DISCRIMINANTS AND DIFFERENTS

Contents

1. Introduction

In this chapter we study the different and discriminant of locally quasi-finite morphisms of schemes. A good reference for some of this material is [Kun86].

Given a quasi-finite morphism $f : Y \to X$ of Noetherian schemes there is a relative dualizing module $\omega_{Y/X}$. In Section 2 we construct this module from scratch, using Zariski's main theorem and étale localization methods. The key property is that given a diagram

$$
\begin{array}{ccc}
Y' & \xrightarrow{\ g'\ } & Y \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
X' & \xrightarrow{\ g\ } & X
\end{array}
$$

with $g : X' \to X$ flat, $Y' \subset X' \times_X Y$ open, and $f' : Y' \to X'$ finite, then there is a canonical isomorphism

$$
f'_*(g')^*\omega_{Y/X} = \mathcal{H}om_{\mathcal{O}_{X'}}(f'_*\mathcal{O}_{Y'}, \mathcal{O}_{X'})
$$

as sheaves of $f'_*\mathcal{O}_{Y'}$-modules. In Section 4 we prove that if f is flat, then there is a canonical global section $\tau_{Y/X} \in H^0(Y, \omega_{Y/X})$ which for every commutative diagram as above maps $(g')^*\tau_{Y/X}$ to the trace map of Section 3 for the finite locally free

morphism f'. In Section 9 we define the different for a flat quasi-finite morphism of Noetherian schemes as the annihilator of the cokernel of $\tau_{Y/X} : \mathcal{O}_X \to \omega_{Y/X}$.

The main goal of this chapter is to prove that for quasi-finite syntomic[1] f the different agrees with the Kähler different. The Kähler different is the zeroth fitting ideal of $\Omega_{Y/X}$, see Section 7. This agreement is not obvious; we use a slick argument due to Tate, see Section 12. On the way we also discuss the Noether different and the Dedekind different.

Only in the end of this chapter, see Sections 15 and 16, do we make the link with the more advanced material on duality for schemes.

2. Dualizing modules for quasi-finite ring maps

Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. By Zariski's main theorem (Algebra, Lemma 123.14) there exists a factorization $A \to B' \to B$ with $A \to B'$ finite and $B' \to B$ inducing an open immersion of spectra. We set

$$(2.0.1) \qquad\qquad \omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$$

in this situation. The reader can think of this as a kind of relative dualizing module, see Lemmas 15.1 and 2.12. In this section we will show by elementary commutative algebra methods that $\omega_{B/A}$ is independent of the choice of the factorization and that formation of $\omega_{B/A}$ commutes with flat base change. To help prove the independence of factorizations we compare two given factorizations.

Lemma 2.1. *Let $A \to B$ be a quasi-finite ring map. Given two factorizations $A \to B' \to B$ and $A \to B'' \to B$ with $A \to B'$ and $A \to B''$ finite and $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ and $\mathrm{Spec}(B) \to \mathrm{Spec}(B'')$ open immersions, there exists an A-subalgebra $B''' \subset B$ finite over A such that $\mathrm{Spec}(B) \to \mathrm{Spec}(B''')$ an open immersion and $B' \to B$ and $B'' \to B$ factor through B'''.*

Proof. Let $B''' \subset B$ be the A-subalgebra generated by the images of $B' \to B$ and $B'' \to B$. As B' and B'' are each generated by finitely many elements integral over A, we see that B''' is generated by finitely many elements integral over A and we conclude that B''' is finite over A (Algebra, Lemma 36.5). Consider the maps

$$B = B' \otimes_{B'} B \to B''' \otimes_{B'} B \to B \otimes_{B'} B = B$$

The final equality holds because $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is an open immersion (and hence a monomorphism). The second arrow is injective as $B' \to B$ is flat. Hence both arrows are isomorphisms. This means that

$$
\begin{array}{ccc}
\mathrm{Spec}(B''') & \longleftarrow & \mathrm{Spec}(B) \\
\downarrow & & \downarrow \\
\mathrm{Spec}(B') & \longleftarrow & \mathrm{Spec}(B)
\end{array}
$$

is cartesian. Since the base change of an open immersion is an open immersion we conclude. $\qquad\square$

Lemma 2.2. *The module (2.0.1) is well defined, i.e., independent of the choice of the factorization.*

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[1] AKA flat and lci.

Proof. Let B', B'', B''' be as in Lemma 2.1. We obtain a canonical map

$$\omega''' = \operatorname{Hom}_A(B''', A) \otimes_{B'''} B \longrightarrow \operatorname{Hom}_A(B', A) \otimes_{B'} B = \omega'$$

and a similar one involving B''. If we show these maps are isomorphisms then the lemma is proved. Let $g \in B'$ be an element such that $B'_g \to B_g$ is an isomorphism and hence $B'_g \to (B''')_g \to B_g$ are isomorphisms. It suffices to show that $(\omega''')_g \to \omega'_g$ is an isomorphism. The kernel and cokernel of the ring map $B' \to B'''$ are finite A-modules and g-power torsion. Hence they are annihilated by a power of g. This easily implies the result. $\square$

Lemma 2.3. *Let $A \to B$ be a quasi-finite map of Noetherian rings.*
  1. *If $A \to B$ factors as $A \to A_f \to B$ for some $f \in A$, then $\omega_{B/A} = \omega_{B/A_f}$.*
  2. *If $g \in B$, then $(\omega_{B/A})_g = \omega_{B_g/A}$.*
  3. *If $f \in A$, then $\omega_{B_f/A_f} = (\omega_{B/A})_f$.*

Proof. Say $A \to B' \to B$ is a factorization with $A \to B'$ finite and $\operatorname{Spec}(B) \to \operatorname{Spec}(B')$ an open immersion. In case (1) we may use the factorization $A_f \to B'_f \to B$ to compute ω_{B/A_f} and use Algebra, Lemma 10.2. In case (2) use the factorization $A \to B' \to B_g$ to see the result. Part (3) follows from a combination of (1) and (2). $\square$

Let $A \to B$ be a quasi-finite ring map of Noetherian rings, let $A \to A_1$ be an arbitrary ring map of Noetherian rings, and set $B_1 = B \otimes_A A_1$. We obtain a cocartesian diagram

$$\begin{array}{ccc} B & \longrightarrow & B_1 \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_1 \end{array}$$

Observe that $A_1 \to B_1$ is quasi-finite as well (Algebra, Lemma 122.8). In this situation we will define a canonical B-linear base change map

$$(2.3.1) \qquad\qquad \omega_{B/A} \longrightarrow \omega_{B_1/A_1}$$

Namely, we choose a factorization $A \to B' \to B$ as in the construction of $\omega_{B/A}$. Then $B'_1 = B' \otimes_A A_1$ is finite over A_1 and we can use the factorization $A_1 \to B'_1 \to B_1$ in the construction of ω_{B_1/A_1}. Thus we have to construct a map

$$\operatorname{Hom}_A(B', A) \otimes_{B'} B \longrightarrow \operatorname{Hom}_{A_1}(B' \otimes_A A_1, A_1) \otimes_{B'_1} B_1$$

Thus it suffices to construct a B'-linear map $\operatorname{Hom}_A(B', A) \to \operatorname{Hom}_{A_1}(B' \otimes_A A_1, A_1)$ which we will denote $\varphi \mapsto \varphi_1$. Namely, given an A-linear map $\varphi : B' \to A$ we let φ_1 be the map such that $\varphi_1(b' \otimes a_1) = \varphi(b')a_1$. This is clearly A_1-linear and the construction is complete.

Lemma 2.4. *The base change map (2.3.1) is independent of the choice of the factorization $A \to B' \to B$. Given ring maps $A \to A_1 \to A_2$ the composition of the base change maps for $A \to A_1$ and $A_1 \to A_2$ is the base change map for $A \to A_2$.*

Proof. Omitted. Hint: argue in exactly the same way as in Lemma 2.2 using Lemma 2.1. $\square$

Lemma 2.5. *If $A \to A_1$ is flat, then the base change map (2.3.1) induces an isomorphism $\omega_{B/A} \otimes_B B_1 \to \omega_{B_1/A_1}$.*

Proof. Assume that $A \to A_1$ is flat. By construction of $\omega_{B/A}$ we may assume that $A \to B$ is finite. Then $\omega_{B/A} = \text{Hom}_A(B, A)$ and $\omega_{B_1/A_1} = \text{Hom}_{A_1}(B_1, A_1)$. Since $B_1 = B \otimes_A A_1$ the result follows from More on Algebra, Lemma 65.4. $\square$

Lemma 2.6. *Let $A \to B \to C$ be quasi-finite homomorphisms of Noetherian rings. There is a canonical map $\omega_{B/A} \otimes_B \omega_{C/B} \to \omega_{C/A}$.*

Proof. Choose $A \to B' \to B$ with $A \to B'$ finite such that $\text{Spec}(B) \to \text{Spec}(B')$ is an open immersion. Then $B' \to C$ is quasi-finite too. Choose $B' \to C' \to C$ with $B' \to C'$ finite and $\text{Spec}(C) \to \text{Spec}(C')$ an open immersion. Then the source of the arrow is

$$\text{Hom}_A(B', A) \otimes_{B'} B \otimes_B \text{Hom}_B(B \otimes_{B'} C', B) \otimes_{B \otimes_{B'} C'} C$$

which is equal to

$$\text{Hom}_A(B', A) \otimes_{B'} \text{Hom}_{B'}(C', B) \otimes_{C'} C$$

This indeed comes with a canonical map to $\text{Hom}_A(C', A) \otimes_{C'} C = \omega_{C/A}$ coming from composition $\text{Hom}_A(B', A) \times \text{Hom}_{B'}(C', B) \to \text{Hom}_A(C', A)$. $\square$

Lemma 2.7. *Let $A \to B$ and $A \to C$ be quasi-finite maps of Noetherian rings. Then $\omega_{B \times C/A} = \omega_{B/A} \times \omega_{C/A}$ as modules over $B \times C$.*

Proof. Choose factorizations $A \to B' \to B$ and $A \to C' \to C$ such that $A \to B'$ and $A \to C'$ are finite and such that $\text{Spec}(B) \to \text{Spec}(B')$ and $\text{Spec}(C) \to \text{Spec}(C')$ are open immersions. Then $A \to B' \times C' \to B \times C$ is a similar factorization. Using this factorization to compute $\omega_{B \times C/A}$ gives the lemma. $\square$

Lemma 2.8. *Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. Then $\text{Ass}_B(\omega_{B/A})$ is the set of primes of B lying over associated primes of A.*

Proof. Choose a factorization $A \to B' \to B$ with $A \to B'$ finite and $B' \to B$ inducing an open immersion on spectra. As $\omega_{B/A} = \omega_{B'/A} \otimes_{B'} B$ it suffices to prove the statement for $\omega_{B'/A}$. Thus we may assume $A \to B$ is finite.

Assume $\mathfrak{p} \in \text{Ass}(A)$ and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p}$. Let $x \in A$ be an element whose annihilator is $\mathfrak{p}$. Choose a nonzero $\kappa(\mathfrak{p})$ linear map $\lambda : \kappa(\mathfrak{q}) \to \kappa(\mathfrak{p})$. Since $A/\mathfrak{p} \subset B/\mathfrak{q}$ is a finite extension of rings, there is an $f \in A$, $f \notin \mathfrak{p}$ such that $f\lambda$ maps $B/\mathfrak{q}$ into $A/\mathfrak{p}$. Hence we obtain a nonzero A-linear map

$$B \to B/\mathfrak{q} \to A/\mathfrak{p} \to A, \quad b \mapsto f\lambda(b)x$$

An easy computation shows that this element of $\omega_{B/A}$ has annihilator $\mathfrak{q}$, whence $\mathfrak{q} \in \text{Ass}(\omega_{B/A})$.

Conversely, suppose that $\mathfrak{q} \subset B$ is a prime ideal lying over a prime $\mathfrak{p} \subset A$ which is not an associated prime of A. We have to show that $\mathfrak{q} \notin \text{Ass}_B(\omega_{B/A})$. After replacing A by $A_{\mathfrak{p}}$ and B by $B_{\mathfrak{p}}$ we may assume that $\mathfrak{p}$ is a maximal ideal of A. This is allowed by Lemma 2.5 and Algebra, Lemma 63.16. Then there exists an $f \in \mathfrak{m}$ which is a nonzerodivisor on A. Then f is a nonzerodivisor on $\omega_{B/A}$ and hence $\mathfrak{q}$ is not an associated prime of this module. $\square$

Lemma 2.9. *Let $A \to B$ be a flat quasi-finite homomorphism of Noetherian rings. Then $\omega_{B/A}$ is a flat A-module.*

Proof. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$. We will show that the localization $\omega_{B/A,\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$. This suffices by Algebra, Lemma 39.18. By Algebra, Lemma 145.2 we can find an étale ring map $A \to A'$ and a prime ideal $\mathfrak{p}' \subset A'$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}') = \kappa(\mathfrak{p})$ and such that

$$B' = B \otimes_A A' = C \times D$$

with $A' \to C$ finite and such that the unique prime $\mathfrak{q}'$ of $B \otimes_A A'$ lying over $\mathfrak{q}$ and $\mathfrak{p}'$ corresponds to a prime of C. By Lemma 2.5 and Algebra, Lemma 100.1 it suffices to show $\omega_{B'/A',\mathfrak{q}'}$ is flat over $A'_{\mathfrak{p}'}$. Since $\omega_{B'/A'} = \omega_{C/A'} \times \omega_{D/A'}$ by Lemma 2.7 this reduces us to the case where B is finite flat over A. In this case B is finite locally free as an A-module and $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ is the dual finite locally free A-module. $\square$

Lemma 2.10. *If $A \to B$ is flat, then the base change map (2.3.1) induces an isomorphism $\omega_{B/A} \otimes_B B_1 \to \omega_{B_1/A_1}$.*

Proof. If $A \to B$ is finite flat, then B is finite locally free as an A-module. In this case $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ is the dual finite locally free A-module and formation of this module commutes with arbitrary base change which proves the lemma in this case. In the next paragraph we reduce the general (quasi-finite flat) case to the finite flat case just discussed.

Let $\mathfrak{q}_1 \subset B_1$ be a prime. We will show that the localization of the map at the prime $\mathfrak{q}_1$ is an isomorphism, which suffices by Algebra, Lemma 23.1. Let $\mathfrak{q} \subset B$ and $\mathfrak{p} \subset A$ be the prime ideals lying under $\mathfrak{q}_1$. By Algebra, Lemma 145.2 we can find an étale ring map $A \to A'$ and a prime ideal $\mathfrak{p}' \subset A'$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}') = \kappa(\mathfrak{p})$ and such that

$$B' = B \otimes_A A' = C \times D$$

with $A' \to C$ finite and such that the unique prime $\mathfrak{q}'$ of $B \otimes_A A'$ lying over $\mathfrak{q}$ and $\mathfrak{p}'$ corresponds to a prime of C. Set $A'_1 = A' \otimes_A A_1$ and consider the base change maps (2.3.1) for the ring maps $A \to A' \to A'_1$ and $A \to A_1 \to A'_1$ as in the diagram

$$\begin{array}{ccc} \omega_{B'/A'} \otimes_{B'} B'_1 & \longrightarrow & \omega_{B'_1/A'_1} \\ \uparrow & & \uparrow \\ \omega_{B/A} \otimes_B B'_1 & \longrightarrow & \omega_{B_1/A_1} \otimes_{B_1} B'_1 \end{array}$$

where $B' = B \otimes_A A'$, $B_1 = B \otimes_A A_1$, and $B'_1 = B \otimes_A (A' \otimes_A A_1)$. By Lemma 2.4 the diagram commutes. By Lemma 2.5 the vertical arrows are isomorphisms. As $B_1 \to B'_1$ is étale and hence flat it suffices to prove the top horizontal arrow is an isomorphism after localizing at a prime $\mathfrak{q}'_1$ of B'_1 lying over $\mathfrak{q}$ (there is such a prime and use Algebra, Lemma 39.17). Thus we may assume that $B = C \times D$ with $A \to C$ finite and $\mathfrak{q}$ corresponding to a prime of C. In this case the dualizing module $\omega_{B/A}$ decomposes in a similar fashion (Lemma 2.7) which reduces the question to the finite flat case $A \to C$ handled above. $\square$

Remark 2.11. Let $f : Y \to X$ be a locally quasi-finite morphism of locally Noetherian schemes. It is clear from Lemma 2.3 that there is a unique coherent $\mathcal{O}_Y$-module $\omega_{Y/X}$ on Y such that for every pair of affine opens $\mathrm{Spec}(B) = V \subset Y$,

$\mathrm{Spec}(A) = U \subset X$ with $f(V) \subset U$ there is a canonical isomorphism

$$H^0(V, \omega_{Y/X}) = \omega_{B/A}$$

and where these isomorphisms are compatible with restriction maps.

Lemma 2.12. *Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. Let $\omega^\bullet_{B/A} \in D(B)$ be the algebraic relative dualizing complex discussed in Dualizing Complexes, Section 25. Then there is a (nonunique) isomorphism $\omega_{B/A} = H^0(\omega^\bullet_{B/A})$.*

Proof. Choose a factorization $A \to B' \to B$ where $A \to B'$ is finite and $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is an open immersion. Then $\omega^\bullet_{B/A} = \omega^\bullet_{B'/A} \otimes^{\mathbf{L}}_B B'$ by Dualizing Complexes, Lemmas 24.7 and 24.9 and the definition of $\omega^\bullet_{B/A}$. Hence it suffices to show there is an isomorphism when $A \to B$ is finite. In this case we can use Dualizing Complexes, Lemma 24.8 to see that $\omega^\bullet_{B/A} = R\,\mathrm{Hom}(B, A)$ and hence $H^0(\omega^\bullet_{B/A}) = \mathrm{Hom}_A(B, A)$ as desired. $\qquad\square$

3. Discriminant of a finite locally free morphism

Let X be a scheme and let $\mathcal{F}$ be a finite locally free $\mathcal{O}_X$-module. Then there is a canonical *trace* map

$$\mathrm{Trace} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \longrightarrow \mathcal{O}_X$$

See Exercises, Exercise 22.6. This map has the property that $\mathrm{Trace}(\mathrm{id})$ is the locally constant function on $\mathcal{O}_X$ corresponding to the rank of $\mathcal{F}$.

Let $\pi : X \to Y$ be a morphism of schemes which is finite locally free. Then there exists a canonical *trace for π* which is an $\mathcal{O}_Y$-linear map

$$\mathrm{Trace}_\pi : \pi_* \mathcal{O}_X \longrightarrow \mathcal{O}_Y$$

sending a local section f of $\pi_* \mathcal{O}_X$ to the trace of multiplication by f on $\pi_* \mathcal{O}_X$. Over affine opens this recovers the construction in Exercises, Exercise 22.7. The composition

$$\mathcal{O}_Y \xrightarrow{\pi^\sharp} \pi_* \mathcal{O}_X \xrightarrow{\mathrm{Trace}_\pi} \mathcal{O}_Y$$

equals multiplication by the degree of π (which is a locally constant function on Y). In analogy with Fields, Section 20 we can define the trace pairing

$$Q_\pi : \pi_* \mathcal{O}_X \times \pi_* \mathcal{O}_X \longrightarrow \mathcal{O}_Y$$

by the rule $(f, g) \mapsto \mathrm{Trace}_\pi(fg)$. We can think of Q_π as a linear map $\pi_* \mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_Y}(\pi_* \mathcal{O}_X, \mathcal{O}_Y)$ between locally free modules of the same rank, and hence obtain a determinant

$$\det(Q_\pi) : \wedge^{top}(\pi_* \mathcal{O}_X) \longrightarrow \wedge^{top}(\pi_* \mathcal{O}_X)^{\otimes -1}$$

or in other words a global section

$$\det(Q_\pi) \in \Gamma(Y, \wedge^{top}(\pi_* \mathcal{O}_X)^{\otimes -2})$$

The *discriminant of π* is by definition the closed subscheme $D_\pi \subset Y$ cut out by this global section. Clearly, D_π is a locally principal closed subscheme of Y.

Lemma 3.1. *Let $\pi : X \to Y$ be a morphism of schemes which is finite locally free. Then π is étale if and only if its discriminant is empty.*

Proof. By Morphisms, Lemma 36.8 it suffices to check that the fibres of π are étale. Since the construction of the trace pairing commutes with base change we reduce to the following question: Let k be a field and let A be a finite dimensional k-algebra. Show that A is étale over k if and only if the trace pairing $Q_{A/k} : A \times A \to k$, $(a, b) \mapsto \mathrm{Trace}_{A/k}(ab)$ is nondegenerate.

Assume $Q_{A/k}$ is nondegenerate. If $a \in A$ is a nilpotent element, then ab is nilpotent for all $b \in A$ and we conclude that $Q_{A/k}(a, -)$ is identically zero. Hence A is reduced. Then we can write $A = K_1 \times \ldots \times K_n$ as a product where each K_i is a field (see Algebra, Lemmas 53.2, 53.6, and 25.1). In this case the quadratic space $(A, Q_{A/k})$ is the orthogonal direct sum of the spaces $(K_i, Q_{K_i/k})$. It follows from Fields, Lemma 20.7 that each K_i is separable over k. This means that A is étale over k by Algebra, Lemma 143.4. The converse is proved by reading the argument backwards. $\square$

4. Traces for flat quasi-finite ring maps

The trace referred to in the title of this section is of a completely different nature than the trace discussed in Duality for Schemes, Section 7. Namely, it is the trace as discussed in Fields, Section 20 and generalized in Exercises, Exercises 22.6 and 22.7.

Let $A \to B$ be a finite flat map of Noetherian rings. Then B is finite flat as an A-module and hence finite locally free (Algebra, Lemma 78.2). Given $b \in B$ we can consider the *trace* $\mathrm{Trace}_{B/A}(b)$ of the A-linear map $B \to B$ given by multiplication by b on B. By the references above this defines an A-linear map $\mathrm{Trace}_{B/A} : B \to A$. Since $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ as $A \to B$ is finite, we see that $\mathrm{Trace}_{B/A} \in \omega_{B/A}$.

For a general flat quasi-finite ring map we define the notion of a trace as follows.

Definition 4.1. Let $A \to B$ be a flat quasi-finite map of Noetherian rings. The *trace element* is the unique[2] element $\tau_{B/A} \in \omega_{B/A}$ with the following property: for any Noetherian A-algebra A_1 such that $B_1 = B \otimes_A A_1$ comes with a product decomposition $B_1 = C \times D$ with $A_1 \to C$ finite the image of $\tau_{B/A}$ in ω_{C/A_1} is Trace_{C/A_1}. Here we use the base change map (2.3.1) and Lemma 2.7 to get $\omega_{B/A} \to \omega_{B_1/A_1} \to \omega_{C/A_1}$.

We first prove that trace elements are unique and then we prove that they exist.

Lemma 4.2. *Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Then there is at most one trace element in $\omega_{B/A}$.*

Proof. Let $\mathfrak{q} \subset B$ be a prime ideal lying over the prime $\mathfrak{p} \subset A$. By Algebra, Lemma 145.2 we can find an étale ring map $A \to A_1$ and a prime ideal $\mathfrak{p}_1 \subset A_1$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}_1) = \kappa(\mathfrak{p})$ and such that

$$B_1 = B \otimes_A A_1 = C \times D$$

with $A_1 \to C$ finite and such that the unique prime $\mathfrak{q}_1$ of $B \otimes_A A_1$ lying over $\mathfrak{q}$ and $\mathfrak{p}_1$ corresponds to a prime of C. Observe that $\omega_{C/A_1} = \omega_{B/A} \otimes_B C$ (combine Lemmas 2.5 and 2.7). Since the collection of ring maps $B \to C$ obtained in this manner is a jointly injective family of flat maps and since the image of $\tau_{B/A}$ in ω_{C/A_1} is prescribed the uniqueness follows. $\square$

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[2]Uniqueness and existence will be justified in Lemmas 4.2 and 4.6.

Here is a sanity check.

Lemma 4.3. *Let $A \to B$ be a finite flat map of Noetherian rings. Then $\text{Trace}_{B/A} \in \omega_{B/A}$ is the trace element.*

Proof. Suppose we have $A \to A_1$ with A_1 Noetherian and a product decomposition $B \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Of course in this case $A_1 \to D$ is also finite. Set $B_1 = B \otimes_A A_1$. Since the construction of traces commutes with base change we see that $\text{Trace}_{B/A}$ maps to Trace_{B_1/A_1}. Thus the proof is finished by noticing that $\text{Trace}_{B_1/A_1} = (\text{Trace}_{C/A_1}, \text{Trace}_{D/A_1})$ under the isomorphism $\omega_{B_1/A_1} = \omega_{C/A_1} \times \omega_{D/A_1}$ of Lemma 2.7. $\square$

Lemma 4.4. *Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Let $\tau \in \omega_{B/A}$ be a trace element.*

(1) *If $A \to A_1$ is a map with A_1 Noetherian, then with $B_1 = A_1 \otimes_A B$ the image of τ in ω_{B_1/A_1} is a trace element.*
(2) *If $A = R_f$, then τ is a trace element in $\omega_{B/R}$.*
(3) *If $g \in B$, then the image of τ in $\omega_{B_g/A}$ is a trace element.*
(4) *If $B = B_1 \times B_2$, then τ maps to a trace element in both $\omega_{B_1/A}$ and $\omega_{B_2/A}$.*

Proof. Part (1) is a formal consequence of the definition.

Statement (2) makes sense because $\omega_{B/R} = \omega_{B/A}$ by Lemma 2.3. Denote τ' the element τ but viewed as an element of $\omega_{B/R}$. To see that (2) is true suppose that we have $R \to R_1$ with R_1 Noetherian and a product decomposition $B \otimes_R R_1 = C \times D$ with $R_1 \to C$ finite. Then with $A_1 = (R_1)_f$ we see that $B \otimes_A A_1 = C \times D$. Since $R_1 \to C$ is finite, a fortiori $A_1 \to C$ is finite. Hence we can use the defining property of τ to get the corresponding property of τ'.

Statement (3) makes sense because $\omega_{B_g/A} = (\omega_{B/A})_g$ by Lemma 2.3. The proof is similar to the proof of (2). Suppose we have $A \to A_1$ with A_1 Noetherian and a product decomposition $B_g \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Set $B_1 = B \otimes_A A_1$. Then $\text{Spec}(C) \to \text{Spec}(B_1)$ is an open immersion as $B_g \otimes_A A_1 = (B_1)_g$ and the image is closed because $B_1 \to C$ is finite (as $A_1 \to C$ is finite). Thus we see that $B_1 = C \times D_1$ and $D = (D_1)_g$. Then we can use the defining property of τ to get the corresponding property for the image of τ in $\omega_{B_g/A}$.

Statement (4) makes sense because $\omega_{B/A} = \omega_{B_1/A} \times \omega_{B_2/A}$ by Lemma 2.7. Suppose we have $A \to A'$ with A' Noetherian and a product decomposition $B \otimes_A A' = C \times D$ with $A' \to C$ finite. Then it is clear that we can refine this product decomposition into $B \otimes_A A' = C_1 \times C_2 \times D_1 \times D_2$ with $A' \to C_i$ finite such that $B_i \otimes_A A' = C_i \times D_i$. Then we can use the defining property of τ to get the corresponding property for the image of τ in $\omega_{B_i/A}$. This uses the obvious fact that $\text{Trace}_{C/A'} = (\text{Trace}_{C_1/A'}, \text{Trace}_{C_2/A'})$ under the decomposition $\omega_{C/A'} = \omega_{C_1/A'} \times \omega_{C_2/A'}$. $\square$

Lemma 4.5. *Let $A \to B$ be a flat quasi-finite map of Noetherian rings. Let $g_1, \ldots, g_m \in B$ be elements generating the unit ideal. Let $\tau \in \omega_{B/A}$ be an element whose image in $\omega_{B_{g_i}/A}$ is a trace element for $A \to B_{g_i}$. Then τ is a trace element.*

Proof. Suppose we have $A \to A_1$ with A_1 Noetherian and a product decomposition $B \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. We have to show that the image of τ in ω_{C/A_1} is Trace_{C/A_1}. Observe that $g_1, \ldots, g_m$ generate the unit ideal in $B_1 = B \otimes_A A_1$ and

that τ maps to a trace element in $\omega_{(B_1)_{g_i}/A_1}$ by Lemma 4.4. Hence we may replace A by A_1 and B by B_1 to get to the situation as described in the next paragraph.

Here we assume that $B = C \times D$ with $A \to C$ is finite. Let τ_C be the image of τ in $\omega_{C/A}$. We have to prove that $\tau_C = \mathrm{Trace}_{C/A}$ in $\omega_{C/A}$. By the compatibility of trace elements with products (Lemma 4.4) we see that τ_C maps to a trace element in $\omega_{C_{g_i}/A}$. Hence, after replacing B by C we may assume that $A \to B$ is finite flat.

Assume $A \to B$ is finite flat. In this case $\mathrm{Trace}_{B/A}$ is a trace element by Lemma 4.3. Hence $\mathrm{Trace}_{B/A}$ maps to a trace element in $\omega_{B_{g_i}/A}$ by Lemma 4.4. Since trace elements are unique (Lemma 4.2) we find that $\mathrm{Trace}_{B/A}$ and τ map to the same elements in $\omega_{B_{g_i}/A} = (\omega_{B/A})_{g_i}$. As $g_1, \ldots, g_m$ generate the unit ideal of B the map $\omega_{B/A} \to \prod \omega_{B_{g_i}/A}$ is injective and we conclude that $\tau_C = \mathrm{Trace}_{B/A}$ as desired. $\square$

Lemma 4.6. *Let $A \to B$ be a flat quasi-finite map of Noetherian rings. There exists a trace element $\tau \in \omega_{B/A}$.*

Proof. Choose a factorization $A \to B' \to B$ with $A \to B'$ finite and $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ an open immersion. Let $g_1, \ldots, g_n \in B'$ be elements such that $\mathrm{Spec}(B) = \bigcup D(g_i)$ as opens of $\mathrm{Spec}(B')$. Suppose that we can prove the existence of trace elements τ_i for the quasi-finite flat ring maps $A \to B_{g_i}$. Then for all i, j the elements τ_i and τ_j map to trace elements of $\omega_{B_{g_i g_j}/A}$ by Lemma 4.4. By uniqueness of trace elements (Lemma 4.2) they map to the same element. Hence the sheaf condition for the quasi-coherent module associated to $\omega_{B/A}$ (see Algebra, Lemma 24.1) produces an element $\tau \in \omega_{B/A}$. Then τ is a trace element by Lemma 4.5. In this way we reduce to the case treated in the next paragraph.

Assume we have $A \to B'$ finite and $g \in B'$ with $B = B'_g$ flat over A. It is our task to construct a trace element in $\omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$. Choose a resolution $F_1 \to F_0 \to B' \to 0$ of B' by finite free A-modules F_0 and F_1. Then we have an exact sequence

$$0 \to \mathrm{Hom}_A(B', A) \to F_0^\vee \to F_1^\vee$$

where $F_i^\vee = \mathrm{Hom}_A(F_i, A)$ is the dual finite free module. Similarly we have the exact sequence

$$0 \to \mathrm{Hom}_A(B', B') \to F_0^\vee \otimes_A B' \to F_1^\vee \otimes_A B'$$

The idea of the construction of τ is to use the diagram

$$B' \xrightarrow{\mu} \mathrm{Hom}_A(B', B') \leftarrow \mathrm{Hom}_A(B', A) \otimes_A B' \xrightarrow{ev} A$$

where the first arrow sends $b' \in B'$ to the A-linear operator given by multiplication by b' and the last arrow is the evaluation map. The problem is that the middle arrow, which sends $\lambda' \otimes b'$ to the map $b'' \mapsto \lambda'(b'')b'$, is not an isomorphism. If B' is flat over A, the exact sequences above show that it is an isomorphism and the composition from left to right is the usual trace $\mathrm{Trace}_{B'/A}$. In the general case, we consider the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_A(B', A) \otimes_A B' & \longrightarrow & \mathrm{Hom}_A(B', A) \otimes_A B'_g \\
\psi \nearrow \quad \downarrow & & \downarrow \\
B' \dashrightarrow[\mu] \mathrm{Hom}_A(B', B') & \longrightarrow & \mathrm{Ker}(F_0^\vee \otimes_A B'_g \to F_1^\vee \otimes_A B'_g)
\end{array}
$$

By flatness of $A \to B'_g$ we see that the right vertical arrow is an isomorphism. Hence we obtain the unadorned dotted arrow. Since $B'_g = \mathrm{colim}\, \frac{1}{g^n} B'$, since colimits commute with tensor products, and since B' is a finitely presented A-module we can find an $n \geq 0$ and a B'-linear (for right B'-module structure) map $\psi : B' \to \mathrm{Hom}_A(B', A) \otimes_A B'$ whose composition with the left vertical arrow is $g^n \mu$. Composing with ev we obtain an element $ev \circ \psi \in \mathrm{Hom}_A(B', A)$. Then we set

$$\tau = (ev \circ \psi) \otimes g^{-n} \in \mathrm{Hom}_A(B', A) \otimes_{B'} B'_g = \omega_{B'_g/A} = \omega_{B/A}$$

We omit the easy verification that this element does not depend on the choice of n and ψ above.

Let us prove that τ as constructed in the previous paragraph has the desired property in a special case. Namely, say $B' = C' \times D'$ and $g = (f, h)$ where $A \to C'$ flat, D'_h is flat, and f is a unit in C'. To show: τ maps to $\mathrm{Trace}_{C'/A}$ in $\omega_{C'/A}$. In this case we first choose n_D and $\psi_D : D' \to \mathrm{Hom}_A(D', A) \otimes_A D'$ as above for the pair (D', h) and we can let $\psi_C : C' \to \mathrm{Hom}_A(C', A) \otimes_A C' = \mathrm{Hom}_A(C', C')$ be the map seconding $c' \in C'$ to multiplication by c'. Then we take $n = n_D$ and $\psi = (f^{n_D} \psi_C, \psi_D)$ and the desired compatibility is clear because $\mathrm{Trace}_{C'/A} = ev \circ \psi_C$ as remarked above.

To prove the desired property in general, suppose given $A \to A_1$ with A_1 Noetherian and a product decomposition $B'_g \otimes_A A_1 = C \times D$ with $A_1 \to C$ finite. Set $B'_1 = B' \otimes_A A_1$. Then $\mathrm{Spec}(C) \to \mathrm{Spec}(B'_1)$ is an open immersion as $B'_g \otimes_A A_1 = (B'_1)_g$ and the image is closed as $B'_1 \to C$ is finite (since $A_1 \to C$ is finite). Thus $B'_1 = C \times D'$ and $D'_g = D$. We conclude that $B'_1 = C \times D'$ and g over A_1 are as in the previous paragraph. Since formation of the displayed diagram above commutes with base change, the formation of τ commutes with the base change $A \to A_1$ (details omitted; use the resolution $F_1 \otimes_A A_1 \to F_0 \otimes_A A_1 \to B'_1 \to 0$ to see this). Thus the desired compatibility follows from the result of the previous paragraph. $\square$

Remark 4.7. Let $f : Y \to X$ be a flat locally quasi-finite morphism of locally Noetherian schemes. Let $\omega_{Y/X}$ be as in Remark 2.11. It is clear from the uniqueness, existence, and compatibility with localization of trace elements (Lemmas 4.2, 4.6, and 4.4) that there exists a global section

$$\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$$

such that for every pair of affine opens $\mathrm{Spec}(B) = V \subset Y$, $\mathrm{Spec}(A) = U \subset X$ with $f(V) \subset U$ that element $\tau_{Y/X}$ maps to $\tau_{B/A}$ under the canonical isomorphism $H^0(V, \omega_{Y/X}) = \omega_{B/A}$.

Lemma 4.8. *Let k be a field and let A be a finite k-algebra. Assume A is local with residue field k'. The following are equivalent*

    (1) *$\mathrm{Trace}_{A/k}$ is nonzero,*
    (2) *$\tau_{A/k} \in \omega_{A/k}$ is nonzero, and*
    (3) *k'/k is separable and $\mathrm{length}_A(A)$ is prime to the characteristic of k.*

Proof. Conditions (1) and (2) are equivalent by Lemma 4.3. Let $\mathfrak{m} \subset A$. Since $\dim_k(A) < \infty$ it is clear that A has finite length over A. Choose a filtration

$$A = I_0 \supset \mathfrak{m} = I_1 \supset I_2 \supset \dots I_n = 0$$

by ideals such that $I_i/I_{i+1} \cong k'$ as A-modules. See Algebra, Lemma 52.11 which also shows that $n = \text{length}_A(A)$. If $a \in \mathfrak{m}$ then $aI_i \subset I_{i+1}$ and it is immediate that $\text{Trace}_{A/k}(a) = 0$. If $a \notin \mathfrak{m}$ with image $\lambda \in k'$, then we conclude

$$\text{Trace}_{A/k}(a) = \sum\nolimits_{i=0,\ldots,n-1} \text{Trace}_k(a : I_i/I_{i-1} \to I_i/I_{i-1}) = n\text{Trace}_{k'/k}(\lambda)$$

The proof of the lemma is finished by applying Fields, Lemma 20.7. $\qquad\square$

5. Finite morphisms

In this section we collect some observations about the constructions in the previous sections for finite morphisms. Let $f : Y \to X$ be a finite morphism of locally Noetherian schemes. Let $\omega_{Y/X}$ be as in Remark 2.11.

The first remark is that

$$f_*\omega_{Y/X} = \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$$

as sheaves of $f_*\mathcal{O}_Y$-modules. Since f is affine, this formula uniquely characterizes $\omega_{Y/X}$, see Morphisms, Lemma 11.6. The formula holds because for $\text{Spec}(A) = U \subset X$ affine open, the inverse image $V = f^{-1}(U)$ is the spectrum of a finite A-algebra B and hence

$$H^0(U, f_*\omega_{Y/X}) = H^0(V, \omega_{Y/X}) = \omega_{B/A} = \text{Hom}_A(B, A) = H^0(U, \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X))$$

by construction. In particular, we obtain a canonical evaluation map

$$f_*\omega_{Y/X} \longrightarrow \mathcal{O}_X$$

which is given by evaluation at 1 if we think of $f_*\omega_{Y/X}$ as the sheaf $\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$.

The second remark is that using the evaluation map we obtain canonical identifications

$$\text{Hom}_Y(\mathcal{F}, f^*\mathcal{G} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \text{Hom}_X(f_*\mathcal{F}, \mathcal{G})$$

functorially in the quasi-coherent module $\mathcal{F}$ on Y and the finite locally free module $\mathcal{G}$ on X. If $\mathcal{G} = \mathcal{O}_X$ this follows immediately from the above and Algebra, Lemma 14.4. For general $\mathcal{G}$ we can use the same lemma and the isomorphisms

$$f_*(f^*\mathcal{G} \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{G})$$

of $f_*\mathcal{O}_Y$-modules where the first equality is the projection formula (Cohomology, Lemma 54.2). An alternative is to prove the formula affine locally by direct computation.

The third remark is that if f is in addition flat, then the composition

$$f_*\mathcal{O}_Y \xrightarrow{f_*\tau_{Y/X}} f_*\omega_{Y/X} \longrightarrow \mathcal{O}_X$$

is equal to the trace map Trace_f discussed in Section 3. This follows immediately by looking over affine opens.

The fourth remark is that if f is flat and X Noetherian, then we obtain

$$\text{Hom}_Y(K, Lf^*M \otimes_{\mathcal{O}_Y} \omega_{Y/X}) = \text{Hom}_X(Rf_*K, M)$$

for any K in $D_{QCoh}(\mathcal{O}_Y)$ and M in $D_{QCoh}(\mathcal{O}_X)$. This follows from the material in Duality for Schemes, Section 12, but can be proven directly in this case as follows. First, if X is affine, then it holds by Dualizing Complexes, Lemmas 13.1 and 13.9[3]

---

[3]There is a simpler proof of this lemma in our case.

and Derived Categories of Schemes, Lemma 3.5. Then we can use the induction principle (Cohomology of Schemes, Lemma 4.1) and Mayer-Vietoris (in the form of Cohomology, Lemma 33.3) to finish the proof.

6. The Noether different

There are many different differents available in the literature. We list some of them in this and the next sections; for more information we suggest the reader consult [Kun86].

Let $A \to B$ be a ring map. Denote

$$\mu : B \otimes_A B \longrightarrow B, \quad b \otimes b' \longmapsto bb'$$

the multiplication map. Let $I = \mathrm{Ker}(\mu)$. It is clear that I is generated by the elements $b \otimes 1 - 1 \otimes b$ for $b \in B$. Hence the annihilator $J \subset B \otimes_A B$ of I is a B-module in a canonical manner. The *Noether different* of B over A is the image of J under the map $\mu : B \otimes_A B \to B$. Equivalently, the Noether different is the image of the map

$$J = \mathrm{Hom}_{B \otimes_A B}(B, B \otimes_A B) \longrightarrow B, \quad \varphi \longmapsto \mu(\varphi(1))$$

We begin with some obligatory lemmas.

Lemma 6.1. *Let $A \to B_i$, $i = 1, 2$ be ring maps. Set $B = B_1 \times B_2$.*

(1) *The annihilator J of $\mathrm{Ker}(B \otimes_A B \to B)$ is $J_1 \times J_2$ where J_i is the annihilator of $\mathrm{Ker}(B_i \otimes_A B_i \to B_i)$.*
(2) *The Noether different $\mathfrak{D}$ of B over A is $\mathfrak{D}_1 \times \mathfrak{D}_2$, where $\mathfrak{D}_i$ is the Noether different of B_i over A.*

Proof. Omitted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 6.2. *Let $A \to B$ be a finite type ring map. Let $A \to A'$ be a flat ring map. Set $B' = B \otimes_A A'$.*

(1) *The annihilator J' of $\mathrm{Ker}(B' \otimes_{A'} B' \to B')$ is $J \otimes_A A'$ where J is the annihilator of $\mathrm{Ker}(B \otimes_A B \to B)$.*
(2) *The Noether different $\mathfrak{D}'$ of B' over A' is $\mathfrak{D}B'$, where $\mathfrak{D}$ is the Noether different of B over A.*

Proof. Choose generators $b_1, \ldots, b_n$ of B as an A-algebra. Then

$$J = \mathrm{Ker}(B \otimes_A B \xrightarrow{b_i \otimes 1 - 1 \otimes b_i} (B \otimes_A B)^{\oplus n})$$

Hence we see that the formation of J commutes with flat base change. The result on the Noether different follows immediately from this. $\qquad\qquad\square$

Lemma 6.3. *Let $A \to B' \to B$ be ring maps with $A \to B'$ of finite type and $B' \to B$ inducing an open immersion of spectra.*

(1) *The annihilator J of $\mathrm{Ker}(B \otimes_A B \to B)$ is $J' \otimes_{B'} B$ where J' is the annihilator of $\mathrm{Ker}(B' \otimes_A B' \to B')$.*
(2) *The Noether different $\mathfrak{D}$ of B over A is $\mathfrak{D}'B$, where $\mathfrak{D}'$ is the Noether different of B' over A.*

Proof. Write $I = \mathrm{Ker}(B \otimes_A B \to B)$ and $I' = \mathrm{Ker}(B' \otimes_A B' \to B')$. As $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is an open immersion, it follows that $B = (B \otimes_A B) \otimes_{B' \otimes_A B'} B'$. Thus we see that $I = I'(B \otimes_A B)$. Since I' is finitely generated and $B' \otimes_A B' \to B \otimes_A B$ is flat, we conclude that $J = J'(B \otimes_A B)$, see Algebra, Lemma 40.4. Since the $B' \otimes_A B'$-module structure of J' factors through $B' \otimes_A B' \to B'$ we conclude that (1) is true. Part (2) is a consequence of (1). $\qquad\square$

Remark 6.4. Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings. Let J be the annihilator of $\mathrm{Ker}(B \otimes_A B \to B)$. There is a canonical B-bilinear pairing

$$(6.4.1) \qquad\qquad \omega_{B/A} \times J \longrightarrow B$$

defined as follows. Choose a factorization $A \to B' \to B$ with $A \to B'$ finite and $B' \to B$ inducing an open immersion of spectra. Let J' be the annihilator of $\mathrm{Ker}(B' \otimes_A B' \to B')$. We first define

$$\mathrm{Hom}_A(B', A) \times J' \longrightarrow B', \quad (\lambda, \sum b_i \otimes c_i) \longmapsto \sum \lambda(b_i) c_i$$

This is B'-bilinear exactly because for $\xi \in J'$ and $b \in B'$ we have $(b \otimes 1)\xi = (1 \otimes b)\xi$. By Lemma 6.3 and the fact that $\omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$ we can extend this to a B-bilinear pairing as displayed above.

Lemma 6.5. *Let $A \to B$ be a quasi-finite homomorphism of Noetherian rings.*

(1) *If $A \to A'$ is a flat map of Noetherian rings, then*

$$
\begin{array}{ccc}
\omega_{B/A} \times J & \longrightarrow & B \\
\downarrow & & \downarrow \\
\omega_{B'/A'} \times J' & \longrightarrow & B'
\end{array}
$$

is commutative where notation as in Lemma 6.2 and horizontal arrows are given by (6.4.1).

(2) *If $B = B_1 \times B_2$, then*

$$
\begin{array}{ccc}
\omega_{B/A} \times J & \longrightarrow & B \\
\downarrow & & \downarrow \\
\omega_{B_i/A} \times J_i & \longrightarrow & B_i
\end{array}
$$

is commutative for $i = 1, 2$ where notation as in Lemma 6.1 and horizontal arrows are given by (6.4.1).

Proof. Because of the construction of the pairing in Remark 6.4 both (1) and (2) reduce to the case where $A \to B$ is finite. Then (1) follows from the fact that the contraction map $\mathrm{Hom}_A(M, A) \otimes_A M \otimes_A M \to M$, $\lambda \otimes m \otimes m' \mapsto \lambda(m)m'$ commuted with base change. To see (2) use that $J = J_1 \times J_2$ is contained in the summands $B_1 \otimes_A B_1$ and $B_2 \otimes_A B_2$ of $B \otimes_A B$. $\qquad\square$

Lemma 6.6. *Let $A \to B$ be a flat quasi-finite homomorphism of Noetherian rings. The pairing of Remark 6.4 induces an isomorphism $J \to \mathrm{Hom}_B(\omega_{B/A}, B)$.*

Proof. We first prove this when $A \to B$ is finite and flat. In this case we can localize on A and assume B is finite free as an A-module. Let $b_1, \ldots, b_n$ be a basis of B as an A-module and denote $b_1^\vee, \ldots, b_n^\vee$ the dual basis of $\omega_{B/A}$. Note that $\sum b_i \otimes c_i \in J$ maps to the element of $\mathrm{Hom}_B(\omega_{B/A}, B)$ which sends $b_i^\vee$ to c_i. Suppose $\varphi : \omega_{B/A} \to B$ is B-linear. Then we claim that $\xi = \sum b_i \otimes \varphi(b_i^\vee)$ is an element of J. Namely, the B-linearity of φ exactly implies that $(b \otimes 1)\xi = (1 \otimes b)\xi$ for all $b \in B$. Thus our map has an inverse and it is an isomorphism.

Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$. We will show that the localization
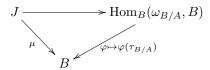
$$J_\mathfrak{q} \longrightarrow \mathrm{Hom}_B(\omega_{B/A}, B)_\mathfrak{q}$$

is an isomorphism. This suffices by Algebra, Lemma 23.1. By Algebra, Lemma 145.2 we can find an étale ring map $A \to A'$ and a prime ideal $\mathfrak{p}' \subset A'$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}') = \kappa(\mathfrak{p})$ and such that

$$B' = B \otimes_A A' = C \times D$$

with $A' \to C$ finite and such that the unique prime $\mathfrak{q}'$ of $B \otimes_A A'$ lying over $\mathfrak{q}$ and $\mathfrak{p}'$ corresponds to a prime of C. Let J' be the annihilator of $\mathrm{Ker}(B' \otimes_{A'} B' \to B')$. By Lemmas 2.5, 6.2, and 6.5 the map $J' \to \mathrm{Hom}_{B'}(\omega_{B'/A'}, B')$ is gotten by applying the functor $- \otimes_B B'$ to the map $J \to \mathrm{Hom}_B(\omega_{B/A}, B)$. Since $B_\mathfrak{q} \to B'_{\mathfrak{q}'}$ is faithfully flat it suffices to prove the result for $(A' \to B', \mathfrak{q}')$. By Lemmas 2.7, 6.1, and 6.5 this reduces us to the case proved in the first paragraph of the proof. $\square$

Lemma 6.7. *Let $A \to B$ be a flat quasi-finite homomorphism of Noetherian rings. The diagram*

commutes where the horizontal arrow is the isomorphism of Lemma 6.6. Hence the Noether different of B over A is the image of the map $\mathrm{Hom}_B(\omega_{B/A}, B) \to B$.

Proof. Exactly as in the proof of Lemma 6.6 this reduces to the case of a finite free map $A \to B$. In this case $\tau_{B/A} = \mathrm{Trace}_{B/A}$. Choose a basis $b_1, \ldots, b_n$ of B as an A-module. Let $\xi = \sum b_i \otimes c_i \in J$. Then $\mu(\xi) = \sum b_i c_i$. On the other hand, the image of ξ in $\mathrm{Hom}_B(\omega_{B/A}, B)$ sends $\mathrm{Trace}_{B/A}$ to $\sum \mathrm{Trace}_{B/A}(b_i)c_i$. Thus we have to show

$$\sum b_i c_i = \sum \mathrm{Trace}_{B/A}(b_i)c_i$$

when $\xi = \sum b_i \otimes c_i \in J$. Write $b_i b_j = \sum_k a_{ij}^k b_k$ for some $a_{ij}^k \in A$. Then the right hand side is $\sum_{i,j} a_{ij}^j c_i$. On the other hand, $\xi \in J$ implies

$$(b_j \otimes 1)(\sum_i b_i \otimes c_i) = (1 \otimes b_j)(\sum_i b_i \otimes c_i)$$

which implies that $b_j c_i = \sum_k a_{jk}^i c_k$. Thus the left hand side is $\sum_{i,j} a_{ij}^i c_j$. Since $a_{ij}^k = a_{ji}^k$ the equality holds. $\square$

Lemma 6.8. *Let $A \to B$ be a finite type ring map. Let $\mathfrak{D} \subset B$ be the Noether different. Then $V(\mathfrak{D})$ is the set of primes $\mathfrak{q} \subset B$ such that $A \to B$ is not unramified at $\mathfrak{q}$.*

Proof. Assume $A \to B$ is unramified at $\mathfrak{q}$. After replacing B by B_g for some $g \in B$, $g \notin \mathfrak{q}$ we may assume $A \to B$ is unramified (Algebra, Definition 151.1 and Lemma 6.3). In this case $\Omega_{B/A} = 0$. Hence if $I = \mathrm{Ker}(B \otimes_A B \to B)$, then $I/I^2 = 0$ by Algebra, Lemma 131.13. Since $A \to B$ is of finite type, we see that I is finitely generated. Hence by Nakayama's lemma (Algebra, Lemma 20.1) there exists an element of the form $1 + i$ annihilating I. It follows that $\mathfrak{D} = B$.

Conversely, assume that $\mathfrak{D} \not\subset \mathfrak{q}$. Then after replacing B by a principal localization as above we may assume $\mathfrak{D} = B$. This means there exists an element of the form $1 + i$ in the annihilator of I. Conversely this implies that $I/I^2 = \Omega_{B/A}$ is zero and we conclude. $\square$

7. The Kähler different

Let $A \to B$ be a finite type ring map. The *Kähler different* is the zeroth fitting ideal of $\Omega_{B/A}$ as a B-module. We globalize the definition as follows.

Definition 7.1. Let $f : Y \to X$ be a morphism of schemes which is locally of finite type. The *Kähler different* is the 0th fitting ideal of $\Omega_{Y/X}$.

The Kähler different is a quasi-coherent sheaf of ideals on Y.

Lemma 7.2. *Consider a cartesian diagram of schemes*

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
X' & \xrightarrow{\;g\;} & X
\end{array}
$$

with f locally of finite type. Let $R \subset Y$, resp. $R' \subset Y'$ be the closed subscheme cut out by the Kähler different of f, resp. f'. Then $Y' \to Y$ induces an isomorphism $R' \to R \times_Y Y'$.

Proof. This is true because $\Omega_{Y'/X'}$ is the pullback of $\Omega_{Y/X}$ (Morphisms, Lemma 32.10) and then we can apply More on Algebra, Lemma 8.4. $\square$

Lemma 7.3. *Let $f : Y \to X$ be a morphism of schemes which is locally of finite type. Let $R \subset Y$ be the closed subscheme defined by the Kähler different. Then $R \subset Y$ is exactly the set of points where f is not unramified.*

Proof. This is a copy of Divisors, Lemma 10.2. $\square$

Lemma 7.4. *Let A be a ring. Let $n \geq 1$ and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$. Set $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. The Kähler different of B over A is the ideal of B generated by $\det(\partial f_i / \partial x_j)$.*

Proof. This is true because $\Omega_{B/A}$ has a presentation

$$
\bigoplus_{i=1,\ldots,n} B f_i \xrightarrow{\mathrm{d}} \bigoplus_{j=1,\ldots,n} B \mathrm{d}x_j \to \Omega_{B/A} \to 0
$$

by Algebra, Lemma 131.9. $\square$

8. The Dedekind different

Let $A \to B$ be a ring map. We say *the Dedekind different is defined* if A is Noetherian, $A \to B$ is finite, any nonzerodivisor on A is a nonzerodivisor on B, and $K \to L$ is étale where $K = Q(A)$ and $L = B \otimes_A K$. Then $K \subset L$ is finite étale and

$$\mathcal{L}_{B/A} = \{x \in L \mid \mathrm{Trace}_{L/K}(bx) \in A \text{ for all } b \in B\}$$

is the Dedekind complementary module. In this situation the *Dedekind different* is

$$\mathfrak{D}_{B/A} = \{x \in L \mid x\mathcal{L}_{B/A} \subset B\}$$

viewed as a B-submodule of L. By Lemma 8.1 the Dedekind different is an ideal of B either if A is normal or if B is flat over A.

Lemma 8.1. *Assume the Dedekind different of $A \to B$ is defined. Consider the statements*

    (1) *$A \to B$ is flat,*
    (2) *A is a normal ring,*
    (3) *$\mathrm{Trace}_{L/K}(B) \subset A$,*
    (4) *$1 \in \mathcal{L}_{B/A}$, and*
    (5) *the Dedekind different $\mathfrak{D}_{B/A}$ is an ideal of B.*

Then we have $(1) \Rightarrow (3)$, $(2) \Rightarrow (3)$, $(3) \Leftrightarrow (4)$, and $(4) \Rightarrow (5)$.

Proof. The equivalence of (3) and (4) and the implication $(4) \Rightarrow (5)$ are immediate.

If $A \to B$ is flat, then we see that $\mathrm{Trace}_{B/A} : B \to A$ is defined and that $\mathrm{Trace}_{L/K}$ is the base change. Hence (3) holds.

If A is normal, then A is a finite product of normal domains, hence we reduce to the case of a normal domain. Then K is the fraction field of A and $L = \prod L_i$ is a finite product of finite separable field extensions of K. Then $\mathrm{Trace}_{L/K}(b) = \sum \mathrm{Trace}_{L_i/K}(b_i)$ where $b_i \in L_i$ is the image of b. Since b is integral over A as B is finite over A, these traces are in A. This is true because the minimal polynomial of b_i over K has coefficients in A (Algebra, Lemma 38.6) and because $\mathrm{Trace}_{L_i/K}(b_i)$ is an integer multiple of one of these coefficients (Fields, Lemma 20.3). $\qquad\square$

Lemma 8.2. *If the Dedekind different of $A \to B$ is defined, then there is a canonical isomorphism $\mathcal{L}_{B/A} \to \omega_{B/A}$.*

Proof. Recall that $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ as $A \to B$ is finite. We send $x \in \mathcal{L}_{B/A}$ to the map $b \mapsto \mathrm{Trace}_{L/K}(bx)$. Conversely, given an A-linear map $\varphi : B \to A$ we obtain a K-linear map $\varphi_K : L \to K$. Since $K \to L$ is finite étale, we see that the trace pairing is nondegenerate (Lemma 3.1) and hence there exists a $x \in L$ such that $\varphi_K(y) = \mathrm{Trace}_{L/K}(xy)$ for all $y \in L$. Then $x \in \mathcal{L}_{B/A}$ maps to φ in $\omega_{B/A}$. $\quad\square$

Lemma 8.3. *If the Dedekind different of $A \to B$ is defined and $A \to B$ is flat, then*

    (1) *the canonical isomorphism $\mathcal{L}_{B/A} \to \omega_{B/A}$ sends $1 \in \mathcal{L}_{B/A}$ to the trace element $\tau_{B/A} \in \omega_{B/A}$, and*
    (2) *the Dedekind different is $\mathfrak{D}_{B/A} = \{b \in B \mid b\omega_{B/A} \subset B\tau_{B/A}\}$.*

Proof. The first assertion follows from the proof of Lemma 8.1 and Lemma 4.3. The second assertion is immediate from the first and the definitions. $\qquad\square$

9. The different

The motivation for the following definition is that it recovers the Dedekind different in the finite flat case as we will see below.

Definition 9.1. Let $f : Y \to X$ be a flat locally quasi-finite morphism of locally Noetherian schemes. Let $\omega_{Y/X}$ be the relative dualizing module and let $\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$ be the trace element (Remarks 2.11 and 4.7). The annihilator of

$$\mathrm{Coker}(\mathcal{O}_Y \xrightarrow{\tau_{Y/X}} \omega_{Y/X})$$

is the *different* of Y/X. It is a coherent ideal $\mathfrak{D}_f \subset \mathcal{O}_Y$.

We will generalize this in Remark 14.2 below. Observe that $\mathfrak{D}_f$ is locally generated by one element if $\omega_{Y/X}$ is an invertible $\mathcal{O}_Y$-module. We first state the agreement with the Dedekind different.

Lemma 9.2. *Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. Let $V = \mathrm{Spec}(B) \subset Y$, $U = \mathrm{Spec}(A) \subset X$ be affine open subschemes with $f(V) \subset U$. If the Dedekind different of $A \to B$ is defined, then*

$$\mathfrak{D}_f|_V = \widetilde{\mathfrak{D}_{B/A}}$$

as coherent ideal sheaves on V.

Proof. This is clear from Lemmas 8.1 and 8.3. $\qquad\square$

Lemma 9.3. *Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. Let $V = \mathrm{Spec}(B) \subset Y$, $U = \mathrm{Spec}(A) \subset X$ be affine open subschemes with $f(V) \subset U$. If $\omega_{Y/X}|_V$ is invertible, i.e., if $\omega_{B/A}$ is an invertible B-module, then*

$$\mathfrak{D}_f|_V = \widetilde{\mathfrak{D}}$$

as coherent ideal sheaves on V where $\mathfrak{D} \subset B$ is the Noether different of B over A.

Proof. Consider the map

$$\mathcal{H}om_{\mathcal{O}_Y}(\omega_{Y/X}, \mathcal{O}_Y) \longrightarrow \mathcal{O}_Y, \quad \varphi \longmapsto \varphi(\tau_{Y/X})$$

The image of this map corresponds to the Noether different on affine opens, see Lemma 6.7. Hence the result follows from the elementary fact that given an invertible module ω and a global section τ the image of $\tau : \mathcal{H}om(\omega, \mathcal{O}) = \omega^{\otimes -1} \to \mathcal{O}$ is the same as the annihilator of $\mathrm{Coker}(\tau : \mathcal{O} \to \omega)$. $\qquad\square$

Lemma 9.4. *Consider a cartesian diagram of Noetherian schemes*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{g} & X \end{array}$$

with f flat and quasi-finite. Let $R \subset Y$, resp. $R' \subset Y'$ be the closed subscheme cut out by the different $\mathfrak{D}_f$, resp. $\mathfrak{D}_{f'}$. Then $Y' \to Y$ induces a bijective closed immersion $R' \to R \times_Y Y'$. If g is flat or if $\omega_{Y/X}$ is invertible, then $R' = R \times_Y Y'$.

Proof. There is an immediate reduction to the case where X, X', Y, Y' are affine. In other words, we have a cocartesian diagram of Noetherian rings

$$
\begin{array}{ccc}
B' & \longleftarrow & B \\
\uparrow & & \uparrow \\
A' & \longleftarrow & A
\end{array}
$$

with $A \to B$ flat and quasi-finite. The base change map $\omega_{B/A} \otimes_B B' \to \omega_{B'/A'}$ is an isomorphism (Lemma 2.10) and maps the trace element $\tau_{B/A}$ to the trace element $\tau_{B'/A'}$ (Lemma 4.4). Hence the finite B-module $Q = \mathrm{Coker}(\tau_{B/A} : B \to \omega_{B/A})$ satisfies $Q \otimes_B B' = \mathrm{Coker}(\tau_{B'/A'} : B' \to \omega_{B'/A'})$. Thus $\mathfrak{D}_{B/A}B' \subset \mathfrak{D}_{B'/A'}$ which means we obtain the closed immersion $R' \to R \times_Y Y'$. Since $R = \mathrm{Supp}(Q)$ and $R' = \mathrm{Supp}(Q \otimes_B B')$ (Algebra, Lemma 40.5) we see that $R' \to R \times_Y Y'$ is bijective by Algebra, Lemma 40.6. The equality $\mathfrak{D}_{B/A}B' = \mathfrak{D}_{B'/A'}$ holds if $B \to B'$ is flat, e.g., if $A \to A'$ is flat, see Algebra, Lemma 40.4. Finally, if $\omega_{B/A}$ is invertible, then we can localize and assume $\omega_{B/A} = B\lambda$. Writing $\tau_{B/A} = b\lambda$ we see that $Q = B/bB$ and $\mathfrak{D}_{B/A} = bB$. The same reasoning over B' gives $\mathfrak{D}_{B'/A'} = bB'$ and the lemma is proved. $\square$

Lemma 9.5. *Let $f : Y \to X$ be a finite flat morphism of Noetherian schemes. Then $\mathrm{Norm}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$ maps $f_*\mathfrak{D}_f$ into the ideal sheaf of the discriminant D_f.*

Proof. The norm map is constructed in Divisors, Lemma 17.6 and the discriminant of f in Section 3. The question is affine local, hence we may assume $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and f given by a finite locally free ring map $A \to B$. Localizing further we may assume B is finite free as an A-module. Choose a basis $b_1, \ldots, b_n \in B$ for B as an A-module. Denote $b_1^\vee, \ldots, b_n^\vee$ the dual basis of $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ as an A-module. Since the norm of b is the determinant of $b : B \to B$ as an A-linear map, we see that $\mathrm{Norm}_{B/A}(b) = \det(b_i^\vee(bb_j))$. The discriminant is the principal closed subscheme of $\mathrm{Spec}(A)$ defined by $\det(\mathrm{Trace}_{B/A}(b_ib_j))$. If $b \in \mathfrak{D}_{B/A}$ then there exist $c_i \in B$ such that $b \cdot b_i^\vee = c_i \cdot \mathrm{Trace}_{B/A}$ where we use a dot to indicate the B-module structure on $\omega_{B/A}$. Write $c_i = \sum a_{il} b_l$. We have

$$
\begin{aligned}
\mathrm{Norm}_{B/A}(b) &= \det(b_i^\vee(bb_j)) \\
&= \det((b \cdot b_i^\vee)(b_j)) \\
&= \det((c_i \cdot \mathrm{Trace}_{B/A})(b_j)) \\
&= \det(\mathrm{Trace}_{B/A}(c_ib_j)) \\
&= \det(a_{il})\det(\mathrm{Trace}_{B/A}(b_lb_j))
\end{aligned}
$$

which proves the lemma. $\square$

Lemma 9.6. *Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. The closed subscheme $R \subset Y$ defined by the different $\mathfrak{D}_f$ is exactly the set of points where f is not étale (equivalently not unramified).*

Proof. Since f is of finite presentation and flat, we see that it is étale at a point if and only if it is unramified at that point. Moreover, the formation of the locus of ramified points commutes with base change. See Morphisms, Section 36 and especially Morphisms, Lemma 36.17. By Lemma 9.4 the formation of R commutes set theoretically with base change. Hence it suffices to prove the lemma when X

is the spectrum of a field. On the other hand, the construction of $(\omega_{Y/X}, \tau_{Y/X})$ is local on Y. Since Y is a finite discrete space (being quasi-finite over a field), we may assume Y has a unique point.

Say $X = \operatorname{Spec}(k)$ and $Y = \operatorname{Spec}(B)$ where k is a field and B is a finite local k-algebra. If $Y \to X$ is étale, then B is a finite separable extension of k, and the trace element $\operatorname{Trace}_{B/k}$ is a basis element of $\omega_{B/k}$ by Fields, Lemma 20.7. Thus $\mathfrak{D}_{B/k} = B$ in this case. Conversely, if $\mathfrak{D}_{B/k} = B$, then we see from Lemma 9.5 and the fact that the norm of 1 equals 1 that the discriminant is empty. Hence $Y \to X$ is étale by Lemma 3.1. $\qquad\square$

Lemma 9.7. *Let $f : Y \to X$ be a flat quasi-finite morphism of Noetherian schemes. Let $R \subset Y$ be the closed subscheme defined by $\mathfrak{D}_f$.*

(1) *If $\omega_{Y/X}$ is invertible, then R is a locally principal closed subscheme of Y.*
(2) *If $\omega_{Y/X}$ is invertible and f is finite, then the norm of R is the discriminant D_f of f.*
(3) *If $\omega_{Y/X}$ is invertible and f is étale at the associated points of Y, then R is an effective Cartier divisor and there is an isomorphism $\mathcal{O}_Y(R) = \omega_{Y/X}$.*

Proof. Proof of (1). We may work locally on Y, hence we may assume $\omega_{Y/X}$ is free of rank 1. Say $\omega_{Y/X} = \mathcal{O}_Y \lambda$. Then we can write $\tau_{Y/X} = h\lambda$ and then we see that R is defined by h, i.e., R is locally principal.

Proof of (2). We may assume $Y \to X$ is given by a finite free ring map $A \to B$ and that $\omega_{B/A}$ is free of rank 1 as B-module. Choose a B-basis element λ for $\omega_{B/A}$ and write $\operatorname{Trace}_{B/A} = b \cdot \lambda$ for some $b \in B$. Then $\mathfrak{D}_{B/A} = (b)$ and D_f is cut out by $\det(\operatorname{Trace}_{B/A}(b_i b_j))$ where $b_1, \ldots, b_n$ is a basis of B as an A-module. Let $b_1^\vee, \ldots, b_n^\vee$ be the dual basis. Writing $b_i^\vee = c_i \cdot \lambda$ we see that $c_1, \ldots, c_n$ is a basis of B as well. Hence with $c_i = \sum a_{il} b_l$ we see that $\det(a_{il})$ is a unit in A. Clearly, $b \cdot b_i^\vee = c_i \cdot \operatorname{Trace}_{B/A}$ hence we conclude from the computation in the proof of Lemma 9.5 that $\operatorname{Norm}_{B/A}(b)$ is a unit times $\det(\operatorname{Trace}_{B/A}(b_i b_j))$.

Proof of (3). In the notation above we see from Lemma 9.6 and the assumption that h does not vanish in the associated points of Y, which implies that h is a nonzerodivisor. The canonical isomorphism sends 1 to $\tau_{Y/X}$, see Divisors, Lemma 14.10. $\qquad\square$

10. Quasi-finite syntomic morphisms

This section discusses the fact that a quasi-finite syntomic morphism has an invertible relative dualizing module.

Lemma 10.1. *Let $f : Y \to X$ be a morphism of schemes. The following are equivalent*

(1) *f is locally quasi-finite and syntomic,*
(2) *f is locally quasi-finite, flat, and a local complete intersection morphism,*
(3) *f is locally quasi-finite, flat, locally of finite presentation, and the fibres of f are local complete intersections,*
(4) *f is locally quasi-finite and for every $y \in Y$ there are affine opens $y \in V = \operatorname{Spec}(B) \subset Y$, $U = \operatorname{Spec}(A) \subset X$ with $f(V) \subset U$ an integer n and $h, f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ such that $B = A[x_1, \ldots, x_n, 1/h]/(f_1, \ldots, f_n)$,*

(5) *for every $y \in Y$ there are affine opens $y \in V = \mathrm{Spec}(B) \subset Y$, $U = \mathrm{Spec}(A) \subset X$ with $f(V) \subset U$ such that $A \to B$ is a relative global complete intersection of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$,*

(6) *f is locally quasi-finite, flat, locally of finite presentation, and $NL_{X/Y}$ has tor-amplitude in $[-1, 0]$, and*

(7) *f is flat, locally of finite presentation, $NL_{X/Y}$ is perfect of rank 0 with tor-amplitude in $[-1, 0]$,*

Proof. The equivalence of (1) and (2) is More on Morphisms, Lemma 62.8. The equivalence of (1) and (3) is Morphisms, Lemma 30.11.

If $A \to B$ is as in (4), then $B = A[x, x_1, \ldots, x_n]/(xh - 1, f_1, \ldots, f_n]$ is a relative global complete intersection by see Algebra, Definition 136.5. Thus (4) implies (5). It is clear that (5) implies (4).

Condition (5) implies (1): by Algebra, Lemma 136.13 a relative global complete intersection is syntomic and the definition of a relative global complete intersection guarantees that a relative global complete intersection on n variables with n equations is quasi-finite, see Algebra, Definition 136.5 and Lemma 122.2.

Either Algebra, Lemma 136.15 or Morphisms, Lemma 30.10 shows that (1) implies (5).

More on Morphisms, Lemma 62.17 shows that (6) is equivalent to (1). If the equivalent conditions (1) – (6) hold, then we see that affine locally $Y \to X$ is given by a relative global complete intersection $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ with the same number of variables as the number of equations. Using this presentation we see that

$$NL_{B/A} = \left((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \longrightarrow \bigoplus_{i=1,\ldots,n} B \mathrm{d}x_i \right)$$

By Algebra, Lemma 136.12 the module $(f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2$ is free with generators the congruence classes of the elements $f_1, \ldots, f_n$. Thus $NL_{B/A}$ has rank 0 and so does $NL_{Y/X}$. In this way we see that (1) – (6) imply (7).

Finally, assume (7). By More on Morphisms, Lemma 62.17 we see that f is syntomic. Thus on suitable affine opens f is given by a relative global complete intersection $A \to B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, see Morphisms, Lemma 30.10. Exactly as above we see that $NL_{B/A}$ is a perfect complex of rank $n - m$. Thus $n = m$ and we see that (5) holds. This finishes the proof. $\qquad\square$

Lemma 10.2. *Invertibility of the relative dualizing module.*

(1) *If $A \to B$ is a quasi-finite flat homomorphism of Noetherian rings, then $\omega_{B/A}$ is an invertible B-module if and only if $\omega_{B \otimes_A \kappa(\mathfrak{p})/\kappa(\mathfrak{p})}$ is an invertible $B \otimes_A \kappa(\mathfrak{p})$-module for all primes $\mathfrak{p} \subset A$.*

(2) *If $Y \to X$ is a quasi-finite flat morphism of Noetherian schemes, then $\omega_{Y/X}$ is invertible if and only if $\omega_{Y_x/x}$ is invertible for all $x \in X$.*

Proof. Proof of (1). As $A \to B$ is flat, the module $\omega_{B/A}$ is A-flat, see Lemma 2.9. Thus $\omega_{B/A}$ is an invertible B-module if and only if $\omega_{B/A} \otimes_A \kappa(\mathfrak{p})$ is an invertible $B \otimes_A \kappa(\mathfrak{p})$-module for every prime $\mathfrak{p} \subset A$, see More on Morphisms, Lemma 16.7. Still using that $A \to B$ is flat, we have that formation of $\omega_{B/A}$ commutes with base change, see Lemma 2.10. Thus we see that invertibility of the relative dualizing

module, in the presence of flatness, is equivalent to invertibility of the relative dualizing module for the maps $\kappa(\mathfrak{p}) \to B \otimes_A \kappa(\mathfrak{p})$.

Part (2) follows from (1) and the fact that affine locally the dualizing modules are given by their algebraic counterparts, see Remark 2.11. $\square$

Lemma 10.3. *Let k be a field. Let $B = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ be a global complete intersection over k of dimension 0. Then $\omega_{B/k}$ is invertible.*

Proof. By Noether normalization, see Algebra, Lemma 115.4 we see that there exists a finite injection $k \to B$, i.e., $\dim_k(B) < \infty$. Hence $\omega_{B/k} = \operatorname{Hom}_k(B, k)$ as a B-module. By Dualizing Complexes, Lemma 15.8 we see that $R\operatorname{Hom}(B, k)$ is a dualizing complex for B and by Dualizing Complexes, Lemma 13.3 we see that $R\operatorname{Hom}(B, k)$ is equal to $\omega_{B/k}$ placed in degree 0. Thus it suffices to show that B is Gorenstein (Dualizing Complexes, Lemma 21.4). This is true by Dualizing Complexes, Lemma 21.7. $\square$

Lemma 10.4. *Let $f : Y \to X$ be a morphism of locally Noetherian schemes. If f satisfies the equivalent conditions of Lemma 10.1 then $\omega_{Y/X}$ is an invertible $\mathcal{O}_Y$-module.*

Proof. We may assume $A \to B$ is a relative global complete intersection of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ and we have to show $\omega_{B/A}$ is invertible. This follows in combining Lemmas 10.2 and 10.3. $\square$

Example 10.5. Let $n \geq 1$ and $d \geq 1$ be integers. Let T be the set of multi-indices $E = (e_1, \ldots, e_n)$ with $e_i \geq 0$ and $\sum e_i \leq d$. Consider the ring

$$A = \mathbf{Z}[a_{i,E}; 1 \leq i \leq n, E \in T]$$

In $A[x_1, \ldots, x_n]$ consider the elements $f_i = \sum_{E \in T} a_{i,E} x^E$ where $x^E = x_1^{e_1} \ldots x_n^{e_n}$ as is customary. Consider the A-algebra

$$B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

Denote $X_{n,d} = \operatorname{Spec}(A)$ and let $Y_{n,d} \subset \operatorname{Spec}(B)$ be the maximal open subscheme such that the restriction of the morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A) = X_{n,d}$ is quasi-finite, see Algebra, Lemma 123.13.

Lemma 10.6. *With notation as in Example 10.5 the schemes $X_{n,d}$ and $Y_{n,d}$ are regular and irreducible, the morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite and syntomic, and there is a dense open subscheme $V \subset Y_{n,d}$ such that $Y_{n,d} \to X_{n,d}$ restricts to an étale morphism $V \to X_{n,d}$.*

Proof. The scheme $X_{n,d}$ is the spectrum of the polynomial ring A. Hence $X_{n,d}$ is regular and irreducible. Since we can write

$$f_i = a_{i,(0,\ldots,0)} + \sum_{E \in T, E \neq (0,\ldots,0)} a_{i,E} x^E$$

we see that the ring B is isomorphic to the polynomial ring on $x_1, \ldots, x_n$ and the elements $a_{i,E}$ with $E \neq (0, \ldots, 0)$. Hence $\operatorname{Spec}(B)$ is an irreducible and regular scheme and so is the open $Y_{n,d}$. The morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite and syntomic by Lemma 10.1. To find V it suffices to find a single point where $Y_{n,d} \to X_{n,d}$ is étale (the locus of points where a morphism is étale is open by definition). Thus it suffices to find a point of $X_{n,d}$ where the fibre of $Y_{n,d} \to$

$X_{n,d}$ is nonempty and étale, see Morphisms, Lemma 36.15. We choose the point corresponding to the ring map $\chi : A \to \mathbf{Q}$ sending f_i to $1 + x_i^d$. Then

$$B \otimes_{A,\chi} \mathbf{Q} = \mathbf{Q}[x_1, \ldots, x_n]/(x_1^d - 1, \ldots, x_n^d - 1)$$

which is a nonzero étale algebra over $\mathbf{Q}$. $\square$

Lemma 10.7. *Let $f : Y \to X$ be a morphism of schemes. If f satisfies the equivalent conditions of Lemma 10.1 then for every $y \in Y$ there exist n, d and a commutative diagram*

$$
\begin{array}{ccccc}
Y & \longleftarrow & V & \longrightarrow & Y_{n,d} \\
\downarrow & & \downarrow & & \downarrow \\
X & \longleftarrow & U & \longrightarrow & X_{n,d}
\end{array}
$$

where $U \subset X$ and $V \subset Y$ are open, where $Y_{n,d} \to X_{n,d}$ is as in Example 10.5, and where the square on the right hand side is cartesian.

Proof. By Lemma 10.1 we can choose U and V affine so that $U = \mathrm{Spec}(R)$ and $V = \mathrm{Spec}(S)$ with $S = R[y_1, \ldots, y_n]/(g_1, \ldots, g_n)$. With notation as in Example 10.5 if we pick d large enough, then we can write each g_i as $g_i = \sum_{E \in T} g_{i,E} y^E$ with $g_{i,E} \in R$. Then the map $A \to R$ sending $a_{i,E}$ to $g_{i,E}$ and the map $B \to S$ sending $x_i \to y_i$ give a cocartesian diagram of rings

$$
\begin{array}{ccc}
S & \longleftarrow & B \\
\uparrow & & \uparrow \\
R & \longleftarrow & A
\end{array}
$$

which proves the lemma. $\square$

11. Finite syntomic morphisms

This section is the analogue of Section 10 for finite syntomic morphisms.

Lemma 11.1. *Let $f : Y \to X$ be a morphism of schemes. The following are equivalent*

(1) *f is finite and syntomic,*
(2) *f is finite, flat, and a local complete intersection morphism,*
(3) *f is finite, flat, locally of finite presentation, and the fibres of f are local complete intersections,*
(4) *f is finite and for every $x \in X$ there is an affine open $x \in U = \mathrm{Spec}(A) \subset X$ an integer n and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ such that $f^{-1}(U)$ is isomorphic to the spectrum of $A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$,*
(5) *f is finite, flat, locally of finite presentation, and $NL_{X/Y}$ has tor-amplitude in $[-1, 0]$, and*
(6) *f is finite, flat, locally of finite presentation, and $NL_{X/Y}$ is perfect of rank 0 with tor-amplitude in $[-1, 0]$,*

Proof. The equivalence of (1), (2), (3), (5), and (6) and the implication (4) $\Rightarrow$ (1) follow immediately from Lemma 10.1. Assume the equivalent conditions (1), (2), (3), (5), (6) hold. Choose a point $x \in X$ and an affine open $U = \mathrm{Spec}(A)$ of x in X and say x corresponds to the prime ideal $\mathfrak{p} \subset A$. Write $f^{-1}(U) = \mathrm{Spec}(B)$. Write $B = A[x_1, \ldots, x_n]/I$. Since $NL_{B/A}$ is perfect of tor-amplitude in $[-1, 0]$ by (6) we

see that I/I^2 is a finite locally free B-module of rank n. Since $B_{\mathfrak{p}}$ is semi-local we see that $(I/I^2)_{\mathfrak{p}}$ is free of rank n, see Algebra, Lemma 78.7. Thus after replacing A by a principal localization at an element not in $\mathfrak{p}$ we may assume I/I^2 is a free B-module of rank n. Thus by Algebra, Lemma 136.6 we can find a presentation of B over A with the same number of variables as equations. In other words, we may assume $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. This proves (4). $\square$

Example 11.2. Let $d \geq 1$ be an integer. Consider variables a_{ij}^l for $1 \leq i, j, l \leq d$ and denote

$$A_d = \mathbf{Z}[a_{ij}^k]/J$$

where J is the ideal generated by the elements

$$\begin{cases} \sum_l a_{ij}^l a_{lk}^m - \sum_l a_{il}^m a_{jk}^l & \forall i, j, k, m \\ a_{ij}^k - a_{ji}^k & \forall i, j, k \\ a_{i1}^j - \delta_{ij} & \forall i, j \end{cases}$$

where δ_{ij} indices the Kronecker delta function. We define an A_d-algebra B_d as follows: as an A_d-module we set

$$B_d = A_d e_1 \oplus \ldots \oplus A_d e_d$$

The algebra structure is given by $A_d \to B_d$ mapping 1 to e_1. The multiplication on B_d is the A_d-bilinar map

$$m : B_d \times B_d \longrightarrow B_d, \quad m(e_i, e_j) = \sum a_{ij}^k e_k$$

It is straightforward to check that the relations given above exactly force this to be an A_d-algebra structure. The morphism

$$\pi_d : Y_d = \operatorname{Spec}(B_d) \longrightarrow \operatorname{Spec}(A_d) = X_d$$

is the "universal" finite free morphism of rank d.

Lemma 11.3. *With notation as in Example 11.2 there is an open subscheme $U_d \subset X_d$ with the following property: a morphism of schemes $X \to X_d$ factors through U_d if and only if $Y_d \times_{X_d} X \to X$ is syntomic.*

Proof. Recall that being syntomic is the same thing as being flat and a local complete intersection morphism, see More on Morphisms, Lemma 62.8. The set $W_d \subset Y_d$ of points where π_d is Koszul is open in Y_d and its formation commutes with arbitrary base change, see More on Morphisms, Lemma 62.21. Since π_d is finite and hence closed, we see that $Z = \pi_d(Y_d \setminus W_d)$ is closed. Since clearly $U_d = X_d \setminus Z$ and since its formation commutes with base change we find that the lemma is true. $\square$

Lemma 11.4. *With notation as in Example 11.2 and U_d as in Lemma 11.3 then U_d is smooth over $\operatorname{Spec}(\mathbf{Z})$.*

Proof. Let us use More on Morphisms, Lemma 12.1 to show that $U_d \to \operatorname{Spec}(\mathbf{Z})$ is smooth. Namely, suppose that $\operatorname{Spec}(A) \to U_d$ is a morphism and $A' \to A$ is a small extension. Then $B = A \otimes_{A_d} B_d$ is a finite free A-algebra which is syntomic over A (by construction of U_d). By Smoothing Ring Maps, Proposition 3.2 there exists a syntomic ring map $A' \to B'$ such that $B \cong B' \otimes_{A'} A$. Set $e_1' = 1 \in B'$. For $1 < i \leq d$ choose lifts $e_i' \in B'$ of the elements $1 \otimes e_i \in A \otimes_{A_d} B_d = B$. Then $e_1', \ldots, e_d'$ is a basis for B' over A' (for example see Algebra, Lemma 101.1). Thus we can write $e_i' e_j' = \sum \alpha_{ij}^l e_l'$ for unique elements $\alpha_{ij}^l \in A'$ which satisfy the

relations $\sum_l \alpha_{ij}^l \alpha_{lk}^m = \sum_l \alpha_{il}^m \alpha_{jk}^l$ and $\alpha_{ij}^k = \alpha_{ji}^k$ and $\alpha_{i1}^j - \delta_{ij}$ in A'. This determines a morphism $\mathrm{Spec}(A') \to X_d$ by sending $a_{ij}^l \in A_d$ to $\alpha_{ij}^l \in A'$. This morphism agrees with the given morphism $\mathrm{Spec}(A) \to U_d$. Since $\mathrm{Spec}(A')$ and $\mathrm{Spec}(A)$ have the same underlying topological space, we see that we obtain the desired lift $\mathrm{Spec}(A') \to U_d$ and we conclude that U_d is smooth over $\mathbf{Z}$. $\qquad\square$

Lemma 11.5. *With notation as in Example 11.2 consider the open subscheme $U'_d \subset X_d$ over which π_d is étale. Then U'_d is a dense subset of the open U_d of Lemma 11.3.*

Proof. By exactly the same reasoning as in the proof of Lemma 11.3, using Morphisms, Lemma 36.17, there is a maximal open $U'_d \subset X_d$ over which π_d is étale. Moreover, since an étale morphism is syntomic, we see that $U'_d \subset U_d$. To finish the proof we have to show that $U'_d \subset U_d$ is dense. Let $u : \mathrm{Spec}(k) \to U_d$ be a morphism where k is a field. Let $B = k \otimes_{A_d} B_d$ as in the proof of Lemma 11.4. We will show there is a local domain A' with residue field k and a finite syntomic A' algebra B' with $B = k \otimes_{A'} B'$ whose generic fibre is étale. Exactly as in the previous paragraph this will determine a morphism $\mathrm{Spec}(A') \to U_d$ which will map the generic point into U'_d and the closed point to u, thereby finishing the proof.

By Lemma 11.1 part (4) we can choose a presentation $B = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. Let d' be the maximum total degree of the polynomials $f_1, \ldots, f_n$. Let $Y_{n,d'} \to X_{n,d'}$ be as in Example 10.5. By construction there is a morphism $u' : \mathrm{Spec}(k) \to X_{n,d'}$ such that
$$\mathrm{Spec}(B) \cong Y_{n,d'} \times_{X_{n,d'},u'} \mathrm{Spec}(k)$$
Denote $A = \mathcal{O}^h_{X_{n,d'},u'}$ the henselization of the local ring of $X_{n,d'}$ at the image of u'. Then we can write
$$Y_{n,d'} \times_{X_{n,d'}} \mathrm{Spec}(A) = Z \amalg W$$
with $Z \to \mathrm{Spec}(A)$ finite and $W \to \mathrm{Spec}(A)$ having empty closed fibre, see Algebra, Lemma 153.3 part (13) or the discussion in More on Morphisms, Section 41. By Lemma 10.6 the local ring A is regular (here we also use More on Algebra, Lemma 45.10) and the morphism $Z \to \mathrm{Spec}(A)$ is étale over the generic point of $\mathrm{Spec}(A)$ (because it is mapped to the generic point of $X_{d,n'}$). By construction $Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \cong \mathrm{Spec}(B)$. This proves what we want except that the map from residue field of A to k may not be an isomorphism. By Algebra, Lemma 159.1 there exists a flat local ring map $A \to A'$ such that the residue field of A' is k. If A' isn't a domain, then we choose a minimal prime $\mathfrak{p} \subset A'$ (which lies over the unique minimal prime of A by flatness) and we replace A' by $A'/\mathfrak{p}$. Set B' equal to the unique A'-algebra such that $Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A') = \mathrm{Spec}(B')$. This finishes the proof. $\qquad\square$

Remark 11.6. Let $\pi_d : Y_d \to X_d$ be as in Example 11.2. Let $U_d \subset X_d$ be the maximal open over which $V_d = \pi_d^{-1}(U_d)$ is finite syntomic as in Lemma 11.3. Then it is also true that V_d is smooth over $\mathbf{Z}$. (Of course the morphism $V_d \to U_d$ is not smooth when $d \geq 2$.) Arguing as in the proof of Lemma 11.4 this corresponds to the following deformation problem: given a small extension $C' \to C$ and a finite syntomic C-algebra B with a section $B \to C$, find a finite syntomic C'-algebra B' and a section $B' \to C'$ whose tensor product with C recovers $B \to C$. By Lemma 11.1 we may write $B = C[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ as a relative global complete intersection. After a change of coordinates we may assume $x_1, \ldots, x_n$ are in the kernel of $B \to C$. Then the polynomials f_i have vanishing constant terms.

Choose any lifts $f_i' \in C'[x_1, \ldots, x_n]$ of f_i with vanishing constant terms. Then $B' = C'[x_1, \ldots, x_n]/(f_1', \ldots, f_n')$ with section $B' \to C'$ sending x_i to zero works.

Lemma 11.7. *Let $f : Y \to X$ be a morphism of schemes. If f satisfies the equivalent conditions of Lemma 11.1 then for every $x \in X$ there exist a d and a commutative diagram*

$$
\begin{array}{ccccccc}
Y & \longleftarrow & V & \longrightarrow & V_d & \longrightarrow & Y_d \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_d} \\
X & \longleftarrow & U & \longrightarrow & U_d & \longrightarrow & X_d
\end{array}
$$

with the following properties

    (1) $U \subset X$ *is open, $x \in U$, and $V = f^{-1}(U)$,*
    (2) $\pi_d : Y_d \to X_d$ *is as in Example 11.2,*
    (3) $U_d \subset X_d$ *is as in Lemma 11.3 and $V_d = \pi_d^{-1}(U_d) \subset Y_d$,*
    (4) *where the middle square is cartesian.*

Proof. Choose an affine open neighbourhood $U = \mathrm{Spec}(A) \subset X$ of x. Write $V = f^{-1}(U) = \mathrm{Spec}(B)$. Then B is a finite locally free A-module and the inclusion $A \subset B$ is a locally direct summand. Thus after shrinking U we can choose a basis $1 = e_1, e_2, \ldots, e_d$ of B as an A-module. Write $e_i e_j = \sum \alpha_{ij}^l e_l$ for unique elements $\alpha_{ij}^l \in A$ which satisfy the relations $\sum_l \alpha_{ij}^l \alpha_{lk}^m = \sum_l \alpha_{il}^m \alpha_{jk}^l$ and $\alpha_{ij}^k = \alpha_{ji}^k$ and $\alpha_{i1}^j - \delta_{ij}$ in A. This determines a morphism $\mathrm{Spec}(A) \to X_d$ by sending $a_{ij}^l \in A_d$ to $\alpha_{ij}^l \in A$. By construction $V \cong \mathrm{Spec}(A) \times_{X_d} Y_d$. By the definition of U_d we see that $\mathrm{Spec}(A) \to X_d$ factors through U_d. This finishes the proof. $\qquad\square$

12. A formula for the different

In this section we discuss the material in [MR70, Appendix A] due to Tate. In our language, this will show that the different is equal to the Kähler different in the case of a flat, quasi-finite, local complete intersection morphism. First we compute the Noether different in a special case.

Lemma 12.1. *Let $A \to P$ be a ring map. Let $f_1, \ldots, f_n \in P$ be a Koszul regular sequence. Assume $B = P/(f_1, \ldots, f_n)$ is flat over A. Let $g_1, \ldots, g_n \in P \otimes_A B$ be a Koszul regular sequence generating the kernel of the multiplication map $P \otimes_A B \to B$. Write $f_i \otimes 1 = \sum g_{ij} g_j$. Then the annihilator of $\mathrm{Ker}(B \otimes_A B \to B)$ is a principal ideal generated by the image of $\det(g_{ij})$.*

Proof. The Koszul complex $K_\bullet = K(P, f_1, \ldots, f_n)$ is a resolution of B by finite free P-modules. The Koszul complex $M_\bullet = K(P \otimes_A B, g_1, \ldots, g_n)$ is a resolution of B by finite free $P \otimes_A B$-modules. There is a map of complexes

$$ K_\bullet \longrightarrow M_\bullet $$

which in degree 1 is given by the matrix (g_{ij}) and in degree n by $\det(g_{ij})$. See More on Algebra, Lemma 28.3. As B is a flat A-module, we can view $M_\bullet$ as a complex of flat P-modules (via $P \to P \otimes_A B$, $p \mapsto p \otimes 1$). Thus we may use both complexes to compute $\mathrm{Tor}_*^P(B, B)$ and it follows that the displayed map defines a quasi-isomorphism after tensoring with B. It is clear that $H_n(K_\bullet \otimes_P B) = B$. On the other hand, $H_n(M_\bullet \otimes_P B)$ is the kernel of

$$ B \otimes_A B \xrightarrow{\ g_1, \ldots, g_n\ } (B \otimes_A B)^{\oplus n} $$

Since $g_1, \ldots, g_n$ generate the kernel of $B \otimes_A B \to B$ this proves the lemma. $\qquad\square$

Lemma 12.2. *Let A be a ring. Let $n \geq 1$ and $h, f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$. Set $B = A[x_1, \ldots, x_n, 1/h]/(f_1, \ldots, f_n)$. Assume that B is quasi-finite over A. Then*

 (1) *B is flat over A and $A \to B$ is a relative local complete intersection,*
 (2) *the annihilator J of $I = \mathrm{Ker}(B \otimes_A B \to B)$ is free of rank 1 over B,*
 (3) *the Noether different of B over A is generated by $\det(\partial f_i/\partial x_j)$ in B.*

Proof. Note that $B = A[x, x_1, \ldots, x_n]/(xh - 1, f_1, \ldots, f_n)$ is a relative global complete intersection over A, see Algebra, Definition 136.5. By Algebra, Lemma 136.13 we see that B is flat over A.

Write $P' = A[x, x_1, \ldots, x_n]$ and $P = P'/(xh - 1) = A[x_1, \ldots, x_n, 1/g]$. Then we have $P' \to P \to B$. By More on Algebra, Lemma 33.4 we see that $xh - 1, f_1, \ldots, f_n$ is a Koszul regular sequence in P'. Since $xh - 1$ is a Koszul regular sequence of length one in P' (by the same lemma for example) we conclude that $f_1, \ldots, f_n$ is a Koszul regular sequence in P by More on Algebra, Lemma 30.14.

Let $g_i \in P \otimes_A B$ be the image of $x_i \otimes 1 - 1 \otimes x_i$. Let us use the short hand $y_i = x_i \otimes 1$ and $z_i = 1 \otimes x_i$ in $A[x_1, \ldots, x_n] \otimes_A A[x_1, \ldots, x_n]$ so that g_i is the image of $y_i - z_i$. For a polynomial $f \in A[x_1, \ldots, x_n]$ we write $f(y) = f \otimes 1$ and $f(z) = 1 \otimes f$ in the above tensor product. Then we have

$$P \otimes_A B/(g_1, \ldots, g_n) = \frac{A[y_1, \ldots, y_n, z_1, \ldots, z_n, \frac{1}{h(y)h(z)}]}{(f_1(z), \ldots, f_n(z), y_1 - z_1, \ldots, y_n - z_n)}$$

which is clearly isomorphic to B. Hence by the same arguments as above we find that $f_1(z), \ldots, f_n(z), y_1 - z_1, \ldots, y_n - z_n$ is a Koszul regular sequence in $A[y_1, \ldots, y_n, z_1, \ldots, z_n, \frac{1}{h(y)h(z)}]$. The sequence $f_1(z), \ldots, f_n(z)$ is a Koszul regular in $A[y_1, \ldots, y_n, z_1, \ldots, z_n, \frac{1}{h(y)h(z)}]$ by flatness of the map

$$P \longrightarrow A[y_1, \ldots, y_n, z_1, \ldots, z_n, \tfrac{1}{h(y)h(z)}], \quad x_i \longmapsto z_i$$

and More on Algebra, Lemma 30.5. By More on Algebra, Lemma 30.14 we conclude that $g_1, \ldots, g_n$ is a regular sequence in $P \otimes_A B$.

At this point we have verified all the assumptions of Lemma 12.1 above with P, $f_1, \ldots, f_n$, and $g_i \in P \otimes_A B$ as above. In particular the annihilator J of I is freely generated by one element δ over B. Set $f_{ij} = \partial f_i/\partial x_j \in A[x_1, \ldots, x_n]$. An elementary computation shows that we can write

$$f_i(y) = f_i(z_1 + g_1, \ldots, z_n + g_n) = f_i(z) + \sum_j f_{ij}(z) g_j + \sum_{j,j'} F_{ijj'} g_j g_{j'}$$

for some $F_{ijj'} \in A[y_1, \ldots, y_n, z_1, \ldots, z_n]$. Taking the image in $P \otimes_A B$ the terms $f_i(z)$ map to zero and we obtain

$$f_i \otimes 1 = \sum_j \left(1 \otimes f_{ij} + \sum_{j'} F_{ijj'} g_{j'} \right) g_j$$

Thus we conclude from Lemma 12.1 that $\delta = \det(g_{ij})$ with $g_{ij} = 1 \otimes f_{ij} + \sum_{j'} F_{ijj'} g_{j'}$. Since $g_{j'}$ maps to zero in B, we conclude that the image of $\det(\partial f_i/\partial x_j)$ in B generates the Noether different of B over A. $\qquad\square$

Lemma 12.3. *Let $f : Y \to X$ be a morphism of Noetherian schemes. If f satisfies the equivalent conditions of Lemma 10.1 then the different $\mathfrak{D}_f$ of f is the Kähler different of f.*

Proof. By Lemmas 9.3 and 10.4 the different of f affine locally is the same as the Noether different. Then the lemma follows from the computation of the Noether different and the Kähler different on standard affine pieces done in Lemmas 7.4 and 12.2. $\square$

Lemma 12.4. *Let A be a ring. Let $n \geq 1$ and $h, f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$. Set $B = A[x_1, \ldots, x_n, 1/h]/(f_1, \ldots, f_n)$. Assume that B is quasi-finite over A. Then there is an isomorphism $B \to \omega_{B/A}$ mapping $\det(\partial f_i/\partial x_j)$ to $\tau_{B/A}$.*

Proof. Let J be the annihilator of $\operatorname{Ker}(B \otimes_A B \to B)$. By Lemma 12.2 the map $A \to B$ is flat and J is a free B-module with generator ξ mapping to $\det(\partial f_i/\partial x_j)$ in B. Thus the lemma follows from Lemma 6.7 and the fact (Lemma 10.4) that $\omega_{B/A}$ is an invertible B-module. (Warning: it is necessary to prove $\omega_{B/A}$ is invertible because a finite B-module M such that $\operatorname{Hom}_B(M, B) \cong B$ need not be free.) $\square$

Example 12.5. Let A be a Noetherian ring. Let $f, h \in A[x]$ such that

$$B = (A[x]/(f))_h = A[x, 1/h]/(f)$$

is quasi-finite over A. Let $f' \in A[x]$ be the derivative of f with respect to x. The ideal $\mathfrak{D} = (f') \subset B$ is the Noether different of B over A, is the Kähler different of B over A, and is the ideal whose associated quasi-coherent sheaf of ideals is the different of $\operatorname{Spec}(B)$ over $\operatorname{Spec}(A)$.

Lemma 12.6. *Let S be a Noetherian scheme. Let X, Y be smooth schemes of relative dimension n over S. Let $f : Y \to X$ be a locally quasi-finite morphism over S. Then f is flat and the closed subscheme $R \subset Y$ cut out by the different of f is the locally principal closed subscheme cut out by*

$$\wedge^n(df) \in \Gamma(Y, (f^*\Omega^n_{X/S})^{\otimes -1} \otimes_{\mathcal{O}_Y} \Omega^n_{Y/S})$$

If f is étale at the associated points of Y, then R is an effective Cartier divisor and

$$f^*\Omega^n_{X/S} \otimes_{\mathcal{O}_Y} \mathcal{O}(R) = \Omega^n_{Y/S}$$

as invertible sheaves on Y.

Proof. To prove that f is flat, it suffices to prove $Y_s \to X_s$ is flat for all $s \in S$ (More on Morphisms, Lemma 16.3). Flatness of $Y_s \to X_s$ follows from Algebra, Lemma 128.1. By More on Morphisms, Lemma 62.10 the morphism f is a local complete intersection morphism. Thus the statement on the different follows from the corresponding statement on the Kähler different by Lemma 12.3. Finally, since we have the exact sequence

$$f^*\Omega_{X/S} \xrightarrow{\mathrm{d}f} \Omega_{Y/S} \to \Omega_{Y/X} \to 0$$

by Morphisms, Lemma 32.9 and since $\Omega_{X/S}$ and $\Omega_{Y/S}$ are finite locally free of rank n (Morphisms, Lemma 34.12), the statement for the Kähler different is clear from the definition of the zeroth fitting ideal. If f is étale at the associated points of Y, then $\wedge^n \mathrm{d}f$ does not vanish in the associated points of Y, which implies that the local equation of R is a nonzerodivisor. Hence R is an effective Cartier divisor. The canonical isomorphism sends 1 to $\wedge^n \mathrm{d}f$, see Divisors, Lemma 14.10. $\square$

13. The Tate map

In this section we produce an isomorphism between the determinant of the relative cotangent complex and the relative dualizing module for a locally quasi-finite syntomic morphism of locally Noetherian schemes. Following [Gar84, 1.4.4] we dub the isomorphism the Tate map. Our approach is to avoid doing local calculations as much as is possible.

Let $Y \to X$ be a locally quasi-finite syntomic morphism of schemes. We will use all the equivalent conditions for this notion given in Lemma 10.1 without further mention in this section. In particular, we see that $NL_{Y/X}$ is a perfect object of $D(\mathcal{O}_Y)$ with tor-amplitude in $[-1, 0]$. Thus we have a canonical invertible module $\det(NL_{Y/X})$ on Y and a global section

$$\delta(NL_{Y/X}) \in \Gamma(Y, \det(NL_{Y/X}))$$

See Derived Categories of Schemes, Lemma 39.1. Suppose given a commutative diagram of schemes

$$\begin{array}{ccc} Y' & \xrightarrow{\quad b \quad} & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

whose vertical arrows are locally quasi-finite syntomic and which induces an isomorphism of Y' with an open of $X' \times_X Y$. Then the canonical map

$$Lb^* NL_{Y/X} \longrightarrow NL_{Y'/X'}$$

is a quasi-isomorphism by More on Morphisms, Lemma 13.16. Thus we get a canonical isomorphism $b^* \det(NL_{Y/X}) \to \det(NL_{Y'/X'})$ which sends the canonical section $\delta(NL_{Y/X})$ to $\delta(NL_{Y'/X'})$, see Derived Categories of Schemes, Remark 39.2.

Remark 13.1. Let $Y \to X$ be a locally quasi-finite syntomic morphism of schemes. What does the pair $(\det(NL_{Y/X}), \delta(NL_{Y/X}))$ look like locally? Choose affine opens $V = \mathrm{Spec}(B) \subset Y$, $U = \mathrm{Spec}(A) \subset X$ with $f(V) \subset U$ and an integer n and $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ such that $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. Then

$$NL_{B/A} = \left((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \longrightarrow \bigoplus_{i=1,\ldots,n} B \, \mathrm{d}x_i \right)$$

and $(f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2$ is free with generators the classes $\overline{f}_i$. See proof of Lemma 10.1. Thus $\det(L_{B/A})$ is free on the generator

$$\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \otimes (\overline{f}_1 \wedge \ldots \wedge \overline{f}_n)^{\otimes -1}$$

and the section $\delta(NL_{B/A})$ is the element

$$\delta(NL_{B/A}) = \det(\partial f_j/\partial x_i) \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \otimes (\overline{f}_1 \wedge \ldots \wedge \overline{f}_n)^{\otimes -1}$$

by definition.

Let $Y \to X$ be a locally quasi-finite syntomic morphism of locally Noetherian schemes. By Remarks 2.11 and 4.7 we have a coherent $\mathcal{O}_Y$-module $\omega_{Y/X}$ and a canonical global section

$$\tau_{Y/X} \in \Gamma(Y, \omega_{Y/X})$$

which affine locally recovers the pair $\omega_{B/A}, \tau_{B/A}$. By Lemma 10.4 the module $\omega_{Y/X}$ is invertible. Suppose given a commutative diagram of locally Noetherian schemes

$$\begin{array}{ccc} Y' & \xrightarrow{\ b\ } & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

whose vertical arrows are locally quasi-finite syntomic and which induces an isomorphism of Y' with an open of $X' \times_X Y$. Then there is a canonical base change map

$$b^* \omega_{Y/X} \longrightarrow \omega_{Y'/X'}$$

which is an isomorphism mapping $\tau_{Y/X}$ to $\tau_{Y'/X'}$. Namely, the base change map in the affine setting is (2.3.1), it is an isomorphism by Lemma 2.10, and it maps $\tau_{Y/X}$ to $\tau_{Y'/X'}$ by Lemma 4.4 part (1).

Proposition 13.2. *There exists a unique rule that to every locally quasi-finite syntomic morphism of locally Noetherian schemes $Y \to X$ assigns an isomorphism*

$$c_{Y/X} : \det(NL_{Y/X}) \longrightarrow \omega_{Y/X}$$

satisfying the following two properties

(1) *the section $\delta(NL_{Y/X})$ is mapped to $\tau_{Y/X}$, and*
(2) *the rule is compatible with restriction to opens and with base change.*

Proof. Let us reformulate the statement of the proposition. Consider the category $\mathcal{C}$ whose objects, denoted Y/X, are locally quasi-finite syntomic morphism $Y \to X$ of locally Noetherian schemes and whose morphisms $b/a : Y'/X' \to Y/X$ are commutative diagrams

$$\begin{array}{ccc} Y' & \xrightarrow{\ b\ } & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\ a\ } & X \end{array}$$

which induce an isomorphism of Y' with an open subscheme of $X' \times_X Y$. The proposition means that for every object Y/X of $\mathcal{C}$ we have an isomorphism $c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$ with $c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X}$ and for every morphism $b/a : Y'/X' \to Y/X$ of $\mathcal{C}$ we have $b^* c_{Y/X} = c_{Y'/X'}$ via the identifications $b^* \det(NL_{Y/X}) = \det(NL_{Y'/X'})$ and $b^* \omega_{Y/X} = \omega_{Y'/X'}$ described above.

Given Y/X in $\mathcal{C}$ and $y \in Y$ we can find an affine open $V \subset Y$ and $U \subset X$ with $f(V) \subset U$ such that there exists some isomorphism

$$\det(NL_{Y/X})|_V \longrightarrow \omega_{Y/X}|_V$$

mapping $\delta(NL_{Y/X})|_V$ to $\tau_{Y/X}|_V$. This follows from picking affine opens as in Lemma 10.1 part (5), the affine local description of $\delta(NL_{Y/X})$ in Remark 13.1, and Lemma 12.4. If the annihilator of the section $\tau_{Y/X}$ is zero, then these local maps are unique and automatically glue. Hence if the annihilator of $\tau_{Y/X}$ is zero, then there is a unique isomorphism $c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$ with $c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X}$. If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}$ and the annihilator of $\tau_{Y'/X'}$ is zero as well, then $b^* c_{Y/X}$ is the unique isomorphism $c_{Y'/X'} : \det(NL_{Y'/X'}) \to \omega_{Y'/X'}$ with $c_{Y'/X'}(\delta(NL_{Y'/X'})) = \tau_{Y'/X'}$. This follows formally from the fact that $b^* \delta(NL_{Y/X}) = \delta(NL_{Y'/X'})$ and $b^* \tau_{Y/X} = \tau_{Y'/X'}$.

We can summarize the results of the previous paragraph as follows. Let $\mathcal{C}_{nice} \subset \mathcal{C}$ denote the full subcategory of Y/X such that the annihilator of $\tau_{Y/X}$ is zero. Then we have solved the problem on $\mathcal{C}_{nice}$. For Y/X in $\mathcal{C}_{nice}$ we continue to denote $c_{Y/X}$ the solution we've just found.

Consider morphisms

$$Y_1/X_1 \xleftarrow{b_1/a_1} Y/X \xrightarrow{b_2/a_2} Y_2/X_2$$

in $\mathcal{C}$ such that Y_1/X_1 and Y_2/X_2 are objects of $\mathcal{C}_{nice}$. **Claim.** $b_1^* c_{Y_1/X_1} = b_2^* c_{Y_2/X_2}$. We will first show that the claim implies the proposition and then we will prove the claim.

Let $d, n \geq 1$ and consider the locally quasi-finite syntomic morphism $Y_{n,d} \to X_{n,d}$ constructed in Example 10.5. Then $Y_{n,d}$ is an irreducible regular scheme and the morphism $Y_{n,d} \to X_{n,d}$ is locally quasi-finite syntomic and étale over a dense open, see Lemma 10.6. Thus $\tau_{Y_{n,d}/X_{n,d}}$ is nonzero for example by Lemma 9.6. Now a nonzero section of an invertible module over an irreducible regular scheme has vanishing annihilator. Thus $Y_{n,d}/X_{n,d}$ is an object of $\mathcal{C}_{nice}$.

Let Y/X be an arbitrary object of $\mathcal{C}$. Let $y \in Y$. By Lemma 10.7 we can find $n, d \geq 1$ and morphisms

$$Y/X \leftarrow V/U \xrightarrow{b/a} Y_{n,d}/X_{n,d}$$

of $\mathcal{C}$ such that $V \subset Y$ and $U \subset X$ are open. Thus we can pullback the canonical morphism $c_{Y_{n,d}/X_{n,d}}$ constructed above by b to V. The claim guarantees these local isomorphisms glue! Thus we get a well defined global isomorphism $c_{Y/X} : \det(NL_{Y/X}) \to \omega_{Y/X}$ with $c_{Y/X}(\delta(NL_{Y/X})) = \tau_{Y/X}$. If $b/a : Y'/X' \to Y/X$ is a morphism of $\mathcal{C}$, then the claim also implies that the similarly constructed map $c_{Y'/X'}$ is the pullback by b of the locally constructed map $c_{Y/X}$. Thus it remains to prove the claim.

In the rest of the proof we prove the claim. We may pick a point $y \in Y$ and prove the maps agree in an open neighbourhood of y. Thus we may replace Y_1, Y_2 by open neighbourhoods of the image of y in Y_1 and Y_2. Thus we may assume there are morphisms

$$Y_{n_1,d_1}/X_{n_1,d_1} \leftarrow Y_1/X_1 \quad \text{and} \quad Y_2/X_2 \to Y_{n_2,d_2}/X_{n_2,d_2}$$

These are morphisms of $\mathcal{C}_{nice}$ for which we know the desired compatibilities. Thus we may replace Y_1/X_1 by $Y_{n_1,d_1}/X_{n_1,d_1}$ and Y_2/X_2 by $Y_{n_2,d_2}/X_{n_2,d_2}$. This reduces us to the case that Y_1, X_1, Y_2, X_2 are of finite type over $\mathbf{Z}$. (The astute reader will realize that this step wouldn't have been necessary if we'd defined $\mathcal{C}_{nice}$ to consist only of those objects Y/X with Y and X of finite type over $\mathbf{Z}$.)
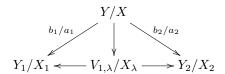
Assume Y_1, X_1, Y_2, X_2 are of finite type over $\mathbf{Z}$. After replacing Y, X, Y_1, X_1, Y_2, X_2 by suitable open neighbourhoods of the image of y we may assume Y, X, Y_1, X_1, Y_2, X_2 are affine. We may write $X = \lim X_\lambda$ as a cofiltered limit of affine schemes of finite type over $X_1 \times X_2$. For each λ we get

$$Y_1 \times_{X_1} X_\lambda \quad \text{and} \quad X_\lambda \times_{X_2} Y_2$$

If we take limits we obtain

$$\lim Y_1 \times_{X_1} X_\lambda = Y_1 \times_{X_1} X \supset Y \subset X \times_{X_2} Y_2 = \lim X_\lambda \times_{X_2} Y_2$$

By Limits, Lemma 4.11 we can find a λ and opens $V_{1,\lambda} \subset Y_1 \times_{X_1} X_\lambda$ and $V_{2,\lambda} \subset X_\lambda \times_{X_2} Y_2$ whose base change to X recovers Y (on both sides). After increasing λ we may assume there is an isomorphism $V_{1,\lambda} \to V_{2,\lambda}$ whose base change to X is the identity on Y, see Limits, Lemma 10.1. Then we have the commutative diagram

$$
\begin{array}{ccc}
& Y/X & \\
{}^{b_1/a_1}\swarrow & \downarrow & \searrow^{b_2/a_2} \\
Y_1/X_1 \longleftarrow V_{1,\lambda}/X_\lambda & \longrightarrow & Y_2/X_2
\end{array}
$$

Thus it suffices to prove the claim for the lower row of the diagram and we reduce to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$. Write $X = \mathrm{Spec}(A)$, $X_i = \mathrm{Spec}(A_i)$. The ring map $A_1 \to A$ corresponding to $X \to X_1$ is of finite type and hence we may choose a surjection $A_1[x_1, \ldots, x_n] \to A$. Similarly, we may choose a surjection $A_2[y_1, \ldots, y_m] \to A$. Set $X_1' = \mathrm{Spec}(A_1[x_1, \ldots, x_n])$ and $X_2' = \mathrm{Spec}(A_2[y_1, \ldots, y_m])$. Set $Y_1' = Y_1 \times_{X_1} X_1'$ and $Y_2' = Y_2 \times_{X_2} X_2'$. We get the following diagram

$$
Y_1/X_1 \leftarrow Y_1'/X_1' \leftarrow Y/X \to Y_2'/X_2' \to Y_2/X_2
$$

Since $X_1' \to X_1$ and $X_2' \to X_2$ are flat, the same is true for $Y_1' \to Y_1$ and $Y_2' \to Y_2$. It follows easily that the annihilators of $\tau_{Y_1'/X_1'}$ and $\tau_{Y_2'/X_2'}$ are zero. Hence Y_1'/X_1' and Y_2'/X_2' are in $\mathcal{C}_{nice}$. Thus the outer morphisms in the displayed diagram are morphisms of $\mathcal{C}_{nice}$ for which we know the desired compatibilities. Thus it suffices to prove the claim for $Y_1'/X_1' \leftarrow Y/X \to Y_2'/X_2'$. This reduces us to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$ and $X \to X_1$ and $X \to X_2$ are closed immersions. Consider the open embeddings $Y_1 \times_{X_1} X \supset Y \subset X \times_{X_2} Y_2$. There is an open neighbourhood $V \subset Y$ of y which is a standard open of both $Y_1 \times_{X_1} X$ and $X \times_{X_2} Y_2$. This follows from Schemes, Lemma 11.5 applied to the scheme obtained by glueing $Y_1 \times_{X_1} X$ and $X \times_{X_2} Y_2$ along Y; details omitted. Since $X \times_{X_2} Y_2$ is a closed subscheme of Y_2 we can find a standard open $V_2 \subset Y_2$ such that $V_2 \times_{X_2} X = V$. Similarly, we can find a standard open $V_1 \subset Y_1$ such that $V_1 \times_{X_1} X = V$. After replacing Y, Y_1, Y_2 by V, V_1, V_2 we reduce to the case discussed in the next paragraph.

Assume Y, X, Y_1, X_1, Y_2, X_2 are affine of finite type over $\mathbf{Z}$ and $X \to X_1$ and $X \to X_2$ are closed immersions and $Y_1 \times_{X_1} X = Y = X \times_{X_2} Y_2$. Write $X = \mathrm{Spec}(A)$, $X_i = \mathrm{Spec}(A_i)$, $Y = \mathrm{Spec}(B)$, $Y_i = \mathrm{Spec}(B_i)$. Then we can consider the affine schemes
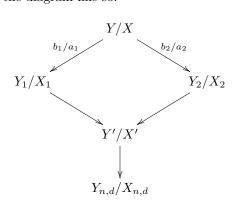
$$
X' = \mathrm{Spec}(A_1 \times_A A_2) = \mathrm{Spec}(A') \quad \text{and} \quad Y' = \mathrm{Spec}(B_1 \times_B B_2) = \mathrm{Spec}(B')
$$

Observe that $X' = X_1 \amalg_X X_2$ and $Y' = Y_1 \amalg_Y Y_2$, see More on Morphisms, Lemma 14.1. By More on Algebra, Lemma 5.1 the rings A' and B' are of finite type over $\mathbf{Z}$. By More on Algebra, Lemma 6.4 we have $B' \otimes_A A_1 = B_1$ and $B' \times_A A_2 = B_2$. In particular a fibre of $Y' \to X'$ over a point of $X' = X_1 \amalg_X X_2$ is always equal to either a fibre of $Y_1 \to X_1$ or a fibre of $Y_2 \to X_2$. By More on Algebra, Lemma 6.8 the ring map $A' \to B'$ is flat. Thus by Lemma 10.1 part (3) we conclude that

Y'/X' is an object of $\mathcal{C}$. Consider now the commutative diagram

$$
\begin{array}{ccc}
 & Y/X & \\
{\scriptstyle b_1/a_1}\swarrow & & \searrow{\scriptstyle b_2/a_2} \\
Y_1/X_1 & & Y_2/X_2 \\
\searrow & & \swarrow \\
 & Y'/X' &
\end{array}
$$

Now we would be done if Y'/X' is an object of $\mathcal{C}_{nice}$. Namely, then pulling back $c_{Y'/X'}$ around the two sides of the square, we would obtain the desired conclusion. Now, in fact, it is true that Y'/X' is an object of $\mathcal{C}_{nice}$[4]. But it is amusing to note that we don't even need this. Namely, the arguments above show that, after possibly shrinking all of the schemes $X, Y, X_1, Y_1, X_2, Y_2, X', Y'$ we can find some $n, d \geq 1$, and extend the diagram like so:

$$
\begin{array}{ccc}
 & Y/X & \\
{\scriptstyle b_1/a_1}\swarrow & & \searrow{\scriptstyle b_2/a_2} \\
Y_1/X_1 & & Y_2/X_2 \\
\searrow & & \swarrow \\
 & Y'/X' & \\
 & \downarrow & \\
 & Y_{n,d}/X_{n,d} &
\end{array}
$$

and then we can use the already given argument by pulling back from $c_{Y_{n,d}/X_{n,d}}$. This finishes the proof. $\square$

14. A generalization of the different

In this section we generalize Definition 9.1 to take into account all cases of ring maps $A \to B$ where the Dedekind different is defined and $1 \in \mathcal{L}_{B/A}$. First we explain the condition "$A \to B$ maps nonzerodivisors to nonzerodivisors and induces a flat map $Q(A) \to Q(A) \otimes_A B$".

Lemma 14.1. *Let $A \to B$ be a map of Noetherian rings. Consider the conditions*

(1) *nonzerodivisors of A map to nonzerodivisors of B,*
(2) *(1) holds and $Q(A) \to Q(A) \otimes_A B$ is flat,*
(3) *$A \to B_{\mathfrak{q}}$ is flat for every $\mathfrak{q} \in Ass(B)$,*
(4) *(3) holds and $A \to B_{\mathfrak{q}}$ is flat for every $\mathfrak{q}$ lying over an element in $Ass(A)$.*

---

[4] Namely, the structure sheaf $\mathcal{O}_{Y'}$ is a subsheaf of $(Y_1 \to Y')_* \mathcal{O}_{Y_1} \times (Y_2 \to Y')_* \mathcal{O}_{Y_2}$.

Then we have the following implications

$$
\begin{array}{ccc}
(1) & \Longleftarrow & (2) \\
\Big\Uparrow & & \Big\Downarrow \\
(3) & \Longleftarrow & (4)
\end{array}
$$

If going up holds for $A \to B$ then (2) and (4) are equivalent.

Proof. The horizontal implications in the diagram are trivial. Let $S \subset A$ be the set of nonzerodivisors so that $Q(A) = S^{-1}A$ and $Q(A) \otimes_A B = S^{-1}B$. Recall that $S = A \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p}$ by Algebra, Lemma 63.9. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$.

Assume (2). If $\mathfrak{q} \in \mathrm{Ass}(B)$ then $\mathfrak{q}$ consists of zerodivisors, hence (1) implies the same is true for $\mathfrak{p}$. Hence $\mathfrak{p}$ corresponds to a prime of $S^{-1}A$. Hence $A \to B_{\mathfrak{q}}$ is flat by our assumption (2). If $\mathfrak{q}$ lies over an associated prime $\mathfrak{p}$ of A, then certainly $\mathfrak{p} \in \mathrm{Spec}(S^{-1}A)$ and the same argument works.

Assume (3). Let $f \in A$ be a nonzerodivisor. If f were a zerodivisor on B, then f is contained in an associated prime $\mathfrak{q}$ of B. Since $A \to B_{\mathfrak{q}}$ is flat by assumption, we conclude that $\mathfrak{p}$ is an associated prime of A by Algebra, Lemma 65.3. This would imply that f is a zerodivisor on A, a contradiction.

Assume (4) and going up for $A \to B$. We already know (1) holds. If $\mathfrak{q}$ corresponds to a prime of $S^{-1}B$ then $\mathfrak{p}$ is contained in an associated prime $\mathfrak{p}'$ of A. By going up there exists a prime $\mathfrak{q}'$ containing $\mathfrak{q}$ and lying over $\mathfrak{p}$. Then $A \to B_{\mathfrak{q}'}$ is flat by (4). Hence $A \to B_{\mathfrak{q}}$ is flat as a localization. Thus $A \to S^{-1}B$ is flat and so is $S^{-1}A \to S^{-1}B$, see Algebra, Lemma 39.18. $\square$

Remark 14.2. We can generalize Definition 9.1. Suppose that $f : Y \to X$ is a quasi-finite morphism of Noetherian schemes with the following properties

(1) the open $V \subset Y$ where f is flat contains $\mathrm{Ass}(\mathcal{O}_Y)$ and $f^{-1}(\mathrm{Ass}(\mathcal{O}_X))$,
(2) the trace element $\tau_{V/X}$ comes from a section $\tau \in \Gamma(Y, \omega_{Y/X})$.

Condition (1) implies that V contains the associated points of $\omega_{Y/X}$ by Lemma 2.8. In particular, τ is unique if it exists (Divisors, Lemma 2.8). Given τ we can define the different $\mathfrak{D}_f$ as the annihilator of $\mathrm{Coker}(\tau : \mathcal{O}_Y \to \omega_{Y/X})$. This agrees with the Dedekind different in many cases (Lemma 14.3). However, for non-flat maps between non-normal rings, this generalization no longer measures ramification of the morphism, see Example 14.4.

Lemma 14.3. *Assume the Dedekind different is defined for $A \to B$. Set $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. The generalization of Remark 14.2 applies to the morphism $f : Y \to X$ if and only if $1 \in \mathcal{L}_{B/A}$ (e.g., if A is normal, see Lemma 8.1). In this case $\mathfrak{D}_{B/A}$ is an ideal of B and we have*

$$
\mathfrak{D}_f = \widetilde{\mathfrak{D}_{B/A}}
$$

as coherent ideal sheaves on Y.

Proof. As the Dedekind different for $A \to B$ is defined we can apply Lemma 14.1 to see that $Y \to X$ satisfies condition (1) of Remark 14.2. Recall that there is

a canonical isomorphism $c : \mathcal{L}_{B/A} \to \omega_{B/A}$, see Lemma 8.2. Let $K = Q(A)$ and $L = K \otimes_A B$ as above. By construction the map c fits into a commutative diagram

$$
\begin{array}{ccc}
\mathcal{L}_{B/A} & \longrightarrow & L \\
{\scriptstyle c} \downarrow & & \downarrow \\
\omega_{B/A} & \longrightarrow & \mathrm{Hom}_K(L, K)
\end{array}
$$

where the right vertical arrow sends $x \in L$ to the map $y \mapsto \mathrm{Trace}_{L/K}(xy)$ and the lower horizontal arrow is the base change map (2.3.1) for $\omega_{B/A}$. We can factor the lower horizontal map as

$$\omega_{B/A} = \Gamma(Y, \omega_{Y/X}) \to \Gamma(V, \omega_{V/X}) \to \mathrm{Hom}_K(L, K)$$

Since all associated points of $\omega_{V/X}$ map to associated primes of A (Lemma 2.8) we see that the second map is injective. The element $\tau_{V/X}$ maps to $\mathrm{Trace}_{L/K}$ in $\mathrm{Hom}_K(L, K)$ by the very definition of trace elements (Definition 4.1). Thus τ as in condition (2) of Remark 14.2 exists if and only if $1 \in \mathcal{L}_{B/A}$ and then $\tau = c(1)$. In this case, by Lemma 8.1 we see that $\mathfrak{D}_{B/A} \subset B$. Finally, the agreement of $\mathfrak{D}_f$ with $\mathfrak{D}_{B/A}$ is immediate from the definitions and the fact $\tau = c(1)$ seen above. $\square$

Example 14.4. Let k be a field. Let $A = k[x, y]/(xy)$ and $B = k[u, v]/(uv)$ and let $A \to B$ be given by $x \mapsto u^n$ and $y \mapsto v^m$ for some $n, m \in \mathbf{N}$ prime to the characteristic of k. Then $A_{x+y} \to B_{x+y}$ is (finite) étale hence we are in the situation where the Dedekind different is defined. A computation shows that

$$\mathrm{Trace}_{L/K}(1) = (nx + my)/(x + y), \quad \mathrm{Trace}_{L/K}(u^i) = 0, \quad \mathrm{Trace}_{L/K}(v^j) = 0$$

for $1 \le i < n$ and $1 \le j < m$. We conclude that $1 \in \mathcal{L}_{B/A}$ if and only if $n = m$. Moreover, a computation shows that if $n = m$, then $\mathcal{L}_{B/A} = B$ and the Dedekind different is B as well. In other words, we find that the different of Remark 14.2 is defined for $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ if and only if $n = m$, and in this case the different is the unit ideal. Thus we see that in nonflat cases the nonvanishing of the different does not guarantee the morphism is étale or unramified.

15. Comparison with duality theory

In this section we compare the elementary algebraic constructions above with the constructions in the chapter on duality theory for schemes.

Lemma 15.1. *Let $f : Y \to X$ be a quasi-finite separated morphism of Noetherian schemes. For every pair of affine opens $\mathrm{Spec}(B) = V \subset Y$, $\mathrm{Spec}(A) = U \subset X$ with $f(V) \subset U$ there is an isomorphism*

$$H^0(V, f^! \mathcal{O}_Y) = \omega_{B/A}$$

where $f^!$ is as in Duality for Schemes, Section 16. These isomorphisms are compatible with restriction maps and define a canonical isomorphism $H^0(f^! \mathcal{O}_X) = \omega_{Y/X}$ with $\omega_{Y/X}$ as in Remark 2.11. Similarly, if $f : Y \to X$ is a quasi-finite morphism of schemes of finite type over a Noetherian base S endowed with a dualizing complex $\omega_S^\bullet$, then $H^0(f^!_{new} \mathcal{O}_X) = \omega_{Y/X}$.

Proof. By Zariski's main theorem we can choose a factorization $f = f' \circ j$ where $j : Y \to Y'$ is an open immersion and $f' : Y' \to X$ is a finite morphism, see More on Morphisms, Lemma 43.3. By our construction in Duality for Schemes, Lemma 16.2

we have $f^! = j^* \circ a'$ where $a' : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{Y'})$ is the right adjoint to Rf'_* of Duality for Schemes, Lemma 3.1. By Duality for Schemes, Lemma 11.4 we see that $\Phi(a'(\mathcal{O}_X)) = R\mathcal{H}om(f'_*\mathcal{O}_{Y'}, \mathcal{O}_X)$ in $D^+_{QCoh}(f'_*\mathcal{O}_{Y'})$. In particular $a'(\mathcal{O}_X)$ has vanishing cohomology sheaves in degrees < 0. The zeroth cohomology sheaf is determined by the isomorphism

$$f'_* H^0(a'(\mathcal{O}_X)) = \mathcal{H}om_{\mathcal{O}_X}(f'_*\mathcal{O}_{Y'}, \mathcal{O}_X)$$

as $f'_*\mathcal{O}_{Y'}$-modules via the equivalence of Morphisms, Lemma 11.6. Writing $(f')^{-1}U = V' = \mathrm{Spec}(B')$, we obtain

$$H^0(V', a'(\mathcal{O}_X)) = \mathrm{Hom}_A(B', A).$$

As the zeroth cohomology sheaf of $a'(\mathcal{O}_X)$ is a quasi-coherent module we find that the restriction to V is given by $\omega_{B/A} = \mathrm{Hom}_A(B', A) \otimes_{B'} B$ as desired.

The statement about restriction maps signifies that the restriction mappings of the quasi-coherent $\mathcal{O}_{Y'}$-module $H^0(a'(\mathcal{O}_X))$ for opens in Y' agrees with the maps defined in Lemma 2.3 for the modules $\omega_{B/A}$ via the isomorphisms given above. This is clear.

Let $f : Y \to X$ be a quasi-finite morphism of schemes of finite type over a Noetherian base S endowed with a dualizing complex $\omega_S^\bullet$. Consider opens $V \subset Y$ and $U \subset X$ with $f(V) \subset U$ and V and U separated over S. Denote $f|_V : V \to U$ the restriction of f. By the discussion above and Duality for Schemes, Lemma 20.9 there are canonical isomorphisms

$$H^0(f^!_{new}\mathcal{O}_X)|_V = H^0((f|_V)^!\mathcal{O}_U) = \omega_{V/U} = \omega_{Y/X}|_V$$

We omit the verification that these isomorphisms glue to a global isomorphism $H^0(f^!_{new}\mathcal{O}_X) \to \omega_{Y/X}$. $\square$

Lemma 15.2. *Let $f : Y \to X$ be a finite flat morphism of Noetherian schemes. The map*

$$\mathrm{Trace}_f : f_*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$$

of Section 3 corresponds to a map $\mathcal{O}_Y \to f^!\mathcal{O}_X$ (see proof). Denote $\tau_{Y/X} \in H^0(Y, f^!\mathcal{O}_X)$ the image of 1. Via the isomorphism $H^0(f^!\mathcal{O}_X) = \omega_{X/Y}$ of Lemma 15.1 this agrees with the construction in Remark 4.7.

Proof. The functor $f^!$ is defined in Duality for Schemes, Section 16. Since f is finite (and hence proper), we see that $f^!$ is given by the right adjoint to pushforward for f. In Duality for Schemes, Section 11 we have made this adjoint explicit. In particular, the object $f^!\mathcal{O}_X$ consists of a single cohomology sheaf placed in degree 0 and for this sheaf we have

$$f_* f^!\mathcal{O}_X = \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$$

To see this we use also that $f_*\mathcal{O}_Y$ is finite locally free as f is a finite flat morphism of Noetherian schemes and hence all higher Ext sheaves are zero. Some details omitted. Thus finally

$$\mathrm{Trace}_f \in \mathrm{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X) = \Gamma(X, f_* f^!\mathcal{O}_X) = \Gamma(Y, f^!\mathcal{O}_X)$$

On the other hand, we have $f^!\mathcal{O}_X = \omega_{Y/X}$ by the identification of Lemma 15.1. Thus we now have two elements, namely Trace_f and $\tau_{Y/X}$ from Remark 4.7 in

$$\Gamma(Y, f^!\mathcal{O}_X) = \Gamma(Y, \omega_{Y/X})$$

and the lemma says these elements are the same.

Let $U = \operatorname{Spec}(A) \subset X$ be an affine open with inverse image $V = \operatorname{Spec}(B) \subset Y$. Since f is finite, we see that $A \to B$ is finite and hence the $\omega_{Y/X}(V) = \operatorname{Hom}_A(B, A)$ by construction and this isomorphism agrees with the identification of $f_* f^! \mathcal{O}_Y$ with $\mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{O}_X)$ discussed above. Hence the agreement of Trace_f and $\tau_{Y/X}$ follows from the fact that $\tau_{B/A} = \operatorname{Trace}_{B/A}$ by Lemma 4.3. $\qquad\square$

16. Quasi-finite Gorenstein morphisms

This section discusses quasi-finite Gorenstein morphisms.

Lemma 16.1. *Let $f : Y \to X$ be a quasi-finite morphism of Noetherian schemes. The following are equivalent*

(1) *f is Gorenstein,*
(2) *f is flat and the fibres of f are Gorenstein,*
(3) *f is flat and $\omega_{Y/X}$ is invertible (Remark 2.11),*
(4) *for every $y \in Y$ there are affine opens $y \in V = \operatorname{Spec}(B) \subset Y$, $U = \operatorname{Spec}(A) \subset X$ with $f(V) \subset U$ such that $A \to B$ is flat and $\omega_{B/A}$ is an invertible B-module.*

Proof. Parts (1) and (2) are equivalent by definition. Parts (3) and (4) are equivalent by the construction of $\omega_{Y/X}$ in Remark 2.11. Thus we have to show that (1)-(2) is equivalent to (3)-(4).

First proof. Working affine locally we can assume f is a separated morphism and apply Lemma 15.1 to see that $\omega_{Y/X}$ is the zeroth cohomology sheaf of $f^! \mathcal{O}_X$. Under both assumptions f is flat and quasi-finite, hence $f^! \mathcal{O}_X$ is isomorphic to $\omega_{Y/X}[0]$, see Duality for Schemes, Lemma 21.6. Hence the equivalence follows from Duality for Schemes, Lemma 25.10.

Second proof. By Lemma 10.2, we see that it suffices to prove the equivalence of (2) and (3) when X is the spectrum of a field k. Then $Y = \operatorname{Spec}(B)$ where B is a finite k-algebra. In this case $\omega_{B/A} = \omega_{B/k} = \operatorname{Hom}_k(B, k)$ placed in degree 0 is a dualizing complex for B, see Dualizing Complexes, Lemma 15.8. Thus the equivalence follows from Dualizing Complexes, Lemma 21.4. $\qquad\square$

Remark 16.2. Let $f : Y \to X$ be a quasi-finite Gorenstein morphism of Noetherian schemes. Let $\mathfrak{D}_f \subset \mathcal{O}_Y$ be the different and let $R \subset Y$ be the closed subscheme cut out by $\mathfrak{D}_f$. Then we have

(1) $\mathfrak{D}_f$ is a locally principal ideal,
(2) R is a locally principal closed subscheme,
(3) $\mathfrak{D}_f$ is affine locally the same as the Noether different,
(4) formation of R commutes with base change,
(5) if f is finite, then the norm of R is the discriminant of f, and
(6) if f is étale in the associated points of Y, then R is an effective Cartier divisor and $\omega_{Y/X} = \mathcal{O}_Y(R)$.

This follows from Lemmas 9.3, 9.4, and 9.7.

Remark 16.3. Let S be a Noetherian scheme endowed with a dualizing complex $\omega_S^\bullet$. Let $f : Y \to X$ be a quasi-finite Gorenstein morphism of compactifyable schemes over S. Assume moreover Y and X Cohen-Macaulay and f étale at the

generic points of Y. Then we can combine Duality for Schemes, Remark 23.4 and Remark 16.2 to see that we have a canonical isomorphism

$$\omega_Y = f^*\omega_X \otimes_{\mathcal{O}_Y} \omega_{Y/X} = f^*\omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(R)$$

of $\mathcal{O}_Y$-modules. If further f is finite, then the isomorphism $\mathcal{O}_Y(R) = \omega_{Y/X}$ comes from the global section $\tau_{Y/X} \in H^0(Y, \omega_{Y/X})$ which corresponds via duality to the map $\mathrm{Trace}_f : f_*\mathcal{O}_Y \to \mathcal{O}_X$, see Lemma 15.2.

17. Other chapters

References

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