COHOMOLOGY OF ALGEBRAIC SPACES

Contents

1. Introduction

In this chapter we write about cohomology of algebraic spaces. Although we prove some results on cohomology of abelian sheaves, we focus mainly on cohomology of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapter "Cohomology of Schemes". Some of the results in this chapter can be found in [Knu71].

An important missing ingredient in this chapter is the *induction principle*, i.e., the analogue for quasi-compact and quasi-separated algebraic spaces of Cohomology of Schemes, Lemma 4.1. This is formulated precisely and proved in detail in Derived Categories of Spaces, Section 9. Instead of the induction principle, in this chapter we use the alternating Čech complex, see Section 6. It is designed to prove vanishing statements such as Proposition 7.2, but in some cases the induction principle is a more powerful and perhaps more "standard" tool. We encourage the reader to take a look at the induction principle after reading some of the material in this section.

2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Higher direct images

Let S be a scheme. Let X be a representable algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent module on X (see Properties of Spaces, Section 29). By Descent, Proposition 9.3 the cohomology groups $H^i(X, \mathcal{F})$ agree with the usual cohomology group computed in the Zariski topology of the corresponding quasi-coherent module on the scheme representing X.

More generally, let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spaces X and Y. Let $\mathcal{F}$ be a quasi-coherent module on X. By Descent, Lemma 9.5 the sheaf $R^if_*\mathcal{F}$ agrees with the usual higher direct image computed for the Zariski topology of the quasi-coherent module on the scheme representing X mapping to the scheme representing Y.

More generally still, suppose $f : X \to Y$ is a representable, quasi-compact, and quasi-separated morphism of algebraic spaces over S. Let V be a scheme and let $V \to Y$ be an étale surjective morphism. Let $U = V \times_Y X$ and let $f' : U \to V$ be the base change of f. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

$$R^if'_*(\mathcal{F}|_U) = (R^if_*\mathcal{F})|_V, \tag{3.0.1}$$

see Properties of Spaces, Lemma 26.2. And because $f' : U \to V$ is a quasi-compact and quasi-separated morphism of schemes, by the remark of the preceding paragraph we may compute $R^if'_*(\mathcal{F}|_U)$ by thinking of $\mathcal{F}|_U$ as a quasi-coherent sheaf on the scheme U, and f' as a morphism of schemes. We will frequently use this without further mention.

Next, we prove that higher direct images of quasi-coherent sheaves are quasi-coherent for any quasi-compact and quasi-separated morphism of algebraic spaces. In the proof we use a trick; a "better" proof would use a relative Čech complex, as discussed in Sheaves on Stacks, Sections 18 and 19 ff.

Lemma 3.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is quasi-compact and quasi-separated, then R^if_* transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_Y$-modules.*

Proof. Let $V \to Y$ be an étale morphism where V is an affine scheme. Set $U = V \times_Y X$ and denote $f' : U \to V$ the induced morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. By Properties of Spaces, Lemma 26.2 we have $R^if'_*(\mathcal{F}|_U) = (R^if_*\mathcal{F})|_V$. Since the property of being a quasi-coherent module is local in the étale topology on Y (see Properties of Spaces, Lemma 29.6) we may replace Y by V, i.e., we may assume Y is an affine scheme.

Assume Y is affine. Since f is quasi-compact we see that X is quasi-compact. Thus we may choose an affine scheme U and a surjective étale morphism $g : U \to X$, see Properties of Spaces, Lemma 6.3. Picture

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[rd, "f \circ g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

The morphism $g : U \to X$ is representable, separated and quasi-compact because X is quasi-separated. Hence the lemma holds for g (by the discussion above the lemma). It also holds for $f \circ g : U \to Y$ (as this is a morphism of affine schemes).

In the situation described in the previous paragraph we will show by induction on n that IH_n: for any quasi-coherent sheaf $\mathcal{F}$ on X the sheaves $R^i f\mathcal{F}$ are quasi-coherent for $i \leq n$. The case $n = 0$ follows from Morphisms of Spaces, Lemma 11.2. Assume IH_n. In the rest of the proof we show that IH_{n+1} holds.

Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_U$-module. Consider the Leray spectral sequence

$$E_2^{p,q} = R^p f_* R^q g_* \mathcal{H} \Rightarrow R^{p+q}(f \circ g)_* \mathcal{H}$$

Cohomology on Sites, Lemma 14.7. As $R^q g_* \mathcal{H}$ is quasi-coherent by IH_n all the sheaves $R^p f_* R^q g_* \mathcal{H}$ are quasi-coherent for $p \leq n$. The sheaves $R^{p+q}(f \circ g)_* \mathcal{H}$ are all quasi-coherent (in fact zero for $p + q > 0$ but we do not need this). Looking in degrees $\leq n + 1$ the only module which we do not yet know is quasi-coherent is $E_2^{n+1,0} = R^{n+1} f_* g_* \mathcal{H}$. Moreover, the differentials $d_r^{n+1,0} : E_r^{n+1,0} \to E_r^{n+1+r,1-r}$ are zero as the target is zero. Using that $QCoh(\mathcal{O}_X)$ is a weak Serre subcategory of $Mod(\mathcal{O}_X)$ (Properties of Spaces, Lemma 29.7) it follows that $R^{n+1} f_* g_* \mathcal{H}$ is quasi-coherent (details omitted).

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Set $\mathcal{H} = g^* \mathcal{F}$. The adjunction mapping $\mathcal{F} \to g_* g^* \mathcal{F} = g_* \mathcal{H}$ is injective as $U \to X$ is surjective étale. Consider the exact sequence

$$0 \to \mathcal{F} \to g_* \mathcal{H} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is the cokernel of the first map and in particular quasi-coherent. Applying the long exact cohomology sequence we obtain

$$R^n f_* g_* \mathcal{H} \to R^n f_* \mathcal{G} \to R^{n+1} f_* \mathcal{F} \to R^{n+1} f_* g_* \mathcal{H} \to R^{n+1} f_* \mathcal{G}$$

The cokernel of the first arrow is quasi-coherent and we have seen above that $R^{n+1} f_* g_* \mathcal{H}$ is quasi-coherent. Thus $R^{n+1} f_* \mathcal{F}$ has a 2-step filtration where the first step is quasi-coherent and the second a submodule of a quasi-coherent sheaf. Since $\mathcal{F}$ is an arbitrary quasi-coherent $\mathcal{O}_X$-module, this result also holds for $\mathcal{G}$. Thus we can choose an exact sequence $0 \to \mathcal{A} \to R^{n+1} f_* \mathcal{G} \to \mathcal{B}$ with $\mathcal{A}$, $\mathcal{B}$ quasi-coherent $\mathcal{O}_Y$-modules. Then the kernel $\mathcal{K}$ of $R^{n+1} f_* g_* \mathcal{H} \to R^{n+1} f_* \mathcal{G} \to \mathcal{B}$ is quasi-coherent, whereupon we obtain a map $\mathcal{K} \to \mathcal{A}$ whose kernel $\mathcal{K}'$ is quasi-coherent too. Hence $R^{n+1} f_* \mathcal{F}$ sits in an exact sequence

$$R^n f_* g_* \mathcal{H} \to R^n f_* \mathcal{G} \to R^{n+1} f_* \mathcal{F} \to \mathcal{K}' \to 0$$

with all modules quasi-coherent except for possibly $R^{n+1} f_* \mathcal{F}$. We conclude that $R^{n+1} f_* \mathcal{F}$ is quasi-coherent, i.e., IH_{n+1} holds as desired. $\square$

Lemma 3.2. *Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any affine object V of $Y_{étale}$ we have*

$$H^q(V \times_Y X, \mathcal{F}) = H^0(V, R^q f_*\mathcal{F})$$

for all $q \in \mathbf{Z}$.

Proof. Since formation of Rf_* commutes with étale localization (Properties of Spaces, Lemma 26.2) we may replace Y by V and assume $Y = V$ is affine. Consider the Leray spectral sequence $E_2^{p,q} = H^p(Y, R^q f_*\mathcal{F})$ converging to $H^{p+q}(X, \mathcal{F})$, see Cohomology on Sites, Lemma 14.5. By Lemma 3.1 we see that the sheaves $R^q f_*\mathcal{F}$ are quasi-coherent. By Cohomology of Schemes, Lemma 2.2 we see that $E_2^{p,q} = 0$ when $p > 0$. Hence the spectral sequence degenerates at E_2 and we win. □

4. Finite morphisms

Here are some results which hold for all abelian sheaves (in particular also quasi-coherent modules). We **warn** the reader that these lemmas do not hold for finite morphisms of schemes and the Zariski topology.

Lemma 4.1. *Let S be a scheme. Let $f : X \to Y$ be an integral (for example finite) morphism of algebraic spaces. Then $f_* : Ab(X_{étale}) \to Ab(Y_{étale})$ is an exact functor and $R^p f_* = 0$ for $p > 0$.*

Proof. By Properties of Spaces, Lemma 18.12 we may compute the higher direct images on an étale cover of Y. Hence we may assume Y is a scheme. This implies that X is a scheme (Morphisms of Spaces, Lemma 45.3). In this case we may apply Étale Cohomology, Lemma 43.5. For the finite case the reader may wish to consult the less technical Étale Cohomology, Proposition 55.2. □

Lemma 4.2. *Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y with lifts $\overline{x}_1, \ldots, \overline{x}_n$ in X. Then*

$$(f_*\mathcal{F})_{\overline{y}} = \prod\nolimits_{i=1,\ldots,n} \mathcal{F}_{\overline{x}_i}$$

for any sheaf $\mathcal{F}$ on $X_{étale}$.

Proof. Choose an étale neighbourhood $(V, \overline{v})$ of $\overline{y}$. Then the stalk $(f_*\mathcal{F})_{\overline{y}}$ is the stalk of $f_*\mathcal{F}|_V$ at $\overline{v}$. By Properties of Spaces, Lemma 18.12 we may replace Y by V and X by $X \times_Y V$. Then $Z \to X$ is a finite morphism of schemes and the result is Étale Cohomology, Proposition 55.2. □

Lemma 4.3. *Let S be a scheme. Let $\pi : X \to Y$ be a finite morphism of algebraic spaces over S. Let $\mathcal{A}$ be a sheaf of rings on $X_{étale}$. Let $\mathcal{B}$ be a sheaf of rings on $Y_{étale}$. Let $\varphi : \mathcal{B} \to \pi_*\mathcal{A}$ be a homomorphism of sheaves of rings so that we obtain a morphism of ringed topoi*

$$f = (\pi, \varphi) : (Sh(X_{étale}), \mathcal{A}) \longrightarrow (Sh(Y_{étale}), \mathcal{B}).$$

For a sheaf of $\mathcal{A}$-modules $\mathcal{F}$ and a sheaf of $\mathcal{B}$-modules $\mathcal{G}$ the canonical map

$$\mathcal{G} \otimes_{\mathcal{B}} f_*\mathcal{F} \longrightarrow f_*(f^*\mathcal{G} \otimes_{\mathcal{A}} \mathcal{F}).$$

is an isomorphism.

Proof. The map is the map adjoint to the map

$$f^*\mathcal{G} \otimes_{\mathcal{A}} f^*f_*\mathcal{F} = f^*(\mathcal{G} \otimes_{\mathcal{B}} f_*\mathcal{F}) \longrightarrow f^*\mathcal{G} \otimes_{\mathcal{A}} \mathcal{F}$$

coming from $\text{id} : f^*\mathcal{G} \to f^*\mathcal{G}$ and the adjunction map $f^*f_*\mathcal{F} \to \mathcal{F}$. To see this map is an isomorphism, we may check on stalks (Properties of Spaces, Theorem 19.12). Let $\overline{y}$ be a geometric point of Y and let $\overline{x}_1, \ldots, \overline{x}_n$ be the geometric points of X lying over $\overline{y}$. Working out what our maps does on stalks, we see that we have to show

$$\mathcal{G}_{\overline{y}} \otimes_{\mathcal{B}_{\overline{y}}} \left(\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{F}_{\overline{x}_i}\right) = \bigoplus\nolimits_{i=1,\ldots,n} (\mathcal{G}_{\overline{y}} \otimes_{\mathcal{B}_{\overline{x}}} \mathcal{A}_{\overline{x}_i}) \otimes_{\mathcal{A}_{\overline{x}_i}} \mathcal{F}_{\overline{x}_i}$$

which holds true. Here we have used that taking tensor products commutes with taking stalks, the behaviour of stalks under pullback Properties of Spaces, Lemma 19.9, and the behaviour of stalks under pushforward along a closed immersion Lemma 4.2. $\square$

We end this section with an insanely general projection formula for finite morphisms.

Lemma 4.4. *With S, X, Y, π, $\mathcal{A}$, $\mathcal{B}$, φ, and f as in Lemma 4.3 we have*

$$K \otimes_{\mathcal{B}}^{\mathbf{L}} Rf_*M = Rf_*(Lf^*K \otimes_{\mathcal{A}}^{\mathbf{L}} M)$$

in $D(\mathcal{B})$ for any $K \in D(\mathcal{B})$ and $M \in D(\mathcal{A})$.

Proof. Since f_* is exact (Lemma 4.1) the functor Rf_* is computed by applying f_* to any representative complex. Choose a complex $\mathcal{K}^\bullet$ of $\mathcal{B}$-modules representing K which is K-flat with flat terms, see Cohomology on Sites, Lemma 17.11. Then $f^*\mathcal{K}^\bullet$ is K-flat with flat terms, see Cohomology on Sites, Lemma 18.1. Choose any complex $\mathcal{M}^\bullet$ of $\mathcal{A}$-modules representing M. Then we have to show

$$\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{B}} f_*\mathcal{M}^\bullet) = f_*\text{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{A}} \mathcal{M}^\bullet)$$

because by our choices these complexes represent the right and left hand side of the formula in the lemma. Since f_* commutes with direct sums (for example by the description of the stalks in Lemma 4.2), this reduces to the equalities

$$\mathcal{K}^n \otimes_{\mathcal{B}} f_*\mathcal{M}^m = f_*(f^*\mathcal{K}^n \otimes_{\mathcal{A}} \mathcal{M}^m)$$

which are true by Lemma 4.3. $\square$

5. Colimits and cohomology

The following lemma in particular applies to diagrams of quasi-coherent sheaves.

Lemma 5.1. *Let S be a scheme. Let X be an algebraic space over S. If X is quasi-compact and quasi-separated, then*

$$\text{colim}_i\, H^p(X, \mathcal{F}_i) \longrightarrow H^p(X, \text{colim}_i\, \mathcal{F}_i)$$

is an isomorphism for every filtered diagram of abelian sheaves on $X_{étale}$.

Proof. This follows from Cohomology on Sites, Lemma 16.1. Namely, let $\mathcal{B} \subset \text{Ob}(X_{spaces,étale})$ be the set of quasi-compact and quasi-separated spaces étale over X. Note that if $U \in \mathcal{B}$ then, because U is quasi-compact, the collection of finite coverings $\{U_i \to U\}$ with $U_i \in \mathcal{B}$ is cofinal in the set of coverings of U in $X_{spaces,étale}$. By Morphisms of Spaces, Lemma 8.10 the set $\mathcal{B}$ satisfies all the assumptions of Cohomology on Sites, Lemma 16.1. Since $X \in \mathcal{B}$ we win. $\square$

Lemma 5.2. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ be a filtered colimit of abelian sheaves on $X_{\acute{e}tale}$. Then for any $p \geq 0$ we have*

$$R^p f_* \mathcal{F} = \mathrm{colim}\, R^p f_* \mathcal{F}_i.$$

Proof. Recall that $R^p f_* \mathcal{F}$ is the sheaf on $Y_{spaces,\acute{e}tale}$ associated to $V \mapsto H^p(V \times_Y X, \mathcal{F})$, see Cohomology on Sites, Lemma 7.4 and Properties of Spaces, Lemma 18.8. Recall that the colimit is the sheaf associated to the presheaf colimit. Hence we can apply Lemma 5.1 to $H^p(V \times_Y X, -)$ where V is affine to conclude (because when V is affine, then $V \times_Y X$ is quasi-compact and quasi-separated). Strictly speaking this also uses Properties of Spaces, Lemma 18.6 to see that there exist enough affine objects. $\square$

The following lemma tells us that finitely presented modules behave as expected in quasi-compact and quasi-separated algebraic spaces.

Lemma 5.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let I be a directed set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation. Then we have*

$$\mathrm{colim}_i\, \mathrm{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \mathrm{Hom}_X(\mathcal{G}, \mathrm{colim}_i\, \mathcal{F}_i).$$

In particular, $\mathrm{Hom}_X(\mathcal{G}, -)$ commutes with filtered colimits in $QCoh(\mathcal{O}_X)$.

Proof. The displayed equality is a special case of Modules on Sites, Lemma 27.12. In order to apply it, we need to check the hypotheses of Sites, Lemma 17.8 part (4) for the site $X_{\acute{e}tale}$. In order to do this, we will check hypotheses (2)(a), (2)(b), (2)(c) of Sites, Remark 17.9. Namely, let $\mathcal{B} \subset \mathrm{Ob}(X_{\acute{e}tale})$ be the set of affine objects. Then

(1) Since X is quasi-compact, there exists a $U \in \mathcal{B}$ such that $U \to X$ is surjective (Properties of Spaces, Lemma 6.3), hence $h_U^\# \to *$ is surjective.
(2) For $U \in \mathcal{B}$ every étale covering $\{U_i \to U\}_{i \in I}$ of U can be refined by a finite étale covering $\{U_j \to U\}_{j=1,\ldots,m}$ with $U_j \in \mathcal{B}$ (Topologies, Lemma 4.4).
(3) For $U, U' \in \mathrm{Ob}(X_{\acute{e}tale})$ we have $h_U^\# \times h_{U'}^\# = h_{U \times_X U'}^\#$. If $U, U' \in \mathcal{B}$, then $U \times_X U'$ is quasi-compact because X is quasi-separated, see Morphisms of Spaces, Lemma 8.10 for example. Hence we can find a surjective étale morphism $U'' \to U \times_X U'$ with $U'' \in \mathcal{B}$ (Properties of Spaces, Lemma 6.3). In other words, we have morphisms $U'' \to U$ and $U'' \to U'$ such that the map $h_{U''}^\# \to h_U^\# \times h_{u'}^\#$ is surjective.

For the final statement, observe that the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ commutes with colimits and that finitely presented modules are quasi-coherent. See Properties of Spaces, Lemma 29.7. $\square$

6. The alternating Čech complex

Let S be a scheme. Let $f : U \to X$ be an étale morphism of algebraic spaces over S. The functor

$$j : U_{spaces,\acute{e}tale} \longrightarrow X_{spaces,\acute{e}tale}, \quad V/U \longmapsto V/X$$

induces an equivalence of $U_{spaces,\acute{e}tale}$ with the localization $X_{spaces,\acute{e}tale}/U$, see Properties of Spaces, Section 27. Hence there exist functors

$$f_! : Ab(U_{\acute{e}tale}) \longrightarrow Ab(X_{\acute{e}tale}), \quad f_! : Mod(\mathcal{O}_U) \longrightarrow Mod(\mathcal{O}_X),$$

which are left adjoint to

$$f^{-1} : Ab(X_{\acute{e}tale}) \longrightarrow Ab(U_{\acute{e}tale}), \quad f^* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_U)$$

see Modules on Sites, Section 19. Warning: This functor, a priori, has nothing to do with cohomology with compact supports! We dubbed this functor "extension by zero" in the reference above. Note that the two versions of $f_!$ agree as $f^* = f^{-1}$ for sheaves of $\mathcal{O}_X$-modules.

As we are going to use this construction below let us recall some of its properties. Given an abelian sheaf $\mathcal{G}$ on $U_{\acute{e}tale}$ the sheaf $f_!$ is the sheafification of the presheaf

$$V/X \longmapsto f_!\mathcal{G}(V) = \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_X(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U),$$

see Modules on Sites, Lemma 19.2. Moreover, if $\mathcal{G}$ is an $\mathcal{O}_U$-module, then $f_!\mathcal{G}$ is the sheafification of the exact same presheaf of abelian groups which is endowed with an $\mathcal{O}_X$-module structure in an obvious way (see loc. cit.). Let $\overline{x} : \mathrm{Spec}(k) \to X$ be a geometric point. Then there is a canonical identification

$$(f_!\mathcal{G})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathcal{G}_{\overline{u}}$$

where the sum is over all $\overline{u} : \mathrm{Spec}(k) \to U$ such that $f \circ \overline{u} = \overline{x}$, see Modules on Sites, Lemma 38.1 and Properties of Spaces, Lemma 19.13. In the following we are going to study the sheaf $f_!\underline{\mathbf{Z}}$. Here $\underline{\mathbf{Z}}$ denotes the constant sheaf on $X_{\acute{e}tale}$ or $U_{\acute{e}tale}$.

Lemma 6.1. *Let S be a scheme. Let $f_i : U_i \to X$ be étale morphisms of algebraic spaces over S. Then there are isomorphisms*

$$f_{1,!}\underline{\mathbf{Z}} \otimes_{\mathbf{Z}} f_{2,!}\underline{\mathbf{Z}} \longrightarrow f_{12,!}\underline{\mathbf{Z}}$$

where $f_{12} : U_1 \times_X U_2 \to X$ is the structure morphism and

$$(f_1 \amalg f_2)_!\underline{\mathbf{Z}} \longrightarrow f_{1,!}\underline{\mathbf{Z}} \oplus f_{2,!}\underline{\mathbf{Z}}$$

Proof. Once we have defined the map it will be an isomorphism by our description of stalks above. To define the map it suffices to work on the level of presheaves. Thus we have to define a map

$$\left(\bigoplus\nolimits_{\varphi_1 \in \mathrm{Mor}_X(V,U_1)} \mathbf{Z}\right) \otimes_{\mathbf{Z}} \left(\bigoplus\nolimits_{\varphi_2 \in \mathrm{Mor}_X(V,U_2)} \mathbf{Z}\right) \longrightarrow \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_X(V, U_1 \times_X U_2)} \mathbf{Z}$$

We map the element $1_{\varphi_1} \otimes 1_{\varphi_2}$ to the element $1_{\varphi_1 \times \varphi_2}$ with obvious notation. We omit the proof of the second equality. $\square$

Another important feature is the trace map

$$\mathrm{Tr}_f : f_!\underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}}.$$

The trace map is adjoint to the map $\mathbf{Z} \to f^{-1}\underline{\mathbf{Z}}$ (which is an isomorphism). If $\overline{x}$ is above, then Tr_f on stalks at $\overline{x}$ is the map

$$(\mathrm{Tr}_f)_{\overline{x}} : (f_!\underline{\mathbf{Z}})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathbf{Z} \longrightarrow \mathbf{Z} = \underline{\mathbf{Z}}_{\overline{x}}$$

which sums the given integers. This is true because it is adjoint to the map $1 : \mathbf{Z} \to f^{-1}\underline{\mathbf{Z}}$. In particular, if f is surjective as well as étale then Tr_f is surjective.

Assume that $f : U \to X$ is a surjective étale morphism of algebraic spaces. Consider the *Koszul complex* associated to the trace map we discussed above

$$\ldots \to \wedge^3 f_!\underline{\mathbf{Z}} \to \wedge^2 f_!\underline{\mathbf{Z}} \to f_!\underline{\mathbf{Z}} \to \underline{\mathbf{Z}} \to 0$$

Here the exterior powers are over the sheaf of rings $\underline{\mathbf{Z}}$. The maps are defined by the rule

$$e_1 \wedge \ldots \wedge e_n \longmapsto \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \text{Tr}_f(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n$$

where $e_1, \ldots, e_n$ are local sections of $f_!\underline{\mathbf{Z}}$. Let $\overline{x}$ be a geometric point of X and set $M_{\overline{x}} = (f_!\underline{\mathbf{Z}})_{\overline{x}} = \bigoplus_{\overline{u}} \mathbf{Z}$. Then the stalk of the complex above at $\overline{x}$ is the complex

$$\ldots \to \wedge^3 M_{\overline{x}} \to \wedge^2 M_{\overline{x}} \to M_{\overline{x}} \to \mathbf{Z} \to 0$$

which is exact because $M_{\overline{x}} \to \mathbf{Z}$ is surjective, see More on Algebra, Lemma 28.5. Hence if we let $K^\bullet = K^\bullet(f)$ be the complex with $K^i = \wedge^{i+1} f_!\underline{\mathbf{Z}}$, then we obtain a quasi-isomorphism

$$K^\bullet \longrightarrow \underline{\mathbf{Z}}[0] \tag{6.1.1}$$

We use the complex $K^\bullet$ to define what we call the alternating Čech complex associated to $f : U \to X$.

Definition 6.2. Let S be a scheme. Let $f : U \to X$ be a surjective étale morphism of algebraic spaces over S. Let $\mathcal{F}$ be an object of $Ab(X_{étale})$. The *alternating Čech complex*[1] $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F})$ associated to $\mathcal{F}$ and f is the complex

$$\text{Hom}(K^0, \mathcal{F}) \to \text{Hom}(K^1, \mathcal{F}) \to \text{Hom}(K^2, \mathcal{F}) \to \ldots$$

with Hom groups computed in $Ab(X_{étale})$.

The reader may verify that if $U = \coprod U_i$ and $f|_{U_i} : U_i \to X$ is the open immersion of a subspace, then $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F})$ agrees with the complex introduced in Cohomology, Section 23 for the Zariski covering $X = \bigcup U_i$ and the restriction of $\mathcal{F}$ to the Zariski site of X. What is more important however, is to relate the cohomology of the alternating Čech complex to the cohomology.

Lemma 6.3. *Let S be a scheme. Let $f : U \to X$ be a surjective étale morphism of algebraic spaces over S. Let $\mathcal{F}$ be an object of $Ab(X_{étale})$. There exists a canonical map*

$$\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D(Ab)$. Moreover, there is a spectral sequence with E_1-page

$$E_1^{p,q} = \text{Ext}^q_{Ab(X_{étale})}(K^p, \mathcal{F})$$

converging to $H^{p+q}(X, \mathcal{F})$ where $K^p = \wedge^{p+1} f_!\underline{\mathbf{Z}}$.

Proof. Recall that we have the quasi-isomorphism $K^\bullet \to \underline{\mathbf{Z}}[0]$, see (6.1.1). Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(X_{étale})$. Consider the double complex $\text{Hom}(K^\bullet, \mathcal{I}^\bullet)$ with terms $\text{Hom}(K^p, \mathcal{I}^q)$. The differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential $K^{p+1} \to K^p$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential $\mathcal{I}^q \to \mathcal{I}^{q+1}$. Denote $\text{Tot}(\text{Hom}(K^\bullet, \mathcal{I}^\bullet))$ the associated total complex, see Homology, Section 18. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 25.

Because $K^\bullet$ is a resolution of $\underline{\mathbf{Z}}$ we see that the complexes

$$\text{Hom}(K^\bullet, \mathcal{I}^q) : \text{Hom}(K^0, \mathcal{I}^q) \to \text{Hom}(K^1, \mathcal{I}^q) \to \text{Hom}(K^2, \mathcal{I}^q) \to \ldots$$

[1] This may be nonstandard notation

are acyclic in positive degrees and have H^0 equal to $\Gamma(X, \mathcal{I}^q)$. Hence by Homology, Lemma 25.4 the natural map

$$\mathcal{I}^\bullet(X) \longrightarrow \text{Tot}(\text{Hom}(K^\bullet, \mathcal{I}^\bullet))$$

is a quasi-isomorphism of complexes of abelian groups. In particular we conclude that $H^n(\text{Tot}(\text{Hom}(K^\bullet, \mathcal{I}^\bullet))) = H^n(X, \mathcal{F})$.

The map $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \to R\Gamma(X, \mathcal{F})$ of the lemma is the composition of $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \to \text{Tot}(\text{Hom}(K^\bullet, \mathcal{I}^\bullet))$ with the inverse of the displayed quasi-isomorphism.

Finally, consider the spectral sequence $('E_r, 'd_r)$. We have

$$E_1^{p,q} = q\text{th cohomology of } \text{Hom}(K^p, \mathcal{I}^0) \to \text{Hom}(K^p, \mathcal{I}^1) \to \text{Hom}(K^p, \mathcal{I}^2) \to \ldots$$

This proves the lemma. $\square$

It follows from the lemma that it is important to understand the ext groups $\text{Ext}_{Ab(X_{\acute{e}tale})}(K^p, \mathcal{F})$, i.e., the right derived functors of $\mathcal{F} \mapsto \text{Hom}(K^p, \mathcal{F})$.

Lemma 6.4. *Let S be a scheme. Let $f : U \to X$ be a surjective, étale, and separated morphism of algebraic spaces over S. For $p \geq 0$ set*

$$W_p = U \times_X \ldots \times_X U \setminus \textit{all diagonals}$$

where the fibre product has $p+1$ factors. There is a free action of S_{p+1} on W_p over X and

$$\text{Hom}(K^p, \mathcal{F}) = S_{p+1}\textit{-anti-invariant elements of } \mathcal{F}(W_p)$$

functorially in $\mathcal{F}$ where $K^p = \wedge^{p+1} f_!\underline{\mathbf{Z}}$.

Proof. Because $U \to X$ is separated the diagonal $U \to U \times_X U$ is a closed immersion. Since $U \to X$ is étale the diagonal $U \to U \times_X U$ is an open immersion, see Morphisms of Spaces, Lemmas 39.10 and 38.9. Hence W_p is an open and closed subspace of $U^{p+1} = U \times_X \ldots \times_X U$. The action of S_{p+1} on W_p is free as we've thrown out the fixed points of the action. By Lemma 6.1 we see that

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} = f^{p+1}_!\underline{\mathbf{Z}} = (W_p \to X)_!\underline{\mathbf{Z}} \oplus Rest$$

where $f^{p+1} : U^{p+1} \to X$ is the structure morphism. Looking at stalks over a geometric point $\overline{x}$ of X we see that

$$\left(\bigoplus\nolimits_{\overline{u} \mapsto \overline{x}} \mathbf{Z}\right)^{\otimes p+1} \longrightarrow (W_p \to X)_!\underline{\mathbf{Z}}_{\overline{x}}$$

is the quotient whose kernel is generated by all tensors $1_{\overline{u}_0} \otimes \ldots \otimes 1_{\overline{u}_p}$ where $\overline{u}_i = \overline{u}_j$ for some $i \neq j$. Thus the quotient map

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} \longrightarrow \wedge^{p+1} f_!\underline{\mathbf{Z}}$$

factors through $(W_p \to X)_!\underline{\mathbf{Z}}$, i.e., we get

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} \longrightarrow (W_p \to X)_!\underline{\mathbf{Z}} \longrightarrow \wedge^{p+1} f_!\underline{\mathbf{Z}}$$

This already proves that $\text{Hom}(K^p, \mathcal{F})$ is (functorially) a subgroup of

$$\text{Hom}((W_p \to X)_!\underline{\mathbf{Z}}, \mathcal{F}) = \mathcal{F}(W_p)$$

To identify it with the S_{p+1}-anti-invariants we have to prove that the surjection $(W_p \to X)_!\underline{\mathbf{Z}} \to \wedge^{p+1} f_!\underline{\mathbf{Z}}$ is the maximal S_{p+1}-anti-invariant quotient. In other words, we have to show that $\wedge^{p+1} f_!\underline{\mathbf{Z}}$ is the quotient of $(W_p \to X)_!\underline{\mathbf{Z}}$ by the subsheaf generated by the local sections $s - \text{sign}(\sigma)\sigma(s)$ where s is a local section of $(W_p \to X)_!\underline{\mathbf{Z}}$. This can be checked on the stalks, where it is clear. $\square$

Lemma 6.5. *Let S be a scheme. Let W be an algebraic space over S. Let G be a finite group acting freely on W. Let $U = W/G$, see Properties of Spaces, Lemma 34.1. Let $\chi : G \to \{+1, -1\}$ be a character. Then there exists a rank 1 locally free sheaf of $\mathbf{Z}$-modules $\underline{\mathbf{Z}}(\chi)$ on $U_{étale}$ such that for every abelian sheaf $\mathcal{F}$ on $U_{étale}$ we have*

$$H^0(W, \mathcal{F}|_W)^\chi = H^0(U, \mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi))$$

Proof. The quotient morphism $q : W \to U$ is a G-torsor, i.e., there exists a surjective étale morphism $U' \to U$ such that $W \times_U U' = \coprod_{g \in G} U'$ as spaces with G-action over U'. (Namely, $U' = W$ works.) Hence $q_*\underline{\mathbf{Z}}$ is a finite locally free $\mathbf{Z}$-module with an action of G. For any geometric point $\overline{u}$ of U, then we get G-equivariant isomorphisms

$$(q_*\underline{\mathbf{Z}})_{\overline{u}} = \bigoplus\nolimits_{\overline{w} \mapsto \overline{u}} \mathbf{Z} = \bigoplus\nolimits_{g \in G} \mathbf{Z} = \mathbf{Z}[G]$$

where the second $=$ uses a geometric point $\overline{w}_0$ lying over $\overline{u}$ and maps the summand corresponding to $g \in G$ to the summand corresponding to $g(\overline{w}_0)$. We have

$$H^0(W, \mathcal{F}|_W) = H^0(U, \mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}})$$

because $q_*\mathcal{F}|_W = \mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}}$ as one can check by restricting to U'. Let

$$\underline{\mathbf{Z}}(\chi) = (q_*\underline{\mathbf{Z}})^\chi \subset q_*\underline{\mathbf{Z}}$$

be the subsheaf of sections that transform according to χ. For any geometric point $\overline{u}$ of U we have

$$\underline{\mathbf{Z}}(\chi)_{\overline{u}} = \mathbf{Z} \cdot \sum\nolimits_g \chi(g) g \subset \mathbf{Z}[G] = (q_*\underline{\mathbf{Z}})_{\overline{u}}$$

It follows that $\underline{\mathbf{Z}}(\chi)$ is locally free of rank 1 (more precisely, this should be checked after restricting to U'). Note that for any $\mathbf{Z}$-module M the χ-semi-invariants of $M[G]$ are the elements of the form $m \cdot \sum_g \chi(g) g$. Thus we see that for any abelian sheaf $\mathcal{F}$ on U we have

$$(\mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}})^\chi = \mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$$

because we have equality at all stalks. The result of the lemma follows by taking global sections. ☐

Now we can put everything together and obtain the following pleasing result.

Lemma 6.6. *Let S be a scheme. Let $f : U \to X$ be a surjective, étale, and separated morphism of algebraic spaces over S. For $p \geq 0$ set*

$$W_p = U \times_X \ldots \times_X U \setminus \textit{all diagonals}$$

(with $p + 1$ factors) as in Lemma 6.4. Let $\chi_p : S_{p+1} \to \{+1, -1\}$ be the sign character. Let $U_p = W_p/S_{p+1}$ and $\underline{\mathbf{Z}}(\chi_p)$ be as in Lemma 6.5. Then the spectral sequence of Lemma 6.3 has E_1-page

$$E_1^{p,q} = H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$$

and converges to $H^{p+q}(X, \mathcal{F})$.

Proof. Note that since the action of S_{p+1} on W_p is over X we do obtain a morphism $U_p \to X$. Since $W_p \to X$ is étale and since $W_p \to U_p$ is surjective étale, it follows that also $U_p \to X$ is étale, see Morphisms of Spaces, Lemma 39.2. Therefore an injective object of $Ab(X_{étale})$ restricts to an injective object of $Ab(U_{p,étale})$, see Cohomology on Sites, Lemma 7.1. Moreover, the functor $\mathcal{G} \mapsto \mathcal{G} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is an auto-equivalence of $Ab(U_p)$, whence transforms injective objects into injective objects and is exact (because $\underline{\mathbf{Z}}(\chi_p)$ is an invertible $\underline{\mathbf{Z}}$-module). Thus given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(X_{\acute{e}tale})$ the complex

$$\Gamma(U_p, \mathcal{I}^0|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \Gamma(U_p, \mathcal{I}^1|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \Gamma(U_p, \mathcal{I}^2|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \ldots$$

computes $H^*(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$. On the other hand, by Lemma 6.5 it is equal to the complex of S_{p+1}-anti-invariants in

$$\Gamma(W_p, \mathcal{I}^0) \to \Gamma(W_p, \mathcal{I}^1) \to \Gamma(W_p, \mathcal{I}^2) \to \ldots$$

which by Lemma 6.4 is equal to the complex

$$\mathrm{Hom}(K^p, \mathcal{I}^0) \to \mathrm{Hom}(K^p, \mathcal{I}^1) \to \mathrm{Hom}(K^p, \mathcal{I}^2) \to \ldots$$

which computes $\mathrm{Ext}^*_{Ab(X_{\acute{e}tale})}(K^p, \mathcal{F})$. Putting everything together we win. □

7. Higher vanishing for quasi-coherent sheaves

In this section we show that given a quasi-compact and quasi-separated algebraic space X there exists an integer $n = n(X)$ such that the cohomology of any quasi-coherent sheaf on X vanishes beyond degree n.

Lemma 7.1. *With S, W, G, U, χ as in Lemma 6.5. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, then so is $\mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$.*

Proof. The $\mathcal{O}_U$-module structure is clear. To check that $\mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$ is quasi-coherent it suffices to check étale locally. Hence the lemma follows as $\underline{\mathbf{Z}}(\chi)$ is finite locally free as a $\underline{\mathbf{Z}}$-module. □

The following proposition is interesting even if X is a scheme. It is the natural generalization of Cohomology of Schemes, Lemma 4.2. Before we state it, observe that given an étale morphism $f : U \to X$ from an affine scheme towards a quasi-separated algebraic space X the fibres of f are universally bounded, in particular there exists an integer d such that the fibres of $|U| \to |X|$ all have size at most d; this is the implication $(\eta) \Rightarrow (\delta)$ of Decent Spaces, Lemma 5.1.

Proposition 7.2. *Let S be a scheme. Let X be an algebraic space over S. Assume X is quasi-compact and separated. Let U be an affine scheme, and let $f : U \to X$ be a surjective étale morphism. Let d be an upper bound for the size of the fibres of $|U| \to |X|$. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^q(X, \mathcal{F}) = 0$ for $q \geq d$.*

Proof. We will use the spectral sequence of Lemma 6.6. The lemma applies since f is separated as U is separated, see Morphisms of Spaces, Lemma 4.10. Since X is separated the scheme $U \times_X \ldots \times_X U$ is a closed subscheme of $U \times_{\mathrm{Spec}(\mathbf{Z})} \ldots \times_{\mathrm{Spec}(\mathbf{Z})} U$ hence is affine. Thus W_p is affine. Hence $U_p = W_p/S_{p+1}$ is an affine scheme by Groupoids, Proposition 23.9. The discussion in Section 3 shows that cohomology of quasi-coherent sheaves on W_p (as an algebraic space) agrees with the cohomology of the corresponding quasi-coherent sheaf on the underlying affine scheme, hence vanishes in positive degrees by Cohomology of Schemes, Lemma 2.2. By Lemma 7.1 the sheaves $\mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)$ are quasi-coherent. Hence $H^q(W_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero when $q > 0$. By our definition of the integer d we see that $W_p = \emptyset$ for $p \geq d$. Hence also $H^0(W_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero when $p \geq d$. This proves the proposition. □

In the following lemma we establish that a quasi-compact and quasi-separated algebraic space has finite cohomological dimension for quasi-coherent modules. We are explicit about the bound only because we will use it later to prove a similar result for higher direct images.

Lemma 7.3. *Let S be a scheme. Let X be an algebraic space over S. Assume X is quasi-compact and quasi-separated. Then we can choose*

1. *an affine scheme U,*
2. *a surjective étale morphism $f : U \to X$,*
3. *an integer d bounding the degrees of the fibres of $U \to X$,*
4. *for every $p = 0, 1, \ldots, d$ a surjective étale morphism $V_p \to U_p$ from an affine scheme V_p where U_p is as in Lemma 6.6, and*
5. *an integer d_p bounding the degree of the fibres of $V_p \to U_p$.*

Moreover, whenever we have (1) – (5), then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^q(X, \mathcal{F}) = 0$ for $q \geq \max(d_p + p)$.

Proof. Since X is quasi-compact we can find a surjective étale morphism $U \to X$ with U affine, see Properties of Spaces, Lemma 6.3. By Decent Spaces, Lemma 5.1 the fibres of f are universally bounded, hence we can find d. We have $U_p = W_p/S_{p+1}$ and $W_p \subset U \times_X \ldots \times_X U$ is open and closed. Since X is quasi-separated the schemes W_p are quasi-compact, hence U_p is quasi-compact. Since U is separated, the schemes W_p are separated, hence U_p is separated by (the absolute version of) Spaces, Lemma 14.5. By Properties of Spaces, Lemma 6.3 we can find the morphisms $V_p \to W_p$. By Decent Spaces, Lemma 5.1 we can find the integers d_p.

At this point the proof uses the spectral sequence

$$E_1^{p,q} = H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \Rightarrow H^{p+q}(X, \mathcal{F})$$

see Lemma 6.6. By definition of the integer d we see that $U_p = 0$ for $p \geq d$. By Proposition 7.2 and Lemma 7.1 we see that $H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero for $q \geq d_p$ for $p = 0, \ldots, d$. Whence the lemma. $\square$

8. Vanishing for higher direct images

We apply the results of Section 7 to obtain vanishing of higher direct images of quasi-coherent sheaves for quasi-compact and quasi-separated morphisms. This is useful because it allows one to argue by descending induction on the cohomological degree in certain situations.

Lemma 8.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

1. *f is quasi-compact and quasi-separated, and*
2. *Y is quasi-compact.*

Then there exists an integer $n(X \to Y)$ such that for any algebraic space Y', any morphism $Y' \to Y$ and any quasi-coherent sheaf $\mathcal{F}'$ on $X' = Y' \times_Y X$ the higher direct images $R^i f'_* \mathcal{F}'$ are zero for $i \geq n(X \to Y)$.

Proof. Let $V \to Y$ be a surjective étale morphism where V is an affine scheme, see Properties of Spaces, Lemma 6.3. Suppose we prove the result for the base change $f_V : V \times_Y X \to V$. Then the result holds for f with $n(X \to Y) = n(X_V \to V)$. Namely, if $Y' \to Y$ and $\mathcal{F}'$ are as in the lemma, then $R^i f'_* \mathcal{F}'|_{V \times_Y Y'}$ is equal to

$R^i f'_{V,*}\mathcal{F}'|_{X'_V}$ where $f'_V : X'_V = V \times_Y Y' \times_Y X \to V \times_Y Y' = Y'_V$, see Properties of Spaces, Lemma 26.2. Thus we may assume that Y is an affine scheme.

Moreover, to prove the vanishing for all $Y' \to Y$ and $\mathcal{F}'$ it suffices to do so when Y' is an affine scheme. In this case, $R^i f'_*\mathcal{F}'$ is quasi-coherent by Lemma 3.1. Hence it suffices to prove that $H^i(X', \mathcal{F}') = 0$, because $H^i(X', \mathcal{F}') = H^0(Y', R^i f'_*\mathcal{F}')$ by Cohomology on Sites, Lemma 14.6 and the vanishing of higher cohomology of quasi-coherent sheaves on affine algebraic spaces (Proposition 7.2).

Choose $U \to X$, d, $V_p \to U_p$ and d_p as in Lemma 7.3. For any affine scheme Y' and morphism $Y' \to Y$ denote $X' = Y' \times_Y X$, $U' = Y' \times_Y U$, $V'_p = Y' \times_Y V_p$. Then $U' \to X'$, $d' = d$, $V'_p \to U'_p$ and $d'_p = d$ is a collection of choices as in Lemma 7.3 for the algebraic space X' (details omitted). Hence we see that $H^i(X', \mathcal{F}') = 0$ for $i \geq \max(p + d_p)$ and we win. $\square$

Lemma 8.2. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then $R^i f_*\mathcal{F} = 0$ for $i > 0$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. Recall that an affine morphism of algebraic spaces is representable. Hence this follows from (3.0.1) and Cohomology of Schemes, Lemma 2.3. $\square$

Lemma 8.3. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$ for all $i \geq 0$.*

Proof. Follows from Lemma 8.2 and the Leray spectral sequence. See Cohomology on Sites, Lemma 14.6. $\square$

9. Cohomology with support in a closed subspace

This section is the analogue of Cohomology, Sections 21 and 34 and Étale Cohomology, Section 79 for abelian sheaves on algebraic spaces.

Let S be a scheme. Let X be an algebraic space over S and let $Z \subset X$ be a closed subspace. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. We let

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \text{Supp}(s) \subset Z\}$$

be the sections with support in Z (Properties of Spaces, Definition 20.3). This is a left exact functor which is not exact in general. Hence we obtain a derived functor

$$R\Gamma_Z(X, -) : D(X_{\acute{e}tale}) \longrightarrow D(Ab)$$

and cohomology groups with support in Z defined by $H^q_Z(X, \mathcal{F}) = R^q\Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on $X_{\acute{e}tale}$. Let $U \subset X$ be the open subspace which is the complement of Z. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Cohomology on Sites, Lemma 12.6) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D(X_{\acute{e}tale})$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(Ab)$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H^i_Z(X, K) \to H^i(X, K) \to H^i(U, K) \to H^{i+1}_Z(X, K) \to \ldots$$

for any K in $D(X_{\acute{e}tale})$.

For an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we can consider the *subsheaf of sections with support in* Z, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset U \times_X Z\}$$

Here we use the support of a section from Properties of Spaces, Definition 20.3. Using the equivalence of Morphisms of Spaces, Lemma 13.5 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on $Z_{étale}$. Thus we obtain a functor

$$Ab(X_{étale}) \longrightarrow Ab(Z_{étale}), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F})$$

which is left exact, but in general not exact.

Lemma 9.1. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on $Z_{étale}$.*

Proof. Observe that for any abelian sheaf $\mathcal{G}$ on $Z_{étale}$ we have

$$\text{Hom}_Z(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \text{Hom}_X(i_*\mathcal{G}, \mathcal{F})$$

because after all any section of $i_*\mathcal{G}$ has support in Z. Since i_* is exact (Lemma 4.1) and as $\mathcal{I}$ is injective on $X_{étale}$ we conclude that $\mathcal{H}_Z(\mathcal{I})$ is injective on $Z_{étale}$. □

Denote

$$R\mathcal{H}_Z : D(X_{étale}) \longrightarrow D(Z_{étale})$$

the derived functor. We set $\mathcal{H}^q_Z(\mathcal{F}) = R^q\mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$. By the lemma above we have a Grothendieck spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow H^{p+q}_Z(X, \mathcal{F})$$

Lemma 9.2. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{G}$ be an injective abelian sheaf on $Z_{étale}$. Then $\mathcal{H}^p_Z(i_*\mathcal{G}) = 0$ for $p > 0$.*

Proof. This is true because the functor i_* is exact (Lemma 4.1) and transforms injective abelian sheaves into injective abelian sheaves (Cohomology on Sites, Lemma 14.2). □

Lemma 9.3. *Let S be a scheme. Let $f : X \to Y$ be an étale morphism of algebraic spaces over S. Let $Z \subset Y$ be a closed subspace such that $f^{-1}(Z) \to Z$ is an isomorphism of algebraic spaces. Let $\mathcal{F}$ be an abelian sheaf on X. Then*

$$\mathcal{H}^q_Z(\mathcal{F}) = \mathcal{H}^q_{f^{-1}(Z)}(f^{-1}\mathcal{F})$$

as abelian sheaves on $Z = f^{-1}(Z)$ and we have $H^q_Z(Y, \mathcal{F}) = H^q_{f^{-1}(Z)}(X, f^{-1}\mathcal{F})$.

Proof. Because f is étale an injective resolution of $\mathcal{F}$ pulls back to an injective resolution of $f^{-1}\mathcal{F}$. Hence it suffices to check the equality for $\mathcal{H}_Z(-)$ which follows from the definitions. The proof for cohomology with supports is the same. Some details omitted. □

Let S be a scheme and let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. We denote $D_T(X_{étale})$ the strictly full saturated triangulated subcategory of $D(X_{étale})$ consisting of objects whose cohomology sheaves are supported on T.

Lemma 9.4. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. The map $Ri_* = i_* : D(Z_{\acute{e}tale}) \to D(X_{\acute{e}tale})$ induces an equivalence $D(Z_{\acute{e}tale}) \to D_{|Z|}(X_{\acute{e}tale})$ with quasi-inverse*

$$i^{-1}|_{D_Z(X_{\acute{e}tale})} = R\mathcal{H}_Z|_{D_{|Z|}(X_{\acute{e}tale})}$$

Proof. Recall that i^{-1} and i_* is an adjoint pair of exact functors such that $i^{-1}i_*$ is isomorphic to the identify functor on abelian sheaves. See Properties of Spaces, Lemma 19.9 and Morphisms of Spaces, Lemma 13.5. Thus $i_* : D(Z_{\acute{e}tale}) \to D_Z(X_{\acute{e}tale})$ is fully faithful and i^{-1} determines a left inverse. On the other hand, suppose that K is an object of $D_Z(X_{\acute{e}tale})$ and consider the adjunction map $K \to i_*i^{-1}K$. Using exactness of i_* and i^{-1} this induces the adjunction maps $H^n(K) \to i_*i^{-1}H^n(K)$ on cohomology sheaves. Since these cohomology sheaves are supported on Z we see these adjunction maps are isomorphisms and we conclude that $D(Z_{\acute{e}tale}) \to D_Z(X_{\acute{e}tale})$ is an equivalence.

To finish the proof we have to show that $R\mathcal{H}_Z(K) = i^{-1}K$ if K is an object of $D_Z(X_{\acute{e}tale})$. To do this we can use that $K = i_*i^{-1}K$ as we've just proved this is the case. Then we can choose a K-injective representative $\mathcal{I}^\bullet$ for $i^{-1}K$. Since i_* is the right adjoint to the exact functor i^{-1}, the complex $i_*\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 31.9). We see that $R\mathcal{H}_Z(K)$ is computed by $\mathcal{H}_Z(i_*\mathcal{I}^\bullet) = \mathcal{I}^\bullet$ as desired. $\square$

10. Vanishing above the dimension

Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. In this case $|X|$ is a spectral space, see Properties of Spaces, Lemma 15.2. Moreover, the dimension of X (as defined in Properties of Spaces, Definition 9.2) is equal to the Krull dimension of $|X|$, see Decent Spaces, Lemma 12.5. We will show that for quasi-coherent sheaves on X we have vanishing of cohomology above the dimension. This result is already interesting for quasi-separated algebraic spaces of finite type over a field.

Lemma 10.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Assume $\dim(X) \leq d$ for some integer d. Let $\mathcal{F}$ be a quasi-coherent sheaf $\mathcal{F}$ on X.*

1. *$H^q(X, \mathcal{F}) = 0$ for $q > d$,*
2. *$H^d(X, \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective for any quasi-compact open $U \subset X$,*
3. *$H^q_Z(X, \mathcal{F}) = 0$ for $q > d$ for any closed subspace $Z \subset X$ whose complement is quasi-compact.*

Proof. By Properties of Spaces, Lemma 22.5 every algebraic space Y étale over X has dimension $\leq d$. If Y is quasi-separated, the dimension of Y is equal to the Krull dimension of $|Y|$ by Decent Spaces, Lemma 12.5. Also, if Y is a scheme, then étale cohomology of $\mathcal{F}$ over Y, resp. étale cohomology of $\mathcal{F}$ with support in a closed subscheme, agrees with usual cohomology of $\mathcal{F}$, resp. usual cohomology with support in the closed subscheme. See Descent, Proposition 9.3 and Étale Cohomology, Lemma 79.5. We will use these facts without further mention.

By Decent Spaces, Lemma 8.6 there exist an integer n and open subspaces

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a quasi-compact separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism.

As $U_n = V_n$ is a scheme, our initial remarks imply the cohomology of $\mathcal{F}$ over U_n vanishes in degrees $> d$ by Cohomology, Proposition 22.4. Suppose we have shown, by induction, that $H^q(U_{p+1}, \mathcal{F}|_{U_{p+1}}) = 0$ for $q > d$. It suffices to show $H^q_{T_p}(U_p, \mathcal{F})$ for $q > d$ is zero in order to conclude the vanishing of cohomology of $\mathcal{F}$ over U_p in degrees $> d$. However, we have

$$H^q_{T_p}(U_p, \mathcal{F}) = H^q_{f_p^{-1}(T_p)}(V_p, \mathcal{F})$$

by Lemma 9.3 and as V_p is a scheme we obtain the desired vanishing from Cohomology, Proposition 22.4. In this way we conclude that (1) is true.

To prove (2) let $U \subset X$ be a quasi-compact open subspace. Consider the open subspace $U' = U \cup U_n$. Let $Z = U' \setminus U$. Then $g : U_n \to U'$ is an étale morphism such that $g^{-1}(Z) \to Z$ is an isomorphism. Hence by Lemma 9.3 we have $H^q_Z(U', \mathcal{F}) = H^q_Z(U_n, \mathcal{F})$ which vanishes in degree $> d$ because U_n is a scheme and we can apply Cohomology, Proposition 22.4. We conclude that $H^d(U', \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective. Assume, by induction, that we have reduced our problem to the case where U contains U_{p+1}. Then we set $U' = U \cup U_p$, set $Z = U' \setminus U$, and we argue using the morphism $f_p : V_p \to U'$ which is étale and has the property that $f_p^{-1}(Z) \to Z$ is an isomorphism. In other words, we again see that

$$H^q_Z(U', \mathcal{F}) = H^q_{f_p^{-1}(Z)}(V_p, \mathcal{F})$$

and we again see this vanishes in degrees $> d$. We conclude that $H^d(U', \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective. Eventually we reach the stage where $U_1 = X \subset U$ which finishes the proof.

A formal argument shows that (2) implies (3). $\square$

11. Cohomology and base change, I

Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Suppose further that $g : Y' \to Y$ is a morphism of algebraic spaces over S. Denote $X' = X_{Y'} = Y' \times_Y X$ the base change of X and denote $f' : X' \to Y'$ the base change of f. Also write $g' : X' \to X$ the projection, and set $\mathcal{F}' = (g')^*\mathcal{F}$. Here is a diagram representing the situation:

(11.0.1)
$$\begin{tikzcd} \mathcal{F}' = (g')^*\mathcal{F} & X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] & \mathcal{F} \\ Rf'_*\mathcal{F}' & Y' \arrow[r, "g"] & Y & Rf_*\mathcal{F} \end{tikzcd}$$

Here is the simplest case of the base change property we have in mind.

Lemma 11.1. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. In this case $f_*\mathcal{F} \cong Rf_*\mathcal{F}$ is a quasi-coherent sheaf, and for every diagram (11.0.1) we have*

$$g^*f_*\mathcal{F} = f'_*(g')^*\mathcal{F}.$$

Proof. By the discussion surrounding (3.0.1) this reduces to the case of an affine morphism of schemes which is treated in Cohomology of Schemes, Lemma 5.1. $\square$

Lemma 11.2 (Flat base change)**.** *Let S be a scheme. Consider a cartesian diagram of algebraic spaces*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module with pullback $\mathcal{F}' = (g')^*\mathcal{F}$. Assume that g is flat and that f is quasi-compact and quasi-separated. For any $i \geq 0$

(1) *the base change map of Cohomology on Sites, Lemma 15.1 is an isomorphism*
$$g^*R^if_*\mathcal{F} \longrightarrow R^if'_*\mathcal{F}',$$
(2) *if $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(B)$, then $H^i(X, \mathcal{F}) \otimes_A B = H^i(X', \mathcal{F}')$.*

Proof. The morphism g' is flat by Morphisms of Spaces, Lemma 30.4. Note that flatness of g and g' is equivalent to flatness of the morphisms of small étale ringed sites, see Morphisms of Spaces, Lemma 30.9. Hence we can apply Cohomology on Sites, Lemma 15.1 to obtain a base change map

$$g^*R^pf_*\mathcal{F} \longrightarrow R^pf'_*\mathcal{F}'$$

To prove this map is an isomorphism we can work locally in the étale topology on Y'. Thus we may assume that Y and Y' are affine schemes. Say $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(B)$. In this case we are really trying to show that the map

$$H^p(X, \mathcal{F}) \otimes_A B \longrightarrow H^p(X_B, \mathcal{F}_B)$$

is an isomorphism where $X_B = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X$ and $\mathcal{F}_B$ is the pullback of $\mathcal{F}$ to X_B. In other words, it suffices to prove (2).

Fix $A \to B$ a flat ring map and let X be a quasi-compact and quasi-separated algebraic space over A. Note that $g' : X_B \to X$ is affine as a base change of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Hence the higher direct images $R^i(g')_*\mathcal{F}_B$ are zero by Lemma 8.2. Thus $H^p(X_B, \mathcal{F}_B) = H^p(X, g'_*\mathcal{F}_B)$, see Cohomology on Sites, Lemma 14.6. Moreover, we have

$$g'_*\mathcal{F}_B = \mathcal{F} \otimes_{\underline{A}} \underline{B}$$

where $\underline{A}$, $\underline{B}$ denotes the constant sheaf of rings with value A, B. Namely, it is clear that there is a map from right to left. For any affine scheme U étale over X we have

$$\begin{aligned} g'_*\mathcal{F}_B(U) &= \mathcal{F}_B(\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U) \\ &= \Gamma(\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U, (\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U \to U)^*\mathcal{F}|_U) \\ &= B \otimes_A \mathcal{F}(U) \end{aligned}$$

hence the map is an isomorphism. Write $B = \mathrm{colim}\, M_i$ as a filtered colimit of finite free A-modules M_i using Lazard's theorem, see Algebra, Theorem 81.4. We deduce that

$$\begin{aligned}
H^p(X, g'_*\mathcal{F}_B) &= H^p(X, \mathcal{F} \otimes_{\underline{A}} \underline{B}) \\
&= H^p(X, \operatorname{colim}_i \mathcal{F} \otimes_{\underline{A}} \underline{M_i}) \\
&= \operatorname{colim}_i H^p(X, \mathcal{F} \otimes_{\underline{A}} \underline{M_i}) \\
&= \operatorname{colim}_i H^p(X, \mathcal{F}) \otimes_A M_i \\
&= H^p(X, \mathcal{F}) \otimes_A \operatorname{colim}_i M_i \\
&= H^p(X, \mathcal{F}) \otimes_A B
\end{aligned}$$

The first equality because $g'_*\mathcal{F}_B = \mathcal{F} \otimes_{\underline{A}} \underline{B}$ as seen above. The second because $\otimes$ commutes with colimits. The third equality because cohomology on X commutes with colimits (see Lemma 5.1). The fourth equality because M_i is finite free (i.e., because cohomology commutes with finite direct sums). The fifth because $\otimes$ commutes with colimits. The sixth by choice of our system. □

12. Coherent modules on locally Noetherian algebraic spaces

This section is the analogue of Cohomology of Schemes, Section 9. In Modules on Sites, Definition 23.1 we have defined coherent modules on any ringed topos. We use this notion to define coherent modules on locally Noetherian algebraic spaces. Although it is possible to work with coherent modules more generally we resist the urge to do so.

Definition 12.1. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. A quasi-coherent module $\mathcal{F}$ on X is called *coherent* if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module on the site $X_{étale}$ in the sense of Modules on Sites, Definition 23.1.

This definition is compatible with the already existing notion of a coherent module on a locally Noetherian scheme; see assertion (5) of Properties of Spaces, Section 30 (or more directly Descent, Lemma 8.10). Thus from now on, if X is a locally Noetherian scheme over S, we will not distinguish between a coherent module on X viewed as a scheme or a coherent module on X viewed as an algebraic space; this is compatible with the corresponding identifications of categories of quasi-coherent modules discussed in Properties of Spaces, Section 29.

Having said the above, the following lemma gives an understandable characterization of coherent modules on locally Noetherian algebraic spaces.

Lemma 12.2. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is coherent,*
2. *$\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module,*
3. *$\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module,*
4. *for any étale morphism $\varphi : U \to X$ where U is a scheme the pullback $\varphi^*\mathcal{F}$ is a coherent module on U, and*
5. *there exists a surjective étale morphism $\varphi : U \to X$ where U is a scheme such that the pullback $\varphi^*\mathcal{F}$ is a coherent module on U.*

In particular $\mathcal{O}_X$ is coherent, any invertible $\mathcal{O}_X$-module is coherent, and more generally any finite locally free $\mathcal{O}_X$-module is coherent.

Proof. To be sure, if X is a locally Noetherian algebraic space and $U \to X$ is an étale morphism, then U is locally Noetherian, see Properties of Spaces, Section 7. The lemma then follows from the points (1) – (5) made in Properties of Spaces, Section 30 and the corresponding result for coherent modules on locally Noetherian schemes, see Cohomology of Schemes, Lemma 9.1. □

Lemma 12.3. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. The category of coherent $\mathcal{O}_X$-modules is abelian. More precisely, the kernel and cokernel of a map of coherent $\mathcal{O}_X$-modules are coherent. Any extension of coherent sheaves is coherent.*

Proof. Choose a scheme U and a surjective étale morphism $f : U \to X$. Pullback f^* is an exact functor as it equals a restriction functor, see Properties of Spaces, Equation (26.1.1). By Lemma 12.2 we can check whether an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent by checking whether $f^*\mathcal{F}$ is coherent. Hence the lemma follows from the case of schemes which is Cohomology of Schemes, Lemma 9.2. □

Coherent modules form a Serre subcategory of the category of quasi-coherent $\mathcal{O}_X$-modules. This does not hold for modules on a general ringed topos.

Lemma 12.4. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Any quasi-coherent submodule of $\mathcal{F}$ is coherent. Any quasi-coherent quotient module of $\mathcal{F}$ is coherent.*

Proof. Choose a scheme U and a surjective étale morphism $f : U \to X$. Pullback f^* is an exact functor as it equals a restriction functor, see Properties of Spaces, Equation (26.1.1). By Lemma 12.2 we can check whether an $\mathcal{O}_X$-module $\mathcal{G}$ is coherent by checking whether $f^*\mathcal{H}$ is coherent. Hence the lemma follows from the case of schemes which is Cohomology of Schemes, Lemma 9.3. □

Lemma 12.5. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S,. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent.*

Proof. Via Lemma 12.2 this follows from the result for schemes, see Cohomology of Schemes, Lemma 9.4. □

Lemma 12.6. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $\overline{x}$ be a geometric point of X lying over $x \in |X|$.*

1. *If $\mathcal{F}_{\overline{x}} = 0$ then there exists an open neighbourhood $X' \subset X$ of x such that $\mathcal{F}|_{X'} = 0$.*
2. *If $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is injective, then there exists an open neighbourhood $X' \subset X$ of x such that $\varphi|_{X'}$ is injective.*
3. *If $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is surjective, then there exists an open neighbourhood $X' \subset X$ of x such that $\varphi|_{X'}$ is surjective.*
4. *If $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is bijective, then there exists an open neighbourhood $X' \subset X$ of x such that $\varphi|_{X'}$ is an isomorphism.*

Proof. Let $\varphi : U \to X$ be an étale morphism where U is a scheme and let $u \in U$ be a point mapping to x. By Properties of Spaces, Lemmas 29.4 and 22.1 as well as More on Algebra, Lemma 45.1 we see that $\varphi_{\overline{x}}$ is injective, surjective, or bijective if and only if $\varphi_u : \varphi^*\mathcal{F}_u \to \varphi^*\mathcal{G}_u$ has the corresponding property. Thus we can apply the schemes version of this lemma to see that (after possibly shrinking U) the map $\varphi^*\mathcal{F} \to \varphi^*\mathcal{G}$ is injective, surjective, or an isomorphism. Let $X' \subset X$ be the open subspace corresponding to $|\varphi|(|U|) \subset |X|$, see Properties of Spaces, Lemma 4.8. Since $\{U \to X'\}$ is a covering for the étale topology, we conclude that $\varphi|_{X'}$ is injective, surjective, or an isomorphism as desired. Finally, observe that (1) follows from (2) by looking at the map $\mathcal{F} \to 0$. $\square$

Lemma 12.7. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{F}$ and $\mathcal{G}$ the quasi-coherent $\mathcal{O}_Z$-module such that $i_*\mathcal{G} = \mathcal{F}$, see Morphisms of Spaces, Definition 15.4. Then $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module.*

Proof. The statement of the lemma makes sense as a coherent module is in particular of finite type. Moreover, as $Z \to X$ is a closed immersion it is locally of finite type and hence Z is locally Noetherian, see Morphisms of Spaces, Lemmas 23.7 and 23.5. Finally, as $\mathcal{G}$ is of finite type it is a coherent $\mathcal{O}_Z$-module by Lemma 12.2 $\square$

Lemma 12.8. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of locally Noetherian algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor i_* induces an equivalence between the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ and the category of coherent $\mathcal{O}_Z$-modules.*

Proof. The functor is fully faithful by Morphisms of Spaces, Lemma 14.1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module annihilated by $\mathcal{I}$. By Morphisms of Spaces, Lemma 14.1 we can write $\mathcal{F} = i_*\mathcal{G}$ for some quasi-coherent sheaf $\mathcal{G}$ on Z. To check that $\mathcal{G}$ is coherent we can work étale locally (Lemma 12.2). Choosing an étale covering by a scheme we conclude that $\mathcal{G}$ is coherent by the case of schemes (Cohomology of Schemes, Lemma 9.8). Hence the functor is fully faithful and the proof is done. $\square$

Lemma 12.9. *Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S with Y locally Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume f is finite and Y locally Noetherian. Then $R^pf_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F}$ is coherent.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $V \times_Y X \to V$ is a finite morphism of locally Noetherian schemes. By (3.0.1) we reduce to the case of schemes which is Cohomology of Schemes, Lemma 9.9. $\square$

13. Coherent sheaves on Noetherian spaces

In this section we mention some properties of coherent sheaves on Noetherian algebraic spaces.

Lemma 13.1. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The ascending chain condition holds for quasi-coherent submodules of $\mathcal{F}$. In other words, given any sequence*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$$

of quasi-coherent submodules, then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$.

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 23.5). If $\mathcal{F}_n|_U = \mathcal{F}_{n+1}|_U = \ldots$ then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 10.1. ☐

Lemma 13.2. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals corresponding to a closed subspace $Z \subset X$. Then there is some $n \geq 0$ such that $\mathcal{I}^n\mathcal{F} = 0$ if and only if $Supp(\mathcal{F}) \subset Z$ (set theoretically).*

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 23.5). Note that $\mathcal{I}^n\mathcal{F}|_U = 0$ if and only if $\mathcal{I}^n\mathcal{F} = 0$ and similarly for the condition on the support. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 10.2. ☐

Lemma 13.3 (Artin-Rees)**.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}$ be a quasi-coherent subsheaf. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then there exists a $c \geq 0$ such that for all $n \geq c$ we have*

$$\mathcal{I}^{n-c}(\mathcal{I}^c\mathcal{F} \cap \mathcal{G}) = \mathcal{I}^n\mathcal{F} \cap \mathcal{G}$$

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 23.5). The equality of the lemma holds if and only if it holds after restricting to U. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 10.3. ☐

Lemma 13.4. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Denote $Z \subset X$ the corresponding closed subspace and set $U = X \setminus Z$. There is a canonical isomorphism*

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n\mathcal{G}, \mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular we have an isomorphism

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}).$$

Proof. Let W be an affine scheme and let $W \to X$ be a surjective étale morphism (see Properties of Spaces, Lemma 6.3). Set $R = W \times_X W$. Then W and R are Noetherian schemes, see Morphisms of Spaces, Lemma 23.5. Hence the result hold for the restrictions of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{I}$, U, Z to W and R by Cohomology of Schemes, Lemma 10.5. It follows formally that the result holds over X. ☐

14. Devissage of coherent sheaves

This section is the analogue of Cohomology of Schemes, Section 12.

Lemma 14.1. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $Supp(\mathcal{F}) = Z \cup Z'$ with Z, Z' closed. Then there exists a short exact sequence of coherent sheaves*

$$0 \to \mathcal{G}' \to \mathcal{F} \to \mathcal{G} \to 0$$

with $Supp(\mathcal{G}') \subset Z'$ and $Supp(\mathcal{G}) \subset Z$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals defining the reduced induced closed subspace structure on Z, see Properties of Spaces, Lemma 12.3. Consider the subsheaves $\mathcal{G}'_n = \mathcal{I}^n\mathcal{F}$ and the quotients $\mathcal{G}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$. For each n we have a short exact sequence

$$0 \to \mathcal{G}'_n \to \mathcal{F} \to \mathcal{G}_n \to 0$$

For every geometric point $\overline{x}$ of $Z' \setminus Z$ we have $\mathcal{I}_{\overline{x}} = \mathcal{O}_{X,\overline{x}}$ and hence $\mathcal{G}_{n,\overline{x}} = 0$. Thus we see that $\text{Supp}(\mathcal{G}_n) \subset Z$. Note that $X \setminus Z'$ is a Noetherian algebraic space. Hence by Lemma 13.2 there exists an n such that $\mathcal{G}'_n|_{X\setminus Z'} = \mathcal{I}^n\mathcal{F}|_{X\setminus Z'} = 0$. For such an n we see that $\text{Supp}(\mathcal{G}'_n) \subset Z'$. Thus setting $\mathcal{G}' = \mathcal{G}'_n$ and $\mathcal{G} = \mathcal{G}_n$ works. □

In the following we will freely use the scheme theoretic support of finite type modules as defined in Morphisms of Spaces, Definition 15.4.

Lemma 14.2. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that the scheme theoretic support of $\mathcal{F}$ is a reduced $Z \subset X$ with $|Z|$ irreducible. Then there exist an integer $r > 0$, a nonzero sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$, and an injective map of coherent sheaves*

$$i_*\left(\mathcal{I}^{\oplus r}\right) \to \mathcal{F}$$

whose cokernel is supported on a proper closed subspace of Z.

Proof. By assumption there exists a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with support Z and $\mathcal{F} \cong i_*\mathcal{G}$, see Lemma 12.7. Hence it suffices to prove the lemma for the case $Z = X$ and $i = \text{id}$.

By Properties of Spaces, Proposition 13.3 there exists a dense open subspace $U \subset X$ which is a scheme. Note that U is a Noetherian integral scheme. After shrinking U we may assume that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$ (for example by Cohomology of Schemes, Lemma 12.2 or by a direct algebra argument). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals whose associated closed subspace is the complement of U in X (see for example Properties of Spaces, Section 12). By Lemma 13.4 there exists an $n \geq 0$ and a morphism $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) \to \mathcal{F}$ which recovers our isomorphism over U. Since $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) = (\mathcal{I}^n)^{\oplus r}$ we get a map as in the lemma. It is injective: namely, if σ is a nonzero section of $\mathcal{I}^{\oplus r}$ over a scheme W étale over X, then because X hence W is reduced the support of σ contains a nonempty open of W. But the kernel of $(\mathcal{I}^n)^{\oplus r} \to \mathcal{F}$ is zero over a dense open, hence σ cannot be a section of the kernel. □

Lemma 14.3. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. There exists a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exists a reduced closed subspace $Z_j \subset X$ with $|Z_j|$ irreducible and a sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (Z_j \to X)_*\mathcal{I}_j$$

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{matrix} T \subset |X| \text{ closed such that there exists a coherent sheaf } \mathcal{F} \\ \text{with } \text{Supp}(\mathcal{F}) = T \text{ for which the lemma is wrong} \end{matrix} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because $|X|$ is Noetherian (Properties of Spaces, Lemma 24.2) we can choose a minimal element $T \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is T and for which the lemma does not hold. Clearly $T \neq \emptyset$ since the only sheaf whose support is empty is the zero sheaf for which the lemma does hold (with $m = 0$).

If T is not irreducible, then we can write $T = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than T. Then we can apply Lemma 14.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\text{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of T each of $\mathcal{G}_i$ has a filtration as in the statement of the lemma. By considering the induced filtration on $\mathcal{F}$ we arrive at a contradiction. Hence we conclude that T is irreducible.

Suppose T is irreducible. Let $\mathcal{J}$ be the sheaf of ideals defining the reduced induced closed subspace structure on T, see Properties of Spaces, Lemma 12.3. By Lemma 13.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{I}^n\mathcal{F} \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does. In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

Assume T is irreducible and $\mathcal{J}\mathcal{F} = 0$ where $\mathcal{J}$ is as above. Then the scheme theoretic support of $\mathcal{F}$ is T, see Morphisms of Spaces, Lemma 14.1. Hence we can apply Lemma 14.2. This gives a short exact sequence

$$0 \to i_*(\mathcal{I}^{\oplus r}) \to \mathcal{F} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is a proper closed subset of T. Hence we see that $\mathcal{Q}$ has a filtration of the desired type by minimality of T. But then clearly $\mathcal{F}$ does too, which is our final contradiction. $\square$

Lemma 14.4. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

(1) *For any short exact sequence of coherent sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) *For every reduced closed subspace $Z \subset X$ with $|Z|$ irreducible and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $i_*\mathcal{I}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from the property (1) for $\mathcal{P}$. On the other hand, by Lemma 14.3 we can filter any $\mathcal{F}$ with successive subquotients as in (2). Hence the lemma follows. $\square$

Here is a more useful variant of the lemma above.

Lemma 14.5. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

(1) *For any short exact sequence of coherent sheaves*
$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$
if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.
(2) *If $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for $\mathcal{F}$.*
(3) *For every reduced closed subspace $i : Z \to X$ with $|Z|$ irreducible there exists a coherent sheaf $\mathcal{G}$ on Z such that*
    (a) *$Supp(\mathcal{G}) = Z$,*
    (b) *for every nonzero quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{I}\mathcal{G}$ such that $Supp(\mathcal{G}/\mathcal{G}')$ is proper closed in $|Z|$ and such that $\mathcal{P}$ holds for $i_*\mathcal{G}'$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. Consider the collection
$$\mathcal{T} = \left\{ \begin{matrix} T \subset |X| \text{ nonempty closed such that there exists a coherent sheaf} \\ \mathcal{F} \text{ with } \mathrm{Supp}(\mathcal{F}) = T \text{ for which the lemma is wrong} \end{matrix} \right\}$$
We are trying to show that $\mathcal{T}$ is empty. If not, then because $|X|$ is Noetherian (Properties of Spaces, Lemma 24.2) we can choose a minimal element $T \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is T and for which the lemma does not hold.

If T is not irreducible, then we can write $T = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than T. Then we can apply Lemma 14.1 to get a short exact sequence of coherent sheaves
$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$
with $\mathrm{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of T each of $\mathcal{G}_i$ has $\mathcal{P}$. Hence $\mathcal{F}$ has property $\mathcal{P}$ by (1), a contradiction.

Suppose T is irreducible. Let $\mathcal{J}$ be the sheaf of ideals defining the reduced induced closed subspace structure on T, see Properties of Spaces, Lemma 12.3. By Lemma 13.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration
$$0 = \mathcal{J}^n\mathcal{F} \subset \mathcal{J}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{J}\mathcal{F} \subset \mathcal{F}$$
each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does by (1). In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

Assume T is irreducible and $\mathcal{J}\mathcal{F} = 0$ where $\mathcal{J}$ is as above. Denote $i : Z \to X$ the closed subspace corresponding to $\mathcal{J}$. Then $\mathcal{F} = i_*\mathcal{H}$ for some coherent $\mathcal{O}_Z$-module $\mathcal{H}$, see Morphisms of Spaces, Lemma 14.1 and Lemma 12.7. Let $\mathcal{G}$ be the coherent sheaf on Z satisfying (3)(a) and (3)(b). We apply Lemma 14.2 to get injective maps
$$\mathcal{I}_1^{\oplus r_1} \to \mathcal{H} \quad \text{and} \quad \mathcal{I}_2^{\oplus r_2} \to \mathcal{G}$$
where the support of the cokernels are proper closed in Z. Hence we find an nonempty open $V \subset Z$ such that
$$\mathcal{H}_V^{\oplus r_2} \cong \mathcal{G}_V^{\oplus r_1}$$
Let $\mathcal{I} \subset \mathcal{O}_Z$ be a quasi-coherent ideal sheaf cutting out $Z \setminus V$ we obtain (Lemma 13.4) a map
$$\mathcal{I}^n\mathcal{G}^{\oplus r_1} \longrightarrow \mathcal{H}^{\oplus r_2}$$

which is an isomorphism over V. The kernel is supported on $Z \setminus V$ hence annihilated by some power of $\mathcal{I}$, see Lemma 13.2. Thus after increasing n we may assume the displayed map is injective, see Lemma 13.3. Applying (3)(b) we find $\mathcal{G}' \subset \mathcal{I}^n\mathcal{G}$ such that

$$(i_*\mathcal{G}')^{\oplus r_1} \longrightarrow i_*\mathcal{H}^{\oplus r_2} = \mathcal{F}^{\oplus r_2}$$

is injective with cokernel supported in a proper closed subset of Z and such that property $\mathcal{P}$ holds for $i_*\mathcal{G}'$. By (1) property $\mathcal{P}$ holds for $(i_*\mathcal{G}')^{\oplus r_1}$. By (1) and minimality of $T = |Z|$ property $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r_2}$. And finally by (2) property $\mathcal{P}$ holds for $\mathcal{F}$ which is the desired contradiction. $\square$

Lemma 14.6. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

1. *For any short exact sequence of coherent sheaves on X if two out of three have property $\mathcal{P}$ so does the third.*
2. *If $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for $\mathcal{F}$.*
3. *For every reduced closed subspace $i : Z \to X$ with $|Z|$ irreducible there exists a coherent sheaf $\mathcal{G}$ on X whose scheme theoretic support is Z such that $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. We will show that conditions (1) and (2) of Lemma 14.4 hold. This is clear for condition (1). To show that (2) holds, let

$$\mathcal{T} = \left\{ \begin{matrix} i : Z \to X \text{ reduced closed subspace with } |Z| \text{ irreducible such} \\ \text{that } i_*\mathcal{I} \text{ does not have } \mathcal{P} \text{ for some quasi-coherent } \mathcal{I} \subset \mathcal{O}_Z \end{matrix} \right\}$$

If $\mathcal{T}$ is nonempty, then since X is Noetherian, we can find an $i : Z \to X$ which is minimal in $\mathcal{T}$. We will show that this leads to a contradiction.

Let $\mathcal{G}$ be the sheaf whose scheme theoretic support is Z whose existence is assumed in assumption (3). Let $\varphi : i_*\mathcal{I}^{\oplus r} \to \mathcal{G}$ be as in Lemma 14.2. Let

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \text{Coker}(\varphi)$$

be a filtration as in Lemma 14.3. By minimality of Z and assumption (1) we see that $\text{Coker}(\varphi)$ has property $\mathcal{P}$. As φ is injective we conclude using assumption (1) once more that $i_*\mathcal{I}^{\oplus r}$ has property $\mathcal{P}$. Using assumption (2) we conclude that $i_*\mathcal{I}$ has property $\mathcal{P}$.

Finally, if $\mathcal{J} \subset \mathcal{O}_Z$ is a second quasi-coherent sheaf of ideals, set $\mathcal{K} = \mathcal{I} \cap \mathcal{J}$ and consider the short exact sequences

$$0 \to \mathcal{K} \to \mathcal{I} \to \mathcal{I}/\mathcal{K} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{J} \to \mathcal{J}/\mathcal{K} \to 0$$

Arguing as above, using the minimality of Z, we see that $i_*\mathcal{I}/\mathcal{K}$ and $i_*\mathcal{J}/\mathcal{K}$ satisfy $\mathcal{P}$. Hence by assumption (1) we conclude that $i_*\mathcal{K}$ and then $i_*\mathcal{J}$ satisfy $\mathcal{P}$. In other words, Z is not an element of $\mathcal{T}$ which is the desired contradiction. $\square$

15. Limits of coherent modules

A colimit of coherent modules (on a locally Noetherian algebraic space) is typically not coherent. But it is quasi-coherent as any colimit of quasi-coherent modules on an algebraic space is quasi-coherent, see Properties of Spaces, Lemma 29.7. Conversely, if the algebraic space is Noetherian, then every quasi-coherent module is a filtered colimit of coherent modules.

Lemma 15.1. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Every quasi-coherent $\mathcal{O}_X$-module is the filtered colimit of its coherent submodules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another coherent $\mathcal{O}_X$-submodule which contains both of them (see Lemmas 12.3 and 12.4). In this way we see that the system is directed. Hence it now suffices to show that $\mathcal{F}$ can be written as a filtered colimit of coherent modules, as then we can take the images of these modules in $\mathcal{F}$ to conclude there are enough of them.

Let U be an affine scheme and $U \to X$ a surjective étale morphism. Set $R = U \times_X U$ so that $X = U/R$ as usual. By Properties of Spaces, Proposition 32.1 we see that $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$. Hence we reduce to showing the corresponding thing for $QCoh(U, R, s, t, c)$. Thus the result follows from the more general Groupoids, Lemma 15.4. $\square$

Lemma 15.2. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S with Y Noetherian. Then every quasi-coherent $\mathcal{O}_X$-module is a filtered colimit of finitely presented $\mathcal{O}_X$-modules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Write $f_*\mathcal{F} = \operatorname{colim} \mathcal{H}_i$ with $\mathcal{H}_i$ a coherent $\mathcal{O}_Y$-module, see Lemma 15.1. By Lemma 12.2 the modules $\mathcal{H}_i$ are $\mathcal{O}_Y$-modules of finite presentation. Hence $f^*\mathcal{H}_i$ is an $\mathcal{O}_X$-module of finite presentation, see Properties of Spaces, Section 30. We claim the map

$$\operatorname{colim} f^*\mathcal{H}_i = f^*f_*\mathcal{F} \to \mathcal{F}$$

is surjective as f is assumed affine, Namely, choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = X \times_Y V$. Then U is a scheme, $f' : U \to V$ is affine, and $U \to X$ is surjective étale. By Properties of Spaces, Lemma 26.2 we see that $f'_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ and similarly for pullbacks. Thus the restriction of $f^*f_*\mathcal{F} \to \mathcal{F}$ to U is the map

$$f^*f_*\mathcal{F}|_U = (f')^*(f_*\mathcal{F})|_V) = (f')^*f'_*(\mathcal{F}|_U) \to \mathcal{F}|_U$$

which is surjective as f' is an affine morphism of schemes. Hence the claim holds.

We conclude that every quasi-coherent module on X is a quotient of a filtered colimit of finitely presented modules. In particular, we see that $\mathcal{F}$ is a cokernel of a map

$$\operatorname{colim}_{j \in J} \mathcal{G}_j \longrightarrow \operatorname{colim}_{i \in I} \mathcal{H}_i$$

with $\mathcal{G}_j$ and $\mathcal{H}_i$ finitely presented. Note that for every $j \in I$ there exist $i \in I$ and a morphism $\alpha : \mathcal{G}_j \to \mathcal{H}_i$ such that

$$\begin{tikzcd}
\mathcal{G}_j \arrow[r, "\alpha"'] \arrow[d] & \mathcal{H}_i \arrow[d] \\
\operatorname{colim}_{j \in J} \mathcal{G}_j \arrow[r] & \operatorname{colim}_{i \in I} \mathcal{H}_i
\end{tikzcd}$$

commutes, see Lemma 5.3. In this situation $\operatorname{Coker}(\alpha)$ is a finitely presented $\mathcal{O}_X$-module which comes endowed with a map $\operatorname{Coker}(\alpha) \to \mathcal{F}$. Consider the set K of triples (i, j, α) as above. We say that $(i, j, \alpha) \leq (i', j', \alpha')$ if and only if $i \leq i'$, $j \leq j'$, and the diagram

$$\begin{tikzcd} \mathcal{G}_j \arrow[r, "\alpha"'] \arrow[d] & \mathcal{H}_i \arrow[d] \\ \mathcal{G}_{j'} \arrow[r, "\alpha'"] & \mathcal{H}_{i'} \end{tikzcd}$$

commutes. It follows from the above that K is a directed partially ordered set,

$$\mathcal{F} = \operatorname{colim}_{(i,j,\alpha) \in K} \operatorname{Coker}(\alpha),$$

and we win. $\square$

16. Vanishing of cohomology

In this section we show that a quasi-compact and quasi-separated algebraic space is affine if it has vanishing higher cohomology for all quasi-coherent sheaves. We do this in a sequence of lemmas all of which will become obsolete once we prove Proposition 16.7.

Situation 16.1. Here S is a scheme and X is a quasi-compact and quasi-separated algebraic space over S with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. We set $A = \Gamma(X, \mathcal{O}_X)$.

We would like to show that the canonical morphism

$$p : X \longrightarrow \operatorname{Spec}(A)$$

(see Properties of Spaces, Lemma 33.1) is an isomorphism. If M is an A-module we denote $M \otimes_A \mathcal{O}_X$ the quasi-coherent module $p^*\tilde{M}$.

Lemma 16.2. *In Situation 16.1 for an A-module M we have $p_*(M \otimes_A \mathcal{O}_X) = \widetilde{M}$ and $\Gamma(X, M \otimes_A \mathcal{O}_X) = M$.*

Proof. The equality $p_*(M \otimes_A \mathcal{O}_X) = \widetilde{M}$ follows from the equality $\Gamma(X, M \otimes_A \mathcal{O}_X) = M$ as $p_*(M \otimes_A \mathcal{O}_X)$ is a quasi-coherent module on $\operatorname{Spec}(A)$ by Morphisms of Spaces, Lemma 11.2. Observe that $\Gamma(X, \bigoplus_{i \in I} \mathcal{O}_X) = \bigoplus_{i \in I} A$ by Lemma 5.1. Hence the lemma holds for free modules. Choose a short exact sequence $F_1 \to F_0 \to M$ where F_0, F_1 are free A-modules. Since $H^1(X, -)$ is zero the global sections functor is right exact. Moreover the pullback p^* is right exact as well. Hence we see that

$$\Gamma(X, F_1 \otimes_A \mathcal{O}_X) \to \Gamma(X, F_0 \otimes_A \mathcal{O}_X) \to \Gamma(X, M \otimes_A \mathcal{O}_X) \to 0$$

is exact. The result follows. $\square$

The following lemma shows that Situation 16.1 is preserved by base change of $X \to \operatorname{Spec}(A)$ by $\operatorname{Spec}(A') \to \operatorname{Spec}(A)$.

Lemma 16.3. *In Situation 16.1.*

1. *Given an affine morphism $X' \to X$ of algebraic spaces, we have $H^1(X', \mathcal{F}') = 0$ for every quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$.*
2. *Given an A-algebra A' setting $X' = X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A')$ the morphism $X' \to X$ is affine and $\Gamma(X', \mathcal{O}_{X'}) = A'$.*

Proof. Part (1) follows from Lemma 8.2 and the Leray spectral sequence (Cohomology on Sites, Lemma 14.5). Let $A \to A'$ be as in (2). Then $X' \to X$ is affine because affine morphisms are preserved under base change (Morphisms of Spaces, Lemma 20.5) and the fact that a morphism of affine schemes is affine. The equality $\Gamma(X', \mathcal{O}_{X'}) = A'$ follows as $(X' \to X)_*\mathcal{O}_{X'} = A' \otimes_A \mathcal{O}_X$ by Lemma 11.1 and thus

$$\Gamma(X', \mathcal{O}_{X'}) = \Gamma(X, (X' \to X)_*\mathcal{O}_{X'}) = \Gamma(X, A' \otimes_A \mathcal{O}_X) = A'$$

by Lemma 16.2. □

Lemma 16.4. *In Situation 16.1. Let $Z_0, Z_1 \subset |X|$ be disjoint closed subsets. Then there exists an $a \in A$ such that $Z_0 \subset V(a)$ and $Z_1 \subset V(a-1)$.*

Proof. We may and do endow Z_0, Z_1 with the reduced induced subspace structure (Properties of Spaces, Definition 12.5) and we denote $i_0 : Z_0 \to X$ and $i_1 : Z_1 \to X$ the corresponding closed immersions. Since $Z_0 \cap Z_1 = \emptyset$ we see that the canonical map of quasi-coherent $\mathcal{O}_X$-modules

$$\mathcal{O}_X \longrightarrow i_{0,*}\mathcal{O}_{Z_0} \oplus i_{1,*}\mathcal{O}_{Z_1}$$

is surjective (look at stalks at geometric points). Since $H^1(X, -)$ is zero on the kernel of this map the induced map of global sections is surjective. Thus we can find $a \in A$ which maps to the global section $(0, 1)$ of the right hand side. □

Lemma 16.5. *In Situation 16.1 the morphism $p : X \to \mathrm{Spec}(A)$ is universally injective.*

Proof. Let $A \to k$ be a ring homomorphism where k is a field. It suffices to show that $\mathrm{Spec}(k) \times_{\mathrm{Spec}(A)} X$ has at most one point (see Morphisms of Spaces, Lemma 19.6). Using Lemma 16.3 we may assume that A is a field and we have to show that $|X|$ has at most one point.

Let's think of X as an algebraic space over $\mathrm{Spec}(k)$ and let's use the notation $X(K)$ to denote K-valued points of X for any extension K/k, see Morphisms of Spaces, Section 24. If K/k is an algebraically closed field extension of large transcendence degree, then we see that $X(K) \to |X|$ is surjective, see Morphisms of Spaces, Lemma 24.2. Hence, after replacing k by K, we see that it suffices to prove that $X(k)$ is a singleton (in the case $A = k$).

Let $x, x' \in X(k)$. By Decent Spaces, Lemma 14.4 we see that x and x' are closed points of $|X|$. Hence x and x' map to distinct points of $\mathrm{Spec}(k)$ if $x \neq x'$ by Lemma 16.4. We conclude that $x = x'$ as desired. □

Lemma 16.6. *In Situation 16.1 the morphism $p : X \to \mathrm{Spec}(A)$ is separated.*

Proof. By Decent Spaces, Lemma 9.2 we can find a scheme Y and a surjective integral morphism $Y \to X$. Since an integral morphism is affine, we can apply Lemma 16.3 to see that $H^1(Y, \mathcal{G}) = 0$ for every quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$. Since $Y \to X$ is quasi-compact and X is quasi-compact, we see that Y is quasi-compact. Since Y is a scheme, we may apply Cohomology of Schemes, Lemma 3.1 to see that Y is affine. Hence Y is separated. Note that an integral morphism is affine and universally closed, see Morphisms of Spaces, Lemma 45.7. By Morphisms of Spaces, Lemma 9.8 we see that X is a separated algebraic space. □

Proposition 16.7. *A quasi-compact and quasi-separated algebraic space is affine if and only if all higher cohomology groups of quasi-coherent sheaves vanish. More precisely, any algebraic space as in Situation 16.1 is an affine scheme.*

Proof. Choose an affine scheme $U = \text{Spec}(B)$ and a surjective étale morphism $\varphi : U \to X$. Set $R = U \times_X U$. As p is separated (Lemma 16.6) we see that R is a closed subscheme of $U \times_{\text{Spec}(A)} U = \text{Spec}(B \otimes_A B)$. Hence $R = \text{Spec}(C)$ is affine too and the ring map

$$B \otimes_A B \longrightarrow C$$

is surjective. Let us denote the two maps $s, t : B \to C$ as usual. Pick $g_1, \ldots, g_m \in B$ such that $s(g_1), \ldots, s(g_m)$ generate C over $t : B \to C$ (which is possible as $t : B \to C$ is of finite presentation and the displayed map is surjective). Then $g_1, \ldots, g_m$ give global sections of $\varphi_*\mathcal{O}_U$ and the map

$$\mathcal{O}_X[z_1, \ldots, z_n] \longrightarrow \varphi_*\mathcal{O}_U, \quad z_j \longmapsto g_j$$

is surjective: you can check this by restricting to U. Namely, $\varphi^*\varphi_*\mathcal{O}_U = t_*\mathcal{O}_R$ (by Lemma 11.2) hence you get exactly the condition that $s(g_i)$ generate C over $t : B \to C$. By the vanishing of H^1 of the kernel we see that

$$\Gamma(X, \mathcal{O}_X[x_1, \ldots, x_n]) = A[x_1, \ldots, x_n] \longrightarrow \Gamma(X, \varphi_*\mathcal{O}_U) = \Gamma(U, \mathcal{O}_U) = B$$

is surjective. Thus we conclude that B is a finite type A-algebra. Hence $X \to \text{Spec}(A)$ is of finite type and separated. By Lemma 16.5 and Morphisms of Spaces, Lemma 27.5 it is also locally quasi-finite. Hence $X \to \text{Spec}(A)$ is representable by Morphisms of Spaces, Lemma 51.1 and X is a scheme. Finally X is affine, hence equal to $\text{Spec}(A)$, by an application of Cohomology of Schemes, Lemma 3.1. □

Lemma 16.8. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Assume that for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. Then X is an affine scheme.*

Proof. The assumption implies that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ by Lemmas 15.1 and 5.1. Then X is affine by Proposition 16.7. □

Lemma 16.9. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n \geq 1$ such that $H^1(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$. Then X is a scheme and $\mathcal{L}$ is ample on X.*

Proof. Let $s \in H^0(X, \mathcal{L}^{\otimes d})$ be a global section. Let $U \subset X$ be the open subspace over which s is a generator of $\mathcal{L}^{\otimes d}$. In particular we have $\mathcal{L}^{\otimes d}|_U \cong \mathcal{O}_U$. We claim that U is affine.

Proof of the claim. We will show that $H^1(U, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$. This will prove the claim by Proposition 16.7. Denote $j : U \to X$ the inclusion morphism. Since étale locally the morphism j is affine (by Morphisms, Lemma 11.10) we see that j is affine (Morphisms of Spaces, Lemma 20.3). Hence we have

$$H^1(U, \mathcal{F}) = H^1(X, j_*\mathcal{F})$$

by Lemma 8.2 (and Cohomology on Sites, Lemma 14.6). Write $j_*\mathcal{F} = \text{colim}\, \mathcal{F}_i$ as a filtered colimit of coherent $\mathcal{O}_X$-modules, see Lemma 15.1. Then

$$H^1(X, j_*\mathcal{F}) = \text{colim}\, H^1(X, \mathcal{F}_i)$$

by Lemma 5.1. Thus it suffices to show that $H^1(X, \mathcal{F}_i)$ maps to zero in $H^1(U, j^*\mathcal{F}_i)$. By assumption there exists an $n \geq 1$ such that

$$H^1(X, \mathcal{F}_i \otimes_{\mathcal{O}_X} (\mathcal{O}_X \oplus \mathcal{L} \oplus \ldots \oplus \mathcal{L}^{\otimes d-1}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$$

Hence there exists an $a \geq 0$ such that $H^1(X, \mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad}) = 0$. On the other hand, the map

$$s^a : \mathcal{F}_i \longrightarrow \mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad}$$

is an isomorphism after restriction to U. Contemplating the commutative diagram

$$\begin{tikzcd}
H^1(X, \mathcal{F}_i) \arrow[r] \arrow[d, "s^a"'] & H^1(U, j^*\mathcal{F}_i) \arrow[d, "\cong"] \\
H^1(X, \mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad}) \arrow[r] & H^1(U, j^*(\mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes ad}))
\end{tikzcd}$$

we conclude that the map $H^1(X, \mathcal{F}_i) \to H^1(U, j^*\mathcal{F}_i)$ is zero and the claim holds.

Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 14.6 we can represent x by a closed immersion $i : \mathrm{Spec}(k) \to X$ (this also uses that a quasi-separated algebraic space is decent, see Decent Spaces, Section 6). Thus $\mathcal{O}_X \to i_*\mathcal{O}_{\mathrm{Spec}(k)}$ is surjective. Let $\mathcal{I} \subset \mathcal{O}_X$ be the kernel and choose $d \geq 1$ such that $H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$. Then

$$H^0(X, \mathcal{L}^{\otimes d}) \to H^0(X, i_*\mathcal{O}_{\mathrm{Spec}(k)} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = H^0(\mathrm{Spec}(k), i^*\mathcal{L}^{\otimes d}) \cong k$$

is surjective by the long exact cohomology sequence. Hence there exists an $s \in H^0(X, \mathcal{L}^{\otimes d})$ such that $x \in U$ where U is the open subspace corresponding to s as above. Thus x is in the schematic locus (see Properties of Spaces, Lemma 13.1) of X by our claim.

To conclude that X is a scheme, it suffices to show that any open subset of $|X|$ which contains all the closed points is equal to $|X|$. This follows from the fact that $|X|$ is a Noetherian topological space, see Properties of Spaces, Lemma 24.3. Finally, if X is a scheme, then we can apply Cohomology of Schemes, Lemma 3.3 to conclude that $\mathcal{L}$ is ample. $\square$

17. Finite morphisms and affines

This section is the analogue of Cohomology of Schemes, Section 13.

Lemma 17.1. *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume f is finite, surjective and X locally Noetherian. Let $i : Z \to X$ be a closed immersion. Denote $i' : Z' \to Y$ the inverse image of Z (Morphisms of Spaces, Section 13) and $f' : Z' \to Z$ the induced morphism. Then $\mathcal{G} = f'_*\mathcal{O}_{Z'}$ is a coherent $\mathcal{O}_Z$-module whose support is Z.*

Proof. Observe that f' is the base change of f and hence is finite and surjective by Morphisms of Spaces, Lemmas 5.5 and 45.5. Note that Y, Z, and Z' are locally Noetherian by Morphisms of Spaces, Lemma 23.5 (and the fact that closed immersions and finite morphisms are of finite type). By Lemma 12.9 we see that $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module. The support of $\mathcal{G}$ is closed in $|Z|$, see Morphisms of Spaces, Lemma 15.2. Hence if the support of $\mathcal{G}$ is not equal to $|Z|$, then after replacing X by an open subspace we may assume $\mathcal{G} = 0$ but $Z \neq \emptyset$. This would mean that $f'_*\mathcal{O}_{Z'} = 0$. In particular the section $1 \in \Gamma(Z', \mathcal{O}_{Z'}) = \Gamma(Z, f'_*\mathcal{O}_{Z'})$

would be zero which would imply $Z' = \emptyset$ is the empty algebraic space. This is impossible as $Z' \to Z$ is surjective. $\square$

Lemma 17.2. *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals on X. If f is affine then $\mathcal{I}f_*\mathcal{F} = f_*(f^{-1}\mathcal{I}\mathcal{F})$ (with notation as explained in the proof).*

Proof. The notation means the following. Since f^{-1} is an exact functor we see that $f^{-1}\mathcal{I}$ is a sheaf of ideals of $f^{-1}\mathcal{O}_X$. Via the map $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ on $Y_{\acute{e}tale}$ this acts on $\mathcal{F}$. Then $f^{-1}\mathcal{I}\mathcal{F}$ is the subsheaf generated by sums of local sections of the form as where a is a local section of $f^{-1}\mathcal{I}$ and s is a local section of $\mathcal{F}$. It is a quasi-coherent $\mathcal{O}_Y$-submodule of $\mathcal{F}$ because it is also the image of a natural map $f^*\mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{F} \to \mathcal{F}$.

Having said this the proof is straightforward. Namely, the question is étale local on X and hence we may assume X is an affine scheme. In this case the result is a consequence of the corresponding result for schemes, see Cohomology of Schemes, Lemma 13.2. $\square$

Lemma 17.3. *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume*

1. *f finite,*
2. *f surjective,*
3. *Y affine, and*
4. *X Noetherian.*

Then X is affine.

Proof. We will prove that under the assumptions of the lemma for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. This implies that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ by Lemmas 15.1 and 5.1. Then it follows that X is affine from Proposition 16.7.

Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow H^1(X, \mathcal{F}) = 0.$$

We are going to apply Lemma 14.5. Thus we have to verify (1), (2) and (3) of that lemma for $\mathcal{P}$. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves. Property (2) follows since $H^1(X, -)$ is an additive functor. To see (3) let $i : Z \to X$ be a reduced closed subspace with $|Z|$ irreducible. Let $i' : Z' \to Y$ and $f' : Z' \to Z$ be as in Lemma 17.1 and set $\mathcal{G} = f'_*\mathcal{O}_{Z'}$. We claim that $\mathcal{G}$ satisfies properties (3)(a) and (3)(b) of Lemma 14.5 which will finish the proof. Property (3)(a) we have seen in Lemma 17.1. To see (3)(b) let $\mathcal{I}$ be a nonzero quasi-coherent sheaf of ideals on Z. Denote $\mathcal{I}' \subset \mathcal{O}_{Z'}$ the quasi-coherent ideal $(f')^{-1}\mathcal{I}\mathcal{O}_{Z'}$, i.e., the image of $(f')^*\mathcal{I} \to \mathcal{O}_{Z'}$. By Lemma 17.2 we have $f_*\mathcal{I}' = \mathcal{I}\mathcal{G}$. We claim the common value $\mathcal{G}' = \mathcal{I}\mathcal{G} = f'_*\mathcal{I}'$ satisfies the condition expressed in (3)(b). First, it is clear that the support of $\mathcal{G}/\mathcal{G}'$ is contained in the support of $\mathcal{O}_Z/\mathcal{I}$ which is a proper subspace of $|Z|$ as $\mathcal{I}$ is a nonzero ideal sheaf on the reduced and irreducible algebraic space Z. The morphism f' is affine, hence $R^1f'_*\mathcal{I}' = 0$ by Lemma 8.2. As Z' is affine (as a closed subscheme of an affine scheme) we have $H^1(Z', \mathcal{I}') = 0$. Hence the Leray spectral sequence (in the form Cohomology on Sites, Lemma 14.6) implies that $H^1(Z, f'_*\mathcal{I}') = 0$. Since $i : Z \to X$

is affine we conclude that $R^1 i_* f'_* \mathcal{I}' = 0$ hence $H^1(X, i_* f'_* \mathcal{I}') = 0$ by Leray again. In other words, we have $H^1(X, i_* \mathcal{G}') = 0$ as desired. $\square$

18. A weak version of Chow's lemma

In this section we quickly prove the following lemma in order to help us prove the basic results on cohomology of coherent modules on proper algebraic spaces.

Lemma 18.1. *Let A be a ring. Let X be an algebraic space over* $\text{Spec}(A)$ *whose structure morphism $X \to$* $\text{Spec}(A)$ *is separated of finite type. Then there exists a proper surjective morphism $X' \to X$ where X' is a scheme which is H-quasi-projective over* $\text{Spec}(A)$.

Proof. Let W be an affine scheme and let $f : W \to X$ be a surjective étale morphism. There exists an integer d such that all geometric fibres of f have $\leq d$ points (because X is a separated algebraic hence reasonable, see Decent Spaces, Lemma 5.1). Picking d minimal we get a nonempty open $U \subset X$ such that $f^{-1}(U) \to U$ is finite étale of degree d, see Decent Spaces, Lemma 8.1. Let

$$V \subset W \times_X W \times_X \ldots \times_X W$$

(d factors in the fibre product) be the complement of all the diagonals. Because $W \to X$ is separated the diagonal $W \to W \times_X W$ is a closed immersion. Since $W \to X$ is étale the diagonal $W \to W \times_X W$ is an open immersion, see Morphisms of Spaces, Lemmas 39.10 and 38.9. Hence the diagonals are open and closed subschemes of the quasi-compact scheme $W \times_X \ldots \times_X W$. In particular we conclude V is a quasi-compact scheme. Choose an open immersion $W \subset Y$ with Y H-projective over A (this is possible as W is affine and of finite type over A; for example we can use Morphisms, Lemmas 39.2 and 43.11). Let

$$Z \subset Y \times_A Y \times_A \ldots \times_A Y$$

be the scheme theoretic image of the composition $V \to W \times_X \ldots \times_X W \to Y \times_A \ldots \times_A Y$. Observe that this morphism is quasi-compact since V is quasi-compact and $Y \times_A \ldots \times_A Y$ is separated. Note that $V \to Z$ is an open immersion as $V \to Y \times_A \ldots \times_A Y$ is an immersion, see Morphisms, Lemma 7.7. The projection morphisms give d morphisms $g_i : Z \to Y$. These morphisms g_i are projective as Y is projective over A, see material in Morphisms, Section 43. We set

$$X' = \bigcup g_i^{-1}(W) \subset Z$$

There is a morphism $X' \to X$ whose restriction to $g_i^{-1}(W)$ is the composition $g_i^{-1}(W) \to W \to X$. Namely, these morphisms agree over V hence agree over $g_i^{-1}(W) \cap g_j^{-1}(W)$ by Morphisms of Spaces, Lemma 17.8. Claim: the morphism $X' \to X$ is proper.

If the claim holds, then the lemma follows by induction on d. Namely, by construction X' is H-quasi-projective over $\text{Spec}(A)$. The image of $X' \to X$ contains the open U as V surjects onto U. Denote T the reduced induced algebraic space structure on $X \setminus U$. Then $T \times_X W$ is a closed subscheme of W, hence affine. Moreover, the morphism $T \times_X W \to T$ is étale and every geometric fibre has $< d$ points. By induction hypothesis there exists a proper surjective morphism $T' \to T$ where T' is a scheme H-quasi-projective over $\text{Spec}(A)$. Since T is a closed subspace of X we see that $T' \to X$ is a proper morphism. Thus the lemma follows by taking the proper surjective morphism $X' \amalg T' \to X$.

Proof of the claim. By construction the morphism $X' \to X$ is separated and of finite type. We will check conditions (1) – (4) of Morphisms of Spaces, Lemma 42.5 for the morphisms $V \to X'$ and $X' \to X$. Conditions (1) and (2) we have seen above. Condition (3) holds as $X' \to X$ is separated (as a morphism whose source is a separated algebraic space). Thus it suffices to check liftability to X' for diagrams

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & V \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & X \end{tikzcd}$$

where R is a valuation ring with fraction field K. Note that the top horizontal map is given by d pairwise distinct K-valued points $w_1, \ldots, w_d$ of W. In fact, this is a complete set of inverse images of the point $x \in X(K)$ coming from the diagram. Since $W \to X$ is surjective, we can, after possibly replacing R by an extension of valuation rings, lift the morphism $\mathrm{Spec}(R) \to X$ to a morphism $w : \mathrm{Spec}(R) \to W$, see Morphisms of Spaces, Lemma 42.4. Since $w_1, \ldots, w_d$ is a complete collection of inverse images of x we see that $w|_{\mathrm{Spec}(K)}$ is equal to one of them, say w_i. Thus we see that we get a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & Z \arrow[d, "g_i"] \\ \mathrm{Spec}(R) \arrow[r, "w"] & Y \end{tikzcd}$$

By the valuative criterion of properness for the projective morphism g_i we can lift w to $z : \mathrm{Spec}(R) \to Z$, see Morphisms, Lemma 43.5 and Schemes, Proposition 20.6. The image of z is in $g_i^{-1}(W) \subset X'$ and the proof is complete. $\square$

19. Noetherian valuative criterion

We prove a version of the valuative criterion for properness using discrete valuation rings. More precise (and therefore more technical) versions can be found in Limits of Spaces, Section 21.

Lemma 19.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

1. *Y is locally Noetherian,*
2. *f is locally of finite type and quasi-separated,*
3. *for every commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where A is a discrete valuation ring and K its fraction field, there is at most one dotted arrow making the diagram commute.

Then f is separated.

Proof. We have to show that the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. We already know Δ is representable, separated, a monomorphism, and locally of finite type, see Morphisms of Spaces, Lemma 4.1. Choose an affine scheme U and an étale morphism $U \to X \times_Y X$. Set $V = X \times_{\Delta, X \times_Y X} U$. It suffices to show that $V \to U$ is a closed immersion (Morphisms of Spaces, Lemma 12.1). Since $X \times_Y X$ is locally of finite type over Y we see that U is Noetherian (use Morphisms of Spaces, Lemmas 23.2, 23.3, and 23.5). Note that V is a scheme as Δ is representable. Also, V is quasi-compact because f is quasi-separated. Hence $V \to U$ is of finite type. Consider a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & V \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & U \end{tikzcd}$$

of morphisms of schemes where A is a discrete valuation ring with fraction field K. We can interpret the composition $\mathrm{Spec}(A) \to U \to X \times_Y X$ as a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ agreeing as morphisms into Y and equal when restricted to $\mathrm{Spec}(K)$. Hence our assumption (3) guarantees $a = b$ and we find the dotted arrow in the diagram. By Limits, Lemma 15.3 we conclude that $V \to U$ is proper. In other words, Δ is proper. Since Δ is a monomorphism, we find that Δ is a closed immersion (Étale Morphisms, Lemma 7.2) as desired. $\square$

Lemma 19.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *Y is locally Noetherian,*
(2) *f is of finite type and quasi-separated,*
(3) *for every commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a discrete valuation ring and K its fraction field, there is a unique dotted arrow making the diagram commute.

Then f is proper.

Proof. It suffices to prove f is universally closed because f is separated by Lemma 19.1. To do this we may work étale locally on Y (Morphisms of Spaces, Lemma 9.5). Hence we may assume $Y = \mathrm{Spec}(A)$ is a Noetherian affine scheme. Choose $X' \to X$ as in the weak form of Chow's lemma (Lemma 18.1). We claim that $X' \to \mathrm{Spec}(A)$ is universally closed. The claim implies the lemma by Morphisms of Spaces, Lemma 40.7. To prove this, according to Limits, Lemma 15.4 it suffices to prove that in every solid commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X' \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[ru, dashed, "a"] \arrow[rru, dashed, "b"'] & & Y \end{tikzcd}$$

where A is a dvr with fraction field K we can find the dotted arrow a. By assumption we can find the dotted arrow b. Then the morphism $X' \times_{X,b} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is a proper morphism of schemes and by the valuative criterion for morphisms of schemes we can lift b to the desired morphism a. ☐

Remark 19.3 (Variant for complete discrete valuation rings)**.** In Lemmas 19.1 and 19.2 it suffices to consider complete discrete valuation rings. To be precise in Lemma 19.1 we can replace condition (3) by the following condition: Given any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a complete discrete valuation ring with fraction field K there exists at most one dotted arrow making the diagram commute. Namely, given any diagram as in Lemma 19.1 (3) the completion $A^\wedge$ is a discrete valuation ring (More on Algebra, Lemma 43.5) and the uniqueness of the arrow $\mathrm{Spec}(A^\wedge) \to X$ implies the uniqueness of the arrow $\mathrm{Spec}(A) \to X$ for example by Properties of Spaces, Proposition 17.1. Similarly in Lemma 19.2 we can replace condition (3) by the following condition: Given any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

where A is a complete discrete valuation ring with fraction field K there exists an extension $A \subset A'$ of complete discrete valuation rings inducing a fraction field extension $K \subset K'$ such that there exists a unique arrow $\mathrm{Spec}(A') \to X$ making the diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] \arrow[rru] & \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

commute. Namely, given any diagram as in Lemma 19.2 part (3) the existence of any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(L) \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(B) \arrow[r] \arrow[rru] & \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

for *any* extension $A \subset B$ of discrete valuation rings will imply there exists an arrow $\mathrm{Spec}(A) \to X$ fitting into the diagram. This was shown in Morphisms of Spaces, Lemma 41.4. In fact, it follows from these considerations that it suffices to look for dotted arrows in diagrams for any class of discrete valuation rings such that, given any discrete valuation ring, there is an extension of it that is in the class. For example, we could take complete discrete valuation rings with algebraically closed residue field.

20. Higher direct images of coherent sheaves

In this section we prove the fundamental fact that the higher direct images of a coherent sheaf under a proper morphism are coherent. First we prove a helper lemma.

Lemma 20.1. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[dr, "f"'] & \mathbf{P}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

of algebraic spaces over S. Assume i is a closed immersion and Y Noetherian. Set $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_Y}(1)$. Let $\mathcal{F}$ be a coherent module on X. Then there exists an integer d_0 such that for all $d \geq d_0$ we have $R^pf_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$.

Proof. Checking whether $R^pf_*(\mathcal{F}\otimes\mathcal{L}^{\otimes d})$ is zero can be done étale locally on Y, see Equation (3.0.1). Hence we may assume Y is the spectrum of a Noetherian ring. In this case X is a scheme and the result follows from Cohomology of Schemes, Lemma 16.2. $\square$

Lemma 20.2. *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y locally Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^if_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module for all $i \geq 0$.*

Proof. We first remark that X is a locally Noetherian algebraic space by Morphisms of Spaces, Lemma 23.5. Hence the statement of the lemma makes sense. Moreover, computing $R^if_*\mathcal{F}$ commutes with étale localization on Y (Properties of Spaces, Lemma 26.2) and checking whether $R^if_*\mathcal{F}$ coherent can be done étale locally on Y (Lemma 12.2). Hence we may assume that $Y = \mathrm{Spec}(A)$ is a Noetherian affine scheme.

Assume $Y = \mathrm{Spec}(A)$ is an affine scheme. Note that f is locally of finite presentation (Morphisms of Spaces, Lemma 28.7). Thus it is of finite presentation, hence X is Noetherian (Morphisms of Spaces, Lemma 28.6). Thus Lemma 14.6 applies to the category of coherent modules of X. For a coherent sheaf $\mathcal{F}$ on X we say $\mathcal{P}$ holds if and only if $R^if_*\mathcal{F}$ is a coherent module on $\mathrm{Spec}(A)$. We will show that conditions (1), (2), and (3) of Lemma 14.6 hold for this property thereby finishing the proof of the lemma.

Verification of condition (1). Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of higher direct images

$$R^{p-1}f_*\mathcal{F}_3 \to R^pf_*\mathcal{F}_1 \to R^pf_*\mathcal{F}_2 \to R^pf_*\mathcal{F}_3 \to R^{p+1}f_*\mathcal{F}_1$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then the higher direct images of the third are sandwiched in this exact complex between two coherent sheaves. Hence these higher direct images are also coherent by Lemmas 12.3 and 12.4. Hence property $\mathcal{P}$ holds for the third as well.

Verification of condition (2). This follows immediately from the fact that $R^if_*(\mathcal{F}_1 \oplus \mathcal{F}_2) = R^if_*\mathcal{F}_1 \oplus R^if_*\mathcal{F}_2$ and that a summand of a coherent module is coherent (see lemmas cited above).

Verification of condition (3). Let $i : Z \to X$ be a closed immersion with Z reduced and $|Z|$ irreducible. Set $g = f \circ i : Z \to \text{Spec}(A)$. Let $\mathcal{G}$ be a coherent module on Z whose scheme theoretic support is equal to Z such that $R^pg_*\mathcal{G}$ is coherent for all p. Then $\mathcal{F} = i_*\mathcal{G}$ is a coherent module on X whose scheme theoretic support is Z such that $R^pf_*\mathcal{F} = R^pg_*\mathcal{G}$. To see this use the Leray spectral sequence (Cohomology on Sites, Lemma 14.7) and the fact that $R^qi_*\mathcal{G} = 0$ for $q > 0$ by Lemma 8.2 and the fact that a closed immersion is affine. (Morphisms of Spaces, Lemma 20.6). Thus we reduce to finding a coherent sheaf $\mathcal{G}$ on Z with support equal to Z such that $R^pg_*\mathcal{G}$ is coherent for all p.

We apply Lemma 18.1 to the morphism $Z \to \text{Spec}(A)$. Thus we get a diagram

$$\begin{tikzcd} Z \arrow[dr, "g"'] & Z' \arrow[l, "\pi"'] \arrow[r, "i"] \arrow[d, "g'"] & \mathbf{P}^n_A \arrow[dl] \\ & \text{Spec}(A) & \end{tikzcd}$$

with $\pi : Z' \to Z$ proper surjective and i an immersion. Since $Z \to \text{Spec}(A)$ is proper we conclude that g' is proper (Morphisms of Spaces, Lemma 40.4). Hence i is a closed immersion (Morphisms of Spaces, Lemmas 40.6 and 12.3). It follows that the morphism $i' = (i, \pi) : \mathbf{P}^n_A \times_{\text{Spec}(A)} Z' = \mathbf{P}^n_Z$ is a closed immersion (Morphisms of Spaces, Lemma 4.6). Set

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_A}(1) = (i')^*\mathcal{O}_{\mathbf{P}^n_Z}(1)$$

We may apply Lemma 20.1 to $\mathcal{L}$ and π as well as $\mathcal{L}$ and g'. Hence for all $d \gg 0$ we have $R^p\pi_*\mathcal{L}^{\otimes d} = 0$ for all $p > 0$ and $R^p(g')_*\mathcal{L}^{\otimes d} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes d}$. By the Leray spectral sequence (Cohomology on Sites, Lemma 14.7) we have

$$E_2^{p,q} = R^pg_*R^q\pi_*\mathcal{L}^{\otimes d} \Rightarrow R^{p+q}(g')_*\mathcal{L}^{\otimes d}$$

and by choice of d the only nonzero terms in $E_2^{p,q}$ are those with $q = 0$ and the only nonzero terms of $R^{p+q}(g')_*\mathcal{L}^{\otimes d}$ are those with $p = q = 0$. This implies that $R^pg_*\mathcal{G} = 0$ for $p > 0$ and that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes d}$. Applying Cohomology of Schemes, Lemma 16.3 we see that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes d}$ is coherent.

We still have to check that the support of $\mathcal{G}$ is Z. This follows from the fact that $\mathcal{L}^{\otimes d}$ has lots of global sections. We spell it out here. Note that $\mathcal{L}^{\otimes d}$ is globally generated for all $d \geq 0$ because the same is true for $\mathcal{O}_{\mathbf{P}^n}(d)$. Pick a point $z \in Z'$ mapping to the generic point ξ of Z which we can do as π is surjective. (Observe that Z does indeed have a generic point as $|Z|$ is irreducible and Z is Noetherian, hence quasi-separated, hence $|Z|$ is a sober topological space by Properties of Spaces, Lemma 15.1.) Pick $s \in \Gamma(Z', \mathcal{L}^{\otimes d})$ which does not vanish at z. Since $\Gamma(Z, \mathcal{G}) = \Gamma(Z', \mathcal{L}^{\otimes d})$ we may think of s as a global section of $\mathcal{G}$. Choose a geometric point $\overline{z}$ of Z' lying over z and denote $\overline{\xi} = g' \circ \overline{z}$ the corresponding geometric point of Z. The adjunction map

$$(g')^*\mathcal{G} = (g')^*g'_*\mathcal{L}^{\otimes d} \longrightarrow \mathcal{L}^{\otimes d}$$

induces a map of stalks $\mathcal{G}_{\overline{\xi}} \to \mathcal{L}_{\overline{z}}$, see Properties of Spaces, Lemma 29.5. Moreover the adjunction map sends the pullback of s (viewed as a section of $\mathcal{G}$) to s (viewed as a section of $\mathcal{L}^{\otimes d}$). Thus the image of s in the vector space which is the source of the arrow

$$\mathcal{G}_{\overline{\xi}} \otimes \kappa(\overline{\xi}) \longrightarrow \mathcal{L}_{\overline{z}}^{\otimes d} \otimes \kappa(\overline{z})$$

isn't zero since by choice of s the image in the target of the arrow is nonzero. Hence ξ is in the support of $\mathcal{G}$ (Morphisms of Spaces, Lemma 15.2). Since $|Z|$ is irreducible and Z is reduced we conclude that the scheme theoretic support of $\mathcal{G}$ is all of Z as desired. $\square$

Lemma 20.3. *Let A be a Noetherian ring. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is finite A-module for all $i \geq 0$.*

Proof. This is just the affine case of Lemma 20.2. Namely, by Lemma 3.1 we know that $R^if_*\mathcal{F}$ is a quasi-coherent sheaf. Hence it is the quasi-coherent sheaf associated to the A-module $\Gamma(\mathrm{Spec}(A), R^if_*\mathcal{F}) = H^i(X, \mathcal{F})$. The equality holds by Cohomology on Sites, Lemma 14.6 and vanishing of higher cohomology groups of quasi-coherent modules on affine schemes (Cohomology of Schemes, Lemma 2.2). By Lemma 12.2 we see $R^if_*\mathcal{F}$ is a coherent sheaf if and only if $H^i(X, \mathcal{F})$ is an A-module of finite type. Hence Lemma 20.2 gives us the conclusion. $\square$

Lemma 20.4. *Let A be a Noetherian ring. Let B be a finitely generated graded A-algebra. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Set $\mathcal{B} = f^*\widetilde{B}$. Let $\mathcal{F}$ be a quasi-coherent graded $\mathcal{B}$-module of finite type. For every $p \geq 0$ the graded B-module $H^p(X, \mathcal{F})$ is a finite B-module.*

Proof. To prove this we consider the fibre product diagram

$$\begin{tikzcd}
X' = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X \arrow[r, "\pi"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
\mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(A)
\end{tikzcd}$$

Note that f' is a proper morphism, see Morphisms of Spaces, Lemma 40.3. Also, B is a finitely generated A-algebra, and hence Noetherian (Algebra, Lemma 31.1). This implies that X' is a Noetherian algebraic space (Morphisms of Spaces, Lemma 28.6). Note that X' is the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ by Morphisms of Spaces, Lemma 20.7. Since $\mathcal{F}$ is a quasi-coherent $\mathcal{B}$-module we see that there is a unique quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$ such that $\pi_*\mathcal{F}' = \mathcal{F}$, see Morphisms of Spaces, Lemma 20.10. Since $\mathcal{F}$ is finite type as a $\mathcal{B}$-module we conclude that $\mathcal{F}'$ is a finite type $\mathcal{O}_{X'}$-module (details omitted). In other words, $\mathcal{F}'$ is a coherent $\mathcal{O}_{X'}$-module (Lemma 12.2). Since the morphism $\pi : X' \to X$ is affine we have

$$H^p(X, \mathcal{F}) = H^p(X', \mathcal{F}')$$

by Lemma 8.2 and Cohomology on Sites, Lemma 14.6. Thus the lemma follows from Lemma 20.3. $\square$

21. Ample invertible sheaves and cohomology

Here is a criterion for ampleness on proper algebraic spaces over affine bases in terms of vanishing of cohomology after twisting.

Lemma 21.1. *Let R be a Noetherian ring. Let X be a proper algebraic space over R. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The following are equivalent*

(1) *X is a scheme and $\mathcal{L}$ is ample on X,*
(2) *for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n_0 \geq 0$ such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$, and*
(3) *for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n \geq 1$ such that $H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Cohomology of Schemes, Lemma 17.1. The implication (2) $\Rightarrow$ (3) is trivial. The implication (3) $\Rightarrow$ (1) is Lemma 16.9. $\square$

Lemma 21.2. *Let R be a Noetherian ring. Let $f : Y \to X$ be a morphism of algebraic spaces proper over R. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume f is finite and surjective. The following are equivalent*

(1) *X is a scheme and $\mathcal{L}$ is ample, and*
(2) *Y is a scheme and $f^*\mathcal{L}$ is ample.*

Proof. Assume (1). Then Y is a scheme as a finite morphism is representable (by schemes), see Morphisms of Spaces, Lemma 45.3. Hence (2) follows from Cohomology of Schemes, Lemma 17.2.

Assume (2). Let P be the following property on coherent $\mathcal{O}_X$-modules $\mathcal{F}$: there exists an n_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$. We will prove that P holds for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, which implies $\mathcal{L}$ is ample by Lemma 21.1. We are going to apply Lemma 14.5. Thus we have to verify (1), (2) and (3) of that lemma for P. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves and the fact that tensoring with an invertible sheaf is an exact functor. Property (2) follows since $H^p(X, -)$ is an additive functor.

To see (3) let $i : Z \to X$ be a reduced closed subspace with $|Z|$ irreducible. Let $i' : Z' \to Y$ and $f' : Z' \to Z$ be as in Lemma 17.1 and set $\mathcal{G} = f'_*\mathcal{O}_{Z'}$. We claim that $\mathcal{G}$ satisfies properties (3)(a) and (3)(b) of Lemma 14.5 which will finish the proof. Property (3)(a) we have seen in Lemma 17.1. To see (3)(b) let $\mathcal{I}$ be a nonzero quasi-coherent sheaf of ideals on Z. Denote $\mathcal{I}' \subset \mathcal{O}_{Z'}$ the quasi-coherent ideal $(f')^{-1}\mathcal{I}\mathcal{O}_{Z'}$, i.e., the image of $(f')^*\mathcal{I} \to \mathcal{O}_{Z'}$. By Lemma 17.2 we have $f_*\mathcal{I}' = \mathcal{I}\mathcal{G}$. We claim the common value $\mathcal{G}' = \mathcal{I}\mathcal{G} = f'_*\mathcal{I}'$ satisfies the condition expressed in (3)(b). First, it is clear that the support of $\mathcal{G}/\mathcal{G}'$ is contained in the support of $\mathcal{O}_Z/\mathcal{I}$ which is a proper subspace of $|Z|$ as $\mathcal{I}$ is a nonzero ideal sheaf on the reduced and irreducible algebraic space Z. Recall that f'_*, i_*, and i'_* transform coherent modules into coherent modules, see Lemmas 12.9 and 12.8. As Y is a scheme and $\mathcal{L}$ is ample we see from Lemma 21.1 that there exists an n_0 such that

$$H^p(Y, i'_*\mathcal{I}' \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0$$

for $n \geq n_0$ and $p > 0$. Now we get

$$\begin{aligned} H^p(X, i_*\mathcal{G}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) &= H^p(Z, \mathcal{G}' \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n}) \\ &= H^p(Z, f'_*\mathcal{I}' \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n})) \\ &= H^p(Z, f'_*(\mathcal{I}' \otimes_{\mathcal{O}_{Z'}} (f')^*i^*\mathcal{L}^{\otimes n})) \\ &= H^p(Z, f'_*(\mathcal{I}' \otimes_{\mathcal{O}_{Z'}} (i')^*f^*\mathcal{L}^{\otimes n})) \\ &= H^p(Z', \mathcal{I}' \otimes_{\mathcal{O}_{Z'}} (i')^*f^*\mathcal{L}^{\otimes n})) \\ &= H^p(Y, i'_*\mathcal{I}' \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0 \end{aligned}$$

Here we have used the projection formula and the Leray spectral sequence (see Cohomology on Sites, Sections 50 and 14) and Lemma 4.1. This verifies property (3)(b) of Lemma 14.5 as desired. □

22. The theorem on formal functions

This section is the analogue of Cohomology of Schemes, Section 20. We encourage the reader to read that section first.

Situation 22.1. Here A is a Noetherian ring and $I \subset A$ is an ideal. Also, $f : X \to \text{Spec}(A)$ is a proper morphism of algebraic spaces and $\mathcal{F}$ is a coherent sheaf on X.

In this situation we denote $I^n\mathcal{F}$ the quasi-coherent submodule of $\mathcal{F}$ generated as an $\mathcal{O}_X$-module by products of local sections of $\mathcal{F}$ and elements of I^n. In other words, it is the image of the map $f^*\widetilde{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$.

Lemma 22.2. *In Situation 22.1. Set $B = \bigoplus_{n \geq 0} I^n$. Then for every $p \geq 0$ the graded B-module $\bigoplus_{n \geq 0} H^p(X, I^n\mathcal{F})$ is a finite B-module.*

Proof. Let $\mathcal{B} = \bigoplus I^n\mathcal{O}_X = f^*\widetilde{B}$. Then $\bigoplus I^n\mathcal{F}$ is a finite type graded $\mathcal{B}$-module. Hence the result follows from Lemma 20.4. □

Lemma 22.3. *In Situation 22.1. For every $p \geq 0$ there exists an integer $c \geq 0$ such that*

(1) *the multiplication map $I^{n-c} \otimes H^p(X, I^c\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is surjective for all $n \geq c$, and*
(2) *the image of $H^p(X, I^{n+m}\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is contained in the submodule $I^{m-c}H^p(X, I^n\mathcal{F})$ for all $n \geq 0$, $m \geq c$.*

Proof. By Lemma 22.2 we can find $d_1, \ldots, d_t \geq 0$, and $x_i \in H^p(X, I^{d_i}\mathcal{F})$ such that $\bigoplus_{n \geq 0} H^p(X, I^n\mathcal{F})$ is generated by $x_1, \ldots, x_t$ over $B = \bigoplus_{n \geq 0} I^n$. Take $c = \max\{d_i\}$. It is clear that (1) holds. For (2) let $b = \max(0, n - c)$. Consider the commutative diagram of A-modules

$$\begin{tikzcd} I^{n+m-c-b} \otimes I^b \otimes H^p(X, I^c\mathcal{F}) \arrow[r] \arrow[d] & I^{n+m-c} \otimes H^p(X, I^c\mathcal{F}) \arrow[r] & H^p(X, I^{n+m}\mathcal{F}) \arrow[d] \\ I^{n+m-c-b} \otimes H^p(X, I^n\mathcal{F}) \arrow[rr] & & H^p(X, I^n\mathcal{F}) \end{tikzcd}$$

By part (1) of the lemma the composition of the horizontal arrows is surjective if $n + m \geq c$. On the other hand, it is clear that $n + m - c - b \geq m - c$. Hence part (2). □

Lemma 22.4. *In Situation 22.1. Fix $p \geq 0$.*

(1) *There exists a $c_1 \geq 0$ such that for all $n \geq c_1$ we have*
$$\mathrm{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) \subset I^{n-c_1}H^p(X, \mathcal{F}).$$
(2) *The inverse system*
$$\left(H^p(X, \mathcal{F}/I^n\mathcal{F})\right)_{n \in \mathbf{N}}$$
satisfies the Mittag-Leffler condition (see Homology, Definition 31.2).

(3) *In fact for any p and n there exists a $c_2(n) \geq n$ such that*
$$\mathrm{Im}(H^p(X, \mathcal{F}/I^k\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$
for all $k \geq c_2(n)$.

Proof. Let $c_1 = \max\{c_p, c_{p+1}\}$, where c_p, c_{p+1} are the integers found in Lemma 22.3 for H^p and H^{p+1}. We will use this constant in the proofs of (1), (2) and (3).

Let us prove part (1). Consider the short exact sequence
$$0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$
From the long exact cohomology sequence we see that
$$\mathrm{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, I^n\mathcal{F}) \to H^p(X, \mathcal{F}))$$
Hence by our choice of c_1 we see that this is contained in $I^{n-c_1}H^p(X, \mathcal{F})$ for $n \geq c_1$.

Note that part (3) implies part (2) by definition of the Mittag-Leffler condition.

Let us prove part (3). Fix an n throughout the rest of the proof. Consider the commutative diagram
$$\begin{tikzcd}
0 \arrow[r] & I^n\mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}/I^n\mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & I^{n+m}\mathcal{F} \arrow[r] \arrow[u] & \mathcal{F} \arrow[r] \arrow[u] & \mathcal{F}/I^{n+m}\mathcal{F} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$
This gives rise to the following commutative diagram
$$\begin{tikzcd}
H^p(X, I^n\mathcal{F}) \arrow[r] & H^p(X, \mathcal{F}) \arrow[r] & H^p(X, \mathcal{F}/I^n\mathcal{F}) \arrow[r, "\delta"'] & H^{p+1}(X, I^n\mathcal{F}) \\
H^p(X, I^{n+m}\mathcal{F}) \arrow[r] \arrow[u] & H^p(X, \mathcal{F}) \arrow[r] \arrow[u, "1"] & H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) \arrow[r] \arrow[u] & H^{p+1}(X, I^{n+m}\mathcal{F}) \arrow[u, "a"]
\end{tikzcd}$$
If $m \geq c_1$ we see that the image of a is contained in $I^{m-c_1}H^{p+1}(X, I^n\mathcal{F})$. By the Artin-Rees lemma (see Algebra, Lemma 51.3) there exists an integer $c_3(n)$ such that
$$I^N H^{p+1}(X, I^n\mathcal{F}) \cap \mathrm{Im}(\delta) \subset \delta\left(I^{N-c_3(n)}H^p(X, \mathcal{F}/I^n\mathcal{F})\right)$$
for all $N \geq c_3(n)$. As $H^p(X, \mathcal{F}/I^n\mathcal{F})$ is annihilated by I^n, we see that if $m \geq c_3(n) + c_1 + n$, then
$$\mathrm{Im}(H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$
In other words, part (3) holds with $c_2(n) = c_3(n) + c_1 + n$. $\square$

Theorem 22.5 (Theorem on formal functions)**.** *In Situation 22.1. Fix $p \geq 0$. The system of maps*

$$H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}/I^n\mathcal{F})$$

define an isomorphism of limits

$$H^p(X, \mathcal{F})^\wedge \longrightarrow \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

where the left hand side is the completion of the A-module $H^p(X, \mathcal{F})$ with respect to the ideal I, see Algebra, Section 96. Moreover, this is in fact a homeomorphism for the limit topologies.

Proof. In fact, this follows immediately from Lemma 22.4. We spell out the details. Set $M = H^p(X, \mathcal{F})$ and $M_n = H^p(X, \mathcal{F}/I^n\mathcal{F})$. Denote $N_n = \mathrm{Im}(M \to M_n)$. By the description of the limit in Homology, Section 31 we have

$$\lim_n M_n = \{(x_n) \in \prod M_n \mid \varphi_i(x_n) = x_{n-1},\ n = 2, 3, \ldots\}$$

Pick an element $x = (x_n) \in \lim_n M_n$. By Lemma 22.4 part (3) we have $x_n \in N_n$ for all n since by definition x_n is the image of some $x_{n+m} \in M_{n+m}$ for all m. By Lemma 22.4 part (1) we see that there exists a factorization

$$M \to N_n \to M/I^{n-c_1}M$$

of the reduction map. Denote $y_n \in M/I^{n-c_1}M$ the image of x_n for $n \geq c_1$. Since for $n' \geq n$ the composition $M \to M_{n'} \to M_n$ is the given map $M \to M_n$ we see that $y_{n'}$ maps to y_n under the canonical map $M/I^{n'-c_1}M \to M/I^{n-c_1}M$. Hence $y = (y_{n+c_1})$ defines an element of $\lim_n M/I^nM$. We omit the verification that y maps to x under the map

$$M^\wedge = \lim_n M/I^nM \longrightarrow \lim_n M_n$$

of the lemma. We also omit the verification on topologies. □

Lemma 22.6. *Let A be a ring. Let $I \subset A$ be an ideal. Assume A is Noetherian and complete with respect to I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Let $\mathcal{F}$ be a coherent sheaf on X. Then*

$$H^p(X, \mathcal{F}) = \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

for all $p \geq 0$.

Proof. This is a reformulation of the theorem on formal functions (Theorem 22.5) in the case of a complete Noetherian base ring. Namely, in this case the A-module $H^p(X, \mathcal{F})$ is finite (Lemma 20.3) hence I-adically complete (Algebra, Lemma 97.1) and we see that completion on the left hand side is not necessary. □

Lemma 22.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S and let $\mathcal{F}$ be a quasi-coherent sheaf on Y. Assume*

1. *Y locally Noetherian,*
2. *f proper, and*
3. *$\mathcal{F}$ coherent.*

Let $\overline{y}$ be a geometric point of Y. Consider the "infinitesimal neighbourhoods"

$$\begin{tikzcd} X_n = \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{y}}^n) \times_Y X \arrow[r, "i_n"'] \arrow[d, "f_n"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{y}}^n) \arrow[r, "c_n"] & Y \end{tikzcd}$$

of the fibre $X_1 = X_{\overline{y}}$ and set $\mathcal{F}_n = i_n^*\mathcal{F}$. Then we have

$$(R^pf_*\mathcal{F})^\wedge_{\overline{y}} \cong \lim_n H^p(X_n, \mathcal{F}_n)$$

as $\mathcal{O}^\wedge_{Y,\overline{y}}$-modules.

Proof. This is just a reformulation of a special case of the theorem on formal functions, Theorem 22.5. Let us spell it out. Note that $\mathcal{O}_{Y,\overline{y}}$ is a Noetherian local ring, see Properties of Spaces, Lemma 24.4. Consider the canonical morphism $c : \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}) \to Y$. This is a flat morphism as it identifies local rings. Denote $f' : X' \to \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}})$ the base change of f to this local ring. We see that $c^*R^pf_*\mathcal{F} = R^pf'_*\mathcal{F}'$ by Lemma 11.2. Moreover, we have canonical identifications $X_n = X'_n$ for all $n \geq 1$.

Hence we may assume that $Y = \mathrm{Spec}(A)$ is the spectrum of a strictly henselian Noetherian local ring A with maximal ideal $\mathfrak{m}$ and that $\overline{y} \to Y$ is equal to $\mathrm{Spec}(A/\mathfrak{m}) \to Y$. It follows that

$$(R^pf_*\mathcal{F})_{\overline{y}} = \Gamma(Y, R^pf_*\mathcal{F}) = H^p(X, \mathcal{F})$$

because $(Y, \overline{y})$ is an initial object in the category of étale neighbourhoods of $\overline{y}$. The morphisms c_n are each closed immersions. Hence their base changes i_n are closed immersions as well. Note that $i_{n,*}\mathcal{F}_n = i_{n,*}i_n^*\mathcal{F} = \mathcal{F}/\mathfrak{m}^n\mathcal{F}$. By the Leray spectral sequence for i_n, and Lemma 12.9 we see that

$$H^p(X_n, \mathcal{F}_n) = H^p(X, i_{n,*}\mathcal{F}) = H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F})$$

Hence we may indeed apply the theorem on formal functions to compute the limit in the statement of the lemma and we win. ☐

Here is a lemma which we will generalize later to fibres of dimension > 0, namely the next lemma.

Lemma 22.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Assume*

1. *Y locally Noetherian,*
2. *f is proper, and*
3. *$X_{\overline{y}}$ has discrete underlying topological space.*

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^pf_*\mathcal{F})_{\overline{y}} = 0$ for all $p > 0$.

Proof. Let $\kappa(\overline{y})$ be the residue field of the local ring of $\mathcal{O}_{Y,\overline{y}}$. As in Lemma 22.7 we set $X_{\overline{y}} = X_1 = \mathrm{Spec}(\kappa(\overline{y})) \times_Y X$. By Morphisms of Spaces, Lemma 34.8 the morphism $f : X \to Y$ is quasi-finite at each of the points of the fibre of $X \to Y$ over $\overline{y}$. It follows that $X_{\overline{y}} \to \overline{y}$ is separated and quasi-finite. Hence $X_{\overline{y}}$ is a scheme by Morphisms of Spaces, Proposition 50.2. Since it is quasi-compact its underlying topological space is a finite discrete space. Then it is an affine scheme by Schemes, Lemma 11.8. By Lemma 17.3 it follows that the algebraic spaces X_n are affine schemes as well. Moreover, the underlying topological of each X_n is the same as that of X_1. Hence it follows that $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > 0$. Hence we see that $(R^p f_*\mathcal{F})^\wedge_{\overline{y}} = 0$ by Lemma 22.7. Note that $R^p f_*\mathcal{F}$ is coherent by Lemma 20.2 and hence $R^p f_*\mathcal{F}_{\overline{y}}$ is a finite $\mathcal{O}_{Y,\overline{y}}$-module. By Algebra, Lemma 97.1 this implies that $(R^p f_*\mathcal{F})_{\overline{y}} = 0$. □

Lemma 22.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Assume*

1. *Y locally Noetherian,*
2. *f is proper, and*
3. $\dim(X_{\overline{y}}) = d$.

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^p f_*\mathcal{F})_{\overline{y}} = 0$ for all $p > d$.

Proof. Let $\kappa(\overline{y})$ be the residue field of the local ring of $\mathcal{O}_{Y,\overline{y}}$. As in Lemma 22.7 we set $X_{\overline{y}} = X_1 = \mathrm{Spec}(\kappa(\overline{y})) \times_Y X$. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same as that of $X_{\overline{y}}$. Hence $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > d$ by Lemma 10.1. Hence we see that $(R^p f_*\mathcal{F})^\wedge_{\overline{y}} = 0$ by Lemma 22.7 for $p > d$. Note that $R^p f_*\mathcal{F}$ is coherent by Lemma 20.2 and hence $R^p f_*\mathcal{F}_{\overline{y}}$ is a finite $\mathcal{O}_{Y,\overline{y}}$-module. By Algebra, Lemma 97.1 this implies that $(R^p f_*\mathcal{F})_{\overline{y}} = 0$. □

23. Applications of the theorem on formal functions

We will add more here as needed.

Lemma 23.1. *(For a more general version see More on Morphisms of Spaces, Lemma 35.1). Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian. The following are equivalent*

1. *f is finite, and*
2. *f is proper and $|X_k|$ is a discrete space for every morphism* $\mathrm{Spec}(k) \to Y$ *where k is a field.*

Proof. A finite morphism is proper according to Morphisms of Spaces, Lemma 45.9. A finite morphism is quasi-finite according to Morphisms of Spaces, Lemma 45.8. A quasi-finite morphism has discrete fibres X_k, see Morphisms of Spaces, Lemma 27.5. Hence a finite morphism is proper and has discrete fibres X_k.

Assume f is proper with discrete fibres X_k. We want to show f is finite. In fact it suffices to prove f is affine. Namely, if f is affine, then it follows that f is integral by Morphisms of Spaces, Lemma 45.7 whereupon it follows from Morphisms of Spaces, Lemma 45.6 that f is finite.

To show that f is affine we may assume that Y is affine, and our goal is to show that X is affine too. Since f is proper we see that X is separated and quasi-compact. We will show that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. This implies that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ by Lemmas 15.1 and 5.1. Then it follows that X is affine from Proposition 16.7. By Lemma 22.8 we conclude that the stalks of $R^1 f_*\mathcal{F}$ are zero for all geometric points of Y. In other words, $R^1 f_*\mathcal{F} = 0$. Hence we see from the Leray Spectral Sequence for f that $H^1(X, \mathcal{F}) = H^1(Y, f_*\mathcal{F})$. Since Y is affine, and $f_*\mathcal{F}$ is quasi-coherent (Morphisms of Spaces, Lemma 11.2) we conclude $H^1(Y, f_*\mathcal{F}) = 0$ from Cohomology of Schemes, Lemma 2.2. Hence $H^1(X, \mathcal{F}) = 0$ as desired. □

As a consequence we have the following useful result.

Lemma 23.2. *(For a more general version see More on Morphisms of Spaces, Lemma 35.2). Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Assume*

1. *Y is locally Noetherian,*
2. *f is proper, and*
3. *$|X_{\overline{y}}|$ is finite.*

Then there exists an open neighbourhood $V \subset Y$ of $\overline{y}$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. The morphism f is quasi-finite at all the geometric points of X lying over $\overline{y}$ by Morphisms of Spaces, Lemma 34.8. By Morphisms of Spaces, Lemma 34.7 the set of points at which f is quasi-finite is an open subspace $U \subset X$. Let $Z = X \setminus U$. Then $\overline{y} \notin f(Z)$. Since f is proper the set $f(Z) \subset Y$ is closed. Choose any open neighbourhood $V \subset Y$ of $\overline{y}$ with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence $f^{-1}(V) \to V$ has discrete fibres X_k (Morphisms of Spaces, Lemma 27.5) which are quasi-compact hence finite. Thus $f^{-1}(V) \to V$ is finite by Lemma 23.1. □

24. Other chapters

References

[Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, vol. 203, Springer-Verlag, 1971.