Proposition 20.7. Then $P = H^1(P)[-1]$ and $H^1(P)$ is finite locally free for example by More on Algebra, Lemma 75.6. Since everything is compatible with base change we conclude. $\square$

21. More vanishing results

Continuation of Section 6.

Lemma 21.1. *In Situation 6.2 assume X is integral and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a 0-dimensional closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. If $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero, then*

$$\deg(\mathcal{L}) \leq 2g - 2 + \deg(Z)$$

with strict inequality unless $\mathcal{I}\mathcal{L} \cong \omega_X$.

Proof. Any curve, e.g. X, is Cohen-Macaulay. If $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero, then there is a nonzero map $\mathcal{I}\mathcal{L} \to \omega_X$, see Lemma 4.2. Since $\mathcal{I}\mathcal{L}$ is torsion free, this map is injective. Since a field is Gorenstein and X is reduced, we find that the Gorenstein locus $U \subset X$ of X is nonempty, see Duality for Schemes, Lemma 24.4. This lemma also tells us that $\omega_X|_U$ is invertible. In this way we see we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{L} \to \omega_X \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is zero dimensional. Hence we have

$$\begin{aligned} 0 &\leq \dim \Gamma(X, \mathcal{Q}) \\ &= \chi(\mathcal{Q}) \\ &= \chi(\omega_X) - \chi(\mathcal{I}\mathcal{L}) \\ &= \chi(\omega_X) - \deg(\mathcal{L}) - \chi(\mathcal{I}) \\ &= 2g - 2 - \deg(\mathcal{L}) + \deg(Z) \end{aligned}$$

by Lemmas 5.1 and 5.2, by (8.1.1), and by Varieties, Lemmas 33.3 and 44.5. We have also used that $\deg(Z) = \dim_k \Gamma(Z, \mathcal{O}_Z) = \chi(\mathcal{O}_Z)$ and the short exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$. The lemma follows. $\square$

Lemma 21.2. *In Situation 6.2 assume X is integral and has genus g. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a 0-dimensional closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. If $\deg(\mathcal{L}) > 2g - 2 + \deg(Z)$, then $H^1(X, \mathcal{I}\mathcal{L}) = 0$ and one of the following possibilities occurs*

(1) $H^0(X, \mathcal{I}\mathcal{L}) \neq 0$, *or*
(2) $g = 0$ *and* $\deg(\mathcal{L}) = \deg(Z) - 1$.

In case (2) if $Z = \emptyset$, *then* $X \cong \mathbf{P}^1_k$ *and* $\mathcal{L}$ *corresponds to* $\mathcal{O}_{\mathbf{P}^1}(-1)$.

Proof. The vanishing of $H^1(X, \mathcal{I}\mathcal{L})$ follows from Lemma 21.1. If $H^0(X, \mathcal{I}\mathcal{L}) = 0$, then $\chi(\mathcal{I}\mathcal{L}) = 0$. From the short exact sequence $0 \to \mathcal{I}\mathcal{L} \to \mathcal{L} \to \mathcal{O}_Z \to 0$ we conclude $\deg(\mathcal{L}) = g - 1 + \deg(Z)$. Thus $g - 1 + \deg(Z) > 2g - 2 + \deg(Z)$ which implies $g = 0$ hence (2) holds. If $Z = \emptyset$ in case (2), then $\mathcal{L}^{-1}$ is an invertible sheaf of degree 1. This implies there is an isomorphism $X \to \mathbf{P}^1_k$ and $\mathcal{L}^{-1}$ is the pullback of $\mathcal{O}_{\mathbf{P}^1}(1)$ by Lemma 10.2. $\square$

Lemma 21.3. *In Situation 6.2 assume* X *is integral and has genus* g. *Let* $\mathcal{L}$ *be an invertible* $\mathcal{O}_X$*-module. If* $\deg(\mathcal{L}) \geq 2g$, *then* $\mathcal{L}$ *is globally generated.*

Proof. Let $Z \subset X$ be the closed subscheme cut out by the global sections of $\mathcal{L}$. By Lemma 21.2 we see that $Z \neq X$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf cutting out Z. Consider the short exact sequence

$$0 \to \mathcal{I}\mathcal{L} \to \mathcal{L} \to \mathcal{O}_Z \to 0$$

If $Z \neq \emptyset$, then $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero as follows from the long exact sequence of cohomology. By Lemma 4.2 this gives a nonzero and hence injective map

$$\mathcal{I}\mathcal{L} \longrightarrow \omega_X$$

In particular, we find an injective map $H^0(X, \mathcal{L}) = H^0(X, \mathcal{I}\mathcal{L}) \to H^0(X, \omega_X)$. This is impossible as

$$\dim_k H^0(X, \mathcal{L}) = \dim_k H^1(X, \mathcal{L}) + \deg(\mathcal{L}) + 1 - g \geq g + 1$$

and $\dim H^0(X, \omega_X) = g$ by (8.1.1). $\square$

Lemma 21.4. *In Situation 6.2 assume* X *is integral and has genus* g. *Let* $\mathcal{L}$ *be an invertible* $\mathcal{O}_X$*-module. Let* $Z \subset X$ *be a nonempty* 0*-dimensional closed subscheme. If* $\deg(\mathcal{L}) \geq 2g - 1 + \deg(Z)$, *then* $\mathcal{L}$ *is globally generated and* $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_Z)$ *is surjective.*

Proof. Global generation by Lemma 21.3. If $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of Z, then $H^1(X, \mathcal{I}\mathcal{L}) = 0$ by Lemma 21.1. Hence surjectivity. $\square$

Lemma 21.5. *In Situation 6.2, assume* X *is geometrically integral over* k *and has genus* g. *Let* $\mathcal{L}$ *be an invertible* $\mathcal{O}_X$*-module. If* $\deg(\mathcal{L}) \geq 2g + 1$, *then* $\mathcal{L}$ *is very ample.*

Proof. By Lemma 21.3, $\mathcal{L}$ is globally generated, and so it determines a morphism $f : X \to \mathbf{P}^n_k$ where $n = h^0(X, \mathcal{L}) - 1$. To show that $\mathcal{L}$ is very ample means to show that f is a closed immersion. It suffices to check that the base change of f to an algebraic closure $\overline{k}$ of k is a closed immersion (Descent, Lemma 23.19). So we may assume that k is algebraically closed; X remains integral, by assumption. Lemma 21.4 gives that for every 0-dimensional closed subscheme $Z \subset X$ of degree 2, the restriction map $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}|_Z)$ is surjective. By Varieties, Lemma 23.2, $\mathcal{L}$ is very ample. $\square$

Lemma 21.6. *Let* k *be a field. Let* X *be a proper scheme over* k *which is reduced, connected, and of dimension* 1. *Let* $\mathcal{L}$ *be an invertible* $\mathcal{O}_X$*-module. Let* $Z \subset X$ *be a*

0-dimensional closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. If $H^1(X, \mathcal{I}\mathcal{L}) \neq 0$, then there exists a reduced connected closed subscheme $Y \subset X$ of dimension 1 such that

$$\deg(\mathcal{L}|_Y) \leq -2\chi(Y, \mathcal{O}_Y) + \deg(Z \cap Y)$$

where $Z \cap Y$ is the scheme theoretic intersection.

Proof. If $H^1(X, \mathcal{I}\mathcal{L})$ is nonzero, then there is a nonzero map $\varphi : \mathcal{I}\mathcal{L} \to \omega_X$, see Lemma 4.2. Let $Y \subset X$ be the union of the irreducible components C of X such that φ is nonzero in the generic point of C. Then Y is a reduced closed subscheme. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Y. Since $\mathcal{J}\mathcal{I}\mathcal{L}$ has no embedded associated points (as a submodule of $\mathcal{L}$) and as φ is zero in the generic points of the support of $\mathcal{J}$ (by choice of Y and as X is reduced), we find that φ factors as

$$\mathcal{I}\mathcal{L} \to \mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \omega_X$$

We can view $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L}$ as the pushforward of a coherent sheaf on Y which by abuse of notation we indicate with the same symbol. Since $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$ by Lemma 4.5 we find a map

$$\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \omega_Y$$

of $\mathcal{O}_Y$-modules which is injective in the generic points of Y. Let $\mathcal{I}' \subset \mathcal{O}_Y$ be the ideal sheaf of $Z \cap Y$. There is a map $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \mathcal{I}'\mathcal{L}|_Y$ whose kernel is supported in closed points. Since ω_Y is a Cohen-Macaulay module, the map above factors through an injective map $\mathcal{I}'\mathcal{L}|_Y \to \omega_Y$. We see that we get an exact sequence

$$0 \to \mathcal{I}'\mathcal{L}|_Y \to \omega_Y \to \mathcal{Q} \to 0$$

of coherent sheaves on Y where $\mathcal{Q}$ is supported in dimension 0 (this uses that ω_Y is an invertible module in the generic points of Y). We conclude that

$$0 \leq \dim \Gamma(Y, \mathcal{Q}) = \chi(\mathcal{Q}) = \chi(\omega_Y) - \chi(\mathcal{I}'\mathcal{L}) = -2\chi(\mathcal{O}_Y) - \deg(\mathcal{L}|_Y) + \deg(Z \cap Y)$$

by Lemma 5.1 and Varieties, Lemma 33.3. If Y is connected, then this proves the lemma. If not, then we repeat the last part of the argument for one of the connected components of Y. $\square$

Lemma 21.7. *Let k be a field. Let X be a proper scheme over k which is reduced, connected, and of dimension 1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that for every reduced connected closed subscheme $Y \subset X$ of dimension 1 we have*

$$\deg(\mathcal{L}|_Y) \geq 2 \dim_k H^1(Y, \mathcal{O}_Y)$$

Then $\mathcal{L}$ is globally generated.

Proof. By induction on the number of irreducible components of X. If X is irreducible, then the lemma holds by Lemma 21.3 applied to X viewed as a scheme over the field $k' = H^0(X, \mathcal{O}_X)$. Assume X is not irreducible. Before we continue, if k is finite, then we replace k by a purely transcendental extension K. This is allowed by Varieties, Lemmas 22.1, 44.2, 6.7, and 8.4, Cohomology of Schemes, Lemma 5.2, Lemma 4.4 and the elementary fact that K is geometrically integral over k.

Assume that $\mathcal{L}$ is not globally generated to get a contradiction. Then we may choose a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $H^0(X, \mathcal{I}\mathcal{L}) = H^0(X, \mathcal{L})$ and such that $\mathcal{O}_X/\mathcal{I}$ is nonzero with support of dimension 0. For example, take $\mathcal{I}$ the ideal sheaf of any closed point in the common vanishing locus of the global sections of $\mathcal{L}$. We consider the short exact sequence

$$0 \to \mathcal{I}\mathcal{L} \to \mathcal{L} \to \mathcal{L}/\mathcal{I}\mathcal{L} \to 0$$

Since the support of $\mathcal{L}/\mathcal{I}\mathcal{L}$ has dimension 0 we see that $\mathcal{L}/\mathcal{I}\mathcal{L}$ is generated by global sections (Varieties, Lemma 33.3). From the short exact sequence, and the fact that $H^0(X, \mathcal{I}\mathcal{L}) = H^0(X, \mathcal{L})$ we get an injection $H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^1(X, \mathcal{I}\mathcal{L})$.

Recall that the k-vector space $H^1(X, \mathcal{I}\mathcal{L})$ is dual to $\mathrm{Hom}(\mathcal{I}\mathcal{L}, \omega_X)$. Choose $\varphi : \mathcal{I}\mathcal{L} \to \omega_X$. By Lemma 21.6 we have $H^1(X, \mathcal{L}) = 0$. Hence

$$\dim_k H^0(X, \mathcal{I}\mathcal{L}) = \dim_k H^0(X, \mathcal{L}) = \deg(\mathcal{L}) + \chi(\mathcal{O}_X) > \dim_k H^1(X, \mathcal{O}_X) = \dim_k H^0(X, \omega_X)$$

We conclude that φ is not injective on global sections, in particular φ is not injective. For every generic point $\eta \in X$ of an irreducible component of X denote $V_\eta \subset \mathrm{Hom}(\mathcal{I}\mathcal{L}, \omega_X)$ the k-subvector space consisting of those φ which are zero at η. Since every associated point of $\mathcal{I}\mathcal{L}$ is a generic point of X, the above shows that $\mathrm{Hom}(\mathcal{I}\mathcal{L}, \omega_X) = \bigcup V_\eta$. As X has finitely many generic points and k is infinite, we conclude $\mathrm{Hom}(\mathcal{I}\mathcal{L}, \omega_X) = V_\eta$ for some η. Let $\eta \in C \subset X$ be the corresponding irreducible component. Let $Y \subset X$ be the union of the other irreducible components of X. Then Y is a nonempty reduced closed subscheme not equal to X. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Y. Please keep in mind that the support of $\mathcal{J}$ is C.

Let $\varphi : \mathcal{I}\mathcal{L} \to \omega_X$ be arbitrary. Since $\mathcal{J}\mathcal{I}\mathcal{L}$ has no embedded associated points (as a submodule of $\mathcal{L}$) and as φ is zero in the generic point η of the support of $\mathcal{J}$, we find that φ factors as

$$\mathcal{I}\mathcal{L} \to \mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \omega_X$$

We can view $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L}$ as the pushforward of a coherent sheaf on Y which by abuse of notation we indicate with the same symbol. Since $\omega_Y = \mathcal{H}om(\mathcal{O}_Y, \omega_X)$ by Lemma 4.5 we find a factorization

$$\mathcal{I}\mathcal{L} \to \mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \xrightarrow{\varphi'} \omega_Y \to \omega_X$$

of φ. Let $\mathcal{I}' \subset \mathcal{O}_Y$ be the image of $\mathcal{I} \subset \mathcal{O}_X$. There is a surjective map $\mathcal{I}\mathcal{L}/\mathcal{J}\mathcal{I}\mathcal{L} \to \mathcal{I}'\mathcal{L}|_Y$ whose kernel is supported in closed points. Since ω_Y is a Cohen-Macaulay module on Y, the map φ' factors through a map $\varphi'' : \mathcal{I}'\mathcal{L}|_Y \to \omega_Y$. Thus we have commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}\mathcal{L} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{L}/\mathcal{I}\mathcal{L} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{I}'\mathcal{L}|_Y \arrow[r] & \mathcal{L}|_Y \arrow[r] & \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y \arrow[r] & 0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{I}\mathcal{L} \arrow[r, "\varphi"'] \arrow[d] & \omega_X \\ \mathcal{I}'\mathcal{L}|_Y \arrow[r, "\varphi''"] & \omega_Y \arrow[u] \end{tikzcd}$$

Now we can finish the proof as follows: Since for every φ we have a φ'' and since $\omega_X \in \mathit{Coh}(\mathcal{O}_X)$ represents the functor $\mathcal{F} \mapsto \mathrm{Hom}_k(H^1(X, \mathcal{F}), k)$, we find that $H^1(X, \mathcal{I}\mathcal{L}) \to H^1(Y, \mathcal{I}'\mathcal{L}|_Y)$ is injective. Since the boundary $H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^1(X, \mathcal{I}\mathcal{L})$ is injective, we conclude the composition

$$H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^0(X, \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y) \to H^1(X, \mathcal{I}'\mathcal{L}|_Y)$$

is injective. Since $\mathcal{L}/\mathcal{I}\mathcal{L} \to \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y$ is a surjective map of coherent modules whose supports have dimension 0, we see that the first map $H^0(X, \mathcal{L}/\mathcal{I}\mathcal{L}) \to H^0(X, \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y)$ is surjective (and hence bijective). But by induction we have that $\mathcal{L}|_Y$ is globally generated (if Y is disconnected this still works of course) and hence the boundary map

$$H^0(X, \mathcal{L}|_Y/\mathcal{I}'\mathcal{L}|_Y) \to H^1(X, \mathcal{I}'\mathcal{L}|_Y)$$

cannot be injective. This contradiction finishes the proof. $\square$

22. Contracting rational tails

In this section we discuss the simplest possible case of contracting a scheme to improve positivity properties of its canonical sheaf.

Example 22.1 (Contracting a rational tail)**.** Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. A *rational tail* will be an irreducible component $C \subset X$ (viewed as an integral closed subscheme) with the following properties

(1) $X' \neq \emptyset$ where $X' \subset X$ is the scheme theoretic closure of $X \setminus C$,
(2) the scheme theoretic intersection $C \cap X'$ is a single reduced point x,
(3) $H^0(C, \mathcal{O}_C)$ maps isomorphically to the residue field of x, and
(4) C has genus zero.

Since there are at least two irreducible components of X passing through x, we conclude that x is a node. Set $k' = H^0(C, \mathcal{O}_C) = \kappa(x)$. Then k'/k is a finite separable extension of fields (Lemma 19.7). There is a canonical morphism

$$c : X \longrightarrow X'$$

inducing the identity on X' and mapping C to $x \in X'$ via the canonical morphism $C \to \mathrm{Spec}(k') = x$. This follows from Morphisms, Lemma 4.6 since X is the scheme theoretic union of C and X' (as X is reduced). Moreover, we claim that

$$c_*\mathcal{O}_X = \mathcal{O}_{X'} \quad \text{and} \quad R^1c_*\mathcal{O}_X = 0$$

To see this, denote $i_C : C \to X$, $i_{X'} : X' \to X$ and $i_x : x \to X$ the embeddings and use the exact sequence

$$0 \to \mathcal{O}_X \to i_{C,*}\mathcal{O}_C \oplus i_{X',*}\mathcal{O}_{X'} \to i_{x,*}\kappa(x) \to 0$$

of Morphisms, Lemma 4.6. Looking at the long exact sequence of higher direct images, it follows that it suffices to show $H^0(C, \mathcal{O}_C) = k'$ and $H^1(C, \mathcal{O}_C) = 0$ which follows from the assumptions. Observe that X' is also a proper scheme over k, of dimension 1 whose singularities are at-worst-nodal (Lemma 19.17) has $H^0(X', \mathcal{O}_{X'}) = k$, and X' has the same genus as X. We will say $c : X \to X'$ is the *contraction of a rational tail.*

Lemma 22.2. *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational tail (Example 22.1). Then* $\deg(\omega_X|_C) < 0$.

Proof. Let $X' \subset X$ be as in the example. Then we have a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C \cap X'} \to 0$$

See Lemmas 4.6, 19.16, and 19.17. With k' as in the example we see that $\deg(\omega_C) = -2[k' : k]$ as $C \cong \mathbf{P}^1_{k'}$ by Proposition 10.4 and $\deg(C \cap X') = [k' : k]$. Hence $\deg(\omega_X|_C) = -[k' : k]$ which is negative. $\square$

Lemma 22.3. *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational tail (Example 22.1). For any field extension K/k the base change $C_K \subset X_K$ is a finite disjoint union of rational tails.*

Proof. Let $x \in C$ and $k' = \kappa(x)$ be as in the example. Observe that $C \cong \mathbf{P}^1_{k'}$ by Proposition 10.4. Since k'/k is finite separable, we see that $k' \otimes_k K = K'_1 \times \ldots \times K'_n$ is a finite product of finite separable extensions K'_i/K. Set $C_i = \mathbf{P}^1_{K'_i}$ and denote $x_i \in C_i$ the inverse image of x. Then $C_K = \coprod C_i$ and $X'_K \cap C_i = x_i$ as desired. $\square$

Lemma 22.4. *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. If X does not have a rational tail (Example 22.1), then for every reduced connected closed subscheme $Y \subset X$, $Y \neq X$ of dimension 1 we have $\deg(\omega_X|_Y) \geq \dim_k H^1(Y, \mathcal{O}_Y)$.*

Proof. Let $Y \subset X$ be as in the statement. Then $k' = H^0(Y, \mathcal{O}_Y)$ is a field and a finite extension of k and $[k' : k]$ divides all numerical invariants below associated to Y and coherent sheaves on Y, see Varieties, Lemma 44.10. Let $Z \subset X$ be as in Lemma 4.6. We will use the results of this lemma and of Lemmas 19.16 and 19.17 without further mention. Then we get a short exact sequence

$$0 \to \omega_Y \to \omega_X|_Y \to \mathcal{O}_{Y \cap Z} \to 0$$

See Lemma 4.6. We conclude that

$$\deg(\omega_X|_Y) = \deg(Y \cap Z) + \deg(\omega_Y) = \deg(Y \cap Z) - 2\chi(Y, \mathcal{O}_Y)$$

Hence, if the lemma is false, then

$$2[k' : k] > \deg(Y \cap Z) + \dim_k H^1(Y, \mathcal{O}_Y)$$

Since $Y \cap Z$ is nonempty and by the divisiblity mentioned above, this can happen only if $Y \cap Z$ is a single k'-rational point of the smooth locus of Y and $H^1(Y, \mathcal{O}_Y) = 0$. If Y is irreducible, then this implies Y is a rational tail. If Y is reducible, then since $\deg(\omega_X|_Y) = -[k' : k]$ we find there is some irreducible component C of Y such that $\deg(\omega_X|_C) < 0$, see Varieties, Lemma 44.6. Then the analysis above applied to C gives that C is a rational tail. $\square$

Lemma 22.5. *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Assume X does not have a rational tail (Example 22.1). If*

1. *the genus of X is 0, then X is isomorphic to an irreducible plane conic and $\omega_X^{\otimes -1}$ is very ample,*
2. *the genus of X is 1, then $\omega_X \cong \mathcal{O}_X$,*
3. *the genus of X is ≥ 2, then $\omega_X^{\otimes m}$ is globally generated for $m \geq 2$.*

Proof. By Lemma 19.16 we find that X is Gorenstein, i.e., ω_X is an invertible $\mathcal{O}_X$-module.

If the genus of X is zero, then $\deg(\omega_X) < 0$, hence if X has more than one irreducible component, we get a contradiction with Lemma 22.4. In the irreducible case we see that X is isomorphic to an irreducible plane conic and $\omega_X^{\otimes -1}$ is very ample by Lemma 10.3.

If the genus of X is 1, then ω_X has a global section and $\deg(\omega_X|_C) = 0$ for all irreducible components. Namely, $\deg(\omega_X|_C) \geq 0$ for all irreducible components C by Lemma 22.4, the sum of these numbers is 0 by Lemma 8.3, and we can apply Varieties, Lemma 44.6. Then $\omega_X \cong \mathcal{O}_X$ by Varieties, Lemma 44.13.

Assume the genus g of X is greater than or equal to 2. If X is irreducible, then we are done by Lemma 21.3. Assume X reducible. By Lemma 22.4 the inequalities of Lemma 21.7 hold for every $Y \subset X$ as in the statement, except for $Y = X$. Analyzing the proof of Lemma 21.7 we see that (in the reducible case) the only inequality used for $Y = X$ are

$$\deg(\omega_X^{\otimes m}) > -2\chi(\mathcal{O}_X) \quad \text{and} \quad \deg(\omega_X^{\otimes m}) + \chi(\mathcal{O}_X) > \dim_k H^1(X, \mathcal{O}_X)$$

Since these both hold under the assumption $g \geq 2$ and $m \geq 2$ we win. □

Lemma 22.6. *Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Consider a sequence*

$$X = X_0 \to X_1 \to \ldots \to X_n = X'$$

of contractions of rational tails (Example 22.1) until none are left. Then

1. *if the genus of X is 0, then X' is an irreducible plane conic,*
2. *if the genus of X is 1, then $\omega_{X'} \cong \mathcal{O}_X$,*
3. *if the genus of X is > 1, then $\omega_{X'}^{\otimes m}$ is globally generated for $m \geq 2$.*

If the genus of X is ≥ 1, then the morphism $X \to X'$ is independent of choices and formation of this morphism commutes with base field extensions.

Proof. We proceed by contracting rational tails until there are none left. Then we see that (1), (2), (3) hold by Lemma 22.5.

Uniqueness. To see that $f : X \to X'$ is independent of the choices made, it suffices to show: any rational tail $C \subset X$ is mapped to a point by $X \to X'$; some details omitted. If not, then we can find a section $s \in \Gamma(X', \omega_{X'}^{\otimes 2})$ which does not vanish in the generic point of the irreducible component $f(C)$. Since in each of the contractions $X_i \to X_{i+1}$ we have a section $X_{i+1} \to X_i$, there is a section $X' \to X$ of f. Then we have an exact sequence

$$0 \to \omega_{X'} \to \omega_X \to \omega_X|_{X''} \to 0$$

where $X'' \subset X$ is the union of the irreducible components contracted by f. See Lemma 4.6. Thus we get a map $\omega_{X'}^{\otimes 2} \to \omega_X^{\otimes 2}$ and we can take the image of s to get a section of $\omega_X^{\otimes 2}$ not vanishing in the generic point of C. This is a contradiction with the fact that the restriction of ω_X to a rational tail has negative degree (Lemma 22.2).

The statement on base field extensions follows from Lemma 22.3. Some details omitted. □

23. Contracting rational bridges

In this section we discuss the next simplest possible case (after the case discussed in Section 22) of contracting a scheme to improve positivity properties of its canonical sheaf.

Example 23.1 (Contracting a rational bridge)**.** Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. A *rational bridge* will be an irreducible component $C \subset X$ (viewed as an integral closed subscheme) with the following properties

1. $X' \neq \emptyset$ where $X' \subset X$ is the scheme theoretic closure of $X \setminus C$,
2. the scheme theoretic interesection $C \cap X'$ has degree 2 over $H^0(C, \mathcal{O}_C)$, and
3. C has genus zero.

Set $k' = H^0(C, \mathcal{O}_C)$ and $k'' = H^0(C \cap X', \mathcal{O}_{C \cap X'})$. Then k' is a field (Varieties, Lemma 9.3) and $\dim_{k'}(k'') = 2$. Since there are at least two irreducible components of X passing through each point of $C \cap X'$, we conclude these points are nodes of X and smooth points on both C and X' (Lemma 19.17). Hence k'/k is a finite separable extension of fields and k''/k' is either a degree 2 separable extension of fields or $k'' = k' \times k'$ (Lemma 19.7). By Section 14 there exists a pushout

$$\begin{tikzcd} C \cap X' \arrow[r] \arrow[d] & X' \arrow[d, "a"] \\ \mathrm{Spec}(k') \arrow[r] & Y \end{tikzcd}$$

with many good properties (all of which we will use below without futher mention). Let $y \in Y$ be the image of $\mathrm{Spec}(k') \to Y$. Then

$$\mathcal{O}_{Y,y}^\wedge \cong k'[[s,t]]/(st) \quad \text{or} \quad \mathcal{O}_{Y,y}^\wedge \cong \{f \in k''[[s]] : f(0) \in k'\}$$

depending on whether $C \cap X'$ has 2 or 1 points. This follows from Lemma 14.1 and the fact that $\mathcal{O}_{X',p} \cong \kappa(p)[[t]]$ for $p \in C \cap X'$ by More on Algebra, Lemma 38.4. Thus we see that $y \in Y$ is a node, see Lemmas 19.7 and 19.4 and in particular the discussion of Case II in the proof of (2) ⇒ (1) in Lemma 19.4. Thus the singularities of Y are at-worst-nodal.

We can extend the commutative diagram above to a diagram

$$\begin{tikzcd} C \cap X' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "a"] & X \arrow[dl, "c"] & C \arrow[l] \arrow[dl] \\ \mathrm{Spec}(k') \arrow[r] & Y & \mathrm{Spec}(k') \arrow[l] \end{tikzcd}$$

where the two lower horizontal arrows are the same. Namely, X is the scheme theoretic union of X' and C (thus a pushout by Morphisms, Lemma 4.6) and the morphisms $C \to Y$ and $X' \to Y$ agree on $C \cap X'$. Finally, we claim that

$$c_*\mathcal{O}_X = \mathcal{O}_Y \quad \text{and} \quad R^1c_*\mathcal{O}_X = 0$$

To see this use the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_C \oplus \mathcal{O}_{X'} \to \mathcal{O}_{C \cap X'} \to 0$$

of Morphisms, Lemma 4.6. The long exact sequence of higher direct images is

$$0 \to c_*\mathcal{O}_X \to c_*\mathcal{O}_C \oplus c_*\mathcal{O}_{X'} \to c_*\mathcal{O}_{C \cap X'} \to R^1c_*\mathcal{O}_X \to R^1c_*\mathcal{O}_C \oplus R^1c_*\mathcal{O}_{X'}$$

Since $c|_{X'} = a$ is affine we see that $R^1c_*\mathcal{O}_{X'} = 0$. Since $c|_C$ factors as $C \to \mathrm{Spec}(k') \to X$ and since C has genus zero, we find that $R^1c_*\mathcal{O}_C = 0$. Since $\mathcal{O}_{X'} \to \mathcal{O}_{C \cap X'}$ is surjective and since $c|_{X'}$ is affine, we see that $c_*\mathcal{O}_{X'} \to c_*\mathcal{O}_{C \cap X'}$ is surjective. This proves that $R^1c_*\mathcal{O}_X = 0$. Finally, we have $\mathcal{O}_Y = c_*\mathcal{O}_X$ by the exact sequence and the description of the structure sheaf of the pushout in More on Morphisms, Proposition 67.3.

All of this means that Y is also a proper scheme over k having dimension 1 and $H^0(Y, \mathcal{O}_Y) = k$ whose singularities are at-worst-nodal (Lemma 19.17) and that Y has the same genus as X. We will say $c : X \to Y$ is the *contraction of a rational bridge*.

Lemma 23.2. *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational bridge (Example 23.1). Then $\deg(\omega_X|_C) = 0$.*

Proof. Let $X' \subset X$ be as in the example. Then we have a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C \cap X'} \to 0$$

See Lemmas 4.6, 19.16, and 19.17. With $k''/k'/k$ as in the example we see that $\deg(\omega_C) = -2[k' : k]$ as C has genus 0 (Lemma 5.2) and $\deg(C \cap X') = [k'' : k] = 2[k' : k]$. Hence $\deg(\omega_X|_C) = 0$. ☐

Lemma 23.3. *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume the singularities of X are at-worst-nodal. Let $C \subset X$ be a rational bridge (Example 23.1). For any field extension K/k the base change $C_K \subset X_K$ is a finite disjoint union of rational bridges.*

Proof. Let $k''/k'/k$ be as in the example. Since k'/k is finite separable, we see that $k' \otimes_k K = K'_1 \times \ldots \times K'_n$ is a finite product of finite separable extensions K'_i/K. The corresponding product decomposition $k'' \otimes_k K = \prod K''_i$ gives degree 2 separable algebra extensions K''_i/K'_i. Set $C_i = C_{K'_i}$. Then $C_K = \coprod C_i$ and therefore each C_i has genus 0 (viewed as a curve over K'_i), because $H^1(C_K, \mathcal{O}_{C_K}) = 0$ by flat base change. Finally, we have $X'_K \cap C_i = \mathrm{Spec}(K''_i)$ has degree 2 over K'_i as desired. ☐

Lemma 23.4. *Let $c : X \to Y$ be the contraction of a rational bridge (Example 23.1). Then $c^*\omega_Y \cong \omega_X$.*

Proof. You can prove this by direct computation, but we prefer to use the characterization of ω_X as the coherent $\mathcal{O}_X$-module which represents the functor $\textit{Coh}(\mathcal{O}_X) \to \textit{Sets}$, $\mathcal{F} \mapsto \mathrm{Hom}_k(H^1(X, \mathcal{F}), k) = H^1(X, \mathcal{F})^\vee$, see Lemma 4.2 or Duality for Schemes, Lemma 22.5.

To be precise, denote $\mathcal{C}_Y$ the category whose objects are invertible $\mathcal{O}_Y$-modules and whose maps are $\mathcal{O}_Y$-module homomorphisms. Denote $\mathcal{C}_X$ the category whose objects are invertible $\mathcal{O}_X$-modules $\mathcal{L}$ with $\mathcal{L}|_C \cong \mathcal{O}_C$ and whose maps are $\mathcal{O}_Y$-module homomorphisms. We claim that the functor

$$c^* : \mathcal{C}_Y \to \mathcal{C}_X$$

is an equivalence of categories. Namely, by More on Morphisms, Lemma 72.8 it is essentially surjective. Then the projection formula (Cohomology, Lemma 54.2) shows $c_*c^*\mathcal{N} = \mathcal{N}$ and hence c^* is an equivalence with quasi-inverse given by c_*.

We claim ω_X is an object of $\mathcal{C}_X$. Namely, we have a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C \cap X'} \to 0$$

See Lemma 4.6. Taking degrees we find $\deg(\omega_X|_C) = 0$ (small detail omitted). Thus $\omega_X|_C$ is trivial by Lemma 10.1 and ω_X is an object of $\mathcal{C}_X$.

Since $R^1c_*\mathcal{O}_X = 0$ the projection formula shows that $R^1c_*c^*\mathcal{N} = 0$ for $\mathcal{N} \in \mathrm{Ob}(\mathcal{C}_Y)$. Therefore the Leray spectral sequence (Cohomology, Lemma 13.6) the diagram

$$\begin{tikzcd} \mathcal{C}_Y \arrow[rr, "c^*"] \arrow[dr, "H^1(Y,-)^\vee"'] & & \mathcal{C}_X \arrow[dl, "H^1(X,-)^\vee"] \\ & \textit{Sets} & \end{tikzcd}$$

of categories and functors is commutative. Since $\omega_Y \in \text{Ob}(\mathcal{C}_Y)$ represents the south-east arrow and $\omega_X \in \text{Ob}(\mathcal{C}_X)$ represents the south-east arrow we conclude by the Yoneda lemma (Categories, Lemma 3.5). □

Lemma 23.5. *Let k be a field. Let X be a proper scheme over k having dimension 1 and $H^0(X, \mathcal{O}_X) = k$. Assume*

1. *the singularities of X are at-worst-nodal,*
2. *X does not have a rational tail (Example 22.1),*
3. *X does not have a rational bridge (Example 23.1),*
4. *the genus g of X is ≥ 2.*

Then ω_X is ample.

Proof. It suffices to show that $\deg(\omega_X|_C) > 0$ for every irreducible component C of X, see Varieties, Lemma 44.15. If $X = C$ is irreducible, this follows from $g \geq 2$ and Lemma 8.3. Otherwise, set $k' = H^0(C, \mathcal{O}_C)$. This is a field and a finite extension of k and $[k' : k]$ divides all numerical invariants below associated to C and coherent sheaves on C, see Varieties, Lemma 44.10. Let $X' \subset X$ be the closure of $X \setminus C$ as in Lemma 4.6. We will use the results of this lemma and of Lemmas 19.16 and 19.17 without further mention. Then we get a short exact sequence

$$0 \to \omega_C \to \omega_X|_C \to \mathcal{O}_{C \cap X'} \to 0$$

See Lemma 4.6. We conclude that

$$\deg(\omega_X|_C) = \deg(C \cap X') + \deg(\omega_C) = \deg(C \cap X') - 2\chi(C, \mathcal{O}_C)$$

Hence, if the lemma is false, then

$$2[k' : k] \geq \deg(C \cap X') + 2\dim_k H^1(C, \mathcal{O}_C)$$

Since $C \cap X'$ is nonempty and by the divisiblity mentioned above, this can happen only if either

(a) $C \cap X'$ is a single k'-rational point of C and $H^1(C, \mathcal{O}_C) = 0$, and
(b) $C \cap X'$ has degree 2 over k' and $H^1(C, \mathcal{O}_C) = 0$.

The first possibility means C is a rational tail and the second that C is a rational bridge. Since both are excluded the proof is complete. □

Lemma 23.6. *Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal and that X has no rational tails. Consider a sequence*

$$X = X_0 \to X_1 \to \ldots \to X_n = X'$$

of contractions of rational bridges (Example 23.1) until none are left. Then $\omega_{X'}$ ample. The morphism $X \to X'$ is independent of choices and formation of this morphism commutes with base field extensions.

Proof. We proceed by contracting rational bridges until there are none left. Then $\omega_{X'}$ is ample by Lemma 23.5.

Denote $f : X \to X'$ the composition. By Lemma 23.4 and induction we see that $f^*\omega_{X'} = \omega_X$. We have $f_*\mathcal{O}_X = \mathcal{O}_{X'}$ because this is true for contraction of a rational bridge. Thus the projection formula says that $f_*f^*\mathcal{L} = \mathcal{L}$ for all invertible $\mathcal{O}_{X'}$-modules $\mathcal{L}$. Hence

$$\Gamma(X', \omega_{X'}^{\otimes m}) = \Gamma(X, \omega_X^{\otimes m})$$

for all m. Since X' is the Proj of the direct sum of these by Morphisms, Lemma 43.17 we conclude that the morphism $X \to X'$ is completely canonical.

Let K/k be an extension of fields, then ω_{X_K} is the pullback of ω_X (Lemma 4.4) and we have $\Gamma(X, \omega_X^{\otimes m}) \otimes_k K$ is equal to $\Gamma(X_K, \omega_{X_K}^{\otimes m})$ by Cohomology of Schemes, Lemma 5.2. Thus formation of $f : X \to X'$ commutes with base change by K/k by the arguments given above. Some details omitted. □

24. Contracting to a stable curve

In this section we combine the contraction morphisms found in Sections 22 and 23. Namely, suppose that k is a field and let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal. Composing the morphism of Lemma 22.6 with the morphism of Lemma 23.6 we get a morphism

$$c : X \longrightarrow Y$$

such that Y also is a proper scheme over k of dimension 1 whose singularities are at worst nodal, with $k = H^0(Y, \mathcal{O}_Y)$ and having genus g, such that $\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$, and such that ω_Y is ample on Y. Lemma 24.2 shows these conditions in fact characterize this morphism.

Lemma 24.1. *Let k be a field. Let $c : X \to Y$ be a morphism of proper schemes over k Assume*

(1) $\mathcal{O}_Y = c_*\mathcal{O}_X$ *and* $R^1c_*\mathcal{O}_X = 0$,
(2) X *and* Y *are reduced, Gorenstein, and have dimension* 1,
(3) $\exists\ m \in \mathbf{Z}$ *with* $H^1(X, \omega_X^{\otimes m}) = 0$ *and* $\omega_X^{\otimes m}$ *generated by global sections.*

Then $c^*\omega_Y \cong \omega_X$.

Proof. The fibres of c are geometrically connected by More on Morphisms, Theorem 53.4. In particular c is surjective. There are finitely many closed points $y = y_1, \ldots, y_r$ of Y where X_y has dimension 1 and over $Y \setminus \{y_1, \ldots, y_r\}$ the morphism c is an isomorphism. Some details omitted; hint: outside of $\{y_1, \ldots, y_r\}$ the morphism c is finite, see Cohomology of Schemes, Lemma 21.1.

Let us carefully construct a map $b : c^*\omega_Y \to \omega_X$. Denote $f : X \to \mathrm{Spec}(k)$ and $g : Y \to \mathrm{Spec}(k)$ the structure morphisms. We have $f^!k = \omega_X[1]$ and $g^!k = \omega_Y[1]$, see Lemma 4.1 and its proof. Then $f^! = c^! \circ g^!$ and hence $c^!\omega_Y = \omega_X$. Thus there is a functorial isomorphism

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{F}, \omega_X) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rc_*\mathcal{F}, \omega_Y)$$

for coherent $\mathcal{O}_X$-modules $\mathcal{F}$ by definition of $c^!$[7]. This isomorphism is induced by a trace map $t : Rc_*\omega_X \to \omega_Y$ (the counit of the adjunction). By the projection formula (Cohomology, Lemma 54.2) the canonical map $a : \omega_Y \to Rc_*c^*\omega_Y$ is an isomorphism. Combining the above we see there is a canonical map $b : c^*\omega_Y \to \omega_X$ such that

$$t \circ Rc_*(b) = a^{-1}$$

In particular, if we restrict b to $c^{-1}(Y \setminus \{y_1, \ldots, y_r\})$ then it is an isomorphism (because it is a map between invertible modules whose composition with another gives the isomorphism a^{-1}).

[7]As the restriction of the right adjoint of Duality for Schemes, Lemma 3.1 to $D^+_{QCoh}(\mathcal{O}_Y)$.

Choose $m \in \mathbf{Z}$ as in (3) consider the map

$$b^{\otimes m} : \Gamma(Y, \omega_Y^{\otimes m}) \longrightarrow \Gamma(X, \omega_X^{\otimes m})$$

This map is injective because Y is reduced and by the last property of b mentioned in its construction. By Riemann-Roch (Lemma 5.2) we have $\chi(X, \omega_X^{\otimes m}) = \chi(Y, \omega_Y^{\otimes m})$. Thus

$$\dim_k \Gamma(Y, \omega_Y^{\otimes m}) \geq \dim_k \Gamma(X, \omega_X^{\otimes m}) = \chi(X, \omega_X^{\otimes m})$$

and we conclude $b^{\otimes m}$ induces an isomorphism on global sections. So $b^{\otimes m} : c^*\omega_Y^{\otimes m} \to \omega_X^{\otimes m}$ is surjective as generators of $\omega_X^{\otimes m}$ are in the image. Hence $b^{\otimes m}$ is an isomorphism. Thus b is an isomorphism. □

Lemma 24.2. *Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal. There is a unique morphism (up to unique isomorphism)*

$$c : X \longrightarrow Y$$

of schemes over k having the following properties:

1. *Y is proper over k, $\dim(Y) = 1$, the singularities of Y are at-worst-nodal,*
2. *$\mathcal{O}_Y = c_*\mathcal{O}_X$ and $R^1c_*\mathcal{O}_X = 0$, and*
3. *ω_Y is ample on Y.*

Proof. Existence: A morphism with all the properties listed exists by combining Lemmas 22.6 and 23.6 as discussed in the introduction to this section. Moreover, we see that it can be written as a composition

$$X \to X_1 \to X_2 \ldots \to X_n \to X_{n+1} \to \ldots \to X_{n+n'}$$

where the first n morphisms are contractions of rational tails and the last n' morphisms are contractions of rational bridges. Note that property (2) holds for each contraction of a rational tail (Example 22.1) and contraction of a rational bridge (Example 23.1). It is easy to see that this property is inherited by compositions of morphisms.

Uniqueness: Let $c : X \to Y$ be a morphism satisfying conditions (1), (2), and (3). We will show that there is a unique isomorphism $X_{n+n'} \to Y$ compatible with the morphisms $X \to X_{n+n'}$ and c.

Before we start the proof we make some observations about c. We first observe that the fibres of c are geometrically connected by More on Morphisms, Theorem 53.4. In particular c is surjective. For a closed point $y \in Y$ the fibre X_y satisfies

$$H^1(X_y, \mathcal{O}_{X_y}) = 0 \quad \text{and} \quad H^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$$

The first equality by More on Morphisms, Lemma 72.1 and the second by More on Morphisms, Lemma 72.4. Thus either $X_y = x$ where x is the unique point of X mapping to y and has the same residue field as y, or X_y is a 1-dimensional proper scheme over $\kappa(y)$. Observe that in the second case X_y is Cohen-Macaulay (Lemma 6.1). However, since X is reduced, we see that X_y must be reduced at all of its generic points (details omitted), and hence X_y is reduced by Properties, Lemma 12.4. It follows that the singularities of X_y are at-worst-nodal (Lemma 19.17). Note that the genus of X_y is zero (see above). Finally, there are only a finite number of points y where the fibre X_y has dimension 1, say $\{y_1, \ldots, y_r\}$, and $c^{-1}(Y \setminus \{y_1, \ldots, y_r\})$ maps isomorphically to $Y \setminus \{y_1, \ldots, y_r\}$ by c. Some details omitted; hint: outside of $\{y_1, \ldots, y_r\}$ the morphism c is finite, see Cohomology of Schemes, Lemma 21.1.

Let $C \subset X$ be a rational tail. We claim that c maps C to a point. Assume that this is not the case to get a contradiction. Then the image of C is an irreducible component $D \subset Y$. Recall that $H^0(C, \mathcal{O}_C) = k'$ is a finite separable extension of k and that C has a k'-rational point x which is also the unique intersection of C with the "rest" of X. We conclude from the general discussion above that $C \setminus \{x\} \subset c^{-1}(Y \setminus \{y_1, \ldots, y_r\})$ maps isomorphically to an open V of D. Let $y = c(x) \in D$. Observe that y is the only point of D meeting the "rest" of Y. If $y \notin \{y_1, \ldots, y_r\}$, then $C \cong D$ and it is clear that D is a rational tail of Y which is a contradiction with the ampleness of ω_Y (Lemma 22.2). Thus $y \in \{y_1, \ldots, y_r\}$ and $\dim(X_y) = 1$. Then $x \in X_y \cap C$ and x is a smooth point of X_y and C (Lemma 19.17). If $y \in D$ is a singular point of D, then y is a node and then $Y = D$ (because there cannot be another component of Y passing through y by Lemma 19.17). Then $X = X_y \cup C$ which means $g = 0$ because it is equal to the genus of X_y by the discussion in Example 22.1; a contradiction. If $y \in D$ is a smooth point of D, then $C \to D$ is an isomorphism (because the nonsingular projective model is unique and C and D are birational, see Section 2). Then D is a rational tail of Y which is a contradiction with ampleness of ω_Y.

Assume $n \geq 1$. If $C \subset X$ is the rational tail contracted by $X \to X_1$, then we see that C is mapped to a point of Y by the previous paragraph. Hence $c : X \to Y$ factors through $X \to X_1$ (because X is the pushout of C and X_1, see discussion in Example 22.1). After replacing X by X_1 we have decreased n. By induction we may assume $n = 0$, i.e., X does not have a rational tail.

Assume $n = 0$, i.e., X does not have any rational tails. Then $\omega_X^{\otimes 2}$ and $\omega_X^{\otimes 3}$ are globally generated by Lemma 22.5. It follows that $H^1(X, \omega_X^{\otimes 3}) = 0$ by Lemma 6.4. By Lemma 24.1 applied with $m = 3$ we find that $c^*\omega_Y \cong \omega_X$. We also have that $\omega_X = (X \to X_{n'})^*\omega_{X_{n'}}$ by Lemma 23.4 and induction. Applying the projection formula for both c and $X \to X_{n'}$ we conclude that

$$\Gamma(X_{n'}, \omega_{X_{n'}}^{\otimes m}) = \Gamma(X, \omega_X^{\otimes m}) = \Gamma(Y, \omega_Y^{\otimes m})$$

for all m. Since $X_{n'}$ and Y are the Proj of the direct sum of these by Morphisms, Lemma 43.17 we conclude that there is a canonical isomorphism $X_{n'} = Y$ as desired. We omit the verification that this is the unique isomorphism making the diagram commute. $\square$

Lemma 24.3. *Let k be a field. Let X be a proper scheme over k of dimension 1 with $H^0(X, \mathcal{O}_X) = k$ having genus $g \geq 2$. Assume the singularities of X are at-worst-nodal and ω_X is ample. Then $\omega_X^{\otimes 3}$ is very ample and $H^1(X, \omega_X^{\otimes 3}) = 0$.*

Proof. Combining Varieties, Lemma 44.15 and Lemmas 22.2 and 23.2 we see that X contains no rational tails or bridges. Then we see that $\omega_X^{\otimes 3}$ is globally generated by Lemma 22.6. Choose a k-basis $s_0, \ldots, s_n$ of $H^0(X, \omega_X^{\otimes 3})$. We get a morphism

$$\varphi_{\omega_X^{\otimes 3}, (s_0, \ldots, s_n)} : X \longrightarrow \mathbf{P}^n_k$$

See Constructions, Section 13. The lemma asserts that this morphism is a closed immersion. To check this we may replace k by its algebraic closure, see Descent, Lemma 23.19. Thus we may assume k is algebraically closed.

Assume k is algebraically closed. We will use Varieties, Lemma 23.2 to prove the lemma. Let $Z \subset X$ be a closed subscheme of degree 2 over Z with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. We have to show that

$$H^0(X, \mathcal{L}) \to H^0(Z, \mathcal{L}|_Z)$$

is surjective. Thus it suffices to show that $H^1(X, \mathcal{I}\mathcal{L}) = 0$. To do this we will use Lemma 21.6. Thus it suffices to show that

$$3\deg(\omega_X|_Y) > -2\chi(Y, \mathcal{O}_Y) + \deg(Z \cap Y)$$

for every reduced connected closed subscheme $Y \subset X$. Since k is algebraically closed and Y connected and reduced we have $H^0(Y, \mathcal{O}_Y) = k$ (Varieties, Lemma 9.3). Hence $\chi(Y, \mathcal{O}_Y) = 1 - \dim H^1(Y, \mathcal{O}_Y)$. Thus we have to show

$$3\deg(\omega_X|_Y) > -2 + 2\dim H^1(Y, \mathcal{O}_Y) + \deg(Z \cap Y)$$

which is true by Lemma 22.4 except possibly if $Y = X$ or if $\deg(\omega_X|_Y) = 0$. Since ω_X is ample the second possibility does not occur (see first lemma cited in this proof). Finally, if $Y = X$ we can use Riemann-Roch (Lemma 5.2) and the fact that $g \geq 2$ to see that the inquality holds. The same argument with $Z = \emptyset$ shows that $H^1(X, \omega_X^{\otimes 3}) = 0$. □

25. Vector fields

In this section we study the space of vector fields on a curve. Vector fields correspond to infinitesimal automorphisms, see More on Morphisms, Section 9, hence play an important role in moduli theory.

Let k be an algebraically closed field. Let X be a finite type scheme over k. Let $x \in X$ be a closed point. We will say an element $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ *fixes* x if $D(\mathcal{I}) \subset \mathcal{I}$ where $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of x.

Lemma 25.1. *Let k be an algebraically closed field. Let X be a smooth, proper, connected curve over k. Let g be the genus of X.*

1. *If $g \geq 2$, then $Der_k(\mathcal{O}_X, \mathcal{O}_X)$ is zero,*
2. *if $g = 1$ and $D \in Der_k(\mathcal{O}_X, \mathcal{O}_X)$ is nonzero, then D does not fix any closed point of X, and*
3. *if $g = 0$ and $D \in Der_k(\mathcal{O}_X, \mathcal{O}_X)$ is nonzero, then D fixes at most 2 closed points of X.*

Proof. Recall that we have a universal k-derivation $d : \mathcal{O}_X \to \Omega_{X/k}$ and hence $D = \theta \circ d$ for some $\mathcal{O}_X$-linear map $\theta : \Omega_{X/k} \to \mathcal{O}_X$. Recall that $\Omega_{X/k} \cong \omega_X$, see Lemma 4.1. By Riemann-Roch we have $\deg(\omega_X) = 2g - 2$ (Lemma 5.2). Thus we see that θ is forced to be zero if $g > 1$ by Varieties, Lemma 44.12. This proves part (1). If $g = 1$, then a nonzero θ does not vanish anywhere and if $g = 0$, then a nonzero θ vanishes in a divisor of degree 2. Thus parts (2) and (3) follow if we show that vanishing of θ at a closed point $x \in X$ is equivalent to the statement that D fixes x (as defined above). Let $z \in \mathcal{O}_{X,x}$ be a uniformizer. Then dz is a basis element for $\Omega_{X,x}$, see Lemma 12.3. Since $D(z) = \theta(dz)$ we conclude. □

Lemma 25.2. *Let k be an algebraically closed field. Let X be an at-worst-nodal, proper, connected 1-dimensional scheme over k. Let $\nu : X^\nu \to X$ be the normalization. Let $S \subset X^\nu$ be the set of points where ν is not an isomorphism. Then*

$$Der_k(\mathcal{O}_X, \mathcal{O}_X) = \{D' \in Der_k(\mathcal{O}_{X^\nu}, \mathcal{O}_{X^\nu}) \mid D' \textit{ fixes every } x^\nu \in S\}$$

Proof. Let $x \in X$ be a node. Let $x', x'' \in X^\nu$ be the inverse images of x. (Every node is a split node since k is algebriacally closed, see Definition 19.10 and Lemma 19.11.) Let $u \in \mathcal{O}_{X^\nu, x'}$ and $v \in \mathcal{O}_{X^\nu, x''}$ be uniformizers. Observe that we have an exact sequence

$$0 \to \mathcal{O}_{X,x} \to \mathcal{O}_{X^\nu, x'} \times \mathcal{O}_{X^\nu, x''} \to k \to 0$$

This follows from Lemma 16.3. Thus we can view u and v as elements of $\mathcal{O}_{X,x}$ with $uv = 0$.

Let $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$. Then $0 = D(uv) = vD(u) + uD(v)$. Since (u) is annihilator of v in $\mathcal{O}_{X,x}$ and vice versa, we see that $D(u) \in (u)$ and $D(v) \in (v)$. As $\mathcal{O}_{X^\nu, x'} = k + (u)$ we conclude that we can extend D to $\mathcal{O}_{X^\nu, x'}$ and moreover the extension fixes x'. This produces a D' in the right hand side of the equality. Conversely, given a D' fixing x' and x'' we find that D' preserves the subring $\mathcal{O}_{X,x} \subset \mathcal{O}_{X^\nu, x'} \times \mathcal{O}_{X^\nu, x''}$ and this is how we go from right to left in the equality. □

Lemma 25.3. *Let k be an algebraically closed field. Let X be an at-worst-nodal, proper, connected 1-dimensional scheme over k. Assume the genus of X is at least 2 and that X has no rational tails or bridges. Then $Der_k(\mathcal{O}_X, \mathcal{O}_X) = 0$.*

Proof. Let $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$. Let X^ν be the normalization of X. Let $D' \in \text{Der}_k(\mathcal{O}_{X^\nu}, \mathcal{O}_{X^\nu})$ be the element corresponding to D via Lemma 25.2. Let $C \subset X^\nu$ be an irreducible component. If the genus of C is > 1, then $D'|_{\mathcal{O}_C} = 0$ by Lemma 25.1 part (1). If the genus of C is 1, then there is at least one closed point c of C which maps to a node on X (since otherwise $X \cong C$ would have genus 1). By the correspondence this means that $D'|_{\mathcal{O}_C}$ fixes c hence is zero by Lemma 25.1 part (2). Finally, if the genus of C is zero, then there are at least 3 pairwise distinct closed points $c_1, c_2, c_3 \in C$ mapping to nodes in X, since otherwise either X is C with two points glued (two points of C mapping to the same node), or C is a rational bridge (two points mapping to different nodes of X), or C is a rational tail (one point mapping to a node of X). These three possibilities are not permitted since C has genus ≥ 2 and has no rational bridges, or rational tails. Whence $D'|_{\mathcal{O}_C}$ fixes c_1, c_2, c_3 hence is zero by Lemma 25.1 part (3). □

26. Other chapters

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