

# PROPERTIES OF ALGEBRAIC SPACES

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## 1. Introduction

Please see Spaces, Section 1 for a brief introduction to algebraic spaces, and please read some of that chapter for our basic definitions and conventions concerning algebraic spaces. In this chapter we start introducing some basic notions and properties of algebraic spaces. A fundamental reference for the case of quasi-separated algebraic spaces is [Knu71].

The discussion is somewhat awkward at times since we made the design decision to first talk about properties of algebraic spaces by themselves, and only later about properties of morphisms of algebraic spaces. We make an exception for this rule regarding *étale morphisms* of algebraic spaces, which we introduce in Section 16. But until that section whenever we say a morphism has a certain property, it automatically means the source of the morphism is a scheme (or perhaps the morphism is representable).

Some of the material in the chapter (especially regarding points) will be improved upon in the chapter on decent algebraic spaces.

## 2. Conventions

The standing assumption is that all schemes are contained in a big fppf site  $Sch_{fppf}$ . And all rings  $A$  considered have the property that  $\mathrm{Spec}(A)$  is (isomorphic) to an object of this big site.

Let  $S$  be a scheme and let  $X$  be an algebraic space over  $S$ . In this chapter and the following we will write  $X \times_S X$  for the product of  $X$  with itself (in the category of algebraic spaces over  $S$ ), instead of  $X \times X$ . The reason is that we want to avoid confusion when changing base schemes, as in Spaces, Section 16.

## 3. Separation axioms

In this section we collect all the “absolute” separation conditions of algebraic spaces. Since in our language any algebraic space is an algebraic space over some definite base scheme, any absolute property of  $X$  over  $S$  corresponds to a conditions imposed on  $X$  viewed as an algebraic space over  $\mathrm{Spec}(\mathbf{Z})$ . Here is the precise formulation.

**Definition 3.1.** (Compare Spaces, Definition 13.2.) Consider a big fppf site  $Sch_{fppf} = (Sch/\mathrm{Spec}(\mathbf{Z}))_{fppf}$ . Let  $X$  be an algebraic space over  $\mathrm{Spec}(\mathbf{Z})$ . Let  $\Delta : X \rightarrow X \times X$  be the diagonal morphism.

- (1) We say  $X$  is *separated* if  $\Delta$  is a closed immersion.
- (2) We say  $X$  is *locally separated*<sup>1</sup> if  $\Delta$  is an immersion.
- (3) We say  $X$  is *quasi-separated* if  $\Delta$  is quasi-compact.
- (4) We say  $X$  is *Zariski locally quasi-separated*<sup>2</sup> if there exists a Zariski covering  $X = \bigcup_{i \in I} X_i$  (see Spaces, Definition 12.5) such that each  $X_i$  is quasi-separated.

Let  $S$  is a scheme contained in  $Sch_{fppf}$ , and let  $X$  be an algebraic space over  $S$ . Then we say  $X$  is *separated*, *locally separated*, *quasi-separated*, or *Zariski locally quasi-separated* if  $X$  viewed as an algebraic space over  $\mathrm{Spec}(\mathbf{Z})$  (see Spaces, Definition 16.2) has the corresponding property.

<sup>1</sup>In the literature this often refers to quasi-separated and locally separated algebraic spaces.

<sup>2</sup>This notion was suggested by B. Conrad.

It is true that an algebraic space  $X$  over  $S$  which is separated (in the absolute sense above) is separated over  $S$  (and similarly for the other absolute separation properties above). This will be discussed in great detail in Morphisms of Spaces, Section 4. We will see in Lemma 6.6 that being Zariski locally separated is independent of the base scheme (hence equivalent to the absolute notion).

**Lemma 3.2.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . We have the following implications among the separation axioms of Definition 3.1:*

- (1) *separated implies all the others,*
- (2) *quasi-separated implies Zariski locally quasi-separated.*

**Proof.** Omitted. □

**Lemma 3.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The following are equivalent*

- (1)  *$X$  is a quasi-separated algebraic space,*
- (2) *for  $U \rightarrow X$ ,  $V \rightarrow X$  with  $U$ ,  $V$  quasi-compact schemes the fibre product  $U \times_X V$  is quasi-compact,*
- (3) *for  $U \rightarrow X$ ,  $V \rightarrow X$  with  $U$ ,  $V$  affine the fibre product  $U \times_X V$  is quasi-compact.*

**Proof.** Using Spaces, Lemma 16.3 we see that we may assume  $S = \text{Spec}(\mathbf{Z})$ . Since  $U \times_X V = X \times_{X \times X} (U \times V)$  and since  $U \times V$  is quasi-compact if  $U$  and  $V$  are so, we see that (1) implies (2). It is clear that (2) implies (3). Assume (3). Choose a scheme  $W$  and a surjective étale morphism  $W \rightarrow X$ . Then  $W \times W \rightarrow X \times X$  is surjective étale. Hence it suffices to show that

$$j : W \times_X W = X \times_{(X \times X)} (W \times W) \rightarrow W \times W$$

is quasi-compact, see Spaces, Lemma 5.6. If  $U \subset W$  and  $V \subset W$  are affine opens, then  $j^{-1}(U \times V) = U \times_X V$  is quasi-compact by assumption. Since the affine opens  $U \times V$  form an affine open covering of  $W \times W$  (Schemes, Lemma 17.4) we conclude by Schemes, Lemma 19.2. □

**Lemma 3.4.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The following are equivalent*

- (1)  *$X$  is a separated algebraic space,*
- (2) *for  $U \rightarrow X$ ,  $V \rightarrow X$  with  $U$ ,  $V$  affine the fibre product  $U \times_X V$  is affine and*

$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(U \times_X V)$$

*is surjective.*

**Proof.** Using Spaces, Lemma 16.3 we see that we may assume  $S = \text{Spec}(\mathbf{Z})$ . Since  $U \times_X V = X \times_{X \times X} (U \times V)$  and since  $U \times V$  is affine if  $U$  and  $V$  are so, we see that (1) implies (2). Assume (2). Choose a scheme  $W$  and a surjective étale morphism  $W \rightarrow X$ . Then  $W \times W \rightarrow X \times X$  is surjective étale. Hence it suffices to show that

$$j : W \times_X W = X \times_{(X \times X)} (W \times W) \rightarrow W \times W$$

is a closed immersion, see Spaces, Lemma 5.6. If  $U \subset W$  and  $V \subset W$  are affine opens, then  $j^{-1}(U \times V) = U \times_X V$  is affine by assumption and the map  $U \times_X V \rightarrow U \times V$  is a closed immersion because the corresponding ring map is surjective. Since the affine opens  $U \times V$  form an affine open covering of  $W \times W$  (Schemes, Lemma 17.4) we conclude by Morphisms, Lemma 2.1. □

#### 4. Points of algebraic spaces

As is clear from Spaces, Example 14.8 a point of an algebraic space should not be defined as a monomorphism from the spectrum of a field. Instead we define them as equivalence classes of morphisms of spectra of fields exactly as explained in Schemes, Section 13.

Let  $S$  be a scheme. Let  $F$  be a presheaf on  $(Sch/S)_{fppf}$ . Let  $K$  be a field. Consider a morphism

$$\mathrm{Spec}(K) \longrightarrow F.$$

By the Yoneda Lemma this is given by an element  $p \in F(\mathrm{Spec}(K))$ . We say that two such pairs  $(\mathrm{Spec}(K), p)$  and  $(\mathrm{Spec}(L), q)$  are *equivalent* if there exists a third field  $\Omega$  and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \longrightarrow & \mathrm{Spec}(L) \\ \downarrow & & \downarrow q \\ \mathrm{Spec}(K) & \xrightarrow{p} & F. \end{array}$$

In other words, there are field extensions  $K \rightarrow \Omega$  and  $L \rightarrow \Omega$  such that  $p$  and  $q$  map to the same element of  $F(\mathrm{Spec}(\Omega))$ . We omit the verification that this defines an equivalence relation.

**Definition 4.1.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . A *point* of  $X$  is an equivalence class of morphisms from spectra of fields into  $X$ . The set of points of  $X$  is denoted  $|X|$ .

Note that if  $f : X \rightarrow Y$  is a morphism of algebraic spaces over  $S$ , then there is an induced map  $|f| : |X| \rightarrow |Y|$  which maps a representative  $x : \mathrm{Spec}(K) \rightarrow X$  to the representative  $f \circ x : \mathrm{Spec}(K) \rightarrow Y$ .

**Lemma 4.2.** *Let  $S$  be a scheme. Let  $X$  be a scheme over  $S$ . The points of  $X$  as a scheme are in canonical 1-1 correspondence with the points of  $X$  as an algebraic space.*

**Proof.** This is Schemes, Lemma 13.3. □

**Lemma 4.3.** *Let  $S$  be a scheme. Let*

$$\begin{array}{ccc} Z \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

*be a cartesian diagram of algebraic spaces over  $S$ . Then the map of sets of points*

$$|Z \times_Y X| \longrightarrow |Z| \times_{|Y|} |X|$$

*is surjective.*

**Proof.** Namely, suppose given fields  $K, L$  and morphisms  $\mathrm{Spec}(K) \rightarrow X$ ,  $\mathrm{Spec}(L) \rightarrow Z$ , then the assumption that they agree as elements of  $|Y|$  means that there is a common extension  $M/K$  and  $M/L$  such that  $\mathrm{Spec}(M) \rightarrow \mathrm{Spec}(K) \rightarrow X \rightarrow Y$  and  $\mathrm{Spec}(M) \rightarrow \mathrm{Spec}(L) \rightarrow Z \rightarrow Y$  agree. And this is exactly the condition which says you get a morphism  $\mathrm{Spec}(M) \rightarrow Z \times_Y X$ . □

**Lemma 4.4.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $f : T \rightarrow X$  be a morphism from a scheme to  $X$ . The following are equivalent*

- (1)  *$f : T \rightarrow X$  is surjective (according to Spaces, Definition 5.1), and*
- (2)  *$|f| : |T| \rightarrow |X|$  is surjective.*

**Proof.** Assume (1). Let  $x : \text{Spec}(K) \rightarrow X$  be a morphism from the spectrum of a field into  $X$ . By assumption the morphism of schemes  $\text{Spec}(K) \times_X T \rightarrow \text{Spec}(K)$  is surjective. Hence there exists a field extension  $K'/K$  and a morphism  $\text{Spec}(K') \rightarrow \text{Spec}(K) \times_X T$  such that the left square in the diagram

$$\begin{array}{ccccc} \text{Spec}(K') & \longrightarrow & \text{Spec}(K) \times_X T & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(K) & \xlongequal{\quad} & \text{Spec}(K) & \xrightarrow{x} & X \end{array}$$

is commutative. This shows that  $|f| : |T| \rightarrow |X|$  is surjective.

Assume (2). Let  $Z \rightarrow X$  be a morphism where  $Z$  is a scheme. We have to show that the morphism of schemes  $Z \times_X T \rightarrow T$  is surjective, i.e., that  $|Z \times_X T| \rightarrow |Z|$  is surjective. This follows from (2) and Lemma 4.3.  $\square$

**Lemma 4.5.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $X = U/R$  be a presentation of  $X$ , see Spaces, Definition 9.3. Then the image of  $|R| \rightarrow |U| \times |U|$  is an equivalence relation and  $|X|$  is the quotient of  $|U|$  by this equivalence relation.*

**Proof.** The assumption means that  $U$  is a scheme,  $p : U \rightarrow X$  is a surjective, étale morphism,  $R = U \times_X U$  is a scheme and defines an étale equivalence relation on  $U$  such that  $X = U/R$  as sheaves. By Lemma 4.4 we see that  $|U| \rightarrow |X|$  is surjective. By Lemma 4.3 the map

$$|R| \longrightarrow |U| \times_{|X|} |U|$$

is surjective. Hence the image of  $|R| \rightarrow |U| \times |U|$  is exactly the set of pairs  $(u_1, u_2) \in |U| \times |U|$  such that  $u_1$  and  $u_2$  have the same image in  $|X|$ . Combining these two statements we get the result of the lemma.  $\square$

**Lemma 4.6.** *Let  $S$  be a scheme. There exists a unique topology on the sets of points of algebraic spaces over  $S$  with the following properties:*

- (1) *if  $X$  is a scheme over  $S$ , then the topology on  $|X|$  is the usual one (via the identification of Lemma 4.2),*
- (2) *for every morphism of algebraic spaces  $X \rightarrow Y$  over  $S$  the map  $|X| \rightarrow |Y|$  is continuous, and*
- (3) *for every étale morphism  $U \rightarrow X$  with  $U$  a scheme the map of topological spaces  $|U| \rightarrow |X|$  is continuous and open.*

**Proof.** Let  $X$  be an algebraic space over  $S$ . Let  $p : U \rightarrow X$  be a surjective étale morphism where  $U$  is a scheme over  $S$ . We define  $W \subset |X|$  is open if and only if  $|p|^{-1}(W)$  is an open subset of  $|U|$ . This is a topology on  $|X|$  (it is the quotient topology on  $|X|$ , see Topology, Lemma 6.2).

Let us prove that the topology is independent of the choice of the presentation. To do this it suffices to show that if  $U'$  is a scheme, and  $U' \rightarrow X$  is an étale morphism, then the map  $|U'| \rightarrow |X|$  (with topology on  $|X|$  defined using  $U \rightarrow X$

as above) is open and continuous; which in addition will prove that (3) holds. Set  $U'' = U \times_X U'$ , so that we have the commutative diagram

$$\begin{array}{ccc} U'' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

As  $U \rightarrow X$  and  $U' \rightarrow X$  are étale we see that both  $U'' \rightarrow U$  and  $U'' \rightarrow U'$  are étale morphisms of schemes. Moreover,  $U'' \rightarrow U'$  is surjective. Hence we get a commutative diagram of maps of sets

$$\begin{array}{ccc} |U''| & \longrightarrow & |U'| \\ \downarrow & & \downarrow \\ |U| & \longrightarrow & |X| \end{array}$$

The lower horizontal arrow is surjective (see Lemma 4.4 or Lemma 4.5) and continuous by definition of the topology on  $|X|$ . The top horizontal arrow is surjective, continuous, and open by Morphisms, Lemma 36.13. The left vertical arrow is continuous and open (by Morphisms, Lemma 36.13 again.) Hence it follows formally that the right vertical arrow is continuous and open.

To finish the proof we prove (2). Let  $a : X \rightarrow Y$  be a morphism of algebraic spaces. According to Spaces, Lemma 11.6 we can find a diagram

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{a} & Y \end{array}$$

where  $U$  and  $V$  are schemes, and  $p$  and  $q$  are surjective and étale. This gives rise to the diagram

$$\begin{array}{ccc} |U| & \xrightarrow{\alpha} & |V| \\ p \downarrow & & \downarrow q \\ |X| & \xrightarrow{a} & |Y| \end{array}$$

where all but the lower horizontal arrows are known to be continuous and the two vertical arrows are surjective and open. It follows that the lower horizontal arrow is continuous as desired.  $\square$

**Definition 4.7.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The underlying *topological space* of  $X$  is the set of points  $|X|$  endowed with the topology constructed in Lemma 4.6.

It turns out that this topological space carries the same information as the small Zariski site  $X_{Zar}$  of Spaces, Definition 12.6.

**Lemma 4.8.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ .*

- (1) *The rule  $X' \mapsto |X'|$  defines an inclusion preserving bijection between open subspaces  $X'$  (see Spaces, Definition 12.1) of  $X$ , and opens of the topological space  $|X|$ .*

- (2) A family  $\{X_i \subset X\}_{i \in I}$  of open subspaces of  $X$  is a Zariski covering (see Spaces, Definition 12.5) if and only if  $|X| = \bigcup |X_i|$ .

In other words, the small Zariski site  $X_{\text{Zar}}$  of  $X$  is canonically identified with a site associated to the topological space  $|X|$  (see Sites, Example 6.4).

**Proof.** In order to prove (1) let us construct the inverse of the rule. Namely, suppose that  $W \subset |X|$  is open. Choose a presentation  $X = U/R$  corresponding to the surjective étale map  $p : U \rightarrow X$  and étale maps  $s, t : R \rightarrow U$ . By construction we see that  $|p|^{-1}(W)$  is an open of  $U$ . Denote  $W' \subset U$  the corresponding open subscheme. It is clear that  $R' = s^{-1}(W') = t^{-1}(W')$  is a Zariski open of  $R$  which defines an étale equivalence relation on  $W'$ . By Spaces, Lemma 10.2 the morphism  $X' = W'/R' \rightarrow X$  is an open immersion. Hence  $X'$  is an algebraic space by Spaces, Lemma 11.3. By construction  $|X'| = W$ , i.e.,  $X'$  is a subspace of  $X$  corresponding to  $W$ . Thus (1) is proved.

To prove (2), note that if  $\{X_i \subset X\}_{i \in I}$  is a collection of open subspaces, then it is a Zariski covering if and only if the  $U = \bigcup U \times_X X_i$  is an open covering. This follows from the definition of a Zariski covering and the fact that the morphism  $U \rightarrow X$  is surjective as a map of presheaves on  $(\text{Sch}/S)_{\text{fppf}}$ . On the other hand, we see that  $|X| = \bigcup |X_i|$  if and only if  $U = \bigcup U \times_X X_i$  by Lemma 4.5 (and the fact that the projections  $U \times_X X_i \rightarrow X_i$  are surjective and étale). Thus the equivalence of (2) follows.  $\square$

**Lemma 4.9.** *Let  $S$  be a scheme. Let  $X, Y$  be algebraic spaces over  $S$ . Let  $X' \subset X$  be an open subspace. Let  $f : Y \rightarrow X$  be a morphism of algebraic spaces over  $S$ . Then  $f$  factors through  $X'$  if and only if  $|f| : |Y| \rightarrow |X|$  factors through  $|X'| \subset |X|$ .*

**Proof.** By Spaces, Lemma 12.3 we see that  $Y' = Y \times_X X' \rightarrow Y$  is an open immersion. If  $|f|(|Y|) \subset |X'|$ , then clearly  $|Y'| = |Y|$ . Hence  $Y' = Y$  by Lemma 4.8.  $\square$

**Lemma 4.10.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $U$  be a scheme and let  $f : U \rightarrow X$  be an étale morphism. Let  $X' \subset X$  be the open subspace corresponding to the open  $|f|(|U|) \subset |X|$  via Lemma 4.8. Then  $f$  factors through a surjective étale morphism  $f' : U \rightarrow X'$ . Moreover, if  $R = U \times_X U$ , then  $R = U \times_{X'} U$  and  $X'$  has the presentation  $X' = U/R$ .*

**Proof.** The existence of the factorization follows from Lemma 4.9. The morphism  $f'$  is surjective according to Lemma 4.4. To see  $f'$  is étale, suppose that  $T \rightarrow X'$  is a morphism where  $T$  is a scheme. Then  $T \times_X U = T \times_{X'} U$  as  $X' \rightarrow X$  is a monomorphism of sheaves. Thus the projection  $T \times_{X'} U \rightarrow T$  is étale as we assumed  $f$  étale. We have  $U \times_X U = U \times_{X'} U$  as  $X' \rightarrow X$  is a monomorphism. Then  $X' = U/R$  follows from Spaces, Lemma 9.1.  $\square$

**Lemma 4.11.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $p : \text{Spec}(K) \rightarrow X$  and  $q : \text{Spec}(L) \rightarrow X$  be morphisms where  $K$  and  $L$  are fields. Assume  $p$  and  $q$  determine the same point of  $|X|$  and  $p$  is a monomorphism. Then  $q$  factors uniquely through  $p$ .*

**Proof.** Since  $p$  and  $q$  define the same point of  $|X|$ , we see that the scheme

$$Y = \text{Spec}(K) \times_{p, X, q} \text{Spec}(L)$$

is nonempty. Since the base change of a monomorphism is a monomorphism this means that the projection morphism  $Y \rightarrow \operatorname{Spec}(L)$  is a monomorphism. Hence  $Y = \operatorname{Spec}(L)$ , see Schemes, Lemma 23.11. We conclude that  $q$  factors through  $p$ . Uniqueness comes from the fact that  $p$  is a monomorphism.  $\square$

**Lemma 4.12.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Consider the map*

$$\{\operatorname{Spec}(k) \rightarrow X \text{ monomorphism where } k \text{ is a field}\} \longrightarrow |X|$$

*This map is injective.*

**Proof.** This follows from Lemma 4.11.  $\square$

We will see in Decent Spaces, Lemma 11.1 that the map of Lemma 4.12 is a bijection when  $X$  is decent.

## 5. Quasi-compact spaces

**Definition 5.1.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . We say  $X$  is *quasi-compact* if there exists a surjective étale morphism  $U \rightarrow X$  with  $U$  quasi-compact.

**Lemma 5.2.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Then  $X$  is quasi-compact if and only if  $|X|$  is quasi-compact.*

**Proof.** Choose a scheme  $U$  and an étale surjective morphism  $U \rightarrow X$ . We will use Lemma 4.4. If  $U$  is quasi-compact, then since  $|U| \rightarrow |X|$  is surjective we conclude that  $|X|$  is quasi-compact. If  $|X|$  is quasi-compact, then since  $|U| \rightarrow |X|$  is open we see that there exists a quasi-compact open  $U' \subset U$  such that  $|U'| \rightarrow |X|$  is surjective (and still étale). Hence we win.  $\square$

**Lemma 5.3.** *A finite disjoint union of quasi-compact algebraic spaces is a quasi-compact algebraic space.*

**Proof.** This is clear from Lemma 5.2 and the corresponding topological fact.  $\square$

**Example 5.4.** The space  $\mathbf{A}_{\mathbf{Q}}^1/\mathbf{Z}$  is a quasi-compact algebraic space.

**Lemma 5.5.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Every point of  $|X|$  has a fundamental system of open quasi-compact neighbourhoods. In particular  $|X|$  is locally quasi-compact in the sense of Topology, Definition 13.1.*

**Proof.** This follows formally from the fact that there exists a scheme  $U$  and a surjective, open, continuous map  $U \rightarrow |X|$  of topological spaces. To be a bit more precise, if  $u \in U$  maps to  $x \in |X|$ , then the images of the affine neighbourhoods of  $u$  will give a fundamental system of quasi-compact open neighbourhoods of  $x$ .  $\square$

## 6. Special coverings

In this section we collect some straightforward lemmas on the existence of étale surjective coverings of algebraic spaces.

**Lemma 6.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . There exists a surjective étale morphism  $U \rightarrow X$  where  $U$  is a disjoint union of affine schemes. We may in addition assume each of these affines maps into an affine open of  $S$ .*



**Proof.** Let  $V \rightarrow X$  be a surjective étale morphism. Let  $V = \bigcup_{i \in I} V_i$  be a Zariski open covering such that each  $V_i$  maps into an affine open of  $S$ . Then set  $U = \coprod_{i \in I} V_i$  with induced morphism  $U \rightarrow V \rightarrow X$ . This is étale and surjective as a composition of étale and surjective representable transformations of functors (via the general principle Spaces, Lemma 5.4 and Morphisms, Lemmas 9.2 and 36.3).  $\square$

**Lemma 6.2.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . There exists a Zariski covering  $X = \bigcup X_i$  such that each algebraic space  $X_i$  has a surjective étale covering by an affine scheme. We may in addition assume each  $X_i$  maps into an affine open of  $S$ .*

**Proof.** By Lemma 6.1 we can find a surjective étale morphism  $U = \coprod U_i \rightarrow X$ , with  $U_i$  affine and mapping into an affine open of  $S$ . Let  $X_i \subset X$  be the open subspace of  $X$  such that  $U_i \rightarrow X$  factors through an étale surjective morphism  $U_i \rightarrow X_i$ , see Lemma 4.10. Since  $U = \bigcup U_i$  we see that  $X = \bigcup X_i$ . As  $U_i \rightarrow X_i$  is surjective it follows that  $X_i \rightarrow S$  maps into an affine open of  $S$ .  $\square$

**Lemma 6.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Then  $X$  is quasi-compact if and only if there exists an étale surjective morphism  $U \rightarrow X$  with  $U$  an affine scheme.*

**Proof.** If there exists an étale surjective morphism  $U \rightarrow X$  with  $U$  affine then  $X$  is quasi-compact by Definition 5.1. Conversely, if  $X$  is quasi-compact, then  $|X|$  is quasi-compact. Let  $U = \coprod_{i \in I} U_i$  be a disjoint union of affine schemes with an étale and surjective map  $\varphi : U \rightarrow X$  (Lemma 6.1). Then  $|X| = \bigcup \varphi(|U_i|)$  and by quasi-compactness there is a finite subset  $i_1, \dots, i_n$  such that  $|X| = \bigcup \varphi(|U_{i_j}|)$ . Hence  $U_{i_1} \cup \dots \cup U_{i_n}$  is an affine scheme with a finite surjective morphism towards  $X$ .  $\square$

The following lemma will be obsoleted by the discussion of separated morphisms in the chapter on morphisms of algebraic spaces.

**Lemma 6.4.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $U$  be a separated scheme and  $U \rightarrow X$  étale. Then  $U \rightarrow X$  is separated, and  $R = U \times_X U$  is a separated scheme.*

**Proof.** Let  $X' \subset X$  be the open subscheme such that  $U \rightarrow X$  factors through an étale surjection  $U \rightarrow X'$ , see Lemma 4.10. If  $U \rightarrow X'$  is separated, then so is  $U \rightarrow X$ , see Spaces, Lemma 5.4 (as the open immersion  $X' \rightarrow X$  is separated by Spaces, Lemma 5.8 and Schemes, Lemma 23.8). Moreover, since  $U \times_{X'} U = U \times_X U$  it suffices to prove the result after replacing  $X$  by  $X'$ , i.e., we may assume  $U \rightarrow X$  surjective. Consider the commutative diagram

$$\begin{array}{ccc} R = U \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

In the proof of Spaces, Lemma 13.1 we have seen that  $j : R \rightarrow U \times_S U$  is separated. The morphism of schemes  $U \rightarrow S$  is separated as  $U$  is a separated scheme, see Schemes, Lemma 21.13. Hence  $U \times_S U \rightarrow U$  is separated as a base change, see Schemes, Lemma 21.12. Hence the scheme  $U \times_S U$  is separated (by the same lemma). Since  $j$  is separated we see in the same way that  $R$  is separated. Hence

$R \rightarrow U$  is a separated morphism (by Schemes, Lemma 21.13 again). Thus by Spaces, Lemma 11.4 and the diagram above we conclude that  $U \rightarrow X$  is separated.  $\square$

**Lemma 6.5.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . If there exists a quasi-separated scheme  $U$  and a surjective étale morphism  $U \rightarrow X$  such that either of the projections  $U \times_X U \rightarrow U$  is quasi-compact, then  $X$  is quasi-separated.*

**Proof.** We may think of  $X$  as an algebraic space over  $\mathbf{Z}$ . Consider the cartesian diagram

$$\begin{array}{ccc} U \times_X U & \longrightarrow & X \\ j \downarrow & & \downarrow \Delta \\ U \times U & \longrightarrow & X \times X \end{array}$$

Since  $U$  is quasi-separated the projection  $U \times U \rightarrow U$  is quasi-separated (as a base change of a quasi-separated morphism of schemes, see Schemes, Lemma 21.12). Hence the assumption in the lemma implies  $j$  is quasi-compact by Schemes, Lemma 21.14. By Spaces, Lemma 11.4 we see that  $\Delta$  is quasi-compact as desired.  $\square$

**Lemma 6.6.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The following are equivalent*

- (1)  $X$  is Zariski locally quasi-separated over  $S$ ,
- (2)  $X$  is Zariski locally quasi-separated,
- (3) there exists a Zariski open covering  $X = \bigcup X_i$  such that for each  $i$  there exists an affine scheme  $U_i$  and a quasi-compact surjective étale morphism  $U_i \rightarrow X_i$ , and
- (4) there exists a Zariski open covering  $X = \bigcup X_i$  such that for each  $i$  there exists an affine scheme  $U_i$  which maps into an affine open of  $S$  and a quasi-compact surjective étale morphism  $U_i \rightarrow X_i$ .

**Proof.** Assume  $U_i \rightarrow X_i \subset X$  are as in (3). To prove (4) choose for each  $i$  a finite affine open covering  $U_i = U_{i1} \cup \dots \cup U_{in_i}$  such that each  $U_{ij}$  maps into an affine open of  $S$ . The compositions  $U_{ij} \rightarrow U_i \rightarrow X_i$  are étale and quasi-compact (see Spaces, Lemma 5.4). Let  $X_{ij} \subset X_i$  be the open subspace corresponding to the image of  $|U_{ij}| \rightarrow |X_i|$ , see Lemma 4.10. Note that  $U_{ij} \rightarrow X_{ij}$  is quasi-compact as  $X_{ij} \subset X_i$  is a monomorphism and as  $U_{ij} \rightarrow X$  is quasi-compact. Then  $X = \bigcup X_{ij}$  is a covering as in (4). The implication (4)  $\Rightarrow$  (3) is immediate.

Assume (4). To show that  $X$  is Zariski locally quasi-separated over  $S$  it suffices to show that  $X_i$  is quasi-separated over  $S$ . Hence we may assume there exists an affine scheme  $U$  mapping into an affine open of  $S$  and a quasi-compact surjective étale morphism  $U \rightarrow X$ . Consider the fibre product square

$$\begin{array}{ccc} U \times_X U & \longrightarrow & U \times_S U \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X \end{array}$$

The right vertical arrow is surjective étale (see Spaces, Lemma 5.7) and  $U \times_S U$  is affine (as  $U$  maps into an affine open of  $S$ , see Schemes, Section 17), and  $U \times_X U$  is quasi-compact because the projection  $U \times_X U \rightarrow U$  is quasi-compact as a base change of  $U \rightarrow X$ . It follows from Spaces, Lemma 11.4 that  $\Delta_{X/S}$  is quasi-compact as desired.

Assume (1). To prove (3) there is an immediate reduction to the case where  $X$  is quasi-separated over  $S$ . By Lemma 6.2 we can find a Zariski open covering  $X = \bigcup X_i$  such that each  $X_i$  maps into an affine open of  $S$ , and such that there exist affine schemes  $U_i$  and surjective étale morphisms  $U_i \rightarrow X_i$ . Since  $U_i \rightarrow S$  maps into an affine open of  $S$  we see that  $U_i \times_S U_i$  is affine, see Schemes, Section 17. As  $X$  is quasi-separated over  $S$ , the morphisms

$$R_i = U_i \times_{X_i} U_i = U_i \times_X U_i \longrightarrow U_i \times_S U_i$$

as base changes of  $\Delta_{X/S}$  are quasi-compact. Hence we conclude that  $R_i$  is a quasi-compact scheme. This in turn implies that each projection  $R_i \rightarrow U_i$  is quasi-compact. Hence, applying Spaces, Lemma 11.4 to the covering  $U_i \rightarrow X_i$  and the morphism  $U_i \rightarrow X_i$  we conclude that the morphisms  $U_i \rightarrow X_i$  are quasi-compact as desired.

At this point we see that (1), (3), and (4) are equivalent. Since (3) does not refer to the base scheme we conclude that these are also equivalent with (2).  $\square$

The following lemma will turn out to be quite useful.

**Lemma 6.7.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $U$  be a scheme. Let  $\varphi : U \rightarrow X$  be an étale morphism such that the projections  $R = U \times_X U \rightarrow U$  are quasi-compact; for example if  $\varphi$  is quasi-compact. Then the fibres of*

$$|U| \rightarrow |X| \quad \text{and} \quad |R| \rightarrow |X|$$

*are finite.*

**Proof.** Denote  $R = U \times_X U$ , and  $s, t : R \rightarrow U$  the projections. Let  $u \in U$  be a point, and let  $x \in |X|$  be its image. The fibre of  $|U| \rightarrow |X|$  over  $x$  is equal to  $s(t^{-1}(\{u\}))$  by Lemma 4.3, and the fibre of  $|R| \rightarrow |X|$  over  $x$  is  $t^{-1}(s(t^{-1}(\{u\})))$ . Since  $t : R \rightarrow U$  is étale and quasi-compact, it has finite fibres (as its fibres are disjoint unions of spectra of fields by Morphisms, Lemma 36.7 and quasi-compact). Hence we win.  $\square$

## 7. Properties of Spaces defined by properties of schemes

Any étale local property of schemes gives rise to a corresponding property of algebraic spaces via the following lemma.

**Lemma 7.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{P}$  be a property of schemes which is local in the étale topology, see Descent, Definition 15.1. The following are equivalent*

- (1) *for some scheme  $U$  and surjective étale morphism  $U \rightarrow X$  the scheme  $U$  has property  $\mathcal{P}$ , and*
- (2) *for every scheme  $U$  and every étale morphism  $U \rightarrow X$  the scheme  $U$  has property  $\mathcal{P}$ .*

*If  $X$  is representable this is equivalent to  $\mathcal{P}(X)$ .*

**Proof.** The implication (2)  $\Rightarrow$  (1) is immediate. For the converse, choose a surjective étale morphism  $U \rightarrow X$  with  $U$  a scheme that has  $\mathcal{P}$  and let  $V$  be an étale  $X$ -scheme. Then  $U \times_X V \rightarrow V$  is an étale surjection of schemes, so  $V$  inherits  $\mathcal{P}$  from  $U \times_X V$ , which in turn inherits  $\mathcal{P}$  from  $U$  (see discussion following Descent, Definition 15.1). The last claim is clear from (1) and Descent, Definition 15.1.  $\square$

**Definition 7.2.** Let  $\mathcal{P}$  be a property of schemes which is local in the étale topology. Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . We say  $X$  has property  $\mathcal{P}$  if any of the equivalent conditions of Lemma 7.1 hold.

**Remark 7.3.** Here is a list of properties which are local for the étale topology (keep in mind that the fpqc, fppf, syntomic, and smooth topologies are stronger than the étale topology):

- (1) locally Noetherian, see Descent, Lemma 16.1,
- (2) Jacobson, see Descent, Lemma 16.2,
- (3) locally Noetherian and  $(S_k)$ , see Descent, Lemma 17.1,
- (4) Cohen-Macaulay, see Descent, Lemma 17.2,
- (5) Gorenstein, see Duality for Schemes, Lemma 24.6,
- (6) reduced, see Descent, Lemma 18.1,
- (7) normal, see Descent, Lemma 18.2,
- (8) locally Noetherian and  $(R_k)$ , see Descent, Lemma 18.3,
- (9) regular, see Descent, Lemma 18.4,
- (10) Nagata, see Descent, Lemma 18.5.

Any étale local property of germs of schemes gives rise to a corresponding property of algebraic spaces. Here is the obligatory lemma.

**Lemma 7.4.** *Let  $\mathcal{P}$  be a property of germs of schemes which is étale local, see Descent, Definition 21.1. Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$  be a point of  $X$ . Consider étale morphisms  $a : U \rightarrow X$  where  $U$  is a scheme. The following are equivalent*

- (1) *for any  $U \rightarrow X$  as above and  $u \in U$  with  $a(u) = x$  we have  $\mathcal{P}(U, u)$ , and*
- (2) *for some  $U \rightarrow X$  as above and  $u \in U$  with  $a(u) = x$  we have  $\mathcal{P}(U, u)$ .*

*If  $X$  is representable, then this is equivalent to  $\mathcal{P}(X, x)$ .*

**Proof.** Omitted. □

**Definition 7.5.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$ . Let  $\mathcal{P}$  be a property of germs of schemes which is étale local. We say  $X$  has property  $\mathcal{P}$  at  $x$  if any of the equivalent conditions of Lemma 7.4 hold.

**Remark 7.6.** Let  $P$  be a property of local rings. Assume that for any étale ring map  $A \rightarrow B$  and  $\mathfrak{q}$  is a prime of  $B$  lying over the prime  $\mathfrak{p}$  of  $A$ , then  $P(A_{\mathfrak{p}}) \Leftrightarrow P(B_{\mathfrak{q}})$ . Then we obtain an étale local property of germs  $(U, u)$  of schemes by setting  $\mathcal{P}(U, u) = P(\mathcal{O}_{U, u})$ . In this situation we will use the terminology “the local ring of  $X$  at  $x$  has  $P$ ” to mean  $X$  has property  $\mathcal{P}$  at  $x$ . Here is a list of such properties  $P$ :

- (1) Noetherian, see More on Algebra, Lemma 44.1,
- (2) dimension  $d$ , see More on Algebra, Lemma 44.2,
- (3) regular, see More on Algebra, Lemma 44.3,
- (4) discrete valuation ring, follows from (2), (3), and Algebra, Lemma 119.7,
- (5) reduced, see More on Algebra, Lemma 45.4,
- (6) normal, see More on Algebra, Lemma 45.6,
- (7) Noetherian and depth  $k$ , see More on Algebra, Lemma 45.8,
- (8) Noetherian and Cohen-Macaulay, see More on Algebra, Lemma 45.9,
- (9) Noetherian and Gorenstein, see Dualizing Complexes, Lemma 21.8.

There are more properties for which this holds, for example G-ring and Nagata. If we every need these we will add them here as well as references to detailed proofs of the corresponding algebra facts.

## 8. Constructible sets

**Lemma 8.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $E \subset |X|$  be a subset. The following are equivalent*

- (1) *for every étale morphism  $U \rightarrow X$  where  $U$  is a scheme the inverse image of  $E$  in  $U$  is a locally constructible subset of  $U$ ,*
- (2) *for every étale morphism  $U \rightarrow X$  where  $U$  is an affine scheme the inverse image of  $E$  in  $U$  is a constructible subset of  $U$ ,*
- (3) *for some surjective étale morphism  $U \rightarrow X$  where  $U$  is a scheme the inverse image of  $E$  in  $U$  is a locally constructible subset of  $U$ .*

**Proof.** By Properties, Lemma 2.1 we see that (1) and (2) are equivalent. It is immediate that (1) implies (3). Thus we assume we have a surjective étale morphism  $\varphi : U \rightarrow X$  where  $U$  is a scheme such that  $\varphi^{-1}(E)$  is locally constructible. Let  $\varphi' : U' \rightarrow X$  be another étale morphism where  $U'$  is a scheme. Then we have

$$E'' = \text{pr}_1^{-1}(\varphi^{-1}(E)) = \text{pr}_2^{-1}((\varphi')^{-1}(E))$$

where  $\text{pr}_1 : U \times_X U' \rightarrow U$  and  $\text{pr}_2 : U \times_X U' \rightarrow U'$  are the projections. By Morphisms, Lemma 22.1 we see that  $E''$  is locally constructible in  $U \times_X U'$ . Let  $W' \subset U'$  be an affine open. Since  $\text{pr}_2$  is étale and hence open, we can choose a quasi-compact open  $W'' \subset U \times_X U'$  with  $\text{pr}_2(W'') = W'$ . Then  $\text{pr}_2|_{W''} : W'' \rightarrow W'$  is quasi-compact. We have  $W' \cap (\varphi')^{-1}(E) = \text{pr}_2(E'' \cap W'')$  as  $\varphi$  is surjective, see Lemma 4.3. Thus  $W' \cap (\varphi')^{-1}(E) = \text{pr}_2(E'' \cap W'')$  is locally constructible by Morphisms, Theorem 22.3 as desired.  $\square$

**Definition 8.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $E \subset |X|$  be a subset. We say  $E$  is *étale locally constructible* if the equivalent conditions of Lemma 8.1 are satisfied.

Of course, if  $X$  is representable, i.e.,  $X$  is a scheme, then this just means  $E$  is a locally constructible subset of the underlying topological space.

## 9. Dimension at a point

We can use Descent, Lemma 21.2 to define the dimension of an algebraic space  $X$  at a point  $x$ . This will give us a different notion than the topological one (i.e., the dimension of  $|X|$  at  $x$ ).

**Definition 9.1.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$  be a point of  $X$ . We define the *dimension of  $X$  at  $x$*  to be the element  $\dim_x(X) \in \{0, 1, 2, \dots, \infty\}$  such that  $\dim_x(X) = \dim_u(U)$  for any (equivalently some) pair  $(a : U \rightarrow X, u)$  consisting of an étale morphism  $a : U \rightarrow X$  from a scheme to  $X$  and a point  $u \in U$  with  $a(u) = x$ . See Definition 7.5, Lemma 7.4, and Descent, Lemma 21.2.

Warning: It is **not** the case that  $\dim_x(X) = \dim_x(|X|)$  in general. A counter example is the algebraic space  $X$  of Spaces, Example 14.9. Namely, let  $x \in |X|$  be a point not equal to the generic point  $x_0$  of  $|X|$ . Then we have  $\dim_x(X) = 0$  but

$\dim_x(|X|) = 1$ . In particular, the dimension of  $X$  (as defined below) is different from the dimension of  $|X|$ .

**Definition 9.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The *dimension*  $\dim(X)$  of  $X$  is defined by the rule

$$\dim(X) = \sup_{x \in |X|} \dim_x(X)$$

By Properties, Lemma 10.2 we see that this is the usual notion if  $X$  is a scheme. There is another integer that measures the dimension of a scheme at a point, namely the dimension of the local ring. This invariant is compatible with étale morphisms also, see Section 10.

## 10. Dimension of local rings

The dimension of the local ring of an algebraic space is a well defined concept.

**Lemma 10.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$  be a point. Let  $d \in \{0, 1, 2, \dots, \infty\}$ . The following are equivalent*

- (1) *for some scheme  $U$  and étale morphism  $a : U \rightarrow X$  and point  $u \in U$  with  $a(u) = x$  we have  $\dim(\mathcal{O}_{U,u}) = d$ ,*
- (2) *for any scheme  $U$ , any étale morphism  $a : U \rightarrow X$ , and any point  $u \in U$  with  $a(u) = x$  we have  $\dim(\mathcal{O}_{U,u}) = d$ .*

*If  $X$  is a scheme, this is equivalent to  $\dim(\mathcal{O}_{X,x}) = d$ .*

**Proof.** Combine Lemma 7.4 and Descent, Lemma 21.3. □

**Definition 10.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$  be a point. The *dimension of the local ring of  $X$  at  $x$*  is the element  $d \in \{0, 1, 2, \dots, \infty\}$  satisfying the equivalent conditions of Lemma 10.1. In this case we will also say  *$x$  is a point of codimension  $d$  on  $X$* .

Besides the lemma below we also point the reader to Lemmas 22.4 and 22.5.

**Lemma 10.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The following quantities are equal:*

- (1) *The dimension of  $X$ .*
- (2) *The supremum of the dimensions of the local rings of  $X$ .*
- (3) *The supremum of  $\dim_x(X)$  for  $x \in |X|$ .*

**Proof.** The numbers in (1) and (3) are equal by Definition 9.2. Let  $U \rightarrow X$  be a surjective étale morphism from a scheme  $U$ . The supremum of  $\dim_x(X)$  for  $x \in |X|$  is the same as the supremum of  $\dim_u(U)$  for points  $u$  of  $U$  by definition. This is the same as the supremum of  $\dim(\mathcal{O}_{U,u})$  by Properties, Lemma 10.2. This in turn is the same as (2) by definition. □

## 11. Generic points

Let  $T$  be a topological space. According to the second edition of EGA I, a *maximal point* of  $T$  is a generic point of an irreducible component of  $T$ . If  $T = |X|$  is the topological space associated to an algebraic space  $X$ , there are at least two notions of maximal points: we can look at maximal points of  $T$  viewed as a topological space, or we can look at images of maximal points of  $U$  where  $U \rightarrow X$  is an étale morphism and  $U$  is a scheme. The second notion corresponds to the set of points

of codimension 0 (Lemma 11.1). The codimension 0 points are easier to work with for general algebraic spaces; the two notions agree for quasi-separated and more generally decent algebraic spaces (Decent Spaces, Lemma 20.1).

**Lemma 11.1.** *Let  $S$  be a scheme and let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$ . Consider étale morphisms  $a : U \rightarrow X$  where  $U$  is a scheme. The following are equivalent*

- (1)  *$x$  is a point of codimension 0 on  $X$ ,*
- (2) *for some  $U \rightarrow X$  as above and  $u \in U$  with  $a(u) = x$ , the point  $u$  is the generic point of an irreducible component of  $U$ , and*
- (3) *for any  $U \rightarrow X$  as above and any  $u \in U$  mapping to  $x$ , the point  $u$  is the generic point of an irreducible component of  $U$ .*

*If  $X$  is representable, this is equivalent to  $x$  being a generic point of an irreducible component of  $|X|$ .*

**Proof.** Observe that a point  $u$  of a scheme  $U$  is a generic point of an irreducible component of  $U$  if and only if  $\dim(\mathcal{O}_{U,u}) = 0$  (Properties, Lemma 10.4). Hence this follows from the definition of the codimension of a point on  $X$  (Definition 10.2).  $\square$

**Lemma 11.2.** *Let  $S$  be a scheme and let  $X$  be an algebraic space over  $S$ . The set of codimension 0 points of  $X$  is dense in  $|X|$ .*

**Proof.** If  $U$  is a scheme, then the set of generic points of irreducible components is dense in  $U$  (holds for any quasi-sober topological space). Thus if  $U \rightarrow X$  is a surjective étale morphism, then the set of codimension 0 points of  $X$  is the image of a dense subset of  $|U|$  (Lemma 11.1). Since  $|X|$  has the quotient topology for  $|U| \rightarrow |X|$  we conclude.  $\square$

## 12. Reduced spaces

We have already defined reduced algebraic spaces in Section 7. Here we just prove some simple lemmas regarding reduced algebraic spaces.

**Lemma 12.1.** *Let  $S$  be a scheme. Let  $Z \rightarrow X$  be an immersion of algebraic spaces. Then  $|Z| \rightarrow |X|$  is a homeomorphism of  $|Z|$  onto a locally closed subset of  $|X|$ .*

**Proof.** Let  $U$  be a scheme and  $U \rightarrow X$  a surjective étale morphism. Then  $Z \times_X U \rightarrow U$  is an immersion of schemes, hence gives a homeomorphism of  $|Z \times_X U|$  with a locally closed subset  $T'$  of  $|U|$ . By Lemma 4.3 the subset  $T'$  is the inverse image of the image  $T$  of  $|Z| \rightarrow |X|$ . The map  $|Z| \rightarrow |X|$  is injective because the transformation of functors  $Z \rightarrow X$  is injective, see Spaces, Section 12. By Topology, Lemma 6.4 we see that  $T$  is locally closed in  $|X|$ . Moreover, the continuous map  $|Z| \rightarrow T$  is a homeomorphism as the map  $|Z \times_X U| \rightarrow T'$  is a homeomorphism and  $|Z \times_Y U| \rightarrow |Z|$  is submersive.  $\square$

The following lemma will help us construct (locally) closed subspaces.

**Lemma 12.2.** *Let  $S$  be a scheme. Let  $j : R \rightarrow U \times_S U$  be an étale equivalence relation. Let  $X = U/R$  be the associated algebraic space (Spaces, Theorem 10.5). There is a canonical bijection*

*$R$ -invariant locally closed subschemes  $Z'$  of  $U \leftrightarrow$  locally closed subspaces  $Z$  of  $X$*   
*Moreover, if  $Z \rightarrow X$  is closed (resp. open) if and only if  $Z' \rightarrow U$  is closed (resp. open).*

**Proof.** Denote  $\varphi : U \rightarrow X$  the canonical map. The bijection sends  $Z \rightarrow X$  to  $Z' = Z \times_X U \rightarrow U$ . It is immediate from the definition that  $Z' \rightarrow U$  is an immersion, resp. closed immersion, resp. open immersion if  $Z \rightarrow X$  is so. It is also clear that  $Z'$  is  $R$ -invariant (see Groupoids, Definition 19.1).

Conversely, assume that  $Z' \rightarrow U$  is an immersion which is  $R$ -invariant. Let  $R'$  be the restriction of  $R$  to  $Z'$ , see Groupoids, Definition 18.2. Since  $R' = R \times_{s,U} Z' = Z' \times_{U,t} R$  in this case we see that  $R'$  is an étale equivalence relation on  $Z'$ . By Spaces, Theorem 10.5 we see  $Z = Z'/R'$  is an algebraic space. By construction we have  $U \times_X Z = Z'$ , so  $U \times_X Z \rightarrow Z$  is an immersion. Note that the property “immersion” is preserved under base change and fppf local on the base (see Spaces, Section 4). Moreover, immersions are separated and locally quasi-finite (see Schemes, Lemma 23.8 and Morphisms, Lemma 20.16). Hence by More on Morphisms, Lemma 57.1 immersions satisfy descent for fppf covering. This means all the hypotheses of Spaces, Lemma 11.5 are satisfied for  $Z \rightarrow X$ ,  $\mathcal{P}$  = “immersion”, and the étale surjective morphism  $U \rightarrow X$ . We conclude that  $Z \rightarrow X$  is representable and an immersion, which is the definition of a subspace (see Spaces, Definition 12.1).

It is clear that these constructions are inverse to each other and we win.  $\square$

**Lemma 12.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $T \subset |X|$  be a closed subset. There exists a unique closed subspace  $Z \subset X$  with the following properties: (a) we have  $|Z| = T$ , and (b)  $Z$  is reduced.*

**Proof.** Let  $U \rightarrow X$  be a surjective étale morphism, where  $U$  is a scheme. Set  $R = U \times_X U$ , so that  $X = U/R$ , see Spaces, Lemma 9.1. As usual we denote  $s, t : R \rightarrow U$  the two projection morphisms. By Lemma 4.5 we see that  $T$  corresponds to a closed subset  $T' \subset |U|$  such that  $s^{-1}(T') = t^{-1}(T')$ . Let  $Z' \subset U$  be the reduced induced scheme structure on  $T'$ . In this case the fibre products  $Z' \times_{U,t} R$  and  $Z' \times_{U,s} R$  are closed subschemes of  $R$  (Schemes, Lemma 18.2) which are étale over  $Z'$  (Morphisms, Lemma 36.4), and hence reduced (because being reduced is local in the étale topology, see Remark 7.3). Since they have the same underlying topological space (see above) we conclude that  $Z' \times_{U,t} R = Z' \times_{U,s} R$ . Thus we can apply Lemma 12.2 to obtain a closed subspace  $Z \subset X$  whose pullback to  $U$  is  $Z'$ . By construction  $|Z| = T$  and  $Z$  is reduced. This proves existence. We omit the proof of uniqueness.  $\square$

**Lemma 12.4.** *Let  $S$  be a scheme. Let  $X, Y$  be algebraic spaces over  $S$ . Let  $Z \subset X$  be a closed subspace. Assume  $Y$  is reduced. A morphism  $f : Y \rightarrow X$  factors through  $Z$  if and only if  $f(|Y|) \subset |Z|$ .*

**Proof.** Assume  $f(|Y|) \subset |Z|$ . Choose a diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & U \\ b \downarrow & & \downarrow a \\ Y & \xrightarrow{f} & X \end{array}$$

where  $U, V$  are schemes, and the vertical arrows are surjective and étale. The scheme  $V$  is reduced, see Lemma 7.1. Hence  $h$  factors through  $a^{-1}(Z)$  by Schemes, Lemma 12.7. So  $a \circ h$  factors through  $Z$ . As  $Z \subset X$  is a subsheaf, and  $V \rightarrow Y$  is a surjection of sheaves on  $(Sch/S)_{fppf}$  we conclude that  $X \rightarrow Y$  factors through  $Z$ .  $\square$



**Definition 12.5.** Let  $S$  be a scheme, and let  $X$  be an algebraic space over  $S$ . Let  $Z \subset |X|$  be a closed subset. An *algebraic space structure on  $Z$*  is given by a closed subspace  $Z'$  of  $X$  with  $|Z'|$  equal to  $Z$ . The *reduced induced algebraic space structure* on  $Z$  is the one constructed in Lemma 12.3. The *reduction  $X_{\text{red}}$*  of  $X$  is the reduced induced algebraic space structure on  $|X|$ .

### 13. The schematic locus

Every algebraic space has a largest open subspace which is a scheme; this is more or less clear but we also write out the proof below. Of course this subspace may be empty, for example if  $X = \mathbf{A}_{\mathbf{Q}}^1/\mathbf{Z}$  (the universal counter example). On the other hand, if  $X$  is for example quasi-separated, then this largest open subscheme is actually dense in  $X$ !

**Lemma 13.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . There exists a largest open subspace  $X' \subset X$  which is a scheme.*

**Proof.** Let  $U \rightarrow X$  be an étale surjective morphism, where  $U$  is a scheme. Let  $R = U \times_X U$ . The open subspaces of  $X$  correspond 1 – 1 with open subschemes of  $U$  which are  $R$ -invariant. Hence there is a set of them. Let  $X_i$ ,  $i \in I$  be the set of open subspaces of  $X$  which are schemes, i.e., are representable. Consider the open subspace  $X' \subset X$  whose underlying set of points is the open  $\bigcup |X_i|$  of  $|X|$ . By Lemma 4.4 we see that

$$\coprod X_i \longrightarrow X'$$

is a surjective map of sheaves on  $(\text{Sch}/S)_{\text{fppf}}$ . But since each  $X_i \rightarrow X'$  is representable by open immersions we see that in fact the map is surjective in the Zariski topology. Namely, if  $T \rightarrow X'$  is a morphism from a scheme into  $X'$ , then  $X_i \times_{X'} T$  is an open subscheme of  $T$ . Hence we can apply Schemes, Lemma 15.4 to see that  $X'$  is a scheme.  $\square$

In the rest of this section we say that an open subspace  $X'$  of an algebraic space  $X$  is *dense* if the corresponding open subset  $|X'| \subset |X|$  is dense.

**Lemma 13.2.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . If there exists a finite, étale, surjective morphism  $U \rightarrow X$  where  $U$  is a quasi-separated scheme, then there exists a dense open subspace  $X'$  of  $X$  which is a scheme. More precisely, every point  $x \in |X|$  of codimension 0 in  $X$  is contained in  $X'$ .*

**Proof.** Let  $X' \subset X$  be the maximal open subspace which is a scheme (Lemma 13.1). Let  $x \in |X|$  be a point of codimension 0 on  $X$ . By Lemma 11.2 it suffices to show  $x \in X'$ . Let  $U \rightarrow X$  be as in the statement of the lemma. Write  $R = U \times_X U$  and denote  $s, t : R \rightarrow U$  the projections as usual. Note that  $s, t$  are surjective, finite and étale. By Lemma 6.7 the fibre of  $|U| \rightarrow |X|$  over  $x$  is finite, say  $\{\eta_1, \dots, \eta_n\}$ . By Lemma 11.1 each  $\eta_i$  is the generic point of an irreducible component of  $U$ . By Properties, Lemma 29.1 we can find an affine open  $W \subset U$  containing  $\{\eta_1, \dots, \eta_n\}$  (this is where we use that  $U$  is quasi-separated). By Groupoids, Lemma 24.1 we may assume that  $W$  is  $R$ -invariant. Since  $W \subset U$  is an  $R$ -invariant affine open, the restriction  $R_W$  of  $R$  to  $W$  equals  $R_W = s^{-1}(W) = t^{-1}(W)$  (see Groupoids, Definition 19.1 and discussion following it). In particular the maps  $R_W \rightarrow W$  are finite étale also. It follows that  $R_W$  is affine. Thus we see that  $W/R_W$  is a scheme, by Groupoids, Proposition 23.9. On the other hand,  $W/R_W$  is an open subspace of  $X$  by Spaces, Lemma 10.2 and it contains  $x$  by construction.  $\square$

We will improve the following proposition to the case of decent algebraic spaces in Decent Spaces, Theorem 10.2.

**Proposition 13.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . If  $X$  is Zariski locally quasi-separated (for example if  $X$  is quasi-separated), then there exists a dense open subspace  $X'$  of  $X$  which is a scheme. More precisely, every point  $x \in |X|$  of codimension 0 on  $X$  is contained in  $X'$ .*

**Proof.** The question is local on  $X$  by Lemma 13.1. Thus by Lemma 6.6 we may assume that there exists an affine scheme  $U$  and a surjective, quasi-compact, étale morphism  $U \rightarrow X$ . Moreover  $U \rightarrow X$  is separated (Lemma 6.4). Set  $R = U \times_X U$  and denote  $s, t : R \rightarrow U$  the projections as usual. Then  $s, t$  are surjective, quasi-compact, separated, and étale. Hence  $s, t$  are also quasi-finite and have finite fibres (Morphisms, Lemmas 36.6, 20.9, and 20.10). By Morphisms, Lemma 51.1 for every  $\eta \in U$  which is the generic point of an irreducible component of  $U$ , there exists an open neighbourhood  $V \subset U$  of  $\eta$  such that  $s^{-1}(V) \rightarrow V$  is finite. By Descent, Lemma 23.23 being finite is fpqc (and in particular étale) local on the target. Hence we may apply More on Groupoids, Lemma 6.4 which says that the largest open  $W \subset U$  over which  $s$  is finite is  $R$ -invariant. By the above  $W$  contains every generic point of an irreducible component of  $U$ . The restriction  $R_W$  of  $R$  to  $W$  equals  $R_W = s^{-1}(W) = t^{-1}(W)$  (see Groupoids, Definition 19.1 and discussion following it). By construction  $s_W, t_W : R_W \rightarrow W$  are finite étale. Consider the open subspace  $X' = W/R_W \subset X$  (see Spaces, Lemma 10.2). By construction the inclusion map  $X' \rightarrow X$  induces a bijection on points of codimension 0. This reduces us to Lemma 13.2.  $\square$

#### 14. Obtaining a scheme

We have used in the previous section that the quotient  $U/R$  of an affine scheme  $U$  by an equivalence relation  $R$  is a scheme if the morphisms  $s, t : R \rightarrow U$  are finite étale. This is a special case of the following result.

**Proposition 14.1.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume*

- (1)  $s, t : R \rightarrow U$  finite locally free,
- (2)  $j = (t, s)$  is an equivalence, and
- (3) *for a dense set of points  $u \in U$  the  $R$ -equivalence class  $t(s^{-1}(\{u\}))$  is contained in an affine open of  $U$ .*

*Then there exists a finite locally free morphism  $U \rightarrow M$  of schemes over  $S$  such that  $R = U \times_M U$  and such that  $M$  represents the quotient sheaf  $U/R$  in the fppf topology.*

**Proof.** By assumption (3) and Groupoids, Lemma 24.1 we can find an open covering  $U = \bigcup U_i$  such that each  $U_i$  is an  $R$ -invariant affine open of  $U$ . Set  $R_i = R|_{U_i}$ . Consider the fppf sheaves  $F = U/R$  and  $F_i = U_i/R_i$ . By Spaces, Lemma 10.2 the morphisms  $F_i \rightarrow F$  are representable and open immersions. By Groupoids, Proposition 23.9 the sheaves  $F_i$  are representable by affine schemes. If  $T$  is a scheme and  $T \rightarrow F$  is a morphism, then  $V_i = F_i \times_F T$  is open in  $T$  and we claim that  $T = \bigcup V_i$ . Namely, fppf locally on  $T$  we can lift  $T \rightarrow F$  to a morphism  $f : T \rightarrow U$  and in that case  $f^{-1}(U_i) \subset V_i$ . Hence we conclude that  $F$  is representable by a scheme, see Schemes, Lemma 15.4.  $\square$

For example, if  $U$  is isomorphic to a locally closed subscheme of an affine scheme or isomorphic to a locally closed subscheme of  $\text{Proj}(A)$  for some graded ring  $A$ , then the third assumption holds by Properties, Lemma 29.5. In particular we can apply this to free actions of finite groups and finite group schemes on quasi-affine or quasi-projective schemes. For example, the quotient  $X/G$  of a quasi-projective variety  $X$  by a free action of a finite group  $G$  is a scheme. Here is a detailed statement.

**Lemma 14.2.** *Let  $S$  be a scheme. Let  $G \rightarrow S$  be a group scheme. Let  $X \rightarrow S$  be a morphism of schemes. Let  $a : G \times_S X \rightarrow X$  be an action. Assume that*

- (1)  $G \rightarrow S$  is finite locally free,
- (2) the action  $a$  is free,
- (3)  $X \rightarrow S$  is affine, or quasi-affine, or projective, or quasi-projective, or  $X$  is isomorphic to an open subscheme of an affine scheme, or  $X$  is isomorphic to an open subscheme of  $\text{Proj}(A)$  for some graded ring  $A$ , or  $G \rightarrow S$  is radicial.

*Then the fppf quotient sheaf  $X/G$  is a scheme and  $X \rightarrow X/G$  is an fppf  $G$ -torsor.*

**Proof.** We first show that  $X/G$  is a scheme. Since the action is free the morphism  $j = (a, \text{pr}) : G \times_S X \rightarrow X \times_S X$  is a monomorphism and hence an equivalence relation, see Groupoids, Lemma 10.3. The maps  $s, t : G \times_S X \rightarrow X$  are finite locally free as we've assumed that  $G \rightarrow S$  is finite locally free. To conclude it now suffices to prove the last assumption of Proposition 14.1 holds. Since the action of  $G$  is over  $S$  it suffices to prove that any finite set of points in a fibre of  $X \rightarrow S$  is contained in an affine open of  $X$ . If  $X$  is isomorphic to an open subscheme of an affine scheme or isomorphic to an open subscheme of  $\text{Proj}(A)$  for some graded ring  $A$  this follows from Properties, Lemma 29.5. If  $X \rightarrow S$  is affine, or quasi-affine, or projective, or quasi-projective, we may replace  $S$  by an affine open and we get back to the case we just dealt with. If  $G \rightarrow S$  is radicial, then the orbits of points on  $X$  under the action of  $G$  are singletons and the condition trivially holds. Some details omitted.

To see that  $X \rightarrow X/G$  is an fppf  $G$ -torsor (Groupoids, Definition 11.3) we have to show that  $G \times_S X \rightarrow X \times_{X/G} X$  is an isomorphism and that  $X \rightarrow X/G$  fppf locally has sections. The second part is clear from the fact that  $X \rightarrow X/G$  is surjective as a map of fppf sheaves (by construction). The first part follows from the isomorphism  $R = U \times_M U$  in the conclusion of Proposition 14.1 (note that  $R = G \times_S X$  in our case).  $\square$

**Lemma 14.3.** *Notation and assumptions as in Proposition 14.1. Then*

- (1) *if  $U$  is quasi-separated over  $S$ , then  $U/R$  is quasi-separated over  $S$ ,*
- (2) *if  $U$  is quasi-separated, then  $U/R$  is quasi-separated,*
- (3) *if  $U$  is separated over  $S$ , then  $U/R$  is separated over  $S$ ,*
- (4) *if  $U$  is separated, then  $U/R$  is separated, and*
- (5) *add more here.*

*Similar results hold in the setting of Lemma 14.2.*

**Proof.** Since  $M$  represents the quotient sheaf we have a cartesian diagram

$$\begin{array}{ccc} R & \xrightarrow{j} & U \times_S U \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \times_S M \end{array}$$

of schemes. Since  $U \times_S U \rightarrow M \times_S M$  is surjective finite locally free, to show that  $M \rightarrow M \times_S M$  is quasi-compact, resp. a closed immersion, it suffices to show that  $j : R \rightarrow U \times_S U$  is quasi-compact, resp. a closed immersion, see Descent, Lemmas 23.1 and 23.19. Since  $j : R \rightarrow U \times_S U$  is a morphism over  $U$  and since  $R$  is finite over  $U$ , we see that  $j$  is quasi-compact as soon as the projection  $U \times_S U \rightarrow U$  is quasi-separated (Schemes, Lemma 21.14). Since  $j$  is a monomorphism and locally of finite type, we see that  $j$  is a closed immersion as soon as it is proper (Étale Morphisms, Lemma 7.2) which will be the case as soon as the projection  $U \times_S U \rightarrow U$  is separated (Morphisms, Lemma 41.7). This proves (1) and (3). To prove (2) and (4) we replace  $S$  by  $\text{Spec}(\mathbf{Z})$ , see Definition 3.1. Since Lemma 14.2 is proved through an application of Proposition 14.1 the final statement is clear too.  $\square$

## 15. Points on quasi-separated spaces

Points can behave very badly on algebraic spaces in the generality introduced in the Stacks project. However, for quasi-separated spaces their behaviour is mostly like the behaviour of points on schemes. We prove a few results on this in this section; the chapter on decent spaces contains many more results on this, see for example Decent Spaces, Section 12.

**Lemma 15.1.** *Let  $S$  be a scheme. Let  $X$  be a Zariski locally quasi-separated algebraic space over  $S$ . Then the topological space  $|X|$  is sober (see Topology, Definition 8.6).*

**Proof.** Combining Topology, Lemma 8.8 and Lemma 6.6 we see that we may assume that there exists an affine scheme  $U$  and a surjective, quasi-compact, étale morphism  $U \rightarrow X$ . Set  $R = U \times_X U$  with projection maps  $s, t : R \rightarrow U$ . Applying Lemma 6.7 we see that the fibres of  $s, t$  are finite. It follows all the assumptions of Topology, Lemma 19.8 are met, and we conclude that  $|X|$  is Kolmogorov<sup>3</sup>.

It remains to show that every irreducible closed subset  $T \subset |X|$  has a generic point. By Lemma 12.3 there exists a closed subspace  $Z \subset X$  with  $|Z| = |T|$ . Note that  $U \times_X Z \rightarrow Z$  is a quasi-compact, surjective, étale morphism from an affine scheme to  $Z$ , hence  $Z$  is Zariski locally quasi-separated by Lemma 6.6. By Proposition 13.3 we see that there exists an open dense subspace  $Z' \subset Z$  which is a scheme. This means that  $|Z'| \subset T$  is open dense. Hence the topological space  $|Z'|$  is irreducible, which means that  $Z'$  is an irreducible scheme. By Schemes, Lemma 11.1 we conclude that  $|Z'|$  is the closure of a single point  $\eta \in |Z'| \subset T$  and hence also  $T = \{\eta\}$ , and we win.  $\square$

**Lemma 15.2.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . The topological space  $|X|$  is a spectral space.*

<sup>3</sup>Actually we use here also Schemes, Lemma 11.1 (sobriety schemes), Morphisms, Lemmas 36.12 and 25.9 (generalizations lift along étale morphisms), Lemma 4.5 (points on an algebraic space in terms of a presentation), and Lemma 4.6 (openness quotient map).

**Proof.** By Topology, Definition 23.1 we have to check that  $|X|$  is sober, quasi-compact, has a basis of quasi-compact opens, and the intersection of any two quasi-compact opens is quasi-compact. By Lemma 15.1 we see that  $|X|$  is sober. By Lemma 5.2 we see that  $|X|$  is quasi-compact. By Lemma 6.3 there exists an affine scheme  $U$  and a surjective étale morphism  $f : U \rightarrow X$ . Since  $|f| : |U| \rightarrow |X|$  is open and continuous and since  $|U|$  has a basis of quasi-compact opens, we conclude that  $|X|$  has a basis of quasi-compact opens. Finally, suppose that  $A, B \subset |X|$  are quasi-compact open. Then  $A = |X'|$  and  $B = |X''|$  for some open subspaces  $X', X'' \subset X$  (Lemma 4.8) and we can choose affine schemes  $V$  and  $W$  and surjective étale morphisms  $V \rightarrow X'$  and  $W \rightarrow X''$  (Lemma 6.3). Then  $A \cap B$  is the image of  $|V \times_X W| \rightarrow |X|$  (Lemma 4.3). Since  $V \times_X W$  is quasi-compact as  $X$  is quasi-separated (Lemma 3.3) we conclude that  $A \cap B$  is quasi-compact and the proof is finished.  $\square$

The following lemma can be used to prove that an algebraic space is isomorphic to the spectrum of a field.

**Lemma 15.3.** *Let  $S$  be a scheme. Let  $k$  be a field. Let  $X$  be an algebraic space over  $S$  and assume that there exists a surjective étale morphism  $\mathrm{Spec}(k) \rightarrow X$ . If  $X$  is quasi-separated, then  $X \cong \mathrm{Spec}(k')$  where  $k/k'$  is a finite separable extension.*

**Proof.** Set  $R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k)$ , so that we have a fibre product diagram

$$\begin{array}{ccc} R & \xrightarrow{s} & \mathrm{Spec}(k) \\ \downarrow t & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & X \end{array}$$

By Spaces, Lemma 9.1 we know  $X = \mathrm{Spec}(k)/R$  is the quotient sheaf. Because  $\mathrm{Spec}(k) \rightarrow X$  is étale, the morphisms  $s$  and  $t$  are étale. Hence  $R = \coprod_{i \in I} \mathrm{Spec}(k_i)$  is a disjoint union of spectra of fields, and both  $s$  and  $t$  induce finite separable field extensions  $s, t : k \subset k_i$ , see Morphisms, Lemma 36.7. Because

$$R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k) = (\mathrm{Spec}(k) \times_S \mathrm{Spec}(k)) \times_{X \times_S X, \Delta} X$$

and since  $\Delta$  is quasi-compact by assumption we conclude that  $R \rightarrow \mathrm{Spec}(k) \times_S \mathrm{Spec}(k)$  is quasi-compact. Hence  $R$  is quasi-compact as  $\mathrm{Spec}(k) \times_S \mathrm{Spec}(k)$  is affine. We conclude that  $I$  is finite. This implies that  $s$  and  $t$  are finite locally free morphisms. Hence by Groupoids, Proposition 23.9 we conclude that  $\mathrm{Spec}(k)/R$  is represented by  $\mathrm{Spec}(k')$ , with  $k' \subset k$  finite locally free where

$$k' = \{x \in k \mid s_i(x) = t_i(x) \text{ for all } i \in I\}$$

It is easy to see that  $k'$  is a field.  $\square$

**Remark 15.4.** Lemma 15.3 holds for decent algebraic spaces, see Decent Spaces, Lemma 12.8. In fact a decent algebraic space with one point is a scheme, see Decent Spaces, Lemma 14.2. This also holds when  $X$  is locally separated, because a locally separated algebraic space is decent, see Decent Spaces, Lemma 15.2.

## 16. Étale morphisms of algebraic spaces

This section really belongs in the chapter on morphisms of algebraic spaces, but we need the notion of an algebraic space étale over another in order to define

the small étale site of an algebraic space. Thus we need to do some preliminary work on étale morphisms from schemes to algebraic spaces, and étale morphisms between algebraic spaces. For more about étale morphisms of algebraic spaces, see Morphisms of Spaces, Section 39.

**Lemma 16.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $U, U'$  be schemes over  $S$ .*

- (1) *If  $U \rightarrow U'$  is an étale morphism of schemes, and if  $U' \rightarrow X$  is an étale morphism from  $U'$  to  $X$ , then the composition  $U \rightarrow X$  is an étale morphism from  $U$  to  $X$ .*
- (2) *If  $\varphi : U \rightarrow X$  and  $\varphi' : U' \rightarrow X$  are étale morphisms towards  $X$ , and if  $\chi : U \rightarrow U'$  is a morphism of schemes such that  $\varphi = \varphi' \circ \chi$ , then  $\chi$  is an étale morphism of schemes.*
- (3) *If  $\chi : U \rightarrow U'$  is a surjective étale morphism of schemes and  $\varphi' : U' \rightarrow X$  is a morphism such that  $\varphi = \varphi' \circ \chi$  is étale, then  $\varphi'$  is étale.*

**Proof.** Recall that our definition of an étale morphism from a scheme into an algebraic space comes from Spaces, Definition 5.1 via the fact that any morphism from a scheme into an algebraic space is representable.

Part (1) of the lemma follows from this, the fact that étale morphisms are preserved under composition (Morphisms, Lemma 36.3) and Spaces, Lemmas 5.4 and 5.3 (which are formal).

To prove part (2) choose a scheme  $W$  over  $S$  and a surjective étale morphism  $W \rightarrow X$ . Consider the base change  $\chi_W : W \times_X U \rightarrow W \times_X U'$  of  $\chi$ . As  $W \times_X U$  and  $W \times_X U'$  are étale over  $W$ , we conclude that  $\chi_W$  is étale, by Morphisms, Lemma 36.18. On the other hand, in the commutative diagram

$$\begin{array}{ccc} W \times_X U & \longrightarrow & W \times_X U' \\ \downarrow & & \downarrow \\ U & \longrightarrow & U' \end{array}$$

the two vertical arrows are étale and surjective. Hence by Descent, Lemma 14.4 we conclude that  $U \rightarrow U'$  is étale.

To prove part (3) choose a scheme  $W$  over  $S$  and a morphism  $W \rightarrow X$ . As above we consider the diagram

$$\begin{array}{ccccc} W \times_X U & \longrightarrow & W \times_X U' & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & U' & \longrightarrow & X \end{array}$$

Now we know that  $W \times_X U \rightarrow W \times_X U'$  is surjective étale (as a base change of  $U \rightarrow U'$ ) and that  $W \times_X U \rightarrow W$  is étale. Thus  $W \times_X U' \rightarrow W$  is étale by Descent, Lemma 14.4. By definition this means that  $\varphi'$  is étale.  $\square$

**Definition 16.2.** Let  $S$  be a scheme. A morphism  $f : X \rightarrow Y$  between algebraic spaces over  $S$  is called *étale* if and only if for every étale morphism  $\varphi : U \rightarrow X$  where  $U$  is a scheme, the composition  $f \circ \varphi$  is étale also.

If  $X$  and  $Y$  are schemes, then this agrees with the usual notion of an étale morphism of schemes. In fact, whenever  $X \rightarrow Y$  is a representable morphism of algebraic spaces, then this agrees with the notion defined via Spaces, Definition 5.1. This follows by combining Lemma 16.3 below and Spaces, Lemma 11.4.

**Lemma 16.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . The following are equivalent:*

- (1)  $f$  is étale,
- (2) there exists a surjective étale morphism  $\varphi : U \rightarrow X$ , where  $U$  is a scheme, such that the composition  $f \circ \varphi$  is étale (as a morphism of algebraic spaces),
- (3) there exists a surjective étale morphism  $\psi : V \rightarrow Y$ , where  $V$  is a scheme, such that the base change  $V \times_X Y \rightarrow V$  is étale (as a morphism of algebraic spaces),
- (4) there exists a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where  $U, V$  are schemes, the vertical arrows are étale, and the left vertical arrow is surjective such that the horizontal arrow is étale.

**Proof.** Let us prove that (4) implies (1). Assume a diagram as in (4) given. Let  $W \rightarrow X$  be an étale morphism with  $W$  a scheme. Then we see that  $W \times_X U \rightarrow U$  is étale. Hence  $W \times_X U \rightarrow V$  is étale as the composition of the étale morphisms of schemes  $W \times_X U \rightarrow U$  and  $U \rightarrow V$ . Therefore  $W \times_X U \rightarrow Y$  is étale by Lemma 16.1 (1). Since also the projection  $W \times_X U \rightarrow W$  is surjective and étale, we conclude from Lemma 16.1 (3) that  $W \rightarrow Y$  is étale.

Let us prove that (1) implies (4). Assume (1). Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where  $U \rightarrow X$  and  $V \rightarrow Y$  are surjective and étale, see Spaces, Lemma 11.6. By assumption the morphism  $U \rightarrow Y$  is étale, and hence  $U \rightarrow V$  is étale by Lemma 16.1 (2).

We omit the proof that (2) and (3) are also equivalent to (1). □

**Lemma 16.4.** *The composition of two étale morphisms of algebraic spaces is étale.*

**Proof.** This is immediate from the definition. □

**Lemma 16.5.** *The base change of an étale morphism of algebraic spaces by any morphism of algebraic spaces is étale.*

**Proof.** Let  $X \rightarrow Y$  be an étale morphism of algebraic spaces over  $S$ . Let  $Z \rightarrow Y$  be a morphism of algebraic spaces. Choose a scheme  $U$  and a surjective étale

morphism  $U \rightarrow X$ . Choose a scheme  $W$  and a surjective étale morphism  $W \rightarrow Z$ . Then  $U \rightarrow Y$  is étale, hence in the diagram

$$\begin{array}{ccc} W \times_Y U & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z \times_Y X & \longrightarrow & Z \end{array}$$

the top horizontal arrow is étale. Moreover, the left vertical arrow is surjective and étale (verification omitted). Hence we conclude that the lower horizontal arrow is étale by Lemma 16.3.  $\square$

**Lemma 16.6.** *Let  $S$  be a scheme. Let  $X, Y, Z$  be algebraic spaces. Let  $g : X \rightarrow Z$ ,  $h : Y \rightarrow Z$  be étale morphisms and let  $f : X \rightarrow Y$  be a morphism such that  $h \circ f = g$ . Then  $f$  is étale.*

**Proof.** Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\chi} & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where  $U \rightarrow X$  and  $V \rightarrow Y$  are surjective and étale, see Spaces, Lemma 11.6. By assumption the morphisms  $\varphi : U \rightarrow X \rightarrow Z$  and  $\psi : V \rightarrow Y \rightarrow Z$  are étale. Moreover,  $\psi \circ \chi = \varphi$  by our assumption on  $f, g, h$ . Hence  $U \rightarrow V$  is étale by Lemma 16.1 part (2).  $\square$

**Lemma 16.7.** *Let  $S$  be a scheme. If  $X \rightarrow Y$  is an étale morphism of algebraic spaces over  $S$ , then the associated map  $|X| \rightarrow |Y|$  of topological spaces is open.*

**Proof.** This is clear from the diagram in Lemma 16.3 and Lemma 4.6.  $\square$

Finally, here is a fun lemma. It is not true that an algebraic space with an étale morphism towards a scheme is a scheme, see Spaces, Example 14.2. But it is true if the target is the spectrum of a field.

**Lemma 16.8.** *Let  $S$  be a scheme. Let  $X \rightarrow \mathrm{Spec}(k)$  be étale morphism over  $S$ , where  $k$  is a field. Then  $X$  is a scheme.*

**Proof.** Let  $U$  be an affine scheme, and let  $U \rightarrow X$  be an étale morphism. By Definition 16.2 we see that  $U \rightarrow \mathrm{Spec}(k)$  is an étale morphism. Hence  $U = \coprod_{i=1, \dots, n} \mathrm{Spec}(k_i)$  is a finite disjoint union of spectra of finite separable extensions  $k_i$  of  $k$ , see Morphisms, Lemma 36.7. The  $R = U \times_X U \rightarrow U \times_{\mathrm{Spec}(k)} U$  is a monomorphism and  $U \times_{\mathrm{Spec}(k)} U$  is also a finite disjoint union of spectra of finite separable extensions of  $k$ . Hence by Schemes, Lemma 23.11 we see that  $R$  is similarly a finite disjoint union of spectra of finite separable extensions of  $k$ . This  $U$  and  $R$  are affine and both projections  $R \rightarrow U$  are finite locally free. Hence  $U/R$  is a scheme by Groupoids, Proposition 23.9. By Spaces, Lemma 10.2 it is also an open subspace of  $X$ . By Lemma 13.1 we conclude that  $X$  is a scheme.  $\square$



### 17. Spaces and fpqc coverings

Let  $S$  be a scheme. An algebraic space over  $S$  is defined as a sheaf in the fppf topology with additional properties. Hence it is not immediately clear that it satisfies the sheaf property for the fpqc topology (see Topologies, Definition 9.12). In this section we give Gabber's argument showing this is true. However, when we say that the algebraic space  $X$  satisfies the sheaf property for the fpqc topology we really only consider fpqc coverings  $\{f_i : T_i \rightarrow T\}_{i \in I}$  such that  $T, T_i$  are objects of the big site  $(Sch/S)_{fppf}$  (as per our conventions, see Section 2).

**Proposition 17.1** (Gabber). *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Then  $X$  satisfies the sheaf property for the fpqc topology.*

**Proof.** Since  $X$  is a sheaf for the Zariski topology it suffices to show the following. Given a surjective flat morphism of affines  $f : T' \rightarrow T$  we have:  $X(T)$  is the equalizer of the two maps  $X(T') \rightarrow X(T' \times_T T')$ . See Topologies, Lemma 9.13 (there is a little argument omitted here because the lemma cited is formulated for functors defined on the category of all schemes).

Let  $a, b : T \rightarrow X$  be two morphisms such that  $a \circ f = b \circ f$ . We have to show  $a = b$ . Consider the fibre product

$$E = X \times_{\Delta_{X/S}, X \times_S X, (a,b)} T.$$

By Spaces, Lemma 13.1 the morphism  $\Delta_{X/S}$  is a representable monomorphism. Hence  $E \rightarrow T$  is a monomorphism of schemes. Our assumption that  $a \circ f = b \circ f$  implies that  $T' \rightarrow T$  factors (uniquely) through  $E$ . Consider the commutative diagram

$$\begin{array}{ccc} T' \times_T E & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ T' & \longrightarrow & T \end{array}$$

Since the projection  $T' \times_T E \rightarrow T'$  is a monomorphism with a section we conclude it is an isomorphism. Hence we conclude that  $E \rightarrow T$  is an isomorphism by Descent, Lemma 23.17. This means  $a = b$  as desired.

Next, let  $c : T' \rightarrow X$  be a morphism such that the two compositions  $T' \times_T T' \rightarrow T' \rightarrow X$  are the same. We have to find a morphism  $a : T \rightarrow X$  whose composition with  $T' \rightarrow T$  is  $c$ . Choose an affine scheme  $U$  and an étale morphism  $U \rightarrow X$  such that the image of  $|U| \rightarrow |X|$  contains the image of  $|c| : |T'| \rightarrow |X|$ . This is possible by Lemmas 4.6 and 6.1, the fact that a finite disjoint union of affines is affine, and the fact that  $|T'|$  is quasi-compact (small argument omitted). Since  $U \rightarrow X$  is separated (Lemma 6.4), we see that

$$V = U \times_{X,c} T' \longrightarrow T'$$

is a surjective, étale, separated morphism of schemes (to see that it is surjective use Lemma 4.3 and our choice of  $U \rightarrow X$ ). The fact that  $c \circ \text{pr}_0 = c \circ \text{pr}_1$  means that we obtain a descent datum on  $V/T'/T$  (Descent, Definition 34.1) because

$$\begin{aligned} V \times_{T'} (T' \times_T T') &= U \times_{X, \text{copr}_0} (T' \times_T T') \\ &= (T' \times_T T') \times_{\text{copr}_1, X} U \\ &= (T' \times_T T') \times_{T'} V \end{aligned}$$

The morphism  $V \rightarrow T'$  is ind-quasi-affine by More on Morphisms, Lemma 66.8 (because étale morphisms are locally quasi-finite, see Morphisms, Lemma 36.6). By More on Groupoids, Lemma 15.3 the descent datum is effective. Say  $W \rightarrow T$  is a morphism such that there is an isomorphism  $\alpha : T' \times_T W \rightarrow V$  compatible with the given descent datum on  $V$  and the canonical descent datum on  $T' \times_T W$ . Then  $W \rightarrow T$  is surjective and étale (Descent, Lemmas 23.7 and 23.29). Consider the composition

$$b' : T' \times_T W \longrightarrow V = U \times_{X,c} T' \longrightarrow U$$

The two compositions  $b' \circ (\text{pr}_0, 1), b' \circ (\text{pr}_1, 1) : (T' \times_T T') \times_T W \rightarrow T' \times_T W \rightarrow U$  agree by our choice of  $\alpha$  and the corresponding property of  $c$  (computation omitted). Hence  $b'$  descends to a morphism  $b : W \rightarrow U$  by Descent, Lemma 13.7. The diagram

$$\begin{array}{ccc} T' \times_T W & \longrightarrow & W \xrightarrow{b} U \\ \downarrow & & \downarrow \\ T' & \xrightarrow{c} & X \end{array}$$

is commutative. What this means is that we have proved the existence of a étale locally on  $T$ , i.e., we have an  $a' : W \rightarrow X$ . However, since we have proved uniqueness in the first paragraph, we find that this étale local solution satisfies the glueing condition, i.e., we have  $\text{pr}_0^* a' = \text{pr}_1^* a'$  as elements of  $X(W \times_T W)$ . Since  $X$  is an étale sheaf we find a unique  $a \in X(T)$  restricting to  $a'$  on  $W$ .  $\square$

## 18. The étale site of an algebraic space

In this section we define the small étale site of an algebraic space. This is the analogue of the small étale site  $S_{\text{étale}}$  of a scheme. Lemma 16.1 implies that in the definition below any morphism between objects of the étale site of  $X$  is étale, and that any scheme étale over an object of  $X_{\text{étale}}$  is also an object of  $X_{\text{étale}}$ .

**Definition 18.1.** Let  $S$  be a scheme. Let  $Sch_{\text{fppf}}$  be a big fppf site containing  $S$ , and let  $Sch_{\text{étale}}$  be the corresponding big étale site (i.e., having the same underlying category). Let  $X$  be an algebraic space over  $S$ . The *small étale site*  $X_{\text{étale}}$  of  $X$  is defined as follows:

- (1) An object of  $X_{\text{étale}}$  is a morphism  $\varphi : U \rightarrow X$  where  $U \in \text{Ob}((Sch/S)_{\text{étale}})$  is a scheme and  $\varphi$  is an étale morphism,
- (2) a morphism  $(\varphi : U \rightarrow X) \rightarrow (\varphi' : U' \rightarrow X)$  is given by a morphism of schemes  $\chi : U \rightarrow U'$  such that  $\varphi = \varphi' \circ \chi$ , and
- (3) a family of morphisms  $\{(U_i \rightarrow X) \rightarrow (U \rightarrow X)\}_{i \in I}$  of  $X_{\text{étale}}$  is a covering if and only if  $\{U_i \rightarrow U\}_{i \in I}$  is a covering of  $(Sch/S)_{\text{étale}}$ .

A consequence of our choice is that the étale site of an algebraic space in general does not have a final object! On the other hand, if  $X$  happens to be a scheme, then the definition above agrees with Topologies, Definition 4.8.

The above is our default site, but there are a couple of variants which we will also use. Namely, we can consider all *algebraic spaces*  $U$  which are étale over  $X$  and this produces the site  $X_{\text{spaces}, \text{étale}}$  we define below or we can consider all *affine schemes*  $U$  which are étale over  $X$  and this produces the site  $X_{\text{affine}, \text{étale}}$  we define below. The first of these two notions is used when discussing functoriality of the small étale site, see Lemma 18.8.

**Definition 18.2.** Let  $S$  be a scheme. Let  $Sch_{fppf}$  be a big fppf site containing  $S$ , and let  $Sch_{\acute{e}tale}$  be the corresponding big étale site (i.e., having the same underlying category). Let  $X$  be an algebraic space over  $S$ . The site  $X_{spaces,\acute{e}tale}$  of  $X$  is defined as follows:

- (1) An object of  $X_{spaces,\acute{e}tale}$  is a morphism  $\varphi : U \rightarrow X$  where  $U$  is an algebraic space over  $S$  and  $\varphi$  is an étale morphism of algebraic spaces over  $S$ ,
- (2) a morphism  $(\varphi : U \rightarrow X) \rightarrow (\varphi' : U' \rightarrow X)$  of  $X_{spaces,\acute{e}tale}$  is given by a morphism of algebraic spaces  $\chi : U \rightarrow U'$  such that  $\varphi = \varphi' \circ \chi$ , and
- (3) a family of morphisms  $\{\varphi_i : (U_i \rightarrow X) \rightarrow (U \rightarrow X)\}_{i \in I}$  of  $X_{spaces,\acute{e}tale}$  is a covering if and only if  $|U| = \bigcup \varphi_i(|U_i|)$ .

As usual we choose a set of coverings of this type, including at least the coverings in  $X_{\acute{e}tale}$ , as in Sets, Lemma 11.1 to turn  $X_{spaces,\acute{e}tale}$  into a site.

Since the identity morphism of  $X$  is étale it is clear that  $X_{spaces,\acute{e}tale}$  does have a final object. Let us show right away that the corresponding topos equals the small étale topos of  $X$ .

**Lemma 18.3.** *The functor*

$$X_{\acute{e}tale} \longrightarrow X_{spaces,\acute{e}tale}, \quad U/X \longmapsto U/X$$

*is a special cocontinuous functor (Sites, Definition 29.2) and hence induces an equivalence of topoi  $Sh(X_{\acute{e}tale}) \rightarrow Sh(X_{spaces,\acute{e}tale})$ .*

**Proof.** We have to show that the functor satisfies the assumptions (1) – (5) of Sites, Lemma 29.1. It is clear that the functor is continuous and cocontinuous, which proves assumptions (1) and (2). Assumptions (3) and (4) hold simply because the functor is fully faithful. Assumption (5) holds, because an algebraic space by definition has a covering by a scheme.  $\square$

**Remark 18.4.** Let us explain the meaning of Lemma 18.3. Let  $S$  be a scheme, and let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be a sheaf on the small étale site  $X_{\acute{e}tale}$  of  $X$ . The lemma says that there exists a unique sheaf  $\mathcal{F}'$  on  $X_{spaces,\acute{e}tale}$  which restricts back to  $\mathcal{F}$  on the subcategory  $X_{\acute{e}tale}$ . If  $U \rightarrow X$  is an étale morphism of algebraic spaces, then how do we compute  $\mathcal{F}'(U)$ ? Well, by definition of an algebraic space there exists a scheme  $U'$  and a surjective étale morphism  $U' \rightarrow U$ . Then  $\{U' \rightarrow U\}$  is a covering in  $X_{spaces,\acute{e}tale}$  and hence we get an equalizer diagram

$$\mathcal{F}'(U) \longrightarrow \mathcal{F}(U') \rightrightarrows \mathcal{F}(U' \times_U U').$$

Note that  $U' \times_U U'$  is a scheme, and hence we may write  $\mathcal{F}$  and not  $\mathcal{F}'$ . Thus we see how to compute  $\mathcal{F}'$  when given the sheaf  $\mathcal{F}$ .

**Definition 18.5.** Let  $S$  be a scheme. Let  $Sch_{fppf}$  be a big fppf site containing  $S$ , and let  $Sch_{\acute{e}tale}$  be the corresponding big étale site (i.e., having the same underlying category). Let  $X$  be an algebraic space over  $S$ . The site  $X_{affine,\acute{e}tale}$  of  $X$  is defined as follows:

- (1) An object of  $X_{affine,\acute{e}tale}$  is a morphism  $\varphi : U \rightarrow X$  where  $U \in \text{Ob}((Sch/S)_{\acute{e}tale})$  is an affine scheme and  $\varphi$  is an étale morphism,
- (2) a morphism  $(\varphi : U \rightarrow X) \rightarrow (\varphi' : U' \rightarrow X)$  of  $X_{affine,\acute{e}tale}$  is given by a morphism of schemes  $\chi : U \rightarrow U'$  such that  $\varphi = \varphi' \circ \chi$ , and

- (3) a family of morphisms  $\{\varphi_i : (U_i \rightarrow X) \rightarrow (U \rightarrow X)\}_{i \in I}$  of  $X_{\text{affine}, \text{étale}}$  is a covering if and only if  $\{U_i \rightarrow U\}$  is a standard étale covering, see Topologies, Definition 4.5.

As usual we choose a set of coverings of this type, as in Sets, Lemma 11.1 to turn  $X_{\text{affine}, \text{étale}}$  into a site.

**Lemma 18.6.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The functor  $X_{\text{affine}, \text{étale}} \rightarrow X_{\text{étale}}$  is special cocontinuous and induces an equivalence of topoi from  $Sh(X_{\text{affine}, \text{étale}})$  to  $Sh(X_{\text{étale}})$ .*

**Proof.** Omitted. Hint: compare with the proof of Topologies, Lemma 4.11.  $\square$

**Definition 18.7.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The *étale topos* of  $X$ , or more precisely the *small étale topos* of  $X$  is the category  $Sh(X_{\text{étale}})$  of sheaves of sets on  $X_{\text{étale}}$ .

By Lemma 18.3 we have  $Sh(X_{\text{étale}}) = Sh(X_{\text{spaces}, \text{étale}})$ , so we can also think of this as the category of sheaves of sets on  $X_{\text{spaces}, \text{étale}}$ . Similarly, by Lemma 18.6 we see that  $Sh(X_{\text{étale}}) = Sh(X_{\text{affine}, \text{étale}})$ . It turns out that the topos is functorial with respect to morphisms of algebraic spaces. Here is a precise statement.

**Lemma 18.8.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ .*

- (1) *The continuous functor*

$$Y_{\text{spaces}, \text{étale}} \longrightarrow X_{\text{spaces}, \text{étale}}, \quad V \longmapsto X \times_Y V$$

*induces a morphism of sites*

$$f_{\text{spaces}, \text{étale}} : X_{\text{spaces}, \text{étale}} \rightarrow Y_{\text{spaces}, \text{étale}}.$$

- (2) *The rule  $f \mapsto f_{\text{spaces}, \text{étale}}$  is compatible with compositions, in other words  $(f \circ g)_{\text{spaces}, \text{étale}} = f_{\text{spaces}, \text{étale}} \circ g_{\text{spaces}, \text{étale}}$  (see Sites, Definition 14.5).*  
 (3) *The morphism of topoi associated to  $f_{\text{spaces}, \text{étale}}$  induces, via Lemma 18.3, a morphism of topoi  $f_{\text{small}} : Sh(X_{\text{étale}}) \rightarrow Sh(Y_{\text{étale}})$  whose construction is compatible with compositions.*  
 (4) *If  $f$  is a representable morphism of algebraic spaces, then  $f_{\text{small}}$  comes from a morphism of sites  $X_{\text{étale}} \rightarrow Y_{\text{étale}}$ , corresponding to the continuous functor  $V \mapsto X \times_Y V$ .*

**Proof.** Let us show that the functor described in (1) satisfies the assumptions of Sites, Proposition 14.7. Thus we have to show that  $Y_{\text{spaces}, \text{étale}}$  has a final object (namely  $Y$ ) and that the functor transforms this into a final object in  $X_{\text{spaces}, \text{étale}}$  (namely  $X$ ). This is clear as  $X \times_Y Y = X$  in any category. Next, we have to show that  $Y_{\text{spaces}, \text{étale}}$  has fibre products. This is true since the category of algebraic spaces has fibre products, and since  $V \times_Y V'$  is étale over  $Y$  if  $V$  and  $V'$  are étale over  $Y$  (see Lemmas 16.4 and 16.5 above). OK, so the proposition applies and we see that we get a morphism of sites as described in (1).

Part (2) you get by unwinding the definitions. Part (3) is clear by using the equivalences for  $X$  and  $Y$  from Lemma 18.3 above. Part (4) follows, because if  $f$  is

representable, then the functors above fit into a commutative diagram

$$\begin{array}{ccc} X_{\acute{e}tale} & \longrightarrow & X_{spaces, \acute{e}tale} \\ \uparrow & & \uparrow \\ Y_{\acute{e}tale} & \longrightarrow & Y_{spaces, \acute{e}tale} \end{array}$$

of categories.  $\square$

We can do a little bit better than the lemma above in describing the relationship between sheaves on  $X$  and sheaves on  $Y$ . Namely, we can formulate this in terms of  $f$ -maps, compare Sheaves, Definition 21.7, as follows.

**Definition 18.9.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a sheaf of sets on  $X_{\acute{e}tale}$  and let  $\mathcal{G}$  be a sheaf of sets on  $Y_{\acute{e}tale}$ . An  $f$ -map  $\varphi : \mathcal{G} \rightarrow \mathcal{F}$  is a collection of maps  $\varphi_{(U,V,g)} : \mathcal{G}(V) \rightarrow \mathcal{F}(U)$  indexed by commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & X \\ g \downarrow & & \downarrow f \\ V & \longrightarrow & Y \end{array}$$

where  $U \in X_{\acute{e}tale}$ ,  $V \in Y_{\acute{e}tale}$  such that whenever given an extended diagram

$$\begin{array}{ccccc} U' & \longrightarrow & U & \longrightarrow & X \\ g' \downarrow & & g \downarrow & & \downarrow f \\ V' & \longrightarrow & V & \longrightarrow & Y \end{array}$$

with  $V' \rightarrow V$  and  $U' \rightarrow U$  étale morphisms of schemes the diagram

$$\begin{array}{ccc} \mathcal{G}(V) & \xrightarrow{\varphi_{(U,V,g)}} & \mathcal{F}(U) \\ \text{restriction of } \mathcal{G} \downarrow & & \downarrow \text{restriction of } \mathcal{F} \\ \mathcal{G}(V') & \xrightarrow{\varphi_{(U',V',g')}} & \mathcal{F}(U') \end{array}$$

commutes.

**Lemma 18.10.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a sheaf of sets on  $X_{\acute{e}tale}$  and let  $\mathcal{G}$  be a sheaf of sets on  $Y_{\acute{e}tale}$ . There are canonical bijections between the following three sets:

- (1) The set of maps  $\mathcal{G} \rightarrow f_{small,*}\mathcal{F}$ .
- (2) The set of maps  $f_{small}^{-1}\mathcal{G} \rightarrow \mathcal{F}$ .
- (3) The set of  $f$ -maps  $\varphi : \mathcal{G} \rightarrow \mathcal{F}$ .

**Proof.** Note that (1) and (2) are the same because the functors  $f_{small,*}$  and  $f_{small}^{-1}$  are a pair of adjoint functors. Suppose that  $\alpha : f_{small}^{-1}\mathcal{G} \rightarrow \mathcal{F}$  is a map of sheaves on  $Y_{\acute{e}tale}$ . Let a diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ g \downarrow & & \downarrow f \\ V & \longrightarrow & Y \end{array}$$

as in Definition 18.9 be given. By the commutativity of the diagram we also get a map  $g_{small}^{-1}(j_V)^{-1}\mathcal{G} \rightarrow (j_U)^{-1}\mathcal{F}$  (compare Sites, Section 25 for the description of the localization functors). Hence we certainly get a map  $\varphi_{(V,U,g)} : \mathcal{G}(V) = (j_V)^{-1}\mathcal{G}(V) \rightarrow (j_U)^{-1}\mathcal{F}(U) = \mathcal{F}(U)$ . We omit the verification that this rule is compatible with further restrictions and defines an  $f$ -map from  $\mathcal{G}$  to  $\mathcal{F}$ .

Conversely, suppose that we are given an  $f$ -map  $\varphi = (\varphi_{(U,V,g)})$ . Let  $\mathcal{G}'$  (resp.  $\mathcal{F}'$ ) denote the extension of  $\mathcal{G}$  (resp.  $\mathcal{F}$ ) to  $Y_{spaces, \acute{e}tale}$  (resp.  $X_{spaces, \acute{e}tale}$ ), see Lemma 18.3. Then we have to construct a map of sheaves

$$\mathcal{G}' \longrightarrow (f_{spaces, \acute{e}tale})_* \mathcal{F}'$$

To do this, let  $V \rightarrow Y$  be an étale morphism of algebraic spaces. We have to construct a map of sets

$$\mathcal{G}'(V) \rightarrow \mathcal{F}'(X \times_Y V)$$

Choose an étale surjective morphism  $V' \rightarrow V$  with  $V'$  a scheme, and after that choose an étale surjective morphism  $U' \rightarrow X \times_U V'$  with  $U'$  a scheme. We get a morphism of schemes  $g' : U' \rightarrow V'$  and also a morphism of schemes

$$g'' : U' \times_{X \times_Y V} U' \longrightarrow V' \times_V V'$$

Consider the following diagram

$$\begin{array}{ccccc} \mathcal{F}'(X \times_Y V) & \longrightarrow & \mathcal{F}(U') & \rightrightarrows & \mathcal{F}(U' \times_{X \times_Y V} U') \\ \uparrow \text{dotted} & & \uparrow \varphi_{(U', V', g')} & & \uparrow \varphi_{(U'', V'', g'')} \\ \mathcal{G}'(X \times_Y V) & \longrightarrow & \mathcal{G}(V') & \rightrightarrows & \mathcal{G}(V' \times_V V') \end{array}$$

The compatibility of the maps  $\varphi \dots$  with restriction shows that the two right squares commute. The definition of coverings in  $X_{spaces, \acute{e}tale}$  shows that the horizontal rows are equalizer diagrams. Hence we get the dotted arrow. We leave it to the reader to show that these arrows are compatible with the restriction mappings.  $\square$

If the morphism of algebraic spaces  $X \rightarrow Y$  is étale, then the morphism of topoi  $Sh(X_{\acute{e}tale}) \rightarrow Sh(Y_{\acute{e}tale})$  is a localization. Here is a statement.

**Lemma 18.11.** *Let  $S$  be a scheme, and let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Assume  $f$  is étale. In this case there is a functor*

$$j : X_{\acute{e}tale} \rightarrow Y_{\acute{e}tale}, \quad (\varphi : U \rightarrow X) \mapsto (f \circ \varphi : U \rightarrow Y)$$

*which is cocontinuous. The morphism of topoi  $f_{small}$  is the morphism of topoi associated to  $j$ , see Sites, Lemma 21.1. Moreover,  $j$  is continuous as well, hence Sites, Lemma 21.5 applies. In particular  $f_{small}^{-1}\mathcal{G}(U) = \mathcal{G}(jU)$  for all sheaves  $\mathcal{G}$  on  $Y_{\acute{e}tale}$ .*

**Proof.** Note that by our very definition of an étale morphism of algebraic spaces (Definition 16.2) it is indeed the case that the rule given defines a functor  $j$  as indicated. It is clear that  $j$  is cocontinuous and continuous, simply because a covering  $\{U_i \rightarrow U\}$  of  $j(\varphi : U \rightarrow X)$  in  $Y_{\acute{e}tale}$  is the same thing as a covering of  $(\varphi : U \rightarrow X)$  in  $X_{\acute{e}tale}$ . It remains to show that  $j$  induces the same morphism of

topoi as  $f_{small}$ . To see this we consider the diagram

$$\begin{array}{ccc} X_{\acute{e}tale} & \longrightarrow & X_{spaces,\acute{e}tale} \\ \downarrow j & & \downarrow j_{spaces} \\ Y_{\acute{e}tale} & \longrightarrow & Y_{spaces,\acute{e}tale} \end{array} \quad \left( \begin{array}{c} \uparrow v: V \mapsto X \times_Y V \\ \downarrow \end{array} \right)$$

of categories. Here the functor  $j_{spaces}$  is the obvious extension of  $j$  to the category  $X_{spaces,\acute{e}tale}$ . Thus the inner square is commutative. In fact  $j_{spaces}$  can be identified with the localization functor  $j_X : Y_{spaces,\acute{e}tale}/X \rightarrow Y_{spaces,\acute{e}tale}$  discussed in Sites, Section 25. Hence, by Sites, Lemma 27.2 the cocontinuous functor  $j_{spaces}$  and the functor  $v$  of the diagram induce the same morphism of topoi. By Sites, Lemma 21.2 the commutativity of the inner square (consisting of cocontinuous functors between sites) gives a commutative diagram of associated morphisms of topoi. Hence, by the construction of  $f_{small}$  in Lemma 18.8 we win.  $\square$

The lemma above says that the pullback of  $\mathcal{G}$  via an étale morphism  $f : X \rightarrow Y$  of algebraic spaces is simply the restriction of  $\mathcal{G}$  to the category  $X_{\acute{e}tale}$ . We will often use the short hand

$$(18.11.1) \quad \mathcal{G}|_{X_{\acute{e}tale}} = f_{small}^{-1} \mathcal{G}$$

to indicate this. Note that the functor  $j : X_{\acute{e}tale} \rightarrow Y_{\acute{e}tale}$  of the lemma in this situation is faithful, but not fully faithful in general. We will discuss this in a more technical fashion in Section 27.

**Lemma 18.12.** *Let  $S$  be a scheme. Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

*be a cartesian square of algebraic spaces over  $S$ . Let  $\mathcal{F}$  be a sheaf on  $X_{\acute{e}tale}$ . If  $g$  is étale, then*

- (1)  $f'_{small,*}(\mathcal{F}|_{X'}) = (f_{small,*}\mathcal{F})|_{Y'}$  in  $Sh(Y'_{\acute{e}tale})^4$ , and
- (2) if  $\mathcal{F}$  is an abelian sheaf, then  $R^i f'_{small,*}(\mathcal{F}|_{X'}) = (R^i f_{small,*}\mathcal{F})|_{Y'}$ .

**Proof.** Consider the following diagram of functors

$$\begin{array}{ccc} X'_{spaces,\acute{e}tale} & \xrightarrow{j} & X_{spaces,\acute{e}tale} \\ \uparrow V' \mapsto V' \times_{Y'} X' & & \uparrow V \mapsto V \times_Y X \\ Y'_{spaces,\acute{e}tale} & \xrightarrow{j} & Y_{spaces,\acute{e}tale} \end{array}$$

The horizontal arrows are localizations and the vertical arrows induce morphisms of sites. Hence the last statement of Sites, Lemma 28.1 gives (1). To see (2) apply (1) to an injective resolution of  $\mathcal{F}$  and use that restriction is exact and preserves injectives (see Cohomology on Sites, Lemma 7.1).  $\square$

<sup>4</sup>Also  $(f')_{small}^{-1}(\mathcal{G}|_{Y'}) = (f_{small}^{-1}\mathcal{G})|_{X'}$  because of commutativity of the diagram and (18.11.1)

The following lemma says that you can think of a sheaf on the small étale site of an algebraic space as a compatible collection of sheaves on the small étale sites of schemes étale over the space. Please note that all the comparison mappings  $c_f$  in the lemma are isomorphisms, which is compatible with Topologies, Lemma 4.20 and the fact that all morphisms between objects of  $X_{\text{étale}}$  are étale.

**Lemma 18.13.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . A sheaf  $\mathcal{F}$  on  $X_{\text{étale}}$  is given by the following data:*

- (1) *for every  $U \in \text{Ob}(X_{\text{étale}})$  a sheaf  $\mathcal{F}_U$  on  $U_{\text{étale}}$ ,*
- (2) *for every  $f : U' \rightarrow U$  in  $X_{\text{étale}}$  an isomorphism  $c_f : f_{\text{small}}^{-1}\mathcal{F}_U \rightarrow \mathcal{F}_{U'}$ .*

*These data are subject to the condition that given any  $f : U' \rightarrow U$  and  $g : U'' \rightarrow U'$  in  $X_{\text{étale}}$  the composition  $c_g \circ g_{\text{small}}^{-1}c_f$  is equal to  $c_{f \circ g}$ .*

**Proof.** We may interpret  $g_{\text{small}}^{-1}$  as in Lemma 18.11. Then the lemma follows from a general fact about sites, see Sites, Lemma 26.6.  $\square$

Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $X = U/R$  be a presentation of  $X$  coming from any surjective étale morphism  $\varphi : U \rightarrow X$ , see Spaces, Definition 9.3. In particular, we obtain a groupoid  $(U, R, s, t, c, e, i)$  such that  $j = (t, s) : R \rightarrow U \times_S U$ , see Groupoids, Lemma 13.3.

**Lemma 18.14.** *With  $S$ ,  $\varphi : U \rightarrow X$ , and  $(U, R, s, t, c, e, i)$  as above. For any sheaf  $\mathcal{F}$  on  $X_{\text{étale}}$  the sheaf<sup>5</sup>  $\mathcal{G} = \varphi^{-1}\mathcal{F}$  comes equipped with a canonical isomorphism*

$$\alpha : t^{-1}\mathcal{G} \longrightarrow s^{-1}\mathcal{G}$$

*such that the diagram*

$$\begin{array}{ccc}
 & pr_1^{-1}t^{-1}\mathcal{G} & \xrightarrow{pr_1^{-1}\alpha} pr_1^{-1}s^{-1}\mathcal{G} \\
 & \parallel & \parallel \\
 pr_0^{-1}s^{-1}\mathcal{G} & & c^{-1}s^{-1}\mathcal{G} \\
 & \nwarrow pr_0^{-1}\alpha & \nearrow c^{-1}\alpha \\
 & pr_0^{-1}t^{-1}\mathcal{G} & = c^{-1}t^{-1}\mathcal{G}
 \end{array}$$

*is commutative. The functor  $\mathcal{F} \mapsto (\mathcal{G}, \alpha)$  defines an equivalence of categories between sheaves on  $X_{\text{étale}}$  and pairs  $(\mathcal{G}, \alpha)$  as above.*

**First proof of Lemma 18.14.** Let  $\mathcal{C} = X_{\text{spaces}, \text{étale}}$ . By Lemma 18.11 and its proof we have  $U_{\text{spaces}, \text{étale}} = \mathcal{C}/U$  and the pullback functor  $\varphi^{-1}$  is just the restriction functor. Moreover,  $\{U \rightarrow X\}$  is a covering of the site  $\mathcal{C}$  and  $R = U \times_X U$ . The isomorphism  $\alpha$  is just the canonical identification

$$(\mathcal{F}|_{\mathcal{C}/U})|_{\mathcal{C}/U \times_X U} = (\mathcal{F}|_{\mathcal{C}/U})|_{\mathcal{C}/U \times_X U}$$

and the commutativity of the diagram is the cocycle condition for glueing data. Hence this lemma is a special case of glueing of sheaves, see Sites, Section 26.  $\square$

<sup>5</sup>In this lemma and its proof we write simply  $\varphi^{-1}$  instead of  $\varphi_{\text{small}}^{-1}$  and similarly for all the other pullbacks.



**Second proof of Lemma 18.14.** The existence of  $\alpha$  comes from the fact that  $\varphi \circ t = \varphi \circ s$  and that pullback is functorial in the morphism, see Lemma 18.8. In exactly the same way, i.e., by functoriality of pullback, we see that the isomorphism  $\alpha$  fits into the commutative diagram. The construction  $\mathcal{F} \mapsto (\varphi^{-1}\mathcal{F}, \alpha)$  is clearly functorial in the sheaf  $\mathcal{F}$ . Hence we obtain the functor.

Conversely, suppose that  $(\mathcal{G}, \alpha)$  is a pair. Let  $V \rightarrow X$  be an object of  $X_{\text{étale}}$ . In this case the morphism  $V' = U \times_X V \rightarrow V$  is a surjective étale morphism of schemes, and hence  $\{V' \rightarrow V\}$  is an étale covering of  $V$ . Set  $\mathcal{G}' = (V' \rightarrow V)^{-1}\mathcal{G}$ . Since  $R = U \times_X U$  with  $t = \text{pr}_0$  and  $s = \text{pr}_1$  we see that  $V' \times_V V' = R \times_X V$  with projection maps  $s', t' : V' \times_V V' \rightarrow V'$  equal to the pullbacks of  $t$  and  $s$ . Hence  $\alpha$  pulls back to an isomorphism  $\alpha' : (t')^{-1}\mathcal{G}' \rightarrow (s')^{-1}\mathcal{G}'$ . Having said this we simply define

$$\mathcal{F}(V) = \text{Equalizer}(\mathcal{G}(V') \rightrightarrows \mathcal{G}(V' \times_V V')).$$

We omit the verification that this defines a sheaf. To see that  $\mathcal{G}(V) = \mathcal{F}(V)$  if there exists a morphism  $V \rightarrow U$  note that in this case the equalizer is  $H^0(\{V' \rightarrow V\}, \mathcal{G}) = \mathcal{G}(V)$ .  $\square$

## 19. Points of the small étale site

This section is the analogue of Étale Cohomology, Section 29.

**Definition 19.1.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ .

- (1) A *geometric point* of  $X$  is a morphism  $\bar{x} : \text{Spec}(k) \rightarrow X$ , where  $k$  is an algebraically closed field. We often abuse notation and write  $\bar{x} = \text{Spec}(k)$ .
- (2) For every geometric point  $\bar{x}$  we have the corresponding “image” point  $x \in |X|$ . We say that  $\bar{x}$  is a *geometric point lying over*  $x$ .

It turns out that we can take stalks of sheaves on  $X_{\text{étale}}$  at geometric points exactly in the same way as was done in the case of the small étale site of a scheme. In order to do this we define the notion of an étale neighbourhood as follows.

**Definition 19.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\bar{x}$  be a geometric point of  $X$ .

- (1) An *étale neighborhood* of  $\bar{x}$  of  $X$  is a commutative diagram

$$\begin{array}{ccc} & U & \\ \bar{u} \nearrow & \downarrow \varphi & \\ \bar{x} \xrightarrow{\bar{x}} & X & \end{array}$$

where  $\varphi$  is an étale morphism of algebraic spaces over  $S$ . We will use the notation  $\varphi : (U, \bar{u}) \rightarrow (X, \bar{x})$  to indicate this situation.

- (2) A *morphism of étale neighborhoods*  $(U, \bar{u}) \rightarrow (U', \bar{u}')$  is an  $X$ -morphism  $h : U \rightarrow U'$  such that  $\bar{u}' = h \circ \bar{u}$ .

Note that we allow  $U$  to be an algebraic space. When we take stalks of a sheaf on  $X_{\text{étale}}$  we have to restrict to those  $U$  which are in  $X_{\text{étale}}$ , and so in this case we will only consider the case where  $U$  is a scheme. Alternately we can work with the site  $X_{\text{space}, \text{étale}}$  and consider all étale neighbourhoods. And there won't be any difference because of the last assertion in the following lemma.

**Lemma 19.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\bar{x}$  be a geometric point of  $X$ . The category of étale neighborhoods is cofiltered. More precisely:*

- (1) *Let  $(U_i, \bar{u}_i)_{i=1,2}$  be two étale neighborhoods of  $\bar{x}$  in  $X$ . Then there exists a third étale neighborhood  $(U, \bar{u})$  and morphisms  $(U, \bar{u}) \rightarrow (U_i, \bar{u}_i)$ ,  $i = 1, 2$ .*
- (2) *Let  $h_1, h_2 : (U, \bar{u}) \rightarrow (U', \bar{u}')$  be two morphisms between étale neighborhoods of  $\bar{x}$ . Then there exist an étale neighborhood  $(U'', \bar{u}'')$  and a morphism  $h : (U'', \bar{u}'') \rightarrow (U, \bar{u})$  which equalizes  $h_1$  and  $h_2$ , i.e., such that  $h_1 \circ h = h_2 \circ h$ .*

Moreover, given any étale neighbourhood  $(U, \bar{u}) \rightarrow (X, \bar{x})$  there exists a morphism of étale neighbourhoods  $(U', \bar{u}') \rightarrow (U, \bar{u})$  where  $U'$  is a scheme.

**Proof.** For part (1), consider the fibre product  $U = U_1 \times_X U_2$ . It is étale over both  $U_1$  and  $U_2$  because étale morphisms are preserved under base change and composition, see Lemmas 16.5 and 16.4. The map  $\bar{u} \rightarrow U$  defined by  $(\bar{u}_1, \bar{u}_2)$  gives it the structure of an étale neighborhood mapping to both  $U_1$  and  $U_2$ .

For part (2), define  $U''$  as the fibre product

$$\begin{array}{ccc} U'' & \longrightarrow & U \\ \downarrow & & \downarrow (h_1, h_2) \\ U' & \xrightarrow{\Delta} & U' \times_X U' \end{array}$$

Since  $\bar{u}$  and  $\bar{u}'$  agree over  $X$  with  $\bar{x}$ , we see that  $\bar{u}'' = (\bar{u}, \bar{u}')$  is a geometric point of  $U''$ . In particular  $U'' \neq \emptyset$ . Moreover, since  $U'$  is étale over  $X$ , so is the fibre product  $U' \times_X U'$  (as seen above in the case of  $U_1 \times_X U_2$ ). Hence the vertical arrow  $(h_1, h_2)$  is étale by Lemma 16.6. Therefore  $U''$  is étale over  $U'$  by base change, and hence also étale over  $X$  (because compositions of étale morphisms are étale). Thus  $(U'', \bar{u}'')$  is a solution to the problem posed by (2).

To see the final assertion, choose any surjective étale morphism  $U' \rightarrow U$  where  $U'$  is a scheme. Then  $U' \times_U \bar{u}$  is a scheme surjective and étale over  $\bar{u} = \text{Spec}(k)$  with  $k$  algebraically closed. It follows (see Morphisms, Lemma 36.7) that  $U' \times_U \bar{u} \rightarrow \bar{u}$  has a section which gives us the desired  $\bar{u}'$ .  $\square$

**Lemma 19.4.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\bar{x} : \text{Spec}(k) \rightarrow X$  be a geometric point of  $X$  lying over  $x \in |X|$ . Let  $\varphi : U \rightarrow X$  be an étale morphism of algebraic spaces and let  $u \in |U|$  with  $\varphi(u) = x$ . Then there exists a geometric point  $\bar{u} : \text{Spec}(k) \rightarrow U$  lying over  $u$  with  $\bar{x} = \varphi \circ \bar{u}$ .*

**Proof.** Choose an affine scheme  $U'$  with  $u' \in U'$  and an étale morphism  $U' \rightarrow U$  which maps  $u'$  to  $u$ . If we can prove the lemma for  $(U', u') \rightarrow (X, x)$  then the lemma follows. Hence we may assume that  $U$  is a scheme, in particular that  $U \rightarrow X$  is representable. Then look at the cartesian diagram

$$\begin{array}{ccc} \text{Spec}(k) \times_{\bar{x}, X, \varphi} U & \xrightarrow{\text{pr}_2} & U \\ \text{pr}_1 \downarrow & & \downarrow \varphi \\ \text{Spec}(k) & \xrightarrow{\bar{x}} & X \end{array}$$

The projection  $\text{pr}_1$  is the base change of an étale morphisms so it is étale, see Lemma 16.5. Therefore, the scheme  $\text{Spec}(k) \times_{\bar{x}, X, \varphi} U$  is a disjoint union of finite separable

extensions of  $k$ , see Morphisms, Lemma 36.7. But  $k$  is algebraically closed, so all these extensions are trivial, so  $\mathrm{Spec}(k) \times_{\bar{x}, X, \varphi} U$  is a disjoint union of copies of  $\mathrm{Spec}(k)$  and each of these corresponds to a geometric point  $\bar{u}$  with  $\varphi \circ \bar{u} = \bar{x}$ . By Lemma 4.3 the map

$$|\mathrm{Spec}(k) \times_{\bar{x}, X, \varphi} U| \longrightarrow |\mathrm{Spec}(k)| \times_{|X|} |U|$$

is surjective, hence we can pick  $\bar{u}$  to lie over  $u$ .  $\square$

**Lemma 19.5.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\bar{x}$  be a geometric point of  $X$ . Let  $(U, \bar{u})$  an étale neighborhood of  $\bar{x}$ . Let  $\{\varphi_i : U_i \rightarrow U\}_{i \in I}$  be an étale covering in  $X_{\mathrm{spaces}, \mathrm{étale}}$ . Then there exist  $i \in I$  and  $\bar{u}_i : \bar{x} \rightarrow U_i$  such that  $\varphi_i : (U_i, \bar{u}_i) \rightarrow (U, \bar{u})$  is a morphism of étale neighborhoods.*

**Proof.** Let  $u \in |U|$  be the image of  $\bar{u}$ . As  $|U| = \bigcup_{i \in I} \varphi_i(|U_i|)$  there exists an  $i$  and a point  $u_i \in U_i$  mapping to  $u$ . Apply Lemma 19.4 to  $(U_i, u_i) \rightarrow (U, u)$  and  $\bar{u}$  to get the desired geometric point.  $\square$

**Definition 19.6.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be a presheaf on  $X_{\mathrm{étale}}$ . Let  $\bar{x}$  be a geometric point of  $X$ . The *stalk* of  $\mathcal{F}$  at  $\bar{x}$  is

$$\mathcal{F}_{\bar{x}} = \mathrm{colim}_{(U, \bar{u})} \mathcal{F}(U)$$

where  $(U, \bar{u})$  runs over all étale neighborhoods of  $\bar{x}$  in  $X$  with  $U \in \mathrm{Ob}(X_{\mathrm{étale}})$ .

By Lemma 19.3, this colimit is over a filtered index category, namely the opposite of the category of étale neighborhoods in  $X_{\mathrm{étale}}$ . More precisely Lemma 19.3 says the opposite of the category of all étale neighbourhoods is filtered, and the full subcategory of those which are in  $X_{\mathrm{étale}}$  is a cofinal subcategory hence also filtered.

This means an element of  $\mathcal{F}_{\bar{x}}$  can be thought of as a triple  $(U, \bar{u}, \sigma)$  where  $U \in \mathrm{Ob}(X_{\mathrm{étale}})$  and  $\sigma \in \mathcal{F}(U)$ . Two triples  $(U, \bar{u}, \sigma), (U', \bar{u}', \sigma')$  define the same element of the stalk if there exists a third étale neighbourhood  $(U'', \bar{u}'')$ ,  $U'' \in \mathrm{Ob}(X_{\mathrm{étale}})$  and morphisms of étale neighbourhoods  $h : (U'', \bar{u}'') \rightarrow (U, \bar{u})$ ,  $h' : (U'', \bar{u}'') \rightarrow (U', \bar{u}')$  such that  $h^* \sigma = (h')^* \sigma'$  in  $\mathcal{F}(U'')$ . See Categories, Section 19.

This also implies that if  $\mathcal{F}'$  is the sheaf on  $X_{\mathrm{spaces}, \mathrm{étale}}$  corresponding to  $\mathcal{F}$  on  $X_{\mathrm{étale}}$ , then

$$(19.6.1) \quad \mathcal{F}_{\bar{x}} = \mathrm{colim}_{(U, \bar{u})} \mathcal{F}'(U)$$

where now the colimit is over all the étale neighbourhoods of  $\bar{x}$ . We will often jump between the point of view of using  $X_{\mathrm{étale}}$  and  $X_{\mathrm{spaces}, \mathrm{étale}}$  without further mention.

In particular this means that if  $\mathcal{F}$  is a presheaf of abelian groups, rings, etc then  $\mathcal{F}_{\bar{x}}$  is an abelian group, ring, etc simply by the usual way of defining the group structure on a directed colimit of abelian groups, rings, etc.

**Lemma 19.7.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\bar{x}$  be a geometric point of  $X$ . Consider the functor*

$$u : X_{\mathrm{étale}} \longrightarrow \mathrm{Sets}, \quad U \longmapsto |U_{\bar{x}}|$$

*Then  $u$  defines a point  $p$  of the site  $X_{\mathrm{étale}}$  (Sites, Definition 32.2) and its associated stalk functor  $\mathcal{F} \mapsto \mathcal{F}_p$  (Sites, Equation 32.1.1) is the functor  $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$  defined above.*

**Proof.** In the proof of Lemma 19.5 we have seen that the scheme  $U_{\bar{x}}$  is a disjoint union of schemes isomorphic to  $\bar{x}$ . Thus we can also think of  $|U_{\bar{x}}|$  as the set of geometric points of  $U$  lying over  $\bar{x}$ , i.e., as the collection of morphisms  $\bar{u} : \bar{x} \rightarrow U$  fitting into the diagram of Definition 19.2. From this it follows that  $u(X)$  is a singleton, and that  $u(U \times_V W) = u(U) \times_{u(V)} u(W)$  whenever  $U \rightarrow V$  and  $W \rightarrow V$  are morphisms in  $X_{\text{étale}}$ . And, given a covering  $\{U_i \rightarrow U\}_{i \in I}$  in  $X_{\text{étale}}$  we see that  $\coprod u(U_i) \rightarrow u(U)$  is surjective by Lemma 19.5. Hence Sites, Proposition 33.3 applies, so  $p$  is a point of the site  $X_{\text{étale}}$ . Finally, the our functor  $\mathcal{F} \mapsto \mathcal{F}_{\bar{s}}$  is given by exactly the same colimit as the functor  $\mathcal{F} \mapsto \mathcal{F}_p$  associated to  $p$  in Sites, Equation 32.1.1 which proves the final assertion.  $\square$

**Lemma 19.8.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\bar{x}$  be a geometric point of  $X$ .*

- (1) *The stalk functor  $PAb(X_{\text{étale}}) \rightarrow Ab$ ,  $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$  is exact.*
- (2) *We have  $(\mathcal{F}^{\#})_{\bar{x}} = \mathcal{F}_{\bar{x}}$  for any presheaf of sets  $\mathcal{F}$  on  $X_{\text{étale}}$ .*
- (3) *The functor  $Ab(X_{\text{étale}}) \rightarrow Ab$ ,  $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$  is exact.*
- (4) *Similarly the functors  $PSh(X_{\text{étale}}) \rightarrow Sets$  and  $Sh(X_{\text{étale}}) \rightarrow Sets$  given by the stalk functor  $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$  are exact (see Categories, Definition 23.1) and commute with arbitrary colimits.*

**Proof.** This result follows from the general material in Modules on Sites, Section 36. This is true because  $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$  comes from a point of the small étale site of  $X$ , see Lemma 19.7. See the proof of Étale Cohomology, Lemma 29.9 for a direct proof of some of these statements in the setting of the small étale site of a scheme.  $\square$

We will see below that the stalk functor  $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$  is really the pullback along the morphism  $\bar{x}$ . In that sense the following lemma is a generalization of the lemma above.

**Lemma 19.9.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ .*

- (1) *The functor  $f_{small}^{-1} : Ab(Y_{\text{étale}}) \rightarrow Ab(X_{\text{étale}})$  is exact.*
- (2) *The functor  $f_{small}^{-1} : Sh(Y_{\text{étale}}) \rightarrow Sh(X_{\text{étale}})$  is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 23.1.*
- (3) *For any étale morphism  $V \rightarrow Y$  of algebraic spaces we have  $f_{small}^{-1}h_V = h_{X \times_Y V}$ .*
- (4) *Let  $\bar{x} \rightarrow X$  be a geometric point. Let  $\mathcal{G}$  be a sheaf on  $Y_{\text{étale}}$ . Then there is a canonical identification*

$$(f_{small}^{-1}\mathcal{G})_{\bar{x}} = \mathcal{G}_{\bar{y}}.$$

where  $\bar{y} = f \circ \bar{x}$ .

**Proof.** Recall that  $f_{small}$  is defined via  $f_{spaces, small}$  in Lemma 18.8. Parts (1), (2) and (3) are general consequences of the fact that  $f_{spaces, \text{étale}} : X_{spaces, \text{étale}} \rightarrow Y_{spaces, \text{étale}}$  is a morphism of sites, see Sites, Definition 14.1 for (2), Modules on Sites, Lemma 31.2 for (1), and Sites, Lemma 13.5 for (3).

Proof of (4). This statement is a special case of Sites, Lemma 34.2 via Lemma 19.7. We also provide a direct proof. Note that by Lemma 19.8. taking stalks commutes with sheafification. Let  $\mathcal{G}'$  be the sheaf on  $Y_{spaces, \text{étale}}$  whose restriction to  $Y_{\text{étale}}$  is

$\mathcal{G}$ . Recall that  $f_{spaces, \acute{e}tale}^{-1} \mathcal{G}'$  is the sheaf associated to the presheaf

$$U \longrightarrow \operatorname{colim}_{U \rightarrow X \times_Y V} \mathcal{G}'(V),$$

see Sites, Sections 13 and 5. Thus we have

$$\begin{aligned} (f_{spaces, \acute{e}tale}^{-1} \mathcal{G}')_{\bar{x}} &= \operatorname{colim}_{(U, \bar{u})} f_{spaces, \acute{e}tale}^{-1} \mathcal{G}'(U) \\ &= \operatorname{colim}_{(U, \bar{u})} \operatorname{colim}_{a: U \rightarrow X \times_Y V} \mathcal{G}'(V) \\ &= \operatorname{colim}_{(V, \bar{v})} \mathcal{G}'(V) \\ &= \mathcal{G}'_{\bar{y}} \end{aligned}$$

in the third equality the pair  $(U, \bar{u})$  and the map  $a : U \rightarrow X \times_Y V$  corresponds to the pair  $(V, a \circ \bar{u})$ . Since the stalk of  $\mathcal{G}'$  (resp.  $f_{spaces, \acute{e}tale}^{-1} \mathcal{G}'$ ) agrees with the stalk of  $\mathcal{G}$  (resp.  $f_{small}^{-1} \mathcal{G}$ ), see Equation (19.6.1) the result follows.  $\square$

**Remark 19.10.** This remark is the analogue of Étale Cohomology, Remark 56.6. Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\bar{x} : \operatorname{Spec}(k) \rightarrow X$  be a geometric point of  $X$ . By Étale Cohomology, Theorem 56.3 the category of sheaves on  $\operatorname{Spec}(k)_{\acute{e}tale}$  is equivalent to the category of sets (by taking a sheaf to its global sections). Hence it follows from Lemma 19.9 part (4) applied to the morphism  $\bar{x}$  that the functor

$$Sh(X_{\acute{e}tale}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\bar{x}}$$

is isomorphic to the functor

$$Sh(X_{\acute{e}tale}) \longrightarrow Sh(\operatorname{Spec}(k)_{\acute{e}tale}) = Sets, \quad \mathcal{F} \longmapsto \bar{x}^* \mathcal{F}$$

Hence we may view the stalk functors as pullback functors along geometric morphisms (and not just some abstract morphisms of topoi as in the result of Lemma 19.7).

**Remark 19.11.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$ . We claim that for any pair of geometric points  $\bar{x}$  and  $\bar{x}'$  lying over  $x$  the stalk functors are isomorphic. By definition of  $|X|$  we can find a third geometric point  $\bar{x}''$  so that there exists a commutative diagram

$$\begin{array}{ccc} \bar{x}'' & \longrightarrow & \bar{x}' \\ \downarrow & \searrow \bar{x}'' & \downarrow \bar{x}' \\ \bar{x} & \longrightarrow & X. \end{array}$$

Since the stalk functor  $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$  is given by pullback along the morphism  $\bar{x}$  (and similarly for the others) we conclude by functoriality of pullbacks.

The following theorem says that the small étale site of an algebraic space has enough points.

**Theorem 19.12.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . A map  $a : \mathcal{F} \rightarrow \mathcal{G}$  of sheaves of sets is injective (resp. surjective) if and only if the map on stalks  $a_{\bar{x}} : \mathcal{F}_{\bar{x}} \rightarrow \mathcal{G}_{\bar{x}}$  is injective (resp. surjective) for all geometric points of  $X$ . A sequence of abelian sheaves on  $X_{\acute{e}tale}$  is exact if and only if it is exact on all stalks at geometric points of  $S$ .*

**Proof.** We know the theorem is true if  $X$  is a scheme, see Étale Cohomology, Theorem 29.10. Choose a surjective étale morphism  $f : U \rightarrow X$  where  $U$  is a scheme. Since  $\{U \rightarrow X\}$  is a covering (in  $X_{spaces, \acute{e}tale}$ ) we can check whether a map of sheaves is injective, or surjective by restricting to  $U$ . Now if  $\bar{u} : \text{Spec}(k) \rightarrow U$  is a geometric point of  $U$ , then  $(\mathcal{F}|_U)_{\bar{u}} = \mathcal{F}_{\bar{x}}$  where  $\bar{x} = f \circ \bar{u}$ . (This is clear from the colimits defining the stalks at  $\bar{u}$  and  $\bar{x}$ , but it also follows from Lemma 19.9.) Hence the result for  $U$  implies the result for  $X$  and we win.  $\square$

The following lemma should be skipped on a first reading.

**Lemma 19.13.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $p : \text{Sh}(pt) \rightarrow \text{Sh}(X_{\acute{e}tale})$  be a point of the small étale topos of  $X$ . Then there exists a geometric point  $\bar{x}$  of  $X$  such that the stalk functor  $\mathcal{F} \mapsto \mathcal{F}_p$  is isomorphic to the stalk functor  $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$ .*

**Proof.** By Sites, Lemma 32.7 there is a one to one correspondence between points of the site and points of the associated topos. Hence we may assume that  $p$  is given by a functor  $u : X_{\acute{e}tale} \rightarrow \text{Sets}$  which defines a point of the site  $X_{\acute{e}tale}$ . Let  $U \in \text{Ob}(X_{\acute{e}tale})$  be an object whose structure morphism  $j : U \rightarrow X$  is surjective. Note that  $h_U$  is a sheaf which surjects onto the final sheaf. Since taking stalks is exact we see that  $(h_U)_p = u(U)$  is not empty (use Sites, Lemma 32.3). Pick  $x \in u(U)$ . By Sites, Lemma 35.1 we obtain a point  $q : \text{Sh}(pt) \rightarrow \text{Sh}(U_{\acute{e}tale})$  such that  $p = j_{small} \circ q$ , so that  $\mathcal{F}_p = (\mathcal{F}|_U)_q$  functorially. By Étale Cohomology, Lemma 29.12 there is a geometric point  $\bar{u}$  of  $U$  and a functorial isomorphism  $\mathcal{G}_q = \mathcal{G}_{\bar{u}}$  for  $\mathcal{G} \in \text{Sh}(U_{\acute{e}tale})$ . Set  $\bar{x} = j \circ \bar{u}$ . Then we see that  $\mathcal{F}_{\bar{x}} \cong (\mathcal{F}|_U)_{\bar{u}}$  functorially in  $\mathcal{F}$  on  $X_{\acute{e}tale}$  by Lemma 19.9 and we win.  $\square$

## 20. Supports of abelian sheaves

First we talk about supports of local sections.

**Lemma 20.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be a subsheaf of the final object of the étale topos of  $X$  (see Sites, Example 10.2). Then there exists a unique open  $W \subset X$  such that  $\mathcal{F} = h_W$ .*

**Proof.** The condition means that  $\mathcal{F}(U)$  is a singleton or empty for all  $\varphi : U \rightarrow X$  in  $\text{Ob}(X_{spaces, \acute{e}tale})$ . In particular local sections always glue. If  $\mathcal{F}(U) \neq \emptyset$ , then  $\mathcal{F}(\varphi(U)) \neq \emptyset$  because  $\varphi(U) \subset X$  is an open subspace (Lemma 16.7) and  $\{\varphi : U \rightarrow \varphi(U)\}$  is a covering in  $X_{spaces, \acute{e}tale}$ . Take  $W = \bigcup_{\varphi : U \rightarrow S, \mathcal{F}(U) \neq \emptyset} \varphi(U)$  to conclude.  $\square$

**Lemma 20.2.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be an abelian sheaf on  $X_{spaces, \acute{e}tale}$ . Let  $\sigma \in \mathcal{F}(U)$  be a local section. There exists an open subspace  $W \subset U$  such that*

- (1)  $W \subset U$  is the largest open subspace of  $U$  such that  $\sigma|_W = 0$ ,
- (2) for every  $\varphi : V \rightarrow U$  in  $X_{\acute{e}tale}$  we have

$$\sigma|_V = 0 \Leftrightarrow \varphi(V) \subset W,$$

- (3) for every geometric point  $\bar{u}$  of  $U$  we have

$$(U, \bar{u}, \sigma) = 0 \text{ in } \mathcal{F}_{\bar{s}} \Leftrightarrow \bar{u} \in W$$

where  $\bar{s} = (U \rightarrow S) \circ \bar{u}$ .

**Proof.** Since  $\mathcal{F}$  is a sheaf in the étale topology the restriction of  $\mathcal{F}$  to  $U_{Zar}$  is a sheaf on  $U$  in the Zariski topology. Hence there exists a Zariski open  $W$  having property (1), see Modules, Lemma 5.2. Let  $\varphi : V \rightarrow U$  be an arrow of  $X_{\acute{e}tale}$ . Note that  $\varphi(V) \subset U$  is an open subspace (Lemma 16.7) and that  $\{V \rightarrow \varphi(V)\}$  is an étale covering. Hence if  $\sigma|_V = 0$ , then by the sheaf condition for  $\mathcal{F}$  we see that  $\sigma|_{\varphi(V)} = 0$ . This proves (2). To prove (3) we have to show that if  $(U, \bar{u}, \sigma)$  defines the zero element of  $\mathcal{F}_{\bar{s}}$ , then  $\bar{u} \in W$ . This is true because the assumption means there exists a morphism of étale neighbourhoods  $(V, \bar{v}) \rightarrow (U, \bar{u})$  such that  $\sigma|_V = 0$ . Hence by (2) we see that  $V \rightarrow U$  maps into  $W$ , and hence  $\bar{u} \in W$ .  $\square$

Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$ . Let  $\mathcal{F}$  be a sheaf on  $X_{\acute{e}tale}$ . By Remark 19.11 the isomorphism class of the stalk of the sheaf  $\mathcal{F}$  at a geometric point lying over  $x$  is well defined.

**Definition 20.3.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be an abelian sheaf on  $X_{\acute{e}tale}$ .

- (1) The *support* of  $\mathcal{F}$  is the set of points  $x \in |X|$  such that  $\mathcal{F}_{\bar{x}} \neq 0$  for any (some) geometric point  $\bar{x}$  lying over  $x$ .
- (2) Let  $\sigma \in \mathcal{F}(U)$  be a section. The *support* of  $\sigma$  is the closed subset  $U \setminus W$ , where  $W \subset U$  is the largest open subset of  $U$  on which  $\sigma$  restricts to zero (see Lemma 20.2).

**Lemma 20.4.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be an abelian sheaf on  $X_{\acute{e}tale}$ . Let  $U \in \text{Ob}(X_{\acute{e}tale})$  and  $\sigma \in \mathcal{F}(U)$ .

- (1) The support of  $\sigma$  is closed in  $|X|$ .
- (2) The support of  $\sigma + \sigma'$  is contained in the union of the supports of  $\sigma, \sigma' \in \mathcal{F}(U)$ .
- (3) If  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is a map of abelian sheaves on  $X_{\acute{e}tale}$ , then the support of  $\varphi(\sigma)$  is contained in the support of  $\sigma \in \mathcal{F}(U)$ .
- (4) The support of  $\mathcal{F}$  is the union of the images of the supports of all local sections of  $\mathcal{F}$ .
- (5) If  $\mathcal{F} \rightarrow \mathcal{G}$  is surjective then the support of  $\mathcal{G}$  is a subset of the support of  $\mathcal{F}$ .
- (6) If  $\mathcal{F} \rightarrow \mathcal{G}$  is injective then the support of  $\mathcal{F}$  is a subset of the support of  $\mathcal{G}$ .

**Proof.** Part (1) holds by definition. Parts (2) and (3) hold because they hold for the restriction of  $\mathcal{F}$  and  $\mathcal{G}$  to  $U_{Zar}$ , see Modules, Lemma 5.2. Part (4) is a direct consequence of Lemma 20.2 part (3). Parts (5) and (6) follow from the other parts.  $\square$

**Lemma 20.5.** The support of a sheaf of rings on the small étale site of an algebraic space is closed.

**Proof.** This is true because (according to our conventions) a ring is 0 if and only if  $1 = 0$ , and hence the support of a sheaf of rings is the support of the unit section.  $\square$

## 21. The structure sheaf of an algebraic space

The structure sheaf of an algebraic space is the sheaf of rings of the following lemma.

**Lemma 21.1.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The rule  $U \mapsto \Gamma(U, \mathcal{O}_U)$  defines a sheaf of rings on  $X_{\acute{e}tale}$ .

**Proof.** Immediate from the definition of a covering and Descent, Lemma 8.1.  $\square$

**Definition 21.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The *structure sheaf* of  $X$  is the sheaf of rings  $\mathcal{O}_X$  on the small étale site  $X_{\text{étale}}$  described in Lemma 21.1.

According to Lemma 18.13 the sheaf  $\mathcal{O}_X$  corresponds to a system of étale sheaves  $(\mathcal{O}_X)_U$  for  $U$  ranging through the objects of  $X_{\text{étale}}$ . It is clear from the proof of that lemma and our definition that we have simply  $(\mathcal{O}_X)_U = \mathcal{O}_U$  where  $\mathcal{O}_U$  is the structure sheaf of  $U_{\text{étale}}$  as introduced in Descent, Definition 8.2. In particular, if  $X$  is a scheme we recover the sheaf  $\mathcal{O}_X$  on the small étale site of  $X$ .

Via the equivalence  $Sh(X_{\text{étale}}) = Sh(X_{\text{spaces}, \text{étale}})$  of Lemma 18.3 we may also think of  $\mathcal{O}_X$  as a sheaf of rings on  $X_{\text{spaces}, \text{étale}}$ . It is explained in Remark 18.4 how to compute  $\mathcal{O}_X(Y)$ , and in particular  $\mathcal{O}_X(X)$ , when  $Y \rightarrow X$  is an object of  $X_{\text{spaces}, \text{étale}}$ .

**Lemma 21.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Then there is a canonical map  $f^\sharp : f_{\text{small}}^{-1} \mathcal{O}_Y \rightarrow \mathcal{O}_X$  such that*

$$(f_{\text{small}}, f^\sharp) : (Sh(X_{\text{étale}}), \mathcal{O}_X) \longrightarrow (Sh(Y_{\text{étale}}), \mathcal{O}_Y)$$

*is a morphism of ringed topoi. Furthermore,*

- (1) *The construction  $f \mapsto (f_{\text{small}}, f^\sharp)$  is compatible with compositions.*
- (2) *If  $f$  is a morphism of schemes, then  $f^\sharp$  is the map described in Descent, Remark 8.4.*

**Proof.** By Lemma 18.10 it suffices to give an  $f$ -map from  $\mathcal{O}_Y$  to  $\mathcal{O}_X$ . In other words, for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ g \downarrow & & \downarrow f \\ V & \longrightarrow & Y \end{array}$$

where  $U \in X_{\text{étale}}$ ,  $V \in Y_{\text{étale}}$  we have to give a map of rings  $(f^\sharp)_{(U, V, g)} : \Gamma(V, \mathcal{O}_V) \rightarrow \Gamma(U, \mathcal{O}_U)$ . Of course we just take  $(f^\sharp)_{(U, V, g)} = g^\sharp$ . It is clear that this is compatible with restriction mappings and hence indeed gives an  $f$ -map. We omit checking compatibility with compositions and agreement with the construction in Descent, Remark 8.4.  $\square$

**Lemma 21.4.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The following are equivalent*

- (1)  *$X$  is reduced,*
- (2) *for every  $x \in |X|$  the local ring of  $X$  at  $x$  is reduced (Remark 7.6).*

*In this case  $\Gamma(X, \mathcal{O}_X)$  is a reduced ring and if  $f \in \Gamma(X, \mathcal{O}_X)$  has  $X = V(f)$ , then  $f = 0$ .*

**Proof.** The equivalence of (1) and (2) follows from Properties, Lemma 3.2 applied to affine schemes étale over  $X$ . The final statements follow the cited lemma and fact that  $\Gamma(X, \mathcal{O}_X)$  is a subring of  $\Gamma(U, \mathcal{O}_U)$  for some reduced scheme  $U$  étale over  $X$ .  $\square$



## 22. Stalks of the structure sheaf

This section is the analogue of Étale Cohomology, Section 33.

**Lemma 22.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\bar{x}$  be a geometric point of  $X$ . Let  $(U, \bar{u})$  be an étale neighbourhood of  $\bar{x}$  where  $U$  is a scheme. Then we have*

$$\mathcal{O}_{X, \bar{x}} = \mathcal{O}_{U, \bar{u}} = \mathcal{O}_{U, u}^{sh}$$

where the left hand side is the stalk of the structure sheaf of  $X$ , and the right hand side is the strict henselization of the local ring of  $U$  at the point  $u$  at which  $\bar{u}$  is centered.

**Proof.** We know that the structure sheaf  $\mathcal{O}_U$  on  $U_{\text{étale}}$  is the restriction of the structure sheaf of  $X$ . Hence the first equality follows from Lemma 19.9 part (4). The second equality is explained in Étale Cohomology, Lemma 33.1.  $\square$

**Definition 22.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\bar{x}$  be a geometric point of  $X$  lying over the point  $x \in |X|$ .

- (1) The *étale local ring of  $X$  at  $\bar{x}$*  is the stalk of the structure sheaf  $\mathcal{O}_X$  on  $X_{\text{étale}}$  at  $\bar{x}$ . Notation:  $\mathcal{O}_{X, \bar{x}}$ .
- (2) The *strict henselization of  $X$  at  $\bar{x}$*  is the scheme  $\text{Spec}(\mathcal{O}_{X, \bar{x}})$ .

The isomorphism type of the strict henselization of  $X$  at  $\bar{x}$  (as a scheme over  $X$ ) depends only on the point  $x \in |X|$  and not on the choice of the geometric point lying over  $x$ , see Remark 19.11.

**Lemma 22.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The small étale site  $X_{\text{étale}}$  endowed with its structure sheaf  $\mathcal{O}_X$  is a locally ringed site, see Modules on Sites, Definition 40.4.*

**Proof.** This follows because the stalks  $\mathcal{O}_{X, \bar{x}}$  are local, and because  $S_{\text{étale}}$  has enough points, see Lemmas 22.1 and Theorem 19.12. See Modules on Sites, Lemma 40.2 and 40.3 for the fact that this implies the small étale site is locally ringed.  $\square$

**Lemma 22.4.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$  be a point. Let  $d \in \{0, 1, 2, \dots, \infty\}$ . The following are equivalent*

- (1) *the dimension of the local ring of  $X$  at  $x$  (Definition 10.2) is  $d$ ,*
- (2)  *$\dim(\mathcal{O}_{X, \bar{x}}) = d$  for some geometric point  $\bar{x}$  lying over  $x$ , and*
- (3)  *$\dim(\mathcal{O}_{X, \bar{x}}) = d$  for any geometric point  $\bar{x}$  lying over  $x$ .*

**Proof.** The equivalence of (2) and (3) follows from the fact that the isomorphism type of  $\mathcal{O}_{X, \bar{x}}$  only depends on  $x \in |X|$ , see Remark 19.11. Using Lemma 22.1 the equivalence of (1) and (2)+(3) comes down to the following statement: Given any local ring  $R$  we have  $\dim(R) = \dim(R^{sh})$ . This is More on Algebra, Lemma 45.7.  $\square$

**Lemma 22.5.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be an étale morphism of algebraic spaces over  $S$ . Let  $x \in X$ . Then (1)  $\dim_x(X) = \dim_{f(x)}(Y)$  and (2) the dimension of the local ring of  $X$  at  $x$  equals the dimension of the local ring of  $Y$  at  $f(x)$ . If  $f$  is surjective, then (3)  $\dim(X) = \dim(Y)$ .*

**Proof.** Choose a scheme  $U$  and a point  $u \in U$  and an étale morphism  $U \rightarrow X$  which maps  $u$  to  $x$ . Then the composition  $U \rightarrow Y$  is also étale and maps  $u$  to  $f(x)$ . Thus the statements (1) and (2) follow as the relevant integers are defined in terms

of the behaviour of the scheme  $U$  at  $u$ . See Definition 9.1 for (1). Part (3) is an immediate consequence of (1), see Definition 9.2.  $\square$

**Lemma 22.6.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$  be a point. The following are equivalent*

- (1) *the local ring of  $X$  at  $x$  is reduced (Remark 7.6),*
- (2)  *$\mathcal{O}_{X,\bar{x}}$  is reduced for some geometric point  $\bar{x}$  lying over  $x$ , and*
- (3)  *$\mathcal{O}_{X,\bar{x}}$  is reduced for any geometric point  $\bar{x}$  lying over  $x$ .*

**Proof.** The equivalence of (2) and (3) follows from the fact that the isomorphism type of  $\mathcal{O}_{X,\bar{x}}$  only depends on  $x \in |X|$ , see Remark 19.11. Using Lemma 22.1 the equivalence of (1) and (2)+(3) comes down to the following statement: a local ring is reduced if and only if its strict henselization is reduced. This is More on Algebra, Lemma 45.4.  $\square$

### 23. Local irreducibility

A point on an algebraic space has a well defined étale local ring, which corresponds to the strict henselization of the local ring in the case of a scheme. In general we cannot see how many irreducible components of a scheme or an algebraic space pass through the given point from the étale local ring. We can only count the number of geometric branches.

**Lemma 23.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$  be a point. The following are equivalent*

- (1) *for any scheme  $U$  and étale morphism  $a : U \rightarrow X$  and  $u \in U$  with  $a(u) = x$  the local ring  $\mathcal{O}_{U,u}$  has a unique minimal prime,*
- (2) *for any scheme  $U$  and étale morphism  $a : U \rightarrow X$  and  $u \in U$  with  $a(u) = x$  there is a unique irreducible component of  $U$  through  $u$ ,*
- (3) *for any scheme  $U$  and étale morphism  $a : U \rightarrow X$  and  $u \in U$  with  $a(u) = x$  the local ring  $\mathcal{O}_{U,u}$  is unibranch,*
- (4) *for any scheme  $U$  and étale morphism  $a : U \rightarrow X$  and  $u \in U$  with  $a(u) = x$  the local ring  $\mathcal{O}_{U,u}$  is geometrically unibranch,*
- (5)  *$\mathcal{O}_{X,\bar{x}}$  has a unique minimal prime for any geometric point  $\bar{x}$  lying over  $x$ .*

**Proof.** The equivalence of (1) and (2) follows from the fact that irreducible components of  $U$  passing through  $u$  are in 1-1 correspondence with minimal primes of the local ring of  $U$  at  $u$ . Let  $a : U \rightarrow X$  and  $u \in U$  be as in (1). Then  $\mathcal{O}_{X,\bar{x}}$  is the strict henselization of  $\mathcal{O}_{U,u}$  by Lemma 22.1. In particular (4) and (5) are equivalent by More on Algebra, Lemma 106.5. The equivalence of (2), (3), and (4) follows from More on Morphisms, Lemma 36.2.  $\square$

**Definition 23.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$ . We say that  $X$  is *geometrically unibranch at  $x$*  if the equivalent conditions of Lemma 23.1 hold. We say that  $X$  is *geometrically unibranch* if  $X$  is geometrically unibranch at every  $x \in |X|$ .

This is consistent with the definition for schemes (Properties, Definition 15.1).

**Lemma 23.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$  be a point. Let  $n \in \{1, 2, \dots\}$  be an integer. The following are equivalent*

- (1) for any scheme  $U$  and étale morphism  $a : U \rightarrow X$  and  $u \in U$  with  $a(u) = x$  the number of minimal primes of the local ring  $\mathcal{O}_{U,u}$  is  $\leq n$  and for at least one choice of  $U, a, u$  it is  $n$ ,
- (2) for any scheme  $U$  and étale morphism  $a : U \rightarrow X$  and  $u \in U$  with  $a(u) = x$  the number irreducible components of  $U$  passing through  $u$  is  $\leq n$  and for at least one choice of  $U, a, u$  it is  $n$ ,
- (3) for any scheme  $U$  and étale morphism  $a : U \rightarrow X$  and  $u \in U$  with  $a(u) = x$  the number of branches of  $U$  at  $u$  is  $\leq n$  and for at least one choice of  $U, a, u$  it is  $n$ ,
- (4) for any scheme  $U$  and étale morphism  $a : U \rightarrow X$  and  $u \in U$  with  $a(u) = x$  the number of geometric branches of  $U$  at  $u$  is  $n$ , and
- (5) the number of minimal prime ideals of  $\mathcal{O}_{X,\bar{x}}$  is  $n$ .

**Proof.** The equivalence of (1) and (2) follows from the fact that irreducible components of  $U$  passing through  $u$  are in 1-1 correspondence with minimal primes of the local ring of  $U$  at  $u$ . Let  $a : U \rightarrow X$  and  $u \in U$  be as in (1). Then  $\mathcal{O}_{X,\bar{x}}$  is the strict henselization of  $\mathcal{O}_{U,u}$  by Lemma 22.1. Recall that the (geometric) number of branches of  $U$  at  $u$  is the number of minimal prime ideals of the (strict) henselization of  $\mathcal{O}_{U,u}$ . In particular (4) and (5) are equivalent. The equivalence of (2), (3), and (4) follows from More on Morphisms, Lemma 36.2.  $\square$

**Definition 23.4.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$ . The number of geometric branches of  $X$  at  $x$  is either  $n \in \mathbf{N}$  if the equivalent conditions of Lemma 23.3 hold, or else  $\infty$ .

## 24. Noetherian spaces

We have already defined locally Noetherian algebraic spaces in Section 7.

**Definition 24.1.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . We say  $X$  is *Noetherian* if  $X$  is quasi-compact, quasi-separated and locally Noetherian.

Note that a Noetherian algebraic space  $X$  is not just quasi-compact and locally Noetherian, but also quasi-separated. This does not conflict with the definition of a Noetherian scheme, as a locally Noetherian scheme is quasi-separated, see Properties, Lemma 5.4. This does not hold for algebraic spaces. Namely,  $X = \mathbf{A}_k^1/\mathbf{Z}$ , see Spaces, Example 14.8 is locally Noetherian and quasi-compact but not quasi-separated (hence not Noetherian according to our definitions).

A consequence of the choice made above is that an algebraic space of finite type over a Noetherian algebraic space is not automatically Noetherian, i.e., the analogue of Morphisms, Lemma 15.6 does not hold. The correct statement is that an algebraic space of finite presentation over a Noetherian algebraic space is Noetherian (see Morphisms of Spaces, Lemma 28.6).

A Noetherian algebraic space  $X$  is very close to being a scheme. In the rest of this section we collect some lemmas to illustrate this.

**Lemma 24.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ .

- (1) If  $X$  is locally Noetherian then  $|X|$  is a locally Noetherian topological space.
- (2) If  $X$  is quasi-compact and locally Noetherian, then  $|X|$  is a Noetherian topological space.

**Proof.** Assume  $X$  is locally Noetherian. Choose a scheme  $U$  and a surjective étale morphism  $U \rightarrow X$ . As  $X$  is locally Noetherian we see that  $U$  is locally Noetherian. By Properties, Lemma 5.5 this means that  $|U|$  is a locally Noetherian topological space. Since  $|U| \rightarrow |X|$  is open and surjective we conclude that  $|X|$  is locally Noetherian by Topology, Lemma 9.3. This proves (1). If  $X$  is quasi-compact and locally Noetherian, then  $|X|$  is quasi-compact and locally Noetherian. Hence  $|X|$  is Noetherian by Topology, Lemma 12.14.  $\square$

**Lemma 24.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . If  $X$  is Noetherian, then  $|X|$  is a sober Noetherian topological space.*

**Proof.** A quasi-separated algebraic space has an underlying sober topological space, see Lemma 15.1. It is Noetherian by Lemma 24.2.  $\square$

**Lemma 24.4.** *Let  $S$  be a scheme. Let  $X$  be a Noetherian algebraic space over  $S$ . Let  $\bar{x}$  be a geometric point of  $X$ . Then  $\mathcal{O}_{X,\bar{x}}$  is a Noetherian local ring.*

**Proof.** Choose an étale neighbourhood  $(U, \bar{u})$  of  $\bar{x}$  where  $U$  is a scheme. Then  $\mathcal{O}_{X,\bar{x}}$  is the strict henselization of the local ring of  $U$  at  $u$ , see Lemma 22.1. By our definition of Noetherian spaces the scheme  $U$  is locally Noetherian. Hence we conclude by More on Algebra, Lemma 45.3.  $\square$

## 25. Regular algebraic spaces

We have already defined regular algebraic spaces in Section 7.

**Lemma 25.1.** *Let  $S$  be a scheme. Let  $X$  be a locally Noetherian algebraic space over  $S$ . The following are equivalent*

- (1)  *$X$  is regular, and*
- (2) *every étale local ring  $\mathcal{O}_{X,\bar{x}}$  is regular.*

**Proof.** Let  $U$  be a scheme and let  $U \rightarrow X$  be a surjective étale morphism. By assumption  $U$  is locally Noetherian. Moreover, every étale local ring  $\mathcal{O}_{X,\bar{x}}$  is the strict henselization of a local ring on  $U$  and conversely, see Lemma 22.1. Thus by More on Algebra, Lemma 45.10 we see that (2) is equivalent to every local ring of  $U$  being regular, i.e.,  $U$  being a regular scheme (see Properties, Lemma 9.2). This is equivalent to (1) by Definition 7.2.  $\square$

We can use Descent, Lemma 21.4 to define what it means for an algebraic space  $X$  to be regular at a point  $x$ .

**Definition 25.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$  be a point. We say  $X$  is regular at  $x$  if  $\mathcal{O}_{U,u}$  is a regular local ring for any (equivalently some) pair  $(a : U \rightarrow X, u)$  consisting of an étale morphism  $a : U \rightarrow X$  from a scheme to  $X$  and a point  $u \in U$  with  $a(u) = x$ .

See Definition 7.5, Lemma 7.4, and Descent, Lemma 21.4.

**Lemma 25.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $x \in |X|$  be a point. The following are equivalent*

- (1)  *$X$  is regular at  $x$ , and*
- (2) *the étale local ring  $\mathcal{O}_{X,\bar{x}}$  is regular for any (equivalently some) geometric point  $\bar{x}$  lying over  $x$ .*

**Proof.** Let  $U$  be a scheme,  $u \in U$  a point, and let  $a : U \rightarrow X$  be an étale morphism mapping  $u$  to  $x$ . For any geometric point  $\bar{x}$  of  $X$  lying over  $x$ , the étale local ring  $\mathcal{O}_{X,\bar{x}}$  is the strict henselization of a local ring on  $U$  at  $u$ , see Lemma 22.1. Thus we conclude by More on Algebra, Lemma 45.10.  $\square$

**Lemma 25.4.** *A regular algebraic space is normal.*

**Proof.** This follows from the definitions and the case of schemes See Properties, Lemma 9.4.  $\square$

## 26. Sheaves of modules on algebraic spaces

If  $X$  is an algebraic space, then a sheaf of modules on  $X$  is a sheaf of  $\mathcal{O}_X$ -modules on the small étale site of  $X$  where  $\mathcal{O}_X$  is the structure sheaf of  $X$ . The category of sheaves of modules is denoted  $\text{Mod}(\mathcal{O}_X)$ .

Given a morphism  $f : X \rightarrow Y$  of algebraic spaces, by Lemma 21.3 we get a morphism of ringed topoi and hence by Modules on Sites, Definition 13.1 we get well defined pullback and direct image functors

$$(26.0.1) \quad f^* : \text{Mod}(\mathcal{O}_Y) \longrightarrow \text{Mod}(\mathcal{O}_X), \quad f_* : \text{Mod}(\mathcal{O}_X) \longrightarrow \text{Mod}(\mathcal{O}_Y)$$

which are adjoint in the usual way. If  $g : Y \rightarrow Z$  is another morphism of algebraic spaces over  $S$ , then we have  $(g \circ f)^* = f^* \circ g^*$  and  $(g \circ f)_* = g_* \circ f_*$  simply because the morphisms of ringed topoi compose in the corresponding way (by the lemma).

**Lemma 26.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be an étale morphism of algebraic spaces over  $S$ . Then  $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$ , and  $f^*\mathcal{G} = f_{\text{small}}^{-1}\mathcal{G}$  for any sheaf of  $\mathcal{O}_Y$ -modules  $\mathcal{G}$ . In particular,  $f^* : \text{Mod}(\mathcal{O}_Y) \rightarrow \text{Mod}(\mathcal{O}_X)$  is exact.*

**Proof.** By the description of inverse image in Lemma 18.11 and the definition of the structure sheaves it is clear that  $f_{\text{small}}^{-1}\mathcal{O}_Y = \mathcal{O}_X$ . Since the pullback

$$f^*\mathcal{G} = f_{\text{small}}^{-1}\mathcal{G} \otimes_{f_{\text{small}}^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

by definition we conclude that  $f^*\mathcal{G} = f_{\text{small}}^{-1}\mathcal{G}$ . The exactness is clear because  $f_{\text{small}}^{-1}$  is exact, as  $f_{\text{small}}$  is a morphism of topoi.  $\square$

We continue our abuse of notation introduced in Equation (18.11.1) by writing

$$(26.1.1) \quad \mathcal{G}|_{X_{\text{étale}}} = f^*\mathcal{G} = f_{\text{small}}^{-1}\mathcal{G}$$

in the situation of the lemma above. We will discuss this in a more technical fashion in Section 27.

**Lemma 26.2.** *Let  $S$  be a scheme. Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

*be a cartesian square of algebraic spaces over  $S$ . Let  $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$ . If  $g$  is étale, then  $f'_*(\mathcal{F}|_{X'}) = (f_*\mathcal{F})|_{Y'}$ <sup>6</sup> and  $R^if'_*(\mathcal{F}|_{X'}) = (R^if_*\mathcal{F})|_{Y'}$  in  $\text{Mod}(\mathcal{O}_{Y'})$ .*

**Proof.** This is a reformulation of Lemma 18.12 in the case of modules.  $\square$

<sup>6</sup>Also  $(f')^*(\mathcal{G}|_{Y'}) = (f^*\mathcal{G})|_{X'}$  by commutativity of the diagram and (26.1.1)

**Lemma 26.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . A sheaf  $\mathcal{F}$  of  $\mathcal{O}_X$ -modules is given by the following data:*

- (1) *for every  $U \in \text{Ob}(X_{\text{étale}})$  a sheaf  $\mathcal{F}_U$  of  $\mathcal{O}_U$ -modules on  $U_{\text{étale}}$ ,*
- (2) *for every  $f : U' \rightarrow U$  in  $X_{\text{étale}}$  an isomorphism  $c_f : f_{\text{small}}^* \mathcal{F}_U \rightarrow \mathcal{F}_{U'}$ .*

*These data are subject to the condition that given any  $f : U' \rightarrow U$  and  $g : U'' \rightarrow U'$  in  $X_{\text{étale}}$  the composition  $c_g \circ g_{\text{small}}^* c_f$  is equal to  $c_{f \circ g}$ .*

**Proof.** Combine Lemmas 26.1 and 18.13, and use the fact that any morphism between objects of  $X_{\text{étale}}$  is an étale morphism of schemes.  $\square$

## 27. Étale localization

Reading this section should be avoided at all cost.

Let  $X \rightarrow Y$  be an étale morphism of algebraic spaces. Then  $X$  is an object of  $Y_{\text{spaces}, \text{étale}}$  and it is immediate from the definitions, see also the proof of Lemma 18.11, that

$$(27.0.1) \quad X_{\text{spaces}, \text{étale}} = Y_{\text{spaces}, \text{étale}} / X$$

where the right hand side is the localization of the site  $Y_{\text{spaces}, \text{étale}}$  at the object  $X$ , see Sites, Definition 25.1. Moreover, this identification is compatible with the structure sheaves by Lemma 26.1. Hence the ringed site  $(X_{\text{spaces}, \text{étale}}, \mathcal{O}_X)$  is identified with the localization of the ringed site  $(Y_{\text{spaces}, \text{étale}}, \mathcal{O}_Y)$  at the object  $X$ :

$$(27.0.2) \quad (X_{\text{spaces}, \text{étale}}, \mathcal{O}_X) = (Y_{\text{spaces}, \text{étale}} / X, \mathcal{O}_Y|_{Y_{\text{spaces}, \text{étale}} / X})$$

The localization of a ringed site used on the right hand side is defined in Modules on Sites, Definition 19.1.

Assume now  $X \rightarrow Y$  is an étale morphism of algebraic spaces and  $X$  is a scheme. Then  $X$  is an object of  $Y_{\text{étale}}$  and it follows that

$$(27.0.3) \quad X_{\text{étale}} = Y_{\text{étale}} / X$$

and

$$(27.0.4) \quad (X_{\text{étale}}, \mathcal{O}_X) = (Y_{\text{étale}} / X, \mathcal{O}_Y|_{Y_{\text{étale}} / X})$$

as above.

Finally, if  $X \rightarrow Y$  is an étale morphism of algebraic spaces and  $X$  is an affine scheme, then  $X$  is an object of  $Y_{\text{affine}, \text{étale}}$  and

$$(27.0.5) \quad X_{\text{affine}, \text{étale}} = Y_{\text{affine}, \text{étale}} / X$$

and

$$(27.0.6) \quad (X_{\text{affine}, \text{étale}}, \mathcal{O}_X) = (Y_{\text{affine}, \text{étale}} / X, \mathcal{O}_Y|_{Y_{\text{affine}, \text{étale}} / X})$$

as above.

Next, we show that these localizations are compatible with morphisms.

**Lemma 27.1.** *Let  $S$  be a scheme. Let*

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

be a commutative diagram of algebraic spaces over  $S$  with  $p$  and  $q$  étale. Via the identifications (27.0.2) for  $U \rightarrow X$  and  $V \rightarrow Y$  the morphism of ringed topoi

$$(g_{spaces, \acute{e}tale}, g^\sharp) : (Sh(U_{spaces, \acute{e}tale}), \mathcal{O}_U) \longrightarrow (Sh(V_{spaces, \acute{e}tale}), \mathcal{O}_V)$$

is 2-isomorphic to the morphism  $(f_{spaces, \acute{e}tale, c}, f_c^\sharp)$  constructed in Modules on Sites, Lemma 20.2 starting with the morphism of ringed sites  $(f_{spaces, \acute{e}tale}, f^\sharp)$  and the map  $c : U \rightarrow V \times_Y X$  corresponding to  $g$ .

**Proof.** The morphism  $(f_{spaces, \acute{e}tale, c}, f_c^\sharp)$  is defined as a composition  $f' \circ j$  of a localization and a base change map. Similarly  $g$  is a composition  $U \rightarrow V \times_Y X \rightarrow V$ . Hence it suffices to prove the lemma in the following two cases: (1)  $f = \text{id}$ , and (2)  $U = X \times_Y V$ . In case (1) the morphism  $g : U \rightarrow V$  is étale, see Lemma 16.6. Hence  $(g_{spaces, \acute{e}tale}, g^\sharp)$  is a localization morphism by the discussion surrounding Equations (27.0.1) and (27.0.2) which is exactly the content of the lemma in this case. In case (2) the morphism  $g_{spaces, \acute{e}tale}$  comes from the morphism of ringed sites given by the functor  $V_{spaces, \acute{e}tale} \rightarrow U_{spaces, \acute{e}tale}$ ,  $V'/V \mapsto V' \times_V U/U$  which is also what the morphism  $f'$  is defined by, see Sites, Lemma 28.1. We omit the verification that  $(f')^\sharp = g^\sharp$  in this case (both are the restriction of  $f^\sharp$  to  $U_{spaces, \acute{e}tale}$ ).  $\square$

**Lemma 27.2.** *Same notation and assumptions as in Lemma 27.1 except that we also assume  $U$  and  $V$  are schemes. Via the identifications (27.0.4) for  $U \rightarrow X$  and  $V \rightarrow Y$  the morphism of ringed topoi*

$$(g_{small}, g^\sharp) : (Sh(U_{\acute{e}tale}), \mathcal{O}_U) \longrightarrow (Sh(V_{\acute{e}tale}), \mathcal{O}_V)$$

is 2-isomorphic to the morphism  $(f_{small, s}, f_s^\sharp)$  constructed in Modules on Sites, Lemma 22.3 starting with  $(f_{small}, f^\sharp)$  and the map  $s : h_U \rightarrow f_{small}^{-1} h_V$  corresponding to  $g$ .

**Proof.** Note that  $(g_{small}, g^\sharp)$  is 2-isomorphic as a morphism of ringed topoi to the morphism of ringed topoi associated to the morphism of ringed sites  $(g_{spaces, \acute{e}tale}, g^\sharp)$ . Hence we conclude by Lemma 27.1 and Modules on Sites, Lemma 22.4.  $\square$

Finally, we discuss the relationship between sheaves of sets on the small étale site  $Y_{\acute{e}tale}$  of an algebraic space  $Y$  and algebraic spaces étale over  $Y$ . Let  $S$  be a scheme and let  $Y$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be an object of  $Sh(Y_{\acute{e}tale})$ . Consider the functor

$$X : (Sch/S)_{fppf}^{opp} \longrightarrow Sets$$

defined by the rule

$$X(T) = \{(y, s) \mid y : T \rightarrow Y \text{ is a morphism over } S \text{ and } s \in \Gamma(T, y_{small}^{-1} \mathcal{F})\}$$

Given a morphism  $g : T' \rightarrow T$  the restriction map sends  $(y, s)$  to  $(y \circ g, g_{small}^{-1} s)$ . This makes sense as  $y_{small} \circ g_{small} = (y \circ g)_{small}$  by Lemma 18.8.

**Lemma 27.3.** *Let  $S$  be a scheme and let  $Y$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be a sheaf of sets on  $Y_{\acute{e}tale}$ . Provided a set theoretic condition is satisfied (see proof) the functor  $X$  associated to  $\mathcal{F}$  above is an algebraic space and there is an étale morphism  $f : X \rightarrow Y$  of algebraic spaces such that  $\mathcal{F} = f_{small, *}^*$  where  $*$  is the final object of the category  $Sh(X_{\acute{e}tale})$  (constant sheaf with value a singleton).*

**Proof.** Let us prove that  $X$  is a sheaf for the fppf topology. Namely, suppose that  $\{g_i : T_i \rightarrow T\}$  is a covering of  $(Sch/S)_{fppf}$  and  $(y_i, s_i) \in X(T_i)$  satisfy the glueing condition, i.e., the restriction of  $(y_i, s_i)$  and  $(y_j, s_j)$  to  $T_i \times_T T_j$  agree. Then since

$Y$  is a sheaf for the fppf topology, we see that the  $y_i$  give rise to a unique morphism  $y : T \rightarrow Y$  such that  $y_i = y \circ g_i$ . Then we see that  $y_{i,small}^{-1}\mathcal{F} = g_{i,small}^{-1}y_{small}^{-1}\mathcal{F}$ . Hence the sections  $s_i$  glue uniquely to a section of  $y_{small}^{-1}\mathcal{F}$  by Étale Cohomology, Lemma 39.2.

The construction that sends  $\mathcal{F} \in \text{Ob}(Sh(Y_{\text{étale}}))$  to  $X \in \text{Ob}((Sch/S)_{fppf})$  preserves finite limits and all colimits since each of the functors  $y_{small}^{-1}$  have this property. Of course, if  $V \in \text{Ob}(Y_{\text{étale}})$ , then the construction sends the representable sheaf  $h_V$  on  $Y_{\text{étale}}$  to the representable functor represented by  $V$ .

By Sites, Lemma 12.5 we can find a set  $I$ , for each  $i \in I$  an object  $V_i$  of  $Y_{\text{étale}}$  and a surjective map of sheaves

$$\coprod h_{V_i} \longrightarrow \mathcal{F}$$

on  $Y_{\text{étale}}$ . The set theoretic condition we need is that the index set  $I$  is not too large<sup>7</sup>. Then  $V = \coprod V_i$  is an object of  $(Sch/S)_{fppf}$  and therefore an object of  $Y_{\text{étale}}$  and we have a surjective map  $h_V \rightarrow \mathcal{F}$ .

Observe that the product of  $h_V$  with itself in  $Sh(Y_{\text{étale}})$  is  $h_{V \times_Y V}$ . Consider the fibre product

$$h_V \times_{\mathcal{F}} h_V \subset h_{V \times_Y V}$$

There is an open subscheme  $R$  of  $V \times_Y V$  such that  $h_V \times_{\mathcal{F}} h_V = h_R$ , see Lemma 20.1 (small detail omitted). By the Yoneda lemma we obtain two morphisms  $s, t : R \rightarrow V$  in  $Y_{\text{étale}}$  and we find a coequalizer diagram

$$h_R \rightrightarrows h_V \longrightarrow \mathcal{F}$$

in  $Sh(Y_{\text{étale}})$ . Of course the morphisms  $s, t$  are étale and define an étale equivalence relation  $(t, s) : R \rightarrow V \times_S V$ .

By the discussion in the preceding two paragraphs we find a coequalizer diagram

$$R \rightrightarrows V \longrightarrow X$$

in  $(Sch/S)_{fppf}$ . Thus  $X = V/R$  is an algebraic space by Spaces, Theorem 10.5. The other statements follow readily from this; details omitted.  $\square$

## 28. Recovering morphisms

In this section we prove that the rule which associates to an algebraic space its locally ringed small étale topoi is fully faithful in a suitable sense, see Theorem 28.4.

**Lemma 28.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . The morphism of ringed topoi  $(f_{small}, f^\#)$  associated to  $f$  is a morphism of locally ringed topoi, see Modules on Sites, Definition 40.9.*

**Proof.** Note that the assertion makes sense since we have seen that  $(X_{\text{étale}}, \mathcal{O}_{X_{\text{étale}}})$  and  $(Y_{\text{étale}}, \mathcal{O}_{Y_{\text{étale}}})$  are locally ringed sites, see Lemma 22.3. Moreover, we know that  $X_{\text{étale}}$  has enough points, see Theorem 19.12. Hence it suffices to prove that  $(f_{small}, f^\#)$  satisfies condition (3) of Modules on Sites, Lemma 40.8. To see this take a point  $p$  of  $X_{\text{étale}}$ . By Lemma 19.13  $p$  corresponds to a geometric point  $\bar{x}$  of  $X$ . By Lemma 19.9 the point  $q = f_{small} \circ p$  corresponds to the geometric point

<sup>7</sup>It suffices if the supremum of the cardinalities of the stalks of  $\mathcal{F}$  at geometric points of  $Y$  is bounded by the size of some object of  $(Sch/S)_{fppf}$ .



$\bar{y} = f \circ \bar{x}$  of  $Y$ . Hence the assertion we have to prove is that the induced map of étale local rings

$$\mathcal{O}_{Y, \bar{y}} \longrightarrow \mathcal{O}_{X, \bar{x}}$$

is a local ring map. You can prove this directly, but instead we deduce it from the corresponding result for schemes. To do this choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where  $U$  and  $V$  are schemes, and the vertical arrows are surjective étale (see Spaces, Lemma 11.6). Choose a lift  $\bar{u} : \bar{x} \rightarrow U$  (possible by Lemma 19.5). Set  $\bar{v} = \psi \circ \bar{u}$ . We obtain a commutative diagram of étale local rings

$$\begin{array}{ccc} \mathcal{O}_{U, \bar{u}} & \longleftarrow & \mathcal{O}_{V, \bar{v}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{X, \bar{x}} & \longleftarrow & \mathcal{O}_{Y, \bar{y}}. \end{array}$$

By Étale Cohomology, Lemma 40.1 the top horizontal arrow is a local ring map. Finally by Lemma 22.1 the vertical arrows are isomorphisms. Hence we win.  $\square$

**Lemma 28.2.** *Let  $S$  be a scheme. Let  $X, Y$  be algebraic spaces over  $S$ . Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $t$  be a 2-morphism from  $(f_{small}, f^\#)$  to itself, see Modules on Sites, Definition 8.1. Then  $t = id$ .*

**Proof.** Let  $X'$ , resp.  $Y'$  be  $X$  viewed as an algebraic space over  $\text{Spec}(\mathbf{Z})$ , see Spaces, Definition 16.2. It is clear from the construction that  $(X_{small}, \mathcal{O})$  is equal to  $(X'_{small}, \mathcal{O})$  and similarly for  $Y$ . Hence we may work with  $X'$  and  $Y'$ . In other words we may assume that  $S = \text{Spec}(\mathbf{Z})$ .

Assume  $S = \text{Spec}(\mathbf{Z})$ ,  $f : X \rightarrow Y$  and  $t$  are as in the lemma. This means that  $t : f_{small}^{-1} \rightarrow f_{small}^{-1}$  is a transformation of functors such that the diagram

$$\begin{array}{ccc} f_{small}^{-1} \mathcal{O}_Y & \xleftarrow{t} & f_{small}^{-1} \mathcal{O}_Y \\ & \searrow f^\# & \swarrow f^\# \\ & \mathcal{O}_X & \end{array}$$

is commutative. Suppose  $V \rightarrow Y$  is étale with  $V$  affine. Write  $V = \text{Spec}(B)$ . Choose generators  $b_j \in B$ ,  $j \in J$  for  $B$  as a  $\mathbf{Z}$ -algebra. Set  $T = \text{Spec}(\mathbf{Z}[\{x_j\}_{j \in J}])$ . In the following we will use that  $\text{Mor}_{Sch}(U, T) = \prod_{j \in J} \Gamma(U, \mathcal{O}_U)$  for any scheme  $U$  without further mention. The surjective ring map  $\mathbf{Z}[x_j] \rightarrow B$ ,  $x_j \mapsto b_j$  corresponds to a closed immersion  $V \rightarrow T$ . We obtain a monomorphism

$$i : V \longrightarrow T_Y = T \times Y$$

of algebraic spaces over  $Y$ . In terms of sheaves on  $Y_{\text{étale}}$  the morphism  $i$  induces an injection  $h_i : h_V \rightarrow \prod_{j \in J} \mathcal{O}_Y$  of sheaves. The base change  $i' : X \times_Y V \rightarrow T_X$  of  $i$  to  $X$  is a monomorphism too (Spaces, Lemma 5.5). Hence  $i' : X \times_Y V \rightarrow T_X$  is a monomorphism, which in turn means that  $h_{i'} : h_{X \times_Y V} \rightarrow \prod_{j \in J} \mathcal{O}_X$  is an injection

of sheaves. Via the identification  $f_{small}^{-1}h_V = h_{X \times_Y V}$  of Lemma 19.9 the map  $h_i$  is equal to

$$f_{small}^{-1}h_V \xrightarrow{f^{-1}h_i} \prod_{j \in J} f_{small}^{-1}\mathcal{O}_Y \xrightarrow{\prod f^\#} \prod_{j \in J} \mathcal{O}_X$$

(verification omitted). This means that the map  $t : f_{small}^{-1}h_V \rightarrow f_{small}^{-1}h_V$  fits into the commutative diagram

$$\begin{array}{ccccc} f_{small}^{-1}h_V & \xrightarrow{f^{-1}h_i} & \prod_{j \in J} f_{small}^{-1}\mathcal{O}_Y & \xrightarrow{\prod f^\#} & \prod_{j \in J} \mathcal{O}_X \\ \downarrow t & & \downarrow \prod t & & \downarrow \text{id} \\ f_{small}^{-1}h_V & \xrightarrow{f^{-1}h_i} & \prod_{j \in J} f_{small}^{-1}\mathcal{O}_Y & \xrightarrow{\prod f^\#} & \prod_{j \in J} \mathcal{O}_X \end{array}$$

The commutativity of the right square holds by our assumption on  $t$  explained above. Since the composition of the horizontal arrows is injective by the discussion above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on  $Y_{\acute{e}tale}$  admits a surjection from a (huge) coproduct of sheaves of the form  $h_V$  with  $V$  affine (combine Lemma 18.6 with Sites, Lemma 12.5). Thus we conclude that  $t : f_{small}^{-1} \rightarrow f_{small}^{-1}$  is the identity transformation as desired.  $\square$

**Lemma 28.3.** *Let  $S$  be a scheme. Let  $X, Y$  be algebraic spaces over  $S$ . Any two morphisms  $a, b : X \rightarrow Y$  of algebraic spaces over  $S$  for which there exists a 2-isomorphism  $(a_{small}, a^\#) \cong (b_{small}, b^\#)$  in the 2-category of ringed topoi are equal.*

**Proof.** Let  $t : a_{small}^{-1} \rightarrow b_{small}^{-1}$  be the 2-isomorphism. We may equivalently think of  $t$  as a transformation  $t : a_{spaces, \acute{e}tale}^{-1} \rightarrow b_{spaces, \acute{e}tale}^{-1}$  since there is not difference between sheaves on  $X_{\acute{e}tale}$  and sheaves on  $X_{spaces, \acute{e}tale}$ . Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{a} & Y \end{array}$$

where  $U$  and  $V$  are schemes, and  $p$  and  $q$  are surjective étale. Consider the diagram

$$\begin{array}{ccc} h_U & \xrightarrow{\alpha} & a_{spaces, \acute{e}tale}^{-1} h_V \\ \parallel & & \downarrow t \\ h_U & \cdots \cdots \cdots & b_{spaces, \acute{e}tale}^{-1} h_V \end{array}$$

Since the sheaf  $b_{spaces, \acute{e}tale}^{-1} h_V$  is isomorphic to  $h_{V \times_{Y, b} X}$  we see that the dotted arrow comes from a morphism of schemes  $\beta : U \rightarrow V$  fitting into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\beta} & V \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{b} & Y \end{array}$$

We claim that there exists a sequence of 2-isomorphisms

$$\begin{aligned}
 (\alpha_{small}, \alpha^\sharp) &\cong (\alpha_{spaces, \acute{e}tale}, \alpha^\sharp) \\
 &\cong (a_{spaces, \acute{e}tale, c}, a_c^\sharp) \\
 &\cong (b_{spaces, \acute{e}tale, d}, b_d^\sharp) \\
 &\cong (\beta_{spaces, \acute{e}tale}, \beta^\sharp) \\
 &\cong (\beta_{small}, \beta^\sharp)
 \end{aligned}$$

The first and the last 2-isomorphisms come from the identifications between sheaves on  $U_{spaces, \acute{e}tale}$  and sheaves on  $U_{\acute{e}tale}$  and similarly for  $V$ . The second and fourth 2-isomorphisms are those of Lemma 27.1 with  $c : U \rightarrow X \times_{a, Y} V$  induced by  $\alpha$  and  $d : U \rightarrow X \times_{b, Y} V$  induced by  $\beta$ . The middle 2-isomorphism comes from the transformation  $t$ . Namely, the functor  $a_{spaces, \acute{e}tale, c}^{-1}$  corresponds to the functor

$$(\mathcal{H} \rightarrow h_V) \mapsto (a_{spaces, \acute{e}tale}^{-1} \mathcal{H} \times_{a_{spaces, \acute{e}tale}^{-1} h_V, \alpha} h_U \rightarrow h_U)$$

and similarly for  $b_{spaces, \acute{e}tale, d}^{-1}$ , see Sites, Lemma 28.3. This uses the identification of sheaves on  $Y_{spaces, \acute{e}tale}/V$  as arrows  $(\mathcal{H} \rightarrow h_V)$  in  $Sh(Y_{spaces, \acute{e}tale})$  and similarly for  $U/X$ , see Sites, Lemma 25.4. Via this identification the structure sheaf  $\mathcal{O}_V$  corresponds to the pair  $(\mathcal{O}_Y \times h_V \rightarrow h_V)$  and similarly for  $\mathcal{O}_U$ , see Modules on Sites, Lemma 21.3. Since  $t$  switches  $\alpha$  and  $\beta$  we see that  $t$  induces an isomorphism

$$t : a_{spaces, \acute{e}tale}^{-1} \mathcal{H} \times_{a_{spaces, \acute{e}tale}^{-1} h_V, \alpha} h_U \longrightarrow b_{spaces, \acute{e}tale}^{-1} \mathcal{H} \times_{b_{spaces, \acute{e}tale}^{-1} h_V, \beta} h_U$$

over  $h_U$  functorially in  $(\mathcal{H} \rightarrow h_V)$ . Also,  $t$  is compatible with  $a_c^\sharp$  and  $b_d^\sharp$  as  $t$  is compatible with  $a^\sharp$  and  $b^\sharp$  by our description of the structure sheaves  $\mathcal{O}_U$  and  $\mathcal{O}_V$  above. Hence, the morphisms of ringed topoi  $(\alpha_{small}, \alpha^\sharp)$  and  $(\beta_{small}, \beta^\sharp)$  are 2-isomorphic. By Étale Cohomology, Lemma 40.3 we conclude  $\alpha = \beta$ ! Since  $p : U \rightarrow X$  is a surjection of sheaves it follows that  $a = b$ .  $\square$

Here is the main result of this section.

**Theorem 28.4.** *Let  $X, Y$  be algebraic spaces over  $\text{Spec}(\mathbf{Z})$ . Let*

$$(g, g^\sharp) : (Sh(X_{\acute{e}tale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{\acute{e}tale}), \mathcal{O}_Y)$$

*be a morphism of locally ringed topoi. Then there exists a unique morphism of algebraic spaces  $f : X \rightarrow Y$  such that  $(g, g^\sharp)$  is isomorphic to  $(f_{small}, f^\sharp)$ . In other words, the construction*

$$Spaces/\text{Spec}(\mathbf{Z}) \longrightarrow \text{Locally ringed topoi}, \quad X \longrightarrow (X_{\acute{e}tale}, \mathcal{O}_X)$$

*is fully faithful (morphisms up to 2-isomorphisms on the right hand side).*

**Proof.** The uniqueness we have seen in Lemma 28.3. Thus it suffices to prove existence. In this proof we will freely use the identifications of Equation (27.0.4) as well as the result of Lemma 27.2.

Let  $U \in \text{Ob}(X_{\acute{e}tale})$ , let  $V \in \text{Ob}(Y_{\acute{e}tale})$  and let  $s \in g^{-1}h_V(U)$  be a section. We may think of  $s$  as a map of sheaves  $s : h_U \rightarrow g^{-1}h_V$ . By Modules on Sites, Lemma

22.3 we obtain a commutative diagram of morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(X_{\acute{e}tale}/U), \mathcal{O}_U) & \xrightarrow{(j, j^\#)} & (Sh(X_{\acute{e}tale}), \mathcal{O}_X) \\ (g_s, g_s^\#) \downarrow & & \downarrow (g, g^\#) \\ (Sh(V_{\acute{e}tale}), \mathcal{O}_V) & \xrightarrow{\quad} & (Sh(Y_{\acute{e}tale}), \mathcal{O}_Y). \end{array}$$

By Étale Cohomology, Theorem 40.5 we obtain a unique morphism of schemes  $f_s : U \rightarrow V$  such that  $(g_s, g_s^\#)$  is 2-isomorphic to  $(f_{s, small}, f_s^\#)$ . The construction  $(U, V, s) \rightsquigarrow f_s$  just explained satisfies the following functoriality property: Suppose given morphisms  $a : U' \rightarrow U$  in  $X_{\acute{e}tale}$  and  $b : V' \rightarrow V$  in  $Y_{\acute{e}tale}$  and a map  $s' : h_{U'} \rightarrow g^{-1}h_{V'}$  such that the diagram

$$\begin{array}{ccc} h_{U'} & \xrightarrow{s'} & g^{-1}h_{V'} \\ a \downarrow & & \downarrow g^{-1}b \\ h_U & \xrightarrow{s} & g^{-1}h_V \end{array}$$

commutes. Then the diagram

$$\begin{array}{ccc} U' & \xrightarrow{f_{s'}} & u(V') \\ a \downarrow & & \downarrow u(b) \\ U & \xrightarrow{f_s} & u(V) \end{array}$$

of schemes commutes. The reason this is true is that the same condition holds for the morphisms  $(g_s, g_s^\#)$  constructed in Modules on Sites, Lemma 22.3 and the uniqueness in Étale Cohomology, Theorem 40.5.

The problem is to glue the morphisms  $f_s$  to a morphism of algebraic spaces. To do this first choose a scheme  $V$  and a surjective étale morphism  $V \rightarrow Y$ . This means that  $h_V \rightarrow *$  is surjective and hence  $g^{-1}h_V \rightarrow *$  is surjective too. This means there exists a scheme  $U$  and a surjective étale morphism  $U \rightarrow X$  and a morphism  $s : h_U \rightarrow g^{-1}h_V$ . Next, set  $R = V \times_Y V$  and  $R' = U \times_X U$ . Then we get  $g^{-1}h_R = g^{-1}h_V \times g^{-1}h_V$  as  $g^{-1}$  is exact. Thus  $s$  induces a morphism  $s \times s : h_{R'} \rightarrow g^{-1}h_R$ . Applying the constructions above we see that we get a commutative diagram of morphisms of schemes

$$\begin{array}{ccc} R' & \xrightarrow{f_{s \times s}} & R \\ \downarrow \downarrow & & \downarrow \downarrow \\ U & \xrightarrow{f_s} & V \end{array}$$

Since we have  $X = U/R'$  and  $Y = V/R$  (see Spaces, Lemma 9.1) we conclude that this diagram defines a morphism of algebraic spaces  $f : X \rightarrow Y$  fitting into an obvious commutative diagram. Now we still have to show that  $(f_{small}, f^\#)$  is 2-isomorphic to  $(g, g^\#)$ . Let  $t_V : f_{s, small}^{-1} \rightarrow g_s^{-1}$  and  $t_R : f_{s \times s, small}^{-1} \rightarrow g_{s \times s}^{-1}$  be the 2-isomorphisms which are given to us by the construction above. Let  $\mathcal{G}$  be a sheaf on  $Y_{\acute{e}tale}$ . Then we see that  $t_V$  defines an isomorphism

$$f_{small}^{-1} \mathcal{G}|_{U_{\acute{e}tale}} = f_{s, small}^{-1} \mathcal{G}|_{V_{\acute{e}tale}} \xrightarrow{t_V} g_s^{-1} \mathcal{G}|_{V_{\acute{e}tale}} = g^{-1} \mathcal{G}|_{U_{\acute{e}tale}}.$$

Moreover, this isomorphism pulled back to  $R'$  via either projection  $R' \rightarrow U$  is the isomorphism

$$f_{small}^{-1}\mathcal{G}|_{R'_{\acute{e}tale}} = f_{s \times s, small}^{-1}\mathcal{G}|_{R_{\acute{e}tale}} \xrightarrow{t_R} g_{s \times s}^{-1}\mathcal{G}|_{R_{\acute{e}tale}} = g^{-1}\mathcal{G}|_{R'_{\acute{e}tale}}.$$

Since  $\{U \rightarrow X\}$  is a covering in the site  $X_{spaces, \acute{e}tale}$  this means the first displayed isomorphism descends to an isomorphism  $t : f_{small}^{-1}\mathcal{G} \rightarrow g^{-1}\mathcal{G}$  of sheaves (small detail omitted). The isomorphism is functorial in  $\mathcal{G}$  since  $t_V$  and  $t_R$  are transformations of functors. Finally,  $t$  is compatible with  $f^\sharp$  and  $g^\sharp$  as  $t_V$  and  $t_R$  are (some details omitted). This finishes the proof of the theorem.  $\square$

**Lemma 28.5.** *Let  $X, Y$  be algebraic spaces over  $\mathbf{Z}$ . If*

$$(g, g^\sharp) : (Sh(X_{\acute{e}tale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{\acute{e}tale}), \mathcal{O}_Y)$$

*is an isomorphism of ringed topoi, then there exists a unique morphism  $f : X \rightarrow Y$  of algebraic spaces such that  $(g, g^\sharp)$  is isomorphic to  $(f_{small}, f^\sharp)$  and moreover  $f$  is an isomorphism of algebraic spaces.*

**Proof.** By Theorem 28.4 it suffices to show that  $(g, g^\sharp)$  is a morphism of locally ringed topoi. By Modules on Sites, Lemma 40.8 (and since the site  $X_{\acute{e}tale}$  has enough points) it suffices to check that the map  $\mathcal{O}_{Y,q} \rightarrow \mathcal{O}_{X,p}$  induced by  $g^\sharp$  is a local ring map where  $q = f \circ p$  and  $p$  is any point of  $X_{\acute{e}tale}$ . As it is an isomorphism this is clear.  $\square$

## 29. Quasi-coherent sheaves on algebraic spaces

In Descent, Sections 8, 9, and 10 we have seen that for a scheme  $U$ , there is no difference between a quasi-coherent  $\mathcal{O}_U$ -module on  $U$ , or a quasi-coherent  $\mathcal{O}$ -module on the small étale site of  $U$ . Hence the following definition is compatible with our original notion of a quasi-coherent sheaf on a scheme (Schemes, Section 24), when applied to a representable algebraic space.

**Definition 29.1.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . A *quasi-coherent*  $\mathcal{O}_X$ -module is a quasi-coherent module on the ringed site  $(X_{\acute{e}tale}, \mathcal{O}_X)$  in the sense of Modules on Sites, Definition 23.1. The category of quasi-coherent sheaves on  $X$  is denoted  $QCoh(\mathcal{O}_X)$ .

Note that as being quasi-coherent is an intrinsic notion (see Modules on Sites, Lemma 23.2) this is equivalent to saying that the corresponding  $\mathcal{O}_X$ -module on  $X_{spaces, \acute{e}tale}$  is quasi-coherent.

As usual, quasi-coherent sheaves behave well with respect to pullback.

**Lemma 29.2.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . The pullback functor  $f^* : Mod(\mathcal{O}_Y) \rightarrow Mod(\mathcal{O}_X)$  preserves quasi-coherent sheaves.*

**Proof.** This is a general fact, see Modules on Sites, Lemma 23.4.  $\square$

Note that this pullback functor agrees with the usual pullback functor between quasi-coherent sheaves of modules if  $X$  and  $Y$  happen to be schemes, see Descent, Proposition 9.4. Here is the obligatory lemma comparing this with quasi-coherent sheaves on the objects of the small étale site of  $X$ .

**Lemma 29.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . A quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  is given by the following data:*

- (1) for every  $U \in \text{Ob}(X_{\text{étale}})$  a quasi-coherent  $\mathcal{O}_U$ -module  $\mathcal{F}_U$  on  $U_{\text{étale}}$ ,
- (2) for every  $f : U' \rightarrow U$  in  $X_{\text{étale}}$  an isomorphism  $c_f : f_{\text{small}}^* \mathcal{F}_U \rightarrow \mathcal{F}_{U'}$ .

These data are subject to the condition that given any  $f : U' \rightarrow U$  and  $g : U'' \rightarrow U'$  in  $X_{\text{étale}}$  the composition  $c_g \circ g_{\text{small}}^* c_f$  is equal to  $c_{f \circ g}$ .

**Proof.** Combine Lemmas 29.2 and 26.3.  $\square$

**Lemma 29.4.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module. Let  $x \in |X|$  be a point and let  $\bar{x}$  be a geometric point lying over  $x$ . Finally, let  $\varphi : (U, \bar{u}) \rightarrow (X, \bar{x})$  be an étale neighbourhood where  $U$  is a scheme. Then*

$$(\varphi^* \mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\bar{x}} = \mathcal{F}_{\bar{x}}$$

where  $u \in U$  is the image of  $\bar{u}$ .

**Proof.** Note that  $\mathcal{O}_{X,\bar{x}} = \mathcal{O}_{U,u}^{sh}$  by Lemma 22.1 hence the tensor product makes sense. Moreover, from Definition 19.6 it is clear that

$$\mathcal{F}_{\bar{x}} = \text{colim}(\varphi^* \mathcal{F})_u$$

where the colimit is over  $\varphi : (U, \bar{u}) \rightarrow (X, \bar{x})$  as in the lemma. Hence there is a canonical map from left to right in the statement of the lemma. We have a similar colimit description for  $\mathcal{O}_{X,\bar{x}}$  and by Lemma 29.3 we have

$$((\varphi')^* \mathcal{F})_{u'} = (\varphi^* \mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{U',u'}$$

whenever  $(U', \bar{u}') \rightarrow (U, \bar{u})$  is a morphism of étale neighbourhoods. To complete the proof we use that  $\otimes$  commutes with colimits.  $\square$

**Lemma 29.5.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{G}$  be a quasi-coherent  $\mathcal{O}_Y$ -module. Let  $\bar{x}$  be a geometric point of  $X$  and let  $\bar{y} = f \circ \bar{x}$  be the image in  $Y$ . Then there is a canonical isomorphism*

$$(f^* \mathcal{G})_{\bar{x}} = \mathcal{G}_{\bar{y}} \otimes_{\mathcal{O}_{Y,\bar{y}}} \mathcal{O}_{X,\bar{x}}$$

of the stalk of the pullback with the tensor product of the stalk with the local ring of  $X$  at  $\bar{x}$ .

**Proof.** Since  $f^* \mathcal{G} = f_{\text{small}}^{-1} \mathcal{G} \otimes_{f_{\text{small}}^{-1} \mathcal{O}_Y} \mathcal{O}_X$  this follows from the description of stalks of pullbacks in Lemma 19.9 and the fact that taking stalks commutes with tensor products. A more direct way to see this is as follows. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{a} & Y \end{array}$$

where  $U$  and  $V$  are schemes, and  $p$  and  $q$  are surjective étale. By Lemma 19.4 we can choose a geometric point  $\bar{u}$  of  $U$  such that  $\bar{x} = p \circ \bar{u}$ . Set  $\bar{v} = \alpha \circ \bar{u}$ . Then we

see that

$$\begin{aligned}
 (f^*\mathcal{G})_{\bar{x}} &= (p^*f^*\mathcal{G})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\bar{x}} \\
 &= (\alpha^*q^*\mathcal{G})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\bar{x}} \\
 &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u} \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\bar{x}} \\
 &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{X,\bar{x}} \\
 &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{Y,\bar{y}} \otimes_{\mathcal{O}_{Y,\bar{y}}} \mathcal{O}_{X,\bar{x}} \\
 &= \mathcal{G}_{\bar{y}} \otimes_{\mathcal{O}_{Y,\bar{y}}} \mathcal{O}_{X,\bar{x}}
 \end{aligned}$$

Here we have used Lemma 29.4 (twice) and the corresponding result for pullbacks of quasi-coherent sheaves on schemes, see Sheaves, Lemma 26.4.  $\square$

**Lemma 29.6.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_X$ -modules. The following are equivalent*

- (1)  $\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_X$ -module,
- (2) there exists an étale morphism  $f : Y \rightarrow X$  of algebraic spaces over  $S$  with  $|f| : |Y| \rightarrow |X|$  surjective such that  $f^*\mathcal{F}$  is quasi-coherent on  $Y$ ,
- (3) there exists a scheme  $U$  and a surjective étale morphism  $\varphi : U \rightarrow X$  such that  $\varphi^*\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_U$ -module, and
- (4) for every affine scheme  $U$  and étale morphism  $\varphi : U \rightarrow X$  the restriction  $\varphi^*\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_U$ -module.

**Proof.** It is clear that (1) implies (2) by considering  $\text{id}_X$ . Assume  $f : Y \rightarrow X$  is as in (2), and let  $V \rightarrow Y$  be a surjective étale morphism from a scheme towards  $Y$ . Then the composition  $V \rightarrow X$  is surjective étale as well and by Lemma 29.2 the pullback of  $\mathcal{F}$  to  $V$  is quasi-coherent as well. Hence we see that (2) implies (3).

Let  $U \rightarrow X$  be as in (3). Let us use the abuse of notation introduced in Equation (26.1.1). As  $\mathcal{F}|_{U_{\text{étale}}}$  is quasi-coherent there exists an étale covering  $\{U_i \rightarrow U\}$  such that  $\mathcal{F}|_{U_{i,\text{étale}}}$  has a global presentation, see Modules on Sites, Definition 17.1 and Lemma 23.3. Let  $V \rightarrow X$  be an object of  $X_{\text{étale}}$ . Since  $U \rightarrow X$  is surjective and étale, the family of maps  $\{U_i \times_X V \rightarrow V\}$  is an étale covering of  $V$ . Via the morphisms  $U_i \times_X V \rightarrow U_i$  we can restrict the global presentations of  $\mathcal{F}|_{U_{i,\text{étale}}}$  to get a global presentation of  $\mathcal{F}|_{(U_i \times_X V)_{\text{étale}}}$ . Hence the sheaf  $\mathcal{F}$  on  $X_{\text{étale}}$  satisfies the condition of Modules on Sites, Definition 23.1 and hence is quasi-coherent.

The equivalence of (3) and (4) comes from the fact that any scheme has an affine open covering.  $\square$

**Lemma 29.7.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The category  $QCoh(\mathcal{O}_X)$  of quasi-coherent sheaves on  $X$  has the following properties:*

- (1) Any direct sum of quasi-coherent sheaves is quasi-coherent.
- (2) Any colimit of quasi-coherent sheaves is quasi-coherent.
- (3) The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.
- (4) Given a short exact sequence of  $\mathcal{O}_X$ -modules  $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$  if two out of three are quasi-coherent so is the third.
- (5) Given two quasi-coherent  $\mathcal{O}_X$ -modules the tensor product is quasi-coherent.
- (6) Given two quasi-coherent  $\mathcal{O}_X$ -modules  $\mathcal{F}, \mathcal{G}$  such that  $\mathcal{F}$  is of finite presentation (see Section 30), then the internal hom  $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$  is quasi-coherent.

**Proof.** If  $X$  is a scheme, then this is Descent, Lemma 10.3. We will reduce the lemma to this case by étale localization.

Choose a scheme  $U$  and a surjective étale morphism  $\varphi : U \rightarrow X$ . Our notation will be that  $\text{Mod}(\mathcal{O}_U) = \text{Mod}(U_{\text{étale}}, \mathcal{O}_U)$  and  $QCoh(\mathcal{O}_U) = QCoh(U_{\text{étale}}, \mathcal{O}_U)$ ; in other words, even though  $U$  is a scheme we think of quasi-coherent modules on  $U$  as modules on the small étale site of  $U$ . By Lemma 29.2 we have a commutative diagram

$$\begin{array}{ccc} QCoh(\mathcal{O}_X) & \xrightarrow{\varphi^*} & QCoh(\mathcal{O}_U) \\ \downarrow & & \downarrow \\ \text{Mod}(\mathcal{O}_X) & \xrightarrow{\varphi^*} & \text{Mod}(\mathcal{O}_U) \end{array}$$

The bottom horizontal arrow is the restriction functor (26.1.1)  $\mathcal{G} \mapsto \mathcal{G}|_{U_{\text{étale}}}$ . This functor has both a left adjoint and a right adjoint, see Modules on Sites, Section 19, hence commutes with all limits and colimits. Moreover, we know that an object of  $\text{Mod}(\mathcal{O}_X)$  is in  $QCoh(\mathcal{O}_X)$  if and only if its restriction to  $U$  is in  $QCoh(\mathcal{O}_U)$ , see Lemma 29.6. With these preliminaries out of the way we can start the proof.

Proof of (1). Let  $\mathcal{F}_i$ ,  $i \in I$  be a family of quasi-coherent  $\mathcal{O}_X$ -modules. By the discussion above we have

$$\left( \bigoplus \mathcal{F}_i \right) |_{U_{\text{étale}}} = \bigoplus \mathcal{F}_i |_{U_{\text{étale}}}$$

Each of the modules  $\mathcal{F}_i |_{U_{\text{étale}}}$  is quasi-coherent. Hence the direct sum is quasi-coherent by the case of schemes. Hence  $\bigoplus \mathcal{F}_i$  is quasi-coherent as a module restricting to a quasi-coherent module on  $U$ .

Proof of (2). Let  $\mathcal{I} \rightarrow QCoh(\mathcal{O}_X)$ ,  $i \mapsto \mathcal{F}_i$  be a diagram. Then

$$(\text{colim } \mathcal{F}_i) |_{U_{\text{étale}}} = \text{colim } \mathcal{F}_i |_{U_{\text{étale}}}$$

by the discussion above and we conclude in the same manner.

Proof of (3). Let  $a : \mathcal{F} \rightarrow \mathcal{F}'$  be an arrow of  $QCoh(\mathcal{O}_X)$ . Then we have  $\text{Ker}(a) |_{U_{\text{étale}}} = \text{Ker}(a|_{U_{\text{étale}}})$  and  $\text{Coker}(a) |_{U_{\text{étale}}} = \text{Coker}(a|_{U_{\text{étale}}})$  and we conclude in the same manner.

Proof of (4). The restriction  $0 \rightarrow \mathcal{F}_1|_{U_{\text{étale}}} \rightarrow \mathcal{F}_2|_{U_{\text{étale}}} \rightarrow \mathcal{F}_3|_{U_{\text{étale}}} \rightarrow 0$  is short exact. Hence we have the 2-out-of-3 property for this sequence and we conclude as before.

Proof of (5). Let  $\mathcal{F}$  and  $\mathcal{G}$  be in  $QCoh(\mathcal{O}_X)$ . Then we have

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_{U_{\text{étale}}} = \mathcal{F}|_{U_{\text{étale}}} \otimes_{\mathcal{O}_U} \mathcal{G}|_{U_{\text{étale}}}$$

and we conclude as before.

Proof of (6). Let  $\mathcal{F}$  and  $\mathcal{G}$  be in  $QCoh(\mathcal{O}_X)$  with  $\mathcal{F}$  of finite presentation. We have

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})|_{U_{\text{étale}}} = \text{Hom}_{\mathcal{O}_U}(\mathcal{F}|_{U_{\text{étale}}}, \mathcal{G}|_{U_{\text{étale}}})$$

Namely, restriction is a localization, see Section 27, especially formula (27.0.4)) and formation of internal hom commutes with localization, see Modules on Sites, Lemma 27.2. Thus we conclude as before.  $\square$



It is in general not the case that the pushforward of a quasi-coherent sheaf along a morphism of algebraic spaces is quasi-coherent. We will return to this issue in Morphisms of Spaces, Section 11.

### 30. Properties of modules

In Modules on Sites, Sections 17, 23, and Definition 28.1 we have defined a number of intrinsic properties of modules of  $\mathcal{O}$ -module on any ringed topos. If  $X$  is an algebraic space, we will apply these notions freely to modules on the ringed site  $(X_{\acute{e}tale}, \mathcal{O}_X)$ , or equivalently on the ringed site  $(X_{spaces, \acute{e}tale}, \mathcal{O}_X)$ .

Global properties  $\mathcal{P}$ :

- (a) *free*,
- (b) *finite free*,
- (c) *generated by global sections*,
- (d) *generated by finitely many global sections*,
- (e) *having a global presentation*, and
- (f) *having a global finite presentation*.

Local properties  $\mathcal{P}$ :

- (g) *locally free*,
- (f) *finite locally free*,
- (h) *locally generated by sections*,
- (i) *locally generated by  $r$  sections*,
- (j) *finite type*,
- (k) *quasi-coherent* (see Section 29),
- (l) *of finite presentation*,
- (m) *coherent*, and
- (n) *flat*.

Here are some results which follow immediately from the definitions:

- (1) In each case, except for  $\mathcal{P}$  = “coherent”, the property is preserved under pullback, see Modules on Sites, Lemmas 17.2, 23.4, and 39.1.
- (2) Each of the properties above (including coherent) are preserved under pullbacks by étale morphisms of algebraic spaces (because in this case pullback is given by restriction, see Lemma 18.11).
- (3) Assume  $f : Y \rightarrow X$  is a surjective étale morphism of algebraic spaces. For each of the local properties (g) – (m), the fact that  $f^*\mathcal{F}$  has  $\mathcal{P}$  implies that  $\mathcal{F}$  has  $\mathcal{P}$ . This follows as  $\{Y \rightarrow X\}$  is a covering in  $X_{spaces, \acute{e}tale}$  and Modules on Sites, Lemma 23.3.
- (4) If  $X$  is a scheme,  $\mathcal{F}$  is a quasi-coherent module on  $X_{\acute{e}tale}$ , and  $\mathcal{P}$  any property except “coherent” or “locally free”, then  $\mathcal{P}$  for  $\mathcal{F}$  on  $X_{\acute{e}tale}$  is equivalent to the corresponding property for  $\mathcal{F}|_{X_{Zar}}$ , i.e., it corresponds to  $\mathcal{P}$  for  $\mathcal{F}$  when we think of it as a quasi-coherent sheaf on the scheme  $X$ . See Descent, Lemma 8.10.
- (5) If  $X$  is a locally Noetherian scheme,  $\mathcal{F}$  is a quasi-coherent module on  $X_{\acute{e}tale}$ , then  $\mathcal{F}$  is coherent on  $X_{\acute{e}tale}$  if and only if  $\mathcal{F}|_{X_{Zar}}$  is coherent, i.e., it corresponds to the usual notion of a coherent sheaf on the scheme  $X$  being coherent. See Descent, Lemma 8.10.

### 31. Locally projective modules

Recall that in Properties, Section 21 we defined the notion of a locally projective quasi-coherent module.

**Lemma 31.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module. The following are equivalent*

- (1) *for some scheme  $U$  and surjective étale morphism  $U \rightarrow X$  the restriction  $\mathcal{F}|_U$  is locally projective on  $U$ , and*
- (2) *for any scheme  $U$  and any étale morphism  $U \rightarrow X$  the restriction  $\mathcal{F}|_U$  is locally projective on  $U$ .*

**Proof.** Let  $U \rightarrow X$  be as in (1) and let  $V \rightarrow X$  be étale where  $V$  is a scheme. Then  $\{U \times_X V \rightarrow V\}$  is an fppf covering of schemes. Hence if  $\mathcal{F}|_U$  is locally projective, then  $\mathcal{F}|_{U \times_X V}$  is locally projective (see Properties, Lemma 21.3) and hence  $\mathcal{F}|_V$  is locally projective, see Descent, Lemma 7.7.  $\square$

**Definition 31.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module. We say  $\mathcal{F}$  is *locally projective* if the equivalent conditions of Lemma 31.1 are satisfied.

**Lemma 31.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $\mathcal{G}$  be a quasi-coherent  $\mathcal{O}_Y$ -module. If  $\mathcal{G}$  is locally projective on  $Y$ , then  $f^*\mathcal{G}$  is locally projective on  $X$ .*

**Proof.** Choose a surjective étale morphism  $V \rightarrow Y$  with  $V$  a scheme. Choose a surjective étale morphism  $U \rightarrow V \times_Y X$  with  $U$  a scheme. Denote  $\psi : U \rightarrow V$  the induced morphism. Then

$$f^*\mathcal{G}|_U = \psi^*(\mathcal{G}|_V)$$

Hence the lemma follows from the definition and the result in the case of schemes, see Properties, Lemma 21.3.  $\square$

### 32. Quasi-coherent sheaves and presentations

Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $X = U/R$  be a presentation of  $X$  coming from any surjective étale morphism  $\varphi : U \rightarrow X$ , see Spaces, Definition 9.3. In particular, we obtain a groupoid  $(U, R, s, t, c)$ , such that  $j = (t, s) : R \rightarrow U \times_S U$ , see Groupoids, Lemma 13.3. In Groupoids, Definition 14.1 we have defined the notion of a quasi-coherent sheaf on an arbitrary groupoid. With these notions in place we have the following observation.

**Proposition 32.1.** *With  $S$ ,  $\varphi : U \rightarrow X$ , and  $(U, R, s, t, c)$  as above. For any quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  the sheaf  $\varphi^*\mathcal{F}$  comes equipped with a canonical isomorphism*

$$\alpha : t^*\varphi^*\mathcal{F} \longrightarrow s^*\varphi^*\mathcal{F}$$

*which satisfies the conditions of Groupoids, Definition 14.1 and therefore defines a quasi-coherent sheaf on  $(U, R, s, t, c)$ . The functor  $\mathcal{F} \mapsto (\varphi^*\mathcal{F}, \alpha)$  defines an equivalence of categories*

$$\begin{array}{ccc} \text{Quasi-coherent} & & \text{Quasi-coherent modules} \\ \mathcal{O}_X\text{-modules} & \longleftrightarrow & \text{on } (U, R, s, t, c) \end{array}$$

**Proof.** In the statement of the proposition, and in this proof we think of a quasi-coherent sheaf on a scheme as a quasi-coherent sheaf on the small étale site of that scheme. This is permissible by the results of Descent, Sections 8, 9, and 10.

The existence of  $\alpha$  comes from the fact that  $\varphi \circ t = \varphi \circ s$  and that pullback is functorial in the morphism, see discussion surrounding Equation (26.0.1). In exactly the same way, i.e., by functoriality of pullback, we see that the isomorphism  $\alpha$  satisfies condition (1) of Groupoids, Definition 14.1. To see condition (2) of the definition it suffices to see that  $\alpha$  is an isomorphism which is clear. The construction  $\mathcal{F} \mapsto (\varphi^* \mathcal{F}, \alpha)$  is clearly functorial in the quasi-coherent sheaf  $\mathcal{F}$ . Hence we obtain the functor from left to right in the displayed formula of the lemma.

Conversely, suppose that  $(\mathcal{F}, \alpha)$  is a quasi-coherent sheaf on  $(U, R, s, t, c)$ . Let  $V \rightarrow X$  be an object of  $X_{\text{étale}}$ . In this case the morphism  $V' = U \times_X V \rightarrow V$  is a surjective étale morphism of schemes, and hence  $\{V' \rightarrow V\}$  is an étale covering of  $V$ . Moreover, the quasi-coherent sheaf  $\mathcal{F}$  pulls back to a quasi-coherent sheaf  $\mathcal{F}'$  on  $V'$ . Since  $R = U \times_X U$  with  $t = \text{pr}_0$  and  $s = \text{pr}_0$  we see that  $V' \times_V V' = R \times_X V$  with projection maps  $V' \times_V V' \rightarrow V'$  equal to the pullbacks of  $t$  and  $s$ . Hence  $\alpha$  pulls back to an isomorphism  $\alpha' : \text{pr}_0^* \mathcal{F}' \rightarrow \text{pr}_1^* \mathcal{F}'$ , and the pair  $(\mathcal{F}', \alpha')$  is a descent datum for quasi-coherent sheaves with respect to  $\{V' \rightarrow V\}$ . By Descent, Proposition 5.2 this descent datum is effective, and we obtain a quasi-coherent  $\mathcal{O}_V$ -module  $\mathcal{F}_V$  on  $V_{\text{étale}}$ . To see that this gives a quasi-coherent sheaf on  $X_{\text{étale}}$  we have to show (by Lemma 29.3) that for any morphism  $f : V_1 \rightarrow V_2$  in  $X_{\text{étale}}$  there is a canonical isomorphism  $c_f : \mathcal{F}_{V_1} \rightarrow \mathcal{F}_{V_2}$  compatible with compositions of morphisms. We omit the verification. We also omit the verification that this defines a functor from the category on the right to the category on the left which is inverse to the functor described above.  $\square$

**Proposition 32.2.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ .*

- (1) *The category  $QCoh(\mathcal{O}_X)$  is a Grothendieck abelian category. Consequently,  $QCoh(\mathcal{O}_X)$  has enough injectives and all limits.*
- (2) *The inclusion functor  $QCoh(\mathcal{O}_X) \rightarrow Mod(\mathcal{O}_X)$  has a right adjoint<sup>8</sup>*

$$Q : Mod(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

*such that for every quasi-coherent sheaf  $\mathcal{F}$  the adjunction mapping  $Q(\mathcal{F}) \rightarrow \mathcal{F}$  is an isomorphism.*

**Proof.** This proof is a repeat of the proof in the case of schemes, see Properties, Proposition 23.4. We advise the reader to read that proof first.

Part (1) means  $QCoh(\mathcal{O}_X)$  (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 10. By Lemma 29.7 colimits in  $QCoh(\mathcal{O}_X)$  exist and agree with colimits in  $Mod(\mathcal{O}_X)$ . By Modules on Sites, Lemma 14.2 filtered colimits are exact. Hence (a) and (b) hold.

To construct a generator, choose a presentation  $X = U/R$  so that  $(U, R, s, t, c)$  is an étale groupoid scheme and in particular  $s$  and  $t$  are flat morphisms of schemes. Pick a cardinal  $\kappa$  as in Groupoids, Lemma 15.7. Pick a collection  $(\mathcal{E}_t, \alpha_t)_{t \in T}$  of  $\kappa$ -generated quasi-coherent modules on  $(U, R, s, t, c)$  as in Groupoids, Lemma 15.6. Let  $\mathcal{F}_t$  be the quasi-coherent module on  $X$  which corresponds to the quasi-coherent module  $(\mathcal{E}_t, \alpha_t)$  via the equivalence of categories of Proposition 32.1. Then we see

<sup>8</sup>This functor is sometimes called the *coherator*.

that every quasi-coherent module  $\mathcal{H}$  is the directed colimit of its quasi-coherent submodules which are isomorphic to one of the  $\mathcal{F}_t$ . Thus  $\bigoplus_t \mathcal{F}_t$  is a generator of  $QCoh(\mathcal{O}_X)$  and we conclude that (c) holds. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 11.7 and Lemma 13.2.

Proof of (2). To construct  $Q$  we use the following general procedure. Given an object  $\mathcal{F}$  of  $Mod(\mathcal{O}_X)$  we consider the functor

$$QCoh(\mathcal{O}_X)^{opp} \longrightarrow Sets, \quad \mathcal{G} \longmapsto \text{Hom}_X(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 13.1. Thus there exists a quasi-coherent sheaf  $Q(\mathcal{F})$  and a functorial isomorphism  $\text{Hom}_X(\mathcal{G}, \mathcal{F}) = \text{Hom}_X(\mathcal{G}, Q(\mathcal{F}))$  for  $\mathcal{G}$  in  $QCoh(\mathcal{O}_X)$ . By the Yoneda lemma (Categories, Lemma 3.5) the construction  $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$  is functorial in  $\mathcal{F}$ . By construction  $Q$  is a right adjoint to the inclusion functor. The fact that  $Q(\mathcal{F}) \rightarrow \mathcal{F}$  is an isomorphism when  $\mathcal{F}$  is quasi-coherent is a formal consequence of the fact that the inclusion functor  $QCoh(\mathcal{O}_X) \rightarrow Mod(\mathcal{O}_X)$  is fully faithful.  $\square$

### 33. Morphisms towards schemes

Here is the analogue of Schemes, Lemma 6.4.

**Lemma 33.1.** *Let  $X$  be an algebraic space over  $\mathbf{Z}$ . Let  $T$  be an affine scheme. The map*

$$\text{Mor}(X, T) \longrightarrow \text{Hom}(\Gamma(T, \mathcal{O}_T), \Gamma(X, \mathcal{O}_X))$$

*which maps  $f$  to  $f^\#$  (on global sections) is bijective.*

**Proof.** We construct the inverse of the map. Let  $\varphi : \Gamma(T, \mathcal{O}_T) \rightarrow \Gamma(X, \mathcal{O}_X)$  be a ring map. Choose a presentation  $X = U/R$ , see Spaces, Definition 9.3. By Schemes, Lemma 6.4 the composition

$$\Gamma(T, \mathcal{O}_T) \rightarrow \Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_U)$$

corresponds to a unique morphism of schemes  $g : U \rightarrow T$ . By the same lemma the two compositions  $R \rightarrow U \rightarrow T$  are equal. Hence we obtain a morphism  $f : X = U/R \rightarrow T$  such that  $U \rightarrow X \rightarrow T$  equals  $g$ . By construction the diagram

$$\begin{array}{ccc} \Gamma(U, \mathcal{O}_U) & \longleftarrow & \Gamma(X, \mathcal{O}_X) \\ & \nwarrow g^\# & \uparrow \varphi \\ & & \Gamma(T, \mathcal{O}_T) \end{array} \quad \begin{array}{c} \uparrow f^\# \\ \varphi \end{array}$$

commutes. Hence  $f^\#$  equals  $\varphi$  because  $U \rightarrow X$  is an étale covering and  $\mathcal{O}_X$  is a sheaf on  $X_{\text{étale}}$ . The uniqueness of  $f$  follows from the uniqueness of  $g$ .  $\square$

### 34. Quotients by free actions

Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $G$  be an abstract group. Let  $a : G \rightarrow \text{Aut}(X)$  be a homomorphism, i.e.,  $a$  is an *action* of  $G$  on  $X$ . We will say the action is *free* if for every scheme  $T$  over  $S$  the map

$$G \times X(T) \longrightarrow X(T)$$

is free. (We cannot use a criterion as in Spaces, Lemma 14.3 because points may not have well defined residue fields.) In case the action is free we're going to

construct the quotient  $X/G$  as an algebraic space. This is a special case of the general Bootstrap, Lemma 11.7 that we will prove later.

**Lemma 34.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $G$  be an abstract group with a free action on  $X$ . Then the quotient sheaf  $X/G$  is an algebraic space.*

**Proof.** The statement means that the sheaf  $F$  associated to the presheaf

$$T \longmapsto X(T)/G$$

is an algebraic space. To see this we will construct a presentation. Namely, choose a scheme  $U$  and a surjective étale morphism  $\varphi : U \rightarrow X$ . Set  $V = \coprod_{g \in G} U$  and set  $\psi : V \rightarrow X$  equal to  $a(g) \circ \varphi$  on the component corresponding to  $g \in G$ . Let  $G$  act on  $V$  by permuting the components, i.e.,  $g_0 \in G$  maps the component corresponding to  $g$  to the component corresponding to  $g_0 g$  via the identity morphism of  $U$ . Then  $\psi$  is a  $G$ -equivariant morphism, i.e., we reduce to the case dealt with in the next paragraph.

Assume that there exists a  $G$ -action on  $U$  and that  $U \rightarrow X$  is surjective, étale and  $G$ -equivariant. In this case there is an induced action of  $G$  on  $R = U \times_X U$  compatible with the projection mappings  $t, s : R \rightarrow U$ . Now we claim that

$$X/G = U / \coprod_{g \in G} R$$

where the map

$$j : \coprod_{g \in G} R \longrightarrow U \times_S U$$

is given by  $(r, g) \mapsto (t(r), g(s(r)))$ . Note that  $j$  is a monomorphism: If  $(t(r), g(s(r))) = (t(r'), g'(s(r')))$ , then  $t(r) = t(r')$ , hence  $r$  and  $r'$  have the same image in  $X$  under both  $s$  and  $t$ , hence  $g = g'$  (as  $G$  acts freely on  $X$ ), hence  $s(r) = s(r')$ , hence  $r = r'$  (as  $R$  is an equivalence relation on  $U$ ). Moreover  $j$  is an equivalence relation (details omitted). Both projections  $\coprod_{g \in G} R \rightarrow U$  are étale, as  $s$  and  $t$  are étale. Thus  $j$  is an étale equivalence relation and  $U / \coprod_{g \in G} R$  is an algebraic space by Spaces, Theorem 10.5. There is a map

$$U / \coprod_{g \in G} R \longrightarrow X/G$$

induced by the map  $U \rightarrow X$ . We omit the proof that it is an isomorphism of sheaves.  $\square$

### 35. Other chapters

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